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*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

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WAITING TIMES AND GENERALIZED FIBONACCI SEQUENCES

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1. INTRODUCTION AND SUMMARY

Suppose we consider the following experiment: Toss a coin until we observe two heads in succession for the first time. One may ask for the probability of this event. Intuitively, one feels that the solution to this problem may be related to the Fibonacci sequence; and, in fact, this is so. More generally, one may be interested in finding the probability distribution of the waiting time to find r heads in succession for the first time. As one may guess, these results contain generalized Fibonacci, Tribonacci, ..., sequences. This problem was studied by Turner [8], who expressed the probability distribution in terms of generalized Fibonacci-T sequences which, in turn, were expressed in terms of generalized Pascal-T triangles. In this paper, we will express the probability distribution of this waiting time as a difference of two sums (Proposition 2.1). This result enables us to express Fibonacci numbers, Tribonacci numbers, etc., and their generalizations as sums of weighted binomial coefficients.

In probability literature (Feller [2]), the probability generating functions of waiting times of this type are well known. We derive Proposition 2.1 from one of these generating functions. In Section 3 we illustrate how one can obtain further generalizations of Fibonacci-T sequences by using the probability generating functions of the waiting times associated with different events of interest. Finally, starting with the generating function, we obtain new formulas for Tribonacci numbers.

2. THE PROBABILITY DISTRIBUTIONS OF WAITING TIMES

Suppose there are k possible outcomes on each trial (denoted by E_1, E_2, ..., E_k) with probabilities π_1, π_2, ..., π , respectively, such that $\pi_i \geqslant 0$ and $\pi_1 + \pi_2 + \cdots + \pi_k = 1$. At each trial, exactly one of the outcomes is observed. After n independent trials, we are interested in finding the

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probability of the first occurrence of r specified outcomes in succession. Let $\underline{E}_r$ denote this event, and W_r denote the waiting time for the first occurrence of $\underline{E}_r$. We are interested in the distributional properties of W_r.

Suppose $\underline{E}_r = \{E_1E_1 \ldots E_1\}$, which corresponds to the occurrence of the same outcome E_1, r times in a row. Then we have the following:

Proposition 2.1

The probability distribution of the discrete random variable W_r, denoted by f_{n+r}, is given by

$$P[W_r = n + r] = \pi_1^r \sum_{j=0}^{\infty} (-1)^j \binom{n - jr}{j} ((1 - \pi_1)\pi_1^r)^j$$
$$- \pi_1^{r+1} \sum_{j=0}^{\infty} (-1)^j \binom{n - 1 - jr}{j} ((1 - \pi_1)\pi_1^r)^j, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

where we define $\binom{m}{k} = 0$ if $m < k$ or $m < 0$.

The derivation of this proposition will be given in a later section. We discuss the generalities of this result now. If there are two possible outcomes (i.e., $k = 2$) with $\pi_1 = \pi_2 = \frac{1}{2}$, then we define

$$\beta_{n,r} = \begin{cases} 1 & n = 0 \\ 2^{n+r}P[W_r = n + r] = A_{n,r} - A_{n-1,r}, & n \geqslant 1 \end{cases} \tag{2.2}$$

where
$$A_{n,r} = 2^n \sum_{j=0}^{\infty} (-1)^j \binom{n - jr}{r} (1/2^{(r+1)j}),$$

with
$$A_{j,r} = 2^j, \text{ for } 0 \leqslant j \leqslant r. \tag{2.3}$$

We shall show later that the sequences $\{\beta_{n,r}\}$ are generalized Fibonacci sequences. Specifically, for $r = 2$, $\{\beta_{n,2}\}$ is the Fibonacci sequence given by 1, 1, 2, 3, 5, 8, 13, ... For $r = 3$, we have the so-called Tribonacci sequence (Feinberg [1]), given by 1, 1, 2, 4, 7, 13, 24, 44, ... For $r = 4$, one can verify that

$$\beta_{n+4,4} = \beta_{n+3,4} + \beta_{n+2,4} + \beta_{n+1,4} + \beta_{n,4}, \tag{2.4}$$

and the sequence $\{\beta_{n,4}\}$ is given by 1, 1, 2, 4, 8, 15, ... For general r, we have

$$\beta_{n+r,r} = \beta_{n+r-1,r} + \beta_{n+r-2,r} + \cdots + \beta_{n,r}, \tag{2.5}$$

which is an rth order Fibonacci-T sequence.

If we leave k unspecified but still require $\pi_1 = \pi_2 = \cdots = \pi_k = 1/k$, then we can define

$$\beta_{n,r}^{(k)} = k^{n+r} [W_r = n + r] \tag{2.6}$$

so that, using Proposition 2.1, we get

$$\beta_{n,r}^{(k)} = A_{n,r}^{(k)} - A_{n+1,r}^{(k)} \tag{2.7}$$

where
$$A_{n,r}^{(k)} = k^n \sum_{j=0}^{\infty} (-1)^j \binom{n - jr}{j} \left[\frac{(k-1)}{(k^{r+1})}\right]^j. \tag{2.8}$$

We prove in Section 3 that

$$\beta_{n+r,r}^{(k)} = (k-1)\left[\beta_{n+r-1,r}^{(k)} + \beta_{n+r-2,r}^{(k)} + \cdots + \beta_{n,r}^{(k)}\right] \tag{2.9}$$

with the boundary conditions

$$\beta_{r,r}^{(k)} = 1 \quad \text{and} \quad \beta_{s,r}^{(k)} = 0 \quad \text{for } s < r;$$

and for the special case $k = 2$, (2.9) gives the recursion satisfied by the rth order Fibonacci-T sequence given in (2.5). For $r = 2$ and $k = 3$, the sequence $\{\beta_{n,2}^{(3)}\}$ is given by 1, 2, 6, 16, 44, 120, ... For $r = 3$ and $k = 3$, the sequence $\{\beta_{n,3}^{(3)}\}$ is given by 1, 2, 6, 18, 52, 152, 444, ...

3. THE PROBABILITY GENERATING FUNCTIONS OF WAITING TIMES

In this section we shall give a derivation of Proposition 2.1, starting from the probability generating function of the waiting times for recurrent events and then prove equation (2.9). Following Feller [2], the generating function given for binomial processes can easily be extended to multinomial processes, for the events of type $\underline{E_r}$ considered in this paper. In particular, the probability generating function of the first occurrence of $\underline{E_r}$ discussed in Section 2, is given by

$$\begin{aligned} F(s) &= \sum_{n=0}^{\infty} s^{n+r} P[W_r = n + r] = \frac{\pi_1^r s^r (1 - \pi_1 s)}{1 - s + (1 - \pi_1)\pi_1^r s^{r+1}} \\ &= \frac{\pi_1^r s^r}{1 - s + (1 - \pi_1^r)\pi_1 s^{r+1}} - \frac{\pi_1^{r+1} s^{r+1}}{1 - s + (1 - \pi_1^r)\pi_1 s^{r+1}}, \\ &= \text{(i)} - \text{(ii)}. \end{aligned} \tag{3.1}$$

Let $\theta = (1 - \pi_1)\pi_1^r$, then

$$\begin{aligned} \text{(i)} &= \frac{\pi_1^r s^r}{1 - s(1 - s^r\theta)} = \pi_1^r s^r [1 + s(1 - s^r\theta) + s^2(1 - s^r\theta)^2 + \cdots \\ &\quad + s^r(1 - s^r\theta)^r + \cdots + s^{(j-1)r}(1 - s^r\theta)^{(j-1)r} + \cdots]. \end{aligned} \tag{3.2}$$

In (3.2), s^{jr} appears only in the following $(j - 1)$ terms:

$$\pi_1^r s^{jr}(1 - s^r\theta)^{(j-1)r},\ \pi_1^r s^{(j-1)r}(1 - s^r\theta)^{(j-2)r},\ \ldots\ \pi_1^r s^{2r}(1 - s^r\theta)^r;$$

and the coefficient of s^{jr} in (i) is given by

$$\left\{\binom{(j-1)r}{0} - \binom{(j-2)r}{1}\theta + \binom{(j-3)r}{2}\theta^2 \cdots + (-1)^{j-2}\binom{r}{j-2}\theta^{j-2}\right\}\pi_1^r. \tag{3.3}$$

More generally, $s^{jr+\ell}$, $0 \leqslant \ell \leqslant r-1$, appears in (3.2) only in the following $(j-1)$ terms:

$$\pi_1^r s^{jr+\ell}(1 - s^r\theta)^{(j-1)r+\ell},\ \pi_1^r s^{(j-1)r+\ell}(1 - s^r\theta)^{(j-2)r+\ell},$$

$$\pi_1^r s^{(j-2)r+\ell}(1 - s^r\theta)^{(j-3)r+\ell},\ \ldots,\ \pi_1^r s^{2r+\ell}(1 - s^r\theta)^{r+\ell};$$

and the coefficient of $s^{jr+\ell}$, $0 \leqslant \ell \leqslant r-1$, in (3.2) is given by

$$\left\{\binom{(j-1)r+\ell}{0} - \binom{(j-2)r+\ell}{1}\theta + \binom{(j-3)r+\ell}{2}\theta^2 \right.$$
$$\left. \cdots + (-1)^{j-2}\binom{r+\ell}{j-2}\theta^{j-2}\right\}\pi_1^r. \tag{3.4}$$

Since f_{n+r} is equal to the sum of the coefficients of s^{n+r} in (i) and (ii), taking $n = (j-1)r+\ell$ in the above, we obtain:

$$f_{n+r} = \left\{\binom{n}{0} - \binom{n-r}{1}\theta + \binom{n-2r}{2}\theta^2 \cdots\right\}\pi_1^r$$
$$- \left\{\binom{n-1}{0} - \binom{n-1-r}{1}\theta + \binom{n-1-2r}{2}\theta^2 \cdots\right\}\pi_1^{r+1}, \tag{3.5}$$

which proves Proposition 2.1.

The probability generating function given by (3.1) can also be written in the form

$$F(s) = 1\Big/\left[1 + (1-s)\left[\frac{1}{s\pi_1} + \left(\frac{1}{s\pi_1}\right)^2 + \cdots + \left(\frac{1}{s\pi_1}\right)^r\right]\right], \tag{3.6}$$

which may be recognized as a special case of the probability generating function discussed by Johnson [5] and Johnson & Kotz [6]. In order to summarize these results, we need to introduce some notation.

Returning to the situation introduced in Section 2, suppose we are interested in a specific event $\underline{E}_r$ of length r (or r independent outcomes). We shall now obtain the probability generating function for the waiting time, W_r, which denotes the first occurrence associated with the event $\underline{E}_r$. As a first step, we introduct the definition of the critical points of $\underline{E}_r$, as defined by Johnson [5].

Definition: A critical point of $\underline{E}_r$ is defined as the position between two labels, such that the subsequence of labels up to that position is identical to the subsequence of labels of the same length concluding the pattern. Also, a critical point always follows the last trial at which event $\underline{E}_r$ occurs.

As an illustration, suppose we toss a coin so that we have the two possible outcomes, Heads and Tails, denoted by the labels H and T, respectively. For a given pattern like HTHTH, we can observe three critical points. Since the last trial completes the pattern, it precedes a critical point. At the third trial of this pattern, we have a H, and the subsequence HTH up to the third trial is the same as the subsequence at the end of the pattern and, hence, the third trial precedes a critical point. And finally, at the first trial of this pattern we have a H, and we have a H at the end, so the first trial also precedes a critical point.

Let us consider another pattern E_7, defined by HHTHHHT, which has only two critical points. For this pattern, the seventh trial (by definition) and the third trial precede the two critical points.

More generally, let the event of interest, $\underline{E}_r$, have $c(1 \leqslant c \leqslant r)$ critical points. Let $a_{\alpha t}$ denote the number of outcomes E_α observed up to the tth critical point, for $\alpha = 1, 2, \ldots, k$ and $t = 1, 2, \ldots, c$. Then the probability generating function $F(s)$ of W_r, as given by Johnson [5], is

$$F(s) = 1 \Big/ \left[1 + (1 - s)\sum_{t=1}^{c} \frac{1}{\left(s^{a_{1t} + a_{2t} + \cdots + a_{kt}}\right)}\left\{\prod_{\alpha=1}^{k} \pi_\alpha^{-a_{\alpha t}}\right\}\right]. \tag{3.7}$$

Special Cases

(1) When the event of interest $\underline{E}_r$ is given by a succession of r identical events E_1, then there are r critical points associated with this event; and associated with the first critical point, we have

$$a_{11} = 1,\ a_{21} = 0,\ \ldots,\ a_{k1} = 0,$$

and associated with the tth critical point, we have

$$a_{1t} = t,\ a_{\alpha t} = 0,\ \alpha = 2,\ \ldots,\ k \text{ for } t = 2,\ \ldots,\ r.$$

In this case, the probability generating function of the event of length r, given by $E_1E_1 \ldots E_1$ reduces to

$$F(s) = 1 \Big/ \left[1 + (1 - s)\sum_{t=1}^{r} \frac{1}{s^t}\frac{1}{\pi_1^t}\right], \tag{3.8}$$

which agrees with (3.6).

Next, taking $\pi_1 = 1/k$, we shall derive (2.9). We have

$$\begin{aligned} F(s) &= \sum_{n=0}^{\infty} s^{n+r} P[W_r = n + r] \\ &= \sum_{n=0}^{\infty} (s/k)^{n+r}\beta_{n,r}^{(k)} && \text{from (2.6)} \\ &= 1 \Big/ \left[1 + (1 - s)\left(\frac{k}{s} + \frac{k^2}{s^2} + \cdots + \frac{k^r}{s^r}\right)\right] && \text{from (3.8)} \\ &= 1 \Big/ \left[\frac{k^r}{s^r} - (k - 1)\left(1 + \frac{k}{s} + \frac{k^2}{s^2} + \cdots + \frac{k^{r-1}}{s^{r-1}}\right)\right]. \end{aligned}$$

Therefore, we have the relation

$$\left[\sum_{n=0}^{\infty} (s/k)^{n+r}\beta_{n,r}^{(k)}\right]\left[\frac{k^r}{s^r} - (k - 1)\left(1 + \frac{k}{s} + \cdots + \frac{k^{r-1}}{s^{r-1}}\right)\right] = 1,$$

From which it follows that

$$\sum_{n=0}^{\infty}(s/k)^n \beta_{n,r} = 1 + (k-1)\left(1 + \frac{k}{s} + \cdots + \frac{k^{r-1}}{s^{r-1}}\right)\sum_{n=0}^{\infty}(s/k)^{n+r}\beta_{n,r}^{(k)}.$$

Equating the coefficients of s^{n+r}, on both sides, we find

$$\beta_{n+r,r}^{(k)} = (k-1)\left[\beta_{n+r-1,r}^{(k)} + \beta_{n+r-2,r}^{(k)} + \cdots + \beta_{n,r}^{(k)}\right],$$

which proves (2.9).

(2) Let the event of interest be $E_1E_2 \ldots E_k$, which is of length k, and the outcomes occur in the specified order. This event has only one critical point, and

$$a_{11} = 1 = a_{21} = \cdots = a_{k1},$$

and all others are zero. In this special case, the probability generating function is given by

$$F(s) = 1\Big/\left[1 + (1-s)\left(\frac{1}{s^k\pi_1 \ldots \pi_k}\right)\right]. \tag{3.9}$$

(3) Let $k = 2$ and the event of interest be $E_1E_1E_1$ (of length 3) and let $\pi_1 = \frac{1}{2} = \pi_2$. In this case, there are $c = 3$ critical points and

$$a_{11} = 1,\ a_{12} = 2,\ a_{13} = 3,$$

$$a_{21} = 0,\ a_{22} = 0,\ a_{23} = 0.$$

With these values,

$$\begin{aligned}\sum_{n=0}^{\infty} s^{n+3}P[W_3 = n+3] = F(s) &= 1\Big/\left[1 + (1-s)\sum_{t=1}^{3}(2/s)^t\right] \\ &= 1\Big/\left[1 + (1-s)\left(\frac{2}{s} + \frac{4}{s^2} + \frac{8}{s^3}\right)\right] \qquad (3.10) \\ &= \frac{s^3}{s^3 + 2(1-s)(s^2+2s+4)}.\end{aligned}$$

From this, we obtain

$$\sum_{n=0}^{\infty} t^{n+3}2^{n+3}P[W_3 = n+3] = F(2t) = t^3/[1 - t - t^2 - t^3],$$

and, as defined in (2.2),

$$2^{n+3}P[W_3 = n+3] = \beta_{n,3}, \tag{3.11}$$

which are the Tribonacci numbers.

From this generating function of the Tribonacci numbers, we obtain a representation for $\beta_{n,3}$ in terms of trigonometric functions, which is stated in the following proposition.

Proposition 3.1

The Tribonacci numbers $\beta_{n,3}$ are given by

$$\beta_{n,3} = \frac{1}{(c-1)(c+3)}\left[c^{1+(n/2)}\left\{\frac{\sin(n+1)\theta}{\sin\theta} - \frac{c^{3/2}\sin n\theta}{\sin\theta}\right\} - \frac{1}{c^{n-1}}\right]$$

for $n = 2, 3, \ldots,$ where

$$c = (1/3)[(\sqrt{297}+17)^{1/3} - (\sqrt{297}-17)^{1/3} - 1]$$

and $\theta = \pi - \text{Arc}\sin(\sqrt{3 - c^2})/2$, and $\beta_{0,3}$ and $\beta_{1,3}$ are defined to be equal to 1. From (3.11), we note that $\beta_{n,3}$ is given by the coefficient of t^{n-1} in $1/(1 - t - t^2 - t^3)$. In order to find this coefficient, we use partial fractions given by

$$\frac{1}{1 - t - t^2 - t^3} = \frac{C}{(c-t)} + \frac{D}{(d-t)} + \frac{G}{(g-t)}.$$

Let c, d, and g denote the real and the complex conjugate roots of the cubic $1 - t - t^2 - t^3 = 0$, given by

$$c = (1/3)(\gamma - \delta - 1),$$

$$d = (-1/6)(\gamma - \delta - 2) + (\sqrt{3}/6)i(\gamma + \delta) = (1/\sqrt{c})e^{i\theta},$$

and $$g = (1/\sqrt{c})e^{-i\theta}$$

where $\gamma = (\sqrt{297}+17)^{1/3}$, $\delta = (\sqrt{297}-17)^{1/3}$, and $i = \sqrt{-1}$. Now, C, D, and G can be expressed in terms of c, d, and g, and we obtain

$$\begin{aligned}\frac{1}{1 - t - t^2 - t^3} &= \frac{1}{(d-c)(g-c)c}\left[1 + \frac{t}{c} + \frac{t^2}{c^2} + \frac{t^3}{c^3} + \cdots + \frac{t^{n-1}}{c^{n-1}} + \cdots\right]\\ &+ \frac{1}{(c-d)(d-g)d}\left[1 + \frac{t}{d} + \frac{t^2}{d^2} + \cdots + \frac{t^{n-1}}{d^{n-1}} + \cdots\right]\\ &+ \frac{1}{(c-g)(d-g)g}\left[1 + \frac{t}{g} + \frac{t^2}{g^2} + \cdots + \frac{t^{n-1}}{g^{n-1}} + \cdots\right].\end{aligned}$$

Therefore, $\beta_{n,3}$ can be obtained as the coefficient of t^{n-1}, given by

$$\begin{aligned}\beta_{n,3} &= \frac{1}{c(c-d)(g-c)}\left[-\frac{1}{c^{n-1}} + \frac{c(g-c)}{(g-d)d^n} - \frac{c(c-d)}{(d-g)g^n}\right]\\ &= \frac{-1}{c(c-d)(g-c)}\left[\frac{1}{c^{n-1}} + \frac{(cg-c^2)}{(d-g)}c^n g^n + \frac{(c^2-cd)}{(d-g)}c^n d^n\right]\\ &\qquad\text{(here we use the fact that } cdg = 1)\\ &= \frac{-1}{c(c-d)(g-c)}\,\frac{1}{c^{n-1}}\\ &\quad + \frac{1}{c(c-d)(g-c)}\left[\frac{c^{n+1}(g^{n+1} - d^{n+1})}{(g-d)} - \frac{c^{n+2}(g^n - d^n)}{(g-d)}\right].\end{aligned}$$

The following identities between the roots c, d, and g can be verified.

$$\text{(i)} \quad c(c - d)(g - c) = (c - 1)(c + 3)$$

and

$$\text{(ii)} \quad \frac{\sin(k + 1)\theta}{\sin \theta} = \frac{d^{k+1} - g^{k+1}}{d - g},$$

where θ is as defined in Proposition (3.1).

Using these properties, we find

$$(c - 1)(c + 3)\beta_{n,3} = c^{1+(n/2)}\left\{\frac{\sin(n + 1)\theta}{\sin \theta} - \frac{c^{3/2} \sin n\theta}{\sin \theta}\right\} - \frac{1}{c^{n-1}},$$

for $n = 2, 3, \ldots$. This representation corresponds to the "Golden Number" representation of the Fibonacci numbers.

4. REMARKS

We wish to thank a referee for bringing to our attention the article by Philippou & Muwafi [6], which also deals with the waiting time problem for the kth consecutive success of a Bernoulli process. There is not much of an overlap with our results, and the references cited by these authors may be of historical interest to the reader.

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POWERS OF T AND SODDY CIRCLES

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(Submitted April 1982)

1. INTRODUCTION

T is the real root of the equation $T^3 - T^2 - T - 1 = 0$, and is approximately equal to 1.8392867... T has the property:

$$T^{n-3} + T^{n-2} + T^{n-1} = T^n,$$

which is similar to the formula that defines the Tribonacci numbers:

$$t(n - 3) + t(n - 2) + t(n - 1) = t(n).$$

In fact, T has a relationship to the Tribonacci numbers similar to that between ϕ and the Fibonacci numbers. Binet's formula for calculating the value of the nth Fibonacci number is

$$f(n) = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}.$$

Since $\phi^{-1} = .618\ldots < 1$, we can see that the ratio between two adjacent Fibonacci numbers is a close approximation to ϕ, and moreso as the value of n increases:

$$f(n + 1)/f(n) = (\phi^{n+1} - (-\phi)^{-(n+1)})/(\phi^n - (-\phi)^{-n}) \to \phi \text{ as } n \to \infty.$$

Similarly, given Binet's formula for deriving a Tribonacci number $t(n)$:

$$t(n) = \alpha T^n + r^n(\beta \cos n\theta + \gamma \sin n\theta) \qquad \text{(see [1])},$$

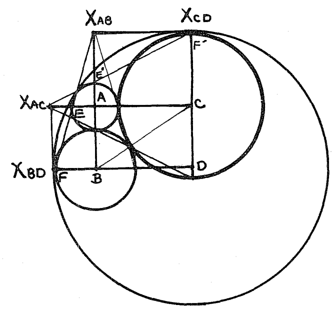
and since $|r| = .7374\ldots < 1$, we can see that the value of the ratio of two adjacent Tribonacci numbers is a close approximation to T, and moreso as the value of n increases:

$$t(n + 1)/t(n) = (\alpha T^{n+1} + r^{n+1}(\beta \cos n\theta + \gamma \sin n\theta))/$$
$$(\alpha T^n + r^n(\beta \cos n\theta + \gamma \sin n\theta)) \to T \text{ as } n \to \infty.$$

2. A GEOMETRIC APPLICATION OF T

If three circles are externally tangent to each other, and the radii of each are three successive powers of T, then a fourth circle, internally tangent to all three has a radius equal to the next higher power of T.

