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CONTENTS

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

MATRIX AND OTHER SUMMATION TECHNIQUES FOR PELL POLYNOMIALS

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1. INTRODUCTION

Pell polynomials $P_n(x)$ and *Pell-Lucas polynomials* $Q_n(x)$ are defined in [7], [9], and [10] by the recurrence relations

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x), \quad P_0(x) = 0, \; P_1(x) = 1, \tag{1.1}$$

and

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x), \quad Q_0(x) = 2, \; Q_1(x) = 2^x, \tag{1.2}$$

with integer n unrestricted.

Equation (1.1) may be written in the form

$$P_r(x) = \{P_{r+1}(x) - P_{r-1}(x)\}/2x. \tag*{(1.1)$'$}$$

Binet forms are

$$P_n(x) = (\alpha^n - \beta^n)/(\alpha - \beta) \tag{1.3}$$

and

$$Q_n(x) = \alpha^n + \beta^n, \tag{1.4}$$

where α and β are the roots of the characteristic equation of (1.1) and (1.2), namely,

$$t^2 - 2xt - 1 = 0 \tag{1.5}$$

so that

$$\begin{cases} \alpha = x + \sqrt{x^2 + 1} \\ \beta = x - \sqrt{x^2 + 1} \end{cases} \text{with } \alpha + \beta = 2x, \; \alpha\beta = -1, \; \alpha - \beta = 2\sqrt{x^2 + 1}. \tag{1.6}$$

Explicit summation representations for $P_n(x)$ and $Q_n(x)$, and relations among them, are established in [7], [9], and [10].

Emphasis in this paper will be given to matrix methods so we introduce the matrix P which generates Pell polynomials and many of their properties ([7], [9]). Historical information about the background of this matrix is provided in [9].

Let

$$P = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \tag{1.7}$$

so that, by induction,

$$P^n = \begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix} \tag{1.8}$$

Hence,

$$\begin{bmatrix} P_{n+1}(x) \\ P_n(x) \end{bmatrix} = P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1.9}$$

and so

$$P_n(x) = [1 \quad 0]\, P^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{1.10}$$

From [7, (2.1)], we deduce

$$\begin{bmatrix} Q_{n+1}(x) \\ Q_n(x) \end{bmatrix} = P^n \begin{bmatrix} 2x \\ 2 \end{bmatrix} \tag{1.11}$$

and

$$Q_n(x) = [1 \quad 0]\, P^n \begin{bmatrix} 2x \\ 2 \end{bmatrix}. \tag{1.12}$$

Although some summation formulas for $P_n(x)$ and $Q_n(x)$ are recorded in [7], it is thought desirable to investigate the summation problem more fully. Initially, some well-established techniques are utilized to produce simple summations. More complicated techniques are derived to achieve a higher degree of completeness.

As an example of the usage of the matrix (and determinant) approach, we demonstrate the Simson formula for Pell polynomials, [7, (2.5)], namely,

$$P_{n+1}(x)P_{n-1}(x) - P_n^2(x) = (-1)^n, \tag{1.13}$$

which may, of course, be established by means of the Binet form (1.3).

More generally in the first instance, consider

$$P_n^2(x) - P_{n+r}(x)P_{n-r}(x) = \begin{vmatrix} P_n(x) & P_{n+r}(x) \\ P_{n-r}(x) & P_n(x) \end{vmatrix} \tag{1.14}$$

$$= \left| \begin{bmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{bmatrix} P^{n-r} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \vdots \begin{bmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{bmatrix} P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right|$$

... by (1.8), [7, (3.14)]

$$= \begin{vmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{vmatrix} \left| P^{n-r} \right| \begin{vmatrix} 1 & P_{r+1}(x) \\ 0 & P_r(x) \end{vmatrix}, \text{ by (1.9)}, = (-1)^{n-r} P_r^2(x).$$

Putting $r = 1$ in this generalized Simson formula, we obtain the Pell-analogue (1.13) of the Simson formula for Fibonacci numbers.

Because of its importance and subsequent use, we append the difference equation [7, (3.28)]

$$P_{mn+r}(x) = \begin{cases} P_m(x)Q_{(n-1)m+r}(x) + (-1)^m P_{(n-2)m+r}(x) \\ Q_m(x)P_{(n-1)m+r}(x) + (-1)^{m-1}P_{(n-2)m+r}(x) \end{cases} \tag{1.15}$$

and the Pell-Lucas analogue [7, (3.29)]

$$Q_{mn+r}(x) = Q_m(x)Q_{(n-1)m+r}(x) + (-1)^{m-1}Q_{(n-2)m+r}(x). \tag{1.16}$$

A result needed in Section 8, which is not specifically given in [7], is

$$Q_n(x)Q_{n+1}(x) - 4(x^2 + 1)P_n(x)P_{n+1}(x) = 4x(-1)^n, \tag{1.17}$$

which may be proved by using (1.3) and (1.4).

2. SOME SUMMATION TECHNIQUES

A. Consider the *series of matrices* [cf. (1.8)]

$$A = I + P + P^2 + \cdots + P^{n-2} + P^{n-1}.$$

Then

$$PA = P + P^2 + P^3 + \cdots + P^{n-1} + P^n,$$

whence

$$\begin{aligned} (P - I)A &= P^n - I \\ A &= (P - I)^{-1}(P^n - I) \\ &= \frac{1}{2x}\begin{bmatrix} 1 & 1 \\ 1 & 1 - 2x \end{bmatrix}\begin{bmatrix} P_{n+1}(x) - 1 & P_n(x) \\ P_n(x) & P_{n-1}(x) - 1 \end{bmatrix} \quad \text{by (1.8)} \\ &= \frac{1}{2x}\begin{bmatrix} P_{n+1}(x) + P_n(x) - 1 & P_n(x) + P_{n-1}(x) - 1 \\ P_n(x) + P_{n-1}(x) - 1 & P_{n-2}(x) + P_{n-1}(x) + 2x - 1 \end{bmatrix}. \end{aligned}$$

Now

$$\sum_{r=1}^{n} P_r(x) = [1 \quad 0]A\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ by (1.10)}.$$

Hence

$$\sum_{r=1}^{n} P_r(x) = (P_{n+1}(x) + P_n(x) - 1)/2x. \tag{2.1}$$

B. Using the Binet form (1.4), we have

$$\sum_{r=1}^{n} Q_r(x) = \sum_{r=1}^{n} (\alpha^r + \beta^r)$$

which, with the application of the summation formula for a geometric series and the properties of α and β, reduces to

$$\sum_{r=1}^{n} Q_r(x) = (Q_{n+1}(x) + Q_n(x) - 2x - 2)/2x. \tag{2.2}$$

Clearly, the matrix technique A could be used here also.

C. Next, we use *difference equations* derived from the recurrence relation (1.1), namely,

$$\begin{aligned} 2xP_1(x) &= P_2(x) - P_0(x) \\ 2xP_3(x) &= P_4(x) - P_2(x) \\ &\cdots\cdots\cdots\cdots\cdots \\ 2xP_{2n-1}(x) &= P_{2n}(x) - P_{2n-2}(x) \end{aligned}$$

whence, on addition and simplification,

$$\sum_{r=1}^{n} P_{2r-1}(x) = P_{2n}(x)/2x. \tag{2.3}$$

Summation formulas for

$$\sum_{r=1}^{n} P_{2r}(x), \quad \sum_{r=1}^{n} Q_{2r-1}(x), \quad \text{and} \quad \sum_{r=1}^{n} Q_{2r}(x)$$

are given in [9], as indeed are (2.1), (2.2), and (2.3).

D. Fourthly, we utilize an extension of technique C. In this method, our aim is to find sums of series of Pell polynomials with subscripts in arithmetic progression.

Let

$$S_1 = \sum_{i=1}^{n} P_{im}(x), \quad S_2 = \sum_{i=1}^{n} P_{im-1}(x),$$

$$S_3 = \sum_{i=1}^{n} P_{im-2}(x), \quad \ldots, \quad S_m = \sum_{i=1}^{n} P_{im-(m-1)}(x). \tag{2.4}$$

Then, the set of equations connecting the members of $\{S_i\}$ in (2.4) may be shown to be:

$$\left\{ \begin{array}{l} 2xS_1 + S_2 \cdots\cdots\cdots\cdots\cdots\cdots -S_m = P_{nm+1}(x) - P_1(x) \\ -S_1 + 2xS_2 + S_3 \cdots\cdots\cdots\cdots\cdots = 0 \\ \qquad -S_2 + 2xS_3 + S_4 \cdots\cdots\cdots\cdots = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots -S_{m-2} + 2xS_{m-1} + S_m = 0 \\ S_1 \cdots\cdots\cdots\cdots\cdots -S_{m-1} + 2xS_m = P_{nm}(x) - P_0(x) \end{array} \right. \tag{2.5}$$

Next, write:

$$\mathscr{E}_1 = \begin{bmatrix} 2x \\ -1 \\ 0 \\ \\ \vdots \\ \\ 1 \end{bmatrix} \quad \mathscr{E}_2 = \begin{bmatrix} 1 \\ 2x \\ -1 \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix} \quad \mathscr{E}_3 = \begin{bmatrix} 0 \\ 1 \\ 2x \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \ldots, \ \mathscr{E}_m = \begin{bmatrix} -1 \\ 0 \\ \\ \vdots \\ \\ 1 \\ 2x \end{bmatrix} \tag{2.6}$$

$$\mathscr{F} = \begin{bmatrix} P_{nm+1}(x) - P_1(x) \\ 0 \\ 0 \\ \vdots \\ 0 \\ P_{nm}(x) - P_0(x) \end{bmatrix},$$

where $\mathscr{E}_i$ and $\mathscr{F}$ are $m \times 1$ matrices.

Denote by e_{ij} the element in the i^{th} row of $\mathscr{E}_j$.

Matrices in (2.6) are then defined by:

$$\begin{cases} e_{ii} = 2x & \text{for } i = 1, 2, \ldots, m \\ e_{1m} = -1 & \\ e_{m1} = 1 & \\ e_{i,i+1} = 1 & \text{for } i = 1, 2, \ldots, m-1 \\ e_{i-1,i} = -1 & \text{for } i = 2, 3, \ldots, m \\ e_{ij} = 0 & \text{otherwise.} \end{cases} \tag{2.6$'$}$$

All the entries in $\mathscr{F}$, except those in the first and last rows, are zero. Write

$$\psi_m(x) = \left| \mathscr{E}_1 \ \vdots \ \mathscr{E}_2 \ \vdots \ \cdots \ \vdots \ \mathscr{E}_m \right|. \tag{2.7}$$

Designate by $\psi_m^{(i)}(x)$ the determinant obtained from $\psi_m(x)$ in (2.7) by replacing the i^{th} column by $\mathscr{F}$ in (2.6).

Cramer's Rule then gives the solution of the system of equations (2.5) as

$$S_i = \frac{\psi_m^{(i)}(x)}{\psi_m(x)}. \tag{2.8}$$

Comparing this result with (2.10) below leads us to the identity [compare (3.15, (3.16)]

$$\psi_m(x) = Q_m(x) - 1 + (-1)^{m+1}, \tag{2.9}$$

which may be proved by induction.

One may use whichever of the above techniques, A-D, is most appropriate to the occasion.

This brief illustration of four simple techniques is by no means exhaustive. Other methods will be suggested later.

More generally, let

$$\mathscr{P} = P_{m+k}(x) + P_{2m+k}(x) + P_{3m+k}(x) + \cdots + P_{mn+k}(x).$$

$$\therefore \quad -Q_m(x)\mathscr{P} = -Q_m(x)P_{m+k}(x) - Q_m(x)P_{2m+k}(x) - Q_m(x)P_{3m+k}(x) - \cdots \cdots - Q_m(x)P_{mn+k}(x)$$

$$(-1)^m\mathscr{P} = (-1)^m P_{m+k} + (-1)^m P_{2m+k}(x) + (-1)^m P_{3m+k}(x) + \cdots \cdots + (-1)^m P_{mn+k}(x).$$

Add and use equation (1.15) to obtain, with care,

$$\sum_{r=1}^{n} P_{mr+k}(x) = \frac{(-1)^m\{P_{mn+k}(x) - P_k(x)\} - \{P_{m(n+1)+k}(x) - P_{m+k}(x)\}}{1 - Q_m(x) + (-1)^m}. \tag{2.10}$$

Similarly,

$$\sum_{r=1}^{n} Q_{mr+k}(x) = \frac{(-1)^m\{Q_{mn+k}(x) - Q_k(x)\} - \{Q_{m(n+1)+k}(x) - Q_{m+k}(x)\}}{1 - Q_m(x) + (-1)^m}. \tag{2.11}$$

Results (2.10) and (2.11) could be obtained laboriously by other means, e.g., by using the Binet form or the matrix P.

Various specializations of (2.10) and (2.11) appearing in [9] are of interest, as, e.g.,

$$\sum_{r=1}^{n} P_{3r}(x) = \{P_{3n+3}(x) + P_{3n}(x) - P_3(x)\}/Q_3(x). \tag{2.12}$$

Several interesting simplifications arise when $m = 4a$ and $m = 4a + 2$, e.g., after manipulation,

$$\sum_{r=1}^{n} P_{4ar+k}(x) = P_{2a(n+1)+k}(x)P_{2an}(x)/P_{2a}(x). \tag{2.13}$$

Details are given in [9].

3. DETERMINANTAL GENERATION

Following the ideas and notation in [7], let us define the determinants of order n below, where d_{ij} is the entry in row i and column j:

$$\Delta_{n,m}(x) : \begin{cases} d_{ii} = Q_m(x) & i = 1, 2, \ldots, n \\ d_{i,i+1} = 1 & i = 1, 2, \ldots, n-1 \\ d_{i,i-1} = (-1)^m & i = 2, \ldots, n \\ d_{ij} = 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

$$\delta_{n,m}(x) : \text{as for } \Delta_{n,m}(x) \text{ except that } d_{i,i+1} = -1,\ d_{i,i-1} = -(-1)^m. \tag{3.2}$$

$$\Delta^*_{n,m}(x) : \text{as for } \Delta_{n,m}(x) \text{ except that } d_{12} = 2. \tag{3.3}$$

$$\delta^*_{n,m}(x) : \text{as for } \delta_{n,m}(x) \text{ except that } d_{12} = -2. \tag{3.4}$$

Using the method of induction, we can establish that

$$\Delta_{n,m}(x) = P_{(n+1)m}(x)/P_m(x). \tag{3.5}$$

When $m = 1$, (3.5) becomes equation (5.5) in [7]. For $m = k + 1$, we use equation (1.15) to validate (3.5).

Similarly, we demonstrate with the aid of (1.16) that

$$\delta_{n,m}(x) = P_{(n+1)m}(x)/P_m(x). \tag{3.6}$$

In a similar vein, we may show that

$$\Delta^*_{n,m}(x) = Q_{mn}(x) \tag{3.7}$$

and

$$\delta^*_{n,m}(x) = Q_{mn}(x). \tag{3.8}$$

Suitable expansion down columns or along rows yields:

$$\Delta_{n,m}(x) = Q_m(x)\Delta_{n-1,m}(x) + (-1)^{m+1}\Delta_{n-2,m}(x); \tag{3.9}$$

$$\delta_{n,m}(x) = Q_m(x)\delta_{n-1,m}(x) + (-1)^{m+1}\delta_{n-2,m}(x); \tag{3.10}$$

$$\begin{aligned}\Delta^*_{n,m}(x) &= Q_m(x)\Delta^*_{n-1,m}(x) + (-1)^{m+1}\Delta^*_{n-2,m}(x)\\ &= Q_m(x)\Delta_{n-1,m}(x) + 2(-1)^{m+1}\Delta_{n-2,m}(x);\end{aligned} \tag{3.11}$$

$$\begin{aligned}\delta^*_{n,m}(x) &= Q_m(x)\delta^*_{n-1,m}(x) + (-1)^{m+1}\delta^*_{n-2,m}(x)\\ &= Q_m(x)\delta_{n-1,m}(x) + 2(-1)^{m+1}\delta_{n-2,m}(x).\end{aligned} \tag{3.12}$$

Putting $m = 1$ in (3.5)-(3.8), and in (3.9) and (3.11), we readily obtain the equations (5.5)-(5.8), and (5.9) and (5.10), respectively, in [7]. Moreover, $\Delta_{n,1}(1) = \delta_{n,1}(1) = P_{n+1}$ and $\Delta^*_{n,1}(1) = \delta^*_{n,1}(1) = Q_n$, where P_{n+1} and Q_n are *Pell numbers* and *Pell-Lucas numbers*, respectively, occurring when $x = 1$.

Variations, though small, of the determinants (3.1)-(3.4) above and of their specializations when $m = 1$, as given in [7], are used in [9] to obtain (3.5)-(3.12). Mahon, in [9], conceived these determinants with some complex entries as extensions of a determinant utilized in [2] and [8].

Next, consider the determinant $\omega_{n,m}(x)$ of order n defined by

$$\omega_{n,m}(x): \begin{cases} d_{ii} = Q_m(x) & i = 1, 2, \ldots n \\ d_{i,i+1} = -1 & i = 1, 2, \ldots, n-1 \\ d_{i,i-1} = -(-1)^m & i = 2, 3, \ldots, n \\ d_{n1} = (-1)^m & \\ d_{1n} = 1 & \\ d_{ij} = 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Careful evaluation of this determinant, with appeal to (3.8) and (3.12) gives us

$$\omega_{n,m}(x) = Q_{mn}(x) + (-1)^m + (-1)^{m(n-1)}. \tag{3.14}$$

In particular, when $m = 1$, and writing $\omega_n(x) \equiv \omega_{n,1}(x)$, we get:

$$\omega_n(x) = Q_n(x) - 1 + (-1)^{n+1}; \tag{3.15}$$

$$\omega_{2n-1}(x) = Q_{2n-1}(x); \tag{3.16}$$

$$\omega_{4n}(x) = 4(x^2 + 1)P_{2n}^2(x), \text{ by equation (2.18) in [7]}; \tag{3.17}$$

$$\omega_{4n+2}(x) = Q_{2n+1}^2(x); \tag{3.18}$$

where, to obtain (3.18), we may use result (3.25) in [7] in which n and r are both replaced by $2n + 1$ $[Q_0(x) = 2]$.

Observe that $-\omega_n(x)$ in (3.15) is precisely the form of the denominator in (2.10) and (2.11). [Cf. (2.9).] Indeed, it was in this context that the need to investigate the determinants $\omega_n(x)$ arose.

4. ALTERNATING AND RELATED SERIES

To avoid tedium and to save some space, we will as a rule hereafter merely give the results of the more important summations which we desire to record. Some of the proofs are quite difficult.

$$\sum_{r=1}^{n} rP_r(x) = [nxP_{n+1}(x) + \{(n-1)x - 1\}P_n(x) - P_{n-1}(x) + 1]/2x^2. \tag{4.1}$$

Proving this is straightforward. From (1.1)′, we have

$$2xP_1(x) = P_2(x) - P_0(x).$$

Multiply this by 2, 3, ..., n in turn, add, and use (2.1). Then (4.1) results.

Similarly, we establish

$$\sum_{r=1}^{n} rQ_r(x) = [nxQ_{n+1}(x) + \{(n-1)x - 1\}Q_n(x) - Q_{n-1}(x) + 2]/2x^2; \tag{4.2}$$

$$\sum_{r=1}^{n} (-1)^r rP_r(x) = [(-1)^n nxP_{n+1}(x) + (-1)^{n-1}P_n(x)\{(n-1)x + 1\} + (-1)^n P_{n-1}(x) - 1]/2x^2; \tag{4.3}$$

$$\sum_{r=1}^{n} (-1)^r rQ_r(x) = [(-1)^n nxQ_{n+1}(x) + (-1)^{n-1}Q_n(x)\{(n-1)x + 1\} + (-1)^n Q_{n-1}(x) - Q_1(x) + Q_0(x)(1 + x)]/2x^2. \tag{4.4}$$

More generally, suppose we write

$$F(n, x, y) = \sum_{r=1}^{n} P_{mr+k}(x)y^r \tag{4.5}$$

and

$$G(n, x, y) = \sum_{r=1}^{n} Q_{mr+k}(x)y^r. \tag{4.6}$$

Now use (1.15) and (1.16) for $P_{m+k}(x)$, $P_{2m+k}(x)$, ..., $P_{mn+k}(x)$ and $Q_{m+k}(x)$, $Q_{2m+k}(x)$, ..., $Q_{mn+k}(x)$, add and obtain explicit expressions for $F(n, x, y)$ and $G(n, x, y)$. Details of these calculations are left to the reader. If we then put $y = 1$, we derive formulas for

$$\sum_{r=1}^{n} P_{mr+k}(x) \quad \text{and} \quad \sum_{r=1}^{n} Q_{mr+k}(x).$$

On the other hand, $y = -1$ leads to formulas for

$$\sum_{r=1}^{n}(-1)^r P_{mr+k}(x) \quad \text{and} \quad \sum_{r=1}^{n}(-1)^r Q_{mr+k}(x).$$

Differentiating with respect to y in (4.5) and (4.6) gives

$$\sum_{r=1}^{n} rP_{mr+k}(x) = F'(n, x, 1), \tag{4.7}$$

$$\sum_{r=1}^{n} rQ_{mr+k}(x) = G'(n, x, 1), \tag{4.8}$$

$$\sum_{r=1}^{n}(-1)^{r-1} rP_{mr+k}(x) = F'(n, x, -1), \tag{4.9}$$

$$\sum_{r=1}^{n}(-1)^{r-1} rQ_{mr+k}(x) = G'(n, x, -1), \tag{4.10}$$

in which the prime denotes the derivative with respect to y. When $m = 1$, $k = 0$ in (4.7)-(4.10), (4.1)-(4.4) occur.

Next, consider $P_1(x) = \{P_2(x) - P_0(x)\}/2x$ from the recurrence (1.1)'. Multiply this equation by $2^2, 3^2, \ldots, n^2$ in turn, add, and use (4.1). Then

$$\sum_{r=1}^{n} r^2 P_r(x) = [2n^2x^2P_{n+1}(x) + 2(n-1)x\{(n-1)x - 2\}P_n(x) - 4\{(n-2)x - 1\}P_{n-1}(x) + 4P_{n-2}(x) - 4]/4x^3. \tag{4.11}$$

Similarly,

$$\sum_{r=1}^{n} r^2 Q_r(x) = [2n^2x^2Q_{n+1}(x) + 2(n-1)x\{(n-1)x - 2\}Q_n(x) - 4\{(n-2)x - 1\}Q_{n-1}(x) + 4Q_{n-2}(x) - 4x^2 - 8]/4x^3, \tag{4.12}$$

$$\sum_{r=1}^{n}(-1)^r r^2 P_r(x) = [(-1)^n 2x^2n^2P_{n+1}(x) + (-1)^{n-1}2x(n-1)P_n(x)\{x(n-1) + 2\} + 4(-1)^{n-2}P_{n-1}(x)\{1 + (n-2)x\} + 4(-1)^{n-1}P_{n-2}(x) - 4]/4x^3, \tag{4.13}$$

$$\sum_{r=1}^{n}(-1)^r r^2 Q_r(x) = [(-1)^n 2x^2n^2Q_{n+1}(x) + (-1)^{n-1}2x(n-1)Q_n(x)\{x(n-1) + 2\} + 4(-1)^{n-2}Q_{n-1}(x)\{1 + (n-2)x\} + 4(-1)^{n-1}Q_{n-2}(x) + 4x^2 + 8]/4x^3. \tag{4.14}$$

Other methods for obtaining the above results in this section are available, for example the difference equation technique employed in [9], although this involves a great deal of complicated algebraic manipulation. Of the various approaches open to us for obtaining the summations, perhaps the most powerful and most appealing procedure is that using difference equations. Indeed, by employing one such difference equation, Mahon [9] has found formulas involving the generalized summations

$$\sum_{r=1}^{n} r^t P_{mr+k}(x) \quad \text{and} \quad \sum_{r=1}^{n} r^t Q_{mr+k}(x),$$

but the results are not a pretty sight!.

To give a flavor for these difference equations, we record one used in the construction of the formula (4.11) by this method, namely,

$$\begin{aligned}&(r+1)^2P_{m(r+1)+k}(x) - Q_m(x)r^2P_{mr+k}(x) + (-1)^m(r-1)^2P_{m(r-1)+k}(x)\\&= 2rP_m(x)Q_{mr+k}(x) + Q_m(x)P_{mr+k}(x).\end{aligned}$$

Many similar complicated results are given in [9].

To conclude this section, we append some sums of cubes of $P_n(x)$ and $Q_n(x)$ obtained with the aid of the Binet formulas (1.3) and (1.4).

$$\begin{aligned}\sum_{r=1}^{n} P_r^3(x) = [&P_{3n+3}(x) + P_{3n}(x) - 3(4x^2+3)\{(-1)^n(P_{n+1}(x) - P_n(x)\}\\&+ 8(x^2+1)]/4(x^2+1)Q_3(x).\end{aligned} \tag{4.16}$$

$$\begin{aligned}\sum_{r=1}^{n} Q_r^3(x) = [&Q_{3n+3}(x) + Q_{3n}(x) - Q_3(x) - Q_0(x) + 3(4x^2+3)\{(-1)^nQ_{n+1}(x)\\&- Q_n(x)) - Q_1(x) + Q_0(x)\}]/Q_3(x).\end{aligned} \tag{4.17}$$

$$\begin{aligned}\sum_{r=1}^{n} (-1)^rP_r^3(x) = [&(-1)^n\{P_{3n+3}(x) - P_{3n}(x)\} - P_3(x) - 3(4x^2+3)\{P_{n+1}(x)\\&+ P_n(x) - 1\}]/4(x^2+1)Q_3(x).\end{aligned} \tag{4.18}$$

$$\begin{aligned}\sum_{r=1}^{n} (-1)^rQ_r^3(x) = [&(-1)^n\{Q_{3n+3}(x) - Q_{3n}(x)\} - \{Q_3(x) - Q_0(x)\}\\&+ 3(4x^2+3)\{Q_{n+1}(x) + Q_n(x) - Q_1(x) - Q_0(x)\}]/Q_3(x).\end{aligned} \tag{4.19}$$

5. SERIES OF SQUARES AND PRODUCTS OF $P_n(x)$ AND $Q_n(x)$

Multiply both sides of (1.1)′ by $P_r(x)$ and add. Then

$$\sum_{r=1}^{n} P_r^2(x) = P_{n+1}(x)P_n(x)/2x. \tag{5.1}$$

Similarly,

$$\sum_{r=1}^{n} Q_r^2(x) = \{Q_{n+1}(x)Q_n(x) - 4x\}/2x. \tag{5.2}$$

Again, in this development, the method of difference equations has general applicability. For instance, after much algebraic maneuvering, one can obtain the difference equation appropriate to (5.1), namely,

$$P_{n+1}^2(x) - (4x^2+2)P_n^2(x) + P_{n-1}^2(x) = 2(-1)^n. \tag{5.1a}$$

More generally, difference equations can be applied to find formulas for

$$\sum_{r=1}^{n} P_{mr+k}^2(x) \quad \text{and} \quad \sum_{r=1}^{n} Q_{mr+k}^2(x).$$

For the former summation, for instance, the difference equation is

$$P_{m(r+1)+k}^2(x) - Q_{2m}(x)P_{mr+k}(x) + P_{m(r-1)+k}^2(x) = 2P_m^2(x)(-1)^{mr+k}, \tag{5.1b}$$

which reduces to the simpler form (5.1a) when $m = 1$, $k = 0$ (and r is replaced by n).

