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The Fibonacci Quarterly

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*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

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ADDITIVE PARTITIONS OF THE POSITIVE INTEGERS AND GENERALIZED FIBONACCI REPRESENTATIONS

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ABSTRACT

The generalized Fibonacci numbers $\{u_n\}$,

$$u_{n+2} = u_{n+1} + u_n,\ u_1 = a,\ u_2 = b,\ (a,\ b) = 1,$$

induce a unique additive partition of the set of positive integers formed by two disjoint subsets such that no two distinct elements of either subset have u_n as their sum. We examine the values of a special function

$$E_n(m) = mu_{n-1} - u\,[mu_{n-1}/u_n],\ m = 1,\ 2,\ \ldots,\ u_n - 1,\ n \geqslant 2,$$

and find relationships to the additive partition of N as well as to Wythoff's pairs and to representations of integers using the double-ended sequence $\{u_n\}_{-\infty}^{\infty}$ and the extended sequence $\{u_n\}_{-m}^{\infty}$. We write a Zeckendorf theorem for double-ended sequences and show completeness for the extended sequences.

1. TABULATION OF $E_n(m)$ FOR THE FIBONACCI AND LUCAS SEQUENCES

We begin with the ordinary Fibonacci sequence $\{F_n\}$, where $F_1 = 1$, and $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$. We tabulate and examine a special function $E_n(m)$, defined by

$$E_n(m) = mF_{n-1} - F_n[mF_{n-1}/F_n],\ m = 1,\ 2,\ \ldots,\ F_n - 1,\ n \geqslant 2, \tag{1.1}$$

where $[x]$ is the greatest integer function. Notice that $n = 2$ gives the trivial $E_2(m) = 0$ for all m, while $E_3(m)$ is 1 for m odd and 0 for m even.

The table of values for $E_n(m)$ (Table 1.1) reveals many immediate patterns. First, $E_n(m)$ is periodic with period F_n, and the rth term in the cycle is $E_n(r) = rF_{n-1} \pmod{F_n}$. We could easily show, using properties of modular arithmetic and of the greatest integer function, that

$$E_n(1) = F_{n-1},\ E_n(2) = F_{n-3},\ E_n(3) = L_{n-2},\ E_n(4) = 2F_{n-3}.$$

Also, counting from the end of a cycle, we have

$$E_n(-1) = F_{n-2},\ E_n(-2) = 2F_{n-2},\ E_n(-3) = F_{n-4},\ E_n(-4) = L_{n-3},$$

which also can be established by elementary methods, but these apparent patterns are not the main thrust of this paper.

TABLE 1.1

VALUES OF $E_n(m)$ FOR THE FIBONACCI SEQUENCE

	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
$E_n(m)$:	$2m - 3\left[\frac{2m}{3}\right]$	$3m - 5\left[\frac{3m}{5}\right]$	$5m - 8\left[\frac{5m}{8}\right]$	$8m - 13\left[\frac{8m}{13}\right]$	$13m - 21\left[\frac{13m}{21}\right]$
$m = 1$	2	3	5	8	13
$m = 2$	1	1	2	3	5
$m = 3$	0	4	7	11	18
$m = 4$	2	2	4	6	10
$m = 5$	1	0	1	1	2
$m = 6$	0	3	6	9	15
$m = 7$	2	1	3	4	7
$m = 8$	1	4	0	12	20
$m = 9$	0	2	5	7	12
$m = 10$	2	0	2	2	4
$m = 11$	1	3	7	10	17
$m = 12$	0	1	4	5	9
$m = 13$	2	4	1	0	1
$m = 14$	1	2	6	8	14
$m = 15$	0	0	3	3	6
$m = 16$	2	3	0	11	19
$m = 17$	1	1	5	6	11
$m = 18$	0	4	2	1	3
$m = 19$	2	2	7	9	16
$m = 20$	1	0	4	4	8
$m = 21$	0	3	1	12	0

We need two other number sequences, derived from the Fibonacci numbers. We write the disjoint sets $\{A_n\}$ and $\{B_n\}$, which are formed by making a partition of the positive integers such that no two distinct members from A and no two distinct members from B have a sum which is a Fibonacci number. We also write the first few Wythoff pairs (a_n, b_n) [1] for inspection.

n	A_n	B_n
1	1	2
2	3	4
3	6	5
4	8	7
5	9	10
6	11	12
7	14	13
8	16	15
9	17	18
10	19	20
...	...	...

n	a_n	b_n
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26
...	...	...

We note that the Wythoff pairs are given by

$$a_n = [n\alpha] \quad \text{and} \quad b_n = [n\alpha^2], \tag{1.2}$$

where $[x]$ is the greatest integer contained in x and $\alpha = (1+\sqrt{5})/2$ is the Golden Section ratio. Also, $b_n = a_n + n$, and a_n is the smallest integer not yet used. It is also true that no two distinct b_n's have a Fibonacci number as their sum, and that $\{b_n\} \subset \{B_n\}$.

Now, examine the periods of the values of $E_n(m)$:

$n = 5$:

$$\left.\begin{array}{r}3\\1\\4\end{array}\right\} a_n \qquad \left.\begin{array}{r}3\\1\end{array}\right\} A_n$$

$$\left.\begin{array}{r}2\end{array}\right\} b_n \qquad \left.\begin{array}{r}4\\2\end{array}\right\} B_n$$

$n = 6$:

$$\left.\begin{array}{r}5\\2\\7\end{array}\right\} b_n \qquad \left.\begin{array}{r}5\\2\\7\\4\end{array}\right\} B_n$$

$$\left.\begin{array}{r}4\\1\\6\\3\end{array}\right\} a_n \qquad \left.\begin{array}{r}1\\6\\3\end{array}\right\} A_n$$

$n = 7$:

$$\left.\begin{array}{r}8\\3\\11\\6\\1\\9\\4\\12\end{array}\right\} a_n \qquad \left.\begin{array}{r}8\\3\\11\\6\\1\\9\end{array}\right\} A_n$$

$$\left.\begin{array}{r}7\\2\\10\\5\end{array}\right\} b_n \qquad \left.\begin{array}{r}4\\12\\7\\2\\10\\5\end{array}\right\} B_n$$

$n = 8$:

$$\left.\begin{matrix}13\\5\\18\\10\\2\\15\\7\\20\end{matrix}\right\} b_n \qquad \left.\begin{matrix}13\\5\\18\\10\\2\\15\\7\\20\\12\\4\end{matrix}\right\} B_n \qquad \left.\begin{matrix}12\\4\\17\\9\\1\\14\\6\\19\\11\\3\\16\\8\end{matrix}\right\} a_n \qquad \left.\begin{matrix}17\\9\\1\\14\\6\\19\\11\\3\\16\\8\end{matrix}\right\} A_n$$

Notice that the integers 1, 2, 3, ..., $F_n - 1$, all appear, but not in natural order. Each cycle is made up of early values of $\{a_n\}$ and $\{b_n\}$, and of early values of $\{A_n\}$ and $\{B_n\}$, not in order, but without omissions.

If we apply $E_n(m)$ to the Lucas numbers L_n, defined by $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1} + L_n$, so that we consider

$$E_n(m) = mL_{n-1} - L_n[mL_{n-1}/L_n],\; m = 1, 2, \ldots, L_n - 1, \tag{1.3}$$

then we get the integers 1, 2, 3, ..., $L_n - 1$ in some order. Recall our generalized Wythoff numbers a_n, b_n, and c_n [1, p. 200]. We obtain within each cycle a segment of $\{a_n\}$, a segment of $\{c_n\}$, and a segment of $\{b_n\}$, where each segment is complete (the first few terms of each sequence without omission, but not in order). This same cycle contains the first few terms of $\{A_n\}$, out of order, but without omissions, followed by the first few terms of $\{B_n\}$, where $\{A_n\}$ and $\{B_n\}$ is the unique split of the positive integers induced by the Lucas sequence such that no two elements of $\{A_n\}$, and no two elements of $\{B_n\}$, have a Lucas number for their sum.

To illustrate the Lucas case, we write the first twelve values of the generalized Wythoff numbers, and early values of the partition sets:

n	a_n	b_n	c_n
1	1	3	2
2	4	7	6
3	5	10	9
4	8	14	13
5	11	18	17
6	12	21	20

n	A_n	B_n
1	1	2
2	4	3
3	5	6
4	8	7
5	9	10
6	11	13

n	a_n	b_n	c_n
7	15	25	24
8	16	28	27
9	19	32	31
10	22	36	35
11	23	39	38
12	26	43	42

n	A_n	B_n
7	12	14
8	15	17
9	16	18
10	19	20
11	22	21
12	23	24
13	26	25
14	27	28

Now, examine the periods of values of $E_n(m)$ for the Lucas sequence:

$n = 4$: $4m - 7[4m/7]$

$$\left.\begin{array}{r}4\\1\\5\end{array}\right\}a_n \quad \left.\begin{array}{r}2\\6\end{array}\right\}c_n \quad \left.\begin{array}{r}3\end{array}\right\}b_n \qquad\qquad \left.\begin{array}{r}4\\1\\5\end{array}\right\}A_n \quad \left.\begin{array}{r}2\\6\\3\end{array}\right\}B$$

$n = 5$: $7m - 11[7m/11]$

$$\left.\begin{array}{r}7\\3\\10\end{array}\right\}b_n \quad \left.\begin{array}{r}6\\2\\9\end{array}\right\}c_n \quad \left.\begin{array}{r}5\\1\\8\\4\end{array}\right\}a_n \qquad\qquad \left.\begin{array}{r}7\\3\\10\\6\\2\end{array}\right\}B_n \quad \left.\begin{array}{r}9\\5\\1\\8\\4\end{array}\right\}A_n$$

$n = 6$: $11m - 18[11m/18]$

$$\left.\begin{array}{r}11\\4\\15\\8\\1\\12\\5\\16\end{array}\right\}a_n \quad \left.\begin{array}{r}9\\2\\13\\6\\17\end{array}\right\}c_n \quad \left.\begin{array}{r}10\\3\\14\\7\end{array}\right\}b_n \qquad\qquad \left.\begin{array}{r}11\\4\\15\\8\\1\\12\\5\\16\\9\end{array}\right\}A_n \quad \left.\begin{array}{r}2\\13\\6\\17\\10\\3\\14\\7\end{array}\right\}B_n$$

$n = 7$: $18m - 29[18m/29]$

$$\left.\begin{array}{r}18\\7\\25\\14\\3\\21\\10\\28\end{array}\right\}b_n \quad \left.\begin{array}{r}17\\6\\24\\13\\2\\20\\9\\27\end{array}\right\}c_n \quad \left.\begin{array}{r}16\\5\\23\\12\\1\\19\\8\\26\\15\\4\\22\end{array}\right\}a_n \qquad\qquad \left.\begin{array}{r}18\\7\\25\\14\\3\\21\\10\\28\\17\\6\\24\\13\\2\\20\end{array}\right\}B_n \quad \left.\begin{array}{r}9\\27\\16\\5\\23\\12\\1\\19\\8\\26\\15\\4\\22\end{array}\right\}A_n$$

For comparison, the generalized Wythoff numbers are formed by letting a_n be the smallest positive integer not yet used, letting $c_n = b_n - 1$, and forming $b_n = a_n + d_n$, where $d_n \neq b_k + 1$. Letting the generalized Wythoff numbers be denoted with an asterisk, we can express them in terms of Wythoff pair numbers as

$$a_n^* = 2a_n - n, \quad b_n^* = b_n + n = a_n + 2n, \quad c_n^* = a_n + 2n - 1 = a_{a_n} + n.$$

It is also true that $a_i^* + a_j^* \neq L_m$, $b_i^* + b_j^* \neq L_m$, and the Lucas generalized Wythoff numbers and the Lucas partition sets have the subset relationships $\{a_n\} \subset \{A_n\}$ and $\{b_n\} \subset \{B_n\}$.

2. ZECKENDORF THEOREM FOR DOUBLE-ENDED SEQUENCES

Before considering representations and additive partitions regarding the generalized Fibonacci sequence $\{u_n\}_{-m}^{\infty}$, where $u_1 = a$ and $u_2 = b$, $u_{n+2} = u_{n+1} + u_n$, we consider representations of the integers in terms of specialized $\{u_n\}$, where $u_1 = 1$ and $u_2 = p$.

Theorem 2.1 (Zeckendorf Theorem for double-ended sequences): Let $p \geqslant 1$ be a positive integer, and let $u_{n+2} = u_{n+1} + u_n$, $u_1 = 1$, $u_2 = p$. Then every positive integer has a representation from $\{u_n\}_{-\infty}^{\infty}$, provided that no two consecutive u_j are in the same representation.

Proof: We need to recall two major results from earlier work. David Klarner [2] has proved

Klarner's Theorem: Given the nonnegative integers A and B, there exists a unique set of integers $\{k_1, k_2, k_3, \ldots, k_r\}$ such that

$$A = F_{k_1} + F_{k_2} + \cdots + F_{k_r}$$
$$B = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1}$$

for $|k_i - k_j| \geqslant 2$, $i \neq j$, where each F_i is an element of the sequence $\{F_i\}_{-\infty}^{\infty}$.

When $u_1 = 1$ and $u_2 = p$, we know from earlier work that

$$u_{n+1} = pF_n + F_{n-1},$$

for all integral n. Next, if we wish a representation of an integer $m > 0$, we merely solve

$$A = 0 = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1}$$

$$B = m = F_{k_1} + F_{k_2} + \cdots + F_{k_r}$$

which has a unique solution by Klarner's Theorem. A constructive method of solution is given in [3]. Thus,

$$\begin{aligned} m &= u_{k_1+1} + u_{k_2+1} + \cdots + u_{k_r+1} \\ &= p(F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1}) + (F_{k_1} + F_{k_2} + \cdots + F_{k_r}) \end{aligned}$$

We note in passing that the representation we now have is independent of the explicit $p > 0$.

Theorem 2.2: The Fibonacci extended sequence is complete with respect to the integers.

Proof: Since 1, 2, 3, 5, 8, 13, ..., is complete with respect to the positive integers, one notes

$$F_{-n} = (-1)^{n+1}F_n,$$

and, therefore, one can pick out an arbitrarily large negative Fibonacci number. Consider M an arbitrary negative integer, and there exists a Fibonacci number F_{-k} such that $F_{-k} < M < 0$. Now, $M - F_{-k} = N$, which is positive and has a Zeckendorf representation using Fibonacci numbers, and $M = N + F_{-k}$ is the representation we seek.

Since $u_{n+2} = u_{n+1} + u_n$, if it consists of positive integers as $n \to \infty$, then, as $n \to -\infty$, the terms become alternating and negatively very large. Thus, the same thing holds for the generalized Fibonacci numbers once we know that they are complete with respect to the positive integers, finishing the proof of Theorem 2.1.

Completeness of the generalized sequence $\{u_n\}_{-\infty}^{\infty}$ is equivalent to showing that every positive integer is expressible as the sum of a subsequence $\{u_n\}_{-m}^{\infty}$, $m > 0$, where m is independent of the integer chosen. We show some special cases:

Case 1:	$p = 1$	1, 1, 2, 3, 5, ...	Already complete.
Case 2:	$p = 2$	1, 2, 3, 5, 8, ...	Already complete.
Case 3:	$p = 3$	1, 3, 4, 7, 11, ... Complete when $L_0 = 2$ is added to the sequence.	

Case 4: $p = 4$ 1, 4, 5, 9, 14, 23, ... Complete when $u_0 = 3$ and $u_{-1} = -2$ are added to the sequence.

Case 5: $p = 5$ 1, 5, 6, 11, 17, ... Complete when $u_0 = 4$ and $u_{-1} = -3$ are added to the sequence.

Case 6: $p = 6$ 1, 6, 7, 13, 20, ... Complete when $u_0 = 5$, $u_{-1} = -4$, and $u_{-2} = 9$ are added.

Case 7: $p = 7$ 1, 7, 8, 15, 23, ... Complete when $u_0 = 6$, $u_{-1} = -5$, $u_{-2} = 11$, and $u_{-3} = -16$ are added.

Case 8: $p = 8$ 1, 8, 9, 17, 26, ... Becomes complete when $u_0 = 7$, $u_{-1} = -6$, $u_{-2} = 13$, and $u_{-3} = -19$ are added.

Next we consider the generalized Fibonacci sequence.

Theorem 2.3: Let $u_{n+2} = u_{n+1} + u_n$, where $u_1 = a$, $u_2 = b$, and $(a, b) = 1$, $b \geqslant a \geqslant 1$. Then, every positive integer has a representation from $\{u_n\}_{-\infty}^{\infty}$ provided that no two consecutive u_j are in the same representation.

Proof: It is known that the generalized Fibonacci numbers are related to the ordinary Fibonacci numbers by

$$u_{n+1} = bF_n + aF_{n-1}. \tag{2.1}$$

Let m be a positive integer, $m \geqslant b$. Then we can always write

$$m = bA + aB$$

for some integers A and B, since $(a, b) = 1$. If both A and B are nonnegative, we are done, since the dual representation of A and B, by Klarner's Theorem, leads to a representation of m via (2.1). If A or B is negative, notice that, since the ordered pair (A, B) is a lattice point for a line with slope $-b/a$ and y-intercept m/a, if we can add an arbitrarily large integer to m, then we can raise the line so that it crosses the first quadrant and we will have nonnegative values for A and B. Thus, choose $u_{-k} < 0$ with an absolute value sufficiently large, and we represent

$$m - u_{-k} = bA^* + aB^*$$

for A^* and B^* nonnegative. We then represent $m - u_{-k}$ via Klarner's Theorem, and add u_{-k} to that representation to represent m. Similarly, if $m < b$, since the negatively subscripted terms of u_n become negatively as large as we please, choose $u_{-k} < 0$ so that $m - u_{-k} > b$, represent $m - u_{-k}$

as above, and then add u_{-k} to the representation.

3. A PATTERN ARISING FROM KLARNER'S DUAL ZECKENDORF REPRESENTATION

Recall the Klarner dual Zeckendorf representation given in Section 2, where

$$A = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1} = 0$$

$$B = F_{k_1} + F_{k_2} + \cdots + F_{k_r} = n,$$

where $n = 1, 2, 3, \ldots, \; |k_i - k_j| \geqslant 2, \; i \neq j,$ and F_j comes from $\{F_j\}_{-\infty}^{\infty}$. The constructive method for solving for the subscripts k_j to represent A and B described in our earlier work [3] leads to a symbolic display with a generous sprinkling of Lucas numbers. Here we use only two basic formulas,

$$u_{n+2} = u_{n+1} + u_n \quad \text{and} \quad 2u_n = u_{n+1} + u_{n-2}.$$

This allows us to push both right and left. We continue to add $F_{-1} = 1$ at each step, using the rules given to simplify the result. For example, for $n = 1$, we have $F_{-1} = 1$. For $n = 2$, $F_{-1} + F_{-1} = 2F_{-1} = F_0 + F_{-3} = 2$. For $n = 3$, $F_{-1} + F_0 + F_{-3}$ becomes $F_1 + F_{-3} = 1 + 2 = 3$. We display Table 3.1.

Strangely enough, the Wythoff pairs sequences enter into this again. The basic column centers under F_{-1}. The display is for expressions for B only; A is a translation of one space to the right. At each step, $B = n$, and $A = 0$.

From Table 3.1, many patterns are discernible. There are always the same number of successive entries in a given column. Under F_{-2} there are L_1; under F_{-3}, L_2; under F_{-4}, L_3; and under F_{-5}, L_4. Under F_{-6} there are L_5 successive entries, starting with $B = 30$, and under F_{-7} there are L_6 successive entries. On the line for $B = 47$, there are only two entries, one corresponding to $F_{-9} = 34$ and one to $F_7 = 13$, so that $34 + 13 = 47$ as required, while $F_{-8} = -21$ and $F_8 = 21$ have a zero sum as required.

The columns to the right of F_{-1} (under F_0, for instance) have $L_n \pm 1$ alternately successive entries, but the same numbers of successive entries always appear in the columns. Once we have all spaces cleared except the extreme edges in the pattern being built, we start again in the middle, as in line 48 or line 19.

TABLE 3.1

B	F_{-9}	F_{-8}	F_{-7}	F_{-6}	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6
1									x							
2							x			x						
3							x				x					
4							x		x		x					
5					x			x				x				
6					x					x		x				
7					x								x			
8					x				x				x			
9					x		x			x			x			
10					x		x				x		x			
11					x		x		x		x		x			
12			x			x		x						x		
13			x			x				x				x		
14			x			x					x			x		
15			x			x			x		x			x		
16			x					x				x		x		
17			x							x		x		x		
18			x												x	
19			x						x						x	
20			x				x			x					x	
21			x				x				x				x	
22			x				x		x		x				x	
23			x		x			x				x			x	
24			x		x					x		x			x	
25			x		x								x		x	
26			x		x				x				x		x	
27			x		x		x			x			x		x	
28			x		x		x				x		x		x	
29			x		x		x		x		x		x		x	
30	x			x		x		x								x

4. REPRESENTATIONS AND ADDITIVE PARTITIONS FOR THE SEQUENCE 1, 4, 5, 9, 14, 23, ...

We make the following array:

$$\begin{aligned} A_n &= \text{first positive integer not yet used} \\ B_1(n) &= B_n - 2 \\ B_2(n) &= B_n - 1 \\ B_n &= A_n + d_n, \end{aligned}$$

where $d_n \neq A_j$ and goes through the complement of $\{A_n\}$ in order, except we do not use $B_1(n)$ opposite the second of a consecutive pair of A_n; i.e., we do not use $B_1(n)$ if $A_n = A_{n-1} + 1$. The underlined numbers in the following table cannot be used for d_n.

n	A_n	$B_1(n)$	$B_2(n)$	B_n	d_n
0	0				
1	$\underline{1}$	$\underline{2}$	3	4	3
2	$\underline{5}$	$\underline{7}$	8	9	4
3	$\underline{6}$	$\underline{11}$	12	13	7
4	$\underline{10}$	$\underline{16}$	17	18	8
5	$\underline{14}$	21	22	23	9
6	$\underline{15}$	$\underline{25}$	26	27	12
7	$\underline{19}$	$\underline{30}$	31	32	13
8	$\underline{20}$	$\underline{34}$	35	36	16
9	$\underline{24}$	$\underline{39}$	40	41	17
10	$\underline{28}$	44	45	46	18
...	...	...	...	...	...

We now have the following constant differences, where (a_n, b_n) is a Wythoff pair:

$$B_{n+1} - B_n = \begin{cases} 5, & n = a_i \\ 4, & n = b_j \end{cases} \tag{4.1}$$

$$A_{n+1} - A_n = \begin{cases} 4, & n = a_i \\ 1, & n = b_j \end{cases} \tag{4.2}$$

Alternately,

$$A_n = 3a_n - 2n = (2n - a_n) \cdot 1 + (a_n - n) \cdot 4 \tag{4.3}$$

$$B_n = a_n + 3n = (2n - a_n) \cdot 4 + (a_n - n) \cdot 5 \tag{4.4}$$

Apparently, $d_n \neq B_j + 1$ and $d_n \neq B_j + 2$.

This extends for the sequence $1, p, p+1, \ldots, u_{n+2} = u_{n+1} + u_n$.

5. REPRESENTATIONS AND ADDITIVE PARTITIONS USING THE GENERALIZED FIBONACCI NUMBERS

We consider the general case for $(a, b) = 1$, and

$$u_1 = a,\ u_2 = b,\ u_{n+2} = u_{n+1} + u_n.$$

First we have a unique additive partition and the function

$$E_n(m) = mu_{n-1} - u_n[mu_{n-1}/u_n],\ m = 1, 2, \ldots, u_n - 1 \tag{5.1}$$

generates 1, 2, 3, ..., $u_n - 1$, but, of course, not in natural order. One set of the additive partition includes $1 \leqslant m \leqslant u_n/2$, while the other has $u_n/2 < m \leqslant u_n - 1$. Suppose $0 < a^* < b^*$; then the values of $E_n(m)$ are split into b^* disjoint sets whose first elements are 1, 2, 3, ..., a^*, ..., b^*. The elements of the sets to the left of a^* are, correspondingly, 1 less, 2 less, 3 less, ..., as we go to the left, while the sets between a^* and b^* have their values 1 less, 2 less, 3 less, ..., than b^*. Each element satisfies

$$\begin{aligned} a^*_{n+1} - a^*_n &= b,\ n \in \{a_k\},\ a_k = [k\alpha],\ \alpha = \frac{1+\sqrt{5}}{2}, \\ a^*_{n+1} - a^*_n &= a,\ n \in \{b_k\},\ b_k = [k\alpha^2]. \end{aligned} \tag{5.2}$$

The a^*_n are the representations using

$$a + \alpha_2 b + \alpha_3 u_3 + \ldots,\ \alpha_i = 0 \text{ or } 1,$$

while we can show

$$a^*_n = (2n - a_n)a + (a_n - n)b \tag{5.3}$$

$$b^*_n = (2n - a_n)b + (a_n - n)(a + b) \tag{5.4}$$

because of the formula

$$u_{n+1} = bF_n + aF_{n-1}. \tag{5.5}$$

Let $a^* = \{a + \alpha_2 u_2 + \alpha_3 u_3 + \cdots\}$ in natural order. Then

$$\begin{aligned} a^*_n &= bF_0 + aF_{-1} + \alpha_2(bF_1 + aF_0) + \cdots \\ &= b(F_0 + \alpha_2 F_1 + \alpha_3 F_2 + \cdots) + a(F_{-1} + \alpha_2 F_0 + \cdots) \\ &= a(2n - a_n) + b(a_n - n) \end{aligned}$$

since

$$a_n \to n \to a_n - n \to 2n - a_n.$$

Thus, once we know that $a_{n+1} - a_n = 2$ for $n = a_j$ and $a_{n+1} - a_n = 1$ for $n \neq a_j$, and $u_{n+1} = bF_n + aF_{n-1}$, we have (5.3) and (5.4). Further, these a^*_n and b^*_n are the generalizations of the Wythoff pair numbers $a_n = [n\alpha]$ and $b_n = [n\alpha^2]$ themselves $\left(\alpha = \dfrac{1+\sqrt{5}}{2}\right)$.

