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CONTENTS

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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

THE GENERATION OF HIGHER-ORDER LINEAR RECURRENCES FROM SECOND-ORDER LINEAR RECURRENCES

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Let $\{u_n\}$ be a Lucas sequence of the first kind defined by the second-order recursion relation

$$u_{n+2} = au_{n+1} + bu_n,$$

where a and b are integers and $u_0 = 0$, $u_1 = 1$. By the Binet formulas

$$u_n = (\alpha^n - \beta^n)/(\alpha - \beta),$$

where α and β are roots of the characteristic polynomial

$$x^2 - ax - b.$$

Let

$$D = (\alpha - \beta)^2 = a^2 + 4b$$

be the discriminant of the characteristic polynomial of $\{u_n\}$. We shall prove the following theorem which demonstrates that the quotients of specified terms of the second-order recurrence $\{u_n\}$ satisfy a higher-order relation.

Theorem 1: Consider the sequence

$$\{w_n\}_{n=1}^{\infty} = \{u_{nk}/u_n\}_{n=1}^{\infty},$$

where k is a fixed positive integer, $\alpha\beta \neq 0$, and α/β is not a root of unity. Then $\{w_n\}$ satisfies a k^{th}-order linear integral recursion relation. Further, the order k is minimal.

Along the lines of this theorem, Selmer [1] has shown how one can form a higher-order linear recurrence consisting of the term-wise products of two other linear recurrences. In particular, let $\{s_n\}$ be an m^{th}-order and $\{t_n\}$ be a p^{th}-order linear integral recurrence with the associated polynomials $s(x)$ and $t(x)$, respectively. Let α_i, $i = 1, 2, \ldots, m$, and β_j, $j = 1, 2, \ldots, p$, be the roots of the polynomials $s(x)$ and $t(x)$, respectively, and assume that each polynomial has no repeated roots. Then, the sequence

$$\{h_n\} = \{s_n t_n\}$$

satisfies a linear integral recurrence of order at most mp, whose characteristic polynomial $h(x)$ has roots consisting of the r distinct elements of the set $\{\alpha_i\beta_j\}$, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant p$. Note that the coefficients of $h(x)$ are integral because they are symmetric in the conjugate algebraic integers $\alpha_i\beta_j$. However, $\{h_n\}$ may satisfy a recursion relation of order lower than r.

Selmer's proof depends on the fact that the recurrences $\{s_n\}$ and $\{t_n\}$ can be expressed in terms of their characteristic roots by means of the formulas

$$s_n = \sum_{i=1}^{m} \gamma_i \alpha_i^n, \quad t_n = \sum_{j=1}^{p} \delta_j \beta_j^n. \tag{1}$$

This follows from the fact that the sequences $\{\alpha_i^n\}$, $1 \leqslant i \leqslant m$, and $\{\beta_j^n\}$, $1 \leqslant j \leqslant p$, satisfy the same recursion relations as $\{s_n\}$ and $\{t_n\}$, respectively. Further, a linear combination of sequences satisfying the same linear recursion relation also satisfies that linear recursion relation. By means of Cramer's rule, one can then solve (1) for s_n, $1 \leqslant n \leqslant m$, and t_n, $1 \leqslant n \leqslant p$, in terms of α_i^n, $1 \leqslant i \leqslant m$, and β_j^n, $1 \leqslant j \leqslant p$, respectively. The fact that the roots α_i, $1 \leqslant i \leqslant m$, and β_j, $1 \leqslant j \leqslant p$, are distinct guarantees unique solutions in terms of α_i^n and β_j^n. Now,

$$h_n = s_n t_n = \left(\sum_{i=1}^{m} \gamma_i \alpha_i^n\right)\left(\sum_{j=1}^{p} \delta_j \beta_j^n\right) = \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant p}} \gamma_i \delta_j (\alpha_i \beta_j)^n,$$

and each $\alpha_i \beta_j$ is a root of the polynomial $h(x)$.

Before proving our main result, we will need the following lemma. A proof of this lemma is given by Willett [2].

Lemma 1: Consider the sequence $\{s_n\}$. We introduce the determinant

$$D_r(t) = \begin{vmatrix} s_t & s_{t+1} & \cdots & s_{t+r-1} \\ s_{t+1} & s_{t+2} & & s_{t+r} \\ \cdots & \cdots & \cdots & \cdots \\ s_{t+r-1} & s_{t+r} & & s_{t+2r-2} \end{vmatrix}$$

Then $\{s_n\}$ satisfies a recursion relation of minimal order k if and only if

$$D_k(0) \neq 0$$

and

$$D_r(0) = 0 \text{ for } r > k.$$

We are now ready for the proof of the main result. The first part of the proof will show that $\{w_n\}$ satisfies a k^{th}-order linear integral recursion relation. The second part of the proof will establish the minimality of k. The simple proof of minimality was suggested by Professor Ernst S. Selmer.

Proof of Theorem 1: First, we claim that $u_n \neq 0$ for $n \geqslant 1$ and $\{w_n\}$ is well-defined. If $u_n = 0$, then $\alpha^n - \beta^n = 0$ and $(\alpha/\beta)^n = 1$, since $\beta \neq 0$. This is impossible because α/β is not a root of unity. Note that

$$w_n = \sum_{i=0}^{k-1} \alpha^{(k-1-i)n} \cdot \beta^{in}.$$

The k algebraic integers $\alpha^{k-1-i}\beta^i$, $0 \leqslant i \leqslant k-1$, are all distinct because α/β is not a root of unity. If α and β are rational integers, then the numbers $\alpha^{k-1-i}\beta^i$, $0 \leqslant i \leqslant k-1$, certainly satisfy a monic polynomial of degree k over the rational integers. If α and β are irrational, then α and β are conjugate in the algebraic number field $K = Q(\alpha, \beta) = Q(\alpha)$, where Q denotes the rational numbers. Thus, $\alpha^{k-1-i}\beta^i$ and $\alpha^i\beta^{k-1-i}$ are conjugate in K. Hence, the numbers $\alpha^{k-1-i}\beta^i$, $0 \leqslant i \leqslant k-1$, satisfy a polynomial of degree k which is a product of monic, integral quadratic polynomials and at most one monic, integral linear polynomial. By our discussion preceding the statement of Lemma 1, the sequences $\{(\alpha^{k-1-i}\beta^i)^n\}_{n=1}^{\infty}$, $0 \leqslant i \leqslant k-1$, all satisfy the same linear integral recursion relation of order k. Thus, $\{w_n\}_{n=1}^{\infty}$, the sum of these k sequences, also satisfies this same recursion relation.

To prove the minimality of k, we first note that $\{w_n\}$ may also be defined for $n = 0$ if we put $w_0 = k$. Replacing $D_r(t)$ of Lemma 1 by $D_r(s_n, t)$, the minimality will follow if we can show that $D_k(w_n, 0) \neq 0$. To illustrate the method, let us take $k = 3$ as an example, when

$$D_k(w_n, 0) = \begin{vmatrix} 3 & \alpha^2 + \alpha\beta + \beta^2 & \alpha^4 + \alpha^2\beta^2 + \beta^4 \\ \alpha^2 + \alpha\beta + \beta^2 & \alpha^4 + \alpha^2\beta^2 + \beta^4 & \alpha^6 + \alpha^3\beta^3 + \beta^6 \\ \alpha^4 + \alpha^2\beta^2 + \beta^4 & \alpha^6 + \alpha^3\beta^3 + \beta^6 & \alpha^8 + \alpha^4\beta^4 + \beta^8 \end{vmatrix}.$$

The corresponding matrix may be written as the product

$$\begin{pmatrix} 1 & 1 & 1 \\ \alpha^2 & \alpha\beta & \beta^2 \\ \alpha^4 & \alpha^2\beta^2 & \beta^4 \end{pmatrix} \cdot \begin{pmatrix} 1 & \alpha^2 & \alpha^4 \\ 1 & \alpha\beta & \alpha^2\beta^2 \\ 1 & \beta^2 & \beta^4 \end{pmatrix}$$

Thus, $D_k(w_n, 0)$ is the square of a Vandermonde determinant:

$$D_k(w_n, 0) = \begin{vmatrix} 1 & \alpha^2 & \alpha^4 \\ 1 & \alpha\beta & \alpha^2\beta^2 \\ 1 & \beta^2 & \beta^4 \end{vmatrix}^2 = [(\alpha\beta - \alpha^2)(\beta^2 - \alpha^2)(\beta^2 - \alpha\beta)]^2.$$

Since we assume $\alpha\beta \neq 0$ and α/β is not a root of unity, we have $D_k(w_n, 0) \neq 0$, as required.

In the general case, we similarly get

$$D_k(w_n, 0) = \begin{vmatrix} 1 & \alpha^{k-1} & (\alpha^{k-1})^2 & \cdots & (\alpha^{k-1})^{k-1} \\ 1 & \alpha^{k-2}\beta & (\alpha^{k-2}\beta)^2 & \cdots & (\alpha^{k-2}\beta)^{k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \beta^{k-1} & (\beta^{k-1})^2 & \cdots & (\beta^{k-1})^{k-1} \end{vmatrix}^2 \neq 0,$$

and the proof of the minimality is completed.

As a final remark, we note the condition for α/β not to be a root of unity. When $\alpha\beta = -b \neq 0$, then $z = \alpha/\beta$ is the root of a quadratic equation

$$p(z) = z^2 + \left(\frac{a^2}{b} + 2\right)z + 1 = 0.$$

If α/β shall not be a root of unity, we must have $z \neq \pm 1$, and $p(z)$ cannot be one of the quadratic cyclotomic polynomials $z^2 + 1$, $z^2 \pm z + 1$. Hence, we must demand that

$$\frac{a^2}{b} + 2 \neq \pm 2,\ 0,\ \pm 1 \quad \text{or} \quad -a^2 \neq 0,\ b,\ 2b,\ 3b,\ 4b.$$

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THE SOLUTION OF AN ITERATED RECURRENCE

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1. INTRODUCTION

Hofstadter [1, p. 137] defines the following iterated recurrence,

$$g(0) = 0,\ g(n) = n - g^r(n - 1),\ n = 1, 2, \ldots,$$

where $g^r(n)$ denotes the iterated function

$$\overbrace{g(g(\ldots(g(n))\ldots)}^{r \text{ levels}}.$$

He does not show how to determine the values of this irregular function. In this paper, we will show that the solution to the iterated recurrence can be given as a simple truncation function on numbers written in a generalized Fibonacci base.

First, for convenience, we will change the iterated recurrence by a translation of the origin. The iterated recurrence to be studied is the following:

$$g(0) = 0 \tag{1a}$$

$$g(n) = n - 1 - g^r(n - 1) \tag{1b}$$

The values of $g(n)$ for r = 1, 2, 3, and 7 are tabulated below.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$g(n)$	0	0	1	1	2	2	3	3	4	4	5	5	6

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$g(n)$	0	0	1	2	2	3	3	4	5	5	6	7	7

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$g(n)$	0	0	1	2	3	3	4	4	5	6	6	7	8

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$g(n)$	0	0	1	2	3	4	5	6	7	7	8	8	9

If $r = 1$, it is clear that $g(n)$ is the integer part of $\frac{1}{2}n$, but for larger r it is more irregular. However, in the next section we will show that, if n is expressed in the appropriate Fibonacci base, then $g(n)$ is a simple truncation function.

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2. SOLUTION OF THE RECURRENCE

Define the "radix" b_i as:

$$b_i = i,\ i = 1, 2, \ldots, r \tag{2a}$$

$$b_i = b_{i-1} + b_{i-r},\ i = r + 1, r + 2, \ldots . \tag{2b}$$

If r is 1, then $b_i = 2^i$, and we get the binary base in which all numbers have a unique positional representation of zeros and ones. If $r = 2$, then we have, for example, the following representations:

$$9 = 10001 = 1 \times 8 + 0 \times 5 + 0 \times 3 + 0 \times 2 + 1 \times 1$$

$$10 = 10010 = 1 \times 8 + 0 \times 5 + 0 \times 3 + 1 \times 2 + 0 \times 1.$$

And for $r = 3$:

$$9 = 100000 = 1 \times 9 + 0 \times 6 + 0 \times 4 + 0 \times 3 + 0 \times 2 + 0 \times 1$$

$$10 = 100001 = 1 \times 9 + 0 \times 6 + 0 \times 4 + 0 \times 3 + 0 \times 2 + 1 \times 1.$$

Note that if $r > 1$, the representation is not unique. When $r = 3$, for example, 10 could also be expressed as:

$$10 = 11000 = 1 \times 6 + 1 \times 4 + 0 \times 3 + 0 \times 2 + 0 \times 1.$$

However, the representation can be made unique as follows. To represent a positive number n, find the largest b_i that is less than or equal to n. The representation of n will have a one in the i^{th} digit. Now find the largest b_j less than or equal to $n - b_i$. The representation will also have a one in the j^{th} digit. This process of reduction is continued until n equals a sum of distinct "radix" numbers b_i. Then n will be represented in this base by $a_k a_{k-1} \ldots a_2 a_1$ where a_i, $i = 1, 2, \ldots, k$ is one or zero, depending on whether or not b_i is present in the sum. This will be called the normalized form of the number in this base.

The recurrence (2b) generates what have been called "generalized Fibonacci numbers." So we will call these bases "generalized Fibonacci bases."

A function which removes or truncates the last digit of a number n represented in a generalized Fibonacci base will be denoted by $T(n)$. If $n = a_k a_{k-1} \ldots a_2 a_1$ or, equivalently,

$$n = \sum_{i=1}^{k} a_i b_i,$$

then

$$T(n) = a_k a_{k-1} \ldots a_2 = \sum_{i=1}^{k-1} a_{i+1} b_i.$$

We will define $T(0)$ to be 0.

For example, if $n = 10$, then in the Fibonacci base with $r = 2$, $10 = 10001$ and

$$T(10001) = 1000 = 1 \times 5 + 0 \times 3 + 0 \times 2 + 0 \times 1 = 5.$$

In the binary base with $r = 1$,

$$T(10001) = 1000 = 8.$$

Theorem: The solution to the iterated recurrence

$$g(0) = 0$$
$$g(n) = n - 1 - g^r(n - 1),\ n \geqslant 1,$$

is $g(n) = T(n)$, where $T(n)$ is the truncation function.

3. PROOF OF THEOREM

The function $T(n)$ obviously satisfies the condition (1a). To satisfy (1b), $T(n)$ must equal $n - 1 - T^r(n - 1)$. The following lemma shows this equality.

Lemma: If m and $m + 1$ are written in a generalized Fibonacci base, then

$$T(m + 1) = m - T^r(m). \tag{3}$$

Proof: Let m be represented in normalized form by

$$a_k a_{k-1} \ldots a_{r+1} a_r a_{r-1} \ldots a_2 a_1 \text{ (k digits).} \tag{4}$$

Writing

$$m = \sum_{i=1}^{k} a_i b_i = \sum_{i=1}^{r} a_i b_i + \sum_{i=r+1}^{k} a_i b_i,$$

the relation (2b) can be used on the second sum to show

$$m = \sum_{i=1}^{r} a_i b_i + \sum_{i=r}^{k-1} a_{i+1} b_i + \sum_{i=1}^{k-r} a_{r+i} b_i.$$

Since the last sum is the value of $T^r(m)$, the right-hand side of (3) equals

$$a_k a_{k-1} \ldots a_{r+2}(a_{r+1} + a_r) a_{r-1} \ldots a_2 a_1 \text{ ($k - 1$ digits).} \tag{5}$$

Note that this number might not be in normalized form.

The representation of $m + 1$ can be found by first noting that at most one of the a_i, $i = 1, 2, \ldots, r$ is a 1 in (4). Three cases to consider are: $a_i = 0$ for all $i = 1, 2, \ldots, r$; $a_i = 1$ for some $i < r$; and $a_r = 1$. In the first case, the representation of $m + 1$ will be like (4) but with $a_1 = 1$. This representation is in normalized form, so $T(m + 1)$ is

$$a_k a_{k-1} \ldots a_{r+2} a_{r+1} a_r \ldots a_3 a_2 \quad \text{($k - 1$ digits).}$$

Since $a_i = 0$, $i = 2, 3, \ldots, r$, this is identical to (5). In the second case, $a_i = 1$ for some $i < r$, and $m + 1$ has a one in the $i + 1^{\text{st}}$ digit. Now $T(m + 1)$ can be found even though $m + 1$ as described is not necessarily normalized. It has representation (5). In the third case, where $a_r = 1$, $m + 1$ has the digits 1 to r all zeros and a one is added to the digit a_{r+1}. Thus $T(m + 1)$ is again as shown in (5). ∎

4. CONCLUDING REMARKS

If $g(n)$, for some large n, has to be calculated, the straightforward recursive method for doing so requires the calculation of all $g(i)$ numbers for $i \leqslant n$.

However, it can be done efficiently by calculating the "radix" b_i numbers using (2), finding the representation of the number as described in ¶2 of §2, using $T(n)$ to obtain the answer, and then converting the answer back to base 10. If

$$Z = \lim_{1 \to \infty} \frac{b_{i+1}}{b_i},$$

then this method takes approximately 3 $\log_Z n$ steps.

A closed form solution for (1) seems impossible to obtain for $r \geqslant 2$, but a good approximation to $g(n)$ is n/Z.

Finally, the theorem can be generalized by noting that the iterated recurrence

$$g(A) = A, \ A \text{ an integer}$$

$$g(n) = n - 1 + A - g^r(n - 1), \ n \geqslant A + 1$$

has solution $g(n) = T(n - A) + A$, for $n \geqslant A$. For $A = 1$, this gives a solution to Hofstadter's original problem.

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DIGIT FUNCTIONS OF INTEGER SEQUENCES

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1. SHIFTED INTEGER SEQUENCES

It was noticed by Benford [1] that the first nonzero digit in certain sets of real numbers is not uniformly distributed among the integers 1 through 9; in fact, the probability that this first, leftmost digit equals β is equal to

$$\log_{10}(1 + \beta^{-1}).$$

He extended the analysis to the frequency of digits beyond the first for numbers obeying a particular probability law: the logarithmic distribution. This phenomenon of nonuniform distribution of digits has generated a considerable mathematical literature, particularly for the first digit, and has been shown to apply to the Fibonacci numbers [2], [3], [4].

The purpose of this paper is to examine the probabilistic structure of the entire set of digits from certain integer sequences. The Fibonacci sequence provides one example.

The essential results are that, for a large class of probability laws, the digits are not equiprobable and their values are correlated; but in the limit, as the ordinal number of the digits goes to infinity, the digit values approach equiprobability and their correlation goes to zero. However, under certain conditions, this limiting behavior does not occur; rather, the nonuniform behavior persists for all digits. In particular, subsequences of the Fibonacci sequence exist which exhibit "persistent Benford" behavior.

Let $\omega = \{a_n\}$ be a sequence of positive integers. Define a shifted sequence $\hat{\omega}$ of rationals $\hat{a}_n \in U_b = [b^{-1}, 1]$, for integer base $b \geqslant 2$, by

$$\hat{a}_n = a_n b^{-v(a_n)}$$

where

$$v(a_n) = [\log_b a_n] + 1$$

is the number of digits in the b-adic representation of a_n, with $[\cdot]$ the greatest integer function.

The asymptotic distribution function (a.d.f.) $g : U_b \to E^1$ is defined for $\hat{\omega}$ as usual by

$$g(x) = \lim_{N \to \infty} \frac{A([b^{-1}, x); N; \hat{\omega})}{N} \tag{1}$$

when this limit exists. Here A is the counting function which records the number among the first N terms of $\hat{\omega}$ that lie in the interval $[b^{-1}, x)$. Note that g is left-continuous.

Theorem 1: If $a_n = \alpha^n$, $\alpha > 1$ and not a rational power of b, then the a.d.f. g of $\{\hat{a}_n\}$ exists and

$$g(x) = 1 + \log_b x. \tag{2}$$

Proof: Since $\hat{a}_n \leqslant x$ if and only if $1 + \log_b \hat{a}_n \leqslant 1 + \log_b x$,

$$g(x) = \lim_{N \to \infty} \frac{A([0,\ 1 + \log_b x);\ N;\ \{1 + \log_b \hat{a}_n\})}{N}$$

if the limit exists. But, since α is not a rational power of b,

$$\{1 + \log_b \hat{a}_n\} = \{1 + n\xi\},\ \xi \text{ irrational},$$

is uniformly distributed mod 1, thus yielding the theorem. ■

It can be shown that (2) is also the a.d.f. of the shifted sequence $\{\hat{F}_n\}$ of Fibonacci numbers F_n because

$$F_n \sim \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^{n+1}$$

(see also [5]). In fact, this a.d.f. holds for any integer sequence defined by a recurrence relation.

An example of an important sequence of integers that does not have an a.d.f. is the sequence of primes. It was shown by Wintner [6] that the limit (1) does not exist in this case. However, the relative logarithmic density does exist [7].

Theorem 2: If $\{\hat{a}_n\}$ has a continuous a.d.f. g, then for every Riemann-integrable function $f : U_b \to E^1$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(\hat{a}_n) = \int_{b^{-1}}^{1} f(x)dg(x).$$

Proof: Immediate from Theorem 7.2 of [8]. ■

Theorem 2 provides us the means to apply the standard facts of probability theory to the study of digit functions of integer sequences.

2. DIGIT FUNCTIONS AND ASYMPTOTIC EQUIPROBABILITY

Let the digit function d_k be defined such that $d_k(x)$ equals the k^{th} digit of x so that

$$x = \sum_{k=1}^{\infty} d_k(x) b^{-k}.$$

Define

$$T[\beta(k)] = \{x \in U_b \,|\, d_k(x) = \beta(k)\} \subseteq U_b,$$

where $\beta(k) \in Z_b = \{0,\ \ldots,\ b - 1\}$. Then, the joint probability p_g that

$$d_{k_1}(x) = \beta(k_1),\ \ldots,\ d_{k_s}(x) = \beta(k_s)$$

is given by the Lebesgue-Stieltjes integral

$$p_g[\beta(k_1),\ \ldots,\ \beta(k_s)] = \int_{b^{-1}}^{1} I_{T[\beta(k_1)]} \cdots I_{T[\beta(k_s)]} dg(x), \tag{3}$$

where I_G is the indicator function of the set $G \subseteq U_b$. Allowing some abuse of notation, the same symbol p_g will be used for all such probability functions, regardless of the dimensionality of the domain. Also, when no confusion will result, the argument k of β will be suppressed.

When g is the logarithmic distribution (2),

$$p_g[\beta(k_1), \ldots, \beta(k_s)] = \sum_{\beta(1)=1}^{b-1} \sum_{\beta(2)=0}^{b-1} \cdots \sum_{\beta(k_s-1)=0}^{b-1} \log_b \left[1 + \frac{b^{-k_s}}{\sum_{m=1}^{k_s} \beta(m) b^{-m}}\right], \tag{4}$$

where the sums over $\beta(k_j)$ for $j = 1, \ldots, s - 1$ are to be excluded.

The relative frequency of digit values will be derived by setting $s = 1$ in (3) and (4). The succeeding section uses $s = 2$ to infer dependence properties between digits.

Definition 1: The a.d.f. g is *asymptotically equiprobable* with respect to b if and only if

$$\lim_{k\to\infty} p_g[\beta(k)] = b^{-1} \text{ for all } \beta \in Z_b. \tag{5}$$

It can be shown that g is asymptotically equiprobable if a density function f exists for g. Furthermore, for a sufficiently smooth a.d.f., such as the logarithmic distribution (2), the rate of approach can also be displayed, as in Theorem 3. When f exists, p_g and p_f will be used interchangeably to denote the function defined in (3), as suits the occasion, with the symbol f being reserved for the density function and g for the a.d.f.

Theorem 3: If $f \in C^2[b^{-1}, 1]$, then

$$p_f[\beta(k)] = b^{-1} + h(\beta)b^{-k} + 0(b^{-2k}) \text{ for all } \beta \in Z_b,$$

where

$$h(\beta) = \frac{1}{2}\left(\frac{2\beta + 1}{b} - 1\right)[f(1) - f(b^{-1})].$$

Proof: Let $q_i[\beta(k)]$ be the b-adic rationals defined by

$$T[\beta(k)] = \bigcup_{i=1}^{M} [q_i[\beta(k)], q_i[\beta(k)] + b^{-k}] \tag{6}$$

with

$$M = \begin{cases} 1, & k = 1, \\ (b - 1)b^{k-2}, & k > 1. \end{cases} \tag{7}$$

Then, writing q_i for $q_i[\beta(k)]$,

$$\begin{aligned} p_f[\beta(k)] &= \int_{b^{-1}}^{1} I_{T[\beta(k)]} f(x)dx = \sum_{i=1}^{M} \int_{q_i}^{q_i+b^{-k}} f(x)dx \\ &= \sum_{i=1}^{M} \frac{1}{2} b^{-k} [f(q_i) + f(q_i + b^{-k})] + 0(b^{-2k}), \end{aligned}$$

where the last equality follows from the trapezoidal rule of integration [9]. The two ordinate sums in this last equation can be converted into integrals, with remainders, by use of the Euler-Maclaurin formula [10]. For $k > 1$,

$$\sum_{i=1}^{M} \frac{1}{2} b^{-k} f(q_i) = \frac{1}{2b} \sum_{i=1}^{M} b^{-k+1} f[b^{-1} + (i-1)b^{-k+1} + \beta b^{-k}]$$
$$= \frac{1}{2b} \int_{b^{-1}}^{1} f(x)dx + \frac{b^{-k+1}}{2b}\left(\frac{\beta}{b} - \frac{1}{2}\right)[f(1) - f(b^{-1})].$$

For $k = 1$, $q_i = \beta b^{-1}$, and the same result is obtained. Calculating a similar expression for the term involving $f(q_i + b^{-k})$ and using the fact that

$$\int_{b^{-1}}^{1} f(x)dx = 1$$

yield the theorem. ■

Using Theorem 3, the expected value of the k^{th} digit of x is

$$E(d_k) = \frac{b-1}{2} + b^{-k}[f(1) - f(b^{-1})]\frac{b^2 - 1}{12} + 0(b^{-2k}),$$

which is approximately $(b - 1)/2$ for large k (as expected!).

To denote the special case of the density function corresponding to the logarithmic distribution (applicable to the Fibonacci sequence), r will be used in place of f; that is,

$$r(x) = \frac{d \log_b(x)}{dx} = \frac{1}{x \ln b},$$

which has been termed the "reciprocal density function" [11]. Theorem 3 applies and gives

$$p_r[\beta(k)] = b^{-1} + h(\beta)b^{-k} + 0(b^{-2k}).$$

Theorem 4:

$$p_r[\beta(k)] = \sum_{i=1}^{M} \log_b\left(1 + \frac{b^{-k}}{q_i}\right),$$

where q_i is defined by (6) and M by (7).

Proof:

$$p_r[\beta(k)] = \int_{b^{-1}}^{1} I_{T[\beta(k)]} r(x)dx = \sum_{i=1}^{M} \int_{q_i}^{q_i + b^{-k}} \frac{dx}{x \ln b}$$
$$= \sum_{i=1}^{M} \frac{1}{\ln b}[\ln(q_i + b^{-k}) - \ln(q_i)],$$

which yields the theorem. ■

For the special case $b = 10$, the relative frequencies, obtained from Theorem 4, of values of the first four digits are given in the accompanying table. The last digit in each entry has been rounded and not truncated. Columns 1 and 2 contain Benford's original results. For subsequent digits, the rapid approach of these data to b^{-1} is readily apparent when plotted as in Figure 1.

Figure 2 provides samples of the convergence of the relative frequency of second-digit values for the Fibonacci sequence to their theoretical limits (cf.

column 2 of the table). The fraction of the first N Fibonacci numbers with second digit equal to β is plotted against N for five values of β.

Probability that Digit k Equals β for the Logarithmic Distribution (Base 10)

β \ k	1	2	3	4
0	–	.11968	.10178	.10018
1	.30103	.11389	.10138	.10014
2	.17609	.10882	.10097	.10010
3	.12494	.10433	.10057	.10006
4	.09691	.10031	.10018	.10002
5	.07918	.09668	.09979	.09998
6	.06695	.09337	.09940	.09994
7	.05799	.09035	.09902	.09990
8	.05115	.08757	.09864	.09986
9	.04576	.08500	.09827	.09982

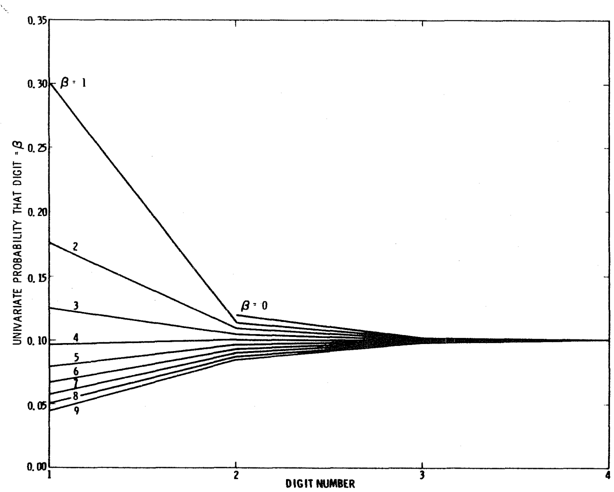

Fig. 1. Approach of Relative Frequency of Digits to b^{-1}. Logarithmic Distribution with $b = 10$

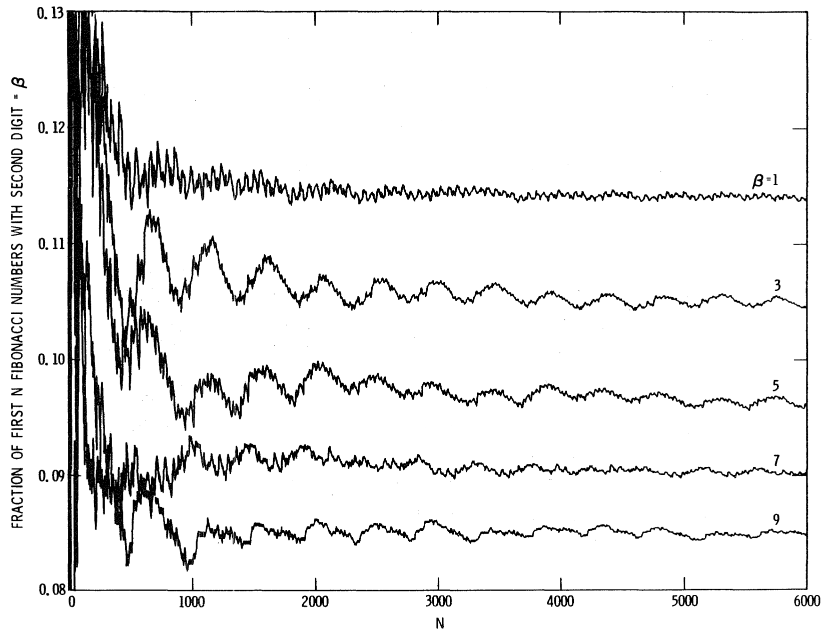

Fig. 2. Convergence of Relative Frequencies to Theoretical Values for Second Digit of Fibonacci Numbers

There exist integer sequences for which asymptotic equiprobability does not hold (for the a.d.f.). For example, Benford's first-digit frequencies can be retained for all subsequent digits for certain subsequences of the Fibonacci sequence, and, in the next theorem, conditions are given for the existence of integer sequences which possess specified digit properties, a reversal of the approach used thus far.

