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CONTENTS

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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

A THIRD-ORDER ANALOG OF A RESULT OF L. CARLITZ

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(Submitted March 1983)

1. INTRODUCTION

In 1966, L. Carlitz [1] employed a technique based on a generating function to solve completely the second-order difference equation

$$f_{n+2}(x) = (x + 2n + p + 1)f_{n+1}(x) - (n^2 + pn + q)f_n(x), \quad (n = 0, 1, 2, \ldots),$$

with the initial conditions

$$f_0(x) = 0, \; f_1(x) = 1,$$

and p, q are parameters subject only to the restriction

$$p^2 - 4q \neq 0.$$

The polynomials $f_n(x)$ are known to be orthogonal on the real line with respect to some weight function.

Though the difference equation considered by Carlitz is of a special form, by studying Carlitz's proof, it is evident that his technique can also be used to solve analogous difference equations of higher order. It is our purpose here to illustrate this by way of solving completely the following third-order difference equation:

$$\begin{aligned} f_{n+3}(x) = {} & (x^2 + 3pn + q)f_{n+2}(x) + \{-3p^2n^2 + (3p^2 - 2pq)n + r\}f_{n+1}(x) \\ & + \{p^3n^3 + (-3p^2 + p^2q)n^2 + (2p^3 - p^2q - pr)n + s\}f_n(x), \\ & (n = 0, 1, 2, \ldots), \end{aligned} \tag{1}$$

with the initial conditions

$$f_0(x) = f_1(x) = 0, \; f_2(x) = 1, \tag{2}$$

and p, q, r, s are arbitrary parameters subject to the following three restrictions:

I. $p \neq 0$,

II. all three roots λ_1, λ_2, λ_3 of the equation
$$p^3\lambda^3 + (3p^3 - p^2q)\lambda^2 + (2p^3 - p^2q - pr)\lambda - s = 0$$
are distinct and none is a nonpositive integer,

III. both roots μ_1 and μ_2 of the equation
$$p^3\mu^2 + (3\lambda p^3 + 3p^3 - p^2q)\mu + (3\lambda^2 + 6\lambda + 2)p^3 - (2\lambda + 1)p^2q - pr = 0,$$
where λ denotes any one of λ_1, λ_2, or λ_3 from II, are nonpositive integers.

2. THE METHOD

Let

$$F(t) := F(t, x) = \sum_{n=0}^{\infty} f_n(x)\frac{t^n}{n!} \tag{3}$$

be a generating function for $f_n(x)$. From (1), (2), and (3) we get

$$(1 - pt)^3 F'''(t) - q(1 - pt)^2 F''(t) - r(1 - pt)F'(t) - sF(t) = x^2 F''(t).$$

We remark here that, save the right-hand side, this differential equation resembles the well-known Euler linear differential equation (see, e.g., Ince [2], pp. 141-143).

Next, we define an operator

$$\Delta := (1 - pt)^3 D^3 - q(1 - pt)^2 D^2 - r(1 - pt)D - s, \quad (D = d/dt).$$

Then our differential equation becomes

$$\Delta F(t) = x^2 F''(t).$$

We expect three independent solutions of this differential equation to be of the form

$$\phi(t, \lambda) := \phi(t, \lambda, x) = \sum_{k=0}^{\infty} T_k x^k (1 - pt)^{-\lambda - k},$$

where λ is any one of λ_1, λ_2, λ_3. Thus, we must compute $T_k = T_k(\lambda)$.

By direct computation, we get

$$\frac{\Delta(1 - pt)^{-\lambda - k}}{(1 - pt)^{-\lambda - k}} = (\lambda + k)(\lambda + k + 1)(\lambda + k + 2)p^3 - (\lambda + k)(\lambda + k + 1)p^2 q - (\lambda + k)pr - s.$$

Equating the coefficients of $x^k(1 - pt)^{-\lambda - k}$ for $k \geqslant 2$ in

$$\Delta\phi(t, \lambda) = x^2\phi''(t, \lambda), \tag{4}$$

we get

$$T_k = \frac{(\lambda + k - 2)(\lambda + k + 1)p^2}{(\lambda + k)(\lambda + k + 1)(\lambda + k + 2)p^3 - (\lambda + k)(\lambda + k + 1)p^2 q - (\lambda + k)pr - s} T_{k-2}.$$

Making use of restriction II that λ is a (nonpositive integer) root of

$$p^3\lambda^3 + (3p^3 - p^2 q)\lambda^2 + (2p^3 - p^2 q - pr)\lambda - s = 0,$$

we have

$$T_k = \frac{(\lambda + k - 2)(\lambda + k - 1)p^2}{k[p^3 k^2 + (3p^3 + 3p^3 - p^2 q)k + \{(3\lambda^2 + 6\lambda + 2)p^3 - (2\lambda + 1)p^2 q - pr\}]} T_{k-2}.$$

Also, making use of condition III that both roots μ of

$$p^3\mu^2 + (3p^3\lambda + 3p^3 - p^2 q)\mu + \{(3\lambda^2 + 6\lambda + 2)p^3 - (2\lambda + 1)p^2 q - pr\} = 0$$

are nonpositive integers, we arrive at the fact that

$$T_k = \frac{(\lambda + k - 2)(\lambda + k - 1)}{k(k - \mu_1)(k - \mu_2)p} T_{k-2}$$

is well defined. Consequently,

$$T_{2k} = T_0 p^{-k} \prod_{\ell=1}^{k} \frac{(2\ell - 2 + \lambda)(2\ell - 1 + \lambda)}{2\ell(2\ell - \mu_1)(2\ell - \mu_2)} = \frac{\left(\frac{\lambda}{2}\right)_k \left(\frac{\lambda}{2} + \frac{1}{2}\right)_k}{p^k 2^k k! \left(1 - \frac{\mu_1}{2}\right)_k \left(1 - \frac{\mu_2}{2}\right)_k} T_0,$$

where $(y)_k = y(y + 1) \cdots (y + k - 1)$, and

$$T_{2k+1} = \frac{2^k k! \left(\frac{\lambda}{2} + \frac{1}{2}\right)_k \left(\frac{\lambda}{2} + 1\right)_k}{p^k (2k + 1)! \left(\frac{3}{2} - \frac{\mu_1}{2}\right)_k \left(\frac{3}{2} - \frac{\mu_2}{2}\right)_k} T_1.$$

Thus,

$$\phi(t, \lambda) = \sum_{k=0}^{\infty} \{T_{2k} x^{2k} (1 - pt)^{-\lambda - 2k} + T_{2k+1} x^{2k+1} (1 - pt)^{-\lambda - 2k - 1}\}.$$

Since the degree (in x) of $f_n(x)$ is even, we must choose $T_1 = 0$. Also, we have to adjust the initial conditions; equating the coefficients of $x^0(1 - pt)^{-\lambda - 0}$ in (4) and using restriction II, we may take $T_0 = 1$. Thus,

$$\phi(t, \lambda) = \sum_{k=0}^{\infty} T_{2k} x^{2k} (1 - pt)^{-\lambda - 2k} = \sum_{k=0}^{\infty} T_{2k} x^{2k} \sum_{n=0}^{\infty} (\lambda + 2k)_n p^n \frac{t^n}{n!},$$

where

$$T_{2k} = \frac{\left(\frac{\lambda}{2}\right)_k \left(\frac{\lambda}{2} + \frac{1}{2}\right)_k}{p^k 2^k k! \left(1 - \frac{\mu_1}{2}\right)_k \left(1 - \frac{\mu_2}{2}\right)_k}, \quad (k = 0, 1, 2, \ldots).$$

Let $c_n(\lambda) := c_n(\lambda, x)$ be the coefficient of $t^n/n!$ in $\phi(t, \lambda)$. Then

$$c_n(\lambda) = \sum_{k=0}^{\infty} T_{2k} (\lambda + 2k)_n p^n x^{2k}.$$

Hence, we have the general solution to (1) as

$$f_n(x) = w_1 c_n(x, \lambda_1) + w_2 c_n(x, \lambda_2) + w_3 c_n(x, \lambda_3),$$

where

$$w_i = w_i(x, \lambda_1, \lambda_2, \lambda_3), \quad (i = 1, 2, 3)$$

are to be chosen so that the initial conditions (2) are fulfilled, namely:

$$0 = w_1 c_0(\lambda_1) + w_2 c_0(\lambda_2) + w_3 c_0(\lambda_3);$$
$$0 = w_1 c_1(\lambda_1) + w_2 c_1(\lambda_2) + w_3 c_1(\lambda_3);$$
$$1 = w_1 c_2(\lambda_1) + w_2 c_2(\lambda_2) + w_3 c_2(\lambda_3).$$

Solving this system of equations, we get

$$Dw_1 = c_0(\lambda_2) c_1(\lambda_3) - c_0(\lambda_3) c_1(\lambda_2)$$
$$Dw_2 = c_0(\lambda_3) c_1(\lambda_1) - c_0(\lambda_1) c_1(\lambda_3),$$
$$Dw_3 = c_0(\lambda_1) c_1(\lambda_2) - c_0(\lambda_2) c_1(\lambda_1),$$

where

$$D := D(x, \lambda_1, \lambda_2, \lambda_3)$$

$$= \det \begin{bmatrix} c_0(\lambda_1) & c_0(\lambda_2) & c_0(\lambda_3) \\ c_1(\lambda_1) & c_1(\lambda_2) & c_1(\lambda_3) \\ c_2(\lambda_1) & c_2(\lambda_2) & c_2(\lambda_3) \end{bmatrix}.$$

It can be verified that $D \not\equiv 0$. With these values, we have completely solved our difference equation.

3. AN EXAMPLE

In closing, we give a more specific example to our result. Take $p = 1$, $q = 4$, $r = -3$, $s = 1$. The difference equation (1) then becomes

$$f_{n+3}(x) = (x^2 + 3n + 4)f_{n+2}(x) + (-3n^2 - 5n - 3)f_{n+1}(x) + (n^3 + n^2 + n + 1)f_n(x).$$

The three roots of

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

are

$$\lambda_1 = 1,\ \lambda_2 = i = \sqrt{-1},\ \lambda_3 = -i.$$

The roots of

$$\mu^2 + (3\lambda - 1)\mu + (3\lambda^2 - 2\lambda + 1) = 0$$

for the corresponding λ are

$$\lambda_1 = 1 : \mu_{11} = \sqrt{2}\exp\left(\frac{3\pi i}{4}\right),\ \mu_{12} = \sqrt{2}\exp\left(\frac{5\pi i}{4}\right),$$

$$\lambda_2 = i : \mu_{21} = \sqrt{2}\exp\left(\frac{7\pi i}{4}\right),\ \mu_{22} = \sqrt{2}\exp\left(\frac{3\pi i}{2}\right),$$

$$\lambda_3 = -i : \mu_{31} = \sqrt{2}\exp\left(\frac{\pi i}{4}\right),\ \mu_{32} = \sqrt{2}\exp\left(\frac{\pi i}{2}\right).$$

Also,

$$T_{2k}(\lambda_1) = T_{2k}(1) = \frac{(2k)!}{2^k k! \prod_{j=1}^{k}[(2j+1)^2 + 1]},\ (k = 0, 1, 2, \ldots),$$

$$T_{2k}(\lambda_2) = T_{2k}(i) = \frac{(i/2)_k}{k!2^k(1+i)_k},$$

$$T_{2k}(\lambda_3) = T_{2k}(-i) = \frac{(-i/2)_k}{k!2^k(1-i)_k},$$

$$c_n(\lambda_1) = c_n(1) = \sum_{k=0}^{\infty} \frac{(2k+n)!x^k}{2^k k! \prod_{j=1}^{k}[(2j+1)^2 + 1]},\ (n = 0, 1, 2, \ldots),$$

$$c_n(\lambda_2) = c_n(i) = \sum_{k=0}^{\infty} \frac{(i/2)_k (i + 2k)_n}{k!2^k(1 + i)_k} x^{2k},$$

$$c_n(\lambda_3) = c_n(-i) = \sum_{k=0}^{\infty} \frac{(-i/2)_k (-i + 2k)_n}{k!2^k(1 - i)_k} x^{2k}.$$

If we consider the case where $x = 0$, then we get

$$c_n(\lambda_1, 0) = n!, \; c_n(\lambda_2, 0) = (i)_n,$$
$$c_n(\lambda_3, 0) = (-i)_n, \; (n = 0, 1, 2, \ldots),$$

and

$$w = \frac{1}{2}, \; w = \frac{1}{4}(-1 + i), \; w = \frac{1}{4}(-1 - i).$$

Hence,

$$f_n(0) = \frac{1}{2}n! + \frac{1}{4}(-1 + i)(i)_n + \frac{1}{4}(-1 - i)(-i)_n.$$

This solution can be directly checked via the differential equation

$$(1 - t)^3 F'''(t) - 4(1 - t)^2 F''(t) + 3(1 - t)F'(t) - F(t) = 0,$$

which is the familiar Euler linear differential equation.

The solution with initial conditions

$$f(0) = F(0) = 0, \; f_1(0) = F'(0) = 0, \; f_2(0) = F''(0) = 1$$

is given by (see, e.g., Ince [2], pp. 140-141)

$$F(t) = \frac{1}{2}(1 - t)^{-1} + \frac{1}{4}(-1 + i)(1 - t)^{-i} + \frac{1}{4}(-1 - i(1 - t))^{i}$$

and it can be immediately verified that this agrees with the solution found above.

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2. E. L. Ince. *Ordinary Differential Equations*. New York: Dover, 1956.

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GENERALIZED FIBONACCI NUMBERS AND SOME DIOPHANTINE EQUATIONS

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(Submitted May 1983)

1. INTRODUCTION

The object of this paper is to generalize the results of Finkelstein [3], [4], and Robbins [8] about the Fibonacci and Lucas numbers of the form $z^2 \pm 1$, by using the method of Cohn [2]. Some results which contain the Fibonacci and Lucas numbers of the form $2z^2 \pm 1$ as special cases are also given.

In all cases we obtain information about the solution of an infinite class of biquadratic diophantine equations, with the exception of Theorems 8 and 10, where it is not known if the class considered is finite or infinite [5].

The following notation will be used:

- F_m, L_m for the (usual) Fibonacci, Lucas numbers.
- $a \equiv b \pmod{c}$ or $a \equiv b(c)$ for congruences.
- (a/b) for the Jacobi quadratic symbol.
- The solutions $(\pm x, \pm y)$ of a diophantine equation are counted *once* if x and y possess only even exponents.

2. PRELIMINARIES

Definition 1: Let $d \in \mathbf{N}$, $d \neq 0$, and d not be a square.

(i) d will be called of the *first kind* if the Pellian equation $x^2 - dy^2 = -4$ has a solution with both x and y odd integers.

(ii) d will be called of the *second kind* if d is not of the first kind and the Pellian equation $x^2 - dy^2 = 4$ has a solution with both x and y odd integers.

Remark: A necessary but not sufficient condition for d to be of the first or second kind is $d \equiv 5(8)$. A counterexample is $d = 37$.

Definition 2: Let $d \in \mathbf{N}$ be of the first or the second kind with $d = 5 + 8\nu$. Let $\alpha = \frac{1}{2}(a + b\sqrt{d})$ be the fundamental solution (see [7]) of $x^2 - dy^2 = -4$ or $x^2 - dy^2 = 4$ and $\beta = \frac{1}{2}(a - b\sqrt{d})$. We define, for all integers n,

$$\begin{cases} U_n = d^{-1/2}(\alpha^n - \beta^n) \\ V_n = \alpha^n + \beta^n. \end{cases}$$

It is easy to see that $U_0 = 0$, $U_1 = b$, $V_0 = 2$, $V_1 = a$, and U_n, V_n are integers for each $n \in \mathbf{Z}$.

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The terms of the sequence $\{U_n\}$, $n \in \mathbf{N}$ ($\{V_n\}$, $n \in \mathbf{N}$) will be called *generalized Fibonacci (Lucas)* numbers.

Remarks: (i) From Definitions 1 and 2, it follows that both a and b must be odd.

(ii) If $b = 1$, then our definition of generalized Fibonacci numbers agrees with the Fibonacci polynomials $U_n = F_n(a)$, a odd, but in general, b can be different from one as for example in the case $d = 61$, $a = 39$, $b = 5$.

From now on, d will always be of the first kind with the fundamental solution $\frac{1}{2}(a + b\sqrt{d})$ of the corresponding Pellian equation $x^2 - dy^2 = -4$. According to [2], the following identities hold:

$$U_{n+2} = aU_{n+1} + U_n, \tag{1}$$

$$V_{n+2} = aV_{n+1} + V_n, \tag{2}$$

$$U_{-n} = (-1)^{n-1}U_n, \tag{3}$$

$$V_{-n} = (-1)^n V_n, \tag{4}$$

$$2U_{m+n} = U_m V_n + U_n V_m, \tag{5}$$

$$2V_{m+n} = dU_m U_n + V_m V_n, \tag{6}$$

$$(-1)^n 4 = V_n^2 - dU_n^2, \tag{7}$$

$$V_n^2 = V_{2n} + (-1)^n \cdot 2, \tag{8}$$

$$2|U_n \text{ iff } 2|V_n \text{ iff } 3|n, \tag{9}$$

$$(U_n, V_n) = \begin{cases} 1 & \text{if } 3\nmid n \\ 2 & \text{if } 3|n, \end{cases} \tag{10}$$

$$V_{n+12} \equiv V_n \pmod 8, \tag{11}$$

$$2U_{m+2N} \equiv (-1)^{N-1}2U_m \pmod{V_N}, \tag{12}$$

$$2V_{m+2N} \equiv (-1)^{N-1}2V_m \pmod{V_N}, \tag{13}$$

$$2U_{m+2N} \equiv (-1)^N 2U_m \pmod{U_N}, \tag{14}$$

$$2V_{m+2N} \equiv (-1)^N 2V_m \pmod{U_N}, \tag{15}$$

$$V_n \equiv 2 \pmod a \text{ if } 2|n, \tag{16}$$

$$V_n \equiv (-1)^{n/2} \cdot 2 \pmod b \text{ if } 2|n, \tag{17}$$

$$b \equiv 1(4), \tag{18}$$

and, furthermore, for $k \in \mathbf{Z}$, with $2|k$, $3\nmid k$,

$$V_k > 0 \text{ and } V_k \equiv \begin{cases} 3(8) & \text{if } k \equiv 2(4) \\ 7(8) & \text{if } 4|k, \end{cases} \tag{19}$$

$$\left(\frac{2}{V_k}\right) = (-1)^{k/2}, \tag{20}$$

$$U_{m+2k} \equiv -U_m \pmod{V_k}, \tag{21}$$

$$V_{m+2k} \equiv -V_m \pmod{V_k}, \tag{22}$$

$$\left(\frac{a}{V_k}\right) = \left(\frac{-2}{a}\right), \tag{23}$$

$$\left(\frac{V_3}{V_k}\right) = \left(\frac{-2}{a}\right), \tag{24}$$

$$\left(\frac{V_k}{U_5}\right) = -\left(\frac{2}{b}\right) \text{ provided that } 5 \nmid k, \tag{25}$$

the general solution of $x^2 - dy^2 = -4$ is $x = V_{2n+1}$, $y = U_{2n+1}$, (26)

the general solution of $x^2 - dy^2 = 4$ is $x = V_{2n}$, $y = U_{2n}$, (27)

$$\text{if } V_n = x^2, \text{ then } \begin{cases} n = 1 & \text{if } a = t^2 \text{ and } d \neq 5 \\ n = 1,\ 3 & \text{if } d = 5 \\ n = 3 & \text{if } d = 13, \end{cases} \tag{28}$$

$$\text{if } V_n = 2x^2, \text{ then } \begin{cases} n = 0 \\ \text{and} \\ n = \pm 6 & \text{if } d = 5,\ 29, \end{cases} \tag{29}$$

$$\text{if } U_n = x^2, \text{ then } \begin{cases} n = 0 \\ n = 12 & \text{if } d = 5 \\ n = 2 & \text{if } a = t^2 \text{ and } b = r^2 \\ n = \pm 1 & \text{if } b = r^2, \end{cases} \tag{30}$$

$$\text{if } U_n = 2x^2, \text{ then } \begin{cases} n = 0 \\ n = 6 \quad \text{if } d = 5 \\ \text{and possibly the solutions } n = \pm 3. \end{cases} \tag{31}$$

We also need some values for U_n and V_n:

n	U_n	V_n
0	0	2
1	b	a
2	ab	$a^2 + 2$
3	$(a^2 + 1)b$	$a^3 + 3a$
4	$(a^3 + 2a)b$	$a^4 + 4a^2 + 2$
5	$(a^4 + 3a^2 + 1)b$	$a^5 + 5a^3 + 5a$
6	$(a^5 + 4a^3 + 3a)b$	$a^6 + 6a^4 + 9a^2 + 2$

3. GENERALIZED FIBONACCI NUMBERS OF THE FORM $\mu z^2 + \nu$

Theorem 1: Let $a \equiv 1,\ 3(8)$ and $b \equiv 1(8)$. Then the equation

$$U_m = az^2 + b,\ m \equiv 1(2),$$

has

(a) the solutions $m = \pm 1$, ± 3, and ± 5 if $d = 5$,

(b) the solutions $m = \pm 1$, ± 5 if $d = 13$,

(c) the solutions $m = \pm 1$, ± 3 if a and b are both perfect squares, $d \neq 5$,

(d) only the solutions $m = \pm 1$ in all other cases.

Proof: It is sufficient by (3) to consider only the cases $m \equiv 1(8)$, $3(16)$, and $5(16)$.

Case 1. Let $m \equiv 1(8)$. For $m = 1$, $z = 0$ is a solution. If $m \neq 1$, then we write $m = 1 + 2 \cdot 3^s \cdot n$, where $4|n$, $3\nmid n$, and $az^2 + b = U_m \equiv -U_1 \pmod{V_n}$ by (21). Thus $(az)^2 \equiv -2ab \pmod{V_n}$. But

$$\left(\frac{-2ab}{V_n}\right) = -1$$

by (19), (20), (16), (17), and the assumption. Hence, $U_m \neq az^2 + b$.

Case 2. Let $m \equiv 3(16)$. If $m = 3$, then $az^2 + b = (a^2 + 1)b$ iff $z^2 = ab$ iff a and b are both perfect squares, since $(a, b) = 1$.

If $m \neq 3$, then we write $m = 3 + 2 \cdot 3^s \cdot n$, where $8|n$, $3\nmid n$, and $az^2 + b = U_m \equiv -U_3 \pmod{V_n} \equiv -(a^2 + 1)b \pmod{V_n}$, by (21). Thus $(az)^2 \equiv -abV_2 \pmod{V_n}$.

By applying (13) repeatedly, we obtain

$$2V_n \equiv -2V_{n-4} \equiv 2V_{n-8} \equiv \cdots \equiv 2V_0 \equiv 4 \pmod{V_2}, \tag{32}$$

which by (19) implies $V_n \equiv 2 \pmod{V_2}$. Thus $(V_n, V_2) = (2, V_2) = 1$ and

$$\left(\frac{V_2}{V_n}\right) = -\left(\frac{V_n}{V_2}\right) = -\left(\frac{2}{V_2}\right) = \pm 1.$$

Now $(-abV_2/V_n)$ can be calculated to be -1 by using (19), (16), (17), (33), and the assumption. Hence, $U_m \neq az^2 + b$.

Case 3. Let $m \equiv 5(16)$. If $m = 5$, then there exists a solution iff $az^2 + b = (a^4 + 3a^2 + 1)b$ iff $z^2 = a(a^2 + 3)b$. Since b is odd and $b|U_3$,

$$(b, V_3)/(U_3, V_3) = 2,$$

which implies $(b, V_3) = 1$. Hence,

$$z^2 = a(a^2 + 3)b = V_3 b \text{ iff } b = r^2 \text{ and } a(a^2 + 3) = z_1^2.$$

By [1], the last equation has only the solutions $(z_1, a) = (0, 0), (\pm 2, 1), (\pm 6, 3), (\pm 42, 12)$. Since we have $a \equiv 1(2)$, the only possible solutions are $(z_1, a) = (\pm 2, 1), (\pm 6, 3)$. For $a = 1$, we have $b = 1 = r^2$ and $d = 5$. For $a = 3$, we have $b = 1 = r^2$ and $d = 13$.

If $m \neq 5$, then $m = 5 + 2 \cdot 3^s \cdot n$ with $8|n$, $3\nmid n$, and thus

$$U_m \equiv -U_5 \pmod{V_n} \equiv -(a^4 + 3a^2 + 1)b \pmod{V_n} \text{ by (21).}$$

Applying (15) repeatedly and using (4), we have

$$2V_n \equiv -2V_{n-6} \equiv 2V_{n-12} \equiv \cdots \equiv \pm 2V_2 \pmod{U_3}. \tag{34}$$

Since $(V_n, V_2) = 1$ implies $(2V_n, U_3) = 2$, we see that

$$\left(\frac{U_3/2}{V_n}\right) = \left(\frac{(a^2+1)/2}{V_n}\right)\left(\frac{b}{V_n}\right) = \left(\frac{V_n}{(a^2+1)/2}\right)\left(\frac{b}{V_n}\right)$$
$$= \left(\frac{\pm V_2}{(a^2+1)/2}\right)\left(\frac{b}{V_n}\right) = \left(\frac{b}{V_n}\right). \tag{35}$$

Now, if $az^2 + b = U_m$, we have

$$(ax)^2 \equiv -a(a^4 + 3a^2 + 2)b \equiv -abV_2U_3 \pmod{V_n},$$

which is impossible because $(-abV_2U_3/V_n) = -1$ by (19), (16), (17), (33), (35), and the assumption. Hence, $U_m \neq az^2 + b$.

Corollary 1: The diophantine equation $x^2 = a^2dz^4 + 2abdz^2 + a^2$ with $a \equiv 1, 3(8)$ and $b \equiv 1(8)$, has

(a) three solutions $(x, y) = (\pm 1, 0), (\pm 4, \pm 1), (\pm 11, \pm 2)$ if $d = 5$,

(b) two solutions $(x, z) = (\pm 3, 0), (\pm 393, 16)$ if $d = 13$,

(c) two solutions $(x, z) = (\pm a, 0), (\pm a(a^2 + 3), \pm tr)$, where $a = t^2$ and $b = r^2$ are both perfect squares, $d \neq 5$,

(d) only one solution $(x, z) = (\pm a, 0)$ in all other cases.

Proof: This follows directly from (26), Theorem 1, and Definition 2.

Following the arguments of Theorem 1 and Corollary 1, we have

Theorem 2: Let $b \equiv 1(8)$. Then the equation $U_m = z^2 + b$, $m \equiv 1(2)$, has

(a) the solutions $m = \pm 1, \pm 3, \pm 5$, if $d = 5$,

(b) the solutions $m = \pm 1, \pm 3$, if $b = r^2$, $d \neq 5$,

(c) only the solutions $m = \pm 1$ in all other cases,

and

Corollary 2: The diophantine equation $x^2 = dz^4 + 2dbz^2 + a^2$ with $b \equiv 1(8)$ has

(a) three solutions $(x, z) = (\pm 1, 0), (\pm 4, \pm 1), (\pm 11, \pm 2)$, if $d = 5$,

(b) two solutions $(x, z) = (\pm a, 0), (\pm a(a^2 + 3), \pm ar)$ if $b = r^2$, $d \neq 5$,

(c) only one solution $(x, z) = (\pm a, 0)$ in all other cases.

We now show the following results, which are similar to the above but with m even.

Theorem 3: Let $a \equiv 1, 3(8)$ and $b \equiv 1(8)$ or $a \equiv 5, 7(8)$ and $b \equiv 5(8)$. Then the equation $U_m = z^2 + ab$, $m \equiv 0(2)$, has only the solution $m = 2$.

Proof:

Case 1. Let $m \equiv 0(4)$. No solution exists for $m = 0$; but if $m \neq 0$, then we write $m = 2 \cdot 3^s \cdot n$ with $2|n$, $3\nmid n$, and thus $U_m \equiv 0 \pmod{V_n}$ by (21). If $U_m = z^2 + ab$ for some m, then we have $z^2 \equiv -ab \pmod{V_n}$, which is impossible, since $(-ab/V_n) = -1$ by (19), (16), (17), and the assumption.

Case 2: Let $m \equiv 2(8)$. For $m = 2$, we have the solution $z = 0$. If $m \neq 2$, then we write $m = 2 + 2 \cdot 3^s \cdot n$ with $4|n$, $3\nmid n$, and thus

$$U_m \equiv -U_2 \pmod{V_n} \equiv -ab \pmod{V_n} \text{ by (21),}$$

Thus, if $U_m = z^2 + ab$, we should have $z^2 \equiv -2ab \pmod{V_n}$, which is impossible, since $(-2ab/V_n) = -1$ by (19), (20), (16), (17), and the assumption.

Case 3: Let $m = 6(8)$. If $m = 6$, we have a solution iff

$$z^2 + ab = (a^5 + 4a^3 + 3a)b \text{ iff } z^2 = a(a^4 + 4a^2 + 2)b = aV_4b.$$

But $b|U_4$; hence,

$$(b, V_4)/(U_4, V_4) = 1 \text{ by (10).}$$

Therefore, it follows that $b = t^2$, $a = r^2$, and $a^4 + 4a^2 + 2 = V_4 = s^2$, which is impossible mod 4.

If $m \neq 6$, then we write $m = 6 + 2 \cdot 3^s \cdot n$ with $4|n$, $3 \nmid n$, and thus

$U_m \equiv -U_6 \pmod{V_n} \equiv -(a^5 + 4a^3 + 3a)b \pmod{V_n}$ by (21).

Hence, if $U_m = z^2 + ab$, we have $z^2 \equiv -ab(a^4 + 4a^2 + 4) \equiv -ab(a^2 + 2)^2 \pmod{V_n}$, which is impossible since

$$\left(\frac{-ab(a^2+2)^2}{V_n}\right) = \left(\frac{-ab}{V_n}\right) = -1 \text{ by (19), (16), (17), and the assumption.}$$

Applying Theorem 1(a) and Theorem 3, we now have

Corollary 3: (Theorem of Finkelstein [3], [9], [1])

$F_m = z^2 + 1$ iff $m = \pm 1, 2, \pm 3, \pm 5$.

Using an argument similar to that of Theorem 3, we have Theorem 4 and two immediate corollaries.

