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CONTENTS

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The Fibonacci Quarterly

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Br. Alfred Brousseau, and I.D. Ruggles

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

SOME CONGRUENCE PROPERTIES OF GENERALIZED LUCAS INTEGRAL SEQUENCES

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1. INTRODUCTION

Let $\{L_n\}$ be a sequence on integers defined as

$$L_0 = 2,\ L_1 = 1, \text{ and } L_n = L_{n-1} + L_{n-2}, \text{ for } n \geqslant 2.$$

This is the famous Lucas sequence. In [1], Hoggatt and Bicknell proved that $L_p \equiv L_1 \pmod{p}$ if p is a prime, together with its generalization $L_{kp} \equiv L_k \pmod{p}$. It is interesting to note that these properties are not lost in generalization of the sequence. The purpose of this paper is to prove these results for generalized Lucas integral sequences defined in §2 below. One more generalization of $L_p \equiv L_1 \pmod{p}$ has also been proved. In light of these results, the sequences given in [2] have been discussed.

2. DEFINITIONS

A generalized Lucas integral sequence of order m is defined as

$$L_n = \alpha_1^n + \alpha_2^n + \cdots + \alpha_m^n, \tag{2.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the roots of the equation

$$x^m = a_1x^{m-1} + a_2x^{m-2} + \cdots + a_m \tag{2.2}$$

with integral coefficients and $a_m \neq 0$.

These L_n's are easily obtained in terms of the a_i's by Newton's well-known formula:

$$\begin{aligned} &L_0 = m,\ L_n = a_1L_{n-1} + a_2L_{n-2} + \cdots + a_{n-1}L_1 + na_n, \text{ if } n = 1, 2, \ldots, m-1, \\ &L_n = a_1L_{n-1} + a_2L_{n-2} + \cdots + a_mL_{n-m}, \text{ for } n \geqslant m. \end{aligned} \tag{2.3}$$

Equation (2.2) is called the characteristic equation of (2.3).

3. MAIN RESULTS

First, we shall prove a lemma for each theorem. The monomial symmetric functions

$$\sum \alpha_1^{t_1}\alpha_2^{t_2}\ldots\alpha_n^{t_n},$$

where $t_1, t_2, \ldots, t_n$ are integers as defined in [3]. Equation (3.1), used in the proofs of the lemmas, is given in [3].

Lemma 3.1

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the roots of (2.2). Then $\sum \alpha_1^{t_1}\alpha_2^{t_2}\ldots\alpha_n^{t_n}$, with different indices for α's, is an integer.

Proof: We prove the lemma by mathematical induction on n. Since

$$\sum \alpha_1^{t_1} = \alpha_1^{t_1} + \alpha_2^{t_1} + \cdots + \alpha_m^{t_1} = L_{t_1},$$

an integer, therefore, the lemma is true for $n = 1$. Suppose the lemma is true for $n = s - 1$. As all the indices for α's are different, we have:

$$\begin{aligned}\left(\sum \alpha_1^{t_1}\right)&\left(\sum \alpha_1^{t_2}\alpha_2^{t_3} \ldots \alpha_{s-1}^{t_s}\right)\\ &= \sum \alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_s^{t_s} + \sum \alpha_1^{t_2+t_1}\alpha_2^{t_3} \ldots \alpha_{s-1}^{t_s}\\ &\quad + \sum \alpha_1^{t_2}\alpha_2^{t_3+t_1} \ldots \alpha_{s-1}^{t_s} + \cdots + \sum \alpha_1^{t_2}\alpha_2^{t_3} \ldots \alpha_{s-1}^{t_s+t_1}\end{aligned} \tag{3.1}$$

Using the induction hypothesis and the fact that $\sum \alpha_1^{t_1}$ is an integer, we find that

$$\sum \alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_s^{t_s}$$

is an integer; i.e., the lemma is true for $n = s$. So, by induction, the lemma is completely proved.

Theorem 3.1

Let $\{L_n\}$ be a generalized Lucas integral sequence and p be a prime number. Then

$$L_p \equiv L_1 \pmod{p}.$$

Proof: By using the multinomial theorem, we have

$$(\alpha_1 + \alpha_2 + \cdots + \alpha_m)^p = \sum \frac{p!}{t_1!t_2! \ldots t_m!}\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_m^{t_m}, \tag{3.2}$$

where $t_1, t_2, \ldots, t_m$ are nonnegative integers such that $t_1 + t_2 + \cdots + t_m = p$ and all indices of α's are different.

From (3.2), we have

$$\begin{aligned}(\alpha_1 &+ \alpha_2 + \cdots + \alpha_m)^p\\ &= \alpha_1^p + \alpha_2^p + \cdots + \alpha_m^p + \sum \frac{p!}{t_1! \ldots t_m!}\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_m^{t_m},\end{aligned} \tag{3.3}$$

with the above conditions on t_i's and no $t_i = p$. With these conditions on the t_i's, we have that

$$\frac{p!}{t_1! \ldots t_m!}$$

is an integral multiple of p. Since for each set of possible values of $t_1, t_2, \ldots, t_m$ all $\sum \alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_m^{t_m}$'s are integers, by our Lemma 3.1 we have, from (3.3) and (2.1),

$$(L_1)^p = L_p + p\lambda, \text{ where } \lambda \text{ is an integer.}$$

Using Fermat's little theorem, we get

$$L_p \equiv L_1 \pmod{p}.$$

This completes the proof of Theorem 3.1.

Lemma 3.2

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the roots of (2.2). Then, for different indices of α's, $\sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_n^{t_n})^k$ is an integer for every positive integer k.

Proof: Simply write kt_i for t_i everywhere in the proof of Lemma 3.1.

Theorem 3.2

Let $\{L_n\}$ be a generalized Lucas integral sequence and p be a prime number. Then, for every positive integer k,

$$L_{pk} \equiv L_k \pmod p.$$

Proof: $(\alpha_1^k + \alpha_2^k + \cdots + \alpha_m^k)^p$

$$= \alpha_1^{pk} + \alpha_2^{pk} + \cdots + \alpha_m^{pk} + \sum \frac{p!}{t_1!t_2! \ldots t_m!}(\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_m^{t_m})^k.$$

$\dfrac{p!}{t_1! \ldots t_m!}$ is a multiple of p and $\sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_m^{t_m})$ is an integer for every given set of values of $t_1, \ldots, t_m$ by Lemma 3.2. Therefore,

$$(L_k)^p = L_{pk} + p\lambda_1, \text{ where } \lambda_1 \text{ is an integer}$$

or $L_{pk} \equiv L_k \pmod p$, by Fermat's little theorem,

$$L_k^p \equiv L_k \pmod p.$$

Lemma 3.3

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the roots of (2.2). Then, for different indices of α's,

$$\sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_n^{t_n})^{kp^r} \equiv \sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_n^{t_n})^{kp^{r-1}} \pmod{p^r}.$$

Proof: We shall prove the lemma by induction on r. In order to prove the lemma for $r = 1$, we have to prove

$$\sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_n^{t_n})^{kp} \equiv \sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_n^{t_n})^{k} \pmod p. \tag{3.4}$$

The congruence (3.4) may be proved by induction on n. Since

$$\sum (\alpha_1^{t_1})^{kp} - \sum (\alpha_1^{t_1})^k = L_{t_1kp} - L_{t_1k}$$
$$\equiv 0 \pmod p \quad \text{(by Theorem 3.2)},$$

or

$$\sum (\alpha_1^{t_1})^{kp} \equiv \sum (\alpha_1^{t_1})^k \pmod p. \tag{3.5}$$

Therefore, (3.4) is true for $n = 1$. Consider the equation

$$\left(\sum \alpha_1^{t_1kp}\right)\left(\sum (\alpha_1^{t_2}\alpha_2^{t_3} \ldots \alpha_{s-1}^{t_s})^{kp}\right)$$
$$= \sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_s^{t_s})^{kp} + \sum (\alpha_1^{t_2+t_1}\alpha_2^{t_3} \ldots \alpha_{s-1}^{t_s})^{kp}$$
$$+ \sum (\alpha_1^{t_2}\alpha_2^{t_3+t_1} \ldots \alpha_{s-1}^{t_s})^{kp} + \cdots + \sum (\alpha_1^{t_2}\alpha_2^{t_3} \ldots \alpha_{s-1}^{t_s+t_1})^{kp}.$$

Using the induction hypothesis and (3.5), we have

$$\left(\sum \alpha_1^{t_1 k}\right)\left(\sum (\alpha_1^{t_2}\alpha_2^{t_3} \ldots \alpha_{s-1}^{t_s})^k\right)$$

$$\equiv \sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_s^{t_s})^{kp} + \sum (\alpha_1^{(t_2+t_1)}\alpha_2^{t_3} \ldots \alpha_{s-1}^{t_s})^k$$

$$+ \sum (\alpha_1^{t_2}\alpha_2^{t_3+t_1} \ldots \alpha_{s-1}^{t_s})^k + \cdots + \sum (\alpha_1^{t_2}\alpha_2^{t_3} \ldots \alpha_{s-1}^{t_s+t_1})^k \pmod{p}$$

or

$$\sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_s^{t_s})^{kp} \equiv \sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_s^{t_s})^k \pmod{p}.$$

This proves that (3.4) is true for $n = s$. Thus, induction completes the proof of (3.4).

Next, we suppose that our lemma is true for $r = s$. That is,

$$\sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_n^{t_n})^{kp^s} \equiv \sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_n^{t_n})^{kp^{s-1}} \pmod{p^s}$$

or

$$\lambda_1^{kp^s} + \cdots + \lambda_q^{kp^s} \equiv \lambda_1^{kp^{s-1}} + \cdots + \lambda_q^{kp^{s-1}} \pmod{p^s},$$

where q is the number of terms in the expansion of

$$\sum \alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_n^{t_n}$$

and each λ is the product of powers of the α's. Therefore,

$$(\lambda_1^{kp^s} + \cdots + \lambda_q^{kp^s})^p \equiv (\lambda_1^{kp^{s-1}} + \cdots + \lambda_q^{kp^{s-1}})^p \pmod{p^{s+1}}$$

or

$$\lambda_1^{kp^{s+1}} + \cdots + \lambda_q^{kp^{s+1}} + \sum \frac{p!}{\mu_1! \ldots \mu_q!}(\lambda_1^{\mu_1}\lambda_2^{\mu_2} \ldots \lambda_q^{\mu_q})^{kp^s}$$

$$\equiv \lambda_1^{kp^s} + \cdots + \lambda_q^{kp^s} + \sum \frac{p!}{\mu_1! \ldots \mu_q!}(\lambda_1^{\mu_1}\lambda_2^{\mu_2} \ldots \lambda_q^{\mu_q})^{kp^{s-1}}.$$

Since $\dfrac{p!}{\mu_1! \ldots \mu_q!}$ is a multiple of p and

$$\sum (\lambda_1^{\mu_1}\lambda_2^{\mu_2} \ldots \lambda_q^{\mu_q})^{kp^s} \equiv \sum (\lambda_1^{\mu_1}\lambda_2^{\mu_2} \ldots \lambda_q^{\mu_q})^{kp^{s-1}} \pmod{p^s}$$

by the induction hypothesis, we have

$$\lambda_1^{kp^{s+1}} + \cdots + \lambda_q^{kp^{s+1}} \equiv \lambda_1^{kp^s} + \cdots + \lambda_q^{kp^s} \pmod{p^{s+1}}$$

or

$$\sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_n^{t_n})^{kp^{s+1}} \equiv \sum (\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_n^{t_n})^{kp^s} \pmod{p^{s+1}},$$

which shows that the lemma is true for $r = s + 1$. Thus, the lemma is proved completely by induction.

Theorem 3.3

Let $\{L_n\}$ be a generalized Lucas integral sequence and p be a prime number. Then, for positive integers k and r,

$$L_{kp^r} \equiv L_{kp^{r-1}} \pmod{p^r}.$$

Proof: The proof will be given by induction on r. By Theorem 3.2,

$$L_{kp} \equiv L_k \pmod{p}.$$

This proves the theorem for $r = 1$.

Suppose that the theorem is true for $r = s - 1$, i.e.,

$$L_{kp^{s-1}} \equiv L_{kp^{s-2}} \pmod{p^{s-1}},$$

which implies

$$L^p_{kp^{s-1}} \equiv L^p_{kp^{s-2}} \pmod{p^s}. \tag{3.6}$$

Now,

$$(\alpha_1^{kp^{s-1}} + \cdots + \alpha_m^{kp^{s-1}})^p = \alpha_1^{kp^s} + \cdots + \alpha_m^{kp^s} + \sum \frac{p!}{t_1! \ldots t_m!}(\alpha_1^{t_1} \ldots \alpha_m^{t_m})^{kp^{s-1}}$$

or

$$L^p_{kp^{s-1}} = L_{kp^s} + \sum \frac{p!}{t_1! \ldots t_m!}(\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_m^{t_m})^{kp^{s-1}}.$$

Similarly,

$$L^p_{kp^{s-2}} = L_{kp^{s-1}} + \sum \frac{p!}{t_1! \ldots t_m!}(\alpha_1^{t_1}\alpha_2^{t_2} \ldots \alpha_m^{t_m})^{kp^{s-2}}.$$

On subtracting, we get

$$L^p_{kp^{s-1}} - L^p_{kp^{s-2}}$$

$$= L_{kp^s} - L_{kp^{s-1}} + \sum \frac{p!}{t_1! \ldots t_m!}[(\alpha_1^{t_1} \ldots \alpha_m^{t_m})^{kp^{s-1}} - (\alpha_1^{t_1} \ldots \alpha_m^{t_m})^{kp^{s-2}}].$$

Using (3.6), $\dfrac{p!}{t_1! \ldots t_m!}$ is a multiple of p, and Lemma 3.3, we have

$$L_{kp^s} \equiv L_{kp^{s-1}} \pmod{p^s},$$

which shows that the theorem is true for $r = s$. Therefore, the theorem is completely proved by induction.

Note: Theorem 3.3 is a generalization of our previous theorems. The beauty of this theorem is that multiplying the index of each term of the difference

$$L_{kp^r} - L_{kp^{r-1}}$$

by p produces one more factor p to the new difference. It is observed that

$$L_{kp^s} \not\equiv L_{kp^{s-1}} \pmod{p^{s+1}}$$

in most of the cases. In some cases, there exist primes where this incongruence relation fails. For example, we take the sequence

$$L_0 = 3,\ L_1 = 1,\ L_2 = 5, \text{ and } L_n = L_{n-1} + 2L_{n-2} + L_{n-3}, \text{ for } n \geqslant 3.$$

Writing a few initial terms of the sequence,

$$3, 1, 5, 10, 21, 46, \ldots,$$

we find that there exist primes 2 and 3 such that

$$L_2 \equiv L_1 \pmod{4} \quad\text{and}\quad L_3 \equiv L_1 \pmod{9}.$$

4. SEQUENCES WHERE $p|L_p$ FOR EVERY PRIME p

Sequences of this type have been considered in [2]. First, let us prove the following simple theorem.

Theorem 4.1

Let $\{L_n\}$ be a generalized Lucas integral sequence. Then, for every prime p, $p|L_p \iff L_1 = 0$.

Proof: Suppose $L_1 = 0$. Therefore, by Theorem 3.1,

$$L_p \equiv 0 \pmod{p}, \text{ i.e., } p|L_p \text{ for every prime } p.$$

Conversely, suppose $p|L_p$ for every prime p. We find, again from Theorem 3.1,

$$L_1 \equiv 0 \pmod{p} \text{ for every prime } p.$$

This implies that $L_1 = 0$. Hence, the theorem is proved.

Note: In light of this theorem, we conclude that for making such sequences we need $L_1 = 0$. Ensuring the right start as pointed out in [2] is not needed. As a matter of fact, this right start is a consequence of $L_1 = 0$. Moreover, it will be an appropriate place to point out a shortcoming in Lehmer's proof presented in [2]. He first takes integers x, y, z, and t, and then allows $x = \alpha$, $y = \beta$, $z = \gamma$, and $t = \delta$, which are not integers because α, β, γ, and δ are the roots of the characteristic equation $x^4 = 2x^2 + 2x + 1$. Consequently, one cannot argue that $F_p(x, y, z, t)$ is an integer implies $F_p(\alpha, \beta, \gamma, \delta)$ is also an integer. In fact, $F_p(\alpha, \beta, \gamma, \delta)$ is an integer, as we see in our Theorem 3.1, with the help of Lemma 3.1.

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ON WEIGHTED STIRLING AND OTHER RELATED NUMBERS AND SOME COMBINATORIAL APPLICATIONS

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1. INTRODUCTION

The signless (absolute) Stirling numbers of the first kind

$$S_1(m, n) = (-1)^{m-n} s(m, n)$$

and the Stirling numbers of the second kind

$$S(m, n)$$

may be defined by

$$S_1(m, n) = (-1)^{m-n}\frac{1}{n!}[D^n(x)_m]_{x=0}, \quad S(m, n) = \frac{1}{n!}[\Delta^n x^m]_{x=0},$$

where $(x)_m = x(x-1)\ \dots\ (x-m+1)$ denotes the falling factorial of degree m, D the differential operator, and Δ the difference operator. The numbers

$$C(m, n, r) = \frac{1}{n!}[\Delta^n(rx)_m]_{x=0}, \ r \text{ a real number},$$

which first arose as coefficients in the n-fold convolution of zero-truncated binomial (with r a positive integer) and negative binomial (with r a negative integer) distributions (see [1]) and have subsequently been studied systematically by the present author in [6], [7], and [8], are closely related to the Stirling numbers. This was the reason why Carlitz in [2] called the numbers

$$S_1(m, n|\lambda) = (-1)^{m-n}\lambda^{-n}C(m, n, \lambda), \quad S(m, n|\lambda) = \lambda^m C(m, n, \lambda^{-1})$$

degenerate Stirling numbers of the first and second kind, respectively.

Recently, Carlitz introduced and studied in [3] and [4] weighted Stirling numbers $\bar{S}_1(m, n, \lambda)$ and $\bar{S}(m, n, \lambda)$ by considering suitable combinatorial interpretations of $S_1(m, n)$ and $S(m, n)$, respectively. Several properties of these numbers and the related numbers

$$R_1(m, n, \lambda) = \bar{S}_1(m, n+1, \lambda) + S_1(m, n),$$

and

$$R(m, n, \lambda) = S(m, n+1, \lambda) + S(m, n)$$

were obtained.

In the present paper, by considering suitable combinatorial interpretations of the number $C(m, n, r)$ when r is a positive or negative integer, we introduce the weighted C-number, $\bar{C}(m, n; r, s)$, with r an integer and s a real number. Certain properties of these numbers are obtained in §2.

The related numbers

$$G(m, n; r, s) = \bar{C}(m, n+1; r, s) + C(m, n, r)$$

are shown to be equal to

$$G(m, n; r, s) = \frac{1}{n!}[\Delta^n(rx+s)_m]_{x=0}.$$

These numbers have been systematically studied in [9]. A representation of

$$G(m,\ m - n;\ r,\ s)$$

as the sum of binomial coefficients is obtained and certain properties of

$$G_m(r,\ s) = \sum_{n=0}^{m} G(m,\ n;\ r,\ s)$$

are derived in §3.

Combinatorial applications of the numbers

$$R_1(m,\ n,\ \lambda),\quad R(m,\ n,\ \lambda),\ \text{and}\ G(m,\ n;\ r,\ s)$$

are discussed in §4.

2. THE NUMBERS $\bar{C}(m,\ n;\ r,\ s)$

The C-numbers

$$C(m,\ n,\ r) = \frac{1}{n!}[\Delta^n (rx)_m]_{x=0}$$

may be expressed in the form (see [7]):

$$C(m,\ n,\ r) = \frac{m!}{n!} \sum_{\pi(m,\ n)} C(m;\ k_1,\ k_2,\ \ldots,\ k_m;\ r), \tag{2.1}$$

where

$$C(m;\ k_1,\ k_2,\ \ldots,\ k_m;\ r) = \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1!\ k_2!\ \ldots\ k_m!}\binom{r}{1}^{k_1}\binom{r}{2}^{k_2} \cdots \binom{r}{m}^{k_m} \tag{2.2}$$

and the summation is over all partitions $\pi(m,\ n)$ of m in n parts, that is, all nonnegative integer solutions $(k_1,\ k_2,\ \ldots,\ k_m)$ of the equations

$$k_1 + 2k_2 + \cdots + mk_m = m,\quad k_1 + k_2 + \cdots + k_m = n. \tag{2.3}$$

Note that $C(m;\ k_1,\ k_2,\ \ldots,\ k_m;\ r)$, r a positive integer, is a distribution of (number of ways of putting) m like balls into $k_1 + k_2 + \cdots + k_m$ different cells, each of which has r different compartments of capacity limited to one ball, such that k_j cells contain exactly j balls each, $j = 1,\ 2,\ \ldots,\ m$. When the capacity of each cell is unlimited, the corresponding number is equal to

$$|C(m;\ k_1,\ k_2,\ \ldots,\ k_m;\ -r)| = (-1)^m C(m;\ k_1,\ k_2,\ \ldots,\ k_m;\ -r),$$

where r is a positive integer.

The expression (2.1) leads to the following combinatorial interpretations of the C-numbers:

$$\frac{m!}{n!}\ C(m,\ n,\ r),\ r \text{ a positive integer},$$

is the number of ways of putting m like balls into n different cells, each of which has r different compartments of capacity limited to one ball, with no cell empty. When the capacity of each compartment is unlimited, the corresponding number is equal to

$$\frac{m!}{n!}|C(m,\ n,\ -r)| = (-1)^m \frac{n!}{m!} C(m,\ n,\ -r),\ r \text{ a positive integer}.$$

Consider the weighted number of distributions

$$C(m;\, k_1,\, k_2,\, \ldots,\, k_m;\, r,\, s)$$
$$= \frac{m!}{(k_1 + k_2 + \cdots + k_m)!} \sum (k_1 w_1 + k_2 w_2 + \cdots + k_m w_m) \tag{2.4}$$

where the weights

$$w_j = w_j(r,\, s) = (s)_j/(r)_j,\; j = 1,\, 2,\, \ldots,\, m,\; r \text{ a positive integer, } s \text{ a real number,}$$

and the summation is over all distributions of m like balls into $k_1 + k_2 + \cdots + k_m$ different cells, each of which has r different compartments of capacity limited to one ball, such that k_j cells contain exactly j balls each, $j = 1, 2, \ldots, m$, and

$$\overline{C}(m;\, k_1,\, k_2,\, \ldots,\, k_m;\, -r,\, s)$$
$$= \frac{m!}{(k_1 + k_2 + \cdots + k_m)!} \sum (k_1 v_1 + k_2 v_2 + \cdots + k_m v_m) \tag{2.5}$$

where the weights

$$v_j = v_j(-r,\, s) = (s)_j/(-r)_j,\; j = 1,\, 2,\, \ldots,\, m,\; r \text{ a positive integer, } s \text{ a real number,}$$

and the summation is over all distributions of m like balls into $k_1 + k_2 + \cdots + k_m$ different cells, each of which has r different compartments of unlimited capacity, such that k_j cells contain exactly j balls each, $j = 1, 2, \ldots, m$.

Let

$$\overline{C}(m,\, n;\, r,\, s) = \sum_{\pi(m,\, n)} \overline{C}(m;\, k_1,\, k_2,\, \ldots,\, k\;;\, r,\, s),\; r \text{ an integer, } s \text{ a real number,} \tag{2.6}$$

where the summation is over all partitions $\pi(m, n)$ of m in n parts. The numbers

$$C(m,\, n;\, r,\, s) = \frac{1}{n}\,\overline{C}(m,\, n;\, r,\, s) \tag{2.7}$$

may be called *weighted C-numbers*.

Putting $s = r$ in (2.4) and (2.6), with $w_j = 1$, $j = 1, 2, \ldots, m$, we obtain

$$C(m,\, n;\, r,\, r) = C(m,\, n,\, r), \tag{2.8}$$

while putting $s = -r$ in (2.5) and (2.6), with $v_j = 1$, $j = 1, 2, \ldots, m$, we get

$$(-1)^m C(m,\, n;\, -r,\, -r) = (-1)^m C(m,\, n,\, -r) = |C(m,\, n,\, -r)|. \tag{2.9}$$

Now consider the generating function

$$\overline{F}(t,\, u_1,\, u_2,\, \ldots;\, r,\, s) = \sum_{m=0}^{\infty} \sum_{\pi(m)} \overline{C}(m;\, k_1,\, k_2,\, \ldots,\, k_m;\, r,\, s)\, \frac{t^m}{m!} u_1^{k_1} u_2^{k_2} \ldots\, u_m^{k_m},$$
$$r \text{ an integer, } s \text{ a real number,}$$

where the inner summation is over all partitions $\pi(m)$ of m, that is, over all nonnegative integer solutions $(k_1, k_2, \ldots, k_m)$ of the equation

$$k_1 + 2k_2 + \cdots + mk_m = m.$$

Using (2.4) when r is a positive integer and (2.5) when r is a negative integer, we get

$$\bar{F}(t, u_1, u_2, \ldots; r, s)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\pi(m)} (k_1 w_1 + k_2 w_2 + \cdots + k_m w_m) \frac{m!}{k_1! k_2! \ldots k_m!} \left[\binom{r}{1} u_1 t\right]^{k_1} \left[\binom{r}{2} u_2 t\right]^{k_2} \cdots \left[\binom{r}{m} u_m t\right]^{k_m}$$

$$= \left\{\binom{s}{1} u_1 t + \binom{s}{2} u_2 t^2 + \cdots + \binom{s}{m} u_m t^m + \cdots\right\} \exp\left\{\binom{r}{1} u_1 t + \binom{r}{2} u_2 t^2 + \cdots + \binom{r}{m} u_m t^m + \cdots\right\}.$$

The generating function

$$\begin{aligned} \bar{F}(t, u; r, s) &= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \bar{C}(m, n; r, s) \frac{t^m}{m!} u^n \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \bar{C}(m, n; r, s) \frac{t^m}{m!} u^n \end{aligned} \tag{2.10}$$

may be obtained from $\bar{F}(t, u_1, u_2, \ldots; r, s)$ by putting $u_j = u$, $j = 1, 2, \ldots$. We get

$$\bar{F}(t, u; r, s) = u[(1 + t)^s - 1]\exp\{u[(1 + t)^r - 1]\}, \tag{2.11}$$

and

$$\bar{f}\,(t; r, s) = \sum_{m=n}^{\infty} \bar{C}(m, n; r, s) \frac{t^m}{m!} = \frac{1}{(n-1)!}[(1+t)^s - 1][(1+t)^r - 1]^{n-1}. \tag{2.12}$$

The corresponding generating function of the usual C-numbers is ([7]):

$$f_n(t; r) = \sum_{m=n}^{\infty} C(m, n, r) \frac{t^m}{m!} = \frac{1}{n!}[(1 + t)^r - 1]^n. \tag{2.13}$$

Since

$$\bar{f}_n(t; r, s) = [(1 + t)^s - 1] f_{n-1}(t; r),$$

we find

$$\bar{C}(m, n; r, s) = \sum_{j=1}^{m-n+1} \binom{m}{j} (s)_j \, C(m - j, n - 1, r). \tag{2.14}$$

Note that (2.12) for $s = r$ reduces to

$$\bar{f}_n(t; r, s) = n f_n(t; r),$$

which implies (2.8) and (2.9).

