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CONTENTS

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The Fibonacci Quarterly

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Br. Alfred Brousseau, and I.D. Ruggles

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

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A PATH COUNTING PROBLEM IN DIGRAPHS

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1. INTRODUCTION

In this paper, we consider only directed graphs without loops or multiple edges. Our terminology and notation will be standard except as noted. A good reference for any undefined terms is [1].

Our main goal is to determine the maximum possible number of directed paths between a pair of vertices in an acyclic digraph with m edges (and any number of vertices). Denoting this maximum possible number by $N(m)$, we will establish that

$$N(m) = \begin{cases} F_{(m+1)/2} & \text{for } m \text{ odd} \\ 1 & \text{for } m = 2 \\ 2F_{(m/2)-1} & \text{for } m \geqslant 4 \text{ and even} \end{cases}$$

where F satisfies the recurrence relation

$$F_k = F_{k-1} + F_{k-2},\ F_1 = 1,\ F_2 = 2.$$

The actual proof of this formula will be preceded by a sequence of five easy lemmas.

We then conclude with a brief discussion of the following related question: Given a positive integer k, what is the least number of edges in an acyclic digraph having *exactly* k directed paths between a pair of vertices.

2. PROOFS OF THE LEMMAS AND MAIN RESULT

Lemma 1

Let D be an acyclic digraph. Then D contains vertices a and z such that indegree a = outdegree z = 0. (We call a and z, respectively, a source and a sink of D.)

Proof: Let $x \in V(D)$. Consider a longest path directed away from x, say from x to z. Then outdegree $z = 0$ (since any edge leaving z would yield either a longer directed path away from x or a directed cycle in D).

The proof that indegree $a = 0$ for some $a \in V(D)$ is entirely analogous. ■

Lemma 2

Let D be an acyclic digraph. Then the vertices of D can be ordered, say $1, 2, \ldots, n$, such that every edge in D is of the form (i, j), where $i < j$.

Proof: We proceed by induction on $n = |V(D)|$. The result is trivially true for $n = 2$. For the induction step, choose any $z \in V(D)$ with outdegree $z = 0$ (one exists by Lemma 1), and consider the digraph $D - z$. By the induction hypothesis, the vertices of $D - z$ can be ordered, say $1, 2, \ldots, n - 1$, in the manner described. If we let z be the n^{th} vertex, we have the desired ordering of $V(D)$. ■

In what follows, we assume D is an acyclic digraph with vertices ordered $1, 2, \ldots, n$ such that every edge of D is of the form (i, j), where $i < j$.

For any $x \in V(D)$, let $p_D(x)$ denote the number of directed paths from 1 to x in D. [When D is clear from context, we will use just $p(x)$ for this number.]

Lemma 3

Suppose D has a set of vertices $S = \{i < \cdots < j < k\}$, with $1 < i < k \leqslant n$, which induces a tournament (i.e., a digraph with every pair of vertices joined by precisely one edge). Then

$$p(k) \geqslant p(i) + \cdots + p(j).$$

Proof: For each $x \in S$, let $P(x)$ denote the set of directed paths from 1 to x. If $x \neq k$, let $P'(x)$ denote the set of directed paths from 1 to k obtained by taking a path from 1 to x together with the edges (x, k). Then, clearly,

$$P'(i) \cup \cdots \cup P'(j) \subseteq P(k),$$

and the sets on the left side are disjoint. Since

$$|P'(x)| = |P(x)| = p(x),$$

it follows at once that

$$p(i) + \cdots + p(j) \leqslant p(k). \blacksquare$$

Let $N(m)$ denote the maximum possible number of directed paths between two vertices of an acyclic digraph with m edges. Certainly $N(m)$ is a nondecreasing function of m. Let us call an acyclic digraph on m edges having precisely $N(m)$ directed paths between some pair of vertices a *path maximum m-graph*. It is easily seen that there will be a path maximum m-graph D with the vertices ordered as in Lemma 2 such that 1 and n are joined by precisely $N(m)$ directed paths, and 1 (resp., n) is the unique source (resp., sink) in D. We will assume this property for the path maximum m-graphs we consider in what follows.

Lemma 4

There exists a path maximum m-graph D in which

$$\{x \in V(D) \mid (x, n) \in E(D)\}$$

(i.e., the predecessors of n in D) induce a tournament.

Proof: Otherwise, let i, j be two predecessors of n (with say $i < j$) such that $(i, j) \notin E(D)$. Form the digraph

$$D' = D - (i, n) + (i, j).$$

To each directed path in D from 1 to n containing the edge (i, n) there corresponds uniquely a directly path in D' from 1 to n containing the edges (i, j) and (j, n). Hence, $p_{D'}(n) \geqslant p_D(n)$, and so D' is also a path maximum m-graph in which n has one less predecessor than in D. We simply iterate this procedure until we obtain a path maximum m-graph with the desired properties. ■

Lemma 5

If $m \geqslant 3$, there exists a path maximum m-graph in which n has indegree 2.

Proof: Let D be a path maximum m-graph in which the predecessors of n (ordered say $1 < \cdots < j < k$) induce a tournament. By Lemma 3,

$$p(k) \geqslant p(i) + \cdots + p(j).$$

Hence,

$$2p(k) \geqslant p(i) + \cdots + p(j) + p(k) = p(n) = N(m).$$

If indegree $n \geqslant 3$, we can construct a new acyclic digraph D' with m edges, as shown in Figure 1. Note that

$$p_{D'}(n') = 2p(k) \geqslant N(m),$$

and hence D' is also a path maximum m-graph. But indegree $_{D'}n' = 2$, and the proof is complete. ∎

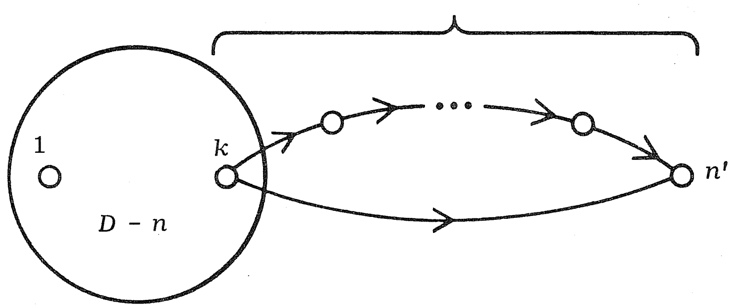

Figure 1. The Digraph D'

We now state and prove our main result.

Theorem

Let m be a positive inteter. Then

$$N(m) = \begin{cases} F_{(m+1)/2} & \text{for } m \text{ odd} \\ 1 & \text{for } m = 2 \\ 2F_{(m/2)-1} & \text{for } m \geqslant 4 \text{ and even} \end{cases}$$

where F_k is the Fibonacci number satisfying $F_k = F_{k-1} + F_{k-2}$, $F_1 = 1$, $F_2 = 2$.

Proof: It is readily verified that

$$N(1) = N(2) = 1,\ N(3) = N(4) = 2,\ N(5) = 3,\ N(6) = 4,$$

and so the formula holds for $m \geqslant 6$. We thus proceed by induction on $m \geqslant 7$.

Since the digraphs in Figure 2 contain m edges, and have as many dipaths from 1 to n as the number specified in the formula, it suffices to show the numbers in the formula are upper bounds for $N(m)$.

By Lemma 5 there is a path maximum m-graph D in which the indegree of n is 2. Let x, y denote the predecessors of n in D, with say $x < y$. We then have precisely three possibilities:

(i) $(x, y) \notin E(D)$ (Using the construction in the proof of Lemma 4, we could obtain a path maximum m-graph in which n has indegree 1.)

(ii) $(x, y) \in E(D)$, and x is the only predecessor of y.

(iii) $(x, y) \in E(D)$, and x is not the only predecessor of y.

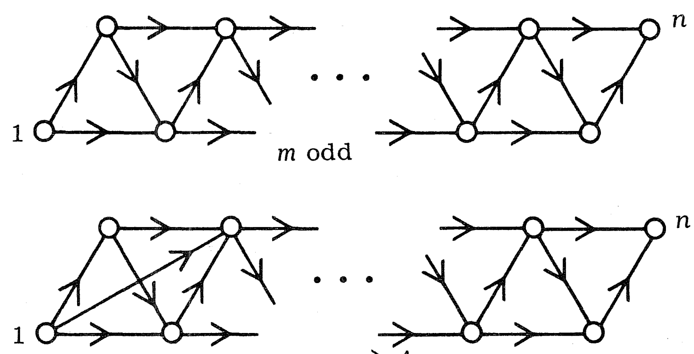

Figure 2. Path Maximum m-Graphs

By considering the maximum possible number of dipaths from the source to x and y in cases (i), (ii), and (iii), respectively, we get

$$N(m) \leqslant \max\{N(m-1),\ 2N(m-3),\ N(m-2) + N(m-4)\}.$$

Using the induction hypothesis, and the fact that $m \geqslant 7$, we obtain

$$N(m) \leqslant \begin{cases} \max\{2F_{(m-3)/2},\ 4F_{(m-5)/2},\ F_{(m-1)/2} + F_{(m-3)/2}\} = F_{(m+1)/2}, & \text{if } m \text{ odd}, \\ \max\{F_{(m/2)},\ 2F_{(m/2)-1},\ 2F_{(m/2)-2} + 2F_{(m/2)-3}\} = 2F_{(m/2)-1}, & \text{if } m \text{ even}. \end{cases}$$

The inductive step, and hence the proof of the theorem, are now complete. ■

3. A RELATED PROBLEM

The authors have also considered the following problem: Given a positive integer k, what is the least number of edges in an acyclic digraph having *exactly* k paths between some pair of vertices? Noting the $N(m)$ is nondecreasing in m, it seems reasonable to conjecture that if $N(m-1) < k \leqslant N(m)$, then m is the minimum number of edges required. This conjecture is indeed true for $k \leqslant 32$. However, $N(14) < 33 < N(15)$, and we have shown that at least 16 edges are needed in any digraph having exactly 33 directed paths between a pair of vertices. Although it appears that a complete solution to this problem may be very difficult, we have the following conjecture to offer:

Conjecture: Let k_n be the smallest integer such that $N(m-1) < k_n < N(m)$, but at least $m + n$ edges are needed in any digraph with precisely k_n directed paths between a pair of vertices. Then k_n satisfies the recurrence relation $k_n = 34k_{n-1} + 21$, $k_1 = 33$.

REFERENCE

1. M. Behzad, G. Chartrand, & L. Lesniak-Foster. *Graphs and Digraphs*. Boston, Mass.: Prindle, Weber and Schmidt, 1979.

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PELL AND PELL-LUCAS POLYNOMIALS

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1. INTRODUCTION

The object of this paper is to record some properties of *Pell polynomials* $P_n(x)$ and *Pell-Lucas polynomials* $Q_n(x)$ defined by the recurrence relations

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x) \qquad P_0(x) = 0,\ P_1(x) = 1 \tag{1.1}$$

and

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x) \qquad Q_0(x) = 2,\ Q_1(x) = 2x. \tag{1.2}$$

Initially, the polynomials are defined for $n \geqslant 0$ but their existence for $n < 0$ is readily extended, yielding

$$P_{-n}(x) = (-1)^{n+1}P_n(x) \tag{1.3}$$

and

$$Q_{-n}(x) = (-1)^n Q_n(x). \tag{1.4}$$

Some of these polynomials are:

$$\begin{cases} P_2(x) = 2x, \quad P_3(x) = 4x^2 + 1, \quad P_4(x) = 8x^3 + 4x, \\ P_5(x) = 16x^4 + 12x^2 + 1, \quad P_6(x) = 32x^5 + 32x^3 + 6x, \ \ldots; \end{cases} \tag{1.5}$$

$$\begin{cases} Q_2(x) = 4x^2 + 2, \quad Q_3(x) = 8x^3 + 6x, \quad Q_4(x) = 16x^4 + 16x^2 + 2, \\ Q_5(x) = 32x^5 + 40x^3 + 10x, \quad Q_6(x) = 64x^6 + 96x^4 + 36x^2 + 2, \ \ldots . \end{cases} \tag{1.6}$$

Important special numerical cases are: $P_n(1) = P_n$, the n^{th} *Pell number*; $Q_n(1) = Q_n$, the n^{th} *Pell-Lucas number*; $P_n(\frac{1}{2}) = F_n$, the n^{th} *Fibonacci number*; and $Q_n(\frac{1}{2}) = L_n$, the n^{th} *Lucas number*. Furthermore, $P_n(\frac{1}{2}x) = F_n(x)$, the n^{th} *Fibonacci polynomial*, and $Q_n(\frac{1}{2}x) = L_n(x)$, the n^{th} *Lucas polynomial* (see [2]).

Following standard procedures, we easily obtain the *Binet forms*

$$P_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.7}$$

and

$$Q_n(x) = \alpha^n + \beta^n, \tag{1.8}$$

where

$$\begin{cases} \alpha = x + \sqrt{x^2 + 1} \\ \beta = x - \sqrt{x^2 + 1} \end{cases} \tag{1.9}$$

are the roots of

$$\lambda^2 - 2x\lambda - 1 = 0, \tag{1.10}$$

so that

$$\alpha + \beta = 2x, \ \alpha - \beta = 2\sqrt{x^2 + 1}, \ \alpha\beta = -1. \tag{1.11}$$

The generating functions for the infinite sets of polynomials $\{P_n(x)\}$ and $\{Q_n(x)\}$ are found in the usual way to be

$$\sum_{r=0}^{\infty} P_{r+1}(x)y^r = \frac{1}{1 - 2xy - y^2} \tag{1.12}$$

and

$$\sum_{r=0}^{\infty} Q_{r+1}(x)y^r = \frac{2x + 2y}{1 - 2xy - y^2}. \tag{1.13}$$

Results involving these generating functions are not developed here.

2. ELEMENTARY PROPERTIES OF $P_n(x)$, $Q_n(x)$

Important elementary relationships involving $P_n(x)$ and $Q_n(x)$ follow without difficulty with the aid of (1.7)-(1.11). Some of these are:

$$P_{n+1}(x) + P_{n-1}(x) = Q_n(x) = 2xP_n(x) + 2P_{n-1}(x) \tag{2.1}$$

$$Q_{n+1}(x) + Q_{n-1}(x) = 4(x^2 + 1)P_n(x) \tag{2.2}$$

$$P_n(x)Q_n(x) = P_{2n}(x) \tag{2.3}$$

$$Q_{2n}(x) = \tfrac{1}{2}\{Q_n^2(x) + 4(x^2 + 1)P_n^2(x)\} \tag{2.4}$$

$$\left.\begin{aligned} &P_{n+1}(x)P_{n-1}(x) - P_n^2(x) = (-1)^n && (2.5)\\ &Q_{n+1}(x)Q_{n-1}(x) - Q_n^2(x) = (-1)^{n-1}4(x^2 + 1) && (2.6)\end{aligned}\right\}\ \textit{Simson formulas}$$

$$P_{n+1}^2(x) - P_{n-1}^2(x) = 2xP_{2n}(x) \quad \text{by (1.1), (2.1), (2.3)} \tag{2.7}$$

$$4(x^2 + 1)P_n^2(x) - Q_n^2(x) = 4(-1)^{n-1} \tag{2.8}$$

Formula (2.3) is useful in establishing divisibility properties of the polynomials. Geometrical paradoxes can be constructed from (2.5) when numerical values of x are inserted.

Summations of an elementary nature are obtained in the usual manner. The simplest are:

$$\sum_{r=1}^{n} P_{2r}(x) = (P_{2n+1}(x) - 1)/2x \tag{2.9}$$

$$\sum_{r=1}^{n} P_{2r-1}(x) = P_{2n}(x)/2x \tag{2.10}$$

$$\sum_{r=1}^{n} P_r(x) = (P_{n+1}(x) + P_n(x) - 1)/2x \quad \text{by (2.9), (2.10)} \tag{2.11}$$

$$\sum_{r=1}^{n} Q_{2r}(x) = (Q_{2n+1}(x) - 2x)/2x \tag{2.12}$$

$$\sum_{r=1}^{n} Q_{2r-1}(x) = (Q_{2n}(x) - 2)/2x \tag{2.13}$$

$$\sum_{r=1}^{n} Q_r(x) = (Q_{n+1}(x) + Q_n(x) - 2 - 2x)/2x \quad \text{by (2.12), (2.13)} \tag{2.14}$$

Extensions and variations of these finite summations, e.g., $\sum_{r=1}^{n} r^2P_r(x)$ and $\sum_{r=1}^{n}(-1)^rQ_r(x)$, are omitted in this treatment of the polynomials.

Induction can be used, with a little effort, to establish the explicit expressions

$$P_n(x) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \binom{n-m-1}{m}(2x)^{n-2m-1} \tag{2.15}$$

and

$$Q_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-m}\binom{n-m}{m}(2x)^{n-2m}, \quad n \neq 0, \tag{2.16}$$

where, in (2.16) we used the combinatorial identity

$$\frac{n}{n-m}\binom{n-m}{m} + \frac{n-1}{n-m}\binom{n-m}{m-1} = \frac{n+1}{n-m+1}\binom{n-m+1}{m}.$$

We proceed to prove (2.15).

Proof of (2.15): The formula is trivially true for $n = 1$ and $n = 2$. Assume it is true for $n = k$ and $n = k - 1$ where $k \geqslant 3$. Then we have

$$\begin{aligned} P_{k+1}(x) &= 2xP_k(x) + P_{k-1}(x) \qquad \text{by (1.1)} \\ &= \sum_{m=0}^{\left[\frac{k-1}{2}\right]} \binom{k-m-1}{m}(2x)^{k-2m} + \sum_{m=0}^{\left[\frac{k-2}{2}\right]} \binom{k-m-2}{m}(2x)^{k-2m-2}. \end{aligned}$$

If $k = 2t$, this becomes

$$\begin{aligned} &\sum_{m=0}^{t-1} \binom{2t-m-1}{m}(2x)^{2t-2m} + \sum_{m=0}^{t-1} \binom{2t-m-2}{m}(2x)^{2t-2m-2} \\ &= \binom{2t-1}{0}(2x)^{2t} + \binom{2t-2}{1}(2x)^{2t-2} + \binom{2t-3}{2}(2x)^{2t-4} + \cdots + \binom{t}{t-1}(2x)^2 \\ &\qquad + \binom{2t-2}{0}(2x)^{2t-2} + \binom{2t-3}{1}(2x)^{2t-4} + \cdots + \binom{t}{t-2}(2x)^2 + \binom{t-1}{t-1} \\ &= \sum_{m=0}^{t} \binom{2t-m}{m}(2x)^{2t-2m} = \sum_{m=0}^{[k/2]} \binom{k-m}{m}(2x)^{k-2m} \end{aligned}$$

by using Pascal's formula. Similarly, it holds if k is odd, and the proof is completed.

Basic relationships involving $P_n(x)$ and $Q_n(x)$ may be obtained from these combinatorial formulas, but the calculations required are tedious. Binet forms produce the same results more quickly.

In passing, we note the differential calculus result:

$$\frac{dQ_n(x)}{dx} = 2nP_n(x). \tag{2.17}$$

Later, in (6.20), we shall see that the first derivative of $P_n(x)$ is given in terms of a (complex) Gegenbauer polynomial.

Because $P_n(x)$ and $Q_n(x)$ are generalizations of F_n and L_n, the collection of miscellaneous results for F_n and L_n given in [7] may be generalized; e.g.,

$$Q_{4n}(x) - 2 = 4(x^2 + 1)P_{2n}^2(x), \tag{2.18}$$

$$P_{n-1}(x)P_{n+1}(x) + Q_{n-1}(x)Q_{n+1}(x) = (4x^2+5)P_n^2(x) + (-1)^{n-1}(4x^2-1), \tag{2.19}$$

and

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} P_{2k+p}(x) = [4(x^2+1)]^n Q_{2n+p+1}(x). \tag{2.20}$$

3. MATRIX GENERATION OF FORMULAS

We demonstrate that the matrix

$$P = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \tag{3.1}$$

generates Pell polynomials and Pell-Lucas polynomials, and use it to establish some elementary properties of these polynomials.

Induction, with (1.1), leads to

$$P^n = \begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix} \tag{3.2}$$

whence

$$\begin{bmatrix} P_{n+1}(x) \\ P_n(x) \end{bmatrix} = P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.3}$$

and

$$P_n(x) = [1 \quad 0] P^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.4}$$

The characteristic equation of P is

$$\lambda^2 - 2x\lambda - 1 = 0 \tag{3.5}$$

with eigenvalues

$$\begin{cases} \alpha = x + \sqrt{x^2+1} \\ \beta = x - \sqrt{x^2+1} \end{cases} \tag{3.6}$$

By the division algorithm for polynomials,

$$\lambda^n = (\lambda^2 - 2x\lambda - 1)f(\lambda) + m\lambda + k, \tag{3.7}$$

where $f(\lambda)$ is of degree $n-2$ in λ and m, k are functions of x.

Put $\lambda = \alpha$ in (3.7). Then

$$\alpha^n = m\alpha + k. \tag{3.8}$$

Similarly,

$$\beta^n = m\beta + k. \tag{3.9}$$

Solving (3.8) and (3.9) yields

$$m = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad k = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}. \tag{3.10}$$

From (3.8)

$$P^n = mP + kI. \tag{3.11}$$

Equate the top right elements in (3.11) to obtain $m = P_n(x)$ so that the Binet form (1.7) for $P_n(x)$ is again produced from (3.10).

Use of (2.1) gives

$$\begin{bmatrix} Q_{n+1}(x) \\ Q_n(x) \end{bmatrix} = P^n \begin{bmatrix} 2x \\ 2 \end{bmatrix} \tag{3.12}$$

and

$$Q_n(x) = [1 \quad 0]P^{n-1} \begin{bmatrix} 2x \\ 2 \end{bmatrix}. \tag{3.13}$$

To illustrate the matrix technique, we prove

$$P_{m+n}(x) = P_{m-1}(x)P_n(x) + P_m(x)P_{n+1}(x) \tag{3.14}$$

for

$$\begin{aligned} P_{m-1}(x)P_n(x) + P_m(x)P_{n+1}(x) &= [P_m(x), \quad P_{m-1}(x)] \begin{bmatrix} P_{n+1}(x) \\ P_n(x) \end{bmatrix} \\ &= [P_m(x), \quad P_{m-1}(x)]P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ by (3.3)} \\ &= [1 \quad 0]P^{m+n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ by (3.3) and } P^mP^n = P^{m+n} \\ &= P_{m+n}(x) \quad \text{by (3.4).} \end{aligned}$$

Similarly

$$Q_{m+n}(x) = P_{m-1}(x)Q_n(x) + P_m(x)Q_{n+1}(x). \tag{3.15}$$

From (3.14) and (3.15) with (3.2) and (3.12), we derive

$$\begin{bmatrix} P_{n+r}(x) \\ P_n(x) \end{bmatrix} = \begin{bmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{bmatrix} P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.16}$$

and

$$\begin{bmatrix} Q_{n+r}(x) \\ Q_n(x) \end{bmatrix} = \begin{bmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{bmatrix} P^n \begin{bmatrix} 2x \\ 2 \end{bmatrix}. \tag{3.17}$$

Equation (3.14), including an interchange of m and n, in conjunction with (2.1) gives

$$P_{m+n}(x) = \tfrac{1}{2}\{P_m(x)Q_n(x) + P_n(x)Q_m(x)\}, \tag{3.18}$$

while (3.15), including a replacement of m by $m + 1$ and n by $n - 1$, with (2.1) and (2.2) gives

$$Q_{m+n}(x) = \tfrac{1}{2}\{Q_m(x)Q_n(x) + 4(x^2+1)P_m(x)P_n(x)\}. \tag{3.19}$$

Putting $m = n$ in (3.18) and (3.19) yields (2.3) and (2.4). Further,

$$P_{n+1}^2(x) + P_n^2(x) = P_{2n+1}(x) \tag{3.20}$$

since

$$\begin{aligned} P_{n+1}^2(x) + P_n^2(x) &= [P_{n+1}(x), \quad P_n(x)]\begin{bmatrix} P_{n+1}(x) \\ P_n(x) \end{bmatrix} \\ &= [1 \quad 0]P^{2n}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{by (3.2) and (3.3)} \\ &= P_{2n+1}(x) \quad \text{by 3.4.} \end{aligned}$$

Result (3.20) also follows directly from (3.14) with $m = n + 1$.

Similarly,

$$Q_{n+1}^2(x) + Q_n^2(x) = 4(x^2+1)P_{2n+1}(x). \tag{3.21}$$

All the above results can, of course, be derived by using the Binet forms (1.7) and (1.8). Techniques employed in these sections give rise to the following formulas:

$$P_{n+r}(x) + P_{n-r}(x) = \begin{cases} P_n(x)Q_r(x) & \text{if } r \text{ is even} \\ Q_n(x)P_r(x) & \text{if } r \text{ is odd} \end{cases} \tag{3.22}$$

$$Q_{n+r}(x) + Q_{n-r}(x) = \begin{cases} Q_n(x)Q_r(x) & r \text{ even} \\ 4(x^2+1)P_n(x)P_r(x) & r \text{ odd} \end{cases} \tag{3.23}$$

$$P_{n+r}(x) - P_{n-r}(x) = \begin{cases} Q_n(x)P_r(x) & r \text{ even} \\ P_n(x)Q_r(x) & r \text{ odd} \end{cases} \tag{3.24}$$

$$Q_{n+r}(x) - Q_{n-r}(x) = \begin{cases} 4(x^2+1)P_n(x)P_r(x) & r \text{ even} \\ Q_n(x)Q_r(x) & r \text{ odd} \end{cases} \tag{3.25}$$

$$P_{n+r}^2(x) - P_{n-r}^2(x) = P_{2n}(x)P_{2r}(x) \quad \text{by (3.22), (3.24) and (2.3)} \tag{3.26}$$

$$Q_{n+r}^2(x) - Q_{n-r}^2(x) = 4(x^2+1)P_{2n}(x)P_{2r}(x) \quad \text{by (3.23), (3.25), and (2.3)} \tag{3.27}$$

$$P_{mn+r}(x) = \begin{cases} P_n(x)Q_{(m-1)n+r}(x) + (-1)^n P_{(m-2)n+r}(x) \\ P_{(m-1)n+r}(x)Q_n(x) + (-1)^{n-1}P_{(m-2)n+r}(x) \end{cases} \tag{3.28}$$

$$Q_{mn+r}(x) = Q_{(m-1)n+r}(x)Q_n(x) + (-1)^{n-1}Q_{(m-2)n+r} \tag{3.29}$$

$$\left.\begin{aligned} P_n^2(x) - P_{n+r}(x)P_{n-r}(x) &= (-1)^{n-r}P_r^2(x) & (3.30) \\ Q_n^2(x) - Q_{n+r}(x)Q_{n-r}(x) &= (-1)^{n-r+1}4(x^2+1)P_r^2(x) & (3.31) \end{aligned}\right\} \textit{Simson formulas}$$

$$P_{n+h}(x)P_{n+k}(x) - P_n(x)P_{n+h+k}(x) = (-1)^n P_h(x)P_k(x) \tag{3.32}$$

$$Q_{n+h}(x)Q_{n+k}(x) - Q_n(x)Q_{n+h+k}(x) = (-1)^{n-1}4(x^2+1)P_h(x)P_k(x) \tag{3.33}$$

$$P_{n+h}(x)Q_{n+k}(x) - P_n(x)Q_{n+h+k}(x) = (-1)^n P_h(x)Q_k(x) \tag{3.34}$$

Finally, we offer two relationships that can be described as being of the *de Moivre type*:

$$\{Q_n(x) + 2\sqrt{x^2+1}P_n(x)\}^r = 2^{r-1}\{Q_{nr}(x) + 2\sqrt{x^2+1}P_{nr}(x)\} \tag{3.35}$$

and

$$\{Q_n(x) - 2\sqrt{x^2+1}P_n(x)\}^r = 2^{r-1}\{Q_{nr}(x) - 2\sqrt{x^2+1}P_{nr}(x)\}. \tag{3.36}$$

When $x = \frac{1}{2}$, (3.35) and (3.36) reduce to

$$\left\{\frac{L_n + \sqrt{5}F_n}{2}\right\}^r = \frac{L_{nr} + \sqrt{5}F_{nr}}{2} \tag{3.37}$$

and

$$\left\{\frac{L_n - \sqrt{5}F_n}{2}\right\}^r = \frac{L_{nr} - \sqrt{5}F_{nr}}{2}, \tag{3.38}$$

respectively, the first of which is given in [7, p. 60].

Results involving $P_n(x)$ and $Q_n(x)$ are as multitudinous as the sands of the seashore, and one can gather these grains *ad infinitum, ad nauseam.*

4. PASCAL ARRAYS GENERATING $P_n(x)$, $Q_n(x)$

Consider the following table.

Table 1: Pell Polynomials from Rising Diagonals

$n \backslash m$	1	2	3	4	5	6 ...
1	1					
2	$2x$	1				
3	$4x^2$	$4x$	1			
4	$8x^3$	$12x^2$	$6x$	1		
5	$16x^4$	$32x^3$	$24x^2$	$8x$	1	
6	$32x^5$	$80x^4$	$80x^3$	$40x^2$	$10x$	1
⋮						

(4.1)

Denote the coefficient of the power of x in the m^{th} row and n^{th} column by (m, n).

It is now shown that the rising diagonals presented in Table 1 produce the Pell polynomial (1.5).

Define the entries in row m as the terms in the expansion $(2x+1)^{m-1}$, that is

$$\sum_{n=1}^{m}(m, n)x^{m-n} = (2x+1)^{m-1} \qquad m \geqslant n. \tag{4.2}$$

Hence,

$$(m,\ n) = \binom{m-1}{m-n}2^{m-n} \qquad m \geqslant n. \tag{4.3}$$

Now the rising diagonal function $R_m(x)$ of degree m in x in Table 1 is:

$$\begin{aligned}
R_m(x) &= \sum_{n=1}^{\left[\frac{m+1}{2}\right]} (m+1-n,\ n)x^{m+1-2n} \qquad (m \geqslant 1) & (4.4)\\
&= \sum_{n=1}^{\left[\frac{m+1}{2}\right]} \binom{m-n}{m+1-2n}(2x)^{m+1-2n} \qquad \text{by } (4.3)\\
&= \sum_{n=1}^{\left[\frac{m+1}{2}\right]} \binom{m-n}{n-1}(2x)^{m+1-2n}\\
&= \sum_{n=0}^{\left[\frac{m-1}{2}\right]} \binom{m-n-1}{n}(2x)^{m-1-2n}\\
&= P_m(x) \qquad \text{from } (2.15)
\end{aligned}$$

Now consider Table 2.

Table 2: Pell-Lucas Polynomials from Rising Diagonals

$n \backslash m$	1	2	3	4	5	6	7 ...
1	$\cancel{2x}$	$\cancel{2}$					
2	$\cancel{4x^2}$	$\cancel{6x}$	$\cancel{2}$				
3	$\cancel{8x^3}$	$\cancel{16x^2}$	$\cancel{10x}$	$\cancel{2}$			
4	$\cancel{16x^4}$	$\cancel{40x^3}$	$\cancel{36x^2}$	$\cancel{14x}$	2		
5	$\cancel{32x^5}$	$\cancel{96x^4}$	$\cancel{112x^3}$	$64x^2$	$18x$	2	
6	$\cancel{64x^6}$	$\cancel{224x^5}$	$320x^4$	$240x^3$	$100x^2$	$22x$	2
⋮							

(4.5)

Let $[m,\ n]$ denote the coefficient of the power of x in the m^{th} row and n^{th} column.