Theorem 4: Let $b \equiv 1(8)$. Then, the equation $U_m = az^2 + ab$, $m \equiv 0(2)$, has only the solution $m = 2$.

Corollary 4: Let $d = a^2 + 4$, $2 \nmid a$. Then, the equation $U_m = az^2 + a$ has only the solution $m = 2$.

Corollary 5: The diophantine equation $x^2 = a^2dz^4 + 2a^2dbz^2 + (a^2 + 2)^2$ with $b \equiv 1(8)$ has only the solution $(x, y) = (\pm(a^2 + 2), 0)$.

An argument similar to Theorem 3 will also give us the following extended result of Theorem 1.

Theorem 5: Let $a \equiv 1, 3(8)$ and $b \equiv 1(8)$. Then, each of the equations

$U_m = 2az^2 + b$, $U_m = 2z^2 + b$, $m \equiv 1(2)$,

has only the solutions $m = \pm 1$.

Corollary 6: Let $a \equiv 1, 3(8)$ and $b \equiv 1(8)$. Then, the equations

$x^2 = 4a^2dz^4 + 4abdz^2 + a^2$ and $x^2 = 4dz^4 + 4dbz^2 + a^2$

have only the solution $(x, z) = (\pm a, 0)$.

The following is an extended result of Theorem 3 and is similar to Theorem 5 but with m even.

Theorem 6: Let $a \equiv 1, 3(8)$ and $b \equiv 1(8)$, or $a \equiv 5, 7(8)$ and $b \equiv 5(8)$. Then, the equation $U_m = 2z^2 + ab$, $m \equiv 0(2)$ has

(a) the solutions $m = 2, 4$ if $d = 5$,

(b) only the solution $m = 2$ in all other cases.

Proof:

Case 1. Let $m \equiv 0(8)$. If $m = 0$, $2z^2 + ab = 0$ is impossible. If $m \neq 0$, we write $m = 2 \cdot 3^s \cdot n$ with $4|n$, $3\nmid n$, and therefore $U_m \equiv 0 \pmod{V_n}$ by (21). Thus, if $2z^2 + ab = U_m$, we have $(2z)^2 \equiv -2ab \pmod{V_n}$, which is impossible, since

$\left(\frac{-2ab}{V_n}\right) = -1$ by (19), (20), (16), (17), and the assumption.

Case 2. Let $m \equiv 4(8)$. If $m = 4$, then there exists a solution iff $2z^2 = ab(a^2 + 1)$. Since $a^2 - db^2 = -4$, we have $(b, a^2 + 1) = 1$ or 3. But $a^2 + 1 \not\equiv 0(3)$; therefore, $(b, a^2 + 1) = 1$. It is obvious that $(a, b) = (a, a + 1) = 1$. So we must have $a = t^2$, $b = r^2$, and $a^2 + 1 = 2\lambda^2$, so that $t^4 + 1 = 2\lambda^2$. In [6] W. Ljunggren proved that the diophantine equation $Ax^2 - By^4 = 1$ has at most one solution in positive numbers x and y. In our case, this is $(t, \lambda) = (\pm 1, \pm 1)$, which corresponds to $a = 1$, so $b = 1 = r^2$ and $d = 5$.

If $m \neq 4$, then we write $m = 4 + 2 \cdot 3^s \cdot n$ with $4|n$, $3\nmid n$, and therefore,

$U_m \equiv -(a^3b + 2ab) \pmod{V_n}$ by (21).

Hence, if $2z^2 + ab = U_m$, we have $2z^2 \equiv -ab(a^2 + 3) \equiv -2bV_3 \pmod{V_n}$, which is impossible, since

$\left(\frac{-2bV_3}{V_n}\right) = -1$ by (19), (20), (16), (17), (24), and the assumption.

Case 3. Let $m \equiv 2(4)$. If $m = 2$, then $z = 0$ is a solution. If $m \neq 2$, then we write $m = 2 + 2 \cdot 3^s \cdot n$, with $2|n$, $3\nmid n$, and thus,

$U_m \equiv -ab \pmod{V_n}$ by (21).

Hence, if $2z^2 + ab = U_m$, we have $(2z)^2 \equiv -4ab \pmod{V_n}$, which is impossible, since

$\left(\frac{-4ab}{V_n}\right) = -1$ by (19), (16), (17), and the assumption.

The following corollaries are direct results of the previous theorems. Hence, the proofs are omitted.

Corollary 7: Let $a \equiv 1, 3(8)$ and $b \equiv 1(8)$, or $a \equiv 5, 7(8)$ and $b \equiv 7(8)$. Then, the equation $x^2 = 4dz^4 + 4abdz^2 + (a^2 + 2)^2$ has

(a) two solutions $(x, z) = (\pm 3, 0), (\pm 7, \pm 1)$ if $d = 5$,

(b) only the one solution $(x, z) = (\pm(a^2 + 2), 0)$ in all other cases.

Corollary 8: $F_m = 2z^2 + 1$ iff $m = \pm 1, 2, 4$.

4. GENERALIZED FIBONACCI NUMBERS OF THE FORM $\mu z^2 - \nu$

Lemma 1: The generalized Fibonacci numbers U_m have the form

$U_{2n+1} = b(f_{2n+1}(a^2) + 1), \quad U_{2m} = abf_{2n}(a^2)$

and the generalized Lucas numbers V_m have the form

$V_{2n+1} = ag_{2n+1}(a^2), \quad V_{2n} = g_{2n}(a^2) + 2,$

where f_m, $g_m \in \mathbf{Z}[a^2]$ for each $m \in \mathbf{Z}$ and f_{2n+1}, g_{2n} have no constant term.

Proof: $U_{2n+1} = b(f_{2n+1}(a^2) + 1)$. The proof is by induction on n. If $n = 0$, we have $U_1 = b$, and the relation is true for $f_1(a^2) \equiv 0$. Let us now assume the proposition is true for all values less than or equal to n. Then we have

$$\begin{aligned} U_{2n+3} &= aU_{2n+2} + U_{2n+1} && \text{by (1)} \\ &= a(aU_{2n+1} + U_{2n}) + U_{2n+1} \\ &= (a^2 + 1)b(f_{2n+1}(a^2) + 1) + aU_{2n} && \text{by assumption} \\ &= (a^2 + 1)b(f_{2n+1}(a^2) + 1) + a(aU_{2n-1} + U_{2n-2}) \\ &= (a^2 + 1)b(f_{2n+1}(a^2) + 1) + a^2b(f_{2n-1}(a^2) + 1) + aU_{2n-2} && \text{by assumption} \\ &= \cdots = b(f_{2n+3}(a^2) + 1) + aU_0 = b(f_{2n+3}(a^2) + 1), \end{aligned}$$

with $f_{2n+3}(a^2)$ having no constant term.

In the same way, we can prove the other cases.

Lemma 2: The following identities hold:

$$U_{4n\pm1} = U_{2n\pm1}V_{2n} - b \tag{36}$$
$$U_{4n} = U_{2n-1}V_{2n+1} - ab \tag{37}$$
$$U_{4n} = U_{2n+1}V_{2n-1} + ab \tag{38}$$
$$U_{4n-2} = U_{2n}V_{2n-2} - ab \tag{39}$$
$$U_{4n-2} = U_{2n-2}V_{2n} + ab \tag{40}$$
$$bV_{m+n} = U_{m-1}V_n + U_mV_{n+1} \tag{41}$$
$$V_{2n+1} = V_nV_{n+1} - (-1)^na \tag{42}$$

Proof of (36): We have $2U_{4n\pm1} = U_{2n\pm1}V_{2n} + U_{2n}V_{2n\pm1}$ by (5); thus,

$$U_{4n\pm1} + b = \frac{U_{2n\pm1}V_{2n} + U_{2n}V_{2n\pm1} + 2b}{2}.$$

It is therefore sufficient to show that

$$U_{2n}V_{2n+1} + 2b = U_{2n+1}V_{2n} \tag{43}$$

and

$$U_{2n}V_{2n-1} + 2b = U_{2n-1}V_{2n}. \tag{44}$$

We will prove (43) by induction on n. For $n = 0$, (43) is true, because $U_0V_{\pm1} + 2b = U_{\pm1}V_0$. Under the assumption that (43) is true for n, it is enough to show that $U_{2n+2}V_{2n+3} + 2b = U_{2n+3}V_{2n}$. By using (1) and (2), we find that it is equivalent to $U_{2n}V_{2n+1} + 2b = U_{2n+1}V_{2n}$, which holds by assumption. In the same way, (44) can be proved.

Proof of (37): By using (5), it is enough to show that

$$U_{2n}V_{2n} = U_{2n-1}V_{2n+1} - ab, \tag{45}$$

which can be proved by induction on n with the aid of (1) and (2). Similarly, (38), (39), and (40) can be proved.

Proof of (41): We again use induction on n. For $n = 0$, it must first be proved that $bV_m = U_{m-1}V_0 + U_mV_1 = 2U_{m-1} + aU_m$. This can be proved by induction on m. The remainder of the proof is straightforward.

Proof of (42): This follows by induction on n using (8) and (2).

Lemma 3: If $b = 1$, then $(U_m, V_{m \pm n}) | V_n$.

Proof: By (4), it suffices to show that $g | V_n$, where $g = (U_m, V_{m+n})$. By (41), $g | U_{m-1}V_n$. If $g_1 = (g, U_{m-1})$, then $g_1 | U_m$ and $g_1 | U_{m-1}$, so that $g_1 | U_{m-2}$. Hence, $g_1 | b$. But $b = 1$. Therefore, $g_1 = 1$ and $g | V_n$.

Corollary 9: If $b = 1$, then $(U_{2n \pm 1}, V_{2n}) = 1$.

Proof: Let g be as in Lemma 3, with $m = 2n \pm 1$ and $n = \mp 1$, then $g | V_{\pm 1}$ or $g | a$. Since $g | U_{2n \pm 1}$ and $g | a$, Lemma 1 implies $g | b$. However, $(a, b) = 1$. Hence, $g = 1$.

Theorem 7: Let $b = 1$. Then, the equation $U_m = z^2 - b$, $m \equiv 1(2)$, has no solution.

Proof: By (36), we have $U_{2n \pm 1}V_{2n} = z^2$. Hence, Corollary 9 implies that $U_{2n \pm 1} = z_1^2$ and $V_{2n} = z_2^2$, which is impossible by (28).

Theorem 8: Let $b = 1$ and $a^2 + 2 = p$, p a prime. Then, the equation

$$U_m = z^2 - a, \; m \equiv 0(2),$$

has

(a) the solutions $m = -2, 0, 4, 6$, if $d = 5$,

(b) the solutions $m = -2, 4$, if $d = 13$,

(c) the solutions $m = -2, 0, 6$, if a is a perfect square, $d \neq 5$,

(d) only the solution $m = -2$ in all other cases.

Proof:

Case 1. Let $m = 4n - 2$. By (39), $U_{2n}V_{2n-2} = z^2$. Lemma 3 implies that

$$(U_{2n}, V_{2n-2}) | p.$$

Hence, we have two possibilities:

(a) $U_{2n} = W_1^2$ and $V_{2n-2} = W_2^2$ or (b) $U_{2n} = pW_1^2$ and $V_{2n-2} = pW_2^2$.

The first is impossible by (28). The second can be written by (5) as

$$U_nV_n = pW_1^2, \; V_{2n-2} = pW_2^2.$$

Let $n \not\equiv 0(3)$. Then equation (10) implies that $(U_n, V_n) = 1$, and so

$$U_n = pt^2, \; V_n = r^2, \; V_{2n-2} = pW_2^2 \tag{46}$$

$$U_n = t^2, \; V_n = pr^2, \; V_{2n-2} = pW_2^2. \tag{47}$$

Equation (46) does not possess any solution, since the possible values of n, by (28), in order for V_n to be a perfect square, do not yield a solution of $U_n = pt^2$.

By using (30) and direct computation, we find that (47) has only one solution, which is $n = 2$ or $m = 6$ provided a is a perfect square.

Let $n \equiv 0(3)$. Equation (10) implies that $(U_n, V_n) = 2$, and so we have to check the following subcases:

$$U_{3\lambda} = 2pt^2,\ V_{3\lambda} = 2r^2,\ V_{2n-2} = pW_2^2, \tag{48}$$

or

$$U_{3\lambda} = 2t^2,\ V_{3\lambda} = 2pr^2,\ V_{2n-2} = pW_2^2,\ (n = 3\lambda). \tag{49}$$

By (29) and the assumption, $V_{3\lambda} = 2r^2$ is possible only for $\lambda = 0$ or $\lambda = \pm 2$ in the case $d = 5$. The value $\lambda = 0$ implies $n = 0$ or $m = -2$, which gives a solution to (48). The values $\lambda = \pm 2$, $d = 5$, do not give a solution, since $F_{\pm 6} = \pm 8 \neq 2pt^2$.

According to (31), the only values of λ for which a solution of (49) may exist are $\lambda = 2$ if $d = 5$, or $\lambda = 0$ and $\lambda = \pm 1$. Now, $\lambda = 0$ does not give any solution, because we would have $pr^2 = 1$. Similarly, $\lambda = \pm 1$ does not give any solution, since we would have $V_{\pm 3} = \pm a(a^2 + 3) = 2pt^2$, which is impossible because $p \nmid a$ and $p \nmid (a^2 + 3)$ when $a^2 + 3 = p + 1$. Finally, $\lambda = 2$, $d = 5$, does not give any solution, since $L_6 = 18 \neq 2 \cdot 3r^2$.

Case 2. Let $m = 4n$. By (37), $U_{2n-1}V_{2n+1} = z^2$. Now Lemma 3 implies that $(U_{2n-1}, V_{2n+1}) | p$, so we have two possibilities, which are

$$U_{2n-1} = W_1^2,\ V_{2n+1} = W_2^2 \tag{50}$$

or

$$U_{2n-1} = pt^2 = V_2t^2,\ V_{2n+1} = V_2r^2. \tag{51}$$

By using (28) and (30), we find that (50) has only the solutions:

(a) $m = 0, 4$, if $d = 5$,

(b) $m = 4$, if $d = 13$,

(c) $m = 0$, if a is a perfect square, $d \neq 5$.

Using (13) for $2n + 1 = 4\lambda \pm 1$, we have

$$2V_{2n\pm 1} \equiv -2V_{4\lambda - 4 \pm 1} \equiv \cdots \equiv \pm 2V_{\pm 1} \pmod{V_2}.$$

Therefore, since $V_{2n+1} = pr^2 = V_2r^2$, we have $(a^2 + 2) | V_{\pm 1}$ or $p | a$, which is impossible. Thus, (51) has no solution.

Corollary 10: For each $d = a^2 + 4$, $a \equiv 1(2)$, the diophantine equation

$$x^2 = dz^4 - 2dz^2 + a^2$$

has no solution.

Corollary 11: Let $d = a^2 + 4$ and $a^2 + 2 = p$, where p is a prime. Then, the diophantine equation $x^2 = dz^4 - 2adz^2 + (a^2 + 2)^2$ has:

(a) Four solutions, $(x, z) = (\pm 3, 0), (\pm 2, \pm 1), (\pm 7, \pm 2), (\pm 18, \pm 3)$, if $d = 5$.

(b) Two solutions, $(x, z) = (\pm 11, 0), (\pm 119, \pm 6)$, if $d = 13$.

(c) Three solutions, $(x, z) = (\pm(a^2 + 2), 0), (\pm 2, \pm t), (\pm(a^6 + 6a^4 + 9a^2 + 2), \pm t(a^2 + 2))$, if $a = t^2$ is a perfect square.

(d) Only the solution $(x, z) = (\pm(a^2 + 2), 0)$ in all other cases.

When $a = 1$ in Theorem 8, we have the following result, found in [8].

Corollary 12: $F_m = z^2 - 1$ iff $m = -2, 0, 4, 6$.

The next result is an extension of Theorem 7.

Theorem 9: Let $b = 1$. Then, the equation $U_m = 2z^2 - b$, $m \equiv 1(2)$, has only the solutions $m = \pm 1$.

Proof: Equation (36) implies that $U_{2n\pm 1}V_{2n} - b = 2z^2 - b$, for $m = 4n \pm 1$. Hence, $U_{2m\pm 1}V_{2n} = 2z^2$. By Corollary 9,

$$U_{2n\pm 1} = 2t^2,\ V_{2n} = r^2 \quad \text{or} \quad U_{2n\pm 1} = t^2,\ V_{2n} = 2r^2.$$

Now $V_{2n} = r^2$ is impossible by (28) and the second case implies, using (30) and (29), that $n = 0$ or $m = \pm 1$.

The following result is an extended parallel of Theorem 8.

Theorem 10: Let $b = 1$ and $a^2 + 2 = p$, where p is a prime. Then, the equation $U_m = 2z^2 - a$, $m \equiv 0(2)$ has

(a) the solutions $m = -2, 2$ if a is a perfect square,

(b) only the solution $m = -2$ in all other cases.

Proof:

Case 1. Let $m = 4n - 2$. Equation (39) implies that $U_{2n}V_{2n-2} = 2z^2$. But, by Lemma 3, $(U_{2n}, V_{2n-2})|V_2$, where $V_2 = p$, so that $(U_{2n}, V_{2n-2}) = 1$ or p. If $(U_{2n}, V_{2n-2}) = 1$, then we must have

$$U_{2n} = 2t^2,\ V_{2n-2} = r^2 \quad \text{or} \quad U_{2n} = t^2,\ V_{2n-2} = 2r^2.$$

The first case is impossible by (28). The second case has, by (30) and (29), only the solution $n = 1$ or $m = 2$ if a is a perfect square.

Now, let $(U_{2n}, V_{2n-2}) = p$. We then have to check two possibilities:

$$U_{2n} = pt^2,\ V_{2n-2} = 2pr^2 \quad \text{or} \quad U_{2n} = 2pt^2,\ V_{2n-2} = pr^2.$$

In the first case we must have, by (9), $n \equiv 1(3)$, say $n = 3\lambda + 1$. By (5), we also have $U_nV_n = pt^2$. But $(U_n, V_n) = 1$; therefore, we have

$$U_n = pW_1^2,\ V_n = W_2^2,\ V_{2n-2} = 2pr^2, \tag{52}$$

or

$$U_n = W_1^2,\ V_n = pW_2^2,\ V_{2n-2} = 2pr^2. \tag{53}$$

Equation (52) has no solution since, by (28), the only solution of $V_n = W_2^2$ is $n = 1$, for which $U_n = pW_1^2$ is impossible. Equation (53) has no solution either since, by (30), the only possible value for n of $U = W_1^2$ is $n = 1$, but then $V_1 = a = pW_2^2$, which is impossible.

For the second case we must have, by (9), $3|n$, say $n = 3\lambda$. By (5), we have $U_{3\lambda}V_{3\lambda} = 2pt^2$. Since, by (10), $(U_{3\lambda}, V_{3\lambda}) = 2$, we must check the following subcases:

$$U_{3\lambda} = 4pr_1^2,\ V_{3\lambda} = 2r_2^2,\ V_{2n-2} = pr^2; \tag{54}$$

$$U_{3\lambda} = (2r_1)^2,\ V_{3\lambda} = 2pr_2^2,\ V_{2n-2} = pr^2; \tag{55}$$

$$U_{3\lambda} = 2pr_1^2,\ V_{3\lambda} = (2r_2)^2,\ V_{2n-2} = pr^2; \tag{56}$$

$$U_{3\lambda} = 2r_1^2,\ V_{3\lambda} = 4pr_2^2,\ V_{2n-2} = pr^2. \tag{57}$$

By (29), the only possible solutions of (54) are $\lambda = 0$ for each d, and $\lambda = \pm 2$ if $d = 5$. We know $\lambda = 0$ is a solution, since $U_0 = 0 = 4pr_1^2$ with $r_1 = 0$ and $V_{-2} = pr^2 = V_2 r^2$ with $r = \pm 1$.

Since $F_{\pm 6} = \pm 8 \neq 4 \cdot 3 \cdot r_1^2$, $\lambda = \pm 2$ is not a solution of (54). By (30), the only possible solutions of (55) are $\lambda = 0$, and $\lambda = 4$ if $d = 5$. It is obvious that $\lambda = 0$ is not a solution, since $V_0 = 2 \neq 2 \cdot V_2^2$. Neither is $\lambda = 4$ a solution, since $L_{12} = 322 \neq 2 \cdot 3 \cdot r_2^2$. In the same way, we can prove that (56) and (57) have no solutions. The possible values $\lambda = \pm 1$ in (57) do not yield a solution, since $p = a^2 + 2 \nmid a(a^2 + 3) = V_{\pm 3}$.

Case 2. Let $m = 4n$. By (37), $U_{2n-1}V_{2n+1} = 2z^2$. Using Lemma 3 and the assumption, $(U_{2n-1}, V_{2n+1}) = 1$ or p.

If $(U_{2n-1}, V_{2n+1}) = 1$, we have

$$U_{2n-1} = 2t^2,\ V_{2n+1} = r^2 \tag{58}$$

or

$$U_{2n-1} = t^2,\ V_{2n+1} = 2r^2. \tag{59}$$

By (31) and (28), (58) has no solution. By (29), (59) has no solution.

If $(U_{2n-1}, V_{2n+1}) = p$, we have

$$U_{2n-1} = 2pz_1^2,\ V_{2n+1} = pz_2^2 \tag{60}$$

or

$$U_{2n-1} = pz_1^2,\ V_{2n+1} = 2pz_2^2. \tag{61}$$

Neither (60) nor (61) has a solution by using a proof similar to that given at the end of Theorem 8.

The following are immediate consequences of the preceding theorems.

Corollary 13: If $d = a^2 + 4$, $a \equiv 1(2)$, then the equation $x^2 = 4dz^4 - 4dz^2 + a^2$ has only the solution $(x, z) = (\pm a, 0)$.

Corollary 14: Let $d = a^2 + 4$ and $a^2 + 2 = p$, where p is a prime. Then, the equation $x^2 = 4dz^4 - 4adz^2 + (a^2 + 2)^2$ has

(a) two solutions, $(x, z) = (\pm(a^2 + 2), 0)$, $(\pm(a^2 + 2), \pm r)$ if a is a perfect square, $a = r^2$,

(b) only the one solution $(x, z) = (\pm(a^2 + 2), 0)$ in all other cases.

Corollary 15: $F_m = 2z^2 - 1$ iff $m = \pm 1, \pm 2$.

5. GENERALIZED LUCAS NUMBERS OF THE FORM $\mu z^2 \pm \nu$

Theorem 11: The equation $V_m = z^2 + a$, $m \equiv 1(2)$, has only the solution $m = 1$.

Proof:

Case 1. Let $m = 4n - 1$. By (42), $V_{2n-1}V_{2n} = z^2$. Since $(V_{2n-1}, V_{2n}) = 1$, we have $V_{2n-1} = t^2$, $V_{2n} = r^2$, which is impossible by (28).

Case 2. Let $m = 4n + 1$. By (42), $V_{2n}V_{2n+1} - 2a = z^2$. Hence, using (8) and (42), we have

$$\{V_n^2 - 2(-1)^n\}\{V_nV_{n+1} - (-1)^n a\} - 2a = z^2,$$

which implies that $V_n M_n = z^2$ with $M_n = V_n^2 V_{n+1} - (-1)^n aV_n - 2(-1)^n V_{n+1}$. Let p be an odd prime and let $p^e \| V_n$. Since $(V_{n+1}, V_n) = \cdots = (V_1, V_0) = (a, 2) = 1$, it follows that $p \nmid M_n$. This implies $e \equiv 0(2)$ and therefore $V_n = t^2$ or $V_n = 2t^2$. Using (28) and (29), we find that the possible solutions are $m = 1, 5, 13, 25, -23$ if $d = 5$, $m = 1, 13$ if $d = 13$, $m = 1, 5, 25, -23$ if $d = 29$, $m = 1, 5$ if $a = t^2$ and $d \neq 5$, $m = 1$ otherwise. Obviously, $m = 1$ is a solution. For $m = 5$ and $a = t^2$, we have $(a^2 + 2)^2 + a^2 = r^2$, which is impossible because both a and $a^2 + 2$ are odd. By a direct computation of each corresponding V_m in all other cases, we see that no other solutions exist. Note that for $d = 29$,

$$V_{25} = 766628450142675125.$$

Following an argument similar to Theorem 11, we can prove Theorem 12.

Theorem 12: The equation $V_m = z^2 - a$, $m \equiv 1(2)$ has only the solution $m = -1$.

Corollary 16: If $b = 1$, then the diophantine equations

$$dy^2 = z^4 + 2az^2 + a^2 + 4 \quad \text{and} \quad dy^2 = z^4 - 2az^2 + a^2 + 4$$

have only the solution $(y, z) = (\pm 1, 0)$.

The next two theorems are similar to the last two, but m is even.

Theorem 13: Let p be an odd prime. Then, the equation $V_m = z^2 + (p - 2)$, $m \equiv 0(2)$ has

(a) the solution $m = 0$ if $p = 3$,

(b) the solutions $m = \pm 2, \pm 4$ if $d = 5$ and $p = 5$,

(c) at most $\prod_{i=1}^{r} (s_i + 1) + 1$ solutions if

$$p - 4 = q_1^{s_1} \cdot q_2^{s_2} \cdot \cdots \cdot q_r^{s_r}$$

as its unique factorization.

Proof:

Case 1. Let $m = 4n$. By (8), $V_{2n}^2 - z^2 = p$, which implies that

$$V_{2n} = \pm \frac{p + 1}{2} \quad \text{or} \quad V_{2n} = \frac{p + 1}{2} \quad \text{by (19).}$$

If $p = 3$, then $V_{2n} = 2$, which implies that $n = 0$ or $m = 0$ is a solution with $z = 0$. If $p = 5$, then $V_{2n} = 3$, which can only be true if $n = \pm 1$ and $d = 5$ or $m = \pm 4$ and $d = 5$. If $p > 5$, there exists at most one solution.

Case 2. Let $m = 4n + 2$. By (8), $V_{2n+1}^2 - z^2 = p - 4$. If $p = 3$, then $V_{2n+1} = 0$, which is impossible. If $p = 5$, then $V_{2n+1} = \pm 1$ and the only possibilities for solutions are $n = 0$ or -1 and $d = 5$ or $m = \pm 2$ and $d = 5$. If $p > 5$, then

$$V_{2n+1} = \pm \frac{d_1 + d_2}{2}, \; d_1 > 0, \; d_2 > 0,$$

where (d_1, d_2) runs over all the divisors of $p - 4$ with $d_1 d_2 = p - 4$. Since the number of divisors of $p - 4$ is $\prod_{i=1}^{r}(s_i + 1)$, the theorem is proved.

In the same way, we can prove

Theorem 14: Let p be an odd prime. Then, the equation $V_m = z^2 - (p - 2)$, $m \equiv 0(2)$, has

(a) the solutions $m = \pm 2$, $d = 5$, if $p = 3$,

(b) no solution if $p = 5$,

(c) at most $\begin{cases} \frac{1}{2}\left[\prod_{i=1}^{r}(s_i + 1) - 1\right] + 2 \text{ solutions if } p - 4 \text{ is a perfect square} \\ \frac{1}{2}\prod_{i=1}^{r}(s_i + 1) + 2 \text{ solutions if } p - 4 \text{ is not a perfect square,} \end{cases}$

where $p - 4 = q_1^{s_1}q_2^{s_2} \ldots q_r^{s_r}$ as its unique factorization.

Corollary 17:

(i) The diophantine equation $z^4 + 2(p - 2)z^2 + p(p - 4) = dy^2$ has

(a) one solution for each d if $p = 3$,

(b) four solutions for $d = 5$ if $p = 5$,

(c) at most $\prod_{i=1}^{r}(s_i + 1) + 1$ solutions if $p > 5$ and $p - 4 = q_1^{s_1} \ldots q_r^{s_r}$ as its unique factorization.

(ii) The diophantine equation $z^4 - 2(p - 2)z^2 + p(p - 4) = dy^2$ has

(a) one solution for each d is $p = 3$,

(b) no solution for each d if $p = 5$,

(c) at most $\begin{cases} \frac{1}{2}\left[\prod_{i=1}^{r}(s_i + 1) - 1\right] + 2 \text{ solutions if } p - 4 \text{ is a perfect square} \\ \frac{1}{2}\prod_{i=1}^{r}(s_i + 1) + 2 \text{ solutions if } p - 4 \text{ is not a perfect square,} \end{cases}$

where $p > 5$ and $p - 4 = q_1^{s_1} \ldots q_r^{s_r}$ as its unique factorization.

Corollary 18: The following can be found in [4] and [8]:

$L_m = z^2 + 1$ iff $m = 0, 1$,

$L_m = z^2 - 1$ iff $m = -1, \pm 2$.

By an argument similar to Theorems 11 and 12, we can prove

Theorem 15:

(i) The equation $V_m = 2z^2 + a$, $m \equiv 1(2)$, has only the solution $m = 1$.

(ii) The equation $V_m = 2z^2 - a$, $m \equiv 1(2)$, has

(a) the solutions $m = \pm 1$ is a is a perfect square,

(b) only the solution $m = -1$ in all other cases.

By using the method of Cohn, as before, we can also prove

Theorem 16: $L_m = 2z^2 + 1$, $m \equiv 0(2)$, iff $m = \pm 2$,
$L_m = 2z^2 - 1$, $m \equiv 0(2)$, iff $m = \pm 4$.

Corollary 19: $L_m = 2z^2 + 1$ iff $m = \pm 2$, 1,
$L_m = 2z^2 - 1$ iff $m = \pm 1$, ± 4.

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ZIGZAG POLYNOMIALS

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1. INTRODUCTION

The object of this paper is to obtain some basic properties of certain polynomials which we choose to call *zigzag* polynomials. These arise in a specified way from the diagonal terms of the Pascal-type array of polynomials generated by a given second-order recurrence relation.