Using the relation ([7]),

$$(s)_j = \sum_{k=1}^{j} C(j, k, r)(s/r)_k.$$

(2.14) may be written as

$$\begin{aligned} \bar{C}(m, n; r, s) &= \sum_{j=1}^{m-n+1} \binom{m}{j} \left\{\sum_{k=1}^{j} C(j, k, r)(s/r)_k\right\} C(m - j, n - 1, r) \\ &= \sum_{k=1}^{m-n-1} \left\{\sum_{j=k}^{m} \binom{m}{j} C(j, k, r) C(m - j, n - 1, r)\right\} (s/r)_k. \end{aligned}$$

From (2.13), we have

$$\binom{k+n}{k} f_{k+n}(t;\ r) = f_k(t;\ r) f_n(t;\ r),$$

which implies

$$\binom{k+n}{k} C(m,\ k+n,\ r) = \sum_{j=k}^{m} \binom{m}{j} C(j,\ k,\ r) C(m-j,\ n,\ r).$$

Therefore,

$$\overline{C}(m,\ n;\ r,\ s) = \sum_{k=1}^{m-n-1} \binom{n+k-1}{k} C(m,\ n+k-1,\ r)(s/r)_k. \tag{2.15}$$

Using the generating functions (see [3]),

$$\overline{g}_n(t,\ \lambda) = \sum_{m=n}^{\infty} \overline{S}_1(m,\ n,\ \lambda)\frac{t^m}{m!} = \frac{1}{(n-1)!}[(1-t)^{-\lambda} - 1][-\log(1-t)]^{n-1}, \tag{2.16}$$

and

$$h_n(t) = \sum_{m=n}^{\infty} S(m,\ n)\frac{t^m}{m!} = \frac{1}{n!}(e^t - 1)^n.$$

(2.12) may be expressed as

$$\begin{aligned}\overline{f}_n(t;\ r,\ s) &= \sum_{m=n}^{\infty} \overline{C}(m,\ n;\ r,\ s)\frac{t^m}{m!} = \frac{1}{(n-1)!}[(1+t)^s - 1][e^{r\log(1+t)} - 1]^{n-1} \\ &= \sum_{k=n}^{\infty} S(k-1,\ n-1)r^{k-1}\left\{\frac{1}{(k-1)!}[(1+t)^s - 1][\log(1+t)]^{k-1}\right\} \\ &= \sum_{k=n}^{\infty} r^{k-1}S(k-1,\ n-1)\sum_{m=n}^{\infty}(-1)^{m-k-1}\overline{S}_1(m,\ k,\ -s)\frac{t^m}{m!} \\ &= \sum_{m=n}^{\infty}\left\{\sum_{k=n}^{m}(-1)^{m-k-1}r^{k-1}\overline{S}_1(m,\ k,\ -s)S(k-1,\ n-1)\right\}\frac{t^m}{m!};\end{aligned}$$

hence,

$$\overline{C}(m,\ n;\ r,\ s) = \sum_{k=n}^{m}(-1)^{m-k+1}r^{k-1}\overline{S}_1(m,\ k,\ -s)S(k-1,\ n-1). \tag{2.17}$$

Again from (2.12) we have

$$\lim_{r\to 0} r^{-n+1}\overline{f}_n(t;\ r,\ s) = \frac{1}{(n-1)!}[(1+t)^s - 1][\log(1+t)]^{n-1}$$

and

$$\lim_{r\to\infty} \overline{f}_n(t/r;\ r,\ s) = \frac{1}{(n-1)!}(e^{\lambda t} - 1)(e^t - 1)^{n-1}, \text{ if } \lim_{r\to\infty}\frac{s}{r} = \lambda,$$

which, by virtue of the generating functions of the weighted Stirling numbers, (2.16), and (see [3])

$$\overline{h}(t,\ \lambda) = \sum_{m=n}^{\infty} S(m,\ n,\ \lambda)\frac{t^m}{m!} = \frac{1}{(n-1)!}(e^{\lambda t} - 1)(e^t - 1)^{n-1}, \tag{2.18}$$

imply

$$\lim_{r\to 0} r^{-n+1}\overline{C}(m,\ n;\ r,\ s) = (-1)^{m-n+1}S_1(m,\ n,\ -s) \tag{2.19}$$

and

$$\lim_{r\to\infty} r^{-m}\overline{C}(m,\ n;\ r,\ s) = S(m,\ n,\ \lambda), \text{ if } \lim_{r\to\infty}\frac{s}{r} = \lambda, \tag{2.20}$$

respectively.

3. THE NUMBERS $G(m, n; r, s)$

Let

$$G(m, n; r, s) = \bar{C}(m, n+1; r, s) + C(m, n, r). \tag{3.1}$$

Then (2.14) implies

$$G(m, n; r, s) = \sum_{j=0}^{m-n} \binom{m}{j} (s)_j C(m-j, n, r). \tag{3.2}$$

Since

$$C(m, n, r) = \frac{1}{n!}[\Delta^n (rx)_m]_{x=0}, \quad n = 0, 1, 2, \ldots, m, \ m = 0, 1, 2, \ldots$$

and

$$C(m, n, r) = 0 \text{ for } m < n,$$

it follows that

$$G(m, n; r, s) = \sum_{j=0}^{m} \binom{m}{j} (s)_j C(m-j, n, r) = \frac{1}{n!}\Delta^n \left[\sum_{j=0}^{m} \binom{m}{j} (s)_j (rx)_{m-j} \right]_{x=0}$$

and, by virtue of Vandermonde's convolution formula,

$$G(m, n; r, s) = \frac{1}{n!}[\Delta^n (rx+s)_m]_{x=0} = \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (rk+s)_m .$$

These numbers, shown as coefficients in a generalization of the Hermite polynomials considered by Gould and Hopper, were systematically studied in [9]. A representation of $G(m, m-n; r, s)$ as the sum of binomial coefficients will be discussed here.

The numbers $G(m, n; r, s)$ satisfy the triangular recurrence relation

$$G(m+1, n; r, s) = (rn + s - m)G(m, n; r, s) + rG(m, n-1, r) \tag{3.3}$$

with initial conditions

$$G(0, n; r, s) = \delta_{0n}, \ G(m, 0; r, s) = (s)_m, \text{ and } G(m, n; r, s) = 0 \text{ for } m < n.$$

Putting $n = m + 1$, we get

$$G(m+1, m+1; r, s) = rG(m, m; r, s), \ m = 0, 1, 2, \ldots$$

and

$$G(m, m; r, s) = r^m. \tag{3.4}$$

If we put $n = 1$ in (3.3), we find

$$G(m+1, 1; r, s) = (r + s - m)G(m, 1; r, s) + r(s)_m$$

and, in particular,

$$G(2, 1; r, s) = (r + s - 1)r + rs = r(r + 2s - 1).$$

Again, if we put $n = m - k + 1$ in (3.3), we obtain

$$\begin{aligned} &G(m+1, m+1-k; r, s) - rG(m, m-k; r, s) \\ &= [r(m-k+1) + s - m]G(m, m-k+1; r, s) \end{aligned}$$

or

$$\begin{aligned} &\Delta_m r^{-m+k} G(m, m-k; r, s) \\ &= r^{-m+k-1}[(r-1)m - r(k-1) + s]G(m, m-k+1; r, s). \end{aligned} \tag{3.5}$$

For $k = 1$, we have

$$\Delta_m r^{-m+1} G(m, m-1; r, s) = (r-1)m + s$$

and

$$r^{-m+1}G(m,\ m-1;\ r,\ s) = \Delta_m^{-1}[(r-1)m+s] = (r-1)\binom{m}{2} + s\binom{m}{1} + K.$$

Since $G(2,\ 1;\ r,\ s) = r(r+2s-1)$, $K=0$, and

$$r^{-m+1}G(m,\ m-1;\ r,\ s) = (r-1)\binom{m}{2} + s\binom{m}{1}. \tag{3.6}$$

Taking $k = 2$ in (3.5), we get

$$r^{-m+2}G(m,\ m-2;\ r,\ s) = \Delta_m^{-1}\left\{[(r-1)m+s-r]\left[(r-1)\binom{m}{2} + s\binom{m}{1}\right]\right\},$$

which on using the relations

$$\Delta^{-1}\binom{m}{j} = \binom{m}{j+1},$$

$$\Delta_m^{-1}\left\{m\binom{m}{j}\right\} = m\binom{m}{j+1} - \binom{m+1}{j+2} = (j+1)\binom{m}{j+2} + j\binom{m}{j+1},$$

gives

$$r^{-m+2}G(m,\ m-2;\ r,\ s) = 3(r-1)^2\binom{m}{4} + (r-1)(r+3s-2)\binom{m}{3} + s(s-1)\binom{m}{2}.$$

Hence, $r^{-m+2}G(m,\ m-2;\ r,\ s)$ is a polynomial of m of degree 4. Consequently, $r^{-m+n}G(m,\ n-n;\ r,\ s)$ will be a polynomial of m of degree $2n$. Let us write it as follows:

$$r^{-m+n}G(m,\ m-n;\ r,\ s) = \sum_{k=0}^{2n} H(n,\ k;\ r,\ s)\binom{m}{2n-k}.$$

Multiplying both numbers by $[(r-1)m - rn + s]$ and using (3.5), we have

$$\Delta_m r^{-m+n+1}G(m,\ m-n-1;\ r,\ s) = \sum_{k=0}^{2n} H(n,\ k;\ r,\ s)[(r-1)m - rn + s]\binom{m}{2n-k},$$

and since

$$\Delta_m^{-1}[(r-1)m - rn + s]\binom{m}{2n-k}$$

$$= (r-1)(2n-k+1)\binom{m}{2n-k+2} + [(r-1)(n-k) - n + s]\binom{m}{2n-k+1},$$

we get

$$r^{-m+n+1}G(m,\ m-n-1;\ r,\ s)$$

$$= \sum_{k=0}^{2n}(2n-k+1)(r-1)H(n,\ k;\ r,\ s)\binom{m}{2n-k+2}$$

$$+ \sum_{k=0}^{2n}[(r-1)(n-k) - n + s]H(n,\ k;\ r,\ s)\binom{m}{2n-k+1} + K$$

and

$$\sum_{k=0}^{2n+2} H(n+1,\ k;\ r,\ s)\binom{m}{2n-k+2}$$

$$= \sum_{k=0}^{2n}(2n-k+1)(r-1)H(n,\ k;\ r,\ s)\binom{m}{2n-k+2}$$

$$+ \sum_{k=1}^{2n+1}[(n-k+1)(r-1) - n + s]H(n,\ k-1;\ r,\ s)\binom{m}{2n-k+2} + K.$$

Therefore,

$$H(n+1,\ k;\ r,\ s) = (2n - k + 1)(r - 1)H(n,\ k;\ r,\ s) + [(n - k + 1)(r - 1) - n + s]H(n,\ k - 1;\ r,\ s) \tag{3.7}$$

and

$$H(n+1,\ 2n+2;\ r,\ s) = K.$$

From (3.6), it follows that

$H(1,\ 0;\ r,\ s) = r - 1$, $H(1,\ 1;\ r,\ s) = s$, and $H(1,\ k;\ r,\ s) = 0$ for $k > 1$.

Putting successively $n = 1, 2, \ldots$ in (3.7), we conclude that

$$H(n,\ k;\ r,\ s) = 0 \text{ if } k > n,$$

and hence,

$$r^{-m+n}G(m,\ m - n;\ r,\ s) = \sum_{k=0}^{n} H(n,\ k;\ r,\ s)\binom{m}{2n - k}. \tag{3.8}$$

Using (3.7), we may easily deduce that

$$H(n,\ n;\ r,\ s) = (s)_n,\ n = 1, 2, \ldots, \tag{3.9}$$

and

$$H(n,\ 0;\ r,\ s) = (r - 1)^n 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1) = (r - 1)^n \frac{(2n)!}{n!2^n} \tag{3.10}$$

Moreover, for

$$S_n(r,\ s) = \sum_{k=0}^{n} (-1)^{n-k} H(n,\ k;\ r,\ s)$$

we get

$$S_n(r,\ s) = [(s - r + 1) - r(n - 1)]S_{n-1}(r,\ s),\ n = 2, 3, \ldots,$$

and

$$S_1(r,\ s) = -H(1,\ 0;\ r,\ s) + H(1,\ 1;\ r,\ s) = s - r + 1.$$

Therefore,

$$S_n(r,\ s) = \sum_{k=0}^{n} (-1)^{n-k} H(n,\ k;\ r,\ s) = r^n\left(\frac{s - r + 1}{r}\right)_n. \tag{3.11}$$

Multiplying both members of (3.8) by $(-1)^{m+j}\binom{2n - j}{m}$ and summing for $m = n$, $n + 1, \ldots, 2n - j$, we obtain the relation

$$H(n,\ j;\ r,\ s) = \sum_{m=n}^{2n-j} (-1)^{m+j}\binom{2n - j}{m} r^{-m+n}G(m,\ m - n;\ r,\ s), \tag{3.12}$$

which leads to interesting combinatorial interpretations for these numbers or, more precisely, for the numbers

$$\begin{aligned} G_2(m,\ n;\ r,\ s) &= r^n H(m - n,\ m - 2n;\ r,\ s) \\ &= \sum_{k=0}^{n} (-1)^k \binom{m}{k} r^k G(m - k,\ n - k;\ r,\ s). \end{aligned} \tag{3.13}$$

Since (see [9])

$$\sum_{m=n}^{\infty} G(m,\ n;\ r,\ s)\frac{t^m}{m!} = \frac{1}{n!}(1 + t)^s[(1 + t)^r - 1]^n$$

it follows that

$$\sum_{m=n}^{\infty} G_2(m,\ n;\ r,\ s)\frac{t^m}{m!} = \sum_{m=n}^{\infty}\left\{\sum_{k=0}^{n} (-1)^k \binom{m}{k} r^k G(m - k,\ n - k;\ r,\ s)\right\}\frac{t^m}{m!}$$

(continued)

$$= \sum_{k=0}^{n} (-1)^k \frac{(rt)^k}{k!} \sum_{m=n}^{\infty} G(m - k;\ n - k,\ r,\ s) \frac{t^{m-k}}{(m-k)!}$$

$$= \frac{1}{n!}(1 + t)^s \sum_{k=0}^{n} \binom{n}{k} [(1 + t)^r - 1]^{n-k}(-rt)^k,$$

and

$$\sum_{m=n}^{\infty} G_2(m,\ n;\ r,\ s)\frac{t^m}{m!} = \frac{1}{n!}(1 + t)^s[(1 + t)^r - 1 - rt]^n. \tag{3.14}$$

Consider n different cells of r different compartments each and a (control) cell of s different compartments. The compartments may be of limited capacity or not (Riorday [11, Ch. 5]). From (3.14), it follows that the number of ways of putting m like balls into these cells such that each cell among the first n contains at least two balls is equal to

$$\frac{n!}{m!} G_2(m,\ n;\ r,\ s)$$

when the capacity of each compartment is limited to one ball and to

$$(-1)^m \frac{n!}{m!} G_2(m,\ n;\ -r,\ -s)$$

when the capacity of each compartment is unlimited.

It is worth noting that the expression (3.8) may be written in the form

$$r^{-m+n}G(m,\ m - n;\ r,\ s) = \sum_{j=0}^{n} L(n,\ j;\ r,\ s)\binom{m + j}{2n}, \tag{3.15}$$

where, on using the relation

$$\binom{m + j}{2n} = \sum_{k=0}^{j} \binom{j}{k}\binom{m}{2n - k}$$

the coefficients $L(n,\ j;\ r,\ s)$ are related to the coefficients $H(n,\ k;\ r,\ s)$ by

$$H(n,\ k;\ r,\ s) = \sum_{j=k}^{n} \binom{j}{k} L(n,\ j;\ r,\ s), \tag{3.16}$$

$$L(n,\ j;\ r,\ s) = \sum_{k=j}^{n} (-1)^{k-j}\binom{k}{j} H(n,\ k;\ r,\ s). \tag{3.17}$$

Moreover, $L(n,\ j;\ r,\ s)$ satisfy the recurrence relation

$$\begin{aligned} L(n + 1,\ j;\ r,\ s) &= [(r - 1)(n + j + 1) + n - s]L(n,\ j;\ r,\ s) \\ &\quad + [(r - 1)(n - j + 1) - n + s]L(n,\ j - 1;\ r,\ s), \end{aligned} \tag{3.18}$$

with initial conditions

$L(1,\ 0;\ r,\ s) = r - s - 1$, $L(1,\ 1;\ r,\ s) = s$, and $L(n,\ j;\ r,\ s) = 0$ if $j > n$.

Also, by (3.9), (3.10), and (3.11),

$$L(n,\ n;\ r,\ s) = H(n,\ n;\ r,\ s) = (s)_n,\ n = 1,\ 2,\ \ldots, \tag{3.19}$$

$$L(n,\ 0;\ r,\ s) = \sum_{k=0}^{n} (-1)^k H(n,\ k;\ r,\ s) = (-1)^n r^n \left(\frac{s - r + 1}{r}\right)_n, \tag{3.20}$$

$$\sum_{j=0}^{n} L(n,\ j;\ r,\ s) = H(n,\ 0;\ r,\ s) = (r - 1)^n \frac{(2n)!}{n!\,2^n} \tag{3.21}$$

We conclude this section by considering the sum

$$G_m(r,\ s) = \sum_{n=0}^{m} G(m,\ n;\ r,\ s), \tag{3.22}$$

which for $s = 0$ reduces to

$$C_m(r) = \sum_{n=0}^{m} C(m,\ n,\ r). \tag{3.23}$$

This sum has been studied in [5] and also by Carlitz in [2] as

$$A_m(\lambda) = \sum_{n=0}^{m} S(m,\ n|\lambda) = \sum_{n=0}^{m} \lambda^m C(m,\ n,\ 1/\lambda) = \lambda^m C_m(1/\lambda).$$

Note that, since (see [7])

$$\lim_{r\to\infty} r^{-m}C(m,\ n,\ r) = S(m,\ n), \tag{3.24}$$

it follows that

$$\lim_{r\to\infty} r^{-m}C_m(r) = \sum_{n=0}^{m} S(m,\ n) = B_m, \tag{3.25}$$

where B_m denotes the Bell number. Also from (3.1) we get, on using (2.20) and (3.24),

$$\begin{aligned}\lim_{r\to\infty} r^{-m}G(m,\ n;\ r,\ s) &= \overline{S}(m,\ n+1,\ \lambda) + S(m,\ n)\\ &= R(m,\ n,\ \lambda),\ \lambda = \lim_{r\to\infty}\frac{s}{r}.\end{aligned}$$

Hence,

$$\lim_{r\to\infty} r^{-m}G_m(r,\ s) = \sum_{n=0}^{m} R(m,\ n,\ \lambda) = B_m(\lambda),\ \lambda = \lim_{r\to\infty}\frac{s}{r}, \tag{3.26}$$

where the number $B_m(\lambda)$ has been discussed by Carlitz in [3].

Now, from (3.22), (3.23), and (3.2), it follows that

$$G_m(r,\ s) = \sum_{n=0}^{m}\sum_{j=0}^{m-n}\binom{m}{j}(s)_j\, C(m-j,\ n,\ r) = \sum_{j=0}^{m}\binom{m}{j}(s)_j\sum_{n=0}^{m-j} C(m-j,\ n,\ r),$$

$$G_m(r,\ s) = \sum_{j=0}^{m}\binom{m}{j}(s)_j\, C_{m-j}(r), \tag{3.27}$$

and

$$\begin{aligned}F(t;\ r,\ s) &= \sum_{m=0}^{\infty} G_m(r,\ s)\frac{t^m}{m!} = \sum_{j=0}^{s}\binom{s}{j}t^j\sum_{m=0}^{\infty} C_m(r)\frac{t^m}{m!}\\ &= (1+t)^s\exp\{(1+t)^r - 1\},\end{aligned} \tag{3.28}$$

since (see [5] or [2])

$$F(t;\ r) = \sum_{m=0}^{\infty} C_m(r)\frac{t}{m!} = \exp\{(1+t)^r - 1\}.$$

We have

$$F(t;\ r,\ s+1) = (1+t)F(t;\ r,\ s)$$

and, hence,

$$G_m(r,\ s+1) = G_m(r,\ s) + mG_{m-1}(r,\ s),\ m = 1,\ 2,\ \ldots,\ G_0(r,\ s) = 1. \tag{3.29}$$

Differentiation of (3.29) gives the differential equation

$$(1 + t)\frac{d}{dt} F(t;\ r,\ s) = sF(t;\ r,\ s) + r(1 + t)^r F(t;\ r,\ s),$$

which implies

$$G_{m+1}(r,\ s) = (s - m)G_m(r,\ s) + r\sum_{j=0}^{m}\binom{m}{j}(r)_j G_{m-j}(r,\ s). \tag{3.30}$$

Writing the generating function $F(t;\ r,\ s)$ in the form

$$F(t;\ r,\ s) = e^{-1}(1 + t)^s \exp\{(1 + t)^r\} = e^{-1}\sum_{k=0}^{\infty}\frac{(1 + t)^{rk+s}}{k!}$$

$$= e^{-1}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{m=0}^{\infty}(rk + s)_m\frac{t^m}{m!},$$

we find

$$G_m(r,\ s) = e^{-1}\sum_{k=0}^{\infty}\frac{(rk + s)_m}{k!} \tag{3.31}$$

which should be compared to Dobinski's formula for the Bell number:

$$B_m = e^{-1}\sum_{k=0}^{\infty}\frac{k^m}{k!}. \tag{3.32}$$

From (3.31) we obtain, on using (3.32) and the relation (see Carlitz [3]),

$$(rk + s)_m = \sum_{n=0}^{m}(-1)^{m-n}R_1(m,\ n,\ -s)r^n k^n,$$

$$G_m(r,\ s) = \sum_{n=0}^{m}(-1)^{m-n}R_1(m,\ n,\ -s)r^n B_n. \tag{3.33}$$

4. COMBINATORIAL APPLICATIONS

4.1 Modified Occupancy Stirling Distributions of the First Kind

Consider an urn containing r identical balls from each of $n + v$ different kinds (colors). Suppose that m balls are drawn one after the other and after each drawing the chosen ball is returned togather with another ball of the same kind (color). Let X be the number of kinds (colors) among n specified appearing in the sample. The probability function of X, on using the sieve (inclusion-exclusion) formula, may be obtained as

$$p_1(k;\ m,\ n,\ r,\ v) = Pr(X + k)$$

$$= \binom{n}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{rj + rv + m - 1}{m}\Big/\binom{rn + rv + m - 1}{m}$$

$$= \frac{(n)_k}{(rn + rv + m - 1)_m}|G(m,\ k;\ -r,\ -rv)|, \tag{4.1}$$

$$k = 1,\ 2,\ \ldots,\ \min\{m,\ n\}.$$

Now, consider the case where the number m of balls is not fixed but balls are sequentially drawn and after each drawing the chosen ball is returned together with another ball of the same kind until a predetermined number k of kinds among the n specified is represented in the sample. Let Y be the number of balls required. Then the probability function of Y is given by

$$\begin{aligned} q_1(m;\ k,\ n,\ r,\ v) &= p_1(k-1;\ m-1,\ n,\ r,\ v)\frac{r(n-k+1)}{rn+rv+m-1} \\ &= \frac{(n)_{k-1}}{(rn+rv+m-2)_{m-1}}|G(m-1,\ k-1;\ -r,\ -rv)|\frac{r(n-k+1)}{rn+rv+m-1} \\ &= \frac{r(n)_k}{(rn+rv+m-1)_m}|G(m-1,\ k-1;\ -r,\ -rv)|, \qquad (4.2) \\ &\qquad m = k,\ k+1,\ \ldots\ . \end{aligned}$$

Suppose that $\lim_{r\to 0} rn = \theta$ and $\lim_{r\to 0} rv = \lambda$, then since (see [9])

$$\lim_{r\to 0} r^{-k}|G(m,\ k;\ -r,\ -rv)| = |s(m,\ k,\ \lambda)| = S_1(m,\ k,\ \lambda)$$

it follows from (4.1) and (4.2) that

$$p_1(k;\ m,\ \theta,\ \lambda) = \lim_{r\to 0} p_1(k;\ m,\ n,\ r,\ v) = \frac{(\theta)_k}{(\theta+\lambda+m-1)_m} S_1(m,\ k,\ \lambda), \qquad (4.3)$$

and

$$\begin{aligned} q_1(m;\ k,\ \theta,\ \lambda) &= \lim_{r\to 0} q_1(m;\ k,\ n,\ r,\ v) \\ &= \frac{(\theta)_k}{(\theta+\lambda+m-1)_m} S_1(m-1,\ k-1,\ \lambda). \qquad (4.4) \end{aligned}$$

Note that (4.3) gives in particular $\lambda = 0$ the occupancy Stirling distribution of the first kind (cf. Johson and Kotz [10, p. 246]).

4.2 Modified Occupancy Stirling Distributions of the Second Kind

Suppose that m distinct balls are randomly allocated into $n + r$ different cells and let X be the number of occupied cells (by at least one ball) among n specified. Since $R(m,\ k,\ r)$ is the number of ways of putting the m balls into the $n + r$ cells such that k cells among the n specified are occupied (by at least one ball), it follows that

$$Pr(X = k) = \frac{(n)_k}{(n+r)^m} R(m,\ k,\ r),\ k = 1,\ 2,\ \ldots,\ \min\{m,\ n\}. \qquad (4.5)$$

The factorial moments of X may be obtained in terms of the number $R(m, k, r)$ as follows:

$$\begin{aligned} \mu_{(j)} &= \sum_{k=j}^{n}(k)_j Pr(X = k) = \frac{1}{(n+r)^m}\sum_{k=r}^{n}(k)_j(n)_k R(m,\ k,\ r) \\ &= \frac{\binom{n}{j}}{(n+r)^m}\sum_{k=j}^{n}\binom{n-j}{k-j}\frac{k!}{j!} R(m,\ k,\ r) \end{aligned}$$

(continued)

$$= \frac{\binom{n}{j}}{(n + r)^m} \sum_{i=0}^{n-j} \binom{n - j}{i} (i + j)_i R(m, i + j, r).$$

Since

$$\sum_{i=0}^{n-j} \binom{n - j}{i} (i + j)_i R(m, i + j, r) = \frac{1}{j!} \sum_{i=0}^{n-j} \binom{n - j}{i} \Delta^{i+j} r^m = \frac{1}{j!} \Delta^j E^{n-j} r^m$$

$$= \frac{1}{j!} \Delta^j (r + n - j)^m = R(m, j, r + n - j),$$

$$\mu_{(j)} = \frac{1}{(n + r)^m} \binom{n}{j} R(m, j, r + n - j). \tag{4.6}$$

Now, consider the case where the number of balls is not fixed but balls are sequentially (one after the other) allocated into the $n + r$ different cells until a predetermined number k of cells among the n specified are occupied. Let Y be the number of balls required. Then,

$$Pr(Y = m) = \frac{(n)_{k-1}}{(n + r)^{m-1}} R(m - 1, k - 1, r) \frac{n - k + 1}{n + r}$$

$$= \frac{(n)_k}{(n + r)^m} R(m - 1, k - 1, r), \; m = k, k + 1, \ldots .$$

Since $\sum_{m=k}^{\infty} Pr(Y = m) = 1$, we must have

$$\sum_{m=k}^{\infty} R(m - 1, k - 1, r) \frac{1}{(n + r)^m} = \frac{1}{(n)_k}.$$

This relation holds in the more general case where r is any real number and n real number different from 0, 1, 2, ..., $k - 1$. Indeed from Carlitz [3],

$$\sum_{m=k}^{\infty} R(m, k, \lambda) z^m = \frac{z^k}{(1 - \lambda z)(1 - (\lambda + 1)z) \ldots (1 - (\lambda + k)z)}$$

it follows that

$$\sum_{m=k}^{\infty} R(m - 1, k - 1, r) z^{m-1} = \frac{1}{(z^{-1} - \lambda)(z^{-1} - \lambda - 1) \ldots (z^{-1} - \lambda - k + 1)} \quad \frac{1}{(z^{-1} - \lambda)_k}$$

and putting $z^{-1} - \lambda = n$, $z = (n + \lambda)^{-1}$, we obtain

$$\sum_{m=k}^{\infty} R(m - 1, k - 1, \lambda) \frac{1}{(m + \lambda)^m} = \frac{1}{(n)_k}.$$

Remark 4.1

The distribution (4.5) with r not necessarily a positive integer arose in the following randomized occupancy problem (see [10, p. 139]). Suppose that m balls are randomly allocated into n different cells and that each ball has probability p of staying in its cell and probability $q = 1 - p$ of leaking. Let X be the number of occupied cells. Then, the probability function of X may be obtained by using the sieve (inclusion-exclusion) formula in the form

$$Pr(X = k) = \binom{n}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (q + p(k - j)/n)^m$$
$$= \frac{(n)_k}{(n + \lambda)^m} R(m, k, \lambda),\ k = 1, 2, \ldots, \min\{m, n\},\ \lambda = nq/p.$$

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ON A FAMILY OF NESTED RECURRENCES

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1. INTRODUCTION

A recursive definition of a function f is called *nested* if, in the definition body, the function f is called with an argument whose evaluation involves yet another call to function f. A famous example of such a nested recursive definition is "McCarthy's 91-function"

$$f(n) = \begin{cases} f(f(n+11)) & 0 \leqslant n \leqslant 100 \\ n - 10 & n > 100 \end{cases}$$

whose solution

$$f(n) = \begin{cases} 91 & 0 \leqslant n \leqslant 100 \\ n - 10 & n > 100 \end{cases}$$

is described in [2, p. 373]. Such recurrences seem difficult to understand and solve, and general solution techniques are lacking.

In this paper a complete solution is developed for the family of nested recurrences (one for each integer $k > 0$) given by

$$g_k(n) = \begin{cases} n - g_k(g_k(n-k)) & n \geqslant 1 \\ 0 & n \leqslant 0. \end{cases} \tag{1.1}$$

For the case $k = 1$, this recurrence is mentioned in [1, p. 137], where its behavior is described diagramatically.

The functions $g_1(n)$ and $g_2(n)$ are plotted in Figures 1 and 2.

Recently Meek and van Rees [3] have examined the recurrence family

$$f_r(n) = n - f_r(f_r(\cdots(f_r(n-1))\cdots)),\ n \geqslant 1$$

where f_r is nested to r levels and $f_r(0) = 0$. In [3] the solution for $f_r(n)$ is expressed indirectly through a transformation: n is represented as a generalized Fibonacci base numeral (dependent on r), the least significant digit of this representation is truncated, and the resulting Fibonacci base numeral represents $f_r(n)$. In this paper we give a closed form solution for $f_2(n)$, which is $g_1(n)$ in our notation. The problem of finding a closed form for $f_r(n)$, $r \geqslant 3$, remains open.

The approximate behavior of $g_k(n)$ is easy to describe. Figures 1 and 2 suggest looking for an asymptotic approximation to the solution having the form $g_k(n) = An + O(1)$. Substituting this into (1.1) and equating coefficients of n on both sides yields

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$$g_k(n) = \psi n + O(1) \text{ as } n \to \infty, \tag{1.2}$$

where $\psi = (\sqrt{5} - 1)/2$ is the reciprocal of the "golden ratio" ϕ.* The relationship between $g_k(n)$ and the line ψn is even closer than the asymptotic estimate (1.2) would indicate, since Theorem 1 states

$$g_k(n) = \sum_{i=0}^{k-1} \left\lfloor \psi \left\lfloor \frac{n+i}{k} \right\rfloor + \psi \right\rfloor. \tag{1.3}$$

Thus, $g_1(n) = \lfloor \psi(n+1) \rfloor$ is the function described in [1]. This function has an interesting number-theoretic property: Theorem 2 shows that the points at which $g_1(n)$ increases form a Beatty sequence.

