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CONTENTS

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The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

EXPANSION OF THE FIBONACCI NUMBERS F_{mn+r} IN THE m^{th} POWERS OF FIBONACCI OR LUCAS NUMBERS

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1. INTRODUCTION

It is known, see [1, p. 77], that

$$F_{2a}F_{2n} = F_{a+n}^2 - F_{a-n}^2$$

and, see [2, p. 43], that

$$F_a F_{2a} F_{3n} = F_{a+n}^3 + (-1)^{a+1} L_a F_n^3 + (-1)^{n+1} F_{a-n}^3$$

for arbitrary integers a and n. These identities suggest the possible existence of a general identity of the form

$$wF_{mn} = \sum_{t=1}^{k} b_t [F_{ta+n}^m + (-1)^{mn+1} F_{ta-n}^m] + b, \tag{1}$$

where m, n, and a are integers with $m > 0$, and where w and b_t, $1 \leqslant t \leqslant k$, are integral expressions free of the variable n, and b is an integral expression. Gladwin [3] has given existence proofs for some general identities of a similar type. An example of the kind of identity that we shall obtain is

$$\begin{aligned} F_a^2 F_{2a}^2 F_{3a} F_{4a} F_{5a} F_{6a} F_{6n} &= F_a F_{2a} F_{3a} F_{n+3a}^6 + (-1)^{a+1} F_{2a}^2 F_{6a} F_{n+2a}^6 \\ &+ (-1)^a F_a F_{5a} F_{6a} F_{n+a}^6 + (-1)^{a+1} F_a F_{5a} F_{6a} F_{n-a}^6 \\ &+ (-1)^a F_{2a}^2 F_{6a} F_{n-2a}^6 - F_a F_{2a} F_{3a} F_{n-3a}^6 \end{aligned}$$

for arbitrary integers a and n. In the sequel we shall use the following well-known results: for all integers a and n,

$$(L_n \pm \sqrt{5}F_n)^m = 2^{m-1}(L_{mn} \pm \sqrt{5}F_{mn}) \tag{2}$$

where m is a positive integer,

$$F_{-n} = (-1)^{n+1}F_n \quad \text{and} \quad L_{-n} = (-1)^n L_n, \tag{3}$$

$$L_n^2 = 5F_n^2 + (-1)^n 4, \tag{4}$$

$$2F_{a+n} = F_a L_n + L_a F_n, \tag{5}$$

$$2L_{a+n} = 5F_a F_n + L_a L_n. \tag{6}$$

2. PRELIMINARY LEMMAS

Lemma 1: $L_m^2/F_m^2 - L_n^2/F_n^2 = (-1)^{n+1}4F_{m+n}F_{m-n}/F_m^2F_n^2$ for $m \neq 0$ and $n \neq 0$.

The proof of Lemma 1 follows from equations (3) and (5).

In the sequel, let a be a nonzero integer.

Lemma 2: For $m > 0$,

$$\text{(i)}\quad 2^{m-1}F_{mn} = \sum_{i=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2i-1} 5^{i-1}F_n^{2i-1}L_n^{m+1-2i},$$

$$\text{(ii)}\quad 2^{m-1}L_{mn} = \sum_{i=1}^{\left[\frac{m+2}{2}\right]} \binom{m}{2i-2} 5^{i-1}F_n^{2i-2}L_n^{m+2-2i},$$

$$\text{(iii)}\quad 2^{m-1}[F_{a+n}^m + (-1)^{mn+1}F_{a-n}^m] = \sum_{i=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2i-1} F_n^{2i-1}L_n^{m+1-2i}F_a^{m+1-2i}L_a^{2i-1},$$

$$\text{(iv)}\quad 2^{m-1}[F_{a+n}^m + (-1)^{mn}F_{a-n}^m] = \sum_{i=1}^{\left[\frac{m+2}{2}\right]} \binom{m}{2i-2} F_n^{2i-2}L_n^{m+2-2i}F_a^{m+2-2i}L_a^{2i-2},$$

$$\text{(v)}\quad 2^{m-1}[L_{a+n}^m + (-1)^{mn}L_{a-n}^m] = \sum_{i=1}^{\left[\frac{m+2}{2}\right]} \binom{m}{2i-2} 5^{2i-2}F_n^{2i-2}L_n^{m+2-2i}F_a^{2i-2}L_a^{m+2-2i},$$

$$\text{(vi)}\quad 2^{m-1}[L_{a+n}^m + (-1)^{mn+1}L_{a-n}^m] = \sum_{i=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2i-1} 5^{2i-1}F_n^{2i-1}L_n^{m+1-2i}F_a^{2i-1}L_a^{m+1-2i}.$$

Proof: We shall prove formulas (i) and (iii). The remaining formulas have similar proofs. By equation (2),

$$2^{m-1}(L_{mn} + \sqrt{5}F_{mn}) - 2^{m-1}(L_{mn} - \sqrt{5}F_{mn}) = (L_n + \sqrt{5}F_n)^m - (L_n - \sqrt{5}F_n)^m.$$

That is,

$$\begin{aligned} 2^m\sqrt{5}F_{mn} &= \sum_{i=1}^{m} \binom{m}{i} L_n^{m-i}(\sqrt{5}F_n)^i[1 + (-1)^{i+1}] \\ &= 2\sum_{i=1,\, i \text{ odd}}^{2\left[\frac{m+1}{2}\right]-1} \binom{m}{i} L_n^{m-i}(\sqrt{5}F_n)^i \\ &= 2\sum_{i=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2i-1} L_n^{m-2i+1}(\sqrt{5}F_n)^{2i-1}. \end{aligned}$$

Formula (i) can now be obtained by dividing through by $2\sqrt{5}$.

Now, by equations (5) and (3),

$$\begin{aligned}2^m[F^m_{a+n} + (-1)^{mn+1}F^m_{a-n}] &= (F_aL_n + L_aF_n)^m + (-1)^{mn+1}(F_aL_{-n} + L_aF_{-n})^m\\ &= (F_aL_n + L_aF_n)^m - (F_aL_n - L_aF_n)^m\\ &= \sum_{i=1}^{m}\binom{m}{i}F_a^{m-i}L_n^{m-i}L_a^iF_n^i[1 + (-1)^{i+1}]\\ &= 2\sum_{i=1,\, i \text{ odd}}^{2\left[\frac{m+1}{2}\right]-1}\binom{m}{i}F_n^iL_n^{m-i}F_a^{m-i}L_a^i\\ &= 2\sum_{i=1}^{\left[\frac{m+1}{2}\right]}\binom{m}{2i-1}F_n^{2i-1}L_n^{m-2i+1}F_a^{m-2i+1}L_a^{2i-1}.\end{aligned}$$

Formula (iii) is obtained by dividing through by 2.

Let $V_k = (x_t^{i-1})$ for $1 \leqslant i,\ t \leqslant k$, denote the Vandermonde matrix. From [4, pp. 15, 16] it follows that for $k > 1$ and $t = 1, 2, \ldots, k$,

$$|V_k| = (V_k)_{kt}\prod_{\substack{i=1\\ i\neq t}}^{k}(x_t - x_i), \tag{7}$$

where $(V_k)_{kt}$ is the cofactor of x_t^{k-1} in $|V_k|$.

Lemma 3: For $k > 1$ and any constant $c \neq x_t$, $t = 1, 2, \ldots, k$,

$$\sum_{t=1}^{k}(V_k)_{kt}\Big/(c - x_t) = |V_k|\Big/\prod_{i=1}^{k}(c - x_i).$$

Proof: Let $C_k = [c_{it}]$, where $c_{it} = 1$ if $i = t$, $c_{it} = -c$ if $i = t + 1$, and $c_{it} = 0$ otherwise. Then,

$$\begin{aligned}|V_k| = |C_k|\cdot|V_k| = |C_kV_k| = \sum_{t=1}^{k}(C_kV_k)_{1t} &= \sum_{t=1}^{k}\left[\prod_{\substack{i=1\\ i\neq t}}^{k}(c - x_i)\right](V_k)_{kt}\\ &= \prod_{i=1}^{k}(c - x_i)\sum_{t=1}^{k}(V_k)_{kt}\Big/(c - x_t).\end{aligned}$$

In the sequel, let $x_t \equiv L^2_{ta}/F^2_{ta}$ for $t = 1, 2, \ldots, k$.

Lemma 4: For $k > 1$ and $t = 1, 2, \ldots, k$,

$$(-1)^{ta+1}2^{2k-2}(V_k)_{kt}\prod_{i=k+1}^{k+t}F_{ia} = L_{ta}F_{ta}^{2k-2}|V_k|\prod_{i=k-t+1}^{k}F_{-ia}.$$

Proof: By equation (7) and Lemma 1, for $k > 1$ and $t = 1, 2, \ldots, k$,

$$\begin{aligned}|V_k|/(V_k)_{kt} = \prod_{\substack{i=1\\ i\neq t}}^{k}(x_t - x_i) &= \prod_{\substack{i=1\\ i\neq t}}^{k}[(-1)^{ia+1}4F_{ta+ia}F_{ta-ia}/F^2_{ta}F^2_{ia}]\\ &= (-1)^{ta+1}(F^4_{ta}/4F_{2ta})\left[\prod_{i=1}^{k}[(-1)^{ia+1}4F_{ta+ia}/F^2_{ta}F^2_{ia}]\right]\prod_{\substack{i=1\\ i\neq t}}^{k}F_{ta-ia}\end{aligned}$$

$$= (-1)^{ta+1}(4^{k-1}F_{ta}^3/L_{ta})\left[\prod_{i=t+1}^{t+k} F_{ia}\right]\left[\prod_{i=1}^{t-1} F_{ta-ia}\right]\left[\prod_{i=t+1}^{k} F_{ta-ia}\right]$$

$$\div F_{ta}^{2k}\left[\prod_{i=1}^{k} F_{-ia}\right]\left[\prod_{i=1}^{k} F_{ia}\right]$$

$$= (-1)^{ta+1}(4^{k-1}/L_{ta})\left[\prod_{i=t}^{t+k} F_{ia}\right]\left[\prod_{i=1}^{t-1} F_{ia}\right]\left[\prod_{i=1}^{k-t} F_{-ia}\right]$$

$$\div F_{ta}^{2k-2}\left[\prod_{i=1}^{k} F_{-ia}\right]\left[\prod_{i=1}^{k} F_{ia}\right]$$

$$= (-1)^{ta+1}2^{2k-2}\left[\prod_{i=k+1}^{k+t} F_{ia}\right] \Big/ L_{ta}F_{ta}^{2k-2}\prod_{i=k-t+1}^{k} F_{-ia}$$

since, by equation (5), $F_{2ta} = F_{ta}L_{ta}$.

Lemma 5: For $k > 1$,

$$\sum_{t=1}^{k}(-1)^{ta+1}F_{ta}^2(V_k)_{kt} = (1/2^{2k-2})|V_k|\left[\prod_{i=1}^{k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}.$$

Proof: For $t = 1, 2, \ldots, k$,

$$x_t \equiv L_{ta}^2/F_{ta}^2 = (5F_{ta}^2 + (-1)^{ta}4)/F_{ta}^2 = 5 + (-1)^{ta}4/F_{ta}^2.$$

It follows that $(-1)^{ta+1}F_{ta}^2 = 4/(5 - x_t)$. Therefore, by Lemma 3,

$$\sum_{t=1}^{k}(-1)^{ta+1}F_{ta}^2(V_k)_{kt} = \sum_{t=1}^{k}(4/(5 - x_t))(V_k)_{kt} = 4|V_k| \Big/ \prod_{i=1}^{k}(5 - x_i)$$

$$= 4|V_k| \Big/ \prod_{i=1}^{k}((-1)^{ia+1}4/F_{ia}^2)$$

$$= (1/4^{k-1})|V_k|\left[\prod_{i=1}^{k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}.$$

Lemma 6: Let u be a positive integer and let z_t be a real number for $t = 1, 2, \ldots, k$.

$$\sum_{t=1}^{k} z_t x_t^i = 5^i\sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant u,$$

if and only if

$$\sum_{t=1}^{k} z_t x_t^i = 5\sum_{t=1}^{k} z_t x_t^{i-1} \quad \text{for each } i,\ 1 \leqslant i \leqslant u.$$

Proof: Let

$$\sum_{t=1}^{k} z_t x_t^i = 5^i\sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant u.$$

Then, for $i = 1$,

$$\sum_{t=1}^{k} z_t x_t = 5\sum_{t=1}^{k} z_t$$

and, for $2 \leqslant i \leqslant u$,

$$\sum_{t=1}^{k} z_t x_t^i = 5 \cdot 5^{i-1} \sum_{t=1}^{k} z_t = 5 \sum_{t=1}^{k} z_t x_t^{i-1}.$$

Conversely, we use mathematical induction on u. The case $u = 1$ is true. For $q \geqslant 1$, assume that

$$\sum_{t=1}^{k} z_t x_t^i = 5 \sum_{t=1}^{k} z_t x_t^{i-1} \quad \text{for each } i,\ 1 \leqslant i \leqslant q,$$

implies

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant q.$$

Now let

$$\sum_{t=1}^{k} z_t x_t^i = 5 \sum_{t=1}^{k} z_t x_t^{i-1} \quad \text{for each } i,\ 1 \leqslant i \leqslant q + 1.$$

Then

$$\sum_{t=1}^{k} z_t x_t^i = 5 \sum_{t=1}^{k} z_t x_t^{i-1} \quad \text{for each } i,\ 1 \leqslant i \leqslant q$$

and

$$\sum_{t=1}^{k} z_t x_t^{q+1} = 5 \sum_{t=1}^{k} z_t x_t^q.$$

Therefore, by the induction hypothesis,

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant q$$

and

$$\sum_{t=1}^{k} z_t x_t^{q+1} = 5 \sum_{t=1}^{k} z_t x_t^q.$$

Hence

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant q,$$

and

$$\sum_{t=1}^{k} z_t x_t^{q+1} = 5 \cdot 5^q \sum_{t=1}^{k} z_t = 5^{q+1} \sum_{t=1}^{k} z_t.$$

Thus

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant q + 1.$$

The proof is complete by mathematical induction.

Lemma 7: Let z_t be a real number for $t = 1, 2, \ldots, k$, and let j be a fixed integer.

$$\sum_{t=1}^{k} (-1)^{ta} z_t x_t^{j-1}/F_{ta}^2 = 0 \text{ if and only if } \sum_{t=1}^{k} z_t x_t^j = 5 \sum_{t=1}^{k} z_t x_t^{j-1}.$$

Proof:

$$\begin{aligned}\sum_{t=1}^{k} z_t x_t^j &= \sum_{t=1}^{k} z_t (L_{ta}^2/F_{ta}^2) x_t^{j-1} \\ &= \sum_{t=1}^{k} z_t ((5F_{ta}^2 + (-1)^{ta} 4)/F_{ta}^2) x_t^{j-1} \\ &= 5 \sum_{t=1}^{k} z_t x_t^{j-1} + 4 \sum_{t=1}^{k} (-1)^{ta} z_t x_t^{j-1}/F_{ta}^2.\end{aligned}$$

Corollary 1: Let z_t be a real number for $t = 1, 2, \ldots, k$.

$$\sum_{t=1}^{k} (-1)^{ta} z_t x_t^{i-1} / F_{ta}^2 = 0 \quad \text{for each } i,\ 1 \leqslant i \leqslant u$$

if and only if

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t \quad \text{for each } i,\ 1 \leqslant i \leqslant u.$$

Proof: Apply Lemma 6 and Lemma 7.

Corollary 2: Let $k > 1$. Then

$$\sum_{t=1}^{k} (-1)^{ta} F_{ta}^2 x_t^{i-1} (V_k)_{kt} = 5^{i-1} \sum_{t=1}^{k} (-1)^{ta} F_{ta}^2 (V_k)_{kt}$$

for each i, $1 \leqslant i \leqslant k$.

Proof: In Corollary 1, let $z_t = (-1)^{ta} F_{ta}^2 (V_k)_{kt}$ for $t = 1, 2, \ldots, k$, and let $u = k - 1$. Then

$$\sum_{t=1}^{k} (-1)^{ta} z_t x_t^{i-1} / F_{ta}^2 = \sum_{t=1}^{k} x_t^{i-1} (V_k)_{kt} = 0$$

for each i, $1 \leqslant i \leqslant k - 1$, since a determinant with two identical rows has numerical value zero. By Corollary 1,

$$\sum_{t=1}^{k} z_t x_t^i = 5^i \sum_{t=1}^{k} z_t$$

is true for each i, $1 \leqslant i \leqslant k - 1$, and clearly is true for $i = 0$. Therefore,

$$\sum_{t=1}^{k} (-1)^{ta} F_{ta}^2 x_t^{i-1} (V_k)_{kt} = 5^{i-1} \sum_{t=1}^{k} (-1)^{ta} F_{ta}^2 (V_k)_{kt}$$

for each i, $1 \leqslant i \leqslant k$.

3. THEOREMS

Theorem 1: For any positive integer k,

$$\text{(i)} \quad 5^{j-1} \left[\prod_{i=1}^{2k} F_{ia} \right] \prod_{i=1}^{k} F_{-ia} = \sum_{t=1}^{k} \left[\left[\prod_{i=k+t+1}^{2k} F_{ia} \right] \prod_{i=k-t+1}^{k} F_{-ia} \right] F_{ta}^{2k+2-2j} L_{ta}^{2j-1}$$

for $1 \leqslant j \leqslant k$, and

$$\text{(ii)} \quad 5^{k} \left[\prod_{i=1}^{2k} F_{ia} \right] \prod_{i=1}^{k} F_{-ia} = \sum_{t=1}^{k} \left[\left[\prod_{i=k+t+1}^{2k} F_{ia} \right] \prod_{i=k-t+1}^{k} F_{-ia} \right] L_{ta}^{2k+1} + 2^{2k} \prod_{i=k+1}^{2k} F_{ia},$$

and (iii)

$$\left[\prod_{i=1}^{2k} F_{ia} \right] \prod_{i=1}^{k} F_{-ia} = 5 \sum_{t=1}^{k} \left[\left[\prod_{i=k+t+1}^{2k} F_{ia} \right] \prod_{i=k-t+1}^{k} F_{-ia} \right] F_{ta}^{2k+2} L_{ta}^{-1}$$
$$+ (-1)^k 2^{2k} \left[\prod_{i=1}^{k} F_{ia} \right] \left[\prod_{i=1}^{2k} F_{ia} \right] \Big/ \prod_{i=1}^{k} L_{ia}^2.$$

Proof: The three identities are easily verified for $k = 1$. Assume that $k > 1$. Denote

$$A = \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia} \qquad \text{and} \qquad A_t = \left[\prod_{i=k+t+1}^{2k} F_{ia}\right] \prod_{i=k-t+1}^{k} F_{-ia}$$

for $t = 1, 2, \ldots, k$, and

$$K = -|V_k| \Big/ 2^{2k-2} \prod_{i=k+1}^{2k} F_{ia}.$$

By lemma 5,

$$KA = \sum_{t=1}^{k} (-1)^{ta} F_{ta}^2 (V_k)_{kt} \tag{8}$$

and, by Lemma 4,

$$KA_t = (-1)^{ta} (V_k)_{kt} / L_{ta} F_{ta}^{2k-2} \tag{9}$$

for $t = 1, 2, \ldots, k$. Now, by Corollary 2 and equations (8) and (9) we have, for each j, $1 \leqslant j \leqslant k$,

$$\begin{aligned} 5^{j-1} KA &= 5^{j-1} \sum_{t=1}^{k} (-1)^{ta} F_{ta}^2 (V_k)_{kt} = \sum_{t=1}^{k} (-1)^{ta} F_{ta}^2 x_t^{j-1} (V_k)_{kt} \\ &= \sum_{t=1}^{k} KA_t x_t^{j-1} L_{ta} F_{ta}^{2k} = \sum_{t=1}^{k} KA_t (L_{ta}^{2j-2} / F_{ta}^{2j-2}) L_{ta} F_{ta}^{2k}. \end{aligned}$$

Therefore, for each j, $1 \leqslant j \leqslant k$,

$$5^{j-1} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia} = \sum_{t=1}^{k} \left[\left[\prod_{i=k+t+1}^{2k} F_{ia}\right] \prod_{i=k-t+1}^{k} F_{-ia}\right] F_{ta}^{2k-2j+2} L_{ta}^{2j-1}. \tag{10}$$

The proof of (i) is complete.

From equation (10), we obtain, for $j = k$,

$$\begin{aligned} 5^{k-1} A &= \sum_{t=1}^{k} A_t F_{ta}^2 L_{ta}^{2k-1} = (1/5) \sum_{t=1}^{k} A_t (L_{ta}^2 + (-1)^{ta+1} 4) L_{ta}^{2k-1} \\ &= (1/5) \sum_{t=1}^{k} A_t L_{ta}^{2k+1} - (4/5) \sum_{t=1}^{k} (-1)^{ta} A_t L_{ta}^{2k-1} \\ &= (1/5) \sum_{t=1}^{k} A_t L_{ta}^{2k+1} - (4/5K) \sum_{t=1}^{k} (L_{ta}^{2k-2} / F_{ta}^{2k-2}) (V_k)_{kt} \end{aligned}$$

by equation (9). Thus,

$$\begin{aligned} 5^{k-1} A &= (1/5) \sum_{t=1}^{k} A_t L_{ta}^{2k+1} - (4/5K) \sum_{t=1}^{k} x_t^{k-1} (V_k)_{kt} \\ &= (1/5) \sum_{t=1}^{k} A_t L_{ta}^{2k+1} - (4/5K) |V_k| \end{aligned}$$

$$= (1/5)\sum_{t=1}^{k} A_t L_{ta}^{2k+1} + (2^{2k}/5)\prod_{i=k+1}^{2k} F_{ia}.$$

The proof of (ii) is complete.

From equation (10) we obtain, for $j = 1$,

$$A = \sum_{t=1}^{k} A_t F_{ta}^{2k} L_{ta} = \sum_{t=1}^{k} A_t F_{ta}^{2k}(5F_{ta}^2 + (-1)^{ta}4)/L_{ta}$$

$$= 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + 4\sum_{t=1}^{k} A_t (-1)^{ta} F_{ta}^{2k} L_{ta}^{-1}$$

$$= 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + (4/K)\sum_{t=1}^{k} (F_{ta}^2/L_{ta}^2)(V_k)_{kt}$$

$$= 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + (4/K)\sum_{t=1}^{k} (1/x_t)(V_k)_{kt}.$$

Therefore, by Lemma 3,

$$A = 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + (4/K)(-1)|V_k| \Big/ \prod_{i=1}^{k}(-x_i)$$

$$= 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + (-1)^k 2^{2k}\left[\prod_{i=k+1}^{2k} F_{ia}\right]\left[\prod_{i=1}^{k} F_{ia}^2\right] \Big/ \prod_{i=1}^{k} L_{ia}^2$$

$$= 5\sum_{t=1}^{k} A_t F_{ta}^{2k+2} L_{ta}^{-1} + (-1)^k 2^{2k}\left[\prod_{i=1}^{2k} F_{ia}\right]\left[\prod_{i=1}^{k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2.$$

The proof of (iii) is complete.

Lemma 8: Let a and n be nonzero integers, let k and m be positive integers, and let $\varepsilon = 0$ or $\varepsilon = 1$.

(i) For $m \leqslant 2k + 2 + 2\varepsilon$, $2^{m-1}F_{mn}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k} F_{ta}^{2k+1-m+2\varepsilon} L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]\left[F_{ta+n}^m + (-1)^{mn+1}F_{ta-n}^m\right]$$

$$+ (-1)^k\binom{m}{2\varepsilon - 1}2^{2k}F_n L_n^{m-1}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \binom{m}{2k + 1 + 2\varepsilon}5^{\varepsilon}2^{2k}F_n^{2k+1+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k} F_{ia},$$

(ii) For $m \leqslant 2k + 2 + 2\varepsilon$, $5^{k+\varepsilon}2^{m-1}F_{mn}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$

$$= 2^{m-1}\sum_{t=1}^{k} F_{ta}^{1-2\varepsilon} L_{ta}^{2k-m+2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]\left[L_{ta+n}^m + (-1)^{mn+1}L_{ta-n}^m\right]$$

$$+ \binom{m}{2\varepsilon - 1}5 \cdot 2^{2k}F_n L_n^{m-1}\prod_{i=k+1}^{2k} F_{ia} +$$

(continued)

$$+ (-1)^k \binom{m}{2k+1+2\varepsilon} 5^{2k+2\varepsilon} 2^{2k} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon} \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2,$$

(iii) For $m \leqslant 2k + 1 + 2\varepsilon$, $5^{k-1+\varepsilon} 2^{m-1} L_{mn} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$

$$= 2^{m-1} \sum_{t=1}^{k} F_{ta}^{2-2\varepsilon} L_{ta}^{2k-m-1+2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]\left[L_{ta+n}^m + (-1)^{mn} L_{ta-n}^m\right]$$

$$+ \binom{m}{2\varepsilon-2} 2^{2k} L_n^m \prod_{i=k+1}^{2k} F_{ia}$$

$$+ (-1)^k \binom{m}{2k+2\varepsilon} 5^{2k-1+2\varepsilon} 2^{2k} F_n^{2k+2\varepsilon} L_n^{m-2k-2\varepsilon} \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2,$$

(iv) For $m \leqslant 2k + 1 + 2\varepsilon$, $2^{m-1} L_{mn} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$

$$= 5^{\varepsilon} \cdot 2^{m-1} \sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon} L_{ta}^{1-2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]\left[F_{ta+n}^m + (-1)^{mn} F_{ta-n}^m\right]$$

$$+ (-1)^k \binom{m}{2\varepsilon-2} 2^{2k} L_n^m \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \binom{m}{2k+2\varepsilon} 5^{\varepsilon} 2^{2k} F_n^{2k+2\varepsilon} L_n^{m-2k-2\varepsilon} \prod_{i=k+1}^{2k} F_{ia}.$$

Proof: (i) Let $F_{ta}^{2k+1-m+2\varepsilon} L_{ta}^{-2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right] \prod_{i=k-t+1}^{k} F_{-ia} = b_t$ for $1 \leqslant t \leqslant k$. Then, by Theorem 1 (i),

$$5^{j-1} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia} = \sum_{t=1}^{k} b_t F_{ta}^{m+1-2j-2\varepsilon} L_{ta}^{2j-1+2\varepsilon} \quad \text{for } 1 \leqslant j \leqslant k.$$

Thus,

$$5^{j-1-\varepsilon} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia} = \sum_{t=1}^{k} b_t F_{ta}^{m+1-2j} L_{ta}^{2j-1} \quad \text{for } 1 + \varepsilon \leqslant j \leqslant k + \varepsilon.$$

So

$$\binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} 5^{j-1} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon} \sum_{t=1}^{k} b_t \binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} F_{ta}^{m+1-2j} L_{ta}^{2j-1} \quad \text{for } 1 + \varepsilon \leqslant j \leqslant k + \varepsilon.$$

Since, by hypothesis, $m \leqslant 2k + 2 + 2\varepsilon$, we have $[(m-1)/2] \leqslant k + \varepsilon$. Therefore,

$$\sum_{j=1+\varepsilon}^{\left[\frac{m-1}{2}\right]} \binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} 5^{j-1} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon} \sum_{t=1}^{k} b_t \sum_{j=1+\varepsilon}^{\left[\frac{m-1}{2}\right]} \binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} F_{ta}^{m+1-2j} L_{ta}^{2j-1}.$$

By Theorem 1 (ii),

$$5^k\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia} = \sum_{t=1}^{k} b_t F_{ta}^{m-2k-1-2\varepsilon} L_{ta}^{2k+1+2\varepsilon} + 2^{2k}\prod_{i=k+1}^{2k} F_{ia},$$

and by Theorem 1 (iii),

$$5^{-1}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$
$$= \sum_{t=1}^{k} b_t F_{ta}^{m+1-2\varepsilon} L_{ta}^{2\varepsilon-1} + (-1)^k 2^{2k}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ 5\prod_{i=1}^{k} L_{ia}^2.$$

Therefore,

$$\binom{m}{2k+1+2\varepsilon} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon} 5^{k+\varepsilon}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$
$$= 5^{\varepsilon}\sum_{t=1}^{k} b_t \binom{m}{2k+1+2\varepsilon} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon} F_{ta}^{m-2k-1-2\varepsilon} L_{ta}^{2k+1+2\varepsilon}$$
$$+ \binom{m}{2k+1+2\varepsilon} 5^{\varepsilon} 2^{2k} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon} \prod_{i=k+1}^{2k} F_{ia}$$

and

$$\binom{m}{2\varepsilon-1} F_n^{2\varepsilon-1} L_n^{m+1-2\varepsilon} 5^{\varepsilon-1}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$
$$= 5^{\varepsilon}\sum_{t=1}^{k} b_t \binom{m}{2\varepsilon-1} F_n^{2\varepsilon-1} L_n^{m+1-2\varepsilon} F_{ta}^{m+1-2\varepsilon} L_{ta}^{2\varepsilon-1}$$
$$+ (-1)^k\binom{m}{2\varepsilon-1} 5^{\varepsilon} 2^{2k} F_n^{2\varepsilon-1} L_n^{m+1-2\varepsilon}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ 5\prod_{i=1}^{k} L_{ia}^2.$$

Since, by hypothesis, $m \leqslant 2k + 2 + 2\varepsilon$, we have $[(m+1)/2] \leqslant k + 1 + \varepsilon$, and we have $\binom{m}{2k+1+2\varepsilon} = 0$ if and only if $[(m+1)/2] < k + 1 + \varepsilon$. Therefore,

$$\binom{m}{2k+1+2\varepsilon} F_n^{2[(m+1)/2]-1} L_n^{m+1-2[(m+1)/2]} 5^{[(m-1)/2]}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$
$$= 5^{\varepsilon}\sum_{t=1}^{k} b_t \binom{m}{2k+1+2\varepsilon} (F_n L_{ta})^{2[(m+1)/2]-1} (L_n F_{ta})^{m+1-2[(m+1)/2]}$$
$$+ \binom{m}{2k+1+2\varepsilon} 5^{\varepsilon} 2^{2k} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon} \prod_{i=k+1}^{2k} F_{ia}.$$

Since $\binom{m}{-1} = 0$, we have

$$\sum_{j=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} 5^{j-1}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$
$$= 5^{\varepsilon}\sum_{t=1}^{k} b_t \sum_{j=1}^{\left[\frac{m+1}{2}\right]} \binom{m}{2j-1} F_n^{2j-1} L_n^{m+1-2j} F_{ta}^{m+1-2j} L_{ta}^{2j-1} +$$

(continued)

$$+ (-1)^k \binom{m}{2\varepsilon - 1} 2^{2k} F_n L_n^{m-1} \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \binom{m}{2k + 1 + 2\varepsilon} 5^{\varepsilon} 2^{2k} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon} \prod_{i=k+1}^{2k} F_{ia}.$$

By Lemma 2 (i) and Lemma 2 (iii),

$$2^{m-1} F_{mn} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon} 2^{m-1} \sum_{t=1}^{k} b_t [F_{ta+n}^m + (-1)^{mn+1} F_{ta-n}^m]$$

$$+ (-1)^k \binom{m}{2\varepsilon - 1} 2^{2k} F_n L_n^{m-1} \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \binom{m}{2k + 1 + 2\varepsilon} 5^{\varepsilon} 2^{2k} F_n^{2k+1+2\varepsilon} L_n^{m-2k-1-2\varepsilon} \prod_{i=k+1}^{2k} F_{ia}.$$

After substitution for b_t, $1 \leqslant t \leqslant k$, the proof of (i) is complete. The proofs of (ii), (iii), and (iv) are similar.