Theorem 5: For each $k = 1, 2, \ldots,$ let t_k be a function from the Cartesian product of Z_b with itself k times to $[0, 1]$ and satisfying the three consistency conditions:

$$t_k[\beta(1), \ldots, \beta(k)] \geqslant 0; \qquad \sum_{\beta(1)\in Z_b} t_1[\beta(1)] = 1;$$

and

$$\sum_{\beta(k+1)\in Z_b} t_{k+1}[\beta(1), \ldots, \beta(k), \beta(k+1)] = t_k[\beta(1), \ldots, \beta(k)].$$

Then, for any integer sequence ω with $\hat{\omega}$ dense in U_b, there exists a subsequence τ with a.d.f. g such that $p_g = t_k$.

Proof: By Billingsley's theorem [12] (a consequence of Kolmogorov's existence theorem), the three conditions on t_k insure the existence of a probability measure μ on the Borel sets of U_b such that, for each k,

$$\mu(T[\beta(1)] \cap \cdots \cap T[\beta(k)]) = t_k[\beta(1), \ldots, \beta(k)]$$

for all $\beta(1), \ldots, \beta(k)$ in Z_b.

Then define a distribution $g : U_b \to [0, 1]$ by $g(x) = \mu[b^{-1}, x)$. By Theorem 4.3 of [8], there exists a sequence $\hat{\sigma}$ in U_b with a.d.f. $g_\sigma = g$.

Let $\hat{\sigma} = \{s_j\}$. Since $\hat{\omega}$ is dense in U_b, there exists a subsequence τ of ω with $\hat{\tau} = \{v_j\}$ such that $v_j = s_j + \Delta_j$, where $\Delta_j \geqslant 0$ and $\lim_{j\to\infty} \Delta_j = 0$.

Since $\Delta_j \geqslant 0$,

$$A([b^{-1}, x); N; \hat{\tau}) \leqslant A([b^{-1}, x); N; \hat{\sigma}). \tag{8}$$

For $\varepsilon > 0$, choose N_0 such that $\Delta_j < \varepsilon$ for $j \geqslant N_0$. Then

$$A([b^{-1}, x_\varepsilon); N; \{s_j\}_{N_0}^\infty) \leqslant A([b^{-1}, x); N; \{v_j\}_{N_0}^\infty), \tag{9}$$

where $x_\varepsilon = \min\{b^{-1}, x - \varepsilon\}$.

By (1), there exists k_N such that

$$\frac{A([b^{-1}, x); N; \{s_j\}_{N_0}^\infty)}{N} = g_\sigma(x) + k_N(x),$$

where $\lim_{N\to\infty} k_N(x) = 0$ for every $x \in U_b$.

Using (8) and (9):

$$g_\sigma(x - \varepsilon) - g_\sigma(x) + k_N(x - \varepsilon) \leqslant \frac{A([b^{-1}, x); N; \{v_j\}_{N_0}^\infty)}{N} - g_\sigma(x) \leqslant k_N(x).$$

Letting N go to ∞ gives

$$g_\sigma(x - \varepsilon) - g_\sigma(x) \leqslant g_\tau(x) - g_\sigma(x) \leqslant 0.$$

Since g_σ is continuous from the left and ε is arbitrary, $g_\tau = g_\sigma = g$, and the theorem is established. ■

Definition 2: An integer sequence ω is said to be *absolutely equiprobable* with respect to b if and only if

$$\lim_{N\to\infty} \frac{A(T[\beta(k)]; N; \hat{\omega})}{N} = \begin{cases} (b-1)^{-1}, & k = 1 \\ b^{-1}, & k > 1 \end{cases}, \text{ for all } \beta \in Z_b.$$

Corollary 5.1: For every $b \geqslant 2$, there exists a subsequence of the Fibonacci numbers that is absolutely equiprobable with respect to b.

Proof: Let $t_k[\beta(1), \ldots, \beta(k)] = (b - 1)^{-1}b^{-k+1}$. Then, by Theorem 5, there exists a subsequence τ of $\{F_n\}$ with a.d.f. g such that $p_g = t_k$ for all k.

Since

$$A(T[\beta(k)]; N; \hat{\tau}) = \sum_i A([q_i, q_i + b^{-k}); N; \hat{\tau}),$$

then

$$\begin{aligned}\lim_{N\to\infty} \frac{A(T[\beta(k)]; N; \hat{\tau})}{N} &= \sum_i [g(q_i + b^{-k}) - g(q_i)] = p_g(\beta(k)) \\ &= \sum{}' p_g(\beta(1), \ldots, \beta(k-1), \beta(k)) \\ &= \sum{}' t_k(\beta(1), \ldots, \beta(k-1), \beta(k)) \\ &= b^{-1}\sum{}' t_{k-1}(\beta(1), \ldots, \beta(k-1)),\end{aligned}$$

where Σ' denotes the sum over all $\beta(j)$ for $j < k$. Then, $k - 2$ applications of the third consistency condition of Theorem 5, followed by use of the second condition, yields the corollary. The case $k = 1$ is trivial. Thus,

$$\lim_{N\to\infty} \frac{A(T[\beta(k)];\ N;\ \hat{\tau})}{N} = b^{-1}$$

as required. ■

Definition 3: An integer sequence ω is said to be a *persistent Benford sequence* with respect to b if and only if

$$\lim_{N\to\infty} \frac{A(T[\beta(k)];\ N;\ \hat{\omega})}{N} = \begin{cases} \log_b(1 + \beta^{-1}(k)), & \beta(k) > 0 \\ 0, & \beta(k) = 0, \end{cases}$$

for all $k \geqslant 1$ and all $\beta(k) \in Z_b$.

Corollary 5.2: For every $b \geqslant 2$, there exists a subsequence of the Fibonacci numbers that is persistent Benford with respect to b.

Proof: A calculation similar to that contained in the proof of Corollary 5.1 serves here and, in fact, for any t_k defined as the product of univariate density functions. ■

3. WEAK DEPENDENCE OF DIGIT FUNCTIONS

Dependence between digit functions is demonstrated by showing that they are correlated random variables.

First, an expression for the bivariate density function is derived.

Theorem 6: If $f \in C^2[b^{-1}, 1]$ and $k_2 > k_1$, then

$$\begin{aligned} p_f[\beta(k_1),\ \beta(k_2)] = b^{-1}p_f[\beta(k_1)] &+ h[\beta(k_2)]b^{-k_2-1} \\ &+ \tilde{h}[\beta(k_1),\ \beta(k_2)]b^{-k_1-k_2} + 0(b^{-\min\{2k_1+k_2,\ 2k_2\}}) \end{aligned}$$

where the function h is defined in Theorem 3 and

$$\begin{aligned} \tilde{h}[\beta(k_1),\ \beta(k_2)] = \frac{b}{4}&\left[B_1\left(\frac{\beta(k_2)}{b}\right) + B_1\left(\frac{\beta(k_2)+1}{b}\right)\right] \\ &\times \left[B_2\left(\frac{\beta(k_1)+1}{b}\right) - B_2\left(\frac{\beta(k_1)}{b}\right)\right][f'(1) - f'(b^{-1})] \end{aligned}$$

with B_1, B_2 Bernoulli polynomials and the prime denoting differentiation.

Proof: Let $u_i(\beta(k_1),\ \beta(k_2))$ be the b-adic rationals defined by

$$T[\beta(k_1)] \cap T[\beta(k_2)] = \bigcup_{i=1}^{ML} [u_i(\beta(k_1),\ \beta(k_2)),\ u_i(\beta(k_1),\ \beta(k_2)) + b^{-k_2}],$$

where M is defined in (7), $L = b^{k_2-k_1-1}$ and $i = (i_1 - 1)L + i_2$. Then, writing u_i for $u_i(\beta(k_1),\ \beta(k\ ))$,

$$p_f[\beta(k_1),\ \beta(k_2)] = \sum_{i_1=1}^{M} \sum_{i_2=1}^{L} \int_{u_i}^{u_i+b^{-k_2}} f(x)dx.$$

Using the trapezoidal rule,

$$p_f[\beta(k_1),\ \beta(k_2)] = \sum_{i_1=1}^{M}\sum_{i_2=1}^{L}\frac{b^{-k_2}}{2}[f(u_i) + f(u_i + b^{-k_2})] + O(b^{-3k_2}).$$

Substituting $u_i = q_{i_1} + (i_2 - 1)b^{-k_2+1} + \beta(k_2)b^{-k_2}$ in this expression and applying the Euler-Maclaurin formula, as in the proof of Theorem 3, to the sums over i_2 gives

$$\begin{aligned}p_f[\beta(k_1),\ \beta(k_2)] &= \frac{1}{b}\sum_{i_1=1}^{M}\int_{q_{i_1}}^{q_{i_1}+b^{-k_1}} f(x)dx + \frac{1}{2b}\sum_{i_1=1}^{M} b^{-k_2+1}\\ &\times\Bigg[\left(B_1\left(\frac{\beta(k_2)}{b}\right) + B_1\left(\frac{\beta(k_2)+1}{b}\right)\right)[f(q_{i_1} + b^{-k_1}) - f(q_{i_1})]\\ &+ \frac{b^{-k_2+1}}{2}\left(B_2\left(\frac{\beta(k_2)}{b}\right) + B_2\left(\frac{\beta(k_2)+1}{b}\right)\right)\\ &\times [f'(q_{i_1} + b^{-k_1}) - f'(q_{i_1})]\Bigg] + O(b^{-k_1-3k_2}).\end{aligned}$$

Recognizing the univariate expression for digit k_1 in the first term and again applying the Euler-Maclaurin formula to each of the four sums inherent in the second term yields

$$\begin{aligned}p_f[\beta(k_1),\ \beta(k_2)] &= \frac{1}{b}p_f[\beta(k_1)] + \frac{1}{2b}b^{-k_2+1}\left[B_1\left(\frac{\beta(k_2)}{b}\right) + B_1\left(\frac{\beta(k_2)+1}{b}\right)\right]\\ &\times\Bigg[\left[B_1\left(\frac{\beta(k_1)+1}{b}\right) - B_1\left(\frac{\beta(k_1)}{b}\right)\right][f(1) - f(b^{-1})]\\ &+ \frac{b^{-k_1+1}}{2}\left[B_2\left(\frac{\beta(k_1)+1}{b}\right) - B_2\left(\frac{\beta(k_1)}{b}\right)\right][f'(1) - f'(b^{-1})]\Bigg]\\ &+ \frac{1}{2b}\frac{b^{-2k_2+2}}{2}\left[B_2\left(\frac{\beta(k_2)}{b}\right) + B_2\left(\frac{\beta(k_2)+1}{b}\right)\right]\\ &\times\Bigg[\left[B_1\left(\frac{\beta(k_1)+1}{b}\right) - B_1\left(\frac{\beta(k_1)}{b}\right)\right][f'(1) - f'(b^{-1})]\\ &+ \frac{b^{-k_1+1}}{2}\left[B_2\left(\frac{\beta(k_1)+1}{b}\right) - B_2\left(\frac{\beta(k_1)}{b}\right)\right][f''(1) - f''(b^{-1})]\Bigg]\\ &+ O(b^{-2k_1-k_2}),\end{aligned}$$

which reduces to the theorem. ∎

Corollary 6.1: If $f \in C^2[b^{-1}, 1]$ and $k_2 > k_1$, then

$$p_f[\beta(k_1),\ \beta(k_2)] = b^{-2} + O(b^{-k_1}).$$

Theorem 7: If $f \in C^2[b^{-1}, 1]$ and $k_2 > k_1$, then

$$\operatorname{cov}_f(d_{k_1},\ d_{k_2}) = c_f b^{-k_1-k_2} + O(b^{-\min\{2k_1+k_2,2k_2\}}),$$

where

$$c_f = \left[\frac{(b-1)(b+1)}{12}\right]^2 [f'(1) - f'(b^{-1}) - (f(1) - f(b^{-1}))^2].$$

Proof: Write

$$\operatorname{cov}_f(d_{k_1}, d_{k_2}) = \sum_{\beta(k_1)=1}^{b-1} \sum_{\beta(k_2)=1}^{b-1} \beta(k_1)\beta(k_2)[p_f(\beta(k_1), \beta(k_2)) - p_f(\beta(k_1))p_f(\beta(k_2))].$$

Using the univariate and bivariate expressions of Theorems 3 and 6, respectively:

$$\begin{aligned}\operatorname{cov}_f(d_{k_1}, d_{k_2}) = b^{-k_1-k_2} &\sum_{\beta(k_1)=1}^{b-1} \sum_{\beta(k_2)=1}^{b-1} \beta(k_1)\beta(k_2)\frac{1}{4}\left(\frac{2\beta(k_2)+1}{b} - 1\right) \\ &\times \left(\frac{2\beta(k_1)+1}{b} - 1\right)\left[[f'(1) - f'(b^{-1})] - [f(1) - f(b^{-1})]^2\right] \\ &+ 0(b^{-\min\{2k_1+k_2,\, 2k_2\}}).\end{aligned}$$

Then, performing the two indicated sums yields the theorem. ■

Corollary 7.1: If $f \in C^2[b^{-1}, 1]$ and $k_2 > k_1$, then

$$\lim_{k_1+k_2\to\infty} \operatorname{cov}_f(d_{k_1}, d_{k_2}) = 0.$$

A second indicator of the weakening of dependence for large-digit numbers exists because it can be shown that the sequence $\{d_k\}$ of digit functions is *-mixing in the sense of Blum, Hanson, and Koopmans [13] when $f \in C^2[b^{-1}, 1]$ and $1/f$ is bounded above.

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WIEFERICHS AND THE PROBLEM $z(p^2) = z(p)$

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INTRODUCTION

1. Let $z(n)$ be the index of the first Fibonacci number divisible by the natural number n. At this writing, there has not been found a prime p whose square enters the Fibonacci sequence at the same index as does p. This does not occur for $p < 10^6$ [2].

The problem is related to the following one. For what relatively prime p, b, is it true that $p^2 | b^{p-1} - 1$? Apparently, this question was first asked by Abel. Dickson [1] devotes a chapter to related results. For $b = 2$, the conforming p^2 values are the well-known Wieferich squares, which enter in the solution of Fermat's Last Theorem. The only two Wieferich squares with $p < 3 \cdot 10^9$ are 1093^2 and 3511^2 [6, p. 229]. These phenomena are rare but, to a degree, predictable. An investigation of this predictability sheds some light on the Fibonacci phenomenon.

2.1 Notation. Define $n \| b^x - 1$ as meaning $n | b^x - 1$, and $n \nmid b^y - 1$ for $y < x$ (i.e., b belongs to the exponent x, modulo n).

2.2 The following are well known. For p prime, $(b, p) = 1$; if $p \| b^\alpha - 1$, then $p | b^\beta - 1$ if and only if $\beta = k \cdot \alpha$. Since $p | b^{p-1} - 1$ (Fermat), it follows that $\alpha | p - 1$. For q prime, $(b, q) = 1$; if $q \| b^\gamma - 1$, then $pq \| b^{\text{lcm}(\alpha,\gamma)} - 1$. The multiplicative properties are similar to those of the Euler ϕ function. Indeed, $p^2 | b^{p\alpha} - 1$ as $\phi(p^2) = p\phi(p)$. However, here we have a deviation: $p^2 \| b^{p\alpha} - 1$, unless $p^2 \| b^\alpha - 1$. (In terms of decimals of reciprocals of integers, the first prime > 3, such that $1/p^2$ has a period the same length as $1/p$, i.e., $p^2 | 10^{p-1}$, is 487. Its period is of length 486.) It can be shown that this deviation occurs if and only if $p^2 | b^{p-1} - 1$. If such is the case, and imitating Shanks's flair for coinage of such terms, we say p is a wieferich, modulo b.

2.3 Consider the solutions to $x^{p-1} \equiv 1 \pmod{p^2}$. Gauss [3, art. 85] assures us that there are $p - 1$ distinct solutions, x, between 1 and $p^2 - 1$.

For each b, $1 \leqslant b < p$, there is a distinct k such that

$$(b + kp)^{p-1} \equiv 1 \pmod{p^2}.$$

These provide the $p - 1$ solutions:

$$(b + kp)^{p-1} - 1 \equiv b^{p-1} - 1 + (p - 1)b^{p-2}kp \pmod{p^2}$$

and

$$\left(\frac{b^{p-1} - 1}{p}\right) - b^{-1}k \equiv 0 \pmod{p}, \text{ yielding } k \equiv b\left(\frac{b^{p-1} - 1}{p}\right) \pmod{p}.$$

If x is a solution, so too is $p^2 - x$. $x = 1$ is always a solution; therefore, $(p - 3)/2$ solutions are scattered from $x = 2$ to $x = (p^2 - 1)/2$. If randomly distributed, the probability that a particular $x = b$ is a solution is

$(p - 3)/(p^2 - 3)$. Holding b fixed and letting p range, the expected number of solutions encountered $\leqslant P$ is $\Sigma_p^P(p - 3)/(p^2 - 3)$. Since the series is divergent ($\Sigma_{p \leqslant x} 1/p = \ln\ln x + c + O(1/\log x)$ [5, Th. 50, p. 120]), but diverges slowly, the relative scarcity of these wieferichs, modulo b, is not surprising.

THE MAIN THEOREMS

3.1 In [4], information about the entry points of the Fibonacci sequence was obtained by imbedding the sequence in a family of sequences with similar properties. Specifically, let $\{\Gamma_n\}$ be a linear recursive sequence with n^{th} term given by

$$\Gamma_n(c, q) = \frac{\Psi^n - \overline{\Psi}^n}{R} = \begin{cases} \dfrac{(c + \sqrt{q})^n - (c - \sqrt{q})^n}{2\sqrt{q}} & \text{for } q \not\equiv c^2 \pmod 4 \\[2ex] \dfrac{\left(\dfrac{c + \sqrt{q}}{2}\right)^n - \left(\dfrac{c - \sqrt{q}}{2}\right)^n}{\sqrt{q}} & \text{for } q \equiv c^2 \pmod 4 \end{cases}$$

yielding the sequences defined by

$$\Gamma_n = \begin{cases} 2c\Gamma_{n-1} + (q - c^2)\Gamma_{n-2} \\ c\Gamma_{n-1} + \dfrac{q - c^2}{4}\Gamma_{n-2} \end{cases}$$

with initial values 1, $2c$ or 1, c. For $c = 1$, $q = 5$, we have the Fibonacci sequence.

Let $e = (q/p)$ be the Legendre symbol.

With $q \not\equiv c^2$, $c \not\equiv 0$, $q \not\equiv 0 \pmod p$, we have $p \mid \Gamma_{p-e}$.

If $p \| \Gamma_\alpha$, then $p \mid \Gamma_\beta$ if and only if $\beta = k\alpha$. Also, $\alpha \mid p - e$, [4].

3.2 *Theorem:* Let $p \| \Gamma_\alpha$. Then, $p^2 \| \Gamma_\alpha$ if and only if $p^2 \mid \Gamma_{p-e}$ (paralleling the result mentioned in ¶2.2). Proof is by means of Lemmas 3.2.1, 3.2.2, and 3.2.3 below.

3.2.1 *Lemma:* If $p^2 \| \Gamma_\alpha$, then $p^2 \mid \Gamma_x$ if and only if x is a multiple of α. Consider:

$$\Gamma_{k\alpha} = \frac{\Psi^{k\alpha} - \overline{\Psi}^{k\alpha}}{R} = \left(\frac{\Psi^\alpha - \overline{\Psi}^\alpha}{R}\right)(\Psi^{(k-1)\alpha} + \Psi^{(k-2)\alpha}\overline{\Psi} + \cdots + \overline{\Psi}^{(k-1)\alpha}).$$

Since $p^2 \left| \dfrac{\Psi^\alpha - \overline{\Psi}^\alpha}{R} \right.$, and $\Psi^n + \overline{\Psi}^n$ and $(\Psi\overline{\Psi})^n$ are integers, it follows that $p^2 \mid \Gamma_{k\alpha}$.

Suppose $p^2 \mid \Gamma_{k\alpha+r}$, $0 < r < \alpha$, and that this is the smallest such index not a multiple of α. Dividing $\Gamma_{k\alpha+r}$ by $\Gamma_{k\alpha}$, we obtain

$$\frac{\Psi^{k\alpha+r} - \overline{\Psi}^{k\alpha+r}}{R} = \Psi^r\left(\frac{\Psi^{k\alpha} - \overline{\Psi}^{k\alpha}}{R}\right) + \overline{\Psi}^{k\alpha}\left(\frac{\Psi^r - \overline{\Psi}^r}{R}\right)$$

or

$$\Gamma_{k\alpha+r} = \Psi^r\Gamma_{k\alpha} + \overline{\Psi}^{k\alpha}\Gamma_r.$$

From 3.1, $q \not\equiv c^2 \pmod{p}$, so $p \nmid \overline{\Psi}^{k\alpha}$ and, thus, $p^2 | \Gamma_r$. But this contradicts the hypothesis that α was the smallest such index.

3.2.2 ***Lemma:*** If $p \| \Gamma_\alpha$, then $p^2 | \Gamma_{p\alpha}$. Consider:

$$(\Gamma_\alpha)^p = \left(\frac{\Psi^\alpha - \overline{\Psi}^\alpha}{R}\right)^p.$$

Noting that R^{p-1} is an integer,

$$R^{p-1}(\Gamma_\alpha)^p = \frac{\Psi^{p\alpha} - \overline{\Psi}^{p\alpha}}{R} + \sum_{s=1}^{\frac{p-1}{2}} (-1)^s (\Psi\overline{\Psi})^{s\alpha} \binom{p}{s} \left[\frac{\Psi^{(p-2s)\alpha} - \overline{\Psi}^{(p-2s)\alpha}}{R}\right].$$

p^2 divides all terms but $\dfrac{\Psi^{p\alpha} - \overline{\Psi}^{p\alpha}}{R} = \Gamma_{p\alpha}$, so it must divide it also.

3.2.3 ***Lemma:*** If $p \| \Gamma_\alpha$ but $p^2 \| \Gamma_{t\alpha}$, $1 < t < p$, then, since $p^2 | \Gamma_{kt\alpha}$ (from 3.2.1) and $p^2 | \Gamma_{p\alpha}$ (from 3.2.2), it follows that $t | p$; but p is prime, so

$$p^2 \| \Gamma_\alpha \quad \text{or} \quad p^2 \| \Gamma_{p\alpha}.$$

In the former case, $p | \Gamma_{p\pm1}$; in the latter, since $p \pm 1$ is not a multiple of $p\alpha$, $p^2 \nmid \Gamma_{p\pm1}$. This establishes the result.

3.3 We next consider Ψ, $\overline{\Psi}$ with $c = c_1 + \xi p$ and $q = q_1 + \zeta p$, expand and reduce $\dfrac{\Psi^{p\pm1} - \overline{\Psi}^{p\pm1}}{R} \pmod{p^2}$. The result is linear in ξ and ζ. Thus, for given c, q, for $\dfrac{\Psi^{p\pm1} - \overline{\Psi}^{p\pm1}}{R} \equiv 0 \pmod{p^2}$, each ξ, $0 \leqslant \xi < p$, generates one ζ, $0 \leqslant \zeta < p$.

Fix c. Let q range from 1 to $(p - 1)$. One of these pairs (c, q), that with $q \equiv c^2 \pmod{p}$, will produce a sequence not containing an entry point for p [4]. The other $p - 2$ pairs will each generate a solution $\xi = 0$, $\zeta = \theta$ yielding a sequence with Ψ associated with $c + \sqrt{q + \theta p}$ such that $z(p) = z(p^2)$. When $c = 1$, $q = 5$, we have the Fibonacci sequence. If the solutions θ are randomly distributed over $0, 1, 2, \ldots, p - 1$, the probability $\theta = 0$ is $1/p$. The expected number of such phenomena, $p \leqslant P$, is $\Sigma_p^P 1/p$, whose series diverges (§2.3). On the basis of random distribution, the phenomenon should occur before $p > 10^6$. On the other hand, $\ln \ln 10^6$ is not yet 3, perhaps not too wide a miss?

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MULTIPARAMETER STIRLING AND C-NUMBERS: RECURRENCES AND APPLICATIONS

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1. INTRODUCTION

Stirling numbers of the first and second kind are less known among statisticians than among people who deal with combinatorics or finite differences. Only recently have they made their appearance in distribution theory and statistics. They emerge in the distribution of a sum of zero-truncated classical discrete distributions: those of the second kind, $S(m, n)$, in the case of a Poisson distribution truncated away from zero, Tate & Goen [13], Cacoullos [2], the signless (absolute-value) Stirling numbers of the first kind, $|s(m, n)|$, in the logarithmic series distribution, Patil [9]. In general, such distributional problems are essential in the construction of minimum variance unbiased estimators (mvue) for parametric functions of a left-truncated power series distribution (PSD).

Analogous considerations for binomial and negative binomial distributions truncated away from zero motivated the introduction of a new kind of numbers, called C-numbers by Cacoullos & Charalambides [5]. These three-parameter C-numbers, $C(m, n, k)$, were further studied by Charalambides [8], who gave the representation

$$C(m, n, k) = \sum_{r=n}^{m} k^r s(m, r)S(r, n)$$

in terms of Stirling numbers of the first kind, $s(m, r)$, and the second kind, $S(r, n)$. Interestingly enough, this representation in a disguised form was, in effect, used by Shumway & Gurland [11] to tabulate C-numbers, involved in the calculation of Poisson-binomial probabilities.

The so-called generalized Stirling and C-numbers emerged as a natural extension of the corresponding simple ones in the study of the mvue problem for a PSD truncated on the left at an arbitrary (known or unknown) point (Charalambides [7]). It should be mentioned that, in particular, the generalized Stirling numbers of the second kind were independently rediscovered and tabulated by Sobel *et al.* [12], in connection with the Incomplete Type I-Dirichelt integral.

The multiparameter Stirling and C-numbers are the analogues of generalized Stirling and C-numbers in a multi-sample situation where the underlying PSD is multiply truncated on the left (Cacoullos [3], [4]).

Recurrence relations for ratios of Stirling and C-numbers are necessary, because the mvue of certain parametric functions of left-truncated logarithmic series, Poisson, binomial and negative binomial distributions are expressed in terms of such ratios. These recurrences bypass the computational difficulties which come from the fact that the numbers themselves (but not the ratios of interest) grow very fast with increasing arguments. Recurrences for ratios of simple Stirling numbers of the second kind were developed by Berg [1].

The main purpose of this paper is to provide recurrences for certain ratios of multiparameter Stirling and C-numbers, thus unifying several special results, including those of Berg [1]. For the development of the topic, we found the use of exponential generating functions (egf) most appropriate both for introducing the numbers themselves and for deriving recurrences. Without claiming completeness, we included certain basic recurrences. As observed elsewhere, Cacoullos [3] and [4], it is emphasized here, once more, that in the study of PSDs the egf approach is the one suggested by the probability function itself in its truncated form. Also, we found it appropriate to include certain asymptotic relations between Stirling and C-numbers, which reflect corresponding relations between binomial and Poisson distributions or logarithmic series and negative binomial distributions.

A typical result, which involves ratios considered here, is the following: Let x_{ij}, $j = 1, \ldots, n_i$, be a random sample from a left-truncated one-parameter PSD distribution with p.f.

$$p(x; \theta) = \frac{a_i(x)\theta^x}{f_i(\theta, r_i)}, \quad x = r_i, r_i + 1, \ldots, \tag{1.1}$$

where

$$f_i(\theta, r_i) = \sum_{x=r_i}^{\infty} a_i(x)\theta^x, \quad i = 1, \ldots, k.$$

If the truncation point $\underset{\sim}{r} = (r_1, \ldots, r_k)$ is known and $a_i(x) > 0$ for every $x > r_i$, $i = 1, \ldots, k$, then, according to Cacoullos [4], for every $j = 1, 2, \ldots$, θ_j is estimable and its (unique) mvue, based on all k independent samples $\{x_{ij}\}$, is given by

$$\hat{\theta}_j(m) = (m)_j \frac{a(m - j; \underset{\sim}{n}, \underset{\sim}{r})}{a(m; \underset{\sim}{n}, \underset{\sim}{r})}, \tag{1.2}$$

where $\underset{\sim}{n} = (n_1, \ldots, n_k)$, $\underset{\sim}{r} = (r_1, \ldots, r_k)$, $(m)_j = m(m-1)\cdots(m-j+1)$ and

$$a(m; \underset{\sim}{n}, \underset{\sim}{r}) = \frac{m!}{n_1! \ldots n_k!} \sum_m \prod_{i=1}^{k} \prod_{j=1}^{n_i} a_i(x_{ij}), \tag{1.3}$$

where the summation extends over all ordered N-tuples $(N = n_1 + \cdots + n_k)$ of integers x_{ij} satisfying $x_{ij} \geqslant r_i$,

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} = m.$$

In the cases of interest (Poisson, binomial, and so on), the numbers (integers) $a(m; \underset{\sim}{n}, \underset{\sim}{r})$ turn out to be Stirling or C-numbers, depending on the series function f_i in (1.1), which at the same time suggests the corresponding egf of these numbers.

2. MULTIPARAMETER STIRLING NUMBERS OF THE FIRST KIND: DEFINITION-GENERAL PROPERTIES

Let $r_1, \ldots, r_k$ and $n_1, \ldots, n_k$ be nonnegative integers $(k \geqslant 1)$. The multiparameter Stirling numbers of the first kind with parameters $\underset{\sim}{r} = (r_1, r_2, \ldots, r_k)$ and $\underset{\sim}{n} = (n_1, n_2, \ldots, n_k)$, to be denoted by $s(m; \underset{\sim}{n}, \underset{\sim}{r})$, can be defined (cf. Cacoullos [3]) by the egf,

$$g_{\underset{\sim}{n}}(t;\ \underset{\sim}{r}) = \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty} s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})t^m/m!$$
$$= \prod_{i=1}^{k} \frac{1}{n_i!}\left[\log(1 + t) - \sum_{j=1}^{r_i-1} (-1)^{j-1}\frac{t^j}{j}\right]^{n_i}, \tag{2.1}$$

where we set $m = \underset{\sim}{r}'\underset{\sim}{n} = r_1n_1 + \cdots + r_kn_k$.

The special case $k = 1$, $r_1 = r$, $n_1 = n$ yields the generalized Stirling numbers of the first kind, $s(m;\ n,\ r)$, defined by Charalambides [6], while $k = 1$, $r = 1$ gives the simple Stirling numbers of the first kind, $s(m,\ n)$. Propositions 2.1 - 2.3 summarize basic properties and recurrences for $s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})$ and facilitate their computation.