Consider the sequence of *generalized Pell polynomials* $\{A_n(x)\}$ defined by

$$A_n(x) = 2xA_{n-1}(x) + A_{n-2}(x),\; A_0(x) = q,\; A_1(x) = p \quad (n \geqslant 2). \tag{1.1}$$

Special cases of $A_n(x)$ which will concern us are:

the *Pell polynomials* $P_n(x)$ occurring when $p = 1$, $q = 0$, (1.2)

the *Pell-Lucas polynomials* $Q_n(x)$ occurring when $p = 2x$, $q = 2$. (1.3)

The explicit *Binet form* for $A_n(x)$ is given in [4], namely,

$$A_n(x) = \frac{(p - q\beta)\alpha^n - (p - q\alpha)\beta^n}{\alpha - \beta}, \tag{1.4}$$

where α, β are the roots of $y^2 - 2xy - 1 = 0$ ($\alpha = x + \sqrt{x^2 + 1}$, $\beta = x - \sqrt{x^2 + 1}$). From (1.4), the Binet forms of $P_n(x)$ and $Q_n(x)$ are readily derived using (1.2) and (1.3).

The generating function for $\{A_n(x)\}$ is

$$\sum_{n=0}^{\infty} A_{n+1}(x)t^n = (p + qt)[1 - (2xt + t^2)]^{-1}. \tag{1.5}$$

Generating functions for $P_n(x)$ and $Q_n(x)$ are then, from (1.2), (1.3), and (1.5),

$$\sum_{n=0}^{\infty} P_{n+1}(x)t^n = [1 - (2xt + t^2)]^{-1} \tag{1.6}$$

and

$$\sum_{n=0}^{\infty} Q_{n+1}(x)t^n = (2x + 2t)[1 - (2xt + t^2)]^{-1}, \tag{1.7}$$

as given in [3].

Results (1.4)-(1.7) will not be used in this paper. Nevertheless, we append them here for reasons of completeness and comparison.

Though it will not interest us for the purpose of this paper, the curious reader may wish to investigate the special, simple case of (1.1) arising from the values $p = 1$, $q = 1$.

Background information for the theory about to be developed is to be found in [1] and [2].

2. ZIGZAG RISING DIAGONAL POLYNOMIALS

From (1.1), we form the Pascal-type array (Table 1).

$$\begin{cases} A_0(x) = q \\ A_1(x) = p \\ A_2(x) = 2px + q \\ A_3(x) = 4px^2 + 2qx + p \\ A_4(x) = 8px^3 + 4qx^2 + 4px + q \\ A_5(x) = 16px^4 + 8qx^3 + 12px^2 + 4qx + p \\ A_6(x) = 32px^5 + 16qx^4 + 32px^3 + 12qx^2 + 6px + q \\ A_7(x) = 64px^6 + 32qx^5 + 80px^4 + 32qx^3 + 24px^2 + 6qx + p \\ A_8(x) = 128px^7 + 64qx^6 + 192px^5 + 80qx^4 + 80px^3 + 24qx^2 + 8px + q \\ A_9(x) = 256px^8 + 128qx^7 + 448px^6 + 192qx^5 + 240px^4 + 80qx^3 + 40px^2 + 8qx + p \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \tag{2.1}$$

Table 1. Zigzag Rising Diagonal Polynomials of $\{A_n(x)\}$

Let us agree to call the polynomials in Table 1 that arise upward in step-like formation from the left (indicated by lines) the *zigzag polynomials* (or *echelon polynomials*) associated with $\{A_n(x)\}$. At each level in the step-like formation, other than the first, the terms are paired in the second and third columns, the fourth and fifth columns, ..., where this is appropriate.

As will be evident in the next section, the value of this pairing technique is that specializations can be quickly visualized and obtained from the general pattern, e.g., by the disappearance of the first column of a pair when $p = 1$, $q = 0$ (the Pell polynomials), and by the amalgamation of corresponding elements in a pair of columns when $p = 2x$, $q = 2$, i.e., $p = qx$ (the Pell-Lucas polynomials).

Designate the zigzag polynomials by $Z_n(x)$. Start with $Z_0(x) = q$. Then, the first few zigzag polynomials are, from (2.1):

$$\begin{cases} Z_0(x) = q,\ Z_1(x) = p,\ Z_2(x) = 2px,\ Z_3(x) = 4px^2 + q, \\ Z_4(x) = 8px^3 + 2qx + p,\ Z_5(x) = 16px^4 + 4qx^2 + 4px, \\ Z_6(x) = 32px^5 + 8qx^3 + 12px^2 + q,\ Z_7(x) = 64px^6 + 16qx^4 + 32px^3 + 4qx + p, \\ Z_8(x) = 128px^7 + 32qx^5 + 80px^4 + 12qx^2 + 6px,\ \ldots \end{cases} \tag{2.2}$$

Using (1.1) and the nature of the formation of the $Z_n(x)$, we observe that

$$Z_n(x) = 2xZ_{n-1}(x) + Z_{n-3}(x). \tag{2.3}$$

Elementary methods applied to (2.3) produce the generating function for $Z_n(x)$, namely (when $n > 0$),

$$\sum_{n=1}^{\infty} Z_n(x)t^{n-1} = (p + qt^2)[1 - (2xt + t^3)]^{-1} \equiv Z(x, t). \tag{2.4}$$

Explicit formulation of an expression for $Z\ (x)$ can be obtained by comparison of coefficients of t in (2.4). Computation yields

$$Z_n(x) = p\sum_{i=0}^{\left[\frac{n-1}{3}\right]}\binom{n-1-2i}{i}(2x)^{n-1-3i} + q\sum_{i=0}^{\left[\frac{n-3}{3}\right]}\binom{n-3-2i}{i}(2x)^{n-3-3i}, \quad n \geqslant 3, \tag{2.5}$$

where $[n/3]$ is the integral part of $n/3$.

Certain differential equations are satisfied by the zigzag polynomials. These include the partial differential equation

$$2t\frac{\partial}{\partial t}Z(x,\ t) - (2x + 3t^2)\frac{\partial}{\partial x}Z(x,\ t) = 4qt^2[1 - (2xt + t^3)]^{-1} \tag{2.6}$$

and the ordinary differential equation

$$2x\frac{d}{dx}Z_{n+2}(x) + 3\frac{d}{dx}Z_n(x) = 2(n+1)Z_{n+2}(x) - 4qR_n(x), \tag{2.7}$$

where $R_n(x)$ is to be defined in the next section.

In deriving the results (2.5), (2.6), and (2.7), we have been guided by similar specialized results established in [2] for the rising diagonal polynomials $R_n(x)$ and $r_n(x)$. To these polynomials we now turn our attention.

3. SPECIALIZATIONS

Using (1.1), (1.2), and (1.3), we form Tables 2 and 3 for the polynomial sequences $\{P_n(x)\}$ and $\{Q_n(x)\}$:

$$R_0(x),\ R_1(x),\ R_2(x),\ R_3(x),\ R_4(x),\ R_5(x),\ R_6(x),\ R_7(x),\ R_8(x)$$

$$\begin{cases} P_0(x) = 0 \\ P_1(x) = 1 \\ P_2(x) = 2x \\ P_3(x) = 4x^2 + 1 \\ P_4(x) = 8x^3 + 4x \\ P_5(x) = 16x^4 + 12x^2 + 1 \\ P_6(x) = 32x^5 + 32x^3 + 6x \\ P_7(x) = 64x^6 + 80x^4 + 24x^2 + 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases} \tag{3.1}$$

Table 2. Rising Diagonal Polynomials of $\{P_n(x)\}$

Tables 2 and 3, it may be noted, are special cases of arrays given in [2]. Allowing for the necessary change of notation from [2] to this paper, denote the rising diagonal polynomials in Tables 2 and 3 by $R_n(x)$ and $r_n(x)$, respectively, commencing with $R_0(x) = 0$, $r_0(x) = 2$.

$$
\begin{cases}
 & r_0(x) \\
Q_0(x) = 2 & r_1(x) \\
Q_1(x) = 2x & r_2(x) \quad r_3(x) \\
Q_2(x) = 4x^2 + 2 & r_4(x) \\
Q_3(x) = 8x^3 + 6x & r_5(x) \quad r_6(x) \\
Q_4(x) = 16x^4 + 16x^2 + 2 & r_7(x) \\
Q_5(x) = 32x^5 + 40x^3 + 10x & \\
Q_6(x) = 64x^6 + 96x^4 + 36x^2 + 2 & \\
Q_7(x) = 128x^7 + 224x^5 + 112x^3 + 14x & \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &
\end{cases}
\tag{3.2}
$$

Table 3. Rising Diagonal Polynomials of $\{Q_n(x)\}$

Observe the relationships (cf. [2]), subject to the restriction $n \geqslant 3$,

$$
\begin{cases}
R_n(x) = 2xR_{n-1}(x) + R_{n-3}(x) \\
r_n(x) = 2xr_{n-1}(x) + r_{n-3}(x) \\
r_{n-1}(x) = R_n(x) + R_{n-3}(x).
\end{cases}
\tag{3.3}
$$

The formal structural equivalence of (2.3) and the first two equations in (3.3) is, of course, expected and essential.

Substituting the appropriate values from (1.2) and (1.3) in (2.5), we derive the explicit forms

$$R_n(x) = \sum_{i=0}^{\left[\frac{n-1}{3}\right]} \binom{n-1-2i}{i}(2x)^{n-1-3i}, \quad n \geqslant 1, \tag{3.4}$$

and

$$r_n(x) = \sum_{i=0}^{\left[\frac{n-1}{3}\right]} \binom{n-1-2i}{i}(2x)^{n-3i} + 2\sum_{i=0}^{\left[\frac{n-3}{3}\right]} \binom{n-3-2i}{i}(2x)^{n-3-3i}, \quad n \geqslant 3. \tag{3.5}$$

Generating functions are, from (1.2), (1.3), and (2.4), when $n > 0$,

$$\sum_{n=1}^{\infty} R_n(x)t^{n-1} = [1 - (2xt + t^3)]^{-1} \equiv R(x, t) \tag{3.6}$$

and

$$\sum_{n=1}^{\infty} r_n(x)t^{n-1} = 2(x + t^2)[1 - (2xt + t^3)]^{-1} \equiv r(x, t). \tag{3.7}$$

Furthermore, on applying (1.2) to (2.6) and (2.7) in succession, we deduce that

$$2t\,\frac{\partial R}{\partial t}(x, t) - (2x + 3t^2)\,\frac{\partial R}{\partial x}(x, t) = 0$$

and

$$2x \frac{d}{dx} R_{n+2}(x) + 3 \frac{d}{dx} R_n(x) = 2(n + 1)R_{n+2}(x).$$

But we cannot apply (1.3) to (2.6) and (2.7) because, in (2.6) and (2.7), p and q were implicitly assumed to be constants, whereas in (1.3), $p = 2x$ and $q = 2$, i.e., p is a function of x.

Guided by the appropriate results in [2] and carrying out the processes of differentiation, *mutatis mutandis*, we arrive at the differential equations

$$2t \frac{\partial}{\partial t} r(x, t) - (2x + 3t^2) \frac{\partial}{\partial x} r(x, t) = r(x, t) - 6xR(x, t) \tag{3.10}$$

and

$$2x \frac{d}{dx} r_{n+2}(x) + 3 \frac{d}{dx} r_n(x) = 2(n - 1)r_{n+2}(x) + 6R_{n+3}(x), \tag{3.11}$$

which should be compared with the corresponding results in [2].

Equations (3.3)-(3.9) occur in [2], slightly modified where necessary to take into account the minor differences in notation in [2] and in this paper.

In passing, it might be observed that a marginally neater form of (3.7) exists if the summation is allowed to commence with $n = 2$, instead of with $n = 1$ in conformity with (2.4). [Had our summation in (2.4) begun with $n = 0$, we would have obtained a slightly less simple form of the generating function than that given in (2.4).]

While there may be other mathematically interesting instances of $\{A_n(x)\}$, we have limited our attention to the two well-known and related sequences $\{P_n(x)\}$ and $\{Q_n(x)\}$. Properties of $\{A_n(x)\}$ are an amalgam of their separate properties.

4. ORDINARY (NON-ZIGZAG) RISING DIAGONAL POLYNOMIALS

Consider next the ordinary (non-zigzag) rising diagonal polynomials in Table 1, which must not be confused with the $Z_n(x)$.

Denote these non-zigzag polynomials by the suggestive notation $\not{Z}_n(x)$, beginning with $\not{Z}_0(x) = q$.

Some of these polynomials are:

$$\begin{cases} \not{Z}_0(x) = q,\ \not{Z}_1(x) = p,\ \not{Z}_2(x) = 2px,\ \not{Z}_3(x) = 4px^2 + q, \\ \not{Z}_4(x) = 8px^3 + 2qx,\ \not{Z}_5(x) = 16px^4 + 4qx^2 + p, \\ \not{Z}_6(x) = 32px^5 + 8qx^3 + 4px,\ \not{Z}_7(x) = 64px^6 + 16qx^4 + 12px^2 + q, \\ \not{Z}_8(x) = 128px^7 + 32qx^5 + 32px^3 + 4qx,\ \ldots \end{cases} \tag{4.1}$$

Observe that the recurrence relation for $\{\not{Z}_n(x)\}$ is

$$\not{Z}_n(x) = 2x\not{Z}_{n-1}(x) + \not{Z}_{n-4}(x). \tag{4.2}$$

Using elementary procedures, we may demonstrate that the (somewhat ungainly) generating function for $\not{Z}_n(x)$ is

$$\sum_{n=0}^{\infty} \not{Z}_n(x) t^n = \{q + (p - 2qx)t + qt^3\}[1 - (2xt + t^4)]^{-1}. \tag{4.3}$$

An explicit expression for the elements of $\{\not{Z}\ (x)\}$ may be established, namely,

$$\bar{Z}_n(x) = p\sum_{i=0}^{\left[\frac{n-3}{3}\right]}\binom{n-1-3i}{i}(2x)^{n-1-4i} + q\sum_{i=0}^{\left[\frac{n-5}{3}\right]}\binom{n-3-3i}{i}(2x)^{n-3-4i}, \quad n \geqslant 5. \tag{4.4}$$

Finally, we emphasize that the rising diagonals $R_n(x)$ and $r_n(x)$ for $\{P_n(x)\}$ and $\{Q_n(x)\}$ in (3.1) and (3.2) are special cases of $Z_n(x)$, not $\bar{Z}_n(x)$, as a little thought reveals.

5. ZIGZAG DESCENDING DIAGONAL POLYNOMIALS

Just as the rising zigzag diagonal polynomials are constructed from Table 1, so the corresponding zigzag polynomials for descending diagonals may be generated, i.e., by proceeding downward in step-like fashion from the left.

To avoid repetitious waste of space, we invite the reader to refer to Table 1 and to compose the following list of descending diagonal *zigzag polynomials* (or echelon polynomials) $z_n(x)$, with initial value $z_0(x) = q$:

$$\begin{cases} z_0(x) = q, \ z_1(x) = p + q, \ z_2(x) = (p + q)(2x + 1), \\ z_3(x) = (p + q)(2x + 1)^2, \ z_4(x) = (p + q)(2x + 1)^3, \\ z_5(x) = (p + q)(2x + 1)^4, \ z_6(x) = (p + q)(2x + 1)^5, \ \ldots \end{cases} \tag{5.1}$$

The pattern is crystal clear. One does not have to be psychic to deduce immediately the recurrence relation from the geometric progression, namely,

$$z_{n+1}(x) = (2x + 1)z_n(x), \ n \geqslant 1, \tag{5.2}$$

with general term

$$z_n(x) = (p + q)(2x + 1)^{n-1}, \ n \geqslant 1. \tag{5.3}$$

The generating function for $z_n(x)$ (if $n > 0$) is obviously

$$z(x, t) \equiv \sum_{n=1}^{\infty} z_n(x)t^{n-1} = (p + q)[1 - (2x + 1)t]^{-1}. \tag{5.4}$$

Mathematical calculations involving $z_n(x)$ will be manifestly simpler than those associated with $Z_n(x)$. In particular, the following differential equations flow easily from (5.3) and (5.4):

$$2t\frac{\partial}{\partial t} z(x, t) - (2x + 1)\frac{\partial}{\partial x} z(x, t) = 0 \tag{5.5}$$

$$(2x + 1)\frac{d}{dx} z_n(x) - 2(n - 1)z_n(x) = 0. \tag{5.6}$$

Specializations of (5.3)-(5.6) for $\{P_n(x)\}$ and $\{Q_n(x)\}$ are readily obtained. Thus, for the descending diagonal polynomials $D_n(x)$ of the Pell polynomial array in Table 2, with initial conditions $D_0(x) = 0$ and $D_1(x) = 1$, we derive

$$D_n(x) = (2x + 1)^{n-1}, \ n \geqslant 1, \tag{5.7}$$

$$D(x, t) \equiv \sum_{n=1}^{\infty} D_n(x)t^{n-1} = [1 - (2x + 1)t]^{-1}, \tag{5.8}$$

$$2t\frac{\partial}{\partial t} D(x, t) - (2x + 1)\frac{\partial}{\partial x} D(x, t) = 0, \tag{5.9}$$

$$(2x + 1)\frac{d}{dx} D_n(x) - 2(n - 1)D_n(x) = 0, \tag{5.10}$$

while, for the descending diagonal polynomials $d_n(x)$ of the Pell-Lucas polynomial array in Table 3, we deduce

$$d_n(x) = 2(x + 1)(2x + 1)^{n-1}, \; n \geqslant 1, \tag{5.11}$$

$$d(x, t) \equiv \sum_{n=1}^{\infty} d_n(x)t^{n-1} = 2(x + 1)[1 - (2x + 1)t]^{-1}. \tag{5.12}$$

Initially, $d_0(x) = 2$.
Observe that

$$d_n(x) = D_n(x) + D_{n+1}(x). \tag{5.13}$$

Equations (5.5) and (5.6) cannot be applied directly to $d_n(x)$ since, in this case, $p = 2x$ is not a constant (although $q = 2$ is). However, the results for $d(x, t)$ and $d_n(x)$ corresponding to those for $D(x, t)$ and $D_n(x)$ in (5.9) and (5.10), respectively, may be established without too much difficulty if we permit ourselves to be assisted by similar results in [2]. They are:

$$2t\frac{\partial}{\partial t}d(x, t) - (2x + 1)\left[\frac{\partial}{\partial x}d(x, t) - 2D(x, t)\right] = 0 \tag{5.14}$$

$$2(x + 1)\frac{d}{dx}(d_{n+1}(x)) - 2d_{n+1}(x) - 8n(x + 1)^2D_n(x) = 0. \tag{5.15}$$

The above specializations should be compared with analogous derivations in [2], modified as demanded by the circumstances. Variations that occur between a result in [2] and a corresponding result in this paper exist because of the different starting points, i.e., different values of $d_1(x)$.

Earlier results obtained in [1] relating to material in this paper might also be consulted.

6. CONCLUDING COMMENTS

This completes what we wished to say about the zigzag polynomials at this stage. Various generalizations of aspects of this paper suggest themselves, but, as we believe these developments go beyond the unity of this paper, they are left for possible further consideration.

Finally, it might be observed that results (2.3), (3.3), (4.2), (5.2), (5.7) and (5.11) are readily established by using the rule of formation and the generating functions for the columns of the respective arrays. In Table 1, for instance, the generating functions for the first, second, third, ..., pair of columns are $(1 - 2x)^{-1}$, $(1 - 2x)^{-2}$, $(1 - 2x)^{-3}$, ..., with appropriate multipliers p and q.

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ON A FIBONACCI ARITHMETICAL TRICK

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1. INTRODUCTION

A standard arithmetical trick for school children is to ask them to choose two positive integers, to extend this to a sequence of 10 numbers by adding any two to obtain the next in the Fibonacci manner, and then to add up the numbers in the sequence. When the exercise is complete the teacher, having unobtrusively noted the seventh number in each student's sequence while checking around the room to see that each is proceeding properly, can mystify the students by announcing the sum each has achieved. Given that the students did the arithmetic correctly, the sum is just 11 times the seventh number in their original sequence. If, for example, a student chooses 5 and 1, his sequence is

5, 1, 6, 7, 13, 20, 33, 53, 86, 139

and the sum is $363 = 11 \cdot 33$.

Of course, as the reader will expect, this is just a special case of more general results which we now examine.

2. SOME GENERAL RESULTS

Let F_n and L_n denote, respectively, the n^{th} Fibonacci and Lucas numbers so that

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1} \text{ for } n \geqslant 1,$$

and

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1} \text{ for } n \geqslant 1.$$

Also, define sequences H_n and K_n for integers a and b by

$$H_1 = a,\ H_2 = b,\ H_{n+2} = H_{n+1} + H_n \text{ for } n \geqslant 1,$$

and

$$K_1 = -a + 2b,\ K_2 = 2a + b,\ K_{n+2} = K_{n+1} + K_n \text{ for } n \geqslant 1.$$

Then the following theorem holds.

Theorem 1: For $n \geqslant 1$,

$$\text{(i)}\quad \sum_{i=1}^{4n-2} H_i = L_{2n-1}H_{2n+1}, \qquad \sum_{i=1}^{4n} H_i = F_{2n}K_{2n+2},$$

$$\text{(ii)}\quad \sum_{i=1}^{4n-2} K_i = L_{2n-1}K_{2n+1}, \qquad \sum_{i=1}^{4n} K_i = 5F_{2n}H_{2n+2}.$$

The arithmetical trick described above derives from the first formula of part (i) of the theorem with $n = 3$. For $n = 4$, it would say that the sum of the first 14 integers in the sequence is divisible by the ninth number in the sequence, and so on.

The proof of Theorem 1 depends on the following well-known results which we state for completeness.

Lemma 1: For $n \geqslant 1$,

$$H_n = aF_{n-2} + bF_{n-1} \quad \text{and} \quad K_n = aL_{n-2} + bL_{n-1}.$$

Lemma 2: For $n \geqslant 1$,

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \quad \text{and} \quad \sum_{i=1}^{n} L_i = L_{n+2} - 3.$$

Lemma 3: For integers r and s,

$$\text{(i)} \quad F_{r+2s} - F_r = \begin{cases} F_s L_{r+s} & s \text{ even,} \\ L_s F_{r+s} & s \text{ odd,} \end{cases}$$

$$\text{(ii)} \quad L_{r+2s} - L_r = \begin{cases} 5F_s F_{r+s} & s \text{ even,} \\ L_s L_{r+s} & s \text{ odd,} \end{cases}$$

$$\text{(iii)} \quad F_{r+2s} + F_r = \begin{cases} L_s F_{r+s} & s \text{ even,} \\ F_s L_{r+s} & s \text{ odd,} \end{cases}$$

$$\text{(iv)} \quad L_{r+2s} + L_r = \begin{cases} L_s L_{r+s} & s \text{ even,} \\ 5F_s F_{r+s} & s \text{ odd.} \end{cases}$$

Note that Lemmas 1 and 2 are easily proved by induction and that Lemma 3 follows from Binet's formulas. Alternatively, Lemmas 1 and 2 follow from (7) and (6), page 456 of [2] for suitable choices of p and q, and Lemma 3 follows from (5)-(12), page 115 of [1] by setting $r = n - k$ and $s = k$. In fact, Theorem 1 can also be deduced from (6), page 456 of [2] and Lemma 3. However, for ease of reading, we give an independent proof.

Proof of Theorem 1: Since all the arguments are similar, we prove only part (iv). By Lemmas 1, 2, and 3,

$$\begin{aligned}
\sum_{i=1}^{4n} K &= \sum_{i=1}^{4n} (aL_{i-2} + bL_{i-1}) \\
&= aL_{-1} + aL_0 + a\sum_{i=3}^{4n} L_{i-2} + bL_0 + b\sum_{i=2}^{4n} L_{i-1} \\
&= -a + 2a + a(L_{4n} - 3) + 2b + b(L_{4n+1} - 3) \\
&= a(L_{4n} - L_0) + b(L_{4n+1} - L_1) \\
&= 5aF_{2n}^2 + 5bF_{2n}F_{2n+1} \\
&= 5F_{2n}(aF_{2n} + bF_{2n+1}) \\
&= 5F_{2n}H_{2n+2}
\end{aligned}$$

as claimed.

Setting $a = b = 1$, we obtain the following immediate corollary to Theorem 1.

Corollary 1: For $n \geqslant 1$,

$$\text{(i)} \quad \sum_{i=1}^{4n-2} F_i = L_{2n-1}F_{2n+1}, \qquad \sum_{i=1}^{4n} F_i = F_{2n}L_{2n+2},$$

$$\text{(ii)} \quad \sum_{i=1}^{4n-2} L_i = L_{2n-1}L_{2n+1}, \qquad \sum_{i=1}^{4n} L_i = 5F_{2n}F_{2n+2}.$$

Now Lemma 1 and Theorem 1 suggest a further generalization. Define the sequences P, Q, R, S, T, U, V, and W for $n \geqslant 1$ by

$$\begin{aligned} P_n &= aF_{n-2} + bL_{n-1}, & Q_n &= aL_{n-2} + bF_{n-1}, \\ R_n &= aL_{n-2} + 5bF_{n-1}, & S_n &= 5aL_{n-2} + bF_{n-1}, \\ T_n &= aF_{n-2} + 5bL_{n-1}, & U_n &= 5aF_{n-2} + bL_{n-1}, \\ V_n &= aL_{n-2} + 5^2bF_{n-1}, & W_n &= 5^2aF_{n-2} + bL_{n-1}. \end{aligned}$$

Then the following results hold.

Theorem 2: For $n \geqslant 1$,

$$\text{(i)} \quad \sum_{i=1}^{4n-2} P_i = L_{2n-1}P_{2n+1}, \qquad \sum_{i=1}^{4n} P_i = F_{2n}R_{2n+2},$$

$$\text{(ii)} \quad \sum_{i=1}^{4n-2} Q_i = L_{2n-1}Q_{2n+1}, \qquad \sum_{i=1}^{4n} Q_i = F_{2n}U_{2n+2},$$

$$\text{(iii)} \quad \sum_{i=1}^{4n-2} R_i = L_{2n-1}R_{2n+1}, \qquad \sum_{i=1}^{4n} R_i = 5F_{2n}P_{2n+2},$$

$$\text{(iv)} \quad \sum_{i=1}^{4n-2} S_i = L_{2n-1}S_{2n+1}, \qquad \sum_{i=1}^{4n} S_i = F_{2n}W_{2n+2},$$

$$\text{(v)} \quad \sum_{i=1}^{4n-2} T_i = L_{2n-1}T_{2n+1}, \qquad \sum_{i=1}^{4n} T_i = F_{2n}V_{2n+2},$$

$$\text{(vi)} \quad \sum_{i=1}^{4n-2} U_i = L_{2n-1}U_{2n+1}, \qquad \sum_{i=1}^{4n} U_i = 5F_{2n}Q_{2n+2},$$

$$\text{(vii)} \quad \sum_{i=1}^{4n-2} V_i = L_{2n-1}V_{2n+1}, \qquad \sum_{i=1}^{4n} V_i = 5F_{2n}T_{2n+2},$$

$$\text{(viii)} \quad \sum_{i=1}^{4n-2} W_i = L_{2n-1}W_{2n+1}, \qquad \sum_{i=1}^{4n} W_i = 5F_{2n}S_{2n+2}.$$

We omit the proof, since it is similar to that of Theorem 1.

3. MORE GENERAL RESULTS

We may generalize the results of Section 2 as follows. Define the sequences $\{f_n\}_{n\geqslant 0} = \{f_n(x)\}_{n\geqslant 0}$ and $\{\ell_n\}_{n\geqslant 0} = \{\ell_n(x)\}_{n\geqslant 0}$ by

$$f_0 = 0,\ f_1 = 1,\ f_{n+1} = af_n + f_{n-1}$$
and
$$\ell_0 = 2,\ \ell_1 = a,\ \ell_{n+1} = a\ell_n + \ell_{n-1},$$

where $a = a(x)$ is an arbitrary function of x. Then it is easily shown, as with the Fibonacci and Lucas sequences, that

$$f_n = \frac{\rho^n - \sigma^n}{\sqrt{a^2 + 4}} \tag{1}$$

and

$$\ell_n = \rho^n + \sigma^n \tag{2}$$

for all n where

$$\rho = \frac{a + \sqrt{a^2 + 4}}{2} \quad \text{and} \quad \sigma = \frac{a - \sqrt{a^2 + 4}}{2}.$$

Also,

$$\sum_{i=1}^{n} f_i = \frac{f_{n+1} + f_n - 1}{a}, \tag{3}$$

$$\sum_{i=1}^{n} \ell_i = \frac{\ell_{n+1} + \ell_n - a - 2}{a}, \tag{4}$$

$$f_{-1} = 1 \quad \text{and} \quad \ell_{-1} = -a. \tag{5}$$

In addition, we have the following generalization of Lemma 3.

Lemma 4: For integers r and s,

$$\text{(i)} \quad f_{r+2s} - f_r = \begin{cases} f_s \ell_{r+s} & s \text{ even,} \\ \ell_s f_{r+s} & s \text{ odd,} \end{cases}$$

$$\text{(ii)} \quad \ell_{r+2s} - \ell_r = \begin{cases} (a^2 + 4) f_s f_{r+s} & s \text{ even,} \\ \ell_s \ell_{r+s} & s \text{ odd,} \end{cases}$$

$$\text{(iii)} \quad f_{r+2s} + f_r = \begin{cases} \ell_s f_{r+s} & s \text{ even,} \\ f_s \ell_{r+s} & s \text{ odd,} \end{cases}$$

$$\text{(iv)} \quad \ell_{r+2s} + \ell_r = \begin{cases} \ell_s \ell_{r+s} & s \text{ even,} \\ (a^2 + 4) f_s f_{r+s} & s \text{ odd.} \end{cases}$$

Equations (1), (2), (3), and (4) can all be proved by induction, and Lemma 4 follows as before from the Binet formulas (1) and (2). Alternatively, (1) and (2) are essentially special cases of (53) and (54), page 119 of [1] and Lemma 4 is, in the same sense, a special case of (56)-(63) of [1].

