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CONTENTS

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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

A NOTE ON SOMER'S PAPER ON LINEAR RECURRENCES

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(Submitted January 1983)

In a recent paper [1], Somer uses the second-order linear recursion relation

$$s_{n+2} = as_{n+1} + bs_n, \; a, \; b \in \mathbf{Z}, \tag{1}$$

to generate higher-order linear recurrences. The purpose of this note is to extend Somer's results. In what follows, the notation in [1] is used without further comment.

We assume $\alpha\beta \neq 0$, α/β not a root of unity, and ask under what conditions the *rational* sequence

$$\{t_n\}_{n=0}^{\infty} = \{s_{nk}/s_n\}_{n=0}^{\infty} \tag{2}$$

satisfies a linear recursion relation of *minimal* order k.

Somer gives the solution $\{s_n\} = \{u_n\}$, where $u_0 = 0$, $u_1 = 1$. We can argue similarly for $\{s_n\} = \{v_n\}$, where $v_0 = 2$, $v_1 = a$, and $v_n = \alpha^n + \beta^n$, in the case when k is *odd*. Then

$$t_n = \frac{v_{nk}}{v_n} = \frac{\alpha^{nk} + \beta^{nk}}{\alpha^n + \beta^n} = \sum_{i=0}^{k-1} \alpha^{(k-1-i)n}(-1)^i \beta^{in}$$

is a rational integer, and $\{t_n\}$ clearly satisfies the same k^{th}-order linear recursion relation as $\{w_n\} = \{u_{nk}/u_n\}$. The proof of the minimality runs as for $\{w_n\}$: In the *first* matrix factor of $D_k(w_n, 0)$, we just change the sign of every odd-numbered column.

The general solution $s_n \neq s_1 u_n$ of (1) may be written as

$$s_n = \frac{A\alpha^n + B\beta^n}{A + B},$$

if we "normalize" to $s_0 = 1$. The above result for $\{v_n\}$ then follows from the fact that $-B/A = -1$ is a primitive square root of unity. In general, put $-B/A = \rho$, where ρ is a primitive m^{th} root of unity, and assume that

$$k \equiv 1 \pmod{m}.$$

Then

$$t_n = \frac{s_{nk}}{s_n} = \frac{\alpha^{nk} - \rho\beta^{nk}}{\alpha^n - \rho\beta^n} = \frac{\alpha^{nk} - (\rho\beta^n)^k}{\alpha^n - \rho\beta^n} = \sum_{i=0}^{k-1} \alpha^{(k-1-i)n}\rho^i \beta^{in}.$$

The question of minimality is settled as above: To obtain $D_k(t_n, 0)$, we multiply the successive columns of the first matrix factor of $D_k(w_n, 0)$ by 1, ρ, ρ^2, ..., ρ^{k-1}, respectively.

For $m > 2$, however, the *rationality* of $\{s_n\}$ imposes severe conditions. In particular,

$$s_1 = \frac{A\alpha + B\beta}{A + B} = \frac{\alpha - \rho\beta}{1 - \rho}$$

should be rational, showing that $\rho = \sqrt[m]{1}$ must be a quadratic irrationality, so $m = 3, 4$, or 6. But even in these cases, we get conditions on the coefficients a and b.

We illustrate the method in the case $m = 4$, $\rho = \pm i$. With

$$\alpha = \frac{a + \sqrt{D}}{2}, \ \beta = \frac{a - \sqrt{D}}{2}, \ D = a^2 + 4b,$$

this gives $s_1 = (a \pm i\sqrt{D})/2$, which is rational if and only if $D = -c^2$, $c \in \mathbf{Z}$. Then

$$s_1 = \frac{a + c}{2} = \frac{v_1 + cu_1}{2} \quad \left(\text{and} \quad s_0 = \frac{v_0 + cu_0}{2} = 1\right).$$

To get b integral, both a and c must be even. To get $\alpha\beta \neq 0$ and α/β not a root of unity, we must have $ac \neq 0$ and $a \neq \pm c$. Consequently, we have shown that if

$$c \in \mathbf{Z}, \ b = -\frac{a^2 + c^2}{4}, \ 2|a, \ 2|c, \ ac \neq 0, \ a \neq \pm c,$$

then the integral sequence

$$\left\{s_n\right\}_{n=0}^{\infty} = \left\{\frac{v_n + cu_n}{2}\right\}_{n=0}^{\infty}$$

has the property (2) when $k \equiv 1 \pmod 4$.

We only state the corresponding results for $m = 3$ and $m = 6$. Let

$$c \in \mathbf{Z}, \ b = -\frac{a^2 + 3c^2}{4},$$

a and c be of the same parity, $ac \neq 0$, $a \neq \pm c$, $\pm 3c$. Then the following integral sequences have the property (2):

$$\left\{s_n\right\}_{n=0}^{\infty} = \left\{\frac{v_n + cu_n}{2}\right\}_{n=0}^{\infty} \text{ if } k \equiv 1 \pmod 3,$$

$$\left\{s_n\right\}_{n=0}^{\infty} = \left\{\frac{v_n + 3cu_n}{2}\right\}_{n=0}^{\infty} \text{ if } k \equiv 1 \pmod 6.$$

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1. L. Somer. "The Generation of Higher-Order Linear Recurrences from Second-Order Linear Recurrences." *The Fibonacci Quarterly* 22, no. 2 (1984):98-100.

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MULTILEVEL FIBONACCI CONVERSION AND ADDITION

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I. INTRODUCTION

Recently we have been making studies [1, 2] on the Fibonacci number system that appears to be of considerable importance to the Fibonacci computer [3]. A specific result that appears to be of particular interest is that through multilevel coefficients on a Fibonacci radix system, efficient extension of representations can occur. For example, through a ternary coefficient system, a doubling of the range with half the number of digits over a binary system has been shown, while in fact allowing a restriction to only even-subscripted Fibonacci numbers in the radix [1]. Consequently, further investigation of various other properties and extensions has seemed warranted. This has led us to our present studies which indicate that the Fibonacci computer may have added features, over those of redundancy for error detection and correction already reported [4]. One such added feature lies in efficient processing techniques with these being based upon the conversion of numbers into special forms, including even- and odd-subscripted Fibonacci radix systems. With this in mind, we develop in this paper several new Fibonacci number representations, in particular even- and odd-subscripted ones, a signed ternary one, and conversions between them. These ideas are developed in Sections II and III. In particular, we give the details of conversions among the various ternary Fibonacci radix representations.

For reference purposes, we recall the defining recursion relation of the Fibonacci numbers [5]

$$F_i = F_{i-1} + F_{i-2},\ F_0 = 0,\ F_1 = 1. \tag{1.1}$$

At times we will use this expressed in the alternate form $F_i = F_{i+1} - F_{i-1}$. If $M = F_{m+2} - 2$, then any nonnegative integer N, $0 \leqslant N \leqslant M$, can be expanded in a Fibonacci representation using $(m - 1)$ binary coefficients on the Fibonacci numbers as a complete base or radix set. Previously, we have shown how these expansions can be made in terms of some ternary and quaternary coefficient sets [1]; we will extend these latter expansions here as needed for conversions and arithmetic operations.

II. BINARY TO MULTILEVEL CONVERSIONS

Let an Unsigned Binary Fibonacci Representation, UBFR, be

$$N = \sum_{j=2}^{m} b_j F_j,\ b_j = \{0, 1\} \tag{2.1}$$

in the range $(0, M)$, where

$$M = \sum_{j=2}^{m} F_j = F_{m+2} - 2.$$

Such a representation can be derived for a given number N using the conversion algorithm previously presented [2].

An Unsigned Quaternary Fibonacci Representation using only even-subscripted Fibonacci numbers, $UQFR_e$, is derived by the following formula [1]:

$$q^e_{2i} = b_{2i-1} + b_{2i} - b_{2i+1} \tag{2.2}$$

where

$$q^e_{2i} \in \{-1, 0, 1, 2\}, \; i = 1, 2, \ldots, k,$$

and

$$N = \sum_{j=2}^{m} b_j F_j = \sum_{i=1}^{k} q^e_{2i} F_{2i}. \tag{2.3}$$

The even-subscripted coefficients q^e_{2i} take four possible values according to the conversion Table 2.1. As is known, see [1], several Unsigned Ternary Fibonacci Representations, UTFR, exist. Two of these that use only even-subscripted Fibonacci numbers, called $UTFR_e\{-1, 0, 1\}$ and $UTFR_e\{0, 1, 2\}$ may be derived from Table 2.1 by applying (1.1) on a UBFR, and thus eliminating either case 6 or case 1 by a prior conversion of the UBFR into the "minimum," UBFR(min), or the "maximum," UBFR(max), form respectively [6]. Of these two ternary representations, the one derived from the UBFR(min) is of particular interest because it allows positive-to-negative number conversion by a simple form of complementation, namely $-1 \leftrightarrow 1$ and $0 \leftrightarrow 0$. In this Signed Ternary Fibonacci Representation, $STFR_e\{-1, 0, 1\}$, the sign of the number is determined by the sign of the most significant nonzero coefficient [5, p. 56]. In the above, we note that the binary representation of (2.1) holds only for nonnegative numbers; hence, we have called it "unsigned," as well as those representations coming from it by the quaternary transformation (2.2). However, upon making the complementation just mentioned within a UTFR $\{-1, 0, 1\}$, negative numbers are contained within the system, which we consequently have called "signed" specifically to point out its broader nature.

TABLE 2.1

Binary-to-Quaternary Coefficient Conversion

Case	b_{2i-1}	b_{2i}	b_{2i+1}	$q^e_{2i} = b_{2i-1} + b_{2i} - b_{2i+1}$
0	0	0	0	0
1	0	0	1	-1
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	0
6	1	1	0	2
7	1	1	1	1

Thus, the UBFR(min)-to-$STFR_e\{-1, 0, 1\}$ conversion is simply effected by

$$t^e_{2i} = b_{2i-1} + b_{2i} - b_{2i+1} \tag{2.4}$$

in the relation

$$N = \sum_{j=2}^{m} b_j F_j = \sum_{i=1}^{k} t^e_{2i} F_{2i}, \tag{2.5}$$

where $k = [(m+1)/2]$ is the integer portion of $(m+1)/2$. The range now becomes $(-M, M)$, where $M = F_{m+2} - 2 = F_{2k+2} - 2$ when m is even.

The complete odd-subscripted Fibonacci representations may also be derived easily from a UBFR, in a manner analogous to the one discussed above for even subscripts. Indeed, a UBFR-to-UQFR$_o${-1, 0, 1, 2}, as also the UBFR(min)-to-UTFR$_o${-1, 0, 1} and UBFR(max)-to-UTFR$_o${0, 1, 2} conversions, are all effected simply by almost identical conversion formulas [subtract "one" from all i subscripts in relations (2.2) through (2.5)]. Again notice that the complementation operation $-1 \leftrightarrow 1$ and $0 \leftrightarrow 0$ allows for the positive-negative conversion, and thus provides the STFR$_o${-1, 0, 1}.

III. TERNARY CONVERSIONS AND ADDITION

In this section, we will discuss techniques for conversion between several ternary representations, the ones of interest being the full-, STFR$_f${-1, 0, 1},

$$N = \sum_{j=1}^{n} t_j F_j \tag{3.1a}$$

the even-, STFR$_e${-1, 0, 1},

$$N = \sum_{i=1}^{k} t^{e}_{2i} F_{2i} \tag{3.1b}$$

and the odd-, STFR$_o${-1, 0, 1},

$$N = \sum_{i=1}^{k} t^{o}_{2i-1} F_{2i-1} \tag{3.1c}$$

subscripted representations. In all cases, the coefficients t_i, t^e_i, t^o_i are in the set {-1, 0, 1}.

Before discussion of the actual conversions, we show their use in terms of addition.

A. Ternary Addition

Addition becomes a very simple matter if we assume the availability of numbers in the full-, even-, and odd-subscripted ternary representations of (3.1). Thus, let two numbers N_1 and N_2 be given, one in the even- and the other in the odd-subscripted form:

$$N_1 = \sum_{i=1}^{k} t^{e}_{2i} F_{2i} \quad \text{and} \quad N_2 = \sum_{i=1}^{k} t^{o}_{2i-1} F_{2i-1};$$

then their sum has the full representation with

$$t_j = \begin{cases} t^e_j, & j = \text{even}, \\ t^o_j, & j = \text{odd}. \end{cases} \tag{3.2}$$

That is, addition (and with it subtraction, because of readily executable complementation) occurs through the interleaving of the digits of the numbers being summed. The process, as illustrated in Figure 1, is especially convenient for hardware implementations.

B. Full- to Even-/Odd-Subscript Conversion

As seen by the addition technique just discussed, it is convenient to be able to convert numbers from a full-subscripted form to a form using only even

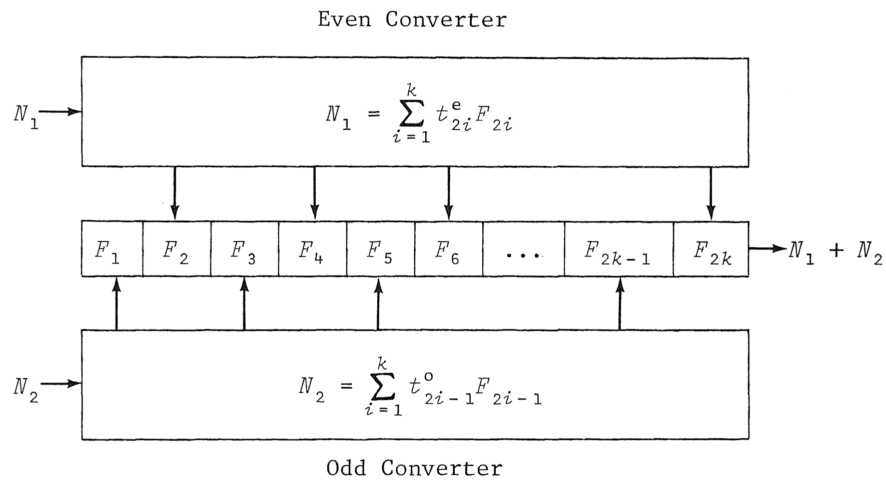

FIGURE 1. Ternary Addition

subscripts as well as to one using only odd subscripts. As the principles are identical for obtaining the odd-subscripted form, we will concentrate here on obtaining the even one.

Thus, let there be given a full-subscripted ternary representation. By substituting for odd-subscripted Fibonacci numbers

$$F_{2i-1} = F_{2i} - F_{2i-2} \tag{$*$}$$

and writing $k = [(n+1)/2]$ for the integer part of $(n+1)/2$, we obtain:

$$N = \sum_{j=1}^{n} t_j F_j = \sum_{i=1}^{k} t_{2i} F_{2i} + \sum_{i=1}^{k} t_{2i-1}(F_{2i} - F_{2i-2}) \tag{3.3a}$$

$$= \sum_{i=1}^{k} (t_{2i-1} + t_{2i} - t_{2i+1}) F_{2i}. \tag{3.3b}$$

Here we again assume $t_j = 0$ for $j < 1$ and $j > n$. Thus, the coefficients

$$t^{e}_{2i} = t_{2i-1} + t_{2i} - t_{2i+1} \tag{3.3c}$$

are those of an even-subscripted representation. However, if (3.3c) were to be applied directly to an arbitrary full-subscripted representation, it would lead to an even-subscripted representation with coefficients in the range of integers $\{\pm 3, \pm 2, \pm 1, 0\}$, i.e., a septenary rather than a ternary one. Table 3.1 shows these $3^3 = 27$ possibilities, where the fourth column gives the result of applying (3.3c) directly. As is seen in Table 3.1, there are eight possible out-of-code (i.e., nonternary) cases. In each out-of-code case, though, a preliminary preparation will bring the relevant coefficient back into code, the necessary preparation transformations being given in the final column of Table 3.1. That is, by making an appropriate substitution via a Fibonacci number identity in the original full-subscripted representation, an out-of-code converted coefficient can be brought back into code. Since some of the preparation transformations affect neighboring coefficients which could be brought out-of-code after transformation, it is necessary to continue the preparation until all coefficients become ternary when (3.3c) is finally applied. Since

TABLE 3.1

Full- to Even-Subscripted Conversion*
($t^e_{2i} = t_{2i-1} + t_{2i} - t_{2i+1}$)

Case	t_{2i-1}	t_{2i}	t_{2i+1}	t^e_{2i}	Revert to Case	By
1	0	0	0	0		Preliminary Preparation Transformations
2	1	0	0	1		
3	-1	0	0	-1		
4	0	1	0	1		
5	0	-1	0	-1		
6	0	0	1	-1		
7	0	0	-1	1		
8	1	-1	0	0		
9	-1	1	0	0		
10	1	0	1	0		
11	-1	0	-1	0		
12	0	1	1	0		
13	0	-1	-1	0		
14	1	1	1	1		
15	1	-1	1	-1		
16	1	-1	-1	1		
17	-1	1	1	-1		
18	-1	-1	-1	1		
19	-1	1	-1	1		
20	1	1	0	2	6	$F_{2i-1} + F_{2i} = F_{2i+1}$
21	-1	-1	0	-2	7	$-F_{2i-1} - F_{2i} = -F_{2i+1}$
22	1	0	-1	2	5	$F_{2i-1} - F_{2i+1} = -F_{2i}$
23	-1	0	1	-2	4	$-F_{2i-1} + F_{2i+1} = F_{2i}$
24	0	1	-1	2	3	$F_{2i} - F_{2i+1} = -F_{2i-1}$
25	0	-1	1	-2	2	$F_{2i} + F_{2i+1} = F_{2i-1}$
26	1	1	-1	3	1	$F_{2i-1} + F_{2i} - F_{2i+1} = 0$
27	-1	-1	1	-3	1	$-F_{2i-1} - F_{2i} + F_{2i+1} = 0$

*For a full- to odd-subscripted conversion, all subscripts are shifted down by one in the table entries.

each preparatory transformation reduces by at least one the number of nonzero ternary coefficients in the full-subscripted representation being converted, successive applications of the preparatory transformations eventually will lead to a termination with only ternary t^e_{2k} calculated according to (3.3c). It should be noted that preparatory transformations can be applied simultaneously to nonoverlapping strings of three consecutive coefficients, but that simultaneous application to overlapping strings should be avoided.

As an example, consider (see Table 3.1 for case numbers):

$$N = -3 = (F_1 - F_3) + F_4 - (F_5 - F_7) + F_8 - F_9, \quad \text{cases 22 \& 23} \tag{3.4a}$$

$$= -F_2 + F_4 + F_6 + (F_8 - F_9), \quad \text{case 24} \tag{3.4b}$$

$$= -F_2 + F_4 + (F_6 - F_7), \text{ case } 24 \tag{3.4c}$$

$$= -F_2 + (F_4 - F_5), \quad \text{case } 24 \tag{3.4d}$$

$$= -F_2 - F_3 \tag{3.4e}$$

$$= -F_4, \text{ by Eq. (1.1)} \tag{3.4f}$$

The ripple effect of the preparatory transformations is well-exhibited by this example.

It should be noted that none of the ternary representations of (3.1) need be unique, for example:

$$N = 4 = F_2 - F_3 + F_5 = F_2 + F_4 = -F_4 + F_6 = -F_1 + F_5 = F_1 - F_3 + F_5. \tag{3.5}$$

Conversion from full- to odd-subscripted representations uses identical techniques, the only difference being that one is subtracted from the indices in (3.3c).

C. Even-/Odd-Subscript Conversion

The conversion of an odd- to an even-subscripted representation, or vice versa, is really a special case of the full- to even-subscripted, or odd-subscripted conversion. Thus, Table 3.1 applies. However, since the $t_{2k} = 0$ in the present case of Table 3.1, only $3^2 = 9$ of the cases occur with only two of these requiring preliminary preparation. This is illustrated in Table 3.2, where, also, in columns 2 and 6 we give the possible range of adjacent coefficients. In the two cases requiring preliminary preparation, we can actually carry out the adjustment after application of the conversion formula (3.3c) by using $3F_i = F_{i-2} + F_{i+2}$ [5, p. 59] for the replacement

$$2F_i = 3F_i - F_i = F_{i-2} - F_i + F_{i+2}. \tag{3.6}$$

It is seen in the three right-hand columns of Table 3.2 that adjacent coefficients are brought back into code. However, if two adjacent coefficients would become out-of-code by application of (3.3c), then (3.6) should only be applied to one of them so that the correction will remain in code. An example will illustrate this technique:

$$N = -24 = F_3 - F_5 + F_7 - F_9, \quad \text{given} \tag{3.7a}$$

$$= -F_2 + 2F_4 - 2F_6 + 2F_8 - F_{10}, \quad \text{by } (*) \tag{3.7b}$$

$$= -F_2 + (F_2 - F_4 + F_6) - 2F_6 + (F_6 - F_8 + F_{10}) - F_{10}, \text{ by (3.6)} \tag{3.7c}$$

$$= -F_4 - F_8. \tag{3.7d}$$

This last also results from one preliminary transformation on the given representation using Table 3.1.

In the case of even- to odd-subscripted conversions, or vice versa, it is seen that the "ripples" associated with the preliminary preparations can be avoided by one cycle of application of (3.6) after the use of (3.3c).

IV. Discussion

Conversion between several Fibonacci number representations has been discussed here with special emphasis upon those representations which appear most useful for multivalued logic realizations of the Fibonacci computers. Of special interest along this line is the addition technique that is seen, via Figure 1, to be a simple matter of register loading when even- and odd-subscripted ternary representations are on hand. Besides this rather considerable advantage of the ternary Fibonacci representations, it is clear that they also

TABLE 3.2

Odd to Even Conversion Table
(for Even to Odd, interchange t^e and t^o)

Cases	t^e_{2i-2}	t^o_{2i-1}	$t^e_{2i} = t^o_{2i-1} - t^o_{2i+1}$	t^o_{2i+1}	t^e_{2i+2}	Followed by (3.6) to give: t^e_{2i-2}	t_{2i}	t^e_{2i+2}
1	$\{0, 1, 2\}$	-1	0	-1	$\{-2, -1, 0\}$			
2	$\{0, 1, 2\}$	-1	-1	0	$\{-1, 0, 1\}$			
3	$\{0, 1, 2\}$	-1	-2	1	$\{0, 1, 2\}$	$\{-1, 0, 1\}$	1	$\{-1, 0, 1\}$
4	$\{-1, 0, 1\}$	0	1	-1	$\{-2, -1, 0\}$			
5	$\{-1, 0, 1\}$	0	0	0	$\{-1, 0, 1\}$			
6	$\{-1, 0, 1\}$	0	-1	1	$\{0, 1, 2\}$			
7	$\{-2, -1, 0\}$	1	2	-1	$\{-2, -1, 0\}$	$\{-1, 0, 1\}$	-1	$\{-1, 0, 1\}$
8	$\{-2, -1, 0\}$	1	1	0	$\{-1, 0, 1\}$			
9	$\{-2, -1, 0\}$	1	0	1	$\{0, 1, 2\}$			

conveniently allow numbers and their negatives to be represented without use of negative subscripts. The negative of a number is easily formed by the inversion of each coefficient in any ternary representation, also an advantage for hardware implementations using three-state devices or two-bits per cell binary equivalents [7]. Ternary Fibonacci representations also allow the sign of a number to be determined conveniently by observation of the most significant bit as mentioned in Section II.

The "ripples" that occur in the preliminary preparation transformations of Table 3.1 should be investigated for minimization and hardware implementation, as well as for interfacing the Fibonacci processor with conventional binary processors and trade-offs between the two kinds of processors.

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SUMS OF FIBONACCI NUMBERS BY MATRIX METHODS

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Recently, Kalman [2] derives a number of closed-form formulas for the generalized Fibonacci sequence by matrix methods. In this note, we extend the matrix representation and show that the sums of the generalized Fibonacci numbers could be derived directly using this representation.

Define k sequences of the generalized order-k Fibonacci numbers as shown:

$$g_n^i = \sum_{j=1}^{k} c_j g_{n-j}^i, \text{ for } n > 0 \text{ and } 1 \leqslant i \leqslant k, \tag{1}$$

with boundary conditions

$$g_n^i = \begin{cases} 1, & i = 1 - n, \\ & \qquad\qquad \text{for } 1 - k \leqslant n \leqslant 0, \\ 0, & \text{otherwise,} \end{cases}$$

where c_j, $1 \leqslant j \leqslant k$, are constant coefficients, and g_n^i is the n^{th} term of the i^{th} sequence. When $k = 2$, the generalized order-k Fibonacci sequence is reduced to the conventional Fibonacci sequence.

Following the approach taken by Kalman [2], we define a $k \times k$ square matrix A as follows:

$$A = \begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_{k-1} & c_k \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Then, by a property of matrix multiplication, we have

$$\left[g_{n+1}^i \; g_n^i \; \dots \; g_{n-k+2}^i\right]^T = A\left[g_n^i \; g_{n-1}^i \; \dots \; g_{n-k+1}^i\right]^T. \tag{2}$$

To deal with the k sequences of the generalized order-k Fibonacci series simultaneously, we define a $k \times k$ square matrix G_n as follows:

$$G_n = \begin{bmatrix} g_n^1 & g_n^2 & \dots & g_n^k \\ g_{n-1}^1 & g_{n-1}^2 & \dots & g_{n-1}^k \\ \vdots & \vdots & & \vdots \\ g_{n-k+1}^1 & g_{n-k+1}^2 & \dots & g_{n-k+1}^k \end{bmatrix}.$$

Generalizing Eq. (2), we derive

$$G_{n+1} = AG_n. \tag{3}$$

Then, by an inductive argument, we may rewrite it as

$$G_{n+1} = A^n G_1. \tag{4}$$

Now, it can be readily seen that, by Definition (1), $G_1 = A$; therefore, $G_n = A^n$. We may thus rewrite Eqs. (3) and (4) as shown:

$$G_{n+1} = G_1 G_n = G_n G_1. \tag{5}$$

In other words, G_1 is commutative under matrix multiplication. Hence, we have:

$$\begin{aligned} g_{n+1}^i &= c_i\, g_n^1 + g_n^{i+1}, \text{ for } 1 \leqslant i \leqslant k-1, \\ g_{n+1}^k &= c_k\, g_n^1. \end{aligned} \tag{6}$$

More generally, we may write Eq. (5) as $G_{r+c} = G_r G_c$. Consequently, an element of G_{r+c} is the product of a row of G_r and a column of G_c:

$$g_{r+c}^i = \sum_{j=1}^{k} g_r^j g_{c-j+1}^i.$$

In particular, when $r = c = n$, we have $G_{2n} = G_n^2$; this provides us with a means of evaluating G_n in an order of $\log_2 n$ steps.