Remark 2.1: In the sequel, in order to avoid unnecessary complications in the recurrences, we assume that all $n_i > 0$, some n_i, say ν, are zero, then the parameter k becomes $k' = k - \nu$ and the necessary modifications are obvious.

Proposition 2.1: The multiparameter Stirling numbers of the first kind $s(m;\underset{\sim}{n},\underset{\sim}{r})$ have the following representation

$$s(m;\ \underset{\sim}{n},\ r) = (-1)^{m-N}\frac{m!}{n_1!\ \ldots\ n_k!}\sum_{m}\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{1}{x_{ij}}, \tag{2.2}$$

where $N = n_1 + \cdots + n_k$ and the summation extends over all ordered N-tuples of integers x_{ij} satisfying the relations

$$x_{ij} \geqslant r_i,\ i = 1,\ \ldots,\ k \quad \text{and} \quad \sum_{i=1}^{k}\sum_{j=1}^{n_i} x_{ij} = m.$$

Proof: We have

$$\gamma(t,\ r_i) = \log(1 + t) - \sum_{k=1}^{r_i-1}(-1)^{k-1}\frac{t^k}{k}$$
$$= \sum_{k=r_i}^{\infty}(-1)^{k-1}\frac{t^k}{k},\ i = 1,\ \ldots,\ k. \tag{2.3}$$

Forming the Cauchy product of series, we find, by virtue of (2.1),

$$g_{\underset{\sim}{n}}(t;\ \underset{\sim}{r})\prod_{i=1}^{k}n_i! = \prod_{i=1}^{k}[\gamma(t,\ r_i)]^{n_i}$$
$$= \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty}(-1)^{m-N}t^m\sum_{m}\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{1}{x_{ij}}, \tag{2.4}$$

where $\sum_m$ has the same meaning as above. Comparing (2.4) with (2.1), we get (2.2).

To obtain recurrence relations, we make use of the easily verified difference/differential equation, satisfied by the egf $g_{\underset{\sim}{n}}(t;\ \underset{\sim}{r})$ in (2.1), namely,

$$(1 + t)\frac{d}{dt}g_{\underset{\sim}{n}}(t;\ \underset{\sim}{r}) = \sum_{i=1}^{k}(-1)^{r_i-1}t^{r_i-1}g_{\underset{\sim}{n}-\underset{\sim}{e}_i}(t;\ \underset{\sim}{r}), \tag{2.5}$$

where $\underset{\sim}{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, i.e., a k-component vector with zero components except the i^{th} component, which is equal to 1.

Proposition 2.2: $(m; \underset{\sim}{n})$-wise relations: The numbers $s(m; \underset{\sim}{n}, \underset{\sim}{r})$ satisfy the recurrence relation

$$s(m+1; \underset{\sim}{n}, \underset{\sim}{r}) + ms(m; \underset{\sim}{n}, \underset{\sim}{r})$$
$$= \sum_{i=1}^{k} (-1)^{r_i - 1} (m)_{r_i - 1}\, s(m - r_i + 1; \underset{\sim}{n} - \underset{\sim}{e}_i, \underset{\sim}{r}) \tag{2.6}$$

with initial conditions

$$s(0; 0, \underset{\sim}{r}) = 1,\ s(0; \underset{\sim}{n}, \underset{\sim}{r}) = 0 \text{ whenever } \sum_{i=1}^{k} r_i n_i > 0,$$
$$s(m; \underset{\sim}{n}, \underset{\sim}{r}) = 0 \text{ if } m < \underset{\sim}{r}'\underset{\sim}{n}.$$

Proof: Equation (2.5) by virtue of (2.1) can be written as

$$(1+t) \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty} s(m; \underset{\sim}{n}, \underset{\sim}{r}) \frac{t^{m-1}}{(m-1)!}$$
$$= \sum_{i=1}^{k} \sum_{m=\underset{\sim}{r}'\underset{\sim}{n} - r_i}^{\infty} (-1)^{r_i - 1} s(m; \underset{\sim}{n} - \underset{\sim}{e}_i, \underset{\sim}{r}) \frac{t^{m+r_i-1}}{m!}. \tag{2.7}$$

Equating the coefficients of $t^m/m!$ in (2.7) yields (2.6). Note that equation (2.6) for $k = 1$, $r_1 = 1$ gives the well-known recurrence for the simple Stirling numbers of the first kind

$$s(m+1, n) = s(m, n-1) - ms(m, n). \tag{2.8}$$

Proposition 2.3: $(m; \underset{\sim}{n}, \underset{\sim}{r})$-wise relations: The numbers $s(m; \underset{\sim}{n}, \underset{\sim}{r})$ satisfy

$$s(m; \underset{\sim}{n}, \underset{\sim}{r} + \underset{\sim}{e}_i) = \sum_{j=0}^{n_i} (-1)^{jr_i} \frac{(m)_{jr}}{j!\,(r_i)^j} s(m - jr_i; \underset{\sim}{n} - j\underset{\sim}{e}_i, \underset{\sim}{r}),\ i = 1, \ldots, k. \tag{2.9}$$

Proof: We have, using also (2.3),

$$g_{\underset{\sim}{n}}(t; \underset{\sim}{r} + \underset{\sim}{e}_i) = \frac{1}{n_i!}\left[\gamma(t; \underset{\sim}{r}) + (-1)^{r_i}\frac{t^{r_i}}{r_i}\right]^{n_i} \prod_{\substack{j \neq i\\ j=1}}^{k} \frac{1}{n_j!}[\gamma(t, r_j)]^{n_j}; \tag{2.10}$$

and using the binomial expansion

$$\left[\gamma(t, r_i) + (-1)^{r_i}\frac{t^{r_i}}{r_i}\right]^{n_i} = \sum_{j=0}^{n_i} \binom{n_i}{j} (n_i - j)!\, g_{n_j - j}(t, r_i)(-1)^{jr_i} \frac{t^{jr_i}}{r_i^j}, \tag{2.11}$$

we can write (2.10) as

$$\sum_{m=\underset{\sim}{r}'\underset{\sim}{n}+n_i}^{\infty} s(m; \underset{\sim}{n}, \underset{\sim}{r} + \underset{\sim}{e}_i)\frac{t^m}{m!} = \sum_{j=0}^{n_i} \frac{(-1)^{jr_i}}{j!\,r_i^j} \sum_{m=\underset{\sim}{r}'\underset{\sim}{n} - jr_i} s(m; \underset{\sim}{n} - j\underset{\sim}{e}_i, \underset{\sim}{r}) \frac{t^{m+jr_i}}{m!}. \tag{2.12}$$

Hence, equating the coefficients of $t^m/m!$, we obtain (2.9).

Signless Multiparameter Stirling Numbers

From the recurrence relation (2.6), it follows that the numbers $s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})$ are integers. Moreover, from the representation in (2.2), we conclude that $s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})$ is an integer with sign $(-1)^{N-m}$, where $N = n_1 + \cdots + n_k$. Therefore, if we multiply (2.6) by $(-1)^{m-N+1}$, we obtain

$$|s(m+1;\ \underset{\sim}{n},\ \underset{\sim}{r})| = m|s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})| + \sum_{i=1}^{k} (m)_{r_i - 1} |s(m - r_i + 1;\ \underset{\sim}{n} - \underset{\sim}{e}_i,\ \underset{\sim}{r})|. \tag{2.13}$$

We call $|s(m;\ n,\ r)|$ the signless (positive) multiparameter (k-parameter)

Stirling Number of the First Kind. We will show

Proposition 2.4: The egf of $|s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})|$ is given by

$$g^*_{\underset{\sim}{n}}(t;\ \underset{\sim}{r}) = \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty} |s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})| \frac{t^m}{m!} = \prod_{i=1}^{k} \frac{1}{n_i!} \left[-\log(1-t) - \sum_{j=1}^{r_i - 1} \frac{t^j}{j} \right]^{n_i}. \tag{2.14}$$

Proof: From the difference equation (2.13), it is easily verified that the egf $g^*_{\underset{\sim}{n}}(t;\ \underset{\sim}{r})$ satisfies the difference/differential equation

$$(1-t)\frac{d}{dt} h_{\underset{\sim}{n}}(t;\ \underset{\sim}{r}) = \sum_{i=1}^{k} t^{r_i - 1} g^*_{\underset{\sim}{n} - \underset{\sim}{e}_i}(t;\ \underset{\sim}{r}), \tag{2.15}$$

which, in turn, yields (2.14).

Alternatively, (2.14) leads to the representation of $|s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})|$ as obtained from (2.2).

3. RATIOS OF MULTIPARAMETER STIRLING NUMBERS OF THE FIRST KIND

We define, as a ratio of multiparameter Stirling numbers of the first kind with respect to argument m, the function

$$R_1(m;\ \underset{\sim}{n},\ \underset{\sim}{r}) = \frac{s(m+1;\ \underset{\sim}{n},\ \underset{\sim}{r})}{s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})}. \tag{3.1}$$

Ratios with respect to the arguments $n_i, r_i, i = 1, \ldots, k$, can also be defined The main reason for considering ratios with respect to m is seen from (1.1), which actually involves reciprocals of R_1, when we are concerned with the parameter of a logarithmic series distribution.

Proposition 3.1: A recurrence relation for the ratio $R_1(m;\ \underset{\sim}{n},\ \underset{\sim}{r})$, independent of the multiparameter Stirling numbers of the first kind, is given by

$$R_1(m;\ \underset{\sim}{n},\ \underset{\sim}{r}) + m = \frac{\displaystyle\sum_{j=1}^{k} \frac{(m)_{r_j - 1} r_j n_j}{(\underset{\sim}{r}'\underset{\sim}{n})_{r_j}} \prod_{i=1}^{m+1-\underset{\sim}{r}'\underset{\sim}{n}} R_1(m - r_j + 1 - i;\ \underset{\sim}{n} - \underset{\sim}{e}_j,\ \underset{\sim}{r})}{\displaystyle\prod_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_1(m - i;\ \underset{\sim}{n},\ \underset{\sim}{r})} \tag{3.2}$$

for $\underset{\sim}{n} > \underset{\sim}{1}$ and $m > \underset{\sim}{r}'\underset{\sim}{n}$, with the boundary conditions

$$R_1(m, 1, r) = -m \tag{3.3}$$

and

$$R_1(\underset{\sim}{r}'\underset{\sim}{n}; \underset{\sim}{n}, \underset{\sim}{r}) = (-1)(\underset{\sim}{r}'\underset{\sim}{n} + 1)\sum_{j=1}^{k}\frac{r_j n_j}{(r_j + 1)} \tag{3.4}$$

Proof: Using equation (3.1), it can easily be seen that

$$\prod_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_1(m - i; \underset{\sim}{n}, \underset{\sim}{r}) = \frac{s(m; \underset{\sim}{n}, \underset{\sim}{r})}{s(\underset{\sim}{r}'\underset{\sim}{n}; \underset{\sim}{n}, \underset{\sim}{r})}. \tag{3.5}$$

But equation (2.2), for $m = \underset{\sim}{r}'\underset{\sim}{n}$, $m_{i1} = m_{i2} = \cdots = m_{in_i} = r_i$, becomes

$$s(\underset{\sim}{r}'\underset{\sim}{n}; \underset{\sim}{n}, \underset{\sim}{r}) = (-1)^{\underset{\sim}{r}'\underset{\sim}{n}-N}\frac{(\underset{\sim}{r}'\underset{\sim}{n})!}{\prod_{i=1}^{k} n_i!\prod_{i=1}^{k} r_i^{n_i}}. \tag{3.6}$$

Consequently, equation (3.5) becomes

$$s(m; \underset{\sim}{n}, \underset{\sim}{r}) = \frac{(-1)^{\underset{\sim}{r}'\underset{\sim}{n}-N}(\underset{\sim}{r}'\underset{\sim}{n})!\prod_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_1(m - i; \underset{\sim}{n}, \underset{\sim}{r})}{\prod_{i=1}^{k} n_i!\prod_{i=1}^{k} r_i^{n_i}} \tag{3.7}$$

From equations (2.6) and (3.1) we have

$$R_1(m; \underset{\sim}{n}, \underset{\sim}{r}) + m = \frac{\sum_{j=1}^{k}(-1)^{r_j - 1}(m)_{r_j - 1}s(m - r_j + 1; \underset{\sim}{n} - \underset{\sim}{e}_j, \underset{\sim}{r})}{s(m; \underset{\sim}{n}, \underset{\sim}{r})} \tag{3.8}$$

and substituting for $s(m - r_j + 1; \underset{\sim}{n} - \underset{\sim}{e}_j, \underset{\sim}{r})$ and $s(m; \underset{\sim}{n}, \underset{\sim}{r})$ from (3.7) yields (3.2). By definition,

$$R_1(\underset{\sim}{r}'\underset{\sim}{n}; \underset{\sim}{n}, \underset{\sim}{r}) = \frac{s(\underset{\sim}{r}'\underset{\sim}{n} + 1; \underset{\sim}{n}, \underset{\sim}{r})}{s(\underset{\sim}{r}'\underset{\sim}{n}; \underset{\sim}{n}, \underset{\sim}{r})}, \tag{3.9}$$

and since equation (2.2), for $m = \underset{\sim}{r}'\underset{\sim}{n} + 1$, becomes

$$s(\underset{\sim}{r}'\underset{\sim}{n} + 1; \underset{\sim}{n}, \underset{\sim}{r}) = (-1)^{\underset{\sim}{r}'\underset{\sim}{n}+1-N}\frac{(\underset{\sim}{r}'\underset{\sim}{n} + 1)!}{\prod_{\substack{i=1\\ j\neq i}}^{k} n_i!}\sum_{i=1}^{k}\frac{n_j}{r_j^{n_j - 1}(r_j + 1)\prod_{i\neq 1}^{k} r_i^{n_i}}, \tag{3.10}$$

by using equation (3.6), the required formula (3.4) is easily obtained.

The special case $k = 1$ yields

Proposition 3.2: A recurrence relation for the ratio $R_1(m, n, r)$, independent of the generalized Stirling numbers of the first kind, is given by

$$R_1(m, n, r) + m = \frac{\dfrac{rn(m)_{r-1}}{(rn)_r} \displaystyle\prod_{i=1}^{m+1-rn} R_1(m - r + 1 - i, n - 1, r)}{\displaystyle\prod_{i=1}^{m-rn} R_1(m - i, n, r)} \tag{3.11}$$

for $n > 1$ and $m > rn$, with

$$R_1(m, 1, r) = -m \tag{3.12}$$

$$R_1(rn, n, r) = -\frac{rn(rn + 1)}{r + 1} \tag{3.13}$$

Also, for $k = 1$, $r = 1$, we obtain

Proposition 3.3: A recurrence relation for the ratio $R_1(m, n)$, independent of the simple Stirling numbers of the first kind, is given by

$$R_1(m, n) + m = \frac{\displaystyle\prod_{i=1}^{m+1-n} R_1(m - i, n - 1)}{\displaystyle\prod_{i=1}^{m-n} R_1(m - i, n)} \tag{3.14}$$

for $n > 1$ and $m > n$, with

$$R_1(m, 1) = -m \tag{3.15}$$

$$R_1(n, n) = -n(n + 1)/2 \tag{3.16}$$

Proposition 3.4: An alternative recurrence relation for the ratio $R_1(m, n, r)$ is given by

$$R_1(m, n, r) + m = \frac{m}{m - r + 1} \frac{[R_1(m-1, n, r) + m - 1]R_1(m-r, n-1, r)}{R_1(m - 1, n, r)} \tag{3.17}$$

for $n > 1$ and $m > rn$. $R_1(m, 1, r)$ and $R_1(rn, n, r)$ are given by (3.12) and (3.13), respectively.

Proof: Using equation (2.6) with $k = 1$, we have

$$R_1(m, n, r) + m = \frac{(-1)^{r-1}(m)_{r-1}s(m - r + 1, n - 1, r)}{s(m, n, r)} \tag{3.18}$$

from which equation (3.17) can easily be derived.

Applying Proposition 3.4 with $r = 1$ gives

Proposition 3.5: An alternative recurrence relation for the ratio $R_1(m, n)$ is given by

$$R_1(m, n) + m = \frac{[R_1(m - 1, n) + m - 1]R_1(m - 1, n - 1)}{R_1(m - 1, n)} \tag{3.19}$$

for $n > 1$ and $m > n$.

4. MULTIPARAMETER STIRLING NUMBERS OF THE SECOND KIND

The multiparameter Stirling numbers of the second kind $S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ are defined by their egf

$$f_{\underset{\sim}{n}}(t;\, \underset{\sim}{r}) = \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty} S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})\frac{t^m}{m!} = \prod_{i=1}^{k} \frac{1}{n_i!}\left[e^t - \sum_{j=0}^{r_i-1} \frac{t^j}{j!}\right]^{n_i}. \tag{4.1}$$

Taking $k = 1$, $r_1 = r$ gives the generalized Stirling numbers of the second kind, $S(m,\, n,\, r)$ (Charalambides [6]; taking $k = 1$, $r = 1$ defines the simple Sterling numbers $S(m,\, n)$. The following properties of $S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ can easily be established (cf. §2).

a) They have the representation

$$S(m;\, \underset{\sim}{n},\, \underset{\sim}{r}) = \frac{m!}{n_1!\, \ldots\, n_k!}\sum_{m} \prod_{i=1}^{k} \prod_{j=1}^{n_i} \frac{1}{x_{ij}!}, \tag{4.2}$$

where the summation extends over all ordered N-tuples ($N = n_1 + \cdots + n_k$) of integers x_{ij} satisfying

$$x_{ij} \geqslant r_i,\ i = 1,\, \ldots,\, k \quad \text{and} \quad \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} = m.$$

b) They satisfy the following recurrence relations,

$$S(m + 1;\, \underset{\sim}{n},\, \underset{\sim}{r}) = NS(m;\, \underset{\sim}{n},\, \underset{\sim}{r}) + \sum_{i=1}^{k} \binom{m}{r_i - 1} S(m - r_i + 1;\, \underset{\sim}{n} - \underset{\sim}{e}_i,\, \underset{\sim}{r}) \tag{4.3}$$

and

$$S(m;\, \underset{\sim}{n},\, \underset{\sim}{r} + \underset{\sim}{e}_i) = \sum_{j=0}^{n_i} (-1)^j \frac{(m)_{jr_i}}{j!\,(r_i!)^j} S(m - jr_i;\, \underset{\sim}{n} - j\underset{\sim}{e}_i,\, \underset{\sim}{r}), \tag{4.4}$$

with initial conditions

$$S(0;\, 0,\, \underset{\sim}{r}) = 1,\ S(0;\, \underset{\sim}{n},\, \underset{\sim}{r}) = 0 \text{ whenever } \sum r_i n_i > 0$$

and

$$S(m;\, \underset{\sim}{n},\, \underset{\sim}{r}) = 0 \text{ if } m < \underset{\sim}{r}'\underset{\sim}{n}. \tag{4.5}$$

These follow from the difference/differential equation

$$\frac{d}{dt} f_{\underset{\sim}{n}}(t;\, \underset{\sim}{r}) = Nf_{\underset{\sim}{n}}(t,\, \underset{\sim}{r}) + \sum_{i=1}^{k} f_{\underset{\sim}{n} - \underset{\sim}{e}_i}(t;\, \underset{\sim}{r})t^{r_i - 1}/(r_i - 1)! \tag{4.6}$$

It can easily be seen that the representation (4.2) provides the following combinatorial interpretation in terms of occupancy numbers.

Proposition 4.1: The number of ways of placing m distinguishable balls into $N = n_1 + \cdots + n_k$ cells so that each cell of the i^{th} group of n_i cells contains at least r_i balls for $i = 1,\, \ldots,\, k$ is equal to $n_1!\, \ldots\, n_k!S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ if the N cells are distinguishable, and is equal to $S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ if only cells belonging to different groups are distinguishable (and cells in the same group are alike).

If is easily concluded from Proposition 4.1, or from (4.3)-(4.5), that the numbers $S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ are nonnegative integers.

5. RATIOS OF MULTIPARAMETER STIRLING NUMBERS OF THE SECOND KIND

We define, as a ratio of multiparameter Stirling numbers of the second kind with respect to argument m, the function

$$R_2(m; \underset{\sim}{n}, \underset{\sim}{r}) = \frac{S(m+1; \underset{\sim}{n}, \underset{\sim}{r})}{S(m; \underset{\sim}{n}, \underset{\sim}{r})}. \tag{5.1}$$

Working as for Proposition 3.1, we obtain

Proposition 5.1: A recurrence relation for the ratio $R_2(m; \underset{\sim}{n}, \underset{\sim}{r})$, independent of the multiparameter Stirling numbers of the second kind, is given by

$$R_2(m; \underset{\sim}{n}, \underset{\sim}{r}) - N = \frac{\displaystyle\sum_{j=1}^{k} \frac{\binom{m}{r_j - 1} r_j!\, n_j}{(\underset{\sim}{r}'\underset{\sim}{n})_{r_j}} \prod_{i=1}^{m+1-\underset{\sim}{r}'\underset{\sim}{n}} R_2(m - r_j + 1 - i, \underset{\sim}{n} - \underset{\sim}{e}_j, \underset{\sim}{r})}{\displaystyle\prod_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_2(m - i, \underset{\sim}{n}, \underset{\sim}{r})}, \tag{5.2}$$

for $\underset{\sim}{n} > \underset{\sim}{1}$ and $m > \underset{\sim}{r}'\underset{\sim}{n}$, with

$$R_2(m, 1, r) = k \tag{5.3}$$

and

$$R_2(\underset{\sim}{r}'\underset{\sim}{n}; \underset{\sim}{n}, \underset{\sim}{r}) = (\underset{\sim}{r}'\underset{\sim}{n} + 1)\sum_{j=1}^{k} \frac{n_j}{(r_j + 1)}. \tag{5.4}$$

The special case $k = 1$ yields

Proposition 5.2: A recurrence relation for the ratio $R_2(m, n, r)$, independent of the generalized Stirling numbers of the second kind, is given by

$$R_2(m, n, r) - n = \frac{\displaystyle\frac{n\binom{m}{r-1} r!}{(rn)_r} \prod_{i=1}^{m+1-rn} R_2(m - r + 1 - i, n - 1, r)}{\displaystyle\prod_{i=1}^{m-rn} R_2(m - i, n, r)} \tag{5.5}$$

for $n > 1$ and $m > rn$, with

$$R_2(m, 1, r) = 1 \tag{5.6}$$

and

$$R_2(rn, n, r) = n(rn + 1)/(r + 1). \tag{5.7}$$

Also for $k = 1$, $r = 1$ we obtain

Proposition 5.3: A recurrence relation for the ratio $R_2(m, n)$, independent of the usual Stirling numbers of the second kind, is given by

$$R_2(m, n) - n = \frac{\displaystyle\prod_{i=1}^{m+1-n} R_2(m - i, n - 1)}{\displaystyle\prod_{i=1}^{m-n} R_2(m - i, n)}, \tag{5.8}$$

for $n > 1$ and $m > n$, with

$$R_2(m, 1) = 1 \tag{5.9}$$

and

$$R_2(n, n) = n(n+1)/2. \tag{5.10}$$

Proposition 5.4: An alternative recurrence relation for the ratio $R_2(m, n, r)$ is given by

$$R_2(m, n, r) - n = \frac{m}{m-r+1}\,\frac{[R_2(m-1, n, r) - n]R_2(m-r, n-1, r)}{R_2(m-1, n, r)}, \tag{5.11}$$

for $n > 1$ and $m > rn$.

Applying Proposition 5.4 with $r = 1$ gives

Proposition 5.5: An alternative recurrence relation for the ratio $R_2(m, n)$, is given by

$$R_2(m, n) - n = \frac{[R_2(m-1, n) - n]R_2(m-1, n-1)}{R_2(m-1, n)}, \tag{5.12}$$

for $n > 1$ and $m > n$.

The last relation was also derived by Berg [1].

6. MULTIPARAMETER C-NUMBERS

The multiparameter C-numbers, $C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})$, are defined by their egf

$$\varphi_{\underset{\sim}{n}}(t; \underset{\sim}{s}, \underset{\sim}{r}) = \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty} C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})\frac{t^m}{m!} = \prod_{i=1}^{k}\frac{1}{n_i!}\left[(1+t)^{s_i} - \sum_{j=0}^{r_i-1}\binom{s_i}{j}t^j\right]^{n_i}, \tag{6.1}$$

where the $s_i \neq 0$, $i = 1, \ldots, k$, are any real numbers.

Taking $k = 1$ gives the generalized C-numbers (Charalambides [6]) and $k = 1$, $r_1 = 1$ defines the simple C-numbers (Cacoullos and Charalambides [5], Charalambides [8].

The following properties of $C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})$ are easily verified.

a) They have the representation

$$C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r}) = \frac{m!}{n_1! \ldots n_k!}\sum_m \prod_{i=1}^{k}\prod_{j=1}^{n_i}\binom{s_i}{x_{ij}}, \tag{6.2}$$

where the summation extends over all ordered N-tuples $(N = n_1 + \cdots + n_k)$ of integers x_{ij} satisfying

$$x_{ij} \geqslant r_i,\ i = 1, \ldots, k \quad \text{and} \quad \sum_{i=1}^{k}\sum_{j=1}^{n_i} x_{ij} = m.$$

b) They satisfy the following recurrence relations,

$$\begin{aligned} C(m+1; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r}) &= (\underset{\sim}{s}'\underset{\sim}{n} - m)C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r}) \\ &\quad + \sum_{i=1}^{k}\binom{m}{r_i - 1}(s_i)_{r_i}\, C(m - r_i + 1; \underset{\sim}{n} - \underset{\sim}{e}_i, \underset{\sim}{s}, \underset{\sim}{r}) \end{aligned} \tag{6.3}$$

and

$$C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r} + \underset{\sim}{e}_i) = \sum_{j=0}^{n_i} (-1)^j \frac{(m)_{jr_i}}{j!}\binom{s_i}{r_i}^j C(m - jr_i;\ \underset{\sim}{n} - j\underset{\sim}{e}_i,\ \underset{\sim}{s},\ \underset{\sim}{r}), \tag{6.4}$$

with initial conditions

$$C(0;\ 0,\ \underset{\sim}{s},\ \underset{\sim}{r}) = 1,\ C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r}) = 0 \text{ when } m < \underset{\sim}{r}'\underset{\sim}{n}.$$

They are obtained from the difference/differential equation

$$(1+t)\frac{d}{dt}\varphi_{\underset{\sim}{n}}(t;\ \underset{\sim}{s},\ \underset{\sim}{r}) = \underset{\sim}{s}'\underset{\sim}{n}\varphi_{\underset{\sim}{n}}(t;\ \underset{\sim}{s},\ \underset{\sim}{r}) + \sum_{i=1}^{k} \frac{(s_i)_{r_i}}{(r_i - 1)!} t^{r_i - 1}\varphi_{\underset{\sim}{n} - \underset{\sim}{e}_i}(t;\ \underset{\sim}{s},\ \underset{\sim}{r}). \tag{6.5}$$

The representation (6.2) leads us to the following interpretation of the $C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r})$-numbers in the framework of coupon-collecting problems.

Consider an urn containing k groups (sets) of distinguishable balls; the i^{th} group consists of $s_i n_i$ balls and is divided into equal subgroups (subsets) of s_i balls each bearing the numbers $1, \ldots, n_i$; moreover, suppose the balls of the k groups are distinguished by different colors so that each ball in the urn is distinguished by its color and number. Now it is easily seen from (6.2) that

Proposition 6.1: The number of ways of selecting m balls out of an urn with

$$\underset{\sim}{s}'\underset{\sim}{n} = \sum_{i=1}^{k} s_i n_i$$

distinguishable balls, divided into k groups by color and number as above into n_i subsets of size s_i within the i^{th} subgroup, so that each number $1, \ldots, n_i$ of the i^{th} subgroup (color) appears at least r_i times is equal to

$$\frac{n_1! \ \ldots \ n_k!}{m!} C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r}). \tag{6.6}$$

Here it was assumed that s_i is a positive integer. If s_i is a negative integer, say $s_i = -s_i^*$, then

$$\prod_{i=1}^{k}\prod_{j=1}^{n_i}\binom{s_i}{x_{ij}} = \prod_{i=1}^{k}\prod_{j=1}^{n_i}\binom{-s_i^*}{x_{ij}} = \prod_{i=1}^{k}\prod_{j=1}^{n_i}(-1)^{x_{ij}}\binom{s_i^* + x_{ij} - 1}{x_{ij}} \tag{6.7}$$

and from (6.2) it can be concluded that the sign of $C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r})$ is the same as $(-1)^m$. Furthermore, we may deduce

Proposition 6.2: The number of ways of distributing m $(m \geqslant \underset{\sim}{r}'\underset{\sim}{n})$ nondistinguishable balls into $\underset{\sim}{s}^{*\prime}\underset{\sim}{n}$ cells, divided into k groups of cells with $s_i n_i$ cells in the i^{th} group and n_i subgroups each of s_i cells in the i^{th} group, so that each subgroup of the i^{th} group contains at least r_i balls is equal to

$$\frac{n_1! \ \ldots \ n_k!}{m!} |C(m;\ \underset{\sim}{n},\ -\underset{\sim}{s}^*,\ \underset{\sim}{r})|. \tag{6.8}$$

As an indication of the applicability of the multiparameter C-numbers in occupancy problems, we refer to a problem posed by Sobel *et al.* [12, p. 52].

Signless Multiparameter C-Numbers

From the basic recurrence relation (6.3) or from the last two propositions, we conclude that:

a) for $s_i > 0$ an integer, the numbers $C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})$ are nonnegative integers; they are positive for $\underset{\sim}{r}'\underset{\sim}{n} \leqslant m \leqslant \underset{\sim}{s}'\underset{\sim}{n}$; otherwise zero;

b) for $s_i < 0$ an integer, the numbers $C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})$ are integers having the sign of $(-1)^m$.

Thus, as in the case of the Stirling numbers of the first kind, Riordan [10], the positive numbers

$$|C(m; \underset{\sim}{n}, -\underset{\sim}{s}^*, \underset{\sim}{r})| = (-1)^m C(m; \underset{\sim}{n}, -\underset{\sim}{s}^*, \underset{\sim}{r}) \tag{6.9}$$

will be called signless multiparameter C-numbers.

