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TWENTY-FIFTH ANNIVERSARY YEAR

CONTENTS

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PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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THE FIBONACCI QUARTERLY seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

A SHORT HISTORY OF *THE FIBONACCI QUARTERLY*

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This volume marks the 25th year of publication of *The Fibonacci Quarterly*, prompting memories of just how it all started. As a long-time observer and participant, I was asked to write a short history of the early organization.

In the beginning, the Fibonacci Association grew out of the bond of friendship formed by those sharing an interest in the Fibonacci numbers. Professor Verner E. Hoggatt, Jr., San Jose State College, had become interested in the Fibonacci sequence in the late 1950s. Vern's colleague Dmitri Thoro introduced him to Brother Alfred Brousseau, St. Mary's College, in the early 1960s. Vern and Brother Alfred began a long friendship and met frequently to discuss Fibonacci numbers and often sang songs, accompanied by Brother Alfred's accordion. (I recall a ballad written by Brother Alfred, "Do What Comes Fibernaturally!", to the tune of "The Blue-Tail Fly.")

As time went on, their intense interest in the Fibonacci sequence began to take a more organized direction. Brother Alfred, for example, compiled a bibliography of more than 700 Fibonacci references, ranging from recreational to serious research, to disseminate to interested initiates. Both took any and every opportunity to lecture on the sequence, so much so that Vern soon became fondly known as "Professor Fibonacci."

By December of 1962, the group also included Professor Paul Byrd, I. Dale Ruggles, Stanley L. Basin, and Terrance A. Brennan. It was this group of men who founded the Fibonacci Association to provide an opportunity for those who shared an interest in the Fibonacci numbers to exchange ideas.

So much interest in the Fibonacci numbers was apparent to the "founding fathers" that they decided to publish *The Fibonacci Quarterly*, despite limited support and all the other problems that beset a new venture. Vern and Brother Alfred wanted a journal to provide rapid dissemination of the ever expanding research on the Fibonacci numbers and to invite teachers and students to share their enthusiasm for mathematics.

With a very small amount of money from subscriptions and donations, and a large amount of volunteer labor from students, friends, and family, the first issue of *The Fibonacci Quarterly* was published in February 1963, with Editor Verner E. Hoggatt, Jr., and Managing Editor Brother U. Alfred.

Due to shoestring economics, the first issue was typed by Brother Alfred; after that, several professional technical typists came and went. Keeping a good typist almost caused Vern to have a nervous breakdown, until he met someone who needed him to complete a golf foursome and discovered a technical typist in the course of getting acquainted!

The first printer was a photocopy shop with a small press, but this proved inadequate and costly. Then Brother Alfred approached William Descalso, who had done printing for St. Mary's College since 1948, to take on the printing of the *Quarterly*. Descalso had a large web press which could print 16 pages at one time. (This explains why we had 80, 96, or 112 pages, but never 89.) These signatures and the cover were put into a folding machine, and the journal was assembled, stapled, and trimmed in one operation. Mr. Descalso took special interest in the *Quarterly* for many years, and I suspect that he helped us to continue by making personal sacrifices. Also, he used to deliver the *Quarterly* to Brother Alfred for mailing, then bring the reprints to Vern's home in a big truck for stapling and mailing.

At first, subscriptions came in slowly (59 on January 31, 1963), but with some advertising and favorable notices in various magazines, especially *Scientific American* (June 1963, p. 152), the tempo increased. As a result, by September 1963 there were 659 subscribers, and 915 subscribers by the end of the first year of publication. From this point on, it was a matter of maintaining the momentum. While researching this article, I found a handwritten page entitled "back-sliders" among Vern's notes; he had personally called every person who failed to renew his or her subscription for the second year!

The Fibonacci Quarterly slowly began to draw attention. While at the first meeting in December 1962, Professor Paul Byrd had wondered how we would obtain enough material for such a specialized journal. Ironically, the problem, over the years, turned out to be a superabundance of material. Vern answered all of the many inquiries addressed to the *Quarterly* personally, in longhand. Brother Alfred wrote and published the booklet, *Fibonacci Discovery*, as an aid to beginners and as another source of income for the Association. Many articles were written especially to interest beginners in the study of Fibonacci numbers. (Subsequently, these early articles were collected together and published as *A Primer for the Fibonacci Numbers.*) *The Fibonacci Quarterly* was mentioned in Martin Gardner's column in *Scientific American* in March 1969, and Brother Alfred and Vern were interviewed in an article in *Time*, April 4, 1969, pages 48 and 50. Vern was asked to write a series of articles for *Math Log*, published by Mu Alpha Theta, and his book, *Fibonacci and Lucas Numbers*, was published by Houghton Mifflin in 1969. (I know that he had to write two complete drafts of this book because I typed both versions!) With a little fame, Vern was given a small grant by San Jose State College, and a semester-long sabbatical leave.

In those early days, the Editor carried everyone's address, telephone number, and research paper in his head. Although carrying a full teaching load, Vern still answered all correspondence personally, often writing more than 50 letters a week. He carried on such a prolific correspondence on Fibonacci matters that he frequently slept for only four hours a night. While I lived only across town, I would receive two or three letters each week because Vern wanted to put his thoughts on paper. Then he would call me for feedback, often before I had received the letters! Vern put his family to work stapling reprints and mailing them to the authors, and gave his graduate students proofreading, typing, and other tasks. I once spent many hours proofreading the first 571 Fibonacci numbers (F_{571} has 119 digits) in an attempt to make the project perfect; however, the printer's helper dropped the tray of lead characters, transposing 50 digits of F_{521} and F_{522}! Nevertheless, that article, which appeared in the October 1962 issue of *Recreational Mathematics Magazine*, was a good source of publicity for the soon-to-appear *Fibonacci Quarterly*. I also remember that he had such a concern for struggling foreign authors that he asked me to do a bit of ghost-writing because he didn't have the heart to reject their papers.

As Managing Editor, Brother Alfred kept track of all subscription and book orders and the mailing list. He mailed everything from St. Mary's College and soon had an entire basement devoted to storing Fibonacci magazines and books. When the fifty pound boxes of magazines arrived from the printer, he had to carry them to the basement and then haul them back upstairs to mail them. Because of the large volume of manuscripts, whenever the Association could raise extra money, they published an extra issue, so there were five or six issues a year at times after 1966. Storage space kept filling up; when the back issues and books were transferred to Santa Clara University in 1975, there were 257 boxes. (A Fermat number!)

Brother Alfred wrote a number of elementary articles to interest and stimulate beginners, teachers, and students, and compiled several books of tables which are still available from The Fibonacci Association. He could generate new pages for the books at such a prodigious rate that I found it difficult to keep up with the proofreading. He gave lectures at nearly every meeting of mathematics teachers in California for years. And, of course, all of this was in addition to his teaching load.

Brother Alfred seemed always to have a new Fibonacci-related problem or a new approach to present. He was interested in phyllotaxis and collected more than 6000 pinecones, including cones from the twenty native pine trees of California, because the Fibonacci sequence occurred in the spirals of the cones. Vern once sent him a "Lucas" sunflower that exhibited Lucas numbers instead of the expected Fibonacci sequence; Vern had grown the sunflower himself especially to count its spirals.

Verner E. Hoggatt, Jr., and Brother Alfred Brousseau

October 20, 1973

A SHORT HISTORY OF *THE FIBONACCI QUARTERLY*

In January 1968, the Board of The Fibonacci Association was formed to set policy and to provide continuity for The Fibonacci Association and its publications. The members of the original Board of The Fibonacci Association were: Brother Alfred Brousseau, Verner E. Hoggatt, Jr., G. L. Alexanderson, George Ledin, I. Dale Ruggles, and myself. For many years, a research conference was held annually, and a special conference for high school teachers and their students was held at the University of San Francisco for five consecutive years.

By 1972, *The Fibonacci Quarterly* was listed regularly in both *Mathematics Reviews* and *Zentralblatt für Mathematik*, and a fine article entitled "A Magic Ratio Recurs Throughout Art and Nature" appeared in the December 1975 issue of *Smithsonian*. Also, Vern was invited to write an article for the 1977 Yearbook of *Encyclopaedia Britannica*, in "Science and the Future," pp. 177-192.

Brother Alfred continued as Managing Editor for 13 years, until his retirement in 1975, and Vern Hoggatt served as Editor for 18 years, until his death on August 11, 1980. It is hard to imagine *The Fibonacci Quarterly* having been published for so long if it had not been for the propitious meeting and enduring friendship of two such talented men and their interest in an obscure mathematical sequence, 1, 1, 2, 3, 5, 8, ...

The 1987 volume marks the twenty-fifth year of publication of *The Fibonacci Quarterly*, which has evolved into a research journal with international subscribers. (There are over 200 foreign subscribers, mostly from West Germany, Canada, Japan, Australia, The United Kingdom, Greece, and Italy, but representing 36 other countries as well.)

Long live Fibonacci!

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A SYSTEMATIC SEARCH FOR UNITARY HYPERPERFECT NUMBERS

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1. INTRODUCTION

If m and t are natural numbers, we say that m is a unitary hyperperfect number of order t if

$$m = 1 + t[\sigma^*(m) - m - 1], \tag{1}$$

where $\sigma^*(m)$ denotes the sum of the unitary divisors of m. m is said to be a hyperperfect number of order t if

$$m = 1 + t[\sigma(m) - m - 1], \tag{2}$$

where σ is the usual divisor sum function. Hyperperfect numbers (HP's) were first studied by D. Minoli & R. Bear [4], while the study of unitary hyperperfect numbers (UHP's) was initiated by the present author [3]. H. J. J. te Riele [6] has found *all* (151) HP's less than 10^8 as well as many larger ones having more than two prime factors. D. Buell [2] has found all (146) UHP's less than 10^8. More recently, W. Beck & R. Najar [1] have studied the properties of HP's and UHP's. One of the results they obtained was the following.

Proposition 1: If m is a unitary hyperperfect number of order t, then $(m, t) = 1$ and m *and* t are of opposite parity.

The purpose of the present paper is to develop a search procedure, different from that employed by Buell, which can be used to find all of the unitary hyperperfect numbers less than a specified bound with a specified number of distinct prime factors (provided the necessary computer time is available).

2. THE GENERAL PROCEDURE

Suppose that $m = ar^\gamma s^\lambda$, where r and s are distinct primes, $\gamma\lambda \neq 0$, and $(a, rs) = 1$. If m is a unitary hyperperfect number of order t, then, since σ^* is multiplicative and $\sigma^*(r^\gamma) = 1 + r^\gamma$, it follows from (1) that

$$[a - t(\sigma^*(a) - a)]r^\gamma s^\lambda - t\sigma^*(a)[r^\gamma + s^\lambda] = 1 + t[\sigma^*(a) - 1].$$

Multiplying this equality by $a - t(\sigma^*(a) - a)$ and then adding $[t\sigma^*(a)]^2$ to each side, we obtain

$$\begin{aligned}&\{[a - t(\sigma^*(a) - a)]r^\gamma - t\sigma^*(a)\}\{[a - t(\sigma^*(a) - a)]s^\lambda - t\sigma^*(a)\}\\ &= [a - t(\sigma^*(a) - a)][1 + t(\sigma^*(a) - 1)] + [t\sigma^*(a)]^2.\end{aligned} \tag{3}$$

If AB, where $1 \leqslant A < B$, is the "correct" factorization of the right-hand member of (3), then we see that

$$r^\gamma = [t\sigma^*(a) + A]/[a - t(\sigma^*(a) - a)],$$
$$s^\lambda = [t\sigma^*(a) + B]/[a - t(\sigma^*(a) - a)]. \tag{4}$$

Since the steps just described are reversible, given values of a and t, *if* a factorization AB of the right-hand member of (3) can be found for which the right-hand members of (4) are distinct prime powers relatively prime to a, then the integer $ar^\gamma s^\lambda$ is a unitary hyperperfect number of order t. Of course, for most values of a and t the right-hand members of (4) will not both be integers, let alone prime powers. It should be mentioned that the above derivation of (4) is basically due to Euler via H. J. J. te Riele (see [5]).

3. THE CASE $a = 1$

If, in (4), we set $a = 1$, then, since $\sigma^*(1) = 1$, it follows that $r^\gamma = t + A$ and $s^\lambda = t + B$, where, from (3), $AB = 1 + t^2$. Suppose that t is odd. Then $AB \equiv 2 \pmod 8$ and it follows that A and B are of opposite parity. Therefore, without loss of generality, $r = 2$ and, since $3|ts^\lambda$ (see Fact 1 in [3]), we have proved the following result.

Proposition 2: If $m = r^\gamma s^\lambda$ is a unitary hyperperfect number of odd order t, then $2|m$ and either $m = 2^\gamma 3^\lambda$ or $3|t$.

Using the CDC CYBER 750 at the Temple University Computing Center, a search was made for all unitary hyperperfect numbers less than 10^{14} of the form $2^\gamma 3^\lambda$. Only two were found:

$$2 \cdot 3 \ (t = 1) \quad \text{and} \quad 2^5 \cdot 3^2 \ (t = 7).$$

The search required less than one second.

We now drop the restriction that t be odd.

Proposition 3: If $m = r^\gamma s^\lambda = RS$ is a unitary hyperperfect number of order t, then $m > 4t^2$.

Proof: $RS = 1 + t(\sigma^*(RS) - RS - 1) = 1 + t(R + S)$. Therefore, $R > t(1 + R/S)$. Similarly, $S > t(1 + S/R)$, and it follows that

$$RS > t^2(1 + R/S + S/R + 1) > 4t^2.$$

From Proposition 3, we see that all unitary hyperperfect numbers less than 10^{10} and of the form $r^\gamma s^\lambda$ can be found by decomposing $1 + t^2$, for $1 \leqslant t < 50000$, into two factors A and B and then testing $t + A$ and $t + B$ to see if each is a prime power. This was done, and 822 UHP's less than 10^{10} with two components were found. 790 were square-free and, therefore, also HP's. Of the remaining 32 "pure" UHP's, all but one, $3^2 \cdot 2^5$ $(t = 7)$, were of the form $r^\gamma s$ or rs^λ. t was odd for only ten of the 822 numbers, the two largest being

$$2^{13} \cdot 33413 \ (t = 6579) \quad \text{and} \quad 2^{15} \cdot 238037 \ (t = 28803).$$

The complete search took about five minutes of computer time.

4. AN IMPORTANT INEQUALITY

In this section, we shall generalize the inequality of Proposition 3.

Proposition 4: Suppose that m is a unitary hyperperfect (*or* a hyperperfect) number of order t with exactly n prime-power components. Then $m > (nt)^n$.

Proof: Suppose first that $n = 3$ and $m = p^\alpha q^\beta r^\gamma = PQR$, where $P > Q > R$. From (1) [and (2)], it follows easily that

$$PQR > t(PQ + PR + QR).$$

If $A = P/Q$ and $B = P/R$, then

$$P > t(1 + A + B),\ Q > t(1 + B/A + 1/A),\text{ and } R > t(1 + A/B + 1/B).$$

Therefore,

$$m = PQR > t^3(1 + A + B)^3/AB. \tag{5}$$

If $F(x, y) = (1 + x + y)^3/xy$, where $x > 0$ and $y > 0$, then

$$\partial F/\partial x = (1 + x + y)^2(2x - y - 1)/x^2y$$

and

$$\partial F/\partial y = (1 + x + y)^2(2y - x - 1)/xy^2.$$

It follows easily that, if $x > 0$ and $y > 0$, then $F(x, y) \geqslant F(1, 1) = 3^3$. From (5), we have $m > (3t)^3$.

Now suppose that $n = 4$ and $m = p^\alpha q^\beta r^\gamma s^\lambda = PQRS$, where $P > Q > R > S$. From (1) [or (2)],

$$PQRS > t(PQR + PQS + PRS + QRS).$$

If $A = P/Q$, $B = P/R$, and $C = P/S$, then

$$P > t(1 + A + B + C),\ Q > t(1 + B/A + C/A + 1/A),$$

$$R > t(1 + A/B + C/B + 1/B),\text{ and } S > t(1 + A/C + B/C + 1/C).$$

Therefore,

$$m = PQRS > t^4(1 + A + B + C)^4/ABC. \tag{6}$$

If $G(x, y, z) = (1 + x + y + z)^4/xyz$, where $x > 0$, $y > 0$, $z > 0$, then

$$\partial G/\partial x = (1 + x + y + z)^3(3x - y - z - 1)/x^2yz,$$

$$\partial G/\partial y = (1 + x + y + z)^3(3y - x - z - 1)/y^2xz,$$

and

$$\partial G/\partial z = (1 + x + y + z)^3(3z - x - y - 1)/z^2xy.$$

It follows that $G(x, y, z)$ has a minimum at $(1, 1, 1)$ *and* that $G(x, y, z) \geqslant 4^4$ if $x > 0$, $y > 0$, $z > 0$. From (6), we see that $m > (4t)^4$.

A similar argument can be used for any value of n that exceeds 4.

5. THE CASE $a = p^\alpha$

If, in (4) and the right-hand member of (3), we set $a = p^\alpha$, then, since $\sigma^*(p^\alpha) = p^\alpha + 1$, it follows that

$$r^\gamma = (t\sigma^*(p^\alpha) + A)/(p^\alpha - t) \quad \text{and} \quad s^\lambda = (t\sigma^*(p^\alpha) + B)/(p^\alpha - t) \tag{7}$$

where

$$AB = (p^\alpha - t)(1 + tp^\alpha) + t^2(p^\alpha + 1)^2. \tag{8}$$

If $m = p^\alpha r^\gamma s^\lambda$ is a UHP of order t such that $m < 10^9$, then it is easy to see that if p^α is the smallest prime-power component of m, $p^\alpha < 1000$. From Proposition 4, $t < 1000/3$. All solutions of (7) and (8) (with $A < B$) were sought with $2 \leqslant p^\alpha \leqslant 997$, $1 \leqslant t \leqslant 333$, and $p^\alpha r^\gamma s^\lambda < 10^9$. The search yielded nine UHP's less than 10^9. Five of these were given in [2]. The four new ones are:

$$2^6 \cdot 659 \cdot 2693 \ (t = 57); \ 67 \cdot 643 \cdot 79^2 \ (t = 60);$$

$$547 \cdot 569 \cdot 1259 \ (t = 228); \ 7^2 \cdot 79 \cdot 119971 \ (t = 30).$$

The search required about thirty minutes of computer time.

6. THE UHP's LESS THAN 10^9

Let M_n denote the set of all unitary hyperperfect numbers m such that $m < 10^9$ and m has exactly n distinct prime divisors. From Fact 2 in [3], M_1 is empty and, from the searches described in Sections 3 and 5, M_2 and M_3 have 330 and 9 elements, respectively. Since $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 > 10^9$, we see that M_n is empty if $n > 9$. If $n = 8$ or 9, then, from Proposition 4, it follows easily that $t = 1$ so that, if $m \in M_8$ or $m \in M_9$, then m is a unitary perfect number ($\sigma^*(m) = 2m$). Since there are no unitary perfect numbers less than 10^9 with 8 or 9 prime-power components (see [7]), it follows that both M_8 and M_9 are empty.

If $m < 10^9$, then, from Proposition 4, if $n = 4$, then $t \leqslant 44$, if $n = 5$, then $t \leqslant 12$, if $n = 6$, then $t \leqslant 5$, if $n = 7$, then $t \leqslant 2$. Subject to these restrictions on t, and with a restricted so that r^γ is greater than every prime-power component of a while $ar^\gamma s^\lambda < 10^9$, a search was made for solutions of (4). This search required two-and-one-half hours of computer time, and it was found that M_4, M_6, and M_7 are empty, while M_5 has one element, $2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ ($t = 1$). Thus, there are exactly 340 UHP's less than 10^9.

It should, perhaps, be mentioned that while M_4 is empty, one UHP with four prime-power components *was* found: $59 \cdot 149 \cdot 29077 \cdot 10991483959$ ($t = 42$) is both a UHP and an HP (since it is square free). It does not appear in te Riele's lists of HP's and may be the smallest HP with exactly four distinct prime factors.

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CYCLIC COUNTING TRIOS

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(Submitted February 1985)

In this paper, we extend the concept of mutually counting sequences discussed in [1] to the case of three sequences of the same length. Specifically, given the positive integer $n > 1$, we define three sequences,

A: $a(0), a(1), \ldots, a(n-1)$,

B: $b(0), b(1), \ldots, b(n-1)$,

C: $c(0), c(1), \ldots, c(n-1)$,

where $a(i)$ is the multiplicity of i in B, $b(j)$ is the multiplicity of j in C, and $c(k)$ is the multiplicity of k in A. We call the ordered triple (A, B, C) a *cyclic counting trio*, and we make some preliminary observations:

(i) the entries in sequences A, B, and C are nonnegative integers less than n.

(ii) if $S(A) = \sum_{i=0}^{n-1} a(i)$, $S(B) = \sum_{j=0}^{n-1} b(j)$, and $S(C) = \sum_{k=0}^{n-1} c(k)$, then

$$S(A) = S(B) = S(C) = n.$$

(iii) if (A, B, C) is a cyclic counting trio, then so are (B, C, A) and (C, A, B). Such permuted trios will not be considered to be different.

We say that the cyclic counting trio (A, B, C) is *redundant* if A, B, and C are identical. In what follows, we show that there is a unique redundant trio for each $n \geqslant 7$:

$a(0) = n - 4$, $a(1) = 2$, $a(2) = 1$, $a(n-4) = 1$, $a(i) = 0$
for all remaining i.