If we multiply both sides of (1.1) by $P_{r-1}(x)$ and add, then, by Simson's formula (1.14),

$$\sum_{r=1}^{n} P_{r-1}(x)P_r(x) = \{P_n^2(x) - \tfrac{1}{2}(1 - (-1)^n)\}/2x. \tag{5.3}$$

Similarly,

$$\sum_{r=1}^{n} Q_{r-1}(x)Q_r(x) = \{Q_n^2(x) - 4 + 2(x^2 + 1)(1 - (-1)^n)\}/2x. \tag{5.4}$$

Alternating series may be summed using (1.1)′. First, write

$$D = \sum_{r=1}^{n} (-1)^r P_r^2(x), \quad E = \sum_{r=1}^{n} (-1)^{r-1} P_{r-1}(x)P_r(x).$$

Then, multiplying both sides of (1.1)′ by $(-1)^r P_r(x)$ and adding gives

$$2xD - 2E = (-1)^n P_n(x)P_{n+1}(x) \;..........\; \text{(i)}.$$

Next, multiplying both sides of (1.1)′ by $(-1)^{r-1}P_{r-1}(x)$ and adding gives

$$2D + 2xE = (-1)^n P_n^2(x) - n \;.............\; \text{(ii)}.$$

Solve (i) and (ii), and use (2.1) and (2.3) in [7] to obtain

$$\sum_{r=1}^{n} (-1)^r P_r^2(x) = \{(-1)^n Q_{n+1}(x)P_n(x) - 2n\}/4(x^2 + 1) \tag{5.5}$$

and

$$\sum_{r=1}^{n} (-1)^{r-1} P_{r-1}(x)P_r(x) = \{(-1)^{n+1} P_{2n}(x) - 2nx\}/4(x^2 + 1). \tag{5.6}$$

Similarly,

$$\sum_{r=1}^{n} (-1)^r Q_r^2(x) = (-1)^n Q_n(x)P_{n+1}(x) + 2(n - 1) \tag{5.7}$$

and

$$\sum_{r=1}^{n} (-1)^{r-1} Q_{r-1}(x)Q_r(x) = 2nx + (-1)^{n+1} P_{2n}(x). \tag{5.8}$$

Now multiply both sides of (1.1)′ by $(-1)^r rP_r(x)$ and sum. Write

$$D_1 = \sum_{r=1}^{n} (-1)^r rP_r^2 \quad \text{and} \quad E_1 = \sum_{r=1}^{n} (-1)^r (2r - 1)P_{r-1}(x)P_r(x).$$

Then

$$2xD_1 + E_1 = n(-1)^n P_n(x)P_{n+1}(x) \;...........\; \text{(iii)},$$

$$4D_1 - 2xE_1 = (-1)^n (2n + 1)P_n^2(x) - n^2 \;......\; \text{(iv)},$$

where, in (iv), we have multiplied both sides of (1.1)′ by

$$(-1)^{r-1}(2r - 1)P_{r-1}(x)$$

and summed.

Solve (iii) and (iv) to obtain

$$\sum_{r=1}^{n}(-1)^r rP_r^2(x) = [(-1)^n P_n(x)\{nQ_{n+1}(x) + P_n(x)\} - n^2]/4(x^2+1) \tag{5.9}$$

and

$$\sum_{r=1}^{n}(-1)^{r-1}(2r-1)P_{r-1}(x)P_r(x) = [2(-1)^n P_n(x)(xP_n(x) - nQ_n(x)) - 2n^2x]/4(x^2+1). \tag{5.10}$$

Similarly,

$$\sum_{r=1}^{n}(-1)^r rQ_r^2(x) = (-1)^n[nQ_n(x)P_{n+1}(x) + P_n^2(x)] + n^2 \tag{5.11}$$

and

$$\sum_{r=1}^{n}(-1)^{r-1}(2r-1)Q_{r-1}(x)Q_r(x) = 2(-1)^n P_n(x)[xP_n(x) - nQ_n(x)] + 2n^2x. \tag{5.12}$$

Formulas for

$$\sum_{r=1}^{n}(-1)^r P_{mr+k}^2(x) \quad \text{and} \quad \sum_{r=1}^{n}(-1)^r Q_{mr+k}^2(x)$$

may be established by employing appropriate difference equations, e.g., (5.1b) in the first case.

6. COMBINATORIAL SUMMATION IDENTITIES FOR $P_n(x)$ AND $Q_n(x)$

Binomial coefficient factors associated with summations involving $P_n(x)$ and $Q_n(x)$ may be introduced to yield some useful formulas. The techniques for deriving these formulas are varied. Some approaches are indicated below.

Binet formulas (1.3) and (1.4) may be used to derive the following, for which proofs may be found in [9]:

$$\sum_{k=0}^{2n}\binom{2n}{k}P_{k+j}^2(x) = 4^{n-1}(x^2+1)^{n-1}Q_{2n+2j}(x); \tag{6.1}$$

$$\sum_{k=0}^{2n}\binom{2n}{k}Q_{k+j}^2(x) = 4^n(x^2+1)^nQ_{2n+2j}(x); \tag{6.2}$$

$$\sum_{k=0}^{2n}\binom{2n+1}{k}P_{k+j}^2(x) = 4^n(x^2+1)^nP_{2n+2j+1}(x); \tag{6.3}$$

$$\sum_{k=0}^{2n}\binom{2n+1}{k}Q_{k+j}^2(x) = 4^{n+1}(x^2+1)^{n+1}P_{2n+2j+1}(x). \tag{6.4}$$

A considerable number of combinatorial identities relating to $P_n(x)$ and $Q_n(x)$ may be determined. Among these are the general explicit expressions (developments of ideas for Fibonacci numbers in [8]—see also [3]).

$$P_{rn}(x) = \left\{\sum_{k=0}^{[(n-1)/2]}(-1)^{k(r-1)}\binom{n-1-k}{k}Q_r^{n-1-2k}(x)\right\}P_r(x) \tag{6.5}$$

and

$$Q_{rn}(x) = \sum_{k=0}^{[n/2]}(-1)^{k(r-1)}\frac{n}{n-k}\binom{n-k}{k}Q_r^{n-2k}(x), \; n \neq 0. \tag{6.6}$$

Proofs of (6.5) and (6.6) are by the method of mathematical induction.

Putting $r = 1$ in (6.5) and (6.6), we deduce the explicit expressions for $P_n(x)$ and $Q_n(x)$ given in [7] as equations (2.15) and (2.16), respectively. Other summation formulas for $P_n(x)$ are given in [9], where, further, combinatorial expressions are obtained for $P_{(2i+1)r+k}(x)$, $P_{2ir+k}(x)$, $Q_{(2i+1)r+k}(x)$, and $Q_{2ir+k}(x)$.

Bergum and Hoggatt, in [1], found expressions for sums of numbers of recurrence sequences as products of these sequences. It is possible to apply their methods to polynomials.

Two examples of this type of result are herewith given, while many others are derived in [9].

$$\sum_{i=0}^{2^j-1} P_{n+4ki}(x) = P_{n+2(2^j-1)k}(x) \prod_{i=1}^{j} Q_{2^i k}(x) \qquad (k \geqslant 1). \tag{6.7}$$

$$\sum_{i=0}^{2^j-1} Q_{n+(2i-1)k}(x) = Q_{n+2(2^{j-1}-1)k}(x) \prod_{i=0}^{j-1} Q_{2^i k}(x) \qquad (k \text{ even}). \tag{6.8}$$

To establish (6.7), we need equation (3.22) in [7], whereas (6.8) requires (3.23) in [7] together with the result for $Q_n(x)$ corresponding to (6.7) for $P_n(x)$, namely, (6.7) with $P_n(x)$ replaced by $Q_n(x)$.

7. MATRIX SUMMATION METHODS

In Section 1, the matrix P was used to obtain sums of series in which the terms contain Pell polynomials of degree one. Since the particular methods employed there were not especially convenient, we turn our attention to a more fruitful matrix approach, developing an idea expounded in [6]. Applying the Cayley-Hamilton theorem to the matrix P in (1.7), we have

$$P^2 = 2xP + I \tag{7.1}$$

whence

$$P^{2n+j} = (2xP + I)^n P^j. \tag{7.2}$$

Equating appropriate elements on both sides with the aid of (1.8), we obtain the combinatorial summations

$$P_{2n+j}(x) = \sum_{r=0}^{n} \binom{n}{r} (2x)^r P_{r+j}(x) \qquad [2x = P_2(x)] \tag{7.3}$$

and

$$P_{2n+1+j}(x) = \sum_{r=0}^{n} \binom{n}{r} (2x)^r P_{r+1+j}(x). \tag{7.4}$$

Post-multiplying both sides of (7.2) by the column vector $[2x \quad 2]^T$ (the transpose of the corresponding row vector), and appealing to (1.11), we find, on equating appropriate elements, that

$$Q_{2r+j}(x) = \sum_{r=0}^{n} \binom{n}{r} (2x)^r Q_{r+j}(x) \tag{7.5}$$

and

$$Q_{2n+1+j}(x) = \sum_{r=0}^{n} \binom{n}{r} (2x)^r Q_{r+1+j}(x). \tag{7.6}$$

Next, consider

$$\begin{aligned}\sum_{k=0}^{2n}\binom{2n}{k}P^{2k+r} &= P^r(P^2 + I)^{2n} \\ &= P^r\{2(xP + I)\}^{2n} && \text{by (7.1)} \\ &= 2^{2n}P^r(x^2P^2 + 2xP + I)^n \\ &= 2^{2n}(x^2 + 1)^n P^{2n+r} && \text{by (7.1) again,}\end{aligned}$$

whence

$$\sum_{k=0}^{2n}\binom{2n}{k}P_{2k+r}(x) = 2^{2n}(x^2 + 1)^n P_{2n+r}(x). \tag{7.7}$$

Likewise, from (1.11),

$$\sum_{k=0}^{2n}\binom{2n}{k}Q_{2k+r}(x) = 2^{2n}(x^2 + 1)^n Q_{2n+r}(x). \tag{7.8}$$

Similarly,

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}P_{2k+r}(x) = 2^{2n}(x^2 + 1)^n Q_{2n+r+1}(x) \tag{7.9}$$

and

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}Q_{2k+r}(x) = 2^{2n+2}(x^2 + 1)^{n+1} P_{2n+r+1}(x). \tag{7.10}$$

From (7.1) it follows, since $P_2(x) = 2x$, $P_3(x) = 4x^2 + 1$, that

$$P^3 = P_3(x)P + P_2(x)I, \tag{7.11}$$

whence, after calculation,

$$P_{3n+j}(x) = \sum_{r=0}^{n}\binom{n}{r}P_3^{n-r}(x)P_2^r(x)P_{n-r+j}(x). \tag{7.12}$$

Since

$$P^{3n+j}\begin{bmatrix}2x\\2\end{bmatrix} = \sum_{r=0}^{n}\binom{n}{r}P_3^{n-r}(x)P_2^r(x)P^{n-r+j}\begin{bmatrix}2x\\2\end{bmatrix}, \tag{7.13}$$

then

$$Q_{3n+j}(x) = \sum_{r=0}^{n}\binom{n}{r}P_3^{n-r}(x)P_2^r(x)Q_{n-r+j}(x). \tag{7.14}$$

Note in (7.12) and (7.14) the emergence of extra terms in the summation, a fact which was hidden in (7.3) and (7.5) by $P_1(x) = 1$.

More generally, one can show that

$$P_{kn+j}(x) = \sum_{r=0}^{n}\binom{n}{r}P_k^{n-r}(x)P_{k-1}^r(x)P_{n-r+j}(x) \tag{7.15}$$

and

$$Q_{kn+j}(x) = \sum_{r=0}^{n}\binom{n}{r}P_k^{n-r}(x)P_{k-1}^r(x)Q_{n-r+j}(x). \tag{7.16}$$

Special cases of (7.15) and (7.16) occurring when $k = 2$ are given in (7.3) and (7.5), respectively, in equivalent forms.

From (7.11) we deduce

$$P_3(x)P = P^3 - P_2(x)I, \tag{7.17}$$

whence

$$P_3^n(x)P^{n+j} = (P^3 - P_2(x)I)^n P^j, \tag{7.18}$$

from which it follows that

$$P_3^n(x)P_{n+j}(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} P_{3(n-r)+j}(x)P_2^r(x). \tag{7.19}$$

Similarly,

$$P_3^n(x)Q_{n+j}(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q_{3(n-r)+j}(x)P_2^r(x). \tag{7.20}$$

More generally,

$$P_k^n(x)P_{n+j}(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} P_{k(n-r)+j}(x)P_{k-1}^r(x) \tag{7.21}$$

and

$$P_k^n(x)Q_{n+j}(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q_{k(n-r)+j}(x)P_{k-1}^r(x). \tag{7.22}$$

By (1.8) and (1.15), we may prove

$$P^{mr+k} = Q_m(x)P^{m(r-1)k} - (-1)^m P^{m(r-2)+k}. \tag{7.23}$$

Hence

$$P^{(mr+k)n} = P^{\{m(r-2)+k\}n}(Q_m(x)P^m - (-1)^m I)^n. \tag{7.24}$$

Equating appropriate elements yields

$$P_{(mr+k)n}(x) = \sum_{i=0}^{n} (-1)^{i(m+1)} \binom{n}{i} P_{\{m(r-1)+k\}n-mi}(x)Q_m^{n-i}(x). \tag{7.25}$$

Putting $k = 0$ in (7.25) produces a formula for $P_{mrn}(x)$.

Again using (1.15), three times now, we obtain another form of (7.23):

$$P^{mr+k} = Q_{2m}(x)P^{m(r-2)+k} - P^{m(r-4)+k}. \tag{7.26}$$

Following the reasoning outlined in (7.24) and (7.25), we derive alternative formulas for $P_{(mr+k)n}(x)$ and $P_{mnr}(x)$ which closely resembly (7.24) and (7.25).

Equation (7.25) may be generalized further by extension of (7.26) to get

$$P_{(mr+k)n}(x) = \sum_{i=0}^{n} (-1)^{i(ms+1)} \binom{n}{i} Q_{sm}^{n-i}(x)P_{\{m(r-s)+k\}n-msi}(x) \tag{7.27}$$

with a corresponding simplification for $P_{mrn}(x)$ when $k = 0$.

Since, by (7.23),

$$Q_m(x)P^{mr+k} = P^{m(r-1)+k}(P^{2m} + (-1)^m I) \tag{7.28}$$

we may demonstrate that

$$Q_m^n(x)P_{(mr+k)n}(x) = \sum_{i=0}^{n} (-1)^{mi} \binom{n}{i} P_{\{m(r+1)+k\}n-2mi}(x) \tag{7.29}$$

with a specialization when $k = 0$.

Arguments similar to those used to obtain the general result (7.27) may be utilized to prove that

$$Q_{ms}^{n}(x)P_{(mr+k)n} = \sum_{i=0}^{n}(-1)^{msi}\binom{n}{i}P_{\{m(r+s)+k\}n-2msi}(x) \tag{7.30}$$

leading to the simpler form when $k = 0$.

8. THE MATRIX SEQUENCE $\{n^V\}$

Ideas introduced in [5] for Fibonacci numbers are here expanded to apply to Pell polynomials.

Now, a generalization of the matrix P is the matrix

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4x \\ 1 & 2x & 4x^2 \end{bmatrix}. \tag{8.1}$$

Induction demonstrates that

$$S^n = \begin{bmatrix} P_{n-1}^2(x) & P_{n-1}(x)P_n(x) & P_n^2(x) \\ 2P_{n-1}(x)P_n(x) & P_n^2(x) + P_{n-1}(x)P_{n+1}(x) & 2P_{n+1}(x)P_n(x) \\ P_n^2(x) & P_n(x)P_{n+1}(x) & P_{n+1}^2(x) \end{bmatrix} \tag{8.2}$$

The characteristic equation of S is

$$\lambda^3 - (4x^2+1)\lambda^2 - (4x^2+1)\lambda + 1 = 0. \tag{8.3}$$

From the Cayley-Hamilton theorem applied to (8.3), we have the recursion formula

$$S^n[S^3 - (4x^2+1)S(S+I) + I] = 0. \tag{8.4}$$

Corresponding elements in S^{n+3}, S^{n+2}, S^{n+1}, and S^n must satisfy (8.4). Therefore, from (8.2), we have the identities

$$P_{n+3}^2(x) - (4x^2+1)P_{n+2}^2(x) - (4x^2+1)P_{n+1}^2(x) + P_n^2(x) = 0 \tag{8.5}$$

and

$$P_{n+3}(x)P_{n+4}(x) - (4x^2+1)P_{n+2}(x)P_{n+3}(x) - (4x^2+1)P_{n+1}(x)P_{n+2}(x) + P_n(x)P_{n+1}(x) = 0. \tag{8.6}$$

[Parenthetically, we remark that the Cayley-Hamilton theorem may be employed with S to derive the sums given in (5.1) and (5.3).]

Again, after a little algebraic manipulation, the Cayley-Hamilton theorem leads to

$$(S+I)^3 = 4(x^2+1)S(S+I). \tag{8.7}$$

Mathematical induction establishes

$$(S+I)^{2n+1} = 4^n(x^2+1)^nS^n(S+I). \tag{8.8}$$

Now multiply both sides of (8.8) by S^j.

Equate corresponding elements to obtain

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}P_{k+j}^2(x) = 4^n(x^2+1)^n P_{2n+1+2j}(x) \quad \text{by [7, (3.20)].} \tag{8.9}$$

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}P_k(x)P_{k+j+1}(x) = 4^n(x^2+1)^n P_{2n+2+2j}(x). \tag{8.10}$$

By [7, (2.8)] we have

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}Q_k^2(x) = 4^{n+1}(x^2+1)^{n+1}P_{2n+1}(x),$$

while by (1.17) we have

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}Q_k(x)Q_{k+1}(x) = 4^{n+1}(x^2+1)^{n+1}P_{2n+2}(x)$$

with similar results to those in (8.9) and (8.10) when k is replaced by $k+j$ in (8.11) and (8.12).

If, now, in (8.8) we multiply both sides by $(S+I)S^j$, we get

$$\sum_{k=0}^{2n+2}\binom{2n+2}{k}P_{k+j}^2(x) = 4^n(x^2+1)^n Q_{2n+2j+2}(x) \tag{8.13}$$

and

$$\sum_{k=0}^{2n+2}\binom{2n+2}{k}P_{k+j}(x)P_{k+j+1}(x) = 4^n(x^2+1)^n Q_{2n+2j+3}(x). \tag{8.14}$$

When use is made of [7, (2.8)], (1.17), and both sides of the formula for (8.8) multiplied by $(S+I)S^j$, we derive

$$\sum_{k=0}^{2n+2}\binom{2n+2}{k}Q_{k+j}^2(x) = 4^{n+1}(x^2+1)^{n+1}Q_{2n+2j+2}(x) \tag{8.15}$$

and

$$\sum_{k=0}^{2n+2}\binom{2n+2}{k}Q_{k+j}(x)Q_{k+j+1}(x) = 4^{n+1}(x^2+1)^{n+1}P_{2n+2j+3}(x). \tag{8.16}$$

Extending the forms of the matrices P and S further, we have

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 6x \\ 0 & 1 & 4x & 12x^2 \\ 1 & 2x & 4x^2 & 8x^3 \end{bmatrix}$$

for which the characteristic equation is

$$\lambda^4 - (8x^3+4x)\lambda^3 - (16x^4+12x^2+2)\lambda^2 + (8x^3+4x)\lambda + I = 0, \tag{8.18}$$

From which are obtained (see [9]) forms for T^n and formulas for three cubic expressions in Pell polynomials corresponding to the two quadratic ones in (8.5) and (8.6), and an expression for

$$\sum_{r=1}^{n}P_r^3(x)$$

which is a variation of (4.16).

Matrices S and T are elements in a sequence of matrices $\{{}_nV\}$,

$$ {}_1V = [1], \quad {}_2V = \begin{bmatrix} 0 & 1 \\ 1 & 2x \end{bmatrix}, \quad {}_3V = S, \quad {}_4V = T, \ \ldots, \ {}_rV, \ \ldots, \tag{8.19}$$

the order of ${}_rV$ being r.

The element ${}_r\nu_{ij}$ of ${}_rV$ in the i^{th} row and j^{th} column is

$$ {}_r\nu_{ij} = \binom{j-1}{j+i-r-1}(2x)^{i+j-r-1}. \tag{8.20}$$

It is *conjectured* that the characteristic equation of ${}_rV$ is

$$ \sum_{k=0}^{r} (-1)^{[k(k+1)]/2}\{r,\ k\}\lambda^{r-k} = 0, \tag{8.21}$$

where

$$ \{r,\ k\} = \prod_{i=1}^{r}\{P_i(x)\} \Big/ \prod_{i=0}^{k}\{P_i(x)\}\prod_{i=1}^{r-k}\{P_i(x)\}, \quad 0 \leqslant k \leqslant r, \tag{8.22}$$

using the notation (extended) of [4]. That is, the symbol $\{r,\ k\}$ represents a generalization of a binomial coefficient. Following the ideas in [4], we note the results:

$$ \{r,\ k\} = \{r,\ r-k\} \qquad \text{by (8.22);} \tag{8.23}$$

$$ \{r,\ r\} = 1 \qquad \text{by (8.22);} \tag{8.24}$$

$$ \{r,\ 0\} = 1 \qquad \text{by (8.23) and (8.24);} \tag{8.25}$$

$$ \{r,\ 1\} = \{r,\ r-1\} = P_r(x) \qquad \text{by (8.22) and (8.23).} \tag{8.26}$$

Next, we write

$$ \{r,\ k\} = P_r(x)C(x), \tag{8.27}$$

whence

$$ \{r-1,\ k\} = P_{r-k}(x)C(x) \tag{8.28}$$

and

$$ \{r-1,\ k-1\} = P_k(x)C(x). \tag{8.29}$$

Further,

$$ \begin{aligned} \{r,\ k\} &= P_{r-k+k}(x)C(x) \\ &= P_{r-k}(x)P_{k+1}(x)C(x) + P_{r-k-1}(x)P_k(x)C(x) \quad \text{by [7, (2.14)],} \end{aligned} $$

so, by (8.28) and (8.29),

$$ \{r,\ k\} = P_{r-k+1}(x)\{r-1,\ k-1\} + P_{k+1}(x)\{r-1,\ k\}, \tag{8.30}$$

a type of Pascal triangle relationship.

Similarly,

$$ \{r,\ k\} = P_{r-k-1}(x)\{r-1,\ k-1\} + P_{k-1}(x)\{r-1,\ k\}. \tag{8.31}$$

Adding (8.30 and (8.31), and invoking [7, (3.24)], we deduce

$$2\{r,\ k\} = Q_{r-k}(x)\{r-1,\ k-1\} + Q_k(x)\{r-1,\ k\}. \tag{8.32}$$

Going back to conjecture (8.21), we note that the expression for the symbol $\{r,\ k\}$ in (8.21) and (8.22) involves divisibility properties of the Pell polynomials. Although these are not discussed here, they are investigated in some detail in [9]. A key divisibility result proved in [9], for instance, is

$$P_m(x) \mid P_n(x) \quad \text{if and only if} \quad m \mid n. \tag{8.33}$$

The polynomial expressions occurring as powers of λ in (8.3) and (8.18), e.g., are $\{3,\ 1\}$ and $\{3,\ 2\}$, and $\{4,\ 1\}$, $\{4,\ 2\}$, and $\{4,\ 3\} = \{4,\ 1\}$, respectively.

9. CONCLUDING REMARKS

Naturally the consequences of the use of matrix methods in developing combinatorial number-theoretic properties of Pell and Pell-Lucas polynomials are by no means exhausted in our brief account above.

Quite apart from pursuing the discovery of additional formulas by the matrix techniques indicated, we can introduce different matrices to obtain new results.

Another interesting set of problems is to derive the sum of series whose terms are fractional and involve products of Pell or Pell-Lucas polynomials in the denominator, e.g.,

$$\sum_{r=1}^{n} \frac{(-1)^r}{P_r(x)P_{r+1}(x)}.$$

Putting $x = 1$ in the expression and summing to infinity, we may deduce the infinite alternating series summation involving Pell numbers,

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{P_r P_{r+1}} = 1 - \sqrt{2}, \tag{9.1}$$

but enough has been said on our general theme for the moment.

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LETTER TO THE EDITOR

July 1, 1986

Over the years, several articles have appeared in *The Fibonacci Quarterly* relating the Fibonacci numbers to growth patterns in plants. Recently, Roger V. Jean, Professor of Mathematics and research worker in biomathematics at the University of Quebec has written the book *Mathematical Approach to Pattern and Form in Plant Growth* (Wiley & Sons), which should interest many readers of the *Quarterly*.

Dr. Jean addresses the mathematical problems raised by phyllotaxis, the study of relative arrangements of similar parts of plants and of technical concepts related to plant growth. He includes not only recent mathematical developments but also those that have appeared in specialized periodicals since 1830, listing well over 400 references. The book is written as a textbook for an advanced course in plant biology and mathematics or as a reference for workers in biomathematics. Besides that, it is just plain interesting reading.

Sincerely,

Marjorie Bicknell-Johnson

ON THE LEAST COMMON MULTIPLE OF SOME BINOMIAL COEFFICIENTS

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(Submitted September 1984)

Let

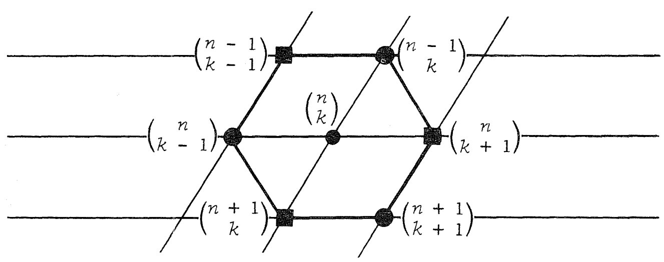
$$a = \binom{n-1}{k-1} \cdot \binom{n+1}{k}, \quad b = \binom{n+1}{k} \cdot \binom{n}{k+1}, \quad c = \binom{n}{k+1} \cdot \binom{n-1}{k-1},$$

$$d = \binom{n}{k-1} \cdot \binom{n+1}{k+1}, \quad e = \binom{n+1}{k+1} \cdot \binom{n-1}{k}, \text{ and } f = \binom{n-1}{k} \cdot \binom{n}{k-1}.$$

We prove that

$$\text{L.C.M.}\{a, b, c\} = \text{L.C.M.}\{d, e, f\},$$

where L.C.M. denotes the least common multiple. The proof technique is due to the late Ernst Straus and rests upon elementary properties of the p-adic valuations of Q, the field of rational numbers. The geometry of the situation is indicated in the figure below.



Multiplying each of the quantities a through f by

$$\frac{k!(k+1)!(n-k)!(n-k+1)!}{(n-1)!n!}$$

produces the six corresponding quantities

$$(n+1)k(k+1), \; n(n+1)(n-k), \; k(n-k)(n-k+1),$$
$$n(n+1)k, \; (n+1)(n-k)(n-k+1), \text{ and } k(k+1)(n-k).$$

Since $|\text{L.C.M.}\{\alpha, \beta\}|_p = \min\{|\alpha|_p, |\beta|_p\}$ for every p-adic valuation $|\;|_p$ of Q, the original problem is equivalent to proving that $m_1(n, k) = m_2(n, k)$ for all (finite) primes p, provided we define

$$m_1(n, k) = \min\{|(n+1)k(k+1)|_p, \; |n(n+1)(n-k)|_p, \; |k(n-k)(n-k+1)|_p\}$$

and

$$m_2(n,\ k) = \min\{|n(n+1)k|_p,\ |(n+1)(n-k)(n-k+1)|_p,\ |k(k+1)(n-k)|_p\}.$$

We first establish that $m_1(n,\ k) \geqslant m_2(n,\ k)$. In each of the three steps of this argument we make repeated use of the following standard facts concerning p-adic valuations of Q:

(1) the ultrametric inequality: $|\alpha + \beta|_p \leqslant \max\{|\alpha|_p,\ |\beta|_p\}$;

(2) $|\alpha + \beta|_p = \max\{|\alpha|_p,\ |\beta|_p\}$ if $|\alpha|_p \neq |\beta|_p$;

(3) $|z|_p \leqslant 1$, for every integer z and for every (finite) prime p;

(4) $|z|_p < 1$ if and only if the integer z is divisible by the prime p (equivalently, $|z|_p = 1$ if and only if the integer z is not divisible by the prime p).

We provide a detailed proof of the first step of the argument and then give somewhat abbreviated arguments for the remaining two steps.

Step 1. Assume that $|(n+1)k(k+1)|_p < m_2(n,\ k)$, that is,

(i) $|k+1|_p < |n|_p$,

(ii) $|k(k+1)|_p < |(n-k)(n-k+1)|_p$, and

(iii) $|n+1|_p < |n-k|_p$.

From (1) and (3), it follows that $|k+1|_p < 1$ so that, from (4), $p \mid k+1$. Since $(k,\ k+1) = 1$, it follows that $p \nmid k$, which can be rewritten using (4) as $|k|_p = 1$. From (iii) and (3), it follows that $|n+1|_p < 1 = |k|_p$ which, in conjunction with (2), allows us to conclude that

$$|n-k+1|_p = |(n+1) - k|_p = \max\{|n+1|_p,\ |k|_p\} = 1.$$

Going to (ii) and making use of the fact that $|k|_p = 1$ and $|n-k+1|_p = 1$, we get

$$|k(k+1)|_p = |k+1|_p < |(n-k)(n-k+1)|_p = |n-k|_p.$$

Finally

$$|n-k|_p = |(n+1) - (k+1)|_p \leqslant \max\{|n+1|_p,\ |k+1|_p\} < |n-k|_p,$$

from (1), and we have our desired contradiction.

Step 2. If $|n(n+1)(n-k)|_p < m_2(n,\ k)$, then we have

$$|n-k|_p < |k|_p,\ |n|_p < |n-k+1|_p, \text{ and } |n(n+1)|_p < |k(k+1)|_p.$$

Hence $|n-k+1| = |n+1| = 1$. Now,

$$|k|_p = |(n-k) - n|_p \leqslant \max\{|n-k|_p,\ |n|_p\} < |k|_p,$$

a contradiction. Here we made use of the fact that $|n|_p < |k(k+1)|_p \leqslant |k|_p$.