From equations (5) and (6), we obtain the following four identities:

$$L_n + F_n = 2F_{n+1},\quad L_n - F_n = 2F_{n-1},\quad 5F_n + L_n = 2L_{n+1},\quad 5F_n - L_n = 2L_{n-1}$$

for all integers n.

Corollary 3: Let a and n be nonzero integers and let k and m be positive integers and let $\varepsilon = 0$ or $\varepsilon = 1$. For $m \leqslant 2k + 1 + 2\varepsilon$,

(i) $$2^{m-1} F_{mn+1} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon} 2^{m-1} \sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon} L_{ta}^{-2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]\left[F_{ta+1} F_{ta+n}^m + (-1)^{mn} F_{ta-1} F_{ta-n}^m\right]$$

$$+ (-1)^k \left[\binom{m}{2\varepsilon - 2} L_n + \binom{m}{2\varepsilon - 1} F_n\right] 2^{2k-1} L_n^{m-1} \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \left[\binom{m}{2k + 2\varepsilon} L_n + \binom{m}{2k + 1 + 2\varepsilon} F_n\right] 5^{\varepsilon} 2^{2k-1} F_n^{2k+2\varepsilon} L_n^{m-2k-1-2\varepsilon} \prod_{i=k+1}^{2k} F_{ia},$$

and

(ii) $$2^{m-1} L_{mn+1} \left[\prod_{i=1}^{2k} F_{ia}\right] \prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon} 2^{m-1} \sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon} L_{ta}^{-2\varepsilon} \left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right]\left[L_{ta+1} F_{ta+n}^m + (-1)^{mn+1} L_{ta-1} F_{ta-n}^m\right]$$

$$+ (-1)^k \left[\binom{m}{2\varepsilon - 2} L_n + \binom{m}{2\varepsilon - 1} 5F_n\right] 2^{2k-1} L_n^{m-1} \left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Big/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \left[\binom{m}{2k + 2\varepsilon} L_n + \binom{m}{2k + 1 + 2\varepsilon} 5F_n\right] 5^{\varepsilon} 2^{2k-1} F_n^{2k+2\varepsilon} L_n^{m-2k-1-2\varepsilon} \prod_{i=k+1}^{2k} F_{ia},$$

and

(iii) $$5^{k+\varepsilon}2^{m-1}F_{mn+1}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$$

$$= 2^{m-1}\sum_{t=1}^{k}F_{ta}^{1-2\varepsilon}L_{ta}^{2k-m-1+2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]\left[L_{ta+1}L_{ta+n}^{m}+(-1)^{mn}L_{ta-1}L_{ta-n}^{m}\right]$$

$$+\left[\binom{m}{2\varepsilon-2}L_n+\binom{m}{2\varepsilon-1}F_n\right]5\cdot 2^{2k-1}L_n^{m-1}\prod_{i=k+1}^{2k}F_{ia}+(-1)^k\left[\binom{m}{2k+2\varepsilon}L_n\right.$$

$$\left.+\binom{m}{2k+1+2\varepsilon}F_n\right]5^{2k+2\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2,$$

and

(iv) $$5^{k-1+\varepsilon}2^{m-1}L_{mn+1}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$$

$$= 2^{m-1}\sum_{t=1}^{k}F_{ta}^{1-2\varepsilon}L_{ta}^{2k-m-1+2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]\left[F_{ta+1}L_{ta+n}^{m}+(-1)^{mn+1}F_{ta-1}L_{ta-n}^{m}\right]$$

$$+\left[\binom{m}{2\varepsilon-2}L_n+\binom{m}{2\varepsilon-1}5F_n\right]2^{2k-1}L_n^{m-1}\prod_{i=k+1}^{2k}F_{ia}+(-1)^k\left[\binom{m}{2k+2\varepsilon}L_n\right.$$

$$\left.+\binom{m}{2k+1+2\varepsilon}5F_n\right]5^{2k-1+2\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2.$$

Proof: (1) and (ii) follow from Lemma 8, parts (i) and (iv). (iii) and (iv) follow from Lemma 8, parts (ii) and (iii).

Theorem 2: Let a and n be nonzero integers, let k and m be positive integers, let $\varepsilon = 0$ or $\varepsilon = 1$, and let r be an integer. For $m \leqslant 2k + 1 + 2\varepsilon$,

(i) $$2^{m-1}F_{mn+r}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k}F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]$$

$$\times\left[F_{ta+r}F_{ta+n}^{m}+(-1)^{mn+1+r}F_{ta-r}F_{ta-n}^{m}\right]$$

$$+(-1)^k\left[\binom{m}{2\varepsilon-2}F_rL_n+\binom{m}{2\varepsilon-1}L_rF_n\right]2^{2k-1}L_n^{m-1}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2$$

$$+\left[\binom{m}{2k+2\varepsilon}F_rL_n+\binom{m}{2k+1+2\varepsilon}L_rF_n\right]5^{\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k}F_{ia},$$

and

(ii) $$2^{m-1}L_{mn+r}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k}F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]\left[L_{ta+r}F_{ta+n}^{m}+(-1)^{mn+r}L_{ta-r}F_{ta-n}^{m}\right]$$

$$+ (-1)^k\left[\binom{m}{2\varepsilon-2}L_rL_n + \binom{m}{2\varepsilon-1}5F_rF_n\right]2^{2k-1}L_n^{m-1}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2$$

$$+ \left[\binom{m}{2k+2\varepsilon}L_rL_n + \binom{m}{2k+1+2\varepsilon}5F_rF_n\right]5^{\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k}F_{ia},$$

and

(iii) $$5^{k+\varepsilon}2^{m-1}F_{mn+r}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$$

$$= 2^{m-1}\sum_{t=1}^{k}F_{ta}^{1-2\varepsilon}L_{ta}^{2k-m-1+2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right] \times$$

$$\times \left[L_{ta+r}L_{ta+n}^{m} + (-1)^{mn+1+r}L_{ta-r}L_{ta-n}^{m}\right]$$

$$+ \left[\binom{m}{2\varepsilon-2}F_rL_n + \binom{m}{2\varepsilon-1}L_rF_n\right]5\cdot 2^{2k-1}L_n^{m-1}\prod_{i=k+1}^{2k}F_{ia} + (-1)^k\left[\binom{m}{2k+2\varepsilon}F_rL_n\right.$$

$$\left.+ \binom{m}{2k+1+2\varepsilon}L_rF_n\right]5^{2k+2\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2,$$

and

(iv) $$5^{k-1+\varepsilon}2^{m-1}L_{mn+r}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$$

$$= 2^{m-1}\sum_{t=1}^{k}F_{ta}^{1-2\varepsilon}L_{ta}^{2k-m-1+2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]\left[F_{ta+r}L_{ta+n}^{m} + (-1)^{mn+r}F_{ta-r}L_{ta-n}^{m}\right]$$

$$+ \left[\binom{m}{2\varepsilon-2}L_rL_n + \binom{m}{2\varepsilon-1}5F_rF_n\right]2^{2k-1}L_n^{m-1}\prod_{i=k+1}^{2k}F_{ia} + (-1)^k\left[\binom{m}{2k+2\varepsilon}L_rL_n\right.$$

$$\left.+ \binom{m}{2k+1+2\varepsilon}5F_rF_n\right]5^{2k-1+2\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\left[\prod_{i=1}^{k}F_{ia}\right]\left[\prod_{i=1}^{2k}F_{ia}\right]\Big/\prod_{i=1}^{k}L_{ia}^2.$$

Proof: To prove (i), we use mathematical induction on r. The cases $r = 0$ and $r = 1$ are true by Lemma 8 (i) and Corollary 3 (i). Assume that the hypothesis is true for $r = q$ and for $r = q + 1$, where q is an integer. Then

$$2^{m-1}F_{mn+q+2}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$$

$$= 2^{m-1}F_{mn+q+1}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia} + 2^{m-1}F_{mn+q}\left[\prod_{i=1}^{2k}F_{ia}\right]\prod_{i=1}^{k}F_{-ia}$$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k}F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right](F_{ta+q+1} + F_{ta+q})F_{ta+n}^{m}$$

$$+ 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k}F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k}F_{ia}\right]\left[\prod_{i=k-t+1}^{k}F_{-ia}\right]$$

$$\times (-1)^{mn+q+1}(-F_{ta-q-1} + F_{ta-q})F_{ta-n}^{m} + (-1)^k\left[\binom{m}{2\varepsilon-2}(F_{q+1} + F_q)L_n + \right.$$

$$+ \binom{m}{2\varepsilon - 1}(L_{q+1} + L_q)F_n\Bigg] 2^{2k-1}L_n^{m-1}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Bigg/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \left[\binom{m}{2k + 2\varepsilon}(F_{q+1} + F_q)L_n\right.$$

$$\left. + \binom{m}{2k + 1 + 2\varepsilon}(L_{q+1} + L_q)F_n\right] 5^{\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k} F_{ia}$$

by the induction hypothesis. Therefore,

$$2^{m-1}F_{mn+q+2}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right] F_{ta+q+2}F_{ta+n}^{m}$$

$$+ 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right] (-1)^{mn+1+q}F_{ta-q-2}F_{ta-n}^{m}$$

$$+ (-1)^k\left[\binom{m}{2\varepsilon - 2}F_{q+2}L_n + \binom{m}{2\varepsilon - 1}L_{q+2}F_n\right] 2^{2k-1}L_n^{m-1}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Bigg/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \left[\binom{m}{2k + 2\varepsilon}F_{q+2}L_n + \binom{m}{2k + 1 + 2\varepsilon}L_{q+2}F_n\right] 5^{\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k} F_{ia}.$$

Similarly,

$$2^{m-1}F_{mn+q-1}\left[\prod_{i=1}^{2k} F_{ia}\right]\prod_{i=1}^{k} F_{-ia}$$

$$= 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right] F_{ta+q-1}F_{ta+n}^{m}$$

$$+ 5^{\varepsilon}2^{m-1}\sum_{t=1}^{k} F_{ta}^{2k-m+2\varepsilon}L_{ta}^{-2\varepsilon}\left[\prod_{i=k+t+1}^{2k} F_{ia}\right]\left[\prod_{i=k-t+1}^{k} F_{-ia}\right] (-1)^{mn+q}F_{ta-q+1}F_{ta-n}^{m}$$

$$+ (-1)^k\left[\binom{m}{2\varepsilon - 2}F_{q-1}L_n + \binom{m}{2\varepsilon - 1}L_{q-1}F_n\right] 2^{2k-1}L_n^{m-1}\left[\prod_{i=1}^{k} F_{ia}\right]\left[\prod_{i=1}^{2k} F_{ia}\right] \Bigg/ \prod_{i=1}^{k} L_{ia}^2$$

$$+ \left[\binom{m}{2k + 2\varepsilon}F_{q-1}L_n + \binom{m}{2k + 1 + 2\varepsilon}L_{q-1}F_n\right] 5^{\varepsilon}2^{2k-1}F_n^{2k+2\varepsilon}L_n^{m-2k-1-2\varepsilon}\prod_{i=k+1}^{2k} F_{ia}.$$

The proof of (i) is complete by mathematical induction. The proofs of (ii), (iii), and (iv) are similar.

The three identities given as examples in the introduction can be obtained as special cases of Theorem 2 (i) by using the ordered 6-tuple $(\varepsilon, k, m, n, a, r)$ in the forms $(0,1,2,n,a,0)$, $(0,1,3,n,a,0)$, and $(0,3,6,n,a,0)$, respectively. A special case of Theorem 2 (ii) with the ordered 6-tuple $(0,1,3,n,a,0)$ can be found in [6].

The author thanks the referee for the type of proof used in Lemma 3 and for reference number [4] and for suggestions which led to major simplifications of

several of the proofs, such as the proof of Lemma 2, and which brought the statement of Theorem 1 out of the realm of unintelligibility.

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SOME COMBINATORIAL SEQUENCES

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1. INTRODUCTION

We will enumerate the different $m \times m$ matrices $B_r(n)$, $n = 1, 2, 3, \ldots, r = 1, 2, 3, \ldots, x_n$, having elements from the set [0, 1], where the allowed column vectors B_j and some conditions between elements b_{ij} are specified. That is,

$$\text{C1:} \quad b_{ij} = 1 \Rightarrow b_{i,j-1} = 0,$$

$$\text{C2:} \quad b_{ij} = 1 \Rightarrow \begin{cases} b_{i-1,j} = 0 \\ b_{i+1,j} = 0, \ m > i > 1, \end{cases}$$

and

$$b_{1j} = 1 \Rightarrow b_{2j} = 0,$$

$$b_{mj} = 1 \Rightarrow b_{m-1,j} = 0.$$

The number of different matrices $B_r(n)$ is called x_n and is the general term of a combinatorial sequence $\{x_n : n = 1, 2, 3, \ldots\}$. The vector $B_j = P_j$ is one of the p distinct column vectors in an $m \times p$ matrix P called the primitive matrix. The vector P_j is named in accordance with the following rules:

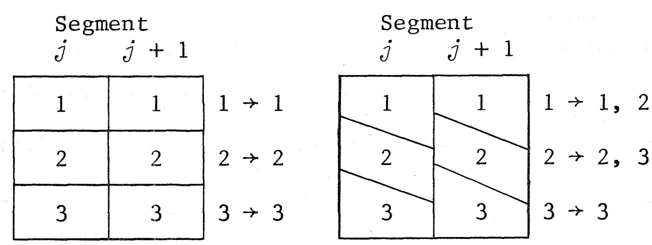
1. The name of the zero vector is 0; the remaining vectors may be identified by the positions of 1's in them.
2. The numbers in these names, if more than one, are conveniently given in increasing order with a bar placed over them.
3. The dimension m of B_j is greater than or equal to the largest number in its name.

EXAMPLES

Name of P_j	P_j
0	0 0 0 ... 0
1	1 0 0 ... 0
2	0 1 0 ... 0
$\overline{12}$	1 1 0 ... 0
$\overline{13}$	1 0 1 0 ... 0
$\overline{123}$	1 1 1 0 ... 0

m	Some Primitive Matrices P	
	Under C2	Unrestricted
1	(0 1)	(0 1)
2	(0 1 2)	(0 1 $\overline{12}$ 2)
3	(0 1 $\overline{13}$ 2 3)	(0 1 $\overline{12}$ $\overline{123}$ $\overline{13}$ 2 $\overline{23}$ 3)
4	(0 1 $\overline{13}$ $\overline{14}$ 2 $\overline{24}$ 3 4)	(0 1 $\overline{12}$ $\overline{123}$ $\overline{1234}$ $\overline{124}$ $\overline{13}$ $\overline{134}$ $\overline{14}$ 2 $\overline{23}$ $\overline{234}$ $\overline{24}$ 3 $\overline{34}$ 4)
Size	$m \times F_{m+2}$	$m \times 2^m$

Any figure consisting of a succession of like segments each of which is divided into m cells which can be occupied by either a 1 or a 0 under given conditions may be represented by a matrix $B_r(n)$ in which n is the number of segments in the figure. The cells in any segment must be numbered in a given way (1, 2, 3, ..., m) and correspond to the row numbers in $B_r(n)$. Figures in which only cells of like number in adjacent segments are adjacent are said to be *regular*. This adjacency condition (AC) is symbolized by $b_i \to b_i$. Figures in which at least one cell b_{ij} in the j^{th} segment is adjacent to more than one cell in the $(j+1)^{\text{st}}$ segment ($b_{s,j+1}$, $b_{t,j+1}$, ...) are said to be *irregular*. This AC is symbolized by $b_i \to b_s, b_t, \ldots$ (see Fig. 1).



(a) Regular Figure

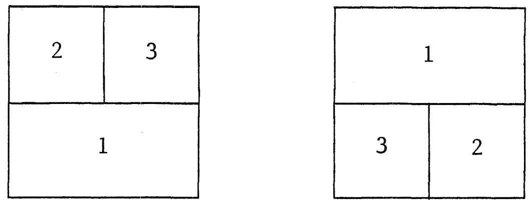
(b) Irregular Figure

Figure 1

Consider a prism of n segments formed of segments of unit height on bases A or B (Figure 2). If the segments have equal bases A or B, P = (0 1 2 3) is a possible primitive matrix and $b_i \to b_i$. If the successive segments have bases that alternate between A and B, P may be unchanged but $1 \to 2, 3$; $2 \to 1$; $3 \to 1$.

Condition 1 may be replaced by the more general condition C3: any two adjacent cells, each from a different segment cannot both contain the number 1.

The matrix P has a companion matrix $\overline{P}$ in which the column P_j has a counterpart $\overline{P}_j$ in $\overline{P}$ obtained by applying the given AC, $b_i \to b_s, b_t, \ldots,$ to each number i in the name of P_j and ordering the resulting numbers without repetition. A bar is placed over these numbers to distinguish the columns of $\overline{P}$. That is, if P = (1 $\overline{12}$ $\overline{13}$ 2 3) in Figure 1(b), then $\overline{P}$ = ($\overline{12}$ $\overline{123}$ $\overline{123}$ $\overline{23}$ 3).



Base A Base B

Figure 2

Define the $(p+1) \times 1$ set matrix $M(1)$ with elements consisting of sets of matrices such that $m_1(1) = \emptyset$, the empty set, and $m_i(1) = [P_{i-1}]$, where $p + 1 \geqslant i > 1$ and L is the $(p+1) \times (p+1)$ partitioned matrix

$$L = \left(\begin{array}{c|c} \multicolumn{2}{c}{U^T} \\ \hline 0 & K \end{array}\right),$$

where 0 is the $p \times 1$ zero vector, U is $(p+1) \times 1$ with $u_1 = 0$, and $u_i = 1$ if $p + 1 \geqslant i > 1$. A matrix K, called the kernel, is $p \times p$ with $K_{ij} \in [0, 1]$ and is a function of P and the given AC as described later.

A special product is defined for L and a conforming set matrix generating another set matrix as a product.

$$L \cdot M(n - 1) = M(n),\ n > 1, \tag{1.1}$$

hence

$$(L.)^{n-1}M(1) = M(n). \tag{1.2}$$

The expression $\ell_{ji}m_i(n-1)(P_{j-1})$ represents the result of augmenting each member of the set $m_i(n-1)$ by appending the vector P_{j-1} on the right if $\ell_{ji} = 1$. If $\ell_{ji} = 0$, this expression represents $\emptyset$.

$$m_1(n) = \bigcup_2^{p+1} \ell_{1i}m_i(n - 1)$$

$$m_j(n) = \bigcup_2^{p+1} \ell_{ji}m_i(n - 1)(P_{j-1}),\ j > 1.$$

Define $N(1)$ as the vector with $n_1(1) = 0$ and $n_j(1) = 1$ if $p + 1 \geqslant j > 1$. Let

$$LN(n - 1) = N(n),\ n > 1. \tag{1.3}$$

The sets $m_j(n)$, $p + 1 \geqslant j > 1$ are disjoint, and their cardinality is unchanged by appending columns to their matrix elements. It can be shown by mathematical induction that $N(n)$ is a vector with $n_1(n) = x_{n-1}$ and that $n_j(n)$ is the number of matrices $B_r(n)$ having P_{j-1} for the n^{th} column.

Let $\mathbf{N}_n$ be the $p \times 1$ matrix with $\mathbf{n}_i(n) = n_{i+1}(n)$, $p \geqslant i > 1$, then

$$x_n = n_1(n + 1) = \sum_2^{p+1} n_i(n) = \sum_1^{p} \mathbf{n}_i(n). \tag{1.4}$$

Example: Let $B_r(n)$ represent a $2 \times n$ matrix with $P = [0 \quad 1]$. If C1 holds,

$$k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M(1) = \begin{bmatrix} \emptyset \\ [0] \\ [1] \end{bmatrix}, \quad N(1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } L = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$LM(n-1) = M(n) \qquad \text{and} \qquad LN(n-1) = N(n),$$

so

$$M(1) = \begin{bmatrix} \emptyset \\ [0] \\ [1] \end{bmatrix}, \qquad N(1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$M(2) = \begin{bmatrix} [0,\ 1] \\ [00,\ 10] \\ [01] \end{bmatrix}, \qquad N(2) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad x_1 = 2,$$

$$M(3) = \begin{bmatrix} [00,\ 10,\ 01] \\ [000,\ 100,\ 010] \\ [001,\ 101] \end{bmatrix}, \qquad N(3) = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}, \quad x_2 = 3,$$

$$x_n = F_{n+2}.$$

Equation (1.3) implies

$$K\mathbf{N}(n) = \mathbf{N}(n+1) \tag{1.5}$$

so

$$K^n\mathbf{N}(1) = \mathbf{N}(n+1). \tag{1.6}$$

Let kernel K_r yield a value $n_1(n+1) = x_{rn}$, then if K_1 and K_2 yield $x_{1n} = x_{2n}$ they are said to be *virtually equivalent* and $K_1 \approx K_2$. Virtual equivalence is an equivalence relation.

Let Q_r represent a $p \times p$ permutation matrix, i.e., a square matrix whose elements in any row or column are all zero except for one element which is one. There are $p!$ such matrices and since $Q_rQ_r^T = I$, $Q_r^T = Q_r^{-1}$. From Equation (1.6), $K^{n-1}\mathbf{N}(1) = \mathbf{N}(n)$ and, if K is replaced by $Q_rKQ_r^{-1}$,

$$(Q_rKQ_r^{-1})^{n-1}\mathbf{N}(1) = Q_rK^{n-1}Q_r^T\mathbf{N}(1) = Q_rK^{n-1}\mathbf{N}(1) = Q_r\mathbf{N}(n).$$

From Equation (1.4), $x_n = \sum_1^p \mathbf{n}_i(n)$ for K and for $Q_rKQ_r^T$; the $\mathbf{n}_i(n)$ are summed in possibly a different order. The result is the same, so

$$Q_rKQ_r^T \approx K. \tag{1.7}$$

Let K_r be a $p_r \times p_r$ kernel, $r = 1, 2, 3$, and define the direct sum

$$K_1 \oplus K_2 = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

Permutation matrices Q_s and Q_s^T can be constructed so that

$$Q_s(K_1 \oplus K_2)Q_s^T = K_2 \oplus K_1.$$

If $q_{ij} \in Q_s$, then $q_{ij} = 1$ if

i	1	2	$\cdots$	p_2	p_2+1	p_2+2	$\cdots$	p_2+p_1
j	p_1+1	p_1+2	$\cdots$	p_1+p_2	1	2	$\cdots$	p_1

and $q_{ij} = 0$ otherwise. Let $p_1 = 2$ and $p_2 = 3$, then

$$Q_s = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

From Equation (1.7),

$$K_1 \oplus K_2 \approx K_2 \oplus K_1. \tag{1.8}$$

Define the direct product $K_1 \times K_2$ as the partitioned matrix

$$K_1 \times K_2 = \begin{bmatrix} k_{111}K_2 & k_{112}K_2 & \cdots & k_{11p_1}K_2 \\ k_{121}K_2 & k_{122}K_2 & \cdots & k_{12p_1}K_2 \\ \cdots & \cdots & \cdots & \cdots \\ k_{1p_11}K_2 & k_{1p_12}K_2 & \cdots & k_{1p_1p_1}K_2 \end{bmatrix}$$

in which $k_{1rs} \in K_1$ and $k_{2tu} \in K_2$.

Let

$$k_{1rs}k_{2tu} = \begin{cases} k'_{iv} \in K_1 \times K_2 \\ k''_{jw} \in K_2 \times K_1 \end{cases},$$

then

$$i = (r - 1)p_2 + t \tag{a}$$

and

$$j = (t - 1)p_1 + r. \tag{b}$$

From Equation (a),

$$t - 1 = (i - 1) \bmod p_2 \tag{c}$$

and

$$r - 1 = \left[\frac{i - 1}{p_2}\right], \tag{d}$$

in which $[x]$ represents the greatest integer in the number x. Substituting Equations (c) and (d) in (b),

$$j = p_1((i - 1) \bmod p_2) + \left[\frac{i - 1}{p_2}\right] + 1. \tag{e}$$

If i, j, r, and t are replaced by v, w, s, and u, respectively, Equations (a)-(e) still hold and Equation (e) becomes

$$w = p_1((v - 1) \bmod p_2) + \left[\frac{v - 1}{p_2}\right] + 1. \tag{f}$$

Consider a matrix Q where $q_{ij} = 1$ if Equation (e) is satisfied and $q_{ij} = 0$ otherwise. From Equation (a) if i is given, r and t are uniquely defined, and from Equation (b) j is uniquely defined. Conversely, if j is given, then i is uniquely defined. This implies that every row and column of Q has just one element 1 and all other elements are zero. Q is then a permutation matrix.

Consider the matrix Q' where $q'_{vw} = 1$ if Equation (f) is satisfied and $q'_{vw} = 0$ otherwise. By a similar argument, Q' is also a permutation matrix and since j and i may replace w and v, respectively, in Equation (f) to produce Equation (e), then we let $Q_p = Q' = Q$ so that

$$Q_p(K_1 \times K_2)Q_p^T = K_2 \times K_1.$$

From Equation (1.7),

$$K_1 \times K_2 \approx K_2 \times K_1. \tag{1.9}$$

For example, if $p_1 = 2$ and $p_2 = 3$,

$$Q_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let

$$K_3 = K_1 \oplus K_2 = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \quad \text{and} \quad \mathbf{N}_3(1) = \begin{bmatrix} \mathbf{N}_1(1) \\ \mathbf{N}_2(1) \end{bmatrix}$$

then

$$K_3^{n-1} = \begin{bmatrix} K_1^{n-1} & 0 \\ 0 & K_2^{n-1} \end{bmatrix}$$

and, by Equation (1.6)

$$\mathbf{N}_3(n) = \begin{bmatrix} \mathbf{N}_1(n) \\ \mathbf{N}_2(n) \end{bmatrix}$$

Applying Equation (1.4),

$$x_{3n} = x_{1n} + x_{2n} \quad \text{if } K_3 = K_1 \oplus K_2. \tag{1.10}$$

Suppose $K_3 = K_1 \times K_2$ with $\mathbf{N}_3(1) = \mathbf{N}_1(1) \times \mathbf{N}_2(1)$, a $p_1p_2 \times 1$ matrix of 1's. Then, by Equation (1.4), $x_{31} = x_{11}x_{21}$. Assume that $\mathbf{N}_3(r) = \mathbf{N}_1(r) \times \mathbf{N}_2(r)$ for any $r > 0$, then

$$K_3\mathbf{N}_3(r) = (K_1 \times K_2)(\mathbf{N}_2(r) \times \mathbf{N}_1(r)) = \sum_{j=1}^{p_1} k_{1ij}n_{1ji}(r)K_2\mathbf{N}_2(r), \ i = 1, 2, \ldots, p_1,$$

or $K_3\mathrm{N}_3(r) = K_1\mathrm{N}_1(r) \times K_2\mathrm{N}_2(r)$, and by Equation (1.5),

$$\mathrm{N}_3(r + 1) = \mathrm{N}_1(r + 1) \times \mathrm{N}_2(r + 1).$$

It follows by mathematical induction that $\mathrm{N}_3(n) = \mathrm{N}_1(n) \times \mathrm{N}_2(n)$ for all n and, from Equation (1.4),

$$x_{3n} = x_{1n}x_{2n} \quad \text{if } K_3 = K_1 \times K_2. \tag{1.11}$$

From definitions

$$(K_1 \oplus K_2) \times K_3 = (K_1 \times K_3) \oplus (K_2 \times K_3), \tag{1.12}$$

32 virtual equivalences may be deduced using the commutative laws for $\oplus$ and $\times$.

2. EVALUATION OF K

Theorem 2.1: If C3 holds, and if $\overline{P}_i$ and P_j have one or more numbers common in their names, then $k_{ij} = 0$; if $\overline{P}_i$ and P_j have no numbers common in their names, then $k_{ij} = 1$.

Proof: From Equation (1.1), $L \cdot M(n - 1) = M(n)$, and by renumbering elements,

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & k_{11} & k_{12} & \cdots & k_{1p} \\ 0 & k_{21} & k_{22} & \cdots & k_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & k_{p1} & k_{p2} & \cdots & k_{pp} \end{bmatrix} \begin{bmatrix} m_0(n-1) \\ m_1(n-1) \\ m_2(n-1) \\ \vdots \\ m_p(n-1) \end{bmatrix} = \begin{bmatrix} m_0(n) \\ m_1(n) \\ m_2(n) \\ \vdots \\ m_p(n) \end{bmatrix}.$$

Through multiplication,

$$m_0(n) = \bigcup_1^p m_i(n-1)(\emptyset) = \bigcup_1^p m_i(n-1),$$
$$m_j(n) = \bigcup_1^p k_{ji}m_i(n-1)(P_j), \quad j = 1, 2, \ldots, p,$$

where $m_i(n - 1)(P_j)$ represents the set $m_i(n - 1)$ in which each element $B_r(n - 1)$ has P_i as the terminal column and is augmented by the vector P_j to form a matrix $B_r'(n)$. The last two columns of $B_r'(n)$ are P_i and P_j. If P_i has one or more elements of value one adjacent to a like element in P_j, the name of $\overline{P}_i$ must have one or more numbers in common with the name of P_j, and C3 implies $B_r'(n) \notin m_j(n)$, hence $K_{ij} = 0$. If P_i has no elements of value one adjacent to a like element in P_j, the name of $\overline{P}_i$ and the name of P_j must have no numbers in common and C3 implies $B_r'(n) \in m_j(n)$, so $K_{ij} = 1$. ∎

Let $R = \overline{P}_i^TP_j = (r_{11})$, a 1×1 matrix. Then

Corollary 2.1: If C3 holds and $r_{11} = 0$, $k_{ij} = 1$; if $r_{11} > 0$, $k_{ij} = 0$.

Corollary 2.2: If C1 holds, K is symmetric.