2. SOLUTION

Let us first give a solution for the function $g_1(n)$. From it, we generalize the solution of (1.1).

Figure 1 shows that while the line ψn must miss all the integral lattice points, the values of $g_1(n)$ fall on lattice points near the line. This behavior suggests looking for a solution of the form $g_1(n) = \lfloor \psi n + C \rfloor$. If one substitutes this form into (1.1) with $k = 1$ and performs calculations similar to those in Lemma 1 below, it emerges that a choice of $C = \psi$ will cause the equation to balance. Turning this calculation around into a proof yields the following Lemma, which shows that $g_1(n) = \lfloor \psi n + \psi \rfloor$. This result is also needed in the proof of Theorem 1.

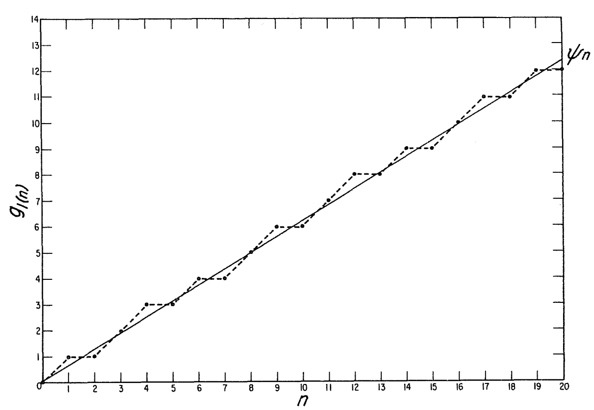

FIG. 1. Plot of $g_1(n)$ for $0 \leqslant n \leqslant 20$. The values of the function are indicated by heavy dots. The dashed lines are present only to facilitate interpretation. Superimposed on the function is the straight line ψn.

Lemma 1

For all $n \geqslant 0$, $\lfloor \psi n + \psi \rfloor = n - \lfloor \psi \lfloor \psi n \rfloor + \psi \rfloor$. $\quad$ (2.1)

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*ϕ *is the positive root of* $\phi^2 - \phi - 1 = 0$, *while* $\psi = \phi^{-1} = \phi - 1$ *is the positive root of* $\psi^2 + \psi - 1 = 0$.

Proof: Let $\psi n = \lfloor \psi n \rfloor + \varepsilon$, where $\varepsilon = \psi n \bmod 1$, the fractional part of ψn. First, we note that ε can never equal ψ^2. For suppose $\psi n = \lfloor \psi n \rfloor + \psi^2$ for some n. Then $\psi n - \psi^2 = \psi n - (1 - \psi) = \psi(n + 1) - 1$ is an integer, and $\psi(n + 1)$ is an integer for some n, which is clearly impossible.

Now (2.1) is equivalent to the assertion

$$\lfloor \psi n + \psi \rfloor + \lfloor \psi(\psi n - \varepsilon) + \psi \rfloor = n.$$

This is equivalent to

$$\lfloor \psi n + \psi \rfloor + \lfloor \psi^2 n - \psi\varepsilon + \psi \rfloor = n.$$

Since $\psi^2 n = n - \psi n$, we may cancel the integer n, yielding

$$\lfloor \psi n + \psi \rfloor + \lfloor -\psi n - \psi\varepsilon - \psi \rfloor = 0,$$

or

$$\lfloor \lfloor \psi n \rfloor + \varepsilon + \psi \rfloor + \lfloor -\lfloor \psi n \rfloor - \varepsilon - \psi\varepsilon + \psi \rfloor = 0.$$

Cancelling the integers from inside the floor functions, this is equivalent to

$$\lfloor \varepsilon + \psi \rfloor + \lfloor \psi - \varepsilon(1 + \psi) \rfloor = 0.$$

This last identity can be seen to hold for all $\varepsilon \neq \psi^2$ in the interval $(0, 1)$ as follows: The argument of the second floor term is linear in ε, decreasing over $(0, 1)$, with a zero at $\varepsilon = \psi^2$. In case $\varepsilon < \psi^2$, both terms yield zero, because the arguments of each floor are positive and less than 1. In case $\varepsilon > \psi^2$, the first term is 1 and the second is -1. ■

Next, we turn to the solution of g_2, defined by $g_2(n) = n - g_2(g_2(n - 2))$. At even arguments $n = 2m$, we have

$$g_2(2m) = 2m - g_2(g_2(2(m - 1))). \tag{2.2}$$

Define the function h via

$$g_2(2i) = 2h(i). \tag{2.3}$$

Then (2.2) can be written

$$2h(m) = 2m - g(2h(m - 1)) = 2m - 2h(h(m - 1)),$$

by using (2.3) again. Thus

$$h(m) = m - h(h(m - 1))$$

with $h(0) = 0$ and so $h(m) = g_1(m) = \lfloor \psi m + \psi \rfloor$. Putting this into (2.3) and using $n = 2m$ yields finally

$$g_2(n) = 2\left\lfloor \psi \frac{n}{2} + \psi \right\rfloor, \ n \text{ even}. \tag{2.4}$$

To solve for odd arguments n is not so straightforward. But an examination of Figure 2 shows that the values of $g_2(n)$ at odd n seem to lie on a straight line between the neighboring values at even arguments. This observation suggests that the solution is the "average" of the two nearest even argument values, or

$$g_2(n) = \left\lfloor \psi \left\lfloor \frac{n}{2} \right\rfloor + \psi \right\rfloor + \left\lfloor \psi \left\lfloor \frac{n + 1}{2} \right\rfloor + \psi \right\rfloor, \ n \geqslant 0.$$

This expression is certainly consistent with (2.4). That this is indeed the solution is established by an induction argument. In fact, the "natural" generalization of this expression, given by (2.7), will be shown in Theorem 1 to satisfy the general recurrence (1.1).

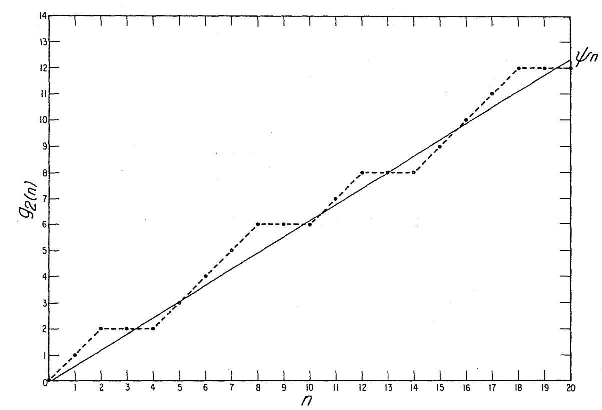

FIG. 2. Plot of $g_2(n)$ for $0 \leqslant n \leqslant 20$. The values of the function are indicated by heavy dots. The dashed lines are present only to facilitate interpretation. Superimposed on the function is the straight line ψn. This function appears to be a "scaled up" version of Figure 1.

The following lemma is needed for the induction of Theorem 1.

Lemma 2

For all $n \geqslant 0$, $0 \leqslant g_k(n) \leqslant n$. (2.5)

Proof: By induction on n. The base $0 \leqslant n < k$ is easily checked, since

$$g_k(n) = n$$

for arguments in this range. Assume that $n \geqslant k$ and that (2.5) holds for all $0 \leqslant i < n$. We will establish (2.5) for n. Now

$$g_k(n) = n - g_k(g_k(n - k)). \tag{2.6}$$

Let $i = g_k(n-k)$. By the induction hypothesis for $n-k$, we have $0 \leqslant i \leqslant n - k$, and so by the induction hypothesis for i, $0 \leqslant g_k(i) \leqslant i$, that is,

$$0 \leqslant g_k(g_k(n - k)) \leqslant g_k(n - k).$$

Using this inequality with (2.6) yields

$$n - g_k(n - k) \leqslant g_k(n) \leqslant n,$$

and, since $n - k - g_k(n - k) \geqslant 0$ by the induction hypothesis, the result (2.5) follows for n. ■

Now to the main result.

Theorem 1

The solution to (1.1) is given by

$$g_k(n) = \sum_{i=0}^{k-1} \left\lfloor \psi \left\lfloor \frac{n + i}{k} \right\rfloor + \psi \right\rfloor, \; n \geqslant 0. \tag{2.7}$$

Proof: By induction on n. The base $0 \leqslant n < k$ can be checked directly, as $g_k(n) = n$ for arguments in this range. Assume that $n \geqslant k$ and that (2.7) holds for all $0 \leqslant i < n$. We will establish (2.7) for n.

By the induction hypothesis,

$$g_k(n - k) = \sum_{i=0}^{k-1} \left\lfloor \psi \left\lfloor \frac{n + i}{k} \right\rfloor \right\rfloor. \tag{2.8}$$

Suppose that $n = qk + r$ with remainder $0 \leqslant r < k$. Then the first $k - r$ of the quotients

$$\left\lfloor \frac{n}{k} \right\rfloor, \left\lfloor \frac{n+1}{k} \right\rfloor, \ldots, \left\lfloor \frac{n+k-1}{k} \right\rfloor$$

are equal to q and the remaining r quotients are equal to $q + 1$. Thus,

$$g_k(n - k) = (k - r)\lfloor \psi q \rfloor + r\lfloor \psi q + \psi \rfloor. \tag{2.9}$$

and similarly

$$\sum_{i=0}^{k-1} \left\lfloor \psi \left\lfloor \frac{n + i}{k} \right\rfloor + \psi \right\rfloor = (k - r)\lfloor \psi q + \psi \rfloor + r\lfloor \psi q + 2\psi \rfloor. \tag{2.10}$$

We would like to show that $g_k(n)$ is equal to (2.10). There are two cases to consider.

Case $\lfloor \psi q + \psi \rfloor = \lfloor \psi q \rfloor$: Then it follows that

$$\lfloor \psi q + 2\psi \rfloor = 1 + \lfloor \psi q + \psi \rfloor, \tag{2.11}$$

and by (2.9), $g_k(n - k) = k\lfloor \psi q \rfloor$. But then all the quotients

$$\left\lfloor \frac{g_k(n - k) + i}{k} \right\rfloor, \quad 0 \leqslant i < k \tag{2.12}$$

are identically equal to $\lfloor \psi q \rfloor$. Since $g_k(n - k) \leqslant n - k$ by Lemma 2, the induction hypothesis (2.7) holds with argument set to $g_k(n - k)$, and so using the equality of all the quotients (2.12)

$$g_k(g_k(n - k)) = k\lfloor \psi \lfloor \psi q \rfloor + \psi \rfloor. \tag{2.13}$$

By Lemma 1, the right side of (2.13) is $k(q - \lfloor \psi q + \psi \rfloor)$, and using this fact in (1.1):

$$g_k(n) = n - g_k(g_k(n - k)) = qk + r - g_k(g_k(n - k)) = r + k\lfloor \psi q + \psi \rfloor. \tag{2.14}$$

Using (2.11) in (2.10) gives agreement with the expression for $g_k(n)$ in (2.14), establishing the step in this case.

Case $\lfloor \psi q + \psi \rfloor = \lfloor \psi q \rfloor + 1$: In this case (2.9) yields

$$g_k(n - k) = k\lfloor \psi q \rfloor + r. \tag{2.15}$$

Because of (2.15), we obtain

$$\begin{aligned} \left\lfloor \frac{g_k(n - k) + i}{k} \right\rfloor &= \lfloor \psi q \rfloor, \qquad 0 \leqslant i < k - r \\ \left\lfloor \frac{g_k(n - k) + i}{k} \right\rfloor &= \lfloor \psi q \rfloor + 1, \quad k - r \leqslant i < k \end{aligned} \tag{2.16}$$

Lemma 2 guarantees that $g_k(n - k) \leqslant n - k$, so the induction hypothesis (2.7) holds with argument $g_k(n - k)$. Along with identities (2.16), this gives

$$g_k(g_k(n - k)) = (k - r)\lfloor\psi\lfloor\psi q\rfloor + \psi\rfloor + r\lfloor\psi\lfloor\psi q\rfloor + 2\psi\rfloor,$$

which in light of the case assumption can be rewritten

$$g_k(g_k(n - k)) = (k - r)\lfloor\psi\lfloor\psi q\rfloor + \psi\rfloor + r\lfloor\psi\lfloor\psi(q + 1)\rfloor + \psi\rfloor. \tag{2.17}$$

Now apply Lemma 1 to each of the terms in (2.17), and simplify to obtain

$$g_k(g_k(n - k)) = kq + r - (k - r)\lfloor\psi q + \psi\rfloor - r\lfloor\psi q + 2\psi\rfloor. \tag{2.18}$$

From this, using the recurrence (1.1),

$$g_k(n) = n - g_k(g_k(n - k)) = (k - r)\lfloor\psi q + \psi\rfloor + r\lfloor\psi q + 2\psi\rfloor. \tag{2.19}$$

and this is seen to be just (2.10), as required. This case completes the induction and the proof of (2.7). ■

3. THE DISTRIBUTION OF TRANSITION POINTS FOR $g_1(n)$

Let

$$\nabla f(n) = f(n) - f(n - 1),\ n = 1, 2, 3, \ldots$$

be the "backward difference" sequence of the function $f(n)$, $n = 0, 1, 2, \ldots$. The values of n for which $\nabla f(n) \neq 0$ are called the *transition points* of f and the sequence T_f of the values of n for which $\nabla f(n) \neq 0$ is called the *transition sequence* for f.

Successive values for $g_1(n)$ clearly can differ by at most one. That is, ∇g_1 is a sequence of zeros and ones. As observed in Figure 1, the distribution of transition points for $g_1(n)$ also shows considerable regularity. In fact, Theorem 2 establishes that T_{g_1} is the Beatty sequence [4, pp. 29-30] for the "golden ratio" $\phi = \psi + 1$.

Beatty sequences are defined as follows: if α and β are positive irrationals such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

then the two sequences

$$B_\alpha = \{\lfloor\alpha\rfloor, \lfloor 2\alpha\rfloor, \lfloor 3\alpha\rfloor, \ldots\} \quad \text{and} \quad B_\beta = \{\lfloor\beta\rfloor, \lfloor 2\beta\rfloor, \lfloor 3\beta\rfloor, \ldots\}$$

are mutually exclusive and together contain all the positive integers without repetition. A proof may be found in [5, §12.2].

If $\alpha = \phi$, then $\beta = \phi + 1$ and the two complementary sequences are

$$B_\phi = \{1, 3, 4, 6, 8, 9, 11, 12, 14, 16, \ldots\}$$

and

$$B_{\phi+1} = \{2, 5, 7, 10, 13, 15, 18, \ldots\}.$$

In order to show that $T_{g_1} = B_\phi$, we establish the following identities. The first states that the function $g_1(n)$ is the inverse of Beatty's function $\lfloor\phi n\rfloor$, and that transitions do occur at points in the sequence B_ϕ.

<u>Lemma 3</u>

$$g_1(\lfloor\phi n\rfloor) = n$$
$$g_1(\lfloor\phi n\rfloor - 1) = n - 1$$

Proof: $\lfloor \phi n \rfloor = \lfloor (\psi + 1)n \rfloor = \lfloor \psi n \rfloor + n$, hence

$$g_1(\lfloor \phi n \rfloor) = g_1(\lfloor \psi n \rfloor + n) = \lfloor \psi(\lfloor \psi n \rfloor + n + 1) \rfloor$$

by Theorem 1. Using $\psi n = \lfloor \psi n \rfloor + \varepsilon$,

$$g_1(\lfloor \phi n \rfloor) = \lfloor \psi^2 n + \psi n + \psi(1 - \varepsilon) \rfloor = \lfloor n + \psi(1 - \varepsilon) \rfloor = n,$$

where we have used $\psi^2 + \psi = 1$.

For the second identity, note that

$$\lfloor \phi n \rfloor - 1 = \lfloor (\psi + 1)n - 1 \rfloor = \lfloor \psi n \rfloor + n - 1$$

so that

$$g_1(\lfloor \phi n \rfloor - 1) = \lfloor \psi(\lfloor \psi n \rfloor + n) \rfloor = \lfloor \psi^2 n + \psi n - \psi(\varepsilon) \rfloor$$
$$= \lfloor n - \psi(\varepsilon) \rfloor = n - 1. \blacksquare$$

Lemma 4

Let $\varepsilon = \psi n \bmod 1$ be the fractional part of ψn. Then for all n,

$$\lfloor \psi n + \psi(1 - \varepsilon) \rfloor = \lfloor \psi n \rfloor = \lfloor \psi n - \psi\varepsilon \rfloor. \tag{3.1}$$

Proof: Obviously $0 < \psi(1 - \varepsilon) < 1 - \varepsilon$, and so

$$\psi n < \psi n + \psi(1 - \varepsilon) < \psi n + 1 - \varepsilon.$$

Since $\psi n = \lfloor \psi n \rfloor + \varepsilon$,

$$\lfloor \psi n \rfloor + \varepsilon < \psi n + \psi(1 - \varepsilon) < \lfloor \psi n \rfloor + 1,$$

and so it follows that $\lfloor \psi n \rfloor = \lfloor \psi n + \psi(1 - \varepsilon) \rfloor$, establishing the first equality.

Next, notice that

$$\psi n - \psi\varepsilon = \lfloor \psi n \rfloor + \varepsilon(1 - \psi).$$

Since $0 < \varepsilon(1 - \psi) < 1$, the second equality follows. $\blacksquare$

The next lemma gives information about the points where g_1 does not have a transition.

Lemma 5

$$g_1(\lfloor (\phi + 1)n \rfloor) = g_1(\lfloor (\phi + 1)n \rfloor - 1) = \lfloor \phi n \rfloor. \tag{3.2}$$

Proof: Consider the first equality. Since $\phi = \psi + 1$, by Theorem 1 this is equivalent to showing that

$$\lfloor \psi \lfloor \psi n + 2n \rfloor + \psi \rfloor = \lfloor \psi \lfloor \psi n + 2n \rfloor \rfloor. \tag{3.3}$$

Now (3.3) is equivalent to showing

$$\lfloor \psi \lfloor \psi n \rfloor + 2n\psi + \psi \rfloor = \lfloor \psi \lfloor \psi n \rfloor + 2n\psi \rfloor. \tag{3.4}$$

Let $\psi n = \lfloor \psi n \rfloor + \varepsilon$ where $\varepsilon = \psi n \bmod 1$. Substituting this into (3.4) and simplifying using $\psi^2 + \psi = 1$ shows that (3.4) is equivalent to

$$\lfloor n + \psi n + \psi(1 - \varepsilon) \rfloor = \lfloor n + \psi n - \psi\varepsilon \rfloor. \tag{3.5}$$

By Lemma 4, these expressions are equal, proving that (3.3) holds.

Consider the second equality. By Theorem 1 this is equivalent to showing that

$$\lfloor \psi \lfloor \psi n + 2n \rfloor \rfloor = \lfloor \psi n + n \rfloor, \tag{3.6}$$

which is equivalent to

$$\lfloor \psi \lfloor \psi n \rfloor + 2\psi n \rfloor = \lfloor \psi n + n \rfloor. \tag{3.7}$$

Using the substitution $\lfloor \psi n \rfloor = \psi n - \varepsilon$ in (3.7), and simplifying using $\psi^2 + \psi = 1$ shows this is equivalent to

$$\lfloor n + \psi n - \psi\varepsilon \rfloor = \lfloor \psi n + n \rfloor. \tag{3.8}$$

By Lemma 4, this last equality holds, and so (3.6) holds. ■

The connection with the Beatty sequence can now be made.

Theorem 2

$T_{g_1} = B_\phi$.

Proof: By Lemma 3,

$$g_1(\lfloor \phi n \rfloor) - g_1(\lfloor \phi n \rfloor - 1) = 1$$

so that $\lfloor \phi n \rfloor$ are transition points corresponding to B_ϕ, while by Lemma 5

$$g_1(\lfloor (\phi + 1)n \rfloor) - g_1(\lfloor (\phi + 1) \rfloor n - 1) = 0,$$

so that the nontransition points $\lfloor (\phi + 1)n \rfloor$ correspond to $B_{\phi+1}$. By the properties of Beatty sequences, B_ϕ and $B_{\phi+1}$ include all the positive integers. ■

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AN ORDER-THEORETIC REPRESENTATION OF THE POLYGONAL NUMBERS

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1. INTRODUCTION

Polygonal numbers received their name from their standard geometric realization. In this geometric realization one considers sequences of regular polygons that share a common angle and have points at equal distances along each side. The total number of points on a sequence of the regular polygons is a sequence of polygonal numbers. For example (see Fig. 1), if the polygon is a triangle, we get the triangular numbers 1, 3, 6, 10, 15, ..., and if the polygon is a pentagon, we get the pentagonal numbers 1, 5, 12, 22, 35, ... More information on the polygonal numbers may be found in L. E. Dickson's *History of Number Theory* [4, Vol. II, Ch. 1]. We also recommend the discussion of "figurirte oder vieleckigte Zahlen" by L. Euler [5, p. 159].

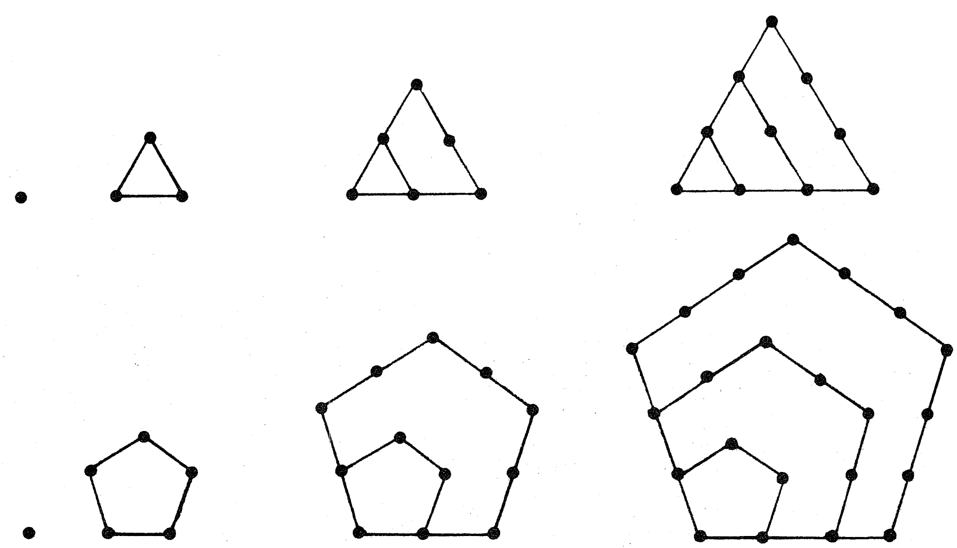


FIGURE 1

In this paper we describe an order-theoretic realization of the polygonal numbers. We represent the polygonal numbers as the cardinalities of sequences of modular lattices that can be glued together from simple building blocks. The construction of these lattices is described in the first part of §3, the main result is formulated in Theorem 3.3. It is interesting to note that, in the case of the triangular numbers and of the square numbers, the diagrams in our lattice-theoretic representation (Figure 3 and Figure 4) look just like the usual illustrations in the standard geometric realization. For all other polygonal numbers, however, the diagrams are very different.

In §2 we introduce some essential terminology and necessary facts about function lattices. For a more complete treatment of these topics, we refer the reader to the standard textbooks [1], [2], [6], and to [3].

2. FUNCTION LATTICES

Let P and Q be partially ordered sets. A mapping $f{:}P \to Q$ is order-preserving if $x \leqslant y$ in P implies $f(x) \leqslant f(y)$ in Q for all $x, y \in P$. An order-isomorphism is a mapping f that is one-to-one, onto, and has the property that $x \leqslant y$ in P if and only if $f(x) \leqslant f(y)$ in Q, for all $x, y \in P$. The set Q^P of all the order-preserving mappings from P to Q can be partially ordered by $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in P$. If $f, g \in Q^P$, then the supremum of f and g, $f \vee g$, exists in Q^P if and only if the supremum of $f(x)$ and $g(x)$ exists in Q for all $x \in P$, and $(f \vee g)(x) = f(x) \vee g(x)$. Since the same is true for the infimum of f and g, it follows that Q^P is a lattice whenever Q is a lattice, P may be an arbitrary partially ordered set. It can be shown that the function lattice Q^P is a distributive or modular lattice provided that Q is a distributive or modular lattice, respectively.

For integers $n \geqslant 0$, $\underline{n} = \{1, 2, \ldots, n\}$ denotes the totally ordered chain of n elements ordered in their natural order, $\underline{0}$ the empty chain, and $\underline{m}^{\underline{n}}$ the distributive function lattice of order-preserving mappings from the n-element chain $\underline{n}$ into the m-element chain $\underline{m}$. $M(n)$ is the modular lattice of length 2 with n atoms, $M(0) = \underline{2}$, $M(1) = \underline{3}$.

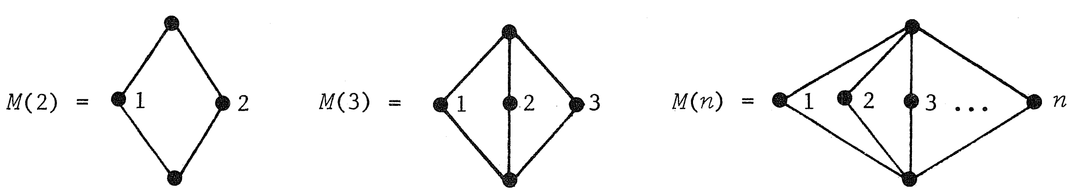


FIGURE 2

An element a in a lattice is join-irreducible if $a = b \vee c$ implies $a = b$ or $a = c$; it is meet-irreducible if $a = b \wedge c$ implies $a = b$ or $a = c$. A doubly irreducible element is an element which is both join- and meet-irreducible. Chains of doubly irreducible elements will play an important role in the construction in 3. As examples we shall now determine the sequences of function lattices $\underline{3}^{\underline{n}} = M(1)^{\underline{n}}$ (Fig. 3) and $M(2)^{\underline{n}}$ (Fig. 4) for $n \geqslant 0$. In $\underline{3}^{\underline{n}}$, the doubly irreducible elements are circled where the function $f{:}\underline{n} \to \underline{3}$ is represented by its image vector, i.e., 1223 stands for the function $f{:}\underline{4} \to \underline{3}$ given by $f(1) = 1$, $f(2) = f(3) = 2$, and $f(4) = 3$.

Obviously, the cardinalities of the lattices in Figure 3 are the triangular numbers, the cardinalities of the lattices in Figure 4 are the square numbers. This, of course, raises the question: Is it possible to represent all polygonal numbers as function lattices?

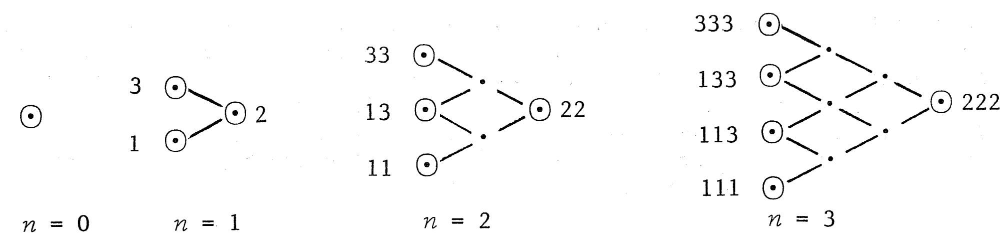


FIGURE 3

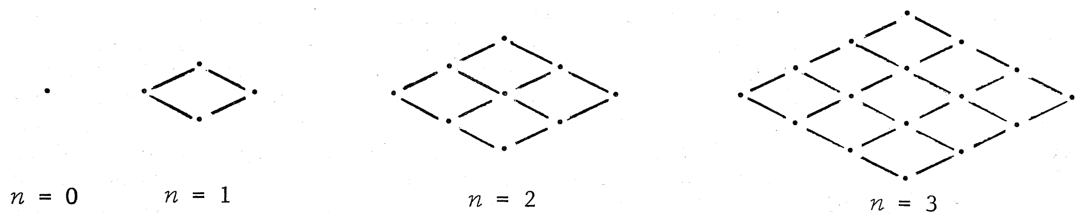


FIGURE 4

3. MODULAR PADDLEWHEELS AND POLYGONAL NUMBERS

Let $C = \{c_0 < c_1 < \cdots < c_n\}$ be a chain and let L_i, $1 \leqslant i \leqslant k$, be partially ordered sets with least and largest elements, z_i and e_i, which admit order-isomorphisms $\phi : C \to L_i$ into L_i so that $\phi_i(c_0) = z_i$ and $\phi_i(c_n) = e_i$ for each i. On the disjoint union of the L_i, $1 \leqslant i \leqslant k$, we define a relation R by $(x, y) \in R$ if and only if

$$\phi_i^{-1}(x) = \phi_j^{-1}(y) \text{ for some } i \text{ and } j, \text{ or } x = y.$$

R is an equivalence relation and the factorization of $\cup\{L_i \mid 1 \leqslant i \leqslant k\}$ with respect to this equivalence relation, denoted by $M = M(L_1, \ldots, L_k; C)$, is a partially ordered set where the order of each piece L_i is the given order, and if $x \in L_i$ and $y \in L_j$, $i \neq j$, then $x \leqslant_M y$ if and only if there is $0 \leqslant s \leqslant n$ so that $x \leqslant \phi_i(c_s)$ and $\phi_j(c_s) \leqslant y$. Moreover, if we let

$$m = \min\{t \mid x \leqslant \phi_i(c_t) \text{ and } y \leqslant \phi_j(c_t)\},$$

then either $\quad x \nleqslant \phi_i(c_{m-1}) \quad$ and $\quad y \nleqslant \phi_j(c_{m-1})$

or $\quad x \leqslant \phi_i(c_{m-1}) \quad$ and $\quad y \nleqslant \phi_j(c_{m-1})$

or $\quad x \nleqslant \phi_i(c_{m-1}) \quad$ and $\quad y \leqslant \phi_j(c_{m-1})$.