Step 3. If $|k(n-k)(n-k+1)|_p < m_2(n,\ k)$, then we have

$$|(n-k)(n-k+1)|_p < |n(n+1)|_p,\ |k|_p < |n+1|_p, \text{ and}$$

$$|n - k + 1|_p < |k + 1|_p.$$

Since $|n - k + 1| < 1$, we have $|n - k| = 1$, and so we get

$$|n - k + 1|_p < |n(n + 1)|_p \leqslant |n + 1|_p.$$

However,

$$|n - k + 1|_p = |(n + 1) - k|_p = \max\{|n + 1|_p, |k|_p\} = |n + 1|_p,$$

since $|k|_p < |n + 1|_p$. Hence, once again we have a contradiction.

Since $m_2(n, k) = m_1(-k - 1, -n - 1)$, and since $m_1(n, k) \geqslant m_2(n, k)$ has already been established, we can finish the proof using the following chain of inequalities:

$$\begin{aligned} m_1(n, k) \geqslant m_2(n, k) &= m_1(-k - 1, -n - 1) \geqslant m_2(-k - 1, -n - 1) \\ &= m_1(-(-n - 1) - 1, -(-k - 1) - 1) \\ &= m_1(n, k). \end{aligned}$$

Remarks: The result of this note can alternatively be deduced from the following previously established (see, respectively, [1], [2], and [3]) results:

(1) $\binom{n-1}{k} \cdot \binom{n}{k-1} \cdot \binom{n+1}{k+1} = \binom{n-1}{k-1} \cdot \binom{n}{k+1} \cdot \binom{n+1}{k}$

(2) $\text{G.C.D.}\left\{\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right\} = \text{G.C.D.}\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\}$

where G.C.D. denotes the greatest common divisor.

(3) $xyz = \text{G.C.D.}\{x, y, z\} \cdot \text{L.C.M.}\{xy, yz, zx\}$, valid for arbitrary positive integers x, y, and z. A more involved result can be obtained using the fact (see [3]) that

$$\begin{aligned} xyz = \text{G.C.D.}\{x, y, z\} \cdot \text{L.C.M.}\{&\text{G.C.D.}\{x, y\}, \text{G.C.D.}\{y, z\}, \\ &\text{G.C.D.}\{z, x\}\} \cdot \text{L.C.M.}\{x, y, z\}. \end{aligned}$$

Finally, we ask whether such results have any combinatorial interpretation.

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SIDNEY'S SERIES

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Sidney's series of numbers may qualify in two ways for being considered a part of the world of pure mathematics. This series is, as far as this author knows, without practical application, and is very beautiful. The series was discovered by the author's daughter Sidney Larison in 1968 when she was about age fifteen.

Using two-digit numerals, five series can be produced (or six if you count zero). Using three-digit numerals, nineteen series can be produced (or twenty if you count zero). Using four-digit numerals, eleven series can be produced (or twelve if you count zero).

To produce the series using two-digit numerals, start with any two-digit numeral, for example,

23.

Add them together and affix their sum, as,

235.

Add the last two digits together and affix their sum, as,

2358.

Add the last two digits together and affix their sum, modulo 10, *always dropping from the sum the digit in tens place if there is one*, as,

23583, and then, 235831... .

Continue the process until the first two digits repeat.

The first series in the set is now complete.

To produce the second series in the set of six, start with any two-digit numeral not included in the first series and repeat the process.

To produce the third, fourth, and fifth series in the set, select any as-yet-unused two-digit numeral and repeat the process.

The sixth series in the set simply contains zero.

These six series of numbers contain all of the two-digit numerals from 00 through 99 and none will appear more than once. Each two-digit numeral can fit one series and no other.

Series utilizing numerals of three, four, or any desired number of digits may be produced. To produce the set of twenty series using three-digit numerals, select any three-digit numeral, add the digits and affix their sum, modulo 10, as,

123 6 1 0 7 8 5 0 3 .... .

When the first three digits repeat, that series in the set is complete.

The twenty series in the set using three-digit numerals utilize every numeral from 000 through 999 and none is used more than once. Each three-digit numeral appears in one series and no other.

Completing the set based on four-digit numerals proved to be too large a task to be accomplished by hand so the computer was used. William G. Sjostrom of Modesto, California, wrote in BASIC the necessary programs to write the set of four-digit series. There turn out to be only twelve series in the set—six sets of 1560 digits each, two sets of 312 digits each, three sets of 5 digits each, and zero.

When the six series of numbers based on two-digit numerals are equally spaced in a set of six concentric circles, some interesting properties become apparent. Any series which contains more than one zero will contain four of them and they will be equally spaced around the circle. Pairs of digits which are directly opposite each other in the circle will add up to either zero or ten.

No attempt has as yet been made to place the ten thousand digits of the four-digit series in a set of twelve concentric circles, but an inspection of the lists shows that those series containing 000 more than once will contain it four times and they will be equally spaced around the circle. As in the series based on two-digit numerals, single digits directly opposite each other in the circle will have as their sum either zero or ten.

The twenty series of numbers based on three-digit numerals when equally spaced in a set of twenty concentric circles exhibit no interesting properties in relation to zero. Nor do digits directly opposite each other in the circle add up to ten or zero. However, a study of this series in a search for interesting properties revealed a fascinating property shared by all series so far tested.

To examine this property, proceed as follows:

List, horizontally, a string of digits as they occur in any series from any set, as, from the set based on three digits,

6095487940...

Under it write another series from the same set, as,

6095487940...
<u>2035869380...</u>

Add, modulo 10.

Your result, in this case 8020246220..., will follow all the rules for producing a series from that number of digits and will, indeed, be another series from that set!

It works without fail! Add together, in order, the digits from two or more series from the same set and the result will be a series in the same set!

Multiply, modulo 10, in order, the digits from any series by the same numeral, and your result will be a series in the same set.

For example, take

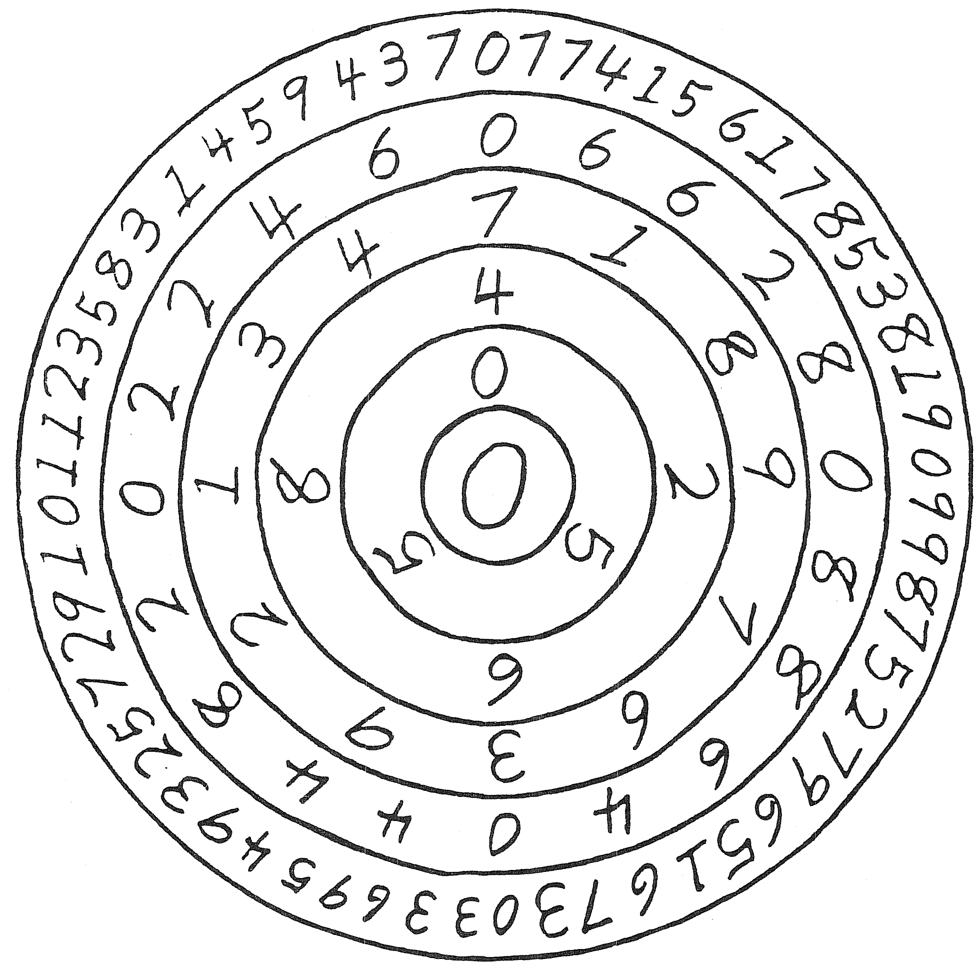
6 0 9 5 4 8 7 9 4 0 ... from the three-digit set.

Multiply by <u>3 3 3 3 3 3 3 3 3 3</u> ... Your result, in this case

8 0 7 5 2 4 1 7 2 0 ... follows all the rules and is a member of the three-digit set.

There may be other interesting properties to be discovered in these series of numbers. No one knows, for example, how many series will be required to complete the set based on five-digit numerals or what properties they will display.

The author predicts that the set based on five-digit numerals will display the same properties as the other sets in relation to addition and multiplication, and forty series will be required to complete the set.



THE SET OF SERIES BASED ON TWO-DIGIT NUMERALS

Please turn to page 361.

A SOLUTION TO A TANTALIZING PROBLEM

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INTRODUCTION

In a recent paper, R. Backstrom [1] computed various sums of reciprocal Fibonacci and Lucas numbers. By a strange limit process, he also gets an estimate (to the seventh decimal place) of the sum

$$\sum_0^\infty \frac{1}{L_{2n}+2} \approx \frac{1}{8} + \frac{1}{4\log\alpha}, \text{ where } \alpha = \frac{\sqrt{5}+1}{2}$$

(here $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$). An even better estimate is the formula

$$\sum_0^\infty \frac{1}{L_{2n}+2} \approx \frac{1}{8} + \frac{1}{4\log\alpha} + \frac{\pi^2}{(\log\alpha)^2} \cdot \frac{1}{e^{\pi^2/\log\alpha} - 2},$$

which has at least thirty correct decimal places. But both these formulas are just the first terms in a very rapidly converging series, that is, a quotient of two theta functions.

This paper contains no new results. On the contrary, most of the results are approximately 150 years old, mostly due to Jacobi. The formulas for the sums of reciprocal Fibonacci and Lucas numbers are obtained by substituting $q = \alpha^{-1}$ or $q = \alpha^{-2}$ in identities valid for formal power series or for series converging for $|q| < 1$.

Probably all the results in Backstrom's paper can be obtained by specializing to $q = \alpha^{-1}$ or $q = \alpha^{-2}$ in sums of telescoping series. For example, let us look at Theorem I in [1]:

$$\sum_{n=0}^\infty \frac{1}{F_{2n+1} + F_{2r+1}} = \sqrt{5}\left(r + \frac{1}{2}\right)\Big/ L_{2r+1}.$$

We have

$$\frac{1}{F_{2n+1}+F_{2r+1}} = \frac{\sqrt{5}}{L_{2r+1}}\left(\frac{1}{1+\alpha^{-2(n+r+1)}} - \frac{1}{1+\alpha^{-2(n-r)}}\right)$$
$$= \frac{\sqrt{5}}{L_{2r+1}}\left(\frac{1}{1+q^{n+r+1}} - \frac{1}{1+q^{n-r}}\right), \text{ where } q = \alpha^{-2}.$$

Hence, it is sufficient to show that

$$\sum_{n=0}^\infty \left(\frac{1}{1+q^{n+r+1}} - \frac{1}{1+q^{n-r}}\right) = r + \frac{1}{2} \text{ for } 0 < |q| < 1.$$

Now,

$$\sum_{n=0}^{\infty}\left(\frac{1}{1+q^{n+r+1}}-\frac{1}{1+q^{n-r}}\right)=\sum_{\nu=N-r+1}^{N+r+1}\frac{1}{1+q^{\nu}}-\sum_{\nu=-r}^{r}\frac{1}{1+q^{\nu}}$$

$$\rightarrow 2r+1-\left(r+\frac{1}{2}\right)=r+\frac{1}{2}\quad \text{as } N\rightarrow\infty,$$

since

$$\frac{1}{1+q^{\nu}}+\frac{1}{1+q^{-\nu}}=1 \text{ for } \nu\neq 0.$$

Here we never used the fact that $q=\alpha^{-2}$, so the summation of the inner series has nothing to do with Fibonacci numbers.

We hope to get some of the Fibonacci enthusiasts interested in theta functions. An excellent text is Rademacher's lecture notes [6]. They pair German thoroughness with elegance. On the other side of the spectrum is Bellman's very thin book [2], which contains almost no proofs, only the most important results and some applications.

1. THETA FUNCTIONS

We have the following theta functions (the summation is over all n in $\mathbb{Z}$):

$$\vartheta_1(x,q)=\frac{1}{i}\sum_n(-1)^n q^{[(n+(1/2)]^2}e^{i(2n+1)\pi x};$$

$$\vartheta_2(x,q)=\sum_n q^{[n+(1/2)]^2}e^{i(2n+1)\pi x};$$

$$\vartheta_3(x,q)=\sum_n q^{n^2}e^{i2n\pi x};$$

$$\vartheta_4(x,q)=\sum_n(-1)^n q^{n^2}e^{i2n\pi x}. \tag{1}$$

We make the substitution $q=e^{\pi iz}$ and get the following *functional equations*:

$$\vartheta_1\left(\frac{x}{z},\ -\frac{1}{z}\right)=\frac{1}{i}\sqrt{\frac{z}{i}}\,e^{(\pi ix^2/z)}\vartheta_1(x,z);$$

and

$$\vartheta_{3+\nu}\left(\frac{x}{z},\ -\frac{1}{z}\right)=\sqrt{\frac{z}{i}}\,e^{(\pi ix^2/z)}\vartheta_{3-\nu}(x,z)\ \text{for } \nu=-1,\ 0,\ 1. \tag{2}$$

($\sqrt{i}$ is taken in the first quadrant.)

This was essentially proved in 1823 by Poisson in the form:

$$\frac{1}{\sqrt{x}}=\frac{1+2e^{-\pi x}+2e^{-4\pi x}+2e^{-9\pi x}+\cdots}{1+2e^{-\pi/x}+2e^{-4\pi/x}+2e^{-9\pi/x}+\cdots}.$$

Notation: In the sequel we will only have to consider the case $x=0$. We will write

$$\vartheta_\nu=\vartheta_\nu(0,q),$$

$$\vartheta_\nu^{(n)}=\left(\frac{\partial}{\partial x}\right)^n\vartheta(0,q).$$

We have many formulas, as follows:

$$\vartheta_1' = \pi \sum_n (-1)^n (2n + 1) q^{[n+(1/2)]^2};$$

$$\vartheta_2 = \sum_n q^{[n+(1/2)]^2};$$

$$\vartheta_3 = \sum_n q^{n^2};$$

$$\vartheta_4 = \sum_n (-1)^n q^{n^2}, \quad \vartheta_4(-q) = \vartheta_3(q). \tag{3}$$

$$\vartheta_1''' = -\pi^3 \sum_n (-1)^n (2n + 1)^3 q^{[n+(1/2)]^2};$$

$$\vartheta_2'' = -\pi^2 \sum_n (2n + 1)^2 q^{[n+(1/2)]^2};$$

$$\vartheta_3'' = -4\pi^2 \sum_n n^2 q^{n^2};$$

$$\vartheta_4'' = -4\pi^2 \sum_n (-1)^n n^2 q^{n^2}. \tag{4}$$

By using the transformed formulas, we get:

$$\vartheta_1' = \frac{2\pi^2}{\log q} \sqrt{-\frac{\pi}{\log q}} \sum_n (-1)^n \left(n - \frac{1}{2}\right) e^{\frac{\pi^2}{\log q}[n-(1/2)]^2};$$

$$\vartheta_2 = \sqrt{-\frac{\pi}{\log q}} \sum_n (-1)^n e^{\frac{\pi^2}{\log q}(n^2)};$$

$$\vartheta_3 = \sqrt{-\frac{\pi}{\log q}} \sum_n e^{\frac{\pi^2 n^2}{\log q}};$$

$$\vartheta_4 = \sqrt{-\frac{\pi}{\log q}} \sum_n e^{\frac{\pi^2[n-(1/2)]^2}{\log q}};$$

$$\vartheta_2'' = \frac{2\pi^2}{\log q} \sqrt{-\frac{\pi}{\log q}} \sum_n \left(1 + \frac{2\pi^2 n^2}{\log q}\right) e^{\pi^2 n^2/\log q} (-1)^n;$$

$$\vartheta_3'' = \frac{2\pi^2}{\log q} \sqrt{-\frac{\pi}{\log q}} \sum_n \left(1 + \frac{2\pi^2 n^2}{\log q}\right) e^{\pi^2 n^2/\log q};$$

$$\vartheta_4'' = \frac{2\pi^2}{\log q} \sqrt{-\frac{\pi}{\log q}} \sum_n \left(1 + \frac{2\pi^2}{\log q}\left(n - \frac{1}{2}\right)^2\right) e^{\frac{\pi^2[n-(1/2)]^2}{\log q}}. \tag{5}$$

2. COMPUTATION OF THE SUM $\sum_0^\infty \frac{1}{L_{2n} + 2}$

We will go through the computation of the above sum in detail. We put, as usual,

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Then $\alpha\beta = -1$ and $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, $L_n = \alpha^n + \beta^n$. If we put $q = \alpha^{-1}$, we get:

$$S = \sum_0^\infty \frac{1}{L_{2n} + 2} = \frac{1}{4} + \sum_1^\infty \frac{1}{\alpha^{2n} + \alpha^{-2n} + 2} = \frac{1}{4} + \sum_1^\infty \frac{q^{2n}}{(1+q^{2n})^2}.$$

By by formulas in Tannery and Molk [7, II, pp. 250 and 260] we have

$$\frac{\vartheta_2''}{\vartheta_2} = -\pi^2\left(1 + 8\sum_1^\infty \frac{q^{2n}}{(1+q^{2n})^2}\right).$$

Hence,

$$S = \frac{1}{8}\left(1 - \frac{1}{\pi^2}\cdot\frac{\vartheta_2''}{\vartheta_2}\right),$$

and if we use formulas (3) and (4), we get:

$$S = \frac{1}{8}\left(1 + \frac{\sum_{n=1}^\infty (2n+1)^2 \alpha^{-[n+(1/2)]^2}}{\sum_1^\infty \alpha^{-[n+(1/2)]^2}}\right).$$

This series converges very rapidly (10 terms will give about 20 decimal places) but it does not contain $\log \alpha$ as Backstrom's approximation does. By using the functional equation and the formulas in (5) we can improve the rate of convergence.

$$S = \frac{1}{8}\left(1 - \frac{2}{\log q}\,\frac{\sum_n (-1)^n \left(1 + \frac{2\pi^2 n^2}{\log q}\right) e^{\pi^2 n^2/\log q}}{\sum_n (-1)^n e^{\pi^2 n^2/\log q}}\right).$$

Putting $q = \alpha^{-1}$, we obtain the final formula:

$$\sum_0^\infty \frac{1}{L_{2n} + 2} = \frac{1}{8} + \frac{1}{4\log\alpha}\left[1 - \frac{4\pi^2}{\log\alpha}\cdot\frac{\sum_1^\infty (-1)^n n^2 e^{-\pi^2 n^2/\log\alpha}}{1 + 2\sum_1^\infty (-1)^n e^{-\pi^2 n^2/\log\alpha}}\right].$$

This series converges extremely rapidly. We have $e^{-\pi^2/\log\alpha} \approx e^{-20} \approx 10^{-9}$, so taking just one term ($n = 1$) will give over 30 correct decimal places. Ten terms will give around 900 correct decimal places.

3. A CATALOGUE OF FORMULAS

In this section we collect some formulas connecting sums of reciprocals of Fibonacci and Lucas numbers and theta functions. We leave it to the reader to derive the final formulas as in the last section. The formulas are found in Tannery and Molk [7, II, pp. 250, 260, 258; IV, pp. 108, 107], Jacobi [4, pp. 159-167], and Hancock [3, p. 407].

I. Put $q = \alpha^{-1}$. Then:

(a) $\displaystyle\sum_1^\infty \frac{(-1)^n}{L_{2n} + 2} = \frac{1}{8}(\vartheta_3^2\vartheta_4^2 - 1)$

(b) $\displaystyle\sum_1^\infty \frac{(-1)^n}{L_{2n}} = \frac{1}{4}(\vartheta_3\vartheta_4 - 1)$

(c) $\displaystyle\sum_1^\infty \frac{n^3}{F_{2n}} = \frac{\sqrt{5}}{256}\vartheta_2^8$

(d) $\displaystyle\sum_1^\infty \frac{2n - 1}{L_{2n-1}} = \frac{1}{16}\vartheta_2^4$

II. Put $q = \alpha^{-2}$. Then:

(a) $\displaystyle\sum_1^\infty \frac{1}{F_{2n-1}} = \frac{\sqrt{5}}{4}\vartheta_2^2$

(b) $\displaystyle\sum_1^\infty \frac{1}{L_{2n}} = \frac{1}{4}(\vartheta_3^2 - 1)$

(c) $\displaystyle\sum_1^\infty \frac{(-1)^{n-1}(2n - 1)}{F_{2n-1}} = \frac{\sqrt{5}}{4}\vartheta_2^2\vartheta_4^2$

(d) $\displaystyle\sum_1^\infty \frac{2n - 1}{L_{2n-1}} = \frac{1}{4}\vartheta_2^2\vartheta_4^2$

(e) $\displaystyle\sum_1^\infty \frac{(-1)^n}{L_{2n}^2} = \frac{1}{8}(\vartheta_3^2\vartheta_4^2 - 1)$

(f) $\displaystyle\sum_1^\infty \frac{1}{F_{2n}^2} = \frac{5}{24}\left(1 + \frac{1}{\pi^2}\frac{\vartheta_1''}{\vartheta_1'}\right)$

(g) $\displaystyle\sum_1^\infty \frac{1}{L_{2n}^2} = -\frac{1}{8}\left(1 + \frac{1}{\pi^2}\frac{\vartheta_2''}{\vartheta_2}\right)$

(h) $\displaystyle\sum_1^\infty \frac{1}{F_{2n-1}^2} = -\frac{5}{8\pi^2}\frac{\vartheta_3''}{\vartheta_3}$

(i) $\displaystyle\sum_1^\infty \frac{1}{L_{2n-1}^2} = \frac{1}{8\pi^2}\frac{\vartheta_4''}{\vartheta_4}$

(j) $\displaystyle\sum_1^\infty \frac{n}{F_{2n}} = \frac{\sqrt{5}}{8\pi^2}\frac{\vartheta_4''}{\vartheta_4}$

4. SOME IDENTITIES

There are numerous identities among theta functions. Specializing to $q = \alpha^{-1}$ or $q = \alpha^{-2}$ will give identities among sums of Fibonacci and Lucas numbers. We will give a few examples.

(a) Formulas II(i) and (j) give two expressions for $\vartheta_4''/\vartheta_4$:

$$\sum_1^\infty \frac{n}{F_{2n}} = \sqrt{5}\sum_1^\infty \frac{1}{L_{2n-1}^2}$$

(b) Formulas I(d) and II(a) give, with $q = \alpha^{-2}$,

$$\left(\sum_1^\infty \frac{1}{F_{2n-1}}\right)^2 = 5\sum_1^\infty \frac{2n - 1}{L_{4n-2}}$$

(c) The identity (Tannery and Molk [7, II, p. 250])

$$3\sum_1^\infty \frac{q^{2n}}{(1 - q^{2n})^2} = \sum_1^\infty \frac{q^{2n-1}}{(1 - q^{2n-1})^2} - \sum_1^\infty \frac{q^{2n-1}}{(1 + q^{2n-1})^2} - \sum_1^\infty \frac{q^{2n}}{(1 + q^{2n})^2}$$

gives, with $q = \alpha^{-2}$:

$$3\sum_1^\infty \frac{1}{F_{2n}^2} + \sum_1^\infty \frac{1}{F_{2n-1}^2} = 5\left(\sum_1^\infty \frac{1}{L_{2n-1}^2} - \sum_1^\infty \frac{1}{L_{2n}^2}\right)$$

(d) We have $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$, which implies:

$$\left(1 + 4\sum_1^\infty \frac{1}{L_{2n}}\right)^2 = \frac{16}{5}\left(\sum_1^\infty \frac{1}{F_{2n-1}}\right)^2 + \left(1 + 4\sum_1^\infty \frac{(-1)^n}{L_{2n}}\right)^2$$

5. A NEW TANTALIZING QUESTION

Unfortunately, we have not been able to find an expression for the sum

$$\sum_1^\infty \frac{1}{F_n}.$$

Since we know from II(a) that

$$\sum_1^\infty \frac{1}{F_{2n-1}} = \frac{\pi\sqrt{5}}{8 \log \alpha}\left\{1 + 2\sum_1^\infty (-1)^n e^{-\pi^2 n^2/2 \log \alpha}\right\}^2$$

we only need to compute $\sum_1^\infty \frac{1}{F_{2n}}$. For this, we need (wtih $q = \alpha^{-2}$)

$$f(q) = \sum_1^\infty \frac{q^n}{1 - q^{2n}} = \sum_1^\infty \frac{q^{2n-1}}{1 - q^{2n-1}} = \sum_1^\infty T_0(n) q^n,$$

where $T_0(n)$ is the number of odd divisors of n. Since T_0 is multiplicative, i.e., $T_0(mn) = T_0(m)T_0(n)$ if $(m, n) = 1$, we can compute the Dirichlet series (for Re $s > 1$).

$$\Phi(s) = \sum_{n=1}^\infty \frac{T_0(n)}{n^s} = \prod_p \left(\sum_{\nu \geqslant 0} \frac{T_0(p^\nu)}{p^{\nu s}}\right),$$

where the product is taken over all prime numbers. We have

$$T_0(2^\nu) = 1 \quad \text{and} \quad T_0(p^\nu) = \nu + 1 \quad \text{if } p \geqslant 3.$$

Hence, putting $t = p^{-s}$, we have

$$\sum_0^\infty T_0(2^\nu)t^\nu = \frac{1}{1 - t}$$

and

$$\sum_0^\infty T_0(p^\nu)t^\nu = \sum_0^\infty (\nu + 1)t^\nu = \frac{1}{(1 - t)^2} \quad \text{for } p \geqslant 3.$$

It follows that

$$\Phi(s) = \frac{1}{1 - 2^{-s}} \prod_{p \geqslant 3} \frac{1}{(1 - p^{-s})^2} = (1 - 2^{-s})\zeta(s)^2,$$

where

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s}$$

is the Riemann ζ-function.

It is possible, at least theoretically, to recover f from Φ by Mellin inversion (see Ogg [5, p. I.6]); however, we have not been able to compute the integral.

We end by giving some formulas due to Clausen (see Jacobi [4, I, p. 239]): Put

$$h(q) = \sum_1^\infty \frac{q^n}{1 - q^n} = \sum_1^\infty q^{n^2} \frac{1 + q^n}{1 - q^n}$$

What we need is

$$\sum_1^\infty \frac{q^{2n-1}}{1 - q^{2n-1}} = h(q) - h(q^2) = \sum_1^\infty \left(q^{n^2} \frac{1 + q^n}{1 - q^n} - q^{2n^2} \frac{1 + q^{2n}}{1 - q^{2n}} \right)$$

which converges very rapidly when $q = \alpha^{-2}$.

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TENTH ROOTS AND THE GOLDEN RATIO

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1. INTRODUCTION

The ratio of the radius of a circle and a side of the inscribed regular decagon equals the golden ratio τ. In the complex plane the spines of a regular decagon inscribed in a circle of unit radius are the vector representations of the complex tenth roots of -1, if the decagon is appropriately turned. These two observations motivate an interest in expressing the tenth roots of -1 in terms of the golden ratio. The roots themselves may be derived using either the polar representation of -1, for it is known that they are expressible as $e^{\pi i r/10}$ when r is an integer, or they may be obtained algebraically, since when 5 divides n, the field of the n^{th} roots of -1 contains $\sqrt{5}$ and hence contains τ.

2. RÉSUMÉ ON THE GOLDEN RATIO

The golden ratio is the limiting ratio of two successive Fibonacci numbers. This limiting ratio satisfies the quadratic equation,

$$\tau^2 - \tau - 1 = 0 \tag{1}$$

in which the first root

$$\tau = \frac{1 + \sqrt{5}}{2} \tag{2}$$

is the golden ratio and the second root is

$$\frac{1 - \sqrt{5}}{2} = -\frac{1}{\tau}; \tag{3}$$

see [1].