We may define the entries in row m as the terms in the expansion of

$$(2x+1)^m + (2x+1)^{m-1} = (2x+1)^{m-1}(2x+2),$$

that is,

$$\sum_{n=1}^{m+1} [m,\ n]x^{m+1-n} = (2x+1)^{m-1}(2x+2) \tag{4.6}$$

and so

$$[m, n] = 2(m, n) + 2(m, n-1) = 2(m, n) + (m, n-1) + (m, n-1)$$
$$= (m+1, n) + (m, n-1). \tag{4.7}$$

Denote the rising diagonal function of degree m in x in Table 2 by $S_m(x)$. Then

$$\begin{aligned} S_m(x) &= \sum_{n=1}^{\left[\frac{m+2}{2}\right]} [m+1-n, n]x^{m+2-2n} \\ &= \sum_{n=1}^{\left[\frac{m+2}{2}\right]} \{(m+2-n, n) + (m+1-n, n-1)\}x^{m+2-2n} && \text{by (4.7)} \\ &= \sum_{n=1}^{\left[\frac{m+2}{2}\right]} \left\{\binom{m+1-n}{n-1} + \binom{m-n}{n-2}\right\}(2x)^{m+2-2n} && \text{by (4.3)} \\ &= \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{m}{m-n}\binom{m-n}{n}(2x)^{m-2n} && \text{on simplification} \\ &= Q_m(x) && \text{by (2.16)} \end{aligned}$$

Thus, we have demonstrated that Pell and Pell-Lucas polynomials are generated by the rising diagonals in Table 1 and Table 2, respectively.

Next, arrange the coefficients of the powers of x in $P_n(x)$, (1.5), in the following Pascal-like display.

Table 3: Pell Polynomial Coefficients

Coeffs. in $P_n(x)$ \ Powers	0	1	2	3	4	5	6	7	8	9 ...
1	1									
2	0	2								
3	1	0	4							
4	0	4	0	8						
5	1	0	12	0	16					
6	0	6	0	32	0	32				
7	1	0	24	0	80	0	64			
8	0	8	0	80	0	192	0	128		
9	1	0	40	0	240	0	448	0	256	
10	0	10	0	160	0	672	0	1024	0	512
⋮										

Designate the entry in the r^{th} row and c^{th} column of Table 3 by $\{r, c\}$. From the table and (2.15), we have:

$$\{2r, 2c\} = 0 \tag{4.8}$$

$$\{2r,\ 2c-1\} = \begin{cases} \binom{r+c-1}{r-c} 2^{2c-1} & c = 1, 2, \ldots, r \\ 0 & c > r \end{cases} \tag{4.9}$$

$$\{2r-1,\ 2c-1\} = 0 \tag{4.10}$$

$$\{2r-1,\ 2c\} = \begin{cases} \binom{r+c-1}{r-c-1} 2^{2c} & c = 0, 1, 2, \ldots, r-1 \\ 0 & c \geqslant r \end{cases} \tag{4.11}$$

Using (4.8)-(4-11), we can prove:

$$\sum_{i=0}^{r-1} \{2r-1-i,\ i\} = 3^{r-1} \tag{4.12}$$

$$\sum_{i=1}^{2r} \{i,\ 2c-1\} = \tfrac{1}{2}\{2r+1,\ 2c\} \tag{4.13}$$

$$\sum_{i=1}^{2r} \{i,\ 2c\} = \tfrac{1}{2}\{2r,\ 2c+1\} \tag{4.14}$$

$$\sum_{i=1}^{2r-1} \{i,\ 2c-1\} = \tfrac{1}{2}\{2r-1,\ 2c\} \tag{4.15}$$

$$\sum_{i=1}^{2r-1} \{i,\ 2c\} = \tfrac{1}{2}\{2r,\ 2c+1\} \tag{4.16}$$

Proof of (4.12)

$$\begin{aligned}\sum_{i=0}^{r-1} \{2r-1-i,\ i\} &= \{2r-1,\ 0\} + \{2r-2,\ 1\} + \cdots + \{r,\ r-1\} \\ &= \binom{r-1}{r-1}2^0 + \binom{r-1}{r-2}2^1 + \cdots + \binom{r-1}{0}2^{r-1} \qquad \text{by (4.9) and (4.11)} \\ &= (1+2)^{r-1} = 3^{r-1}\end{aligned}$$

Proof of (4.13)

$$\begin{aligned}\sum_{i=1}^{2r} \{i,\ 2c-1\} &= \{2,\ 2c-1\} + \{4,\ 2c-1\} + \cdots + \{2r,\ 2c-1\} \qquad \text{by (4.10)} \\ &= \{2c,\ 2c-1\} + \{2c+2,\ 2c-1\} + \cdots + \{2r,\ 2c-1\} \qquad \text{by (4.9)} \\ &= 2^{2c-1}\left(\binom{2c-1}{0} + \binom{2c}{1} + \cdots + \binom{r+c-1}{r-c}\right) \qquad \text{by (4.9)} \\ &= 2^{2c-1}\left(\binom{2c-1}{2c-1} + \binom{2c}{2c-1} + \cdots + \binom{r+c-1}{2c-1}\right) \\ &= 2^{2c-1}\binom{r+c}{2c} \qquad \text{by identity (1.52) in [6]} \\ &= \tfrac{1}{2}\{2r+1,\ 2c\} \qquad \text{by (4.11)}\end{aligned}$$

If a similar table for $Q_n(x)$ is constructed, and if we designate the element in row r and column c by $\overline{r,\ c}$, we have from (2.1) that

$$\overline{r,\ c} = \{r + 1,\ c\} + \{r - 1,\ c\} = 2\{r,\ c - 1\} + 2\{r - 1,\ c\}. \tag{4.17}$$

Properties of $\overline{r,\ c}$ may then be developed on the basis of (4.8)-(4.11).

From (2.2), we derive

$$\overline{r + 1,\ c} + \overline{r - 1,\ c} = 4\{r,\ c\} + 4\{r,\ c - 2\}. \tag{4.18}$$

To conclude this section, we establish a relationship between $(m,\ n)$ and $\{r,\ c\}$ in Tables 1 and 3, respectively (both relating to the Pell polynomials). A relationship between $[m,\ n]$ and $\overline{r,\ c}$ will also be formulated for the Pell-Lucas polynomials.

Now in (4.9), $2c - 1$ is the power of x in $P_{2r}(x)$. Comparing the coefficient of the term x^{2c-1} in (2.15) with that in (4.3), where we recall that

$$\binom{m-1}{m-n} = \binom{m-1}{n-1}$$

we deduce that

$$\{2r,\ 2c - 1\} = (r + c,\ r - c + 1) \tag{4.19}$$

and so

$$(r,\ c) = \{r + c - 1,\ r - c\}. \tag{4.20}$$

A similar argument applied to (2.15) and (4.3) for (4.1) yields

$$\{2r - 1,\ 2c\} = (r + c,\ r - c)$$

whence (4.20) results again.

Lastly, consider $\overline{2r,\ 2c}$, the coefficient of x^{2c} in $Q_{2r}(x)$. From (4.17),

$$\overline{2r,\ 2c} = \left(\binom{r+c}{r-c} + \binom{r+c-1}{r-c-1}\right)2^{2c}.$$

Using (4.7) with (4.3), we find

$$[m,\ n] = \left(\binom{m}{n-1} + \binom{m-1}{n-2}\right)2^{m-n+1}$$

whence, by comparison of the two forms,

$$\overline{2r,\ 2c} = [r + c,\ r - c + 1]. \tag{4.21}$$

Reversely,

$$[r,\ c] = \overline{r + c - 1,\ r - c + 1}. \tag{4.22}$$

A similar formula to (4.21) is

$$\overline{2r - 1,\ 2c + 1} = [r + c,\ r - c]$$

whence (4.22) results again.

5. DETERMINANTAL GENERATION OF $P_n(x)$, $Q_n(x)$

Write d_{ij} for the element in the i^{th} row and j^{th} column of an $n \times n$ determinant.

Let $\Delta_n(x)$ be the $n \times n$ determinant defined by

$$\Delta_n(x): \begin{cases} d_{ii} = 2x & i = 1,\ 2,\ \ldots,\ n \\ d_{i,\,i+1} = 1 & i = 1,\ \ldots,\ n - 1 \\ d_{i,\,i-1} = -1 & i = 2,\ \ldots,\ n \\ d_{ij} = 0 & \text{otherwise} \end{cases} \tag{5.1}$$

From $\Delta_n(x)$, the determinants $\delta_n(x)$, $\Delta_n^*(x)$, and $\delta_n^*(x)$ are defined as follows:

$\delta_n(x)$: as for $\Delta_n(x)$ except that $d_{i,i+1} = -1$, $d_{i,i-1} = 1$ (5.2)

$\Delta_n^*(x)$: as for $\Delta_n(x)$ except that $d_{12} = 2$, $d_{i,i+1} = 1$ $(i = 2, \ldots, n-1)$ (5.3)

$\delta_n^*(x)$: as for $\Delta_n(x)$ except that $d_{12} = -2$, $d_{i,i+1} = -1$, $d_{i,i-1} = 1$. $(i = 2, \ldots, n-1)$ (5.4)

Induction and expansion along the first row, together with basic properties of $P_n(x)$ and $Q_n(x)$, e.g., (1.1), (2.1), yield

$$\Delta_n(x) = P_{n+1}(x) \tag{5.5}$$

$$\delta_n(x) = P_{n+1}(x) \tag{5.6}$$

$$\Delta_n^*(x) = Q_n(x) \tag{5.7}$$

$$\delta_n^*(x) = Q_n(x). \tag{5.8}$$

In the process of expansion, we derive recurrence relations such as

$$\Delta_k(x) = 2x\Delta_{k-1}(x) + \Delta_{k-2}(x) \qquad k \geqslant 3 \tag{5.9}$$

and

$$\Delta_k^*(x) = 2x\Delta_{k-1}^*(x) + 2\Delta_{k-2}^*(x) \qquad k \geqslant 3. \tag{5.10}$$

6. RELATIONS OF $P_n(x)$, $Q_n(x)$ TO OTHER FUNCTIONS

Perhaps the simplest results relating $P_n(x)$ to other functions are found in [4]:

$$\left.\begin{aligned} P_{2n}(x) &= \sinh 2nt/\cosh t & (6.1)\\ P_{2n+1}(x) &= \cosh(2n+1)t/\cosh t & (6.2)\end{aligned}\right\} x = \sinh t$$

Hence

$$\left.\begin{aligned} Q_{2n}(x) &= 2\cosh 2nt & (6.3)\\ Q_{2n+1}(x) &= 2\sinh(2n+1)t & (6.4)\end{aligned}\right\} x = \sinh t$$

Comparison of the explicit summation formulas for $P_n(x)$ and $Q_n(x)$ given in (2.15) and (2.16) with the explicit summation formulas for $U_n(x)$ and $T_n(x)$, the Chebyshev polynomials of the second and first kinds, respectively (see [11]), shows that

$$P_n(x) = (-i)^{n-1}U_{n-1}(ix) \tag{6.5}$$

and

$$Q_n(x) = 2(-i)^n T_n(ix) \tag{6.6}$$

i.e., $P_n(x)$ and $Q_n(x)$ are modified Chebyshev polynomials in a complex variable. To reconcile the form in [11] with (2.16) we had to replace the Gamma function, namely, $\Gamma(n-m) = (n-m-1)!$

Because of (6.5) and (6.6), $P_n(x)$ and $Q_n(x)$ would have [9] complex hypergeometric representations. Other representations also exist in view of the many forms the expressions for $U_n(x)$ and $T_n(x)$ can take.

In particular, we may record that

$$P_n(i\cosh x) = i^{n-1}\sinh nx/\sinh x \tag{6.7}$$

and

$$Q_n(i\cosh x) = 2i^n\cosh nx. \tag{6.8}$$

From (1.1) we observe that

$$P_{n+1}(ix) + P_{n-1}(ix) = Q_n(ix)$$

leads, with the help of (6.5) and (6.6), to

$$U_n(ix) - U_{n-2}(ix) = 2T_n(ix), \tag{6.9}$$

which is a complex version of a basic relationship between the two kinds of Chebyshev polynomials. Similarly, other Chebyshev relationships may be tied to corresponding relationships involving $P_n(x)$ and $Q_n(x)$.

Finally, we allude to the *Gegenbauer* (*ultraspherical*) *polynomial* of degree n and order ν, $C_n^\nu(x)$, defined by

$$\sum_{n=0}^{\infty} C_n^\nu(x)t^n = (1 - 2xt + t^2)^{-\nu} \qquad (\nu > 0, \quad |t| < 1). \tag{6.10}$$

with explicit forms

$$C_n^0(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r}{n-r}\binom{n-r}{r}(2x)^{n-2r} \qquad C_0^0(x) = 1 \quad (\nu = 0) \tag{6.11}$$

and

$$C_n^\nu(x) = \frac{1}{\Gamma(\nu)} \sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r \frac{\Gamma(n-r+\nu)}{\Gamma(n-r+1)}\binom{n-r}{r}(2x)^{n-2r} \qquad (\nu > -\tfrac{1}{2};\ \nu \neq 0). \tag{6.12}$$

A recurrence relation for $C_n^\nu(x)$ is

$$(n+2)C_{n+2}^\nu(x) = 2(n+\nu+1)xC_{n+1}^\nu(x) - (n+2\nu)C_n^\nu(x) \tag{6.13}$$

which, for $\nu = 1$, reduces to

$$C_{n+2}^1(x) = 2xC_{n+1}^1(x) - C_n^1(x) \tag{6.14}$$

with

$$C_0^1(x) = 1, \quad C_1^1(x) = 2x. \tag{6.15}$$

Clearly, $C_n^1(x) = U_n(x)$, and by (6.5),

$$P_n(x) = (-i)^{n-1}C_{n-1}^1(ix). \tag{6.16}$$

When $\nu = 0$, (6.11), where $C_1^0(x) = 2x$, gives

$$C_n^0(x) = \frac{2}{n}T_n(x),$$

so that (6.6) gives

$$Q_n(x) = n(-i)\ C_n^0(ix) \qquad (n \geqslant 1) \tag{6.17}$$

i.e., $P_n(x)$, $Q_n(n)$ are modified Gegenbauer polynomials in a complex variable.

As the Fibonacci and Lucas numbers arise from $P_n(x)$ and $Q_n(x)$ when $x = \frac{1}{2}$, we have, from (6.16) and (6.17),

$$F_1 = C_0^1\left(\frac{i}{2}\right) = 1, \quad F_n = (-i)^{n-1}C_{n-1}^1\left(\frac{i}{2}\right) \tag{6.18}$$

and

$$L_0 = 2C_0^0\left(\frac{i}{2}\right) = 2, \quad L_n = n(-1)^n C_n^0\left(\frac{i}{2}\right) \qquad n \geqslant 1. \tag{6.19}$$

Using the known [9] result $dT_n(x)/dx = nU_{n-1}(x)$ from [11] with (6.5) and (6.6), we can arrive back at (2.17), viz., $dQ_n(x)/dx = 2nP_n(x)$.

Differentiating in (2.15) and applying (6.12) in the case $\nu = 2$, we deduce that

$$\frac{dP_n(x)}{dx} = 2(-i)^{n-2}C_{n-2}^2(ix). \tag{6.20}$$

Alternatively, we may differentiate in (6.16) and invoke the result [11]

$$\frac{dC_n^\nu(x)}{dx} = 2\nu C_{n-1}^{\nu+1}(x)$$

to obtain (6.20).

Some of the above results, e.g., (6.16), were generalized in [12] for the sequence of polynomials $\{A_k(x)\}$ defined by

$$A_{n+2}(x) = 2xA_{n+1}(x) + A_n(x) \qquad A_0(x) = s, \quad A_1(x) = r. \tag{6.21}$$

Of course, $\{A_n(x)\}$ is a special case of the sequence $\{W_n(p, q; a, b)\}$, some of whose properties are documented in [8].

Information related to some aspects of the above ideas can be found in [1], [2], [3], [4], [5], [9], and [10].

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A NEW PERSPECTIVE TO THE GENERALIZATION OF THE FIBONACCI SEQUENCE

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I

Let the arbitrary real numbers a, b, c, and d be given. Construct two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ for which

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \alpha_{n+2} = \beta_{n+1} + \beta_n, \quad n \geqslant 0 \\ \beta_{n+2} = \alpha_{n+1} + \alpha_n, \quad n \geqslant 0 \end{cases} \tag{1}$$

Clearly, if we set $a = b$ and $c = d$, then the sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ will coincide with each other and with the sequence $\{F_i(a, d)\}_{i=0}^{\infty}$. The first ten terms of the sequences defined in (1) are:

n	α_n	β_n
0	a	b
1	c	d
2	$b + d$	$a + c$
3	$a + c + d$	$b + c + d$
4	$a + b + 2c + d$	$a + b + c + 2d$
5	$a + 2b + 2c + 3d$	$2a + b + 3c + 2d$
6	$3a + 2b + 4c + 4d$	$2a + 3b + 4c + 4d$
7	$4a + 4b + 7c + 6d$	$4a + 4b + 6c + 7d$
8	$6a + 7b + 10c + 11d$	$7a + 6b + 11c + 10d$
9	$11a + 10b + 17c + 17d$	$10a + 11b + 17c + 17d$

A careful examination of the corresponding terms in each column leads one immediately to

Theorem 1.1

(a) $\alpha_{3n} + \beta_0 = \beta_{3n} + \alpha_0, \quad n \geqslant 0$

(b) $\alpha_{3n+1} + \beta_1 = \beta_{3n+1} + \alpha_1, \quad n \geqslant 0$

(c) $\alpha_{3n+2} + \alpha_0 + \alpha_1 = \beta_{3n+2} + \beta_0 + \beta_1, \quad n \geqslant 0$

Proof of (a): The statement is obviously true if $n = 0$. Assume the statement is true for some integer $n \geqslant 1$. Then

$$\alpha_{3n+3} + \beta_0 = \beta_{3n+2} + \beta_{3n+1} + \beta_0 \qquad \text{by (1)}$$

(continued)

$$\begin{aligned}
&= \alpha_{3n+1} + \alpha_{3n} + \beta_{3n+1} + \beta_0 && \text{by (1)}\\
&= \alpha_{3n+1} + \beta_{3n} + \beta_{3n+1} + \alpha_0 && \text{by induction hypothesis}\\
&= \alpha_{3n+1} + \alpha_{3n+2} + \alpha_0 && \text{by (1)}\\
&= \beta_{3n+3} + \alpha_0 && \text{by (1).}
\end{aligned}$$

Hence, the statement is true for all integers $n \geqslant 0$. Similar proofs can be given for parts (b) and (c).

Adding the first n terms of each sequence $\{\alpha_i\}$ and $\{\beta_i\}$ yields a result similar to that obtained by adding the first n Fibonacci numbers. That is,

Theorem 1.2. For all integers $k \geqslant 0$, we have:

(a) $\alpha_{3k+2} = \sum_{i=0}^{3k} \beta_i + \beta_1$

(b) $\alpha_{3k+3} = \sum_{i=0}^{3k+1} \alpha_i + \beta_1$

(c) $\alpha_{3k+4} = \sum_{i=0}^{3k+2} \beta_i + \alpha_1$

(d) $\beta_{3k+2} = \sum_{i=0}^{3k} \alpha_i + \alpha_1$

(e) $\beta_{3k+3} = \sum_{i=0}^{3k+1} \beta_i + \alpha_1$

(f) $\beta_{3k+4} = \sum_{i=0}^{3k+2} \alpha_i + \beta_1$

Because the proofs of each part are very similar, we give only a proof of part (e).

Proof of (e): If $k = 0$ the statement is obviously true, since

$$\sum_{i=0}^{1} \beta_i + \alpha_1 = \beta_0 + \beta_1 + \alpha_1 = \alpha_2 + \alpha_1 = \beta_3.$$

Assume (e) is true for some integer $k \geqslant 1$, then

$$\begin{aligned}
\beta_{3k+6} &= \alpha_{3k+5} + \alpha_{3k+4} && \text{by (1)}\\
&= \beta_{3k+4} + \beta_{3k+3} + \alpha_{3k+4} && \text{by (1)}\\
&= \beta_{3k+4} + \sum_{i=0}^{3k+1} \beta_i + \alpha_1 + \beta_{3k+3} + \beta_{3k+2} && \text{by induction hypothesis and (1)}\\
&= \sum_{i=0}^{3k+4} \beta_i + \alpha_1.
\end{aligned}$$

Hence, (e) is true for all integers $k \geqslant 0$.

Adding the first n terms with even or odd subscripts for each sequence $\{\alpha_i\}$ and $\{\beta_i\}$, we obtain more results which are similar to those obtained when one adds the first n terms of the Fibonacci sequence with even or odd subscripts. That is,

Theorem 1.3. For all integers $k \geqslant 0$, we have:

(a) $\alpha_{6k+5} = \sum_{i=0}^{3k+2} \beta_{2i} - \alpha_0 + \beta_1$

(b) $\alpha_{6k+6} = \sum_{i=1}^{3k+3} \beta_{2i-1} + \alpha_0$

(c) $\alpha_{6k+7} = \sum_{i=0}^{3k+3} \beta_{2i} - \beta_0 + \alpha_1$

(d) $\alpha_{6k+8} = \sum_{i=1}^{3k+4} \beta_{2i-1} + \beta_0$

(e) $\alpha_{6k+9} = \sum_{i=0}^{3k+4} \beta_{2i} - \beta_0 + \beta_1$

(f) $\alpha_{6k+10} = \sum_{i=1}^{3k+5} \beta_{2i-1} + \alpha_0 + \alpha_1 - \beta_1$

(g) $\beta_{6k+5} = \sum_{i=0}^{3k+2} \alpha_{2i} - \beta_0 + \alpha_1$

(h) $\beta_{6k+6} = \sum_{i=1}^{3k+3} \alpha_{2i-1} + \beta_0$

(i) $\beta_{6k+7} = \sum_{i=0}^{3k+3} \alpha_{2i} - \alpha_0 + \beta_1$

(j) $\beta_{6k+8} = \sum_{i=1}^{3k+4} \alpha_{2i-1} + \alpha_0$

(k) $\beta_{6k+9} = \sum_{i=0}^{3k+4} \alpha_{2i} - \alpha_0 + \alpha_1$

(l) $\beta_{6k+10} = \sum_{i=1}^{3k+5} \alpha_{2i-1} + \beta_0 - \alpha_1 + \beta_1$

Proof of (g): If $k = 0$ the statement is obviously true, since

$$\sum_{i=0}^{2} \alpha_{2i} - \beta_0 + \alpha_1 = \alpha_0 + \alpha_2 + \alpha_4 - \beta_0 + \alpha_1 = 2a + b + 3c + 2d = \beta_5.$$

Assume (g) is true for some integer $k \geqslant 1$, then

$$\begin{aligned}
\beta_{6k+11} &= \alpha_{6k+10} + \alpha_{6k+9} && \text{by (1)} \\
&= \alpha_{6k+10} + \beta_{6k+9} + \alpha_0 - \beta_0 && \text{by Theorem 1.1, part (a)} \\
&= \alpha_{6k+10} + \alpha_{6k+8} + \alpha_{6k+7} + \alpha_0 - \beta_0 && \text{by (1)} \\
&= \alpha_{6k+10} + \alpha_{6k+8} + \beta_{6k+6} + \sum_{i=0}^{3k+2} \alpha_{2i} + \alpha_1 + \alpha_0 - 2\beta_0 && \\
& && \text{by (1) and induction hypothesis} \\
&= \alpha_{6k+10} + \alpha_{6k+8} + \sum_{i=0}^{3k+3} \alpha_{2i} + \alpha_1 - \beta_0 && \text{by Theorem 1.1, part (a)} \\
&= \sum_{i=0}^{3k+5} \alpha_{2i} + \alpha_1 - \beta_0.
\end{aligned}$$

Hence, (g) is true for all integers $k \geqslant 0$. A similar proof can be given for each of the remaining eleven parts of the theorem.

The following result is an interesting relationship which follows immediately from Theorems 1.1 and 1.2. Therefore, the proofs are omitted.

Theorem 1.4. If $k \geqslant 0$, then

(a) $\sum_{i=0}^{3k} (\alpha_i - \beta_i) = \alpha_0 - \beta_0$

(b) $\sum_{i=0}^{3k+1} (\alpha_i - \beta_i) = \beta_2 - \alpha_2$

(c) $\sum_{i=0}^{3k+2} (\alpha_i - \beta_i) = 0.$

As one might suspect, there should be a relationship between the new sequence and the Fibonacci numbers. The next theorem establishes one of these relationships.

Theorem 1.5. If $n \geqslant 0$, then

$$\alpha_{n+2} + \beta_{n+2} = F_{n+1}(\alpha_0 + \beta_0) + F_{n+2}(\alpha_1 + \beta_1).$$

Proof: The statement is obviously true if $n = 0$ and $n = 1$. Assume that the statement is true for all integers less than or equal to some $n \geqslant 2$. Then

$$\begin{aligned} \alpha_{n+3} + \beta_{n+3} &= \beta_{n+2} + \beta_{n+1} + \alpha_{n+2} + \alpha_{n+1} && \text{by (1)} \\ &= F_{n+1}(\alpha_0 + \beta_0) + F_{n+2}(\alpha_1 + \beta_1) + F_n(\alpha_0 + \beta_0) \\ &\qquad + F_{n+1}(\alpha_1 + \beta_1) && \text{by induction hypothesis} \\ &= F_{n+2}(\alpha_0 + \beta_0) + F_{n+3}(\alpha_1 + \beta_1). \end{aligned}$$

Hence, the statement is true for all integers $n \geqslant 0$.

At this point, one could continue to establish properties for the two sequences $\{\alpha_i\}$ and $\{\beta_i\}$ which are similar to those of the Fibonacci sequence. However, we have chosen another route.

II

Express the members of the sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$, when $n \geqslant 0$, as follows:

$$\left.\begin{aligned} \alpha_n &= \Gamma_n^1 a + \Gamma_n^2 b + \Gamma_n^3 c + \Gamma_n^4 d \\ \beta_n &= \delta_n^1 a + \delta_n^2 b + \delta_n^3 c + \delta_n^4 d \end{aligned}\right\} \tag{2}$$

In this way we obtain the eight sequences $\{\Gamma_i^j\}_{i=0}^{\infty}$, $\{\delta_i^j\}_{i=0}^{\infty}$, $(j = 1,2,3,4)$. The purpose of this section is to show how these eight sequences are related to each other and to the Fibonacci numbers with the major intent of finding a direct formula for calculating α_n and β_n for any n.

Theorem 2.1 establishes a relationship between these eight sequences and the Fibonacci numbers.

Theorem 2.1

(a) $\Gamma_n^1 + \delta_n^1 = F_{n-1}, \quad n \geqslant 0$

(b) $\Gamma_n^2 + \delta_n^2 = F_{n-1}, \quad n \geqslant 0$

(c) $\Gamma_n^3 + \delta_n^3 = F_n, \quad n \geqslant 0$

(d) $\Gamma_n^4 + \delta_n^4 = F_n, \quad n \geqslant 0.$

Proof of (a): This is obviously true if $n = 0$ and 1, since

$$\Gamma_0^1 + \delta_0^1 = 1 + 0 = F_{-1} \quad \text{and} \quad \Gamma_1^1 + \delta_1^1 = 0 + 0 = 0 = F_0.$$

Assume this is true for all integers less than or equal to some integer $n \geqslant 2$. Then

$$\Gamma^1_{n+1} + \delta^1_{n+1} = \delta^1_n + \delta^1_{n-1} + \Gamma^1_n + \Gamma^1_{n-1} = F_{n-1} + F_{n-2} = F_n,$$

and (a) is true for all integers $n \geqslant 0$. Similarly, one can prove parts (b), (c), and (d).

The next step is to show how the above eight sequences are related to each other.

Theorem 2.2. If $k \geqslant 0$, then

(a) $\Gamma^1_{3k} = \delta^1_{3k} + 1$

(b) $\Gamma^1_{3k+1} = \delta^1_{3k+1}$

(c) $\Gamma^1_{3k+2} = \delta^1_{3k+2} - 1$

(d) $\Gamma^2_{3k} = \delta^2_{3k} - 1$

(e) $\Gamma^2_{3k+1} = \delta^2_{3k+1}$

(f) $\Gamma^2_{3k+2} = \delta^2_{3k+2} + 1$

(g) $\Gamma^3_{3k} = \delta^3_{3k}$

(h) $\Gamma^3_{3k+1} = \delta^3_{3k+1} + 1$

(i) $\Gamma^3_{3k+2} = \delta^3_{3k+2} - 1$

(j) $\Gamma^4_{3k} = \delta^4_{3k}$

(k) $\Gamma^4_{3k+1} = \delta^4_{3k+1} - 1$

(l) $\Gamma^4_{3k+2} = \delta^4_{3k+2} + 1$

Proof of (j): It is obvious that (j) is true if $k = 0$, since $\Gamma^4_0 = \delta^4_0 = 0$. Assume the statement is true for some integer $k \geqslant 1$. Then

$$\begin{aligned}
\Gamma^4_{3k+3} &= \delta^4_{3k+2} + \delta^4_{3k+1} && \text{by (1)}\\
&= \Gamma^4_{3k+1} + \Gamma^4_{3k} + \delta^4_{3k+1} && \text{by (1)}\\
&= \Gamma^4_{3k+1} + \delta^4_{3k} + \delta^4_{3k+1} && \text{by induction hypothesis}\\
&= \Gamma^4_{3k+1} + \Gamma^4_{3k+2} = \delta^4_{3k+3} && \text{by (1)}
\end{aligned}$$

and the statement is proved. The remaining parts are proved in a similar way.

We now show

Theorem 2.3. If $n \geqslant 0$, then

(a) $\Gamma^1_n + \Gamma^2_n = \delta^1_n + \delta^2_n$

(b) $\Gamma^3_n + \Gamma^4_n = \delta^3_n + \delta^4_n$

Proof of (a): This is obviously true if $n = 0$ and $n = 1$. Assume true for all integers less than or equal to some integer $n \geqslant 2$. Then

$$\begin{aligned}
\Gamma^1_{n+1} + \Gamma^2_{n+1} &= \delta^1_n + \delta^1_{n-1} + \delta^2_n + \delta^2_{n-1} && \text{by (1)}\\
&= \Gamma^1_n + \Gamma^2_n + \Gamma^1_{n-1} + \Gamma^2_{n-1} && \text{by induction hypothesis}\\
&= \delta^1_{n+1} + \delta^2_{n+1} && \text{by (1)}
\end{aligned}$$

Similarly, one can prove part (b).

Before stating and proving our main result for this section, we need the following three theorems.

Theorem 2.4. If $n \geqslant 0$, then

(a) $\delta_n^1 = \Gamma_n^2$

(b) $\delta_n^2 = \Gamma_n^1$

(c) $\delta_n^3 = \Gamma_n^4$

(d) $\delta_n^4 = \Gamma_n^3$

(e) $\Gamma_n^3 = \Gamma_{n+1}^2$

(f) $\Gamma_n^4 = \Gamma_{n+1}^1$

(g) $\delta_n^3 = \delta_{n+1}^2$

(h) $\delta_n^4 = \delta_{n+1}^1$

Proof of (a): The statement is trivially true for $n = 0, 1, 2$, so assume it is true for all integers less than or equal to n where $n \geqslant 3$. Then

$$\begin{aligned}
\delta_{n+1}^1 &= \Gamma_n^1 + \Gamma_{n-1}^1 && \text{by (1)} \\
&= \delta_{n-1}^1 + \delta_{n-2}^1 + \delta_{n-2}^1 + \delta_{n-3}^1 && \text{by (1)} \\
&= \Gamma_{n-1}^2 + \Gamma_{n-2}^2 + \Gamma_{n-2}^2 + \Gamma_{n-3}^2 && \text{by induction hypothesis}
\end{aligned}$$

Two applications of (1) will complete the proof of part (a) of the theorem. The other parts are proved by similar arguments.