In the first case, $x \vee_M y = \phi_i(c_m)$ holds. In the second case, any common upper bound $z \in M$ of x and y such that $z \ngeqslant \phi_i(c_m)$ is in the piece L_j; hence, $x \vee_M y$ exists in the piece L_j if L_j has suprema. In the third case, $x \vee_M y$ exists in L_i if suprema exist in L_i. Of course $x \wedge_M y$ behaves in a similar way. Therefore, $M = M(L_1, \ldots, L_k; C)$ is a lattice whenever each L_i is a lattice.

We will use this construction only in the case where $L_i = L_j$ and $\phi_i = \phi_j$, for all i and j, and we indicate that we have k copies of the same lattice L in the abbreviated notation $M = M(k(L); C)$.

If all $L_i = L$, a three-dimensional illustration of $M(k(L); C)$ looks like a paddlewheel with k paddles, with the chain C as the vertical axis, and the k copies of the lattice L as the paddles, equally spaced around a circle and glued to the chain C by the mappings $\phi = \phi_i$, for all $1 \leqslant i \leqslant k$.

As an example, let

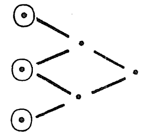
$L = \underline{3}^{\underline{2}} =$ 

FIGURE 5

We want to construct $M(4(L); 3)$. $\underline{3}^{\underline{2}}$ contains an order-isomorphic copy of $\underline{3}$, namely, a three-element chain of doubly irreducible elements, circled in the diagram above. Four copies of L are glued to this chain and we get

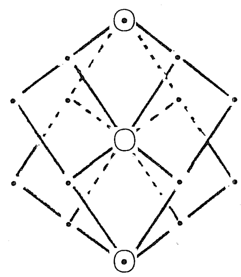


FIGURE 6

The following theorem will show that this lattice is $M(4)^{\underline{2}}$. The proof of the theorem requires some knowledge of the irreducible elements in $\underline{3}^{\underline{n}}$.

Every function $f:\underline{n} \to \underline{m}$ is piecewise constant and may be written as an increasing tuple of m values. A convenient notation is

$$1^{k_1} 2^{k_2} \ldots m^{k_m}$$

with $k_i \geqslant 0$, $1 \leqslant i \leqslant m$, and $k_1 + \cdots + k_m = n$, where the exponents k_i count the number of occurrences of the value i for the function $f(x) = i$ if and only if

$$k_1 + \cdots + k_{i-1} < x \leqslant k_1 + \cdots + k_i.$$

Now, there are two types of doubly irreducible elements in $\underline{m}^{\underline{n}}$, the constant mappings where $k_i = n$ for exactly one i, and $k_j = 0$ for all $j \neq i$, and those whose only values are the extremal elements of $\underline{m}$. The latter are of the form

$$1^{k_1} m^{k_m}, \text{ where } k_i = 0 \text{ for all } 1 < i < m.$$

The constant mappings obviously form a chain of m elements in $\underline{m}^{\underline{n}}$. For the second type of doubly irreducible elements, we have $k_1 + k_m = n$; hence, the possibilites $k_m = 0, 1, \ldots, n$, and therefore these doubly irreducible elements form a chain of $n + 1$ elements in $\underline{m}^{\underline{n}}$. This is the chain that we want to use for our paddlewheel construction. So in Theorem 3.1, $\underline{n} \oplus \underline{1}$ may be interpreted as the chain of these doubly irreducible elements in $\underline{3}^{\underline{n}}$, with $\phi:\underline{n} \oplus \underline{1} \to \underline{3}^{\underline{n}}$ the identity mapping.

Theorem 3.1

$M = M(k(\underline{3}^{\underline{n}});\ \underline{n} \oplus \underline{1})$ is the modular lattice $M(k)^{\underline{n}}$, for $k \geqslant 1$, $n \geqslant 0$.

Proof: An element in $\underline{3}^{\underline{n}}$ may be represented as $z^\alpha u^\beta e^\gamma$, where $\alpha + \beta + \gamma = n$, $0 \leqslant \alpha, \beta, \gamma \leqslant n$, and where $z < u < e$ is the chain $\underline{3}$. Similarly, we represent elements in M as $(z^\alpha u^\beta e^\gamma)_i$ for $1 \leqslant i \leqslant k$, where the index i indicates that the element is in the i^{th} of the k copies of $\underline{3}^{\underline{n}}$. Elements in $M(k)^{\underline{n}}$ are of the form $z^p j^r e^t$, where $p + r + t = n$, $0 \leqslant p, r, t \leqslant n$, and j is the j^{th} of the k atoms of $M(k)$.

We now define a mapping $\psi: M \to M(k)^{\underline{n}}$ by

$$\psi((z^\alpha u^\beta e^\gamma)_i) = z^\alpha i^\beta e^\gamma.$$

Should $(z^\alpha u^\beta e^\gamma)_i$ be in the chain $\underline{n} \oplus \underline{1}$ of M, i.e., i is not uniquely determined, then it is doubly irreducible with $\beta = 0$ and its image under ψ is then of the form $z^\alpha e^\gamma$ with $\alpha + \gamma = n$; in other words, it is independent of i. ψ is thus well defined, and it is rather straightforward to show that ψ is an order-isomorphism. ∎

Note that for $k = 1$ we have

$$M(1(\underline{3}^{\underline{n}});\ \underline{n} \oplus \underline{1}) \backsim \underline{3}^{\underline{n}} \text{ (see Fig. 3)},$$

and for $k = 2$ we have

$$M(2(\underline{3}^{\underline{n}});\ \underline{n} \oplus \underline{1}) \backsim M(2)^{\underline{n}}.$$

In the latter case, the two copies of $\underline{3}^{\underline{n}}$ are glued together so that we get a planar diagram symmetric on its vertical axis (see Fig. 4). This representation theorem provides a procedure to calculate $|M(k)^{\underline{n}}|$, the number of elements of $M(k)$, from the number of elements in $\underline{3}^{\underline{n}}$. But

$$|\underline{3}^{\underline{n}}| = \binom{n+2}{n},$$

which can be easily verified by induction on n.

Theorem 3.2

$$|M(k)^{\underline{n}}| = \left(k \cdot \frac{n}{2} + 1\right) \cdot (n + 1) \text{ for all } n, k > 0.$$

Proof: For $k = 0$, $|M(0)^{\underline{n}}| = |\underline{2}^{\underline{n}}| = n + 1$. In all other cases, it follows from the representation in Theorem 3.1,

$$M(k)^{\underline{n}} = M(k(\underline{3}^{\underline{n}});\ \underline{n} \oplus \underline{1}),$$

that $|M(k)^{\underline{n}}| = k \cdot |\underline{3}^{\underline{n}}| - (k - 1) \cdot (n + 1)$. Since

$$|\underline{3}^{\underline{n}}| = \binom{n+2}{n},$$

we get

$$|M(k)^{\underline{n}}| = k \cdot \binom{n+2}{n} - (k - 1) \cdot (n + 1) = \left(k \cdot \frac{n}{2} + 1\right) \cdot (n + 1). \ ∎$$

It is now easy to see that the numbers $P_{n,k} = |M(k)^{\underline{n}}|$ also satisfy the recursion formula

$$\begin{aligned} P_{n,k} &= P_{n,k-1} + P_{n-1,1} && \text{for } n, k > 0 \\ P_{n,0} &= n + 1 && \text{for } n \geqslant 0 \\ P_{0,k} &= 1 && \text{for } k \geqslant 0. \end{aligned}$$

However, this recursion defines the polygonal numbers [4, Vol. II, Ch. 1]. So we find that the modular lattices $M(k)^{\underline{n}}$ are order-theoretic realizations of the polygonal numbers.

Theorem 3.3

The cardinalities of the sequence of modular lattices $M(k)^{\underline{n}}$ for increasing $n \geqslant 0$ and for $k \geqslant 0$ are the polygonal numbers.

To illustrate the connection between $|M(k)^{\underline{n}}|$ and polygonal numbers, we list them in the following table for $n, k \leqslant 5$. For example, the horizontal line with entry $k = 3$ contains, from left to right, the numbers

$$1 = |M(3)^{\underline{0}}|,\ 5 = |M(3)^{\underline{1}}|,\ 12 = |M(3)^{\underline{2}}|, \text{ etc.}$$

These are the pentagonal numbers, listed in [7] as sequence number 1562.

$k \backslash n$	0	1	2	3	4	5	Name	Sloan Number
0	1	2	3	4	5	6	natural numbers	# 173
1	1	3	6	10	15	21	triangular numbers	# 1002
2	1	4	9	16	25	36	squares	# 1350
3	1	5	12	22	35	51	pentagonal numbers	# 1562
4	1	6	15	28	45	66	hexagonal numbers	# 1705
5	1	7	18	34	55	81	heptagonal numbers	# 1826

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PELL NUMBERS AND COAXAL CIRCLES

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1. INTRODUCTION

The purpose of this note is to generalize the results in [2] and to apply them to the particular case of Pell numbers. An acquaintance with [2] is desirable.

Define the generalized sequence $\{W_n\}$ by

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = r,\ W_1 = r + s \tag{1.1}$$

for all integral n, where p, q, r, and s are arbitrary, but will generally be thought of as integers.

Then, from [1], *mutatis mutandis*,

$$W_n = \frac{(r + s - r\beta)\alpha^n - \{(r + s) - r\alpha\}\beta^n}{\Delta}, \tag{1.2}$$

where α and β are the roots of $x^2 - px + q = 0$, so that $\alpha + \beta = p$, $\alpha\beta = q$, and $\alpha - \beta = \Delta = \sqrt{p^2 - 4q}$.

The generalized sequence $\{H_n\}$ in [2] occurs when

$$p = 1,\ q = -1,\ \Delta = \sqrt{5},\ r = 2b,\ \text{and } s = a - b,$$

with the special cases of the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ arising when $a = 1$, $b = 0$ (i.e., $r = 0$, $s = 1$) and $a = 0$, $b = 1$ (i.e., $r = 2$, $s = -1$), respectively.

Our particular concern in this note is with the case $p = 2$, $q = -1$, where $\alpha = 1 + \sqrt{2}\ (>0)$, $\beta = 1 - \sqrt{2}\ (<0)$, i.e., $\Delta = 2\sqrt{2}$.

Writing W_n' for W_n when $p = 2$, $q = -1$, we have from (1.2) that

$$W_n' = sP_n + \frac{r}{2}Q_n, \tag{1.3}$$

where

$$P_n = (\alpha^n - \beta^n)/2\sqrt{2} \tag{1.4}$$

and

$$Q_n = \alpha^n + \beta^n \tag{1.5}$$

are the n^{th} Pell and the n^{th} "Pell-Lucas" numbers, respectively, occurring in (1.1), (1.2), and (1.3) when $r = 0$, $s = 1$ (for P_n) and $r = 2$, $s = 0$ (for Q_n).

From (1.4) and (1.5), we have

$$2\sqrt{2}P_n < Q_n \quad \text{when } n \text{ is even,} \tag{1.6}$$

and

$$2\sqrt{2}P_n > Q_n \quad \text{when } n \text{ is odd.} \tag{1.7}$$

2. COAXAL CIRCLES FOR $\{W_n\}$

Consider the point $(x, 0)$ in the Euclidean plane with

$$x = [(r + s - r\beta)\alpha^{2n} + (-(r + s) + r\alpha)\cos(n - 1)\pi]/\Delta\alpha^n \tag{2.1}$$

The circle CW_n having

$$\text{center} \quad \bar{x}(W_n) = \frac{(r + s - r\beta)}{\Delta}\,\alpha^n, \; \bar{y}(W_n) = 0, \tag{2.2}$$

and

$$\text{radius} \quad r(W_n) = \left|\frac{-(r + s) + r\alpha)}{\Delta\alpha^n}\right| \tag{2.3}$$

has the equation

$$\left(x - \frac{(r + s - r\beta)}{\Delta}\,\alpha^n\right)^2 + y^2 = \left(\frac{-(r + s) + r\alpha}{\Delta\alpha^n}\right)^2, \tag{2.4}$$

so that

$$\bar{x}(W_n)/\bar{x}(W_{n-1}) = \alpha \tag{2.5}$$

and

$$r(W_n)/r(W_{n-1}) = \frac{1}{\alpha}. \tag{2.6}$$

The points of intersection of CW_n and the x-axis are given by

$$\begin{aligned} x(W_n) &= \frac{(r + s - r\beta)\alpha^n}{\Delta} \pm \frac{(-(r + s) + r\alpha)}{\Delta\alpha^n} \\ &= \left\{(r + s)\left\{\alpha^n \mp \frac{\beta^n}{q^n}\right\} - rq\left\{\alpha^{n-1} \mp \frac{\beta^{n-1}}{q^n}\right\}\right\}\Big/\Delta. \end{aligned} \tag{2.7}$$

Highest points on CW_n lie on the upper branch of the rectangular hyperbola

$$xy = (r + s - r\beta)\,|(r + s - r\alpha)|/\Delta^2.$$

3. COAXAL CIRCLES FOR $\{P_n\}$ AND $\{Q_n\}$

Proceeding now to the Pell numbers P_n (1.4) and Pell-Lucas numbers Q_n (1.5) we can tabulate results corresponding to the more general results (2.1)-(2.8) as follows.

Eq.	$\{P_n\}$	$\{Q_n\}$
(3.1)	$\begin{cases} x = \{\alpha^{2n} - \cos(n - 1)\pi\}/2\sqrt{2}\alpha^n \\ y = 0 \end{cases}$	$\begin{cases} x = \{\alpha^{2n} + \cos(n - 1)\pi\}/\alpha^n \\ y = 0 \end{cases}$
(3.2)	$\bar{x}(P_n) = \alpha^n/2\sqrt{2},\ \bar{y}(P_n) = 0$	$\bar{x}(Q_n) = \alpha^n,\ \bar{y}(Q_n) = 0$
(3.3)	$r(P_n) = 1/2\sqrt{2}\alpha^n$	$r(Q_n) = 1/\alpha^n$
(3.4)	$CP_n: \left\{x - \dfrac{\alpha^n}{2\sqrt{2}}\right\}^2 + y^2 = \dfrac{1}{8\alpha^{2n}}$	$CQ_n: (x - \alpha^n)^2 + y^2 = \dfrac{1}{\alpha^{2n}}$
(3.5)	$\bar{x}(P_n)/\bar{x}(P_{n-1}) = \alpha$	$\bar{x}(Q_n)/\bar{x}(Q_{n-1}) = \alpha$
(3.6)	$r(P_n)/r(P_{n-1}) = \dfrac{1}{\alpha}$	$r(Q_n)/r(Q_{n-1}) = \dfrac{1}{\alpha}$
(3.7)	$x(P_n) = P_n,\ \dfrac{Q_n}{2\sqrt{2}}$	$x(Q_n) = Q_n,\ 2\sqrt{2}P_n$
(3.8)	$xy = \dfrac{1}{8}$	$xy = 1$

Remarks about the circle-generation of Pell and Pell-Lucas numbers, similar to those made about results (3.7) in the tabulation in [2], may now be made about results (3.7) in the preceding table.

It is worth noting that the same locus $xy = 1$ in (3.8) arises from both the Lucas numbers L_n [2] and the Pell-Lucas numbers Q_n, although the two sequences of points on the hyperbola are different.

There do not appear to be any really interesting geometrical relations among the circles associated with F_n, L_n, P_n, and Q_n.

In passing, we note that in (3.7) we use

$$P_n + P_{n-1} = \frac{1}{2}Q_n$$

$$Q_n + Q_{n-1} = 4Q_n$$

both of which may be easily derived from (1.4) and (1.5).

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BINET'S FORMULA FOR THE RECURSIVE SEQUENCE OF ORDER K

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(Submitted February 1983)

1. INTRODUCTION

The terms of a recursive sequence are usually defined by a recurrence procedure; that is, any term is the sum of preceding terms. Such a definition might not be entirely satisfactory, because the computation of any term could require the computation of all of its predecessors. An alternative definition gives any term of a recursive sequence as a function of the index of the term. Binet's formulas for the two simplest nontrivial recursive sequences are known. For the recursive sequence of order 2, the Fibonacci sequence, the formula

$$u_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})$$

defines any Fibonacci number as a function of n and two constants α and β [1]. Similarly, for the recursive sequence of order 3, the Tribonacci sequence, the formula

$$u_n = \frac{\rho^2}{\rho^2 - 2\rho r \cos\theta + r^2}\rho^n + \frac{r(r - 2\rho\cos\theta)}{\rho^2 - 2\rho r\cos\theta + r^2}r^n \cos n\theta + \frac{r^2\cos\theta - \rho r(1 - 2\sin^2\theta)}{\sin\theta(\rho^2 - 2\rho r\cos\theta + r^2)}r^n \sin n\theta$$

defines any Tribonacci number as a function of n and three constants ρ, r, and θ [2].

In this paper, an analog of Binet's formula for the recursive sequence of order k ($k \geqslant 3$) is derived. The recursive sequence of order k is defined as follows:

$$u_n = 1 \qquad n = 0$$

$$u_n = \sum_{i=0}^{n-1} u_i \qquad 1 \leqslant n \leqslant k - 1$$

$$u_n = \sum_{i=n-k}^{n-1} u_i \qquad n \geqslant k.$$

The analog of Binet's formula defines any term of the recursive sequence of order k as a function of the index of the term and k constants.

2. BINET'S FORMULA FOR THE RECURSIVE SEQUENCE OF ORDER k

Binet's formula for the recursive sequence of order k is derived by solving the system of difference equations:

$$u_0 = 1$$

$$u_n = 2^{n-1} \qquad 1 \leqslant n \leqslant k-1$$

$$u_{n+1} = \sum_{n-k+1}^{n} u_i \qquad n \geqslant k-1.$$

Let $f(x) = \sum_{i=0}^{\infty} u_i x^i$ be the generating function for the solution. Then,

$$\left(1 - \sum_{j=1}^{k} x^j\right) f(x) = 1$$

or

$$f(x) = \frac{1}{1 - \sum_{j=1}^{k} x^j} = \frac{1}{\prod_{j=0}^{k-1} (1 - \alpha_j x)} = \frac{1}{p_k(x)},$$

where $1/\alpha$ is a zero of $1 - \sum_{j=1}^{k} x^j = 0$. Miller (see [3]) proved that the zeros of $p_k(1/x)$ are simple, consequently the roots of $p_k(x)$ are simple. Hence, $f(x)$ may be expressed by partial fractions as

$$f(x) = \frac{1}{\prod_{j=0}^{k-1} (1 - \alpha_j x)} = \sum_{j=0}^{k-1} \frac{A_j}{1 - \alpha_j x}, \text{ where } A_j = \frac{1}{\prod_{\substack{m=0 \\ m \neq j}}^{k-1} \left[1 - \alpha_m \left(\frac{1}{\alpha_j}\right)\right]}.$$

Further, since $\dfrac{1}{\prod_{\substack{m=0 \\ m \neq j}}^{k-1} \left[1 - \alpha_m \left(\frac{1}{\alpha_j}\right)\right]} = \dfrac{-\alpha_j}{p_k'\left(\frac{1}{\alpha_j}\right)}$, it follows that $A_j = \dfrac{-\alpha_j}{p_k'\left(\frac{1}{\alpha_j}\right)}$. Hence,

$$f(x) = \sum_{j=0}^{k-1} \frac{-\alpha_j}{p_k'\left(\frac{1}{\alpha_j}\right)} \cdot \frac{1}{1 - \alpha_j x} = \sum_{i=0}^{\infty} \left[\sum_{j=0}^{k-1} \frac{-\alpha_j}{p_k'\left(\frac{1}{\alpha_j}\right)}\right] \alpha_j^i x^i.$$

Therefore, $u_n = \sum_{j=0}^{k-1} \dfrac{-\alpha_j (\alpha_j)^n}{p_k'\left(\frac{1}{\alpha_j}\right)}$. Since $p_k'\left(\frac{1}{\alpha_j}\right) = -\sum_{m=1}^{k} m \left(\frac{1}{\alpha_j}\right)^{m-1}$ and

$$\left[1 - \sum_{m=1}^{k} \left(\frac{1}{\alpha_j}\right)^m\right] = 0 \text{ for } 0 \leqslant j \leqslant k-1,$$

then $-\left(1 - \frac{1}{\alpha_j}\right) p_k'\left(\frac{1}{\alpha_j}\right) = 2 - (k+1)\left(\frac{1}{\alpha_j}\right)^k$, and it follows that

$$u_n = \sum_{j=0}^{k-1} \frac{\alpha_j (\alpha_j)^n \left(1 - \frac{1}{\alpha_j}\right)}{2 - (k+1)\left(\frac{1}{\alpha_j}\right)^k}.$$

Multiplying by $(\alpha_j/\alpha_j)^k$, yields $u_n = \sum_{j=0}^{k-1} \dfrac{\left(\alpha_j^{k+1} - \alpha_j^k\right)\alpha_j^n}{2\alpha_j^k - (k+1)}$.

Let

$$r_k(x) = x^k - \sum_{j=0}^{k-1} x^j.$$

Miller [3] showed that $r_k(x) = 0$ has one real zero λ such that $1 < \lambda < 2$, with the remaining zeros inside the unit circle of the complex plane. Considering

$$q_k(x) = (x - 1) \cdot r_k(x) = 0$$

and using Descartes' Rule of Signs, it follows that $r_k(x) = 0$ has exactly one positive real zero when k is odd and that $r_k(x) = 0$ has exactly one positive real zero and exactly one negative real zero when k is even. Therefore, all other zeros of $r_k(x) = 0$ are complex and appear in conjugate pairs.

Now let

$$t = \left[\!\left[\frac{k-3}{2}\right]\!\right],$$

where $[\![\ ]\!]$ denotes the greatest integer function. Further, let α_j be a complex zero of $r_k(x) = 0$ if

$$0 \leqslant j \leqslant 2t + 1$$

and α_j be a real zero of $r_k(x) = 0$ if

$$2t + 2 \leqslant j \leqslant k - 1.$$

Also order the subscripts of the zeros so that $\alpha_m = \bar{\alpha}_j$ for

$$m = t + j + 1,\ 0 \leqslant j \leqslant t.$$

Consequently,

$$u_n = \sum_{j=0}^{t} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)} + \sum_{j=t+1}^{2t+1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)} + \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)}$$

$$= \sum_{j=0}^{t} \left[\frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)} + \frac{(\bar{\alpha}_j^{k+1} - \bar{\alpha}_j^k)\bar{\alpha}_j^n}{2\bar{\alpha}_j^k - (k+1)}\right] + \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)}$$

$$= \sum_{j=0}^{t} \frac{[2\bar{\alpha}_j^k - (k+1)](\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n + [2\alpha_j^k - (k+1)](\bar{\alpha}_j^{k+1} - \bar{\alpha}_j^k)\bar{\alpha}_j^n}{|2\alpha_j^k - (k+1)|^2}$$

$$+ \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)}$$

$$= \sum_{j=0}^{t} \frac{[2|\alpha_j|^{2k}\alpha_j - 2|\alpha_j|^{2k} - (k+1)\alpha_j^{k+1} + (k+1)\alpha_j^k]\alpha_j^n}{|2\alpha_j^k - (k+1)|^2}$$

$$+ \sum_{j=0}^{t} \frac{[2|\alpha_j|^{2k}\bar{\alpha}_j - 2|\alpha_j|^{2k} - (k+1)\bar{\alpha}_j^{k+1} + (k+1)\bar{\alpha}_j^k]\bar{\alpha}_j^n}{|2\alpha_j^k - (k+1)|^2}$$

$$+ \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)}$$

Applying Euler's formula,

$$\alpha_j = r_j(\cos\theta_j + i\sin\theta_j),\ \bar{\alpha}_j = r_j(\cos\theta_j - i\sin\theta_j),$$

the relation $|\alpha_j|^2 = r_j^2$, and simplifying yields

$$u_n = \sum_{j=0}^{t} r_j^n [A(k, j)\cos n\,\theta_j + B(k, j)\sin n\,\theta_j] + \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)},$$

where

$$A(k, j) = \frac{2r_j^k[2r_j^{k+1}\cos\theta_j - 2r_j^k - (k+1)r_j\cos(k+1)\theta_j + (k+1)\cos k\,\theta_j]}{4r_j^{2k} - 4(k+1)r_j^k\cos k\,\theta_j + (k+1)^2}$$

and

$$B(k, j) = \frac{-2r_j^k[2r_j^{k+1}\sin\theta_j - (k+1)r_j\sin(k+1)\theta_j + (k+1)\sin k\,\theta_j]}{4r_j^{2k} - 4(k+1)r_j^k\cos k\,\theta_j + (k+1)^2}$$

A form more suitable for computation of a single term is:

$$u_n = \sum_{j=0}^{t} \frac{\begin{array}{c}2r_j^{n+k}[2r_j^{k+1}\cos(n+1)\theta_j - (k+1)r_j\cos(n+k+1)\theta_j \\ + (k+1)\cos(n+k)\theta_j - 2r_j^k\cos n\,\theta_j]\end{array}}{4r_j^{2k} - 4(k+1)r_j^k\cos k\,\theta_j + (k+1)^2} + \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)}.$$

3. SOME NUMERICAL RESULTS

Let

$$C(k, j) = \frac{\alpha_j^{k+1} - \alpha_j^k}{2\alpha_j^k - (k+1)} \quad \text{for } (2t+2) \leqslant j \leqslant (k-1).$$

Then approximate values for the constants in the Binet formula

$$u_n = \sum_{j=0}^{t} r_j^n [A(k, j)\cos n\,\theta_j + B(k, j)\sin n\,\theta_j] + \sum_{j=2t+2}^{k-1} C(k, j)\alpha_j^n$$

for $2 \leqslant k \leqslant 10$ are given in the following table:

CONSTANTS IN BINET'S FORMULA FOR $2 \leqslant k \leqslant 10$

k	j	m	$\alpha_j,\ \bar{\alpha}_j$	r_j	θ_j	$A(k, j)$	$B(k, j)$	$C(k, j)$
2	0	–	1.6180	–	–	–	–	0.7236
	1	–	–0.6180					0.2764
3	0	1	$-0.4196 \pm 0.6063i$	0.7374	2.1762	0.3816	0.0374	–
	2	–	1.8393	–	–	–	–	0.6184
4	0	1	$-0.0764 \pm 0.8147i$	0.8183	1.6643	0.2842	0.0563	–
	2	–	–0.7748	–	–	–	–	0.1495
	3	–	1.9276	–	–	–	–	0.5663

CONSTANTS IN BINET'S FORMULA FOR $2 \leqslant k \leqslant 10$ (continued)

k	j	m	α_j, $\bar{\alpha}_j$	r_j	θ_j	$A(k, j)$	$B(k, j)$	$C(k, j)$
5	0	2	$-0.6784 \pm 0.4585i$	0.8188	2.5472	0.2421	0.0178	-
	1	3	$0.1954 \pm 0.8489i$	0.8710	1.3446	0.2200	0.0654	-
	4	-	1.9659	-	-	-	-	0.5379
6	0	2	$-0.4619 \pm 0.7191i$	0.8547	2.1418	0.2012	0.0279	-
	1	3	$0.3903 \pm 0.8179i$	0.9062	1.1255	0.1741	0.0689	-
	4	-	-0.8403	-	-	-	-	0.1029
	5	-	1.9835	-	-	-	-	0.5218
7	0	3	$-0.7842 \pm 0.3600i$	0.8629	2.7112	0.1765	0.0103	-
	1	4	$-0.2407 \pm 0.8492i$	0.8826	1.8469	0.1703	0.0340	-
	2	5	$0.5289 \pm 0.7653i$	0.9303	0.9661	0.1398	0.0691	-
	6	-	1.9920	-	-	-	-	0.5125
8	0	3	$-0.6416 \pm 0.6064i$	0.8828	2.3844	0.1550	0.0168	-
	1	4	$0.6287 \pm 0.7085i$	0.9472	0.8450	0.1132	0.0672	-
	2	5	$-0.0469 \pm 0.9030i$	0.9042	1.6227	0.1461	0.0377	-
	6	-	-0.8763	-	-	-	-	0.0785
	7	-	1.9960	-	-	-	-	0.5071
9	0	4	$-0.8397 \pm 0.2948i$	0.8900	2.8040	0.1401	0.0067	-
	1	5	$0.1143 \pm 0.9140i$	0.9211	1.4464	0.1266	0.0398	-
	2	6	$0.7019 \pm 0.6539i$	0.9593	0.7500	0.0924	0.0641	-
	3	7	$-0.4755 \pm 0.7637i$	0.8996	2.1276	0.1368	0.0212	-
	8	-	1.9980	-	-	-	-	0.5040
10	0	4	$0.2462 \pm 0.9013i$	0.9344	1.3041	0.1106	0.0408	-
	1	5	$-0.3130 \pm 0.8584i$	0.9137	1.9205	0.1218	0.0242	-
	2	6	$0.7567 \pm 0.6039i$	0.9682	0.6735	0.0759	0.0602	-
	3	7	$-0.7399 \pm 0.5168i$	0.9025	2.5319	0.1259	0.0113	-
	8	-	-0.8990	-	-	-	-	0.0635
	9	-	1.9990	-	-	-	-	0.5022

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SOME PREDICTABLE PIERCE EXPANSIONS

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I. INTRODUCTION

In 1929, T. A. Pierce discussed an algorithm for expanding real numbers $x \in (0, 1)$ in the form

$$x = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} \cdots, \tag{1}$$

where the a_i form a strictly increasing sequence of positive integers.