Proof: Given the three circles with centers A, B, and C:



$$rA = T^n$$
$$rB = T^{n+1}$$
$$rC = T^{n+2}$$

Since $$(AB)^2 = (T^n + T^{n+1})^2 = T^{2n} + 2T^{2n+1} + T^{2n+2}$$

and $$(AC)^2 = (T^n + T^{n+2})^2 = T^{2n} + 2T^{2n+2} + T^{2n+4},$$

then $$(AB)^2 + (AC)^2 = 2(T^{2n} + T^{2n+1} + T^{2n+2}) + T^{2n+2} + T^{2n+4}$$
$$= T^{2n+2} + 2T^{2n+3} + T^{2n+4}.$$

And since $$(BC)^2 = (T^{n+1} + T^{n+2})^2 = T^{2n+2} + 2T^{2n+3} + T^{2n+4},$$

then $$(AB)^2 + (AC)^2 = (BC)^2.$$

Triangle ABC is a right triangle; angle BAC = 90 degrees. Extend CA to E on the circumference of circle A. Draw BF parallel to AC; F is on the circumference of circle B. Extend FE to meet AB extended at X_{AB}, which is the external center of similitude for circles A and B.

Then, if $X_{AB}A = X$, an unknown, and

$$AE/FB = X/(X + AB)$$

and given the aforementioned values for AB, $AE = rA$, and $FB = rB$, then

$$T^n/T^{n+1} = X/(X + T^n + T^{n+1})$$
$$XT^{n+1} = XT^n + T^{2n} + T^{2n+1}$$
$$X(T^{n+1} - T^n) = T^{2n} + T^{2n+1}.$$

If we define

$$d = T^n/(T^{n+1} - T^n) = T^n/(T^{n-1} + T^{n-2}),$$

then

$$T^{n+1} - T^n = T^{n-1} + T^{n-2} = T^n/d$$

and

$$T^{2n} + T^{2n+1} = T^{2n+2}/d;$$

therefore,

$$X = (T^{2n} + T^{2n+1})/(T^{n+1} - T^n) = (T^{2n+2}/d)(T^n/d) = T^{n+2} = rC.$$

Where a tangent from X_{AB} touches the circumference of circle C is the external center of similitude between circle C and the fourth circle (X_{CD}), which is where they are internally tangent; a line drawn from X_{CD} through C will contain the center of the fourth circle, D. Since $X_{AB}A$ is perpendicular to AC and equal to rC, $X_{CD}C$ is parallel to AB and also perpendicular to AC.

We can also construct the point X_{BD} in the same manner; $X_{BD}B$ will be found to be perpendicular to AB and parallel to AC. So D is located at a point such that BD is parallel and equal to AC and perpendicular to AB and CD; AB and CD are in turn parallel and equal to each other.

The definition of the construction of this fourth circle, D, is that it is tangent to each of the other three circles at a point where a line from the external center of similitude of the other two circles in each case is tangent to it. We do not need to construct point X_{AD} to locate point D.

Therefore, since

$$rD = rC + CD = rB + BD,$$

and having shown that

$$CD = AB = rA + rB$$

and that

$$BD = AC = rA + rC,$$

then

$$rD = rA + rB + rC = T^n + T^{n+1} + T^{n+2} = T^{n+3}.$$

Q.E.D.

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ON SOME DIVISIBILITY PROPERTIES OF FIBONACCI AND RELATED NUMBERS

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(Submitted April 1982)

1. Let x be an arbitrary natural number. We define, recursively, the following two sequences of rational integers.

$$S_{-1}(x) = -1,\ S_0(x) = 0,\ S_n(x) = xS_{n-1}(x) - S_{n-2}(x),\ n \geqslant 1. \tag{1}$$

$$R_{-1}(x) = 1,\ R_0(x) = 0,\ R_n(x) = xR_{n-1}(x) + R_{n-2}(x),\ n \geqslant 1 \tag{2}$$

If $x = 1$ and $n \geqslant 0$, then $R_n(x)$ is the nth Fibonacci number. By mathematical induction, we immediately obtain

$$R_{2n}(x) = xS_n(x^2 + 2) \tag{3}$$

and

$$R_{2n-1}(x) = S_n(x^2 + 2) - S_{n-1}(x^2 + 2),\text{ where } n \in \mathbf{N} \cup \{0\}. \tag{4}$$

The purpose of this note is to look at some divisibility properties of the natural numbers $R_n(x)$ that are of great interest to some subgroup problems for the general linear group $GL(2, \mathbf{Z})$.

Of the many papers dealing with divisibility properties for Fibonacci numbers, perhaps the most useful are those of Bicknell [1], Bicknell & Hoggatt [2], Hairullin [4], Halton [5], Hoggatt [6], Somer [9], and the papers which are cited in these. Numerical results are given in [3]. Some of our results are known or are related to known results but are important for our purposes. As far as I know, the other results presented here are new or are at least generalizations of known results.

2. Let p be a prime number. Let $n(p, x)$ be the subscript of the first positive number $R_n(x)$, $n \geqslant 1$, divisible by p.

If p divides x, then

$$n(p, x) = 2.$$

If $p = 2$ and x is odd, then

$$n(p, x) = 3.$$

Henceforth, let p always be an odd prime number that does not divide x. Then it is known that $n(p, x)$ divides $p - \varepsilon$, $\varepsilon = 0, 1$, or -1, where

$$\varepsilon = \left(\frac{x^2 + 4}{p}\right)$$

is Legendre's symbol (cf., for instance, [7]).

We want to prove some more intrinsic results about $n(p, x)$. For this we make use of the next five identities; the proof of these identities is computational.

$$R_{n+3}(x) = (x^2 + 2)R_{n+1}(x) - R_{n-1}(x); \tag{5}$$

$$R_{kn}(x) = S_k\big(R_{n+1}(x) + R_{n-1}(x)\big) \cdot R_n(x) \text{ if } n \text{ is even}, \tag{6a}$$

$$R_{kn}(x) = R_k\big(R_{n+1}(x) + R_{n-1}(x)\big) \cdot R_n(x) \text{ if } n \text{ is odd}; \tag{6b}$$

$$R_{n+1}(x)R_{n-1}(x) - R_n^2(x) = (-1)^n; \tag{7}$$

$$R_{n+2}^2(x) - R_{n+4}(x)R_n(x) = (-1)^n x^2; \tag{8}$$

$$R_{2n-1}(x) = R_n^2(x) + R_{n-1}^2(x), \tag{9a}$$

$$xR_{2n}(x) = R_{n+1}^2(x) - R_{n-1}^2(x); \tag{9b}$$

where $n \in \mathbf{N} \cup \{0\}$.

3. The case $n(p, x)$ odd.

Let $n(p, x) = 2m - 1$, $m \in \mathbf{N}$; it is $m \geqslant 2$.

Proposition 1

a. $R_{2m+1}(x) \equiv -R_{2m-3}(x) \pmod p$.

b. $R_{2m-3}^2(x) \equiv -x^2 \pmod p$.

c. $R_{2m-2}^2(x) \equiv -1 \pmod p$.

d. $R_{2m-1-k}(x) \equiv (-1)^{k+1}R_k(x)R_{2m-2}(x) \pmod p$ for all integers k with $0 \leqslant k \leqslant 2m - 1$.

Proof: Statements (a), (b), and (c) follow directly from (3), (5), (7), and (8).

We now prove statement (d) by mathematical induction. Statement (d) is true for $k = 0$ and $k = 1$ because $R_{2m-1}(x) \equiv 0 \pmod p$ and $R_1(x) = 1$. Now we suppose that statement (d) is true for all integers ℓ with $0 \leqslant \ell \leqslant k$, where $1 \leqslant k < 2m - 1$.

For $1 \leqslant k < 2m - 1$ and k even, we obtain

$$\begin{aligned} R_{2m-1-(k+1)}(x) &\equiv -xR_{2m-1-k}(x) + R_{2m-1-(k-1)}(x) \\ &\equiv \big(xR_k(x) + R_{k-1}(x)\big) \cdot R_{2m-2}(x) \\ &\equiv (-1)^{k+2}R_{k+1}(x)R_{2m-2}(x) \pmod p. \end{aligned}$$

For $1 \leqslant k < 2m - 1$ and k odd, we obtain

$$\begin{aligned} R_{2m-1-(k+1)}(x) &\equiv \big(-xR_k(x) - R_{k-1}(x)\big) \cdot R_{2m-2}(x) \\ &\equiv (-1)^{k+2}R_{k+1}(x)R_{2m-2}(x) \pmod p. \end{aligned}$$

Q.E.D.

Corollary 1

$p \equiv 1 \pmod 4$.

Proof: Proposition 1 gives that -1 is a quadratic residue mod p. That means

$$1 = \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2},$$

and, therefore, $p \equiv 1 \pmod 4$. Q.E.D.

Proposition 2

If $p \equiv 1 \pmod 4$, then there is a natural number z such that

$$z^2 + 1 \equiv 0 \pmod p$$

and

$$(xz + 1)R_{m-1}^2(x) \equiv z^{2m} \pmod p.$$

Proof: From (9) we get

$$R_m^2(x) \equiv -R_{m-1}^2(x) \pmod p.$$

Then there is a natural number z such that

$$z^2 + 1 \equiv 0 \pmod p$$

and

$$R_m(x) \equiv zR_{m-1}(x) \pmod p.$$

Therefore,

$$R_{m+1}(x) \equiv xR_m(x) + R_{m-1}(x) \equiv (xz + 1)R_{m-1}(x) \pmod p$$

and

$$z^{2m} \equiv (-1)^m \equiv R_{m+1}(x)R_{m-1}(x) - R_m^2(x) \equiv (xz + 2)R_{m-1}^2(x) \pmod p$$

by (7). Q.E.D.

The following corollary is an immediate consequence.

Corollary 2

If $p \equiv 1 \pmod p$, then there is a natural number z such that

$$z^2 + 1 \equiv 0 \pmod p$$

and $xz + 2$ is a quadratic residue mod p.

Remark concerning Proposition 2: If $p = 4q + 1$, $q \geqslant 1$, and g is a primitive root mod p, then $z \equiv \pm g^q \pmod p$. But unfortunately, no direct method is known for calculating primitive roots in general without a great deal of computation, especially for large p.

Proposition 3

Let $n \geqslant 1$ be a natural number such that p divides $R_{2n-1}(x)$. Then

$$R_{2(k+1)-1}(x) \cdot S_{n-k}(x^2+2) \equiv R_{2k-1}(x) \cdot S_{n-(k+1)}(x^2+2) \pmod{p},$$

for all integers k with $0 \leqslant k \leqslant n$.

Proof by mathematical induction: The statement is true for $k = 0$, since

$$S_n(x^2+2) \equiv S_{n-1}(x^2+2) \pmod{p} \quad \text{[by (4)]}.$$

Now suppose the statement is true for an integer k with $0 \leqslant k < n$. Then we obtain

$$R_{2k-1}(x) \cdot S_{n-(k+1)}(x^2+2) \equiv R_{2k+1}(x) \cdot S_{n-k}(x^2+2)$$

$$\equiv \left((x^2+2)S_{n-(k+1)}(x^2+2) - S_{n-(k+2)}(x^2+2)\right) \cdot R_{2k+1}(x) \pmod{p}.$$

This gives

$$R_{2(k+1)-1}(x) \cdot S_{n-(k+2)}(x^2+2)$$

$$\equiv \left((x^2+2)R_{2k+1}(x) - R_{2k-1}(x)\right) \cdot S_{n-(k+1)}(x^2+2)$$

$$\equiv R_{2(k+1)-1}(x) \cdot S_{n-(k+1)}(x^2+2) \pmod{p} \quad \text{[by (5)]. Q.E.D.}$$

Corollary 3

a. $0 \not\equiv R_{2(m-1)-1}(x) \cdot S_{m-k}(x^2+2) \equiv R_{2k-1}(x) \pmod{p}$ for all integers k with $0 \leqslant k \leqslant m-1$.

b. $R_{2(k+\ell)-1}(x) \cdot S_{m-k}(x^2+2) \equiv R_{2k-1}(x) \cdot S_{m-(k+\ell)}(x^2+2) \pmod{p}$ for all integers k and ℓ with $0 \leqslant k$, $0 \leqslant \ell$, and $0 \leqslant k + \ell \leqslant m$.

Proof: Statement (b) is obviously true for $k = m$ (if $k = m$ then $\ell = 0$); statements (a) and (b) are also obviously true for $k = m - 1$. Now, letting $0 \leqslant k \leqslant m - 2$, we obtain (from Proposition 1)

$$R_{2k-1}(x) \cdot R_{2k+1}(x) \cdot S_{m-(k+2)}(x^2+2)$$

$$\equiv R_{2k-1}(x) \cdot R_{2k+3}(x) \cdot S_{m-(k+1)}(x^2+2)$$

$$\equiv R_{2k+3}(x) \cdot R_{2k+1}(x) \cdot S_{m-k}(x^2+2) \pmod{p},$$

which gives

$$R_{2(k+2)-1}(x) \cdot S_{m-k}(x^2+2) \equiv R_{2k-1}(x) \cdot S_{m-(k+2)}(x^2+2) \pmod{p}$$

because $R_{2k+1}(x) \not\equiv 0 \pmod{p}$.

Now, by mathematical induction, we obtain

$$R_{2(k+\ell)-1}(x) \cdot S_{m-k}(x^2+2) \equiv R_{2k-1}(x) \cdot S_{m-(k+\ell)}(x^2+2) \pmod{p}$$

for all integers k and ℓ with $0 \leqslant k$, $0 \leqslant \ell$, and $0 \leqslant k + \ell \leqslant m$ (this statement is trivial for $\ell = 0$ and just Proposition 3 for $\ell = 1$). Now statement (b) is proved; statement (a) follows for $k + \ell = m - 1$. Q.E.D.

4. The case $n(p, x)$ even. Let $n(p, x) = 2m$, $m \in \mathbf{N}$; it is $m \geqslant 2$ because p does not divide x. Moreover, $S_m(x^2 + 2) \equiv 0 \pmod p$ by (3).

Proposition 4

$$(x^2 + 4)R_{m-1}^2(x) \equiv (-1)^{m+1}x^2 \pmod p.$$

Proof: From (6), we get

$$-R_{m-1}(x) \equiv R_{m+1}(x) \equiv xR_m(x) + R_{m-1}(x) \pmod p$$

and

$$xR_m(x) \equiv -2R_{m-1}(x) \pmod p$$

because $n(p, x)$ is minimal. Therefore,

$$(-1)^m x^2 \equiv x^2(R_{m+1}(x)R_{m-1}(x) - R_m^2(x)) \equiv -(x^2 + 4)R_{m-1}^2(x) \pmod p$$

by (7). Q.E.D.

Corollary 4

If $p \equiv 1 \pmod 4$, then $x^2 + 4$ is a quadratic residue mod p.

Proof: If $p \equiv 1 \pmod 4$, then $\left(\frac{-1}{p}\right) = 1$ and the statement follows immediately from Proposition 4. Q.E.D.

If we ask for prime numbers p' with $p' \equiv 1 \pmod 4$ and $\left(\frac{x^2+4}{p'}\right) = -1$, we obtain the following.

Corollary 5 (Special Cases)

a. If $x = 1$, then $p \not\equiv q \pmod{20}$, where $q = 13$ or 17.

b. If $x = 2$ or 4, then $p \not\equiv 5 \pmod 8$.

c. If $x = 3$, then $p \not\equiv q \pmod{52}$, where $q = 5, 21, 33, 37, 41$, or 45.

d. If $x = 5$, then $p \not\equiv q \pmod{116}$, where $q = 17, 21, 37, 41, 61, 69, 73, 77, 85, 89, 97, 101, 105$, or 113.

Analogous to Proposition 1, Proposition 3, and Corollary 3, we obtain the following results.

Proposition 5

a. $R_{2m+2}(x) \equiv -R_{2m-2}(x) \pmod p$.

b. $R_{2m-2}^2(x) \equiv x^2 S_{m-1}(x^2 + 2) \equiv x^2 \pmod p$.

c. $R_{2m-1}^2(x) \equiv 1 \pmod p$.

d. $R_{2m-k}(x) \equiv (-1)^{k+1}R_k(x)R_{2m-1}(x) \pmod p$
for all integers k with $0 \leqslant k \leqslant 2m$.

Proposition 6

Let $n \geqslant 1$ be a natural number such that p divides $R_{2n}(x)$. Then

$$R_{2k}(x) \cdot S_{n-(k+1)}(x^2+2) \equiv R_{2(k+1)}(x) \cdot S_{n-k}(x^2+2) \pmod{p}$$

for all integers k with $0 \leqslant k \leqslant n$.

Corollary 6

a. $0 \not\equiv R_{2(m-1)}(x) \cdot S_{m-k}(x^2+2) \equiv R_{2k}(x) \pmod{p}$
for all integers k with $0 \leqslant k \leqslant m-1$.

b. $R_{2(k+\ell)}(x) \cdot S_{m-k}(x^2+2) \equiv R_{2k}(x) \cdot S_{m-(k+\ell)}(x^2+2) \pmod{p}$
for all integers k and ℓ with $0 \leqslant k$, $0 \leqslant \ell$, and $0 \leqslant k+\ell \leqslant m$.

5. Final Remark. I wish to thank the referee for two relevant references that were not included in the original version of the paper. He also noted that some results of this paper are special cases of results of Somer [9] for the sequence

$$T_0(x, y) = 0,\ T_1(x, y) = 1,\ T_n(x, y) = xT_{n-1}(x, y) + yT_{n-2}(x, y),\ n \geqslant 2,$$

where x and y are arbitrary rational integers. Proposition 1(c) is a special case of Somer's Theorem 8(i); Proposition 2 is a special case of his Lemma 3(i) and the proof of his Lemma 4 when one takes into account the hypothesis that $(-1)/(p) = 1$; Corollary 4 is a special case of Somer's Lemma 3(ii) and (iii); finally, Proposition 5(c) is a special case of his Theorem 8(i).

But, on the other side, some results of Somer's paper follow directly from known results about the numbers $S_n(x)$ and $R_n(x)$. For, let x and y now be arbitrary complex numbers with $y \neq 0$. Let $S_n(x)$, $R_n(x)$, and $T_n(x, y)$ be analogously defined as above. Then

$$T_n(x, y) = (\sqrt{-y})^{n-1} S_n\left(\frac{x}{\sqrt{-y}}\right) = (\sqrt{y})^{n-1} R_n\left(\frac{x}{\sqrt{y}}\right),\quad n \geqslant 0,$$

where $\sqrt{y}$ and $\sqrt{-y}$ are suitably determined (see, for instance, [7]).

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LETTER TO THE EDITOR

JOHN BRILLHART
July 14, 1983

In the February 1983 issue of this Journal, D. H. and Emma Lehmer introduced a set of polynomials and, among other things, derived a partial formula for the discriminant of those polynomials (Vol. 21, no. 1, p. 64). I am writing to send you the complete formula; namely,

$$D(P_n(x)) = 5^{n-1}n^{2n-4}F_n^{2n-4},$$

where F_n is the nth Fibonacci number. This formula was derived using the Lehmers' relationship

$$(x^2 - x - 1)P_n(x) = x^{2n} - L_n x^n + (-1)^n,$$

where L_n is the Lucas number. Central to this standard derivation is the nice formula by Phyllis Lefton published in the December 1982 issue of this Journal (Vol. 20, no. 4, pp. 363-65) for the discriminant of a trinomial.

The entries in the Lehmers' paper for $D(P_4(x))$ and $D(P_6(x))$ should be corrected to read

$$2^8 \cdot 3^4 \cdot 5^3 \quad \text{and} \quad 2^{32} \cdot 3^8 \cdot 5^5,$$

respectively.

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ON THE SOLUTION OF $\left\{E^2 + (\lambda p - 2)E + (1 - \lambda p - \lambda^2 q)\right\}^m G_n = n^k$, BY EXPANSIONS AND OPERATORS

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(Submitted September 1981)

I. INTRODUCTION

This paper continues the work initiated in the author's joint paper [1] with A. Qadir, in which the authors found the particular solution of the difference equation $(E^2 - E - 1)G_n = n^k$, using two methods, that is, the usual operator method and the method of expansions, eventually establishing an identity involving the Fibonacci numbers F_n defined recursively by $F_1 = F_2 = 1$ and

$$F_{n+2} = F_{n+1} + F_n, \quad n \geqslant 1,$$

the Lucas numbers L_n given by $L_0 = 2$, $L_1 = 1$, and

$$L_{n+2} = L_{n+1} + L_n, \quad n \geqslant 0,$$

and the Sterling numbers of the second kind.

In this paper, the author uses the same two methods to solve a more general difference equation, namely,

$$\{E^2 + (\lambda p - 2)E + (1 - \lambda p - \lambda^2 q)\}^m G_n = n^k,$$

getting an identity involving the Sterling numbers of the second kind, the mth convolved Fibonacci numbers, $F_n^m(p, q)$, where

$$\frac{1}{(1 - px - qx^2)^m} = \sum_{i=0}^{\infty} F_i^m(p, q)x^i$$

and the generalized Lucas numbers, where

$$L_{n+2}(p, q) = pL_{n+1}(p, q) + qL_n(p, q), \; L_0(p, q) = 2, \; L_1(p, q) = p.$$

The plan for this work is as follows. First, in II we find the particular solution of the above-mentioned difference equation by the usual operator method. Then, in III we find the particular solution of the same equation by the method of expansions. Finally, in IV we compare the coefficients of similar powers of n and those of λ, which finally results in the aforesaid identities.

II. PARTICULAR SOLUTION BY THE METHOD OF OPERATORS

From [1] it is known that

$$\frac{n^k}{E - a} = \sum_{i=0}^{k} \sum_{r=0}^{i} \frac{(-1)^r \binom{k}{i} (r)! S(i,\ r) n^{k-i}}{(1 - a)^{r+1}}$$

$$= \sum_{i=0}^{k} \sum_{r=0}^{k} \frac{(-1)^r \binom{k}{i} (r)! S(i,\ r) n^{k-i}}{(1 - a)^{r+1}}.$$

Where $S(i,\ r)$ are the Sterling numbers of the second kind, the shift operator E is defined as

$$Ef(n) = f(n + 1)$$

and the difference operator Δ is defined as

$$\Delta f(n) = f(n + 1) - f(n) = (E - 1)f(n).$$

That is, $\Delta = E - 1$.