To calculate the sums S_n, $n \geqslant 0$, of the generalized order-k Fibonacci numbers, defined by

$$S_n = \sum_{i=0}^{n} g_i^1, \tag{7}$$

let B be a $(k+1) \times (k+1)$ square matrix, such that

$$B = \begin{bmatrix} 1 & 0 & 0 \ldots 0 \\ 1 & & \\ 0 & & A \\ \vdots & & \\ 0 & & \end{bmatrix}.$$

Further, let E_n also be a $(k+1) \times (k+1)$ square matrix, such that

$$E_n = \begin{bmatrix} 1 & 0 & 0 \ldots 0 \\ S_{n-1} & & \\ S_{n-2} & & G_n \\ \vdots & & \\ S_{n-k} & & \end{bmatrix}.$$

Then, by Eq. (6) and

$$S_{n+1} = g_{n+1}^1 + S_n, \tag{8}$$

we derive a recurrence equation

$$E_{n+1} = E_n B. \tag{9}$$

Inductively, we also have

$$E_{n+1} = E_1 B^n. \tag{10}$$

Since $S_{-i} = 0$, $1 \leqslant i \leqslant k$, we thus infer $E_1 = B$, and in general, $E_n = B^n$. So, from Eqs. (9) and (10), we reach the following equation:

$$E_{n+1} = E_1 E_n = E_n E_1, \tag{11}$$

which shows that E_1 is commutative as well under matrix multiplication. By an application of Eq. (11), the sums of the generalized order-k Fibonacci numbers satisfy the following recurrence relation:

$$S_n = 1 + \sum_{i=1}^{k} S_{n-i}. \tag{12}$$

Substituting $S_n = g_n^1 + S_{n-1}$, an instance of Eq. (8), into Eq. (12), we may express g_n^1 in terms of the sums of the generalized order-k Fibonacci numbers:

$$g_n^1 = 1 + \sum_{i=2}^{k} S_{n-i}. \tag{13}$$

When $k = 2$, this equation is reduced to

$$g_n^1 = 1 + S_{n-2}.$$

If $c_1 = c_2 = 1$, we derive the well-known result [1]:

$$F_n = 1 + \sum_{i=0}^{n-2} F_i, \tag{14}$$

where F_n is the n^{th} term of the standard Fibonacci sequence. Equation (14) is also apparent from the Fibonacci number system viewpoint. Let

$$W = \{b_m \ \ldots \ b_2 b_1 b_0\}$$

be a bit pattern, where b_i is either 0 or 1 associating with a weight F_i. Thus, by an analogy of the binary number system, any natural number N may be defined as

$$N = \sum_{i=0}^{m} b_i F_i,$$

where m is sufficiently large. Since

$$S_n = \sum_{i=0}^{n} F_i,$$

the bit pattern of S_n consists of $(n + 1)$ 1's, that is, $\{1\}_0^n$. By Zeckendorf's theorem [1, p. 74], the bit pattern can be normalized to a pattern made up of 1's at b_{n+2-i}, where i is odd, and 0's at other positions. If a 1 is added to this pattern and, after the same normalization, the whole bit pattern consists of a 1 at b_{n+2} and 0's at other positions; the value is clearly equal to F_{n+2}. By induction, Eq. (14) holds.

Further, Eq. (11) could be generalized to $E_{r+c} = E_r E_c$. If $r = c = n$, we have $E_{2n} = E_n^2$. Thus, E_n may be evaluated in an order of $\log_2 n$ steps, too.

ACKNOWLEDGMENT

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FIBONACCI AND RELATED SEQUENCES IN DIGITAL FILTERING

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1. INTRODUCTION

In many communication and signal-processing systems, desired signals (sequences) are embedded in noise. Linear filters have been the primary tool for smoothing or recovering the desired signal from the degraded signal. Linear filters perform particularly well where the spectrum of the desired signal is significantly different from that of the interference. In many situations, however, the spectrum of the signal and of the noise are mixed in the same range and the performance of linear filters is very poor. Median filters can be used to circumvent these problems. Tukey [1] is generally credited with the idea of introducing nonlinear filters based on moving sample medians of the input signal. In this paper, we do not address the filtering problem, but we analyze the signal (sequence) set of median filtered binary sequences. To best explain the goal of this paper, the implementation of the median filter is described first.

To begin, take a binary sequence of length n; across this signal we slide a window that spans $2s - 1$ samples of the binary sequence, for $s = 2, 3, \ldots$. At each point of the sequence, the median of the samples within the window of the filter is computed and the output of the filter at the center sample is set equal to the computed median. To account for start-up and end effects at the two endpoints of the n-length sequence, $s - 1$ samples are appended to the beginning and end of the sequence. The value of the appended samples to the beginning is equal to the value of the first sample; similarly, the value of the appended samples to the end of the sequence equals the value of the last sample of the sequence. Figure 1(a) shows a binary signal of length 10 being filtered by a filter of window of size 3. The filtered signal is shown below. The appended samples are shown as crosses (X). Figure 1(b) shows the same sequence filtered by a filter of window size 5. Figure 1(c) shows similar results with a larger window. An interesting observation is that there exist sequences that are not modified by the median filter. Moreover, it has been shown that any finite input sequence, after repeated median filtering, will be reduced to one of these invariant sequences [2]. A sequence that is not modified by the filtering process is called a "root" sequence. The following theorem provides the upper bound on the number of successive filter passes necessary to reduce an input sequence to a root sequence [2]:

Theorem

Upon successive median filter passes, any nonroot sequence will become a root sequence after a maximum of $(n - 2)/2$ successive filterings, where n is the sequence length.

If we observe the structure of binary root sequences, we can see that they consist of identically-valued segments of at least s samples. These segments of at least s consecutive equal-valued samples are called "constant neighborhoods."

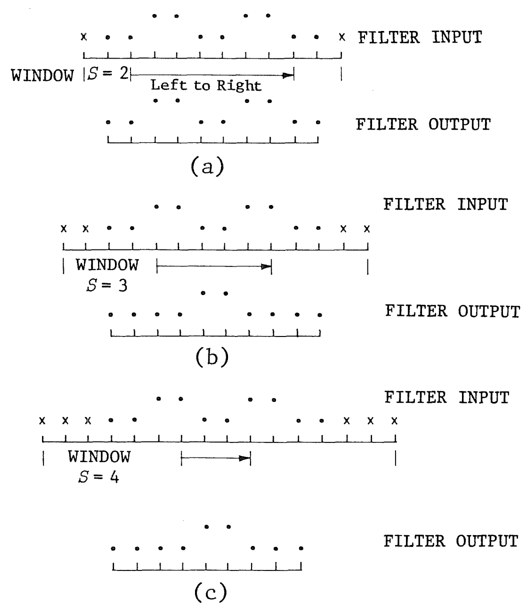


FIGURE 1. Signal Filtered by Three Different Median Filters:
(a) $S = 2$ (b) $S = 3$ (c) $S = 4$

Any sequence that does not consist only of constant neighborhoods will be modified by the filter. As an example, consider a window of size 3 (i.e., $s = 2$); if a sequence contains the segment "...11011...," then, clearly, this sequence will be modified when the window is centered at the "0" sample. In this case, the shortest constant neighborhood we can have is two.

The problem addressed in this paper is concerned with the binary root sequence space of median filters. In particular, for a median filter of window size $2s - 1$, how many possible binary root sequences can we have for a given sequence length? For instance, for a window of size 3 and sequence length 4, the only possible root sequences are:

sequence 1	0 0 0 0
sequence 2	0 0 0 1
sequence 3	1 0 0 0
sequence 4	1 1 0 0
sequence 5	0 0 1 1
sequence 6	0 1 1 0
sequence 7	1 0 0 1
sequence 8	1 1 1 0
sequence 9	0 1 1 1
sequence 10	1 1 1 1

There are only 10 possible root sequences of length 4, compared to 16 possible binary sequences we can obtain if no restriction is imposed on the sequences. Thus, for a particular window size and sequence length n, we are interested in finding $R(n)$, the number of possible root signals.

2. TREE STRUCTURE FOR A WINDOW OF SIZE THREE

Consider a window of size 3 ($s = 2$). As mentioned above, the minimum constant neighborhood for this filter is 2. Now, Let us build a root signal (a signal that will not be modified by the filter). The first sample can take any arbitrary values; for purposes of illustration, let us choose the first sample to be a "0." Next, for filtering purposes, we append a sample to the left of the first "0" sample. So far the sequence is "0 0" (appended + root sequence). The second sample of the sequence can be either a "0" or a "1." Let us pick a "1" for the second sample; the entire sequence is now: "0 0 1." The third sample of the root sequence (fourth of the entire sequence) is of decisive importance; if we let it be a "1," the entire sequence would consist of two different constant neighborhoods satisfying the property of being invariant to the filter. On the other hand, if we let the third sample be a "0," then a nonallowed structure occurs and the resultant sequence would be affected by the filter. Figure 2 shows every allowable path that the root signal can take.

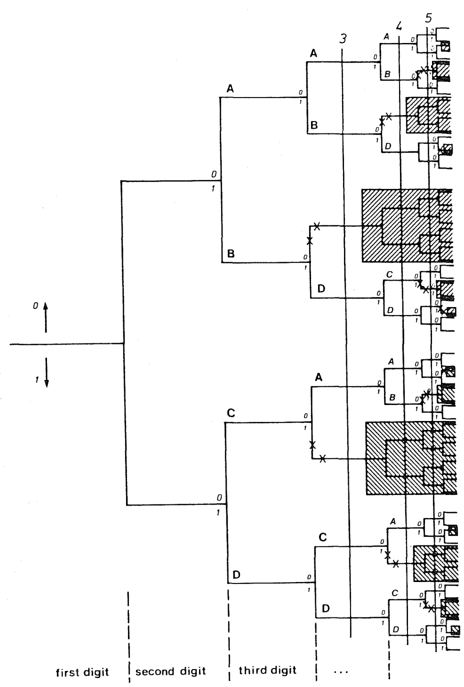


FIGURE 2. Tree Structure for a Filter of Window Size 3

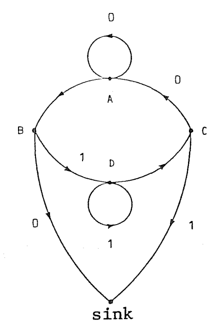
These paths branch in a tree structure fashion. If we take a close look at the tree structure, we can distinguish that sections of the tree repeat themselves as the tree propagates. This observation gives us the concept of the existence of discrete states. As is shown next, this is in fact true. These states are shown in Figure 2 and are denoted A, B, C, and D. Each state is determined by a sequence to two consecutive digits; for the filter of size 3, these states are:

$$A = \{0,\ 0\},\ B = \{0,\ 1\},\ C = \{1,\ 0\},\ D = \{1,\ 1\}.$$

Figure 2 shows how these states propagate as the sequence length increases.

Each state will generate other states; this can be seen in Figure 3, where a state transition diagram shows the state propagation. Notice that states B and C have only one allowable path. The nonallowed path is denoted by the "sink" in Figure 3. State A generates another state A plus a state B, state B generates a state D only, state C generates a state A, and finally state D generates a state D and a state C. Notice that the pattern of growth is predictable, in other words, given the number of states A, B, C, D at a given stage of the tree, we can predict the number of A, B, C, D states at the next stage. Let n denote the n^{th} stage (root sequence of length n), and let $A(n)$ be the number of A states that the tree structure has at this n^{th} stage. From the properties of the states, previously mentioned, we can write:

$$\begin{aligned} A(n+1) &= A(n) + C(n) \\ B(n+1) &= A(n) \\ C(n+1) &= D(n) \\ D(n+1) &= D(n) + B(n). \end{aligned} \tag{1}$$



State Diagram
Window = 3

FIGURE 3. State Diagram for a Filter with Window Size 3

Refer to the tree structure in Figure 2 and randomly select any stage, say stage 3. At that stage, we have two A states, two D states, one C state, and one B state; a total of 6 states (6 branches or possible roots). For a sequence of length 4, we have 10 states (or 10 possible root sequences). In general, the number of root sequences at the n stage is simply

$$R(n) = A(n) + B(n) + C(n) + D(n), \tag{2}$$

and at the $n + 1$ stage is

$$R(n+1) = A(n+1) + B(n+1) + C(n+1) + D(n+1).$$

Replacing (1) into (2), we obtain the recursive expression for $R(n + 1)$:

$$R(n+1) = 2A(n) + 2D(n) + C(n) + B(n), \tag{3}$$

with the initial conditions $A(2) = B(2) = C(2) = D(2) = 1$. Using this expression, a recursion table for the number of different states and number of roots is obtained and shown in Table 1.

Although the recursion table gives us a way to obtain the number of roots at any sequence length, a closed form solution for $R(n)$ is more desirable. From (3) and (2), we obtain

$$R(n+1) = R(n) + A(n) + D(n), \tag{4}$$

but, referring to the state diagram,

TABLE 1

Recursion Table for $R(n)$, Window = 3

Sequence Length n	$A(n)$	$B(n)$	$C(n)$	$D(n)$	$R(n)$
2	1	1	1	1	4
3	2	1	1	2	6
4	3	2	2	3	10
5	5	3	3	5	16
6	8	5	5	8	26
7	13	8	8	13	42
8	21	13	13	21	68
9	34	21	21	34	110
10	55	34	34	55	178
11	89	55	55	89	288
⋮	⋮	⋮	⋮	⋮	⋮

$$A(n) = A(n-1) + C(n-1),$$

and,

$$D(n) = D(n-1) + B(n-1).$$

Replacing these expressions for $A(n)$ and $D(n)$ into (4), we obtain

$$R(n+1) = R(n) + R(n-1).$$

We have obtained a difference equation for the number of roots of a binary sequence for a filter with window size 3 and initial conditions

$$R(1) = 2 \quad \text{and } R(2) = 4.$$

The solution is simply $R(n) = 2F(n+1)$, where $F(n)$ is the Fibonacci sequence

$$F(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \text{ for } n \geqslant 1.$$

3. TREE STRUCTURE FOR THE GENERAL WINDOW

Let us see what happens if we increase the window size to 5; later on we will generalize the window size to $2s - 1$. For this window, the minimum constant neighborhood length is 3. By using the same procedure as before, we obtain a tree structure for this size window and it is shown in Figure 4. The difference between the tree structures for the filters of size 3 and 5 is that for the latter we have two similar states B and two similar states C. For the filter with window size 5, the states are specified as follows:

$$A = \{0, 0, 0\}, B1 = \{0, 0, 1\}, B2 = \{0, 1, 1\},$$

$$C1 = \{1, 1, 0\}, C2 = \{1, 0, 0\}, \text{ and } D = \{1, 1, 1\}.$$

The similarity between states $C1$ and $C2$ is that both sequences start a neighborhood of value "0," the difference is in that $C1$ is a delay state (will generate a state $C2$ only). Similar observations can be made for states $B1$ and $B2$. Figure 5 shows the state diagram for the filter of size 5, and the delay states

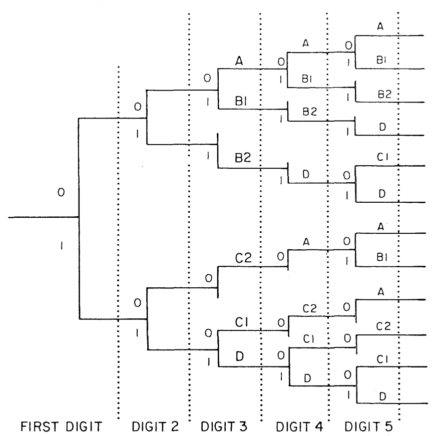


FIGURE 4. Tree Structure for a Filter with Window Size 5

can clearly be seen there. From the state diagram, we obtain the recursive expressions:

$$
\begin{aligned}
A(n) &= A(n-1) + C2(n-1) \\
B1(n) &= A(n-1) \\
B2(n) &= B1(n-1) \\
C1(n) &= D(n-1) \\
C2(n) &= C1(n-1) \\
D(n) &= D(n-1) + B2(n-1).
\end{aligned}
\tag{5}
$$

As before,

$$R(n) = A(n) + B1(n) + B2(n) + C1(n) + C2(n) + D(n). \tag{6}$$

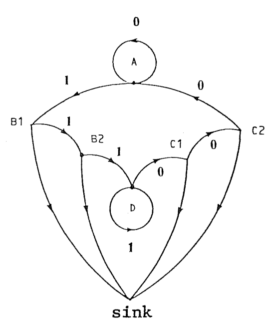


FIGURE 5. State Diagram for a Window of Size 5

Substituting (5) into (6), and after some manipulations, we find that

$$R(n+1) = R(n) + R(n-2). \tag{7}$$

Naturally, for a given sequence length, the number of roots decreases as we increase the window size. We have seen that, if we increase the window size, only delay states are added to the state diagram. Although by following the same procedure we could obtain the difference equation for larger window sizes, a general recursive expression for a general size filter is a more convenient result. This relation will be obtained next.

Figure 6(a) shows a state diagram for a filter of arbitrary window size $2s - 1$. The dotted line separates the diagram in odd symmetric parts. The odd symmetric correspondence is not only in a position sense, but in the multiplicity of the given states also (i.e., # of $B1$ states = # of $C1$ states, etc.). States Bi and Ci are delay states (each Ci or Bi state will be transformed into only one other state as we move along the diagram). On the other hand, states A and D not only have the previous property, but, also, they will generate another state of their own kind. Hence, for this $2s - 1$ window size filter, the number of roots is

$$\begin{aligned} R(n) = {} & A(n) + B1(n) + B2(n) + \cdots + B[s-1](n) \\ & + C1(n) + C2(n) + \cdots + C[s-1](n) + D(n), \end{aligned} \tag{8}$$

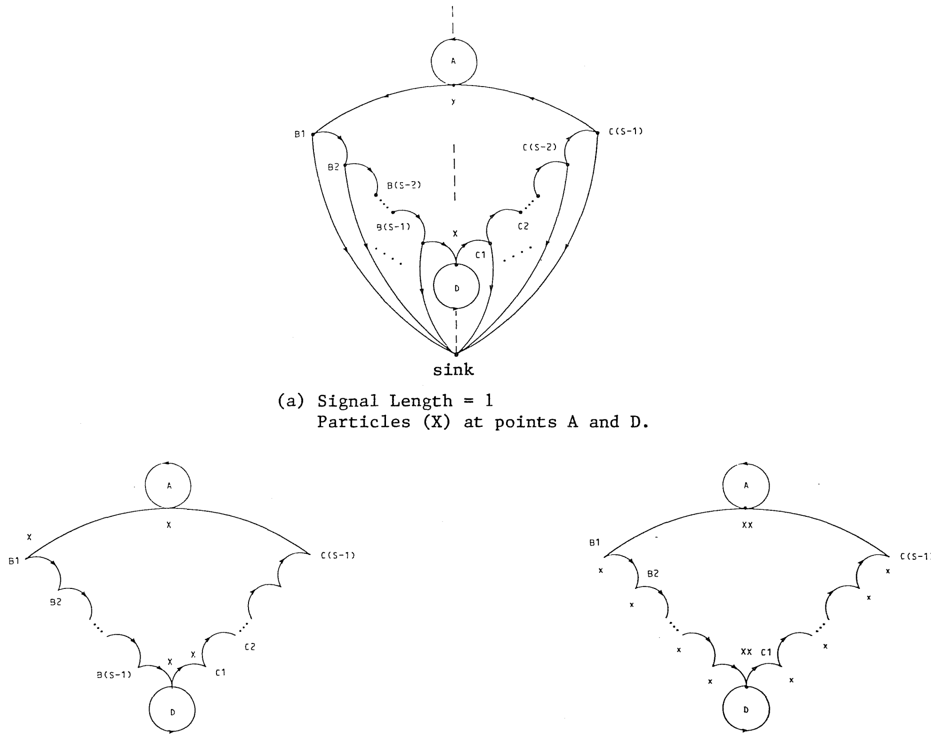
and

$$\begin{aligned} A(n) &= D(n) \\ B1(n) &= A(n-1) \\ B2(n) &= A(n-2) \\ &\;\;\vdots \\ B[s-1](n) &= A(n-[s-1]) \\ C1(n) &= B1(n) \\ C2(n) &= B2(n) \\ &\;\;\vdots \\ C[s-1](n) &= B[s-1](n). \end{aligned} \tag{9}$$

Therefore, $R(n)$ can be represented in terms of a recursion relation of the A states only. It is important to recall that s is the minimum constant neighborhood for a window of size $2s - 1$. We find that $R(n)$ can be written as

$$R(n) = 2\sum_{i=0}^{s-1} A(n-1). \tag{10}$$

Let us now describe some properties of the multiplicity of A states. Refer to the state diagram for the general window size filter, Figure 6(a). Think of the state diagram as describing the propagation of particles in space. [Particles in Figure 6(a, b, c) are shown as X's.) A particle at point A represents a state A; if at a given time there would be 5 particles at point D, this would imply that there would be 5 states D, and so on. At a sequence of length 1, we have 1 state A and 1 state D; this is shown in Figure 6(a). Increasing the sequence length to 2, state will generate another state A and also generates a state $B1$. Similarly, state D generates a state D and also a state $C1$. As we can see in Figure 6(b), with a sequence of length 2, the number of states is the same as it was at a sequence of length 1. The states generated at point D move toward point A; this process goes on until the first state generated at point D gets to point A. As we can see in Figure 6(c), when the first particle generated by D reaches the A point, a particle in point A not only generates a new state A by itself, but also, it receives another state from the particle that has propagated from state A along points $C1$, $C2$, ..., $C[s-1]$. In other words, point A has to wait s discrete intervals until the number of states in that location increases by the number of particles at point D, s intervals ago.



(a) Signal Length = 1
Particles (X) at points A and D.

(b) Signal Length = 2
(sink omitted)

(c) Signal Length = $S+1$
(sink omitted)

FIGURE 6. State Propagation for a Filter with Window Size $2s - 1$

Since the number of particles at the D point is the same as the number of particles at the A point at any time, the previous observation can be written as

$$A(n) = A(n - 1) + A(n - n). \tag{11}$$

Replacing (11) into (10), we find, after some manipulations, that

$$R(n) = R(n - 1) + R(n - s) \tag{12}$$

is the recursive expression for the number of root sequences of a filter with window size $2s - 1$, for any sequence length n. Letting

$$R(n + i - 1) = x_i(n), \tag{13}$$

we can see from (12) and (13) that

$$\begin{aligned} x_1(n+1) &= x_2(n) \\ x_2(n+1) &= x_3(n) \\ x_{s-1}(n+1) &= x_s(n) \\ x_s(n+1) &= x_s(n) + x_1(n). \end{aligned} \tag{14}$$

We can represent (14) in vector notation as

$$X(n + 1) = AX(n), \tag{15}$$

where

$$X(n) = [x_1(n), x_2(n), \ldots, x_s(n)]^T,$$

and where A is the bottom companion matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 1 \\ 1 & 0 & 0 & & 0 & 0 & 0 \end{bmatrix}.$$

From (13), $R(n) = [1, 0, 0, \ldots, 0]X(n)$, where $X(n)$ is the solution of (15),

$$X(n) = A^n X(0), \tag{17}$$

and where $X(0)$ are the initial conditions obtained from the tree structure or recursion table; hence,

$$R(n) = [1, 0, 0, \ldots, 0]A^n X(0). \tag{18}$$

The characteristic equation of the A matrix in (16) is obtained to be

$$\lambda^s - \lambda^{s-1} - 1 = 0. \tag{19}$$

With the help of Sturm's theorem [3], we can show that (19) does not have repeated eigenvalues; hence, we can find $R(n)$ from (18) as

$$R(n) = [1, 0, 0, \ldots, 0]MD^n M^{-1}X(0), \tag{20}$$

where M is the matrix that diagonalizes A as $M^{-1}AM = D$. In this case,

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & & \\ 0 & & & \lambda_s \end{bmatrix}, \tag{21}$$

and

$$M = \begin{array}{c} \begin{array}{ccc} 1 & 1 & 1 \end{array} \\ \begin{bmatrix} \lambda_2 & \lambda_2 & \lambda_s \\ \lambda_1^2 & \lambda_2^2 & \lambda_s^2 \\ \vdots & \vdots & \vdots \\ \lambda_1^{s-1} & \lambda_2^{s-1} & \lambda_s^{s-1} \end{bmatrix} \end{array}, \tag{22}$$

where $\lambda_1, \ldots, \lambda_s$ are the s distinct eigenvalues of A. Replacing (21) and (22) into (20), we obtain the general solution for $R(n)$:

$$R(n) = [\lambda_1, \lambda_2, \ldots, \lambda\ ]M^{-1}X(0).$$

4. CONCLUSION

We have developed a tree structure for the root sequence set of median filters of binary signals. This structure has the characteristic that the number of branches it has at each stage is described by a simple recursive expression. In the case of the filter with window = 3, the number of branches is related to the Fibonacci sequence. In general, it is shown that the number of roots $R(n)$ for a sequence of length n and window size $2s - 1$ is represented by the recurrence relation

$$R(n) = R(n - 1) + R(n - s).$$

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EULER'S INTEGERS

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Euler's integers are less known than the classic Eulerian numbers, though, in figurate form, they appear since antiquity.*

First, we shall look at their origin and find their general expression; then we shall establish some of their properties and give various combinatoric applications. Several results may not have been published previously.

The notation of periodic numbers and the notion of arithmetic polynomials will be useful tools.

I. GENERAL EXPRESSION OF EULER'S INTEGERS

Consider the infinite product

$$\pi(x) = (1 - x)(1 - x^2)(1 - x^3)\ldots$$

which Euler encountered in relation to the problem of the partition of integers. For instance, he showed that the number $p(n)$ of partitions of n into integers, distinct or not, is generated by the function

$$\frac{1}{\pi(n)} = 1 + \sum_{n>0} p(n)x^n.$$

If we develop $\pi(x)$ in series, we expect *a priori* to find increasing coefficients. But, surprisingly, all coefficients are +1 or -1, isolated in gaps of zero coefficients, gaps which, on the whole, increase and tend to infinity. More precisely,

$$\pi(x) = 1 - x^{a_1} - x^{a_2} + x^{a_3} + x^{a_4} - x^{a_5} - x^{a_6} + \cdots + \varepsilon_n x^{a_n} + \cdots \tag{1}$$

The coefficients are pairwise alternately -1 and +1. In order to see the behavior of the exponents, we shall examine some initial values of *Euler's integers* a_n:

TABLE 1

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	1	2	4	7	12	15	22	26	35	40	51	57	70	77	92	100
Δ_n	1	3	2	5	3	7	4	9	5	11	6	13	7	15	8	

The integers a_n seem to follow a complicated law, since their rate of increase oscillates. But if we form the differences $\Delta_n = a_{n+1} - a_n$, we see that they are the integers for n odd, and the odd numbers (beginning with 3) for n even.

Now, we shall try to express the general term $\varepsilon_n x^{a_n}$ of the series (1) in a simple form.

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*The two kinds of Eulerian numbers E_n and $A(n, k)$ are defined by:

$$\frac{1}{\cosh x} = \sum_{n\geqslant 0} E_n \frac{x^n}{n!} \quad \text{and} \quad x^n = \sum_{1\le k\le n} A(n, k)\binom{x + k - 1}{n}.$$

Definitions.

(1) A *periodic number* $u_n = [u_1, u_2, \ldots, u_k]$ is equal to the u_i in the brackets, such that $i \equiv n$, modulo k. So we represent a series of period k by its k first terms. For instance, $u_n = [a, b]$ equals a or b, according to whether n is odd or even, and $u_n = [4, -1, 0]$ is the n^{th} term of the series 4, -1, 0, 4, -1, 0, 4, -1, 0, ...

(2) An *arithmetic polynomial* $P(n)$ is defined only for positive integers and takes only integer values. Contrary to an ordinary polynomial, some of its coefficients are periodic numbers. Example: $3n^2 - [4, -1, 0]n + [5, 7]$.

We shall admit the following theorem, easy to establish [1].

Theorem 1

For $u_i = [a, b], \sum_{i=1}^{n} u_i = \dfrac{(a + b)n + [a - b, 0]}{2}$;

for $u_i = [a, b]i, \sum_{i=1}^{n} u_i = \dfrac{(a + b)n^2 + [2a, 2b]n + [a - b, 0]}{4}$.

Clearly,

$$\Delta_i = \frac{i + 1}{2}[1, 2] = \frac{1}{2}[1, 2]i + \frac{1}{2}[1, 2].$$

So we can calculate

$$a_n = 1 + \sum_{i=1}^{n-1} \Delta_i$$

by Theorem 1. Paying attention in brackets to the difference in parity of $n - 1$ and n, we find

$$a_n = 1 + \frac{1}{2}\cdot\frac{3(n - 1) + [0, 1]}{2} + \frac{1}{2}\cdot\frac{3(n - 1)^2 + [4, 2](n - 1) + [0, -1]}{4},$$

and, after simplification:

Theorem 2

The n^{th} Eulerian integer is the arithmetic trinomial

$$a_n = \frac{3n^2 + [4, 2]n + [1, 0]}{8} = \left\| \frac{n}{8}(3n + [4, 2]) \right\|, \tag{2}$$

where the double bars indicate the nearest integer.