It can easily be verified that

Proposition 6.3: The egf of the signless multiparameter C-numbers

$$|C(m; \underset{\sim}{n}, -\underset{\sim}{s}, \underset{\sim}{r})|,$$

$s_i > 0$, $i = 1, \ldots, k$, is given by

$$\varphi_n^*(t; -\underset{\sim}{s}, \underset{\sim}{r}) = \prod_{i=1}^{k} \frac{1}{n_i!}\left[(1 - t)^{-s_i} - \sum_{j=0}^{r_i - 1} (-1)^j \binom{-s_i}{j} t^j\right]^{n_i}. \tag{6.10}$$

Remark: It should be observed that this is exactly the egf required for the treatment of the mvue problem in the negative binomial case when the probability function of the i^{th} sample is

$$\begin{aligned} P(X = x_{ij}) &= \frac{1}{g(\theta, r_i)}\binom{s_i + x_{ij} - 1}{x_{ij}} \theta^{x_{ij}}(1 - \theta)^{s_i} \\ &= (-1)^{x_{ij}}\binom{-s_i}{x_{ij}}\theta^{x_{ij}}(1 - \theta)^{s_i} \end{aligned} \tag{6.11}$$

with

$$g(\theta, r_i) = (1 - \theta)^{-s_i} - \sum_{j=0}^{r_i - 1} (-1)^j \binom{-s_i}{j}\theta^j, \quad i = 1, \ldots, k. \tag{6.12}$$

7. RATIOS OF MULTIPARAMETER C-NUMBERS

We define, as a ratio of multiparameter C-numbers with respect to argument m, the function

$$R_3(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r}) = \frac{C(m + 1; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})}{C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})} \tag{7.1}$$

Proposition 7.1: A recurrence relation for the ratio $R_3(m, \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})$, independent of the multiparameter C-numbers, is given by

$$R_3(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r}) + m - \underset{\sim}{s}'\underset{\sim}{n}$$

$$= \frac{\displaystyle\sum_{j=1}^{k} \frac{\binom{m}{r_j - 1}(s_j)_{r_j}\, n_j}{(\underset{\sim}{r}'\underset{\sim}{n})_{r_j}\binom{s_j}{r_j}} \prod_{i=1}^{m+1-\underset{\sim}{r}'\underset{\sim}{n}} R_3(m - r_j + 1 - i, \underset{\sim}{n} - \underset{\sim}{e}_j, \underset{\sim}{s}, \underset{\sim}{r})}{\displaystyle\prod_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_3(m - i, \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})} \tag{7.2}$$

for $\underset{\sim}{n} > \underset{\sim}{1}$ and $m > \underset{\sim}{r}'\underset{\sim}{n}$, with

$$R_3(m, 1, s, r) = s - m \tag{7.3}$$

and

$$R_3(\underset{\sim}{r}'\underset{\sim}{n}; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r}) = (\underset{\sim}{r}'\underset{\sim}{n} + 1) \sum_{j=1}^{k} \frac{n_j(s_j - r_j)}{(r_j + 1)}. \tag{7.4}$$

Proposition 7.2: A recurrence relation for the ratio $R_3(m, n, s, r)$, independent of the generalized C-numbers (case $k = 1$), is given by

$$R_3(m, n, s, r) + m - sn = \frac{\displaystyle\frac{\binom{m}{r-1}(s)_r\, n}{(rn)_r\binom{s}{r}} \prod_{i=1}^{m+1-rn} R_3(m - r + 1 - i, n - 1, s, r)}{\displaystyle\prod_{i=1}^{m-rn} R_3(m - i, n, s, r)}, \tag{7.5}$$

for $n > 1$ and $m > rn$, with

$$R_3(m, 1, s, r) = s - m \tag{7.6}$$

and

$$R_3(rn, n, s, r) = n(rn + 1)(s - r)/(r + 1). \tag{7.7}$$

Proposition 7.3: A recurrence relation for the ratio $R_3(m, n, s)$, independent of the usual C-numbers (case $r = 1$), is given by

$$R_3(m, n, s) + m - sn = \frac{\displaystyle\prod_{i=1}^{m+1-n} R_3(m - i, n - 1, s)}{\displaystyle\prod_{i=1}^{m-n} R_3(m - i, n, s)}, \tag{7.8}$$

for $n > 1$ and $m > n$, with

$$R_3(m, 1, s) = s - m \tag{7.9}$$

and

$$R_3(n, n, s) = (s - 1)n(n + 1)/2. \tag{7.10}$$

Proposition 7.4: An alternative recurrence relation for the ratio $R_3(m, n, s, r)$ is given by

$$R_3(m, n, s, r) + m - sn$$

$$= \frac{m}{m - r + 1} \frac{[R_3(m - 1, n, s, r) + m - sn - 1]R_3(m - r, n - 1, s, r)}{R_3(m - 1, n, s, r)}, \tag{7.11}$$

for $n > 1$ and $m > rn$.

Proposition 7.5: An alternative recurrence relation for the ratio $R_3(m, n, s)$ is given by

$$R_3(m, n, s) + m - sn = \frac{[R_3(m-1, n, s) + m - sn - 1]R_3(m-1, n-1, s)}{R_3(m-1, n, s)}, \tag{7.12}$$

for $n > 1$ and $m > n$.

8. RELATIONS BETWEEN THE STIRLING AND C-NUMBERS

It was observed in Cacoullos and Charalambides [5] that

$$\lim_{s_i \to \pm\infty} s^{-m} C(m, n, s) = S(m, n); \tag{8.1}$$

i.e., the C-numbers can be approximated by the Stirling numbers of the second kind for large s, a fact that reflects the corresponding well-known convergence of the binomial to the Poisson ($s \to \infty$, $p \to 0$, i.e., $\theta = p/q \to 0$ and, hence, sp or $s\theta$ converges to the Poisson parameter λ). The above property extends to the case of multiparameter Stirling numbers of the second kind and multiparameter C-numbers; namely,

$$\lim_{s_i \to \pm\infty} s_i^{-m} C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r}) = S(m; \underset{\sim}{n}, \underset{\sim}{r}), \; i = 1, \ldots, k. \tag{8.2}$$

This can be verified by using the corresponding representations (4.2) and (6.2) of these numbers and noting that

$$\lim_{s_i \to \pm\infty} s^{-k} \binom{s}{k} = 1/k!. \tag{8.3}$$

A relationship between the signless multiparameter Stirling numbers of the first kind and the multiparameter C-numbers reflects the limiting relationship between the negative binomial and the logarithmic series distributions:

$$\lim_{s_i \to 0} s_i^{-N} |C(m; \underset{\sim}{n}, -\underset{\sim}{s}, \underset{\sim}{r})| = |s(m; \underset{\sim}{n}, \underset{\sim}{r})|, \; N = n_1 + \cdots + n_k. \tag{8.4}$$

This can be seen, e.g., by showing that the egf of the $s_i^{-N}|C(m, \underset{\sim}{n}, -\underset{\sim}{s}, \underset{\sim}{r})|$-numbers converge to the egf of the $|s(m; \underset{\sim}{n}, \underset{\sim}{r})|$-numbers; i.e.,

$$\lim_{s_i \to 0} \frac{1}{s_i^N} \prod_{i=1}^{k} \frac{1}{n_i!} \left[(1-t)^{-s_i} - \sum_{j=0}^{r_i - 1} (-1)^j \binom{-s_i}{j} t^j \right]^{n_i}$$
$$= \prod_{i=1}^{k} \frac{1}{n_i!} \left[-\log(1-t) - \sum_{j=1}^{r_i - 1} \frac{t^j}{j} \right]^{n_i}. \tag{8.5}$$

For this, note that

$$\frac{1}{s}(-1)^j \binom{-s}{j} t^j \xrightarrow[s \to 0]{} \frac{t^j}{j}. \tag{8.6}$$

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THE MATRICES OF FIBONACCI NUMBERS

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1. INTRODUCTION

In a recent paper, Kalman [3] derives many interesting properties of generalized Fibonacci numbers. In this paper, we take a different approach and derive some other interesting properties of matrices of generalized Fibonacci numbers. As an application of such properties, we construct an efficient algorithm for computing matrices of generalized Fibonacci numbers.

The topic of generalized Fibonacci sequences discussed here is related to the theory of polyphase sorting in an interesting way; in fact, it is used in optimizing the polyphase sort (see [1] and [7]). The theory of polyphase sorting, in return, helps shape the construction of a fast algorithm for computing the order-k Fibonacci numbers in $0(k^2 \log n)$ steps (see [2] and [5]).

2. DEFINITIONS

Define k sequences of generalized order-k Fibonacci numbers, for some $k \geqslant 2$, as follows:

$$F_t^n = \sum_{i=1}^{k} F_t^{n-i}, \text{ for } 1 \leqslant t \leqslant k, \tag{1}$$

where F_t^n is the n^{th} Fibonacci number of the t^{th} sequence. We may arrange these k sequences in k columns extending to infinity in both directions. Define the *window* at level n, W_n, to be the $k \times k$ matrix of generalized Fibonacci numbers such that

$$W_n = (a_{ij}^n), \text{ for } 1 \leqslant i,\ j \leqslant k, \tag{2}$$

where $a_{ij}^n = F_j^{n-k+i}$.

A set of initial values of these k sequences, defined by (1), may be given by

$$F_t^n = \begin{cases} 1, & t = n + k \\ 0, & \text{otherwise} \end{cases}, \text{ for } 1 - k \leqslant n \leqslant 0. \tag{3}$$

In other words, W_0 is the $k \times k$ identity matrix.

By an application of (1)-(3), we deduce that

$$F_t^1 = 1, \text{ for } 1 \leqslant t \leqslant k. \tag{4}$$

In consequence, we have

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix} \tag{5}$$

To derive the n^{th} Fibonacci number such that $n \leqslant -k$, we simply invert (1):

$$F_t^n = F_t^{n+k} - \sum_{i=1}^{k-1} F_t^{n+i}, \text{ for } 1 \leqslant t \leqslant k. \tag{6}$$

In this way, the k columns can be extended to infinity in both directions, starting from the identity matrix, W_0.

3. SOME PROPERTIES

By the definition of generalized order-k Fibonacci numbers, we have

$$W_n = W_1 W_{n-1}. \tag{7}$$

In other words, W_1 may be viewed as a row operator as it shifts a window vertically by one level. From (7), we derive

$$W_n = W_1^n W_0. \tag{8}$$

Since $W_0 = I$, we have just derived

$$W_n = W_1^n. \tag{9}$$

Abbreviating W_1 as W, we may write W_n for W^n.

As a consequence of (9), we have

$$W_n = W_1 W_{n-1} = W_{n-1} W_1. \tag{10}$$

The above equation shows that matrix multiplication of windows is commutative. Indeed, $\{W_n | n \in Z\}$ is an infinite cyclic group and satisfies the usual laws of exponents.

From $W_n = W_{n-1}W_1$, we obtain the following two equations relating elements of two adjacent rows;

$$F_t^n = F_k^{n-1} + F_{t-1}^{n-1}, \text{ for } 2 \leqslant t \leqslant k, \tag{11}$$

and

$$F_1^n = F_k^{n-1}. \tag{12}$$

Interestingly, these two equations are foundational to the basic theory of polyphase sorting [1]. The n^{th} row of Fibonacci numbers is precisely the so-called *ideal distribution* in the sorting context.

More interestingly, the column and row recursions of windows can be interpreted as follows. Multiplying by W on the left of any window has the effect of rolling the window down, exposing a window at the next level. More generally, multiplying two windows at levels r and c, respectively, may be viewed as rolling the window at level c down by r levels, with the resulting window placed at level $(r + c)$, i.e.,

$$W^{r+c} = W^r W^c, \tag{13}$$

where r and c are any integers. In contrast, multiplying any window by W on the right has the effect of bringing the row recursion into play. If

$$R_n = [F_1^n F_2^n \dots F_k^n]$$

is the n^{th} row of Fibonacci numbers (therefore, the last row of W_n), then the $(n+1)^{\text{st}}$ row,

$$R_{n+1} = R_n W, \tag{14}$$

may be obtained by:

a) shifting each element of R_n one position to the right (filling the vacant position with a zero and truncating the element, F_k^n, moved out of place); and

b) adding the truncated element, F_k^n, to each entry.

These two steps may be illustrated as follows:

$$[F_1^n \; F_2^n \ldots F_k^n] \to [0 \; F_1^n \ldots F_{k-1}^n] \quad (F_k^n \text{ drops out})$$
$$\to [F_k^n \; F_k^n + F_1^n \ldots F_k^n + F_{k-1}^n].$$

We see, from the above discussions, that matrix W contains the mechanisms for computing (1), (11), and (12). Surprisingly, to compute generalized Fibonacci numbers, (1) need not be used directly; instead, (11), (12), and (13) are used.

4. APPLICATIONS

As an application of the interesting properties of windows, discussed previously, we describe the construction of a fast algorithm for computing generalized Fibonacci numbers. Paradoxically, when n is large, it is faster to compute the n^{th} Fibonacci number by using the matrix method discussed in the previous two sections than by using (1) alone (see [2] and [5]). As shown in equation (13), it is possible to increase the exponent of W through matrix multiplication, by treating each window as a single entity. In hand calculation or in computer implementation, it is desired to keep $r = c$ so that (1) only one $k \times k$ matrix needs to be maintained during the computation and (2) the destination level can be reached in the shortest time.

Note that any positive integer n can be expressed in terms of the binary representation:

$$n = \sum x_i 2^i, \tag{15}$$

where $x_i = 0$ or 1. Therefore, we may write

$$W^n = \prod_{x_i = 1} W^{2^i}. \tag{16}$$

If an algorithm starts with the window at level 1 and doubles the window level each time, then W^n can be reached in $O(\log n)$ steps. However, this approach requires two matrix multiplications: one for matrix squaring, another for accumulating the result by applying (16) (see Urbanek's implementation [5]). We now give an algorithm for computing the generalized order-k Fibonacci numbers, which is better than the algorithm given by Urbanek because it requires only one matrix multiplication per cycle.

Note that (15) can be rewritten as follows:

$$n = (\ldots((1 * 2 + x_{j-1}) * 2 + x_{j-2}) * 2 \ldots) * 2 + x_0, \tag{17}$$

where j is the smallest positive integer such that $n < 2^{j+1}$, and $x_i = 0$ or 1.

Consequently, we have

$$\begin{aligned} W^n &= W^{(\ldots((1*2+x_{j-1})*2+x_{j-2})*2\ldots)*2+x_0} \\ &= (\ldots((W^2 * W^{x_{j-1}})^2 * W^{x_{j-2}})^2 * W^{x_0}. \end{aligned} \tag{18}$$

For instance,

$$\begin{aligned} W^{25} &= W^{(((1*2+1)*2+0)*2+0)*2+1} \\ &= (((W^2 * W)^2)^2)^2 W. \end{aligned}$$

Equation (18) shows that, by working from the central parenthetical quantity outwards, W^n can be computed through successive steps requiring either matrix squaring or matrix squaring followed by multiplying by W. Fortunately, multiplying by W can be accomplished by applying (11) and (12) without the need of matrix multiplication.

An efficient algorithm for implementing the ideas described above is best based on the following recursive expressions:

$$W^n = \begin{cases} (W^{n/2})^2, & n \text{ is even} \\ (W^{\lfloor n/2 \rfloor})^2 W, & n \text{ is odd,} \end{cases}$$

and

$$W^1 = W.$$

The details of the algorithm are presented below using a programming notation commonly used in Computer Science (see [6]). Note that (n div 2) = $\lfloor n/2 \rfloor$, and that A[1,] is row 1 of A.

```
function Fibonacci (n, k : integer) : integer;
    {Given n ε Z and k ⩾ 2, this function returns F_k^n as a result.}
var  A : k x k matrix;
procedure  Window (n : integer);
    {Compute W^n.}
    var  R : 1 x k matrix;
    begin
        if n = 1 then A := W^1
        else begin
            Window (n div 2);
            R := A[1,] * A; {R = W^m[1,] * W^m, m = n div 2}
            if odd(n) then
                A[1,] := R * W^1 {A[1,] = W^2m[1,] * W}
            else A[1,] := R;     {A[1,] = W^2m[1,]}
            Compute rows 2 to k of A from previous rows
            end
    end {Window};
```

```
begin
    if n = 0 then return (1);
    if n < 0 then
        begin
        Window(-n);
        Inverse {Inverse matrix A}
        end
    else Window (n);
    return (A[k,k])
end {Fibonacci};
```

The procedure "Window" is called recursively for achieving the effect of starting the computation from the innermost pair of brackets of (18). It halves the value of n per recursion, truncating the remainder for odd n, and terminates the recursion when n is reduced to 1. In the last recursive activation, matrix A is initialized to W^1. Thus, the number of activations of Window is $O(\log n)$. In contrast, a direct application of (1) takes $O(n)$ steps.

Note that every row of a window satisfies (14). Therefore, in squaring a window, it is unnecessary to compute the value of every element of the resulting window by matrix multiplication (where a total of k^3 multiplicative operations would be required). Instead, we compute the first row of the resulting window as $A[1,] * A$ (see the procedure Window), and then compute the remaining rows by using (11) and (12). In this case, k^2 multiplicative operations are needed for squaring a window. Note further that, if the level of a window is odd, a fine adjustment of the window by multiplying it by W is required. Again this operation can be carried out economically by using (11) and (12). If such an adjustment is required, it is more economical to carry it out immediately after $A[1,] * A$ is computed than otherwise; hence, the test for odd(n), and $R * W^1$ in the procedure Window. Thus, the total number of multiplicative operations per procedure call of Window is k^2.

Since the cost of computation of additions is negligible in comparing with that of multiplications, it is ignored in the calculation of cost. Thus, the overall running time of this algorithm is $O(k^2 \log n)$. In contrast, Urbanek's implementation [5] requires two matrix multiplications. Since the probability of carrying out the second matrix multiplication is 0.5, the overall running time for his algorithm is $O(1.5k^2 \log n)$, taking into account that matrix multiplications could be done in $O(k^2)$ steps. Our algorithm thus runs 33% faster than Urbanek's algorithm. Moreover, our algorithm supports the computation of $-n^{\text{th}}$ Fibonacci numbers, as seen in the procedure Fibonacci, which is not addressed in [2] and [5]. Alternatively, it is computationally faster by making procedure Window take the initial window as a second parameter. If $n < 0$, W^{-1} is passed as a second parameter to Window; whereas, if $n > 0$, W^1 is passed as a parameter.

For an interesting application of the generalized order-k Fibonacci numbers to the polyphase sorting, the reader is referred to [1].

5. REMARKS

The material presented here could easily be adapted to computing solutions of linear difference equations with constant coefficients [4]. This is left as an exercise for the interested reader.

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LOWER BOUNDS FOR UNITARY MULTIPERFECT NUMBERS

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1. INTRODUCTION

Throughout this paper n and k will denote positive integers that exceed 2. With or without a subscript, p will denote a prime, and the i^{th} odd prime will be symbolized by P_i. If d is a positive integer such that $d \mid n$ and $(d, n/d) = 1$, then d is said to be a unitary divisor of n. The sum of all of the unitary divisors of n is symbolized by $\sigma^*(n)$. If $n = p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}$, where the p_i are distinct and $a_i > 0$ for all i, then it is easy to see that

$$\sigma^*(n) = \prod_{i=1}^{s} (1 + p_i^{a_i}) \tag{1}$$

and that σ^* is a multiplicative function.

Subbarao and Warren [2] have defined n to be a unitary perfect number if $\sigma^*(n) = 2n$. Five unitary perfect numbers have been found (see [3]). The smallest is 6, the largest has 24 digits. Harris and Subbarao [1] have defined n to be a *unitary multiperfect number* (UMP) if $\sigma^*(n) = kn$, where $k > 2$. We know of no example of a unitary multiperfect number and, as we shall see, if one exists it must be very large.

Suppose first that $n = p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}$, where n is odd and $\sigma^*(n) = kn$. Assume that $k = 2^c M$, where $2 \nmid M$ and $c \geqslant 0$. Then, since

$$2 \mid (1 + p_i^{a_i}) \text{ for } i = 1, 2, \ldots, s,$$

it follows from (1) that $s \leqslant c$. Also,

$$2^c M = k = \sigma^*(n)/n = \prod_{i=1}^{s} (1 + p_i^{-a_i}) < 2^s \leqslant 2^c,$$

which is a contradiction. We have proved

Theorem 1: There are no odd unitary multiperfect numbers.

This result was stated in [1]. Its proof is included here for the sake of completeness.

2. LOWER BOUNDS FOR UNITARY MULTIPERFECT NUMBERS

We assume from now on that

$$n = 2^\alpha \prod_{i=1}^{t} p_i^{a_i}, \text{ where } \alpha a_i > 0 \text{ and } 3 \leqslant p_1 < p_2 < \cdots < p_t. \tag{2}$$

Also, $\sigma^*(n) = kn$, so that

$$k = \sigma^*(n)/n = (1 + 2^{-\alpha}) \prod_{i=1}^{t} (1 + p_i^{-a_i}). \tag{3}$$

Since $2|(1 + p_i^{\alpha_i})$, it follows from (1) and (2) that $t \leqslant \alpha + 2$ if $k = 4$, and $t \leqslant \alpha + 1$ if $k = 6$. Therefore, since $1 + x^{-1}$ is monotonic decreasing for $x > 0$, it follows from (3), if $k = 4$ or 6, that

$$4 \leqslant k \leqslant (1 + 2^{-\alpha}) \prod_{i=1}^{t} (1 + P_i^{-1}) \leqslant (1 + 2^{-\alpha}) \prod_{i=1}^{\alpha+2} (1 + P_i^{-1}) = F(\alpha).$$

A computer run showed that $F(\alpha) < 4$ for $\alpha \leqslant 48$. Therefore, $\alpha \geqslant 49$ if $k = 4$ or 6. Also, from (3),

$$4 \leqslant k \leqslant (1 + 2^{-49}) \prod_{i=1}^{t} (1 + P_i^{-1}) = G(t).$$

Since $G(50) < 4$, we see that $t \geqslant 51$. Thus

$$n \geqslant 2^{49} \prod_{i=1}^{51} P_i > 10^{110} \text{ if } k = 4 \text{ or } 6.$$

If $k \geqslant 8$, then

$$8 \leqslant k \leqslant 1.5 \prod_{i=1}^{t} (1 + P_i^{-1}) = H(t).$$

A computer run showed that $H(t) < 8$ for $t \leqslant 246$. Therefore, if $k \geqslant 8$, $t \geqslant 247$ and

$$n \geqslant 2 \prod_{i=1}^{247} P_i > 10^{663}.$$

Now suppose that k is odd and $k \geqslant 5$. Since $2|(1 + p_i^{\alpha_i})$, we see that $t \leqslant \alpha$. Also, from (3),

$$5 \leqslant k \leqslant (1 + 2^{-\alpha}) \prod_{i=1}^{\alpha} (1 + P_i^{-1}) = J(\alpha);$$

and since $J(\alpha) < 5$ for $\alpha \leqslant 165$, it follows that $\alpha \geqslant 166$. Moreover,

$$5 \leqslant k \leqslant (1 + 2^{-166}) \prod_{i=1}^{t} (1 + P_i^{-1}) = K(t),$$

and since $K(165) < 5$, we see that $t \geqslant 166$. Therefore, if $k \geqslant 5$ and k is odd, then

$$n \geqslant 2^{166} \prod_{i=1}^{166} P_i > 10^{461}.$$

Theorem 2: Suppose that n is a UMP with t distinct odd prime factors and that $\sigma^*(n) = kn$. If $k \geqslant 8$, then $n > 10^{663}$ and $t \geqslant 247$. If $k = 4$ or 6, then $n > 10^{110}$, $t \geqslant 51$, and $2^{49}|n$. If k is odd and $k \geqslant 5$, then $n > 10^{461}$, $t \geqslant 166$, and $2^{166}|n$.

3. UNITARY TRIPERFECT NUMBERS

If $\sigma^*(n) = 3n$, n will be said to be a *unitary triperfect number*. Throughout this section we assume that n is such a number. We shall denote by q_i the i^{th} prime congruent to 1 modulo 3 and by Q_i the i^{th} prime congruent to 2 modulo 3. If $3 \nmid n$, then $t \leqslant \alpha$ and, from (3),

$$3 \leqslant (1 + 2^{-\alpha}) \prod_{i=2}^{\alpha+1} (1 + P_i^{-1}) = L(\alpha).$$

Since $L(\alpha) < 3$ for $\alpha \leqslant 49$, we see that $\alpha \geqslant 50$. Also,

$$3 \leqslant (1 + 2^{-50}) \prod_{i=2}^{t+1} (1 + P_i^{-1}) = M(t),$$

and since $M(49) < 3$, it follows that $t \geqslant 50$. And, finally, since $3 \| \sigma^*(n)$ and $3 | (1 + p)$ if $p \equiv 2 \pmod 3$, we see that

$$n \geqslant 2^{50} 5^2 11^2 17^2 23 \prod_{i=1}^{46} q_i > 10^{105}. \quad \text{(Note that } q_{46} = 523.)$$

If $3 \| n$, then $t \leqslant \alpha - 1$, since

$$3 \cdot 2^\alpha \prod_{i=1}^{t} p_i^{a_i} = (1 + 2^\alpha)(4) \prod_{i=2}^{t} (1 + p_i^{a_i}).$$

From (3),

$$3 = (1 + 2^{-\alpha})(4/3) \prod_{i=2}^{t} (1 + p_i^{-a_i}) \leqslant (1 + 2^{-\alpha}) \prod_{i=1}^{\alpha-1} (1 + P_i^{-1}) = N(\alpha),$$

and since $N(\alpha) < 3$ for $\alpha \leqslant 16$, we see that $\alpha \geqslant 17$. Also, $3^2 \| \sigma^*(n)$ and $3 | (1 + p)$ if $p \equiv 2 \pmod 3$. Therefore, since $1 + x^{-1}$ is monotonic decreasing for $x > 0$, *and* since

$$(1 + 2^{-17})(4/3)(6/5)(12/11)(290/17^2) \prod_{i=1}^{40} (1 + q_i^{-1}) < 3,$$

it follows from (3) that $t \geqslant 45$. Thus, $\alpha \geqslant 46$ and

$$n \geqslant 2^{46} 3 \cdot 5 \cdot 11 \cdot 17^2 \prod_{i=1}^{41} q_i > 10^{107}. \quad \text{(Note that } q_{41} = 439.)$$

If $3^2 \| n$, then $t \leqslant \alpha$ and, from (3),

$$3 \leqslant (1 + 2^{-\alpha})(10/9) \prod_{i=2}^{\alpha} (1 + P^{-1}) = R(\alpha).$$

$\alpha \geqslant 32$, since $R(\alpha) < 3$ for $\alpha \leqslant 31$. Also, $3^3 \| \sigma^*(n)$ and $3 | (1 + p)$ if $p \equiv 2$ (mod 3). Therefore, since

$$(1 + 2^{-32})(10/9)(6/5)(12/11)(24/23)(290/17^2) \prod_{j=5}^{8} (1 + Q_j^{-2}) \prod_{i=1}^{227} (1 + q_i^{-1}) < 3,$$

we see that $t \geqslant 237$. ($Q_8 = 53$ and $q_{227} = 3307$.) Thus, $\alpha \geqslant 237$ and

$$n \geqslant 2^{237}(5 \cdot 11 \cdot 23)(3 \cdot 17 \cdot 29 \cdot 41 \cdot 47 \cdot 53)^2 \prod_{i=1}^{228} q_i > 10^{779}.$$

If $3^3 \| n$, then $t \leqslant \alpha - 1$ and

$$3 \leqslant (1 + 2^{-\alpha})(28/27) \prod_{i=2}^{\alpha-1} (1 + P_i^{-1}) = S(\alpha).$$

Since $S(\alpha) < 3$ for $\alpha \leqslant 43$, we see that $\alpha \geqslant 44$. Also, $3^4 \| \sigma^*(n)$ and $3 | (1 + p)$ if $p \equiv 2 \pmod 3$. Therefore, since

$$(1 + 2^{-44})(28/27)(6/5)(12/11)(18/17)\prod_{j=4}^{12}(1 + Q_j^{-2})\prod_{i=1}^{530}(1 + q_i^{-1}) < 3,$$

we conclude that $t \geqslant 544$. ($Q_{12} = 89$ and $q_{530} = 8623$.) Thus, $\alpha \geqslant 545$ and

$$n \geqslant 2^{545}3^3 \cdot 5 \cdot 11 \cdot 17 \prod_{j=4}^{12} Q_j^2 \prod_{i=1}^{531} q_i > 10^{2026}.$$

If $3^4 | n$, then $t \leqslant \alpha$ and

$$3 \leqslant (1 + 2^{-\alpha})(82/81)\prod_{i=2}^{\alpha}(1 + P_i^{-1}) = T(\alpha).$$

Since $T(\alpha) < 3$ for $\alpha \leqslant 47$, it follows that $\alpha \geqslant 48$. From (3),

$$3 \leqslant (1 + 2^{-48})(82/81)\prod_{i=2}^{t}(1 + P_i^{-1}) = U(t),$$

and since $U(47) < 3$, we conclude that $t \geqslant 48$ and

$$n \geqslant 2^{48}3^4 \prod_{i=2}^{48} P_i > 10^{102}.$$

We summarize these results in the following theorem.

Theorem 3: Suppose that n is a unitary triperfect number with t distinct odd prime factors. Then $t \geqslant 45$, $n > 10^{102}$, and $2^{46} | n$. If $3^2 \| n$, then $t \geqslant 237$, $n > 10^{779}$, and $2^{237} | n$. If $3^3 \| n$, then $t \geqslant 544$, $n > 10^{2026}$, and $2^{545} | n$.

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ON THE ASYMPTOTIC PROPORTIONS OF ZEROS AND ONES IN FIBONACCI SEQUENCES

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By "Fibonacci sequence" we mean a binary sequence such that no two one's, say, are consecutive, with unrestricted first entry; hence, the number of such sequences of length n is f_{n+1}.

It is understood that

$$f_n = c\alpha^n + \overline{c}\overline{\alpha}^n \text{ with } \alpha = \frac{1+\sqrt{5}}{2} \text{ (the "golden ratio")}, \tag{1}$$
$$c = \frac{5+\sqrt{5}}{10},\ \overline{\alpha} = 1 - \alpha \text{ and } \overline{c} = 1 - c.$$

We denote by p and q the asymptotic proportions of zeros and ones, respectively, in Fibonacci sequences, so that $p + q = 1$. We will show

Theorem:

$$p = c \text{ and } q = \overline{c}. \tag{2}$$

Let ω_n be the total number of ones in all Fibonacci sequences of length n; hence, $\omega_0 = 0$ and $\omega_1 = 1$. Since the total number of ones in all n-sequences is the number in all $(n - 1)$-sequences, with zeros appended to the ends, plus the number in all $(n - 2)$-sequences, with zero-ones appended, plus the number of ones in those zero-ones, we have

$$\omega_n = \omega_{n-1} + \omega_{n-2} + f_{n-1}. \tag{3}$$

We know that such a recursion [1, p. 101] gives

$$\omega_{n+1} = \sum_{k=0}^{n} f_k f_{n-k}. \tag{4}$$

The proportion of ones is the number of ones divided by the number of entries—n per sequence times f_{n+1} sequences—so we define

$$q_n = \frac{\omega_n}{nf_{n+1}} \quad \text{and} \quad q = \lim_{n\to\infty} q_n. \tag{5}$$

Clearly, the limit exists and is less than 1/2, as the ones are restricted but the zeros are not.

