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CONTENTS

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The Fibonacci Quarterly

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*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

THE FIRST DIGIT PROPERTY FOR EXPONENTIAL SEQUENCES IS INDEPENDENT OF THE UNDERLYING DISTRIBUTION

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(Submitted July 1982)

The natural density in the set $R \equiv \{cr^k : k = 0, 1, 2, \ldots\}$, where $c > 0$, $r > 1$, and $\log_{10} r$ is irrational, of the elements beginning with the first digit ℓ is known to be

$$\log_{10}\left(\frac{1+\ell}{\ell}\right).$$

We show that this property persists for *any* finitely additive, translation invariant density on sets of the form

$$E \equiv \{e_k \equiv (cr^k + a_k) : a_k = o(r^k),\ k = 0, 1, \ldots\},$$

where $c > 0$ and $\log_{10} r$ is irrational.

In particular, this includes the Fibonacci sequences.

Let c and r be real numbers, such that $c > 0$ and $r > 1$, but $r \neq 10^q$ for q a rational number. Define

$$R \equiv \{cr^k : k = 0, 1, 2, \ldots\}$$

and let $R(\ell)$ be the subset of R whose members begin with the string of digits ℓ in the decimal representation, e.g., if $c = 3$ and $r = 7$, then $147 \in R(1)$ (147 begins with digit 1); 147 is also in $R(14)$ (147 begins with a two-digit string 14), and $147 \in R(147)$. If A is any subset of R, define its indicator function as follows:

$$\chi(k;\ A) = \begin{cases} 1 & \text{if } cr^{k-1} \in A \\ 0 & \text{if } cr^{k-1} \notin A \end{cases} \qquad k = 1, 2, 3, \ldots .$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi(k;\ R(\ell)) = \log_{10}\left(\frac{1+\ell}{\ell}\right),$$

which is a consequence of the fact that the set

$$\{(\log_{10} cr^k) \bmod 1 : k = 0, 1, 2, \ldots\}$$

is uniformly distributed in the interval $[0, 1)$. (See [4].)

When the limit exists,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi(k;\ A)$$

is called the natural density of A with respect to R. Although the natural density exists for each $R(\ell)$, there are subsets of R which do not have natural density. Nevertheless, the natural density can be extended to all subsets of R in a way which preserves finite additivity and translation invariance [defined below as properties (D1) and (D2)]. However, even with added restrictions such as scale invariance, such extensions are not unique. (See [1].)

Now consider any density d on R which satisfies the following two properties:

(D1) For all $A, B \subset R$, $d(A \cup B) = d(A) + d(B) - d(A \cap B)$ (finite additivity).

(D2) For all $A \subset R$, $d(A) = d(A^+)$, where A^+ is the "successor set" defined by $A^+ \equiv \{cr^k : cr^{k-1} \in A\}$ (translation invariance).

Subsequent successor sets to A will be denoted by

$$A^{+h} = (A^{+h-1})^+ = \{cr^k : cr^{k-h} \in A\}.$$

Notice that $A^+ = rA$ and $A^{+h} = r^h A$. Note also that (D2) implies that $d(A) = d(A^{+h})$ for all $h = 2, 3, 4, \ldots$, and that $d(A) = 0$ if A is finite [since $d(R) = 1$].

Naturally, the natural density satisfies (D1) and (D2).

We remark that any density defined on an algebra of subsets of R which includes the single point sets, $\{cr^k\}$ for each $k = 0, 1, 2, \ldots$, and which satisfies (D1) and (D2), can be extended to all subsets of R. We presume that any density considered in Theorems I and II is defined on the entire power set. Also, since finite sets and sets of density zero are unimportant in the sequel, we adopt the following definitions:

If $A, B \subset R$, say

(i) $A =_d B$ if and only if $d(A) = d(A \cap B) = d(B)$, and
(ii) $A \subset_d B$ if and only if $d(A) = d(A \cap B) \leqslant d(B)$.

Theorem I: For any density d on R which satisfies properties (D1) and (D2),

$$d(R(\ell)) = \log_{10}\left(\frac{1+\ell}{\ell}\right).$$

Proof of Theorem I: There are two key observations to be made about the first digit sets, $R(\ell)$. The first observation is that

$$\begin{aligned} R(1) &=_d R(10) \cup R(11) \cup R(12) \cup \cdots \cup R(19) \\ &=_d R(100) \cup R(101) \cup \cdots \cup R(199), \end{aligned}$$

$$\begin{aligned} R(2) &=_d R(20) \cup R(21) \cup \cdots \cup R(29) \\ &=_d R(200) \cup R(201) \cup \cdots \cup R(299), \end{aligned}$$

etc. Since $R = R(1) \cup R(2) \cup \cdots \cup R(9)$ and $R(j) \cap R(\ell) = \emptyset$ for $1 \leqslant j < \ell \leqslant 9$, it follows that

$$\sum_{j=10^k}^{10^{k+1}-1} d(R(j)) = 1 \quad \text{for } k = 0, 1, 2, \ldots . \tag{1}$$

The second key observation concerns the successor sets of the first digit sets. In the case in which c and r are integers, they have the form:

$$R(1)^+ =_d R(r) \cup R(r+1) \cup \cdots \cup R(2r-1)$$
$$R(2)^+ =_d R(2r) \cup R(2r+1) \cup \cdots \cup R(3r-1)$$
$$R(\ell)^+ =_d \bigcup_{j=\ell r}^{(\ell+1)r-1} R(j). \tag{2}$$

Then

$$d(R(\ell)) = \sum_{j=\ell r}^{(\ell+1)r-1} d(R(j)) \quad \text{for } \ell = 1, 2, 3, \ldots . \tag{3}$$

The idea of the proof is to tie together formula (1) and formula (3). However, if the decimal expansion of r does not terminate, $R(r)$ is no longer a well-defined object; thus, before proceeding further, it is necessary to generalize the notion of first digit sets.

If $1 \leqslant x \leqslant y \leqslant 10x$, define

$$R(x, y) \equiv \{u \in R : x \leqslant 10\ u < y \text{ for some integer } j\}.$$

Note that $R(\ell) = R(\ell, \ell + 1)$.

For notational simplicity, assume $r < 10$. Otherwise, in what follows replace r by $\overline{r}$, defined by

$$\overline{r} \equiv r10^{-[\log_{10} r]},$$

where the brackets denote the greatest integer function, e.g., $[3.76] = 3$. Then

$$R(1, r)^+ =_d R(r, r^2), \quad R(1, r)^{+h} =_d R(r^h, r^{h+1}),$$

and equation (2) generalizes to

$$R(x, y)^+ = R(xr, yr). \tag{4}$$

By assumption (D2) of translation invariance,

$$md(R(1, r)) = \sum_{h=0}^{m-1} d(R(1, r)^{+h}) = \sum_{h=0}^{m-1} d(R(r^h, r^{h+1})). \tag{5}$$

By assumption (D1) of finite additivity, and the fact that $r < 10$,

$$d(R(1, r)) + d(R(r, r^2)) + \cdots + d(R(r^{m-1}, r^m))$$
$$= \sum_{\ell=1}^{[r^m]-1} d(R(\ell)) + d(R([r^m], r^m)). \tag{6}$$

Combining equations (1), (5), and (6) yields

$$[md(R(1, r))] = \sum_{\ell=1}^{[r^m]-1} d(R(\ell)) + d(R([r^m], r^m)) = [m \log_{10} r]. \tag{7}$$

Since equation (7) must be true for any choice of m, it follows that

$$d(R(1, r)) = \log_{10} r.$$

Now let $1 \leqslant x \leqslant 10$. We show that $d(R(1, r)) = d(R(x, xr))$.

Case 1: $1 < r \leqslant x \leqslant 10$.

$$d(R(1, x)) = d(R(1, r)) + d(R(r, x))$$

and

$$d(R(r, rx)) = d(R(r, x)) + d(R(x, xr)).$$

By (D2), $d(R(1, x)) = d(R(r, rx))$, so the result follows.

Case 2: $1 \leqslant x \leqslant r < 10$.

Again using $d(R(1, x)) = d(R(r, rx))$, we have

$$\begin{aligned} d(R(1, r) &= d(R(1, x)) + d(R(x, r)) \\ &= d(R(r, rx)) + d(R(x, r)) = d(R(x, rx)). \end{aligned}$$

Hence, by repeated use of (D2),

$$\log_{10} r = d(R(1, r)) = d(R(xr^j, xr^{j+1})) \quad \text{for any } j \geqslant 0,$$

so that

$$md(R(1, r)) = \sum_{j=0}^{m-1} d(R(xr^j, xr^{j+1})),$$

from which it follows that

$$md(R(1, r)) + d(R(1, x)) = \sum_{\ell=1}^{[xr^m]-1} d(R(\ell)) + d(R([xr^m], xr^m)),$$

which implies

$$[m \log_{10} r + d(R(1, x))] = [m \log_{10} r + \log_{10} x].$$

Thus

$$d(R(1, x)) = \log_{10} x. \tag{8}$$

Since $d(R(x, y)) = d(R(10^j x, 10^j y))$ by the definition of $R(x, y)$, for all integers j, the results

$$d(R(x, y)) = \log_{10}(y/x) \quad \text{for } 1 \leqslant x \leqslant y \leqslant 10x$$

and

$$d(R(\ell)) = \log_{10}\left(\frac{\ell + 1}{\ell}\right)$$

follow easily from equation (8) and assumption (D1). Q.E.D.

Now consider real numbers c and r as above and real numbers a_k for $k = 0, 1, 2, \ldots$, such that $a_k = o(r^k)$. Define

$$E \equiv \{e_k \equiv (cr^k + a_k) : k = 0, 1, 2, \ldots\},$$

and a corresponding set

$$R_E \equiv \{(e_k - a_k) : k = 0, 1, 2, \ldots\}.$$

Define a bijective function $f: E \to R_E$ by

$$f(e_k) \equiv e_k - a_k = cr^k.$$

Let the sets $E(x, y)$, $E(\ell)$, $R_E(x, y)$, $R_E(\ell)$ be defined as above.

Assumptions (D1) and (D2), and the notions of a successor set, $=_d$, and $\subset_d$ all extend to E in a natural fashion (although it is no longer true that $A^+ = rA$ for the successor set of $A \subset E$). Sets of type E include linear recursive sequences of the form

$$w_{n+1} = \alpha_0 w_n + \alpha_1 w_{n-1} + \cdots + \alpha_k w_{n-k}$$

whenever the characteristic equation has a unique highest root. In particular, the classic Fibonacci numbers $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$ occur when

$$c = \frac{1}{\sqrt{5}}, \quad r = \frac{1 + \sqrt{5}}{2}, \quad a_k = -\frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^k.$$

Note that $\log_{10}\left(\frac{1 + \sqrt{5}}{2}\right)$ is indeed irrational.

Theorem II: Let d be a density on E satisfying assumptions (D1) and (D2), as they extend to E. Then

$$d(E(\ell)) = \log_{10}\left(\frac{1 + \ell}{\ell}\right).$$

Proof of Theorem II: The density d gives rise to a corresponding density d_R on R_E, defined by

$$d_R(A) \equiv d(f^{-1}(A)) \quad \text{for } A \subset R_E.$$

Theorem I applies to d_R.

Since $a_k = o(r^k)$, it is evident that, for any $\varepsilon > 0$,

$$f^{-1}(R_E(x + \varepsilon, y - \varepsilon)) \subset_d E(x, y) \subset_d f^{-1}(R_E(x - \varepsilon, y + \varepsilon)).$$

Hence

$$\log_{10}\left(\frac{y - \varepsilon}{x + \varepsilon}\right) = d_R(R_E(x + \varepsilon, y - \varepsilon) \leqslant d_R(R_E(x - \varepsilon, y + \varepsilon)) = \log_{10}\left(\frac{y + \varepsilon}{x - \varepsilon}\right)$$

and the result follows. Q.E.D.

These results can also be obtained using the measure-theoretic techniques developed in [1]. For a review of the literature on the First Digit Problem, see [5]. It should be noted that the base 10 logarithmic behavior is due to the convention of writing numbers in decimal form. If the numbers were written in base b, then

$$d(R(\ell)) = \log_b\left(\frac{1 + \ell}{\ell}\right).$$

Another example of a density which satisfies (D1) and (D2) is the logarithmic density

$$d_{\log}(A) \equiv \lim_{n\to\infty} \frac{\displaystyle\sum_{k=1}^{n} \frac{\chi(k;\, A)}{k}}{\displaystyle\sum_{k=1}^{n} \frac{1}{k}}.$$

Like the natural density, there exist sets which do not have logarithmic density. The logarithmic density agrees with the natural density wherever the natural density exists, but there are sets which have logarithmic density which do not have natural density. This raises the following questions: Does every density which satisfies (D1) and (D2) agree with the natural density on sets which have natural density? with the logarithmic density? with other summability methods?

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GENERALIZED ZIGZAG POLYNOMIALS

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(Submitted July 1983)

1. INTRODUCTION

The purpose of this paper is to extend and generalize the results established in [5] for a category of polynomials described therein as "zigzag." These arise in a specified way from a given polynomial sequence generated by a second-order recurrence relation.

Consider the sequence of polynomials $\{W_n(x)\}$ defined by the second-order recurrence relation

$$W_{n+2}(x) = kxW_{n+1}(x) + mW_n(x) \qquad (n \geqslant 0) \tag{1.1}$$

with initial values

$$W_0(x) = h, \quad W_1(x) = kx, \tag{1.2}$$

wherein h, k, and m are real numbers, usually integers.

We have represented these polynomials in abbreviated form by $W_n(x)$ though the parametric symbolism $W_n(h,\ kx;\ kx,\ m)$ more fully describes them. Note that a characteristic feature of the definition (1.1) and (1.2) is that the initial value $W_1(x) = kx$ in (1.2) must be the same as the coefficient of $W_{n+1}(x)$ in the recurrence (1.1).

Standard methods enable us to derive the generating function for $\{W_n(x)\}$, namely,

$$\sum_{n=0}^{\infty} W_n(x)t^n = \{h + kx(1-h)t\}[1 - (kxt + mt^2)]^{-1} \tag{1.3}$$

or, equivalently,

$$\sum_{n=0}^{\infty} W_{n+1}(x)t^n = (kx + mht)[1 - (kxt + mt^2)]^{-1}. \tag{1.3$'$}$$

An explicit form of $W_n(x)$ $(n \geqslant 2)$ is, in the usual notation,

$$W_n(x) = kx\sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-i}{i} m^i (kx)^{n-1-2i} + mh \sum_{i=0}^{\left[\frac{n-2}{2}\right]} \binom{n-2-i}{i} m^i (kx)^{n-2-2i}. \tag{1.4}$$

This formula will be essential when we prove (3.3).

At this point, we stress that $W_n(h,\ kx;\ kx,\ m)$ defined above is a polynomial variation of the $W_n(a,\ b;\ p,\ q)$, wherein $a = h$, $b = p = kx$, $q = m$, whose basic and special properties have been discussed in [7] and [8]. Therefore, no further consideration of its salient features is required here.

Special cases of $W_n(h,\ kx;\ kx,\ m)$ which interest us are (when $h = 2$):

POLYNOMIALS	h	k	m
Lucas	2	1	1
Pell-Lucas	2	2	1
Chebyshev (2nd kind)	2	2	−1
Fermat	2	1	−2

(1.5)

More will be said about these special cases in Section 4.

2. RISING DIAGONAL ZIGZAG POLYNOMIALS

The first few members of the polynomial set $\{W_n(x)\}$ are, from (1.1) with (1.2):

Table 1. Rising Diagonal Zigzag Polynomials for $\{W_n(x)\}$

$$\left\{\begin{array}{l} Z_0(x),\ Z_1(x),\ Z_2(x),\ Z_3(x),\ Z_4(x),\ Z_5(x),\ Z_6(x),\ Z_7(x),\ Z_8(x),\ Z_9(x),\ Z_{10}(x) \\ W_0(x) = h \\ W_1(x) = kx \\ W_2(x) = (kx)^2 + mh \\ W_3(x) = (kx)^3 + \underline{mh(kx) + m(kx)} \\ W_4(x) = (kx)^4 + \underline{mh(kx)^2 + 2m(kx)^2} + m^2h \\ W_5(x) = (kx)^5 + \underline{mh(kx)^3 + 3m(kx)^3} + \underline{2m^2h(kx) + m^2(kx)} \\ W_6(x) = (kx)^6 + \underline{mh(kx)^4 + 4m(kx)^4} + \underline{3m^2h(kx)^2 + 3m^2(kx)^2} + m^3h \\ W_7(x) = (kx)^7 + \underline{mh(kx)^5 + 5m(kx)^5} + \underline{4m^2h(kx)^3 + 6m^2(kx)^3} + \underline{3m^3h(kx) + m^3(kx)} \\ W_8(x) = (kx)^8 + \underline{mh(kx)^6 + 6m(kx)^6} + \underline{5m^2h(kx)^4 + 10m^2(kx)^4} + \underline{6m^3h(kx)^2 + 4m^3(kx)^2} + m^4h \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right. \tag{2.1}$$

In Table 1, pair terms in columns 2 and 3, columns 4 and 5, ..., to form the *rising diagonal generalized zigzag polynomials* $Z_n(x)$ as indicated by the lines, beginning with $Z_0(x) = h$. For example, some of these generalized zigzag polynomials are:

$$\left\{\begin{array}{l} Z_0(x) = h,\ Z_1(x) = kx,\ Z_2(x) = (kx)^2,\ Z_3(x) = (kx)^3 + mh, \\ Z_4(x) = (kx)^4 + mk(kx) + m(kx),\ Z_5(x) = (kx)^5 + mh(kx)^2 + 2m(kx)^2, \\ Z_6(x) = (kx)^6 + mh(kx)^3 + 3m(kx)^3 + m^2h,\ \ldots . \end{array}\right. \tag{2.2}$$

Previously, in [5], we mentioned that the virtue of the pairing technique by which the zigzag polynomials are produced is that specializations may be readily obtained. In the case of Table 1 this is achieved by the amalgamation of corresponding elements in appropriate pairs of columns.

For example, the rising diagonal polynomials for Pell-Lucas polynomials (1.5), already given in [5], are obtained by adding like terms in columns 2 and 3, columns 4 and 5, ... (as appropriate), in Table 1 when $h = 2$, $k = 2$, $m = 1$, to give, for instance, the special expression for $Z_6(x)$ in (2.2) as

$$64x^6 + 40x^3 + 2$$

(which is the polynomial $r_6(x)$ in [5]).

Correspondingly, for the Fermat polynomials (1.5) the rising diagonal polynomial is $x^6 - 10x^3 + 8$ (represented in [3] by $R'(x)$).

Before proceeding to establish some properties of $Z_n(x)$, we introduce the companion polynomials $X_n(x)$, defined by

$$X_n(x) = Z_n(x)\Big|_{h=1}, \tag{2.3}$$

i.e., $X_n(x)$ are the rising diagonal zigzag polynomials of the set of polynomials $\{W_n(x)\}$ defined in (1.1) for which $h = 1$.

Thus, if we consider the four special cases of $W_n(2, kx; kx, m)$ which are listed in (1.5), yielding particular instances of the $Z_n(x)$ when $h = 2$ [the polynomials $Y_n(x)$ defined in (2.11) below), then the corresponding polynomials $X_n(x)$ are associated with the four special cases of $W_n(1, kx; kx, m)$ corresponding to those in (1.5), but with $h = 1$. These are the *Fibonacci polynomials*, the *Pell polynomials*, the *Chebyshev polynomials of the first kind*, and the *companion Fermat polynomials* ("Fermat polynomials of the first kind"), respectively.

From (2.2) and (2.3) we have the expressions for the simplest polynomials $X_n(x)$:

$$\begin{cases} X_0(x) = 1,\ X_1(x) = kx,\ X_2(x) = (kx)^2,\ X_3(x) = (kx)^3 + m, \\ X_4(x) = (kx)^4 + 2m(kx),\ X_5(x) = (kx)^5 + 3m(kx)^2, \\ X_6(x) = (kx)^6 + 4m(kx)^3 + m^2,\ \ldots\ . \end{cases} \tag{2.4}$$

The recurrence relation, the generating function, and the explicit form for $X_n(x)$ corresponding to (2.5)-(2.7), and the differential equations corresponding to (2.8) and (2.9) which $X_n(x)$ satisfy, may all be readily derived by simple substitution.

Following procedures already established in [5], we derive, without much effort, the results exhibited below.

RECURRENCE RELATION

$$Z_n(x) = kxZ_{n-1}(x) + mZ_{n-3}(x) \qquad (n \geqslant 3) \tag{2.5}$$

GENERATING FUNCTION

$$\sum_{n=1}^{\infty} Z_n(x)t^{n-1} = (kx + mht^2)[1 - (kxt + mt^3)]^{-1} \equiv Z(x, t) \tag{2.6}$$

EXPLICIT FORM

$$Z_n(x) = kx\sum_{i=0}^{\left[\frac{n-1}{3}\right]}\binom{n-1-2i}{i}m^i(kx)^{n-1-3i} + mh\sum_{i=0}^{\left[\frac{n-3}{3}\right]}\binom{n-3-2i}{i}m^i(kx)^{n-3-3i} \qquad (n \geqslant 3) \tag{2.7}$$

DIFFERENTIAL EQUATIONS

$$kt\frac{\partial}{\partial t}Z(x, t) - (kx + 3mt^2)\frac{\partial}{\partial x}Z(x, t) = k\{(2h-3)mt^2 - kx\}[1 - (kxt + mt^3)]^{-1} \tag{2.8}$$

$$kx\frac{d}{dx}Z_{n+2}(x) + 3m\frac{d}{dx}Z_n(x) = k\{(n-1)Z_{n+2}(x) + 3X_{n+2}(x)\} \tag{2.9}$$

Alternative and equivalent forms exist in some of the above results. For example, the bracketed factor on the right-hand side of (2.9) may be equally well expressed as

$$(n+2)Z_{n+2}(x) - 3m(h-1)X_{n-1}(x).$$

The equality of these two forms arises from the relationship

$$Z_n(x) = X_n(x) + m(h-1)X_{n-3}(x) \qquad (n \geqslant 3), \tag{2.10}$$

which may be readily demonstrated. Substitution of $h = 1$ in (2.10) produces $Z_n(x) = X_n(x)$, of course, in accord with (2.3).

Another alternative expression occurs in the right-hand side of (2.8), which can be made to simplify to $k\{2Z(x, t) - 3X(x, t)\}$ where the symbol

$$X(x, t) = Z(x, t)\Big|_{h=1}.$$

Next, for completion, we introduce the related polynomial $Y_n(x)$, defined by

$$Y_n(x) = Z_n(x)\Big|_{h=2}, \tag{2.11}$$

i.e., the $Y_n(x)$ are the particular cases of $Z_n(x)$ occurring when $h = 2$.

Expressions for some of the $Y_n(x)$ are, by (2.2) and (2.11):

$$\begin{cases} Y_0(x) = 2,\ Y_1(x) = kx,\ Y_2(x) = (kx)^2,\ Y_3(x) = (kx)^3 + 2m, \\ Y_4(x) = (kx)^4 + 3m(kx),\ Y_5(x) = (kx)^5 + 4m(kx)^2, \\ Y_6(x) = (kx)^6 + 5m(kx)^3 + 2m^2,\ \ldots, \end{cases} \tag{2.12}$$

whence, by (2.4) and (2.12),

$$Y_n(x) = X_n(x) + mX_{n-3}(x). \tag{2.13}$$

Corresponding to (2.5)-(2.9), the recurrence relation, the generating function, and the explicit form for $Y_n(x)$, along with the differential equations satisfied by $Y_n(x)$, are easily deducible.

Subtraction of (2.13) from (2.10) reveals that

$$Z_n(x) = Y_n(x) + m(h-2)X_{n-3}(x). \tag{2.14}$$

When $h = 2$, (2.14) leads to $Z_n(x) = Y_n(x)$ in accord with (2.11).

3. DESCENDING DIAGONAL ZIGZAG POLYNOMIALS

Re-organize the material in Table 1, as indicated in Table 2 below, to produce the *descending diagonal generalized zigzag polynomials*:

Table 2. Descending Diagonal Zigzag Polynomials for $\{W_n(x)\}$

$$\begin{cases} W_0(x) = h & \\ & z_0(x) \\ W_1(x) = kx & \\ W_2(x) = (kx)^2 + mh & z_1(x) \\ W_3(x) = (kx)^3 + mh(kx) + m(kx) & \\ W_4(x) = (kx)^4 + mh(kx)^2 + 2m(kx)^2 + m^2h & z_2(x) \\ W_5(x) = (kx)^5 + mh(kx)^3 + 3m(kx)^3 + 2m^2h(kx) + m^2(kx) & \\ W_6(x) = (kx)^6 + mh(kx)^4 + 4m(kx)^4 + 3m^2h(kx)^2 + 3m^2(kx)^2 + m^3h & z_3(x) \\ W_7(x) = (kx)^7 + mh(kx)^5 + 5m(kx)^5 + 4m^2h(kx)^3 + 6m^2(kx)^3 + 3m^3h(kx) + m^3(kx) & \\ W_8(x) = (kx)^8 + mh(kx)^6 + 6m(kx)^6 + 5m^2h(kx)^4 + 10m^2(kx)^4 + 6m^3h(kx)^2 + 4m^3(kx)^2 + m^4h & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{cases} \tag{3.1}$$

Designate these polynomials by $z_n(x)$. Then, as we learned from experience to expect, we derive the relatively simple expressions

$$\begin{cases} z_0(x) = h, \; z_1(x) = kx + mh, \; z_2(x) = (kx + mh)(kx + m), \\ z_3(x) = (kx + mh)(kx + m)^2, \; z_4(x) = (kx + mh)(kx + m)^3, \; \ldots, \end{cases} \tag{3.2}$$

and in general

$$z_n(x) = (kx + mh)(kx + m)^{n-1} \qquad (n = 1, 2, 3, \ldots), \tag{3.3}$$

so that

$$\frac{z_{n+1}(x)}{z_n(x)} = kx + mh. \tag{3.4}$$

As result (3.3) is crucial, we proceed to demonstrate its validity.

Proof of (3.3): Temporarily, write $W_n(x) = kxP(x) + mhQ(x)$ in (1.4), wherein $P(x)$ and $Q(x)$ stand for the appropriate summations.

Let typical values of i in $P(x)$ and $Q(x)$ be represented by p and q respectively ($p = 0, 1, \ldots, n - 1$; $q = 0, 1, \ldots, n - 1$).

Each value of n in the $W_n(x)$ giving rise to a specified $z_n(x)$ in Table 2 requires a pair of values (p, q).

For

$$W_n(x),\; W_{n+1}(x),\; W_{n+2}(x),\; \ldots,\; W_{2n-1}(x),\; W_{2n}(x),$$

these are

$$(0, -),\; (1, 0),\; (2, 1),\; \ldots,\; (n - 1, n - 2),\; (-, n - 1),$$

respectively, in which the dash (-) signifies nonoccurrence.

Then, from (1.4), we have, after the necessary simplifications:

$$\begin{aligned} z_n(x) &= kx\left\{\binom{n-1}{0}(kx)^{n-1}m^0 + \binom{n-1}{1}(kx)^{n-2}m^1 + \binom{n-1}{2}(kx)^{n-3}m^2 + \cdots \right. \\ &\quad \left. + \binom{n-1}{n-1}(kx)^0 m^{n-1}\right\} + mh\left\{\binom{n-1}{0}(kx)^{n-1}m^0 + \binom{n-1}{1}(kx)^{n-2}m^1 \right. \\ &\quad \left. + \binom{n-1}{2}(kx)^{n-3}m^2 + \cdots + \binom{n-1}{n-1}(kx)^0 m^{n-1}\right\} \\ &= kx(kx+m)^{n-1} + mh(kx+m)^{n-1} \\ &= (kx+mh)(kx+m)^{n-1}. \end{aligned}$$

The generating function for $z_n(x)$ $(n > 0)$ is

$$z(x, t) \equiv \sum_{n=1}^{\infty} z_n(x)t^{n-1} = (kx + mh)[1 - (kx + m)t]^{-1}. \tag{3.5}$$

Differential equations satisfied by the descending diagonal zigzag polynomials are, from (3.3) and (3.5),

$$kt\frac{\partial}{\partial t}z(x, t) - (kx + m)\frac{\partial}{\partial x}(x, t) + k\frac{(kx + m)}{(kx + mh)}z(x, t) = 0 \tag{3.6}$$

and

$$(kx + m)\frac{d}{dx}z_n(x) - k(n - 1)z_n(x) - k(kx + m)^n = 0. \tag{3.7}$$

Just as we have the specialized forms (2.3) and (2.11) of $Z_n(x)$ occurring when $h = 1$ and $h = 2$ respectively, so we have the specialized forms of $z_n(x)$:

$$x_n(x) = z_n(x)\Big|_{h=1} \tag{3.8}$$

and

$$y_n(x) = z_n(x)\Big|_{h=2}. \tag{3.9}$$

Consequently,

$$x_n(x) = (kx + m)^n \tag{3.10}$$

and

$$y_n(x) = (kx + 2m)(kx + m)^{n-1}. \tag{3.11}$$

Result (3.7) may then, by (3.10), have the factor of k in the last term replaced by $x_n(x)$.