If we now define the sequences h and k by

$$h_1 = c,\ h_2 = d,\ h_{n+1} = ah_n + h_{n-1} \tag{6}$$

and

$$k_1 = -ac + 2d,\ k_2 = ad + 2c,\ k_{n+1} = ak_n + k_{n-1}, \tag{7}$$

where $c = c(x)$ and $d = d(x)$ are also arbitrary functions of x, then it can be shown by induction that

$$h_n = cf_{n-2} + df_{n-1} \tag{8}$$

and

$$k_n = c\ell_{n-2} + d\ell_{n-1} \tag{9}$$

for all n. Finally, by analogy with Section 2, we define the sequences p, q, r, s, t, u, v, and w by

$$\begin{aligned}
p_n &= cf_{n-2} + d\ell_{n-1}\\
q_n &= c\ell_{n-2} + df_{n-1}\\
r_n &= c\ell_{n-2} + (a^2 + 4)df_{n-1}\\
s_n &= (a^2 + 4)c\ell_{n-2} + df_{n-1}\\
t_n &= cf_{n-2} + (a^2 + 4)d\ell_{n-1}\\
u_n &= (a^2 + 4)cf_{n-2} + d\ell_{n-1}\\
v_n &= c\ell_{n-2} + (a^2 + 4)^2df_{n-1}\\
w_n &= (a^2 + 4)^2cf_{n-2} + d\ell_{n-1}
\end{aligned}$$

for all n. Then, as before, we have the following result that generalizes both Theorem 1 and Theorem 2.

<u>Theorem 3</u>: For $n \geqslant 1$,

(i) $$\sum_{i=1}^{4n-2} h_i = \frac{\ell_{2n-1}(h_{2n-1} + h_{2n})}{a}, \qquad \sum_{i=1}^{4n} h_i = \frac{f_{2n}(k_{2n+1} + k_{2n})}{a},$$

(ii) $$\sum_{i=1}^{4n-2} k_i = \frac{\ell_{2n-1}(k_{2n-1} + k_{2n})}{a}, \qquad \sum_{i=1}^{4n} k_i = \frac{(a^2 + 4)f_{2n}(h_{2n+1} + h_{2n})}{a},$$

(iii) $$\sum_{i=1}^{4n-2} p_i = \frac{\ell_{2n-1}(p_{2n-1} + p_{2n})}{a}, \qquad \sum_{i=1}^{4n} p_i = \frac{f_{2n}(r_{2n+1} + r_{2n})}{a},$$

(iv) $$\sum_{i=1}^{4n-2} q_i = \frac{\ell_{2n-1}(q_{2n-1} + q_{2n})}{a}, \qquad \sum_{i=1}^{4n} q_i = \frac{f_{2n}(u_{2n+1} + u_{2n})}{a},$$

(v) $$\sum_{i=1}^{4n-2} r_i = \frac{\ell_{2n-1}(r_{2n-1} + r_{2n})}{a}, \qquad \sum_{i=1}^{4n} r_i = \frac{(a^2 + 4)f_{2n}(p_{2n+1} + p_{2n})}{a},$$

(vi) $$\sum_{i=1}^{4n-2} s_i = \frac{\ell_{2n-1}(s_{2n-1} + s_{2n})}{a}, \qquad \sum_{i=1}^{4n} s_i = \frac{f_{2n}(w_{2n+1} + w_{2n})}{a},$$

(vii) $$\sum_{i=1}^{4n-2} t_i = \frac{\ell_{2n-1}(t_{2n-1} + t_{2n})}{a}, \qquad \sum_{i=1}^{4n} t_i = \frac{f_{2n}(v_{2n+1} + v_{2n})}{a},$$

$$\text{(viii)} \quad \sum_{i=1}^{4n-2} u_i = \frac{\ell_{2n-1}(u_{2n-1} + u_{2n})}{a}, \qquad \sum_{i=1}^{4n} u_i = \frac{(a^2+4)f_{2n}(q_{2n+1} + q_{2n})}{a},$$

$$\text{(ix)} \quad \sum_{i=1}^{4n-2} v_i = \frac{\ell_{2n-1}(v_{2n-1} + v_{2n})}{a}, \qquad \sum_{i=1}^{4n} v_i = \frac{(a^2+4)f_{2n}(t_{2n+1} + t_{2n})}{a},$$

$$\text{(x)} \quad \sum_{i=1}^{4n-2} w_i = \frac{\ell_{2n-1}(w_{2n-1} + w_{2n})}{a}, \qquad \sum_{i=1}^{4n} w_i = \frac{(a^2+4)f_{2n}(s_{2n+1} + s_{2n})}{a}.$$

Proof: The proofs of these formulas are all similar to those of Theorem 1 and require the use of (3), (4), and Lemma 4 in the obvious places. To illustrate, we prove the first result in (i). Since $f_0 = 0$, we have that

$$\begin{aligned}
\sum_{i=1}^{4n-2} h_i &= \sum_{i=1}^{4n-2} (cf_{i-2} + df_{i-1}) = cf_{-1} + c\sum_{i=3}^{4n-2} f_{i-2} + d\sum_{i=2}^{4n-2} f_{i-1} \\
&= c + c\,\frac{f_{4n-3} + f_{4n-4} - 1}{a} + d\,\frac{f_{4n-2} + f_{4n-3} - 1}{a} \\
&= \frac{c(f_{4n-3} + f_{4n-4} + a - 1) + d(f_{4n-3} + f_{4n-3} - 1)}{a} \\
&= \frac{c(f_{4n-3} - f_1 + f_{4n-4} + f_2) + d(f_{4n-2} - f_0 + f_{4n-3} - f_1)}{a} \\
&= \frac{c(f_{2n-2}\ell_{2n-1} + f_{2n-3}\ell_{2n-1}) + d(f_{2n-1}\ell_{2n-1} + f_{2n-2}\ell_{2n-1})}{a} \\
&= \frac{\ell_{2n-1}[(cf_{2n-2} + df_{2n-1}) + (cf_{2n-3} + df_{2n-2})]}{a} \\
&= \frac{\ell_{2n-1}(h_{2n} + h_{2n-1})}{a}.
\end{aligned}$$

The formulas in Theorem 3 are still neat and tidy though not so simple as those in Theorems 1 and 2. The difficulty is that $H_{2n} + H_{2n-1} = H_{2n+1}$ in Theorem 1, whereas here we require $h_{2n} + ah_{2n-1} = h_{2n+1}$. Of course, if $a = 1$, the results coincide.

4. STILL MORE GENERAL RESULTS

It is natural to ask if the results can be generalized even further. Indeed, it would be reasonable to define sequences $\{\bar{f}_n\}_{n\geqslant 0} = \{\bar{f}_n(x)\}_{n\geqslant 0}$ and $\{\bar{\ell}_n\}_{n\geqslant 0} = \{\bar{\ell}_n(x)\}_{n\geqslant 0}$ by

$$\bar{f}_0 = 0,\ \bar{f}_1 = 1,\ \bar{f}_{n+1} = a\bar{f}_n + b\bar{f}_{n-1}$$

and

$$\bar{\ell}_0 = 2,\ \bar{\ell}_1 = a,\ \bar{\ell}_{n+1} = a\bar{\ell}_n + b\bar{\ell}_{n-1},$$

where $a = a(x)$ and $b = b(x)$ are arbitrary functions of x. Setting

$$\bar{\rho} = \frac{a + \sqrt{a^2 + 4b}}{2} \quad \text{and} \quad \bar{\sigma} = \frac{a - \sqrt{a^2 + 4b}}{2},$$

we obtain as before (see [1], p. 119),

$$\bar{f}_n = \frac{\bar{\rho}^n - \bar{\sigma}^n}{\sqrt{a^2 + 4b}}, \tag{18}$$

$$\bar{\ell}_n = \bar{\rho}^n - \bar{\sigma}^n, \tag{19}$$

$$\sum_{i=1}^{n} \bar{f}_i = \frac{\bar{f}_{n+1} + b\bar{f}_n - 1}{a + b - 1}, \tag{20}$$

$$\sum_{i=1}^{n} \bar{\ell}_i = \frac{\bar{\ell}_{n+1} + b\bar{\ell}_n - a - 2b}{a + b - 1}, \tag{21}$$

and the following lemma.

Lemma 5: For integers r and s,

$$\text{(i)} \quad \bar{f}_{r+2s} - b^s\bar{f}_r = \begin{cases} \bar{f}_s\bar{\ell}_{r+s} & s \text{ even}, \\ \bar{\ell}_s\bar{f}_{r+s} & s \text{ odd}, \end{cases}$$

$$\text{(ii)} \quad \bar{\ell}_{r+2s} - b^s\bar{\ell}_r = \begin{cases} (a^2 + 4b)\bar{f}_s\bar{f}_{r+s} & s \text{ even}, \\ \bar{\ell}_s\bar{\ell}_{r+s} & s \text{ odd}, \end{cases}$$

$$\text{(iii)} \quad \bar{f}_{r+2s} + b^s\bar{f}_r = \begin{cases} \bar{\ell}_s\bar{f}_{r+s} & s \text{ even}, \\ \bar{f}_s\bar{\ell}_{r+s} & s \text{ odd}, \end{cases}$$

$$\text{(iv)} \quad \bar{\ell}_{r+2s} + b^s\bar{\ell}_r = \begin{cases} \bar{\ell}_s\bar{\ell}_{r+s} & s \text{ even}, \\ (a^2 + 4b)\bar{f}_s\bar{f}_{r+s} & s \text{ odd}. \end{cases}$$

Continuing, if we define $\bar{h}_i$ and $\bar{k}_i$ by

$$\bar{h}_1 = c,\ \bar{h}_2 = d,\ \bar{h}_{n+1} = a\bar{h}_n + b\bar{h}_{n-1} \tag{22}$$

and

$$\bar{k}_1 = 2d - ac,\ \bar{k}_2 = ad + 2bc,\ \bar{k}_{n+1} = a\bar{k}_n + b\bar{k}_{n-1} \tag{23}$$

where $c = c(x)$ and $d = d(x)$ as above, we prove as before that

$$\bar{h}_n = bc\bar{f}_{n-2} + d\bar{f}_{n-1} \tag{24}$$

and

$$\bar{k}_n = bc\bar{\ell}_{n-2} + d\bar{\ell}_{n-1}. \tag{25}$$

If, by analogy with (10)-(17), we now define sequences $\bar{p}_n$, $\bar{q}_n$, $\bar{r}_n$, $\bar{s}_n$, $\bar{t}_n$, $\bar{u}_n$, $\bar{v}_n$, and $\bar{w}_n$ by

$$\bar{p}_n = bc\bar{f}_{n-2} + d\bar{\ell}_{n-1}, \tag{26}$$

$$\bar{q}_n = bc\bar{\ell}_{n-2} + d\bar{f}_{n-1}, \tag{27}$$

$$\bar{r}_n = bc\bar{\ell}_{n-2} + (a^2 + 4b)d\bar{f}_{n-1}, \tag{28}$$

$$\overline{s}_n = (a^2 + 4b)bc\overline{\ell}_{n-2} + d\overline{f}_{n-1}, \tag{29}$$

$$\overline{t}_n = bc\overline{f}_{n-2} + (a^2 + 4b)d\overline{\ell}_{n-1}, \tag{30}$$

$$\overline{u}_n = (a^2 + 4b)bc\overline{f}_{n-2} + d\overline{\ell}_{n-1}, \tag{31}$$

$$\overline{v}_n = bc\overline{\ell}_{n-2} + (a^2 + 4b)\ d\overline{f}_{n-1}, \tag{32}$$

and

$$\overline{w}_n = (a^2 + 4b)^2bc\overline{f}_{n-2} + d\overline{\ell}_{n-1}, \tag{33}$$

we can then prove the following theorems that contain all the preceding results as special cases. Of course, the formulas are less elegant, but they still exhibit a nice symmetry.

Theorem 4: For $n \geqslant 1$,

(i) $$\sum_{i=1}^{4n-2} \overline{f}_i + \frac{1 - b^{2n-1}}{a + b - 1} = \frac{\overline{\ell}_{2n-1}(\overline{f}_{2n} + b\overline{f}_{2n-1})}{a + b - 1},$$

(ii) $$\sum_{i=1}^{4n} \overline{f}_i + \frac{1 - b^{2n}}{a + b - 1} = \frac{\overline{f}_{2n}(\overline{\ell}_{2n+1} + b\overline{\ell}_{2n})}{a + b - 1},$$

(iii) $$\sum_{i=1}^{4n-2} \overline{\ell}_i - \frac{(a + 2b)(1 - b^{2n-1})}{a + b - 1} = \frac{\overline{\ell}_{2n-1}(\overline{\ell}_{2n} + b\overline{\ell}_{2n-1})}{a + b - 1},$$

(iv) $$\sum_{i=1}^{4n} \overline{\ell}_i - \frac{(a + 2b)(1 - b^{2n})}{a + b - 1} = \frac{(a^2 + 4b)\overline{f}_{2n}(\overline{f}_{2n+1} + b\overline{f}_{2n})}{a + b - 1}.$$

The proof is similar to that of Theorem 5 and will be omitted.

We note that Theorem 4 specializes to Corollary 1 if we set $a = b = c = d = 1$.

Theorem 5: Let

$$A = \frac{c + d - ac}{a + b - 1}, \qquad B = \frac{c(2b + a^2 - a) + d(2 - a)}{a + b - 1},$$

$$C = \frac{c(1 - a) + d(2 - a)}{a + b - 1}, \qquad D = \frac{c(2b + a^2 - a) + d}{a + b - 1},$$

$$E = \frac{c(2b + a^2 - a) + d(a^2 + 4b)}{a + b - 1}, \qquad F = \frac{c(a^2 + 4b)(2b + a^2 - a) + d}{a + b - 1},$$

$$G = \frac{c(1 - a) + d(a^2 + 4b)(2 - a)}{a + b - 1}, \qquad H = \frac{c(1 - a)(a^2 + 4b) + d(2 - a)}{a + b - 1},$$

$$I = \frac{c(2b + a^2 - a) + d(a^2 + 4b)}{a + b - 1}, \qquad J = \frac{c(1 - a)(a^2 + 4b)^2 + d(2 - a)}{a + b - 1}.$$

Then, for $n \geqslant 1$,

(i) $$\sum_{i=1}^{4n-2} \overline{h}_i + A(1 - b^{2n-1}) = \frac{\overline{\ell}_{2n-1}(\overline{h}_{2n} + b\overline{h}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{h}_i + A(1 - b^{2n}) = \frac{\bar{f}_{2n}(\bar{k}_{2n+1} + b\bar{k}_{2n})}{+ \quad - 1},$$

(ii) $$\sum_{i=1}^{4n-2} \bar{k}_i + B(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{k}_{2n} + b\bar{k}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{k}_i + B(1 - b^{2n}) = \frac{(a^2 + 4b)\bar{f}_{2n}(\bar{h}_{2n+1} + b\bar{h}_{2n})}{a + b - 1},$$

(iii) $$\sum_{i=1}^{4n-2} \bar{p}_i + C(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{p}_{2n} + b\bar{p}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{p}_i + C(1 - b^{2n}) = \frac{\bar{f}_{2n}(\bar{r}_{2n+1} + b\bar{r}_{2n})}{a + b - 1},$$

(iv) $$\sum_{i=1}^{4n-2} \bar{q}_i + D(1 - b^{2n-1}) + \frac{\bar{\ell}_{2n-1}(\bar{q}_{2n} + b\bar{q}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{q}_i + D(1 - b^{2n}) = \frac{\bar{f}_{2n}(\bar{u}_{2n+1} + b\bar{u}_{2n})}{a + b - 1},$$

(v) $$\sum_{i=1}^{4n-2} \bar{r}_i + E(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{r}_{2n} + b\bar{r}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{r}_i + E(1 - b^{2n}) = \frac{(a^2 + 4b)\bar{f}_{2n}(\bar{p}_{2n+1} + b\bar{p}_{2n})}{a + b - 1},$$

(vi) $$\sum_{i=1}^{4n-2} \bar{s}_i + F(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{s}_{2n} + b\bar{s}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{s}_i + F(1 - b^{2n}) = \frac{\bar{f}_{2}(\bar{w}_{2n+1} + b\bar{w}_{2n})}{a + b - 1},$$

(vii) $$\sum_{i=1}^{4n-2} \bar{t}_i + G(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{t}_{2n} + b\bar{t}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{t}_i + G(1 - b^{2n}) = \frac{\bar{f}_{2n}(\bar{v}_{2n+1} + b\bar{v}_{2n})}{a + b - 1},$$

(viii) $$\sum_{i=1}^{4n-2} \bar{u}_i + H(1 - b^{2n-1}) = \frac{\ell_{2n-1}(\bar{u}_{2n} + b\bar{u}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{u}_i + H(1 - b^{2n}) = \frac{(a^2 + 4b)\bar{f}_{2n}(\bar{q}_{2n+1} + b\bar{q}_{2n})}{a + b - 1},$$

(ix) $$\sum_{i=1}^{4n-2} \bar{v}_i + I(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{v}_{2n} + b\bar{v}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{v}_i + I(1 - b^{2n}) = \frac{(a^2 + 4b)\bar{f}_{2n}(\bar{t}_{2n+1} + b\bar{t}_{2n})}{a + b - 1}$$

(x) $$\sum_{i=1}^{4n-2} \bar{w}_i + J(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{w}_{2n} + b\bar{w}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{w}_i + J(1 - b^{2n}) = \frac{(a^2 + 4b)\bar{f}_{2n}(\bar{s}_{2n+1} + b\bar{s}_{2n})}{a + b - 1}.$$

Proof: Again, since the proofs are similar, we prove only the first part of (ii). Since $\bar{f}_{-1} = 1/b$, $\bar{f}_0 = 0$, and $\bar{\ell}_0 = 2$, we have from (20), (21), and Lemma 5, that

$$\begin{aligned}
\sum_{i=1}^{4n-2} \bar{p}_i &= \sum_{i=1}^{4n-2} (bc\bar{f}_{i-2} + d\bar{\ell}_{i-1}) = bc\left(\frac{1}{b}\right) + bc\sum_{i=1}^{4n-2} \bar{f}_{i-2} + 2d + d\sum_{i=1}^{4n-2} \bar{\ell}_{i-1} \\
&= c + \frac{bc(\bar{f}_{4n-3} + b\bar{f}_{4n-4} - 1)}{a + b - 1} + 2d + \frac{d(\bar{\ell}_{4n-2} + b\bar{\ell}_{4n-3} - a - 2b)}{a + b - 1} \\
&= \frac{ac - c + bc\bar{f}_{4n-3} + b^2c\bar{f}_{4n-4}}{a + b - 1} + \frac{ad - 2d + d\bar{\ell}_{4n-2} + db\bar{\ell}_{4n-3}}{a + b - 1} \\
&= \frac{bc(\bar{f}_{4n-3} - b^{2n-2}\bar{f}_1) + b^2c(\bar{f}_{4n-4} + b^{2n-3}\bar{f}_2)}{a + b - 1} \\
&\quad + \frac{d(\bar{\ell}_{4n-2} - b^{2n-1}\bar{\ell}_0) + db(\bar{\ell}_{4n-3} + p^{2n-2}\bar{\ell}_1)}{a + b - 1} \\
&\quad + \frac{c(a - 1) + d(a - 2) + b^{2n-1}c(1 - a) + b^{2n-1}d(2 - a)}{a + b - 1} \\
&= \frac{bc\bar{f}_{2n-2}\bar{\ell}_{2n-1} + b^2c\bar{f}_{2n-3}\bar{\ell}_{2n-1} + d\,\bar{\ell}_{2n-1}^2 + db\bar{\ell}_{2n-2}\bar{\ell}_{2n-1}}{a + b - 1} \\
&\quad + \frac{[c(a - 1) + d(a - 2)][1 - b^{2n-1}]}{a + b - 1} \\
&= \frac{\bar{\ell}_{2n-1}[bc\bar{f}_{2n-2} + d\bar{\ell}_{2n-1} + b(bc\bar{f}_{2n-3} + d\bar{\ell}_{2n-2})]}{a + b - 1} \\
&\quad + \frac{[c(a - 1) + d(a - 2)][1 - b^{2n-1}]}{a + b - 1} \\
&= \frac{\bar{\ell}_{2n-1}(\bar{p}_{2n} + b\bar{p}_{2n-1})}{a + b - 1} + \frac{[c(a - 1) + d(a - 2)][1 - p^{2n-1}]}{a + b - 1}
\end{aligned}$$

by definition of $\bar{p}_n$. But this implies the desired result.

Of course, if $b = 1$, these yield the formulas of Theorem 3 as they should.

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A FIBONACCI SEQUENCE OF DISTRIBUTIVE LATTICES

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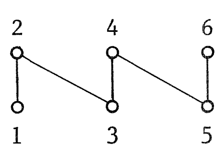
In this paper we describe an order-theoretic realization of the Fibonacci numbers 1, 2, 3, 5, 8, 13, ... and of the Bisection Lucas numbers 3, 7, 18, 47, 123, ... The Bisection Lucas numbers are part of the Lucas sequence and are obtained from the Lucas numbers 2, 1, 3, 4, 7, 11, ... by deleting 2, 1, 4, and then every second number after that. We represent the Fibonacci numbers and the Bisection Lucas numbers as the cardinalities of sequences of distributive lattices that we glue together from simple building blocks. The gluing process is described in Section 2, and the main results are formulated in Section 3 as Theorem 3.1, Theorem 3.4, and their corollaries. In Section 1, we introduce some essential terminology and necessary facts about function lattices. For a more complete treatment of these topics, we refer the reader to the standard textbooks [1], [2], [5], and to [3]. For a related recursive construction of a sequence of modular lattices whose cardinalities are the polygonal numbers, we refer the reader to [6]. It should be noted that the construction discussed in [6] is very different from the construction discussed here in Section 2.

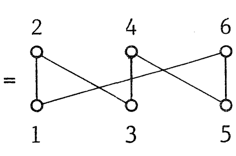
1. FENCES, CROWNS, AND FUNCTION LATTICES

Let P be a partially ordered set, then $|P|$ is the cardinality of P and P^* is the dual of P. For integers $n \geqslant 0$, $\mathbf{n} = \{1, 2, \ldots, n\}$ is the totally ordered chain of n elements ordered in their natural order, $\mathbf{0}$ is the empty chain. The partially ordered set $F(n) = \{i \mid 1 \leqslant i \leqslant n\}$ for $n \geqslant 1$ is a fence if it has the following order:

$$\begin{aligned} &i < i + 1 \quad \text{if } i \text{ is odd,}\\ &i > i + 1 \quad \text{if } i \text{ is even.}\end{aligned} \tag{1.1}$$

From the $2n$-element fence $F(2n)$, for $n \geqslant 2$, we construct the $2n$-element crown $C(2n)$ by introducing exactly one additional order relation, namely $1 < 2n$. For example,

$F(6)$ = 

and $C(6)$ = 

We extend the definitions to include $C(0) = F(0) = \mathbf{0}$ and $C(2) = F(2) = \mathbf{2}$.

For partially ordered sets P, Q, we define Q^P to be the set of all order-preserving mappings $f: P \to Q$ partially ordered by

$$f \leqslant g \text{ if and only if } f(x) \leqslant g(x) \text{ for all } x \in P. \tag{1.2}$$

If $f, g \in Q^P$, then the supremum of f and g, $f \vee g$, exists in Q^P if and only if the supremum of $f(x)$ and $g(x)$ exists in Q for all $x \in P$ and

$$(f \vee g)(x) = f(x) \vee g(x).$$

Since the same is true for the infimum of f and g, it follows that Q^P is a lattice whenever Q is a lattice, P may be an arbitrary partially ordered set. It can be easily verified that Q^P is a distributive or modular lattice, provided that Q is a distributive or modular lattice. All of the partially ordered sets of the form Q^P that we study in this paper are distributive lattices. We are particularly interested in the distributive lattices $\mathbf{2}^{F(n)}$ and $\mathbf{2}^{C(n)}$, for $n \geqslant 0$. Note that $\mathbf{2}^{F(0)} = \mathbf{2}^{C(0)} = \mathbf{1}$, $\mathbf{2}^{F(1)} = \mathbf{2}$, and $\mathbf{2}^{F(2)} = \mathbf{2}^{C(2)} = \mathbf{3}$. As a convenient notation for an order-preserving function $f: F(n) \to \mathbf{2}$, we use its representation by its image vector, i.e., 11212 stands for the function $f: F(5) \to \mathbf{2}$ given by $f(1) = f(2) = f(4) = 1 \in \mathbf{2}$ and $f(3) = f(5) = 2 \in \mathbf{2}$.

A list of arithmetical rules for the exponentiation of arbitrary partially ordered sets P, Q, R may be found in [2] and [3]. We restate here only two that will be needed later.

$$(Q^P)^R \cong Q^{P \times R} \cong (Q^R)^P \tag{1.3}$$

$$(Q^P)^* \cong (Q^*)^{P^*} \tag{1.4}$$

Since we want to recursively construct the lattices $\mathbf{2}^{F(n)}$ and $\mathbf{2}^{C(n)}$ for increasing n, we shall first describe a process of gluing for lattices that is the basis of our recursive construction.

2. A LATTICE CONSTRUCTION

Let L be a lattice. An ideal in L is a nonempty subset $I \subset L$ such that for $x, y \in I$ also $x \vee y \in I$, and for $a \in I$, $x \in L$, $x \leqslant a$ implies $x \in I$. The dual concept is called a filter or a dual ideal in L. Now let L be a lattice and let $I \subset L$ be an ideal. We glue an order-isomorphic copy I' of I below I to L as follows: Let M be the disjoint union of L and I' with the order defined as

$$\begin{aligned} x \leqslant_M y \text{ if any only if } & x \leqslant_L y \\ \text{or } & x \leqslant_{I'} y \\ \text{or } & x = i' < i \leqslant_L y \quad \text{for some } i \in I. \end{aligned} \tag{2.1}$$

With this order M is a lattice where the lattice operations are the given ones on L and on I' and in addition we have $x \vee_M i' = x \vee_L i$ and $x \wedge_M i' = (x \wedge_L i)'$. With this structure, M will be denoted by $L \downarrow I$. Similarly, if $F \subset L$ is a filter, we can glue a copy F' of F above F to the lattice L. The order on the disjoint union K of L and of F' is then defined as

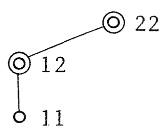
$$\begin{aligned} x \leqslant_K y \text{ if and only if } & x \leqslant_L y \\ \text{or } & x \leqslant_{F'} y \\ \text{or } & x \leqslant_L f < f' = y \quad \text{for some } f \in F, \end{aligned} \tag{2.2}$$

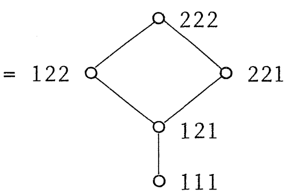
and the lattice operations are defined accordingly. With this structure, K will be denoted as $L \uparrow F$. $L \uparrow F$ and $L \downarrow I$ are distributive or modular lattices whenever L is a distributive or modular lattice, and L is a sublattice of both $L \uparrow F$ and $L \downarrow I$. Moreover, since the gluing constructions are duals of each other, we have the De Morgan properties

$$\begin{aligned}(L \downarrow I)^* &\cong L^* \uparrow I^* \\ (L \uparrow F)^* &\cong L^* \downarrow F^*\end{aligned} \tag{2.3}$$

for any lattice L, ideal $I \subset L$ and filter $F \subset L$.

To illustrate how we will use this construction in the next section, let us look at $\mathbf{2}^{F(2)} \uparrow \mathbf{2}^{F(1)} = \mathbf{3} \uparrow \mathbf{2}$, where the elements of the dual ideal $\mathbf{2}$ in $\mathbf{3}$ are circled:

$\mathbf{2}^{F(2)} = \mathbf{3} =$ 

$\mathbf{3} \uparrow \mathbf{2} =$ 

But the latter is $\mathbf{2}^{F(3)}$ with the mappings indicated in the diagram, so we get that $\mathbf{2}^{F(3)} = \mathbf{2}^{F(2)} \uparrow \mathbf{2}^{F(1)}$.

This construction can, in a rather loose sense, be considered an opposite of a construction used in [4]. In our case, a separate copy of an ideal I or filter F of a lattice L is added to L and the new lattice has cardinality

$$|L| + |I| \quad \text{or} \quad |L| + |F|,$$

whereas in [4] a filter F in a lattice L_1 is identified with an isomorphic ideal I in a lattice L_2 and the new lattice has cardinality

$$|L_1| + |L_2| - |F| = |L_1| + |L_2| - |I|.$$

In both constructions, modularity and distributivity are preserved and the old lattices are sublattices of the new ones.

3. A FIBONACCI SEQUENCE OF DISTRIBUTIVE LATTICES

We are now ready to recursively construct the sequence of distributive lattices whose cardinalities are the Fibonacci numbers.

Theorem 3.1: (1) $\mathbf{2}^{F(n)} \cong \mathbf{2}^{F(n-1)} \downarrow \mathbf{2}^{F(n-2)}$ if n is even, $n \geqslant 2$.

(2) $\mathbf{2}^{F(n)} \cong \mathbf{2}^{F(n-1)} \uparrow \mathbf{2}^{F(n-2)}$ if n is odd, $n \geqslant 2$.

Proof: (1) If $n \geqslant 2$ and even, n is a maximal element in $F(n)$, and the subset A of $\mathbf{2}^{F(n)}$ where n gets mapped to $2 \in \mathbf{2}$ is order-isomorphic to $\mathbf{2}^{F(n-1)}$. In $\mathbf{2}^{F(n-1)}$ we find the set B of all the mappings where $n - 1$ gets mapped to $1 \in \mathbf{2}$. B is an ideal in $\mathbf{2}^{F(n-1)}$ and B is order-isomorphic to $\mathbf{2}^{F(n-2)}$. Therefore, we can define the bijection $\phi : \mathbf{2}^{F(n-1)} \downarrow \mathbf{2}^{F(n-2)} \to \mathbf{2}^{F(n)}$ as follows:

$$\begin{aligned}\phi(f) = g \text{ if and only if } & g|F(n-1) = f \text{ and } g(n) = 2, \text{ if } f \in \mathbf{2}^{F(n-1)} \\ & g|F(n-2) = f \text{ and } g(n-1) = g(n) = 1, \text{ if } f \in \mathbf{2}^{F(n-2)}.\end{aligned}$$

For any $f \in \mathbf{2}^{F(n-2)}$, the extension $\bar{f} \in \mathbf{2}^{F(n)}$ of f defined by $\bar{f}|F(n-2) = f$ and $\bar{f}(n-1) = 1$ and $\bar{f}(n) = 2$ is a direct upper neighbor of $\phi(f)$ in $\mathbf{2}^{F(n)}$; conversely, for each $g, h \in \mathbf{2}^{F(n)}$ with $f = \phi^{-1}(g) \in \mathbf{2}^{F(n-2)}$ and $\phi^{-1}(h) \in \mathbf{2}^{F(n-1)}$ and $g < h$, the extension $\bar{f}$ of f with $\bar{f}(n-1) = 1$ in $\mathbf{2}^{F(n-1)}$ is a direct upper neighbor of

f and $\bar{f} \leqslant \phi^{-1}(h)$ in $2^{F(n-1)}$. Straightforward calculations will complete the proof that ϕ is an order-isomorphism.