From Theorems 2.1 and 2.4, we have the following.

Theorem 2.5

(a) $\Gamma_n^1 + \Gamma_n^2 = \delta_n^1 + \delta_n^2 = F_{n-1} \qquad (n \geqslant 0)$

(b) $\Gamma_n^3 + \Gamma_n^4 = \delta_n^3 + \delta_n^4 = F_n \qquad (n \geqslant 0)$

Finally, we have the following statement.

Theorem 2.6. If $n \geqslant 2$, then

(a) $\Gamma_n^1 = \Gamma_{n-1}^1 + \Gamma_{n-2}^1 + 3\left[\frac{n}{3}\right] - n + 1$

(b) $\Gamma_n^2 = \Gamma_{n-1}^2 + \Gamma_{n-2}^2 + n - 3\left[\frac{n}{3}\right] - 1$

(c) $\Gamma_n^1 = \Gamma_n^2 + 3\left[\frac{n}{3}\right] - n + 1$

(d) $\Gamma_n^3 = \Gamma_{n-1}^3 + \Gamma_{n-2}^3 + n - 3\left[\frac{n+1}{3}\right]$

(e) $\Gamma_n^4 = \Gamma_{n-1}^4 + \Gamma_{n-2}^4 + 3\left[\frac{n+1}{3}\right] - n$

(f) $\Gamma_n^3 = \Gamma_n^4 + n - 3\left[\frac{n+1}{3}\right]$

Proof of (a): The statement is obviously true if n equals 2 or 3. Assume the statement true for all integers less than or equal to $n \geqslant 4$. Then

$$\begin{aligned}
\Gamma_{n+1}^1 &= \delta_n^1 + \delta_{n-1}^1 = \Gamma_n^2 + \Gamma_{n-1}^2 && \text{by (1) and Theorem 2.4, part (a)} \\
&= \delta_{n-1}^2 + \delta_{n-2}^2 + \delta_{n-2}^2 + \delta_{n-3}^2 && \text{by (1)} \\
&= \Gamma_{n-1}^1 + \Gamma_{n-2}^1 + \Gamma_{n-2}^1 + \Gamma_{n-3}^1 && \text{by Theorem 2.4, part (b)}
\end{aligned}$$

(continued)

$$\begin{aligned}
&= \Gamma_n^1 - 3\left[\frac{n}{3}\right] + n - 1 + \Gamma_{n-1}^1 - 3\left[\frac{n-1}{3}\right] + n - 2 \qquad \text{by induction hypothesis}\\
&= \Gamma_n^1 + \Gamma_{n-1}^1 + 2n - 3 - 3\left[\frac{n}{3}\right] - 3\left[\frac{n-1}{3}\right]\\
&= \Gamma_n^1 + \Gamma_{n-1}^1 + 2n - 3 + 3\left[\frac{n+1}{3}\right] - 3n + 3\\
&= \Gamma_n^1 + \Gamma_{n-1}^1 + 3\left[\frac{n+1}{3}\right] - (n-1) + 1
\end{aligned}$$

and part (a) is proved. (It can be shown that $[(n+1)/3] + [n/3] + [(n-1)/3] = n - 1$, $n \geqslant 1$.) Similarly, one can prove parts (b), (d), and (e).

The proof of part (c) above follows directly from part (a) of Theorem 2.6, (1), and part (a) of Theorem 2.4. The proof of part (f) follows by a similar argument.

Adding the equations of part (a) of both Theorems 2.5 and 2.6, we have, for $n \geqslant 0$,

$$\begin{aligned}
\Gamma_{n+2}^1 &= \frac{1}{2}\left(F_{n+1} - \Gamma_{n+2}^2 + \Gamma_{n+1}^1 + \Gamma_n^1 + 3\left[\frac{n+2}{3}\right] - n - 1\right)\\
&= \frac{1}{2}\left(F_{n+1} - \Gamma_{n+2}^2 + \delta_{n+2}^1 + 3\left[\frac{n+2}{3}\right] - n - 1\right) \qquad \text{by (1)}\\
&= \frac{1}{2}\left(F_{n+1} + 3\left[\frac{n+2}{3}\right] - n - 1\right) \qquad \text{by (a) of Theorem 2.4}\\
&= \delta_{n+2}^2 \qquad \text{by (a) of Theorem 2.4}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\Gamma_{n+2}^2 &= \frac{1}{2}\left(F_{n+1} - 3\left[\frac{n+2}{3}\right] + n + 1\right) = \delta_{n+2}^1\\
\Gamma_{n+2}^3 &= \frac{1}{2}\left(F_{n+2} - 3\left[\frac{n}{3}\right] + n - 1\right) = \delta_{n+2}^4\\
\Gamma_{n+2}^4 &= \frac{1}{2}\left(F_{n+2} + 3\left[\frac{n}{3}\right] - n + 1\right) = \delta_{n+2}^3.
\end{aligned}$$

Substiting these four equations into (2), we have our

<u>BASIC THEOREM</u>. If $n \geqslant 0$, then

$$\begin{aligned}
\alpha_{n+2} &= \frac{1}{2}\Bigg\{\left(F_{n+1} + 3\left[\frac{n+2}{3}\right] - n - 1\right)a + \left(F_{n+1} + n + 1 - 3\left[\frac{n+2}{3}\right]\right)b\\
&\qquad + \left(F_{n+2} + n - 3\left[\frac{n}{3}\right] - 1\right)c + \left(F_{n+2} + 3\left[\frac{n}{3}\right] + 1 - n\right)d\Bigg\}\\
&= \frac{1}{2}\Bigg\{(a+b)F_{n+1} + (c+d)F_{n+2} + \left(3\left[\frac{n+2}{3}\right] - n - 1\right)(a-b)\\
&\qquad + \left(n - 3\left[\frac{n}{3}\right] - 2\right)(c-d)\Bigg\};\\
\beta_{n+2} &= \frac{1}{2}\Bigg\{\left(F_{n+1} + n + 1 - 3\left[\frac{n+2}{3}\right]\right)a + \left(F_{n+1} + 3\left[\frac{n+2}{3}\right] - n - 1\right)b\\
&\qquad + \left(F_{n+2} + 3\left[\frac{n}{3}\right] + 1 - n\right)c + \left(F_{n+2} + n - 3\left[\frac{n}{3}\right] - 1\right)d\Bigg\}
\end{aligned}$$

(continued)

$$= \frac{1}{2}\left\{(a+b)F_{n+1} + (c+d)F_{n+2} + \left(3\left[\frac{n+2}{3}\right] - n - 1\right)(b-a) + \left(n - 3\left[\frac{n}{3}\right] - 1\right)(d-c)\right\}.$$

III

The sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ can also be expressed as follows [similarly to (1)]:

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \left.\begin{aligned} \alpha_{n+2} &= \alpha_{n+1} + \alpha_n \\ \beta_{n+2} &= \beta_{n+1} + \beta_n \end{aligned}\right\} \quad (n \geqslant 0) \end{cases} \tag{3}$$

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \left.\begin{aligned} \alpha_{n+2} &= \beta_{n+1} + \alpha_n \\ \beta_{n+2} &= \alpha_{n+1} + \beta_n \end{aligned}\right\} \quad (n \geqslant 0) \end{cases} \tag{4}$$

$$\begin{cases} \alpha_0 = a, \quad \alpha_1 = c, \quad \beta_0 = b, \quad \beta_1 = d \\ \left.\begin{aligned} \alpha_{n+2} &= \alpha_{n+1} + \beta_n \\ \beta_{n+2} &= \beta_{n+1} + \alpha_n \end{aligned}\right\} \quad (n \geqslant 0) \end{cases} \tag{5}$$

The sequences (3) are actually two independent Fibonacci sequences of the form $\{F_i(a, c)\}_{i=0}^{\infty}$ and $\{F_i(b, d)\}_{i=0}^{\infty}$. It is easily seen that the sequences (4) can be expressed through the sequences $\{F_i(a, d)\}_{i=0}^{\infty}$ and $\{F_i(b, c)\}_{i=0}^{\infty}$, namely, $\alpha_{2n} = F_{2n}(a, d)$, $\alpha_{2n+1} = F_{2n+1}(b, c)$, $\beta_{2n} = F_{2n}(b, c)$, $\beta_{2n+1} = F(a, d)$, $n \geqslant 1$.

In the case of (5), two sequences are introduced whose members are related similarly to those discussed in I and II. Therefore, we shall discuss them no further here.

Numerous similar pairs of sequences can be constructed. However, the ones introduced here stand most closely to the very spirit of the Fibonacci sequence and its generalization rules.

We are deeply thankful to the referee for his thorough discussion.

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GENERATING FUNCTIONS OF FIBONACCI-LIKE SEQUENCES AND DECIMAL EXPANSIONS OF SOME FRACTIONS

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1. In this note I respond to two earlier notes [1] and [2] on the decimal expansion of some fractions that are related to the Fibonacci numbers F_n and the Lucas numbers L_n. The simplest example is

$$\frac{1}{89} = \begin{array}{l} .0112358 \\ \quad\quad\;\; 13 \\ \quad\quad\;\;\; 21 \\ \quad\quad\;\;\;\; \cdots \end{array} = \sum_{n=1}^{\infty} F_{n-1}10^{-n}.$$

I propose to put these expansions into a context from which more examples can be drawn in abundance. The recently studied Tribonacci numbers (see [3], [4]) will also fit into this context.

The Fibonacci and Lucas numbers can be defined by the recursions

$$F_0 = 0,\ F_1 = 1,\ L_0 = 2,\ L_1 = 1,\ F_{n+1} = F_n + F_{n-1},\ L_{n+1} = L_n + L_{n-1},$$

for $n \geqslant 1$, or equivalently, by the formulas

$$F_n = \frac{1}{\sqrt{5}}(\omega^n - \tilde{\omega}^n), \quad L_n = \omega^n + \tilde{\omega}^n, \tag{1}$$

where $\omega = \frac{1}{2}(1 + \sqrt{5})$, $\tilde{\omega} = \frac{1}{2}(1 - \sqrt{5})$. Taking this as a definition of F_n and L_n for arbitrary integers n, it follows from

$$\omega\tilde{\omega} = -1 \tag{2}$$

that $F_{-n} = (-1)^{n+1}F_n$, $L_{-n} = (-1)^n L_n$.

First, I shall restate and prove Theorem 2 of [2] in the following form:

Theorem 1. Let A, B, a_0, a_1 be arbitrary complex numbers. Define the sequence $(a_n)_n$ by the recursion $a_{n+1} = Aa_n + Ba_{n-1}$. Then the formula

$$\sum_{n=1}^{\infty} \frac{a_{n-1}}{z^n} = \frac{a_0 z + (a_1 - Aa_0)}{z^2 - Az - B} \tag{3}$$

holds for all complex z such that $|z|$ is larger than the absolute values of the zeros of $z^2 - Az - B$.

Corollary 2. Let a rational function

$$f(z) = \frac{a_0 z + b_1}{z^2 - Az - B}$$

with arbitrary complex numbers A, B, a_0, b_1 be given. Then formula (3) holds for sufficiently large $|z|$, where the coefficients a_n are uniquely determined by the recursion $a_1 = b_1 + Aa_0$, $a_{n+1} = Aa_n + Ba_{n-1}$.

Proof: From the recursion, it is clear that $a_n = O(c^n)$ for some $c > 0$. Therefore, the power series converges for $|z| > c$. Let

$$S = \sum_{n=1}^{\infty} a_{n-1}z^{-n}.$$

Then it follows that

$$(Az + B)S = \sum_{n=1}^{\infty}\left(\frac{Aa_{n-1}}{z^{n-1}} + \frac{Ba_{n-1}}{z^n}\right) = Aa_0 + \sum_{n=1}^{\infty}\frac{Aa_n + Ba_{n-1}}{z^n}$$

$$= Aa_0 + \sum_{n=1}^{\infty}\frac{a_{n+1}}{z^n} = Aa_0 + z^2S - a_0z - a_1.$$

This implies (3). As a power series expansion, (3) is valid in the largest annulus $|z| > r$ which does not contain a pole of the function represented. This proves the theorem, and the corollary follows immediately.

2. As an application, I shall prove a result which shows that all decimal expansions in [1] can be regarded as special instances of Theorem 1 and, therefore, of Theorem 2 in [2]. Moreover, I believe that this result clarifies the question of convergence in [1].

Theorem 3. Let k and ℓ be integers, $k \geqslant 1$. Then the formula

$$\sum_{n=1}^{\infty}\frac{F_{k(n-1)+\ell}}{z^n} = \frac{F_\ell z + (-1)^\ell F_{k-\ell}}{z^2 - L_k z + (-1)^k} \tag{4}$$

holds for all complex z that satisfy $|z| > \omega^k$.

Proof: This is a direct consequence of (1), (2), (3), and the geometric sum formula:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{F_{k(n-1)+\ell}}{z^n} &= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{\omega^{kn+\ell} - \tilde{\omega}^{kn+\ell}}{z^{n+1}}\\ &= \frac{1}{\sqrt{5}}\left(\frac{\omega^\ell}{z - \omega^k} - \frac{\tilde{\omega}^\ell}{z - \tilde{\omega}^k}\right)\\ &= \frac{1}{\sqrt{5}}\left(\frac{(\omega^\ell - \tilde{\omega}^\ell)z + (\omega^k\tilde{\omega}^\ell - \omega^\ell\tilde{\omega}^k)}{z^2 - (\omega^k + \tilde{\omega}^k)z + (\omega\tilde{\omega})^k}\right)\\ &= \frac{F_\ell z + (-1)^\ell F_{k-\ell}}{z^2 - L_k z + (-1)^k}.\end{aligned}$$

Corollary 2 now implies the recursion

$$a_{n+1} = L_k a_n + (-1)^{k+1}a_{n-1} \quad \text{for } a_n = F_{kn+\ell}. \tag{5}$$

One can also prove (5) directly and then obtain Theorem 3 as a consequence of Theorem 1.

3. Examples: For $\ell = 0$, formula (5) reads

$$\sum_{n=1}^{\infty} \frac{F_{k(n-1)}}{z^n} = \frac{F_k}{z^2 - L_k z + (-1)^k} \quad \text{for } |z| > \omega^k. \tag{6}$$

This looks simpler than (5.1) and (5.2) in [1], and because of

$$L_1 + (L_2 + L_4 + L_6 + \cdots + L_{2m}) = L_{2m+1},$$

and

$$L_2 + (L_3 + L_5 + \cdots + L_{2m-1}) = L_{2m},$$

it is in fact equivalent with those formulas. All decimal expansions in [1] are special instances of (6) when z is a power of 10. I shall now write some instances of (4) with $\ell > 0$.

(a) Choose $z = 10^2$, $\ell = 1$, $k = 2, 3$. This yields

$$\frac{99}{9701} = \frac{10^2F_1 - F_1}{10^4 - 10^2L_2 + 1} = .010205133489\ldots,$$

$$\frac{99}{9599} = \frac{10^2F_1 - F_2}{10^4 - 10^2L_3 - 1} = \begin{array}{l} .01031355 \\ \qquad\quad 233 \\ \qquad\qquad 987 \\ \qquad\qquad\quad \ldots . \end{array}$$

For $z = 10^2$, the condition $|z| > \omega^k$ is satisfied for $k \leqslant 9$, and therefore with $\ell = 1$ there are similar expansions of the fractions 98/9301, 97/8899, 95/8201, 92/7099, 87/5301, and 79/2399.

(b) Choose $z = 10^3$, $k = 5$, and let ℓ run from 1 to 4. With

$$N = 10^6 - 10^3L_5 - 1 = 988999,$$

this yields

$$\frac{997}{988999} = (10^3F_1 - F_4)/N = .001008089987\ldots,$$

$$\frac{1002}{988999} = (10^3F_2 + F_3)/N = \begin{array}{l} .001013144 \\ \qquad\qquad 1597 \\ \qquad\qquad\quad \ldots , \end{array}$$

$$\frac{1999}{988999} = (10^3F_3 - F_2)/N = \begin{array}{l} .002021233 \\ \qquad\qquad 2584 \\ \qquad\qquad\quad \ldots , \end{array}$$

$$\frac{3001}{988999} = (10^3F_4 + F_1)/N = \begin{array}{l} .003034377 \\ \qquad\qquad 4181 \\ \qquad\qquad\quad \ldots . \end{array}$$

For $z = 10^3$, the series (4) converges if $k \leqslant 14$. Generally, if z is fixed and $|z|$ is large, the range of values of k for which Theorem 3 applies is easily read from a table of Lucas numbers because, by (1) and $|\tilde{\omega}| < 2/3$, L_n is a good approximation for ω^n.

Remark: The reasoning in the proof of Theorem 3 can also be applied to the Lucas numbers. The result is

$$\sum_{n=1}^{\infty} \frac{L_{k(n-1)+\ell}}{z^n} = \frac{L_\ell z - (-1)^\ell L_{k-\ell}}{z^2 - L_k z + (-1)^k} \quad \text{for } |z| > \omega^k, \tag{7}$$

$$a_{n+1} = L_k a_n + (-1)^{k+1} a_{n-1} \quad \text{for } a_n = L_{kn+\ell}. \tag{8}$$

4. Theorem 1 and its proof can easily be generalized for sequences with a more complicated recursion, and any rational function can be dealt with in this way.

Theorem 4. Let arbitrary complex numbers $A_0, A_1, \ldots, A_m, \ a_0, a_1, \ldots, a_m$ be given. Define the sequence $(a_n)_n$ by the recursion

$$a_{n+1} = A_0 a_n + A_1 a_{n-1} + \cdots + A_m a_{n-m}. \tag{9}$$

Then for all complex z such that $|z|$ is larger than the absolute values of all zeros of

$$q(z) = z^{m+1} - A_0 z^m - A_1 z^{m-1} - \cdots - A_m, \tag{10}$$

the formula

$$\sum_{n=1}^{\infty} \frac{a_{n-1}}{z^n} = \frac{p(z)}{q(z)} \tag{11}$$

holds with

$$\begin{aligned} p(z) &= a_0 z^m + b_1 z^{m-1} + \cdots + b_m, \\ b_k &= a_k - \sum_{j=0}^{k-1} A_j a_{k-1-j} \quad \text{for } 1 \leqslant k \leqslant m. \end{aligned} \tag{12}$$

Corollary 5. Let any rational function $f(z) = p(z)/q(z)$ be given such that the degree of the polynomial p is less than that of q. Then there are complex numbers $A_0, A_1, \ldots, A_m, \ a_0, a_1, \ldots, a_m$ such that, for $|z|$ sufficiently large, formula (11) holds with the sequence $(a_n)_n$ defined by the recursion (9).

Proof: From (9) it follows that $a_n = 0(c^n)$ for some $c > 0$. Therefore, the power series in (11) converges for $|z| > c$. With

$$S = \sum_{n=1}^{\infty} a_{n-1} z^{-n},$$

it follows that

$$(A_0 z^m + A_1 z^{m-1} + \cdots + A_m)S = \sum_{n=1}^{\infty} (A_0 z^m + A_1 z^{m-1} + \cdots + A_m) a_{n-1} z^{-n}$$

$$= \sum_{n=1}^{\infty} (A_0 a_{n+m-1} + A_1 a_{n+m-2} + \cdots + A_m a_{n-1}) z^{-n} + A_0 (a_0 z^{m-1} + a_1 z^{m-2} + \cdots + a_{m-1})$$

$$+ A_1 (a_0 z^{m-2} + a_1 z^{m-3} + \cdots + a_{m-2}) + \cdots + A_{m-1} a_0$$

(continued)

$$= \sum_{n=1}^{\infty} a_{m+n} z^{-n} + A_0 a_0 z^{m-1} + (A_0 a_1 + A_1 a_0) z^{m-2} + \cdots$$

$$= z^{m+1} S - a_0 z^m - b_1 z^{m-1} - b_2 z^{m-2} - \cdots - b_m,$$

where the b_k are defined as in (12). This implies (11). As a power series expansion, (11) is valid in the largest annulus $|z| > r$ which does not contain a pole of the function p/q. This proves the theorem. The corollary follows at once, because the constants $A_0, A_1, \ldots, A_m, \ a_0, a_1, \ldots, a_m$ can be read from (10) and (12).

The coefficients a_n are uniquely determined by the function p/q. The recursion (9), however, is not unique unless one requires m to be minimal.

5. One must ask for good examples to illustrate Theorem 4 and its corollary. In view of (1), one may think of units in cubic number fields. An example of this kind is provided by the so-called Tribonacci numbers T_n (see [3], [4]). I will discuss these numbers briefly in section 6.

As a first example, I choose

$$q(z) = z^3 - z - 1$$

for the denominator in (11). This means that I consider sequences $(a_n)_n$ that satisfy the recursion

$$a_n = a_{n-2} + a_{n-3}. \tag{13}$$

There are a real zero $\omega_1 = 1.32471\ldots$ and a pair of conjugate complex zeros ω_2, $\omega_3 = \overline{\omega_2}$ of the polynomial q. Define

$$\lambda_n = \omega_1^n + \omega_2^n + \omega_3^n \quad \text{for } n \text{ any integer.} \tag{14}$$

Since λ_n is symmetric in the roots of q that are algebraic units, it is plain that all λ_n must be rational integers. This can also be shown as follows. The roots of q satisfy

$$\omega_1 + \omega_2 + \omega_3 = 0, \quad \omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1 = -1, \quad \omega_1\omega_2\omega_3 = 1. \tag{15}$$

This implies

$$\lambda_2 = \omega_1^2 + \omega_2^2 + (\omega_1 + \omega_2)^2 = 2(\omega_1^2 + \omega_2^2) + 2\omega_1\omega_2$$

$$= 2(\lambda_2 - \omega_3^2) + \frac{2}{\omega_3} = 2\lambda_2 - 2\omega_3^2 + 2(\omega_3^2 - 1) = 2\lambda_2 - 2,$$

whence $\lambda_2 = 2$, and from $\omega_\nu^3 = \omega_\nu + 1$ it follows that $\lambda_n = \lambda_{n-2} + \lambda_{n-3}$ for all n. Thus, the λ_n satisfy recursion (13), the starting values being $\lambda_0 = 3$, $\lambda_1 = 0$, $\lambda_2 = 2$. The λ_n may be regarded as an analogue to the Lucas numbers. A short table of these numbers is shown below.

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
λ_n	-3	2	1	-1	3	0	2	3	2	5	5	7	10	12	17	22

Note that $\omega_1\omega_2\omega_3 = 1$ implies

$$\lambda_{-n} = (\omega_2\omega_3)^n + (\omega_3\omega_1)^n + (\omega_1\omega_2)^n.$$

The table indicates that $\lambda_{n+5} - \lambda_{n+4} = \lambda_n$; this is easily shown for any sequence $(a_n)_n$ that satisfies (13). Another consequence from (15) is $|\omega_2|^2 = 1/\omega_1 < 1$. Therefore, the power series $\sum_{n=1}^{\infty} \lambda_{n-1} z^{-n}$ converges for $|z| > \omega_1$, and the following analogue to Theorem 3 has a wider range of validity than Theorem 3:

Theorem 6. Let λ_n be defined as in (14), and let k and ℓ be integers, $k \geqslant 1$. Then the formula

$$\sum_{n=0}^{\infty} \frac{\lambda_{kn+\ell}}{z^{n+1}} = \frac{\lambda_\ell z^2 + (\lambda_{k+\ell} - \lambda_k \lambda_\ell) z + \lambda_{\ell-k}}{z^3 - \lambda_k z^2 + \lambda_{-k} z - 1} \tag{16}$$

holds for all complex z that satisfy $|z| > \omega_1^k$. The numbers $c_n = \lambda_{kn+\ell}$ satisfy the recursion

$$c_n = \lambda_k c_{n-1} - \lambda_{-k} c_{n-2} + c_{n-3}. \tag{17}$$

Proof: We proceed exactly as in the proof of Theorem 3, using the geometric sum formula and the relations (15) to obtain (16). Recursion (17) then follows from Corollary 5.

For numerical examples, choose $z = 10^2$, $k = 3$, $\ell = 0, 1, 2$. This yields

$$\frac{29402}{970199} = \sum_{n=0}^{\infty} \frac{\lambda_{3n}}{10^{2(n+1)}} = .030305122968\,158\ldots,$$

$$\frac{201}{970199} = \sum_{n=0}^{\infty} \frac{\lambda_{3n+1}}{10^{2(n+1)}} = .000207173990\,209\ldots,$$

$$\frac{19899}{970199} = \sum_{n=0}^{\infty} \frac{\lambda_{3n+2}}{10^{2(n+1)}} = .0205102251\,119\ldots.$$

The particular choice of the numbers λ_n is not essential for the conclusion in Theorem 6. In fact, let arbitrary complex numbers a_0, a_1, a_2 be given. Then the system of three linear equations

$$d_1\omega_1^n + d_2\omega_2^n + d_3\omega_3^n = a_n \qquad (n = 0,\ 1,\ 2) \tag{18}$$

has the unique solution

$$d_1 = \frac{\omega_3 - \omega_2}{\sqrt{D}}\left(\frac{a_0}{\omega_1} + a_1\omega_1 + a_2\right) \quad \text{etc.},$$

where $D = -23$ is the discriminant of q. Use (18) to define a_n for all integers n. Then, from $\omega_\nu^3 = \omega_\nu + 1$, it follows that the a_n satisfy (13). Thus, any sequence $(a_n)_n$ which obeys (13) can be represented in the form (18). Therefore, we may proceed as in the proof of Theorem 3, and the result is

$$\sum_{n=0}^{\infty} \frac{a_{kn+\ell}}{z^{n+1}} = \frac{a_\ell z^2 + (a_{k+\ell} - \lambda_k a_\ell) z + a_{\ell-k}}{z^3 - \lambda_k z^2 + \lambda_{-k} z - 1} \quad \text{for } |z| > \omega_1^k. \tag{19}$$

It suffices to state and prove (19) for $\ell = 0$, since the case of a general ℓ can be reduced to $\ell = 0$ by a modification of a_0, a_1, a_2.

6. The validity of a result like (19) does not depend on the particular choice of the polynomial q. Let

$$q(z) = z^{m+1} - A_0 z^m - A_1 z^{m-1} - \cdots - A_m$$

be any polynomial with only simple zeros $\omega_1, \ldots, \omega_{m+1}$. Then it follows as in

(18) that any sequence $(a_n)_n$ which satisfies recursion (9) can be represented in the form

$$a_n = d_1\omega_1^n + \cdots + d_{m+1}\omega_{m+1}^n$$

with uniquely-determined coefficients $d_1, \ldots, d_{m+1}$. Thus, an analogue to formula (19) must hold for any such sequence.

As a final example, let me discuss the polynomial

$$q(z) = z^3 - z^2 - z - 1$$

and sequences $(a_n)_n$ which obey

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}. \tag{20}$$

The numbers T_n that satisfy $T_0 = 1$, $T_1 = 1$, $T_2 = 2$ and the recursion (20) (with a_n replaced by T_n) have been called the Tribonacci numbers in [3] and [4]. An equivalent of formula (11) for this particuler sequence $(T_n)_n$ has been proved in [4]. The zeros of $q(1/z)$ have been computed in [3]; $q(z)$ has a real zero $\zeta_1 = 1.83928\ldots$ and a pair of conjugate complex zeros ζ_2, $\zeta_3 = \overline{\zeta_2}$. An appropriate analogue to L_n and λ_n are the numbers

$$\Lambda_n = \zeta_1^n + \zeta_2^n + \zeta_3^n;$$

they satisfy $\Lambda_0 = 3$, $\Lambda_1 = 1$, $\Lambda_2 = 3$ and the recursion (20) (with a_n replaced by Λ_n). The corresponding formula for the Tribonacci numbers is

$$T_n = d_1\zeta_1^n + d_2\zeta_2^n + d_3\zeta_3^n,$$

where

$$d_1 = \frac{\zeta_3 - \zeta_2}{\sqrt{D}} \cdot \zeta_1^2, \text{ etc.,}$$

and $D = -44$ is the discriminant of q. The analogue to (19) reads

$$\sum_{n=0}^{\infty} \frac{a_{kn}}{z^{n+1}} = \frac{a_0 z^2 + (a_k - \Lambda_k a_0)z + a_{-k}}{z^3 - \Lambda_k z^2 + \Lambda_{-k} z - 1} \quad \text{for } |z| > \zeta_1^k$$

and any sequence $(a_n)_n$ that satisfies (20).

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ON $P_{r,k}$ SEQUENCES

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INTRODUCTION

Definition 1: Let k be a given positive integer. Two integers α and β are said to have the property p_k (resp. p_{-k}) if $\alpha\beta + k$ (resp. $\alpha\beta - k$) is a perfect square. A set of integers is said to be a P_k set if every pair of distinct elements in the set has the property p_k. A sequence of integers is said to be a $P_{r,k}$ sequence if every r consecutive terms of the sequence constitute a P_k set.

Given a positive integer k, we can always find two integers α and β having the property p_k. Conversely, given two integers α and β, we can always find a positive integer k such that α and β have the property p_k. If S is a given P_k set and j is a given integer, then by multiplying all the elements of S by j, we obtain a P_{kj^2} set. Suppose we are given two numbers $a_1 < a_2$ with property p_k and we want to extend the set $\{a_1, a_2\}$ such that the resulting set is also a P_k set. Toward this end, in this paper we construct a $P_{3,k}$ sequence $\{a_n\}$.

ASSOCIATED $P_{3,k}$ SEQUENCES

Suppose

$$a_1a_2 + k = b_1^2 \tag{1}$$

and let $a_3 \in \{a_1, a_2, \ldots\}$, a P_k set. Then we have

$$a_1a_3 + k = x^2 \tag{2}$$

and

$$a_2a_3 + k = y^2 \tag{3}$$

for some integers x and y. Eliminating a_3 from (2) and (3), we obtain

$$X^2 - a_1a_2Y^2 = ka_2(a_2 - a_1), \tag{4}$$

where $X = a_2x$, $Y = y$. Using (1) in (4), we obtain

$$X^2 - (b_1^2 - k)Y^2 = k(a_2^2 - b_1^2 + k). \tag{5}$$

One can check that $X = a_2(a_1 + b_1)$, $Y = a_2 + b_1$, is always a solution of (5). When $b_1^2 - k$ is positive and square free, (5) has an infinite number of solutions. Henceforth, we concentrate on the solution $X = a_2(a_1 + b_1)$, $Y = a_2 + b_1$ of (5). This gives

$$a_2a_3 + k = b_2^2,$$

$$a_1a_3 + k = c_1^2,$$

with

AMS (MOS) Subject Classifications (1980) 10 B 05.

$$b_2 = a_2 + b_1, \quad c_1 = a_1 + b_1, \quad a_3 = b_2 + c_1.$$

In what follows, we construct three sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, where a_1, a_2, a_3, b_1, b_2, and c_1 are as above. We say that $\{b_n\}$ and $\{c_n\}$ are the sequences associated with $\{a_n\}$. Taking

$$b_3 = a_3 + b_2, \quad c_2 = a_2 + b_2, \quad a_4 = b_3 + c_2,$$

we can see that $2(a_3 + a_2) - a_1 = a_4$. Using this fact, we obtain

$$a_2a_4 + k = c_2^2 \quad \text{and} \quad a_3a_4 + k = b_3^2.$$

For the construction of the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, the following diagram can be helpful.