Obviously, (3.10) and (3.11) together yield

$$\frac{x_n(x)}{y_n(x)} = \frac{kx + m}{kx + 2m} \tag{3.12}$$

and

$$y_n(x) = mx_{n-1}(x) + x_n(x). \tag{3.13}$$

4. SPECIAL CASES

Recall that our generalization in this paper relates specifically to the situations in which

$$W_1(x) = kx = \text{the coefficient of } W_{n+1}(x) \text{ in the definition (1.1).}$$

This leads to some interesting and familiar polynomials which have been listed in (1.5).

Details concerning the results for the rising and descending diagonal polynomials cataloged in (1.5) are to be found in a chain of papers in the following sources:

POLYNOMIAL	REFERENCE	
Lucas	[2]	
Pell-Lucas	[5]	(4.1)
Chebyshev	[1], [3], [10]	
Fermat	[3], [4]	

where the reference numbers are those in the bibliographical references below.

Results for these specialized polynomials should be compared with the corresponding generalized results in this paper. Allowance must, however, be duly made on occasion for slight variations in notation, especially where these involve the initial conditions.

These principles are now carefully illustrated for the case of the Fermat polynomials ("of the second kind") in (1.5) for which $h = 2$. The companion Fermat polynomials ("of the first kind") for which $h = 1$ will also be required. In the illustration, we verify that equation (2.8) above does indeed reduce to equation (39) in [4] for the Fermat polynomials.

Illustration (Fermat Polynomials): For the Fermat polynomials we have, by substitution in (2.6),

$$\begin{aligned} Y = Y(x,\ t) &= (x - 4t^2)[1 - (xt - 2t^3)]^{-1} \\ &= Y_1(x) + Y_2(x)t + Y_3(x)t^2 + \cdots, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} X = X(x,\ t) &= (x - 2t^2)[1 - (xt - 2t^3)]^{-1} \\ &= X_1(x) + X_2(x)t + X_3(x)t^2 + \cdots, \end{aligned} \tag{4.3}$$

using a simplified notation.

Now in [3] and [4] the following notation was employed [wherein the dash (') does not indicate differentiation]:

$$R = [1 - (xt - 2t^3)]^{-1} = R_1(x) + R_2(x)t + R_3(x)t^2 + \cdots \equiv R(x,\ t); \tag{4.4}$$

$$R' = (1 - 2t^3)[1 - (xt - 2t^3)]^{-1} = 1 + R_2'(x)t + R_3'(x)t^2 + \cdots \equiv R'(x,\ t). \tag{4.5}$$

But

$$X_n(x) = R_{n+1}(x) \tag{4.6}$$

and

$$Y_n(x) = R_{n+1}'(x). \tag{4.7}$$

Hence (4.2)-(4.7) give

$$X = \frac{R - 1}{t} \tag{4.8}$$

and

$$Y = \frac{R' - 1}{t}.$$

Substitution in (2.8) from (1.5) for Fermat polynomials leads to

$$t\frac{\partial Y}{\partial t} - (x - 6t^2)\frac{\partial Y}{\partial x} = (-x - 2t^2)[1 - (xt - 2t^3)]^{-1},$$

i.e., by (4.9),

$$t\left(\frac{1}{t}\,\frac{\partial R'}{\partial t} - \frac{R' - 1}{t^2}\right) - (x - 6t^2)\frac{\partial R'}{\partial x} = (-x - 2t^2)[1 - (xt - 2t^3)]^{-1},$$

$$t\frac{\partial R'}{\partial t} - (x - 6t^2)\frac{\partial R'}{\partial x} = t(-x - 2t^2)[1 - (xt - 2t^3)]^{-1} + R' - 1$$

$$= -6t^2[1 - (xt - 2t^3)]^{-1}$$

$$= 3(R' - R) \qquad \text{by (4.4) and (4.5).}$$

This is equation (39) in [4], which we set out to verify.

In addition to the comments preceding the illustration, we remark that corresponding properties are developed for the polynomials $W_n(2, px; px, q)$ in [4], while in [6] and [9] analogous properties of the Gegenbauer polynomials, which are closely related to the Chebyshev polynomials, are investigated. (Brief mention is also made in [4] of the generalized Humbert polynomial of which the Gegenbauer and Chebyshev polynomials are particular cases.)

Some interesting number sequences result if appropriate values of x (e.g., $x = \frac{1}{2}$, $x = 1$) are substituted in the various rising and descending diagonal polynomials discussed in the above papers.

Thus, we have presented a summary and a synthesis of the basic thrust of the material in papers [1]-[6] and [9] by the author, along with that in [10] by Jaiswal.

5. POSSIBLE EXTENSIONS

One would like to be able to extend some of the ideas which have been applied in this paper to recurrence relations of higher order, particularly to the case of third-order recurrence relations. In order to produce the most worthwhile results, it would be necessary to choose the most fertile initial polynomials (including constants) to generate the required polynomial set.

Given such a fruitful selection of initial conditions, it might be possible to discover some geometrical results in three dimensions (Euclidean space) which would be analogous to, or extensions of, similar results about circles (in the Euclidean plane) by the author in other papers which are not listed in the References. These investigations could be extended to three-dimensional surfaces corresponding to the conics in the plane.

Hopefully (if tediously), such considerations could be further extended to hyper-surfaces in multi-dimensional Euclidean space.

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SUMMATION OF RECIPROCAL SERIES OF NUMERICAL FUNCTIONS OF SECOND ORDER

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This paper is an extension of the results of G. E. Bergum and V. E. Hoggatt, Jr. [1] concerning the problem of summation of reciprocals of products of Fibonacci and Lucas polynomials. The method used here will also allow us to generalize some formulas of R. Backstrom [2] related to sums of reciprocal series of Fibonacci and Lucas numbers.

1

The general numerical functions of second order which, following the notation of Horadam [3], we write as $\{w_n(a, b; p, q)\}$ may be defined by

$$w_n = pw_{n-1} - qw_{n-2},\ n \geqslant 2,\ w_0 = a,\ w_1 = b,$$

with

$$w_n = w_n(a, b; p, q),$$

where a and b are arbitrary integers.

We are interested in the sequences

$$u_n = w_n(0, 1; p, q) \tag{1}$$

and

$$v_n = w_n(2, p; p, q) \tag{2}$$

that can be expressed in the form

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geqslant 1, \tag{3}$$

and

$$v_n = \alpha^n + \beta^n, \quad n \geqslant 1, \tag{4}$$

where

$$\alpha = (p + \sqrt{p^2 - 4q})/2,\ \beta = (p - \sqrt{p^2 - 4q})/2,\ \alpha + \beta = p,\ \alpha\beta = q,$$

and $\alpha - \beta = \delta = \sqrt{\Delta}$.

Using (3) and (4), we obtain

$$2\alpha^n = v_n + \delta u_n$$

and

$$4\alpha^{m+n} = v_m v_n + \Delta u_m u_n + \delta(u_m v_n + u_m v_n),$$

from which it follows that

$$u_{s+r}v_s - u_s v_{s+r} = 2q^s u_r \tag{5}$$

and

$$v_{s+r}v_s - \Delta u_s u_{s+r} = 2q^s v_r. \tag{6}$$

2

From relation (5), we have

$$\frac{v_s}{u_s} - \frac{v_{s+r}}{u_{s+r}} = 2q^s \frac{u_r}{u_s u_{s+r}}.$$

If we replace s here by s, $s + r$, $s + 2r$, ..., $s + (n - 1)r$, successively, and add the results, we obtain, due to the telescoping effect,

$$S_n(p, q; r, s) = \sum_{k=1}^{n} \frac{q^{(k-1)r}}{u_{s+(k-1)r}u_{s+kr}} = \left(\frac{v_s}{u_s} - \frac{v_{s+nr}}{u_{s+nr}}\right)\frac{1}{2q^s u_r} = \frac{u_{nr}}{u_r u_s u_{s+nr}}. \tag{7}$$

Similarly, again using (5), we also have

$$\sigma_n(p, q; r, s) = \sum_{k=1}^{n} \frac{q^{(k-1)r}}{v_{s+(k-1)r}v_{s+kr}} = \left(\frac{u_{s+nr}}{v_{s+nr}} - \frac{u_s}{v_s}\right)\frac{1}{2q^s u_r} = \frac{u_{nr}}{u_r v_s v_{s+nr}}. \tag{8}$$

Because

$$\lim_{n\to\infty} \frac{u_n}{u_{n+r}} = \begin{cases} \alpha^{-r}, & |\beta/\alpha| < 1 \\ \beta^{-r}, & |\alpha/\beta| < 1, \end{cases}$$

and

$$\lim_{n\to\infty} \frac{u_n}{v_{n+r}} = \begin{cases} \alpha^{1-r}/(\alpha^2 - q), & |\beta/\alpha| < 1 \\ \beta^{1-r}/(\beta^2 - q), & |\alpha/\beta| < 1, \end{cases}$$

we obtain

$$S(p, q; r, s) = \sum_{k=1}^{\infty} \frac{q^{(k-1)r}}{u_{s+(k-1)r}u_{s+kr}} = \begin{cases} \dfrac{\alpha^{-s}}{u_r u_s}, & |\beta/\alpha| < 1 \\[2ex] \dfrac{\beta^{-s}}{u_r u_s}, & |\alpha/\beta| < 1, \end{cases} \tag{9}$$

$$\sigma(p, q; r, s) = \sum_{k=1}^{\infty} \frac{q^{(k-1)r}}{v_{s+(k-1)r}v_{s+kr}} = \begin{cases} \dfrac{\alpha^{1-s}}{\alpha^2 - q}\dfrac{1}{u_r v_s}, & |\beta/\alpha| < 1 \\[2ex] \dfrac{\beta^{1-s}}{\beta^2 - q}\dfrac{1}{u_r v_s}, & |\alpha/\beta| < 1. \end{cases} \tag{10}$$

In particular, with $r = s$, we have

$$S(p, q; r, r) = \begin{cases} \alpha^{r-2}\left(\dfrac{\alpha^2 - q}{\alpha^{2r} - q^r}\right)^2, & |\beta/\alpha| < 1 \\[2ex] \beta^{r-2}\left(\dfrac{\beta^2 - q}{\beta^{2r} - q^r}\right)^2, & |\alpha/\beta| < 1, \end{cases} \tag{11}$$

and

$$\sigma(p, q; r, r) = \begin{cases} \alpha^r/(\alpha^{4r} - q^{2r}), & |\beta/\alpha| < 1 \\ \beta^r/(\beta^{4r} - q^{2r}), & |\alpha/\beta| < 1. \end{cases} \tag{12}$$

3. SPECIAL CASES

It is not difficult to obtain the formulas of Bergum and Hoggatt from (9) and (10). Indeed, if we let $p = x$ and $q = -1$ in (1) and (2), these relations define the sequences of the Fibonacci polynomials $\{F_k(x)\}_{k=1}^{\infty}$ and the Lucas polynomials $\{L_k(x)\}_{k=1}^{\infty}$. In this case,

$$\alpha(x) = (x + \sqrt{x^2 + 4})/2, \quad \beta(x) = (x - \sqrt{x^2 + 4})/2,$$

where

$$-1 < \alpha(x) < 1 \quad \text{and} \quad \beta(x) > 1 \quad \text{when } x > 0,$$

$$0 < \alpha(x) < 1 \quad \text{and} \quad \beta(x) < 1 \quad \text{when } x < 0.$$

Hence, (9) and (10) become

$$S(x, -1; r, s) = \lim_{n\to\infty} S_n(x, -1; r, s) = \begin{cases} \dfrac{1}{\alpha^s(x)} \dfrac{1}{F_r(x)F_s(x)}, & x > 0, \\ \dfrac{1}{\beta^s(x)} \dfrac{1}{F_r(x)F_s(x)}, & x < 0, \end{cases} \tag{13}$$

and

$$\sigma(x, -1; r, s) = \lim_{n\to\infty} \sigma_n(x, -1; r, s) = \begin{cases} \dfrac{\alpha^{1-s}(x)}{1 + \alpha^2(x)} \dfrac{1}{F_r(x)L_s(x)}, & x > 0 \\ \dfrac{\beta^{1-s}(x)}{1 + \beta^2(x)} \dfrac{1}{F_r(x)L_s(x)}, & x < 0. \end{cases} \tag{14}$$

Comparing the results of Bergum and Hoggatt [1, p. 149, formulas (9) and (17)] with our (13) and (14) above, we find that

$$U(q, a, b, x) = (-1)^b F_k(x)F_q(x)S(x, -1; q, b) \tag{15}$$

and

$$V(q, a, b, x) = (-1)^b F_k(x)F_q(x)(x^2 + 4)\sigma(x, -1; q, b), \tag{16}$$

when $q = b - a + k$.

As particular cases, we give:

$$S(x, -1; 2, 2) = \sum_{k=1}^{\infty} \frac{1}{F_{2k}(x)F_{2(k+1)}(x)} = \begin{cases} \beta^2(x)/x^2, & x > 0, \\ \alpha^2(x)/x^2, & x < 0, \end{cases}$$

and

$$\sigma(x, -1; 2, 2) = \sum_{k=1}^{\infty} \frac{1}{L_{2k}(x)L_{2(k+1)}(x)} = \begin{cases} \alpha^2(x)/(\alpha^8(x) - 1), & x > 0, \\ \beta^2(x)/(\beta^8(x) - 1), & x < 0. \end{cases}$$

4

Using the relations (5) and (6) with $u_{-n} = -q^{-n}u_n$ and $v_{-n} = q^{-n}v_n$, we have

$$v_{2r} - q^{r-s}v_{2s} = \Delta u_{r-s}u_{r+s}.$$

Then, by the method used to obtain (7), we have

$$\Delta \sum_{k=0}^{n} \frac{q^{kr}}{v_{(2k+1)r+2s} - q^{s+kr}v_r} = \frac{1}{u_s u_r}\frac{u_{(n+1)r}}{u_{s+(n+1)r}} \tag{17}$$

so that

$$\Delta \sum_{k=0}^{\infty} \frac{q^{s+kr}}{v_{(2k+1)r+2s} - q^{s+kr}v_r} = \begin{cases} \dfrac{\beta^s}{u_r u_s}, & |\beta/\alpha| < 1, \\ \dfrac{\alpha^s}{u_r u_s}, & |\alpha/\beta| < 1. \end{cases} \tag{18}$$

Similarly, from

$$v_{2r} + q^{r-s}v_{2s} = v_{r-s}v_{r+s},$$

using (8) we obtain

$$\sum_{k=0}^{n} \frac{q^{kr}}{v_{(2k+1)r+2s} + q^{s+kr}v_r} = \frac{1}{u_r v}\frac{u_{(n+1)r}}{v_{s+(n+1)r}} \tag{19}$$

or

$$\sum_{k=0}^{\infty} \frac{q^{s+kr}}{v_{(2k+1)r+2s} + q^{s+kr}v_r} = \begin{cases} \beta^{s+1}/(q - \beta^r)u_r v_s, & |\beta/\alpha| < 1, \\ \alpha^{s+1}/(q - \alpha^r)u_r v_s, & |\alpha/\beta| < 1. \end{cases} \tag{20}$$

In particular, if we put $p = -q = 1$ in (17)-(20), we obtain the formulas of Backstrom [2] concerning the Lucas numbers. These are

$$\sum_{k=0}^{n} \frac{1}{L_{(2k+1)r+2s} + L_r} = \begin{cases} \dfrac{1}{5F_r F_s}\dfrac{F_{(n+1)r}}{F_{(n+1)r+s}}, & s \text{ odd}, \\ \dfrac{1}{F_r L_s}\dfrac{F_{(n+1)r}}{L_{(n+1)r+s}}, & s \text{ even}, \end{cases}$$

and

$$\sum_{k=0}^{\infty} \frac{1}{L_{(2k+1)r+2s} + L_r} = \begin{cases} \left(\dfrac{-1+\sqrt{5}}{2}\right)^s \dfrac{1}{5F_r F_s}, & s \text{ odd}, \\ \left(\dfrac{\sqrt{5}-1}{2}\right)^s \dfrac{1}{5F_r L_s}, & s \text{ even}, \end{cases}$$

where r is an even integer satisfying $-r \leqslant 2s \leqslant r - 2$.

We notice that, from

$$u_r^2 - q^{r-s}u_s^2 = u_{r-s}u_{r+s},$$

it follows that

$$\sum_{k=1}^{n} \frac{q^{2(n-1)r}}{u^2_{(2k-1)r+s} - q^{s+2kr}u^2_r} = S_n(p,\ q;\ 2r,\ s) \tag{21}$$

and

$$\sum_{k=1}^{\infty} \frac{q^{2(n-1)r}}{u^2_{(2k-1)r+s} - q^{s+2kr}u^2_r} = \begin{cases} \beta^s/u_{2r}u_s, & |\beta/\alpha| < 1, \\ \alpha^s/u_{2r}u_s, & |\alpha/\beta| < 1. \end{cases}$$

Similarly,

$$\Delta \sum_{k=1}^{n} \frac{q^{2(k-1)r}}{v^2_{(2k-1)r+s} - q^{s+2kr}v^2_r} = S_n(p,\ q;\ 2r,\ s).$$

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METRIC THEORY OF PIERCE EXPANSIONS

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(Submitted October 1983)

1. INTRODUCTION

It is well known that every real number admits an essentially unique expansion as a *continued fraction* in the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

where the a_i are positive integers (except for a_0, which may be negative or 0).

Many mathematicians have been interested in the length of such expressions; in particular, if $x = p/q$ is rational, the expansion terminates with a_n as the last *partial quotient*, and it is not difficult to show that

$$n = O(\log q).$$

See, for example, [14]. This type of result is of particular interest because continued fractions are closely linked to Euclid's algorithm to compute the greatest common divisor.

Another question that has received attention is how the a_i are related to x, in particular, by equating probabilities with Lebesgue measure, we can consider the $a_i = a_i(x)$ to be random variables, and ask:

1. How are the $a_i(x)$ distributed? What are the means and variances of these distributions?

2. Are the $a_i(x)$ independent, or "almost" independent? What does the distribution of $a_i(x)$ look like as $i \to \infty$?

We could also restate these questions in terms of iteration of an appropriate function. For example, if

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \quad \text{and} \quad g(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor,$$

then it is easy to see that

$$g(x) = \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}$$

so that $g(x)$ may be viewed as a "shift" operator. Here $\lfloor x \rfloor$ is the greatest integer function.

This so-called "metric theory" of continued fractions has been studied extensively by Kuzmin [16], Lévy [17], Khintchine [12], and others.

We can ask similar questions of other algorithms for expressing real numbers. Engel's series

$$x = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots$$

was investigated thoroughly by Erdös, Rényi, and Szüsz [7], and later by Rényi [21] and Deheuvels [5].

Cantor's product

$$x = \left(1 + \frac{1}{a_1}\right)\left(1 + \frac{1}{a_2}\right)\left(1 + \frac{1}{a_3}\right) \cdots$$

was investigated by Rényi [22].

There are also results for Sylvester's series [7] and other expansions of Cantor. For a summary of some of these results, see [9].

The subject of this paper is an expansion that has not received much attention; it is of the form

$$x = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \cdots \tag{1}$$

and is due to Pierce [19], who briefly examined its properties. Remez [20] attributes the expansion to M. V. Ostrogradskij and proves some elementary results. There are some metric theory results in [24], but they do not overlap with our results. We call an expansion of the form (1) a *Pierce expansion*, and in this paper we will demonstrate a connection between these expansions and Stirling numbers of the first kind. We obtain some new identities for Stirling numbers, and give a new derivation of a series for $\zeta(3)$. We discuss the distribution of the $a_i = a_i(x)$, and the behavior of the related function

$$f(x) = 1 \bmod x = 1 - x\lfloor 1/x \rfloor,$$

where by $a \bmod b$ we mean $a - b\lfloor a/b \rfloor$.

We also obtain some results on the lengths of finite Pierce expansions.

2. ELEMENTARY CONSIDERATIONS

In this section, we sketch some of the simple properties of Pierce expansions. The proofs are easy and all details are not given.

Any real number $x \in (0, 1]$ can be written uniquely in the form

$$x = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \cdots \tag{2}$$

where the a_i form a strictly increasing sequence of positive integers, and the expansion may or may not terminate. If the expansion does terminate with

$$\frac{(-1)^{n+1}}{a_1 a_2 \cdots a_n}$$

as the last term, we impose the additional restriction

$$a_{n-1} < a_n - 1.$$

This is to ensure uniqueness, since we could write $1/k$ as

$$\frac{1}{k - 1} - \frac{1}{(k - 1)k}.$$

We will sometimes abbreviate the expansion (2) as

$$x = \{a_1, a_2, a_3, \ldots\}$$

where appropriate.

Given a real number x, we can obtain the terms of the Pierce expansion using the following algorithm:

[Pierce expansion algorithm]: Given a real number $x \in (0, 1]$, this algorithm produces the sequence of a_i such that $x = \{a_1, a_2, \ldots\}$.

P1. [Initialize]. Set $x_0 \leftarrow x$, set $i \leftarrow 1$.

P2. [Iterate]. Set $a_i \leftarrow \lfloor 1/x_{i-1} \rfloor$; set $x_i \leftarrow 1 - a_i x_{i-1}$.

P3. [All done?]. If $x_i = 0$, stop. Otherwise set $i \leftarrow i + 1$ and return to P2.

If we run this algorithm on the rational number $x = p/q$, it is easy to see that in step P2 we sill replace p by q mod p; this is less than p, and so eventually $x_i = 0$ and the algorithm terminates. On the other hand, if the algorithm terminates, we have

$$x = \{a_1, a_2, \ldots, a_n\}$$

and so x must be rational.

(This argument provides simple irrationality proofs for some numbers of interest. For example, using the Taylor series for e^x, $\sin x$, and $\cos x$, we find:

$$1 - e^{-1/a} = \{a, 2a, 3a, 4a, \ldots\},$$
$$\sin(1/a) = \{a, 6a^2, 20a^2, 42a^2, \ldots\},$$
$$\cos(1/a) = \{1, 2a^2, 12a^2, 30a^2, \ldots\}.$$

Since the expansions do not terminate, these functions take irrational values for any positive integer a.)

Now choose x uniformly from $(0, 1]$, and let $\Pr[X = c]$ be the probability that the random variable X equals c (thinking of probability as Lebesgue measure). Let

$$x = \{a_1, a_2, \ldots\}$$

be the Pierce expansion of x. Then

Theorem 1:

$$\Pr[a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n] = \frac{1}{b_1 b_2 \cdots b_n(b_n + 1)}$$

Proof: Let $b_1, \ldots, b_n$ be chosen. Now it is easy to see that the numbers whose expansions begin $\{b_1, b_2, \ldots, b_n\}$ form a half-open interval whose endpoints are the two numbers

$$x_1 = \{b_1, b_2, \ldots, b_{n-1}, b_n\}$$

and

$$x_2 = \{b_1, b_2, \ldots, b_{n-1}, b_n + 1\}.$$

(The first point is included, but the second is not.) The measure of this interval is just

$$|x_1 - x_2| = \frac{1}{b_1 b_2 \cdots b_n(b_n + 1)}$$

and the result follows.

Theorem 2:

$$\Pr[a_{n+1} = k | a_n = j] = \frac{j + 1}{k(k + 1)}$$

(Compare this with the result in [7] for Engel's series.)

Proof: To prove this, we show it is true for all x that have Pierce expansions that begin $\{b_1, b_2, \ldots, b_{n-1}, j\}$ where the b_i are specified constants. Then

$$\Pr[a_{n+1} = k | a_1 = b_1, \ldots, a_{n-1} = b_{n-1}, a_n = j]$$

$$= \frac{\Pr[a_1 = b_1, \ldots, a_{n-1} = b_{n-1}, a_n = j, a_{n+1} = k]}{\Pr[a_1 = b_1, \ldots, a_{n-1} = b_{n-1}, a_n = j]}$$

$$= \frac{b_1 b_2 \cdots b_{n-1} j(j + 1)}{b_1 b_2 \cdots b_{n-1} jk(k + 1)} = \frac{j + 1}{k(k + 1)}.$$

Now this conditional probability is the SAME for any specified prefix $b_1, \ldots, b_{n-1}$; hence, it is equal to

$$\frac{j + 1}{k(k + 1)}$$

if the b_i are left unspecified. In particular, the conditional probability in this theorem shows that the $a_i = a_i(x)$, considered as a sequence of random variables, form a homogeneous Markov chain.

Theorem 3:

$$\Pr[a_n = k] = \frac{\left[{k \atop n}\right]}{(k + 1)!}$$

where $\left[{k \atop n}\right]$ is a Stirling number of the first kind. See, e.g., [14] or [11].

Proof: By Theorem 1, we can compute the measure of the set of x whose Pierce expansions begin with a specified prefix. Let us fix $a_n = k$, and sum over all possible prefixes, i.e., all strictly increasing sequences of positive integers of length n whose largest element is k.

$$\Pr[a_n = k] = \sum_{1 < a_1 < \cdots < a_{n-1} < k} \frac{1}{a_1 a_2 \cdots a_{n-1} k(k + 1)}$$

(continued)

$$= \sum_{\substack{A \subset \{1, 2, \ldots, k-1\} \\ |A| = n-1}} \frac{1}{\Pi A} \cdot \frac{1}{k(k+1)} = \sum_{\substack{B \subset \{1, 2, \ldots, k-1\} \\ |B| = k-n}} \frac{\Pi B}{(k-1)!} \cdot \frac{1}{k(k+1)}$$

$$= \frac{1}{(k+1)!} \sum_{\substack{B \subset \{1, 2, \ldots, k-1\} \\ |B| = k-n}} \Pi B$$

and the proof is now complete if we observe that the sum over the product of elements of B is in fact the coefficient of x^n in the polynomial

$$x(x+1)(x+2) \cdots (x+k-1)$$

which is just $\begin{bmatrix} k \\ n \end{bmatrix}$, a Stirling number of the first kind.

(Some brief comments about the notation: in the proof above, A and B are sets. $|A|$ is the cardinality of A. The sum is over all subsets with specified cardinality, and ΠA means the product of all elements in A.)

We get two interesting corollaries: using a theorem of Jordan [11] we can estimate the distribution of the a_n. We have

$$\begin{bmatrix} k \\ n \end{bmatrix} \sim \frac{(k-1)!}{(n-1)!}(\log k + \gamma)^{n-1}$$

and so we get

$$\mathbf{Pr}[a_n = k] \sim \frac{(\log k + \gamma)^{n-1}}{k(k+1)(n-1)!}$$

where n is fixed and $k \to \infty$ and γ is Euler's constant. Compare this with the similar result of Békéssy [2] for Engel's series. More detailed asymptotic results can be obtained by using the results of Moser and Wyman [18].

Also, we observe that the events $a_n = 1$, $a_n = 2$, ... are all disjoint and exhaust the space of events. Therefore,

$$\sum_{k=0}^{\infty} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = 1, \tag{3}$$

which is another derivation of the formula due to Jordan [11, p. 165].

In the next section, we derive some results on series involving Stirling numbers.

3. IDENTITIES ON STIRLING NUMBERS

Theorem 4:

$$\sum_{j=1}^{\infty} \frac{\begin{bmatrix} j \\ n \end{bmatrix}}{j \cdot j!} = \zeta(n+1)$$

where $\zeta(k)$ is Riemann's zeta function.

Proof: This is a result due to Jordan [11, pp. 164, 194, 339].