There are also two redundant trios when $n = 4$, one when $n = 5$, and no others. Furthermore, we show that a nonredundant trio results *only* when $n = 7$:

$a(0) = 4$, $a(1) = 1$, $a(3) = 2$, $a(2) = a(4) = a(5) = a(6) = 0$;

$b(0) = 3$, $b(1) = 3$, $b(4) = 1$, $b(2) = b(3) = b(5) = b(6) = 0$;

$c(0) = 4$, $c(1) = c(2) = c(4) = 1$, $c(3) = c(5) = c(6) = 0$.

As a way to become familiar with the problem, we invite the interested reader to investigate the existence of cyclic counting trios when $n < 7$. We will therefore proceed under the assumption that (A, B, C) is a cyclic counting trio and that $n \geqslant 7$. For future reference, we let

$$n^* = n - \left[\frac{n}{2}\right],$$

and note that

$$n^* = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even,} \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Since $n \geqslant 7$, it follows that $n^* \geqslant 4$.

<u>I. For each $N \geqslant n^*$, $a(N) = 0$ or 1, $b(N) = 0$ or 1, and $c(N) = 0$ or 1.</u>

If $a(N) \geqslant 2$, then N appears at least twice in B. So

$$n = S(B) \geqslant 2N \geqslant 2n^* = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd,} \end{cases}$$

which is only possible when n is even. In this case,

$$N = n^* = \frac{n}{2} \quad \text{and} \quad a\left(\frac{n}{2}\right) = 2,$$

which implies that $n/2$ appears exactly twice in B. Thus, 0 must appear exactly $n - 2$ times in B. Then

$a(0) = n - 2$, $a\left(\frac{n}{2}\right) = 2$, and the $n - 2$ remaining entries of A are 0

$\Rightarrow c(0) = n - 2$, $c(2) = 1$, $c(n - 2) = 1$, and the $n - 3$ remaining entries of C are 0

$\Rightarrow b(0) = n - 3$, $b(1) = 2$, $b(n - 2) = 1$, and the $n - 3$ remaining entries of B are 0

$\Rightarrow a(0) = n - 3$, a contradiction.

Conclude that $a(N) = 0$ or 1, and use a similar argument to show that $b(N) = 0$ or 1 and $c(N) = 0$ or 1.

<u>II. $a(j) = 1$ for at most one $j \geqslant n^*$, $b(k) = 1$ for at most one $k \geqslant n^*$, and $c(\ell) = 1$ for at most one $\ell \geqslant n^*$.</u>

Let N and N' be distinct integers, each $\geqslant n^*$, and suppose that

$$a(N) = a(N') = 1.$$

Then

$$n = S(B) \geqslant N + N' > 2n^* = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd,} \end{cases} \quad \text{a contradiction.}$$

Conclude that there is at most one $j \geqslant n^*$ such that $a(j) = 1$. Similarly, there is at most one $k \geqslant n^*$ such that $b(k) = 1$ and at most one $\ell \geqslant n^*$ such that $c(\ell) = 1$. Note that this result implies that 0 appears at least

$$n - n^* - 1 = \left[\frac{n}{2}\right] - 1$$

times in A, B, and C, so that

$$a(0) \geqslant \left[\frac{n}{2}\right] - 1,\ b(0) \geqslant \left[\frac{n}{2}\right] - 1, \text{ and } c(0) \geqslant \left[\frac{n}{2}\right] - 1.$$

III. If $a(j) = 1$ for some $j \geqslant n^*$, then $b(0) = j$.

Assume that $a(j) = 1$ for some $j \geqslant n^*$. Then j appears exactly once in B, so that $b(j^*) = j$ for some integer j^*. This means that j^* appears j times in C.

$$\text{If } j^* \geqslant 2, \text{ then } n = S(C) \geqslant j^*j \geqslant 2j \geqslant 2n^* = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd,} \end{cases}$$

which is only possible when n is even, $j^* = 2$, and $j = n/2$. Hence, 2 appears $n/2$ times in C, and since $n = S(C)$, it follows that 0 appears $n/2$ times in C as well. Thus, $b(0) = n/2$, $b(2) = n/2$, and the $n - 2$ remaining entries of B are 0. This implies that $a(0) = n - 2$, $a(n/2) = 2$, and the $n - 2$ remaining entries of A are 0, contradicting the assumption that $a(j) = 1$ for some $j \geqslant n^*$. Thus, either $j^* = 1$ or $j^* = 0$.

Assume that $j^* = 1$. Then $b(1) = j$, so that

$$n = S(B) \geqslant b(0) + b(1) \geqslant \left[\frac{n}{2}\right] - 1 + j \geqslant \left[\frac{n}{2}\right] - 1 + n^* = n - 1.$$

This tells us that $b(0) + b(1) = n$ or $b(0) + b(1) = n - 1$. If $b(0) + b(1) = n$, then

$b(0) = n - j$, $b(1) = j$, and the $n - 2$ remaining entries of B are 0

$\Rightarrow a(0) = n - 2$, $a(j) = 1$, $a(n - j) = 1$, and the $n - 3$ remaining entries of A are 0

[If $n - j$ and j were equal, then $a(j) = 2$, a contradiction.]

$\Rightarrow c(0) = n - 3$, $c(1) = 2$, $c(n - 2) = 1$, and the $n - 3$ remaining entries of C are 0

$\Rightarrow b(1) = 1$.

This means that $j = 1$, contradicting the fact that $j \geqslant n^* \geqslant 4$.

If $b(0) + b(1) = n - 1$, then

$$b(0) = n - j - 1,\ b(1) = j,$$

one of the remaining entries of B is 1, and the other $n - 3$ remaining entries of B are 0. If $n - j - 1 = j$, then $a(j) = 2$, a contradiction. If $n - j - 1 = 1$ or 0, then $b(0) = 1$ or 0, contradicting the fact that

$$b(0) \geqslant \left[\frac{n}{2}\right] - 1 \geqslant 2.$$

Hence, the integers 0, 1, j, and $n - j - 1$ are all distinct. This means that 1, j, and $n - j - 1$ each appear once in B, and the $n - 3$ remaining entries of B are 0. So

$a(0) = n - 3$, $a(1) = 1$, $a(n - j - 1) = 1$, $a(j) = 1$,
and the $n - 4$ remaining entries of A are 0

$\Rightarrow c(0) = n - 4,\ c(1) = 3,\ c(n - 3) = 1,$ and the $n - 3$ remaining entries of C are 0

$\Rightarrow b(1) = 1.$

Once again, this means that $j = 1$, a contradiction.

Therefore, $j^* \neq 1$. Conclude that $j^* = 0$, so that if $a(j) = 1$ for some $j \geqslant n^*$, then $b(0) = j$.

IV. If $n > 7$, there exists $j \geqslant n^*$ such that $a(j) = 1$.

Assume that $a(N) = 0$ for all $N \geqslant n^*$. Since $b(0) \geqslant \left[\frac{n}{2}\right] - 1$, two possibilities exist: either $b(0) = \left[\frac{n}{2}\right] - 1$ or $b(0) = \left[\frac{n}{2}\right]$ when n is odd. $\left(\text{If } b(0) = \left[\frac{n}{2}\right] \text{ when } n \text{ is even or if } b(0) > \left[\frac{n}{2}\right], \text{ then } a(N) \neq 0 \text{ for some } N \geqslant n^*.\right)$

Suppose first that $b(0) = \left[\frac{n}{2}\right] - 1$. Then 0 appears exactly $\left[\frac{n}{2}\right] - 1$ times in C, so that there are $n - \left(\left[\frac{n}{2}\right] - 1\right) = n^* + 1$ nonzero entries in C. Consequently,

$$n = S(A) \geqslant \sum_{i=0}^{n^*} i = \frac{n^*(n^* + 1)}{2}.$$

If n is even, then this inequality becomes

$$n \geqslant \frac{\frac{n}{2}\left(\frac{n}{2} + 1\right)}{2}, \text{ which is false for even } n > 6.$$

If n is odd, then this inequality becomes

$$n \geqslant \frac{\left(\frac{n + 1}{2}\right)\left(\frac{n + 1}{2} + 1\right)}{2}, \text{ which is false for odd } n > 3.$$

Suppose next that $b(0) = \left[\frac{n}{2}\right]$ when n is odd. Then 0 appears exactly $\left[\frac{n}{2}\right]$ times in C, so that there are $n - \left[\frac{n}{2}\right] = n^*$ nonzero entries in C. Therefore,

$$n = S(A) \geqslant \sum_{i=0}^{n^*-1} i = \frac{(n^* - 1)n^*}{2} = \frac{\left(\frac{n + 1}{2} - 1\right)\left(\frac{n + 1}{2}\right)}{2},$$

which is false for odd $n > 7$.

The conclusion follows.

V. If $n = 7$, $a(N) = 0$ for all $N \geqslant n^* = 4$, and $b(0) = \left[\frac{n}{2}\right] = 3$, then two cyclic counting trios exist, one of which is nonredundant. (These represent the only set of circumstances that did *not* lead to a contradiction in IV.)

Since $b(0) = 3$ and $S(B) = 7$, it follows that

$$\sum_{k=1}^{6} b(k) = 4.$$

Furthermore, $S(C) = 7$ implies that

$$\sum_{k=1}^{6} kb(k) = 7.$$

For convenience, we will let $\{k_1, k_2, k_3, k_4, k_5, k_6\}$ represent some permutation of $\{1, 2, 3, 4, 5, 6\}$. From II, we know that

$$a(0) \geqslant \left[\frac{n}{2}\right] - 1 = 2.$$

$$a(0) = 2 \Rightarrow b(k_1) = b(k_2) = b(k_3) = b(k_4) = 1,\ b(k_5) = b(k_6) = 0$$
$$\Rightarrow 7 = k_1 + k_2 + k_3 + k_4 \geqslant 10, \quad \text{a contradiction.}$$

$$a(0) = 3 \Rightarrow b(k_1) = 2,\ b(k_2) = b(k_3) = 1,\ b(k_4) = b(k_5) = b(k_6) = 0$$
$$\Rightarrow 7 = 2k_1 + k_2 + k_3 \Rightarrow k_1 = 1,\ k_2 = 2,\ k_3 = 3$$
$$\Rightarrow b(1) = 2,\ b(2) = b(3) = 1,\ b(4) = b(5) = b(6) = 0.$$

Recalling that $b(0) = 3$, we find that

$$a(0) = 3,\ a(1) = 2,\ a(2) = a(3) = 1,\ a(4) = a(5) = a(6) = 0,$$

which, in turn, implies that

$$c(0) = 3,\ c(1) = 2,\ c(2) = c(3) = 1,\ c(4) = c(5) = c(6) = 0.$$

This is the redundant trio predicted for $n = 7$.

$$a(0) = 4 \Rightarrow b(k_1) + b(k_2) = 4,\ b(k_3) = b(k_4) = b(k_5) = b(k_6) = 0.$$

If $b(k_1) = b(k_2) = 2$, then $2k_1 + 2k_2 = 7$, a contradiction. If $b(k_1) = 3$ and $b(k_2) = 1$, then $3k_1 + k_2 = 7$, so that either $k_1 = 2$ and $k_2 = 1$ or $k_1 = 1$ and $k_2 = 4$. In the first case, $b(0) = 3$, $b(1) = 1$, $b(2) = 3$, and the four remaining entries of B are $0 \Rightarrow a(0) = 4$, $a(1) = 1$, $a(3) = 2$, and the four remaining entries of A are $0 \Rightarrow c(0) = 4$, $c(1) = 1$, $c(2) = 1$, $c(4) = 1$, and the three remaining entries of C are $0 \Rightarrow b(1) = 3$, a contradiction.

In the second case, $b(0) = 3$, $b(1) = 3$, $b(4) = 1$, and the four remaining entries of B are $0 \Rightarrow a(0) = 4$, $a(1) = 1$, $a(3) = 2$, and the four remaining entries of A are $0 \Rightarrow c(0) = 4$, $c(1) = 1$, $c(2) = 1$, $c(4) = 1$, and the three remaining entries of C are 0. This is the nonredundant trio predicted at the outset for $n = 7$.

$$a(0) = 5 \Rightarrow b(k_1) = 4,\ b(k_2) = b(k_3) = b(k_4) = b(k_5) = b(k_6) = 0$$
$$\Rightarrow 4k_1 = 7, \text{ a contradiction.}$$

$$a(0) = 6 \Rightarrow b(k_1) = b(k_2) = b(k_3) = b(k_4) = b(k_5) = b(k_6) = 0$$
$$\Rightarrow 0 = 4, \text{ a contradiction.}$$

If $n = 7$ and $a(j) = 1$ for some $j \geqslant n^* = 4$, then it is easy to verify that j must be 4. The cyclic counting trios that subsequently result are permuted versions of the nonredundant one just found. As a result, we may now continue under the assumption that $n > 7$.

VI. $a(n^* - 1) = 0;\ c(0) \geqslant \left[\frac{n}{2}\right]$.

Suppose that $a(n^* - 1) \neq 0$. Then $n^* - 1$ appears at least once in B. Since $b(0) = j$ and since $j \geqslant n^*$ implies $j \neq n^* - 1$, we find that

$$n = S(B) \geqslant j + (n^* - 1) \geqslant n^* + (n^* - 1)$$
$$= 2n^* - 1 = \begin{cases} n - 1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

This tells us that $a(n^* - 1) = 1$, i.e., $n^* - 1$ appears exactly once in B.

If n is even, then some other entry of B is 1 and the $n - 3$ remaining entries of B are 0. Therefore,

$$a(0) = n - 3,\ a(1) = 3, \text{ and the } n - 2 \text{ remaining entries of } A \text{ are } 0$$
$$\Rightarrow c(0) = n - 2,\ c(3) = 1,\ c(n - 3) = 1, \text{ and the } n - 3 \text{ remaining entries of } C \text{ are } 0$$
$$\Rightarrow b(1) = 2, \text{ a contradiction.}$$

If n is odd, then the $n - 2$ remaining entries of B are 0. Therefore,

$$a(0) = n - 2,\ a(1) = 2, \text{ and the } n - 2 \text{ remaining entries of } A \text{ are } 0$$
$$\Rightarrow c(0) = n - 2,\ c(2) = 1,\ c(n - 2) = 1, \text{ and the } n - 3 \text{ remaining entries of } C \text{ are } 0$$
$$\Rightarrow b(1) = 2, \text{ again a contradiction.}$$

Hence, we conclude that $a(n^* - 1) = 0$. Using this fact and the observation following II, we can now assert that 0 appears at least $\left(\left[\frac{n}{2}\right] - 1\right) + 1 = \left[\frac{n}{2}\right]$ times in A, so that $c(0) \geqslant \left[\frac{n}{2}\right]$.

VII. If $c(0) = \left[\frac{n}{2}\right]$, then the only cyclic counting trio that results is the redundant one for $n = 8$.

Since $c(0) = \left[\frac{n}{2}\right]$, it follows that $a(i) \neq 0$ for $1 \leqslant i \leqslant n^* - 2$. Thus, each positive integer less than or equal to $n^* - 2$ appears at least once in B. Recalling that j appears once in B as well, we get

$$n = S(B) \geqslant j + \sum_{i=1}^{n^*-2} i \geqslant n^* + \frac{(n^* - 2)(n^* - 1)}{2},$$

i.e.,

$$n \geqslant \frac{(n^*)^2 - n^* + 2}{2}.$$

If n is odd, then $n^* = (n + 1)/2$ and this inequality leads to $n^2 - 8n + 7 \leqslant 0$, a contradiction for odd $n > 7$. If n is even, then $n^* = n/2$ and this inequality leads to $n^2 - 10n + 8 \leqslant 0$, a contradiction for even $n > 8$.

The case in which $n = 8$ produces the redundant cyclic counting trio with $a(0) = 4$, $a(1) = 2$, $a(2) = 1$, $a(4) = 1$, and $a(i) = 0$ for all remaining i.

VIII. If $c(0) > \left[\frac{n}{2}\right]$, then $b(n^* - 1) = 0$ and $a(0) \geqslant \left[\frac{n}{2}\right]$.

The fact that $c(0) > \left[\frac{n}{2}\right]$ implies that $c(0) \geqslant n^*$. Therefore, $b(k) = 1$ for exactly one integer $k \geqslant n^*$ and $c(0) = k$. If $b(n^* - 1) \neq 0$, then $n^* - 1$ appears at least once in C. Since k appears in C as well, and since

$$k + (n^* - 1) > \left[\frac{n}{2}\right] + (n^* - 1) = n - 1,$$

it follows from $S(C) = n$ that the $n - 2$ remaining entries of C must be 0 and that

$$k = c(0) = \left[\frac{n}{2}\right] + 1.$$

Thus,

$$b(0) = n - 2,\ b\left(\left[\frac{n}{2}\right] + 1\right) = 1,\ b(n^* - 1) = 1,$$
and the $n - 3$ remaining entries of B are 0

$$\Rightarrow a(0) = n - 3,\ a(1) = 2,\ a(n - 2) = 1,$$
and the $n - 3$ remaining entries of A are 0

$$\Rightarrow c(0) = n - 3,\ c(1) = 1,\ c(2) = 1,\ c(n - 3) = 1,$$
and the $n - 4$ remaining entries of C are 0, contradicting the fact that $b(0) = n - 2$.

As a result, we conclude that $b(n^* - 1) = 0$, so that (as in VI), $a(0) \geqslant \left[\frac{n}{2}\right]$.

IX. If $a(0) = \left[\frac{n}{2}\right]$, no cyclic counting trio can be produced; if $a(0) > \left[\frac{n}{2}\right]$, then $c(n^* - 1) = 0$.

The argument used in VII can be employed to show that no cyclic counting trio results when $a(0) = \left[\frac{n}{2}\right]$. (The only possibility, the redundant trio for $n = 8$, is disqualified because $c(0) > \left[\frac{n}{2}\right]$.) If $a(0) > \left[\frac{n}{2}\right]$, then $a(0) \geqslant n^*$. Thus, $c(\ell) = 1$ for exactly one integer $\ell \geqslant n^*$, and $a(0) = \ell$. As in VIII, we can conclude that $c(n^* - 1) = 0$.

At this point, we are left with one case to consider:

$$a(j) = 1,\ b(0) = j;\ b(k) = 1,\ c(0) = k;$$
$$c(\ell) = 1,\ a(0) = \ell, \text{ where } j,\ k,\ \ell \geqslant n^*.$$

X. $j = k = \ell$.

For convenience, let us write $j = n - r$, $k = n - s$, and $\ell = n - t$, where $1 \leqslant r,\ s,\ t \leqslant \left[\frac{n}{2}\right]$.

If $r = 1$, then $j = n - 1$, so $b(0) = n - 1$. This means that $n - 1$ entries of C are 0, contradicting the fact that $c(0) = k$ and $c(\ell) = 1$. If $r = 2$, then $j = n - 2$, so $b(0) = n - 2$. Since $c(0) = k$ and $c(\ell) = 1$, all remaining entries of C must be 0. Then $n = S(C) = k + 1$, implying that $k = n - 1$. Hence, $c(0) = n - 1$, so that $n - 1$ entries of A are 0, contradicting the fact that $a(0) = \ell$ and $a(j) = 1$. Therefore, $r \neq 1$ or 2. Similarly, $s \neq 1$ or 2 and $t \neq 1$ or 2.

Suppose that $a(i) \neq 0$ for some integer $i \geqslant r - 1$, where $i \neq j$. (Note that $i \geqslant 2$.) Then

$$n = S(B) \geqslant i + j + 1 \geqslant r - 1 + j + 1 = r + j = n,$$

which implies that $i = r - 1$ and that the $n - 3$ remaining entries of B are 0. Hence,

$$a(0) = n - 3,\ a(1) = 1,\ a(j) = 1,\ a(r - 1) = 1, \text{ and the } n - 4 \text{ remaining entries of } A \text{ are } 0$$

$$\Rightarrow c(0) = n - 4,\ c(1) = 3,\ c(n - 3) = 1, \text{ and the } n - 3 \text{ remaining entries of } C \text{ are } 0$$

$$\Rightarrow b(0) = n - 3,\ b(1) = 1,\ b(3) = 1,\ b(n - 4) = 1, \text{ and the } n - 4 \text{ remaining entries of } B \text{ are } 0$$

$$\Rightarrow a(0) = n - 4, \text{ a contradiction.}$$

Consequently, $a(i) = 0$ for all integers $i \geqslant r - 1$, where $i \neq j$. In a similar manner, we can show that

$$b(i) = 0 \text{ for all integers } i \geqslant s - 1, \text{ where } i \neq k,$$

and

$$c(i) = 0 \text{ for all integers } i \geqslant t - 1, \text{ where } i \neq \ell.$$

Thus,

$$c(0) \geqslant ((n-1) - (r-1) + 1) - 1 = n - r, \quad \Rightarrow k \geqslant j$$
$$a(0) \geqslant ((n-1) - (s-1) + 1) - 1 = n - s, \quad \Rightarrow \ell \geqslant k$$
$$b(0) \geqslant ((n-1) - (t-1) + 1) - 1 = n - t, \quad \Rightarrow j \geqslant \ell$$

These three inequalities together imply that $j = k = \ell$.

