THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION


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# The Fibonacci Quarterly 

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# ON SOME POLYGONAL NUMBERS Which ARE, at the same time, <br> THE SUMS, DIFFERENCES, AND PRODUCTS OF TWO OTHER POLYGONAL NUMBERS 

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(Submitted December 1981)

We denote the $n$th $g$-gonal number by

$$
P_{n, g}=n\{(g-2) n-(g-4)\} / 2 .
$$

For $g=3,5,6$, and 8 , we denote $P_{n, g}$ by $T_{n}$, the triangular numbers, $P_{n}^{\prime}$, the pentagonal numbers, $H_{n}$, the hexagonal numbers, and $O_{n}$, the octagonal numbers, respectively. We denote $P_{n, g}$ by $P_{n}$ whenever there is no danger of confusion.

Sierpiński [18] has proved that "there exist an infinite number of triangular numbers which are, at the same time, the sums, differences and products of two other triangular numbers>1." Ando [1] proved that "there exist an infinite number of $g$-gonal numbers that can be expressed as the sum and difference of two other $g$-gonal numbers at the same time." It was also shown in [6] that there are an infinite number of $g$-gonal numbers that can be expressed as the product of two other $g$-gonal numbers.

The present paper will show that there are infinitely many $g$-gonal numbers ( $g=5,6$, and 8 ) which are at the same time the sums, differences, and products of two other $g$-gonal numbers.

## 1. THE EQUATION $P_{u+w}+P_{v+w}=P_{u+v+w}$

If $P_{x}+P_{y}=P_{z}$, by putting $u=z-y, v=z-x$, and $w=x+y-z$, we have $x=u+w, y=v+w$, and $z=u+v+w$. However, a little algebra shows that $P_{u+w}+P_{v+w}=P_{u+v+w}$ implies $2(g-2) u v=(g-2) w(w-1)+2 w$. Hence

Theorem 1: Any solution $x, y, z$ of the equation $P_{x}+P_{y}=P_{z}$ can be expressed as $x=u+w, y=v+w, z=u+v+w$, where

$$
w \equiv 0(\bmod g-2)
$$

and

$$
u v=\left\{(g-2) w^{2}-(g-4) w\right\} / 2(g-2)
$$

Using this theorem, which is a generalization of the work of Fauquembergue [7] and of Shah [15] on triangular numbers, we can obtain the solutions of the equation $P_{x}+P_{y}=P_{z}$ in an efficient way. For example, we have the following table for $g=5$.

Table 1. $P_{x}^{\prime}+P_{y}^{\prime}=P_{z}^{\prime}(w \leqslant 9, u \leqslant v)$

| $w$ | $\left(3 w^{2}-w\right) / 6$ | $u$ | $v$ | $x$ | $y$ | $z$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 4 | 2 | 2 | 5 | 5 | 7 |
|  |  | 1 | 4 | 4 | 7 | 8 |
| 6 | 17 | 1 | 17 | 7 | 23 | 24 |
| 9 | 39 | 3 | 13 | 12 | 22 | 25 |
| 1 | 39 | 10 | 48 | 49 |  |  |

If we put $v+w=w^{\prime}$ in $P_{1+v+w}=P_{1+w}+P_{v+w}$ and $P_{1+w^{\prime}}=P_{1+v^{\prime}+w^{\prime}}-P_{v^{\prime}+w^{\prime}}$, then we obtain $g$-gonal numbers that can be expressed as the sum and difference of two other $g$-gonal numbers at the same time.

Corollary: If $w \equiv 0\left(\bmod (g-2)^{2}\right)$ and $v=\left\{(g-2) w^{2}-(g-4) w\right\} / 2(g-2)$, then we have

$$
\begin{aligned}
P_{v+w+1} & =P_{w+1}+P_{v+w}=P_{a}-P_{b}, \text { where } \\
a & =\left\{(g-2)(v+w)^{2}-(g-4)(v+w)\right\} / 2(g-2)+v+w+1
\end{aligned}
$$

and

$$
b=\left\{(g-2)(v+w)^{2}-(g-4)(v+w)\right\} / 2(g-2)+v+w
$$

Putting $w=x-1$ for $g=3$, we obtain a result of Sierpiński [18]; putting $w=9 n$ for $g=5, w=16 n$ for $g=6, w=25 k$ for $g=7$, and $w=36 n$ for $g=8$, we obtain the results of Hansen [9], 0'Donne11 [13], Hindin [10], and 0'Donnell [14], respectively.

## 2. THE EQUATION $P_{a t-d}+P_{b t-e}=P_{c t-f}$

In this section we study somewhat more general second-degree sequences than $P_{n}$, and obtain necessary and sufficient conditions for certain infinite families of representations to exist. We then specialize to polygonal numbers. To this end, let $F(\alpha, \beta ; n)=n(\alpha n-\beta)$, where $\alpha, \beta$ are integers with $(\alpha, \beta)=1$ and $\alpha>0$.

Theorem 2: Let $a, b, c, d, e$, and $f$ be integers with $a, b$, and $c$ positive and $(a, b, c)=1$. A necessary and sufficient condition for the identity in $t$,

$$
F(\alpha, \beta ; a t-d)+F(\alpha, \beta ; b t-e)=F(\alpha, \beta ; c t-f),
$$

to hold is that there exist integers $p, q, r$, and $s$ that satisfy equations (0) and (I), or (0) and (II):

$$
\left\{\begin{array}{l}
a=(p+q)(p-q), b=2 p q, c=p^{2}+q^{2}  \tag{0}\\
(p, q)=1, p>q>0, p+q \equiv 1(\bmod 2)
\end{array}\right.
$$

$$
\begin{align*}
& \left\{\begin{array}{l}
d=(p+q) r, e=\frac{q}{\alpha} s, f=(p-q) r+\frac{q}{\alpha} s, \\
q \equiv 0(\bmod \alpha), 2 \alpha p r-(p-q) s=-\beta
\end{array}\right.  \tag{I}\\
& \left\{\begin{array}{l}
d=\frac{p-q}{\alpha} r, e=p s, f=\frac{p-q}{\alpha} r+q s \\
p \equiv q(\bmod \alpha), 2 q r-\alpha(p+q) s=\beta .
\end{array}\right. \tag{II}
\end{align*}
$$

Proof: In order for the desired identity in $t$,

$$
(a t-d)(\alpha a t-\alpha d-\beta)+(b t-e)(\alpha b t-\alpha e-\beta)=(c t-f)(\alpha c t-\alpha f-\beta)
$$

to hold, it is necessary and sufficient that the equations

$$
\begin{align*}
& a^{2}+b^{2}=c^{2}  \tag{1}\\
& (2 \alpha d+\beta) a+(2 \alpha e+\beta) b=(2 \alpha f+\beta) c  \tag{2}\\
& (\alpha d+\beta) d+(\alpha e+\beta) e=(\alpha f+\beta) f \tag{3}
\end{align*}
$$

be valid.
From (2),

$$
\begin{equation*}
c f=a d+b e+\frac{\beta(a+b-c)}{2 \alpha}, \tag{4}
\end{equation*}
$$

and from (1), (3), and (4), we obtain

$$
\begin{aligned}
\left(\alpha^{2}\right. & \left.+b^{2}\right)\{(\alpha d+\beta) d+(\alpha e+\beta) e\} \\
& =c^{2}(\alpha f+\beta) f \\
& =\alpha(c f)^{2}+\beta c(c f) \\
& =\alpha\left\{a d+b e+\frac{\beta(a+b-c)}{2 \alpha}\right\}^{2}+\beta c\left\{a d+b e+\frac{\beta(\alpha+b-c)}{2 \alpha}\right\}
\end{aligned}
$$

Expanding and transforming the above, we have

$$
\alpha(b d-a e)^{2}-\beta(a-b)(b d-a e)-\frac{\beta^{2} a b}{2 \alpha}=0
$$

Hence,

$$
\left\{\begin{array}{l}
(a) \quad b d-a e=\frac{\beta(a-b-c)}{2 \alpha}, \text { or }  \tag{5}\\
(b) \quad b d-a e=\frac{\beta(a-b+c)}{2 \alpha} .
\end{array}\right.
$$

Now, for positive integers $a, b$, and $c$ with $(a, b, c)=1$ and $b$ even, the solutions of (1) are given by
(0)

$$
\left\{\begin{array}{l}
a=(p+q)(p-q), b=2 p q, c=p^{2}+q^{2}, \text { where } \\
p \text { and } q \text { are positive integral parameters with } \\
(p, q)=1, p>q>0, \text { and } p+q \equiv 1(\bmod 2) .
\end{array}\right.
$$

Equations (6) and (7) below are necessary for (4) and (5) to hold.

$$
\begin{equation*}
\beta(a+b-c)=2 \beta q(p-q) \equiv 0(\bmod 2 \alpha) \tag{6}
\end{equation*}
$$

$$
\left\{\begin{array}{l}
(\mathrm{a}) \quad \beta(a-b-c)=-2 \beta q(p+q) \equiv 0(\bmod 2 \alpha), \text { or }  \tag{7}\\
(\mathrm{b}) \quad \beta(a-b+c)=2 \beta p(p-q) \equiv 0(\bmod 2 \alpha) .
\end{array}\right.
$$

Since $(\alpha, \beta)=1$ and $(p, q)=1,(6)$ and (7) hold only if

$$
\left\{\begin{array}{l}
(\mathrm{a})  \tag{8}\\
(\mathrm{b}) \\
p \equiv 0(\bmod \alpha), \text { or } \\
(\bmod \alpha) .
\end{array}\right.
$$

(I) If $q \equiv 0(\bmod \alpha),(5)(a)$ becomes

$$
2 p q d-(p+q)(p-q) e=-\beta \frac{q}{\alpha}(p+q)
$$

so that we have

$$
2 \alpha p r-(p-q) s=-\beta, \text { where } d=(p+q) r \text { and } e=\frac{q}{\alpha} s
$$

Substituting this into (4), we have $f=(p-q) r+\frac{q}{\alpha} s$.
(II) If $p \equiv q(\bmod \alpha)$, (5) (b) becomes

$$
2 p q d-(p+q)(p-q) e=\beta p \cdot \frac{p-q}{\alpha}
$$

so that we have

$$
2 q r-\alpha(p+q) s=\beta, \text { where } d=\frac{p-q}{\alpha} r \text { and } e=p s
$$

Substituting this into (4), we have $f=\frac{p-q}{\alpha} p+q s$. Thus, we have the equivalence relation

$$
(1) \cdot(2) \cdot(3) \Leftrightarrow(0) \cdot(4) \cdot(5) \Leftrightarrow(0) \cdot(I) \text { or }(0) \cdot(I I)
$$

which proves Theorem 2.
Corollary: Solutions of $P_{x}+P_{y}=P_{z}$ are obtained by $x=a t-d, y=b t-e$, $z=c t-f$. We use Theorem 2 by putting

$$
\begin{aligned}
& P_{n, g}=\frac{1}{2} F(g-2, g-4 ; n) \text { for } g \text { odd, and } \\
& P_{n, g}=F\left(\frac{g-2}{2}, \frac{g-4}{2} ; n\right) \text { for } g(\neq 4) \text { even. }
\end{aligned}
$$

In the case $g=4$, we obtain $a, b$, and $c$ from Theorem 2 (0) by putting $d=e=$ $f=0$.

Example: If $g=5$, then $\alpha=3, \beta=1$. Since $q \equiv 0(\bmod 3)$, or $p \equiv q(\bmod 3)$, and $(p, q)=1, p>q>0, p+q \equiv 1(\bmod 2)$, we have

$$
\begin{aligned}
q & =1 ; p=4,10,16, \ldots \\
q & =2 ; p=5,11,17, \ldots \\
q & =3 ; p=4,8,10,14,16, \ldots
\end{aligned}
$$

When $p=4, q=1,2 q r-\alpha(p+q) s=\beta$ becomes $2 r-15 s=1$, where one solution is $r=8, s=1$. Using these values in (0) • (II), we obtain

$$
a=15, b=8, c=17, d=8, e=4, f=9
$$

and

$$
P_{15 t-8}^{\prime}+P_{8 t-4}^{\prime}=P_{17 t-9}^{\prime}
$$

Changing $t$ into $8 t-3$ and $17 t-7$, we have

$$
P_{136 t-60}^{\prime}=P_{120 t-53}^{\prime}+P_{64 t-28}^{\prime}=P_{289 t-128}^{\prime}-P_{255 t-113}^{\prime}
$$

Table 2. $P_{a t-d}^{\prime}+P_{b t-e}^{\prime}=P_{c t-f}^{\prime}, P_{x}^{\prime}+P_{y}^{\prime}=P_{z}^{\prime} \quad(z \leqslant 30)$

|  | $p$ | $q$ | $r$ | $s$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $t$ | $x$ | $y$ | $z$ |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| (II) | 4 | 1 | 8 | 1 | 15 | 8 | 17 | 8 | 4 | 9 | 1 | 7 | 4 | 8 |
| 22 | 12 | 25 |  |  |  |  |  |  |  |  |  |  |  |  |
| (I) | 4 | 3 | 0 | 1 | 7 | 24 | 25 | 0 | 1 | 1 | 1 | 7 | 23 | 24 |
| (II) | 5 | 2 | 16 | 3 | 21 | 20 | 29 | 16 | 15 | 22 | 1 | 5 | 5 | 7 |
| (I) | 8 | 3 | 3 | 29 | 55 | 48 | 73 | 33 | 29 | 44 | 1 | 22 | 19 | 29 |

Table 3. Correspondence of the Solutions of $P_{x}+P_{y}=P_{z}$ in [1]
Ex. 1

| $g$ | Parity | Case | $p$ | $q$ | $r$ | $s$ | $t$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k:$ even |  | (I) | $\frac{(k-2)^{2}}{2} t+1$ | $\frac{(k-2)^{2}}{2} t$ | 0 | $\frac{k-4}{2}$ | 1 |
| $k:$ odd | $t:$ even | (I) | $\frac{(k-2)^{2}}{2} t+1$ | $\frac{(k-2)^{2}}{2} t$ | 0 | $k-4$ | 1 |
|  | $t:$ odd | (II) | $(k-2)^{2} t+1$ | 1 | $\frac{(k-2)^{3} t+(3 k-8)}{2}$ | 1 | 1 |

3. THE EQUATIONS $P_{z}=P_{x}+P_{y}=P_{u}-P_{v}=P_{r} P_{s}$

For $g \neq 4$, if $(g-2) P_{n}-(g-4)=2 P_{m}$, we conjecture that $P_{P_{n}}=P_{n} P_{m}$ can be expressed as the sum and difference of two other $g$-gonal numbers. But we cannot prove this. However, we have

Theorem 3: There exist an infinite number of hexagonal numbers that can be expressed as the sum-difference-product of two other hexagonal numbers.

Proof: If we assume $H_{n}=H_{3} H_{m}$, then we have $(4 n-1)^{2}-15(4 m-1)^{2}=-14$. By putting $N=4 n-1, M=4 m-1$, we get $N^{2}-15 M^{2}=-14$. Its complete solution is given by the formulas
(i) $N_{i}+\sqrt{15} M_{i}= \pm(1+\sqrt{15})(4+\sqrt{15})^{i}$
and
(ii) $N_{i}+\sqrt{15} M_{i}= \pm(-1+\sqrt{15})(4+\sqrt{15})^{i}$,
where $i=0, \pm 1, \pm 2, \pm 3, \ldots$.
In (i), if $N_{i}+\sqrt{15} M_{i}>0, i>0$, and $i \equiv 2(\bmod 4)$, then $N_{i} \equiv M_{i} \equiv-1(\bmod$ 4). $\quad N_{i}$ satisfies a recurrence relation

$$
N_{i+2}=8 N_{i+1}-N_{i},
$$

which leads to $N_{i+4}=62 N_{i+2}-N_{i}$. Also, by repetition, $N_{i+8}=3842 N_{i+4}-N_{i}$. From $4 n_{i+8}-1=3842\left(4 n_{i+4}-1\right)-\left(4 n_{i}-1\right)$, it follows that $n_{i+8}=3842 n_{i+4}-$ $n_{i}$ - 960. Changing $4 i-2$ into $i$, it becomes
$n_{i+2}=3842 n_{i+1}-n_{i}-960$,
with initial values $n_{1}=38, n_{2}=145058$. Similarly, we get

$$
m_{i+2}=3842 m_{i+1}-m_{i}-960,
$$

with initial values $m_{1}=10, m_{2}=37454$.
For all $i$, we have

$$
\begin{aligned}
H_{n_{i}}=H_{3} H_{m_{i}} & =15 m_{i}\left(2 m_{i}-1\right) \\
& =\left(4 m_{i}-1\right)(8 m-3)-\left(m_{i}-1\right)\left(2 m_{i}-3\right) \\
& =H_{4 m_{i}-1}-H_{m_{i}-1} .
\end{aligned}
$$

For $i \equiv 1(\bmod 7)$, we have $n_{i} \equiv-1(\bmod 13)$. On taking $t=\left(n_{i}+1\right) / 13$ in

$$
H_{13 t-1}=H_{5 t}+H_{12 t-1},
$$

we get

$$
H_{n_{i}}=H_{\left(5 n_{i}+5\right) / 13}+H_{\left(12 n_{i}-1\right) / 13} .
$$

Thus, for $i \equiv 1(\bmod 7), H_{n_{i}}$ is expressed as the sum-difference-product of two other hexagonal numbers. If we put $i=1$, then we have

$$
H_{38}=H_{15}+H_{35}=H_{39}-H_{9}=H_{3} H_{10} .
$$

In a similar way, we obtain
Theorem 4: For $g=5$ and 8, there exist an infinite number of $g$-gonal numbers that can be expressed as the sum-difference-product of two other $g$-gonal numbers.

Proof: If we put

$$
\left.\begin{array}{l}
n_{1}=4, n_{2}=600912, n_{i+2}=155234 n_{i+1}-n_{i}-25872 \\
m_{1}=1, m_{2}=128115, m_{i+2}=155234 m_{i+1}-m_{i}-25872
\end{array}\right\} i=1,2,3, \ldots,
$$

then, for $i \equiv 9(\bmod 14)$, we have $n_{i} \equiv 7(\bmod 29)$ and $m_{i} \equiv 1(\bmod 2)$, so that

## ON SOME POLYGONAL NUMBERS

$$
\begin{aligned}
P_{n_{i}}^{\prime}=P_{\left(21 n_{i}-2\right) / 29}^{\prime}+P_{\left(20 n_{i}+5\right) / 29}^{\prime} & =P_{\left(23 m_{i}-7\right) / 2}^{\prime}-P_{\left(21 m_{i}-7\right) / 2}^{\prime} \\
& =P_{4}^{\prime} P_{m_{i}}^{\prime} .
\end{aligned}
$$

Also, if we put

$$
\left.\begin{array}{l}
n_{1}=304, n_{2}=1345421055984, \\
n_{i+2}=4430499842 n_{i+1}-n_{i}-1476833280 \\
m_{1}=38, m_{2}=166878943590, \\
m_{i+2}=4430499842 m_{i+1}-m_{i}-1476833280
\end{array}\right\} \quad i=1,2,3, \ldots,
$$

then, for $i \equiv 0,1(\bmod 7)$, we have $n \equiv 14(\bmod 29)$, so that

$$
O_{n_{i}}=O_{\left(21 n_{i}-4\right) / 29}+O_{\left(20 n_{i}+10\right) / 29}=O_{9 m_{i}-4}-O_{4 m_{i}-4}=O_{5} O_{m_{i}}
$$

Here, if we put $i=1$, then we have

$$
O_{304}=O_{220}+O_{210}=O_{338}-O_{148}=O_{5} O_{38}
$$

## ACKNOWLEDGMENT

I am grateful to Professor Koichi Yamamoto of Tokyo Woman's Christian University for his valuable comments on this manuscript.

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## 

## LETTER TO THE EDITOR

$$
19 \text { December } 1985
$$

## Dear Editor:

Before the publication of my article, "Generators of Unitary Amicable Numbers," in the May 1985 issue of The Fibonacci Quarterly, Dr. H. J. J. te Riele and $I$ exchanged letters concerning unitary amicable numbers. He pointed out that his report, NW 2/78, published by the Matematisch Centrum in Amsterdam (with which he is affiliated), contains many of the results in my paper, albeit from a slightly different point of view. Both references to these letters and to report NW $2 / 78$ were inadvertently omitted from my article.

The Centrum's address:
Stichting Matematisch Centrum
Kruislaan 4131098 SJ Amsterdam
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The Netherlands

> Sincerely yours,
O. William McClung

# SOME RESULTS CONCERNING PYTHAGOREAN TRIPLETS 

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(Submitted December 1983; revised August 1984)

## I. INTRODUCTION

In connection with the discussion in my earlier paper [1] entitled: "A Corollary to Iterated Exponentiation," in which I have presented a new conjecture concerning Fermat's Last Theorem, it occurred to me that it is of interest to make a systematic study of the sets of three integers $x, y$, $z$ which satisfy the condition

$$
\begin{equation*}
x^{2}+y^{2}=z^{2} \tag{1}
\end{equation*}
$$

Such a triplet of integers ( $x, y, z$ ) is commonly referred to as a "Pythagorean triplet," for which we shall also use the abbreviation $P$-triplet.

The actual motivation of the present work is to explore as thoroughly as possible the two cases, $n=1$ and $n=2$, for which the Diophantine equation of Fermat has solutions, namely,

$$
\begin{equation*}
x^{n}+y^{n}=z^{n} \quad(n=1,2) \tag{2}
\end{equation*}
$$

This interest is, in turn, derived from my earlier conjecture [1] that because $n=1$ and $n=2$ are the only two positive integers that are smaller than $e$, (2) holds only for $n=1$ and $n=2$ when $x, y$, and $z$ are restricted to being positive integers. Most of the discussion in the present paper will be devoted to the case in which $n=2$.

## II. PYTHAGOREAN DECOMPOSITIONS

By using a computer program devised by M. Creutz, we were able to determine all Pythagoeran triplets for which $z \leqslant 300$. At this point, a distinction must must be made between $P$-triplets for which $x, y$, and $z$ have no common divisor [the so-called "primitive solutions" of (1)] and P-triplets which are related to the primitive solutions by multiplication by a common integer factor $k$. So, if $x_{i}, y_{i}, z_{i}$ are rela'tively prime and obey (1), it is obvious that the derived triplet ( $k x_{i}, k y_{i}, k z_{i}$ ) will also satisfy (1).

The original computer program was therefore modified to print out only the primitive solutions, and was extended up to $z \leqslant 3000$. To anticipate one of my results, the number of primitive solutions in any interval of 100 in $z$ is approximately constant and equal to $\approx 16$. Thus there are 80 primitive solutions (PS) between $z=1$ and 500, and 477 PS in the entire interval $1 \leqslant z \leqslant 3000$. We

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## SOME RESULTS CONCERNING PYTHAGOREAN TRIPLETS

will make the convention to denote by $x$ the larger of the two numbers in the left-hand side of (1), i.e., $x>y$.

In Table 1, I have tabulated all primitive solutions for $1 \leqslant z \leqslant 500$. The triplets are presented in the order $x_{i}, y_{i}, z_{i}$. When a value of $z_{i}$ is underlined, this indicates that it is not prime. The nonunderlined $z_{i}$ values are primes which we will call "Pythagorean primes" or P-primes. In this work, and also for the region $501 \leqslant z \leqslant 2000$, the tables of primes and prime factors given in the Handbook of Chemistry and Physics [2] were essential.

When the $z_{i}$ of the primitive solution is not a prime, I have underlined it, and the underlined number is usually followed by a subscript 1 or 2 , which has the following significance. Already in the work for $z \leqslant 300$ (with all triplets listed), I have noticed the following rule: If $z_{p, i}$ and $z_{p, j}$ belong to two different primitive solutions, the product

$$
\begin{equation*}
z_{p, k}=z_{p, i} z_{p, j} \tag{3}
\end{equation*}
$$

belongs to two new primitive solutions, namely,

$$
\begin{equation*}
\left(x_{1, k}, y_{1, k}, z_{p, k}\right) \quad \text { and } \quad\left(x_{2, k}, y_{2, k}, z_{p, k}\right) \tag{4}
\end{equation*}
$$

These two new $P$ decompositions are relatively prime and are also prime with respect to the expected decomposition obtained by taking the product of $z_{p, j}$ with the decomposition $\left(x_{p}, i, y_{p, i}, z_{p, i}\right)$ and that obtained by taking the product of $z_{p, i}$ with the decomposition $\left(x_{p, j}, y_{p, j}, z_{p, j}\right)$. Thus, there are four linearly independent $P$ decompositions for the number $z_{p, k}$ of (3). To take an example, according to Table 1 , the number 65 has the decompositions ( 56,33 , 65 ) and $(63,16,65)$, and, in addition, $(52,39,65)$ and $(60,25,65)$ obtained from (4, 3, 5) and (12, 5, 13), respectively.

This rule is satisfied in all decompositions of products $z_{p, i} z_{p, j}$ provided that the prime factors of $z_{p, i}$ and $z_{p, j}$ are different. On the other hand, if $z_{p, i}$ and $z_{p, j}$ are merely powers of the same prime $p_{i}$, then there will be just one additional linearly independent Pythagorean decomposition for

$$
\begin{equation*}
z_{p, k}=p_{i}^{\alpha_{i}} p_{i}^{\alpha_{i^{\prime}}}=p_{i}^{\alpha_{i}+\alpha_{i^{\prime}}} . \tag{5}
\end{equation*}
$$

As an example, the number $25=5^{2}$ has one additional $P$ decomposition, namely, ( $24,7,25$ ) besides that derived from (4, 3, 5), namely, (20, 15, 25). Similarly, the number $125=5^{3}$ has one new $P$ decomposition, namely, (117, 44, 125) in addition to the two decompositions derived from the $P$ decompositions for 5 and 25 , namely, $(100,75,125)$ and $(120,35,125)$, respectively.

We may notice that the square $5^{2}=25$ has two $P$ decompositions and the cube $5^{3}=125$ has three $P$ decompositions. Thus, in general, a power $p_{i}^{\alpha_{i}}$ will have $\alpha_{i}$ Pythagorean decompositions, where $p_{i}$ is a Pythagorean prime (such as 5, 13, 17, etc.). In Table 1, I have indicated the factors $z_{p, i}$ and $z_{p, j}$ which give rise to the new double primitive solution, when $z_{p, k}$ is a product of two different $z_{p, i}$ and $z_{p, j}$ which are relatively prime to each other. When a single power $p_{i}^{\alpha_{i}}$ is involved, this has also been noted, e.g., $13^{2}=169$ has the new $P$ decomposition (120, 119, 169), in addition to the one expected from (12, 5, 13), namely, (156, 65, 169).

The total number of primitive solutions in the successive intervals of 100 in Table 1 are: 16 from 1 to 100,16 from 101 to 200,15 from 201 to 300 , 16 from 301 to 400 , and 17 from 401 to 500, giving a total of

$$
\begin{equation*}
\sum n_{p}=16+16+15+16+17=80 \tag{6}
\end{equation*}
$$

Table 1. Listing of the Pythagorean primitive decompositions for the integers in the range $1 \leqslant N \leqslant 500$. The values of $z$ which are not prime numbers are underlined, and the subscripts 1 and 2 indicate the two new primitive solutions associated with such numbers. An exception occurs when the number $N_{i}$ is a power of a single $P$-prime number, $p_{i}^{\alpha_{i}}$, in which case only one new primitive solution arises. For the numbers which are underlined (non-primes), the prime decomposition is indicated.

| $\nu_{i}$ | $x_{i}, y_{i}, z_{i}$ | $v_{i}$ | $x_{i}, y_{i}, z_{i}$ | $\nu_{i}$ | $x_{i}, y_{i}, z_{i}$ | $v_{i}$ | $x_{i}, y_{i}, z_{i}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 4,3,5 | 21 | 105,88,137 | 41 | $247,96, \underline{265} 1=5 \times 53$ | 61 | $352,135, \underline{377}{ }_{2}=13 \times 29$ |
| 2 | 12,5,13 | 22 | $143,24,145_{1}=5 \times 29$ | 42 | $264,23, \underline{265} 2=5 \times 53$ | 62 | 340,189,389 |
| 3 | 15,8,17 | 23 | $144,17, \underline{145}_{2}=5 \times 29$ | 43 | 260,69,269 | 63 | 325,228,397 |
| 4 | $24,7, \underline{25}=5^{2}$ | 24 | 140,51,149 | 44 | 252,115,277 | 64 | 399,40,401 |
| 5 | 21,20,29 | 25 | 132,85,157 | 45 | 231,160,281 | 65 | 391,120,409 |
| 6 | 35,12,37 | 26 | $120,119,169=13^{2}$ | 46 | $240,161, \underline{289}=17^{2}$ | 66 | 420,29,421 |
| 7 | 40,9,41 | 27 | 165,52,173 | 47 | 285,68,293 | 67 | 304,297, $\underline{-25}_{1}=5 \times 85$ |
| 8 | 45,28,53 | 28 | 180,19,181 | 48 | $224,207, \underline{305}_{1}=5 \times 61$ | 68 | $416,87, \underline{425} 2=5 \times 85$ |
| 9 | 60,11,61 | 29 | $153,104,185_{1}=5 \times 37$ | 49 | $273,136, \underline{305}_{2}=5 \times 61$ | 69 | 408,145,433 |
| 10 | 56,33, $\underline{65}_{1}=5 \times 13$ | 30 | $176,57, \underline{185}_{2}=5 \times 37$ | 50 | 312,25,313 | 70 | 396,203, $\underline{445}_{1}=5 \times 89$ |
| 11 | $63,16, \underline{65}_{2}=5 \times 13$ | 31 | 168,95,193 | 51 | 308,75,317 | 71 | $437,84, \underline{445} 2=5 \times 89$ |
| 12 | 55,48,73 | 32 | 195,28,197 | 52 | 253, $204, \underline{-325}_{1}=5 \times 65$ | 72 | 351,280,449 |
| 13 | $77,36, \underline{85}_{1}=5 \times 17$ | 33 | $156,133, \underline{205}_{1}=5 \times 41$ | 53 | 323, $36, \underline{325}_{2}=5 \times 65$ | 73 | 425,168,457 |
| 14 | $84,13, \underline{85}_{2}=5 \times 17$ | 34 | $187,84, \underline{\underline{205}} 2=5 \times 41$ | 54 | 288,175,337 | 74 | 380,261,461 |
| 15 | 80,39,89 | 35 | $171,140, \underline{221}_{1}=13 \times 17$ | 55 | 299,180,349 | 75 | 360,319, $\underline{481}_{1}=13 \times 37$ |
| 16 | 72,65,97 | 36 | $220,21,221_{2}=13 \times 17$ | 56 | 272,225,353 | 76 | $480,31, \underline{481}_{2}=13 \times 37$ |
| 17 | 99,20,101 | 37 | 221,60,229 | 57 | 357, $76, \underline{365} 1=5 \times 73$ | 77 | $476,93, \underline{485} 1=5 \times 97$ |
| 18 | 91,60,109 | 38 | 208,105,233 | 58 | $364,27,365_{2}=5 \times 73$ | 78 | $483,44,485{ }_{2}=5 \times 97$ |
| 19 | 112,15,113 | 39 | 209,120,241 | 59 | 275,252,373 | 79 | $468,155, \underline{493} 1=17 \times 29$ |
| 20 | $117,44,125=5^{3}$ | 40 | 255,32,257 | 60 | $345,152, \underline{377} 1=13 \times 29$ | 80 | $475,132, \underline{493} 2=17 \times 29$ |

In Table 1 the numbers $z_{i}$ that are not underlined are the primes for which a Pythagorean decomposition is possible. We will call them Pythagorean primes or $P$ primes. The other primes (which are not $P$-decomposable) will be called non-Pythagorean primes or $N P$ primes, e.g., 2, 3, 7, 11, 19, 23, 31, 43, and 47 are the $N P$ primes below $N=50$.

As mentioned above, all of the primitive solutions up to $N=3000$ have been obtained with the computer program. (The total running time on the CDC-7600 Computer was less than 30 seconds.) However, I have limited the main analysis to the numbers $N \leqslant 2000$.

In the discussion below, I will derive a general formula for the number $n_{d}$ of Pythagorean decompositions for an arbitrary integer.

In connection with the results of (3) and (4), it was noted and proved by M. Creutz [3] that when the triplets $\left(x_{1}, y_{1}, z_{1}\right)$ and $\left(x_{2}, y_{2}, z_{2}\right)$ are multiplied by each other, the additional primitive solutions mentioned in (4) have the following form:

$$
\begin{array}{ll}
X_{1}=x_{1} y_{2}+y_{1} x_{2}, & y_{1}=\left|x_{1} x_{2}-y_{1} y_{2}\right| ; \\
X_{2}=\left|x_{1} y_{2}-y_{1} x_{2}\right|, & Y_{2}=x_{1} x_{2}+y_{1} y_{2} .
\end{array}
$$

## SOME RESULTS CONCERNING PYTHAGOREAN TRIPLETS

Here we have omitted the subscript $p$ for simplicity of notation. To prove the validity of (7) and (8), we note that

$$
\begin{align*}
X_{1}^{2}+Y_{1}^{2} & =x_{1}^{2} y_{2}^{2}+y_{1}^{2} x_{2}^{2}+2 x_{1} x_{2} y_{1} y_{2}+x_{1}^{2} x_{2}^{2}+y_{1}^{2} y_{2}^{2}-2 x_{1} x_{2} y_{1} y_{2} \\
& =\left(x_{1}^{2}+y_{1}^{2}\right)\left(x_{2}^{2}+y_{2}^{2}\right)=z_{1}^{2} z_{2}^{2}=\left(z_{1} z_{2}\right)^{2}=z^{2} \tag{9}
\end{align*}
$$

thus verifying that $Z \equiv z_{1} z_{2}$ has the $P$ decomposition ( $X_{1}, Y_{1}, Z$ ). A similar equation is obtained by calculating $X_{2}^{2}+Y_{2}^{2}=z_{1}^{2} z_{2}^{2}=Z^{2}$, thus confirming the new $P$ triplet ( $X_{2}, Y_{2}, Z$ ).

As an example, for $Z=65$, we have $x_{1}=4, y_{1}=3, z_{1}=5$ and $x_{2}=12, y_{2}=5$, $z_{2}=13$, which gives $X_{1}=56, Y_{1}=33$, leading to the triplet (56, 33, 65) listed in Table 1. Furthermore, equations (8) give $X_{2}=16, Y_{2}=63$, which is equivalent to the second triplet, $(63,16,65)$, also listed in Table 1.