Theorem 5:

$$\sum_{k=j}^{\infty} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = 1 - \sum_{k=0}^{j-1} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = \frac{1}{j!}\sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix} \quad \text{for } n \geqslant 1,\ j \geqslant 1.$$

Proof: The proof of the first equality is just formula (3) above. To verify the second, we use induction on j, holding n fixed. It is easy to verify the case $j = 1$. Now assume true for j; we show the identity holds for $j + 1$. We have

$$1 - \sum_{k=0}^{j-1} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = \frac{1}{j!}\sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix}.$$

Now subtract $\dfrac{\begin{bmatrix} j \\ n \end{bmatrix}}{(j+1)!}$ from both sides to get:

$$\begin{aligned} 1 - \sum_{k=0}^{j} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} &= \left(\frac{1}{j!}\sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix}\right) - \frac{\begin{bmatrix} j \\ n \end{bmatrix}}{(j+1)!} \\ &= \left(\frac{1}{(j+1)!}\sum_{i=1}^{n} (j+1)\begin{bmatrix} j \\ i \end{bmatrix}\right) - \frac{\begin{bmatrix} j \\ n \end{bmatrix}}{(j+1)!} \\ &= \left(\frac{1}{(j+1)!}\sum_{i=1}^{n} j\begin{bmatrix} j \\ i \end{bmatrix}\right) + \left(\frac{1}{(j+1)!}\sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix}\right) - \frac{\begin{bmatrix} j \\ n \end{bmatrix}}{(j+1)!} \\ &= \frac{1}{(j+1)!}\sum_{i=1}^{n}\left(\begin{bmatrix} j+1 \\ i \end{bmatrix} - \begin{bmatrix} j \\ i-1 \end{bmatrix} + \begin{bmatrix} j \\ i \end{bmatrix}\right) - \frac{\begin{bmatrix} j \\ n \end{bmatrix}}{(j+1)!} \\ &= \frac{1}{(j+1)!}\sum_{i=1}^{n} \begin{bmatrix} j+1 \\ i \end{bmatrix}, \end{aligned}$$

where we have used telescoping cancellation and the well-known identity on Stirling numbers

$$j\begin{bmatrix} j \\ i \end{bmatrix} = \begin{bmatrix} j+1 \\ i \end{bmatrix} - \begin{bmatrix} j \\ i-1 \end{bmatrix}.$$

This completes the proof of Theorem 5. This is apparently a new identity on Stirling numbers.

Michael Luby made the following clever observation (personal communication): It is possible to prove Theorem 5 without the use of induction, by interpreting the left and right sides combinatorially, in terms of the a_n. The left side, in fact, is just

$$\Pr[a_n \geqslant j]$$

while the right side can be shown to be

$$\Pr[(a_1 \geqslant j) \text{ or } (a_1 < j \text{ and } a_2 \geqslant j) \text{ or } (a_1, a_2 < j \text{ and } a_3 \geqslant j) \cdots].$$

Theorem 6:

$$\sum_{k=1}^{\infty} H(k)\frac{\left[\begin{matrix}k\\n\end{matrix}\right]}{(k+1)!} = \zeta(2) + \zeta(3) + \cdots + \zeta(n+1)$$

where $H(k)$ is the k^{th} harmonic number,

$$H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}.$$

Proof:

$$\sum_{k=1}^{\infty} H(k)\frac{\left[\begin{matrix}k\\n\end{matrix}\right]}{(k+1)!} = \sum_{k=1}^{\infty} \frac{\left[\begin{matrix}k\\n\end{matrix}\right]}{(k+1)!}\sum_{j=1}^{k}\frac{1}{j} = \sum_{j=1}^{\infty}\frac{1}{j}\sum_{k=j}^{\infty}\frac{\left[\begin{matrix}k\\n\end{matrix}\right]}{(k+1)!}$$

$$= \sum_{j=1}^{\infty}\frac{1}{j\cdot j!}\sum_{i=1}^{n}\left[\begin{matrix}j\\i\end{matrix}\right] = \sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{\left[\begin{matrix}j\\i\end{matrix}\right]}{j\cdot j!} = \sum_{i=1}^{n}\zeta(i+1),$$

where we have used Theorems 4 and 5.

The author would like to express his thanks to Richard Fateman and the Vaxima version of the MACSYMA computer algebra system—an early version of Theorem 6 was suggested by experimentation with Vaxima!

Theorem 7:

$$\sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{\left[\begin{matrix}k\\n\end{matrix}\right]}{(k+1)!} = \zeta(n+1) - 1$$

Proof: See [11, p. 339].

We can now give a new derivation of a formula for $\zeta(3)$ due to Briggs et al. [3]. Noting that

$$\left[\begin{matrix}k\\2\end{matrix}\right] = H(k-1)(k-1)!$$

we get

$$\zeta(3) = \sum_{k=1}^{\infty}\left(1 + \frac{1}{k}\right)\frac{\left[\begin{matrix}k\\2\end{matrix}\right]}{(k+1)!} = \sum_{k=1}^{\infty}\frac{H(k-1)}{k^2}$$

or, adding $\sum_{k=1}^{\infty}\frac{1}{k^3}$ to both sides, we get

$$2\zeta(3) = \sum_{k=1}^{\infty}\frac{H(k)}{k^2}.$$

Many similar formulas can be given; for example, by appealing to Theorem 6, we can obtain

$$\sum_{k=1}^{\infty}\frac{H(k)(H(k-1)-1)}{k(k+1)} = \zeta(3).$$

See also [4], [10], [13], and [23].

Theorem 8:

$$\sum_{k=1}^{\infty} H(k+1)\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!} = n + 1$$

Proof:

$$\sum_{k=1}^{\infty} H(k+1)\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!} = \sum_{k=1}^{\infty}\sum_{j=1}^{k+1}\frac{1}{j}\cdot\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!}$$

$$= \sum_{k=1}^{\infty}\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!} + \sum_{j=2}^{\infty}\sum_{k=j-1}^{\infty}\frac{1}{j}\cdot\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!}$$

$$= 1 + \sum_{j=2}^{\infty}\frac{1}{j}\sum_{i=1}^{n}\frac{\begin{bmatrix}j-1\\i\end{bmatrix}}{(j-1)!}$$

$$= 1 + \sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{\begin{bmatrix}j\\i\end{bmatrix}}{(j+1)!} = n + 1$$

In the next section, we use these identities on Stirling numbers to derive estimates for the expected value and variance of quantities connected with a_n.

4. EXPECTED VALUES AND VARIANCES

We will use $\mathsf{E}[X]$ and $\mathsf{Var}[X]$ for the expected value and variance of the random variable X.

We are interested in how the a_n are distributed. However, the a_n are distributed such that $\mathsf{E}[a_n] = \infty$ for every n. It is reasonable to expect that the quantity $\log a_n$ rather than a_n gives more information.

Theorem 9:

(a) $\mathsf{E}[H(a_n)] = \zeta(2) + \zeta(3) + \cdots + \zeta(n+1)$

(b) $\mathsf{E}[\log a_n] = n + 1 - \gamma + O(2^{-n})$

Proof:

(a) $$\mathsf{E}[H(a_n)] = \sum_{k=1}^{\infty} H(k)\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!} = \zeta(2) + \zeta(3) + \cdots + \zeta(n+1)$$

using Theorem 6.

(b) To prove part (b) we use the famous estimate

$$H(k) = \log k + \gamma + O\left(\frac{1}{k}\right),$$

and therefore, using Theorems 6 and 7,

$$\mathsf{E}[\log a_n] = \mathsf{E}[H(a_n)] - \gamma + O\left(\mathsf{E}\left[\frac{1}{a_n}\right]\right)$$

$$= \zeta(2) + \zeta(3) + \cdots + \zeta(n+1) - \gamma + O(\zeta(n+1) - 1).$$

Now it is easily shown that

$$\zeta(2) + \zeta(3) + \cdots + \zeta(k) = k + \theta(2^{-k}) \tag{4}$$

and

$$\zeta(k) = 1 + \theta(2^{-k});$$

so, by substitution, we obtain the desired result.

Similar techniques allow us to calculate the variance.

Theorem 10:

(a) $\mathrm{Var}[H(a_n)] = n + O(1)$

(b) $\mathrm{Var}[\log a_n] = n + O(1)$

Proof: We find first that

(a) $$\mathrm{E}[H(a_n)^2] = \sum_{k=1}^{\infty} H(k)^2 \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} \frac{2H(j-1)}{j} + \frac{1}{j^2} \right) \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!}$$

$$= \sum_{j=1}^{\infty} \left(\frac{2H(j-1)}{j} + \frac{1}{j^2} \right) \sum_{k=j}^{\infty} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!}$$

$$= \sum_{j=1}^{\infty} \left(\frac{2H(j-1)}{j} + \frac{1}{j^2} \right) \frac{1}{j!} \sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix}, \tag{5}$$

where we have used the fact that

$$H(j)^2 = H(j-1)^2 + \frac{2H(j-1)}{j} + \frac{1}{j^2}$$

and Theorem 5. Note that $H(0) = 0$ by definition.

On the other hand, we have already seen that

$$\sum_{k=1}^{\infty} \frac{H(k)}{k+1} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{k!} = \sum_{k=1}^{\infty} H(k) \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = n + 1 + O(2^{-n+1}),$$

and therefore,

$$\sum_{k=1}^{\infty} \frac{H(k)}{k+1} \frac{1}{k!} \sum_{i=1}^{n} \begin{bmatrix} k \\ i \end{bmatrix} = \sum_{i=1}^{n} (i + 1 + O(2^{-i+1})) = \frac{n^2 + 3n}{2} + O(1).$$

Hence, we find

$$\sum_{j=1}^{\infty} \frac{2H(j)}{j+1} \frac{1}{j!} \sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix} = n^2 + 3n + O(1).$$

The left side of this equation looks very much like the right side of equation (5). In fact, it is easy to show that their difference is bounded by a constant that is independent of n. We have

$$\sum_{j=1}^{\infty} \left(\frac{2H(j-1)}{j} + \frac{1}{j^2} - \frac{2H(j)}{j+1} \right) \frac{1}{j!} \sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix} \leqslant \sum_{j=1}^{\infty} \left(\frac{2H(j-1)}{j} + \frac{1}{j^2} - \frac{2H(j)}{j+1} \right) \tag{6}$$

since

$$\sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix} \leqslant j!.$$

Now the sum on the right side of (6) can be computed exactly:

$$\sum_{j=1}^{\infty}\left(\frac{2H(j-1)}{j} + \frac{1}{j^2} - \frac{2H(j)}{j+1}\right) = \sum_{j=1}^{\infty}\left(\frac{2H(j-1)}{j} + \frac{1}{j^2} - \frac{2H(j-1)}{j+1} - \frac{2}{j(j+1)}\right)$$

$$= \sum_{j=1}^{\infty}\left(\frac{2H(j-1)}{j(j+1)} + \frac{1}{j^2} - \frac{2}{j(j+1)}\right)$$

$$= \zeta(2) = O(1).$$

Thus, we conclude that

$$\mathsf{E}[H(a_n)^2] = n^2 + 3n + O(1).$$

On the other hand, from Theorem 9, we see that

$$\mathsf{E}^2[H(a_n)] = n^2 + 2n + 1 + O(n2^{-n})$$

and therefore,

$$\mathsf{Var}[H(a_n)] = n + O(1)$$

which is the desired result.

(b) To prove part (b), we use the fact that

$$H(k) = \log k + O(1)$$

to get

$$\mathsf{Var}[\log a_n] = \mathsf{Var}[H(a_n)] + O(\mathsf{Var}[1]) = n + O(1).$$

This completes the proof.

In a similar fashion, we can obtain theorems about the expected values of various functions of the a_n. We give some unusual examples.

Let $f(x) = 1 \bmod x = 1 - x\lfloor 1/x \rfloor$. Then it is easy to see that if

$$x = \{a_1, a_2, \ldots\}$$

then

$$f(x) = \{a_2, a_3, \ldots\}.$$

Let us write $f^{(2)}(x) = f(f(x))$, etc. Then we have

Theorem 11:

$$\mathsf{E}[f^{(n)}(x)] = \frac{1}{2}(n + 1 - \zeta(2) - \zeta(3) - \cdots - \zeta(n+1)) = \theta(2^{-n+2})$$

Proof: Suppose $a_n = k$. What is the expected value of $f^{(n)}(x)$? If we restrict our attention to the half-open interval that contains all numbers whose Pierce expansions begin

$$\{a_1, a_2, \ldots, a_{n-1}, k\},$$

then it is easily seen that $f^{(n)}(x)$ is linear on this interval. The minimum and maximum values that $f^{(n)}(x)$ attains are 0 and $1/(k+1)$ respectively; hence the expected value of $f^{(n)}(x)$ on this specified interval is $1/[2(k+1)]$. But this is independent of the choice of $a_1, a_2, \ldots, a_{n-1}$; hence the expected value of $f^{(n)}(x)$ given that $a_n = k$ is $1/[2(k+1)]$. Therefore,

$$\begin{aligned}\mathsf{E}[f^{(n)}(x)] &= \sum_{k=1}^{\infty} \frac{1}{2(k+1)} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = \frac{1}{2}\left(\sum_{k=1}^{\infty}(H(k+1) - H(k))\frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!}\right)\\ &= \frac{1}{2}\left(\sum_{k=1}^{\infty} H(k+1)\frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} - \sum_{k=1}^{\infty} H(k)\frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!}\right)\\ &= \frac{1}{2}(n + 1 - \zeta(2) - \zeta(3) - \cdots - \zeta(n+1)),\end{aligned}$$

where we have used Theorems 7 and 8.

From equation (4), this quantity is $\theta(2^{-n+2})$, and the proof is complete.

It is of some interest to note that Theorem 11 is a generalization of a result of Dirichlet [6]. He stated that

$$\sum_{k=1}^{n} k\left\lfloor \frac{n}{k} \right\rfloor \sim \frac{\pi^2 n^2}{12}.$$

We can derive this easily. From Theorem 11, we have

$$\begin{aligned}\frac{n}{2}(2 - \zeta(2)) &= n\int_0^1 1 \bmod x\, dx = \int_0^1 n \bmod nx\, dx = \frac{1}{n}\int_0^n n \bmod x\, dx\\ &= \frac{1}{n}\int_0^n n - x\lfloor n/x \rfloor dx = n - \frac{1}{n}\int_0^n x\lfloor n/x \rfloor dx,\end{aligned}$$

and we get the desired result by approximating the integral with a sum.

Theorem 12:

$$\sum_{k=1}^{\infty} \frac{1}{a_k}$$

converges for almost all x (i.e., for all but a set of measure 0). The expected value of the sum is 1. The set of exceptions

$$\left\{x \,\middle|\, \sum_{k=1}^{\infty} \frac{1}{a_k} \text{ diverges}\right\}$$

is uncountable and dense.

Proof: From Theorem 7, we have

$$\mathsf{E}\left[\frac{1}{a_n}\right] = \zeta(n+1) - 1 < 2^{1-n},$$

and it is easily seen that the variance $\mathsf{Var}\left[\frac{1}{a_n}\right]$ is also $< 2^{1-n}$.

Then, by Chebyshev's inequality,

$$\Pr\left[\frac{1}{a_n} - 2^{1-n} \geqslant 2^{-n/4}\right] \leqslant \frac{2^{1-n}}{2^{-n/2}}$$

Now, by the Borel-Cantelli lemma, with probability 1 only finitely many of the events

$$\frac{1}{a_n} - 2^{1-n} \geqslant 2^{-n/4}$$

occur, and so the series converges almost everywhere.

We also have

$$E\left[\sum_{k=1}^{\infty} \frac{1}{a_n}\right] = \sum_{k=1}^{\infty} (\zeta(k+1) - 1) = 1.$$

(See, e.g., [11, p. 340].) This proves the result on the expected value.

Now we show that the set of exceptions is uncountable. Let the real number x in the interval $(0, 1)$ be written in base two notation,

$$x = e_1 e_2 e_3 \ldots,$$

where each $e_i = 1$ or 0. Then associate with each such x the real number whose Pierce expansion is given by

$$h(x) = \{1 + e_1,\ 3 + e_2,\ 5 + e_3,\ \ldots\}.$$

Then each of these numbers $h(x)$ is distinct by the uniqueness of Pierce expansions, and for each $h(x)$ we have

$$\sum_{k=1}^{n} \frac{1}{a_k} \geqslant \sum_{k=1}^{n} \frac{1}{2k}$$

and so the series diverges.

The proof that the set of exceptions is dense is left to the reader.

Theorem 13:

$$\sum_{k=1}^{\infty} f^{(k)}(x)$$

converges for almost all x. The expected value of the sum is $\frac{\pi^2}{12} - \frac{1}{2}$. The set of exceptions

$$\left\{x \,\middle|\, \sum_{k=1}^{\infty} f^{(k)}(x) \text{ diverges}\right\}$$

is uncountable and dense.

Proof: We prove only the result on the expected value, leaving the rest to the reader.

$$E\left[\sum_{k=1}^{\infty} f^{(k)}(x)\right] = \sum_{k=1}^{\infty} \frac{1}{2}\left(k + 1 - \sum_{j=2}^{k+1} \zeta(j)\right) = \frac{1}{2} \sum_{k=1}^{\infty} \left(1 - \sum_{j=2}^{k+1} (\zeta(j) - 1)\right)$$

(continued)

$$= \frac{1}{2}\sum_{k=1}^{\infty}\left(1 - \sum_{j=2}^{k+1}\sum_{i=2}^{\infty}\frac{1}{i^j}\right) = \frac{1}{2}\sum_{k=1}^{\infty}\left(1 - \sum_{i=2}^{\infty}\frac{1/i - 1/i^{k+1}}{i - 1}\right)$$

$$= \frac{1}{2}\sum_{k=1}^{\infty}\sum_{i=2}^{\infty}\frac{1}{(i - 1)i^{k+1}} = \frac{1}{2}\sum_{i=2}^{\infty}\frac{1}{i - 1}\sum_{k=2}^{\infty}\frac{1}{i^{k+1}} = \frac{1}{2}\sum_{i=2}^{\infty}\frac{1}{i(i - 1)^2}$$

$$= \frac{1}{2}(\zeta(2) - 1),$$

which is the desired result.

5. DISTRIBUTION OF THE a_n: METHOD OF RÉNYI

So far we have shown that $\log a_n$ has an expected value that tends to $n + 1 - \gamma$ as n approaches ∞. We have also seen that the variance is small. In fact, it is possible to prove much stronger results; for example, that

$$\lim_{n\to\infty} a^{1/n} = e$$

for almost all x. We will use a method employed by Rényi in his analysis of Engel's series [21]. We start by identifying some new random variables and we show they are independent.

Define

$$\varepsilon_k(x) = \begin{cases} 1 & \text{if } k \text{ appears in the Pierce expansion of } x, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

Theorem 14: $\mathrm{E}[\varepsilon_k(x)] = \dfrac{1}{k + 1}$

Proof:

$$\mathrm{E}[\varepsilon_k(x)] = \sum_{n=1}^{\infty}\frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k + 1)!} = \frac{1}{(k + 1)!}\sum_{n=1}^{\infty}\begin{bmatrix} k \\ n \end{bmatrix} = \frac{k!}{(k + 1)!} = \frac{1}{k + 1},$$

since the events $a_i = k$ and $a_j = k$ are disjoint if $i \neq j$.

Theorem 15: The random variables $\varepsilon_k(x)$ are independent.

Proof: Let

$$\varepsilon_1 = \delta_1,\ \varepsilon_2 = \delta_2,\ \ldots,\ \varepsilon_n = \delta_n$$

represent an assignment of 0's and 1's for the values of ε_i. Let b_i $(1 \leqslant i \leqslant k)$ be such that $\delta_{b_i} = 1$ and all other values of δ_j are 0. Without loss of generality, assume that $\delta_n = 1$. Then the probability that the events

$$\varepsilon_1 = \delta_1,\ \varepsilon_1 = \delta_2,\ \ldots,\ \varepsilon_n = \delta_n$$

simultaneously occur is just the probability that the Pierce expansion for x begins $b_1, b_2, \ldots, b_k$, which we have seen is equal to

$$\frac{1}{b_1 b_2 \cdots b_k(b_k + 1)}.$$

On the other hand, we have

$$\mathbf{Pr}[\varepsilon_i = \delta_i] = \begin{cases} \dfrac{i}{i+1} & \text{if } \delta_i = 0, \\[2ex] \dfrac{1}{i+1} & \text{if } \delta_i = 1. \end{cases}$$

Let us compute

$$\prod_{i=1}^{n} \mathbf{Pr}[\varepsilon_i = \delta_i]. \tag{7}$$

In the numerator of (7) we have those i corresponding to the δ_i that equal 0; in the denominator we have $(n + 1)!$. By canceling in the numerator and denominator, we see that the value of the product (7) is just

$$\frac{1}{b_1 b_2 \cdots b_k(b_k + 1)},$$

which shows the independence of the ε_i.

It is also easy to see that

$$\mathbf{Var}[\varepsilon_k] = \frac{1}{k+1} - \frac{1}{(k+1)^2}$$

Now, let $\mu_N = \mu_N(x)$ denote the number of terms of the sequence $a_n = a_n(x)$ that are $\leqslant N$. In other words, put

$$\mu_N = \sum_{k=1}^{N} \varepsilon_k.$$

Then we see immediately that

$$\mathbf{E}[\mu_N] = \sum_{k=1}^{N} \frac{1}{k+1} = \log N + \gamma - 1 + O\left(\frac{1}{N}\right)$$

and

$$\mathbf{Var}[\mu_N] = \sum_{k=1}^{N} \left(\frac{1}{k+1} - \frac{1}{(k+1)^2}\right) = \log N + \gamma - \frac{\pi^2}{6} + O\left(\frac{1}{N}\right).$$

We can prove the strong law of large numbers for the random variables ε_k. We need the following general form of this law [21]:

If $\xi_1, \xi_2, \ldots$ are independent nonnegative random variables with finite expectation $E_k = \mathbf{E}[\xi_k]$ and variance $V_k = \mathbf{Var}[\xi_k]$ and if putting

$$A_N = \sum_{k=1}^{N} E_k$$

one has

$$\lim_{N \to \infty} A_N = \infty$$

and also

$$\sum_{N=1}^{\infty} \frac{V_N}{A_N^2} < \infty,$$

then with probability 1 we have

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} \xi_k}{A_N} = 1.$$

The conditions of this theorem are fulfilled for $\xi_k = \varepsilon_k$, since

$$\sum_{N=1}^{\infty} \frac{\dfrac{1}{N+1} - \dfrac{1}{N+1^2}}{(\log N + \gamma - 1)^2}$$

converges by comparison with the integral

$$\int \frac{1}{x(\log x)^2}\, dx = \frac{-1}{\log x}.$$

Thus we obtain

Theorem 16: For almost all x we have

$$\lim_{N\to\infty} \frac{\mu_N}{\log N + \gamma - 1} = 1$$

Using $\mu_{a_n} = n$, we obtain

$$\lim_{n\to\infty} a^{1/n} = e$$

for almost all x.

[We can easily get a similar result for iterates of $f(x) = 1 \bmod x$. Since

$$\frac{1}{1 + a_{n+1}} < f^{(n)}(x) \leqslant \frac{1}{a_{n+1}},$$

we find

$$\lim_{n\to\infty} (f^{(n)}(x))^{1/n} = \frac{1}{e}$$

for almost all x.]

We can use Ljapunov's condition [8] to obtain a central limit theorem for the a_n. We have

$$\mathbf{E}[\varepsilon_k^3] = \frac{1}{k+1}$$

and

$$\frac{\mathbf{E}[\varepsilon_k^3]}{\mathbf{Var}[\mu_k]}$$

is bounded. Also $\sqrt{\mathbf{Var}[\mu_k]} \to \infty$. Hence, we find

Theorem 17:

$$\lim_{N\to\infty} \mathbf{Pr}\left[\frac{\mu_N - \log N}{\sqrt{\log N}} < y\right] = \Phi(y),$$

where $\Phi(y)$ is the normal distribution given by $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2}\, du$.

Now, noting that

$$\Pr[\mu_N < n] = \Pr[a_n > N],$$

we see that an equivalent statement of Theorem 17 is

$$\lim_{n\to\infty} \Pr\left[\frac{\log a_n - n}{\sqrt{n}} < y\right] = \Phi(y).$$

As a corollary, we get

$$\lim_{n\to\infty} \sum_{k=e^{n+\alpha\sqrt{n}}}^{k=e^{n+\beta\sqrt{n}}} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = \Phi(\beta) - \Phi(\alpha).$$

This is similar to the result

$$\lim_{n\to\infty} \sum_{k=\log n+\alpha\sqrt{\log n}}^{k=\log n+\beta\sqrt{\log n}} \frac{\begin{bmatrix} n \\ k \end{bmatrix}}{n!} = \Phi(\beta) - \Phi(\alpha)$$

given in [8].

Similarly, as the conditions given by Kolmogoroff [15] for the law of the iterated logarithm are fulfilled for the variables ε_k, we get

Theorem 18: For almost all x,

$$\limsup_{N\to\infty} \frac{\mu_N - \log N}{\sqrt{2\log N \cdot \log\log\log N}} = 1$$

and

$$\liminf_{N\to\infty} \frac{\mu_N - \log N}{\sqrt{2\log N \cdot \log\log\log N}} = -1$$

or, stated equivalently,

$$\limsup_{n\to\infty} \frac{\log a_n - n}{\sqrt{2n \cdot \log\log n}} = 1$$

and

$$\liminf_{n\to\infty} \frac{\log a_n - n}{\sqrt{2n \cdot \log\log n}} = -1.$$

6. SOME RESULTS ON FINITE PIERCE EXPANSIONS

In [7], Erdös et al. put $E_1(a, b) = n$, where

$$\frac{a}{b} = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \cdots + \frac{1}{q_1 q_2 \cdots q_n}$$

(an expansion into Engel's series) and ask for a nontrivial estimation of $E_1(a, b)$.

We prove two results on the length of finite Pierce expansions. Unfortunately, it does not seem possible to use our techniques for Engel's series.

Let us put $L(p, q) = n$, where

$$\frac{p}{q} = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{n+1}}{a_1 a_2 \cdots a_n}.$$

Then we have

Theorem 19: $L(p, q) < 2\sqrt{q}$.

Proof: Let us write

$$\frac{p}{q} = \{a_1, a_2, \ldots, a_n\}$$

and, as in the Pierce Expansion Algorithm, put $p_1 = p$ and

$$a_i = \lfloor q/p_i \rfloor,$$

$$p_{i+1} = q - a_i p_i.$$

Without loss of generality, we may assume that $a_1 = 1$. For otherwise we have

$$\frac{q - p}{q} = \{1, a_1, a_2, \ldots, a_n\},$$

which is a longer Pierce expansion.

Then suppose $p_n \geqslant a_n$. Since $a_n p_n = q$, we have $a_n \leqslant \sqrt{q}$. But the a_i are strictly increasing, so $n \leqslant \sqrt{q}$.

Now suppose $p_n < a_n$. Since the p_i are strictly decreasing, and the a_i are strictly increasing, we see that $p_i - a_i$ is a strictly decreasing sequence. But $p_1 - a_1 \geqslant 0$ since $a_1 = 1$, and $p_n - a_n < 0$ by hypothesis. Hence, there must be a unique subscript k such that

$$p_k - a_k \geqslant 0$$

but

$$p_{k+1} - a_{k+1} < 0.$$

Then, since $p_i a_i \leqslant q$ for all i, we see that

$$a_k \leqslant \sqrt{q} \quad \text{and} \quad p_{k+1} < \sqrt{q}.$$

By the monotonicity of these sequences, we see that $k \leqslant \sqrt{q}$ and $n - k < \sqrt{q}$. We add these inequalities to get $n < 2\sqrt{q}$, which is the desired result.

Unfortunately, this bound is not very tight. For example,

$$\frac{470}{743} = \{1, 2, 3, 4, 5, 10, 11, 14, 17, 61, 67, 123, 148, 247, 371, 743\}.$$

This is the longest Pierce expansion with $q < 1000$. We see that $n = 16$, but our estimate guarantees just $n < 54$.

It seems likely that $L(p, q) = O(\log q)$; we cannot expect a much better lower bound. For example, we have the following theorem.

Theorem 20: There exist infinitely many q with $L(p, q) > \dfrac{\log q}{\log \log q}$.

Proof: The proof is constructive. Let $q = n!$, and set

$$p = n!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n+1}}{n!}\right).$$

Then we have

$$\frac{p}{q} = \{1, 2, 3, \ldots, n - 3, n - 2, n\},$$

and therefore, $L(p, q) = n - 1$.