Proof: If C1 holds, $P_i = \overline{P}_i$, so $R = (r_{11}) = R^T$ and $\overline{P}_i^TP_j = P_i^TP_j = P_j^TP_i = \overline{P}_j^TP_i$. By Corollary 2.1, if $r_{11} = 0$, $k_{ij} = k_{ji} = 1$; if $r_{11} > 0$, $k_{ij} = k_{ji} = 0$. Since

$r_{11} \geqslant 0$, $k_{ij} = k_{ji}$ and K is symmetric. ■

Corollary 2.3: If C3 holds and $P_i = 0$, then $k_{ij} = 1$ for all j; $P_i \neq 0$ implies $k_{ii} = 0$.

Corollary 2.4: If C3 holds, then K can have at most one row of 1's.

Let $X = [x_i : i = 1, 2, 3, \ldots, r]$, $m \geqslant r > 0$, be the set of all the different numbers appearing in the names of the columns of P and in the AC, and let $Y = [y_i =: i = 1, 2, 3, \ldots, r]$ be any other set of r distinct numbers, then

Corollary 2.5: K is unchanged by replacing x_i by y_i, $i = 1, 2, 3, \ldots, r$, in P and in the AC under C3.

Definition: A proper K is a K in which there is at most one row of 1's.

Theorem 2.2: Every proper K may be derived from some P under $C3$ and AC.

Proof: Given $k_{ij} \in [0, 1]$. If a row K_i consists only of 1's, it is named 0 and the remaining rows are named $1, 2, 3, \ldots, p - 1$. If no such row exists, name the rows $1, 2, 3, \ldots, p$. Then P consists of columns P_j which are in the same sequence as the named rows of K and have the same names. Suppose K_i has an element $k_{ij} = 0$, then the AC must include $i \to j$; if $k_{ij} = 1$, then $i \nrightarrow j$. Since K is proper, there is at most one row of 1's which is named 0. All columns of P have names which are unique. ■

The AC under C3 may sometimes be simplified by changing the columns of P without altering K. Let d, e, and f represent three distinct cells in a segment B_j of $B_r(n)$ and let r, s, and $r \cup s$ represent sets of cells in B_{j+1} adjacent to d, e, and f, respectively. The adjacency conditions are represented by the set $[d \to r,\ e \to s,\ f \to r \cup s]$, and f may be replaced by $\overline{de}$ in the names of P_j and in AC forming P' and the set $\overline{AC} = [d \to r,\ c \to s]$ which, by Theorem 2.1, yields the same K.

Example: Let

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

By Theorem 2.2, K may be derived from $P = (1\ 2\ 3\ 4\ 5\ 6\ 7)$ under C3 with the AC

$1 \to 1, 2, 3, 4$
$2 \to 1, 2, 3, 4, 5, 6$
$3 \to 1, 2, 3, 4, 5, 6, 7$
$4 \to 1, 2, 3, 4, 6, 7$
$5 \to 2, 3, 5, 6$
$6 \to 2, 3, 4, 5, 6, 7$
$7 \to 3, 4, 6, 7.$

The AC may be simplified as follows:

Consider:

$$1 \to 2,\ 3,\ 1,\ 4$$
$$2 \to 2,\ 3,\ 1,\ 4,\ 5,\ 6$$
$$5 \to 2,\ 3,\ 5,\ 6$$

We can then replace 2 by $\overline{15}$. Similarly we can replace 3 by $\overline{24}$, 4 by $\overline{17}$, and 6 by $\overline{57}$, so P becomes $P' = (1\ \ \overline{15}\ \ \overline{157}\ \ \overline{17}\ \ 5\ \ \overline{57}\ \ 7)$. By renumbering in accordance with Corollary 2.5, $P' = (1\ \ \overline{12}\ \ \overline{123}\ \ \overline{13}\ \ 2\ \ \overline{23}\ \ 3)$ with

$$AC = [1 \to 1;\ 2 \to 2;\ 3 \to 3].$$

Further examples giving P, AC, K, x_n, and recurrence relations are:

#1 $P = (0\ \ 1)$
$AC = [1 \to 1]$
$x_n = \{2, 3, 5, 8, 13, \ldots\}$
$x_{n+2} - x_{n+1} - x_n = 0$

$$K = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

#2 $P = (0\ \ 1\ \ 2)$
$AC = [1 \to 1;\ 2 \to 2]$
$x_n = \{3, 7, 17, 41, 99, 239, \ldots\}$
$x_{n+2} - 2x_{n+1} - x_n = 0$

$$K = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

#3 $P = (1\ \ 2)$
$AC = [1 \to 1, 2;\ 2 \to 1, 2]$
$x_n = \{2, 0, 0, \ldots\}$
$x_{n+1} = 0$

$$K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

#4 $P = (0\ \ 1\ \ 2)$
$AC = [1 \to 1, 2;\ 2 \to 1, 2]$
$x_n = \{3, 5, 11, 21, 43, 85, \ldots\}$
$x_{n+2} - x_{n+1} - 2x_n = 0$

$$K = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

#5 $P = (1\ \ 2\ \ 3\ \ 4\ \ 5)$
$AC = [1 \to 3, 4, 5;\ 2 \to 2, 3, 4, 5;$
$3 \to 1, 2;\ 4 \to 1, 2, 4;$
$5 \to 1, 2, 5]$
$x_n = \{5, 10, 22, 49, 112, 260, \ldots\}$
$x_{n+4} - 3x_{n+3} + 3x_{n+1} + x_n = 0$

$$K = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

#6 $P = (0\ \ 1\ \ 2\ \ 3)$
$AC = [1 \to 1, 3;\ 2 \to 2, 3;\ 3 \to 1, 2, 3$
also $P = (0\ \ 1\ \ 2\ \ \overline{12})$
$AC = [1 \to 1;\ 2 \to 2]$
$x_n = \{4, 9, 25, 64, 169, 441, \ldots\}$
$x_{n+4} - x_{n+3} - 4x_{n+2} - x_{n+1} + x_n = 0$

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Example #1 represents the sequence $x_n = F_{n+2}$. Examples #2 and #4 represent sequences of Winthrop and Horadam [2], $x_n = w_n(1, 3; 2, -1)$ and $x_n = w_n(1, 3; 1, -2)$, respectively, where $w\ (a, b; p, q)$ has $w_0 = a$, $w_1 = b$, and $w_n = pw_{n-1} - qw_{n-1}$, $n \geqslant 2$. Example #5 illustrates $K_3 = K_1 \oplus K_2$ with $x_n = F_{n+2} + w_n(1, 3; 2, -1)$, and Example #6 illustrates $K_3 = K_1 \times K_2$ with $x_n = (F_{n+2})^2$ in which two values for P and the corresponding AC are given.

3. RECURRENCE RELATIONS

The characteristic function of K is $f(y) = |yI - K|$ and its characteristic equation is

$$f(y) = \sum_{0}^{p} c_i y^i = 0. \tag{3.1}$$

Theorem 3.1:

$$\sum_{0}^{p} c_i x_{n+i} = 0$$

is a recurrence relation for the sequence $\{x_n : n = 1, 2, 3, \ldots\}$.

Proof: Apply the Cayley-Hamilton theorem to Equation (3.1), giving

$$\sum_{0}^{p} c_i K^i = 0.$$

Multiply each side of this on the right by $K^{n-1}\mathbf{N}(1)$, giving

$$\sum_{0}^{p} c_i K^{n-1+i}\mathbf{N}(1).$$

Then, by Equation (1.6),

$$\sum_{0}^{p} c_i \mathbf{N}(n + i) = 0.$$

Multiply on the left by $\mathbf{U}^T$, a $1 \times p$ matrix with $\mathbf{u}_{1i} = 1$, giving

$$\sum_{0}^{p} c_i \sum_{0}^{p} \mathbf{n}_j(n + i) = 0,$$

and by Equation (1.4),

$$\sum_{0}^{p} c_i x_{n+i} = 0.$$

This is a recurrence relation for the sequence $\{x_n : n = 1, 2, 3, \ldots\}$. ■

Corollary 3.1: If the characteristic equation of K is

$$(y - d)\sum_{0}^{p-1} c_i y^i = 0$$

and if $K - dI$ is nonsingular, then

$$\sum_{0}^{p-1} c_i x_{n+i} = 0$$

is a recurrence relation for $\{x_n : n = 1, 2, 3, \ldots\}$.

Proof: By the Cayley-Hamilton theorem,

$$(K - dI)\sum_0^{p-1} c_i K^i = 0.$$

If $K - dI$ is nonsingular, apply its inverse to both sides of the equation, so

$$\sum_0^{p-1} c_i x^i = 0.$$

Proceed as in Theorem 3.1 to show that $\sum_0^{p-1} c_i x_{n+i} = 0$ is the desired recurrence relation. ∎

Note that if $\mathbf{N}(1)$, in which $\mathbf{n}_{i1} = 1$, were defined as some other vector of size $p \times 1$, the new sequence $\{x_n\}$ would still possess the same recurrence relation.

Let

$$f_j(y) = \sum_0^{p_j} c_{jq} y^q = 0$$

represent the characteristic equation for $K_j : j = 1, 2, 3$.

Theorem 3.2: If $K_3 = K_1 \oplus K_2$, a recurrence relation for the sequence $\{x_{3n} : n = 1, 2, 3, \ldots\}$ is

$$\sum_0^{p_1+p_2} \sum_{q+r=i} c_{1q} c_{2r} x_{3(n+1)} = 0.$$

Proof: $$\sum_0^{p_3} c_{3i} y^i = \begin{vmatrix} yI - K_1 & 0 \\ 0 & yI - K_2 \end{vmatrix} = |yI - K_1||yI - K_2|$$

$$= \sum_0^{p_1} c_{1q} y^q \sum_0^{p_2} c_{2r} y^r,$$

then

$$c_{3i} = \sum_{q+r=i} c_{1q} c_{2r}$$

and, from Theorem 3.1, the recurrence relation for the sequence $\{x_{3n} : n = 1, 2, 3, \ldots\}$ is

$$\sum_0^{p_1+p_2} \sum_{q+r=i} c_{1q} c_{2r} x_{3(n+i)} = 0. \blacksquare$$

Corollary 3.2: If $K_3 = 2K_1$, the recurrence relation for x_{3n} is

$$\sum_0^{p_1} c_{1i} x_{n+i} = 0.$$

Consider the direct product $K_3 = K_1 \times K_2$. Let K_1 be partitioned into four square matrices.

$$K_1 = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad K_1 \times K_2 = \begin{bmatrix} A_1 \times K_2 & A_2 \times K_2 \\ A_3 \times K_2 & A_4 \times K_2 \end{bmatrix}.$$

Let $Q = yI - K_3$, then

$$Q = \begin{bmatrix} yI - A_1 \times K_2 & -A_2 \times K_2 \\ -A_3 \times K_2 & yI - A_4 \times K_2 \end{bmatrix}.$$

Multiply the top row of Q by $(A_3 \times K_2)(yI - A_1 \times K_2)^{-1}$ and add this to the second row [1], then

$$|Q| = \begin{vmatrix} yI - A_1 \times K_2 & -A_2 \times K_2 \\ 0 & yI - A_4 \times K_2 - (A_3 \times K_2)(yI - A_1 \times K_2)^{-1}(A_2 \times K_2) \end{vmatrix}.$$

If A_1 and A_3 commute, then

$$|Q| = |(yI - A_1 \times K_2)(yI - A_4 \times K_2) - (A_3 \times K_2)(A_2 \times K_2)| = 0$$

is the characteristic equation for K_3. This reduces to

$$|y^2 I - y(A_1 + A_4) \times K_2 + (A_1 A_4 - A_3 A_2) \times K_2^2| = 0. \tag{3.2}$$

The recurrence relation for the sequence $\{x_{3n} : n = 1, 2, 3, \ldots\}$ may then be derived if K_1 and K_2 are known.

Example: Let $K_1 = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ where $K_2 = A_1 = A_2 = A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_4 = 0$. From Equation (3.2), the characteristic equation is

$$|yI - K_3| = \left| y^2 I - y \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right| = 0$$

or

$$y^8 - y^7 - 13y^6 - 8y^5 + 20y^4 + 8y^3 - 13y^2 + y + 1 = 0.$$

The recurrence relation for the sequence $\{x_n = (F_{n+2})^3\}$ is

$$x_{n+8} - x_{n+7} - 13x_{n+6} - 8x_{n+5} + 20x_{n+4} + 8x_{n+3} - 13x_{n+2} + x_{n+1} + x_n = 0.$$

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HYPERSURFACES ASSOCIATED WITH SIMSON FORMULA ANALOGUES

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1. INTRODUCTION

The *Simson formula* for the Fibonacci numbers F_n defined by

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1, \tag{1.1}$$

is

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \tag{1.2}$$

which may be expressed in determinant form as

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n. \tag{1.2$'$}$$

For the numbers w_n defined by the generalized second-order recurrence relation

$$w_{n+2} = pw_{n+1} - qw_n,\ w_0 = a,\ w_1 = b, \tag{1.3}$$

a Simson formula was obtained in [3]. If, in this generalized Simson formula, we write $w_n = x$, $w_{n+1} = y$, then various conics—ellipses and rectangular hyperbolas—in the Euclidean plane arise as loci of the points (x, y). An analysis of these conics was made in [4] for the special cases of (1.3) which give the Fibonacci, Lucas, Pell, Fermat, and Chebyshev sequences of numbers (and also for the degenerate case when the conic breaks up).

Further developments of this theme were made by Bergum [1].

It is a natural desire to want to extend the geometrical aspect of Simson's formula (1.2) to higher dimensions. This was partly achieved in [4] for a third-order recurrence relation where a suitable analogue to Simson's formula (Waddill and Sacks [5]) was used to produce a corresponding cubic surface in three-dimensional Euclidean space. However, as this analogue had not been extended to higher-order recurrences, it was not possible to proceed to higher geometrical dimensions.

What was required was a technique, an algorithm, for determining an analogue to Simson's formula for recurrence relations of arbitrary order r.

Happily, such a method was already in existence (Hoggatt and Bicknell [2]).

After a brief, but necessary, recapitulation in the next part of this paper of the work done in [4] on the situation in three dimensions, we will proceed to employ the Hoggatt-Bicknell results [2] exclusively in the further development of our theme.

Before doing this, however, we introduce some definitions and notation.

In r-dimensional Euclidean space ($r \geqslant 2$), a locus of points whose coordinates satisfy an equation of degree m will be called a *hypersurface* of order m with dimension $r - 1$. It may be represented by the symbol L_{r-1}^m.

When the equation is linear ($m = 1$), L_{r-1}^1 is the symbol for a *hyperplane* in r dimensions, i.e., a "flat" space of maximum dimension in the containing space.

2. A CUBIC SURFACE IN THREE DIMENSIONS

Consider the third-order recurrence analogue of (1.1) for the number sequence $\{P_n\}$ defined by

$$P_{n+3} = P_{n+2} + P_{n+1} + P_n \tag{2.1}$$

with initial conditions (Waddill and Sacks [5])

$$P_0 = 0,\ P_1 = 1,\ P_2 = 1. \tag{2.2}$$

The first few numbers in this sequence are:

$$\begin{cases} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & P_{10} & P_{11} & \cdots \\ 1 & 1 & 2 & 4 & 7 & 13 & 24 & 44 & 81 & 149 & 274 & \cdots \end{cases} \tag{2.3}$$

Waddill and Sacks [5] obtained a Simson formula analogue for $\{P_n\}$ which, not unexpectedly, was of the third degree.

Putting $P_n = x$, $P_{n+1} = y$, $P_{n+2} = z$ in their formula, the author [4] derived the cubic equation

$$x^3 + 2y^3 + z^3 + 2x^2y + 2xy^2 - 2yz^2 + x^2z - xz^2 - 2xyz = 1. \tag{2.4}$$

Interpreting x, y, and z as Cartesian coordinates, we see that the points (x,y,z) lie on the cubic surface (2.4) in Euclidean space of three dimensions. For example, the point (1,1,2) in (2.3) lies on this L_2^3 (2.4), as may be easily verified.

Sections of the cubic surface (2.4) by the coordinate planes L_2^1 are the cubic curves L_1^3:

$$\begin{cases} x = 0: & 2y^3 + z^3 - 2yz^2 = 1 \\ y = 0: & x^3 + z^3 + x^2z - xz^2 = 1 \\ z = 0: & x^3 + 2y^3 + 2x^2y + 2xy^2 = 1. \end{cases} \tag{2.5}$$

A close study of these L_1^3 (2.5) might give us some insight into the nature and appearance of the L_2^3 (2.4), but no detailed investigation is undertaken here.

It must be clearly understood that the locus (2.4) and its other-dimensional analogues contain only the infinitude of points for which they are defined, i.e., within the context of this article these loci are not continuous. For instance, the point with coordinates $(0, 2^{-1/3}, 0)$ lies on the L_2^3 since $(0, 2^{-1/3}, 0)$ satisfies equation (2.4), yet the triplet $0, 2^{-1/3}, 0$ does not belong to the infinite set of numbers of the sequence $\{P_n\}$. Despite the lacunary nature of our geometrical loci, it is nevertheless sometimes worthwhile considering them as continuous entities [as for the sectional loci (2.5), for example].

In addition to the sequence (2.3) and the corresponding Simson formula analogue, Waddill and Sacks [5] discussed a closely related sequence for which the author [4] obtained a cubic equation almost identical to (2.4). However, this sequence is irrelevant to our purposes here and no further reference will be made to it. The true Fibonacci-type pattern which generalizes (1.1) and (1.2)′ is that given in (2.3), as we shall see.

Equation (2.4) of the cubic surface in three dimensions L_2^3 may also be established by a different approach using the "interesting determinant identity" of Hoggatt and Bicknell [2]. This identity, which has the structural appearance of an extension of (1.2)$'$, and which relates to the sequence (2.3) with $P_{-1} = 0$ is, in our notation,

$$\begin{vmatrix} P_{n+2} & P_{n+1} & P_n \\ P_{n+1} & P_n & P_{n-1} \\ P_n & P_{n-1} & P_{n-2} \end{vmatrix} = -1. \tag{2.6}$$

Let us now write $P_n = x$, $P_{n+1} = y$, $P_{n+2} = z$, and observe from (2.1) that

$$\begin{cases} P_{n-1} = P_{n+2} - P_{n+1} - P_n = z - x - y \\ P_{n-2} = 2P_{n+1} - P_{n+2} = 2y - z \end{cases} \tag{2.7}$$

Expanding (2.6) with the aid of (2.7), we derive

$$x^3 + 2y^3 + z^3 + 2x^2y + 2xy^2 - 2yz^2 + x^2z - xz^2 - 2xyz = 1, \tag{2.8}$$

which is identical to equation (2.4)

Thus, the same cubic surface L_2^3 in three-dimensional Euclidean space is produced both from the Waddill and Sacks [5] cubic equation and from the Hoggatt and Bicknell [2] determinant identity.

3. HYPERSPACES IN FOUR DIMENSIONS

Next, introduce a fourth-order recurrence relation for numbers Q_n (in our notation):

$$Q_{n+4} = Q_{n+3} + Q_{n+2} + Q_{n+1} + Q_n \tag{3.1}$$

with initial conditions

$$Q_0 = 0,\ Q_1 = 1,\ Q_2 = 1,\ Q_3 = 2 \quad (Q_{-1} = 0,\ Q_{-2} = 0). \tag{3.2}$$

Then the sequence $\{Q_n\}$ looks like this:

$$\begin{cases} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 & Q_7 & Q_8 & Q_9 & Q_{10} & \cdots \\ 1 & 1 & 2 & 4 & 8 & 15 & 29 & 56 & 108 & 208 & \cdots \end{cases} \tag{3.3}$$

Following the method by which (2.6) was established, Hoggatt and Bicknell [2] exhibited the neat determinantal identity

$$\begin{vmatrix} Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \end{vmatrix} = (-1)^{n+1}. \tag{3.4}$$

Write $Q_n = x$, $Q_{n+1} = y$, $Q_{n+2} = z$, $Q_{n+3} = t$. Observe that, from (3.1), we may deduce that

$$\begin{cases} Q_{n-1} = Q_{n+3} - Q_{n+2} - Q_{n+1} - Q_n = t - x - y - z \\ Q_{n-2} = 2Q_{n+2} - Q_{n+3} = 2z - t \\ Q_{n-3} = 2Q_{n+1} - Q_{n+2} = 2y - z. \end{cases} \tag{3.5}$$

Expand (3.4) along the first row. Then, the locus of the point (x,y,z,t) in four-dimensional Euclidean space is the quartic hypersurface L_3^4 (in fact, two such loci depending on the evenness or oddness of n):

$$\begin{cases} x[x\{y(t-x-y-z)-x^2\}-y\{y(2z-t)-x(t-x-y-z)\} \\ \qquad\qquad + z\{x(2z-t)-(t-x-y-z)^2\}] \\ -y[(t-x-y-z)\{y(t-x-y-z)-x^2\}-y\{y(2y-z)-x(2z-t)\} \\ \qquad\qquad + z\{x(2y-z)-(2z-t)(t-x-y-z)\}] \\ +z[(t-x-y-z)\{y(2z-y)-x(t-x-y-z)\}-x\{y(2y-z)-x(2z-t)\} \\ \qquad\qquad + z\{(2y-z)(t-x-y-z)-(2z-t)^2\}] \\ -t[(t-x-y-z)\{x(2z-t)-(t-x-y-z)^2\} \\ \qquad\qquad - x\{x(2y-z)-(2z-t)(t-x-y-z)\} \\ \qquad\qquad + y\{(2y-z)(t-x-y-z)-(2z-t)^2\}] \\ = (-1)^n. \end{cases} \tag{3.6}$$

Discretion seems the better part of valor here, so we will leave the equations in this form which is useful for deducing the sectional loci in (3.7). However, the interested reader may care to expand the expressions in (3.6) still further. It certainly bears out the author's trepidation [4] about the cumbersome algebraic manipulation involved in the fourth-order recurrence case.

Before expanding along the first row, one might secure a slightly simpler form of the determinant by adding to the fourth row the sum of the first three rows. But, in all probability, perhaps no great economy of effort in exhibiting (3.6) is thereby effected.

Planar sections (quartic curves L_1^4) of the hypersurface (3.6) by pairs of three-dimenaional coordinate hyperplanes (L_3^1) are readily obtainable, namely:

$$\begin{cases} x=0,\ y=0: & -3y^4+2z^3t+2z^2t^2-3zt^3+t^4=(-1)^n \\ x=0,\ z=0: & y^4+3y^3t-2yt^3+t^4=(-1)^n \\ x=0,\ t=0: & y^4-3y^3z-7y^2z^2-5yz^3-3z^4=(-1)^n \\ y=0,\ z=0: & -x^4-x^3t+3x^2t^2-3xt^3+t^4=(-1)^n \\ y=0,\ t=0: & -x^4-2x^3z-xz^3-3z^4=(-1)^n \\ z=0,\ t=0: & -x^4-3x^3y-4x^2y^2-2xy^3+y^4=(-1)^n. \end{cases} \tag{3.7}$$

Superficially, there does not appear to be anything memorable about these quartic plane curves.

One must be struck, in comparing (1.2)′, (2.6), and (3.4), which relate to $r = 2$, 3, and 4, respectively, by the fact that when r is even the value (± 1) of the determinant depends on the evenness or oddness of n, whereas in the case of r odd (= 3) this is not so, the value being -1 always.

These variations raise obvious questions. Is the incipient result for r = 2, 4 a true pattern for r even generally? Might we reasonably expect the determinantal value for $r = 5$ to be +1, and will the incipient pattern for r odd prove to be valid for r odd generally?

Answering these questions constitutes an interesting part of the overall problem.

4. HYPERSURFACES IN HIGHER DIMENSIONS

Extending the pattern of the ideas used for lower-order recurrence relations, Hoggatt and Bicknell [2] defined the sequence $\{R_n\}$ of order r by

$$R_{n+r} = R_{n+r-1} + R_{n+r-2} + \cdots + R_n \tag{4.1}$$

with initial conditions

$$R_0 = 0,\ R_1 = 1 \tag{4.2}$$

and

$$R_{-(r-2)} = R_{-(r-3)} = \cdots = R_{-1} = 0. \tag{4.3}$$

For these numbers R_n generated by the r-order recurrence relation (4.1), they established the determinantal identity

$$\begin{vmatrix} R_{n+r-1} & R_{n+r-2} & \cdots & R_{n+1} & R_n \\ R_{n+r-2} & R_{n+r-3} & \cdots & R_n & R_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{n+1} & R_n & \cdots & R_{n-r+3} & R_{n-r+2} \\ R_n & R_{n-1} & \cdots & R_{n-r+2} & R_{n-r+1} \end{vmatrix} = (-1)^{(r-1)n+[(r-1)/2]}, \tag{4.4}$$

which specializes to the determinantal results (1.2)′, (2.6), and (3.4) already given for small values of r, namely, r = 2, 3, and 4, respectively. In (4,4), the notation $[(r - 1)/2]$ refers to the greatest integer function.

[It should be noted that a small typographical aberration occurs in the power of (-1) on the right-hand side of (4.4) as given in [2].]

Putting $R_n = x_1$, $R_{n+1} = x_2$, $R_{n+2} = x_3$, ..., $R_{n+r-1} = x_r$ in (4.4), and substituting by means of (4.1)-(4.3) for elements below the reverse diagonal, we could theoretically obtain the locus of points $(x_1, x_2, x_3, \ldots, x_r)$ in r-dimensional Euclidean space satisfying equation (4.4).

By analogy with (2.8) and (3.6), this locus is a L^r_{r-1}, a hypersurface (dimension $r - 1$) of order r. Sections by sets of $r - 2$ coordinate hyperplanes ("flat" hyperspaces L^1_{r-1} of dimension $r - 1$) from the total set

$$\{x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ \ldots,\ x_r = 0\}$$

of such hyperplanes give the planar curves L^r_1 of order r in two dimensions corresponding to the conics (L^2_1), cubics (L^3_1), and quartics (L^4_1) in the lower-dimensional cases.

For example, in six-dimensional Euclidean space (r = 6), the section of the sextic hypersurface L^6_5 by the four coordinate hyperplanes $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, $x_6 = 0$ is a plane sextic curve L^6_1.

A representative instance of (4.4) is, for r = 5, n = 7 (say),

$$\begin{vmatrix} 464 & 236 & 120 & 61 & 31 \\ 236 & 120 & 61 & 31 & 16 \\ 120 & 61 & 31 & 16 & 8 \\ 61 & 31 & 16 & 8 & 4 \\ 31 & 16 & 8 & 4 & 2 \end{vmatrix} = +1 \qquad \text{(on calculation)}$$

$$= (-1)^{28+2} = (-1)^{30} \text{ in accord with (4.4).}$$

For various values of r and n, the determinantal values in (4.4), i.e., +1 or -1, may be summarized in the following table:

Table 1: Determinantal Values in (4.4)

<table>
<tr><th rowspan="2">r / n</th><th colspan="2">Even</th><th colspan="2">Odd</th></tr>
<tr><th>2,6,10,...</th><th>4,8,12,...</th><th>3,7,11,...</th><th>5,9,13,...</th></tr>
<tr><td>Odd
1,3,5,7,...</td><td>-1</td><td>+1</td><td rowspan="2">-1</td><td rowspan="2">+1</td></tr>
<tr><td>Even
2,4,6,...</td><td>+1</td><td>-1</td></tr>
</table>

Or, expressed symbolically: If

$$r = 4k + 2,\ 4k + 3,\ 4k + 4,\ 4k + 5 \quad (k \geqslant 0),$$

then

$$(-1)^{(r-1)n+[(r-1)/2]} = (-1)^n,\ -1,\ (-1)^{n+1},\ 1,$$

respectively.

Thus, for each odd value of r, there is just one hypersurface irrespective or the value of n, while, for each even value of r, there are two "companion" hypersurfaces which depend on the evenness or oddness of n.

Now, in [4] it was stated that, when $r = 2$, a hyperbola for which n is odd (even) may be transformed into its companion hyperbola occurring when n is even (odd) by a reflection in the line $y = x$ followed by a reflection in the y-axis (x-axis).

Remembering that in two dimensions ($r = 2$), a line (a L_1^1) is a hyperplane, one may speculate whether a similar, though more complicated, system of geometrical reflections in higher even-dimensional spaces ($r = 4,6,\ldots$) will produce a transformation of one hypersurface into another. Further, one wonders whether any self-transformation of a hypersurface is possible for an odd value of r.

With these reflections, we leave the geometry.

A concluding comment on nomenclature is appropriate. Numbers, and their polynomial extensions, defined in (2.1)-(2.2), (3.1)-(3.2), and (4.1)-(4.3) are sometimes referred to in the literature as Tribonacci, Quadranacci, and r-bonacci respectively. While these adjectives are suggestive and useful, they do not appeal to the author and consequently have not been utilized in this article.

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DETERMINANTAL HYPERSURFACES FOR LUCAS SEQUENCES OF ORDER r, AND A GENERALIZATION

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1. INTRODUCTION

In Hoggatt and Bicknell [1], the Fibonacci sequence $\{R_n\}$ of order $r\ (\geqslant 2)$ was defined by

$$R_{n+r} = R_{n+r-1} + R_{n+r-2} + \cdots + R_n, \quad R_1 = 1,\ R_2 = 1, \tag{1.1}$$

with

$$R_{-(r-2)} = R_{-(r-3)} = \cdots = -R_1 = R_0 = 0. \tag{1.2}$$

Using the method of a generating matrix for $\{R_n\}$, they obtained the determinantal identity

$$\begin{vmatrix} R_{n+r-1} & R_{n+r-2} & \cdots & R_{n+1} & R_n \\ R_{n+r-2} & R_{n+r-3} & \cdots & R_n & R_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{n+1} & R_n & & R_{n-r+3} & R_{n-r+2} \\ R_n & R_{n-1} & & R_{n-r+2} & R_{n-r+1} \end{vmatrix} = (-1)^{(r-1)n+[(r-1)/2]} \tag{1.3}$$

which is an extension of the Simson formula (identity) for the simplest case $r = 2$ for Fibonacci numbers.

Carrying these numbers R_n through to coordinate notation (writing $x_1 = R_n$, $x_2 = R_{n+1}$, $x_3 = R_{n+2}$, $\ldots$, $x_r = R_{n+r-1}$), the author [4] showed that (1.3) could be interpreted as one or more hypersurfaces in Euclidean space of r dimensions (the number of hypersurface loci depending on n). The cases $r = 2, 3, 4$ were delineated in a little detail ([3], [4]).

It is now proposed to extend the results in [3] and [4] to the case of a Lucas sequence $\{S_n\}$ or order r, i.e., to construct a determinant analogous to (1.3) and to interpret it geometrically as a locus in r-space.

From experience, we should expect the algebraic aspects of $\{S_n\}$ to resemble those of $\{R_n\}$. Nevertheless, there are sufficient variations from the Fibonacci case to make the algebraic maneuvers, which constitute the main part of this article, a challenging and absorbing exercise.