Corollary

The general term of the series $\pi(x)$ is

$$\varepsilon_n x^{a_n} = [-1, -1, 1, 1]x^{\frac{1}{8}(3n^2 + [4, 2]n + [1, 0])}.$$

We define Euler's integers a_n by Table 1, indefinitely extended by means of the two arithmetic progressions mixed in Δ_n, and then deduce (2). But we have admitted (1) without proof, and so did Euler for ten years. In an article entitled "Discovery of a Most Extraordinary Law of Numbers in Relation to the Sum of Their Divisors," he said:

> I have now multiplied many factors, and I have found this progression. . . . One may attempt this multiplication and continue it as far as one wishes, in order to be convinced of the truth of this series. . . . A long time I vainly searched for a rigorous demonstration . . . and I proposed this research to some of my friends, whose competence in such questions I know; they all agreed with me on the truth of this conversion, but could not discover any source of demonstration. So it will be a known, but not yet proven truth.

Nevertheless, he finally proved it in a letter to Goldbach (1750). In the next century, various demonstrations were found, especially by Legendre [2], Cauchy, Jacobi, and Sylvester.

II. PROPERTIES OF EULER'S INTEGERS

First a quite simple question: What is the parity of the n^{th} Eulerian integer? If one observes Table 1, it seems that the same parities reappear with period 8. That is true, for

$$a_{n+8} = \frac{3(n+8)^2 + [4,\ 2](n+8) + [1,\ 0]}{8} = a_n + 6n + [28,\ 26]$$

whether $6n + [28, 26]$ is even. Likewise, we find:

Theorem 3

Modulo k, the Eulerian integer $a_n \equiv a_{n+4k}$ or $a_n \equiv a_{n+2k}$, according to whether k is even or odd. Particularly,

$$a_n \equiv [1,\ 0,\ 1,\ 1,\ 0,\ 1,\ 0,\ 0],\ \text{mod } 2;$$

$$a_n \equiv [1,\ -1,\ -1,\ 1,\ 0,\ 0],\ \text{mod } 3.$$

Now a more important question: Find a characteristic property of the integers a_n. An integer N is Eulerian, if the equation in n,

$$N = \frac{9n^2 + [4,\ 2]n + [1,\ 0]}{8} \quad \text{or} \quad 3n^2 + [4,\ 2]n + [1,\ 0] - 8N = 0,$$

has an integer and positive root. Therefore, its discriminator

$$[2,\ 1]^2 - 3[1,\ 0] + 24N = [4,\ 1] - [3,\ 0] + 24N = 24N + 1$$

must be a square. Conversely, if

$$24N + 1 = k^2,$$

k has the form $3n + 2$ or $3n + 1$. But Eq. (2) gives

$$24a_n + 1 = 9n^2 + 3[4,\ 2]n + [4,\ 1] = (3n + [2,\ 1])^2.$$

So N is the n^{th} Eulerian integer.

Theorem 4

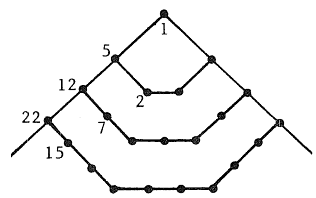
An integer N is Eulerian iff $24N + 1$ is a square k^2. Then its rank is $\left[\frac{k}{3}\right]$ (the greatest integer $\leqslant k/3$).

The integers a_n have a second characteristic property, an arithmogeometric one. If in (2) we distinguish n odd and n even, we have:

a) $n = 2k - 1$: $a_n = \dfrac{3k^2 - k}{3} = 1,\ 5,\ 12,\ 22,\ \ldots;$

b) $n = 2k$: $a_n = \dfrac{3k^2 + k}{2} = 2, 7, 15, 26, \ldots$

The integers $\dfrac{3k^2 - k}{2}$ are the pentagonal numbers, known since antiquity, and they count the dots of the *closed pentagons* below.



The integers $\dfrac{3k^2 + k}{2}$ also have a simple figurative signification: they count the dots of the *open pentagons*. Therefore, we call them *second-class pentagonal numbers*. Note that we also get them by $\dfrac{3k^2 - k}{2}$ for $k = -1, -2, -3, \ldots$.

Theorem 5

Eulerian integers and pentagonal numbers are identical.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 ...
a_n	1	2	5	7	12	15	22	26	35	40	51	57	70	77	92	100 ...

Do the integers a_n satisfy a recurrence relation? Yes, for a_n is an arithmetic polynomial. We know [1] that such a polynomial a_n of characteristics (d, g, p) (we shall define this notion directly) verifies the linear recurrence relation

$$\{(1 - a)^{d-g}(1 - a^p)^{g+1}\} = 0,$$

the exterior braces meaning that in the developed polynomial each power a^i will be replaced by a_{n-i}. For our trinomial a_n of (2), the *degree* $d = 2$, the *grade* $g = 1$ (i.e., that n^1 is the highest power with periodic coefficient) and the *pseudoperiod* $p = 2$ (the least common multiple of the periods of the coefficients). So

$$\{(1 - a)(1 - a^2)^2\} = \{1 - a - 2a^2 + 2a^3 + a^4 - a^5\} = 0.$$

Theorem 6

The Eulerian integers verify the recurrence relation.

$$a_n - a_{n-1} - 2a_{n-2} + 2a_{n-3} + a_{n-4} - a_{n-5} = 0.$$

We know [1] that every arithmetic polynomial a_n whose recurrence relation is $\{F(a)\} = 0$ is generated by a rational fraction $f(x)/F(x)$, where $f(x)$ is of lower degree than $F(x)$. So the Eulerian integer a_n is generated by a fraction

$$\frac{f(x)}{(1 - x)(1 - x^2)^2} = 1 + x + 2x^2 + 5x^3 + 7x^4 + \cdots + a_nx^n + \cdots,$$

where $f(x)$ is of degree 4 at most. Hence,

$$f(x) = 1 - x^2 + 3x^3 + x^4.$$

Theorem 7

Euler's integers are generated by the fraction

$$\frac{1 - x^2 + 3x^3 + x^4}{(1 - x)(1 - x^2)^2} = 1 + \sum_{n>0} a_n x^n.$$

As application, we now shall see Euler's integers in relation to the Eulerian function σ_n and the partitions.

III. EULER'S FUNCTION $\sigma(n)$

As usual, $\sigma(n)$ indicates the sum of the divisors of the integer n. Hence, $\sigma(8) = 1 + 2 + 4 + 8 = 15$, and $\sigma(n) = 1 + n$, iff n is prime. Descartes already noted that $\sigma(nm) = \sigma(n)\sigma(m)$, iff n and m are relatively prime. The first values of $\sigma(n)$ are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\sigma(n)$	1	3	4	7	6	12	8	15	13	18	12	28	14	24	24	31

With respect to this table, far prolonged, Euler observed: "The irregularity of the series of the prime numbers is here intermingled. . . . It seems even that this progression is much more whimsical." Indeed the values of $\sigma(n)$ present an infinity of irregular oscillations. But Euler discovered an unexpected law in their capricious succession.

Theorem 8

The function $\sigma(n)$ verifies the recursive relation

$$\sigma(n) = \sigma(n - a_1) + \sigma(n - a_2) - \sigma(n - a_3) - \sigma(n - a_4) + \cdots \tag{3}$$

with the convention

$$\sigma(k) = \begin{cases} n & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

The a_i are Euler's integers and the signs alternate pairwise.

Example: $\sigma(7) = \sigma(6) + \sigma(5) - \sigma(2) - \sigma(0) = 12 + 6 - 3 - 7 = 8.$

Admire the master's ingenious demonstration:

Take the logarithmic derivative of the two members of (1) and multiply them by $(-x)$:

$$y = \frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} + \cdots = \frac{a_1 x^{a_1} + a_2 x^{a_2} - a_3 x^{a_3} - a_4 x^{a_4} + \cdots}{\pi(x)} = \frac{f(x)}{\pi(x)}.$$

Develop in series the fractions of the first member:

$$\begin{array}{lllllllllll}
y = x + & x^2 + & x^3 + & x^4 + & x^5 + & x^6 + & x^7 + & x^8 & + \cdots \\
 & + 2x^2 & & + 2x^4 & & + 2x^6 & & + 2x^8 & + \cdots \\
 & & + 3x^3 & & & + 3x^6 & & & + \cdots \\
 & & & + 4x^4 & & & & + 4x^8 & + \cdots \\
 & & & & + 5x^5 & & & & + \cdots \\
 & & & & & + 6x^6 & & & + \cdots \\
 & & & & & & + 7x^7 & & + \cdots \\
 & & & & & & & + 8x^8 & + \cdots \\
 & & & & & & & & - - -
\end{array}$$

Hence,

$$y = \sigma(1)x + \sigma(2)x^2 + \sigma(3)x^3 + \cdots .$$

The identity $0 \equiv -f(x) + y\pi(x)$ then gives:

$$\begin{aligned}
0 \equiv -x \quad - \quad 2x^2 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad + \quad 5x^5 \quad\quad\quad\quad\quad + \quad 7x^7 + \cdots \\
+\sigma(1)x + \sigma(2)x^2 + \sigma(3)x^3 + \sigma(4)x^4 + \sigma(5)x^5 + \sigma(6)x^6 + \sigma(7)x^7 + \cdots \\
-\sigma(1)x^2 - \sigma(2)x^3 - \sigma(3)x^4 - \sigma(4)x^5 - \sigma(5)x^6 - \sigma(6)x^7 + \cdots \\
-\sigma(1)x^3 - \sigma(2)x^4 - \sigma(3)x^5 - \sigma(4)x^6 - \sigma(5)x^7 + \cdots \\
+\sigma(1)x^6 + \sigma(2)x^7 + \cdots
\end{aligned}$$

Relation (3) states that the coefficient of x^n in the second member of the preceding identity if zero. We see it clearly when we look at the coefficient of x^7 for example.

The Series $u = \dfrac{\sigma(n)}{n}$

We proved that *the function* $\dfrac{\sigma(k!)}{k!}$ *increases and we shall see that it tends to infinity with* k.

Let $P_1, P_2, \ldots, P_r$ be the prime numbers up to P_r. Then

$$\frac{\sigma(P_i)}{P_i} = 1 + \frac{1}{P_i} \quad \text{and} \quad \frac{\sigma(P_r!)}{P_r!} > \left(1 + \frac{1}{P_1}\right)\left(1 + \frac{1}{P_2}\right) \cdots \left(1 + \frac{1}{P_r}\right).$$

Hence,

$$L\frac{(P_r!)}{P_r!} > L\left(1 + \frac{1}{P_1}\right) + L\left(1 + \frac{1}{P_2}\right) + \cdots + L\left(1 + \frac{1}{P_r}\right) = \sum\frac{1}{P_i} - \frac{1}{2}\sum\frac{1}{P_i^2} + \frac{1}{3}\sum\frac{1}{P_i^3} \ldots,$$

the sums being taken from $i = 1$ to $i = r$. We know that $\sum\frac{1}{P_i} \to \infty$ with r, while the other sums converge. Therefore, $\dfrac{\sigma(P!)}{P!} \to \infty$ with P, and also $\dfrac{\sigma(k!)}{k!} \to \infty$ with k.

What a curious series is $u_n = \dfrac{\sigma(n)}{n}$! Obviously, $u_n \geqslant 1$. It oscillates irregularly—probably between 1 and 6 for $n < 10^{17}$—but it presents an initial regularity: it has a relative extremum for each $n < 62$. The extreme example is likely

$$n = 2^{12} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \simeq 0.998 \times 10^{17}$$

with $u_n \simeq 5999$. It contains at once a decreasing series $u(P_i)$, which tends to 1, a constant series $u(E_k) = 2$, where E_k is the k^{th} Euclidean integer, increasing series $u(a^k)$, which tend to finite numbers if $k \to \infty$, and an increasing series $u(k!)$, which tends to infinity. Furthermore, $u_{nm} = u_n u_m$ if n and m are relatively prime, and $u_{nm} < u_n u_m$ if not. For a prime P and an arbitrary integer k, $u(P^k) < 2$. While

$$u_1 = 1,\ u_6 = 2,\ u_{120} = 3,\ u_{30240} = 4,$$

the least known n for $u_n = 6$ exceeds 10^{28} and the least n for $u_n = 8$ is gigantic [3]. Descartes, Fermat, and others, assiduously searched for values of n for which u_n is an integer. All the found values, save 1 and 6, are multiples of 4.

Perfect numbers can be defined by $u_n = 2$. Euler proved that *the only even perfect numbers are the Euclidean integers* $p(p+1)/2$, where $p = 2^{k+1} - 1$ is prime. *Can an odd perfect number exist?* Nobody knows. But we know that the order of such an odd n would be at least 10^{200} [3]. The difficulty of this millenary question has been compared to that of the transcendency of π (previously, to Lindemann's historical demonstration) or that of Fermat's open problem. More generally:

Conjecture

For an odd n, save 1, the number u_n is never an integer.

Here are some initial values of $u(k!)$, approached for $k > 5$:

k	1	2	3	4	5	6	7	8	9	10 ---	13 ---	20 ---	30
$u(k!)$	1	1.5	2	2.5	3	3.36	3.84	3.95	4.08	4.22	--4.99	--5.52	--5.95

Generally, $u_n < u(k!)$ for $n < k!$. But never: $30240 < 8!$, although $u(30240) = 4$ (found by Descartes) exceeds $u(8!) \simeq 3.95$.

IV. PARTITIONS INTO DISTINCT OR UNRESTRICTED PARTS

Another Eulerian formula is strangely similar to (3). It concerns the number $p(n)$ of partitions of n, into integers distinct or not, whose first values are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$p(n)$	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231

Theorem 9

The number of unrestricted partitions of verifies the recursive relation

$$p(n) = p(n - a_1) + p(n - a_2) - p(n - a_3) - p(n - a_4) + \cdots$$

with the convention

$$p(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

The a_i are Euler's integers and the signs alternate pairwise.

This formula results directly from the fact, mentioned at the beginning, that $p(n)$ is generated by $1/\pi(x)$.

Is it not fabulous that two beings, so disparate as $\sigma(n)$ and $p(n)$ (sum of the divisors of n and number of its partitions) follow the same recursive law (aside from a slight detail: $\sigma_0 = n$, $p_0 = 1$)?

Could a similar recursive law exist, perhaps not linear, for the prime numbers P_n?

Recently D. R. Hickerson found an interesting relation between the numbers of distinct or unrestricted partitions [4]:

Theorem 10

The number p_n of unrestricted partitions of n and the number q_n of its partitions into distinct parts are related by

$$q_n = p_n - p_{n-2a_1} - p_{n-2a_2} + p_{n-2a_3} + p_{n-2a_4} - \cdots$$

with the convention

$$p = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

The signs alternate pairwise.

Starting from the generating functions of p_n and σ_n, we have established an unexpected relation between them:

Theorem 11

The arithmetic functions p_n and σ_n are related by

$$\sigma_n = a_1 p_{n-a_1} + a_2 p_{n-a_2} - a_3 p_{n-a_3} - a_4 p_{n-a_4} + \cdots$$

with the convention

$$p_k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

The signs alternate pairwise.

Conjecture

For $n > 6$, the function $Lp_n/\sqrt{n}$ increases and Lp_n/n decreases. But

$$\frac{Lp_{20}}{\sqrt{20}} > 1.44 \quad \text{and} \quad \frac{Lp_{20}}{20} < 0.33.$$

Hence,

$$e^{1.44\sqrt{n}} < p_n < e^{0.33n} \text{ for } n > 20. \tag{4}$$

Remarks

1) An asymptotical value for p_n was found by Hardy and Ramanujan:

$$p_n \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n} \sim \frac{e^{2.57\sqrt{n}}}{6.93n}.$$

Consequently (4) *is proved for n great.*

2) We know that *the number of partitions of n into unrestricted parts is* 2^{n-1}, *if the order of the summands is relevant.*

Example: For $n = 3 = 1 + 2 = 2 + 1 = 1 + 1 + 1$, this number is 2^2.

Therefore,

$$p_n < 2^{n-1} \text{ for } n > 2.$$

Theorem 12

Let q_n'' and q_n', respectively, be the numbers of partitions of n into an even or an odd number of distinct parts. If the integer n is not Eulerian, $q_n' = q_n''$; for a Eulerian integer a_n, $q_{a_n}'' = q_{a_n}' + [-1, -1, 1, 1]$, the periodic number being related to the rank n of a_n.

Corollary

The number of partitions of an integer n into distinct parts is odd iff is Eulerian. Euler stated that this number equals the number of partitions of n in which all parts, distinct or not, are odd.

The coefficient of x^N is the same in the series

$$(1-x)(1-x^2)(1-x^3)\cdots = 1 + c_1x + c_2x^2 + \cdots + c_Nx^N + \cdots$$

and in the polynomial

$$(1-x)(1-x^2)(1-x^3)\cdots(1-x^N) = 1 + c_1x + c_2x^2 + \cdots + c_Nx^N + x^{N+1}P(x). \quad (5)$$

By developing the product (5) without reducing similar terms, we get, with coefficient (+1), every x^N whose exponent appears as a partition of N in an even number of distinct terms, and with coefficient (-1), each x^N whose exponent appears as a partition of N in an odd number of distinct integers. Therefore,

$$c_N = q_N'' - q_N'.$$

But in (1), $\varepsilon_n = [-1, -1, 1, 1]$ or 0, according to whether N is Eulerian or not.

Remarks

1) Although Theorem 12 follows easily from identity (1), Legendre seems to have been the first to state it [2].

2) Now the great gaps in the series (1) are explained: they simply signify that generally an integer has as many partitions in an even as in an odd number of distinct parts.

3) An odd q_n is characteristic of Eulerian integers, *as an odd* σ_n *is characteristic of squares or double squares*. But the problem of the parity of p_n is still open.

V. PARTITIONS INTO PARTS OF GIVEN VALUES

The following text of Euler shows with charming simplicity his enthusiasm for his amazing formula (3). His integers seem to be still a little mysterious to him.

> We are the more surprised by this beautiful property, as we see no relation between the composition of our formula and the divisors whose sums concern the proposition. The progression of the numbers 1, 2, 5, 7, 12, 15, ... not only seems to have no relation to the subject, but—as the law of their numbers is interrupted and they are a mixture of two different progressions: 1, 5, 12, 22, 35, 51, ... and 2, 7, 15, 26, 40, 57, ...—it almost seems that such an irregularity could not exist in analysis.

So Euler was surprised that a_n takes its values from two progressions, trinomials of the second degree. However, notwithstanding what he believed, one often meets in analysis series of integers that take their values from several polynomials: *the arithmetic polynomials*, which all have a generating rational fraction and satisfy a linear recurrence relation. It is piquant to see that such series occur, particularly in a question which Euler examined at length: the partition into parts of given values [5].

Example 1

In how many ways can n identical objects be divided in groups of 12, 13, and 17 pieces?

This is equivalent to finding the number j_n of nonnegative integer solutions of the equation

$$12x + 13y + 17z = n. \quad (6)$$

Those problems are solved by a general theorem, whose first part is due to Euler:

Theorem 13

The number j_n of nonnegative solutions of the diophantine equation

$$\sum_{i=1}^{r} \alpha_i x^i = n$$

with positive coefficients, is generated by the fraction

$$\frac{1}{(1 - t^{\alpha_1})(1 - t^{\alpha_2}) \ldots (1 - t^{\alpha_r})} = \sum_{n \geqslant 0} j_n t^n.$$

The function $j(n)$ is an arithmetic polynomial, whose pseudoperiod is the least common multiple of the α_i, its degree $r - 1$ and its grade $m - 1$, m being the greatest number of coefficients α_i that have a common divisor other than 1 [1].

So, for Eq. (6), j_n is an arithmetic trinomial whose characteristics are (2, 0, 2652). More precisely, we know [1] that j_n verifies a relation of the form

$$2(12 \times 13 \times 17)j_n = n^2 + (12 + 13 + 17)n + u_n,$$

where u_n is a number of period $12 \times 13 \times 17 = 2652$.

You may think the 2652 components of the periodic number u_n long to calculate, and the expression of j_n long to write. Not at all. The calculation of u_n is performed in an instant by the computer (with the program for the resolution of a system of linear equations, which every computing center has) and

$$j_n = \left\| \frac{n^2 + 42n + 100(A_n - B_n)}{5304} \right\|,$$

where the periodic numbers

$$A_n = [5, 21, 25, 17, -2, 17, 25, 21, 5, 30, 42, 42, 30]$$

and

$$B_n = [-2, 17, 6, 17, -2, 0, 24, 17, 33, 17, 24, 0]$$

have, respectively, 13 and 12 components.

The error is at most 1, *for*

$$j_n \simeq \left\| \frac{n(n + 42)}{5304} \right\|.$$

Example 2

What is the number of solutions in nonnegative integers of the equation

$$x + 2y + 6z + 3 = 3n?$$

We have shown that *this number is Euler's integer* a_n.

Note that a_n has the characteristics (2, 1, 2), while, for the diophantine equation

$$x + 2y + 6z = n$$

the number of nonnegative solutions is, by Theorem 13, an arithmetic trinomial of characteristics (2, 1, 6).

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THE GENERAL SOLUTION TO THE DECIMAL FRACTION OF FIBONACCI SERIES

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1. INTRODUCTION

After Stancliff [1] noted that 1/89 can be represented as the sum of Fibonacci Series, Long [2] and Hudson & Winans [1] also perceived that there are some other numbers which can be represented as the sum of Fibonacci Series or Lucas Series. Hudson & Winans [1] gave the solution of the series

$$\sum_{i=1}^{\infty} 10^{-k(i+1)} F_{\alpha i}.$$

Long [2] gave some particular solutions for the series

$$\sum_{i=0}^{\infty} (\pm 10)^{-k(i+1)} F_i$$

and for

$$\sum_{i=0}^{\infty} (\pm 10)^{-k(i+1)} L_i.$$

In this paper, a method similar to that employed by Hudson & Winans is used to obtain the general solution for all such series.

2. THE SERIES $\sum_{i=1}^{\infty} 10^{-k(i+1)} F_{\alpha i}$

According to Hoggatt [4], the n^{th} Fibonacci number and the n^{th} Lucas number can be represented, respectively, by

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \tag{1}$$

and

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{2}$$

Note that we have

$$L_n + \sqrt{5}F_n = 2^{1-n}(1+\sqrt{5})^n, \tag{3}$$

and

$$L_n - \sqrt{5}F_n = 2^{1-n}(1-\sqrt{5})^n. \tag{4}$$

Using these, we obtain:

$$\sum_{i=1}^{\infty} 10^{-k(i+1)} F_{\alpha i} = \sum_{i=1}^{\infty} 10^{-k(i+1)} \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{\alpha i} - \left(\frac{1-\sqrt{5}}{2}\right)^{\alpha i}\right]$$

(continued)

$$= \sum_{i=1}^{\infty} 10^{-k(i+1)} \frac{1}{\sqrt{5}}\left[\left(\frac{L_\alpha + \sqrt{5}F_\alpha}{2}\right)^i - \left(\frac{L_\alpha - \sqrt{5}F_\alpha}{2}\right)^i\right]$$

$$= \sum_{i=1}^{\infty} \frac{1}{10^k\sqrt{5}}\left[\left(\frac{L_\alpha + \sqrt{5}F_\alpha}{2 \cdot 10^k}\right)^i - \left(\frac{L_\alpha - \sqrt{5}F_\alpha}{2 \cdot 10^k}\right)^i\right].$$

Since $r + r^2 + r^3 + \cdots = \sum_{n=1}^{\infty} r^n$, $\sum_{n=1}^{\infty} r^n$ converges to $\frac{r}{1-r}$ iff $|r| < 1$.

Consequently, for values of α and k for which the series converges, we have

$$\sum_{i=1}^{\infty} 10^{-k(i+1)} F_{\alpha i} = \frac{4 \cdot 10^k \cdot F_\alpha\sqrt{5}}{10^k\sqrt{5}[4 \cdot 10^{2k} - 4 \cdot 10^k \cdot L_\alpha + L_\alpha^2 - 5F_\alpha^2]}$$

$$= \frac{4F_\alpha}{4[10^{2k} - 10^k \cdot L_\alpha + (-1)^\alpha]}$$

$$= \frac{F_\alpha}{10^{2k} - 10^k \cdot L_\alpha + (-1)^\alpha} \tag{5}$$

Equation (5) agrees with (1.1) and (1.2) of [1] obtained by Hudson, noting that $(\alpha + 1)/2$ in (1.2) of [1] is a misprint and should read $(\alpha - 2)/2$.

Using the same method, we have

$$\sum_{i=0}^{\infty} 10^{-k(i+1)} L_{\alpha i} = \frac{2 \cdot 10^k - L_\alpha}{10^{2k} - 10^k \cdot L_\alpha + (-1)^\alpha} \tag{6}$$

$$\sum_{i=1}^{\infty} (-10^{-k})^{i+1} F_{\alpha i} = \frac{F_\alpha}{10^{2k} + 10^k \cdot L_\alpha + (-1)^\alpha} \tag{7}$$

$$\sum_{i=0}^{\infty} (-10^{-k})^{i+1} L_{\alpha i} = -\frac{2 \cdot 10^k + L_\alpha}{10^{2k} + 10^k \cdot L_\alpha + (-1)^\alpha} \tag{8}$$

(k, α, i, n are integers).