However, it is easily shown that, for n sufficiently large,

$$n - 1 > \frac{\log n!}{\log \log n!}$$

and the desired result easily follows.

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NEWTON'S METHOD AND SIMPLE CONTINUED FRACTIONS

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1. INTRODUCTION

Let N be a positive integer that is not a perfect square. The Newton approximations to $\sqrt{N}$ will be obtained by fixing x_0 and setting

$$x_{n+1} = (x_n^2 + N)/(2x_n).$$

For example, one possible list of Newton approximations to $\sqrt{2}$ is

$$x_0 = 1,\ x_1 = 3/2,\ x_2 = 17/12,\ x_3 = 577/408,\ \ldots .$$

Let $(a_0, a_1, a_2, \ldots)$ represent the simple continued fraction for $\sqrt{N}$ with a_0, a_1, a_2, ... as partial quotients. Designate $c_n = p_n/q_n$, $(p_n, q_n) = 1$, as the n^{th} convergent of the continued fraction for $\sqrt{N}$. Thus, for example, $\sqrt{2} = (1, 2, 2, \ldots)$ has convergents

$$c_0 = 1,\ c_1 = 3/2,\ c_2 = 7/5,\ c_3 = 17/12,\ c_4 = 41/29,$$
$$c_5 = 99/70,\ c_6 = 239/169,\ c_7 = 577/408,\ \ldots .$$

Comparing the two lists of approximations, we see that each of the Newton approximations obtained in the manner above is a convergent of the continued fraction for $\sqrt{2}$; in fact, it appears that $x_n = c_{2^n - 1}$. This is indeed the case and follows from Theorem 1 below (cf. [1, p. 468], [2], [3], [4]). We give a proof which appears to be simpler than those in the literature.

Theorem 1: If the continued fraction for $\sqrt{N}$ has period k, then for any positive integer m, Newton's method applied to c_{mk-1} results in c_{2mk-1}.

Proof: The s^{th} positive solution to the equation $x^2 - Ny^2 = \pm 1$ can be found in the following two ways:

(i) Write $(p_{k-1} + \sqrt{N}q_{k-1})^s$ in the form $u + \sqrt{N}v$, where u and v are integers; then $(x, y) = (u, v)$ is the s^{th} solution.

(ii) Calculate c_{sk-1}; then (p_{sk-1}, q_{sk-1}) is the s^{th} solution.

Letting $s = 2m$ gives

$$p_{2mk-1} + \sqrt{N}q_{2mk-1} = [(p_{k-1} + \sqrt{N}q_{k-1})^m]^2 = (p_{mk-1} + \sqrt{N}q_{mk-1})^2$$
$$= p_{mk-1}^2 + Nq_{mk-1}^2 + \sqrt{N}(2p_{mk-1}q_{mk-1})$$

so that

$$p_{2mk-1}/q_{2mk-1} = (p_{mk-1}^2 + Nq_{mk-1}^2)/(2p_{mk-1}q_{mk-1}),$$

finishing the proof.

From Theorem 1, we see that whenever the continued fraction for $\sqrt{N}$ has period one, Newton's method applied to a convergent of the continued fraction for $\sqrt{N}$ results in a convergent. An identical result holds when the continued fraction for $\sqrt{N}$ has period two and follows as a corollary of the next theorem which we state without proof (cf. [3]).

Theorem 2: If the continued fraction for $\sqrt{N}$ has an even period $k = 2r$, then for any positive integer m, Newton's method applied to c_{mr-1} results in c_{2mr-1}.

We now know that if the continued fraction for $\sqrt{N}$ has period one or two, and if x_0 is a convergent of the continued fraction for $\sqrt{N}$, then we can conclude that all the successive approximations x_n are convergents of the continued fraction for $\sqrt{N}$. The following example shows that the conclusion is possible even when x_0 is rational but not a convergent. Let $N = 2$ and $x_0 = 2$; then we have $x_1 = 3/2$, $x_2 = 17/12$, $x_3 = 577/408$, ... which results in the same sequence we had with $x_0 = 1$. In the next two sections, we shall examine more closely the connection between Newton approximations and convergents in the cases when the continued fraction for $\sqrt{N}$ has period one or two.

2. CONTINUED FRACTION FOR $\sqrt{N}$ WITH PERIOD ONE

If the continued fraction for $\sqrt{N}$ has period one, we can tell for what rational x_0 the sequence $\{x_n\}$ of Newton approximations to $\sqrt{N}$ contains convergents and how many x_n are convergents. We note that if $x_0 = N/c_m$, then

$$x_1 = (x_0^2 + N)/(2x_0) = ([N/c_m]^2 + N)/(2N/c_m) = (c_m^2 + N)/(2c_m),$$

which is the same Newton approximation obtained if $x_0 = c_m$. Since x_{n+1} depends only on x_n (and N), we see that the entire sequence $\{x_n\}_{n=1}^{\infty}$ of Newton approximations to $\sqrt{N}$ is the same whether we begin with $x_0 = c_m$ or $x_0 = N/c_m$. This explains why we get the same Newton approximations to $\sqrt{2}$ when we begin with $x_0 = 1$ and when we begin with $x_0 = 2$.

Theorem 3: If the continued fraction for $\sqrt{N}$ has period one and if $\{x_n\}_{n=1}^{\infty}$ is the sequence of Newton approximations to $\sqrt{N}$ beginning with any rational number $x_0 \neq 0$, then either $\{x_n\}$ consists entirely of convergents or $\{x_n\}$ contains no convergents at all. Furthermore, $\{x_n\}$ consists entirely of convergents if and only if x_0 is a convergent or N times the reciprocal of a convergent.

Proof: We have already seen that if $x_0 = c_m$ or N/c_m for some nonnegative integer m, then $\{x_n\}_{n=1}^{\infty}$ consists entirely of convergents; therefore, it suffices to show that if x_0 is neither c_m nor N/c_m for any m, then $\{x_n\}$ contains no convergents. We begin with such an x_0 and prove by induction that every subsequent Newton approximation is of the same form. This is clearly the case if $x_0 < 0$, since for such an x_0 we have $\{x_n\}$ contains only negative numbers. Now consider $x_0 > 0$. Suppose that we have shown that x_n is neither c_m nor N/c_m for any m. Then

$$x_{n+1} = (x_n^2 + N)/(2x_n),$$

which is equivalent to

$$x_n^2 - 2x_{n+1}x_n + N = 0 \tag{1}$$

or

$$x_n = x_{n+1} \pm (x_{n+1}^2 - N)^{1/2}. \tag{2}$$

Assume $x_{n+1} = c_m$ for some nonnegative integer m. Since x_0 is rational and therefore real, so is x_n, whence, by (2), $x_{n+1} > \sqrt{N}$; this means that x_{n+1} must be an odd convergent. By Theorem 1, taking $k = 1$, we see that Newton's method applied to $c_{n/2}$ results in x_{n+1}. Consequently, Newton's method applied to $N/c_{n/2}$ also results in x_{n+1}. Since $\sqrt{N}$ is irrational, $c_{n/2} \neq N/c_{n/2}$. Hence, $c_{n/2}$ and $N/c_{n/2}$ must be the two distinct roots of (1) so that, contrary to the induction hypothesis, $x_n = c_{n/2}$ or $N/c_{n/2}$.

Assume now that $x_{n+1} = N/c_m$ for some m. Then (2) becomes

$$x_n = N/c_m \pm (\{N(Nq_m^2 - p_m^2)\}^{1/2}/p_m). \tag{3}$$

Since x_n is rational, we must have $\{N(Nq_m^2 - p_m^2)\}^{1/2}$ rational. But the continued fraction for $\sqrt{N}$ has period one, so that $Nq_m^2 - p_m^2 = \pm 1$, and therefore,

$$\{N(Nq_m^2 - p_m^2)\}^{1/2} = \{\pm N\}^{1/2},$$

which is not rational. Hence, $x_{n+1} \neq N/c_m$ for any m, completing the proof.

3. CONTINUED FRACTION FOR $\sqrt{N}$ WITH PERIOD TWO

When the continued fraction for $\sqrt{N}$ has period two, a theorem analogous to Theorem 3 does not exist. To see this, consider $N = 12$ and $x_0 = 6$. We have

$$\sqrt{12} = (3, 2, 6, 2, 6, \ldots),$$

with convergents 3, 7/2, 45/13, ..., so that x_0 is not a convergent. Also, $x_0 = 12/2$ so that x_0 is not $12/c_m$ for any m. But

$$(6^2 + 12)/(2 \cdot 6) = 12/3 = 12/c_0$$

which means, by an argument similar to that used at the beginning of Section 2, Newton's method applied twice to x_0 yields a convergent, namely $c_1 = 7/2$. We shall see, in fact, that there are infinitely many N such that the continued fraction for $\sqrt{N}$ has period two and, for some rational x_0 that is neither a c_m nor an N/c_m, the resulting sequence $\{x_n\}$ contains infinitely many convergents. On the other hand, we shall see that there are infinitely many N such that the continued fraction for $\sqrt{N}$ has period two and, for any rational x_0 that is neither a c_m nor an N/c_m, the resulting sequence $\{x_n\}$ contains no convergents. Before we begin, we note that some of the results of Section 2 carry over immediately into this section, namely Newton's method applied to c_m is identical to Newton's method applied to N/c_m, and the first part of the induction proof for Theorem 3 works here by using Theorem 2 rather than Theorem 1.

Theorem 4: Let S be the set of all $s = kx^2$ or $4kx^2$ where $x^2 - ky^2 = 1$ for some positive integers x, y, and k. If $N \in S$, then the continued fraction for $\sqrt{N}$ has period two and there is a rational x_0 not of the form c_m or N/c_m such that the sequence $\{x_n\}$ of Newton approximations to $\sqrt{N}$, beginning with x_0, contains infinitely many convergents. Also, if $N \notin S$ and the continued fraction for $\sqrt{N}$ has period two, then for any rational x_0 that is neither a c_m nor an N/c_m, the resulting sequence contains no convergents.

Proof: Let T be the set of all N such that the continued fraction for $\sqrt{N}$ has period two and, for any rational x_0 not of the type c_m or N/c_m, the resulting sequence $\{x_n\}$ of Newton approximations to $\sqrt{N}$, beginning with x_0, contains no convergent of the continued fraction for $\sqrt{N}$. Consider some N such that the continued fraction for $\sqrt{N}$ has period two. We show first that $N \notin T$ if and only $\{N(N - p_0^2)\}^{1/2}$ is rational. Assume $\{N(N - p_0^2)\}^{1/2}$ is rational. Set

$$x_0 = N/c_0 \pm (\{N(N - p_0^2)\}^{1/2}/p_0). \tag{4}$$

Since $q_0 = 1$, (4) is precisely (3) with $n = m = 0$. Thus, $x_1 = N/c_0$. Since the continued fraction for $\sqrt{N}$ has period two, there are positive integers a and b such that $b|2a$, $N = a^2 + (2a/b)$ and $\sqrt{N} = (a, b, 2a, b, 2a, \ldots)$, so the first two convergents of the continued fraction for $\sqrt{N}$ are a and $(ab + 1)/b$. Also, $x_1 = N/c_0 = (a^2 + 2a/b)/a = (ab + 2)/b$. Thus, x_1 is not a convergent. Therefore, x_0 is different from c_m and N/c_m for all m, but the sequence $\{x_n\}$ contains infinitely many convergents of the continued fraction for $\sqrt{N}$, namely all x_k for $k \geqslant 2$. Thus, $N \notin T$.

Now assume $\{N(N - p_0^2)\}^{1/2}$ is not rational. Suppose x_n is the n^{th} Newton approximation to $\sqrt{N}$ starting from some rational x_0 and is given by (2) and (3) where $x_{n+1} = N/c_m$ for some m. From (2) and the fact that x_n is rational, we have that $x_{n+1} > \sqrt{N}$ so that $c_m < \sqrt{N}$ and m is even. Thus,

$$p_m^2 - Nq_m^2 = p_0^2 - Nq_0^2 = p_0^2 - N,$$

so that by (3),

$$x_n = N/c_m \pm (\{N(N - p_0^2)\}^{1/2}/p_m),$$

which is not rational by assumption, giving a contradiction. The induction argument given in the proof of Theorem 3 now works here, and we may conclude that $N \in T$, which finishes what we first set out to show.

To complete the proof of the theorem we need only show that the continued fraction for $\sqrt{N}$ has period two and $\{N(N - p_0^2)\}^{1/2}$ is rational if and only if $N \in S$. Consider N such that the continued fraction for $\sqrt{N}$ has period two and write, as before, $N = a^2 + (2a/b)$ where $b|(2a)$. Assume that $\{N(N - p_0^2)\}^{1/2}$ is rational. We have $N - p_0^2 = N - a^2 = 2a/b$ so that $N(N - p_0^2) = N(2a/b) = d^2$ for some positive integer d. Then we consider two possible cases.

Case 1. b is odd.

Here $b|a$. Set $a_1 = a/b$ so that $N = b^2a_1^2 + 2a_1$. Therefore,

$$d^2 = 2a_1^2(b^2a_1 + 2). \tag{5}$$

Thus, a_1 is even and $(2a_1)|d$. Writing $a_1 = 2a_2$ and $d = 2a_1d_1$, (5) becomes

$$d_1^2 - a_2b^2 = 1,$$

and, therefore,

$$N = b^2a_1^2 + 2a_1 = 4a_2(b^2a_2 + 1) = 4a_2d_1^2.$$

Hence, $N \in S$.

Case 2. b is even.

Here $b = 2b_1$, where $b_1|a$, so that $a = b_1a_1$ and $d = a_1d_1$ for some integers a_1 and d_1 with

$$d_1^2 - a_1b_1^2 = 1. \tag{6}$$

We conclude that $N = b_1^2a_1^2 + a_1 = a_1d_1^2$ and, therefore, $N \in S$.

Now suppose $N \in S$. Then $N = 4kx^2$ or kx^2 for some positive integers x, y, and k such that $x^2 - ky^2 = 1$.

Case 1. $N = 4kx^2$.

Here set $a = 2ky$ and $b = y$. Then

$$a^2 + (2a/b) = 4k(ky^2 + 1) = 4kx^2 = N \text{ and } b|(2a).$$

Also, $b \neq 2a$, since $y < 4ky$. Thus, the continued fraction for $\sqrt{N}$ has period two. Also, we get $N(N - p_0^2) = (4kx)^2$.

Case 2. $N = kx^2$.

Here set $a = ky$ and $b = 2y$. Then $a^2 + (2a/b) = N$ and $b|2a$, so that the continued fraction for $\sqrt{N}$ has period two (note that $b \neq 2a$ since $x^2 - y^2 \neq 1$). Also, $N(N - p_0^2) = (kx)^2$.

This completes the proof.

Corollary 1: If the continued fraction for $\sqrt{N}$ has period two and N is square-free, then $N \notin S$.

Proof: Suppose $N \in S$. Then $N = kx^2$ or $4kx^2$ for some positive integers x and k. Thus, $x = 1$ and $1 - ky^2 = 1$ for some positive integer y, giving a contradiction.

Corollary 2: If $N = (2d)^2 + 2$, where d is the denominator of an odd convergent of the continued fraction for $\sqrt{2}$, then $N \in S$. On the other hand, if $N = a^2 + 2$ for any positive integer a not twice the denominator of an odd convergent of the continued fraction for $\sqrt{2}$, then $N \notin S$. In particular, if N is odd and of the form $a^2 + 2$, then $N \notin S$.

Proof: Consider $N = a^2 + (2a/b)$, where $b = a$. From the proof of Theorem 4, we know that $N \in S$ if and only if $2N = N(2a/b) = d_1^2$ for some positive integer d_1 if and only if $a^2 + 2 = N = 2d_2^2$ for some positive integer d_2 if and only if $a = 2d$ for some positive integer d and $d_2^2 - 2d^2 = 1$ if and only if $N = a^2 + 2$, where $a = 2d$ and d_2/d is an odd convergent of the continued fraction for $\sqrt{2}$, which proves the first part of the corollary. The last statement follows from the observation that if N is odd, then a is odd.

Corollary 3: There exist infinitely many $N \in S$ and infinitely many N such that the continued fraction for $\sqrt{N}$ has period two and $N \notin S$.

Proof: Take N of the form $a^2 + 2$, and use Corollary 2.

Finally, we note that the only $N \in S$ less than 1000 are

12, 18, 48, 72, 147, 150, 240, 288, 405, 448, 578, 588, 600, 960.

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LETTER FROM THE EDITOR

The editor wishes to express his gratitude to those who have agreed to referee papers for *The Fibonacci Quarterly* during 1986. A complete list of these referees will be given in the May 1986 issue.

G. E. Bergum

SKEW CIRCULANTS AND THE THEORY OF NUMBERS

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1. SKEW CIRCULICES AND SKEW DISCRETE FOURIER TRANSFORMS

The matrices

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

provide a familiar representation of complex numbers $x + iy$. Let us consider the more general matrix

$$X = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{t-1} \\ -x_{t-1} & x_0 & x_1 & \cdots & x_{t-2} \\ -x_{t-2} & -x_{t-1} & x_0 & \cdots & x_{t-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -x_2 & -x_3 & -x_4 & \cdots & x_1 \\ -x_1 & -x_2 & -x_3 & \cdots & x_0 \end{bmatrix}, \tag{1}$$

where t is a positive integer, and $x_0, x_1, \ldots, x_{t-1}$ are real numbers. The determinant of X is called a *skew circulant* by Muir [7, p. 442] and by Davis [1, pp. 83-85], and we call X a *skew circulix*. We can write X in the form

$$X = x_0 I + x_1 J + \cdots + x_{t-1} J^{t-1}, \tag{2}$$

where I is the $t \times t$ unit matrix and

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ -1 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}. \tag{3}$$

Since $J^t = -I$, it follows that all polynomials in J can be expressed as polynomials of degree at most $t - 1$, and every such polynomial is a skew circulix. Hence, all skew circulices commute with each other.

On a point of terminology, a skew circulix is not in general a skew-symmetric matrix although it can be. For example, J^2 is both a skew circulix and a skew-symmetric matrix when $t = 4$.

The eigenvalues of X are

$$x_s^\dagger = \sum_{r=0}^{t-1} x_r j^{r(2s+1)} \qquad (s = 0,\ 1,\ \ldots,\ t - 1), \tag{4}$$

where $j = e^{\pi i/t}$. By analogy with the discrete Fourier transform, we may call the sequence $(x_0^\dagger,\ x_1^\dagger,\ \ldots,\ x_{t-1}^\dagger)$ the skew discrete Fourier transform (skew DFT) of $(x_0,\ x_1,\ \ldots,\ x_{t-1})$. The eigenvectors of all skew circulices are the columns of the matrix

$$\{j^{r(2s+1)}\} \qquad (r,\ s = 0,\ 1,\ \ldots,\ t - 1).$$

We now list a few further properties of skew circulices to emphasize their mathematical respectability. See also Section 4.

Just as the ordinary discrete Fourier transform is associated with sequences of period t, the skew DFT is associated with sequences of antiperiod t; that is, doubly infinite sequences such that $x_{r+t} = -x_r$ for all integers r. This is so in the sense that, if (x_r) has antiperiod t, then the sum (4) is unchanged if r runs through any complete set of residues modulo t, not necessarily from 0 to $t - 1$. Antiperiodicity is a natural concept because, if a sequence has period $2t$, it can be readily expressed as the sum of two sequences, one with period t and one with antiperiod t.

The skew DFT has the inversion formula

$$x_r = \frac{1}{t}\sum_{s=0}^{t-1} x_s^\dagger j^{-r(2s+1)}, \tag{5}$$

an application of which is mentioned in Section 4. The skew DFT also has the convolution property that, if

$$z_q = \sum_{r=0}^{t-1} x_r y_{q-r} \qquad (q = 0,\ 1,\ \ldots,\ t - 1),$$

where either (x_r) or (y_r) has antiperiod t, then

$$z_s^\dagger = x_s^\dagger y_s^\dagger \qquad (s = 0,\ 1,\ \ldots,\ t - 1). \tag{6}$$

Under the same circumstances, and if (x_r) is real, the skew DFT of $\sum_r x_r y_{q+r}$ is $\bar{x}_s^\dagger y_s^\dagger$, where the bar denotes complex conjugacy. In particular, the skew DFT of $\sum_q x_q x_{q+r}$ is $|x_s^\dagger|^2$ so that, by the inversion formula, we have a "Parseval" formula,

$$\sum x_q^2 = \frac{1}{t}\sum |x_s^\dagger|^2. \tag{7}$$

Exercise: The skew circulant with top row $(1,\ x,\ x^2,\ \ldots,\ x^{t-1})$ is equal to $(x^t + 1)^{t-1}$.

2. CYCLOTOMOUS INTEGERS

A *cyclotomic integer* is defined (for example, by Edwards [2, pp. 81-88]) as a number of the form

$$\sum_{r=0}^{m-1} c_r \omega^r \qquad (\omega = e^{2\pi i/m}), \tag{8}$$

where $c_0, c_1, \ldots, c_{m-1}$ are ordinary integers, positive, negative, or zero, and where m is prime. However, if we generalize the definition by allowing m to be even, and write $m = 2t$, then (8) becomes

$$\sum_{r=0}^{t-1}(c_r - c_{r+t})j^r \qquad (j = e^{\pi i/t}).$$

Accordingly, for any positive integer t,

$$\sum_{r=0}^{t-1} a_r j^r = \sum_{r=0}^{t-1} a_r e^{\pi i r/t}, \tag{9}$$

where each a_r is an integer, will be called a *cyclotomous integer* (with respect to t). It is cyclotomic with respect to $2t$, under the generalized use of the expression "cyclotomic."

When $t = 1$, the cyclotomous integers are the ordinary integers, and when $t = 2$ they are the Gaussian integers (for example, LeVeque [6, pp. 129-131]).

Definition: We say that t is *ausgezeichnet* if the corresponding cyclotomous integers are "unique," that is, if the equation

$$\sum_{r=0}^{t-1} a_r j^r = \sum_{r=0}^{t-1} b_r j^r$$

implies that $a_r = b_r$ ($r = 0, 1, \ldots, t - 1$); or, in other words, if

$$\sum_{r=0}^{t-1} a_r j^r = 0 \tag{10}$$

only if $a_r = 0$ ($r = 0, 1, \ldots, t - 1$).

Theorem 1: The ausgezeichnet integers are the powers of 2, namely 1, 2, 4, 8, ... The others are unausgezeichnet.

Proof: If t is not a power of 2, then it has an odd factor $k > 1$. Write $t = ck$, where $1 \leqslant c < t$. Then

$$0 = 1 + j^t = (1 + j^c)(1 - j^c + j^{2c} - \cdots + j^{(k-1)c}).$$

Therefore, $1 - j^c + j^{2c} - \cdots + j^{(k-1)c}$ is a cyclotomous integer that vanishes, so t is unausgezeichnet.

To prove the theorem for $t = 2^n$ ($n = 0, 1, 2, \ldots$), we note first that the result is obvious when $n = 0$ or 1 (and very easily proved when $n = 2$), and we shall proceed by mathematical induction, assuming $n \geqslant 2$, so that $t \geqslant 4$. Our inductive assumption is that $\frac{1}{2}t$ is ausgezeichnet.

Suppose that equation (10) is satisfied for some "vector" (a_r). Then

$$a_0 + (a_1 - a_{t-1})\cos\frac{\pi}{t} + (a_2 - a_{t-2})\cos\frac{2\pi}{t} + \cdots$$
$$+ (a_{\frac{1}{2}t-1} - a_{\frac{1}{2}t+1})\cos\frac{(\frac{1}{2}t - 1)\pi}{t} = 0 \tag{11}$$

and

$$(a_1 + a_{t-1})\sin\frac{\pi}{t} + (a_2 + a_{t-2})\sin\frac{2\pi}{t} + \cdots$$
$$+ (a_{\frac{1}{2}t-1} + a_{\frac{1}{2}t+1})\sin\frac{(\frac{1}{2}t - 1)\pi}{t} = 0. \tag{12}$$

Equation (11) can be rewritten as

$$\left[a_0 + (a_2 - a_{t-2})\cos\frac{2\pi}{t} + (a_4 - a_{t-4})\cos\frac{4\pi}{t} + \cdots\right]$$
$$+ \left[(a_1 - a_{t-1})\cos\frac{\pi}{t} + (a_3 - a_{t-3})\cos\frac{3\pi}{t} + \cdots\right] = 0. \tag{13}$$

Now, if m is any positive integer, $\cos(2m\pi/t)$ is a polynomial in $\cos(2\pi/t)$ with integer coefficients, while $\cos[(2m + 1)\pi/t]$ is of the form

$$\cos\frac{(2m + 1)\pi}{t} = R\left[\cos\frac{2\pi}{t}\right]\cos\frac{\pi}{t}, \tag{14}$$

where R, with or without a subscript, denotes a rational function (with rational coefficients), not necessarily the same function on each occasion. Equation (14) can be deduced, for example, from Hobson [5, p. 106, formula (6)], where in fact the rational function is a polynomial with integral coefficients. It then follows from (13) that either both bracketed expressions vanish or else $\cos(\pi/t)$ is a rational function of $\cos(2\pi/t)$. Therefore, if we can rule out the latter possibility, we see from our inductive hypothesis that

$$a_0 = a_1 - a_{t-1} = a_2 - a_{t-2} = \cdots = 0. \tag{15}$$

Similarly, on rewriting (12) as

$$(a_1 + a_{t-1})\cos\frac{(\frac{1}{2}t - 1)\pi}{t} + \cdots + (a_{\frac{1}{2}t-1} + a_{\frac{1}{2}t+1})\cos\frac{\pi}{t} = 0, \tag{16}$$

we infer that

$$a_1 + a_{t-1} = a_2 + a_{t-2} = \cdots = 0 \tag{17}$$

provided, once again, that, when $t = 4, 8, 16, \ldots,$

$$\cos(\pi/t) \text{ is not a rational function of } \cos(2\pi/t). \tag{18}$$

Thus, if we can prove (18), it will follow that (15) and (17) are both true and, therefore, $a_0 = a_1 = a_2 = \cdots = a_{t-1} = 0$, which would complete the inductive proof of our theorem. It remains to prove statement (18). To do so, we formulate a slightly more general result because the increased generality enables the method of induction to work.

Theorem 2: When $t = 2^n$ ($n = 2, 3, 4, \ldots$) neither $\cos(\pi/t)$ nor $\sin(\pi/t)$ is of the form $R[\cos(2\pi/t)]$.

For its historical interest, we mention in passing that the product of all the cosines is $2/\pi$, as François Viète or Franciscus Vieta, an eminent mathematician, lawyer, and cryptanalyst, discovered in the seventeenth century. (See Hobson [5, p. 128] for its proof.) Vieta's formula is often expressed in the form

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2 + \sqrt{2}}}{2}\cdot\frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2}\cdot \cdots .$$

Proof of Theorem 2: When $t = 4$, the result is obvious, so we take $n > 2$ and proceed by induction. Suppose that $\cos(\pi/t) = R[\cos(2\pi/t)]$ and try to arrive at a contradiction. The rational function of $\cos(2\pi/t)$ is of the form

$$\frac{c_0 + c_1 \cos(2\pi/t) + \cdots + c_p \cos^p(2\pi/t)}{d_0 + d_1 \cos(2\pi/t) + \cdots + d_q \cos^q(2\pi/t)}$$

$$= \frac{P_1[\cos(4\pi/t)] + \cos(2\pi/t)P_2[\cos(4\pi/t)]}{P_3[\cos(4\pi/t)] + \cos(2\pi/t)P_4[\cos(4\pi/t)]},$$

where $P_1, \ldots, P_4$ are polynomials with integer coefficients. [See the remarks following equation (14).] Multiply the numerator and denominator by

$$P_3[\cos(4\pi/t)] - \cos(2\pi/t)P_4[\cos(4\pi/t)]$$

which, by the inductive hypothesis, does not vanish, and we obtain an equation of the form

$$\cos(\pi/t) = R_1[\cos(4\pi/t)] + \cos(2\pi/t)R_2[\cos(4\pi/t)].$$

Squaring both sides gives, after dropping the arguments of R_1 and R_2 for the sake of brevity,

$$\tfrac{1}{2} + \tfrac{1}{2}\cos(2\pi/t) = R_1^2 + [\tfrac{1}{2} + \tfrac{1}{2}\cos(4\pi/t)]R_2^2 + 2\cos(2\pi/t)R_1R_2.$$

However, by the inductive hypothesis, $\cos(2\pi/t)$ is not a rational function of $\cos(4\pi/t)$, so

$$\tfrac{1}{2} = R_1^2 + \cos^2(2\pi/t)R_2^2$$

and

$$1 = 4R_1R_2.$$

Therefore,

$$\tfrac{1}{2} = R_1^2 + \frac{\cos^2(2\pi/t)}{16R_1^2}.$$

Therefore,

$$R_1^4 - \tfrac{1}{2}R_1^2 + \frac{1}{16}\cos^2(2\pi/t) = 0.$$

Therefore,

$$R_1^2 = \tfrac{1}{2} \pm \sqrt{[\tfrac{1}{4} - \tfrac{1}{4}\cos^2(2\pi/t)} = \tfrac{1}{2}[1 \pm \sin(2\pi/t)].$$

Therefore, $\sin(2\pi/t)$ is a rational function of $\cos(4\pi/t)$, which is false by the inductive hypothesis. So $\cos(\pi/t)$ is not a rational function of $\cos(2\pi/t)$.