6. REPRESENTATIONS AND ADDITIVE PARTITIONS ARISING FROM TWO SUCCESSIVE FIBONACCI NUMBERS

It is well known that if we start with 1 and 2, we get the Wythoff pairs and have a unique additive partition of the positive integers. Next, to see something else, take 2 and 3. Since (2, 3) = 1, we still have the same additive partition of the positive integers, and the function $E_n(m)$ of (5.1) and (1.1) still yields the residues mod F_n, but our array changes in an interesting way.

TABLE 6.1

2, 3, 5, 8, 13, 21, ...

			A_n	B_n
1	2	3	1	2
4	5	8	3	4
6	7	11	6	5
9	10	16	8	7
12	13	21	9	10
14	15		11	12
...	...	...	14	13
			16	15
a_{a_n}	b_n	a_{b_n}	17	18
$W = 2$		$W = 3$	19	20

Note that, if we give a weight one and b weight two, we have the weights abbreviated by W above. The successive values of $E_n(m)$ are the same as before, but we now have a different split to look at. Note that A and B are formed so that no two elements of either set have 2, 3, 5, 8, 13, ..., as their sum. Now let us look at $E_n(m)$ for $13 = F_{n-1}$ and $21 = F_n$; i.e.,

$$E_n(m) = 13m - 21[13m/21].$$

The twenty values in the cycle are:

13, 5, 18, 10, 2, 15, 7, 20, 12, 4, 17, 9, 1, 14, 6, 19, 11, 3, 16, 8.

The first 10 are elements of B_n; the other 10 are A_n. The first 8 have the form b_n; the next 8 have the form a_{a_n}; the last 4 have the form a_{b_n}.

Now look at the array induced by 3 and 5 as a starting pair.

TABLE 6.2

3, 5, 8, 13, 21, 34, ...

					A_n	B_n
1	2	3	4	5	1	2
6	7	8	12	13	3	4
9	10	11	17	18	6	5
14	15	16	25	26	8	7
19	20	21	33	34	9	10
22	23	24	38	39	11	12
27	28	29	46	47	14	13
			51	52	16	15
...	...	...	...	...	17	18
$a_{a_{a_n}}$	b_{a_n}	a_{b_n}	$a_{a_{b_n}}$	b_{b_n}	19	20
					...	...
$W = 3$			$W = 4$			

The A_n and B_n are the same as before.

Return to the values of $E_n(m)$ for 13 and 21 given above. Notice that the first 3 values—13, 5, 18—come from b_{b_n}; the next 5 from b_{a_n}; the next 3 from $a_{a_{b_n}}$; then five from $a_{a_{a_n}}$; and, lastly, 4 from a_{b_n}. We begin to see familiar patterns emerging [4], [5].

We write the array induced by 5 and 8 in Table 6.3.

TABLE 6.3

5, 8, 13, 21, 34, 55, ...

								A_n	B_n
1	2	3	4	5	6	7	8	1	2
9	10	11	12	13	19	20	21	3	4
14	15	16	17	18	27	28	29	6	5
22	23	24	25	26	40	41	42	8	7
30	31	32	33	34	53	54	55	9	10
35	36	37	38	39	61	62	63	11	12
43	44	45	46	47	74	75	76	14	13
...	...	...	...	...	..	...	...	16	15
$a_{a_{a_a}}$	$b_{a_{a_n}}$	$a_{b_{a_n}}$	$a_{a_{b_n}}$	b_{b_n}	$a_{a_{a_{b_n}}}$	$b_{a_{b_n}}$	$a_{b_{b_n}}$	17	18
Five of weight 4					Three of weight 5			19	20
								...	...

Since a_n and b_n are elements in complementary sets, the array on the left covers the positive integers. Note that the additive partition sequence A_n and B_n is the same for (1, 2), (2, 3), (3, 5), (5, 8), and for all consecutive Fibonacci pairs.

Now, the weights mentioned under the arrays from (2, 3), (3, 5), and (8, 13) are precisely the unshortened sequence of 1's and 2's in the compositions of W (the weight) as laid out by our scheme in [4]. As each must end in a $1 = a_1$ or a $2 = b_1$, to get the proper representation, we simply replace 1 in each case by n and let $n = 1, 2, 3, \ldots$. This is a wonderful application of Wythoff pairs, Fibonacci representations, additive partitions of the positive integers, and the function $E_n(m)$.

Before we prove all of this, we need some results for Wythoff's pairs from [1] and [5]. For Wythoff's pairs (a_n, b_n),

$$a_{b_n+1} - a_{b_n} = 1 \quad \text{and} \quad a_{a_n+1} - a_{a_n} = 2; \tag{6.1}$$

$$a_{a_n} + 1 = b_n. \tag{6.2}$$

Return to the weights given in the array induced by 3 and 5 in Table 6.2. From (6.2), replacing n by a_n, we get immediately that

$$a_{a_{a_n}} + 1 = b_{a_n}. \tag{6.3}$$

Now,

$$a_{a_{a_n}} + 1 = b_{a_n} \quad \text{and} \quad a_{a_{a_n}+1} = a_{a_{a_n}} + 2$$

from (6.1) rewritten as

$$a_{a_n+1} = a_{a_n} + 2.$$

Thus,

$$b_{a_n} + 1 = a_{a_{a_n}+1} = a_{b_n}$$

as required.

We obtain

$$a_{a_{b_n}} + 1 = b_{b_n} \tag{6.4}$$

by replacing n by b_n in (6.2). These are all the weights appearing in Table 6.2.

Now, we move to Table 6.3, the array induced by 5 and 8, and examine the weights. From (6.2), by replacing n by a_{a_n}, we easily obtain

$$a_{a_{a_{a_n}}} + 1 = b_{a_{a_n}}.$$

Then,

$$a_{a_{a_{a_n}}} + 2 = b_{a_{a_n}} + 1 = a_{a_{a_{a_n}}} + 1 = a_{b_{a_n}}.$$

Next, again from (6.2) with n replaced by b_n and then n replaced by a_{b_n}, we have

$$a_{a_{b_n}} + 1 = b_{b_n} \qquad \text{and} \qquad a_{a_{a_{b_n}}} + 1 = b_{a_{b_n}}.$$

Again using (6.2), we can also write

$$b_{a_{b_n}} + 1 = a_{a_{a_{b_n}}+1} = a_{b_{b_n}}.$$

This undoubtedly continues.

To get our next line of weighted 1's, we simply add a_n to end each 1 of weight 4, and take those of weight 5 together with these. All of the following are of weight 5:

$$a_{a_{a_{a_n}}} \qquad b_{a_{a_n}} \qquad a_{b_{a_n}} \qquad a_{a_{b_{a_n}}} \qquad b_{b_{a_n}} \qquad a_{a_{a_{b_n}}} \qquad b_{a_{b_n}} \qquad a_{b_{b_n}}$$

The five on the left end in a_n, and came from adding an a_n to each 1 of weight 4; the three on the left ending in b_n are of weight 5 already.

To get the next five of weight 6 to the right, we add a_{b_n} to the end of the weight 3 a_{a_n}, b_{a_n}, and a_{b_n}, then add b_n only to the weight 4 $a_{a_{b_n}}$ and b_{b_n} of Table 6.2, to form

$$a_{a_{a_{b_n}}} \qquad b_{a_{a_{b_n}}} \qquad a_{b_{a_{b_n}}} \quad \text{and} \quad a_{a_{b_{b_n}}} \qquad b_{b_{b_n}}.$$

Now, we would like to have

$$b_{b_{a_n}} + 1 = a_{a_{a_{b_n}}}. \tag{6.5}$$

From $a_{a_{b_{a_n}}} + 1 = b_{b_{a_n}}$, we have

$$a_{a_{b_{a_n}}} + 2 = b_{b_{a_n}} + 1 = a_{a_{b_{a_n}}+1}.$$

Now, $a_{b_{a_n}} + 1 = a_{b_{a_n}+1}$, so that

$$a_{b_{a_n}} + 1 = a_{b_{a_n}+1} = a_{a_{a_n}+2} = a_{a_{a_n}+1} = a_{a_{b_n}}.$$

Thus, $a_{a_{b_{a_n}}+1} = a_{a_{a_{b_n}}}$, establishing (6.5).

Finally, we write a complete proof based on (6.1) and (6.2). Notice that we have to show that the differences between successive columns are always 1, except for the transition that comes between the columns headed by F_{n-1} and $F_{n-1} + 1$ in the array. Also, we need a rule for formuation.

The rule of formation of one array from the preceding array is as follows: To get the array with F_{n+1} columns, build up the left F_{n-1} and the right F_{n-2} columns of the array for F_n columns by extending the subscripts. Add a_n to the bottom of each subscript in the left part, copy down the right part next as is, and then copy down the old left part with b_n added to the bottom to get the rew right part.

$$
\begin{array}{lccccc|ccc}
 & & & & \text{Left Part} & & \text{Right Part} & & \\
\text{Line 1} & & & & & a_n & b_n & & \\
\text{Line 2} & & & & a_{a_n} & b_n & a_{b_n} & & \\
\text{Line 3} & & & a_{a_{a_n}} & b_{a_n} & a_{b_n} & a_{a_{b_n}} & b_{b_n} & \\
\text{Line 4} & a_{a_{a_{a_n}}} & b_{a_{a_n}} & a_{b_{a_n}} & a_{a_{b_n}} & b_{b_n} & a_{a_{a_{b_n}}} & b_{a_{b_n}} & a_{b_{b_n}}
\end{array}
$$

From [4, p. 315], $$F_{2n+2} = \underbrace{a_{b_{b_{\cdot_{\cdot_{\cdot_{b_1}}}}}}}_{n\ b\text{'s}} \quad\text{and}\quad F_{2n+1} = \underbrace{b_{b_{b_{\cdot_{\cdot_{\cdot_{b_1}}}}}}}_{n\ b\text{'s}}$$

Now, the entries before the dashed line in Lines 1, 2, 3, and 4 above are alternately odd- and even-subscripted Fibonacci numbers, while the entries on the far right are the next higher Fibonacci numbers if we replace n by 1. Thus, we have the sequence of representations in natural order.

We show that the columns always differ by one within the left part and within the right part. We count each a subscript 1 and each b subscript 2. Then the left part of Line 1 has weight 1 and the right part weight 2, and the left part of Line 2 has weight 2 and the right part has weight 3. The columns in the left part of Line 2 differ by 1 according to (6.2),

$$a_{a_n} + 1 = b_n,$$

which generalizes to

$$a_{a_{A_n}} + 1 = b_{A_n} \tag{6.6}$$

for any A_n. Next, for Line 3,

$$\begin{array}{ccc|cc} a_{a_{a_n}} & b_{a_n} & a_{b_n} & a_{a_{b_n}} & b_{b_n} \\ & \text{Weight } 3 & & \text{Weight } 4 & \end{array}$$

we have, from (6.2),

$$b_{a_n} + 1 = a_{a_{a_n}} + 2 = a_{a_{a_n}+1} = a_{b_n},$$

and from (6.1) and (6.6),

$$b_{a_{A_n}} + 1 = a_{a_{a_{A_n}}} + 2 = a_{a_{a_{A_n}}+1} = a_{b_{A_n}}, \tag{6.7}$$

so that

$$a_{a_{b_n}} + 1 = b_{b_n},$$

follows for $A_n = b_n$.

Now, for

$$\begin{array}{ccccc|ccc} a_{a_{a_{a_n}}} & b_{a_{a_n}} & a^*_{b_{a_n}} & a_{a_{b_n}} & b_{b_n} & a_{a_{a_{b_n}}} & b_{a_{b_n}} & a_{b_{b_n}} \\ & & \text{Weight } 4 & & & & \text{Weight } 5 & \end{array}$$

note that all cases follow from earlier cases, except transition case #1, marked with an asterisk above:

$$a_{b_{a_n}} + 1 = a_{a_{b_n}}. \tag{6.8}$$

But, $a_{b_{a_n}} + 1 = a_{b_{a_n}+1} = a_{a_{a_{a_n}}+2} = a_{a_{a_{a_n}+1}} = a_{a_{b_n}}$.

We now display all of one more case:

$$\begin{array}{cccccccc|ccccc} a_{a_{a_{a_{a_n}}}} & b_{a_{a_{a_n}}} & a_{b_{a_{a_n}}} & a_{a_{b_{a_n}}} & b^*_{b_{a_n}} & a_{a_{a_{b_n}}} & b_{a_{b_n}} & a_{b_{b_n}} & a_{a_{a_{a_{b_n}}}} & b_{a_{a_{b_n}}} & a_{b_{a_{b_n}}} & a_{a_{b_{b_n}}} & b_{b_{b_n}} \\ & & & \text{Weight } 5 & & & & & & & \text{Weight } 6 & & \end{array}$$

Note that all these columns differ by one within the left and right parts from earlier results, except transition case #2, marked with an asterisk above,

$$b_{b_{a_n}} + 1 = a_{a_{a_{b_n}}}, \tag{6.9}$$

which is proved as follows:

$$b_{b_{a_n}} + 1 = a_{a_{b_{a_n}}} + 1 + 1 = a_{a_{b_{a_n}}} + 2 = a_{a_{b_{a_n}}+1} = a_{a_{a_{b_n}}}$$

from (6.2) and (6.8), which was transition case #1 above.

When we write the next line, our transition case will again be like #1, as

$$a_{b_{b_{a_n}}} + 1 = a_{a_{a_{a_{b_n}}}}, \tag{6.10}$$

proved from (6.1) and (6.9), which was transition case #2 above, as

$$a_{b_{b_{a_n}}} + 1 = a_{b_{b_{a_n}}} = a_{a_{a_{b_n}}}.$$

Next, the transition will again by like #2 above,

$$b_{b_{b_{a_n}}} + 1 = a_{a_{a_{a_{a_{b_n}}}}}, \tag{6.11}$$

proved from transition case #1 given in (6.10):

$$b_{b_{b_{a_n}}} + 1 = a_{a_{b_{b_{a_n}}}} + 2 = a_{a_{b_{b_{a_n}}}+1} = a_{a_{a_{a_{a_{b_n}}}}}.$$

The proof is now complete, by the principle of mathematical induction. That is, if, for the earlier cases, each term of the sequence plus 1 is the next one to the right, then, from the formation rules and general result (6.1), we get that it holds for the next case, but we have to prove the transition cases, since we get the results for each left and right part separately. To get a new left section, we add a_n to the bottom of the old left section subscripts and use general result (6.7) and just copy down the right section as is, and these two parts are, separately, okay. The transition from left to right in the new left section is now proved in four cases by mathematical induction. The new right part, which is the former left part with b_n added on the end, yields to general result (6.1). This completes the discussion.

Suppose the line array, at some level, produces sequences whose elements cover the positive integers without overlap. Since a_n is added to the left portion to form the left part of the new left part and the b_n is added to the left portion to form the new right part, these two pieces cover all the integers together that we covered by the left part before, and the old right side is left intact, ao that the new line again covers the positive integers without overlap.

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ADDENDA TO GEOMETRY OF A GENERALIZED SIMSON'S FORMULA

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1. INTRODUCTION

In [3], the author considers the loci in the Euclidean plane satisfied by points whose Cartesian coordinates are pairs of successive members in recurrence sequences of a special type. The purpose of this paper is to extend that discussion.

We begin as in [1] and [3] by defining the general term of the sequence $\{w_n(a, b; p, q)\}$ as

$$w_{n+2} = pw_{n+1} - qw_n, \ w_0 = a, \ w_1 = b, \tag{1.1}$$

where a, b, p, q belong to some number system, but are generally thought of as integers. In this paper, they will always be integers.

In [1], we find

$$w_n w_{n+2} - w_{n+1}^2 = eq^n, \tag{1.2}$$

where

$$e = pab - qa^2 - b^2. \tag{1.3}$$

Combining (1.1) and (1.2) as in [3], we obtain

$$qw_n^2 - pw_n w_{n+1} + w_{n+1}^2 + eq^n = 0, \tag{1.4}$$

which, with $w_n = x$ and $w_{n+1} = y$, becomes

$$qx^2 - pxy + y^2 + eq^n = 0. \tag{1.5}$$

The graph of (1.5) is a hyperbola if $p^2 - 4q > 0$, an ellipse (or circle) if $p^2 - 4q < 0$, and a parabola if $p^2 - 4q = 0$ (degenerate cases excluded). The xy term can be eliminated by performing a counterclockwise rotation of the axes through the angle θ, where

$$\cot 2\theta = \frac{1 - q}{p}, \tag{1.6}$$

using

$$\begin{aligned} x &= \bar{x}\cos\theta - \bar{y}\sin\theta \\ y &= \bar{x}\sin\theta + \bar{y}\cos\theta. \end{aligned} \tag{1.7}$$

When $q = 1$, $\theta = \pi/4$, and (1.5) becomes

$$(2 - p)\bar{x}^2 + (p + 2)\bar{y}^2 + 2e = 0 \tag{1.8}$$

with

$$e = pab - a^2 - b^2. \tag{1.9}$$

When $q \neq 1$, and therefore $\theta \neq \pi/4$, we let

$$r = \sqrt{p^2 + (q - 1)^2}. \tag{1.10}$$

Substituting (1.7) into (1.5) and using the double angle formulas for $\cos 2\theta$ and $\sin 2\theta$, we find that

$$\bar{x}^2\left(\frac{q + 1 + (q - 1)\cos 2\theta - p\sin 2\theta}{2}\right)$$
$$+ \bar{y}^2\left(\frac{q + 1 - (q - 1)\cos 2\theta + p\sin 2\theta}{2}\right) + eq^n = 0.$$

Now, by (1.1), depending upon the values of q and p, we have

$$\begin{cases} \left(\dfrac{q + 1 + r}{2}\right)\bar{x}^2 + \left(\dfrac{q + 1 - r}{2}\right)\bar{y}^2 + eq^n = 0 \text{ if } p < 0 \\ \left(\dfrac{q + 1 - r}{2}\right)\bar{x}^2 + \left(\dfrac{q + 1 + r}{2}\right)\bar{y}^2 + eq^n = 0 \text{ if } p > 0. \end{cases} \tag{1.11}$$

We now consider the special cases when $q = 1$ (§2) and when $q = -1$ (§3).

2. THE SPECIAL CASES WHEN $q = 1$

If $p = 2$, we have from (1.8) the degenerate conic $2\bar{y}^2 = -e = (a - b)^2$ which gives rise to the parallel lines $x - y = b - a$ and $x - y = a - b$. The sequence of terms associated with this degenerate conic is

$$a,\ b,\ 2b - a,\ 3b - 2a,\ 4b - 3a,\ 5b - 4a,\ \ldots\ . \tag{2.1}$$

Since none of the successive pairs of (2.1) satisfy $x - y = b - a$, we see that all pairs (w_n, w_{n+1}) of (2.1) lie on the line $x - y = a - b$.

If $p = -2$, the degenerate conic is $2\bar{x}^2 = -e = (a + b)^2$, the sequence is

$$a,\ b,\ -2b - a,\ 3b + 2a,\ -4b - 3a,\ 5b + 4a,\ \ldots, \tag{2.2}$$

and the successive pairs (w_n, w_{n+1}) of (2.2) satisfy $x + y = a + b$ if n is even and $x + y = -(a + b)$ when n is odd.

If $p = 0$, the sequence $\{w_n(a, b; 0, 1)\}$ is

$$a,\ b,\ -a,\ -b,\ a,\ b,\ -a,\ -b,\ \ldots, \tag{2.3}$$

so that for (2.3) the only distinct pairs of successive coordinates on the circle $x^2 + y^2 = -e = a^2 + b^2$ are (a, b), $(b, -a)$, $(-a, -b)$, $(-b, a)$.

If $p = 1$, then equation (1.8) becomes $\bar{x}^2 + 3\bar{y}^2 = 2(a^2 + b^2 - ab)$. But $a^2 + b^2 - ab > 0$ if a and b are not both zero, so the graph of (1.5) is always an ellipse with the equation $x^2 + y^2 - xy = a^2 + b^2 - ab$. The sequence $\{w_n(a, b; 1, 1)\}$ is

$$a,\ b,\ b - a,\ -a,\ -b,\ -b + a,\ a,\ b,\ \ldots\ . \tag{2.4}$$

The only distinct pairs of successive coordinates on the ellipse for (2.4) are (a, b), $(b, b - a)$, $(b - a, -a)$, $(-a, -b)$, $(-b, -b + a)$, $(-b + a, a)$.

When $p = -1$, equation (1.8) becomes $3\bar{x}^2 + \bar{y}^2 = -2e = 2(a^2 + b^2 + ab)$. If a and b are not both zero, then $a^2 + b^2 + ab > 0$, so that the graph of (1.5) with equation $x^2 + xy + y^2 = a^2 + b^2 + ab$ is an ellipse. The sequence $\{w_n(a, b; -1, 1)\}$ is

$$a,\ b,\ -b - a,\ a,\ b,\ -b - a,\ \ldots, \tag{2.5}$$

so that the only pairs of successive coordinates of (2.5) on the ellipse are (a, b), $(b, -b - a)$, $(-b - a, a)$.

One might wonder about the case $e = 0$. Under this condition, since a and p are integers, we have $p = \pm 2$, which has already been discussed, or $a = b = 0$, which is a trivial case.

For all other values of p, the graph of (1.8), and hence of (1.5), is a hyperbola. Thus, there exists an infinte number of distinct pairs of integers (w_n, w_{n+1}) lying on each hyperbola for a given p. The following facts help to characterize the hyperbola for a given p.

If $p > 2$ and $e < 0$ or $p < -2$ and $e > 0$, then the asymptotes for (1.5) are

$$y = \frac{p \pm \sqrt{p^2 - 4}}{2}\, x, \tag{2.6}$$

the vertices are

$$\left(\sqrt{\frac{-e}{p+2}}, -\sqrt{\frac{-e}{p+2}}\right) \quad \text{and} \quad \left(-\sqrt{\frac{-e}{p+2}}, \sqrt{\frac{-e}{p+2}}\right) \tag{2.7}$$

the eccentricity is

$$\sqrt{\frac{2p}{p-2}}, \tag{2.8}$$

the foci are

$$\left(\sqrt{\frac{-2pe}{p^2-4}}, -\sqrt{\frac{-2pe}{p^2-4}}\right) \quad \text{and} \quad \left(-\sqrt{\frac{-2pe}{p^2-4}}, \sqrt{\frac{-2pe}{p^2-4}}\right) \tag{2.9}$$

and the endpoints of the latera recta are

$$\left(\frac{s-t}{p^2-4}, \frac{s+t}{p^2-4}\right), \left(\frac{s+t}{p^2-4}, \frac{s-t}{p^2-4}\right), \left(\frac{s+t}{4-p^2}, \frac{s-t}{4-p^2}\right), \left(\frac{s-t}{4-p^2}, \frac{s+t}{4-p^2}\right), \tag{2.10}$$

where $s = (p+2)\sqrt{-e(p+2)}$ and $t = -\sqrt{2pe(p^2-4)}$.