(2) For odd n, n is a minimal element in $F(n)$ and we look for the subset $A \subset 2^{F(n)}$ of functions that map n to $1 \in \mathbf{2}$. As in part (1), A is order-isomorphic to $2^{F(n-1)}$, and in $2^{F(n-1)}$ we find the set B of functions that map $n-1$ to $2 \in \mathbf{2}$. This set B is a filter in $2^{F(n-1)}$. Dualizing the argument of part (1) completes the proof.

Since $2^{F(0)} \cong \mathbf{1}$ and $2^{F(1)} \cong \mathbf{2}$, we have an obvious consequence.

<u>Corollary</u>: The cardinalities of the sequence of distributive lattices $2^{F(n)}$ for increasing $n \geqslant 0$ are the Fibonacci numbers 1, 2, 3, 5, 8, 13, ...

It is possible to give an alternate recursive representation of the lattices $2^{F(n)}$ which uses only the operator $\uparrow$. In essentially the same fashion as in Theorem 3.1 one proves

<u>Theorem 3.2</u>: For any $n \geqslant 2$, $2^{F(n)} \cong A \uparrow 2^{F(n-2)}$, where

$A = (2^{F(n-1)})^*$ if n is even,

and

$A = 2^{F(n-1)}$ if n is odd.

<u>Proof</u>: Let A be the set of all functions that map $1 \in F(n)$ to $1 \in \mathbf{2}$. Then this set is order-isomorphic to $(2^{F(n-1)})^*$. The rest of the proof is as that for Theorem 3.1.

Since $F(2n)$ is a self-dual partially ordered set, every lattice $2^{F(2n)}$, $n \geqslant 0$, is self-dual also. The two theorems, 3.1 and 3.2, and De Morgan's laws (2.3) explain how this self-duality appears in every other step of the recursive construction. Obviously $2^{F(0)} \cong \mathbf{1}$ and $2^{F(2)} \cong \mathbf{3}$ are self-dual and, for $n > 0$, an induction on n establishes

$$\begin{aligned} 2^{F(2n)} &\cong 2^{F(2n-1)} \downarrow 2^{F(2n-1)} \cong (2^{F(2n-1)})^* \uparrow 2^{F(2n-2)} \\ &\cong (2^{F(2n-1)} \downarrow 2^{F(2n-2)})^* \cong (2^{F(2n)})^*. \end{aligned}$$

In fact, this self-duality is a consequence of the following general theorem, which is proved in the same manner.

<u>Theorem 3.3</u>: Let A and B be lattices so that $B \subset A$ is a self-dual ideal of A. The following statements are equivalent:

(1) $A \downarrow B \cong A^* \uparrow B$.

(2) $A \downarrow B$ is self-dual.

Finally, it should be noted that $2^{F(3)}$ is not self-dual.

<u>Theorem 3.4</u>: $2^{C(2n)} \cong 2^{F(2n-1)} \downarrow (2^{F(2n-3)})^*$

$$\cong (2^{F(2n-2)} \uparrow 2^{F(2n-3)}) \downarrow (2^{F(2n-3)})^* \quad \text{for } n \geqslant 2.$$

<u>Proof</u>: The subset A of $2^{C(2n)}$ where the element $2n \in C(2n)$ gets mapped onto $2 \in \mathbf{2}$ is order-isomorphic to $2^{F(2n-1)}$. In $2^{F(2n-1)}$ we find the set B of all those mappings where 1 and also $2n-1$ get mapped onto $1 \in \mathbf{2}$. B is an ideal in

$\mathbf{2}^{F(2n-1)}$, and it is order-isomorphic to the dual of $\mathbf{2}^{F(2n-3)}$ by (1.4). All maps f in B can be extended to maps $\bar{f}: C(2n) \to \mathbf{2}$ by defining

$$\bar{f}(2n) = 1 \quad \text{and} \quad f|F(2n-1) = f.$$

These are the direct lower neighbors of the maps that have the same images on $F(2n-1)$ but map $2n$ to 2. Clearly, $\mathbf{2}^{C(2n)}$ is the disjoint union of A and an order-isomorphic copy of B, and its order structure is that of

$$\mathbf{2}^{F(2n-1)} \downarrow (\mathbf{2}^{F(2n-3)})^*.$$

For the cardinalities of the lattices $\mathbf{2}^{C(2n)}$, we have

$$|\mathbf{2}^{C(2n)}| = |\mathbf{2}^{F(2n-1)}| + |\mathbf{2}^{F(2n-3)}|, \tag{3.1}$$

and we know already that $|\mathbf{2}^{F(n)}|$ for $n \geqslant 0$ are the Fibonacci numbers. The sum of the n^{th} and the $(n+2)^{\text{nd}}$ Fibonacci numbers generates another Fibonacci sequence which is part of the Lucas sequence 2, 1, 3, 4, 7, 11, ... From the Lucas sequence, the Bisection Lucas sequence ([7], p. 101, #1067) is generated by deleting 2, 1, 4, and every second number after that. Since $|\mathbf{2}^{C(2)}| = 3$, and because of (3.1), we have the following

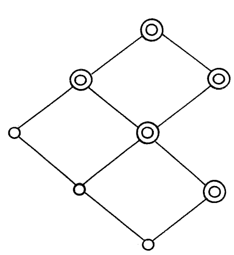
<u>Corollary</u>: The cardinalities of the sequence of distributive lattices $\mathbf{2}^{C(2n)}$ for increasing $n \geqslant 1$ are the Bisection Lucas numbers 3, 7, 18, 47, 123, ...

For an interesting extension of the corollaries to Theorem 3.1 and Theorem 3.4, we replace the two-element chain in the base of our function lattices by the Boolean algebra $\mathbf{2}^k$, $k \geqslant 1$ denoting a k-element antichain. Then, $(\mathbf{2}^k)^{F(n)} \cong (\mathbf{2}^{F(n)})^k$ by (1.3) and, therefore, we have as a consequence of the corollary to Theorem 3.1 that the cardinalities of $(\mathbf{2}^k)^{F(n)}$ for $n \geqslant 0$ are given by the k^{th} powers of the Fibonacci numbers, $1^k, 2^k, 3^k, 5^k, 8^k, \ldots$. Similarly, $(\mathbf{2}^k)^{C(2n)} \cong (\mathbf{2}^{C(2n)})^k$ and, as a consequence of the corollary to Theorem 3.4, the cardinalities of $(\mathbf{2}^k)^{C(2n)}$, $n \geqslant 0$, are the k^{th} powers of the Bisection Lucas numbers, $3^k, 7^k, 18^k, 47^k, \ldots$.

We conclude the paper with an example which illustrates our construction. We show that our method of gluing provides a completely symmetrical construction of the free distributive lattice on three generators, that is, the lattice $\mathbf{2}^{C(6)}$ which has 18 elements. We construct $\mathbf{2}^{C(6)}$ as follows:

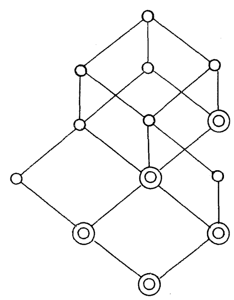
$$\mathbf{2}^{C(6)} \cong \mathbf{2}^{F(5)} \downarrow (\mathbf{2}^{F(3)})^* \cong (\mathbf{2}^{F(4)} \uparrow \mathbf{2}^{F(3)}) \downarrow (\mathbf{2}^{F(3)})^*.$$

The circled elements in the figure below are those of the filter $\mathbf{2}^{F(3)}$ in $\mathbf{2}^{F(4)}$, consisting of the maps where $4 \in F(4)$ is mapped to $2 \in \mathbf{2}$.



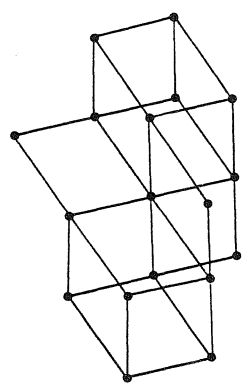
$$= \mathbf{2}^{F(4)} = \mathbf{2}^{F(3)} \downarrow \mathbf{2}^{F(2)}$$

To get $\mathbf{2}^{F(5)} \cong \mathbf{2}^{F(4)} \uparrow \mathbf{2}^{F(3)}$, we glue a copy of $\mathbf{2}^{F(3)}$ above $\mathbf{2}^{F(3)}$ as shown in the following figure. Here the mappings where 1 and 5 in $F(5)$ both go to $2 \in \mathbf{2}$ are circled. This circled set is an ideal in $\mathbf{2}^{F(5)}$ and it is an isomorphic copy of the dual of $\mathbf{2}^{F(3)}$.



$= 2^{F(5)}$

Finally, we attach a copy of the circled ideal in the figure for $\mathbf{2}^{F(5)}$ and get the free distributive lattice $\mathbf{2}^{C(6)}$.



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A NOTE ON THE SUMS OF FIBONACCI AND LUCAS POLYNOMIALS

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(Submitted September 1983)

Recently, G. E. Bergum and V. E. Hoggatt, Jr. [1] have shown that

$$\sum_{n=0}^{\infty} F_{2^n k}^{-1}(x) = \frac{1}{F_k(x)} + \begin{cases} (\alpha^2(x) + 1)/\alpha(x)(\alpha^{2k}(x) - 1), & x > 0, \\ (\beta^2(x) + 1)/\beta(x)(\beta^{2k}(x) - 1), & x < 0, \end{cases} \tag{1}$$

where $\{F_k(x)\}_{k=1}^{\infty}$ is the sequence of Fibonacci polynomials, defined recursively by

$$F_1(x) = 1,\ F_2(x) = x,\ F_{k+2}(x) = xF_{k+1}(x) + F_k(x),\ k \geqslant 1,$$

and $\alpha(x) = (x + \sqrt{x^2+4})/2$, $\beta(x) = (x - \sqrt{x^2+4})/2$. Evidently, for $x = 1$ it is the known formula for the Fibonacci numbers [2].

In this paper we give, by an elementary method, an extension of the result (1). Namely, we show that

$$\sum_{n=0}^{\infty} (-1)^{r^n k} \frac{F_{(r-1)r^n k}(x)}{F_{r^n k}(x) F_{r^{n+1} k}(x)} = \begin{cases} \beta^k(x)/F_k(x), & x > 0, \\ \alpha^k(x)/F_k(x), & x < 0. \end{cases} \tag{2}$$

Obviously, for $r = 2$, we obtain (1) from (2).

Furthermore, we find that

$$\sum_{n=0}^{\infty} \frac{2^n \beta^{2^n k}(x)}{L_{2^n k}(x)} = \begin{cases} \dfrac{\alpha(x)}{\alpha^2(x) + 1} \dfrac{\beta^k(x)}{F_k(x)}, & x > 0, \\ \dfrac{\alpha(x)}{\alpha^2(x) + 1} \dfrac{\alpha^k(x)}{F_k(x)}, & x < 0, \end{cases} \tag{3}$$

where $L_k(k)$ is the Lucas polynomial defined by $L_k(x) = F_{k+1}(x) + F_{k-1}(x)$.

From the identity

$$\sum_{r=0}^{n} \frac{x^{p^r} - x^{p^{r+1}}}{(1 - x^{p^r})(1 - x^{p^{r+1}})} = \frac{x - x^{p^{n+1}}}{(1 - x)(1 - x^{p^{n+1}})}$$

if we put $x = \beta^k(x)/\alpha^k(x)$ we obtain

$$\sum_{r=0}^{m} (-1)^{n^r k} \frac{F_{(n-1)n^r k}(x)}{F_{n^r k}(x) F_{n^{r+1} k}(x)} = (-1)^k \frac{F_{(n^{m+1}-1)k}(x)}{F_k(x) F_{n^{m+1} k}(x)}. \tag{4}$$

Using the facts that $|\beta(x)/\alpha(x)| < 1$ if $x > 0$ and that $\beta(x)/\alpha(x) < -1$ if $x < 0$, from (4), when $m \to \infty$, we have (2).

Similarly, from

$$\sum_{r=0}^{\infty} \frac{2^r x^{2^r}}{1 + x^{2^r}} = \frac{x}{1 - x},$$

if we put $x = \beta^k(x)/\alpha^k(x)$, we find (3).

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JACOBSTHAL POLYNOMIALS AND A CONJECTURE CONCERNING FIBONACCI-LIKE MATRICES

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1. THE CONJECTURE

In the March 1983 issue of the *Mathematical Gazette* [9], Mr. Moore conjectures that if one lets

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 + x \end{bmatrix}, \tag{1.1}$$

raises Q to powers and scales each such matrix down by making the leading entry 1, then the scaled down sequence of matrices approaches

$$\begin{bmatrix} 1 & \phi \\ \phi & \phi^2 \end{bmatrix}, \tag{1.2}$$

as $n \to \infty$, where $\phi = (x + \sqrt{x^2 + 4})/2$.

The purpose of this paper is to show that the conjecture is true if $x > -2$, while the limit is

$$\begin{bmatrix} 1 & -\phi^{-1} \\ -\phi^{-1} & \phi^{-2} \end{bmatrix}, \tag{1.3}$$

if $x < -2$ and does not exist if $x = -2$.

It is worthwhile to mention at this point that the conjecture was first brought to the editor's attention by a letter from Mr. Moore in October 1982. The proofs of Theorems 1 to 6 were completed by Professor Bergum in November 1982. Due to the pressure of other work, the publication of these results was delayed. Several months later, the information on Jacobsthal polynomials arrived from Professor Horadam along with an alternate proof of Theorem 4. Professor Bennett joined the group by showing that (2.14) does not have a limit as n approaches infinity. The combined results are what is to follow.

If one carefully examines the way we multiply matrices, then it is quite obvious that the elements of the powers of Q satisfy linear recurrences. Examining the first five or six powers of Q, we are led to believe that

$$Q^n = \begin{bmatrix} H_n & M_n \\ M_n & N_n \end{bmatrix}, \tag{1.4}$$

where we define the sequences $\{H_n\}$, $\{M_n\}$, and $\{N_n\}$ recursively by

$$H_{n+2} = (x+2)H_{n+1} - xH_n,\quad H_1 = 1,\quad H_2 = 2, \tag{1.5}$$

$$M_{n+2} = (x+2)M_{n+1} - xM_n,\quad M_1 = 1,\quad M_2 = x+2, \tag{1.6}$$

$$N_{n+2} = (x+2)N_{n+1} - xN_n,\quad N_1 = x+1,\quad N_2 = x^2 + 2x + 2. \tag{1.7}$$

Before proving the validity of (1.4), we first establish the following results.

Theorem 1: (a) $H_n + M_n = H_{n+1}$, (c) $(x+1)M_n + H_n = M_{n+1}$,

(b) $M_n + N_n = M_{n+1}$, (d) $(x+1)N_n + M_n = N_{n+1}$.

Proof: Since the proofs are very similar, we prove only part (c).

When $n = 1$ we have $(x+1)M_1 + H_1 = x + 1 + 1 = x + 2 = M_2$, and when $n = 2$ we have $(x+1)M_2 + H_2 = (x+1)(x+2) + 2 = x^2 + 3x + 4 = M_3$; so that (c) is true for $n = 1$ and 2. Now assume the statement is true for all positive integers less than k where $k \geqslant 3$. Then by (1.6), (1.5), and the induction hypothesis, we have

$$\begin{aligned}(x+1)M_k + H_k &= (x+1)[(x+2)M_{k-1} - xM_{k-2}] + [(x+2)H_{k-1} - xH_{k-2}] \\ &= (x+2)[(x+1)M_{k-1} + H_{k-1}] - x[(x+1)M_{k-2} + H_{k-2}] \\ &= (x+2)M_k - xM_{k-1} = M_{k+1},\end{aligned}$$

and (c) is proved.

The proof of (1.4) follows directly from Theorem 1 by mathematical induction giving

Theorem 2: If $Q = \begin{bmatrix} 1 & 1 \\ 1 & 1+x \end{bmatrix}$ then $Q^n = \begin{bmatrix} H_n & M_n \\ M_n & N_n \end{bmatrix}$ for all integers $n \geqslant 1$.

Now we scale down Q^n and obtain a new sequence of matrices $\{R_n\}$ where

$$R_n = \begin{bmatrix} 1 & M_n/H_n \\ M_n/H_n & N_n/H_n \end{bmatrix}, \tag{1.8}$$

and then ask: What happens as $n \to \infty$? To answer this question, we first apply (1.6) and (1.7) found in [6] and obtain

$$H_n = \frac{(2-\beta)\alpha^{n-1} - (2-\alpha)\beta^{n-1}}{\alpha - \beta},\quad n \geqslant 1, \tag{1.9}$$

$$M_n = \frac{(x + 2 - \beta)\alpha^{n-1} - (x + 2 - \alpha)\beta^{n-1}}{\alpha - \beta}, \quad n \geqslant 1, \tag{1.10}$$

$$N_n = \frac{[x^2 + 2x + 2 - (x + 1)\beta]\alpha^{n-1} - [x^2 + 2x + 2 - (x + 1)\alpha]\beta^{n-1}}{\alpha - \beta}, \quad n \geqslant 1, \tag{1.11}$$

where

$$\alpha = \frac{(x + 2) + \sqrt{x^2 + 4}}{2} \quad \text{and} \quad \beta = \frac{(x + 2) - \sqrt{x^2 + 4}}{2}. \tag{1.12}$$

are the roots of the characteristic equation arising from the recurrences (1.5), (1.6), and (1.7). Next, we analyze the range of β/α and α/β, as this is needed before we can find

$$\lim_{n \to \infty} \frac{M_n}{H_n} \quad \text{and} \quad \lim_{n \to \infty} \frac{N_n}{H_n}.$$

If $x > -2$, then $0 < 2(x^2 + 4) + 2(x + 2)\sqrt{x^2 + 4}$, so that

$$4x < [x^2 + 4x + 4 + 2(x + 2)\sqrt{x^2 + 4} + x^2 + 4] = [(x + 2) + \sqrt{x^2 + 4}]^2$$

or

$$1 > 4x/[(x + 2) + \sqrt{x^2 + 4}]^2.$$

When $x \neq 0$ we can multiply and divide the right side of the last inequality by $(x + 2 - \sqrt{x^2 + 4})$ to obtain

$$1 > \frac{x + 2 - \sqrt{x^2 + 4}}{x + 2 + \sqrt{x^2 + 4}} = \frac{\beta}{\alpha}.$$

If $x = 0$, then $\beta = 0$ and $\alpha = 2$, so that $\beta/\alpha = 0 < 1$. Since $x > -2$, we also have

$$0 < x + 2 + \sqrt{x^2 + 4} \quad \text{or} \quad 0 < 2(x + 2)^2 + 2(x + 2)\sqrt{x^2 + 4},$$

so that

$$-4x < 2x^2 + 4x + 8 + 2(x + 2)\sqrt{x^2 + 4}.$$

Hence,

$$4x > -[(x + 2) + \sqrt{x^2 + 4})^2 \quad \text{or} \quad -1 < 4x/[(x + 2) + \sqrt{x^2 + 4}]^2.$$

Operating as before when $x \neq 0$, we see that

$$-1 < \frac{(x + 2) - \sqrt{x^2 + 4}}{(x + 2) + \sqrt{x^2 + 4}} = \frac{\beta}{\alpha},$$

which is also true if $x = 0$. Therefore,

$$-1 < \frac{\beta}{\alpha} < 1, \quad \text{if } x > -2. \tag{1.13}$$

When $x < -2$, we have

$$x + 2 < \sqrt{x^2 + 4} \quad \text{or} \quad 2(x + 2)^2 > 2(x + 2)\sqrt{x^2 + 4},$$

so that

$$2x^2 + 4x + 8 - 2(x + 2)\sqrt{x^2 + 4} = (x + 2 - \sqrt{x^2 + 4})^2 > -4x.$$

Hence,

$$-1 < 4x/(x + 2 - \sqrt{x^2 + 4})^2 = \frac{x + 2 + \sqrt{x^2 + 4}}{x + 2 - \sqrt{x^2 + 4}} = \frac{\alpha}{\beta}.$$

Since $x < 0$,

$$x^2 + 4x + 4 < x^2 + 4 \quad \text{or} \quad \sqrt{(x + 2)^2} < \sqrt{x^2 + 4}.$$

Therefore,

$$|x + 2| < \sqrt{x^2 + 4} \quad \text{and} \quad x + 2 > -\sqrt{x^2 + 4},$$

so that $\alpha > 0$. However, $\beta < 0$ and we get

$$-1 < \frac{\alpha}{\beta} < 0, \quad \text{if } x < -2. \tag{1.14}$$

When $x = -2$, we have $\alpha/\beta = \beta/\alpha = -1$. Combining these results, we obtain

<u>Theorem 3</u>: If $\alpha = \dfrac{x + 2 + \sqrt{x^2 + 4}}{2}$ and $\beta = \dfrac{x + 2 - \sqrt{x^2 + 4}}{2}$, then

(a) $\dfrac{\alpha}{\beta} = \dfrac{\beta}{\alpha} = -1$, if $x = -2$,

(b) $-1 < \dfrac{\alpha}{\beta} < 0$, if $x < -2$,

(c) $-1 < \dfrac{\beta}{\alpha} < 1$, if $x > -2$.

Let $x > -2$ and $x \neq 0$; then by Theorem 3(c), substitution of β, and rationalization

$$\lim_{n\to\infty} \frac{M_n}{H_n} = \frac{x + 2 - \beta}{2 - \beta} = \frac{x + 2 + \sqrt{x^2 + 4}}{(2 - x) + \sqrt{x^2 + 4}} = \frac{x + \sqrt{x^2 + 4}}{2} = \phi.$$

Also, using similar steps, we have

$$\lim_{n\to\infty} \frac{N_n}{H_n} = \frac{x^2 + 2x + 2 - (x + 1)\beta}{2 - \beta} = \frac{x^2 + x + 2 + (x + 1)\sqrt{x^2 + 4}}{(2 - x) + \sqrt{x^2 + 4}}$$

$$= \frac{2x^2 + 4 + 2x\sqrt{x^2 + 4}}{4} = \phi^2.$$

If $x = 0$, then $M_n/H_n = N_n/H_n = 1$ for all n, so that

$$\lim_{n\to\infty} \frac{M_n}{H_n} = \lim_{n\to\infty} \frac{N_n}{H_n} = 1.$$

Hence, we have

<u>Theorem 4</u>: If $x > -2$, then $\lim\limits_{n\to\infty} R_n = \begin{bmatrix} 1 & \phi \\ \phi & \phi^2 \end{bmatrix}$.

Let us now assume that $x < -2$; then reasoning as above, we have

$$\lim_{n\to\infty} \frac{M_n}{H_n} = \frac{x + 2 - \alpha}{2 - \alpha} = \frac{x + 2 - \sqrt{x^2 + 4}}{2 - x - \sqrt{x^2 + 4}} = -\frac{2}{x + \sqrt{x^2 + 4}} = -\phi^{-1}.$$

Similarly,

$$\lim_{n\to\infty} \frac{N_n}{H_n} = \frac{x^2 + 2x + 2 - (x + 1)\alpha}{2 - \alpha} = \phi^{-2},$$

and we have in (1.8)

__Theorem 5__: If $x < -2$, then $\lim\limits_{n\to\infty} R_n = \begin{bmatrix} 1 & -\phi^{-1} \\ -\phi^{-1} & \phi^{-2} \end{bmatrix}$.

When $x = -2$, $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ in (1.1) so that $R_n = \begin{cases} Q, & \text{if } n \text{ is odd} \\ I, & \text{if } n \text{ is even} \end{cases}$, where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, we obtain in (1.8)

__Theorem 6__: If $x = -2$, then $\lim\limits_{n\to\infty} R_n$ does not exist.

Observe that when $x = -1$, (1.5), (1.6), and (1.7) all reduce to the definition for the sequence of Fibonacci numbers and (1.1) becomes

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

which is discussed in [1], [2], [3], and [4].

2. JACOBSTHAL POLYNOMIALS AND MATRICES

The *Jacobsthal polynomials* $J_n(x) \equiv J_n$ are defined in [7] by the recurrence relation

$$J_{n+2} = J_{n+1} + xJ_n \qquad (J_0 = 0,\ J_1 = 1) \tag{2.1}$$

and the first few term of $\{J_n\}$ are

$$\begin{array}{ccccccc} J_1 & J_2 & J_3 & J_4 & J_5 & J_6 & \cdots \\ 1 & 1 & 1 + x & 1 + 2x & 1 + 3x + x^2 & 1 + 4x + 3x^2 & \cdots \end{array} \tag{2.2}$$

The matrix (1.1) can now be expressed as

$$J = \begin{bmatrix} J_1 & J_2 \\ J_2 & J_3 \end{bmatrix}, \tag{2.3}$$

and justifiably called a *Jacobsthal matrix*.

Powers of this matrix obviously do not have Jacobsthal polynomials as their entries.

Therefore, two questions arise:

(i) How may the Jacobsthal matrices $\begin{bmatrix} J_n & J_{n+1} \\ J_{n+1} & J_{n+2} \end{bmatrix}$, $n > 2$, be generated?

(ii) What is the result if we scale these matrices down as in (1.8) and let $n \to \infty$?

The answer to (i) is associated with the matrix H [$\equiv H(x)$]

$$H = \begin{bmatrix} 0 & 1 \\ x & 1 \end{bmatrix}. \tag{2.4}$$

Using (2.1)-(2.4) and induction, we readily obtain

$$H^n J = \begin{bmatrix} J_{n+1} & J_{n+2} \\ J_{n+2} & J_{n+3} \end{bmatrix}, \tag{2.5}$$

so question (i) is answered.

Let the matrices generated by powers of H in (2.5) be represented as

$$\mathscr{J}_n = H^n J. \tag{2.6}$$

We call the set of matrices $\{\mathscr{J}_n\}$ the *Jacobsthal matrices*, since all their entries are Jacobsthal polynomials.

Scaling down the Jacobsthal matrices, we have

$$\mathscr{J}_n^* = \begin{bmatrix} 1 & \dfrac{J_{n+2}}{J_{n+1}} \\ \dfrac{J_{n+2}}{J_{n+1}} & \dfrac{J_{n+3}}{J_{n+1}} \end{bmatrix}. \tag{2.7}$$

Now, the Binet form for J_n can be found by routine measures (see [2] and [8]) to be

$$J_n = \frac{\gamma^n - \delta^n}{\sqrt{1 + 4x}}, \quad x \neq -\frac{1}{4}, \tag{2.8}$$

where

$$\gamma = \frac{1 + \sqrt{1 + 4x}}{2}, \quad \delta = \frac{1 - \sqrt{1 + 4x}}{2} \tag{2.9}$$

are the roots of the characteristic equation

$$\lambda^2 - \lambda - x = 0 \tag{2.10}$$

for the recurrence relation (2.1).

Let $x > -1/4$. Elementary calculations reveal that $|\delta/\gamma| < 1$. Hence,

$$\lim_{n\to\infty} \frac{J_{n+1}}{J_n} = \gamma, \tag{2.11}$$

and

$$\lim_{n\to\infty} \frac{J_{n+2}}{J_n} = \gamma^2, \tag{2.12}$$

so that the limiting form of $\mathcal{J}_n^*$ is

$$\begin{bmatrix} 1 & \gamma \\ \gamma & \gamma^2 \end{bmatrix} \tag{2.13}$$

When $x = -1/4$, $\gamma = \delta = 1/2$. Hence, $J_n = n/2^{n-1}$ by standard methods of difference equations where the roots of the characteristic equation are equal. Therefore, (2.13) still holds.

If $x < -1/4$, then from (2.1)

$$J_n = \frac{2(\sqrt{-x})^n}{\sqrt{-1 - 4x}} \sin(n\tau)$$

where $\cos\tau = 1/2\sqrt{-x}$ and $\sin\tau = \sqrt{-1 - 4x}/2\sqrt{-x}$. Therefore,

$$\frac{J_{n+1}}{J_n} = \sqrt{-x}\,\frac{\sin(n+1)\tau}{\sin(n\tau)} = \left(\frac{1}{2} + \frac{(\cot n\tau)\sqrt{-1 - 4x}}{2}\right). \tag{2.14}$$

<u>Theorem 7</u>: There is no real number τ having the property that

$\lim_{n\to\infty} \cot(n\tau)$ exists as a finite real number or $\pm\infty$.

<u>Case I</u>. Suppose that τ is a rational multiple of π, say $\tau = (p/q)\pi$, where p is an integer and q is a natural number. Then $\cot(n\tau)$ is not even defined for integers n that are multiples of q.

In each of the cases to follow, it will be assumed that τ is *not* a rational multiple of π. Then $\sin\tau \neq 0$ and $\sin(n\tau) \neq 0$ for any positive integer n. So the formula

$$\cot(n+1)\tau = \frac{\cot(n\tau)\cot\tau - 1}{\cot(n\tau) + \cot\tau} \tag{2.15}$$

is valid. Note also that $\cot\tau \neq 0$. Furthermore, $\cot(n\tau) \neq 0$ for any positive integer n since this would imply that τ is a rational multiple of π.

<u>Case II</u>. If $\lim\cot(n\tau) = \pm\infty$, then (2.15) yields

$$\infty = \lim_{n\to\infty}\cot(n\tau) = \lim_{n\to\infty}\cot(n+1)\tau = \lim_{n\to\infty}\frac{\cot\tau - \dfrac{1}{\cot(n\tau)}}{1 + \dfrac{\cot\tau}{\cot(n\tau)}} = \cot\tau,$$

which is impossible.

<u>Case III</u>. Suppose that $\lim_{n\to\infty}\cot(n\tau) = r$, where r is some real number. Set $s = \cot\tau$. If $r + s \neq 0$, then from (2.15),

$$r = \frac{rs - 1}{r + s},$$

$$r^2 + rs = rs - 1,$$

and

$$r^2 = -1, \text{ which is impossible.}$$

If $r + s = 0$, then in order to obtain a finite limit in (2.15), it must follow that $rs - 1 = 0$. Thus,

$$r = -s = \frac{1}{s}$$

or

$$s^2 = -1, \text{ which is impossible.}$$

It has now been shown that, for all possible choices of τ, $\lim_{n\to\infty} \cot(n\tau)$ cannot exist. Hence, $\operatorname{Lim}_{n\to\infty}(J_{n+1}/J_n)$ does not exist.