The idea is to introduce the quantities (2) and (3) into the expressions calculated below for the tenth roots of -1.

3. THE POLAR APPROACH TO THE TENTH ROOTS OF -1

Since -1 has unit modulus and an argument of 180°, its polar representation is

$$-1 = \cos\theta_n + i\sin\theta_n \tag{4}$$

with

$$\theta_n = 180° + 360° n, \tag{5}$$

where $n = 0, \pm 1, \pm 2, \ldots$ accounts for the periodicity of circular measure, and $i^2 = -1$. The tenth roots of -1 are then given by

$$\cos\frac{\Theta_n}{10} + i\sin\frac{\Theta_n}{10}$$

which in complex rectangular form are, successively,

$$\begin{aligned} Z_{0,1} &= i, \\ Z_{2,3,4,5} &= \pm(\cos 18^\circ \pm i \sin 18^\circ), \\ Z_{6,7,8,9} &= \pm(\cos 54^\circ \pm i \sin 54^\circ), \end{aligned} \tag{6}$$

each subscript denoting a different choice of algebraic sign.

The golden ratio is introduced by expressing the trigonometric ratios in (6) as surds. To do this, first use the result

$$\sin(2 \times 18^\circ) = \sin 36^\circ = \cos(90^\circ - 36^\circ) = \cos 54^\circ = \cos(3 \times 18^\circ) \tag{7}$$

to obtain, with the respective double and triple angle formulas for the sine and cosine,

$$2 \sin 18^\circ \cos 18^\circ = 4 \cos^3 18^\circ - 3 \cos 18^\circ. \tag{8}$$

Next, divide both sides of (8) by $\cos 18^\circ$ to reduce it to a quadratic equation in $\sin 18^\circ$, viz,

$$4 \sin^2 18^\circ + 2 \sin 18^\circ - 1 = 0, \tag{9}$$

with positive root

$$\sin 18^\circ = \frac{-1 + \sqrt{5}}{4} = \frac{1}{2\tau}. \tag{10}$$

Then we can write

$$\begin{aligned} \cos 18^\circ &= \sqrt{1 - \sin^2 18^\circ} = \sqrt{1 - \left(\frac{1}{2\tau}\right)^2} = \frac{\sqrt{3 + 4\tau}}{2\tau} \\ &= \frac{(1 + \tau)\sqrt{3 - \tau}}{2\tau} = \frac{\tau\sqrt{3 - \tau}}{2}, \end{aligned} \tag{11}$$

where equation (1) has also been used. Furthermore,

$$\cos 54^\circ = \sin 36^\circ = 2 \sin 18^\circ \cos 18^\circ = \frac{\sqrt{3 - \tau}}{2}$$

and

$$\sin 54^\circ = \sqrt{1 - \cos^2 54^\circ} = \sqrt{1 - \left(\frac{3 - \tau}{4}\right)} = \frac{\sqrt{1 + \tau}}{2} = \frac{\tau}{2}. \tag{12}$$

According to these expressions, the tenth roots of -1 become

$$\begin{aligned} Z_{0,1} &= \pm i, \\ Z_{2,3,4,5} &= \pm\frac{1}{2}\left(\tau\sqrt{3 - \tau} \pm i\,\frac{1}{\tau}\right). \\ Z_{6,7,8,9} &= \pm\frac{1}{2}(\sqrt{3 - \tau} + i\,\tau), \end{aligned} \tag{13}$$

and they may be sketched in the Argand plane as in Figure 1.

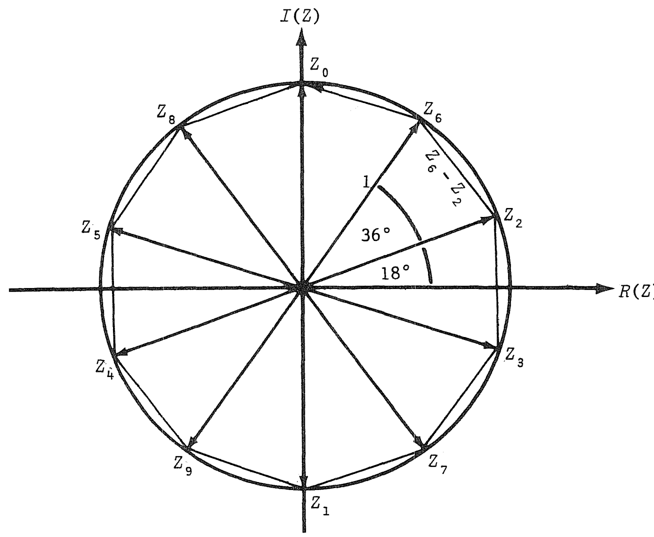

Figure 1

Since the side of the corresponding decagon is the modulus of the difference of two successive roots, we see from the figure that the ratio alluded to in the Introduction is typically

$$1/|Z_6 - Z_2| = \frac{1}{2 \sin 18°} = \tau \tag{14}$$

from (10).

4. ALGEBRAIC APPROACH

The tenth roots of −1 satisfy

$$Z^{10} + 1 = 0 \tag{15}$$

or, replacing Z^2 by ξ, say,

$$\xi^5 + 1 = 0. \tag{16}$$

This shows that the golden ratio is also relevant to an investigation of the fifth roots of −1. The golden ratio arises, for instance, in the geometry of the regular five-pointed star.

Equation (16) can be factorized to

$$(\xi + 1)(\xi^4 - \xi^3 + \xi^2 - \xi + 1) = 0, \tag{17}$$

showing that $\xi = -1$ is a root of the quintic in (16) and confirming that

$$Z = \pm\sqrt{-1} = \pm i \tag{18}$$

are two roots of the corresponding "dectic" in (15).

Dividing the remaining quartic factor in (17) by ξ^2 gives

$$\xi^2 + \frac{1}{\xi^2} - \left(\xi + \frac{1}{\xi}\right) + 1 = 0, \tag{19}$$

which on substituting

$$\eta = \xi + \frac{1}{\xi} \tag{20}$$

reduces to

$$\eta^2 - \eta - 1 = 0, \tag{21}$$

with roots as in (1), viz,

$$\eta_{1,\,2} = \tau,\ \ -\frac{1}{\tau}. \tag{22}$$

From (20), we also have

$$\xi^2 - \eta\xi + 1 = 0 \tag{23}$$

with roots

$$\xi = \frac{\eta \pm \sqrt{\eta^2 - 4}}{2} = \frac{\eta \pm \sqrt{\eta - 3}}{2}. \tag{24}$$

Inserting the appropriate values of η given in (22), we obtain the complex fifth roots of -1 as

$$\xi_{1,\,2} = \frac{1}{2}(\tau \pm i\sqrt{3 - \tau})$$

and

$$\xi_{3,\,4} = \frac{1}{2}\left(-\frac{1}{\tau} \pm i\sqrt{3 - \tau}\right), \tag{25}$$

from which required complex tenth roots follow with, for instance,

$$Z = \pm\sqrt{\xi}. \tag{26}$$

These square roots are found by proceeding typically as follows. Let

$$a + jb = \sqrt{\frac{1}{2}(\tau + j\sqrt{3 - \tau})}. \tag{27}$$

Since the right-hand side is a root of -1, we have

$$a^2 + b^2 = 1. \tag{28}$$

Also, squaring both sides in (27) and equating real and imaginary parts in the result, we arrive at, with a little help from (1),

$$a^2 - b^2 = \frac{\tau}{2} \tag{29}$$

and

$$ab = \frac{\sqrt{3 - \tau}}{2}. \tag{30}$$

These indicate that the product of a and b is positive, meaning that a and b are together either both positive or both negative. Solving (28) and (29) simultaneously gives

$$a^2 = \frac{1}{2}\left(1 + \frac{\tau}{2}\right) = \frac{2 + \tau}{4} = \frac{(2 + \tau)\tau^2}{4\tau^2} = \frac{4\tau + 3}{4\tau^2} = \frac{[\tau^2(3 - \tau)]}{4},$$

from which

$$a = \pm\frac{\tau\sqrt{3 - \tau}}{2}$$

and

$$b^2 = \frac{1}{2}\left(1 - \frac{\tau}{2}\right) = \frac{2 - \tau}{4} - \frac{(2 - \tau)}{4\tau^2}\tau^2 = \frac{(2 - \tau)(1 + \tau)}{4\tau^2} = \frac{1}{4\tau^2},$$

$$\therefore b = \pm\frac{1}{2\tau}.$$

Thus, from the square root in (27), we obtain two of the tenth roots in (13), namely,

$$\sqrt{\frac{1}{2}(\tau + j\sqrt{3 - \tau})} = \pm\frac{1}{2}\left(\tau\sqrt{3 - \tau} + j\left(\frac{1}{\tau}\right)\right). \tag{31}$$

The other tenth roots in (13) can be obtained similarly from the fifth roots in (25).

Of course, the same procedure outlined here is applicable to the problem of expressing the fifth and tenth roots of unity (i.e., +1 rather than -1), in terms of the golden ratio; however, this is left as an exercise for the interested reader.

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A NOTE CONCERNING THE NUMBER OF ODD-ORDER MAGIC SQUARES

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INTRODUCTION

A square array of consecutive integers $1, 2, \ldots, n^2$ is called *magic* of *order* n if the rows, columns, and diagonals all add up to the same number. If in addition, the sum of the numbers in each broken diagonal is also the same number, then the magic square is said to be *pandiagonal*. Let M be a magic square of order n and let its entries by denoted by (i, j), $(1 \leqslant i, j \leqslant n)$. Then M is *symmetrical* if $(i, j) + (n - i + 1, n - j + 1) = n^2 + 1$. Let D_4 denote the dihedral group of order 8. Then, two magic squares M and M' are said to be *equivalent* if there is a σ in D_4 such that $\sigma(M) = M'$.

Let

$\sigma_0(n)$ = number of inequivalent magic squares of order n.

$\delta_0(n)$ = number of inequivalent pandiagonal magic squares of order n.

$\rho_0(n)$ = number of inequivalent symmetrical magic squares of order n.

$\gamma_0(n)$ = number of inequivalent pandiagonal and symmetrical magic squares of order n.

While it is not difficult to construct, for any $n \geqslant 3$, a magic square of order n, it seems formidable to determine $\sigma_0(n)$ or $\delta_0(n)$ for $n \geqslant 6$ (see [1] and [2]). In [4], it is shown that $\delta_0(4) = 48$ and in [5] that $\delta_0(5) = 3600$. In this note, we shall show that, given an odd-order pandiagonal magic square, we can use it to generate a finite iterative sequence of pandiagonal magic squares of the same order. We show that the number of terms in this sequence is always even. It is observed that, if we start with a non-pandiagonal magic square of odd order, then magic squares and non-magic squares occur alternatively in the sequence. It is also observed that if the initial square is symmetrical, then so is the next one. We then determine the number of terms in the above iterative sequences, thereby showing that each of $\sigma_0(n)$, $\rho_0(n)$, $\delta_0(n)$, and $\gamma_0(n)$ is a multiple of the number of terms in its respective sequence. Finally, we note that our results may be combined with others to yield stronger results.

RESULTS

Let M be a pandiagonal magic square of order n. Obtain from M a square $\varphi(M)$ whose entries $\varphi(i, j)$, $(1 \leqslant i, j \leqslant n)$, are given by

$$\varphi(i, j) = (m + 1 + i - j, m + i + j),$$

where $m = (n - 1)/2$ and the operations are taken modulo n. Then it is routine to verify that $\varphi(M)$ is magic and pandiagonal (see [3]). Further, if M is symmetrical, then so is $\varphi(M)$. For $r > 1$, define, inductively,

$$\varphi^r(M) = \varphi(\varphi^{r-1}(M)).$$

Thus, we obtain a sequence M, $\varphi(M)$, ..., of pandiagonal magic squares of order n. Note that φ is one-to-one and onto and hence its inverse exists. Lemma 1, below, asserts that the sequences generated by M and σM under φ are equivalent. Further, there exists r such that $\sigma\varphi^r(M) = M$ for some $\sigma \in D_4$. We wish to determine the smallest such r.

Lemma 1: Let $\sigma \in D_4$. Then $\varphi(\sigma(M)) = \pi\varphi(M)$ for some $\pi \in D_4$.

Proof: If σ is a 90° clockwise rotation, then $\sigma(i, j) = (j, n - i + 1)$. It is routine to verify that $\varphi\sigma^k(i, j) = \sigma^k\varphi(i, j)$, where $k = 0, 1, 2, 3$. If σ is the reflection along the central vertical (horizontal), then $\sigma(i, j) = (i, n - j + 1)$ [respectively, $(n - i + 1, j)$]. Choose π to be the diagonal reflection with $\pi(i, j) = (j, i)$ [respectively, $(n - j + 1, n - i + 1)$]. If σ is a diagonal reflection, then $\sigma(i, j) = (j, i)$ or $\sigma(i, j) = (n - j + 1, n - i + 1)$, in which case let π be the reflection along the central horizontal and central vertical, respectively. This completes the proof.

Let the entries of $\varphi^r(M)$ by denoted by $\varphi^r(i, j)$. Then it is easy to verify that $\varphi^r(i, j)$ is given by the following:

If $r = 2s$, $s \geqslant 1$, then

$$\varphi^r(i, j) = \begin{cases} (m + 1 + 2^{s-1} - 2^s j,\ m + 1 - 2^{s-1} + 2^s i) & s \equiv 1 \pmod 4, \\ (m + 1 + 2^{s-1} - 2^s i,\ m + 1 + 2^{s-1} - 2^s i) & s \equiv 2 \pmod 4, \\ (m + 1 - 2^{s-1} + 2^s j,\ m + 1 + 2^{s-1} - 2^s i) & s \equiv 3 \pmod 4, \\ (m + 1 - 2^{s-1} + 2^s i,\ m + 1 - 2^{s-1} + 2^s j) & s \equiv 0 \pmod 4. \end{cases}$$

If $r = 2s + 1$, $s \geqslant 0$, then

$$\varphi^r(i, j) = \begin{cases} (m + 1 + 2^s - 2^s(i + j),\ m + 1 + 2^s(i - j)) & s \equiv 1 \pmod 4, \\ (m + 1 - 2^s(i - j),\ m + 1 + 2^s - 2^s(i + j)) & s \equiv 2 \pmod 4, \\ (m + 1 - 2^s + 2^s(i + j),\ m + 1 - 2^s(i - j)) & s \equiv 3 \pmod 4, \\ (m + 1 + 2^s(i - j),\ m + 1 - 2^s + 2^s(i + j)) & s \equiv 0 \pmod 4. \end{cases}$$

The proof of the following lemma is straightforward and so is omitted.

Lemma 2: Suppose n is odd and $n = 2m + 1$.

(i) If $2^s \equiv 1 \pmod n$, then $m + 1 - 2^{s-1} \equiv 0 \pmod n$ and $m + 1 + 2^{s-1} \equiv 1 \pmod n$.

(ii) If $2^s \equiv -1 \pmod n$, then $m + 1 - 2^{s-1} \equiv 1 \pmod n$ and $m + 1 + 2^{s-1} \equiv 0 \pmod n$.

Proposition 1: Let n be odd. Then $\delta_0(n) \equiv 0 \pmod 2$ and $\gamma_0(n) \equiv 0 \pmod 2$.

Proof: If r is odd, then (1, 1) will be an entry in the central column or central row of $\varphi^r(M)$. This means that there is no σ in D_4 such that $\sigma\varphi^r(1, 1) = (1, 1)$. The result thus follows.

Proposition 2: Let n be an odd number. Suppose s is the smallest integer such that $2^s \equiv 1 \pmod{n}$ or $2^s \equiv -1 \pmod{n}$. Then

(i) $\sigma_0(n) \equiv 0 \pmod{s}$,

(ii) $\rho_0(n) \equiv 0 \pmod{s}$,

(iii) $\delta_0(n) \equiv 0 \pmod{2s}$,

and (iv) $\gamma_0(n) \equiv 0 \pmod{2s}$.

Proof: We shall prove (iii). The proof of (iv) then follows immediately; that of (i) follows from Proposition 1 and the fact that if M is magic but not pandiagonal then $\varphi(M)$ is not magic but $\varphi^2(M)$ is magic; (ii) follows from the fact that $\varphi(M)$ is symmetrical if M is.

Let $r = 2k$.

Now $\varphi^r(1, 1)$ is one of:

$$(m + 1 - 2^{k-1}, m + 1 + 2^{k-1}),\ (m + 1 - 2^{k-1}, m + 1 - 2^{k-1}),$$
$$(m + 1 + 2^{k-1}, m + 1 - 2^{k-1}),\ (m + 1 + 2^{k-1}, m + 1 + 2^{k-1}).$$

If $r < 2s$, then we see that $m + 1 + 2^{k-1}$ cannot be $0 \pmod{n}$ or $1 \pmod{n}$. Likewsie, $m + 1 - 2^{k-1}$ cannot be $0 \pmod{n}$ or $1 \pmod{n}$. So there is no σ in D_4 such that $\sigma\varphi^r(1, 1) = (1, 1)$.

Suppose $r = 2s$.

Now if $2^s \equiv 1 \pmod{n}$, then, by Lemma 2,

$$m + 1 - 2^{s-1} \equiv 0 \pmod{n} \quad \text{and} \quad m + 1 + 2^{s-1} \equiv 1 \pmod{n}.$$

If $2^s \equiv -1 \pmod{n}$, then

$$m + 1 - 2^{s-1} \equiv 1 \pmod{n} \quad \text{and} \quad m + 1 + 2^{s-1} \equiv 0 \pmod{n}.$$

In either case, $\varphi^r(i, j)$ is one of

$$(j, n - i + 1),\ (i, j),\ (n - j + 1, i),\ (n - i + 1, n - j + 1).$$

Certainly, there is a σ in D_4 such that $\sigma\varphi^r(i, j) = (i, j)$ and the result follows.

REMARKS

Note that there are other operations which will also generate finite sequences of inequivalent magic squares of the same order. For example:

(A) Cyclic permutation of the rows and/or columns of a pandiagonal magic square will produce an inequivalent pandiagonal magic square. Hence $\delta_0(n) \equiv 0 \pmod{n^2}$.

(B) Let $n = 2m + 1$. Then any permutation of the numbers 1, 2, ..., m applied to the first m rows and columns and to the last m rows and columns (in reverse order) of a magic square of order n will result in an inequivalent magic square. Further, if we start with a symmetrical square, then so are all other squares generated in this manner. Hence $\sigma_0(n) \equiv 0 \pmod{m!}$ and $\rho_0(n) \equiv 0 \pmod{m!}$.

More magic squares may be obtained by combining the operation φ with that of (A) or (B).

Proposition 3: Let $n = 2m + 1$ and suppose s satisfies the conditions of Proposition 2. Then

(i) $\sigma_0(n) \equiv 0 \pmod{s \cdot m!}$,

(ii) $\rho_0(n) \equiv 0 \pmod{s \cdot m!}$

and (iii) $\delta_0(n) \equiv 0 \pmod{2sn^2}$.

Proof: Let M be a pandiagonal magic square of order n. For each $\varphi^r(M)$, we apply the operation in (A) to get n^2 inequivalent pandiagonal magic squares. It remains to show that these n^2 squares are not equivalent to any of those generated by φ. To see this, it suffices to note that $(m + 1, m + 1)$ is always fixed under φ, while in the operation (A) it is being transferred to other positions. This proves (iii).

To prove (i) and (ii), let M be a magic square of order n. For each $\varphi^{2k}(M)$ (which is magic), we apply the operation in (B) to get $m!$ inequivalent magic squares. We shall show that these $m!$ squares are not equivalent to any one of those generated by φ. Since the operation φ transfers the central row and the central column of M to the main diagonals of the resulting square, it follows that we need only consider $\varphi^{2k}(M)$. Consider the entries $(i, m + 1)$, where $i = 1, 2, \ldots, m$. If k is odd, then $\varphi^{2k}(i, m + 1) = (m + 1, x + 2yi)$ for some integers x and y. If k is even, then $\varphi^{2k}(i, m + 1) = (x - 2yi, m + 1)$. However, under the operation in (B), the entries $(i, m + 1)$, where $i = 1, 2, \ldots, m$ go to $(\sigma(i), m + 1)$ for some permutation σ of the numbers $1, 2, \ldots, m$. This means that $\varphi^{2k}(M)$ cannot be equivalent to any one of the squares generated by the operation in (B).

ACKNOWLEDGMENT

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A CONGRUENCE RELATION FOR CERTAIN RECURSIVE SEQUENCES

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Recently, the first author [1] showed that

$$F_{n+5} \equiv F_n + F_{n-5} \pmod{10}, \tag{1}$$

where F_n is the n^{th} Fibonacci number, defined by $F_{n+1} = F_n + F_{n-1}$, $n \geqslant 2$, with $F_1 = F_2 = 1$. It was also shown [1] that this result generalizes to a sequence $\{S_n\}_1^\infty$ defined by

$$S_{n+1} = S_n + S_{n-1},\ n \geqslant 2,$$

with $S_1 = c$, $S_2 = d$, where c and d are nonnegative integers. The nonnegative restriction was imposed in order to guarantee that each member of the sequence is a positive number. However, the result is, in fact, valid for any integers c and d.

The purpose of this paper is to generalize (1) further. We will see that the role played by the integer 5 in (1) can, in the generalization, be played by any prime $p \geqslant 5$.

We begin by introducing a more general sequence $\{T_n\}_{-\infty}^\infty$ defined by

$$T_{n+1} = aT_n - bT_{n-1},\ \text{with } T_1 = c,\ T_2 = d, \tag{2}$$

where a, b, c, and d are integers with the restriction $b \neq 0$ (and exclusion of the trivial case where $c = d = 0$). We write $\{\alpha, \beta\}$ to denote the set of solutions of the quadratic equation $x^2 - ax + b = 0$. Two particular choices of c and d in (2) give rise to sequences $\{T_n\}$ of special interest to us. We denote these by $\{U_n\}_{-\infty}^\infty$ and $\{V_n\}_{-\infty}^\infty$, where

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta) \tag{3}$$

and

$$V_n = \alpha^n + \beta^n. \tag{4}$$

For $\{U_n\}$, $c = 1$ and $d = a$ while, for $\{V_n\}$, $c = a$ and $d = a^2 - 2b$. These sequences have been studied by Horadam [4]. [If $\alpha = \beta$, we replace (3) and (4) by the limiting forms $U_n = n\alpha^{n-1}$ and $V_n = 2\alpha^n$, respectively. Note that, in this case, $b = a^2/4$ and $\alpha = a/2$.] For the special case of (2) where $a = -b = 1$, the sequences $\{U_n\}$ and $\{V_n\}$ are, respectively, the Fibonacci and Lucas numbers for which (3) and (4) are the well-known Binet forms. We will write $\{L_n\}$ to denote the Lucas sequence.

Using $\alpha\beta = b$, we readily deduce from (3) and (4) that

$$U_{-n} = -b^{-n}U_n \tag{5}$$

and

$$V_{-n} = b^{-n}V_n.$$

We will require (5) later. We also need two lemmas connecting the sequences $\{U_n\}$ and $\{V_n\}$. The Fibonacci-Lucas forms of these (corresponding to $a = -b = 1$) are given in Hoggatt [3].

Lemma 1: For all integers k,

$$U_{k+1} - bU_{k-1} = V_k. \tag{6}$$

Proof: This is proved by induction or directly by using the generalized Binet forms (3) and (4).

Lemma 2: For all integers n and k,

$$U_{n+k} + b^k U_{n-k} = U_n V_k. \tag{7}$$

Proof: The proof may again be completed either by induction or by direct verification using (3) and (4). For the induction proof, we begin by verifying (7) for $n = 0$ and 1, with the aid of (5).

We generalize this last result to the sequence $\{T_n\}$ defined by (2).

Lemma 3: For all integers n and k,

$$T_{n+k} + b^k T_{n-k} = T_n V_k. \tag{8}$$

Proof: We show by induction that

$$T_n = dU_{n-1} - bcU_{n-2}, \tag{9}$$

and hence verify (8) directly from (7).

The results which we have obtained thus far are, in fact, valid when a, b, c, and d in (2) are real. However, for the divisibility results which follow, we require integer sequences; hence, we require a, b, c, and d to be integers. Also, in view of (5), we need to restrict $\{T_n\}$ to nonnegative n unless $|b| = 1$.

We now prove our first divisibility result.

Lemma 4: For any prime p,

$$V_p \equiv a \pmod{p}. \tag{10}$$

Proof: We need to treat the case $p = 2$ separately.

Since $V_2 = a^2 - 2b$,

$$V_2 - a = a(a - 1) - 2b \equiv 0 \pmod{2}$$

for any choice of integers a and b.

If p is an odd prime,

$$a^p = (\alpha + \beta)^p = \sum_{r=0}^{p} \binom{p}{r} \alpha^{p-r}\beta^r.$$

From $\alpha\beta = b$, we obtain

$$\alpha^{p-r}\beta^r + \alpha^r\beta^{p-r} = b^r(\alpha^{p-2r} + \beta^{p-2r}).$$

and thus

$$a^p = V_p + \sum_{r=1}^{(p-1)/2} \binom{p}{r} b^r V_{p-2r}.$$

In the latter summation, we note that

$$\binom{p}{r} \equiv 0 \pmod{p}$$

for each r and the proof is completed by applying Fermat's theorem

$$a^p \equiv a \pmod{p}.$$

For the Fibonacci-Lucas case (where $a = -b = 1$), Lemma 4 yields

$$L_p \equiv 1 \pmod{p}$$

for any prime p. This special case, although not quoted explicitly, is easily deduced from congruence results for the Fibonacci numbers given in Hardy and Wright [2].

We now state the first of our main results.

Theorem 1: For all $n \geqslant p$ and all primes p,

$$T_{n+p} \equiv aT_n - bT_{n-p} \pmod{p}. \tag{11}$$

Proof: The proof follows from Lemmas 3 and 4 and Fermat's theorem. If $|b| = 1$, then (11) holds for all values of n.

Observe how the congruence relation (11) mimics the pattern of the recurrence relation (2).

To strengthen Theorem 1 for primes greater than 3, we first require:

Lemma 5: If $k \not\equiv 0 \pmod{3}$, then for all choices of a and b,

$$V_k \equiv a \pmod{2}. \tag{12}$$

Proof: In verifying (12) for all possible choices of a and b, it suffices to consider $\{a, b\} = \{0, 1\}$. If a is even and b is even or odd, V_k is even for all k and (12) holds. If a is odd and b is even, V_k is odd for all k and again (12) holds. Finally, if both a and b are odd, then V_k is even if and only if $k \equiv 0 \pmod{3}$, and the lemma is established.

Theorem 2: For all $n \geqslant p$, where p is any prime greater than 3,

$$T_{n+p} \equiv aT_n - bT_{n-p} \pmod{2p}. \tag{13}$$

[We note that (1) is the special case of (13) obtained by taking $p = 5$ and $a = -b = c = d = 1$.]

Proof: From the result of Theorem 1, it remains only to show that

$$T_{n+p} - aT_n + bT_{n-p} \equiv 0 \pmod{2}. \tag{14}$$

Using Lemma 3, the left side of (14) may be expressed as

$$(V_p - a)T_n + (b - b^p)T_{n-p}.$$

Observe that $b - b^p \equiv 0 \pmod 2$ and Lemma 5 shows that $V_p - a \equiv 0 \pmod 2$ for p any prime greater than 3, which completes the proof.

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A NOTE ON THE REPRESENTATION OF INTEGERS AS A SUM OF DISTINCT FIBONACCI NUMBERS

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1. INTRODUCTION AND GENERALITIES

It is known that every positive integer can be represented uniquely as a finite sum of *F-addends* (distinct nonconsecutive Fibonacci numbers). A series of papers published over the past years deal with this subject and related problems [1, 2, 3, 4]. Our purpose in this note is to investigate some minor aspects of this property of the Fibonacci sequence. More precisely, for a given integer $k \geqslant 3$, we consider the set $\mathcal{N}_k$ of all positive integers n less than F_k (as usual F_k and L_k are the k^{th} Fibonacci and Lucas numbers, respectively), and for these integers we determine:

(i) the asymptotic value of the average number of F-addends;

(ii) the most probable number of F-addends;

(iii) the greatest number m_k of F-addends, selected from the set $\mathcal{N}_k$, and the integers representable as a sum of m_k F-addends.

Setting

$$m_k = [(k - 1)/2], \quad (k \geqslant 3) \tag{1}$$

(here and in the following the symbol $[x]$ denotes the greatest integer not exceeding x) and denoting by $f(n, k)$ the number of F-addends the sum of which represents a generic integer $n \in \mathcal{N}_k$, we state the following theorems.

Theorem 1: $1 \leqslant f(n, k) \leqslant m_k$.