He showed that these expansions (which we call *Pierce expansions*) are essentially unique. The Pierce expansion for x terminates if and only if x is rational. See [3] and [5] for details.

In this note, we give formulas for the a_i in the case where

$$x = \frac{c - \sqrt{c^2 - 4}}{2}$$

and $c \geqslant 3$ is an integer. For these numbers, Pierce expansions provide extremely rapidly converging series.

II. FINDING REAL ROOTS OF POLYNOMIALS

To save space, we sill sometimes write equation (1) in the form

$$x = \{a_1, a_2, a_3, \ldots\},$$

where the braces denote a Pierce expansion.

Let

$$p_1(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

be a polynomial with integer coefficients and a single real zero α in the interval $(0, 1)$. We want to find the first term in the Pierce expansion of α. From equation (1) it is easy to see that $a_1 = \lfloor 1/\alpha \rfloor$. Consider the polynomial $q_1(x) = x^n p_1(1/x)$; this is a polynomial with integer coefficients that has $1/\alpha$ as a zero. Through a simple binary search procedure, it is easy to find d_1 such that

$$\text{sign}(q_1(d_1)) = \text{sign}(q_1(d_1 + 1));$$

this shows that $d_1 = \lfloor 1/\alpha \rfloor$ and so we can take $\alpha_1 = d_1$.

Now consider the polynomial

$$p_2(x) = a_1^n p_1\left(\frac{1 - x}{a_1}\right).$$

This again is a polynomial with integer coefficients. It is easily verified that if β is a zero of $p_2(x)$, then

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1}\beta$$

so

$$\beta = \frac{1}{a_2} - \frac{1}{a_2 a_3} + \cdots .$$

By repeating this procedure on the polynomial $p_2(x)$, we generate the coefficient α_2 in the Pierce expansion of α, and by continuing in the same fashion, we can generate as many terms of the Pierce expansion for α as desired:

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots .$$

Now let us specify our polynomial to be

$$p\ (x) = x^2 - cx + 1,$$

where $c \geqslant 3$ is an integer. Let α be the smaller positive zero, so

$$\alpha = \frac{c - \sqrt{c^2 - 4}}{2}. \tag{2}$$

Now $q_1(x) = x^2 p_1(1/x) = x^2 - cx + 1$. We find $q_1(c - 1) = 2 - c$, which is negative, and $q_1(c) = 1$, which is positive. Hence, we see that $a_1 = c - 1$.

Now

$$p_2(x) = (c - 1)^2 p_1\left(\frac{1 - x}{c - 1}\right);$$

hence,

$$p_2(x) = x^2 + (c^2 - c - 2)x + 2 - c.$$

We find

$$q_2(x) = x^2 p_2(1/x) = (2 - c)x^2 + (c^2 - c - 2)x + 1.$$

Now $q_2(c + 1) = 1$, which is positive; but $q_2(c + 2) = 5 - c^2$, which is negative. Hence, we see that $a_2 = c + 1$.

Now

$$p_3(x) = x^2 p_2\left(\frac{1 - x}{c + 1}\right),$$

so we see

$$p_3(x) = x^2 - (c^3 - 3c)x + 1.$$

So far we have been following the algorithm. But now we notice that $p_3(x)$ is essentially just $p_1(x)$ with $c^3 - 3c$ playing the role of c. We have found

$$\alpha = \frac{1}{c - 1} - \frac{1}{(c - 1)(c + 1)} + \frac{1}{(c - 1)(c + 1)}\gamma,$$

where γ is the root of $x^2 - (c^3 - 3c)x + 1 = 0$. By continuing this process, we get:

<u>Theorem</u>

Let α be as in equation (2). Then,

$$\alpha = \{c_0 - 1,\ c_0 + 1,\ c_1 - 1,\ c_1 + 1,\ c_2 - 1,\ c_2 + 1,\ \ldots\},$$

where $c_0 = c$, $c_{k+1} = c_k^3 - 3c_k$.

For example, let $c = 3$. Then we find

$$\frac{3 - \sqrt{5}}{2} = \{2, 4, 17, 19, 5777, 5779, \ldots\}.$$

Another example: let $c = 6$. Then, after some manipulation, we find

$$\sqrt{2} - 1 = \{2, 5, 7, 197, 199, 7761797, 7761799, \ldots\}.$$

Ironically, both Pierce [3] and Salzer [4] gave the first four terms of this expansion, but apparently neither detected the general pattern!

III. THE COEFFICIENTS c_k

The recurrence $c_{k+1} = c_k^3 - 3c_k$ is an interesting one which has been previously studied ([1], [2]). Some brief comments are in order.

If we let α and β be the roots of the quadratic

$$x^2 - cx + 1 = 0,$$

with $\alpha < \beta$, and define

$$V(n) = \alpha^n + \beta^n; \; U(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

then it is easy to show by induction that

$$V(n) = cV(n-1) - V(n-2); \; U(n) = cU(n-1) - U(n-2),$$

where

$$V(0) = 2, \; V(1) = c; \; U(0) = 0, \; U(1) = 1.$$

We can also show that $V(3k) = V(k)^3 - 3V(k)$; hence, by induction, $c_k = V(3^k)$. This gives the following closed form for the c_k:

$$c_k = \left(\frac{c + \sqrt{c^2 - 4}}{2}\right)^{3^k} + \left(\frac{c - \sqrt{c^2 - 4}}{2}\right)^{3^k}.$$

Similarly, it can be shown by induction that

$$\frac{U(3^k - 1)}{U(3^k)} = \{c_0 - 1, c_0 + 1, c_1 - 1, c_1 + 1, \ldots, c_{k-1} - 1, c_{k-1} + 1\}. \quad (3)$$

Here is a sketch of the induction step. Assuming (3) holds, we find

$$\{c_0 - 1, c_0 + 1, c_1 - 1, c_1 + 1, \ldots, c_k - 1, c_k + 1\}$$

$$= \frac{U(3^k - 1)}{U(3^k)} + \frac{1}{U(3^k)}\left(\frac{1}{c_k - 1} - \frac{1}{(c_k - 1)(c_k + 1)}\right)$$

$$= \frac{U(3^k - 1)}{U(3^k)} + \frac{1}{U(3^k)} \frac{c_k}{c_k^2 - 1}$$

$$= \frac{U(3^k - 1)(V(3^k)^2 - 1) + V(3^k)}{U(3^k)(V(3^k)^2 - 1)} \quad (4)$$

Now, using the fact that

$$U(3n) = U(n)(V(n)^2 - 1)$$

and

$$U(3n - 1) = U(n - 1)(V(n)^2 - 1) + V(n),$$

we see that the right side of (4) equals

$$\frac{U(3^{k+1} - 1)}{U(3^{k+1})}$$

which completes the induction step.

Equation (3) gives us an alternative proof of our Theorem above. By letting $k \to \infty$, we see that

$$\{c_0 - 1,\ c_0 + 1,\ c_1 - 1,\ c_1 + 1,\ \ldots\} = \lim_{k \to \infty} \frac{U(3^k - 1)}{U(3^k)} = \frac{1}{\beta} = \alpha.$$

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GENERALIZED PELL POLYNOMIALS AND OTHER POLYNOMIALS

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1. INTRODUCTION

Following some of the techniques in [1] and [2], Walton [8] and [9] discussed several properties of the polynomial sequence $\{A_n(x)\}$ defined by the second-order recurrence relation

$$A_{n+2}(x) = 2xA_{n+1}(x) + A_n(x),\ A_0(x) = q,\ A_1(x) = p. \tag{1.1}$$

The first few terms of $\{A_n(x)\}$ are:

$$\begin{cases} A_0(x) = q,\ A_1(x) = p,\ A_2(x) = 2px + q,\ A_3(x) = 4px^2 + 2qx + p, \\ A_4(x) = 8px^3 + 4qx^2 + 4px + q,\ A_5(x) = 16px^4 + 8qx^3 + 12px^2 + 4qx + p, \end{cases} \tag{1.2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Using standard techniques, we easily obtain the Binet form

$$A_n(x) = \frac{(p - q\beta)\alpha^n - (p - q\alpha)\beta^n}{\alpha - \beta}, \tag{1.3}$$

where

$$\begin{cases} \alpha = x + \sqrt{x^2 + 1} \\ \beta = x - \sqrt{x^2 + 1} \end{cases} \tag{1.4}$$

are the roots of

$$t^2 - 2xt - 1 = 0 \tag{1.5}$$

so that

$$\alpha + \beta = 2x,\ \alpha - \beta = 2\sqrt{x^2 + 1},\ \alpha\beta = -1. \tag{1.6}$$

In this paper we relate part of the work in [8] and [9] to other well-known polynomials. Thus, only some basic features of $\{A_n(x)\}$ will be examined.

It should be noted in passing that the expression for $\{A_n(x)\}$ in (1.3) is in agreement with the form for the n^{th} term of more general sequences of polynomials considered in [6]. Properties of the general sequence of numbers $\{W_n\}$ given in [4] are also readily generalized to yield properties of $\{A_n(x)\}$.

Note that when $x = 1/2$ in (1.1) we obtain the generalized Fibonacci number sequence $\{H_n\}$ whose basic properties are described in [3]. Furthermore, if we also let $p = 1$, $q = 0$ in (1.1), then we derive the sequence $\{F_n\}$ of Fibonacci numbers. Letting $p = 1$, $q = 2$ in (1.1) with $x = 1/2$, we obtain the sequence $\{L_n\}$ of Lucas numbers.

For unspecified x, the Pell polynomials $P_n(x)$ occur when $p = 1$ and $q = 0$ in (1.1), while for $p = 2x$ and $q = 2$ the Pell-Lucas polynomials $Q_n(x)$ arise. Relationships among $P_n(x)$ and $Q_n(x)$ are developed in [5]. Hence, polynomials of the sequence $\{A_n(x)\}$ may be called *generalized Pell polynomials*.

Readers may find some interest in specializing the results for $\{A_n(x)\}$ to the polynomial sequences $\{P_n(x)\}$ and $\{Q_n(x)\}$, and to the number sequences $\{H_n\}$, $\{F_n\}$, and $\{L_n\}$. Some of the specialized formulas for $\{H_n\}$ are, in fact, supplied in [8] and [9].

Though it is not strictly pertinent to this article, we wish to record an important formula for $\{A_n(x)\}$ which was not included in [9], namely, *Simson's formula*:

$$A_n^2(x) - A_{n+1}(x)A_{n-1}(x) = (-1)^n(q^2 - p^2 + 2px). \tag{1.7}$$

2. $A_n(x)$ AND CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

In [8] and [9] it is shown that

$$A_n(x) = q\sum_{m=0}^{[n/2]}\binom{n-m}{m}(2x)^{n-2m} + (p - 2qx)\sum_{m=0}^{\left[\frac{n-1}{2}\right]}\binom{n-1-m}{m}(2x)^{n-1-2m} \tag{2.1}$$

with $n \geqslant 1$. Furthermore, from [5] and [7], we have, respectively, the Pell polynomials given by

$$P_n(x) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]}\binom{n-m-1}{m}(2x)^{n-2m-1} \tag{2.2}$$

and the Chebyshev polynomials of the second kind given by

$$U_n(x) = \sum_{m=0}^{[n/2]}(-1)^m\binom{n-m}{m}(2x)^{n-2m}. \tag{2.3}$$

Letting x be replaced by ix in (2.3), we see that

$$\sum_{m=0}^{[n/2]}\binom{n-m}{m}(2x)^{n-2m} = (-i)^n U_n(ix) = P_{n+1}(x), \tag{2.4}$$

so that (2.1) can be rewritten as

$$\begin{aligned} A_n(x) &= q(-i)^n U_n(ix) + (p - 2qx)(-i)^{n-1}U_{n-1}(ix) \\ &= qP_{n+1}(x) + (p - 2qx)P_n(x) \\ &= pP_n(x) + qP_{n-1}(x), \end{aligned} \tag{2.5}$$

which is another form of (1.1), which could also have been obtained by using the generating functions for $A_n(x)$ (given in [9]) and $P_n(x)$ (given in [5]) or their respective Binet forms.

3. HYPERBOLIC FUNCTIONS AND $A_n(x)$

Elementary methods enable us to derive, when $x = \sinh w = (e^w - e^{-w})/2$,

$$A_{2k}(x) = \{p \sinh 2kw + q \cosh(2k - 1)w\}/\cosh w \tag{3.1}$$

and

$$A_{2k+1}(x) = \{p \cosh(2k+1)w + q \sinh 2kw\}/\cosh w. \tag{3.2}$$

To achieve these results, we use the Binet form (1.3) and

$$\alpha = e^{w},\ \beta = -e^{-w},\ \alpha - \beta = 2\cosh w = e^{w} + e^{-w}.$$

If we now use formulas (6.1) and (6.2) of [5], then (3.1) and (3.2) become (2.5) for the cases $n = 2k$ and $n = 2k + 1$, respectively.

4. GEGENBAUER POLYNOMIALS AND $A_n(x)$

The Gegenbauer polynomials C_n^k for $k > -\frac{1}{2}$, $k \neq 0$, are given in [7] by

$$C_n^k(x) = \frac{1}{\Gamma(k)} \sum_{m=0}^{[n/2]} (-1)^m \frac{\Gamma(n-m+k)}{\Gamma(n-m+1)} \binom{n-m}{m} (2x)^{n-2m}, \tag{4.1}$$

where $\Gamma(x)$ is the Gamma function. With $k = 1$, we have

$$C_n^1(x) = \sum_{m=0}^{[n/2]} (-1)^m \binom{n-m}{m} (2x)^{n-2m} = U_n(x), \tag{4.2}$$

so that by (2.5) we obtain

$$A_n(x) = q(-i)^n C_n^1(ix) + (p - 2qx)(-i)^{n-1} C_{n-1}^1(ix). \tag{4.3}$$

5. DETERMINANTAL GENERATION OF $A_n(x)$

Let us define two functional determinants $\Delta_{n-1}(x)$ and $\delta_{n-1}(x)$ of order $n-1$ as follows, where d_{ij} denotes the element in the i^{th} row and j^{th} column:

$$\Delta_{n-1}(x): \begin{cases} d_{ii} = 2px + q & i = 1, 2, \ldots, n-1 \\ d_{i,i+1} = p & i = 1, 2, \ldots, n-2 \\ d_{i,i-1} = -1 & i = 2, 3, \ldots, n-1 \\ d_{ij} = 0 & \text{otherwise} \end{cases} \tag{5.1}$$

$$\delta_{n-1}(x): \text{ as for } \Delta_{n-1}(x) \text{ except that } d_{i,i+1} = -p,\ d_{i,i-1} = 1. \tag{5.2}$$

Expansion along the first row then yields:

$$\begin{aligned} \Delta_{n-1}(x) &= (2px + q)\Delta_{n-2}(x) + p\Delta_{n-3}(x) && (5.3) \\ &= p\{2xP_{n-1}(x) + P_{n-2}(x)\} + qP_{n-1}(x) && \text{by (5.5) of [5]} \\ &= pP_n(x) + qP_{n-1}(x) && \text{by (1.1) of [5]} \\ &= A_n(x) && \text{by (2.5).} \end{aligned}$$

Similarly,

$$\delta_{n-1}(x) = A_n(x). \tag{5.4}$$

As mentioned at the end of §2, a generating function for $A_n(x)$ is given in [9].

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ON PELL NUMBERS OF THE FORM PX^2, WHERE P IS PRIME

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INTRODUCTION

Let n be a natural number, p a prime. Following Lucas [4], let A and B be integers such that

(i) $(A, B) = 1$ and $D = A^2 + 4B \neq 0.$

Let the roots of (ii) $x^2 = Ax + B$ be

(iii) $a = \frac{1}{2}(A + D^{1/2}), \quad b = \frac{1}{2}(A - D^{1/2}).$

Consider the sequences

(iv) $u_n = (a^n - b^n)/(a - b), \quad v_n = a^n + b^n.$

If $A = B = 1$, then u_n, v_n are the Fibonacci and Lucas sequences, respectively. If $A = 3$ and $B = -2$, then u_n, v_n are the Mersenne and Fermat sequences, respectively. If $A = 2$ and $B = 1$ (so that $D = 8$), then u_n is called the Pell sequence (see [4, p. 187]), and is denoted P_n; v_n may be called the secondary Pell sequence, and denoted R_n, following [7]. For the sake of convenience, we occasionally write $u(n)$ instead of u_n and $P(n)$ instead of P_n. Table 1, below, lists P_n and R_n for $1 \leqslant n \leqslant 50$.

TABLE 1

n	P_n	R_n
1	1	2
2	2	6
3	5	14
4	12	34
5	29	82
6	70	198
7	169	478
8	408	1154
9	985	2786
10	2378	6726
11	5741	16238
12	13860	39202
13	33461	94642
14	80782	228486
15	195025	551614
16	470832	1331714
17	1136689	3215042
18	2744210	7761798
19	6625109	18738638
20	15994428	45239074
21	38613965	109216786

TABLE 1 (continued)

n	P_n	R_n
22	93222358	263672646
23	225058681	636562078
24	543339720	1536796802
25	1311738121	3710155682
26	3166815962	8957108166
27	7645370045	21624372014
28	18457556052	52205852194
29	44560482149	126036076402
30	107578520350	304278004998
31	259717522849	734592086398
32	627013566048	1773462177794
33	1513744654945	4281516441986
34	3654502875938	10336495061766
35	8822750406821	24954506565518
36	21300003689580	60245508192802
37	51422757785981	145445522951122
38	124145519261542	351136554095046
39	299713796309065	847718631141214
40	723573111879672	2046573816377474
41	1746860020068409	4940866263896162
42	4217293152016490	11928306344169798
43	10181446324101389	28797478952235758
44	24580185800219268	69523264248641314
45	59341817924539925	167844007449518386
46	143263821649299118	405211279147678086
47	345869461223138161	978266565744874558
48	835002744095575440	2361744410637427202
49	2015874949414289041	5701755387019728962
50	4866752642924153522	13765255184676885126

All solutions of the Pell equations $x^2 - 2y^2 = \pm 1$ such that $x \geqslant y \geqslant 0$ are given, respectively, by

$$(x_n, y_n) = \left(\frac{1}{2}R_{2n}, P_{2n}\right), \left(\frac{1}{2}R_{2n-1}, P_{2n-1}\right).$$

Furthermore, if $(x, x+1, z)$ is a Pythagorean triple, then there exists n such that $z = P_{2n+1}$, while

$$\{x, x+1\} = \{P_{n+1}^2 - P_n^2, 2P_nP_{n+1}\}.$$

These results follow from [8, pp. 44-48 and 94-98].

In [3], W. Ljunggren proved that if $x \geqslant y \geqslant 0$ and $x^2 - 2y^4 = -1$, then

$$(x, y) = (1, 1) \text{ or } (239, 13).$$

From this result, it follows that if $P_n = x^2$ with $x > 0$, then

$$(n, x) = (1, 1) \text{ or } (7, 13).$$

In this article, we consider the equation:

$$P_n = px^2. \tag{*}$$

We obtain all solutions such that $p \equiv 3 \pmod 4$ or $p < 1000$. The method used here is similar to the method used in [6] to find Fibonacci numbers of the same form. (m/p) is the Legendre symbol.

Definition 1: $z(n) = \min\{k : n|u_k\}$; $z^*(n) = \min\{k : n|P_k\}$.

Definition 2: $y(p)$ is the least prime divisor of $z(p)$.

PRELIMINARY RESULTS

(1) $z^*(2) = 2$

(2) $z^*(3) = 4$

(3) $z^*(5) = 3$

(4) $z^*(7) = 6$

(5) $z^*(13) = z^*(13^2) = 7$

(6) $z^*(29) = 5$

(7) If $D \neq s^2$, then $z(p)|(p - e)$, where $e = \begin{cases} (D/p) & \text{if } p \nmid D \\ 0 & \text{if } p|D \end{cases}$

(8) $p|u_n$ iff $z(p)|n$; $p|P_n$ iff $z^*(p)|n$

(9) $P_{2n+1} = P_n^2 + P_{n+1}^2$

(10) $(P_m, P_n) = P_{(m,n)}$

(11) $P_{2n} = P_n R_n$

(12) $(P_n, R_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$

(13) $R_n \equiv 2 \pmod 4$ for all n

(14) $\left(\frac{1}{2}R_n\right)^2 - 2P_n^2 = (-1)^n$

(15) $u_n|u_{kn}$; $P_n|P_{kn}$

(16) If m is odd, then $R_n \neq ms^2$

(17) If $p \equiv 3 \pmod 4$, then $z^*(p)$ is even

(18) $(u_n, u_{kn}/u_n)|k$; $(P_n, P_{kn}/P_n)|k$

(19) If $x^4 - 2y^2 = (-1)^n$, then n is odd and $x^2 = y^2 = 1$

(20) $R_n = 2x^2$ implies $n = 1$

(21) $P_n = x^2$ implies $n = 1$ or 7

(22) If p is odd and $p^k \| u_n$, then $p^{k+1} \| u_{pn}$

Remarks: Results (1) through (6) may be verified by examining the first seven entries in Table 1. (7) through (15) are elementary and/or well known. (16) follows from (13). (17) follows from (9), (10), Definition 1, and [2, Theorem 367, p. 299]. (18) is Theorem 2 in [5]. (19) is proved in [8, p. 98]. (20) follows from (14) and (19). (21) follows from (14) and the result of Ljunggren mentioned above. (22) follows from [1, Theorem X, p. 42].

THE MAIN THEOREMS

Theorem 1

$P_n = 2x^2$ implies $n = 2$.

Proof: Hypothesis, (1), and (8) imply $n = 2m$, so that (11) implies

$$P_m R_m = 2x^2.$$

If m is even, then (12) implies

$$\left(\frac{1}{2}P_m, \frac{1}{2}R_m\right) = 1.$$

But

$$\left(\frac{1}{2}P_m\right)\left(\frac{1}{2}R_m\right) = 2\left(\frac{1}{2}x\right)^2.$$

Now (13) implies $\frac{1}{2}R_m = s^2$, so that (20) implies $m = 1$, a contradiction. If m is odd, then (12) and (16) imply $P_m = r^2$, $R_m = 2s^2$. Now (20) implies $m = 1$, so $n = 2$.

Theorem 2

If p is odd and $P_{2m} = px^2$, then $p = 3$ and $2m = x^2 = 4$.

Proof: Hypothesis and (11) imply $P_m R_m = px^2$. If m is odd, then (12) implies $R_m = s^2$ or ps^2, contradicting (16). If m is even, then (12) implies

$$\left(\frac{1}{2}P_m, \ \frac{1}{2}R_m\right) = 1.$$

But

$$\left(\frac{1}{2}P_m\right)\left(\frac{1}{2}R_m\right) = p\left(\frac{1}{2}x\right)^2.$$

Therefore, P_m or $R_m = 2s^2$. Now (20) and Theorem 1 imply $m = 2$, so that

$$P_{2m} = P_4 = 12 = 3(2)^2.$$

Corollary 1

If p is odd, $z^*(p)$ is even, and $P_n = px^2$, then $p = 3$ and $n = x^2 = 4$.

Proof: Hypothesis and (8) imply n is even, so that the conclusion follows from Theorem 2.

Corollary 2

If $p \equiv 3 \pmod 4$ and $P_n = px^2$, then $p = 3$ and $n = x^2 = 4$.

Proof: Follows from hypothesis, (17), and Corollary 1.

If $p \geqslant 5$, then the investigation of (*) is facilitated by Lemmas 1 and 2 below, which hold for general sequences u_n, v_n which satisfy (i) through (iv) above, where $D \neq s^2$.

Lemma 1

Suppose p is odd, $p \nmid m$, and $c_i = u(mp^i)/u(mp^{i-1})$ for $i \geqslant 1$. If $i < j$, then

$$(c_i, c_j) = \begin{cases} p & \text{if } p \mid D \\ 1 & \text{if } p \nmid D. \end{cases}$$

Proof: Let $d = (c_i, c_j)$, where $i < j$. Therefore, $d \mid c_i$, $d \mid c_j$, and $d \mid u(mp^i)$. Now hypothesis and (15) imply $d \mid u(mp^{j-1})$, so

$$d \mid (u(mp^{j-1}), u(mp^j)/u(mp^{j-1})).$$

Therefore, (18) implies $d \mid p$. If $p \nmid D$, then (7) and (8) imply $p \nmid u(p)$, so that (15) implies $p \nmid u(mp^i)$. Therefore, $p \nmid d$, so $d = 1$. If $p \mid D$, then (7) and (8) imply $p \mid u(p)$. Now (22) implies $p \mid c_i$ and $p \mid c_j$, so $p \mid d$, and $d = p$.

Lemma 2

If $u_n = px^r$, $y(p) = q$, a prime, and $(pq, D) = 1$, then $n = q^k m$, where $k \geqslant 1$ and $(s, m) = 1$ for all primes, s, such that $s \leqslant q$, and, furthermore, $p \nmid u_m$. If also $q \nmid u_m$, then $u_m = c^r$ and there is an integer, t, such that $1 \leqslant t \leqslant k$, and for all j such that $1 \leqslant j \leqslant k$, we have

$$u(mq^j)/u(mq^{j-1}) = \begin{cases} px_j^r & \text{if } j = t \\ x_j^r & \text{if } j \neq t. \end{cases}$$

Proof: Hypothesis, (8), and Definitions 1 and 2 imply $n = q^k m$, $k \geqslant 1$, and $(s, m) = 1$ for all primes, s, such that $s \leqslant q$. (8) implies $p \nmid u_m$. Let

$$d = (u_m, u_n/u_m).$$

If $q \nmid u_m$, then (18) implies $d = 1$. Since $(u_m)(u_n/u_m) = px^r$, we have $u_m = c^r$ and $u_n/u_m = pw^r$. For each j such that $1 \leqslant j \leqslant k$, let $a_j = u(mq^j)/u(mq^{j-1})$. Now

$$u_n/u_m = \prod_{j=1}^{k} a_j,$$

so that

$$(1/p)\prod_{j=1}^{k} a_j = w^r.$$

Lemma 1 implies the factors on the left side of this last equation are pairwise coprime; the conclusion now follows.

Theorem 3

If $P_n = 5x^2$, then $n = 3$.

Proof: Hypothesis, (3), (8), and Lemma 2 imply $n = 3^k m$ and $(6, m) = 1$. Therefore, (2) and (8) imply $3 \nmid P_m$. Now Lemma 2 implies $P_m = s^2$, so (21) implies $m = 1$ or 7. Lemma 2 implies $P_{3m}/P_m = s^2$ or $5s^2$. Since $P_{21}/P_7 = 5*45697 \neq s^2$, $5s^2$, we must have $m = 1$. If $k \geqslant 2$, then Lemma 2 implies $197 = P_9/P_3 = s^2$, an impossibility. Therefore, $k = 1$, so $n = 3$.

Theorem 4

If $P_n = 29x^2$, then $n = 5$.

Proof: Hypothesis, (6), (8), and Lemma 2 imply $n = 5^k m$ and $(30, m) = 1$. Therefore, (3) and (8) imply $5 \nmid P_m$. Now Lemma 2 implies $P_m = s^2$, so (21) implies $m = 1$ or 7. Lemma 2 implies $P_{5m}/P_m = s^2$ or $29s^2$. Since

$$P_{35}/P_7 = 29*1800193921 \neq s^2,\ 29s^2,$$

we must have $m = 1$. If $k \geqslant 2$, then Lemma 2 implies $45232349 = P_{25}/P_5 = s^2$, an impossibility. Therefore, $k = 1$, so $n = 5$.

Lemma 3

If $P_n = px^2$, where $n = 7^k m$, $k \geqslant 1$, and $(14, m) = 1$, then $P_m = px_1^2$.

Proof: Let $d = (P_m, P_n/P_m)$. Hypothesis, (4), and (8) imply $7 \nmid P_m$, so $7 \nmid d$. Now (18) implies $d = 1$, so $P_m = x_1^2$ or px_1^2. If $P_m = x_1^2$, then hypothesis and (21) imply $m = 1$, so $n = 7^k$. Since

$$P_7 = 13^2 \neq px_1^2,$$

we must have $k \geqslant 2$. But then Lemma 2 implies

$$293*40710764977973 = P_{49}/P_7 = x_2^2 \text{ or } px_2^2,$$

an impossibility. Therefore, we must have $P_m = px_1^2$.

Corollary 3

$P_n \neq 13x^2$.

Proof: If $P_n = 13x^2$, then (5) and (8) imply $n = 7^k m$, $7 \nmid m$. Theorem 2 implies m is odd, so Lemma 3 implies $P_m = 13x_1^2$, contradicting (5) and (8).

Theorem 5

Let $P_n = px^2$, where p and $z^*(p)$ are odd. Then there exists a prime, t, such that $P_t = py^2$. In fact, $t = z^*(p)$.