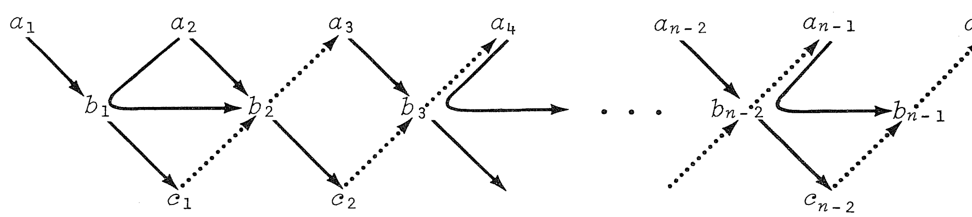

Diagram 1

Explanation for the diagram: Write $b_1 = \sqrt{a_1a_2 + k}$ in the second row, in the space between a_1 and a_2; and write $c_1 = \sqrt{a_1a_3 + k}$ in the third row, in the space beneath a_2. Along the arrows shown by thick lines, sum the elements of of the first and second rows to obtain the elements of the third row. Along the curved arrows, sum the elements of the first and second rows to obtain the elements of the second row. Along the arrows shown by dotted lines, sum the elements of the second and third rows to obtain the elements of the first row. The preceding discussion shows that the scheme provided in the diagram is valid for a_1, a_2, a_3, a_4, b_1, b_2, b_3, c_1, and c_2. Let $n > 2$. Assuming the validity of Diagram 1 for $a_1, \ldots, a_n, b_1, \ldots, b_{n-1}$, and $c_1, \ldots, c_{n-2}$, it can be proved without much difficulty that

$$2(a_n + a_{n-1}) - a_{n-2} = a_{n+1}, \tag{6}$$

and that the scheme is valid for $a_1, \ldots, a_{n+1}, b_1, \ldots, b_n$, and $c_1, \ldots, c_{n-1}$.

Theorem 1. The three sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ have the same recurrence relation.

Proof: We have $a_{n+1} = 2(a_n + a_{n-1}) - a_{n-2}$ [see (6)]. Now

$$\begin{aligned} b_{n+1} &= a_{n+1} + b_n = c_{n-1} + 2b_n = a_{n-1} + b_{n-1} + 2b_n \\ &= 2b_n + b_{n-1} + (b_{n-1} - b_{n-2}) = 2(b_n + b_{n-1}) - b_{n-2}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} c_{n+1} &= a_{n+1} + b_{n+1} = 2a_{n+1} + b_n = 2(c_{n-1} + b_n) + b_n \\ &= 2c_{n-1} + b_n + 2(c_n - a_n) = 2(c_n + c_{n-1}) + (a_n + b_{n-1}) - 2a_n \\ &= 2(c_n + c_{n-1}) - c_{n-2}. \end{aligned} \tag{8}$$

Hence, the theorem is proved.

We shall now obtain additional relations. First, using

$$a_{n+1} = c_{n+1} - b_{n+1} \quad \text{and} \quad a_{n+2} = c_n + b_{n+1},$$

we have

$$a_{n+1} + a_{n+2} = c_n + c_{n+1};$$

that is,

$$a_{n+1} - c_n = -(a_{n+2} - c_{n+1}). \tag{9}$$

Next, from

$$b_n = c_n - a_n \quad \text{and} \quad b_n = b_{n+1} - a_{n+1},$$

we obtain

$$2b_n = (c_n + b_{n+1}) - a_{n+1} - a_n,$$

which yields

$$2b_n = a_{n+2} - a_{n+1} - a_n. \tag{10}$$

Next,

$$\begin{aligned} a_{n+2} - a_{n+1} + a_n &= (b_{n+1} + c_n) - (b_{n+1} - b_n) + a_n \\ &= c_n + b_n + a_n \\ &= 2c_n. \end{aligned} \tag{11}$$

From (10), we obtain

$$a_{n+2} = a_{n+1} + a_n + 2\sqrt{a_n a_{n+1} + k},$$

and from (6) we have

$$a_{n+2} = 2(a_{n+1} + a_n) - a_{n-1}.$$

Hence,

$$a_{n+1} + a_n - a_{n-1} = 2\sqrt{a_n a_{n+1} + k},$$

which gives the relation

$$a_{n+1}^2 + a_n^2 + a_{n-1}^2 - 2a_{n-1}a_n - 2a_{n-1}a_{n+1} - 2a_n a_{n+1} = 4k. \tag{12}$$

FIBONACCI RELATIONSHIPS

Next we shall exhibit a relationship between either of the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ and the Fibonacci sequence $\{F_n\}$. The Fibonacci sequence $\{F_n\}$ is defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n.$$

V. E. Hoggatt, Jr., and G. E. Bergum [1] have shown that the even-subscripted Fibonacci numbers constitute a $P_{3,1}$ sequence. We can set

$$a_{n-1} = F_{2n}, \quad a_n = F_{2n+2}, \quad \text{and} \quad a_{n+1} = F_{2n+4}$$

in (12) and obtain

$$F_{2n}^2 + F_{2n+2}^2 + F_{2n+4}^2 - 2F_{2n}F_{2n+2} - 2F_{2n+2}F_{2n+4} - 2F_{2n}F_{2n+4} = 4.$$

<u>Theorem 2</u>. Any sequence $\{a_n\}$ satisfying (6) is given by

$$a_n = -F_{n-3}F_{n-2}a_1 + F_{n-3}F_{n-1}a_2 + F_{n-2}F_{n-1}a_3, \quad n \geqslant 4. \tag{13}$$

Proof: From (6), we get

$$a_4 = 2(a_3 + a_2) - a_1 = -F_1F_2a_1 + F_1F_3a_2 + F_2F_3a_3,$$
$$a_5 = 2(a_4 + a_3) - a_2 = -2a_1 + 3a_2 + 6a_3 = -F_2F_3a_1 + F_2F_4a_2 + F_3F_4a_3,$$
$$a_6 = 2(a_5 + a_4) - a_3 = -6a_1 + 10a_2 + 15a_3 = -F_3F_4a_1 + F_3F_5a_2 + F_4F_5a_3.$$

So the theorem is true for $n = 4, 5, 6$. Let $n \geqslant 4$ and assume that the theorem is true for all integers j up to n. Using (6) we have

$$\begin{aligned} a_{n+1} = {} & 2(-F_{n-3}F_{n-2}a_1 + F_{n-3}F_{n-1}a_2 + F_{n-2}F_{n-1}a_3) \\ & + 2(-F_{n-4}F_{n-3}a_1 + F_{n-4}F_{n-2}a_2 + F_{n-3}F_{n-2}a_3) \\ & - (-F_{n-5}F_{n-4}a_1 + F_{n-5}F_{n-3}a_2 + F_{n-4}F_{n-3}a_3); \end{aligned}$$

that is,

$$\begin{aligned} a_{n+1} = {} & (-2F_{n-3}F_{n-2} - 2F_{n-4}F_{n-3} + F_{n-5}F_{n-4})a_1 \\ & + (2F_{n-3}F_{n-1} + 2F_{n-4}F_{n-2} - F_{n-5}F_{n-3})a_2 \\ & + (2F_{n-2}F_{n-1} + 2F_{n-3}F_{n-2} - F_{n-4}F_{n-3})a_3. \end{aligned} \tag{14}$$

The coefficient of a_1 in (14) is given by

$$\begin{aligned} -[2F_{n-3}(F_{n-2} + F_{n-4}) - F_{n-4}(F_{n-3} - F_{n-4})] &= -(2F_{n-3}F_{n-2} + F_{n-3}F_{n-4} + F_{n-4}^2) \\ &= -(2F_{n-3}F_{n-2} + F_{n-4}F_{n-2}) \\ &= -F_{n-2}(2F_{n-3} + F_{n-4}) \\ &= -F_{n-2}(F_{n-3} + F_{n-2}) \\ &= -F_{n-2}F_{n-1}. \end{aligned}$$

Similarly, upon simplification, we have the coefficients of a_2 and a_3 in (14) equal to $F_{n-2}F_n$ and $F_{n-1}F_n$, respectively. This proves Theorem 2.

Remark 1. The relations (6), (7), and (8) imply that (13) remains true if the a's are replaced by b's or by c's.

Now we express b's in terms of a_1, a_2, a_3. We have

$$2b_2 = -a_1 + a_2 + a_3.$$

Using $a_4 = 2(a_3 + a_2) - a_1$, we obtain

$$2b_3 = -a_2 + a_3 + a_4 = -a_1 + a_2 + 3a_3,$$
$$2b_4 = -a_2 + a_3 + 3a_4 = -3a_1 + 5a_2 + 7a_3.$$

Suppose $2b_n = -r_na_1 + s_na_2 + t_na_3$. Then

$$2b_{n+1} = -r_na_2 + s_na_3 + t_na_4 = -t_na_1 + 2(t_n - r_n)a_2 + (2t_n + s_n)a_3.$$

Hence, $2b_{n+1} = -r_{n+1}a_1 + s_{n+1}a_2 + t_{n+1}a_3$, where

$$\begin{aligned} & t_2 = 1, \quad t_3 = 3, \quad t_4 = 7, \\ & r_{n+1} = t_n, \\ & s_{n+1} = 2t_n - t_{n-1}, \\ & t_{n+1} = 2(t_n + t_{n-1}) - t_{n-2} \qquad (n \geqslant 4). \end{aligned} \tag{15}$$

Similarly, we have $2c_{n+1} = -u_{n+1}a_1 + v_{n+1}a_2 + w_{n+1}a_3$, where

$$\begin{aligned} &w_1 = w_2 = 1, \quad w_3 = 5, \\ &u_{n+1} = w_n, \\ &v_{n+1} = 2w_n - w_{n-1}, \\ &w_{n+1} = 2(w_n + w_{n-1}) - w_{n-2} \qquad (n \geqslant 3). \end{aligned} \tag{16}$$

Thus, the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{t_n\}$, and $\{w_n\}$ have the same recurrence relation.

Next we consider the possibility for the coincidence of the sequences $\{a_n\}$ and $\{c_n\}$. In this regard, we have the following:

Theorem 3. Let $\{a_n\}$ be a $P_{3,k}$ sequence with the associated sequences $\{b_n\}$ and $\{c_n\}$. The following statements are equivalent:

(i) $a_{n+1} = c_n$ for some integer $n \geqslant 1$

(ii) $a_{n+1} = c_n$ for all integers n

(iii) $a_{n+1} = b_n + c_n$ for all integers n

(iv) $c_{n+1} = b_{n+1} + c_n$ for all integers n

(v) $a_{n+1} = a_n + b_n$ for all integers n

(vi) $b_{n+2} = 3b_{n+1} - b_n$ for all integers n

(vii) $c_{n+2} = 3c_{n+1} - c_n$ for all integers n

(viii) $a_{n+2} = 3a_{n+1} - a_n$ for all integers n

(ix) $k = a_{n+1}^2 - 3a_na_{n+1} + a_n^2$ for all integers n

(x) $-k = b_{n+1}^2 - 3b_nb_{n+1} + b_n^2$ for all integers n

(xi) $k = c_{n+1}^2 - 3c_nc_{n+1} + c_n^2$ for all integers n

(xii) $a_n = -F_{2n-4}a_1 + F_{2n-2}a_2$ for all integers n

and

$b_n = -F_{2n-3}a_1 + F_{2n-1}a_2$ for all integers $n \geqslant 3$

(xiii) b_n is a $P_{3,-k}$ sequence with the associated sequences $\{a_n\}$ and $\{b_n\}$ (where $b_nb_{n+1} - k = a_{n+1}^2$).

Proof: The following scheme may be adopted.

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (ii) $\Rightarrow$ (i),

(v) $\Rightarrow$ (ix) $\Rightarrow$ (viii), (v) $\Rightarrow$ (x) $\Rightarrow$ (vi);

(ii) $\Rightarrow$ (xi) $\Rightarrow$ (vii);

(ii) $\Rightarrow$ (xii) $\Rightarrow$ (ii) and (x) $\Rightarrow$ (xiii) $\Rightarrow$ (x).

The proof itself is left to the reader.

F-TYPE SEQUENCES

Definition 2: Let $\{a_n\}$ be a $P_{3,k}$ sequence together with associated sequences $\{b_n\}$ and $\{c_n\}$. We say that $\{a_n\}$ is an F-type sequence if the sequence

$$\{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\},$$

obtained by juxtaposing the two sequences $\{a_n\}$ and $\{b_n\}$, is of Fibonacci type, i.e., $f_1 = a_1$, $f_2 = b_1$, and $f_n = f_{n-1} + f_{n-2}$, $n \geqslant 3$.

Theorem 4. A $P_{3,k}$ sequence $\{a_n\}$ with the associated sequences $\{b_n\}$ and $\{c_n\}$ for which any one of the equivalent statements in Theorem 3 holds is an F-type sequences. Conversely, given a Fibonacci-type sequence

$$T = \{g, h, g + h, g + 2h, \ldots\},$$

where g and h are two positive integers with $g < h$, if $\{a_n\}$ and $\{b_n\}$ are the sequences formed by taking the terms in the odd and even places, respectively, of T, in the same order as they appear in T, there is an integer k such that $\{a_n\}$ is an F-type $P_{3,k}$ sequence for which the equivalent statements in Theorem 3 hold.

Proof: ($\Rightarrow$) Using $c_{n-1} = a_{n-1} + b_{n-1}$, we obtain $a_n = a_{n-1} + b_{n-1}$ for $n \geqslant 2$. We have that $b_n = a_{n-1} + b_{n-1}$ for $n \geqslant 2$. Hence, the sequence $\{a_1, b_1, a_2, b_2, \ldots\}$ is of the Fibonacci type.

($\Leftarrow$) We have

$$a_1 = g, \quad b_1 = h,$$

$$a_n = F_{2n-3}g + F_{2n-2}h, \quad b_n = F_{2n-2}g + F_{2n-1}h \qquad (n \geqslant 2), \tag{17}$$

where $\{F_n\}$ is the Fibonacci sequence. One can check that

$$a_n + a_{n+2} = 3a_{n+1} \text{ for all } n \geqslant 1. \tag{18}$$

Now

$$\begin{aligned}
&(a_{n+2}^2 - 3a_{n+1}a_{n+2} + a_{n+1}^2) - (a_{n+1}^2 - 3a_n a_{n+1} + a_n^2)\\
&= (a_{n+2}^2 - a_n^2) - 3a_{n+1}(a_{n+2} - a_n)\\
&= (a_{n+2} - a_n)(a_{n+2} + a_n - 3a_{n+1}) = 0 \quad \text{for all } n \geqslant 1.
\end{aligned}$$

Hence, we have

$$a_{n+1}^2 - 3a_n a_{n+1} + a_n^2 = a_{n+2}^2 - 3a_{n+1}a_{n+2} + a_{n+1}^2 = \text{constant, for all } n.$$

Let $a_{n+1}^2 - 3a_n a_{n+1} + a_n^2 = k$. In particular, putting $n = 1$, we get

$$k = h^2 - gh - g^2. \tag{19}$$

We have, using (19),

$$\begin{aligned}
a_n a_{n+1} + k = (F_{2n-3}F_{2n-1} - 1)g^2 &+ (F_{2n-3}F_{2n} + F_{2n-2}F_{2n-1} - 1)gh\\
&+ (F_{2n-2}F_{2n} + 1)h^2.
\end{aligned}$$

It can be seen that $F_{2n-3}F_{2n} - 1 = F_{2n-2}F_{2n-1}$. Therefore,

$$a_n a_{n+1} + k = F_{2n-2}^2 g^2 + 2F_{2n-2}F_{2n-1}gh + F_{2n-1}^2 h^2 = b_n^2.$$

Next,

$$\begin{aligned}
a_{n-1}a_n + k = (F_{2n-5}F_{2n-1} - 1)g^2 &+ (F_{2n-5}F_{2n} + F_{2n-4}F_{2n-1} - 1)gh\\
&+ (F_{2n}F_{2n-4} + 1)h^2.
\end{aligned}$$

After some calculation, we have

$$a_{n-1}a_n + k = F_{2n-3}^2 g^2 + 2F_{2n-2}F_{2n-3}gh + F_{2n-2}^2 h^2 = a_n^2.$$

Consequently, the sequence $\{a_n\}$ is an F-type $P_{3,k}$ sequence with the associated c-sequence given by $c_n = a_{n+1}$ for all integers $n \geqslant 1$.

ASSOCIATED DIOPHANTINE EQUATIONS

Theorem 5. Given a positive integer k, an F-type $P_{3,k}$ sequence exists if and only if the Diophantine equation

$$x^2 - 5y^2 = 4k \tag{20}$$

is solvable in integers.

Proof: ($\Rightarrow$) Let $\{a_n\}$ be an F-type $P_{3,k}$ sequence with the associated sequence $\{b_n\}$ so that $\{a_1, b_1, a_2, b_2, \ldots\}$ is a sequence of the Fibonacci type wherein the relations are given by (17). Then

$$k = h^2 - gh - g^2;$$

that is,

$$h^2 - gh - (g^2 + k) = 0.$$

Treating this as a quadratic equation in h, we obtain $h = \dfrac{g \pm \sqrt{5g^2 + 4k}}{2}$. This implies

$$5g^2 + 4k = A^2$$

for some integer A. Hence, equation (20) is solvable in integers.

($\Leftarrow$) Let (x, y) be an integral solution of (20). Then $x \equiv y \pmod 2$. Form the Fibonacci-type sequence $\{a_1, b_1, a_2, b_2, \ldots\}$ by taking $a_1 = y$, $b_1 = (x + y)/2$. Then by Theorem 4 there is an integer k' such that $\{a_n\}$ is an F-type $P_{3,k'}$ sequence. We have $k' = a_2^2 - 3a_1a_2 + a_1^2$. Since

$$a_2 = a_1 + b_2 = \frac{x + 3y}{2}$$

we obtain

$$k' = \frac{x^2 - 5y^2}{4} = k.$$

Theorem 6. Given a positive integer k, a necessary condition for the existence of an F-type $P_{3,k}$ sequence is that

$$k \not\equiv 2, 3, 6, 7, 8, 10, 12, 13, 14, 17, 18 \pmod{20}$$

and

$$k \not\equiv 10, 15, 35, 40, 60, 65, 85, 90 \pmod{100}.$$

We omit the proof.

To prove our next result, we need the following:

Theorem 7. (Nagell [4]) If $u + v\sqrt{D}$ and $u' + v'\sqrt{D}$ are two given solutions of the equation

$$u^2 - Dv^2 = C \quad (D\text{: positive, square free}),$$

a necessary and sufficient condition for these two solutions to belong to the same class is that the two numbers $(uu' - vv'D)/C$ and $(vu' - uv')/C$ be integers.

In the following theorem, we prove a result for the Diophantine equation (20) by considering the terms of the corresponding F-type $P_{3,k}$ sequence.

Theorem 8. Given a positive integer k, the number of distinct classes of solutions of equation (20) is divisible by 3.

Proof: If (20) is not solvable in integers, then the theorem holds trivially. Assume the solvability of (20). Let (x_1, y_1) be an integral solution of (20). Take $a_1 = y_1$, $b_1 = (x_1 + y_1)/2$ and $a_2 = a_1 + b_1$; i.e., $a_2 = (x_1 + 3y_1)/2$. Then by Theorem 5 we have

$$k = a_2^2 - 3a_1a_2 + a_1^2$$

and $\{a_n\}$ is an F-type $P_{3,k}$ sequence. We have

$$b_2 = a_2 + b_1 = x_1 + 2y_1,$$

$$a_3 = a_2 + b_2 = \frac{3x_1 + 7y_1}{2}, \quad b_3 = a_3 + b_2 = \frac{5x_1 + 11y_1}{2},$$

$$a_4 = a_3 + b_3 = 4x_1 + 9y_1, \quad b_4 = a_4 + b_3 = \frac{13x_1 + 29y_1}{2}.$$

Choose x_i, y_i $(i = 2, 3, 4)$ such that $y_i = a_i$ and $(x_i + y_i)/2 = b_i$; i.e., $x_i = 2b_i - y_i$. Then $x_2 = (3x_1 + 5y_1)/2$, $x_3 = (7x_1 + 15y_1)/2$, $x_4 = (9x_1 + 20y_1)/2$. One can easily check that $x_i + \sqrt{5}y_i$ $(i = 2, 3, 4)$ are solutions of (20). Since

$$\frac{x_1y_2 - y_1x_2}{4k} = \frac{1}{2}, \quad \frac{x_1y_3 - y_1x_3}{4k} = \frac{3}{2}, \quad \text{and} \quad \frac{x_2y_3 - y_2x_3}{4k} = \frac{1}{2},$$

by Theorem 7 it follows that each $x_i + \sqrt{5}y_i$ $(i = 1, 2, 3)$ belongs to a distinct class of solutions of (20). Now

$$x_4 + \sqrt{5}y_4 = (9x_1 + 20y_1) + \sqrt{5}(4x_1 + 9y_1) = (x_1 + \sqrt{5}y_1)(9 + 4\sqrt{5})^n.$$

Since $9 + 4\sqrt{5}$ is the fundamental solution of the equation

$$u^2 - 5v^2 = 1,$$

it follows that $x_1 + \sqrt{5}y_1$ and $x_4 + \sqrt{5}y_4$ belong to the same class of solutions of (20). Thus, given a solution $x_1 + \sqrt{5}y_1$ of (20), we obtain three consecutive terms a_i $(i = 1, 2, 3)$ of an F-type $P_{3,k}$ sequence which in turn yield two more solutions $x_i + \sqrt{5}y_i$ $(i = 2, 3)$ of (20) such that $x_i + \sqrt{5}y_i$ $(i = 1, 2, 3)$ belong to different classes of solutions of (20). Further, it follows by simple induction that, for any integers i, i', j, the terms a_{3i+j} and $a_{3i'+j}$ $(j = 0, 1, 2)$ yield solutions of (20) which belong to the same class. Hence, every F-type $P_{3,k}$ sequence contributes exactly three distinct classes of solutions of (20). Consequently, the number of distinct classes of solutions of (20) is divisible by 3.

Definition 3: Given a positive integer k, two $P_{3,k}$ sequences $\{a_n\}$ and $\{a'_n\}$ are said to be distinct if there do not exist integers r and s such that $a_r = a'_s$.

Theorem 9. Given a positive integer k, the number of distinct F-type $P_{3,k}$ sequences is equal to 1/3 of the number of distinct classes of solutions of (20).

Proof: Follows from Theorem 8.

CONCLUDING COMMENTS

Our next investigation is on $P_{r,k}$ sequences with $r \geqslant 4$. Regarding this, we prove the following theorem.

Theorem 10. If $k \equiv 2 \pmod 4$, then there is no $P_{r,k}$ sequence with $r \geqslant 4$.

Proof: We follow the reasoning given by S. Mohanty [3]. Let $k \equiv 2 \pmod 4$ and let $\{a_n\}$ be a $P_{4,k}$ sequence. Then, for any two integers i, j satisfying $|j - i| \leqslant 3$, we have

$$a_i a_j + k = B^2 \tag{21}$$

for some integer B. If $a_i \equiv 0 \pmod 4$ or if $a_j \equiv 0 \pmod 4$, then (21) implies $B^2 \equiv 2 \pmod 4$, which is impossible. Hence, neither a_i nor a_j is 0 (mod 4). If $a_i \equiv a_j \pmod 4$, we have a contradiction; thus the elements a_i, a_{i+1}, a_{i+2}, and a_{i+3} do not share the property p_k.

The foregoing complements the work of Horadam, Loh, and Shannon [2], whose Pellian sequence $\{Q_n(N)\}$ is a $P_{3,N-2}$ sequence which is there also related to the even-subscripted Fibonacci numbers, to perfect squares, and to Diophantine equations.

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A CONGRUENCE FOR A CLASS OF EXPONENTIAL NUMBERS

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1. INTRODUCTION

A sequence of exponential numbers, say P_n, is defined by its exponential generating function as

$$\sum_{n=0}^{\infty} P_n x^n/n! = \exp\{g(x)\}$$

for some (formal) power series $g(x)$ with constant term zero.

As regards Bell numbers $[g(x) = e^x - 1]$, Lunnon, Pleasants, and Stephens [6] showed that for each positive integer n, there exist integers $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ such that, for all $m \geqslant 0$,

$$P_{m+n} + \alpha_{n-1}P_{m+n-1} + \cdots + \alpha_0 P_m \equiv 0 \pmod{n!}.$$

In this paper, we show a similar congruence for the exponential numbers P_n when $g(x)$ is a certain series function (Section 2). Special cases include numbers P_n equal to the number of permutations of n elements having cycles with given maximal and minimal size or equal to the sum of the horizontal entries of the table of Jordan [5, p. 223], also for P_n equal to the generalized derangement numbers.

2. THE CONGRUENCE

Theorem. Suppose

$$g(x) = \sum_{j=1}^{\infty} b_j \frac{x^j}{j}$$

where the b_j are integers. Let

$$e^{g(x)} = \sum_{n=0}^{\infty} P_n \frac{x^n}{n!} \tag{1}$$

and let

$$\frac{y^k}{k!} e^{-g(y)} = \sum_{n=k}^{\infty} D_{n,k} \frac{y^n}{n!}. \tag{2}$$

Then, for each $m, n \geqslant 0$,

$$\sum_{k=0}^{n} D_{n,k} P_{m+k} \equiv 0 \pmod{n!}.$$

Proof: Let $f(x) = e^{g(x)}$. Then

$$e^{-g(y)}f(x + y) = \sum_{k=0}^{\infty} e^{-g(y)} \frac{y^k}{k!} f^{(k)}(x) = \sum_{m,n=0}^{\infty} \frac{x^m}{m!} \frac{y^n}{n!} \sum_{k=0}^{n} D_{n,k} P_{m+k}.$$

Thus, it is sufficient to show that the coefficient of $x^m/m!$ in $e^{-g(y)}f(x + y)$ is a power series in y with integer coefficients.

Now we have

$$e^{-g(y)}f(x + y) = \exp\left[\sum_{i=1}^{\infty} g^{(i)}(y)\frac{x^i}{i!}\right].$$

Since $g'(y) = \sum_{j=0}^{\infty} b_{j+1}y^j$, $g^{(i)}(y)$ is a power series in y with integer coefficients, $\sum_{i=1}^{\infty} g^{(i)}(y)x^i/i!$ is a Hurwitz series in x (in the sense that the coefficient of $x_i/i!$ is a power series with integer coefficients). Thus,

$$\exp\left[\sum_{i=1}^{\infty} g^{(i)}(y)\frac{x^i}{i!}\right]$$

is also a Hurwitz series in x, which proves the theorem.

Remarks: We have that $g(x)$ is a Hurwitz series. Using the fact that $[g(x)]^k/k!$ is also a Hurwitz series for any nonnegative integer k, we define the integers $A(n, k)$ by

$$\sum_{n=k}^{\infty} A(n, k)x^n/n! = [g(x)]^k/k!. \tag{3}$$

Then, from (1), we have

$$P_n = \sum_{k=0}^{n} A(n, k), \quad P_0 = 1. \tag{4}$$

From (2), we have

$$\begin{aligned}\sum_{n=k}^{\infty} D_{n,k}y^n/n! &= (y^k/k!) \sum_{i=0}^{\infty} (-1)^i \{g(y)\}^i/i! \\ &= \sum_{i=0}^{\infty} (-1)^i \sum_{j=i}^{\infty} A(j, i)y^{j+k}/k!j! \\ &= \sum_{i=0}^{\infty} (-1)^i \sum_{n=i+k}^{\infty} A(n-k, i)\binom{n}{k} y^n/n! \\ &= \sum_{n=k}^{\infty} \sum_{i=0}^{n-k} \binom{n}{k} (-1)^i A(n-k, i)y^n/n!,\end{aligned}$$

and consequently,

$$D_{n,k} = \binom{n}{k} \sum_{i=0}^{n-k} (-1)^i A(n-k, i).$$

For tabulation purposes, we may obtain a recurrence relation for the integers $D_{n,k}$. Using (2), we have

$$D(u, y) = \sum_{n,k} D_{n,k}u^k y^n/n! = e^{-g(y)+uy}. \tag{5}$$

By differentiating both sides of (5) with respect to y, we obtain

$$\frac{\partial}{\partial y}D(u, y) = -e^{-g(y)}g'(y)e^{uy} + e^{-g(y)}e^{uy}u = D(u, y)\{-g'(y) + u\}.$$

Equating coefficients of $u^k y^n/n!$, we obtain

$$D_{n+1,k} = D_{n,k-1} - \sum_{i=0}^{n}\binom{n}{i}b_{n-i+1}(n - i + 1)!D_{i,k} \quad \text{for } n, k \geqslant 0,$$

with $D_{0,0} = 1$ and $D_{n,k} = 0$ for $k > n$ or $k < 0$.

It may be noted that $f(x) = e^{g(x)}$ counts permutations in which a cycle of length j is weighted b_j.

3. SPECIAL CASES

We shall now give some special cases of $g(x)$ for which the numbers P_n are of great interest in Combinatorics.

a. $g(x) = \sum_{j \in S} x^j/j$ where S is any set of positive integers.

Then $f(x) = e^{g(x)}$ counts permutations with all cycle lengths in S. For $S = \{1, 2\}$, $g(x) = x + x^2/2$, and the numbers

$$P_n = t_n = \sum_{k=[n/2]}^{n} A(n, k)$$

have been studied by Moore [3], Moser and Wyman [7], and others. From [4], we have a congruence for t_n which is a special case of our theorem.

b. $g(x) = \sum_{j=r}^{s} \frac{(s)_j}{(j-1)!}\frac{x^j}{j} = (1 + x)^s - \sum_{j=0}^{r-1}\frac{(s)_j}{(j-1)!}\frac{x^j}{j}$,

r, s integers, $1 \leqslant r < s$.

Then $A(n, k)$ have occurred as coefficients in the k-fold convolution of binomial distributions truncated at the point $r - 1$ (see [1]). In the case in which $r = 1$, $A(n, k) = (1/n!)[\Delta^k(sx)_n]_{x=0}$ (see [2]), and the numbers

$$P_n = \sum_{k=[n/s]}^{n} A(n, k)$$

occur in combinatorial analysis being in fact P_n is equal to the sum of the horizontal entries of the table of Jordan (see [5, p. 223]).

c. $g(x) = (s - 1)x + s\sum_{j=2}^{\infty} x^j/j = -x - s\log(1 - x)$, s an integer, $s \geqslant 1$.

Then P_n is equal to the generalized derangement numbers $d(n, s)$ [for $s = 1$, we have the derangement number $d(n)$].

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REGENERATION POINTS IN RANDOM PERMUTATIONS

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1. INTRODUCTION

In this paper, we study a sequence of positive integers defined by recurrence that have applications in combinatorics and probability theory.

Let σ be a permutation of $\mathbb{N}_n = \{1, \ldots, n\}$, i.e., a bijection $\mathbb{N}_n \to \mathbb{N}_n$. Then $k \in \mathbb{N}_n$ is a regeneration point of σ if $\sigma(\mathbb{N}_k) = \mathbb{N}_k$. Here σ will be a random permutation, i.e., we consider σ to be chosen at random from the set S_n of permutations of $\mathbb{N}_n$. Equivalently, we define a probability measure P_n on the power set of S_n by $P_n(\{\sigma_0\}) = P_n(\sigma = \sigma_0) = 1/n!$, $\sigma_0 \in S_n$. Expectation with respect to P_n will be denoted by E_n.