Therefore,

$$\frac{n^k}{(E - 1 + \lambda a)} = \sum_{i=0}^{k} \sum_{r=0}^{k} \frac{(-1)^r \binom{k}{i} (r)! S(i,\ r) n^{k-i}}{\lambda^{r+1} a^{r+1}}$$

Also,

$$\frac{n^k}{(E - 1 + \lambda a)(E - 1 + \lambda b)}$$

$$= \sum_{i=0}^{k} \sum_{r=0}^{k} \sum_{s=0}^{k} \sum_{t=0}^{k} \frac{(-1)^{r+t} \binom{k}{i} \binom{k - i}{s} (r)!(t)! S(i,\ r) S(s,\ t) n^{k-i-s}}{\lambda^{2+r+t} a^{r+1} b^{t+1}}$$

Letting $\ell = i + s$ implies $\min(\ell) = 0$, $\max(\ell) = k$, so that

$$\frac{n^k}{(E - 1 + \lambda a)(E - 1 + \lambda b)}$$

$$= \sum_{i=0}^{k} \sum_{r=0}^{k} \sum_{\ell=0}^{k} \sum_{t=0}^{k} \frac{(-1)^{r+t} \binom{k}{i} \binom{k - i}{\ell - i} (r)!(t)! S(i,\ r) S(\ell - i,\ t) n^{k-\ell}}{\lambda^{2+r+t} a^{1+r} b^{1+t}}$$

Putting $j = r + t$, we have $\min(j) = 0$ and $\max(j) = k$. Now, recall that

$$\binom{k}{i}\binom{k - 1}{\ell - 1} = \binom{k}{\ell}\binom{\ell}{i}$$

and change ℓ to i_2, i to i_1, r to j_1, and j to j_2, to get

$$\frac{n^k}{(E - 1 + \lambda a)(E - 1 + \lambda b)}$$

$$= \sum_{i_1=0}^{k} \sum_{i_2=0}^{k} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} \frac{(-1)^{j_2} \prod_{t=1}^{2} \binom{i_{t+1}}{i_t} (j_t - j_{t-1})!S(i_t - i_{t-1}, j_t - j_{t-1}) n^{k-i_2}}{\lambda^{2+j_2} a^{1+j_1} b^{1+j_2-j_1}}$$

where $i_3 = k$, $i_0 = 0 = j_0$.

Using induction on m, it can be proved that

$$\frac{n^k}{(E - 1 + \lambda a)^m (E - 1 + \lambda b)^m} \tag{2.1}$$

$$= \sum_{i_1=0}^{k} \sum_{i_{2m}=0}^{k} \sum_{j_1=0}^{k} \sum_{j_{2m}=0}^{k} \frac{(-1)^{j_{2m}} \prod_{t=1}^{2m} \binom{i_{t+1}}{i_t} (J_t)!S(I_t, J_t) n^{k-i_{2m}}}{\lambda^{2m+j_{2m}} a^{m-T_{2m-1}} b^{m+T_{2m}}}$$

where $i_{2m+1} = k$, $i_0 = 0 = j_0$, $T_m = \sum_{i=1}^{m} (-1)^i j_i$,

$I_t = i_t - i_{t-1}$ and $J_t = j_t - j_{t-1}$ for every $t > 0$.

Let $G(n, m, k)$ be the particular solution of the difference equation

$$\{E^2 + (\lambda p - 2)E + (1 - \lambda p - \lambda^2 q)\}^m G_n = n^k,$$

and let a, b be the roots of $x^2 = px + q$.

Noting that the left-hand side of (2.1) is symmetric in a, b, we interchange a and b in (2.1) and add the resulting equation to (2.1). Using the fact that $a + b = p$ and $ab = -q$, we get, after a little manipulation,

$$G(n, m, k) \tag{2.2}$$

$$= \frac{1}{2} \sum_{i_1=0}^{k} \sum_{i_{2m}=0}^{k} \sum_{j_1=0}^{k} \sum_{j_{2m}=0}^{k} \frac{(-1)^{j_{2m}} \prod_{t=1}^{2m} \binom{i_{t+1}}{i_t} (J_t)!S(I_t, J_t) L_{T_{2m}+T_{2m-1}} n^{k-i_{2m}}}{\lambda^{2m+j_{2m}} (-q)^{m+T_{2m}}}$$

where $L_s = L_s(p, q)$.

Interchanging a, b in (2.1) and subtracting the resulting equation from (2.1) and dividing both sides by $a - b$, we also have

$$\sum_{i_1=0}^{k} \sum_{i_{2m-1}=0}^{k} \sum_{j_1=0}^{k} \sum_{j_{2m-1}=0}^{k} \frac{\prod_{t=1}^{2m-1} \binom{i_{t+1}}{i_t} (J_t)!S(I_t, J_t) F_{T_{2m}+T_{2m-1}}}{(-q)^{m+T_{2m}}} = 0 \tag{2.3}$$

where $F_s = F_s(p, q)$.

3. PARTICULAR SOLUTION BY THE METHOD OF EXPANSIONS

A Particular Solution of $G(n, m, k)$ is given by

$$G(n, m, k) = \frac{n^k}{(E - 1 + \lambda a)^m (E - 1 + \lambda b)^m} = \frac{n^k}{(\Delta + \lambda a)^m (\Delta + \lambda b)^m}.$$

That is,

$$G(n, m, k) = \frac{n^k}{(\Delta^2 + \lambda p \Delta - \lambda^2 q)^m}, \tag{3.1}$$

where a, b are the roots of $x^2 = px + q$. Since $a + b = p$, $ab = -q$, (3.1) becomes

$$G(n, m, k) = \frac{n^k}{(\Delta^2 + \lambda p \Delta - \lambda^2 q)^m} = \frac{(-1)^m n^k}{\lambda^{2m} q^m \left\{1 - p\left(\frac{\Delta}{q\lambda}\right) - q\left(\frac{\Delta}{q\lambda}\right)^2\right\}^m}$$

$$= \frac{(-1)^m}{\lambda^{2m} q^m} \sum_{i=0}^{\infty} F_i^m(p, q)\left(\frac{\Delta}{q\lambda}\right)^i \cdot n^k$$

where $F_i^m(p, q)$ are the mth convolved Fibonacci numbers.

Therefore,

$$G(n, m, k) = \frac{(-1)^m}{\lambda^{2m} q^m} \sum_{i=0}^{k} \frac{F_i^m(p, q)\Delta^i}{\lambda^i q^i} \sum_{j=0}^{k} S(k, j) \cdot n^{(j)},$$

where $S(k, j)$ are the Sterling numbers of the second kind and

$$n^{(j)} = n(n - 1) \ldots (n - j + 1), \text{ for all } j \geqslant 1, \; n^{(0)} \equiv 1.$$

Therefore,

$$G(n, m, k) = \sum_{i=0}^{k} \sum_{j=0}^{k} \frac{(-1)^m F_i^m(p, q)(j)^{(i)} S(k, j) n^{(j-i)}}{q^{m+i} \lambda^{2m+i}}$$

$$= \sum_{i=0}^{k} \sum_{j=i}^{k} \frac{(-1)^m (j)^{(i)} F_i^m(p, q) S(k, j) n^{(j-i)}}{q^{m+i} \lambda^{2m+i}}$$

$$= \sum_{i=0}^{k} \sum_{j=0}^{k-i} \frac{(-1)^m \binom{j+i}{i} (i)! F_i^m(p, q) S(k, j+i) n^{(j)}}{q^{m+i} \lambda^{2m+i}}.$$

Now, change j to $k - i - j$ in order to reverse the order of summation of j. Then, putting $i + j = \ell$ implies that $\min(\ell) = 0$, $\max(\ell) = k$, so that

$$G(n, m, k) = \sum_{i=0}^{k} \sum_{\ell=0}^{k} \frac{(-1)^m (i)! \binom{k - \ell + i}{i} F_i^m(p, q) S(k, k - \ell + i) n^{(k-\ell)}}{q^{m+i} \lambda^{2m+i}}$$

$$= \sum_{i=0}^{k} \sum_{\ell=0}^{k} \sum_{t=0}^{k-\ell} \frac{(-1)^m (i)! \binom{k - \ell + i}{i} F_i^m(p, q) S(k, k - \ell + i) S_t^{k-\ell} n^t}{q^{m+i} \lambda^{2m+i}}$$

where $S_t^{k-\ell}$ are the Sterling numbers of the first kind.

Let us once again reverse the order of summation of t by changing t to $k - \ell - t$. We then let $\ell + t = r$ so that $\min(r) = 0$ and $\max(r) = k$. Then

$$G(n, m, k) = \sum_{i=0}^{k} \sum_{\ell=0}^{k} \sum_{r=0}^{k} \frac{(-1)^m (i)! \binom{k - \ell + i}{i} F_i^m(p, q) S(k, k - \ell + i) S_{k-r}^{k-\ell} n^{k-r}}{q^{m+i} \lambda^{2m+i}}.$$

Now, replace ℓ by $k - \ell$ in order to reverse the summation of ℓ. Next, note that

$$S(k, \ell + i) = 0 \text{ if } \ell > k - i \text{ and } S_{k-r}^{\ell} = 0 \text{ if } \ell < k - r.$$

Also, from [2], we have

$$\sum_{\ell=k-r}^{\ell=k-i} \binom{\ell + i}{i} S(k, \ell + i) S_{k-r}^{\ell} = \binom{k}{r} S_{(r, i)}$$

Hence, writing i_{2m} for r and j_{2m} for i, we obtain

$$G(n, m, k) = \sum_{i_{2m}=0}^{k} \sum_{j_{2m}=0}^{k} \frac{(-1)^m F_{j_{2m}}^m (p, q) \binom{k}{i_{2m}} (j_{2m})! S(i_{2m}, j_{2m}) n^{k-i_{2m}}}{q^{m+j_{2m}} \lambda^{2m+j_{2m}}}. \tag{3.2}$$

4. THE DERIVATION OF THE IDENTITY

Equating the coefficients of similar powers of n from (2.2) and (3.2), and dividing both sides of the resulting equation by the common factor

$$\binom{k}{i_{2m}}$$

we have

$$\frac{1}{2} \sum_{i_1=0}^{k} \sum_{i_{2m}=0}^{k} \sum_{j_1=0}^{k} \sum_{j_{2m}=0}^{k} \frac{(-1)^{j_{2m}} \prod_{t=1}^{2m-i} \binom{i_{t+1}}{i_t} (J_t)! S(I_t, J_t) L_{T_{2m} + T_{2m-1}}}{\lambda^{2m+j_{2m}} (-q)^{m+T_{2m}}} \tag{4.1}$$

$$= \sum_{j_{2m}=0}^{k} \frac{(-1)^m (j_{2m})! \binom{k}{i_{2m}} S(i_{2m}, j_{2m}) F_{j_{2m}}^m}{q^{m+j_{2m}} \lambda^{2m+j_{2m}}}$$

where $F_{j_{2m}}^m \equiv F_{j_{2m}}^m (p, q)$.

Finally, equating the coefficients of similar powers of λ in (4.1), we obtain

$$\sum_{i_1=0}^{k} \sum_{i_{2m-1}=0}^{k} \sum_{j_1=0}^{k} \sum_{j_{2m-1}=0}^{k} \frac{\prod_{t=1}^{2m-1} \binom{i_{t+1}}{i_t} (J_t)!(-1)^{T_{2m-1}} S(I_t, J_t) L_{T_{2m}+T_{2m-1}}}{q^{T_{2m-1}}} \tag{4.2}$$

$$= 2(j_{2m})!S(i_{2m}, j_{2m})F^m_{j_{2m}}.$$

Equating (2.3) and (4.2) gives the identities we wanted to derive.

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ON FIBONACCI NUMBERS OF THE FORM PX^2, WHERE P IS PRIME

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INTRODUCTION

Let p denote a prime, n a natural number, $F(n)$ the nth Fibonacci number. Consider the equation:

$$F(n) = px^2. \tag{*}$$

In [3], J. H. E. Cohn proved that for $p = 2$, the only solutions of (*) are

(i) $n = 3$, $x^2 = 1$

and

(ii) $n = 6$, $x^2 = 4$.

In [8], R. Steiner proved that for $p = 3$, the only solution of (*) is $n = 4$, $x^2 = 1$. Call a solution of (*) trivial if $x = 1$. In this article, we solve (*) for all odd p such that $p \equiv 3 \pmod 4$ or $p < 10{,}000$. Except for $p = 3{,}001$, all solutions obtained are trivial. $L(n)$ denotes the nth Lucas number.

Definition 1

$z(p)$ is the Fibonacci entry point of p, that is,

$$z(p) = \min\{k : k > 0 \text{ and } p \mid F(k)\}.$$

Definition 2

$y(p)$ is the least prime factor of $z(p)$.

PRELIMINARY RESULTS

(1) $F(2m) = F(m)L(m)$

(2) $(F(m),\ L(m)) \mid 2$

(3) If $\prod_{i=1}^{m} a_i = b$ and the a_i are pairwise coprime, then each $a_i = b_i^n$, where the b_i are pairwise coprime and $\prod_{i=1}^{m} b_i = b$.

(4) $p \mid F(n)$ iff $z(p) \mid n$

(5) $F(m) = x^2$ implies $m = 1$, 2, or 12

(6) $F(m) = 2x^2$ implies $m = 3$ or 6

(7) $L(m) = x^2$ implies $m = 1$ or 3

(8) $L(m) = 2x^2$ implies $m = 6$

(9) $p \geqslant 5$ implies $y(p) \leqslant p$

(10) $(F(m),\ F(km)/F(m))|k$

(11) $p \equiv 3 \pmod 4$ implies $z(p)$ is even

(12) $p|F(p)$ iff $p = 5$

Remarks: (5) through (8) are Theorems 1 through 4 in [3]. (9) follows from Theorem 3 in [7]. (10) is Lemma 16, p. 224 in [6]. The other preliminary results are elementary or well known.

THE MAIN THEOREMS

Theorem 1

If $n = 2m$, then the unique solution of (*) is $p = 3$, $n = 4$, $x^2 = 1$.

Proof: Hypothesis and (1) imply $F(m)L(m) = px^2$. Now (2) and (3) imply $F(m)$ or $L(m)$ is a square or twice a square. By (5), (6), (7), and (8), we have $m = 1$, 2, 3, 6, or 12. The only case which yields a solution of (*) is $m = 2$, so that $n = 4$, $p = 3$, $x^2 = 1$.

Corollary 1

If $p \equiv 3 \pmod 4$, then the unique solution of (*) is $p = 3$, $n = 4$, $x^2 = 1$.

Proof: Follows from hypothesis, (11), (4), and Theorem 1.

Theorem 2

If n is odd, then any solution of (*) requires that $n = z(p) = q$, a prime, unless $n = x^2 = 25$ and $p = 3{,}001$.

Proof: Hypothesis and (4) imply that $z(p)$ is odd. By [5, pp. 643-45], we have $n = z(p) \equiv \pm 1 \pmod 6$, so that $n = q^k m$, where $q \geqslant 5$, $k \geqslant 1$, and each prime factor of m exceeds q. If $q|F(m)$, then (4) and Definition 2 imply $y(q)|m$. But (9) implies $y(q) \leqslant q$, a contradiction. Therefore,

$$(q,\ F(m)) = 1.$$

Now (*) implies $px^2 = F(m) * F(q^k m)/F(m)$. Let $d = (F(m),\ F(q^k m)/F(m))$. (10) implies $d|q^k$. Therefore, the only possible prime divisor of d is q. But, since $(q,\ F(m)) = 1$, we have $d = 1$. Since $m < n$, (4) implies $(p,\ F(m)) = 1$, so $F(m)$ is a square. Since m is odd, (5) implies $m = 1$, so that $n = q^k$.

Therefore, $$px^2 = F(q^{k-1}) * F(q^k)/F(q^{k-1}).$$

Let $d' = (F(q^{k-1}), F(q^k)/F(q^{k-1}))$. (10) implies $d' = 1$ or q. If $d' = 1$, then, since $n = z(p) = q^k$, we have $(p, F(q^{k-1})) = 1$. Once again, this implies that $F(q^{k-1}) = 1$, hence $q^{k-1} = 1$ and $k = 1$. If $d' = q$, then (12) implies $q = 5$, so that (3) implies $F(5^{k-1}) = 5x_1^2$. We have

$$x_1^2 = F(5^{k-1})/F(5) = P_{5^{k-2}}(11)$$

in the notation of [5]. By Theorem 3 in [4], this implies $5^{k-2} = 1$, i.e., $n = 25$, and thus $p = 3{,}001$.

Theorem 3

If $2 < p < 10{,}000$, then the unique nontrivial solution of (*) is $p = 3{,}001$, $n = x^2 = 25$; all other solutions are trivial with

$$(n, p) = (4, 3), (5, 5), (11, 89), (13, 233), \text{ or } (17, 1{,}571).$$

Proof: If n is even, then Theorem 1 implies $(n, p) = (4, 3)$. If n is odd and $p \neq 3{,}001$, then Theorem 2 implies $n = z(p) = q$, a prime. We therefore consider all p such that $5 \leqslant p < 10{,}000$, $p \neq 3{,}001$, and $q = z(p)$ is an odd prime. If $q \leqslant 229$, namely for p = 5, 13, 37, 73, 89, 113, 149, 157, 193, 233, 269, 277, 313, 353, 389, 397, 457, 557, 677, 953, 1,069, 1,597, 2,221, 2,417, 2,749, 2,789, 4,013, 4,513, 5,737, 6,673, or 8,689, the conclusion follows from the examination of the prime factorization of $F(q)$ in [2]. According to [1], there are 101 primes, p, such that $p < 10{,}000$ and $q = z(p)$ is a prime exceeding 229. For each such p, to show that $F(q) \neq px^2$, it suffices to find an odd prime modulus, t, such that $F(q)/p$ is a quadratic nonresidue (mod t). The results are listed in Table 1. For each p, the corresponding t is the least required prime modulus. In each case, $t \leqslant 19$.

TABLE 1

p	q	t	$F(q)$ (mod t)	p (mod t)	$1/p$ (mod t)	$F(q)/p$ (mod t)
613	307	3	2	1	1	2
673	337	19	5	8	12	3
733	367	7	1	5	3	3
757	379	3	2	1	1	2
877	439	5	1	2	3	3
997	499	3	2	1	1	2
1093	547	3	2	1	1	2
1153	577	7	1	5	3	3
1213	607	11	2	3	4	8
1237	619	3	2	1	1	2
1453	727	7	6	4	2	5
1657	829	3	2	1	1	2
1753	877	3	2	1	1	2
1873	937	7	6	4	2	5
1877	313	3	1	2	2	2

TABLE 1 (continued)

p	q	t	$F(q)$ (mod t)	p (mod t)	$1/p$ (mod t)	$F(q)/p$ (mod t)
1933	967	7	6	1	1	6
1949	487	3	1	2	2	2
1993	997	3	2	1	1	2
2017	1009	5	4	2	3	2
2137	1069	3	2	1	1	2
2237	373	7	5	4	2	3
2309	577	3	1	2	2	2
2333	389	5	4	3	2	3
2437	1237	3	2	1	1	2
2557	1279	5	1	2	3	3
2593	1297	7	1	3	5	5
2777	463	3	1	2	2	2
2797	1399	5	1	2	3	3
2857	1429	3	2	1	1	2
2909	727	3	1	2	2	2
2917	1459	3	2	1	1	2
3217	1609	5	4	2	3	2
3253	1627	3	2	1	1	2
3313	1657	7	6	2	4	3
3517	1759	5	1	2	3	3
3557	593	3	1	2	2	2
3733	1867	3	2	1	1	2
4057	2029	3	2	1	1	2
4177	2089	5	4	2	3	2
4273	2137	11	2	5	9	7
4349	1087	3	1	2	2	2
4357	2179	3	2	1	1	2
4637	773	13	11	9	3	7
4733	263	3	1	2	2	2
4909	409	7	6	2	4	3
4933	2467	3	2	1	1	2
5009	313	3	1	2	2	2
5077	2539	3	2	1	1	2
5113	2557	3	2	1	1	2
5189	1297	3	1	2	2	2
5233	2617	7	6	4	2	5
5297	883	7	2	5	3	6
5309	1327	3	1	2	2	2
5381	269	7	2	5	3	6
5413	2707	3	2	1	1	2
5437	2719	5	1	2	3	3
5653	257	17	5	9	2	10
5897	983	3	1	2	2	2
6037	3019	3	2	1	1	2
6073	3037	3	2	1	1	2
6133	3067	3	2	1	1	2
6217	3109	3	2	1	1	2
6269	1567	3	1	2	2	2
6337	3169	5	4	2	3	2

TABLE 1 (continued)

p	q	t	$F(q)$ (mod t)	p (mod t)	$1/p$ (mod t)	$F(q)/p$ (mod t)
6373	3187	3	2	1	1	2
6397	457	13	8	1	1	8
6637	3319	5	1	2	3	3
6737	1123	7	2	3	5	3
6917	1153	3	1	2	2	2
6997	3499	3	2	1	1	2
7057	3529	5	4	2	3	2
7109	1777	3	1	2	2	2
7213	3607	17	13	5	7	6
7393	3697	11	2	1	1	2
7417	3709	3	2	1	1	2
7477	3739	3	2	1	1	2
7537	3769	5	4	2	3	2
7753	3877	3	2	1	1	2
7817	1303	3	1	2	2	2
7933	3967	13	8	3	9	7
8053	4027	3	2	1	1	2
8317	4159	5	1	2	3	3
8353	4177	11	2	4	3	6
8369	523	5	2	4	4	3
8573	1429	5	4	3	2	3
8677	4339	3	2	1	1	2
8713	4357	3	2	1	1	2
8753	1459	5	1	3	2	2
8861	443	5	2	1	1	2
8893	4447	7	1	3	5	5
9013	4507	3	2	1	1	2
9133	4567	11	2	3	4	8
9277	4639	5	1	2	3	3
9377	521	3	1	2	2	2
9397	4699	3	2	1	1	2
9497	1583	3	1	2	2	2
9677	1613	7	2	3	5	3
9697	373	3	2	1	1	2
9817	4909	3	2	1	1	2
9949	829	3	2	1	1	2
9973	4987	3	2	1	1	2

CONCLUDING REMARKS

According to [2], additional trivial solutions exist (corresponding to larger p) for n = 23, 29, 43, 83, 131, 137, 359, 431, 433, 449, 509, and 569. It remains to be decided whether (i) any nontrivial solutions exist apart from those already known, and/or (ii) infinitely many p exist having trivial solutions.

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A GENERAL METHOD FOR DETERMINING A CLOSED FORMULA FOR THE NUMBER OF PARTITIONS OF THE INTEGER n INTO m POSITIVE INTEGERS FOR SMALL VALUES OF m

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The function $p\ (n)$ is defined as the number of partitions of the integer n into exactly m nonzero positive integers where the order is irrelevant. A general method for determining a formula for $p_m(n)$ for small values of m is given. The formulas are simpler in form than any previously given.