Much more can be said about other properties of the Jacobsthal polynomials J_n. They are, in fact, a special case of the $w_n(a, b; p, q)$ discussed in [6], where $p = 1$, $q = -x$. See the Historical Note below for Jacobsthal's original contributions and [5] for additional properties.

3. HISTORICAL NOTE

The recurrence relation (2.1) is associated with the name of Jacobsthal [7] who, in 1919, seems to be the first to record it. His notation is related to ours by the correspondence where $F_n(x)$ are the Fibonacci polynomials defined by $F_1(x) = 1$, $F_2(x) = 1$, $F_{n+2}(x) = xF_{n+1}(x) + F\ (x)$.

Using methods different from ours, Jacobsthal established the Binet form (2.8). Among other basic results demonstrated by him are, in his notation,

(a) the explicit summation formula

$$F_n(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k$$

and

(b) the extension of the definition of $F_n(x)$ to negative values of n. That is,

$$F_{-n}(x) = (-1)^n \frac{F_{n-2}(x)}{x^{n-1}}, \quad n \geqslant 1.$$

Both of the above results can be readily converted, with due care, into our J-notation by means of the stated correspondence.

Although Jacobsthal alludes to the polynomials (2.2) as "Fibonacci polynomials," they are now known by his name; in fairness, then, the matrices whose entries are Jacobsthal polynomials must also bear his name.

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INTEGERS RELATED TO THE BESSEL FUNCTION $J_1(z)$

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1. INTRODUCTION

Let $J_\nu(z)$ denote the Bessel function of the first kind and let $j_{\nu,r}$ denote the zeros of $z^{-\nu}J_\nu(z)$, with $|R(j_{\nu,r})| \leqslant |R(j_{\nu,r+1})|$. The Rayleigh function of order $2n$, $\sigma_{2n}(\nu)$, is defined by

$$\sigma_{2n}(\nu) = \sum_{r=1}^{\infty} (j_{\nu,r})^{-2n} \qquad (n = 1, 2, 3, \ldots).$$

The early history of this function can be found in [10, p. 502]; more recently it has been investigated by Kishore [5], [6] and others. The first twelve Rayleight functions have been computed by Lehmer [8].

It is known that

$$\sigma_{2n}(1/2) = (-1)^{n-1}\frac{2^{2n-1}}{(2n)!}B_{2n},$$

$$\sigma_{2n}(-1/2) = (-1)^{n}\frac{2^{2n-2}}{(2n)!}G_{2n},$$

where B_{2n} is the $2n^{\text{th}}$ Bernoulli number and G_{2n} is the Genocchi number, i.e.,

$$G_{2n} = 2(1 - 2^{2n})B_{2n}.$$

A few other special cases have been examined. The writer [2], [3], and [4] has studied the cases $\nu = \pm 3/2$ and Carlitz [1] has investigated the integers a_r defined by

$$\sigma_{2r}(0) = \frac{2^{-2r}}{r!(r-1)!}a_r. \tag{1.1}$$

Carlitz points out that in view of the known arithmetic properties of the Bernoulli and Genocchi numbers, it is of interest to look for arithmetic properties of $\sigma_{2n}(\nu)$ for other values of ν.

In the present paper we define integers b_r by means of

$$\sigma_{2r}(1) = \frac{2^{-2r}}{r!(r+1)!}b_r, \tag{1.2}$$

and examine their arithmetic properties. A summary of these properties, along with a possible generalization of (1.1) and (1.2), is given in Section 4. A listing of the first 24 values of b_n is presented in section 5.

2. PRELIMINARIES

Using formulas (6), (14), and (22) in [5], we can write a generating function and recurrence formulas for b_n. We have

$$\frac{-x}{2}\frac{J_1'(x)}{J_1(x)} + \frac{1}{2} = \sum_{n=1}^{\infty} \frac{2^{-2n}}{n!(n+1)!} b_n x^{2n}, \tag{2.1}$$

$$(-1)^n(n+1)b_n = -n(n+1) + \sum_{r=1}^{n-1}(-1)^{r-1}\binom{n+1}{r+1}\binom{n+1}{r} b_r, \tag{2.2}$$

$$(n+1)^2 b_n = \sum_{r=1}^{n-1}\binom{n+1}{r+1}\binom{n+1}{r} b_r b_{n-r}. \tag{2.3}$$

It follows from (1.2) that $b_1 = 1$, $b_2 = 1$, $b_3 = 3$, $b_4 = 16$. In some of our proofs it will be convenient to rewrite (2.2) in the following way:

$$(-1)^n(n+1)b_n = -n(n+1) + \sum_{r=1}^{n-1} A(n, r), \tag{2.4}$$

where

$$A(n, r) = (-1)^{r-1}\binom{n+1}{r+1}\binom{n+1}{r} b_r.$$

To derive properties of b_n from (2.2) and (2.3) we need the following lemmas, the first due to Lucas [9] and the second due to Kummer [7]. In Lemma 2.2, and throughout this paper, we use the notation $p^m \| h$ to mean $p^m | h$ and $p^{m+1} \nmid h$.

Lemma 2.1: If p is a prime number and

$$n = n_0 + n_1 p + \cdots + n_k p^k \qquad (0 \leqslant n_i < p)$$
$$r = r_0 + r_1 p + \cdots + r_k p^k \qquad (0 \leqslant r_i < p),$$

then

$$\binom{n}{r} \equiv \binom{n_0}{r_0}\binom{n_1}{r_1} \cdots \binom{n_k}{r_k} \pmod{p}.$$

Lemma 2.2: With the hypotheses of Lemma 2.1, let $n - r = s_0 + s_1 p + \cdots + s_k p^k$ with $0 \leqslant s_i < p$, and suppose

$$\begin{aligned} r_0 + s_0 &= u_0 p + c_0 && (0 \leqslant c_0 < p) \\ u_0 + r_1 + s_1 &= u_1 p + c_1 && (0 \leqslant c_1 < p) \\ &\vdots \\ u_{k-1} + r_k + s_k &= u_k p + c_k && (0 \leqslant c_k < p). \end{aligned}$$

Then

$$p^N \left\| \binom{n}{r}\right., \text{ where } N = u_0 + u_1 + \cdots + u_k.$$

It follows from Lemma 2.2 that, if $r_j > n_j$ and $r_{j+t} \geqslant n_{j+t}$ for $t = 1, \ldots, q - 1$, then

$$\binom{n}{r} \equiv 0 \pmod{p^q}.$$

It may be of interest to note the following relationship between the numbers defined by (1.1) and (1.2). This formula follows easily from Eq. (20) in [5]: for $n > 1$,

$$na_n = \sum_{r=1}^{n-1}\binom{n}{r}\binom{n}{r+1} b_r a_{n-r}.$$

3. PROPERTIES OF b_n

Since

$$\binom{n+1}{r+1}\binom{n+1}{r}\Big/(n+1)$$

is always an integer, it is evident from (2.2) that the b_n are positive integers. Our first five theorems are concerned with determining the prime factors of b_n.

Theorem 3.1: Let $n = 2^k m$, $k \geqslant 0$, m odd. Then $b_n \equiv 0 \pmod{m}$.

Proof: The proof is by induction on n. Using the table in Section 5, we can verify the theorem for $n = 1, 2, \ldots, 24$. Assume it is true for $n = 1, \ldots, j - 1$ and suppose $p^s \| j$, $p > 2$. In (2.4) replace n by j and suppose $p^t \| r$ for a fixed r. If $s < t$, then $b_r \equiv 0 \pmod{p^s}$ by the induction hypothesis. If $0 < t < s$, then

$$b_r \equiv 0 \pmod{p^t} \quad \text{and} \quad \binom{j+1}{r+1} \equiv 0 \pmod{p^{s-t}} \quad \text{by Lemma 2.2.}$$

If $t = 0$, then

$$\text{either } \binom{j+1}{r} \equiv 0 \pmod{p^s} \quad \text{or} \quad \binom{j+1}{r+1} \equiv 0 \pmod{p^s} \quad \text{by Lemma 2.2.}$$

In all cases, $A(j, r) \equiv 0 \pmod{p^s}$, and by (2.4) we see that $b_j \equiv 0 \pmod{p^s}$. This completes the proof.

It follows that if p is an odd prime then $b_p \equiv 0 \pmod{p}$. Also, if we replace n by $p - 1$ in (2.2) and observe that

$$\binom{p}{r+1}\binom{p}{r} \equiv 0 \pmod{p^2} \text{ for } r = 1, \ldots, p-2,$$

we have

$$b_{p-1} \equiv 1 \pmod{p}, \tag{3.1}$$

where p is an odd prime. The next two theorems give more results along this line.

Theorem 3.2: Let p be an odd prime and $0 \leqslant k < p - 2$. Then $b_{mp+k} \equiv 0 \pmod{p}$ for all $m \geqslant 1$.

Proof: We first show the theorem is true for $m = 1$. It is true for $m = 1$, $k = 0$, by Theorem 3.1. Assume it is true for $m = 1$ and $k = 0, \ldots, j - 1$, with $j < p - 2$. Then by (2.4) and Lemma 2.1, we have

$$\begin{aligned}(-1)^{p+j}(p+j+1)b_{p+j} &= -(p+j)(p+j+1) + \sum_{r=1}^{p+j-1} A(p+j, r)\\ &\equiv -j(j+1) + \sum_{r=1}^{j} A(p+j, r) \pmod{p}\\ &\equiv -j(j+1) + \sum_{r=1}^{j} A(j, r) \pmod{p} \equiv 0 \pmod{p},\end{aligned}$$

the last congruence following from (2.4). Thus, the theorem is true for $m = 1$. Now assume it is true for $m = 1, \ldots, h - 1$. We know $b_{hp} \equiv 0 \pmod{p}$ by Theorem 3.1, so we also assume the theorem is true for $m = h$ and $k = 0, \ldots, j - 1$, with $j < p - 2$. Then, as in the first part of the proof, we have

$$(-1)^{hp+j}(hp + j + 1)b_{hp+j} \equiv -j(j + 1) + \sum_{r=1}^{j} A(j, r) \equiv 0 \pmod{p},$$

which completes the proof.

Theorem 3.2 tells us that if $n > p - 1$ and $n \not\equiv -1$, $n \not\equiv -2 \pmod{p}$, then $b_n \equiv 0 \pmod{p}$. The cases $n \equiv -1$, $n \equiv -2 \pmod{p}$ are examined in the following theorem.

Theorem 3.3: Let p be an odd prime. Then for all $m > 1$, $b_{mp-1} \equiv b_{mp-2} \equiv a_m \pmod{p}$, where a_m is defined by (1.1).

Proof: In (2.2), we replace n by $mp - 1$ and divide out p. Then, by Lemma 2.1, Lemma 2.2, and Theorem 3.2,

$$(-1)^{mp-1}b_{mp-1} \equiv 1 + \sum_{r=1}^{m-1}(-1)^r\binom{m}{r}\binom{m-1}{r-1}b_{rp-1} \pmod{p},$$

with $b_{p-1} \equiv 1 \pmod{p}$. In [1] it is shown that $a_1 = 1$ and

$$(-1)^{m-1}a_m = 1 + \sum_{r=1}^{m-1}(-1)^r\binom{m}{r}\binom{m-1}{r-1}a_r. \tag{3.2}$$

It follows that $b_{mp-1} \equiv a_m \pmod{p}$. Now, in (2.2), replace n by $mp - 2$. Then we have

$$(-1)^{m-1}b_{mp-2} \equiv -2 + \sum_{r=1}^{p-2}A(p-2, r) + \sum_{r=2}^{m-1}(-1)^r\binom{m-1}{r-1}^2 b_{rp-2}$$
$$+ \sum_{r=1}^{m-1}(-1)^r\binom{m-1}{r}\binom{m-1}{r-1}b_{rp-1} \pmod{p}. \tag{3.3}$$

Note that $-2 + \sum A(p - 2, r) \equiv 0$ by (2.4). We see from (3.3) that

$$b_{2p-2} \equiv 1 \equiv a_2 \equiv b_{2p-1} \pmod{p};$$

we now proceed to show $b_{mp-2} \equiv a \pmod{p}$ by using induction on m in (3.3). If Theorem 3.3 is true for $m = 2, \ldots, j - 1$, then by (3.3) we have

$$(-1)^{j-1}b_{jp-2} \equiv \sum_{r=2}^{j-1}(-1)^r a_r\binom{j-1}{r-1}\left[\binom{j-1}{r-1} + \binom{j-1}{r}\right] - j + 1$$
$$\equiv 1 + \sum_{r=1}^{j-1}(-1)^r\binom{j}{r}\binom{j-1}{r-1}a_r \equiv a_j \pmod{p}.$$

This completes the proof of Theorem 3.3.

Carlitz [1] has shown that, if $n = mp^r$, then $a_n \equiv a_m \pmod{p}$ for $r = 0, 1, 2, \ldots$. Therefore, we have the following corollary.

Corollary: If p is an odd prime and $n = mp^r - 1$ or $n = mp^r - 2$, then $b_n \equiv a_m \pmod{p}$ for $r = 1, 2, 3, \ldots$.

It follows from the corollary that, if $m > p$ and $p \nmid m$, then $b_n \equiv 0 \pmod{p}$ for $n = mp^r - 1$ or $n = mp^r - 2$.

We next show that Theorem 3.3 is valid for $p = 2$.

Theorem 3.4: For $m \geqslant 1$, $b_{2m+1} \equiv b_{2m} \equiv a_{m+1} \pmod{2}$.

Proof: We first show that $b_{4m} \equiv 0 \pmod{2}$ for all $m \geqslant 1$. It is clear from Lemma 2.1 that

$$\binom{4m+1}{r}\binom{4m+1}{r+1} \equiv 0 \pmod{2} \text{ for } r \equiv 1, 2, \text{ or } 3 \pmod{4}.$$

Therefore, by (2.2), we have

$$b_{4m} \equiv \sum_{r=1}^{m-1} \binom{4m+1}{4r}\binom{4m+1}{4r+1} b_{4r} \pmod{2}.$$

Since $b_4 = 16$, we can now easily prove by induction that $b_{4m} \equiv 0 \pmod{2}$. Now we replace n by $2m + 1$ in (2.2) and divide out $2m + 2$. Then we have

$$\begin{aligned} b_{2m+1} &\equiv 1 + \sum_{r=1}^{m} \binom{m}{r}\binom{m+1}{r} b_{2r} + \sum_{r=1}^{m} \binom{m+1}{r}\binom{m}{r-1} b_{2r-1} \\ &\equiv 1 + \sum_{r=1}^{m} \binom{m+1}{r}\binom{m}{r-1} b_{2r-1} \pmod{2}, \end{aligned}$$

because $b_{4k} \equiv 0 \pmod{2}$ and because

$$\binom{m}{r}\binom{m+1}{r} \equiv 0 \pmod{2} \text{ if } r \text{ is odd.}$$

Since $b_1 = 1$, we now see by (3.2) that $b_{2m+1} \equiv a_{m+1} \pmod{2}$.

Next assume that $b_{2m} \equiv a_{m+1} \pmod{2}$ for $m = 1, \ldots, j - 1$. Replace n by $2j$ in (2.2) to obtain

$$b_{2j} \equiv \sum_{r=1}^{j-1} \binom{j}{r}^2 a_{r+1} + \sum_{r=1}^{j} \binom{j}{r}\binom{j}{r-1} a_r \equiv -1 + \sum_{r=1}^{j} \binom{j+1}{r}\binom{j}{r-1} a_r \pmod{2}.$$

By (3.2), we now have $b_{2j} \equiv a_{j+1} \pmod{2}$, which completes the proof.

It follows that, if $n = 2^k - 1$ or $n = 2^k - 2$, then b_n is odd, $k = 1, 2, 3, \ldots$. Otherwise b_n is even. These facts enable us to extend Theorem 3.1.

Theorem 3.5: $b_n \equiv 0 \pmod{n}$ unless $n = 2^j$, $j = 2, 3, \ldots$. If $n = 2^j - 2$, then $b_n \equiv 0 \pmod{n/2}$.

Proof: We use induction on n. Theorem 3.5 is valid for $n = 1, 2, \ldots, 24$; assume it is true for $n = 1, \ldots, k - 1$. We assume k is even and $k \neq 2^j - 2$, since otherwise, by Theorem 3.1, there is nothing to prove. Assume $2^s \| k$ and $2^t \| r$ for a fixed r, $1 \leqslant r \leqslant k - 1$. If $t > s$, then $b_r \equiv 0 \pmod{2^s}$ by induction hypothesis, and $A(k, r) \equiv 0 \pmod{2^s}$. If $1 < t < s$, then

$$\binom{k+1}{r+1}\binom{k+1}{r} \equiv 0 \pmod{2^{2s-2t}}$$

and $b_r \equiv 0 \pmod{2^t}$, so $A(k, r) \equiv 0 \pmod{2^s}$. If $1 < t < s$, then

$$\binom{k+1}{r+1}\binom{k+1}{r} \equiv 0 \pmod{2^{2s-2}},$$

and $A(k, r) \equiv 0 \pmod{2^s}$. Thus, if $t > 0$ and $s > 1$, $A(k, r) \equiv 0 \pmod{2^s}$. It is now easy to see that, if $s > 1$, we have, by (2.4) and Lemma 2.2,

$$b_k \equiv A(k, 1) + A(k, 2^s - 1) \equiv 2^{s-1} + 2^{s-1} \equiv 0 \pmod{2^s}.$$

If $s = 1$, let $2^{m+1} \| (k+2)$, $m \geqslant 1$. Then by (2.4),

$$b_k \equiv \sum_{i=1}^{m} A(k, 2^i - 1) + \sum_{i=2}^{m+1} A(k, 2^i - 2) \equiv 2m \equiv 0 \pmod 2,$$

and the proof is complete.

If we replace n by an odd prime in (2.2), then since

$$\binom{p+1}{r+1}\binom{p+1}{r} \equiv 0 \pmod{p^2} \text{ for } r = 2, \ldots, p-2,$$

it is easy to see that

$$b_p \equiv p \pmod{p^2}. \tag{3.4}$$

In the same way, we can show that if $p > 3$, then

$$b_{p+1} \equiv \frac{7}{6}p \pmod{p^2}. \tag{3.5}$$

If we set $b_{p+n} \equiv pd_n \pmod{p^2}$, we can find a simple generating function for d_n.

Theorem 3.6: Let p be an odd prime and let $0 \leqslant n \leqslant p-3$. Then $b_{p+n} \equiv pd_n$ (mod p^2), where

$$1 + \sum_{n=0}^{\infty} \frac{d_n(x/2)^{2n+2}}{n!(n+1)!} = \left(\frac{x}{2J_1(x)}\right)^2.$$

Proof: Define $d_n^{(p)}$ by $b_{p+n} \equiv pd_n^{(p)} \pmod{p^2}$ for $0 \leqslant n \leqslant p-3$, and replace n by $p+n$ in (2.3). Using Lemma 2.1, we see that $d_n^{(p)} \equiv d_n \pmod p$, where

$$(n+1)^2 d_n = 2\sum_{r=1}^{n} \binom{n+1}{r+1}\binom{n+1}{r} b_r d_{n-r} + \frac{2b_{n+1}}{n+2} \tag{3.6}$$

with $d_0 = 1$. We multiply both sides of (3.6) by $(x/2)^{2n+2}$ and sum, beginning at $n = 0$, to obtain

$$\frac{x}{2} D'(x) = 2B(x)D(x), \tag{3.7}$$

where

$$D(x) = 1 + \sum_{n=0}^{\infty} \frac{d_n(x/2)^{2n+2}}{n!(n+1)!},$$

$$B(x) = \sum_{n=1}^{\infty} \frac{b_n(x/2)^{2n}}{n!(n+1)!} = -\frac{x}{2}\,\frac{J'(x)}{J\ (x)} + \frac{1}{2},$$

the last equation following from (2.1). Thus,

$$\frac{D'(x)}{D(x)} = -2\,\frac{J_1'(x)}{J_1(x)} + \frac{2}{x}. \tag{3.8}$$

After integrating both sides of (3.8) and plugging in $x = 0$ to determine the constant, we have

$$D(x) = \left(\frac{x}{2J_1(x)}\right)^2,$$

which completes the proof.

Theorem 3.5 can be compared to a similar result for the a_n. Carlitz [1] has shown that for $1 \leqslant n < p$, $a_{p+n} \equiv c_n p \pmod{p^2}$, where the c_n are defined by

$$1 + \sum_{n=1}^{\infty} \frac{c_n (x/2)^{2n}}{(n-1)!(n-1)!} = (J_0(x))^{-2}.$$

Theorem 3.7: If p is a prime number and $n = p^s$, $s \geqslant 3$, then $b_n \equiv p^s \pmod{p^{s+1}}$. If p is odd, the congruence is valid for $s \geqslant 1$.

Proof: First, assume p is odd. Theorem 3.1 tells us that, if $p^t | r$, then $b_r \equiv 0 \pmod{p^t}$; we also note that, if $j = p^s - 1$, then $b_j \equiv 1 \pmod{p}$ by the corollary to Theorem 3.3. Now, in (2.4), replace n by p^s. It is clear from Lemma 2.2 and the above comments that $A(p^s, r) \equiv 0 \pmod{p^{s+1}}$ for $r = 2, \ldots, p^s - 2$. We therefore have, for $n = p^s$,

$$\begin{aligned}(p^s + 1)b_n &\equiv (p^s + 1)p^s + A(p^s, 1) + A(p^s, p^s - 1)\\ &\equiv (p^s + 1)p^s \pmod{p^{s+1}}.\end{aligned}$$

This proof is valid for $s \geqslant 1$.

For $p = 2$, the situation is more complicated. We first show that, if $m = 2^s - 1$ with $s > 2$, then $b_m \equiv 1 \pmod{4}$. In (2.4), replace n by $2^s - 1$, $s > 2$. It is easy to see by Lemma 2.2 and Theorem 3.5 that $A(2^s - 1, r) \equiv 0 \pmod{2^{s+2}}$ for each r except $r = 2^{s-1} - 1$; in that case, $A(2^s - 1, 2^{s-1} - 1) \equiv 0 \pmod{2^{s+1}}$. After dividing both sides of (2.4) by 2 , we have, for $m = 2^s - 1$,

$$b_m \equiv -1 + A(2^s - 1, 2^{s-1} - 1)/2^s \equiv -1 + 2 \equiv 1 \pmod{4}.$$

Now, replace n by 2^s in (2.4). For $r = 1, \ldots, 2^s - 1$, it is easy to see, by Lemma 2.2 and Theorem 3.5, that $A(2^s, r) \equiv 0 \pmod{2^{s+1}}$ if $2^t \| r$ with $t \geqslant 1$. If $t = 0$, then $A(2^s, r) \equiv 0 \pmod{2^{s+1}}$ except for $r = 1$, $2^s - 1$, and $2^{s-1} - 1$. We therefore have, by (2.4) with $w = 2^s$,

$$\begin{aligned}b_w &\equiv 2^s + A(2^s, 1) + A(2^s, 2^s - 1) + A(2^s, 2^{s-1} - 1)\\ &\equiv 2^s + 2^{s-1} + 2^{s-1} + 2^s \equiv 2^s \pmod{2^{s+1}}.\end{aligned}$$

4. SUMMARY

We have shown that the integers b_n defined by (1.2) have the following properties:

$b_n \equiv 0 \pmod{n}$ unless $n = 2^j - 2$, $j = 2, 3, \ldots$. If $n = 2^j - 2$, then $b_n \equiv 0 \pmod{n/2}$. (4.1)

$b_{mp+k} \equiv 0 \pmod{p}$ if p is an odd prime, $0 \leqslant k \leqslant p - 3$, and $m \geqslant 1$. (4.2)

$b_{mp-1} \equiv b_{mp-2} \equiv a_m \pmod{p}$ if p is any prime number, $m > 1$ and a_m is defined by (1.1). (4.3)

$b_{p+n} \equiv pd_n \pmod{p^2}$, if p is an odd prime, $0 \leqslant n \leqslant p - 3$, and d_n is defined by

$$1 + \sum_{n=0}^{\infty} \frac{d_n(x/2)^{2n+2}}{n!\,(n+1)!} = \left(\frac{x}{2J_1(x)}\right)^2. \tag{4.4}$$

$b_n \equiv p^s \pmod{p^{s+1}}$ if $n = p^s$, p any prime number, and $s \geqslant 3$.
If p is odd, the congruence is valid for $s \geqslant 1$. (4.5)

To generalize (1.1) and (1.2), we can define the numbers $a_{k,n}$ by

$$\sigma_{2n}(k) = \frac{2^{-2n}}{(n+k)!\,(n+k-1)!}\, a_{k,n}.$$

It is evident that $a_{0,n} = a_n$ and $a_{1,n} = b_n$. Also, $a_{k,1} = a_{k,2} = (k!)^2$. Formulas analogous to (2.1), (2.2), and (2.3) can be written down, but properties such as (4.1)-(4.5) do not appear to be obvious or easily proved.

5. TABLE OF VALUES

The following table of values for b_n was computed by Elmer Hayashi of Wake Forest University. The writer is grateful to Professor Hayashi for his assistance. The writer also wishes to thank John Baxley of Wake Forest and Sam Wagstaff of Purdue University for their help in proving that all the factors listed below are prime numbers.

Table of Values for b_n

$b_1 = 1$
$b_2 = 1$
$b_3 = 3$
$b_4 = 2^4$
$b_5 = 2 \cdot 5 \cdot 13$
$b_6 = 3^3 \cdot 5 \cdot 11$
$b_7 = 5 \cdot 7 \cdot 647$
$b_8 = 2^3 \cdot 7^2 \cdot 11 \cdot 103$
$b_9 = 2^2 \cdot 3^2 \cdot 7 \cdot 79 \cdot 547$
$b_{10} = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 777013$
$b_{11} = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 195407$
$b_{12} = 2^5 \cdot 3^2 \cdot 5 \cdot 11 \cdot 163 \cdot 193189$
$b_{13} = 2^2 \cdot 3 \cdot 11 \cdot 13 \cdot 449 \cdot 1229 \cdot 26119$
$b_{14} = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 677 \cdot 15473 \cdot 44983$
$b_{15} = 3^2 \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 2897 \cdot 9208057$
$b_{16} = 2^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 85619815212829$
$b_{17} = 2^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 263 \cdot 331 \cdot 379 \cdot 25452443$
$b_{18} = 2^3 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot 181 \cdot 827 \cdot 22338511427$
$b_{19} = 2^3 \cdot 3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 4974009342476711903$
$b_{20} = 2^5 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 19 \cdot 137 \cdot 315195497 \cdot 7249259477$
$b_{21} = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19^2 \cdot 395001666315568761311$
$b_{22} = 2^2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot 463 \cdot 1339414102904792813 3$
$b_{23} = 2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 47 \cdot 151 \cdot 60443 \cdot 3308491075235249$
$b_{24} = 2^5 \cdot 3^3 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 23 \cdot 24917 \cdot 21854261271093057456989$

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THE NUMBER OF SPANNING TREES IN THE SQUARE OF A CYCLE

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INTRODUCTION

A classic result known as the *Matrix Tree Theorem* expresses the number of spanning trees $t(G)$ of a graph G as the value of a certain determinant. There are special graphs G for which the value of this determinant is known to be obtained from a simple formula. Herein, we prove the formula $t(\mathscr{C}_n^2) = nF_n^2$, where F_n is a Fibonacci number, and $\mathscr{C}_n^2$ is the square of the n vertex cycle $\mathscr{C}_n$ using Kirchoff's matrix free theorem [7].

In this work graphs are undirected and, unless otherwise noted, assumed to have no multiple edges or self-loops. We shall follow the terminology and notation of the book by Harary [5]. The graph that consists of exactly one cycle on all its vertices is denoted by $\mathscr{C}_n$. The square G^2 of a graph G has the same vertices of G but u and v are adjacent in G^2 whenever the distance between u and v in G does not exceed 2.

The number of spanning trees of a graph G, denoted by $t(G)$, is the total number of distinct spanning subgraphs that are trees. The problem of finding the number of spanning trees of a graph arises in a variety of applications. In particular, it is of interest in the analysis of electric networks. It was in this context that Kirchhoff [7] obtained a classic result known as the matrix tree theorem. To state the result, we introduce the following matrices. The *Kirchhoff matrix* M of n-vertex graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ matrix $[m_{ij}]$ where $m_{ij} = -1$ if v_i and v_j are adjacent, and m_{ii} equals the degree of vertex i.

KIRCHHOFF'S MATRIX TREE THEOREM

For any graph with two or more vertices, all the cofactors of M are equal, and the value of each cofactor equals $t(G)$.

Clearly, the matrix tree theorem solves the problem of finding the number of spanning trees of a graph. Furthermore, we note that this is an effective result from a computational standpoint, as their are efficient algorithms for evaluating a determinant. However, for certain special cases, it is possible to give an explicit, simple formula for the number of spanning trees. For example, it is easy to see that this number is n if G is $\mathscr{C}_n$. Also, if G is the complete graph K_n, then a classic result known as *Cayley's tree formula* states that $t(K_n) = n^{n-2}$ (see Harary [5] for a proof). Another graph of special interest is the *wheel* W_n which consists of a single cycle $\mathscr{C}_n$ having an additional

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vertex, called the *center*, joined by an edge to each vertex on the cycle. In the case of wheels, there is a fascinating connection between the number of spanning trees, Lucas numbers, and Fibonacci numbers. Many authors including Harary, O'Neil, Read, and Schwenk [6], Sedláček [12], Rebman [10], and Bedrosian [1] have obtained results regarding this connection. The classic result is due to Sedláček who showed that

$$t(W_n) = ((3 + \sqrt{5})/2)^n + ((3 - \sqrt{5})/2)^n - 2 \text{ for } n \geqslant 3.$$

Another simple graph, which is a variant of a cycle, is $\mathscr{C}_n^2$ the square of a cycle.