If $p > 2$ and $e > 0$ or $p < -2$ and $e < 0$, then the asymptotes and eccentricity are found by using (2.6) and (2.8). The vertices are

$$\left(\sqrt{\frac{e}{p-2}}, \sqrt{\frac{e}{p-2}}\right), \left(-\sqrt{\frac{e}{p-2}}, -\sqrt{\frac{e}{p-2}}\right), \tag{2.11}$$

the foci are

$$\left(\sqrt{\frac{2pe}{p^2-4}}, \sqrt{\frac{2pe}{p^2-4}}\right), \left(-\sqrt{\frac{2pe}{p^2-4}}, -\sqrt{\frac{2pe}{p^2-4}}\right), \tag{2.12}$$

and the endpoints of the latera recta are

$$\begin{gathered}\left(\frac{s_1-t_1}{p^2-4}, \frac{s_1+t_1}{p^2-4}\right), \left(\frac{s_1+t_1}{p^2-4}, \frac{s_1-t_1}{p^2-4}\right), \\ \left(\frac{s_1+t_1}{4-p^2}, \frac{s_1-t_1}{4-p^2}\right), \left(\frac{s_1-t_1}{4-p^2}, \frac{s_1+t_1}{4-p^2}\right),\end{gathered} \tag{2.13}$$

where $s_1 = \sqrt{2pe(p^2-4)}$ and $t_1 = (p-2)\sqrt{e(p-2)}$.

3. THE SPECIAL CASES WHEN $q = -1$

Letting $q = -1$ in (1.11) and simplifying, we have

$$\overline{x}^2 - \overline{y}^2 = \frac{2e(-1)^n}{r}, \; p > 0, \tag{3.1}$$

and

$$\bar{y}^2 - \bar{x}^2 = \frac{2e(-1)^n}{r}, \; p < 0, \tag{3.2}$$

where the values of e and r from (1.3) and (1.10) are now

$$e = pab + a^2 - b^2 \quad \text{and} \quad r = \sqrt{p^2 + 4}. \tag{3.3}$$

The case $p = 0$ is trivial and, therefore, omitted.

Since (3.1) and (3.2) are always the equations of a hyperbola, unless $e = 0$, which is a trivial case, $a = b = 0$, there are always an infinite number of distinct pairs of integers (w_n, w_{n+1}) which lie on the original hyperbola of (1.5) for any value of p, unless the sequence is cyclic. The following facts characterize the hyperbola for different values of p, e, and n.

The asymptotes of (1.5) which are perpendicular are always given by

$$y = \frac{p \pm r}{2} x, \tag{3.4}$$

and the eccentricity is always 2, giving a rectangular hyperbola. These cases are in accord with the cases $p = 1$ and $p = 2$ given in [3].

If $p > 0$ and $e(-1)^n > 0$, then the vertices are

$$(u, v), \; (-u, -v), \tag{3.5}$$

the foci are

$$(u\sqrt{2}, v\sqrt{2}), \; (-u\sqrt{2}, -v\sqrt{2}), \tag{3.6}$$

and the endpoints of the latera recta are

$$\begin{aligned} &(u\sqrt{2} - v, \; u + v\sqrt{2}), \; (u\sqrt{2} + v, \; -u + v\sqrt{2}), \\ &(-u\sqrt{2} - v, \; u - v\sqrt{2}), \; (-u\sqrt{2} + v, \; -u - v\sqrt{2}), \end{aligned} \tag{3.7}$$

where $u = \frac{1}{r}\sqrt{e(-1)^n(r + 2)}$ and $v = \frac{1}{r}\sqrt{e(-1)^n(r - 2)}$.

If $p > 0$ and $e(-1)^n < 0$, then the vertices are

$$(v_1, -u_1), \; (-v_1, u_1), \tag{3.8}$$

the foci are

$$(-v_1\sqrt{2}, u_1\sqrt{2}), \; (v_1\sqrt{2}, -u_1\sqrt{2}), \tag{3.9}$$

and the endpoints of the latera recta are

$$\begin{aligned}&(u_1 - v_1\sqrt{2},\ u_1\sqrt{2} + v_1),\ (u_1 + v_1\sqrt{2},\ -u_1\sqrt{2} + v_1),\\ &(-u_1 - v_1\sqrt{2},\ u_1\sqrt{2} - v_1),\ (-u_1 + v_1\sqrt{2},\ -u_1\sqrt{2} - v_1),\end{aligned} \tag{3.10}$$

where $u_1 = \frac{1}{r}\sqrt{e(-1)^{n+1}(r + 2)}$ and $v_1 = \frac{1}{r}\sqrt{e(-1)^{n+1}(r - 2)}$.

If $p < 0$ and $e(-1)^n > 0$, then the vertices are

$$(-u,\ v),\ (u,\ -v), \tag{3.11}$$

the foci are

$$(-u\sqrt{2},\ v\sqrt{2}),\ (u\sqrt{2},\ -v\sqrt{2}), \tag{3.12}$$

and the endpoints of the latera recta are

$$\begin{aligned}&(v - u\sqrt{2},\ v\sqrt{2} + u),\ (v + u\sqrt{2},\ -v\sqrt{2} + u),\\ &(-v - u\sqrt{2},\ v\sqrt{2} - u),\ (-v + u\sqrt{2},\ -v\sqrt{2} - u),\end{aligned} \tag{3.13}$$

where u and v are as before.

If $p < 0$ and $e(-1)^n < 0$, then the vertices are

$$(v_1,\ u_1),\ (-v_1,\ -u_1), \tag{3.14}$$

the foci are

$$(v_1\sqrt{2},\ u_1\sqrt{2}),\ (-v_1\sqrt{2},\ -u_1\sqrt{2}), \tag{3.15}$$

and the endpoints of the latera recta are

$$\begin{aligned}&(v_1\sqrt{2} - u_1,\ v_1 + u_1\sqrt{2}),\ (v_1\sqrt{2} + u_1,\ -v_1 + u_1\sqrt{2}),\\ &(-v_1\sqrt{2} - u_1,\ v_1 - u_1\sqrt{2}),\ (-v_1\sqrt{2} + u_1,\ -v_1 - u_1\sqrt{2}),\end{aligned} \tag{3.16}$$

where u_1 and v_1 are as before.

4. CONCLUDING REMARKS

Consider $p > 0$. Note that the hyperbola for $e < 0$ and n odd is the same as the hyperbola for $e > 0$ and n even for any pairs (a, b) giving the same value of e, while the hyperbola for $e < 0$ and n even is the same as the hyperbola for $e > 0$ and n odd for any pair (a, b) giving the same e. A similar statement holds if $p < 0$.

For the sequence $Q = \{w_n(a, b; p, -1)\}$, we know from (1.4) that

$$pw_nw_{n+1} + w_n^2 - w_{n+1}^2 = \pm e,$$

depending on whether n is even or odd. Let $e < 0$ for $n = 0$ and

$$R = \{w_n(w_{2m},\ w_{2m+1};\ p,\ -1)\}.$$

The successive pairs of Q and R lie on $\overline{y}^2 - \overline{x}^2 = \frac{-2e}{r}$ if n is even and on $\overline{x}^2 - \overline{y}^2 = \frac{-2e}{r}$ if n is odd. Let

$$S = \{w_n(w_{2m+1},\ w_{2m+2};\ p,\ -1)\}.$$

then the successive pairs of S with n even lie on the same hyperbola as the successive pairs of Q with n odd. That is, they lie on

$$\overline{x}^2 - \overline{y}^2 = \frac{2e(-1)^n}{r}.$$

Furthermore, the successive pairs of S with n odd lie on the same hyperbola as the successive pairs of Q with n even. That is, they lie on

$$\overline{y}^2 - \overline{x}^2 = \frac{2e(-1)^{n+1}}{r}.$$

We close by mentioning that the vertices for the Fibonacci sequence $\{w_n(0,\ 1;\ 1,\ -1)\}$ are

$$\left(\sqrt{\frac{\sqrt{5}+2}{5}},\ \sqrt{\frac{\sqrt{5}-2}{5}}\right),\ \left(-\sqrt{\frac{\sqrt{5}+2}{5}},\ -\sqrt{\frac{\sqrt{5}-2}{5}}\right)$$

when n is odd and

$$\left(\sqrt{\frac{\sqrt{5}-2}{5}},\ -\sqrt{\frac{\sqrt{5}+2}{5}}\right),\ \left(-\sqrt{\frac{\sqrt{5}-2}{5}},\ \sqrt{\frac{\sqrt{5}+2}{5}}\right)$$

when n is even. Furthermore, all the pairs $(F_{2n},\ F_{2n+1})$ lie on the right half of the positive branch of $\overline{y}^2 - \overline{x}^2 = 2/r$ when $n > 0$, and on the left half of the positive branch of $\overline{y}^2 - \overline{x}^2 = 2/r$ when $n < 0$, so that no points $(F_n,\ F_{n+1})$ lie on the negative branch of the hyperbola. A similar remark holds for $(F_{2n+1},\ F_{2n+2})$ and $\overline{x}^2 - \overline{y}^2 = 2/r$.

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GENERAL SOLUTION OF A FIBONACCI-LIKE RECURSION RELATION AND APPLICATIONS

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1. INTRODUCTION: THE COMBINATORICS FUNCTION TECHNIQUE

In a series of papers published over the past few years, [1], [2], a method called the combinatorics function technique, or CFT, was perfected to obtain the solution of any linear partial difference equation subject to a set of initial values. Fibonacci-like recursion relations are a special case of difference equations that could be solved by the CFT method. Although many applications of the CFT have been published elsewhere, [3], [4], the study here leads to original results and provides a natural generalization of the problem investigated by Hock and McQuistan in "Occupational Degeneracy for λ-Bell Particles on a Saturated $\lambda \times N$ Lattice Space" [5].

For the reader who is not familiar with the CFT method, we summarize briefly the results of [2]. Consider a function B depending on n variables $(m_1, m_2, m_3, \ldots, m_n)$. The evaluation point, M, in the associated n-dimensional space whose coordinates are $(m_1, m_2, \ldots, m_n)$ and vector $\mathbf{M}$, whose components are the same as the coordinates of point M, will be used interchangeably for convenience. The multivariable function B is said to satisfy a partial *difference* equation when its value at point M, $B(\mathbf{M})$, is linearly related to its values at shifted arguments such as $\mathbf{M} - \mathbf{A}_k$, i.e.,

$$B(\mathbf{M}) = \sum_{k=1}^{N} f_{A_k}(\mathbf{M}) B(\mathbf{M} - \mathbf{A}_k) + I(\mathbf{M}). \tag{1.1}$$

The coefficients $f_{A_k}(\mathbf{M})$ and the inhomogeneous term $I(\mathbf{M})$ are assumed to be known and may not necessarily be constant. The problem to be investigated here is a difference equation with no inhomogeneous term, i.e.,

$$I(\mathbf{M}) = 0, \tag{1.2}$$

one-dimensional ($n = 1$), and with $N = 2$, thus corresponding to a three-term recursion relation.

The set formed by the N shifts or N displacement vectors, $\mathbf{A}_k$, is denoted $\mathscr{S}$;

$$\mathscr{A} = \{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N\}. \tag{1.3}$$

Generally, equation (1.1) holds for points M in the n-dimensional space belonging to a certain region, $\mathscr{R}$, that may not necessarily be finite. The values of a function B at a boundary, called $\mathscr{J}$, of region $\mathscr{R}$, are also generally known as

$$B(\mathbf{J}_\ell) = \Lambda_\ell;\ \ell = 1, 2, \ldots \text{ and } \mathbf{J}_\ell \in \mathscr{J}. \tag{1.4}$$

The evaluation points, $\mathbf{J}_\ell$, exhibited in equation (1.4) are referred to as boundary points for obvious reasons. Similarly, the region $\mathscr{J}$ containing the boundary points J_ℓ is interchangeably referred to as the set of boundary points or, simply, boundary set, while region $\mathscr{R}$ is the set of points M for which equation (1.1) must hold.

Equation (1.1) and its boundary value conditions, equation (1.4), were first discussed by the author and collaborators in earlier papers [1] for the special case of a single-variable function B and therefore defined on a one-dimensional space ($n = 1$). Some applications were also discussed in connection with the Schrödinger equation with a central linear potential [3]. Equation (1.1) is not necessarily consistent with the boundary value condition (1.2). To obtain the consistency condition, it was essential to introduce in [2] a set, $\mathscr{M}$, containing all points M in the n-dimensional space having the following property:

> Every possible path reaching any point belonging to set $\mathscr{M}$ by successive discrete displacement vectors $\mathbf{A}_k \in \mathscr{A}$ should contain at least one point belonging to the boundary set $\mathscr{J}$.

When such a relationship exists between two sets $\mathscr{M}$ and $\mathscr{J}$, then $\mathscr{J}$ is called a *full boundary* of $\mathscr{M}$ with respect to $\mathscr{A}$. Also essential to the consistency problem was the notion of restricted discrete paths connecting a boundary point, say $\mathbf{J}_\ell$, to a point M belonging to set $\mathscr{M}$. Such a restricted path, if it exists, does not contain any boundary point other than $\mathbf{J}_\ell$. In [2], we were able to show that there exists one and only one set, $\mathscr{J}_0$, called the *minimal full boundary* of $\mathscr{M}$ with respect to $\mathscr{A}$, such that each and every element of $\mathscr{J}_0$ can be connected to at least one element of $\mathscr{M}$ by at least one restricted path. It then follows that:

(i) equation (1.1) is consistent with equation (1.4) provided $\mathscr{R} \subset \mathscr{M}$ and,

(ii) if this is the case, its solution is unique and depends only on those ℓ's corresponding to boundary points $J_\ell \subset \mathscr{J}_0$.

The CFT method gives an explicit and systematic way of constructing the solution of equation (1.1) in terms of the so-called combinatorics functions of the *second kind*. An early version of the combinatorics functions of the second kind can be found in [1], and their applications to some physical problems are discussed in [3] and [4]. An extended and more complete version of the e functions was obtained in [2]. We now give the definition of the combinatorics functions of the second kind leading to the construction of the solution of equation (1.1).

For every boundary point $J_\ell \in \mathscr{J}_0$ and evaluation point $M \in \mathscr{M}$, one considers all possible paths connecting J_ℓ to M by discrete displacement vectors $\boldsymbol{\delta}_j \in \mathscr{A}$. A given path is identified by two labels ω and q. Label ω refers to the total number of displacements on a discrete path. Label q is used to distinguish among various paths, having the same number, ω, of displacement vectors. Corresponding to each (ωq)-path with displacement vectors $(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \ldots, \boldsymbol{\delta}_\omega)$, ordered sequentially from the boundary point J_ℓ to the evaluation point M, intermediate points, S_i, on the path are generated and represented by vectors $\mathbf{S}_i$, according to:

$$\mathbf{S}_i = \mathbf{J}_\ell + \sum_{j=1}^{i} \delta_j ; \; i = 1, \ldots, \omega; \; \mathbf{S}_\omega \equiv \mathbf{M}. \tag{1.5}$$

With each (ωq)-path one associates the functional

$$F_\omega^q(\mathbf{J}_\ell ; \mathbf{M}) = \prod_{i=1}^{\omega} W(\mathbf{S}_i) f_{\delta_i}(\mathbf{S}_i). \tag{1.6}$$

Here, the f_δ's are the coefficients appearing in the difference equation (1.1), and $W(\mathbf{S}_i)$ is a weight function that may take the values 0 or 1, according to:

$$\begin{aligned} W(\mathbf{S}_i) &= 0, \text{ if } \mathbf{S}_i \in \mathscr{J}_0; \\ W(\mathbf{S}_i) &= 1, \text{ otherwise.} \end{aligned} \tag{1.7}$$

That is, F_ω^q vanishes whenever a path connecting J_ℓ to M contains an intermediate point S_i belonging to the boudary set $\mathscr{J}_0$. In other words, restricted paths are automatically selected and F_ω^q would otherwise be identically zero. The combinatorics function of the second kind associated with the boundary point J_ℓ and the evaluation point M are then defined as

$$C(\mathbf{J}_\ell; \mathbf{M}) = \sum_\omega \sum_q F_\omega^q(\mathbf{J}_\ell; \mathbf{M}). \tag{1.8}$$

Finally, the solution of a homogeneous ($I \equiv 0$) difference equation (1.1), subject to the initial conditions (1.4), when it exists, is given by

$$B(\mathbf{M}) = \sum_\ell \Lambda_\ell C(\mathbf{J}_\ell; \mathbf{M}). \tag{1.9}$$

The problem we intend to discuss is the three-term Fibonacci-like recursion relation,

$$B_m = aB_{m-1} + bB_{m-\lambda}, \tag{1.10}$$

where a and b are constant parameters and λ is a positive integer greater than unity. The case $\lambda = 2$ was discussed in detail elsewhere using the CFT (see [4]). The case $a = b = 1$ with an unspecified value of λ describes exactly the occupational degeneracy for λ-bell particles on a saturated $\lambda \times m$ lattice space when equation (1.10) is subject to a special set of initial conditions, as studied by Hock and McQuistan [5]. It is the purpose of this article to develop a unified approach to the problems of [4] and [5] that will be based on the generating function of Hock and McQuistan combined with our CFT method.

Section 2 develops the general solution of equation (1.10) subject to the unspecified initial value conditions

$$B_{-\lambda+j} = \Lambda_j \text{ for } j = 1, \ldots, \lambda - 1 \tag{1.11a}$$

$$B_0 = \Lambda_0 \tag{1.11b}$$

The choice Λ_0 for B_0 instead of Λ_λ, as might be suggested by (1.11a), is motivated by nicer looking equations appearing later in the paper.

Section 3 presents a class of generating functions that may be associated with equation (1.10). The method used there is somewhat more general than the one presented in [5].

Section 4 makes the comparison between the CFT and generating function methods, thus leading to a very interesting sum rule that is a generalization of the sum rules obtained in [4] and [5].

The conclusion of the paper is presented in Section 5.

2. THE CFT SOLUTION

The Fibonacci-like recursion relation,

$$B_m = aB_{m-1} + bB_{m-\lambda}, \ \lambda \geqslant 2, \tag{2.1}$$

will be solved for a general set of initial values:

$$\begin{aligned} B_{-\lambda+j} &= \Lambda_j, \text{ for } j = 1, \ldots, \lambda - 1, \\ B_0 &= \Lambda_0. \end{aligned} \tag{2.2}$$

This one-dimensional problem has boundary points $J_0, J_1, \ldots, J_j, \ldots, J_{\lambda-1}$ of abscissae $0, -\lambda + 1, \ldots, -\lambda + j, \ldots, -1$, respectively. The one-dimensional region, $\mathscr{M}$, consistent with the boundary region $\mathscr{J}$ consists of points M of positive integer abscissae. Paths connecting an evaluation point M of abscissa m ($m > 0$) to a boundary point J_j of abscissa $-\lambda + j$, if $j \neq 0$, or 0, if $j = 0$, are made of displacements or steps of lengths 1 and λ. No intermediate point on these paths should belong to the boundary region $\mathscr{J}$. Boundary and evaluation points are represented on Figure 1.

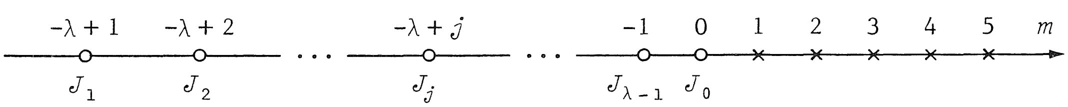

FIGURE 1

The points represented by circles "o" are boundary points and those represented by crosses "x" are evaluation points

We now proceed along the lines set by the CFT method for the construction of the combinatorics function $C(J_j\,;\,M)$. In any path, a displacement by +1 produces a factor $f_{a_1} \equiv a$ and a displacement by λ produces a factor $f_{a_2} \equiv b$. Thus, for any given path connecting J_j to M, we count the number of displacements +1 and the number of displacements λ. This path is then represented, according to the CFT method, by the product

$$F_\omega^q = a^{(\text{number of displacements } +1)} \times b^{(\text{number of displacements } \lambda)} \tag{2.3}$$

It is convenient to discuss separately the construction of $C(J_0;\, M)$ and $C(J_j;\, M)$ for $j = 1, \ldots, \lambda - 1$.

$\underline{C(J_0;\, M)}$

Figure 2 indicates that none of the distinct paths made of displacements +1 and λ leaving boundary point J_0 and reaching evaluation point M contains a boundary point other than J_0. In this case, none of the weight functions $W(S\ )$ is zero. Let $\left[\frac{m}{\lambda}\right]$ refer to integer divisions and $\overline{m}$ to the remainder, so that

$$m = \lambda\left[\frac{m}{\lambda}\right] + \overline{m}. \tag{2.4}$$

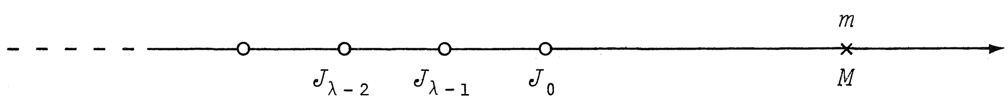

FIGURE 2

None of the distinct paths made either of displacements +1 or λ leaving J_0 and reaching the evaluation point M contains a boundary point other than J_0

Clearly, the maximum number of displacements of length λ from the origin J_0 to M is $\left[\frac{m}{\lambda}\right]$ corresponding to a minimum number of displacements of length +1 equal to $\overline{m}$. If in a path connecting J_0 to M there are k displacements of length λ $\left(0 \leqslant k \leqslant \left[\frac{m}{\lambda}\right]\right)$, then the number of displacements of length +1 must be $(m - \lambda k)$ since the length of segment J_0M is precisely equal to m. The total number of displacements in such a path is $\omega = k + (m - \lambda k)$. If q is the label of this particular path having ω displacements, then

$$F_\omega^q(J_0;\, M) = a^{m-\lambda k} \times b^k. \tag{2.5}$$

For a given total number of displacements ω, distinct paths may be generated by a reshuffling of the order of displacements of different lengths. Clearly, the distinct number of arrangements of these displacements for a given value of ω, i.e., for a given value of k $\left(0 \leqslant k \leqslant \left[\frac{m}{\lambda}\right]\right)$ is the binomial

$$q_{\max}(\omega) = \binom{m - k(\lambda - 1)}{k}. \tag{2.6}$$

Thus, the combinatorics function $C(J_0;\, M)$ is given by

$$C(J_0;\, M) = \sum_{\omega} \sum_{q=1}^{q_{\max}(\omega)} F_{\omega}^{q}(J_0;\, M)$$

$$\sum_{k=0}^{[m/\lambda]} \sum_{q=1}^{q_{\max}} a^{m-k} b^{k} = \sum_{k=0}^{[m/\lambda]} a^{m-\lambda k} b^{k} \binom{m - k(\lambda - 1)}{k} \tag{2.7}$$

<u>$C(J_j;\, M)$ for $j = 1$ to $\lambda - 1$</u>

Figure 3 shows that, in order to avoid intermeriate boundary points on any path that may connect J_j to point M, the first displacement, when leaving J_j, must be of length λ, thus reaching point M_j of abscissa j. Clearly, if the abscissa, m, of the evaluation point M is less than j, then every possible path contains at least one boundary point as an intermediate point, and the associated combinatorics function vanishes, i.e.,

$$C(J_j;\, M) \equiv 0, \text{ for } 0 < m < j. \tag{2.8}$$

On the other hand, if $m \geqslant j$, then all possible paths that contain no boundary points other than J_j have the same first displacement λ.

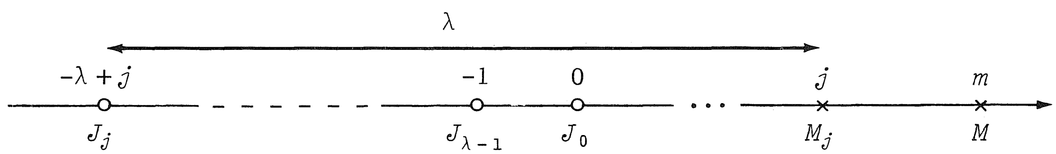

FIGURE 3

Here $j = 1$ to $\lambda - 1$. In order to avoid intermediate boundary points when leaving point J_j to reach the evaluation point M by displacements of lengths +1 and λ, the first displacement must be of length λ, thus reaching point M_j, of abscissa j

This first displacement contributes to the functional F_{ω}^{q} by producing an overall factor b. It is then straightforward to show that

$$F_{\omega}^{q}(J_j;\, M) = bF_{\omega-1}^{q}(M_j;\, M), \tag{2.9}$$

where $F_{\omega-1}(M_j;\, M)$ refers to the functional associated with the qth path cojnecting M_j to M and having $(\lambda - 1)$ dispalcements of length 1 and λ.