Proof: Since $F_1 = F_2$ and since the F-addends are distinct, they can be chosen in the set $\mathcal{F}_k = \{F_2, F_3, \ldots, F_{k-1}\}$ the cardinality of which is $|\mathcal{F}_k| = k - 2$. Moreover, since the F-addends are nonconsecutive Fibonacci numbers, they can be in number at most either $|\mathcal{F}_k|/2$ (for $|\mathcal{F}_k|$ even) or $(|\mathcal{F}_k| + 1)/2$ (for $|\mathcal{F}_k|$ odd). Q.E.D.

Theorem 2: The number $N_{k,m}$ of integers belonging to $\mathcal{N}_k$ which can be represented as a sum of m F-addends is given by

$$N_{k,m} = \binom{k - m - 1}{m}.$$

Proof: Setting $M = |\mathcal{F}_k| = k - 2$, it is evident that $N_{k,m}$ equals the number $B_{M,m}$ of distinct binary sequences of length M containing m nonadjacent 1's and $M - m$ 0's. The number $B_{M,m}$ can be obtained by considering the string

$$\{v\ 0\ v\ 0\ v \cdots v\ 0\ v\}$$

constituted by $M - m$ 0's and $M - m + 1$ empty elements v, and by replacing, in all possible ways, m empty elements by m 1's:

$$B_{M,m} = \binom{M - m + 1}{m}.$$

Replacing M by $k - 2$ in the above relation, the theorem is proved. Q.E.D.

From Theorem 2, we derive immediately the following

Remark:

$$N_{k,m} = \begin{cases} k - 2, & \text{for } m = 1 \\ 0, & \text{for } m > m_k. \end{cases} \tag{2}$$

2. THE AVERAGE VALUE OF $f(n, k)$

In this section, we calculate the limit of the ratio between the average value of $f(n, k)$ and k as k tends to infinity.

From Theorem 2, it is immediately seen that the average value $\overline{f}(n, k)$ of the number of F-addends the sum of which represents the integers belonging to $\mathcal{N}_k$ is given by

$$\overline{f}(n, k) = \frac{1}{|\mathcal{N}_k|}\sum_{m=1}^{m_k} mN_{k,m} = \frac{1}{F_k - 1}\sum_{m=1}^{\left[\frac{k-1}{2}\right]} m\binom{k - m - 1}{m}. \tag{3}$$

Moreover, it is known [5] that the identity

$$\sum_{m=0}^{m_k} (k - m)N_{k,m} = U_k \tag{4}$$

holds, where

$$U_k = \sum_{m=0}^{k-1} F_{m+1}F_{k-m}; \tag{5}$$

from (4), the relation

$$U_k = k\sum_{m=0}^{m_k} N_{k,m} - \sum_{m=0}^{m_k} mN_{k,m}$$

is obtained from which, by virtue of the well-known representation of the Fibonacci numbers as sums of binomial coefficients [6], we get

$$U_k = kF_k - \sum_{m=0}^{m_k} mN_{k,m}.$$

Consequently, we can write

$$\sum_{m=0}^{m_k} mN_{k,m} = \sum_{m=1}^{m_k} mN_{k,m} = kF_k - U_k. \tag{6}$$

The numbers U_k defined by (5) satisfy the recurrence stated in the following theorem.

Theorem 3: $U_k = kF_k - U_{k-2}$, with $U_1 = 1$, $U_2 = 2$.

Proof: Using the well-known identity $F_{s+t} = F_{s+1}F_t + F_sF_{t-1}$ and setting $m = s$, $k - m = t$, we can write the identity

$$F_k = F_{m+k-m} = F_{m+1}F_{k-m} + F_mF_{k-m-1}$$

thus getting $F_{m+1}F_{k-m} = F_k - F_mF_{k-m-1}$. Therefore, from (5), we have

$$U_k = \sum_{m=0}^{k-1}(F_k - F_mF_{k-m-1}) = kF_k - \sum_{m=0}^{k-1}F_mF_{k-m-1} = kF_k - \sum_{m=1}^{k-2}F_mF_{k-m-1}.$$

Setting $r = m - 1$, from the previous relation we obtain

$$U_k = kF_k - \sum_{r=0}^{k-3}F_{r+1}F_{k-r-2} = kF_k - U_{k-2}. \quad \text{Q.E.D.}$$

From Theorem 3, the further expression of U_k is immediately derived:

$$\begin{aligned} U_k &= kF_k - (k-2)F_{k-2} + \cdots + (-1)^{m_k}(k - 2m_k)F_{k-2m_k} \\ &= \sum_{i=0}^{m_k}(-1)^i(k - 2i)F_{k-2i}, \end{aligned} \tag{7}$$

where, as usual, $m_k = [(k - 1)/2]$.

Denoting by α and β the roots of the equation $x^2 - x - 1 = 0$, the following theorem can be stated.

Theorem 4: $\bar{f}(n, k)$ is asymptotic to $\dfrac{1}{1 + \alpha^2}$.

Proof: From (3) and (6), we can write

$$\bar{f}(n, k)/k = \left(\frac{1}{F_k - 1}(kF_k - U_k)\right)\Big/k$$

and calculate the limit

$$\lim_{k\to\infty}\bar{f}(n, k)/k = \lim_{k\to\infty}\left(k - \frac{U_k}{F_k}\right)\Big/k = \lim_{k\to\infty} 1 - \frac{U_k}{kF_k}$$

which, from (7), can be rewritten as

$$\lim_{k\to\infty}\bar{f}(n, k)/k = \lim_{k\to\infty}\left(kF_k - kF_k + \sum_{i=1}^{m_k}(-1)^{i-1}(k - 2i)F_{k-2i}\right)\Big/(kF_k).$$

Finally, using the Binet form for F_k, we get

$$\begin{aligned} \lim_{k\to\infty}\bar{f}(n, k)/k &= \lim_{k\to\infty}\frac{\sum_{i=1}^{m_k}(-1)^{i-1}(k - 2i)(\alpha^{k-2i} - \beta^{k-2i})}{k(\alpha^k - \beta^k)} \\ &= \lim_{k\to\infty}\frac{\sum_{i=1}^{m_k}(-1)^{i-1}(1 - 2i/k)\alpha^{k-2i}}{\alpha^k} = \sum_{i=1}^{\infty}(-1)^{i-1}\alpha^{-2i} = \frac{1}{1 + \alpha^2} \approx 0.2764. \end{aligned}$$

Q.E.D.

The behavior of $\bar{f}(n, k)/k$ versus k has been obtained using a computer calculation and is shown in Figure 1 for $3 \leqslant k \leqslant 100$.

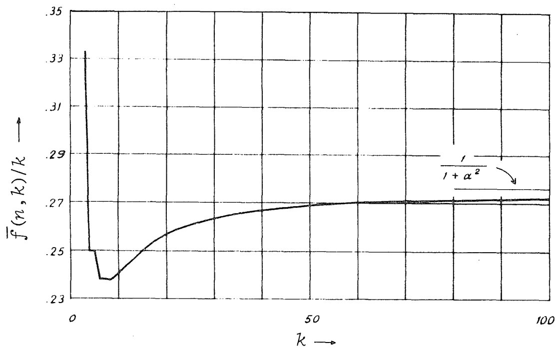

Figure 1. Behavior of $\bar{f}(n, k)/k$ versus k

3. THE MOST PROBABLE VALUE OF $f(n, k)$

In this section, it is shown that the most probable number $\hat{f}(n, k)$ of F-addends the sum of which represents the integers belonging to $\mathscr{N}_k$, can assume at most two (consecutive) values. The value of $\hat{f}(n, k)$ for a given k together with the values of k for which two $\hat{f}(n, k)$'s occur, are worked out.

From Theorem 2, it is immediately seen that $\hat{f}(n, k)$ equals the value(s) of m which maximize the binomial coefficient $N_{k,m}$; consequently let us investigate the behavior of the discrete function

$$\binom{h-n}{n} \tag{8}$$

as n varies, looking for the value(s) $\hat{n}_h$ of n which maximize it. It is evident that $\hat{n}_h$ is the value(s) of n for which the inequalities

$$\binom{h-n}{n} \geqslant \binom{h-n+1}{n-1} \tag{9}$$

and

$$\binom{h-n}{n} \geqslant \binom{h-n-1}{n+1} \tag{10}$$

are simultaneously verified. Using the factorial representation of the binomial coefficients and omitting the intermediate steps for the sake of brevity, the inequality

$$5n^2 - (5h + 7)n + h^2 + 3h + 2 \geqslant 0 \tag{11}$$

is obtained from (9); the roots of the associate equation are

$$\begin{cases} n_1 = (5h + 7 - \sqrt{\Delta})/10, \\ n_2 = (5h + 7 + \sqrt{\Delta})/10, \end{cases} \tag{12}$$

where $\Delta = 5h^2 + 10h + 9$. From (11), we have

$$n_2 \leqslant n \leqslant n_1. \tag{13}$$

Analogously, from (10), we obtain the inequality

$$5n^2 - (5h - 3)n + h^2 - 2h \leqslant 0, \tag{14}$$

from which the roots

$$\begin{cases} n_1' = (5h - 3 - \sqrt{\Delta})/10 \\ n_2' = (5h - 3 + \sqrt{\Delta})/10 \end{cases} \tag{15}$$

are derived. From (14), we have

$$n_1' \leqslant n \leqslant n_2'. \tag{16}$$

Since, for $h \geqslant 2$, the inequality $n_1 < n_2'$ holds, the inequalities (13) and (16) are simultaneously verified within the interval $[n_1', n_1]$. Therefore, we have $n_1' \leqslant \hat{n}_h \leqslant n_1$. Since $n_1 - n_1' = 1$, the value

$$\hat{n}_h = [n_1'] + 1 = [n_1] \tag{17}$$

is unique, provided that n_1' (and n_1) is not an integer. If and only if n_1' is an integer is the binomial coefficient (8) maximized by two consecutive values $\hat{n}_{h,1}$ and $\hat{n}_{h,2}$ of n; that is,

$$\begin{cases} \hat{n}_{h,1} = n_1', \\ \hat{n}_{h,2} = n_1' + 1 = n_1. \end{cases} \tag{17$'$}$$

Now we can state the following theorem.

Theorem 5: $\hat{f}(n, k) = \left[\dfrac{5k - 8 - (5k^2 + 4)^{1/2}}{10}\right] + 1.$

Proof: The proof is derived directly from (17), (17′), and (15) after replacing h by $k - 1$ and n by m in (8). Q.E.D.

On the basis of (17′) and (15), we determine the values of k for which the quantity

$$R_k = (5k - 8 - (5k^2 + 4)^{1/2})/10$$

is integral, i.e., the values of k for which two consecutive values of m maximize $N_{k,m}$ thus yielding the following two values of $\hat{f}(n, k)$:

$$\begin{cases} \hat{f}_1(n, k) = R_k, & (18) \\ \hat{f}_2(n, k) = R_k + 1. & (18') \end{cases}$$

Theorem 6: The most probable values of $f(n, k)$ are both $\hat{f}_1(n, k)$ and $\hat{f}_2(n, k)$, if and only if $k = F_{4s}$, $s = 1, 2, \ldots$.

Proof: For R_k to be integral, the quantity $5k^2 + 4$ must necessarily be the square of an integer, i.e., the equation

$$x^2 - 5k^2 = 4 \tag{19}$$

must be solved in integers. On the basis of [7, p. 100, pp. 197-198] and by induction on r, it is seen that, if $\{x_1, k_1\}$ is a pair of positive integers x, k with minimal x satisfying (19), then all pairs of positive integers $\{x_r, k_r\}$ satisfying this equation are defined by

$$x_r \pm \sqrt{5}k_r = \frac{(x_1 \pm \sqrt{5}k_1)^r}{2^{r-1}}, \quad r = 1, 2, \ldots . \tag{20}$$

Since it is found that $x_1 = 3$ and $k_1 = 1$, from (20), we can write

$$x_r + \sqrt{5}k_r = \frac{(3 + \sqrt{5})^r}{2^{r-1}} = 2\alpha^{2r}. \tag{21}$$

From (19) and (21), we get the relation

$$(5k_r^2 + 4)^{1/2} = 2\alpha^{2r} - \sqrt{5}k_r$$

from which, squaring both sides, we obtain

$$k_r = \frac{1}{\sqrt{5}}\frac{\alpha^{4r} - 1}{\alpha^{2r}} = \frac{1}{\sqrt{5}}(\alpha^{2r} - \alpha^{-2r}) = F_{2r}.$$

Replacing k by F_{2r}, R_k reduces to $(L_{2r-1} - 4)/5$; therefore, to prove the theorem, it is sufficient to prove that, iff r is even, then the congruence $L_{2r-1} \equiv 4 \pmod 5$ holds.

Using Binet's form for L_r, we obtain

$$L_{2r-1} = \frac{1 + S}{2^{2(r-1)}},$$

where

$$S = \sum_{t=1}^{r-1}\binom{2r-1}{2t}(\sqrt{5})^{2t} = 5\sum_{t=1}^{r-1}\binom{2r-1}{2t}(\sqrt{5})^{2(t-1)}.$$

Therefore, we can write the following equivalent congruences,

$$2^{-2(r-1)}(1 + S) \equiv 4 \pmod 5,$$

$$1 + S \equiv 2^{2r} \pmod 5,$$

$$1 \equiv 2^{2r} \pmod 5,$$

which, for Fermat's little theorem, hold iff $r = 2s$, $s = 1, 2, \ldots$. Q.E.D.

4. THE INTEGERS REPRESENTABLE AS A SUM OF m_k F-ADDENDS

In this section, the set of all integers $n \in \mathcal{N}_k$ which can be represented as a sum of m_k F-addends [i.e., all integers such that $f(n, k) = m_k$] is determined.

From Theorem 2 and (1), the following corollary is immediately derived.

Corollary 1:

$$N_{k, m_k} = \begin{cases} k/2, & \text{for even } k, \\ 1, & \text{for odd } k. \end{cases}$$

The following identities are used to prove Theorems 7 and 8.

Identity 1: $\sum_{j=1}^{h} F_{2j} = F_{2h+1} - 1.$

Identity 2: $\sum_{j=1}^{h} F_{2j+1} = F_{2(h+1)} - 1.$

Identity 3: $\sum_{j=0}^{m-1} F_{2j+n} = F_{2m+n-1} - F_{n-1}.$

The proofs of Identities 1, 2, and 3 are obtained by mathematical induction and are omitted here for the sake of brevity.

Theorem 7: $f(F_k - 1) = m_k$.

Proof: (i) Even k.

For even k, we have $m_k = (k - 2)/2$; it follows that $k = 2(m_k + 1)$ and, from Identity 2,

$$F_k - 1 = F_{2(m_k+1)} - 1 = \sum_{i=1}^{m_k} F_{2i+1}.$$

(ii) Odd k.

For odd k, we have $m_k = (k - 1)/2$; it follows that $k = 2(m_k + 1)$ and, from Identity 1,

$$F_k - 1 = F_{2m_k+1} - 1 = \sum_{i=1}^{m_k} F_{2i}.$$

In both cases, $F_k - 1$ can be represented as a sum of m_k F-addends. Q.E.D.

From Theorem 7 and Corollary 1, it is evident that, for odd k, the only integer $n \in \mathcal{N}_k$ such that $f(n, k) = m_k$ is $n = F_k - 1$. Moreover, it is seen that, for even k, the integers $n \in \mathcal{N}_k$ such that $f(n, k) = m_k = (k - 2)/2$ are $k/2$ in number ($F_k - 1$ inclusive); let us denote these integers by

$$A_{k,i},\ i = 1, 2, \ldots, k/2.$$

Theorem 8: $A_{k,i} = F_k - F_{k-2i} - 1,\ i = 1, 2, \ldots, k/2.$

Proof: For a given even k, the integers $A_{k,i}$ can be obtained by means of the following procedure:

$$\begin{aligned}
A_{k,1} &= F_2 + F_4 + F_6 + \cdots + F_{k-6} + F_{k-4} + F_{k-2}\\
A_{k,2} &= F_2 + F_4 + F_6 + \cdots + F_{k-6} + F_{k-4} + (F_{k-1})\\
A_{k,3} &= F_2 + F_4 + F_6 + \cdots + F_{k-6} + (F_{k-3} + F_{k-1})\\
&\vdots\\
A_{k,k/2-2} &= F_2 + F_4 + (F_7 + \cdots + F_{k-5} + F_{k-3} + F_{k-1})\\
A_{k,k/2-1} &= F_2 + (F_5 + F_7 + \cdots + F_{k-5} + F_{k-3} + F_{k-1})\\
A_{k,k/2} &= (F_3 + F_5 + F_7 + \cdots + F_{k-5} + F_{k-3} + F_{k-1})
\end{aligned}$$

The mechanism of choice of the F-addends from two disjoint subsets of $\mathscr{F}_k$ [namely, $\{F_{2t}\}$ and $\{F_{2t+1}\}$, $t = 1, 2, \ldots, (k-2)/2$] illustrated in the previous table yields the following expression of $A_{k,i}$,

$$A_{k,i} = \sum_{r=0}^{k/2-i-1} F_{2+2r} + \sum_{s=0}^{i-2} F_{k-2i+2s+3},$$

from which, by virtue of Identity 3, we obtain

$$\begin{aligned} A_{k,i} &= F_{2(k/2-i)+1} - F_1 + F_{2(i-1)+k-2i+2} - F_{k-2i+2} \\ &= F_{k-2i+1} - 1 + F_k - F_{k-2i+2} = F_k - F_{k-2i} - 1. \quad \text{Q.E.D.} \end{aligned}$$

The following corollary is derived from Theorem 8.

Corollary 2: $A_{k,1} = F_{k-1} - 1,$ (22)

$A_{k,2} = L_{k-2} - 1,$ (23)

$A_{k,k/2} = F_k - 1.$ (24)

Proof: Identities (22) and (24) are obtained directly from Theorem 8. Identity (23) requires some manipulations; that is,

$$\begin{aligned} A_{k,2} &= F_k - F_{k-4} - 1 = F_k - (5F_k - 3F_{k+1}) - 1 \\ &= -F_k + 3(F_{k+1} - F_k) - 1 = -F_k + 3F_{k-1} - 1 \\ &= 2F_{k-1} - F_{k-2} - 1 = F_{k-1} + F_{k-3} - 1 = L_{k-2} - 1. \quad \text{Q.E.D.} \end{aligned}$$

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SIMSON'S FORMULA AND AN EQUATION OF DEGREE 24

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1. INTRODUCTION

In the January 23, 1985, issue of a local (Armidale) newspaper, L. Wilson, of Brisbane, announced that, if $x = F_n$, $y = F_{n+1}$ $(F_{n+2} = x + y)$ are successive numbers of the Fibonacci sequence $\{F_n\}$, then x, y $(>x)$ satisfy the equation of degree 24:

$$((x^5y - x^4y^2 - x^3y^3 + 3x^2y^4 - 3xy^5 + y^6)^2 - 4x^8 - 13x^4 - 1)^2 - 144x^{12} - 144x^8 - 36x^4 = 0. \tag{1}$$

This is a slight simplification of the equation announced three weeks earlier by him in the same newspaper.

Wilson offered no proof of his assertion.

It is the purpose of this paper to outline a proof of Wilson's result by analyzing the structure of (1).

We exclude $n = 0$ from our considerations to accord with the commencing Fibonacci number $F_1 = 1$ used by Wilson [although $x = 0$, $y = 1$ do satisfy (1)].

First, observe that Simson's formula for $\{F_n\}$, namely,

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1} \tag{2}$$

may be written

$$x^2 + xy - y^2 = 1, \; n \text{ odd}, \tag{3}$$

$$x^2 + xy - y^2 = -1, \; n \text{ even}. \tag{4}$$

Simson's formula will be the basic knowledge used in our proof.

2. PROOF OF THE ASSERTION

After a little elementary algebraic manipulation, the left-hand side of (1) factorizes as

$$(y^2A^2 - B_1^2)(y^2A^2 - B_2^2), \tag{5}$$

where

$$\begin{cases} A = x^5 - x^4y - x^3y^2 + 3x^2y^3 - 3xy^4 + y^5, \\ B_1 = 2x^4 - 3x^2 + 1, \\ B_2 = 2x^4 + 3x^2 + 1. \end{cases} \tag{6}$$

Numerical checking with small values of n establishes that the first factor in (5) vanishes for n odd, while the second factor in (5) vanishes for n even. This arithmetical evidence suggests that we may associate this first factor (and therefore B_1) with equation (3) and the second factor (and therefore B_2) with equation (4).

Accordingly, from (3), we have immediately $(x^2 - 1)^2 = (y^2 - xy)^2$ which, after tidying up and applying (3) again, gives us

$$B_1 = y(2y^3 - 4y^2x + 2x^2y - x + y). \tag{7}$$

Similarly,

$$B_2 = y(2y^3 - 4y^2x + 2x^2y + x - y). \tag{8}$$

Now $y - x$ is a factor of A, B_1, B_2. So (6) becomes

$$\begin{cases} A = (y - x)(y^4 - 2xy^3 + x^2y^2 - x^4) = (y - x)a, \\ B_1 = y(y - x)(2y^2 - 2xy + 1) \qquad = y(y - x)b_1, \\ B_2 = y(y - x)(2y^2 - 2xy - 1) \qquad = y(y - x)b_2. \end{cases} \tag{9}$$

Repeated multiplicative maneuvering with (3), followed by substitution in (9) and simplification, yields

$$b_1 = -a. \tag{10}$$

Appealing to B_2 and (4) by a similar argument, we find

$$b_2 = a. \tag{11}$$

From (9), it follows that (5) reduces to

$$y^4(y - x)^4(a^2 - b_1^2)(a^2 - b_2^2), \tag{12}$$

whence, by (10) and (11),

$$y^4(y - x)^4(a^2 - b_1^2)(a^2 - b_2^2) = 0, \tag{13}$$

i.e.,

$$(a^2 - b_1^2)(a^2 - b_2^2) = 0. \tag{14}$$

Thus, the validity of (1) is demonstrated.

Variations, perhaps simplifications, of the above reasoning no doubt exist.

3. REMARKS

Rearranging the four factors in (14) leads to

$$\{(a + b_1)(a - b_2)\}\{(a - b_1)(a + b_2)\} = 0. \tag{15}$$

By (10) and (11),

$$(a + b_1)(a - b_2) = 0. \tag{16}$$

Equation (16), which is of degree 8 in y, is thus also satisfied by successive pairs of Fibonacci numbers.

Even more ponderous and complicated equations of higher, but appropriate, degrees are suggested by (14). For instance,

$$(a^4 - b_1^4)(a^4 - b_2^4) = 0,$$

of degree 32 in y, is satisfied by the Fibonacci conditions.

Only the Fibonacci numbers provide the structure for (1). While similar patterns in (2), (3), and (4) exist for Lucas and Pell numbers, equations different from (1) would be germane to them.

Regarding the factors in (13) involving the fourth power, we remark that $y = 0$ if $n = -1$ (excluded), while $y - x = 0$ if $n = 1$, i.e., when $F_1 = F_2 = 1$.

Finally, we comment that (3) and (4) form the nucleus of a geometrical article on conics [2] by one of the authors, which was followed by an extension [1] by Bergum. One is prompted to speculate on the possibility of some arcane geometry of curves being obscured by the symbolism of (1).

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DIFFERENCES BETWEEN SQUARES AND POWERFUL NUMBERS

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A number P is *powerful* if, whenever a prime p divides P, then p also divides P. In [2] McDaniel proves that each nonzero integer can be written in infinitely many ways as the difference between two relatively prime powerful numbers. (Golomb [1] had conjectured that infinitely many integers could not be represented as the difference between powerful numbers.) An examination of McDaniel's paper shows that he actually proves that, if $n \not\equiv 2 \pmod 4$, then n can be written in infinitely many ways as $S - P$, where S is a square, P is powerful, and $(S, P) = 1$.

In this paper we take care of the case $n \equiv 2 \pmod 4$, to prove

Theorem: If n is any nonzero integer, then n can be written in infinitely many ways as $n = S - P$, where S is a square, P is powerful, and $(S, P) = 1$.

Proof: For compactness, we assume the reader is familiar with [2]. In Theorem 2 of that paper it is proved that if n is a positive integer and $n \not\equiv 2 \pmod 4$ then $x^2 - Dy^2 = n$ has infinitely many relatively prime solutions X, Y such that D divides Y. Clearly, each represents n in the desired way. The method of proof is to show that there exist integers D, p, q, x_0, and y_0 such that

$$D > 0 \text{ and } D \text{ is not a square,} \tag{1}$$

$$p \text{ and } q \text{ satisfy } p^2 - Dq^2 = n \text{ and } (p, q) = 1, \tag{2}$$

$$x_0 \text{ and } y_0 \text{ satisfy } x^2 - Dy^2 = \pm 1, \tag{3}$$

$$(2py_0, D) \text{ divides } q. \tag{4}$$

Although McDaniel assumes $n > 0$ in the proof of his Theorem 2, the arguments he gives work just as well for negative values of n. Thus, only the case $n \equiv 2 \pmod 4$ remains. Let $n = 8k \pm 2$.

Case 1. $n = 8k + 2$ or $3 \nmid n$.

If $n = 2$, then $D = 7$, $p = 3$, $q = 1$, $x_0 = 8$, and $y_0 = 3$ can be checked to satisfy (1) through (4). Likewise, if $n = 10$, then $D = 39$, $p = 7$, $q = 1$, $x_0 = 25$, and $y_0 = 4$ work.

Otherwise, we take $D = (2k - 1)^2 \mp 2$, $p = 2k + 1$, $q = 1$, $x_0 = D \pm 1$, and $y_0 = 2k - 1$. Since $n = 2$ and $n = 10$ have been excluded, we see that $D > 1$ and D is odd. Conditions (2) and (3) are easily checked. Note that because $p^2 - D = n$, we have $p^2 - D - 4p = \pm 2 - 4 = -2$ or -6. Since D is odd, $(p, D) = 1$ or 3, with the latter a possibility only if we take the bottom signs. However, $(p, D) = 3$ implies $3|n$, contrary to our assumption. Thus, $(p, D) = 1$. Also, $y_0^2 - D = \pm 2$, so $(y_0, D) = 1$. This proves (4).

Case 2: $n = 8k - 2$ and $3|n$.

We take $p = 6k - 1$, $q = 1$, $D = p^2 - n = 36k^2 - 20k + 3$, $x_0 = 9D - 1$, and $y_0 = 3(18k - 5)$. It can be checked that $D > 1$ and that D is strictly between

p^2 and $(p-1)^2$ for any value of k. We calculate that $y_0^2 - 81D = -18$, and so $x_0^2 - Dy_0^2 = (9D-1)^2 - D(81D-18) = 1$, while (2) is immediate. Note that $3 \nmid p$ but $3 | n$, so $3 \nmid D$. Since D is odd, we see that $(y_0, D) = 1$. Finally,

$$3(p^2 - D) - 4p = 3n - 4p = -2,$$

and so $(p, D) = 1$ also.

To compute solutions to $S - P = n$, we can follow McDaniel and define integers x_j, y_j for $j > 0$ by

$$x_j + y_j\sqrt{D} = (x_0 + y_0\sqrt{D})^{cj},$$

where $c = 2$, then take

$$S = (px_j + Dqy_j)^2 \quad \text{and} \quad P = D(py_j + qx_j)^2,$$

where j is any positive solution to $(cpy_0)j \equiv -qx_0 \pmod{D}$. If $x_0^2 - Dy_0^2 = +1$, however, such as in the present case and in McDaniel's treatment of the case $n = 4k + 1$, sometimes a smaller solution may be found by taking $c = 1$ in the above discussion. This gives a smaller solution when the least positive solution to $(py_0)j \equiv -qx_0 \pmod{D}$ is less than twice the least positive solution to $(2py_0)j \equiv -qx_0 \pmod{D}$, and, in any case (when $x_0^2 - Dy_0^2 = 1$), more solutions are obtained this way. If $n = 14$, for example, we generate solutions

$$S = (5x + 11y)^2 \quad \text{and} \quad P = 11(5y + x)^2,$$

where x and y are defined so that

$$x + y\sqrt{11} = (10 + 3\sqrt{11})^{3+11t} \text{ or } (10 + 3\sqrt{11})^{2(7+11t)},\ t \geqslant 0,$$

depending on whether we take $c = 1$ or 2.

It has been proved by McDaniel [3] and Mollin and Walsh [4, 5] that every nonzero integer can be written in infinitely many ways as the difference of two relatively prime powerful numbers, *neither* of which is a square.