XI. A unique redundant cyclic counting trio exists for $n > 7$.

From X, we now know that for some $j \geqslant n^*$,

$$a(j) = b(j) = c(j) = 1 \quad \text{and} \quad a(0) = b(0) = c(0) = j.$$

Since $b(i) = 0$ whenever $i \geqslant r - 1$ and $i \neq j$, this accounts for $n - r = j$ zeros in B. Because $a(0) = j$, it follows that $b(i) \neq 0$ for $1 \leqslant i \leqslant r - 2$. Then

$$n = S(B) = j + 1 + \sum_{i=1}^{r-2} b(i),$$

which implies that

$$\sum_{i=1}^{r-2} b(i) = n - j - 1 = r - 1.$$

If $r = 3$, then $b(1) = 2$, so that B consists of one entry of $j = n - 3$, one entry of 1, one entry of 2, and $n - 3$ entries of 0. Therefore,

$$a(0) = n - 3,\ a(1) = 1,\ a(2) = 1,\ a(n-3) = 1,$$
$$\text{and the } n - 4 \text{ remaining entries of } A \text{ are } 0$$
$$\Rightarrow c(0) = n - 4, \text{ contradicting the fact that } c(0) = j = n - 3.$$

So $r > 3$. Then

$$\sum_{i=1}^{r-2} b(i) = r - 1$$

implies that one of the terms in the sum is 2 and each of the $r - 3$ others is 1. Thus, B consists of one entry of j, one entry of 2, $r - 2$ entries of 1, and j entries of 0. Then

$$a(0) = j,\ a(1) = r - 2,\ a(2) = 1,\ a(j) = 1,$$
$$\text{and the } n - 4 \text{ remaining entries of } A \text{ are } 0,$$

which implies that $c(0) = n - 4$.

If $j \neq n - 4$, then the resulting contradiction indicates that no cyclic counting trio can be produced; if $j = n - 4$ (i.e., if $r = 4$), we have

$$a(0) = n - 4,\ a(1) = 2,\ a(2) = 1,\ a(n-4) = 1,$$
$$\text{and the } n - 4 \text{ remaining entries of } A \text{ are } 0$$
$$\Rightarrow c(0) = n - 4,\ c(1) = 2,\ c(2) = 1,\ c(n-4) = 1,$$
$$\text{and the } n - 4 \text{ remaining entries of } C \text{ are } 0$$

$b(0) = n - 4$, $b(1) = 2$, $b(2) = 1$, $b(n - 4) = 1$,
and the $n - 4$ remaining entries of B are 0.

This is the previously mentioned cyclic counting trio for $n > 7$.

REFERENCE

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PELL POLYNOMIAL MATRICES

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1. INTRODUCTION

By defining certain matrices of order 2, we are enabled to derive fresh properties of Pell polynomials $P_n(x)$ and Pell-Lucas polynomials $Q_n(x)$ additional to those obtained by us in [5]. Our work, in summarized form, is an adaptation and extension of some ideas of Walton [6], based on earlier work by Hoggatt and Bicknell-Johnson [2].*

The Pell and Pell-Lucas polynomials which are defined, respectively, by the recurrence relations

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x),\ P_0(x) = 0,\ P_1(x) = 1 \tag{1.1}$$

and

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x),\ Q_0(x) = 2,\ Q_1(x) = 2x \tag{1.2}$$

and some of their basic properties which will be assumed without specific reference, are discussed by us in [3].

To conserve space, we offer our results in a condensed form. This approach has the added virtue of emphasizing techniques.

Convention: For visual ease and simplicity, we abbreviate the functional notation, e.g., $P_n(x) = P_n$, $Q_n(x) = Q_n$.

2. THE ASSOCIATED MATRICES J AND L

Let

$$J = \begin{bmatrix} P_4 & P_2 \\ -P_2 & -P_0 \end{bmatrix}, \tag{2.1}$$

whence, by induction,

$$J^n = P_2^{n-1} \begin{bmatrix} P_{2n+2} & P_{2n} \\ -P_{2n} & -P_{2n-2} \end{bmatrix}. \tag{2.2}$$

Equating corresponding elements in $J^{m+n} = J^m J^n$ gives

$$P_2 P_{2(m+n)} = P_{2(m+1)} P_{2n} - P_{2m} P_{2(n-1)}. \tag{2.3}$$

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*Walton was given a copy of the Hoggatt and Bicknell-Johnson paper while he was writing his thesis. This paper was only published in 1980.

The characteristic equation of J is

$$\lambda^2 - P_4\lambda + P_2^2 = 0, \tag{2.4}$$

so, by the Cayley-Hamilton theorem,

$$J^2 = P_4J - P_2^2I. \tag{2.5}$$

Extending (2.5), we have

$$J^{2n+j} = (P_4J - P_2^2I)^n J^j, \tag{2.6}$$

whence, by (2.2),

$$P_{4n+2j} = \sum_{r=0}^{n}(-1)^r\binom{n}{r} Q_2^{n-r}P_{2n-2r+2j}. \tag{2.7}$$

From (2.5),

$$P_4^nJ^n = (J^2 + P_2^2I)^n. \tag{2.8}$$

Equating corresponding matrix elements and simplifying, we get

$$\sum_{r=0}^{n}\binom{n}{r}P_{4r} = Q_2^nP_{2n}. \tag{2.9}$$

Consider, with appeal to (2.5),

$$(J + P_2I)^2 = (P_4 + 2P_2)J = 8x(x^2 + 1)J. \tag{2.10}$$

Hence,

$$\{8x(x^2 + 1)\}^nJ^n = \sum_{r=0}^{2n}\binom{2n}{r}P_2^{2n-r}J^r. \tag{2.11}$$

Now equate corresponding elements. Simplification then yields

$$\sum_{r=0}^{2n}\binom{2n}{r}P_{2r} = 4^n(x^2 + 1)^nP_{2n}. \tag{2.12}$$

Next write

$$L = \begin{bmatrix} P_3 & P_1 \\ -P_1 & -P_{-1} \end{bmatrix} \quad (\text{so } |L| = |J| = -4x^2). \tag{2.13}$$

Then, by (2.2) and (2.13),

$$J^nL = P_2^n\begin{bmatrix} P_{2n+3} & P_{2n+1} \\ -P_{2n+1} & -P_{2n-1} \end{bmatrix}, \tag{2.14}$$

whence

$$J^{2n+j}L = \sum_{r=0}^{n} (-1)^r \binom{n}{r} P_2^{2n} P_4^{n-r} J^{n-r+j} L, \tag{2.15}$$

and so [cf. (2.7)]

$$P_{4n+2j+1} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q_2^{n-r} P_{2n-2r+2j+1}. \tag{2.16}$$

From (2.5),

$$P_4^n J^n L = \sum_{r=0}^{n} \binom{n}{r} P_2^{2n-2r} J^{2r} L, \tag{2.17}$$

whence, by (2.14),

$$\sum_{r=0}^{n} \binom{n}{r} P_{4r+1} = Q_2^n P_{2n+1}. \tag{2.18}$$

Equation (2.10) leads to

$$(J + P_2 I)^{2n} L = \{8x(x^2 + 1)\}^n J^n L, \tag{2.19}$$

from which

$$\sum_{r=0}^{2n} \binom{2n}{r} P_{2r+1} = 4^n (x^2 + 1)^n P_{2n+1}. \tag{2.20}$$

Again from (2.10),

$$(J + P_2 I)^{2n+1} = \{8x(x^2 + 1)\}^n J^n (J + P_2 I). \tag{2.21}$$

Corresponding entries, when equated, produce

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} P_{2r} = 4^n (x^2 + 1)^n Q_{2n+1}. \tag{2.22}$$

Multiply both sides of (2.21) by L. In the usual way,

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} P_{2r+1} = 4^n (x^2 + 1)^n Q_{2n+2}. \tag{2.23}$$

Next, from (2.5), after some algebraic manipulation,

$$\{J - (4x^3 + 2x)I\}^{2n} = (4x^4)^n \cdot 4^n (x^2 + 1)^n I, \tag{2.24}$$

so that

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (2x^2 + 1)^r P_{4n-2r} = 0 \tag{2.25}$$

and

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (2x^2 + 1)^r P_{4n-2r+2} = P_2^{2n+1} (x^2 + 1)^n. \tag{2.26}$$

Now multiply (2.24) by L. Consequently,

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (2x^2 + 1)^r P_{4n-2r+1} = x^{2n} \{4(x^2 + 1)\}^n. \tag{2.27}$$

Next, multiply both sides of (2.24) by $J - (4x^3 + 2x)I$. It follows that

$$\sum_{r=0}^{2n+1}(-1)^r\binom{2n+1}{r}(2x^2+1)^r P_{4n-2r+3} = \tfrac{1}{2}(2x)^{2n+2}(x^2+1)^n. \tag{2.28}$$

Other results for P_n, some of them quite complicated, may be found in [4], e.g., formulas obtained by considering J^{ns+j} and $J^{ns}L$. One such formula is

$$P_{2n}^s P_{2s+1} = \sum_{r=0}^{s}\binom{s}{r}P_2^{s+r}P_{2n-2}^r P_{2n(s-r)+1}. \tag{2.29}$$

Observe, in passing, that induction leads to

$$L^n = P_2^{n-1}\begin{bmatrix} P_{n+2} & P_n \\ -P_n & -P_{n-2}\end{bmatrix}. \tag{2.30}$$

3. THE MATRICES K AND M

We are able to derive other identities by defining

$$K = \begin{bmatrix} P_8 & P_4 \\ -P_4 & -P_0\end{bmatrix},\quad M = \begin{bmatrix} P_5 & P_1 \\ -P_1 & -P_{-3}\end{bmatrix}, \tag{3.1}$$

and following the techniques used above. The results are listed:

$$K^n = P_4^{n-1}\begin{bmatrix} P_{4n+4} & P_{4n} \\ -P_{4n} & -P_{4n-4}\end{bmatrix} \tag{3.2}$$

$$P_4P_{4(m+n)} = P_{4(m+1)}P_{4n} - P_{4m}P_{4(n-1)} \tag{3.3}$$

$$K^{2n} = (P_8K - P_4^2I)^n \tag{3.4}$$

$$P_4^nP_{8n} = \sum_{r=0}^{n}(-1)^r\binom{n}{r}P_8^{n-r}P_4^rP_{4(n-r)} \tag{3.5}$$

$$P_4^nP_{8n+4} = \sum_{r=0}^{n}(-1)^r\binom{n}{r}P_8^{n-r}P_4^rP_{4(n+1-r)} \tag{3.6}$$

$$P_8^nP_{4n} = P_4^n\sum_{r=0}^{n}\binom{n}{r}P_{8r} \tag{3.7}$$

$$\sum_{r=0}^{2n}\binom{2n}{r}P_{4r} = Q_2^{2n}P_{4n} \tag{3.8}$$

$$\sum_{r=0}^{2n+1}\binom{2n+1}{r}P_{4r} = Q_2^{2n+1}P_{4n+2} \tag{3.9}$$

$$K^nM = P_4^n\begin{bmatrix} P_{4n+5} & P_{4n+1} \\ -P_{4n+1} & -P_{4n-3}\end{bmatrix} \tag{3.10}$$

$$\sum_{r=0}^{2n}\binom{2n}{r}P_{4r+1} = Q_2^{2n}P_{4n+1} \tag{3.11}$$

$$\sum_{r=0}^{2n+1}\binom{2n+1}{r}P_{4r+1} = Q_2^{2n+1}P_{4n+3} \tag{3.12}$$

$$M^n = P_4^{n-1}\begin{bmatrix} P_{n+4} & P_n \\ -P_n & -P_{n-4} \end{bmatrix} \tag{3.13}$$

Additional information on the matrix K is given in Mahon [4].

4. THE MATRICES N AND U

In like manner, by defining the matrices

$$N = \begin{bmatrix} P_6 & P_2 \\ -P_2 & -P_{-2} \end{bmatrix}, \quad U = \begin{bmatrix} P_7 & P_3 \\ -P_3 & -P_{-1} \end{bmatrix}, \tag{4.1}$$

and again using techniques similar to those above, we prove further identities which are listed:

$$K^nN = P_4^n\begin{bmatrix} P_{4n+6} & P_{4n+2} \\ -P_{4n+2} & -P_{4n-2} \end{bmatrix} \tag{4.2}$$

$$\sum_{r=0}^{2n}\binom{2n}{r}P_{4r+2} = Q_2^{2n}P_{4n+2} \tag{4.3}$$

$$\sum_{r=0}^{2n+1}\binom{2n+1}{r}P_{4r+2} = Q_2^{2n+1}P_{4n+4} \tag{4.4}$$

$$K^nU = P_4^n\begin{bmatrix} P_{4n+7} & P_{4n+3} \\ -P_{4n+3} & -P_{4n-1} \end{bmatrix} \tag{4.5}$$

$$\sum_{r=0}^{2n}\binom{2n}{r}P_{4r+3} = Q_2^{2n}P_{4n+3} \tag{4.6}$$

$$\sum_{r=0}^{2n+1}\binom{2n+1}{r}P_{4r+3} = Q_2^{2n+1}P_{4n+5} \tag{4.7}$$

See [4] for further, more complicated results.

From what has been said in the above sections, it appears that there is a chain of matrices of the type given which would produce formulas of (perhaps) minor interest.

5. THE MATRIX W

We now introduce a matrix having the property of generating Pell and Pell-Lucas polynomials simultaneously. It was suggested by a problem proposed by Ferns [1].

$$W = \begin{bmatrix} 2x & 1 \\ 4(x^2 + 1) & 2x \end{bmatrix} \qquad (|W| = -4). \tag{5.1}$$

Induction leads to

$$W^n = 2^{n-1} \begin{bmatrix} Q_n & P_n \\ 4(x^2 + 1)P_n & Q_n \end{bmatrix}. \tag{5.2}$$

Then

$$W^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2^n \begin{bmatrix} P_n \\ Q_n \end{bmatrix}. \tag{5.3}$$

Now

$$W^{m+n} = 2^{m+n-1} \begin{bmatrix} Q_{m+n} & P_{m+n} \\ 4(x^2 + 1)P_{m+n} & Q_{m+n} \end{bmatrix} \text{ by } (5.2) \tag{5.4}$$

$$= 2^{m+n-2} \begin{bmatrix} Q_m & P_m \\ 4(x^2 + 1)P_m & Q_m \end{bmatrix} \begin{bmatrix} Q_n & P_n \\ 4(x^2 + 1)P_n & Q_n \end{bmatrix} \text{ by (5.2) also.}$$

Corresponding entries give formulas (3.18) and (3.19) for P_{m+n} and Q_{m+n}, respectively, appearing in [3].

The characteristic equation for W is

$$\lambda^2 - 4x\lambda - 4 = 0, \tag{5.5}$$

whence, by the Cayley-Hamilton theorem,

$$W^2 - 4xW - 4I = 0, \tag{5.6}$$

so

$$W^{2n} = 4^n(xW + I)^n. \tag{5.7}$$

Algebraic manipulation, after multiplication by W^j, produces the formulas for P_{2n+j} and Q_{2n+j}, (3.28) and (3.29), in [3].

Induction, with the aid of (5.6), yields

$$W^n = 2^{n-1}(P_n W + 2P_{n-1}I). \tag{5.8}$$

Considering W^{ns+j} and tidying up, we have

$$W^{ns+j} = 2^{(n-1)s} \sum_{r=0}^{s} \binom{s}{r} P_n^r P_{n-1}^{s-r} 2^{s-r} W^{r+j}, \tag{5.9}$$

giving

$$P_{ns+j} = \sum_{r=0}^{s} \binom{s}{r} P_n^r P_{n-1}^{s-r} P_{r+j}, \tag{5.10}$$

and

$$Q_{ns+j} = \sum_{r=0}^{s} \binom{s}{r} P_n^r P_{n-1}^{s-r} Q_{r+j}. \tag{5.11}$$

Further,

$$\begin{aligned}\sum_{r=0}^{2n} \binom{2n}{r} (xW)^{r+j} 2^{2n-r} &= (xW + 2I)^{2n} W^j \\ &= (x^2W^2 + 4xW + 4I)^n W^j \\ &= (x^2 + 1)^n W^{2n+j}, \quad \text{by (5.6).}\end{aligned} \tag{5.12}$$

Accordingly,

$$\sum_{r=0}^{2n} \binom{2n}{r} x^r P_{r+j} = (x^2 + 1)^n P_{2n+j} \tag{5.13}$$

and

$$\sum_{r=0}^{2n} \binom{2n}{r} x^r Q_{r+j} = (x^2 + 1)^n Q_{2n+j}. \tag{5.14}$$

From (5.12),

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} (xW)^r 2^{2n+1-r} = (x^2 + 1)^n W^{2n} (xW + 2I) \tag{5.15}$$

and we deduce

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} x^r P_r = \tfrac{1}{2}(x^2 + 1)^n Q_{2n+1} \tag{5.16}$$

and

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} x^r Q_r = 2(x^2 + 1)^{n+1} P_{2n+1}. \tag{5.17}$$

Also, from (5.6),

$$(4xW)^n = (W^2 - 4I)^n, \tag{5.18}$$

whence

$$(2x)^n P_n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} P_{2n-2r} \tag{5.19}$$

and

$$(2x)^n Q_n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q_{2n-2r}. \tag{5.20}$$

Let us revert momentarily to (5.8).

Rearrange (5.8) and raise to the s^{th} power to obtain

$$2^{(n-1)s} P_n^s W^s = \sum_{r=0}^{s} (-1)^r \binom{s}{r} 2^{nr} P_{n-1}^r W^{n(s-r)}. \tag{5.21}$$

Identities such as

$$P_n^s Q_s = \sum_{r=0}^{s} (-1)^r \binom{s}{r} P_{n-1}^r Q_{n(s-r)} \tag{5.22}$$

and

$$P^s P_{s+j} = \sum_{r=0}^{s} (-1)^r \binom{s}{r} P_{n-1}^r P_{n(s-r)+j} \tag{5.23}$$

flow from (5.21).

The above information, together with complementary material in [5], offers some details of the finite summation of Pell and Pell-Lucas polynomials by means of matrices. Clearly, the topics treated are far from complete. For instance, (5.1) extends naturally to

$$W_m = \begin{bmatrix} Q_m & 1 \\ Q_m^2 + 4(-1)^{m-1} & Q_m \end{bmatrix} \quad [|W_m| = 4(-1)^m], \tag{5.24}$$

from which new properties of our polynomials may be derived. Enough has been said, however, to indicate techniques for further development.

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REPRESENTING $\binom{2n}{n}$ AS A SUM OF SQUARES

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INTRODUCTION

A well-known theorem of Lagrange [4, p. 302] states that every natural number can be represented as a sum of at most four squares. For each integer, k, such that $1 \leqslant k \leqslant 4$, let S_k be the set of natural numbers, n, such that $\binom{2n}{n}$ is a sum of k (but not fewer) squares. We show that S_1 is empty, $S_2 = \{1, 3\}$, while S_3 and S_4 are both infinite.

PRELIMINARIES

Let p denote a prime.

Definition 1: $o_p(n) = k$ if $p^k | n$, $p^{k+1} \nmid n$

Definition 2: $t_p(n) = \sum_{i=0}^{r} a_i$ if $n = \sum_{i=0}^{r} a_i p^i$, with $0 \leqslant a_i < p$ for each i.

$$o_p(ab) = o_p(a) + o_p(b) \tag{1}$$

$$o_p(n!) = \frac{n - t_p(n)}{p - 1} \tag{2}$$

$$o_p\left(\binom{n}{k}\right) = \frac{t_p(k) + k_p(n - k) - t_p(n)}{p - 1} \tag{3}$$

$$t_p(ap^j) = t_p(a) \text{ for all } a, j \tag{4}$$

$$o_2\left(\binom{2n}{n}\right) = t_2(n) \tag{5}$$

$$n \neq a^2 + b^2 + c^2 \text{ iff } n = 2^{2k}(8m + 7) \text{ with } k \geqslant 0, m \geqslant 0 \tag{6}$$

$n \neq a^2 + b^2$ iff there is a prime, p, such that

$$p \equiv 3 \pmod 4 \text{ and } o_p(n) \text{ is odd.} \tag{7}$$

Remarks: (1) follows from Definition 1. (2) is [2, p. 131, Problem 7]. (3) follows from (1) and (2). (4) follows from Definition 2. (5) follows from (3) and (4). (6) is stated in [4, p. 311]. (7) is [4, p. 299, Theorem 366]. $t_2(n)$ is denoted $\#_1(n)$ in [5].

THE MAIN THEOREMS

Theorem 1: If $n \neq 1, 3$, then there is a prime, p, such that $p \equiv 3 \pmod 4$ and $n < p < 2n$.

Proof: Breusch [1] proved the conclusion for $n \geqslant 7$. If $n = 2$, then $p = 3$; if $4 \leqslant n \leqslant 6$, then $p = 7$.

Theorem 2: S_1 is empty; $S_2 = \{1, 3\}$.

Proof: If $2 \leqslant n < p < 2n$, then $2n < 2p$, so $o_p\left(\binom{2n}{n}\right) = 1$. Therefore, (7) and Theorem 1 imply $S_1 \cup S_2 \subseteq \{1, 3\}$. Since

$$\binom{2}{1} = 1^2 + 1^2, \text{ and } \binom{6}{3} = 4^2 + 2^2,$$

the conclusion now follows.

Remark: That S_1 is empty also follows from the theorem of P. Erdos [3], which states that $\binom{n}{k}$ is not a power if $k > 3$.