It is also obvious from (7) and (8) that if $x_{1}=x_{2}, y_{1}=y_{2}$, i.e., $z_{p, k}=$ $z_{p, i}^{2}$ in the notation of (3), then

$$
X_{1}=2 x_{1} y_{1}, \quad y_{1}=\left|x_{1}^{2}-y_{1}^{2}\right|
$$

which gives rise to only one new $P$ triplet, since for the other solution, $X_{2}=$ $0, Y_{2}=x_{1}^{2}+y_{1}^{2}=z_{p, i}^{2}=z_{p, k}$. For the case $x_{1}=x_{2}=4, y_{1}=y_{2}=3$, we have

$$
x_{1}=2 x_{1} y_{1}=24, \quad y_{1}=4^{2}-3^{2}=7,
$$

giving the one new triplet, $(24,7,25)$.
In Table 2, all the Pythagorean primes from $N=1$ to $N=2000$ are listed. Successive intervals of 100 are separated by semicolons.

Table 2. List of all Pythagorean primes for $1 \leqslant N \leqslant 2000$, i.e., primes which satisfy (1) where $x$ and $y$ are positive integers. Those primes which are underlined belong to a set of twin primes, i.e., primes $p_{i}$ and $p_{j}$ such that $\left|p_{i}-p_{j}\right|=2$. For each set of twin primes $p_{i}, p_{j}$, one and only one is a $P$-prime. The primes in successive intervals of 100 are separated by a semicolon.

5, 13, 17, 29, $37,41,53, \underline{61}, \underline{73}, 89,97 ; 101,109,113,137,149,157,173$, 181, 193, 197; 229, 233, 241, 257, 269, 277, 281, 293; 313, 317, 337, 349, $353,373,389,397 ; 401,409,421,433,449,457,461$;
$509,521,541,557,569,577,593 ; ~ 601,613, \underline{617}, \underline{641}, 653,661,673,677$;
$701,709,733,757,761,769,773,797 ; 809, \underline{821}, \underline{829}, 853,857,877,881$;
929, 937, 941, 953, 977, 997;
1009, 1013, 1021, 1033, 1049, 1061, 1069, 1093, 1097; 1109, 1117, 1129, 1153, 1181,$1193 ; 1201,1213,1217,1229,1237,1249,1277,1289,1297$; 1301, 1321, 1361, 1373, 1381; 1409, 1429, 1433, 1453, 1481, 1489, 1493;

1549, 1553, 1597; 1601, 1609, 1613, 1621, 1637, 1657, 1669, 1693, 1697; 1709,
1721, 1733, 1741, 1753, 1777, 1789; 1801, 1861, 1873, 1877, 1889; 1901, 1913, 1933, 1949, 1973, 1993, 1997.

## III. CONNECTIONS WITH THE TWIN PRIMES

Note that many of the Pythagorean primes in Table 2 are underlined. These are the primes which belong to a set of twin primes, i.e., primes $p_{i}$ and $p_{j}$, which are separated by 2 , i.e., such that $\left|p_{i}-p_{j}\right|=2$. As an example, 17 is part of the twin prime set (17, 19); similarly, 41 is part of the twin prime set (41, 43). By a survey of all twin primes $N_{i}<2000$, it was found that in all cases, for each set of twin primes, one of them is a $P$-prime ( $P$-decomposable), while the other is a non-P-prime. This result can be shown to follow naturally from a theorem due to Fermat, according to which all primes $p_{i} \equiv 1$ (mod 4) are $P$-primes, while all primes $q_{j} \equiv 3(\bmod 4)$ are non-P-primes. Actually, what Fermat proved is that all primes $p \equiv 1(\bmod 4)$ can be written in the form $p_{i}=a^{2}+b^{2}$, and this is, according to an elementary theorem due to Diophantos, the necessary and sufficient condition for $p_{i}^{2}=x_{i}^{2}+y_{i}^{2}$ to be satisfied [4]. Here, $x_{i}=a^{2}-b^{2}$ and $y_{i}=2 a b$, and the result follows naturally from the following equation:

$$
\begin{equation*}
p_{i}^{2} \equiv\left(a^{2}+b^{2}\right)^{2}=\left(a^{2}-b^{2}\right)^{2}+(2 a b)^{2}=a^{4}+b^{4}-2 a^{2} b^{2}+4 a^{2} b^{2} \tag{10}
\end{equation*}
$$

Obviously, $p_{i} \equiv 1(\bmod 4)$ means that $p_{i}$ can be written as $4 n+1$. Then, if $p_{j}$ is either 2 units larger or smaller than $p_{i}$, it is given by $4 n^{\prime}+3$, and $p_{j} \equiv 3$ (mod 4).

Of the 147 P-primes listed in Table 2, 60 are twin primes. The remaining 87=147-60 P-primes are "isolated" primes, i.e., they do not belong to a twin set. If we consider successive intervals of 500 , we find a total of $44 P$-primes between 1 and 500; $36 P$-primes between 501 and 1000; $36 P$-primes between 1001 and 1500; and 31 P-primes between 1501 and 2000. Incidentally, there is a total of 302 prime numbers between 1 and 2000 , so that the overall fraction of $P_{-}$ primes is $147 / 302=0.487 \approx 49 \%$, close to $50 \%$, as would be expected from Fermat's Theorem concerning $p_{i} \equiv 1(\bmod 4)$.

The approximate equality of the number $n_{P}$ of $P$-primes and $n_{N P}$ of non- $P-$ primes indicates that the Pythagorean primes have an intimate connection with the entire system of positive integers and, in addition, this connection indicates that we may expect that very approximately on the order of one-half of all integers are $P$-decomposable in at least one way ( $n_{d} \geqslant 1$ ), while the other half is not Pythagorean-decomposable. These integers will be called $P$-numbers and non $-P$ or $N P$-numbers, respectively. Numerical results for the fractions of $P$-numbers in three different intervals for $N \leqslant 2000$ will be given below. Obviously, for an integer $N_{i}$ to be $P$-decomposable in at least one way, it is necessary and sufficient that $N_{i}$ can be written as

$$
\begin{equation*}
N_{i}=p_{i} J, \tag{11}
\end{equation*}
$$

where $p_{i}$ is an arbitrary $P$-prime and $J$ is a positive integer.
IV. THE DECOMPOSITION FORMULA FOR $n_{d}$

The most general integer can be written as

$$
\begin{align*}
N_{k} & =p_{1}^{\alpha_{1}} p_{2}^{\alpha_{2}} p_{3}^{\alpha_{3}} \cdots q_{1}^{\beta_{1}} q_{2}^{\beta_{2}} q_{3}^{\beta_{3}} \cdots \\
& =\prod_{i=1}^{n_{a}} p_{i}^{\alpha_{i}} \prod_{j=1}^{n_{b}} q_{j}^{\beta_{j}} \equiv A_{k} B_{k}, \tag{12}
\end{align*}
$$

where the $p_{i}$ and $P$-primes are the $\alpha_{i}$ are the corresponding powers, and similarly, the $q_{j}$ are the non- $P$-primes and the $\beta_{j}$ are the corresponding powers. In the second row of (12), $n_{a}$ denotes the number of different $P$-primes in $N_{k}$ and $n_{b}$ denotes the number of different non- $P$-primes in the prime decomposition of $N_{k}$; finally, $A_{k}$ and $B_{k}$ represent the two products involving $p_{i}^{\alpha_{i}}$ and $q_{j}^{\beta_{j}}$, respectively.

Theorem: The total number of Pythagorean decompositions $n_{d}$ corresponding to $N_{k}$ of (12) is given by:

$$
\begin{align*}
n_{d}=\sum_{i=1}^{n_{a}} \alpha_{i} & +2 \sum_{i<j}^{n_{a}} \alpha_{i} \alpha_{j}+4 \sum_{i<j<k}^{n_{a}} \alpha_{i} \alpha_{j} \alpha_{k}+8 \sum_{i<j<k<\ell}^{n_{a}} \alpha_{i} \alpha_{j} \alpha_{k} \alpha_{\ell}+\cdots \\
& +2^{n_{a}-1} \alpha_{1} \alpha_{2} \alpha_{3} \cdots \alpha_{n_{a}} . \tag{13}
\end{align*}
$$

Here, the first sum extends over all $\alpha_{i}$, the second sum extends over all possible products of pairs of $\alpha_{i}$, the third sum extends over all possible products $\alpha_{i} \alpha_{j} \alpha_{k}$, where three $\alpha_{i}$ 's are involved, etc. As an example, for the number 65 of Table 1 , we have $65=5^{1} \times 13^{1}$, so that $\alpha_{1}=\alpha_{2}=1$, and (13) gives

$$
\begin{equation*}
n_{d}=1+1+2(1)(1)=4 \tag{14}
\end{equation*}
$$

Similarly, for $N_{k}=325=5^{2} \times 13$, with $\alpha_{1}=2, \alpha_{2}=1$, we find

$$
\begin{equation*}
n_{d}=2+1+2(2)(1)=7 \tag{15}
\end{equation*}
$$

In order to illustrate equation (13), we consider the number $1625=5^{3} \times 13$. First, we will count the number of ways in which 1625 can be written without mixing up the 5's and the 13 in the decomposition. We use the notation ( $p_{i}^{\alpha_{i}}$ ) with parentheses to indicate the decomposition of $p_{i}^{\alpha_{i}}$. Now, there are clearly $\alpha_{1}=3$ decompositions pertaining to the powers of 5 alone; they are ( $5^{3}$ ), ( $5^{2}$ ), and (5), where ( $5^{3}$ ) stands for ( $117,44,125$ ) (see Table 1), ( $5^{2}$ ) stands for $(24,7,25)$, and $(5) \equiv(4,3,5)$. Thus, three decompositions of 1625 can be written as $\left(5^{3}\right) \times 13,\left(5^{2}\right) \times 65$, and $(5) \times 325$, where the multiplication applies to the three integers $x_{i}, y_{i}$, and $z_{i}$ listed above for each case. In addition, there is the decomposition (13) $\times 125$, where ( 13 ) $\equiv(12,5,13$ ). These four decompositions correspond to $\alpha_{1}+\alpha_{2}=3+1=4$. Next, we consider the cases in which a product of a power of 5 times 13 appears inside the parentheses. These cases are $\left(5^{3} \times 13\right),\left(5^{2} \times 13\right) \times 5$, and $(5 \times 13) \times 25$. According to the rule of equations (3) and (4) for $z_{p, i}$ and $z_{p, j}$ having different prime factors, there are two new primitive solutions for each such case, e.g.,

$$
(325) \times 5=(253,204,325) \times 5 \text { and }(323,36,325) \times 5,
$$

where $325=5^{2} \times 13$ (see Table 1). There are $\alpha_{1} \alpha_{2}=(3)(1)=3$ such cases, and they contribute $2 \alpha_{1} \alpha_{2}=6$ decompositions. Thus, the total

$$
n_{d}=4+6=10=\alpha_{1}+\alpha_{2}+2 \alpha_{1} \alpha_{2}
$$

as given by (13). This illustration can be generalized to give the various terms of (13) and to provide the proof by induction. In each case, the factor $2,4,8$ in the second, third, and fourth terms, respectively, of (13) corresponds to the doubling of the primitive solutions described above, where more
than one prime is involved. For another example of (13), consider the number

$$
\begin{equation*}
N=\left(5^{2}\right)(13)(17)=5525 \tag{16}
\end{equation*}
$$

It has 22 decompositions of the type

$$
\begin{equation*}
5525^{2}=x^{2}+y^{2} \tag{17}
\end{equation*}
$$

since $\alpha_{1}=2, \alpha_{2}=\alpha_{3}=1$, and, from (13),

$$
\begin{equation*}
n_{d}=(2+1+1)+2(2+2+1)+4(2)=4+10+8=22 . \tag{18}
\end{equation*}
$$

Using (13), we have obtained the number of decompositions $n_{d}$ for three sets of 51 integers, namely those extending from $N=50$ to $N=100$, those extending from $N=950$ to 1000, and those extending from $N=1950$ to 2000. The results are presented in Table 3, which lists $n_{P}$, the number of Pythagorean numbers (for which $n_{d} \geqslant 1$ ), $n_{N P}$, the number of non- $P$-prime numbers (for which $n_{d}=0$ ), the total $\sum n_{d} / n_{P}$ and, finally, the ratio of $n_{P}$ to the total number 51. It is seen that while $n_{P} / a l l N=0.49$ for the first set ( $50-100$ ), for the other two sets, $n_{P} / a 11 N$ is constant at a value of $\approx 0.61$. However, the total number of decompositions, $\sum n_{d}$, increases from 34 (for $N=50-100$ ) to 58 (for $N=1950-$ 2000), and the average $\sum n_{d} / n_{P}$ also increases from 1.36 to 1.87 per Pythagorean number. It thus appears that the fraction of all numbers that are $P$-decomposable reaches a plateau value of -0.61 for large $N$, at least in the range of $N=1000-2000$.

Table 3. For three ranges of $N: 50-100,950-1000,1950-2000$, I have tabulated the total number of Pythagorean numbers $n_{P}$, the total number of non-$P$-numbers $n_{N P}$, the total number of $P$-decompositions $\Sigma n_{d}$, and the ratios $\sum n_{d} / n_{P}$ and $n_{P} / 51$, where 51 is the total number of integers in each range.

| $N$ range | $n_{P}$ | $n_{N P}$ | $\Sigma n_{d}$ | $\sum n / n_{P}$ | $n_{P} / 51$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $50-100$ | 25 | 26 | 34 | 1.36 | 0.490 |
| $950-1000$ | 31 | 20 | 53 | 1.71 | 0.608 |
| $1950-2000$ | 31 | 20 | 58 | 1.87 | 0.608 |

We note that for very large numbers $N_{k}$ (say $N_{k} \sim 10^{20}$ ) which have many factors $p_{i}^{\alpha_{i}}$ [see (12)], the use of (13) for $n_{d}$ becomes cumbersome. For this reason, I have derived a simpler formula for $n_{d}$ which can be readily evaluated for large $N_{k}$. This formula is presented in Appendix A of this paper [see equation (A25)].

As a final remark regarding (12), we note that we may define a Pythagorean congruence ( $P$-congruence) as follows: Referring to (12), it is seen that the product $A_{k}$ determines completely the type and the number $n_{d}$ of $P$-decompositions as given by (13). Therefore, we can write

$$
\begin{equation*}
N_{k} \equiv A_{k}(P), \tag{19}
\end{equation*}
$$

and all numbers $N_{k_{i}}$ with the same product $A_{k}$ (but different values of $B_{k}$ ) will

## SOME RESULTS CONCERNING PYTHAGOREAN TRIPLETS

have the same $P$-decompositions, except for a different cofactor $B_{k_{i}}$. The congruence (19) holds under the operation of multiplication, i.e., if we have two integers $N_{k}$ and $N_{k}$, with different values of $A_{k}$ and $B_{k}$, then the product $N_{k} N_{k}$, can be written as follows,

$$
\begin{equation*}
N_{k} N_{k},=\left(A_{k} A_{k}^{\prime}\right) B_{k} B_{k^{\prime}}, \tag{20}
\end{equation*}
$$

and the $P$-decompositions of $N_{k} N_{k}$, will be uniquely determined by the product $A_{k} A_{k}$, except for the cofactor $B_{k} B_{k}$, which multiplies all decompositions ( $x_{i}$, $y_{i}, z_{i}$ ). Therefore, $N_{k} N_{k}$, is $P$-congruent to $A_{k} A_{k}$ :

$$
\begin{equation*}
N_{k} N_{k}, \equiv A_{k} A_{k},(P) \tag{21}
\end{equation*}
$$

As examples of Pythagorean congruence, we mention three cases: $84 \equiv 1(P)$, since 84 is not $P$-decomposable, and $84=2^{2} \times 3 \times 7$ is a product of non- $P$-primes only; similarly, $6630 \equiv 1105(P)=5 \times 13 \times 17(P)$, where 5,13 , and 17 are $P$-primes. Finally, $929 \equiv 929(P)$, since 929 is a $P$-prime.

## v. CONCLUDING COMMENTS

Of particular interest among the $P$-triplets, are those for which $x=z-1$ (see Table 1 for examples). In this case, it is easily seen that $y$ must be an odd integer, which can therefore be written as

$$
\begin{equation*}
y=2 v+1 \tag{22}
\end{equation*}
$$

where $v$ is an arbitrary positive integer. We can now write:

$$
\begin{align*}
x^{2}+y^{2} & =(z-1)^{2}+(2 v+1)^{2} \\
& =z^{2}-2 z+1+4 \nu^{2}+4 v+1=z^{2} \tag{23}
\end{align*}
$$

Upon subtracting $z^{2}$ from the last two expressions in (23), and dividing by 2 , we obtain

$$
\begin{equation*}
-z+1+2 v^{2}+2 v=0 \tag{24}
\end{equation*}
$$

which gives

$$
\begin{equation*}
z=2 v(v+1)+1 \tag{25}
\end{equation*}
$$

and, therefore, $x=z-1=2 \nu(\nu+1)$, and a suitable ( $x, y, z$ ) triplet exists for any choice of $\nu(>0)$, i.e., for any odd integer except $y=1$. [In the latter case, $x=0$ and equation (1) is trivially satisfied.] Thus, the ensemble of numbers $y$ includes all odd numbers $\geqslant 3$, and hence, obviously, all prime numbers except $y=1$ and $y=2$. An example of such a triplet (from Table 1) is ( $40,9,41$ ), in which case $v=4, z=(2)(4)(5)+1=41, x=z-1=40$. Thus, the set of $y^{\prime}$ s for $x=z-1$ contains all prime numbers larger than $e$. We see again the privileged position of the numbers $y=1$ and $y=2$ (cf.[1]) that are not included among the $y_{i}$ 's in the $P$-triplets, in complete similarity to the exponents $n=1$ and $n=2$ for which Fermat's Last Theorem is satisfied [i.e., equation (2)]. I should also note that I can amplify the statement made in [1] concerning the Diophantine equation

$$
\begin{equation*}
F(x, y) \equiv x^{y}-y^{x}=0 \tag{26}
\end{equation*}
$$

In [1], I stated that the only nontrivial solution of (26) for integer $x$ and $y$ is $x=2, y=4$. However, if we do not demand that $y$ be an integer, but if we consider a limiting process for $x$ and $y$, then another nontrivial solution exists for $x \rightarrow 1$, i.e., the limit of $y$ as $x$ approaches 1 from above $(x=1+\varepsilon$, $\varepsilon \rightarrow 0$ ) is $y=\infty$. Specifically, I have calculated the values of $y$ determined by (26) for $x=1.1, x=1.01$, and $x=1.001$, with the following results:

$$
\begin{array}{ll}
y(x=1.1)=43.56, & x^{y}=y^{x}=63.53 ; \\
y(x=1.01)=658.81, & x^{y}=y^{x}=703.0 ; \\
y(x=1.001)=9133.4, & x^{y}=y^{x}=9217.05 \tag{29}
\end{array}
$$

It is clear from these results that the limit of $y$ as $x$ approaches 1 from above is infinity, i.e.,

$$
\begin{equation*}
\lim _{x \rightarrow 1} y=\infty . \tag{30}
\end{equation*}
$$

Thus, equation (26) is essentially satisfied for both $x=1$ and $x=2$, analogous to Fermat's Last Theorem, which is satisfied only for $n=1$ and $n=2$.

Parenthetically, I may note that for $x=0$, (26) cannot be satisfied for any positive $y$, since

$$
\begin{equation*}
F(0, y)=0^{y}-y^{0}=-1 \tag{31}
\end{equation*}
$$

for all $y$. Analogous to this result, Fermat's Last Theorem, equation (2), also has no solution for $n=0$, since the left-hand side $x^{0}+y^{0}=2$, whereas the right-hand side $z^{0}=1$.

In summary, I have shown that the Pythagorean decompositions of $z$ according to (1) provide a new classification of the number system into: (a) $P$-numbers $N_{P, i}$ [see (11)] that are $P$-decomposable in at least one way ( $n_{d} \geqslant 1$ ); (b) non-$P$-numbers $N_{N P, i}$ that cannot be decomposed according to (11) and (12), i.e., for which all of the $\alpha_{i}$ exponents of (12) are zero. The system of integers is approximately evenly divided between $P$-numbers and non- $P$-numbers in the range $50<N_{i}<100$, although for large $N_{i}$ in the range of $\sim 900-2000$, the $P$-numbers predominate slightly, to the extent of $60 \%$ of all integers.

The set of $P$-primes $p_{i}$ and products or powers of the $p_{i}$, i.e., $p_{i} p_{j}$ or $p_{i}^{\alpha}$ give rise to the primitive solutions ( $x_{i}, y_{i}, z_{i}$ ) for which (1) is satisfied. As described by equations (3) and (4), and (7)-(9), for each pair of primitive solutions ( $x_{p, i}, y_{p, i}, z_{p, i}$ ) and ( $x_{p, j}, y_{p, j}, z_{p, j}$ ), the product $z_{p, k} \equiv z_{p, i} z_{p, j}$ contributes two new primitive solutions (provided the prime factors of $z_{p}, i$ and $z_{p, j}$ are different).

The total number of Pythagorean decompositions for a given $P$-number $N_{p, i}$ increases rapidly with the number $n_{a}$ of $p_{i}$ primes [see equation (12)] and with the powers $\alpha_{i}$ associated with each $p_{i}$. I have obtained a general expression for $n_{d}$ in terms of the $\alpha_{i}$ and $n_{a}$ [see equation (13)]. Furthermore, (13) has been proven by induction in the discussion which follows (15). An equivalent formula for (13) will be derived in Appendix A. The results given in Appendix A provide the means for a rapid evaluation of $n_{d}$ when the integer $N_{k}$ [see (12)] is large, so that there is a large number $n_{a}$ of $P$-primes $p_{i}$ in the prime decomposition of $N_{k}$.

Concerning the primitive solutions, I have noticed empirically from the decomposition tables that the density of primitive solutions, i.e., their frequency, is almost constant in going from $N \sim 0-100$ to $N=3000$. Thus, generally, for each additional interval of 100 in $N$, we obtain sixteen additional
primitive solutions. As an example, the total number of primitive solutions included in Table 1 for $1 \leqslant N \leqslant 500$ is exactly $80=5 \times 16$ [equation (6)]. For $1 \leqslant N \leqslant 1000$, the total number of primitive solutions is 158 , and for the entire sample with $1 \leqslant N \leqslant 3000$, the total number of primitive solutions is 477, almost equal to the expected number $16 \times 30=480$. At present, I have no explanation for the remarkable constancy of the density (frequency) of primitive solutions as a function of $N$.

As a final comment, it is not clear at present to what extent the results reported in this paper for the case $n=2$ will help in the ultimate proof of Fermat's Last Theorem. Nevertheless, my previous suggestion about the values of $n>e$ [1] and its amplification as presented in this paper [equations (26)(30)] may offer a guideline to a complete proof. In any case, the interesting discovery of the doubling of the primitive solutions [equations (3), (4)] and the derivation of the resulting decomposition formula [equation (13)] will perhaps shed new light on the nature of our integer number systme. Additional results on the evaluation of (13) and on the case $n=1$ in (2) will be given in Appendix A and Appendix B, respectively.

## APPENDIX A

## EVALUATION OF EQUATION (13)

In connection with (13) for the number $n_{d}$ of Pythagorean decompositions of an arbitrary integer $N_{k}$ as given by (12), it seems of interest to tabulate typical values of $n_{d}$ for integers with relatively low values of the exponents $\alpha_{i}$. Table 4 shows a systematic listing of the numbers of decompositions $n_{d}$ for all cases for which $\sum \alpha_{i} \leqslant 6$. Obviously, the table can be subdivided into subtables pertaining to those cases for which any given number of $P$-primes $p_{i}$ are involved. Thus, the top part of the table pertains to $\alpha_{1}>0, \alpha_{2}=\alpha_{3}=\alpha_{4}=$ $\alpha_{5}=\alpha_{6}=0$ (i.e., the case $n_{a}=1$ ). The next panel of the table pertains to cases for which two Pythagorean primes occur ( $n_{a}=2$ ) in the decomposition of $N_{k}$ [equation (12)], and these will be denoted $\alpha_{1}$ and $\alpha_{2}$, i.e., $\alpha_{3}, \ldots, \alpha_{6}=0$. In this panel I have arbitrarily assumed that $\alpha_{1} \geqslant \alpha_{2}$ and, of course, all cases are subject to the limitation that $\alpha_{1}+\alpha_{2} \leqslant 6$. The third, fourth, fifth, and sixth panels of the table are similarly constructed.

The next-to-the-last column of the table lists the values of $n_{d}$, while the last column lists the values of $N_{\min }$, the smallest integer $N_{k}$ for which the particular decomposition as given in the first six columns exists. In addition, the prime decomposition of $N_{\min }$ is listed after the value of $N_{\min }$. Obviously, in order to obtain the lowest $N_{k}$ value consistent with the set $\left\{\alpha_{i}\right\}$, we must assume that all of the $\beta_{j}$ in (12) are zero, i.e., $B_{k}=1$. Furthermore, it is necessary to choose for the $P$-prime with the largest $\alpha_{i}$ the value 5 , then the value 13 for the $P$-prime with the next largest $\alpha_{i}$, and so forth.

Several results are apparent from a study of Table 4 and of (13):

1. Consider equation (13) and a particular $\alpha_{i}$, say $\alpha_{i, 0}$. Because the particular $\alpha_{i, 0}$ appears linearly in all of the terms of (13), $n_{d}$ depends linearly on $\alpha_{i, 0}$, and in particular, for equally spaced values of $\alpha_{i}$, e.g.,

$$
\alpha_{i, 0}, \quad \alpha_{i, 0}+1, \quad \text { and } \quad \alpha_{i, 0}, \quad \alpha_{i, 0}-1,
$$

we find

$$
\begin{equation*}
n_{d}\left(\alpha_{i, 0}+1\right)-n_{d}\left(\alpha_{i, 0}\right)=n_{d}\left(\alpha_{i, 0}\right)-n_{d}\left(\alpha_{i, 0}-1\right), \tag{A1}
\end{equation*}
$$

Table 4. Listing of special cases of (13) for the number of Pythagorean decompositions as a function of the $\alpha_{i}$ 's and $n_{a}$. I have tabulated all cases for which $\sum_{i=1}^{6} \alpha_{i} \leqslant 6$. The seventh column of the table gives the values of $n_{d}\left\{\alpha_{i}\right\}$ as obtained from (13). The last column gives the smallest number $N_{\min }$ for which the listed exponents $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}, \alpha_{5}$, and $\alpha_{6}$ are realized. The prime decomposition of $N_{\min }$ is listed for each $N_{\text {min }}$. The blank spaces in the columns for $\alpha_{i}$ correspond to values of $\alpha_{i}=0$.

| $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{5}$ | $\alpha_{6}$ | $n_{d}\left\{\alpha_{i}\right\}$ | $N_{\text {min }}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 |  |  |  |  |  | 1 | 5 |
| 2 |  |  |  |  |  | 2 | 25 |
| 3 |  |  |  |  |  | 3 | 125 |
| 4 |  |  |  |  |  | 4 | 625 |
| 5 |  |  |  |  |  | 6 | 3125 |
| 6 |  |  |  |  |  | 15,625 |  |
| 1 | 1 |  |  |  |  | 4 | $65=5 \times 13$ |
| 2 | 1 |  |  |  |  | 7 | $325=25 \times 13$ |
| 2 | 2 |  |  |  |  | 12 | $4225=25 \times 169$ |
| 3 | 1 |  |  |  |  | 10 | $1625=125 \times 13$ |
| 3 | 2 |  |  |  |  | 24 | $27,125=125 \times 169$ |
| 3 | 3 |  |  |  |  | 13 | $8125=625=125 \times 2197$ |
| 4 | 1 |  |  |  |  | 22 | $105,625=625 \times 169$ |
| 4 | 2 |  |  |  |  | 16 | $40,625=3125 \times 13$ |
| 5 | 1 |  |  |  |  | 13 | $1105=5 \times 13 \times 17$ |
| 1 | 1 | 1 |  |  |  | 22 | $5525=25 \times 13 \times 17$ |
| 2 | 1 | 1 |  |  |  | 37 | $71,825=25 \times 169 \times 17$ |
| 2 | 2 | 1 |  |  |  | 62 | $1,221,025=25 \times 169 \times 289$ |
| 2 | 2 | 2 |  |  |  | 31 | $27,625=125 \times 13 \times 17$ |
| 3 | 1 | 1 |  |  |  | 52 | $359,125=125 \times 169 \times 17$ |
| 3 | 2 | 1 |  |  |  | 40 | $138,125=625 \times 13 \times 17$ |
| 4 | 1 | 1 |  |  |  |  |  |
| 1 | 1 | 1 | 1 |  |  | 40 | $32,045=5 \times 13 \times 17 \times 29$ |
| 2 | 1 | 1 | 1 |  |  | 67 | $160,225=25 \times 13 \times 17 \times 29$ |
| 2 | 2 | 1 | 1 |  |  | 112 | $2,082,925=25 \times 169 \times 17 \times 29$ |
| 3 | 1 | 1 | 1 |  |  | 94 | $801,125=125 \times 13 \times 17 \times 29$ |
| 1 | 1 | 1 | 1 | 1 |  | 121 | $1,185,665=5 \times 13 \times 17 \times 29 \times 37$ |
| 2 | 1 | 1 | 1 | 1 |  | 202 | $5,928,325=25 \times 13 \times 17 \times 29 \times 37$ |
| 1 | 1 | 1 | 1 | 1 | 1 | 364 | $48,612,265=5 \times 13 \times 17 \times 29 \times 37 \times 41$ |
|  |  |  |  |  |  |  |  |

and, indeed, for any two values of $\alpha_{i}$ which differ by 1 , the differences

$$
n_{d}\left(\alpha_{i}\right)-n_{d}\left(\alpha_{i}-1\right)
$$

will be the same. Of course, in applying (Al), one must keep all of the other 1986]
$\alpha_{j}$ values constant. Equation (Al) can be used to check the correctness of the entries of Table 4. As an example,

$$
\begin{align*}
n_{d}(2,2)-n_{d}(2,1)=12-7 & =n_{d}(2,1)-n_{d}(2,0) \\
& =7-2=5 . \tag{A2}
\end{align*}
$$

Similarly,

$$
\begin{align*}
n_{d}(3,1,1,1)-n_{d}(2,1,1,1) & =94-67 \\
& =n_{d}(2,1,1,1)-n_{d}(1,1,1,1) \\
& =67-40=27 \tag{A3}
\end{align*}
$$

Here I have used the notation $n_{d}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}\right)$ and $n_{d}\left(\alpha_{1}, \alpha_{2}\right)$ for the corresponding entries in Table 4.
2. Next, we consider the cases where all of the $\alpha_{i}$ are 1 , e.g.,

$$
n_{d}(1,1,1)=13, \quad n_{d}(1,1,1,1,1)=121, \text { etc. }
$$

For simplicity, $n_{d}(1,1, \ldots, 1)$ with $\xi$ l's will be simply denoted by $n_{d}\left[1_{\xi}\right]$. We note that the $n_{d}\left[l_{\xi}\right]$ satisfy the recursion relations

$$
\begin{equation*}
n_{d}\left[1_{\xi+1}\right]=3 n_{d}\left[1_{\xi}\right]+1 \tag{A4}
\end{equation*}
$$

As an example, $n_{d}\left[1_{6}\right]=364 ; n_{d}\left[1_{5}\right]=121$, and we have

$$
\begin{equation*}
n_{d}\left[1_{6}\right]=3 n_{d}\left[1_{5}\right]+1=364=(3 \times 121)+1 \tag{A5}
\end{equation*}
$$

Equation (A4) together with the additional condition $n_{d}\left[1_{1}\right]=1$ can be used to derive all of the $n_{d}\left[l_{\xi}\right]$ values of Table 4 , namely, $4\left\{=n_{d}\left[1_{2}\right]\right\}, 13,40,121$, and 364.