1. INTRODUCTION

If $p_m(n)$ is the number of partitions of the integer n into exactly m positive integers and if $p_m^{\star}(n)$ is the number of partitions into at most m parts and $p(m)$ is the usual partition function, then there are some simple known relationships between them.

$$p_m(n) - p_m(n - m) = p_{m-1}(n - 1)$$
$$p_m^{\star}(n) = p_m(n + m)$$
$$p(m) = p_m(2m)$$

The first recurrence relationship can be solved sequentially starting with $m = 2$ to determine the exact solution for small values of m. The method is given in Section 2. The procedure is to determine the complementary function and the particular solution to satisfy the m initial conditions $p_m(n) = 0$ for $0 \leqslant n \leqslant m - 1$ starting with $p_1(n) = 1$. This leads to the following forms.

$$p_2(n) = \left[\frac{n}{2!1!}\right] \qquad p_3(n) = \left[\frac{n^2 + 3}{3!2!}\right]$$

$$p_4(n) = \left[\frac{n^3 + 3n^2 + \frac{1}{2}\{9n(-1)^n - 9n\} + 32}{4!3!}\right]$$

$$p_5(n) = \left[\frac{n^4 + 10n^3 + 10n^2 - 75n - 45n(-1)^n + 905}{5!4!}\right]$$

$$p_6(n) = \left[\frac{n^5+22\frac{1}{2}n^4+126\frac{2}{3}n^3-112\frac{1}{2}n^2-1599\frac{1}{6}n+112\frac{1}{2}(-1)^n(n^2+9n)+1066\frac{2}{3}n\cos\frac{2n\pi}{3}+19224}{6!5!}\right].$$

HISTORICAL NOTE

Exact determinations of $p_m(n)$ for small m have been given in a variety of forms by many writers. See Dickson's *History of the Theory of Numbers*, Vol. 2, and *The Royal Society Mathematical Tables*, Vol. 4, by H. Gupta and others for extensive details of previous work together with references. De Morgan (1843) gives formulas for $p_3(n)$ and $p_4(n)$ which are equivalent to the above forms (see Dickson, p. 115). In Gupta (p. xvi), formulas are quoted in the form below, where $p(n, m) = p_m(n + m)$.

$$p(n,\ 1) = 1$$

$$p(n,\ 2) = \frac{1}{2}\left(n + \frac{3}{2}\right) + \frac{1}{4}(-1)^n$$

$$p(n,\ 3) = \frac{1}{12}\left(n^2 + 6n + \frac{47}{6}\right) + \frac{1}{8}(-1)^n + \frac{1}{9}(\alpha_3^n + \alpha_3^{2n})$$

$$p(n,\ 4) = \frac{1}{144}\left(n^3 + 15n^2 + \frac{135n}{2} + \frac{175}{2}\right) + \frac{1}{32}(n+5)(-1)^n + \frac{i}{9\sqrt{3}}(\alpha_3^{n-1} - \alpha_3^{2n-2}) + \frac{1}{16}(i^n + i^{3n})$$

where $\alpha_3 = \exp\dfrac{2i\pi}{3}$ is a cube root of unity.

This development is essentially due to J. W. L. Glaisher (1908) (see Gupta and Dickson, p. 117). Glaisher obtained complete results to $m = 10$ and the results are given to $m = 12$ in Gupta, but the formulas obtained are very complicated.

Further results are given in Gupta, but all the exact formulas given for small m are more complicated than those given here.

SECTION 2

Write the recurrence equation in the form

$$p_m(n + m) - p_m(n) = p_{m-1}(n + m - 1).$$

The solution of this equation is composed of two parts.

1. The complementary function given by the solution of

$$p_m(n + m) - p_m(n) = 0.$$

This simply gives the form

$$a_1\alpha_1^n + a_2\alpha_2^n + \cdots + a_m\alpha_m^n,$$

where the a_i are constants and the α_i are the mth roots of unity where $\alpha_1 = 1$ (say).

2. The particular solution is determined apart from the arbitrary constant which is included in (1) by the solution of the equation

$$\Delta(m)\{p_m(n)\} = p_{m-1}(n + m - 1),$$

where $\Delta(m)$ is an operator such that

$$\Delta(m)\{p_m(n)\} = p_m(n + m) - p_m(n).$$

Thus we can write formally

$$p_m(n) = \frac{1}{\Delta(m)}\{p_{m-1}(n + m - 1)\},$$

where $\frac{1}{\Delta(m)}$ is the inverse operator to $\Delta(m)$.

To Determine the Action of $\frac{1}{\Delta(m)}$

2.1 Let $p(n)$ be any polynomial function in n with constant coefficients. Then

$$\frac{1}{\Delta(m)}\{p(n)\} = \left(\frac{1}{mD} + B_1 + \frac{B_2 mD}{2!} + \frac{B_4 m^3 D^3}{4!} + \frac{B_6 m^5 D^5}{6!} + \cdots\right)\{p(n)\},$$

where the B are the Bernoulli numbers and the right-hand side is finite as $p(n)$ is a polynomial. This is a well-known result.

2.2 Consider $\Delta(m)\left\{\frac{\alpha^n}{\alpha^m - 1}\right\}$, where $\alpha^m \neq 1$

$$= \frac{\alpha^{n+m} - \alpha^n}{\alpha^m - 1} = \alpha^n$$

$$\therefore \quad \frac{1}{\Delta(m)}\{\alpha^n\} = \frac{\alpha^n}{\alpha^m - 1} \quad \text{when } \alpha^m \neq 1.$$

2.3 Consider $\Delta(m)\left\{\frac{n\alpha^n}{m}\right\}$, where $\alpha^m = 1$

$$= \frac{(n + m)\alpha^{n+m} - n\alpha^n}{m} = \alpha^n$$

$$\therefore \quad \frac{1}{\Delta(m)}\{\alpha^n\} = \frac{n\alpha^n}{m} \quad \text{when } \alpha^m = 1.$$

2.4 Let $f(n)$ and $g(n)$ be any functions of n; then

$$\begin{aligned}\Delta(m)\{f(n)g(n)\} &= f(n + m)g(n + m) - f(n)g(n)\\ &= f(n + m)g(n + m) - f(n)g(n + m) + f(n)g(n + m)\\ &\qquad - f(n)g(n)\end{aligned}$$

(continued)

$$= g(n+m)\Delta(m)\{f(n)\} + f(n)\Delta(m)\{g(n)\}$$

$$\therefore \quad f(n)g(n) = \frac{1}{\Delta(m)}\Big\{g(n+m)\Delta(m)\{f(n)\}\Big\} + \frac{1}{\Delta(m)}\Big\{f(n)\Delta(m)\{g(n)\}\Big\}$$

$$\therefore \quad \frac{1}{\Delta(m)}\Big\{f(n)\Delta(m)\{g(n)\}\Big\} = f(n)g(n) - \frac{1}{\Delta(m)}\Big\{g(n+m)\Delta(m)\{f(n)\}\Big\}$$

Put $\Delta(m)\{g(n)\} = \alpha^n$.

$$\therefore \quad g(n) - \frac{1}{\Delta(m)}\{\alpha^n\} = \frac{\alpha^n}{\alpha^m - 1} \text{ if } \alpha^m \neq 1$$

$$\therefore \quad \frac{1}{\Delta(m)}\{f(n)\alpha^n\} = f(n) \cdot \frac{\alpha^n}{\alpha^m - 1} - \frac{1}{\Delta(m)}\left\{\frac{\alpha^{n+m}}{\alpha^m - 1}\Delta(m)\{f(n)\}\right\}$$

$$= \frac{f(n)\alpha^n}{\alpha^m - 1} - \frac{\alpha^m}{\alpha^m - 1}\,\frac{1}{\Delta(m)}\Big\{\alpha^n\Delta(m)\{f(n)\}\Big\}.$$

Thus, if $f(n)$ is a polynomial in n, then this is a reduction formula that can be successively applied to determine the left-hand side. From which it follows that if $\alpha^m \neq 1$ and $f(n)$ is a polynomial of degree p, we have

$$\frac{1}{\Delta(m)}\{\alpha^n f(n)\} = \frac{\alpha^n}{\alpha^m - 1}\left(1 + \frac{\alpha^m}{\alpha^m - 1}\Delta(m)\right)^{-1}\{f(n)\}$$

$$= \frac{\alpha^n}{\alpha^m - 1}\left(1 - \frac{\alpha^m}{\alpha^m - 1}\Delta + \left(\frac{\alpha^m}{\alpha^m - 1}\right)^2\Delta^2 - \cdots + (-1)^p\left(\frac{\alpha^m}{\alpha^m - 1}\right)^p\Delta^p\right)\{f(n)\}.$$

2.5 Consider $\Delta(m)\{f(n)\alpha^n\}$, where $\alpha^m = 1$.

$$\therefore \quad \Delta(m)\{f(n)\alpha^n\} = f(n+m)\alpha^{n+m} - f(n)\alpha^n$$

$$= \alpha^n(f(n+m) - f(n))$$

$$= \alpha^n\Delta(m)\{f(n)\}$$

$$\therefore \quad f(n)\alpha^n = \frac{1}{\Delta(m)}\Big\{\alpha^n\Delta(m)\{f(n)\}\Big\}.$$

Put $\Delta(m)\{f(n)\} = p(n)$.

$$\therefore \quad f(n) = \frac{1}{\Delta(m)}\{p(n)\}$$

$$\therefore \quad \frac{1}{\Delta(m)}\{\alpha^n p(n)\} = \alpha^n f(n) = \alpha^n\frac{1}{\Delta(m)}\{p(n)\} \text{ if } \alpha^m = 1.$$

Thus, if $p(n)$ is a polynomial and $\alpha^m = 1$, we have

$$\frac{1}{\Delta(m)}\{\alpha^n p(n)\} = \alpha^n\left(\frac{1}{mD} + B_1 + \frac{B_2 mD}{2!} + \frac{B_4 m^3 D^3}{4!} + \cdots\right)\{p(n)\}.$$

This determines the action of $\dfrac{1}{\Delta(m)}$ in all cases. Thus, for

$$p_m(n + m) - p_m(n) = p_{m-1}(n + m - 1),$$

we have that

$$p_m(n) = a_1\alpha_1^n + a_2\alpha_2^n + \cdots + a_m\alpha_m^m + \frac{1}{\Delta(m)}\{p_{m-1}(n + m - 1)\},$$

where the a_i are constants and the α_i are the mth roots of unity with $\alpha_1 = 1$. We have the m conditions $p_m(n) = 0$ for $0 \leqslant n \leqslant m - 1$ for the determination of the m constants.

Thus, the $p_m(n)$ can be determined sequentially for values of m starting with $m = 2$.

Now, $p_1(n) = 1$,

$$\therefore \quad p_2(n + 2) - p_2(n) = p_1(n + 1) = 1$$

$$\therefore \quad p_2(n) = a_1(1)^n + a_2(-1)^n + \frac{1}{\Delta(2)}\{1\}$$

$$= a_1 + a_2(-1)^n + \frac{n}{2}$$

Now, $p_2(0) = a_1 + a_2 = 0$

$$p_2(1) = a_1 - a_2 + \frac{1}{2} = 0$$

$$\therefore \quad a_1 = -\frac{1}{4}, \quad a_2 = \frac{1}{4}$$

$$\therefore \quad p_2(n) = -\frac{1}{4} + \frac{1}{4}(-1)^n + \frac{n}{2}$$

Now $p_2(n)$ is an integer for all positive integral n. Now

$$\max\left\{-\frac{1}{4} + \frac{1}{4}(-1)^n\right\} = 0, \text{ for } n = 2 \text{ (say)}.$$

Thus, we can write $p_2(n) = \left[\dfrac{n}{2}\right]$.

<u>$m = 3$</u>

$$\therefore \quad p_3(n + 3) - p_3(n) = p_2(n + 2) = -\frac{1}{4} + \frac{1}{4}(-1)^n + \frac{n + 2}{2}$$

$$\therefore \quad p_3(n) = a_1 + a_2\left(\frac{-1 + i\sqrt{3}}{2}\right)^n + a_3\left(\frac{-1 - i\sqrt{3}}{2}\right)^n - \frac{n}{12} = \frac{1}{8}(-1)^n + \frac{(n + 2)^2}{12} - \frac{(n + 2)}{4}$$

This gives $a_1 = \frac{5}{72}$, $a_2 = a_3 = \frac{8}{72}$.

$$\therefore\quad p_3(n) = \frac{n^2}{12} - \frac{7}{72} - \frac{1}{8}(-1)^n + \frac{8}{72}\left(\frac{-1 + i\sqrt{3}}{2}\right)^n + \frac{8}{72}\left(\frac{-1 - i\sqrt{3}}{2}\right)^n$$

$$= \frac{n^2}{12} + \frac{1}{72}\left(16\cos\left(\frac{2n\pi}{3}\right) - 7 - 9(-1)^n\right).$$

But $p_3(n)$ is an integer for all n, and so as

$$\max 16\left(\cos\left(\frac{2n\pi}{3}\right) - 7 - 9(-1)^n\right) = 18, \text{ for } n = 3 \text{ (say,}$$

we have $p_3(n) = \left[\frac{n^2}{12} + \frac{18}{72}\right] = \left[\frac{n^2 + 3}{12}\right]$.

$\underline{m = 4}$

$$\therefore\quad p_4(n) = a_1 + a_2(-1)^n + a_3(i)^n + a_4(-i)^n + \frac{(n+3)^3}{144} - \frac{7n}{288}$$

$$- \frac{(n+3)^2}{24} + \frac{(n+3)}{18} + \frac{1}{8}\cdot\frac{n(-1)^n}{4} + \frac{8}{72}\cdot\frac{\left(\frac{-1 + i\sqrt{3}}{2}\right)^n}{\left(\frac{-1 + i\sqrt{3}}{2} - 1\right)}$$

$$+ \frac{8}{72}\cdot\frac{\left(\frac{-1 - i\sqrt{3}}{2}\right)^n}{\left(\frac{-1 - i\sqrt{3}}{2} - 1\right)}$$

which can be reduced to

$$p_4(n) = a_1 + a_2(-1)^n + a_3(i)^n + a_4(-i)^n + \left(\frac{2n^3 + 6n^2 - 9n + 9n(-1)^n - 6}{288}\right)$$

$$+ \frac{1}{54}\left(-6\cos\frac{2n\pi}{3} + 2\sqrt{3}\sin\frac{2n\pi}{3}\right).$$

Whence $a_1 = -\frac{7}{288}$, $a_2 = \frac{9}{288}$, $a_3 = a_4 = \frac{1}{16}$.

$$\therefore\quad p_4(n) = -\frac{7}{288} + \frac{9}{288}(-1)^n + \frac{1}{16}(i)^n + \frac{1}{16}(-i)^n$$

$$+ \left(\frac{2n^3 + 6n^2 - 9n + 9n(-1)^n - 6}{288}\right)$$

$$+ \frac{1}{54}\left(-6\cos\frac{2n\pi}{3} + 2\sqrt{3}\sin\frac{2n\pi}{3}\right).$$

Now following the previous technique, since $p_4(n)$ is an integer for all n, we have, for $n = 4$ (say):

$$\max\left(\frac{9}{288}(-1)^n + \frac{1}{16}(i)^n + \frac{1}{16}(-i)^n - \frac{6}{54}\cos\frac{2n\pi}{3} + \frac{2\sqrt{3}}{54}\sin\frac{2n\pi}{3}\right) = \frac{77}{288}.$$

$$\therefore \quad p_4(n) = \left[\frac{n^3 + 3n^2 + \frac{1}{2}(9n(-1)^n - 9n) + 32}{144}\right]$$

It is clear from the above form for $p_4(n)$ which contains $\cos \frac{2n\pi}{3}$ and $\sin \frac{2n\pi}{3}$ that we need to convert formula 2.4 to a form which encompasses this type in order to proceed to determine $p_m(n)$ for $m \geqslant 5$ exactly. The resulting formulas are in themselves interesting. If $\alpha^s = 1$, then

$$\alpha = \cos\left(\frac{2k\pi}{s}\right) + i \sin\left(\frac{2k\pi}{s}\right) \quad \text{and} \quad \alpha^n = \cos\left(\frac{2kn\pi}{s}\right) + i \sin\left(\frac{2kn\pi}{s}\right),$$
$$0 \leqslant k \leqslant s - 1.$$

We have from 2.4 that

$$\frac{1}{\Delta(m)}\{\alpha^n f\} = \frac{\alpha^n}{\alpha^m - 1}\left(1 - \left(\frac{\alpha^m}{\alpha^m - 1}\right)\Delta + \left(\frac{\alpha^m}{\alpha^m - 1}\right)^2 \Delta^2 - \cdots\right)\{f\}, \text{ if } \alpha^m \neq 1.$$

Then it can be shown that

$$\frac{1}{\Delta(m)}\left\{\cos\left(\frac{2kn\pi}{s}\right)f(n)\right\} = \sum_{r=0}^{p} \frac{\operatorname{cosec}^{r+1}\left(\frac{km\pi}{s}\right)}{2^{r+1}} \sin\left(\frac{k\pi}{s}(2n - m + rm) - \frac{r\pi}{2}\right)(-\Delta)^r\{f(n)\},$$

where $f(n)$ is a polynomial of degree p and $\alpha^s = 1$ but $\alpha^m \neq 1$ and $1 \leqslant k \leqslant s - 1$, $k \neq 0$. The proof is easy but lengthy.

Similarly,

$$\frac{1}{\Delta(m)}\left\{\sin\left(\frac{2kn\pi}{s}\right)f(n)\right\} = -\sum_{r=0}^{p} \frac{\operatorname{cosec}^{r+1}\left(\frac{km\pi}{s}\right)}{2^{r+1}} \cos\left(\frac{k\pi}{s}(2n - m + rm) - \frac{r\pi}{2}\right)(-\Delta)^r\{f(n)\}.$$

$\underline{m = 5}$

Thus returning to $p_5(m)$ it can be shown using the previous formulas that

$$\frac{1}{\Delta(5)}\{p_4(n+4)\} = \frac{1}{288}\left(\frac{n^4}{10} + n^3 + n^2 - 7\tfrac{1}{2}n + \frac{9(n+4)(-1)^n}{-2} + \frac{45(-1)^5}{2} \cdot \frac{(-1)^n}{(-1)^5 - 1}\right)$$
$$+ \frac{9}{288} \cdot \frac{(-1)^n}{-2} + \frac{i^n(-1 - i)}{32} + \frac{(-i)^n(-1 + i)}{32} - \frac{1}{18} \cos \frac{2n\pi}{3}$$
$$+ \frac{1}{18}\left(-\frac{1}{\sqrt{3}}\right)\sin \frac{2n\pi}{3} - \frac{\sqrt{3}}{54} \sin\left(\frac{2n\pi}{3}\right) + \frac{1}{54} \cos\left(\frac{2n\pi}{3}\right).$$

Using

$$\frac{1}{32}\left(i^n(-1 - i) + (-i)^n(-1 + i)\right) = \frac{1}{16}\left(\sin \frac{n\pi}{2} - \cos \frac{n\pi}{2}\right)$$

(continued)

$$= \frac{1}{288}\left(\frac{n^4}{10} + n^3 + n^2 - 7\tfrac{1}{2}n - \frac{9(-1)^n}{4}(2n + 5)\right) + \frac{1}{16}\left(\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right)$$
$$- \frac{1}{27}\cos\frac{2n\pi}{3} - \frac{\sqrt{3}}{27}\sin\frac{2n\pi}{3}.$$

$\therefore\ p_5(n) = \text{C.F.} + \text{P.S.}$, where the complementary function is

$$\sum_{k=0}^{4} a_k\left(\cos\frac{2kn\pi}{5} + i\sin\frac{2kn\pi}{5}\right),$$

which by modifying the constants a_k can clearly be written in the form

$$C_0 + C_1\cos\frac{2n\pi}{5} + C_2\cos\frac{4n\pi}{5} + S_1\sin\frac{2n\pi}{5} + S_2\sin\frac{4n\pi}{5}.$$

The method is clearly general.

$n = 0$

$$\therefore\ C_0 + C_1 + C_2 = \frac{2395}{17{,}280} = \beta_0 \text{ (say)}$$

$n = 1$

$$\therefore\ C_0 + C_1\cos\frac{2\pi}{5} + C_2.-\cos\frac{\pi}{5} + S_1\sin\frac{2\pi}{5} + S_2\sin\frac{\pi}{5} = -\frac{1061}{17{,}280} = \beta_1$$

$n = 2$

$$\therefore\ C_0 + C_1.-\cos\frac{\pi}{5} + C_2\cos\frac{2\pi}{5} + S_1\sin\frac{\pi}{5} + S_2.-\sin\frac{2\pi}{5} = -\frac{1061}{17{,}280} = \beta_2$$

$n = 3$

$$\therefore\ C_0 + C_1.-\cos\frac{\pi}{5} + C_2\cos\frac{2\pi}{5} + S_1.-\sin\frac{\pi}{5} + S_2\sin\frac{2\pi}{5} = -\frac{1061}{17{,}280} = \beta_3$$

$n = 4$

$$\therefore\ C_0 + C_1\cos\frac{2\pi}{5} + C_2.-\cos\frac{\pi}{5} + S_1.-\sin\frac{2\pi}{5} + S_2.-\sin\frac{\pi}{5} = -\frac{1061}{17{,}280} = \beta_4$$

Thus if we add the equations we have immediately

$$C_0 = \frac{1}{5}(\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4) = \frac{-1849}{17{,}280 \times 5}.$$

As we are concerned with the mth roots of unity this form will be quite general for C_0. The solution is

$$C_1 = \frac{6912}{17{,}208 \times 5} = C_2 \text{ and } S_1 = S_2 = 0.$$

$$\therefore\ p_5(n) = \frac{1}{2880}(n^4 + 10n^3 + 10n^2 - 75n - 45n(-1)^n) + \left\{\frac{1}{288}\left(\frac{-45(-1)^n}{4}\right)\right.$$
$$+ \frac{1}{16}\left(\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right) - \frac{1}{27}\cos\frac{2n\pi}{3} - \frac{\sqrt{3}}{27}\sin\frac{2n\pi}{3} - \frac{1849}{17{,}280 \times 5}$$
$$\left. + \frac{6912}{17{,}280 \times 5}\cos\frac{2n\pi}{5} + \frac{6912}{17{,}280 \times 5}\cos\frac{4n\pi}{5}\right\}.$$

Again we have that the part within braces is purely trigonometric and has a maximum value given by $n = 5$ (say), which is 905/2880.

$$\therefore \quad p_5(n) = \left[\frac{n^4 + 10n^3 + 10n^2 - 75n - 45n(-1)^n + 905}{2880}\right]$$

It would appear from previous work that we have to determine a solution to a set of linear equations each time we determine $p_m(n)$. But this is not the case as the constants C_i and S_i can be determined explicitly in terms of the β_i as follows.

We have for the Complementary function

$$\sum_{k=0}^{m-1} a_k e^{i\frac{2kn\pi}{m}}$$

and for the Complete Solution, we have

$$n = 0 \qquad a_0 + a_1 + a_2 + \cdots + a_{m-1} = \beta_0 \text{ (say)}$$

$$n = 1 \qquad a_0 + a_1 e^{i\frac{2\pi}{m}} + a_2 e^{i\frac{4\pi}{m}} + \cdots + a_{m-1} e^{i\frac{2(m-1)\pi}{m}} = \beta_1$$

$$n = m-1 \quad a_0 + a_1 e^{i\frac{2(m-1)\pi}{m}} + a_2 e^{i\frac{2(m-1)2\pi}{m}} + \cdots + a_{m-1} e^{i\frac{2(m-1)(m-1)\pi}{m}} = \beta_{m-1}$$

Now $1 + e^{i\frac{2\pi r}{m}} + e^{i\frac{2r2\pi}{m}} + \cdots + e^{i\frac{2r(m-1)\pi}{m}} = \dfrac{e^{i2\pi r} - 1}{e^{i\frac{2r\pi}{m}} - 1} = 0$, as r is an integer $< m$.