Definition 3: If $n = 2^k m$, $k \geqslant 0$, m odd, then $f(n)$ is the least positive residue of $m \pmod 8$.

Lemma 1: If m is odd, then $f(m) \equiv m \pmod 8$.

Proof: The proof follows from the hypothesis and Definition 3.

Lemma 2: If $f(a) \equiv f(b) \pmod 8$, then $f(a) = f(b)$.

Proof: The proof follows from the hypothesis and Definition 3.

Lemma 3: $f(ab) \equiv f(a)f(b) \pmod 8$.

Proof: Let $a = 2^c j$, $b = 2^d k$, with $c \geqslant 0$, $d \geqslant 0$, jk odd. Lemma 1 implies

$$f(jk) \equiv jk \equiv f(j)f(k) \pmod 8.$$

Now $f(ab) = f(2^{c+d}jk) = f(jk)$, while $f(a)f(b) = f(j)f(k)$, so

$$f(ab) \equiv f(a)f(b) \pmod 8.$$

Lemma 4: If $f(b) = 1$, then $f(ab) = f(a)$.

Proof: The proof follows from the hypothesis and Lemmas 3 and 2.

Lemma 5: $f(n^2) = 1$.

Proof: If $n = 2^k m$, $k \geqslant 0$, m odd, then $f(n^2) = f(2^{2k}m^2) = f(m^2)$. Now, Lemma 1 implies $f(m^2) \equiv m^2 \equiv 1 \pmod 8$. But $f(1) = 1$, so we have $f(n^2) \equiv f(1) \pmod 8$. Now, Lemma 2 implies $f(n^2) = f(1) = 1$.

Lemma 6: $f\left(\binom{2n}{n}\right) = f((2n)!)$.

Proof: The proof follows from Lemmas 4 and 5, since $(2n)! = \binom{2n}{n}(n!)^2$.

Definition 4: Let $g(n) = f(n!)$.

Table 1 lists $g(n)$ and $t_2(n)$ for each n such that $1 \leqslant n \leqslant 200$.

Table 1

n	$g(n)$	$t_2(n)$	n	$g(n)$	$t_2(n)$	n	$g(n)$	$t_2(n)$	n	$g(n)$	$t_2(n)$
1	1	1	51	5	4	101	1	4	151	7	5
2	1	1	52	1	3	102	3	4	152	5	3
3	3	2	53	5	4	103	5	5	153	5	4
4	3	1	54	7	4	104	1	3	154	1	4
5	7	2	55	1	5	105	1	4	155	3	5
6	5	2	56	7	3	106	5	4	156	5	4
7	3	3	57	7	4	107	7	5	157	1	5
8	3	1	58	3	4	108	5	4	158	7	5
9	3	2	59	1	5	109	1	5	159	1	6
10	7	2	60	7	4	110	7	5	160	5	2
11	5	3	61	3	5	111	1	6	161	5	3
12	7	2	62	5	5	112	7	3	162	5	3
13	3	3	63	3	6	113	7	4	163	7	4
14	5	3	64	3	1	114	7	4	164	7	3
15	3	4	65	3	2	115	5	5	165	3	4
16	3	1	66	3	2	116	1	4	166	1	4
17	3	2	67	1	3	117	5	5	167	7	5
81	3	2	68	1	2	118	7	5	168	3	3
91	1	3	69	5	3	119	1	7	169	3	4
20	5	2	70	7	3	120	7	4	170	7	4
21	1	3	71	1	4	121	7	5	171	5	5
22	3	3	72	1	2	122	3	5	172	7	4
23	5	4	73	1	3	123	1	6	173	3	5
24	7	2	74	5	3	124	7	5	174	5	5
25	7	3	75	7	4	125	3	6	175	3	6
26	3	3	76	5	3	126	5	6	176	1	3
27	1	4	77	1	4	127	3	7	177	1	4
28	7	3	78	7	4	128	3	1	178	1	4
29	3	4	79	1	5	129	3	2	179	3	5
30	5	4	80	5	2	130	3	2	180	7	4
31	3	5	81	5	3	131	1	3	181	3	5
32	3	1	82	5	3	132	1	2	182	1	5
33	3	2	83	7	4	133	5	3	183	7	6
34	3	2	84	3	3	134	7	3	184	1	4
35	1	3	85	7	4	135	1	4	185	1	5
36	1	2	86	5	4	136	1	2	186	5	5
37	5	3	87	3	5	137	1	3	187	7	6
38	7	3	88	1	3	138	5	3	188	1	5
39	1	4	89	1	4	139	7	4	189	5	6
40	5	2	90	5	4	140	5	3	190	3	6
41	5	3	91	7	5	141	1	4	191	5	7
42	1	3	92	1	4	142	7	4	192	7	2
43	3	4	93	5	5	143	1	5	193	7	3
44	1	3	94	3	5	144	1	2	194	7	3
45	5	4	95	5	6	145	1	3	195	5	4
46	3	4	96	7	2	146	1	3	196	5	3
47	5	5	97	7	3	147	3	4	197	1	4
48	7	2	98	7	3	148	7	3	198	3	4
49	7	3	99	5	4	149	3	4	199	5	5
50	7	3	100	5	3	150	1	4	200	5	3

Theorem 3: $\binom{2n}{n} \neq a^2 + b^2 + c^2$ iff $t_2(n)$ is even and $g(2n) = 7$.

Proof: The proof follows from (5), (6), Lemma 6, and Definition 4.

Theorem 4: Let k be a nonnegative integer. Then

(a) $g(8k) = g(4k)$; (b) $g(8k + 2) = g(4k + 1)$;

(c) $g(8k + 4) \equiv 3g(4k + 2) \pmod 8$; (d) $g(8k + 6) = 8 - g(4k + 3)$.

Proof of (a): By Definition 4 and Lemma 4, it suffices to show that

$$f\left(\frac{(8k)!)}{(4k)!)}\right) = 1 \text{ for all } k \geqslant 0.$$

We proceed by induction on k. The statement is trivially true for $k = 0$. Now

$$f\left(\frac{(8(k+1)!)}{(4(k+1)!)}\right) = f\left(\frac{(8k+8)!)}{(4k+4)!)}\right) = f\left(\frac{(8k+8)!(4k)!(8k)!}{(8k)!(4k+4)!(4k)!}\right)$$

$$= f\left(\frac{(8k+8)!(4k)!}{(8k)!(4k+4)!}\right)$$

by induction hypothesis and Lemma 4. But

$$f\left(\frac{(8k+8)!(4k)!}{(8k)!(4k+4)!}\right)$$

$$= f\left(\frac{(8k+8)(8k+7)(8k+6)(8k+5)(8k+4)(8k+3)(8k+2)(8k+1)}{(4k+4)(4k+3)(4k+2)(4k+1)}\right)$$

$$= f(2^4(8k+7)(8k+5)(8k+3)(8k+1)) = f(7 \cdot 5 \cdot 3 \cdot 1) = f(105) = 1.$$

Parts (b), (c), and (d) may be proved in similar fashion.

Theorem 5: $$g(2m) = \begin{cases} g(m) & \text{if } m \equiv 1 \pmod 4, \\ 8 - g(m) & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Proof: The proof follows from Theorem 4.

Theorem 6: If either (i) $m \equiv 1 \pmod 4$ and $g(m) = 5$, or (ii) $m \equiv -1 \pmod 4$ and $g(m) = 3$, then $g(2m) = 5$ and $g(4m) = 7$.

Proof: The hypothesis and Theorem 5 imply $g(2m) = 5$. Now $m = 4r \pm 1$, so

$$g(4m) = g(4(4r \pm 1)) = g(8(2r) \pm 4) \equiv 3g(4(2r) \pm 2) \equiv 3g(2(4r \pm 1)),$$

$$3g(2m) \equiv 3 \cdot 5 \equiv 7 \pmod 8,$$

by Theorem 4(c). Therefore, $g(4m) = 7$.

Theorem 7: If m is odd and $g(2m) = 5$, then $g(2^k m) = 7$ for all $k \geqslant 2$.

Proof: (Induction on k.) By Theorem 6, the statement is true for $k = 2$. If $k > 2$, then $g(2^k m) = g(8(2^{k-3}m)) = g(4(2^{k-3}m)) = g(2^{k-1}m) = 7$, by Theorem 4(a) and the induction hypothesis.

Theorem 8: S_3 is infinite, that is, there exist infinitely many n such that

$$\binom{2n}{n} = a^2 + b^2 + c^2.$$

Proof: If $m \geqslant 2$, then $t_2(2^{2m-1} - 1) = 2m - 1$, and $2^{2m-1} - 1 > 3$, so that Theorems 2 and 3 imply that $2^{2m-1} - 1$ belongs to S_3.

Theorem 9: S_4 is infinite, that is, there exist infinitely many n such that

$$\binom{2n}{n} \neq a^2 + b^2 + c^2.$$

Proof: By Theorems 3, 6, and 7, it suffices to find an m such that (i) $t_2(m)$ is even, and either (ii) $m \equiv 1 \pmod 4$ and $g(m) = 5$, or (iii) $m \equiv 3 \pmod 4$ and $g(m) = 3$. Examining Table 1, we find the following such $m < 200$:

$$m \in \{3, 15, 43, 53, 63, 147, 153, 175, 189\}.$$

Concluding Remarks: Let d_n be the asymptotic density of S_n, where $1 \leqslant n \leqslant 4$. Since $S_1 \cup S_2$ is finite, by Theorem 2, we have $d_1 = d_2 = 0$, so that $d_3 + d_4 = 1$. If n is a randomly chosen natural number, let A be the event that $t_2(n)$ is even; let B be the event that $g(2n) = 7$. It is easily seen that $\Pr(A) = \frac{1}{2}$. Now $d_4 = \Pr(n \in S_4) = \Pr(A \cap B) \leqslant \Pr(A) = \frac{1}{2}$. Therefore, $d_3 \geqslant \frac{1}{2}$. Table 1 suggests that A and B are independent, and that $\Pr(B) = \frac{1}{4}$. Therefore,

Conjecture: $d_4 = 1/8$, $d_3 = 7/8$.

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ON NONSQUARE POWERFUL NUMBERS

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1. INTRODUCTION

As introduced by Golomb in [1], a *powerful number* n is a positive integer which has no prime appearing to the first power in its canonical prime decomposition; i.e., if a prime p divides n, then p^2 divides n. If n and m are powerful numbers, then $n - m$ is said to be a *proper difference* of powerful numbers if g.c.d. $(n, m) = 1$. Golomb [1] conjectured that there are infinitely many integers which are not proper differences of powerful numbers. This was disproved by McDaniel in [3], wherein he gave an existence proof for the fact that *every* nonzero integer is representable in infinitely many ways as a proper difference of two powerful numbers. We provided a simple proof of this result plus an effective algorithm for finding such representations in [4]. However, in both our proof and McDaniel's proof one of the powerful numbers in such a representation is always a perfect square, except possibly when $n \equiv 2 \pmod 4$. Recently, Vanden Enyden [6] proved that also in the $n \equiv 2 \pmod 4$ case, one of the powerful numbers is always a square. We established in [4] that every *even* integer is representable in infinitely many ways as a *proper nonsquare difference of powerfuls*; i.e., as a proper difference of two powerful numbers neither of which is a perfect square. At this time, the only odd integer known to have such a representation is the integer 1 (see [7]). It is the purpose of this paper to complete the task; viz., to prove that every odd integer greater than 1 (hence every integer) is a proper nonsquare difference of powerfuls, and to provide an algorithm for finding such representations. Therefore, this paper establishes the fact that every nonzero integer is representable in infinitely many ways as a proper difference of two powerful numbers where either one of the powerful numbers is a perfect square and the other is not, or neither one of them is a perfect square.

For other work done on powerful numbers we refer the reader to our list of references.

2. NONSQUARE POWERFUL NUMBERS

To prove our main result, we will need the following lemma, which we state without proof since it is immediate from the binomial theorem.

Lemma: If B is an integer which is not a perfect square and $(T + U\sqrt{B})^i = T_i + U_i\sqrt{B}$, then

$$T_i = \sum_{k=0} \binom{i}{2k} T^{i-2k} U^{2k} B^k \quad \text{and} \quad U_i = \sum_{k=1} \binom{i}{2k-1} T^{i+2k+1} U^{2k-1} B^{k-1},$$

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where ( ) denotes the binomial coefficient.

We are now in a position to prove the main result.

Theorem: Every nonzero integer is representable in infinitely many ways as a proper difference of two powerful numbers neither of which is a perfect square.

Proof: For the case where n is even see [4], and for the case where $n = 1$ see [7]. This leaves the case where $n > 1$ is odd. We break the proof down into two parts. We note that it suffices to prove the result for *either* n *or* $-n$.

Case (i): $n \not\equiv 0 \pmod 5$

Let $D = rs$, where

$$r = (n^2 - 2n + 5)/4 \quad \text{and} \quad s = (n^2 + 2n + 5)/4.$$

Let $T = (n^2 + 3)/4$, then $T^2 - D = -1$. If $(T + \sqrt{D})^i = (T_i + U_i\sqrt{D})$, then

$$T_i^2 - U_i^2 D = \pm 1.$$

Therefore,

$$\pm n = n(T_i^2 - DU_i^2) = sF_i^2 - rE_i^2,$$

where

$$E_i = T_i + sU_i \quad \text{and} \quad F_i = T_i + rU_i.$$

Now we show that, for an appropriate choice of i, we can achieve $E_i \equiv 0 \pmod r$ and $F_i \equiv 0 \pmod s$. To see this, we invoke the Lemma to get

$$E_i \equiv T^i + siT^{i-1} \pmod r.$$

Since $n \not\equiv 0 \pmod 5$, r and s are relatively prime, so we may choose

$$i \equiv -T(s)^{-1} \pmod r$$

which guarantees that $E_i \equiv 0 \pmod r$. Similarly, by choosing

$$i \equiv -T(r)^{-1} \pmod s$$

we guarantee $F_i \equiv 0 \pmod s$.

In order to complete Case (i), it remains to show that E_i and F_i are relatively prime. Suppose that there is a prime p such that:

$$E_i = T_i + sU_i = pt \tag{1}$$

for some integer t, and

$$F_i = T_i + rU_i = pu \tag{2}$$

for some integer u. Multiplying (1) by T_i and (2) by sU_i, then subtracting, we get

$$\pm 1 = T_i^2 - rsU_i^2 = p(tT_i - suU_i)$$

a contradiction.

Case (ii): $n \equiv 0 \pmod 5)$

Let $D = n^2 + 1$, $T = n$, $U = -1$, and $(T + U\sqrt{D})^i = T_i + U_i\sqrt{D}$. Let $A_i = T_i + U_iD$ and $B_i = T_i + U_i$. Our plan of attack for this case is to show that for an appropriate choice of i we get $A_i^2 - B_i^2D = n^2$ with $(A_i \pm n)/2$ being powerful. First we observe that if $B_i \equiv 0 \pmod{2D}$ and g.c.d. $(A_i, n) = 1$, then $(A_i \pm n)/2$ are powerful. We prove g.c.d. $(A_i, n) = 1$ by contradiction. If there is a prime p such that $A_i = T_i + U_iD \equiv 0 \pmod p$ and $n \equiv 0 \pmod p$, then $T_i + U_i \equiv 0 \pmod p$. Therefore,

$$\pm 1 = T_i^2 - U_i^2D \equiv T_i^2 - U_i^2 \equiv 0 \pmod p,$$

a contradiction. Now, by choosing $i \equiv n \pmod{2D}$, we get by the Lemma that:

$$B_i = T_i + U_i \equiv T^i - iT^{i-1} \equiv 0 \pmod{2D}.$$

Hence, we have shown that $(A_i \pm n)/2$ are powerful. It remains to show that neither of these is a perfect square. To do this, we use the following fact. Since $n \equiv 0 \pmod 5$, D must contain, in its prime decomposition, a prime $p > 2$ to an odd exponent; i.e., $D \neq 2d^2$ for any integer d.

We observe that $A_i^2 - n^2 = B_i^2D = 2^5ef^2$, where e is odd. Therefore, whichever of $(A_i \pm n)/2$ is even cannot be a perfect square. It remains to show that $(A_i + n) \not\equiv 0 \pmod{4p}$ and $(A_i - n) \not\equiv 0 \pmod{4p}$; i.e., whichever of $(A_i \pm n)/2$ is odd cannot be a perfect square, since it contains the odd power of p.

Suppose $A_i + n \equiv 0 \pmod{4p}$. Therefore, $T_i + U_iD + n \equiv 0 \pmod{4p}$, which implies

$$T_i \equiv n^i \equiv -n \pmod p.$$

Hence, $n^{i-1} \equiv -1 \equiv n^2 \pmod p$, which implies

$$i \equiv 3 \equiv n \pmod 4.$$

Now, by the Lemma, $T_i \equiv 1 \pmod 4$ and $U_i \equiv 3 \pmod 4$. Thus,

$$0 \equiv T_i + U_iD + n \equiv 1 + 6 + 3 \pmod 4,$$

a contradiction.

Finally, assume $A_i - n \equiv 0 \pmod{4p}$. Therefore, $T_i + U_iD - n \equiv 0 \pmod{4p}$, which implies $T_i \equiv n^i \equiv n \pmod p$, and so $i \equiv 1 \equiv n \pmod 4$. Hence,

$$0 \equiv T_i + U_iD - n \equiv 1 + 6 - 1 \pmod 4,$$

a contradiction which secures the Theorem.

We note that the proof of the Theorem yields an effective algorithm, via the choice of i, for infinitely many representations of a given odd integer as a proper nonsquare difference of powerful numbers. The following examples illustrate the process.

Example 1: Let $i \equiv 1 \pmod{10}$ and $(3 + \sqrt{10})^i = T_i + U_i\sqrt{10}$. Thus,

$$3 = 2(T_i + 5U_i)^2 - 5(T_i + 2U_i)^2$$

with $T_i + 2U_i \equiv 0 \pmod 5$ and $T_i + 5U_i$ even. In particular, if $i = 1$, then $3 = 2^7 - 5^3$.

Example 2: Let $i \equiv 5 \pmod{52}$ and $(5 - \sqrt{26})^i = T_i + U_i\sqrt{26}$. Then

$$(T_i + 26U_i)^2 - 26(T_i + U_i)^2 = 25$$

with $(T_i + 26U_i \pm 5)/2$ nonsquare powerful numbers. In particular, if $i = 5$, then $5 = 7^2 \cdot 13^3 - 2^7 \cdot 29^2$.

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AN UPPER BOUND FOR THE GENERAL RESTRICTED PARTITION PROBLEM

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The function $p^*(p_1, p_2, \ldots, p_m; n)$ is defined as the number of partitions of the integer n into at most m positive integers $p_1, p_2, \ldots, p_m$, where the order is irrelevant. An upper bound for the number of partitions is given. This upper bound is then compared with two known particular cases. An upper bound for the function $p^*(p_1, p_2, \ldots, p_m; \leqslant n)$ is also given. This last function represents the number of partitions of all integers between 0 and n into at most m positive integers $p_1, p_2, \ldots, p_m$.

1. INTRODUCTION

The number of partitions as defined above is equal to the number of solutions of the Diophantine equation

$$p_1x_1 + p_2x_2 + \cdots + p_mx_m = n$$

in integers $x_i \geqslant 0$, where the p_i are given positive integers which need not be distinct. If $(p_1, p_2, \ldots, p_m) = d > 1$, then $p^*(p_1, p_2, \ldots, p_m; n) = 0$ unless d divides n, in which case the factor d can be removed from the above equation without altering the number of partitions. That is,

$$p^*(p_1, p_2, \ldots, p_m; n) = p^*\left(\frac{p_1}{d}, \frac{p_2}{d}, \ldots, \frac{p_m}{d}; \frac{n}{d}\right),$$

where

$$\left(\frac{p_1}{d}, \frac{p_2}{d}, \ldots, \frac{p_m}{d}\right) = 1 \text{ when } d/n.$$

Thus, we can assume that the equation is reduced and that $(p_1, p_2, \ldots, p_m) = 1$ for the rest of this paper. We can also assume without loss of generality that

$$p_1 \leqslant p_2 \leqslant p_3 \leqslant \ldots \leqslant p_m,$$

where there must be at least one strict inequality if $(p_1, p_2, \ldots, p_m) = 1$ unless $p_1 = p_2 = \cdots = p_m = 1$. The number of partitions of n into exactly the parts $p_1, p_2, \ldots, p_m$ will be denoted by the function

$$p(p_1, p_2, \ldots, p_m; n).$$

This is equal to the number of solutions of the equation

$$p_1x_1 + p_2x_2 + \cdots + p_mx_m = n$$

in integers $x_i \geqslant 1$.

It is known that

$$p(p_1, p_2, \ldots, p_m; n) = p^*(p_1, p_2, \ldots, p_m; n - (p_1 + p_2 + \cdots + p_m)) \tag{1.1}$$

and that the function $p^*(p_1, p_2, \ldots, p_m; n)$ satisfies the recurrence equation

$$\begin{aligned} p^*(p_1, p_2, \ldots, p_m; n) &- p^*(p_1, p_2, \ldots, p_m; n - p_m) \\ &= p^*(p_1, p_2, \ldots, p_{m-1}; n), \end{aligned} \tag{1.2}$$

where $p^*(p_1, p_2, \ldots, p_m; 0) = 1$.