Since the length of segment M_jM is $m - j$, one may write an equation similar to (2.4):

$$m - j = \lambda\left[\frac{m - j}{\lambda}\right] + \overline{m - j}, \tag{2.10}$$

where the square bracket [ ], and the top bar, ——, still have the same meaning as in (2.4). The analysis for the paths connecting J_0 to M may be reproduced for the paths connecting M_j to M. The functional associated with a path having k displacements λ and $(m - j - \lambda k)$ displacements of length +1, when leaving M_j to reach M, is

$$F^q_{\omega-1}(M_j\,;\,M) = a^{m-j-\lambda k} \times b^k, \tag{2.11}$$

where k may vary from 0 to $\left[\frac{m - j}{\lambda}\right]$. Combining equations (2.9) and (2.11), one finally obtains

$$\begin{aligned} C(J_j\,;\,M) &= \sum_{\omega}\sum_{q} bF^q_{\omega-1}(M_j\,;\,M) \\ &= \sum_{k=0}^{\left[\frac{m-j}{k}\right]} a^{m-j-\lambda k} \times b^{k+1}\binom{m - j - k(\lambda - 1)}{k}; \text{ for } j = 1, \ldots, \lambda - 1 \\ &\qquad\qquad \text{and } m \geqslant j. \end{aligned} \tag{2.12}$$

The domain of definition of (2.12) can easily be extended to include the region $0 < m < j$ with the understanding that the result of the operation,

$$\sum_{k=0}^{\left[\frac{m-j}{k}\right]}, \text{ for } 0 < m < j,$$

is exactly zero, so as to recover equation (2.8). With this definition in mind, the general solution of the Fibonacci-like recursion relation (2.1) subject to the boundary conditions (2.2) is

$$\begin{aligned} B_m &= \sum_{j=0}^{\lambda-1} \Lambda_j C(J_j\,;\,M) \\ &= \sum_{j=0}^{\lambda-1} \Lambda_j \sum_{k=0}^{\left[\frac{m-j}{\lambda}\right]} a^{m-j-k\lambda} b^{k+1-\delta_{0j}} \binom{m - j - k(\lambda - 1)}{k}. \end{aligned} \tag{2.13}$$

In (2.13), δ_{0j} is Kronecker's symbol, which is zero for $j \neq 0$, and unity otherwise.

3. THE STANDARD SOLUTION AND GENERATING FUNCTIONS

The standard solution of equation (2.1) is obtained by searching for special solutions of the form [4]:

$$B_m = R^m. \tag{3.1}$$

By requiring expression (3.1) to satisfy the recursion relation (2.1), one finds that the only possible values of R are the roots of the so-called characteristic equation, which, in this case, is of order λ; namely,

$$R^\lambda - aR^{\lambda-1} - b = 0. \tag{3.2}$$

This equation has λ roots we refer to as R_k, with index k varying from 1 to λ. The general solution of equation (2.1) is then presented in the form

$$B_m = \sum_{k=1}^{\lambda} L_k R_k^m,$$

where L_k are unspecified parameters.

Next, we will be developing a class of generating functions to the series of coefficients B_m. Following Hock and McQuistan [5], we consider functions $u_m(x)$ satisfying the recurrence relations

$$u_m(x) = A(x)u_{m-1}(x) + B(x)u_{m-\lambda}(x). \tag{3.4}$$

$A(x)$ and $B(x)$ are some chosen functions of x restricted to take on the values

$$A(x_0) = a, \qquad B(x_0) = b, \tag{3.5}$$

when the variable $x = x_0$. Furthermore, for any value of the variable x, one also requires

$$\begin{aligned} u_{-\lambda+j}(x) &\equiv \Lambda_j, \text{ for } j = 1, \ldots, \lambda - 1 \\ u_0(x) &\equiv \Lambda_0. \end{aligned} \tag{3.6}$$

Clearly, $u_m(x)$ evaluated at $x = x_0$ is precisely B_m, whose explicit expression was obtained via the CFT method, namely, equation (2.13). The bivariant generating function $u(x, y)$ is given by [5]:

$$u(x, y) = \sum_{m=0}^{\infty} u_m(x)y^m. \tag{3.7}$$

One separates the summation over m into two pieces as follows:

$$u(x,\ y) = \sum_{m=0}^{\lambda-1} u_m(x)y^m + \sum_{m=\lambda}^{\infty} u_m(x)y^m. \tag{3.8}$$

Next, one replaces $u_m(x)$ appearing in the second summation by the right-hand side of the recurrence relation (3.4),

$$\sum_{m=\lambda}^{\infty} u_m(x)y^m = \sum_{m=\lambda}^{\infty} A(x)u_{m-1}(x)y^m + \sum_{m=\lambda}^{\infty} B(x)u_{m-\lambda}(x)y^m. \tag{3.9}$$

It is straightforward to recognize that

$$\begin{aligned}\sum_{m=1}^{\infty} A(x)u_{m-1}(x)y^m &= \sum_{m=1}^{\infty} A(x)u_{m-1}(x)u_{m-1}(x)y^m - \sum_{m=1}^{\lambda-1} A(x)u_{m-1}(x)y^m \\ &= A(x)y^m u(x,\ y) - \sum_{m=1}^{\lambda-1} A(x)u_{m-1}(x)y^m,\end{aligned} \tag{3.10}$$

and, also, that

$$\sum_{m=\lambda}^{\infty} B(x)u_{m-\lambda}(x)y^m = B(x)y^{\lambda}u(x,\ y). \tag{3.11}$$

Combining (3.8), (3.9), and (3.10) and (3.11), one finds

$$u(x,\ y)\ [1 - A(x)y - B(x)y^{\lambda}] = \sum_{m=0}^{\lambda-1} u_m(x)y^m - \sum_{m=1}^{\lambda-1} A(x)u_{m-1}(x)y^m. \tag{3.12}$$

A last manipulation on the right-hand side of equation (3.12) is possible to obtain the explicit form of the bivariant generating function $u(x,\ y)$; namely,

$$\sum_{m=1}^{\lambda-1} u_m(x)y^m - \sum_{m=1}^{\lambda-1} A(x)u_{m-1}(x)y^m = u_0(x) + \sum_{m=1}^{\lambda-1} [u_m(x) - A(x)u_{m-1}(x)]y^m. \tag{3.13}$$

This is followed by the use of the recurrence relation (3.4) and its initial conditions (3.6):

$$\sum_{m=1}^{\lambda-1} [u_m(x) - A(x)u_{m-1}(x)] = \sum_{m=1}^{\lambda-1} B(x)u_{m-\lambda}(x)y^m = \sum_{m-1}^{\lambda-1} B(x)\Lambda_m y^m. \tag{3.14}$$

Finally, combining equations (3.12), (3.13), (3.14), and, again, (3.6), one finds

$$u(x,\ y) = \frac{\Lambda_0 + B(x)\sum_{j=1}^{\lambda-1} \Lambda_j y^j}{1 - A(x)y + B(x)y^{\lambda}}. \tag{3.15}$$

Equation (3.15) is a generalization of the bivariant generating function of [5], where $\Lambda_j = 0$ for $j = 1$ to $\lambda - 1$, $\Lambda_0 = 1$, $A(x) = 1$, and $B(x) = x$.

4. SUM RULES

The two approaches presented in Sections 2 and 3 must be equivalent. By making use of this equivalence, a sum rule will naturally emerge. One sets $x = x_0$ in equations (3.7) and (3.15), and takes into account the restrictions $A(x_0) = a$, $B(x_0) = b$, and the property $u_m(x_0) = B_m$. It then follows that

$$\sum_{m=0}^{\infty} B_m y^m \equiv \frac{\Lambda_0 + b\sum_{j=1}^{\lambda-1}\Lambda_j y^j}{1 - ay - by^\lambda} = \frac{\sum_{j=0}^{\lambda-1} b^{1-\delta_{0j}}\Lambda_j y^j}{1 - ay - by^\lambda} \tag{4.1}$$

Let

$$f(y) = 1 - ay - by^\lambda. \tag{4.2}$$

This polynomial is of order λ; it has λ roots we call y_k, $k = 1, \ldots, \lambda$. When comparing equations (4.2) and (3.2), it is evident that the roots R_k of equation (3.2) are the inverses of the roots y_k of equation (4.2), i.e.,

$$y_k = \frac{1}{R_k}. \tag{4.3}$$

The standard expension of $\frac{1}{f(y)}$ in terms of the zeros of the function $f(y)$ is

$$\frac{1}{f(y)} = \sum_{k=1}^{\lambda}\frac{D_k}{y - y_k} = -\sum_{k=1}^{\lambda}\frac{(D_k R_k)}{1 - yR_k}, \tag{4.4}$$

and the residue D_k is obtained in the usual manner as

$$D_k = \lim_{y\to y_k}\frac{y - y_k}{f(y)} = -\frac{1}{f'(y_k)} = -\frac{1}{a + b\lambda R_k^{1-\lambda}} \tag{4.5}$$

Thus, equation (4.1) yields

$$\sum_{m=0}^{\infty} B_m y^m \equiv \sum_{j=0}^{\lambda-1} b^{1-\delta_{0j}}\Lambda_j y^j \sum_{k=1}^{\lambda}\frac{R_k}{a + b\lambda R_k^{1-\lambda}}\quad\frac{1}{1 - yR_k} \tag{4.6}$$

The left-hand side (lhs) of equation (4.6) contains B_m whose explicit dependence on Λ_j has been derived in Section 2, equation (2.13). It can be written as

$$(\text{lhs}) = \sum_{j=0}^{\lambda-1} \Lambda_j \sum_{m=0}^{\infty} y^m \sum_{k=0}^{\left[\frac{m-j}{\lambda}\right]} a^{m-j-k\lambda} b^{k+1-\delta_{j0}} \binom{m-j-k(\lambda-1)}{k}. \tag{4.7}$$

The right-hand side (rhs) of equation (4.6) can be rearranged using the power series expansion of $(1 - yR_k)^{-1}$ to yield

$$(\text{rhs}) = \sum_{j=0}^{\lambda-1} b^{1-\delta_{0j}} \Lambda_j \sum_{m=0}^{\infty} y^{j+m} \sum_{k=1}^{\lambda} \frac{R_k^{m-1}}{a + b\lambda R_k^{1-\lambda}} \tag{4.8}$$

Recalling that $\sum_{k=0}^{\left[\frac{m-j}{\lambda}\right]} \equiv 0$ for $0 < m < j$, then

$$(\text{lhs}) = \sum_{j=0}^{\lambda-1} \Lambda\, b^{1-\delta_{0j}} \sum_{m=j}^{\infty} y \sum_{k=0}^{\left[\frac{m-j}{\lambda}\right]} a^{m-j-k\lambda} b^{k} \binom{m-j-k(\lambda-1)}{k}. \tag{4.9}$$

Also, making the shift $m \to m - j$ in the summation index m of (4.8), one has

$$(\text{rhs}) = \sum_{j=0}^{\lambda-1} b^{1-\delta_{0j}} \Lambda_j \sum_{m=j}^{\infty} y \sum_{k=1} \frac{R_k^{m+1-j}}{a + b\lambda R_k^{1-\lambda}}. \tag{4.10}$$

Clearly, since (lhs) and (rhs) are equivalent expressions, and, since Λ_j are completely arbitrary parameters, one necessarily has the identity:

$$\sum_{k=0}^{\left[\frac{m-j}{\lambda}\right]} a^{m-k\lambda} b^{k} \binom{m-j-k(\lambda-1)}{k} \equiv \sum_{k=1}^{\lambda} \frac{R_k^{m+1-j}}{a + b\lambda R_k^{1-\lambda}}, \tag{4.11}$$

which holds for $m \geqslant j \geqslant 0$, or, simply,

$$\sum_{k=0}^{[m/\lambda]} a^{m-k\lambda} b^{k} \binom{m-k(\lambda-1)}{k} \equiv \sum_{k=1}^{\lambda} \frac{R_k^{m-1}}{a + b\lambda R_k^{1-\lambda}}, \text{ for } m \geqslant 0. \tag{4.12}$$

This identity reproduces the one derived by Hock and McQuistan [5] for $a = b = 1$ and any value of λ, and the sum rule derived by Phares and Simmons [4] for $\lambda = 2$ and arbitrary values of a and b. Indeed, for $\lambda = 2$, the two roots R_1 and R_2 of equations (3.2) are (see [4]):

$$\begin{aligned} R_1 &= (1/2)[a + (a^2 + 4b)^{\frac{1}{2}}], \\ R_2 &= (1/2)[a - (a^2 + 4b)^{\frac{1}{2}}]. \end{aligned} \tag{4.13}$$

It is then easy to check that the left-hand side of equation (4.11), with $\lambda = 2$, and, the values of R_1 and R_2 given by equation (4.13), becomes

$$\sum_{k=1}^{2} \frac{R_k^{m+1}}{a + 2bR_k^{-1}} = \frac{[a + (a^2 + 4b)^{\frac{1}{2}}] - [a - (a^2 + 4b)^{\frac{1}{2}}]^{m+1}}{2^{m+1}(a^2 + 4b)^{\frac{1}{2}}}. \tag{4.14}$$

Equation (4.14) shows that equation (4.12) reduces, for $\lambda = 2$, to equation (4.10) of [4].

5. CONCLUSION

A Fibonacci recurrence relation with constant coefficients has been solved exactly for arbitrary initial conditions using the combinatorics function techniques. A class of generating functions involving two arbitrarily chosen functions $A(x)$ and $B(x)$ has been obtained. The method of Hock and McQuistan applied to the generating function and combined with the CFT solution leads to a sum rule that reproduces the two special cases discussed in [4] and [5].

The flexibility shown in the application of the CFT method in the simple case presented here is not an exception. More complicated problems have been solved involving two-dimensional, homogeneous, three-term difference equations with variable coefficients. The interested reader may refer to [6].

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PARTITIONS, COMPOSITIONS AND CYCLOMATIC NUMBER OF FUNCTION LATTICES

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1. INTRODUCTION

By a poset we mean a partially ordered set. If G, H are posets, then their **cardinal power** G^H is defined by Birkhoff (see [1], p. 55) as a set of all order-preserving mappings of the poset H into the poset G with an ordering defined as follows.

For $f, g \in G^H$ there holds $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for every $x \in H$.

If G is a lattice, then G^H is usually called a **function lattice.** (It is easy to prove that if G is a lattice, or modular lattice, or distributive one, then so is G^H (see [1], p. 56).

Let A be a poset and let $a, b \in A$ with $a < b$. If no $x \in A$ exists such that $a < x < b$, then b is said to be a **successor** of the element a. Let $n(a)$ denote the number of all the successors of the element $a \in A$. Further, let $c(A)$ denote the number of all the components of the poset A, i.e., the number of its maximal continuous subsets.

Finally, if X is a set, then $|X|$ is its cardinal number.

Now we can introduce the following definition.

Definition: Let $A \neq \phi$ be a finite poset. Put

(a) $$n(A) = \sum_{a \in A} n(a) \tag{1.1}$$

(b) $$d(A) = \frac{n(A)}{|A|} \tag{1.2}$$

(c) $$v(A) = n(A) - |A| + c(A) \tag{1.3}$$

The number $d(A)$ is called the **density** of the poset A, the number $v(A)$ is called the **cyclomatic number** of the poset A.

It is evident that, for a finite poset A, $n(A)$ is equal to the number of edges in the Hasse diagram of the poset A, thus $v(A)$ is the cyclomatic number of the mentioned Hasse diagram in the sense of graph theory.

Now our aim is to determine the density and the cyclomatic number of functional lattices G^H, where G, H are finite chains.

2. PARTITIONS AND COMPOSITIONS

The symbols N, N_0, will always denote, respectively, the positive integers, the nonnegative integers.

Let $k, n, s \in N$. By a **partition** of n into k summands, we mean, as usual, a k-tuple $a_1, a_2, \ldots, a_k$ such that each $a_i \in N$, $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_k$, and

$$a_1 + \cdots + a_k = n. \tag{2.1}$$

Let $P(n, k)$ denote the set of all the partitions of the number n into k summands. Let $P(n, k, s)$ denote the set of those partitions of n into k summands, in which the summands are not greater than the number s, i.e., such that $s \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant a_k \geqslant 1$.

By a **composition** of the number n into k summands, we mean an ordered k-tuple $(a_1, \ldots, a_k)$, with $a_i \in N$, satisfying (2.1). Let $C(n, k)$ denote the set of all these compositions.

Finally, let $D(n, k)$ denote the set of all the compositions of the number n into k summands $a_i \in N_0$ [so that $C(n, k) \subseteq D(n, k)$].

It is easy to determine the number of elements of the sets $P(n, k)$, $P(n, k, s)$, $C(n, k)$, and $D(n, k)$—see, e.g., [2], [3], and [6].

Theorem 1: For k, n, $s \in N$,

(a) $$|P(n, k)| = \sum_{i=1}^{k} |P(n - k, i)| \tag{2.2}$$

(b) $$|P(n, k, s + 1)| = \sum_{i=1}^{k} |P(n - k, i, s)| \tag{2.3}$$

(c) $$|C(n, k)| = \binom{n - 1}{k - 1} \tag{2.4}$$

(d) $$|D(n, k)| = \binom{n + k - 1}{k - 1} \tag{2.5}$$

Definition: Define the binary relation ρ on the set $D(n, k)$ as follows:

If $\alpha, \beta \in D(n, k)$, $\alpha = (a_1, \ldots, a_k)$, $\beta = (b_1, \ldots, b_k)$, then $\alpha\rho\beta$ if and only if $i \in \{2, 3, \ldots, k\}$ exists such that

$$b_{i-1} = a_{i-1} + 1,\ b_i = a_i - 1, \text{ and } b_j = a_j \text{ for remaining } j.$$

Further, put, for $\alpha \in D(n, k)$,

$$\Gamma(\alpha) = \{\beta \in D(n, k);\ \alpha\rho\beta\}.$$

Remark: Thus, for $\alpha = (a_1, \ldots, a_k) \in D(n, k)$, the elements from $\Gamma(\alpha)$ are all the compositions of the form

$$(a_1, \ldots, a_{i-1} + 1, a_i - 1, a_{i+1}, \ldots, a_k).$$

From the definitions of the set $D(n, k)$ and the relation ρ, it follows that $|\Gamma(\alpha)|$ is equal to the number of nonzero summands a_i, $i = 2, \ldots, k$, in α.

Definition: For $i \in N_0$, we denote

$$D^i(n, k) = \{\alpha \in D(n, k);\ |\Gamma(\alpha)| = i\}. \tag{2.6}$$

Theorem 2: For $k, n \in N$, $i \in N_0$,

$$|D^i(n, k)| = \binom{n}{i}\binom{k-1}{i}. \tag{2.7}$$

Proof: Let $\alpha = (a_1, \ldots, a_k) \in D^i(n, k)$. Therefore, according to the above Remark, there are only i numbers that are nonzeros from all the summands $a_2, \ldots, a_k$, a_1 being arbitrary. If now $a_1 = j$, then by (2.4) there exist precisely

$$\binom{k-1}{i}\binom{n-j-1}{i-1}$$

compositions of the required form. Hence,

$$|D^i(n, k)| = \binom{k-1}{i}\left[\binom{n-1}{i-1} + \binom{n-2}{i-1} + \cdots + \binom{i-1}{i-1}\right] = \binom{k-1}{i}\binom{n}{i}.$$

Remark: It is evident that

$$D^i(n, k) \neq \phi \text{ if and only if } i \leqslant \min[k-1, n].$$

Let Y denote the set of all nonincreasing sequences $(a_1, a_2, \ldots, a_n, \ldots)$ of nonnegative integers in which there are only finitely many $a_i \neq 0$, i.e., such that

$$\sum_{i=1}^{\infty} a_i < \infty.$$

We define the ordering $\leqslant$ on the set Y by:

$(a_1, a_2, \ldots) \leqslant (b_1, b_2, \ldots)$ if and only if $a_i \leqslant b_i$ for every $i \in N$.

Then the poset Y is evidently a distributive lattice. It is the so-called **Young lattice**. For more details on its properties see, e.g., [4] and [5].

Identifying $(a_1, \ldots, a_k) \in P(n, k)$ with $(a_1, \ldots, a_k, 0, 0, \ldots) \in Y$, we henceforth consider the partitions as elements of the Young lattice.

The elements with a height n in Y are evidently all the partitions of the number n. [The element with the height 0 is obviously the sequence

$(0, 0, \ldots)$].

Definition: For $\alpha \in Y$, the **principal ideal** $Y(\alpha)$ is given by

$$Y(\alpha) = \{\beta \in Y;\ \beta \leqslant \alpha\}.$$

Definition: Let σ denote the **covering relation** on the lattice Y, i.e., for $\alpha, \beta \in Y$,

$\alpha\sigma\beta$ if and only if β is a successor of the element α.

The next result follows immediately from the definition of the Young lattice.

Theorem 3: Let $\alpha, \beta \in Y$, $\alpha = (a_1, a_2, \ldots)$, $\beta = (b_1, b_2, \ldots)$. Then $\alpha\sigma\beta$ if and only if there exists $i \in N$ such that $b_i = a_i + 1$, and $a_j = b_j$ for $j \in N$, $j \neq i$.

Definition: Let $\alpha = (a_1, a_2, \ldots) \in Y$, let r be the number $a_i \in \alpha$, with $a_i \neq 0$. Then **canonical mapping** $f : Y(\alpha) \to D(r, 1 + a_1)$ is defined as follows:

For $\beta = (b_1, b_2, \ldots) \in Y(\alpha)$, the image $f(\beta)$ is the composition

$$(c_1, c_2, \ldots, c_{1+a_1}) \in D(r, 1 + a_1)$$

for which c_i is the number of values of j for which $b_j = a_1 + 1 - i$ in the r-tuple $(b_1, \ldots, b_r)$.

Remark: If $\beta = (b_1, \ldots, b_r, 0, 0, \ldots) \in Y(\alpha)$, then evidently $0 \leqslant b_i \leqslant a_i$ for all $i = 1, \ldots, r$. The image of the sequence β under the canonical mapping f is the composition $(c_1, \ldots, c_{1+a_1})$ with the following properties:

c_1 is the number of integers a_1 in $(b_1, \ldots, b_r)$, c_2 is the number of values of j for which $b_j = a_1 - 1$ in $(b_1, \ldots, b_r)$, etc., until c_{1+a_1} is the number of zeros in $(b_1, \ldots, b_r)$.

Theorem 4: Let $\alpha = (a_1, \ldots, a_r, 0, 0, \ldots) \in Y$, with $a_1 = \cdots = a_r = k > 0$. Then

$$(Y(\alpha), \sigma) \cong [D(r, k+1), \rho]. \tag{2.8}$$

Proof: Let $f: Y(\alpha) \to D(r, k+1)$ be the canonical mapping. Then f is evidently a bijection. Let $\beta, \gamma \in Y(\alpha)$. If $\beta = (b_1, b_2, \ldots)$ and if $\beta\sigma\gamma$, then by Theorem 3, $\gamma = (b_1, \ldots, b_i + 1, b_{i+1}, \ldots)$ for some i. Denote b_i by t. Then there is in the sequence γ one less t and one more $t + 1$ than in the sequence β. Combining this fact with the definition of the relation ρ on $D(r, k+1)$, we have

$$\beta\sigma\gamma \text{ if and only if } f(\beta)\rho f(\gamma). \tag{2.9}$$

Thus, the canonical mapping f is an isomorphism from $(Y(\alpha), \sigma)$ on

$$[D(r, k+1), \rho].$$

3. DENSITY AND CYCLOMATIC NUMBER OF FUNCTION LATTICES

Let P, Q be arbitrary posets. If $P = \phi$, $Q \neq \phi$, then $P^Q = \phi$. If $Q = \phi$, then $P^Q = \{\phi\}$, P being arbitrary. Henceforth, we shall consider only such functional lattices P^Q, where $P \neq \phi \neq Q$.

The basic properties of the functional lattices P^Q, where P, Q are finite chains, are described in [6]. Namely, there holds

Theorem 5: Let $p, q \in N$, let P, Q be chains such that $|P| = p$, $|Q| = q$. Then

(a) $$|P^Q| = \binom{p + q - 1}{q} \tag{3.1}$$

(b) For $i \in N_0$, the number of elements in P^Q with height i, is equal to $|P(q + i, q, p)|$.

(c) $P^Q \cong Y(\alpha)$, where $\alpha = (a_1, \ldots, a_q, 0, 0, \ldots)$, $a_1 = \cdots = a_q = p - 1$.

Proof: The assertion (a) is trivial. The proof of the assertion (b) is in [6], p. 9. The assertion (c) results from the following: Put

$$P = \{0 < 1 < \cdots < p - 1\},\ Q = \{1 < 2 < \cdots < q\}.$$

The isomorphism $F : P^Q \to Y(\alpha)$ is given by

$$F(f) = (f(q), f(q - 1), \ldots, f(1), 0, 0, \ldots),$$

for each $f \in P^Q$.