We note that:

$$F_\alpha = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^\alpha - \left(\frac{1-\sqrt{5}}{2}\right)^\alpha\right],$$

$$F_\alpha^2 = \frac{1}{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2\alpha} + \left(\frac{1-\sqrt{5}}{2}\right)^{2\alpha} - 2\left(\frac{1+\sqrt{5}}{2}\right)^\alpha\left(\frac{1-\sqrt{5}}{2}\right)^\alpha\right].$$

$$\therefore\ 5F_\alpha^2 = \left(\frac{1+\sqrt{5}}{2}\right)^{2\alpha} + \left(\frac{1-\sqrt{5}}{2}\right)^{2\alpha} + 2\left(\frac{1+\sqrt{5}}{2}\right)^\alpha\left(\frac{1-\sqrt{5}}{2}\right)^\alpha - 4\left(\frac{1+\sqrt{5}}{2}\right)^\alpha\left(\frac{1-\sqrt{5}}{2}\right)^\alpha$$

$$= L_\alpha^2 - 4(-1)^\alpha.$$

$$\therefore\ L_\alpha^2 - 5F_\alpha^2 = 4(-1)^\alpha.$$

3. SOME PARTICULAR VALUES FOR THE ABOVE SERIES

TABLE 1. Some Values of $\sum_{i=1}^{\infty} 10^{-k(i+1)} F_{\alpha i}$

k \ α	1	2	3	4	5	6	7	8	9	10
1	$\frac{1}{89}$	$\frac{1}{71}$	$\frac{2}{59}$	$\frac{3}{31}$						
2	$\frac{1}{9899}$	$\frac{1}{9701}$	$\frac{2}{9599}$	$\frac{3}{9301}$	$\frac{5}{8899}$	$\frac{8}{8201}$	$\frac{13}{7099}$	$\frac{21}{5301}$	$\frac{34}{2399}$	
3	$\frac{1}{998999}$	$\frac{1}{997001}$	$\frac{2}{995999}$	$\frac{3}{993001}$	$\frac{5}{988999}$	$\frac{8}{982001}$	$\frac{13}{970999}$	$\frac{21}{953001}$	$\frac{34}{923999}$	$\frac{55}{877001}$
	$\frac{89}{800999}$	$\frac{144}{678001}$	$\frac{233}{478999}$	$\frac{377}{157001}$						

TABLE 2. Some Values of $\sum_{i=1}^{\infty} 10^{-k(i+1)} L_{\alpha i}$

k \ α	1	2	3	4	5	6	7	8	9	10
1	$\frac{19}{89}$	$\frac{17}{71}$	$\frac{16}{59}$	$\frac{13}{31}$						
2	$\frac{199}{9899}$	$\frac{197}{9701}$	$\frac{196}{9599}$	$\frac{193}{9301}$	$\frac{189}{8899}$	$\frac{182}{8201}$	$\frac{171}{7099}$	$\frac{153}{5301}$	$\frac{124}{2399}$	
3	$\frac{1999}{998999}$	$\frac{1997}{997001}$	$\frac{1996}{995999}$	$\frac{1993}{993001}$	$\frac{1989}{988999}$	$\frac{1982}{982001}$	$\frac{1971}{970999}$	$\frac{1953}{953001}$	$\frac{1924}{923999}$	$\frac{1877}{877001}$
	$\frac{1801}{800999}$	$\frac{1678}{678001}$	$\frac{1479}{478999}$	$\frac{1157}{157001}$						

TABLE 3. Some Values of $\sum_{i=1}^{\infty} (-10)^{-k(i+1)} F_{\alpha i}$

k \ α	1	2	3	4	5	6	7	8	9	10
1	$\frac{1}{109}$	$\frac{1}{131}$	$\frac{2}{139}$	$\frac{3}{171}$						
2	$\frac{1}{10099}$	$\frac{1}{10301}$	$\frac{2}{10399}$	$\frac{3}{10701}$	$\frac{5}{11099}$	$\frac{8}{11801}$	$\frac{13}{12899}$	$\frac{21}{14701}$	$\frac{34}{17599}$	
3	$\frac{1}{1000999}$	$\frac{1}{1003001}$	$\frac{2}{1003999}$	$\frac{3}{1007001}$	$\frac{5}{1010999}$	$\frac{8}{1018001}$	$\frac{13}{1028999}$	$\frac{21}{1047001}$	$\frac{34}{1075999}$	$\frac{55}{1123001}$
	$\frac{89}{1198999}$	$\frac{144}{1322001}$	$\frac{233}{1520999}$	$\frac{377}{1843001}$						

TABLE 4. Some Values of $\sum_{i=0}^{\infty} (-10)^{-k(i+1)} L_{\alpha i}$

$k \backslash \alpha$	1	2	3	4	5	6	7	8	9	10
1	$-\frac{21}{109}$	$-\frac{23}{131}$	$-\frac{24}{139}$	$-\frac{27}{171}$						
2	$-\frac{201}{10099}$	$-\frac{203}{10301}$	$-\frac{204}{10309}$	$-\frac{207}{10701}$	$-\frac{211}{11099}$	$-\frac{218}{11801}$	$-\frac{229}{12899}$	$-\frac{247}{14701}$	$-\frac{276}{17599}$	
3	$-\frac{2001}{1000999}$	$-\frac{2003}{1003001}$	$-\frac{2004}{1003999}$	$-\frac{2007}{1007001}$	$-\frac{2011}{1010999}$	$-\frac{2018}{1018001}$	$-\frac{2029}{1028999}$	$-\frac{2047}{1047001}$	$-\frac{2076}{1075999}$	$-\frac{2123}{1123001}$
	$-\frac{2199}{1198999}$	$-\frac{2322}{1322001}$	$-\frac{2521}{1520999}$	$-\frac{2843}{1843001}$						

4. EXTENSION TO GENERALIZED FIBONACCI NUMBERS

A general Fibonacci number can be represented as

$$T_n = aT_{n-1} + bT_{n-2} \text{ with } T_0 = c,\ T_1 = d. \tag{9}$$

Long [2] has given the form of the general Fibonacci number as

$$T_n = \left(\frac{c}{2} + \frac{2d - ca}{2\sqrt{a^2 + 4b}}\right)\left(\frac{a + \sqrt{a^2 + 4b}}{2}\right)^n + \left(\frac{c}{2} - \frac{2d - ca}{2\sqrt{a^2 + 4b}}\right)\left(\frac{a - \sqrt{a^2 + 4b}}{2}\right)^n \tag{10}$$

Here, if $c = 0$, $a = b = d = 1$, then T_n can be reduced to F_n, and if $c = 2$, $a = b = d = 1$, then T_n can be reduced to F_n.

Using the above method, we obtain

$$S_n \pm \sqrt{a^2 + 4b}R_n = 2^{1-n}(a \pm \sqrt{a^2 + 4b})^n \tag{11}$$

where

$$S_n = \left(\frac{a + \sqrt{a^2 + 4b}}{2}\right)^n + \left(\frac{a - \sqrt{a^2 + 4b}}{2}\right)^n \tag{12}$$

$$R_n = \frac{1}{\sqrt{a^2 + 4b}}\left[\left(\frac{a + \sqrt{a^2 + 4b}}{2}\right)^n - \left(\frac{a - \sqrt{a^2 + 4b}}{2}\right)^n\right] \tag{13}$$

Then we can get

$$\sum_{i=0}^{\infty} 10^{-k(i+1)} T_{\alpha i} = \frac{\frac{c}{2}(2 \cdot 10^k - S_\alpha) + \frac{2d - ca}{2}R_\alpha}{10^{2k} - 10^k \cdot S_\alpha + (-b)^\alpha} \tag{14}$$

$$\sum_{i=0}^{\infty} (-10)^{-k(i+1)} T_{\alpha i} = \frac{-\frac{c}{2}(2 \cdot 10^k + S_\alpha) + \frac{2d - ca}{2}R_\alpha}{10^{2k} + 10^k \cdot S_\alpha + (-b)^\alpha} \tag{15}$$

for values of α and k with

$$\frac{(a + \sqrt{a^2 + 4b})^\alpha}{2 \cdot 10^k} < 1,$$

or, equivalently, with

$$\frac{S_\alpha + \sqrt{a^2 + 4b}R_\alpha}{2 \cdot 10^k} < 1 \tag{16}$$

As an example, if $a = 1$, $b = 3$, $c = 2$, $d = 5$, some values of S_α, R_α, T_α, $\sum_{i=0}^{\infty} (10)^{-k(i+1)}T_{\alpha i}$, and $\dfrac{S_\alpha + R_\alpha\sqrt{13}}{2 \cdot 10^k}$ are shown in Tables 5, 6, and 7 for different α and k.

TABLE 5. Some Values of S_α, R_α, T_α

Series \ α	1	2	3	4	5	6	7	8	9	10
S_α	1	7	10	31	61	154	337	799	1810	4207
R_α	1	1	4	7	19	40	97	217	508	1159
T_α	5	11	26	59	137	314	725	1667	3842	8843

TABLE 6. Some Values of $\sum_{i=0}^{\infty} 10^{-k(i+1)}T_{\alpha i}$

k \ α	1	2	3	4	5	6	7	8
1	$\frac{23}{87}$	$\frac{17}{39}$						
2	$\frac{203}{9897}$	$\frac{197}{9309}$	$\frac{206}{8973}$	$\frac{197}{6981}$	$\frac{215}{3657}$			
3	$\frac{2003}{998997}$	$\frac{1997}{993009}$	$\frac{1916}{989973}$	$\frac{1997}{969081}$	$\frac{2015}{938757}$	$\frac{2006}{846729}$	$\frac{2051}{660813}$	$\frac{2069}{207561}$

TABLE 7. Some Values of $\dfrac{S_\alpha + R_\alpha\sqrt{13}}{2 \cdot 10^k}$

k \ α	1	2	3	4	5	6	7	8	9
1	0.2303	0.5303	1.2211						
2	0.0230	0.0530	0.1221	0.2812	0.6475	1.4911			
3	0.0023	0.0053	0.0122	0.0281	0.0648	0.1491	0.3434	0.7907	1.8208

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AN EASY PROOF OF THE GREENWOOD-GLEASON EVALUATION OF THE RAMSEY NUMBER $R(3, 3, 3)$

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1. INTRODUCTION

In 1955, Greenwood & Gleason proved that the Ramsey number $R(3, 3, 3) = 17$ by constructing a triangle-free, edge-chromatic graph in three colors of order 16. Their method employed finite fields. This result was obtained later by another method. Here, we give yet another method which can be called "group-theoretical" or, merely, "adding binary codes."

2. THE METHOD

Consider the set of 16 binary codes {0000, 0001, 0010, 0011, ..., 1111}; if we add them componentwise with $0 + 0 = 0$, $1 + 0 = 0 + 1 = 1$, and $1 + 1 = 0$, then this set G under $+$ is isomorphic to the elementary abelian group of order 16. Partition the 15 nonidentity elements into three sets G_1, G_2, G_3 so that no two elements in any of the three sets add up to an element in the same set. Then, we identify the vertices of a graph Γ with the elements of this group G. We 3-color the edges as follows: join the vertices x and y by an edge of color i if $x + y \in G_i$; join 0000 with x by an edge of color i if $x \in G_i$.

3. THE CONSTRUCTION

Partition the 15 nonidentity elements into 3 sets:

$$G_1 = \{1100, 0011, 1001, 1110, 1000\},$$
$$G_2 = \{1010, 0101, 0110, 1101, 0100\},$$
$$G_3 = \{0001, 0010, 0111, 1011, 1111\}.$$

Obviously, no two elements in G_i add up to be an element in G_i. We thus obtain:

4. THE GRAPH

Using solid lines for color 1, dot-dash lines for color 2, and dotted lines for color 3, the triangle-free, edge-chromatic graph in three colors of order 16 is shown in Figures 1-4.

5. EXTENSION OF THE METHOD

This method *can* be used to find the lower bound of other Ramsey numbers. to this end, one first finds an appropriate group, partitions the group elements into several subsets, making sure that in each subset the product of two elements is never in it. The sharpness of the bound depends on the choice of the group.

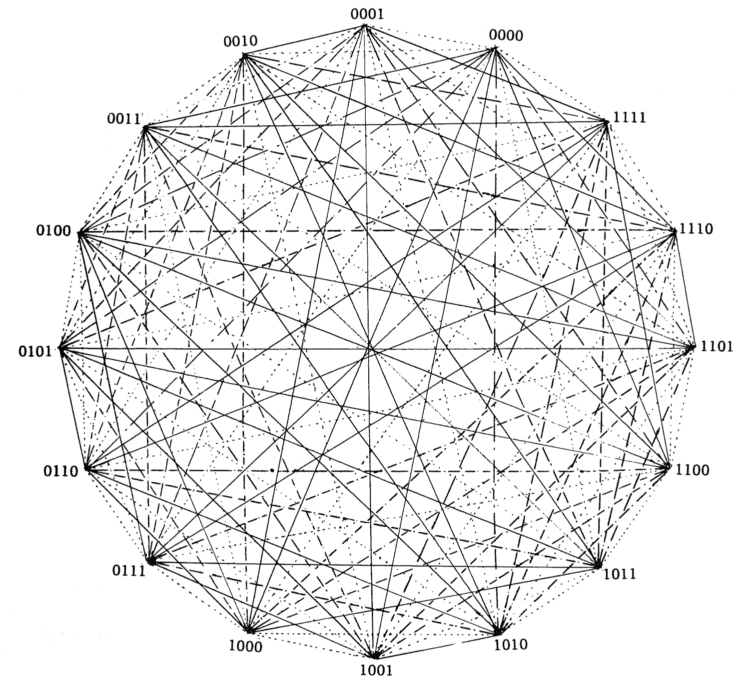

FIG. 1. The Triangle-Free, Edge-Chromatic Graph in Three Colors of Order 16

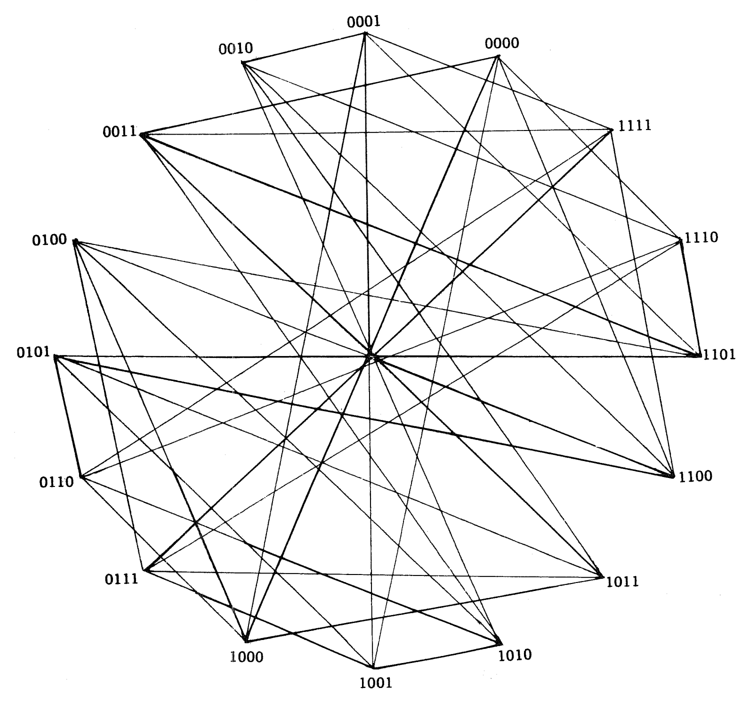

FIG. 2. Subgraph of Solid Lines

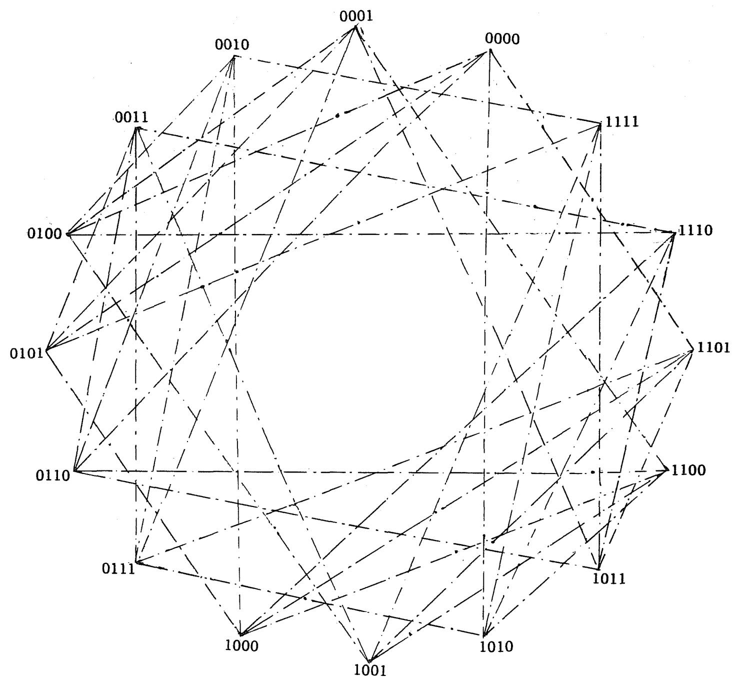

FIG. 3. Subgraph of Dot-Dash Lines

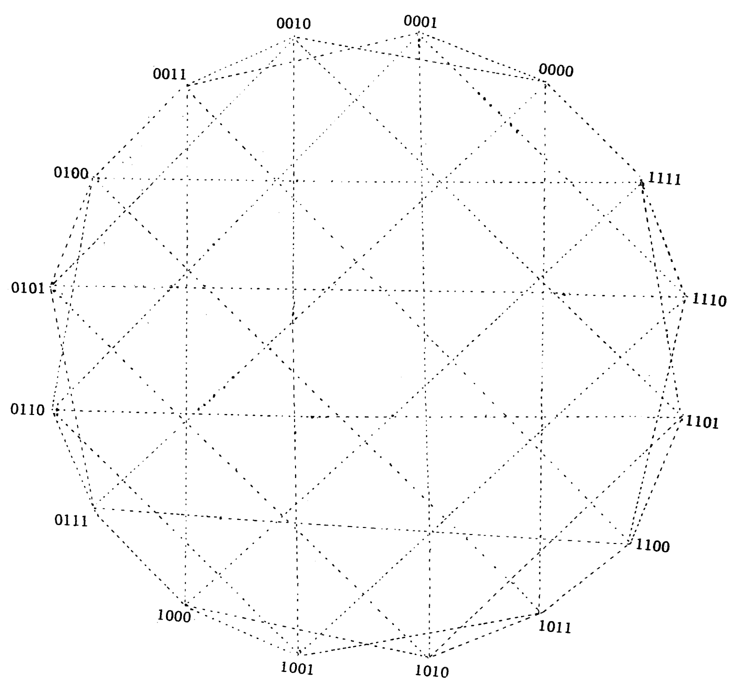

FIG. 4. Subgraph of Dotted lines

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SOME ASYMPTOTIC PROPERTIES OF GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

Horadam [1] has generalized two theorems of Subba Rao [3] which deal with some asymptotic properties of Fibonacci numbers. Horadam defined a sequence

$$\{w_n^{(2)}\} = \{w_n(w_0, w_1; P_{21}, P_{22})\}$$

which satisfies the second-order recurrence relation

$$w_{n+2} = P_{21}w_{n+1} - P_{22}w_n, \text{ with } w_n = A_{21}\alpha_{21}^n + A_{22}\alpha_{22}^n,$$

where α_{21}, α_{22} are the roots of $x^2 - P_{21}x + P_{22} = 0$. We shall let

$$d = \alpha_{22} - \alpha_{21}.$$

Horadam established two theorems for $\{w_n\}$:

I. The number of terms of $\{w_n\}$ not exceeding N is asymptotic to

$$\underline{\log}(Nd/(P_{22}w_0 - \alpha_{21}w_1)).$$

II. The range, within which the rank n of w_n lies, is given by

$$\underline{\log}\, w_n + \underline{\log}(X - d)/x < n + 1 < \underline{\log}\, w_n + \underline{\log}(Y - d)/x,$$

where

$$X = y/(w_{-1} + 2x),\ Y = y/(w_{-1} - 2x),$$
$$x = w_0 - \alpha_{22}w_{-1},\ y = w_0 - \alpha_{21}w_{-1},$$

and in which $\underline{\log}$ stands for logarithm to the base α_{r1}; $r = 2$ in this case.

These were generalizations of two theorems which Subba Rao had proved for

$$\{f_n\} : f_n = w_n(1,\ 1;\ 1,\ -1),$$

the ordinary Fibonacci numbers.

It is proposed here to explore generalizations of the Horadam-Subba Rao theorems to sequences, the elements of which satisfy linear recurrence relations of artitrary order. To this end, we define $\{w_n^{(r)}\}$:

$$w_{n+r}^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj}\, w_{n+r-j}^{(r)},\ n \geqslant 0, \tag{1.1}$$

with suitable initial values $w_n^{(r)}$, $n = 0, 1, \ldots, r - 1$, and where the P_{rj} are arbitrary integers. Thus, $\{w_n^{(2)}\}$ represents Horadam's generalized sequence of integers.

We can suppose then that

$$w_n^{(r)} = \sum_{j=1}^{r} A_{rj}\alpha_{rj}^n, \tag{1.2}$$

in which the A_{rj} depend on the initial values of $\{w_n^{(r)}\}$, and the α_{rj} are the roots (assumed distinct) of

$$x^r - \sum_{j=1}^{r} (-1)^{j+1} P_{rj}\, x^{r-j} = 0. \tag{1.3}$$

In fact, $A_{ri} = d_i/d$, where

$$d = \prod_{\substack{i,j=1\\ i>j}}^{r} (\alpha_{ri} - \alpha_{rj})$$

is the Vandermonde of the roots α_{ri}, and d_i is obtained from d on replacement of its i^{th} column by the r initial terms of $\{w_n^{(r)}\}$ (Jarden [2]).

2. ASYMPTOTIC BEHAVIOR

Where convenient in this section, we follow the reasoning of Horadam or of Subba Rao.

Theorem A

The number of terms of $\{w_n^{(r)}\}$ not exceeding N is asymptotic to

$$\underline{\log}(N/A_{r1}\alpha_{r1}).$$

Proof: Suppose $w_n^{(r)} \leqslant N < w_{n+1}^{(r)}$. (2.1)

Then the left-hand side yields

$$w_n^{(r)} = \sum_{j=1}^{r} A_{rj}\,\alpha_{rj}^n \leqslant N.$$

Suppose further that $|\alpha_{r1}| > |\alpha_{r2}| > \cdots > |\alpha_{rr}| > 0$, so that for $m > 1$,

$$(\alpha_{rm}/\alpha_{r1})^n \to 0 \text{ as } n \to \infty,$$

and

$$A_{r1} \leqslant N/\alpha_{r1}^n.$$

Hence,

$$n \log \alpha_{r1} + \log A_{r1} \leqslant \log N$$

and

$$n \leqslant \underline{\log}(N/A_{r1}). \tag{2.2}$$

The right-hand side of the first inequality (2.1) yields

$$N < \sum_{j=1}^{r} A_{rj}\,\alpha_{rj}^{n+1};$$

whence

$$n + 1 > \underline{\log}(N/A_{r1}). \tag{2.3}$$

Thus, from inequalities (2.2) and (2.3):

$$n - 1 \leqslant \underline{\log}(N/A_{r1}) - 1 < n,$$

or

$$n \sim \underline{\log}(N/A_{r1}\alpha_{r1}) \text{ as required.}$$

This is a generalization of Theorem I of Horadam, because when $r = 2$ and $w_0^{(2)} = a$, $w_1^{(2)} = b$,

$$A_{21}\alpha_{21} = d_1\alpha_{21}/d = (a\alpha_{22} - b)\alpha_{21}/d = (aP_{22} - \alpha_{21}b)/d),$$

which agrees with Horadam.

3. RANK

To obtain a partial generalization of Theorem II of Horadam and the corresponding proposition of Subba Rao, we first define $\{U_n^{(r)}\}$, a fundamental sequence of order r; we illustrate its fundamental nature by showing that any linear recursive sequence of order r can be expressed in terms of $\{U_n^{(r)}\}$.

We define $\{U_n^{(r)}\}$ by means of

$$\begin{aligned} \alpha_{rj}^n &= \frac{1}{r} U_n^{(r)} \sum_{k=1}^{r} D^k/\omega^{kj}, \; n \geqslant 0, \\ &= 0, \; n < 0. \end{aligned} \tag{3.1}$$

where $D = \sum_{j=1}^{r} \omega^j \alpha_{rj}$, in which $\omega = \exp(2\pi i/r)$ and α_{rj} satisfies (1.3).

It follows that

$$U_n^{(r)} = D^{-1} \sum_{j=1}^{r} \omega^j \alpha_{rj}^n, \; n \geqslant 0, \tag{3.2}$$

Proof: $\sum_{j=1}^{r} \omega^j \alpha_{rj}^n = \frac{1}{r} U_n^{(r)} \sum_{k=1}^{r} D^k \sum_{j=1}^{r} \omega^{(1-k)j} = \frac{1}{r} U_n^{(r)} Dr,$

which gives the result, since

$$\frac{1}{r} \sum_{j=1}^{r} \omega^{ij} = \delta_{i0}, \text{ the Kronecker delta.}$$

For example, when $r = 2$, $\omega = -1$, and we get, from (3.2), that

$$\begin{aligned} U_0^{(2)} &= D^{-1}(-1 + 1) = 0, \\ U_1^{(2)} &= D^{-1}(-\alpha_{21} + \alpha_{22}) = 1, \\ U_2^{(2)} &= D^{-1}(-\alpha_{21} + \alpha_{22}) = P_{21}, \end{aligned}$$

so that $U_n^{(2)} = u_{n-1}$ defined in (1.8) of Horadam, because, for $n > 1$, $U_n^{(r)}$ satisfies the recurrence relation (1.1).

Proof: The right-hand side of this recurrence relation is

$$\begin{aligned} \sum_{j=1}^{r} (-1)^{j+1} P_{rj} U_{n-j}^{(r)} &= \sum_{k=1}^{r} \sum_{j=1}^{r} (-1)^{j+1} P_{rj} D^{-1} \alpha_{rk}^{n-j} \omega^k \\ &= \frac{1}{D} \sum_{k=1}^{r} \left(\sum_{j=1}^{r} (-1)^{j+1} P_{rj} \alpha_{rk}^{r-j} \right) \alpha_{rk}^{n-r} \omega^k \\ &= \frac{1}{D} \sum_{k=1}^{r} \alpha_{rk}^r \alpha_{rk}^{n-r} \omega^k \\ &= U_n^{(r)} \end{aligned}$$

Our next result is quite important in that it justifies our finding the rank of $U_n^{(r)}$ instead of that of $w_n^{(r)}$ because every $w_n^{(r)}$ can be expressed in terms of the fundamental $U_n^{(r)}$.

To prove this, we look first at the set

$$P = P(P_{r1}, P_{r2}, \ldots, P_{rr})$$

of all sequences of order r which satisfy

$$\sum_{j=0}^{r}(-1)^{j}P_{rj}\,w_{n-j}^{(r)} = 0,\ P_{r0} = -1.$$

P is closed with respect to "addition" and "scalar multiplication" and

$$\{U_{n-k}^{(r)}\} \in P,\ k < n+1,$$

so we may seek to express the elements of P as a linear combination of the fundamental sequence:

$$w^{(r)} = \sum_{k=0}^{r-1} b_k U_{n-k+1}^{(r)},\ n \geqslant 0. \tag{3.3}$$

The first r of these relations (3.3) may be considered as a system of simultaneous equations in the b_k as unknowns; since the determinant of the system is

$$\begin{vmatrix} U_1^{(r)} & 0 & \dots & 0 \\ U_2^{(r)} & U_1^{(r)} & \dots & 0 \\ U_r^{(r)} & U_{r-1}^{(r)} & \dots & U_1^{(r)} \end{vmatrix} = 1,\ (U_1^{(r)} = 1),$$

the solution always exists, is unique, and can be expressed easily in determinant form. To obtain a simpler expression, we calculate for $n < r$ that

$$\begin{aligned}\sum_{j=0}^{n}(-1)^{j}P_{rj}\,w_{n-j}^{(r)} &= \sum_{j=0}^{n}\sum_{k=0}^{r-1}(-1)^{j}P_{rj}\,b_k U_{n-j-k+1}^{(r)}\\ &= \sum_{k=0}^{r-1} b_k \sum_{j=0}^{n}(-1)^{j}P_{rj}\,U_{n-k-j+1}^{(r)}\\ &= -b_n,\end{aligned}$$

since

$$\sum_{j=0}^{n}(-1)^{j}P_{rj}\,U_{n-k-j+1}^{r} = P_{r0}\,\delta_{nk},$$

where δ_{nk} is again the Kronecker delta; this follows from the facts that

$$U_n^{(r)} = 0 \quad \text{if } n \leqslant 0,$$

$$\sum_{j=0}^{r}(-1)^{j}P_{rj}\,U_{n-j+1}^{(r)} = 0 \quad \text{if } n > 0,$$

and

$$U_1^{(r)} = 1.$$

Thus,

$$\begin{aligned}w_n^{(r)} &= -\sum_{k=0}^{r-1}\sum_{i=0}^{k}(-1)^{i+1}P_{ri}\,w_{k-i}^{(r)}U_{n-k+1}^{(r)}\\ &= -\sum_{j=0}^{r-1}\left(\sum_{k=j}^{r-1}(-1)^{k-j+1}P_{r,k-j}U_{n-k+1}^{(r)}\right)w^{(r)}.\end{aligned}$$

That is, $w_n^{(r)}$ depends on the initial values $w_0^{(r)}$, $w_1^{(r)}$, ..., $w_{r-1}^{(r)}$, and $U_n^{(r)}$, and so the properties of $w_n^{(r)}$ depend on the $U_n^{(r)}$.

We now seek the rank of $U_n^{(r)}$ instead of $w_n^{(r)}$; this will be a generalization of Subba Rao rather than Horadam. From Eq. (3.1), we have that

$$\alpha_{r1}^n = \frac{1}{r} U_n^{(r)} \sum_{k=1}^{r} D^k \Big/ \omega^k.$$

So

$$\alpha_{r1}^n > \frac{1}{r} U_n^{(r)} \sum_{k=1}^{r} D^{\frac{1}{2}k} \omega^{-k}$$

and

$$\alpha_{r1}^n < \frac{1}{r} U_n^{(r)} \sum_{k=1}^{r} D^k.$$

Thus,

$$n + 1 < \underline{\log}\, U_n^{(r)} \left(\alpha_{r1} \sum_{k=1}^{r} D^k\right) \Big/ r$$

and

$$n + 1 > \underline{\log}\, U_n^{(r)} \left(\alpha_{r1} \sum_{k=1}^{r} D^{\frac{1}{2}k} \omega^{-k}\right) \Big/ r,$$

which yield:

<u>Theorem B</u>

$$\underline{\log}\, U_n^{(r)} + \underline{\log} \left(\alpha_{r1} \sum_{k=1}^{r} D^{\frac{1}{2}k} \omega^{-k} \Big/ r\right) < n + 1 < \underline{\log}\, U_n^{(r)} + \underline{\log} \left(\alpha_{r1} \sum_{k=1}^{r} D^k \Big/ r\right),$$

and this gives the range within which the rank n of $U_n^{(r)}$ lies.