I also note that the difference $n_{d}\left[1_{\xi}+1\right]-n_{c}\left[1_{\xi}\right]$ obeys the equation

$$
\begin{equation*}
n_{d}\left[1_{\xi+1}\right]-n_{d}\left[1_{\xi}\right]=3^{\xi} \tag{A6}
\end{equation*}
$$

As an example: $n_{d}\left[1_{6}\right]-n_{d}\left[1_{5}\right]=364-121=243=3^{5}$.
Therefore, I find:

$$
\begin{equation*}
n_{d}\left[1_{\xi}\right]=\sum_{\eta=0}^{\xi-1} 3^{n} . \tag{A7}
\end{equation*}
$$

3. A similar relation is obtained when we calculate differences between values of $n_{d}(2,1, \ldots, 1)$. For simplicity, we write $n_{d}(2,1, \ldots, 1)$ with $\gamma$ l's as $n_{d}\left[2,1_{\gamma}\right]$. We note that

$$
\begin{align*}
& n_{d}(2,1,1)-n_{d}(2,1)=22-7=15  \tag{A8}\\
& n_{d}(2,1,1,1)-n_{d}(2,1,1)=67-22=45 \tag{A9}
\end{align*}
$$

and also

$$
\begin{equation*}
n_{d}(2,1)-n_{d}(2)=7-2=5 \tag{A10}
\end{equation*}
$$

These results suggest the relation:

$$
\begin{equation*}
n_{d}\left[2, l_{\gamma}\right]-n_{d}\left[2,1_{\gamma-1}\right]=5 \times 3^{\gamma-1} \tag{All}
\end{equation*}
$$

In fact, for $\gamma=4$, we find

$$
\begin{equation*}
n_{d}\left[2,1_{4}\right]-n_{d}\left[2,1_{3}\right]=5 \times 3^{3}=135=202-67 . \tag{Al2}
\end{equation*}
$$

Moreover, I have found that

$$
\begin{equation*}
n_{d}\left[2,1_{\gamma}\right]-n_{d}\left[1,1_{\gamma}\right]=3^{\gamma}, \tag{Al3}
\end{equation*}
$$

and, therefore, in view of (A7), and generalizing to $n_{d}\left[k, l_{\gamma}\right]$,

$$
\begin{equation*}
n_{d}\left[k, l_{\gamma}\right]=\sum_{\eta=0}^{\gamma} 3^{\eta}+(k-1) 3^{\gamma}, \tag{A14}
\end{equation*}
$$

where $k$ is an arbitrary positive integer.
Finally, as a generalization of (A7), I have found that the $n_{d}\left[k_{\xi}\right]$ for an arbitrary number $\xi$ of integers $k$, e.g., $n_{d}[2,2,2]=n_{d}\left[2_{3}\right]$, are given by the following expression:

$$
\begin{equation*}
n_{d}\left[k_{\xi}\right]=k \sum_{n=0}^{\xi-1}(2 k+1)^{n} . \tag{A15}
\end{equation*}
$$

As an example: $n_{d}[2,2,2]=n_{d}\left[2_{3}\right]$ is given by

$$
\begin{equation*}
n_{d}\left[2_{3}\right]=2 \sum_{n=0}^{2}(5)^{n}=2\left(1+5+5^{2}\right)=62, \tag{A16}
\end{equation*}
$$

in agreement with the corresponding entry in Table 4. The generalized recursion relation which pertains to (A15) is

$$
\begin{equation*}
n_{d}\left[k_{\xi+1}\right]=(2 k+1) n_{d}\left[k_{\xi}\right]+k . \tag{A17}
\end{equation*}
$$

A more general formula which is based on (A14) and (A15) gives

$$
\begin{equation*}
n_{d}\left[k, k_{\gamma}^{\prime}\right]=k^{\prime} \sum_{n=0}^{\gamma-1}\left(2 k^{\prime}+1\right)^{\eta}+k\left(2 k^{\prime}+1\right)^{\gamma} . \tag{A18}
\end{equation*}
$$

(A18) gives $n_{d}$ for $\gamma$ powers $\alpha_{i}$ equal to $k^{\prime}$ and a single power $\alpha_{j}$ equal to $k$. In an attempt to simplify the evaluation of (A15) and (A18), we note that the sum in (A18) can be written as follows:

$$
\begin{equation*}
\sum_{n=0}^{\gamma-1}\left(2 k^{\prime}+1\right)^{\eta}=\left(2 k^{\prime}+1\right)^{\gamma-1}\left[1+\frac{1}{2 k^{\prime}+1}+\frac{1}{\left(2 k^{\prime}+1\right)^{2}}+\cdots+\frac{1}{\left(2 k^{\prime}+1\right)^{\gamma-1}}\right] . \tag{A19}
\end{equation*}
$$

The expression in square brackets is the major part of the infinite series

$$
\begin{equation*}
\frac{1}{1-1 /\left(2 k^{\prime}+1\right)}=1+\frac{1}{2 k^{\prime}+1}+\frac{1}{\left(2 k^{\prime}+1\right)^{2}}+\cdots . \tag{A20}
\end{equation*}
$$

The left-hand side of (A20) can be rewritten as follows:

$$
\begin{equation*}
\frac{1}{1-1 /\left(2 k^{\prime}+1\right)}=\frac{2 k^{\prime}+1}{2 k^{\prime}} . \tag{A21}
\end{equation*}
$$

Therefore, the sum of (A19) is approximately given by

$$
\begin{equation*}
\sum_{n=0}^{\gamma-1}\left(2 k^{\prime}+1\right)^{\eta} \cong\left(2 k^{\prime}+1\right)^{\gamma} / 2 k^{\prime} . \tag{A22}
\end{equation*}
$$

The part of the expression (A20) which is not included in the sum of (A19) can be shown to result in a negative contribution to $n_{d}\left[k, k_{j}^{\prime}\right]$, which is given by

$$
\begin{equation*}
\Delta n_{d}=-k^{\prime}\left[\frac{1}{1-1 /\left(2 k^{\prime}+1\right)}-1\right]=-k^{\prime}\left(\frac{2 k^{\prime}+1}{2 k^{\prime}}-1\right)=-\frac{1}{2} . \tag{A23}
\end{equation*}
$$

Upon inserting these results in (A18), we obtain:

$$
\begin{align*}
n_{d}\left[k, k_{\gamma}^{\prime}\right] & =k^{\prime}\left(2 k^{\prime}+1\right)^{\gamma} / 2 k^{\prime}-\frac{1}{2}+k\left(2 k^{\prime}+1\right)^{\gamma} \\
& =\frac{1}{2}\left(2 k^{\prime}+1\right)^{\gamma}(2 k+1)-\frac{1}{2} . \tag{A24}
\end{align*}
$$

Equation (A24) suggests a natural generalization to an arbitrary number of different $k_{i}^{\prime}$ s, since each $k_{i}$ gives rise to a power $\left(2 k_{i}+1\right)^{\gamma_{i}}$ in the expression for $n_{d}$. We therefore obtain:

$$
\begin{equation*}
n_{d}\left(\left\{\alpha_{i}\right\}\right)=\frac{1}{2} \prod_{i=1}^{i_{\max }}(2 k+1)^{\gamma_{i}}-\frac{1}{2} . \tag{A25}
\end{equation*}
$$

This equation permits a rapid evaluation of $n_{d}\left(\left\{\alpha_{i}\right\}\right)$ and is completely equivalent to the much more complicated equation (13) from which it is ultimately derived. I may note that we have the additional relation

$$
\begin{equation*}
\sum_{i=1}^{i_{\max }} \gamma_{i}=n_{a} \tag{A26}
\end{equation*}
$$

where $n_{a}$ is the number of different $P$-primes, as used in (12). As an example, I consider the following number,

$$
\begin{align*}
N\left[2,1_{13}\right] & \equiv 5^{2} \times 13 \times 17 \times 29 \times 37 \times 41 \times 53 \times 61 \times 73 \times 89 \times 97 \times 101 \times 109 \times 113 \\
& \cong 6.1605 \times 10^{23}, \tag{A27}
\end{align*}
$$

which is close to Avogadro's number

$$
N_{\mathrm{Av}}=6.02204 \times 10^{23}
$$

The notation $N\left[2,1_{13}\right]$ obviously means that the lowest $P$-prime, $p_{1}=5$, was squared and the next $13 P$-primes (power $k_{i}=1$ ) were multiplied in the order of increasing $p_{i}$ (see Table 2).

According to (A25), the number of Pythagorean decompositions of $N\left[2,1_{13}\right]$ is

$$
\begin{equation*}
n_{d}\left(\left\{\alpha_{i}\right\}\right)=\frac{1}{2}(5)\left(3^{13}\right)-\frac{1}{2}=3,985,807 \tag{A28}
\end{equation*}
$$

In general, we may try to calculate numbers $N_{k}$ which in a given range have the largest number of $P$-decompositions $n_{d}$. This is usually accomplished by multiplying an appropriate number $\gamma_{1}$ of $P$-primes, all taken linearly ( $k_{1}=1$ ), i.e., to the first power. This conclusion was derived from the results of Table 4 which show, for example, that $N[1,1,1,1,1]=N\left[1_{5}\right]=1,185,665$ has
$n_{d}=121$-decompositions, whereas the slightly larger $N[2,2,2]=N\left[2_{3}\right]=$ $1,221,025$ has only $n_{d}=62 P$-decompositions.

In view of this result, I have made a study of the numbers $N\left[l_{\gamma}\right]$, where $N\left[1_{\gamma}\right]$ denotes the product of the first $\gamma$ primes in Table 2. As an example,

$$
\begin{align*}
N\left[1_{14}\right] & =5 \times 13 \times 17 \times 29 \times 37 \times 41 \times 53 \times 61 \times 73 \times 89 \times 97 \times 101 \times 109 \times 113 \\
& \cong 1.2321 \times 10^{23} \tag{A29}
\end{align*}
$$

has $n_{d}\left[1_{14}\right]$ Pythagorean decompositions, where [from (A25)]:

$$
\begin{equation*}
n_{d}\left[1_{14}\right]=\frac{1}{2}\left(3^{14}-1\right)=2,391,484 \tag{A30}
\end{equation*}
$$

For several values of $\gamma$ up to $\gamma=25$, Table 5 gives the values of $N\left[1_{\gamma}\right]$, the corresponding $n_{d}\left[1_{\gamma}\right]$ [cf. (A30)], and the exponent $\sigma(\gamma)$, which will be defined presently. I noticed that $n_{d}\left[1_{\gamma}\right]$ is, in all cases, of the order of

$$
\left\{N\left[1_{\gamma}\right]\right\}^{1 / 3} \text { to }\left\{N\left[1_{\gamma}\right]\right\}^{1 / 4}
$$

so that an accurate inverse power, denoted by $1 / \sigma$, can be defined for each $\gamma$, such that

$$
\begin{equation*}
n_{d}\left[1_{\gamma}\right]=\left\{N\left[1_{\gamma}\right]\right\}^{1 / \sigma} \tag{A31}
\end{equation*}
$$

$\sigma(\gamma)$ is a slowly varying function of $\gamma$ that increases from $\sigma=2.732$ for $\gamma=3$ to $\sigma=4.145$ for $\gamma=25$. Below $\gamma=3, \sigma(\gamma)$ increases to $\sigma=3.011$ for $\gamma=2$ and to $\infty$ for $\gamma=1$, since the first $P$-prime, $p_{1}=5$, has a single $P$-decomposition, and $5^{0}=1$. The resulting curve of $\sigma(\gamma) v s \gamma$ is shown in Figure 1 .


Figure 1. The inverse exponent $\sigma$ as a function of $\gamma$ for the $n_{d}$ values pertaining to $N\left[l_{\gamma}\right]$ [see (A31)].

Table 5. Values of $\sigma(\gamma), N\left[l_{\gamma}\right]$, and $n_{d}\left[1_{\gamma}\right]$ for selected values of $\gamma$ in the range $1 \leqslant \gamma \leqslant 25$ [see (A31)].

| $\gamma$ | $\sigma(\gamma)$ | $N\left[1_{\gamma}\right]$ | $n_{d}\left[1_{\gamma}\right]$ |
| ---: | :--- | :--- | :--- |
| 1 | $\infty$ | 5 | 1 |
| 2 | 3.011 | 65 | 4 |
| 3 | 2.732 | 1105 | 13 |
| 4 | 2,813 | 32,045 | 40 |
| 5 | 2.916 | $1,185,665$ | 121 |
| 6 | 3.001 | $48,612,265$ | 364 |
| 8 | 3.184 | $1.572 \times 10^{11}$ | 3,280 |
| 10 | 3.358 | $1.021 \times 10^{15}$ | 29,524 |
| 12 | 3.503 | $1.004 \times 10^{19}$ | 265,720 |
| 14 | 3.620 | $1.232 \times 10^{23}$ | $2,391,484$ |
| 17 | 3.789 | $3.949 \times 10^{29}$ | $64,570,081$ |
| 20 | 3.936 | $2.286 \times 10^{36}$ | $1.743 \times 10^{9}$ |
| 22 | 4.024 | $1.076 \times 10^{41}$ | $1.569 \times 10^{10}$ |
| 25 | 4.145 | $1.553 \times 10^{48}$ | $4.236 \times 10^{11}$ |

## APPENDIX B

THE CASE $n=1$ OF EQUATION (2) AND COMMENTS ABOUT GOLDBACH'S CONJECTURE
It is obvious that the case $n=1$ of (2), namely

$$
\begin{equation*}
x+y=z \tag{B1}
\end{equation*}
$$

always has a solution with integers $x, y$, and $z$. We will assume, for definiteness, that $x \geqslant y$. Then (B1) has $z / 2$ linearly independent solutions when $z$ is even, and $(z-1) / 2$ linearly independent solutions when $z$ is odd. As an example for $z=11$, we have the following (11-1)/2 = 5 linear decompositions of $z: 10+1,9+2,8+3,7+4$, and $6+5$.

There is a well-known conjecture, namely Goldbach's Conjecture, that any even $z$ can be written as the sum of two prime numbers $x$ and $y$. To my knowledge, this conjecture has not yet been proven in the general case, i.e., for an arbitrary even z. In this Appendix I have made a systematic study of the linear decompositions [equation (B1)] of all the even numbers $z \leqslant 100$ in terms of sums of two primes $x$ and $y$.

It can be shown that the total number of linearly independent decompositions of an even $z$ into a sum of two odd numbers according to (B1) is $z / 4$ for $z=4 \nu$ (divisible by 4) and $(z+2) / 4$ for $z=4 v+2$ (not divisible by 4). According to the above-mentioned program, I am led to consider all of the linear decompositions of $z$ as a sum $x+y$, where $x$ and $y$ are restricted to being
prime numbers. It will be seen shortly that in this endeavor, the concepts of a Pythagorean prime ( $P$-prime) and a non-P-prime are of great importance.

In Table 6, I have listed all of the prime decompositions for even $z$ in the range from 2 to 100 . The number $z$ is also denoted by $N$. In the prime decompositions, I have underlined the value of $x_{i}$ or $y_{i}$ in those cases where $x_{i}$ or $y_{i}$ is a Pythagorean prime. The most striking result of this table (aside from the large number of prime decompositions as $z=N$ increases) is that there are two types of cases, depending upon whether $N$ is or is not divisible by 4: (a) If $N$ is divisible by 4, i.e., $N=4 \nu(\nu=$ positive integer), then each decomposition is the sum of a $P$-prime and a non- $P$-prime. (The only apparent exception occurs for $4=2+2$, and this decomposition will be discussed further below.) (b) If $N$ is not divisible by 4, i.e., for $N=4 v+2$, the prime decompositions involve either the sum of two $P$-primes (both $x$ and $y$ underlined) or the sum of two non-P-primes (neither $x$ nor $y$ underlined). As an example, $N=16=13+3=11+5$. By contrast, $N=10=7+3=\underline{5}+\underline{5}$.

These two rules can be derived from the theorem of Fermat [see the discussion preceding equation (10)] that all primes $p_{i} \equiv 1$ (mod 4) are Pythagorean primes, while all primes $q_{i} \equiv 3(\bmod 4)$ are non-P-primes. Thus, we can write:

$$
\begin{align*}
& p_{i}=4 v_{i}+1  \tag{B2}\\
& q_{j}=4 v_{j}-1 \tag{B3}
\end{align*}
$$

from which it follows that

$$
\begin{equation*}
p_{i}+q_{j}=4\left(v_{i}+v_{j}\right)=4 v \tag{B4}
\end{equation*}
$$

for numbers $N=4 \nu$ that are divisible by 4 . On the other hand,

$$
\begin{align*}
& p_{i}+p_{i^{\prime}}=4 v_{i}+4 v_{i^{\prime}}+2=4\left(v_{i}+v_{i^{\prime}}\right)+2=4 v+2  \tag{B5}\\
& q_{j}+q_{j^{\prime}}=4 v_{j}+4 v_{j^{\prime}}-2=4\left(v_{j}+v_{j^{\prime}}-1\right)+2=4 \bar{v}+2 \tag{B6}
\end{align*}
$$

for even numbers that are not divisible by 4 , i.e., $N=4 \nu+2$ or $4 \bar{\nu}+2$.
It may be noted that, in constructing Table 6, I have underlined the number 1 , i.e., I have treated 1 as a Pythagorean prime (with the decomposition $1^{2}=$ $1^{2}+0^{2}$ ). This is essentially a matter of definition, but it is mandated by the result that the decompositions which involve 1 obey the rules (a) and (b) described above, provided that 1 is regarded as a $P$-prime for the present purposes. I will also note that to regard 1 as a $P$-prime in cases where a direct addition is involved makes good sense, whereas in the arguments leading to the decomposition formula, (13), if $I$ had introduced an arbitrary factor $1_{0}^{\alpha_{0}}$ in the expression for $N_{k}$ of (12), this would have invalidated (13) for the total number of decompositions $n_{d}$, unless $\alpha_{0}=0$.

The decomposition $4=2+2$ is an apparent exception to rules (a) and (b) given above. It does not seem to conform to the rule that one of the pair ( $x$, $y)$ be a $P$-prime, whereas the other of the pair $(x, y)$ should be a non-P-prime. One way to obviate this contradiction is to specify that rules (a) and (b) apply only when the prime numbers $x$ and $y$ are odd. Another way of looking at the situation with respect to both 1 and 2 is that, as was emphasized repeatedly in [1] and in this paper, both 1 and 2 are special integers to which some of the rules governing other primes ( $\geqslant 3$ ) do not apply; see especially the last two paragraphs of [1] and the discussion following (26) above. This privileged position of 1 and 2 has been correlated with the special properties of the powers $n=1$ and $n=2$ in the original Fermat equation, (2). Finally, a third
and more speculative way to describe the status of the integer 2 in connection with $4=2+2$ is that just as $y=1$ had to be defined as a $P$-prime in connection with Table 6, but as a non-P-prime in connection with (13), so $x=2$ or $y=2$ behaves half of the time as a $P$-prime (with the decomposition $2^{2}=2^{2}+0^{2}$ ) and half of the time as a non-P-prime which has no decomposition $2^{2}=x^{2}+y^{2}$, where $x, y>0$. According to this interpretation, we could write $4=3+\underline{1}=$ $2+\underline{2}$ in Table 6.

Table 6. Linear decompositions of all even numbers $2 \leqslant N \leqslant 100$. For each $N=z$, all of the linear decompositions into a sum of prime numbers $z=x+y$ are listed. Values of $x$ and $y$ which correspond to Pythagorean primes are underlined; the nonunderlined values correspond to non-P-primes. Note that when $N$ is divisible by 4, i.e., $N=4 V$ ( $V=$ positive integer), one of the pair $(x, y)$ is a $P$-prime whereas the other number in the sum is a non-P-prime. When $N$ is divisible by 2 , but not by 4 , i.e., for $N=4 v+2$, either both $x$ and $y$ are $P$-primes, or both $x$ and $y$ are non-P-primes. A possible exception occurs for the decomposition of $4=2+2$ (see discussion in text). We assume that $x \geqslant y$.

| $N$ | $x_{i}+y_{i}$ |  |
| :---: | :---: | :---: |
| 2 | $\underline{1}+\underline{1}$ |  |
| 4 | $3+1,2+2$ |  |
| 6 | $\underline{5}+\underline{1}, 3+3$ |  |
| 8 | $7+\underline{1}, \underline{5}+3$ |  |
| 10 | $7+3, \underline{5}+\underline{5}$ |  |
| 12 | $11+\underline{1}, 7+\underline{5}$ |  |
| 14 | $\underline{13+1}, 11+3,7+7$ |  |
| 16 | $\underline{13}+3,11+\underline{5}$ |  |
| 18 | $\underline{17}+\underline{1}, \underline{13}+\underline{5}, 11+7$ |  |
| 20 | $19+\underline{1}, \underline{17}+3, \underline{13}+7$ |  |
| 22 | $19+3, \underline{17}+\underline{5}, 11+11$ |  |
| 24 | $23+\underline{1}, 19+\underline{5}, \underline{17}+7,13+11$ |  |
| 26 | $23+3,19+7, \underline{13}+\underline{13}$ |  |
| 28 | $23+\underline{5}, \underline{17}+11$ |  |
| 30 | $\underline{29}+\underline{1}, 23+7,19+11, \underline{17}+\underline{13}$ |  |
| 32 | $31+\underline{1}, \underline{29}+3,19+\underline{13}$ |  |
| 34 | $31+3,29+\underline{5}, 23+11, \underline{17}+\underline{17}$ |  |
| 36 | $31+\underline{5}, \underline{29}+7,23+\underline{13}, 19+\underline{17}$ |  |
| 38 | $\underline{37}+\underline{1}, 31+7,19+19$ |  |
| 40 | $\underline{37}+3, \underline{29}+11,23+\underline{17}$ |  |
| 42 | $\underline{41}+\underline{1}, \underline{37}+\underline{5}, 31+11,29+13,23+19$ |  |
| 44 | $43+1,41+3,37+7,31+\underline{13}$ |  |
| 124 |  | [May |

Table 6. continued

| $N$ | $x_{i}+y_{i}$ |
| :---: | :---: |
| 46 | $43+3, \underline{41}+\underline{5}, \underline{29}+\underline{17}, 23+23$ |
| 48 | $47+\underline{1}, 43+\underline{5}, \underline{41}+7, \underline{37}+11,31+\underline{17}, \underline{29}+19$ |
| 50 | $47+3,43+7, \underline{37}+\underline{13}, 31+19$ |
| 52 | $47+\underline{5}, \underline{41}+11, \underline{29}+23$ |
| 54 | $\underline{53}+\underline{1}, 47+7,43+11, \underline{41}+\underline{13}, \underline{37}+\underline{17}, 31+23$ |
| 56 | $\underline{53}+3,43+\underline{13}, \underline{37}+19$ |
| 58 | $\underline{53}+\underline{5}, 47+11, \underline{41}+\underline{17}, \underline{29}+\underline{29}$ |
| 60 | $59+\underline{1}, \underline{53}+7,47+\underline{13}, 43+\underline{17}, \underline{41}+19, \underline{37}+23,31+\underline{29}$ |
| 62 | 61+1, $59+3,43+19,31+31$ |
| 64 | $\underline{61}+3,59+\underline{5}, \underline{53}+11,47+\underline{17}, \underline{41}+23$ |
| 66 | $\underline{61}+\underline{5}, 59+7, \underline{53}+\underline{13}, 47+19,43+23, \underline{37}+\underline{29}$ |
| 68 | $67+\underline{1}, \underline{61}+7, \underline{37}+31$ |
| 70 | $67+3,59+11, \underline{53}+\underline{17}, 47+23, \underline{41}+\underline{29}$ |
| 72 | $71+\underline{1}, 67+\underline{5}, \underline{61}+11,59+\underline{13}, \underline{53}+19,43+\underline{29}, \underline{41}+31$ |
| 74 | $\underline{73}+\underline{1}, 71+3,67+7, \underline{61}+\underline{13}, 43+31, \underline{37}+\underline{37}$ |
| 76 | $\underline{73}+3,71+\underline{5}, 59+\underline{17}, \underline{53}+23,47+\underline{29}$ |
| 78 | $\underline{73}+\underline{5}, 71+7,67+11, \underline{61}+\underline{17}, 59+19,47+31, \underline{41}+\underline{37}$ |
| 80 | $79+\underline{1}, \underline{73}+7,67+\underline{13}, \underline{61}+19,43+\underline{37}$ |
| 82 | $79+3,71+11,59+23, \underline{53}+\underline{29}, \underline{41}+\underline{41}$ |
| 84 | $83+\underline{1}, 79+\underline{5}, \underline{73}+11,71+\underline{13}, 67+\underline{17}, \underline{61}+23, \underline{53}+31,47+\underline{37}, 43+\underline{41}$ |
| 86 | $83+3,79+7, \underline{73}+\underline{13}, 67+19,43+43$ |
| 88 | $83+\underline{5}, 71+\underline{17}, 59+\underline{29}, 47+\underline{41}$ |
| 90 | $\frac{89}{47}+\underline{1}, 83+7,79+11, \underline{73}+\underline{17}, 71+19,67+23, \underline{61}+\underline{29}, 59+31, \underline{53}+\underline{37},$ |
| 92 | $\underline{89}+3,79+\underline{13}, \underline{73}+19, \underline{61}+31$ |
| 94 | $\underline{89}+\underline{5}, 83+11,71+23, \underline{53}+\underline{41}, 47+47$ |
| 96 | $\underline{89}+7,83+\underline{13}, 79+\underline{17}, \underline{73}+23,67+\underline{29}, 59+\underline{37}, \underline{53}+43$ |
| 98 | $\underline{97}+\underline{1}, 79+19,67+31, \underline{61}+\underline{37}$ |
| 100 | $\underline{97}+3, \underline{89}+11,83+\underline{17}, 71+\underline{29}, 59+\underline{41}, \underline{53}+47$ |

The number of prime linear decompositions $n_{\ell d}$, (B1), varies somewhat sporadically in going from a specific $N, N_{i}$, to its neighbors $N_{i}+2, N_{i}+4$, etc. However, there is a definite trend of an increasing number of prime decompositions $n_{\ell d}$ with increasing $N$, as would be expected because of the increasing number of integers $x, y$ which are smaller than $N$, as $N$ increases. We note, in particular, that $n_{\ell d}=10$ for $N=90$ (see Table 6). Since the total number of all linear decompositions of $N=90$ into a sum of two odd numbers is

$$
(N+2) / 4=23
$$

we see that the percentage of the linear decompositions which consist of sums of primes is $10 / 23=43 \%$.

In Table 7 I have tabulated the total number of linear prime decompositions (ld) $n_{\ell d}$ for all even numbers $N$ in the range $2 \leqslant N \leqslant 100$. For the cases where $N$ is not divisible by 4, I have also 1 isted the partial $n_{\ell d}$ 's for two $P$-primes $(x, y)$, denoted by $n_{\ell d, 2}$, and for $n o$ P-prime, denoted by $n_{\ell d, 0}$. Obviously, when $N$ is not divisible by 4 , we have

$$
\begin{equation*}
n_{\ell d}=n_{\ell d, 2}+n_{\ell d, 0} . \tag{B7}
\end{equation*}
$$

At the bottom of the table, I have 1 isted the total number of ld's $\sum n_{\ell d}$ in the range $2 \leqslant N \leqslant 50$ and $52 \leqslant N \leqslant 100$, and for the complete range $2 \leqslant N \leqslant 100$. It is seen that $\sum n_{\ell d}$ increases from 78 for the first half of the table ( $N \leqslant 50$ ) to $\Sigma n_{\ell d}=135$ for the second half of the table ( $52 \leqslant N \leqslant 100$ ), showing the increase of the average $\Sigma n_{\ell d} / 25$ from 3.12 to 5.40 .

Similar tabulations have been made for $\sum n_{\ell d, 0}$ and $\sum n_{\ell d, 2}$. It is seen that the total number of $\ell d$ 's with $n_{P \text {-primes }}=0$ slightly predominates over the total number of $\ell d$ 's with $n_{p-\text { primes }}=2$. The ratio for the complete sample of 108 decompositions (up to $N=100$ ) is $60 / 48=1.25$.

I have also written down the prime decompositions for eight even integers in the range $102 \leqslant N \leqslant 200$. The results are:

$$
\begin{aligned}
& n_{\ell d}(N=116)=6, \quad n_{\ell d}(130)=7, \quad n_{\ell d}(150)=13, \quad n_{\ell d}(164)=6, \\
& n_{\ell d}(180)=15, \quad n_{\ell d}(182)=7, n_{\ell d}(184)=8, \text { and } n_{\ell d}(200)=9 .
\end{aligned}
$$

Finally, I wish to point out an important correlation which is as simple as the one derived by Fermat concerning $p_{i}=4 v+1$ for a $P$-prime and $q_{j}=4 v+3$ for a non- $P$-prime. It is well known that any prime number $p_{i}$ can be written in the form

$$
\begin{equation*}
p_{i}=6 v_{i}+1 \text { or } 6 v_{i}-1 \tag{B8}
\end{equation*}
$$

where $v_{i}$ is an arbitrary positive integer. (This equation does not, however, apply to the prime numbers 2 and 3 , and for $p_{i}=1$ we must use $\nu_{i}=0$.). The argument for (B8) goes as follows: Consider a specific $\nu_{i}$. Then $6 \nu_{i}+1$ is divisible by neither 2 nor 3, and therefore may be a prime; $6 v_{i}+2$ is divisible by $2 ; 6 v_{i}+3$ is divisible by $3 ; 6 v_{i}+4$ is again divisible by $2 ; 6 v_{i}+5=$ $6\left(\nu_{i}+1\right)-1$ is divisible by neither 2 nor 3, and therefore is a candidate for being a prime number.

Table 7. For all even integers $N$ in the range from 2 to $100, n_{\ell d}$ is the number of linear decompositions of $N$ into a sum of primes $N=x_{i}+y_{i}$, as given in Eq. (B1). For the integers $N$ which are divisible by 2 but not by 4, i.e., for values $N=4 \nu+2$, I have also listed the number of linear decompositions into a sum of two $P$-primes, denoted by $n_{\ell d, 2}$, and the number of linear decompositions into a sum of two non-P-primes, denoted by $n_{\ell d, 0}$. Obviously, for values of $N=4 \nu+2$, we have $n_{\ell d}=$ $n_{\ell d, 2}+n_{\ell d, 0}$. The sum of all $n_{\ell d}$ and $n_{\ell d, \alpha}(\alpha=0$ or 2$)$ is listed at the end of the table for the intervals $2 \leqslant N \leqslant 50$ and $52 \leqslant N \leqslant 100$, and also for the total range $2 \leqslant N \leqslant 100$.

| $N$ | $n_{l d}$ | $n_{\ell d, 2}$ | $n_{\ell d, 0}$ | N | $n_{\text {ld }}$ | $n_{\text {ld, } 2}$ | $n_{\ell d, 0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 1 | 1 | 0 | 52 | 3 |  |  |
| 4 | 2 |  |  | 54 | 6 | 3 | 3 |
| 6 | 2 | 1 | 1 | 56 | 3 |  |  |
| 8 | 2 |  |  | 58 | 4 | 3 | 1 |
| 10 | 2 | 1 | 1 | 60 | 7 |  |  |
| 12 | 2 |  |  | 62 | 4 | 1 | 3 |
| 14 | 3 | 1 | 2 | 64 | 5 |  |  |
| 16 | 2 |  |  | 66 | 6 | 3 | 3 |
| 18 | 3 | 2 | 1 | 68 | 3 |  |  |
| 20 | 3 |  |  | 70 | 5 | 2 | 3 |
| 22 | 3 | 1 | 2 | 72 | 7 |  |  |
| 24 | 4 |  |  | 74 | 6 | 3 | 3 |
| 26 | 3 | 1 | 2 | 76 | 5 |  |  |
| 28 | 2 |  |  | 78 | 7 | 3 | 4 |
| 30 | 4 | 2 | 2 | 80 | 5 |  |  |
| 32 | 3 |  |  | 82 | 5 | 2 | 3 |
| 34 | 4 | 2 | 2 | 84 | 9 |  |  |
| 36 | 4 |  |  | 86 | 5 | 1 | 4 |
| 38 | 3 | 1 | 2 | 88 | 4 |  |  |
| 40 | 3 |  |  | 90 | 10 | 4 | 6 |
| 42 | 5 | 3 | 2 | 92 | 4 |  |  |
| 44 | 4 |  |  | 94 | 5 | 2 | 3 |
| 46 | 4 | 2 | 2 | 96 | 7 |  |  |
| 48 | 6 |  |  | 98 | 4 | 2 | 2 |
| 50 | 4 | 1 | 3 | 100 | 6 |  |  |
| $\begin{aligned} & \sum n_{\ell d}(2 \leqslant N \leqslant 50) . \\ & \sum n_{\ell d}(52 \leqslant N \leqslant 100) \\ & \sum n_{\ell d}(2 \leqslant N \leqslant 100) \end{aligned}$ |  |  |  |  | 78 | 19 | 22 |
|  |  |  |  |  | 135 | 29 | 38 |
|  |  |  |  |  | 213 | 48 | 60 |

Now the correlation which can be derived from Fermat's $p_{i}=4 v+1$ theorem is that all Pythagorean primes are of the form

$$
\begin{equation*}
p_{i}=6 v_{i}+1 \text {, if } v_{i} \text { is even, } \tag{B9}
\end{equation*}
$$

and

$$
\begin{equation*}
p_{i}=6 v_{i}-1, \text { if } v_{i} \text { is odd. } \tag{B10}
\end{equation*}
$$

Thus, $37=(6)(6)+1$ is an example of (B9) (even $\left.\nu_{i}=6\right) ; 89=(6)(15)-1$ is an example of (B10).

In view of (B9) and (B10), the non-P-primes (except 2 and 3) are of the form

$$
\begin{align*}
& q_{j}=6 v_{j}-1, \text { if } v_{j} \text { is even, }  \tag{B11}\\
& q_{j}=6 v_{j}+1, \text { if } v_{j} \text { is odd. }
\end{align*}
$$

and

It should perhaps be noted that not all $\nu_{i}$ or $\nu_{j}$ give rise to $P-$ or non- $P-$ primes. The first few $\nu_{i}$ values which do not give rise to a prime number are: $v_{i}=20,24,31,34,36,41$, etc. The preceding equations signify only that if a given number is a $P$-prime $p_{i}$ or a non- $P$-prime $q_{j}$, then it can be expressed by (B9) or (B10), and (B11) or (B12), respectively.

Referring to the results of Table 7, I wish to note that the total number $n_{\text {ld }}$ of prime decompositions has maxima when $N$ is divisible by 6 ( $N=6 \mathrm{~V}$ ), at least starting with $N=24$. This trend is particularly noticeable when $N$ lies in the range from 72 to 96 . Thus, $n_{\ell d}(90)=10$ is considerably larger than $n_{\ell d}(88)=4$ and $n_{\ell d}(92)=4$. Similarly, $n_{\ell d}(84)=9$ characterizes a peak in the $n_{\ell d}$ values as a function of $N$ since, for the neighboring $N=82$ and $N=86$, we find $n_{\ell d}(82)=5$ and $n_{\ell d}(86)=5$. This property may be caused by the fact that, when $N=6 v$, we have two primes such that one of them is of the form $6 v_{1}+1$ and the other prime can be written as $6 v_{2}-1$, and in taking the sum, we obtain $N=6\left(\nu_{1}+\nu_{2}\right)=6 \nu$. It is also interesting that in several cases, particularly for $N=6 \mathrm{v}$, both members of each of two twin prime sets are involved, e.g.,

$$
78=\underline{73}+\underline{5}=71+7=\underline{61}+\underline{17}=59+19 .
$$

Note also that
and

$$
84=73+11=71+\underline{13}=\underline{41}+43
$$

$$
90=\underline{73}+\underline{17}=71+19=\underline{61}+\underline{29}=59+31 .
$$

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# ON THE MINIMUM OF A TERNARY CUBIC FORM 

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(Submitted May 1984)

Let

$$
\begin{equation*}
f_{a}=f_{a}(x, y, z)=x^{3}+y^{3}+z^{3}+3 a x y z \tag{1}
\end{equation*}
$$

$\alpha$ an arbitrary real constant, and denote, for a lattice $\Lambda$ in $\mathbb{R}^{3}$, by $\mu_{a}(\Lambda)$ the infimum of $\left|f_{a}\right|$ if $(x, y, z)$ runs through all lattice points of $\Lambda$ except ( 0,0 , 0 ). It is the objective of the present paper to estimate, from the above, the supremum $M_{a}$ of $\mu_{a}(\Lambda)$, taken over all lattices $\Lambda$ with lattice constant 1 . (Since any homogeneous ternary cubic polynomial can be transformed into the shape (1) by a suitable linear transformation, there is no loss of generality in starting from this canonical form.)

Classical work on this topic has been done by Mordell [6] (on the basis of his method of reducing the problem to a two-dimensional one) and by Davenport [1], [2]. Significant progress has been achieved in the special case $a=0$. For arbitrary $a$ however, the results obtained were not very sharp, as was noted by Golser [3], who improved upon Mordell's estimate for the general case, by a refined variant of his method. Later on, in [4], he observed that, for a certain range of the constant $a$, the bound can be improved further by the simple idea of inscribing a sphere into the star body $\left|f_{a}\right| \leqslant 1$.