Proof: If n is prime, then $t = n$ and $x^2 = y^2$. Therefore, assume n is composite. Hypothesis and Theorem 2 imply n is odd. (1) and (8) imply P_n is odd, so x is odd. If $n = 7^k m$, $7 \nmid m$, then Hypothesis and Lemma 3 imply $P_m = px_1^2$. So without loss of generality assume $7 \nmid n$, so that if $d \mid n$, then $d \neq 7$.

Case 1 Suppose there exists d such that $d \mid n$, $1 < d < n$, and $z^*(p) \nmid d$. Then (8) implies $p \nmid P_d$. Since $d \neq 7$, (21) implies $P_d \neq s^2$. Therefore, there exists a prime, q_1, such that $q_1 \neq p$ and $q_1^{2j_1-1} \,\|\, P_d$. Now, (15) implies $q_1^{2j_1-1} \mid P_n$, so that $q_1^{2j_1-1} \mid x^2$. This implies that $q_1^{2j_1} \mid x^2$, so that $q_1^{2j_1} \mid P_n$. But (22) implies $q_1^{j_1} \,\|\, P_{dq_1}$. Therefore, $q_1^{2j_1} \mid (P_n, P_{dq_1})$. Now, (10) implies $q_1^{2j_1} \mid P_{(n, dq_1)}$. Since $q_1^{2j_1-1} \,\|\, P_d$, we must have $(n, dq_1) > d$, so that $(n/d, q_1) > 1$. Therefore, $q_1 \mid n/d$ and $q_1 \mid n$. Since $q_1 \neq 7$, (21) implies $P(q_1) \neq s^2$. Thus, there exists a prime, q_2, such that $q_2^{2j_2-1} \,\|\, P(q_1)$. If the only such prime is p, then

$$P(q_1) = p^{2j_2-1}s^2 = p(p^{j_2-1}s)^2,$$

so that $t = q_1$. If $q_2 \neq p$, then $q_2^{2j_2-1} \mid x^2$, so that by reasoning as above we obtain $q_2 \mid n$. Continuing in like fashion, we obtain a sequence of primes: q_1, q_2, q_3, etc., such that $q_i \mid P(q_{i-1})$ and either $q_i \mid n$ or $q_i = p$ for $i \geqslant 2$. Since the q_i are all odd, (7) and (8) imply $q_i \neq q_{i-1}$. Now (10) implies that the q_i are all distinct. Since n has only finitely many divisors, there must exist r such that $q_r = p$, and thus $q_{r-1} = t$.

Case 2 Suppose that $z^*(p) \mid d$ for all d such that $d \mid n$ and $1 < d < n$. Then $z^*(p) = q$ is a prime and $n = q^k$. Now Lemma 2 implies $P_q = x_1^2$ or px_1^2. (21) implies $P_q \neq x_1^2$, so $P_q = px_1^2$ and $t = q$. In either case, since $p \mid P_t$, (8) implies $z^*(p) \mid t$. Since t is prime, we must have $z^*(p) = t$.

Lemma 4

Suppose $z^*(p) = q$, a prime, and $q > 3$. If $p \equiv \pm 2 \pmod 5$, then

$$\left(\frac{p^{-1}P_q}{5}\right) = -1;$$

if $p \equiv 3$, 5, or 6 (mod 7), then

$$\left(\frac{p^{-1}P_q}{7}\right) = -1.$$

Proof: Hypothesis implies $q \equiv \pm 1 \pmod 6$, so that $P_q \equiv \pm 1 \pmod 5$ and $P_q \equiv 1 \pmod 7$. If $p \equiv \pm 2 \pmod 5$, then

$$\left(\frac{p^{-1}P_q}{5}\right) = \left(\frac{p^{-1}}{5}\right)\left(\frac{P_q}{5}\right) = \left(\frac{p}{5}\right)\left(\frac{P_q}{5}\right) = (-1)1 = -1.$$

If $p \equiv 3$, 5, or 6 (mod 7), then

$$\left(\frac{p^{-1}P_q}{7}\right) = \left(\frac{p^{-1}}{7}\right)\left(\frac{P_q}{7}\right) = \left(\frac{p}{7}\right)\left(\frac{P_q}{7}\right) = (-1)1 = -1.$$

Lemma 5

Suppose $z^*(p) = q$, a prime, and $q > 3$. If either

(i) $\left(\frac{p}{11}\right) = -1$ and $q \equiv \pm 1$ or $\pm 7 \pmod{24}$, or

(ii) $\left(\frac{p}{11}\right) = 1$ and $q \equiv \pm 5$ or $\pm 11 \pmod{24}$,

then $\left(\frac{p^{-1}P_q}{11}\right) = -1$.

Proof: If (i) holds, then $P_q \equiv 1$ or 4 (mod 11), so $\left(\frac{P_q}{11}\right) = 1$; if (ii) holds, then $P_q \equiv 7$ or 10 (mod 11), so $\left(\frac{P_q}{11}\right) = -1$. Therefore,

$$\left(\frac{p^{-1}P_q}{11}\right) = \left(\frac{p^{-1}}{11}\right)\left(\frac{P_q}{11}\right) = \left(\frac{p}{11}\right)\left(\frac{P_q}{11}\right) = (-1)1 \text{ or } 1(-1) = -1.$$

Theorem 6

If $P_n = px^2$ and $p < 1000$, then $(n, p) = (2, 2), (4, 3), (3, 5)$, or $(5, 29)$.

Proof: By Theorems 1, 3, 4, and 5, and Corollaries 2 and 3, we need only consider those primes p, such that $37 \leqslant p < 1000$, $p \equiv 1 \pmod 4$, and $z^*(p) = q$ is prime. Examining Table 2 below, we see that these primes are:

37, 61, 137, 157, 229, 277, 397, 421, 541, 569, 593,
613, 661, 677, 733, 757, 821, 853, 857, 877, 997.

Lemma 4 implies that $p^{-1}P_q$ is a quadratic nonresidue (mod 5) or (mod 7) except for $p = 421, 541, 569$, and 821. In each of these four latter cases, Lemma 5 implies that $p^{-1}P_q$ is a quadratic nonresidue (mod 11). Therefore, in no case does $P_q = px^2$.

TABLE 2

PELL ENTRY POINTS OF PRIMES, p, SUCH THAT $p \equiv 1 \pmod 4$, $p < 1000$

p	$z^*(p)$	p	$z^*(p)$	p	$z^*(p)$	p	$z^*(p)$
5	3	197	9	433	216	709	355
13	7	229	23	449	224	733	367
17	8	233	116	457	114	757	379
29	5	241	40	461	231	761	190
37	19	257	64	509	255	769	384
41	10	269	15	521	65	773	129
53	27	277	139	541	271	797	399
61	31	281	140	557	279	809	202
73	36	293	49	569	71	821	137
89	44	313	78	577	16	829	415
97	48	317	159	593	37	853	61
101	51	337	28	601	60	857	107
109	55	349	175	613	307	877	439
113	28	353	22	617	308	881	220
137	17	373	187	641	320	929	464
149	75	389	39	653	327	937	468
157	79	397	199	661	331	941	471
173	87	401	200	673	336	953	119
181	91	409	102	677	113	977	488
193	96	421	211	701	351	997	499

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A FIBONACCI-LIKE SEQUENCE OF ABUNDANT NUMBERS

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Let $\sigma(n)$ denote the sum of the divisors of n. An integer n is said to be *abundant* if $\sigma(n) > 2n$, *perfect* if $\sigma(n) = 2n$, or *deficient* if $\sigma(n) < 2n$. It is known [2] that if the greatest common divisor of the integers a and b is deficient, then there exist infinitely many deficient integers $n \equiv a \pmod{b}$. Fibonacci buffs might expect an analogous result for generalized Fibonacci numbers, something along the lines of "if $x_{n+1} = x_n + x_{n-1}$ and gcd (x_1, x_2) is deficient, then the sequence $\{x_n\}$ contains infinitely many deficient terms." In this note we shatter any such expectations by constructing a Fibonacci-like sequence $\{x_n\}$ with all terms abundant and having gcd (x_1, x_2) deficient.

Vital to the construction are two easily proved theorems:

(1) Any multiple of an abundant number is abundant.

(2) If p is an odd prime, then $2^a p$ is abundant if $p < 2^{a+1} - 1$,
perfect if $p = 2^{a+1} - 1$,
and deficient if $p > 2^{a+1} - 1$.

Graham [1] defined a Fibonacci-like sequence by

$$g_1 = 1786\ 772701\ 928802\ 632268\ 715130\ 455793,$$

$$g_2 = 1059\ 683225\ 053915\ 111058\ 165141\ 686995,$$

and $g_{n+1} = g_n + g_{n-1}$. Graham's sequence has the remarkable property that even though gcd $(g_1, g_2) = 1$, every term is composite. More specifically, every term is divisible by at least one of the primes

(3) 2, 3, 5, 7, 11, 17, 19, 31, 41, 47, 53, 61, 109, 1087, 2207, 2521, 4481, 5779.

Now, define a sequence $\{x_n\}$ by

$$x_n = 2^{12}8209 \cdot g_n,$$

where $\{g_n\}$ is Graham's sequence. Since $5779 < 2^{13} = 8192$, $2^{12}q$ is abundant for each odd q listed in (3), and $2^{13}8209$ is abundant since $8209 < 2^{14} - 1$. Therefor, each x_n is abundant. But

$$\gcd(x_1, x_2) = 2^{12}8209$$

is deficient since $8209 > 2^{13} - 1$.

Clearly, in the construction above, we may replace 8209 by any prime p such that $2^{13} < p < 2^{14}$.

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A NOTE ON THE CYCLE INDICATOR

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1. INTRODUCTION

Let the cycle indicator

$$C_n(t) \equiv C_n(t_1, \ldots, t_n) = \sum \frac{n!}{k_1! \ldots k_n!}\left(\frac{t_1}{1}\right)^{k_1} \cdots \left(\frac{t_n}{n}\right)^{k_n}, \tag{1}$$

where the summation is over all nonnegative integral values of $k_1, \ldots, k_n$ such that $k_1 + 2k_2 + \cdots + nk_n = n$.

The exponential generating function of $C_n(t)$ is (see [2, Ch. 4]:

$$\exp uC = \sum_{n=0}^{\infty} C_n(t)\frac{u^n}{n} = \exp\left\{\sum_{k=1}^{\infty} \frac{t_k}{k}u^k\right\}, \quad |u| < 1. \tag{2}$$

Applying a Tauberian theorem [1, Th. 5, p. 447] to (2), we will be able to derive a limiting expression of $C_n(t)/n!$, as $n \to \infty$, under certain conditions.

2. A LIMIT THEOREM

Before we state and prove the main theorem, we shall prove the following lemma, which will be useful in the sequel.

Lemma 1

If

$$\frac{1}{n}\sum_{k=1}^{n} t_k \to t, \quad 0 < t < \infty,$$

and the sequence $\{t_n\}$, $n = 1, 2, \ldots$, is monotonic, then the sequence

$$\left\{\frac{C_n(t)}{n!}\right\}, \quad n = 1, 2, \ldots,$$

is monotonic for $n > N$, where N is a fixed number.

Proof: Using the well-known recurrence relation of the cycle indicator, we have:

$$\begin{aligned}\frac{C_{n+1}(t)}{(n+1)!} &= \frac{1}{(n+1)!}\sum_{k=0}^{n} (n)_k\, t_{k+1}C_{n-k}(t)\\ &= \frac{1}{n+1}t_1\frac{C_n(t)}{n!} + \frac{1}{n+1}\left\{t_2\frac{C_{n-1}(t)}{(n-1)!} + \cdots + t_{n+1}\right\}.\end{aligned} \tag{3}$$

Supposing that $\{t_n\}$, $n = 1, 2, \ldots$, is monotonic decreasing, equation (3) is written:

$$\frac{C_{n+1}(t)}{(n+1)!} < \frac{1}{n+1} t_1 \frac{C_n(t)}{n!} + \frac{1}{n+1} \frac{C_n(t)}{n!} = \frac{t_1 + 1}{n+1} \frac{C_n(t)}{n!}. \tag{4}$$

Since $\{t_n\}$, $n = 1, 2, \ldots$, is bounded, equation (4) is bounded by

$$\frac{C_{n+1}(t)}{(n+1)!} < \left(\frac{N+1}{n+1}\right)\frac{C_n(t)}{n!} \quad \text{or} \quad \frac{C_{n+1}(t)}{(n+1)!} < \frac{C_n(t)}{n!} \quad \text{for all } n > N.$$

Theorem 1

If $\frac{1}{n} \sum_{k=1}^{n} t_k \to t$, as $n \to \infty$, $0 < t < \infty$, then

$$\exp uC \sim \frac{1}{(1-u)^t} L\left(\frac{1}{1-t}\right), \text{ as } u \uparrow 1-, \tag{5}$$

where L is a slowly varying function at infinity.

Furthermore, equation (5) implies that

$$\sum_{k=0}^{n-1} \frac{C_k(t)}{k!} \sim \frac{1}{\Gamma(t+1)} n^t L(n), \text{ as } n \to \infty. \tag{6}$$

If, additionally, $\{t_n\}$, $n = 1, 2, \ldots$, is monotonic, then equation (5) is equivalent to

$$\frac{C_n(t)}{n!} \sim \frac{1}{\Gamma(t)} n^{t-1} L(n), \text{ as } n \to \infty. \tag{7}$$

Proof: Using the relation

$$\sum_{k=1}^{\infty} \frac{u^k}{k} = \log \frac{1}{1-u}, \text{ for } 0 < u < 1,$$

equation (2) is written

$$\exp uC = \frac{1}{(1-u)^t} \exp\left\{\sum_{k=1}^{\infty} \frac{u^k}{k}(t_k - t)\right\}. \tag{8}$$

Letting

$$L\left(\frac{1}{1-u}\right) = \exp\left\{\sum_{k=1}^{\infty} \frac{u^k}{k}(t_k - t)\right\}, \tag{9}$$

and making the substitutions

$$\frac{1}{1-u} = x \quad \text{and} \quad t_k - t = y_k,$$

equation (9) is written

$$L(x) = \exp\left\{\sum_{k=1}^{\infty} \left(1 - \frac{1}{x}\right)^k \frac{y_k}{k}\right\},$$

which is a slowly varying function at infinity, according to [1, Cor. p. 282]. So equation (5) has been proved. Now, applying Theorem 5 of [1, p. 447], we get equation (6).

Using Lemma 1 and the same Theorem we have that (5) is equivalent to (7).

Corollary 1

If
$$\frac{1}{n}\sum_{k=1}^{n} t_k \to t \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{n} s_k \to s,$$
and the sequences $\{t_n\}$, $\{s_n\}$, $n = 1, 2, \ldots$, are monotonic, then
$$\frac{C_n(t+s)}{n!} \sim \frac{1}{\Gamma(t+s)} n^{t+s-1}L(n), \text{ as } n \to \infty. \tag{10}$$

Proof: Since the $C_n(t)$ is of the binomial type, we have:
$$C_n(t+s) = \sum_{k=0}^{n} \binom{n}{k} C_k(t)C_{n-k}(s). \tag{11}$$

Applying equation (7) to (11), we get
$$\frac{C_n(t+s)}{n!} = \sum_{k=0}^{n} \frac{1}{\Gamma(t)} k^{t-1}L(k)\frac{1}{\Gamma(s)}(n-k)^{s-1}L(n-k) + o(k^{t-1}, (n-k)^{s-1}), \tag{12}$$
where $o(k^{t-1}, (n-k)^{s-1})$ is such that
$$\frac{o(k^{t-1}, (n-k)^{s-1})}{k^{t-1} \cdot (n-k)^{s-1}} \to 0$$
uniformly in k and n as the $\min(k, n-k) \to \infty$.

Equation (12) is equivalent to
$$\frac{C_n(t+s)}{n!} = \frac{n^{t+s-1}}{\Gamma(t)\Gamma(s)} L^2(n) \sum_{\frac{k}{n}=0}^{1} n^{-1}\left(\frac{k}{n}\right)^{t-1}\left(1-\frac{k}{n}\right)^{s-1} \frac{L\left(n\frac{k}{n}\right)}{L(n)} \frac{L\left(n\left(1-\frac{k}{n}\right)\right)}{L(n)}$$
$$+ o\left(\left(\frac{k}{n}\right)^{t-1}, \left(1-\frac{k}{n}\right)^{s-1}\right). \tag{13}$$

By the definition of slowly varying function at infinity, we have that
$$\frac{L\left(n\frac{k}{n}\right)L\left(n\left(1-\frac{k}{n}\right)\right)}{L(n)L(n)} \to 1 \text{ as } n \to \infty.$$

Thus, interpreting the sum in (13) as the approximation to a Riemann integral as $n \to \infty$, we get
$$\frac{C_n(t+s)}{n!} \sim \frac{n^{t+s-1}}{\Gamma(t)\Gamma(s)} L^2(n)\int_0^1 x^{t-1}(1-x)^{s-1}dx$$
or
$$\frac{C_n(t+s)}{n!} \sim \frac{n^{t+s-1}}{\Gamma(t)\Gamma(s)} L(n)B(t, s), \tag{14}$$

where $B(t+s)$ is the Beta function. Since it is well known that

$$B(t,\ s) = \frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)},$$

equation (14) implies (10).

Corollary 2

If $t_k = t$ for $k = 1, 2, \ldots, 0 < t < 1$, then

$$\sum_{k=0}^{n-1} \frac{C_k(t)}{n!} \sim \frac{1}{\Gamma(t+1)} n^t, \text{ as } n \to \infty, \tag{15}$$

and

$$\frac{C_n(n)}{n!} \sim \frac{1}{\Gamma(t)} n^{t-1}, \text{ as } n \to \infty. \tag{16}$$

Proof: In this case, the exponential generating function of $C_n(t, \ldots, t)$ is written

$$\exp uC = (1-u)^{-t} \tag{17}$$

as it is well known [2, p. 70].

Applying Theorem 5 [1, p. 447] to (17) we get (15), and since the sequence

$$\left\{\frac{C_n(t)}{n!}\right\},\ n = 1, 2, \ldots,$$

is monotonic decreasing [2, (11), p. 71], relation (17) is equivalent to (16).

Remark 1: Concerning the same probability problem as that in [2, p. 71], $C_n(t)/n!$ is the generating function of certain probabilities.

Using equation (16), we can easily verify by differentiating that

$$\mu \sim \log(n) + \gamma, \text{ as } n \to \infty,$$

where γ is Euler's constant and

$$\sigma^2 \sim \log n + \gamma + \zeta(2),$$

where $\zeta(2)$ is the Riemann Zeta function which, in this special case, is equal to $\pi^2/6$. Both these results agree with those obtained in [2, p. 72].

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HARMONIC, GEOMETRIC, AND ARITHMETIC MEANS IN GENERALIZED FIBONACCI SEQUENCES

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The Fibonacci sequence of numbers, F_i, can be defined as the sequence whose first two terms are unity and whose n^{th} term (for $n > 2$) is equal to the sum of the $(n - 1)^{\text{st}}$ term and the $(n - 2)^{\text{nd}}$ term. The ratio of increase between successive terms rapidly approaches a constant value, the positive root of the equation

$$x^2 - x - 1 = 0, \tag{1}$$

which is $\phi = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618034$.

Fibonacci sequences can be generalized by increasing the number of previous terms that are summed to produce subsequent terms. Therefore, the Tribonacci sequence, T_i, has its n^{th} term equal to the sum of its $(n - 1)^{\text{st}}$, $(n - 2)^{\text{nd}}$, and $(n - 3)^{\text{rd}}$ terms. The ratio of increase between successive terms in the Tribonacci sequence is the real, positive root of the equation

$$x^3 - x^2 - x - 1 = 0, \tag{2}$$

which is $\tau = 1.839287$ (see [1], [2], and [3]).

In general, the n-bonacci sequence has its i^{th} term equal to the sum of the previous n terms. The ratio if increase is then the real, positive root of the equation

$$x^n - x^{n-1} - \cdots - x - 1 = 0. \tag{3}$$

For $n \geqslant 2$, $\phi \leqslant x < 2$.

Because such generalized Fibonacci sequences soon approximate geometric sequences, all of the terms in those sequences (aside from initial values) are approximately equal to the geometric means of the immediately preceding and immediately following terms. At the same time, because of the Fibonacci nature of those sequences, each term is also approximately equal to the harmonic mean and exactly equal to the arithmetic mean of neighboring terms. Those properties are the focus of the present paper.

The harmonic, geometric, and arithmetic means of two positive numbers, a and b, are defined as

$$HM(a, b) = \frac{2ab}{a + b} \tag{4}$$

$$GM(a, b) = \sqrt{ab}, \tag{5}$$

and

$$AM(a, b) = \frac{a + b}{2}, \tag{6}$$

respectively. They are related by the classical chain of inequalities

$$HM(a, b) \leqslant GM(a, b) \leqslant AM(a, b).$$

Now consider the question of finding a geometric sequence (i.e., a sequence of terms where the i^{th} term is equal to the $(i - 1)^{\text{st}}$ term times a constant) in which any four consecutive terms can be written in the form

$$a,\ HM(a, b),\ AM(a, b),\ b.$$

If we set $a = 1$ and denote the ratio by x, we must solve the set of equations

$$x = \frac{2b}{1+b}, \quad x^2 = \frac{1+b}{2}, \text{ and } x^3 = b, \tag{7}$$

which are consistent and reduce to

$$x^3 - 2x^2 + 1 = 0. \tag{8}$$

By inspection, $x = 1$ is a root of equation (8), indicating that a sequence of identical terms satisfies the stated conditions. To exclude that trivial solution, we divide equation (8) by $(x - 1)$ and find

$$x^2 - x - 1 = 0, \tag{9}$$

the equation for the ratio of the Fibonacci sequence. Thus, the integers in a Fibonacci sequence approximate the harmonic and arithmetic means of nearby Fibonacci numbers. Table 1 shows the first 15 Fibonacci numbers and indicates that the arithmetic mean relationship is exact from $n = 2$ onward, while the harmonic mean relationship is correct to within ±0.1 as early as $n = 6$.

TABLE 1

HARMONIC AND ARITHMETIC MEANS IN THE FIBONACCI SEQUENCE

n	Fibonacci Number F_n	Harmonic Mean of F_{n-1}, F_{n+2}	Arithmetic Mean of F_{n-2}, F_{n+1}	Ratio F_n/F_{n-1}
1	1	-	-	-
2	1	1.500	-	1.000
3	2	1.667	2	2.000
4	3	3.200	3	1.500
5	5	4.875	5	1.667
6	8	8.077	8	1.600
7	13	12.952	13	1.625
8	21	21.029	21	1.615
9	34	33.982	34	1.619
10	55	55.011	55	1.618
11	89	88.993	89	1.618
12	144	144.004	144	1.618
13	233	232.997	233	1.618
14	377	377.002	377	1.618
15	610	609.999	610	1.618

The same approach can be applied to finding a geometric series where successive terms can be written in the form

$$a,\ HM(a, b),\ GM(a, b),\ AM(a, b),\ b.$$

With $a = 1$ and ratio x, the equations are

$$x = \frac{2b}{1+b}, \quad x^2 = \sqrt{b}, \quad x^3 = \frac{1+b}{2}, \text{ and } x^4 = b, \tag{10}$$

which are consistent and reduce to

$$x^4 - 2x^3 + 1 = 0. \tag{11}$$

If we again divide by $(x - 1)$ to eliminate the trivial solution, we have

$$x^3 - x^2 - x - 1 = 0, \tag{12}$$

the equation for the ratio of the Tribonacci sequence. Table 2 shows the first 15 Tribonacci numbers, and again the mean relationships emerge quite quickly.

TABLE 2

HARMONIC, GEOMETRIC, AND ARITHMETIC MEANS IN THE TRIBONACCI SEQUENCE

n	Tribonacci Number T_n	Harmonic Mean of T_{n-1}, T_{n+3}	Geometric Mean of T_{n-2}, T_{n+2}	Arithmetic Mean of T_{n-3}, T_{n+1}	Ratio T_n/T_{n-1}
1	1	-	-	-	-
2	1	1.750	-	-	1.000
3	2	1.857	2.646	-	2.000
4	4	3.692	3.606	4	2.000
5	7	7.333	6.928	7	1.750
6	13	12.886	13.266	13	1.857
7	24	23.914	23.812	24	1.846
8	44	44.134	44.011	44	1.833
9	81	80.934	81.093	81	1.841
10	149	148.982	148.916	149	1.840
11	274	274.051	274.020	274	1.839
12	504	503.967	504.029	504	1.839
13	927	927.000	926.965	927	1.839
14	1705	1705.018	1705.014	1705	1.839
15	3136	3135.985	3136.007	3136	1.839

Let us now formally generalize the above relationships between Fibonacci sequences and harmonic and geometric means.

Theorem

If positive, real numbers 1 and b are the m^{th} and $(m + n + 1)^{\text{st}}$ terms in a geometric sequence with ratio $x > 1$, and the $(m + 1)^{\text{st}}$ term is $HM(1, b)$ and the $(m + n)^{\text{th}}$ term is $AM(1, b)$, then the ratio of that geometric sequence is equal to the ratio of the corresponding n-bonacci sequence, i.e., the real, positive root of the equation

$$x^n - x^{n-1} - \cdots - x - 1 = 0.$$

Proof: The terms in the geometric sequence must satisfy the equations

$$x = \frac{2b}{1 + b}, \quad x^n = \frac{1 + b}{2}, \quad x^{n+1} = b. \tag{13}$$

which are consistent and which reduce to

$$x^{n+1} - 2x^n + 1 = 0. \tag{14}$$

Eliminating the root $x = 1$ by dividing equation (14) by $(x - 1)$ yields the n-bonacci equation

$$x^n - x^{n-1} - \cdots - x - 1 = 0. \tag{15}$$

Thus, in any n-bonacci sequence S, the term S_m is approximately equal to the harmonic mean of terms S_{m-1} and S_{m+n}, and exactly equal to the arithmetic mean of terms S_{m-n} and S_{m+1}.

One might ask whether all of the n terms between S_m and S_{m+n+1} in an n-bonacci sequence can be expressed in terms of generalized means, where a generalized mean of order t, $M(t)$, is given by

$$M(t) = \left[\frac{1}{2}(a^t + b^t)\right]^{1/t}. \tag{16}$$

When $t = 1$ equation (16) yields the arithmetic mean, when $t = -1$ it yields the harmonic mean, and in the limit as t goes to 0, it yields the geometric mean [4, p. 10]. The answer, in general, is no, as in shown in the following paragraph.

Let us examine the case where $n = 4$, i.e., the Tetranacci sequence. If six consecutive terms can be expressed in the form

$$a,\ HM(a,\ b),\ M(-t),\ M(t),\ AM(a,\ b),\ b,$$

then we have the equations (with $a = 1$):

$$x = \frac{2b}{1 + b},\ x^2 = \left[\frac{1}{2}(1 + b^{-t})\right]^{-1/t},\ x^3 = \left[\frac{1}{2}(1 + b^t)\right]^{1/t},\ x^4 = \frac{1 + b}{2},\ x^5 = b. \tag{17}$$

The first and fourth equations, for the harmonic and arithmetic means, reduce to

$$x^5 - 2x^4 + 1 = 0. \tag{18}$$

The second and third equations are consistent (because the values $-t$ and t are used), and reduce to

$$x^{5t} - 2x^{3t} + 1 = 0. \tag{19}$$

Equations (18) and (19) are clearly inconsistent, however, as there is no value of t that can simultaneously satisfy the conditions $5t = 5$ and $3t = 4$. Thus, aside from the trivial solution $x = 1$, it is not possible to represent four consecutive terms of a Tetranacci sequence as generalized means of the two adjacent terms.

In summary, harmonic and arithmetic means naturally arise in Fibonacci-type sequences. In the geometric series that forms the limit of every n-bonacci sequence, the m^{th} term will be equal to the harmonic mean of the $(m - 1)^{\text{st}}$ and the $(m + n)^{\text{th}}$ terms and the arithmetic mean of the $(m - n)^{\text{th}}$ and $(m + 1)^{\text{st}}$ terms. The aesthetic appeal of Fibonacci proportions may be due, in part, to their natural blending of harmonic, geometric, and arithmetic means.

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n^{th} POWER RESIDUES CONGRUENT TO ONE

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(Submitted March 1983)

It has been proved [7, Lemma 3] that an integer has the property that

$$(x, m) = 1 \text{ implies } x^2 \equiv 1 \pmod{m} \text{ iff } m|24.$$

To generalize this result, we make the following definition.

Definition 1

Let n be a positive integer. The integer m has *property* $P(n)$ if and only if $(x, m) = 1$ implies $x^n \equiv 1 \pmod{m}$.

In §1 we shall determine, for $n \geqslant 1$, all integers which have property $P(n)$; in §2 we shall prove some consequences of an integer having property $P(n)$ or a similar property.

1. INTEGERS HAVING PROPERTY $P(n)$

In Theorem 2, we shall show that the only integers having property $P(n)$, where n is an odd positive integer, are -2, -1, 1, and 2. In Theorem 3, we shall determine the integers which have property $P(n)$, where n is an even positive integer. In particular, we shall show that:

m has property $P(4)$ iff m divides $240 = 2^4 3 \cdot 5$

m has property $P(6)$ iff m divides $504 = 2^3 3^2 \cdot 7$

m has property $P(8)$ iff m divides $480 = 2^5 \cdot 3 \cdot 5$

m has property $P(10)$ iff m divides $264 = 2^3 3 \cdot 11$

m has property $P(12)$ iff m divides $65{,}520 = 2^4 3^2 5 \cdot 7 \cdot 13$

Theorem 2

Let n be an odd positive integer. The integer m has property $P(n)$ iff $m|2$.