Thus, if we add,

$$a_0 = \frac{\beta_0 + \beta_1 + \cdots + \beta_{m-1}}{m}.$$

To determine a_1, we can essentially do the same thing. Multiply equation (2) by $e^{-i\frac{2\pi}{m}}$, (3) by $e^{-i\frac{4\pi}{m}}$, ..., (m) by $e^{-i\frac{2(m-1)\pi}{m}}$. Thus, the coefficients in the a_1 column are all one. Then add the equations by columns again and we have

$$ma_1 = \beta_0 + \beta_1 e^{-i\frac{2\pi}{m}} + \cdots + \beta_{m-1} e^{-i\frac{2(m-1)\pi}{m}}.$$

In general,

$$ma_k = \beta_0 + \beta_1 e^{-i\frac{2k\pi}{m}} + \cdots + \beta_{m-1} e^{-i\frac{2(m-1)k\pi}{m}}.$$

Thus, we have the form

$$\frac{1}{m}\sum_{k=0}^{m-1}\left(\beta_0 + \beta_1 e^{-i\frac{2k\pi}{m}} + \cdots + \beta_{m-1} e^{-i\frac{2(m-1)k\pi}{m}}\right) e^{i\frac{2kn\pi}{m}}.$$

This is the Complementary function but not in an explicit real form, but the terms can be grouped to give the real form.

If m is odd $\geqslant 3$,

$$= \frac{1}{m}(\beta_0 + \beta_1 + \cdots + \beta_{m-1}) + \frac{2}{m}\sum_{k=0}^{m-1}\beta_k\left\{\cos\left(\frac{2n\pi}{m} - \frac{2k\pi}{m}\right) + \cdots + \cos\left(\frac{(m-1)n\pi}{m} - \frac{(m-1)k\pi}{m}\right)\right\}$$

If m is even $\geqslant 4$, there is a root (-1) in the form, and we have

$$= \frac{1}{m}(\beta_0 + \cdots + \beta_{m-1}) + \frac{(-1)^n}{m}(\beta_0 - \beta_1 + \beta_2 - \cdots - \beta_{m-1})$$
$$+ \frac{2}{m}\sum_{k=0}^{m-1}\beta_k\left\{\cos\left(\frac{2n\pi}{m} - \frac{2k\pi}{m}\right) + \cdots + \cos\left(\frac{(m-2)n\pi}{m} - \frac{(m-2)k\pi}{m}\right)\right\}.$$

Or finally, by regrouping, we have for m odd $\geqslant 3$:

$$= \frac{1}{m}(\beta_0 + \cdots + \beta_{m-1})$$
$$+ \frac{2}{m}\sum_{k=1}^{\frac{m-1}{2}}\left(\beta_0 + \beta_1 \cos\frac{2k\pi}{m} + \cdots + \beta_{m-1}\cos\frac{2(m-1)k\pi}{m}\right)\cos\frac{2nk\pi}{m}$$
$$+ \frac{2}{m}\sum_{k=1}^{\frac{m-1}{2}}\left(\beta_1 \sin\frac{2k\pi}{m} + \cdots + \beta_{m-1}\sin\frac{2(m-1)k\pi}{m}\right)\sin\frac{2nk\pi}{m}.$$

For m even $\geqslant 4$,

$$= \frac{1}{m}(\beta_0 + \cdots + \beta_{m-1}) + \frac{(-1)^n}{m}(\beta_0 - \beta_1 + \cdots - \beta_{m-1})$$
$$+ \frac{2}{m}\sum_{k=1}^{\frac{m-2}{2}}\left(\beta_0 + \beta_1 \cos\frac{2k\pi}{m} + \cdots + \beta_{m-1}\cos\frac{2(m-1)k\pi}{m}\right)\cos\frac{2nk\pi}{m}$$
$$+ \frac{2}{m}\sum_{k=1}^{\frac{m-2}{2}}\left(\beta_1 \sin\frac{2k\pi}{m} + \cdots + \beta_{m-1}\sin\frac{2(m-1)k\pi}{m}\right)\sin\frac{2nk\pi}{m}.$$

Thus, returning to $p_6(n)$, we have that the particular solution is

$$\frac{1}{2880 \times 30}\left(n^5 + \frac{45n^4}{2} + \frac{380n^3}{3} - \frac{225n^2}{2} - 1599\tfrac{1}{6}n\right)$$
$$+ \frac{3(-1)^n}{8 \times 288}(n^2 + 9n - 39) - \frac{1}{32}\left(\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right) + \frac{n\cos\frac{2n\pi}{3}}{81}$$
$$+ \frac{6912}{17{,}280 \times 10}\left(-\operatorname{cosec}\frac{\pi}{5}\sin\frac{\pi}{5}(2n-6)\right) + \left(\operatorname{cosec}\frac{2\pi}{5}\sin\frac{2\pi}{5}(2n-6)\right).$$

The Complementary function is

$$C_0 + C_1 \cos\frac{2n\pi}{6} + C_2 \cos\frac{4n\pi}{6} + C_3 \cos\frac{6n\pi}{6} + S_1 \sin\frac{2n\pi}{6} + S_2 \sin\frac{4n\pi}{6}.$$

The coefficients C_i are

$$C_0 = -\frac{756\frac{3}{4}}{86,400}, \quad C_1 = C_2 = \frac{4800}{86,400}, \quad C_3 = \frac{5156\frac{1}{4}}{86,400}$$

$$S_1 = 0, \qquad S_2 = \frac{1066\frac{2}{3}}{\sqrt{3} \times 86,400}$$

Thus $p_6(n)$ is the sum of the two forms. Again the maximum value of the purely trigonometric part—that is, the part that does not contain any algebraic powers of n, is given when $n = 6$ and is 19,224/86,400. Hence,

$$p_6(n) = \left[\frac{n^5 + 22\frac{1}{2}n^4 + 126\frac{2}{3}n^3 - 112\frac{1}{2}n^2 - 1599\frac{1}{6}n + 112\frac{1}{2}(-1)^n(n^2 + 9n) + 1066\frac{2}{3}n\cos\frac{2n\pi}{3} + 19224}{6!5!}\right].$$

The method can of course be continued; I simply state the result for $p_7(n)$.

$$p_7(n) = \left[\frac{\left(n^6 + 42n^5 + 560n^4 + 1960n^3 - 8725\frac{1}{2}n^2 - 45,325n - (-1)^n \cdot 2362\frac{1}{2}(n^2 + 14n) + 22,400n \operatorname{cosec}\frac{\pi}{3}\sin\frac{\pi}{3}(2n - 7) + 1,029,154\right)}{7!6!}\right].$$

Having determined the explicit form for $p_6(n)$, it is time for some general remarks. Looking at the method of production, we can see that the leading terms are purely algebraic and that this property of the formulas will continue under the operator $\frac{1}{\Delta(m)}$. The leading nonalgebraic power of n or, more precisely, its coefficient increases when (-1) is a root of the operator $\frac{1}{\Delta(m)}$, as we see from formula 2.5.

That is for all even powers of m. Thus for $m = 7$ we have that the first four powers are purely algebraic, that is, for n^6, n^5, n^4, and n^3. For $n = 8$ we have that n^7, n^6, n^5, and n^4 will be, but not n^3.

The pattern is quite clear, and we can see that the first $\left[\frac{m+1}{2}\right]$ powers are purely algebraic in $p_m(n)$. We can go further than this and say that $p_m(n)$ contains a purely algebraic part which is a polynomial in n of degree $(m - 1)$ with rational coefficients as the Bernoulli numbers B_i are rational. Let this polynomial of degree $(m - 1)$ be denoted by $q_m(n)$ (say) and the trigonometric or nonpolynomial part by $t_m(n)$. Thus

$$p_m(n) = q_m(n) + t_m(n),$$

where the polynomials $q_m(n)$ naturally satisfy

$$q_m(n) - q_m(n - m) = q_{m-1}(n - 1).$$

From the forms so far determined, we have

$$q_2(n) = \frac{1}{2!1!}(n - \tfrac{1}{2})$$

$$q_3(n) = \frac{1}{3!2!}(n^2 - 1\tfrac{1}{6})$$

$$q_4(n) = \frac{1}{4!3!}(n^3 + 3n^2 - 4\tfrac{1}{2}n - 6\tfrac{1}{2})$$

$$q_5(n) = \frac{1}{5!4!}(n^4 + 10n^3 + 10n^2 - 75n - 61\tfrac{19}{30})$$

$$q_6(n) = \frac{1}{6!5!}(n^5 + 22\tfrac{1}{2}n^4 + 126\tfrac{2}{3}n^3 - 112\tfrac{1}{2}n^2 - 1599\tfrac{1}{6}n - 756\tfrac{3}{4})$$

where the constant term is just the value of C_0. As the first $\left[\frac{m+1}{2}\right]$ terms agree with $p_m(n)$, an examination of the general form of these leading terms is required.

3. A SERIES EXPANSION FOR $q_m(n)$

The general form for the leading terms of $q_m(n)$ are given in [1], where I also consider the problem of determining an upper bound for $p_m(n)$ for arbitrary m and n, together with some numerical examples. For the sake of completeness, I simply quote the expansion of $q_m(n)$ given in that paper.

$$\begin{aligned} q_m(n) &= \frac{n^{m-1}}{m!(m-1)!} + \frac{1}{m!(m-2)!}\left(\frac{m^2 - 3m}{4 \cdot 1!}\right)n^{m-2} \\ &+ \frac{1}{m!(m-3)!}\left(\frac{m^4 - \frac{58}{9}m^3 + \frac{75}{9}m^2 - \frac{2}{9}m}{4^2 \cdot 2!}\right)n^{m-3} \\ &+ \frac{1}{m!(m-4)!}\left(\frac{m^6 - \frac{31}{3}m^5 + 29m^4 - \frac{65}{3}m^3 + 2m^2}{4^3 \cdot 3!}\right)n^{m-4} \\ &+ \frac{1}{m!(m-5)!}\left[\frac{\left(m^8 - 14\frac{2}{3}m^7 + 66\frac{16}{27}m^6 - 107\frac{29}{225}m^5 + 55\frac{134}{135}m^4 - 10\frac{54}{135}m^3 + \frac{4}{27}m^2 - \frac{16}{225}m\right)}{4^4 \cdot 4!}\right]n^{m-5} \end{aligned}$$

where the first $\left[\frac{m+1}{2}\right]$ terms in the expansion of $p_m(n)$ are algebraic and agree with the terms above if $\left[\frac{m+1}{2}\right] \geqslant 5$ or $m \geqslant 9$. The polynomials can be generated by means of a computer program where the summations are effected using the Bernoulli polynomials. This expansion, although of some interest, is of little use for calculating $p_m(n)$ unless n is large compared with m. J. W. L. Glaisher gives an expansion for $q_m(n)$ based on the "waves" of J. J. Sylvester (see Gupta [3]).

Looking at the action of the operator $1/\Delta(m)$ in formulas 2.2 and 2.3, it is easy to see the form of the leading term in $t_m(n)$, the nonpolynomial part of $p_m(n)$. We have

$$t_m(n) = \frac{(-1)^{m+n} n^{\left[\frac{m-2}{2}\right]}}{2^m \left[\frac{m}{2}\right]! \left[\frac{m-2}{2}\right]!} \ldots \text{ for } m \geqslant 4.$$

4. CONCLUSION

The method not only yields closed formulas for small values of but also illustrates the general structure of $p_m(n)$. The method is perfectly general but clearly, as the formulas are calculated recurvisely, the computations become increasingly lengthy. The method can also be used to determine closed formulas for partitioning into an arbitrary small set of integers. The recurrence relationship is

$$p^*(p_1, p_2, \ldots, p_m; n) - p^*(p_1, p_2, \ldots, p_m; n - p_m) = p^*(p_1, p_2, \ldots, p_{m-1}; n)$$

where $p^*(p_1, p_2, \ldots, p_m; n)$ means the number of partitions of n into at most parts $p_1, p_2, \ldots, p_m$ or, equivalently, the number of solutions in integers $\geqslant 0$ of the Diophantine equation

$$p_1x_1 + p_2x_2 + \cdots + p_mx_m = n.$$

For example, the method yields

$$p^*(1, 2, 3, 5; n) = \left[\frac{n^3 + 16\frac{1}{2}n^2 + 81n + 180}{1 \cdot 2 \cdot 3 \cdot 5 \cdot 3!}\right].$$

This more general problem will be explored in a future paper.

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n-DIMENSIONAL FIBONACCI NUMBERS AND THEIR APPLICATIONS

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INTRODUCTION

In one of his papers [3] Bernstein investigated the $F(n)$ function. This function was derived from a special kind of numbers which could well be defined as 3-dimensional Fibonacci numbers. The original Fibonacci numbers should then be called 2-dimensional Fibonacci numbers. The present paper deals with n-dimensional Fibonacci numbers in a sense to be explained in the sequel. In a later paper [4] Bernstein derived an interesting identity that was based on 3-dimensional Fibonacci numbers. Also Carlitz in his paper [5] deals with this subject.

If we remember that the original Fibonacci numbers are generated by the formula

$$F(n) = \sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n-i}{i}, \quad n = 1, 2, \ldots,$$

then the function

$$F(n) = \sum_{i=0} (-1)^i \binom{n-2i}{i}$$

can be regarded as a generalization of the first, and the author thought that

$$F(n) = \sum_{i=0} (-1)^i \binom{n-ki}{i}, \quad k = 1, 2, \ldots,$$

could serve as a k - 1-dimensional generalization of the original Fibonacci numbers, but, regretfully, this consideration led nowhere. From the fact that the Fibonacci numbers are derived from the periodic expansion by the Euclidean algorithm of $\sqrt{5}$, there is opened a new horizon for the wanted generalization.

In a previous paper [1], the author had followed the ideas of Perron [9] and of Bernstein [4] and stated a general Algorithm that leads to an n-dimensional generalization of Fibonacci numbers.

In this paper, the author is introducing the GEA (Generalized Euclidean Algorithm) to investigate the various properties and applications of her k-dimensional Fibonacci numbers. It first turns out that these k-dimensional Fibonacci numbers are most useful for a good approximation of algebraic irrationals by rational integers. Further, the author proceeded to investigate higher-degree Diophantine equations and to state identities of a larger magnitude than those investigated before, in an explicit and simple form.

n-DIMENSIONAL FIBONACCI NUMBERS AND THEIR APPLICATIONS

1. THE GEA

Let w be the irrational

$$\begin{cases} w = \sqrt[n]{D^n + 1};\ n \geqslant 2,\ D \in N;\ x^{(v)} = (x_1^{(v)}(w),\ \ldots,\ x_{n-1}^{(v)}(w)), \\ \langle a^{(v)} \rangle,\ \langle b^{(v)} \rangle \text{ sequences of the form } x^{(v)},\ v = 0,\ 1,\ \ldots\ . \end{cases} \tag{1.1}$$

The GEA of the fixed vector $a^{(0)}$ is the sequence $\langle a^{(v)} \rangle$ obtained by the recurrency formula

$$\begin{cases} a^{(v+1)} = (a_1^{(v)} - b_1^{(v)})^{-1}(a_2^{(v)} - b_2^{(v)},\ \ldots,\ a_{n-1}^{(v)} - b_{n-1}^{(v)},\ 1) \\ \quad b_i^{(v)} = a_i^{(v)}(D);\ i = 1,\ \ldots,\ n-1;\ v = 0,\ 1,\ \ldots;\ a_1^{(v)} \neq b_1^{(v)}. \end{cases} \tag{1.2}$$

The GEA of $a^{(0)}$ is called purely periodic if there exists a number m such that

$$\begin{cases} a^{(0)} = a^{(m)};\ m \text{ is called the length of} \\ \qquad\qquad \text{the primitive period} \end{cases} \tag{1.3}$$

The following formulas were proved in [2]. Let

$$\begin{cases} A_s^{(v+n)} = \sum_{k=0}^{n-1} b_k^{(v)} A_s^{(v+k)};\ v = 0,\ 1,\ \ldots \\ \quad A_i^{(j)} = \delta_i^j;\ \delta_i^j \text{ the Kronecker delta,} \\ \qquad\qquad i,\ j = 0,\ 1,\ \ldots,\ n-1;\ s = 0,\ 1,\ \ldots,\ n-1; \\ \quad b_k^{(v)} = a_k^{(v)}(D);\ k = 0,\ 1,\ \ldots,\ n-1;\ a_0^{(v)} = b_0^{(v)} = 1; \end{cases} \tag{1.4}$$

$A_s^{(v)}$ are called the matricians of GEA; then the three formulas hold:

$$\begin{vmatrix} A_0^{(v)} & A_0^{(v+1)} & \ldots & A_0^{(v+n-1)} \\ A_1^{(v)} & A_1^{(v+1)} & \ldots & A_1^{(v+n-1)} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n-1}^{(v)} & A_{n-1}^{(v+1)} & \ldots & A_{n-1}^{(v+n-1)} \end{vmatrix} = (-1)^{v(n-1)} \tag{1.5}$$

$$\begin{cases} a_s^{(0)} = \dfrac{\sum_{k=0}^{n-1} a_k^{(v)} A_s^{(v+k)}}{\sum_{k=0}^{n-1} a_k^{(v)} A_0^{(v+k)}},\ v = 0,\ 1,\ \ldots;\ s = 0,\ \ldots,\ n-1. \end{cases} \tag{1.6}$$

$$\prod_{k=1}^{v} a_{n-1}^{(k)} = \sum_{k=0}^{n-1} a_k^{(v)} A_0^{(v+k)}. \tag{1.7}$$

Perron proved the following theorem which, under the conditions of the GEA ($D \geqslant 1$), becomes

Theorem 1

The GEA is convergent in the sense that

$$\left\{ a_s^{(0)} = \frac{\lim_{v \to \infty} A_0^{(v)}}{\lim_{v \to \infty} A_0^{(v)}}, \quad s = 1, \ldots, n - 1 \right. \tag{1.8}$$

$A_s^{(v)} : A_0^{(v)}$ is called the vth convergent of GEA.

In [1], the author proved

Theorem 2

If the GEA of $a^{(0)}$ is purely periodic with m = length of the primitive period, then

$$\left\{ \begin{array}{l} \prod_{k=0}^{m-1} a_{n-1}^{(k)} = \sum_{k=0}^{n-1} a_k^{(m)} A_0^{(m+k)}, \\ \text{is a unit in } Q(w). \end{array} \right. \tag{1.9}$$

From (1.9) the formula follows, in virtue of (1.7),

$$\left(\prod_{k=0}^{m-1} a_{n-1}^{(k)} \right)^v = \sum_{k=0}^{n-1} a_k^{(m)} A_0^{(vm+k)}, \quad v = 1, 2, \ldots . \tag{1.10}$$

2. A PERIODIC GEA

In this section, we construct a periodic GEA, with length of primitive period $m = 1$. The fixed vector $a^{(0)}$ must be chosen accordingly, and this may look complicated at first. We prove

Theorem 3

The GEA of the fixed vector

$$\left\{ \begin{array}{l} a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \ldots, a_s^{(0)}, \ldots, a_{n-1}^{(0)}) \\ a_s^{(0)} = \sum_{i=0}^{s} \binom{n - s - 1 + i}{i} w^{s-i} D^i \\ s = 1, \ldots, n - 1 \end{array} \right. \tag{2.1}$$

is purely periodic and the length of its primitive period $m = 1$.

Proof: We shall first need the formula

$$\sum_{i=0}^{s} \binom{n - s - 1 + i}{i} = \binom{n}{s}, \quad s = 1, \ldots, n - 1. \tag{2.2}$$

This is proved by induction. The proof is left to the reader. We have, from (2.1), the following components of $a^{(0)}$ which we shall use later:

$$a_1^{(0)} = w + (n - 1)D; \; a_{n-1}^{(0)} = \sum_{i=0}^{n-1} w^{n-1-i}D^i. \tag{2.3}$$

Since $w^n - D^n = 1$, we also have

$$\sum_{i=0}^{n-1} w^{n-1-i}D^i = (w - D)^{-1}. \tag{2.4}$$

The vectors $b_i^{(v)}$ $(i = 1, \ldots, n - 1; \; v = 0, 1, \ldots)$ obtained from $a_i^{(v)}(w)$ by the defining rule (1.2) are called their corresponding companion vectors. We shall calculate the companion vector $b^{(0)}$ of $a^{(0)}$ and have

$$b_s^{(0)} = \sum_{i=0}^{s} \binom{n - s - 1 + i}{i} D^{s-i}D^i = D^s \sum_{i=0}^{s} \binom{n - s - 1 + i}{i},$$

so that, by (2.2),

$$b_s^{(0)} = \binom{n}{s} D^s, \; s = 1, 2, \ldots, n - 1. \tag{2.5}$$

Thus,

$$b^{(0)} = \left(\binom{n}{1}D, \binom{n}{2}D^2, \ldots, \binom{n}{n-1}D^{n-1}\right).$$

We shall now calculate the vector $a^{(1)}$. From (1.2), it follows that

$$a^{(1)} = (a_1^{(0)} - b_1^{(0)})^{-1}(a_2^{(0)} - b_2^{(0)}, \ldots, a_{n-1}^{(0)} - b_{n-1}^{(0)}, 1). \tag{2.6}$$

From (2.3), (2.4), and (2.5), we obtain:

$$\begin{cases} a_1^{(0)} - b_1^{(0)} = w + (n - 1)D - \binom{n}{1}D = w - D, \\ a_{n-1}^{(1)} = (w - D)^{-1} = \sum_{i=0}^{n-1} w^{n-1-i}D^i = a_{n-1}^{(0)}. \end{cases} \tag{2.7}$$

We can prove the relation

$$(a_s^{(0)} - b_s^{(0)})(a_1^{(0)} - b_1^{(0)})^{-1} = a_{s-1}^{(1)}, \; s = 2, \ldots, n - 1. \tag{2.8}$$

Since the proof is elementary, we leave it to the reader.

From (2.6), it follows that

$$\begin{cases} (a^{(1)} = \; a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-2}^{(0)}, a_{n-1}^{(0)}) = a^{(0)}, \\ a^{(v)} = a^{(0)}, \; v = 1, 2, \ldots . \end{cases} \tag{2.9}$$

This proves Theorem 3.