Lemma: For $k, n \in N$,

$$\text{(a)}\quad \sum_{i=0}^{n} \binom{k}{i}\binom{n}{i} = \binom{k+n}{n} \tag{3.2}$$

$$\text{(b)}\quad \sum_{i=0}^{n} i\binom{k}{i}\binom{n}{i} = \frac{kn}{k+n}\binom{k+n}{n} \tag{3.3}$$

Proof: (a) The assertion (3.2) is well known.

(b) In [8], Hagen states without proofs many combinatorial identities. As the 17th there is stated:

$$\sum_{i=0}^{n} \frac{a + bi}{(p - id)(q + id)}\binom{p - id}{n - i}\binom{q + id}{i} = \frac{a(p + q - ni) + bnq}{q(p + q)(p - id)}\binom{p + q}{n}. \tag{3.4}$$

The first very complicated proof of formula (3.4) was given by Jensen in 1902. The simplest of the known proofs is given in [9].

Substituting $a = 0$, $b = 1$, $p = n$, $q = k$, $d = 0$ into (3.4), we obtain

$$\sum_{i=0}^{n} \frac{i}{kn}\binom{n}{n - i}\binom{k}{i} = \frac{kn}{(k + n)kn}\binom{k + n}{n},$$

by which formula (3.3) is proved.

Theorem 6: Let $p, q \in N$, let P, Q be chains such that $|P| = p$, $|Q| = q$. Then

$$n(P^Q) = \frac{q(p - 1)}{p + q - 1} \cdot \binom{p + q - 1}{q}. \tag{3.5}$$

Proof: If $p = 1$, then $|P^Q| = 1$ so that $n(P^Q) = 0$ and (3.5) is evidently valid. Thus let $p > 1$. By Theorem 5(c), we have $P^Q \cong Y(\alpha)$, where

$\alpha = (a_1, \ldots, a_q, 0, 0, \ldots)$, $a_1 = \cdots = a_q = p - 1$. Let $f: Y(\alpha) \to D(p, q)$ be the canonical mapping. For $\beta \in Y(\alpha)$, $n(\beta) = |\Gamma[f(\beta)]|$ by Theorem 4. Combining this fact with (2.7) and (3.3), we obtain

$$n(P^Q) = \sum_{\beta \in Y(\alpha)} n(\beta) = \sum_{i=0}^{q} i\binom{q}{i}\binom{p-1}{i} = \frac{q(p-1)}{p+q-1}\binom{p+q-1}{i}.$$

Remark: Combining (3.1), (3.5), and the proof of Theorem 6, we have

$$n(P^Q) = \frac{q(p-1)}{p+q-1}\binom{p+q-1}{q} = \frac{q(p-1)}{p+q-1} \cdot |P^Q| = \sum_{i=1}^{q} i\binom{q}{i}\binom{p-1}{i}. \quad (3.6)$$

Now it is easy to determine the density and also the cyclomatic number of the functional lattice P^Q.

Theorem 7: Let $p, q \in N$, let P, Q be chains such that $|P| = p$, $|Q| = q$. Then

(a) $$d(P^Q) = \frac{q(p-1)}{p+q-1} \quad (3.7)$$

(b) $$\nu(P^Q) = \sum_{i=1}^{q} (i-1)\binom{q}{i}\binom{p-1}{i} \quad (3.8)$$

Proof: (a) The assertion (3.7) follows from (1.2) and (3.6).

(b) If A is a connected poset, then $c(A) = 1$. Combining this fact with (1.3), (3.6), (3.1), and (3.2), we obtain

$$\begin{aligned}
\nu(P^Q) &= \sum_{i=1}^{q} i\binom{q}{i}\binom{p-1}{i} - \binom{p+q-1}{q} + 1 \\
&= \sum_{i=1}^{q} i\binom{q}{i}\binom{p-1}{i} - \sum_{i=0}^{q}\binom{q}{i}\binom{p-1}{i} + 1 \\
&= \sum_{i=1}^{q} i\binom{q}{i}\binom{p-1}{i} - \sum_{i=1}^{q}\binom{q}{i}\binom{p-1}{i} = \sum_{i=1}^{q}(i-1)\binom{q}{i}\binom{p-1}{i}.
\end{aligned}$$

Remark: Combining (3.6), (3.8), and (3.1), we obtain

$$\nu(P^Q) = \sum_{i=1}^{q}(i-1)\binom{q}{i}\binom{p-1}{i} = \frac{q(p-1)}{p+q-1}\binom{p+q-1}{q} - \binom{p+q-1}{q} + 1. \quad (3.9)$$

Let $p, q, r, s \in N$, and let P, Q, R, S be chains such that $|P| = p$, $|Q| = q$, $|R| = r$, $|S| = s$. By (3.7) and (3.8),

$$\text{if } r = q + 1,\ s = p - 1, \text{ then } d(P^Q) = d(R^S),\ \nu(P^Q) = \nu(R^S). \quad (3.10)$$

But in [7] we have proved that for $p > 1$,

$$P^Q \cong R^S \text{ if and only if } p = r,\ q = s \text{ or } r = q + 1,\ s = p - 1.$$

(3.10) is now evident.

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RULES FOR CONSTRUCTING HYPERPERFECT NUMBERS

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1. INTRODUCTION

As usual, let $\sigma(n)$ denote the sum of all the divisors of n [with $\sigma(1) = 1$] and let $\omega(n)$ denote the number of different prime factors of n [with $\omega(1) := 0$]. The set of prime numbers will be denoted by $\mathscr{P}$. The set of hyperperfect numbers (HP's) is the set $M := \bigcup_{n=1}^{\infty} M_n$, where

$$M_n := \{m \in \mathbf{N} \mid m = 1 + n[\sigma(m) - m - 1]\}. \tag{1}$$

We also define the sets

$$_kM_n := \{m \in M_n \mid \omega(m) = k\},\ k,\ n \in \mathbf{N}, \tag{2}$$

and $_kM := \bigcup_{n=1}^{\infty} {_kM_n}$; clearly, we have $M_n = \bigcup_{k=1}^{\infty} {_kM_n}$. We will also use the related set $M^* := \bigcup_{n=1}^{\infty} M_n^*$, where

$$M_n^* := \{m \in \mathbf{N} \mid m = 1 + n[\sigma(m) - m]\}, \tag{3}$$

and the sets

$$_kM_n^* := \{m \in M_n^* \mid \omega(m) = k\},\ k \in \mathbf{N} \cup \{0\},\ n \in \mathbf{N}, \tag{4}$$

and $_kM^* := \bigcup_{n=1}^{\infty} {_kM_n^*}$, so that also $M_n^* = \bigcup_{k=0}^{\infty} {_kM_n^*}$.

It is not difficult to verify that $_1M_n = \emptyset$, $\forall n \in \mathbf{N}$, and that

$$\begin{cases} {_0M_n^*} = \{1\},\ \forall n \in \mathbf{N} \text{ and} \\ {_1M_n^*} = \begin{cases} \{(n+1)^\alpha,\ \alpha \in \mathbf{N}\}, & \text{if } n+1 \in \mathscr{P}, \\ \emptyset, & \text{if } n+1 \notin \mathscr{P}. \end{cases} \end{cases} \tag{5}$$

M_1 is the set of perfect numbers [for which $\sigma(m) = 2m$]. The n-hyperperfect numbers M_n, introduced by Minoli and Bear [1], are a meaningful generalization of the even perfect numbers because of the following rule.

RULE 0 (from [2]): If $p \in \mathscr{P}$, $\alpha \in \mathbf{N}$, and if $q := p^{\alpha+1} - p + 1 \in \mathscr{P}$, then $p^\alpha q \in M_{p-1}$.

There are 71 hyperperfect numbers below 10^7 (see [2], [3], [4], [5]). Only one of them belongs to ${}_3M$, all others are in ${}_2M$. In [6] and [7] the present author has constructively computed several elements of ${}_3M$ and two of ${}_4M$.

In Section 2 of this paper, we shall give rules by which one may find (with enough computer time) an element of ${}_{(k+2)}M_n$ and of ${}_{(k+1)}M_n$ from an element of ${}_kM_n^*$ ($k \geqslant 0$), and an element of ${}_kM_n^*$ from an element of ${}_{(k-2)}M_n^*$ ($k \geqslant 2$). Because of (5), this suggests the possibility to construct HP's with k different prime factors for any positive integer $k \geqslant 2$. By actually applying the rules, we have found many elements of ${}_3M$, seven elements of ${}_4M$, and one element of ${}_5M$.[1]

In Section 3, necessary and sufficient conditions are given for numbers of the form $p^\alpha q$, $\alpha \in \mathbf{N}$, to be hyperperfect. For example, for $\alpha \geqslant 3$, these conditions imply that there are no other HP's of the form $p^\alpha q$ than those characterized by Rule 0. The results of this section enable us to compute very cheaply *all* HP's of the form $p^\alpha q$ below a given bound. Unfortunately, we have not been able to extend these results to more complicated HP's like those of the form $p^\alpha q^\beta$, $\alpha \geqslant 2$ and $\beta \geqslant 2$, or $p^\alpha q^\beta r^\gamma$ with $\alpha \geqslant 1$, $\beta \geqslant 1$ and $\gamma \geqslant 1$, etc. (However, these numbers are extremely scarce compared to HP's of the form $p^\alpha q$, and no HP's of the form $p^\alpha q^\beta$ and $p^\alpha q^\beta r^\gamma$ with $\alpha \geqslant 2$ and $\beta \geqslant 2$ have been found to date.)

Because of the importance of the set M^* for the construction of hyperperfect numbers, we given in Section 4 the results of an exhaustive search for all $m \in M^*$ with $m \leqslant 10^8$ and $\omega(m) \geqslant 2$. It turned out that elements of ${}_3M^*$ are very rare compared with ${}_2M^*$, in analogy with the sets ${}_3M$ and ${}_2M$. This search also gave all elements $\leqslant 10^8$ of M, at very low cost, because of the similarity of the equations defining M^* and M. See note 1 below.

The paper concludes with a few remarks, in Section 5, on a possible generalization of hyperfect numbers to so-called hypercycles, special cases of which are the ordinary perfect numbers and the amicable number pairs.

[1] *Lists of these numbers may be obtained from the author on request.*

Remark: After completing this paper, the author computed, with the rules given in Section 2, 860 HP's below the bound 10^{10}. See note 1 above.

2. RULES FOR CONSTRUCTING HYPERPERFECT NUMBERS

We have found the following rules [we write $\overline{a}$ for $\sigma(a)$]:

RULE 1: Let $k \in \mathbf{N}$, $n \in \mathbf{N}$, $a \in {}_kM_n^*$, and $p := n\overline{a} + 1 - n$; if $p \in \mathscr{P}$, then $ap \in {}_{(k+1)}M_n$.

RULE 2: Let $k \in \mathbf{N} \cup \{0\}$, $n \in \mathbf{N}$, $a \in {}_kM_n^*$, and $p := n\overline{a} + A$, $q := n\overline{a} + B$, where $AB = 1 - n + n\overline{a} + n^2\overline{a}^{\,2}$; if $p \in \mathscr{P}$ and $q \in \mathscr{P}$, then $apq \in {}_{(k+2)}M_n$.

RULE 3: Let $k \in \mathbf{N} \cup \{0\}$, $n \in \mathbf{N}$, $a \in {}_kM_n^*$, and $p := n\overline{a} + A$, $q := n\overline{a} + B$, where $AB = 1 + n\overline{a} + n^2\overline{a}^{\,2}$; if $p \in \mathscr{P}$ and $q \in \mathscr{P}$, then $apq \in {}_{(k+2)}M_n^*$.

The proofs of these rules don't require much more than the application of the definitions, and are therefore left to the reader. In fact, the proof of Rule 2 was already given in [7], although the rule itself was formulated there less explicitly.

Rule 1 can be applied for $k \geqslant 1$, but not for $k = 0$, since ${}_0M_n^* = \{1\}$ and $a = 1$ gives $p = 1 \notin \mathscr{P}$. For $k = n = 1$, Rule 1 reads:

$$\text{If } p := 2^{\alpha+1} - 1 \in \mathscr{P}, \text{ then } 2^{\alpha}p \in {}_2M_1,$$

which is Euclid's rule for finding even perfect numbers. For $k = 1$, Rule 1 is equivalent to Rule 0, given in Section 1.

Rules 2 and 3 can both be applied for $k \geqslant 0$. For instance, for $k = 0$, Rule 2 reads:

$$\text{Let } n \in \mathbf{N} \text{ be given; if } p := n + A \in \mathscr{P} \text{ and } q := n + B \in \mathscr{P}, \text{ where } AB = 1 + n^2, \text{ then } pq \in {}_2M_n.$$

For $n = 1$, 2, and 6, this yields the hyperperfect numbers 2×3, 3×7, and 7×43, respectively. Rule 3 reads, for $k = 0$:

$$\text{Let } n \in \mathbf{N} \text{ be given; if } p := n + A \in \mathscr{P} \text{ and } q := n + B \in \mathscr{P}, \text{ where } AB = 1 + n + n^2, \text{ then } pq \in {}_2M_n^*.$$

For $n = 4$ and $n = 10$, we find that $7 \times 11 \in {}_2M_4^*$ and $13 \times 47 \in {}_2M_{10}^*$, respectively.

Rule 3 shows a rather curious "side-effect" for $k \geqslant 1$: if both the numbers p and q in this rule are prime, then not only $apq \in {}_{(k+2)}M_n^*$, but also the number $b := pq$ is an element of ${}_2M_{n\overline{a}}^*$. Indeed, we have

$$\frac{b-1}{\sigma(b)-b} = \frac{pq-1}{p+q+1} = \frac{n^2\overline{a}^2 + n\overline{a}(A+B) + AB - 1}{2n\overline{a} + A + B + 1}$$

$$= \frac{n^2\overline{a}^2 + n\overline{a}(A+B) + n\overline{a} + n^2\overline{a}^2}{2n\overline{a} + A + B + 1} = n\overline{a} \in \mathbf{N}.$$

For example, we know that $7 \times 11 \in {}_2M_4^*$. From Rule 3 with $k = 2$, $n = 4$, and $a = 7 \times 11$, we find that $7 \times 11 \times 547 \times 1291 \in {}_4M_4^*$; the side-effect is that

$$547 \times 1291 \in {}_2M^*_{(4\times 8\times 12)} = {}_2M^*_{384}.$$

In [6] we gave the following additional rule.

RULE 4: Let $t \in \mathbf{N}$ and $p := 6t - 1$, $q := 12t + 1$; if $p \in \mathscr{P}$ and $q \in \mathscr{P}$, then $p^2q \in {}_2M_{(4t-1)}$.

For example, $t = 1$ and $t = 3$ give $5^213 \in {}_2M_3$ and $17^237 \in {}_2M_{11}$, respectively. In Section 3 we will prove that with Rules 1, 2, and 4 it is possible to find all HP's of the form $p^\alpha q$, $\alpha \in \mathbf{N}$, below a given bound. We leave it to interested readers to discover why there is no rule (at least for $k \geqslant 1$), analogous to Rule 1, for finding an element of ${}_{(k+1)}M_n^*$ from an element of ${}_kM_n^*$.

From Rules 1-3, it follows that elements of ${}_kM_n$ for some given $k \in \mathbf{N}$ may be found from ${}_{(k-1)}M_n^*$ (with Rule 1) and from ${}_{(k-2)}M_n^*$ (with Rule 2) provided that sufficiently many elements of ${}_{(k-1)}M_n^*$ resp. ${}_{(k-2)}M_n^*$ are available; these can be found with Rule 3 and the "starting" sets ${}_0M_n^*$ and ${}_1M_n^*$ given in (5). We have carried out this "program" for the constructive computation of HP's with three, four, and five different prime factors.

(i) *Construction of elements of* ${}_3M_n$. With Rule 1, we found 34 HP's of the form pqr, from numbers $pq \in {}_2M_n^*$:

the smallest is $61 \times 229 \times 684433 \in {}_3M_{48}$;

the largest one is $9739 \times 13541383 \times 1283583456107389 \in {}_3M_{9732}$.

The elements of ${}_2M_n^*$ were "generated" with Rule 3 from ${}_0M_n^* = \{1\}$. Using Rule 2 we found, from prime powers $p^\alpha \in {}_1M_n^*$, 67 HP's of the form pqr:

five of the smallest are given in [6],

the largest is $8929 \times 79727051 \times 577854714897923 \in {}_3M_{8928}$;

48 HP's of the form p^2qr,

the smallest five are given in [6],

the largest is $7459^2 414994003583 \times 3444400460163740816321 9 \in {}_3M_{7458}$;

9 of the form p^3qr,

the smallest is given in [6],

the largest is $811^3 432596915921 \times 89927962885420066391 \in {}_3M_{810}$;

4 of the form p^4qr,

the smallest is $7^4 30893 \times 36857 \in {}_3M_6$,

the largest is $223^4 553821371657 \times 130059326113901 \in {}_3M_{222}$;

and, furthermore,

$7^6 1340243 \times 2136143 \in {}_3M_6$,

$13^7 815787979 \times 11621986347871 \in {}_3M_{12}$,

and

$19^8 322687706723 \times 11640844402910006759 \in {}_3M_{18}$.

(ii) *Construction of elements of* ${}_4M_n$. In order to construct elements of ${}_4M_n$ with Rule 1, sufficiently many elements of ${}_3M_n^*$ had to be available. This was realized with Rule 3, starting with elements $p^\alpha \in {}_1M_{(p+1)}$, $p \in \mathscr{P}$. The following four HP's with four different prime factors were found:

$3049 \times 9297649 \times 69203101249 \times 5981547458963067824996953 \in {}_4M_{3048}$,

$4201 \times 17692621 \times 7061044981 \times 220478637088071105410940 1 \in {}_4M_{4200}$,

$181^2 5991031 \times 579616291 \times 2059102068590772565038 1 \in {}_4M_{180}$,

$181^3 1108889497 \times 33425259193 \times 39781151786825440683346549261 \in {}_4M_{180}$.

By means of Rules 2 and 3, the following three additional elements of ${}_4M_n$ were found:

$1327 \times 6793 \times 10020547039 \times 17769709449589 \in {}_4M_{1110}$ (is in [6]),

$1873 \times 24517 \times 79947392729 \times 80855915754575789 \in {}_4M_{1740}$ (is in [7]),

$5791 \times 10357 \times 222816095543 \times 482764219012881017 \in {}_4M_{3714}$.

(iii) *Construction of an element of* ${}_5M_n$. We have also constructively computed one element of ${}_5M_n$ with Rule 1. The elements of ${}_4M_n^*$ needed for this purpose were computed from ${}_0M_n^*$ by twice applying Rule 3 (first yielding elements of ${}_2M_n^*$, then elements of ${}_4M_n^*$). The HP found is the largest one we know of (apart from the ordinary perfect numbers). It is the 87-digit number:

$$\begin{aligned}&2095497171870781405883328851321934328974054074379064142367649255383173390207087865907 93\\&= 4783 \times 83563 \times 1808560287211 \times 297705496733220305347\\&\times 9737620193207006500935201284805753200507613 01 \in {}_5M_{4524}.\end{aligned}$$

3. CHARACTERIZATION OF ALL HP'S OF THE FORM $p^\alpha q$

The hyperperfect numbers of the form $p^\alpha q$ are characterized by the following theorem.

Theorem: Let $m := p^\alpha q$ ($\alpha \in \mathbf{N}, p \in \mathscr{P}, q \in \mathscr{P}$) be a hyperperfect number, then

(i) $\alpha = 1 \Rightarrow (\exists n \in \mathbf{N}$ with $m \in {}_2M_n$ such that $p = n + A$, $q = n + B$, with $AB = 1 + n^2)$;

(ii) $\alpha = 2 \Rightarrow (\exists t \in \mathbf{N}$ with $m \in {}_2M_{(4t-1)}$ and $p = 6t - 1$ and $q = 12t + 1)$ $\vee$ ($m \in {}_2M_{(p-1)}$ with $q = p^3 - p + 1$);

(iii) $\alpha > 2 \Rightarrow$ ($m \in {}_2M_{(p-1)}$ with $q = p^{\alpha+1} - p + 1$).

Proof: (i) This case follows immediately from Rule 2 (with $k = 0$). (ii) If p^2q is hyperperfect, then the number $(p^2q - 1)/((p + 1)(p + q))$ must be a positive integer. Consider the function

$$f(x, y) := \frac{x^2y - 1}{(x + 1)(x + y)}, \; x, y \in \mathbf{N}.$$

To characterize all pairs x, y for which $f(x, y) \in \mathbf{N}$, we can safely take $x \geqslant 2$ and $y \geqslant 2$. Let $x \geqslant 2$ be fixed, then we have for all $y \geqslant 2$,

$$f(x, y) < \frac{x^2y}{(x + 1)(x + y)} < \frac{x^2}{x + 1} = x - 1 + \frac{1}{x + 1}.$$

Hence, the largest integral value which could possibly be assumed by f is $x - 1$, and one easily checks that this value is actually assumed for $y = x^3 - x + 1$. So we have found

$$f(x, x^3 - x + 1) = x - 1, \; x \in \mathbf{N}, \; x \geqslant 2. \tag{6}$$

One also easily checks that f is monotonically increasing in y (x fixed), so that

$$2 \leqslant y \leqslant x^3 - x + 1. \tag{7}$$

Now, in order to have $f \in \mathbf{N}$, it is necessary that $x + 1$ divides $x^2y - 1$, or, equivalently, that $x + 1$ divides $y - 1$, since

$$\frac{x^2y - 1}{x + 1} = y(x - 1) + \frac{y - 1}{x + 1}.$$

Therefore, we have $y = k(x + 1) + 1$, with $k \in \mathbf{N}$ and $1 \leqslant k \leqslant x(x - 1)$ by (7). Substitution of this into f yields

$$f(x, y) = \frac{kx^2 + x - 1}{(k + 1)(x + 1)} = x - 1 - \frac{x^2 - x - k}{(k + 1)(x + 1)} =: x - 1 - g(x, k).$$

It follows that $x + 1$ must divide $x^2 - x - k$, or, equivalently, that $x + 1$ must divide $k - 2$. Hence, $k = j(x + 1) + 2$, with $j \in \mathbf{N} \cup \{0\}$ and $0 \leqslant j \leqslant x - 2$. Substitution of this into g yields

$$g(x, j(x + 1) + 2) = \frac{x - 2 - j}{j(x + 1) + 3}.$$

This function is decreasing in j, and for $j = 0, 1, \ldots, x - 2$ it assumes the values:

$$\begin{aligned} g(x, 2) &= (x - 2)/3, \\ g(x, x + 3) &= \frac{x - 3}{x - 4} < 1, \\ &\vdots \\ g(x, x(x - 1)) &= 0. \end{aligned}$$

It follows that there is precisely one more possibility [in addition to (6)] for f to be a positive integer, viz., when $j = 0$, $k = 2$, $y = 2x + 3$, and $x \pmod 3 = 2$. So we have found

$$f(3t - 1, 6t + 1) = 2t - 1, \ t \in \mathbf{N}. \tag{8}$$

The statement in the Theorem now easily follows from (6) and (8).

(iii) As in the proof of (ii), we now have to find out for which values of x, $y \in \mathbf{N}$, $x \geqslant 2$, and $y \geqslant 2$, the function $f(x, y) \in \mathbf{N}$, where

$$f(x, y) := \frac{x^{\alpha}y - 1}{(x^{\alpha-1} + \cdots + 1)(x + y)}, \ \alpha > 2.$$

For fixed $x \geqslant 2$, we have

$$f(x, y) < \frac{x^{\alpha}}{x^{\alpha-1} + \cdots + 1} = x - 1 + \frac{1}{x^{\alpha-1} + \cdots + 1}.$$

As in the proof of (ii) we find that $f(x, y) = x - 1$ for $y = x^{\alpha+1} - x + 1$ and that $2 \leqslant y \leqslant x^{\alpha+1} - x + 1$. Furthermore, $x^{\alpha-1} + \cdots + 1$ must divide $x^{\alpha}y - 1$, so that $y = k(x^{\alpha-1} + \cdots + 1) + 1$, with $1 \leqslant k \leqslant x(x - 1)$. Substitution of this into f yields a certain function g, in the same way as in the proof of (ii), but in this case g can only assume integral values for $k = x(x - 1)$. This implies the statement in the Theorem, case (iii). Q.E.D.

It is easy to see that the characterizations given in this Theorem are equivalent to Rule 2 ($k = 0$) when $\alpha = 1$, to Rule 4 or Rule 1 ($k = 1$) when $\alpha = 2$, and to Rule 1 ($k = 1$) when $\alpha > 2$.