For example, when $r = 2$, $D = d$, $d^2 = 5$, $\omega = -1$, $P_{21} = -P_{22} = 1$, $\alpha_{21} \doteqdot 1.6$, $\alpha_{22} \doteqdot -0.6$, we get for the Fibonacci number $F_3^{(2)} = 2$ that

$$\frac{\log 2}{\log 1.6} + \frac{\log 2.9}{\log 1.6} < 3 + 1 < \frac{\log 2}{\log 1.6} + \frac{\log 5.8}{\log 1.6} \quad \text{or} \quad 3.7 < 5 < 5.3.$$

This is not quite as good as Subba Rao's result, because there is just one number in the corresponding range for his result, but we do have an acceptable range.

Thus, in Theorem A we have generalized Horadam's result, and in Theorem B we have generalized Subba Rao's result; we have also established a link between the more generalized sequence in Theorem A and the fundamental generalized sequence in Theorem B.

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A MODIFIED TRIBONACCI SEQUENCE

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(Submitted November 1982)

1. INTRODUCTION

The Tribonacci sequence [1] is generated by the recurrence relation

$$U_{n+3} = U_{n+2} + U_{n+1} + U_n, \tag{1}$$
$$\text{with } U_0 = 0, \text{ and } U_1 = U_2 = 1.$$

Part of the charm of the original Fibonacci sequence $\{F_n\}$ is the ease with which new relations can be found, and a wealth of applications. However, (1) is rather unweildy and does not yield relations too readily. This article suggests a modification so that a development analogous to the Fibonacci sequence can be made. In addition, higher-order sequences can be constructed.

2. RECURRENCE RELATIONS FOR THE MODIFIED TRIBONACCI SEQUENCE

Consider $\{T_n\}$ generated by the recurrence relation

$$T_{2n} = T_{2n-1} + T_{2n-3}, \tag{2a}$$
$$T_{2n+1} = T_{2n-1} + T_{2n-2}, \tag{2b}$$
$$\text{where } n > 2, \text{ and } T_1 = T_2 = T_3 = 1.$$

The numerical sequence that emerges using (2a) and (2b) is:

1, 1, 1, 2, 2, 3, 4, 6, 7, 11, 13, 20, 24, 37, 44, 68, 81, ...

Note that $\{T_n\}$ resembles $\{F_n\}$ in its mode of definition.

However, successively odd and even terms are defined separately—note also that each odd term is the sum of the three previous odd terms, and, similarly, for the even terms. In this latter respect, the sequence resembles Tribonacci.

3. SOME PROPERTIES OF (T_n)

We can now go on to develop properties of $\{T_n\}$, some of which are analogous in form to $\{F_n\}$. These are presented without proof, as they are all elementary; no claim to completenes of the list is made.

$$T_{2n+5} = T_{2n+3} + T_{2n+1} + T_{2n-1}, \; n \geqslant 2; \tag{3}$$
$$T_{2n+6} = T_{2n+4} + T_{2n+2} + T_{2n}, \; n \geqslant 2; \tag{4}$$
$$T_2 + T_4 + \cdots + T_{2n} = T_{2n+3} - 1, \; n \geqslant 2; \tag{5}$$
$$T_1 + T_3 + \cdots + T_{2n-1} = (T_{2n} + T_{2n+2} - 1)/2, \; n \geqslant 2; \tag{6}$$
$$T_{2n+1}^2 - T_{2n-1}^2 = T_{2n+2} \cdot T_{2n-2}, \; n \geqslant 2; \tag{7}$$
$$T_2T_6 + T_4T_8 + \cdots + T_{2n-2} \cdot T_{2n+2} = T_{2n+1}^2 - 1, \; n \geqslant 2; \tag{8}$$
$$T_1T_3 + T_3T_5 + T_5T_7 + \cdots + T_{2n+1} \cdot T_{2n+3} = (T_{2n+4}^2 + T_{2n+2}^2 - 1)/4, \; n \geqslant 2; \tag{9}$$

$$T^2_{2n+2} + T^2_{2n-2} = 2(T^2_{2n+1} + T^2_{2n-1}),\ n \geqslant 2. \tag{10}$$

4. A GENERATING FUNCTION FOR $\{T_n\}$

A generating function corresponding to the development in [2] is now presented. We first consider the odd and even series separately:

$$F_e(x) = 1 + T_2x^2 + T_4x^4 + T_6x^6 + T_8x^8 \ \ldots \text{ when the subscript is even} \tag{11}$$

and

$$F_o(x) = T_1x + T_3x^3 + T_5x^5 + T_7x^7 + T_9x^9 \ldots \text{ when the subscript is odd.}$$

Therefore,

$$(1 - x^2 - x^4 - x^6) \cdot F_e(x) = 1 - x^6, \text{ by (4)} \tag{12}$$

or

$$F_e(x) = (1 - x^6)/(1 - x^2 - x^4 - x^6) \text{ when the subscript is even.}$$

Similarly, we have

$$F_o(x) = x/(1 - x^2 - x^4 - x^6), \text{ by (3), when the subscript is odd.}$$ Hence,

$$F(x) = (1 + x - x^6)/(1 - x^2 - x^4 - x^6) \tag{13}$$

is the required generating function.

5. AN ALTERNATIVE PRESENTATION

Consider the original Fibonacci sequence $\{F_n\}$, with

$$F_0 = 0 \quad \text{and} \quad F_1 = 1,$$

then

$$F_{n+2} = F_{n+1} + F_n,\ n \geqslant 0. \tag{14}$$

It is well known that if

$$x^2 = 1 + x, \tag{15}$$

then

$$x^{n+1} = F_{n-1} + F_nx. \tag{16}$$

We see that the Fibonacci sequence is generated in this way. Similarly, we can generate $\{T_n\}$ by considering

$$x^3 = T_1 + T_2x + T_3x \ = 1 + x + x^2. \tag{17}$$

This gives

$$x^4 = T_3 + T_4x + T_5x^2 \tag{18}$$

$$x^5 = T_5 + T_6x + T_7x^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

leading to

$$x^{n+3} = T_{2n+1} + T_{2n+2}x + T_{2n+3}x^2,\ n \geqslant 2. \tag{19}$$

6. GENERALIZATIONS

By considering the method of Section 5 applied to

$$x^4 = 1 + x + x^2 + x^3, \tag{20}$$

we can construct the sequence $\{Q\ \}$, defined by

$$Q_1 = Q_2 = Q_3 = Q_4 = 1, \tag{21}$$

and (for $n \geqslant 1$),

$$Q_{3n+2} = Q_{3n+1} + Q_{3n-2}, \qquad (22)$$
$$Q_{3n+3} = Q_{3n+1} + Q_{3n-1},$$
$$Q_{3n+4} = Q_{3n+1} + Q_{3n}.$$

leading to

$$X^{n+4} = Q_{3n+1} + Q_{3n+2}x + Q_{3n+3}x^2 + Q_{3n+4}x^3. \qquad (23)$$

This sequence has the form

$$1, 1, 1, 1, 2, 2, 2, 3, 4, 4, 6, 7, 8, 12, 14, 15, 23, 27, \ldots \qquad (24)$$

We note that three Fibonacci-like recurrence relations are interwoven, and the feature

$$Q_{3n} = Q_{3n-3} + Q_{3n-6} + Q_{3n-9} + Q_{3n-12}, \; n \geqslant 4, \qquad (25)$$

is retained. Further properties of this sequence can then be considered, as well as higher-order sequences.

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INFINITE CLASSES OF SEQUENCE-GENERATED CIRCLES

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1. INTRODUCTION

In a previously published paper on the geometry of a generalized Simson's formula, Horadam [2] considered the loci in the Euclidean plane satisfied by points whose Cartesian coordinates are pairs of consecutive elements of a generalized Fibonacci sequence. A Simson's formula as generalized by Horadam [1] was employed in obtaining the loci.

In this paper, we also utilize the same Simson's formula to develop a generalized "Fibonacci circle"; that is, we show how the locus of a point generated by three consecutive elements of the generalized Fibonacci sequence $\{w_n\}$, defined below, approximates a circle *for large* n, subject to special restrictions.

We define the sequence $\{w_n\}$ by

$$w_{n+2} = pw_{n+1} - qw_n, \quad w_0 = a, \; w_1 = b, \tag{1.1}$$

where a, b, p, and q belong to some number system but are usually thought of as integers [1].

It is common knowledge that the terms of $\{w_n\}$ are related to the roots of the equation

$$\lambda^2 - p\lambda + q = 0. \tag{1.2}$$

We denote the roots by

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}$$

and assume throughout the remainder of this paper that

$$\begin{aligned}
&\text{(a)} \quad p^2 > 4q,\\
&\text{(b)} \quad p^2 - 4q \neq t^2\\
&\text{(c)} \quad |q| \leqslant 1\\
&\text{(d)} \quad \alpha < 1 + \sqrt{2}\\
&\text{(e)} \quad \{w_n\} \text{ is strictly increasing.}
\end{aligned} \tag{1.3}$$

Now $\alpha\beta = q$, so parts (c) and (d) of (1.3) tell us that $|\beta| < 1$. Therefore, from Horadam [1, 3.1], we know

$$\lim_{n \to \infty} \frac{w_n}{w_{n-1}} = \alpha. \tag{1.4}$$

In closing, we observe that part (b) of (1.3) guarantees that $p \neq 1 + q$, which is enough to show that $\alpha \neq 1$. Part (b) with (e) is also enough to show that

$$\lim_{n\to\infty} w_n = \infty. \tag{1.5}$$

2. PRELIMINARIES

Let k, ℓ, and m be three consecutive terms of $\{w_n\}$ with $k = w_n$.

Since w_n is strictly increasing and $w_n \to \infty$, we may as well consider throughout the rest of the paper only those terms of w_n that are greater than 0.

From [1, 4.3 & 1.9], we know that

$$\begin{aligned} \ell^2 - mk &= -eq^n \\ &= -(pab - qa^2 - b^2)q^n \\ &= (w_1^2 - w_0w_2)q^n \quad \text{by (1.1)} \\ &< M \quad \text{by (1.3), part (c)} \end{aligned} \tag{2.1}$$

for some positive integer M. We also have

$$\lim_{n\to\infty}(\ell - k) = \lim_{n\to\infty} k\left(\frac{\ell}{k} - 1\right) = \infty, \tag{2.2}$$

by (1.4) and (1.5). Hence, for n sufficiently large,

$$\ell^2 - mk < \ell - k \tag{2.3}$$

or, with r as the midpoint of $\frac{\ell - 1}{k}$ and $\frac{m - 1}{\ell}$,

$$\frac{\ell - 1}{k} < r = \frac{\ell^2 + km - \ell - k}{2k\ell} < \frac{m - 1}{\ell}. \tag{2.4}$$

From (2.4), we immediately have

$$\ell - rk < 1 < m - r\ell. \tag{2.5}$$

Using (2.1), (2.4), and (1.4), we see that

$$\lim_{n\to\infty}(\ell - rk) = \lim_{n\to\infty} \frac{\ell^2 - km + k + \ell}{2\ell} = \frac{\alpha + 1}{2\alpha} \tag{2.6}$$

and

$$\lim_{n\to\infty}(m - r\ell) = \lim_{n\to\infty} \frac{km - \ell^2 + \ell + k}{2k} = \frac{\alpha + 1}{2}. \tag{2.7}$$

Since $\alpha > 0$, we can now strengthen (2.5) using (2.6) to

$$0 < \ell - rk < 1 < m - r\ell, \ n \text{ sufficiently large.} \tag{2.8}$$

Another obvious conclusion of (2.6) and (2.7) is

$$\lim_{n\to\infty} \frac{m - r\ell}{\ell - rk} = \alpha. \tag{2.9}$$

In conclusion, using (2.6) and (2.7) with part (d) of (1.3), let us observe that

$$\lim_{n\to\infty}(\ell - rk + 1 - m + r\ell) = \frac{1 + 2\alpha - \alpha^2}{2\alpha} > 0 \tag{2.10}$$

so that for n sufficiently large

$$\ell - rk + 1 > m - r\ell. \tag{2.11}$$

3. THE GEOMETRY

Throughout this section, we assume n is sufficiently large. We let

$$\begin{aligned} AB &= 1 \\ QA &= \ell - rk \\ QB &= m - r\ell \end{aligned} \tag{3.1}$$

and locate the origin of our system by setting

$$OA = 1/(\alpha^2 - 1) \tag{3.2}$$

and by extending BA to O.

We let D be the foot of the perpendicular form Q to OB. By (2.8) and (2.11) this construction is legitimate and gives us the triangle QAB (see Figure 1).

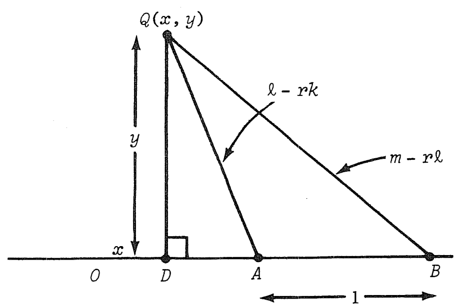

FIGURE 1

Now,

$$\begin{aligned} \text{area } QAB &= \frac{1}{2}DQ \\ &= \sqrt{(s(s - QB)(s - QA)(s - AB))} \end{aligned} \tag{3.3}$$

where s is the semi-perimeter of the triangle QAB.

For notational convenience, let

$$QA = u. \tag{3.4}$$

Then, for sufficiently large n, for which

$$QB = \alpha \cdot QA = \alpha u, \text{ by (2.9), (3.4)} \tag{3.5}$$

we have

$$s = \frac{1}{2}(\alpha u + u + 1), \text{ by (3.1), (3.4), (3.5)} \tag{3.6}$$

and so

$$\begin{aligned} 4DQ^2 &= (\alpha u + u + 1)(-\alpha u + u + 1)(\alpha u - u + 1)(\alpha u + u - 1), \\ &\qquad \text{by (3.1), (3.3), (3.4), (3.5), (3.6)} \\ &= ((\alpha u + u)^2 - 1)(1 - (\alpha u - u)^2) \\ &= 2u^2(\alpha^2 + 1) - 1 - u^4(\alpha^2 - 1)^2. \end{aligned} \tag{3.7}$$

Then,

$$\begin{aligned} 4DA^2 &= 4QA^2 - 4DQ^2 \quad \text{by the Pathagorean Theorem} \\ &= -2u^2(\alpha^2 - 1) + 1 + u^4(\alpha^2 - 1)^2, \text{ by (3.4), (3.7)} \\ &= (u^2(\alpha^2 - 1) - 1)^2. \end{aligned}$$

Whence

$$2DA = u^2(\alpha^2 - 1) - 1. \tag{3.8}$$

Now OD and DQ are the x- and y-coordinates, respectively, of Q, so that

$$\begin{aligned} x^2 + y^2 &= OD^2 + DQ^2 \\ &= (OA - DA)^2 + DQ^2 \\ &= OA^2 + DA^2 + DQ^2 - 2OA \cdot DA \\ &= OA^2 + QA^2 - OA(2DA) \quad \text{by the Pathagorean Theorem} \\ &= \frac{1}{(\alpha^2 - 1)^2} + u^2 - \frac{1}{(\alpha^2 - 1)}(u^2(\alpha^2 - 1) - 1), \\ &\qquad \text{by (3.2), (3.4), (3.8),} \\ &= \frac{1}{(\alpha^2 - 1)^2} + \frac{1}{\alpha^2 - 1} \\ &= \frac{\alpha^2}{(\alpha^2 - 1)^2}. \end{aligned}$$

That is,

$$x^2 + y^2 = \left(\frac{\alpha}{\alpha^2 - 1}\right)^2. \tag{3.9}$$

The locus of Q as n increases is, therefore, a circle with center 0 and radius $\alpha/(\alpha^2 - 1)$.

As p, q (and, consequently, α) vary, the corresponding sequences clearly generate an infinite set of concentric circles.

4. FIBONACCI-TYPE CIRCLES

For the sequence of ordinary Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, ..., we have

$$p = -q = 1, \ \alpha^2 = \alpha + 1, \text{ and } \alpha = \frac{1}{2}(1 + \sqrt{5}),$$

so the circle given by (3.9) becomes the unit circle.

Moreover, all sequences for which $p = -q = 1$ [and so for which $\alpha^2 = \alpha + 1$, $\alpha = (1/2)(1 + \sqrt{5})$], e.g., the Lucas sequence 2, 1, 3, 4, 7, 11, 18, 29, ..., give rise to this unit circle.

The following table illustrates the result for the Fibonacci numbers.

n	F_n	F_{n+1}	$x^2 + y^2$
2	1	2	.763932
3	2	3	.328550
4	3	5	.914537
5	5	8	.698798
6	8	13	1.003089
7	13	21	.878930
8	21	34	1.044630
9	34	55	.952913
10	55	89	1.029224
11	89	144	.981894
12	144	233	1.011208
13	233	377	.993066
14	377	610	1.004288
15	610	987	.997349
16	987	1597	1.001639
17	1597	2584	.998987
18	2584	4181	1.000626
19	4181	6765	.999613
20	6765	10946	1.000239
21	10946	17711	.999852
22	17711	28657	1.000091
23	28657	46368	.999944
24	46368	75025	1.000035
25	75025	121393	.999978
26	121393	196418	1.000013
27	196418	317811	.999992
28	317811	514229	1.000005
29	514229	832040	.999997
30	832040	1346269	1.000002

Gratitude is expressed to Wilson [3], whose Fibonacci circle, derived from five successive large Fibonacci numbers, was useful in the development of this theory.

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THE GOOSE THAT LAID THE GOLDEN EGG

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(Submitted December 1982)

Last year when I was 13 and we were studying elementary algebra, I learned that $x + y = 1$ could be graphed as a line with x- and y-intercepts of (1, 0) and (0, 1) and that $x^2 + y^2 = 1$ could be graphed as a circle of radius 1 with its center at (0, 0). This year we studied functions of the form

$$f(x) = Ax^2 + Bx + C$$

and saw that their graphs were parabolas. Since this shape was so different from a circle, with what did not appear to me to be an enormous difference in mathematical form, I wondered what other curves of the form $x^n + y^n = 1$ would look like. Fortunately, I have an Atari 800 computer at home which allows me the opportunity to make such an investigation relatively simple.

Eventually, I became bored with integral exponents and, since I had been working with the golden ratio for a math project, I wondered what $x^{\phi^2} + y^{\phi^2} = 1$, where ϕ = the Golden Ratio, 1.618033989..., would look like. Inasmuch as all other facets of this ratio that I had investigated were so special, I thought that graphs using it should have very interesting shapes. I was correct. As soon as I looked at the shape generated by $x^{\phi^2} + y^{\phi^2} = 1$, I recognized it as one end of an egg. This was an amazement to me. Knowing that eggs do not have two axes of symmetry, I wondered whether I could combine the curve generated with the curve of a slightly altered function to create the rest of a realistically-shaped egg. My hypothesis was that there is an egg shape (which I called the "golden egg") whose configuration is directly related to the golden ratio. It is a composite shape, different from a circle, ellipse, or oblate spheroid. The left portion of the egg is the graph of the function generated by the golden ratio exponential $x^{\phi^2} + y^{\phi^2} = 1$. The right portion of the egg is the graph of the function generated by the golden ratio exponential:

$$x^{\phi} + \frac{1}{\phi^{\phi}}(y)^{\phi} = 1 \quad \text{(see Figure 1).}$$

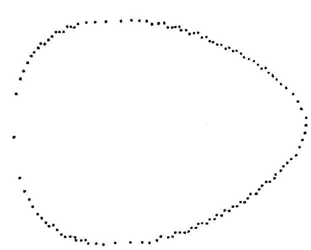
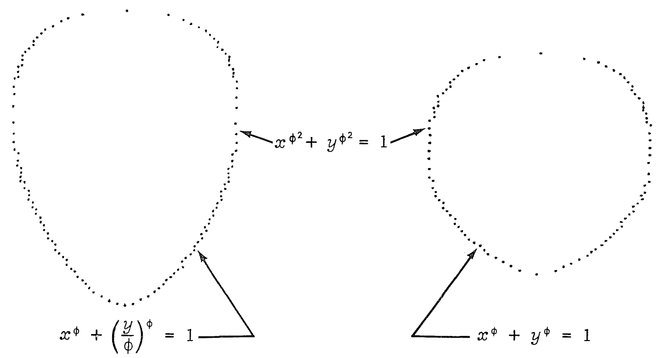


FIGURE 1. Picture of Golden Egg Generated by Atari 800 Computer

I was so pleased with the result of my graph and my development and analysis of the golden egg that I wondered whether Fibonacci-related exponential functions would generate other configurations. I began to experiment with the coefficients and the positions of x and y. By rotating the golden egg, I noted that the curve showed a strong resemblance to the shape of an adult head. A change in the y-coefficient created the outline of an infant's head. (See Figure 2.)



Adult Shape Infant Shape

FIGURE 2

With additional changes to the coefficients and constants, carrots, acorns, pine cones, and other figures appeared on my computer console. These figures, with descriptions of the equations used appear below as Figures 3 through 7.

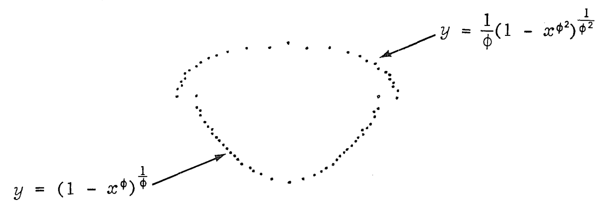


FIGURE 3. Acorn

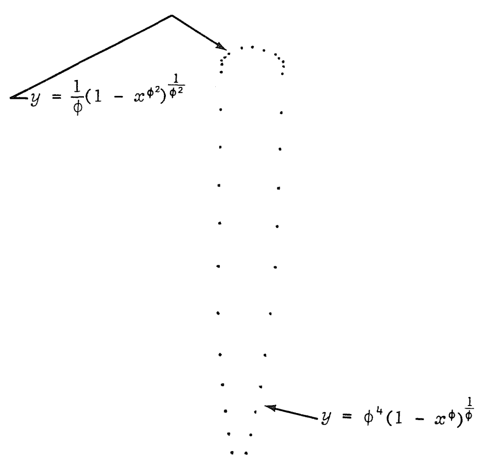


FIGURE 4. Carrot

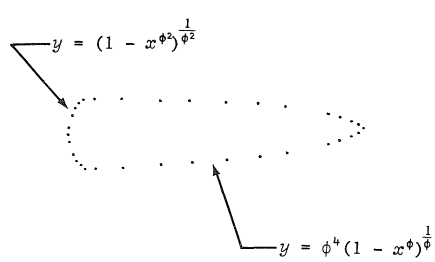
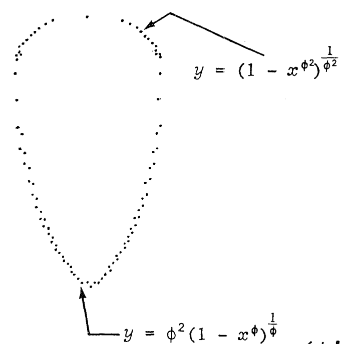
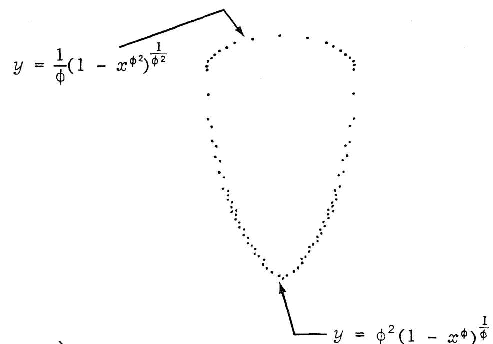


FIGURE 5. Corn on the Cob





(ϕ'*N* Cones)

FIGURE 6. Round-Top Pine Cone FIGURE 7. Flat-Top Pine Cone

It is interesting that Brother Alfred Brousseau found Fibonacci numbers in pine cones, and now we find pine cones in Fibonacci-related functions [1].

I have defined equations of the type used to generate the previous configurations as Golden Functions (i.e., equations that are functions of variables raised to a power that is a function of ϕ). One might wonder whether the creation of the Golden Functions and these shapes is merely an academic exercise and an accident. I choose to believe not and leave the investigation of equations of the form $Ax^{\phi} + Bx + C = 1$ and $x^{\phi}y = 1$ to the reader.

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SOME IDENTITIES ARISING FROM THE FIBONACCI NUMBERS OF CERTAIN GRAPHS

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Tichy and Prodinger [5] have defined the Fibonacci number of a graph G to be the number of independent vertex sets I in G; recall that I is independent if no two of its vertices are adjacent. Following Tichy and Prodinger, we denote the Fibonacci number of G by $F(G)$. If k is a nonnegative integer, we will denote the k-element independent vertex sets in G by $F_k(G)$. It is clear that $\sum F_k(G) = F(G)$. Kreweras [4] (see also [3]) has introduced the notion of the Fibonacci polynomial,

$$F(x) = \sum_{k \geqslant 0} \binom{n-k}{k} x^k.$$

We define the more general concept of the Fibonacci polynomial of a graph G, denoted $F_G(x)$. In case G is a path on n vertices,

$$F_G(x) = \sum_{k \geqslant 0} \binom{n-k+1}{k} x^k,$$

which closely resembles Kreweras' polynomial. Before defining $F_G(x)$, we compute $F_k(P_n)$, P_n the path on n vertices, and $F_k(C_n)$, C_n the cycle on n vertices.

Proposition 1

(i) $F_0(P_n) = 1$;

(ii) $F_1(P_n) = n$;

(iii) $F_k(P_{n+1}) = F_k(P_n) + F_{k-1}(P_{n-1})$ for $1 \leqslant k \leqslant \left[\frac{n+2}{2}\right]$;

(iv) $F_k(P_n) = \binom{n-k-1}{k}$ for $0 \leqslant k \leqslant \left[\frac{n+1}{2}\right]$.

Proof: The first two statements are obvious. To verify (iii), consider those k-element independent sets that contain the initial point of the path and those that do not. Finally, (iv) may be verified using (iii) and induction on n. ∎

Proposition 1 provides a natural graph-theoretic interpretation of the well-known formula

$$\sum_{k \geqslant 0} \binom{n-k+1}{k} = F_{n+1},$$

the $n+1^{\text{th}}$ Fibonacci number. The right side of the equality is the number of independent sets of a path with n vertices. The left side is the sum over all k of the number of k-element independent sets. The following proposition will enable us to give an analogous identity involving Lucas numbers, and a graph-theoretic interpretation of that identity.

Proposition 2

(i) $F_0(C_n) = 1$;

(ii) $F_1(C_n) = n$;

(iii) $F_k(C_n) = F_k(P_{n-1}) + F_{k-1}(P_{n-3})$ for $1 \leqslant k \leqslant \left[\frac{n}{2}\right]$ and $n \geqslant 3$;

(iv) $F_k(C_n) = \frac{n}{k}\binom{n-k-1}{k-1}$ for $1 \leqslant k \leqslant \left[\frac{n}{2}\right]$ and $n \geqslant 3$.