From (1) and (4), we have

$$\omega_{n+1} = \sum_{k=0}^{n} (c\alpha^k + \overline{c}\overline{\alpha}^k)(c\alpha^{n-k} + \overline{c}\overline{\alpha}^{n-k}) \tag{6a}$$

$$\omega_{n+1} = (n+1)(c^2\alpha^n + \overline{c}^2\overline{\alpha}^n) + c\overline{c}\sum_{k=0}^{n}(\alpha^k\overline{\alpha}^{k-n} + \alpha^{n-k}\overline{\alpha}^k). \tag{6b}$$

As $\alpha\overline{\alpha} = -1$, the indexed sum on the right of (6b) is

$$c\bar{c}\sum_{k=0}^{n}[(-1)^k\bar{\alpha}^{n-2k} + (-1)^k\alpha^{n-2k}] \tag{7a}$$

and by inverting the order of summation on the left,

$$2c\bar{c}\sum_{k=0}^{n}(-1)^{n-k}\alpha^{n-2k} \tag{7b}$$

which is clearly less than

$$\frac{\alpha^{2n+2} - 1}{\alpha^n(\alpha^2 - 1)}\, 2c\bar{c} = o(n\alpha^n) \tag{7c}$$

where $f(n) = o(g(n))$ means $\lim\limits_{n\to\infty}\dfrac{f(n)}{g(n)} = 0$.

Substituting (7c) into (6b), and thence into (5), we have

$$q_{n+1} = \frac{c^2\alpha^n + \bar{c}^2\bar{\alpha}^n}{c\alpha^{n+2} + \bar{c}\bar{\alpha}^{n+2}} + o(1); \tag{8}$$

as $\alpha > 1$ and $|\bar{\alpha}| < 1$, and taking $n \to \infty$,

$$q = \frac{c}{\alpha^2} = \frac{5+\sqrt{5}}{10} \cdot \frac{2}{3+\sqrt{5}} = \frac{5-\sqrt{5}}{10} = \bar{c}, \tag{9}$$

and hence the theorem.

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FIBONACCI k-SEQUENCES, PASCAL-T TRIANGLES, AND k-IN-A-ROW PROBLEMS

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1. INTRODUCTION AND SUMMARY

In what follows, we use the Fibonacci sequences of order k, as for example in Philippou and Muwafi [2] (although modified somewhat here), and the Pascal-T triangles, as in Turner [6], to solve a number of enumeration problems involving the number of binary numbers of length n which have (or do not have) a string of k consecutive ones, subject to various auxilliary conditions (no k consecutive ones, exactly k, at least k, and so on). Collectively, these kinds of problems might be labelled k-in-a-row problems, and they have a number of interpretations and applications (a few of which are discussed in §4): combinatorics (ménage problems), statistics (runs problems), probability (reliability theory), number theory (compositions with specified largest part). Generating functions, inclusion-exclusion arguments, and the like, are perhaps most commonly used in these problems, but the methods developed here are simple, surprisingly effective, and computationally efficient. Finally, we note that although the string length n is fixed here, some of our results will also apply to parts of [2], [5], [6] (cf., e.g., the Corollary to Theorem 3.1), which discuss the problem of waiting for the k^{th} consecutive success, since the situation there is in some respects essentially that of having a fixed string length of size $n + k$.

Definitions and constructions are in §2, the enumeration theorems are in §3, and §4 gives several examples of their use.

2. MODIFIED k-SEQUENCES, AND TRIANGLES T

We need a slightly altered version of the usual definition of a Fibonacci sequence of order k, one that omits the leading 0, 1.

Definition 2.1: The sequence $\{f_k(n)\}_{n=0}^{\infty}$, $k \geqslant 0$, is said to be the modified Fibonacci sequence of order k if $f_0(n) \equiv 0$, $f_1(n) \equiv 1$, and for $k \geqslant 2$,

$$f_k(n) = \begin{cases} 2^n, & 0 \leqslant n \leqslant k-1 \\ \sum_{i=n-k}^{n-1} f_k(i), & n \geqslant k \end{cases} \quad . \tag{2.1}$$

It will prove convenient to have a notation for the corresponding Pascal-T triangles of order k.

Definition 2.2: For any $k \geqslant 0$, T_k is the array whose rows are indexed by $N = 0, 1, 2, \ldots$, and columns by $K = 0, 1, 2, \ldots$, and whose entries are obtained as follows:

a) T_0 is the all-zero array;

b) T_1 is the array all of whose rows consist of a one followed by zeros;

c) T_k, $k \geqslant 2$, is the array whose $N = 0$ row is a one followed by zeros, whose $N = 1$ row is k ones followed by zeros, and any of whose entries in subsequent rows is the sum of the k entries just above and to the left in the preceding row (with zeros making up any shortage near the left-hand edge).

Definition 2.3: In T_k, denote the entry at the intersection of row N and column K by $C_k(N, K)$.

T_2 is of course *the* Pascal triangle, and we will denote its entries by $\binom{N}{K}$. We note that the T_k can be tabulated for moderate values of N, K and can be considered as available as a binomial table. For $k > 0$, by construction there are $(N(k-1)+1)$ nonzero entries in each row, the symmetry relation among the C_k is

$$C_k(N, K) = C_k(N, N(k-1) - K), \quad 0 \leqslant K \leqslant N(k-1), \tag{2.2}$$

and the relation among the C_k in adjacent rows is

$$C_k(N, K) = \sum_{j=0}^{k-1} C_k(N-1, K-j): \tag{2.3}$$

here N, K, and k are nonnegative, an empty sum is taken to be zero, and any C_k with either argument negative is zero. That is, (2.3) just expresses property (c) of the definition of T_k. Also by construction, the relation between the f_k and the C_k is

$$f_k(n) = \sum_{j=0}^{n} C_k(n-j+1, j), \tag{2.4}$$

so that the $f_k(n)$ are also given by the successive southwest-northeast diagonals of T_k [starting with the (1, 0) entry]. This follows from the recurrence in the definition of $f_k(n)$ and that fact that, by (2.3), each element in the diagonal making up $f_k(n)$ is a sum of k preceding elements.

Definition 2.4: Denote by β_{pk} the number of binary numbers of length n which have a total of p ones and a longest string of exactly k consecutive ones. For any $k \geqslant 2$, define the B_k-array to be

$$\begin{array}{llll}
\beta_{kk} & & & \\
\beta_{k+1,k} & \beta_{k+1,k+1} & & \\
\vdots & \vdots & & \\
\beta_{n,k} & \beta_{n,k+1} & \cdots & \beta_{nn}
\end{array}$$

in which the row elements are associated with a fixed total number of ones, and the column elements with a fixed number of consecutive ones.

3. ENUMERATION THEOREMS

Theorem 3.1: The number of binary numbers of length n which have no k consecutive ones is given by $f_k(n)$, $n \geqslant 0$.

Proof: Let $g(n)$ enumerate the numbers having the property stated. Then we have, schematically.

$$\underbrace{\boxed{\ n\ }}_{g(n)} = \underbrace{\boxed{n-1}}_{g(n-1)}\boxed{0} + \underbrace{\boxed{n-2}}_{g(n-2)}\boxed{01} + \cdots + \underbrace{\boxed{n-k}}_{g(n-k)}\boxed{011\ldots1},$$

with $g(n)$ satisfying the same initial conditions as in (2.1), and so $g(n)$ is just $f_k(n)$.

Corollary 3.1: The number of binary numbers of length n which end with k consecutive ones but have no other string of k consecutive ones is given by

$$f_k(n - k - 1),\ n \geqslant k,$$

and with $f_k(-1) = 1$.

Proof: We know that $f_k(n - 1)$ enumerates the binary strings of length $(n - 1)$ with no k consecutive ones. These, however, form (with one zero at the end) the "first half" of the strings of length n. Thus, passing from $f_k(n)$ to $f_k(n - 1)$ amounts to stripping the last one from the strings in the "last half" of the strings of length n. Continuing the argument in this way, we come to $f_k(n - k - 1)$, which enumerates the strings of length $n - k$ that end with a zero. But, when a string of k consecutive ones is appended, these are precisely the configurations we wish to count.

Remark 3.1: These two results can also be obtained from the work of Philippou and Muwafi [2]. For $k \geqslant 2$, our $f_k(n)$ is their sequence $f_{n+2}^{(k)}$, $n \geqslant 0$. Then, Theorem 3.1 follows from the results in [2], since their $a_n^{(k)}$ is

$$a_n^{(k)} = a_{n+1}^{(k)}(f) = A_{n+1}^{(k)} = f_{n+2}^{(k)} = f_k(n),\ n \geqslant 0,$$

and Corollary 3.1 is equivalent to their Lemma 2.2.

Theorem 3.2: The number of binary numbers of length n which have a longest string of exactly k consecutive ones is given by $f_{k+1}(n) - f_k(n)$, $n \geqslant 1$.

Proof: By Theorem 3.1, $(2^n - f_k(n))$ is the number of configurations with k or more consecutive ones; $(2^n - f_{k+1}(n))$ is the number with $(k + 1)$ or more consecutive ones. Their difference is the number with exactly k.

Corollary 3.2: The column sums of the B_k-array are given by the numbers

$$f_{k+1}(n) - f_k(n).$$

Theorem 3.3: The number of binary numbers of length n that have a total of j ones, no k consecutive is given by $C_k(n - j + 1, j)$.

Proof: Let $g_k(n, j)$ enumerate these numbers. For $0 \leqslant j \leqslant k - 1$, we have, by definition, and because we are in T_k,

$$g_k(n, j) = \binom{n}{j} = C_k(n - j + 1, j),$$

and for $n \geqslant k$, $g_k(n, n) = C_k(1, n) = 0$. Now let $k \leqslant j \leqslant n$. The numbers we want that end in 0 are enumerated by

$$g_k(n - 1, j);$$

those that end in 01 are enumerated by

$$g_k(n - 2, j - 1);$$

those that end in 011 are enumerated by

$$g_k(n - 3, j - 2), \ldots; \text{ and so on.}$$

Then we have the recurrence

$$g_k(n, j) = g_k(n - 1, j) + g_k(n - 2, j - 1) + \cdots + g_k(n - k, j - k + 1). \qquad (*)$$

The conclusion can be proved by induction: the hypothesis asserts that

$$g_k(n - 1, j) = C_k(n - j, j),\ g_k(n - 2, j - 1) = C_k(n - j, j - 1), \ldots,$$
$$g_k(n - k, j - k + 1) = C_k(n - j, j - k + 1).$$

But this implies

$$\begin{aligned} g_k(n, j) &= C_k(n - j, j) + C_k(n - j, j - 1) + \cdots + C_k(n - j, j - k + 1), \text{ by } (*) \\ &= C_k(n - j + 1, j), \text{ by } (2.3). \end{aligned}$$

Corollary 3.3: The number of binary numbers of length n that have a total of j ones and a string of ones of length at least k is given by

$$\binom{n}{j} - C_k(n - j + 1, j).$$

Corollary 3.4: The row sums of the B_k-array are given by the formula of Corollary 3.3.

Theorem 3.4: The columns of the B_k-array (the elements β_{pk} that give the number of binary numbers of length n with a total of p ones and a longest string of exactly k consecutive ones) are given by:

$$\begin{aligned}
\beta_{jj} &= \binom{n}{j} && - C_j(n - (j - 1), j) \\
\beta_{j+1,j} &= C_{j+1}(n - j, j + 1) && - C_j(n - j, j + 1) \\
\beta_{j+2,j} &= C_{j+1}(n - (j + 1), j + 2) && - C_j(n - (j + 1), j) \qquad 2 \leqslant k \leqslant j \leqslant n \\
&\vdots && \vdots \\
\beta_{n-1,j} &= C_{j+1}(2, n - 1) && - C_j(2, n - 1) \\
\beta_{n,j} &= C_{j+1}(1, n) && - C_j(1, n)
\end{aligned}$$

Proof: Having the row sums of B_k for any k by Corollary 3.4, we can obtain B_k column by column.

For completeness, we mention that although B_k was initially defined for $k \geqslant 2$, B_0 and B_1 can also be formed. The $k = 0$ column is a one followed by zeros, and the $k = 1$ column consists of the numbers $\binom{n - p + 1}{p}$, $1 \leqslant p \leqslant n$. The corresponding column sums are $f_1(n) - f_0(n) = 1$ and $f_2(n) - f_1(1)$, and the row sums in this case are just the $\binom{n}{p}$.

4. APPLICATIONS

In this section we give three brief examples that are quite straightforward but nevertheless give some idea of the variety of possible interpretations and uses of the previous material.

Example 4.1: Given n objects arranged in a row, the number of ways of choosing j objects from among the n such that among the j chosen no k are consecutive is, by Theorem 3.3, $C_k(n - j + 1, j)$. For $k = 2$, this is just $\binom{n-j+1}{j}$, a result which is one of the principal steps in the solution of the ménage problem [4, p. 33].

Example 4.2: Engineers often increase the reliability of a system by making the conditions under which it fails more stringent. An example from reliability theory is what is called a "consecutive-k-out-of-n:F system" [1]. This is a system of n independent, linearly ordered components, each of which operates (fails) with probability $p(q)$, such that the system fails when and only when k consecutive components fail. What needs to be calculated is the system failure probability, $P_f(n, k)$. If we let a one stand for a failure, then by Corollary 3.3, if we put

$$r_j = \binom{n}{j} - C_k(n - j + 1, j)$$

(j total 1's and at least k consecutive 1's), the failure probability is given by

$$P_f(n, k) = \sum_{j=k}^{n} r_j^{(k)} p^{n-j} q^j.$$

Example 4.3: In number theory, an ordered partition of n is called a composition of n. Let $a(n, k)$ denote the number of compositions of n in which the largest part equals k. There is a natural one-to-one correspondence between the compositions $a(n, k)$ and the number of binary numbers of length n beginning with a zero, and containing the pattern 1...1 with $k - 1$ ones but not the pattern 1...1 with k ones; that is, any integer m in the composition is represented by the pattern 01...1 with $m - 1$ ones. But if the string of length n must begin with a zero, we are just considering the "first half" of all the strings of length n. This is equivalent to considering strings of length $n - 1$ that have a largest consecutive-ones substring of length $k - 1$, and so Theorem 3.2 solves the problem of enumerating the $a(n, k)$; i.e., $a(n, k) = f_k(n - 1) - f_{k-1}(n - 1)$, $n \geq 1$, $1 \leq k \leq n$. A short table follows:

$a(n, k)$:

$k \backslash n$	1	2	3	4	5	6
1	1	1	1	1	1	1
2		1	2	4	7	12
3			1	2	5	11
4				1	2	5
5					1	2
6						1

(For a generating function approach to this enumeration, see John Riordan [3, Ch. 6].)

It seems fair to say that the generalized Fibonacci-sequence/Pascal-triangle approach, as well as being interesting in its own right, is quite useful and a reasonable alternative to the generating function or multinomial methods often used in these kinds of problems.

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THE n-NUMBER GAME

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INTRODUCTION AND BACKGROUND

Given an ordered set of n nonnegative integers,

$$S_1 = (a, b, \ldots, m, n),$$

define a linear transformation $T: S_1 \to S_2$, where

$$S_2 = (|a - b|, |b - c|, \ldots, |m - n|, |n - a|).$$

Upon iteration of this transformation, a sequence of n-tuples of nonnegative integers is created. This sequence is called "the n-number game."

The n-number game has been considered primarily in the form of the four-number game. For example,

$$\begin{aligned} S_1 &= (37, 17, 97, 28), \\ S_2 &= (20, 80, 69, 9), \\ S_3 &= (60, 11, 60, 11), \\ S_4 &= (49, 49, 49, 49), \\ S_5 &= (0, 0, 0, 0). \end{aligned}$$

It is well known that the 2^m-number game will always terminate (i.e., reach a 2^m-tuple of all zeros) [7], [10].

The domain of the elements of the n-number game can be extended to the reals with some interesting consequences (e.g., see [3], [6]). However, such extensions will not be dealt with in this paper.

Let us note three well-known properties of the n-number game and make a definition.

(1) There exists a positive integer k such that S_k is an n-tuple all of elements are 0's and α's (e.g., see [1]).

Using Property (1), we will make the following definition.

Definition: The length of the sequence beginning with S_1, $|S_1|$, is $m - 1$, where m is the smallest integer such that the elements of S_{m+k} are all 0's and α's for $k \geqslant 0$.

If $\alpha \neq 0$, we will say that the sequence cycles.

(2) If $S_1 = (ad, bd, \ldots, nd) = d(a, b, \ldots, n) = dS_1^*$, then $|S_1| = |S_1^*|$. (This is easily proven.)

(3) The necessary and sufficient condition for a "parent" to exist for a given n-tuple S_1 is that S_1 can be partitioned into two subsets where the sum of the elements in each subset is the same (e.g., see [1]).

Property (3) implies that, with the exception of the trivial case, the odd-number game will always cycle. To see this, assume that the odd-number game terminates and work backward. If $\beta \neq 0$, a simple parity argument shows that $(\beta, \beta, \ldots, \beta)$ cannot have a parent if n is odd.

An n-tuple will be called an "orphan" if it has no parent. From Property (3), if S_1 is not an orphan, no permutation of S_1 is one; the converse is also true. In the example, $S_1 = (37, 17, 97, 28)$ is an orphan. If S_1 is not an orphan, a parent that is not an orphan is not necessarily unique. An example of this is the case where $S_1 = (5, 3, 8)$, $S_0 = (13, 8, 5)$ or $(3, 8, 11)$.

With these observations in mind, let us proceed to a more systematic study of the three-number game.

THE THREE-NUMBER GAME

In the study of the three-number game, we will use the convention that the first triple in any three-number sequence, $S_1 = (a, b, c)$ is not an orphan and and that $a \geqslant b \geqslant c$. However, by Property (3), $a = b + c$. Therefore,

$$S_1 = (b + c, b, c).$$

Each triple in the cycle of the three-number game is of the form $(0, d, d)$. Since the order of the elements in each triple is of no consequence, no distinction will be made between permutations of a given triple.

Theorem 1: If $S = (0, d, d)$, d is the greatest common divisor (g.c.d.) of the elements of S_1 $(d \neq 0)$.

Proof: Let d be the g.c.d. of the elements of $S_1 = (b + c, b, c)$. Let

$$S_1^* = (1/d)S_1 = ((b + c)/d, b/d, c/d) = (b^* + c^*, b^*, d^*).$$

Then $(b^*, c^*) = 1$ and

$$S_2^* = (c^*, b^* - c^*, b^*),$$

where the g.c.d. of the elements of S_2^* is also 1. By induction, the g.c.d. of the elements of S_m^* is 1 for all $m \geqslant 1$. Therefore, if $S_k^* = (0, d^*, d^*)$, $d^* = 1$ (since $d^* \neq 0$ by assumption) and $S_k = (0, d, d)$.

THE LENGTH OF THE THREE-NUMBER GAME

Let us consider an example of the three-number game:

$$\begin{aligned} S_1 &= (17, 37, 20), & S_6 &= (3, 8, 5),\\ S_2 &= (20, 17, 3), & S_7 &= (5, 3, 2),\\ S_3 &= (3, 14, 17), & S_8 &= (2, 1, 3),\\ S_4 &= (11, 3, 14), & S_9 &= (1, 2, 1),\\ S_5 &= (8, 11, 3) & S_{10} &= (1, 1, 0).\end{aligned}$$

The number of appearances of $c = 17$ is three. The generalization of this observation is in Theorem 2.

Theorem 2: If $S_1 = (b + c, b, c)$, where $b = qc + r$, $0 < r < c$, the number of appearances of c is $q + 2 = [b/c] + 2$, where [ ] is the greatest integer function.

Proof: We have

$$\begin{aligned} S_1 &= ((q + 1)c + r, qc + r, c),\\ S_2 &= (qc + r, (q - 1)c + r, c),\end{aligned}$$

and, by simple induction,

$$S_q = (2c + r,\ c + r,\ c).$$

Thus,

$$S_{q+1} = (c + r,\ c,\ r),\ S_{q+2} = (c,\ r,\ c - r),\ \text{and}\ S_{q+3} = (r,\ c - r,\ |c - 2r|).$$

All the elements of S_{q+3} are less than c and, since the transformation T cannot make $\max(S_{k+1}) > \max(S_k)$, the number of appearances of c is $q + 2$.

Note that c is not the g.c.d. of the elements of S_1 in Theorem 2 by the assumption that $0 < r < c$. That case is dealt with in Theorem 3.

Theorem 3: If $S_1 = (b + c,\ b,\ c)$, where $b = qc$, then $|S_1| = q = \frac{b}{c}$ $(c \neq 0)$.

Proof: $S_1 = ((q + 1)c,\ qc,\ c)$ and simple induction gives $S_q = (2c,\ c,\ c)$. Thus, $S_{q+1} = (c,\ c,\ 0)$ and $|S_1| = q$.

With these two results, we have Theorem 4, which gives the length of the three-number game and clarifies what is indeed occurring in the sequence.

Theorem 4: If $S_1 = (b + c,\ b,\ c)$, $c \neq 0$, then

$$|S_1| = \sum_{i=1}^{k} q_i,$$

where the q_i, $1 \leqslant i \leqslant k$, are all the quotients in the Euclidean Algorithm for b and c.

Proof: Let $b = q_1c + r_1$, $0 < r_1 < c$. By Theorem 2,

$$S_{q_1+1} = (c + r_1,\ c,\ r_1).$$

Let $c = q_2r_1 + r_2$, $0 < r_2 < r_1$, and repeat this process in the style of the Euclidean Algorithm until $r_k = 0$. If

$$t = 1 + \sum_{i=1}^{k-1} q_i,$$

then

$$S_t = (r_{k-1} + r_{k-2},\ r_{k-2},\ r_{k-1}),$$

where $r_{k-2} = q_kr_{k-1}$. By Theorem 3, we have the desired result.

From Theorem 4, we see that the length of the three-number game is greater than or equal to the length of the corresponding Euclidean Algorithm (with equality if and only if all the quotients in the Euclidean Algorithm are ones, i.e., if and only if $b = c$).

There is a special case of the three-number game where the length is very easy to calculate. This case is given in Theorem 5.

Theorem 5: If $S_1 = (b + c,\ b,\ c)$ and d is the g.c.d. of the elements of S_1 and $b \equiv d \pmod{c}$, then

$$|S_1| = \frac{b - d}{c} + \frac{c}{d}.$$

Proof: If $b - kc = d$, Theorem 2 gives

$$S_{k+1} = (b - kc, b - (k - 1)c, c) = (d + c, c, d).$$

By Theorem 3,

$$|S_{k+1}| = \frac{c}{d}.$$

Thus,

$$|S_1| = k + \frac{c}{d} = \frac{b - d}{c} + \frac{c}{d}.$$

REMARKS

It is important to note that although all the theorems in this paper refer to S_1, they can be applied to any suitable triple in a sequence by neglecting previous triples.

The three-number game affords a method for finding the g.c.d. of two positive integers b and c [using $S_1 = (b + c, b, c)$ and finding d]. By Theorem 4, the length of the algorithm is small (relative to the size of b and c) if b and c are Fibonacci numbers, while the length of the corresponding Euclidean Algorithm is maximized. In this case, the three-number game takes one more step. In general, however, the three-number game is not a viable method for finding the g.c.d. For a computer that can only add or subtract, it might be useful.

It is known that the length of the four-number game is nearly maximized if the initial entrants are Tribonacci numbers [9]. Can we define "$(n-1)$onacci" numbers that strongly influence the length of the n-number game?

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WEIGHTED ASSOCIATED STIRLING NUMBERS

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1. INTRODUCTION

The Stirling numbers of the first and second kind can be defined by

$$\begin{cases} (-\log(1 - x))^k = k! \sum_{n=k}^{\infty} S_1(n, k)x^n/n! \\ (e^x - 1)^k = k! \sum_{n=k}^{\infty} S(n, k)x^n/n! \end{cases} \tag{1.1}$$

These numbers are well known and have been studied extensively. There are many good references for them, including [4, Ch. 5] and [9, Ch. 4, pp. 32-38].

Not as well known are the *associated* Stirling numbers of the first and second kind, which can be defined by

$$\begin{cases} (-\log(1 - x) - x)^k = k! \sum_{n=2k}^{\infty} d(n, k)x^n/n! \\ (e^x - x - 1)^k = k! \sum_{n=2k}^{\infty} b(n, k)x^n/n! \end{cases} \tag{1.2}$$

We are using the notation of Riordan [9] for these numbers. One reason they are of interest is their relationship to the Stirling numbers:

$$\begin{cases} S_1(n, n - k) = \sum_{j=0}^{k} d(2k - j, k - j)\binom{n}{2k - j} \\ S(n, n - k) = \sum_{j=0}^{k} b(2k - j, k - j)\binom{n}{2k - j} \end{cases} \tag{1.3}$$

Equations (1.3) prove that $S_1(n, n - k)$ and $S(n, n - k)$ are both polynomials in n of degree $2k$. Combinatorially, $d(n, k)$ is the number of permutations of $Z_n = \{1, 2, \ldots, n\}$ having exactly k cycles such that each cycle has at least two elements; $b(n, k)$ is the number of set partitions of Z_n consisting of exactly k blocks such that each block contains at least two elements. Tables for $d(n, k)$ and $b(n, k)$ can be found in [9, pp. 75-76].

Carlitz [1], [2], has generalized $S_1(n, k)$ and $S(n, k)$ by defining *weighted* Stirling numbers $\overline{S}_1(n, k, \lambda)$ and $\overline{S}(n, k, \lambda)$, where λ is a parameter. Carlitz has also investigated the related functions

$$\begin{cases} R_1(n, k, \lambda) = \overline{S}_1(n, k + 1, \lambda) + S_1(n, k) \\ R(n, k, \lambda) = \overline{S}(n, k + 1, \lambda) + S(n, k) \end{cases} \tag{1.4}$$

For all of these numbers, Carlitz has found generating functions, combinatorial interpretations, recurrence formulas, and other properties. See [1] and [2] for details.

The purpose of this paper is to define, in an appropriate way, the weighted associated Stirling numbers $\bar{d}(n, k, \lambda)$ and $\bar{b}(n, k, \lambda)$, and to examine their properties. In particular, we have the following relationships to $\bar{S}_1(n, k, \lambda)$ and $\bar{S}(n, k, \lambda)$:

$$\begin{cases} \bar{S}_1(n, n-k, \lambda) = \displaystyle\sum_{j=0}^{k} \binom{n}{2k-j+1} \bar{d}(2k-j+1, k-j+1, \lambda) \\ \qquad\qquad\qquad + n\lambda S_1(n-1, n-1-k) \\ \bar{S}(n, n-k, \lambda) = \displaystyle\sum_{j=0}^{k} \binom{n}{2k-j+1} \bar{b}(2k-j+1, k-j+1, \lambda) \\ \qquad\qquad\qquad + n\lambda S(n-1, n-1-k) \end{cases} \tag{1.5}$$

We also define and investigate related functions $Q_1(n, k, \lambda)$ and $Q(n, k, \lambda)$, which are analogous to $R_1(n, k, \lambda)$ and $R(n, k, \lambda)$. In particular, we define $Q_1(n, k, \lambda)$ and $Q(n, k, \lambda)$ so that

$$\begin{cases} R_1(n, n-k, \lambda) = \displaystyle\sum_{j=0}^{k} Q_1(2k-j, k-j, \lambda)\binom{n}{2k-j} \\ R(n, n-k, \lambda) = \displaystyle\sum_{j=0}^{k} Q(2k-j, k-j, \lambda)\binom{n}{2k-j} \end{cases} \tag{1.6}$$

which can be compared to (1.3).

The development of the weighted associated Stirling numbers will parallel as much as possible the analogous work in [1] and [2]. In addition to the relationships mentioned above, we shall find generating functions, combinatorial interpretations, recurrence formulas, and other properties of $\bar{d}(n, k, \lambda)$, $\bar{b}(n, k, \lambda)$, $Q_1(n, k, \lambda)$, and $Q(n, k, \lambda)$.

2. THE FUNCTIONS $\bar{d}(n, k, \lambda)$ AND $\bar{b}(n, k, \lambda)$

Let n, k be positive integers, $n \geqslant k$, and $k_2, k_3, \ldots, k_n$ nonnegative such that

$$\begin{cases} k = k_2 + k_3 + \cdots + k_n \\ n = 2k_2 + 3k_3 + \cdots + nk_n. \end{cases} \tag{2.1}$$

Put

$$b(n; k_2, \ldots, k_n; \lambda) = \sum (k_2\lambda^2 + k_3\lambda^3 + \cdots + k_n\lambda^n) \tag{2.2}$$

where the summation is over all the partitions of $Z_n = \{1, 2, \ldots, n\}$ into k_2 blocks of cardinality 2, k_3 blocks of cardinality 3, ..., k_n blocks of cardinality n. Then, following the method of Carlitz [1], we sum on both sides of (2.2) and obtain, after some manipulation,

$$\sum_{n=1}^{\infty} \frac{x}{n!} \sum_{k_1, k_2, \cdots} b(n; k_2, k_3, \ldots; \lambda)y^k = y(e^{\lambda x} - \lambda x - 1)\exp\{y(e^x - x - 1)\}. \tag{2.3}$$

Now we define

$$b(n, k, \lambda) = \sum\sum (k_2\lambda^2 + k_3\lambda^3 + \cdots + k_n\lambda^n), \tag{2.4}$$

(2.4)

where the inner summation is over all partitions of Z_n into k_2 blocks of cardinality 2, k_3 blocks of cardinality 3, ..., k_n blocks of cardinality n; the outer summation is over all $k_2, k_3, \ldots, k_n$ satisfying (2.1).