3. EXPLICIT MATRICIANS $A_0^{(v+n)}$

We shall proceed to find an explicit formula for the "zero-degree matricians" $A_0^{(v+n)}$, $v = 0, 1, \ldots$, and shall make use, for this purpose, of the defining formula (1.4), and the fact that the GEA is purely periodic with length of the primitive period $m = 1$. Taking into account (2.5) and (2.9), we have

$$A_0^{(v+n)} = \sum_{s=0}^{n-1} \binom{n}{s} D^s A_0^{(v+s)}; \quad v = 0, 1, \ldots. \tag{3.1}$$

We shall now make use of Euler's generating function. We have

$$\begin{aligned}
\sum_{i=0}^{\infty} A_0^{(i)} x^i &= x^0 A_0^{(0)} + \sum_{i=1}^{n-1} A_0^{(i)} x^i + \sum_{i=n}^{\infty} A_0^{(i)} x^i \\
&= 1 + \sum_{i=0}^{\infty} x^{i+n}\left(A_0^{(i)} + \binom{n}{1} D A_0^{(i+1)} + \binom{n}{2} D^2 A_0^{(i+2)} + \cdots + \binom{n}{n-1} D^{n-1} A_0^{i+n-1}\right) \\
&= 1 + x^n \sum_{i=0}^{\infty} A_0^{(i)} x^i + x^{n-1} \sum_{i=0}^{\infty} \binom{n}{1} D A_0^{(i+1)} x^{i+1} + x^{n-2} \sum_{i=0}^{\infty} \binom{n}{2} D^2 A_0^{(i+2)} x^{(i+2)} \\
&\quad + \cdots + x \sum_{i=0}^{\infty} \binom{n}{n-1} D^{n-1} A_0^{(i+n-1)} x^{i+n-1} \\
&= 1 + \left(x^n + \binom{n}{1} D x^{n-1} + \binom{n}{2} D^2 x^{n-2} + \cdots + \binom{n}{n-1} D^{n-1} x\right) \sum_{i=0}^{\infty} A_i^{(0)} x^i \\
&\quad - \left(\binom{n}{1} D x^{n-1} + \binom{n}{2} D^2 x^{n-2} + \cdots + \binom{n}{n-1} D^{n-1} x\right) = \sum_{i=0}^{\infty} A_0^{(i)} x^i,
\end{aligned}$$

$$\left[1 - \left(x^n + \sum_{k=1}^{n-1} \binom{n}{k} D^k x^{n-k}\right)\right] \sum_{i=0}^{\infty} A_0^{(i)} x^i = 1 - \sum_{k=1}^{n-1} \binom{n}{k} D^k x^{n-k},$$

$$\sum_{i=0}^{\infty} A_0^{(i)} x^i = \frac{1 - \left(x^n + \sum_{k=1}^{n} \binom{n}{k} D^k x^{n-k}\right) + x^n}{1 - \left(x^n + \sum_{k=1}^{n-1} \binom{n}{k} D^k x^{n-k}\right)}$$

$$= \frac{x^n}{1 - \sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k}} + 1,$$

$$A_0^{(0)} + \sum_{i=1}^{\infty} A_0^{(i)} x^i = x^n \sum_{t=0}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k}\right)^t + 1,$$

$$x A_0^{(1)} + A_0^{(2)} x^2 + \cdots + A_0^{(n-1)} x^{n-1} + \sum_{i=n}^{\infty} A_0^i x^i = x^n \sum_{t=0}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k}\right)^t,$$

For x sufficiently small. Thus, since $A_0^{(1)} = \cdots = A_0^{(n-1)} = 0$, we have

$$\sum_{i=n}^{\infty} A_0^{(i)} x^i = x^n \sum_{t=0}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k} \right)^t,$$

$$\sum_{i=0}^{\infty} A_0^{(n+i)} x^{n+i} = x^n \sum_{t=0}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k} \right)^t,$$

$$\sum_{i=0}^{\infty} A_0^{(n+i)} x^i = \sum_{t=0}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k} \right)^t, \tag{3.2}$$

and comparing coefficients of powers x^v on both sides of (3.2), we obtain

$$A_0^{(v+n)} = \sum_{ny_1 + (n-1)y_2 + \cdots + 2y_{n-1} + y_n = v} \binom{y_1 + y_2 + \cdots + y_n}{y_1,\ y_2,\ \ldots,\ y_n} \prod_{k=0}^{n-1} \left(\binom{n}{k} D^k \right)^{y_{k+1}} \tag{3.3}$$

or

$$A_0^{(v+n)} = \sum_{\sum_{i=0}^{n-1}(n-i)y_{i+1} = v} \binom{y_1 + y_2 + \cdots + y_n}{y_1,\ y_2,\ \ldots,\ y_n} D^{\sum_{j=1}^{n-1} j y_{j+1}} \prod_{k=0}^{n-1} \binom{n}{k}^{y_{k+1}}, \tag{3.4}$$

$$v = 0, 1, \ldots .$$

Formula (3.4) looks very complicated. $A_0^{(v+n)}$ can also be calculated by the recurrency relation (1.4). It is conjectured that it is easier to do so by formula (3.4), and would be a challenging computer problem.

4. MATRICIANS OF DEGREE s, $s = 1, 2, \ldots, n-1$

In this section, we shall express "s-degree matricians,"

$$A_s^{(v)}, \quad s = 1, \ldots, n-1,$$

by means of zero-degree matricians. This is not an easy task. Now we shall prove a very important theorem.

Theorem 4

The s-degree matricians are expressed through the zero-degree matricians by means of the relation

$$A_s^{(v+n-1)} = \sum_{k=0}^{s} \binom{n}{k} D^k A_0^{(v+n-s+k-1)}, \quad v = 0, 1, \ldots; \quad s = 1, \ldots, n-1. \tag{4.1}$$

Proof: From formula (1.6) it follows that

$$a_s^{(0)} \sum_{k=0}^{n-1} a_k^{(0)} A^{(v+k)} = \sum_{k=0}^{n-1} a_k^{(0)} A_0^{(v+k)}, \quad s = 1, 2, \ldots, n-1; \quad v = 0, 1, \ldots . \tag{4.2}$$

Or, writing a_i for $a_i^{(0)}$, $i = 0, \ldots, n - 1$, and substituting their values from (2.1), we obtain

$$\sum_{i=0}^{s} \binom{n - s - 1 + i}{i} w^{s-i} D^i \sum_{k=0}^{n-1} a_k A_0^{(v+k)} = \sum_{k=0}^{n-1} a_k A_s^{(v+k)}. \tag{4.3}$$

We shall now compare coefficients of w^{n-1} on both sides of (4.3). The power of w^{n-1} appears, on the right side only in

$$a_{n-1} = w^{n-1} + Dw^{n-2} + \cdots + D^{n-1},$$

and its coefficients is

$$A_s^{(v+n-1)}. \tag{4.4}$$

So the whole problem is to find the coefficient of w^{n-1} on the left side, and this is *the* problem. We shall start with the first power of w in a_s, which is w^s (in the left side). Now in

$$\sum_{k=0}^{n-1} a_k A_0^{(n+k)}$$

we have to look for those a_k's which have the powers w^{n-s-1}; this appears in

$$\begin{aligned}
&a_{n-s-1} \left(\text{first term, coefficient} = A_0^{(v+n-s-1)}\right)\\
&a_{n-s} \left(\text{second term, coefficient} = \binom{s}{1} DA_0^{(v+n-s)}\right)\\
&a_{n-s+1} \left(\text{third term, coefficient} = \binom{s}{2} D^2 A_0^{(v+n-s+1)}\right)\\
&\vdots\\
&a_{n-1} \left((1+s)\text{th term, coefficient} = \binom{s}{s} D^s A_0^{(v+n-1)}\right).
\end{aligned}$$

Thus, we have obtained the partial sum of coefficients of w^{n-1} in the left side.

$$A_0^{(v+n-s-1)} + \binom{s}{1} DA_0^{(v+n-s)} + \binom{s}{2} D^2 A_0^{(v+n-s+1)} + \cdots + \binom{s}{s} D^s A_0^{(v+n-1)}.$$

Now the next power of a_s on the left side is w^{s-1} with coefficient

$$\binom{n - s - 1 + 1}{1} D = \binom{n - s}{1} D.$$

To obtain w^{n-1}, w^{s-1} must be multiplied by $n - s$, so we must start with the first term of a_{n-s}, the second term of a_{n-s+1}, $\ldots$, etc. Compared with the previous sum, s has to be replaced by $s - 1$. The sum will then be multiplied by $\binom{n - s}{1} D$, and the number of summands will be smaller by one. We then obtain the partial sum:

$$\binom{n-s}{1} D \left[A_0^{(v+n-} + \binom{s-1}{1} DA_0^{(v+n-s+1)} + \binom{s-1}{2} D^2 A_0^{(v+n-s+2)} + \cdots + \binom{s-1}{s-1} D^{s-1} A_0^{(v+n-1)} \right].$$

Proceeding in this way, we obtained the partial sums:

$$
\left\{
\begin{aligned}
&A_0^{(v+n-s-1)} + \binom{s}{1}DA_0^{(v+n-s)} + \binom{s}{2}D^2A_0^{(v+n-s+1)} + \binom{s}{3}D^3A_0^{(v+n-s+2)} + \binom{s}{4}D^4A_0^{(v+n-s+3)} + \cdots + \binom{s}{k}D^kA_0^{(v+n-s-1+k)}\\
&+ \binom{n-s}{1}DA_0^{(v+n-s)} + \binom{n-s}{1}\binom{s-1}{1}D^2A_0^{(v+n-s+1)} + \binom{n-s}{1}\binom{s-1}{2}D^3A_0^{(v+n-s+2)} + \binom{n-s}{1}\binom{s-1}{3}D^4A_0^{(v+n-s+3)} + \cdots + \binom{n-s}{1}\binom{s-1}{k-1}D^kA_0^{(v+n-s-1+k)}\\
&+ \binom{n-s+1}{2}\binom{s-2}{0}D^2A_0^{(v+n-s+1)} + \binom{n-s+1}{2}\binom{s-2}{1}D^3A_0^{(v+s-n+2)} + \binom{n-s+1}{2}\binom{s-2}{2}D^4A_0^{(v+n-s+3)} + \cdots + \binom{n-s+1}{2}\binom{s-2}{k-2}D^kA_0^{(v+n-s-1+k)}\\
&+ \binom{n-s+2}{3}\binom{s-3}{0}D^3A_0^{(v+n-s+2)} + \binom{n-s+2}{3}\binom{s-3}{1}D^4A_0^{(v+n-s+3)} + \cdots + \binom{n-s+2}{3}\binom{s-3}{k-3}D^kA_0^{(v+n-s-1+k)}\\
&+ \cdots\cdots\cdots\cdots + \binom{n-s-1+k}{k}\binom{s-k}{k-k}D^kA_0^{(v+n-s-1+k)}\\
&+ \cdots\cdots\cdots\cdots\\
&\text{(continued)}\\
&+ \cdots + \binom{s}{s-1}D^{s-1}A_0^{(v+n-2)} + \binom{s}{s}D^sA_0^{(v+n-1)}\\
&+ \cdots + \binom{n-s}{1}\binom{s-1}{s-2}D^{s-1}A_0^{(v+n-2)} + \binom{n-s}{1}\binom{s-1}{s-1}D^sA_0^{(v+n-1)}\\
&+ \cdots + \binom{n-s+1}{2}\binom{s-2}{s-3}D^{s-1}A_0^{(v+n-2)} + \binom{n-s+1}{2}\binom{s-2}{s-2}D^sA_0^{(v+n-1)}\\
&+ \cdots + \binom{n-s+2}{3}\binom{s-3}{s-4}D^{s-1}A_0^{(v+n-2)} + \binom{n-s+2}{3}\binom{s-3}{s-3}D^sA_0^{(v+n-1)}\\
&+ \cdots + \binom{n-s-1+k}{s-k-1}\binom{s-k}{s-k-1}D^{s-1}A_0^{(v+n-2)} + \binom{n-s-1+k}{k}\binom{s-k}{s-k}D^sA_0^{(v+n-1)}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&+ \binom{n-s-1+s}{s}\binom{s-s}{s-s}D^sA_0^{(v+n-1)}
\end{aligned}
\right.
\tag{4.5}
$$

Thus the general term in the sum of coefficients of w^{n-1} on the left side of (4.3) which contains $D^k A_0^{(v+n-s-1+k)}$ as a constant factor has the form, adding up in (4.5) the column with this factor,

$$\sum_{i=0}^{k} \binom{n-s-1+i}{i} \binom{s-i}{k-i} D^k A_0^{(v+n-s-k+k)}. \tag{4.6}$$

The following formula is well known:

$$\sum_{i=0}^{k} \binom{n-s-1+i}{i} \binom{s-i}{k-i} = \binom{n}{k}, \tag{4.7}$$

which becomes formula (2.2) for $k = s$. Now, since in

$$a_s = \sum_{i=0}^{s} \binom{n-s-1+i}{i} w^{s-i} D^i,$$

the exponent of D sums from $i = 0$ to $i = s$, we have, finally,

$$A_s^{(v+n-1)} = \sum_{k=0}^{s} \binom{n}{k} D^k A_0^{(v+n-1-s+k)}$$

which is formula (4.1) and proves Theorem 4. From formula (4.1), we have the single cases

$$A_1^{(v+n-1)} = A_0^{(v+n-2)} + \binom{n}{1} D A_0^{(v+n-1)}, \tag{4.8}$$

and

$$A_{n-1}^{(v+n-1)} = A_0^{(v+n)}. \tag{4.9}$$

(4.9) is a very surprising relation and will be applied in the next section. Similarly,

$$A_2^{(v+n-1)} = A_0^{(v+n-3)} + \binom{n}{1} D A_0^{(v+n-2)} + \binom{n}{2} D^2 A_0^{(v+n-s)}, \text{ etc.} \tag{4.10}$$

5. APPROXIMATION OF IRRATIONALS BY RATIONALS

We shall investigate especially the case $D = 1$, but produce first formulas for any value of D. We obtain from (4.8) and (1.6),

$$a_1^{(0)} = \frac{\lim\limits_{v\to\infty} A_1^{(v+n-1)}}{\lim\limits_{v\to\infty} A_0^{(v+n-1)}} = \lim_{v\to\infty} \frac{A_0^{(v+n-2)} + nDA_0^{(v+n-1)}}{A_0^{(v+n-1)}},$$

$$w + (n-1)D = nD + \lim_{v\to\infty} \frac{A_0^{(v+n-2)}}{A_0^{(v+n-1)}},$$

$$w = D + \lim_{v\to\infty} \frac{A_0^{(v+n-2)}}{A_0^{(v+n-1)}} = D + \lim_{v\to\infty} \frac{A_0^{(v+n-1)}}{A_0^{(v+n)}}. \tag{5.1}$$

For $D = 1$, $w = \sqrt[n]{2}$, and from (3.4) and (5.1) we obtain the approximation formula

$$\sqrt[n]{2} \approx \left\{ \frac{\displaystyle\sum_{\sum(n-i)y_{i+1} = v,\ i=0,\ldots,n-1} \binom{y_1 + y_2 + \cdots + y_n}{y_1,\ y_2,\ \ldots,\ y_n} \prod_{k=0}^{n-1} b^{y_{k+1}}}{\displaystyle\sum_{\sum(n-i)y_{i+1} = v+1,\, i=0,\ldots,n-1} \binom{y_1 + y_2 + \cdots + y_n}{y_1,\ y_2,\ \ldots,\ y_n} \prod_{k=0}^{n-1} b^{y_{k+1}}} \right.$$

$$b_k = \binom{n}{k},\ k = 0,\ \ldots,\ n-1;\ b_0 = 1.$$

The approximations are not very close, and we would have to continue a few steps further to get a closer approximation. Formula (4.9), surprisingly simple as it is, does not yield any news. It enables us to calculate w^{n-1} by means of the powers w_k, $k = 1, \ldots, n-2$.

We have approximately, expanding $\sqrt[n]{2} = (1+1)^{1/n}$ by the binomial series,

$$\sqrt[n]{2} \approx 1 + \frac{1}{n}.$$

According to our approximation formula (5.1) with $D = 1$,

$$\sqrt[n]{2} = w \approx 1 + \frac{A_0^{(n)}}{A_0^{(n+1)}};$$

$$A_0^{(n+1)} = A_0^{(1)} + \binom{n}{1}A_0^{(2)} + \cdots + \binom{n}{n-1}A_0^{(n)} = \binom{n}{n-1}A_0^{(n)} = nA_0^{(n)} = n,$$

since $A_0^{(n)} = A_0^{(0)} + A_0^{(1)} + \cdots + \binom{n}{n-1}A_0^{(n-1)} = A_0^{(0)} = 1$, $\sqrt[n]{2} \approx 1 + \frac{1}{n}$, as should be.

6. DIOPHANTINE EQUATIONS

We shall construct two types of Diophantine equations of degree n in n unknowns and state their explicit solutions, which are infinite in number. We have from (1.5)

$$\begin{vmatrix} A_0^{(v+n)} & A_0^{(v+n+1)} & A_0^{(v+n+2)} & \cdots & A_0^{(v+n+n-1)} \\ A_1^{(v+n)} & A_1^{(v+n+1)} & A_1^{(v+n+2)} & & A_1^{(v+n+n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n-1}^{(v+n)} & A_{n-1}^{(v+n-1)} & A_{n-1}^{(v+n+2)} & & A_{n-1}^{(v+n+n-1)} \end{vmatrix} = (-1)^{(n-1)v}, \tag{6.1}$$

$$v = 0, 1, \ldots.$$

Substituting in (6.1) the values of $A_s^{(t)}$ from (4.1) we obtain, after simple row rearrangements,

$$\begin{vmatrix} A_0^{(v+n)} & A_0^{(v+n+1)} & A_0^{(v+n+2)} & \cdots & A_0^{(v+n+n-1)} \\ A_0^{(v+n-1)} & A_0^{(v+n)} & A_0^{(v+n+1)} & \cdots & A_0^{(v+n+n-2)} \\ A_0^{(v+n-2)} & A_0^{(v+n-1)} & A_0^{(v+n)} & \cdots & A_0^{(v+n+n-3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_0^{(v+3)} & A_0^{(v+4)} & A_0^{(v+5)} & \cdots & A_0^{(v+n+2)} \\ A_0^{(v+2)} & A_0^{(v+3)} & A_0^{(v+4)} & \cdots & A_0^{(v+n+1)} \\ A_0^{(v+1)} & A_0^{(v+2)} & A_0^{(v+3)} & \cdots & A_0^{(v+n)} \end{vmatrix} = (-1)^{(n-1)v} \tag{6.3}$$

We introduce the notations

$$X_{v,k} = A_0^{(v+k)}, \quad k = 1, 2, \ldots, n. \tag{6.3}$$

$$\begin{cases} A_0^{(v+k)} = A_0^{(v+k-n)} + b_1 A_0^{(v+k-n+1)} + b_2 A_0^{(v+k+2-n)} + \cdots + b_{n-1} A_0^{(v+k-1)} \\ b_k = \binom{n}{k} D^k, \quad k = 0, 1, \ldots, n-1, \quad v = 1, 2, \ldots . \end{cases} \tag{6.4}$$

We introduce these notations in (6.2) and then make the following manipulations in this determinant.

From the first row we subtract the b_1 multiple of the first row from below, then the b_2 multiple of the second row from below, ..., then the b_kth multiple of the kth row from below, $k = 1, \ldots, n-1$.

Then (6.2) takes the form, in virtue of (6.4),

$$\begin{vmatrix} X_{v,n} - \sum_{k=1}^{n-1} b_k X_{v,k} & X_{v,1} & X_{v,2} & \cdots & X_{v,n-1} \\ A_0^{(v+n-1)} & A_0^{(v+n)} & A_0^{(v+n+1)} & \cdots & A_0^{(v+n+n-2)} \\ A_0^{(v+n-2)} & A_0^{(v+n-1)} & A_0^{(v+n)} & \cdots & A_0^{(v+n+n-3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_0^{(v+2)} & A_0^{(v+3)} & A_0^{(v+4)} & \cdots & A_0^{(v+n+1)} \\ X_{v,1} & X_{v,2} & X_{v,3} & \cdots & X_{v,n} \end{vmatrix} = (-1)^{(n-1)v} \tag{6.5}$$

We further subtract from the second row the b_2 multiple of the first row from below, the b_3 multiple of the second row from below, ..., the b_k multiple of the $(k-1)$th row from below; the determinant (6.5) then takes the form ($k = 2, \ldots, n-2$):

$$\left\{\begin{vmatrix} X_{v,n} - \sum_{k=1}^{n-1} b_k X_{v,k} & X_{v,1} & X_{v,2} & X_{v,3} & \cdots & X_{v,n-1} \\ X_{v,n-1} - \sum_{k=1}^{n-2} b_{k+1} X_{v,k} & X_{v,n} - \sum_{k=1}^{n-2} b_{k+1} X_{v,k+1} & X_{v,1} + b_1 X_{v,2} & X_{v,2} + b_1 X_{v,3} & \cdots & X_{v,n-2} + b_1 X_{v,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ V_{v,1} & X_{v,2} & X_{v,3} & X_{v,4} & \cdots & X_{v,n} \end{vmatrix} \right. = (-1)^{(n-1)v} \tag{6.6}$$

Continuing this process by another step, the third row of determinant (6.6) will have the form

$$X_{v,n-2} - \sum_{k=1}^{n-3} b_{k+2} X_{v,k} \quad X_{v,n-1} - \sum_{k=1}^{n-3} b_{k+2} X_{v,k+1} \quad X_{v,n} - \sum_{k=1}^{n-3} b_{k+2} X_{v,k+2}$$

$$X_{v,1} + b_1 X_{v,2} + b_2 X_{v,3} \quad X_{v,2} + b_1 X_{v,3} + b_2 X_{v,4} \quad \cdots$$

$$X_{v,n-3} + b_1 X_{v,n-2} + b_2 X_{v,n-1}.$$

Generally we subtract from the th row in (6.2) the b_i multiple of the first row from below, then the b_{i+1} multiple of the second row from below, ..., the b_{n-1} multiple of the $(n-i)$th row from below ($i = 1, \ldots, n-1$). The reader can verify, that by these operations the determinant (6.2) transforms into one containing only the unknowns $X_{v,i}$ ($i = 1, \ldots, n$), which yields the Diophantine equation of degree n in these unknowns.

7. MORE DIOPHANTINE EQUATIONS

The GEA of $a^{(0)}$ is purely periodic with length of the primitive period $m = 1$. Since

$$a_{n-1}^{(0)} = \sum_{i=0}^{n-1} \binom{n-1-(n-1)+i}{i} w^{n-1-i} D^i = \sum_{i=0}^{n-1} w^{n-1-i} D^i$$

we have by Theorem 2 and formula (1.10),

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^v = \sum_{i=0}^{n-1} a_i^{(0)} A_0^{(v+i)}, \quad v = 1, 2, \ldots. \tag{7.1}$$

We find the norm of $(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^v$. We have

$$\left\{\begin{aligned} & D^n - w^n = -1, \\ & D^n - w^n = -\sum_{k=0}^{n-1} (D - \rho_k w) = -N(D - w), \\ & \rho_k = e^{2\pi i k/n}, \ k = 0, 1, \ldots, n-1. \end{aligned}\right. \tag{7.2}$$

But $w^{n-1} + Dw^{n-2} + \cdots + D^{n-1} = -(D - w)^{-1}$; hence,

$$N[(w^{n-1}+Dw^{n-2}+\cdots+D^{n-1})^v] = (-1)^{(n-1)v}, \quad v = 1, 2, \ldots; \quad n = 2, 3, \ldots . \tag{7.3}$$

We have

$$\begin{aligned}\sum_{i=0}^{n-1} a_i^{(0)} A_0^{(v+i)} = A_0^{(v)} &+ \left[w + \binom{n-1}{1}D\right]A_0^{(v+1)} + \left[w^2 + \binom{n+2}{1}Dw + \binom{n-1}{2}D^2\right]A_0^{(v+2)} \\ &+ \left[w^3 + \binom{n-3}{1}w^2D + \binom{n-2}{2}wD^2 + \binom{n-1}{3}D^3\right]A_0^{(v+3)} + \cdots \\ &+ \left[w^{n-1} + \binom{1}{1}w^{n-1}D + \binom{2}{2}w^{n-2}D^2 + \cdots + \binom{n-1}{n-1}D^{n-1}\right]A_0^{(v+n-1)}.\end{aligned}$$

Denoting

$$\begin{cases} X_{v,k} = \displaystyle\sum_{s=0}^{n-1-k} \binom{n-1-k}{s} A_0^{(v+s+k)} D^s, \\ k = 0, 1, \ldots, n-s. \end{cases} \tag{7.4}$$

This $X_{v,k}$ is not the $X_{v,k}$ from (6.4). We have from (7.1),

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^v = \sum_{k=0}^{n-1} X_{v,k}w^k = e^v, \; e \text{ a unit.} \tag{7.5}$$

We shall find the field equation of

$$\sum_{k=0}^{n-1} X_{v,k}w^k.$$

The free member of it is the norm of e^v, and since e^v is a unit with the norm $(-1)^{(n-1)v}$, according to (7.3), we find easily, by known methods, that

$$\begin{vmatrix} X_{v,0} & X_{v,1} & X_{v,2} & \cdots & X_{v,n-2} & X_{v,n-1} \\ mX_{v,n-1} & X_{v,0} & X_{v,1} & \cdots & X_{v,n-3} & X_{v,n-2} \\ mX_{v,n-2} & mX_{v,n-1} & X_{v,0} & \cdots & X_{v,n-4} & X_{v,n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ mX_{v,2} & mX_{v,3} & mX_{v,4} & \cdots & X_{v,0} & X_{v,1} \\ mX_{v,1} & mX_{v,2} & mX_{v,3} & \cdots & mX_{v,n-1} & X_{v,1} \end{vmatrix} = (-1)^{(n-1)v} \tag{7.6}$$

It is not difficult to see that, in the case $n = 2m + 1$ $(m = 1, 2, \ldots)$, the highest powers of the n unknowns of the discriminant (7.6) as

$$X_{v,0}^n, \; mX_{v,1}^n, \; m^2X_{v,2}^n, \; \ldots, \; m^{n-1}X_{v,n-1}^n,$$

while the last unknown, $X_{v,n-1}$ does not have the exponent n, but a smaller one. In the case $n = 2m$ $(m = 1, 2, \ldots)$ these $n - 1$ powers are the same, but with alternating signs, viz.,

$$X_{v,0}^n, \ -mX_{v,1}^n, \ +m^2X_{v,2}^n, \ \ldots .$$

In the case $n = 2$, the expanded discriminat (7.6) had the form

$$X_v^2 - mY_v^2 = \pm 1,$$

and in the case $n = 3$, it had the form

$$X^3 + mY^3 + m^2Z^3 - 3mXYZ = 1.$$

The first is Pell's equation.