Proof: Again, the first two statements are obvious. To verify (iii), fix a vertex x in C_n. Consider those k-element independent sets that contain x and those that do not; use (iv) of Proposition 1. To verify (iv), we use (iii):

$$\begin{aligned} F_k(C_n) &= F_k(P_{n-1}) + F_{k-1}(P_{n-3}) \\ &= \binom{n-k}{k} + \binom{n-k-1}{k-1} \\ &= \frac{n}{k}\binom{n-k-1}{k-1}. \blacksquare \end{aligned}$$

We now use Proposition 2 to obtain an identity analogous to that following Proposition 1. L_n denotes the n^{th} Lucas number.

Proposition 3

For $n \geqslant 3$, $1 + \sum_{k \geqslant 1} \frac{n}{k}\binom{n-k-1}{k-1} = L_n$.

Proof: The right side is the number of independent sets in C_n (see [5]). The left side is the sum over k of the number of k-element independent subsets. ■

We now pause to establish some notation and state a definition. If G and H are graphs, we will denote by $G \cdot H$ the standard composition or lexicographic product (see [1]). That is, $G \cdot H$ is the graph constructed by replacing each vertex v of G by an isomorphic copy H_v of H, and by joining each vertex of H_v to each vertex of H_w whenever v is adjacent to w in G. We define the Fibonacci polynomial of G, F_G, by $F_G(x) = F(G \cdot k_x)$ for positive integers x. As usual, k_x is the complete graph on x vertices. That F_G is a polynomial follows from the next proposition.

Proposition 4

Let G be a graph, and let $F_k = F_k(G)$ for $k \geqslant 0$. Then $F_G(x) = \sum_{k \geqslant 0} F_k x^k$.

Proof: To obtain a k-element independent set in $G \cdot k_x$, one must first choose a k-element independent set in G, and then choose one of the x vertices in each of the k chosen copies of k_x. ■

The study of the Fibonacci polynomial of G thus reduced to the study of the coefficients $F_k(G)$. For example, the constant term of $F_G(x)$ is 1, the linear term is nx, and the coefficient of x^2 is $\binom{n}{2} - m$, where m is the number of edges of G. The degree of $F_G(x)$ is the independence number of G, that is, the number of vertices in the largest independent set.

We obtain some combinatorial identities by expanding the Fibonacci polynomials of paths and cycles.

Theorem 5

Let x be a positive integer, and let n be a nonnegative integer. Let ℓ be $\frac{1}{2}(1 \pm \sqrt{1+4x})$. Then,

$$\sum_{k \geq 0} \binom{n-k+1}{k} x^k = \frac{1}{2\ell - 1}(\ell^{n+2} - (1-\ell)^{n+2}).$$

Proof: We compute the Fibonacci polynomial of P_n in two ways. First, use Proposition 4 and Proposition 1 to get

$$\sum_{k \geq 0} \binom{n-k+1}{k} x^k.$$

As a second approach, we derive and solve a second-order linear recursion for $a_n = F(P_n \circ k_x)$. Clearly, $a_0 = 1$ and $a_1 = x + 1$. Divide the independent sets in $P_n \circ k_x$ into those that contain a vertex in the last stalk and those that do not. There are xa_{n-2} of the first type, and a_{n-1} of the second type. Hence, $a_n = a_{n-1} + xa_{n-2}$. This recursion has characteristic equation $\lambda^2 - \lambda - x = 0$. Solving this equation, subject to the initial conditions, yields

$$a_n = F(P_n \circ k_x) = \frac{1}{2\ell - 1}(\ell^{n+2} - (1-\ell)^{n+2}). \blacksquare$$

Note that the identity in Theorem 5 is true for infinitely many values of x. Hence, it is in fact true for all complex numbers x. The same remark applies to the following theorem.

Theorem 6

Let x be a positive integer, and let n be a nonnegative integer. Let ℓ be $\frac{1}{2}(1 \pm \sqrt{1+4x})$. Then,

$$1 + \sum_{k \geq 1} \frac{n}{k}\binom{n-k-1}{k-1} x^k = \ell^n + (1-\ell)^n.$$

Proof: We compute the Fibonacci polynomial of C_n in two ways. First, we use Propositions 2 and 4 to get

$$1 + \sum_{k \geq 1} \frac{n}{k}\binom{n-k-1}{k-1} x^k.$$

Now we use Theorem 5. Let S be a fixed stalk in $C_n \circ k_x$. Divide the independent sets in $C_n \circ k_x$ into those that contain a vertex in S and those that do not. There are

$$x\left(\frac{1}{2\ell - 1}\right)(\ell^{n-1} - (1-\ell)^{n-1})$$

independent sets of the first type and

$$\frac{1}{2\ell - 1}(\ell^{n+1} - (1-\ell)^{n+1})$$

of the second type. Adding, and substituting $x = \ell^2 - \ell$ yields the theorem. $\blacksquare$

The identity of Theorem 5 is known. See, for example, [2, p. 76]. But our approach seems to provide a new interpretation for this identity. We believe that new identities may be obtained by expanding Fibonacci polynomials of graphs.

ACKNOWLEDGMENT

We wish to thank the referee for his helpful suggestions about this paper.

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ON THE NUMBERS OF THE FORM $an^2 + bn$

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It is clear that for any given positive integer N there are infinitely many square numbers which can be represented as the difference of square numbers in at least N different ways.

For instance, if $n = 4p_1p_2 \ldots p_r$, where $p_1, p_2, \ldots, p_r$ are the smallest r odd primes such that $r \geqslant \log_2 N$, then for each subset S of $\{1, 2, 3, \ldots, r\}$, n^2 has the expression

$$n^2 = (h^2 + k^2)^2 - (h^2 - k^2)^2,$$

where

$$h = 2\prod_{i \in S} p_i, \quad k = \prod_{i \in \overline{S}} p_i,$$

with the convention that an empty product means 1 and the notation $\overline{S}$ for the complement of S, giving $2^r \geqslant N$ distinct expressions.

Thus, we can choose n in such a way that

$$n = O(e^{c \log N \operatorname{loglog} N}) \tag{1}$$

for large values of N, where c is a constant.

In this paper we prove a similar theorem concerning the sequence of numbers $A_n = an^2 + bn$ for any integers a and b with $a > 0$, which includes the earlier result [1] as the special case of $N = 2$.

Theorem

For any given positive integer N, there exist an infinite number of A_n's which can be expressed as the difference of two numbers of the same type in at least N different ways. We can choose an n for each N in such a way that it satisfies (1) as N tends to infinity.

Proof: It is enough to prove that for any sufficiently large N, there is an A_n which has at least N such expressions. Since

$$A_n = A_h - A_k \tag{2}$$

is equivalent to

$$n(an + b) = (h - k)(ah + ak + b),$$

in order to get the expression (2) for given n, it is sufficient to find a decomposition of n into two factors s and t; $n = st$, for which

$$h - k = s, \; a(h + k) + b = t(an + b) \tag{3}$$

has positive integral solutions h and k.

Let $p_1, p_2, \ldots, p_r$ be the smallest r distinct prime numbers in the arithmetic progression consisting of positive integers congruent to 1 modulo $2a$, and let

$$n = 2p_1p_2 \ldots p_r.$$

For each proper subset S of $\{1, 2, \ldots, r\}$, there corresponds a distinct decomposition of n into two factors

$$s = 2\prod_{i \in S} p_i \quad \text{and} \quad t = \prod_{i \in \bar{S}} p_i,$$

where t can be expressed as $t = 1 + 2au$ for a positive integer u, and we have

$$h + k = st + 2u(an + b)$$

from the second equation of (3).

If n is sufficiently large so that it will satisfy $an + b > 0$, then Eq. (3) gives distinct pairs h, k for different decompositions $n = st$ of n.

In this case, however, two different h's may give the same A_h if b/a is a negative integer. Since at most four pairs of h, k give the same expression, we have at least N distinct expressions (2) of A_n if r satisfies

$$2^r - 1 \geqslant 4N,$$

and N is sufficiently large so that corresponding n will satisfy $an + b > 0$.

If we take r that satisfies

$$\log_2(4N + 1) \leqslant r < \log_2(4N + 1) + 1,$$

then for large values of N we have

$$\log n = \log 2 + \log p_1 + \cdots + \log p_r = O(p_r) = O(r \log r),$$

from which we obtain

$$n = O(e^{c \log N \log\log N})$$

for a constant c, completing the proof.

If we do not care about the size of n, we can take simpler forms for s and t in (3); if b/a is not a negative integer,

$$s = 2(1 + 2a)^i, \; t = (1 + 2a)^{N-i}, \; (i = 1, 2, \ldots, N - 1)$$

give N distinct expressions of the form (2) for h and k determined by (3), and if b/a is a negative integer, N will be substituted by $4N$.

These results apparently cover the case of polygonal numbers of any order.

Examples

For tiagonal numbers $t_n = \frac{1}{2}(n^2 + n)$, we have $t_n = t_h - t_k$, where

$$n = 2 \times 3^N, \; h = 3^i + 3^{2N-i} + \tfrac{1}{2}(3^{N-i} - 1), \; k = -3^i + 3^{2N-i} + \tfrac{1}{2}(3^{N-i} - 1)$$

for $i = 1, 2, \ldots, N - 1$.

For hexagonal numbers $h_n = 2n^2 - n$, we have $h_n = h_h - h_k$, where

$$n = 2 \times 5^N, \; h = 5^i + 5^{2N-i} - \tfrac{1}{4}(5^{N-i} - 1), \; k = -5^i + 5^{2N-i} - \tfrac{1}{4}(5^{N-i} - 1)$$

for $i = 1, 2, \ldots, N - 1$.

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ON CERTAIN SERIES OF RECIPROCALS OF FIBONACCI NUMBERS

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The purpose of this note is to give an alternative, shorter proof of a result of R. P. Backstrom concerning the sums of series whose terms are reciprocals of Fibonacci numbers, a problem on which much interest has recently been focused.

Furthermore, the method used here gives the possibility of obtaining new formulas related to the Fibonacci and Lucas numbers.

In fact, we establish in explicit form series of the form

$$\sum_{n=0}^{\infty}\frac{1}{F_{an+b}\pm c},\ \sum_{n=0}^{\infty}\frac{1}{F_{an+b}F_{cn+d}},\ \sum_{n=0}^{\infty}\frac{1}{F_{an+b}^2\pm F_{cn+d}^2},$$

for certain values of a, b, c, and d.

We start with the identity

$$F_n - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2, \tag{1}$$

which, by replacing n with $(2n+1)r+2k$, becomes

$$F_{2(n+1)r+2k}^2 - F_r^2 = F_{2nr+2k}F_{2(n+1)r+2k}. \tag{2}$$

Then

$$\frac{1}{F_{2(n+1)r+2k}+F_r} = \frac{F_{(2n+1)r+2k}-F_r}{F_{2nr+2k}F_{2(n+1)r+2k}} \tag{3}$$

with $-(r-1)\leqslant 2k\leqslant r-1$.

Since

$$L_rF_{2(nr+k)+r} = F_{2(nr+k)+2r} + (-1)^rF_{2(nr+k)},$$

from (3) we obtain

$$\frac{1}{F_{(2n+1)r+2k}+F_r} = \frac{1}{L_r}\left(\frac{1}{F_{2nr+2k}}+\frac{(-1)^r}{F_{2(n+1)r+2k}}\right) - \frac{F_r}{F_{2nr+2k}F_{2(n+1)r+2k}}.$$

Now, consider the sum

$$\begin{aligned}S_N(r,k) &= \sum_{n=0}^{N}\frac{1}{F_{(2n+1)r+2k}+F_r}\\ &= \frac{1}{L_r}\sum_{n=0}^{N}\left(\frac{1}{F_{2nr+2k}}+\frac{(-1)^r}{F_{2(n+1)r+2k}}\right) - F_r\sum_{n=0}^{N}\frac{1}{F_{2nr+2k}F_{2(n+1)r+2k}}.\end{aligned}$$

We have

$$\sum_{n=0}^{N}\left(\frac{1}{F_{2nr+2k}}+\frac{(-1)^r}{F_{2(n+1)r+2k}}\right) = \frac{1}{F_{2k}} - \frac{1}{F_{2(N+1)r+2k}},$$

for an odd integer r, and

$$\sum_{n=0}^{N}\frac{1}{F_{2nr+2k}F_{2(n+1)r+2k}} = \frac{1}{2F_{2r}}\left(\frac{L_{2k}}{F_{2k}} - \frac{L_{2(N+1)r+2k}}{F_{2(N+1)r+2k}}\right), \tag{4}$$

which follows from the identity

$$\frac{L_{2k}}{F_{2k}} - \frac{L_{2k+2r}}{F_{2k+2r}} = \frac{2F_{2r}}{F_{2k}F_{2k+2r}},$$

if we successively replace k by k, $k+r$, ..., $k+Nr$, and sum the obtained equations.

Therefore,

$$S_N(r, k) = \frac{1}{2L_r}\left(\frac{2 - L_{2k}}{F_{2k}} + \frac{L_{2(N+1)r+2k} - 2}{F_{2(N+1)r+2k}}\right).$$

Using the relations

$$L_{2n} = L_n^2 - 2(-1)^n = 5F_n^2 + 2(-1)^n,$$

it follows that (for odd integer r)

$$S_N(r, k) = \sum_{n=0}^{N} \frac{1}{F_{(2n+1)r+2k} + F_r} = \begin{cases} \left(\dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}} - \dfrac{5F_k}{L_k}\right)\Big/ 2L_r, & N\text{-even, } k\text{-even,} \\[2ex] \left(\dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}} - \dfrac{L_k}{F_k}\right)\Big/ 2L_r, & N\text{-even, } k\text{-odd,} \\[2ex] \left(\dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}} - \dfrac{5F_k}{L_k}\right)\Big/ 2L_r, & N\text{-odd, } k\text{-even,} \\[2ex] \left(\dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}} - \dfrac{L_k}{F_k}\right)\Big/ 2L_r, & N\text{-odd, } k\text{-odd.} \end{cases} \tag{5}$$

Letting $N \to \infty$, we have

$$S(r, k) = \sum_{n=0}^{\infty} \frac{1}{F_{(2n+1)r+2k} + F_r} = \begin{cases} \dfrac{1}{2L_r}\left(\sqrt{5} - \dfrac{5F_k}{L_k}\right), & k\text{-even,} \\[2ex] \dfrac{1}{2L_r}\left(\sqrt{5} - \dfrac{L_k}{F_k}\right), & k\text{-odd.} \end{cases} \tag{5a}$$

Summing $S(r, k)$ over the r values of k finally yields

$$S(r) = \sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + F_r} = \frac{r\sqrt{5}}{2L_r},$$

by using the relations $F_{-n} = (-1)^n F_n$ and $L_{-n} = (-1)^n L_n$.

Following arguments similar to the above for obtaining (5), we have

$$\overline{S}_N(r, k) = \sum_{n=0}^{N} \frac{1}{F_{(2n+1)r+2k} - F_r} \tag{6}$$

$$= \frac{1}{2L}\left(\frac{2 + L_{2k}}{F_{2k}} - \frac{L_{2(N+1)r+2k} + 2}{F_{2(N+1)r+2k}}\right)$$

(continued)

$$= \begin{cases} \left(\dfrac{5F_k}{L_k} - \dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}}\right)\Big/ 2L_r, & N\text{-even, } k\text{-odd,} \\ \left(\dfrac{L_k}{F_k} - \dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}}\right)\Big/ 2L_r, & N\text{-even, } k\text{-even,} \\ \left(\dfrac{5F_k}{L_k} - \dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}}\right)\Big/ 2L_r, & N\text{-odd, } k\text{-odd,} \\ \left(\dfrac{L_k}{F_k} - \dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}}\right)\Big/ 2L_r, & N\text{-odd, } k\text{-even.} \end{cases}$$

whenever $k \geqslant 1$ and r is odd.

Comparing (5) and (6) by letting k be even, $k = 2s$ in (6), and k be odd, $k = 2t - 1$ in (5), we see that if r is odd, then

$$\overline{S}_N(r,\ 2s) = -S_N(r,\ 2t - 1).$$

Similarly, with $k = 2s - 1$ in (6) and $k = 2t$ in (5), we have, for r odd, that

$$\overline{S}_N(r,\ 2s - 1) = -S_N(r,\ 2t).$$

Letting $N \to \infty$ in (6), we have, for odd r, that

$$\overline{S}(r,\ k) = \sum_{n=0}^{\infty} \frac{1}{F_{2(n+1)r+2k} - F_r} = \begin{cases} \left(\dfrac{5F_k}{L_k} - \sqrt{5}\right)\Big/ 2L_r, & k\text{-odd,} \\ \left(\dfrac{L_k}{F_k} - \sqrt{5}\right)\Big/ 2L_r, & k\text{-even.} \end{cases} \tag{6a}$$

Comparing (5a) and (6a) as above, we see that if $k = 2s - 1$ in (6a) and $k = 2t$ in (5a), then for r odd,

$$\overline{S}(r,\ 2s - 1) = -S(r,\ 2t),$$

while $k = 2s$ in (6a) and $k = 2t - 1$ in (5a) yields

$$\overline{S}(r,\ 2s) = -S(r,\ 2t - 1), \text{ if } r \text{ is odd.}$$

We note that, from (1), it follows that

$$\frac{1}{F_{2(n+1)+r} - F_r} - \frac{1}{F_{2(n+1)+r} + F_r} = \frac{2F_r}{F_{2(n+1)+2r}F_{2(n+1)}}.$$

Hence, we have

$$\sum_{n=0}^{N} \frac{1}{F_{2(n+1)+r} - F_r} = \sum_{n=0}^{N} \frac{1}{F_{2(n+1)+r} + F_r} + 2F_r \sum_{n=0}^{N} \frac{1}{F_{2(n+1)}F_{2(n+1)+2r}}.$$

Taking (4) into consideration along with the fact that $\lim\limits_{n\to\infty} \dfrac{L_n}{F_n} = \sqrt{5}$, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{F_{2nr+2k}F_{2(n+1)r+2k}} = \frac{1}{2F_{2r}}\left(\frac{L_{2k}}{F_{2k}} - \sqrt{5}\right), \text{ if } r \text{ is odd.}$$

Similarly, for the Lucas numbers, starting from

$$L_n^2 - 5F_{n-r}F_{n+r} = (-1)^{n-r}L_r^2,$$

we find that

$$G(r, k) = \sum_{n=0}^{N} \frac{1}{L_{(2n+1)r+2k} + L_r} = \frac{1}{10F_r}\left(\frac{2 - L_{2k}}{F_{2k}} + \frac{L_{2(N+1)r+2k} - 2}{F_{2(N+1)r+2k}}\right),$$

with $-r \leqslant 2k \leqslant r - 2$ and r an even integer.

Following the methods used above, we obtain, for r even,

$$G_N(r, k) = \begin{cases} \dfrac{1}{10F_r}\left(\dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}} - \dfrac{L_k}{F_k}\right), & k\text{-odd}, \\[2ex] \dfrac{1}{10F_r}\left(\dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}} - \dfrac{5F_k}{L_k}\right), & k\text{-even}. \end{cases} \tag{7}$$

Letting $N \to \infty$, this relation yeilds, for r even,

$$G(r, k) = \sum_{n=0}^{\infty} \frac{1}{L_{(2n+1)r+2k} + L_r} = \begin{cases} \dfrac{1}{10F_r}\left(\sqrt{5} - \dfrac{L_k}{F_k}\right), & k\text{-odd}, \\[2ex] \dfrac{1}{10F_r}\left(\sqrt{5} - \dfrac{5F_k}{L_k}\right), & k\text{-even}. \end{cases} \tag{7a}$$

Summing the last equation over the $r - 1$ values of k, leads to

$$G(r) = \sum_{n=0}^{\infty} \frac{1}{L_{2n} + L_r} = \begin{cases} \dfrac{r\sqrt{5}}{10F_r} + \dfrac{1}{10F_{r/2}^2}, & r/2\text{-odd} \\[2ex] \dfrac{r\sqrt{5}}{10F_r} + \dfrac{1}{2L_{r/2}^2}, & r/2\text{-even}. \end{cases}$$

when r is even.

Similarly, for r even,

$$\overline{G}_N(r, k) = \sum_{n=0}^{N} \frac{1}{L_{(2n+1)r+2k} - L} = \begin{cases} \dfrac{1}{10F_r}\left(\dfrac{L_k}{F_k} - \dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}}\right), & k\text{-even}, \\[2ex] \dfrac{1}{10F_r}\left(\dfrac{5F_k}{L_k} - \dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}}\right), & k\text{-odd}. \end{cases} \tag{8}$$

so that

$$\overline{G}(r, k) = \sum_{n=0}^{\infty} \frac{1}{L_{2(n+1)r+2k} - L_r} = \begin{cases} \dfrac{1}{10F_r}\left(\dfrac{5F_k}{L_k} - \sqrt{5}\right), & k\text{-odd}, \\[2ex] \dfrac{1}{10F_r}\left(\dfrac{L_k}{F_k} - \sqrt{5}\right), & k\text{-even}. \end{cases} \tag{8a}$$

Comparing (7a) and (8a) as we did (5a) and (6a), we have, for r even, $r = 2s - 1$ in (8a) and $k = 2t$ in (7a), that

$$G(r, 2s - 1) = -\overline{G}(r, 2t)$$

while, for r even, $k = 2s$ in (8a) and $k = 2t - 1$ in (7a), we have

$$\overline{G}(r,\ 2s) = -G(r,\ 2t - 1).$$

By similar methods, the relations (1) and (4) can also be used to show that

$$\sum_{n=0}^{N} \frac{1}{F^2_{(2n+1)r+k} - (-1)^k F_r^2} = \frac{F_{2(N+1)r}}{F_k F_{2r} F_{2(N+1)r+k}}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{F^2_{(2n+1)r+k} - (-1)^k F_r^2} = \frac{(\sqrt{5} - 1)^k}{2^k F_k F_{2r}}.$$

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AN APPLICATION OF THE RECIPROCITY THEOREM FOR DEDEKIND SUMS

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1. Put

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & (x \neq \text{integer}), \\ 0 & (x = \text{integer}). \end{cases} \tag{1.1}$$

The Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{r \pmod{k}} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right). \tag{1.2}$$

It is well known that $s(h, k)$ satisfies the reciprocity theorem

$$s(h, k) = s(k, h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right), \tag{1.3}$$

where $(h, k) = 1$. For references, see [1, Ch. 2].

In this note, we shall show that (1.3) implies the following result.

Theorem 1

Let h, h', k, k' denote positive integers. (a) The system

$$\begin{cases} hh' \equiv 1 \pmod{k}, & hh' \equiv 1 \pmod{k'} \\ kk' \equiv 1 \pmod{h}, & kk' \equiv 1 \pmod{h'} \end{cases} \tag{1.4}$$

has no solutions with $h \neq h'$, $k \neq k'$. (b) The solutions of

$$\begin{cases} hh' \equiv -1 \pmod{k}, & hh' \equiv -1 \pmod{k'} \\ kk' \equiv 1 \pmod{h}, & kk' \equiv 1 \pmod{h'} \end{cases} \tag{1.5}$$

with $k \neq k'$ satisfy

$$kk' - hh' = 1, \tag{1.6}$$

and conversely.

The auxiliary inequalities in hypotheses (a) and (b) cannot be dispensed with. Thus, for example, (1.4) is satisfied by

$$(h, h', k, k') = (2, 3, 5, 5) \text{ and } (2, 4, 7, 7);$$

(1.5) is satisfied by

$$(h, h', k, k') = (3, 5, 4, 4) \text{ and } (2, 3, 7, 7).$$

Note that (3, 5, 4, 4) satisfies (1.6), but (2, 3, 7, 7) does not.

The congruences (1.4) and (1.5) suggest that it may be of interest to consider the following, more general, situation.

$$\begin{cases} hh' \equiv \alpha \pmod{k}, & hh' \equiv \beta \pmod{k'} \\ kk' \equiv \gamma \pmod{h}, & kk' \equiv \delta \pmod{h'}, \end{cases} \tag{1.7}$$

where each of α, β, γ, and δ is equal to ± 1. We find that the method used in proving Theorem 1 applies, provided $\alpha\delta = \beta\gamma$ and $\alpha = \beta$. Thus, there are just four cases to consider. The cases $\alpha = \gamma = 1$ and $\alpha = -1$, $\gamma = 1$ are covered by Theorem 1. The case $\alpha = 1$, $\gamma = -1$ is essentially the same as $\alpha = -1$, $\gamma = 1$. The one remaining case is covered by the following:

Theorem 2

The system of congruences

$$\begin{cases} hh' \equiv -1 \pmod{k}, & hh' \equiv -1 \pmod{k'} \\ kk' \equiv -1 \pmod{h}, & kk' \equiv -1 \pmod{h'} \end{cases} \tag{1.8}$$

has no solutions in positive integers h, h', k, k'.

Note that it is now not necessary to assume either $k' \neq k$ or $h' \neq h$.

2. It follows from (1.1) that

$$((-x)) = -((x)). \tag{2.1}$$

Thus,

$$s(-h, k) = -s(h, k). \tag{2.2}$$

In the next place, if $hh' \equiv 1 \pmod{k}$, then, by (1.2),

$$s(h', k) = \sum_{r \pmod{k}} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{h'r}{k}\right)\right) = \sum_{t \pmod{k}} \left(\left(\frac{ht}{k}\right)\right)\left(\left(\frac{t}{k}\right)\right),$$

on replacing r by ht and using the periodicity of $((x))$. Hence,

$$s(h', k) = s(h, k) \qquad [hh' \equiv 1 \pmod{k}]. \tag{2.3}$$

Similarly,

$$s(h', k) = -s(h, k) \qquad [hh' \equiv -1 \pmod{k}]. \tag{2.4}$$

Now let h, h', k, k' be positive integers that satisfy the system of congruences

$$\begin{cases} hh' \equiv 1 \pmod{k}, & hh' \equiv 1 \pmod{k'} \\ kk' \equiv 1 \pmod{h}, & kk' \equiv 1 \pmod{h'}. \end{cases} \tag{2.5}$$

Thus,

$$(h, k) = (h, k') = (h', k) = (h', k') = 1.$$

Therefore, we may apply the reciprocity theorem (1.3) to get the following set of equations:

$$\begin{cases} s(h, k) + s(k, h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) \\ s(h', k) + s(k, h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) \\ s(h, k') + s(k', h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) \\ s(h', k') + s(k', h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) \end{cases} \tag{2.6}$$

In view of (2.3), we have

$$\begin{cases} s(h,\ k) + s(k,\ h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) \\ s(h,\ k) + s(k,\ h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) \\ s(h,\ k') + s(k,\ h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) \\ s(h,\ k') + s(k,\ h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) \end{cases} \tag{2.7}$$

Multiplying the first and fourth equations in (2.6) by +1, the second and third by -1, and adding the resulting equations, we get

$$\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) - \left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) - \left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) + \left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) = 0,$$

or better,

$$h'k'(h^2 + 1 + k^2) - hk'(h'^2 + 1 + k^2) - h'k(h^2 + 1 + k'^2) + hk(h'^2 + 1 + k'^2) = 0.$$

A little manipulation yields

$$(h' - h)(k' - k)(1 - hh' - kk') = 0. \tag{2.8}$$

Now, assuming that $h' \neq h$ and $k' \neq k$, (2.8) reduces to

$$hh' + kk' = 1. \tag{2.9}$$

Since (2.9) obviously has no solutions in positive integers $h,\ h',\ k,\ k'$, we have proved the first half of Theorem 1.