Proof: Assume that m has property $P(n)$, where n is an odd positive integer. Thus, since $(-1, m) = 1$,

$$1 \equiv (-1)^n \equiv -1 \pmod{m}.$$

Therefore, $m|2$. Clearly, $m|2$ implies that m has property $P(n)$.

Theorem 3

Let n be an even positive integer and let the distinct odd primes p which are such that $\phi(p)|n$ be denoted by $p_1, p_2, \ldots, p_t$. Choose e such that $2^e \| n$, and for $i = 1, 2, \ldots, t$, choose e_i such that $\phi(p^{e_i}) \mid n$ and $\phi(p^{e_i+1}) \nmid n$. The integer m has property $P(n)$ iff $m|2^{e+2}p_1^{e_1}p_2^{e_2} \ldots p^{e_t}$.

On page 47 of [2], it is stated that the integer

$$2^{e+2}p_1^{e_1}p_2^{e_2} \ldots p^{e_t}$$

defined in Theorem 3 is the largest integer to have property $P(n)$. Given a positive integer n, Theorems 2 and 3 enable us to find all integers m that have property $P(n)$. Given an integer m, Theorem 2 of [1] and its proof enable us to find all positive integers n such that m has property $P(n)$. An earlier reference is Theorem 4-9 of [4].

We shall need the following two lemmas to prove Theorem 3.

Lemma 4

Let d, m, n be integers with n positive. If m has property $P(n)$ and $d|m$, then d has property $P(n)$.

Proof: Without loss of generality, assume $d > 1$ and $m > 1$. Let

$$m = q_1^{e_1}q_2^{e_2} \ldots q_t^{e_t},$$

where $q_1, q_2, \ldots, q_t$ are distinct primes and $e_1, e_2, \ldots, e_t$ are positive integers. Also let $q_1, q_2, \ldots, q_j$, where $1 \leqslant j \leqslant t$, be the distinct primes that divide d. We shall now prove that d has property $P(n)$. Thus, let $(a, d) = 1$. Choose b such that

$$b \equiv a \pmod{q_1^{e_1}q_2^{e_2} \ldots q_j^{e_j}} \quad \text{and} \quad b \equiv 1 \pmod{q_{j+1}^{e_{j+1}} \ldots q_t^{e_t}}.$$

Since $(b, m) = 1$ and m has property $P(n)$, $b^n \equiv 1 \pmod{m}$. Therefore, since $a \equiv b \pmod{d}$ and $d|m$, $a^n \equiv b^n \equiv 1 \pmod{d}$.

A proof of the next lemma can be found, for example, in [6, pp. 104-105].

Lemma 5

Let e be a positive integer. We have that:

(i) $a^{2^e} \equiv 1 \pmod{2^{e+2}}$ for all odd integers a.

(ii) 5 belongs to the exponent 2^e modulo 2^{e+2}.

Proof of Theorem 3: First assume that the integer m has property $P(n)$. We shall show that

$$m \,|\, 2^{e+2}p_1^{e_1}p_2^{e_2} \ldots p_t^{e_t}$$

by showing that:

(i) 2^{e+3} does not divide m,

(ii) for $i = 1, 2, \ldots, t$, $p_i^{e_i+1}$ does not divide m, and

(iii) the only odd primes that may possibly divide m are $p_1, p_2, \ldots, p_t$.

If $2^{e+3}|m$, then by Lemma 4, $5^n \equiv 1 \pmod{2^{e+3}}$. But since 5 belongs to the exponent 2^{e+1} modulo 2^{e+3}, we have the contradiction $2^{e+1}|n$.

Now suppose $p_i^{e_i+1}|m$ and let x be a primitive root modulo $p_i^{e_i+1}$. By Lemma 4, $x^n \equiv 1 \pmod{p_i^{e_i+1}}$. But this is impossible since $\phi(p_i^{e_i+1})$ does not divide n.

Similarly, suppose there is an odd prime p such that $p|m$ and $p \neq p_i$ for $i = 1, 2, \ldots, t$, and let x be a primitive root modulo p. By Lemma 4, $x^n \equiv 1 \pmod{p}$. But this is impossible since $\phi(p)$ does not divide n.

Conversely, assume

$$m | 2^{e+2} p_1^{e_1} p_2^{e_2} \ldots p_t^{e_t}.$$

Thus, by Lemma 4, it is sufficient to prove that $2^{e+2} p_1^{e_1} p_2^{e_2} \ldots p_t^{e_t}$ has property $P(n)$. So assume

$$(a, 2^{e+2} p_1^{e_1} p_2^{e_2} \ldots p_t^{e_t}) = 1.$$

Thus $(a, 2) = 1$, so by Lemma 5, $a^{2^e} \equiv 1 \pmod{2^{e+2}}$. Also, for $i = 1, 2, \ldots, t$, $(a, p_i^{e_i}) = 1$, so by the Euler-Fermat theorem,

$$a^{\phi(p_i^{e_i})} \equiv 1 \pmod{p_i^{e_i}}.$$

Since $2^e | n$ and $\phi(p_i^{e_i}) | n$ for $i = 1, 2, \ldots, t$, $a^n \equiv 1 \pmod{2^{e+2}}$ and $a^n \equiv 1 \pmod{p_i^{e_i}}$ for $i = 1, 2, \ldots, t$. Therefore,

$$a^n \equiv 1 \pmod{2^{e+2} p_1^{e_1} p_2^{e_2} \ldots p_t^{e_t}}.$$

2. SOME CONSEQUENCES OF $P(n)$

We shall now consider some consequences of an integer m having property $P(n)$ or a similar property. Our first result shows that an integer m having property $P(n)$ puts a restriction not just on the n^{th} powers of the integers relatively prime to m but on the n^{th} powers of all integers.

Theorem 6

Let m and n be integers with $n > 2$. The following four conditions are equivalent:

I. m has property $P(n)$.

II. For all integers a, b, k, where k is positive,

$$a^{kn} + b^{kn} \equiv a^{kn} b^{kn} + (a, b)^{kn} \pmod{m}.$$

III. For all integers a,

$$a^n \equiv (a, m)^n \pmod{m}.$$

IV. For all integers a and b, if $(ab, m) = (b, m)$, then, for all positive integers k,

$$a^{kn} b \equiv b \pmod{m}.$$

Theorem 6 is not true for $n = 2$; for $n = 2$, $m = 24$, $k = 1$, $a = 10$, and $b = 14$. I is true but II is false.

For Theorem 6, we clearly have that III implies I. Also, by letting $b = m$ and $k = 1$ in II, we see that II implies III and, by letting $b = 1$ and $k = 1$ in IV, we see that IV implies I. We shall complete the proof of Theorem 6 by showing that I implies II and that I implies IV. To show that I implies II, we shall need the following lemma, which, for the case $ab \equiv 0 \pmod{m}$ and $k = 1$, was proved in Theorem 13 of [1].

Lemma

Let n be a positive integer. If m has property $P(n)$ and $(a, b) = 1$, then, for all positive integers k,

$$a^{kn} + b^{kn} \equiv a^{kn}b^{kn} + 1 \pmod{m}.$$

Proof: Choose d and e such that

$$de = m,\ (d, e) = 1,\ (a, d) = 1, \text{ and } (b, e) = 1.$$

We can do this as follows: If $(b, m) = 1$, let $d = 1$ and $e = m$. Otherwise, let $p_1, p_2, \ldots, p_t$ be the distinct primes that divide both b and m and, for $i = 1, 2, \ldots, t$, choose $e_1, e_2, \ldots, e_t$ such that $p_i^{e_i} \| m$. Just let

$$d = p_1^{e_1}p_2^{e_2}\ldots p_t^{e_t} \quad \text{and} \quad e = \frac{m}{d}.$$

Since $d|m$, d has property $P(n)$. Thus, $a^{kn} \equiv 1 \pmod{d}$. Similarly $b^{kn} \equiv 1 \pmod{e}$. Therefore,

$$0 \equiv (a^{kn} - 1)(b^{kn} - 1) \equiv a^{kn}b^{kn} - a^{kn} - b^{kn} + 1 \pmod{m}.$$

That is,

$$a^{kn} + b^{kn} \equiv a^{kn}b^{kn} + 1 \pmod{m}.$$

Proof that I Implies II

Assume that m has property $P(n)$ and let a, b, k be integers with k positive. Let $p_1, p_2, \ldots, p_t$ be the distinct primes that divide all three of a, b, m and, for $i = 1, 2, \ldots, t$, choose e_i such that $p_i^{e_i} \| m$. Thus, there is an integer c such that

$$m = p_1^{e_1}p_2^{e_2}\ldots p_t^{e_t}c,\ (a, b, c) = 1, \text{ and } \left(c, \frac{m}{c}\right) = 1.$$

In addition, since m has property $P(n)$ and $n > 2$, $e_i \leqslant n$ for $i = 1, 2, \ldots, t$. We shall prove that I implies II by showing that

$$a^{kn} + b^{kn} \quad \text{and} \quad a^{kn}b^{kn} + (a, b)^{kn}$$

are congruent modulo c and modulo m/c.

Since c has property $P(n)$, the preceding lemma implies that

$$\frac{a^{kn}}{(a, b)^{kn}} + \frac{b^{kn}}{(a, b)^{kn}} \equiv \frac{a^{kn}b^{kn}}{(a, b)^{2kn}} + 1 \pmod{c}$$

and $((a, b), c) = 1$ implies that

$$(a, b)^{kn} \equiv 1 \pmod{c}.$$

These two congruences imply that

$$a^{kn} + b^{kn} \equiv a^{kn}b^{kn} + (a, b)^{kn} \pmod{c}.$$

Since, for $i = 1, 2, \ldots, t$, $p_i|(a, b)$ and $e_i \leqslant n \leqslant kn$, $(a, b)^{kn} \equiv 0 \pmod{m/c}$. Hence, a^{kn}, b^{kn}, and $a^{kn}b^{kn}$ are also congruent to 0 modulo m/c. Thus,

$$a^{kn} + b^{kn} \equiv 0 \equiv a^{kn}b^{kn} + (a, b)^{kn} \pmod{m/c}.$$

Proof that I Implies IV

Assume that m has property $P(n)$ and that $(ab, m) = (b, m)$. Since

$$(b, m) = (ab, m) = (ab, m(a, 1)) = (ab, am, m)$$
$$= (a(b, m), m) = (b, m)\left(a, \frac{m}{(b, m)}\right),$$

we have that $1 = \left(a, \frac{m}{(b, m)}\right)$. Thus,

$$a^{kn} \equiv 1 \left(\text{mod } \frac{m}{(b, m)}\right),$$

so

$$a^{kn}b \equiv b \left(\text{mod } \frac{mb}{(b, m)}\right).$$

Therefore, $a^{kn}b \equiv b \pmod{m}$.

The equivalence, for $k = 1$, of I and III in Proposition 7, below, implies Corollary 3.1 of [5].

Proposition 7

Let m, n, r be integers where n and r are positive and m has property $P(n)$. The following three conditions are equivalent:

I. m is $(r + 1)$ power-free.

II. For all integers a, $(a^r, m) = (a^{r+1}, m)$.

III. For all integers a and all positive integers k, $a^{kn+r} \equiv a^r \pmod{m}$.

Proof: It is easy to show that I and II are equivalent. Now, II implying III follows from the equivalence of Theorem 6(I) and Theorem 6(IV) with $b = a^r$. To prove that III implies II, assume that $a^{n+r} \equiv a^r \pmod{m}$. Therefore,

$$(a^r, m) = (a^{n+r}, m) \geqslant (a^{r+1}, m) \geqslant (a^r, m).$$

Proposition 8

Let k, m, n be integers such that k and n are positive, m has property $P(n)$, and m is $(k, n) + 1$ power-free. For every integer a, if the congruence

$$x^{(k,n)} \equiv a \pmod{m}$$

has a solution, then congruence $x^k \equiv a \pmod{m}$ has a solution.

Proof: Let a be an integer and assume that the congruence

$$x^{(k,n)} \equiv a \pmod{m}$$

has a solution, say $x = b$. There are positive integers u and w such that

$$ku = nw + (k, n).$$

Thus, by Proposition 7,

$$b^{ku} = b^{nw+(k,n)} \equiv b^{(k,n)} \equiv a \pmod{m}.$$

Therefore, the congruence $x^k \equiv a \pmod{m}$ has a solution, for example, $x = b^u$.

The restriction "m is $(k, n) + 1$ power-free" is needed in Proposition 8. In general, for a prime p, if $p^{(k,n)+1}$ divides m and $k > (k, n)$, then the congruence

$$x^{(k,n)} \equiv p^{(k,n)} \pmod{m}$$

will have a solution, but the congruence

$$x^k \equiv p^{(k,n)} \pmod{m}$$

will not have a solution. This is so because, for p a prime,

$$p^{(k,n)+1} | m,\ x^k \equiv p^{(k,n)} \pmod{m}, \text{ and } k > (k, n)$$

imply the contradiction

$$p^{(k,n)+1} \leqslant (x^k, m) = (p^{(k,n)}, m) = p^{(k,n)}.$$

Our next result is a generalization of Theorem 1 of [3].

Theorem 9

Let c, d, m, n be integers with n positive and $(cd, m) = 1$. The following two conditions are equivalent.

I. For all integers t, if $(t, m) = 1$, then
$$(t^n - c^n)(t^n - d^n) \equiv 0 \pmod{m}.$$

II. For all integers a and b, if $ab \equiv cd \pmod{m}$, then
$$a^n + b^n \equiv c^n + d^n \pmod{m}.$$

Proof: First assume I and assume $ab \equiv cd \pmod{m}$. Thus,

$$(a, m) \leqslant (ab, m) = (cd, m) = 1.$$

Hence, by I,

$$\begin{aligned} 0 &\equiv (a^n - c^n)(a^n - d^n) = a^{2n} - a^n d^n - a^n c^n + c^n d^n \\ &\equiv a^{2n} - a^n d^n - a^n c^n + a^n b^n = a^n(a^n - d^n - c^n + b^n) \pmod{m}. \end{aligned}$$

Therefore, since $(a, m) = 1$,

$$a^n + b^n \equiv c^n + d^n \pmod{m}.$$

Conversely, assume II and assume $(t, m) = 1$. Thus, there is an integer a such that $at \equiv cd \pmod{m}$. Hence, by II,

$$a^n + t^n \equiv c^n + d^n \pmod{m}.$$

Therefore,

$$\begin{aligned} 0 = 0t^n &\equiv (t^n - d^n - c^n + a^n)t^n = t^{2n} - d^n t^n - c^n t^n + a^n t^n \\ &\equiv t^{2n} - d^n t^n - c^n t^n + c^n d^n = (t^n - c^n)(t^n - d^n) \pmod{m}. \end{aligned}$$

Theorem 10

If an integer m has property $P(2k)$, where k is a positive integer, then there is an integer c such that $(t, m) = 1$ implies

$$(t^k - c^k)(t^k - 1^k) \equiv 0 \pmod{m}.$$

Proof: Assume $m = p_1^{e_1} p_2^{e_2} \ldots p_j^{e_j}$ has property $P(2k)$. We can choose c such that $c \equiv c_i \pmod{p_i^{e_i}}$ for $i = 1, 2, \ldots, j$, where $c_1, c_2, \ldots, c_j$ are chosen as follows:

For $p_i = 2$, $c_i = 1$ if k is an even integer and $c_j = 3$ if k is an odd integer. For p_i an odd prime, $c_i = 1$ if $p_i^{e_i}$ has property $P(k)$; otherwise, choose c_i such that $c_i^k \equiv -1 \pmod{p_i^{e_i}}$.

The converse of Theorem 10 is false. A counterexample is $k = 2$ and $m = 64$. We do have that $(t, 64) = 1$ implies that

$$(t^2 - 1)(t^2 - 1) \equiv 0 \pmod{64},$$

but 64 does not have property $P(4)$. The reason $(t^2 - 1)(t^2 - 1) \equiv 0 \pmod{64}$ is because t odd implies $8 | (t^2 - 1)$.

The next theorem is a generalization of Theorem 2 of [3].

Theorem 11

Let k be an odd positive integer. The following two conditions are equivalent.

I. There is an integer d such that if $ab \equiv d \pmod{m}$, then

$$a^k + b^k \equiv 1 + d^k \pmod{m}.$$

II. m has property $P(2k)$.

Proof: Assume I and assume $(x, m) = 1$. Thus, there is an integer y such that $xy \equiv d \pmod{m}$. Since $xy \equiv d \pmod{m}$ and $(-1)(-d) \equiv d \pmod{m}$, by I,

$$x^k + y^k \equiv 1 + d^k \pmod{m} \tag{1}$$

and

$$-1 - d^k \equiv (-1)^k + (-d)^k \equiv 1 + d^k \pmod{m}. \tag{2}$$

If m is an odd integer, then by (2), $d^k \equiv -1 \pmod{m}$. Hence, by (1),

$$x^k \equiv -y^k \pmod{m}.$$

Therefore,

$$x^{2k} \equiv -x^k y^k \equiv -d^k \equiv 1 \pmod{m}.$$

If m is an even integer, then since $(x, m) = 1$ and by (2), 2 divides $x^k - 1$ and $m/2$ divides $d^k + 1$. Thus,

$$0 \equiv (d^k + 1)(x^k - 1) = d^k x^k - d^k + x^k - 1 \pmod{m}. \tag{3}$$

Therefore, by (1) and (3),

$$\begin{aligned} x^{2k} &\equiv x^k(1 + d^k - y^k) = x^k + d^k x^k - x^k y^k \\ &\equiv x^k + d^k x^k - d^k \equiv 1 \pmod{m}. \end{aligned}$$

Now assume m has property $P(2k)$. To prove I, we will prove that if $ab \equiv -1 \pmod{m}$, then $a^k + b^k \equiv 0 \pmod{m}$. Therefore, assume $ab \equiv -1 \pmod{m}$. Hence, $(a, m) = 1$. Thus,

$$0 \equiv a^{2k} - 1 \equiv a^{2k} + (ab)^k = a^k(a^k + b^k) \pmod{m}.$$

Since $(a, m) = 1$, this implies that $a^k + b^k \equiv 0 \pmod{m}$.

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LETTER TO THE EDITOR

Dear Dr. Bergum:

A paper by Charles R. Wall entitled "Unitary Harmonic Numbers" appeared in the February 1983 issue of *The Fibonacci Quarterly*. We thought you might be interested in knowing that a paper with the same title and similar content was published by us (P. Hagis & G. Lord) in the *Proceedings of The American Mathematical Society*, v. 51, 1975, pp. 1-7. Comparing Wall's results with ours, you will see that both of Wall's theorems contain minor errors. Thus, there are 45 (not 43) unitary harmonic numbers less than 10^6, including $1512 = 2^3 3^3 7$ and 791700, both of which were missed by Wall. And, since $\omega(1512) = 3$, there are 24 (not 23) unitary harmonic numbers n for which $\omega(n) \leqslant 4$.

It should also be mentioned that Wall's conjecture that "there are only finitely many unitary harmonic numbers with $\omega(n)$ fixed" is Theorem 2 in our paper.

Sincerely,

Peter Hagis, Jr.

Graham Lord

RESPONSE

Dear Dr. Bergum:

Professors Hagis and Lord are correct in their observations. The omission of 1512 and 791700 resulted from an oversight which is entirely my responsibility. The duplication of their earlier work was unfortunate but done in innocence; it is doubly unfortunate that neither the referee nor I was aware of the earlier paper.

Independent but duplicate results are inevitable. One hopes that a reinvented wheel is in some way superior; in this case, alas, the earlier model was better in all respects. I apologize to you and to readers of *The Fibonacci Quarterly*.

Sincerely,

Charles R. Wall

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ON LINEAR RECURRENCES AND DIVISIBILITY BY PRIMES

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(Submitted March 1983)

I. INTRODUCTION

Recently Neumann & Wilson [6] and Shannon & Horadam [8] have discussed the sequence of numbers given by the linear recurrence

$$T_k = T_{k-2} + T_{k-3};\ T_0 = 3,\ T_1 = 0,\ T_2 = 2.$$

This sequence has the following interesting property:

$$\text{If } p \text{ is a prime, then } p \mid T_p. \tag{1}$$

The sequence $\{T_k\}$ has been discussed several times before; for example, see [1], [2], [3], [4], [5], and [7]. In particular, Perrin [7] asks if the converse to (1) is true, that is:

Does $p \mid T_p$ imply that p is prime?

Neumann & Wilson call a counterexample to the converse a *pseudoprime*. They did not find any pseudoprimes for the sequence $\{T_k\}$.

Unfortunately, the converse is false; the first example being

$$271441 = 521^2.$$

The only other composite n less than 1000000 for which $n \mid T_n$ is

$$904631 = 7 \cdot 13 \cdot 9941.$$

These numbers were found using a computer program written in APL and were checked independently by John Hughes using a FORTRAN program.

It can be shown that the sequence $\{T_k\}$ is, essentially, exponential in growth. In particular, for large k we have

$$T_k \sim \alpha^k,$$

where α is the real root of $x^3 - x - 1 = 0$ and $\alpha = 1.32$, approximately.

In [8], Shannon & Horadam remark that the sequence $\{T_k\}$ "is possibly the slowest growing integer sequence for which $p \mid T_p$ for all primes p." This is clearly false, as simple examples like

$$A_k = k \cdot |\log k|$$

or even

$$A_k = k$$

will show. These examples might be dismissed as trivial. In this note we will show that there exist nontrivial sequences $\{T_k\}$ given by a linear recurrence having the property (1) that have rates of growth like

$$T_k \sim \alpha^k,$$

where $\alpha - 1$ is a positive number arbitrarily close to 0.

II. SLOWLY-GROWING SEQUENCES

Let $n \geqslant 3$ be a positive integer and define

$$f(x) = x^n - x - 1.$$

Let the roots of $f(x) = 0$ be

$$\alpha_1, \alpha_2, \ldots, \alpha_n$$

and put

$$T_k = \alpha_1^k + \alpha_2^k + \cdots + \alpha_n^k.$$

Then it is easy to see that

$$T_k = T_{k+1-n} + T_{k-n},$$

where the starting values are given by

$$T_0 = n,\ T_1 = 0,\ T_2 = 0,\ \ldots,\ T_{n-2} = 0,\ T_{n-1} = n - 1.$$

By Theorem 2 of [6], the sequence $\{T_k\}$ has the property of (1).

We have the following:

Theorem

Let $f(x) = x^n - x - 1$. Then:

(1) All zeros of f are smaller in magnitude than $3^{1/n}$.

(2) All zeros of f are of multiplicity 1.

(3) f has exactly 1 real zero if n is odd and exactly 2 real zeros if n is even.

(4) f has a real zero α satisfying $2^{1/n} < \alpha < 3^{1/n}$. If n is even, there is in addition a real zero β satisfying $-1 < \beta < 0$.

(5) The positive real zero α is in fact the zero of f largest in magnitude.

Proof:

(1) Let α be the zero of f which is largest in magnitude. Then, for some integer $k \geqslant 0$, we have

$$k^{1/n} \leqslant |\alpha| < (k+1)^{1/n}.$$

Now $\alpha^n = \alpha + 1$, so

$$|\alpha^n| = |\alpha + 1| \leqslant |\alpha| + 1 < (k+1)^{1/n} + 1,$$

whereas $k \leqslant |\alpha^n|$. Hence

$$k < (k+1)^{1/n} + 1$$

and so certainly $k < 3$.

(2) Put $g(x) = nf(x) - xf'(x)$. Now, if there were a repeated zero of f, it would be a zero of f' and hence also a zero of g. But g is linear; in fact,

$$g(x) = (1 - n)x - n.$$

It is easily verified that the zero of g, namely $n/(1-n)$, is not a zero of f'. This gives us the desired contradiction.

(3) Suppose n is even. Then $f'(n) = 0$ has only one real root, namely

$$n^{-1/(n-1)}.$$

It is easily verified that $f(x) \to +\infty$ as $x \to \pm\infty$. Hence, f attains its minimum at $x = n^{-1/(n-1)}$. It is easily verified that this minimum is negative. Hence, f has two real zeros.

Now suppose n is odd. Then $f'(x) = 0$ has two real roots, namely

$$\pm n^{-1/(n-1)}.$$

Now $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$, so f attains a local maximum at $-n^{-1/(n-1)}$ and attains a local minimum at $n^{-1/(n-1)}$. It is easily verified that f is negative at both these points, so f has only one real zero.

(4) It is easily verified that $f(2^{1/n}) < 0$, while $f(3^{1/n}) > 0$. Also, if n is even, then $f(-1) = 1$ but $f(0) = -1$.

(5) Let $y_0 = r_0 e^{i\theta}$ be a complex zero of f. Then

$$f(y_0) = (r_0 e^{i\theta})^n - r_0 e^{i\theta} - 1 = 0.$$

Hence, $r_0 = |r_0 e^{i\theta} + 1| < r_0 + 1$. Thus, $f(r_0) = r_0^n - r_0 - 1 < 0$. However, r_0 is positive; and from parts (3) and (4) above, we see that if r_0 is positive and $f(r_0) < 0$, then $r_0 < \alpha$. Hence, $|y_0| < \alpha$.

This completes the proof of our Theorem. □

This theorem implies that if

$$T_k = \alpha_1^k + \alpha_2^k + \cdots + \alpha_n^k,$$

and if $\alpha_1 = \alpha$, the positive real zero of $x^n - x - 1$, then the other zeros are smaller in magnitude, and hence for large k we have

$$T_k \sim \alpha^k.$$

From part (4) of the theorem, we know that

$$2^{1/n} < \alpha < 3^{1/n},$$

so by choosing n sufficiently large, we can make α as close to 1 as desired. For example, if we choose $n = 4$, we get a sequence with property (1) that grows approximately like 1.22^k.

The authors thank the referee for detailed comments and a shorter proof of part (2) of the theorem.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

Assistant Editor
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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-532 *Proposed by Herta T. Freitag, Roanoke, VA*

Find a, b, and c in terms of n so that

$$a^3(b - c) + b^3(c - a) + c^3(a - b) = 2F_nF_{n+1}F_{n+2}F_{n+3}.$$

B-533 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $g(a, b, c) = a^4(b^2 - c^2) + b^4(c^2 - a^2) + c^4(a^2 - b^2)$. Determine an infinitude of choices for a, b, and c such that $g(a, b, c)$ is the product of five Fibonacci numbers.

B-534 *Proposed by A. B. Patel, V. S. Patel College of Arts & Sciences, Bilimora, India*

One obtains the lengths of the sides of a Pythagorean triangle by letting

$$a = u^2 - v^2,\ b = 2uv,\ c = u^2 + v^2,$$

where u and v are integers with $u > v > 0$. Prove that the area of such a triangle is not a perfect square when $u = F_{n+1}$, $v = F_n$, and $n \geqslant 2$.

B-535 *Proposed by L. Cseh & I. Merenyi, Cluj, Romania*

Prove that there is no positive integer n for which

$$F_1 + F_2 + F_3 + \cdots + F_{3n} = 16!.$$

B-536 *Proposed by L. Kuipers, Sierre, Switzerland*

Find all solutions in integers x and y of

$$x^4 + 2x^3 + 2x^2 + x + 1 = y^2.$$

B-537 *Proposed by L. Kuipers, Sierre, Switzerland*

Find all solutions in integers x and y of

$$x^4 + 3x^3 + 3x^2 + x + 1 = y^2.$$

SOLUTIONS

Application of the Bertrand-Chebyshev Theorem

B-508 *Proposed by Philip L. Mana, Albuquerque, NM*

Find all n in $\{1, 2, 3, \ldots, 200\}$ such that the sum $n! + (n + 1)!$ of successive factorials is the square of an integer.

I. *Solution by Paul S. Bruckman, Fair Oaks, CA*

Let $\theta_n \equiv n! + (n + 1)! = (n + 2)n!$. We will show that $n = 4$ is the only integer $n \in \{1, 2, 3, \ldots, 200\}$ such that θ_n is square.

Proof: We easily verify that $\theta_1 = 3$, $\theta_2 = 8$, $\theta_3 = 30$, while $\theta_4 = 144 = 12^2$. If $p \leqslant n \leqslant 2p - 3$, where p is any odd prime, then $p|\theta_n$ but $p^2 \nmid \theta_n$; hence, θ_n cannot be a square in this range. Also, if p and q are any two consecutive primes in the sequence of primes, with $5 \leqslant p \leqslant 103$, it is easy to verify that $7 \leqslant q \leqslant 2p - 3 \leqslant 203$. Thus, the range $\{5, 6, 7, \ldots, 200\}$ is spanned by at least one prime p with $p|\theta_n$ but with $p^2 \nmid \theta_n$; this shows that θ_n is not square in this range.

II. *Solution by J. Suck, Essen, Germany*

$n! + (n + !)!$ is a square only for $n = 4$ and a cube only for $n = 2$.

Proof: Bertrand's "postulate" as proved by Chebyshev states that for every integer $k > 3$, there is a prime p satisfying $k < p < 2k - 2$. (See, e.g., Hardy and Wright, *An Introduction to the Theory of Numbers*, 4th ed., p. 373.) Now, let $n = 2m$ or $2m - 1$, $m > 2$. We have a prime p then with $m + 1 < p < 2m$, so that $p|n!$. However, because $2p > 2m + 2 \geqslant n + 2$, p^2 is not a divisor of $n!(n + 2) = n! + (n + 1)!$.