For $n \geqslant 5$, the squared cycle $\mathscr{C}_n^2$ has all its vertices of degree 4. For $n = 5$, $\mathscr{C}_5^2 = K_5$; for $n = 4$, $\mathscr{C}_4^2 = K_4$; however, the vertices of K_4 have degree 3. In the case $n \geqslant 5$, the matrix M can be permuted into a circulant matrix form. Here we are assuming that an $n \times n$ circulant matrix K is one in which each row is a one-element shift of the previous row, i.e., $k_{ij} = k_{i+1,j+1}$, where the indices are taken modulo n. Namely for $\mathscr{C}_n^2$, $m_{ii} = 4$, $m_{ij} = -1$ if $|i - j| = 1, 2, n - 1$, or $n - 2$, and $m_{ij} = 0$ otherwise. Alternatively, as M is a circulant, it could be specified by its first row $(4, -1, -1, 0, 0, \ldots, 0, -1, -1)$.

Recently, Boesch and Wang [2] conjectured, without knowledge of [8], that $t(\mathscr{C}_n^2) = nF_n^2$, F_n being the Fibonacci numbers $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Herein, we prove that this formula is indeed correct. Clearly, by Kirchhoff's Theorem, if u_n denotes $t(\mathscr{C}_n^2)$, then u_n is the determinant of the $(n-1) \times (n-1)$ matrix V_{n-1}, where V_n is the following $k \times k$ matrix:

$$\begin{bmatrix}
4 & -1 & -1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -1 \\
-1 & 4 & -1 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
-1 & -1 & 4 & -1 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & -1 & -1 & 4 & -1 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\
\vdots & & & & & & & & & & \vdots \\
0 & \cdot & \cdot & \cdot & 0 & -1 & -1 & 4 & -1 & -1 & 0 \\
0 & \cdot & \cdot & \cdot & 0 & 0 & -1 & -1 & 4 & -1 & -1 \\
0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -1 & -1 & 4 & -1 \\
-1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -1 & -1 & 4
\end{bmatrix} = V_k.$$

For convenience of the proof, we introduce the following family of matrices, all of size $k \times k$:

A_k is the matrix obtained by deleting the first row and first column of V_{k+1}, whereas

$$B_k = \left[\begin{array}{c|c}
-1 & -1 \quad 0 \ldots 0 \\ \hline
-1 & \\
-1 & \\
0 & A_{k-1} \\
\vdots & \\
0 &
\end{array}\right], \qquad
C_k = \left[\begin{array}{c|c}
-1 \quad -1 \quad 0 \ \ldots & 0 \\ \hline
 & \vdots \\
A_{k-1} & 0 \\
 & -1 \\
 & -1
\end{array}\right],$$

$$D_k = \left[\begin{array}{c|ccc} -1 & -1 & 0 \ldots & 0 \\ \hline 4 & & & \\ -1 & & & \\ 0 & & B_{k-1} & \\ \vdots & & & \\ 0 & & & \end{array}\right]$$

Let a_k, b_k, c_k, d_k, v_k be respectively the determinants of A_k, B_k, C_k, D_k, V_k. Note that $u_n = v_{n-1}$.

Lemma 1: $v_n = a_n - a_{n-2} + 2(-1)^n c_{n-1}$.

Proof: We use the following simple identity:

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = (-1)^{n+1} a_{n1} \cdot \det \begin{bmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n} \end{bmatrix}$$

$$+ \det \begin{bmatrix} a_{11} & & \cdots & a_{1n} \\ \vdots & & & \\ a_{n-1,1} & & & \vdots \\ 0 & a_{n,2} & \cdots & a_{nn} \end{bmatrix} \tag{1}$$

Applying this to v_n, we obtain:

$$v_n = (-1)^n \det \left[\begin{array}{ccc|c} -1 \quad -1 & 0 \ldots & 0 & -1 \\ \hline & & & 0 \\ & A_{n-2} & & \vdots \\ & & & 0 \\ & & & -1 \\ & & & -1 \end{array}\right] + \det \left[\begin{array}{c|ccc} 4 & -1 \quad -1 & 0 \ldots 0 & -1 \\ \hline -1 & & & \\ -1 & & A_{n-1} & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{array}\right] \tag{2}$$

Now, applying the transpose version of (1) to each of the two matrices in (2), where M^t is the transpose of M, we get

$$v_n = (-1)^n c_{n-1} + (-1)^n (-1)^{n+1} a_{n-2} + (-1)^n \det C_{n-1}^t + a_n. \quad \square$$

We now proceed to ascertain the recursions that a_n, b_n, c_n, and d_n satisfy.

Lemma 2: (i) $a_n = 4a_{n-1} + b_{n-1} - d_{n-1}$

(ii) $b_n = b_{n-1} - a_{n-1}$

(iii) $d_n = 5b_{n-2} - b_{n-3} - 5b_{n-1}$

(iv) $c_n = -c_{n-1} + 4c_{n-2} - c_{n-3} - c_{n-4}$

Proof: (i) is obtained by expanding A_n with respect to the first column.

(ii) If we expand B_n with respect to the first row, we get

$$b_n = -a_{n-1} + \det(B_{n-1}^t) = -a_{n-1} + b_{n-1}.$$

(iii) We expand D_n with respect to the first row:

$$d_n = -b_{n-1} + \det\left[\begin{array}{c|c} 4 & -1 \quad 0 \ \dots \ 0 \\ \hline -1 & \\ 0 & A_{n-2} \\ \vdots & \\ 0 & \end{array}\right],$$

and by expanding further with respect to the first row,

$$d_n = -b_{n-1} + 4a_{n-2} + \det\left[\begin{array}{c|c} -1 & -1 \quad -1 \quad 0 \ \dots \ 0 \\ \hline 0 & \\ \vdots & A_{n-3} \\ 0 & \end{array}\right],$$

which is $d_n = -b_{n-1} + 4a_{n-2} - a_{n-3}$. Now, by using (ii) to substitute for a_{n-2} and a_{n-3}, we obtain the desired result.

(iv) We expand C_n with respect to the first row:

$$c_n = -c_{n-1} + \det\left[\begin{array}{c|c} 4 & -1 \quad 0 \ \dots \ 0 \\ \hline -1 & \\ -1 & \\ 0 & C_{n-2} \\ \vdots & \\ 0 & \end{array}\right]$$

$$= -c_{n-1} + 4c_{n-2} + \det\left[\begin{array}{c|c} -1 & -1 \quad 0 \ \dots \ 0 \\ \hline -1 & \\ 0 & C_{n-3} \\ \vdots & \\ 0 & \end{array}\right]$$

$$= -c_{n-1} + 4c_{n-2} - c_{n-3} + \det \begin{bmatrix} -1 & -1 \quad 0 \ldots 0 \\ 0 & \\ \vdots & C_{n-4} \\ 0 & \end{bmatrix}$$

or $c_n = -c_{n-1} + 4c_{n-2} - c_{n-3} - c_{n-4}$ as desired. □

We now establish that the sequence $\{v_n\}$ (and thus $\{u_n\}$) satisfies the same recursion as nF_n^2. For convenience, we use the following terminology. If we have a sequence $\{x_n\}$ and a recursion

$$\lambda_k x_{n+k} + \lambda_{k-1} x_{n+k-1} + \cdots + \lambda_0 x_0 = 0,$$

then we say $\{x_n\}$ fulfills the recursion given by

$$\lambda_k E^k + \lambda_{k-1} E^{k-1} + \cdots + \lambda_0 E^0 = 0,$$

where E is the shift operator $Ex_n = x_{n+1}$, $E^0 = 1$, and $\lambda_0, \lambda_1, \ldots, \lambda_k$ are constants.

Lemma 3: The sequence $\{v_n\}$ fulfills

$$(E + 1)^2(E^2 - 3E + 1)^2 = E^6 - 4E^5 + 10E^3 - 4E + 1 = 0.$$

Proof: By Lemma 1, $v_n = a_n - a_{n-2} + 2(-1)^n c_{n-1}$.

We shall first determine the recursion for b_n and, from this, determine a recursion for a_n. Then, by obtaining a recursion for c_n, we get a recursion for v_n.

By (ii) of Lemma 2 with $n = n + 1$, and by (iii) of Lemma 2 with $n = n - 1$, we obtain, by substitution in (i) of Lemma 2, that

$$b_n - b_{n+1} = a_n = 4a_{n-1} + b_{n-1} - 5b_{n-3} + b_{n-4} + 5b_{n-2}.$$

Now, substituting for a_{n-1} its value from (ii) of Lemma 2, we get

$$b_{n+1} - 5b_n + 5b_{n-1} + 5b_{n-2} - 5b_{n-3} + b_{n-4} = 0.$$

Hence, shifting the index so $b_{n+1} \to b_{n+5}$, we see that $\{b_n\}$ fulfills

$$p(E) = E^5 - 5E^4 + 5E^3 + 5E^2 - 5E + 1 = (E^2 - 3E + 1)^2(E + 1) = 0.$$

Since $a_n = b_n - b_{n+1}$, $\{a_n\}$ fulfills the same recursion.

By Lemma 2, the sequence $\{c_n\}$ fulfills

$$q(E) = E^4 + E^3 - 4E^2 + E + 1 = (E - 1)^2(E^2 + 3E + 1) = 0$$

and $(-1)^n c_n$ fulfills the recursion where E is to be replaced by $-E$. Which is

$$q(-E) = (E + 1)^2(E^2 - 3E + 1) = 0.$$

Since

$$v_n = a_n - a_{n-2} + 2(-1)^n c_{n-1},$$

and $(E + 1)^2(E^2 - 3E + 1)^2$ is a common multiple of $p(E)$ and $q(-E)$, v_n fulfills this recursion. □

Lemma 4: The sequence nF^2 fulfills

$$E^6 - 4E^5 + 10E^3 - 4E + 1 = 0.$$

Proof: Since

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

we obtain

$$nF_n^2 = \frac{n}{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2(-1)^n\right],$$

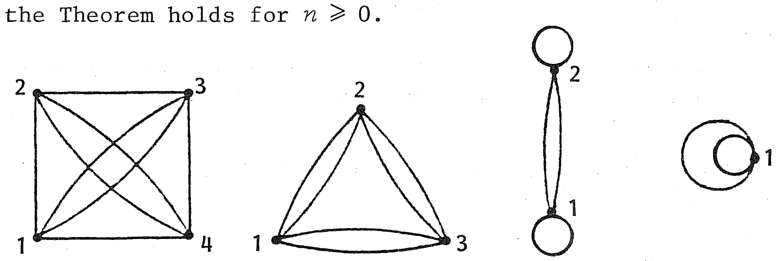
Now by the standard methods for finding the solution of a linear recursion relation via its characteristic polynomial, we see that nF_n^2 fulfills

$$\left(E - \frac{3+\sqrt{5}}{2}\right)^2 \cdot \left(E - \frac{3-\sqrt{5}}{2}\right)^2 \cdot (E+1)^2 = (E^2 - 3E + 1)^2(E+1)^2 = 0. \;\square$$

So we see that v_n, u_n, and nF_n^2 fulfill the same recursion. Since the computer computations of Boesch and Wang [2] tell us that $u_i = iF_i^2$, $5 \leqslant i \leqslant 16$, we know that the sequences coincide and have proved the following Theorem.

Theorem: The number of spanning trees of the square of the cycle $\mathscr{C}_n$, for $n \geqslant 5$, is given by nF_n^2.

Remarks: If we consider the square of a cycle for $n < 5$, which means that we consider the edge set to be a multiset, we have multiple edges and loops and the Theorem holds for $n \geqslant 0$.



$\mathscr{C}_4^2$ $4 \cdot 3^2 = 36$ $\mathscr{C}_3^2$ $3 \cdot 2^2 = 12$ $\mathscr{C}_2^2$ $2 \cdot 1^2 = 2$ $\mathscr{C}_1^2$ $1 \cdot 1^2 = 1$ $\mathscr{C}_0^2$ $0 \cdot 0^2 = 0$

Figure 1

In closing, we note that there is an alternative approcah to finding $t(\mathscr{C}_n^2)$ that uses the properties of circulant matrices. First, we note that M can be written as $4I - A$, where I is the identity matrix and A is the adjacency matrix of $\mathscr{C}_n^2$. If the maximum eigenvalue of the real, symmetric matrix A is denoted by λ_n, then a result of Sachs [11] states that

$$t(\mathscr{C}^2) = \frac{1}{n}\prod_{i=1}^{n-1}(4 - \lambda_i),$$

where λ_i are the eigenvalues of A. Now, using the explicit formulas for the eigenvalues of a circulant matrix (see, for example, Marcus and Minc [9]), one obtains

$$nt(\mathscr{C}^2) = \prod_{k=1}^{n-1} 4\sin^2\frac{\pi k}{n}\left(1 + 4\cos^2\frac{\pi k}{n}\right).$$

Thus, the Theorem could be proved by showing that the above product is n^2F^2. However, we have not found this approach to be any simpler than the one given here.

The authors would like to point out that reference [8] gives a purely combinatorial proof of our result, which was conjectured by Bedrosian in [1]. Furthermore, the paper by Kleitman and Golden was not discovered until after our paper had been refereed and accepted for publication.

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A RATIO ASSOCIATED WITH $\phi(x) = n$

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1. INTRODUCTION

Let $\phi(x)$ be Euler's totient function. The literature on solving the equation $\phi(x) = n$ (see [1, pp. 221-223], [2-5], [6, pp. 50-55, problems B36-B42], [7-11], [12, pp. 228-256], and the references therein) can be viewed as a collection of open problems. For $n = 2^\alpha$, we essentially have the problem of factoring the Fermat numbers. Another notorious example is Carmichael's conjecture [3, 7] that if a solution exists it is not unique. Some results (e.g., Example 15 of [12, pp. 238-239]) can be established on the basis of Schinzel's Conjecture H [12, p. 128] of which the twin prime conjecture is a very special case. See also [10, 11].

Here we define a new ratio $R(n)$ that is associated with this equation in a very natural way. Our main result, Theorem 3 of §3, is that $R(n)$ can be arbitrarily large. This can be read independently of §2, where the highest power of 2 dividing $R(n)$ is studied.

To define $R(n)$, let L_n be the least common multiple of all solutions of $\phi(x) = n$. Then, let G_n be the greatest common divisor of all numbers $a^n - 1$, where a is in the reduced residue system modulo L_n given by

$$1 \leqslant a \leqslant L_n, \qquad (a, L_n) = 1, \tag{1.1}$$

Since

$$a^n - 1 = a^{\phi(x)} - 1 \equiv 0 \bmod x \tag{1.2}$$

for any solution x, we have

$$a^n - 1 \equiv 0 \bmod L_n. \tag{1.3}$$

Hence, the ratio $R(n)$ defined by

$$R(n) = G_n / L_n \tag{1.4}$$

is an integer. For example, if $n = 2$, then x is 3, 4, or 6, so

$$L_2 = 12,\ G_2 = (1^2 - 1, 5^2 - 1, 7^2 - 1, 11^2 - 1) = 24, \tag{1.5}$$

and hence $R(2) = 2$.

Our L_n, G_n resemble Carmichael's L and M on pp. 221-222 of [1]. In fact, Carmichael very briefly alludes to the ratio M/L on p. 222. However, his table on p. 222 shows that his $M = M_n$ is often astronomical in comparison to our G_n, and that M_n/G_n need not be an integer.

We write $(m)_p$ for the highest power of the prime p in m, and $(m)_{\text{odd}}$ for $m/(m)_2$. Thus, $(m)_2 = 2^e$ is equivalent to $2^e \| m$. Theorem 3 of §3 asserts that,

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for every prime p and every $M > 0$, there is an $n = n(p, M)$ such that

$$(R(n))_p > M.$$

2. RESULTS ON PARITY

By means of induction, the binomial theorem, and the identity

$$z^2 - 1 = (z - 1)(z + 1),$$

it is easy to prove the following lemma.

Lemma 1: If $\alpha \geqslant 1$ is an integer, then

$$2^{\alpha+2} \| 11^{2^\alpha} - 1, \tag{2.1}$$

$$2^{\alpha+2} \| (8m + 5)^{2^\alpha} - 1, \tag{2.2}$$

and

$$2^{\alpha+2} | (2k + 1)^{2^\alpha} - 1. \tag{2.3}$$

Propositions 1-3 and Theorems 1 and 2 are consequences of this Lemma. We give the details of the proof for Theorem 2 only; the others are similar.

Write Φ for the set of all n such that $\phi(x) = n$ has a solution, and Φ' for the complement of this set.

Proposition 1: If $n \geqslant 2$, then $2|L_n$. If $n = 2n'$, where $n \in \Phi$ and $n' \in \Phi'$, then $2\|L_n$.

It is harder to show that infinitely often *every* solution is even; this is proved in [12, p. 238, Example 14].

Proposition 2: If $n \geqslant 2$, then $(R(n))_2 \geqslant 2$.

Proposition 3: If $(n)_2 = 2^\alpha$, then $(R(n))_2 \leqslant 2^{\alpha+1}$.

In the case of $n = 136 = 8 \cdot 17$, for example, the bound of Proposition 3 is exact.

Theorem 1: Let $s \geqslant 1$ be a fixed integer. If $t \geqslant 0$ is minimal, such that

$$n = 2^t(2s + 1) \in \Phi, \tag{2.4}$$

then

$$(R(n))_2 = 2^{t+1}. \tag{2.5}$$

We observe that again $n = 136 = 8 \cdot 17$ illustrates this result, since 17, 34, and 68 all belong to Φ'. Theorem 1 is proved with the aid of Proposition 3 which, in turn, is proved with the assistance of (2.2) of Lemma 1.

Corollary 1: If $s \geqslant 1$ is an integer and $n = 2(2s + 1) \in \Phi$, then $(R(n))_2 = 4$.

Proof: Clearly, $2s + 1 \in \Phi'$.

Corollary 2: Infinitely often $(R(n))_2 = 4$.

Proof: If p is any prime of the form $4s + 3$, then

$$4s + 2 = p - 1 = \phi(p). \tag{2.6}$$

We may vary s so that p runs over the primes of the form

$$p = 2^{t+1}s + 2^t + 1; \tag{2.7}$$

this implies that

$$\phi(p) = 2^t(2s + 1) \in \Phi. \tag{2.8}$$

However, it does *not* follow directly from crude density considerations and the prime number theorem for arithmetic progressions that the $2^h(2s + 1)$ for $1 \leqslant h < t$ will sometimes all lie in Φ'. In fact, Erdös [4] has proved that, for any $M > 0$, the number of elements of Φ not exceeding x is

$$\gg \frac{x}{\log x}(\log \log x)^M. \tag{2.9}$$

Corollary 3: Schinzel's Conjecture H [12, p. 128] implies that, for any fixed $t \geqslant 0$, the equality $(R(n))_2 = 2^{t+1}$ holds infinitely often.

Proof: For $t = 0, 1$, this follows unconditionally from Theorem 2 and Theorem 1, Corollary 2. For $t \geqslant 3$, we first show that there are infinitely many s for which the two polynomials

$$2s + 1, \quad 2^{t+1}s + 2^t + 1 \tag{2.10}$$

are simultaneously prime, whereas the $t - 1$ polynomials

$$2(2s + 1), \quad 2^2(2s + 1), \ \ldots, \ 2^{t-1}(2s + 1) + 1 \tag{2.11}$$

are all composite. In fact, for $(A, B) = 1$ and $A > 0$, the greatest common divisor of the infinite set

$$(2x + 1)[2A(2x + 1) + B], \qquad x = 1, 2, 3, \ldots, \tag{2.12}$$

is unity (a trivial exercise in [12, p. 130]). Hence, "condition S" of Conjecture H is satisfied for the first two polynomials, and the above assertion follows from [10] (use statement C_{13}, p. 1). Now write $p = 2^{t+1}s + 2^t + 1$ so

$$\phi(p) = 2^t(2s + 1) \in \Phi. \tag{2.13}$$

If

$$\phi(x) = 2^h(2s + 1), \qquad 0 \leqslant h < t, \tag{2.14}$$

then x must be divisible by a non-Fermat prime q such that

$$\phi(q) \mid 2^h(2s + 1). \tag{2.15}$$

Hence,

$$q - 1 = 2^g(2s + 1), \qquad 0 \leqslant g \leqslant h, \tag{2.16}$$

a contradiction. Hence, t satisfies the hypothesis of Theorem 1, and the result follows. C. Pomerance's proof does not use H.

Theorem 2: If $\alpha \geqslant 1$ and $n = 2^\alpha$, then $(R(n))_2 = 2$.

Proof: Since $\phi(2^{\alpha+1}) = n$, we have $2^{\alpha+1} \mid L_n$. Since for any odd m,

$$\phi(2^{\alpha+2}m) \geqslant 2^{\alpha+1} > 2^\alpha, \tag{2.17}$$

we have $2^{\alpha+1} \| L_n$.

For any integer s, we have $10|\phi(11s)$, so $\phi(11s) \neq 2^\alpha$. Hence (since $L_n \geqslant 12$ is true for $n \leqslant 12$, and is obvious for $n > 12$), the number 11 is in the reduced residue system. Thus,

$$G_n | 11^{2^\alpha} - 1 \tag{2.18}$$

and, by (2.1) of Lemma 1,

$$(G_n)_2 \leqslant 2^{\alpha+2}. \tag{2.19}$$

Because every element of the reduced residue system is odd, (2.3) of Lemma 1 yields $2^{\alpha+2} | (G_n)_2$. Hence, $(G_n)_2 = 2^{\alpha+2}$ and the result follows.

Remark: We know of no other cases in which $(R(n))_2 = 2$. For $\ell(\alpha) = [\log_2 \alpha] \leqslant 4$, numerical calculations suggest, for $n = 2^\alpha$, that

$$L_n = 2n \prod_{m=0}^{\ell(\alpha)} F_m \quad \text{and} \quad G_n = 2L_n, \tag{2.20}$$

where F_m is the Fermat number

$$F_m = 2^{2^m} + 1. \tag{2.21}$$

However, this simply reflects the fact that the Fermat numbers F_m are prime for $m \leqslant 4$, and (2.20) must fail for $\ell(\alpha) \geqslant 5$; see [12, pp. 237-238, Example 13]. It is possible that $(R(n))_{\text{odd}} > 1$ for infinitely many $n = 2^\alpha$. C. Pomerance has proved the converse of Theorem 2.

3. ARBITRARILY LARGE $R(n)$

Observe that

$$\phi(x) = 2 \iff x = 3,\ 4,\ \text{or}\ 6, \tag{3.1}$$

and

$$\phi(x) = 44 \iff x = 3 \cdot 23,\ 4 \cdot 23,\ \text{or}\ 6 \cdot 23. \tag{3.2}$$

We say that 23 is a *prime replicator* of 2.

Definition: The prime p is a *prime replicator* of m if all solutions of

$$\phi(x) = m(p - 1) \tag{3.3}$$

are given by $b_1 p, \ldots, b_r p$, where $b_1, \ldots, b_r$ are all solutions of

$$\phi(x) = m. \tag{3.4}$$

Theorem E: Given $m \geqslant 2$, all but $o(x/\log x)$ of the primes are prime replicators of m.

Proof: This is a result of Erdős [5, pp. 15-16]. His proof [5, pp. 15-18] uses Brun's method.

It follows by the prime number theorem for arithmetic progressions that every arithmetic progression containing infinitely many primes has infinitely many prime replicators of m.

Theorem 3: Let q be any prime, and $e \geqslant 1$ an integer. Then, for some n,

$$(R(n))_q \geqslant q^e. \tag{3.5}$$

Proof: Set $m = \phi(q^e)$. Let $b_1, \ldots, b_r$ be all solutions of $\phi(x) = m$. Set

$$B = [b_1, \ldots, b_r] \quad \text{and} \quad q^f = (B)_q. \tag{3.6}$$

Clearly, $f \geqslant e$. By Theorem E, we can choose k so that

$$p = q^f\phi(q^{2f})k + 1 > B \tag{3.7}$$

is a prime replicator of m. Then all solutions to

$$\phi(x) = n = m(p - 1) = q^f\phi(q^{2f})mk \tag{3.8}$$

are $b_1p, \ldots, b_rp$, so

$$L_n = [b_1, \ldots, b_r]p = Bp. \tag{3.9}$$

If a is in the reduced residue system, then

$$a = q^fh + t, \quad 0 \leqslant t < q^f, \quad (t, q) = 1. \tag{3.10}$$

Hence, for $Q = q^{2f}$, we have

$$\begin{aligned} a^n - 1 = (t + q^fh)^n - 1 &= t^n + nt^{n-1}q^fh + \cdots - 1 \\ &\equiv t^n - 1 \bmod Q \equiv s^{\phi(Q)} - 1 \bmod Q, \end{aligned} \tag{3.11}$$

where $(s, Q) = 1$. By Euler's generalization of Fermat's simple theorem, the above is congruent to zero, and hence

$$(G_n/L_n) = (G_n)_q/q^f \geqslant q^{2f}/q^f \geqslant q^e. \tag{3.12}$$

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HYPERPERFECT AND UNITARY HYPERPERFECT NUMBERS

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INTRODUCTION

In this paper, k, m, and n will represent arbitrary natural numbers; p, q, r, s, primes; and a, b, c, d, natural number exponents. σ is the sum-of-divisors function; σ^*, the sum-of-unitary divisors function; and τ, the count-of-prime-factors function.

Definition 1 [6]: A number m is said to be n-hyperperfect, n-HP, if it satisfies

$$m = 1 + n[\sigma(m) - m - 1]. \tag{1}$$

Definition 2 [2]: A number m is said to be n-unitary hyperperfect, n-UHP, if it satisfies

$$m = 1 + n[\sigma^*(m) - m - 1]. \tag{2}$$

For $n = 1$, the definitions reduce to those of the usual perfect and unitary perfect numbers. The two definitions agree on square-free numbers. To speak of both concepts simultaneously, we subsume equations (1) and (2) into

$$m = 1 + n[\Sigma(m) - m - 1] \tag{3}$$

and speak of n-(unitary) hyperperfect numbers, n-(U)HP.

1. PARITY

Theorem 1: Let m be n-(U)HP. Then:

(a) $(m, n) = 1$;
(b) If m is even, n is odd;
(c) If n is even, m is odd;
(d) $(m, \Sigma(m) - m - 1) = 1$;
(e) $(m, \Sigma(m) - 1) = 1$;
(f) $\tau(m) > 1$.

Proof: (a-e) Follow directly from (3).

(f) By contradiction. If $m = p^a$, $a > 1$, then

$p \ (m, \Sigma(m) - 1)$

which contradicts (e). ■

The possibility that both m and n are odd is not addressed in this theorem. The table of hyperperfect numbers in [3] includes odd m for odd n. For example, 325 is 3-HP. In the unitary case, we have a complete result.

Theorem 2: If m is n-UHP, then not both m and n are odd.

Proof: By contradiction. Assume $m = 2s + 1$, $n = 2t + 1$. Equation (2) becomes

$$2s + 1 = 1 + (2t + 1)[\sigma^*(m) - (2s + 1) - 1].$$

Expand and regroup.

$$4s + 2 = (2t + 1)\sigma^*(m) - 4ts - 4t.$$

Reduce modulo 4, remembering that $2t + 1$ is odd.

$$\sigma^*(m) \equiv 2 \bmod 4. \tag{4}$$

For (4) to be true, $\tau(m) = 1$. This contradicts Theorem 1(f). ■

Theorems 1 and 2 say that if m is n-UHP, not only are m and n relatively prime, they must be of opposite parity. The case in which $n = 1$ reduces to an old result.

Corollary 1 [7]: There are no odd unitary perfect numbers.

2. STRUCTURE THEOREMS

Equation (3) can also be written in the form

$$(n + 1)m = n\Sigma(m) - (n - 1). \tag{5}$$

Theorem 3: If m is n-HP, n odd, then m has as a component an odd prime to an odd power.

Proof: Let $m = 2^a m'$, $(2, m') = 1$. Equation (5) becomes

$$(n + 1)2^a m' = n\sigma(2^a)\sigma(m') - (n - 1).$$

The first and third terms are even since n is odd; n and $\sigma(2^a)$ are odd. Therefore $\sigma(m')$ is even. This happens only if an odd prime factor of m occurs to an odd power. ■

This argument yields no information in the unitary case, because $\sigma^*(m')$ is even. Note that the argument does not depend on a; it holds for $a = 0$.

Theorem 4: Let m be n-UHP, $m = p^a m'$, $(p, m') = 1$. Then

$$(p^a - n)(m' - n) \geqslant n^2 + 1.$$

Proof: Equation (5) becomes

$$(n + 1)m = n(p^a + 1)\sigma^*(m') - (n - 1) = np^a\sigma^*(m') + n\sigma^*(m') - (n - 1)$$

$$(n + 1)p^a m' - np^a\sigma^*(m') = n\sigma^*(m') - (n - 1)$$

$$p^a[(n + 1)m' - n\sigma^*(m')] = n\sigma^*(m') - (n - 1)$$

$$p^a = \frac{n\sigma^*(m') - (n - 1)}{(n + 1)m' - n\sigma^*(m')} \tag{6}$$

$\sigma^*(m') \geqslant m' + 1$ implies

$$(n + 1)m' - n\sigma^*(m') \leqslant (n + 1)m' - n(m' + 1); \qquad (7)$$

and

$$\frac{n\sigma^*(m') - (n - 1)}{(n + 1)m' - n\sigma^*(m')} \geqslant \frac{n(m' + 1) = (n - 1)}{(n + 1)m' - n(m' + 1)} = \frac{nm' + 1}{m' - n} = n + \frac{n^2 + 1}{m' - n}.$$

Thus,

$$p^a \geqslant n + \frac{n^2 + 1}{m' - n}$$

or

$$(p^a - n)(m' - n) \geqslant n^2 + 1. \blacksquare$$

Corollary 2: Let m be n-UHP, $m = p^a m'$, $(p, m') = 1$. Then

$$\frac{n + 1}{n} > \frac{\sigma^*(m')}{m'}$$

Proof: In (6), the numerator is positive; hence, so is the denominator: $(n + 1)m' - n\sigma^*(m') > 0$.

The inequality follows immediately. $\blacksquare$

Corollary 3: Let m be n-UHP, $m = p^a m'$, $(p, m') = 1$. Then

$$\frac{n + 1}{n} > \frac{m' + 1}{m'}$$

Proof: $\sigma^*(m') \geqslant m' + 1$. Alternatively, the right side of (7) is positive, as the left side is. $\blacksquare$

Corollary 4: Let m be n-UHP, $m = p^a q^b$. Then

$$(p^a - n)(q^b - n) = n^2 + 1.$$

Proof: In Theorem 4, $m' = q^b$. $\sigma^*(q^b) = q^b + 1$. Equation (7) is an equality. The result follows. $\blacksquare$

Corollary 5: For given n, there are finitely many m of the form $m = p^a q^b$ which are n-UHP.