2. PRELIMINARY RESULTS

In order to determine an upper bound for $p^*(p_1, p_2, \ldots, p_m; n)$ under the most general possible condition, which is $(p_1, p_2, \ldots, p_m) = 1$, we require some preliminary results, which will be stated without proof. The proofs are quite straightforward but in the case of (2.1) rather lengthy. The proofs have been omitted in this revised version to reduce the length of the paper.

If $(p_1, p_2) = \alpha_2$ and $(p_1, p_2, p_3) = \alpha_3$, then, for $n \geqslant 0$,

$$p^*(p_1, p_2, p_3; n) \leqslant \frac{\alpha_3}{2p_1p_2p_3}\left(n + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \frac{\alpha_2 p_3}{\alpha_3}\right)\right)^2. \tag{2.1}$$

If $A > 0$ and $B > 0$ and k is an integer $\geqslant 2$, then

$$\sum_{r=0}^{t} (Ar + B)^{k-1} \leqslant \frac{1}{Ak}\left(A\left(t + \frac{1}{2}\right) + B\right)^k. \tag{2.2}$$

The upper bound in (2.1) cannot be weakened, since it is actually attained under very special circumstances. If we consider $p^*(p_1, p_2, p_3; n)$, where

$$(p_1, p_2, p_3) = 1 \quad \text{and} \quad p_3 = \frac{2p_1p_2}{\alpha_2^2},$$

then $\alpha_3 = 1$ and, for an arbitrary positive integer k, we have, using (2.1), that

$$p^*\left(p_1, p_2, \frac{2p_1p_2}{\alpha_2^2}; \frac{2kp_1p_2}{\alpha_2}\right) \leqslant (k + 1)^2.$$

But it can be shown that in this case we have

$$p^*\left(p_1, p_2, \frac{2p_1p_2}{\alpha_2^2}; \frac{2kp_1p_2}{\alpha_2}\right) = (k + 1)^2$$

and the bound is attained.

3. THE MAIN RESULT

We now state and prove the main result of this paper.

Theorem: If $(p_1, p_2) = \alpha_2$, $(p_1, p_2, p_3) = \alpha_3, \ldots, (p_1, p_2, \ldots, p_m) = \alpha_m$, then for $n \geqslant 0$ and $m \geqslant 3$,

$$p^*(p_1, p_2, \ldots, p_m; n) \tag{3.1}$$
$$\leqslant \frac{\alpha_m}{p_1 p_2 \ldots p_m (m-1)!}\left(n + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3}p_3 + \cdots + \frac{\alpha_{m-1}}{\alpha_m}p_m\right)\right)^{m-1},$$

where, if the partition is reduced, $\alpha_m = 1$.

Proof: Assume the result correct if $m = k$ (say), and consider

$$p^*(p_1, p_2, \ldots, p_{k+1}; n), \text{ where } (p_1, p_2, \ldots, p_{k+1}) = 1.$$

Writing

$$n = ap_{k+1} + b, \text{ where } a = \left[\frac{n}{p_{k+1}}\right] \text{ and } 0 \leqslant b \leqslant p_{k+1} - 1.$$

$$\therefore\ p^*(p_1, p_2, \ldots, p_k, p_{k+1}; n) = \sum_{i=0}^{a} p^*(p_1, p_2, \ldots, p_k; ip_{k+1} + b),$$

using (1.2), since $p^*(p_1, p_2, \ldots, p_{k+1}; b) = p^*(p_1, p_2, \ldots, p_k; b)$, where $(p_1, p_2, \ldots, p_k) = \alpha_k$. Now the sum is zero if $\alpha_k \nmid ip_{k+1} + b$. Consider

$$ip_{k+1} + b \equiv 0 \pmod{\alpha_k}, \text{ where } (p_{k+1}, \alpha_k) = 1$$

as $(p_1, p_2, \ldots, p_k, p_{k+1}) = 1$. Thus, there is a unique solution

$$i = i_0 \pmod{\alpha_k}, \text{ where } 0 \leqslant i_0 \leqslant \alpha_k - 1.$$

$$\therefore\ i = i_0,\ i_0 + \alpha_k,\ i_0 + 2\alpha_k,\ \ldots,\ i_0 + \left[\frac{a - i_0}{\alpha_k}\right]\cdot\alpha_k \leqslant a \text{ if } a - i_0 \geqslant 0.$$

Hence,

$$p^*(p_1, p_2, \ldots, p_{k+1}; n)$$

$$= \begin{cases} \displaystyle\sum_{i \text{ (as above)}} p^*(p_1, p_2, \ldots, p_k; ip_{k+1} + b) & \text{if } a - i_0 \geqslant 0 \\ = 0 & \text{if } a - i_0 < 0 \end{cases}$$

$$\leqslant \sum_i \frac{\alpha_k}{p_1p_2\cdots p_k(k-1)!}\left(ip_{k+1} + b + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3}p_3 + \cdots + \frac{\alpha_{k\mp1}}{\alpha_k}p_k\right)\right)^{k-1}$$

$$= \frac{\alpha_k}{p_1\cdots p_k(k-1)!}\sum_{r=0,1,2,\ldots}^{\left[\frac{a-i_0}{\alpha_k}\right]}\left(p_{k+1}(i_0 + r\alpha_k) + b + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \cdots + \frac{\alpha_{k-1}}{\alpha_k}p_k\right)\right)^{k-1}$$

$$= \frac{\alpha_k}{p_1 \cdots p_k (k-1)!} \sum_{r=0}^{\left[\frac{a-i_0}{\alpha_k}\right]} \left(p_{k+1}\alpha_k r + p_{k+1} i_0 + b + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \cdots + \frac{\alpha_{k-1}}{\alpha_k} p_k \right) \right)^{k-1}$$

$$= \frac{\alpha_k}{p_1 \cdots p_k (k-1)!} \sum_{r=0,1,2,\ldots}^{t} (Ar + B)^{k-1}, \text{ where } t = \left[\frac{a - i_0}{\alpha_k}\right] \geqslant 0$$

$$\text{and } A = p_{k+1}\alpha_k,\ B = p_{k+1} i_0 + b + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \cdots + \frac{\alpha_{k-1}}{\alpha_k} p_k \right),$$

$$\leqslant \frac{\alpha_k}{p_1 \cdots p_k (k-1)!} \cdot \frac{1}{p_{k+1}\alpha_k \cdot k} \left(p_{k+1}\alpha_k \left(t + \frac{1}{2} \right) + p_{k+1} i_0 + b + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \cdots + \frac{\alpha_{k-1}}{\alpha_k} p_k \right) \right)^k, \text{ using 2.2,}$$

$$= \frac{1}{p_1 p_2 \cdots p_{k+1} k!} \left(p_{k+1}\alpha_k t + p_{k+1} i_0 + b + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3} p_3 + \cdots + \frac{\alpha_{k-1}}{\alpha_k} p_k + \frac{\alpha_k}{1} p_{k+1} \right) \right)^k.$$

Now $\alpha_k p_{k+1} t = p_{k+1}\alpha_k \left[\frac{a - i_0}{\alpha_k}\right] \leqslant p_{k+1}(a - i_0)$

$$\leqslant \frac{1}{p_1 p_2 \cdots p_{k+1} k!} \left(a p_{k+1} - p_{k+1} i_0 + p_{k+1} i_0 + b + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \cdots + \frac{\alpha_k}{1} p_{k+1} \right) \right)^k$$

$$= \frac{1}{p_1 p_2 \cdots p_{k+1} k!} \left(n + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3} p_3 + \cdots + \frac{\alpha_{k-1}}{\alpha_k} p_k + \frac{\alpha_k}{1} p_{k+1} \right) \right)^k.$$

Thus, we have that if the result is correct for $m = k$ then it is correct for $m = k + 1$ when $\alpha_{k+1} = 1$. Now assume that $(p_1, p_2, \ldots, p_{k+1}) = \alpha_{k+1}$ (say).

If $\alpha_{k+1} \nmid n$, then $p^*(p_1, \ldots, p_{k+1}; n) = 0$. If $\alpha_{k+1} | n$, then

$$p^*(p_1, p_2, \ldots, p_{k+1}; n) = p^*\left(\frac{p_1}{\alpha_{k+1}}, \frac{p_2}{\alpha_{k+1}}, \ldots, \frac{p_{k+1}}{\alpha_{k+1}}; \frac{n}{\alpha_{k+1}} \right),$$

$$\text{where } \left(\frac{p_1}{\alpha_{k+1}}, \ldots, \frac{p_{k+1}}{\alpha_{k+1}} \right) = 1, \text{ and thus}$$

$$p^*(p_1, \ldots, p_{k+1}; n) \leqslant \frac{1}{\frac{p_1}{\alpha_{k+1}} \cdots \frac{p_{k+1}}{\alpha_{k+1}} k!} \left(\frac{n}{\alpha_{k+1}} + \frac{1}{2}\left(\frac{\frac{2p_1}{\alpha_{k+1}} \cdot \frac{p_2}{\alpha_{k+1}}}{\frac{\alpha_2}{\alpha_{k+1}}} + \frac{\frac{\alpha_2}{\alpha_{k+1}}}{\frac{\alpha_3}{\alpha_{k+1}}} \cdot \frac{p_3}{\alpha_{k+1}} + \cdots + \frac{\frac{\alpha_{k-1}}{\alpha_{k+1}}}{\frac{\alpha_k}{\alpha_{k+1}}} \cdot \frac{p_k}{\alpha_{k+1}} + \frac{\frac{\alpha_k}{\alpha_{k+1}}}{\frac{\alpha_{k+1}}{\alpha_{k+1}}} \cdot \frac{p_{k+1}}{\alpha_{k+1}} \right) \right)^k$$

$$= \frac{\alpha_{k+1}}{p_1 p_2 \cdots p_{k+1} k!}\left(n + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3} p_3 + \cdots + \frac{\alpha_k}{\alpha_{k+1}} p_{k+1}\right)\right)^k.$$

Thus, the result is correct for $m = k + 1$ if it is correct for $m = k$. But, we know that the result is correct for $m = 3$, and hence the result is correct for $m \geqslant 3$. This completes the proof.

4. A COMPARISON WITH KNOWN PARTICULAR CASES

(a) The upper bound given by Rieger [6] is

$$p_m(n) < \frac{1}{m!(m-1)!}\left(n + \frac{m(m-3)}{4}\right)^{m-1} \quad \text{for } n \geqslant 0,\ m \geqslant 4.$$

We have $p_m(n) = p^*(1, 2, 3, \ldots, m; n - m)$ and $\alpha_2 = 1$; thus,

$$p_m(n) \leqslant \frac{1}{m!(m-1)!}\left(n - m + \frac{1}{2}(4 + 3 + 5 + \cdots + m)\right)^{m-1}.$$

$$\therefore \quad \text{Our result} = \frac{1}{m!(m-1)!}\left(n + \frac{m(m-3)}{4} + \frac{1}{2}\right)^{m-1} \quad \text{for } m \geqslant 3.$$

(b) H. Gupta [5] has given the following result for the particular case in which $p_1 = 1$:

$$\frac{\binom{n+m-1}{m-1}}{p_2 p_3 \cdots p_m} \leqslant p^*(1, p_2, p_3, \ldots, p_m; n) \leqslant \frac{\binom{n + p_2 + p_3 + \cdots + p_m}{m-1}}{p_2 p_3 \cdots p_m}$$

For the upper bound, we have

$$\binom{n + p_2 + \cdots + p_m}{m-1} = \frac{(n + p_2 + \cdots + p_m)!}{(m-1)!(n + p_2 + \cdots + p_m - (m-1))!}$$

For large $n + p_2 + \cdots + p_m - (m - 1)$,

$$\sim \frac{1}{(m-1)!} \frac{(n + p_2 + \cdots + p_m)^{n+p_2+\cdots+p_m+\frac{1}{2}} \cdot e^{-(m-1)}}{(n + p_2 + \cdots + p_m - (m-1))^{n+p_2+\cdots+p_m+\frac{1}{2}-(m-1)}}$$

$$= \frac{(n + p_2 + \cdots + p_m - (m-1))^{m-1}}{(m-1)!} \cdot \frac{e^{-(m-1)}}{\left(1 - \frac{(m-1)}{n + p_2 + \cdots + p_m}\right)^{n+p_2+\cdots+p_m+\frac{1}{2}}}$$

$$\sim \frac{(n + p_2 + \cdots + p_m - (m-1))^{m-1}}{(m-1)!}.$$

Thus, Gupta's result for the upper bound is asymptotic to

$$\frac{1}{1 \cdot p_2 \cdots p_m (m-1)!}(n + p_2 + p_3 + \cdots + p_m - (m-1))^{m-1}.$$

Our result with $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 1$ as $p_1 = 1$ is sharper if

$$p_2 + p_3 + \cdots + p_m - (m - 1) > \frac{1}{2}(2p_2 + p_3 + \cdots + p_m)$$

or

$$p_3 + p_4 + \cdots + p_m > 2m - 2.$$

For arbitrarily large p_i, this is obviously satisfied as $\sum_{i=3}^{m} p_i$ will, in general, be much larger than $2m - 2$.

5. AN UPPER BOUND FOR $p^*(p_1, p_2, \ldots, p_m; \leqslant n)$

This function represents the number of solutions of the inequality

$$p_1x_1 + p_2x_2 + \cdots + p_mx_m \leqslant n$$

in integers $x_i \geqslant 0$ for $n \geqslant 0$. Alternatively, this represents the number of lattice points within and on the hypertetrahedron bounded by the planes $x_i = 0$ and the hyperplane

$$p_1x_1 + p_2x_2 + \cdots + p_mx_m = n.$$

We can assume that $(p_1, p_2, \ldots, p_m) = 1$, and thus

$$p^*(p_1, p_2, \ldots, p_m; \leqslant n)$$

$$\leqslant \frac{1}{p_1p_2\ldots p_m(m-1)!}\sum_{r=0}^{n}\left(r + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3}p_3 + \cdots + \frac{\alpha_{m-1}}{1}p_m\right)\right)^{m-1}$$

$$\leqslant \frac{1}{p_1p_2\ldots p_m m!}\left(n + \frac{1}{2} + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3}p_3 + \cdots + \frac{\alpha_{m-1}}{1}p_m\right)\right)^{m}$$

for $n \geqslant 0$ and $m \geqslant 3$, using 2.2.

6. NUMERICAL RESULTS AND ASYMPTOTICS

Consider the example

$$p^*(60, 120, 150, 216, 243, 247; n),$$

where $\alpha_2 = 60$, $\alpha_3 = 30$, $\alpha_4 = 6$, $\alpha_5 = 3$, and $\alpha_6 = 1$. It is clear that α_{k+1} must divide α_k.

It is known [4] that if $(p_1, p_2, \ldots, p_m) = 1$ then $p^*(p_1, p_2, \ldots, p_m; n) > 0$ for sufficiently large n. This implies that there is a largest integer n for which $p^*(p_1, p_2, \ldots, p_m; n) = 0$. This greatest integer is denoted by

$$G(p_1, p_2, \ldots, p_m).$$

The paper [4] gives some upper bounds for $G(p_1, p_2, \ldots, p_m)$. Using these upper bounds and a numerical search, it can be found that

$$G(60, 120, 150, 216, 243, 247) = 1541.$$

For larger n, the partition function will be much smoother and the upper bound will become asymptotically better.

We have, for the previous particular numerical example, the following results.

n	$p^*(;\ n)$	Upper Bound	$p^*(;\leqslant n)$	Upper Bound
1541	0	136	7090	67396
6944	$.11723 \times 10^5$	$.24412 \times 10^5$	$.17050 \times 10^8$	$.34057 \times 10^8$
19760	$.19217 \times 10^7$	$.25387 \times 10^7$	$.68932 \times 10^{10}$	$.89646 \times 10^{10}$
39779	$.61270 \times 10^8$	$.70673 \times 10^8$	$.42470 \times 10^{12}$	$.48535 \times 10^{12}$
44505	$.11307 \times 10^9$	$.12163 \times 10^9$	$.82616 \times 10^{12}$	$.93112 \times 10^{12}$
60000	$.49311 \times 10^9$	$.52036 \times 10^9$	$.48728 \times 10^{13}$	$.53274 \times 10^{13}$
490000	$.16900 \times 10^{14}$	$.17057 \times 10^{14}$	$.13817 \times 10^{19}$	$.13970 \times 10^{19}$

CONCLUSION

An upper bound has been determined for $p^*(p_1, p_2, \ldots, p_m; n)$ and $p^*(p_1, p_2, \ldots, p_m; \leqslant n)$ for all $n \geqslant 0$ and $m \geqslant 3$.

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ORDINARY GENERATING FUNCTIONS FOR PELL POLYNOMIALS

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1. INTRODUCTION

The object of this paper is to investigate, by using a variety of methods, the properties of Pell polynomials $P_n(x)$ and the Pell-Lucas polynomials $Q_n(x)$ [6] which are derivable from their generating functions. Brief acquaintance with the main aspects of [6] is desirable.

In an endeavor to conserve space, we will generally offer only an indication of the potential development, with a minimum of results, so that just a representative sample of the material available is presented. Omitted information will be happily supplied on request. Among the many facets of this exposition, we find the sections numbered 4 and 5 especially appealing.

For visual convenience, the functional notation will be suppressed and an abbreviated notation used, e.g., $P_n(x) \equiv P_n$, $Q_n(x) \equiv Q_n$.

First, we introduce the notation

$$P(j,\ m,\ k,\ x,\ y) = \sum_{r=0}^{\infty} P_{mr+k}^{j} y^{r}, \tag{1.1}$$

$$Q(j,\ m,\ k,\ x,\ y) = \sum_{r=0}^{\infty} Q_{mr+k}^{j} y^{r}. \tag{1.2}$$

Then, for example, by difference equations [6],

$$P(1,\ 1,\ 0,\ x,\ y) = y\Delta \tag{1.3}$$

or, equivalently,

$$P(1,\ 1,\ 1,\ x,\ y) = \Delta = \sum_{r=0}^{\infty} P_{r+1} y^{r}, \tag{1.4}$$

and

$$Q(1,\ 1,\ 0,\ x,\ y) = (2 - 2xy)\Delta \tag{1.5}$$

or, equivalently,

$$Q(1,\ 1,\ 1,\ x,\ y) = (2x + 2y)\Delta = \sum_{r=0}^{\infty} Q_{r+1} y^{r}, \tag{1.6}$$

in all of which

$$\Delta = (1 - 2xy - y^2)^{-1} = \Delta(x,\ y,\ 1,\ 1) \quad [\text{cf. } (1.8)]. \tag{1.7}$$

Result (1.4), for example, may also be obtained using the method of column generators [1] with the aid of binomial coefficient expressions for P_n given in [7]. Matrices and Binet forms may also be utilized (see [7]) in establishing (1.3)-(1.6).

Let us introduce the symbolism

$$\Delta_{(m)}^{(j)} \equiv \Delta(x, y, j, m)$$

[cf. (1.13)] in which the superscript and subscript will be suppressed when $j = 1$ and/or $m = 1$, e.g., $\Delta_{(1)}^{(1)} = \Delta$ [cf. (1.7)] and

$$\Delta_{(m)} = (1 - Q_m y + (-1)^m y^2)^{-1} \equiv \Delta(x, y, 1, m) \tag{1.8}$$

whence (1.7) follows when $m = 1$. Replacing y by $-y$, we write

$$\Delta'_{(m)} \equiv \Delta(x, -y, 1, m). \tag{1.9}$$

Furthermore, with $m = 1$, let

$$\Delta^{(j)} \equiv \Delta(x, y, j, 1) = \left[\sum_{r=0}^{j+1} (-1)^{\frac{r(r+1)}{2}} \{j + 1, r\} y^r\right]^{-1}, \tag{1.10}$$

where the symbol $\{a, b\}$, defined in [8], is

$$\{a, b\} = \prod_{i=1}^{a} P_i \Big/ \left(\prod_{i=1}^{b} P_i\right)\left(\prod_{i=1}^{a-b} P_i\right) \tag{1.11}$$

Thus, in particular, from (1.10) and (1.11),

$$\begin{cases} \Delta = (1 - P_2 y - y^2)^{-1} & \text{as in (1.7)} \\ \Delta^{(2)} = (1 - P_3 y - P_3 y^2 + y^3)^{-1} & \\ \Delta^{(3)} = (1 - P_4 y - (P_3 P_4 / P_1 P_2) y^2 + P_4 y^3 + y^4)^{-1} & \end{cases} \tag{1.12}$$

More generally,

$$\Delta_{(m)}^{(j)} = \left[\sum_{r=0}^{j+1} (-1)^{\frac{r[m(r-1)+2]}{2}} \{j + 1, r\}_m y^r\right]^{-1}, \tag{1.13}$$

in which

$$\{a, b\}_m = \prod_{i=1}^{a} P_{im} \Big/ \left(\prod_{i=1}^{b} P_{im}\right)\left(\prod_{i=1}^{a-b} P_{im}\right). \tag{1.14}$$

The case $j = 1$ occurs in (1.8), while the case $m = 1$ occurs in (1.10). Later, in (6.6), we refer to the case $j = 3$, i.e., to $\Delta_{(m)}^{(3)}$.