Let A_k be the event that k is a regeneration point of the random permutation. Then

$$P_n(A_k) = k!(n-k)!/n! = \binom{n}{k}^{-1}, \quad k \in \mathbb{N}_n. \tag{1.1}$$

For the event that $k_1, \ldots, k_r$, with $1 \leqslant k_1 < \cdots < k_r \leqslant n$, are regeneration points, we have

$$P_n(A_{k_1}A_{k_2} \ldots A_{k_r}) = k_1!(k_2 - k_1)! \ldots (k_r - k_{r-1})!(n - k_r)!/n!. \tag{1.2}$$

Let M be the total number of regeneration points in σ. The (factorial) moments of M can be expressed in terms of (1.2), e.g.,

$$E_nM = 1 + Q_n = 1 + \sum_{k=1}^{n-1} P_n(A_k) = 1 + \sum_{k=1}^{n-1}\binom{n}{k}^{-1}. \tag{1.3}$$

Note that n is always a regeneration point.

The theory of regeneration points is dominated by the numbers c_n or $c(n)$, $n = 1, 2, \ldots$, where c_n is the number of elements of S_n that have only one regeneration point, or

$$P_n(M = 1) = c_n/n!, \quad n = 1, 2, \ldots . \tag{1.4}$$

This will be seen in Section 2. Here we mention the relation

$$P_n(k) = P_n(\nu = k) = c_k(n-k)!/n!, \quad k \in \mathbb{N}_n, \tag{1.5}$$

where ν is the first regeneration point of the random permutation σ. Since $P_n(1) + \cdots + P_n(n) = 1$, we have

$$\sum_{k=1}^{n}(n-k)!c_k/n! = 1, \quad n \geqslant 1. \tag{1.6}$$

The c_n can be computed recursively from (1.6). We find

$$\begin{aligned} &c_1 = c_2 = 1, \quad c_3 = 3, \quad c_4 = 13, \quad c_5 = 71, \\ &c_6 = 461, \quad c_7 = 3447 = g \times 383. \end{aligned} \tag{1.7}$$

From (1.6) we see, by induction on n, that the c_n are odd. Divisibility of the c_n is considered in Section 4. By (1.4), the principle of inclusion and exclusion, and by (1.2),

$$c_n/n! = 1 - P(A_1 \cup \cdots \cup A_{n-1}) = 1 + \sum_{h=1}^{n-1}(-1)^h T_h = \sum_{h=0}^{n-1}(-1)^h T_h. \tag{1.8}$$

Here $T_0 = 1$ and for $h > 0$,

$$T_h = \Sigma' P_n(A_{i_1}A_{i_2} \ldots A_{i_h}) = \Sigma' i_1!(i_2 - i_1)! \ldots (i_h - i_{h-1})!(n - i_h)!/n!$$
$$= \Sigma'' j_1!j_2! \ldots j_{h+1}!/n!,$$

where Σ' sums over all $i_1, \ldots, i_h$ with $1 \leqslant i_1 < \cdots < i_h \leqslant n - 1$ and Σ'' over all $j_1 \geqslant 1, \ldots, j_{h+1} \geqslant 1$ with $j_1 + \cdots + j_{h+1} = n$. In (1.8) this gives, by putting $h = m - 1$.

$$c_n = \sum_{m=1}^{n}(-1)^{m-1}\Sigma^* j_1! \ldots j_m!, \quad n \geqslant 1. \tag{1.9}$$

where Σ^* sums over all $j_1 \geqslant 1, \ldots, j_m \geqslant 1$ with $j_1 + \cdots + j_m = n$.

In Section 2, an integral equation for the exponential generating function of the c_n will be derived. Section 3 studies the asymptotic behavior of c_n for $n \to \infty$. We have $c_n/n! \to 1$, so M tends to 1 in probability as $n \to \infty$. In Section 5, some applications of the c_n in combinatorial probability theory are given.

2. GENERAL FORMULAS

For the total number M of regeneration points we find, by specifying regeneration points only at $j_1, j_1 + j_2, \ldots, j_1 + \cdots + j_m = n$,

$$P_n(M = m) = \Sigma^* c(j_1)c(j_2) \cdots c(j_m)/n!, \quad m \in \mathbb{N}_n, \tag{2.1}$$

where Σ^* is the same as in (1.9). The event $\{M \geqslant m\}$, with $m \geqslant 2$, means that there are at least $m - 1$ regeneration points in $\{1, \ldots, n - 1\}$. This gives, in the same way as (2.1),

$$\begin{aligned} P_n(M \geqslant m) &= \Sigma' c(j_1) \cdots c(j_{m-1})(n - j_1 - \cdots - j_{m-1})!/n! \\ &= \Sigma^* c(j_1) \cdots c(j_{m-1})j_m!/n!, \quad m = 2, \ldots, n, \end{aligned} \tag{2.2}$$

where Σ' sums over all $j_1 \geqslant 1, \ldots, j_{m-1} \geqslant 1$ with $j_1 + \cdots + j_{m-1} \leqslant n - 1$ and Σ^* is the same as in (1.9).

For the first regeneration point ν we have, with (1.5),

$$\begin{aligned} E_n\nu &= \sum_{k=1}^{n} kc_k(n - k)!/n! = \sum_{k=1}^{n}(n + 1)P_n(k) - \sum_{k=1}^{n} c_k(n + 1 - k)!/n! \\ &= (n + 1) - (n + 1)\sum_{k=1}^{n} P_{n+1}(k) = (n + 1)P_{n+1}(n + 1) = c_{n+1}/n!. \end{aligned} \tag{2.3}$$

From the relation $k^2 = (n + 2 - k)(n + 1 - k) + (2n + 3)k - (n + 2)(n + 1)$, we find, in a similar way,

$$E_n\nu^2 = \{2(n + 1)c_{n+1} - c_{n+2}\}/n!. \tag{2.4}$$

Let

$$C(z) = \sum_{k=1}^{\infty} z^k c_k / k!, \quad |z| < 1. \tag{2.5}$$

From (1.6),

$$z(1 - z)^{-1} = \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} c_k (n - k)!/n! = \sum_{k=1}^{\infty} c_k \sum_{n=k}^{\infty} z^n (n - k)!/n!$$

$$= \sum_{k=1}^{\infty} c_k \sum_{j=0}^{\infty} z^{k+j} j!/(k + j)!.$$

With the relation

$$\int_0^z (z - x)^{k-1}(1 - x)^{-1}dx = (k - 1)! \sum_{j=0}^{\infty} z^{k+j} j!/(k + j)!,$$

to be derived by putting $x = zt$ and expanding $(1 - zt)^{-1}$, we see that

$$z(1 - z)^{-1} = \int_0^z C'(z - x)(1 - x)^{-1}dx,$$

and with partial integration, noting that $C(0) = 0$,

$$z(1 - z)^{-1} = C(z) + \int_0^z (1 - x)^{-2} C(z - x)dx, \quad |z| < 1. \tag{2.6}$$

The author was unable to find a solution of (2.6) in closed form. The Neumann series solution gives a series of iterated convolutions which, on expansion into powers of z, leads back to (1.9).

3. ASYMPTOTIC BEHAVIOR

We use the notation for falling factorials

$$(n)_r = n!/(n - r)!, \quad r = 0, \ldots, n, \quad n = 1, 2, \ldots . \tag{3.1}$$

First we consider $Q_n = E_n M - 1$ given by (1.3). Rockett [4] gave an expression for

$$\sum \binom{n}{k}^{-1},$$

but direct use of (1.3) seems better for asymptotic estimates. We have

$$Q_n = 2n^{-1} + 4(n)_2^{-1} + V(n_3)^{-1}, \quad n \geqslant 6, \tag{3.2}$$

$$V_n = \sum_{k=3}^{n-3} k!(n - k)!/(n - 3)!, \quad n \geqslant 6. \tag{3.3}$$

<u>Theorem 1</u>. We have

$$V_n \geqslant 12, \ n \geqslant 7; \quad V_n \leqslant 156/7, \ n \geqslant 6; \tag{3.4}$$

$$V_n = 12 + O(n^{-1}), \ n \to \infty; \tag{3.5}$$

$$V_{n+1} < V_n, \ n \geqslant 11. \tag{3.6}$$

<u>Proof</u>: The first inequality in (3.4) follows by considering the terms with $k = 3$ and $k = n - 3$ in (3.3). The relation (3.5) follows by estimating the

terms in (3.3) with $k = 4$, $k = n - 4$, and $5 \leqslant k \leqslant n - 5$. From (3.3), for $n \geqslant 6$,

$$\begin{aligned} V_{n+1} - V_n &= 6 + \sum_{k=3}^{n-3} k!(n-k)!\{(n+1-k)(n-2)^{-1} - 1\}/(n-3)! \\ &= 6 + 4(n-2)^{-1}V_n - \sum_{k=3}^{n-3}(k+1)!(n-k)!/(n-2)! \\ &= 6 + 4(n-2)^{-1}V_n - \sum_{h=4}^{n-2} h!(n+1-h)!/(n-2)! \\ &= 12 + 4(n-2)^{-1}V_n - V_{n+1}, \end{aligned} \tag{3.7}$$

so that $2V_{n+1} = 12 + (n+2)(n-2)^{-1}V_n$. Substituting this into (3.7) shows that $V_{n+1} < V_n$, for $n \geqslant 6$, if and only if

$$V_n > 12 + 48(n-6)^{-1}. \tag{3.8}$$

From the terms in (3.3) with $k \leqslant 5$ and $k \geqslant n - 5$,

$$V_n \geqslant 12 + 48(n-3)^{-1} + 240(n-3)^{-1}(n-4)^{-1}, \quad n \geqslant 11.$$

Applying this to (3.8) we find (3.6). From (3.6) and direct computation of V_n, $n = 6, \ldots, 11$, we see that max $V_n = 156/7$ is reached for $n = 11$. Better bounds for larger n may be obtained from (3.6) by computing some V_n.

For the study of c_n, we introduce the following notation, see (1.5) and (3.1):

$$H_n = 1 - c_n/n! = P_n(\nu \leqslant n-1) = \sum_{k=1}^{n-1} c_k(n-k)!/n!; \tag{3.9}$$

$$D_n = (n)_3\{H_n - 2n^{-1} - (n)_2^{-1}\}, \quad n \geqslant 3. \tag{3.10}$$

We need some numerical values of nH_n and D_n. By means of (1.6), (3.9), and (3.10), the values of nH_n and D_n for $3 \leqslant n \leqslant 200$ were computed for the author at the University of Groningen Computing Centre. Part of the values are given in Tables 1 and 2, but the most important numerical result is

$$D_{n+1} < D_n, \quad n = 13, \ldots, 199. \tag{3.11}$$

Table 1

n	nH_n	n	nH_n	n	nH_n
1	0.000000	4	1.833333	7	2.212500
2	1.000000	5	2.041667	8	2.227579
3	1.500000	6	2.158333	9	2.220660

Theorem 2. With D_n defined by (3.9) and (3.10),

$$D_n > 4, \; n \geqslant 9; \; D_n < 6, \; n \geqslant 20. \tag{3.12}$$

Proof: Since $c_k \leqslant k!$, we see from (3.9), (1.1), (1.3), (3.2), and (3.4) that $nH_n \leqslant nQ_n < 3$, $n \geqslant 9$.

With Table 1, we then extend this to

$$nH_n < 3, \; n \geqslant 1. \tag{3.13}$$

From (3.9), for $n \geqslant 7$,

$$\begin{aligned} n!H_n \geqslant (n-1)!c_1 + (n-2)!c_2 + (n-3)!c_3 \\ + 6c(n-3) + 2c(n-2) + c(n-1) \end{aligned} \tag{3.14}$$

With (1.7) and (3.13), writing $c_k = k!(1 - H_k)$ for $k \geqslant n - 3$, this gives

$$H_n > 2n^{-1} + 3(n)_3^{-1} - 18(n)_4^{-1}, \quad n \geqslant 7. \tag{3.15}$$

From (3.15) we see that $nH_n > 2$, $n > 9$, and then from Table 1,

$$nH_n > 2, \quad n \geqslant 5. \tag{3.16}$$

From (3.9) for $n \geqslant 9$, with $c_k \leqslant k!$,

$$n!H_n \leqslant \left(\sum_{k=1}^{4} + \sum_{k=n-4}^{n-1}\right)c_k(n-k)! + (n-9)5!(n-5)!. \tag{3.17}$$

With (1.7) and (3.16), writing $c_k = k!(1 - H_k)$ for $k \geqslant n - 4$, we find

$$H_n \leqslant 2n^{-1} + (n)_2^{-1} + h(n)(n)_3^{-1}, \quad n \geqslant 9; \tag{3.18}$$

$$h(n) = 5 + 25(n-3)^{-1} + (120(n-9) - 48)(n-3)^{-1}(n-4)^{-1}. \tag{3.19}$$

Table 2

n	D_n	n	D_n	n	D_n
3	-2.000000	10	6.625992	17	6.687779
4	-3.000000	11	7.376414	18	6.406247
5	-2.500000	12	7.702940	19	6.156020
6	-0.833333	13	7.726892	20	5.939237
7	1.375000	14	7.561317	21	5.754089
8	3.558333	15	7.295355	21	5.596962
9	5.356944	16	6.991231	23	5.463713

By elementary computation we see that $h(n+1) < h(n)$ for $145n > 1876$ or $n \geqslant 13$ and $h(196) < 6$, so that $h(n) < 6$, $n \geqslant 196$. Hence, $D_n < 6$, $n \geqslant 196$, by (3.10). The second inequality in (3.12) then follows from (3.11) and Table 2, and it shows that

$$H_n < 2n^{-1} + 2(n)_2^{-1}, \quad n \geqslant 20. \tag{3.20}$$

From (3.9), for $n \geqslant 9$,

$$n!H_n \geqslant \left(\sum_{k=1}^{4} + \sum_{k=n-4}^{n-1}\right)c_k(n-k)!.$$

Here we apply (1.7) for $k \leqslant 4$ and write $c_k = k!(1 - H_k)$ for $k \geqslant n - 4$. Application of (3.20) for $k = n - 3$, $n - 4$, and of (3.10) with $D_n < 6$ then gives

$$H_n > 2n^{-1} + (n)_2^{-1} + 4(n)_3^{-1} + g(n)(n)_4^{-1}, \quad n \geqslant 24; \tag{3.21}$$

$$g(n) = 17 - 72(n - 4)^{-1} - 48(n - 4)^{-1}(n - 5)^{-1}. \tag{3.22}$$

Since $g(n + 1) > g(n)$, $n \geqslant 6$, and $g(13) > 0$,

$$H_n > 2n^{-1} + (n)_2^{-1} + 4(n)_3^{-1}, \quad n \geqslant 24,$$

i.e., $D_n > 4$, $n \geqslant 24$. The first inequality in (3.12) then follows from Table 2.

Remark 1. By taking into account more terms in (3.9), it was proved in Stam [5] that $D_n = 4 + O(n^{-1})$ and $D_{n+1} < D_n$, $n \geqslant 13$.

Remark 2. For the conditional probability $P(A_1|M \geqslant 2)$, we see, using (1.1), (3.10), and (3.12), that

$$P(A_1|M \geqslant 2) = P(A_1)/P(M \geqslant 2) = n^{-1}H_n^{-1} \to \frac{1}{2}, \quad n \to \infty,$$

and in the same way,

$$P(A_{n-1}|M \geqslant 2) \to \frac{1}{2},$$

so that the regeneration points concentrate near the end points of $\mathbb{N}_n$ as $n \to \infty$.

4. DIVISIBILITY

From (1.6) we have, since m divides $h!$ if $h \geqslant m$, the congruences

$$\sum_{j=0}^{m-1} j!c_{n-j} \equiv 0 \pmod{m}, \quad n \geqslant m. \tag{4.1}$$

Let $d_n = d_n(m)$ be the remainder of c_n on division by m. Then the recurrence (4.1) also holds for the d_i and determines them completely if $d_1, \ldots, d_{m-1}$ are given. Since $d_n \in \{0, \ldots, m - 1\}$, there are at most m^{m-1} possibilities for the sequence $u_k = (d_k, \ldots, d_{k+m-2})$. One of them is $u_k = (0, \ldots, 0)$ and this would give $d_n = 0$, $n \geqslant 1$, which is excluded because $c_1 = 1$. So we must have $u_k = u_{k+p}$ for some k and some minimal $p \leqslant m^{m-1} - 1$. Since any u_k determines all d_n, $n \geqslant 1$, with (4.1) and the coefficients in (4.1) do not depend on n, it follows that the sequence d_n, $n \geqslant 1$ is periodic with period p.

If $m = 3$, then (4.1) becomes

$$c_n + c_{n-1} + 2c_{n-2} \equiv 0 \quad \text{or} \quad c_n + c_{n-1} - c_{n-2} \equiv 0, \bmod 3, \ n \geqslant 3,$$

so that $(-1)^n c_n$ satisfies the same recurrence mod 3 as the Fibonacci numbers, but the initial conditions are different. We find

$$c_n \equiv 1, 1, 0, 1, 2, 2, 0, 2, 1, 1, \bmod 3, \ n = 1, \ldots, 10.$$

So p has its maximal value 8.

If $m = 4$, then (4.1) gives

$$c_n + c_{n-1} + 2c_{n-2} + 6c_{n-3} \equiv c_n + c_{n-1} + 2c_{n-2} + 2c_{n-3} \equiv 0, \bmod 4, \ n \geqslant 4.$$

Since the c_i are odd, this gives

$$c_n + c_{n-1} \equiv 0, \bmod 4, \ n \geqslant 4.$$

With (1.7) we see that $c_n \equiv 1$, mod 4, if n is even and $c_n \equiv 3$, mod 4, if $n \geqslant 3$ is odd.

Since the c_n are odd, we have $c_n \equiv 1$, $c_n \equiv 3$, $c_n \equiv 5$, mod 6, if $c_n \equiv 1$, $c_n \equiv 0$, $c_n \equiv 2$, mod 3, respectively.

The Computing Centre of the University of Groningen computed the sequences for $m = 5$, $m = 7$, and part of the sequence for $m = 11$. For $m = 5$ the period is 62, whereas $5^4 - 1 = 624$. For $m = 7$ the period is 684, whereas $7^6 - 1 = 117649$. For $m = 5$, 7, and 11 all possible values of $c_n \pmod{m}$ occur. It is conjectured that this holds for all prime m.

We note that for m prime the last two coefficients in (4.1) are 1 and -1 (mod m) by Wilson's theorem (see Grosswald [2, Ch. 4.3]).

5. APPLICATIONS IN COMBINATORIAL PROBABILITY THEORY

If σ and τ are independent stochastic elements of S_n and one of them has uniform distribution P_n, then the points $k \in \mathbb{N}_n$ such that $\sigma(\mathbb{N}_k) = \tau(\mathbb{N}_k)$ have the same joint distribution as the regeneration points of a random permutation, since $\sigma(\mathbb{N}_k) = \tau(\mathbb{N}_k)$ if and only if $\sigma^{-1}\tau(\mathbb{N}_k) = \mathbb{N}_k$ and $\sigma^{-1}\tau$ has probability distribution P_n.

Let $X_1, \ldots, X_n$ be independent random variables with common continuous distribution function and $Y_1, \ldots, Y_n$ their increasing order statistics, i.e., the value of Y_k is the k^{th} smallest of the values of $X_1, \ldots, X_k$. Then the stochastic points k in $\mathbb{N}_n$ such that $X_1 + \cdots + X_k = Y_1 + \cdots + Y_k$ have the same joint probability distribution as the regeneration points of a random permutation of $\mathbb{N}_n$. We have $Y_1 < Y_2 < \cdots < Y_n$ with probability 1 and the conditional distribution of $X_1, \ldots, X_n$ given $Y_i = y_i$, $i = 1, \ldots, n$ is the same as the distribution of $\sigma(y_1), \ldots, \sigma(y_n)$, where σ is a random element of S_n (see Rényi [3]). Furthermore, $\sigma(y_1) + \cdots + \sigma(y_k) = y_1 + \cdots + y_k$ if and only if

$$\sigma(\{y_1, \ldots, y_k\}) = \{y_1, \ldots, y_k\}.$$

A deeper application is the following. Let σ and τ be independent random elements of S_n. Dixon [1] defined t_n as the probability that the subgroup $\langle \tau, \sigma \rangle$ of S_n generated by σ and τ is transitive, i.e., has $\mathbb{N}_n$ as the only orbit. This occurs if and only if $\sigma(A) = \tau(A) = A$ for no proper subset A of $\mathbb{N}_n$. Using formal power series, Dixon [1] proved that

$$\sum_{k=1}^{n} (n-k)!k!kt_k = n!n, \quad n \geq 1. \tag{5.1}$$

A slightly shorter proof starts from U_1, the orbit of $\langle \sigma, \tau \rangle$ that contains 1. By the definition of t_k, we have

$$P(U_1 = A) = (k!)^2 t_k((n-k)!)^2/(n!)^2,$$

if $A \subset \mathbb{N}_n$, $1 \in A$, and $|A| = k$. So

$$P(|U_1| = k) = \binom{n-1}{k-1} P(U_1 = A) = (n-k)!k!kt_k(n!n)^{-1}.$$

Equation (5.1) states that these probabilities sum to 1. From (1.6) and (1.7),

$$\sum_{k=1}^{n} (n-k)!c_{k+1} = \sum_{j=2}^{n+1} (n+1-j)!c_j = (n+1)! - n!c_1 = n!n, \quad n \geq 1.$$

Comparing this with (5.1) we see that the sequences c_{n+1} and $n!nt_n$, $n \geq 1$, are determined (uniquely) by the same recurrence. So

$$n!nt_n = c_{n+1}, \quad n \geq 1.$$

The author was unable to find a direct combinatorial proof of this result.

Let $X = (X_1, \ldots, X_n)$ be a random sample with replacement from $\mathbb{N}_m$, or a random function $\mathbb{N}_n \to \mathbb{N}_m$. If $X(\mathbb{N}_k) = \mathbb{N}_k$, then $X_1, \ldots, X_k$ defines a bijection $\mathbb{N}_k \to \mathbb{N}_k$. So the probability that h is the first k with $X(\mathbb{N}_k) = \mathbb{N}_k$ is

$$c_h m^{n-h} m^{-n} = c_h m^{-h}$$

and the probability that there is at least one such k is

$$\sum_{h=1}^{m \wedge n} m^{-h} c_h .$$

When the sample is drawn without replacement, so that $n \leqslant m$, the corresponding probability is

$$\sum_{h=1}^{n} c_h (m - h)!/m!.$$

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ON LUCAS FUNDAMENTAL FUNCTIONS AND CHEBYCHEV POLYNOMIAL SEQUENCES

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1. INTRODUCTION

Let $\{U_n(p, q)\}$ be the sequence of fundamental functions defined by Lucas [2] as follows:

$$U_{n+2} = pU_{n+1} - qU_n \qquad (n \geqslant 0)$$

with initial values $U_0 = 0$, $U_1 = 1$. Further, let $\{S_n(x)\}$ and $\{T_n(x)\}$ denote the Chebychev polynomial sequences of the first and second kind, respectively. In [5], formulas were obtained for

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n+j}}{(3n + j)!}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n S_{3n+j}(x)}{(3n + j)!}, \quad \text{and} \sum_{n=0}^{\infty} \frac{(-1)^n T_{3n+j}(x)}{(3n + j)!}, \quad j = 0, 1, 2.$$

As mentioned in [5, Remark 4], we generalize the above formulas in this paper to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{nr+j}}{(nr + j)!}, \quad j = 0, 1, \ldots, p - 1,$$

and similar formulas for $\{S_n(x)\}$ and $\{T_n(x)\}$.

2. PRELIMINARIES

The generalized circular functions are defined as follows. For any positive integer r,

$$M_{r,j}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{rn+j}}{(rn + j)!}, \quad j = 0, 1, \ldots, r - 1,$$

and

$$N_{r,j}(t) = \sum_{n=0}^{\infty} \frac{t^{rn+j}}{(rn + j)!}, \quad j = 0, 1, \ldots, r - 1.$$

Note that $M_{1,0}(t) = e^{-t}$, $M_{2,0}(t) = \cos t$, $M_{2,1}(t) = \sin t$, and $N_{1,0}(t) = e^t$, $N_{2,0}(t) = \cosh t$, $N_{2,0}(t) = \sinh t$.

The notation and some of the results presented here are found in Pethe and Sharma [4].

Following Barakat [1] and Walton [7], we define generalized trigonometric and hyperbolic functions of any square matrix X by

$$M_{r,j}(X) = \sum_{n=0}^{\infty} \frac{(-1)^n X^{rn+j}}{(rn + j)!}, \quad j = 0, 1, \ldots, r - 1,$$

and

$$N_{r,j}(X) = \sum_{n=0}^{\infty} \frac{X^{rn+j}}{(rn + j)!}, \quad j = 0, 1, \ldots, r - 1.$$

Lemma 1. Let X be a 2×2 matrix given by

$$X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Let $\operatorname{tr} X = p$ and $\det X = q$. Then, for any integer n,

$X^n = U_n X - qU_{n-1} I$,

where U_n is the n^{th} fundamental function and I the unit matrix of order 2.

This is proved in [1].

Lemma 2. We have, for a positive integer r and $j = 0, 1, \ldots, r - 1$,

$$M_{r,j}(x + y) = \sum_{k=0}^{j} M_{r,k}(x)M_{r,j-k}(y) - \sum_{k=j+1}^{r-1} M_{r,k}(x)M_{r,r+j-k}(y).$$

This is proved in [3].

Lemma 3. Let r be a positive integer, and $j = 0, 1, \ldots, r - 1$. Then:

a. For even r,

$$M_{r,j}(x) + M_{r,j}(-x) = \begin{cases} 2M_{r,j}(x), & j \text{ even} \\ 0, & j \text{ odd}, \end{cases} \tag{2.1}$$

and

$$M_{r,j}(x) - M_{r,j}(-x) = \begin{cases} 0, & j \text{ even} \\ 2M_{r,j}(x), & j \text{ odd}. \end{cases} \tag{2.2}$$

b. For odd r,

$$M_{r,j}(x) + M_{r,j}(-x) = \begin{cases} 2N_{2r,j}(x), & j \text{ even} \\ -2N_{2r,r+j}(x), & j \text{ odd}, \end{cases} \tag{2.3}$$

and

$$M_{r,j}(x) - M_{r,j}(-x) = \begin{cases} -2N_{2r,r+j}(x), & j \text{ even} \\ 2N_{2r,j}(x), & j \text{ odd}. \end{cases} \tag{2.4}$$

Proof: We prove (2.1) and (2.4). The proofs of (2.2) and (2.3) are similar.

Let r be even. Now,

$$M_{r,j}(x) + M_{r,j}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{nr+j}}{(nr+j)!}(1 + (-1)^{nr+j}). \tag{2.5}$$

Since r is even, $(-1)^{nr+j} = (-1)^j$. Hence (2.5) becomes

$$M_{r,j}(x) + M_{r,j}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{nr+j}}{(nr+j)!}(1 + (-1)^j) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{2(-1)^n x^{nr+j}}{(nr+j)!}, & j \text{ even} \\ 0, & j \text{ odd}, \end{cases}$$

(continued)

$$= \begin{cases} 2M_{r,j}(x), & j \text{ even} \\ 0, & j \text{ odd}, \end{cases}$$

which proves (2.1).

Now, let r be odd. Then

$$M_{r,j}(x) - M_{r,j}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{nr+j}}{(nr+j)!}(1 - (-1)^{nr+j}). \tag{2.6}$$

Since r is odd, $(-1)^{nr+j} = (-1)^{n(r-1)+n+j} = (-1)^{n+j}$; therefore, (2.6) becomes

$$M_{r,j}(x) - M_{r,j}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{nr+j}}{(nr+j)!}(1 - (-1)^{n+j})$$

$$= \begin{cases} \displaystyle\sum_{n=1,3,\ldots}^{\infty} \frac{2(-1)^n x^{nr+j}}{(nr+j)!}, & j \text{ even} \\ \displaystyle\sum_{n=0,2,\ldots}^{\infty} \frac{2(-1)^n x^{nr+j}}{(nr+j)!}, & j \text{ odd}, \end{cases}$$

$$= \begin{cases} \displaystyle -2\sum_{n=0}^{\infty} \frac{x^{2nr+r+j}}{(2nr+r+j)!}, & j \text{ even} \\ \displaystyle 2\sum_{n=0}^{\infty} \frac{x^{2nr+j}}{(2nr+j)!}, & j \text{ odd}, \end{cases}$$

$$= \begin{cases} -2N_{2r,r+j}(x), & j \text{ even} \\ 2N_{2r,j}(x), & j \text{ odd}, \end{cases}$$

which proves (2.4).

<u>Lemma 4</u>. We have for $j = 0, 1, \ldots, 2r - 1$ and $i = \sqrt{-1}$,

a. $$M_{2r,j}(ix) = \begin{cases} (-1)^{j/2} M_{2r,j}(x), & r \text{ even} \\ (-1)^{j/2} N_{2r,j}(x), & r \text{ odd}, \end{cases} \tag{2.7}$$

b. $$N_{2r,j}(ix) = \begin{cases} (-1)^{j/2} N_{2r,j}(x), & r \text{ even} \\ (-1)^{j/2} M_{2r,j}(x), & r \text{ odd}. \end{cases} \tag{2.8}$$

<u>Proof</u>: By definition,

$$M_{2r,j}(ix) = \sum_{n=0}^{\infty} \frac{(-1)^n i^{2nr+j} x^{2nr+j}}{(2nr+j)!}. \tag{2.9}$$

Now

$$(i)^{2nr+j} = \begin{cases} (i^4)^{nr/2}(i)^j, & r \text{ even} \\ (i^4)^{\frac{1}{2}n(r-1)}(i)^{2n}(i)^j, & r \text{ odd}, \end{cases}$$

so that

$$(i)^{2nr+j} = \begin{cases} (-1)^{j/2}, & r \text{ even} \\ (-1)^{n+j/2}, & r \text{ odd.} \end{cases} \tag{2.10}$$

Using (2.10) in (2.9), we obtain

$$M_{2r,j}(ix) = \begin{cases} (-1)^{j/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2nr+j}}{(2nr+j)!}, & r \text{ even} \\ (-1)^{j/2} \sum_{n=0}^{\infty} \frac{(-1)^{2n} x^{2nr+j}}{(2nr+j)!}, & r \text{ odd,} \end{cases}$$

which proves (2.7). We can prove (2.8) in a similar manner.

3. SUMMATION FORMULAS FOR LUCAS FUNDAMENTAL FUNCTIONS

We shall now prove

Theorem 1. a. For even r and $j = 0, 1, \ldots, r-1$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{nr+j}}{(nr+j)!} = \frac{2}{\delta}\left[\sum_{k=0}^{[\frac{1}{2}(j-1)]} M_{r,2k+m}(p/2)M_{r,\alpha}(\delta/2) - \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(r-2)} M_{r,2k+m}(p/2)M_{r,r+\alpha}(\delta/2)\right] \tag{3.1}$$

b. For odd r and $j = 0, 1, \ldots, r-1$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{nr+j}}{(nr+j)!} = \frac{2}{\delta}\left[\sum_{k=0}^{[\frac{1}{2}(j-1)]} M_{r,2k+m}(p/2)N_{2r,\alpha}(\delta/2) - \sum_{k=0}^{\frac{1}{2}(r-3)+m} M_{r,2k+1-m}(p/2)N_{2r,r+\beta-1}(\delta/2) + \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(p-1)-m} M_{r,2k+m}(p/2)N_{2r,2r+\alpha}(\delta/2)\right] \tag{3.2}$$

where, in both (a) and (b) above and in Theorems 2 and 3 below,

$$\alpha = j - 2k - m, \quad \beta = j - 2k + m, \quad \text{and} \quad m = \begin{cases} 1, & j \text{ even} \\ 0, & j \text{ odd.} \end{cases}$$

Further, $[S]$ = the greatest integer $\leqslant S$ and δ as defined below.