The purpose of this short note is to establish a result that improves upon all known estimates for certain intervals of $\alpha$ (at least for $0.9 \leqslant \alpha \leqslant 2.9$ and for $-6 \leqslant a \leqslant-1.2$; see the tables at the end) by the elementary procedure of inscribing an ellipsoid of the shape

$$
\begin{equation*}
E_{t}(r): x^{2}+y^{2}+z^{2}+2 t(x y+x z+y z) \leqslant r^{2} \tag{2}
\end{equation*}
$$

where $t$ is a parameter with $-\frac{1}{2}<t<1$, into the body $K_{a}:\left|f_{a}\right| \leqslant 1$. Our result reads

Theorem: For arbitrary real $a$ and a parameter $t$ with $-\frac{1}{2}<t<1, t \neq 0$, we have

$$
M_{a} \leqslant \sqrt{2}(1-t) \sqrt{1+2 t} m_{a}(t),
$$

where

$$
\begin{aligned}
m_{a}(t) & :=\max \left\{|1+a|(1+2 t)^{-3 / 2} 3^{-1 / 2}, \phi_{1}(t), \phi_{2}(t)\right\}, \\
\phi_{j}(t) & :=\left(2+2 t+4 t c_{j}+c_{j}^{2}\right)^{-3 / 2}\left|2+3 a c_{j}+c_{j}^{3}\right| \quad(j=1,2), \\
c_{j} & :=(2 t)^{-1}\left(b-2 t+(-1)^{j}\left(b^{2}+4 t+4 b t^{2}\right)^{1 / 2}\right), b=a-1
\end{aligned}
$$

Proof: We first briefly recall some well-known facts from the Geometry of Numbers. The critical determinant $\Delta\left(K_{a}\right)$ of our body $K_{a}$ is defined as the infimum of all lattice constants $d(\Lambda)$ of lattices $\Lambda$ in $\mathbb{R}^{3}$ which have no point in the interior of $K_{a}$ except the origin. For any such lattice $\Lambda$, we put

$$
\Lambda_{1}=d(\Lambda)^{-1 / 3} \Lambda \text {, }
$$

[such that $\left.d\left(\Lambda_{1}\right)=1\right]$ and $\Lambda^{\prime}=\Lambda-\{(0,0,0)\}, \Lambda_{1}^{\prime}=\Lambda_{1}-\{(0,0,0)\}$. Since $f_{a}$ is homogeneous of degree 3 , it follows that

$$
\begin{aligned}
\Delta\left(K_{a}\right) & =\inf \left\{d(\Lambda): \inf _{\Lambda^{\prime}}\left|f_{a}\right| \geqslant 1\right\} \\
& =\inf _{d\left(\Lambda_{1}\right)=1} \inf _{\Lambda_{1}^{\prime}}\left\{d \in \mathbb{R}: \inf _{\Lambda_{1}^{\prime}}\left|f_{a}\right| \geqslant 1 / d\right\} \\
& =\left(\sup _{d\left(\Lambda_{1}\right)=1} \inf _{\Lambda_{1}^{\prime}}\left|f_{a}\right|\right)^{-1},
\end{aligned}
$$

hence $M_{a}=\Delta\left(K_{a}\right)^{-1}$. We further note that the ellipsoid $E_{t}(r)$ can be transformed into the unit sphere by the linear transformation

$$
\begin{aligned}
& x^{\prime}=(x+t y+t z) r^{-1}, y^{\prime}=\left(\sqrt{1-t^{2}} y+\sqrt{t-t^{2}} z\right) r^{-1} \\
& z^{\prime}=\frac{\sqrt{(1-t)(1+2 t)}}{r \sqrt{1+t}} z
\end{aligned}
$$

which is of determinant $(1-t) \sqrt{1+2 t} r^{-3}$. Since the critical determinant of the unit sphere equals $1 / \sqrt{2}$ (see 011erenshaw [7] or [5], p. 259), we conclude that

$$
\begin{equation*}
\Delta\left(E_{t}(r)\right)=r^{3}((1-t) \sqrt{1+2 t} \sqrt{2})^{-1} . \tag{3}
\end{equation*}
$$

If we choose $r$ maximal such that $E_{t}(r) \subset K_{a}$, then obviously $\Delta\left(K_{a}\right) \geqslant \Delta\left(E_{t}(r)\right)$, hence

$$
M_{a}=\Delta\left(K_{a}\right)^{-1} \leqslant r^{-3} \sqrt{2}(1-t) \sqrt{1+2 t}
$$

and, by homogeneity,

$$
\max _{E_{t}(r)}\left|f_{a}\right|=1 \Leftrightarrow \max _{E_{t}(1)}\left|f_{a}\right|=r^{-3} .
$$

Therefore, it suffices to establish the following
Lemma: For arbitrary $t$ with $-\frac{1}{2}<t<1, t \neq 0$, the absolute maximum of $\left|f_{a}\right|$ on $E_{t}(1)$ equals $m_{a}(t)$.

Proof: Since the absolute maximum of $\left|f_{a}\right|$ can be found among the relative extrema of $f_{a}$ on the boundary of $E_{t}(1)$, we determine the latter by Lagrange's rule. We obtain

$$
\begin{align*}
& 3 x^{2}+3 a y z+k(2 x+2 t(y+z))=0  \tag{4}\\
& 3 y^{2}+3 a x z+k(2 y+2 t(x+z))=0  \tag{5}\\
& 3 z^{2}+3 a x y+k(2 z+2 t(x+y))=0  \tag{6}\\
& x^{2}+y^{2}+z^{2}+2 t(x y+x z+y z)=1 \tag{7}
\end{align*}
$$

This system does not have any solution with $x \neq y \neq z \neq x$, for otherwise we could infer from (4) and (5) (subtracting and dividing by $x-y$ ) that

$$
3(x+y)-3 a z+2 k-2 k t=0
$$

and similarly, from (5) and (6), that

$$
3(y+z)-3 a x+2 k-2 k t=0
$$

Again subtracting, we would get the contradiction $x=z$ (at least for $a \neq-1$; the case $a=-1$ then can be settled by an obvious continuity argument).

Furthermore, it is impossible that a solution of our system satisfies $x=$ $y=0$, because this would imply that $k t z=0$ and $z(3+2 k)=0$, hence $z=0$, which contradicts (7). There remain two possibilities (apart from cyclic permutations).

Case 1: $x=y=z \neq 0$. By (7), we have

$$
x=y=z= \pm(1+2 t)^{-1 / 2} 3^{-1 / 2}
$$

and for these values of $x, y$, and $z$,

$$
\begin{equation*}
\left|f_{\alpha}\right|=|1+a|(1+2 t)^{-3 / 2} 3^{-1 / 2} \tag{8}
\end{equation*}
$$

Case 2: $0 \neq x=y \neq z$. Eliminating $k$ from (4) and (6), we get

$$
(2 t-a-a t) x^{3}+(1+a t) x^{2} z+(a-1-t) x z^{2}-t z^{3}=0
$$

This can be divided by $x-z$ and yields

$$
t z^{2}+(1+2 t-a) x z+(2 t-a-a t) x^{2}=0
$$

hence $z / x=c_{j}(j=1,2$, as defined in our theorem). From (7) we deduce that

$$
x=y= \pm\left(2+2 t+4 t c_{j}+c_{j}^{2}\right)^{-1 / 2}, \quad z=c_{j} x
$$

and for these values of $x, y$, and $z$,

$$
\begin{equation*}
\left|f_{a}\right|=\phi_{j}(t) \quad(j=1,2) . \tag{9}
\end{equation*}
$$

Combining (8) and (9), we complete the proof of the lemma and thereby that of our theorem.

Concluding Remarks: Letting $t \rightarrow 0$ in our result, we just obtain Golser's theorem 1 in [4]. However, this choice of $t$ turns out not to be the optimal one. In principle, one could look for an "advantageous" choice of the parameter $t$ (for a given value of the constant $a$ ) by computer calculations, but it can be justified by straightforward monotonicity considerations that it is optimal to choose $t$ such that $\max \left\{\phi_{1}(t), \phi_{2}(t)\right\}$ equals the right-hand side of (8).

We conclude the paper with tables indicating the new upper bounds for $M_{a}$ (for certain values of $\alpha$ ) as well as the corresponding "favorable" values of $t$ and the previously-known best results due to Golser [3], [4].

| $a$ | 0.9 | 1 | 2 | 2.9 |
| :--- | :--- | :--- | :--- | :---: |
| $t$ | 0.02799 | 0.040786 | 0.07973 | 0.07301 |
| $M_{a} \leqslant$ | 1.428 | 1.4483 | 1.9442 | 2.5758 |
| Golser: $M_{a} \leqslant$ | 1.454 | 1.5018 | 2.0597 | 2.5775 |

## ON THE MINIMUM OF A TERNARY CUBIC FORM

| $a$ | -6 | -5 | -4 | -3 | -2 | -1.2 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $t$ | -0.064204 | -0.07892 | -0.101987 | -0.14273 | -0.23042 | -0.41324 |
| $M_{a} \leqslant$ | 4.9848 | 4.1843 | 3.391 | 2.6116 | 1.8634 | 1.33 |
| Golser: $M_{a} \leqslant$ | 5.03779 | 4.31314 | 3.58475 | 2.85169 | 2.1106 | 1.54372 |

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# ON THE COEFFICIENTS OF A RECURSION RELATION FOR THE FIBONACCI PARTITION FUNCTION 

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Let $F=\{1,2,3,5, \ldots\}$ be the set of Fibonacci numbers, where $F_{2}=1$, $F_{3}=2$, and thereafter $F_{n}=F_{n-1}+F_{n-2}$. We shall examine the function $p_{F}(n)$, which we define to be the number of ways to additively partition the integer $n$ into (not necessarily distinct) Fibonacci numbers.

We first consider the generating function for $p_{F}(n)$. By elementary partition theory, we have

$$
\begin{equation*}
\sum_{n \geqslant 0} p_{F}(n) x^{n}=\prod_{a \in F} \frac{1}{1-x^{a}}=\prod_{m \geqslant 2} \frac{1}{1-x^{F_{m}}} . \tag{1}
\end{equation*}
$$

Equivalently,

$$
\begin{equation*}
\left(\prod_{m \geqslant 2}\left(1-x^{F_{m}}\right)\right) \sum_{n \geqslant 0} p_{F}(n) x^{n}=1 . \tag{2}
\end{equation*}
$$

We may expand the infinite product as a power series

$$
\begin{equation*}
\prod_{m \geqslant 2}\left(1-x^{F_{m}}\right)=\sum_{m \geqslant 0} a_{m} x^{m}, \tag{3}
\end{equation*}
$$

where $a_{m}$ counts the number of partitions of $m$ into an even number of distinct Fibonacci numbers, minus the number of partitions of $m$ into an odd number of distinct Fibonacci numbers; we may write this as

$$
\begin{equation*}
a_{m}=p_{F}^{\mathrm{e}}(m)-p_{F}^{\circ}(m) . \tag{4}
\end{equation*}
$$

We shall see later that knowledge of the terms $a_{m}$ will lead us to a recursion relation for $p_{F}(n)$. With this objective in mind, we prove the following.

Theorem: Let

$$
P_{k}=\prod_{m=2}^{k}\left(1-x^{F_{m}}\right) \quad \text { when } k \geqslant 2
$$

and set $P_{1}=1$. Let $L_{n}=F_{n+1}+F_{n-1}$ be the $n^{\text {th }}$ Lucas number. Then

$$
P_{\infty}=\prod_{m \geqslant 2}\left(1-x^{F_{m}}\right)=1-x-x^{2}+\sum_{k \geqslant 3} x^{L_{k}} P_{k-2} .
$$

First proof: This proof is combinatorial in nature. First consider the partial products $P_{k}$. When expanded as a power series of the form

$$
P_{k}=\sum_{m=0}^{k} a_{m}^{(k)} x^{m}
$$

the coefficients $\alpha_{m}^{(k)}$ represent the same thing as the $\alpha_{m}$, except that the partitions are now restricted to Fibonacci numbers not exceeding $F_{k}$. It is evident that $a_{m}^{(k)}=a_{m}$ for all $m$ such that $0 \leqslant m<F_{k+1}$, by inspection of (4). We shall use this fact later.

We may partition an integer $n$ into distinct Fibonacci numbers in one particular way by first writing down the largest Fibonacci number not greater than $n$, subtracting, and iterating this process on the difference. For example,

$$
27=21+5+1
$$

See the references listed at the end of this paper for a more detailed discussion of these points.

For simplicity of notation, we will represent a partition of a number into distinct Fibonacci numbers as a string of l's and 0 's, with the rightmost place corresponding to $F_{2}=1$, and each succeeding place corresponding to the next Fibonacci number. In the above example,

$$
27=1 \cdot 21+0 \cdot 13+0 \cdot 8+1 \cdot 5+0 \cdot 3+0 \cdot 2+1 \cdot 1
$$

which we may write more compactly as
(8)

27: 1001001,
where the (8) signifies that the 1 below it is the coefficient of $F_{8}=21$.
The first few terms $a_{m}$ for $m=0,1,2$, and 3 can be obtained by direct calculation of the first few $P_{k}$; they are seen to be $1,-1,-1$, and 0 , respectively. Now let $n \geqslant 3$; our objective will be to characterize the terms $a_{m}$ in the range $L_{n} \leqslant m<L_{n+1}$. As $L_{3}=4$, this will give us $a_{m}$ for every nonnegative $m$. Since $n \geqslant 3$, we have a partition of $L_{n+1}$ obtained in the above manner (from here on, all partitions are into distinct Fibonacci numbers, unless otherwise stated):

$$
L_{n+1}: 10 \stackrel{(n)}{1} 00000 c c c c
$$

Thus, we have the following nine possibilities for partitions of $m$, where $L_{n} \leqslant m<L_{n+1}$ :
(n)
(d) $0110 x \cdots x$
(e) $0101 \times x \cdots x$
(f) $0100 x \cdots x$
(n)
(a) $1001 x \cdots x$
(b) $10000 x \cdots x$
(g) $00011 x \times x$
(h) $0010 x \cdots x$
(i) $000 x x \cdots x$

The $x$ 's indicate "we don't care which digits go here."
In the above list, we may find a one-to-one correspondence between the partitions in (a) and the partitions in (c). Given a partition beginning with 100 , we may replace these three digits with 011 . Both strings will have equal value because $F_{n+2}=F_{n+1}+F_{n}$. However, out of each of these pairs of partitions, one is a partition of even cardinality, whereas the other is odd, since they are different only in their first three places. Hence, these partitions will cancel each other out when we compute $a_{m}$ using equation (4). Similarly, there is a one-to-one correspondence between partitions of type (b) and of type (d), and they cancel out for the same reason. Partitions of the forms (f) and (g) differ only in the positions corresponding to $F_{n+1}, F_{n}$, and $F_{n-1}$; they cancel each other out in the same way. Partitions of the forms (h) and (i) are excluded from possibility. To see this, recall that

$$
\begin{equation*}
F_{1}+F_{2}+\cdots+F_{k}=F_{k+2}-1 \tag{5}
\end{equation*}
$$

Thus, the largest number expressible in the form (h) is

$$
F_{n}+F_{n}-2<2 F_{n} .
$$

But $L_{n}=F_{n+1}+F_{n-1}$, so

$$
L_{n}=F_{n}+2 F_{n-1}>F_{n}+F_{n-1}+F_{n-2}=2 F_{n} .
$$

Thus, if $m$ is expressible in the form (h), then $m<L_{n}$, contrary to assumption. Similarly, if $m$ is expressible in the form (i), then $m \leqslant F_{n+1}-2$, which is less than $F_{n+1}$, which in turn is less than $L_{n}$, again a contradiction.

Therefore, the only class of partitions of $m$ which will contribute to the right-hand side of (4) are those of the form (e). But the leftmost four places in (e) form our partition of $L_{n}$; therefore, the $x \cdots x$ in (e) must represent a partition of $m-L_{n}$ into distinct Fibonacci numbers of size less than or equal to $n-2$. Conversely, given any such partition of $m-L_{n}$, we can construct a partition of $m$ of the form (e). Since both partitions in this correspondence are of the same parity, i.e., either both are partitions into odd numbers of Fibonacci numbers or both are partitions into even numbers of Fibonacci numbers, we deduce from (4) that

$$
\begin{equation*}
a_{L_{n}+m}=a_{m}^{(n-2)}, \quad \text { whenever } 0 \leqslant m<L_{n-1} . \tag{6}
\end{equation*}
$$

We have thus proved the theorem.
Second proof: This proof is analytical. We require the following two results: Let $A=a_{1}, a_{2}, \ldots$ be an arbitrary set of positive integers. If $|q|<1$, then

$$
\begin{equation*}
\prod_{a \in A} \frac{1}{\left(1-z q^{a}\right)}=1+\sum_{i \geqslant 1} \frac{z q^{a_{i}}}{\left(1-z q^{a_{1}}\right)\left(1-z q^{a_{2}}\right) \cdots\left(1-z q^{a_{i}}\right)} \tag{7}
\end{equation*}
$$

and

$$
\begin{equation*}
\prod_{a \in A}\left(1+z q^{a}\right)=1+\sum_{i \geqslant 1}\left(1+z q^{a_{1}}\right) \cdots\left(1+z q^{a_{i-1}}\right) z q^{a_{i}} \tag{8}
\end{equation*}
$$

Proof of (7): We consider the partial products. Clearly,

$$
\frac{1}{\left(1-z q^{a_{1}}\right)}=1+\frac{z q^{a_{1}}}{\left(1-z q^{a_{1}}\right)}
$$

Now suppose that

$$
\prod_{i=1}^{n} \frac{1}{\left(1-z q^{a_{1}}\right)}=1+\sum_{i=1}^{n} \frac{z q^{a_{i}}}{\left(1-z q^{a_{1}}\right)\left(1-z q^{a_{2}}\right) \cdots\left(1-z q^{a_{i}}\right)}
$$

Then

$$
\begin{aligned}
& \prod_{i=1}^{n+1} \frac{1}{\left(1-z q^{a_{i}}\right)}=\left(\prod_{i=1}^{n} \frac{1}{\left(1-z q^{a_{i}}\right)}\right)\left(1+\frac{z q^{a_{n+1}}}{\left(1-z q^{a_{n+1}}\right)}\right) \\
& =1+\sum_{i=1}^{n} \frac{z q^{a_{i}}}{\left(1-z q^{a_{1}}\right) \cdots\left(1-z q^{a_{i}}\right)}+\frac{z q^{a_{n+1}}}{\left(1-z q^{a_{n+1}}\right)} \prod_{i=1}^{n} \frac{1}{\left(1-z q^{a_{i}}\right)}
\end{aligned}
$$

## ON THE COEFFICIENTS OF A RECURSION RELATION

$$
=1+\sum_{i=1}^{n+1} \frac{z q^{a_{i}}}{\left(1-z q^{a_{1}}\right) \cdots\left(1-z q^{a_{i}}\right)} .
$$

By induction on $n$, the partial products on the left-hand side of (7) are equal to the partial sums on the right-hand side. As $|q|<1$, the sum converges. This proves (7).

Proof of (8): The first partial product is clearly the first partial sum. So suppose that

$$
\prod_{i=1}^{n}\left(1+z q^{a_{i}}\right)=1+\sum_{i=1}^{n}\left(1+z q^{a_{1}}\right) \cdots\left(1+z q^{a_{i-1}}\right) z q^{a_{i}}
$$

Then

$$
\begin{aligned}
& \prod_{i=1}^{n+1}\left(1+z q^{a_{i}}\right)=\left(\prod_{i=1}^{n}\left(1+z q^{a_{i}}\right)\right)\left(1+z q^{a_{n+1}}\right) \\
& =1+\sum_{i=1}^{n}\left(1+z q^{a_{1}}\right) \cdots\left(1+z q^{a_{i-1}}\right) z q^{a_{i}}+z q^{a_{n+1}} \prod_{i=1}^{n}\left(1+z q^{a_{i}}\right) \\
& =1+\sum_{i=1}^{n+1}\left(1+z q^{a_{1}}\right) \cdots\left(1+z q^{a_{i-1}}\right) z q^{a_{i}}
\end{aligned}
$$

By induction on $n$, this proves (8).
The following argument is due to the referee.
In (8) we set $z=-1, q=x$, and $A=F$ :

$$
\begin{aligned}
\prod_{m \geqslant 2}\left(1-x^{F_{m}}\right)= & 1-x-x^{2}+x^{3}-\sum_{m \geqslant 3}\left(1-x^{F_{2}}\right) \cdots\left(1-x^{F_{m}}\right) x^{F_{m+1}} \\
= & 1-x-x^{2}+x^{3}-\sum_{m \geqslant 3}\left(1-x^{F_{2}}\right) \cdots\left(1-x^{F_{m-1}}\right) x^{F_{m+1}} \\
& +\sum_{m \geqslant 3}\left(1-x^{F_{2}}\right) \cdots\left(1-x^{F_{m-1}}\right) x^{F_{m+2}} \\
= & 1-x-x^{2}+x^{3}-\sum_{m \geqslant 3}\left(1-x^{F_{2}}\right) \cdots\left(1-x^{F_{m-2}}\right) x^{F_{m+1}} \\
& +\sum_{m \geqslant 3}\left(1-x^{F_{2}}\right) \cdots\left(1-x^{\left.F_{m-2}\right) x^{L_{m}}}\right. \\
& +\sum_{m \geqslant 3}\left(1-x^{F_{2}}\right) \cdots\left(1-x^{F_{m-1}}\right) x^{F_{m+2}} \\
= & 1-x-x^{2}+\sum_{m \geqslant 3}\left(1-x^{F_{2}}\right) \cdots\left(1-x^{\left.F_{m-2}\right) x^{L_{m}}}\right.
\end{aligned}
$$

This proves the theorem.
Corollary: For all $m \geqslant 0, a_{m}$ is either $1,-1$, or 0 .
Proof: We have already seen that this is true for $m \leqslant 3$. Now the degree of $P_{k-2}$ is $1+2+3+\cdots+F_{k-2}$, or $F_{k}-2$, which is clearly less than $L_{k+1}-L_{k}$, which equals $L_{k-1}$. What this tells us is that the polynomials on the righthand side of the theorem add together without overlapping. Thus, we only need to show that each $P_{k}$ has coefficients $\alpha_{m}^{(k)}=0,1$, or -1 for all $m$.

We do this by induction on $k$. Clearly, it is true for $k=1$, as $P_{1}=1$. Suppose, then, that $a_{m}^{(k)}=1,-1$, or 0 for all $m$ and all $k<n$. By the definition of $P_{n}$, it is clear that the first $F_{n+1}$ coefficients of $P_{n}$ are identical to those of $P_{\infty}$; in other words,

$$
a_{m}=a_{m}^{(n)} \text { for all } m \text { such that } 0 \leqslant m<F_{n+1}
$$

Hence, by the theorem, the first $n+1$ terms $\alpha_{m}^{(n)}$ are the coefficients of the partial products $P_{k}$ with $k$ such that $L_{k}<F_{n+1}$; this includes all $k$ less than $n-2$ because $L_{n-3}<F_{n+1}<L_{n-2}$. By the induction hypothesis, the first $F_{n+1}$ coefficients are either $1,-1$, or 0 .

Now recall that $P_{n}$ is a finite product of "antipalindromic" polynomials of the form ( $1-x^{F_{k}}$ ). Thus, we have

$$
\begin{equation*}
a_{m}^{(n)}=(-1)^{n+1} a_{F_{n+2}-2-m}^{(n)} \tag{9}
\end{equation*}
$$

whenever both subscripts are positive, since the degree of $P_{n}$ is $F_{n+2}-2$. But $F_{n+1}>\frac{1}{2}\left(F_{n+2}-2\right)$, so the first half of the coefficients in $P_{n}$ are 1 , -1 , or 0 . By (9), so are the last half. By induction, all $P_{k}$ 's have coefficients 1 , -1 , or 0 . By the theorem, all terms $a_{m}$ are $1,-1$, or 0 . This proves the corollary.

By equating like terms on both sides of (2), where we have evaluated the product $P$ as the power series (3), we obtain, for all $n \geqslant 0$ :

$$
a_{0} p_{F}(n)+a_{1} p_{F}(n-1)+\cdots+a_{n-1} p_{F}(1)+a_{n} p_{F}(0)=0,
$$

where $p_{F}(0)=1$ in accordance with the power series (1). This yields a recursion for $p_{F}(n)$ with all coefficients $a_{k}$ equal to $1,-1$, or 0 .

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# HIGHER-ORDER FIBONACCI SEQUENCES MODULO M 

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Let $\left\{U_{n}, n \geqslant 0\right\}$ be the ordinary Fibonacci sequence defined by

$$
U_{0}=0, U_{1}=1, U_{n}=U_{n-1}+U_{n-2}, \text { for } n \geqslant 2 .
$$

For any integer $k \geqslant 2$, let $\left\{V_{k}(n), n \geqslant-k+2\right\}$ be the $k^{\text {th }}$-order Fibonacci sequence defined by

$$
\begin{aligned}
& V_{k}(j)=0, \text { for }-k+2 \leqslant j \leqslant 0, \quad V_{k}(1)=1, \\
\text { and } \quad & V_{k}(n)=V_{k}(n-1)+V_{k}(n-2)+\cdots+V_{k}(n-k), \text { for } n \geqslant 2 .
\end{aligned}
$$

It is well known that, for any integer $m \geqslant 2$, the sequence $U_{n}\left[=V_{2}(n)\right] \bmod m$ is periodic, and it is easy to see that this also holds for any sequence $V_{k}(n)$ mod $m$ with $k \geqslant 3$. For any $m \geqslant 2$, let $p(k, m)$ denote the length of the period of the sequence $V_{k}(n) \bmod m$. The proof of the next result is almost identical to that in [3] for the ordinary Fibonacci sequence $V_{2}(n)$, thus is omitted here.

Theorem 1: The sequence $V_{k}(n)$ mod $m$ is simply periodic, i.e., it is periodic and it repeats by returning to its starting values. If $m$ has the prime factorization $m=\pi q_{i}^{s_{i}}$, then $p(k, m)=1 \mathrm{~cm}\left[p\left(k, q_{i}^{s_{i}}\right)\right]$, the least common multiple of the $p\left(k, q_{i}^{s_{i}}\right)$.

In order to prove Theorem 2, we first state Lemma 1, the proof of which is quite simple and, therefore, will be omitted here.

Lemma 1: Let $\left\{W_{i}(n), n \geqslant 0\right\}, i=1,2,3$, be three sequences such that for each $i, W_{i}(n)=W_{i}(n-1)+\cdots+W_{i}(n-k)$ for all $n \geqslant k$. If the equality $W_{3}(n)=$ $W_{1}(n)+W_{2}(n)$ holds for $0 \leqslant n \leqslant k-1$, it also holds for all $n \geqslant k$.

The following result extends the corresponding result [3] for the sequence $V_{2}(n)$ to any sequence $V_{k}(n)$ with $k \geqslant 2$. Our proof is quite different from that in [3], and we do not have a general formula for $V_{k}(n)$.

Theorem 2: Let $q$ be any prime number. If $p\left(k, q^{2}\right) \neq p(k, q)$, then

$$
\begin{equation*}
p\left(k, q^{e}\right)=q^{e-1} p(k, q) \tag{1}
\end{equation*}
$$

for any integer $e \geqslant 2$.
Proof: Let $r=p(k, q)$. For the sake of simpler notation, we shall prove (1) only for $e=2$. The same proof stands, with obvious modifications, for $e>2$. Define the $k$-tuple

$$
T_{0}=\left(V_{k}(-k+2), \ldots, V_{k}(1)\right)=(0, \ldots, 0,1),
$$

and

$$
\begin{aligned}
T_{1} & =\left(V_{k}(-k+2+r), \ldots, V_{k}(1+r)\right)=(0, \ldots, 0,1) \quad \bmod q \\
& =\left(q s_{1}, \ldots, q s_{k-1}, q s_{k}+1\right) \quad \bmod q^{2}
\end{aligned}
$$

where $0 \leqslant s_{j}<q$ for $1 \leqslant j \leqslant k$, and $s_{1}+\cdots+s_{k} \geqslant 1$. The $k$-tuple $T_{1}$ is obtained by moving $T_{0} r$ units to the right.
$T_{1}$ can be decomposed as follows:

$$
\begin{aligned}
T_{1}= & q s_{1}\left(1,1,2, \ldots, 2^{k-2}\right)+q\left(s_{2}-s_{1}\right)\left(0,1,1, \ldots, 2^{k-3}\right) \\
& +q\left(s_{3}-s_{2}-s_{1}\right)\left(0,0,1, \ldots, 2^{k-4}\right)+\ldots \\
& +\left[q\left(s_{k}-s_{k-1}-\cdots-s_{1}\right)+1\right](0,0, \ldots, 0,1) \bmod q^{2} .
\end{aligned}
$$

Applying Lemma 1 , one can obtain the $k$-tuple $T_{2}$ by moving $T_{1} r$ units to the right.

$$
\begin{aligned}
T_{2}= & {\left[q\left(s_{k}-s_{k-1}-\ldots-s_{1}\right)+1\right]\left(q s_{1}, q s_{2}, q s_{3}, \ldots, q s_{k-1}, q s_{k}+1\right)+\cdots } \\
& +q\left(s_{2}-s_{1}\right)\left(q s_{k-1}, q s_{k}+1, q\left(s_{k}+s_{k-1}+s_{k-2}\right)+1, \ldots, q(\ldots)+2^{k-3}\right) \\
& +q s_{1}\left(q s_{k}+1, q\left(s_{k}+s_{k-1}+s_{k-2}\right)+1, \ldots, q(\ldots)+2^{k-2}\right) \bmod q^{2} \\
= & \left(2 q s_{1}, 2 q s_{2}, \ldots, 2 q s_{k-1}, 2 q s_{k}+1\right) \bmod q^{2} .
\end{aligned}
$$

Similarly, one has

$$
T_{j}=\left(j q s_{1}, j q s_{2}, \ldots, j q s_{k-1}, j q s_{k}+1\right) \bmod q^{2}
$$

for $2 \leqslant j \leqslant q$. Since $q$ is a prime number, $T_{j} \neq T_{0}$ for $1 \leqslant j \leqslant q-1$, and since $T_{q}=T_{0} \bmod q^{2}$, we have $p\left(k, q^{2}\right)=q r=q p(k, q)$. This completes the proof.

As a final remark, we note that some simple facts about higher-order Fibonacci sequences can be easily observed. For example, many moduli $m$ have the property that the sequence $V_{k}(n)$ mod $m$ contains a complete system of residue modulo $m$, while $m=8$ and $m=9$ are the smallest moduli which do not have this property in the case $k=3$, and they are said to be defective [2]. For $m=2$ and $m=11$, the sequence $V_{3}(n)$ mod $m$ is uniformly distributed. (See [1] for a definition.) It is interesting to extend the results for ordinary Fibonacci sequences to those of higher order.

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# A NOTE ON PASCAL-T TRIANGLES, MULTINOMIAL COEFFICIENTS, AND PASCAL PYRAMIDS 

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## 1. INTRODUCTION

In what follows we give a formula for the entries in the Pascal- $T$ triangle $T_{m}$ in terms of the multinomial coefficients; this is the counterpart for these arrays of the result of Philippou [1] on the elements of the Fibonacci $k$-sequences in terms of the multinomial coefficients. The proof is direct, and the method also leads to a recurrence relation which gives the elements of a given triangle $T_{m}$ as a combination of certain elements of the "preceeding" triangle $T_{m-1}$, the coefficients in the combination being binomial coefficients. Finally, because the multinomial coefficients provide the connection here, we offer some remarks on those arrays of multinomial coefficients referred to in the literature as "Pascal pyramids."

It will be convenient to recall the definition of the triangle $T_{m}$.
Definition 1.1: For any $m \geqslant 0, T_{m}$ is the array whose rows are indexed by $n=0$, $1,2, \ldots$. and columns by $k=0,1,2, \ldots$, and whose entries are obtained as follows:
a) $T_{0}$ is the all-zero array;
b) $T_{1}$ is the array all of whose rows consist of a one followed by zeros;
c) $T_{m}, m \geqslant 2$, is the array whose $n=0$ row is a one followed by zeros, whose $n=1$ row is $m$ ones followed by zeros, and any of whose entries in subsequent rows is the sum of the $m$ entries just above and to the left in the preceeding row.

The entry in row $n$ and column $k$ is denoted by $C_{m}(n, k)$, although we note that

$$
C_{2}(n, k)=\binom{n}{k}
$$

since $T_{2}$ is the Pascal Triangle. There will be $n(m-1)+1$ nonzero entries in row $n$, and the principal property we need is that these are the coefficients (see, e.g., [2], p. 66) in the expansion

$$
\begin{equation*}
\left(1+t+t^{2}+\cdots+t^{m-1}\right)^{n}=\sum_{k=0}^{n(m-1)} C_{m}(n, k) t^{k} \tag{1.1}
\end{equation*}
$$

Although it is easy to use property (c) to build the array $T_{m}$ by means of the relation

$$
\begin{equation*}
C_{m}(n, k)=\sum_{j=0}^{m-1} C_{m}(n-1, k-j) \tag{1.2}
\end{equation*}
$$

the main result presented here evaluates $C_{m}(n, k)$ directly as a sum of certain multinomial coefficients.

## 2. FORMULAS FOR $C_{m}(n, k)$

Theorem 2.1: If $C_{m}(n, k)$ is the ( $n, k$ )-entry in $T_{m}, m \geqslant 3$, then for any $n \geqslant 0$ and $0 \leqslant k \leqslant n(m-1)$,

$$
\begin{equation*}
C_{m}(n, k)=\sum_{n_{1}, n_{2}, \ldots, n_{m}}\binom{n}{n_{1}, n_{2}, \ldots, n_{m}} \tag{2.1}
\end{equation*}
$$

where the summation is over all m-part compositions $n_{1}, n_{2}, \ldots, n_{m}$ of $n$ such that (1) $n_{1}+n_{2}+\cdots+n_{m}=n$, and (2) $0 n_{1}+1 n_{2}+\cdots+(m-1) n_{m}=k$.

Proof: The proof follows directly from the multinomial theorem, for if in

$$
\begin{equation*}
\left(x_{1}+x_{2}+\cdots+x_{m}\right)^{n}=\sum\binom{n}{n_{1}, \ldots, n_{m}} x_{1}^{n_{1}} x_{2}^{n_{2}} \ldots x_{m}^{n_{m}}, \tag{2.2}
\end{equation*}
$$

where the summation is over all m-part compositions of $n$, we put $x_{i}=t^{i-1}$, $1 \leqslant i \leqslant m$, we have

$$
\begin{equation*}
\left(1+t+\cdots+t^{m-1}\right)^{n}=\sum\binom{n}{n_{1}, \ldots, n_{m}} t^{n_{2}+2 n_{3}+\cdots+(m-1) n_{m}} \tag{2.3}
\end{equation*}
$$

and when the coefficients of $t^{k}$ on the right-hand sides of (1.1) and (2.3) are equated, (2.1) follows from conditions (1) and (2).

$=1+12+6+12=31$
Another application of the multinomial expansion, used partly as a binomial expansion, gives the following theorem.