3. To prove the second part of the theorem, let $h,\ h',\ k,\ k'$ be positive integers that satisfy the congruences

$$\begin{cases} hh' \equiv -1 \pmod{k}, & hh' \equiv -1 \pmod{k'} \\ kk' \equiv 1 \pmod{h}, & kk' \equiv 1 \pmod{h'}. \end{cases} \tag{3.1}$$

Then

$$(h,\ k) = (h,\ k') = (h',\ k) = (h',\ k') = 1,$$

and exactly as above, we get the set of equations (2.6). However, we now use both (2.3) and (2.4). Thus, in place of (2.7), we get

$$\begin{cases} s(h,\ k) + s(k,\ h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) \\ -s(h,\ k) + s(k,\ h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) \\ s(h,\ k') + s(k,\ h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) \\ -s(h,\ k') + s(k,\ h') = -\frac{1}{4} + \frac{1}{12}\left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) \end{cases} \tag{3.2}$$

Multiply the first and second equations by +1, the third and fourth by -1, and add:

$$\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) + \left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) - \left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) - \left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) = 0,$$

or

$$h'k'(h^2 + 1 + k^2) + hh'(h'^2 + 1 + k^2) - h'k(h^2 + 1 + k'^2) - hk(h'^2 + 1 + k'^2) = 0.$$

This reduces to

$$(h' + h)(k' - k)(1 + hh' - kk') = 0.$$

Hence, assuming $k' \neq k$, we get

$$kk' - hh' = 1. \tag{3.3}$$

This completes the proof of the theorem.

4. We now consider the system of congruences

$$\begin{cases} hh' \equiv \alpha \pmod{k}, & hh' \equiv \beta \pmod{k'} \\ kk' \equiv \gamma \pmod{h}, & kk' \equiv \delta \pmod{h'}, \end{cases} \tag{4.1}$$

where each of α, β, γ, δ is equal to ± 1. Then, in place of (2.7), we get

$$\begin{cases} s(h,\,k) + s(k,\,h) \\ \alpha s(h,\,k) + s(k,\,h') \\ s(h,\,k') + \gamma s(k,\,h) \\ \beta s(h,\,k') + \delta s(k,\,h'), \end{cases} \tag{4.2}$$

where, for brevity, we indicate only the left-hand sides.

Now, multiply the first equation in (4.2) by 1, the second by ξ, the third by η, the fourth by ζ. To eliminate the left-hand side, ξ, η, ζ must satisfy

$$1 + \alpha\xi = 0,\ 1 + \gamma\eta = 0,\ \xi + \delta\zeta = 0,\ \eta + \beta\zeta = 0.$$

This gives

$$\xi = -\alpha,\ \eta = -\gamma,\ \zeta = \alpha\delta,\ \zeta = \beta\gamma, \tag{4.3}$$

so that $\alpha\delta = \beta\gamma$, $\delta = \alpha\beta\gamma$. Hence, (4.3) becomes

$$\xi = -\alpha,\ \eta = -\gamma,\ \zeta = \beta\gamma. \tag{4.4}$$

It follows that (4.2) implies

$$\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right) - \alpha\left(\frac{h'}{k} + \frac{1}{h'k} + \frac{k}{h'}\right) - \gamma\left(\frac{h}{k'} + \frac{1}{hk'} + \frac{k'}{h}\right) + \beta\gamma\left(\frac{h'}{k'} + \frac{1}{h'k'} + \frac{k'}{h'}\right) = 0.$$

Simplifying, we get

$$\begin{aligned}(h'k' - \alpha hk' - \gamma h'k + \beta\gamma hk) - \alpha hh'(h'k' - \alpha hk' - \alpha\beta\gamma h'k + \alpha\gamma hk)& \\ - \gamma kk'(h'k' - \beta hk' - \gamma h'k + \alpha\gamma hk) &= 0.\end{aligned}$$

If $\alpha = \beta$, this becomes

$$(h' - \alpha h)(k' - \gamma k)(1 - \alpha hh' - \gamma kk') = 0, \tag{4.5}$$

while (4.1) reduces to

$$\begin{cases} hh' \equiv \alpha \pmod{k}, & hh' \equiv \alpha \pmod{k'} \\ kk' \equiv \gamma \pmod{h}, & kk' \equiv \gamma \pmod{h'}. \end{cases} \tag{4.6}$$

The cases $\alpha = \gamma = 1$ and $\alpha = -1$, $\gamma = 1$ are covered by Thoerem 1. The case $\alpha = 1$, $\gamma = -1$ is essentially the same as $\alpha = -1$, $\gamma = 1$. Thus, the only case to consider is $\alpha = \gamma = -1$. In this case (4.5) is

$$(h' + h)(k' + k)(1 + hh' + kk') = 0. \tag{4.7}$$

Clearly, (4.7) cannot be satisfied in positive integers. It is now not necessary to assume either $k' \neq k$ or $h' \neq h$.

This completes the proof of Theorem 2.

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COAXAL CIRCLES ASSOCIATED WITH RECURRENCE-GENERATED SEQUENCES

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1. INTRODUCTION

Recently, some articles [1], [3], and [4] of a geometrical nature relating Fibonacci numbers to circles, with an extension to conics, have appeared in this journal. Here, we offer another geometrical connection between Fibonacci-type numbers and circles (though this material bears no relation to the other articles). In particular, it is shown how Fibonacci and Lucas numbers, and their generalization, are associated with sets of coaxal circles.

Define the recurrence-generated sequence $\{H_n\}$ for all values of n (integer) by

$$H_{n+2} = H_{n+1} + H_n,\ H_0 = 2b,\ H_1 = a + b, \tag{1.1}$$

where a and b are arbitrary, but may be thought of as integers.

Using [2], equation (δ), we have, *mutatis mutandis*, the explicit Binet form for this generalized sequence

$$H_n = \frac{(a + \sqrt{5}b)\alpha^n - (a - \sqrt{5}b)\beta^n}{\sqrt{5}}, \tag{1.2}$$

where $\alpha = (1 + \sqrt{5})/2\ (> 0)$, $\beta = (1 - \sqrt{5})/2\ (< 0)$ are the roots of $x^2 - x - 1 = 0$ (so that $\alpha\beta = -1$).

From (1.2) it follows that

$$H_n = aF_n + bL_n, \tag{1.3}$$

where

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} \tag{1.4}$$

and

$$L_n = \alpha^n + \beta^n \tag{1.5}$$

are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively, occurring in (1.1), (1,2), and (1.3) when $a = 1$, $b = 0$ (for F_n) and $a = 0$, $b = 1$ (for L_n).

Observe from (1.4) and (1.5) that

$$\sqrt{5}F_n < L_n \text{ when } n \text{ is even}, \tag{1.6}$$

while

$$\sqrt{5}F_n > L_n \text{ when } n \text{ is odd}. \tag{1.7}$$

2. COAXAL CIRCLES FOR $\{H_n\}$

Consider the point with Cartesian coordinates $(x, 0)$ where x is given by

$$x = [(a + \sqrt{5}b)\alpha^{2n} + (a - \sqrt{5}b)\cos(n - 1)\pi]/\sqrt{5}\alpha^n \tag{2.1}$$

[the x-value being another form of H_n in (1.2)].

Elementary calculations show that the circle, denoted by CH_n, with

$$\text{center:}\quad \left(\left(\frac{a+\sqrt{5}b}{\sqrt{5}}\right)\alpha^n,\ 0\right) \equiv (\bar{x}(H_n),\ \bar{y}(H_n)) \tag{2.2}$$

and

$$\text{radius:}\quad r(H_n) = \left|\frac{a-\sqrt{5}b}{\sqrt{5}\alpha^n}\right| \tag{2.3}$$

is

$$\left(x - \left(\frac{a+\sqrt{5}b}{\sqrt{5}}\right)\alpha^n\right)^2 + y^2 = \left(\frac{a-\sqrt{5}b}{\sqrt{5}\alpha^n}\right)^2. \tag{2.4}$$

Clearly,

$$\bar{x}(H_n)/\bar{x}(H_{n-1}) = \alpha \tag{2.5}$$

and

$$r(H_n)/r(H_{n-1}) = 1/\alpha, \tag{2.6}$$

so that the sets $\{\bar{x}(H_n)\}$ and $\{r(H_n)\}$ form geometrical progressions.

The circles CH_n cut the x-axis where

$$\begin{aligned} x(H_n) &= (a+\sqrt{5}b)\alpha^n/\sqrt{5} \pm (a-\sqrt{5}b)/\sqrt{5}\alpha^n \\ &= a(\alpha^n \pm (-1)^n\beta^n)/\sqrt{5} + b(\alpha^n \mp (-1)^n\beta^n), \text{ since } \alpha\beta = -1. \end{aligned}$$

That is,

$$\begin{aligned} x(H_n) &= aF_n + bL_n \quad \text{or} \quad aL_n/\sqrt{5} + \sqrt{5}bF_n \\ &= H_n \quad \text{or} \quad aL_n/\sqrt{5} + \sqrt{5}bF_n \ \text{[by (1.3)]}. \end{aligned} \tag{2.7}$$

The coordinates $x = \bar{x}(H_n)$, $y = r(H_n)$ of the highest point on CH_n lie on the upper branch of the rectangular hyperbola

$$xy = \left(\frac{a+\sqrt{5}b}{5}\right)|a - \sqrt{5}b| \tag{2.8}$$

on making use of (2.2) and (2.3).

Of the other three points of intersection of the circle (2.4) and the rectangular hyperbola (2.8), only one is real, given by the real root of the cubic equation $x^3 - \alpha^n x^2 - \alpha^{-2n}x - \alpha^{-n} = 0$, e.g., in the case of $\{L_n\}$. No obvious geometry follows from the set of these real points [though one might hope that their locus would be a simple curve (another rectangular hyperbola?)].

Similar results apply to the case of the lowest point.

3. COAXAL CIRCLES FOR $\{F_n\}$ AND $\{L_n\}$

Parallel details for the special cases $\{F_n\}$ and $\{L_n\}$ of $\{H_n\}$ arising when $a = 1$, $b = 0$, and $a = 0$, $b = 1$, respectively, can be tabulated, as in the following table, after making appropriate notational adjustments to the results (2.1)-(2.8) in the previous section.

Interesting features of the table appear in (3.7):

(i) the (integer) Fibonacci numbers and the irrational numbers of the Lucas-related sequence $\{L_n\}/\sqrt{5}$ are represented on the x-axis as the points of intersection of the axis and the set of coaxal circles CF_n, and

(ii) the (integer) Lucas numbers and the irrational numbers of the Fibonacci-related sequence $\sqrt{5}\{F_n\}$ are represented on the x-axis as the points of intersection of the axis and the set of coaxal circles CL_n.

If we define the orientation of a circle of the coaxal sets to be that in going (above the x-axis) from the Fibonacci value to the Lucas value in (3.7),

	$\{F_n\}$	$\{L_n\}$
(3.1)	$\begin{cases} x = (\alpha^{2n} + \cos(n-1)\pi)/\sqrt{5}\alpha^n \\ y = 0 \end{cases}$	$\begin{cases} x = (\alpha^{2n} - \cos(n-1)\pi)/\alpha^n \\ y = 0 \end{cases}$
(3.2)	$\bar{x}(F_n) = \alpha^n/\sqrt{5},\ y(F_n) = 0$	$\bar{x}(L_n) = \alpha^n,\ \bar{y}(L_n) = 0$
(3.3)	$r(F_n) = 1/\sqrt{5}\alpha^n$	$r(L_n) = 1/\alpha^n$
(3.4)	$CF_n\colon \left[x - \dfrac{\alpha^n}{\sqrt{5}}\right]^2 + y^2 = \dfrac{1}{5\alpha^{2n}}$	$CL_n\colon (x - \alpha^n)^2 + y^2 = 1/\alpha^{2n}$
(3.5)	$\bar{x}(F_n)/\bar{x}(F_{n-1}) = \alpha$	$\bar{x}(L_n)/\bar{x}(L_{n-1}) = \alpha$
(3.6)	$r(F_n)/r(F_{n-1}) = 1/\alpha$	$r(L_n)/r(L_{n-1}) = 1/\alpha$
(3.7)	$x(F_n) = F_n,\ L_n/\sqrt{5}$	$x(L_n) = L_n,\ \sqrt{5}F_n$
(3.8)	$xy = 1/5$	$xy = 1$

then (1.6) and (1.7) disclose that the orientation is reversed for alternate circles in both coaxal sets.

It is an instructive exercise to draw some of the circles CF_n and CL_n for small integral values of n (<0, =0, >0), but we omit the diagram here in order to conserve space.

4. CONCLUDING REMARKS

This article developed from a brief private communication from L. G. Wilson [5], to whom the author expresses his thanks. Wilson, however, was concerned only with the polar coordinate representation of the points given in Cartesian coordinates (x, y) by x as in (2.1), and $y = (a - \sqrt{5}b)\sin(n-1)\pi/\sqrt{5}\alpha^n$ but with n not restricted to integral values. Our concentration on just two special points on each circle was stimulated by a desire to exhibit the circle generation of the members of $\{F_n\}$ and $\{L_n\}$.

The occurrence of $\alpha^n/\sqrt{5}$ and α^n reminds us that these, by (1.4) and (1.5), are the limiting values of F_n and L_n, respectively. Thus, if n is graphed against $y = \lim\limits_{n\to\infty} F_n$ and $y = \lim\limits_{n\to\infty} L_n$ in turn, we find that the points (F_n, y) and (L_n, y) lie remarkably close to the exponential curves $y = \alpha^n/\sqrt{5}$ and $y = \alpha^n$ even for small values of n.

It seems reasonable to expect an extension, albeit a slightly tedious one, to the more general sequence $\{W_n\}$ defined for all integral n by

$$W_{n+2} = pW_{n+1} - qW_n, \tag{4.1}$$

with specified values for W_0 and W_1. Possibly some worthwhile results for the special cases of the Pell sequences arising from (4.1) when $p = 2$, $q = 1$ might eventuate from this investigation.

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-526 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

Find all ordered pairs (m, n) of positive integers for which there is an integer x satisfying the equation

$$F_mF_nx^2 - [F_m(F_m, F_n) + F_nF_{(m,n)}]x + (F_m, F_n)F_{(m,n)} = 0.$$

Here (r, s) denotes the greatest common divisor of r and s.

B-527 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

Do as in B-526 with the equation replaced by

$$(F_m, F_n)x^2 - (F_m + F_n)x + F_{(m,n)} = 0.$$

B-528 *Proposed by Herta T. Freitag, Roanoke, VA*

For nonnegative integers n, prove that

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{i+1}^2 = 5^n F_{2n+3}.$$

<u>B-529</u> *Proposed by Herta T. Freitag, Roanoke, VA*

For positive integers n, find a compact form for $\sum_{i=0}^{2n} \binom{2n}{i} F_{i+1}^2$.

<u>B-530</u> *Proposed by Michael Eisenstein, San Antonio, TX*

Let $\alpha = (1 + \sqrt{5})/2$. For n an odd positive integer, prove that the continued fraction

$$L_n + \cfrac{1}{L_n + \cfrac{1}{L_n + \cdots}} = \alpha^n.$$

<u>B-531</u> *Proposed by Michael Eisenstein, San Antonio, TX*

For n an even positive integer, prove that

$$L_n - \cfrac{1}{L_n - \cfrac{1}{L_n - \cdots}} = \alpha^n.$$

SOLUTIONS

Even Sum of Fibonacci Products

<u>B-502</u> *Proposed by Herta T. Freitag, Roanoke, VA*

Given that h and k are integers with $h+k$ an integral multiple of 3, prove that $F_k F_{k-h-1} + F_{k+1} F_{k-h}$ is even.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

Letting $t = n + 1$ in (I_{26})—see p. 59 of Verner E. Hoggatt, Jr., *Fibonacci and Lucas Numbers* (Boston: Houghton Mifflin Co., 1969)—yields the following identity:

$$F_{m+t} = F_{m+1}F_t + F_m F_{t-1}. \tag{*}$$

Thus,

$$\begin{aligned} F_k F_{k-h-1} + F_{k+1}F_{k-h} &= F_{k+1}F_{k-h} + F_k F_{k-h-1} \\ &= F_{k+(k-h)} \text{ [by (*)]} \\ &= F_{2k-h} \\ &= F_{3k-(h+k)}. \end{aligned}$$

Because $h + k$ is a multiple of 3, 3 divides $3k - (h + k)$, hence $2 = F_3$ divides $F_{3k-(h+k)}$.

Also solved by Wray G. Brady, Paul S. Bruckman, L. Cseh, M. J. DeLeon, C. Georghiou, Walther Janous, L. Kuipers, Graham Lord, I. Merenyi, Bob Prielipp, Heinz-Jürgen Seiffert, Sahib Singh, and the proposer.

Even Perfect Numbers Mod 7

<u>B-503</u> *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Prove that every even perfect number except 28 is congruent to 1 or -1 modulo 7.

Solution by L. Cseh, Cluj, Romania

It is well known that every even perfect number is of the form

$$2^{p-1}(2^p - 1),$$

where p is prime and so is $(2^p - 1)$. Every prime, except 3 is of the form

$$3k + 1 \quad \text{or} \quad 3k + 2.$$

Thus, we have

$$2^{3k}(2^{3k+1} - 1) \equiv 1 \cdot (1 \cdot 2 - 1) \equiv 1 \pmod 7$$
$$2^{3k+1}(2^{3k+2} - 1) \equiv 2 \cdot (4 - 1) \equiv 6 \equiv -1 \pmod 7,$$

and because for $p = 3$ we obtain 28, the proof is complete.

Also solved by Paul S. Bruckman, M. J. DeLeon, Herta T. Freitag, C. Georghiou, Walther Janous, H. Klauser and M. Wachtel, L. Kuipers, Graham Lord, I. Merenyi, Bob Prielipp, Sahib Singh, and the proposer.

Triangular Fibonacci Numbers Mod 24

B-504 *Proposed by Charles R. Wall*

Prove that if n is an odd integer and F_n is in the set $\{0, 1, 3, 6, 10, \ldots\}$ of triangular numbers, then $n \equiv \pm 1 \pmod{24}$.

Solution by Leonard Dresel, University of Reading, England

If F_n is in the set of triangular numbers, then there is an integer k such that $F_n = \frac{1}{2}k(k + 1)$, so that $8F_n + 1 = (2k + 1)^2$ is a perfect square. Reducing this modulo 9, we have

$$8F_n + 1 \text{ is a quadratic residue modulo } 9.$$

The Fibonacci sequence reduced modulo 9 is periodic with period 24, and for the odd integers n, we have

$$\begin{array}{rcccccccccccccl}
n & \equiv & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & \pmod{24} \\
F_n & \equiv & 1 & 2 & 5 & 4 & 7 & 8 & 8 & 7 & 4 & 5 & 2 & 1 & \pmod 9 \\
8F_n + 1 & \equiv & 0 & 8 & 5 & 6 & 3 & 2 & 2 & 3 & 6 & 5 & 8 & 0 & \pmod 9.
\end{array}$$

By squaring the numbers 0, 1, 2, 3, and 4, we find that the quadratic residues modulo 9 are 0, 1, 4, 7. Hence, the only quadratic residue in the sequence for $8F_n + 1 \pmod 9$ is the number 0, and this occurs only for $n \equiv \pm 1 \pmod{24}$.

We can extend this result in various ways. For example, by reducing the sequence $8F_n + 1$ modulo 11, we obtain the further condition $n \equiv \pm 1 \pmod{10}$.

Also solved by Paul S. Bruckman and the proposer.

Sum of Lucas Products

B-505 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$N = N(m, a) = L_{m-2a}L_m - L_{m+1-2a}L_{m-1},$$

where m and a are positive integers. Prove or disprove that N is: (a) always (exactly) divisible by 5; (b) never divisible by 3, 4, 6, 7, 8, 9, or 11; and (c) divisible by 10 if $a \equiv 2 \pmod 3$.

Solution by C. Georghiou, University of Patras, Greece

When L_n is replaced by $\alpha^n + \beta^n$, we get

$$N = L_{m-2a}L_m - L_{m+1-2a}L_{m-1} = (-1)^m(L_{2a} + L_{2a-2}) = (-1)^m 5F_{2a-1};$$

therefore, N is divisible by 5.

Next, we take the following properties of the Fibonacci numbers as known (otherwise, they can easily be established):

$$F_n \equiv 0 \pmod 3 \quad \text{iff} \quad n \equiv 0 \pmod 4 \tag{1}$$

$$F_n \equiv 0 \pmod 4 \quad \text{iff} \quad n \equiv 0 \pmod 6 \tag{2}$$

$$F_n \equiv 0 \pmod 7 \quad \text{iff} \quad n \equiv 0 \pmod 8 \tag{3}$$

$$F_n \equiv 0 \pmod{11} \quad \text{iff} \quad n \equiv 0 \pmod{10} \tag{4}$$

$$F_n \equiv 0 \pmod 2 \quad \text{iff} \quad n \equiv 0 \pmod 3 \tag{5}$$

Now (1) $\Rightarrow N \not\equiv 0$ (mod 3 or mod 6 or mod 9),
(2) $\Rightarrow N \not\equiv 0$ (mod 4 or mod 8),
(3) $\Rightarrow N \not\equiv 0 \pmod 7$,
(4) $\Rightarrow N \not\equiv 0 \pmod{11}$, and finally,
(5) $\Rightarrow N \equiv 0 \pmod{10}$ iff $2a - 1 \equiv 0 \pmod 3$ or $a \equiv 2 \pmod 3$.

Also solved by Paul S. Bruckman, L. Cseh, M. J. DeLeon, Walther Janous, L. Kuipers, Graham Lord, Bob Prielipp, Sahib Singh, and the proposer.

Fibonacci and Lucas Convolutions

B-506 *Proposed by Heinz-Jürgen Sieffert, student, Berlin, Germany*

Let $G_n = (n+1)F_n$ and $H_n = (n+1)L_n$. Prove that:

(a) $$\sum_{k=0}^{n} G_k G_{n-k} = \frac{(n+2)(n+3)}{30}H_n - \frac{2}{25}H_{n+2} + \frac{4}{25}F_{n+3};$$

(b) $$\sum_{k=0}^{n} H_k H_{n-k} = \frac{(n+2)(n+3)}{6}H_n + \frac{2}{5}H_{n+2} - \frac{4}{5}F_{n+3}.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

Let

$$U(x) = x/(1 - x - x^2) = \frac{1}{\sqrt{5}}((1 - \alpha x)^{-1} - (1 - \beta x)^{-1}) = \sum_{n=0}^{\infty} F_n x^n; \tag{1}$$

$$V(x) = (2 - x)/(1 - x - x^2) = P + Q = \sum_{n=0}^{\infty} L_n x^n,$$

where

$$P = (1 - \alpha x)^{-1} \quad \text{and} \quad Q = (1 - \beta x)^{-1}.$$

Also, let

$$A(x) = (xU(x))', \; B(x) = (xV(x))'. \tag{2}$$

Then

$$A(x) = \sum_{n=0}^{\infty} G_n x^n, \; B(x) = \sum_{n=0}^{\infty} H_n x^n. \tag{3}$$

Note that $(xP)' = P^2$, $(xQ)' = Q^2$. Hence,

$$A(x) = 5^{-1/2}(P^2 - Q^2),\ B(x) = P^2 + Q^2. \tag{4}$$

Let

$$R(x) = P^4 + Q^4,\ S(x) = P^2Q^2. \tag{5}$$

Then

$$R(x) = \sum_{n=0}^{\infty}\binom{n+3}{3}(\alpha^n + \beta^n)x^n,$$

or

$$R(x) = \frac{1}{6}\sum_{n=0}^{\infty}(n+2)(n+3)H_n x^n. \tag{6}$$

Also,

$$S(x) = (1 - x - x^2)^{-2} = \frac{1}{4}(U(x) + V(x))^2$$
$$= \frac{1}{4}\{(1 + 5^{-1/2})P + (1 - 5^{-1/2})Q\}^2 = \frac{1}{5}\{\alpha^2P^2 + 2PQ + \beta^2Q^2\};$$

hence,

$$5S(x) = \alpha^2P^2 + U(x) + V(x) + \beta^2Q^2$$
$$= \sum_{n=0}^{\infty}(n+1)L_{n+2}x^n + \frac{2}{\sqrt{5}}(\alpha P - \beta Q)$$
$$= \sum_{n=0}^{\infty}\{(n+1)L_{n+2} + 2F_{n+1}\}x^n$$
$$= \sum_{n=0}^{\infty}\{(n+3)L_{n+2} + 2(F_{n+1} - L_{n+2})\}x^n,$$

or

$$S(x) = \frac{1}{5}\sum_{n=0}^{\infty}(H_{n+2} - 2F_{n+3})x^n. \tag{7}$$

Now,

$$(A(x))^2 = \left(\sum_{n=0}^{\infty}G_nx^n\right)^2 = \sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}G_kG_{n-k};$$

also, however, from (4) and (5), $5(A(x))^2 = R(x) - 2S(x)$. Using (6) and (7):

$$\sum_{k=0}^{n}G_kG_{n-k} = \frac{1}{30}(n+2)(n+3)H_n - \frac{2}{25}(H_{n+2} - 2F_{n+3}),$$

or

$$\sum_{k=0}^{n}G_kG_{n-k} = \frac{1}{30}(n+2)(n+3)H_n - \frac{2}{25}H_{n+2} + \frac{4}{25}F_{n+3}. \tag{8}$$

Likewise,

$$(B(x))^2 = \left(\sum_{n=0}^{\infty}H_nx^n\right)^2 = \sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}H_kH_{n-k} = R(x) + 2S(x)$$
$$= \frac{1}{6}\sum_{n=0}^{\infty}(n+2)(n+3)H_nx^n + \frac{2}{5}\sum_{n=0}^{\infty}(H_{n+2} - 2F_{n+3})x^n,$$

so

$$\sum_{k=0}^{n}H_kH_{n-k} = \frac{1}{6}(n+2)(n+3)H_n + \frac{2}{5}H_{n+2} - \frac{4}{5}F_{n+3}. \tag{9}$$

Also solved by C. Georghiou, L. Kuipers, J. Suck, Gregory Wulczyn, and the proposer.

Mixed Convolution

B-507 *Proposed by Heinz-Jürgen Sieffert, Berlin, Germany*

Let G_n and H_n be as in B-506. Find a formula for $\sum_{k=0}^{n} G_k H_{n-k}$ similar to the formulas in B-506.

Solution by Paul S. Bruckman, Fair Oaks, CA

We follow the notation introduced in the solution to B-506, and note that

$$A(x)B(x) = \sum_{n=0}^{\infty} G_n x^n \cdot \sum_{n=0}^{\infty} H_n x^n = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} G_k H_{n-k}.$$

On the other hand,

$$A(x)B(x) = 5^{-1/2}(P^4 - Q^4) = 5^{-1/2} \sum_{n=0}^{\infty} \binom{n+3}{3} (\alpha^n - \beta^n) x^n$$

$$= \frac{1}{6} \sum_{n=0}^{\infty} (n+2)(n+3) G_n x^n.$$

Hence,

$$\sum_{k=0}^{n} G_k H_{n-k} = \frac{1}{6}(n+2)(n+3)G_n.$$

Also solved by C. Georghiou, L. Kuipers, J. Suck, Gregory Wulczyn, and the proposer.

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(Continued from page 272)

3. A. F. Horadam. "Geometry of a Generalized Simson's Formula." *The Fibonacci Quarterly* 20, no. 2 (1982):164-68.
4. A. G. Shannon & A. F. Horadam. "Infinite Classes of Sequence-Generated Circles." *The Fibonacci Quarterly* (to appear).
5. L. G. Wilson. "Fibonacci Sequences." Private communication, 1982.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS and SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, the solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-372 *Proposed by M. Wachtel, Zürich, Switzerland*

There exist infinitely many sequences, each with infinitely many solutions of the form:

$\underline{A}\cdot x_1^2 + C = \underline{B}\cdot y_1^2$	$\underline{A} = F_{n+3}$	$\underline{C} = L_n$	$\underline{B} = F_{n+1}$
$\underline{A}\cdot x_2^2 + C = \underline{B}\cdot y_2^2$	$\underline{x_1} = 1$		$\underline{y_1} = 2$
$\underline{A}\cdot x_3^2 + C = \underline{B}\cdot y_3^2$	$\underline{x_2} = F_{n-1}F_n + F_{n+1}^2$		$\underline{y_2} = 2F_{n+1}^2$
....			
$\underline{A}\cdot x_m^2 + C = \underline{B}\cdot y_m^2$	$\underline{x_3} = 2F_{2n+4} + (-1)^n$		$\underline{y_3} = 2F_{2n+5}$

Find a recurrence formula for $x_4/y_4,\ x_5/y_5,\ \ldots,\ x_m/y_m$ (y_m = dependent on x_m).