Because of complications associated with the fact that S_0 [to be defined in (2.1)] is nonzero, whereas $R_0 = 0$, the method used by Hoggatt and Bicknell [1] for $\{R_n\}$ is not applied here for $\{S_n\}$. However, our method is applicable to $\{R_n\}$, as we shall see, provided we add to the definitions in [1] the injunction $R_{-(r-1)} = 1$.

Schematically, this paper consists of two parts. Part I is organized to secure results for the Lucas sequence which correspond to those for the Fibonacci sequence. On the basis of this knowledge, in Part II we briefly generalize the results for a sequence which contains the Fibonacci and Lucas (and other) sequences as special cases.

PART I

2. LUCAS SEQUENCE OF ORDER r

Define $\{S_n\}$, the *Lucas sequence of order* r ($\geqslant 2$) by

$$S_{n+r} = S_{n+r-1} + S_{n+r-2} + \cdots + S_n, \quad S_0 = 2,\ S_1 = 1, \tag{2.1}$$

with other initial conditions

$$\begin{cases} S_{-1} = S_{-2} = \cdots = S_{-(r-2)} = 0 \\ S_{-(r-1)} = -1. \end{cases} \tag{2.2}$$

Simplest special cases of $\{S_n\}$ occur as follows:

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2,\ L_1 = 1,\ L_{-1} = -1; \tag{2.3}$$

$$M_{n+3} = M_{n+2} + M_{n+1} + M_n, \quad M_0 = 2,\ M_1 = 1,\ M_{-1} = 0,\ M_{-2} = -1; \tag{2.4}$$

$$N_{n+4} = N_{n+3} + N_{n+2} + N_{n+1} + N_n, \quad \begin{array}{l} N_0 = 2,\ N_1 = 1, \\ N_{-2} = N_{-1} = 0,\ N_{-3} = -1. \end{array} \tag{2.5}$$

The first few numbers of these sequences are:

$$\begin{cases} L_0 & L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} & \cdots \\ 2 & 1 & 3 & 4 & 7 & 11 & 18 & 29 & 47 & 76 & 123 & \cdots \end{cases} \tag{2.3$'$}$$

$$\begin{cases} M_0 & M_1 & M_2 & M_3 & M_4 & M_5 & M_6 & M_7 & M_8 & M_9 & M_{10} & \cdots \\ 2 & 1 & 3 & 6 & 10 & 19 & 35 & 64 & 118 & 217 & 399 & \cdots \end{cases} \tag{2.4$'$}$$

$$\begin{cases} N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_9 & N_{10} & \cdots \\ 2 & 1 & 3 & 6 & 12 & 22 & 43 & 83 & 160 & 308 & 594 & \cdots \end{cases} \tag{2.5$'$}$$

The determinant of order r (which we may here call the *Lucas determinant of order* r, corresponding to that in (1.3) for the Fibonacci sequence of order r) is

$$\Delta_r = \begin{vmatrix} S_{n+r-1} & S_{n+r-2} & \cdots & S_{n+1} & S_n \\ S_{n+r-2} & S_{n+r-3} & \cdots & S_n & S_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{n+1} & S_n & \cdots & S_{n-r+3} & S_{n-r+2} \\ S_n & S_{n-1} & \cdots & S_{n-r+2} & S_{n-r+1} \end{vmatrix}. \tag{2.6}$$

Notice the cyclical nature of the elements in the columns of Δ_r. Consequently, there is symmetry about the leading diagonal of Δ_r. Both of these properties for $\{S_n\}$ are also features of the Fibonacci sequence $\{R_n\}$.

Special notation: We use the symbol r_i' to mean the operation of subtracting from row i the sum of all the other rows, in a determinant of arbitrary order. An operation such as r_i' may be called a *basic operation*. Clearly, r_i' utilizes the defining recurrence (2.1) with (2.2).

It is now necessary to introduce the concept of a basic Lucas determinant.

3. BASIC LUCAS DETERMINANTS

Let us define the *basic Lucas determinant of order* r, δ_r, as

$$\delta_r = \begin{vmatrix} S_{r-1} & S_{r-2} & S_{r-3} & S_{r-4} & \cdots & S_2 & S_1 & S_0 \\ S_{r-2} & S_{r-3} & S_{r-4} & & & & S_0 & 0 \\ S_{r-3} & S_{r-4} & & & & & & 0 \\ S_{r-4} & & & & & & & 0 \\ \vdots & & & & & & & \vdots \\ S_2 & & & & & & & 0 \\ S_1 & S_0 & & & & \mathbf{0} & & 0 \\ S_0 & 0 & 0 & 0 & \cdots & & & -1 \end{vmatrix}. \tag{3.1}$$

All elements in a given upward slanting line are the same, e.g., all the elements in the reverse (upward) diagonal are S_0 (= 2). Except for the element (= -1) in the bottom right-hand corner, all the elements below the reverse diagonal are zero.

Observe the cyclical nature of elements in the columns, remembering initial conditions (2.2) applying to symbols with negative suffixes.

Of course, (3.1) is only the special case of (2.6) when $n = 0$.

Concerning basic Lucas determinants, we now prove the following theorem (a determinantal recurrence relation).

Theorem: $\delta_r = (-1)^{[r/2]}2^r + (-1)^{r-1}\delta_{r-1}$. (3.2)

Proof: Expand δ_r in (3.1) along the bottom row to obtain

$$\delta_r = (-1)^{[r/2]}2^r - \begin{vmatrix} S_{r-1} & S_{r-2} & S_{r-3} & \cdots & S_2 & S_1 \\ S_{r-2} & S_{r-3} & & & S_1 & S_0 \\ S_{r-3} & & & & & 0 \\ \vdots & & & & & \vdots \\ S_2 & S_1 & & & \mathbf{0} & \\ S_1 & S_0 & 0 & \cdots & & 0 \end{vmatrix} \quad (S_1 = 1,\ S_0 = 2)$$

$$= (-1)^{[r/2]}2^r - \begin{vmatrix} 2 & 0 & 0 & \cdots & 0 & -1 \\ S_{r-2} & S_{r-3} & S_{r-4} & & 1 & 2 \\ S_{r-3} & S_{r-4} & & & 2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ S_2 & 1 & 0 & & & \\ 1 & 2 & 0 & \cdots & \mathbf{0} & 0 \end{vmatrix} \quad \text{by } r_1'$$

$= (-1)^{[r/2]}2^r - (-1)^{r-2}\delta_{r-1}$ after $r - 2$ cyclical row interchanges

$= (-1)^{[r/2]}2^r + (-1)^{r-1}\delta_{r-1}$.

Thus, we have, for $r \geqslant 2$,

$$[r = 2] \quad \delta_2 = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -2^2 - 1 = -5 = \underline{\underline{-(2^3 - 3)}} \tag{3.3}$$

$$[r = 3] \quad \delta_3 = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & -1 \end{vmatrix} = -2^3 + \delta_2 = -2^3 - 2^2 - 1 = -13 = \underline{\underline{-(2^4 - 3)}} \tag{3.4}$$

$$[r = 4] \quad \delta_4 = \begin{vmatrix} 6 & 3 & 1 & 2 \\ 3 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & -1 \end{vmatrix} = 2^4 - \delta_3 = 2^4 + 2^3 + 2^2 + 1 = 29 = \underline{\underline{2^5 - 3}} \tag{3.5}$$

$$[r = 5] \quad \delta_5 = \begin{vmatrix} 12 & 6 & 3 & 1 & 2 \\ 6 & 3 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 \end{vmatrix} = 2^5 + \delta_4 = 2^5 + 2^4 + 2^3 + 2^2 + 1 = 61 = \underline{\underline{2^6 - 3}} \tag{3.6}$$

$$[r = 6] \quad \delta_6 = \begin{vmatrix} 24 & 12 & 6 & 3 & 1 & 2 \\ 12 & 6 & 3 & 1 & 2 & 0 \\ 6 & 3 & 1 & 2 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} = -2^6 - \delta_5 = -2^6 - 2^5 - 2^4 - 2^3 - 2^2 - 1 = -125 = \underline{\underline{-(2^7 - 3)}} \tag{3.7}$$

and so on.

The emerging summation pattern by which the δ_r may be evaluated is clearly discernible. Notice that the term 2^1 (i.e., 2) does not occur in any δ_r summation.

However, before establishing the value of δ_r, we display the following tabulated information, for all possible values of r:

	$r = 4k$	$r = 4k + 1$	$r = 4k + 2$	$r = 4k + 3$
$[r/2]$	$2k$ } even	$2k$ } even	$2k + 1$ } odd	$2k + 1$ } odd
$(r - 1) + \left[\frac{r - 1}{2}\right]$	$6k - 2$ } even	$6k$ } even	$6k + 1$ } odd	$6k + 3$ } odd

(3.8)

From (3.8), we deduce

$$(-1)^{[r/2]} = (-1)^{r - 1 + [(r - 1)/2]}. \tag{3.9}$$

Invoking this result and applying (3.2) repeatedly, we may now calculate the value of δ_r.

Theorem: $\delta_r = (-1)^{[r/2]}(2^{r+1} - 3)$. (3.10)

Proof:
$$\begin{aligned}
\delta_r &= (-1)^{[r/2]}2^r + (-1)^{r-1}\{(-1)^{[(r-1)/2]}2^{r-1} + (-1)^{r-2}\delta_{r-2}\} && \text{by (3.2)}\\
&= (-1)^{[r/2]}\{2^r + 2^{r-1}\} - \delta_{r-2} \;\ldots\ldots\; (\alpha) && \text{by (3.9)}\\
&= (-1)^{[r/2]}\{2^r + 2^{r-1}\} - (-1)^{[(r-2)/2]}\{2^{r-2} + 2^{r-3}\} + \delta_{r-4} && \text{by } (\alpha)\\
&= (-1)^{[r/2]}\{2^r + 2^{r-1} + 2^{r-2} + 2^{r-3}\} + \delta_{r-4}\\
&= (-1)^{[r/2]}\{2^r + 2^{r-1} + 2^{r-2} + \cdots + 2^3 + 2^2 + 1\} && \text{ultimately, by (3.3) or by (3.4)}\\
&= (-1)^{[r/2]}\{2^r + 2^{r-1} + 2^{r-2} + \cdots + 2^3 + 2^2 + 2 + 1 - 2\}\\
&= (-1)^{[r/2]}\left\{\frac{2(1-2^r)}{1-2} - 1\right\} && \text{summing the finite geometric progression}\\
&= (-1)^{[r/2]}(2^{r+1} - 3).
\end{aligned}$$

Checking back shows that the special cases of δ_r listed in (3.3)-(3.7) have values in accord with (3.10), as expected.

4. EVALUATION OF LUCAS DETERMINANTS

Next, we show that [cf. (2.6), (3.1)]

$$\Delta_r = \pm\delta_r.$$

To illustrate the ideas involved in the proof we shall give for this connection between Δ_r and δ_r, suppose we take $r = 5$, $n = 3$, i.e., r is *odd*. This implies that we are dealing with the integer sequence

$$\begin{cases} S_{-4} & S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & \ldots \\ -1 & 0 & 0 & 0 & 2 & 1 & 3 & 6 & 12 & 24 & 46 & 91 & 179 & \ldots \end{cases} \tag{4.1}$$

Perform the basic operations r_1', r_2', r_3' successively on the determinant Δ_5 when $n = 3$ to derive:

$$\underbrace{\begin{vmatrix} 91 & 46 & 24 & 12 & 6 \\ 46 & 24 & 12 & 6 & 3 \\ 24 & 12 & 6 & 3 & 1 \\ 12 & 6 & 3 & 1 & 2 \\ 6 & 3 & 1 & 2 & 0 \end{vmatrix}}_{\Delta_5} = \underbrace{\begin{vmatrix} 3 & 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 \\ 12 & 6 & 3 & 1 & 2 \\ 6 & 3 & 1 & 2 & 0 \end{vmatrix}}_{\delta_5^*} = \underbrace{\begin{vmatrix} 12 & 6 & 3 & 1 & 2 \\ 6 & 3 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 \end{vmatrix}}_{\delta_5} \;\text{[= 61, see (3.6)]} \tag{4.2}$$

In Δ_5, the leading term 91 (= S_7) is reduced to the leading term 12 (= S_4) in δ_5 by the $7 - 4 = 3$ (= n) basic operations specified. Because of the cyclical nature of Δ_5, these basic operations act to produce a determinant $\delta_5^* = \Delta_5$ whose rows are the permutation

$$\begin{bmatrix} 12 & 6 & 3 & 1 & 2 \\ 3 & 1 & 2 & 12 & 6 \end{bmatrix}$$

of the rows of δ_5. Due to the fact that r is odd, this cyclic permutation is even. Expressed otherwise, δ_5^* is transformed to δ_5 by an even number of row interchanges, so the sign associated with δ_5 is +, i.e., $\delta_5^* = \delta_5$. Hence, $\Delta_5 = \delta_5$.

If $n > 5$, the cyclic process of basic operations is continued until the basic determinant δ_5 is reached. Obviously, the actual value of n is irrelevant.

The reasoning inherent in the case $r = 5$ applies equally well to the case when r is arbitrarily odd. Consequently, for r odd, $\Delta_r = \delta_r$ always.

If r is *even*, the situation is a little more complicated.

For illustrative purposes, let us examine the case $r = 4$. Substituting the numbers in (2.5)′ into (2.6) when $n = 3, 4, 5, 6$ in turn, we readily calculate that Δ_4 is reduced to the four δ_4^* whose rows are respectively the permutations

$$\begin{bmatrix} 6 & 3 & 1 & 2 \\ 3 & 1 & 2 & 6 \end{bmatrix}, \begin{bmatrix} 6 & 3 & 1 & 2 \\ 6 & 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 6 & 3 & 1 & 2 \\ 2 & 6 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 6 & 3 & 1 & 2 \\ 1 & 2 & 6 & 3 \end{bmatrix}$$

of the rows of δ_4 (and this is true here for $n = 4k - 1, 4k, 4k + 1, 4k + 2$, respectively ($k = 1, 2, \ldots$).

As these permutations are successively odd, even, odd, even, it follows that $\delta_4^* = -\delta_4, \delta_4, -\delta_4, \delta_4$ in turn. Thus, $\Delta_4 = \pm\delta_4$, depending on n, namely, $\Delta_4 = \delta_4$ when n is even, while $\Delta_4 = -\delta_4$ when n is odd.

Armed with this knowledge, we can now attack the general problem, i.e., when r and n are arbitrary integers.

First, we establish the following result:

Theorem: $\Delta_r = (-1)^m \delta_r$, where $m = n'(r - n')$, $n' \equiv n \bmod r$. (4.3)

Proof: In (2.6), the leading term S_{n+r-1} in Δ_r is diminished to the leading term S_r in δ_r by $n + r - 1 - (r - 1) = n$ basic operations $r_1', r_2', \ldots, r_n'$ which produce the determinant δ_r^*. Simultaneously, S_{r-1} drops n places in the first column of δ_r^*.

To restore the cyclical order in the first column of δ_r^* to the basic cyclical order of the first column in δ_r, beginning with S_{r-1}, it is necessary to effect $n(r - n)$ interchanges of sign to account for the $r - n$ terms below and including S_{r-1}, and the n terms above S_{r-1}.

When $n > r$, we reduce n mod r.

Each interchange accounts for a change of sign in the value of the determinant.

When r is odd, the product $n(r - n)$ is always even, no matter what the value of n is.

But when r is even, the product $n(r - n)$ is odd if n is odd, and even when n is even. [That is, when r is a given odd number there is only one value of Δ_r, whereas for a given even r there are two values of Δ_r depending on the value of n.]

Thus,

$$\Delta_r = (-1)^m \delta_r, \text{ where } m = n'(r - n'),\ n' \equiv n \bmod r.$$

Combining (3.10) and (4.3), we have the following theorem as an immediate deduction.

Theorem: $\Delta_r = (-1)^{m+[r/2]}(2^{r+1} - 3)$, where $m = n'(r - n')$, $n' \equiv n \bmod r$. (4.4)

For example,

$$\text{when } r = 5,\ n = 3{:}\quad \Delta_4 = (-1)^{3\times 2+2}(2^6 - 3) \quad \text{from (4.4)}$$
$$= +61$$

in conformity with (4.2) and (3.6).

On the other hand,

$$\text{when } r = 5,\ n = 4:\quad \Delta_4 = (-1)^{1\times 3+2}(2^5 - 3) \qquad \text{from (4.4)}$$
$$= -29$$

as we have seen in the discussion preceding (4.3).

Applying (4.4) to a random choice $r = 6$, $n = 8$ (but not so random that the computations are unmanageable!), we discover on substitution that

$$\Delta_6 = (-1)^{2\times 4+3}(2^7 - 3) = -125,$$

as may be verified by direct calculation.

Our result (4.4) for a Lucas sequence of order r should be compared with the Hoggatt-Bicknell result (1.3) for the corresponding Fibonacci case.

5. HYPERSURFACES FOR THE LUCAS SEQUENCES

Geometrical interpretations can now be given to the identity (4.4) and its specializations for small values of r. The reader is referred to [3] and [4] for details of the geometry relating to Simson-type identities for Fibonacci sequences of order r.

As this corresponding work for Simson-type identities for Lucas sequences of order r parallels the results in [3] and [4], we will content ourselves here with a fairly brief statement of the main ideas.

Write $x_1 = S_n$, $x_2 = S_{n+1}$, $\ldots$, $x_r = S_{n+r-1}$. Represent a point in r-dimensional Euclidean space by Cartesian coordinates $(x_1, x_2, \ldots, x_r)$.

Then, we interpret (4.4) as the equation of a locus of points in r-space which has maximum dimension $r - 1$ in the containing space. Such a locus is called a *hypersurface*.

Each of the loci given by the simple linear equations $x_1 = 0$, $x_2 = 0$, $\ldots$, $x_r = 0$ is a "flat" (linear) space of dimension $r - 1$, and is called a (coordinate) *hyperplane*.

Hypersurfaces of the simplest kind occur for small values of r. In accord with our theory, there will be one hypersurface when r is odd, and two when r is even.

Examples of hypersurfaces for $\{S_n\}$ are:

$$r = 2 \text{ (conic)}:\quad x_1^2 + x_1x_2 - x_2^2 = 5(-1)^n \tag{5.1}$$

$$r = 3 \text{ (cubic surface)}:\quad x_1^3 + 2x_2^2 + x_3^2 + 2x_1^2x_2 + 2x_1x_2^2 - 2x_2x_3^2 + x_1^2x_3 - x_1x_3^2 - 2x_1x_2x_3 = -13. \tag{5.2}$$

In passing, we note that (5.1) expresses the well-known Simson-type identity for Lucas sequences of order 2, namely,

$$L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n+1}. \tag{5.1$'$}$$

Moreover, the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix},$$

whose determinant δ_2 is associated with identity (5.1)$'$, has several interesting geometrical interpretations (relating to: angle-bisection, reflection, vector mapping). (See Hoggatt and Ruggles [2].)

Observe that if we replace x_1, x_2, x_3 by x, y, z, respectively, in (5.1) and (5.2), we obtain equations whose forms, except for the numbers on the right-hand sides, are identical to those given in [3] and [4]. However, this formal structure camouflages the fact that the corresponding equations of the conics and cubic surfaces, for Fibonacci and Lucas sequences, are satisfied by different sets of numbers.

Carrying further our comparison with the results for Lucas and Fibonacci sequences, we obtain, in the case $r = 4$, a nasty equation (refer to [4]) for a quartic hypersurface in four-dimensional Euclidean space. And so on for hypersurfaces in higher dimensions.

Sections of these loci by coordinate hyperplanes yield plane curves of various orders (cubics, quartics, quintics, sextics, and, generally, curves of order r). Refer here also to [4].

This completes our summarized outline of the geometrical consequences of of the determinantal identity (4.4) for Lucas sequences paralleling those for the Fibonacci sequences.

With the notions of Part I in mind, we are in a position to examine closely a more general sequence of order r which has the Fibonacci and Lucas sequences of order r as special cases.

PART II

Only a brief outline of the ensuing generalization, which parallels the information in Part I, will be given.

6. A GENERALIZED SEQUENCE OF ORDER r

Let us now introduce the generalized sequence $\{H_n\}$ defined by the recurrence relation

$$H_{n+r} = H_{n+r-1} + H_{n+r-2} + \cdots + H_n, \quad H_0 = a,\ H_1 = b, \tag{6.1}$$

with further initial conditions

$$\begin{cases} H_{-1} = H_{-2} = \cdots = H_{-(r-2)} = 0 \\ H_{-(r-1)} = b - a. \end{cases} \tag{6.2}$$

Interested readers might wish to write out the first few terms of these sequences for different values of r. For example, H takes on, in turn, the values $5a + 8b$, $11a + 13b$, $14a + 15b$, $15a + 16b$, and $16a + 16b$ for successive values $r = 2, 3, 4, 5$, and 6.

As in (2.6), the *generalized determinant of order* r is defined to be

$$D_r = \begin{vmatrix} H_{n+r-1} & H_{n+r-2} & \cdots & H_{n+1} & H_n \\ H_{n+r-2} & H_{n+r-3} & \cdots & H_n & H_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_{n+1} & H_n & \cdots & H_{n-r+3} & H_{n-r+2} \\ H_n & H_{n-1} & \cdots & H_{n-r+2} & H_{n-r+1} \end{vmatrix} \tag{6.3}$$

Evaluation of D_r is the object of Part II.

Define, as in (3.1), the *basic generalized determinant of order* r, d_r, to be obtained from (6.3) when $n = 0$. That is,

$$d_r = (D_r)_{n=0}. \tag{6.4}$$

Then, the following simplest basic generalized determinants may be readily calculated by expanding along the bottom row:

$$[r = 2] \quad d_2 = -a^2 + (b - a)b \tag{6.5}$$

$$[r = 3] \quad d_3 = -a^3 - (b - a)d_2 = -a^3 + (b - a)a^2 - (b - a)^2 b \tag{6.6}$$

$$\begin{aligned}[r = 4] \quad d_4 &= a^4 + (b - a)d_3 \\ &= a^4 - (b - a)a^3 + (b - a)^2 b^2 - (b - a)^3 b\end{aligned} \tag{6.7}$$

$$\begin{aligned}[r = 5] \quad d_5 &= a^5 - (b - a)d_4 \\ &= a^5 - (b - a)a^4 + (b - a)^2 b^3 - (b - a)^3 a^2 + (b - a)^4 b\end{aligned} \tag{6.8}$$

$$\begin{aligned}[r = 6] \quad d_6 &= -a^6 + (b - a)d_5 \\ &= -a^6 + (b - a)a^5 - (b - a)^2 a^4 + (b - a)^3 a^3 \\ &\qquad - (b - a)^4 a^2 + (b - a)^5 b\end{aligned} \tag{6.9}$$

and so on.

A developing pattern is clearly discernible.

Calculation of d_r follows the method employed in (3.2).

Although only an outline of the theory in Part II is being offered, it is generally desirable for clarity of exposition to exhibit the main points of the calculation of d_r in a little detail, even at the risk of some possibly superfluous documentation.

Theorem: $d_r = (-1)^{[r/2]}\{a^{r+1} + (-1)^{r-1}(a + b)(b - a)^r\}/b.$ (6.10)

Proof: Expand d_r in (6.3) and (6.4) along the last row to obtain

$$\begin{aligned} d_r &= (-1)^{[r/2]}a^r - (b - a)(-1)^{r-1}d_{r-1} \;\cdots\cdots\; \text{(i)} \\ &= (-1)^{[r/2]}\{a^r - (b - a)a^{r-1}\} - (b - a)^2 d_{r-2} \qquad \text{by (i), (3.9)} \\ &= (-1)^{[r/2]}\{a^r - (b - a)a^{r-1} + (b - a)^2 a^{r-2} - (b - a)^3 a^{r-3} + \cdots \\ &\qquad + (-1)^{r-2}(b - a)^{r-2}a^2 + (-1)^{r-1}(b - a)^{r-1}b \\ &\qquad + [(-1)^{r-1}(b - a)^{r-1}a - (-1)^{r-1}(b - a)^{r-1}a]\} \\ &= (-1)^{[r/2]}\{a^r(1 - (b - a)a^{-1} + (b - a)^2 a^{-2} - (b - a)^3 a^{-3} + \cdots \\ &\qquad + (-1)^r(b - a)^{r-2}a^{-(r-2)} + (-1)^{r-1}(b - a)^{r-1}a^{-(r-1)}) \\ &\qquad + (-1)^{r-1}(b - a)^r\} \\ &= (-1)^{[r/2]}\left\{a^r\frac{[1 - (-(b - a)a^{-1})^r]}{1 - (-(b - a)a^{-1})} + (-1)^{r-1}(b - a)^r\right\} \\ &- (-1)^{[r/2]}\{a^{r+1} + (-1)^{r-1}(a + b)(b - a)^r\}/b. \end{aligned}$$

Repeated use of (i) has been made in the proof. Also, the summation formula for a finite geometric progression has been invoked.

Applying next the arguments used in the evaluation of the Lucas determinant or order r, Δ_r, we easily have

Theorem: $D_r = (-1)^m d_r = (-1)^{m+[r/2]}\{a^{r+1} + (-1)^{r-1}(a+b)(b-a)^r\}/b,$ (6.11)

where $\begin{cases} m = n'(r - n') \\ n' \equiv n \mod r. \end{cases}$

Proof: As for (4.3).

For the Lucas sequence of order r, $\{S_n\}$,

$$a = 2,\ b = 1 \quad (\text{so } a + b = 3,\ b - a = -1).$$

It is easy to verify that, in this case,

$$d_r = \delta_r, \quad D_r = \Delta_r.$$

[Cf. (3.10), (4.4).]

Coming now to the case of the Fibonacci sequence of order r, $\{R_n\}$, we have

$$a = 0,\ b = 1 \quad (\text{so } a + b = 1,\ b - a = 1).$$

It is important to note that, for our theory to be used for $\{R_n\}$, the terms of $\{R_n\}$ with negative suffixes have to be extended by one term in the definition (1.1), (1.2) given by Hoggatt and Bicknell [1], namely,

$$R_{-(r-1)} = 1. \tag{6.12}$$

Augmenting $\{R_n\}$ by this single element enables us to construct *basic Fibonacci determinants* of order r, ∇_r, for $\{R_n\}$ derived from (1.3) analogously to those for $\{S_n\}$ from (2.6). Computation yields

$$\nabla_2 = 1,\ \nabla_3 = -1,\ \nabla_4 = -1,\ \nabla_5 = 1,\ \nabla_6 = 1.$$

To give some appreciation of the appearance of the ∇_r, choose $r = 6$, so

$$\nabla_6 = \begin{vmatrix} 8 & 4 & 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \quad (= 1) \tag{6.13}$$

which is rather simpler than the corresponding form (3.7) for δ_6.

Putting $a = 0$, $b = 1$ in (6.10), we have, with the aid of (3.9)

$$d_r = (-1)^{[r/2]+r-1} = (-1)^{[(r-1)/2]} = \nabla_r.$$

Now it may be shown that $m + [(r-1)/2]$, the power of -1 in (6.11), and $(r-1)n + [(r-1)/2]$, the corrected power of -1 in the Hoggatt-Bicknell [1] evaluation in (1.3), are both even or both odd. That is

$$(-1)^{m+[r/2]} = (-1)^{(r-1)n+[(r-1)/2]}. \tag{6.14}$$

Thus, $a = 0$, $b = 1$ in (6.11) with (3.9) give

$$D_r = (-1)^{m+[(r-1)/2]} = (-1)^{(r-1)n+[(r-1)/2]} = \nabla_r,$$

where ∇_r is the symbol to represent the Hoggatt-Bicknell determinant (1.3).

Suppose we check for $\{R_n\}$ when $r = 6$, $n = 3$, i.e., we are dealing with the sequence

$$\begin{cases} R_{-5} & R_{-4} & R_{-3} & R_{-2} & R_{-1} & R_0 & R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 8 & 16 & 32 & 63 & \ldots \end{cases} \tag{6.15}$$

Then

$$\begin{aligned} (n = 3):\quad D_6 &= (-1)^{3\times 3+2} = -1 && \text{by (6.11), (3.9)} \\ &= (-1)^{5\times 3+2} = -1 && \text{by (1.3)} \\ &= -\nabla_6 \qquad\quad = -1 && \text{on direct calculation.} \end{aligned}$$

Geometrical considerations similar to those in [4] and in Part I of this article are now applicable to the general case of $\{H_n\}$ when a and b are unspecified, and also to the multifarious special cases of $\{H_n\}$ occurring when a and b are given particular values.

But we do not proceed *ad infinitum, ad nauseam* by discussing other classes of sequences. Unsatiated readers, if such there be, may indulge to surfeit in such an algebraic geometry orgy.

One further generalization might be contemplated if, in (6.1), we were to associate with each H_{n+r-j} $(j = 1, 2, \ldots, r)$ a nonzero, nonunity factor p_j. However, the algebra involved makes this a daunting prospect.

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*To the list of references, we take the liberty of adding the monumental text [5] by Room, though its subject is determinantal loci in projective, not Euclidean, spaces.

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ON THE OCCURRENCES OF FIBONACCI SEQUENCES IN THE COUNTING OF MATCHINGS IN LINEAR POLYGONAL CHAINS

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1. INTRODUCTION

The graphs considered here will be finite and will have no multiple edges. Let G be such a graph. A *matching* in G is a spanning subgraph whose components are nodes and edges only. We define a *k-matching* in G to be a matching with k edges. When the matching consists of edges only, it will be called a *perfect matching*. The number of perfect matchings in G will be denoted by $\gamma(G)$. The total number of matchings in G will be denoted by $\tau(G)$.

The following example illustrates the above definitions.

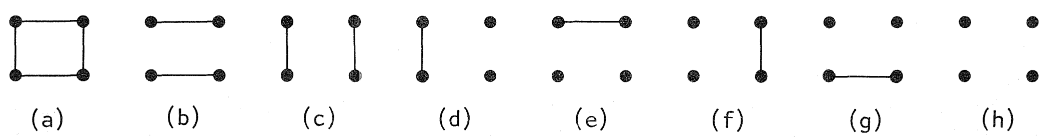


Figure 1

The graph G shown in Figure 1(a) has two perfect matchings [graphs (b) and (c)]. Therefore $\gamma(G) = 2$. G has four 1-matchings [graphs (d), (e), (f), and (g)] and one 0-matching [graph (h)]. Hence G has 7 matchings; i.e., $\tau(G) = 7$.

By a *polygonal chain* $P_{k,n}$, we will mean the graph obtained by concatenating n k-gons in such a manner that adjacent k-gons (*cells*) have exactly one edge in common. Also, for $k > 3$, no three cells have a common node.

If the first and last cells (cells which are adjacent to exactly one other cell) of $P_{k,n}$ are joined together, so that they have exactly one edge in common, the "circular" structure obtained will be called a *long polygonal chain* $C_{k,n}$. n is called the *length* of the chain. Notice that in $C_{k,n}$, every cell will be adjacent to exactly two cells.