Theorem 2.2:

$$
\begin{equation*}
C_{m}(n, k)=\sum_{j=0}^{n}\binom{n}{j} C_{m-1}(j, k-j) . \tag{2.4}
\end{equation*}
$$

Proof: If the left side of (2.2) is grouped as $\left[x_{1}+\left(x_{2}+\cdots+x_{m}\right)\right]^{n}$, expanded as a binomial, and again $t^{i-1}$ is substituted for $x_{i}$, the result is

$$
\begin{equation*}
\sum_{k=0}^{n(m-1)} C_{m}(n, k) t^{k}=\sum_{j=0}^{n}\binom{n}{j} t^{j}\left(1+t+\cdots+t^{m-2}\right)^{j} \tag{2.5}
\end{equation*}
$$

But then the factors $\left(1+t+\cdots+t^{m-2}\right)^{j}$ may be expressed in terms of $C_{m-1}{ }^{\text {' }} \mathrm{s}$, using (1.1). When the coefficients of a given power of $t$ on the right of (2.5) are collected and equated to the corresponding coefficient on the left, then (2.4) follows.

Example: $C_{4}(4,4)=\binom{4}{0} C_{3}(0,4)+\binom{4}{1} C_{3}(1,3)+\binom{4}{2} C_{3}(2,2)$

$$
+\binom{4}{3} C_{3}(3,1)+\binom{4}{4} C_{3}(4,0)
$$

$=1 \cdot 0+4 \cdot 0+6 \cdot 3+4 \cdot 3+1 \cdot 1=31$

## 3. THE PASCAL PYRAMID

The device used in the previous theorem of bracketing off one term of a multinomial in order to expand the result as a binomial can, of course, be repeated with the remaining multinomial parts, eventually running the unexpanded part down to a binomial itself; this offers the possibility of obtaining the multinomials entirely as products of binomial coefficients. In fact, this has been done in [3] and [4] for a trinomial expansion, with the multinomial coefficients appearing in the successive powers of $\left(x_{1}+x_{2}+x_{3}\right)$ being associated with points in triangular arrays, which form successive levels of a pyramidal structure-the so-called Pascal pyramid. For example, Figure 1 shows the first four levels, with each point labelled with both a multinomial coefficient and the composition which gives rise to it (the compositions can be obtained by designating the sides as first, second, third in some fashion, and letting $n_{1}$, $n_{2}, n_{3}$ in the composition measure units of perpendicular distances from the first, second, third sides, respectively). The law of formation for this trinomial case is clear (and also correct, as is easily verified by doing the reduction described earlier): just generate the ordinary Pascal triangle down to level $n$, and then multiply the rows successively upward by the numbers found in the last line. For $n=3$, e.g.,

1

## $1 \quad 1$

$\begin{array}{lll}1 & 2 & 1\end{array}$
$\begin{array}{llll}1 & 3 & 3 & 1\end{array}$
becomes

$n=3$


Figure 1. Levels of the Pascal Pyramid for the Case ( $x_{1}+x_{2}+x_{3}$ )

This works nicely, the idea is not restricted to trinomials (the generalization is not just a bigger triangle, but it is similarly simple), and there have been several comments to the effect that it is surprising that the Pascal pyramids (or hyperpyramids) are not more widely known or used.

Why should this be? An answer would seem to be that as soon as one gets past the trinomials, the method, while still elegant, becomes computationally unwieldy. That is, in the usual expansion of a multinomial $\left(x_{1}+\cdots+x_{m}\right)^{n}$ there are $\binom{n+m-1}{n}$ terms corresponding to the $m$-part compositions of $n$. To see what is required to deal with these in terms of products of binomials, we look at what might be called the Pascal square, in which we tabulate for $m=0$, $1,2, \ldots$ and $n=0,1,2, \ldots$ the number of $m$-part compositions of $n$ (taking the entry for $m=n=0$ to be one). The first several lines are shown below:

Pascal Square

| No. Parts $m$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 1 | 3 | 6 | 10 | 15 | 21 |
| 4 | 1 | 4 | 10 | 20 | 35 | 56 |
| 5 | 1 | 5 | 15 | 35 | 70 | 126 |

Here, the law of formation is that each entry is the sum of $a Z Z$ those entries above and to the left of it in the preceeding row, and we recognize the $m=3$ row as the triangular numbers, the $m=4$ as the pyramidal numbers, and so on. The point here is that the square shows that the trinomials ( $m=3$ ) are simple sequences of products of binomials; as in the example, the ten trinomials in $\left(x_{1}+x_{2}+x_{3}\right)^{3}$ reduce to $4+3+2+1$ products of binomials. But for $m>3$, we find not sequences, but sequences of sequences. The thirty-five terms in $\left(x_{1}+x_{2}+x_{3}+x_{4}\right)^{4}$ [the (4, 4) entry], e.g., have to be obtained using the sequence

| 15 | $=$ | $5+4+3+2+1$, |  |
| ---: | ---: | ---: | ---: |
| 10 | $=$ | $4+3+2+1$, |  |
| 6 | $=$ | $3+2+1$, |  |
| 3 | $=$ | $2+1$, |  |
| 1 |  |  | 1, |

of sequences of products of binomials. It would seem that in spite of the appeal of an array for multinomial coefficients similar to the triangle for binomials, one is better off for most purposes using a convenient algorithm (e.g., [5], pp. 46-51) to generate the $m$-part compositions of $n$, from which the exponents on the $x_{i}$ and the multinomial associated with a given term are immediately available.

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## ANNOUNCEMENT

# SECOND INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS 

August 13-16, 1986<br>San Jose State University<br>San Jose, California 95192

LOCAL COMMITTEE
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Edgar, Hugh, Cochairman
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## CALL FOR PAPERS

The SECOND INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at San Jose State University, San Jose, CA, Aug. 13-16, 1986. This conference is sponsored jointly by The Fibonacci Association and San Jose State University.

Papers on all branches of mathematics and science related to the Fibonacci numbers and their generalizations are welcome. Manuscripts are requested by June 15, 1986. Abstracts and manuscripts should be send to the chairman of the local committee. Invited and contributed papers will appear in the Conference Proceedings, which are expected to be published.

The program for the Conference will be mailed to all participants, and to those individuals who have indicated an interest in attending the conference, by June 15, 1986. All talks should be limited to one hour.

For further information concerning the conference, please contact either of the following:
Professor G.E. Bergum, Editor
Professor Calvin Long
The Fibonacci Quarterly
Department of Mathematics
Department of Mathematics
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Washington State University
Pullman, Wash. 99163
Brookings, S.D. 57007-1297

# AN ENTIRE FUNCTION THAT GIVES THE FIBONACCI numbers at the integers 

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## 1. INTRODUCTION

It is well known that $\Gamma(z)$ is an analytic function of $z$ that gives $n$ ! when $z=n+1$. It is reasonable to look for a similar function for the Fibonacci numbers $F_{n}$. Several such functions are known (see Bunder [2] where further references are given), but the formula we will derive is more general than any of those obtained earlier.

To be specific, we are looking for an $F(z)$ with the following properties:
(a) $F(z)$ is an analytic function (perhaps entire),
(b) $F(z)$ is real valued for all real $z$,
(c) $F(n)=F_{n}$, the $n^{\text {th }}$ Fibonacci number for all integers $n$,
(d) For $z$ in the domain of analyticity we have

$$
\begin{equation*}
F(z+2)=F(z+1)+F(z) . \tag{1}
\end{equation*}
$$

It is clear that if $F(0)=F_{0}=0$ and $F(1)=F_{1}=1$, then equation (1) implies that $F(n)=F_{n}$ for every positive integer $n$. This follows immediately from the defining equations $F_{0}=0, F_{1}=1$, and $F_{n+2}=F_{n+1}+F_{n}$. In fact, this latter relation can be used to define the Fibonacci numbers for negative integers.

If $F(z)$ satisfies the functional equation (1), then so does each derivative $F^{(m)}(z), m=1,2, \ldots$. This suggests that we try $e^{R z}$ as a solution, for some number $R$. When $e^{R z}$ is used in (1), we find that it is a solution if and only if $e^{R}$ is a root of

$$
\begin{equation*}
x^{2}=x+1 \tag{2}
\end{equation*}
$$

Using the standard notation for the roots of (2), we have

$$
\begin{equation*}
\alpha=\frac{1+\sqrt{5}}{2}, \quad \beta=\frac{1-\sqrt{5}}{2}, \tag{3}
\end{equation*}
$$

and hence

$$
R=\ln \alpha \text { or } R=\ln \beta=\ln |\beta|+(2 q+1) \pi i, q=0, \pm 1, \pm 2, \ldots
$$

Using the linearity of (1) (see Spickerman [4]), it is clear that if $p$ and $q$ are integers, and $C_{1}$ and $C_{2}$ are arbitrary real numbers, then

$$
\begin{equation*}
f(z)=C_{1} e^{z(\ln \alpha+2 p \pi i)}+C_{2} e^{z(\ln |\beta|+(2 q+1) \pi i)} \tag{4}
\end{equation*}
$$

satisfies the functional equation (1). Now $f(z)$ is an entire function but it is not real valued for every real $z$. To remedy this defect, we consider
an entire function that gives the fibonacci numbers at the integers

$$
[f(z)+\overline{f(z)}] / 2
$$

This function is not an analytic function, but if we replace $z$ by $x$ we obtain the real function

$$
\begin{equation*}
F(x)=C_{1} e^{x \ln \alpha} \cos 2 p \pi x+C_{2} e^{x \ln |\beta|} \cos (2 q+1) \pi x \tag{5}
\end{equation*}
$$

If we now replace $x$ by $z$ in (5), we have a function that satisfies the conditions (a), (b), and (d). The initial conditions $F(0)=0$ and $F(1)=1$ force the selection $C_{1}=1 / \sqrt{5}$ and $C_{2}=-1 / \sqrt{5}$. Then, finally, the function

$$
\begin{equation*}
F(z)=\frac{1}{\sqrt{5}}\left[e^{z \ln \alpha} \cos 2 p \pi z-e^{z \ln |\beta|} \cos (2 q+1) \pi z\right] \tag{6}
\end{equation*}
$$

has all of the properties (a), (b), (c), and (d) that we wish.
Equation (6) was given earlier by Spickerman [4] and is an entire function that gives the Fibonacci numbers for integral values of $z$.

## 2. THE MAIN THEOREM

Equation (6) gives a countable infinity of functions that satisfy the conditions (a), (b), (c), and (d), and we may ask if we now have all such functions. In fact, we shall soon see that (6) gives only a tiny portion of the functions that satisfy (a), (b), (c); and (d). We first observe that if $\alpha$ and $\beta$ are the roots of (2) and $m$ is an integer, then

$$
\begin{equation*}
G(z)=e^{z \ln \alpha} \sin 2 m \pi z+e^{z \ln |\beta|} \sin (2 m+1) \pi z \tag{7}
\end{equation*}
$$

satisfies the three conditions (a), (b), and (d). Further, $G(n)=0$ for every integer $n$.

We now take linear combinations of the functions $F(z)$ and $G(z)$ defined by (6) and (7). To simplify the presentation, we impose a condition on the coefficients to ensure that we obtain entire functions.

Definition: We say that the real sequences $\left\{A_{m}\right\},\left\{B_{m}\right\},\left\{C_{m}\right\}$, and $\left\{D_{m}\right\}$ satisfy condition $E$ if

$$
\begin{equation*}
\sum_{m=0}^{\infty} C_{m}=1, \quad \sum_{m=0}^{\infty} D_{m}=1 \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{m=0}^{\infty} A_{m} e^{m z}, \quad \sum_{m=0}^{\infty} B_{m} e^{m z}, \quad \sum_{m=0}^{\infty} C_{m} e^{m z}, \quad \sum_{m=0}^{\infty} D_{m} e^{m z} \tag{9}
\end{equation*}
$$

are all entire functions.
These are very weak restrictions. For example, (9) is trivially satisfied if all but a finite number of terms in each sequence are zero. The linearity of equation (1), and our earlier work, immediately give

Theorem 1: Let $\left\{A_{m}\right\},\left\{B_{m}\right\},\left\{C_{m}\right\},\left\{D_{m}\right\}$ satisfy condition $E$, and let $\alpha$ and $\beta$ be defined by (3). Then each one of the functions

$$
\begin{equation*}
F(z)=\frac{1}{\sqrt{5}} \sum_{m=0}^{\infty} C_{m} e^{z \ln \alpha} \cos 2 m \pi z-\frac{1}{\sqrt{5}} \sum_{m=0}^{\infty} D_{m} e^{z \ln |\beta|} \cos (2 m+1) \pi z+ \tag{10}
\end{equation*}
$$ (continued)

## an entire function that gives the fibonacci numbers at the integers

$$
+\sum_{m=0}^{\infty} A_{m} e^{z \ln \alpha} \sin 2 m \pi z+\sum_{m=0}^{\infty} B_{m} e^{z \ln |\beta|} \sin (2 m+1) \pi z
$$

satisfies the conditions (a), (b), (c), and (d).
It is clear that (1) gives an uncountable infinity of suitable functions. We still have an uncountable infinity if we set all coefficients equal to zero except $C_{0}, C_{1}=1-C_{0}, D_{0}$, and $D_{1}=1-D_{0}$.

Do we have all such function? In other words, given a function with properties (a), (b), (c), and (d), is it one of the functions described in Theorem 1? This is an open problem.

The Fibonacci numbers satisfy many interesting relations, see, for example, Bachman [1, II:55-96], Vorob'ev [5], or Wall [6]. Many of these generalize, and we cite only a few here.

If $F(z)$ is any one of the uncountably many functions given in Theorem 1 , then, for all $z$,

$$
\begin{align*}
& \sum_{k=0}^{N} F(z+k)=F(z+N+2)-F(z+1)  \tag{11}\\
& \sum_{k=1}^{N} F(z+2 k-1)=F(z+2 N)-F(z) \tag{12}
\end{align*}
$$

and

$$
\begin{equation*}
\sum_{k=0}^{2 N}(-1)^{k} F(z+k)=F(z+2 N-1)-F(z-2) \tag{13}
\end{equation*}
$$

## 3. A GENERALIZATION

One natural generalization arises when we replace $F_{n+2}=F_{n+1}+F_{n}$ by

$$
F_{n+2}=r F_{n+1}+s F_{n}
$$

and impose the initial conditions $F_{0}=a$ and $F_{1}=b$. To extend the work of §l and §2, we look for entire functions that are real on the real axis, give the generalized Fibonacci numbers at the positive integers, and satisfy the functional equation

$$
\begin{equation*}
F(z+2)=r F(z+1)+s F(z) \tag{14}
\end{equation*}
$$

for all z. Here we restrict $r, s, a$, and $b$ to be real. We preserve the basic notation of $\S 2$ and set

$$
\begin{equation*}
\alpha=\frac{r+\sqrt{r^{2}+4 s}}{2}, \quad \beta=\frac{r-\sqrt{r^{2}+4 s}}{2}, \tag{15}
\end{equation*}
$$

the two roots of

$$
\begin{equation*}
x^{2}=r x+s . \tag{16}
\end{equation*}
$$

[Compare this equation with equation (2).]
For simplicity, we assume that $\alpha$ and $\beta$ are distinct real roots, and this implies that $r^{2}+4 s>0$. We also assume that $s \neq 0$ because, if $s=0$, equation (14) reduces to $F(z+1)=r F(z)$ for all $z$, and the generalized Fibonacci sequence is then a geometric sequence. If $r$ and $s$ are positive, then $\alpha>0>\beta$. We consider this case first.

Theorem 2: Suppose that $\alpha>0>\beta$, where $\alpha$ and $\beta$ are given by (15), $r$ and $s$ are real numbers, and the sequences $\left\{A_{m}\right\},\left\{B_{m}\right\},\left\{C_{m}\right\}$, and $\left\{D_{m}\right\}$ satisfy condition $E$. Set

$$
\begin{align*}
F(z)= & \sum_{m=0}^{\infty} \frac{-a \beta+b}{\alpha-\beta} C_{m} e^{z \ln \alpha} \cos 2 m \pi z+\sum_{m=0}^{\infty} \frac{\alpha \alpha-b}{\alpha-\beta} D_{m} e^{z \ln |\beta|} \cos (2 m+1) \pi z \\
& +\sum_{m=0}^{\infty}\left(A_{m} e^{z \ln \alpha} \sin 2 m \pi z+B_{m} e^{z \ln |\beta|} \sin (2 m+1) \pi z\right) \tag{17}
\end{align*}
$$

Then:
(a) $F(z)$ is an entire function;
(b) $F(z)$ is real on the real axis;
(c) $F(z)$ satisfies the functional equation (14);
(d) for all positive integers $F(n)=F_{n}$, the $n^{\text {th }}$ generalized Fibonacci number defined by $F_{0}=a, F_{1}=b, F_{n+2}=r F_{n+1}+s F_{n}, n=0,1,2, \ldots$.

We omit the proof because it follows the pattern set forth in §2. First, one shows that each individual term satisfies (14), and then one applies the linearity property. A simple computation shows that $F(0)=a$ and $F(1)=b$. Parker [3] obtained a simplified version of (17) in which only two of the coefficients are different from zero.

If $r>0$ and $s<0$, then $\alpha>\beta>0$. In this case, we have
Theorem 3: Suppose that $\alpha>\beta>0$ and the sequences $\left\{A_{m}\right\},\left\{B_{m}\right\},\left\{C_{m}\right\}$, and $\left\{D_{m}\right\}$ satisfy condition $E$. Set

$$
\begin{align*}
F^{\prime}(z)= & \sum_{m=0}^{\infty} \frac{-\alpha \beta+b}{\alpha-\beta} C_{m} e^{z \ln \alpha} \cos 2 m \pi z+\sum_{m=0}^{\infty} \frac{\alpha \alpha-b}{\alpha-\beta} D_{m} e^{z \ln \beta} \cos 2 m \pi z \\
& +\sum_{m=0}^{\infty}\left(A_{m} e^{z \ln \alpha} \sin 2 m \pi z+B_{m} e^{z \ln \beta} \sin 2 m \pi z\right) \tag{18}
\end{align*}
$$

Then $F(z)$ satisfies conditions (a), (b), (c), and (d) of Theorem 2.
The proof is similar to that of Theorem 2; thus, it is omitted here.
If $r<0$ and $s<0$, then $0>\alpha>\beta$. In this case, we replace $\alpha$ and $\beta$ by $|\alpha|$ and $|\beta|$, respectively, in (18). Further, $\cos 2 m \pi z$ is replaced by $\cos (2 m+1) \pi z$ and $\sin 2 m \pi z$ is replaced by $\sin (2 m+1) \pi z$. The details are left to the reader.

In each of the three cases, there is an uncountable infinity of functions, each satisfying the conditions (a), (b), (c), and (d).

## 4. CONCLUDING REMARKS

We return to the original Fibonacci sequence $0,1,1,2,3,5, \ldots$ treated in §§1 and 2. If $\alpha$ and $\beta$ are given by (3), then, as is well known,

$$
\begin{equation*}
F_{n}=\frac{1}{\sqrt{5}}\left(\alpha^{n}-\beta^{n}\right) \tag{19}
\end{equation*}
$$

## AN ENTIRE FUNCTION THAT GIVES THE FIBONACCI NUMBERS AT THE INTEGERS

This formula for $F_{n}$ is called Binet's formula. If we replace the minus sign by a plus sign in (19), we obtain

$$
\begin{equation*}
L_{n}=\alpha^{n}+\beta^{n} \tag{20}
\end{equation*}
$$

These numbers $L_{n}, n=0,1,2, \ldots$. are often called the Lucas numbers [5, 6]. Now $L_{0}=2, L_{1}=1$, and $L_{n+2}=L_{n+1}+L_{n}$ for $n=0,1,2, \ldots$ Consequently, Theorem 2 gives a set of uncountably many entire functions for the Lucas numbers. Indeed, set $a=2$ and $b=1$ in (17) to obtain

$$
\begin{equation*}
\frac{-a \beta+b}{\alpha-\beta}=1 \quad \text { and } \quad \frac{a \alpha-b}{\alpha-\beta}=1 \tag{21}
\end{equation*}
$$

Then $F(n)=L_{n}$ for all $n$.
Finally, we note that Binet's formula can be extended to cover the generalized Fibonacci numbers treated in §3. Let $r, s, a, b, \alpha$, and $\beta$ be real numbers, where $\alpha$ and $\beta$ are given by (15). If $F_{0}=\alpha, F_{1}=b, F_{n+2}=r F_{n+1}+s F_{n}$, for $n=0,1,2, \ldots$, then

$$
\begin{equation*}
F_{n}=\frac{-\alpha \beta+b}{\alpha-\beta} \alpha^{n}+\frac{\alpha \alpha-b}{\alpha-\beta} \beta^{n}, \text { for } n=0,1,2, \ldots . \tag{22}
\end{equation*}
$$

Here, of course, we assume that $r^{2}+4 s>0$ so $\alpha \neq \beta$ and both $\alpha$ and $\beta$ are real numbers. For brevity, we omit the discussion of the special cases (a) $\alpha=\beta$, (b) $\alpha=0>\beta$, and (c) $\alpha>\beta=0$. In these last two cases, equation (16) gives $s=0$. Hence, $F_{n+1}=r F_{n}$ and the sequence $\left\{F_{n}\right\}$ is a geometric sequence.

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# ON THE ENUMERATOR FOR SUMS OF THREE SQUARES 

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## 1. INTRODUCTION

For each nonnegative integer $n, r_{3}(n)$ denotes the cardinal number of the set:

$$
\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{Z}^{3} \mid n=x_{1}^{2}+x_{2}^{2}+x_{3}^{2}\right\}
$$

We here propose to express $r_{3}$ in terms of simple divisor functions, defined as follows.

Definition: For each pair of positive integers $i$, $n$, with $i \leqslant 2, \delta_{i}(n)$ is defined by

$$
\delta_{i}(n)=\sum_{d \equiv i(\bmod 3)}(-1)^{(n / d)-1}
$$

Theorem 1: Let $n$ denote an arbitrary positive integer.

$$
\begin{equation*}
\text { If } n=3 m^{2} \text {, for some positive integer } m \text {, then } \tag{i}
\end{equation*}
$$

$$
\begin{aligned}
r_{3}(n)=2 & +6(-1)^{n}\left[\delta_{2}(n)-\delta_{1}(n)\right] \\
& +12(-1)^{n} \sum_{i=1}(-1)^{n}\left[\delta_{2}\left(n-3 i^{2}\right)-\delta_{1}\left(n-3 i^{2}\right)\right]
\end{aligned}
$$

(ii) If $n$ is not of the form $3 m^{2}$, then

$$
\begin{aligned}
& r_{3}(n)=6(-1)^{n}\left[\delta_{2}(n)-\delta_{1}(n)\right] \\
&+12(-1)^{n} \sum_{i=1}(-1)^{n}\left[\delta_{2}\left(n-3 i^{2}\right)-\delta_{1}\left(n-3 i^{2}\right)\right]
\end{aligned}
$$

In both statements (i) and (ii), summation for the sums indexed by $i$ extends over all values of $i$ for which the arguments of $\delta_{1}$ and $\delta_{2}$ are positive.

In §2, we prove this theorem. Our concluding remarks are concerned with comparison of the present representation of $r_{3}$ with the classical representation due to Dirichlet.

## 2. PROOF OF THEOREM 1

Our proof is predicated on the quintuple-product identity

$$
\begin{align*}
& \prod_{1}^{\infty}\left(1-x^{n}\right)\left(1-a x^{n}\right)\left(1-a^{-1} x^{n-1}\right)\left(1-a^{2} x^{2 n-1}\right)\left(1-a^{-2} x^{2 n-1}\right) \\
& =\sum_{-\infty}^{\infty} x^{n(3 n+1) / 2}\left(a^{3 n}-a^{-3 n-1}\right) \tag{1}
\end{align*}
$$

which (as observed by Carlitz and Subbarao [1]) is derivable from the classical

## ON THE ENUMERATOR FOR SUMS OF THREE SQUARES

triple-product identity

$$
\begin{equation*}
\prod_{1}^{\infty}\left(1-x^{2 n}\right)\left(1+a x^{2 n-1}\right)\left(1+a^{-1} x^{2 n-1}\right)=\sum_{-\infty}^{\infty} x^{n^{2}} a^{n} \tag{2}
\end{equation*}
$$

Both identities are valid for each pair of complex numbers $\alpha, x$ such that $a \neq 0$ and $|x|<1$. We shall also require the following classical identities associated with the names of Euler, Gauss, and Jacobi.

$$
\begin{align*}
& \prod_{1}^{\infty}\left(1-x^{2 n-1}\right)\left(1+x^{n}\right)=1  \tag{3}\\
& \prod_{1}^{\infty}\left(1-x^{2 n}\right)\left(1+x^{2 n-1}\right)^{2}=\sum_{-\infty}^{\infty} x^{n^{2}} \tag{4}
\end{align*}
$$

Identity (4) is an easy special case of (2) (simply set $a=1$ ), but we list it separately to observe that the cube of its right side generates $r_{3}$.

In (1), let $a \rightarrow a^{2}$ and multiply the resulting identity by $a$ to get:

$$
\begin{align*}
& \left(a-a^{-1}\right) \prod_{1}^{\infty}\left(1-x^{n}\right)\left(1-a^{2} x^{n}\right)\left(1-a^{-2} x^{n}\right)\left(1-a^{4} x^{2 n-1}\right)\left(1-a^{-4} x^{2 n-1}\right) \\
& =a \sum_{-\infty}^{\infty} x^{n(3 n+1) / 2} a^{6 n}-a^{-1} \sum_{-\infty}^{\infty} x^{n(3 n+1) / 2} a^{-6 n} \\
& =a \prod_{1}^{\infty}\left(1-x^{3 n}\right)\left(1+a^{6} x^{3 n-1}\right)\left(1+a^{-6} x^{3 n-2}\right) \\
& \quad-a^{-1} \prod_{1}^{\infty}\left(1-x^{3 n}\right)\left(1+a^{-6} x^{3 n-1}\right)\left(1+a^{6} x^{3 n-2}\right) . \tag{5}
\end{align*}
$$

Here we have used (2) to express the infinite series as infinite products. For the sake of brevity, put

$$
\begin{aligned}
& F(a)=F(a, x)=\prod_{1}^{\infty}\left(1-a^{2} x^{n}\right)\left(1-a^{-2} x^{n}\right)\left(1-a^{4} x^{2 n-1}\right)\left(1-a^{-4} x^{2 n-1}\right) \\
& G(a)=G(a, x)=\prod_{1}^{\infty}\left(1+a^{6} x^{3 n-1}\right)\left(1+a^{-6} x^{3 n-2}\right)
\end{aligned}
$$

and

$$
H(a)=G\left(a^{-1}\right)
$$

Hence, (5) becomes

$$
\prod_{1}^{\infty}\left(1-x^{n}\right)\left(\alpha-\alpha^{-1}\right) F(\alpha)=\prod_{1}^{\infty}\left(1-x^{3 n}\right)\left\{a G(\alpha)-a^{-1} H(\alpha)\right\}
$$

We now differentiate the foregoing identity with respect to a to get:

$$
\begin{align*}
& \prod_{1}^{\infty}\left(1-x^{n}\right)\left\{\left(1+\alpha^{-2}\right) F(\alpha)+\left(\alpha-a^{-1}\right) F^{\prime}(\alpha)\right\} \\
= & \prod_{1}^{\infty}\left(1-x^{3 n}\right)\left\{G(\alpha)+a^{-2} H(\alpha)+\alpha G^{\prime}(\alpha)-a^{-1} H^{\prime}(\alpha)\right\} \tag{6}
\end{align*}
$$

Sequentially, we use the technique of logarithmic differentiation to evaluate $G^{\prime}(\alpha)$ and $H^{\prime}(\alpha)$, substitute these evaluations into (6), let $\alpha \rightarrow 1$ in the resulting identity, and finally cancel a factor of 2 to get:

$$
\begin{aligned}
& \prod_{1}^{\infty}\left(1-x^{n}\right)^{3}\left(1-x^{2 n-1}\right)^{2} \\
& =\prod_{1}^{\infty}\left(1-x^{3 n}\right)\left(1+x^{3 n-1}\right)\left(1+x^{3 n-2}\right)\left\{1+6\left(\sum_{1}^{\infty} \frac{x^{3 n-1}}{1+x^{3 n-1}}-\sum_{1}^{\infty} \frac{x^{3 n-2}}{1+x^{3 n-2}}\right)\right\} \\
& =\prod_{1}^{\infty}\left(1-x^{3 n}\right)\left(1+x^{3 n-1}\right)\left(1+x^{3 n-2}\right)\left\{1+6 \sum_{1}^{\infty}\left[\delta_{2}(n)-\delta_{1}(n)\right] x^{n}\right\} .
\end{aligned}
$$

Now,

$$
\begin{aligned}
& \prod_{1}^{\infty} \frac{\left(1-x^{n}\right)^{3}\left(1-x^{2 n-1}\right)^{2}}{\left(1-x^{3 n}\right)\left(1+x^{3 n-1}\right)\left(1+x^{3 n-2}\right)} \\
& =\prod_{1}^{\infty}\left(1-x^{n}\right)^{3}\left(1-x^{2 n-1}\right)^{3} \cdot \frac{\left(1+x^{3 n}\right)\left(1+x^{3 n-1}\right)\left(1+x^{3 n-2}\right)}{\left(1-x^{3 n}\right)\left(1+x^{3 n-1}\right)\left(1+x^{3 n-2}\right)}
\end{aligned}
$$

[by Euler's identity (3)]

$$
=\left\{\sum_{0}^{\infty} r_{3}(n)(-x)^{n}\right\} \cdot \prod_{1}^{\infty} \frac{1+x^{3 n}}{1-x^{3 n}}
$$

Hence,

$$
\begin{aligned}
\sum_{0}^{\infty} r_{3}(n)(-x)^{n} & =\prod_{1}^{\infty} \frac{1-x^{3 n}}{1+x^{3 n}}\left\{1+6 \sum_{1}^{\infty}\left[\delta_{2}(n)-\delta_{1}(n)\right] x^{n}\right\} \\
& =\left\{1+2 \sum_{1}^{\infty}\left(-x^{3}\right)^{n^{2}}\right\}\left\{1+6 \sum_{1}^{\infty}\left[\delta_{2}(n)-\delta_{1}(n)\right] x^{n}\right\} .
\end{aligned}
$$

Now, letting $x \rightarrow-x$, we have

$$
\begin{aligned}
\sum_{0}^{\infty} r_{3}(n) x^{n}=1 & +2 \sum_{m=1}^{\infty} x^{3 m^{2}}+6 \sum_{n=1}^{\infty}(-1)^{n}\left[\delta_{2}(n)-\delta_{1}(n)\right] x^{n} \\
& +12 \sum_{n=1}^{\infty}(-1)^{n} x^{n} \sum_{i=1}(-1)^{i}\left[\delta_{2}\left(n-3 i^{2}\right)-\delta_{1}\left(n-3 i^{2}\right)\right]
\end{aligned}
$$

[Here we adopt the convention that $\delta_{i}(k)=0$ whenever $k<0, i=1,2$.] Equating coefficients of like powers of $x$, we thus prove our theorem. [Note that $r_{3}(0)=1$.]

## CONCLUDING REMARKS

There is a somewhat complicated formula for $r_{3}(n)\left[n \in \mathbb{Z}^{+}\right]$due to Dirichlet. This is:

$$
r_{3}(n)=\frac{16}{\pi} n^{1 / 2} \chi_{2}(n) K(-4 n) \cdot \prod_{p^{2} \mid n}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{\tau-1}}+\frac{1}{p^{\tau}}\left(1-\left(\frac{p^{-2 \tau} n}{p}\right) \frac{1}{p}\right)^{-1}\right)
$$

where the definition of $\tau$ is $p^{2 \tau} \mid n$, but $p^{2(\tau+1)} \mid n$,

$$
K(-4 n)=\sum_{m=1}^{\infty}\left(\frac{-4 n}{m}\right) \frac{1}{m},
$$

Here, and above, $\left(\frac{-4 n}{m}\right)$ is a Jacobi symbol. And

$$
x_{2}(n)= \begin{cases}0 & \text { if } 4^{-a} n \equiv 7(\bmod 8) \\ 2^{-a}, & \text { if } 4^{-a} n \equiv 3(\bmod 8) \\ 3.2^{-1-a}, & \text { if } 4^{-a} n \equiv 1,2,5,6(\bmod 8),\end{cases}
$$

and here the definition of $a$ is $4^{a} \mid n$, but $4^{a+1} \mid n$. This formula (among others) is given by Hua [2, pp. 215-216]. First of all, it is far from obvious that this expression for $r_{3}(n)$ is an integer, whereas our expressions of Theorem 1 are clearly integral. However, Dirichlet's formula permits an easy proof of the fact: $r_{3}(n)>0$, if and only if, $n$ is not of the form $4^{a}(8 m+7)$. At the moment, the author has not seen a way of deducing this fact from Theorem 1.

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$\diamond \diamond \diamond \diamond$

# BERNOULLI NUMBERS AND KUMMER'S CRITERION 

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## 1. INTRODUCTION

There is a large literature concerning various properties of the Bernoulli numbers; see, for example, $[1,12,16,23]$ and their references. According to H. S. Vandiver [23], by 1960 over 1500 papers had been written on the subject. The main thrust of the present paper is to consider several congruence properties of the Bernoulli numbers that extend various results of Vandiver, Nielson, Carlitz, and Stevens; see $[2,16,19,22]$. The Bernoulli numbers $B_{n}(n \geqslant 0)$ are defined by the expansion

$$
\frac{x}{e^{x}-1}=\sum_{n=0}^{\infty} B_{n} \frac{x^{n}}{n!}
$$

which is equivalent to

$$
\begin{equation*}
\sum_{r=0}^{n}\binom{n}{r} B_{r}=B_{n} \quad(n>1) \tag{1.1}
\end{equation*}
$$

together with $B_{0}=1$. It is sometimes convenient to write (1.1) in the form

$$
\begin{equation*}
(B+1)^{n}=B^{n} \quad(n>1) \tag{1.2}
\end{equation*}
$$

where it is understood that, after expansion of the left-hand side, we replace $B^{k}$ by $B_{k}$. It is easy to check that for the first few values of $n$ we have

$$
B_{1}=-1 / 2, \quad B_{2}=1 / 6, \quad B_{4}=-1 / 30
$$

and that in general $B_{2 k+1}=0$ if $k \geqslant 1$.
Bernoulli numbers have numerous interesting properties. For example, if $S_{n}(k)=1^{n}+\ldots+k^{n}$, then $S_{n}(k)=\left(B_{n+1}(k+1)-B_{n+1}\right) /(n+1)$, where $B_{n}(x)=$ $(B+x)^{n}$. The Bernoulli numbers are related to class numbers and to Fermat's Last Theorem. Moreover, they satisfy numerous recurrences and congruences. For further details regarding various properties of the Bernoulli numbers, the reader should consult the papers $[1,12,16,23]$ and their references.