Examples: $(x_1 - x_3)$

$\underline{n = 3}$	(in numbers)
$\underline{F_6}\cdot(1)^2 + \underline{L_3} = \underline{F_4}\cdot(2)^2$	$8\cdot 1 + \underline{4} = \underline{3}\cdot 2^2$
$\underline{F_6}\cdot(F_2F_3 + F_4^2)^2 + \underline{L_3} = \underline{F_4}\cdot(2F_4^2)^2$	$\underline{8}\cdot 11^2 + \underline{4} = \underline{3}\cdot 18^2$
$\underline{F_6}\cdot(2F_{10} - 1)^2 + \underline{L_3} = \underline{F_4}\cdot(2F_{11})^2$	$\underline{8}\cdot 109^2 + \underline{4} = \underline{3}\cdot 178^2$
$\underline{n = 4}$	
$\underline{F_7}\cdot(1)^2 + \underline{L_4} = \underline{F_5}\cdot(2)^2$	$\underline{13}\cdot 1 + \underline{7} = \underline{5}\cdot 2^2$
$\underline{F_7}\cdot(F_3F_4 + F_5^2)^2 + \underline{L_4} = \underline{F_5}\cdot(2F_5^2)^2$	$\underline{13}\cdot 31^2 + \underline{7} = \underline{5}\cdot 50^2$
$\underline{F_7}\cdot(2F_{12} + 1)^2 + \underline{L_4} = \underline{F_5}\cdot(2F_{13})^2$	$\underline{13}\cdot 289^2 + \underline{7} = \underline{5}\cdot 466^2$

H-373 *Proposed by Andreas N. Philippou, University of Patras, Greece*

For any fixed integers $k \geqslant 0$ and $r \geqslant 2$, set

$$f_{n+1,r}^{(k)} = \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\cdots+n_k=n}} \binom{n_1+\cdots+n_k+r-1}{n_1,\ \ldots,\ n_k,\ r-1},\quad n \geqslant 0.$$

Show that

$$f_{n+1,r}^{(k)} = \sum_{\ell=0}^{n} f_{\ell+1,1}^{(k)} f_{n+1-\ell,r-1}^{(k)}, \quad n \geqslant 0.$$

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H-374 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

If $\sigma^*(n)$ is the sum of the unitary divisors of n, then

$$\sigma^*(n) = \prod_{p^e \| n} (1 + p^e),$$

where p^e is the highest power of the prime p that divides n. The ratio $\sigma^*(n)/n$ increases as new primes are introduced as factors of n, but decreases as old prime factors appear more often. As N increases, is $\sigma^*(N!)/N!$ bounded or unbounded?

H-375 *Proposed by Piero Filipponi, Rome, Italy*

Conjecture 1

If $F_k \equiv 0 \pmod{k}$ and $k \neq 5^n$, then $k \equiv 0 \pmod{12}$.

Conjecture 2

Let $m > 1$ be odd. Then, $F_{12m} \equiv 0 \pmod{12m}$ implies either 3 divides m or 5 divides m.

Conjecture 3

Let $p > 5$ be a prime such that $p \nmid F_{24}$, then $F_{12m} \not\equiv 0 \pmod{12m}$.

Conjecture 4

If $L_k \equiv 0 \pmod{k}$, then $k \equiv 0 \pmod{6}$ for $k > 1$.

SOLUTIONS

Lotta Sequences

H-350 *Proposed by M. Wachtel, Zürich, Switzerland*
(Vol. 21, no. 1, February 1983)

There exist an infinite number of sequences, each of which has an infinite number of solutions of the form:

$$\begin{array}{lll}
A \cdot x_1^2 + 1 = 5 \cdot y_1^2 & \underline{A} = 5 \cdot (a^2 + a) + 1 & \underline{a} = 0, 1, 2, 3, \ldots \\
A \cdot x_2^2 + 1 = 5 \cdot y_2^2 & & \\
A \cdot x_3^2 + 1 = 5 \cdot y_3^2 & \underline{x_1} = 2;\ \underline{x_2} = 40(2a+1)^2 - 2 & \\
\cdots\cdots \quad \cdots \quad \cdots\cdots & & \\
A \cdot x_n^2 + 1 = 5 \cdot y_n^2 & \underline{y_1} = 2a + 1;\ \underline{y_2} = (2a+1) \cdot (16A + 1) &
\end{array}$$

Find a recurrence formula for x_3/y_3, x_4/y_4, $\ldots$, x_n/y_n (y_n = dependent on x_n).

Examples

$\underline{a = 0}$ $1 \cdot \left(\frac{L_3}{2}\right)^2 + 1 = 5 \cdot \left(\frac{F_3}{2}\right)^2$

$1 \cdot \left(\frac{L_9}{2}\right)^2 + 1 = 5 \cdot \left(\frac{F_9}{2}\right)^2$

$1 \cdot \left(\frac{L_{15}}{2}\right)^2 + 1 = 5 \cdot \left(\frac{F_{15}}{2}\right)^2$

$1 \cdot \ldots \quad + 1 = 5 \cdot \ldots$

$\underline{a = 1}$ $11 \cdot 2^2 + 1 = 5 \cdot 3^2$

$11 \cdot 358^2 + 1 = 5 \cdot 531^2$

$11 \cdot 63722^2 + 1 = 5 \cdot 94515^2$

$11 \cdot \ldots \quad + 1 = 5 \cdot \ldots$

$\underline{a = 5}$ $151 \cdot 2^2 + 1 = 5 \cdot 11^2$

$151 \cdot 4{,}838^2 + 1 = 5 \cdot 26{,}587^2$

$151 \cdot 11{,}698{,}282^2 + 1 = 5 \cdot 64{,}287{,}355^2$

$151 \quad \ldots \quad + 1 = 5 \quad \ldots$

Solution by Paul S. Bruckman, Carmichael, CA

The general solution of the Diophantine equation:

$$5y^2 - Ax^2 = 1 \tag{1}$$

is given by

$$x_n = \frac{u^{2n-1} - v^{2n-1}}{2\sqrt{A}}, \quad y_n = \frac{u^{2n-1} + v^{2n-1}}{2\sqrt{5}}, \quad n = 1, 2, \ldots, \tag{2}$$

where

$$u = (2a + 1)\sqrt{5} + 2\sqrt{A}, \quad v = (2a + 1)\sqrt{5} - 2\sqrt{A}. \tag{3}$$

Note that $uv = 5(2a + 1)^2 - 4(5(a^2 + a) + 1) = 1$. Also,

$$u^2 = 5(2a + 1)^2 + 4A + 4(2a + 1)\sqrt{5A} = 8A + 1 + 4(2a + 1)\sqrt{5A},$$

and

$$\begin{aligned} u^4 &= (8A + 1)^2 + 80A(2a + 1)^2 + 8(2a + 1)(8A + 1)\sqrt{5A} \\ &= (8A + 1)^2 + 80A\left(1 + \frac{4}{5}(A - 1)\right) + 2(8A + 1)(u^2 - 8A - 1) \\ &= -(8A + 1)^2 + 16A + 64A^2 + 2(8A + 1)u^2 = 2(8A + 1)u^2 - 1. \end{aligned}$$

Note that v satisfies the same relation. Thus,

$$w^4 - 2Bw^2 + 1 = 0, \tag{4}$$

where w denotes either u or v, and $B = 8A + 1 = 40a^2 + 40a + 9$. From (4), we readily deduce the recursions:

$$z_{n+2} - 2Bz_{n+1} + z_n = 0, \; n = 1, 2, \ldots, \tag{5}$$

where z denotes either x or y.

Now, let

$$r_n = \frac{x_n}{y_n}, \; n = 1, 2, \ldots . \tag{6}$$

Then, using (5), we obtain:

$$r_{n+2} = \frac{2Bx_{n+1} - x_n}{2By_{n+1} - y_n} = \frac{2Br_{n+1} - \dfrac{r_n y_n}{y_{n+1}}}{2B - \dfrac{y_n}{y_{n+1}}} = r_{n+1} + \frac{(r_{n+1} - r_n)y_n}{2By_{n+1} - y_n}.$$

Hence,

$$\frac{r_{n+1} - r_n}{r_{n+2} - r_{n+1}} = 2B \cdot \frac{y_{n+1}}{y_n} - 1,$$

or, equivalently:

$$\frac{y_{n+1}}{y_n} = \frac{1}{2B} \cdot \frac{r_{n+2} - r_n}{r_{n+2} - r_{n+1}}. \tag{7}$$

Also, using (5) and (7),

$$\frac{y_{n+2}}{y_{n+1}} = 2B - \frac{y_n}{y_{n+1}} = \frac{1}{2B} \cdot \frac{r_{n+3} - r_{n+1}}{r_{n+3} - r_{n+2}},$$

which implies:

$$4B^2 - 4B^2\left(\frac{r_{n+2} - r_{n+1}}{r_{n+2} - r_n}\right) = \frac{r_{n+3} - r_{n+1}}{r_{n+3} - r_{n+2}},$$

or

$$\frac{r_{n+3} - r_{n+1}}{r_{n+3} - r_{n+2}} = 4B^2\left(\frac{r_{n+1} - r_n}{r_{n+2} - r_n}\right). \tag{8}$$

Solving for r_{n+3} in (8) yields the desired recursion:

$$r_{n+3} = \frac{r_{n+1}(r_{n+2} - r_n) - 4B^2 r_{n+2}(r_{n+1} - r_n)}{r_{n+2} - r_n - 4B^2(r_{n+1} - r_n)}. \tag{9}$$

Also solved by the proposer.

Hats Off

H-352 *Proposed by Stephen Turner, Babson College, Babson Park, MA (Vol. 21, no. 2, May 1983)*

> One night during a national mathematical society convention, n mathematicians decided to gather in a suite at the convention hotel for an "after hours chat." The people in this group share the habit of wearing the same kind of hats, and each brought his hat to the suite. However, the chat was so engaging that at the end of the evening each (being deep in thought and oblivious to the practical side of matters) simply grabbed a hat at random and carried it away by hand to his room.
>
> Use a variation of the Fibonacci sequence for calculating the probability that none of the mathematicians carried his own hat back to his room.

Solution by J. Suck, Essen, Germany

The problem is Montmort's 1708 "problème des rencontres" (see [1], p. 180) and the required solution is an old hat of Euler's. In [2], he proves by an easy-to-find combinatorial argument that the number D_n of derangements (= permutations without fixed point) of $\{1, 2, \ldots, n\}$ satisfies the recurrence

$$D_{n+2} = (n + 1)(D_{n+1} + D_n), \; D_0 = 1, \; D_1 = 0,$$

and hence,

$$D_{n+1} = (n + 1)D_n + (-1)^{n+1}.$$

We can proceed to show by induction, then, the "closed" expression (also known to Euler [3])

$$D_n = n! \sum_{\nu=0}^{n} (-1)^\nu/\nu!,$$

which has to be divided by $n!$ to get the required probability.

Disclosing that k out of the n gathering mathematicians were members of the Fibonacci Association, one is led to ask a bit more generally for the probability that none at least of that group carried back his own hat. Denote by $D_{n,k}$ the numerator, that is, the number of permutations of $\{1, \ldots, n\}$ with no fixed point in $\{1, \ldots, k\}$. We have

$$D_{n,k} = \sum_{\nu=k}^{n} \binom{n-k}{\nu-k} D_\nu = \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} (n-\nu)!$$

where the right-hand side equality was a recent problem by Wang [4]. It pays to look at the triangular array $D_{n,k}$, $0 \leqslant k \leqslant n$. Morgenstern ([5], p. 15, Problem 11) offers the recurrence

$$D_{n,k} = (n-1)D_{n-1,k-1} + (k-1)D_{n-2,k-2}.$$

To get started, note that the first two columns are

$$D_{n,0} = n! \quad \text{and} \quad D_{n,1} = (n-1)!(n-1), \ n \geqslant 1.$$

We may also start from Euler's edge with

$$D_{n,k} = D_{n-1,k} + D_{n,k+1}.$$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	1	0				
2	2	1	1			
3	6	4	3	2		
4	24	18	14	11	9	
5	120	96	78	64	53	44

Denoting by R_n the n^{th} row sum, we find

$$R_{n+2} = (n+2)(R_{n+1} + R_n) \quad \text{and} \quad R_n = (n+1)! - D_{n+1}.$$

And for the n^{th} rising diagonal sum $S_n := D_{n,0} + D_{n-1,1} + \cdots$, we have

$$S_{n+2} = \begin{cases} S_{n+1} - S_n + (n+2)! & \text{for } n \text{ even,} \\ S_{n+1} + S_n + (n+2)! - D_{(n+1)/2} & \text{for } n \text{ odd.} \end{cases}$$

Peeping into the future, we see n growing, so finally, for our probability

$$\lim_{n\to\infty} D_{n,k}/n! = \begin{cases} 1 & \text{if } k \text{ remains constant,} \\ 1/e & \text{if } n-k \text{ does.} \end{cases}$$

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Also solved by P. Bruckman, E. Schmutz and M. Wachtel, N. Saxena and Sridhar Manthani (paper), and the proposer.

Dual Solution

H-353 *Proposed by Jerry Metzger, University of North Dakota, Grand Forks, ND (Vol. 21, no. 2, May 1983)*

For a positive integer n, describe all two-element sets $\{a, b\}$ for which there is a polynomial $f(x)$ such that $f(x) \equiv 0 \pmod{n}$ has solution set exactly $\{a, b\}$.

Solution by L. Kuipers, Switzerland

Let the congruence related to a pair (a, b) be written in the form

$$(x - a)(x - b) \equiv 0 \pmod{n}. \tag{1}$$

If a and b are the only solutions of (1), then (a, b) is called an S_n-pair, or S-pair. We assume throughout $a \neq b$, and distinguish several cases:

(i) Let $n = p$, p prime. Then any pair (a, b) forms a set $\{a, b\}$.

(ii) Let $n = p^2$, p being a prime. Consider the congruence

$$(x - a)(x - b) \equiv 0 \pmod{p^2}. \tag{2}$$

Each factor of the left-hand side of (2) if not zero produces at most one factor p. Hence, if $a \not\equiv b \pmod{p}$, then (a, b) is an S-pair. If $a \equiv b \pmod{p}$, then $x \equiv a$, $x \equiv b \pmod{p^2}$ are not the only solutions of (2). Let $a - b = kp$ $[k \not\equiv 0 \pmod{p}]$. Then take $x = a + p$, and substitution in (2) gives

$$(x - a)(x - b) \equiv p^2(1 + k) \equiv 0 \pmod{p^2}.$$

(iii) Let $n = p^3$, p being a prime. Consider the congruence

$$(x - a)(x - b) \equiv 0 \pmod{p^3}. \tag{3}$$

If $a \not\equiv b \pmod{p}$, then (a, b) is an S-pair. If $a \equiv b \pmod{p^2}$, then (a, b) is an S-pair if and only if $p = 2$. We have here $a = b + p^2$, for in $a - b = kp^2$ we have $k < p$. If $p \geqslant 3$, $a \equiv b \pmod{p^2}$, then there is always a solution of (3) distinct from $x \equiv a$ and $x \equiv b \pmod{p^3}$. Let $a - b = kp^2$, $k < p$. Take $x = a + p^2$, then

$$(x - a)(x - b) \equiv p^2(p^2 + kp^2) \equiv 0 \pmod{p^3}.$$

So (a, b) is not an S-pair.

If $a \equiv b \pmod{p}$, or $a - b = kp$, then, taking $x = a + p^2$, we have to take $k = 1$. In these cases, (a, b) is not an S-pair.

(iv) Let $n = p^4$, p being a prime. Consider the congruence

$$(x - a)(x - b) \equiv 0 \pmod{p^4}. \tag{4}$$

If $a \not\equiv b \pmod{p}$, then (a, b) is an S-pair. Let $a \equiv b \pmod{p}$, or $a - b = kp$ $(k < p^3)$. Now take $x = a + p^3$ in (4).

$$(x - a)(x - b) = p^2(p^2 + kp) \equiv 0 \pmod{p^3}.$$

One then obtains $p^3(p^3 + kp) \equiv 0 \pmod{p^4}$.

In general, if $n = p$ $(k \geqslant 5)$, p being a prime, then $a \not\equiv b \pmod{p}$ yields the sets $\{a, b\}$, while $x = a + p^{k-1}$ gives a third solution to the congruence $(x - a)(x - b) \equiv 0 \pmod{p^k}$ if $a - b \equiv 0 \pmod{p}$.

(v) Let $n = pq$; p, q being primes, $(p, q) = 1$. Consider the congruence

$$(x - a)(x - b) \equiv 0 \pmod{pq}. \tag{5}$$

For a solution of (5), one factor of the left-hand side of (5) must produce p and the second one the factor q. So, consider the system

$$\begin{aligned} x &\equiv a \pmod{p} \\ x &\equiv b \pmod{q}. \end{aligned} \tag{6}$$

Let $qb_1 \equiv 1 \pmod{p}$, $pb_2 \equiv 1 \pmod{q}$. Then a solution of (6) is given by

$$x_0 = qb_1a + pb_2b \pmod{pq}.$$

Now $x_0 \equiv a \pmod{pq}$ implies $a \equiv b \pmod{q}$, and conversely. Thus, if $a \equiv b$ (mod p), then (a, b) is an S-pair, and if $a \equiv b \pmod{q}$, then (a, b) is an S-pair.

Now, assume $a \not\equiv b \pmod{p}$, $a \not\equiv b \pmod{q}$. There are integers x and y such that $xp + yq = 1$. Hence, $a - b = (a - b)xp + (a - b)yq$ or $a - b = kp + \ell q$ or $a - kp = b + \ell q$. Thus, $a - kp$ is another solution of (5), as can be seen by substitution.

(vi) Let $n = m_1m_2$, $(m_1, m_2) = 1$. Consider the congruence

$$(x - a)(x - b) \equiv 0 \pmod{(m_1m_2)}. \tag{7}$$

For an extra solution of (7), it is sufficient that the first factor of the left-hand side of (7), i.e., $(x - a)$, is a multiple of m_1 and the second one is divisible by m_2. So consider the system

$$\begin{aligned} x &\equiv a \pmod{m_1} \\ x &\equiv b \pmod{m_2}. \end{aligned} \tag{8}$$

Let $m_2b_1 \equiv 1 \pmod{m_1}$, $m_1b_2 \equiv 1 \pmod{m_2}$.

Then a solution of (8) is given by

$$x_0 = m_2b_1a + m_1b_2b \pmod{m_1m_2}.$$

Now, $x_0 \equiv a \pmod{m_1m_2}$ implies $a \equiv b \pmod{m_2}$, and conversely. Also $x_0 \equiv b$ (mod m_1m_2) implies $a \equiv b \pmod{m_1}$. Hence, if $a \equiv b \pmod{m_2}$, then (a, b) is an S-pair; if $a \equiv b \pmod{m_2}$, then (a, b) is an S-pair.

Assume now that $m_1 \nmid a - b$ and $m_2 \nmid a - b$. There are integers x and y such that $xm_1 + ym_2 = 1$. Hence,

$$a - b = (a - b)xm_1 + (a - b)ym_2 = km_1 + \ell m_2 \quad \text{or} \quad a - km_1 = b + \ell m_2.$$

Thus, $a - km_1$ is another solution of (7) as follows by substitution.

(vii) After the preceding cases, it is not difficult to deal with the general case

$$n = p_1^{r_1}p_2^{r_2} \ \ldots \ p_t^{r_t}.$$

As soon as $a - b$ is divisible by $p_i^{r_i}$, we have an S-pair, and if $p_i^{r_i} \nmid a - b$, $i = 1, 2, \ldots, t$, then there are, besides a and b, extra solutions of the involved congruence.

Also solved by the proposer.

Not for Squares

H-354 *Proposed by Paul Bruckman, Concord, CA*
(Vol. 21, no. 2, May 1983)

Find necessary and sufficient conditions so that a solution in relatively prime integers x and y can exist for the Diophantine equation:

$$ax^2 - by^2 = c,$$

given that a, b, and c are pairwise relatively prime positive integers, and moreover, a and b are not both perfect squares.

Solution by M. Wachtel, Zürich, Switzerland

1.1 In conformity with Problem H-350, which represents a special case of H-354, the following symbolism is used:

$A \cdot x_1^2 + C = B \cdot y_1^2$	A, B, and C = constant values
$A \cdot x_2^2 + C = B \cdot y_2^2$	A, B = relatively prime
.....	C = dependent on A and B
$A \cdot x_n^2 + C = B \cdot y_n^2$	C, x, y = reciprocally dependent

1.2 These infinite sequences consist of an undeterminable number of groups and classes. Considering the limited space available, only main fragments of the whole issue can be dealt with here.

2.1 First, we have to determine the desired C and the least x_1 and y_1 for a given A and B.

2.2 As to C, we have to distinguish between:

a) C = 1, 2, a prime, a double prime, or a quadruple prime. Then, only one sequence exists, containing all terms possible.

b) If C is one of the remaining composite numbers, then two or more sequences exist. No term in a sequence is identical to a term in another sequence.

2.3 To determine x_2, y_2, there does not (presumably) exist a general formula, but an undeterminable number of different construction rules, according to the group or class to which the sequence belongs. When both x_1, y_1 and x_2, y_2 are found, all other terms are determined. See Section 3 below.

3.1 For x_3, y_3, x_4, y_4, ..., x_n, y_n, the following procedure leads to a recurrence formula which comprehends the whole of the terms in integers that are possible.

3.2 The following applies if: $A < B$.

3.3 Let: $x_2 - x_1 = \underline{u}$ and $y_2 - y_1 = \underline{v}$.

3.4 Divide $\underline{u}$ and $\underline{v}$ by their greatest common divisor $\underline{d}$ and let:

$$\frac{u}{d} = \underline{U} \quad \text{and} \quad \frac{v}{d} = \underline{V}.$$

$\underline{U}$, $\underline{V}$ = *auxiliary constants* relatively prime.

3.5 Let: $U \cdot y_1 - V \cdot x_1 = \underline{D}$. Now, let

$$\frac{x_1 + x_2}{D} = \underline{F} \quad \text{and} \quad \frac{y_1 + y_2}{D} = \underline{G}.$$

$\underline{F}$, $\underline{G}$ = *auxiliary constants*.

3.6 Further, let:

$$\begin{aligned} U \cdot y_1 + V \cdot x_1 &= S_1 \\ U \cdot y_2 + V \cdot x_2 &= S_2 \\ &\cdots\cdots \\ U \cdot y_n + V \cdot x_n &= S_n \end{aligned}$$

3.7 and we obtain the recurrence formula:

$$x_3 = F \cdot S_1 + x_1 \qquad y_3 = G \cdot S_1 - y_1$$
$$x_4 = F \cdot S_2 + x_2 \qquad y_4 = G \cdot S_2 - y_2$$
$$\ldots \qquad \ldots$$
$$x_n = F \cdot S_{n-2} + x_{n-2} \qquad y_n = G \cdot S_{n-2} - y_{n-2}$$

3.8 The auxiliary constants U, V and F, G hold also for any $\underline{C}$ in a sequence corresponding to A, B. That means it suffices to choose an arbitrary $\underline{C}$ (fitted to A, B) to determine U, V and F, G for any sequence A, B.

3.9 If $A > B$, the procedure is similar to that of 3.2 but is omitted here to conserve space.

3.10 Examples (for the sake of brevity and lucidity, the constants A, B, and C are listed only once, and the power "2" above x and y is omitted throughout).

3.11 Example I: $\underline{A} = 21$, $\underline{B} = 31$, $\underline{C} = 19$ (= prime, one sequence only, see 2.2a).

$\underline{21} \cdot$ 6 (x_1)	$\underline{u} = 124$	$\underline{d} = 2$ (see 3.4)	$\underline{v} = 102$	$\underline{31} \cdot$ 5 (y_1)
130 (x_2)				107 (y_2)

$\frac{u}{d} = \underline{U}$	$U \cdot y_1 - V \cdot x_1 = \underline{D}$	$\frac{v}{d} = \underline{V}$
$\frac{124}{2} = \underline{62}$	$62 \cdot 5 - 51 \cdot 6 = \underline{4}$	$\frac{102}{2} = \underline{51}$
$\frac{x_1 + x_2}{D} = \underline{F}$		$\frac{y_1 + y_2}{D} = \underline{G}$
$\frac{136}{4} = \underline{34}$		$\frac{112}{4} = \underline{28}$

20,950 (x_3)	$U \cdot y_1 + V \cdot x_1 = \underline{S_1}$	17,243 (y_3)
$= F \cdot S_1 + x_1$		$= G \cdot S_1 - y_1$
$= 34 \cdot 616 + 6$	62 5 + 51 6 = $\underline{616}$	$= 28 \cdot 616 - 5$
451,106 (x_4)	$U \cdot y_2 + V \cdot x_2 = \underline{S_2}$	371,285 (y_4)
$= F \cdot S_2 + x_2$		$= G \cdot S_2 - y_2$
$= 34 \cdot 13{,}264 + 130$	62 107 + 51 130 = $\underline{13{,}264}$	$= 28 \cdot 13{,}264 - 107$
72,696,494 (x_5)		59,833,205 (y_5)

Example II: $\underline{A} = 21$, $\underline{B} = 31$, $\underline{C} = 82$ (see 2.2a)

$\underline{21} \cdot$ 23 (x_1)	$\underline{31} \cdot$ 19
147 (x_2)	121
79,957 (x_3)	65,809
510,113 (x_4)	419,851

3.12 Example III: $A = 6$, $B = 41$, $C = 1001$ (= composite number = $7 \cdot 11 \cdot 13$ yields four different sequences, see 2.2b), x_1, x_2, x_3.

Sequence (a): x_1, x_2, x_3

6 ·	2	41 ·	5
	983,100		376,081
	1,338,320		511,969

Sequence (b)

6 ·	-80	41 ·	31
	92,002		35,195
	14,300,802		5,470,715

Sequence (c):

6 ·	248	41 ·	95
	29,850		11,419
	44,070,130		16,861,531

Sequence (d)

6 ·	2,052	41 ·	785
	3,610		1,381
	364,459,330		139,422,469

Observe:
$$80 - 2 = 13 \cdot 6$$
$$80 + 2 = 2 \cdot 41$$
$$248 - 80 = 28 \cdot 6$$
$$248 + 80 = 8 \cdot 41$$
$$\ldots$$

4. Some construction rules for x_2, y_2, based on C:

4.1 $C = 1$: $x_2 = x_1(4B \cdot y_1^2 - 1)$ $\quad y_2 = y_1(4A \cdot x_1^2 + 1)$

Example: $A = 23$, $B = 26$, $C = 1$

23 ·	185 (x_1)	x_n, y_n = see 3.7	26 ·	174 (y_1)
	582,510,055 (x_2)			587,873,974 (y_2)

4.2 $C = 2$: $x_2 = x_1(2B \cdot y_1^2 - 1)$ $\quad y_2 = y_1(2A \cdot x_1^2 + 1)$

Example: $A = 33$, $B = 107$, $C = 2$

33 ·	9 (x_1)	x_n, y_n = see 3.7	107 ·	5 (y_1)
	48,141 (x_2)			26,735 (y_2)

4.3 $C = 4$; A, B, x, y = odd

$x_2 = x_1[(B \cdot y_1^2 - 1)(B \cdot y_1^2 - 2) - 1]$ $\quad y_2 = y_1[(B \cdot y_1^2 - 2)(B \cdot y_1^2 - 3) - 1]$

Example: $A = 11$, $B = 47$, $C = 4$

11 ·	31 (x_1)	47 ·	15 (y_1)
	3,465,765,931 (x_2)		1,676,666,325 (y_2)

5. Apart from other formulas for x_2, y_2, based on other values of C, there exist those construction rules for groups (e.g., Problem H-350, and the problem based on F/L numbers I submitted in July 1982). However, this would be a field with no end, thus Problem H-354 has no general solution.

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