Proof: By Sylvester's matrix interpolation formula (see [6]), we have

$$M_{r,j}(X) = \frac{1}{\lambda_1 - \lambda_2}\{[M_{r,j}(\lambda_1) - M_{r,j}(\lambda_2)]X - [\lambda_1 M_{r,j}(\lambda_1) - \lambda_2 M_{r,j}(\lambda_2)]I\}, \tag{3.3}$$

where λ_1, λ_2 are distinct eigenvalues of X as defined in Lemma 1. It is easy to see that $\lambda_1 = (p + \delta)/2$, $\lambda_2 = (p - \delta)/2$, where $\delta = \sqrt{(p^2 - 4q)}$. Now

$$M_{r,j}(\lambda_1) - M_{r,j}(\lambda_2) = M_{r,j}\left(\frac{p+\delta}{2}\right) - M_{r,j}\left(\frac{p-\delta}{2}\right). \tag{3.4}$$

Using Lemma 2, (3.4) becomes

$$\begin{aligned} M_{r,j}(\lambda_1) - M_{r,j}(\lambda_2) &= \sum_{k=0}^{j} M_{r,k}(p/2)[M_{r,j-k}(\delta/2) - M_{r,j-k}(-\delta/2)] \\ &\quad - \sum_{k=j+1}^{r-1} M_{r,k}(p/2)(M_{r,r+j-k}(\delta/2) - M_{r,r+j-k}(-\delta/2)). \end{aligned} \tag{3.5}$$

Let r and j both be even. Breaking the summation on the right side of (3.5) into even and odd values of k and then using (2.2), we obtain

$$\begin{aligned} M_{r,j}(\lambda_1) - M_{r,j}(\lambda_2) &= 2\sum_{k=1,3,\ldots}^{j-1} M_{r,k}(p/2)M_{r,j-k}(\delta/2) \\ &\quad - 2\sum_{k=j+1,j+3,\ldots}^{r-1} M_{r,k}(p/2)M_{r,r+j-k}(\delta/2). \end{aligned}$$

Changing k to $2k + 1$, because k takes only odd values, we obtain

$$\begin{aligned} M_{r,j}(\lambda_1) - M_{r,j}(\lambda_2) &= 2\sum_{k=0}^{\frac{1}{2}(j-2)} M_{r,2k+1}(p/2)M_{r,j-2k-1}(\delta/2) \\ &\quad - 2\sum_{k=j/2}^{\frac{1}{2}(r-2)} M_{r,2k+1}(p/2)M_{r,r+j-2k-1}(\delta/2). \end{aligned} \tag{3.6}$$

Now, by definition of $M_{r,j}(X)$ and Lemma 1, we have

$$M_{r,j}(X) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(nr+j)!}[U_{nr+j}X - qU_{nr+j-1}I]. \tag{3.7}$$

Equating the coefficients of X in (3.7) and (3.3) and then making use of (3.6), we get (3.1) for even j. For odd j, (3.1) and (3.2) are similarly proved.

4. SUMMATION FORMULAS FOR $S_n(x)$

For Chebychev polynomials $S_n(x)$ of the first kind, we prove the following theorem. Let $x = \cos\theta$ and $y = \sin\theta$.

Theorem 2. a. Let r be such that $r/2$ is even, and $j = 0, 1, \ldots, r - 1$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} &= \frac{1}{y}\left\{\sum_{k=0}^{[\frac{1}{2}(j-1)]} (-1)^{\frac{1}{2}(\alpha-1)} M_{r,2k+m}(x)M_{r,\alpha}(y)\right. \\ &\quad \left. - \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(r-2)} (-1)^{\frac{1}{2}(r+\alpha-1)} M_{r,2k+m}(x)M_{r,r+\alpha}(y)\right\}. \end{aligned}$$

b. Let r be such that $r/2$ is odd, and $j = 0, 1, \ldots, r - 1$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \frac{1}{y}\left\{\sum_{k=0}^{[\frac{1}{2}(j-1)]}(-1)^{\frac{1}{2}(\alpha-1)} M_{r,\,2k+m}(x) N_{r,\,\alpha}(y) - \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(r-2)}(-1)^{\frac{1}{2}(r+\alpha-1)} M_{r,\,2k+m}(x) N_{r,\,r+\alpha}(y)\right\}.$$

c. Let r be odd, $j = 0, 1, \ldots, r - 1$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \frac{1}{y}\left\{\sum_{k=0}^{\frac{1}{2}(r-3)+m}(-1)^{\frac{1}{2}(r+\beta)} M_{r,\,2k+1-m}(x) M_{2r,\,r+\beta-1}(y) + \sum_{k=0}^{[\frac{1}{2}(j-1)]}(-1)^{\frac{1}{2}(\alpha-1)} M_{r,\,2k+m}(x) M_{2r,\,\alpha}(y) + \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(r-1)-m}(-1)^{\frac{1}{2}(2r+\alpha-1)} M_{r,\,2k+m}(x) M_{2r,\,2r+\alpha}(y)\right\}.$$

Proof: If we write $x = \cos\theta$ and let $p = 2x$ and $q = 1$, then $U_n(p, q)$ are the Chebychev polynomials of the first kind, i.e.,

$$U_n(2x, 1) = S_n(x) = \frac{\sin n\theta}{\sin\theta} \qquad (n \geqslant 0),$$

where

$$S_{n+2} = 2xS_{n+1} - S_n, \quad \text{with } S_0 = 0 \text{ and } S_1 = 1.$$

We shall prove (a) and (b). Now

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{nr+j}}{(nr+j)!} = \sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \sin(nr+j)\theta}{(nr+j)!\,\sin\theta}$$

$$= \frac{1}{\sin\theta}\sum_{n=0}^{\infty}\frac{(-1)^n}{(nr+j)!}\left[\frac{e^{i(nr+j)\theta} - e^{-i(nr+j)\theta}}{2i}\right]$$

$$= \frac{1}{2i\sin\theta}\sum_{n=0}^{\infty}\frac{(-1)^n}{(nr+j)!}[(e^{i\theta})^{nr+j} - (e^{-i\theta})^{nr+j}]$$

$$= \frac{1}{2i\sin\theta}[M_{r,j}(e^{i\theta}) - M_{r,j}(e^{-i\theta})].$$

Hence,

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \frac{1}{2iy}[M_{r,j}(x+iy) - M_{r,j}(x-iy)]. \tag{4.1}$$

Now, by Lemma 2,

$$M_{r,j}(x+iy) - M_{r,j}(x-iy) = \sum_{k=0}^{j} M_{r,k}(x)[M_{r,j-k}(iy) - M_{r,j-k}(-iy)] - \sum_{k=j+1}^{r-1} M_{r,k}(x)[M_{r,r+j-k}(iy) - M_{r,r+j-k}(-iy)]. \tag{4.2}$$

First, let j be even. Breaking up the right-hand side of (4.2) into summations over even and odd values of k and making use of (2.2), we obtain

$$\begin{aligned} M_{r,j}(x+iy) - M_{r,j}(x-iy) &= \sum_{k=1,3,\ldots}^{j-1} 2M_{r,k}(x)M_{r,j-k}(iy) \\ &\quad - \sum_{k=j+1,j+3,\ldots}^{r-1} 2M_{r,k}(x)M_{r,r+j-k}(iy). \end{aligned} \tag{4.3}$$

Now, since r is even, $r/2$ is an integer that is either even or odd. First, let $r/2$ be *even*. By (2.7), (4.3) then becomes

$$\begin{aligned} M_{r,j}(x+iy) - M_{r,j}(x-iy) &= 2\sum_{k=1,3,\ldots}^{j-1} (i)^{j-k}M_{r,k}(x)M_{r,j-k}(y) \\ &\quad - 2\sum_{k=j+1,j+3,\ldots}^{r-1} (i)^{r+j-k}M_{r,k}(x)M_{r,r+j-k}(y). \end{aligned} \tag{4.4}$$

If $r/2$ is *odd*, then again making use of (2.7), (4.3) becomes

$$\begin{aligned} M_{r,j}(x+iy) - M_{r,j}(x-iy) &= 2\sum_{k=1,3,\ldots}^{j-1} (i)^{j-k}M_{r,k}(x)N_{r,j-k}(y) \\ &\quad - 2\sum_{k=j+1,j+3,\ldots}^{r-1} (i)^{r+j-k}M_{r,k}(x)N_{r,r+j-k}(y). \end{aligned} \tag{4.5}$$

Note that the power of i in all the summations in (4.4) and (4.5) is odd, so that when we substitute (4.4) and (4.5) in (4.1) and cancel i from the numerator and denominator, the remaining power of i will be an even integer. Then (4.1) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \frac{1}{y}\Bigg[&\sum_{k=1,3,\ldots}^{j-1} (-1)^{\frac{1}{2}(j-k-1)}M_{r,k}(x)M_{r,j-k}(y) \\ &- \sum_{k=j+1,j+3,\ldots}^{r-1} (-1)^{\frac{1}{2}(r+j-k-1)}M_{r,k}(x)M_{r,r+j-k}(y)\Bigg] \end{aligned} \tag{4.6}$$

when $r/2$ is *even*, and

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n S_{nr+j}(x)}{(nr+j)!} = \frac{1}{y}\Bigg[&\sum_{k=1,3,\ldots}^{j-1} (-1)^{\frac{1}{2}(j-k-1)}M_{r,k}(x)N_{r,j-k}(y) \\ &- \sum_{k=j+1,j+3,\ldots}^{r-1} (-1)^{\frac{1}{2}(r+j-k-1)}M_{r,k}(x)N_{r,r+j-k}(y)\Bigg] \end{aligned} \tag{4.7}$$

when $r/2$ is *odd*.

Replacing k by $2k+1$ in the right-hand side of (4.6) and (4.7), we finally get (a) and (b) for even j. By adopting similar techniques, we get (a) and (b) for odd j and (c).

5. SUMMATION FORMULAS FOR $T_n(x)$

Theorem 3. For the Chebychev polynomials $T_n(x)$ of the second kind, the following summation formulas hold.

a. Let r be such that $r/2$ is even and $j = 0, 1, \ldots, r-1$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{nr+j}(x)}{(nr+j)!} = \sum_{k=0}^{[j/2]} (-1)^{\frac{1}{2}(\beta-1)} M_{r,\,2k+1-m}(x) M_{r,\,\beta-1}(y)$$
$$- \sum_{k=[\frac{1}{2}(j+2)]}^{\frac{1}{2}(r-2)} (-1)^{\frac{1}{2}(r+\beta-1)} M_{r,\,2k+1-m}(x) M_{r,\,r+\beta-1}(y).$$

b. Let r be such that $r/2$ is odd, $j = 0, 1, \ldots, r-1$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{nr+j}(x)}{(nr+j)!} = \sum_{k=0}^{[j/2]} (-1)^{\frac{1}{2}(\beta-1)} M_{r,\,2k+1-m}(x) N_{r,\,\beta-1}(y)$$
$$- \sum_{k=[\frac{1}{2}(j+2)]}^{\frac{1}{2}(r-2)} (-1)^{\frac{1}{2}(r+\beta-1)} M_{r,\,2k+1-m}(x) N_{r,\,r+\beta-1}(y).$$

c. Let r be odd, $j = 0, 1, \ldots, r-1$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{nr+j}(x)}{(nr+j)!} = \sum_{k=0}^{[j/2]} (-1)^{\frac{1}{2}(\beta-1)} M_{r,\,2k+1-m}(x) M_{2r,\,\beta-1}(y)$$
$$- \sum_{k=0}^{\frac{1}{2}(r-1)-m} (-1)^{\frac{1}{2}(r+\alpha)} M_{r,\,2k+m}(x) M_{2r,\,r+\alpha}(y)$$
$$+ \sum_{k=[\frac{1}{2}(j+2)]}^{\frac{1}{2}(r-3)+m} (-1)^{\frac{1}{2}(2r+\beta-1)} M_{r,\,2k+1-m}(x) M_{2r,\,2r+\beta-1}(y).$$

Proof: The proof follows the same technique as in Theorem 2 and is therefore omitted. Notice that the power of (-1) in each of the above summations is an integer.

Remark. Since

$$S_n(x) = \frac{\sin n\theta}{\sin \theta} \quad \text{and} \quad T_n(x) = \cos n\theta,$$

summation formulas in Theorems 2 and 3 also give those for

$$\sum_{n=0}^{\infty} \frac{(-1)^n \sin(nr+j)\theta}{(nr+j)!} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^n \cos(nr+j)\theta}{(nr+j)!}.$$

For example, formula (a) in Theorem 2 can be expressed as

$$\sum_{n=0}^{\infty} \frac{(-1)^n \sin(nr+j)\theta}{(nr+j)!} = \sum_{k=0}^{[\frac{1}{2}(j-1)]} (-1)^{\frac{1}{2}(\alpha-1)} M_{r,\,2k+m}(\cos\theta) M_{r,\,\alpha}(\sin\theta)$$
$$- \sum_{k=[\frac{1}{2}(j+1)]}^{\frac{1}{2}(r-2)} (-1)^{\frac{1}{2}(r+\alpha-1)} M_{r,\,2k+m}(\cos\theta) M_{r,\,r+\alpha}(\sin\theta).$$

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A LUCAS TRIANGLE PRIMALITY CRITERION DUAL TO THAT OF MANN-SHANKS

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Consider the following array of numbers

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2										
2	1	3	2									
3	1	4	5	2								
4	1	5	9	7	2							
5	1	6	14	16	9	2						
6	1	7	20	30	25	11	2					
7	1	8	27	50	55	36	13	2				
8	1	9	35	77	105	91	49	15	2			
9	1	10	44	112	182	196	140	64	17	2		
10	1	11	54	156	294	378	336	204	81	19	2	
11	1	12	65	210	450	672	714	540	285	100	21	2

where any element in the array is found by the usual Pascal recurrence, i.e.,

$$A(n, k) = A(n-1, k) + A(n-1, k-1), \tag{1}$$

subject to the initial conditions $A(1, 0) = 1$, $A(1, 1) = 2$, with $A(n, k) = 0$ for $k < 0$ or $k > n$. This array has been called a Lucas triangle by Feinberg [1], because rising diagonals sum to give the Lucas numbers 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, ..., in contrast to the rising diagonals in the standard Pascal triangle where rising diagonals sum to give the Fibonacci numbers 1, 1, 2, 3, 5, 8, ... The seventh diagonal in our array is 1, 7, 14, 7; the eleventh diagonal is 1, 11, 44, 77, 55, 11. This suggests the following.

Theorem 1. The number $D \geqslant 2$ is a prime number if and only if every entry that is greater than 1 along the D^{th} rising diagonal in the Lucas triangle is divisible by D.

Before giving a proof, we set down further notation in order to rephrase the theorem.

It is easy to prove directly from (1), or one can quote the general theorem of Gupta [4], that

$$A(n, k) = \binom{n}{k} + \binom{n-1}{k-1}, \tag{2}$$

so that the Lucas triangle is simply a combination of two shifted Pascal triangles. Let D be the diagonal number in question and let j be the position of an entry along that diagonal, then a typical element of the diagonal is given by $A(D - j, j)$, where $0 \leqslant j \leqslant D/2$. We can now rephrase Theorem 1 as follows.

Theorem 2. $D \geqslant 2$ is a prime number if and only if $D \mid A(D-j,\, j)$ for all j such that $1 \leqslant j \leqslant D/2$.

Proof: We have from (2) that

$$A(D-j,\, j) = \binom{D-j}{j} + \binom{D-j-1}{j-1} = D\binom{D-j-1}{D-2j}\Big/ j$$
$$= D(D-j-1)!/j!(D-2j)!.$$

If $D = p$ is a prime $\geqslant 2$, we observe that $(j!,\, p) = 1$ and $((p-2j)!,\, p) = 1$ for $1 \leqslant j \leqslant p/2$ so that surely $j!(p-2j)! \mid (p-j-1)!$ and therefore p is a factor of the number $p \cdot ((p-j-1)!/j!(p-2j)!)$.

Now suppose that D is composite. Then, from the formula for A,

$$D \mid A(D-j,\, j) \text{ if and only if } D \,\Big|\, D\binom{D-j-1}{D-2j}\Big/ j.$$

We will show that for a composite D, some j cannot divide $\binom{D-j-1}{D-2j}$. Recall that for the binomial coefficients we have $\binom{x}{m} = (-1)^m\binom{-x+m-1}{m}$. Therefore

$$j \,\Big|\, \binom{D-j-1}{D-2j} \text{ if and only if } j \,\Big|\, \binom{-j}{D-2j}.$$

we need not consider the question of divisilibity of the entries in any diagonal by D when D is even, since the last entry is always a 2 for even D, so we can restrict our analysis to odd composite $D > 3$. Put $D = p(2k+1)$, where p is an odd prime factor of D, and choose $j = pk$. Then we are concerned with whether

$$pk \,\Big|\, \binom{-pk}{p}.$$

But

$$\frac{1}{pk}\binom{-pk}{p} = \frac{-(-pk-1)(-pk-2)\cdots(-pk-p+1)}{p(p-1)(p-2)\cdots 3\cdot 2\cdot 1}, \tag{3}$$

and we observe that the factors $p-1$, $p-2$, ... 3, 2 cannot affect the divisibility of the numerator by p since $(p,\, p-r) = 1$ for all $1 \leqslant r \leqslant p-1$. Furthermore, p is relatively prime to every factor in the numerator; that is,

$$(p,\, pk+s) = 1 \quad \text{for all } 1 \leqslant s \leqslant p-1,$$

and so the indicated quotient cannot be an integer. This completes the proof.

We now claim that Theorem 2 is a dual to the criterion discovered by Mann and Shanks [7]. In [2] and [3] it is shown that the Mann-Shanks criterion can be restated as follows.

Theorem 3. The number $C \geqslant 2$ is a prime number if and only if

$$R \,\Big|\, \binom{R}{C-2R} \tag{4}$$

for all $R \geqslant 1$ such that $C/3 \leqslant R \leqslant C/2$.

Comparison of our proof of the new prime criterion with that of the Mann-Shanks criterion in [2], [3], and [7] shows that the same considerations have been made using (3), except that the numerator in the earlier proof was

$$(pk-1)(pk-2)\cdots(pk-p+1)$$

and the minus sign made no difference in the argument. In fact, we see that our new criterion may be restated as follows.

Theorem 4. The number $C \geqslant 2$ is a prime number if and only if

$$R \left| \binom{-R}{C - 2R} \right. \tag{5}$$

for all R such that $1 \leqslant R \leqslant C/2$.

The natural display for our criterion is the Lucas triangle, just as the natural display for the Mann-Shanks criterion is their shifted Pascal triangle.

Since the rising diagonals in the Lucas triangle sum to give Lucas numbers, that is, as Feinberg [1] noted,

$$L_n = \sum_{j=0}^{[n/2]} A(n - j, j) = 1 + \sum_{j=1}^{[n/2]} \frac{n}{j}\binom{n - j - 1}{j - 1} \tag{6}$$

where $L_n = \alpha^n + \beta^n$ with α and β the roots of the equation $x^2 - x - 1 = 0$, and $L_{n+1} = L_n + L_{n-1}$, subject to $L_1 = 1$, $L_2 = 3$, then we have an obvious

Corollary. The Lucas numbers satisfy the congruence

$$L_p \equiv 1 \pmod{p} \tag{7}$$

for all primes $p \geqslant 2$.

This corollary is well known and can be found in Lehmer [6] or in [8].

That the converse of (7) does not hold follows from the well-known counterexample of Hoggatt and Bicknell that

$$L_{705} \equiv 1 \ (\text{mod } 705 = 3 \cdot 5 \cdot 47),$$

although Lind [8] used computer calculations to show that, for all $2 \leqslant n < 700$, $L_n \equiv 1 \pmod{n}$ implies that n is prime.

In later papers, we shall exhibit and prove corresponding duals to the extensions of the Mann-Shanks criterion given in [2] and [3].

Remark: It is interesting to compare the criterion discussed here with the familiar fact that

$$n \left| \binom{n}{k} \right. \text{ for all with } 1 \leqslant k < n \text{ if and only if } n \text{ is a prime.}$$

Harborth [5] has shown that "almost all" binomial coefficients $\binom{n}{k}$ are divisible by their row number n.

Finally, we note that the generating function for the A's is clearly

$$(1 + 2x)(1 + x)^{n-1} = \sum_{k=0}^{n} A(n, k)x^k. \tag{8}$$

The results of this paper were first announced in an abstract [9] in 1977. There is now a rather extensive international bibliography on criteria related to the Mann-Shanks theorem, and we hope to summarize this at a later date.

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EXTENSIONS OF SOME RESULTS CONCERNING ODD PERFECT NUMBERS

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1. INTRODUCTION

Throughout this paper we shall suppose that N is an odd perfect number, so that N is an odd integer and $\sigma(N) = 2N$, where σ is the positive-divisor-sum function. There is no known example of an odd perfect number, and it has not been proved that none exists. However, a great number of necessary conditions which must be satisfied by N have been established. The first of these, due to Euler, is that

$$N = p^{\alpha}q_1^{2\beta_1} \ \cdots \ q_t^{2\beta_t}$$

for distinct odd primes $p, q_1, \ldots, q_t$, with $p \equiv \alpha \equiv 1 \pmod 4$. (We shall always assume this form for the prime factor decomposition of N). Many writers have found conditions which must be satisfied by the exponents $2\beta_1, \ldots, 2\beta_t$, and it is our intention here to extend some of those results. We shall find it necessary to call on a number of conditions of other types, some of which have only recently been found. These are outlined in Section 2.

It is known (see [8]) that we cannot have $\beta_i \equiv 1 \pmod 3$ for all i or (see [9]) $\beta_i \equiv 17 \pmod{35}$ for all i. Also, if $\beta_1 = \cdots = \beta_t = \beta$, then: from [6], $\beta \neq 2$; from [4], $\beta \neq 3$; and from [9], $\beta \neq 5$, 12, 24, or 62. We shall prove

Theorem 1. If N as above is an odd perfect number and $\beta_1 = \cdots = \beta_t = \beta$, then $\beta \neq 6$, 8, 11, 14, or 18.

The possibility that $\beta_2 = \cdots = \beta_t = 1$ (with $\beta_1 > 1$) has also been considered. In this case, it is known (see [1]) that $\beta_1 \neq 2$ and (see [7]) that $\beta_1 \neq 3$; by a previously mentioned result [8], we also have that $\beta_1 \not\equiv 1 \pmod 3$. We shall prove

Theorem 2. If N as above is an odd perfect number and $\beta_2 = \cdots = \beta_t = 1$, then $\beta_1 \neq 5$ or 6.

The computations required to prove these two theorems were mostly carried out on the Honeywell 66/40 computer at The New South Wales Institute of Technology. We also made use of some factorizations in [10].

Finally, we shall introduce a theorem whose proof is quite elementary, but it is a result which, to our knowledge, has not been noted previously. Euler's form for N, shown above, follows quickly by considering the equation $\sigma(N) = 2N$, modulo 4. Using the modulus 8 instead, we will obtain

Theorem 3. If N as above is an odd perfect number and x is the number of prime powers $q_i^{2\beta_i}$ in which both $q_i \equiv 1 \pmod 4$ and $\beta_i \equiv 1 \pmod 2$, then

$$p - \alpha \equiv 4x \pmod 8.$$

To obtain the following corollary, we then only need to notice that $x = 0$.

Corollary. If N as above is an odd perfect number and $\beta_i \equiv 0 \pmod 2$ for all i, then $p \equiv \alpha \pmod 8$.

2. PRELIMINARY RESULTS

Since we are assuming that $\sigma(N) = 2N$, it is clear in the first place that any odd divisor of $\sigma(N)$ is also a divisor of N. The proof of Theorem 1 makes use of the following facts.

(i) N is divisible by $(p + 1)/2$ (since α is odd).

(ii) If q and $2\beta + 1 = r$ are primes, then $r|\sigma(q^{2\beta})$ if and only if $q \equiv 1 \pmod r$. Furthermore, if $r|\sigma(q^{2\beta})$, then $r\|\sigma(q^{2\beta})$. If $s|\sigma(q^{2\beta})$ and $s \neq r$, then $s \equiv 1 \pmod r$. (This is a special case of results given, for example, in [9].)

(iii) If $\beta_1 = \cdots = \beta_t = \beta$ and $2\beta + 1 = r$ is prime, then $r^4|N$ and $p \equiv 1 \pmod r$. In particular, $p \neq r$. (See [6] for generalizations of this.)

(iv) If $n|N$, then $\sigma(n)/n \leqslant 2$.

The proof of Theorem 2 uses (i), (ii), and (iv), as well as the following results.

(v) The second greatest prime factor of N is at least 1009 (see [3]) and the greatest at least 100129 (see [5]).

(vi) The equation $q^2 + q + 1 = p^a$ has no solution in primes p and q if a is an integer greater than 1 (see [1]).

3. PROOF OF THEOREM 1

We shall assume that $\beta = 6, 8, 11, 14$, and 18, in turn, and in each case obtain a contradiction, usually along the following lines. In each case, $2\beta + 1 = r$ is prime so that, by (iii), $r^{2\beta}\|N$. Then $\sigma(r^{2\beta})|N$. If s is prime, $s \neq p$ and $s|\sigma(r^{2\beta})$, then $s \equiv 1 \pmod r$ and $s^{2\beta}\|N$, so that $r\|\sigma(s^{2\beta})$, by (ii). Applying the same process to other prime factors of $\sigma(s^{2\beta})$ and repeating it sufficiently often, we find that $r^{2\beta+1}|N$, which is our contradiction.

Except in the case $\beta = 8$, we were not able to carry out sufficiently many factorizations explicitly. (We generally restricted ourselves to seeking prime factors less than 5×10^6.) However, we were able to test whether unfactored quotients were pseudoprime (base 3) or not. Each P below is a pseudoprime and each M is an unfactored quotient which is not a pseudoprime, and hence is not a prime. We checked that each M was not a perfect power so that the existence of two distinct prime factors of each M was assured. We checked also that no M's or P's within each case had any prime factors in common with each other or with known factors of N. In this way, we could distinguish sufficiently many distinct prime factors of N to imply that $r^{2\beta+1}|N$. There is a slightly special treatment required when $\beta = 6$.

We shall give the details of the proof here only in the cases $\beta = 6$ and $\beta = 11$. These illustrate well the methods involved. The other parts of the proof are available from the first named author.

(a) Suppose $\beta = 6$, so that $13^{12}\|N$; $\sigma(13^{12}) = 53 \cdot 264031 \cdot 1803647$. The relevant factorizations are given in Table 1. We distinguish two main cases.

Table 1

	q	Some factors of $\sigma(q^{12})/13$
	53	3297113, P_1
	264031	P_2
	1803647	131, M_1
	131	79, Q
	79	M_2
(A)	131	$Q = M_3$
(B)	131	$Q = q_9$
	q_9	q_{10}

Suppose first that $p \neq 53$. We may assume that $q_{2i-1}q_{2i}|M_i$ ($i = 1, 2$) and $q_{j+4}|P_j$ ($j = 1, 2$). In Table 1, Q is also a pseudoprime (base 3) and we need to consider two distinct alternatives. In (A), we suppose that $Q = M_3$ is composite, so that $q_7q_8|M_3$, say. (We checked that Q was not a perfect power.) In (B), we suppose that Q is prime, so we write $Q = q_9$. If this is so, then $q_9 \neq p$, since $Q \equiv 3 \pmod 4$. Thus, we have 14 primes:

$$53,\ 79,\ 131,\ 264031,\ 1803647,\ 3297113,\ q_i\ (1 \leqslant i \leqslant 6)$$

with q_7 and q_8, or with q_9 and q_{10}. Each of these primes is congruent to 1 (mod 13) and at most one of them might be p. Put

$$\Lambda = \{53, 79, 131, 264031, 1803647, 3297113, M_1, M_2, P_1, P_2, Q,\ (Q^{13}-1)/(Q-1)\}.$$

We checked that no two elements of Λ had a common prime factor; therefore, the 14 primes above are distinct. Hence, $13^{13}|N$, the desired contradiction.

Now suppose that $p = 53$. By (i), $3|N$ and so $\sigma(3^{12}) = 797161|N$. Certainly there is a prime q_{11} dividing $\sigma(797161^{12})/13$. We thus have 13 primes:

$$79,\ 131,\ 264031,\ 797161,\ 1803647,\ q_i\ (1 \leqslant i \leqslant 4),\ q_6,\ q_{11}$$

with q_7 and q_8, or with q_9 and q_{10}. Each of these is congruent to 1 (mod 13), and we checked that no two elements of the set

$$(\Lambda - \{53, 3297113, P_1\}) \cup \{797161, \sigma(797161^{12})/13\}$$

had a common prime factor. Hence, again, $13^{13}|N$.

(b) Suppose $\beta = 11$, so that $23^{22}\|N$, and note that

$$\sigma(23^{22}) = 461 \cdot 1289 \cdot M_1.$$

Now refer to Table 2, where an asterisk signifies that the prime is 1 (mod 4), when that is relevant.

There are three cases to consider. First, suppose that $p = 1289$. By (i), $3 \cdot 5|N$ so that $n_1|N$ where $n_1 = (3 \cdot 5 \cdot 23 \cdot 47)^{22}$; but $\sigma(n_1)/n_1 > 2$, contradicting (iv). Similarly, if $p = 461$, then we have $3 \cdot 7 \cdot 11|N$ so that $n_2|N$ where $n_2 = (3 \cdot 7 \cdot 11 \cdot 23)^{22}$; but $\sigma(n_2)/n_2 > 2$.

Now suppose that $p \neq 461$ and $p \neq 1289$. We may suppose that $q_{2i-1}q_{2i}|M_i$ ($1 \leqslant i \leqslant 7$) and $q_{15}|P$. Thus, N is divisible by the following 24 primes, each 1 (mod 23):

$$47, 139, 461, 1289, 37123, 133723, 281153, 300749, 2258831,\ q_i\ (1 \leqslant i \leqslant 15).$$

Table 2

q	Some factors of $\sigma(q^{22})/23$
461*	139, 133723, P
133723	47, 37123, 2258831, $461 \cdot M_2$
2258831	300749,* M_3
1289*	281153,* M_4
47	M_5
139	M_6
37123	M_7

We checked that the 24 primes given above were distinct. One of them might be p, so $23^{23}|N$, our usual contradiction.

This shows that $\beta \neq 11$. We remark that we also looked at the remaining possible values of β less than 15, namely, 9, 15, 20, 21, and 23, without further success.

4. PROOF OF THEOREM 2

We begin by proving more than is stated in Theorem 2 in the case in which $3 \nmid N$.

Lemma. If N as before is an odd perfect number, $3 \nmid N$ and $\beta_2 = \cdots = \beta_t = 1$, then $\beta_1 \neq 5$, 6, or 8.

Proof: We will show first that, if $\beta_1 = 5$, 6, or 8, then $7 \nmid N$. Notice that $q_i \equiv 2 \pmod 3$ $(2 \leqslant i \leqslant t)$, since, otherwise, $3|\sigma(q_i^2)|N$. In particular, $7^2 \nmid N$, so that $q_1 = 7$ if $7|N$. In that case, we obtain contradictions, as follows.