Also solved by L. Cseh, Walther Janous, Edwin M. Klein, L. Kuipers, Imre Merenyi, J. M. Metzger, Bob Prielipp, Neville Robbins, Sahib Singh, M. Wachtel, and the proposer.

Dedekind Function Inequality

B-509 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Let ψ be Dedekind's function given by

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

For example,

$$\psi(12) = 12(1 + 1/2)(1 + 1/3) = 24.$$

Show that

$$\psi(\psi(\psi(n))) > 2n \text{ for } n = 1, 2, 3, \ldots .$$

Solution by J. M. Metzger, University of N. Dakota, Grand Forks, ND

The statement is false for $n = 1$.

Since $\psi(\psi(\psi(2))) = 6$, the inequality is correct for $n = 2$.

Now assume $n \geqslant 3$. For such n, $\psi(n)$ is clearly even. Note that for all $n \geqslant 2$, $\psi(n) \geqslant n + 1$ because $\psi(n)$ is an integer greater than n. Moreover, if k is even, then

$$\psi(k) = k \prod_{p|k}\left(1 + \frac{1}{p}\right) \geqslant k \cdot \left(1 + \frac{1}{2}\right) = \frac{3k}{2}.$$

It follows that

$$\psi(\psi(\psi(n))) \geqslant \frac{3}{2}\,\psi(\psi(n)) \geqslant \frac{3}{2}\cdot\frac{3}{2}\,\psi(n) \geqslant \frac{9}{4}(n + 1) > 2n.$$

Also solved by Paul S. Bruckman, L. Cseh, Alberto Facchini, C. Georghiou, Walther Janous, L. Kuipers, I. Merenyi, Lawrence Somer, J. Suck, and the proposer.

Inequality on Euler and Dedekind Functions

B-510 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Euler's ϕ function and its companion, Dedekind's ψ function, are defined by

$$\phi(n) = n \prod_{p|n}\left(1 - \frac{1}{p}\right) \quad \text{and} \quad \psi(n) = n \prod_{p|n}\left(1 + \frac{1}{p}\right).$$

(a) Show that $\phi(n) + \psi(n) \geqslant 2n$ for $n > 1$.
(b) When is the inequality strict?

Solution by Alberto Facchini, University of Udine, Italy

Let $p_1, \ldots, p_t$ be the prime factors of n. Then,

$$\begin{aligned}\prod_{p|n}\left(1 \pm \frac{1}{p}\right) = 1 &\pm \left(\frac{1}{p_1} + \cdots + \frac{1}{p_t}\right)\\ &+ \left(\frac{1}{p_1p_2} + \frac{1}{p_1p_3} + \cdots + \frac{1}{p_1p_t} + \frac{1}{p_2p_3} + \cdots + \frac{1}{p_{t-1}p_t}\right)\\ &\pm \left(\frac{1}{p_1p_2p_3} + \cdots\right) + \cdots + (\pm 1)^t\,\frac{1}{p_1p_2\cdots p_t}.\end{aligned}$$

Therefore,

$$\phi(n) + \psi(n) = 2n\left[1 + \left(\frac{1}{p_1p_2} + \frac{1}{p_1p_3} + \cdots + \frac{1}{p_{t-1}p_t}\right) + \cdots\right] \geqslant 2n$$

and the inequality is strict if and only if n has at least two distinct prime factors.

Also solved by Paul S. Bruckman, L. Cseh, C. Georghiou, Walther Janous, L. Kuipers, Vania D. Mascioni, I. Merenyi, J. M. Metzger, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Telescoping Fibonacci Products

B-511 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers with j even. Prove that

$$F_j(F_n + F_{n+2j} + F_{n+4j} + \cdots + F_{n+2jk}) = (L_{n+2jk+j} - L_{n-j})/5.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall show that

$$5F_nF_j + 5F_{n+2j}F_j + 5F_{n+4j}F_j + 5F_{n+6j}F_j + \cdots + 5F_{n+(2k-2)j}F_j + 5F_{n+2jk}F_j$$
$$= L_{n+(2k+1)j} - L_{n-j},$$

which is clearly equivalent to the desired result. From (12) on p. 115 of the April 1975 issue of this journal,

$$5F_sF_t = L_{s+t} - L_{s-t},\ t \text{ even}.$$

Thus, since j is even,

$$5F_nF_j + 5F_{n+2j}F_j + 5F_{n+4j}F_j + 5F_{n+6j}F_j + \cdots + 5F_{n+(2k-2)j}F_j + 5F_{n+2jk}F_j$$
$$= (L_{n+j} - L_{n-j}) + (L_{n+3j} - L_{n+j}) + (L_{n+5j} - L_{n+3j}) + (L_{n+7j} - L_{n+5j})$$
$$+ \cdots + (L_{n+(2k-1)j} - L_{n+(2k-3)j}) + (L_{n+(2k+1)j} - L_{n+(2k-1)j})$$
$$= L_{n+(2k+1)j} - L_{n-j}$$

because telescoping occurs.

Also solved by Paul S. Bruckman, L. Cseh, Herta T. Freitag, C. Georghiou, Walther Janous, L. Kuipers, I. Merenyi, H.-J. Seiffert, A. G. Shannon, J. Suck, Sahib Singh, and the proposer.

Telescoping Fibonacci-Lucas Products

B-512 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers with j odd. Prove that

$$L_j(F_n + F_{n+2j} + F_{n+4j} + \cdots + F_{n+2kj}) = F_{n+2kj+j} - F_{n-j}.$$

Solution by J. Suck, Essen, Germany

Do not use induction. Just telescope the left-hand side by Hoggatt's I_{23}:

$$L_jF_m = F_{m+j} - F_{m-j},\ j \text{ odd}.$$

Also solved by Paul S. Bruckman, L. Cseh, Herta T. Freitag, C. Georghiou, Walther Janous, L. Kuipers, I. Merenyi, Bob Prielipp, H.-J. Seiffert, A. G. Shannon, Sahib Singh, and the proposer.

Fibonacci Convolution and Rising Pascal Diagonals

B-513 *Proposed by Andreas N. Philippou, University of Patras, Greece*

Show that

$$\sum_{k=0}^{n} F_{k+1}F_{n+1-k} = \sum_{k=0}^{[n/2]} (n + 1 - k)\binom{n-k}{k} \text{ for } n = 0, 1, \ldots,$$

where $[x]$ denotes the greatest integer in x.

Solution by C. Georghiou, University of Patras, Greece

Since the generating function of the sequence $\{F_{n+1}\}_{n=0}^{\infty}$ is

$$f(x) = (1 - x - x^2)^{-1},$$

it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{n} F_{k+1}F_{n+1-k}x^n &= (1 - x - x^2)^{-2} \\
&= \sum_{n=0}^{\infty} (-1)^n\binom{-2}{n}(x + x^2)^n, \quad |x| < \frac{1}{\alpha} \\
&= \sum_{n=0}^{\infty} (n + 1)(x + x^2)^n \\
&= \sum_{n=0}^{\infty} \sum_{j+k=n} (j + k + 1)\binom{j+k}{k}x^{j+2k} \\
&= \sum_{n=0}^{\infty} \sum_{j+2k=n} (j + k + 1)\binom{j+k}{k}x^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (n + 1 - k)\binom{n-k}{k}x^n
\end{aligned}$$

from which the assertion is established.

Also solved by Paul S. Bruckman, L. Cseh, Walther Janous, L. Kuipers, H.-J. Seiffert, A. G. Shannon, J. Suck, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-376 *Proposed by H. Klauser, Zurich, Switzerland*

Let (a, b, c, d) be a quadruple of integers with the property that

$$(a^3 + b^3 + c^3 + d^3) = 0.$$

Clearly, at least one integer must be negative.

Examples: (3, 4, 5, -6), (9, 10, -1, -12)

Find a construction rule so that:

1. out of two given quadruples a new quadruple arises;
2. out of the given quadruple a new quadruple always arises.

H-377 *Proposed by Lawrence Somer, Washington, D.C.*

Let $\{w_n\}_{n=0}^{\infty}$ be a k^{th}-order linear integral recurrence satisfying the recursion relation

$$w_{n+k} = a_1 w_{n+k-1} + a_2 w_{n+k-2} + \cdots + a_k w_n.$$

Let t be a fixed positive integer and d a fixed nonnegative integer. Show that the sequence

$$\{s_n\} = \{w_{tn+d}\}_{n=0}^{\infty}$$

also satisfies a k^{th}-order linear integral recursion relation

$$s_{n+k} = a_1^{(t)} s_{n+k-1} + a_2^{(t)} s_{n+k-2} + \cdots + a_k^{(t)} s_n.$$

Show further that the coefficients $a_1^{(t)}, a_2^{(t)}, \ldots, a_k^{(t)}$ depend on t but not on d, and that $a_k^{(t)}$ can be chosen so that

$$a_k^{(t)} = (-1)^{(k+1)(t+1)} a_k^t.$$

H-378 *Proposed by M. Wachtel, Zurich, Switzerland*

For every positive integer x and y, provided that they are *prime to each other*, show that no integral divisor of

$$x^2 - 5y^2$$

is congruent to 3 or 7, modulo 10.

H-379 *Proposed by A. N. Philippou and F. S. Makri, Univ. of Patras, Greece*

For each fixed integer $k \geqslant 2$, let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order $k(1)$. Show that

$$f_{n+2}^{(k)} = \sum_{i=0}^{\infty} \sum_{j=0}^{n} (-1)^i \binom{n - ik}{n - j} \binom{n - j + 1}{i}, \ n \geqslant 0.$$

Reference: A. N. Phliippou & A. A. Muwafi. "Waiting for the k Consecutive Success and the Fibonacci Sequence of Order K." *The Fibonacci Quarterly* 20, no. 1 (1982):28-32.

H-380 *Proposed by Charles R. Wall, Trident Tech. College, Charleston, SC*

The sequence 1, 4, 5, 9, 13, 14, 16, 25, 29, 30, 36, 41, 49, 50, 54, 55, ... of squares and sums of consecutive squares appeared in Problem B-495. Show that this sequence has Schnirelmann density zero.

SOLUTIONS

A reply from M. Wachtel regarding H-335 (May 1983)

In the February 1984 issue, the proposer is claiming that the solution to the above-mentioned problem is incorrect.

Reply: The roots of the polynomial, as split up by the proposer, are:

$$\begin{array}{ll} (x - 1) & x_0 = 1 \\ (x^2 + bx - a^2) & x_{1,2} = \pm\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1 \\ (x^2 + ax - b^2) & x_{3,4} = \pm\sqrt{30 - 6\sqrt{5}} - (\sqrt{5} + 1) \end{array}$$

These roots are exactly identical to those shown in my solution published in the May 1983 issue, with one exception:

> As far as $x_{3,4}$ are concerned, I have erroneously omitted to apply the parentheses $-(\sqrt{5} + 1)$, sorry. Apart from this error, I do not see why this solution should be incorrect. Certainly, the solution by the proposer is more obvious.

Sum Difference!

H-355 *Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA (Vol. 21, no. 2, May 1983)*

Solve the second-order finite difference equation

$$n(n - 1)a_n - \{2rn - r(r + 1)\}a_{n-r} + r^2 a_{n-2r} = 0,$$

r and n are integers. If $n - kr < 0$, $a_{n-kr} = 0$.

Solution by Paul S. Bruckman, Sacramento, CA

Let

$$y = f_r(x) = \sum_{n=0}^{\infty} a_n x^n \text{ (where the } a_n \text{ depend on } r). \tag{1}$$

We deal with four separate cases.

Case I: $r < 0$

Letting $r = -s$, the given recursion becomes

$$n(n - 1)a_n + (2sn + s^{(2)})a_{n+s} + s^2 a_{n+2s} = 0,$$

or, equivalently,

$$s^2 a_n + (2sn - 3s^2 - s)a_{n-s} + (n - 2s)^{(2)} a_{n-2s} = 0. \tag{2}$$

Letting $n = 0, 1, 2, \ldots,$ successively, we find that (2) has only the trivial solution

$$a_n = 0,\ n = 0, 1, 2, \ldots . \tag{3}$$

Case II: $r = 0$

The given recursion becomes $n(n - 1)a_n = 0$. This implies

$$a_n = 0,\ n = 2, 3, \ldots, \text{ with } a_0 \text{ and } a_1 \text{ arbitrary.} \tag{4}$$

Case III: $r = 1$

The given recursion becomes

$$n(n - 1)a_n - 2(n - 1)a_{n-1} + a_{n-2} = 0.$$

Again we find that a_0 and a_1 are arbitrary. Making the substitution $b_n = n!a_n$, then $b_n - 2b_{n-1} + b_{n-2} = 0$, i.e., $\Delta^2 b_n = 0$, $n = 0, 1, \ldots$. Hence, $b_n = A + Bn$ for some constants A and B. To find A and B, note

$$b_0 = A = a_0,\ b_1 = A + B = a_1,$$

so

$$A = a_0,\ B = a_1 - a_0.$$

Hence,

$$a_n = \frac{a_0 + (a_1 - a_0)n}{n!},\ n = 0, 1, 2, \ldots . \tag{5}$$

Case IV: $r \geqslant 2$

We transform the given recursion into a differential equation:

$$x^2 y'' - 2rx^{r+1}y' + (r^2 x^{2r} - r^{(2)}x^r)y = 0. \tag{6}$$

We may verify (6) by using (1) and noting that the left member of (6) becomes:

$$\sum_{n=0}^{\infty} n^{(2)} a_n x^n - 2r\sum_{n=r}^{\infty}(n - r)a_{n-r}x^n + r^2 \sum_{n=2r}^{\infty} a_{n-2r}x^n - r^{(2)} \sum_{n=r}^{\infty} a_{n-r}x^n$$

$$= \sum_{n=0}^{r-1} n^{(2)} a_n x^n + \sum_{n=r}^{2r-1} \{n^{(2)}a_n - (2rn - r(r + 1))a_{n-r}\}x^n$$

$$+ \sum_{n=2r}^{\infty} \{n^{(2)}a_n - (2rn - r(r + 1))a_{n-r} + r^2 a_{n-2r}\}x^n = 0,$$

since each sum vanishes, using the recursion.

To solve (6), we make the fortuitous substitution $y = ue^{x^r}$, where u is some function of x. We find

$$y' = (rx^{r-1}u + u')e^{x^r}, \quad y'' = (r^2x^{2r-2}u + 2rx^{r-1}u' + r^{(2)}x^{r-2}u + u'')e^{x^r}.$$

Then, eliminating the factor e^{x^r} and simplifying, we obtain

$$x^2u'' = 0. \tag{7}$$

Since (7) is to be valid for all x, we may also eliminate the factor x^2. Then $u'' = 0$, which implies $u = A + Bx$ for some constants A and B. Thus,

$$y = f_r(x) = (A + Bx)e^{x^r}. \tag{8}$$

Since $f\ (0) = A = a_0$ and $f_r'(0) = B = a_1$, we have

$$f_r(x) = (a_0 + a_1x)e^{x^r}. \tag{9}$$

Therefore,

$$f_r(x) = a_0\sum_{n=0}^{\infty} x^{rn}/n! + a_1\sum_{n=0}^{\infty} x^{rn+1}/n!.$$

This shows that

$$a_n = \begin{cases} a_0/(n/r)!, & \text{if } r|n; \\ a_1/(n - 1/r)!, & \text{if } r|(n - 1); \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

An equivalent and compact formulation is the following:

$$a_n = \{\delta(r|n) + \delta(r|(n - 1)) - \delta(r|n)\delta(r|(n - 1))\}a_{n-rm}/m!, \tag{11}$$

where

$$m = [n/r] \text{ and } \delta(r|k) = \begin{cases} 1 & \text{if } r|k, \\ 0 & \text{otherwise.} \end{cases}$$

Lotsa Words

H-356 *Proposed by David Singmaster, Polytechnic of the South Bank, London (Vol. 21, no. 3, August 1983)*

Consider a set of r types of letter with n_i occurrences of letter i. How many words can we form, using some or all of these letters?

If we use k_i of letter i, then there are obviously

$$\binom{\Sigma k_i}{k_1, \ldots, k_r}$$

ways to form a word and the desired number is

$$\sum_{k_i \leqslant n_i} \binom{\Sigma k_i}{k_1, \ldots, k_r}.$$

When $r = 2$, this can be readily evaluated using properties of Pascal's triangle to obtain

$$\binom{n_1 + n_2 + 2}{n_1 + 1} - 1.$$

W. O. J. Moser has found a nice combinatorial derivation of this result, but neither approach works for $r > 2$.

Moser's solution for $r = 2$ follows.

In the case $r = 2$,

$$(**)\quad \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} \binom{i+j}{i}$$

is the number of ways of forming words with some of m A's and n B's. Any such word with i A's and j B's can be extended to a word of $m+1$ A's and $n+1$ B's by appending $m+1-i$ A's and $n+1-j$ B's to it. If our original word begins with an A, we append a block of $m+1-i$ A's followed by a block of $n+1-j$ B's at the beginning. If the original word begins with a B, we append the block of B's followed by the block of A's at the beginning. The empty word can be extended in two ways: AA...ABB...A or BB...BAA...A. Otherwise, we have a one-to-one correspondence between our original words and words formed from all of $m+1$ A's and $n+1$ B's. The reverse correspondence is to take any word of $m+1$ A's and $n+1$ B's and delete its first two blocks (i.e., constant subintervals). Since the empty word arises from two extended words, we have

$$\binom{m+n+2}{m+1} - 1$$

of our original words.

As an illustration, let $m = n = 1$:

Original Word	Extended Word
-	AABB or BBAA
A	ABBA
B	BAAB
AB	ABAB

Solution (Partial) by Paul S. Bruckman, Fair Oaks, CA

We let $\underline{n}_r = (n_1, n_2, \ldots, n_r)$ and $\pi(\underline{n}_r) = (n_{i_1}, n_{i_2}, \ldots, n_{i_r})$ denote any permutation of the elements of $\underline{n}_r$. Also, we let

$$S_r(\underline{n}_r) = \sum_{\substack{0 \leqslant i_j \leqslant n_j \\ j=1,2,\ldots,r}} \binom{i_1 + i_2 + \cdots + i_r}{i_1, i_2, \ldots, i_r}, \tag{1}$$

and

$$F_r(\underline{x}_r) = \sum_{\substack{n_j \geqslant 0 \\ j=1,2,\ldots,r}} S_r(\underline{n}_r) x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}, \tag{2}$$

where $\underline{x}_r = (x_1, x_2, \ldots, x_r)$. Then,

$$\begin{aligned} F_r(\underline{x}_r) &= \sum_{\substack{n_j \geqslant 0 \\ j=1,2,\ldots,r}} \sum_{i_j \geqslant 0} \binom{i_1 + i_2 + \cdots + i_r}{i_1, i_2, \ldots, i_r} x_1^{n_1+i_1} x_2^{n_2+i_2} \ldots x_r^{n_r+i_r} \\ &= (1-x_1)^{-1}(1-x_2)^{-1} \ldots (1-x_r)^{-1} \sum_{\substack{i_j \geqslant 0 \\ j=1,2,\ldots,r}} \binom{i_1 + i_2 + \cdots + i_r}{i_1, i_2, \ldots, i_r} x_1^{i_1} x_2^{i_2} \ldots x_r^{i_r} \\ &= (1-x_1)^{-1}(1-x_2)^{-1} \ldots (1-x_r)^{-1} \sum_{n=0}^{\infty} (x_1 + x_2 + \cdots + x_r)^n, \end{aligned}$$

or

$$F_r(\underline{x}_r) = (1 - x_1)^{-1}(1 - x_2)^{-1} \ldots (1 - x_r)^{-1}(1 - x_1 - x_2 - \cdots - x_r)^{-1}. \tag{3}$$

The symmetry inherent in the definition in (1) provides us with the following:

$$S_r(\underline{n}_r) = S_r(\pi(\underline{n}_r)) \text{ for all permutations } \pi. \tag{4}$$

Also, if m is any positive integer less then r, we may set $x_j = 0$ $(m < j \leqslant r)$ in (3) and obtain:

$$S_r(n_1, n_2, \ldots, n_m, \underbrace{0, 0, \ldots, 0}_{r-m}) = S_m(\underline{n}_m). \tag{5}$$

Of course, we may also obtain (5) by setting $n_j = 0$ in (1), $m < j \leqslant r$. Another interesting relationship is obtained by multiplying (3) throughout by the factor $(1 - x_1 - x_2 - \cdots - x_r)$. We then obtain:

$$(1 - x_1)^{-1}(1 - x_2)^{-1} \ldots (1 - x_r)^{-1}$$

$$= (1 - x_1 - x_2 - \cdots - x_r) \sum_{\substack{n_j \geqslant 0 \\ j=1,2,\ldots,r}} S_r(\underline{n}_r)x_1^{n_1}x_2^{n_2}\ldots x_r^{n_r}.$$

Since

$$(1 - x_1)^{-1}(1 - x_2)^{-1} \ldots (1 - x_r)^{-1} = \sum_{\substack{n_j \geqslant 0 \\ j=1,2,\ldots,r}} x_1^{n_1}x_2^{n_2}\ldots x_r^{n_r},$$

this yields the recursion:

$$S_r(n_1, n_2, \ldots, n_r) = 1 + S_r(n_1 - 1, n_2, \ldots, n_r) + S_r(n_1, n_2 - 1, \ldots, n_r) + S_r(n_1, n_2, \ldots, n_r - 1). \tag{6}$$

The special cases $r = 1$ and $r = 2$ have already been noted, and are ready consequences of the relations already derived:

$$S_1(n_1) = n_1 + 1; \tag{7}$$

$$S_2(n_1, n_2) = \binom{n_1 + n_2 + 2}{n_1 + 1} - 1. \tag{8}$$

Even for the next case, $r = 3$, however, in spite of the fact that a generating function for the $S_3(n_1, n_2, n_3)$ is known, the general formula is difficult to obtain. By a change of notation, setting $r = 3$ in (6), we obtain:

$$S_3(u, v, w) = 1 + S_3(u - 1, v, w) + S_3(u, v - 1, w) + S_3(u, v, w - 1). \tag{9}$$

The remainder of this manuscript is devoted to the case $r = 3$, and even this is only imperfectly resolved. For brevity in the sequel, the following notation is adopted:

$$\begin{aligned} U_0 &\equiv S_3(u, v, w); \qquad\qquad (10)\\ U_1 &\equiv S_3(u - 1, v, w) + S_3(u, v - 1, w) + S_3(u, v, w - 1);\\ U_2 &\equiv S_3(u, v - 1, w - 1) + S_3(u - 1, v, w - 1) + S_3(u - 1, v - 1, w);\\ U_3 &\equiv S_3(u - 1, v - 1, w - 1). \end{aligned}$$

Thus, a restatement of (9) would be as follows:

$$U_0 = 1 + U_1. \tag{11}$$

Multiplying (2) and (3) throughout by

$$(1 - x)(1 - y)(1 - z)$$

for the case $r = 3$ (by a change of notation), we obtain, on the one hand:

$$(1 - x - y - z)^{-1} = \sum_{u,\, v,\, w \geq 0} \binom{u + v + w}{u,\ v,\ w} x^u y^v z^w.$$

On the other hand, this is equal to

$$(1 - x - y - z + xy + yz + xz - xyz) \sum_{u,\, v,\, w \geq 0} S_3(u,\ v,\ w) x^u y^v z^w.$$

This yields the relation:

$$U_0 - U_1 + U_2 - U_3 = \binom{u + v + w}{u,\ v,\ w}. \tag{12}$$

We use (11) and (12) to derive another interesting recursion involving $S_3(u,\ v,\ w)$:

$$U_0 + U_3 + 1 = \left\{\frac{(u+v+w+1)^2(u+v+w+2) + uvw}{(u+v+1)(v+w+1)(u+w+1)}\right\}\binom{u + v + w}{u,\ v,\ w}. \tag{13}$$

A derivation of (13) follows:

$$\begin{aligned}
S_3(u,\ v,\ w) &= \sum_{i=0}^{u} \sum_{j=0}^{v} \sum_{k=0}^{w} \binom{i + j + k}{i,\ j,\ k} = \sum_{i=0}^{u} \sum_{j=0}^{v} \binom{i + j}{i} \sum_{k=0}^{w} \binom{i + j + k}{i + j} \\
&= \sum_{i=0}^{u} \sum_{j=0}^{v} \binom{i + j}{i} \binom{i + j + w + 1}{i + j + 1} \\
&= \sum_{i=0}^{u} \sum_{j=0}^{v} \binom{i + j}{i} \left\{ \binom{i + j + w}{i + j} + \binom{i + j + w}{i + j + 1} \right\} \\
&= \sum_{i=0}^{u} \binom{i + w}{w} \sum_{j=0}^{v} \binom{i + w + j}{i + w} + S_3(u,\ v,\ w - 1) \\
&= \sum_{i=0}^{u} \binom{i + w}{w} \binom{i + v + w + 1}{i + w + 1} + S_3(u,\ v,\ w - 1) \\
&= \sum_{i=0}^{u} \binom{i + w}{w} \left\{ \binom{i + v + w}{i + w} + \binom{i + v + w}{i + w + 1} \right\} + S_3(u,\ v,\ w - 1) \\
&= \binom{v + w}{v} \sum_{i=0}^{u} \binom{v + w + i}{v + w} + S_3(u,\ v - 1,\ w) \\
&\qquad - S_3(u,\ v - 1,\ w - 1) + S_3(u,\ v,\ w - 1) \\
&= \binom{v + w}{v} \binom{u + v + w + 1}{v + w + 1} + S_3(u,\ v - 1,\ w) \\
&\qquad - S_3(u,\ v - 1,\ w - 1) + S_3(u,\ v,\ w - 1),
\end{aligned}$$

or

$$\begin{aligned}
&S_3(u,\ v,\ w) + S_3(u,\ v - 1,\ w - 1) \\
&= S_3(u,\ v - 1,\ w) + S_3(u,\ v,\ w - 1) + \frac{(u + v + w + 1)}{(v + w + 1)}\binom{u + v + w}{u,\ v,\ w}.
\end{aligned} \tag{14}$$

Interchanging u, v, and w in (14), and adding the resulting relations, we get:

$$3U_0 + U_2 = 2U_1 + \psi\theta, \tag{15}$$

where

$$\psi \equiv (u + v + w + 1)\left\{\frac{1}{u + v + 1} + \frac{1}{v + w + 1} + \frac{1}{u + w + 1}\right\}, \quad \theta \equiv \binom{u + v + w}{u,\ v,\ w}.$$

Then, eliminating U_1 and U_2 from (15) by means of (11) and (12) and simplifying the result, we obtain (13).

Applying (8) initially, then (13) recursively with succeeding values of w, we may derive the following formulas:

$$S_3(u, v, 1) = \frac{(u+1)(v+1)}{(u+v+3)}\binom{u+v+4}{u+2}; \tag{16}$$

$$S_3(u, v, 2) = \frac{\{(u+1)(v+1)(u+v+5)+4\}(u+v+5)!}{2(u+1)(u+3)(v+1)(v+3)(u+v+3)(u+v+5)u!v!} - 1; \tag{17}$$

$$S_3(u, v, 3) = \frac{\{(u+2)(v+2)(u+v+5)+12\}(u+v+6)!}{6(u+2)(u+4)(v+2)(v+4)(u+v+3)(u+v+5)u!v!}; \tag{18}$$

$$S_3(u, v, 4) = \frac{\begin{array}{c}(u+1)(u+3)(v+1)(v+3)(u+v+5)(u+v+7)\\ +\ 24(u+1)(v+1)(u+v+7)\ +\ 192\end{array}}{24(u+3)(u+5)(v+3)(v+5)(u+v+3)(u+v+5)} \cdot \frac{(u+v+6)!}{(u+1)!(v+1)!} \quad 1; \tag{19}$$

$$S_3(u, v, 5) = \left\{\frac{\begin{array}{c}(u+2)(u+4)(v+2)(v+4)(u+v+5)(u+v+7)\\ +\ 40(u+2)(v+2)(u+v+7)\ +\ 960\end{array}}{120(u+2)(u+4)(u+6)(v+2)(v+4)(v+6)(u+v+3)(u+v+5)(u+v+7)}\right\} \frac{(u+v+8)!}{u!v!}. \tag{20}$$

From the above formulas, we may infer the following general formula for $S_3(u, v, w)$:

$$S_3(u, v, w) = -e_w + \frac{(u+v+w+3)!}{(u+v+3)(u+w+1)(v+w+1)u!v!w!} \cdot \sum_{k=0}^{[\frac{1}{2}w]} \frac{(-1)^k\binom{\frac{1}{2}w}{k}\binom{\frac{1}{2}(w-1)}{k}}{\binom{-\frac{1}{2}(u+v+5)}{k}\binom{\frac{1}{2}(u+w-1)}{k}\binom{\frac{1}{2}(v+w-1)}{k}}, \tag{21}$$

where $e_w = \frac{1}{2}(1+(-1)^w)$.

A proof of (21) was not attempted, although it appears that induction [using (13)] should dispose of it. Unfortunately, the expression in (21) is neither symmetrical in u, v, and w nor in closed form. The more general case, $r \geqslant 3$, appears even more formidable. It seems likely that any fruitful results will require the generating function in (3).

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BOOKS AVAILABLE THROUGH THE FIBONACCI ASSOCIATION

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