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A NOTE ON MOESSNER'S PROCESS

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1. INTRODUCTION

According to Moessner's Theorem [3], [6], the k^{th} powers of the positive integers can be generated in the following interesting way. Delete every k^{th} integer from the sequence of positive integers, form a new sequence by taking partial sums of the original altered sequence, delete every $(k-1)^{\text{st}}$ entry from the sequence of partial sums, and so on. After $k-1$ steps, this process terminates with the deletion of the sequence of k^{th} powers. For example, for $k = 3$, we have

$$\begin{array}{cccccccccccc}
1 & 2 & \cancel{3} & 4 & 5 & \cancel{6} & 7 & 8 & \cancel{9} & 10 & 11 & \cancel{12}\ \ldots \\
1 & \cancel{3} & & 7 & \cancel{12} & & 19 & \cancel{27} & & 37 & \cancel{48}\ \ldots & \\
\cancel{1} & & & \cancel{8} & & & \cancel{27} & & & \cancel{64}\ \ldots & &
\end{array}$$

Note that we can think of the process terminating when we delete the single element at the bottom vertex of each small triangular array. A more general result due to I. Paasche [4] is that, if $\{k_i\}$ is a sequence of nonnegative integers, if the sequence

$$k_1,\ 2k_1 + k_2,\ 3k_1 + 2k_2 + k_3,\ \ldots \tag{1}$$

is deleted from the sequence of positive integers, if the sequence of partial sums is formed, and so on, the process terminates with the sequence

$$1^{k_1},\ 2^{k_1}1^{k_2},\ 3^{k_1}2^{k_2}1^{k_3},\ \ldots\ .$$

For example, if $k_i = 1$ for all i, the numbers deleted are the triangular numbers, and we obtain

$$\begin{array}{ccccccccccc}
\cancel{1} & \cancel{2} & \cancel{3} & 4 & 5 & \cancel{6} & 7 & 8 & 9 & \cancel{10} & 11\ \ldots \\
 & \cancel{2} & & 6 & \cancel{11} & & 18 & 26 & \cancel{35} & & 46\ \ldots \\
 & & & \cancel{6} & & & 24 & \cancel{50} & & & 96\ \ldots \\
 & & & & & & \cancel{24} & & & & 120\ \ldots \\
 & & & & & & & & & & \cancel{120}\ \ldots
\end{array}$$

and

$$1 = 1^1,\ 2 = 2^1 \cdot 1^1,\ 6 = 3^1 \cdot 2^1 \cdot 1^1,\ 24 = 4^1 \cdot 3^1 \cdot 2^1 \cdot 1^1$$

and

$$120 = 5^1 \cdot 4^1 \cdot 3^1 \cdot 2^1 \cdot 1^1.$$

Of course, this is more neatly written as

$$1 = 1!,\ 2 = 2!,\ 6 = 3!,\ 24 = 4!,\ 120 = 5!$$

If we start the Moessner process by deleting the triangular numbers

$$\frac{n(n+1)}{2} = \binom{n+1}{2}$$

we generate the factorials—a truly remarkable result!

It is natural to ask what happens if we commence the process by deleting the terms of other well-known sequences—say the Fibonacci or Lucas numbers, the square numbers, the binomial coefficients

$$\binom{n+k-1}{k}$$

for fixed k, the terms of a geometric progression $\{ar^{n-1}\}$ for positive integers a and $r > 1$, and other sequences the reader might think of. We might also ask what happens if the k's in (1) above are in some well-known sequence. Both of these questions are addressed in what follows. The interested reader will also want to consult [1], [2], [5], and [7].

2. AN INVERSION THEOREM

Let $f(n)$ be any increasing positive integer valued function whose successive values, $f(1)$, $f(2)$, $f(3)$, ..., we want to delete from the sequence of positive integers to initiate Moessner's process. To determine the products generated, it is necessary to determine the nonnegative integers k_i, $i \geqslant 1$, such that

$$f(1) = k_1, \quad f(2) = 2k_1 + k_2, \quad f(3) = 3k_1 + 2k_2 + k_3, \ \ldots,$$

i.e., such that

$$f(n) = \sum_{i=1}^{n} (n+1-i)k_i, \ n \geqslant 1. \tag{2}$$

Of course, the condition that the k's be nonnegative has implications for the growth rate of $f(n)$. Thus,

$$k_1 \leqslant k_1 + k_2 \leqslant k_1 + k_2 + k_3 \leqslant \cdots, \tag{3}$$

and so

$$f(1) \leqslant f(2) - f(1) \leqslant f(3) - f(2) \leqslant \cdots. \tag{4}$$

This will force some adjustments later on, but does not affect the following inversion theorem.

Theorem 1: Formulas (2) hold with

$$k_1 = f(1), \ k_2 = f(2) - 2f(1)$$

and

$$k_i = f(i) - 2f(i-1) + f(i-2), \text{ for } i \geqslant 3.$$

That is to say,

$$f(n) = nf(1) + (n-1)[f(2) - 2f(1)] + \sum_{i=3}^{n} (n+1-i)[f(i) - 2f(i-1) + f(i-2)]. \tag{5}$$

Proof: Clearly, $k_1 = f(1)$ and $k_2 = f(2) - 2k_1 = f(2) - 2f(1)$. Moreover, for $j \geqslant 2$,

$$f(j) - f(j-1) = \sum_{i=1}^{j} (j+1-i) - \sum_{i=1}^{j-1} (j-i)k_i = \sum_{i=1}^{j} k_i,$$

and hence, for $j \geqslant 3$,

$$\begin{aligned} k_j = \sum_{i=1}^{j} k_i - \sum_{i=1}^{j-1} k_i &= f(j) - f(j-1) - [f(j-1) - f(j-2)] \\ &= f(j) - 2f(j-1) + f(j-2) \end{aligned}$$

as claimed.

We now apply Theorem 1 to some interesting sequences, making sure at the same time that (2) and (4) are satisfied.

3. THE FIBONACCI SEQUENCE

If F_n denotes the n^{th} Fibonacci number, then

$$F_{i+1} - F_i = F_{i-1},$$

so the sequence of differences is nondecreasing for $i \geqslant 1$. Since $F_3 \geqslant 2F_2$, we may set $f(n) = F_{n+1}$ and the Moessner process will apply. Also, from Theorem 1, we have

$$k_1 = F_2 = 1,\ k_2 = F_3 - 2F_2 = 0,$$

and

$$k_i = F_{i+1} - 2F_i + F_{i-1} = F_{i-3}$$

for $i \geqslant 3$. Thus, from (5), we have

$$F_{n+1} = n + \sum_{i=3}^{n} (n+1-i)F_{i-3} \tag{6}$$

and if we delete the numbers 1, 2, 3, 5, 8, 13, ... from the sequence of positive integers, the Moessner process generates products with the exponents

$$1,\ 0,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots .$$

That is, the products generated are

$$1^1,\ 2^1 1^0,\ 3^1 2^0 1^0,\ 4^1 3^0 2^0 1^1,\ 5^1 4^0 3^0 2^1 1^1,\ 6^1 5^0 4^0 3^1 2^1 1^2,\ \ldots .$$

4. THE LUCAS SEQUENCE

There is a little difficulty with the Lucas sequence $\{L_n\}$ because of (4). Thus,

$$L_{i+1} - L_i = L_{i-1}$$

and the sequence of differences only increases for $i \geqslant 2$. Also, if we attempt to set $f(i) = L_{i+1}$ as for the Fibonacci sequence, then

$$f(2) = L_3 = 4 \ngeqslant 6 = 2L_2 = 2f(1).$$

This difficulty, however, can be overcome by a slight artifice. Consider the function $f(n) = n$ for $1 \leqslant n \leqslant 2$, and $f(n) = L_{n-1}$ for $n \geqslant 3$. Here the differences are nondecreasing for $i \geqslant 1$ and $f(2) \geqslant 2f(1)$. For this sequence, we have

$$k_1 = f(1) = 1,\ k_2 = f(2) - 2f(1) = 2 - 2 \cdot 1 = 0,$$
$$k_3 = f(3) - 2f(2) + f(1) = 3 - 2 \cdot 2 + 1 = 0,$$
$$k_4 = f(4) - 2f(3) + f(2) = 4 - 2 \cdot 3 + 2 = 0,$$

and for $i \geqslant 5$,

$$k_i = f(i) - 2f(i-1) + f(i-2) = L_{i-1} - 2L_{i-2} + L_{i-3} = L_{i-5}.$$

Thus, from (5), for $n \geqslant 4$, we have

$$\begin{aligned} L_{n-1} &= nk_1 + (n-1)k_2 + (n-2)k_3 + (n-3)k_4 + \sum_{i=5}^{n}(n+1-i)k_i \\ &= n + \sum_{i=5}^{n}(n+1-i)L_{i-5}, \end{aligned} \tag{7}$$

and, if we begin the Moessner process by deleting 1, 2, 3, 4, 7, 11, 18, ..., the exponents in the generated products are 1, 0, 0, 0, 2, 1, 3, 4, 7, ...

5. THE GENERAL SECOND-ORDER RECURRENCE

Consider the general second-order recurrence defined by $g_1 = c$, $g_2 = d$, and $g_{n+2} = ag_{n+1} + bg_n$ for $n \geqslant 1$, where a, b, c, and d are positive integers with $d \geqslant 2c$. The first few terms of $\{g_n\}$ are

$$g_1 = c,\ g_2 = d,\ g_3 = ad + bc,\ g_4 = a^2d + abc + bd,\ \ldots\ .$$

Now define the sequence $\{k_i\}$ by

$$\begin{aligned} k_1 &= g_1 = c, \\ k_2 &= g_2 - 2g_1 = d - 2c, \\ k_i &= g_i - 2g_{i-1} + g_{i-2},\ \text{for } i \geqslant 3, \end{aligned} \tag{8}$$

so that the k_i satisfy Theorem 1 for all $i \geqslant 1$. Then, deleting the sequence $\{g_i\}$ from the sequence of positive integers in the Moessner process generates products whose exponents are successive terms of the sequence $\{k_i\}$. In addition to the above, note that

$$k_3 = ad + bc - 2d + c,\quad k_4 = a^2d + abc + bd - 2ad - 2bc + d,$$

and that

$$k_n = ak_{n-1} + bk_{n-2},\ \text{for } n \geqslant 5.$$

We may also ask what sequence $\{f(i)\}$ should be deleted from the sequence of integers to start a Moessner process that generates products where the expontnes are the sequence $\{g_i\}$. We must determine $f(n)$ such that

$$f(n) = \sum_{i=1}^{n}(n+1-i)g_i. \tag{9}$$

It turns out that the desired function $f(n)$ may be defined by the following second-order, nonlinear recurrence.

$$f(1) = c,\ f(2) = 2c + d,$$
$$f(n + 1) = af(n) + bf(n - 1) - nac + (n + 1)c + nd,\ n \geqslant 2.$$

To see this, we note that

$$f(1) = c = g_1 \quad \text{and} \quad f(2) = 2c + d = 2g_1 + g_2.$$

Now assume that (9) holds for $n = k - 1$ and $n = k$ for some fixed $k \geqslant 2$. Then,

$$\begin{aligned} f(k + 1) &= af(k) + bf(k - 1) - kac + (k + 1)c + kd \\ &= a\sum_{i=1}^{k}(k + 1 - i)g_i + b\sum_{i=1}^{k-1}(k - i)g_i - kac + (k + 1)c + kd \\ &= akg_1 + a\sum_{i=2}^{k}(k + 1 - i)g_i + b\sum_{i=1}^{k-1}(k - i)g_i - kac + (k + 1)c + kd \\ &= akc + a\sum_{j=2}^{k}(k + 1 - j)g_j + b\sum_{j=2}^{k}(k + 1 - j)g_{j-1} \\ &\qquad - kac + (k + 1)c + kd \\ &= \sum_{j=2}^{k}(k + 1 - j)(ag_j + bg_{j-1}) + (k + 1)c + kd \\ &= \sum_{i=1}^{k+1}(k + 2 - i)g_i. \end{aligned}$$

This completes the induction.

Incidentally, it now follows from Theorem 1 that

$$g_i = f(i) - 2f(i - 1) + f(i - 2),\ i \geqslant 3. \tag{10}$$

6. SOME OTHER INTERESTING SEQUENCES

If we start the Moessner process by deleting terms in the arithmetic progression $\{a + (n - 1)d\}$, where $d \geqslant a$ in view of (4), it follows from Theorem 1 that the exponents in the generated products are

$$k_1 = a,$$
$$k_2 = (a + d) - 2a = d - a,$$

and $$k_i = [a + (i - 1)d] - 2[a + (i - 2)d] + [a + (i - 3)d] - 0,$$

for $i \geqslant 3$. Thus, the generated products are simply

$$1^a,\ 2^a1^{d-a},\ 3^a2^{d-a},\ 4^a3^{d-a},\ \ldots\ .$$

If, instead of starting Moessner's process by deleting the terms of an arithemtic progression, we desire that the k's (i.e., the exponents in the resulting products) be in arithmetic progression, we must delete the successive terms of the sequence $\{f(n)\}$, where

$$f(n) = \sum_{i=1}^{n} (n + 1 - i)[a + (i - 1)d] = \binom{n+1}{2}a + \binom{n+1}{3}d. \tag{11}$$

In Section 1 we saw that interesting results were obtained if we began the Moessner process by deleting the triangular numbers

$$\frac{n(n+1)}{2} = \binom{n+1}{2}, \quad n \geqslant 1.$$

This naturally raises the question of deleting the binomial coefficients

$$\binom{n+k+1}{k} \text{ for any fixed integer } k \geqslant 2.$$

Of course, the result follows from Theorem 1 with

$$f(n) = \binom{n+k-1}{k}.$$

We have

$$k_1 = f(1) = \binom{1+k-1}{k} = \binom{k}{k} = \binom{k-2}{k-2},$$

$$k_2 = f(2) - 2f(1) = \binom{2+k-1}{k} - 2\binom{1+k-1}{k} = k - 1 = \binom{k-1}{k-2},$$

and, for $i \geqslant 3$,

$$k_i = \binom{i+k-1}{k} - 2\binom{i-1+k-1}{k} + \binom{i-2+k-1}{k} = \binom{i+k-3}{k-2}.$$

That is,

$$\binom{n+k-1}{k} = \sum_{i=1}^{n} (n + 2 - i)\binom{i+k-3}{k-2}. \tag{12}$$

Thus, if we delete the sequence $\left\{\binom{n+2}{3}\right\}_{n \geqslant 1}$, we generate the products

$$1^{\binom{1}{1}},\ 2^{\binom{1}{1}}1^{\binom{2}{1}},\ 3^{\binom{1}{1}}2^{\binom{2}{1}}1^{\binom{3}{1}},\ \ldots\ .$$

If we delete the sequence $\left\{\binom{n+3}{4}\right\}_{n \geqslant 1}$, we generate the products

$$1^{\binom{2}{2}},\ 2^{\binom{2}{2}}1^{\binom{3}{2}},\ 3^{\binom{2}{2}}2^{\binom{3}{2}}1^{\binom{4}{2}},\ \ldots,$$

and so on.

Finally, we consider the case when $\{f(n)\}$, the sequence of deleted numbers, is the geometric progression $\{ar^{n-1}\}$ with a and r positive integers and $r \geqslant 2$; and also the case where $k_i = ar^{i-1}$.

If $f(n) = ar^{n-1}$, we are starting Moessner's process deleting the terms of a geometric progression and we have from Theorem 1 that the exponents in the generated products are

$$k_1 = f(1) - a,$$

$$k_2 = f(2) - 2f(1) = ar - 2a = a(r - 2),$$

and, for $i \geqslant 3$,

$$\begin{aligned} k_i &= f(i) - 2f(i-1) + f(i-2) \\ &= ar^{i-1} - 2ar^{i-2} + ar^{i-3} \\ &= ar^{i-3}(r^2 - 2r + 1) \\ &= ar^{i-3}(r-1)^2, \end{aligned}$$

again a geometric progression with common ratio r after the first two terms. In any case, we also have that

$$ar^{n-1} = na + (n-1)a(r-2) + a(r-1)^2 \sum_{i=3}^{n} r^{i-3}. \tag{13}$$

If, on the other hand, $k_i = ar^{i-1}$ for $i \geqslant 1$, we must begin Moessner's process by deleting the successive terms of the sequence $\{f(n)\}$, where

$$\begin{aligned} f(n) &= a\sum_{i=1}^{n}(n+i-1)r^{i-1} = a\sum_{i=0}^{n-1}(n-i)r^i \\ &= a\sum_{i=0}^{n-1}\frac{(n-i)r^i r^{n-i-1}}{r^{n-i-1}} = ar^{n-1}\sum_{i=0}^{n-1}(n-i)x^{n-i-1}, \end{aligned}$$

where $x = r^{-1}$. Thus,

$$\begin{aligned} f(n) &= r^{n-1} \cdot \frac{d}{dx}\sum_{i=0}^{n-1}x^{n-i}\bigg|_{x=r^{-1}} = ar^{n-1} \cdot \frac{d}{dx}\frac{x - x^{n+1}}{1-x}\bigg|_{x=r^{-1}} \\ &= \frac{a(r^{n+1} - r + n - rn)}{(r-1)^2}, \end{aligned} \tag{14}$$

and the Moessner process yields the products

$$1^a,\ 2^a \cdot 1^{ar},\ 3^a \cdot 2^{ar} \cdot 1^{ar^2},\ \ldots .$$

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FIBONACCI SEQUENCES OF PERIOD n IN GROUPS

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An unusual application of Fibonacci sequences occurs in a musical composition by Iannis Xenakis. In *Nomos Alpha* the composer uses Fibonacci sequences of group elements to produce "Fibonacci motions," sequences of musical properties such as pitch, volume, and timbre that give the composition its framework (see [1], [4]). This setting suggests some interesting mathematical questions:

1. Given elements a and b in a finite abelian group , what is the period of the Fibonacci sequence $a, b, ab, ab^2, a^2b^3, \ldots$ in G?
2. Given an integer $n > 2$, is there a Fibonacci sequence of period n in a group G, and can such a sequence be readily obtained?

A helpful starting point is the paper entitled "Fibonacci Series Modulo m" by D. D. Wall [3]. With Wall, we let f_n denote the n^{th} member of the sequence of integers $f_0 = a$, $f_1 = b$, ..., where $f_{n+1} = f_n + f_{n-1}$. The symbol $h(m)$ will denote the length of the period of the sequence resulting from reducing each f_n modulo m. The basic Fibonacci sequence will be given by $u_0 = 0$, $u_1 = 1$, ... and the Lucas sequence by $v_0 = 2$, $v_1 = 1$, ... The symbol $k(m)$ will denote the lenght of the period of the basic Fibonacci sequence 0, 1, 2, 3, ... when it is reduced modulo m. Since we will often work in a group setting, we will let $\mathbb{Z}$ and $\mathbb{Z}_m$ represent the group of integers and the group of integers modulo m, respectively.

We summarize some of Wall's results in the following, using a group setting for convenience.

Theorem (Wall): In $\mathbb{Z}_m$, the following hold:

(1) Any Fibonacci sequence is periodic.

(2) If m has prime factorization $\prod p_i^{e_i}$ and if h_i denotes the period of the Fibonacci sequence $f_n \pmod{p_i^{e_i}}$, then $h(m) = \text{lcm}\{h_i\}$.

(3) The terms for which $u_n \equiv 0 \pmod{m}$ have subscripts which form a simple arithmetic progression.

(4) If p is prime and $p = 10x \pm 1$, then $k(p)$ divides $p - 1$.

(5) If p is prime and $p = 10x \pm 3$, then $k(p)$ divides $2p + 2$.

(6) If $k(p^2) \neq k(p)$, then $k(p^e) = p^{e-1}k(p)$ for $e > 1$.

The results in (4) and (5) give upper bounds for $k(p)$, but, as Wall points out, there are many primes for which $k(p)$ is less than the given upper bound. Unfortunately, one must obtain the sequence itself in order to determine $k(p)$. The following theorem provides a method for determining $k(m)$ from the prime factorization of certain u_i and v_i. We note first that in $\mathbb{Z}_2$ the sequence 0, 1, 1, ... has period 3 and in any group G, an element of order 2 yields a sequence 0, a, a, 0, ... of period 3.

Theorem 1: If $m > 2$, the sequence 0, 1, 1, 2, ..., u_n, ... has period $2n$ in $\mathbb{Z}_m$ for $n = \text{minimum}\{n \text{ even and } m|u_n;\ n \text{ odd and } m|v_n\}$.

Proof: Consider the sequence 0, 1, 1, 2, ..., u_n, ... in $\mathbb{Z}_m$. By Wall's Theorem, it is periodic, so we must have

$$0, 1, 1, 2, 3, \ldots, u_n, \ldots, -3, 2, -1, 1, 0, 1, \ldots$$

and the "middle" of the period must have one of the four forms:

(i) $\ldots, u_{n-2}, u_{n-1}, u_{n-1}, -u_{n-2}, \ldots;$

(ii) $\ldots, u_{n-2}, u_{n-1}, -u_{n-1}, u_{n-2}, \ldots;$

(iii) $\ldots, u_{n-2}, u_{n-1}, 0, u_{n-1}, -u_{n-2}, \ldots;$

(iv) $\ldots, u_{n-2}, u_{n-1}, u_n, -u_{n-1}, u_{n-2}, \ldots$.

If (i) occurs, then $u_{n-2} \equiv 0$ and $2u_{n-1} \equiv 0$. Thus, u_{n-1} equals 0 or has order 2 in $\mathbb{Z}_m$, and 0, 0, 0, ... or 0, u_{n-1}, u_{n-1}, 0, ... are the resulting sequences. These cannot occur, since 1 has order m in $\mathbb{Z}_m$.

If (ii) occurs, it is easy to obtain a similar result.

If (iii) occurs, $n - 1$ must be odd (so n is even) and $u_n \equiv 0 \pmod{m}$ so that $m|u_n$. These two conditions are sufficient to imply repetition after $2n$ terms, since we must then have 1, 1, 2, 3, ..., $2u_{n-1}$, $-u_{n-1}$, u_{n-1}, 0, u_{n-1}, u_{n-1}, $2u_{n-1}$, ..., $u_{n-1}u_{n-1} \equiv 1$, 0, ... by symmetry of the terms of odd index.

In (iv), $n - 1$ must be even (so n is odd) and $u_{n-1} + u_{n+1} \equiv 0 \pmod{m}$ so that $v_n \equiv 0 \pmod{m}$ and $m|v_n$. As in (iii), these two conditions imply repetition after $2n$ terms, for they require

$$\begin{aligned} &1, 1, 2, \ldots, u_{n-1}, u_n, -u_{n-1}, u_n - u_{n-1} \\ &= u_{n-2}, -u_{n-3}, \ldots, -u_2, u_{n-(n-1)} \\ &= u_1 \equiv 1, 0, \ldots . \end{aligned}$$

Thus, to find the period of the sequence 1, 1, 2, 3, ... modulo m, we need only locate the smallest n such that $m|u_n$ for even n or $m|v_n$ for odd n. The period of the sequence will equal $2n$.

Since the period is always $2n$, we easily obtain a result of Wall.

Corollary 1: For $m > 2$, the sequence 1, 1, 2, 3, ... modulo m has even period.

Example: In $\mathbb{Z}_{13}$, the sequence 1, 1, 2, 3, ..., u_n, ... has period 28, since $u_{14} = 377$ is the first eligible u_n or v_n divisible by 13. The index 14 is doubled to obtain the period.

For larger m, our search is narrowed by (2), (4), (5), and (6) of Wall's Theorem. Note that (4) becomes $n|(p - 1)/2$ for $p = 10x \pm 1$ and (5) becomes $n|p+1$ for $n = 10x \pm 3$, since our n represents half the period of the sequence.

Example: In $\mathbb{Z}_{47}$, (5) requires that $n|48$, and Theorem 1 yields $n = 16$, since $u_{16} = 987$ is the first eligible u_n or v_n divisible by 47. The period of 1, 1, 2, ..., u , ... in $\mathbb{Z}_{47}$ is therefore 32.

We remind the reader of three known results (see [2]) which are helpful in the search for a minimal n.

(i) $v_n | v_m$ if and only if $m = (2k - 1)n$ for $n > 1$.

(ii) $v_n | u_m$ if and only if $m = 2kn$ for $n > 1$.

(iii) $u_n | u_m$ if and only if $n | m$.

The following related result completes the picture.

(iv) For $n > 1$, u_{2n} does not divide v_k for k odd.

Proof: If $n = 2$, then $u_4 = 3 = v_2$. Thus, by (i), only those v_x with x even are divisible by u_4.

If $n = 3$, then $u_6 = 8$, and it can be shown that no v_k is divisible by 8. (Use the fact that any number with at least 3 digits is divisible by 8 if and only if the number consisting of its last 3 digits is divisible by 8. Then observe that the set of odd multiples of $v_3 = 4$ yields only a finite set of final 3 digits, none of which is divisible by 8.)

For $n > 3$, assume there exists an odd k such that $u_{2n} | v_k$. Then $u_{2n} | u_{2k}$ by (ii), so $2n | 2k$ and $n | k$ so that $u_n | u_k$ by (iii). Since $u_{2n} | v_k$, it follows that $u_n | v_k$. Hence, u_n is a common divisor of both u_n and v_k and thus u_n must equal 1 or 2. This is impossible for $n > 3$.

These four facts and Wall's Theorem make it quite simple to determine the period of Fibonacci sequences of the form 0, 1, 2, 3, ..., u_n, ... modulo m.

In an arbitrary group G, if we use multiplicative notation, we may apply Theorem 1 to the exponents to obtain

Corollary 2: Let G be any group and a an element of order $m > 2$ in G. Then the sequence $a, a, a^2, a^3, \ldots, a^{u_n}, \ldots$ will have period $2n$ for

$$n = \text{minimum}\{n \text{ even and } m | u_n;\ n \text{ odd and } m | v_n\}.$$

Example: If a is an element of order 4 in a group, then the sequence $a, a, a^2, a^5, \ldots, a^{u_n}, \ldots$ has period 6, since 4 divides $v_3 = 4$ and no previous u_n for n or v_n for n odd.

It is evident from Theorem 1 and Corollary 2 that the process of finding n may be reversed. If we are given $n > 2$, we can construct a Fibonacci sequence of period $2n$. If n is even, we can use any element a of order u_n, and if n is odd, an element of order v_n will suffice. We can often do better, since we need only a factor x of u_n or v_n which is not a factor of any previous u_n of even index or v_n of odd index (i.e., n will be the index of the first qualifying term divisible by x). We state this formally.

Corollary 3: A sequence of the form $a, a, a^2, \ldots, a^{u_n}, \ldots$ in a group G will have period $2n > 5$ if a is chosen to have order u_n for n even or v_n for n odd. Furthermore, a may be chosen to have order x where x divides this u_n or v_n but is not a factor of any previous qualifying u_n or v_n.

Example: To find a sequence of period $16 = 2n$, use $u_8 = 21$. Any element of order 21 in a group G will yield a sequence of the form $a, a, a^2, a^3, \ldots, a^{u_n}$, ... which has period 16. Since 7 is a factor of 21 which divides no previous u_n of even index or v_n of odd index, any element of order 7 will also suffice.

We may use the previous results to present a simple method for obtaining primes p for which $k(p)$ is a proper divisor of $p - 1$ for $p = 10x \pm 1$ or of $2p + 2$ for $p = 10x \pm 3$. As mentioned earlier, our minimal n equals $[k(p)/2]$, so we seek primes p such that n does not equal $(p - 1)/2$ or $p + 1$.

First of all, if we are given a prime $p > 5$, set $n = (p - 1)/2$ or $n = p + 1$, depending on whether $p = 10x \pm 1$ or $p = 10x \pm 3$. Then, using previous results, see whether u_n for n even or v_n for n odd is the smallest such u_n or v_n divisible by p. For example, if $p = 31$, set $n = 15$. Since v_{15} is divisible by 31 and no smaller qualifying u_n or v_n is divisible by 31, $n = (p - 1)/2$ works and $k(31) = 30$. However, if we begin with $p = 47$, set $n = 48$. Since 47 divides $u_{16} < u_{48}$, it follows that $k(47) = 32 \neq 96$.

Another approach begins with N rather than p. Given N, find the prime factors $p_1, \ldots, p_k$ of u_N for N even or v_N for N odd. Proceed as above to set $(p_i - 1)/2$ or $p_i + 1$ equal to n_i for each p_i. If $n_i > N$, then $k(p_i) <$ the given upper bound $p_i - 1$ or $2p_i + 2$. If $n_i = N$, check whether p_i divides a previous u_k of even index or v_k of odd index. If so, then $k(p_i) <$ the given upper bound. If not, $k(p_i) =$ the correct upper bound. (If $n_i < N$, disregard the associated p_i.)

Example: For $N = 44$, the prime factors of u_{44} are 3, 43, 307, 89, and 199. We disregard 3 since $n = 4 < 44$. For $p = 43$, $n = 44$ and, in fact, $k(43) = 88$. For $p = 307$, $n = 308 > 44$, so $k(307) \leqslant 88 \neq 616$. For $p = 89$, $n = 44$ and, in fact, $k(89) = 88$. Finally, for $p = 199$, $n = 99 > 44$, so $k(199) \leqslant 88 \neq 198$.

Two more results follow easily from Theorem 1.