Some useful results from [7] are collected here for later reference:

$$Q_{n+r} + Q_{n-r} = \begin{cases} Q_n Q_r & r \text{ even}, \\ 4(x^2 + 1) P_n P_r & r \text{ odd}. \end{cases} \tag{1.15}$$

$$P_{n+1}^2 - (4x^2 + 2) P_n^2 + P_{n-1}^2 = 2(-1)^n. \tag{1.16}$$

$$P_{m(r+1)+k} - Q_m P_{mr+k} + (-1)^m P_{m(r-1)+k} = 0. \tag{1.17}$$

Also important for our matrix treatment are (see [6]):

$$P = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}; \tag{1.18}$$

$$P^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} \quad \text{so } |P^n| = (-1)^n. \tag{1.19}$$

Consult [6], [7], and [8] for details of some of the applications of P.

2. APPLICATIONS OF GENERATING FUNCTIONS

Using (1.17) as a difference equation, we find eventually that

$$P(1,\ m,\ k,\ x,\ y) = [P_k + (-1)^k P_{m-k}y]\Delta_{(m)}. \tag{2.1}$$

Similarly,

$$Q(1,\ m,\ k,\ x,\ y) = [Q_k + (-1)^{k-1} Q_{m-k}y]\Delta_{(m)}. \tag{2.2}$$

The specializations given in (1.3) and (1.5) follow immediately. Numerous other specializations of some interest, e.g., those for

$$P(1,\ 2,\ 0,\ x,\ y),\ P(1,\ 2,\ 1,\ x,\ y),\ P(1,\ 3,\ 3,\ x,\ -y)$$

and $\quad Q(1,\ 2,\ 1,\ x,\ -y),$

are listed in [7].

Differentiating (1.4) with respect to y, we obtain

$$\sum_{r=0}^{\infty} rP_{r+1}y^{r-1} = (2x + 2y)\Delta^2. \tag{2.3}$$

Likewise

$$\sum_{r=0}^{\infty} rQ_{r+1}y^{r-1} = [4x^2 + 2 + 4xy + 2y^2]\Delta^2. \tag{2.4}$$

Replacing y by $-y$ gives generating functions of some importance. Results (2.3) and (2.4) may be extended if we differentiate (2.1) and (2.2) w.r.t. y, but the process is somewhat algebraically messy.

Now, (2.3) leads to an interesting summation. With (1.4) and (1.6) it gives

$$\sum_{r=0}^{\infty} rP_{r+1}y^{r-1} = \left\{\sum_{r=0}^{\infty} P_{r+1}y^n\right\}\left\{\sum_{r=0}^{\infty} Q_{r+1}y^n\right\}. \tag{2.5}$$

Equate coefficients of y^r on both sides, thus obtaining

$$(r + 1)P_{r+2} = \sum_{r=0}^{r+1} P_i Q_{r+2-i}. \tag{2.6}$$

Next, differentiate (1.5) w.r.t. y. Then

$$\sum_{r=0}^{\infty} (r + 1)Q_{r+1}y^r = (2x + 4y - 2xy^2)\Delta^2. \tag{2.7}$$

Combining (1.4) and (2.7), we find

$$\sum_{r=0}^{\infty}(r+1)Q_{r+1}y^r - \sum_{r=0}^{\infty}(r+1)\ P_{r+2}y^r = y(2-2xy)\Delta^2 \tag{2.8}$$
$$= \left\{\sum_{r=0}^{\infty}P_r y^r\right\}\left\{\sum_{r=0}^{\infty}Q_r y^r\right\}$$

by (1.3) and (1.5).

Equate coefficients to get

$$(r+1)(Q_{r+1}-P_{r+2}) = \sum_{i=0}^{r}P_i Q_{r-i}. \tag{2.9}$$

Differentiating in (1.3) w.r.t. y, then multiplying by y, we determine a generating function for rP_r, namely,

$$\sum_{r=0}^{\infty}rP_r y^r = y(1+y^2)\Delta^2. \tag{2.10}$$

Similarly,

$$\sum_{r=0}^{\infty}rQ_r y^r = (2xy + 4y^2 - 2xy^3)\Delta^2. \tag{2.11}$$

Generating functions may be used to derive already known properties of Pell polynomials, e.g.,

$$\sum_{n=0}^{\infty}Q_n y^n = (2-2xy)\Delta \qquad \text{by (1.5)}$$
$$= \Delta + (1-2xy)\Delta$$
$$= \sum_{n=0}^{\infty}P_{n+1}y^n + \sum_{n=0}^{\infty}P_{n-1}y^n \quad \text{by (1.4) and (2.1),}$$

whence $\qquad Q_n = P_{n+1} + P_{n-1} \qquad$ [6, equation (2.1)].

Moreover, we may show that

$$Q(1,\ 1,\ 1,\ x,\ y) + Q(1,\ 1,\ -1,\ x,\ y) = 4(x^2+1)P(1,\ 1,\ 0,\ x,\ y),$$

whence $\qquad Q_{n+1} + Q_{n-1} = 4(x^2+1)P_n \qquad$ [cf. (1.15)].

New, but less elementary, identities may also be established. For instance,

$$\sum_{n=0}^{\infty}\{Q_nQ_{m-1} + Q_{n+1}Q_m\}y^n$$
$$= [(2-2xy)Q_{m-1} + (2x+2y)Q_m]\Delta \qquad \text{by (1.5) and (1.6)}$$
$$= [(2xQ_m + 2Q_{m-1}) + (2Q_m - 2xQ_{m-1})y]\Delta$$
$$= 4(x^2+1)(P_m + yP_{m-1})\Delta$$

by (1.13) and the recurrence relation for Q_m.

Terms in y^n being equated, we derive

$$Q_nQ_{m-1} + Q_{n+1}Q_m = 4(x^2+1)P_{m+n}. \tag{2.12}$$

Following the technique of Serkland [9] for Pell numbers, we can also establish fresh identities involving Pell polynomials. See [7] for details. A representative result incorporating this process is

$$P_u P_v P_w = \sum_{k=0}^{w-1}\left\{P_{u+v+w-k}P_{k+1} - P_{u+k+1}P_{v+w-k}\right\}. \tag{2.13}$$

Finite series may be summed using a generating function. To illustrate this contention, choose

$$\begin{aligned}\sum_{r=1}^{m} P_r y^r &= \sum_{r=0}^{\infty} P_r y^r - \sum_{r=0}^{\infty} P_{r+m+1} y^r \\ &= y\{1 - (P_{m+1} + yP_m)\}\Delta \qquad \text{by (1.3) and (2.1).}\end{aligned}$$

Then, $y = 1$ gives equation (2.11) in [6].

Ideas of Hoggatt [2] in relation to Fibonacci and Lucas numbers may be extended to generators of Pell polynomials. For example,

$$\begin{aligned}&\sum_{k=0}^{\infty} 4^k(x^2+1)^k P_{2k+1} y^{2k+1} \qquad (2.14)\\ &= yP(1, 2, 1, x, 4(x^2+1)y^2) \qquad \text{by (1.1)}\\ &= y^2\{1 - 4(x^2+1)y^2\}\delta_{(2)} \qquad \text{by (2.1)}\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=0}^{\infty} 4^k(x^2+1)^k Q_{2k+2} y^{2k+2} \qquad (2.15)\\ &= y^2Q(1, 2, 2, x, 4(x^2+1)y^2) \qquad \text{by (1.2)}\\ &= y^2\{(4x^2+2) - 8y^2(x^2+1)\}\delta_{(2)} \qquad \text{by (2.2)}\end{aligned}$$

where, in (2.14) and (2.15), $\delta_{(2)}$ means $\Delta_{(2)}$ with y replaced by $4(x^2+1)y$ [cf. (1.8)].

Add (2.14) and (2.15). Simplifying, we are left with

$$\begin{aligned}&\sum_{k=0}^{\infty} 4^k(x^2+1)^k\{P_{2k+1} + yQ_{2k+2}\}y^{2k+1} \qquad (2.16)\\ &= \frac{y - 2y^2}{1 - 4(x^2+1)y + 4(x^2+1)y^2}\end{aligned}$$

Further details appear in [7].

3. ELEMENTARY RELATIONS AMONG GENERATING FUNCTIONS

Analogous relations to those among polynomials may be determined for generating functions. Consider, for instance, the derivation of the recurrence relation

$$\begin{aligned}P(1,\ 1,\ n+2,\ x,\ y) &= (P_{n+2} + yP_{n+1})\Delta \quad \text{by (2.1)} \\ &= (2x\{P_{n+1} + yP_n\} + P_n + yP_{n-1})\Delta \quad \text{by the definition of } P_n \\ &= 2xP(1,\ 1,\ n+1,\ x,\ y) + P(1,\ 1,\ n,\ x,\ y) \quad \text{by (2.1).}\end{aligned} \tag{3.1}$$

Likewise,

$$Q(1,\ 1,\ n+2,\ x,\ y) = 2xQ(1,\ 1,\ n+1,\ x,\ y) + Q(1,\ 1,\ n,\ x,\ y). \tag{3.2}$$

It might be noted that the direct generating function analogue of

$$Q_n = P_{n+1} + P_{n-1}$$

flows almost immediately from (2.1) and (2.2).

Matrix representations of the generating functions are, in the notation of [8] for the matrix P,

$$\begin{bmatrix} P(1,\ 1,\ n,\ x,\ y) \\ P(1,\ 1,\ n-1,\ x,\ y) \end{bmatrix} = P^{n-1} \begin{bmatrix} P(1,\ 1,\ 1,\ x,\ y) \\ P(1,\ 1,\ 0,\ x,\ y) \end{bmatrix}, \tag{3.3}$$

$$\begin{bmatrix} Q(1,\ 1,\ n,\ x,\ y) \\ Q(1,\ 1,\ n-1,\ x,\ y) \end{bmatrix} = P^{n-1} \begin{bmatrix} Q(1,\ 1,\ 1,\ x,\ y) \\ Q(1,\ 1,\ 0,\ x,\ y) \end{bmatrix}, \tag{3.4}$$

$$P(1,\ 1,\ n,\ x,\ y) = [1 \quad 0]P^{n-1} \begin{bmatrix} P(1,\ 1,\ 1,\ x,\ y) \\ P(1,\ 1,\ 0,\ x,\ y) \end{bmatrix}, \tag{3.5}$$

$$Q(1,\ 1,\ n,\ x,\ y) = [1 \quad 0]P^{n-1} \begin{bmatrix} Q(1,\ 1,\ 1,\ x,\ y) \\ Q(1,\ 1,\ 0,\ x,\ y) \end{bmatrix}. \tag{3.6}$$

Now let us apply these matrices to obtain formulas for Pell and Pell-Lucas generating functions. First,

$$\begin{aligned}Q(1,\ 1,\ m+n,\ x,\ y) &= [1 \quad 0]P^{m+n-1} \begin{bmatrix} Q(1,\ 1,\ 1,\ x,\ y) \\ Q(1,\ 1,\ 0,\ x,\ y) \end{bmatrix} \quad \text{by (3.6)} \\ &= [P_m \quad P_{m-1}] \begin{bmatrix} Q(1,\ 1,\ n+1,\ x,\ y) \\ Q(1,\ 1,\ n,\ x,\ y) \end{bmatrix} \quad \text{by (3.4) and (1.19)} \\ &= P_mQ(1,\ 1,\ n+1,\ x,\ y) + P_{m-1}Q(1,\ 1,\ n,\ x,\ y).\end{aligned} \tag{3.7}$$

A similar formula pertains to $P(1,\ 1,\ m+n,\ x,\ y)$, viz.,

$$P(1,1,m+n,x,y) = P_mP(1,\ 1,\ n+1,\ x,\ y) + P_{m-1}P(1,\ 1,\ n,\ x,\ y). \tag{3.8}$$

Of course, (3.1) and (3.2) are special cases of (3.7) and (3.8) when $m = 2$.

Representative of another set of results is

$$P(1,\ 1,\ m+n,\ x,\ y) + (-1)^n P(1,\ 1,\ m-n,\ x,\ y) = Q_n P(1,\ 1,\ m,\ x,\ y) \tag{3.9}$$

Analogues of *Simson's formulas* can be established. Thus,

$$P^2(1,\ 1,\ n,\ x,\ y) - P(1,\ 1,\ n+1,\ x,\ y)P(1,\ 1,\ n-1,\ x,\ y) \tag{3.10}$$

$$= \begin{vmatrix} P(1,\ 1,\ n,\ x,\ y) & P(1,\ 1,\ n+1,\ x,\ y) \\ P(1,\ 1,\ n-1,\ x,\ y) & P(1,\ 1,\ n,\ x,\ y) \end{vmatrix}$$

$$= \left| P^{n-1} \begin{bmatrix} P(1,\ 1,\ 1,\ x,\ y) \\ P(1,\ 1,\ 0,\ x,\ y) \end{bmatrix} \middle| P^{n} \begin{bmatrix} P(1,\ 1,\ 1,\ x,\ y) \\ P(1,\ 1,\ 0,\ x,\ y) \end{bmatrix} \right| \quad \text{by (3.3)}$$

$$= |P^{n-1}| \begin{vmatrix} P(1,\ 1,\ 1,\ x,\ y) & 2xP(1,\ 1,\ 1,\ x,\ y) + P(1,\ 1,\ 0,\ x,\ y) \\ P(1,\ 1,\ 0,\ x,\ y) & P(1,\ 1,\ 1,\ x,\ y) \end{vmatrix} \quad \text{by (1.18)}$$

$$= (-1)^{n-1}\{P^2(1,\ 1,\ 1,\ x,\ y) - P(1,\ 1,\ 2,\ x,\ y)P(1,\ 1,\ 0,\ x,\ y)\} \quad \text{by (3.1)}$$

$$= (-1)^{n-1}(1 - 2xy - y^2)\Delta^2 \quad \text{by (1.3), (1.4), and (2.1)}$$

$$= (-1)^{n-1}P(1,\ 1,\ 1,\ x,\ y) \quad \text{by (2,1).}$$

Similarly,

$$Q^2(1,\ 1,\ n,\ x,\ y) - Q(1,\ 1,\ n+1,\ x,\ y)Q(1,\ 1,\ n-1,\ x,\ y) = 4(x^2+1)P(1,\ 1,\ 1,\ x,\ y). \tag{3.11}$$

More complicated algebra, with the use of the above method, produces the *generalized Simson's formula* analogues, namely,

$$P^2(1,\ 1,\ n,\ x,\ y) - P(1,\ 1,\ n+r,\ x,\ y)P(1,\ 1,\ n-r,\ x,\ y) = (-1)^{n-r}P_r^2P(1,\ 1,\ 1,\ x,\ y) \tag{3.12}$$

and

$$Q^2(1,\ 1,\ n,\ x,\ y) - Q(1,\ 1,\ n+r,\ x,\ y)Q(1,\ 1,\ n-r,\ x,\ y) = (-1)^{n+r+1}4(x^2+1)P_r^2P(1,\ 1,\ 1,\ x,\ y). \tag{3.13}$$

Other interesting results may be established by the methods exhibited, for example,

$$P(1,\ 1,\ 2n,\ x,\ y) = \frac{1}{2}\{P_nQ(1,\ 1,\ n,\ x,\ y) + Q_nP(1,\ 1,\ n,\ x,\ y)\}. \tag{3.14}$$

The above information represents a small sample of knowledge available to us. However, the algebra becomes quite awkward when the more general generating functions (2.1) and (2.2) are exploited in that context.

4. SUMS OF GENERATING FUNCTIONS

Let us now consider series whose terms are generating functions.

Summing in (3.1) used as a difference equation and tidying up, we come to

$$\sum_{r=1}^{n} P(1, 1, r, x, y) = \{P(1, 1, n+1, x, y) + P(1, 1, n, x, y) - P(1, 1, 1, x, y) - P(1, 1, 0, x, y)\}/2x. \tag{4.1}$$

For variation, consider next a matrix approach. Accordingly, by (3.6) applied repeatedly,

$$\begin{aligned}\sum_{r=1}^{n} Q(1, 1, r, x, y) &= [1 \quad 0][I + P + P^2 + \cdots + P^{n-1}]\begin{bmatrix} Q(1, 1, 1, x, y) \\ Q(1, 1, 0, x, y)\end{bmatrix} \\ &= \frac{1}{2x}[1 \quad 0]\begin{bmatrix} P_{n+1} + P_n - 1 & P_n + P_{n-1} - 1 \\ P_n + P_{n-1} - 1 & P_{n-1} + P_{n-2} - 2x - 1\end{bmatrix}\begin{bmatrix} Q(1, 1, 1, x, y) \\ Q(1, 1, 0, x, y)\end{bmatrix} \\ &= \{Q(1, 1, n+1, x, y) + Q(1, 1, n, x, y) - Q(1, 1, 1, x, y) - Q(1, 1, 0, x, y)\}/2x,\end{aligned} \tag{4.2}$$

by (3.7), (1.19), and [6, equation (2.11)].

Parallel treatments produce

$$\sum_{r=1}^{n} (-1)^r P(1, 1, r, x, y) = \{(-1)^n P(1, 1, n+1, x, y) + (-1)^{n-1} P(1, 1, n, x, y) - P(1, 1, 1, x, y) + P(1, 1, 0, x, y)\}/2x \tag{4.3}$$

and

$$\sum_{r=1}^{n} (-1)^r Q(1, 1, r, x, y) = \{(-1)^n Q(1, 1, n+1, x, y) + (-1)^{n-1} Q(1, 1, n, x, y) - Q(1, 1, 1, x, y) + Q(1, 1, 0, x, y)\}/2x. \tag{4.4}$$

Extensions of the above theory may be exhibited (see [7]) for

$$P(1, m, mr+k, x, y, z) = \sum_{r=0}^{\infty} P(1, m, mr+k, x, y) z^r \tag{4.5}$$

with a similar formulation for the Pell-Lucas generating functions.

5. GENERATING FUNCTIONS FOR SECOND POWERS OF PELL POLYNOMIALS

Exploiting (1.16) as a difference equation, we may demonstrate that, ultimately,

$$(1 - Q_2y + y^2)\sum_{r=0}^{\infty} P_r^2 y^r \tag{5.1}$$

$$= -y + 2y - 2y^2 + 2y^3 - \cdots + 2(-1)^{r-1}y^r + \cdots$$

$$= \frac{y - y^2}{1 + y},$$

whence

$$\sum_{r=0}^{\infty} P_r y^r = \frac{y - y^2}{1 - (4x^2 + 1)y - (4x^2 + 1)y^2 + y^3}, \tag{5.2}$$

that is,

$$P(2, 1, 0, x, y) = (y - y^2)\Delta^{(2)} \quad \text{by (1.12).} \tag*{(5.2)$'$}$$

Similarly,

$$Q(2, 1, 0, x, y) = (4 - (12x^2 + 4)y - 4x^2y^2)\Delta^{(2)}. \tag{5.3}$$

One may also show that

$$\sum_{r=0}^{\infty} P_{r+1}P_{r+2}y^r = 2x\Delta^{(2)} \tag{5.4}$$

and

$$\sum_{r=0}^{\infty} Q_{r+1}Q_{r+2}y^r = 2x\{(4x^2 + 2) + 2(4x^2 + 2)y - 2y^2\}\Delta^{(2)}. \tag{5.5}$$

Generalizations of (5.2) and (5.3) to expressions for $P(2, 1, m, x, y)$ and $Q(2, 1, m, x, y)$ are obtainable (see [7]). In particular,

$$P(2, 1, 1, x, y) = (1 - y)\Delta^{(2)}, \tag{5.6}$$

while

$$Q(2, 1, 2, x, y) = \{(4x^2 + 2)^2 + (16x^4 + 4x^2 - 4)y - 4x^2y^2\}\Delta^{(2)}. \tag{5.7}$$

Note in passing the marginally useful result that

$$P(2, 1, 1, x, y) - P(2, 1, 0, x, y) = (1 - y)^2\Delta^{(2)}, \tag{5.8}$$

which has an application in some complicated algebra elsewhere [7].

The theory outlined above extends (though not easily) to $P(2, 1, m, x, y)$ [and $Q(2, 1, m, x, y)$], and more generally to $P(2, m, mr + k, x, y)$. A difference equation resulting from this algebraic maelstrom, and which is useful in deriving fresh information, is

$$P(2, m, m + k, x, y) - Q_{2m}P(2, m, k, x, y) + P(2, m, -m + k, x, y) \tag{5.9}$$

$$= \frac{2(-1)^k P_m^2}{1 + y}.$$

6. GENERATING FUNCTIONS FOR CUBES OF PELL POLYNOMIALS

With care, we may demonstrate the validity of

$$P_{n+1}^3 - Q_3 P_n^3 - P_{n-1}^3 = (-1)^n 6xP_n. \tag{6.1}$$

Use this for summing to derive, first [cf. (1.9) and (1.12)],

$$(1 - Q_3 y - y^2)\sum_{r=0}^{\infty} P_r^3 y^r = y - 6xy^2\Delta', \tag{6.2}$$

whence

$$P(3, 1, 0, x, y) = (y - 4xy^2 - y^3)\Delta^{(3)} \tag{6.3}$$

in which

$$\Delta^{(3)}(1 - Q_3 y - y^2) = \Delta'. \tag{6.4}$$

Similarly,

$$Q(3, 1, 0, x, y) = \{8 - (56x^3 + 32x)y - (64x^4 + 48x^2 + 8)y^2 + 8x^3y^3\}\Delta^{(3)}. \tag{6.5}$$

Indulging in an orgy of algebra, we may construct (see [7]) a generalization of (6.1) relating to P_{mr+k}^3 as leading term. Ultimately, we establish a formula for $P(3, m, k, x, y)$, the generating function for P_{mr+k}^3, although it it not a pretty sight.