Similarly, if $\sin(\pi/t)$ is a rational function of $\cos(2\pi/t)$, we have, as before, in turn,

$$\sin(\pi/t) = R_3[\cos(4\pi/t)] + \cos(2\pi/t)R_4[\cos(4\pi/t)],$$

$$\tfrac{1}{2} - \tfrac{1}{2}\cos(2\pi/t) = R_3^2 + [\tfrac{1}{2} + \tfrac{1}{2}\cos(4\pi/t)]R_4^2 + 2\cos(2\pi/t)R_3R_4,$$

$$\tfrac{1}{2} = R_3^2 + \cos^2(2\pi/t)R_4^2,$$

$$-1 = 4R_3R_4,$$

and

$$\tfrac{1}{2} = R_3^2 + \frac{\cos^2(2\pi/t)}{16R_3^2},$$

and, just as before, we deduce that $\sin(\pi/t)$ cannot be a rational function of $\cos(2\pi/t)$. This completes the proof of Theorem 2 and hence of Theorem 1.

Theorem 3: When t is ausgezeichnet, that is, a power of 2, the degree of the algebraic integer j is t.

Proof: By Theorem 1 we know that 0 cannot be expressed as a cyclotomous integer other than in the obvious manner; that is, j cannot satisfy an equation of degree $t - 1$ or less having integral coefficients. But j does satisfy an equation of degree t, namely $j^t + 1 = 0$, so j is an algebraic integer of degree (precisely) t.

Theorem 4: If j is replaced by j^{2s+1} in (10), where s is a positive integer, then Theorem 1 remains valid.

To see this, note first that the sequence of complex numbers

$$1,\ j^{2s+1},\ j^{2(2s+1)},\ \ldots,\ j^{(t-1)(2s+1)}$$

is merely a permutation of the same sequence with s replaced by 0. Hence, the substitution leaves the class of cyclotomous numbers invariant. The remaining details of the proofs of Theorems 1 and 2 go through with only trivial changes.

Theorem 4 shows that the eigenvalues of the integral skew circulix A with top row $(a_0, a_1, \ldots, a_{t-1})$, where the a's are integers, are all uniquely expressible as cyclotomous integers, when t is a power of 2. These cyclotomous integers are called *associates* of one another and their product is det A, the determinant of A. This determinant is also known as the *norm* of any one of these cyclotomous integers.

When $t = 2$, the cyclotomous integers are the Gaussian integers $a + ib$. The associate of $a + ib$ is $a - ib$ and the norm is $a^2 + b^2$. The so-called *units* of the ring of Gaussian integers are those whose reciprocals are also Gaussian integers, that is, those with norm 1. These units are ± 1 and $\pm i$. It is familiar that in the ring of Gaussian integers the "fundamental theorem of arithmetic" is true, that is, each Gaussian integer has a unique decomposition into prime Gaussian integers, apart from units. For a rigorous statement of this property, and for its proof, see, for example, Hardy and Wright [4, pp. 184-186].

3. THE CYCLOTOMOUS INTEGERS WHEN $t = 4$

Hardy and Wright [4, l.c., pp. 280-281] state the fundamental theorem for the algebraic integers $\alpha + \beta i + \gamma\sqrt{2} + \delta i\sqrt{2}$, where α and β are integers and γ and δ are either both integers or both halves of odd integers. It is readily seen that these are the same as the cyclotomous integers corresponding to $t = 4$, namely $a + bj + cj^2 + dj^3$, where $j = e^{\pi i/4} = (1 + i)/\sqrt{2}$. These again are the same as the cyclotomic integers corresponding to $m = 8$, but the cyclotomous form has the merit of unique representation. The proof of the fundamental theorem in this case can be obtained along the lines of the proof given in [4, §12.8] for the Gaussian integers, which is the case $m = 4$. But when $m = 2t = 16$, or any higher power of 2, this proof does not work, and presumably in these cases the decomposition into cyclotomous primes in not unique.

In the remainder of this section, we assume that $t = 4$. Let a, b, c, and d be integers, and let

$$A = \begin{bmatrix} a & b & c & d \\ -d & a & b & c \\ -c & -d & a & b \\ -b & -c & -d & a \end{bmatrix}. \tag{19}$$

Then

$$\det A = \prod_{s=0}^{3} [a + bj^{2s+1} + cj^{2(2s+1)} + dj^{3(2s+1)}]. \tag{20}$$

By pairing off the complex conjugate pair of factors with $s = 0$ and $s = 3$, and the pair with $s = 1$ and $s = 2$, we see that

$$\det A \geqslant 0. \tag{21}$$

The determinant $\det A$ is also called the norm of $a + bj + cj^2 + dj^3$ and will be denoted by $N(a, b, c, d)$.

The three ways of pairing off the four factors a of (20) ($s = 0, 1, 2, 3$), lead naturally to three ways of writing the norm. Thus:

$$\begin{aligned} a_0^\dagger a_1^\dagger &= (a + bj + cj^2 + dj^3)(a + bj^3 + cj^6 + dj^9) \\ &= a^2 - b^2 + c^2 - d^2 + (j + j^3)(ad + ab - bc + cd) \\ &= a^2 - b^2 + c^2 - d^2 + i\sqrt{2}(ad + ab - bc + cd). \end{aligned}$$

But $a_3^\dagger = \bar{a}_0^\dagger$, $a_2^\dagger = \bar{a}_1^\dagger$, so $a_2^\dagger a_3^\dagger$ is the complex conjugate of $a_0^\dagger a_1^\dagger$ and

$$N(a, b, c, d) = (a^2 - b^2 + c^2 - d^2)^2 + 2(ad + ab - bc + cd)^2. \tag{22}$$

Again

$$\begin{aligned} a_0^\dagger a_3^\dagger = |a_0^\dagger|^2 &= \left| a + \frac{b - d}{\sqrt{2}} + i\left(c + \frac{b + d}{\sqrt{2}}\right)\right|^2 \\ &= \left(a + \frac{b - d}{\sqrt{2}}\right)^2 + \left(c + \frac{b + d}{\sqrt{2}}\right)^2 \\ &= a^2 + b^2 + c^2 + d^2 + \sqrt{2}(-ad + ab + bc + cd), \end{aligned}$$

and

$$N(a, b, c, d) = (a^2 + b^2 + c^2 + d^2)^2 - 2(ad - ab - bc - ca)^2. \tag{23}$$

Finally,

$$\begin{aligned} a_0^\dagger a_2^\dagger &= (a + bj + cj^2 + dj^3)(a + bj^5 + cj^{10} + dj^{15}) \\ &= [a + ci + j(b + di)][a + ci - j(b + di)] = (a + ci)^2 - i(b + di)^2 \\ &= a^2 - c^2 + 2bd - i(b^2 - d^2 - 2ac). \end{aligned}$$

So

$$N(a, b, c, d) = (a^2 - c^2 + 2bd)^2 + (b^2 - d^2 - 2ac)^2. \tag{24}$$

Exercises:

(i) $N(a, b, c, d) = N(-a, -b, -c, -d) = N(d, c, b, a)$.

(ii) $N(-1, x-1, x, x+1) = (x^2+1)^2$. [Form $N(x, 0, 1, 0)N(0, 1, 1, 1)$.]

(iii) The product of the skew circulices whose top rows are $(x, 1, 0, 0)$, $(x, -1, 0, 0)$, $(x, 0, 0, 1)$, and $(x, 0, 0, -1)$ is $(x^4 + 1)I$.

(iv) $N(x, x, x + 2, x + 3) = N(x + 2, x + 3, x + 1, x + 1)$.

(v) $N(1, b, b, 0) - 1$, where b is an integer, is eight times the square of the triangular number $b(b - 1)/2$.

(vi) If a positive integer ν is not of the form $\alpha^2 + 2\beta^2$, then ν^2 is of the form $h^2 + k^2 + 2\ell^2$, where not more than one of the three terms can vanish. (Hint: Use the equality of (23) and (24) combined with Bachet's theorem that every positive integer is the sum of four squares.]

Theorem 5: $N(a, b, c, d)$ vanishes only if $a = b = c = d = 0$.

For, from (23), $N(a, b, c, d) = 0$ implies that

$$a^2 + b^2 + c^2 + d^2 = \pm\sqrt{2}(ad - ab - bc - cd).$$

Therefore, $a^2 + b^2 + c^2 + d^2$, being rational, must vanish, and the result follows. (**Exercise:** The *rational* skew circulices form a field.)

Thus, (21) can be sharpened to

$$\det A \geqslant 1. \tag{25}$$

The units of the ring of cyclotomous integers (with $t = 4$) are the solutions of the Diophantine equation

$$N(a, b, c, d) = 1. \tag{26}$$

We shall adopt the abbreviation (a, b, c, d) for the number

$$a + bj + cj^2 + dj^3.$$

Theorem 6: The units of the ring of cyclotomous integers (with $t = 4$) are:

$$\pm 1,\ \pm j,\ \pm i,\ \pm j^3$$

and

$$(\varepsilon q_n, \pm p_n, \varepsilon q_n, 0),\ (0, \varepsilon q_n, \pm p_n, \varepsilon q_n),\ (-\varepsilon q_n, 0, \varepsilon q_n, \pm p_n)$$

and

$$(\pm p_n, -\varepsilon q_n, 0, \varepsilon q_n)$$

where $\varepsilon = 1$ or $\varepsilon = -1$, and p_n/q_n is the n^{th} convergent in the continued fraction for $\sqrt{2}$; that is,

$$p_n = \frac{(1 + \sqrt{2})^n + (1 - \sqrt{2})^n}{2}, \quad q_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}}. \tag{27}$$

These units are all of the form

$$j^r(1 + j + j^2)^s, \tag{28}$$

where r and s are integers (positive, negative, or zero).

Recall first that the sequences of p_n's and q_n's begin with the values 1, 3, 7, 17, 41, 99, etc., and 1, 2, 5, 12, 29, 70, ..., and satisfy the recurrence relations $p_{n+1} = 2p_n + p_{n-1}$, $q_{n+1} = 2q_n + q_{n-1}$. Moreover, (p_n, q_n) provides the general solutions of the (Fermat-)Pell equations $r^2 = 2s^2 \pm 1$ (see, for example, LeVeque [6, pp. 139-144]). In fact

$$p_{2n}^2 = 2q_{2n}^2 + 1 \quad \text{and} \quad p_{2n-1}^2 = 2q_{2n-1}^2 - 1, \tag{29}$$

which is true even when $n = 0$ if we write $p_0 = 1$, $q_0 = 0$ (as we must if we want to satisfy the recurrence relations when $n = 1$).

To prove Theorem 6 we note, for example, that $N(a, b, a, 0) = (2a^2 - b^2)^2$, from (22), and hence $N(q_n, p_n, q_n, 0) = 1$, from (29). Or we can simply check that (1, 1, 1, 0) is a unit, that its inverse is (1, 0, -1, 1), and then show that all the units defined in (28) are of the forms mentioned in the rest of the statement of the theorem.

That there are no units other than those mentioned in the theorem follows from a deep theorem due to Dirichlet, concerning units in general; see, for example, LeVeque [6, p. 75]. In particular, therefore, $N(a, b, c, d) = 1$ implies $abcd = 0$.

As an example of Theorem 6, we have

$$N(29, 41, 29, 0) = \begin{vmatrix} 29 & 41 & 29 & 0 \\ 0 & 29 & 41 & 29 \\ -29 & 0 & 29 & 41 \\ -41 & -29 & 0 & 29 \end{vmatrix} = 1.$$

Although $N(\varepsilon q_n, \pm p_n, \varepsilon q_n, 0) = 1$, we have $N(q_n, p_n, -q_n, 0) = p_{2n}^2$, so the signs can have a big effect.

By (23), (29), and Theorem 6, we see that the only solutions of the simultaneous Diophantine equations

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= p_{2n}, \\ ad - ab - bc - cd &= \pm q_{2n} \end{aligned} \tag{30}$$

are given by $a = c = \pm q_n$, $b = \pm p_n$, $d = 0$, and the "antirotations" of these solutions listed in the statement of Theorem 6. In particular, there is no solution with $abcd \neq 0$.

An allied question is what integers, and especially what primes, are expressible as integral skew circulants, not necessarily of order 4. For order 2, the problem is the familiar solved one of expressing integers as the sum of two squares. Since the product of two integral skew circulices is a third one, we know that the products of "expressible" numbers are also expressible (as skew circulants of order 4).

If $N(a, b, c, d)$ is prime, then, by (24) it must either be 2, for example, $N(1, 1, 0, 0) = 2$, or it is of the form $4q + 1$. In the latter case, r and s are of opposite parity, where $r = a^2 - c^2 + 2bd$ and $s = b^2 - d^2 - 2ac$. Suppose that r is odd and s is even. Then $a \not\equiv c \pmod 2$ and $b \equiv d \pmod 2$. By trying the four possibilities for the parities of (a, b, c, d) we see that $r^2 + s^2 \equiv 1 \pmod 8$, and the same conclusion is reached if r is even and s is odd. Thus, the only odd primes that $N(a, b, c, d)$ can equal are of the form $8n + 1$. I conjecture that every prime of this form is expressible as $N(a, b, c, d)$, that is, as an integral skew circulant, having found that this is true up to 1033, as shown in Table 1. Call this Conjecture 1.

Table 1. Values of (a, b, c, d) for which $p = N(a, b, c, d)$ where p is prime and $p \equiv 1 \pmod 8$, for all $p \leqslant 1033$. Solutions are given for which a, b, c, and d are all nonnegative.

p	a	b	c	d	p	a	b	c	d	p	a	b	c	d
17	2	1	0	0	337	4	3	0	0	641	5	2	0	0
41	2	1	1	1	353	4	1	1	1	673	3	3	2	3
73	2	2	0	1	401	3	3	1	2	761	4	7	8	2
89	3	1	1	0	409	4	2	0	1	769	4	0	2	3
97	3	2	0	0	433	4	0	2	1	809	4	3	0	2
113	3	0	1	1	449	4	2	3	0	857	1	6	3	1
137	3	3	2	1	457	3	1	3	2	881	5	4	0	0
193	3	1	2	1	521	3	2	1	3	929	5	1	2	1
233	1	4	1	1	569	6	8	7	0	937	5	0	1	3
241	4	2	1	0	577	5	3	1	0	953	5	1	1	2
257	4	1	0	0	593	4	2	1	2	977	5	5	2	1
281	5	5	3	0	601	8	11	9	1	1009	8	9	6	0
313	3	3	3	2	617	4	1	2	2	1033	7	9	8	1

Given a solution of $N(a, b, c, d) = 1$, we can multiply each of a, b, c, d by any number and thus show that all squares are expressible. So Conjecture 1 implies that all numbers of the form $2^q S p_1 p_2 \ldots$, where S is a square and p_1, p_2, ... are primes of the form $8n + 1$, are expressible, and I suspect that no other numbers are. (Conjecture 2.) This conjecture is based on well over one hundred numerical examples.

It is familiar that primes of the form $4n + 1$, and *a fortiori* those of the form $8n + 1$, are expressible in essentially only one way as the sum of two squares. Suppose that a prime $p \equiv 1 \pmod 8$ is $r^2 + s^2$, where we can take $r > 0$ and $s > 0$. Then, if Conjecture 1 is right, integers a, b, c, d exist, so that the two terms in (24) satisfy

$$|a^2 - c^2 + 2bd| = r \text{ or } s \quad \text{and} \quad |b^2 - d^2 - 2ac| = s \text{ or } r. \tag{31}$$

Researches by Gauss, Lagrange, Cauchy, Eisenstein, Jacobi, and Stern (see Smith [9, p. 269] included the remarkable result that a prime p of the form $8n + 1$ is also uniquely expressible in the form $h^2 + 2k^2$, where

$$\pm 2h \equiv \binom{5n}{n} \pmod p. \tag{32}$$

Conjecture 1 would then imply, from (22), that h and k can be written (by no means uniquely) in the forms

$$h = |a^2 - b^2 + c^2 - d^2| \quad \text{and} \quad k = |ad + ab - bc + cd|. \tag{33}$$

We also know, by a theorem due to Gauss (see Smith [9, p. 268]), that $p = r^2 + s^2$, where

$$2r \equiv \binom{4n}{2n} \pmod{p}. \tag{34}$$

Formulas (31)-(34) can be helpful in finding values for a, b, c, d, that is, in expressing a prime of the form $8n + 1$ as a skew circulant.

Table 1 can be used for writing down, for a given prime $p \equiv 1 \pmod 8$, the essentially unique solutions of $p = r^2 + s^2$ and $p = h^2 + 2k^2$ when $p \leqslant 1033$. We can also use the table (up to $p = 1033$), combined with (23), to obtain arbitrarily many solutions of $p = \alpha^2 - 2\beta^2$, because we can multiply (a, b, c, d) by any unit. For example,

$$\begin{aligned}(a, b, c, d) \times (1, 1, 1, 0) &= (a - c - d,\ a + b - d,\ a + b + c,\ b + c + d)\\ &= (a', b', c', d')\end{aligned}$$

say; and we see, by elementary algebra, that

$$a'^2 + b'^2 + c'^2 + d'^2 = 3(a^2 + b^2 + c^2 + d^2) - 4(ad - ab - bc - cd),$$

while

$$a'd' - a'b' - b'c' - c'd' = 3(ad - ab - bc - cd) - 2(a^2 + b^2 + c^2 + d^2).$$

This forces us to notice that if (α_n, β_n) is a solution of $p = \alpha^2 - 2\beta^2$, then another one is $(\alpha_{n-1}, \beta_{n-1})$, where we have the "backward" recursion

$$\alpha_{n-1} = 3\alpha_n - 4\beta_n,\ \beta_{n-1} = 3\beta_n - 2\alpha_n. \tag{35}$$

Likewise, by forming $(a, b, c, d)(1, 0, -1, 1)$, or from (35), we are led to the "forward" recursion

$$\alpha_n = 3\alpha_{n-1} + 4\beta_{n-1},\ \beta_n = 2\alpha_{n-1} + 3\beta_{n-1}. \tag{36}$$

Thus, given one solution, we can generate an unlimited supply (compare LeVeque [6, p. 146], for example), by climbing up and down a ladder infinite in both directions.

One can verify that equations (36) are equivalent to

$$\begin{aligned}\alpha_n &= [(\ell\sqrt{8} - 4m)\lambda^{n+1} + (\ell\sqrt{8} + 4m)\mu^{n+1}](32)^{-1/2}\\ \beta_n &= [(2\ell - m\sqrt{8})\lambda^{n+1} - (2\ell + m\sqrt{8})\mu^{n+1}](32)^{-1/2},\end{aligned} \tag{37}$$

where n is *any* integer, $\lambda = 3 + \sqrt{8}$, $\mu = 3 - \sqrt{8} = \lambda^{-1}$, $\ell^2 - 2m^2 = p$. Indeed, using only the fact that $\lambda\mu = 1$, one can verify directly that if $\ell^2 - 2m^2 = x$, for any x, then $\alpha_n^2 - 2\beta_n^2 = x$ also. For example, when $p = 17$, we can take $\ell = 7$, $m = 4$, giving $\ldots\alpha_{-2} = 37$, $\alpha_{-1} = 7$, $\alpha_0 = 5$, $\alpha_1 = 23$, $\alpha_2 = 133$, $\ldots$, $\beta_{-2} = -26$, $\beta_{-1} = -4$, $\beta_0 = 2$, $\beta_1 = 16$, $\beta_2 = 94$, $\ldots$.

Equations (37), in their turn, are equivalent to

$$\alpha_{n+1} = 6\alpha_n - \alpha_{n-1},\ \beta_{n+1} = 6\beta_n - \beta_{n-1}, \tag{38}$$

which can be used both forward and backward. Equations (37) and (38) are decidedly Fibonaccian, so it is not surprising that the α's and β's have further nice properties. For example,

$$\alpha_n\alpha_{n+k} - 2\beta_n\beta_{n+k} = p\gamma_k,$$

where $\gamma_0 = 1$, $\gamma_1 = 3$, $\gamma_{n+1} = 6\gamma_n - \gamma_{n-1}$. The even-numbered numerators of the simple continued fraction for $\sqrt{2}$ are $\gamma_1, \gamma_2, \gamma_3, \ldots$.

To conclude this section, consider one more conjecture, Conjecture 3: Let $p \equiv 1 \pmod 8$ be prime. Then *all* solutions of $p = \alpha^2 - 2\beta^2$ can be obtained from (37), or recursively from (38), by starting with a single solution. That there *is* a solution would be a consequence of Conjecture 1. The two conjectures combined imply that all solutions are of the form shown in formula (23). (See Table 2.)

Table 2. Some solutions of $p = \alpha^2 - 2\beta^2$ where $p \equiv 1 \pmod 8$, and the top rows of the corresponding skew circulices. The signs preserve the recurrences $\alpha_{n+1} = 6\alpha_n - \alpha_{n-1}$, $\beta_{n+1} = 6\beta_n - \beta_{n-1}$. In each case, $\alpha = a^2 + b^2 + c^2 + d^2$ and $\beta = ad - ab - bc - cd$ as in formula (23).

$p = 17$						$p = 41$					
α	β	a	b	c	d	α	β	a	b	c	d
37	26	−2	4	−4	1	71	50	−1	5	−6	2
7	4	1	1	−2	1	13	8	2	1	−2	2
5	−2	2	1	0	0	7	−2	2	1	1	1
23	−16	2	3	3	1	29	−20	0	2	4	3
133	−94	−2	4	8	7	167	−118	−7	−1	6	9
775	−548	−17	−5	10	19	973	−688	−22	−17	−2	14

4. DISCUSSION OF ALLIED MATTERS

Complicated Numbers

In an unpublished paper, the author called the skew circulix (1) a representation of a "complicated number"

$$x_0 + j_1x_1 + \cdots + j_{t-1}x_{t-1}$$

and developed a theory of functions of a complicated variable (see Good [3]). The theory contained, for example, an easy generalization of the Cauchy-Riemann equations, and a more difficult generalization of Cauchy's residue theorem for integrals over contours encircling flat manifolds of dimension $t - 2$. These manifolds generalize the poles in the usual theory. Generalizations of Liouville's theorem and analytic continuation were also given. The following discussion is extracted from that document to which it was, however, somewhat incidental.

Generalized Trigonometry

The skew DFT is related to the following generalization of trigonometry.

Consider the differential equations

$$D^ty = k^ty, \tag{39}$$

$$D^ty = -k^ty, \tag{40}$$

where k is a positive number and D means d/du. (The case $t = 4$ occurs in the theory of a vibrating elastic bar; see Webster [11, p. 139].) A fundamental set of solutions (39) is given by the generalized hyperbolic functions of Ungar [10]:

$$f_r(u) = \frac{u^r}{r!} + \frac{u^{r+t}}{(r+t)!} + \frac{u^{r+2t}}{(r+2t)!} + \cdots \quad (r = 0, 1, \ldots, t-1), \tag{41}$$

while, for (40), a fundamental set contains the generalized trigonometric functions

$$g_r(u) = \frac{u^r}{r!} - \frac{u^{r+t}}{(r+t)!} + \frac{u^{r+2t}}{(r+2t)!} - \cdots \quad (r = 0, 1, \ldots, t-1). \tag{42}$$

(Compare to Muir [7, pp. 443-444], where the corresponding definitions contain minor errors; and Ramanujan [8].) The solution of (39), with initial values $c_0, c_1, c_2, \ldots, c_{t-1}$ for $y, Dy, \ldots, D^{t-1}y$ at $u = 0$, is $\Sigma c_r f_r(ku)$, while that of (40), with the same initial values, is $\Sigma c_r g_r(ku)$. Let us list some formulas that are satisfied by these generalized trigonometric functions. The reader should mentally consider what they state for the case $t = 2$. We omit most of the similar formulas for the functions $f_r(u)$. The reader might like to verify, however, that

$$\sum_r [f_r(u)]^2 = t^{-1}\sum_{s=0}^{t-1}\exp[2u\cos(2\pi s/t)]. \tag{43}$$

When $t \to \infty$, this gives a familiar formula for the Bessel function $I_0(2u)$ as an integral.

The formula

$$\exp(uj^{2s+1}) = \sum_{r=0}^{t-1} g_r(u)j^{r(2s+1)} \tag{44}$$

[which is true also when j is replaced by j^{2p+1} ($p = 0, 1, 2, \ldots$)], is a direct generalization of "de Moivre's formula," which is the case $t = 2$. As in ordinary trigonometry we can obtain an addition formula by first deducing an expression for $\exp[(u + y)j^{2s+1}]$ from (44), and then taking the inverse skew DRT. Another method, which is closely related, is to note that

$$e^{uJ} = g_0(u)I + g_1(u)J + \cdots + g_{t-1}(u)J^{t-1} \tag{45}$$

is a skew circulix whose eigenvalues are $\exp(uj^{2s+1})$ ($s = 0, 1, \ldots, t-1$). Hence,

$$g_r(u + y) = \sum_{s=0}^{t-1} \varepsilon_{r,s} g_s(u) g_{r+s}(y) \quad (r, s = 0, 1, \ldots, t-1), \tag{46}$$

where $\varepsilon_{r,s} = 1$ if $r \geqslant s$ and $\varepsilon_{r,s} = -1$ if $r < s$. These identities generalize the usual formulas for $\cos(u + y)$ and $\sin(u + y)$. It follows, for example, that $g_r(nu)$ is a homogeneous polynomial in $g_0(u), \ldots, g_{t-1}(u)$ ($n = 1, 2, 3, \ldots$).

It seems fair to conclude that the skew circulix has not previously been given the attention that it merits.

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ON A CLASS OF KNOTS WITH FIBONACCI INVARIANT NUMBERS

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This paper describes how a subclass of the rational knots* may be constructed sequentially, the knots in the sequence having 1, 2, ..., i, ... crossings. For these knots, the values of a certain knot invariant are Fibonacci numbers, the i^{th} knot in the sequence having invariant number F_i.

The knot invariant has a wide number of interpretations and properties, and some of these will be outlined, particularly in relation to knots in the constructed class.

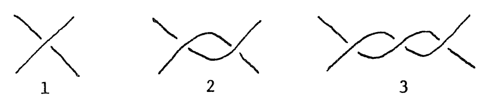
The class will be called the *Fibonacci knot-class*. A generalization of this class will be introduced and briefly discussed.

1. THE RATIONAL KNOTS

J. H. Conway [2] defines the notion of "integer tangle," and gives rules for combining integer tangles to form a large class of alternating knots which he calls *rational knots*. He develops operations by which all knots on a given number of crossings may be constructed and tested for equivalences.

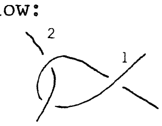
Conway's Notation and Construction of the Rational Knots

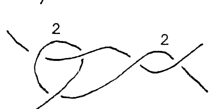
Only an outline of the methods used, proceeding largely by examples, can be given. The following diagrams show the first few *integer tangles* with their designations.

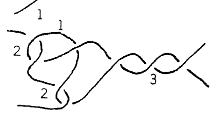


Integer tangles 1, 2, and 3

Integer tangles are combined to form *rational tangles*, as the following examples show:

21 

212 

2123 

Note that to form the tangle $abcd$ (where a, b, c, d represent integer tangles), first a is reflected in a leading diagonal then joined to b. Then the tangle ab is reflected and joined to c. Finally, abc is reflected and joined to d. The manner of joining two tangles is evident from the examples.