It is clear that different polygonal chains will result, according to the manner in which the cells are concatenated. For example, in the following diagram we show four nonisomorphic versions of $P_{5,4}$.

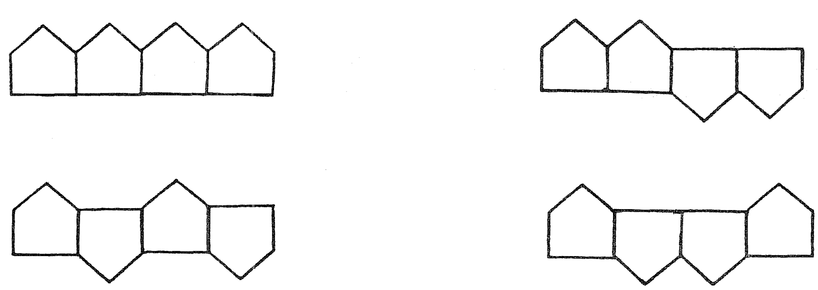


Figure 2

Notice, however, that when $k = 4$ there is only one polygonal chain, $P_{4,n}$. We can also define $P_{4,n}$ as the graph obtained by joining the corresponding nodes of two equal paths with n nodes. We refer to one path as the *upper path* and the other as the *lower path*. The edges in the upper path will be called *upper edges* and those in the lower path will be called *lower edges*.

We define the *linear polygonal chain* $S_{k,n}$ $(k > 3)$ to be the graph obtained from $P_{4,n}$ as follows. If k is even, then every upper and lower edge is replaced by a path of length $(k - 2)/2$. If k is odd, then every upper edge is replaced by a path of length $(k - 1)/2$, and every lower edge is replaced by a path of length $(k - 3)/2$. For n-even, $S_{3,n}$ is obtained from $P_{4,(n/2)}$ by joining diagonally opposite nodes in a consistent direction. For n-odd, $S_{3,n}$ is obtained from $S_{3,n+1}$ by removing a node of valency 2. The *long linear polygonal chain* $L_{k,n}$ is analogously obtained from $S_{k,n}$, as $C_{k,n}$ is obtained from $P_{k,n}$.

Linear polygonal chains have been the subject of numerous investigations. Their matching polynomials were extensively investigated (see [1, 2, 3, 4, 5]). Polygonal chains have also been called animals, and are special cases of the general animal defined in Harary and Palmer [7]. During investigations of the matching polynomials of linear polygonal chains, it was observed that the number of perfect matchings, and in some cases the total number of matchings, were Fibonacci numbers. These observations form the basis for this report. We refer the reader to Harary [6] for the basic definitions in Graph Theory.

2. PRELIMINARY RESULTS

Let G be a graph and xy an edge in G joining nodes x and y. We can partition the perfect matchings in G into two classes: (i) those containing xy and (ii) those not containing xy. The perfect matchings in class (i) will be perfect matchings in the graph G'' obtained from G by removing nodes x and y. Those in class (ii) will be perfect matchings in G', the graph obtained from G by deleting the edge xy. Thus we have the following lemma.

Lemma 1: $\gamma(G) = \gamma(G') + \gamma(G'')$.

Suppose that G consists of two components H and K. Then any perfect matchings in H and K can be combined to yield a perfect matching in G. Conversely, every perfect matching in G can be broken up into a perfect matching in H and a perfect matching in K. Hence we have the following result which generalizes the argument.

Lemma 2: Let G be a graph consisting of r components $H_1, H_2, \ldots, H_r$. Then

$$\gamma(G) = \prod_{i=1}^{r} \gamma(H_i).$$

It is clear that if G is a connected graph with an odd number of nodes, then G cannot have a perfect matching.

Lemma 3: Let G be a graph. If G has an odd number of nodes, then

$$\gamma(G) = 0.$$

Lemma 1 can be very useful for detecting the polygonal chains G for which $\gamma(G)$ is a Fibonacci number. We simply investigate the relations between $\gamma(G')$ and $\gamma(G'')$ and the chains of shorter lengths. Lemma 2 is useful when applying

Lemma 1, since the deletion of an edge from G might yield a disconnected graph. Lemma 3 is useful for reducing the number of graphs to be considered in applications of Lemma 1.

We can use an argument similar to the one preceding Lemmas 1 and 2 to establish the following analogous results.

Lemma 4: $\tau(G) = \tau(G') + \tau(G'')$.

Lemma 5: If G consists of r components $H_1, H_2, \ldots, H_r$, then

$$\tau(G) = \prod_{i=1}^{r} \tau(H_i).$$

Lemmas 1 and 4 yield algorithms for counting perfect matchings and total number of matchings, respectively, in graphs. The algorithms consist of repeated applications of the lemmas until graphs H_i are obtained for which $\gamma(H_i)$ and $\tau(H_i)$, respectively, can be written down. These algorithms will be referred to as *reduction processes*. When applying a reduction process, the graph G' will be referred to as the *edge-deleted graph*. G'' will be referred to as the *node-deleted graph*.

3. THE TRIVIAL CHAINS $S_{1,n}$ AND $L_{1,n}$

We define $P_{1,n}$ to be a tree with nodes of valencies 1 and 2 only. This graph is also called the path or chain P_n. When the end-nodes of P_n are identified, the resulting graph $C_{1,n}$ is called the cycle or n-gon C_n.

Let us apply Lemma 4 to the chain P_n by deleting an edge incident to a node of valency 1. Then G' will contain two components, P_{n-1} and an isolated node P_1. Therefore,

$$\tau(G') = \tau(P_{n-1}) \cdot \tau(P_1) = \tau(P_{n-1}).$$

G'' will be the graph P_{n-2}. Therefore,

$$\tau(G'') = \tau(P_{n-2}).$$

Hence, from Lemma 4, we get

$$\tau(P_n) = \tau(P_{n-1}) + \tau(P_{n-2}).$$

It is clear that $\tau(P_1) = 1$ and $\tau(P_2) = 2$. We define $\tau(P_0) = 1$. Hence we have the following theorem.

Theorem 1: The total number of matchings in the chains P_n form a Fibonacci sequence with initial values $\tau(P_0) = \tau(P_1) = 1$.

Let us apply Lemma 4 to the long chain C_n. In this case, G' will be the graph P_n and G'' will be P_{n-2}. Hence we have

$$\tau(C_n) = \tau(P_n) + \tau(P_{n-2}).$$

Therefore,

$$\tau(C_{n-1}) + \tau(C_{n-2}) = \tau(P_{n-1}) + \tau(P_{n-3}) + \tau(P_{n-2}) + \tau(P_{n-4})$$

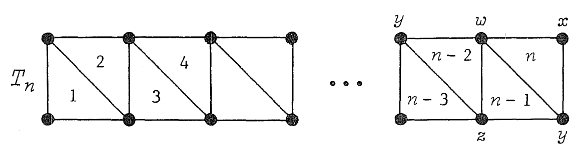
$$= [\tau(P_{n-1}) + \tau(P_{n-2})] + [\tau(P_{n-3}) + \tau(P_{n-4})]$$
$$= \tau(P_n) + \tau(P_{n-2}) = \tau(C_n).$$

Hence we have the following theorem.

Theorem 2: The total number of matchings in the cycles C_n $(n > 2)$ form a Fibonacci sequence with initial values $\tau(C_3) = 4$ and $\tau(C_4) = 7$.

4. TRIANGULAR CHAINS

For brevity of notation, we will denote the linear triangular chain $S_{3,n}$ by T_n. The long triangular chain $L_{3,n}$ (n-even) will be denoted by L_n. The graphs T_n and $L_{3,12}$ are shown below in Figures 3(a) and 3(b), respectively.



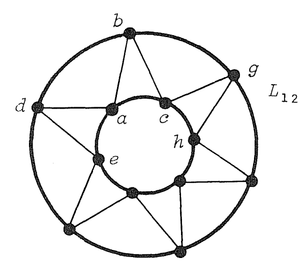


Figure 3

It can be verified that T_n contains $n + 2$ nodes and $2n+1$ edges. Also L_n contains n nodes and $2n$ edges. Therefore, for odd n, T_n and L_n do not have perfect matchings.

Let us apply the reduction process for perfect matchings to the graph T_n (n-even) by deleting the edge xy [see Figure 3(a)]. G' will be T_{n-1} with the edge wx attached to it; G'' will be T_{n-2}. Now, any perfect matching in G' must contain the edge wx since the node x will have valency 1. It follows that the edge zy must also be in every perfect matching of G'. The rest of the perfect matching will be a perfect matching of T_{n-4}. Hence we get

$$\gamma(G') = \gamma(T_{n-4}).$$

Also,

$$\gamma(G'') = \gamma(T_{n-2}).$$

Therefore, from Lemma 1, we get

$$\gamma(T_n) = \gamma(T_{n-2}) + \gamma(T_{n-4}). \tag{1}$$

It can be confirmed that $\gamma(T_2) = 2$ and $\gamma(T_4) = 3$. We define $\tau(T_0)$ to be 1. Hence we have the following theorem.

Theorem 3: The number of perfect matchings in the triangular chains T_n (n-even) form a Fibonacci sequence with boundary values $\gamma(T_0) = 1$ and $\gamma(T_2) = 2$.

Let us apply the reduction process for perfect matchings to the graph L_n by deleting the edge bg [see Figure 3(b)]. G' will be L_n with edge bg removed.

G'' will be L_n with nodes b and g removed. Let us now apply the reduction process to G' by deleting edge bc. Let G_2' be the edge-deleted graph. The graph G_2'' obtained by deleting nodes b and c will be T_{n-4}.

$$\gamma(G_2'') = \gamma(T_{n-4}).$$

Apply the reduction process to G_2' by deleting edge ac. The edge-deleted graph will be T_{n-2}. The node-deleted graph will be T_{n-5} with an edge attached to a node of valency 2. Therefore,

$$\gamma(G_2') = \gamma(T_{n-2}) + \gamma(T_{n-6}).$$

Consider now the graph G''. We can apply the reduction process by deleting edge ac. The edge-deleted graph G_3' will be T_{n-5} with an edge attached to a node of valency 2. Therefore,

$$\gamma(G_3') = \gamma(T_{n-6}).$$

The node-deleted graph will be T_{n-6}. Therefore, we get

$$\gamma(G'') = 2\gamma(T_{n-6}).$$

Hence, by adding the contributions of the final graphs, we obtain the following lemma.

Lemma 6: $\gamma(L_n) = \gamma(T_{n-2}) + \gamma(T_{n-4}) + 3\gamma(T_{n-6})$ (n-even and $n > 4$), with

$$\gamma(T_0) = 1,\ \gamma(T_2) = 2, \text{ and } \gamma(T_4) = 3.$$

The above lemma yields:

$$\begin{aligned}\gamma(L_{n-2}) + \gamma(L_{n-4}) &= \gamma(T_{n-4}) + \gamma(T_{n-6}) + 3\gamma(T_{n-8}) + \gamma(T_{n-6}) \\ &\qquad + \gamma(T_{n-8}) + 3\gamma(T_{n-10}) \\ &= [\gamma(T_{n-4}) + \gamma(T_{n-6})] + [\gamma(T_{n-6}) + \gamma(T_{n-8})] \\ &\qquad + 3[\gamma(T_{n-8}) + \gamma(T_{n-10})] \\ &= \gamma(T_{n-2}) + \gamma(T_{n-4}) + 3\gamma(T_{n-6}), \text{ using Equation (1)} \\ &= \gamma(L_n), \text{ from Lemma 6.}\end{aligned}$$

Thus, we obtain the following result.

Theorem 4: The number of perfect matchings in the long triangular chains L_n (n-even) form a Fibonacci sequence with initial values $\gamma(L_0) = 4$ (by convention), $\gamma(L_2) = 2$, and $\gamma(L_4) = 6$.

5. CHAINS OF HIGHER ORDERS

We will denote by $G - S$ the graph obtained from a graph G by removing a subset $S = \{v_1, v_2, \ldots, v_k\}$ of its nodes. When k is small, we will simply write $G - v_1 - v_2 - \cdots - v_k$.

Let P_r be the path with r nodes. By *attaching* P_r to a connected graph G, we will mean that an end-node of P_r is identified with a node of G to form a graph H_r in which the subgraphs P_r and G are in the same component. We say P_r

is *added* to G when the two end-nodes of P_r are attached to different nodes of G. In this case, we assume that G has more than two nodes. The resulting graph will be denoted by J_r. The nodes of G used in the identification process will be called *nodes of attachment*.

The following lemmas will be useful in the material of this section.

Lemma 7: Let u be the node of attachment of H_r. Then

$$\gamma(H_r) = \begin{cases} \gamma(G - u) & \text{if } r \text{ is even,} \\ \gamma(G) & \text{if } r \text{ is odd.} \end{cases}$$

Proof: Apply the reduction process to H_r by deleting the edge of P_r incident to u. The result follows immediately. ■

Lemma 8: Let u and v be the nodes of attachment of J_r. Then

$$\gamma(J_r) = \begin{cases} \gamma(G) + \gamma(G - u - v) & \text{if } r \text{ is even,} \\ \gamma(G - u) + \gamma(G - v) & \text{if } r \text{ is odd.} \end{cases}$$

Proof: The result follows easily by applying the reduction process by deleting an edge incident to one of the nodes of attachment and then using Lemma 7. ■

The edges of $P_{4,n}$ which join nodes of the upper and lower paths are called *link edges*, and the corresponding nodes are called *link nodes*. A *terminal edge* is a link edge which is incident to link nodes of valency 2. Also, we denote the n^{th} Fibonacci number by F_n: $F_n = F_{n-1} + F_{n-2}$, $F_0 = F_1 = 1$.

Theorem 5: For $n, m, k \geqslant 1$,

(i) $\gamma(S_{4k+2,n}) = n + 1$,

(ii) $\gamma(S_{4k,n}) = F_{n+1}$,

(iii) $\gamma(S_{2k+1,2m+1}) = 0$

(iv) $\gamma(S_{2k+1,2m}) = m + 1$.

Proof:

(i) Apply Lemma 8 to $S_{4k+2,n}$. In this case r is even. We get

$$\gamma(S_{4k+2,n}) = \gamma(S_{4k+2,n-1}) + \gamma(B_1), \tag{2}$$

where B_1 is the graph $S_{4k+2,n-2}$ with P_{2k} attached to the ends of a terminal edge. Using Lemma 7, with r even, we get $\gamma(B_1) = \gamma(B_2)$, where B_2 is $S_{4k+2,n-3}$ with P_{2k} attached to the ends of a terminal edge. By repeated applications of the lemma, we get $\gamma(B_1) = 1$. Therefore, from Equation (2),

$$\gamma(S_{4k+2,n}) = \gamma(S_{4k+2,n-1}) + 1.$$

But $\gamma(S_{4k+2,1}) = \gamma(C_{4k+2}) = 2$. Therefore, we have

$$\gamma(S_{4k+2,n}) = n + 1.$$

(ii) Apply Lemma 8 to $S_{4k,n}$. Again r is even, so we get

$$\gamma(S_{4k,n}) = \gamma(S_{4k,n-1}) + \gamma(A), \tag{3}$$

where A is $S_{4k,n-2}$, with P_{2k-1} attached to the ends of a terminal edge. Using Lemma 7, with r odd, we get

$$\gamma(A) = \gamma(S_{4k,n-2}).$$

Hence, from Equation (3), we obtain

$$\gamma(S_{4k,n}) = \gamma(S_{4k,n-1}) + \gamma(S_{4k,n-2}).$$

Clearly $\gamma(S_{4k,1}) = 2 = F_2$ and $\gamma(S_{4k,2}) = 3 = F_3$. Therefore, we define

$$\gamma(S_{4k,0}) = 1 = F_1.$$

Hence, from Equation (3), we have

$$\gamma(S_{4k,n}) = F_{n+1}.$$

(iii) It can be easily verified that $S_{2k+1,2m+1}$ has an odd number of nodes $[2(2mk+k-m+1)+1]$. Hence, the result follows from Lemma 3.

(iv) First, we will label (in order) the link edges of $S_{2k+1,2m}$ with 1, 2, 3, ..., $2m+1$, beginning with a terminal edge. Let us apply the reduction process to $S_{2k+1,2m}$ by deleting an even labelled link edge. The graph G'' will contain two components; A, consisting of $S_{2k+1,i}$ with the chains P_k and P_{k-1} attached to the ends of a terminal edge, and B, consisting of $S_{2k+1,j}$ with P_k and P_{k-1} attached to the ends of a terminal edge. Clearly, $i + j = 2m - 2$ and both i and j will be even. It can be easily confirmed that A will contain $2ik + 2k - i - 1$ nodes. Since this is odd, for all even values of i, we get

$$\gamma(A) = \gamma(B) = 0 \Rightarrow \gamma(G'') = 0.$$

Hence, no perfect matching contains an even (labelled) link edge. It follows that

$$\gamma(S_{2k+1,2m}) = \gamma(R_m),$$

where R_m is the polygonal chain $P_{4k,m}$ obtained from $P_{4,m}$ by replacing each upper edge with $2k$ edges and each lower edge with $2k-2$ edges.

Apply Lemma 8 to R_m. This gives

$$\gamma(R_m) = \gamma(R_{m-1}) + \gamma(B), \tag{4}$$

where B is the graph R_{m-2} with P_{2k} and P_{2k-2} attached to the ends of a terminal edge. Hence, by an analysis similar to that used in establishing (i), we get

$$\gamma(R_m) = m + 1 = \gamma(S_{2k+1,2m}). \blacksquare$$

We now give bounds for general polygonal chains comprising $(2k+1)$-gons.

Theorem 6: For $m, k \geqslant 1$, $m + 1 \leqslant \gamma(P_{2k+1,2m}) \leqslant F_{m+1}$.

Proof: Let us construct $P_{2k+1,2m}$ from $P_{4,2m}$ by replacing the first pair of upper and lower edges with P_{k+1} and P_k, respectively, the second pair by P_k and P_{k+1}, respectively, the third pair by P_{k+1} and P_k, respectively, and so on.

As we have shown above [(iii) of Theorem 5], no even link edge can belong to a perfect matching. Therefore, we can remove all the even labelled edges to obtain a graph which contains the same number of perfect matchings as $P_{2k+1,2m}$. In this case, the graph will be $S_{4k,m}$. Therefore,

$$\gamma(P_{2k+1,2m}) = \gamma(S_{4k,m}) = F_{m+1}, \text{ by (ii) of Theorem 5.} \tag{5}$$

When $P_{2k+1,2m}$ is the graph $S_{2k+1,2m}$, we get

$$\gamma(P_{2k+1,2m}) = \gamma(S_{2k+1,2m}) = m + 1, \text{ by (i) of Theorem 5.}$$

It can be seen from the proof of Theorem 5(iv) that, in the general case, the minimum value of $\gamma(B)$ in (4) is 1 and the maximum value is $\gamma(P_{2k+1,2m-2})$, and the result follows. ∎

The following theorem is the long-chain analogue of Theorem 5.

Theorem 7: For $k \geqslant 1$, $m \geqslant 2$, and $n \geqslant 3$,

(i) $\gamma(L_{4k+2,n}) = 4$,

(ii) $\gamma(L_{4k,2m}) = \gamma(S_{4k,2m-1}) + \gamma(S_{4k,2m-3}) + 2 = F_{2m} + F_{2m-2} + 2$,

(iii) $\gamma(L_{4k,2m+1}) = \gamma(S_{4k,2m}) + \gamma(S_{4k,2m-1}) = F_{2m+1} + F_{2m-1}$,

(iv) $\gamma(L_{2k+1,2m-1}) = 0$,

(v) $\gamma(L_{2k+1,2m}) = 4$.

Proof:

(i) It can be easily confirmed that no perfect matching in $L_{4k+2,n}$ can contain a link edge. Therefore,

$$\gamma(L_{4k+2,n}) = \gamma(A) = 4,$$

where A is the graph consisting of two disjoint cycles each with $2kn$ nodes.

(ii) and (iii), $k > 1$: Apply the reduction process of $L_{4k,r}$ by deleting the second upper edge (counting from the edge adjacent to a link edge) of a cell. Continue to apply the reduction process in the same nammer to both G' and G'', but this time using the corresponding lower edge. The four resulting graphs will be the following: (1) A_{r-1}, consisting of the graphs $S_{4k,r-1}$ with P_2 attached to each end of a terminal edge and P_{2k-2} attached to the ends of the other terminal edge; (2) B_{r-1}, consisting of the graph $S_{4k,r-1}$ with P_2 and P_{k-2} attached to its two upper terminal nodes and P_{2k-3} attached to the other end of the terminal edge adjacent to an edge of P_{k-2} (note that B_{r-1} will occur twice); and (3) D_{r-1}, the graph $S_{4k,r-1}$ with the odd chain P_{2k-3} attached to the ends of a terminal edge.

It can be confirmed that:

1. $\gamma(A_{r-1}) = \gamma(S_{4k,r-3})$;

2. $\gamma(B_{r-1}) = \begin{cases} 1 & \text{if } r \text{ is even,} \\ 0 & \text{if } r \text{ is odd;} \end{cases}$

3. $\gamma(D_{r-1}) = \gamma(S_{4k,r-1})$.

For $k = 1$, the reduction process can be applied by deleting any upper edge. The graphs corresponding to A_{r-1}, B_{r-1}, and D_{r-1} will be $S_{4k,r-1}$, $S_{4k,r-3}$ with P_2 attached to the two upper terminal nodes, and $S_{4k,r-3}$, respectively. Hence, for $k \geqslant 1$, we get

$$\gamma(L_{4k,r}) = \gamma(S_{4k,r-1}) + \gamma(S_{4k,r-3}) + \delta,$$

where $\delta = \begin{cases} 2 & \text{if } r \text{ is even,} \\ 0 & \text{if } r \text{ is odd.} \end{cases}$

The results (ii) and (iii) then follow from Theorem 5(ii).

(iv) It can easily be verified that $L_{2k+1,2m+1}$ has an odd number of nodes. Therefore, the result follows.

(v) This is similar to Theorem 5(iv). ■

Theorem 8: For $m \geqslant 2$ and $k \geqslant 1$, $4 \leqslant \gamma(C_{2k+1,2m}) \leqslant 2(F_m + F_{m-2})$.

Proof: The proof is similar to that of Theorem 6. It follows by applying the reduction process to $C_{2k+1,2m}$, then using Equation (5) and Thoerem 7(v). ■

Note that Theorems 3 and 4 are special cases of Theorems 6 and 8, respectively, when $k = 1$.

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THE FIBONACCI RATIO IN A THERMODYNAMICAL PROBLEM: A COMBINATORIAL APPROACH

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In a previous contribution to this journal [1], the author showed how the Fibonacci ratio arises in the solution of a particular thermodynamical problem, namely, the calculation of the entropy of a chain of electrons localized onto lattice sites with density one, and with the constraints that half the lattice sites may contain at most two electrons each, while the other half may contain at most only one electron each. The use of the thermodynamical grand-canonical formulation [1], while simplifying the calculation, greatly obscured the purely combinatorial nature of the problem, which we think is by itself a fascinating one, and which we purport here to present.

The problem in [1] might be restated as follows: Given $2N$ different boxes, and $2N$ identical coins, with half the boxes containing at most two coins each, and the other half containing at most one coin each, in how many different ways can one arrange or put the $2N$ coins into the $2N$ boxes, as $N \to \infty$? Although a single coin may be put into a box in two different ways, as head or tail, we shall agree that once we put two coins into a box we shall not inquire as to which is head or which is tail, and shall count that arrangement as only one.

With this understanding, it is straightforward to show, in a purely combinatorial manner, that the total number of arrangements $A(N)$ of the $2N$ coins in the $2N$ boxes, is given by

$$A(N) = 2^{2N} \sum_{k=N}^{2N} 2^{-k} \binom{2N}{k} \binom{N}{2N-k}, \tag{1}$$

for $N = 1, 2, 3, \ldots$. The above expression is exact and holds for any $N \geqslant 1$; a proof of (1) is given in the text.

The Fibonacci ratio arises from (1) through the "entropy" $S(N)$ associated with the number of arrangements $A(N)$, i.e.,

$$S(N) \equiv \ln A(N), \tag{2}$$

and the extensive property of the entropy in the thermodynamic limit ($N \to \infty$), i.e.,

$$\lim_{N \to \infty} \frac{S(N)}{N} = \ln k, \text{ with } k \equiv f^5, \tag{3}$$

a constant independent of N, where here $f \equiv (1 + \sqrt{5})/2$ is the positive Fibonacci ratio.

In the remainder of this paper we give the proofs of Equations (1) and (3). Equation (1) can be proved by the use of the well-known generating function method [2] of combinatorial analysis.

Thus, $A(N)$ will be the coefficient of an x^{2N} in the expansion in powers of x of the appropriate generating or enumerating function,

$$G(x) = (1 + 2x)^N (1 + 2x + x^2)^N = (1 + 2x)^N (1 + x)^{2N}. \tag{4}$$

In the enumerating function $G(x)$ in (4), the enumerating factor $(1 + 2x)^N$ takes account of the N boxes that contain at most one coin each, while the enumerating factor $(1 + x)^{2N}$ takes account of the N boxes that contain at most two coins each. But,

$$G(x) = \left(\sum_{\ell=0}^{N}\binom{N}{\ell}(2x)^{\ell}\right)\cdot\left(\sum_{m=0}^{2N}\binom{2N}{m}x^{m}\right) = \sum_{\ell=0}^{N}\sum_{m=0}^{2N}2^{\ell}\binom{N}{\ell}\binom{2N}{m}x^{\ell+m}. \tag{5}$$

The coefficient of x^{2N} will thus be given by the sum of all the terms in (5), such that $\ell + m = 2N$, i.e., by

$$\sum_{m=N}^{2N} 2^{2N-m}\binom{N}{2N-m}\binom{2N}{m},$$

which proves Equation (1).

We now proceed to give the proof of Equation (3). When $N \to \infty$, each term in the right-hand side of Equation (1), with $N \leqslant k \leqslant 2N$, is a product of a very rapidly increasing function of k, namely $[(2N - k)!]^{-2}$, times a very rapidly decreasing function of k, namely $[2^k k!(k - N)!]^{-1}$. In the thermodynamic limit, $N \to \infty$, the product of the two functions will have an extremely sharp maximum for some value of k, with $N \leqslant k \leqslant 2N$. In the limit $N \to \infty$, the entire right-hand side summation in Equation (1) can thus be replaced by the maximum term in the same summation, in an asymptotically exact manner.

We will then have

$$\frac{S(N)}{N} = 2\ln 2 + \frac{1}{N}\ln\left(\sum_{k=N}^{2N}P(k;\,N)\right), \tag{6}$$

where

$$P(k;\,N) \equiv 2^{-k}\binom{2N}{k}\binom{N}{2N-k},$$

and hence

$$\lim_{N\to\infty}\frac{S(N)}{N} = 2\ln 2 + \lim_{N\to\infty}\frac{1}{N}[\ln P(k;\,N)]_{\text{Max.}}, \tag{7}$$

where we have used $\ln[P(k;\,N)]_{\text{Max.}} = [\ln P(k;\,N)]_{\text{Max.}}$. But

$$\begin{aligned}\ln P(k;\,N) = -k\ln 2 + \ln(2N)! - \ln k! - \ln(2N-k)!\\ + \ln N! - \ln(2N-k)! - \ln(k-N)!,\end{aligned} \tag{8}$$

and by the Stirling approximation, we have that (for $N \to \infty$)

$$\ln N! = N(\ln N - 1) + \frac{1}{2}\ln N + C_N, \tag{9}$$

where C_N is a number of the order of unity.

Then, we will obviously have that

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}[\ln P(k;\,N)]_{\text{Max.}} &= \left[\lim_{N\to\infty}\frac{1}{N}\ln P(k;\,N)\right]_{\text{Max.}}\\ &= \left\{\lim_{N\to\infty}\frac{1}{N}[-k\ln 2 - k(\ln k - 1) - 2(2N-k)(\ln(2N-k) - 1)\right.\\ &\quad \left. - (k-N)(\ln(k-N) - 1) + Q(N)]\right\}_{\text{Max.}} \equiv [\mathscr{P}(k;\,N)]_{\text{Max.}}\,,\end{aligned} \tag{10}$$

where

$$Q(N) = 2N \ln 2 + 3N(\ln N - 1) \tag{11}$$

is a function of N only.

It is important to notice that the terms in

$$\frac{1}{2} \ln N + C_N$$

in the Stirling approximation in (9) and the corresponding ones for $(2N - k)!$, $k!$, and $(k - N)!$ contribute *nothing* to the limit in (10), since, typically,

$$\lim_{N \to \infty} \frac{1}{N}\left(\frac{1}{2} \ln N + C_N\right) = 0.$$

To find the maximum of the function $\mathscr{P}(k; N)$ defined in (10), we find the value of k, such that

$$\frac{d}{dk}\mathscr{P}(k; N) = 0, \tag{12}$$

where N is considered as a parameter. This value of k is then substituted back into $\mathscr{P}(k; N)$ to give $(\mathscr{P}(k; N))_{\text{Max.}}$.

Interchanging the derivative with the limit in (10), Equation (12) leads to

$$-\ln 2 - \ln k + 1 - \frac{k}{k} + 2(\ln(2N - k) - 1)$$
$$+ 2\,\frac{(2N - k)}{(2N - k)} - \ln(k - N) + 1 - \frac{(k - N)}{(k - N)} = 0, \tag{13}$$

or

$$-\ln 2 - \ln k + 2 \ln(2N - k) - \ln(k - N) = 0, \tag{14}$$

for k.

Finally, Equation (14) leads to

$$\frac{(2N - k)^2}{2k(k - N)} = 1,$$

or the quadratic equation

$$\left(\frac{k}{2N}\right)^2 + \left(\frac{k}{2N}\right) - 1 = 0, \tag{15}$$

for k. Because k must be a positive number, the only appropriate solution of (15) is

$$\frac{k}{2N} = \frac{1}{f}, \qquad (N \leqslant k \leqslant 2N),$$

where f is the positive Fibonacci ratio. Hence, we will have

$$(\mathscr{P}(k; N))_{\text{Max.}} = \lim_{N \to \infty} \frac{1}{N}\Bigg[-\frac{2N}{f} \ln 2 - \frac{2N}{f}\left(\ln\left(\frac{2N}{f}\right) - 1\right) - 2\left(2N - \frac{2N}{f}\right)\left(\ln\left(2N - \frac{2N}{f}\right) - 1\right)$$
$$- \left(\frac{2N}{f} - N\right)\left(\ln\left(\frac{2N}{f} - N\right) - 1\right) + Q(N)\Bigg] =$$

(continued)

$$= -\frac{2}{f}\ln 2 - \frac{2}{f}\ln 2 + \frac{2}{f}\ln f + \frac{2}{f} - \frac{4}{f}(f - 1)(\ln 2 + \ln(f - 1) - \ln f - 1)$$

$$- \frac{(2 - f)}{f}(\ln(2 - f) - \ln f - 1) + 2\ln 2 - 3$$

$$= -2\ln 2 + 3\ln f + \frac{4}{f}\ln(f - 1) - 4\ln(f - 1) - \frac{2}{f}\ln(2 - f) + \ln(2 - f). \tag{16}$$

By using the relationships $\ln(f - 1) = -\ln f$ and $\ln(2 - f) = -2\ln f$, which hold true for the Fibonacci ratio f, (16) finally leads to the remarkably simple result

$$(\mathscr{P}(k; N))_{\text{Max.}} = -2\ln 2 + 5\ln f,$$

or, finally,

$$\lim_{N\to\infty}\frac{S(N)}{N} = 5\ln f.$$

This proves Equation (3) and coincides, of course, with the result obtained in [1] through the use of the grand-canonical formalism.