## 2. CONGRUENCE PROPERTIES

If $p$ is a prime, we now consider several congruence properties of sequences of rational numbers where we say that $a / b$ is integral modulo $p$ if $(b, p)=1$.

[^0]Moreover, if $\alpha / b$ and $c / d$ are integral modulo $p$, then

$$
\frac{a}{b} \equiv \frac{c}{d}(\bmod p) \text { if } a d \equiv b c(\bmod p)
$$

We assume throughout this paper that $p$ is an odd prime even though similar results could be obtained for the case in which $p=2$.

In [15] Kummer proved that

$$
\frac{B_{n+1}-p}{n+p-1} \equiv \frac{B_{n}}{n} \quad(\bmod p)
$$

for all $n>1$, where $(p-1) \nmid n$. More generally, one can consider congruences of the form

$$
\begin{equation*}
\sum_{s=0}^{r}(-1)^{s}\binom{r}{s} \frac{B_{n+s(p-1)}}{n+s(p-1)} \equiv 0 \quad\left(\bmod p^{r}\right) \tag{2.1}
\end{equation*}
$$

for $n>r$, where $(p-1) \nmid n$. In [15] Kummer studied congruences similar to the above but in a more general setting in which he proved the following theorem.

Theorem 1 (Kummer): Let $a_{n}$ be integral modulo $p$ and suppose

$$
\begin{equation*}
\sum_{n=0}^{\infty} a_{n} \frac{x^{n}}{n!}=\sum_{n=0}^{\infty} A_{n}\left(e^{x}-1\right)^{n} \tag{2.2}
\end{equation*}
$$

If the $A_{n}$ are integral modulo $p$, then

$$
\begin{equation*}
\sum_{s=0}^{r}(-1)^{s}\binom{r}{s} a_{n+s(p-1)} \equiv 0 \quad\left(\bmod p^{r}\right), \text { for } n \geqslant r \geqslant 1 \tag{2.3}
\end{equation*}
$$

Nie1son showed in [16] that if $a_{n}=B_{n}$, the $n^{\text {th }}$ Bernoulli number, then the Bernoulli numbers themselves satisfy (2.3) if ( $p-1$ ) $\mid n$, where the modulus is replaced by $p^{r-1}$. In attempting to remove the restriction $(p-1) \mid n$, Vandiver [22] showed that if $n=a(p-1)$ and $a_{n}=B_{n}$ then (2.3) holds modulo $p^{r-1}$ provided that $r+a<p-1$. This latter restriction is, however, a rather severe one. In [2] Carlitz showed that the congruence (2.3) holds if $r<p-1$ and that some much weaker congruences hold if $r \geqslant p-1$.

Congruences similar to (2.3) were later studied in a series of papers by Carlitz and Stevens [5-9, 18-21]. Recently, a number of authors have taken renewed interest in the topic of congruences for various sequences of numbers. For example, Rota and Sagan [17], Gessel [13], J. Cowles [10], and J. Cowles, S. Chowla, and M. J. Cowles [11] have used various general combinatorial techniques, such as group actions on sets, to obtain various congruence properties for several sequences of numbers.

If one looks at Kummer's Criterion (2.2) and (2.3), it is easy to see that the condition is sufficient but not necessary. We will make use of the following theorem due to Carlitz [5].

Theorem 2 (Carlitz): Let $a_{n}$ be integral modulo $p$ and suppose

$$
\sum_{n=0}^{\infty} a_{n} \frac{x^{n}}{n!}=\sum_{k=0}^{\infty} A_{k} \frac{\left(e^{x}-1\right)^{k}}{k!}
$$

Then $A_{k} \equiv 0\left(\bmod p^{[k / p]}\right)$ for all $k \geqslant 0$ if and only if

$$
\begin{gathered}
\sum_{s=0}^{r}(-1)^{s}\binom{r}{s} a_{n+s(p-1)} \equiv \\
0 \quad\left(\bmod p^{r}\right), \text { for all } n \geqslant r \geqslant 1 . \\
\text { 3. APPLICATIONS }
\end{gathered}
$$

In this section we apply Theorem 2 to the Bernoulli numbers to obtain several congruences that extend various results of Vandiver, Nielson, Carlitz, and Stevens, see $[1,16,19,22]$. Finally, we use the theorem to obtain an elementary proof of the Staudt-Clausen theorem. Let us put

$$
\frac{x}{e^{x}-1}=\sum_{n=0}^{\infty} B_{n} \frac{x^{n}}{n!}=\frac{\log \left(1+\left(e^{x}-1\right)\right)}{e^{x}-1}=\sum_{n=0}^{\infty}(-1)^{n} \frac{n!}{n+1} \frac{\left(e^{x}-1\right)^{n}}{n!}
$$

so that

$$
A_{n}=\frac{(-1)^{n} n!}{n+1}
$$

Now however, the $A_{n}$ 's do not satisfy the condition of the theorem. If we multiply by $p$, each coefficient in the new series does satisfy the condition, except for the coefficient of

$$
\frac{\left(e^{x}-1\right)^{p^{2}-1}}{\left(p^{2}-1\right)!}
$$

Thus, we have

$$
\sum_{n=0}^{\infty} p B_{n} \frac{x^{n}}{n!}=\frac{(-1)^{p^{2}-1}}{p}\left(e^{x}-1\right)^{p^{2}-1}+C(x)
$$

where $C(x)$ satisfies the condition of the theorem. Hence, if $D$ is the derivative operator, then

$$
\left(D^{p}-D\right)^{r} \sum_{n=0}^{\infty} p B_{n} \frac{x^{n}}{n!} \equiv\left(D^{p}-D\right)^{r} \frac{(-1)^{p^{2}-1}}{p}\left(e^{x}-1\right)^{p^{2}-1} \quad\left(\bmod p^{r}\right)
$$

where we say that

$$
\sum_{n=0}^{\infty} a_{n} \frac{x^{n}}{n!} \equiv \sum_{n=0}^{\infty} b_{n} \frac{x^{n}}{n!} \quad(\bmod m)
$$

if $a_{n} \equiv b_{n}(\bmod m)$ for each $n \geqslant 0$.
We now consider $\left(D^{p}-D\right)^{r}\left(e^{x}-1\right)^{p^{2}-1}\left(\bmod p^{r+1}\right)$. Since

$$
\left(e^{x}-1\right)^{p^{2}-1}=\sum_{j=0}^{p^{2}-1}(-1)^{x^{2}-1-j}\binom{p^{2}-1}{j} e^{j x}
$$

if we apply the operator $\left(D^{p}-D\right)^{r}$, we get after some simplification that, for each $n \geqslant 0$, the coefficient of $x^{n} / n$ ! is

$$
\begin{equation*}
\sum_{j=0}^{p^{2}-1}(-1)^{p^{2}-1-j}\binom{p^{2}-1}{j}\left(j^{p-1}-1\right)^{r} j^{n+r} \tag{3.1}
\end{equation*}
$$

We now break the sum (3.1) into two sums $\Sigma^{\prime}$ and $\Sigma^{\prime \prime}$, where in $\Sigma^{\prime}$ we sum over those $j$ for which $p \ j$, while in $\sum_{j}^{\prime \prime}$ we sum over those $j$ for which $p \mid j$. To compute $\Sigma^{\prime}$, suppose that $j^{p-1}-1=p k(j)$, so that

$$
\left.\Sigma^{\prime}=p^{r} \sum_{j=0}^{p^{2}-1}(-1)^{p^{2}-1-j\left(p^{2}-1\right.}{ }_{j}\right) k(j)^{r} j^{n+r}
$$

We know that

$$
\left(p^{2}-1\right) \equiv(-1)^{j} \quad(\bmod p)
$$

If $j^{\prime} \equiv j(\bmod p)$ so that $j^{\prime}=j+Q p$, then $k\left(j^{\prime}\right) \equiv k(j)-j^{p-2} Q(\bmod p)$, and hence

$$
\begin{align*}
& \sum_{j=0}^{p^{2}-1}(-1)^{p^{2}-1-j}\left(p^{2}-1\right)(k(j))^{r} j^{n+r}  \tag{*}\\
& \equiv(-1)^{p^{2}-1} \sum_{j=1}^{p-1}\left[\sum_{Q=0}^{p-1}\left(k(j)-j^{p-2} Q\right)^{r}\right] j^{n+r}(\bmod p) \\
& \equiv(-1)^{p^{2}-1} \sum_{j=1}^{p-1} j^{n+r} \sum_{Q=0}^{p-1} Q^{r}(\bmod p),
\end{align*}
$$

since the terms in the brackets run through a complete residue system modulo $p$. If $(p-1) \nmid r$, then the inner sum is zero modulo $p$, while if $(p-1) \nmid(n+r)$, then the outer sum is zero modulo $p$. If $(p-1) \mid r$ and $(p-1) \mid(n+r)$, then the left-hand side of (*) is congruent to $(-1)^{p^{2}-1}$ modulo $p$. Hence,

$$
\Sigma^{\prime} \equiv \begin{cases}0\left(\bmod p^{r+1}\right) & \text { if }(p-1) \nmid r \\ 0\left(\bmod p^{r+1}\right) & \text { if }(p-1) \nmid(n+r) \\ (-1)^{p^{2}-1\left(\bmod p^{r+1}\right)} & \text { if }(p-1) \mid r \text { and }(p-1) \mid(n+r)\end{cases}
$$

Along similar lines, we may compute the sum $\Sigma^{\prime \prime}$ to obtain

$$
\Sigma^{\prime \prime} \equiv \begin{cases}0\left(\bmod p^{r+1}\right) & \text { if }(p-1) \nmid(n+r) \\ (-1)^{p^{2}+r} p^{n+r}\left(\bmod p^{r+1}\right) & \text { if }(p-1) \mid(n+r)\end{cases}
$$

Therefore, combining the congruences obtained for $\Sigma^{\prime}$ and $\Sigma^{\prime \prime}$, we see that $p B_{n}$ is integral modulo $p$. Thus, we may apply Theorem 2 to the sequence $a_{n}=p B_{n}$ to obtain
Theorem 3: Let $N=\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s} B_{n+s(p-1)}$
(A) If $(p-1) \nmid n$ where $n \geqslant r \geqslant 1$, then $N \equiv 0\left(\bmod p^{r-1}\right)$.
(B) If $(p-1) \mid n$ and $(p-1) \nmid r$ where $n>r \geqslant 1$, then $N \equiv 0\left(\bmod p^{r-1}\right)$.
(C) If $(p-1) \mid n$ and $(p-1) \mid r$ where $n>p \geqslant 1$, then $N \equiv p^{r-2}\left(\bmod p^{r-1}\right)$ 。
(D) If $n=r$ and $(p-1) \mid n$, then $N \equiv 0\left(\bmod p^{r+1}\right)$.

We note that (A) is a result of Nielson [16], while the result in (B) improves upon results of Vandiver [22] and Carlitz [2].

We now obtain a generalization of these congruences. Since

$$
\begin{aligned}
x^{p^{e}}-y^{p^{e}}=\left(x^{p^{e-1}}\right. & \left.-y^{p^{e-1}}\right)\left(x^{(p-1) p^{e-1}}+x^{(p-2) p^{e-1}} y^{p^{e-1}}\right. \\
& \left.+x^{(p-3) p^{e-1}} y^{2 p^{e-1}}+\cdots+y^{(p-1) p^{e-1}}\right)
\end{aligned}
$$

by induction on $e$ one can prove the following identity:

$$
\begin{equation*}
x^{p^{e-1}}-y^{p^{e-1}}=\sum_{i=0}^{e-1} p^{i}(x-y)^{p^{e-1-i}} f_{i}(x, y) \quad(e \geqslant 1) \tag{3.2}
\end{equation*}
$$

where each $f_{i}(x, y)$ is a polynomial in $x$ and $y$. Let $E$ be the difference operator, and suppose $b \geqslant 1$. Let $x=E^{b(p-1)}$ and $y=1$ in (3.2) and then take the $r^{\text {th }}$ power of both sides. We obtain

$$
\begin{equation*}
\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s} B_{n+s b p^{e-1}(p-1)} \equiv 0\left(\bmod p^{A}\right) \tag{3.3}
\end{equation*}
$$

where $A$ is the minimum of

$$
-1+\sum_{i=1}^{e-1} i \alpha+\sum_{i=0}^{e-1} p^{e-1-i} \alpha_{i} \quad \text { and } \quad \alpha_{0}+\cdots+\alpha_{e-1}=r
$$

This minimum occurs when

$$
\alpha_{0}=\cdots=\alpha_{e-2}=0 \quad \text { and } \quad \alpha_{e-1}=r .
$$

Hence, if $n \geqslant e r$, then $A=e r-1$. We may now state
Theorem 4: Let $b \geqslant 1, e \geqslant 1$, and $M=\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s} B_{n+s b p^{e-1}(p-1)^{*}}$
(A) If $r \geqslant 1, n>e r$, and either $(p-1) \mid n$ or $(p-1) \mid r$, then $M \equiv 0\left(\bmod p^{e r-1}\right)$.
(B) If $n>e r,(p-1) \mid n$, and $(p-1) \mid r$, then $M \equiv 0\left(\bmod p^{e r-2}\right)$.

These results should be compared with Theorem 8 of Stevens [19].
We now apply Theorem 2 to obtain an elementary proof of
Theorem 5 (Staudt-Clausen): If $n \geqslant 1$, then

$$
B_{2 n}=G_{2 n}-\sum_{(p-1) \mid 2 n} \frac{1}{p}
$$

where $G_{2 n}$ is an integer.
Proof: It suffices to show that $p B_{n} \equiv-1(\bmod p)$ if and only if $(p-1) \mid n$. We have

$$
\sum_{k=0}^{\infty} p B_{k} \frac{x^{k}}{k!}=\sum_{k=0}^{\infty}(-1)^{k} \frac{p}{k+1}\left(e^{x}-1\right)^{k}
$$

By induction on $n$ in (1.1), it is easy to show that $p B_{n} \equiv 0(\bmod p)$ if $0 \leqslant n \leqslant$ $p-2$, and hence from (1.1) we have that $p B_{p-1} \equiv-1(\bmod p)$. If $n=\alpha(p-1)$,
then for $r=1$ we have $p B_{a(p-1)} \equiv p B_{p-1}(\bmod p)$ so that $B_{a(p-1)}=-1 / p+Q$ where $Q$ is integral modulo $p$. Similarly, $p B_{n} \equiv 0(\bmod p)$ if $(p-1) \mid n$. Thus $p$ divides the denominator of $B_{n}$ if and only if $(p-1) \mid n$.

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# DISCOVERING FIBONACCI IDENTITIES 

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## 1. INTRODUCTION

One of the more appealing aspects of the Fibonacci sequence, and certainly the most appealing to the uninitiated, is the very large number of remarkable identities that can be found. Discussing identities with Vern Hoggatt several years ago, I pointed out that it was easy to discover new identities simply by varying the pattern of known identities and using inductive reasoning to guess new results. With characteristic enthusiasm, Vern immediately picked up on the idea and suggested that an appropriate paper be written. Shortly after returning home, I received a letter from Vern which began: "There are a surprising number of good ways of expanding the list of identities. Consider ... "" And the last sentence read: "At least some of this is sparkling new, and we are only using observation."

What follows is an account of some of the ideas we were sharing. They are not deep but, like Vern, I find them interesting. Of course, the ideas can be extended to more general recurrent sequences in obvious ways, but we restrict our attention here to the familiar Fibonacci and Lucas sequences defined by

$$
\begin{equation*}
F_{i}=\frac{\alpha^{i}-\beta^{i}}{\sqrt{5}} \quad \text { and } \quad L_{i}=\alpha^{i}+\beta^{i} \tag{1}
\end{equation*}
$$

where $\alpha=(1+\sqrt{5}) / 2, \beta=(1-\sqrt{5}) / 2$, and $i$ is an integer.

## 2. THE GENERAL IDEA

The identities

$$
\begin{equation*}
\sum_{i=1}^{n} F_{i}^{2}=F_{n} F_{n+1} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{i=1}^{n} L_{i}^{2}=L_{n} L_{n+1}-2 \tag{3}
\end{equation*}
$$

are well known (see, for example, [4], p. 55]. Alternatively, for the Lucas sequence, one can easily obtain

$$
\sum_{i=1}^{n} L^{2}= \begin{cases}5 F_{n} F_{n+1} & n \text { even } \\ 5 F_{n} F_{n+1}-4 & n \text { odd }\end{cases}
$$

How might these be generalized? Well, sums of squares might be viewed as sums of terms of the second degree in Fibonacci and Lucas numbers. Thus, one might consider other such sums like, for example,

$$
\sum_{i=1}^{n} F_{i} F_{i+1}, \sum_{i=1}^{n} F_{i} F_{i+2}, \ldots, \sum_{i=1}^{n} F_{i} F_{i+d}
$$

and their Lucas counterparts or the mixed sums

$$
\sum_{i=1}^{n} F_{i} L_{i+1}, \sum_{i=1}^{n} F_{i} L_{i+2}, \ldots, \sum_{i=1}^{n} F_{i} L_{i+d}
$$

One can now proceed formally, or with a little guessing, to obtain, for $d$ any positive integer,

$$
\begin{align*}
& \sum_{i=1}^{n} F_{i} F_{i+d}= \begin{cases}F_{n} F_{n+d+1} & n \text { even } \\
F_{n} F_{n+d+1}-F_{d} & n \text { odd }\end{cases}  \tag{4}\\
& \sum_{i=1}^{n} L_{i} L_{i+d}= \begin{cases}5 F_{n} F_{n+d+1} \\
5 F_{n} F_{n+d+1}-L_{d+3} & n \text { even odd, }\end{cases}  \tag{5}\\
& \sum_{i=1}^{n} F_{i} L_{i+d}= \begin{cases}F_{n} L_{n+d+1} & n \text { even } \\
F_{n} F_{n+d+1}-L_{d} & n \text { odd }\end{cases}  \tag{6}\\
& \sum_{i=1}^{n} L_{i} F_{i+d}= \begin{cases}F_{n} L_{n+d+1} & n \text { even } \\
F_{n} L_{n+d+1}-F_{d+3} & n \text { odd }\end{cases} \tag{7}
\end{align*}
$$

which, as one would expect, exhibit a pleasing symmetry.
The proofs are straightforward utilizing Binet's formulas (1) and the known identities (see [1] and [10])

$$
\begin{align*}
& F_{r+2 s}-F_{r}= \begin{cases}F_{s} L_{r+s} & s \text { even }, \\
L_{s} F_{r+s} & s \text { odd },\end{cases}  \tag{8}\\
& L_{r+2 s}-L_{r}= \begin{cases}5 F_{s} F_{r+s} & s \text { even }, \\
L_{s} L_{r+s} & s \text { odd },\end{cases}  \tag{9}\\
& F_{r+2 s}+F_{r}= \begin{cases}L_{s} F_{r+s} & s \text { even }, \\
F_{s} L_{r+s} & s \text { odd },\end{cases}  \tag{10}\\
& L_{r+2 s}+L_{r}=\left\{\begin{array}{cc}
L_{s} L_{r+s} & s \text { even } \\
5 F_{s} F_{r+s} & s \text { odd }
\end{array}\right. \tag{11}
\end{align*}
$$

As an example of the proofs of (4)-(7), we prove (4). Since $\alpha \beta=-1,1-$ $\alpha^{2}=-\alpha$, and $1-\beta^{2}=-\beta$, we have

$$
\sum_{i=1}^{n} F_{i} F_{i+d}=\sum_{i=1}^{n}\left(\frac{\alpha^{i}-\beta^{i}}{\sqrt{5}}\right)\left(\frac{\alpha^{i+d}-\beta^{i+d}}{\sqrt{5}}\right)
$$

$$
\begin{aligned}
& \quad \text { DISCOVERING FIBONACCI IDENTITIES } \\
&= \frac{\alpha^{d}\left(\alpha^{2}-\alpha^{2 n+2}\right)}{5\left(1-\alpha^{2}\right)}+\frac{\beta^{d}\left(\beta^{2}-\beta^{2 n+2}\right)}{5\left(1-\beta^{2}\right)}-\left(\frac{\alpha^{d}+\beta^{d}}{5}\right) \cdot \sum_{i=1}^{n}(-1)^{i} \\
&= \frac{1}{5}\left\{\alpha^{d}\left(\alpha^{2 n+1}-\alpha\right)+\beta^{d}\left(\beta^{2 n+1}-\beta\right)-L_{d} \cdot \sum_{i=1}^{n}(-1)^{i}\right\} \\
&= \frac{1}{5}\left\{\alpha^{2 n+d+1}+\beta^{2 n+d+1}-\left(\alpha^{d+1}+\beta^{d+1}\right)-L_{d} \cdot \sum_{i=1}^{n}(-1)^{n}\right\} .
\end{aligned}
$$

Therefore, for $n$ even,

$$
\sum_{i=1}^{n} F_{i} F_{i+d}=\frac{1}{5}\left(L_{2 n+d+1}-L_{d+1}\right)=F_{n} F_{n+d+1}
$$

by (9). For $n$ odd, we have

$$
\begin{aligned}
\sum_{i=1}^{n} F_{i} F_{i+d} & =\frac{1}{5}\left(L_{2 n+d+1}-L_{d+1}+L_{d}\right)=\frac{1}{5}\left(L_{2 n+d+1}-L_{d-1}\right) \\
& =\frac{1}{5}\left(L_{2 n+d+1}+L_{d+1}-5 F_{d}\right)=F_{n} F_{n+d+1}-F_{d}
\end{aligned}
$$

by (11), since $L_{d+1}=L_{d}+L_{d-1}$ and $L_{d-1}+L_{d+1}=5 F_{d}$ for all $d$. The other results are proved similarly.

$$
\text { 3. THE IDENTITY } L_{n}^{2}-5 F_{n}^{2}=4(-1)^{n}
$$

As a second example, we consider the identity

$$
\begin{equation*}
L_{n}^{2}-5 F_{n}^{2}=4(-1)^{n} \tag{12}
\end{equation*}
$$

Again the terms on the left are of the second degree and we are led to consider expressions like

$$
L_{n}^{2}-5 F_{n+d}^{2}, L_{n} L_{n+d}-5 F_{n} F_{n+d}, L_{n} F_{n+d}-F_{n} L_{n+d}, L_{n} L_{m}-5 F_{n+d} F_{m+d},
$$

and so on. As before, one can proceed either inductively or formally, or with a combination of both approaches, and it is a meta-theorem that we will not be disappointed. In fact, the following results can be exhibited. Let $m$, $n$, and $d$ be integers. Then

$$
\begin{align*}
& L_{n} L_{m}-5 F_{n+d} F_{m+d}= \begin{cases}5 F_{-d} F_{m+n+d}+2(-1)^{n} L_{m-n} & d \text { even, } \\
L_{-d} L_{m+n+d} & d \text { odd },\end{cases}  \tag{13}\\
& L_{n} L_{m}-L_{n+d} L_{m+d}= \begin{cases}5 F_{-d} F_{m+n+d} & d \text { even, } \\
L_{-d} L_{m+n+d}+2(-1)^{n} L_{m-n} & d \text { odd, }\end{cases}  \tag{14}\\
& F_{n} F_{m}-F_{n+d} F_{m+d}= \begin{cases}F_{-d} F_{m+n+d} & d \text { even, } \\
\frac{1}{5}\left(L_{-d} L_{m+n+d}-2(-1)^{n} L_{m-n}\right) & d \text { odd, },\end{cases} \tag{15}
\end{align*}
$$

$$
\begin{align*}
& L_{n} F_{m}-L_{n+d} F_{m+d}= \begin{cases}F_{-d} L_{m+n+d} & d \text { even }, \\
L_{-d} F_{m+n+d}-2(-1)^{n} F_{m-n} & d \text { odd },\end{cases}  \tag{16}\\
& L_{n} F_{m}-L_{m+d} F_{n+d}= \begin{cases}F_{-d} L_{m+n+d}+2(-1)^{n} F_{m-n} & d \text { even }, \\
L_{-d} F_{m+n+d} & d \text { odd, }\end{cases}  \tag{17}\\
& L_{n} L_{m+d}-5 F_{n} F_{m+d}=2(-1)^{n} L_{m-n+d},  \tag{18}\\
& L_{n} F_{m+d}-L_{m+d} F_{n}=2(-1)^{n} F_{m-n+d},  \tag{19}\\
& L_{n} L_{m}-5 F_{n-d} F_{m+d}=(-1)^{n} L_{-d} L_{m-n+d},  \tag{20}\\
& L_{n} L_{m}-L_{n-d} L_{m+d}=5(-1)^{n+1} F_{-d} F_{m-n+d},  \tag{21}\\
& F_{n} F_{m}-F_{n-d} F_{m+d}=(-1)^{n+1} F_{-d} F_{m-n+d},  \tag{22}\\
& L_{n} F_{m}-L_{n-d} F_{m+d}=(-1)^{n} F_{-d} L_{m-n+d},  \tag{23}\\
& F_{n} L_{m}-L_{n-d} F_{m+d}=(-1)^{n+1} L_{-d} F_{m-n+d} .
\end{align*}
$$

and

Moreover, these identities, or the known identities,

$$
\begin{align*}
& F_{m+n+1}=F_{m} F_{n}+F_{m+1} F_{n+1}  \tag{25}\\
& L_{m+n+1}=L_{m} F_{n}+L_{m+1} F_{n+1}  \tag{26}\\
& 5 F_{m+n+1}=L_{m} L_{n}+L_{m+1} L_{n+1} \tag{27}
\end{align*}
$$

and
suggest that we seek identities like (13)-(24) but with a plus sign on the left in place of the minus sign. Identities indeed exist and, somewhat surprisingly, are exactly the same as before but with the even and odd cases reversed. Thus, for example

$$
F_{n} F_{m}+F_{n+d} F_{m+d}= \begin{cases}\frac{1}{5}\left(L_{-d} L_{n+m+d}-2(-1)^{n} L_{m-n}\right) & d \text { even }  \tag{28}\\ F_{-d} F_{n+m+d} & d \text { odd }\end{cases}
$$

This should be compared with (15) above. Since this is the only change required, we refrain from listing the remaining counterparts to (13)-(24).

The proofs of (13)-(24) and their counterparts with the plus sign on the left-hand side all depend on Binet's formulas, identities (7)-(9) and equivalent identities obtained by replacing $d$ by $-d$, and on the identities

$$
\begin{equation*}
F_{-n}=(-1)^{n-1} F_{n} \tag{29}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{-n}=(-1)^{n} L_{n} \tag{30}
\end{equation*}
$$

for all $n$. As an example, we prove (14). We have

$$
L_{n} L_{m}-L_{n+d} L_{m+d}=\left(\alpha^{n}+\beta^{n}\right)\left(\alpha^{m}+\beta^{m}\right)-\left(\alpha^{n+d}+\beta^{n+d}\right)\left(\alpha^{m+d}+\beta^{m+d}\right)
$$

(continued)

$$
\begin{aligned}
=\alpha^{m+n} & +\beta^{m+n}+(\alpha \beta)^{n}\left(\alpha^{m-n}+\beta^{m-n}\right) \\
& \quad-\alpha^{m+n+2 d}-\beta^{m+n+2 d}-(\alpha \beta)^{n+d}\left(\alpha^{m-n}+\beta^{m-n}\right) \\
= & -\left(L_{m+n+2 d}-L_{m+n}\right)+(-1)^{n}\left[1-(-1)^{d}\right] L_{m-n} .
\end{aligned}
$$

Thus, using (9), (29), and (30), we have, for $d$ even,

$$
L_{n} L_{m}-L_{n+d} L_{m+d}=-5 F_{d} F_{m+n+d}=5 F_{-d} F_{m+n+d}
$$

and, for $d$ odd,

$$
L_{n} L_{m}-L_{n+d} L_{m+d}=-L_{d} L_{m+n+d}+2(-1)^{n} L_{m-n}=L_{-d} L_{m+n+d}+2(-1)^{n} L_{m-n}
$$

as claimed.

## 4. HIGHER-ORDER IDENTITIES

Casting about for other identities to treat in the same way,
and

$$
\begin{equation*}
F_{n+3}^{2}-2 F_{n+2}^{2}-2 F_{n+1}^{2}+F_{n}^{2}=0 \tag{31}
\end{equation*}
$$

$$
\begin{equation*}
F_{n+3} F_{n+4}-2 F_{n+2} F_{n+3}-2 F_{n+1} F_{n+2}+F_{n} F_{n+1}=0 \tag{32}
\end{equation*}
$$

were found in a paper by Hoggatt and Bicknell [6]. Note that (32) is already related to (31) in the manner of this paper, and one would expect such results as

$$
\begin{align*}
& L_{n+3}^{2}-2 L_{n+2}^{2}-2 L_{n+1}^{2}+L_{n}^{2}=0,  \tag{33}\\
& F_{n+3} L_{n+3}-2 F_{n+2} L_{n+2}-2 F_{n+1} L_{n+1}+F_{n} L_{n}=0,  \tag{34}\\
& F_{n+3} L_{m+3}-2 F_{n+2} L_{m+2}-2 F_{n+1} L_{m+1}+F_{n} L_{m}=0, \tag{35}
\end{align*}
$$

and so on. Checking a bit further, I found that these and a good deal more are already known to hold. In [2], T. Brennan shows that

$$
\sum_{r=0}^{n+1}(-1)^{r(r+1) / 2}\left[\begin{array}{c}
n+1  \tag{36}\\
r
\end{array}\right] x^{n+1-r}=0
$$

where

$$
\left[\begin{array}{l}
n  \tag{37}\\
r
\end{array}\right]=\frac{F_{n} F_{n-1} \cdots F_{n-r+1}}{F_{r} F_{r-1} \cdots F_{1}},\left[\begin{array}{l}
n \\
0
\end{array}\right]=1,
$$

is the auxiliary equation for $q_{n}$, where $q_{n}$ is the product of any $n$ sequences satisfying the recurrence $\mu_{n+2}=\mu_{n+1}+\mu_{n}$.

For $n=2$, (36) becomes

$$
\begin{equation*}
x^{3}-2 x^{2}-2 x+1=0, \tag{38}
\end{equation*}
$$

which implies the truth of (33)-(35) and all other such generalizations. For $n=3$, (36) becomes

$$
\begin{equation*}
x^{4}-3 x^{3}-6 x^{2}+3 x+1=0 \tag{39}
\end{equation*}
$$

## DISCOVERING FIBONACCI IDENTITIES

which implies such identities as

$$
\begin{align*}
& F_{n+4}^{3}-3 F_{n+3}^{3}-6 F_{n+2}^{3}+3 F_{n+1}^{3}+F_{n}^{3}=0,  \tag{40}\\
& F_{n+4} F_{m+4} F_{p+4}-3 F_{n+3} F_{m+3} L_{p+3}-6 F_{n+2} F_{m+2} L_{p+2} \\
&  \tag{41}\\
& \quad+3 F_{n+1} F_{m+1} L_{p+1}+F_{n} F_{m} L_{p}=0,
\end{align*}
$$

and so on. Those interested in these and similar matters should also see [3], [5], [7], [8], [9], and [11].

As a final example, we consider the well-known and elegant identity

$$
\begin{equation*}
F_{3 n}=F_{n+1}^{3}+F_{n}^{3}-F_{n-1}^{3} \tag{42}
\end{equation*}
$$

(see [12], p. 11). In the spirit of this paper, there are three immediate generalizations, and one has only to consider a few examples to guess the following:

$$
\begin{align*}
& L_{3 n}=L_{n+1} F_{n+1}^{2}+L_{n} F_{n}^{2}-L_{n-1} F_{n-1}^{2},  \tag{43}\\
& 5 F_{3 n}=F_{n+1} L_{n+1}^{2}+F_{n} L_{n}^{2}-F_{n-1} L_{n-1}^{2},  \tag{44}\\
& 5 L_{3 n}=L_{n+1}^{3}+L_{n}^{3}-L_{n-1}^{3} . \tag{45}
\end{align*}
$$

and

For completeness, we prove each of (42)-(45). They are not difficult, but are a bit subtle, and it is easy to take a wrong turn. We make repeated use of (25), (26), and (27), above.

To prove (42), we use (25), and write

$$
\begin{aligned}
F_{3 n} & =F_{n-1+2 n+1} \\
& =F_{n-1} F_{2 n}+F_{n} F_{2 n+1} \\
& =F_{n-1} F_{n-1+n+1}+F_{n} F_{n+n+1} \\
& =F_{n-1}\left(F_{n-1} F_{n}+F_{n} F_{n+1}\right)+F_{n}\left(F_{n}^{2}+F_{n+1}^{2}\right) \\
& =F_{n-1} F_{n}\left(F_{n-1}+F_{n+1}\right)+F_{n}^{3}+F_{n} F_{n+1}^{2} \\
& =F_{n-1}\left(F_{n+1}-F_{n-1}\right)\left(F_{n+1}+F_{n-1}\right)+F_{n}^{3}+F_{n} F_{n+1}^{2} \\
& =F_{n-1} F_{n+1}^{2}-F_{n-1}^{3}+F_{n}^{3}+F_{n} F_{n+1}^{2} \\
& =F_{n+1}^{2}\left(F_{n-1}+F_{n}\right)+F_{n}^{3}-F_{n-1}^{3} \\
& =F_{n+1}^{3}+F_{n}^{3}-F_{n-1}^{3}
\end{aligned}
$$

as claimed.
For (43), we use (26) and the formulas

$$
F_{2 n}=F_{n+1}^{2}-F_{n-1}^{2} \quad \text { and } \quad F_{2 n+1}=F_{n}^{2}+F_{n+1}^{2}
$$

from the proof of (42) to write

$$
\begin{aligned}
L_{3 n} & =L_{n-1+2 n+1} \\
& =L_{n-1} F_{2 n}+L_{n} F_{2 n+1} \\
& =L_{n-1}\left(F_{n+1}^{2}-F_{n-1}^{2}\right)+L_{n}\left(F_{n}^{2}+F_{n+1}^{2}\right) \\
& =L_{n-1} F_{n+1}^{2}-L_{n-1} F_{n-1}^{2}+\left(L_{n} F_{n}^{2}+L_{n} F_{n+1}^{2}\right.
\end{aligned}
$$

(continued)

$$
\begin{aligned}
& =F_{n+1}^{2}\left(L_{n-1}+L_{n}\right)+L_{n} F_{n}^{2}-L_{n-1} F_{n-1}^{2} \\
& =L_{n+1} F_{n+1}^{2}+L_{n} F_{n}^{2}-L_{n-1} F_{n-1}^{2} .
\end{aligned}
$$

For (44), we use (27) and (26) to write

$$
\begin{aligned}
5 F_{3 n} & =5 F_{n-1+2 n+1} \\
& =L_{n-1} L_{2 n}+L_{n} L_{2 n+1} \\
& =L_{n-1} L_{n-1+n+1}+L_{n} L_{n+n+1} \\
& =L_{n-1}\left(L_{n-1} F_{n}+L_{n} F_{n+1}\right)+L_{n}\left(L_{n} F_{n}+L_{n+1} F_{n+1}\right) \\
& =L_{n-1}^{2} F_{n}+L_{n-1} L_{n} F_{n+1}+L_{n}^{2} F_{n}+L_{n} L_{n+1} F_{n+1} \\
& =L_{n-1}^{2}\left(F_{n+1}-F_{n-1}\right)+L_{n-1} L_{n} F_{n+1}+L_{n}^{2} F_{n}+L_{n}\left(L_{n-1}+L_{n}\right) F_{n+1} \\
& =F_{n+1}\left(L_{n-1}^{2}+2 L_{n-1} L_{n}+L_{n}^{2}\right)+L_{n}^{2} F_{n}-L_{n-1}^{2} F_{n-1} \\
& =F_{n+1}\left(L_{n-1}+L_{n}\right)^{2}+L_{n}^{2} F_{n}-L_{n-1}^{2} F_{n-1} \\
& =F_{n+1} L_{n+1}^{2}+F_{n} L_{n}^{2}-F_{n-1} L_{n-1}^{2} .
\end{aligned}
$$

Finally, to obtain (45), we use (26) and (27) to write

$$
\begin{aligned}
5 L_{3 n} & =5 L_{n-1+2 n+1} \\
& =5\left(L_{n-1} F_{2 n}+L_{n} F_{2 n+1}\right) \\
& =L_{n-1} \cdot 5 F_{n-1+n+1}+L_{n} \cdot 5 F_{n+n+1} \\
& =L_{n-1}\left(L_{n-1} L_{n}+L_{n} L_{n+1}\right)+L_{n}\left(L_{n}^{2}+L_{n+1}^{2}\right) \\
& =L_{n-1}^{2}\left(L_{n+1}-L_{n-1}\right)+L_{n-1} L_{n} L_{n+1}+L_{n}^{3}+L_{n} L_{n+1}^{2} \\
& =L_{n-1}^{2} L_{n+1}-L_{n-1}^{3}+L_{n-1} L_{n} L_{n+1}+L_{n}^{3}+\left(L_{n+1}-L_{n-1}\right) L_{n+1}^{2} \\
& =L_{n+1}^{3}+L_{n}^{3}-L_{n-1}^{3}+L_{n-1} L_{n+1}\left(L_{n-1}+L_{n}-L_{n+1}\right) \\
& =L_{n+1}^{3}+L_{n}^{3}-L_{n-1}^{3}
\end{aligned}
$$

as claimed.