Proof: From Corollary 4,

$$p^a = n + \frac{n^2 + 1}{q^b - n} \quad \text{and} \quad q^b = n + \frac{n^2 + 1}{p^a - n}.$$

There are finitely many solutions for p^a, q^b. $\blacksquare$

Corollary 6: There is exactly one unitary perfect number with two distinct prime divosors.

Proof: In Corollary 5, $n = 1$, $n^2 + 1 = 2$. There is only one solution for p^a, q^b; namely, 2, 3. $m = 6$. $\blacksquare$

Corollary 7: Let m be n-UHP, $p^a \| m$. Then $p^a > n$.

Proof: This is the penultimate inequality in the proof of Theorem 4. ■

The import of Corollary 7 is that, if m is n-UHP, then all unitary divisors of m, except 1, exceed n. In the nonunitary case, every divisor of m, except 1, exceeds n ([6], Theorem 1). Minoli and Bear ([6], Theorem 3) demonstrate bounds on the prime factors of an n-HP number of the form $m = pq$. These bounds can be proved for the unitary case with some generalization.

Corollary 8: Let m be n-UHP, $m = p^a q^b$, $p^a < q^b$. Then:

(a) If $n > 1$, $n < p^a < 2n < q^b \leqslant n^2 + n + 1$;

(b) If $n = 1$, $n < p^a \leqslant 2n < q^b \leqslant n^2 + n + 1$.

Further,

(c) For $n = 1, 2$, there are unique solutions.

Proof: The first inequality is Corollary 7. The last inequality arises from Corollary 4.

$$n^2 + 1 = (p^a - n)(q^b - n) \geqslant q^b - n;$$

thus,

$$q^b \leqslant n^2 + n + 1.$$

For the second inequality, rewrite equation (2) as

$$p^a q^b = 1 + np^a + nq^b < 1 + 2nq^b$$

$$p^a q^b \leqslant 2nq^b$$

$$p^a \leqslant 2n.$$

If $p = 2$, by Theorem 1, n is odd. Thus, equality is possible only for $n = 1$, $p^a = 2$. Equation (2) also yields

$$p^a q^b \geqslant 2np^a$$

$$q^b \geqslant 2n.$$

Again, if $q = 2$, n is odd. Equality is possible only for $n = 1$, $q^b = 2$. Then $\tau(m) = 1$, which contradicts Theorem 1 and the initial assumption. This completes the proof of the inequalities. For $n = 1$, they reduce to

$$1 < p^a \leqslant 2 < q^b \leqslant 3.$$

The only solution is $p^a = 2$; $q^b = 3$, $m = 6$. For $n = 2$,

$$2 < p^a < 4 < q^b \leqslant 7;$$

thus, $p^a = 3$. By Corollary 4, $q^b = 7$. ■

Theorem 5: If m is n-(U)HP, then

$$\frac{n}{n+1} \geqslant \frac{m}{\Sigma(m)} > \left(\frac{n}{n+1}\right)\left(\frac{m-1}{m}\right),$$

with equality on the left if and only if $n = 1$.

Proof: On division by $(n + 1)\Sigma(m)$, equation (5) becomes

$$\frac{m}{\Sigma(m)} = \frac{n}{n+1} - \frac{n-1}{(n+1)\Sigma(m)}. \tag{8}$$

The left inequality is immediate.

As $\Sigma(m) > m$,

$$\frac{n-1}{(n+1)\Sigma(m)} \leqslant \frac{n-1}{(n+1)m} \quad \text{and} \quad -\frac{n-1}{(n+1)\Sigma(m)} \geqslant -\frac{n-1}{(n+1)m}.$$

Equation (8) yields

$$\frac{m}{\Sigma(m)} \geqslant \frac{n}{n+1} - \frac{n-1}{(n+1)m} = \frac{nm-n+1}{(n+1)m} > \frac{nm-n}{(n+1)m} = \left(\frac{n}{n+1}\right)\left(\frac{m-1}{m}\right),$$

which is the inequality on the right. ■

Results on mod 3 properties have appeared before. In particular, Hagis [2] proved the following.

<u>Theorem 6</u>: Let m be n-UHP, then:

(a) If $m \not\equiv 0 \bmod 3$, then $m \equiv 1 \bmod 3$;
(b) If $n \equiv 0 \bmod 3$, then $m \equiv 1 \bmod 3$;
(c) If $n \equiv 1 \bmod 3$, then $\sigma^*(m) \equiv 2m \bmod 3$;
(d) If $n \equiv -1 \bmod 3$, then $\sigma^*(m) \equiv 2 \bmod 3$.

Results (b), (c), and (d) follow immediately from equation (3) and so are valid for the (ordinary) hyperperfect case also.

3. UNITARY HYPERPERFECT NUMBERS

The set of unitary hyperperfect numbers has nonempty intersections with the set of (ordinary) hyperperfect numbers and with the set of unitary perfects. In the first case, the intersection is the set of square-free hyperperfect numbers. In the second, it is the set (see [7], [11]) of 1-unitary hyperperfect numbers. For square-free hyperperfect numbers, see [4], [5], [6], [8], [9], and [10].

Hagis [2] ran a computer search for unitary hyperperfect numbers through 10^6. Buell [1] found 146 unitary hyperperfect numbers less than 10^8.

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ANNOUNCEMENT

The Second International Research Conference on Applications of the Fibonacci Numbers will be held in the San Francisco area immediately following the International Conference at the University of California at Berkeley in August 1986. Currently, we are in the planning stages and would be interested in receiving any comments from those who might consider attending. Send all comments or requests for information to:

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-550 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Show that the powers of -13 form a Fibonacci-like sequence modulo 181, that is, show that

$$(-13)^{n+1} \equiv (-13)^n + (-13)^{n-1} \pmod{181} \text{ for } n = 1, 2, 3, \ldots .$$

B-551 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Generalize on Problem B-550.

B-552 *Proposed by Philip L. Mana, Albuquerque, NM*

Let S be the set of integers n with $10^9 < n < 10^{10}$ and with each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 appearing (exactly once) in n.

(a) What is the smallest integer n in S with $11 | n$?
(b) What is the probability that $11 | n$ for a randomly chosen n in S?

B-553 *Proposed by D. L. Muench, St. John Fisher College, Rochester, NY*

Find a compact form for $\sum_{i=0}^{2n} \binom{2n}{i} L_{i+1}^2$.

B-554 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

For all n in $Z^+ = \{1, 2, \ldots\}$, prove that there exist x and y in Z^+ such that

$$(F_{4n-1} + 1)(F_{4n+1} + 1) = x^2 + y^2.$$

B-555 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

For all n in Z^+, prove that there exist x, y, and z in Z^+ such that

$$(F_{2n-1} + 4)(F_{2n+5} + 1) = x^2 + y^2 + z^2.$$

SOLUTIONS

Quadratic with an Integer Solution

B-526 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

Find all ordered pairs (m, n) of positive integers for which there is an integer x satisfying the equation

$$F_mF_nx^2 - [F_m(F_m, F_n) + F_nF_{(m,n)}]x + (F_m, F_n)F_{(m,n)} = 0.$$

Here (r, s) denotes the greatest common divisor of r and s.

Solution by Paul S. Bruckman, Fair Oaks, CA

We use the well-known relation

$$(F_m, F_n) = F_{(m,n)}. \tag{1}$$

Then, letting $d = F_{(m,n)}$, the given equation becomes

$$(F_mx - d)(F_nx - d) = 0, \tag{2}$$

to be satisfied for some integer x. Since $m \geqslant (m, n)$, $n \geqslant (m, n)$ and $(F_n)_{n=1}^{\infty}$ is an increasing sequence (except for $F_1 = F_2 = 1$), we see that for $x = d/F_m$ to be an integer, we must have one of the following:

(a) $F_m = F_{(m,n)}$ or (b) $F_n = F_{(m,n)}$.

These, in turn, imply at least one of the following:

(i) $m = 1$; (ii) $m = 2$; (iii) $m|n$; (iv) $n = 1$; (v) $n = 2$; (vi) $n|m$.

Some of these cases are redundant, and we can consolidate them as follows: all ordered pairs $\{m, n\}$ with (a) $m|n$; (b) $n|m$; (c) $m = 2$; (d) $n = 2$. (Note that there is still some redundancy, but this is minimal.)

Also solved by Paul S. Bruckman, Laszlo Cseh, A. Di Porto & P. Filipponi, Herta T. Freitag, Walther Janous, L. Kuipers, Bob Prielipp, Sahib Singh, and the proposer.

Another Quadratic with an Integer Solution

B-527 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

Do as in B-526 with the equation replaced by

$$(F_m, F_n)x^2 - (F_m + F_n)x + F_{(m,n)} = 0.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria

The given equation reads

$$F_{(m,n)}x^2 - (F_m + F_n)x + F_{(m,n)} = 0. \tag{1}$$

Since $(m, n)\,|\,m$ and $(m, n)\,|\,n$, it holds that $s_{m,n} = (F_m + F_n)/F_{(m,n)}$ is integral; that is, (1) reads $x^2 - s_{m,n}x + 1 = 0$. This symmetric equation has to have the double root $x_1 = x_2 = 1$, whence $F_m + F_n = 2F_{(m,n)}$.

Because $F_{(m,n)} \leqslant F_m$ and $F_{(m,n)} \leqslant F_n$, it follows that $F_m = F_n = F_{(m,n)}$. Thus, $m = n$ or $m = 1$, $n = 2$ or $m = 2$, $n = 1$.

Also solved by Paul S. Bruckman, A. Di Porto & P. Filipponi, Herta T. Freitag, L. Kuipers, Bob Prielipp, Sahib Singh, and the proposer.

Special Case of a Sum

B-528 *Proposed by Herta T. Freitag, Roanoke, VA*

For nonnegative integers n, prove that

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}F_{i+1}^2 = 5^nF_{2n+3}.$$

Solution by Marjorie Bicknell-Johnson, Santa Clara, CA

Let $p = 1$ in equation (4) on page 30 of the following article: "Some New Fibonacci Identities" by Verner E. Hoggatt, Jr., and Marjorie Bicknell, in *The Fibonacci Quarterly* 2, no. 1 (February 1964):29-32.

Also solved by Wray G. Brady, Paul S. Bruckman, Laszlo Cseh, Leonard A. G. Dresel, Piero Filipponi, C. Georghiou, Walther Janous, L. Kuipers, Graham Lord, George N. Philippou, Bob Prielipp, A. G. Shannon, Sahib Singh, J. Suck, Robert L. Vogel, and the proposer.

Compact Form for a Sum

B-529 *Proposed by Herta T. Freitag, Roanoke, VA*

For positive integers n, find a compact form for

$$\sum_{i=0}^{2n}\binom{2n}{i}F_{i+1}^2.$$

Solution by Leonard A. G. Dresel, University of Reading, England

Let $T = \sum_{i=0}^{2n}\binom{2n}{i}F_{i+1}^2$. Then

$$5T = \sum\binom{2n}{i}(\alpha^{i+1} - \beta^{i+1})^2 = \sum\binom{2n}{i}(\alpha^{2i+2} - 2\alpha\beta(\alpha\beta)^i + \beta^{2i+2})$$

$$= \alpha^2(1 + \alpha^2)^{2n} - 2\alpha\beta(1 + \alpha\beta)^{2n} + \beta^2(1 + \beta^2)^{2n}.$$

Now, since $n > 0$ and $\alpha\beta = -1$, the middle term vanishes, and

$$\alpha^2 + 1 = \alpha(\alpha - \beta) = \sqrt{5}\alpha \quad \text{and} \quad \beta^2 + 1 = \beta(\beta - \alpha) = -\sqrt{5}\beta.$$

Hence,

$$T = 5^{n-1}(\alpha^{2n+2} + \beta^{2n+2}) = 5^{n-1}L_{2n+2}.$$

Also solved by Marjorie Bicknell-Johnson, Wray G. Brady, Paul S. Bruckman, Laszlo Cseh, Piero Filipponi, C. Georghiou, Walther Janous, L. Kuipers, Graham Lord, D. L. Muench, George N. Philippou, Bob Prielipp, A. G. Shannon, Sahib Singh, J. Suck, Robert L. Vogel, and the proposer.

Lucas Continued Fraction

B-530 *Proposed by Michael Eisenstein, San Antonio, TX*

Let $\alpha = (1 + \sqrt{5})/2$. For n an odd positive integer, prove that the continued fraction

$$L_n + \cfrac{1}{L_n + \cfrac{1}{L_n + \dots}} = \alpha^n.$$

Solution by Graham Lord, Princeton, NJ

The simple continued fraction is convergent (see Hardy & Wright, for example). The limit x satisfies the inequality $L_n \leqslant x$, and is a root of the equation $L_n + 1/x = x$. Since n is odd, the latter equation can be rewritten as

$$(x - \alpha^n)(x - \beta^n) = 0,$$

from which, together with the inequality, it follows that α^n is the required value.

Also solved by Wray Brady, Paul S. Bruckman, Laszlo Cseh, Walther Janous, A. Di Porto & P. Filipponi, Leonard A. G. Dresel, Herta T. Freitag, C. Georghiou, L. Kuipers, I. Merenyi, D. L. Muench, Bob Prielipp, Sahib Singh, Robert L. Vogel, and the proposer.

Even Case of Lucas Continued Fraction

B-531 *Proposed by Michael Eisenstein, San Antonio, TX*

For n an even positive integer, prove that

$$L_n - \cfrac{1}{L_n - \cfrac{1}{L_n - \dots}} = \alpha^n.$$

Solution by Graham Lord, Princeton, NJ

The existence of the infinite continued fraction is first established. If x_k is the k^{th} convergent, then easy induction arguments show that

$$L_n - 1 \leqslant x_k \leqslant L_n,$$

and that

$$x_k + \frac{1}{x_k} > L_n;$$

the latter requires use of the identity

$$x_{k+1} + \frac{1}{x_{k+1}} = L_n + \left(x_k + \frac{1}{x_k} - L_n\right)\Big/(x_k L_n - 1).$$

So

$$x_k - x_{k+1} = x_k + \frac{1}{x_k} - L_n > 0.$$

Hence, x is a strictly decreasing sequence which is bounded below by $L_n - 1$; thus, the limit exists.

The value of the limit is a root of the equation $x = L_n - 1/x$, which can be rewritten as $(x - \alpha^n)(x - \beta^n) = 0$, since n is even. Because $x_k > L_n - 1$, the value of the continued fraction is α^n.

Also solved by Paul S. Bruckman, Laszlo Cseh, A. Di Porto & P. Filipponi, Leonard A. G. Dresel, Herta T. Freitag, C. Georghiou, Walther Janous, L. Kuipers, I. Merenyi, D. L. Meunch, Bob Prielipp, Sahib Singh, Robert L. Vogel, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-389 *Proposed by Andreas N. Philippou, University of Patras, Patras, Greece*

Show that

$$F_{n+2}^{(n-i)} = 2^n - 2^i(1 + i/2) \qquad (n \geqslant 2i + 1)$$

for each nonnegative integer i, where $F_{n+2}^{(n-i)}$ is the $n + 2$ Fibonacci number of order $n - i$ [1] and $F_3^{(1)} = 1$.

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order k." *The Fibonacci Quarterly 20*, no. 1 (1982):28-32.

H-390 *Proposed by M. Wachtel, Zurich, Switzerland*

For every m,

$2F_{2-m}F_{5+m} + (-1)^m(F_mF_{m+1} + F_{m+2}^2)$ has the unique value 11.

Find a general formula for analogous constant values, which should represent the terms of an infinite sequence.

Prove that no divisor of any of these terms is congruent to 3 or 7 modulo 10.

H-391 *Proposed by Lawrence Somer, Washington, D.C.*

For every n, show that no integral divisor of L_{2n} is congruent to 11, 13, 17, or 19 modulo 20. (This problem was suggested by Problem H-364 on p. 313 of the November 1983 issue of *The Fibonacci Quarterly*.)

SOLUTIONS

Any More?

H-363 *Proposed by Andreas N. Philippou, University of Patras, Patras, Greece (Vol. 21, no. 4, November 1983)*

For each fixed integer $k \geqslant 2$, let $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k, i.e., $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)}, & \text{if } 2 \leqslant n \leqslant k, \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)}, & \text{if } n \geqslant k+1. \end{cases}$$

Evaluate the series

$$\sum_{n=0}^{\infty} \frac{1}{f_{m^n}^{(k)}} \qquad (k \geqslant 2,\ m \geqslant 2).$$

Remark: The Fibonacci sequence of order k appears in the work of Philippou and Muwafi [*The Fibonacci Quarterly 20* (1982):28-32.]

Comment by Paul S. Bruckman, Carmichael, CA

Letting

$$S(k, m) = \sum_{n=0}^{\infty} (f_{m^n}^{(k)})^{-1},$$

to the best of my knowledge, the only known result (fairly well-known in fact), is

$$S(2, 2) = \sum_{n=0}^{\infty} 1/F_{2^n} = \tfrac{1}{2}(7 - \sqrt{5}) \doteq 2.381966.$$

I would be very surprised—indeed, amazed!—to learn of any other closed form expressions for $S(k, m)$.

Only Two!

H-364 *Proposed by M. Wachtel, Zurich, Switzerland (Vol. 21, no. 4, November 1983)*

For every n, show that no integral divisor of L_{2n+1} is congruent to 3 or 7 modulo 10.

Solution by Paul S. Bruckman, Carmichael, CA

Given any prime p with $p \equiv \pm 3 \pmod{10}$, then $(5/p) = (p/5) = -1$. It is sufficient to prove that p does not divide L_{2n+1} for all n, since any divisor of L_{2n+1} congruent to 3 or 7 (mod 10) must be divisible by such a prime. By the calculus of "complex residues" (see [1]),

$$\alpha^p \equiv \beta,\ \beta^p \equiv \alpha \pmod{p}. \tag{1}$$

This, in turn, inplies $\alpha^{p+1} \equiv \beta^{p+1} \equiv -1 \pmod{p}$; hence,

$$L_{p+1} \equiv -2 \pmod{p}, \qquad F_{p+1} \equiv 0 \pmod{p}.$$

In the sequel all congruences will be understood to be modulo p, and the notation "(mod p)" will be omitted wherever no confusion is likely to arise. We will let $e = e(p)$ denote the "entry point" (if any) of p in the Lucas sequence, i.e., e is the smallest positive integer k (if any) such that $L_k \equiv 0 \pmod{p}$. We consider two distinct cases:

(A) $p \equiv 3$ or $7 \pmod{20}$. Let $s = \frac{1}{4}(p + 1)$, an integer. Then

$$(-1)^{\frac{1}{2}(p+1)} = (-1)^{2s} = 1.$$

Note that $L_{p+1} = L_{4s} = L_{2s}^2 - 2 \equiv -2$. Hence,

$$L_{2s} \equiv 0. \tag{3}$$

Thus, e exists and we must have

$$e \mid 2s. \tag{4}$$

We suppose e is odd. Then, since $L_e \equiv 0$, we must have $L_{me} \equiv 0$ for all odd m, because $L_e \mid L_{me}$ in that case. On the other hand,

$$L_{2e} = L_e^2 + 2 \equiv 2, \quad L_{4e} = L_{2e}^2 - 2 \equiv 2, \quad L_{6e} = L_{3e}^2 + 2 \equiv 2, \text{ etc.},$$

i.e., $L_{me} \equiv 2$ for all even m. Since $2s$ is an even multiple of e, it follows that $L_{2s} \equiv 2$, which is a contradiction of (3); thus, e is even. Now, given any positive k with $L_k \equiv 0$, we have $e \mid k$. Since e is even, so is k. Therefore, the congruence $L_{2n+1} \equiv 0$ is impossible in this case.

(B) $p \equiv 13$ or $17 \pmod{20}$. We will show that $L_k \not\equiv 0$ for all k, in this case, i.e., e does not exist. Let e' denote the entry point of p in the Fibonacci sequence, i.e., e' is the smallest positive integer k with $F_k \equiv 0 \pmod{p}$. It is known (see [2]) that e' always exists and that, if e exists, then $e' = 2e$. We suppose e exists; hence, e' is even.

Let $t = \frac{1}{2}(p + 1)$, an odd number. Then $L_t^2 + 2 = L_{2t} = L_{p+1} \equiv -2$, which implies $L_t \not\equiv 0$. Also, since $F_{p+1} = F_{2t} = F_t L_t \equiv 0$, we have $F_t \equiv 0$. Therefore, $e' \mid t$. However, because t is odd, it cannot be divisible by an even integer. This contradiction establishes that e does not exist. Hence, $L_k \not\equiv 0$ for all k, in this case; a fortiori, the congruence $L_{2n+1} \equiv 0$ is impossible.

Combining the results of (A) and (B), we reach the desired conclusion.

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2. Brother A. Brousseau (compiler). *Fibonacci and Related Number Theoretic Tables*, p. 25. Santa Clara, Calif: The Fibonacci Association, 1972.

Also solved by L. Somer and the proposer.

Poly Nomial

H-366 *Proposed by Stanley Rabinowitz, Digital Equipment Corp. Merrimack, NH (Vol. 22, no. 1, February 1984)*

The *Fibonacci Polynomials* are defined by the recursion

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$$

with the initial conditions $f_1(x) = 1$ and $f_2(x) = x$. Prove that the discriminant of $f_n(x)$ is

$$(-1)^{(n-1)(n-2)/2} 2^{n-1} n^{n-3} \text{ for } n > 1.$$

Remark: The idea of investigating discriminants fo interesting polynomials was suggested by [1]. The definition of the discriminant of a polynomial can be found in [2]. Fibonacci polynomials are well known (see, e.g., [3] and [4]). I ran a computer program to find the discriminant of $f_n(x)$ as n varied from 2 to 11, and by analyzing the results, reached the conjecture given above in the proposed problem. The discriminant was calculated by finding the resultant of $f_n(x)$ and $f_n'(x)$ using a computer algebra system similar to the MACSYMA program as described in [5]. Much useful material can be found in [6] where the problem of finding the discriminant of the Hermite, Laguerre, and Chebyshev polynomials is discussed. The discriminant of the Fibonacci polynomials should be provable using similar techniques; however, I was not able to do so.

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6. D. K. Faddeev & I. S. Sominskii. *Problems in Higher Algebra*. Trans. by J. L. Brenner. San Francisco: Freeman and Company. Problems 833-851.

Solution by Paul S. Bruckman, Carmichael, CA

The Fibonacci polynomials are given by the explicit expression

$$f_n(x) = \frac{u^n - v^n}{u - v}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where

$$u = u(x) = \tfrac{1}{2}(x + \sqrt{x^2 + 4}), \quad v = v(x) = \tfrac{1}{2}(x - \sqrt{x^2 + 4}). \tag{2}$$

From the defining recursion and the initial values, it is easy to see that f_n is a monic polynomial of degree $n - 1$.

We also define the *Lucas polynomials* $g_n(x)$ as follows:

$$g_n(x) = u^n + v^n, \quad n = 0, 1, 2, \ldots . \tag{3}$$

Let

$$D_n = \operatorname{disc}(f_n), \quad n = 2, 3, \ldots . \tag{4}$$

If the zeros of f_n are $x_1, x_2, \ldots, x_{n-1}$, an explicit expression for D_n is given by

$$D_n = \prod_{1 \leqslant r < s \leqslant n-1} (x_r - x_s)^2, \quad n \geqslant 3; \quad \text{also, } D_2 = 1. \tag{5}$$

We also know from higher algebra that, if the x_k's are distinct,

$$|D_n| = \left|\prod_{k=1}^{n-1} f'(x_k)\right|. \tag{6}$$

We will use (5) only to determine the sign of D_n, and (6) to determine its absolute value, using the relation

$$D_n = |D_n| \cdot \operatorname{sgn}(D_n).$$

The x_k are determined by setting the expression in (1) equal to zero. Then

$$(u/v)^n = 1 \Rightarrow u/v = \exp(2ki\pi/n);$$

since $uv = -1$, we have

$$-u^2 = \exp(2ki\pi/n) \Rightarrow u = \pm i \exp(ki\pi/n).$$

Changing the sign in the last expression above is equivalent to replacing k by $(n - k)$, showing that we need to consider only the positive sign. Thus, we may take $u = i \exp(ki\pi/n)$; hence, $v = i \exp(-ki\pi/n)$. Since f_n is of degree $n - 1$, we may take k to vary from 1 through $n - 1$; thus,

$$x_k = u + v = 2i \cos(k\pi/n), \quad k = 1, 2, \ldots, n - 1.$$

Note that the x_k are distinct, which allows the use of (6). Finally, since f_n is monic and a polynomial, we obtain the factorization

$$f_n(x) = \prod_{k=1}^{n-1}(x - 2i \cos(k\pi/n)), \quad n = 2, 3, \ldots . \tag{7}$$

To evaluate the expression in (6), we differentiate (1), noting first that

$$u'(x) = \tfrac{1}{2}(1 + x/\sqrt{x^2 + 4}), \quad v'(x) = \tfrac{1}{2}(1 - x/\sqrt{x^2 + 4})$$

or

$$u'(x) = \frac{u}{u - v}, \quad v'(x) = \frac{-v}{u - v}. \tag{8}$$

Then,

$$f_n'(x) = \frac{(u - v)\left\{\dfrac{nu^{n-1} \cdot u + nv^{n-1} \cdot v}{u - v}\right\} - (u^n - v^n)\left\{\dfrac{u + v}{u - v}\right\}}{(u - v)^2}$$

$$= \frac{n(u^n + v^n) - x\left\{\dfrac{u^n - v^n}{u - v}\right\}}{(u - v)^2},$$

or

$$f_n'(x) = \frac{ng_n(x) - xf_n(x)}{x^2 + 4}. \tag{9}$$

Setting $x = x_k = 2i \cos(k\pi/n)$ in (9), we see that

$$u(x_k) = i \cos(k\pi/n) + \sin(k\pi/n) = i \exp(-ki\pi/n),$$

and

$$v(x_k) = i \exp(ki\pi/n);$$

thus,

$$f_n(x_k) = i^{n-1} \sin(k\pi)/\sin(k\pi/n) = 0$$

as expected, whereas

$$g_n(x_k) = i^n \cdot 2 \cos(k\pi) = 2i^n(-1)^k;$$

or

$$g_n(x_k) = 2\exp(\tfrac{1}{2}i\pi(n - 2k)), \quad k = 1, 2, \ldots, n - 1. \tag{10}$$

Substituting this last expression into (9), we see that

$$f_n'(x_k) = \frac{2n\exp(\frac{1}{2}i\pi(n - 2k))}{4\sin^2(k\pi/n)},$$

or

$$|f_n'(x_k)| = \frac{n}{2\sin^2(k\pi/n)} \tag{11}$$

Therefore, using (6),

$$|D_n| = \prod_{k=1}^{n-1} n/2\sin^2(k\pi/n),$$

or

$$|D_n| = n^{n-1}\left\{\prod_{k=1}^{n-1} 2\sin^2(k\pi/n)\right\}^{-1}. \tag{12}$$

To evaluate the expression in (12), we set $x = 2i$ in (7). Then,

$$f_n(2i) = \prod_{k=1}^{n-1}(2i)(1 - \cos k\pi/n) = (2i)^{n-1}\prod_{k=1}^{n-1} 2\sin^2(k\pi/2n).$$

Replacing k by $(n - k)$ in the last expression yields

$$f_n(2i) = (2i)^{n-1}\prod_{k=1}^{n-1} 2\cos^2(k\pi/2n).$$

Therefore,

$$(f_n(2i))^2 = (-4)^{n-1}\prod_{k=1}^{n-1}\sin^2(k\pi/n),$$

or

$$(f_n(2i))^2 = (-2)^{n-1}\prod_{k=1}^{n-1} 2\sin^2(k\pi/n). \tag{13}$$

On the other hand, $u(2i) = v(2i) = i$. Using (1),

$$f_n(2i) = \lim_{z\to i}\left(\frac{z^n - i^n}{z - i}\right) = \lim_{z\to i} nz^{n-1} = ni^{n-1}.$$

Thus,

$$(f_n(2i))^2 = n^2(-1)^{n-1}. \tag{14}$$

Comparing (13) and (14) generates the identity:

$$\prod_{k=1}^{n-1} 2\sin^2(k\pi/n) = \frac{n^2}{2^{n-1}}, \quad n = 2, 3, \ldots. \tag{15}$$

Substituting this expression in (12) yields

$$|D_n| = 2^{n-1}n^{n-3}. \tag{16}$$

To obtain the sign of D_n, we consider the expression given in (5). Then,

$$D_n = \prod_{1\leqslant r<s\leqslant n-1}(2i)^2(\cos r\pi/n - \cos s\pi/n)^2;$$

hence,

$$\operatorname{sgn}(D_n) = \prod_{1\leqslant r<s\leqslant n-1}(-1) = \prod_{s=2}^{n-1}\prod_{r=1}^{s-1}(-1) = \prod_{s=2}^{n-1}(-1)^{s-1} = (-1)^{(1+2+\cdots+n-2)},$$

or

$$\operatorname{sgn}(D_n) = (-1)^{\binom{n-1}{2}}.$$

Finally, combining (16) and (17), we obtain

$$D_n = (-1)^{\binom{n-1}{2}} 2^{n-1} n^{n-3}, \quad n \geqslant 3. \tag{18}$$

Note also that setting $n = 2$ in (18) yields the correct expression

$$D_2 = 1.$$

Hence, the proposer's conjecture is correct.

Note: The proposer observed that some results regarding discriminants of Chebyshev polynomials (among others) were discussed in reference [6] of the proposed problem. This reference was unavailable to this solver; it may be shown, however, that the f_n' are, in fact, modified Chebyshev polynomials of the second kind, namely,

$$f_n(x) = (-i)^{n-1} U_{n-1}(ix/2) = |U_{n-1}(ix/2)|.$$

This might lead to an alternative (and briefer) derivation of (18).

Also solved by R. Stanley, who used Chebyshev's polynomials.

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Majorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Majorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.

Recurring Sequences by Dov Jarden. Third and enlarged edition. Riveon Lematematika, Israel, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence — 18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Please write to the Fibonacci Association, University of Santa Clara, CA 95053, U.S.A., for current prices.