Corollary 4: Any element whose order is a multiple of 5 will yield a sequence $a, a, a^2, \ldots, a^{u_n}, \ldots$ whose period is a multiple of 4.

Proof: No Lucas number is divisible by 5, so n must be even and $2n$ is therefore divisible by 4.

Corollary 5: Any sequence of the form $a, b, ab, ab^2, \ldots, a^{u_{n-1}}b^{u_n}, \ldots$ in an Abelian group G will have odd period > 3 only if it does not contain the identity element.

Proof: By Corollary 2, any sequence of the form $e, a, a, a^2, \ldots, a^{u_n}, \ldots$ for a of order > 2 has even period.

Corollary 3 allows us to construct Fibonacci sequences of period $2n$ for $n > 2$. Corollary 5 requires us to examine sequences not containing the identity element if we wish to obtain sequences of odd period. We first observe that, if the sequence $a, a, a^2, \ldots, a^{u_i}, \ldots$ has period x and the sequence $b, b, b^2, \ldots, b^{u_i}, \ldots$ has period y in an Abelian group G, then the sequence $a, b, ab, ab^2, \ldots, a^{u_{i-1}}b^{u_i}, \ldots$ will repeat after $\mathrm{lcm}\{x, y\}$ terms. Hence, the period of this sequence will be a divisor of $\mathrm{lcm}\{x, y\}$. (Wall [3] gives some sufficient conditions for $h(m)$ to equal $k(m)$ in $\mathbb{Z}_m$.)

Example: In $\mathbb{Z}_5$, both $a = 1$ and $b = 3$ have order 5, and the sequences

$$0, 1, 1, 2, \ldots \quad \text{and} \quad 0, 3, 3, 6, \ldots$$

each have period 20 (since u_{10} is the first qualifying u_n or v_n divisible by 5). However, the sequence 1, 3, 4, 2, 1, ... has period 4.

Our goal is to construct Fibonacci sequences of odd period and the following theorem provides the means to accomplish this.

Theorem 2: Given any integer $n > 2$, there exists a Fibonacci sequence of period n.

Proof: Consider the sequence of integers

$$u_n,\ 1 - u_{n-1},\ 1 + u_{n-2},\ \ldots,\ u_{k-1} + (-1)^{k-1}u_{n-(k-1)},\ \ldots,$$
$$u_n,\ u_{n+1} + (-1)^{n+1},\ \ldots .$$

This is a Fibonacci sequence of period n provided that

$$1 - u_{n-1} \equiv u_{n+1} + (-1)^{n+1} \quad \text{or} \quad v_n = \begin{cases} 0 \pmod{m} \text{ for } n \text{ odd,} \\ 2 \pmod{m} \text{ for } n \text{ even.} \end{cases}$$

Thus, if n is odd, use the given sequence in $\mathbb{Z}_m$ with $m = v_n$ and, if n is even, use the given sequence in $\mathbb{Z}_m$ with $m = v_n - 2$.

Although Theorem 2 establishes the existence of Fibonacci sequences of period n, in practice the calculations often involve large m. To simplify this, observe that we need only a *divisor* of v_n or $v_n - 2$ which has not appeared as a factor of a previous v_k for k odd or $v_k - 2$ for k even.

Example: Given $n = 7$, the resulting sequence is

$$13,\ -7,\ 6,\ -1,\ 5,\ 4,\ 9,\ 13,\ 22,\ \ldots,$$

where $22 \equiv -7 \pmod{m}$, so $m = 29 = v_7$. Other sequences of period 7 may be obtained by multiplication of this sequence by any nonzero element in $\mathbb{Z}_{29}$.

Example: If $n = 9$, the resulting sequence is

$$34,\ -20,\ 14,\ -6,\ 8,\ 2,\ 10,\ 12,\ 22,\ 34,\ 56,\ \ldots,$$

and $m = v_9 = 76 = 2^2 \cdot 19$. Here, we may use the smaller $m = 19$ to obtain the sequence 15, 18, 14, 13, 8, 2, 10, 12, 3, 15, ... in $\mathbb{Z}_{19}$. (Note that if the original sequence is reduced modulo 4, we obtain 2, 0, 2, 2, 0, ... which has period 3 instead of period 9. The problem here is that 4 has appeared in previous v_k for k odd and $v_k - 2$ for k even.) As in the previous example, multiplication of the sequence of period 9 by any number relatively prime to m will yield a sequence of period 9.

Applying Theorem 2 to exponents, we obtain

Corollary 6: Given $n > 2$, an element a of order v_n for n odd or v_{n-2} for n even in an Abelian group G will yield a sequence

$$a^{u_n},\ a^{1-u_{n-1}},\ \ldots,\ a^{u_{(k-1)}+(-1)^{k-1}u_{n-(k-1)}},\ \ldots$$

of period n.

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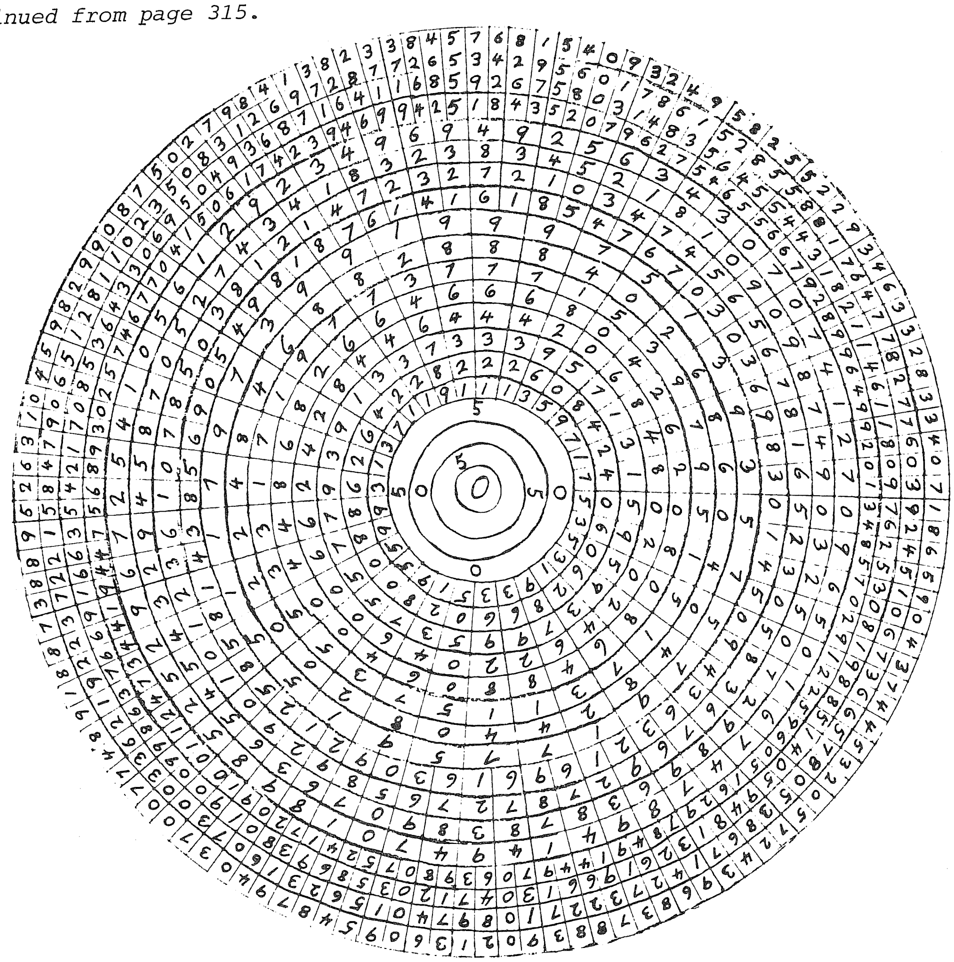

THE SET OF SERIES BASED ON THREE-DIGIT NUMERALS

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ON A SECOND NEW GENERALIZATION OF THE FIBONACCI SEQUENCE

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A new perspective to the generalization of the Fibonacci sequence was introduced in [1]. Here, we take another step in the same direction. In [1] we studied the sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ defined by

$$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \beta_{n+1} + \beta_n, & (n \geqslant 0) \\ \beta_{n+2} = \alpha_{n+1} + \alpha_n, \end{cases} \tag{1}$$

where a, b, c, and d are fixed real numbers. We also utilized the generalization $\{F_i(a, b)\}_{i=0}^{\infty}$, where

$$\begin{cases} F_0(a, b) = a \\ F_1(a, b) = b & (n \geqslant 0) \\ F_{n+2}(a, b) = F_{n+1}(a, b) + F_n(a, b) \end{cases}$$

so that $F_n = F_n(0, 1)$, where $\{F_i\}_{i=0}^{\infty}$ is the Fibonacci sequence.

We shall study here the properties of the sequences for the scheme,

$$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \alpha_{n+1} + \beta_n, & (n \geqslant 0) \\ \beta_{n+2} = \beta_{n+1} + \alpha_n, \end{cases} \tag{2}$$

where a, b, c, and d are fixed real numbers, and will conclude with a theorem, similar to [1]. Since the proofs of the results in this paper are similar to those in [1], we shall only list the results and eliminate the proofs.

Obviously when $a = b$ and $c = d$, the schemes from (2), as well as from (1), coincide with the Fibonacci sequence $\{F_i(a, b)\}_{i=0}^{\infty}$. The first few terms of the sequences from (2) are:

n	α_n	β_n
0	a	b
1	c	d
2	$b + c$	$a + d$
3	$b + c + d$	$a + c + d$
4	$a + b + c + 2d$	$a + b + 2c + d$
5	$2a + b + 2c + 3d$	$a + 2b + 3c + 2d$
6	$3a + 2b + 4c + 4d$	$2a + 3b + 4c + 4d$
7	$4a + 4b + 7c + 6d$	$4a + 4b + 6c + 7d$
8	$6a + 7b + 11c + 10d$	$7a + 6b + 10c + 11d$
9	$10a + 11b + 17c + 17d$	$11a + 10b + 17c + 17d$

Lemma 1: For every $k \geqslant 0$:

(a) $\alpha_{6k} + \beta_0 = \beta_{6k} + \alpha_0$;

(b) $\alpha_{6k+1} + \beta_1 = \beta_{6k+1} + \alpha_1$;

(c) $\alpha_{6k+2} + \alpha_0 + \beta_1 = \beta_{6k+2} + \beta_0 + \alpha_1$;

(d) $\alpha_{6k+3} + \alpha_0 = \beta_{6k+3} + \beta_0$;

(e) $\alpha_{6k+4} + \alpha_1 = \beta_{6k+4} + \beta_1$;

(f) $\alpha_{6k+5} + \beta_0 + \alpha_1 = \beta_{6k+5} + \alpha_0 + \beta_1$.

Lemma 2: For every $n \geqslant 0$:

(a) $\alpha_{n+2} = \sum_{i=0}^{n} \beta_i + \alpha_1$;　　(b) $\beta_{n+2} = \sum_{i=0}^{n} \alpha_i + \beta_1$.

Lemma 3: For every $n \geqslant 0$:

(a) $\sum_{i=0}^{6k} (\alpha_i - \beta_i) = \alpha_0 - \beta_0$;

(b) $\sum_{i=0}^{6k+1} (\alpha_i - \beta_i) = \alpha_0 - \beta_0 + \alpha_1 - \beta_1$;

(c) $\sum_{i=0}^{6k+2} (\alpha_i - \beta_i) = 2\alpha_1 - 2\beta_1$;

(d) $\sum_{i=0}^{6k+3} (\alpha_i - \beta_i) = -\alpha_0 + \beta_0 + 2\alpha_1 - 2\beta_1$;

(e) $\sum_{i=0}^{6k+4} (\alpha_i - \beta_i) = -\alpha_0 + \beta_0 + \alpha_1 - \beta_1$;

(f) $\sum_{i=0}^{6k+5} (\alpha_i - \beta_i) = 0$.

Lemma 4: For every $n \geqslant 0$:

$$\alpha_{n+2} + \beta_{n+2} = F_{n+1} \cdot (\alpha_0 + \beta_0) + F_{n+2} \cdot (\alpha_1 + \beta_1).$$

As in [1], we express the members of the sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ when $n \geqslant 0$, as follows:

$$\begin{cases} \alpha_n = \gamma_n^1 \cdot a + \gamma_n^2 \cdot b + \gamma_n^3 \cdot c + \gamma_n^4 \cdot d \\ \beta_n = \delta_n^1 \cdot a + \delta_n^2 \cdot b + \delta_n^3 \cdot c + \delta_n^4 \cdot d \end{cases}$$

It is interesting to note that Lemmas 5-7 have results identical to those found in [1] for the sequences $\{\gamma_n^1\}_{n=0}^{\infty}$, $\{\gamma_n^2\}_{n=0}^{\infty}$, etc., even though they are different sequences.

Lemma 5: For every $n \geqslant 0$:

(a) $\gamma_n^1 + \delta_n^1 = F_{n-1}$;　　(c) $\gamma_n^3 + \delta_n^3 = F_n$;

(b) $\gamma_n^2 + \delta_n^2 = F_{n-1}$;　　(d) $\gamma_n^4 + \delta_n^4 = F_n$.

Lemma 6: For every $n \geqslant 0$

(a) $\gamma_n^1 + \gamma_n^2 = \delta_n^1 + \delta_n^2$; (b) $\gamma_n^3 + \gamma_n^4 = \delta_n^3 + \delta_n^4$.

Lemma 7: For every $n \geqslant 0$:

(a) $\delta_n^1 = \gamma_n^2$; (e) $\gamma_n^3 = \gamma_{n+1}^2$;

(b) $\delta_n^2 = \gamma_n^1$; (f) $\gamma_n^4 = \gamma_{n+1}^1$;

(c) $\delta_n^3 = \gamma_n^4$; (g) $\delta_n^3 = \delta_{n+1}^2$;

(d) $\delta_n^4 = \gamma_n^3$; (h) $\delta_n^4 = \delta_{n+1}^1$.

Let ψ be the integer function defined for every $k \geqslant 0$ by:

τ	$\psi(6k + \tau)$
0	1
1	0
2	-1
3	-1
4	0
5	1

Obviously, for every $n \geqslant 0$,

$$\psi(n + 3) = -\psi(n). \tag{3}$$

Using the definition of the function ψ, the following are easily proved by induction.

Lemma 8: For every $n \geqslant 0$:

(a) $\gamma_n^1 = \delta_n^1 + \psi(n)$; (c) $\gamma_n^3 = \delta_n^3 + \psi(n + 4)$;

(b) $\gamma_n^2 = \delta_n^2 + \psi(n + 3)$; (d) $\gamma_n^4 = \delta_n^4 + \psi(n + 1)$

Lemma 9: For every $n \geqslant 0$:

(a) $\gamma_{n+2}^1 = \gamma_{n+1}^1 + \gamma_n^1 + \psi(n + 3)$; (d) $\gamma_{n+2}^3 = \gamma_{n+1}^3 + \gamma_n^3 + \psi(n + 1)$;

(b) $\gamma_{n+2}^2 = \gamma_{n+1}^2 + \gamma_n^2 + \psi(n)$; (e) $\gamma_{n+2}^4 = \gamma_{n+1}^4 + \gamma_n^4 + \psi(n + 4)$;

(c) $\gamma_n^1 = \gamma_n^2 + \psi(n)$; (f) $\gamma_n^3 = \gamma_n^4 + \psi(n + 4)$.

From Lemmas 5, 7, 8, and (3), we obtain the equations:

$$\gamma_n^1 = \delta_n^2 = \frac{1}{2}(F_{n-2} + \psi(n));$$

$$\gamma_n^2 = \delta_n^1 = \frac{1}{2}(F_{n-1} + \psi(n + 3));$$

$$\gamma_n^3 = \delta_n^4 = \frac{1}{2}(F_n + \psi(n + 4));$$

$$\gamma_n^4 = \delta_n^3 = \frac{1}{2}(F_n + \psi(n + 1)).$$

Theorem: For every $n \geqslant 0$:

$$\alpha_n = \frac{1}{2}\{(F_{n-1} + \psi(n))a + (F_{n-1} + \psi(n + 3))b + (F_n + \psi(n + 4))c + (F_n + \psi(n + 1))d\}$$

$$= \frac{1}{2}\{(a + b)F_{n-1} + (c + d)F_n + \psi(n)a + \psi(n + 3)b + \psi(n + 4)c + \psi(n + 1)d\}.$$

$$\beta = \frac{1}{2}\{(F_{n-1} + \psi(n + 3))a + (F_{n-1} + \psi(n))b + (F_n + \psi(n + 1))c + (F_n + \psi(n + 4))d\}$$

$$= \frac{1}{2}\{(a + b)F_{n-1} + (c + d)F_n + \psi(n + 3)a + \psi(n)b + \psi(n + 1)c + \psi(n + 4)d\}.$$

On the basis of what has been done in [1] and in this paper, one could be led to generalize and examine sequences of the following types

$$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = p \cdot \beta_{n+1} + q \cdot \beta_n, & (n \geqslant 0) \\ \beta_{n+2} = t \cdot \alpha_{n+1} + s \cdot \alpha_n, \end{cases}$$

$$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = p \cdot \alpha_{n+1} + q \cdot \beta_n, & (n \geqslant 0) \\ \beta_{n+2} = t \cdot \beta_{n+1} + s \cdot \alpha_n, \end{cases}$$

for the fixed real numbers p, q, t, and s.

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EUCLIDEAN COORDINATES AS GENERALIZED FIBONACCI NUMBER PRODUCTS

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1. INTRODUCTION AND DEFINITIONS

In [2], it was shown how to obtain the coordinates of a point in (real) three-dimensional Euclidean space as triple products of Fibonacci numbers.

This was achieved as a development of two-dimensional ideas involving complex numbers, though the three-dimensional extension was devoid of any dependence on complex numbers.

Here, we wish to enlarge these notions to more general recurrence-generated number sequences and then to generalize our result to n-dimensional Euclidean space. To accomplish this objective, we will need to introduce a symbol $G(\ell, m, n)$, originally defined in [2] in relation to Fibonacci numbers only. This symbol represents a number with three components which may be regarded as the coordinates of a point with respect to three rectangular Cartesian axes, x, y, and z, i.e., as Cartesian or "Euclidean" coordinates.

First, we define the recurrence sequence $\{U_n\}$ by

$$U_{n+2} = pU_{n+1} - qU_n,\ U_0 = 0,\ U_1 = 1 \qquad (n \geqslant 0), \tag{1.1}$$

where p and q are generally integers.

Next, for positive integers ℓ, m, n, let

$$\begin{cases} G(\ell + 2, m, n) = pG(\ell + 1, m, n) - qG(\ell, m, n) \\ G(\ell, m + 2, n) = pG(\ell, m + 1, n) - qG(\ell, m, n) \\ G(\ell, m, n + 2) = pG(\ell, m, n + 1) - qG(\ell, m, n) \end{cases} \tag{1.2}$$

with

$$\begin{cases} G(0, 0, 0) = (a, a, a),\ G(1, 0, 0) = (b, 0, 0),\ G(0, 1, 0) \\ \qquad = (0, b, 0), \\ G(0, 0, 1) = (0, 0, b),\ G(1, 1, 0) = p(b, b, 0),\ G(1, 0, 1) \\ \qquad = p(b, 0, b), \\ G(0, 1, 1) = p(0, b, b),\ G(1, 1, 1) = p^2(b, b, b) \end{cases} \tag{1.3}$$

a and b being integers.

2. PROPERTIES OF $G(\ell, m, n)$

Inductive proofs, with appeal to (1.1)-(1.3), readily establish the following (cf. [2]):

$$G(\ell, 0, 0) = U_\ell G(1, 0, 0) - qU_{\ell-1}G(0, 0, 0) \tag{2.1}$$

$$G(\ell, 1, 0) = U_\ell G(1, 1, 0) - qU_{\ell-1}G(0, 1, 0) \tag{2.2}$$

$$G(\ell, m, 0) = U_m G(\ell, 1, 0) - qU_{m-1}G(\ell, 0, 0) \tag{2.3}$$

$$G(\ell,\ 0,\ 1) = U_\ell G(1,\ 0,\ 1) - qU_{\ell-1}G(0,\ 0,\ 1) \tag{2.4}$$
$$G(\ell,\ 1,\ 1) = U_\ell G(1,\ 1,\ 1) - qU_{\ell-1}G(0,\ 1,\ 1) \tag{2.5}$$
$$G(\ell,\ m,\ 1) = U_m G(\ell,\ 1,\ 1) - qU_{m-1}G(\ell,\ 0,\ 1) \tag{2.6}$$
$$G(\ell,\ m,\ n) = U_n G(\ell,\ m,\ 1) = qU_{n-1}G(\ell,\ m,\ 0) \tag{2.7}$$

Then,

$$\begin{aligned} G(\ell,\ m,\ n) &= U_n\{U_m G(\ell,\ 1,\ 1) - qU_{m-1}G(\ell,\ 0,\ 1)\} - qU_{n-1}\{U_m G(\ell,\ 1,\ 0) \\ &\qquad - qU_{m-1}G(\ell,\ 0,\ 0)\} \text{ by (2.3), (2.6), and (2.7)} \\ &= U_m U_n G(\ell,\ 1,\ 1) - qU_{m-1}U_n G(\ell,\ 0,\ 1) - qU_m U_{n-1}G(\ell,\ 1,\ 0) \\ &\qquad + q^2 U_{m-1}U_{n-1}G(\ell,\ 0,\ 0) \\ &= U_m U_n\{p^2 U_\ell(b,\ b,\ b) - pqU_{\ell-1}(0,\ b,\ b)\} \\ &\qquad - qU_{m-1}U_n\{pU_\ell(b,\ 0,\ b) - qU_{\ell-1}(0,\ 0,\ b)\} \\ &\qquad - qU_m U_{n-1}\{pU_\ell(b,\ b,\ 0) - qU_{\ell-1}(0,\ b,\ 0)\} \\ &\qquad + q^2 U_{m-1}U_{n-1}\{U_\ell(b,\ 0,\ 0) \\ &\qquad - qU_{\ell-1}(a,\ a,\ a)\} \text{ by (2.1), (2.2), (2.4), and (2.5).} \end{aligned} \tag{2.8}$$

Further,

$$\begin{aligned} U_\ell U_{m+1}U_{n+1} &= U_\ell(pU_m - qU_{m-1})(pU_n - qU_{n-1}) \quad \text{by (1.1)} \\ &= p^2 U_\ell U_m U_n - pqU_\ell U_m U_{n-1} - pqU_\ell U_{m-1}U_n + q^2 U_\ell U_{m-1}U_{n-1} \end{aligned} \tag{2.9}$$

with similar expressions for $U_{\ell+1}U_m U_{n+1}$ and $U_{\ell+1}U_{m+1}U_n$.

Comparing (2.8) and (2.9), we see that the right-hand side of (2.9) contains precisely those coefficients in (2.8) of coordinate sets with b in the first position, i.e., in the x-direction. Missing is the term in $U_{\ell-1}U_{m-1}U_{n-1}$.

Similar remarks apply to $U_{\ell+1}U_m U_{n+1}$ for b in the second position, and to $U_{\ell+1}U_{m+1}U_n$ for b in the third position, of a coordinate set.

Accordingly, we have established that

$$\begin{aligned} G(\ell,\ m,\ n) = (&p^2 bU_\ell U_{m+1}U_{n+1} - q^3 aU_{\ell-1}U_{m-1}U_{n-1}, \\ &p^2 bU_{\ell+1}U_m U_{n+1} - q^3 aU_{\ell-1}U_{m-1}U_{n-1}, \\ &p^2 bU_{\ell+1}U_{m+1}U_n - q^3 aU_{\ell-1}U_{m-1}U_{n-1}). \end{aligned} \tag{2.10}$$

Equation (2.10) gives the cooedinates of a point in three-dimensional Euclidean space in terms of numbers of the sequence $\{U_n\}$.

When $p = 1$, $q = -1$, $b = 1$, $a = 0$ in (1.1), we obtain the result for Fibonacci numbers F_n given in [2], namely,

$$G(\ell,\ m,\ n) = (F_\ell F_{m+1}F_{n+1},\ F_{\ell+1}F_m F_{n+1},\ F_{\ell+1}F_{m+1}F_n). \tag{2.11}$$

Setting $p = 2$, $q = -1$, $b = 1$, $a = 0$ in (1.1), we have the *Pell numbers* P_n for which (2.10) becomes

$$G(\ell,\ m,\ n) = (4P_\ell P_{m+1}P_{n+1},\ 4P_{\ell+1}P_m P_{n+1},\ 4P_{\ell+1}P_{m+1}P_n). \tag{2.12}$$

Before concluding this section we observe that, say, (2.6) may be expressed in an alternative form as

$$G(\ell,\ m,\ 1) = U_\ell G(1,\ m,\ 1) - qU_{\ell-1}(0,\ m,\ 1). \tag{2.6$'$}$$

3. HIGHER-DIMENSIONAL SPACE

Suppose we now extend the definitions in (1.1)-(1.3) to n dimensions in a natural way as follows. (The use of n here is not to be confused with its use in a different context in the symbol G in the previous section.)

For the n variables ℓ_i ($i = 1, 2, \ldots, n$), we define

$$\begin{cases} G(\ell_1+2, \ell_2, \ell_3, \ldots, \ell_n) = pG(\ell_1+1, \ell_2, \ell_3, \ldots, \ell_n) - qG(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) \\ G(\ell_1, \ell_2+2, \ell_3, \ldots, \ell_n) = pG(\ell_1, \ell_2+1, \ell_3, \ldots, \ell_n) - qG(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) \\ \text{---} \\ G(\ell_1, \ell_2, \ell_3, \ldots, \ell_n+2) = pG(\ell_1, \ell_2, \ell_3, \ldots, \ell_n+1) - qG(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) \end{cases} \tag{3.1}$$

with

$$\begin{cases} G(0, 0, 0, \ldots, 0) = (a, a, a, \ldots, a) \\ G(1, 1, 1, \ldots, 1) = (b, b, b, \ldots, b) \\ G(\text{---------------}) = p^k(\text{-------------}) \end{cases} \tag{3.2}$$

in which G(-----) contains $k + 1$ 1's and $n - (k + 1)$ 0's, and (-----) contains $k + 1$ b's and $n - (k + 1)$ 0's, in corresponding positions.

Mutatis mutandis, similar but more complicated results to those obtained in the previous section now apply to (3.1) and (3.2).

In particular, the result corresponding to (2.10) is

$$\begin{aligned} G(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) = (&p^{n-1}bU_{\ell_1}U_{\ell_2+1}U_{\ell_3+1}\cdots U_{\ell_n+1} + U, \\ &p^{n-1}bU_{\ell_1+1}U_{\ell_2}U_{\ell_3+1}\cdots U_{\ell_n+1} + U, \\ &\text{--------------------------------} \\ &p^{n-1}bU_{\ell_1+1}U_{\ell_2+1}U_{\ell_3+1}\cdots U_{\ell_n} + U) \end{aligned} \tag{3.3}$$

where, for visual and notational ease, we have written

$$U = (-q)^n aU_{\ell_1-1}U_{\ell_2-1}U_{\ell_3-1}\cdots U_{\ell_n-1}. \tag{3.4}$$

Clearly, (3.3) may represent the coordinates of a point in n-dimensional Euclidean space in terms of the numbers of the sequence $\{U_n\}$.

For Fibonacci numbers, $U = 0$, and (3.3) reduces to

$$\begin{aligned} G(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) = (&F_{\ell_1}F_{\ell_2+1}F_{\ell_3+1}\cdots F_{\ell_n+1}, \ldots, \\ &F_{\ell_1+1}F_{\ell_2+1}F_{\ell_3+1}\cdots F_{\ell_n}). \end{aligned} \tag{3.5}$$

Likewise, for Pell numbers, $U = 0$ also, and (3.3) becomes

$$\begin{aligned} G(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) = (&2^{n-1}P_{\ell_1}P_{\ell_2+1}P_{\ell_3+1}\cdots P_{\ell_n+1}, \ldots, \\ &2^{n-1}P_{\ell_1+1}P_{\ell_2+1}P_{\ell_3+1}\cdots P_{\ell_n}). \end{aligned} \tag{3.6}$$

It does not appear that any useful geometrical applications of an elementary nature can be deduced from the above results.

Harman [2] noted that if, in his case for Fibonacci numbers, the three expressions in (1.2) are combined, then the value of $G(\ell + 2, m + 2, n + 2)$ is given by the sum of the values of the symbol G at the eight vertices of the cube diagonally below that point. Similar comments apply to our more general expressions (1.2) with corresponding observations for the extension to n dimensions entailed in (3.1) in connection with the 2^n vertices of a "hypercube."