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## EXPLICIT FORMULAS FOR NUMBERS OF RAMANUJAN

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## 1. INTRODUCTION

In Chapter 3 of his second notebook [1, p. 165], Ramanujan defined numbers $a(n, k)$ such that $a(2,0)=1$ and for $n \geqslant 2$,

$$
\begin{equation*}
a(n+1, k)=(n-1) a(n, k-1)+(2 n-1-k) a(n, k) \tag{1.1}
\end{equation*}
$$

He defined $a(n, k)=0$ when $k<0$ or $k>n-2$. The numbers were used in the following way: Fix $a>1 / e$ and for real $h$ define $x>0$ by the relation

$$
x^{x}=a^{a} e^{h}
$$

Then it can be shown [1, pp. 164-165] that

$$
\frac{x-a}{a}=\sum_{n=1}^{\infty}(-1)^{n-1} \frac{A_{n}}{n!}\left(\frac{h}{a}\right)^{n},
$$

where $|h|$ is sufficiently small, $A_{1}=(1+\log a)^{-1}$,

$$
A_{n}=\sum_{k=0}^{n-2} a(n, k)(1+\log a)^{1+k-2 n}, \quad n \geqslant 2 .
$$

The values of $\alpha(n, k)$ for $2 \leqslant n \leqslant 7$ are given in the following table.
Table 1

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 2 | 1 |  |  |  |  |  |
| 3 | 3 | 1 |  |  |  |  |
| 4 | 15 | 10 | 2 |  |  |  |
| 5 | 105 | 105 | 40 | 6 |  |  |
| 6 | 945 | 1260 | 700 | 196 | 24 |  |
| 7 | 10395 | 17325 | 12600 | 5068 | 1148 | 120 |

The purpose of this paper is to show how $a(n, k)$ can be expressed in terms of Stirling numbers of the first kind and associated Stirling numbers of the second kind. We prove in §2 that

$$
\begin{aligned}
& a(n, n-2)=(n-2)!=(-1)^{n} s(n-1,1), \\
& a(n, n-3)=(-1)^{n}(n-2) s(n-1,1)+(-1)^{n-1} 2 s(n-1,2),
\end{aligned}
$$

and in general, for $k \geqslant 2$,

$$
a(n, n-k)=\sum_{t=1}^{k-1}(-1)^{n-1-t} P_{k, t}(n) s(n-1, t),
$$

where $P_{k, t}(n)$ is a polynomial in $n$ of degree $k-1-t$ and $s(n-1, t)$ is the Stirling number of the first kind. A recurrence formula for the coefficients of $P_{k, t}(n)$ is derived and the values of $P_{k, t}(n)$ for $2 \leqslant k \leqslant 6$ are computed (see Table 2). In §3 we show that

$$
a(n, 0)=b(2 n-2, n-1), a(n, 1)=b(2 n-3, n-2),
$$

and, for $k>1$,

$$
a(n, k)=\sum_{r=3}^{n-k+1} Q_{k}(n, r) \frac{(2 n-k-3)!!}{(2 r-3)!!}(r-1) b(2 r-3, r-2),
$$

where the $Q_{k}(n, r)$ are rational numbers,

$$
n!!=\left\{\begin{array}{lll}
1.3 & \cdots & n  \tag{1.2}\\
2.4 & \text { if } n & \text { is odd } \\
\text { if } n \text { is even }
\end{array}\right.
$$

and $b(n, k)$ is the associated Stirling number of the second kind. A recurrence formula for $Q_{k}(n, r)$ is worked out and the values of $Q_{k}(n, r)$ for $k=2$ and $k=3$ are given. In §4 we prove an identity for the Stirling numbers of the first kind. This identity, interesting in its own right, is used in the proof of Theorem 2.1.

## 2. STIRLING NUMBERS OF THE FIRST KIND

Throughout the paper we use the notation

$$
(x)_{n}=x(x-1) \cdots(x-n+1)
$$

The Stirling number of the first kind, $s(n, k)$, can be defined by means of

$$
\begin{equation*}
(x)_{n}=\sum_{k=0}^{n} s(n, k) x^{k} . \tag{2.1}
\end{equation*}
$$

These numbers are well known and have been extensively studied; a table of values for $1 \leqslant n \leqslant 15$ can be found in [2, p. 310]. In particular,

$$
s(n, 1)=(-1)^{n-1}(n-1)!
$$

By (1.1) and the fact that $\alpha(n, k)=0$ for $k>n-2$, we have

$$
\alpha(n, n-2)=(n-2) \alpha(n-1, n-3)=(n-2)!\alpha(2,0)=(n-2)!,
$$

and therefore

$$
a(n, n-2)=(-1)^{n} s(n-1,1)
$$

Theorem 2.1: For $k \geqslant 2$,

$$
a(n, n-k)=\sum_{t=1}^{k-1} P_{k, t}(n)(-1)^{n-1-t_{s}}(n-1, t)
$$

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where $P_{k, t}(n)$ is a polynomial in $n$ of degree $k-1-t$. The coefficient of $n^{k-1-t}$ is $t!/(k-t-1)!$. If we write

$$
P_{k, t}(n)=\sum_{j=0}^{k-1-t} c_{k}(t, j)(n-1)_{j}=\sum_{j=0}^{k-1-t} d_{k}(t, j)(n-2)_{j},
$$

then, for $k>2, d_{k}(1,0)=0, d_{k}(t, 0)=(k-1) c_{k-1}(t-1,0)$ for $t>1$, and

$$
d_{k}(t, j)=\sum_{m=t}^{k-1-j}(-1)^{m-t}\left(\frac{1}{j}\right)^{m-t+1}\left(d_{k-1}(m, j-1)+(k-1) c_{k-1}(m, j)\right)
$$

for $t \geqslant 1, j>0$.
Proof: We showed above that the theorem is true for $\alpha(n, n-2)$; assume it is true for $a(n, n-(k-1))$, so we can write

$$
\begin{align*}
& a(n, n-(k-1))=\sum_{m=1}^{k-2} P_{k-1, m}(n)(-1)^{n-1-m} s(n-1, m)  \tag{2.2}\\
& P_{k-1, m}(n)=\sum_{j=0}^{k-2-m} c_{k-1}(m, j)(n-1)_{j}=\sum_{j=0}^{k-2-m} d_{k-1}(m, j)(n-2)_{j} \tag{2.3}
\end{align*}
$$

By (1.1), we have the recurrence

$$
\begin{equation*}
a(n, n-k)=(n-2) \alpha(n-1, n-1-k)+(n-3+k) \alpha(n-1, n-k) \tag{2.4}
\end{equation*}
$$

We define the formal power series

$$
A_{k}(x)=\sum_{n=k}^{\infty} a(n, n-k) \frac{x^{n-1}}{(n-2)!}
$$

and sum on both sides of (2.4), after multiplying by $\frac{x^{n-1}}{(n-2)!}$, to obtain

$$
A_{k}(x)=x A_{k}(x)+\sum_{n=k}^{\infty}(n-3+k) a(n-1, n-k) \frac{x^{n-1}}{(n-2)!}
$$

Therefore,

$$
\begin{equation*}
A_{k}(x)=\frac{1}{1-x} \sum_{n=k-1}^{\infty}(n-2+k) a(n, n-(k-1)) \frac{x^{n}}{(n-1)!} \tag{2.5}
\end{equation*}
$$

Comparing coefficients of $x^{n-1}$ in (2.5), we have

$$
\begin{align*}
a(n, n-k)= & \sum_{r=k-1}^{n-1} \frac{(n-2)!}{(r-2)!} a(r, r-(k-1)) \\
& +(k-1) \sum_{r=k-1}^{n-1} \frac{(n-2)!}{(r-1)!} a(r, r-(k-1)) \tag{2.6}
\end{align*}
$$

We now substitute into (2.6) the formula for $a(r, r-(k-1)$ ) given by (2.2) and (2.3). Then (2.6) becomes, after some manipulation,

$$
\begin{align*}
& a(n, n-k)=\sum_{m=1}^{k-2} \sum_{j=0}^{k-2-m}(-1)^{m-n} d_{k-1}(m, j) \sum_{r=j+1}^{n-2} \frac{(n-2)!(-1)^{n-r}}{(r-j-1)!} s(r, m) \\
& +(k-1) \sum_{m=1}^{k-2} \sum_{j=0}^{k-2-m}(-1)^{m-n} c_{k-1}(m, j) \sum_{r=j}^{n-2} \frac{(n-2)!(-1)^{n-r}}{(r-j)!} s(r, m) \tag{2.7}
\end{align*}
$$

At this point we need the following lemma, which we prove in §4.
Lemma 2.1: We have

$$
\sum_{r=j}^{n} \frac{n!(-1)^{n-r}}{(r-j)!} s(r, m)= \begin{cases}-(n)_{j} \sum_{t=1}^{m} s(n+1, t)\left(\frac{1}{j}\right)^{m-t+1} & \text { if } j>0 \\ s(n+1, m+1) & \text { if } j=0\end{cases}
$$

We now substitute the formulas of Lemma 2.1 (with $n$ replaced by $n-2$ and $j$ replaced by $j+1$ ) into (2.7) and change the order of the $m$, $t$ summations. We have

$$
\begin{aligned}
a(n, n-k)= & \sum_{t=1}^{k-2} \sum_{j=0}^{k-1-t} d_{k}(t, j)(n-2)_{j}(-1)^{n-1-t_{s}(n-1, t)} \\
& +d_{k}(k-1,0)(-1)^{n-k_{s}}(n-1, k-1) \\
= & \sum_{t=1}^{k-1} P_{k, t}(n)(-1)^{n-1-t_{s}(n-1, t)}
\end{aligned}
$$

where $d_{k}(1,0)=0, d_{k}(t, 0)=(k-1) c_{k-1}(t-1,0)$ for $t>1$ and

$$
d_{k}(t, j)=\sum_{m=t}^{k-1-j}(-1)^{m-t}\left(\frac{1}{j}\right)^{m-t+1}\left(d_{k-1}(m, j-1)+(k-1) c_{k-1}(m, j)\right)
$$

for $t \geqslant 1, j>0$. It follows that $P_{k, t}(n)$ has degree $k-1-t$ and the coefficient of $n^{k-1-t}$ is

$$
\begin{aligned}
d_{k}(t, k-1-t) & =\frac{1}{k-1-t} d_{k-1}(t, k-2-t) \\
& =\frac{1}{(k-1-t)!} d_{t+1}(t, 0)
\end{aligned}
$$

Since

$$
\begin{aligned}
P_{k, k-1}(n)=d_{k}(k-1,0) & =(k-1) c_{k-1}(k-2,0) \\
& =(k-1) d_{k-1}(k-2,0)=(k-1)!
\end{aligned}
$$

the coefficient of $n^{k-1-t}$ in $P_{k, t}(n)$ is $\frac{t!}{(k-1-t)!}$. This completes the proof of Theorem 2.1.

From Theorem 2.1, we have the following special cases:

$$
\begin{aligned}
& c_{k}(t, j)=0=d_{k}(t, j) \quad \text { if } j>k-1-t \\
& c_{k}(k-2,1)=d_{k}(k-2,1)=(k-2)! \\
& c_{k}(k-2,0)=(k-1)!+(-1)^{k-1} s(k, 2), \\
& d_{k}(k-2,0)=k(k-2)!+(-1)^{k-1} s(k, 2),
\end{aligned}
$$

$$
\begin{aligned}
& c_{k}(t, k-2-t)=\left[t!(k-2)+(-1)^{t}(t+1) s(t+1,2)\right] /(k-2-t)! \\
& d_{k}(t, k-2-t)=\left[t!(k-1)+(-1)^{t}(t+1) s(t+1,2)\right] /(k-2-t)!
\end{aligned}
$$

It follows that

$$
\begin{equation*}
P_{k, k-2}(n)=(k-2)!n+(k-2)(k-2)!+(-1)^{k-1} s(k, 2) \tag{2.8}
\end{equation*}
$$

We have already pointed out that

$$
\begin{equation*}
P_{k, k-1}(n)=(k-1)! \tag{2.9}
\end{equation*}
$$

The evidence seems to indicate that

$$
P_{k, 1}(n)=\binom{n+k-5}{k-2},
$$

but this has not been proved.
Since $(n-1)_{j}=(n-2)_{j}+j(n-2)_{j-1}$, we have the relationship:

$$
\begin{equation*}
d_{k}(t, j)=c_{k}(t, j)+(j+1) c_{k}(t, j+1) \tag{2.10}
\end{equation*}
$$

Since

$$
(n-2)_{j}=\sum_{r=0}^{j}(-1)^{j-r} j!(n-1)_{r} / r!
$$

we have

$$
\begin{equation*}
c_{k}(t, j)=(-1)^{j} \sum_{r=j}^{k-1-t}(-1)^{r} r!d_{k}(t, r) / j! \tag{2.11}
\end{equation*}
$$

Using (2.10) and (2.11), we can obviously write the recurrence for the coefficients $P_{k, t}(n)$ in several different ways.

The following values of $P_{k, t}(n)$ have been worked out using Theorem 2.1.
Table 2

| $k^{t}$ | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 1 |  |  |  |  |
| 3 | $n-2$ |  |  |  |  |
| 4 | $\binom{n-1}{2}$ | $2 n-7$ | 6 |  |  |
| 5 | $\binom{n}{3}$ | $(n-2)(n-4)$ | $6 n-32$ | 24 |  |
| 6 | $\binom{n+1}{4}$ | $2\binom{n-1}{3}-\binom{n-1}{2}$ | $3 n^{2}-29 n+61$ | $24 n-178$ | 120 |

## 3. ASSOCIATED STIRLING NUMBERS

The associated Stirling number of the second kind, $b(n, k)$, can be defined by means of

$$
\left(e^{x}-x-1\right)^{k}=k!\sum_{n=2 k}^{\infty} b(n, k) \frac{x^{n}}{n!}
$$

We are using the notation of Riordan [3, pp. 74-78] for these numbers. They are also discussed in [2, pp. 221-222], where the notation $S_{2}(n, k)$ is used. A recurrence formula is

$$
\begin{equation*}
b(n+1, k)=k b(n, k)+n b(n-1, k-1) \tag{3.1}
\end{equation*}
$$

with $b(0,0)=1$ and $b(n, k)=0$ if $n<2 k$. A table of values for $b(n, k)$, $1 \leqslant n \leqslant 18$, is given in [2, p. 222]. It follows from (3.1) that

$$
b(2 n, n)=1 \cdot 3 \cdot 5 \cdot \cdots \cdot(2 n-1)=(2 n-1)!!,
$$

with the notation of (1.2).
Since $a(n, 0)=(2 n-3) a(n-1,0)=(2 n-3)!!$, we have

$$
\begin{equation*}
a(n, 0)=b(2 n-2, n-1), \quad n \geqslant 2 . \tag{3.2}
\end{equation*}
$$

Also,

$$
\begin{align*}
b(2 n-1, n-1) & =(n-1) b(2 n-2, n-1)+(2 n-2) b(2 n-3, n-2) \\
& =(n-1) a(n, 0)+(2 n-2) b(2 n-3, n-2), \tag{3.3}
\end{align*}
$$

with $b(3,1)=1$. Comparing (3.3) with (1.1), we have

$$
\begin{equation*}
a(n, 1)=b(2 n-3, n-2), \quad n \geqslant 3 . \tag{3.4}
\end{equation*}
$$

Let $F_{k}(x)$ be the formal power series

$$
\sum_{n=0}^{\infty} \frac{a(n+1, k)}{(2 n-k-1)!!} x^{n}
$$

Then from (1.1) we have

$$
\begin{equation*}
F_{k}(x)=\frac{1}{1-x} \sum_{n=0}^{\infty} \frac{(n-1) a(n, k-1)}{(2 n-k-1)!!} x^{n} \tag{3.5}
\end{equation*}
$$

Comparing coefficients of $x^{n-1}$ in (3.5), we have

$$
\begin{equation*}
a(n, k)=\sum_{j=k+1}^{n-1} \frac{(2 n-k-3)!!}{(2 j-k-1)!!}(j-1) a(j, k-1) . \tag{3.6}
\end{equation*}
$$

It follows from (3.4) and (3.6) that

$$
\begin{equation*}
a(n, 2)=\sum_{r=3}^{n-1} \frac{(2 n-5)!!}{(2 r-3)!!}(r-1) b(2 r-3, r-2) . \tag{3.7}
\end{equation*}
$$

Theorem 3.1: For $k \geqslant 2$,

$$
a(n, k)=\sum_{r=3}^{n-k+1} Q_{k}(n, r) \frac{(2 n-k-3)!!}{(2 r-3)!!}(r-1) b(2 r-3, r-2),
$$

where the $Q_{k}(n, r)$ are rational numbers such that $Q_{2}(n, r)=1$ and

$$
Q_{k}(n, r)=\sum_{m=r+k-2}^{n-1} \frac{(2 m-k-2)!!}{(2 m-k-1)!!}(m-1) Q_{k-1}(m, r)
$$

for $3 \leqslant r \leqslant n-1$ and $n \geqslant 4$.

## EXPLICIT FORMULAS FOR NUMBERS OF RAMANUJAN

Proof: According to (3.7), the Theorem is true for $\alpha(n, 2)$; assume it is true for $a(n, k-1)$. The proof for $a(n, k)$ follows immediately when we substitute

$$
a(j, k-1)=\sum_{r=3}^{j-k+2} Q_{k-1}(j, r) \frac{(2 j-k-2)!!}{(2 r-3)!!}(r-1) b(2 r-3, r-2)
$$

into (3.6) and change the order of the summations. This completes the proof.
It is not difficult to evaluate

$$
\begin{aligned}
Q_{3}(n, r) & =\sum_{m=r+1}^{n-1} \frac{(2 m-5)!!}{(2 m-4)!!}(m-1) \\
& =\frac{n(2 n-5)}{3 \cdot 4^{n-3}}\binom{2 n-6}{n-3}-\frac{(r+1)(2 r-3)}{3 \cdot 4^{r-2}}\binom{2 r-4}{r-2}
\end{aligned}
$$

but apparently the formulas for $Q_{k}(n, r)$ for $k>3$ are complicated.

## 4. PROOF OF LEMMA 2.1

The second equality in Lemma 2.1 is proved in [2, p. 215]. To the writer's knowledge, the first equality is new and is of interest in its own right. We shall make use of the generating function

$$
\begin{equation*}
(1+t)^{u}=\sum_{n=0}^{\infty} \sum_{k=1}^{n} s(n, k) u^{k} \frac{t^{n}}{n!}, \tag{4.1}
\end{equation*}
$$

which follows from (2.1) and the MacLaurin series for $(1+t)^{u}$. We have

$$
\begin{equation*}
t^{j}(u)_{j}(1+t)^{u-j}=\sum_{n=j}^{\infty} \sum_{k=1}^{n} s(n, k) u^{k} \frac{t^{n}}{(n-j)!}, \tag{4.2}
\end{equation*}
$$

so

$$
\begin{equation*}
t^{j}(u)_{j}(1+t)^{u-j-1}=\sum_{n=j}^{\infty} \sum_{r=j}^{n}\left(\sum_{k=1}^{r} \frac{(-1)^{n-r} s(r, k) u^{k}}{(r-j)!}\right) \frac{t^{n}}{n!} . \tag{4.3}
\end{equation*}
$$

From (4.1) and the binomial theorem,
so

$$
(1+t)^{u-j-1}=1+\sum_{n=1}^{\infty} \sum_{k=1}^{n} s(n, k) \sum_{r=0}^{k}\binom{k}{r}(-j-1)^{k-r} u^{r} \frac{t^{n}}{n!}
$$

$$
t^{j}(u)_{j}(1+t)^{u-j-1}
$$

$$
\begin{equation*}
=t^{j}(u)_{j}+\sum_{n=1}^{\infty} \sum_{k=1}^{n} s(n, k) \sum_{r=0}^{k}\binom{k}{r}(-j-1)^{k-r}\left(\sum_{m=1}^{j} s(j, m) u^{r+m}\right) \frac{t^{n+j}}{n!} \tag{4.4}
\end{equation*}
$$

Comparing coefficients of $u^{k} t^{n} / n!$ in (4.3) and (4.4), we have

$$
\begin{equation*}
\sum_{r=j}^{n} \frac{(-1)^{n-r} s(r, k) n!}{(r-j)!}=(n)_{j} \sum_{m=1}^{j} s(j, m) \sum_{i=k-m}^{n-j}\binom{i}{k-m}(-j-1)^{i-k+m_{s}(n-j, i)} \tag{4.5}
\end{equation*}
$$

We now obtain the right-hand side of (4.5) in another way. We know
$(n)_{j}(x)_{n+1} /(x-j)$
$=(n)_{j}(x)_{j}(x-j-1)_{n-j}$
$=(n)_{j} \sum_{m=0}^{j} s(j, m) x^{m} \sum_{i=0}^{n-j} s(n-j, i)(x-j-1)^{i}$
$=(n){ }_{j} \sum_{m=1}^{j} s(j, m) \sum_{i=0}^{n-j} s(n-j, i)\left(\sum_{w=0}^{i}\binom{i}{w}(-j-1)^{i-w}\right) x^{w+m}$.
The coefficient of $x^{k}$ on the right side of (4.6) is

$$
(n)_{j} \sum_{m=1}^{j} s(j, m) \sum_{i=k-m}^{n-j}\binom{i}{k-m}(-j-1)^{i-k+m} s(n-j, i),
$$

which can be compared to the right side of (4.5). The left side of (4.6) can be written

$$
(n)_{j}(x)_{n+1} /(x-j)=\frac{-(n)_{j}}{j} \frac{1}{1-\left(\frac{x}{j}\right)} \sum_{m=0}^{\infty} s(n+1, m) x^{m}
$$

so the coefficient of $x^{k}$ is

$$
\begin{equation*}
-(n)_{j} \sum_{m=1}^{k} s(n+1, m)\left(\frac{1}{j}\right)^{k-m+1} \tag{4.7}
\end{equation*}
$$

Comparing (4.7) and the left side of (4.5), we have the first equality of Lemma 2.1. This completes the proof.

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# SKEW CIRCULANTS AND THE THEORY OF NUMBERS: AN ADDENDUM 

I. J. GOOD<br>Virginia Polytechnic Institute and State University, Blacksburg, VA 24061<br>(February 3, 1986)

While correcting the proofs of my article "Skew Circulants and the Theory of Numbers," my interest in the topic was revived, and I have now tracked down some relevant work by the great Jacobi. On pages 277 and 278 of his paper "Uber die complexen Primzahlen, welche in der Theorie der Reste $5^{\text {ten }}, 8^{\text {ten }}$ und $12^{\text {ten }}$ Potenzen zu Betrachten sind" (1839), which is in Volume VI of his collected papers, he shows that any prime of the form $8 n+1$ can be factorized as $\phi(\alpha) \phi\left(\alpha^{3}\right) \phi\left(\alpha^{5}\right) \phi\left(\alpha^{7}\right)$, where $\alpha=\exp (2 \pi i / 8)$ and $\phi(\alpha)$ is of the form $y^{\prime}+y^{\prime \prime} \alpha^{2}+$ $z^{\prime} \alpha+z^{\prime \prime} \alpha^{3}$, and this is equivalent to my first conjecture although Jacobi does not mention skew circulants. His proof depends on Gauss's theory of "biquadratic residues" and on work by Lagrange (presumably Lagrange's Oeuvres III, pp. 693-795). Jacobi's proof is too succinct for me to understand, and I think he may have been slightly careless. For example, he says (in free translation): "One can prove that any number $a+i b$ that divides a number of the form $y^{2}-i z^{2}$ is again of this form itself, and the proof is exactly like that of the analogous fact that any whole number that divides a number of the form $y^{2}+z^{2}$ is itself a sum of two squares. (Without some gloss, this last statement is false; for example, 7 divides $49^{2}+196^{2}$. No doubt $y$ and $z$ are supposed to be mutually prime.) If his paper had been written by a much less eminent mathematician, I might have suspected that his claims were based in part on numerical evidence and not on complete proofs.

The basic idea in Jacobi's proof is to note that much of ordinary number theory can be generalized to the Gaussian integers $a+i b$.

Jacobi states that a similar method can be used to prove that every prime of the form $12 n+1$ can be expressed as a product of four factors each related to a twelfth root of unity. (Also in the forms $a^{2}+b^{2}, c^{2}+3 d^{2}$, and $e^{2}-3 f^{2}$.) This result cannot lead to an expression of $12 n+1$ as a skew carculant of order other than 2 because, for example, 13 and 37 are primes of the form $12 n+1$ but not of the form $8 m+1$. Jacobi mentions further that a prime of the form $5 n+1$ can always be written in the form $a^{2}-5 b^{2}$. The smallest prime that is of all three forms $5 \ell+1,8 m+1$, and $12 n+1$ is 241 and is, therefore, presumably the smallest number that can be expressed in all six of the ways:

$$
a^{2}+b^{2}, c^{2}+2 d^{2}, e^{2}+3 f^{2}, g^{2}-2 h^{2}, k^{2}-3 l^{2}, \text { and } p^{2}-5 q^{2}
$$

Indeed,

$$
\begin{aligned}
241 & =4^{2}+15^{2} \\
& =13^{2}+2 \times 6^{2} \\
& =7^{2}+3 \times 8^{2} \\
& =21^{2}-2 \times 10^{2} \\
& =17^{2}-3 \times 4^{2} \\
& =31^{2}-5 \times 12^{2} .
\end{aligned}
$$

Any prime of the form $120 n+1$ will, of course, have the six representations. Presumably (i) the expressions with positive signs are unique, and (ii) those with negative signs have an infinity of representations.

The main point of this addendum is, of course, that the first conjecture in my paper is equivalent to a result seemingly proved by Jacobi in 1839, although he did not express the result in terms of skew circulants.

I expect that anyone familiar with both Gauss's and Lagrange's work would be able to prove my second conjecture which specified all the integers expressible as skew circulants of order 4. Unfortunately, during the next several months, my other commitments will prevent me from achieving the requisite familiarity, fascinating though this study would undoubtedly be.

# ON FIBONACCI BINARY SEQUENCES 

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(Submitted July 1984)

A Fibonacci binary sequence of degree $k$ is defined as a $\{0,1\}$-sequence such that no $k+11$ 's are consecutive. For $k=1$, we have ordinary Fibonacci sequences [2]. Let $G(k, n)$ denote the number of Fibonacci binary sequences of degree $k$ and length $n$. As was given in [2], it can be easily verified that for $k=1$, we have $G(1,1)=2=F_{2}, G(1,2)=3=F_{3}$, and $G(1, n)=G(1, n-1)+$ $G(1, n-2)=F_{n+1}$ for $n \geqslant 3$, where $F_{n}$ is the $n$th Fibonacci number. In general, we have

$$
\begin{array}{ll}
G(k, n)=2^{n} & \text { for } 1 \leqslant n \leqslant k \\
G(k, n)=\sum_{j=1}^{k+1} G(k, n-j) & \text { for } n \geqslant k+1 \tag{1b}
\end{array}
$$

Thus, for any $k \geqslant 1$, the sequence $\left\{F_{k, n}=G(k, n-1), n \geqslant 0\right\}$ is the $k^{\text {th }}$-order Fibonacci sequence, where we set $G(k,-1)=G(k, 0)=1$ for convenience.

Let $W(k, n)$ denote the total number of 1 's in all binary sequences of degree $k$ and length $n$. Then,

$$
\begin{array}{ll}
W(k, n)=n 2^{n-1} & \text { for } 0 \leqslant n \leqslant k \\
W(k, n)=\sum_{j=0}^{k}[W(k, n-j-1)+j G(k, n-j-1)] \text { for } n \geqslant k+1
\end{array}
$$

The ratio $q(k, n)=W(k, n) / n G(k, n)$ gives the proportion of 1 's in all the binary sequences of degree $k$ and length $n$. It was proved in [2] that the limit

$$
q(k)=\lim _{n \rightarrow \infty} q(k, n)
$$

which is the asymptotic proportion of 1 's in Fibonacci binary sequences of degree $k$, exists for $k=1$, and actually the limit is $q(1)=(5-\sqrt{5}) / 10$. It is interesting to extend this result and solve the problem for all integers $k \geqslant 1$.

Let $\{A(n), n \geqslant-(k+1)\}$ be a sequence of numbers with $A(j)=0$ for $-(k+1) \leqslant j \leqslant-1$. If we define a sequence

$$
B(n)=A(n)-A(n-1)-\cdots-A(n-k-1) \text { for } n \geqslant 0
$$

similar to the result in the case $k=1$, we have the inverse relation

$$
\begin{equation*}
A(n)=\sum_{j=0}^{k} G(k, j-1) B(n-j) \quad \text { for } n \geqslant 0 \tag{3}
\end{equation*}
$$

where the sequence $\{G(k, n), n \geqslant-1\}$ is defined above. From (2), we obtain

$$
\sum_{m=0}^{k} m G(k, n-m-1)=W(k, n)-\sum_{j=0}^{k} W(k, n-j-1) \quad \text { for } n \geqslant k+1
$$

The inverse relation (3) then implies that

$$
\begin{equation*}
W(k, n)=\sum_{j=0}^{n}\left(G(k, j-1)\left[\sum_{m=0}^{k} m G(k, n-j-m-1)\right]\right) \text { for } n \geqslant k+1, \tag{4}
\end{equation*}
$$

where we set $G(k, n)=0$ for $n \leqslant-2$ for convenience.
The characteristic equation for the recursion (1) is

$$
\begin{equation*}
h(x)=x^{k+1}-x^{k}-\cdots-x-1=0 . \tag{5}
\end{equation*}
$$

Let $r_{1}, \ldots, r_{k+1}$ be its solution. We have the expression

$$
\begin{equation*}
G(k, n)=\sum_{j=1}^{k+1} c_{j} r_{j}^{n}, \text { for } n \geqslant 0, \tag{6}
\end{equation*}
$$

where $c_{j}$ are constants [1]. It is known that (5) has exactly one solution, say $r_{1}$, whose norm is not less than 1 [3]. Using Cramer's rule, we obtain an explicit form for $c_{1}$ from (6):

$$
\begin{aligned}
c_{1} & =\frac{\left(2-r_{2}\right)\left(2-r_{3}\right) \ldots\left(2-r_{k+1}\right)\left(r_{2}-r_{3}\right) \ldots\left(r_{k}-r_{k+1}\right)}{\left(r_{1}-r_{2}\right)\left(r_{1}-r_{3}\right) \ldots\left(r_{1}-r_{k+1}\right)\left(r_{2}-r_{3}\right) \cdots\left(r_{k}-r_{k+1}\right)} \\
& =\frac{\left(2-r_{2}\right) \cdots\left(2-r_{k+1}\right)}{\left(r_{1}-r_{2}\right) \ldots\left(r_{1}-r_{k+1}\right)}=\frac{1}{\left(2-r_{1}\right) h^{\prime}\left(r_{1}\right)} .
\end{aligned}
$$

From the equality (4), we get

$$
W(k, n)=\sum_{j=0}^{n}\left[\left(\sum_{\ell=1}^{k+1} c_{\ell} r_{\ell}^{j-1}\right)\left(\sum_{m=1}^{k} m\left(\sum_{p=1}^{k+1} c_{p} r_{p}^{n-j-m-1}\right)\right)\right] .
$$

Since $\left|r_{\ell}\right|<1$ for $2 \leqslant \ell \leqslant n+1$ and $G(k, n)=0\left(r_{1}^{n}\right)$, we have

$$
\begin{aligned}
& \sum_{j=0}^{n} c_{\ell} r_{\ell}^{j-1} m c_{p} r_{p}^{n-j-m-1} \\
& =\left\{\begin{array}{l}
m c_{\ell} c_{p} r_{\ell}^{-1} r_{p}^{-m-1}\left(r_{p}^{n+1}-r_{\ell}^{n+1}\right)\left(r_{p}-r_{\ell}\right)^{-1}=o(n G(k, n)) \text { for } r_{p} \neq r_{\ell} \\
n m c_{\ell} c_{p} r_{\ell}^{n-m-2}=o(n G(k, n)) \text { for } r_{\ell}=r_{p} \neq r_{1} .
\end{array}\right.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
q(k) & =\lim _{n \rightarrow \infty} \frac{W(k, n)}{n G(k, n)}=\lim _{n \rightarrow \infty} \sum_{j=0}^{n} c_{1} r_{1}^{j-1}\left(\sum_{m=1}^{k} m c_{1} r_{1}^{n-j-m-1}\right) / n c_{1} r_{1}^{n} \\
& =c_{1} \sum_{m=1}^{k} m r_{1}^{-m-2}
\end{aligned}
$$

We have established the rollowing result.
Theorem: Let $r$ be the solution of (5) in the interval (1, 2). Then

$$
q(k)=\left(\sum_{j=1}^{k} j r^{-j-2}\right) /\left[(2-r)\left((k+1) r^{k}-\sum_{j=1}^{k} j r^{j-1}\right)\right]
$$

Finally, three numerical examples are presented below.

$$
\begin{aligned}
& \text { For } k=2, r=1.83929, q(2)=0.38158 . \\
& \text { For } k=3, r=1.92756, q(3)=0.43366 . \\
& \text { For } k=4, r=1.96595, q(4)=0.46207 .
\end{aligned}
$$

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## ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by<br>A. P. HILLMAN<br>Assistant Editors<br>GLORIA C. PADILLA and CHARLES R. WALL

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS
The Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ satisfy
and

$$
F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1
$$

$$
L_{n+2}=L_{n+1}+L_{n}, L_{0}=2, L_{1}=1
$$

PROBLEMS PROPOSED IN THIS ISSUE
B-568 Proposed by Wray G. Brady, Slippery Rock University, Slippery Rock, PA
Find a simple curve passing through all of the points

$$
\left(F_{1}, L_{1}\right),\left(F_{3}, L_{3}\right), \ldots,\left(F_{2 n+1}, L_{2 n+1}\right), \ldots .
$$

B-569 Proposed by Wray G. Brady, Slippery Rock University, Slippery Rock, PA
Find a simple curve passing through all of the points

$$
\left(F_{0}, L_{0}\right),\left(F_{2}, L_{2}\right), \ldots,\left(F_{2 n}, L_{2 n}\right), \ldots .
$$

B-570 Proposed by Herta T. Freitag, Roanoke, VA
Let $a, b$, and $c$ be the positive square roots of $F_{2 n-1}, F_{2 n+1}$, and $F_{2 n+3}$, respectively. For $n=1,2$, ..., show that

$$
(a+b+c)(-a+b+c)(a-b+c)(a+b-c)=4
$$

B-571 Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany
Conjecture and prove a simple expression for

$$
\sum_{r=0}^{[n / 2]} \frac{n}{n-r}\binom{n-r}{r}
$$

where $[n / 2]$ is the largest integer $m$ with $2 m \leqslant n$.