This Theorem enables us to find very cheaply all HP's of the form $p^{\alpha}q$, $\alpha \in \mathbf{N}$, below a given bound. For example, to find all HP's in M_n of the form pq below 10^8, we only have to check whether

$$p := n + A \in \mathscr{P} \qquad \text{and} \qquad q := n + B \in \mathscr{P}$$

for all possible factorizations of $AB = 1 + n^2$, for $1 \leqslant n \leqslant 4999$. This range of n follows from the fact that if $pq \in M_n$ then $pq > 4n^2$. The following additional restrictions can be imposed on n:

(i) n should be 1 or even since, if n is odd and $n \geqslant 3$, then $n^2 + 1 \equiv 2 \pmod 4$, so that one of A or B is odd and one of p or q is even and $\geqslant 4$.

(ii) If $n \geqslant 3$, then $n \equiv 0 \pmod 3$, since if $n \equiv 1$ or $2 \pmod 3$, then $n^2 + 1 \equiv 2 \pmod 3$, so that one of A or B is $\equiv 1 \pmod 3$ and the other is $\equiv 2 \pmod 3$; consequently, one of p or q is $\equiv 0 \pmod 3$ and > 3.

Hence, the only values of n to be checked are $n = 1$, $n = 2$, and $n = 6t$, $1 \leqslant t \leqslant 833$. It took about 6 seconds CPU-time on a CDC CYBER 175 computer to check these values of n, and to generate in this way all HP's of the form pq below 10^8.

4. EXHAUSTIVE COMPUTER SEARCHES

From the rules given in Section 2, it follows that it is of importance to know elements of M^* when one wants to find elements of M. Therefore, we have carried out an exhaustive computer search for all elements of M^* below the bound 10^8. Because of (5) the search was restricted to elements with at least two different prime factors. A check was done to determine whether $(m-1)/(\sigma(m)-m) \in \mathbf{N}$, for all $m \leqslant 10^8$ with $\omega(m) \geqslant 2$. Since the most time-consuming part is the computation of $\sigma(m)$, a second check was done to determine whether $(m-1)/(\sigma(m)-m-1) \in \mathbf{N}$ [in the case where $(m-1)/(\sigma(m)-m) \notin \mathbf{N}$]. If so, m was an HP; thus, our program also produced, almost for free, all HP's below 10^8. (The search took about 100 hours of "idle" computer time on a CDC CYBER 175.) The results are as follows.

Apart from the ordinary perfect numbers, there are 146 HP's below 10^8. Only two of them have the form $p^\alpha qr$:

$$13 \times 269 \times 449 \in {}_3M_{12} \quad \text{and} \quad 7^2 383 \times 3203 \in {}_3M_6;$$

these were also found in the searches described in Section 2. All others have the form characterized in Section 3, and could have been found with a search based on that characterization (using the fact that if $p^\alpha q \in {}_2M_n$, then $p > n$ and $q > n$). A question that naturally arises is the following: Are there any HP's that can*not* be constructed with one of Rules 1, 2, or 4?[2]

There are 312 numbers $m \leqslant 10^8$ which belong to M^* and which have $\omega(m) \geqslant 2$. Of these, 306 have the form pq and could have been (and, as a check, actually were) found very cheaply with Rule 3 of Section 2. The others are:

$$7 \times 61 \times 229 \in {}_3M_6^*,\ 113 \times 127 \times 2269 \in {}_3M_{58}^*,$$

$$149 \times 463 \times 659 \in {}_3M_{96}^*,\ 19 \times 373 \times 10357 \in {}_3M_{18}^*,$$

$$151 \times 373 \times 1487 \in {}_3M_{100}^*,\ 7 \times 11 \times 547 \times 1291 \in {}_4M_4^*;$$

the second, third, and fifth numbers could not have been found using Rule 3.

[2] *The referee has answered this question in the affirmative by giving the example* $12161963773 = 191 \times 373 \times 170711 \in M_{126}$.

5. HYPERCYCLES

A possible generalization of hyperperfect numbers can be obtained as follows. Let $n \in \mathbf{N}$ be given, and define the function $f_n : \mathbf{N}\backslash\{1\} \Rightarrow \mathbf{N}$ as

$$f_n(m) := 1 + n[\sigma(m) - m - 1], \quad m \in \mathbf{N}\backslash\{1\}. \tag{9}$$

Starting with some $m_0 \in \mathbf{N}\backslash\{1\}$, one might investigate the sequence

$$m_0,\ f_n(m_0),\ f_n(f_n(m_0)),\ \ldots\ . \tag{10}$$

For $n = 1$, this is the well-known aliquot sequence of m_0, which can have cycles of length 1 (perfect numbers), length 2 (amicable pairs), and others. In order to get some impression of the cyclic behavior for $n > 1$, we have computed, for $2 \leqslant n \leqslant 20$, five terms of all sequences (10) with starting term $m_0 \leqslant 10^6$, and we have registered the cycles with length $\geqslant 2$ and $\leqslant 5$ in the following table.

TABLE 1

HYPERCYCLES*

n	k	$m_0, m_1, \ldots, m_{k-1}$
5	2	$19461 = 3 \times 13 \times 499$, $42691 = 11 \times 3881$
7	3	$925 = 5^2 37$, $1765 = 5 \times 353$, $2507 = 23 \times 109$
8	2	$28145 = 5 \times 13 \times 433$, $66481 = 19 \times 3499$
	3	$238705 = 5 \times 47741$, $381969 = 3^3 7 \times 43 \times 47$, $2350961 = 79 \times 29759$
	4	$94225 = 5^2 3769$, $181153 = 7^2 3697$, $237057 = 3 \times 31 \times 2549$, $714737 = 61 \times 11717$
	2	$3452337 = 3^2 7 \times 54799$, $17974897 = 53 \times 229 \times 1481$
9	2	$469 = 7 \times 67$, $667 = 23 \times 29$
	2	$1315 = 5 \times 263$, $2413 = 19 \times 127$
	2	$1477 = 7 \times 211$, $1963 = 13 \times 151$
	2	$2737 = 7 \times 17 \times 23$, $6463 = 23 \times 281$
10	3	$1981 = 7 \times 283$, $2901 = 3 \times 967$, $9701 = 89 \times 109$
12	2	$697 = 17 \times 41$, $2041 = 13 \times 157$
	2	$3913 = 7 \times 13 \times 43$, $12169 = 43 \times 283$
	2	$54265 = 5 \times 10853$, $130297 = 29 \times 4493$
14	2	$1261 = 13 \times 97$, $1541 = 23 \times 67$
	3	$508453 = 11 \times 17 \times 2719$, $1106925 = 3 \times 5^2 14759$, $10126397 = 281 \times 36037$

*Different numbers $m_0, m_1, \ldots, m_{k-1}$ such that $m_k = m_0$, where $m_{i+1} := f_n(m_i)$, f_n defined in (9).

TABLE 1 (continued)

n	k	$m_0, m_1, \ldots, m_{k-1}$
19	2	$9197 = 17 \times 541$, $10603 = 23 \times 461$
	4	$184491 = 3^3 6833$, $1688493 = 3 \times 562831$, $10693847 = 709 \times 15083$, $300049 = 31 \times 9679$
	2	$5151775 = 5^2 251 \times 821$, $24124073 = 89 \times 271057$

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ON LUCAS POLYNOMIALS AND SOME SUMMATION FORMULAS FOR CHEBYCHEV POLYNOMIAL SEQUENCES VIA THEM

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1. INTRODUCTION

Lucas [2] defined the fundamental and the primordial functions $U_n(p, q)$ and $V_n(p, q)$, respectively, by the second-order recurrence relation

$$W_{n+2} = pW_{n+1} - qW_n \qquad (n \geqslant 0),$$

where

$$\begin{cases} \{W_n\} = \{U_n\} & \text{if } W_0 = 0,\ W_1 = 1, \text{ and} \\ \{W_n\} = \{V_n\} & \text{if } W_0 = 2,\ W_1 = p. \end{cases} \tag{1.1}$$

Let X be a matrix defined by

$$X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{1.2}$$

Taking

$$\text{tr. } X = p \qquad \text{and} \qquad \text{det. } X = q$$

and using matrix exponential functions

$$e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \qquad \text{and} \qquad e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} X^{-n},$$

Barakat [1] obtained summation formulas for

$$\sum_{n=0}^{\infty} \frac{1}{n!} U_n(p, q), \ \sum_{n=0}^{\infty} \frac{1}{n!} V_n(p, q), \text{ and } \sum_{n=0}^{\infty} \frac{1}{n!} U_{n+1}(p, q).$$

Walton [7] extended Barakat's results by using the sine and cosine functions of the matrix X to obtain various other summation formulas for the functions $U_n(p, q)$ and $V_n(p, q)$. Further, using the relation between $\{U_n\}$, $\{V_n\}$, and the Chebychev polynomials $\{S_n\}$ and $\{T_n\}$ of the first and second kinds, respectively, he obtained the following summation formulas:

$$\begin{cases} \sum_{n=0}^{\infty} \frac{(-1)^n \sin 2n\theta}{(2n)!} = -\sin(\cos\theta)\sinh(\sin\theta) \\ \sum_{n=0}^{\infty} \frac{(-1)^n \sin(2n+1)\theta}{(2n+1)!} = \cos(\cos\theta)\sinh(\sin\theta) \\ \sum_{n=0}^{\infty} \frac{(-1)^n \cos 2n\theta}{(2n)!} = \cos(\cos\theta)\cosh(\sin\theta) \\ \sum_{n=0}^{\infty} \frac{(-1)^n \cos(2n+1)\theta}{(2n+1)!} = \sin(\cos\theta)\cosh(\sin\theta) \end{cases} \tag{1.3}$$

The question—Can the summation formulas for U_n and V_n and identities in (1.3) be further extended?—then naturally arises. The object of this paper is to obtain these extensions, if they exist, by using generalized circular functions.

2. GENERALIZED CIRCULAR FUNCTIONS

Pólya and Mikusiński [3] appear to be among the first few mathematicians who studied the generalized circular functions defined as follows.

For any positive integer r,

$$M_{r,j}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{rn+j}}{(rn+j)!}, \quad j = 0, 1, \ldots, r-1$$

and

$$N_{r,j}(t) = \sum_{n=0}^{\infty} \frac{t^{rn+j}}{(rn+j)!}, \quad j = 0, 1, \ldots, r-1.$$

The notation and some of the results used here are according to [4]. Note that

$$M_{1,0}(t) = e^{-t},\ M_{2,0}(t) = \cos t,\ M_{2,1}(t) = \sin t,$$

$$N_{1,0}(t) = e^{t},\ N_{2,0}(t) = \cosh t,\ N_{2,1}(t) = \sinh t.$$

Following Barakat [1] and Walton [7], we define generalized trigonometric and hyperbolic functions of any square matrix X by

$$\begin{cases} M_{r,j}(X) = \sum_{n=0}^{\infty} \frac{(-1)^n X^{rn+j}}{(rn+j)!}, \quad j = 0, 1, \ldots, r-1, \text{ and} \\ N_{r,j}(X) = \sum_{n=0}^{\infty} \frac{X^{rn+j}}{(rn+j)!}, \quad j = 0, 1, \ldots, r-1. \end{cases} \tag{2.1}$$

3. SUMMATION FORMULAS FOR THE FUNDAMENTAL FUNCTION

We use the following Lemmas.

Lemma 1: Let X be the matrix defined in (1.2), and $U_n(p, q)$ the fundamental functions defined by (1.1). Then

$$X^n = U_n X - qU_{n-1}I, \tag{3.1}$$

where I is the 2×2 unit matrix.

This lemma is proved by Barakat [1].

Lemma 2: If $f(t)$ is a polynomial of degree $\leqslant N - 1$, and if $\lambda_1, \ldots, \lambda_N$ are the N distinct eigenvalues of X, then

$$f(X) = \sum_{i=1}^{N} f(\lambda_i) \prod_{\substack{1 \leqslant j \leqslant N \\ j \neq i}} \left[\frac{X - \lambda_i I}{\lambda_i - \lambda_j}\right]. \tag{3.2}$$

This is Sylvester's matrix interpolation formula (see [6]).

Lemma 3: (a) The following identities are proved in [3]:

$$M_{3,0}(x + y) = M_{3,0}(x)M_{3,0}(y) - M_{3,1}(x)M_{3,2}(y) - M_{3,2}(x)M_{3,1}(y),$$
$$M_{3,1}(x + y) = M_{3,0}(x)M_{3,1}(y) + M_{3,1}(x)M_{3,0}(y) - M_{3,2}(x)M_{3,2}(y),$$
$$M_{3,2}(x + y) = M_{3,0}(x)M_{3,2}(y) + M_{3,1}(x)M_{3,1}(y) + M_{3,2}(x)M_{3,0}(y).$$

(b) $N_{r,j}(t) = \omega^{j/2} M_{r,j}(\omega^{-1/2} t)$, where $\omega = e^{2\pi i/r}$.

The proof is straightforward and thus omitted (for notation, see [4]).

Lemma 4: We have

$$M_{3,j}(x) - M_{3,j}(-x) = \begin{cases} -2N_{6,j+3}(x), & j = 0, 2, \\ 2N_{6,1}(x), & j = 1. \end{cases}$$

$$M_{3,j}(x) + M_{3,j}(-x) = \begin{cases} 2N_{6,j}(x), & j = 0, 2, \\ -2N_{6,4}(x), & j = 1. \end{cases}$$

Proof:

$$M_{3,j}(x) - M_{3,j}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+j}}{(3n + j)!}[1 - (-1)^{3n+j}], \quad j = 0, 1, 2.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+j}}{(3n + j)!}[1 - (-1)^{n+j}]$$

$$= \begin{cases} \displaystyle\sum_{n=1,3,\ldots}^{\infty} \frac{2(-1)^n x^{3n+j}}{(3n+j)!}, & j = 0,\ 2 \\ \displaystyle\sum_{n=0,2,\ldots}^{\infty} \frac{2(-1)^n x^{3n+j}}{(3n+j)!}, & j = 1 \end{cases}$$

$$= \begin{cases} \displaystyle -2\sum_{n=0}^{\infty} \frac{x^{6n+3+j}}{(6n+3+j)!}, & j = 0,\ 2 \\ \displaystyle 2\sum_{n=0}^{\infty} \frac{x^{6n+j}}{(6n+j)!}, & j = 1 \end{cases}$$

$$= \begin{cases} -2N_{6,3+j}, & j = 0,\ 2 \\ 2N_{6,j}, & j = 1. \end{cases}$$

The other formula can be similarly proved.

Theorem 1: The following formulas hold for $\{U_n(p, q)\}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n}}{(3n)!} \tag{3.3}$$

$$= -\frac{2}{\delta}\{M_{3,0}(p/2)N_{6,3}(\delta/2) - M_{3,1}(p/2)N_{6,5}(\delta/2) + M_{3,2}(p/2)N_{6,1}(\delta/2)\}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n+1}}{(3n+1)!} \tag{3.4}$$

$$= \frac{2}{\delta}\{M_{3,0}(p/2)N_{6,1}(\delta/2) - M_{3,1}(p/2)N_{6,3}(\delta/2) + M_{3,2}(p/2)N_{6,5}(\delta/2)\}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n+2}}{(3n+2)!} \tag{3.5}$$

$$= -\frac{2}{\delta}\{M_{3,0}(p/2)N_{6,5}(\delta/2) - M_{3,1}(p/2)N_{6,1}(\delta/2) + M_{3,2}(p/2)N_{6,3}(\delta/2)\}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n-1}}{(3n)!} = \frac{1}{\delta q}\{\lambda_1 M_{3,0}(\lambda_1) - \lambda_2 M_{3,0}(\lambda_2)\} \tag{3.6}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n}}{(3n+1)!} = \frac{1}{\delta q}\{\lambda_1 M_{3,1}(\lambda_1) - \lambda_2 M_{3,1}(\lambda_2)\} \tag{3.7}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n+1}}{(3n+2)!} = \frac{1}{\delta q}\{\lambda_1 M_{3,2}(\lambda_1) - \lambda_2 M_{3,2}(\lambda_2)\}. \tag{3.8}$$

Here $p = \text{tr. } X$, $q = \det. X$, where X is the matrix defined in (1.2) and λ_1, λ_2 are its eigenvalues. Further $\delta = \sqrt{p^2 - 4q}$.

Proof: We prove (3.4) and (3.7). The proofs of the others are similar. Since λ_1, λ_2 are the eigenvalues of X, they satisfy its characteristic equation $\lambda^2 - p\lambda + q = 0$. Therefore,

$$\lambda_1 + \lambda_2 = p, \quad \lambda_1\lambda_2 = q, \quad \text{and} \quad \lambda_1 = \frac{p + \delta}{2}, \quad \lambda_2 = \frac{p - \delta}{2}.$$

Now, using (3.2), we have

$$M_{3,1}(X) = \frac{1}{\lambda_1 - \lambda_2}\{(X - \lambda_1 I)M_{3,1}(\lambda_1) - (X - \lambda_2 I)M_{3,1}(\lambda_2)\},$$

i.e.,

$$M_{3,1}(X) = \frac{1}{\delta}\{[M_{3,1}(\lambda_1) - M_{3,1}(\lambda_2)]X - [\lambda_1 M_{3,1}(\lambda_1) - \lambda_2 M_{3,1}(\lambda_2)]I\}. \tag{3.9}$$

Using Lemma 3, we get

$$\begin{aligned} M_{3,1}(\lambda_1) - M_{3,1}(\lambda_2) &= M_{3,1}\left(\frac{p + \delta}{2}\right) - M_{3,1}\left(\frac{p - \delta}{2}\right) \\ &= M_{3,0}(p/2)[M_{3,1}(\delta/2) - M_{3,1}(-\delta/2)] \\ &\quad + M_{3,1}(p/2)[M_{3,0}(\delta/2) - M_{3,0}(-\delta/2)] \\ &\quad - M_{3,2}(p/2)[M_{3,2}(\delta/2) - M_{3,2}(-\delta/2)]. \end{aligned}$$

Now, using Lemma 4, we get

$$\begin{aligned} &M_{3,1}(\lambda_1) - M_{3,1}(\lambda_2) \\ &\quad = 2M_{3,0}(p/2)N_{6,1}(\delta/2) - 2M_{3,1}(p/2)N_{6,3}(\delta/2) + 2M_{3,2}(p/2)N_{6,5}(\delta/2). \end{aligned} \tag{3.10}$$

Substituting (3.10) in (3.9), we get

$$\begin{aligned} M_{3,1}(X) = \frac{2}{\delta}\Big\{&[M_{3,0}(p/2)N_{6,1}(\delta/2) - M_{3,1}(p/2)N_{6,3}(\delta/2) \\ &+ M_{3,2}(p/2)N_{6,5}(\delta/2)]X - \frac{1}{2}[\lambda_1 M_{3,1}(\lambda_1) - \lambda_2 M_{3,1}(\lambda_2)]I\Big\}. \end{aligned} \tag{3.11}$$

Now, by (3.1) and (2.1), we have

$$M_{3,1}(X) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(3n + 1)!}[U_{3n+1}X - qU_{3n}I]. \tag{3.12}$$

Equating the coefficients of X and I in (3.11) and (3.12), we get (3.4) and (3.7).

Starting with $M_{3,0}(X)$ and $M_{3,2}(X)$ and following a similar procedure, we obtain (3.3), (3.6), and (3.5) and (3.8).

Remark 1: The right-hand sides of (3.6)-(3.8) are expressible in terms of p and δ; however, the formulas become messy and serve no better purpose.

4. SUMMATION FORMULAS FOR THE CHEBYCHEV POLYNOMIALS

Theorem 2: The following summation formulas hold for $\{S_n(x)\}$ and $\{T_n(x)\}$. Let $x = \cos\theta$ and $y = \sin\theta$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{3n}(x)}{(3n)!} \tag{4.1}$$

$$= \frac{1}{y}[M_{3,0}(x)M_{6,3}(y) + M_{3,1}(x)M_{6,5}(y) - M_{3,2}(x)M_{6,1}(y)]$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{3n+1}(x)}{(3n+1)!} \tag{4.2}$$

$$= \frac{1}{y}[M_{3,0}(x)M_{6,1}(y) + M_{3,1}(x)M_{6,3}(y) + M_{3,2}(x)M_{6,5}(y)]$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{3n+2}(x)}{(3n+2)!} \tag{4.3}$$

$$= \frac{1}{y}[-M_{3,0}(x)M_{6,5}(y) + M_{3,1}(x)M_{6,1}(y) + M_{3,2}(x)M_{6,3}(y)]$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{3n}(x)}{(3n)!} \tag{4.4}$$

$$= M_{3,0}(x)M_{6,0}(y) + M_{3,1}(x)M_{6,2}(y) + M_{3,2}(x)M_{6,4}(y)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{3n+1}(x)}{(3n+1)!} \tag{4.5}$$

$$= -M_{3,0}(x)M_{6,4}(y) + M_{3,1}(x)M_{6,0}(y) + M_{3,2}(x)M_{6,2}(y)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{3n+2}(x)}{(3n+2)!} \tag{4.6}$$

$$= -M_{3,0}(x)M_{6,2}(y) - M_{3,1}(x)M_{6,4}(y) + M_{3,2}(x)M_{6,0}(y).$$

Proof: If $p = 2x = 2\cos\theta$ and $q = 1$, then $U_n(p, q)$ are the Chebychev polynomials $S_n(x)$ of the first kind; i.e.,

$$U_n(2x, 1) = S_n(x) = \frac{\sin n\theta}{\sin\theta} \qquad (n \geqslant 0),$$

where

$$S_{n+2} = 2xS_{n+1} - S_n,$$

with

$$S_0 = 0,\ S_1 = 1.$$

We prove (4.2) and (4.5) as follows. Now,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n U_{3n+1}}{(3n+1)!} &= \sum_{n=0}^{\infty}\frac{(-1)^n S_{3n+1}(x)}{(3n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n \sin(3n+1)\theta}{(3n+1)!\,\sin\theta}\\ &= \frac{1}{\sin\theta}\sum_{n=0}^{\infty}\frac{(-1)^n}{(3n+1)!}\left[\frac{e^{i(3n+1)\theta} - e^{-i(3n+1)\theta}}{2i}\right]\\ &= \frac{1}{2i\sin\theta}\sum_{n=0}^{\infty}\frac{(-1)^n}{(3n+1)!}[(e^{i\theta})^{3n+1} - (e^{-i\theta})^{3n+1}]\\ &= \frac{1}{2i\sin\theta}[M_{3,1}(e^{i\theta}) - M_{3,1}(e^{-i\theta})].\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty}\frac{(-1)^n U_{3n+1}}{(3n+1)!} = \frac{1}{2iy}[M_{3,1}(x+iy) - M_{3,1}(x-iy)]. \tag{4.7}$$

By Lemma 3,

$$\begin{aligned}M_{3,1}(x+iy) - M_{3,1}(x-iy) = M_{3,0}(x)[M_{3,1}(iy) - M_{3,1}(-iy)]&\\ + M_{3,1}(x)[M_{3,0}(iy) - M_{3,0}(-iy)]&\\ - M_{3,2}(x)[M_{3,2}(iy) - M_{3,2}(-iy)]&,\end{aligned}$$

so that by Lemma 4,

$$\begin{aligned}&M_{3,1}(x+iy) - M_{3,1}(x-iy)\\ &\quad = 2M_{3,0}(x)N_{6,1}(iy) - 2M_{3,1}(x)N_{6,3}(iy) + 2M_{3,2}(x)N_{6,5}(iy).\end{aligned} \tag{4.8}$$

Further, by Lemma 3(b),

$$\begin{aligned}N_{6,k}(iy) &= N_{6,k}(w^{3/2}y),\ \text{where } w = e^{2\pi i/6},\ k = 0, 1, \ldots, 5\\ &= w^{k/2}M_{6,k}(wy)\\ &= w^{k/2}\sum_{n=0}^{\infty}\frac{(-1)^n w^{6n+k}y^{6n+k}}{(6n+k)!},\end{aligned}$$

so that

$$N_{6,k}(iy) = w^{3k/2}M_{6,k}(y). \tag{4.9}$$

Note that $w^{3/2} = i$, $w^{9/2} = -i$, and $w^{15/2} = i$. Hence, substituting (4.9) in (4.8), we get

$$\begin{aligned} &M_{3,1}(x + iy) - M_{3,1}(x - iy) \\ &\quad = 2i[M_{3,0}(x)M_{6,1}(y) + M_{3,1}(x)M_{6,3}(y) + M_{3,2}(x)M_{6,5}(y)]. \end{aligned} \tag{4.10}$$

Substituting (4.10) in (4.7), we get (4.2). It is easy to see that (4.1) and (4.3) can be similarly obtained.