8. IDENTITIES AND UNITS

We return to formulas (7.4) and (7.5), and have

$$\begin{cases} X_{nv,k} = \sum_{s=0}^{n-1-k} \binom{n-1-k}{s} A_0^{(vn+s+k)} D^s \\ \\ k = 0, 1, \ldots, n-1 \\ \\ (w^{n-1} + Dw^{n-2} + \cdots + D^{n-1}) = \sum_{k=0}^{n-1} X_{nv,k}\, w^k. \end{cases} \tag{8.1}$$

We compare powers of w^k $(k = 0, 1, \ldots, n-1)$ on both sides of (8.1) and take into consideration that $w^{nt} = m^t = (D^n + 1)^t$. We have, looking for the rational part of the right side, $k = 0$, and the value of the right side equals $X_{nv,0}$, and by (7.4),

$$X_{nv,0} = \sum_{s=0}^{n-1} \binom{n-1}{s} A^{(nv+s)} D^s, \quad v = 0, 1, \ldots . \tag{8.2}$$

On the left side, we have to look for the coefficients of w^n. Since the highest power in the expression

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^{nv}$$

is $n(n-1)v$, we have the expression

$$\sum_{\substack{\sum_{i=1}^{n-1}(n-i)y_i = sn \leqslant n(n-1)v, \\ \sum_{i=1}^{n-1} iy_{i+1} = n(n-1)v - sn, \ s = 0, 1, \ldots, (n-1)v}} \binom{y_1 + y_2 + \cdots + y_n}{y_1, y_2, \ldots, y_n} w^{\sum_{i=1}^{n-1}(n-i)y_i} D^{\sum_{i=1}^{n-1} iy_{i+1}} = X_{nv}, \tag{8.3}$$

We want to obtain in this way the rational part of

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^{nv}.$$

At the same time

$$\sum_{i=1}^{n-1} iy_{i+1}$$

is the sum of the exponents of the powers of y_{i+1} ($i = 1, \ldots, n - 1$). Since in every summand of

$$w^{n-1} + Dw^{n-2} + \cdots + D^{n-1}$$

the sum of the exponents of $D^i w^{n-1-i}$ ($i = 0, 1, \ldots, n - 1$) is $n - 1$, and the highest exponent in the expansion if $n(n - 1)v$, we have that

$$\sum_{i=1}^{n-1} (n - i)y_i + \sum_{i=1}^{n-1} iy_{i+1} = n(n - 1)v,$$

which explains the left side of (8.3). We further have

$$\sum_{i=1}^{n-1} [(n - i)y_i + iy_{i+1}] = n(n - 1)v,$$

so that

$$y_1 + y_2 + \cdots + y_n = nv. \tag{8.4}$$

Now, taking into account that the exponent of w under the summation sign in (8.3) equals sn, $w^{sn} = m^s$, and $D^n = m - 1$, formula (8.3) takes the form

$$\begin{cases} \displaystyle\sum_{\sum_{i=1}^{n-1}(n-i)y_i = sn} \binom{nv}{y_1,\ y_2,\ \ldots,\ y_n} m^s(m - 1)^{(n-1)v-s} = X_{nv,0} \\ \displaystyle = \sum_{k=0}^{n-1} \binom{n-1}{k} D^k A_0^{(nv+k)}, \\ s = 0, 1, \ldots, (n - 1)v;\ v = 0, 1, \ldots \\ y_1 + y_2 + \cdots + y_n = nv \end{cases} \tag{8.5}$$

(8.5) is an interesting combinatorial identity.

From (8.1), $n - 1$ more identities can be obtained by comparing the coefficients of the powers w^i, $i = 1, \ldots, n - 1$, on both sides of (8.1). The identities have a somewhat complicated form; however, they will express the coefficients of w^t, $t = 1, \ldots, n - 1$, in the expansion of

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^{nv},$$

with $w^n = m = D^n + 1$:

$$\begin{cases} \displaystyle\sum_{\substack{\sum_{i=1}^{n-1}(n-i)y_i = sn+t \leqslant n(n-1)v \\ \sum_{i=1}^{n-1} iy_{i+1} = n(n-1)v-(sn+t)}} \binom{nv}{y_1, y_2, \ldots, y_n} m^s(m-1)^{(n-1)v-s-1}D^{n-t} = X_{nv,t} \\ \displaystyle = \sum_{j=0}^{n-1-t} \binom{n-1-t}{j} A_0^{(nv+j+t)} D^j \\ j = 0, 1, \ldots, (n-1)v-1; \; t = 1, \ldots, n-1. \end{cases} \tag{8.6}$$

We wish to explain the appearance of the factor D^{n-t} under the summation sign on the left side of (8.6). The power of D in the expantion of

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^{vn}$$

equals

$$\begin{aligned} \sum_{i=1}^{n-1} iy_{i+1} &= n(n-1)v - (sn+t) \\ &= n(n-1)v - sn - n + (n-t) \\ &= n[(n-1)v - s - 1] + n - t. \end{aligned}$$

Thus, the power of D equals

$$(D^n)^{n(n-1)v-s-1} \cdot D^{n-t}, \text{ with } D^n = m-1.$$

The power of w is

$$\sum_{i=1}^{n-1}(n-i)y_i = sn + t = (w^n)^s w^t = m^s w^t,$$

so m^s is the coefficient of w^t as desired.

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COUNTING THE PROFILES IN DOMINO TILING

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(Submitted June 1982)

1. INTRODUCTION

Read [2] describes "profiles" that can be formed when one tiles a given rectangle with dominoes. For rectangles of width $m = 2, 3, 4$, the number of profiles $N(m)$ subject to certain rules are shown to be 2, 9, and 12, respectively. In fact, it is not difficult for one to program a computer to produce the following tabulated values for $N(m)$:

m	2	3	4	5	6	7	8	9	10
$N(m)$	2	9	12	50	60	245	280	1134	1260

We notice that values of $N(m)$ grow rather rapidly. Knowing these numbers is helpful in the estimation of execution time and storage requirement if one follows Read's method to calculate the number of domino tilings on a given chessboard.

In this note, we shall sketch a proof of the following formula:

$$N(m) = \begin{cases} \dbinom{m}{m/2} m/2, & \text{if } m \text{ is even} \\[2ex] \dbinom{m+1}{(m+1)/2} m/2, & \text{if } m \text{ is odd.} \end{cases}$$

2. DEFINING THE PROFILES

The profiles in [2] can be seen as patterns on an $m \times 2$ board with certain properties. We label 1 for each square taken by a domino and label 0 for each square not taken by a domino on the profile. For $m = 4$, say, we can represent the 12 profiles in [2] as follows,

00	00	00	11	10	11	11	10	10	11	10	11
00	10	00	00	00	00	11	10	11	11	10	10
00	10	10	00	00	10	00	00	00	11	11	10
00	00	10	00	10	10	00	00	10	00	00	00
A	L	I	B	H	K	D	C	J	G	F	E
	(1)			(2)			(3)			(4)	

where the letters A-L are names of the corresponding profiles given in [2].

Count rows from top to bottom and columns from left to right. Assign Boolean variables $L_1, L_2, \ldots, L_m$ to the corresponding left squares and Boolean variables $R_1, R_2, \ldots, R_m$ to the corresponding right squares. Using the argument of [1], a profile can be defined as a solution of the following system of equations and inequalities,

$$\sum_{i=1}^{m} (-1)^{i+1}(L_i - R_i) = p$$

$$L_i \geqslant R_{i+j}, \; i = 1, \ldots, m; \; j = 0, 1, \ldots, m - i \tag{$*$}$$

$$L_1 + L_2 + \cdots + L_m < m,$$

where $p = 0$ if m is even and $p = 0$ or 1 if m is odd.

3. COUNTING THE PROFILES

We shall indicate how to calculate the number of solutions of the system $(*)$ when $m = 2h$ is even. Consider the cases,

$$C_k : L_k = 0, \text{ and } L_j = 1 \text{ for } j < k$$

for $k = 1, \ldots, m$. Then by the first inequality in $(*)$, $R_{k+j} = 0$ for $j = 0, 1, \ldots, m - k$. For example, when $m = 4$, the four cases are shown in the previous section.

Assume the case C_k. The equation in the system $(*)$ becomes

$$\sum_{i=1}^{k-1} (-1)^{i+1}(1 - R_i) + \sum_{i=k+1}^{m} (-1)^{i+1}L_i = 0.$$

When k is odd, there are

$$\sum_{i=0}^{h-i} \binom{h-1}{i}\binom{h}{i} \tag{1}$$

solutions.

When k is even, there are

$$\sum_{i=0}^{h-i} \binom{h-1}{i}\binom{h}{i+1} \tag{2}$$

solutions.

In either case, the number is independent of k. There are h odd k values and h even k values. The number of solutions of $(*)$ is h times the sum of (1) and (2), which is the number of profiles when m is even.

4. OTHER CONNECTIONS

Klarner and Pollack [1] attacked the domino tiling problem using a different approach. It is interesting to note that the number of profiles is always $m/2$ times the dimension of the graph matrix constructed in [1]. The graph matrix obtained from the profiles has a simpler structure than the one used in [1]. The number of edges of the graph matrix in Read [2] can be calculated by the following formula:

$$E(m) = \begin{cases} N(m) \times 3/2, & \text{if } m \text{ is even} \\ N(m) \times (3/2 - 1/(2m \times m)), & \text{if } m \text{ is odd.} \end{cases}$$

We see that $E(m)$ is close to 3/2 of $N(m)$ when m is large.

ACKNOWLEDGMENT

The author would like to thank Professors B. Coen and J. Malkevitch for their interest and stimulating conversations on the domino problem.

REFERENCES

1. D. A. Klarner & J. Pollack. "Domino Tilings of Rectangles with Fixed Width." *Discrete Mathematics* 32 (1980):53-57.
2. R. C. Read. "A Note on Tiling Rectangles with Dominoes." *The Fibonacci Quarterly* 18, No. 1 (1980):24-27.

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THE FIBONACCI SEQUENCE F_n MODULO L_m

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(Submitted August 1981)

This paper is concerned with determining the length of the period of a Fibonacci series after reducing it by a modulus m. Some of the results established by Wall (see [1]) are used. We investigate further the length of the period.

The Fibonacci sequence is defined with the conditions $f_0 = \alpha$, $f_1 = \beta$ and $f_{n+1} = f_n + f_{n-1}$ for $n > 1$. We will refer to the two special sequences when $\alpha = 0$, $\beta = 1$ and $\alpha = 2$, $\beta = 1$ as (F_n) and (L_n), respectively. (L_n) is often called the Lucas sequence.

The Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, ... reduced modulo 3 is

$$0,\ 1,\ 1,\ 2,\ 0,\ 2,\ 2,\ 1,\ 0,\ 1,\ 1,\ 2,\ \ldots\ .$$

The reduced sequence repeats after 8 terms. We say that the reduced sequence is periodic with period 8. The second half of the period is twice the first half. We refer to the terminology used by Robinson [2] and say that the sequence has a restricted period of 4 with multiplier 2 or -1 (since $2 \equiv -1 \bmod 3$). If the reduced sequence has a value of -1 at F_{k-1} and 0 at F_k, then the sequence is said to have a restricted period of k with multiplier -1. The period of the reduced sequence is $2k$. The $2k$ terms of the period form two sets of k terms. The terms of the second half are -1 times the terms of the first half.

Wall [1] produced many results concerning the length of the period of the recurring sequence obtained by reducing a Fibonacci sequence by a modulus m. The length of the period of the special sequence F_n reduced modulo m will be denoted by $p(m)$.

Theorem 1 (Wall)

$f_n \pmod{m}$ forms a simply periodic series. That is, the series is periodic and repeats by returning to its starting values.

We have (see [3]):

(1) $F_m = (a^m - b^m)/(a - b)$,

(2) $L_m = a^m + b^m = F_{m-1} + F_{m+1}$, where $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$.

Also,

(3) $F_{2m} \equiv 0 \pmod{L_m}$ [follows from (1) and (2)].

Note that

$$ab = \left(\frac{1 + \sqrt{5}}{2}\right)\left(\frac{1 - \sqrt{5}}{2}\right) = -1.$$

Since $(ab)^{m-1} = (-1)^{m-1}$, we have

$$\begin{aligned} a^{2m-1} - b^{2m-1} - (-1)^{m-1}(a - b) &= a^{2m-1} - b^{2m-1} - (ab)^{m-1}(a - b) \\ &= a^{2m-1} - b^{2m-1} - a\, b^{m-1} + a^{m-1}b^m \\ &= (a^{m-1} - b^{m-1})(a^m + b^m). \end{aligned}$$

From this, we have

$$F_{2m-1} - (-1)^{m-1} = F_{m-1}L_m.$$

Hence

(4) $F_{2m-1} \equiv (-1)^{m-1} \pmod{L_m}$.

Theorem 2

For $m \geqslant 2$, the Fibonacci sequence $F_n \pmod{L_m}$ has period $4m$ if m is even and period $2m$ if m is odd.

Proof: Suppose m is odd, and the sequence $F_n \pmod{L_m}$ has period p. It follows from (3) and (4) that the reduced sequence has values 1 at F_{2m-1} and 0 at F_{2m}. Therefore, $2m$ is a multiple of p and $2m = kp$ for some integer $k > 0$. From (2) we have $L_m = F_{m-1} + F_{m+1}$ and $L_m > F_j$ for all $j \leqslant m + 1$, if $m \geqslant 2$. Hence, L_m cannot divide any F_j for $j \leqslant m + 1$, which implies that $F_j \not\equiv 0 \pmod{L_m}$ for any $j \leqslant m$. Therefore, $p > m$, $kp = 2m < 2p$, and $k < 2$. Thus, $k = 1$ and $p(L_m) = 2m$.

Suppose m is even. It follows from (3) and (4) that the reduced sequence has values -1 at F_{2m-1} and 0 at F_{2m}. This implies that the reduced sequence has a restricted period. Let p' be the restricted period. It follows that $2m = k \cdot p'$ for some $k > 0$. Again $m < p'$ since $F_j < L$ for all $j \leqslant m$. This implies that $k < 2$ and, therefore, $k = 1$. Thus, the restricted period is $2m$ and the period is $4m$. ■

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2. D. W. Robinson. "The Fibonacci Matrix Modulo m." *The Fibonacci Quarterly* 1, No. 1 (1963):29035.
3. L. W. Dickson. *History of the Theory of Numbers.* Vol. I, pp. 393-411. Washington, D.C., 1920.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

A. P. HILLMAN

Assistant Editors: Gloria C. Padilla & Charles R. Wall

Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 Solano Dr., S.E.; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date, and proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-508 *Proposed by Philip L. Mana, Albuquerque, NM*

Find all n in $\{1, 2, 3, \ldots, 200\}$ such that the sum $n! + (n + 1)!$ of successive factorials is the square of an integer.

B-509 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Let ψ be Dedekind's function given by

$$\psi(n) = n \prod_{p|n}\left(1 + \frac{1}{p}\right).$$

For example, $\psi(12) = 12\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) = 24$. Show that

$$\psi(\psi(\psi(n))) > 2n \text{ for } n = 1, 2, 3, \ldots .$$

B-510 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Euler's ϕ function and its companion, Dedekind's ψ function are defined by

$$\phi(n) = n \prod_{p|n}\left(1 - \frac{1}{p}\right) \quad \text{and} \quad \psi(n) = n \prod_{p|n}\left(1 + \frac{1}{p}\right).$$

(a) Show that $\phi(n) + \psi(n) \geqslant 2n$ for $n > 1$.
(b) When is the inequality strict?

B-511 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers with j even. Prove that

$$F_j(F_n + F_{n+2j} + F_{n+4j} + \cdots + F_{n+2jk}) = (L_{n+2jk+j} - L_{n-j})/5.$$

B-512 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers with j odd. Prove that

$$L_j(F_n + F_{n+2j} + F_{n+4j} + \cdots + F_{n+2jk}) = F_{n+2jk+j} - F_{n-j}.$$

B-513 *Proposed by Andreas N. Philippou, University of Patras, Greece*

Show that

$$\sum_{k=0}^{n} F_{k+1}F_{n+1-k} = \sum_{k=0}^{[n/2]} (n + 1 - k)\binom{n-k}{k} \text{ for } n = 0, 1, \ldots,$$

where $[x]$ denotes the greatest integer in x.

SOLUTIONS

Correction of a Previously Published "Solution"

B-468 *Proposed by Miha'ly Bencze, Brasov, Romania*

Find a closed form for the nth term a_n of the sequence for which a_1 and a_2 are arbitrary real numbers in the open interval (0, 1) and

$$a_{n+2} = a_{n+1}\sqrt{1 - a_n^2} + a_n\sqrt{1 - a_{n+1}^2}.$$

The formula for a_n should involve Fibonacci numbers if possible.

Solution by Charles R. Wall, Trident Technical College, Charleston, SC

The published solution (*FQ*, Feb. 1983) is clearly erroneous, because it allows negative terms in a sequence of positive numbers. The error apparently arises from $(1 - \sin^2 t)^{\frac{1}{2}} = \cos t$, which is false if $\cos t < 0$.

Let

$$b_n = F_{n-2} \operatorname{Arcsin} a_1 + F_{n-1} \operatorname{Arcsin} a_2$$

and let k be the least positive integer for which $b_k > \pi/2$. Then $k \geqslant 3$, and it is easy to show that $a_n = \sin b_n$ for $n \leqslant k$ (as given in the erroneous solution). However,

$$\begin{aligned} a_{k+1} &= \sin b_k(1 - \sin^2 b_{k-1})^{\frac{1}{2}} + \sin b_{k-1}(1 - \sin^2 b_k)^{\frac{1}{2}} \\ &= \sin b_k(\cos b_{k-1}) + \sin b_{k-1}(-\cos b_k) \\ &= \sin(b_k - b_{k-1}) = \sin b_{k-2} = a_{k-2}. \end{aligned}$$

Also,

$$\begin{aligned} a_{k+2} &= \sin b_{k-2}(1 - \sin^2 b_k)^{\frac{1}{2}} + \sin b_k(1 - \sin^2 b_{k-2})^{\frac{1}{2}} \\ &= \sin b_{k-2}(-\cos b_k) + \sin b_k(\cos b_{k-2}) \\ &= \sin(b_k - b_{k-2}) = \sin b_{k-1} = a_{k-1}. \end{aligned}$$

Then

$$\begin{aligned} a_{k+3} &= a_{k+2}(1 - a_{k+1}^2)^{\frac{1}{2}} + a_{k+1}(1 - a_{k+2}^2)^{\frac{1}{2}} \\ &= a_{k-1}(1 - a_{k-2}^2)^{\frac{1}{2}} + a_{k-2}(1 - a_{k-1}^2)^{\frac{1}{2}} = a_k. \end{aligned}$$

Thus, the sequence eventually repeats in a cycle of three values, so we have

$$a_n = \begin{cases} \sin b_n & \text{if } n \leqslant k \\ \sin b_{k-2} & \text{if } n = k + 3j + 1 \text{ and } j \geqslant 0 \\ \sin b_{k-1} & \text{if } n = k + 3j + 2 \text{ and } j \geqslant 0 \\ \sin b_k & \text{if } n = k + 3j \text{ and } j \geqslant 0 \end{cases}$$

where $\{b_n\}$ and k are defined as above.

Efficient Raising to Powers

B-484 *Proposed by Philip L. Mana, Albuquerque, NM*

For a given x, what is the least number of multiplications needed to calculate x^{98}? (Assume that storage is unlimited for intermediate products.)

Solution by Walther Janous, Universitaet Innsbruck, Austria

Since $96 = 2^6 + 2^5 + 2$, the least number of multiplications needed to calculate x^{98} is $6 + 2 = 8$. This can be achieved as follows:

$$xx = x^2;\ x^2x^2 = x^4;\ x^4x^4 = x^8;\ x^8x^8 = x^{16};\ x^{16}x^{16} = x^{32};$$

$$x^{32}x^{32} = x^{64};\ x^{32}x^{64} = x^{96};\ x^{96}x^2 = x^{98}.$$

In general, the following theorem holds true: If

$$\sum_{i=1}^{k} a_i 2^i,\ a_i \in \{0, 1\},$$

is the dual-representation of the number N, then the least number of multiplications needed to calculate x^N (under assumption of unlimited storage for intermediate products) equals

$$p(N) = k + \#\{i : i < k \text{ and } a_i = 1\}.$$

Also solved by L. Kuipers, Vania D. Mascioni, Samuel D. Moore, John Oman & Bob Prielipp, Stanley Rabinowitz, Sahib Singh, J. Suck, and the proposer.

Difference Equation

B-485 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Find the complete solution u_n to the difference equation

$$u_{n+2} - 5u_{n+1} + 6u_n = 11F_n - 4F_{n+2}.$$

Solution by J. Suck, Essen, Germany

Since

$$u_{n+2} - 5u_{n+1} + 6u_n = 11F_n - 4F_{n+2} = F_{n+2} - 5F_{n+1} + 6F_n,$$

we see that the difference sequence $d_n := u_n - F_n$ has the auxiliary equation $x^2 - 5x + 6 = 0$, of which the roots are 2 and 3. The general solution for d_n is, thus, $d_n = a2^n + b3^n$, and so $u_n = a2^n + b3^n + F_n$ with arbitrary constants a, b [which are $a = 3(u_0 - F_0) - u_1 + F_1$, $b = u_1 - F_1 - 2(u_0 - F_0)$ in terms of initial values].