By (2.3) and (2.4), we have

$$\sum_{n,k} \bar{b}(n, k, \lambda)\frac{x^n}{n!} y^k = y(e^{\lambda x} - \lambda x - 1)\exp\{y(e^x - x - 1)\}, \tag{2.5}$$

and from (2.5) we obtain

$$k!\sum_{n=0}^{\infty} \bar{b}(n, k + 1, \lambda)\frac{x^n}{n!} = (e^{\lambda x} - \lambda x - 1)(e^x - x - 1)^k. \tag{2.6}$$

It follows from (1.2) and (2.6) that

$$\bar{b}(n, k, \lambda) = \sum_{m=2}^{n-2k+2} \binom{n}{m}\lambda^m b(n - m, k - 1). \tag{2.7}$$

For $\lambda = 1$, (2.6) reduces to

$$k!\sum_{n=0}^{\infty} \bar{b}(n, k + 1, 1)\frac{x^n}{n!} = (e^x - x - 1)^{k+1} = (k + 1)!\sum_{n=0}^{\infty} b(n, k + 1)\frac{x^n}{n!}.$$

Thus, we have

$$\bar{b}(n, k, 1) = kb(n, k). \tag{2.8}$$

We also have, by (2.6) and (2.7),

$$\begin{aligned}
\bar{b}(n, 0, \lambda) &= 0,\\
\bar{b}(n, 1, \lambda) &= \lambda \quad \text{if } n \geqslant 2,\\
\bar{b}(n, 2, \lambda) &= \binom{n}{2}\lambda^2 + \binom{n}{3}\lambda^3 + \cdots + \binom{n}{n-2}\lambda^{n-2},\\
\bar{b}(n, k, \lambda) &= 0 \text{ if } n < 2k,\\
\bar{b}(2k, k, \lambda) &= \binom{2k}{2}b(2k - 2, k - 1)\lambda^2.
\end{aligned}$$

The relationship to $\bar{S}(n, k, \lambda)$ is most easily proved by using an extension of a theorem in [7]. In a forthcoming paper [8], we prove the following:

Theorem 2.1: For $r \geqslant 1$ and $f \neq 0$, let

$$F(x) = \sum_{i=r}^{\infty} f_i\frac{x^i}{i!} \quad \text{and} \quad W(x, \lambda) = 1 + \sum_{t=1}^{\infty} w_t(\lambda)\frac{x^t}{t!}$$

be formal power series. Define $B_{n,j}^{(\lambda)}(0, \ldots, 0, f_r, f_{r+1}, \ldots)$ by

$$W(x, \lambda)(F(x))^j = j!\sum_{n=0}^{\infty} B_{n,j}^{(\lambda)}(0, \ldots, 0, f_r, f_{r+1}, \ldots)\frac{x^n}{n!}.$$

Then $\left(\frac{r!}{f_r}\right)^n B_{k+rn,n}^{(\lambda)}(0, \ldots, 0, f_r, f_{r+1}, \ldots) = (k + rn)(k + rn - 1) \ldots (n + 1)$

$$\cdot \sum_{j=0}^{k} \frac{n(n - 1) \ldots (n - j + 1)}{(k + rj)!}\left(\frac{r!}{f_r}\right)^j B_{k+rj,j}^{(\lambda)}(0, \ldots, 0, f_{r+1}, \ldots).$$

It follows from Theorem 2.1 and the generating function for $\overline{S}(n, k, \lambda)$ that if we define

$$(e^{\lambda x} - 1)(e^x - x - 1)^k = k!\sum_{n=0}^{\infty} \overline{a}(n, k+1, \lambda)\frac{x^n}{n!}, \tag{2.9}$$

then

$$\overline{S}(n, n-k, \lambda) = \sum_{j=0}^{k} \binom{n}{2k-j+1} \overline{a}(2k-j+1, k-j+1, \lambda). \tag{2.10}$$

By (2.6) and (2.9),

$$\overline{a}(n, k+1, \lambda) = \overline{b}(n, k+1, \lambda) + \lambda n b(n-1, k),$$

and by (1.3), (2.10) can be wrtiten

$$\overline{S}(n, n-k, \lambda) = \sum_{j=0}^{k} \overline{b}(2k-j+1, k-j+1, \lambda)\binom{n}{2k-j+1} + \lambda n S(n-1, n-1-k), \tag{2.11}$$

which proves $\overline{S}(n, n-k, \lambda)$ is a polynomial in n of degree $2k+1$.

It is convenient to define

$$Q(n, k, \lambda) = \overline{b}(n, k+1, \lambda) + n\lambda b(n-1, k) + b(n, k), \tag{2.12}$$

which implies

$$Q(n, k, \lambda) = \sum_{m=0}^{n-2k} \binom{n}{m} b(n-m, k)\lambda^m. \tag{2.13}$$

Note that $Q(n, k, 0) = b(n, k)$.

A generating function can be found. If we sum on both sides of (2.12), we have

$$\sum_{n,k} Q(n, k, \lambda)\frac{x^n}{n!} y^k = e^{\lambda x}\exp\{y(e^x - x - 1)\}. \tag{2.14}$$

If we differentiate both sides of (2.14) with respect to y and compare the coefficients of $x^n y^k$, we have

$$Q(n, k, \lambda+1) = Q(n, k, \lambda) + (k+1)Q(n, k+1, \lambda) + nQ(n-1, k, \lambda). \tag{2.15}$$

If we differentiate both sides of (2.14) with respect to x, we have

$$Q(n+1, k, \lambda) = \lambda Q(n, k, \lambda) + Q(n, k-1, \lambda+1) - Q(n, k-1, \lambda). \tag{2.16}$$

Combining (2.15) and (2.16), we have our main recurrence formula:

$$Q(n+1, k, \lambda) = (\lambda + k)Q(n, k, \lambda) + nQ(n-1, k-1, \lambda). \tag{2.17}$$

It follows from (3.4) that

$$Q(n, k, 1) = b(n, k) + b(n+1, k).$$

We also have

$$Q(n, 0, \lambda) = \lambda^n,$$

$$Q(n, 1, \lambda) = \binom{n}{0}\lambda^0 + \binom{n}{1}\lambda^1 + \cdots + \binom{n}{n-2}\lambda^{n-2},$$

$$Q(n, k, 0) = b(n, k),$$

$$Q(2k, k, \lambda) = b(2k, k),$$

$$Q(n, k, \lambda) = 0 \text{ if } n < 2k.$$

A small table of values is given below.

$Q(n, k, \lambda)$

$n \backslash k$	0	1	2	3
0	1			
1	λ			
2	λ^2	1		
3	λ^3	$1 + 3\lambda$		
4	λ^4	$1 + 4\lambda + 6\lambda^2$	3	
5	λ^5	$1 + 5\lambda + 10\lambda^2 + 10\lambda^3$	$10 + 15\lambda$	
6	λ^6	$1 + 6\lambda + 15\lambda^2 + 20\lambda^3 + 15\lambda^4$	$25 + 60\lambda + 45\lambda^2$	15

It follows from (2.14) that

$$k!\sum_{n=0}^{\infty} Q(n, k, \lambda)\frac{x^n}{n!} = e^{\lambda x}(e^x - x - 1) . \tag{2.18}$$

By comparing coefficients of x^n on both sides of (2.18), we get an explicit formula for $Q(n, k, \lambda)$:

$$Q(n, k, \lambda) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{t=0}^{k-j}\binom{k-j}{t}(n)_t(\lambda + j)^{n-t}, \tag{2.19}$$

where $(n)_t = n(n-1) \ldots (n-t+1)$.

It follows from Theorem 2.1 and the generating function for $R(n, k, \lambda)$ that

$$R(n, n-k, \lambda) = \sum_{j=0}^{k} Q(2k-j, k-j, \lambda)\binom{n}{2k-j}, \tag{2.20}$$

which shows that $R(n, n-k, \lambda)$ is a polynomial in n of degree $2k$. Equation (2.20) also shows that $R'(n, k, \lambda) = Q(2n-k, n-k, \lambda)$, where $R'(n, k, \lambda)$ is defined by Carlitz in [2].

In [1], Carlitz generalized the Bell number [4, p. 210] by defining

$$B(n, \lambda) = \sum_{k=0}^{n} R(n, k, \lambda). \tag{2.21}$$

This suggests the definition

$$A(n, \lambda) = \sum_{k=0}^{n} Q(n, k, \lambda), \tag{2.22}$$

which for $\lambda = 0$ reduces to

$$A(n) = \sum_{k=0}^{n} b(n, k).$$

The function $A(n)$ appears in [5] and [6].

By (2.13), we have

$$A(n, \lambda) = \sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{n-m} b(n-m, k)\lambda^m = \sum_{m=0}^{n}\binom{n}{m}\lambda^m A(n-m). \tag{2.23}$$

Also by (2.18),

$$\sum_{n=0}^{\infty} A(n,\ \lambda)\frac{x^n}{n!} = e^{\lambda x}\exp(e^x - x - 1), \tag{2.24}$$

and (2.24) implies

$$\sum_{n=0}^{\infty} A(n,\ \lambda)\frac{x^n}{n!} = e^{x(\lambda-1)}\exp(e^x - 1) = \sum_{n=0}^{\infty} B(n,\ \lambda - 1)\frac{x^n}{n!},$$

so

$$A(n,\ \lambda) = B(n,\ \lambda - 1). \tag{2.25}$$

For example, $A(n,\ 1) = B(n,\ 0)$, so

$$\sum_{k=0}^{[(n+1)/2]} (b(n,\ k) + b(n+1,\ k)) = \sum_{k=0}^{n} S(n,\ k).$$

There are combinatorial interpretations of $A(n,\ \lambda)$ and $Q(n,\ k,\ \lambda)$ that are similar to the interpretations of $B(n,\ \lambda)$ and $R(n,\ k,\ \lambda)$ given in [1]. Let λ be a nonnegative integer and let $B_1, B_2, \ldots, B_\lambda$ denote λ open boxes. Let $P(n,\ k,\ \lambda)$ denote the number of partitions of Z_n into k blocks with each block containing at least two elements, with the understanding that an arbitrary number of the elements of Z_n may be placed in any number (possibly none) of the boxes. We shall call these λ_1 partitions. Clearly, $P(n,\ k,\ 0) = b(n,\ k)$.

Now, if i elements are placed in the λ boxes, there are $\binom{n}{i}$ ways to choose the elements, and for each element chosen there are λ choices for a box. The number of such partitions is $\binom{n}{i}\lambda^i b(n - i,\ k)$. Hence,

$$P(n,\ k,\ \lambda) = \sum_{m=0}^{n}\binom{n}{m}\lambda^m b(n - m,\ k) = Q(n,\ k,\ \lambda). \tag{2.26}$$

It is clear from (2.26) that $A(n,\lambda)$ is the number of λ_1 partitions of Z_n.

It is also clear from (2.7) and the above comments that $\overline{b}(n,\ k+1,\ \lambda)$ is the number of λ_1 partitions of Z_n into k blocks such that at least two elements of Z_n are placed in the open boxes. Definition (2.4) furnishes another combinatorial interpretation of $\overline{b}(n,\ k,\ \lambda)$.

Finally, we note that some of the definitions and formulas in this section can be generalized in terms of the r-associated Stirling numbers of the second kind $b_r(n,\ k)$. These numbers are defined by means of

$$\left(e^x - \sum_{i=0} \frac{X^i}{i!}\right)^k = k!\sum_{n=0}^{\infty} b_r(n,\ k)\frac{X^n}{n!},$$

and their properties are examined in [3], [5], and [6]. Using the methods of this section, we can define functions $\overline{b}_r(n,\ k,\ \lambda)$, $Q^{(r)}(n,\ k,\ \lambda)$ and $A^{(r)}(n,\ \lambda)$ which reduce to $\overline{S}(n,\ k,\ \lambda)$, $R(n,\ k,\ \lambda)$, and $B(n,\ \lambda)$ when $r = 0$, and reduce to $\overline{b}(n,\ k,\ \lambda)$, $Q(n,\ k,\ \lambda)$, and $A(n,\ \lambda)$ when $r = 1$. The combinatorial interpretations and formulas (2.4)-(2.7), (2.10), (2.11), (2.17), (2.18), (2.22), (2.23) can all be generalized.

3. THE FUNCTIONS $d(n, k, \lambda)$ AND $Q_1(n, k, \lambda)$

We define $\langle\lambda\rangle_j = \lambda(\lambda + 1) \ldots (\lambda + j - 1)$. Now put

$$d(n;\ k_2,\ \ldots,\ k_n;\ \lambda) = \sum \left(k_2 \frac{\langle\lambda\rangle_2}{1!} + \cdots + k_n \frac{\langle\lambda\rangle_n}{(n-1)!}\right), \tag{3.1}$$

where the summation is over all permutations of Z_n,

$$n = 2k_2 + 3k_3 + \cdots + nk_n,$$

with k_2 cycles of length 2, k_3 cycles of length 3, ..., k_n cycles of length n. Then, as in [1], we sum on both sides of (3.1) and obtain, after some manipulation,

$$\begin{aligned}&\sum_{n=2}^{\infty} \frac{x^n}{n!} \sum_{k_2, k_3, \ldots} d(n;\ k_2,\ k_3,\ \ldots;\ \lambda) y^k \\ &= y((1 - x)^{-\lambda} - \lambda x - 1)\exp\{y(-\log(1 - x) - x)\}.\end{aligned} \tag{3.2}$$

We now define

$$\bar{d}(n,\ k,\ \lambda) = \sum\sum\left(k_2 \frac{\langle\lambda\rangle_2}{1!} + k_3 \frac{\langle\lambda\rangle_3}{2!} + \cdots + k_n \frac{\langle\lambda\rangle_n}{(n-1)!}\right), \tag{3.3}$$

where the inner summation is over all permutations of Z_n with k_2 cycles of length 2, k_3 cycles of length 3, ..., k_n cycles of length n; the outer summation is over all $k_2,\ k_3,\ \ldots,\ k_n$ satisfying (2.1).

By (3.2) and (3.3), we have

$$\begin{aligned}\sum_{n,k} \bar{d}(n,\ k,\ \lambda)\frac{x^n}{n} y^k &= y((1 - x)^{-\lambda} - \lambda x - 1)\exp\{y(-\log(1 - x) - x)\} \\ &= y((1 - x)^{-\lambda} - \lambda x - 1)(1 - x)^{-y} e^{-xy},\end{aligned} \tag{3.4}$$

and from (3.4), we obtain

$$k! \sum_{n=0}^{\infty} \bar{d}(n,\ k + 1,\ \lambda)\frac{x^n}{n} = ((1 - x)^{-\lambda} - \lambda x - 1)(-\log(1 - x) - x)^k. \tag{3.5}$$

It follows from (1.2) and (3.5) that

$$\bar{d}(n,\ k,\ \lambda) = \sum_{m=2}^{n-2k+m} \binom{n}{m} d(n - m,\ k - 1)\langle\lambda\rangle_m. \tag{3.6}$$

For $\lambda = 1$, (3.4) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{d}(n,\ k,\ 1)\frac{x^n}{n} y &= y((1 - x)^{-1} - x - 1)\exp\{y(-\log(1 - x) - x)\} \\ &= \frac{\partial}{\partial x} \exp\{y(-\ln(1 - x) - x)\} - xy \exp\{(y(-\log(1 - x) - x)\} \\ &= \sum_{n,k} d(n + 1,\ k)\frac{x^n}{n!} y^k - \sum_{n,k} nd(n - 1,\ k - 1)\frac{x^n}{n!} y^k.\end{aligned}$$

Thus, we have

$$\bar{d}(n,\ k,\ 1) = d(n + 1,\ k) - nd(n - 1,\ k - 1) = nd(n,\ k). \tag{3.7}$$

We also have, by (3.5) and (3.6),

$$\bar{d}(n, 0, \lambda) = 0$$

$$\bar{d}(n, 1, \lambda) = \langle\lambda\rangle_n \text{ if } n \geqslant 2,$$

$$\bar{d}(n, 2, \lambda) = \binom{n}{2}(n-3)!\langle\lambda\rangle_2 + \binom{n}{3}(n-4)!\langle\lambda\rangle_3 + \cdots + \binom{n}{n-2}1!\langle\lambda\rangle_{n-2},$$

$$\bar{d}(n, k, \lambda) = 0 \text{ if } n < 2k,$$

$$\bar{d}(2k, k, \lambda) = \binom{2k}{2}d(2k-2, k-1)\langle\lambda\rangle_2.$$

To find the relationship to $\overline{S}_1(n, k, \lambda)$, we use Theorem (2.1). We define $\bar{c}(n, k, \lambda)$ by

$$((1-x)^{-\lambda} - 1)(-\log(1-x) - x)^k = k!\sum_{n=0}^{\infty}\bar{c}(n, k+1, \lambda)\frac{x^n}{n!}. \tag{3.8}$$

Then by Theorem 2.1 and the generating function for $\overline{S}_1(n, k, \lambda)$,

$$\overline{S}_1(n, n-k, \lambda) = \sum_{j=0}^{k}\binom{n}{2k-j+1}\bar{c}(2k-j+1, k-j+1, \lambda). \tag{3.9}$$

By (3.5) and (3.8),

$$\bar{c}(n, k+1, \lambda) = \bar{d}(n, k+1, \lambda) + \lambda n d(n-1, k),$$

so by (1.3), equation (3.9) can be written

$$\begin{aligned}\overline{S}_1(n, n-k, \lambda) = \sum_{j=0}^{k}\bar{d}(2k-j+1, k-j+1, \lambda)\binom{n}{2k-j+1}\\ + \lambda n S_1(n-1, n-1-k),\end{aligned} \tag{3.10}$$

which proves $\overline{S}_1(n, n-k, \lambda)$ is a polynomial in n of degree $2k+1$.

We now define the function $Q_1(n, k, \lambda)$ by means of

$$Q_1(n, k, \lambda) = \bar{d}(n, k+1, \lambda) + d(n, k) + n\, d(n-1, k). \tag{3.11}$$

then by (3.6),

$$Q_1(n, k, \lambda) = \sum_{m=0}^{n-2k}\binom{n}{m}d(n-m, k)\langle\lambda\rangle_m. \tag{3.12}$$

Note that $Q_1(n, k, 0) = d(n, k)$.

A generating function can be found by summing on both sides of (3.11). We have

$$\begin{aligned}\sum_{n,k} Q_1(n, k, \quad)\frac{x^n}{n} y &= (1-x)^{-\lambda}\exp\{y(-\log(1-x) - x)\}\\ &= (1-x)^{-\lambda-y}e^{-xy}.\end{aligned} \tag{3.13}$$

If we differentiate (3.13) with respect to x, multiply by $1-x$, and then compare coefficients of $x^n y^k$, we obtain

$$Q_1(n+1, k, \lambda) = (\lambda + n)Q_1(n, k, \lambda) + nQ_1(n-1, k-1, \lambda). \tag{3.14}$$

If we multiply both sides of (3.13) by $1-x$ and compare coefficients $x^n y^k$, we have

$$Q_1(n, k, \lambda - 1) = Q_1(n, k, \lambda) - nQ_1(n - 1, k, \lambda). \tag{3.15}$$

For $\lambda = 1$, (3.14) and (3.15) can be combined to yield

$$d(n + 1, k + 1) = nQ_1(n - 1, k, 1). \tag{3.16}$$

Also, if $\lambda = 0$ in (3.15), we have

$$Q_1(n, k, -1) = d(n, k) - nd(n - 1, k).$$

In addition

$$Q_1(n, 0, \lambda) = \langle\lambda\rangle_n,$$

$$Q_1(n, 1, \lambda) = (n - 1)! + \binom{n}{1}(n - 2)!\langle\lambda\rangle_1 + \cdots + \binom{n}{n-2}1!\langle\lambda\rangle_{n-2},$$

$$Q_1(n, k, 0) = d(n, k),$$

$$Q_1(2k, k, \lambda) = d(2k, k),$$

$$Q_1(n, k, \lambda) = 0 \text{ if } n < 2k.$$

A small table of values is given below.

$Q_1(n, k, \lambda)$

$n \backslash k$	0	1	2	3
0	1			
1	λ			
2	$\langle\lambda\rangle_2$	1		
3	$\langle\lambda\rangle_3$	$2 + 3\lambda$		
4	$\langle\lambda\rangle_4$	$6 + 14\lambda + 6\lambda^2$	3	
5	$\langle\lambda\rangle_5$	$24 + 70\lambda + 50\lambda^2 + 10\lambda^3$	$20 + 15\lambda$	
6	$\langle\lambda\rangle_6$	$120 + 404\lambda + 375\lambda^2 + 130\lambda^3 + 15\lambda^4$	$130 + 65\lambda + 45\lambda^2$	15

It follows from (3.13) that

$$k!\sum_{n=0}^{\infty} Q_1(n, k, \lambda)\frac{x^n}{n!} = (1 - x)^{-\lambda}(-\log(1 - x) - x)^k, \tag{3.17}$$

and from Theorem 2.1, that

$$R_1(n, n - k, \lambda) = \sum_{j=0}^{k} Q_1(2k - j, k - j, \lambda)\binom{n}{2k - j}, \tag{3.18}$$

which shows that $R_1(n, n - k, \lambda)$ is a polynomial in n of degree $2k$. Equation (3.18) also shows that $R'(n, k, \lambda) = Q_1(2n - k, n - k, \lambda)$, where $R_1'(n, k, \lambda)$ is defined by Carlitz in [2].

Letting $y = 1$ in (3.13), we have

$$\sum_{k=0}^{[n/2]} Q_1(n, k, \lambda) = \sum_{t=0}^{n} (-1)^{n-t}\binom{n}{t}\langle\lambda + 1\rangle_t,$$

and more generally,

$$\sum_{k=0}^{[n/2]} Q_1(n, k, \lambda)y^k = \sum_{t=0}^{n} (-y)^{n-t}\binom{n}{t}\langle \lambda + y \rangle_t.$$

A combinatorial interpretation of $Q_1(n, k, \lambda)$ follows. Let λ be a nonnegative integer and let $B_1, B_2, \ldots, B_\lambda$ denote λ open boxes. Let $P_1(n, k, \lambda)$ denote the number of permutations of Z_n with k cycles such that each cycle contains at least two elements, with the understanding that an arbitrary number of elements of Z_n may be placed in any number (possibly none) of the boxes and then permuted in all possible ways in each box. We call these λ_1 permutations. Clearly, $P_1(n, k, 0) = d(n, k)$.

If i elements are placed in the boxes, there are $\binom{n}{i}$ ways to choose the elements and then $\lambda(\lambda + 1)$ $(\lambda + 2) \ldots (\lambda + i - 1)$ ways to place the elements in the boxes. The number of such permutations is $\binom{n}{i}\langle\lambda\rangle_i d(n - i, k)$. Hence,

$$P_1(n, k, \lambda) = \sum_{m=0}^{n}\binom{n}{m}\langle\lambda\rangle_m d(n - m, k) = Q_1(n, k, \lambda). \tag{3.19}$$

It is clear from (3.6) and the above comments that $\bar{d}(n, k + 1, \lambda)$ is the number of λ_1 permutations of Z_n with k cycles such that at least two elements of Z_n are placed in the open boxes. Definition (3.3) furnishes another combinatorial interpretation of $\bar{d}(n, k, \lambda)$.

We note that some of the definitions and formulas in this section can be generalized in terms of the r-associated Stirling numbers of the first kind $d_r(n, k)$. These numbers are defined by means of

$$\left(-\log(1 - x) - \sum_{i=1}^{r}\frac{X^i}{i!}\right)^k = k!\sum_{n=0}^{\infty} d_r(n, k)\frac{X^n}{n!},$$

and their properties are discussed in [3] and [6]. Using the methods of this section, we can define functions $d_r(n, k, \lambda)$ and $Q^{(r)}(n, k, \lambda)$ which reduce to $\bar{S}_1(n, k, \lambda)$ and $R_1(n, k, \lambda)$ when $r = 0$, and to $\bar{d}(n, k, \lambda)$ and $Q_1(n, k, \lambda)$ when $r = 1$. The combinatorial interpretations and formulas (3.3)-(3.6), (3.11)-(3.14), and (3.17) can all be generalized.

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SOME OPERATIONAL FORMULAS FOR THE q-LAGUERRE POLYNOMIALS

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1. INTRODUCTION

L. B. Rédei [7] proved an operational identity for the Laguerre polynomials that was later generalized by Viskov [9]. Viskov's main results were as follows: if $D = d/dx$, then for $n = 0, 1, 2, \ldots$, we have

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} e^x\{(\alpha + 1 + xD)D\}^n e^{-x} = \frac{(-1)^n}{n!} e^x B^n e^{-x} \tag{1.1}$$

and

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\{(1 + \alpha - x + xD)(1 - D)\}^n \cdot 1, \tag{1.2}$$

where $L^{(\alpha)}(x)$ is the n^{th} Laguerre polynomial.

A third formula of a similar nature was given earlier by Carlitz [2]:

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\prod_{k=1}^{n}(xD - x + \alpha + k) \cdot 1. \tag{1.3}$$

Recently, there has been renewed interest in q-identities and operators as well as in the q-Laguerre polynomial (see, e.g., [3], [5], [6]). Therefore, we felt it would also be interesting to discuss q-generalizations of the identities (1.1)-(1.3). In the following, we shall assume always that $|q| < 1$.

We first introduce the following notation:

$[a]_0 = (a;\, q)_0 = 1;$

$[a]_n = (a;\, q)_n = (1 - a)(1 - aq) \ldots (1 - aq^{n-1}) \quad (n = 1, 2, 3, \ldots).$

Also, we shall use $[a]_\infty = (a;\, q)_\infty$ to mean the convergent product

$$\prod_{k=0}^{\infty}(1 - aq^k).$$

It is well known that $[a]_\infty$ is a q-analog of the exponential function. Thus, we have

$$\lim_{q \to 1^-} (-(1 - q)x;\, q)_\infty^{-1} = e^{-x}.$$

For this reason, the more suggestive notation

$$(x;\, q)_\infty^{-1} = e_q(x)$$

is used for a q-analog of the exponential function.

The q-derivatives of a function $f(x)$ is given by

$$D_q f(x) = \frac{f(x) - f(xq)}{x},$$

so whenever f has a derivative at x, we have

$$\lim_{q \to 1} \frac{1}{(1 - q)} D_q f(x) = f'(x).$$

We shall also use the substitution operator $\eta : \eta f(x) = f(qx)$. It is related to the q-derivative by means of $xD_q = I - \eta$, where I is the identity operator. Note that x and D_q do not commute.

We recall that the q-Laguerre polynomials [6] are defined by

$$L_n^{(\alpha)}(x|q) = \frac{[q^{\alpha+1}]n}{[q]_n} \sum_{k=0}^{n} \frac{[q^{-n}]\ q^{\frac{1}{2}k(k+1)+k(n+\alpha)}}{[q]_k [q^{\alpha+1}]_k} x^k \tag{1.4}$$

so that

$$\lim_{q \to 1} L_n^{(\alpha)}((1-q)x|q) = L_n^{(\alpha)}(x), \quad n = 0, 1, 2, \ldots .$$

These polynomials, which are orthogonal and belong to an indetermined Stieltjes moment problem ([3] and [6]), were known to W. Hahn [5]. They have, among other properties, a Rodrigues formula:

$$L_n^{(\alpha)}(x|q) = \frac{x^{-\alpha}[-x]_\infty}{[q]_n} D_q^n \left\{\frac{x^{\alpha+n}}{[-x]_\infty}\right\}. \tag{1.5}$$

Cigler [4] gave the representation

$$L_n^{(\alpha)}(x|q) = \frac{(-1)^n}{[q]_n}(\eta - D_q)^{n+\alpha} x^n = (-1)^n x^{-\alpha} \frac{1}{[q]_n}(\eta - D_q)^n x^{n+\alpha}. \tag{1.6}$$

Representations (1.5) and (1.6) are both of the same nature—the n^{th} iterate of the operator (D_q or $\eta - D_q$, respectively) acts on a function that depends on n also. In some applications, this is a drawback. This is why (1.1) and (1.2) are interesting.

2. A q-ANALOG OF THE REDEI-VISKOV OPERATOR

Put

$$B_q = \{(1 - q^{\alpha+1})I + q^{\alpha+1}xD_q\}D_q. \tag{2.1}$$

Thus, formally, we have

$$\lim_{q \to 1} \frac{B_q}{(1-q)^2} f(x) = (\alpha + 1 + xD)Df(x) = Bf(x),$$

which is the operator that appears in the right-hand side of (1.1).

It is easy to see that

$$B_q x^n = (1 - q^n)(1 - q^{n+\alpha})x^{n-1}, \tag{2.2}$$

from which we can verify another representation for the B_q operator, namely,

$$B_q = (I - q^{\alpha+1}\eta)D_q \tag{2.3}$$

and

$$B_q = x^{-\alpha} D_q x^{\alpha+1} D_q = D_q x^{1-\alpha} D_q x^\alpha. \tag{2.4}$$

The latter representation shows that the operator B_q is also a q-analog of the Bessel operator (see [10]):

$$B = x^{-\alpha} \frac{d}{dx} x^{\alpha+1} \frac{d}{dx}.$$

From the relation $D_q(1 - q^{\alpha+1}\eta) = (1 - q^{\alpha+2}\eta)D_q$, we get, by induction,

$$B_q^n = \left\{ \prod_{k=1}^{n} (1 - q^{\alpha+k}\eta) \right\} D_q^n \qquad (n = 0, 1, 2, \ldots). \tag{2.5}$$

It is easy to see that $D_q\{e_q(-x)f(x)\} = e_q(-x)(D_q - \eta)f(x)$. Thus, we have, for any formal power series $F(x)$,

$$F(D_q)\{e_q(-x)f(x)\} = e_q(-x)F(D_q - \eta)f(x). \tag{2.6}$$

Using mathematical induction and noting that

$$(D_q - \eta)(I - q^{\mu}(1 + x)\eta) = (I - q^{\mu+1}(1 + x)\eta)(D_q - \eta),$$

we get

$$\begin{aligned} B_q^n\{e_q(-x)f(x)\} &= e_q(-x)\{(I - q^{\alpha+1}(1 + x)\eta)(D_q - \eta)\}^n \cdot f(x) \\ &= e_q(-x)\left\{ \prod_{k=1}^{n} (1 - q^{\alpha+k}(1 + x)\eta) \right\} (D_q - \eta)^n \cdot f(x) \\ &= \frac{1}{x^n} e_q(-x) \prod_{k=1}^{n} (1 - q^{\alpha+k-n}(1 + x)\eta)(1 - q^{k-n}(1 + x)\eta) \cdot f(x). \end{aligned}$$

Now, to obtain operational representations for the q-Laguerre polynomials, we first calculate

$$\begin{aligned} B_q^n e_q(-x) &= B_q^n \sum_{k=0}^{\infty} \frac{(-1)^k}{[q]_k} x^k \\ &= \sum_{k=n}^{\infty} \frac{(1-q^k)(1-q^{k-1}) \ldots (1-q^{k-n+1})(1-q^{k+\alpha}) \ldots (1-q^{k-n+\alpha+1})(-x)^{k-n}}{[q]_k} \\ &= (-1)^n [q^{\alpha+1}]_n \sum_{k=0}^{\infty} \frac{[q^{\alpha+n+1}]_k}{[q]_k [q^{\alpha+1}]_k} (-x)^k. \end{aligned}$$

Andrews [1] gave a q-analog of Kummer's Theorem, i.e.,

$$\sum_{k=0}^{\infty} \frac{[\beta]_k [\alpha]_k (-1)^k q^{\frac{1}{2}k(k-1)}}{[q]_k [\gamma]_k [x\alpha]_k} \left(\frac{x\gamma}{\beta}\right)^k = \frac{[x]_\infty}{[x\alpha]_\infty} \sum_{k=0}^{\infty} \frac{[\gamma/\beta]_k [\alpha]_k}{[q]_k [\gamma]_k} x^k \tag{2.8}$$

Putting $\alpha = 0$ in this formula, replacing x by $-x$, and then taking $\gamma = q^{\alpha+1}$, $\beta = q^{-n}$, we get that

$$B_q^n e_q(-x) = \frac{(-1)^n [q^{\alpha+1}]_n}{[-x]_\infty} \sum_{k=0}^{n} \frac{[q^{-n}]_k q^{\frac{1}{2}k(k-1)+k(\alpha+n+1)}}{[q]_k} x^k = \frac{(-1)^n [q]_n}{[-x]_\infty} L_n^{(\alpha)}(x|q).$$

Together with (2.7), this formula gives the following three representations:

$$L_n^{(\alpha)}(x|q) = \frac{(-1)^n}{[q]_n}\{e_q(-x)\}^{-1} B_q^n\{e_q(-x)\}; \tag{2.9}$$

$$= \frac{1}{[q]_n} \prod_{k=1}^{n} (I - q^{\alpha+k}(1 + x)\eta) \cdot 1; \tag{2.10}$$

$$= \frac{(-1)^n}{[q]_n}\{(I - q^{\alpha+1}(1 + x)\eta)(D_q - \eta)\}^n \cdot 1. \tag{2.11}$$

If we let $q \to 1$, then (2.9), after suitable normalization by $(1 - q)$, reduces to (1.1); (2.11) reduces to (1.2); and (2.10) reduces to Carlitz's formula (1.3).