Noting that

$$V_n(2x, 1) = 2T_n(x) = 2 \cos \theta,$$

and using similar techniques, we obtain (4.4)-(4.6).

Remark 2: Since $S_n(x) = \dfrac{\sin n\theta}{\sin \theta}$, and $T_n(x) = \cos n\theta$, (3.13)-(3.18) also give summation formulas for

$$\sum_{n=0}^{\infty} \frac{(-1)^n \sin (3n + j)\theta}{(3n + j)!} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^n \cos (3n + j)\theta}{(3n + j)!}, \; j = 0, 1, 2.$$

For example, from (3.14) we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n \sin (3n + 1)\theta}{(3n + 1)!} &= M_{3,0}(\cos \theta)M_{6,1}(\sin \theta) \\ &+ M_{3,1}(\cos \theta)M_{6,3}(\sin \theta) + M_{3,2}(\cos \theta)M_{6,5}(\sin \theta). \end{aligned}$$

Remark 3: Shannon and Horadam [5] studied the third-order recurrence relation

$$S_n = PS_{n-1} + QS_{n-2} + RS_{n-3} \quad (n \geqslant 4), \quad S_0 = 0,$$

where they write

$$\{S_n\} = \{J_n\} \text{ when } S_1 = 0,\ S_2 = 1, \text{ and } S_3 = P,$$

$$\{S_n\} = \{K_n\} \text{ when } S_1 = 1,\ S_2 = 0, \text{ and } S_3 = Q,$$

and

$$\{S_n\} = \{L_n\} \text{ when } S_1 = 0,\ S_2 = 0, \text{ and } S_3 = R.$$

Following Barakat, and using the matrix exponential function, they then obtained formulas for

$$\sum_{n=0}^{\infty} \frac{J_n}{n!}, \; \sum_{n=0}^{\infty} \frac{K_n}{n!}, \text{ and } \sum_{n=0}^{\infty} \frac{L_n}{n!}$$

in terms of eigenvalues of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Using matrix circular functions and their extensions and following similar techniques could be a matter of discussion for an additional paper on the derivation of the higher-order formulas for $\{J_n\}$, $\{K_n\}$, and $\{L_n\}$.

Remark 4: A question naturally arises as to whether Theorems 1 and 2 can be extended further. This encounters some difficulties, due to the peculiar behavior of $M_{r,j}(x)$ and $N_{r,j}(x)$ for higher values of r. This will be the topic of discussion in our next paper.

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THE GOLDEN MEAN IN THE SOLAR SYSTEM

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The mean distances of planets and satellites from their primary, divided by the next one out, bears a loose resemblance to the golden mean and the Fibonacci sequence. B. A. Read [6] explored this resemblance and related the deviation or offset of the planets from an exact Fibonacci spacing to the density of a planet and that of the next planet inward from it. However, when the aphelion and apogee distances are considered, the resemblance is no longer loose. Instead, as can be seen from the accompanying tables, the resemblance is close enough to reflect some underlying natural law. In Table 1, the observed aphelion distances of the planets from the sun are compared to distances calculated in direct proportion to the Fibonacci sequence as well as to distances calculated in proportion to the golden mean. The golden mean, 1.618034, an irrational number, is the limit that one Fibonacci number, divided by the preceding Fibonacci number, converges towards, which is equal to $(1 + \sqrt{5})/2$. Its reciprocal is 0.618034, which is the form of the golden mean used in this paper.

As can be seen from Table 1, Mercury deviates considerably from calculated distances, as would be expected from tidal interactions, as do the innermost satellites of Jupiter, Saturn, and Uranus in Table 2. The deviations of Jupiter and Saturn are not so easily dispensed with, and the gap between Jupiter and Saturn may be suggestive of a missing planet. However, Pluto's distance fits well, suggesting that Pluto is a normal member of the solar system rather than an asteroidal member. At the bottom of Table 1 is a statistical workup of the various calculated spacings compared with the observed spacings.

In the case of the planets, the Fibonacci sequence gives a better fit than the golden mean; however, the apogee distances of the satellites of Jupiter, Saturn, and Uranus fit the golden mean distances as well as the Fibonacci distances, as can be seen in Table 2. The Fibonacci and golden mean distances are calculated from assumed "true values" which are underlined. The asteroidal satellites of Jupiter form two families, the Himalia group consisting of Ananke, Carme, Sinape, and Pasiphae. The Himalia group satellites are close together and have a weighted apogee mean somewhat under the calculated value such that it appears more reasonable that they are fragments of a shattered moon rather than captured asteriodal objects. Likewise for the retrograde group; however, Ananke's inclusion may be doubtful and, if so, then the weighted mean would be 30360×10^3 Km, which fits better than the weighted mean for all four bodies.

Retrograde bodies may well be normal satellites or fragments of normal satellites. The break from direct to retrograde motion occurs at about the same value of the gravitational gradient for both Jupiter and Saturn. (The gravitational gradient is proportional to mass/distance cubed.) It would not surpirse these writers if both Uranus and Neptune were found to have outer retrograde satellites, and if planets beyond Pluto were found to be retrograde. In the case of the sun, Pluto lies farther out with respect to gravitational gradient than do the retrograde satellites of Jupiter and Saturn; thus, there is probably more to retrograde motion than gravitational gradient.

In Table 3, the aphelion and apogee distances are divided by the distance of the next body outward from the primary. For purposes of comparison and averaging over intermediate and skipped spacings, the resultant ratios in Table 1 are raised to appropriate exponents. Thus, it can be seen that the ratios of closely-spaced satellite orbits of Saturn correspond to the square root, 0.78615, of the golden mean reciprocal. In the statistical workup for the overall mean, the values for the innermost bodies, Mercury, Amalthea, the moonlets, and Miranda, were rejected since they would be the most subject to tidal forces. This workup yields a mean spacing ratio of 0.62103, which comes within 0.5% of the reciprocal of the golden mean. And if Phoebe and the "asteroidal" satellites of Jupiter are also rejected, the overall mean comes to 0.61877, which is within approximately 0.1% of that reciprocal. This golden mean orbital interval corresponds to a constant increase in the gravitational gradient by a factor

TABLE 1

COMPARISON OF OBSERVED APHELION DISTANCES WITH FIBONACCI AND GOLDEN MEAN RATIOS

Planet	Aphelion Distance from Sun $\times 10^6$ Km	Fibonacci Number	Distances Proportional to Fibonacci Numbers $\times 10^6$ Km: With Neptune at "True Value"	With Pluto at "True Value"	With an Adjusted Best Fit	Golden Mean Ratio Best Fit	Titius-Bode Law Numbers and Distances with Uranus at "True Value": Number	Distance $\times 10^6$ Km
MERCURY	69.86	1	50.98	51.22	51.88	61.04	4	61.3
VENUS	108.8	2	101.9	102.4	103.8	98.77	7	107.2
EARTH	152.1	3	152.9	153.6	155.6	159.8	10	153.2
MARS	249.1	5	254.9	256.1	259.3	258.6	16	245.1
ASTER.		8	407.8	409.7	414.8	418.4	28	248.9
JUPITER	815.8	13	662.7	665.8	674.1	677.0	52	796.4
		21	1070	1075	1089	1095	-	-
SATURN	1504	34	1733	1741	1763	1772	100	1532
URANUS	3002	55	2804	2817	2852	2868	196	3002
NEPTUNE	4537	89	4537	4558	4615	4640	-	-
PLUTO	7375	144	7340	7375	7467	7508	388	5943

The above $\dfrac{\text{Calculated Value}}{\text{Observed Value}} - 1$

	With Neptune at "True Value"	With Pluto at "True Value"	With an Adjusted Best Fit	Golden Mean Ratio Best Fit
MERCURY	-0.270	-0.267	-0.257	-0.126
VENUS	-0.063	-0.059	-0.046	-0.093
EARTH	0.006	0.010	0.023	0.051
MARS	0.023	0.028	0.041	0.038
JUPITER	-0.188	-0.184	-0.174	-0.170
SATURN	0.152	0.157	0.172	0.178
URANUS	-0.066	-0.062	-0.050	-0.045
NEPTUNE	0.000	0.005	0.017	0.023
PLUTO	0.005	0.000	0.013	0.018
Sum, less Mercury	-0.131	-0.109	-0.002	0.000

The aphelion distances were taken from Joseph Armento's compilation [1]. The Titius-Bode law relationship, which works best with mean distances is shown for comparison only.

TABLE 2

COMPARISON OF THE APOGEE DISTANCES OF THE SATELLITES OF JUPITER, SATURN, AND URANUS WITH THE FIBONACCI AND GOLDEN MEAN RATIOS

(The satellites are listed in order of increasing apogee distances.)

Satellite	Mean Distance $\times 10^3$ Km	Eccen-tricity	Inclina-tion	Apogee Distance $\times 10^3$ Km	Fibonacci Number	Distance Proportional to Fibonacci	Distance Proportional Golden Mean
JUPITER							
Amalthea	181.3	0.003	0.4	181.8	1	233.7	256.3
Io	421.6	0.000	0.0	241.6	2	447.3	414.7
Europa	670.9	0.0001	0.5	671.0	3	671.0	671.0
Ganymede	1070	0.0014	0.2	1072	5	1118	1086
Callisto	1883	0.0074	0.2	1897	8	1789	1757
Leda	11110	0.1478	26.7	12750			
Lysithea	11710	0.1074	29	12970			
Himalia	11470	0.1580	28	13280			
Elara	11720	0.2072	28	14180			
Weighted mean of the above four . . .				13370	55	12300	12040
Ananke	20700	0.169	147	24200			
Carme	22350	0.207	163	26980			
Sinape	23700	0.275	157	30220			
Pasiphae	23300	0.410	148	32850			
Weighted mean of the above four . . .				29750	144	32210	31530
SATURN							
					1		
Weighted mean apogee distance of A and B ring moonlets 1980S28, S27, S26, S3, and S1				150	$\sqrt{2}$	138	145
Mimas	186	0.020	1.5	190	2	196	184
Enceladus	238	0.005	0.0	239	$2\sqrt{3/2}$	240	235
Tethys	295	0.000	1.1	295	3	294	298
Dione	378	0.002	0.0	379	$3\sqrt{5/3}$	379	379
Rhea	528	0.001	0.3	529	5	489	482
Titan	1223	0.029	0.3	1258	13	1272	1267
Hyperion	1484	0.104	0.6	1638	$13\sqrt{21/13}$	1616	1605
Iapetus	3562	0.028	14.7	3662	34	3327	3304
Phoebe	12960	0.163	150	15070	144	14090	14000
URANUS							
Miranda	130.5	0.00	0.0	130.5	$1/\sqrt{2}$	137.0	145.4
Ariel	191.8	0.003	0.0	192.4	2	193.8	185.0
Umbriel	267.2	0.004	0.0	268.3	3	290.6	299.4
Titania	483.4	0.002	0.0	484.4	5	484.4	484.4
Oberon	586.3	0.001	0.0	586.9	$5\sqrt{8/5}$	612.8	616.2

Weighted means were found by multiplying apogee distances by radii cubed. In the case of the Himalia group, the diameters of Leda, Lysithea, Himalia, and Elara are 8, 19, 170, and 80 Km, respectively. In the retrograde group, the diameters of Ananke, Carme, Sinape, and Pasiphae are 17, 24, 21, and 27 Km, respectively.

Assumed "true values" from which calculations were started are underlined.

Satellite data were obtained from Patrick Moore's compilation [2]. A and B moonlets distance calculated from Robert Burnham's compilation [3].

TABLE 3

ORBITAL RATIOS

Planet or Satellite	$\frac{d_1}{d_2}$	Exponent y	$\left(\frac{d_1}{d_2}\right)^y$
Mercury	0.64180	1	0.64180
Venus	0.71579	1	0.71579
Earth	0.61043	1	0.61043
Mars	0.30537	1/2	0.53260
Jupiter	0.54224	1/2	0.73637
Saturn	0.50112	1	0.50112
Uranus	0.66177	1	0.66177
Neptune	0.61516	1	0.61516
Amalthea	0.43121	1	0.43121
Io	0.62832	1	0.62832
Europa	0.62622	1	0.62622
Ganymede	0.56484	1	0.56484
Callisto	0.14191	1/4	0.61374
Himalia Group	0.44038	1/2	0.66361
Moonlets	0.78947	2	0.62327
Mimas	0.79498	2	0.63199
Enceladus	0.81017	2	0.65637
Tethys	0.77836	2	0.60585
Dione	0.71645	2	0.51329
Rhea	0.42051	1/2	0.64847
Titan	0.76801	2	0.58984
Hyperion	0.44730	2/3	0.58488
Iapetus	0.24300	1/3	0.62400
Miranda	0.67827	2	0.46006
Ariel	0.71711	1	0.71711
Umbriel	0.55388	1	0.55388
Titania	0.82535	2	0.68121
Mean ratio with Mercury, Amalthea, moonlets, and Miranda excluded			0.62103

This table shows the aphelion and apogee distances, d_1, of planets and satellites divided by the distance, d_2, of the next body outward from the primary. The ratios are raised to various powers for purposes of averaging over intermediate spacings or skipped spacings.

of 4.236068, the cube of the golden mean, which is equal to $2 + \sqrt{5}$, going from one orbit inward to the next orbit nearer to the primacy.

Concerning the motions of Mercury and Venus, Robert R. Newton [4] has come up with some interesting observations. He has carefully analyzed astronomical observations since Babylonian times and has noted that Mercury has been persistently gaining energy and, likewise, Venus to a lesser extent. The angular accelerations he has come up with, in seconds of arc per century squared are: Mercury, 4.1520; Venus, 1.6225. These numbers are maximum values; thus, the true values are probably one-half or less of these numbers. These numbers are of the right magnitude to account for the deviation from golden mean positions for these planets. Robert R. Newton [4] has noted a small increase in Saturn's angular motion, but not enough to account for the observed discrepancy. No change has been noted for Jupiter. Possibly the explanation lies in the large mass of Jupiter and Saturn.

The authors conclude that there is some underlying law involving gravitation and the golden mean that determines both aphelion and apogee distances. With respect to some underlying gravitational principle, R. Louise [5] remarked: "that satellite systems mimic the planetary system suggests some possible unsuspected property of gravitation."

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DISTRIBUTION PROPERTY OF RECURSIVE SEQUENCES DEFINED BY $u_{n+1} \equiv u_n + u_n^{-1}$ (MOD m)

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(Submitted July 1982)

1. INTRODUCTION

We shall consider a distribution property of sequences of integers. Let us denote $a = (a_n)_{n \in \mathbf{N}}$ an infinite sequence of integers. For integers $N \geqslant 1$, $m \geqslant 2$, and j $(0 \leqslant j \leqslant m-1)$, let us define $A_N(j, m, a)$ as the number of terms among $a_1, a_2, \ldots, a_N$ satisfying the congruence $a_n \equiv j \pmod{m}$.

A sequence $a = (a_n)_{n \in \mathbf{N}}$ is said to be uniformly distributed modulo m (u.d. mod m) if, for every $j = 0, 1, \ldots, m-1$,

$$\lim_{N \to \infty} \frac{A_N(j, m, a)}{N} = \frac{1}{m}. \tag{1.1}$$

A sequence $a = (a_n)_{n \in \mathbf{N}}$ is said to be uniformly distributed in $\mathbf{Z}$ if, for any integer $m \geqslant 2$, $a = (a_n)_{n \in \mathbf{N}}$ is uniformly distributed modulo m.

This notion was first introduced by Niven [6] and various results are already obtained (see Kuipers & Niederreiter's book [4]), among which the sequence of Fibonacci numbers and its generalizations were investigated with respect to uniform distribution property modulo m. The sequence of generalized Fibonacci numbers is defined by the following linear recurrence formula of second order,

$$h_{n+2} = h_{n+1} + h_n \quad (n \geqslant 1), \tag{1.2}$$

with initial values $h_1 = a$ and $h_2 = b$.

The sequence of Fibonacci numbers $(h_n)_{n \in \mathbf{N}}$ with $h_1 = h_2 = 1$ is not uniformly distributed mod m for any $m \neq 5^k$ $(k = 1, 2, \ldots)$. Any sequence of generalized Fibonacci numbers is not uniformly distributed mod m for any $m \neq 5^k$ $(k = 1, 2, \ldots)$ and even for $m = 5^k$ $(k = 1, 2, \ldots)$ for certain initial values a and b [3].

Various modifications for the recurrence formula (1.2) can be considered. In this note we shall consider the following congruential recurrence formula:

$$u_{n+1} \equiv u_n + u_n^{-1} \pmod{m}. \tag{1.3}$$

Since our interest is the distribution property of integer sequences modulo m, the congruential recurrence will be sufficient for our purpose.

For two given integers s and m, where $m \geqslant 2$ is the modulus and $s = u_1$ is the starting point, we can generate a sequence of integers $u = u(s, m)$ mod m by the recurrence formula (1.3). We give our attention only to infinite sequences, and the set of these starting points is denoted by A_m.

The structure of A_m will be discussed in the next section. Similarly to the notion of uniform distribution modulo m, we define the function $A_N(j, m, u(s, m))$ for j each invertible element in the ring $\mathbf{Z}/m\mathbf{Z}$, and we call $u = u(s, m)$ for $s \in A$ uniformly distributed in $(\mathbf{Z}/m\mathbf{Z})^*$ if, for any invertible element $j \in \mathbf{Z}/m\mathbf{Z}$,

$$\lim_{N \to \infty} \frac{A_N(j, m, u(s, m))}{N} = \frac{1}{\phi(m)},$$

where $\phi(\cdot)$ denotes the Euler function.

It will be proved that recursive sequences $u(s, m)$ are not uniformly distributed in $(\mathbf{Z}/m\mathbf{Z})^*$ except for $m = 3$.

Finally, we generalize the recurrence formula (1.3) as

$$u_{n+1} \equiv au_n + bu_n^{-1} \pmod{m},$$

and a similar result will be given.

2. THE STRUCTURE OF A_m

We consider the solvability of the congruence

$$u_{n+1} \equiv u_n + u_n^{-1} \pmod{m}$$

in $(\mathbf{Z}/m\mathbf{Z})^*$.

Case I: m is even

In this case, invertible elements in $\mathbf{Z}/m\mathbf{Z}$ are odd and their inverses are necessarily odd. Therefore, the sum of an invertible element and its inverse is even. Here we get

Theorem 1: If m is even, then $A_m = \phi$.

Case II: $m = p$ (odd prime)

In this case, the only noninvertible element in $\mathbf{Z}/p\mathbf{Z}$ is 0, so we can start with any starting point s, except 0, the recurrence

$$u_{n+1} \equiv u_n + u_n^{-1} \pmod{p}.$$

We consider the condition on $s \in (\mathbf{Z}/p\mathbf{Z})^*$ for which

$$s + s^{-1} \equiv 0 \pmod{p}. \tag{2.1}$$

This congruence is equivalent to

$$s^2 \equiv -1 \pmod{p}, \tag{2.2}$$

since s and p are relatively prime.

The first complementary law of reciprocity [1] shows that for any odd prime p,

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}, \tag{2.3}$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol. Thus, we have

Theorem 2:

i) For any prime p of the form $4n + 3$,

$$A_p = (\mathbf{Z}/p\mathbf{Z})^*.$$

ii) For any prime p of the form $4n + 1$, no sequences $u(s, p)$ are uniformly distributed in $(\mathbf{Z}/p\mathbf{Z})^*$ for any starting point $s \in (\mathbf{Z}/p\mathbf{Z})^*$.

Case III: m is a power of an odd prime p

In this case, $m = p^\alpha$, $\alpha > 1$, and we shall consider the following congruence,

$$s + s^{-1} \equiv a \pmod{p^\alpha},$$

where $s \in (\mathbf{Z}/p^\alpha\mathbf{Z})^*$ and p divides a. This is equivalent to

$$s^2 \equiv as - 1 \pmod{p^\alpha}, \tag{2.4}$$

since s and p^α are relatively prime.

Letting $f(x) = x^2 - ax + 1$, then $f'(x) = 2x - a$. If the congruence

$$s^2 \equiv as - 1 \pmod{p} \tag{2.5}$$

has a solution s_0, then (2.4) has a solution, since

$$f'(s_0) = 2s_0 - a \equiv 2s_0 \not\equiv 0 \pmod{p}.$$

But (2.5) is identical to (2.2) because p divides a. Thus, we have

Theorem 3: Let $m = p^\alpha$ with $\alpha > 0$ and p an odd prime.

i) If p is of the form $4n + 3$, then $A_{p^\alpha} = (\mathbf{Z}/p^\alpha\mathbf{Z})^*$.

ii) If p is of the form $4n + 1$, then no $u(s, p^\alpha)$ is uniformly distributed in $(\mathbf{Z}/p^\alpha\mathbf{Z})^*$.

Case IV: m is odd

In this case,

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r},$$

where the p_i's are odd primes and $\alpha_i > 0$.

Considering the congruence,

$$s^2 - as + 1 \equiv 0 \pmod{m},$$

where a divides m, the solvability of

$$s^2 - as + 1 \equiv 0 \pmod{p_i}$$

depends on the value $\left(\dfrac{a^2 - 4}{p_i}\right)$. Thus, we cannot conclude, as in previous cases, that the structure of A_m is in a compact form.

3. DISTRIBUTION PROPERTY OF $u(s, m)$

In the preceding section, we saw that for infinitely many m, $A_m \neq \phi$. We shall prove in this section that the distribution property of $u(s, m)$ is quite similar to that of the sequence of Fibonacci numbers.

Direct calculation gives

Theorem 4: For any $s \in A_3 = (\mathbf{Z}/3\mathbf{Z})^*$, $u(s, 3)$ is uniformly distributed in $(\mathbf{Z}/3\mathbf{Z})^*$.

We now present the main statement of the paper as Theorem 5.

Theorem 5: Let m be a positive integer greater than 1 satisfying $A_m \neq \phi$. For any $s \in A_m$, $u(s, m)$ is not uniformly distributed in $(\mathbf{Z}/m\mathbf{Z})^*$, except for $m = 3$.

We now generalize the recurrence formula (1.3) as follows:

$$u_{n+1} \equiv au_n + bu_n^{-1} \pmod{m}, \tag{3.2}$$

where a and b are invertible elements in $\mathbf{Z}/m\mathbf{Z}$. The sequence generated by (3.2) is denoted by $u(s; a, b, m)$, where $s = u_1$ is the invertible starting value, and the set of starting values that generates infinite sequences is written as $A_{m;a,b}$.

Similarly to Theorem 3, for even m, $A_{m;a,b} = \phi$. We do not mention the structure of $A_{m;a,b}$ since the distribution property of $u(s; a, b, m)$ is in question.

Theorem 6: For any s contained in nonempty $A_{m;a,b}$, no sequence $u(s; a, b, m)$ is uniformly distributed in $(\mathbf{Z}/m\mathbf{Z})^*$, except in the case of Theorem 4.

Proof: As Theorem 6 includes Theorem 5, we only give the proof of the latter.

We know that we only have to consider odd m greater than 2. If a sequence generated by (3.2) is uniformly distributed in $G = (\mathbf{Z}/m\mathbf{Z})^*$, then every element of G must appear in the sequence (considered mod m). In particular, for every $c \in G$, there exists $s \in G$ with

$$as + bs^{-1} \equiv c \pmod{m}.$$

Hence the function $f: G \to \mathbf{Z}/m\mathbf{Z}$, defined by

$$f(s) = as + bs^{-1},$$

is a bijection of G. But

$$f(s) = f(ba^{-1}s^{-1})$$

for all $s \in G$, and since f is a bijection, we get

$$s \equiv ba^{-1}s^{-1} \pmod{m};$$

hence,

$$s^2 \equiv ba^{-1} \pmod{m}$$

for all $s \in G$. Setting $s = 1$ gives

$$ba^{-1} \equiv 1 \pmod{m},$$

and setting $s = 2$ gives $m = 3$.

Inspection shows that only the case $a = b = 1$ yields a uniformly distributed sequence in $(\mathbf{Z}/3\mathbf{Z})^*$. Q.E.D.

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FIBONACCI RESEARCH CONFERENCE ANNOUNCEMENT

The Fibonacci Association and the University of Patras, Greece, would like to announce their intentions to jointly sponsor an international conference on Fibonacci numbers and their applications. This conference is tentatively set for late August or early September of 1984. Anyone interested in presenting a paper or attending the conference should contact:

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

A. P. HILLMAN

Assistant Editors

GLORIA C. PADILLA and CHARLES R. WALL

Send all communications regarding ELEMENTARY PROBLEMS and SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 Solano Dr., S.E.; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-514 *Proposed by Philip L. Mana, Albuquerque, NM*

Prove that $\binom{n}{5} + \binom{n+4}{5} \equiv n \pmod 2$ for $n = 5, 6, 7, \ldots$.