A tangle is turned into a knot by joining the two upper strings (loose ends), and then joining the two lower strings.

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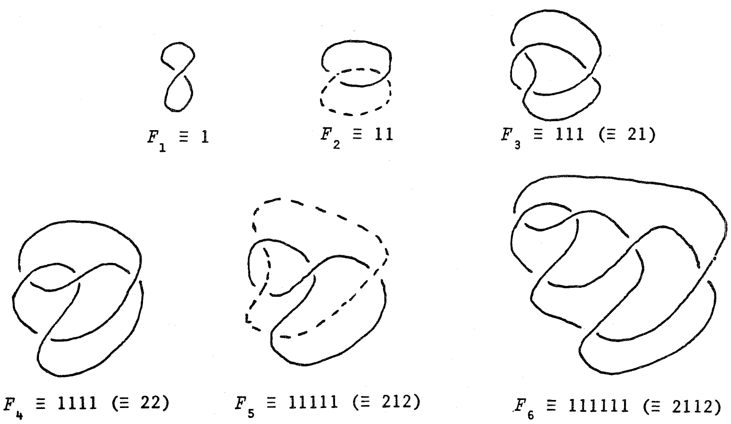
*As in [2], we use "knot" as an inclusive term for "μ-link," $\mu \geqslant 1$.

In [5] a table of diagrams of prime knots and links is given, showing the knots on n crossings, for $n = 2, 3, \ldots, 10$. Conway, in [2], classifies the knots and links through to $n = 11$ crossings.

2. THE FIBONACCI KNOT-CLASS

We now define what we have called the *Fibonacci knot-class* to be the sequence of rational knots which are, in Conway's constructional notation, 1, 11, 111, 1111, ... There is thus one knot in the class for each value of n-crossings; we give diagrams for the first six in the sequence before describing the properties that relate them to the Fibonacci numbers.



$F_1 \equiv 1$ $F_2 \equiv 11$ $F_3 \equiv 111$ ($\equiv 21$)

$F_4 \equiv 1111$ ($\equiv 22$) $F_5 \equiv 11111$ ($\equiv 212$) $F_6 \equiv 111111$ ($\equiv 2112$)

The Fibonacci knots to $n = 6$

In the sequence, each knot corresponds to its Fibonacci number through a certain knot-invariant to be described. Then when F_i is odd the knot is a 1-link, and when F_i is even the knot is a 2-link (where $\{F_i\}$ is the sequence 1, 2, 3, 5, 8, ...).

3. PROPERTIES OF THE FIBONACCI KNOT-CLASS

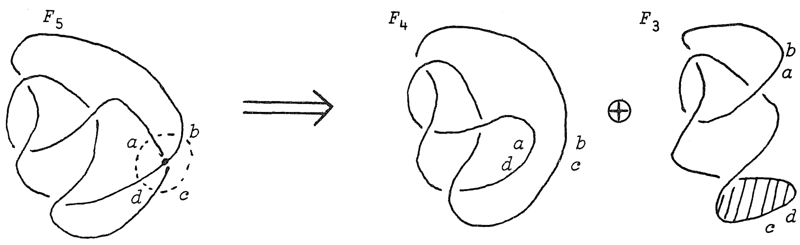
A Vertex-Deletion Operation; Production of "Twins"

If a crossing of a knot diagram is "cut-out" or "deleted," the four cut-ends may be joined again in two ways that lead to a pair of alternating knots, each having one fewer crossing than the original knot. We may call the original knot K, and the associated pair of knots which are obtainable from the vertex-deletion K' and K''; we may speak of K as the *parent* knot, and call (K', K'') a pair of *twins*.

Let us write, formally, that $K = K' \oplus K''$ whenever (K', K'') are twins from parent knot K.

Twins from the Fibonacci Knots

Consider, for example, the Fibonacci knot $F_5 \equiv 11111$. By its construction, the last 1 corresponds to the crossing on the far right of its diagram. We demonstrate that deletion of this vertex leads to the twins (F_4, F_3). Thus:



Cut out vertex

Join a to d and b to c

Join a to b and c to d

The knot on the far right is immediately seen to be equivalent to F_3 once the loop (shown shaded) is removed by twisting it once, out of the plane and back, through 180° clockwise.

To transform the first right-hand knot to the one shown in Section 2 requires two operations: (1) turn the entire knot over in the plane, rotating it about an axis in the plane that runs from NW to SE; (2) rotate the entire knot through 180° in the plane (about an axis perpendicular to the plane).

Similarly, we can show that, if we delete its last vertex, F_6 has twins (F_5, F_4), F_7 has twins (F_6, F_5), and so on. Using the symbol $\oplus$ as described above, we can write, formally,

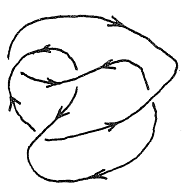
$$F_{n+2} = F_{n+1} \oplus F_n, \qquad n = 1, 2, \ldots,$$

which is the recurrence relation for the Fibonacci series.

The "Tree Number" Knot Invariant

The edges of an alternating knot-graph may be given orientations in such a way that the arrows alternate in direction as the knot is toured from edge to edge. We call this a *balanced alternating orientation*.

For a knot-graph with a balanced alternating orientation, we may count the number of directed spanning trees that emanate from any given vertex. We can show that this number is independent of the vertex chosen as root and, further, that it is a knot-invariant for alternating knots. The first proof of invariance of this *tree number* (τ) may be found in [3].



$\tau = 5$ (whichever vertex is taken as root; and whichever alternating diagram is used to represent the knot).

Example: Knot F_4, with balanced alternating orientation

Computation of τ for the Rational Knots

In [6] we derive the following recurrence equations for

$$\tau(m_1 m_2 \ldots m_c),$$

the tree number of the rational knot $m_1 m_2 \ldots m_c$.

$$\tau(m_1) = m_1$$
$$\tau(m_1 m_2) = m_2 m_1 + 1$$
$$\dots\dots\dots\dots\dots\dots$$
$$\tau(m_1 m_2 \ \dots \ m_c) = m_c \cdot \tau(m_1 m_2 \ \dots \ m_{c-1}) + \tau(m_1 m_2 \ \dots \ m_{c-2}).$$

The tree numbers of the Fibonacci knot-class are given by setting $m_i = 1$, $i = 1, \dots, c$. This gives

$$\tau(F_1) = 1, \quad \tau(F_2) = 2, \quad \dots, \quad \tau(F_i) = \tau(F_{i-1}) + \tau(F_{i-2}).$$

Therefore, in this knot-class the tree numbers follow the Fibonacci sequence.

Consider the rational knot $m_1 m_2 \dots m_c$, and the associated *continued fraction (terminated)* (C.F.):

$$\text{C.F.}(m_1 m_2 \ \dots \ m_c) \equiv m_c + \frac{1}{m_{c-1}} + \frac{1}{m_{c-2}} + \cdots + \frac{1}{m_1}.$$

In view of the recurrence equations, the following is true:

$$\text{C.F.}(m_1 m_2 \ \dots \ m_c) = \frac{\tau(m_1 m_2 \ \dots \ m_c)}{\tau(m_1 m_2 \ \dots \ m_{c-1})}.$$

This gives the following formula for the tree number of the c^{th} Fibonacci knot:

$$\tau(F_c) = \sum_{i=1}^{c} \text{C.F.}(F_i).$$

It should be noted here that Conway derives some interesting topological properties relating to the continued fraction of a rational knot in [2].

Other Interpretations of the Number τ

There are a number of knot invariants which have the same value as τ for any given knot. We list three here; a fuller discussion of them can be found in [6].

Entities equal in value to τ

(1) The torsion number of the two-fold branched cyclic covering space of the knot [1].

(2) The number of Euler circuits on the knot-digraph [4].

(3) The quantity $|\Delta(-1)|$, where $\Delta(x)$ is the Alexander polynomial of the knot [5].

Thus, for the Fibonacci knots, all of these invariant values follow the Fibonacci sequence.

On Parity of Tree Numbers

In [6], we show that τ is odd if and only if the knot-graph is a 1-link (i.e., one string). In the Fibonacci knot sequence, then, the knots F_1, F_3,

F_4, F_6, F_7, ... are 1-links; it is easy to show that every third knot, with even τ, is a 2-link. That is F_2, F_5, F_8, ... are 2-links.

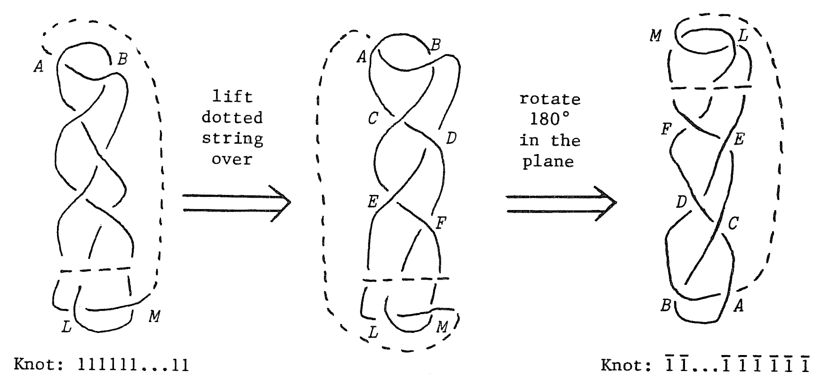
On Amphicheirality

A knot is *amphicheiral* if it can be transformed into its mirror image by a bi-continuous transformation (that is, without cutting and rejoining the string).

In Conway's notation, the mirror image of 1 1 ... 1 is $\bar{1}\,\bar{1}$ ... $\bar{1}$; the symbol $\bar{1}$ denotes a crossing $\diagup\!\!\!\!\diagdown$.

Proposition: F_c is amphicheiral for c = 1, 2, 4, 6, ... (c even after 1).

Proof: For c = 1 and 2, it is easy to note how the transformation can be carried out. For general c, the necessary transformations to carry the knot into its mirror image are as follows:



It is well known that knots with an odd number of crossings cannot be amphicheiral; hence, F_i, where i = 3, 5, ... are not amphicheiral.

4. GENERALIZATIONS

An obvious generalization of the above work would be to study the *knot-classes* $\{F_i^{(m)}\}$, where

$\{F_i^{(1)}\} \equiv \{F_i\}$ is the Fibonacci class,

$\{F_i^{(2)}\}$ is the class of rational knots 2, 22, 222, 2222, ...,

$\{F_i^{(3)}\}$ is the class 3, 33, 333, 3333, ...,

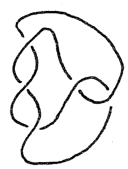
etc.

Knots with i = 2, 4, ... (even) in each sequence are amphicheiral.

The *tree numbers* of knots in these classes satisfy the equations of Section 3. For $m = 1$, they are the Fibonacci numbers; for $m = 2$, the Pell numbers.

Doubtless the properties of these numbers, which form interesting two-way sequences, are well known.

Any rational knot may be represented as a formal sum of knots of type $F_i^{(m)}$, making use of the vertex deletion operation described in Section 3. Such representations are not in general unique (that is, a given knot may have more than one representation), but it is conjectured that any representation is an invariant of that knot. For example, the knot (32) shown below may be represented in the following ways, by various vertex deletions:



$$\Longrightarrow \textit{knot}\ (31) \oplus F_3 \Rightarrow F_3 \oplus F_1 \oplus F_3 \Rightarrow 2F_3 \oplus F_1$$

$$\overset{\text{also}}{\Longrightarrow} \quad F_4 \oplus F_2$$

Knot (32)

Note that to each representation there corresponds a linear decomposition of the knot's tree number into Fibonacci numbers; e.g., for the knot (32) we have $\tau = 7$, with the corresponding decompositions $5 + 2$ and $2 \times 3 + 1$.

It would be exciting if a study of number sequences associated with knot-classes were to lead to methods for counting more general classes of knots. There are virtually no results in this area, to my knowledge.

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ON TWO- AND FOUR-PART PARTITIONS OF NUMBERS EACH PART A SQUARE

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1. INTRODUCTION

For each given pair of positive integers k, n, with $k \leqslant n$, a k-part partition of n is a k-element multi-set of positive integers whose sum is n; e.g., all of the 3-part partitions of 7 are: [5, 1, 1], [4, 2, 1], [3, 3, 1], and [3, 2, 2]. In this paper we are especially interested in k-part partitions of numbers for which $k = 2, 4$ and all of the parts are squares. We briefly refer to these as 2-square and 4-square partitions of a number. Thus, [4, 1] is a 2-square partition of 5. Also, recall that for each positive integer n, $\sigma(n)$ denotes the sum of all positive divisors of n.

We are now prepared to state our results.

Theorem 1: A nonsquare odd number n has an odd number of 2-square partitions if and only if $\sigma(n)$ is twice an odd number, i.e., $n = p^e m^2$, $e, m, p \in \mathbb{Z}^+$, p a prime, $p \nmid m$, and $p \equiv e \equiv 1 \pmod 4$.

Theorem 2: If a is odd and not of the form $j(3j \pm 2)$, then $3a + 1$ has an odd number of 4-square partitions of the form

$$3a + 1 = 3j^2 + (6k \pm 1)^2, \quad j, k \in \mathbb{Z}^+$$

if and only if a is a square.

In Section 2, we prove these theorems, and also deduce Fermat's classical two-square theorem as an immediate corollary of Theorem 1.

2. PROOFS OF THEOREMS 1 AND 2

Our proofs are based on two recurrences for the sum-of-divisors function. These recurrences are best stated with the aid of several auxiliary arithmetical functions, which we now define.

Definition: For each positive integer n, $b(n)$ denotes the exponent of the highest power of 2 dividing n; and, $O(n)$ is then defined by the equation

$$n = 2^{b(n)} O(n).$$

Hence, $b(n)$ is a nonnegative integer and $O(n)$ is odd. We now define the arithmetical functions ω and ρ by:

$$\omega(n) = \sigma(n) + \sigma(O(n)), \quad \rho(n) = 3\sigma(n) - 5\sigma(O(n)).$$

The two recurrences are, for each positive integer m:

$$(1)\quad \sigma(2m-1) - \sum_{k=1}\omega(2m-1-(2k-1)^2) + 2\sum_{k=1}\sigma(2m-1-(2k)^2) = \begin{cases} j^2, & \text{if } 2m-1=j^2,\\ 0, & \text{otherwise.}\end{cases}$$

$$(2)\quad \sigma(2m-1) + \sum_{k=1}(6k+1)\sigma(2m-1-2k(6k+2)) - \sum_{k=1}(6k-1)\sigma(2m-1-2k(6k-2)) + \sum_{k=1}(3k-1)\rho(2m-1-(2k-1)(6k-1)) - \sum_{k=1}(3k-2)\rho(2m-1-(2k-1)(6k-5))$$
$$= \begin{cases} -j(3j+1)(3j+2)/2, & \text{if } 2m-1=j(3j+2),\\ j(3j-2)(3\ \ -1)/2, & \text{if } 2m-1=j(3j-2),\\ 0, & \text{otherwise.}\end{cases}$$

In both (1) and (2), the sums indexed by k extend over all values of k which cause the arguments of σ, ω, and ρ to be positive. For a proof of (1), see [1, pp. 215-217]. (2) is proved in [2, pp. 679-682], where $\rho(n) = \omega(3, -5; n)$.

Proof of Theorem 1: Assume that $2m+1$, with $m \geqslant 0$, is nonsquare. Recurrence (1) then becomes

$$(3)\quad \sigma(2m+1) - \sum_{1}\omega(2m+1-(2k-1)^2) + 2\sum_{1}\sigma(2m+1-(2k)^2) = 0.$$

If $\sigma(2m+1)$ is twice an odd number, say $\sigma(2m+1) = 4a+2$, for some $a \geqslant 0$, then (3) becomes

$$2a+1-\sum_{1}\frac{\omega(2m+1)-(2k-1)^2)}{2} + \sum_{1}\sigma(2m+1-(2k)^2) = 0.$$

Next, owing to the multiplicativity of σ, $\omega(n) = 2^{b(n)+1}\sigma(O(n))$. Hence, for n even, 4 divides $\omega(n)$. It follows that the sum $\Sigma\sigma(2m+1-(2k)^2)$ is odd and, therefore, contains an odd number of odd summands. But, from the well-known fact: $\sigma(n)$ is odd $\Longleftrightarrow n$ is a square or twice a square, it then follows that there is an odd number of pairs $2k$, $2j-1$ ($j, k \in \mathbb{Z}^+$) such that

$$2m+1 = (2k)^2 + (2j-1)^2.$$

In a word, $2m+1$ has an odd number of 2-square partitions.

Conversely, if $2m+1$ has an odd number of 2-square partitions, then recurrence (3) allows us to reverse the steps of the foregoing argument, whence $\sigma(2m+1) \equiv 2 \pmod 4$; i.e., $\sigma(2m+1)$ is twice an odd number.

Corollary (Fermat): Each rational prime p of the form $4m+1$ is expressible as a sum of two squares.

Proof: For such a prime p, $\sigma(p) = p + 1 = 4m + 2 = 2(2m + 1)$. Hence, p has at least one 2-square partition.

Proof of Theorem 2: Assume $2m + 1$, with $m \geqslant 0$, is not of the form $j(3j \pm 2)$. Recurrence (2) then becomes

$$\begin{aligned}(4)\quad \sigma(2m+1) &+ \sum_{k=1}(6k+1)\sigma(2m+1-2k(6k+2)) \\ &- \sum_{k=1}(6k-1)\sigma(2m+1-2k(6k-2)) \\ &+ \sum_{k=1}(3k-1)\rho(2m+1-(2k-1)(6k-1)) \\ &- \sum_{k=1}(3k-2)\rho(2m+1-(2k-1)(6k-5)) = 0.\end{aligned}$$

If $2m + 1$ is a square, then $\sigma(2m + 1)$ is odd. Now,

$$\rho(n) = 2(3 \cdot 2^{b(n)} - 4)\sigma(O(n)).$$

Hence, the sum

$$\sum_{1}(6k+1)\sigma(2m+1-2k(6k+2)) - \sum_{1}(6k-1)\sigma(2m+1-2k(6k-2))$$

is odd and therefore contains an odd number of odd summands. In a word, there exists an odd number of pairs $j, k \in \mathbb{Z}^+$ such that

$$2m + 1 = j^2 + 2k(6k \pm 2),$$

or equivalently,

$$3(2m + 1) + 1 = 3j^2 + (6k \pm 1)^2.$$

Conversely, if $3(2m + 1) + 1$ has an odd number of 4-square partitions of the prescribed form, then recurrence (4) allows us to reverse the steps of the foregoing argument. And, then, $2m + 1$ must be a square.

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RAPIDLY CONVERGING EXPANSIONS WITH FIBONACCI COEFFICIENTS

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1. FIBONACCI NUMBERS AND CHEBYCHEV POLYNOMIALS

Properties of Fibonacci numbers have been known for a very long time. Their origin dates back to the year 1202 with the publication of the *Liber Abaci* by the Italian mathematician Leonardo of Pisa, better known to us by the nickname "Fibonacci," a short form of *Filius Bonacci*, meaning "Son of Bonacci."

Fibonacci seems to have had a sense of humor apart from his mathematical talents: Liber was a Latin God, son of Ceres and brother of Proserpina. The Romans assimilated this God to Bacchus or Dyonisus, the Greek god of wine. Festivals, known as "Liberalia," were celebrated every year honoring *Liber Bacus*. Since *Liber Abaci* means *Book of the Abacus*, Fibonacci may have amused himself by naming his book, at a time of strong domination by the Roman Catholic Church, in a way reminiscent of a pagan god of wine and fertility. We know Fibonacci was fond of play on words. For instance, he signed some of his work "Leonardo Bigollo." *Bigollo* is a work meaning both "traveler," which Fibonacci certainly was, and "blockhead." It has been said that Fibonacci had in mind the latter meaning to tease his contemporaries who had ridiculed him for his interest in Hindu-Arabic numerals and methods. Fibonacci had become a very successful mathematician whith these methods.

Fibonacci did not discover any of the properties of the sequence which bears his name. He simply proposed, and solved, in the *Liber Abaci*, the problem of how many rabbits would be born in one year starting from a given pair. With some natural assumptions about the breeding habits of rabbits, the population of rabbit pairs per month correspond to the elements of the Fibonacci sequence—1, 1, 2, 3, 4, 8, 13, etc.—where, beginning with zero and one, each term of the sequence is the sum of the two preceding ones.

With the passage of time, this sequence would appear in so many areas with no possible connection to the breeding of rabbits that, in 1877, Edward Lucas proposed naming it *Fibonacci Sequence* and its terms *Fibonacci Numbers*. The fertility of this sequence seems to be inexhaustible, and every year new and curious properties of it are discovered. F_n has become the standard symbol for Fibonacci numbers, and their defining relation is

$$F_n = F_{n-1} + F_{n-2},\ F_0 = 0,\ F_1 = 1.$$

In spite of the above preamble, it will, perhaps, appear as surprising to encounter some new, simple, and unexpected relations between Fibonacci numbers and Chebychev polynomials. Let us proceed to their derivation.

The known relation [8] for Chebychev polynomials of the first kind,

$$T_n(x) = \tfrac{1}{2}[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n], \tag{1}$$

gives, with $x = \sqrt{5}/2$,

$$\frac{2}{\sqrt{5}} T_n\left(\frac{\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^n + (-1)^n\left(\frac{1 - \sqrt{5}}{2}\right)^n\right]. \tag{2}$$

For odd n, this relation coincides with Binet's formula [6] for Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{3}$$

Thus, we obtain

$$F_{2n+1} = \frac{2}{\sqrt{5}} T_{2n+1}\left(\frac{\sqrt{5}}{2}\right). \tag{4}$$

In a similar fashion, the known relation for Chebychev polyonmials of the second kind,

$$U_n(x) = \tfrac{1}{2}\left[\frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{\sqrt{(x^2-1)}}\right], \tag{5}$$

gives, with $x = \sqrt{5}/2$, and n replaced by $2n - 1$,

$$F_{2n} = \frac{1}{\sqrt{5}} U_{2n-1}\left(\frac{\sqrt{5}}{2}\right), \quad n \geqslant 1. \tag{6}$$

The relation [8],

$$T_n(x) = U_n(x) - xU_{n-1}(x), \tag{7}$$

gives, after changing n to $2n + 1$, letting $x = \sqrt{5}/2$, and using (4) and (6) and the recurrence relation for F_n,

$$F_{2n} + F_{2n+2} = U_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{8}$$

Equation (7) gives, after changing n by $2n$, letting $x = \sqrt{5}/2$, using (4), (6), and the recurrence relation for F_n

$$\frac{F_{2n+1} + F_{2n-1}}{2} = T_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{9}$$

The relation [12]

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}, \tag{10}$$

which can be proved by induction, gives, after replacing both n and m by $2n+1$, together with (8), the result

$$\frac{F_{4n+2}}{F_{2n+1}} = U_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{11}$$

Replacing both n and m in (10) by $2n$ gives, together with (9), the result

$$\frac{F_{4n}}{F_{2n}} = 2T_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{12}$$

Equations (4), (6), (8) or (11), and (9) or (12) relate all Chebychev polynomials with argument $\sqrt{5}/2$ with Fibonacci numbers.

Identities that relate Chebychev polynomials lead to identities for Fibonacci numbers. For instance, the relation [5]

$$\sum_{m=0}^{n-1} T_{2m+1}(x) = \tfrac{1}{2}U_{2n-1}(x)$$

gives, with $x = \sqrt{5}/2$, the known result [6]

$$\sum_{m=0}^{n-1} F_{2m+1} = F_{2n}.$$

Equation [5]

$$2(1 - x^2)\sum_{m=1}^{n} U_{2m-1}(x) = x - T_{2n+1}(x),$$

gives, with $x = \sqrt{5}/2$, the known result [6]

$$\sum_{m=1}^{n} F_{2m} = F_{2n+1} - 1.$$

Binet's formula (3) gives us

$$\begin{aligned} F_{2n+1} &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n+1}\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n+1}\right] - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2n+1} \\ &= \frac{1}{\sqrt{5}}\, U_{2n}\left(\frac{\sqrt{5}}{2}\right) - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2n+1} \cong \frac{1}{\sqrt{5}}\, U_{2n}\left(\frac{\sqrt{5}}{2}\right). \end{aligned} \tag{13}$$

In an identical fashion, we obtain from (2) and (3) the approximation

$$F_{2n} \cong \frac{2}{\sqrt{5}}\, T_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{14}$$

Equations (8) and (9) combine with equations (13) and (14) to give the following interesting approximate relations

$$\frac{F_{n-1} + F_{n+1}}{\sqrt{5}} \cong F_n. \tag{15}$$

In (10), $m = n$ gives, together with (15), the following approximate relation:

$$\frac{F_{2n}}{\sqrt{5}} \cong F_n^2. \tag{16}$$

The exact relation corresponding to (15), obtainable from (3), is

$$\frac{F_{n-1} + F_{n+1}}{\sqrt{5}} = F_n + \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{17}$$

From (17) we see that (15) approximates F_n by excess for even n, and by defect for odd n.

Equations (13) to (16) give excellent approximations if n is greater than 5.

2. EXPANSIONS WITH FIBONACCI COEFFICIENTS

Chebychev polynomials are special cases of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. The exact relations are [5]:

$$T_n(x) = (g_n)^{-1}P_n^{(-\frac{1}{2},-\frac{1}{2})}(x),$$

$$U_n(x) = (2g_{n+1})^{-1}P_n^{(\frac{1}{2},\frac{1}{2})}(x),$$

with

$$g_n = \frac{(\frac{1}{2})_n}{n!}.$$

Consider the expansion [8], due to Gegenbauer,

$$\exp(xt) = \left(\frac{t}{2}\right)^{-\nu}\Gamma(\nu)\sum_{n=0}^{\infty}(\nu + n)I_{\nu+n}(t)C_n^{\nu}(x), \tag{18}$$

where $I_k(t)$ are modified Bessel functions of the first kind [5], given by

$$I_k(t) = \frac{(\frac{1}{2}t)^k}{\Gamma(k+1)}\,{}_0F_1(-;\ 1+k;\ \tfrac{1}{4}t^2),$$

and $C_n^{\nu}(x)$ are ultraspherical polynomials [8] defined by:

$$C_n^{\nu}(x) = \frac{(2\nu)_n P_n^{(\nu-\frac{1}{2},\ \nu-\frac{1}{2})}(x)}{(\nu+\frac{1}{2})_n}$$

In terms of Gegenbauer polynomials, Chebychev polynomials are given by:

$$U_n(x) = C_n^1(x), \tag{19}$$

$$T_n(x) = \lim_{\nu\to 0}\frac{C_n^{\nu}(x)}{C_n^{\nu}(1)}. \tag{20}$$

In (18), replace x by $-x$, recalling that $C_n^{\nu}(-x) = (-1)^n C_n^{\nu}(x)$, and subtract the resulting series from (18) to obtain

$$\sinh xt = \left(\frac{t}{2}\right)^{-\nu}\Gamma(\nu)\sum_{n=1}^{\infty}(\nu + 2n + 1)I_{\nu+2n+1}(t)C_{2n+1}^{\nu}(x).$$

Now let $\nu = 1$. Replace t by $-it$ and recall that $I_n(-it) = i^{-n}J_n(t)$, where $J_n(t)$ are Bessel functions of the first kind. Let $x = \sqrt{5}/2$, replace n by $n - 1$, and finally let $\sqrt{5}t/2 = \xi$, to obtain, with the help of (19) and (6),

$$\sin\xi = \frac{5}{\xi}\sum_{n=1}^{\infty}(-1)^{n+1}2nF_{2n}J_{2n}(2\xi/\sqrt{5}). \tag{21}$$

Separating the *even* part of (18), instead of the odd, gives, with the use of (8) and (11),

$$\cos\xi = \frac{\sqrt{5}}{\xi}\sum_{n=0}^{\infty}(-1)^n(2n+1)\frac{F_{4n+2}}{F_{2n+1}}J_{2n+1}(2\xi/\sqrt{5}), \tag{22}$$

$$\cos \xi = \frac{5}{\xi} \sum_{n=0}^{\infty} (-1)^n (2n + 1)\left[\frac{F_{2n} + F_{2n+2}}{\sqrt{5}}\right] J_{2n+1}(2\xi/\sqrt{5}). \tag{23}$$

If use is made of equation (17), (23) can be written as

$$\cos \xi = \frac{5}{\xi} \sum_{n=0}^{\infty} (-1)^n (2n + 1) F_{2n+1} J_{2n+1}(2\xi/\sqrt{5})$$
$$+ \frac{2\sqrt{5}}{\xi} \sum_{n=0}^{\infty} (-1)^n (2n + 1)\left(\frac{1 - \sqrt{5}}{2}\right)^{2n+1} J_{2n+1}(2\xi/\sqrt{5}). \tag{24}$$

The terms in the second series in (24) tend to zero very rapidly with increasing n.