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A FURTHER NOTE ON PASCAL GRAPHS

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1. INTRODUCTION

In a recent paper Deo and Quinn [1], in their search for a class of graphs to be used as computer networks, introduced Pascal graphs that are constructed using Pascal's triangle modulo 2 [3]. They derived a number of useful results for Pascal graphs and Pascal matrices and, in the conclusion, they made certain interesting conjectures. The objective of the present note is to find an exact expression for the number of edges in Pascal graphs of different orders and to settle one of the conjectures made in [1].

We have used standard graph theoretic terms [4], [5] in this paper, and the reader is assumed to be familiar with [1].

2. BASIC CONCEPTS

Definition 1: A Pascal matrix PM_n of order n is defined to be an $n \times n$ symmetric binary matrix where the main diagonal entries are all 0's and the lower triangular part of the matrix consists of the first $(n - 1)$ rows of Pascal's triangle modulo 2. $PM_n(i, j)$ denotes the $(i, j)^{\text{th}}$ element of PM_n. A Pascal graph PG_n having n vertices is a graph corresponding to the adjacency matrix PM_n.

Remark: This definition of a Pascal matrix is the same as in [1] in contrast to that in [2].

Definition 2: The generator polynomial of the m^{th} row, $m \geqslant 1$, of a Pascal matrix PM_n of any fixed order $n \geqslant m$ is defined to be a polynomial $f_m(x)$ with binary coefficients such that $PM_n(m, j)$ is given by the coefficient of x^{j-1} in $f_m(x)$, $1 \leqslant j \leqslant n$.

Since $PM_n(m, m) = 0$ by definition, we can write, for a Pascal matrix PM_n, $n \geqslant m$,

$$f_m(x) = \begin{cases} g_m(x) + x^m h_m(x), & \text{for } n > m \\ g_m(x), & \text{for } n = m, \end{cases}$$

where $g_m(x)$ and $h_m(x)$ are the generator polynomials of the lower and the upper triangular parts, respectively, of the m^{th} row in PM_n. By definition, $g_m(x)$ applies only for $m \geqslant 2$.

Definition 3: The B-sequence of a positive integer n is defined as the strictly decreasing sequence $B(n) = (p_1, p_2, \ldots, p_{L_n})$ of L_n nonnegative integers such that

$$n = \sum_{j=1}^{L_n} 2^{p_j},$$

where L_n is the length of the sequence.

Remarks: (1) The B-sequence of any positive integer n gives the positions of 1's in the binary representation of n in decreasing order.

(2) The B-sequence of zero is defined to be a null sequence.

Lemma 1: For any Pascal matrix PM_n with $n \geqslant m$,

(a) For $m \geqslant 2$, $g_m(x) = \prod_{j \in B(m-2)} (1 + x^{2^j})$.

(b) For $m \geqslant 1$, $h_m(x) = \prod_{\substack{j \geqslant 0 \\ j \notin B(m-1)}} (1 + x^{2^j})$.

Proof:

(a) From the definitions of a Pascal matrix and $g_m(x)$, it is apparent that $g_m(x) = (1 + x)^{m-2}$, with the coefficients computed in the modulo 2 field, from which the proof follows.

(b) Since PM_n is symmetric, $h_m(x)$ will contain x^k as a nonzero term iff $g_{m+k+1}(x)$ contains x^{m-1} as a nonzero term, $k \geqslant 0$. This is possible if and only if $B(m+k-1)$ contains $B(m-1)$ as a subsequence, i.e., when there is no element common to both $B(k)$ and $B(m-1)$. Hence the claim.

Example: In a Pascal matrix of order $n = 30$,

$$f_{13}(x) = (1 + x)(1 + x^2)(1 + x^8) + x^{13}(1 + x)(1 + x^2)(1 + x^{16}),$$

$$f_{20}(x) = (1 + x^2)(1 + x^{16}) + x^{20}(1 + x^4)(1 + x^8).$$

Remark: For any m, $m \geqslant 2$, $(1 + x)$ is a factor of $g_m(x)$ iff $(1 + x)$ is also a factor of $h_m(x)$, since $B(n)$, $n > 0$, can contain 0 only when n is odd.

Definition 4: The m^{th} row of PM_n will be called the p^{th} instance of all 1's in the lower triangle if $m = 2^p + 1$, $p \geqslant 1$.

3. NUMBER OF EDGES IN PASCAL GRAPHS

Let $e(n)$ denote the number of edges in PG_n. Deo and Quinn [1] showed that

$$e(n) \leqslant \lfloor (n - 1)^{\log_2 3} \rfloor.$$

In this section we find an exact expression for $e(n)$.

Lemma 2: In a Pascal graph PG_n, where

$$n = (2^p + 1) + i,$$

for some nonnegative integer p and $1 \leqslant i \leqslant 2^p$, the degree $d(n)$ of the n^{th} vertex in PG_n is given by

$$d(n) = 2d(i + 1),$$

where $d(i + 1)$ denotes the degree of the $(i + 1)^{\text{st}}$ vertex in PG_{i+1}.

Proof: In PM_n, the n^{th} row has only its lower triangular part and so does the $(i + 1)^{\text{th}}$ row in PM_{i+1}. Hence, in PM_n,

$$f_n(x) = g_n(x) = (1 + x)^{n-2} = (1 + x)^{2^p} \cdot (1 + x)^{i-1}.$$

Since the coefficients of the polynomials are computed in a modulo 2 field, we get

$$g_n(x) = (1 + x^{2^p}) \cdot g_{i+1}(x).$$

Therefore, since $i \leqslant 2^p$, the number of nonzero terms in $g_n(x)$ is twice that in $g_{i+1}(x)$. Hence $d(n) = 2d(i + 1)$. Q.E.D.

If the n^{th} row of PM_n corresponds to the p^{th} instance $(p \geqslant 1)$ of all 1's in its lower triangular part, i.e., if $n = 2^p + 1$, then we also denote the number of edges in PG_n by $E(p)$, i.e., $E(p) = e(2^p + 1)$.

Lemma 3: $E(p) = 3^p$.

Proof:

$$\begin{aligned} E(p) &= \text{Number of edges in } PG \text{ of order } (2^{p-1} + 1) \\ &\quad + \text{Number of edges added due to addition of extra } 2^{p-1} \text{ vertices} \\ &= E(p - 1) + 2E(p - 1) \qquad \text{[by Lemma 2]} \\ &= 3E(p - 1) \\ &= 3^2E(p - 2) = \cdots = 3^{p-1}E(1). \end{aligned}$$

Now $E(1)$ corresponds to the number of edges in PG_3, which is 3. Hence, we get

$$E(p) = 3^p. \quad \text{Q.E.D.}$$

Theorem 1: The number of edges in PG_n $(n > 1)$ is given by

$$e(n) = \sum_{j=1}^{L_{n-1}} 2^{j-1} \cdot 3^{p_j},$$

where $(p_1, p_2, \ldots, p_{L_{n-1}})$ is the B-sequence $B(n - 1)$ of length L_{n-1}.

Proof: Let $n - 1 = n_1 + n_2 + \cdots + n_k$, where $k = L_{n-1}$, $n_i = 2^{p_i}$, $1 \leqslant i \leqslant k$. Hence the $(n_1 + 1)^{\text{th}}$ row of PM_n corresponds to the $p_1{}^{\text{th}}$ instance of all 1's in the lower triangle, and so by Lemmas 2 and 3,

$$\begin{aligned} e(n) &= E(p_1) + \text{extra edges due to addition of vertices } v_{n_1+2}, \ldots, v_n \\ &\qquad \text{to } PG_{n_1+1} \\ &= 3^{p_1} + 2e(n_2 + n_3 + \cdots + n_k + 1) = 3^{p_1} + 2e(n'), \end{aligned}$$

where $n' = (1 + n_2) + n_3 + \cdots + n_k$. Repeating the process, we get

$$\begin{aligned} e(n) &= 3^{p_1} + 2(3^{p_2} + 2e(n'')) \quad [\text{where } n'' = n_3 + n_4 + \cdots + n_k + 1] \\ &= \cdots \\ &= \sum_{j=1}^{k} 2^{j-1} \cdot 3^{p_j}. \quad \text{Q.E.D.} \end{aligned}$$

4. DETERMINANTS OF PASCAL MATRICES

We now settle one of the conjectures made in [1] regarding the values of n for which the determinant of PM_n will be zero.

Let us consider an integer A which is either 2 or can be expressed as

$$A = 2 + (a_1 + a_2 + \cdots + a_k),$$

where $a_1 = 4$ or 8, and $a_{i+1} = 4a_i$, $1 \leqslant i \leqslant k$.

We now define the *index set* I of A as follows:

$$I = \begin{cases} \varphi, & \text{for } A = 2 \\ \{1, 2, \ldots, k\}, & \text{otherwise} \end{cases}$$

Let the cardinality of I be denoted by t. It can be verified that for $A > 2$, we can write

$$A = 1 + \sum_{j=0}^{t} 4^j, \quad \text{for } a_1 = 4 \tag{1}$$

and

$$A = 2\sum_{j=0}^{t} 4^j, \quad \text{for } a_1 = 8. \tag{2}$$

Both (1) and (2) also apply for $A = 2$, i.e., for $t = 0$ as well. Let $a_i' = 2a_i$, for $1 \leqslant i \leqslant k$. We use an arbitrary subset $I' = \{j_1, j_2, \ldots, j_p\}$ of I to denote different integers generated from A as follows:

$$A_{j_1 j_2 \ldots j_p} = 2 + \sum_{i \in I'} a_i' + \sum_{i \in I - I'} a_i.$$

When $\{j_1, j_2, \ldots, j_p\} = \varphi$, $A_{j_1 j_2 \ldots j_p} = A$ itself.

Let $P(I)$ be the power set of I. We define the expansion set $S(A)$ of A as

$$S(A) = \left\{A_{j_1 j_2 \ldots j_p} \middle| \{j_1, j_2, \ldots, j_p\} \in P(I)\right\}.$$

Example: For $A = 2$, $S(A) = \{2\}$.

For $A = 22$, $a_1 = 4$, $a_2 = 16$, $a_1' = 8$, $a_2' = 32$,

$$\begin{aligned} A_1 &= 2 + a_1' + a_2 = 26, \\ A_2 &= 2 + a_1 + a_2' = 38, \\ A_{12} &= 2 + a_1' + a_2' = 42, \end{aligned}$$

and $\quad S(A) = \{22, 26, 38, 42\}$.

The *r-distant co-expansion set* $T_r(A)$ of A is defined as

$$T_r(A) = \left\{A_{j_1 j_2 \ldots j_p} + r \middle| A_{j_1 j_2 \ldots j_p} \in S(A)\right\}.$$

We construct a set of polynomials of the form $F_{j_1 j_2 \ldots j_q}^{i_1 i_2 \ldots i_p}$, where

$$\{i_1, i_2, \ldots, i_p\} \subseteq I, \ \{j_1, j_2, \ldots, j_q\} \subseteq I, \ 1 \in \{i_1, i_2, \ldots, i_p\}$$

and

$$\{i_1, i_2, \ldots, i_p\} \cap \{j_1, j_2, \ldots, j_q\} = \varphi,$$

using the following recurrence relation:

$$F^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = F^{i_1 i_2 \ldots i_{p-1}}_{j_1 j_2 \ldots j_q} - F^{i_1 i_2 \ldots i_{p-1}}_{j_1 j_2 \ldots j_q i_p},$$

with

$$F^{1}_{j_1 j_2 \ldots j_q} = (f_{A_{j_1 j_2 \ldots j_q}} - f_{A_{j_1 j_2 \ldots j_q}+2}) - (f_{A_{1 j_1 j_2 \ldots j_q}} - f_{A_{1 j_1 j_2 \ldots j_q}+2}),$$

where the f's are the generator polynomials as given in Definition 2.

It may be noted that if $\{j_1, j_2, \ldots, j_q\} = \varphi$, then the polynomial is represented simply as $F^{i_1 i_2 \ldots i_p}$, i.e., without a subscript. Moreover, the superscript set $\{i_1, i_2, \ldots, i_p\}$ can never be empty.

Example: Let $A = 22$. Hence $I = \{1, 2\}$, $A_1 = 26$, $A_2 = 38$, $A_{12} = 42$,

$$F^1 = (f_A - f_{A+2}) - (f_{A_1} - f_{A_1+2}) = (f_{22} - f_{24}) - (f_{26} - f_{28})$$

$$F^1_2 = (f_{38} - f_{40}) - (f_{42} - f_{44})$$

$$F^{12} = F^1 - F^1_2 = (f_{22} - f_{24}) - (f_{26} - f_{28}) - [(f_{38} - f_{40}) - (f_{42} - f_{44})].$$

In particular, the polynomials of the form $F^{i_1 i_2 \ldots i_p}$ will play an important role in proving the conjecture, as we shall see later on. The recursive computation of such polynomials can be visualized easily with the help of a binary tree. Consider, for example, the computation of F^{123}, which is represented by the binary tree as shown in Figure 1. The leaf nodes represent the generator polynomials corresponding to different rows of the Pascal matrix and each of the non-leaf nodes represents the arithmetic subtraction operation. Some of the non-leaf nodes are labelled, e.g., F^1, F^1_2, F^{12}, etc. The inorder traversal [6] of the subtree rooted at any labelled non-leaf node computes the polynomial denoted by that label.

Let $[\alpha_t, \beta_t]$ be a closed interval of integers given by

$$\alpha_t = 2 + 2\sum_{j=0}^{t} 4^j, \quad \beta_t = 1 + \sum_{j=0}^{t+1} 4^j, \quad t \geqslant 0. \tag{3}$$

Theorem 2: In a Pascal matrix of order n, where n lies within the closed interval $[\alpha_t, \beta_t]$, as defined in (3), the 2^{t+1} rows denoted by the expansion set $S(A)$ and the 2-distant co-expansion set $T_2(A)$ of the integer A as given in (1) are linearly dependent, i.e., the determinant of PM_n for such values of n will be zero.

Proof:

<u>Case 1</u>. $t = 0$

In this case, $\alpha_0 = 4$, $\beta_0 = 6$, and $A = 2$. So $S(A) = \{2\}$ and $T_2(A) = \{4\}$. Since the order n of the Pascal matrix is limited by $\beta_0 = 6$, we write

$$f_2 = 1 + x^2(1 + x^2) \quad \text{and} \quad f_4 = (1 + x^2) + x^4.$$

So

$$f_2 - f_4 = 0.$$

<u>Case 2</u>. $t \geqslant 1$

To prove the linear dependence among the different rows of PM_n, it is sufficient to show that any of the polynomials of the form $F^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ will be zero.

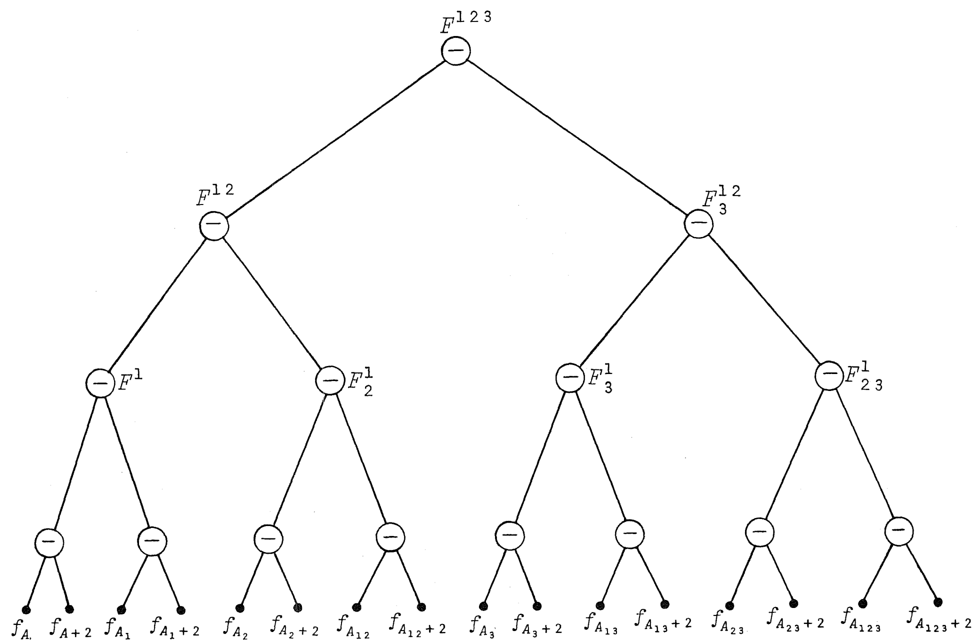

Figure 1

Since the order n of PM_n is limited by β_t, we write

$$f_A = (1 + x^{a_1})(1 + x^{a_2}) \cdots (1 + x^{a_t}) + x^A(1 + x^2)(1 + x^{a_1'})(1 + x^{a_2'}) \cdots (1 + x^{a_t'})$$

and

$$f_{A+2} = (1 + x^2)(1 + x^{a_1}) \cdots (1 + x^{a_t}) + x^{A+2}(1 + x^{a_1'}) \cdots (1 + x^{a_t'}).$$

Hence,

$$f_A - f_{A+2} = -x^2(1 + x^{a_1})(1 + x^{a_2}) \cdots (1 + x^{a_t}) + x^A(1 + x^{a_1'}) \cdots (1 + x^{a_t'}).$$

Similarly,

$$f_{A_1} - f_{A_1+2} = -x^2(1 + x^{a_1'})(1 + x^{a_2}) \cdots (1 + x^{a_t}) + x^{A_1}(1 + x^{a_1})(1 + x^{a_2'}) \cdots (1 + x^{a_t'}).$$

Hence,

$$F^1 = -x^2x^{a_1}(1 - x^{a_1})(1 + x^{a_2}) \cdots (1 + x^{a_t}) + x^A(1 - x^{a_1})(1 + x^{a_2'}) \cdots (1 + x^{a_t'}).$$

Also,

$$\begin{aligned} F_2^1 &= (f_{A_2} - f_{A_2+2}) - (f_{A_{12}} - f_{A_{12}+2}) \\ &= -x^2x^{a_1}(1 - x^{a_1})(1 + x^{a_2'})(1 + x^{a_3}) \cdots (1 + x^{a_t}) \\ &\quad + x^{A_2}(1 - x^{a_1})(1 + x^{a_2})(1 + x^{a_3'}) \cdots (1 + x^{a_t'}) \end{aligned}$$

and

$$\begin{aligned} F^{12} &= F^1 - F_2^1 \\ &= -x^2x^{a_1}x^{a_2}(1 - x^{a_1})(1 - x^{a_2})(1 + x^{a_3}) \cdots (1 + x^{a_t}) \\ &\quad + x^A(1 - x^{a_1})(1 - x^{a_2})(1 + x^{a_3'}) \cdots (1 + x^{a_t'}). \end{aligned}$$

Continuing the process, we get

$$F^{123\cdots t} = -x^{2+a_1+a_2+\cdots+a_t}(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_t})$$
$$+ x^A(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_t}) = 0. \quad \text{Q.E.D.}$$

Let $[\gamma_t, \delta_t]$ be a closed interval of integers given by

$$\gamma_t = 4 + 4\sum_{j=0}^{t} 4^j, \quad \delta_t = 2\sum_{j=0}^{t+1} 4^j, \quad t \geqslant 0. \tag{4}$$

Theorem 3: In a Pascal matrix of order n, where n lies within the closed interval $[\gamma_t, \delta_t]$, as defined in (4), the 2^{t+1} rows denoted by the expansion set $S(A)$ and the 6-distant co-expansion set $T_6(A)$ of the integer A as given in (2) are linearly dependent, i.e., the determinant of PM_n for such values of n will be zero.

Proof: The proof is similar to that of Theorem 2 and is omitted here.

Remarks: (1) $\gamma_t = \beta_t + 2$ and $\alpha_{t+1} = \delta_t + 2$.

(2) For all t, $t \geqslant 0$, $[\alpha_t, \beta_t]$, and $[\gamma_t, \delta_t]$ give two series of intervals of orders of Pascal matrices having zero determinants.

(3) In a Pascal matrix PM_n, where $n = \beta_t + 1$ or $\delta_t + 1$, $t \geqslant 0$, the approach used in the proof of Theorem 2 fails to discover a set of linearly dependent rows. This can be seen as follows:

If $n = \beta_t + 1$, then we must consider terms up to $x^{\beta t}$ in the generator polynomials of the rows of PM_n. Since $\beta_t = A + 4^{t+1} = A + a_{t+1}$, $t \geqslant 0$, only the polynomial f_A will have an added term $(1 + x^{a_{t+1}})$ in its h_A-part; all other polynomials, e.g., f_{A+2}, f_{A_1}, f_{A_1+2}, ..., etc., as given in the proof of Theorem 2, will remain unaltered. Hence, $F^{123\cdots t}$ will not be zero. The case for $n = \delta \;\; + 1$ can be similarly verified.

ACKNOWLEDGMENTS

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ON FIBONACCI k-ARY TREES

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In this paper we extend some results on Fibonacci binary trees to Fibonacci k-ary trees, $k \geqslant 2$. The multinomial coefficients and higher-order Fibonacci numbers are used in our study.

For any integer $k \geqslant 2$, let $\{F_n^k, n \geqslant 0\}$ be a sequence of integers defined by

$$F_0^k = 0,\ F_n^k = 1, \text{ for } 1 \leqslant n \leqslant k,$$

and

$$F_n^k = F_{n-1}^k + F_{n-2}^k + \cdots + F_{n-k}^k, \text{ for } n \geqslant k+1.$$

The sequence $\{F_n^2, n \geqslant 0\}$ is thus the ordinary Fibonacci sequence. For $k \geqslant 3$, the sequence $\{F_n^k, n \geqslant 0\}$ is different from the Fibonacci sequence $\{V_n^k, n \geqslant 0\}$ of order k, which is defined by

$$V_0^k = 0,\ V_1^k = 1,\ V_n^k = 2^{n-2}, \text{ for } 2 \leqslant n \leqslant k,$$

and

$$V_n^k = V_{n-1}^k + V_{n-2}^k + \cdots + V_{n-k}^k, \text{ for } n \geqslant k+1.$$

We also need the following integer sequence. For any integer $k \geqslant 2$ and $1 \leqslant m \leqslant k$, let $\{F_n^{k,m}, n \geqslant -k\}$ be a sequence defined by

$$F_n^{k,m} = 0, \text{ for } n \leqslant 0,$$

$$F_n^{k,m} = 2^{n-1}, \text{ for } 1 \leqslant n \leqslant m,$$

and

$$F_n^{k,m} = F_{n-1}^{k,m} + F_{n-2}^{k,m} + \cdots + F_{n-k}^{k,m}, \text{ for } n \geqslant m+1.$$

It is easy to see that, for any integer $k \geqslant 1$, the sequence $\{F_n^{k,1}, n \geqslant 0\}$ is precisely the Fibonacci sequence of order k, i.e., $F_n^{k,1} = V_n^k$. By induction, it can be shown that, for any $k < n$,

$$\sum_{m=1}^{k} F_{n-k}^{k,m} = F_n^k.$$

For any fixed $k \geqslant 2$ and $n \geqslant 0$, one can obtain multinomial coefficients $c_{n,j}^k$, $0 \leqslant j \leqslant (k-1)n+1$, by expanding the expression

$$(1 + x + x^2 + \cdots + x^{k-1})^n,$$

and obtain the corresponding (generalized) Pascal triangle (see [4], [5], and [6]). For convenience, we set $c_{n,j}^k = 0$, for $j \leqslant -1$ and $j \geqslant (k-1)n+2$. For $k = 2$, one has binary coefficients and the Pascal triangle. For $k = 3$, one has trinomial coefficients $c_{n,j}^3$ and the corresponding generalized Pascal triangle, as shown in Figure 1.

One can draw diagonals in the triangle, and see that the sums of numbers between parallel lines are precisely the 3rd-order Fibonacci numbers V_n^3, just as in the case $k = 2$ [1, p. 245]. In general, by an argument similar to that in [1, p. 246], one has the following relation between Fibonacci numbers V_n^k and multinomial coefficients $c_{n,j}^k$:

$$V_{n+1}^k = \sum_{j=0}^{\lfloor n - n/k \rfloor} c_{n-j,j}^k,$$

where $\lfloor\ \rfloor$ indicates the largest integer function.

1	1										
1	1	1	1								
2	1	2	3	2	1						
4	1	3	6	7	6	3	1				
7	1	4	10	16	19	16	10	4	1		
13	1	5	15	30	45	51	45	30	15	5	1

Figure 1. Generalized Pascal Triangle

For any fixed integer $k \geqslant 2$, we now define the Fibonacci trees T_n^k of order n, $n \geqslant 1$, inductively on n. For $1 \leqslant n \leqslant k$, T_n^k consists of only a root node. For $n \geqslant k + 1$, T_n^k consists of a root node with k ordered sons T_{n-1}^k, T_{n-2}^k, ..., T_{n-k}^k from left to right. For $k = 2$, one has the ordinary Fibonacci trees [2]. For $k = 3$, one has the Fibonacci ternary trees.

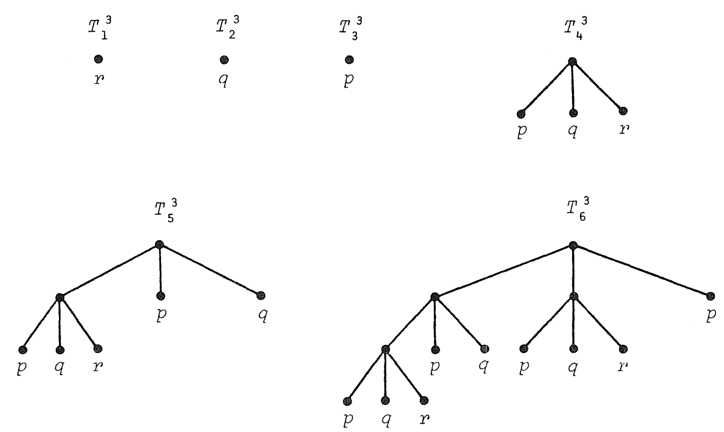

Figure 2. Fibonacci Ternary Trees

In Figure 2, every terminal node of T_n^3 is labelled by p, q, or r. For each $n \geqslant 5$, the tree T_n^3 can be obtained from T_{n-1}^3 by replacing all the labels r and q in T_{n-1}^3 by q and p, respectively, and replacing all the terminal nodes in T_{n-1}^3 with label p by T_4^3. This is a simple rule to grow a Fibonacci tree to a higher order, similar to that given in [3] for Fibonacci binary trees. One can also set a similar rule to grow the trees T_n^k with any fixed $k \geqslant 3$, where, instead of using only three labels, k labels—p_1, p_2, ..., p_k—are needed.

For any $k \geqslant 2$, a k-ary tree is a tree with each internal node containing exactly k ordered sons. We now specify branch costs of a k-ary tree. We will assume that each left-most branch has unit cost 1, each second-to-the-left branch has cost 2, ..., and each right-most branch has cost k. The cost a_i of a node i is the sum of costs of the branches from the root to this node. If the path from the root to a node has ℓ branches, the node is said to be at level ℓ. The average cost of a tree T is defined by

$$s = \sum_{j=1}^{m} a_j/m,$$

where m is the number of terminal nodes in T, and the summation is over all the terminal nodes in T. As in the case $k = 2$ (see [2]), one can see that if a k-ary tree has n internal nodes, then it has $(k - 1)n + 1$ terminal nodes. It is easy to verify the following lemma.

Lemma 1: In a k-ary tree, let a_i be the cost of the terminal node i, and let b_j be the cost of the internal node j. Then

$$s = \frac{\sum_{i=1}^{(k-1)n+1} a_i}{(k - 1)n + 1} = \frac{(k - 1)\sum_{j=1}^{n} b_j + nk(k + 1)/2}{(k - 1)n + 1}.$$

As was stated in [7], one can construct an optimal k-ary tree in the sense of minimum average cost as follows: Suppose that an optimal k-ary tree with $(k - 1)(n - 1) + 1$ terminal nodes is given. To obtain an optimal k-ary tree with $(k - 1)n + 1$ terminal nodes, one can split a terminal node of minimum cost in T to produce k new terminal nodes. This can be verified by using Lemma 1, just as was done in the case $k = 2$ in [2].

It is obvious that each tree T_n^k is a k-ary tree, and that it has F_n^k terminal nodes. As in the case $k = 2$ in [2], we have the following lemma.

Lemma 2: Each Fibonacci tree T_n^k, $n \geqslant k + 1$, has exactly $F_{n-k}^{k,j}$ terminal nodes of cost $n - j$, where $1 \leqslant j \leqslant k \leqslant n - 1$.

Proof: The proof is by induction on n. The tree T_{k+1}^k has k terminal nodes, and it has exactly 1 $(= F_1^{k,j})$ terminal node of cost $k + 1 - j$, where $1 \leqslant j \leqslant k$. Now, we assume that the Lemma holds for all n, $k + 1 \leqslant n \leqslant N$, where $N \geqslant k + 1$ is a fixed integer. The tree T_{N+1}^k has k subtrees $T_N^k, T_{N-1}^k, \ldots, T_{N-k+1}^k$, from left to right. The number of terminal nodes of cost $N + 1 - j$ in T_{N+1}^k is, for $k \geqslant j \geqslant N + 1 - k$,

$$\begin{aligned} F_{N-k}^{k,j} + F_{N-k-1}^{k,j} + \cdots + F_1^{k,j} + 1 &= 2^{N-k-1} + 2^{N-k-2} + \cdots + 1 + 1 \\ &= 2^{N-k} = F_{N+1-k}^{k,j}, \end{aligned}$$

and for $j < N + 1 - k$, the number is

$$F_{N-k}^{k,j} + F_{N-k-1}^{k,j} + \cdots + F_{N-2k+1}^{k,j} = F_{N+1-k}^{k,j}.$$

This completes the proof.

With the branch costs specified as above, for any fixed $k \geqslant 2$, we have the following theorem.