For possible interest we append the expression for $\Delta_{(m)}^{(3)}$, namely, cf. (1.13) also,

$$\Delta_{(m)}^{(3)} = \begin{bmatrix} 1 - \{Q_{3m} + (-1)^m Q_m\}y + (-1)^m\{Q_m Q_{3m} + 2\}y^2 \\ -(-1)^m\{Q_{3m} + (-1)^m Q_m\}y^3 + y^4 \end{bmatrix}^{-1} \tag{6.6}$$

Obviously, the foregoing theory could be developed almost *ad infinitum ad nauseam* for $P(j, m, k, x, y)$. Patience, skill, and motivation would be required for this task.

7. GENERATING FUNCTIONS FOR DIAGONAL FUNCTIONS

Rising diagonal functions R_n for $\{P_n\}$ and r_n for $\{Q_n\}$ were defined in [6]. *Descending diagonal functions* D_n and d_n for these polynomials also exist (see [7]). Work on these types of functions, but for other polynomials, may be found in [3], [4], and [5].

Write

$$D \equiv D(x, y) = \sum_{n=1}^{\infty} D_n y^{n-1}, \tag{7.1}$$

$$d \equiv d(x, y) = \sum_{n=2}^{\infty} d_n y^{n-1}, \tag{7.2}$$

$$R \equiv R(x, y) = \sum_{n=1}^{\infty} R_n y^{n-1}, \tag{7.3}$$

$$r \equiv r(x,\ y) = 1 + \sum_{n=2}^{\infty} r_n y^{n-1}. \tag{7.4}$$

Then, following [3]-[5], we find

$$D = \frac{1}{1 - (2x + 1)y}, \tag{7.5}$$

$$d = \frac{2x + 2}{1 - (2x + 1)y}, \tag{7.6}$$

$$R = \frac{1}{1 - 2xy - y^3}, \tag{7.7}$$

$$r = \frac{1 + y^3}{1 - 2xy - y^3}. \tag{7.8}$$

Furthermore,

$$\sum_{n=1}^{\infty} D_{2n} y^{n-1} = \frac{2x + 1}{1 + (2x + 1)^2 y} \tag{7.9}$$

and

$$\sum_{n=1}^{\infty} D_{2n-1} y^{n-1} = \frac{1}{1 + (2x + 1)^2 y}. \tag{7.10}$$

Partial differentiation yields

$$2y\,\frac{\partial D}{\partial y} - (2x + 1)\frac{\partial D}{\partial x} = 0, \tag{7.11}$$

$$2y\,\frac{\partial d}{\partial y} - (2x + 1)\left(\frac{\partial d}{\partial x} - 2D\right) = 0, \tag{7.12}$$

$$2y\,\frac{\partial R}{\partial y} - (2x + 3y^2)\frac{\partial R}{\partial x} = 0, \tag{7.13}$$

$$2y\,\frac{\partial r}{\partial y} - (2x + 3y^2)\frac{\partial r}{\partial x} - 6(r - R) = 0. \tag{7.14}$$

8. CONCLUDING REMARKS

Information provided above is merely "the tip of the iceberg." Much more lies to be discovered by effort and enterprise.

Clearly, there exists a corresponding investigation involving exponential generating functions.

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THE NAMING OF POPES AND A FIBONACCI SEQUENCE IN TWO NONCOMMUTING INDETERMINATES

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The naming of Popes is a serious matter,
It isn't just one of your holiday games.
I know you may think I'm as mad as a hatter,
But I say that a Pope must have two different names.

—Apologies to T. S. Eliot and
Old Possum's Book of Practical Cats

The year 1978 saw three occupations of the Chair of St. Peter and the second was the shortest reign of modern times. Luciano Albini was acclaimed as the successor to Paul VI but was fated to be Christ's Vicar on Earth for only a month. He nevertheless introduced a novelty. So impressed was he by his two predecessors that he chose the double appellation of John-Paul. The innovation seemed to meet general approval, as it affirmed continuity in Church policy while paying tribute to the two previous pontiffs. However, it was a dangerous precedent and it was fortunate indeed that the present Bishop of Rome did not feel obliged to follow his predecessor's example, but prudently opted simply to extend the line of John-Pauls. Indeed, a moment's reflection will reveal that if John-Paul I had insisted that all his successors should follow his lead in this matter the effect on papal nomenclature would have been catastrophic, although of considerable mathematical interest.

Disaster was averted, but let us look at the mathematics anyway. Suppose that John-Paul I had insisted that each future pope should take as his name the names of his two predecessors in chronological order. Commencing with Pope John XXIII, the "papal sequence," as we shall call it, would begin

$$J,\ P,\ JP,\ PJP,\ JP^2JP,\ PJPJP^2JP,\ JP^2JP^2JPJP^2JP,\ \ldots,$$

where J, P, and P^2 have their obvious meanings. An impossible situation for the popes of the third millenium; each would spend a great deal of time trying to remember his own name. However, this same sequence should delight the heart of any lover of the Golden Ratio because it can be regarded as a Fibonacci sequence in two noncommuting generators, J and P. We shall study this sequence with an eye to finding an efficient algorithm to determine P_n, the name of the n^{th} pope, where we shall take P_1 to be Pope John-Paul himself.

We shall begin with several simple observations. Denote the length of P_n by $|P_n|$, and denote by $|P_n|_J$ and $|P_n|_P$ the number of occurrences of John and Paul, respectively, in P_n. We use F_n to denote the n^{th} Fibonacci number.

Lemma 1: In the papal sequence, for all $n \geqslant 1$,

(i) $|P_n|_J = F_n, \quad |P_n|_P = F_{n+1}$;

(ii) $|P_n| = F_{n+2}$;

(iii) P_m ends in P_n for all $m \geqslant n$;

(iv) P_n does not contain two successive J's nor three successive P's.

Proof: Each of (i), (ii), and (iii) follow immediately from the definition of the papal sequence and induction.

(iv). From (iii) with $n = 1$, it follows that P_m ends in JP for all $m \geqslant 1$. It is obvious, then, that J^2 can never occur in the papal sequence. Next, obseve that P_n begins with PJ or with JP, according as n is even or odd (again this is immediate by induction). Hence, no P_n begins nor ends in P^2, a fact that ensures that P^3 never appears in our sequence.

This lemma allows us to reformulate our problem. Denote the reverse of P_n by $\bar{P}_n$. We associate with the papal sequence an infinite sequence $A = (a_i)_{i \in N}$, in which each a_i is either J, P, or P^2, by defining a_n to be the n^{th} term in $\bar{P}_m$ (read as a word in J, P, and P^2) for all m such that $|\bar{P}_m|$ is sufficiently long for this to make sense. Part (iii) of Lemma 1 guarantees that A is well-defined (to be precise, we should take m such that the length of $\bar{P}_m$, considered as a work in J, P, and P^2, is at least $n + 1$).

Since our problem is now of more mathematical than religious interest, we shall dispense with J, P, and P^2, replacing them by the symbols 0, 1, and 2, respectively. Since $|P_n|$ is known (up to the value of F_{n+2}), the papal sequence can be reconstructed from our sequence A. Furthermore, A begins in 1, and part (iv) of Lemma 1 tells us that A is a sequence in which each 1 and 2 is preceded and followed by 0, while 00 never occurs. Therefore, A can be reconstructed from the sequence B, which is obtained from A by deleting all the 0's (given that A begins in 1).

Our problem, then, is to discover a good way of generating this sequence B, which begins 12122..., the first five numbers corresponding to $\bar{P}_5$.

We introduce a sequence of finite sequences $B_0, B_1, B_2, \ldots$ (each of which, as we shall show, is an initial subsequence of its successor and of B). The sequence is defined recursively beginning $B_0 = 1$. We construct B_{m+1} from B_m by replacing each 1 by 12 and each 2 by 122. The next few B_i's are

B_1: 12,

B_2: 12122,

B_3: 1212212122122,

B_4: 1212212122122121221212212212122122.

Each B_i is an initial subsequence of its immediate (and hence of each) successor. Indeed, we can say more.

Lemma 2: For $n > 1$,

$$B_n = B_{n-1}^2 B_{n-2} B_{n-3} \ldots B_1 2,$$

the product being concatenation of the sequences.

Proof: We denote by Q the operation defined in the recursive definition of $(B_i)_{i \in N^0}$, that is

$$Q(B_i) = B_{i+1}, \; i = 0, 1, 2, \ldots .$$

The result is evidently true for $n = 2$. For $n \geqslant 3$, we obtain

$$\begin{aligned} B_n &= Q(B_{n-1}) = Q(B_{n-2}^2 B_{n-3} \cdots B_1 2) \text{ by the inductive hypothesis,} \\ &= Q(B_{n-2})Q(B_{n-2})Q(B_{n-3}) \ \ldots \ Q(B_1)Q(2) \\ &= B_{n-1}B_{n-1}B_{n-2} \cdots B_2 122 \\ &= B_{n-1}^2 B_{n-2} \cdots B_2 B_1 2. \end{aligned}$$

Remark: We can regard members of the sequence $(B_i)_{i \in N^0}$ as a set of generators for a semigroup S, whose multiplication is defined by concatenation. The operator $Q : S \to S$ is then seen to be an injective semigroup endomorphism.

Henceforth, we shall regard $\overline{P}_n$ as a finite sequence in 0, 1, 2, and, moreover, we shall agree to delete the 0's (as $\overline{P}_n$ can be recovered even if the 0's are deleted), but we shall denote this reduced version of $\overline{P}_n$ by the same symbol.

Lemma 3: For each $n \geqslant 1$, B_n is an initial subsequence of B. In fact,

$$B_n = \overline{P}_{2n+1}.$$

Proof: The proof is by induction. We shall prove the two identities

$$B_n = \overline{P}_{2n+1} \quad \text{and} \quad B_n B_{n-1} \cdots B_1 2 = \overline{P}_{2n} \cdot \overline{P}_{2n+1}, \ n \geqslant 1,$$

where the product on the right-hand side is defined by concatenation, with the understanding that two adjacent 1's are replaced by 2.

For $n = 1$, we have $B_1 = 12 = \overline{P}_3$ (as P_3 is JP^2JP), and $B_1 2 = \overline{P}_2 \cdot \overline{P}_3$, since $B_1 2 = 122$, while $\overline{P}_2 \cdot \overline{P}_3 = 11 \cdot 12 = 122$. Our inductive hypothesis is that

$$B_m = \overline{P}_{2m+1}$$

and

$$B_m B_{m-1} \cdots B_1 2 = \overline{P}_{2m} \cdot P_{2m+1} \text{ for all } 1 \leqslant m < n, \ n > 1.$$

Now, by Lemma 2, we have

$$B_n B_{n-1} \cdots B_1 2 = B_{n-1}^2 B_{n-2} \cdots B_1 2 B_{n-1} B_{n-2} \cdots B_1 2,$$

which, by the inductive hypothesis is equal to

$$\begin{aligned} &\overline{P}_{2n-1} \cdot (\overline{P}_{2n-2} \cdot \overline{P}_{2n-1}) \cdot (\overline{P}_{2n-2} \cdot \overline{P}_{2n-1}) \\ &= (\overline{P}_{2n-1} \cdot \overline{P}_{2n-2}) \cdot (\overline{P}_{2n-1} \cdot \overline{P}_{2n-2}) \cdot \overline{P}_{2n-1} \\ &= \overline{P}_{2n} \cdot (\overline{P}_{2n} \cdot \overline{P}_{2n-1}) = \overline{P}_{2n} \cdot \overline{P}_{2n+1}. \end{aligned}$$

Hence, by Lemma 2,

$$\begin{aligned} B_n &= B_{n-1}^2 B_{n-2} \cdots B_1 2 \\ &= B_{n-1}(B_{n-1}B_{n-2} \cdots B_1 2) \\ &= \overline{P}_{2n-1} \cdot (\overline{P}_{2n-2} \cdot \overline{P}_{2n-1}) = (\overline{P}_{2n-1} \circ \overline{P}_{2n-2}) \cdot \overline{P}_{2n-1} \\ &= \overline{P}_{2n} \cdot \overline{P}_{2n-1} = \overline{P}_{2n+1}, \end{aligned}$$

as required.

Result: Algorithm for constructing the papal sequence.

Odd Case: Suppose $n = 2m + 1$, $m \geqslant 0$.

1. Calculate $B_m = Q^m(1)$.
2. Write $\overline{B}_m$, the reverse of B_m.
3. Write 0 at the beginning and between each pair of symbols of $\overline{B}_m$.
4. Replace each 0, 1, and 2 by J, P, and P^2, respectively.

Even Case: Suppose $n = 2m$, $m \geqslant 1$.

1. Calculate $B_m = Q^m(1)$.
2. Suppose $B_m = b_1b_2\ldots b_k$, say. Truncate B_m at $B_m' = b_1b_2\ldots b_t$, where

$$\sum_{i=1}^{t} b_i = F_{2m+1} + 1.$$

   Replace $b_t = 2$ by 1 in B_m' to give B_m''.
3. Write $\overline{B}_m''$.
4. Insert 0 between each pair of symbols of $\overline{B}_m''$.
5. Replace each 0, 1, and 2 by J, P, and P^2, respectively.

Proof: The algorithm for the odd case is an immediate consequence of Lemma 3 together with the observations made on the occurrences of J in P_n, when n is odd.

On the other hand, if $n = 2m$, then $B_m = \overline{P}_{2m+1}$. But P_{2m+1} ends in P_{2m}, and so some initial subsequence of B_m corresponds to $\overline{P}_{2m}$. The remaining problem is to determine the length of this subsequence. Now, by Lemma 1(i),

$$|P_{2m}|_P = F_{2m+1};$$

thus, we need to truncate $B_m = b_1b_2\ldots b_k$ at b_t, where t is the least integer such that

$$\sum_{i=1}^{t} b_i \geqslant F_{2m+1}.$$

Finally, observe that P_{2m} begins PJ, P_{2m-1} ends in JP, whence $b_t = 2$ and

$$\sum_{i=1}^{t} b_i = F_{2m+1} + 1.$$

The result follows from these observations.

Example: P_4. Here, $m = 2$.

1. $B_2 = Q^2(1) = Q(12) = 12122$.
2. $F_{2m+1} = F_5 = 5$, so $F_{2m+1} + 1 = 6$. Hence, $B_2' = 1212$.
3. $\overline{B}_2'' = 1121$.

4. $1121 \to 1010201$.

5. P_4 is $PJPJP^2JP$.

Example: P_7. Here, $m = 3$.

1. $B_3 = Q^3(1) = Q^2(12) = Q(12122) = 1212212122122$.
2. $\overline{B}_3 = 2212212122121$.
3. $\overline{B}_3 \to 0202010202010201020202010201$.
4. P_7 is $JP^2JP^2JPJP^2JP^2JPJP^2JPJP^2JP^2JPJP^2JP$.

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INCREDIBLE IDENTITIES REVISITED

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Consider the numbers:

$$A = \sqrt{2 \cdot 11 + 2\sqrt{5}} + \sqrt{5};$$

$$B = \sqrt{11 + \sqrt{116}} + \sqrt{(11 + 5) - \sqrt{116} + 2\sqrt{5}(11 - \sqrt{116})}.$$

Although one feels that these numbers couldn't be equal, Shanks [2] assures us that they are. Indeed, Follin (as reported by Spohn [3]) points out that one may take 5, 11, and 116 as *indeterminates* subject only to the identity

$$5 = 11^2 - 116 \tag{1}$$

(which certainly is true for the usual interpretation of these strings of decimal digits). As we shall see, it is only the first 5 in A which needs to be given by the representation (1); the remaining 5's may be treated as a separate indeterminate. The proofs of the equality of A and B given in [2] and [3] seem to be little more than appeals to the principle, attributed to J. Littlewood, that "any identity, once written down, is trivial."

Please ask yourself the following questions before reading further:

1. Why does $A = B$ seem so unlikely?
2. Given that it is true that $A = B$, how can it be proved?

The answers to both questions can be traced to the same source, Book X of Euclid's *Elements* [1]. Indeed, in Proposition 42, it is shown that a number expressible as a sum of two incommensurate square roots of *rational* numbers has a unique such representation up to interchanging the order of the summands. This deals with question 1.

Much of Euclid's work deals with more complicated algebraic numbers, albeit only *constructible* numbers. In this analysis, repeated use is made of the rule

$$\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}, \tag{2}$$

which is employed forward and backward. That is, to take the square root of a quantity like $22 + 2\sqrt{5}$, one solves

$$\begin{aligned} a + b &= 22 \\ ab &= 5 \end{aligned} \tag{3}$$

to obtain a and b as $11 + \sqrt{116}$ and $11 - \sqrt{116}$. At this point, it is clear that our quantities A and B are the two different ways of associating

$$\sqrt{11 + \sqrt{116}} + \sqrt{11 - \sqrt{116}} + \sqrt{5}$$

using (2) to express the first sum that one takes in each case. Q.E.D.

Equation (2) has led to puzzles before. You can discover one by using the method (3) to obtain another expression for

$$\sqrt{2 + 2\sqrt{2}}.$$

One case where the method has a fairly satisfying answer is

$$\sqrt{5 + 2\sqrt{6}}.$$

Finally, while it seems that, in the case of

$$\sqrt{22 + 2\sqrt{5}},$$

the method has caused the complication to *ramify*, it does not lead to *proliferation*. To see this, find

$$\sqrt{11 + 2\sqrt{29}}.$$

Although Euclid's study of algebraic numbers is full of detailed discussion of points which seem to us to be misguided, it is sobering to note that it can lead to a natural explanation of an identity that is not very close to the surface in our modern theory of algebraic numbers.

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AFTERTHOUGHTS

Since composing the article, I have corresponded with Professor Shanks and others whose interest in this topic came to light in that correspondence. It seems that everyone has his own favorite proof of this identity, usually reflecting the individual's background in classical algebra.

It also appears that different types of proofs have different gestation times. The proof in Spohn's letter had multiple independent discoveries at that time, and a proof along the lines of my article was communicated to Shanks by J. G. Wendel of the University of Michigan in October 1984.

In all proofs, two separate parts must be distinguished. First, the quantities A and B can be shown to satisfy the same polynomial with rational coefficients, i.e., to be algebraically conjugate. This is most susceptible to proof by Littlewood's principle. To show that the numbers are actually equal as real numbers relies on special knowledge of the real roots of that polynomial. This is hidden in my proof because I need only distinguish the two square roots of a real number. Another tool which is used in my proof (but could be overlooked) is the fact that the sum of algebraic numbers is algebraic.

Shanks also notes that his proof is really a means of discovery of such identities, and he refers the reader to his article [4].

INCREDIBLE IDENTITIES REVISITED

ADDITIONAL REFERENCE

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ON THE EXISTENCE OF e-MULTIPERFECT NUMBERS

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Dedicated to the memory of Robert Arnold Smith

1. INTRODUCTION

By an exponential divisor (or e-divisor) of a positive integer $N > 1$ with canonical form

$$N = p_1^{a_1} \ldots p_r^{a_r},$$

we mean a divisor d of N of the form

$$d = p_1^{b_1} \ldots p_r^{b_r},\ b_i | a_i,\ i = 1, \ldots, r.$$

The sum of such divisors of N is denoted by $\sigma^{(e)}(N)$, and the number of such divisors by $\tau^{(e)}(N)$. By convention, 1 is an exponential divisor of itself, so that $\sigma^{(e)}(1) = 1$. The functions $\tau^{(e)}(N)$ and $\sigma^{(e)}(N)$ were introduced in [1] and have been studied in [1] and [2].

An integer N is said to be e-perfect whenever $\sigma^{(e)}(N) = 2N$, and e-multiperfect when $\sigma^{(e)}(N) = kN$ for an integer $k > 2$. In [1] and [2], several examples of e-perfect numbers are given. It is also proved in [2] that all e-perfect and all e-multiperfect numbers are even.

Several unsolved problems are listed in [2], and one of them is whether or not there exists an e-multiperfect number. In this paper, we show that if such a number exists, it must indeed be very, very large.

2. NOTATION AND SOME LEMMAS

In all that follows, the positive integer N is assumed to be an e-multiperfect number, so that

$$\sigma^{(e)}(N) = kN \text{ for some integer } k > 2. \tag{2.1}$$

Note that if n is a square-free integer, then $\sigma^{(e)}(n) = n$, so that if $(n, N) = 1$, then Nn is also e-multiperfect. Hence, we assume (as we may) in the future that N is powerful. Also note here that we have used the fact that $\sigma^{(e)}$ is a multiplicative function.

Write

$$N = 2^h(q_1^{a_1} \ldots q_s^{a_s})(p_1^{b_1} \ldots p_t^{b_t}), \tag{2.2}$$

where the p's and q's are distinct primes, and each a_i is a non-square integer $\geqslant 2$, and each b_j is a square integer $\geqslant 4$. It follows then that each $\sigma^{(e)}(q_i^{a_i})$ is even and each $\sigma^{(e)}(p_j^{b_j})$ is odd.

Let $k = 2^{\omega}M$, where M is odd and $\omega \geqslant 0$.

Lemma 2.3: N is even, i.e., $h \geqslant 2$.

This is a consequence of Theorem 2.2 of [2].

Lemma 2.4: $s < \omega + h$.