Series (21) to (24) converge very rapidly and are, to the author's knowledge, completely new results.

Paul Byrd [4] obtained some expressions for the sine and the cosine with Fibonacci coefficients that are very similar to (21) and (24). Byrd's results are:

$$\sin \xi = \frac{1}{\xi} \sum_{n=1}^{\infty} (-1)^{n+1} 2n F_{2n} I_{2n}(2\xi),$$

and

$$\cos \xi = \frac{1}{\xi} \sum_{n=0}^{\infty} (-1)^n (2n + 1) F_{2n+1} I_{2n+1}(2\xi),$$

where $I_n(\xi)$ are modified Bessel functions of the first kind.

From the series

$$\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

we obtain, in an obvious manner, using (3), the interesting expansion

$$\log\left[\frac{1 + \frac{1 + \sqrt{5}}{2} t}{1 + \frac{1 - \sqrt{5}}{2} t}\right] = \sqrt{5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} F_n t^n}{n}. \tag{25}$$

It must be noticed that this is a general technique. Given a function $f(x, t)$ that allows for an expansion of the form

$$f(x, t) = \sum_{n=0}^{\infty} a_n(t) C_n^{\nu}(x), \tag{26}$$

it is necessary only to give appropriate values to ν, and to let x equal $\sqrt{5}/2$, provided $\sqrt{5}/2$ is within the x-region of convergence, to come to an expression such as (21), (23), or (25). References [3] and [8] contain ample information on conditions that guarantee the validity of results such as (26).

It is important to bear in mind that Fibonacci numbers grow very rapidly, for example, $F_{10} = 55$, $F_{20} = 6765$, $F_{30} = 832{,}040$, $F_{40} = 102{,}334{,}155$. *Hence, when an expansion with Fibonacci coefficients is convergent, the* $a_n(t)$ *must decrease very rapidly with increasing* n. If t is not near the boundary of the t-region of convergence, this circumstance makes these series very amenable for numerical work. We will illustrate this fact in the following sections.

3. A SERIES FOR THE ARC TANGENT

Consider the identity,

$$\tan^{-1}\frac{2x\xi}{1-\xi^2} = \tan^{-1}[(x+\sqrt{x^2-1})\xi] + \tan^{-1}[(x-\sqrt{x^2-1})\xi],$$

easily verified by taking the tangent of both sides.

Let us substitute the expansion,

$$\tan^{-1}\xi = \sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1}}{2n+1}, \tag{27}$$

on the two terms on the right-hand side above, and make use of equation (1) to obtain,

$$\tan^{-1}\frac{2x\xi}{1-\xi^2} = 2\sum_{n=0}^{\infty}\frac{(-1)^n T_{2n+1}(x)\xi^{2n+1}}{2n+1}. \tag{28}$$

Series (27), known as Gregory's series, is a special case of series (28) corresponding to $x = 1$, when use is made of the identity for the tangent of the half-angle: $\tan^{-1}[2\xi/(1-\xi^2)] = 2\tan^{-1}\xi$.

In (28), let $x = \sqrt{5}/2$, $\sqrt{5}\xi = t$, and use equation (4) to obtain

$$\tan^{-1}\frac{5t}{5-t^2} = \sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}t^{2n+1}}{5^n(2n+1)}.$$

Now let $5t/(5-t^2) = \alpha > 0$, and choose the smaller of the roots of this quadratic equation, to obtain

$$\tan^{-1}\alpha = \sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}t^{2n+1}}{5^n(2n+1)}, \tag{29}$$

with

$$t = \frac{2\alpha}{1+\sqrt{1+(4\alpha^2/5)}}, \tag{30}$$

a curious and simple series for the arc tangent with odd Fibonacci numbers as coefficients.

4. COMPARISON WITH EULER'S SERIES FOR THE ARC TANGENT

Series (27) discovered by Gregory in 1671, converges very slowly except for very small values of its argument. For $\xi = 1$, for example, it yields Leibniz' celebrates series for $\pi/4$ that requires two thousand terms to give three decimal figures of π.

Let us use Pochhammer's symbol

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), \quad \alpha \neq 0,$$

and the identity $(3/2)_n/(1/2)_n = 2n+1$ to write (27) in hypergeometric form:

$$\tan^{-1}t = tF(1, 1/2; 3/2; -t^2). \tag{31}$$

Now consider the relation

$$F(a, b; c; z) = (1 - z)^{-a}F(a, c - b; c; -z/(1 - z)), \tag{32}$$

valid if $|z| < 1$, and $|z/(1 - z)| < 1$. This relation is an equality among two of Kummer's twenty-four solutions to Gauss's hypergeometric differential equation. In (32), let $a = 1$, $b = 1/2$, $c = 3/2$, and $z = -t^2$, to obtain

$$\tan^{-1}t = tF(1, 1/2; 3/2; -t^2) = [t/(1 + t^2)]F(1, 1; 3/2; t^2/(1 + t^2)).$$

Since $(2n + 1)! = (2)_{2n} = 2^{2n}n!(3/2)_n$, the above equation gives

$$\tan^{-1}t = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n + 1)!} \frac{t^{2n+1}}{(1 + t^2)^{n+1}}. \tag{33}$$

Inasmuch as $t^2/(1 + t^2) < 1$ for every real t, we can conclude that (33) converges for every real value of its argument.

Equation (33) is Euler's famous series for the arc tangent discovered in 1755. This series converges very rapidly for all t, and especially for small values of its argument.

Let $t = \alpha \ll 1$. Using Stirling's formula for the factorial

$$n! \cong \sqrt{2\pi n}(n/e)^n, \quad n \text{ large},$$

we obtain, for the general term a_n of Euler's series, the estimate

$$a_n \cong \frac{e\sqrt{\pi}n^{2n+1}\alpha^{2n+1}}{2(n + \frac{1}{2})^{2n+3/2}}. \tag{34}$$

If n is large, $n + \frac{1}{2} \cong n$, and we have the estimate

$$a_n \cong \frac{e\sqrt{\pi}}{2} \frac{\alpha^{2n+1}}{\sqrt{n}}. \tag{35}$$

To compare this result with the corresponding one for series (29), notice that for α small (30) gives $t \cong \alpha$. For the general term, omitting the sign, b_n of series (29), we then have, recalling equation (3), the estimate

$$b_n \cong \frac{\alpha^{2n+1}}{2n + 1}\left(\frac{1 + 5^{-\frac{1}{2}}}{2}\right)^{2n+1}. \tag{36}$$

Comparison of (34) or (35) with (36), observing that the expression in parentheses above is $0.723606798\ldots < 1$, shows that, for small values of its argument, series (29) converges substantially faster than (33).

The requirement of the argument being small is only an apparent restriction, necessary to simplify the proof above. If α is large, it is simply necessary to use the identity

$$\tan^{-1}\alpha = \frac{\pi}{2} - \tan^{-1}\frac{1}{\alpha}, \quad (\alpha > 1).$$

Series (29) has the added advantage of being an alternating series, which series (33) is not. It is, as is well known, a general property of such series that the remainder after n terms has a value which is between zero and the first term not taken. It is a simple matter, then, to determine the number of terms of (29) needed to obtain a given accuracy.

If in series (28) we let $x = \cos\theta$, $2\xi\cos\theta/(1-\xi^2) = \alpha$, solve for ξ in terms of α and $\cos\theta$, and substitute back into (28), recalling that $T_n(\cos\theta) = \cos n\theta$, we obtain the curious series

$$\tan^{-1}\alpha = 2\sum_{n=0}^{\infty}\frac{(-1)^n\alpha^{2n+1}\cos(2n+1)\theta}{(2n+1)(\cos\theta+\sqrt{\alpha^2+\cos^2\theta})^{2n+1}}, \quad \left(0 \leqslant \theta < \frac{\pi}{2}\right), \tag{37}$$

where the right-hand side is independent of θ. The rapidity of the convergence, though, depends on the choice of θ. Series (37) converges very rapidly if both α and θ are small.

5. ANOTHER SERIES FOR THE ARC TANGENT

Iteration of the method used in Section 3 to obtain equation (28) yields a new series for the arc tangent. In (28), replace ξ by $\xi(x+\sqrt{x^2-1})$ and by $\xi(x-\sqrt{x^2-1})$ and add the two arc tangents to obtain

$$\tan^{-1}\frac{2x\xi(x+\sqrt{x^2-1})}{1-(x+\sqrt{x^2-1})^2\xi^2} + \tan^{-1}\frac{2x\xi(x-\sqrt{x^2-1})}{1-(x-\sqrt{x^2-1})^2\xi^2}$$

$$= 4\sum_{n=0}^{\infty}\frac{(-1)^n T_{2n+1}^2(x)\xi^{2n+1}}{2n+1}.$$

Combining the two arc tangents by means of the identity

$$\tan^{-1}a + \tan^{-1}b = \tan^{-1}\frac{a+b}{1-ab},$$

we obtain

$$\tan^{-1}\frac{4x^2\xi(1-\xi^2)}{1-2(4x^2-1)\xi^2+\xi^4} = 4\sum_{n=0}^{\infty}\frac{(-1)^n T_{2n+1}^2(x)\xi^{2n+1}}{2n+1}. \tag{38}$$

Gregory's series (27) is the special case of (38) corresponding to $x = 1$, if use is made of the identity for the tangent of one-fourth of the angle:

$$\tan^{-1}\xi = \frac{1}{4}\tan^{-1}\frac{4\xi(1-\xi^2)}{1-6\xi^2+\xi^4}.$$

Let the argument of the arc tangent in (38) equal α, and solve the resulting fourth-degree equation for ξ. The solution is easily obtained by dividing through by ξ^2, and making the substitutions $\xi - \xi^{-1} = -2t$, $\xi^2 + \xi^{-2} = 4t^2 + 2$, which reduces it to two quadratics. The results are

$$\xi = (t+\sqrt{t^2+1})^{-1} \quad\text{and}\quad t = \frac{x^2}{\alpha}(1+\sqrt{1+[\alpha^2(2x^2-1)/x^4]}). \tag{39}$$

Now, if we let $x = \sqrt{5}/2$ in (38) and (39), we obtain

$$\tan^{-1}\alpha = 5\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}^2}{(2n+1)(t+\sqrt{t^2+1})^{2n+1}}, \tag{40}$$

with

$$t = \frac{5}{4\alpha}\left(1 + \sqrt{1 + (24\alpha^2/25)}\right). \tag{41}$$

Series (40) converges substantially faster than series (29). If α is small, the n^{th} term, c_n, of series (40), is, apart from the sign, approximately given by

$$c_n \cong \frac{\alpha^{2n+1}}{2n+1}\left(\frac{1 + 5^{-\frac{1}{2}}}{2}\right)^{4n+2}. \tag{42}$$

6. THE NEXT ITERATION

Iteration of formula (38) by the method used in Section 5 gives, after some simple but lengthy algebra, the result

$$\tan^{-1}\frac{8x^3[\xi - (12x^2 - 5)\xi^3 + (12x^2 - 5)\xi^3 - \xi^7]}{1 - 4(12x^4 - 6x^2 + 1)\xi^2 + 2(32x^6 + 24x^4 - 24x \ \ + 3)\xi^4 - 4(12x^4 - 6x^2 + 1)\xi^6 + \xi^8}$$

$$= 8\sum_{n=0}^{\infty}\frac{(-1)^n T^3_{2n+1}(x)\xi^{2n+1}}{2n + 1}. \tag{43}$$

If we let the argument of the arc tangent in (43) equal α we obtain, after dividing through by ξ^4 and setting

$$\xi - \xi^{-1} = -2t/\sqrt{5},\ \xi^2 + \xi^{-2} = \frac{4}{5}t^2 + 2,\ \xi^3 - \xi^{-3} = -\frac{8}{5\sqrt{5}}t^3 - \frac{6}{\sqrt{5}}t,$$

$$\xi^4 + \xi^{-4} = \frac{16}{25}t^4 + \frac{16}{5}t^2 + 2, \text{ and } x = \frac{\sqrt{5}}{2},$$

the result is

$$\tan^{-1}\alpha = \sum_{n=0}^{\infty}\frac{(-1)^n 5^{n+2}F^3_{2n+1}}{(2n + 1)(t + \sqrt{t^2 + 5})^{2n+1}}, \tag{44}$$

with t the largest positive root of

$$8\alpha t^4 - 100t^3 - 450\alpha t^2 + 875t + 625\alpha = 0. \tag{45}$$

This quartic equation is *in principle* solvable by radicals [1] for any value of α. The algorithm, though, does not seem to lead to any manageable combination of radicals, and for its solution we resorted to Newton's iterative method. Several solutions are discussed in the next section.

7. SOME SERIES FOR π

To illustrate the convergence of series (29), (40), and (44), we will obtain some expressions for π. Let $\alpha = 1$ in (30), and substitute into (29) to get

$$\pi = \sqrt{5}\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}2^{2n+3}}{(2n + 1)(3 + \sqrt{5})^{2n+1}}. \tag{46}$$

$\alpha = 1$ in equation (41) when substituted into (40) gives

$$\pi = 20\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}^2}{(2n+1)(3+\sqrt{10})^{2n+1}}. \tag{47}$$

Thirty-two terms of (46) give fifteen decimal places of π, while series (47) requires nineteen terms. For $\alpha = 1$, the largest positive root of (45) is

$$t = 15.63057705819013\ldots .$$

With this value of t fourteen terms of (44) give fifteen decimal figures of π. Euler's series (33) for the same argument and for the same accuracy requires fifty terms.

From equations (36) and (42), we see that rapid convergence of series (29) and (40) depends on our ability to choose the argument of the arc tangent, or, which is the same, the angle, sufficiently small. For example,

$$\frac{\pi}{12} = \tan^{-1}(2-\sqrt{3}).$$

For this argument, series (29) gives

$$\pi = 12\sqrt{5}\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}}{2n+1}\left[\frac{2(2-\sqrt{3})}{\sqrt{5}+\sqrt{1+16(2-\sqrt{3})}}\right]^{2n+1}. \tag{48}$$

The expression in brackets above is 0.1181577543..., which is more than seven times smaller than the corresponding root for equation (46). Series (48) converges very rapidly. Ten terms give fifteen decimal places of π. For this same argument the corresponding value of equation (41) is $t = 9.488217845\ldots$. With this value of t nine terms of (40) give fifteen decimal figures of π. Euler's series (33) for the same argument and for the same accuracy requires thirteen terms.

Use of Machin's formula,

$$\pi = 16\tan^{-1}(1/5) - 4\tan^{-1}(1/239), \tag{49}$$

with α in (41) equal to 1/5 and to 1/239 gives values for t which are approximately,

$$t = 12.61886960\ldots, \quad \text{and} \quad t' = 597.5025107\ldots .$$

Using these values on (49) with series (40), we obtain a very rapidly converging series for π. A computer run with the double-precision routines of the BASIC Level II interpreter of the Radio Shack TRS-80 Model I microcomputer with this combination of arc tangents gave the values shown in the following table for Gregory's series (27), Euler's series (33), and series (40). We see that series (40) consistently gives better approximations than either Euler's or Gregory's series. Series (29) will also, of course, converve more rapidly than Gregory's or Euler's series ($n = 9$).

n	Gregory	Euler	Series (40)
0	3.183263598326360	3.060186968243409	3.148158616418292
1	3.140597029326061	3.139082236428362	3.141554182069219
2	3.141621029325035	3.141509789149037	3.141592944101887
3	3.141591772182177	3.141589818359699	3.141592651171905
4	3.141591682404400	3.141592554401089	3.141592653611002
5	3.141592652615309	3.141592650066872	3.141592653589601
6	3.141592653623555	3.141592653463209	3.141592653589795
7	3.141592653588603	3.141592653585213	3.141592653589793
8	3.141592653589836	3.141592653589626	
9	3.141592653589792	3.141592653589787	
10	3.141592653589794	3.141592653589793	
11	3.141592653589793		

The largest positive root of (45) corresponding to $\alpha = 1/5$ is

$$t = 63.25229744727801\ldots,$$

and the one corresponding to $\alpha = 1/239$ is

$$t' = 2987.51589950963\ldots.$$

Using these values on (49) with series (44) gives fifteen decimal figures of π after seven terms ($n = 6$). We see that with the use of expressions such as Machin's identity (49), iterations beyond the second are not worth the added labor, it being much simpler to work with series (40).

Application of the trigonometric identities

$$\tan^{-1}\frac{1}{a \pm b} = \tan^{-1}\frac{1}{a} \mp \tan^{-1}\frac{b}{a^2 \pm ab + 1}, \tag{50}$$

and

$$\tan^{-1}\frac{1}{a} = 2\tan^{-1}\frac{1}{2a} - \tan^{-1}\frac{1}{4a^3 + 3a} \tag{51}$$

on simpler formulas such as Machin's identity, or on the identity

$$\pi = 20\tan^{-1}(1/7) + 8\tan^{-1}(3/79), \tag{52}$$

due to Euler, give additional expressions for the calculation of π. Repeated application of (51) to Machin's formula, letting a equal 5, 10, 20, and (40), in turn, yields the identity

$$\pi = 256\tan^{-1}(1/80) - 4\tan^{-1}(1/239) - 16\tan^{-1}(1/515) - 32\tan^{-1}(1/4030) - 64\tan^{-1}(1/32{,}060) - 128\tan^{-1}(1/256{,}120), \tag{53}$$

first obtained by Cashmore in [7]. Identity (53) together with (40) provides an extremely rapidly converging series for the calculation of π. Four terms of this series give fifteen decimal figures of π. Euler's series (33) also requires four terms. The computed values are shown in the following table.

n	Euler's Series	Series (40)
0	3.141259656493609	3.141619294232185
1	3.141592611940046	3.141592652964994
2	3.141592653584216	3.141592653589812
3	3.141592653589793	3.141592653589793

Once again we see that (40) converges to its limiting value faster than does Euler's series. As more decimal figures are calculated, though, the difference between the series becomes significant and the tide swings in favor of our series. For the same value of the argument, the tenth term of Euler's series is 4.54×10^6 times bigger than the tenth term of (40). The twentieth term of Euler's series is 2.60×10^{12} times bigger than the corresponding one of (40). The thirtieth term is 1.35×10^{18} times bigger. The one hundredth term is already 1.46×10^{58} times bigger, and the one hundred fiftieth term is 2.25×10^{86} times bigger.

With the combination of arc tangents given in (53), twenty-three terms of (40) give one hundred decimal places of π. Two hundred twenty-six terms will give one thousand decimal places of π. The calculation of the radicals in (40) and (41) can be performed very quickly, because of the smallness of α, with the quadratically converging algorithm given in Rudin [10]. Identity (53) is very amenable for a high-precision calculation of π. It would be of interest to compare (53) against Eugene Salamin's quadratically converging algorithm [11] based on the theory of elliptic integrals.

It should, perhaps, be mentioned that there exist series for the calculation of π which converge faster than any series we have obtained. For example,

$$\frac{1}{\pi} = 2\sqrt{2}\left[\frac{1103}{99^2} + \frac{27493}{99^6}\,\frac{1}{2}\,\frac{1 \cdot 3}{4^2} + \frac{53883}{99^{10}}\,\frac{1 \cdot 3}{2 \cdot 4}\,\frac{1 \cdot 3 \cdot 5 \cdot 7}{4^2 \cdot 8^2} + \cdots\right], \tag{54}$$

due to Ramanujan [9]. The numerators of the first fractions of each term above are in arithmetic progression. Three terms of (54) give seventeen decimal figures of π!

As stated at the beginning of this section, we have used π simply as an illustration of the convergence of the arc tangent series (29), (40), and (44), and these series *do converge faster than any other known arc tangent series*.

It is an interesting historical fact that Fibonacci made an attempt to determine the value of π using Archimedes' method of inscribed and circumscribed polygons. Using a 96-sided polygon, he obtained for π the approximation $864 \div 275$, which gave him the value 3.141818, correct to three decimal places [2]. It seems safe to think that he never suspected that the peculiar sequence he had discovered on the growth of the rabbit population would yield, nearly eight centuries later, a simple and powerful algorithm for the calculation of π with any desired accuracy.

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ANNOUNCEMENT

SECOND INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

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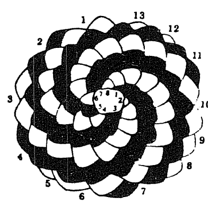

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CALL FOR PAPERS

The SECOND INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at San Jose State University, San Jose, CA, Aug. 13-16, 1986. This conference is sponsored jointly by The Fibonacci Association and San Jose State University.

Papers on all branches of mathematics and science related to the Fibonacci numbers and their generalizations are welcome. Abstracts are to be submitted by March 15, 1986. Manuscripts are requested by May 1, 1986. Abstracts and manuscripts should be sent to the chairman of the local committee. Invited and contributed papers will appear in the Conference Proceedings, which are expected to be published.

The program for the Conference will be mailed to all participants, and to those individuals who have indicated an interest in attending the conference, by June 15, 1986. All talks should be limited to one hour.

For further information concerning the conference, please contact either of the following:

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-562 *Proposed by Herta T. Freitag, Roanoke, VA*

Let c_n be the integer in $\{0, 1, 2, 3, 4\}$ such that

$$c_n \equiv L_{2n} + [n/2] - [(n - 1)/2] \pmod 5,$$

where $[x]$ is the greatest integer in x. Determine c_n as a function of n.

B-563 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_n = \sum_{i=1}^{n} L_{2i+1}L_{2i-2}$. For which values of n is S_n exactly divisible by 4?

B-564 *Proposed by Lászlό Cseh, Cluj, Romania*

Let $a = (1 + \sqrt{5})/2$ and $[x]$ be the greatest integer in x. Prove that

$$[aF_1] + [aF_2] + \cdots + [aF_n] = F_{n+3} - [(n + 4)/2].$$

B-565 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Let $P_0, P_1, \ldots$ be the sequence of Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \in \{2, 3, \ldots\}$. Show that

$$9\sum_{k=0}^{n} P_k F_k = P_{n+2}F_n + P_{n+1}F_{n+2} + P_n F_{n-1} - P_{n-1}F_{n+1}.$$

B-566 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*

Let P_n be as in B-565. Show that

$$9\sum_{k=0}^{n} P_k L_k = P_{n+2}L_n + P_{n+1}L_{n+2} + P_n L_{n-1} - P_{n-1}L_{n+1} - 6.$$

B-567 *Proposed by P. Rubio, Dragados Y Construcciones, Madrid, Spain*

Let $a_0 = a_1 = 1$ and $a_{n+1} = a_n + na_{n-1}$ for n in $Z^+ = \{1, 2, \ldots\}$. Find a simple formula for

$$G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k.$$

SOLUTIONS

Lucas Geometric Progression

B-538 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that $\sqrt{5}g^n = gL_n + L_{n-1}$, where g is the golden ratio $(1 + \sqrt{5})/2$.

Solution by Lâszló Cseh, Cluj, Romania

It is well known that $L_n = g^n + \overline{g}^n$, where $\overline{g} = (1 - \sqrt{5})/2$. Now

$$\begin{aligned} gL_n + L_{n-1} &= g^{n+1} + g \cdot \overline{g} \cdot \overline{g}^{n-1} + g^{n-1} + \overline{g}^{n-1} \\ &= g^{n+1} - \overline{g}^{n-1} + g^{n-1} + \overline{g}^{n-1} \\ &= g^n(g + g^{-1}) = \sqrt{5}g^n. \quad \text{Q.E.D.} \end{aligned}$$

Remark: By a similar argument, it can be proved that $g^n = gF_n + F_{n-1}$.

Also solved by Wray G. Brady, Paul S. Bruckman, L. A. G. Dresel, Russell Euler, Piero Filipponi, C. Georghiou, Walther Janous, Hans Kappus, L. Kuipers, Graham Lord, I. Merenyi, George N. Philippou, Bob Prielipp, Heinz-Jürgen Seiffert, A. G. Shannon, Lawrence Somer, W. R. Utz, and the proposer.

Not Necessarily Golden GP's

B-539 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $g = (1 + \sqrt{5})/2$ and show that

$$\left[1 + 2\sum_{i=1}^{\infty} g^{-3i}\right]\left[1 + 2\sum_{i=1}^{\infty} (-1)^i g^{-3i}\right] = 1.$$

Solution by A. G. Shannon, NSWIT, Sydney, Australia

$$\left[1 + 2\sum_{i=1}^{\infty} g^{-3i}\right]\left[1 + 2\sum_{i=1}^{\infty} (-1)^i g^{-3i}\right] \qquad |g| < 1$$

(continued)

$$= \left[1 + \frac{2}{g^3 - 1}\right]\left[1 - \frac{2}{g^3 + 1}\right] \quad \text{(sums of GPs)}$$

$$= \left[\frac{g^3 + 1}{g^3 - 1}\right]\left[\frac{g^3 - 1}{g^3 + 1}\right] = 1, \text{ as required.}$$

This holds for $|g| < 1$; i.e., g does not have to equal α.

Also solved by Wray G. Brady, Paul S. Bruckman, Lászlő Cseh, L. A. G. Dresel, Russell Euler, Piero Filipponi, C. Georghiou, Walther Janous, L. Kuipers, Graham Lord, I. Merenyi, George N. Philippou, Bob Prielipp, Heinz-Jürgen Seiffert, Lawrence Somer, and the proposer.

Product of 3 Successive Integers

B-540 *Proposed by A. B. Patel, V.S. Patel College of Arts & Sciences, Bilimora, India*

For $n = 2, 3, \ldots$, prove that

$$F_{n-1}F_nF_{n+1}L_{n-1}L_nL_{n+1}$$

is not a perfect square.

Solution by L. A. G. Dresel, University of Reading, England

Using the identities $F_nL_n = F_{2n}$ and $F_{2n-2}F_{2n+2} = F_{2n}^2 - 1$, we have

$$P = F_{n-1}F_nF_{n+1}L_{n-1}L_nL_{n+1} = F_{2n-2}F_{2n}F_{2n+2} = F_{2n}(F_{2n}^2 - 1).$$

Now for $n = 2, 3, \ldots$, we have $F_{2n} > 1$ and, therefore, $(F_{2n}^2 - 1)$ is not a perfect square; furthermore, $F_{2n}^2 - 1 = (F_{2n} - 1)(F_{2n} + 1)$ is coprime to F_{2n} and, therefore, the expression P is not a perfect square.

Also solved by Wray G. Brady, Paul S. Bruckman, Adina Di Porto & Piero Filipponi, Walther Janous, L. Kuipers, Bob Prielipp, A. G. Shannon, Lawrence Somer, and the proposer.

Congruence Modulo 9

B-541 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Show that $P_{n+3} + P_{n+1} + P_n \equiv 3(-1)^nL_n \pmod 9$, where the P_n are the Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and

$$P_{n+2} = 2P_{n+1} + P_n \text{ for } n \text{ in } N = \{0, 1, 2, \ldots\}.$$

Solution by L. A. G. Dresel, University of Reading, England

$$P_{n+3} + P_{n+1} + P_n = 2P_{n+2} + 2P_{n+1} + P_n = 3P_{n+2}.$$

Let $K_n = (-1)^nL_n$. Then since $L_{n+2} = L_{n+1} + L_n$, multiplying by $(-1)^n$ we obtain $K_{n+2} = -K_{n+1} + K_n$, so that $K_{n+2} \equiv 2K_{n+1} + K_n \pmod 3$. Thus, K_n and P_n satisfy the same recurrence relation modulo 3, and furthermore,

$$P_2 = 2P_1 + P_0 = 2 = K_0 \quad \text{and} \quad P_3 = 2P_2 + P_1 = 5 \equiv -1 = K_1 \pmod 3.$$

It follows that $P_{n+2} \equiv K \pmod 3$ for n in $N = \{0, 1, 2, \ldots\}$ and, therefore,

$3P_{n+2} \equiv 3K_n \pmod 9$ for n in N, so that

$$P_{n+3} + P_{n+1} + P_n \equiv 3(-1)^n L_n \pmod 9.$$

Also solved by László Cseh, Herta T. Freitag, C. Georghiou, Walther Janous, L. Kuipers, Imre Merenyi, George N. Philippou, Bob Prielipp, A. G. Shannon, Lawrence Somer, and the proposer.