Theorem 1: The average cost of a Fibonacci tree T_n^k of order n is

$$s_n^k = (n-1) + \left[\sum_{j=2}^{k}(j-1)F_{n-k}^{k,j}\right]\Big/F_n^k,$$

and it is optimal among k-ary trees for each $n \geqslant k+1$.

Proof: By Lemma 2, for $n \geqslant k+1$,

$$\begin{aligned} s_n^k &= \sum_{j=1}^{k}(n-j)F_{n-k}^{k,j}\Big/F_n^k \\ &= (n-1)\sum_{j=1}^{k}F_{n-k}^{k,j}\Big/F_n^k - \sum_{j=1}^{k}(j-1)F_{n-k}^{k,j}\Big/F_n^k \\ &= (n-1) - \sum_{j=2}^{k}(j-1)F_{n-k}^{k,j}\Big/F_n^k. \end{aligned}$$

If $k = 2$, one has

$$s_n^2 = (n-1) - F_{n-2}^{2,2}/F_n^2 = (n-2) + (F_n^2 - F_{n-1}^2)/F_n^2 = (n-2) + F_{n-2}^2/F_n^2,$$

as was shown in Theorem 3 of [2].

For the second assertion, by the rule for constructing optimal k-ary trees mentioned above, we need to show that, by splitting all the terminal nodes of cost $(n-k)$ in T_n^k, we can obtain T_{n+1}^k. As in the proof in [2], we proceed by induction on n. The claim clearly holds for $n = k+1$. We assume it holds for all n, $k+1 \leqslant n \leqslant N-1$, where $N \geqslant k+2$ is a fixed integer. Since the left-most subtree of T_N^k is T_{N-1}^k, by the induction hypothesis, after splitting all the terminal nodes of cost $(N-k)$ in this subtree, we obtain T_N^k. A similar argument applies to all the remaining $(k-1)$ subtrees of T_N^k. Therefore, the resulting tree has k ordered subtrees $T_N^k, T_{N-1}^k, \ldots, T_{N-k+1}^k$, and so it is T_{N+1}^k. This completes the proof.

Our next result generalizes a result in [3] which deals with the number of terminal nodes at each level of a Fibonacci tree.

Theorem 2: At level ℓ in a Fibonacci tree T_n^k, $n \geqslant k+1$, there are $c_{\ell,n-k-\ell}^k$ nodes with label p_1, and $c_{\ell-1,n-k-\ell}^k + c_{\ell-1,n-k-\ell-1}^k + \cdots + c_{\ell-1,n-k-\ell-(k-j)}^k$ nodes with label p_j, $2 \leqslant j \leqslant k$.

Proof: The assertion holds for $n = k+1$. We assume that it holds for some $n \geqslant k+1$, and then prove it for $n+1$. By hypothesis, there are

$$c_{\ell-1,n-k-\ell+1}^k$$

nodes with label p_1 in T_n^k at level $\ell - 1$, and

$$c_{\ell-1,n-k-\ell}^k + \cdots + c_{\ell-1,n-k-\ell-(k-2)}^k$$

nodes with label p_2 in T_n^k at level ℓ. Thus, the number of nodes with label p_1 in T_{n+1}^k at level ℓ is

$$c_{\ell-1,n-k-\ell+1}^k + c_{\ell-1,n-k-\ell}^k + \cdots + c_{\ell-1,n-k-\ell-(k-2)}^k = c_{\ell,n+1-k-\ell}^k.$$

Similarly, one can compute the number of nodes with label p_j, $j \geqslant 2$, in T_{n+1}^k at level ℓ. This completes the proof.

One can see from Theorem 2 that the number of terminal nodes at level ℓ in the tree T_n^k is

$$\sum_{j=1}^{k} jc_{\ell-1,\, n-2k-\ell+j}^{k},$$

and since T_n^k has F_n^k terminal nodes,

$$F_n^k = \sum_{\ell=1}^{n-k} \sum_{j=1}^{k} jc_{\ell-1,\, n-2k-\ell+j}^{k} \text{ for } n \geqslant k+1.$$

Finally, one can see that the trees T_n^k and T_n^2 have average costs s_n^k and s_n^2, respectively. Since the characteristic equations of the recurrence relations for the sequences $\{F_n^k,\ n \geqslant 0\}$ and $\{F_n^{k,j},\ n \geqslant 0\}$ are the same, and have exactly one root x_1 satisfying $|x_1| > 1$, and since the coefficients of the n^{th} power of x_1 in the expressions of F_n^k and $F_n^{k,j}$ are clearly nonzero, the ratio $F_{n-k}^{k,j}/F_n^k$ converges to a finite limit as $n \to \infty$. Using Theorem 1, one has the limit

$$s_n^k/s_n^2 \to 1 \text{ as } n \to \infty.$$

On the other hand, the trees T_n^k and T_n^2 have F_n^k and F_n^2 terminal nodes, respectively. For any $k \geqslant 3$, one has the limit

$$F_n^k/F_n^2 \to \infty \text{ as } n \to \infty.$$

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FIBONACCI NUMBERS AS EXPECTED VALUES IN A GAME OF CHANCE

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Our objective in this note is to introduce an interesting game of chance and show that, when the game is unfair, its expected value is (plus or minus) a Fibonacci number. We prove this in an elegant and unexpected way, with ramifications going beyond the Fibonacci numbers.

1. THE GAME

We assign five payoffs to the vertices of a pentagon. Three of these are \$0, the remaining two are $\$2^N$ and $\$-2^N$, where N is a fixed positive integer (preferably large). A ball moves clockwise around the five positions, and where it stops determines the payoff. The ball is propelled by coin tossing. When a fair coin shows a head, the ball moves one position clockwise. When the coin shows a tail, the ball does not move. The coin is tossed N times. The distribution of the payoffs, the starting position of the ball, and the value of N are immaterial to the mathematics—the Fibonacci numbers are here no matter what. As for the gambler's fortune, that is another story.

The expected value of the game is easily shown to have the form

$$\sum_j \left(\binom{N}{5j+p} - \binom{N}{5j+q}\right), \quad 0 \leqslant p,\ q \leqslant 4, \tag{1}$$

but these integers are not immediately recognizable as positive or negative, let alone Fibonacci numbers.

2. GENERALIZED BINOMIAL COEFFICIENTS

The following is a well-known identity (see, e.g., [1], Chap. 3, Prob. 29).

$$\binom{n}{0}F_0 + \binom{n}{1}F_1 + \binom{n}{2}F_2 + \cdots + \binom{n}{n}F_n = F_{2n}, \tag{2}$$

where $\{F_j\}_{j \geqslant 0} = \{0, 1, 1, 2, \ldots\}$ is the Fibonacci sequence. There are several ways to prove (2), but here is a way which gets to the heart of the relation between the Fibonacci numbers and the binomial coefficients. Let

$$\left\{ {n \atop j} \right\}^* = F_{2n+j}.$$

Observe that

$$\left\{ {n+1 \atop j} \right\}^* = F_{2n+2+j} = F_{2n+j+1} + F_{2n+j} = \left\{ {n \atop j+1} \right\}^* + \left\{ {n \atop j} \right\}^*. \tag{3}$$

Except for the advanced, as opposed to retarded j-argument, (3) states the Pascal recurrence for the coefficients $\left\{ {n \atop j} \right\}^*$. Because of the close connection to the binomial coefficients, there must be a precise statement relating the two. Leaving the particular choice of $\left\{ {n \atop j} \right\}^*$ values behind, but retaining recurrence

(5), below, this is

$$\sum_j \binom{n}{j}\left\{ {r \atop j+i} \right\} = \left\{ {n+r \atop i} \right\}, \tag{4}$$

a formula [2] easily proved by induction on n (fixed $r \geqslant 0$ and $-\infty < i < +\infty$). Setting $\left\{ {k \atop j} \right\} = \left\{ {k \atop j} \right\}^*$ and $r = i = 0$ in (4) yields (2).

The lesson to be learned from this is two-fold. First, (4) depends only on the Pascal-like recurrence

$$\left\{ {n+1 \atop j} \right\} = \left\{ {n \atop j} \right\} + \left\{ {n \atop j+1} \right\} \tag{5}$$

so (2) holds for *any* sequence satisfying the Fibonacci recurrence (e.g., the Lucas numbers). We are motivated to look for more *generalized binomial coefficients* (gbc's) among the Fibonacci numbers, and find them easily:

$$\left\{ {n \atop j} \right\} \text{ could be } F_{m+n-j},\ (-1)^{n+j}2^jF_{m-3n+j},\ 2^{-n-j}(-1)^jF_{m-2n+j},$$
$$(-1)^{n+j}F_{m-n+j},\ 2^{-n-j}F_{m+2n-j},\ 2^jF_{m+3n+j},\ (-1)^jF_{m-2n-j},\ \ldots .$$

Secondly, since the initial conditions

$$\left\{ {0 \atop j} \right\} = c_j$$

are free for us to choose, rewriting (4) as

$$\left\{ {n \atop 0} \right\} = \sum_j \binom{n}{j} c_j$$

gives us a single coefficient which computes entire binomial sums.

Thus, the idea of a generalized binomial coefficient is itself worth generalizing. Let

$$\left[{n \atop j} \right] \equiv (-1)^n \left\{ {n \atop j} \right\} \quad \text{and} \quad \left\langle {n \atop j} \right\rangle \equiv \left[{n \atop j} \right] - \inf_k \left[{n \atop k} \right],\ n \in \mathbb{N},\ j,\ k \in \mathbb{Z}. \tag{6}$$

To assure that the gbc's $\left\langle {n \atop j} \right\rangle$ are well defined, we need only require

$$\sup_k \left| \left\{ {0 \atop k} \right\} \right| < +\infty.$$

3. THE GAME AND THE gbc's

By answering some natural questions about how the coefficients $\left\{ {n \atop j} \right\}$, $\left[{n \atop j} \right]$, and $\left\langle {n \atop j} \right\rangle$ are related, we get immediate answers about the connection between the roulette-like game of Section 1 and the Fibonacci numbers. For example, what type of recurrence do the $\left\langle {n \atop j} \right\rangle$ satisfy? Given

$$\left\{ \left\langle {m \atop k} \right\rangle \right\}_{\substack{0 \leqslant m \leqslant n \\ -\infty < k < +\infty}},$$

how do we recover $\left\{ {n \atop j} \right\}$? The answers are in

Theorem 1: Let $\binom{n}{j}$ denote the binomial coefficients, and $\left\{ {n \atop j} \right\}$ denote any coefficients satisfying (5), $n = 0, 1, \ldots; -\infty < j < +\infty$. With the convention

$$\left\{{0 \atop j}\right\} = \left[{0 \atop j}\right] = \left\langle{0 \atop j}\right\rangle,$$

define $\left[{n \atop j}\right]$ and $\left\langle{n \atop j}\right\rangle$ by (6). Then

$$\left\langle{n+1 \atop j}\right\rangle = \lambda_{n+1} - \left\langle{n \atop j}\right\rangle - \left\langle{n \atop j+1}\right\rangle, \quad \text{with } \lambda_{n+1} = \sup_k\left(\left\langle{n \atop k}\right\rangle + \left\langle{n \atop k+1}\right\rangle\right); \tag{7}$$

$$\left\{{n \atop j}\right\} = (-1)^n\left\langle{n \atop j}\right\rangle + \sum_{k=1}^{n}(-1)^{k-1}2^{n-k}\lambda_k; \tag{8}$$

$$\sum_{j=0}^{n}\binom{n}{j}\left\langle{r \atop j+i}\right\rangle = (-1)^n\left\langle{n+r \atop i}\right\rangle + \sum_{k=1}^{n}(-1)^{k-1}2^{n-k}\lambda_{r+k}. \tag{9}$$

Outline of Proof: A straightforward application of the definitions yields (7).

To obtain (8), let $S_0 = 0$ and

$$S_n = \begin{cases} \inf_k\left\{{n \atop k}\right\}, & n \text{ even} \\ \sup_k\left\{{n \atop k}\right\}, & n \text{ odd}, n > 0. \end{cases}$$

It follows that $S_n = \left\{{n \atop j}\right\} + (-1)^{n+1}\left\langle{n \atop j}\right\rangle$ for all j, and

$$S_{n+1} = 2S_n + (-1)^n\lambda_{n+1}. \tag{10}$$

Solving (10) gives

$$S_n = \sum_{k=1}^{n}(-1)^{k-1}2^{n-k}\lambda_k$$

and (8).

To obtain (9), substitute (8) with the appropriate indices in (4). ■

Here are the important consequences of Theorem 1: Relation (7) is an *algorithm* for constructing an array of gbc's $\left\langle{n \atop j}\right\rangle$. Consideration of (1) shows that we will want to take

$$\ldots 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0 \ldots \tag{11}$$

as our initial row. Secondly, setting $r = 0$ in (9) and choosing i appropriately,

$$\Sigma_j\left(\binom{N}{5j+p} - \binom{N}{5j+q}\right) = (-1)^N\left(\left\langle{N \atop -p}\right\rangle - \left\langle{N \atop -q}\right\rangle\right). \tag{12}$$

The implication is that to know the expected value of our game we need only construct the array of gbc's $\left\langle{n \atop j}\right\rangle$ with initial row (11). Here is where the Fibonacci numbers appear.

$$\begin{array}{c|ccccc|cc}
\ldots\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0\ \ldots \\
\ldots\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1\ \ldots \\
\ldots\ 2 & 1 & 0 & 0 & 1 & 2 & 1 & 0\ \ldots \\
\ldots\ 0 & 2 & 3 & 2 & 0 & 0 & 2 & 3\ \ldots \\
\ldots\ 3 & 0 & 0 & 3 & 5 & 3 & 0 & 0\ \ldots \\
\ldots\ 5 & 8 & 5 & 0 & 0 & 5 & 8 & 5\ \ldots \\
\vdots & & & & & & \vdots &
\end{array} \tag{13}$$

A complete description of array (13) is the concern of

Theorem 2: Array (13) consists of rows of repeating blocks

$$B_n = \left(\left\langle {n \atop j} \right\rangle\right)_{j=0}^{4}$$

which (modulo a shift) have the form

$$M_n = (F_{n+1}, F_n, 0, 0, F_n). \tag{14}$$

Let $R_k(\cdot)$ denote the operator which shifts the elements of a vector k steps to the right with wraparound. Then,

$$B_n = R_{2n(\text{mod } 5)} M_n,\ n = 0, 1, \ldots . \tag{15}$$

Outline of Proof: The fact that the blocks have the form $(b_n, a_n, 0, 0, a_n)$, where eventually $0 < a_n < b_n$, is a simple observation, as is the right-shifting action described by (15).

The fact that the a_n and b_n are the Fibonacci numbers follows from the basic recurrence (7). The latter implies

$$\begin{aligned} b_{n+1} &= b_n + a_n \\ a_{n+1} &= b_{n+1} - a_n, \end{aligned} \tag{16}$$

and (16) implies, in turn, that $b_{n+2} = b_{n+1} + b_n$, $a_{n+2} = a_{n+1} + a_n$. With the initial conditions, we have $b_n = F_{n+1}$, $a_n = F_n$. ■

Corollary: The expected value of the game of Section 1 is zero or (plus or minus) a Fibonacci number.

Proof: Consider (12) in conjunction with (13), (14), (15). The difference of any two elements in (14) is zero or (plus or minus) a Fibonacci number. ■

4. EXTENSIONS

A natural generalization of the game is to assign payoffs to the vertices of an n-gon and ask about the analogues of the Fibonacci numbers in this case. This question is addressed in [3], where we generalize results of Hoggatt and Alexanderson [4].

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SEQUENCES GENERATED BY MULTIPLE REFLECTIONS

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1. We consider the situation of a light ray multiply reflected by a set of parallel glass plates in contact. The ray is assumed to be totally reflected or transmitted at any interface. A sequence is formed by considering the number of distinct ways a ray can be reflected n times before emerging. It is well known that this is the Fibonacci sequence if only two plates are present [1]. Several aspects of the general case for k plates have already been considered: Moser and Wyman [2] place a plane mirror behind the stack of plates, while Hoggatt and Junge [3] tackle the above situation. We will show how the enumerating matrices of [2] and [3] are related, and derive a procedure for evaluating the asymptotic form of the general sequence. In addition, some Fibonacci-like relations of the general sequence are shown.

2. We will restrict ourselves to the cases of two and three plates in this section, with generalizations being obvious to k plates. A scheme for counting the reflections of a given order is shown in Diagrams 1 and 2. A string of digits is used to enumerate the labelled interfaces at which reflections occur.

<u>2 plates</u>: (2, 3), (21, 31, 32), (212, 213, 312, 313, 323), ... (1)

<u>3 plates</u>: (2, 3, 4), (21, 31, 32, 41, 42, 43), (212, 213, 224, ...) (2)

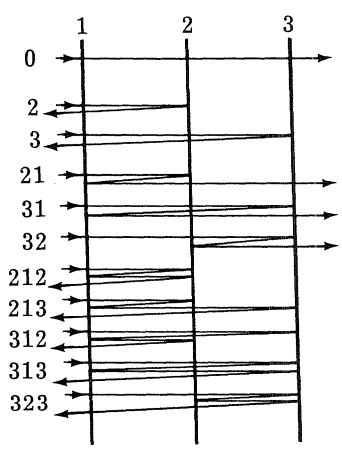

Diagram 1. Some of the labelled reflections from two sheets of glass.

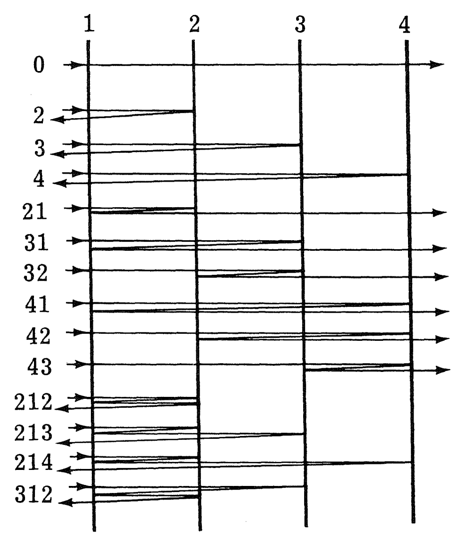

Diagram 2. Labelled reflections from three sheets of glass.

The reflections can now be shown without recourse to drawing them. All the reflections of a given order are placed in parentheses above. The number of reflections of a given order that end on the same final interface are now counted, and arranged in a sequence whose non-zero members are non-decreasing. The zeros arise, of course, because the ray must finally pass out through the first or last face.

The sequence that arises from (1) is:

$$0, 1, 1, 0, 1, 2, 0, 2, 3, 0, 3, 5, 0, 5, 8, 0, 8, 13, 0, 13, 21, 0, 21, 34, 0, \ldots, \tag{3}$$

which is seen to contain the Fibonacci sequence. The sequence that arises from (2) is:

$$0, 1, 1, 1, 0, 1, 2, 3, 0, 3, 5, 6, 0, 6, 11, 14, 0, 14, 25, 31, 0, \ldots . \tag{4}$$

Now, (3) is the sequence generated by the starting conditions:

$$r_0 = 0, \; r_1 = r_2 = 1, \tag{5}$$

together with the recurrence relations:

$$r_{3n} = 0, \; r_{3n+1} = r_{3n-1}, \; r_{3n+2} = r_{3n-1} + r_{3n-2}, \quad \text{for } n \geqslant 1. \tag{6}$$

In the same way, (4) is produced by

$$r_0 = 0, \; r_1 = r_2 = r_3 = 1, \tag{7}$$

where

$$\begin{aligned} &r_{4n} = 0, \; r_{4n+1} = r_{4n-1}, \; r_{4n+2} = r_{4n-1} + r_{4n-2}, \\ &r_{4n+3} = r_{4n-1} + r_{4n-2} + r_{4n-3}, \quad \text{for } n \geqslant 1. \end{aligned} \tag{8}$$

Some simple sequence properties are now listed for the sequence (2). These are all readily proven from the definition (8):

$$r_1 + r_5 + r_9 + \cdots + r_{4n+1} = r_{4n+2}; \tag{9}$$

$$r_3 + r_7 + r_{11} + \cdots + r_{4n+3} = r_{4n+6} - 2; \tag{10}$$

$$r_2 + r_6 + r_{10} + \cdots + r_{4n+2} = r_{4n+6} - r_{4n+2} - 1; \tag{11}$$

$$r_{4n}^2 + r_{4n+1}^2 + r_{4n+2}^2 + r_{4n+3}^2 = r_{2(4n+3)+1}. \tag{12}$$

In establishing (11), the following result is needed:

$$r_{4n+6} - r_{4n+2} = r_{4n+2} + r_{4n-3}. \tag{13}$$

We can use these partial sums to give the sum of all the reflections up to order n:

$$\sum_{i=1}^{4n} r_i = r_{4n-2} + 2 \cdot r_{4n+2} - r_{4n-6} - 2. \tag{14}$$

3. We consider the general case to obtain a procedure for evaluating terms like those on the right-hand side of (14). Note first that the non-zero terms

in the sequence can be generated in the following matrix notation:

$$\begin{bmatrix} r_{nk+1} \\ r_{nk+2} \\ \vdots \\ r_{(n+1)k-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix} \begin{bmatrix} r_{(n-1)k+1} \\ r_{(n-1)k+2} \\ \vdots \\ r_{nk-1} \end{bmatrix}, \tag{15}$$

or

$$\mathbf{r}_n = A\mathbf{r}_{n-1} = A^n \mathbf{r}_0, \tag{16}$$

easily by induction, where $\mathbf{r}_0$ is the starting conditions column vector. As is pointed out in [2], this approach can only be made viable by making use of the eigenvalues (λ) and their corresponding eigenvectors ($\mathbf{u}$) as follows:

Repeated application of A to the eigenvector $\mathbf{u}$ gives

$$A\mathbf{u} = \lambda\mathbf{u},\ A^2\mathbf{u} = \lambda^2\mathbf{u},\ \dots,\ A^n\mathbf{u} = \lambda_n\mathbf{u}. \tag{17}$$

The solution of (16) follows on expressing $\mathbf{r}_0$ as a linear combination of the eigenvectors of A. However, [2] considers the case with the mirror, which involves a different enumerating matrix. This means that all the reflections of odd order are unaffected by the mirror because they proceed to the left in any case, while a reflection of even order is added to the next odd order. The matrix that does this is A^2, where A is defined as in (15).

We now proceed to find the eigenvalues of A from the determinant of order k:

$$D_k(\lambda) = |A - \lambda I| = 0. \tag{18}$$

Now, [3] provides the useful recurrence relation:

$$D_k(\lambda) = (2\lambda^2 - 1)D_{k-2}(\lambda) - \lambda^4 D_{k-4}(\lambda). \tag{19}$$

If we assume a solution to (18) of the form $D_k(\lambda) = P^k$, where P is a polynomial in λ, independent of k, then we find that

$$D_k(\lambda) = c_1a^k + c_2b^k + c_3a^k \cdot (-1)^k + c_4b^k \cdot (-1)^k, \tag{20}$$

where

$$P = \pm(((2\lambda^2 - 1) \pm \Delta)/2)^{1/2} = \pm a,\ \pm b,$$

where a is the root with the positive discriminant and b that with the negative discriminant, while

$$\Delta = (1 - 4\lambda^2)^{1/2}. \tag{21}$$

The coefficients c (i = 1, 2, 3, 4), which are independent of k, can be found using the four characteristic equations of lowest order, i.e.,

$$D_0(\lambda) = 1,\ D_1(\lambda) = -\lambda + 1,\ D_2(\lambda) = \lambda^2 - \lambda - 1, \text{ and}$$
$$D_3(\lambda) = \lambda^3 + 2\lambda^2 + \lambda - 1, \tag{22}$$

as follows:

When k is even,

$$D_k(\lambda) = 0 = (c_1 + c_3)a^k + (c_2 + c_4)b^k, \tag{23}$$

leading to

$$(1 + 2\lambda - \Delta)/(1 + 2\lambda + \Delta) = ((2\lambda^2 - 1 - \Delta)/(2\lambda^2 - 1 + \Delta))^{k/2}, \tag{24}$$

on making use of $D_0(\lambda)$ and $D_2(\lambda)$.

We can readily solve (24) on making the substitutions

$$\lambda = \frac{1}{2}\sin\theta = t/(1 + t^2), \text{ where } t = \tan\theta/2, \tag{25}$$

giving:

$$t^{2k+1} = 1, \text{ with solutions } t = e^{\frac{\pm 2n\pi i}{2k+1}}, \quad n = 0, 1, \ldots, k. \tag{26}$$

Hence, the eigenvalues are given by

$$\lambda = \frac{1}{2}\sec(2n\pi/2k + 1), \ n = 1, 2, \ldots, k. \tag{27}$$

When k is odd, a similar argument leads to solving

$$t^{2k+1} = -1, \tag{28}$$

giving the eigenvalues:

$$\lambda = \frac{1}{2}\sec(2n + 1)\pi/2k + 1. \tag{29}$$

We are now in a position to evaluate (16), which we will briefly show for the case $k = 2$: From (27), the eigenvalues are

$$\lambda_1 = \frac{1}{2}\sec 2\pi/5 \quad \text{and} \quad \lambda_2 = \frac{1}{2}\sec 4\pi/5,$$

with the corresponding eigenvectors

$$\begin{pmatrix}1\\ \mathbf{t}\end{pmatrix}, \quad \begin{pmatrix}\mathbf{t}\\ -1\end{pmatrix}, \tag{30}$$

on writing $\mathbf{t} = \frac{1}{2}\sec 2\pi/5$.

On expressing $\mathbf{r}_0 = \begin{pmatrix}1\\1\end{pmatrix}$ in terms of the eigenvectors, and on using (16), we find:

$$\begin{pmatrix}r_{3n+1}\\ r_{3n+2}\end{pmatrix} = A^2\begin{pmatrix}1\\1\end{pmatrix} = \frac{2\mathbf{t} + 1}{\mathbf{t} + 2}\cdot \mathbf{t}^{n-1}\cdot\begin{pmatrix}1\\ \mathbf{t}\end{pmatrix} + \frac{\mathbf{t} - 2}{\mathbf{t} + 2}\cdot \mathbf{t}^{-n+1}\cdot\begin{pmatrix}\mathbf{t}\\ -1\end{pmatrix}; \tag{31}$$

$k \geqslant 2$ values are best tackled numerically, as the algebra becomes excessive.

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A RELATION FOR THE PRIME DISTRIBUTION FUNCTION

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In this paper, we obtain an interesting duality relationship between the prime distribution function (π-function) and another, less well-known, number theoretic function. The domain of definition throughout is the set of natural numbers.

We recall the definition of the π-function:

$$\pi(n) = \sum_{p \leqslant n} 1, \text{ which counts the number of primes} \leqslant n. \tag{1}$$

Also, we recall the Möbius function, defined as follows:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n \text{ is divisible by a square (or higher power)} \\ & \text{of a prime;} \\ (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct primes.} \end{cases} \tag{2}$$

We also indicate, without proof, a well-known relationship satisfied by the Möbius function:

$$\sum_{d|n} \mu(d) = \delta_{1n} = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1, \end{cases} \tag{3}$$

where the sum is taken over all divisors d of n.

We now introduce another function $\lambda(n)$ which seeks to enumerate all *powers* of primes (including first powers) so that such powers are $\leqslant n$. We may count $\lambda(n)$ by letting k vary from $1, 2, 3, \ldots$ and counting the acceptable k^{th} powers of primes. For a given prime p, the inequality $p^k \leqslant n$ is equivalent to

$$k \leqslant \frac{\log n}{\log p},$$

and is satisfied by

$$k = 1, 2, 3, \ldots, \left[\frac{\log n}{\log p}\right], \text{ i.e., for } \left[\frac{\log n}{\log p}\right] \text{ values.}$$

Summing over all p, we thus obtain:

$$\lambda(n) = \sum_{p \leqslant n} \left[\frac{\log n}{\log p}\right]. \tag{4}$$

An alternative expression for $\lambda(n)$ can be obtained by noting that $p^k \leqslant n$ is equivalent to $p \leqslant [n^{1/k}]$. The component of $\lambda(n)$ that counts all k^{th} powers of primes thus counts all primes $p \leqslant [n^{1/k}]$, and must therefore equal $\pi([n^{1/k}])$. Summing over all possible k, we therefore obtain the relationship:

$$\lambda(n) = \sum_{k=1}^{\infty} \pi([n^{1/k}]). \tag{5}$$

Note that the "infinite" series in (5) actually terminates since, for sufficiently large k, $[n^{1/k}] = 1$ for all n, and $\pi(1) = 0$.

The relationship in (5) may be inverted to yield an expression for $\pi(n)$ in terms of λ, valued at varying arguments. This expression is as follows:

$$\pi(n) = \sum_{k=1}^{\infty} \mu(k)\lambda([n^{1/k}]). \tag{6}$$

A comment similar to that following (5) applies here, too, since $\lambda(1) = 0$.

To prove (6), we resort to a pair of seemingly unrelated lemmas.

Lemma 1: Given positive integers m, n, and r, let

$$\chi(m|n) = \begin{cases} 1, & \text{if } m|n; \\ 0, & \text{if } m \nmid n. \end{cases}$$

Define $r \times r$ matrices $A_r = (a_{ij}^{(r)})$ and $B_r = (b_{ij}^{(r)})$ as follows:

$$a_{ij}^{(r)} = \chi(i|j); \tag{8}$$

$$b_{ij}^{(r)} = \chi(i|j)\mu(j/i), \quad (i, j = 1, 2, 3, \ldots, r). \tag{9}$$

Then

$$A_r B_r = I_r, \text{ i.e., } B_r = A_r^{-1}. \tag{10}$$

Proof of Lemma 1: Let $A_r B_r = C_r = (c_{ij}^{(r)})$. Then

$$c_{ij}^{(r)} = \sum_{k=1}^{r} a_{ik}^{(r)} b_{kj}^{(r)} = \sum_{k=1}^{r} \chi(i|k)\chi(k|j)\mu(j/k).$$

Note that each term of this sum vanishes *unless* $i|k|j$, i.e., unless $i|j$. Thus, $c_{ij}^{(r)} = 0$ if $i \nmid j$. Suppose now that $i|j$, and let $j = id$. Then

$$c_{ij}^{(r)} = \sum_{u=1}^{j/i} \chi(ui|j)\mu(j/ui) = \sum_{u=1}^{d} \chi(u|d)\mu(d/u) = \sum_{d_1|d} \mu(d_1) = \delta_{1d}$$

[using (3)]. Hence,

$$c_{ij}^{(r)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

This is equivalent to $C_r = I_r$. Q.E.D.