If $\beta_1 = 5$, then $7^{10}|N$. But $1123|\sigma(7^{10})|N$ and $p \neq 1123$, so $1123^2 \| N$. But $1123 \equiv 1 \pmod 3$. If $\beta_1 = 6$, then $7^{12} \| N$. Then $r = \sigma(7^{12}) = 16148168401|N$; if $r = p$, then $103|N$, by (i). However, $103 \equiv r \equiv 1 \pmod 3$. If $\beta_1 = 8$, then $7^{16} \| N$, $14009|\sigma(7^{16})|N$. Then $p \neq 14009$, else $3|N$ by (i), so $14009^2 \| N$. But $223|\sigma(14009^2)$ and $223 \equiv 1 \pmod 3$.

Now we can show that $13 \nmid N$ for any of these values of β_1. Since N is not divisible by either 3 or 7, we must have $q_1 = 13$ if $13|N$. Then $\beta_1 \neq 5$, else $23|\sigma(13^{10})|N$ and $7|\sigma(23^2)|N$. Also, $\beta_1 \neq 6$, else $264031|\sigma(13^{12})|N$ and $264031 \equiv 1 \pmod 3$. Similarly, $\beta_1 \neq 8$, else $103|\sigma(13^{16})|N$.

Notice next that, by (ii), divisors of $\sigma(q_i^2)$ $(2 \leqslant i \leqslant t)$ are congruent to $1 \pmod 3$, so that $\sigma(q_i^2) = p^{a_i} q_1^{b_i}$ for some a_i, b_i $(0 \leqslant a_i \leqslant \alpha,\ 0 \leqslant b_i \leqslant 2\beta_1)$ and for each i $(2 \leqslant i \leqslant t)$. There can be at most $2\beta_1$ values of $i \geqslant 2$ such that $q_1|\sigma(q_i^2)$; by (vi), there is at most one value of $i \geqslant 2$ such that $\sigma(q_i^2) = p^c$ $(c \geqslant 1)$. It follows that N has at most $2\beta_1 + 3$ distinct prime factors. Of these, at most two are congruent to $1 \pmod 3$, namely, p and q_1. By (i), certainly $p \equiv 1 \pmod 3$, so that in fact $p \equiv 1 \pmod{12}$.

In our case, when $\beta_1 = 5$, 6, or 8, we must have $p \geqslant 37$ (since $13 \nmid N$) and has at most 19 distinct prime factors. Using (v), we can now obtain the final contradiction which proves the lemma:

$$2 = \frac{\sigma(N)}{N} = \frac{p - p^{-\alpha}}{p - 1} \prod_{i=1}^{t} \frac{q_i - q_i^{-2\beta_i}}{q_i - 1} < \frac{p}{p - 1} \prod_{i=1}^{t} \frac{q_i}{q_i - 1}$$

(continued)

$$< \frac{5}{4}\frac{11}{10}\frac{17}{16}\frac{19}{18}\frac{23}{22}\frac{29}{28}\frac{37}{36}\frac{41}{40}\frac{47}{46}\frac{53}{52}\frac{59}{58}\frac{71}{70}\frac{83}{82}\frac{89}{88}\frac{101}{100}\frac{107}{106}\frac{113}{112}\frac{1009}{1008}\frac{100129}{100128} < 2.$$

We shall give the remaining details only in the case $\beta_1 = 6$; the proof for the case $\beta_1 = 5$ is available from the first named author. By the Lemma, we can assume that $3|N$.

We will assume first that $q_1 = 3$. Then $797161 = \sigma(3^{12})|N$. We cannot have $p = 797161$ because then, by (i), $398581^2\|N$: $1621|\sigma(398581^2)$, $7 \cdot 13|\sigma(1621^2)$, $19|\sigma(7^2)$, and $127|\sigma(19^2)$, so that $n|N$, where $n = 3^{12}(7 \cdot 13 \cdot 19 \cdot 127)^2$; but $\sigma(n)/n > 2$ and (iv) is contradicted. Hence, $797161^2\|N$.

Notice that $\sigma(797161^2) = 3 \cdot 61 \cdot 151 \cdot 22996651$; also note that $7|\sigma(151^2)$ and $19|\sigma(7^2)$. Thus, $7^2 19^2\|N$. Making use of (i), we then see that $p \neq 1693$, since then $(p + 1)/2 = 7 \cdot 11^2$ and $7|\sigma(11^2)$, so that $7^3|N$, and $p \neq 433$, since then $(p + 1)/2 = 7 \cdot 31$, $331|\sigma(31^2)$ and $7|\sigma(331^2)$, so that again $7^3|N$. We now observe that

$$43|\sigma(22996651^2),\quad 631|\sigma(43^2),\quad 433|\sigma(631^2),\quad 1693|\sigma(433^2),\quad 13|\sigma(1693^2),$$

so that $n|N$, where $n = 3^{12}13(7 \cdot 19 \cdot 43)^2$; but $\sigma(n)/n > 2$, contradicting (iv).

Now, we assume that $3^2\|N$, so that we can have at most two values of $i \geqslant 2$ with $q_i \equiv 1 \pmod 3$. We have $13 = \sigma(3^2)|N$.

First, we will suppose that $p = 13$, so that, by (i), $7|N$. We cannot have $q_1 = 7$, because $\sigma(7^{12}) = 16148168401 = r$ is prime, $433|\sigma(r^2)$, $37|\sigma(433^2)$, and $37 \equiv 433 \equiv r \equiv 1 \pmod 3$. Hence, $7^2\|N$, so $19|\sigma(7^2)|N$. Again, $q_1 \neq 19$, because $599 \cdot 29251|\sigma(19^{12})$, $51343|\sigma(599^2)$, and $29251 \equiv 51343 \equiv 1 \pmod 3$. Thus, $19^2\|N$ and for no further values of i can be have $q_i \equiv 1 \pmod 3$. Therefore, we have $127|\sigma(19^2)|N$.

Clearly, $127^2 \nmid N$, so $q_1 = 127$. Setting $q_2 = 7$ and $q_3 = 19$, we must have, for $i \geqslant 4$, $\sigma(q_i^2) = 7^{a_i}13^{b_i}19^{c_i}127^{d_i}$ where $a_i \leqslant 1$, $b_i \leqslant \alpha$, $c_i \leqslant 1$, and $d_i \leqslant 11$, since, by (ii), any other prime divisors of $\sigma(q_i^2)$ would be congruent to 1 (mod 3). Using (vi), as in the proof of the Lemma, it follows that there are at most 14 primes q_i with $i \geqslant 4$. We cannot have $11|N$ [although $\sigma(11^2) = 7 \cdot 19$], since then $n|N$, where $n = 3^2 7^2 11^2 13 \cdot 19^2$; but $\sigma(n)/n > 2$, contradicting (iv). Possibly $107|N$, since $\sigma(107^2) = 7 \cdot 13 \cdot 127$, but we find that no other prime less than 500 can be q_i for some $i \geqslant 4$. Then we have our contradiction: there are 13 primes q, $503 \leqslant q \leqslant 653$, that are congruent to 2 (mod 3); thus,

$$2 = \frac{\sigma(N)}{N} < \frac{\sigma(3^2 7^2 19^2)}{3^2 7^2 19^2}\frac{13}{12}\frac{107}{106}\frac{127}{126}\prod_{\substack{q=503\\ q \equiv 2 \pmod 3}}^{653}\frac{q}{q-1} < 2.$$

This shows that $p \neq 13$.

We cannot have $q_1 = 13$, because $53 \cdot 264031|\sigma(13^{12})$, $p \neq 53$ [else $3^3|N$, by (i)], $\sigma(53^2) = 7 \cdot 409$ and $7 \equiv 409 \equiv 264031 \equiv 1 \pmod 3$. Hence, $13^2\|N$, so we have $62|\sigma(13^2)|N$.

Suppose that $p = 61$, so that, by (i), $31|N$. Then $q_1 \neq 31$, since $\sigma(31^{12}) = 42407 \cdot 2426789 \cdot 7908811$, $43|\sigma(7908811^2)$, and $13 \equiv 43 \equiv 7908811 \equiv 1 \pmod 3$. Thus, $31^2\|N$ and $331|\sigma(31^2)|N$. Since $13 \equiv 31 \equiv 331 \equiv 1 \pmod 3$, then $q_1 = 331$. But $53|\sigma(331^{12})$, $7|\sigma(53^2)$, and $7 \equiv 13 \equiv 31 \equiv 1 \pmod 3$. This shows that $p \neq 61$. Also, $q_1 \neq 61$, since $187123|\sigma(61^{12})$, $19|\sigma(187123^2)$, and $13 \equiv 19 \equiv 187123 \equiv 1 \pmod 3$. Hence, $61^2\|N$, so $97|\sigma(61^2)|N$, and we can have no further values of $i \geqslant 2$ with $q_i \equiv 1 \pmod 3$. In particular, $97^2 \nmid N$.

If $p = 97$, then $7|N$ by (i), so $q_1 = 7$; but $\sigma(7^{12}) = r$ (above) $\equiv 1 \pmod 3$. Thus, $q_1 = 97$. But $79|\sigma(97^{12})$ and $79 \equiv 1 \pmod 3$.

This completes the proof.

5. PROOF OF THEOREM 3

We note first that, modulo 8,

$$\begin{aligned}\sigma(q_i^{2\beta_i}) &= 1 + q_i + q_i^2 + \cdots + q_i^{2\beta_i} \equiv 1 + q_i + 1 + \cdots + q_i + 1\\ &= 1 + \beta_i(q_i + 1),\end{aligned}$$

and, writing $\alpha = 4a + 1$,

$$\sigma(p^\alpha) = 1 + p\sigma(p^{4a}) \equiv 1 + p(1 + 2a(p + 1)) \equiv (2a + 1)(p + 1).$$

Since $\sigma(N) = 2N$, we have

$$(2a + 1)(p + 1)\prod_{i=1}^{t}(1 + \beta_i(q_i + 1)) \equiv 2p \pmod 8,$$

or, since $p \equiv 1 \pmod 4$,

$$(2a + 1)\frac{p + 1}{2}\prod_{i=1}^{t}(1 + \beta_i(q_i + 1)) \equiv 1 \pmod 4.$$

If $q_i \equiv 1 \pmod 4$ and $\beta_i \equiv 1 \pmod 2$, then $1 + \beta_i(q_i + 1) \equiv 3 \pmod 4$; otherwise, $1 + \beta_i(q_i + 1) \equiv 1 \pmod 4$. Thus,

$$3^x(2a + 1)\frac{p + 1}{2} \equiv 1 \pmod 4.$$

We see that $3^x \equiv 2x + 1 \pmod 4$, so now

$$(2a + 2x + 1)\frac{p + 1}{2} \equiv 1 \pmod 4.$$

Considering separately the possibilities $p \equiv 1 \pmod 8$ and $p \equiv 5 \pmod 8$, we find that this is equivalent to

$$a + x \equiv \frac{p - 1}{4} \pmod 2,$$

or $p - \alpha = p - 4a - 1 \equiv 4x \pmod 8$, as required.

Note: Since this paper was prepared for publication, we have noticed that Ewell [2] has also given a form of Theorem 3. Both his statement of the theorem and his proof are more complicated than the above.

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ON TRIANGULAR FIBONACCI NUMBERS

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In Memory of Vern Hoggatt

Let F_n denote the n^{th} Fibonacci number:

$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}.$

Tallman [2] noted that $0 = F_0$, $1 = F_1 = F_2$, $3 = F_4$, $21 = F_8$, and $55 = F_{10}$ are triangular, i.e., of the form $k(k + 1)/2$, and asked if any more Fibonacci numbers are triangular. In this paper, we develop some congruences which must be satisfied by n if F_n is triangular. As a result, we prove that there are no more triangular numbers among the first billion Fibonacci numbers.

Moreover, the congruences developed here are so strikingly similar that they suggest an approach to proving that the known triangular Fibonacci numbers are in fact the only ones. A pattern is strongly suggested, but unfortunately any underlying generality remains elusive, leaving us with a good notion of how to test, but with no assurances that such tests will succeed. Thus, in a sense, the results in this paper constitute mere *number crunching*, albeit on a rather massive scale, given the simplicity of the techniques.

Throughout this paper, let

$A = 2^3 3 \cdot 5 = 120$

$B = 7A = 2^3 3 \cdot 5 \cdot 7 = 840$

$C = 6B = 2^4 3^2 5 \cdot 7 = 5040$

$D = 11C = 2^4 3^2 5 \cdot 7 \cdot 11 = 55{,}440$

$E = 10D = 2^5 3^2 5^2 7 \cdot 11 = 554{,}400$

$F = 13E = 2^5 3^2 5^2 7 \cdot 11 \cdot 13 = 7{,}207{,}200$

$G = 17F = 2^5 3^2 5^2 7 \cdot 11 \cdot 13 \cdot 17 = 122{,}522{,}400$

$H = 19G = 2^5 3^2 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 = 2{,}327{,}925{,}600$

Our approach is to show successively that

$$\text{if } F_n \text{ is triangular, then } n \equiv 0, 1, 2, 4, 8, 10, M/2 \text{ or } M - 1 \pmod{M} \tag{1}$$

for $M = A, \ldots, H$. Once (1) is established for $M = H$, it follows at once that there are no new triangular Fibonacci numbers with subscript less than one billion.

At the heart of what we do here is the simple observation that an integer f is triangular if and only if $8f + 1$ is a square.

If p is an odd prime, let $Z(p)$ be the entry point of p in the Fibonacci sequence. That is, $Z(p)$ is the subscript of the first Fibonacci number divisible by p. Then $p \mid F_n$ if and only if $Z(p) \mid n$. Tables of $Z(p)$ for $p < 10^4$ may be found in [1].

Further, let $k(p)$ be the period of the Fibonacci sequence modulo p. It is known that:

If $Z(p) = 2m + 1$, then $k(p) = 4Z(p)$.

If $Z(p) = 2(2m + 1)$, then $k(p) = Z(p)$.

If $Z(p) = 2^{a}(2m + 1)$ with $a \geqslant 2$, then $k(p) = 2Z(p)$.

What we will do is to find primes p for which $k(p)$ divides the new modulus but not the old one, and then eliminate most choices of n relative to the new modulus by showing that $1 + 8F_n$ is not a quadratic residue modulo p. The same thing may be done with composite moduli for alleged resedues, but it was necessary to do so only once.

Lemma. If F is triangular, then $n \equiv 0, 1, 2, 4, 8, 10, 20, 24$, or $39 \pmod{40}$.

Proof: We cannot have $n \equiv 3, 5, 6$, or $7 \pmod{10}$ or else $1 + 8F_n$ is a nonresidue (mod 11). We rule out $n \equiv 9, 11, 12, 14$, or $18 \pmod{20}$ to avoid having $1 + 8F_n$ be a nonresidue (mod 5). Similarly, we cannot have $n \equiv 3, 5$, or $6 \pmod{8}$ or else $1 + 8F_n$ is a nonresidue (mod 3). Finally, $n \equiv 28 \pmod{40}$ is impossible because $1 + 8F_{28}$ is a nonresidue (mod 41).

Theorem. (1) holds for $M = A, B, C, D, E, F, G$, and H.

Proof: The lemma and Table 1 establish the result for $M = A$; in Table 1 and the following tables, the entry gives a modulus which eliminates F_n as a triangular number. Then Table 2 establishes the result for $M = B$. The proofs for $M = C, D, E, F, G$, and H are given in Tables 3, 4, 5, 6, 7, and 8, respectively.

Table 1

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+20	X+24	X+39
0							31	31	9
40	2521	9	31	61	31	31		2521	9
80	31	9	2521	31	31	2521	2521	61	

Table 2

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+A/2	X+A-1
0							911	29
A	421	29	71	1427	71	911	911	71
2A	911	29	911	71	71	911	13	29
3A	83	29	13	71	281	83		29
4A	911	29	71	83	281	281	421	29
5A	911	71	71	71	911	13	911	29
6A	13	29	71	911	13	911	281	

Table 3

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+B/2	X+B-1
0							19	19
B	19	19	19	19	17	17	7	19
2B	19	19	17	17	19	19	19	167
3B		167	167	7	241	23	19	19
4B	19	19	19	19	17	17	167	19
5B	19	19	17	17	19	19	19	

Table 4

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+C/2	X+C-1
0							881	89
C	89	199	89	89	89	89	43	199
2C	43	199	89	43	881	307	199	199
3C	89	199	89	89	43	199	199	199
4C	331	199	881	661	199	199	199	199
5C	89	43	89	331	199	199		43
6C	881	199	307	199	199	307	89	199
7C	43	199	199	199	331	991	43	199
8C	199	199	199	199	89	89	89	199
9C	199	199	199	331	43	89	331	199
10C	199	89	307	89	89	89	89	

Table 5

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+D/2	X+D-1
0							151	3001
D	101	151	101	3001	101	47	3001	101
2D	3001	101	3001	3001	151	101	3041	101
3D	151	101	3001	3001	3001	47	101	151
4D	3001	3001	101	3001	3001	151	3001	1601
5D		1601	1601	1103	1103	47	151	3001
6D	101	151	101	3001	101	701	3001	101
7D	3001	101	3001	3001	151	101	1103	101
8D	151	101	3001	3001	3001	701	101	151
9D	3001	3001	101	3001	3001	151	3001	

Table 6

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+E/2	X+E-1
0							521	103
E	79	521	79	79	859	521	521	521
2E	1951	521	131	859	521	521	521	521
3E	131	859	2081	233	521	521	521	859
4E	79	233	859	521	521	521	521	103
5E	233	521	521	521	521	521	521	521
6E	521	79	521	521	521	521		79
7E	521	521	521	521	521	521	79	521
8E	521	103	521	521	521	1951	1951	233
9E	521	859	521	521	79	859	131	859
10E	521	521	521	521	2081	3329	79	521
11E	521	521	521	2081	79	79	3121	521
12E	521	103	79	859	131	859	521	

Table 7

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+F/2	X+F-1
0							239	919
F	3571	3571	3469	3571	3571	3469	3571	1597
2F	67	919	919	67	883	919	3469	3571
3F	1597	919	3571	919	1597	3571	1597	67
4F	1597	3469	919	1021	3469	3469	919	3571
5F	3571	3571	3469	3571	3571	1597	3571	67
6F	239	1597	919	67	67	1597	919	373
7F	919	3571	3571	1597	3469	3571	919	1597
8F	919	1597	3469	1597	919	239		1597
9F	1871	1597	919	3571	3571	919	3571	3571
10F	3571	373	373	3469	919	67	67	1597
11F	3469	67	1223	239	1597	3571	1597	3571
12F	1597	3571	3571	3469	1597	239	1597	3469
13F	919	67	3571	919	3571	919	3571	919
14F	3571	3571	1597	3571	3469	1223	239	919
15F	919	1597	1597	3469	919	3571	919	3571
16F	919	919	3571	1597	67	3469	919	

Table 8

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+G/2	X+G-1
0							113	9349
G	113	113	37	9349	9349	229	9349	9349
2G	37	9349	37	9349	9349	37	229	797
3G	229	37	37	9349	37	9349	9349	37
4G	797	9349	9349	113	227	227	9349	9349
5G	191	9349	9349	9349	37	37	37	37
6G	229	229	9349	229	9349	37	2281	9349
7G	9349	37	229	9349	229	229	37	9349
8G	9349	761	9349	9349	113	37	9349	37
9G	9349	37	37	37	37	37		37
10G	113	37	9349	229	419	37	113	761
11G	9349	9349	9349	37	761	9349	37	37
12G	229	9349	191	9349	191	9349	229	229
13G	9349	37	683	227	229	9349	797	9349
14G	9349	9349	37	113	9349	229	191	9349
15G	37	37	9349	37	9349	9349	229	37
16G	2281	797	229	229	9349	37	9349	9349
17G	37	9349	37	229	113	9349	9349	113
18G	9349	9349	37	227	9349	9349	9349	

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GUESSING EXACT SOLUTIONS

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A recent problem [1] in this journal provides a nice illustration of a technique for guessing exact solutions of polynomial equations from approximate solutions. The technique depends on nothing more complicated than the familiar fact that if $ax^2 + bx + c = 0$ has roots s and t, then $s + t = -b/a$ and $st = c/a$.

Problem H-335 asked for exact solutions of the equation

$$x^5 - 5x^3 + 5x - 1 = 0. \tag{1}$$

One of the solutions is $x = 1$, and dividing (1) by $x - 1$ yields

$$x^4 + x^3 - 4x^2 - 4x + 1 = 0. \tag{2}$$

Using bracketing techniques and a calculator, it is relatively easy to see that (2) has rounded solutions: $r_1 = -1.8271$, $r_2 = -1.3383$, $r_3 = 0.2091$, $r_4 = 1.9563$.

Now we seek pairs of these solutions that have recognizable sums and products. Fibonacci fans are certainly familiar with the number $\alpha = (1 + \sqrt{5})/2 = 1.6180\ldots$. Upon noting that $r_2 + r_4 \approx 0.618 \approx \alpha^{-1}$ and $r_2 r_4 \approx -2.618 \approx -\alpha^2$, we suspect that r_2 and r_4 are solutions of

$$x^2 - \alpha^{-1}x - \alpha^2 = 0. \tag{3}$$

Long division, using familiar properties of powers of α, confirms that suspicion as fact, since

$$x^4 + x^3 - 4x^2 - 4x + 1 = (x^2 - \alpha^{-1}x - \alpha^2)(x^2 + \alpha x - \alpha^{-2}).$$

Then we can verify that r_2 and r_4 are indeed solutions of (3), namely,

$$x = \frac{\alpha^{-1} \pm \sqrt{\alpha^{-2} + 4\alpha^2}}{2} = \frac{\alpha - 1 \pm \sqrt{6 + 3\alpha}}{2} = \frac{-1 + \sqrt{5} \pm \sqrt{30 + 6\sqrt{5}}}{4}.$$

Also, r_1 and r_3 are solutions of $x^2 + \alpha x - \alpha^{-2} = 0$, namely,

$$x = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\alpha^{-2}}}{2} = \frac{-\alpha \pm \sqrt{9 - 3\alpha}}{2} = \frac{-1 - \sqrt{5} \pm \sqrt{30 - 6\sqrt{5}}}{4}.$$

(Incidentally, the published solution was incorrect in that r_1 and r_3 were each off by 0.5, because of an incorrect sign in the numerator.)

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ON THE REDUCTION OF A LINEAR RECURRENCE OF ORDER r

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1. INTRODUCTION

J. R. Bastida shows in his paper [1] that, if $u \in R$, $u > 1$, and $(x_n)_{n \geqslant 0}$ is a sequence given by

$$x_{n+1} = ux_n + \sqrt{(u^2 - 1)(x_n^2 - x_0^2) + (x_1 - ux_0)^2}, \quad n \geqslant 0, \tag{1}$$

then $x_{n+2} = 2ux_{n+1} - x_n$, $n \geqslant 0$. So, if the numbers u, x_0, and x_1 are integers, it results that x_n is an integer for any $n \geqslant 0$.

Bastida and DeLeon [2] establish sufficient conditions for the numbers u, t, x_0, and x_1 such that the linear recurrence

$$x_{n+2} = 2ux_{n+1} - tx_n \tag{2}$$

can be reduced to a relation of form (1), between x_n and x_{n+1}. Consequently, the relation's two consecutive terms of Fibonacci, Lucas, and Pell sequences are given in [2].

S. Roy [6] finds this relation for the Fibonacci sequence using hyperbolic functions.

In this paper we shall prove that if a sequence $(x_n)_{n \geqslant 1}$ satisfies a linear recurrence of order $r \geqslant 2$, then there exists a polynomial relation between any r consecutive terms. This shows that the linear recurrence of order r was reduced to a nonlinear recurrence of order $r - 1$.

From a practical point of view, for $r \geqslant 3$, expressing x_n in the function of $x_{n-1}, \ldots, x_{n-r+1}$ is difficult, because we must solve an algebraic equation of degree $\geqslant 3$ and choose the "good solution."

If $r = 2$, we can do this in many important cases. An application of this case is a generalization of the result given in [3].

2. THE MAIN RESULT

Let $(x_n)_{n \geqslant 1}$ be a sequence given by the linear recurrence of order r,

$$x_n = \sum_{k=1}^{r} a_k x_{n-r+k-1}, \quad n \geqslant r + 1, \ x_i = \alpha_i, \quad 1 \leqslant i \leqslant r, \tag{3}$$

where $\alpha_1, \ldots, \alpha_r$ and $a_1, \ldots, a_r$ are given real numbers (they can also be complex numbers or elements of an arbitrary commutative field). Suppose $a_1 \neq 0$.

For $n \geqslant r$, we consider the determinant

$$D_n = \begin{vmatrix} x_{n-r+1} & x_{n-r+2} & \cdots & x_{n-1} & x_n \\ x_{n-r+2} & x_{n-r+3} & \cdots & x_n & x_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1} & x_n & \cdots & x_{n+r-3} & x_{n+r-2} \\ x_n & x_{n+1} & \cdots & x_{n+r-2} & x_{n+r-1} \end{vmatrix} \tag{4}$$

and then prove the following theorem.

Theorem 1. Let $(x_n)_{n\geqslant 1}$ be a sequence given by (3) and let D_n be given by (4). Then, for any $n \geqslant r$, we have the r relation

$$D_n = (-1)^{(r-1)(n-r)} a_1^{n-r} D_r \tag{5}$$

Proof: Following the method of [4], [5], and [7] (for $r = 2$), we introduce the matrix

$$A_n = \begin{bmatrix} x_{n-r+1} & x_{n-r+2} & \cdots & x_{n-1} & x_n \\ x_{n-r+2} & x_{n-r+3} & \cdots & x_n & x_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1} & x_n & \cdots & x_{n+r-3} & x_{n+r-2} \\ x_n & x_{n+1} & \cdots & x_{n+r-2} & x_{n+r-1} \end{bmatrix}. \tag{6}$$

It is easy to see that

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_{r-2} & a_{r-1} & a_r \end{bmatrix} A_n = A_{n+1}, \tag{7}$$

so that

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_{r-2} & a_{r-1} & a_r \end{bmatrix} A_r = A_n. \tag{8}$$

Passing to determinants in (8), we obtain

$$((-1)^{r-1}a_1)^{n-r}D_r = D_n \quad \text{for } n \geqslant r;$$

that is, the relation (5) is true.

Theorem 2. Let $(x_n)_{n\geqslant 1}$ be the sequence given by the linear recurrence (3). There exists a polynomial function of degree r, $F_r: R^r \to R$, such that the relation

$$F_r(x_n, x_{n-1}, \ldots, x_{n-r+1}) = (-1)^{(r-1)(n-r)}a_1^{n-r}F_r(\alpha_r, \alpha_{r-1}, \ldots, \alpha_1) \tag{9}$$

is true for every $n \geqslant r$.

Proof: Observe that, from the recurrence (3), we can compute the value of D_r knowing $\alpha_1, \alpha_2, \ldots, \alpha_r$. Also, from the recurrence (3), we can express successively all elements of D_n as a function of the terms $x_n, x_{n-1}, \ldots, x_{n-r+1}$ of the sequence $(x_n)_{n\geqslant 1}$. Thus there exists a polynomial function of degree r, $F_r: R^r \to R$ such that the relation (9) is true.

If we suppose that the equation

$$F_r(x_n, x_{n-1}, \ldots, x_{n-r+1}) = (-1)^{(r-1)(n-r)}a_1^{n-r}F_r(\alpha_r, \ldots, \alpha_1)$$

can be resolved with respect to x_n, we find that x_n depends only on the terms $x_{n-1}, x_{n-2}, \ldots, x_{n-r+1}$.

If this is possible, the expression of x_n is, in general, very complicated.

When $r = 2$, we obtain

$$F_2(x, y) = x^2 - a_2xy - a_1y^2, \tag{10}$$

and it results that, for the sequence $(x_n)_{n\geqslant 1}$ given by

$$x_n = a_1x_{n-2} + a_2x_{n-1}, \quad n \geqslant 3, \quad x_1 = \alpha_1, \quad x_2 = \alpha_2, \tag{11}$$

the relation $F_2(x_n, x_{n-1}) = (-1)^n a_1^{n-2}F_2(\alpha_2, \alpha_1)$ holds. The last relation is the first result of [2], where it was proved by mathematical induction. If we write this relation explicitly, we obtain

$$(2x_n - a_2x_{n-1})^2 = (a_2^2 + 4a_1)x_{n-1}^2 + 4(-1)^{n-1}a_1^{n-2}(a_1\alpha_1^2 + a_2\alpha_1\alpha_2 - \alpha_2^2). \tag{12}$$

From the relation (12), under some supplementary conditions concerning the sequence $(x_n)_{n\geqslant 1}$, we can express x_n in terms of x_{n-1}.

Again, from (12), it follows that if the sequence satisfies (11), where $a_1, a_2, \alpha_1, \alpha_2 \in N$, then for any $n \geqslant 3$,

$$(a_2^2 + 4a_1)x_{n-1}^2 + 4(-1)^{n-1}a_1^{n-2}(a_1\alpha_1^2 + a_2\alpha_1\alpha_2 - \alpha_2^2)$$

is a square. This result is an extension of [3].

In the particular case $r = 3$, after elementary calculation, we obtain

$$\begin{aligned} F_3(x, y, z) = -x^3 &- (a_1 + a_2a_3)y^3 - a_1^2z^3 + 2a_3x^2y + a_2x^2z \\ &- (a_2^2 + a_1a_3)y^2z - (a_3^2 - a_2)xy^2 \\ &- a_1a_3xz^2 - 2a_1a_2yz^2 + (3a_1 - a_2a_3)xyz. \end{aligned}$$

So from relation (9), we get that, for the linear recurrence

$$x_n = a_1x_{n-3} + a_2x_{n-2} + a_3x_{n-1}, \quad n \geqslant 4, \quad x_1 = \alpha_1, \quad x_2 = \alpha_2, \quad x_3 = \alpha_3, \tag{13}$$

the relation $F_3(x_n, x_{n-1}, x_{n-2}) = a_1^{n-3}F_3(\alpha_3, \alpha_2, \alpha_1)$ is true.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Proposed problems should be accompanied by their solutions. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-538 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that $\sqrt{5}g^n = gL_n + L_{n-1}$, where g is the golden ratio $(1 + \sqrt{5})/2$.

B-539 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $g = (1 + \sqrt{5})/2$ and show that

$$\left[1 + 2\sum_{i=1}^{\infty} g^{-3i}\right]\left[1 + 2\sum_{i=1}^{\infty} (-1)^i g^{-3i}\right] = 1.$$

B-540 *Proposed by A. B. Patel, V. S. Patel College of Arts & Sciences, Bilimora, India*

For $n = 2, 3, \ldots$, prove that

$$F_{n-1}F_nF_{n+1}L_{n-1}L_nL_{n+1}$$

is not a perfect square.

B-541 *Proposed by Heinz-Jürgen Seiffert, student, Berlin, Germany*

Show that $P_{n+3} + P_{n+1} + P_n \equiv 3(-1)^n L_n \pmod 9$, where the P_n are the Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and

$$P_{n+2} = 2P_{n+1} + P_n \quad \text{for } n \text{ in } N = \{0, 1, 2, \ldots\}.$$

B-542 *Proposed by Ioan Tomescu, University of Bucharest, Romania*

Find the sequence satisfying the recurrence relation

$$u(n) = 3u(n-1) - u(n-2) - 2u(n-3) + 1$$

and the initial conditions $u(0) = u(1) = u(2) = 0$.

B-543 *Proposed by P. Rubio, Dragados Y Construcciones, Madrid, Spain*

Let $a_0 = a_1 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for n in $Z^+ = \{1, 2, \ldots\}$. Find a simple formula for

$$G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k.$$

SOLUTIONS

Same Parity

B-514 *Proposed by Philip L. Mana, Albuquerque, N.M.*

Prove that $\binom{n}{5} + \binom{n+4}{5} \equiv n \pmod 2$ for $n = 5, 6, 7, \ldots$.