Using (2.2) and (1.4), we get, for $m = 0, 1, 2, \ldots,$

$$B_q^m L_n^{(\alpha)}(x|q) = (-1)^m \frac{[q^{\alpha+1}]_n}{[q^{\alpha+1}]_{n-m}} q^{m(m+\alpha)} L_{n-m}^{(\alpha)}(q^{2m}x|q). \tag{2.12}$$

Notice that the operation on $L_n^{(\alpha)}(x|q)$ by B_q reduces the degree by one without changing the value of the parameter α.

There is another q-analog of the exponential function e^{-x}, namely,

$$E_q(x) = \sum_{k=0}^{\infty} \frac{(-1)\ q^{\frac{1}{2}k(k-1)}}{[q]_k} x^k = \prod_{j=0}^{\infty}(1 - xq^j).$$

If we repeat the above calculation, we can show that

$$B_q^n E_q(x) = (-1)^n [q^{\alpha+1}]_n q^{\frac{1}{2}n(n-1)} \sum_{j=0}^{n} \frac{(-1)^j q^{\frac{1}{2}j(j-1)+nj}[q^{\alpha+n+1}]_j}{[q]_j [q^{\alpha+1}]_j} x^j.$$

Once again we can transform the right-hand side of this formula by using (2.8) (with $\alpha = 0$, $\beta = q^{\alpha+n+1}$, $\gamma = q^{\alpha+1}$, and $x \to xq^{2n}$), to obtain

$$B_q^n\{E_q(x)\} = (-1)^n q^{\frac{1}{2}n(n-1)+\alpha n}[q]_n E_q(xq^{2n}) L_n^{(\alpha)}(-xq^{2n-1}|q^{-1}). \tag{2.13}$$

Comparison formulas to this are:

$$e_q(-\eta^{-2}B_q)\{x^n\} = (-1)^n q^{-n(n-1)}[q]_n L_n^{(\alpha)}(q^{\alpha+1}x|q) \tag{2.14}$$

and

$$E_q(-B_q)\{x^n\} = (-1)^n [q]_n q^{\frac{1}{2}n(n-1)} L_n^{(\alpha)}(xq^{2n-1}|q^{-1}). \tag{2.15}$$

Both formulas (2.14) and (2.15) reduce in the case $q \to 1$ to the new formula for the ordinary Laguerre polynomials:

$$e^{-B}x^n = (-1)^n n! L_n^{(\alpha)}(x) \qquad (n = 0, 1, 2, \ldots). \tag{2.16}$$

On the other hand, formula (2.13) reduces to (1.1).

If we calculate the right-hand side of (2.9) directly, we get

$$\begin{aligned} B_q^n e_q(-x) &= (-1)^n \prod_{k=1}^{n}(1 - q^{\alpha+1}\eta)e_q(-x) \\ &= \sum_{j=0}^{n}(-1)^{n+j}\begin{bmatrix} n \\ j \end{bmatrix} q^{\frac{1}{2}j(j-1)+(\alpha+1)j}\eta^j e_q(-x) \\ &= e_q(-x)\sum_{j=0}^{n}(-1)^{n+j}\begin{bmatrix} n \\ j \end{bmatrix} q^{\frac{1}{2}j(j+1)+\alpha j}[-x]_j. \end{aligned} \tag{2.17}$$

The second equality is due to the Euler identity

$$\prod_{k=1}^{n}(1 - q^{k-1}x) = \sum_{j=0}^{n}(-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{\frac{1}{2}j(j-1)} x^j,$$

where $\begin{bmatrix} n \\ j \end{bmatrix}$ stands for the q-binomial coefficient, i.e., for 1 if $j = 0$ and for

$$(1 - q)(1 - q^{n-1}) \ldots (1 - q^{n-j+1})/(1 - q)(1 - q^2) \ldots (1 - q^j)$$

if $j \geqslant 1$. Combining (2.9) and (2.17), we obtain another expansion for the q-Laguerre polynomial:

$$L_n^{(\alpha)}(x|q) = \frac{1}{[q]_n} \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} [-x]_j q^{\frac{1}{2}j(j+1)+\alpha j}. \tag{2.18}$$

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THE NUMBER FIELD $Q(\sqrt{5})$ AND THE FIBONACCI NUMBERS

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1. INTRODUCTION

The set of algebraic integers (hereinafter called integers) of the quadratic number field $Q(\sqrt{5})$ is given by

$$Z(\omega) = \{a + b\omega : a,\ b \in Z\},$$

where $\omega = \frac{1}{2}(1 + \sqrt{5})$. It is well known that $Z(\omega)$ is a Euclidean domain [6, pp. 214-15], and that the units of $Z(\omega)$ are given by $\pm\omega^n$, where $n \in Z$ [6, p. 221].

The Binet formula

$$F_n = (\omega^n - \overline{\omega}^n)/(\omega - \overline{\omega}) = (\omega^n - \overline{\omega}^n)/\sqrt{5},$$

where $\overline{\omega} = \frac{1}{2}(1 - \sqrt{5})$ is the conjugate of ω, expresses the n^{th} Fibonacci number in terms of the unit ω. Simiarly, the n^{th} Lucas number is given by $L_n = \omega^n + \overline{\omega}^n$. Also, an elementary induction argument using the result $\omega^2 = \omega + 1$ shows that $\omega^n = F_{n-1} + F_n\omega$ for $n \geqslant 1$. These results suggest that the arithmetic theory of $Z(\omega)$ can be a powerful tool in the investigation of the arithmetical properties of the Fibonacci and Lucas numbers. This is indeed the case, and the articles by Carlitz [4], Lind [10], and Lagarias & Weisser [9] utilize $Z(\omega)$ on a limited scale. In this paper, I further document the utility of $Z(\omega)$ by deriving many of the familiar divisibility properties of the Fibonacci numbers using the arithmetic theory of $Z(\omega)$. Much of the development has been adapted from pages 164-174 of my doctoral dissertation [5], which gives a comprehensive treatment of number theory in $Z(\omega)$.

2. CONVENTIONS AND PRELIMINARIES

We assume it is known that $Z(\omega)$ is a Euclidean domain and that the units of $Z(\omega)$ are given by $\pm\omega^n$. In the proof of Theorem 5, we use some results from quadratic residue theory. Apart from this, only the first notions of elementary number theory are taken for granted.

Throughout this paper, lower case Latin letters denote rational integers (elements of Z), and lower case Greek letters denote elements of $Z(\omega)$. The Fibonacci number F_n is denoted by $F(n)$, and n is called the index of the Fibonacci number $F(n)$. Also, p and q denote rational primes; and m, n, and r denote positive rational integers. A greatest common divisor of α and β is denoted by $\mathrm{GCD}(\alpha, \beta)$. Of course, $\mathrm{GCD}(\alpha, \beta)$ is unique up to associates. We continue to use $\gcd(a, b)$ in the sense of rational integer theory; that is, $\gcd(a, b)$ is the unique largest positive rational integer that divides both a and b. We say that α and β are congruent modulo μ, and write $\alpha \equiv \beta \pmod{\mu}$, provided $\mu \mid (\alpha - \beta)$; that is, $\alpha - \beta = \gamma\mu$ for some γ. We continue to use $a \equiv b$ (mod m) in the traditional rational integer sense. In the present setting this notation is a bit superfluous since $a \equiv b$ (mod m) if and only if $a \equiv b$ (MOD m). As in rational integer theory, $\alpha + \gamma \equiv \beta + \delta$ (MOD μ) and $\alpha\gamma \equiv \beta\delta$ (MOD μ) whenever $\alpha \equiv \beta$ (MOD μ) and $\gamma \equiv \delta$ (MOD μ). Finally, it is clear that $m \mid (c + d\omega)$ in the sense of $Z(\omega)$ if and only if $m \mid c$ and $m \mid d$ in the sense of Z.

3. SIMPLEST DIVISIBILITY PROPERTIES

Our first efforts will be directed toward establishing the classic results listed in Theorem 4. The reader is no doubt familiar with the standard proofs such as found in [6, pp. 148-49] and [11, pp. 29-32]. The attack here is different: An arithmetical function $V(n)$ with values in $Z(\omega)$ and closely related to $F(n)$ is introduced. This function will be shown to have properties analogous to those of $F(n)$ in Theorem 4. Theorem 4 will than follow as a simple corollary.

Definition 1: $V(n) = \omega^{2n} - (-1)^n$.

Theorem 1: $V(n) = \sqrt{5}\omega^n F(n)$.

Proof: By the Binet formula, we have

$$\sqrt{5}\omega^n F(n) = \omega^n(\omega^n - \overline{\omega}^n) = \omega^{2n} - (\omega\overline{\omega})^n = \omega^{2n} - (-1)^n = V(n). \quad \text{Q.E.D.}$$

Theorem 2: If $m|n$, then $V(m)|V(n)$.

Proof: Let $\alpha = \omega^{2m}$, $\beta = (-1)^m$, and $n = mt$, so that

$$V(m) = \alpha - \beta \quad \text{and} \quad V(n) = \alpha^t - \beta^t.$$

Then $\gamma = \alpha^{t-1} + \alpha^{t-2}\beta + \cdots + \beta^{t-1}$ is an integer and

$$V(n) = \alpha^t - \beta^t = (\alpha - \beta)\gamma = V(m) \cdot \gamma.$$

Thus $V(m)|V(n)$. Q.E.D.

Lemma 1: If $\omega^{2n} \equiv (-1)^n \pmod{\mu}$, then $\omega^{2na} \equiv (-1)^{na} \pmod{\mu}$ for any rational integer a.

Proof: If $a \geqslant 0$, the result is immediate. If $a < 0$, $\omega^{-2na} \equiv (-1)^{-na} \pmod{\mu}$. Multiplying both sides of the last congruence by the integer $(-1)^{na}\omega^{2na}$, we obtain $\omega^{2na} \equiv (-1)^{na} \pmod{\mu}$. Q.E.D.

Theorem 3: If $d = \gcd(n, m)$, then $\mathrm{GCD}(V(n), V(m)) = V(d)$.

Proof: Let $\delta = \mathrm{GCD}(V(n), V(m))$. Since $d = \gcd(n, m)$, there exist a and b such that $d = ma + nb$. Now $V(m) \equiv 0 \pmod{\delta}$, so that $\omega^{2m} \equiv (-1)^m \pmod{\delta}$. Similarly, $\omega^{2n} \equiv (-1)^n \pmod{\delta}$. Thus, by Lemma 1,

$$\omega^{2ma} \equiv (-1)^{ma} \pmod{\delta} \quad \text{and} \quad \omega^{2nb} \equiv (-1)^{nb} \pmod{\delta}.$$

Accordingly, $\omega^{2ma+2nb} \equiv (-1)^{ma+nb} \pmod{\delta}$, and since $d = ma + nb$, $\omega^{2d} \equiv (-1)^d \pmod{\delta}$. Consequently, $V(d) = \omega^{2d} - (-1)^d \equiv 0 \pmod{\delta}$; that is, $\delta|V(d)$. Conversely, since $d|n$ and $d|m$, $V(d)|V(n)$ and $V(d)|V(m)$ by Theorem 2; and so $V(d)|\delta$. We thus conclude that $\delta = V(d)$ (up to associates). Q.E.D.

Theorem 4: (i) If $m|n$, then $F(m)|F(n)$.
(ii) If $d = \gcd(m, n)$, then $\gcd(F(m), F(n)) = F(d)$. In particular, if $\gcd(m, n) = 1$, then $\gcd(F(m), F(n)) = 1$.

(iii) If $\gcd(n, m) = 1$, then $F(m) \cdot F(n) \mid F(mn)$.
(iv) If $m > 2$, then $m \mid n$ if and only if $F(m) \mid F(n)$.

Proof: If $m \mid n$, then $V(m) \mid V(n)$ by Theorem 2. Thus, by Theorem 1, $\sqrt{5}\omega^m F(m)$ divides $\sqrt{5}\omega^n F(n)$; and since ω is a unit, $F(m) \mid F(n)$. This establishes (i). By Theorems 1 and 3, we have

$$\begin{aligned}\sqrt{5}\omega^d F(d) = V(d) &= \mathrm{GCD}(V(m),\ V(n)) = \mathrm{GCD}(\sqrt{5}\omega^m F(m),\ \sqrt{5}\omega^n F(n)) \\ &= \sqrt{5}\,\mathrm{GCD}(F(m),\ F(n)).\end{aligned}$$

Thus $F(d) = \mathrm{GCD}(F(m), F(n))$, and so $F(d) = \gcd(F(m), F(n))$. Consequently, (ii) is true. Now (iii) follows from (i) and (ii), because $F(m) \mid F(mn)$, $F(n) \mid F(mn)$, and $\gcd(F(m), F(n)) = 1$. Half of (iv) follows from (i). Suppose $F(m) \mid F(n)$. Then by (ii) we have $F(m) = \gcd(F(m), F(n)) = F(d)$, where $d = \gcd(m, n)$. Thus $F(m) = F(d)$; and if $m > 2$, we have $m = d = \gcd(m, n)$, so that $m \mid n$. Q.E.D.

Corollary 1: $\gcd(F(n), F(n + 1)) = 1$.

Proof: We have $\gcd(n, n + 1) = 1$, and so, by Theorem 4(ii),

$$\gcd(F(n),\ F(n + 1)) = F(1) = 1. \quad \text{Q.E.D.}$$

4. LAW OF APPARITION AND RELATED RESULTS

If $m > 0$ is given, then a classical result states that m divides some Fibonacci number having positive index not exceeding m^2 [7, p. 44]. In this section we deal with various aspects of this problem. The key results we need from the arithmetic theory of $Z(\omega)$ are found in Theorems 5 and 6. Theorem 5 and its proof is a special case of Theorem 258 in Hardy and Wright [6, pp. 222-23]. Theorem 6, although trivial to prove, will be used many times in the remainder of this paper.

Theorem 5: If $p \equiv \pm 2 \pmod 5$ and $q \equiv \pm 1 \pmod 5$, then

$$\text{(i)}\quad \omega^{p+1} \equiv -1 \ (\mathrm{MOD}\ p) \qquad \text{and} \qquad \text{(ii)}\quad \omega^{q-1} \equiv 1 \ (\mathrm{MOD}\ q).$$

Proof: Since $\omega^2 - \overline{\omega} = \omega + 1 - (1 - \omega) = 2\omega$, then $\omega^2 \equiv \overline{\omega}$ (MOD 2). Accordingly, $\omega^3 \equiv \omega\overline{\omega} = -1$ (MOD 2) and the result is true for $p = 2$.

Now let $t \neq 5$ be an odd rational prime. Since $2^t \equiv 2 \pmod t$, by Fermat's theorem for rational integers, we have

$$2\omega^t \equiv (2\omega)^t = (1 + \sqrt{5})^t \equiv 1 + 5^{\frac{1}{2}(t-1)}\sqrt{5} \ (\mathrm{MOD}\ t).$$

By Euler's criterion for quadratic residues, $5^{\frac{1}{2}(t-1)} \equiv (5 \mid t) \pmod t$. Therefore, $2\omega^t \equiv 1 + (5 \mid t)\sqrt{5}$ (MOD t). By quadratic reciprocity, $(5 \mid p) = (p \mid 5) = -1$ and $(5 \mid q) = (q \mid 5) = 1$. Thus

$$2\omega^p \equiv 1 - \sqrt{5} = 2\overline{\omega} \ (\mathrm{MOD}\ p) \qquad \text{and} \qquad 2\omega^q \equiv 1 + \sqrt{5} = 2\omega \ (\mathrm{MOD}\ q).$$

By cancellation, $\omega^p \equiv \overline{\omega}$ (MOD p) and $\omega^q \equiv \omega$ (MOD q). Thus

$$\omega^{p+1} \equiv \omega\overline{\omega} = -1 \ (\mathrm{MOD}\ p) \qquad \text{and} \qquad \omega^{q-1} = \omega^{-1}\omega^q \equiv \omega^{-1}\omega = 1 \ (\mathrm{MOD}\ q). \quad \text{Q.E.D.}$$

Theorem 6: We have that $m|F(n)$ if and only if ω^n is congruent modulo m to a rational integer. Moreover, if $m|F(n)$, then $\omega^n \equiv F(n-1)$ (MOD m).

Proof: If $m|F(n)$, then

$$\omega^n = F(n-1) + F(n)\omega \equiv F(n-1) \text{ (MOD } m).$$

Conversely, if $\omega^n \equiv a$ (MOD m), then $\overline{\omega}^n \equiv a$ (MOD m). Thus, we have $\omega^n - \overline{\omega}^n \equiv 0$ (MOD m); and since

$$\omega^n - \overline{\omega}^n = \sqrt{5}F(n) = (-1 + 2\omega)F(n) = -F(n) + 2F(n)\omega,$$

it follows that $m|F(n)$. Q.E.D.

Theorem 7 (Law of Apparition): If $p \equiv \pm 2 \pmod 5$ and $q \equiv \pm 1 \pmod 5$, then

$$\text{(i)}\quad p|F(p+1),\quad \text{(ii)}\quad q|F(q-1),\ \text{and}\quad \text{(iii)}\quad 5|F(5).$$

Proof: By Theorem 5, $\omega^{p+1} \equiv -1$ (MOD p) and $\omega^{q-1} \equiv 1$ (MOD q). Thus, by Theorem 6, $p|F(p+1)$ and $q|F(q-1)$. Assertion (iii) is immediate, because $F(5) = 5$. Q.E.D.

Theorem 8: If $p^r|F(n)$, then $p^{r+1}|F(np)$.

Proof: Since $p^r|F(n)$, $\omega^n \equiv a$ (MOD p^r) by Theorem 6. Thus $\omega^n = a + p^r\alpha$ and so

$$\omega^{np} = (a + p^r\alpha)^p \equiv a^p + pa^{p-1}p^r\alpha \equiv a^p \text{ (MOD } p^{r+1}).$$

It therefore follows from Theorem 6 that $p^{r+1}|F(np)$. Q.E.D.

Theorem 9: If $p|F(n)$, then $p^r|F(p^{r-1}n)$.

Proof: The proof is by induction on r. By hypothesis, the result holds for $r = 1$; and if $p^r|F(p^{r-1}n)$, then $p^{r+1}|F(p^rn)$ by Theorem 8. Q.E.D.

Theorem 10: If $p \equiv \pm 2 \pmod 5$ and $q \equiv \pm 1 \pmod 5$, then

$$\text{(i)}\quad p^r|F(p^{r-1}(p+1)),\quad \text{(ii)}\quad q^r|F(q^{r-1}(q-1)),\quad \text{(iii)}\quad 5^r|F(5^r).$$

Proof: Immediate from Theorems 7 and 9.

Definition 2: If $p \equiv \pm 2 \pmod 5$ and $q \equiv \pm 1 \pmod 5$, then

$$T(1) = 1,\ T(p^r) = p^{r-1}(p+1),\ T(q^r) = q^{r-1}(q-1),\ T(5^r) = 5^r;$$

and if m has the rational prime decomposition $m = p_1^{e_1}p_2^{e_2}\ \dots\ p_s^{e_s}$, then

$$T(m) = \operatorname{lcm}(T(p_1^{e_1}),\ T(p_2^{e_2}),\ \dots,\ T(p_s^{e_s})).$$

Theorem 11: We have $m|F(T(m))$.

Proof: The result is certainly true if $m = 1$. If $m > 1$, then let m have the rational prime decomposition

$$m = p_1^{e_1}p_2^{e_2}\ \dots\ p_s^{e_s}.$$

Since $T(p_i^{e_i})$ divides $T(m)$, $F(T(p_i^{e_i}))$ divides $F(T(m))$ by Theorem 4(i). Also $p_i^{e_i}$ divides $F(T(p_i^{e_i}))$ by Theorem 10 and Definition 2. Thus, $p_i^{e_i}$ divides $F(T(m))$, And since the $p_i^{e_i}$ are pairwise relatively prime, m divides $F(T(m))$. Q.E.D.

The result mentioned at the beginning of this section is an immediate consequence of Theorem 11, since it is clear that $1 \leqslant T(m) \leqslant m^2$. Theorem 11 is a stronger result in the sense that it exhibits an easily calculated positive index n for which $m|F(n)$.

5. RANK OF APPARITION

Given $m > 0$, it is natural to ask for the smallest $t > 0$ for which $m|F(t)$. We might take $T(m)$ as a tentative guess for t. This guess may not be correct ($T(17) = 18$ and $17|F(9)$), but as we shall presently see in Theorem 13, $t|T(m)$.

Definition 3: The rank of apparition of $m > 0$, denoted by $R(m)$, is the smallest $t > 0$ such that $m|F(t)$. We also say that the index t is the point of entry of m in the Fibonacci numbers.

Tables of $R(p)$ are readily available. Brousseau [1] gives $R(p)$ for each rational prime $p \leqslant 269$, while [2] does the same for $p < 48{,}179$ and [3] does for $48{,}179 \leqslant p < 100{,}000$. Jarden, in [8], gives $R(p)$ for each rational prime $p < 1512$. The following theorem gives a concise formulation of $R(m)$ in terms of the structure of $Z(\omega)$.

Theorem 12: $R(m)$ is the smallest $t > 0$ such that ω^t is congruent modulo m to a rational integer.

Proof: Immediate from Theorem 6. Q.E.D.

It should be noted that the period of m in the Fibonacci numbers also has a concise formulation in $Z(\omega)$. Recall that the period of m in the Fibonacci numbers is the smallest $t > 0$ such that $F(t - 1) \equiv 1 \pmod{m}$ and $F(t) \equiv 0 \pmod{m}$. Thus, since $\omega^t = F(t - 1) + F(t)\omega$, it follows that the period of m in the Fibonacci numbers is the smallest $t > 0$ such that $\omega^t \equiv 1$ (MOD m).

The following trivial lemma paves the way for Theorem 13, the main result of this section.

Lemma 2: The integer $c + d\omega$ is congruent modulo m to a rational integer if and only if $m|d$.

Proof: If $m|d$, then $c + d\omega \equiv c$ (MOD m). Conversely, if $c + d\omega \equiv a$ (MOD m), then $m|(c - a)$ and $m|d$. Q.E.D.

Theorem 13: We have that $m|F(n)$ if and only if $R(m)|n$.

Proof: Let $t = R(m)$. First, suppose that $t|n$. Then, by Theorem 4(i), we have $F(t)|F(n)$; and since $m|F(t)$, it follows that $m|F(n)$. Conversely, suppose that $m|F(n)$. Then $\omega^n \equiv b$ (MOD m) by Theorem 6. Since $n \geqslant t$, then $n = st + x$, where $s > 0$ and $0 \leqslant x < t$. Thus, as $\omega^t \equiv a$ (MOD m) for some a with $\gcd(a, m) = 1$ (Theorem 6), we have $b \equiv \omega^n = \omega^{st+x} \equiv a^s\omega^x$ (MOD m). Suppose $x \neq 0$. Then $\omega^x = c + d\omega$ and $m \nmid d$. [For, if $m|d$, we would have $\omega^x \equiv c$ (MOD m) by Lemma 2, and so $m|F(x)$, a contradiction to the minimality of t.] Thus, $b \equiv ca^s + da^s\omega$ (MOD m). This is impossible by Lemma 2 [$\gcd(a, m) = 1$ and $m \nmid d$; thus $m \nmid da^s$.] Accordingly, $x = 0$, and so $t|n$. Q.E.D.

6. LAW OF REPETITION

We now direct our efforts to establishing the law of repetition (Theorem 15). Along the way, we will establish Theorem 14 and Lemma 3. Theorem 14 is an important result in its own right, whereas Lemma 3 is instrumental in proving the law of repetition. The proof of Lemma 3 will be the last use of the arithmetic theory of $Z(\omega)$ in this paper.

Definition 4: By $p^r \| n$, we mean that $p^r | n$ and $p^{r+1} \nmid n$.

Theorem 14: If $p^r \| F(n)$, then $p^{r+1} | F(nm)$ if and only if $p | m$.

Proof: Suppose $p|m$. We have that $p^{r+1}|F(np)$ by Theorem 8; and since $np|mn$, it follows from Theorem 4(i) that $F(np)|F(nm)$. Now suppose that $p^{r+1}|F(nm)$. Set $a = F(n-1)$ and $bp^r = F(n)$. Since $p^r\|F(n)$, it follows that $\gcd(b, p) = 1$; and $\gcd(a, p) = 1$, since $F(n-1)$ and $F(n)$ are relatively prime. Therefore, $\gcd(ab, p) = 1$. Also,

$$\omega^{nm} = (a + bp^r\omega)^m \equiv a^m + ma^{m-1}bp^r\omega \pmod{p^{r+1}}.$$

Now $p^{r+1}|F(nm)$, and so, by Theorem 6, we have

$$a^m + ma^{m-1}bp^r\omega \equiv c \pmod{p^{r+1}}.$$

But, by Lemma 2, this means that $p|ma^{m-1}b$; and since $\gcd(ab, p) = 1$, it follows that $p|m$. Q.E.D.

Lemma 3: If $p^r\|F(n)$ and $\gcd(m, p) = 1$, then

$$p^{r+1}|F(nmp),$$

and if $p^r \neq 2$, then

$$p^{r+1}\|F(nmp).$$

Proof: Since $n|nm$, then $F(n)|F(nm)$, and so $p^r|F(nm)$. Thus, $p^{r+1}|F(nmp)$, by Theorem 8. Also, since $\gcd(m, p) = 1$, we have $p \nmid m$, so that $p^{r+1} \nmid F(nm)$, by Theorem 14. Accordingly, $p^r\|F(nm)$. Let $x = nm$. Then we have $p^r\|F(x)$, and we are to show that, if $p^r \neq 2$, then $p^{r+1}\|F(xp)$. Of course, we already know that $p^{r+1}|F(xp)$, and so it only remains to show that $p^{r+2}\nmid F(xp)$.

Suppose first that $p > 2$. Set $a = F(x-1)$ and $bp^r = F(x)$. As in the proof of Theorem 14, we have $\gcd(ab, p) = 1$. Also,

$$\omega^{px} = (a + bp^r\omega)^p = a^p + p^{r+1}a^{p-1}b\omega + \alpha p^{r+2}.$$

Thus,

$$\omega^{px} \equiv a^p + p^{r+1}a^{p-1}b\omega \pmod{p^{r+2}},$$

and since $p^{r+2}\nmid p^{r+1}a^{p-1}b$, it follows that ω^{px} is not congruent modulo p^{r+2} to a rational integer (Lemma 2). Therefore, by Theorem 6, $p^{r+2}\nmid F(xp)$.

The proof for the exceptional case $p = 2$, $r > 1$, is exactly the same. (The condition $r > 1$ is needed to obtain the term αp^{r+2}.) Q.E.D.

Theorem 15 (Law of Repetition): If $p^r\|F(n)$ and $\gcd(m, p) = 1$, then, for any $k \geqslant 0$, $p^{r+k}|F(nmp^k)$, and if $p^r \neq 2$, $p^{r+k}\|F(nmp^k)$.

Proof: Straightforward induction on k using Theorem 8 and Lemma 3. Q.E.D.

7. FURTHER DIVISIBILITY RESULTS

We conclude this article by listing in Theorems 16-20 additional well-known divisibility results which readily follow from Theorems 13-15. Since no additional use of the arithmetic theory of $Z(\omega)$ is needed, the proofs are left to the reader.

Theorem 16: If $p \neq 2$, $t = R(p)$, $p^r \| F(t)$, and $k \geq 0$, then $p^{r+k} \| F(n)$ if and only if $n = tmp^k$, where $\gcd(m, p) = 1$.

Theorem 17: (i) $2 \| F(n)$ if and only if $n = 3m$, where $\gcd(m, 2) = 1$.
(ii) If $k \geq 0$, then $2^{3+k} \| F(n)$ if and only if $n = 2^{k+1} \cdot 3 \cdot m$, where $\gcd(m, 2) = 1$.

Theorem 18: If $p \neq 2$, $t = R(p)$, and $p^r \| F(t)$, then

$$R(p^n) = t \cdot p^{\max(0, n-r)} \quad \text{and} \quad p^{r+\max(0, n-r)} \| F(R(p^n)).$$

Theorem 19:

$$R(2^n) = \begin{cases} 3, & n = 1 \\ 2 \cdot 3, & n = 2. \\ 2^{n-2} \cdot 3, & n \geq 3 \end{cases}$$

Furthermore, $2 \| F(3)$, $2^3 \| F(2 \cdot 3)$, and $2^n \| F(2^{n-2} \cdot 3)$ for $n \geq 3$.

Theorem 20: If $m = p_1^{e_1} p_2^{e_2} \ldots p_s^{e_s}$, then

$$R(m) = \operatorname{lcm}(R(p_1^{e_1}), \ldots, R(p_s^{e_s})).$$

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A NOTE ON APERY NUMBERS

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To prove the irrationality of the number

$$\zeta(3) = \sum_{n=1}^{\infty} (1/n^3),$$

Apèry recently introduced the sequence $\{a_n, n \geqslant 0\}$ defined by the recurrence relation

$$a_0 = 1, \; a_1 = 5,$$

and

$$n^3 a_n - (34n^3 - 51n^2 + 27n - 5)a_{n-1} + (n-1)^3 a_{n-2} = 0 \tag{1}$$

for $n \geqslant 2$. Apèry proved that for the pair $(a_0, a_1) = (1, 5)$, all the a_n's are integers, and each a_n has the representation

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

The first six a_n's are:

$$a_0 = 1,\; a_1 = 5,\; a_2 = 73,\; a_3 = 1445,\; a_4 = 33001,\; a_5 = 819005$$

(see [1]).

Some congruence properties of Apèry numbers are established in [1] and [2]. In [1], it is asked if there are values for the pair (a_0, a_1) other than (1, 5) in (1) that would produce a sequence $\{a_n, n \geqslant 0\}$ of integers. In particular, taking $a_0 = 1$, it is also asked if there is a necessary and sufficient condition on a_1 for all the a_n's to be integers. In answering these questions, we first prove the following theorem.