Lemma 2: Suppose n, a, and b are positive integers. Then

$$[[n^{1/a}]^{1/b}] = [n^{1/ab}]. \tag{11}$$

Proof of Lemma 2: Let $u = [n^{1/a}]$. Since $n^{1/a} \geqslant 1$, thus $u \geqslant 1$. Define the integer $v \geqslant 2$ by:

$$1 \leqslant (v-1)^b \leqslant u < v^b - 1.$$

Since $n^{1/a} < u + 1 < v^b$, thus $n^{1/ab} < v$, which implies $[n^{1/ab}] \leqslant v - 1$. However, $v - 1 \leqslant u^{1/b} < (v^b - 1)^{1/b} < v$, which implies $[u^{1/b}] = v - 1$; therefore,

$$[n^{1/ab}] \leqslant [u^{1/b}]. \tag{12}$$

On the other hand, $n^{1/a} \geqslant u \Rightarrow n^{1/ab} \geqslant u^{1/b}$, which implies

$$[n^{1/ab}] \geqslant [u^{1/b}]. \tag{13}$$

It follows from (12) and (13) that

$$[n^{1/ab}] = [u^{1/b}], \tag{14}$$

which is equivalent to (11). Q.E.D.

The proof of (6) follows. Let $f(k) = \pi([n^{1/k}])$, $g(k) = \lambda([n^{1/k}])$, assuming n is given. Applying Lemma 2 and (5) indefinitely, with n replaced successively by $[n^{1/r}]$, $r = 1, 2, 3, \ldots$, the following relationships are evident:

$$g(r) = \sum_{k=1}^{\infty} f(rk), \quad r = 1, 2, 3, \ldots . \tag{15}$$

Let us define the following vectors:

$$\mathbf{f}_r' = (f(1), f(2), \ldots, f(r)), \quad \mathbf{g}_r' = (g(1), g(2), \ldots, g(r)). \tag{16}$$

We may then transform (15) into matrix notation as follows:

$$\mathbf{g}_r = A_r \mathbf{f}_r. \tag{17}$$

Multiplying both sides of (17) by B_r, as given in Lemma 1, yields the desired inversion formula:

$$\mathbf{f}_r = B_r \mathbf{g}_r. \tag{18}$$

Converting (18) back to scalar notation, we obtain:

$$f(r) = \sum_{k=1}^{\infty} \mu(k) g(rk). \tag{19}$$

Now, setting $r = 1$ in (19) yields the desired result in (6). Q.E.D.

Lemma 1 is a very interesting result in its own right, and it provides the basis for the well-known technique of Möbius inversion, of which the dual relationships given in (5) and (6) are special cases.

Note that (6) provides an explicit expression for the prime distribution function, which is an important step in one of the most celebrated of unsolved problems in number theory, namely the discovery of an explicit formula for the n^{th} prime. Before giving vent to undue jubilation, however, it must be noted that the "explicit" expression given by (6) is in terms of another auxiliary number theoretic function, which is itself not readily found in terms of n. Therefore, the pair of relationships in (5) and (6) is apparently only of academic interest insofar as the great unsolved problem is concerned. It may come to pass, nevertheless, that some reader of this paper will find some use for these relationships toward the solution of this or other problem.

We conclude this paper with a brief table of the first few values of the two functions studied herein.

n	$\pi(n)$	$\lambda(n)$
1	0	0
2	1	1
3	2	2
4	2	3
5	3	4
6	3	4
7	4	5
8	4	6
9	4	7
10	4	7
11	5	8
12	5	8
13	6	9
14	6	9
15	6	9
16	6	10
17	7	11
18	7	11
19	8	12
20	8	12
21	8	12
22	8	12
23	9	13
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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to DR. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-574 *Proposed by Valentina Bakinova, Rondout Valley, NY*

Let $a_1, a_2, \ldots$ be defined by $a_1 = 1$ and $a_{n+1} = [\sqrt{s_n}]$, where $s_n = a_1 + a_2 + \cdots + a_n$ and $[x]$ is the integer with $x - 1 < [x] \leqslant x$. Find a_{100}, s_{100}, a_{1000}, and s_{1000}.

B-575 *Proposed by L. A. G. Dresel, Reading, England*

Let R_n and S_n be sequences defined by given values R_0, R_1, S_0, S_1 and the recurrence relations $R_{n+1} = rR_n + tR_{n-1}$ and $S_{n+1} = sS_n + tS_{n-1}$, where r, s, t are constants and $n = 1, 2, 3, \ldots$. Show that

$$(r + s)\sum_{k=1}^{n} R_k S_k t^{n-k} = (R_{n+1}S_n + R_nS_{n+1}) - t^n(R_1S_0 + R_0S_1).$$

B-576 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A = L_{2m+3(4n+1)} + (-1)^m$. Show that A is a product of three Fibonacci numbers for all positive integers m and n.

B-577 *Proposed by Herta T. Freitag, Roanoke, VA*

Let A be as in B-575. Show that $4A/5$ is a difference of squares of Fibonacci numbers.

B-578 *Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy*

It is known (Zeckendorf's theorem) that every positive integer N can be represented as a finite sum of distinct nonconsecutive Fibonacci numbers and that this representation is unique. Let $\alpha = (1 + \sqrt{5})/2$ and $[x]$ denote the greatest integer not exceeding x. Denote by $f(N)$ the number of F-addends in the Zeckendorf representation for N. For positive integers n, prove that $f([\alpha F_n]) = 1$ if n is odd.

B-579 *Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy*

Using the notation of B-578, prove that $f([\alpha F_n]) = n/2$ when n is even.

SOLUTIONS

A Specific Fibonacci-Like Sequence

B-550 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Show that the powers of -13 form a Fibonacci-like sequence modulo 181, that is, show that

$$(-13)^{n+1} \equiv (-13)^n + (-13)^{n-1} \pmod{181} \text{ for } n = 1, 2, 3, \ldots .$$

Solution by L. A. G. Dresel, University of Reading, England

We have

$$(-13)^2 = 169 \equiv -13 + 1 \pmod{181},$$

and multiplying by $(-13)^{n-1}$ we obtain

$$(-13)^{n+1} \equiv (-13)^n + (-13)^{n-1} \pmod{181} \text{ for } n = 1, 2, 3, \ldots .$$

Also solved by Paul S. Bruckman, Herta T. Freitag, C. Georghiou, Hans Kappus, L. Kuipers, Bob Prielipp, Helmut Prodinger, Heinz-Jürgen Seiffert, Sahib Singh, Lawrance Somer, J. Suck, Tad White, and the proposer.

A Generalization

B-551 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Generalize on Problem B-550.

Solution by Lawrence Somer, George Washington University, Washington, D.C.

A generalization would be: Let p be an odd prime. Let a and b be integers. Let x be a nonzero residue modulo p. Then

$$x^{n+1} \equiv ax^n + bx^{n-1} \pmod{p} \text{ for } n = 1, 2, 3, \ldots,$$

if and only if $x \equiv (a \pm \sqrt{a^2 + 4b})/2 \pmod{p}$, where $\sqrt{a^2 + 4b}$ is the least positive residue r such that $r^2 \equiv a^2 + 4b \pmod{p}$ if such a residue exists. This result is proved in [1].

Reference

1. L. Somer. "The Fibonacci Group and a New Proof that $F_{p-(5/p)} \equiv 0 \pmod{p}$." *The Fibonacci Quarterly* **10**, no. 4 (1972):345-348, 354.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Herta T. Freitag, C. Georghiou, Hans Kappus, L. Kuipers, Bob Prielipp, Helmut Prodinger, Heinz-Jürgen Seiffert, Sahib Singh, J. Suck, Tad White, and the proposer.

Permutations of 9876543210 Divisible by 11

B-552 *Proposed by Philip L. Mana, Albuquerque, NM*

Let S be the set of integers n with $10^9 < n < 10^{10}$ and with each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 appearing (exactly once) in n.

(a) What is the smallest integer n is S with $11|n$?
(b) What is the probability that $11|n$ for a randomly chosen n in S?

Solution by L. A. G. Dresel, University of Reading, England

Let us number the digit positions 1 to 10 from left to right, and let P_1 denote the set of odd-numbered positions and P_2 the set of even-numbered positions. For a given $n \in S$, let Q_i be the set of digits occupying P_i, and let q_i be the sum of these digits, for $i = 1, 2$. Since each of the digits 0 to 9 appears exactly once in n, we have $q_1 + q_2 = 45$. But, for divisivility by 11, we require $q_1 \equiv q_2 \pmod{11}$, and therefore we must have $q_1 = 17$ *or* $q_1 = 28$.

(a) Let us assume that the first three digits of the smallest integer n in S which is divisible by 11 are 1, 0, 2, in that order. Then Q_1 contains the digits 1 and 2, and we find that $q_1 = 28$ is not achievable; furthermore, $q_1 = 17$ implies that Q_1 contains the digit 3 as well. Hence, the required smallest n is given by $n = 1024375869$.

(b) Let us enumerate all the sets V_k of five distinct digits with a sum equal to 17. There are exactly 11 such sets, namely:

0 1 2 5 9, 0 1 2 6 8, 0 1 3 4 9, 0 1 3 5 8, 0 1 3 6 7, 0 1 4 5 7,

0 2 3 4 8, 0 2 3 5 7, 0 2 4 5 6, 1 2 3 4 7, 1 2 3 5 6.

For each of these sets V_k ($k = 1, 2, \ldots, 11$), the remaining digits form a complementary set W_k with a sum equal to 28. In the case in which V_k contains the digit 0, there are $4 \times 4!$ ways of placing the digits of V_k in P_1, and $5!$ ways of placing the digits of W_k in P_2, giving in all $4 \times 4! \times 5!$ different numbers of the form (V_k, W_k); but there are also $5!$ ways of placing W_k in P_1, with $5!$ ways of placing V_k in P_2, giving a further $5! \times 5!$ numbers of the form (V_k, W_k). Therefore, the total number of permutations of a particular pair V_k, W_k is $9 \times 4! \times 5!$, and we obtain the same result if the digit 0 is contained in W_k instead of V_k. Now, the total number of integers in S is given by $9 \times 9!$, and of these we have $11 \times 9 \times 4! \times 5!$ divisible by 11. Hence, the probability that $11|n$ is $11 \times 4! \times 5!/(9!)$, which simplifies to $11/126$, and is slightly less than 1 in 11.

Also solved by Paul S. Bruckman, L. Kuipers, J. Suck, Tad White, and the proposer.

Lucas Summation

B-553 *Proposed by D. L. Muench, St. John Fisher College, Rochester, NY*

Find a compact form for $\sum_{i=0}^{2n} \binom{2n}{i} L_{i+1}^2$.

Solution by C. Georghiou, University of Patras, Greece

We have, for $n > 0$, with the help of the Binet formulas,

$$\begin{aligned}\sum_{i=0}^{2n} \binom{2n}{i} L_{i+1}^2 &= \sum_{i=0}^{2n} \binom{2n}{i} [\alpha^{2i+2} + \beta^{2i+2} - 2(-1)^i] \\ &= \alpha^2(1 + \alpha^2)^{2n} + \beta^2(1 + \beta^2)^{2n} \\ &= \alpha^2(\alpha 5^{1/2})^{2n} + \beta^2(\beta 5^{1/2})^{2n} \\ &= 5^n L_{2n+2}.\end{aligned}$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, Herta T. Freitag, Hans Kappus, L. Kuipers, Graham Lord, Bob Prielipp, Helmut Prodinger, Heinz-Jürgen Seiffert, Sahib Singh, J. Suck, Tad White, and the proposer.

Sum of Two Squares

B-554 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

For all n in $Z^+ = \{1, 2, \ldots\}$, prove that there exist x and y in Z^+ such that

$$(F_{4n-1} + 1)(F_{4n+1} + 1) = x^2 + y^2.$$

Solution by Graham Lord, Princeton, NJ

Using the Binet formulas, we have

$$\begin{aligned}(F_{4n-1} + 1)(F_{4n+1} + 1) &= (a^{4n-1} - b^{4n-1} + \sqrt{5})(a^{4n+1} - b^{4n+1} + \sqrt{5})/5 \\ &= \{a^{8n} - 2(ab)^{4n} + b^{8n} + 2 - (a^2 + b^2)(ab)^{4n-1} \\ &\quad - \sqrt{5}[(1 + a^2)a^{4n-1} - (1 + b^2)b^{4n-1}] + 5\}/5 \\ &= (a^{4n} - b^{4n})^2/5 \\ &\quad + \{2 + 3 + 5 + \sqrt{5}[a^{4n}(a - b) + b^{4n}(a - b)]\}/5 \\ &= F_{4n}^2 + L_{2n}^2.\end{aligned}$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, L. Kuipers, Bob Prielipp, Heinz-Jürgen Seiffert, Sahib Singh, J. Suck, Tad White, C. S. Yang & J. F. Wang, and the proposers.

Sum of Three Squares

B-555 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

For all n in Z^+, prove that there exist x, y, and z in Z^+ such that

$$(F_{2n-1} + 4)(F_{2n+5} + 1) = x^2 + y^2 + z^2.$$

Solution by Bob Prielipp, University of Wisconsin, Oshkosh, WI

We shall show that:

(1) $(F_{2n-1} + 4)(F_{2n+5} + 1) = F^2_{2n+2} + F^2_{n+3} + (L_{n+3} - F_{n-2})^2$ if n is even

and

(2) $(F_{2n-1} + 4)(F_{2n+5} + 1) = F^2_{2n+2} + (3F_{n+2})^2 + (F_{n+2} + F_{n+1})^2$ if n is odd.

[The results referred to below (I_{24}, I_{18}, etc.) can be found on pages 56 and 59 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr., Houghton-Mifflin Company, Boston, 1969.]

We begin by establishing the following preliminary results.

Lemma: $F_{2n-1}F_{2n+5} = F^2_{2n+2} + 4.$

Proof: $F_{2n-1}F_{2n+5} = F_{(2n+2)-3}F_{(2n+2)+3} = F^2_{2n+2} + F^2_3$ [by I_{19}] $= F^2_{2n+2} + 4.$

Corollary: $(F_{2n-1} + 4)(F_{2n+5} + 1) = F^2_{2n+2} + 4F_{2n+5} + F_{2n-1} + 8.$

(1) It suffices to prove that

$$4F_{4k+5} + F_{4k-1} + 8 = F^2_{2k+3} + (L_{2k+3} - F_{2k-2})^2.$$

$$\begin{aligned}
F^2_{2k+3} + (L_{2k+3} - F_{2k-2})^2 &= (F^2_{2k+3} + F^2_{2k-2}) - 2L_{2k+3}F_{2k-2} + L^2_{2k+3} \\
&= 5F_{4k+1} - 2(F_{4k+1} - 5) + (L_{4k+6} - 2) \\
&\quad \text{[by } I_{19}, I_{24}, \text{ and } I_{18}, \text{ respectively]} \\
&= 3F_{4k+1} + (F_{4k+6} + 2F_{4k+5}) + 8 \\
&= 3F_{4k+1} + (3F_{4k+5} + F_{4k+4}) + 8 \\
&= 4F_{4k+5} + (3F_{4k+1} - F_{4k+3}) + 8 \\
&= 4F_{4k+5} - (F_{4k+3} - 3F_{4k+1}) + 8 \\
&= 4F_{4k+5} - (F_{4k} - F_{4k+1}) + 8 \\
&= 4F_{4k+5} + F_{4k-1} + 8.
\end{aligned}$$

(2) It suffices to prove that

$$4F_{4k+3} + F_{4k-3} + 8 = (3F_{2k+1})^2 + (F_{2k+1} + L_{2k})^2.$$

$$
\begin{aligned}
(3F_{2k+1})^2 + (F_{2k+1} + L_{2k})^2 &= 2(5F_{2k+1}^2) + 2F_{2k+1}L_{2k} + L_{2k}^2 \\
&= 2(L_{4k+2} + 2) + 2(F_{4k+1} + 1) + (L_{4k} + 2) \\
&\quad [\text{by } I_{17},\ I_{21},\ \text{and } I_{15},\ \text{respectively}] \\
&= 2L_{4k+2} + L_{4k} + 2F_{4k+1} + 8 \\
&= 2(F_{4k+3} + F_{4k+1}) + (F_{4k} + 2F_{4k-1}) \\
&\qquad + 2F_{4k+1} + 8 \\
&= 2F_{4k+3} + 4F_{4k+1} + F_{4k} + 2F_{4k-1} + 8 \\
&= 3F_{4k+3} + 2F_{4k+1} + 2F_{4k-1} + 8 \\
&= 4F_{4k+3} - (F_{4k+2} - F_{4k+1}) + 2F_{4k-1} + 8 \\
&= 4F_{4k+3} - (F_{4k} - F_{4k-1}) + F_{4k-1} + 8 \\
&= 4F_{4k+3} + (F_{4k-1} - F_{4k-2}) + 8 \\
&= 4F_{4k+3} + F_{4k-3} + 8.
\end{aligned}
$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Graham Lord, and the proposers.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-400 *Proposed by Arne Fransen, Stockholm, Sweden*

For natural numbers h, k, with k odd, and an irrational a in the Lucasian sequence $V_{kh} = a^{kh} + a^{-kh}$, define $y_k \equiv V_{kh}$. Put

$$y_k = \sum_{r=0}^{n} c_r^{(2n+1)} y_1^{(2r+1)}, \text{ with } k = 2n + 1.$$

Prove that the coefficients are given by

$$c_r^{(2n+1)} \begin{cases} \equiv 1 \quad \text{for } r = n, \\ = (-1)^{n-r}(2n+1)\displaystyle\sum_{j=1}^{J} \frac{1}{2j-1}\binom{n-j}{2(j-1)}\binom{n-1-3(j-1)}{r-(j-1)} \text{ for } 0 \leqslant r < n, \end{cases}$$

where $J = \min\left(\left[\dfrac{n+2}{3}\right], \left[\dfrac{n+1-r}{2}\right], r+1\right)$.

Also, is there a simpler expression for $c_r^{(2n+1)}$?

H-401 *Proposed by Albert A. Mullin, Huntsville, AL*

It is well known that if $n \neq 4$ and the Fibonacci number F_n is prime then n is prime.

(1) Prove or disprove the complementary result: If $n \neq 8$ and the Fibonacci number F_n is the product of two *distinct* primes then n is either prime or the product of two primes, in which case at least one prime factor of F_n is Fibonacci.

(2) Define the recursions $u_{n+1} = F_{u_n}$, $u_1 = F_m$, $m \geqslant 6$. Prove or disprove that each sequence $\{u_n\}$ represents only finitely many primes and finitely many products of two distinct primes.

H-402 *Proposed by Piero Filipponi, Rome, Italy*

A MATRIX GAME (fom the Italian TV serial *Pentathlon*)

Each element of a square matrix M of order 3 is entered with a symbol chosen randomly (with probability 1/2) between two possible symbols (namely x and y). If M contains at least a row (or a column) entirely formed by x's or by y's, then one wins the game.

Generalize to a matrix of order n and find the win probability.

Remark: By inspection, it is easily seen that

$$P_1 = 1,\ P_2 = 7/8, \text{ and } P_3 = 205/256.$$

A computer experiment gave the following results:

$$\begin{array}{ll} P_3 \doteq .801 & P_7 \doteq .200 \\ P_4 \doteq .637 & P_8 \doteq .111 \\ P_5 \doteq .483 & P_9 \doteq .066 \\ P_6 \doteq .325 & P_{10} \doteq .035 \end{array}$$

The conjecture $\lim_{n\to\infty} P_n = 0$ immediately follows.

SOLUTIONS

Late Acknowledgment: *C. Georghiou solved* H-371.

Somewhat Dependable

H-377 *Proposed by Lawrence Somer, Washington, D.C. (Vol. 22, no. 4, November 1984)*

Let $\{w_n\}_{n=0}^{\infty}$ be a k^{th}-order linear integral recurrence satisfying the recursion relation

$$w_{n+k} = a_1 w_{n+k-1} + a_2 w_{n+k-2} + \cdots + a_k w_n.$$

Let t be a fixed positive integer and d a fixed nonnegative integer. Show that the sequence $\{s_n\} = \{w_{tn+d}\}_{n=0}^{\infty}$ also satisfies a k^{th}-order linear integral recursion relation

$$s_{n+k} = a_1^{(t)} s_{n+k-1} + a_2^{(t)} s_{n+k-2} + \cdots + a_k^{(t)} s_n.$$

Show further that the coefficients $a_1^{(t)}, a_2^{(t)}, \ldots, a_k^{(t)}$ depend on t but not on d, and that $a_k^{(t)}$ can be chosen so that

$$a_k^{(t)} = (-1)^{(k+1)(t+1)} a_k^t.$$

Solution by the proposer

Let

$$f(x) = x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_{k-1} x - a_k \tag{1}$$

be the characteristic polynomial corresponding to the recurrence $\{w_n\}$ with characteristic roots $r_1, r_2, \ldots, r_k$. By a classical result in the theory of finite differences,

$$w_n = \sum_{i=1}^{k} (c_i^{(0)} + c_i^{(1)} n + \cdots + c_i^{(m_i - 1)} n^{(m_i - 1)}) r_i^n, \tag{2}$$

where the $c_i^{(j)}$ are complex constants. By (2),

$$s_n = w_{nt+d} = \sum_{i=1}^{k} (c_i^{(0)} + c_i^{(1)} n + \cdots + c_i^{(m_i - 1)} n^{(m_i - 1)}) r_i^{nt+d}$$

$$= \sum_{i=1}^{k} (c_i^{(0)} r_i^d + c_i^{(1)} r_i^d n + \cdots + c_i^{(m_i - 1)} r_i^d n^{(m_i - 1)}) (r_i^t)^n. \tag{3}$$

Since the roots r_i, $1 \leqslant i \leqslant k$, satisfy a monic polynomial over the integers, it follows that all the algebraic conjugates of a fixed characteristic root r_j appear among the r_i's. It then follows that all the algebraic conjugates of r_j^t appear among the t^{th} powers of the characteristic roots. Thus, r_1^t, r_2^t, ..., r_k^t are the roots of a k^{th}-order integral monic polynomial

$$g(x) = x^k - a_1^{(t)} x^{k-1} - \cdots - a_{k-1}^{(t)} x - a_k^{(t)}. \tag{4}$$

It is evident that the root r_i^t of $g(x)$ appears with a multiplicity of at least m_i and that r_i^t satisfies the k^{th}-order linear integral recurrence

$$h_{n+k} = a_1^{(t)} h_{n+k-1} + a_2^{(t)} h_{n+k-2} + \cdots + a_k^{(t)} h_n \tag{5}$$

for $1 \leqslant i \leqslant k$. It is known and easily verified that if $P_i^{(m_i - 1)}$ is a complex polynomial of degree at most $m_i - 1$, then the sequence $\{c_n\}$ defined by

$$c_n = \sum_{i=1}^{k} P_i^{(m_i - 1)}(n) r_i^t$$

also satisfies the recursion relation given by (5). It thus follows from (3) that $\{s_n\} = \{w_{nt+d}\}$ also satisfies the same recursion relation.

It follows from (4) that, for $1 \leqslant j \leqslant k$,

$$-a_j^{(t)} = \sum (-1)^j r_{i_1}^t r_{i_2}^t \cdots r_{i_j}^t, \tag{6}$$

where one sums over all indices i_1, i_2, ..., i_j such that

$$1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant k.$$

Thus, the coefficients $a_i^{(t)}$, $1 \leqslant i \leqslant k$, clearly depend on t but not on d.

Finally, it follows from (1) that

$$-a_k = (-1)^k r_1 r_2 \cdots r_k. \tag{7}$$

Thus, from (6) and (7), we see that

$$a_k^{(t)} = (-1)^{k+1} r_1^t r_2^t \cdots r_k^t = (-1)^{k+1} (r_1 r_2 \cdots r_k)^t$$

$$= (-1)^{k+1} [(-1)^{k+1} a_k]^t = (-1)^{(k+1)(t+1)} a_k^t.$$

We are now done.

Also solved by P. Bruckman, L. Dresel, and S. Papastavridis.

A Prime Result

H-378 *Proposed by M. Wachtel, Zurich, Switzerland*
(Vol. 22, no. 4, November 1984)

For every positive integer x and y, provided they are prime to each other, show that no integral divisor of $x^2 - 5y^2$ is congruent to 3 or 7, modulo 10.

Solution by J. M. Metzger, Grand Forks, ND

Let p be a prime divisor of $x^2 - 5y^2$. Now p is not a divisor of y for if so it divides x as well, contrary to the assumption that x and y are prime to each other. Since p does not divide y, y has a multiplicative inverse, say z, modulo p. So, from $x^2 - 5y^2 \equiv 0 \pmod{p}$, it follows that $(xz)^2 \equiv 5 \pmod{p}$. Thus, 5 is a quadratic residue modulo p. Hence $p = 2$, 5 or $p \equiv \pm 1 \pmod{10}$. Products of such primes can never be 3 or 7 modulo 10, and so $x^2 - 5y^2$ cannot have divisors congruent to 3 or 7 modulo 10.

Also solved by P. Bruckman, L. Dresel, L. Kuipers, L. Somer, T. White, and the proposer.

Sum Formula!

H-379 *Proposed by Andreas N. Philippou and Frosso S. Makri,*
University of Patras, Patras, Greece
(Vol. 22, no. 4, November 1984)

For each fixed integer $k \geqslant 2$, let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k [1]. Show that

$$f_{n+2}^{(k)} = \sum_{i=0}^{\infty} \sum_{j=0}^{n} (-1)^i \binom{n - ik}{n - j} \binom{n - j + 1}{i}, \quad n \geqslant 0.$$

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order K." *The Fibonacci Quarterly* **20**, no. 1 (1982):28-32.

Solution by the proposers

The problem is trivially true for $n = 0$. It suffices therefore to show it for $n \geqslant 1$. Denote by S_n and L_n, respectively, the number of successes and the length of the longest success run in n ($\geqslant 1$) Bernoulli trials. It has been shown in [1] and [3] that

$$P[L_n \leqslant k - 1 | S_n = j] = \binom{n}{j}^{-1} \sum_{i=0}^{\infty} (-1)^i \binom{n - ik}{n - j} \binom{n - j + 1}{i} \tag{1}$$

and

$$P[L_n \leqslant k - 1 | S_n = j] = \binom{n}{j}^{-1} \sum_{i=0}^{k-1} \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}, \tag{2}$$

where the inner sum is taken over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n - i$ and $n_1 + \cdots + n_k = n - j$. Relations (1) and (2) give

$$\sum_{j=0}^{n}\sum_{i=0}^{k-1}\sum_{\substack{n_1,\ldots,n_k\\ n_1+2n_2+\cdots+kn_k=n-i\\ n_1+\cdots+n_k=n-j}}\binom{n_1+\cdots+n_k}{n_1,\ldots,n_k}=\sum_{i=0}^{\infty}\sum_{j=0}^{n}(-1)^i\binom{n-ik}{n-j}\binom{n-j+1}{i}. \tag{3}$$

Now let $p = 1/2$. Then,

$$P[L_n \leqslant k-1] = \frac{1}{2^n}\sum_{i=0}^{k-1}\sum_{\substack{n_1,\ldots,n_k\\ n_1+2n_2+\cdots+kn_k=n-i}}\binom{n_1+\cdots+n_k}{n_1,\ldots,n_k}, \quad \text{by [2]},$$

$$= \frac{1}{2^n}\sum_{i=0}^{k-1}f_{n-i+1}^{(k)} = f_{n+2}^{(k)}/2^n, \quad \text{by [4]}. \tag{4}$$

Moreover,

$$P[L_n \leqslant k-1] = \sum_{j=0}^{n}P[L_n \leqslant k-1, S_n = j] = \sum_{j=0}^{n}P[L_n \leqslant k-1 | S_n = j]P[S_n = j]$$

$$= \frac{1}{2^n}\sum_{j=0}^{n}\sum_{i=0}^{k-1}\sum_{\substack{n_2,\ldots,n_k\\ n_1+2n_2+\cdots+kn_k=n-i\\ n_1+\cdots+n_k=n-j}}\binom{n_1+\cdots+n_k}{n_1,\ldots,n_k}, \quad \text{by (2)}. \tag{5}$$

The last three relations give

$$f_{n+2}^{(k)} = \sum_{i=0}^{\infty}\sum_{j=0}^{n}(-1)^i\binom{n-ik}{n-j}\binom{n-j+1}{i}, \quad n \geqslant 1,$$

which completes the proof of the problem.

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Also solved by P. Bruckman, C. Georghiou, and S. Papastavridis.

A Sparse Sequence

H-380 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC (Vol. 22, no. 4, November 1984)*

The sequence 1, 4, 5, 9, 13, 14, 16, 25, 29, 30, 36, 41, 49, 50, 54, 55, ... of squares and sums of consecutive squares appeared in Problem B-495. Show that this sequence has Schnirelmann density zero.

Solution by Paul S. Bruckman, Fair Oaks, CA

Let S denote the given sequence. We may characterize S as the sequence of sums of the form

$$q(i, j) = \sum_{k=i}^{i+j-1} k^2, \quad i, j \geqslant 1. \tag{1}$$

Given n, let $P(n)$ denote the number of pairs (i, j) such that $q(i, j) \leqslant n$. Then $\lim\limits_{n\to\infty} P(n)/n$ is the Schnirelmann density of S, which we seek to prove is zero.

Now $q(i, j) = q(1, i + j - 1) - q(1, i - 1)$; after some simplification, we find

$$q(i, j) = ji(i + j - 1) + q(1, j - 1). \tag{2}$$

Assuming j fixed for the time being, we see from (2) that $q(i, j) \leqslant n$ implies $ji^2 \leqslant n$, or $i \leqslant (n/j)^{1/2}$; also,

$$j^3/3 < \frac{1}{6}\, j(j + 1)(2j + 1) = q(1, j) \leqslant q(i, j) \leqslant n,$$

so $j^3/3 < n$, or $j < (3n)^{1/3}$. Therefore,

$$P(n) \leqslant \sum_{j=1}^{m} (n/j)^{1/2}, \quad \text{where } m = [(3n)^{1/3}]. \tag{3}$$

Now consider the sum

$$Z(N) = \sum_{k=1}^{N} k^{-1/2}, \quad \text{where } N \text{ is large.} \tag{4}$$

We see that

$$Z(N) = N^{1/2} \sum_{k=1}^{N} (k/N)^{-1/2} N^{-1} \sim N^{1/2} \int_0^1 x^{-1/2}\, dx \text{ as } N \to \infty.$$

Since

$$\int_0^1 x^{-1/2}\, dx = 2x^{1/2}\Big]_0^1 = 2,$$

thus

$$Z(N) = 0(N^{1/2}) \quad \text{as } N \to \infty. \tag{5}$$

Returning to (3), we see that $P(n) \leqslant Q(n)$, where $Q(n) = 0(n^{1/2} \cdot m^{1/2}) = 0(n^{2/3})$ as $n \to \infty$; hence

$$P(n) = 0(n^{2/3}) \quad \text{as } n \to \infty.$$

Thus, $P(n)/n = 0(n^{-1/3}) = 0(n)$ as $n \to \infty$. Q.E.D.

Also solved by C. Georghiou and the proposer.

SEND IN THOSE PROBLEM PROPOSALS NOW!

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Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

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