Solution by L. Cseh, student, Cluj, Romania

It is well known that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, for every $n > r$. Using this successively, we obtain:

$$\binom{n+4}{5} = \binom{n}{5} + 4\binom{n}{4} + 6\binom{n}{3} + 4\binom{n}{2} + \binom{n}{1}, \text{ for } n \geqslant 5.$$

From here:

$$\binom{n}{5} + \binom{n+4}{5} = 2\binom{n}{5} + 4\binom{n}{4} + 6\binom{n}{3} + 4\binom{n}{2} + n,$$

and so

$$\binom{n}{5} + \binom{n+4}{5} \equiv n \pmod 2 \quad \text{for } n = 5, 6, \ldots .$$

Also solved by Paul S. Bruckman, Adina Di Porto and Piero Filipponi, L. A. G. Dresel, C. Georghiou, Lawrence D. Gould, F. T. Howard, Walther Janous, M. S. Klamkin, H. Klauser, L. Kuipers, Graham Lord, Vania D. Mascioni, Imre Merenyi, George N. Philippou, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Smith, J. Suck, W. R. Utz, and the proposer.

Disguised Lucas Number

B-515 *Proposed by Walter Blumberg, Coral Springs, FL*

Let $Q_0 = 3$, and for $n \geqslant 0$, $Q_{n+1} = 2Q_n^2 + 2Q_n - 1$. Prove that $2Q_n + 1$ is a Lucas number.

Solution by C. Georghiou, University of Patras, Greece

We show that $2Q_n + 1 = L_2 n + 2$. Let $R_n = 2Q_n + 1$. Then $R_0 = 7$, and for $n \geqslant 0$,

$$R_{n+1} = R_n^2 - 2. \qquad (*)$$

Now, using the identity $L_{4n} = L_{2n}^2 - 2$, it is easily verified that $R_n = L_2n + 2$ is a solution of (*). Since $R_0 = 7 = L_2 2$, $R_n = L_2n + 2$ is the unique solution of (*).

Also solved by Paul S. Bruckman, Laszlo Cseh, Adina Di Porto and Piero Filipponi, L. A. G. Dresel, Herta T. Freitag, Walther Janous, M. S. Klamkin, L. Kuipers, Graham Lord, Vania D. Mascioni, Imre Merenyi, George N. Philippou, Bob Prielipp, H.-J. Seiffert, A. G. Shannon, Sahib Singh, P. Smith, Lawrence Somer, J. Suck, M. Wachtel, Gregory Wulczyn, David Zeitlin, and the proposer.

Pell Equation Multiples of 36

B-516 *Proposed by Walter Blumberg, Coral Springs, FL*

Let U and V be positive integers such that $U^2 - 5V^2 = 1$. Prove that UV is divisible by 36.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria

From the theory of Pellian equations, it is very well known that starting from the minimal solution $u_0 = 9$, $v_0 = 4$, all solutions in natural numbers can be obtained via the recursion $u_{n+1} + v_{n+1}\sqrt{5} = (u_n + v_n\sqrt{5})(9 + 4\sqrt{5})$. Thus, the claim $36|UV$ can be shown by induction: $36|u_0v_0 = 36$. Assume that $36|u_nv_n$. Since

$$u_{n+1}v_{n+1} = (9u_n + 20v_n)(4u_n + 9v_n) = 36(u_n^2 + 5v_n^2) + 161u_nv_n,$$

it follows at once that $36|u_{n+1}v_{n+1}$.

Also solved by Paul S. Bruckman, Laszlo Cseh, Adina Di Porto and Piero Filipponi, L. A. G. Dresel, C. Georghiou, Fuchin He, M. S. Klamkin, H. Klauser, Edwin M. Klein, L. Kuipers, Imre Merenyi, Bob Prielipp, H.-J. Seiffert, A. G. Shannon, Sahib Singh, P. Smith, Lawrence Somer, J. Suck, W. R. Utz, M. Wachtel, Gregory Wulczyn, and the proposer.

Square Sum of Adjacent Factorials

B-517 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Find all n such that $n! + (n + 1)! + (n + 2)!$ is the square of an integer.

Solution by Paul S. Bruckman, Fair Oaks, CA

Let $\theta_n = n! + (n + 1)! + (n + 2)!$; then

$$\theta_n = n!(1 + n + 1 + (n + 1)(n + 2)) = n!(n + 2)^2.$$

We see that θ_n is a square iff $n!$ is a square. Note that $\theta_0 = 1 + 1 + 2 = 2^2$ and $\theta_1 = 1 + 2 + 6 = 3^2$.

By Bertrand's Postulate, for any $n \geqslant 1$, there exists a prime p such that $n < p \leqslant 2n$. This, in turn, implies that for any $n \geqslant 2$, there exists a prime p such that $p \leqslant n < 2p$. Hence, if $n \geqslant 2$, $p|n!$ but $kp \nmid n!$ for all $k \geqslant 2$. In particular, $p^2 \nmid n!$. This shows that $n!$ cannot be a square if $n \geqslant 2$. Thus, the only values of n for which θ_n is square are $n = 0$ and $n = 1$.

Also solved by Laszlo Cseh, L. A. G. Dresel, Adina Di Porto and Piero Filipponi, C. Georghiou, Lawrence D. Gould, Fuchin He, Walther Janous, M. S. Klamkin, *Edwin M. Klein, L. Kuipers, Graham Lord, Vania D. Mascioni, Imre Merenyi, George N. Philippou, Bob Prielipp, Sahib Singh, Paul Smith, J. Suck, Gregory Wulczyn, H. Klauser, and the proposer.*

Fibonacci Inradius

B-518 *Proposed by Herta T. Freitag, Roanoke, VA*

Let the measures of the legs of a right triangle be

$$F_{n-1}F_{n+2} \quad \text{and} \quad 2F_nF_{n+1}.$$

What feature of the triangle has $F_{n-1}F_n$ as its measure?

Solution by L. A. G. Dresel, University of Reading, England

The sides of the right-angled triangle are given as

$$a = F_{n-1}F_{n+2} = (F_{n+1} - F_n)(F_{n+1} + F_n) = F_{n+1}^2 - F_n^2,$$
$$b = 2F_nF_{n+1};$$

hence,

$$a^2 + b^2 = (F_{n+1}^2 - F_n^2)^2 + 4F_n^2F_{n+1}^2 = (F_{n+1}^2 + F_n^2)^2$$

so that the third side is $c = F_{n+1}^2 + F_n^2$, and

$$a + b + c = 2F_{n+1}^2 + 2F_nF_{n+1} = 2F_{n+1}F_{n+2},$$

while $F_{n-1}F\ (a + b + c) = ab =$ twice the area of the triangle. It follows that $F_{n-1}F_n$ measures the radius r of the incircle, that is, the circle inscribed in the triangle and touching the three sides.

Also solved by Paul S. Bruckman, Laszlo Cseh, Adina Di Porto and Piero Filipponi, C. Georghiou, Lawrence D. Gould, Walther Janous, M. S. Klamkin, H. Klauser, L. Kuipers, Vania D. Mascioni, Imre Merenyi, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, Gregory Wulczyn, and the proposer.

Lucas Inradius

B-519 *Proposed by Herta T. Freitag, Roanoke, VA*

Do as in B-518 with each Fibonacci number replaced by the corresponding Lucas number.

Solution by L. A. G. Dresel, University of Reading, England

Since the proof for B-518 given above uses only the recurrence relation for the Fibonacci numbers $F_{n+1} = F_n + F_{n-1}$, etc., the corresponding result replacing each F_k by L_k can be proved in exactly the same way.

Also solved by the solvers of B-518 and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-381 *Proposed by Dejan M. Petković, Niš, Yugoslavia*

Let N be the set of all natural numbers and let $m \in N$. Show that

(i) $$\zeta(2m-2) = \frac{(-)^m \overline{u}^{2m-2}(m-1)}{(2m-1)!} + \sum_{i=2}^{m-1} \frac{(-)^i \overline{u}^{2i-2}}{(2i-1)!} \cdot \zeta(2m-2i),\ m \geqslant 2,$$

(ii) $$\beta(2m-1) = \sum_{i=1}^{m-1} \frac{(-)^i \overline{u}^{2i}}{2^{2i}(2i)!} \cdot \beta(2m-2i-1),\ m \geqslant 2,$$

(iii) $$\zeta(2m) = \frac{2^{2m}}{2^{2m}-1} \sum_{i=0}^{m-1} \frac{(-)^i \overline{u}^{2i+1}}{2^{2i+1}(2i+1)!} \cdot \beta(2m-2i-1),\ m \geqslant 1,$$

where

$$\zeta(m) = \sum_{n=1}^{\infty} n^{-m},\ m \geqslant 2, \text{ are Riemann zeta numbers}$$

and

$$\beta(m) = \sum_{n=1}^{\infty} (-)^{n-1}(2n-1)^{-m},\ m \geqslant 1.$$

H-382 *Proposed by Andreas N. Philippou, Patras, Greece*

For each fixed positive integer k, define the sequence of polynomials $A_{n+1}^{(k)}(p)$ by

$$A_{n+1}^{(k)}(p) = \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \left(\frac{1-p}{p}\right)^{n_1 + \cdots + n_k} \quad (n \geqslant 0,\ -\infty < p < \infty), \tag{1}$$

where the summation is taken over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n + 1$. Show that

$$A_{n+1}^{(k)}(p) \leqslant (1-p)p^{-(n+1)}(1-p^k)^{[n/k]} \quad (n \geqslant k-1,\ 0 < p < 1), \tag{2}$$

where $[n/k]$ denotes the greatest integer in (n/k).

It may be noted that (2) reduces to

$$F_n^{(k)} \leqslant 2^n \left(\frac{2^k - 1}{2^k}\right)^{[n/k]} \quad (n \geqslant k-1) \tag{3}$$

and

$$F_n \leqslant 2^n(3/4)^{[n/2]} \quad (n \geqslant 1), \tag{4}$$

where $\{F_n^{(k)}\}_{n=0}^{\infty}$ and $\{F_n\}_{n=0}^{\infty}$ denote the Fibonacci sequence of order k and the usual Fibonacci sequence, respectively, if $p = 1/2$ and $p = 1/2$, $k = 2$.

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1. J. A. Fuchs. Problem B-39. *The Fibonacci Quarterly* **2**, no. 2 (1964):154.
2. A. N. Philippou. Problem H-322. *The Fibonacci Quarterly* **19**, no. 1 (1981): 93.

H-383 *Proposed by Clark Kimberling, Evansville, IN*

For any $x > 0$, let

$$c_1 = 1, \quad c_2 = x, \quad \text{and} \quad c_n = \frac{1}{n}\sum_{i=1}^{n} c_i c_{n-i} \quad \text{for } n = 3, 4, \ldots .$$

Prove or disprove that there exists $y > 0$ such that $\lim_{n\to\infty} y^n c_n = 1$.

H-384 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*

Show that for $n = 0, 1, 2, \ldots,$

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} \prod_{j=0}^{k-1} \left[\left(n + \frac{1}{2}\right)^2 - j^2\right] = \frac{\sqrt{5}}{2} F_{2n+1}$$

SOLUTIONS

Waiting Again

H-358 *Proposed by Andreas N. Philippou, University of Patras, Greece (Vol. 21, no. 3, August, 1983)*

For any fixed integers $k \geqslant 1$ and $r \geqslant 1$, set

$$f_{n+1,r}^{(k)} = \sum_{n_1,\ldots,n_k} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ \ldots, \ n_k, \ r - 1}, \quad n \geqslant 0,$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$ satisfying the relation $n_1 + 2n_2 + \cdots + kn_k = n$. Show that

$$\sum_{n=0}^{\infty} (f_{n+1,r}^{(k)}/2^n) = 2^{rk}.$$

You may note that the present problem reduces to H-322(c) for $r = 1$ (and $k \geqslant 2$), because of Theorem 2.1 of Philippou and Muwafi [1]. In addition, the present problem includes as special cases [for $k = 1$, $r = 1$, and $k = 1$, r $(\geqslant 1)$] the following infinite sums; namely,

$$\sum_{n=0}^{\infty} (1/2^n) = 2 \quad \text{and} \quad \sum_{n=0}^{\infty} \left[\binom{n + r - 1}{n}/2^n\right] = 2^r.$$

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order K." *The Fibonacci Quarterly* **20**, no. 1 (1982):28-32.

Solution by the proposer

Set

$$f_{n+1,r}^{(k)}(p) = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1, 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ \ldots, \ n_k, \ \ r - 1} p^n \left(\frac{1-p}{p}\right)^{n_1 + \cdots + n_k} \tag{1}$$

$$(n \geqslant 0, \ -\infty < p < \infty).$$

It follows, by means of the transformation $n_i = m_i$ $(1 \leqslant i \leqslant k)$ and

$$n = m + \sum_{i=1}^{k} (i-1)m_i,$$

that

$$\sum_{n=0}^{\infty} f_{n+1,r}^{(k)}(p)$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1, 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k + r - 1}{n_1 + \cdots + n_k} \binom{n_1 + \cdots + n_k}{n_1, \ \ldots, \ n_k} p^n \left(\frac{1-p}{p}\right)^{n_1 + \cdots + n_k}$$

$$= \sum_{m=0}^{\infty} \binom{m + r - 1}{m} \left(\frac{1-p}{p}\right)^m \sum_{\substack{m_1, \ldots, m_k \ni \\ m_1 + \cdots + m_k = m}} \binom{m}{m_1, \ \ldots, \ m_k} p^{m_1 + 2m_2 + \cdots + km_k}$$

$$= \sum_{m=0}^{\infty} \binom{m + r - 1}{m} \left(\frac{1-p}{p}\right)^m (p + p^2 + \cdots + p^k)^m, \text{ by the multinomial theorem,}$$

$$= \sum_{m=0}^{\infty} (-1)^m \binom{-r}{m} (1 - p^k)^m = (1 - (1 - p^k))^{-r}, \text{ for } |1 - p^k| < 1,$$

by the binomial theorem,

$$= p^{-kr}, \text{ for } k \text{ odd and } 0 < p < \sqrt[k]{2}, \text{ or } k \text{ even and } -\sqrt[k]{2} < p < \sqrt[k]{2}. \tag{2}$$

For $p = 1/2$, (1) and (2) establish the problem. For $r = 1$, (1) and (2) show H-348.

Also solved by Paul S. Bruckman.

Zetanacci

H-359 *Proposed by Paul S. Bruckman, Carmichael, CA*
(Vol. 21, no. 3, August 1983)

Define the "Zetanacci" numbers $Z(n)$ as follows:

$$Z(n) = \prod_{p^e \| n} F_{e+1}, \ n = 1, 2, 3, \ldots \ [\text{with } Z(1) = 1]. \tag{1}$$

For example, $Z(n) = 1$, $n = 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, \ldots$; $Z(n) = 2$, $n = 4, 9, 12, 18, 20, \ldots$; $Z(8) = 3$, $Z(16) = 5$), $Z(135{,}000) = Z(2^3 3^3 5^4) = 45$, and so forth.

(A) Show that the (Dirichlet) generating function of the Zetanacci numbers is given by:

$$\sum_{n=1}^{\infty} Z(n) n^{-s} = \prod_{p} (1 - p^{-s} - p^{-2s})^{-1},$$

(B) Show that

$$\prod_p (1 - p^{-s} - p^{-2s}) = \sum_{n=1}^{\infty} \mu(P(n)) \cdot |\mu(n/P(n))| \cdot n^{-s},$$

where μ is the Möbius function and

$$P(n) = \prod_{p|n} p \text{ [with } P(1) = 1].$$

Solution by C. Georghiou, University of Patras, Greece

The solution of the problem is based on the following known proposition [see, e.g., G. Polya & G. Szego, *Problems and Theorems in Analysis II* (Springer-Verlag, 1976), pp. 121, 312]:

"Let $f(n)$ be a multiplicative arithmetical function (m.a.f.). Then we have

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p (1 + f(p)p^{-s} + f(p^2)p^{-2s} + f(p^3)p^{-3s} + \cdots) \tag{*}$$

and conversely, if (*) holds, then $f(n)$ is a m.a.f."

(A) From the definition, we note that $Z(n)$ is a m.a.f. and $Z(p^k) = F_{k+1}$ for every prime p. Therefore, from (*), we have

$$\begin{aligned}\sum_{n=1}^{\infty} Z(n)n^{-s} &= \prod_p (1 + F_2p^{-s} + F_3p^{-2s} + F_4p^{-3s} + \cdots)\\ &= \prod_p (1 - p^{-s} - p^{-2s})^{-1},\end{aligned}$$

where we used the fact that the (ordinary) generating function of the sequence $\{F_{n+1}\}_{n=0}^{\infty}$ is $f(x) = (1 - x - x^2)^{-1}$.

(B) We have, according to (*),

$$\begin{aligned}\prod_p (1 - p^{-s}p^{-2s}) &= \prod_p (1 + f(p)p^{-s} + f(p^2)p^{-2s} + f(p^3)p^{-3s} + \cdots)\\ &= \sum_{n=1}^{\infty} f(n)n^{-s},\end{aligned}$$

where $f(n)$ is a m.a.f. and $f(1) = 1$, $f(p) = -1$, $f(p^2) = -1$, and $f(p^k) = 0$ for every prime p and $k > 2$. Thus the problem reduces to that of finding a m.a.f. $f(n)$ with the above-stated properties. By choosing $f(n)$ such that $f(1) = 1$ and

$$f(p^k) = \mu(p) \cdot |\mu(p^{k-1})|,$$

where μ is the Mobius function, for every prime p and $k \geqslant 1$ the above requirements are satisfied. If $n = p_{m_1}^{n_1} p_{m_2}^{n_2} \ldots p_{m_k}^{n_k}$, then since μ is a m.a.f., we have

$$\begin{aligned}f(n) &= \mu(p_{m_1}, p_{m_2}, \ldots, p_{m_k}) \cdot |\mu(n/(p_{m_1}p_{m_2} \ldots p_{m_k}))|\\ &= \mu(p(n) \cdot |\mu(n/P(n))|\end{aligned}$$

from the definition of $P(n)$, and this proves (B).

Also solved by L. Kuipers and the proposer.

Say A

H-360 *Proposed by M. Wachtel, Zurich, Switzerland*
(Vol. 21, no. 4, November, 1983)

Let:
$$F_nF_{n+1} + F_{n+2}^2 = A_1$$
$$F_{n+1}F_{n+2} + F_{n+3}^2 = A_2$$
$$F_{n+2}F_{n+3} + F_{n+4}^2 = A_3$$

Show that:

1. no integral divisor of A is congruent to 3 or 7 modulo 10,
2. $A_1A_2 + 1$, as well as $A_1A_3 + 1$, are products of two consecutive integers.

Solution by Paul S. Bruckman, Fair Oaks, CA

We make a change in notation. Let

$$B_n = F_nF_{n+1} + F_{n+2}^2 \tag{1}$$
$$C_n = B_nB_{n+1} + 1, \tag{2}$$
$$D_n = B_nB_{n+2} + 1,\ n = 0,\ 1,\ 2,\ \ldots . \tag{3}$$

Note that

$$B_n = F_nF_{n+1} + F_{n+3}F_{n+1} + (-1)^{n+1} = F_{n+1}(F_{n+3} + F_n) - (-1)^n$$
$$= F_{n+1}(F_{n+2} + F_{n+1} + F_{n+2} - F_{n+1}) - (-1)^n,$$

or

$$B_n = 2F_{n+1}F_{n+2} - (-1)^n. \tag{4}$$

Proof of Part 1: It is sufficient to prove that no prime p with $p \equiv \pm 3 \pmod{10}$ divides B_n (for all n), since any number congruent to 3 or 7 (mod 10) divisible by such a prime. Note that

$$B_n = F_nF_{n+1} + (F_{n+1} + F_n)^2 = F_{n+1}^2 + 3F_{n+1}F_n + F_n^2,$$

or upon factorization:

$$B_n = (F_{n+1} + \alpha^2F_n)(F_{n+1} + \beta^2F_n), \tag{5}$$

where α and β are the usual Fibonacci constants.

Suppose p is any prime with $p \equiv \pm 3 \pmod{10}$. Then, $(5/p) = (p/5) = -1$. According to the calculus of "complex residues" (see [1]), we can define

$$\alpha \equiv 2^{-1}(1 + \sqrt{5}) \quad \text{and} \quad \beta \equiv 2^{-1}(1 - \sqrt{5}) \pmod{p}$$

as "complex residues" and manipulate such quantities algebraically in a manner analogous to that employed with ordinary complex numbers. In this proof, we assume that all congruences are modulo p and omit the notation "(mod p)" where no confusion is likely to arise.

Assume $B_n \equiv 0 \pmod{p}$. Then one of the two factors indicated in (5) must vanish (mod p). If $F_{n+1} + \alpha^2F_n \equiv 0$, then $\alpha^{n+1} - \beta^{n+1} + \alpha^{n+2} - \beta^{n-2} \equiv 0$, implying

$$\alpha^{n+1}(1 + \alpha) \equiv \beta^{n-2}(\beta^3 + 1) \Rightarrow \alpha^{n+3} \equiv 2\beta^n \Rightarrow \alpha^{2n+3} \equiv 2(-1)^n$$

and

$$\beta^{2n+3} \equiv -2^{-1}(-1)^n.$$

Hence,

$$F_{2n+3} = 5^{-\frac{1}{2}}(\alpha^{2n+3} - \beta^{2n+3}) \equiv (2 + 2^{-1})5^{-\frac{1}{2}}(-1)^n \equiv 2^{-1}5^{\frac{1}{2}}(-1)^n.$$

Similarly, if $F_{n+1} + \beta^2 F_n \equiv 0$, then $F_{2n+3} = -2^{-1}5^{\frac{1}{2}}(-1)^n$. Hence, $B_n \equiv 0$ implies $F_{2n+3} \equiv \pm 2^{-1}5^{\frac{1}{2}}$. However, this is impossible, since F_{2n+3} is "real," while $5^{\frac{1}{2}}$, and thus $\pm 2^{-1}5^{\frac{1}{2}}$ are "imaginary" (mod p). This contradiction establishes that $B_n \not\equiv 0$ (mod p), and hence the desired result.

Proof of Part 2: Using (2) and (4),

$$\begin{aligned} C_n &= (2F_{n+1}F_{n+2} - (-1)^n)(2F_{n+2}F_{n+3} + (-1)^n) + 1 \\ &= 4F_{n+1}F_{n+2}^2F_{n+3} - 2(-1)^n F_{n+2}(F_{n+3} - F_{n+1}) \\ &= 2F_{n+2}^2(2F_{n+1}F_{n+3} - (-1)^n) \\ &= 2F_{n+2}^2\{2(F_{n+2}^2 - (-1)^{n+1}) - (-1)^n\}, \end{aligned}$$

or

$$C_n = 2F_{n+2}^2(2F_{n+2}^2 + (-1)^n). \tag{6}$$

Also,

$$\begin{aligned} D_n &= (2F_{n+1}F_{n+2} - (-1)^n)(2F_{n+3}F_{n+4} - (-1)^n) + 1 \\ &= 4F_{n+1}F_{n+2}F_{n+3}F_{n+4} - 2(-1)^n(F_{n+1}F_{n+2} + F_{n+3}F_{n+4}) + 2 \\ &= 4F_{n+2}F_{n+3}(F_{n+3} - F_{n+2})(F_{n+3} + F_{n+2}) \\ &\qquad - 2(-1)^n\{F_{n+2}(F_{n+3} - F_{n+2}) + F_{n+3}(F_{n+3} + F_{n+2})\} + 2 \\ &= 4F_{n+2}F_{n+3}(F_{n+3}^2 - F_{n+2}^2) - 2(-1)^n(2F_{n+2}F_{n+3} - F_{n+2}^2 + F_{n+3}^2) + 2 \\ &= (F_{n+3}^2 - F_{n+2}^2)(4F_{n+2}F_{n+3} - 2(-1)^n) - (-1)^n(4F_{n+2}F_{n+3} - 2(-1)^n) \\ &= (F_{n+3}^2 - F_{n+2}^2 - (-1)^n)(4F_{n+2}F_{n+3} - 2(-1)^n) \\ &= 2(F_{n+3}^2 - F_{n+3}F_{n+1})(2F_{n+2}F_{n+3} - (-1)^n) \\ &= 2F_{n+3}(F_{n+3} - F_{n+1})(2F_{n+2}F_{n+3} - (-1)^n), \end{aligned}$$

or

$$D_n = 2F_{n+2}F_{n+3}(2F_{n+2}F_{n+3} - (-1)^n). \tag{7}$$

We see from (6) and (7) that C_n and D_n are equal to products of two consecutive integers. Q.E.D.

Reference

1. Paul S. Bruckman. "Some Divisibility Properties of Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **17**, no. 1 (1979):42-49.

Also solved by L. Kuipers and the proposer.

Pell-Mell

H-361 *Proposed by Verner E. Hoggatt, Jr., deceased (Vol. 21, no. 4, November, 1983)*

Let $H_n = P_{2n}/2$, $n > 0$, where P_n denotes the n^{th} Pell number. Show that

$$H_m + H_n = P_k$$
$$H_m + H_n = P_k + P_{k-1}$$

if and only if $m = n + 1$, where $k = 2n + 1$ and

$$P_{2n+2}/2 + P_{2n}/2 = ((2P_{2n+1} + P_{2n}) + P_{2n})/2 = P_{2n+1} + P_{2n}.$$

Editorial Note: Refer to the January 1972 article on the Generalized Zeckendorf Theorem for Pell Numbers.

Solution by Paul S. Bruckman, Fair Oaks, CA

We recall or indicate (without proof) some of the basic definitions and properties of the Pell and "modified Pell" numbers:

$$P_n \equiv \frac{1}{2\sqrt{2}}(\alpha^n - \beta^n); \quad Q_n \equiv \frac{1}{2}(\alpha^n + \beta^n), \; n = 0, 1, 2, \ldots, \tag{1}$$
where $\alpha \equiv 1 + \sqrt{2}$, $\beta \equiv 1 - \sqrt{2}$.

$$P_{n+2} = 2P_{n+1} + P_n; \quad Q_{n+2} = 2Q_{n+1} + Q_n. \tag{2}$$

P_n and Q_n are increasing with n, except for $Q_0 = Q_1 = 1$;
P_n and Q_n are positive, except for $P_0 = 0$. (3)

$P_u | P_v$ iff $u | v$; $Q_u | Q_v \Rightarrow u | v$. (4)
Setting $u = 2$, we see that P_n is even iff n is even.

$$Q_n^2 - 2P_n^2 = (-1)^n; \text{ hence, } Q_n \text{ is odd for all } n. \tag{5}$$

$$P_{(a+1)b} + P_{(a-1)b} = 2P_b Q_{ab}; \quad Q_{(a+1)b} - Q_{(a-1)b} = 2Q_b Q_{ab}, \text{ if } b \text{ is odd.} \tag{6}$$

$$P_n + P_{n-1} = Q_n. \tag{7}$$

$$P_{2m} + P_{2n} = \begin{cases} 2P_{m+n}Q_{m-n}, & \text{if } m + n \text{ is even;} \\ 2P_{m-n}Q_{m+n}, & \text{if } m + n \text{ is odd.} \end{cases} \tag{8}$$

Most of these identities and properties follow readily from the definitions in (1), or are obtainable from the abundant literature on these sequences. Given two positive integers m and n, we define $s \equiv m + n$ and $d \equiv m - n$, where without loss of generality, we can assume $m \geqslant n$. We first note that there is an error in the statement of the problem; the first part of the problem should say:

$$H_m + H_n = P_k \text{ if and only if } m = n, \text{ in which case } k = 2n. \tag{9}$$

Proof of Part 1: The proposed equation is equivalent to the following:

$$P_{2m} + P_{2n} = 2P_k. \tag{10}$$

Hence, P_k is the arithmetic mean of P_{2m} and P_{2n}. Since the P_i's are increasing with i and since $m \geqslant n$, this implies: $2n \leqslant k \leqslant 2m$. We consider two possibilities: $m + n$ is even or $m + n$ is odd.

(a) s is even: Then, using (8), we must solve $P_k = P_s Q_d$. Thus, from (4), $s | k$, or $k = rs$ for some $r \geqslant 1$. Since $2n \leqslant r(m + n) \leqslant 2m$, we must have $r = 1$; hence, since $P_s > 0$, we must have $Q_d = 1$ and $d = 0$ or 1. Since d is even, $d = 0$, i.e., $m = n$, so $k = 2n$. This is the only solution of (10) in this case.

(b) s is odd: Again using (8), we are, therefore, required to solve $P_k = P_d Q_s$. Hence, again using (4), $d | k$, or $k = rd$ for some $r \geqslant 1$. If r is even, so is k; therefore, P_k [using (4)]. But d is odd; hence, P_d and Q_s are odd [by (4) and (5)], making it impossible for P_k to be even. This contradiction shows that r must be odd. Incidentally, this also shows that k must be odd. If $r = 1$, then (since $d \geqslant 1$) we have $Q_s = 1$ and $s = 0$ or 1, which is impossible, because $s \geqslant 3$. Therefore, r must be odd and greater than 2. Now the assumed equation implies

$$P_k = P_{rd} = P_d Q_s = 2P_d Q_{(r-1)d} - P_{(r-2)d},$$

using the first part of (6). Since $r > 2$ and $d \geqslant 1$,

$$P_{(r-2)d} > 0 \quad \text{and} \quad P_d \geqslant 1.$$

Hence, $P_d Q_s < 2P_d Q_{(r-1)d}$, which implies

$$Q_s < 2Q_{(r-1)d} < Q_{(r-1)d+1},$$

using (2). Then, by the property in (3), $s < (r - 1)d + 1$, or equivalently, $2m \leqslant k$. However, since $2n \leqslant k \leqslant 2m$, this implies that $k = 2m$, i.e., k is even: CONTRADICTION! Therefore, no solution of (10) exists in this case. This establishes (9).

Proof of Part 2: We see from (7) that the proposed equation is equivalent to

$$P_{2m} + P_{2n} = 2Q_k. \tag{11}$$

We again consider two cases: s is even or s is odd.

(a) s is even: Then, using (8), we are required to solve $Q_k = P_s Q_d$. Since s is even, so is P_s, hence Q_k. However, this is impossible, since Q_k is odd for all k. This contradiction eliminates any solutions in this case.

(b) s is odd: Now we are required to solve $Q_k = P_d Q_s$. Using (4), we have $s|k$, or $k = rs$ for some $r \geqslant 1$. If $r = 1$, then $Q_k = Q_s > 0$, so $P_d = 1$, implying that $k = 1$. Then, $m = n + 1$ and $k = 2n + 1$. This is a solution to equation (11). Suppose $r \geqslant 2$. Then, since $Q_{rs} - Q_{(r-2)s} = 2Q_s Q_{(r-1)s}$ [from (6)], we have

$$Q_k = Q_{rs} = P_d Q_s > 2Q_s Q_{(r-1)s},$$

implying that $P_d > 2Q_{(r-1)s}$. But clearly $2Q_n > P_n$ for all n [using (7)]. Thus, $P_d > P_{(r-1)s}$, which implies $d > (r - 1)s$, i.e., $(m - n) > (r - 1)(m + n)$. This can be true only if $r = 1$, which contradicts the hypothesis that $r \geqslant 2$.

Hence, $H_m + H_n = Q_k$ if and only if $m = n + 1$, where $k = 2n + 1$. Q.E.D.

Also solved by L. Kuipers.

◆◇◆◇◆

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