Theorem: Let $a_0 = 0$. The condition $a_1 = 0$ is necessary and sufficient for all of the a_n's defined by Apèry recurrence relation to be integers.

Proof: The sufficiency is clear. To prove the necessity we assume, on the contrary, that there exists an integer $k \neq 0$ such that all of the b_n's produced by Apèry recurrence relation with $b_0 = 0$, $b_1 = k$ are integers. Without loss of generality, we assume $k > 0$.

For the sequence $\{b\ , n \geqslant 0\}$, (1) can be written as

$$n^3 b_n = (34n^3 - 51n^2 + 27n - 5)b_{n-1} - (n-1)^3 b_{n-2} \tag{2}$$

and hence

$$b_n - b_{n-1} = \left(33 - \frac{51}{n} + \frac{27}{n^2} - \frac{5}{n^3}\right)b_{n-1} - \left(1 - \frac{3}{n} + \frac{3}{n^2} - \frac{1}{n^3}\right)b_{n-2}.$$

Since we have

$$\left(33 - \frac{51}{n} + \frac{27}{n^2} - \frac{5}{n^3}\right) - \left(1 - \frac{3}{n} + \frac{3}{n^2} - \frac{1}{n^3}\right) = 4\left(2 - \frac{1}{n}\right)^3 > 0$$

for all $n \geqslant 2$, it follows that $b_{n-1} > b_{n-2} \geqslant 0$ implies $b_n > b_{n-1}$. Since $b_1 = k > 0 = b_0$, then, by induction, $b_n > b_{n-1}$ for all $n \geqslant 1$. Similarly, since $a_1 =$

$5 > 1 = a_0$, we also have $a_n > a_{n-1}$ for all $n \geqslant 1$. Thus, $a_n > 0$ and $b_n > 0$ for $n \geqslant 1$.

The equation (2), with $n = 2$, implies that $8b_2 = 117b_1$. Therefore, we have $b_1/a_1 < b_2/a_2$. Now we prove that for each integer $n \geqslant 2$,

$$\frac{b_{n-1}}{a_{n-1}} < \frac{b_n}{a_n}.$$

The Apèry recurrence relation (1) can be written as

$$(34n^3 - 51n^2 + 27n - 5)a_{n-1} = n^3 a_n + (n-1)^3 a_{n-2}. \tag{4}$$

Let $\lambda_i = b_i/a_i$, $i \geqslant 1$. If $\lambda_{n-2} < \lambda_n$, then from (4) we have

$$\begin{aligned}(34n^3 - 51n^2 + 27n - 5)\lambda_{n-1}a_{n-1} &= n^3\lambda_n a_n + (n-1)^3\lambda_{n-2}a_{n-2} \\ &< \lambda_n(n^3 a_n + (n-1)^3 a_{n-2}),\end{aligned}$$

and

$$(34n^3 - 51n^2 + 27n - 5)\lambda_{n-1}a_{n-1} > \lambda_{n-2}(n^3 a_n + (n-1)^3 a_{n-2}).$$

Hence,

$$\lambda_{n-2} < \lambda_n \text{ implies } \lambda_{n-2} < \lambda_{n-1} < \lambda_n.$$

Similarly,

$$\lambda_{n-2} \geqslant \lambda_n \text{ implies } \lambda_{n-2} \geqslant \lambda_{n-1} \geqslant \lambda_n.$$

Therefore, the inequality $\lambda_{n-2} < \lambda_{n-1}$ implies $\lambda_{n-1} < \lambda_n$. Now since (3) holds for $n = 2$, it also holds for all $n \geqslant 2$.

From (4), we get

$$\frac{b_n}{a_n} = \frac{(n+1)^3 b_{n-1} + n^3 b_{n-1}}{(n+1)^3 a_{n-1} + n^3 a_{n-1}}.$$

Hence, clearing the denominator and collecting terms yields

$$(3n^2 + 3n + 1)\left(\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n}\right)a_n a_{n+1} = n^3((a_{n-1}b_n - b_{n-1}a_n) - (a_n b_{n+1} - b_n a_{n+1})).$$

Thus, using (3), we get $a_{n-1}b_n - b_{n-1}a_n > a_n b_{n+1} - b_n a_{n+1}$ for all $n \geqslant 2$; hence,

$$a_n b_{n+1} - b_n a_{n+1} \leqslant (a_{n-1}b_n - b_{n-1}a_n) - 1 \tag{5}$$

for all $n \geqslant 2$. Note that (3) also implies

$$a_n b_{n+1} - b_n a_{n+1} > 0 \tag{6}$$

for all $n \geqslant 2$. Comparing (5) and (6), we can clearly see a contradiction. This completes the proof. ■

We have the following corollary as a consequence of the above theorem.

Corollary: It is necessary and sufficient that the pair $(a_0, a_1) = c(1, 5)$, where c is any integer, for all the a_n's in (1) to be integers.

Proof: The sufficiency follows immediately from the linearity of the relation (1) relative to a_n's. To prove the necessity, suppose $(a_0, a_1) = (c, d)$ is a pair that causes all of the a_n's to be integers. By the linearity of (1), the pair $(0, d - 5c) = (c, d) - c(1, 5)$ is also a pair that causes all of the a_n's to be integers. By the theorem, $d - 5c = 0$, that is, $(c, d) = c(1, 5)$.

As a last comment, we slightly improve a lemma presented in [2].

By multiplying $(6n^2 - 3n + 1)$ to the equation

$$(n + 1)^3 a_{n+1} - (34(n + 1)^3 - 51(n + 1)^2 + 27(n + 1) - 5)a_n + n^3 a_{n-1} = 0,$$

we obtain

$$(6n^2 - 3n + 1)((n^3 + 3n^2 + 3n + 1)a_{n+1} - (34n^3 + 51n^2 + 27n + 5)a_n + n^3 a_{n-1}) = 0,$$

and hence,

$$a_{n+1} \equiv (5 + 12n)a_n \pmod{n^3}$$

for $n \geqslant 2$. The same result was given in [2] with (mod n^2) instead of (mod n^3).

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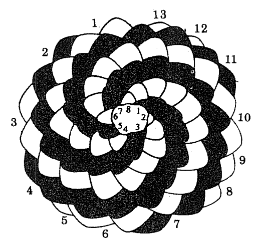

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For further information concerning the conference, please contact either of the cochairmen:

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   Payment may also be made at the beginning of the Conference.

5. Participants who present a paper will have available both an overhead and a slide projector. Presentations will be given at the Central Auditorium of the University. The detailed program of the Conference together with the registration form and further details will be mailed to all participants by July 15, 1984. Preprints of papers will be available, in photocopy form, during the Conference.

The Conference organizers are making every effort to provide a friendly, relaxed environment in which participants can present their ideas and join in friendly discussions with their colleagues. Persons who accompany Conference participants are welcome and will have an opportunity, along with participants, to visit the University, the city of Patras, and neighboring places of historical, cultural, and natural interest.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

Assistant Editors
GLORIA C. PADILLA and CHARLES R. WALL

Send all communications regarding ELEMENTARY PROBLEMS and SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 Solano Dr., S.E.,; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-520 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

(a) Suppose that one has a table for multiplication (mod 10) in which a, b, ..., j have been substituted for 0, 1, ..., 9 in some order. How many decodings of the substitution are possible?

(B) Answer the analogous question for a table of multiplication (mod 12).

B-521 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

See the previous problem. Find all moduli $m > 1$ for which the multiplication (mod m) table can be decoded in only one way.

B-522 *Proposed by Ioan Tomescu, University of Bucharest, Romania*

Find the number $A(n)$ of sequences $(a_1, a_2, \ldots, a_k)$ of integers a_i satisfying $1 \leqslant a_i < a_{i+1} \leqslant n$ and $a_{i+1} - a_i \equiv 1 \pmod 2$ for $i = 1, 2, \ldots, k - 1$. [Here k is variable, but of course $1 \leqslant k \leqslant n$. For example, the three allowable sequences for $n = 2$ are (1), (2), and (1, 2).]

B-523 *Proposed by Laszlo Cseh and Imre Merenyi, Cluj, Romania*

Let $p, a_0, a_1, \ldots, a_n$ be integers with p a positive prime such that

$$\gcd(a_0, p) = 1 = \gcd(a_n, p).$$

Prove that in $\{0, 1, \ldots, p-1\}$ there are as many solutions of the congruence

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \equiv 0 \pmod{p}$$

as there are of the congruence

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a \equiv 0 \pmod{p}.$$

B-524 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_n = F_{2n-1}^2 + F_nF_{n-1}(F_{2n-1} + F_n^2) + 3F_nF_{n+1}(F_{2n-1} + F_nF_{n-1})$. Show that S_n is the square of a Fibonacci number.

B-525 *Proposed by Walter Blumberg, Coral Springs, FL*

Let x, y, and z be positive integers such that $2^x - 1 = y^z$ and $x > 1$. Prove that $z = 1$.

SOLUTIONS

Fibonacci-Lucas Centroid

B-496 *Proposed by Stanley Rabinowitz, Digital Equip. Corp., Merrimack, NH*

Show that the centroid of the triangle whose vertices have coordinates

$$(F_n, L_n), (F_{n+1}, L_{n+1}), (F_{n+6}, L_{n+6})$$

is (F_{n+4}, L_{n+4}).

Solution, independently, by Walter Blumberg, Coral Springs, FL; Wray G. Brady, Slippery Rock, PA; Paul S. Bruckman, Sacramento, CA; Laszlo Cseh, Cluj, Romania; Leonard Dresel, Reading, England; Herta T. Freitag, Roanoke, VA; L. Kuipers, Switzerland; Stanley Rabinowitz, Merimack, NH; Imre Merenyi, Cluj, Romania; John W. Milsom, Butler, PA; Bob Prielipp, Oshkosh, WI; Sahib Singh, Clarion, PA; Lawrence Somer, Washington, CD; Gregory Wulczyn, Lewisburg, PA.

The coordinates (x, y) of the centroid are given by

$$\begin{aligned} 3x &= F_n + F_{n+1} + F_{n+6} \\ &= F_{n+2} + F_{n+4} + F_{n+5} \\ &= F_{n+2} + F_{n+4} + F_{n+3} + F_{n+4} \\ &= 3F_{n+4}, \end{aligned}$$

and similarly,

$$\begin{aligned} 3y &= L_n + L_{n+1} + L_{n+6} \\ &= 3L_{n+4}. \end{aligned}$$

Hence, the centroid is (F_{n+4}, L_{n+4}).

Area of a Fibonacci-Lucas Triangle

B-497 *Proposed by Stanley Rabinowitz, Digital Equip. Corp., Merrimack, NH*

For d an odd positive integer, find the area of the triangle with vertices (F_n, L_n), (F_{n+d}, L_{n+d}), and (F_{n+2d}, L_{n+2d}).

Solution by Paul S. Bruckman, Sacramento, CA

By means of a well-known determinant formula, the area of the given triangle is given by

$$A = \frac{1}{2}\left\|\begin{matrix} 1 & 1 & 1 \\ F_n & F_{n+d} & F_{n+2d} \\ L_n & L_{n+d} & L_{n+2d} \end{matrix}\right\| \tag{1}$$

(In the above expression, the inner bar is the determinant symbol, the outer bar represents absolute value.) Then

$$A = \frac{1}{2}\left|(F_{n+2d}L_{n+d} - F_{n+d}L_{n+2d}) - (F_{n+2d}L_n - F_nL_{n+2d}) + (F_{n+d}L_n - F_nL_{n+d})\right|.$$

Using the relation

$$F_uL_v - F_vL_u = 2(-1)^vF_{u-v}, \tag{2}$$

this becomes:

$$A = (-1)^{n+d}F_d - (-1)^nF_{2d} + (-1)^nF_d = (-1)^dF_d - F_{2d} + F_d,$$

which equals F_{2d} when d is odd (and equals $F_{2d} - 2F_d$ when d is even).

Also solved by Walter Blumberg, Wray G. Brady, Leonard Dresel, Herta T. Freitag, L. Kuipers, Graham Lord, John W. Milsom, Bob Prielipp, Sahib Singh, Gregory Wulczyn, and the proposer.

Fibonacci Recursions Modulo 10

B-498 *Proposed by Herta T. Freitag, Roanoke, VA*

Characterize the positive integers k such that, for all positive integers n, $F_n + F_{n+k} \equiv F_{n+2k} \pmod{10}$.

Solution by Leonard Dresel, University of Reading, England

When k is *odd*, we have the identity $F_{m+k} - F_{m-k} = F_mL_k$. Applying this with $m = n + k$, we have $F_m \equiv F_mL_k \pmod{10}$, and this will be satisfied whenever $L_k \equiv 1 \pmod{10}$.

On the other hand, when k is even, we have $F_{m+k} - F_{m-k} = L_mF_k$, and it is not possible to satisfy the given recurrence for even k.

Returning to the case of *odd* k, the condition $L_k \equiv 1 \pmod{10}$ is equivalent to

$$L_k \equiv 1 \pmod 2 \text{ and } L_k \equiv 1 \pmod 5.$$

The first condition implies that k is *not* divisible by 3; with the help of the Binet formula for L_k, the second condition reduces to $2^{k-1} \equiv 1 \pmod 5$, which gives that $k - 1$ in a multiple of 4. Combining these results, we have

$$k = 12t + 1 \text{ or } k = 12t + 5 \quad (t = 0, 1, 2, 3, \ldots).$$

Also solved by Paul S. Bruckman, Laszlo Cseh, L. Kuipers, Imre Merenyi, Bob Prielipp, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

Lucas Recursions Modulo 12

B-499 *Proposed by Herta T. Freitag, Roanoke, VA*

Do the Lucas numbers analogue of B-498.

Solution by Leonard Dresel, University of Reading, England

We have the identities $L_{m+k} - L_{m-k} = L_m L_k$ when k is odd, and $L_{m+k} - L_{m-k} = 5F_m F_k$ when k is even. Hence, putting $m = n + k$, the relation $L_n + L_{n+k} \equiv L_{n+2k}$ (mod 10) leads, for *odd* k, to $L_m \equiv L_m L_k$ (mod 10), so that we require $L_k \equiv 1$ (mod 10), leading to the same values of k as in B-498 above.

Also solved by Paul S. Bruckman, Laszlo Cseh, L. Kuipers, Imre Merényi, Bob Prielipp, Lawrence Somer, Gregory Wulczyn, and the proposer.

Two Kinds of Divisibility

B-500 *Proposed by Philip L. Mana, Albuquerque, NM*

Let $A(n)$ and $B(n)$ be polynomials of positive degree with integer coefficients such that $B(k)|A(k)$ for all integers k. Must there exist a nonzero integer h and a polynomial $C(n)$ with integer coefficients such that $hA(n) = B(n)C(n)$?

Solution by the proposer.

Using the division algorithm and multiplying by an integer h so as to make all coefficients into integers, one has

$$hA(n) = Q(n)B(n) + R(n), \tag{*}$$

where $Q(n)$ and $R(n)$ are polynomials in n with integral coefficients and $R(n)$ is either the zero polynomial or has degree less than $B(n)$. The hypothesis that $B(n)|A(n)$ and (*) imply that $B(n)|R(n)$ for all integers n. If $R(n)$ is not the zero polynomial, $R(n)$ has lower degree than $B(n)$ and so

$$\lim_{n\to\infty} [R(n)/B(n)] = 0;$$

also $R(n)$ is zero for only a finite number of integers n. Thus $0 < R(n)/B(n) < 1$ for some large enough n, contradicting $B(n)|R(n)$. Hence $R(n)$ is the zero polynomial and (*) shows that the answer is "yes."

Also solved by Paul S. Bruckman and L. Kuipers.

Doubling Back on a Sequence

B-501 *Proposed by J. O. Shallit & J. P. Yamron, U.C. Berkeley, CA*

Let α be the mapping that sends a sequence $X = (x_1, x_2, \ldots, x_{2k})$ of length $2k$ to the sequence of length k,

$$\alpha(X) = (x_1 x_{2k}, x_2 x_{2k-1}, \ldots, x_k x_{k+1}).$$

Let $V = (1, 2, 3, \ldots, 2^h)$, $\alpha^2(V) = \alpha(\alpha(V))$, etc. Prove that $\alpha(V), \alpha^2(V), \ldots, \alpha^{h-1}(V)$ are all strictly increasing sequences.

Solution by Leonard Dresel, University of Reading, England

Suppose the nubmers a_1, a_2, a_3, a_4 form a strictly increasing sequence, subject to the condition $a_1 + a_4 = a_2 + a_3 = S$, then

$$(a_4 - a_1)^2 > (a_3 - a_2)^2$$

and

$$(a_4 - a_1)^2 = (a_3 + a_2)^2$$

gives

$$-4a_4a_1 > -4a_3a_2$$

hence,

$$a_1a_4 < a_2a_3.$$

Now any two consecutive terms of $\alpha(V)$ are of the form a_1a_4, a_2a_3, with

$$a_1 + a_4 = a_3 + a_2 = 1 + 2^h,$$

so that it follows that $\alpha(V)$ is a strictly increasing sequence.

Next, consider $\alpha^2(V)$. To avoid a notational forest, we shall apply our method to the specific case where $h = 4$, with $V = (1, 2, 3, \ldots, 16)$. Then, using a dot to denote multiplication, we have

$$\begin{aligned}\alpha(V) &= (1 \cdot 16,\ 2 \cdot 15,\ 3 \cdot 14,\ \ldots,\ 8 \cdot 9)\\ \alpha^2(V) &= (1 \cdot 16 \cdot 8 \cdot 9,\ 2 \cdot 15 \cdot 7 \cdot 10,\ \ldots,\ 4 \cdot 13 \cdot 5 \cdot 12)\\ &= (1 \cdot 8 \cdot 9 \cdot 16,\ 2 \cdot 7 \cdot 10 \cdot 15,\ \ldots,\ 4 \cdot 5 \cdot 12 \cdot 13)\\ &= (b_1 \cdot c_1,\ b_2 \cdot c_2,\ \ldots,\ b_4 \cdot c_4)\end{aligned}$$

where

$$(b_1, b_2, b_3, b_4) = \alpha(1, 2, 3, \ldots, 8)$$

and

$$(c_1, c_2, c_3, c_4) = \alpha(9, 10, 11, \ldots, 16).$$

By our previous argument, b_i is strictly increasing, and similarly c_i is. Thus $\alpha^2(V) = (b_ic_i)$ is a strictly increasing sequence. Similarly, we can show that $\alpha^3(V) = (d_1e_1f_1g_1,\ d_2e_2f_2g_2)$, where

$$(d_1, d_2) = \alpha(1, 2, 3, 4),\ (e_1, e_2) = \alpha(5, 6, 7, 8), \text{ etc.},$$

is strictly increasing. The above arguments can be generalized to apply to any value of h.

Also solved by Paul S. Bruckman, L. Kuipers, and the proposers.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-368 *Proposed by Andreas N. Philippou, University of Patras, Patras, Greece*

For any fixed integer $k \geqslant 2$,

$$\sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k + 1}{n_1, \ \ldots, \ n_k, \ 1} = \sum_{\ell=0}^{n} f_{\ell+1}^{(k)} f_{n+1-\ell}^{(k)}, \ n \geqslant 0, \tag{A}$$

where $f_n^{(k)}$ are the Fibonacci numbers of order k [1], [2].

In particular, for $k = 2$,

$$\sum_{\ell=0}^{[n/2]} (n + 1 - \ell)\binom{n - \ell}{\ell} = \sum_{\ell=0}^{n} F_{\ell+1}F_{n+1-\ell}, \ n \geqslant 0. \tag{A.1}$$

The problem also includes as a special case ($k = 1$) the following:

$$\binom{n + r - 1}{r - 1} = \sum_{\ell=0}^{n} \binom{n - \ell + r - 2}{r - 2}, \ n \geqslant 0. \tag{B}$$

References

1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order k." *The Fibonacci Quarterly* 20, no. 1 (1983):28-32.
2. A. N. Philippou. "A Note on the Fibonacci Sequence of Order k and Multinomial Coefficients." *The Fibonacci Quarterly* 21. no. 2 (1983):82-86.

H-369 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Call an integer-valued arithmetic function f a *gcd sequence* if

$$\gcd(a, b) = d \quad \text{implies} \quad \gcd(f(a), f(b)) = f(d)$$

for all positive integers a and b. A gcd sequence is *primitive* if it is neither an integer multiple nor a positive integer power of some other gcd sequence. Examples of primitive gcd sequences include:

(1) $f(n) = 1$ (2) $f(n) = n$ (3) $f(n)$ = largest squarefree divisor of n
(4) $f(n) = 2^n - 1$ (5) $f(n) = F_n$ (Fibonacci sequence)

Prove that there are infinitely many primitive gcd sequences.

H-370 *Proposed by M. Wachtel & H. Schmutz, Zurich, Switzerland*

For every positive integer a show that

$$5\cdot[5\cdot(a^2+a)+1]+1 \tag{A}$$

$$5\cdot[5\cdot[5\cdot[5(a^2+a)+1]+1]+1]+1 \tag{B}$$

are *products of two consecutive integers*, and that no integral divisor of

$$5(a^2+a)+1$$

is congruent to 3 or 7, modulo 10.

H-371 *Proposed by Paul S. Bruckman, Carmichael, CA*

Let $[\overline{k}]$ represent the purely periodic continued fraction:

$$k+1/(k+1/(k+\ldots,\ k=1,\ 2,\ 3,\ \ldots\ .$$

Show that

$$[\overline{k}]^3=[\overline{k^3+3k}]. \tag{1}$$

Generalize to other powers.

SOLUTIONS

Give Poly Sum!

H-348 *Proposed by Andreas N. Philippou, Patras, Greece*
(Vol. 20, no. 4, November 1982)

For each fixed integer $k \geqslant 2$, define the sequence of polynomials $a_n^{(k)}(p)$ by

$$a_n^{(k)}(p)=p^{n+k}\sum_{n_1,\ldots,n_k}\binom{n_1+\cdots+n_k}{n_1,\ \cdots,\ n_k}\left(\frac{1-p}{p}\right)^{n_1+\cdots+n_k}\quad(n\geqslant 0,\ -\infty<p<\infty),$$

where the summation is over all nonnegative integers $n_1,\ \ldots,\ n_k$ such that

$$n_1+2n_2+\cdots+kn_k=n.$$

Show that

$$\sum_{n=0}^{\infty}a_n^{(k)}(p)=1\quad(0<p<1).$$

Solution by the proposer.

Using the definition of $\alpha_n^{(k)}(p)$ and the transformation $n_i=m_i$ $(1\leqslant i\leqslant k)$ and

$$n=m+\sum_{i=1}^{k}(i-1)m_i,$$

we get

$$\sum_{n=0}^{\infty}\alpha_n^{(k)}(p)=p^k\sum_{m=0}^{\infty}\sum_{\substack{m_1,\ldots,m_k\ni\\ m_1+\cdots+m_k=m}}\binom{m_1+\cdots+m_k}{m_1,\ \ldots,\ m_k}p^{m_1+2m_2+\cdots+km_k}\left(\frac{1-p}{p}\right)^{m_1+\cdots+m_k}$$

$$=p^k\sum_{m=0}^{\infty}\left(\left(\frac{1-p}{p}\right)(p+p^2+\cdots+p^k)\right)^m,\text{ by the multinomial theorem,}$$

$$= p^k \sum_{m=0}^{\infty} (1 - p^k)^m = 1, \text{ for } |1 - p^k| < 1, \tag{1}$$

which establishes the result. Moreover, (1) implies that for any fixed integer $\ell \geqslant 1$,

$$\sum_{n=0}^{\infty} \alpha_n^{(2\ell)}(p) = 1, \text{ for } -1 < p < 1. \tag{2}$$

Remark: If $p = 1/2$, the problem reduces to showing that

$$\sum_{n=1}^{\infty} (f_n^{(k)}/2^n) = 2^{k-1}, \tag{3}$$

where $f_n^{(k)}$ is the Fibonacci sequence of order k, since it may be seen, [1]-[2], that

$$f_{n+1}^{(k)} = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \quad (n \geqslant 0).$$

A direct proof of (3) is given in [3].

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1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order k." *The Fibonacci Quarterly* 20, no. 1 (1982):28-32.
2. A. N. Philippou. "A Note on the Fibonacci Sequence of Order k and Multinomial Coefficients." *The Fibonacci Quarterly* 21, no. 2 (1983):82-86.
3. A. N. Philippou. Solution of Problem H-322. *The Fibonacci Quarterly* 20, no. 2 (1982):189-90.

Also solved by Paul S. Bruckman and L. Kuipers.

Triggy

H-349 *Proposed by Paul S. Bruckman, Carmichael, CA*
(Vol. 21, no. 1, February 1983)

Define S_n as follows: $S_n \equiv \sum_{k=1}^{n-1} \csc^2 \pi k/n$, $n = 2, 3, \ldots$. Prove $S_n = \dfrac{n^2 - 1}{3}$.

Solution by Ömer Eğecioğlu, University of California, La Jolla, CA

We will prove a slight generalization: Let ξ be a primitive n^{th} root of 1. Then

$$\sum_{k=0}^{n-1} \xi^{km} = \begin{cases} n & \text{if } n|m \\ 0 & \text{otherwise.} \end{cases}$$

For $|t| < 1$, we have

$$\sum_{k=0}^{n-1} \frac{1}{1 - \xi^k t} = \sum_{k=0}^{n-1} \sum_{m=0}^{\infty} \xi^{km} t^m = \sum_{m=0}^{\infty} t^m \sum_{k=0}^{n-1} \xi^{km} = \frac{n}{1 - t^n}.$$

Thus

$$\sum_{k=1}^{n-1} \frac{1}{1 - \xi^k} = \lim_{t \to 1} \frac{n}{1 - t^n} - \frac{1}{1 - t} = \frac{n - 1}{2}.$$

Using the fact that $|1 - \xi^k|^2 = (1 - \xi^k)(1 - \overline{\xi}^k)$ and partial fractions, similar techniques yield the following, more general formula: For $0 < m \leqslant n$, we have

$$\sum_{k=1}^{n-1} \frac{\xi^{km}}{|1 - \xi^k|^2} = \frac{1}{12}(n^2 + 6m(m - n) - 1). \tag{1}$$

Now, writing

$$\xi^k = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n},$$

we obtain

$$|1 - \xi^k|^2 = (1 - \xi^k)(1 - \overline{\xi}^k) = 2(1 - Re\ \xi^k) = 4 \sin^2 \frac{\pi k}{n}.$$

Separating ξ^{km} into its real and imaginary parts, (1) implies

$$\sum_{k=1}^{n-1} \cos \frac{2\pi km}{n} \csc^2 \frac{\pi k}{n} = \frac{1}{3}(n^2 + 6m(m - n) - 1) \tag{1a}$$

$$\sum_{k=1}^{n-1} \sin \frac{2\pi km}{n} \csc^2 \frac{\pi k}{n} = 0 \tag{1b}$$

whenever $0 < m \leqslant n$.

From (1a), we obtain the special case $S_n = \dfrac{n^2 - 1}{3}$ by taking $m = n$. For n even, with $m = n/2$, (1a) yields

$$\sum_{k=1}^{n-1} (-1)^k \csc^2 \frac{\pi k}{n} = -\frac{1}{6}(n^2 + 2).$$

The following identities can also be obtained by arguments similar to the derivation of (1):

$$\sum_{k=1}^{n-1} \frac{\xi^{km}}{1 - \xi^k} = m - \frac{n + 1}{2}; \tag{2}$$

$$\sum_{k=1}^{n-1} \frac{\xi^{km}}{(1 - \xi^k)^2} = -\frac{1}{12}(n^2 + 6n - 6mn + 6m^2 - 12m + 5); \tag{3}$$

$$\sum_{k=1}^{n-1} \frac{1}{|1 - \xi^k|^2 - 1} = 1 - \frac{n}{\sqrt{3}} \cot \frac{n\pi}{6}. \tag{4}$$

These yield further trigonometric identities by separating ξ to its real and imaginary components. For instance, from (4), we obtain

$$\sum_{k=1}^{n-1} \frac{1}{4 \sin^2 \frac{k\pi}{n} - 1} = 1 - \frac{n}{\sqrt{3}} \cot \frac{n\pi}{6},$$

and for n even, (3) with $m = n/2$ gives

$$\sum_{k=1}^{n-1} (-1)^k \cos \frac{2\pi k}{n} \cot \frac{\pi k}{n} = 0.$$

Also solved by P. Bruckman, W. Janous, S. Klein, D. P. Laurie, B. Prielipp, T. J. Rivlin, and J. Suck.

We Have the System

H-351 *Proposed by Verner E. Hoggatt, Jr. (deceased)*
(Vol. 21, No. 1, February 1983)

Solve the following system of equations:

$$\begin{aligned} U_1 &= 1 \\ V_1 &= 1 \\ U_2 &= U_1 + V_1 + F_2 = 3 \\ V_2 &= U_2 + V_1 = 4 \\ &\vdots \\ U_{n+1} &= U_n + V_n + F_{n+1} \qquad (n \geqslant 1) \\ V_{n+1} &= U_{n+1} + V_n \qquad (n \geqslant 1) \end{aligned}$$

Solution by C. Georghiou, University of Patras, Patras, Greece

The generating functions of the sequences

$$\{F_{n+1}\}_{n=0}^{\infty}, \ \{F_{2n}\}_{n=0}^{\infty}, \text{ and } \{F_{2n+1}\}_{n=0}^{\infty}$$

are, respectively,

$$(1 - x - x^2)^{-1}, \ x(1 - 3x + x^2)^{-1}, \text{ and } (1 - x)(1 - 3x + x^2)^{-1}.$$

Let $u(x)$ and $v(x)$ represent the generating functions of the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$, respectively. From the given system we get (since $U_0 = V_0 = 0$):

$$\frac{1}{x}u(x) = u(x) + v(x) + (1 - x - x^2)^{-1} \quad \text{and} \quad \frac{1}{x}v(x) = \frac{1}{x}u(x) + v(x).$$

Then

$$\begin{aligned} v(x) &= x/(1 - x - x^2)(1 - 3x + x^2) = \frac{1}{2}\,\frac{2 - x}{1 - 3x - x^2} - \frac{1}{2}\,\frac{2 + x}{1 - x - x^2} \\ &= \frac{1 - x}{1 - 3x + x^2} + \frac{1}{2}\,\frac{x}{1 - 3x + x^2} - \frac{1}{1 - x - x^2} - \frac{1}{2}\,\frac{x}{1 - x - x^2}. \end{aligned}$$

Therefore

$$V_n = F_{2n+1} + \frac{1}{2}F_{2n} - F_{n+1} - \frac{1}{2}F_n$$

and

$$U_n = V_n - V_{n-1} = \frac{1}{2}F_{2n+2} - \frac{1}{2}F_{n+1}.$$

Also solved by P. Bruckman, W. Janous, L. Kuipers, J. Suck, M. Wachtel, and the proposer.

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SUSTAINING MEMBERS

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Majorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Majorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

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A Collection of Manuscripts Related to the Fibonacci Sequence — 18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

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