B-515 *Proposed by Walter Blumberg, Coral Springs, FL*

Let $Q_0 = 3$, and for $n \geqslant 0$, $Q_{n+1} = 2Q_n^2 + 2Q_n - 1$. Prove that $2Q_n + 1$ is a Lucas number.

B-516 *Proposed by Walter Blumberg, Coral Springs, FL*

Let U and V be positive integers such that $U^2 - 5V^2 = 1$. Prove that UV is divisible by 36.

B-517 *Proposed by Charles R. Wall, Trident Tech. College, Charleston, SC*

Find all n such that $n! + (n + 1)! + (n + 2)!$ is the square of an integer.

B-518 *Proposed by Herta T. Freitag, Roanoke, VA*

Let the measures of the legs of a right triangle be $F_{n-1}F_{n+2}$ and $2F_nF_{n+1}$. What feature of the triangle has $F_{n-1}F_n$ as its measure?

B-519 *Proposed by Herta T. Freitag, Roanoke, VA*

Do as in Problem B-518 with each Fibonacci number replaced by the corresponding Lucas number.

SOLUTIONS

Lucas Addition Formula

B-490 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that the arithmetic mean of $L_{2n}L_{2n+3}$ and $5F_{2n}F_{2n+3}$ is always a Lucas number.

Solution by J. Suck, Essen, GERMANY

This is an instance of the addition formula

$$2L_{m+n} = L_mL_n + 5F_mF_n, \quad m, n \in \mathbf{Z}, \qquad (*)$$

a companion to $2F_{m+n} = F_mL_n + L_mF_n$ (compare Hoggatt's I_{38}). Proof of (*) from the Binet forms $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, $L_n = \alpha^n + \beta^n$, $n \in \mathbf{Z}$:

$$\begin{aligned} L_mL_n + 5F_mF_n &= (\alpha^m + \beta^m)(\alpha^n + \beta^n) + 5(\alpha^m - \beta^m)(\alpha^n - \beta^n)/(\sqrt{5}\sqrt{5}) \\ &= 2\alpha^m\alpha^n + 2\beta^m\beta^n = 2(\alpha^{m+n} + \beta^{m+n}) = 2L_{m+n}. \end{aligned}$$

Also solved by Paul S. Bruckman, C. Georghiou, L. Kuipers, John W. Milsom, Andreas N. Philippou, George N. Philippou, Bob Prielipp, Heinz-Jürgen Seiffert, Sahib Singh, Robert L. Vogel, Charles R. Wall, and the proposer.

Application of the Addition Formula

B-491 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers. Prove that

$$F_kF_{n+j} - F_jF_{n+k} = (L_jL_{n+k} - L_kL_{n+j})/5.$$

Solution by J. Suck, Essen, GERMANY

Using (*) in the above solution to B-490, we have

$$\begin{aligned} 5(F_kF_{n+j} - F_jF_{n+k}) &= 2L_{k+n+j} - L_kL_{n+j} - (2L_{j+n+k} - L_jL_{n+k}) \\ &= L_jL_{n+k} - L_kL_{n+j}. \end{aligned}$$

Also solved by Paul S. Bruckman, Herta T.Freitag, C. Georghiou, L. Kuipers, John W. Milsom, George N. Philippou, Bob Prielipp, Heinz-Jürgen Seiffert, Sahib Singh, Robert L. Vogel, Charles R. Wall, and the proposer.

New Look at Previous Application

B-492 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers. Prove that

$$F_n F_{n+j+k} - F_{n+j} F_{n+k} = (L_{n+j} L_{n+k} - L_n L_{n+j+k})/5.$$

Solution by J. Suck, Essen, GERMANY

The same as B-491: rename $k \leftrightarrow n$, $j \to n + j$.

Remark: A companion problem (from Hoggatt's I_{38}) would have been

$$L_k F_{n+j} - L_j F_{n+k} = F_j L_{n+k} - F_k L_{n+j}, \quad j,\ k,\ n \in \mathbf{Z}.$$

Also solved by Paul S. Bruckman, Herta T. Freitag, C. Georghiou, John W. Milsom, George N. Philippou, Bob Prielipp, Heinz-Jürgen Seiffert, Sahib Singh, Robert L. Vogel, Charles R. Wall, and the proposer.

Exponent of 2 in Sum

B-493 *Proposed by Valentina Bakinova, Rondout Valley, NY*

Derive a formula for the largest integer $e = e(n)$ such that 2^e is an integral divisor of

$$\sum_{i=0}^{\infty} 5^i \binom{n}{2i}$$

where $\binom{n}{k} = 0$ for $k > n$.

Solution by C. Georghiou, University of Patras, GREECE

Note that, for $n \geqslant 0$,

$$\sum_{i=0}^{\infty} 5^i \binom{n}{2i} = \frac{1}{2}[(1+\sqrt{5})^n + (1-\sqrt{5})^n] = 2^{n-1} L_n.$$

From $2 \nmid L_{3n\pm 1}$, $2 | L_{6n}$, $4 \nmid L_{6n}$, $4 | L_{6n+3}$, and $8 \nmid L_{6n+3}$, we get

$$e(n) = \begin{cases} n & \text{if } n \equiv 0 \pmod{6} \\ n+1 & \text{if } n \equiv 3 \pmod{6} \\ n-1 & \text{if } n \not\equiv 0 \pmod{3} \end{cases}$$

Also solved by Paul S. Bruckman, L. Kuipers, Sahib Singh, J. Suck, Charles R. Wall, and the proposer.

Sum of Consecutive Integers

B-494 *Proposed by Philip L. Mana, Albuquerque, NM*

For each positive integer n, find positive integers a_n and b_n such that $101n$ is the following sum of consecutive positive integers:

$$a_n + (a_n + 1) + (a_n + 2) + \cdots + (a_n + b_n).$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We begin by observing that

$$a_n + (a_n + 1) + (a_n + 2) + \cdots + (a_n + b_n) = (b_n + 1)a_n + b_n(b_n + 1)/2.$$

Next, we let

$$a_n = 51 - n \text{ and } b_n = 2n - 1 \text{ for each integer } n,\ 1 \leqslant n \leqslant 50$$

and

$$a_n = n - 50 \text{ and } b_n = 100 \text{ for each integer } n,\ n \geqslant 51.$$

Clearly, a_n and b_n are always positive integers. Also,

(1) if $a_n = 51 - n$ and $b_n = 2n - 1$, then

$$\begin{aligned}(b_n + 1)a_n + b_n(b_n + 1)/2 &= (2n)(51 - n) + (2n - 1)n \\ &= 102n - 2n^2 + 2n^2 - n \\ &= 101n;\end{aligned}$$

(2) if $a_n = n - 50$ and $b_n = 100$, then

$$\begin{aligned}(b_n + 1)a_n + b_n(b_n + 1)/2 &= 101(n - 50) + 50(101) \\ &= 101n - 50(101) + 50(101) \\ &= 101n.\end{aligned}$$

Also solved by Ada Booth, Paul S. Bruckman, M. J. DeLeon, Herta T. Freitag, H. Klauser & E. Schmutz & M. Wachtel, L. Kuipers, Sahib Singh, J. Suck, and the proposer.

Sum of Consecutive Squares

B-495 *Proposed by Philip L. Mana, Albuquerque, NM*

Characterize an infinite sequence whose first 24 terms are:

1, 4, 5, 9, 13, 14, 16, 25, 29, 30, 36, 41, 49, 50, 54, 55,
61, 64, 77, 81, 85, 86, 90, 91, ...

[Note that all perfect squares occur in the sequence.]

Solution by Paul S. Bruckman, Carmichael, CA

The indicated sequence may be characterized as the sequence of positive integers which can be expressed either as squares or as sums of *consecutive* squares, then arranged in increasing order. Equivalently, if the given sequence is denoted by $(x_n)_{n=1}^{\infty}$ and if

$$S_n = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad (\text{with } S_0 \equiv 0),$$

the sequence is characterized as the set of all differences $S_a - S_b$, where $a > b \geqslant 0$, in increasing order.

Also solved by Ada Booth, John W. Milsom, E. Schmutz & M. Wachtel, J. Suck, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-365 *Proposed by Larry Taylor, Rego Park, NY*

Call a Fibonacci-Lucas identity divisible by 5 if every term of the identity is divisible by 5. Prove that, for every Fibonacci-Lucas identity not divisible by 5, there exists another Fibonacci-Lucas identity not divisible by 5 that can be derived from the original identity in the following way:

1) If necessary, restate the original identity in such a way that a derivation is possible.

2) Change one factor in every term of the original identity from F_n to L_n or from L_n to $5F_n$ in such a way that the result is also an identity. If the resulting identity is not divisible by 5, it is the derived identity.

3) If the resulting identity is divisible by 5, change one factor in every term of the original identity from L_n to F_n or from $5F_n$ to L_n in such a way that the result is also an identity. This is equivalent to dividing every term of the first resulting identity by 5. Then, the second resulting identity is the derived identity.

For example, $F_n L_n = F_{2n}$ can be restated as

$$F_n L_n = F_{2n} \pm F_0(-1)^n.$$

This is actually two distinct identities, of which the derived identities are

$$L_n^2 = L_{2n} + L_0(-1)^n$$

and

$$5F_n^2 = L_{2n} - L_0(-1)^n.$$

H-366 *Proposed by Stanley Rabinowitz, Merrimack, NH*

The *Fibonacci polynomials* are defined by the recursion

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$$

with the initial conditions $f_1(x) = 1$ and $f_2(x) = x$. Prove that the discriminant of $f_n(x)$ is $(-1)^{(n-1)(n-2)/2}2^{n-1}n^{n-3}$ for $n > 1$.

Remark: The idea of investigating discriminants of interesting polynomials was suggested by [1]. The definition of the discriminant of a polynomial can be found in [2]. Fibonacci polynomials are well known, see, for example, [3] and [4]. I ran a computer program to find the discriminant of $f_n(x)$ as n varies from 2 to 11, and by analyzing the results, reached the conjecture given in Problem H-366. The discriminant was calculated by finding the resultant of $f_n(x)$ and $f_n'(x)$ using a computer algebra system similar to the MACSYMA program described in [5]. Much useful material can be found in [6] where the problem of finding the discriminant of the Hermite, Laguerre, and Chebyshev polynomials is discussed. The discriminant of the Fibonacci polynomials should be provable using similar techniques; however, I was not able to do so.

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1. Phyllis Lefton. "A Trinomial Discriminant Formula." *The Fibonacci Quarterly* 20 (1982):363-65.
2. Van der Warden. *Modern Algebra*. New York: Ungar, 1940, I, 82.
3. M. N. S. Swamy. Problem B-84. *The Fibonacci Quarterly* 4 (1966):90.
4. S. Rabinowitz. Problem H-129. *The Fibonacci Quarterly* 6 (1968):51.
5. W. A. Martin & R. J. Fateman. "The MACSYMA System." *Proceedings of the 2nd Symposium on Symbolic and Algebraic Manipulation*, Association for Computing Machinery, 1971, pp. 59-75.
6. D. K. Faddeev & I. S. Sominskii. *Problems in Higher Algebra*. Trans. J. L. Brenner. San Francisco: W. H. Freeman, 1965, Problems 833-851.

H-367 *Proposed by M. Wachtel, Zurich, Switzerland*

A. Prove the identity:

$$\sqrt{(L_{2n} - L_{n-2}^2)\cdot(L_{2n+4} - L_n^2) + 30} = 5F_{2n} - 3(-1)^n$$

B. Prove the identities:

$$\left.\begin{array}{l}\sqrt{(F_{n+1}^2 - F_{2n+3})\cdot(F_{n+3}^2 - F_{2n+7})} \\ \sqrt{(F_{n+3}^2 - F_{2n+5})\cdot(F_{n+5}^2 - F_{2n+9})} \\ \sqrt{(F_{n+4}^2 - F_{2n+6})\cdot(F_{n+6}^2 - F_{2n+10})}\end{array}\right\} = F_{n+2}F_{n+4} \quad \text{or} \quad F_{n+3}^2 + (-1)^n$$

SOLUTIONS

Woops!

The published solution to H-335, which appeared in the May 1983 issue of this quarterly is incorrect. The proposer (Paul Bruckman) pointed out that the polynomial in question can be factored as

$$(x - 1)(x^2 + bx - a^2)(x^2 + ax - b^2),$$

where

$$a = (1 + \sqrt{5})/2 \text{ and } b = (1 - \sqrt{5})/2.$$

The desired roots may easily be obtained from this.

Old Timer

H-277 *Proposed by Larry Taylor, Rego Park, NY*
(Vol. 15, No. 4, December 1977)

If $p \equiv \pm 1 \pmod{10}$ is prime and $x \equiv \sqrt{5}$ is of even order (mod p), prove that $x - 3$, $x - 2$, $x - 1$, x, $x + 1$, and $x + 2$ are quadratic nonresidues of p if and only if $p \equiv 39 \pmod{40}$.

Solution by the proposer

Let $f \equiv (x+1)/2 \pmod{p}$. Then $f^3 \equiv x+2 \pmod{p}$. But $(f/p) = (f^3/p)$ and therefore

$$\left(\frac{x+1}{p}\right)\left(\frac{x+2}{p}\right) = (2/p).$$

In other words,

$$(2/p) = 1 \tag{1}$$

is a necessary condition to have the six consecutive quadratic nonresidues of p.

Also, $f^2 \equiv (x+3)/2 \pmod{p}$. But $(2/p) = 1$ has been established and, therefore,

$$\left(\frac{x+3}{p}\right) = 1$$

Since $(x + 3)(x - 3) \equiv -4 \pmod{p}$, we have

$$\left(\frac{x+3}{p}\right)\left(\frac{x-3}{p}\right) = (-1/p).$$

But $\left(\frac{x-3}{p}\right) = -1$ is required and, therefore,

$$(-1/p) = -1 \tag{2}$$

is another necessary condition.

Since $(-1/p) = -1$ has been established and x is of even order (mod p), therefore $(x/p) = -1$. Since $2f^{-1} \equiv x - 1 \pmod{p}$, therefore

$$\left(\frac{x+1}{p}\right) = \left(\frac{x-1}{p}\right).$$

Since $f^{-3} \equiv x - 2 \pmod{p}$, therefore

$$\left(\frac{x+2}{p}\right) = \left(\frac{x-2}{p}\right).$$

In [1], page 24, the following result is given:

$$(\sqrt{p}/5) = \left(\frac{-2x(x+1)}{p}\right).$$

Since $(2/p) = 1$, $(-1/p) = -1$, and $(x/p) = -1$ have been established, and

$$\left(\frac{x+1}{p}\right) = -1$$

is required, therefore

$$(\sqrt{p}/5) = -1 \tag{3}$$

is a third necessary condition.

There are, by inspection, no further necessary conditions. Therefore the logical product of (1), (2), and (3), which is equivalent to $p \equiv 39 \pmod{40}$, is a necessary and sufficient condition that $x - 3, x - 2, x - 1, x, x + 1$, and $x + 2$ are quadratic nonresidues of p.

Reference

1. Emma Lehmer. "Criteria for Cubic and Quartic Residuacity." *Mathematika* 5 (1958):20-29.

Late Acknowledgment: *E. Schmutz and P. Wittwer solved Problem H-333.*

Not in Prime Condition

H-345 *Proposed by Albert A. Mullin, Huntsville, AL (Vol. 20, No. 4, November 1982)*

Prove or disprove: No four consecutive Fibonacci numbers can be products of two distinct primes.

Solution by Lawrence Somer, Washington D.C.

The assertion is true. We, in fact, prove the following more general result:

Theorem: No three consecutive Fibonacci numbers can each be products of two distinct primes, except for the case

$$F_8 = 21 = 3 \cdot 7,\ F_9 = 34 = 2 \cdot 17,\ F_{10} = 55 = 5 \cdot 11.$$

Proof: We first show that a Fibonacci number F_n can be the product of exactly two distinct primes only if $n = 8$ or n is of the form p, $2p$, or p^2, where p is a prime. A prime p is a primitive divisor of F_n if $p \mid F_n$ but $p \nmid F_m$ for $0 < m < n$. R. Carmichael [1] proved that F_n has a primitive prime factor for every n except $n = 1, 2, 6$, or 12. In none of these cases is F_n a product of exactly two distinct primes. It is also known that if $m|n$, then $F_m|F_n$. Thus, if n has two or more distinct proper divisors r and s that are not equal to 1, 2, 6, or 12, then F_n has at least three prime factors—the primitive prime factors of F_r, F_s, and F_{rs}, respectively. Since $F_6 = 8 = 2^3$, it follows that if n is a multiple of 6, then F_n is not a product of exactly two distinct prime factors. Since $F_1 = 1$ is not a product of two prime factors, it follows that if F_n is a product of two distinct prime factors, then F_n is of the form $F_{2^3} = F_8$, F_p, F_{2p}, or F_{p^2}, where p is a prime.

By inspection, one sees that if $n \leqslant 9$, then F_n, F_{n+1}, F_{n+2} are each products of two distinct primes only if $n = 8$. Now assume $n \geqslant 10$. Among the three consecutive integers n, $n + 1$, and $n + 2$, one of these numbers is divisible by 3. Call this number k. If the Fibonacci number F_k is the product of two distinct primes, then k is of the form p, $2p$, or p^2, where p is prime. This is impossible, since $3|k$ and $k > 9$. The theorem is now proved.

Reference

1. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $a^n + b^n$." *Annals of Mathematics* (2nd Ser.) 15 (1913):30-70.

Also solved by P. Bruckman and S. Singh.

Pell-Mell

H-346 *Proposed by Verner E. Hoggatt, Jr. (Deceased)*
(Vol. 20, No. 4, November 1982)

Prove or disprove: Let $P_1 = 1$, $P_2 = 2$, $P_{n+2} = 2P_{n+1} + P_n$ for $n = 1, 2, 3, \ldots$, then $P_7 = 169$ is the largest Pell number which is a square and there are no Pell numbers of the form $2s^2$ for $s > 1$.

Solution by M. Wachtel, Zurich, Switzerland

1.1 The roots of $x^2 - 2x - 1$ are $1 \pm \sqrt{2}$, and the quotient $\dfrac{P_{n+1}}{P_n} = 1 + \sqrt{2}$.

2.1 Pell numbers *with odd index* show the identity: $P_n^2 + P_{n+1}^2 = P_{2n+1}$.

2.2 P_{2n+1} will only be a square if

$$P_n^2 + P_{n+1}^2 = P_{2n+1} \quad \text{[see (2.1)]}$$

is identical with $(2m + 1)^2 + (2m^2 + 2m)^2 = (2m^2 + 2m + 1)^2$.

2.3 Obviously (1.1) and (2.2) will only be satisfied if $m = 2$ and $n = 3$, i.e., $5^2 + 12^2 = 169$. For $m > 2$, the quotient $2m^2 + 2m/2m + 1$ is rising, thus 169 is the greatest Pell number which is a square.

2.4 Using the general formula $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$, and setting $m - n = d$ (odd) yields:

$$(2dm + d^2)^2 + (2m^2 + 2dm)^2 = (2m^2 + 2dm + d^2)^2.$$

To satisfy (1.1), we have to set $m = 2d$, which leads to $5d^2 + 12d^2 = 13d^2$. It is easy to see that (2.2) will only be satisfied if $d = 1$, whereas setting $d = 3, 5, 7, \ldots$ will yield consecutive terms with common divisors, which is contrary to the Pell formula.

2.5 Pell numbers *with odd index* always are *odd* and can never be $2s^2$.

3.1 Pell numbers *with even index* show the *identity*:

$$2P_{n+1}(P_n + P_{n+1}) = P_{2n+2}.$$

3.2 Obviously, P_n and P_{n+1} are coprime, from which it follows that also P_{n+1} and $(P_n + P_{n+1})$ are coprime. Thus, P_{2n+2} can neither be a square nor twice a square.

Also solved by the proposer.

It All Adds Up

H-347 *Proposed by Paul S. Bruckman, Sacramento, CA (Vol. 20, No. 4, November 1982)*

Prove the identity:

$$\left\{\sum_{n=-\infty}^{\infty} \frac{x^n}{1 + x^{2n}}\right\}^2 = \sum_{n=-\infty}^{\infty} \frac{x^n}{(1 + (-x)^n)^2}, \text{ valid for all real } x \neq 0, \pm 1. \quad (1)$$

In particular, prove the identity:

$$\left\{\sum_{n=-\infty}^{\infty} \frac{1}{L_{2n}}\right\}^2 = \sum_{n=-\infty}^{\infty} \frac{1}{L_n^2} \quad (2)$$

Solution by the proposer

Let

$$f(x) \equiv \sum_{n=-\infty}^{\infty} x^{n^2}, \text{ where } -1 < x < 1. \quad (3)$$

Theorems 311 and 312 in [1] state (using our notation):

$$(f(x))^2 = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{1 - x^{2n-1}}; \quad (4)$$

while Theorem 385 in [1] states:

$$(f(x))^2 = 1 + 8\sum \frac{mx^m}{1 - x^m}, \text{ where } m \text{ runs through all positive integral values which are not multiples of 4.} \quad (5)$$

We may rearrange the terms in (4) and (5), since the series are absolutely convergent for $|x| < 1$. Thus,

$$(f(x))^2 = 1 + 4\sum_{m,\,n=1}^{\infty}(-1)^{n-1}x^{(2n-1)m} = 1 + 4\sum_{m=1}^{\infty}x^m\sum_{n=0}^{\infty}(-x^{2m})^n$$

$$= 1 + 4\sum_{m=1}^{\infty}\frac{x^m}{1+x^{2m}}.$$

If

$$u_m(x) = \frac{x^m}{1+x^{2m}},$$

we note that for all $x \neq 0$,

$$u_0(x) = 1/2 \quad \text{and} \quad u_m(x) = u_{-m}(x) = u_m(1/x).$$

Therefore, the *transformed* series for $(f(x))^2$ converges for all real $x \neq 0, \pm 1$. It follows that

$$(f(x))^2 = 2\sum_{n=-\infty}^{\infty}\frac{x^n}{1+x^{2n}}, \text{ for all real } x \neq 0,\ \pm 1. \tag{6}$$

By similar reasoning,

$$\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n} = \sum_{m,\,n=1}^{\infty}nx^{mn} = \sum_{m=1}^{\infty}x^m\sum_{n=0}^{\infty}(n+1)x^{mn} = \sum_{m=1}^{\infty}\frac{x^m}{(1-x^m)^2}.$$

Therefore, using (5),

$$(f(x))^4 = 1 + 8\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n} - 32\sum_{n=1}^{\infty}\frac{nx^{4n}}{1-x^{4n}}$$

$$= 1 + 8\sum_{n=1}^{\infty}\frac{x^n}{(1-x^n)^2} - 32\sum_{n=1}^{\infty}\frac{x^{4n}}{(1-x^{4n})^2}$$

$$= 1 + 8\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(1-x^{2n-1})^2} + 8\sum_{n=1}^{\infty}\frac{x^{2n}}{(1-x^{2n})^2} - 32\sum_{n=1}^{\infty}\frac{x^{4n}}{(1-x^{4n})^2}$$

$$= 1 + 8\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(1-x^{2n-1})^2} + 8\sum_{n=1}^{\infty}\frac{x^{2n}}{(1-x^{4n})^2}\left\{(1+x^{2n})^2 - 4x^{2n}\right\}$$

$$= 1 + 8\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(1-x^{2n-1})^2} + 8\sum_{n=1}^{\infty}\frac{x^{2n}}{(1-x^{4n})^2}(1-x^{2n})^2$$

$$= 1 + 8\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(1-x^{2n-1})^2} + 8\sum_{n=1}^{\infty}\frac{x^{2n}}{(1+x^{2n})^2}$$

$$= 1 + 8\sum_{m=1}^{\infty}\frac{x^m}{(1+(-x)^m)^2}.$$

If

$$v_m(x) = \frac{x^m}{(1-(-x)^m)^2},$$

we note that for all $x \neq 0$,

$$v_0(x) = 1/4 \quad \text{and} \quad v_m(x) = v_{-m}(x) = v_m(1/x).$$

As before, the transformed series for $(f(x))^4$ must therefore converge for all real $x \neq 0, \pm 1$. We then see that

$$(f(x))^4 = 4 \sum_{n=-\infty}^{\infty} \frac{x^n}{(1 + (-x)^n)^2}, \text{ for all real } x = 0, \pm 1. \tag{7}$$

Squaring both sides of (6) and comparing with (7) yields (1). As a special case, we set $x = b^2$, where $b = (1/2)(1 - \sqrt{5})$, and obtain (2).

It should be pointed out that (2) was derived in [2] and given there as relation (59), using elliptic function theory. Indeed, relations (4) and (5) above have their basis in elliptic function theory.

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