## ELEMENTARY PROBLEMS AND SOLUTIONS

B-572 Proposed by Ambati Jaya Krishna, Student, Johns Hopkins University, Baltimore, MD, and Gomathi S. Rao, Orangeburg, SC

Evaluate the continued fraction:

$$
1+\frac{2}{3+\frac{4}{5+\frac{6}{7+\cdots}}}
$$

B-573 Proposed by Charles R. Wall, Trident Technical College, Charleston, SC
For all nonnegative integers $n$, prove that

$$
\sum_{k=0}^{n}\binom{n}{k} L_{k} L_{n-k}=4+5 \sum_{k=0}^{n}\binom{n}{k} F_{k} F_{n-k}
$$

## SOLUTIONS

## Congruence Modulo 12

B-544 Proposed by Herta T. Freitag, Roanoke, VA
Show that $F_{2 n+1}^{2} \equiv L_{2 n+1}^{2}(\bmod 12)$ for all integers $n$.
Solution by Piero Filipponi, Fdn. U. Bordoni, Rome, Italy
First we rewrite the statement as

$$
\begin{equation*}
L_{2 n+1}^{2}-F_{2 n+1}^{2} \equiv 0(\bmod 12), \tag{1}
\end{equation*}
$$

then using Hoggatt's $I_{18}$ and $I_{17}$, we obtain

$$
L_{2 n+1}^{2}-F_{2 n+1}^{2}=4\left(F_{2 n+1}^{2}-1\right)=4\left(F_{2 n+1}+1\right)\left(F_{2 n+1}-1\right)
$$

Since $F_{2 n+1} \equiv \pm 1(\bmod 3)$, it is apparent that congruence (l) is satisfied for all integers $n$.

Also solved by Wray G. Brady, Paul S. Bruckman, L. A. G. Dresel, A. F. Horadam, L. Kuipers, Bob Prielipp, M. Robert Schumann, Heinz-Jürgen Seiffert, A. G. Shannon, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

## Congruences Modulo 5

B-545 Proposed by Herta T. Freitag, Roanoke, VA
Show that there exist integers $a, b$, and $c$ such that

$$
F_{4 n} \equiv a n(\bmod 5) \quad \text { and } \quad F_{4 n+2} \equiv b n+c(\bmod 5)
$$

for all integers $n$.
Solution by Hans Kappus, Rodersdorf, Switzerland
We prove by induction that for $n=0,1,2, .$.

$$
\begin{align*}
& F_{4 n} \equiv 3 n(\bmod 5)  \tag{1}\\
& F_{4 n+2} \equiv 2 n+1(\bmod 5) \tag{2}
\end{align*}
$$

This is obviously true for $n=0,1,2$. Assume (1) and (2) hold for some $n \geqslant 2$. Then for this $n$,

$$
\begin{aligned}
& F_{4 n+1} \equiv F_{4 n+2}-F_{4 n} \equiv 4 n+1(\bmod 5), \\
& F_{4 n+3} \equiv 2 F_{4 n+2}-F_{4 n} \equiv n+2(\bmod 5), \\
& \text { re } \\
& F_{4(n+1)} \equiv 3 F_{4 n+2}-F_{4 n} \equiv 3(n+1)(\bmod 5),
\end{aligned}
$$

hence (1) is true for all $n$. Furthermore,

$$
F_{4(n+1)+2} \equiv 2 F_{4(n+1)}+F_{4 n+3} \equiv 2(n+1)+1(\bmod 5)
$$

The proof is now finished.
Also solved by Wray G. Brady, Paul S. Bruckman, L. A. G. Dresel, A. F. Horadam, L. Kuipers, Bob Prielipp, Heinz-Jürgen Seiffert, A. G. Shannon, Sahib Singh, J. Suck, and the proposer.

## Fibonacci Combinatorial Problem

B-546 Proposed by Stuart Anderson, East Texas State University, Commerce, TX and John Corvin, Amoco Research, Tulsa, OK

For positive integers $a$, let $S_{a}$ be the finite sequence $a_{1}, a_{2}, \ldots, a_{n}$ defined by

$$
\begin{aligned}
& \qquad a_{1}=a \\
& a_{i+1}=a_{i} / 2 \text { if } a_{i} \text { is even, } a_{i+1}=1+a_{i} \text { if } a_{i} \text { is odd, } \\
& \text { the sequence terminates with the earliest term that equals } 1 .
\end{aligned}
$$

For example, $S_{5}$ is the sequence $5,6,3,4,2,1$, of six terms. Let $K_{n}$ be the number of positive integers $a$ for which $S_{a}$ consists of $n$ terms. Does $K_{n}$ equal something familiar?

Solution by Piero Filipponi, Fdn. U. Bordoni, Rome, Italy
It is evident that the only sequence of length 1 is $S_{1}$, the only sequence of length 2 is $S_{2}$, and the only sequence of length 3 is $S_{4}$. That is, we have

$$
\begin{equation*}
k_{1}=k_{2}=k_{3}=1 \tag{1}
\end{equation*}
$$

Let us read the sequences in reverse order so that $\alpha$ is the $n^{\text {th }}$ term of $S_{a}$. By definition, a sequence $S_{a}^{n}$ (of length $n$ ) can generate exactly one(two) sequence(s) $S_{a}^{n+1}$ of length $n+1$, if $a$ is odd (even). Denoting by $e\left(S_{a}^{n}\right)$ and $\circ\left(S_{a}^{n}\right)$ the number of sequences of length $n$ ending with an even term and with an odd term, respectively, we can write

$$
\begin{aligned}
& \mathrm{e}\left(S_{a}^{n+1}\right)=k_{n} \\
& \mathrm{o}\left(S_{a}^{n+1}\right)=\mathrm{e}\left(S_{a}^{n}\right)=k_{n-1},
\end{aligned}
$$

from which we obtain

$$
\begin{equation*}
k_{n+1}=\mathrm{e}\left(S_{a}^{n+1}\right)+\mathrm{o}\left(S_{a}^{n+1}\right)=k_{n}+k_{n-1} \tag{2}
\end{equation*}
$$

From (1) and (2), it is readily seen that

$$
k_{n}=F_{n-1}, \text { for } n>1
$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Ben Freed \& Sahib Singh, Hans Kappus, L. Kuipers, Graham Lord, J. Suck, and the proposer.

## Return Engagement

B-547 Proposed by Philip L. Mana, Albuquerque, NM
For positive integers $p$ and $n$ with $p$ prime, prove that

$$
L_{n p} \equiv L_{n} L_{p}(\bmod p)
$$

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, $P A$
This result has been proved in B-182 (The Fibonacci Quarterly, 1970).
Also solved by Paul S. Bruckman, Odoardo Brugia \& Piero Filipponi, L. A. G. Dresel, L. Kuipers, Bob Prielipp, Lawrence Somer, J. Suck, and the propower.

Number of Squares Needed
B-548 Proposed by Valentina Bakinova, Rondout Valley, NY
Let $D(n)$ be defined inductively for nonnegative integers $n$ by $D(0)=0$ and $D(n)=1+D\left(n-[\sqrt{n}]^{2}\right)$, where $[x]$ is the greatest integer in $x$. Let $n_{k}$ be the smallest $n$ with $D(n)=k$. Then

$$
n_{0}=0, \quad n_{1}=1, \quad n_{2}=2, \quad n_{3}=3, \quad \text { and } \quad n_{4}=7
$$

Describe a recursive algorithm for obtaining $n_{k}$ for $k \geqslant 3$.
Solution by L. A. G. Dresel, University of Reading, England
Let $[\sqrt{n}]=q$, so that $q^{2} \leqslant n \leqslant(q+1)^{2}-1$, and let $R(n)=n-q^{2}$, so that we have $0 \leqslant R(n) \leqslant 2 q$. Suppose now that $n$ is the smallest integer for which $R(n)=r$, and consider the case where $r$ is odd. Then we have $r=2 q-1$ and

$$
n=(q+1)^{2}-2=\frac{1}{4}(r+3)^{2}-2 .
$$

By definition, we have

$$
D\left(n_{k+1}\right)=k+1
$$

and

$$
D\left(n_{k+1}\right)=1+D\left(R\left(n_{k+1}\right)\right) \text {. }
$$

therefore

$$
D\left(R\left(n_{k+1}\right)\right)=k
$$

Since $n_{k}$ is the smallest $n$ for which $D(n)=k$, it follows that
$n_{k+1}$ is the smallest $n$ for which $R(n)=n_{k}$.
Now taking the case where $n \equiv 3(\bmod 4)$, this leads to

$$
n_{k+1} \equiv \frac{1}{4}\left(n_{k}+3\right)^{2}-2
$$

and we have also $n_{k+1} \equiv 3(\bmod 4)$. Hence, starting with $n_{3}=3$, we can use the above recursive algorithm for $k \geqslant 3$.

Also solved by Paul S. Bruckman, Hans Kappus, L. Kuipers, Jerry M. Metzger, Sahib Singh, and the proposer.

Generalized Fibonacci Numbers
B-549 Proposed by George N. Philippou, Nicosia, Cyprus
Let $H_{0}, H_{1}, \ldots$ be defined by $H_{0}=q-p, H_{1}=p$, and $H_{n+2}=H_{n+1}+H_{n}$ for $n=0,1, \ldots$. Prove that, for $n \geqslant m \geqslant 0$,

$$
H_{n+1} H_{m}-H_{m+1} H_{n}=(-1)^{m+1}\left[p H_{n-m+2}-q H_{n-m+1}\right]
$$

Solution by L. A. G. Dresel, University of Reading, England
Define $D(n, m)=H_{n+1} H_{m}-H_{m+1} H_{n}$. Then

$$
\begin{aligned}
D(n, m) & =H_{m}\left(H_{n}+H_{n-1}\right)-H_{n}\left(H_{m}+H_{m-1}\right) \\
& =H_{m} H_{n-1}-H_{n} H_{m-1}=-D(n-1, m-1) .
\end{aligned}
$$

Repeating this reduction step a further $m$ - 2 times, we obtain

$$
\begin{aligned}
D(n, m) & =(-1)^{m-1} D(n-m+1,1) \\
& =(-1)^{m+1}\left(H_{n-m+2} H_{1}-H_{2} H_{n-m+1}\right) \\
& =(-1)^{m+1}\left(p H_{n-m+2}-q H_{n-m+1}\right)
\end{aligned}
$$

Also solved by Paul S. Bruckman, Piero Filipponi, Herta T. Freitag, A. F. Horadam, L. Kuipers, Bob Prielipp, A. G. Shannon, P. D. Siafarikas, Sahib Singh, J. Suck, and the proposer.

## $\bullet \diamond \diamond \diamond$

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

## PROBLEMS PROPOSED IN THIS ISSUE

H-397 Proposed by Paul S. Bruckman, Fair Oaks, CA
For any positive integer $n$, define the function $F_{n}$ on $C$ as follows:

$$
\begin{equation*}
F_{n}(x) \equiv\left(g^{n}-1\right)(x), \tag{1}
\end{equation*}
$$

where $g$ is the operator

$$
\begin{equation*}
g(x) \equiv x^{2}-2 \tag{2}
\end{equation*}
$$

(Thus, $\left.\left.F_{3}(x)=\left\{\left(x^{2}-2\right)\right\}^{2}-2\right\}^{2}-2-x=x^{8}-8 x^{6}+20 x^{4}-16 x^{2}-x+2\right)$. Find all $2^{n}$ zeros of $F_{n}$.

H-398 Proposed by Ambati Jaya Krishna, Freshman, Johns Hopkins University
Let

$$
a+b+c+d+e=\left(\sum_{1}^{\infty}\left(\frac{(-1)^{n+1}}{2 n-1} \frac{2}{3} \cdot 9^{1-n}+7^{1-2 n}\right)\right)^{2}
$$

and

$$
a^{2}+b^{2}+c^{2}+d^{2}+e^{2}=\frac{45}{512} \sum_{1}^{\infty} n^{-4}
$$

$a, b, c, d, e \in \mathbb{R}$. What are the values of $a, b, c, d$, and $e$ if $e$ is to attain its maximum value?

H-399 Proposed by M. Wachtel, Zürich, Switzerland
The twin sequences: $\frac{L_{1+6 n}-1}{2}=0,14,260,4674,83880, \ldots$
and

$$
\frac{L_{5+6 n}-1}{2}=5,99,1785,32039, \ldots
$$

are representable by infinitely many identities, partitioned into several groups of similar structure:

|  | $\frac{L_{1+6 n}-1}{2}=\text { identical to: }$ | $\frac{L_{5+6 n}-1}{2}=\text { identical to: }$ |
| :---: | :---: | :---: |
|  | Group I |  |
| $S_{1}$ | $3 L_{-3+6 n}+\frac{5 F_{-5+6 n}-1}{2}$ | $3 L_{1+6 n}+\frac{5 F_{-1+6 n}-1}{2}$ |
| $S_{2}$ | $61 L_{-9+6 n}+\frac{11 L_{-14+6 n}-1}{2}$ | $61 L_{-5+6 n}+\frac{11 L_{-10+6 n}-1}{2}$ |
| $S_{3}$ | $1103 L_{-15+6 n}+\frac{105 F_{-23+6 n}-1}{2}$ | $1103 L_{-11+6 n}+\frac{105 F_{-19+6 n}-1}{2}$ |
| $S_{4}$ | $19801 L_{-21+6 n}+\frac{199 L_{-32+6 n}-1}{2}$ | $19801 L_{-17+6 n}+\frac{199 L_{-28+6 n}-1}{2}$ |
| $S_{n}$ |  | ... |

Groups II and III (in addition, there are more groups):


Find the construction rules for $S_{n}$ for each group.

SOLUTIONS
Sum Formula!
H-373 Proposed by Andreas N. Philippou, University of Patras, Greece (Vol. 22, no. 3, August 1984)

For any fixed integers $k \geqslant 0$ and $r \geqslant 2$, set

$$
f_{n+1, r}^{(k)}=\sum_{\substack{n_{1}, \ldots, n_{k} \ni \\ n_{1}+2 n_{2}+\cdots+n_{k}=n}}\binom{n_{1}+\cdots+n_{k}+r-1}{n_{1}, \cdots, n_{k}, r-1}, n \geqslant 0
$$

Show that

$$
f_{n+1, r}^{(k)}=\sum_{\ell=0}^{n} f_{\ell+1,1}^{(k)} f_{n+1-\ell, r-1}^{(k)}, n \geqslant 0
$$

Solution by C. Georghiou, University of Patras, Greece
Note that the definition of $f_{n+l, r}^{(k)}$ can be extended to include every positive real number $r$. Define also

$$
f_{n+1}^{(k)}=\delta_{n, 0}, \quad n \geqslant 0
$$

where $\delta_{n, m}$ is the Kronecker symbol. Then we show that

$$
\begin{equation*}
f_{n+1, r}^{(k)}=\sum_{\ell=0}^{n} f_{\ell+1, s}^{(k)} f_{n+\ell-1, r-s}^{(k)} \quad n \geqslant 0 \tag{*}
\end{equation*}
$$

for any fixed positive integer $k$ and any fixed nonnegative real number $r$.
We use generating functions. For fixed $k$ and $r$, let $F_{k, r}(x)$ be the generating function of the sequence $\left\{f_{n+1, r}^{(k)}\right\}_{n=0 \text {. }}^{\infty}$ Then

$$
F_{k, r}(x)=\left(1-x-x^{2}-\cdots-x^{k}\right)^{-r}
$$

Indeed, for some neighborhood of $x=0$, we have

$$
\begin{gathered}
\left(1-x-x^{2}-\cdots-x^{k}\right)^{-r}=\sum_{n=0}^{\infty}\binom{-r}{n}(-1)^{n}\left(x+x^{2}+\cdots+x^{k}\right)^{n} \\
=\sum_{n=0}^{\infty}\binom{r+n-1}{n} \sum_{n_{1}+n_{2}+\cdots+n_{k}=n}\binom{n_{1}+n_{2}+\cdots+n_{k}}{n_{1}, n_{2}, \ldots, n_{k}} x^{n_{1}+2 n_{2}+\cdots+k n_{k}} \\
=\sum_{n=0}^{\infty} \sum_{n_{1}+n_{2}+\cdots+n_{k}=n}\binom{n_{1}+n_{2}+\cdots+n_{k}+r-1}{n_{1}+n_{2}+\cdots+n_{k}} \\
\times\binom{ n_{1}+n_{2}+\cdots+n_{k}}{n_{1}, n_{2}, \ldots, n_{k}} x^{n_{1}+2 n_{2}+\cdots+k n_{k}} \\
=\sum_{n=0}^{\infty} x^{n} n_{n_{1}+2 n_{2}+\cdots+k n_{k}=n}\binom{n_{1}+n_{2}+\cdots+n_{k}+r-1}{n_{1}, n_{2}, \ldots, n_{k}, r-1}
\end{gathered}
$$

## ADVANCED PROBLEMS AND SOLUTIONS

Equation (*) follows from

$$
F_{k, r}(x)=F_{k, s}(x) F_{k, r-s}(x)
$$

Note also that the restriction for $r \geqslant 0$ can be relaxed to $r$ any real number.
Also solved by P. Bruckman.

## Bounds of Joy

H-374 Proposed by Charles R. Wall, Trident Technical College, Charleston, SC (Vol. 22, no. 3, August 1984)

If $\sigma^{*}(n)$ is the sum of the unitary divisors of $n$, then

$$
\sigma^{*}(n)=\prod_{p^{e} \| n}\left(1+p^{e}\right),
$$

where $p^{e}$ is the highest power of the prime $p$ that divides $n$. The ratio $\sigma^{*}(n) / n$ increases as new primes are introduced as factors of $n$, but decreases as old prime factors appear more often. As $N$ increases, is $\sigma^{*}(N!) / N$ ! bounded or unbounded?

Solution by the proposer.
The primes between $N / 2$ and $N$ divide $N$ ! exactly once, and those not exceeding $N / 2$ divide $N$ ! more than once. By considering special cases for $N(\bmod 4)$, it is easy to show by telescoping products that

$$
\prod_{N / 2<p \leqslant N}(p+1) / p<\prod_{N / 2<2 k+1 \leqslant N}(2 k+2) /(2 k+1)<\sqrt{\frac{N+2}{[N / 2]+1}}<1.5
$$

if $N \geqslant 6$. Also

$$
\begin{aligned}
\prod_{\substack{p^{e} \| n \\
p \leqslant N / 2}}\left(1+p^{-e}\right)<\prod_{p \text { prime }}\left(1+p^{-2}\right) & =\prod_{p}\left(1-p^{-4}\right) /\left(1-p^{-2}\right) \\
& =\zeta(2) / \zeta(4)=15 / \pi^{2}<1.52 .
\end{aligned}
$$

Therefore,

$$
\sigma^{*}(N!) / N!<(1.52)(1.5)=2.28
$$

if $N \geqslant 6$. The cases $1 \leqslant N \leqslant 5$ are easily checked, so $\sigma^{*}(N!) / N!<2.28$ for all $N$. (Actually, the best bound is 2, achieved for $N=3$.)

Also solved by P. Bruckman who remarked that $\sigma(N!) / N$ ! is unbounded.

Conjectures No More
H-375 Proposed by Piero Filipponi, Rome, Italy (Vol. 22, no. 3, August 1984)
Conjecture 1
If $F_{k} \equiv 0(\bmod k)$ and $k \neq 5^{n}$, then $k \equiv 0(\bmod 12)$.
Conjecture 2
Let $m>1$ be odd. Then, $F_{12 m} \equiv 0(\bmod 12 m)$ implies either 3 divides $m$ or 5 divides $m$.

## Conjecture 3

Let $p>5$ be a prime such that $p \nmid F_{24}$, then $F_{12 m} \not \equiv 0(\bmod 12 m)$.
Conjecture 4
If $L_{k} \equiv 0(\bmod k)$, then $k \equiv 0(\bmod 6)$ for $k>1$.
Solution by Lawrence Somer, Washington, D.C.
In answering the conjectures, we will make use of several definitions and known results. The rank of apparition of $k$ in $\left\{F_{n}\right\}$, denoted by $\alpha(k)$, is the least positive integer $m$ such that $k \mid F_{m}$. The prime $p$ is a primitive divisor of $F_{n}$ if $p \mid F_{n}$, but $p \nmid F_{m}$ for $1 \leqslant m<n$. The following theorem will be the main result we will use and is given by D. Jarden as Theorem A in his paper "Divisibility of Fibonacci and Lucas Numbers by Their Subscripts" [2, pp. 68-75].

Theorem 1: Let $p_{1}, p_{2}, \ldots, p_{n}$ be the distinct primes dividing $k$, where $k>1$. Then $k \mid F_{k}$ if and only if

$$
\left[a\left(p_{1}\right), a\left(p_{2}\right), \ldots, a\left(p_{n}\right)\right] \mid k
$$

where $[a, b, \ldots]$ denotes the least common multiple of $a, b, \ldots$.
We will also need the following propositions.
Proposition 1: Let $m \geqslant 3$. Then $F_{m} \mid F_{n}$ if and only if $m \mid n$.
Proposition 2: Let

$$
k=\prod_{i=1}^{m} p_{i}^{n_{i}}
$$

be the canonical factorization of $k$ into prime powers. Let $r_{i}$ be the highest power of $p_{i}$ dividing $F_{a\left(p_{i}\right)}$ for $1 \leqslant i \leqslant m$. Then

$$
\alpha(k)=\operatorname{LCM}_{1 \leqslant i \leqslant m}\left\{a\left(p_{i}\right) p_{i}^{\max \left(0, n_{i}-r_{i}\right)}\right\} .
$$

Proposition 3: If $p$ is a prime and $p \neq 2$ or 5, then the prime factors of $a(p)$ are less than $p$.

Proposition 4: If $n \neq 1,2,6$, or 12 , then $F_{n}$ has a primitive prime divisor.
Proposition 1 is well known. Propositions 2 and 3 are given by Jarden in [2, p. 68]. Proposition 4 is proved by Carmichael [1, p. 61].

Conjecture 1: FALSE. There is an infinite number of counterexamples. By Theorem F in Jarden's paper [2, p. 72], if $k \mid F_{k}$, then $12 \mid k$ or $5 \mid k$. Thus, in any counterexample to Conjecture 1,5 must divide $k$. Let $n \geqslant 2$. Let the divisors of $5^{n}$ that are unequal to 5 be denoted by $p_{1}, p_{2}, \ldots, p_{m}$. Since $F_{5}=5$, such prime divisors exist by Proposition 4. Let

$$
\begin{equation*}
k=5^{r_{n}} \sum_{i=1}^{m} p_{i}^{t_{i}} \tag{1}
\end{equation*}
$$

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where $r_{n} \geqslant n$ and at least one of the $t_{i} ' s \geqslant 1$. It follows from Theorem 1 and Propositions 1, 2, and 4 that $k$ is a counterexample to Conjecture 1. Clearly, there is an infinite number of such counterexamples. In particular, by a table of the factorizations of Fibonacci numbers given by Jarden [2, pp. 36-59], the only primitive prime divisor of $F_{25}$ is 3001 and the only primitive prime divisor of $F_{125}$ is 158414167964045700001 . Then, by (1),

$$
\begin{equation*}
k_{1}=5^{r_{1}} 3001^{s_{1}} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
k_{2}=5^{r_{2}} 3001^{s_{2}} 158414167964045700001^{t_{2}} \tag{3}
\end{equation*}
$$

are each counterexamples to Conjecture 1 , where

$$
r_{1} \geqslant 2, s_{1} \geqslant 1, r_{2} \geqslant 3, s_{2} \geqslant 0, \text { and } t_{2} \geqslant 1 .
$$

We now provide another infinite class of counterexamples to Conjecture 1. Suppose that $k$ is a counterexample to Conjecture 1 . Let $q_{1}, q_{2}, \ldots, q_{d}$ be distinct primes such that $q_{i} \nmid k$ and $q_{i}$ is a primitive divisor of $F_{k_{i}}$, where $1 \leqslant i \leqslant d$ and $k_{i} \mid k$. By Proposition 4 , such $q_{i}^{\prime}$ s exist. Then, by Theorem 1 ,

$$
\begin{equation*}
k^{\prime}=k \prod_{i=1}^{d} q_{i}^{n_{i}} \tag{4}
\end{equation*}
$$

is also a counterexample to Conjecture 1 , where at least one of the $n_{i}{ }^{\prime}$ s $\geqslant 1$. One can show that all counterexamples to Conjecture 1 are of the forms given in (1) or (4). Since $k \mid F_{k}$, it follows by (4) and Propositions 1 and 4 that $F_{k}$ is also a counterexample to Conjecture 1. Let $F(n)$ denote $F_{n}, F(F(n))=F^{(2)}(n)$ denote $F_{F_{n}}$ and so on. Then by (2), (3), and Proposition 4,

$$
\begin{equation*}
F^{(r)}\left(5^{r_{1}} 3001^{s_{1}}\right) \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
F^{(s)}\left(5^{r_{2}} 3001^{s_{2}} 158414167964045700001^{t_{2}}\right) \tag{6}
\end{equation*}
$$

are each explicit counterexamples to Conjecture l, where

$$
r_{1} \geqslant 2, r_{2} \geqslant 2, s \geqslant 1, s_{2} \geqslant 0, t_{2} \geqslant 1,
$$

and either it is the case that $r \geqslant 2$ and $s_{1} \geqslant 0$ or it is the case that $s_{1} \geqslant 1$ and $r \geqslant 1$.

Conjecture 2: TRUE. Suppose that Conjecture 2 were false. Then $m>1$ and all the prime factors of $m$ are greater than 5. Let $p$ be the smallest prime factor of $m$. By Theorem 1, $\alpha(p) \mid 12 m$. By Proposition 3, each prime factor of $\alpha(p)$ is less than $p$. It thus follows that $a(p)$ is relatively prime to $m$ and hence, $a(p) \mid$ 12. However, $F_{1}=F_{2}=1$ and the only prime divisors of $F_{3}, F_{4}, F_{6}$, or $F_{12}$ are 2 or 3 . We thus have a contradiction and the result follows.

Conjecture 3: This does not make sense as stated.
Conjecture 4: TRUE, by Theorem F in Jarden's paper [2, p. 72]. Theorem F further states that if $L_{k} \equiv 0(\bmod k)$, then $4 \nmid k$.

## References

1. R. D. Carmichae1. "On the Numerical Factors of the Arithmetic Forms $\alpha^{n} \pm$ $\beta^{n} . "$ Ann. Math. Second Series 15 (1913):30-70.
2. D. Jarden. Recurring Sequences. 3rd ed. Jerusalem: Riveon Lematematika, 1973

Also solved by P. Bruckman and L. Dresel.

## New Construction

H-376 Proposed by H. Klauser, Zürich, Switzerland (Vol. 22, no. 4, November 1984)

Let $(a, b, c, d)$ be a quadruple of integers with the property that

$$
\left(a^{3}+b^{3}+c^{3}+d^{3}\right)=0
$$

Clearly, at least one integer must be negative.
Examples: $(3,4,5,6),(9,10,-1,-12)$
Find a construction rule so that:

1. out of two given quadruples a new quadruple arises;
2. out of the given quadruple a new quadruple arises.

Solution by Paul Bruckman, Fair Oaks, CA
We let $S$ denote the set of all quadruples $(a, b, c, d) \in \mathbb{Z}^{4}$ such that

$$
\begin{equation*}
a^{3}+b^{3}+c^{3}+d^{3}=0 \tag{1}
\end{equation*}
$$

Lemma 1: Given $(a, b, c, d) \in S,\left(a^{\prime}, b^{\prime}, c^{\prime}, d^{\prime}\right) \in S$, let

$$
\begin{align*}
& p=a\left(a^{\prime}\right)^{2}+b\left(b^{\prime}\right)^{2}+c\left(c^{\prime}\right)^{2}+d\left(d^{\prime}\right)^{2} \\
& q=a^{2} a^{\prime}+b^{2} b^{\prime}+c^{2} c^{\prime}+d^{2} d^{\prime} \tag{2}
\end{align*}
$$

Also, let

$$
\begin{equation*}
a^{\prime \prime}=p a-q a^{\prime}, b^{\prime \prime}=p b-q b^{\prime}, c^{\prime \prime}=p c-q c^{\prime}, d^{\prime \prime}=p d-q d^{\prime} \tag{3}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left(a^{\prime \prime}, b^{\prime \prime}, c^{\prime \prime}, d^{\prime \prime}\right) \in S \tag{4}
\end{equation*}
$$

Proof: $\left(a^{\prime \prime}\right)^{3}+\left(b^{\prime \prime}\right)^{3}+\left(c^{\prime \prime}\right)^{3}+\left(d^{\prime \prime}\right)^{3}$

$$
=p^{3}\left(a^{3}+b^{3}+c^{3}+d^{3}\right)-q^{3}\left\{\left(a^{\prime}\right)^{3}+\left(b^{\prime}\right)^{3}+\left(c^{\prime}\right)^{3}+\left(d^{\prime}\right)^{3}\right\}
$$

$-3 p^{2} q\left(a^{2} a^{\prime}+b^{2} b^{\prime}+c^{2} c^{\prime}+d^{2} d^{\prime}\right)$
$+3 p q^{2}\left\{a\left(a^{\prime}\right)^{2}+b\left(b^{\prime}\right)^{2}+c\left(c^{\prime}\right)^{2}+d\left(d^{\prime}\right)^{2}\right\}$
$=p^{3} \cdot 0-q^{3} \cdot 0-3 p^{2} q \cdot q+3 p q^{2} \cdot p=0$.
This shows that ( $\left.a^{\prime \prime}, b^{\prime \prime}, c^{\prime \prime}, d^{\prime \prime}\right) \in S$ given by (2) and (3) may be constructed from the given quadruples $(a, b, c, d) \in S$ and $\left(a^{\prime}, b^{\prime}, c^{\prime}, d^{\prime}\right) \in S$, solving

Part 1 of the problem.
Example: If $(a, b, c, d)=(3,4,5,-6),\left(a^{\prime}, b^{\prime}, c^{\prime}, d^{\prime}\right)=(-1,9,10,-12)$, then $p=-37, q=-47,\left(a^{\prime \prime}, b^{\prime \prime}, c^{\prime \prime}, d^{\prime \prime}\right)=(-158,275,285,-342)$.

Lemma 2: Given $(a, b, c, d) \in S$, 1et

$$
\begin{equation*}
r=a b^{2}+b c^{2}+c d^{2}+d a^{2}, s=a^{2} b+b^{2} c+c^{2} d+d^{2} a \tag{5}
\end{equation*}
$$

Also, let

$$
\begin{equation*}
A=r b-s c, B=r c-s d, C=r d-s a, D=r a-s b \tag{6}
\end{equation*}
$$

Then

$$
\begin{equation*}
(A, B, C, D) \in S \tag{7}
\end{equation*}
$$

Proof: $A^{3}+B^{3}+C^{3}+D^{3}=\left(r^{3}-s^{3}\right)\left(a^{3}+b^{3}+c^{3}+d^{3}\right)$

$$
-3 r^{2} s\left(b^{2} c+c^{2} d+d^{2} a+a^{2} b\right)
$$

$$
+3 r s^{2}\left(b c^{2}+c d^{2}+d a^{2}+a b^{2}\right)
$$

$$
=\left(r^{3}-s^{3}\right) \cdot 0-3 r^{2} s \cdot s+3 r s^{2} \cdot r=0
$$

Thus, ( $A, B, C, D$ ) $\in S$ given by (5) and (6) may be constructed from the given quadruple $(a, b, c, d) \in S$, solving Part 2 of the problem.

Example: If $(a, b, c, d)=(3,4,5,-6)$, then $r=274, s=74,(A, B, C, D)=$ (726, 1814, -1866, 526).

Also solved by the proposer.

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## BOOKS AVAILABLE THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.
A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.
Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

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Recurring Sequences by Dov Jarden. Third and enlarged edition. Riveon Lematematika, Israel, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence-18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Please write to the Fibonacci Association, Santa Clara University, Santa Clara, CA 95033, U.S.A., for current prices.


[^0]:    *Professor Stevens passed away on December 3, 1983. Many of the results in this paper were presented by him to the departmental number theory seminar held on December 1, 1983. The paper, based on results obtained by Professor Stevens, has been written by several departmental colleagues.