Of course, the solution does not depend on $F_0 = 0$, $F_1 = 1$, but only on the Fibonacci recurrence.

Also solved by Wray G. Brady, Paul S. Bruckman, C. Georghiou, Walther Janous, L. Kuipers, John W. Milsom, Bob Prielipp, A. G. Shannon, Sahib Singh, and the proposer.

Monotonic Sequences of Ratios

B486 *Proposed by Valentina Bakinova, Rondout Valley, NY*

Prove or disprove that, for every positive integer k,

$$\frac{F_{k+1}}{F_1} < \frac{F_{k+3}}{F_3} < \frac{F_{k+5}}{F_5} < \cdots < \alpha^k < \cdots < \frac{F_{k+6}}{F_6} < \frac{F_{k+4}}{F_4} < \frac{F_{k+2}}{F_2}.$$

Solution by Vania D. Mascioni, student, Swiss Fed. Inst. of Tech., Zürich

Fix $k > 0$. Using the well-known identity

$$F_{n+k}F_{m-k} - F_nF_m = (-1)^n F_{m-n-k}F_k$$

(see, e.g., Knuth, *The Art of Computer Programming*, I, Ex. 1.2.8.17), we obtain

$$F_{k+2P}F_{2P+2} - F_{k+2P+2}F_{2P} - F_{k+2P+1}F_{2P-1} - F_{k+2P-1}F_{2P+1} = F_k > 0.$$

It is then $\dfrac{F_{k+2P+2}}{F_{2P+2}} < \dfrac{F_{k+2P}}{F_{2P}}$ and $\dfrac{F_{k+2P+1}}{F_{2P+1}} > \dfrac{F_{k+2P-1}}{F_{2P-1}}$ for $P \geqslant 1$.

From $F_n = \left[\frac{\alpha^n}{5} + \frac{1}{2}\right]$, it follows that

$$\lim_{n\to\infty} \frac{F_{n+k}}{F_n} = \alpha^k.$$

Also solved by Paul S. Bruckman, C. Georghiou, Walther Janous, L. Kuipers, Bob Prielipp, Stanley Rabinowitz, A. G. Shannon, Sahib Singh, J. Suck, and the proposer.

Multiple of 50

B-487 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that, for all positive integers n,

$$5L_{4n} - L_{2n}^2 + 6 - 6(-1)^n L_{2n} \equiv 0 \pmod{10F_n^2}.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We will show that the given congruence holds. Since

$$L_{2n} = 5F_n^2 + 2(-1)^n,\ F_{2n} = L_nF_n,\ \text{and}\ L_n^2 - F_n^2 = 4F_n^2 + 4(-1)^n$$

(See Exercises 4, 1, and 10 on p. 29 of *Fibonacci and Lucas Numbers* by V. E. Hoggatt, Jr.),

$$5L_{4n} - L_{2n}^2 + 6 - 6(-1)^n L_{2n} = 25F_{2n}^2 + 10 - [5F_n^2 + 2(-1)^n]^2 + 6 - 6(-1)^n$$

$$[5F_n^2 + 2(-1)^n] = 25F_{2n}^2 + 10 - 25F_n^4 - 20(-1)^n F_n^2 - 4 + 6 - 30(-1)^n F_n^2 - 12$$

$$= 25L_n^2F_n^2 - 25F_n^4 - 50(-1)^n F_n^2 = 25F_n^2(L_n^2 - F_n^2) - 50(-1)^n F_n^2$$

$$= 25F_n^2[4F_n^2 + 4(-1)^n] - 50(-1)^n F_n^2 = 50F_n^2[2F_n^2 + (-1)^n].$$

Clearly the immediately preceding expression is congruent to zero modulo $50F_n^2$ (and hence is congruent to zero modulo $10F_n^2$).

Also solved by Paul S. Bruckman, Walther Janous, L. Kuipers, Stanley Rabinowitz, Heinz-Jürgen Seiffert, Sahib Singh, J. Suck, and the proposer.

Odd Difference

B-488 *Proposed by Herta T. Freitag, Roanoke, VA*

Let a and d be positive integers with d odd. Prove or disprove that for all positive integers h and k,

$$L_{a+hd} + L_{a+hd+d} \equiv L_{a+kd} + L_{a+kd+d} \pmod{L_d}.$$

Solution by Sahib Singh, Clarion State College, Clarion, PA

This congruence is true. The proof follows by using the result of B-479 which states that

$$L_{a+hd} + L_{a+hd+d} \equiv L_{a+d} + L_a \pmod{L_d}.$$

Similarly,

$$L_{a+kd} + L_{a+kd+d} \equiv L_{a+d} + L_a \pmod{L_d}$$

is true.

By subtraction, the required result follows, and we are done.

Also solved by Paul S. Bruckman, Walther Janous, L. Kuipers, Bob Prielipp, J. Suck, and the proposer.

Even Difference

B-489 *Proposed by Herta T. Freitag, Roanoke, VA*

Is there a Fibonacci analogue (or semianalogue) of B-488?

Solution by Walther Janous, Universitaet Innsbruck, Austria

Let a and d be positive integers with d *even*. Then there holds for all positive integers h and k,

$$F_{a+hd} + F_{a+hd+d} \equiv F_{a+kd} + F_{a+kd+d} \pmod{F_d}.$$

As before, it is enough to consider the case $h = k + 1$. Since, for d even, there holds

$$F_{a+(k+2)d} - F_{a+kd} = F_d L_{a+(k+1)d},$$

the claim is proved.

Also solved by Paul S. Bruckman, L. Kuipers, Bob Prielipp, Sahib Singh, J. Suck, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by

RAYMOND E. WHITNEY

Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-360 *Proposed by M. Wachtel, Zürich, Switzerland*

Let

$$\begin{aligned} F_nF_{n+1} + F_{n+2}^2 &= A_1 \\ F_{n+1}F_{n+2} + F_{n+3}^2 &= A_2 \\ F_{n+2}F_{n+3} + F_{n+4}^2 &= A_3 \end{aligned}$$

Show that:

(1) No integral divisor of A is congruent to 3 or 7 modulo 10.
(2) $A_1A_2 + 1$, as well as $A_1A_3 + 1$, are products of two consecutive integers.

H-361 *Proposed by Verner E. Hoggatt, Jr. (deceased)*

Let $H_n = P_{2n}/2$, $n > 0$, where P_n denotes the nth Pell number. Show that

$$H_m + H_n \neq P_k$$
$$H_m + H_n = P_k + P_{k-1}$$

if and only if $m = n + 1$, where $k = 2n + 1$, and

$$P_{2n+2}/2 + P_{2n}/2 = ((2P_{2n+1} + P_{2n}) + P_{2n})/2 = P_{2n+1} + P_{2n}.$$

Editorial Note: *Refer to the January 1972 article on Generalized Zeckendorf Theorem for Pell Numbers.*

H-362 *Proposed by Stanley Rabinowitz, Merrimack, NH*

Let Z be the ring of integers modulo n. A Lucas Number in this ring is a member of the sequence $\{L_k\}$, $k = 0, 1, 2, \ldots$, where $L_0 = 2$, $L_1 = 1$, and $L_{k+2} \equiv L_{k+1} + L_k$ for $k \geqslant 0$. Prove that, for $n > 14$, all members of Z_n are Lucas numbers if and only if n is a power of 3.

Remark: A similar, but more complicated, result is known for Fibonacci numbers. See [1]. I do not have a proof of the above proposal, but I suspect a proof similar to the result in [1] is possible; however, it should be considerably simpler, because there is only one case to consider rather than seven cases.

To verify the conjecture, I ran a computer program that examined Z_n for all n between 2 and 10000 and found that the only cases where all members of Z_n were Lucas numbers were powers of 3, and the exceptional values $n = 2, 4, 6, 7$, and 14 (the same exceptions found in [1]). This is strong evidence for the truth of the conjecture.

Reference

1. S. A. Burr. "On Moduli for Which the Fibonacci Sequence Contains a Complete System of Residues." *The Fibonacci Quarterly* 9 (1971):497.

H-363 *Proposed by Andreas N. Philippou, University of Patras, Greece*

For each fixed integer $k \geqslant 2$, let $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k, i.e., $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)}, & \text{if } 2 \leqslant n \leqslant k, \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)}, & \text{if } n \geqslant k+1. \end{cases}$$

Evaluate the series

$$\sum_{n=0}^{\infty} \frac{1}{f_{m^n}^{(k)}} \quad (k \geqslant 2,\ m \geqslant 2).$$

Remark: The Fibonacci sequence of order k appears in the work of Philippou and Muwafi, *The Fibonacci Quarterly* 20 (1982);28-32.

H-364 *Proposed by M. Wachtel, Zürich, Switzerland*

For every n, show that no integral divisor of L_{2n+1} is congruent to 3 or 7, modulo 10.

SOLUTIONS

The Root of the Problem

H-341 *Proposed by Paul S. Bruckman, Concord, CA*
(Vol. 20, No. 2, May 1982)

Find the real roots, in exact radicals, of the polynomial equation

$$p(x) \equiv x^6 - 4x^5 + 7x^4 - 9x^3 + 7x^2 - 4x + 1 = 0. \tag{1}$$

Solution by the proposer

We note that $p(0) \neq 0$ and $p(x) = x^6 p(1/x)$. Let

$$y = x + x^{-1}. \tag{2}$$

Then $y^2 = x^2 + x^{-2} + 2$ and $y^3 = x^3 + x^{-3} + 3y$; hence,

$$x^{-3}p(x) = x^3 + x^{-3} - 4(x^2 + x^{-2}) + 7(x + x^{-1}) - 9$$

$$= y^3 - 3y - 4(y^2 - 2) + 7y - 9,$$

or

$$y^3 - 4y^2 + 4y - 1 = 0. \tag{3}$$

This polynomial in y may be readily factored, noting that it vanishes for $y = 1$. Thus,

$$(y - 1)(y^2 - 3y + 1) = (y - 1)(y - a^2)(y - b^2) = 0.$$

Now, we may solve for x in terms of y, first multiplying (2) throughout by $x : x^2 - xy + 1 = 0$, or

$$x = \tfrac{1}{2}(y \pm \sqrt{y^2 - 4}). \tag{4}$$

Setting $y = 1$ or $y = b^2$ in (4) yields imaginary roots of (1) (and, moreover, of unit modulus). Setting $y = a^2$, however, yields real roots, which after a little manipulation are found to be as follows:

$$x_1 = \tfrac{1}{4}(3 + \sqrt{5} + \sqrt{6\sqrt{5} - 2}) \doteq 2.1537214, \tag{5}$$

$$x_2 = \tfrac{1}{4}(3 + \sqrt{5} - \sqrt{6\sqrt{5} - 2}) \doteq .46431261 \doteq 1/x_1. \tag{6}$$

Also solved by W. Blumberg, H. Freitag, W. Janous, D. Laurie, D. Russell, C. Shields, and M. Wachtel.

Say A

H-342 *Proposed by Paul S. Bruckman, Corcord, CA*
(Vol. 20, No. 3, August 1982)

Let

$$A_n = \sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{k}\binom{2n - 2k}{n} 4^k, \quad n = 0, 1, 2, \ldots . \tag{1}$$

Prove that

$$\sum_{k=0}^{n} A_k A_{n-k} = 4^n F_{n+1}. \tag{2}$$

Solution by the proposer

Proof #1: The well-known Legendre polynomials are defined by the generating function

$$(1 - 2xz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) z^n \text{ (valid for } |x| < 1,\ |z| < 1), \tag{3}$$

and are given explicitly as

$$P_n(x) = 2^{-n} \sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{k}\binom{2n - 2k}{n}(-1)^k x^{n-2k}. \tag{4}$$

(see, for example, formulas 22.3.8 and 22.9.12 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, ed. Milton Abramowitz & Irene A. Stegun, National Bureau of Standards Applied Mathematics Series 55, issued June 1964, 9th printing, November 1970, with corrections).

In (3) and (4), set $x = \frac{1}{2}i$ and replace z in (3) by $-iz$. Then

$$(1 - z - z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(\tfrac{1}{2}i)(-iz)^n, \tag{5}$$

and, using the definition of A_n in (1):

$$P_n(\tfrac{1}{2}i) = (\tfrac{1}{4}i)^n A_n. \tag{6}$$

Thus,

$$(1 - z - z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} A_n(\tfrac{1}{4}z)^n. \tag{7}$$

Squaring both sides of (7), we obtain the generating function of the Fibonacci numbers:

$$(1 - z - z^2)^{-1} = \sum_{n=0}^{\infty} F_{n+1}z^n = \sum_{n=0}^{\infty} (\tfrac{1}{4}z)^n \sum_{k=0}^{n} A_k A_{n-k}$$

(the last result by convolution). We obtain (2) by comparison of coefficients in the last two expressions. Q.E.D.

The following is a more direct proof of the foregoing result.

Proof #2: Let

$$f(z) \equiv \sum_{n=0}^{\infty} A_n(\tfrac{1}{4}z)^n. \tag{8}$$

Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} (\tfrac{1}{4}z)^n \sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{k}\binom{2n-2k}{n}4^k = \sum_{n,k=0}^{\infty} (\tfrac{1}{4}z)^{n+2k}\binom{n+2k}{k}\binom{2n+2k}{n+2k}4^k \\ &= \sum_{n,k=0}^{\infty} (\tfrac{1}{4})^{n+k} z^{n+2k}\binom{2n+2k}{n+k}\binom{n+k}{k} \\ &= \sum_{n,k=0}^{\infty} (\tfrac{1}{4}z^2)^k \binom{-\frac{1}{2}}{n+k}\binom{n+k}{k}(\tfrac{1}{4}z)^n(-4)^{n+k} \\ &= \sum_{n,k=0}^{\infty} (-z)^n(-z^2)^k\binom{-\frac{1}{2}}{n}\binom{-\frac{1}{2}-n}{k} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-z)^n(1-z^2)^{-\frac{1}{2}-n} \\ &= (1-z^2)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-z)^n(1-z^2)^{-n} = (1-z^2)^{-\frac{1}{2}}\left\{1 - \frac{z}{1-z^2}\right\}^{-\frac{1}{2}}, \end{aligned}$$

or

$$f(z) = (1 - z - z^2)^{-\frac{1}{2}}. \tag{9}$$

The rest of the proof now proceeds as in the first proof, after (7). Q.E.D.

The first few values of $(A_n)_{n=0}^{\infty}$ are as follows: $A_0 = 1$, $A_1 = 2$, $A_2 = 14$, $A_3 = 68$, $A_4 = 406$, $A_5 = 2{,}332$, $A_6 = 13{,}964$, $A_7 = 83{,}848$, etc. The "etc." is puzzling—can any reader discover a closed form expression for A_n?

Also solved by C. Georghiou.

Continue

H-343 *Proposed by Verner E. Hoggatt, Jr. (deceased)*
(Vol. 20, No. 3, August 1982)

Show that every positive integer, m, has a unique representation in the form

$$m = [A_1[A_2[A_3[\ldots[A_n]\ldots],$$

where $A_j = \alpha$ or α^2 for $j = 1, 2, \ldots, n - 1$, and

$$A_n = \alpha^2,$$

where $\alpha = (1 + \sqrt{5})/2$.

Solution by Paul Bruckman, Carmichael, CA

Let $A(k) = [\alpha k]$, $B(k) = [\alpha^2 k]$, $k = 1, 2, 3, \ldots$. Note $A(1) = [\alpha] = 1$ and $B(1) = [\alpha^2] = 2$. Let a "string" denote any composition of functions A or B ending with $B(1)$ [e.g., $A(B(A(B(1))))$]. Let the *length* of a string denote the number n of functions used in the string ($n = 4$ in the example). Let

$$A = (A(k))_{k=1}^{\infty},\ B = (B(k))_{k=1}^{\infty},\ N = (k)_{k=1}^{\infty}.$$

It is a well-known theorem that $A \cup B = N$, $A \cap B = \emptyset$.

The problem is incorrectly stated, since $1 = A(1)$ is *not* representable by a string. We shall prove that all integers > 1 *are* representable.

We first prove that distinct strings represent distinct positive integers. This is trivially true for $n = 1$, since there is only one number of string-length 1, namely $B(1) = 2$. Also, for $n = 2$, we have

$$A(B(1)) = A(2) = 3 \quad \text{and} \quad B(B(1)) = B(2) = 5.$$

Suppose that all distinct strings of length $\leqslant n$ represent distinct positive integers. Then, if k is the integer represented by any string of length n, we have $A(k) \neq B(k)$, since $A \cap B = \emptyset$. Likewise, $A(k) \neq B(j)$, where j is the integer represented by any string of length less than n. If $A(k) = A(j)$ or $B(k) = B(j)$, then $k = j$, since $A(m)$ and $B(m)$ are one-to-one functions. This is, however, contrary to hypothesis. Thus, all distinct strings of length $\leqslant (n + 1)$ represent distinct integers. It follows by induction that distinct strings represent distinct positive integers.

It remains to show that all positive integers $m > 1$ are thus representable. Suppose that all integers k, with $2 \leqslant k \leqslant m$ are representable. Since $A \cup B = N$, thus, $m + 1 = A(j)$ or $B(j)$ for some integer j with $2 \leqslant j \leqslant m$. Therefore, $m + 1$ is also representable. Since $2 = B(1)$, $3 = A(B(1))$, etc., it follows by induction that all integers $m > 1$ are representable. This completes the proof of the problem (as modified).

Also solved by the proposer and by L. Kuipers, who remarked that the solution is contained in this quarterly, Vol 17 (1979):306-07.

Don't Lose Your Identity

H-344 *Proposed by M. D. Agrawal, Government College, Mandsaur, India (Vol. 20, No. 3, August 1982)*

Prove:

1. $L_k L_{k+3m}^2 - L_{k+4m} L_{k+m}^2 = (-1)^k 5^2 F_m^2 F_{2m} F_{k+2m}$ and
2. $L_k L_{k+3m}^2 - L_{k+2m}^3 = 5(-1)^k F_m^2 (L_{k+4m} + 2(-1)^m L_{k+2m})$.

Solution by Bob Prielipp, University of Wisconsin, Oshkosh, WI

Using the Binet formulas

$$L_n = a^n + b^n \quad \text{and } \sqrt{5}F_n = a^n - b^n$$

and the fact that $ab = -1$,

$$\begin{aligned}
L_k L_{k+3m}^2 - L_{k+4m} L_{k+m}^2 &= (a^k + b^k)(a^{k+3m} + b^{k+3m})^2 - (a^{k+4m} + b^{k+4m})(a^{k+m} + b^{k+m})^2 \\
&= (a^k + b^k)(a^{2k+6m} + 2(-1)^{k+m} + b^{2k+6m}) \\
&\quad - (a^{k+4m} + b^{k+4m})(a^{2k+2m} + 2(-1)^{k+m} + b^{2k+2m}) \\
&= a^{3k+6m} + (-1)^k(a^{k+6m}) + 2(-1)^{k+m}(a^k + b^k) + (-1)^k b^{k+6m} \\
&\quad + b^{3k+6m} - a^{3k+6m} - (-1)^k(a^k b^{2m}) - 2(-1)^{k+m}(a^{k+4m} + b^{k+4m}) \\
&\quad - (-1)^k(a^{2m} b^k) - b^{3k+6m} \\
&= (-1)^k[(a^{k+6m} + b^{k+6m}) + 2(-1)^m(a^k + b^k) \\
&\quad - 2(-1)^m(a^{k+4m} + b^{k+4m}) - (a^k b^{2m} + a^{2m} b^k)].
\end{aligned}$$

Also

$$\begin{aligned}
(-1)^k 5^2 F_m^2 F_{2m} F_{k+2m} &= (-1)^k (a^m - b^m)^2 (a^{2m} - b^{2m})(a^{k+2m} - b^{k+2m}) \\
&= (-1)^k (a^{2m} - 2(-1)^m + b^{2m})(a^{k+4m} - b^k - a^k + b^{k+4m}) \\
&= (-1)^k [a^{k+6m} - a^{2m} b^k - a^{k+2m} + b^{k+2m} - 2(-1)^m a^{k+4m} \\
&\quad + 2(-1)^m b^k + 2(-1)^m a^k - 2(-1)^m b^{k+4m} \\
&\quad + a^{k+2m} - b^{k+2m} - a^k b^{2m} + b^{k+6m}] \\
&= (-1)^k [(a^{k+6m} + b^{k+6m}) + 2(-1)^m (a^k + b^k) \\
&\quad - 2(-1)^m (a^{k+4m} + b^{k+4m}) - (a^k b^{2m} + a^{2m} b^k)].
\end{aligned}$$

This establishes the first formula.

Again using the Binet formulas and the fact that $ab = -1$,

$$\begin{aligned}
L_k L_{k+3m}^2 - L_{k+2m}^3 &= (a^k + b^k)(a^{k+3m} + b^{k+3m})^2 - (a^{k+2m} + b^{k+2m})^3 \\
&= (a^k + b^k)(a^{2k+6m} + 2(-1)^{k+m} + b^{2k+6m}) \\
&\quad - (a^{3k+6m} + 3(-1)^k a^{k+2m} + 3(-1)^k b^{k+2m} + b^{3k+6m}) \\
&= a^{3k+6m} + (-1)^k a^{k+6m} + 2(-1)^{k+m}(a^k + b^k) \\
&\quad + (-1)^k b^{k+6m} + b^{3k+6m} - a^{3k+6m} \\
&\quad - 3(-1)^k(a^{k+2m} + b^{k+2m}) - b^{3k+6m} \\
&= (-1)^k[(a^{k+6m} + b^{k+6m}) + 2(-1)^m(a^k + b^k) - 3(a^{k+2m} + b^{k+2m})].
\end{aligned}$$

Also

$$\begin{aligned}
&5(-1)^k F_m^2(L_{k+4m} + 2(-1)^m L_{k+2m}) \\
&= (-1)^k(a^m - b^m)^2[(a^{k+4m} + b^{k+4m}) + 2(-1)^m(a^{k+2m} + b^{k+2m})] \\
&= (-1)^k(a^{2m} - 2(-1)^m + b^{2m})[(a^{k+4m} + b^{k+4m}) + 2(-1)^m(a^{k+2m} + b^{k+2m})] \\
&= (-1)^k[a^{k+6m} + b^{k+2m} + 2(-1)^m a^{k+4m} + 2(-1)^m b^k - 2(-1)^m a^{k+4m} - 2(-1)^m b^{k+4m} \\
&\quad - 4a^{k+2m} - 4b^{k+2m} + a^{k+2m} + b^{k+6m} + 2(-1)^m a^k + 2(-1)^m b^{k+4m}] \\
&= (-1)^k[(a^{k+6m} + b^{k+6m}) + 2(-1)^m(a^k + b^k) - 3(a^{k+2m} + b^{k+2m})].
\end{aligned}$$

This establishes the second formula.

Also solved by P. Bruckman, W. Janous, L. Kuipers, J. Spraggon, and the proposer.

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The Fibonacci Association and the University of Patras, Greece would like to announce their intentions to jointly sponsor an international conference on Fibonacci numbers and their applications. This conference is tentatively set for late August or early September of 1984. Anyone interested in presenting a paper or attending the conference should contact:

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Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

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A Collection of Manuscripts Related to the Fibonacci Sequence — 18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

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