3rd Order Nonhomogeneous Recursion

B-542 *Proposed by Ioan Tomescu, University of Bucharest, Romania*

Find the sequence satisfying the recurrence relation

$$u(n) = 3u(n-1) - u(n-2) - 2u(n-3) + 1$$

and the initial conditions $u(0) = u(1) = u(2) = 0$.

Solution by C. Georghiou, University of Patras, Greece

It is easy to see that the roots of the characteristic polynomial of the homogeneous equation are $r_1 = 2$, $r_2 = a$, and $r_3 = b$ and that a particular solution of the inhomogeneous equation is $u_p(n) = 1$. Therefore, the general solution of the given recurrence relation is

$$u(n) = A2^n + BF_n + CL_n + 1.$$

The initial conditions give $A = 1$, $B = -2$, and $C = -1$, and the solution is

$$u(n) = 2^n - 2F_n - L_n + 1 = 2^n - F_{n+3} + 1.$$

Also solved by Wray G. Brady, Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, László Cseh, L. A. G. Dresel, Russell Euler, Walther Janous, Hans Kappus, L. Kuipers & Peter J. S. Shiue, I. Merenyi, Bob Prielipp, Heinz-Jürgen Seiffert, A. G. Shannon, and the proposer.

Fibonacci Exponential Generating Function

B-543 *Proposed by P. Rubio, Dragados Y Construcciones, Madrid, Spain*

Let $\alpha_0 = \alpha_1 = 1$ and $\alpha_{n+1} = a_n + \alpha_{n-1}$ for n in $Z^+ = \{1, 2, \ldots\}$. Find a simple formula for

$$G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

We see readily that $a_n = F_{n+1}$. Hence,

$$G(x) = \sum_{k=0}^{\infty} F_{k+1} \frac{x^k}{k!} = 5^{-\frac{1}{2}} \sum_{k=0}^{\infty} (\alpha^{k+1} - \beta^{k+1}) \frac{x^k}{k!} = 5^{-\frac{1}{2}} (\alpha e^{\alpha x} - \beta e^{\beta x}).$$

Also solved by Wray G. Brady, O. Brugia & A. Di Porto & P. Filipponi, John R. Burke, László Cseh, L. A. G. Dresel, Russell Euler, C. Georghiou, Walther Janous, Hans Kappus, L. Kuipers, Graham Lord, Imre Merenyi, A. G. Shannon, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-394 *Proposed by Ambati Jaya Krishna, Baltimore, MD, and Gomathi S. Rao, Orangeburg, SC*

Find the value of the continued fraction $1 + \frac{2}{3+} \frac{4}{5+} \frac{6}{7+} \cdots$.

H-395 *Proposed by Heinz Jürgen Seiffert, Berlin, Germany*

Show that for all positive integers m and k,

$$\sum_{n=0}^{m-1} \frac{F_{2k(2n+1)}}{L_{2n+1}} = \sum_{j=0}^{k-1} \frac{F_{2m(2j+1)}}{L_{2j+1}}$$

H-396 *Proposed by M. Wachtel, Zürich, Switzerland*

Establish the identity:

$$\sum_{i=1}^{\infty} \frac{F_{i+n}}{a^i} + \sum_{i=1}^{\infty} \frac{F_{i+n+1}}{a^i} = \sum_{i=1}^{\infty} \frac{F_{i+n+2}}{a^i}$$

$a = 2, 3, 4, \ldots, n = 0, 1, 2, 3, \ldots$.

A reply regarding H-354 by M. Wachtel, Zürich, Switzerland

In a note in the May 1985 issue, the proposer is claiming that my solution which appeared in the August 1984 issue is not a solution.

Reply: After having unsuccessfully attempted to understand the argumentation given in the above note, I might restrict myself to the following:

1. Admittedly, the theories I developed are not a solution in a strict mathematical sense; neither was it intended to evoke this impression, since no proofs were given. I believe, however, that these theories are new ones, and as shown, they lead to the desired solutions of the equation ($Ax^2 + C = By^2$) in integers.

2. The proposer claims that I attempted construction of the solutions to particular cases. Yes, but I designated them clearly as arbitrary examples, and, moreover, I also mentioned in §1.2: "Considering the limited space, only main fragments of the whole issue can be dealt with here." Not one, but many formulas are involved, I surmise.

3. The problems H-350 and H-372, proposed by myself and mentioned by Bruckman, are particular instances of the equation ($Ax^2 + C = By^2$) and solvable by the theories I described.

4. Bruckman states: "Moreover, an explicit formula for all such solutions is known, in terms of the one known solution." Frankly, I cannot imagine that there exists a general formula which would cope with particular values of C, or with the sometimes amazing complexity of the relations of A to B. In §2.3, I outlined: "To determine (x_2, y_2), there does not (presumably) exist a general formula, but an undeterminable number of different construction rules, according to the group or class to which the sequence belongs. When both (x_1, y_1) and (x_2, y_2) are found, all other terms are determined." I have found quite a lot of such construction rules to determine (x_2, y_2).

As to the "explicit formula" for all such solutions, I wonder if, e.g., for ($A = 11(L_5)$, $x_1 = 2$, $C = 3$, $B = 47(L_8)$, $y_1 = 1$) the desired sequence can be established.

5. As an autodidact in mathematics with no high school education, I am, naturally, sometimes unable to observe a strong mathematical way. In conclusion, may I observe that Bruckman in this note quoted my name incorrectly.

SOLUTIONS

Primitive Sequences

H-369 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC (Vol. 22, no. 2, May 1984)*

Call an integer-valued arithmetic function f a *gcd sequence* if

$$\gcd(a, b) = d \quad \text{implies} \quad \gcd(f(a), f(b)) = f(d)$$

for all positive integers a and b. A gcd sequence is *primitive* if it is neither an integer multiple nor a positive integer power of some other gcd sequence. Examples of primitive gcd sequences include:

(1) $f(n) = 1$
(2) $f(n) = n$
(3) $f(n) =$ largest squarefree divisor of n
(4) $f(n) = 2^n - 1$
(5) $f(n) = F_n$ (Fibonacci sequence)

Prove that there are infinitely many primitive gcd sequences.

Solution by Paul S. Bruckman, Fair Oaks, CA

Let G and PG represent the sets of gcd sequences and primitive gcd sequences, respectively. There is a possible misstatement in the definition of of gcd sequence given in the statement of the problem, which requires f to be an *arithmetic* function; recall a function $f: N \to N$ is *arithmetic* if $f(1) = 1$ and $f(mn) = f(m) \quad f(n)$ whenever $\gcd(m, n) = 1$. However, $f_4(n) \equiv 2^n - 1$ and

$f_5(n) \equiv F_n$ are *not* arithmetic functions, even though these functions have the "gcd property" stated in the problem. Assuming that the proposer did not intend the offending word *arithmetic* in the definition of G and its subset PG, no difficulty arises.

Infinitely many sequences $(f_n) \in G$ are then generated by the recursion:

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x),\ n = 0, 1, 2, \ldots;\ f_0(x) = 0,\ f_1(x) = 1, \tag{1}$$

where x is any positive integer.

The $f_n(x)$'s given above are generalized Fibonacci polynomials. It was shown by Hoggatt and Long ["Divisibility Properties of Generalized Fibonacci Polynomials," *The Fibonacci Quarterly* **12**, no. 2 (1974):113-130] that these polynomials have the gcd property, that is,

$$\gcd(f_m(x),\ f_n(x)) = f_{\gcd(m,n)}(x). \tag{2}$$

Hence $(f_n(x)) \in G$. Since $f_1(x) = 1$, $f_n(x)$ is not a multiple of another sequence in G. Also, we may choose $x = f_2(x)$ to be a non-power, in infinitely many ways; with such choices, we see that $(f_n(x))$ cannot be a power of another sequence in G. Hence $(f_n(x))$ PG for infinitely many choices for x. Q.E.D.

Also solved by W. Janous, L. Kuipers, L. Somer, and the proposer.

Lotsa Fives in the Product

H-370 *Proposed by M. Wachtel and H. Schmutz, Zürich, Switzerland (Vol. 22, no. 2, May 1984)*

For every positive integer a show that

(A) $5 \cdot [5 \cdot (a^2 + a) + 1] + 1$

(B) $5 \cdot [5 \cdot [5 \cdot [5 \cdot (a^2 + a) + 1] + 1] + 1] + 1$

are *products of two consecutive integers*, and that no integral divisor of

$$5 \cdot (a^2 + a) + 1$$

is congruent to 3 or 7, modulo 10.

Solution by Lawrence Somer, Washington, D.C.

Expanding expression (A), we obtain

$$25a^2 + 25a + 6,$$

which is the product of the consecutive integers $5a + 2$ and $5a + 3$. Expanding expression (b), we obtain

$$625a^2 + 125a + 156,$$

which is the product of the consecutive integers $25a + 12$ and $25a + 13$.

In general, it can be shown by induction that if

$$S_k(a) = \underbrace{5 \cdot [5 \cdot [5 \cdot \cdots \cdot [5}_{2k\ 5\text{'s}} \cdot (a^2 + a) + \underbrace{1] + 1] + \cdots + 1] + 1}_{2k\ 1\text{'s}},$$

where k is a fixed positive integer, then

$$S_k(a) = (5^k a + (5^k - 1)/2) \cdot (5^k a + ((5^k - 1)/2) + 1).$$

By looking at the ring of integers modulo 10, one sees that an integer is congruent to 3 or 7 modulo 10 if and only if at least one of its prime divisors is congruent to 3 or 7 modulo 10. Thus, to prove the last part of the problem, we need only show that no prime divisor $5(a + a) + 1$ is congruent to 3 or 7 modulo 10. Let p be a prime such that $p = 3$ or 7 modulo 10. Then, by the law of quadratic reciprocity, $(5/p) = -1$, where $(5/p)$ is the Legendre symbol. Suppose

$$5(a^2 + a) + 1 \equiv 0 \pmod{p}.$$

Multiplying both sides by 20 and then adding 5 to both sides, one obtains

$$(100a^2 + 100a + 25) = (10a + 5)^2 \equiv 5 \pmod{p}.$$

However, this is a contradiciton, since $(5/p) = -1$. We are now done.

Also solved by P. Bruckman, O. Brugia & P. Filipponi, L. Dresel, F. He, J. Metzger, B. Prielipp, and the proposers.

Continuing ...

H-371 *Proposed by Paul S. Bruckman, Fair Oaks, CA (Vol. 22, no. 2, May 1984)*

Let $[\overline{k}]$ represent the purely periodic continued fraction:

$$k + 1/(k + 1/(k + \cdots, \; k = 1, 2, 3, \ldots .$$

Show that

$$[\overline{k}]^3 = [\overline{k^3 + 3k}].$$

Generalize to other powers.

Solution by O. Brugia, A. Di Porto, & P. Filipponi, Fdn. U. Bordoni, Rome, Italy

Let δ_m be the m^{th} convergent of $[\overline{k}]$; as known [1], both the numerator P_m and the denominator Q_m of δ_m can be expressed by the same difference equation, $R_m = kR_{m-1} + R_{m-2}$, with initial conditions $R_0 = 1$, $R_1 = k$ for P_m, and $R_0 = 0$, $R_1 = 1$ for Q_m. Since the roots of the corresponding characteristic equation $z^2 - kz - 1 = 0$ are $z_1 = (k - \sqrt{k^2 + 4})/2$ and $z_2 = (k + \sqrt{k^2 + 4})/2$, we get $\delta_m = (z_2^{m+1} + z_1^{m+1})/(z_2^m - z_1^m)$, and hence

$$[\overline{k}] = \lim_{m \to \infty} \delta_m = z_2 = (k + \sqrt{k^2 + 4})/2, \text{ for } k > 0. \tag{1}$$

For every nonnegative integer n we will find, if any, a nonnegative integer h_n such that

$$[\overline{h}_n] = [\overline{k}]^n. \tag{2}$$

From (1), equation (2) can be rewritten as $(h_n + \sqrt{h_n^2 + 4})/2 = z_2^n$ and gives

$$h_n = (z_2^{2n} - 1)/z_2^n = z_2^n - (-z_1)^n$$
$$= ((\sqrt{k^2 + 4} + k)/2)^n - ((\sqrt{k^2 + 4} - k)/2)^n, \tag{3}$$

where use has been made of the relation

$$z_1 z_2 = -1. \tag{4}$$

Because

$$h_n = \begin{cases} 0, & \text{for } n = 0 \\ \dfrac{2}{2^{2\nu}}\sqrt{k^2+4}\displaystyle\sum_{i=0}^{\nu-1}\binom{2\nu}{2i+1}(k^2+4)^i k^{2\nu-2i-1}, & \text{for } n = 2\nu,\ \nu = 1, 2, \ldots \\ \dfrac{1}{2^{2\nu}}\displaystyle\sum_{i=0}^{\nu}\sum_{j=0}^{i}\binom{2\nu+1}{2i}\binom{i}{j}2^{2j}k^{2\nu-2j+1}, & \text{for } n = 2\nu+1,\ \nu = 0, 1, \ldots \end{cases} \tag{5}$$

we see that $h_{2\nu}$ is irrational for $\nu \neq 0$ and $h_{2\nu+1}$ is rational. Moreover, since (5) becomes

$$h_{2\nu+1} = \sum_{\mu=0}^{\nu} \lambda_{2\nu+1,2\mu+1} k^{2\mu+1} \tag{6}$$

where

$$\lambda_{2\nu+1,2\mu+1} = \frac{1}{2^{2\mu}}\sum_{i=\nu-\mu}^{\nu}\binom{2\nu+1}{2i}\binom{i}{\nu-\mu}$$

$$= \frac{1}{2^{2\mu}}\sum_{i=0}^{\mu}\binom{2\nu+1}{2\mu-2i+1}\binom{i+\nu-\mu}{i}, \tag{7}$$

it can be shown that $h_{2\nu+1}$ is a positive integer.

First of all, we observe that the right-hand side of (7) is a polynomial in ν having:

- degree $2\mu + 1$,
- the coefficient of $\nu^{2\mu+1}$ equal to $2/(2\mu+1)!$
- the first $\mu + 1$ roots equal to $-\frac{1}{2}$ and $\nu_r = r$ $(r = 0, 1, \ldots, \mu - 1)$ because either $\binom{2\nu+1}{2\mu-2i+1}$ or $\binom{i+\nu-\mu}{i}$ vanishes for these values of ν and $0 \leqslant i \leqslant \mu$.

To find the remaining μ roots of the above polynomial, we utilize the identity

$$\lambda_{-(2\nu+1),2\mu+1} = -\lambda_{2\nu+1,2\mu+1} \tag{8}$$

derived from (3), (4), and (6). Setting $\nu = -\nu_r - 1$ into (7) and using (8), we have

$$\lambda_{2(-\nu_r-1)+1,2\mu+1} = \lambda_{-(2\nu_r+1),2\mu+1} = -\lambda_{2\nu_r+1,2\mu+1} = 0$$

because the ν_r's are roots of (7), and therefore also $-\nu_r - 1 = -r - 1$ $(r = 0, 1, \ldots, \mu - 1)$ are roots of (7).

On the basis of the previous observations, we have the result:

$$\lambda_{2\nu+1,2\mu+1} = \frac{2}{(2\mu+1)!}\left(\nu+\frac{1}{2}\right)\prod_{r=0}^{\mu-1}(\nu-r)(\nu+r+1)$$

$$= \frac{2\nu+1}{2\mu+1}\binom{\nu+\mu}{2\mu} = 2\binom{\nu+\mu}{2\mu+1} + \binom{\nu+\mu}{2\mu}. \tag{9}$$

Since (9) shows that the $\lambda_{2\nu+1,2\mu+1}$'s are positive integers, we conclude that the $h_{2\nu+1}$'s are positive integers as well. The values of $\lambda_{2\nu+1,2\mu+1}$, for $0 \leqslant \nu \leqslant 4$, are given in the table at the right.

$\nu \backslash \mu$	0	1	2	3	4
0	1				
1	3	1			
2	5	5	1		
3	7	14	7	1	
4	9	30	27	9	1

The second row of the table, together with (6) and (2) shows that

$$[\overline{k}]^3 = [\overline{3k + k^3}].$$

Reference: [1] I. M. Vinogradov. *Elements of Number Theory*. New York: Dover Publications, 1954.

Also solved by P. Bruckman, F. He, W. Janous, L. Kuipers, M. Wachtel, and the proposer.

Recurring Thoughts

H-372 *Proposed by M. Wachtel, Zurich, Switzerland (Vol. 22, no. 3, August 1984)*

There exist infinitely many sequences, each with infinitely many solutions of the form:

$$\begin{array}{l|lll}
\underline{A}\cdot x_1^2 + C = \underline{B}\cdot y_1^2 & \underline{A} = F_{n+3} & \underline{C} = L_n & \underline{B} = F_{n+1} \\
\underline{A}\cdot x_2^2 + C = \underline{B}\cdot y_2^2 & \underline{x_1} = 1 & & \underline{y_1} = 2 \\
\underline{A}\cdot x_3^2 + C = \underline{B}\cdot y_3^2 & & & \\
\cdots\cdots \quad \cdots \quad \cdots\cdots & \underline{x_2} = F_{n-1}F_n + F_{n+1}^2 & & \underline{y_2} = 2F_{n+1}^2 \\
\underline{A}\cdot x_m^2 + C = \underline{B}\cdot y_m^2 & \underline{x_3} = 2F_{2n+4} + (-1)^n & & \underline{y_3} = 2F_{2n+5}
\end{array}$$

Find a recurrence formula for x_4/y_4, x_5/y_5, ..., x_m/y_m (y_m = dependent on x_m).

Examples: $(x_1 - x_3)$

$\underline{n = 3}$	(in numbers)
$\underline{F_6}\cdot(1)^2 + \underline{L_3} = \underline{F_4}\cdot(2)^2$	$8\cdot 1 + \underline{4} = \underline{3}\cdot 2^2$
$\underline{F_6}\cdot(F_2F_3 + F_4^2)^2 + \underline{L_3} = \underline{F_4}\cdot(2F_4^2)^2$	$\underline{8}\cdot 11^2 + \underline{4} = \underline{3}\cdot 18^2$
$\underline{F_6}\cdot(2F_{10} - 1)^2 + \underline{L_3} = \underline{F_4}\cdot(2F_{11})^2$	$\underline{8}\cdot 109^2 + \underline{4} = \underline{3}\cdot 178^2$
$\underline{n = 4}$	
$\underline{F_7}\cdot(1)^2 + \underline{L_4} = \underline{F_5}\cdot(2)^2$	$\underline{13}\cdot 1 + \underline{7} = \underline{5}\cdot 2^2$
$\underline{F_7}\cdot(F_3F_4 + F_5^2)^2 + \underline{L_4} = \underline{F_5}\cdot(2F_5^2)^2$	$\underline{13}\cdot 31^2 + \underline{7} = \underline{5}\cdot 50^2$
$\underline{F_7}\cdot(2F_{12} + 1)^2 + \underline{L_4} = \underline{F_5}\cdot(2F_{13})^2$	$\underline{13}\cdot 289^2 + \underline{7} = \underline{5}\cdot 466^2$

Solution by Paul S. Bruckman, Fair Oaks, CA

Let

$$r_m = x_m/y_m, \ m \in \mathbb{Z}, \tag{1}$$

where (x_m, y_m) are the solutions (if any) of the equation:

$$F_{n+1}y^2 - F_{n+3}x^2 = L_n, \tag{2}$$

and n is a fixed nonnegative integer. This is a particular instance of the general equation:

$$ay^2 - bx^2 = c, \tag{3}$$

where a, b, and c are pairwise relatively prime positive integers, with a and b not both perfect squares.

By a *solution* of (3), we mean any ordered pair of integers (x, y) solving (3), but with $y > 0$. This allows trivial sign variations in the x-coordinate but not in the y-coordinate; the theory is much more elegant with this convention. We then write $(x, y) \in \mathcal{S}(a, b, c)$ iff (x, y) is a solution of (3).

We may infer from the theory of such equations that the solution set $\mathcal{S}(a, b, c)$ (if nonempty) is generated from the set $\mathcal{S}(1, ab, 1)$. More specifically, if $(x_m, y_m) \in \mathcal{S}(a, b, c)$ and $(p_m, q_m) \in \mathcal{S}(1, ab, 1)$, then

$$\begin{aligned} x_m &= ay_0 p_m + x_0 q_m, \\ y_m &= bx_0 p_m + y_0 q_m, \ m \in \mathbb{Z}, \text{ where } (x_0, y_0) \text{ is any solution of (3).} \end{aligned} \tag{4}$$

It is an easy exercise to show that the expressions given by (4) do, in fact, provide solutions of (3), given that $q_m^2 - abp_m^2 = 1$.

For our most specific case, we first show that

$$(1, 2) \in \mathcal{S}(F_{n+1}, F_{n+3}, L_n).$$

For $4F_{n+1} - F_{n+3} = 3F_{n+1} - F_{n+2} = 2F_{n+1} - F_n = F_{n+1} + F_{n-1} = L_n$. Thus, (1, 2) is a solution of (2); clearly, it is the minimal solution. We will find it convenient to choose $(x_0, y_0) = (1, 2)$. We then need to solve the auxiliary equation:

$$v^2 - F_{n+1}F_{n+3}u^2 = 1, \tag{5}$$

then substitute in (4) to obtain all solutions (x_m, y_m) of (2), with

$$x_0 = 1, \ y_0 = 2, \ a = F_{n+1}, \ b = F_{n+3}.$$

Note that $F_{n+1}F_{n+3} = F_{n+2}^2 + (-1)^n$. The general solutions of (5) are generated by expansion of powers of the quantities α and β defined by:

$$\alpha \equiv F_{n+2} + \sqrt{F_{n+1}F_{n+3}}, \quad \beta \equiv F_{n+2} - \sqrt{F_{n+1}F_{n+3}}. \tag{6}$$

Note that

$$\alpha\beta = (-1)^{n+1}, \ \alpha + \beta = 2F_{n+2}, \ \alpha - \beta = 2\sqrt{F_{n+1}F_{n+3}}.$$

If we make the following definitions:

$$u_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}, \quad v_m = \tfrac{1}{2}(\alpha^m + \beta^m), \ m \in \mathbb{Z}, \tag{7}$$

we see that $v_m^2 - F_{n+1}F_{n+3}u_m^2 = \frac{1}{4}(\alpha^m + \beta^m)^2 - \frac{1}{4}(\alpha^m - \beta^m)^2 = \frac{1}{2} \cdot 4(\alpha\beta)^m$, or

$$v_m^2 - F_{n+1}F_{n+3}u_m^2 = (-1)^{m(n+1)}. \tag{8}$$

From the definitions in (7), we may derive the following identities, which are indicated without proof:

$$u_{m+1}v_m - u_m v_{m+1} = (-1)^{(n+1)m}; \tag{9}$$

$$v_m v_{m+1} - F_{n+1}F_{n+3}u_m u_{m+1} = (-1)^{(n+1)m}F_{n+2}; \tag{10}$$

$$u_{m+2}v_m - u_m v_{m+2} = (-1)^{(n+1)m}2F_{n+2}; \tag{11}$$

$$v_m v_{m+2} - F_{n+1}F_{n+3}u_m u_{m+2} = (-1)^{(n+1)m}(2F_{n+2}^2 - 1). \tag{12}$$

We see from (8) that if n is odd, $(u_m, v_m) \in \mathcal{S}(1, F_{n+1}F_{n+3}, 1)$. Setting $a = F_{n+1}$, $b = F_{n+3}$, $x_0 = 1$, $y_0 = 2$, $p_m = u_m$, $q_m = v_m$ in (4), we thus obtain the explicit solutions of (1), if n is odd:

$$x_m = 2F_{n+1}u_m + v_m, \ y_m = F_{n+3}u_m + 2v_m, \ m \in \mathbb{Z}. \tag{13}$$

If n is even, we see from (8) that $(u_m, v_m) \in \mathcal{S}(1, F_{n+1}F_{n+3}, 1)$ iff m is even. Hence, we make the same substitutions in (4) as for the case where n is odd, except that now we set $p_m = u_{2m}$, $q_m = v_{2m}$. Thus, the general solutions of (1) if n is even are as follows:

$$x_m = 2F_{n+1}u_{2m} + v_{2m}, \ y_m = F_{n+3}u_{2m} + 2v_{2m}, \ m \in \mathbb{Z}. \tag{14}$$

Next, we derive a pair of useful relations involving successive values of (x_m, y_m):

$$x_{m+1}y_m - x_m y_{m+1} = \begin{cases} L_n, & n \text{ odd}, \\ 2L_n F_{n+2}, & n \text{ even}; \end{cases} \tag{15}$$

$$F_{n+1}y_m y_{m+1} - F_{n+3}x_m x_{m+1} = \begin{cases} L_n F_{n+2}, & n \text{ odd}, \\ L_n(2F_{n+2} + 1), & n \text{ even}. \end{cases} \tag{16}$$

For brevity, we write $u = u_m$ or u_{2m}, $u' = u_{m+1}$ or u_{2m+2}, depending on whether n is odd or even, respectively, with similar notation for v and v'.

Substituting the expressions in (13) or (14) into (15) and using (9) or (11), as appropriate, the left member of (15) becomes, after simplification:

$$(u'v - uv')(4F_{n+1} - F_{n+3}) = \begin{cases} (-1)^{(n+1)m}L_n = L_n, & n \text{ odd}; \\ (-1)^{(n+1)2m}2F_{n+2}L_n = 2L_nF_{n+2}, & n \text{ even}. \end{cases}$$

Likewise, substituting the expressions in (13) or (14) into (16) and using (10) or (12), as appropriate, the left member of (16) becomes, after simplification:

$$(vv' - F_{n+1}F_{n+3}uu')(4F_{n+1} - F_{n+3}) = \begin{cases} (-1)^{(n+1)m}F_{n+2}L_n = L_nF_{n+2}, & n \text{ odd}; \\ (-1)^{(n+1)2m}(2F_{n+2}^2 - 1)L_n = L_n(2F_{n+2}^2 - 1), & n \text{ even}. \end{cases}$$

This completes the proof of (15) and (16).

Using (15) and (16), we may now derive the desired recursion for the r_m's. Dividing (15) and (16) throughout by $y_m y_{m+1}$, we obtain:

$$r_{m+1} - r_m = A/y_m y_{m+1}, \quad F_{n+1} - F_{n+3}r_{m+1}r_m = B/y_m y_{m+1},$$

where

$$A = \begin{cases} L_n, & n \text{ odd}; \\ 2L_nF_{n+2}, & n \text{ even}; \end{cases} \qquad B = \begin{cases} L_nF_{n+2}, & n \text{ odd}; \\ L_n(2F_{n+2}^2 + 1), & n \text{ even}. \end{cases}$$

Thus,

$$F_{n+1} - F_{n+3}r_{m+1}r_m = (B/A)(r_{m+1} - r_m).$$

Solving for r_{m+1}, we find:

$$r_{m+1} = \frac{Br_m + AF_{n+1}}{AF_{n+3}r_m + B}. \tag{17}$$

In terms of the functions of n, we find that each of the terms in the fraction in (17) contains the constant term L_n, which may be cancelled. Hence, in simplest terms, we obtain the two expressions:

$$r_{m+1} = \begin{cases} \dfrac{F_{n+2}r_m + F_{n+1}}{F_{n+3}r_m + F_{n+2}}, & n \text{ odd}; \\[2ex] \dfrac{(2F_{n+2}^2 + 1)r_m + 2F_{n+1}F_{n+2}}{2F_{n+2}F_{n+3}r_m + 2F_{n+2}^2 + 1}, & n \text{ even}. \end{cases} \tag{18}$$

We may also solve for r_m in terms of r_{m+1}, thus obtaining:

$$\begin{aligned} r_m &= \frac{F_{n+2}r_{m+1} - F_{n+1}}{-F_{n+3}r_{m+1} + F_{n+2}}, \quad n \text{ odd;} \\ r_m &= \frac{(2F_{n+2}^2 + 1)r_{m+1} - 2F_{n+1}F_{n+2}}{-2F_{n+2}F_{n+3}r_m + 2F_{n+2}^2 + 1}, \quad n \text{ even.} \end{aligned} \tag{19}$$

Using (18) and (19), we may extend the sequence

$$(r_m)_{-\infty}^{\infty}$$

in either direction, given $r_0 = \frac{1}{2}$.

Also solved by the proposer

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