By this statement, we mean that when, say, $n = 3$, (1.2) gives

$$\begin{aligned} G(\ell + 2, m + 2, n + 2) = p^3G(\ell + 1, m + 1, n + 1) &- p^2q\{G(\ell + 1, m + 1, n) \\ &+ G(\ell + 1, m, n + 1) + G(\ell, m + 1, n + 1)\} \\ &+ pq^2\{G(\ell + 1, m, n) + G(\ell, m + 1, n) \\ &+ G(\ell, m, n + 1)\} - q^3G(\ell, m, n). \end{aligned} \tag{3.7}$$

In the case of Fibonacci numbers, $p^3 = -p^2q = pq^2 = -q^3 = 1$. For Pell numbers, $p^3 = 8$, $-p^2q = 4$, $pq^2 = 2$, $-q^3 = 1$.

4. CONCLUDING REMARKS

Consider briefly now the two-dimensional aspect of the results in the preceding section, i.e., the case $n = 2$. (Evidently, when $n = 1$, we merely get the numbers U_n strung out on the number axis.)

Writing $\ell_1 = \ell$, $\ell_2 = m$, we find that the truncated forms corresponding to (3.1)-(3.7) are, respectively,

$$\begin{cases} G(\ell + 2, m) = pG(\ell + 1, m) - qG(\ell, m), \\ G(\ell, m + 2) = pG(\ell, m + 1) - qG(\ell, m), \end{cases} \tag{4.1}$$

with

$$\begin{aligned} &G(0, 0) = (a, a),\ G(1, 0) = (b, 0), \\ &G(0, 1) = (0, b),\ G(1, 1) = p(b, b), \end{aligned} \tag{4.2}$$

whence:

$$G(\ell, m) = (pbU_\ell U_{m+1} + aU_{\ell-1}U_{m-1},\ pbU_{\ell+1}U_m + aU_{\ell-1}U_{m-1}), \tag{4.3}$$

$$G(\ell, m) = (F_\ell F_{m+1},\ F_{\ell+1}F_m) \quad \text{for } \{F_n\}, \tag{4.5}$$

$$G(\ell, m) = (2P_\ell P_{m+1},\ 2P_{\ell+1}P_m) \quad \text{for } \{P_n\}, \tag{4.6}$$

$$\begin{aligned} G(\ell + 2, m + 2) = p^2G(\ell + 1, m + 1) &- pqG(\ell + 1, m) \\ &- pqG(\ell, m + 1) + q^2G(\ell, m). \end{aligned} \tag{4.7}$$

Obvious simplifications of (4.7) apply for Fibonacci and Pell numbers.

Some of the above results, for Fibonacci numbers in the real Euclidean plane, should be compared with the corresponding results in the complex (Gaussian) plane obtained in [2]. The present authors [5] have studied the consequences in the complex plane of a natural generalization of the material in [2]. Harman [2], when advancing the innovatory features of his approach, acknowledges the earlier work of [1] and [3], and relates his work to theirs. It might be noted in passing that the introductory comments on quaternions in [3] have been investigated by other authors, e.g., [4]. One wonders whether an application of quaternions to extend the above theory on complex numbers might be at all fruitful.

From the structure provided by the complex Fibonacci numbers, some interesting classical identities involving products are derivable ([2] and [5]). Hopefully, these might give a guide to identities involving triple products of Fibonacci numbers, as conjectured in [2], and products in more general recurrence-generated number systems, as herein envisaged.

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ELEMENTARY PROBLEMS AND SOLUTIONS

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-580 *Proposed by Valentina Bakinova, Rondout Valley, NY*

What are the three smallest positive integers d such that no Lucas number L_n is an integral multiple of d?

B-581 *Proposed by Antal Bege, University of Cluj, Romania*

Prove that, for every positive integer n, there are at least $[n/2]$ ordered 6-tuples (a, b, c, x, y, z) such that

$$F_n = ax^2 + by^2 - cz^2$$

and each of a, b, c, x, y, z is a Fibonacci number. Here $[t]$ is the greatest integer in t.

B-582 *Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy*

It is known that every positive integer N can be represented uniquely as a sum of distinct nonconsecutive positive Fibonacci numbers. Let $f(N)$ be the number of Fibonacci addends in this representation, $\alpha = (1 + \sqrt{5})/2$, and $[x]$ be the greatest integer in x. Prove that

$$f([\alpha F_n^2]) = [(n + 1)/2] \text{ for } n = 1, 2, \ldots .$$

B-583 *Proposed by Dorin Andrica, University of Cluj-Napoca, Romania*

For positive integers n and s, let

$$S_{n,s} = \sum_{k=1}^{n} \binom{n}{k} k^s.$$

Prove that $S_{n,s+1} = n(S_{n,s} - S_{n-1,s})$.

B-584 *Proposed by Dorin Andrica, University of Cluj-Napoca, Romania*

Using the notation of B-583, prove that

$$S_{m+n,s} = \sum_{k=0}^{s} \binom{s}{k} S_{m,k} S_{n,s-k}.$$

B-585 *Proposed by Constantin Gonciulea & Nicolae Gonciulea, Trian College, Drobeta Turnu-Severin, Romania*

For each subset A of $X = \{1, 2, \ldots, n\}$, let $r(A)$ be the number of j such that $\{j, j+1\} \subseteq A$. Show that

$$\sum_{A \subseteq X} 2^{r(A)} = F_{2n+1}.$$

SOLUTIONS

Pattern for Squares

B-556 *Proposed by Valentina Bakinova, Rondout Valley, NY*

State and prove the general result illustrated by

$4^2 = 16$, $34^2 = 1156$, $334^2 = 111556$, $3334^2 = 11115556$.

Solution by Thomas M. Green, Contra Costa College, San Pablo, CA

Let $D_n = 1 + 10 + 10^2 + \cdots + 10^{n-1}$. The general result

$$(3D_n + 1)^2 = 10^n D_n + 5D_n + 1$$

is proved by expanding the left member and observing that $9D_n = 10^n - 1$.

Note: The quantity D_n has several other interesting properties:

(i) $D_n = 111...111$ (n ones)

(ii) $D_n^2 = 123...n...321$ ($n = 1, \ldots, 9$)

(iii) $D_9/9 = 123456789$

(iv) $(b - 1)D_n = b^n - 1$ (b is your number base)

(v) The sequence $D_2^0, D_2^1, D_2^2, \ldots, D_2^n$, is Pascal's triangle (with suitable restrictions on carrying) and the sequences $D_n^0, D_n^1, D_n^2, \ldots, D_n^r$, are Pascal-like triangles where each entry is the sum of the n entries above it.

Also solved by Paul S. Bruckman, Lászlő Cseh, L. A. G. Dresel, Piero Filipponi, J. Foster, Herta T. Freitag, Hans Kappus, H. Klauser, L. Kuipers, Graham Lord, *Imre Merényi, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.*

Not True Any Year

B-557 *Proposed by Imre Merényi, Cluj, Romania*

Prove that there is no integer $n \geqslant 2$ such that

$$F_{3n-6}F_{3n-3}F_{3n+3}F_{3n+6} - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1985^8 + 1.$$

Solution by J. Suck, Essen, Germany

Since $F_{3k} \equiv 2F_k \pmod 3$ [see, e.g., B-182, *The Fibonacci Quarterly* 8 (Dec. 1970), for the more general $F_{pk} \equiv F_pF_k \pmod p$, p a prime], the left-hand side is congruent to $(2^4 - 1)F_{n-2}F_{n-1}F_{n+1}F_{n+2}$, hence to 0. But the right-hand side is not whatever the year may be: if $y \equiv 0, 1, 2$, then $y^8 + 1 \equiv 1, 2, 2$, respectively, mod 3.

Also solved by Paul S. Bruckman, Lászlő Cseh, L. A. G. Dresel, Piero Filipponi, J. Foster, L. Kuipers, Sahib Singh, M. Wachtel, and the proposer.

Impossible Equation

B-558 *Proposed by Imre Merényi, Cluj, Romania*

Prove that there are no positive integers m and n such that

$$F_{4m}^2 - F_{3n} - 4 = 0.$$

Solution by L. A. G. Dresel, University of Reading, England

Since $F_3 = 2$ and $F_6 = 8$, we have $F_{3n} \equiv 2 \pmod 4$ when n is odd, and $F_{3n} \equiv 0$ when n is even. Now consider the equation $F_{4m}^2 = F_{3n} + 4$. Clearly n cannot be odd, since $F^2 \equiv 2 \pmod 4$ is not possible. However, if n is even, $F_{4m}^2 \equiv 4$ (mod 8) and this implies $F_{4m} \equiv \pm 2 \pmod 8$. Hence F_{4m} is even, so that $m = 3k$, where k is an integer, and therefore $F_{4m} = F_{12k}$, which is divisible by 8. This contradicts $F_{4m} = \pm 2 \pmod 8$. Hence there are no integers m and n such that $F_{4m}^2 - F_{3n} - 4 = 0$.

We note that the above argument actually proves the slightly stronger result that there are no integers m and n such that $F_{2m}^2 - F_{3n} - 4 = 0$.

Also solved by Paul S. Bruckman, Lászlő Cseh, Piero Filipponi, L. Kuipers, Sahib Singh, Lawrence Somer, M. Wachtel, and the proposer.

Golden Mean Identity

B-559 *Proposed by Lászlő Cseh, Cluj, Romania*

Let $\alpha = (1 + \sqrt{5})/2$. For positive integers n, prove that

$$[\alpha + .5] + [\alpha^2 + .5] + \cdots + [\alpha^n + .5] = L_{n+2} - 2,$$

where $[x]$ denotes the greatest integer in x.

Solution by J. Foster, Weber State College, Ogden, UT

Since $L_k = \alpha^k + \beta^k$ and, for $k \geqslant 2$, $[.5 - \beta^k] = 0$,

$$\sum_{k=1}^{n}[\alpha^k + .5] = \sum_{k=1}^{n}[L_k - \beta^k + .5] = \sum_{k=1}^{n}(L_k + [.5 - \beta^k])$$

$$= \sum_{k=0}^{n} L_k - L_0 + [.5 - \beta] = (L_{n+2} - 1) - 2 + 1 = L_{n+2} - 2.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, C. Georghiou, Hans Kappus, L. Kuipers, Graham Lord, Imre Merényi, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Another Greatest Integer Identity

B-560 *Proposed by Lászlό Cseh, Cluj, Romania*

Let α and $[x]$ be as in B-559. Prove that

$$[\alpha F_1 + .5] + [\alpha^2 F_2 + .5] + \cdots + [\alpha^n F_n + .5]$$

is always a Fibonacci number.

Solution by C. Georghiou, University of Patras, Greece

We have

$$\alpha^n F_n = \frac{\alpha^{2n} - (-1)^n}{\sqrt{5}} = F_{2n} + \frac{\beta^{2n} - (-1)^n}{\sqrt{5}}$$

and since

$$\left[\frac{\beta^{2n} - (-1)^n}{\sqrt{5}} + .5\right] = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$$

the given sum becomes

$$F_2 + 1 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1}.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, J. Foster, Hans Kappus, L. Kuipers, Imre Merényi, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Q-Matrix Identity

B-561 *Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy*

(i) Let Q be the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. For all integers n, show that

$$Q^n + (-1)^n Q^{-n} = L_n I, \text{ where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(ii) Find a square root of Q, i.e., a matrix A with $A^2 = Q$.

Solution by Sahib Singh, Clarion University, Clarion, PA

(i) If $n = 0$, then $Q^0 + (Q^0)^{-1} = 2I = L_0 I$.

For $n \geqslant 1$, it follows by mathematical induction that:

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

$$Q^{-n} = (-1)^n \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix}$$

Thus, $Q^n + (-1)^n Q^{-n} = \begin{bmatrix} L_n & 0 \\ 0 & L_n \end{bmatrix} = L_n I$ for all $n \geqslant 1$.

Changing n to $-n$, the above equation becomes:

$$Q^{-n} + (-1)^{-n} Q^n = L_{-n} I \quad \text{or} \quad (-1)^n [Q^n + (-1)^n Q^{-n}] = (-1)^n L_n I,$$

so that

$$Q^n + (-1)^n Q^{-n} = L_n I \text{ for } n \leqslant -1.$$

Thus, the result holds for all integers.

(ii) Let a square root of Q be denoted by S where $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $S^2 = Q$ yields

$$a^2 + bc = 1; \ (a + d)b = 1; \ (a + d)c = 1; \ bc + d^2 = 0.$$

Solving these equations, we conclude that

$$a = \frac{1 + b^2}{2b}; \ c = b; \ d = \frac{1 - b^2}{2b}, \text{ where } b \text{ satisfies}$$

$$5b^4 - 2b^2 + 1 = 0.$$

Thus, a square root of Q is $\begin{bmatrix} \frac{1 + b^2}{2b} & b \\ b & \frac{1 - b^2}{2b} \end{bmatrix}$,

where b is a complex number satisfying $5b^4 - 2b^2 + 1 = 0$, which can be solved by the quadratic formula using $x = b^2$.

Also solved by Paul S. Bruckman, László Cseh, L. A. G. Dresel, J. Foster, C. Georghiou, Hans Kappus, L. Kuipers, Bob Prielipp, Lawrence Somer, J. Suck, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-403 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Given p, q real with $p \neq -1 - 2qk$, $k = 0, 1, 2, \ldots$, find a closed form expression for the continued fraction

$$\theta(p, q) \equiv p + \cfrac{p + q}{p + 2q + \cfrac{p + 3q}{p + 4q + \cdots}}. \tag{1}$$

HINT: Consider the *Confluent Hypergeometric* (or *Kummer*) function defined as follows:

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \cdot \frac{z^n}{n!}, \; b \neq 0, -1, -2, \ldots . \tag{2}$$

NOTE: $\theta(1, 1) = 1 + \cfrac{2}{3 + \cfrac{4}{5 + \cdots}}$, which was Problem H-394.

H-404 *Proposed by Andreas N. Philippou & Frosso S. Makri, Patras, Greece*

Show that

(a) $$\sum_{r=0}^{n} \sum_{i=0}^{1} \sum_{\substack{n_1, n_2 \ni \\ n_1 + 2n_2 = n - i \\ n_1 + n_2 = n - r}} \binom{n_1 + n_2}{n_1, n_2} = F_{n+2}, \; n \geq 0;$$

(b) $$\sum_{r=0}^{n} \sum_{i=0}^{k-1} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n - i \\ n_1 + \cdots + n_k = n - r}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} = F_{n+2}^{(k)}, \; n \geq 0, \; k \geq 2,$$

where $n_1, \ldots, n_k$ are nonnegative integers and $\{F_n^{(k)}\}$ is the sequence of Fibonacci-type polynomials of order k [1].

[1] A. N. Philippou, C. Georghiou, & G. N. Philippou, "Fibonacci-Type Polynomials of Order K with Probability Applications," *The Fibonacci Quarterly* **23**, no. 2 (1985):100-105.

H-405 *Proposed by Piero Filipponi, Rome, Italy*

(i) Generalize Problem B-564 by finding a closed form expression for

$$\sum_{n=1}^{N} [\alpha^k F_n], \quad (N = 1, 2, \ldots; k = 1, 2, \ldots)$$

where $\alpha = (1 + \sqrt{5})/2$, F_n is the n^{th} Fibonacci number, and $[x]$ denotes the greatest integer not exceeding x.

(ii) Generalize the above sum to negative values of k.

(iii) Can this sum be further generalized to any rational value of the exponent of α?

Remark: As to (iii), it can be proved that

$$[\alpha^{1/k} F_n] = F_n, \text{ if } 1 \leqslant n \leqslant [(\ln \sqrt{5} - \ln(\alpha^{1/k} - 1))/\ln \alpha].$$

References

1. V. E. Hoggatt, Jr., & M. Bicknell-Johnson, "Representstion of Integers in Terms of Greatest Integer Functions and the Golden Section Ratio," *The Fibonacci Quarterly* 17, no. 4 (1979):306-318.
2. V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers* (Boston: Houghton Mifflin Company, 1969).

SOLUTIONS

Sum Zeta!

H-381 *Proposed by Dejan M. Petkovic, Nis, Yugoslavia*
(Vol. 23, no. 1, February 1985)

Let N be the set of all natural numbers and let $m \in N$. Show that

(i) $$\zeta(2m - 2) = \frac{(-)^m \overline{u}^{2m-2}(m - 1)}{(2m - 1)!} + \sum_{i=2}^{m-1} \frac{(-)^i \overline{u}^{2i-2}}{(2i - 1)!} \cdot \zeta(2m - 2i), m \geqslant 2,$$

(ii) $$\beta(2m - 1) = \sum_{i=1}^{m-1} \frac{(-)^i \overline{u}^{2i}}{2^{2i}(2i)!} \cdot \beta(2m - 2i - 1), m \geqslant 2,$$

(iii) $$\zeta(2m) = \frac{2^{2m}}{2^{2m} - 1} \sum_{i=0}^{m-1} \frac{(-)^i \overline{u}^{2i+1}}{2^{2i+1}(2i + 1)!} \cdot \beta(2m - 2i - 1), m \geqslant 1,$$

where

$$\zeta(m) = \sum_{n=1}^{\infty} n^{-m}, \ m \geqslant 2, \text{ are Riemann zeta numbers}$$

and

$$\beta(m) = \sum_{n=1}^{\infty} (-)^{n-1}(2 \quad - 1)^{-m}, \ m \geqslant 1.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

We use the known expressions

$$\zeta(2m) = \frac{(2\overline{u})^{2m}}{2(2m)!}(-1)^{m-1}B_{2m}, \quad m = 1, 2, \ldots, \tag{1}$$

$$\beta(2m+1) = \frac{(\overline{u}/2)^{2m+1}}{2(2m)!}(-1)^m E_{2m}, \; m = 0, 1, 2, \ldots,$$

where $\overline{u}$ denotes the constant π, and the B_{2m} and E_{2m} are the Bernoulli and Euler numbers, respectively.

These may be defined by the following generating functions:

$$x \cot x = \sum_{m=0}^{\infty} B_{2m}(-1)^m \frac{(2x)^{2m}}{(2m)!}, \tag{3}$$

and

$$\sec x = \sum_{m=0}^{\infty} E_{2m}(-1)^m \frac{x^{2m}}{(2m)!}. \tag{4}$$

Setting $x = \overline{u}z$, then

$$\overline{u}z \cot \overline{u}z = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!}(2\overline{u}z)^{2m} \cdot \frac{2(2m)!\zeta(2m)(-1)^{m-1}}{(2\overline{u})^{2m}},$$

or

$$\overline{u}z \cot \overline{u}z = 1 - 2\sum_{m=1}^{\infty} \zeta(2m)z^{2m}. \tag{5}$$

Also,

$$\sec \overline{u}z = \sum_{m=0}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m}}{(2m)!} \cdot \frac{2(2m)!(-1)^m}{(\overline{u}/2)^{2m+1}} \beta(2m+1),$$

or

$$\sec \overline{u}z = \frac{4}{\overline{u}} \sum_{m=0}^{\infty} \beta(2m+1)(2z)^{2m}. \tag{6}$$

We also use the following well-known expressions:

$$\sin \overline{u}z = \sum_{m=0}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m+1}}{(2m+1)!}; \tag{7}$$

$$\cos \overline{u}z = \sum_{m=0}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m}}{(2m)!}. \tag{8}$$

Multiplying (5) and (7), we obtain:

$$\overline{u}z \cos \overline{u}z = \sum_{m=0}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m+1}}{(2m+1)!} - 2\sum_{m=1}^{\infty} z^{2m+1} \sum_{i=0}^{m-1} (-1)^i \frac{(\overline{u})^{2i+1}}{(2i+1)!} \zeta(2m-2i);$$

on the other hand, from (8),

$$\overline{u}z \cos \overline{u}z = \sum_{m=0}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m+1}}{(2m)!}.$$

Thus,

$$\sum_{m=1}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m+1}}{(2m+1)!}(1 - (2m+1)) = 2\sum_{m=1}^{\infty} z^{2m+1} \sum_{i=0}^{m-1} (-1)^m \frac{(\overline{u})^{2i+1}}{(2i+1)!} \zeta(2m-2i).$$

Comparing coefficients,

$$\frac{-2m(-1)^m(\overline{u})^{2m+1}}{(2m+1)!} = 2\sum_{i=0}^{m-1}(-1)^i \frac{(\overline{u})^{2i+1}}{(2i+1)!}\zeta(2m-2i):$$

replacing m by $m-1$ and dividing by $2\overline{u}$ yields:

$$\begin{aligned}\frac{(m-1)(-1)^m(\overline{u})^{2m-2}}{(2m-1)!} &= \sum_{i=0}^{m-2}(-1)^i \frac{(\overline{u})^{2i}}{(2i+1)!}\zeta(2m-2i-2)\\ &= -\sum_{i=1}^{m-1}(-1)^i \frac{(\overline{u})^{2i-2}}{(2i-1)!}\zeta(2m-2i)\\ &= \zeta(2m-2) - \sum_{i=2}^{m-1}(-1)^i \frac{(\overline{u})^{2i-2}}{(2i-1)!}\zeta(2m-2i).\end{aligned}$$

This is equivalent to the result indicated in (i).

Multiplying (6) and (8), we obtain:

$$1 = 4/\overline{u}\sum_{m=0}^{\infty} z^{2m}\sum_{i=0}^{m}(-1)^i \frac{(\overline{u})^{2i}}{(2i)!}2^{2m-2i}\beta(2m-2i+1);$$

hence, for $m \geqslant 1$,

$$0 = \sum_{i=0}^{m}(-1)^i \frac{(\overline{u})^{2i}}{(2i)!}2^{2m-2i}\beta(2m-2i+1).$$

Replacing m by $m-1$ and dividing by 2^{2m-2} yields:

$$\begin{aligned}0 &= \sum_{i=0}^{m-1}(-1)^i \frac{(\overline{u})^{2i}}{(2i)!}2^{-2i}\beta(2m-2i-1)\\ &= \beta(2m-1) + \sum_{i=1}^{m-1}(-1)^i \frac{(\overline{u})^{2i}}{(2i)!}2^{-2i}\beta(2m-2i-1).\end{aligned}$$

This last result corrects (ii), which is incorrect in the sign of one of its members.

Finally, multiplying (6) and (7) yields:

$$\tan \overline{u}z = 4/\overline{u}\sum_{m=1}^{\infty} z^{2m-1}\sum_{i=0}^{m-1}(-1)^i \frac{(\overline{u})^{2i+1}}{(2i+1)!}2^{2m-2i-2}\beta(2m-2i-1).$$

On the other hand, since $\tan x = \cot x - 2\cot 2x$, we have:

$$\begin{aligned}\tan \overline{u}z &= (\overline{u}z)^{-1}(\overline{u}z \cot \overline{u}z - 2\overline{u}z\cot 2\overline{u}z)\\ &= (\overline{u}z)^{-1}\left\{1 - 2\sum_{m=1}^{\infty}\zeta(2m)z^{2m} - 1 + 2\sum_{m=1}^{\infty}\zeta(2m)(2z)^{2m}\right\}\\ &= 2/\overline{u}\sum_{m=1}^{\infty}\zeta(2m)(2^{2m}-1)z^{2m-1}.\end{aligned}$$

Comparing coefficients,

$$4/\overline{u}\sum_{i=0}^{m-1}(-1)^i \frac{(\overline{u})^{2i+1}}{(2i+1)!}2^{2m-2i-2}\beta(2m-2i-1) = 2/\overline{u}\zeta(2m)(2^{2m}-1),$$

or, equivalently:

$$\zeta(2m) = \frac{2^{2m}}{2^{2m-1}} \sum_{i=0}^{m-1} (-1)^i (\overline{u}/2)^{2i+1} \frac{\beta(2m-2i-1)}{(2i+1)!},$$

which is (iii). Q.E.D.

Also solved by C. Georghiou, S. Papastavridis, P. Siafarikas, P. Sypsas, and the proposer.

H-382 *Proposed by Andreas N. Philippou, Patras, Greece*
(Vol. 23, no. 1, February 1985)

For each fixed positive integer k, define the sequence of polynomials $A_{n+1}^{(k)}(p)$ by

$$A_{n+1}^{(k)}(p) = \sum_{n_1+\cdots+n_k} \binom{n_1+\cdots+n}{n_1, \ldots, n} \left(\frac{1-p}{p}\right)^{n_1, \ldots, n_k} \quad (n \geqslant 0, -\infty < p < \infty), \tag{1}$$

where the summation is taken over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n + 1$. Show that

$$A_{n+1}^{(k)}(p) \leqslant (1-p)p^{-(n+1)}(1-p^k)^{[n/k]} \quad (n \geqslant k-1, 0 < p < 1), \tag{2}$$

where $[n/k]$ denotes the greatest integer in (n/k).

It may be noted that (2) reduces to

$$F_n \leqslant 2^n \left(\frac{2^k-1}{2^k}\right)^{[n/k]} \quad (n \geqslant k-1) \tag{3}$$

and

$$F_n \leqslant 2^n (3/4)^{[n/2]} \quad (n \geqslant 1), \tag{4}$$

where $\{F_n^{(k)}\}_{n=0}^{\infty}$ and $\{F_n\}_{n=0}^{\infty}$ denote the Fibonacci sequence of order k and the usual Fibonacci sequence, respectively, if $p = 1/2$ and $p = 1/2$, $k = 2$.

References

1. J. A. Fuchs. Problem B-39. *The Fibonacci Quarterly 2*, no. 2 (1964):154.
2. A. N. Philippou. Problem H-322. *The Fibonacci Quarterly 19*, no. 1 (1981): 93.

Solution by the proposer

For each fixed positive integer k, let $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ $(x > 0)$ be the sequence of Fibonacci-type polynomials of order k [4] and denote by L_n and $N_n^{(k)}$ the longest success run and the number of success runs of order k, respectively, in n Bernoulli trials. It follows from the definition of $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ that

$$F_2^{(k)}(x) = x$$

and

$$F_{n+2}^{(k)}(x) = (1+x)F_{n+1}^{(k)}(x) = \cdots = (1+x)^n F_2^{(k)}(x) = (1+x)^n x \quad (1 \leqslant n \leqslant k-1, x > 0),$$

which gives

$$F_{n+2}^{(k)}((1-p)/p) = (1-p)p^{-(n+1)} \quad (0 \leqslant n \leqslant k-1,\ 0 < p < 1). \tag{5}$$

Furthermore,

$$\begin{aligned} F_{n+2}^{(k)}((1-p)/p) &= (1-p)p^{-(n+1)}P(L_n \leqslant k-1) \\ &\qquad (n \geqslant k-1,\ 0 < p < 1), \qquad (6) \\ &\qquad \text{by Theorem 2.1(a) of [4]}, \\ &= (1-p)p^{-(n+1)}P(N_n^{(k)} = 0), \\ &\qquad \text{by the definition of } L_n \text{ and } N_n^{(k)}, \\ &= (1-p)p^{-(n+1)}\{1 - P(N_n^{(k)} \geqslant 1)\} \\ &\leqslant (1-p)p^{-(n+1)}\{1 - \{1 - (1-p^k)^{[n/k]}\}\}, \\ &\qquad \text{by Proposition 6.3 of [1]}, \\ &= (1-p)p^{-(n+1)}(1-p^k)^{[n/k]}. \end{aligned}$$

But

$$F_{n+2}^{(k)}((1-p)/p) = A_{n+1}^{(k)}(p) \quad (n \geqslant 0,\ 0 < p < 1), \tag{7}$$

by Lemma 2.2(b) of [4] and (1).

Relations (5)-(7) establish (2), which reduces to (3) and (4), respectively, since

$$F_{n+2}^{(k)} = F_{n+2}^{(k)}(1) = A_{n+1}^{(k)}(1/2) \quad (n \geqslant 0)$$

and

$$F_{n+2}^{(2)} = F_n \quad (n \geqslant 0).$$

It may be noted that inequalities (3) and (4) are sharper than those given in [2] and [3], respectively.

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1. S. M. Berman. *The Elements of Probability.* Reading, Mass.: Addison-Welesly, 1969.
2. J. A. Fuchs. Problem B-39. *The Fibonacci Quarterly* **2**, no. 2 (1964):154.
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LETTER TO THE EDITOR

August 11, 1986

The first written evidence of a knowledge of the relationship between the Fibonacci sequence and division in extreme and mean ratio (the "golden number") has been considered to be a letter written by Kepler in 1608. However, a recently discovered marginal note to theorem II,11 (the geometric construction of d.e.m.r.), in a copy of Paccioli's 1509 edition of the *Elements*, which includes the terms 89, 144, and 233 of the Fibonacci sequence shows that this relationship was already known in the early 16th century. Further, the appearance of the product terms 20736 and 20737 strongly suggests—although the text presents certain difficulties in interpretation—that the author of the note was aware of the result

$$(f_{n+1})^2 - f_n \cdot f_{n+2} = \pm 1.$$

A photograph of the note together with a transcription of the Latin and a translation appear in [1], which also examines existing evidence, and theories proposed in the literature, for a knowledge of the relationship in earlier periods. This text is also discussed in [2, section 31, J], which is entirely devoted to a history of division in extreme and mean ratio including its relationship to the Fibonacci numbers.

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The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

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A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

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