Proof: The relation $\sigma^{(e)}(N) = kN$ gives

$$\sigma^{(e)}(2^h)\left[\prod_{i=1}^{s}\sigma^{(e)}(q_i^{a_i})\right]\left[\prod_{j=1}^{t}\sigma^{(e)}(p_j^{b_j})\right] = 2^{\omega+h}M(q_1^{a_1}\ \cdots\ q_s^{a_s})(p_1^{b_1}\ \cdots\ p_t^{b_t}).$$

Since the only even factors on the left side are $\sigma^{(e)}(q_1^{a_1}), \ldots, \sigma^{(e)}(q_s^{a_s})$, and since $2|\sigma^{(e)}(2^h)$, the result follows.

In what follows, the letter p represents a prime.

Lemma 2.5:

$$\prod_{p \neq 2,\, q_1, \ldots, q_s}\left(1 + \frac{1}{p^2} + \frac{1}{p^3}\right) < (1.27885)\left(1 - \frac{1}{q_1^2}\right)\cdots\left(1 - \frac{1}{q_s^2}\right).$$

Remark: This is a stronger form of Lemma 2.1 of [2], where a similar result is proved with the multiplicative constant on the right being $27/16 \approx 1.6875$. For our present purpose, we need the above stronger result.

Proof of Lemma 2.5:

$$\begin{aligned}
\prod_{p \neq 2,\, q_1, \ldots, q_s}\left(1 + \frac{1}{p^2} + \frac{1}{p^3}\right) &< \prod_{p \neq 2,\, q_1, \ldots, q_s}\left(1 + \frac{1}{p^2}\right)\left(1 + \frac{1}{p^3}\right) \\
&= \prod_{p \neq 2,\, q_1, \ldots, q_s}\left(1 + \frac{1}{p^2}\right)^{-1}\left(1 - \frac{1}{p^4}\right)\left(1 + \frac{1}{p^3}\right) \\
&< \left[\zeta(2)\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{q_1^2}\right)\cdots\left(1 - \frac{1}{q_s^2}\right)\right] \\
&\qquad \cdot \left[\zeta(4)\left(1 - \frac{1}{2^4}\right)\left(1 - \frac{1}{q_1^4}\right)\cdots\left(1 - \frac{1}{q_s^4}\right)\right]^{-1} \\
&\qquad \cdot \left[\zeta(3)\left(1 - \frac{1}{2^3}\right)\left(1 - \frac{1}{q_1^3}\right)\cdots\left(1 - \frac{1}{q_s^3}\right)\right] \\
&< \frac{7}{10}\ \frac{\zeta(2)\zeta(3)}{\zeta(4)}\left(1 - \frac{1}{q_1^2}\right)\cdots\left(1 - \frac{1}{q_s^2}\right),
\end{aligned}$$

on utilizing the result that

$$\left[1 - \frac{1}{q_j^3}\right]\left[1 - \frac{1}{q_j^4}\right]^{-1} < 1,\ j = 1, \ldots, s.$$

Using

$$\zeta(2) < 1.64494,\ \zeta(3) < 1.20206, \text{ and } \zeta(4) < 1.08232$$

([3], p. 811), we obtain the proof of the lemma.

Lemma 2.6:

$$\frac{k}{1.27885} \leqslant \left(1 + \frac{1}{2^{(h-2)/2}}\right)\left[\left(1 + \frac{1}{q_1}\right)\left(1 - \frac{1}{q_1^2}\right) \cdots \left(1 + \frac{1}{q_s}\right)\left(1 - \frac{1}{q_s^2}\right)\right],$$

where $1 + 2^{(h-2)/2}$ is to be taken as $1 + \frac{1}{2}$ for $h = 2, 3$.

Proof:

$$k = \frac{\sigma^{(e)}(N)}{N} = \frac{\sigma^{(e)}(2^h)}{2^h} \cdot \left[\prod_{i=1}^{r} \frac{\sigma^{(e)}(p_i^{a_i})}{p_i^{a_i}}\right]\left[\prod_{j=1}^{s} \frac{\sigma^{(e)}(q_j^{b_j})}{q_j^{b_j}}\right].$$

We note first that, for any prime p, we have

$$\frac{\sigma^{(e)}(p^m)}{p^m} \quad \frac{\sigma^{(e)}(p^2)}{p^2} = 1 + \frac{1}{p}, \ m = 2, 3, \ldots .$$

Also, for $m \geqslant 2$,

$$\begin{aligned}
\frac{\sigma^{(e)}(p^m)}{p^m} &\leqslant (p^m + p^{m/2} + p^{m/3} + \cdots + p)/p^m \\
&< 1 + \frac{1}{p^{m/2}} + \frac{1}{p^{m/2+1}} + \frac{1}{p^{m/2+2}} + \cdots \\
&= 1 + \frac{1}{p^{(m/2-1)}(p - 1)}.
\end{aligned}$$

Thus,

$$\frac{\sigma^{(e)}(2^h)}{2^h} < 1 + \frac{1}{2^{(h/2)-1}} \text{ for } h \geqslant 4; \ \frac{\sigma^{(e)}(2^h)}{2^h} \leqslant 1 + \frac{1}{2}, \ h = 2, 3,$$

and

$$\frac{\sigma^{(e)}(q_j^{b_j})}{q_j^{b_j}} \leqslant 1 + \frac{1}{q_j}, \ j = 1, 2, \ldots, s.$$

Next,

$$\begin{aligned}
\prod_{i=1}^{r}\left[\frac{\sigma^{(e)}(p_i^{a_i})}{p_i^{b_i}}\right] &\leqslant \prod_{i=1}^{r} \frac{\sigma^{(e)}(p_i^4)}{p_i^4} = \prod_{i=1}^{r}\left(1 + \frac{1}{p_i^2} + \frac{1}{p_i^3}\right) \\
&\leqslant \prod_{p \neq 2,\, q_1, \ldots, q_s}\left(1 + \frac{1}{p^2} + \frac{1}{p^3}\right) \\
&< (1.27885)\left(1 - \frac{1}{q^2}\right) \cdots \left(1 - \frac{1}{q_s^2}\right),
\end{aligned}$$

on using Lemma 2.5. The result (2.6) now follows.

3. MAIN RESULTS

Given $k \geqslant 3$, we shall estimate h and s as functions of k and show that

$$\lim_{k\to\infty} h = \lim_{k\to\infty} s = \infty. \tag{3.1}$$

These follow from the results $\omega \leqslant \log k/\log 2$ and

$$h \geqslant s - \omega \geqslant [(1 - \log(32/27)/\log 2)\log k \\ - \log((1.27885)(1.5))]/\log(32.27). \tag{3.2}$$

To obtain (3.2), we utilize Lemmas 2.4 and 2.6. Thus,

$$\frac{k}{1.27885} \leqslant \left(1 + \frac{1}{2}\right)\prod_{i=1}^{s}\left(1 + \frac{1}{q_i}\right)\left(1 - \frac{1}{q_i^2}\right). \tag{3.3}$$

If we take logarithms of both sides and use the estimate that, for all i,

$$\left(1 + \frac{1}{q_i}\right)\left(1 - \frac{1}{q_i^2}\right) \leqslant \left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{3^2}\right) = \frac{32}{27}, \tag{3.4}$$

then, after carrying out routine calculations, we get (3.2) from (3.3).

Actually, the estimate for h in (3.2) can be vastly improved as shown below.

Let $H_0 = H_0(k)$ be the smallest value of h for which N, given by (2.2), is a solution of (2.1). Then we shall show that H_0 increases exponentially with k. In fact, there is a function $H(k)$ such that $H_0(k) \geqslant H(k)$ and $\log\log H \sim \log k$ as $k \to \infty$.

Let $Q_1 = 3$, $Q_2 = 5, \ldots$ be the sequence of odd primes. From (3.2), we have

$$\frac{k}{1.27885} \leqslant \left(1 + \frac{1}{2^{(H-2)/2}}\right)\prod_{i=1}^{H+\omega}\left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right) \tag{3.5}$$

Now let H be the smallest integer satisfying (3.5), so

$$\left(1 + \frac{1}{2^{(H-3)/2}}\right)\prod_{i=1}^{H-1+\omega}\left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right) < \frac{k}{1.27885} \\ \leqslant \left(1 + \frac{1}{2^{(H-2)/2}}\right)\prod_{i=1}^{H+\omega}\left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right). \tag{3.6}$$

It is clear that $H_0(k) \geqslant H(k)$.

Theorem 3.7: $\log\log H \sim \log k$ $(k \to \infty)$.

Proof: Taking logarithms and letting $k \to \infty$ and noting that

$$\log(1 + 2^{-(H-2)/2}) \leqslant \log\left(1 + \frac{1}{2}\right) = O(1) \quad (H \to \infty),$$

and similarly for $\log(1 + 2^{-(H-3)/2})$, and using the result

$$\sum_{i=1}^{t}\log\left(1 - \frac{1}{Q_i^2}\right) = O(1),\ t \to \infty,$$

we get

$$\begin{aligned}&\sum_{i=1}^{H+\omega-1}\log\left(1+\frac{1}{Q_i}\right)+O(1)\leqslant\log k+O(1)\\&\leqslant\sum_{i=1}^{H+\omega}\log\left(1+\frac{1}{Q_i}\right)+O(1).\end{aligned}\tag{3.8}$$

Note that as $k\to\infty$, $H\to\infty$, and

$$\sum_{i=1}^{H}\log\left(1+\frac{1}{Q_i}\right)\sim\log\log H\ (H\to\infty).$$

Thus, (3.8) gives

$$\log\log(H+\omega)\sim\log k\ (k\to\infty).$$

Since $\omega = 0\ (\log k)$, this gives

$$\log\log H\sim\log k\ (k\to\infty).\tag{3.9}$$

Explicit Lower Bounds for N

We shall now give some explicit lower bounds for $N(k)$, the smallest value of N for given values of k that satisfies (2.1).

First, we note the explicit values of $H = H(k)$ for certain small values of k.

Lemma 3.10:

(i) $H(3) = 4$	(iv) $H(6) = 426$
(ii) $H(4) = 41$	(v) $H(7) = 1382$
(iii) $H(5) = 135$	(vi) $H(8) = 4553$

Proof: We recall the definition of H and utilize its characterization given by (3.6). Then a computer calculation gives the above results.

Lemma 3.11: Let $P(x)$ denote the product of all the primes not exceeding x. Then

(i) $\log P(x) > .84x$ for $x \geqslant 101$,

(ii) $\log P(x) > .98x$ for $x \geqslant 7481$.

This follows from Theorem 10 of the estimates given by Rosser and Schoenfeld [4].

Of course, the Prime Number Theorem gives the result that $\log P(x) \sim x$.

Theorem 3.12:

$$N(3) > 2\cdot 10^7 \tag{3.13}$$

$$N(4) > 10^{85} \tag{3.14}$$

$$N(5) > 10^{320} \tag{3.15}$$

$$N(6) > 10^{1210};\ \text{also } N(k) > 10^{1210} \text{ for all even } k \text{ for which } \omega=\omega(k)=1. \tag{3.16}$$

$N(k) > 10^{5270}$ for all odd $k \geqslant 7$. (3.17)

$N(k) > 10^{19884}$ for all even $k \geqslant 8$, for which $\omega = \omega(k) = 3$. (3.18)

Proof: We shall use the results of Lemmas 3.10 and 3.11. We shall illustrate the proof by considering only a few cases.

Let

$$G(H, u) = \left(1 + \frac{1}{2^{(H-2)/2}}\right) \prod_{i=1}^{u} \left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right). \tag{3.19}$$

(i) $\underline{k = 3}$: Since $H(3) = 4$, by Lemma 2.5 and (3.6), we should have

$$G(3, u) \geqslant 3/1.27885.$$

A computer run shows that the smallest value of u for which this inequality holds is $u = 4$. Hence, $s \geqslant 4$ and

$$N(3) \geqslant 2^4 \prod_{i=1}^{4} Q^2 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 = 21344400 > 2 \cdot 10^7.$$

(ii) $\underline{k = 7}$: Since $H(7) = 1382$, $(H - 2)/2 = 691$. We should then have

$$G(7, u) \geqslant 7/1.27885.$$

A computer run shows that the smallest u that satisfies this is $u = 1382$. Thus,

$$N(7) \geqslant 2^{1382} \prod_{i=1}^{1382} Q_i^2 > 10^{5270}$$

on using Lemma 3.11.

(iii) $\underline{k \text{ odd} > 7}$: Then $H(k)$ satisfies

$$k/(1.27885) < \left(1 + \frac{1}{2^{(H(k)-2)/2}}\right) \prod_{i=1}^{H(k)} \left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right).$$

Since $7/1.27885 < k/1.27885$, we have $H(k) > H(7) = 1382$. Hence, the value of u that satisfies

$$G(k, u) > k/1.27885$$

is > 1382, and $N(k) > 10^{5270}$ for all odd $k > 7$.

(iv) $\underline{k = 8}$: We have $\omega = 3$ and $H = H(8) = 4553$. Thus, $(H - 2)/2 = 2276.5$ and

$$\frac{8}{1.27885} \leqslant \left(1 + \frac{1}{2^{2276 \cdot 5}}\right) \prod_{i=1}^{4553} \left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right).$$

A computer run shows that the smallest value of u for which

$$G(8, u) \geqslant 8/1.27885$$

is $u = 4556$. Hence, $s \geqslant 4556$ and

$$N(8) \geqslant 2^{4553} \prod_{i=1}^{4556} Q^2 > 10^{19884}.$$

on using Lemma 3.11 and a computer calculation.

(v) <u>k even and > 8 and $\omega = \omega(k) = 3$</u>: We have

$$\frac{8}{1.27885} < \frac{k}{1.27885} \leqslant \left(1 + \frac{1}{2^{(H(k)-2)/2}}\right) \prod_{i=1}^{H(k)+\omega} \left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right)$$

$$= \left(1 + \frac{1}{2^{(H(k)-2)/2}}\right) \prod_{i=1}^{H(k)+3} \left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right).$$

From this, it is clear that $H(k) > H(8)$ for all even k for which $\omega = \omega(k) = 3$.

Remark 3.20: Though we are unable to prove this, it is very likely that $H(k)$ increases monotonically with k for all $k \geqslant 3$. The numerical evidence supports this; therefore, we make the following conjectures.

Conjecture 3.21: $H(k)$ and $H_0(k)$ are monotonic functions of k for $k \geqslant 3$.

Conjecture 3.22: There are no e-multiperferfect numbers.

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A COMPLETE CHARACTERIZATION OF B-POWER FRACTIONS THAT CAN BE REPRESENTED AS SERIES OF GENERAL n-BONACCI NUMBERS

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1. INTRODUCTION AND MAIN RESULT

In 1953, Fenton Stancliff [5] noted that

$$\frac{1}{89} = \begin{array}{l} .0112358 \\ \quad\quad\;\; 13 \\ \quad\quad\quad 21 \\ \quad\quad\quad\; \cdots \end{array} = \sum_{k=0}^{\infty} 10^{-(k+1)}F_k,$$

where F_k denotes the k^{th} Fibonacci number. Until recently, this expansion was regarded as an anomalous numerical curiosity, possibly related to the fact that 89 is a Fibonacci number (see Remark in [5]), but not generalizing to other fractions in an obvious manner.

In 1980, C. F. Winans [6] showed that the sums $\sum 10^{-(k+1)}F_{\alpha k}$ approximate 1/71, 2/59, and 3/31 for $\alpha = 2$, 3, and 4, respectively. Moreover, he showed that the sums $\sum 10^{-2(k+1)}F_{\alpha k}$ approximate 1/9899, 1/9701, 2/9599, and 3/9301 for $\alpha = 1$, 2, 3, and 4, respectively.

Since then, several authors proved general theorems on fractions that can be represented as series involving Fibonacci numbers and general n-Bonacci numbers [1, 2, 3, 4]. In the present paper we will prove a theorem which includes as special cases all the earlier results. We introduce some notation in order to state our theorem.

Let arbitrary complex numbers $A_0, A_1, \ldots, A_m, W_0, W_1, \ldots, W_m$, and B be given. Construct the sequence W_k by the recursion

$$W_{n+m+1} = \sum_{r=0}^{m} A_r W_{n+m-r}$$

for $n \geqslant 0$ or, equivalently, by the formula

$$W_n = \sum_{r=0}^{m} \lambda_r \omega_r^n$$

for any integer n where ω_r ($r = 0, 1, \ldots, m$) are the zeros of the polynomial

$$q(z) = z^{m+1} - \sum_{r=0}^{m} A_r z^{m-r}$$

and $(\lambda_0, \lambda_1, \ldots, \lambda_m)$ is the unique solution of the system of $m + 1$ linear equations

$$\sum_{r=0}^{m} \lambda_r \omega_r^n = W_n \qquad (n = 0, 1, \ldots, m)$$

(see [2], p. 35). Finally, we introduce, for any integer α,

$$M(m) = \prod_{r=0}^{m} (B - \omega_r^{\alpha}).$$

Theorem: For integers $\alpha \geqslant 1$, $\beta \geqslant 0$, and any complex B satisfying

$$\max_{0 \leqslant r \leqslant m} |\omega_r^{\alpha}/B| < 1,$$

we have the formula

$$M(m) \cdot \sum_{k=0}^{\infty} B^{-k} W_{\alpha k + \beta} = B \cdot \sum_{r=0}^{m} \lambda_r \omega_r^{\beta} \cdot \prod_{\substack{0 \leqslant k \leqslant m \\ k \neq r}} (B - \omega_k^{\alpha}).$$

Remark: In the above formula, $M(m)$ and the right-hand side are in fact integers if B, A_0, A_1, ..., A_m, W_0, W_1, ..., W_m are all integers.

Now we can comment on earlier results in more detail. In 1981, Hudson and Winans [1] handled the case of the ordinary Fibonacci sequence with $\beta = 0$, $B = 10^n$. According to [3] and [4], their result can be written as

$$\sum_{k=1}^{\infty} 10^{-n(k+1)} F_{\alpha k} = \frac{F_{\alpha}}{10^{2n} - 10^n L_{\alpha} - (-1)^{\alpha}},$$

where L_{α} denote the Lucas numbers. Also in 1981, Long [4] treated the case of the general Fibonacci sequence, i.e., $m = 1$ and arbitrary A_0, A_1, W_0, W_1, and B, with the restriction, however, to $\alpha = 1$, $\beta = 0$. In 1985, Köhler [2] gave the generalization for arbitrary m, A_0, A_1, ..., A_m, W_0, W_1, ..., W_m, B, again with the restriction to $\alpha = 1$, $\beta = 0$. His result is

$$\sum_{k=1}^{\infty} B^{-k} W_{k-1} = p(B)/q(B),$$

where p is a polynomial of degree m with explicitly given coefficients. Also in 1985, Lee [3] discussed the cases $m = 1$ and $m = 2$ of general Fibonacci and Tribonacci sequences with arbitrary α and β. The results of [3] will be deduced from our Theorem in Examples 1 and 2 below. For this purpose, we introduce the notation

$$S_n = \sum_{r=0}^{m} \omega_r^n, \quad L(m) = M(m) \cdot \sum_{k=0}^{\infty} B^{-k} W_{\alpha k + \beta}.$$

Proof of Theorem: We have

$$\sum_{k=0}^{\infty} B^{-k} W_{\alpha k + \beta} = \sum_{k=0}^{\infty} B^{-k} \sum_{r=0}^{m} \lambda_r \omega_r^{\alpha k + \beta}$$
$$= \sum_{r=0}^{m} \lambda_r \omega_r^{\beta} \left(\sum_{k=0}^{\infty} (B^{-1} \omega_r^{\alpha})^k \right) = B \sum_{r=0}^{m} \lambda_r \omega_r^{\beta} \cdot \frac{1}{B - \omega_r^{\alpha}}.$$

Convergence is guaranteed by the condition on B. In the same way, we obtain

$$\sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\beta} = \sum_{r=0}^{m} \lambda_r \omega_r^{\beta} \cdot \frac{\omega_r^{\alpha}}{B - \omega_r^{\alpha}}.$$

Multiplying with $M(m)$ yields the Theorem.

Remark: Partial sums of the series in our Theorem can be expressed by these series, according to the formula

$$B^n \cdot \sum_{k=0}^{\infty} B^{-k} W_{\alpha k+\beta} = B^n \cdot \sum_{k=0}^{n} B^{-k} W_{\alpha k+\beta} + \sum_{k=1}^{\infty} B^{-k} W_{\alpha k+(\alpha n+\beta)}.$$

2. EXAMPLES

Example 1: The general Fibonacci sequence. Take $m = 1$. Then we have

$$W_{n+2} = A_0 W_{n+1} + A_1 W_n,$$

$$M(1) = (B - \omega_0^{\alpha})(B - \omega_1^{\alpha}) = B^2 - BS_{\alpha} + (-A_1)^{\alpha},$$

$$\begin{aligned}\sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\beta} &= \{\lambda_0 \omega_0^{\alpha+\beta}(B - \omega_1^{\alpha}) + \lambda_1 \omega_1^{\alpha+\beta}(B - \omega_0^{\alpha})\}/M(1) \\ &= (BW_{\alpha+\beta} - (-A_1)^{\alpha} W_{\beta})/M(1),\end{aligned}$$

$$\sum_{k=0}^{\infty} B^{-k} W_{\alpha k+\beta} = B(BW_{\beta} - (-A_1)^{\alpha} W_{\beta-\alpha})/M(1).$$

As to the partial sums, we get

$$\begin{aligned}\sum_{k=0}^{n} B^{n-k} W_{\alpha k+\beta} &= B^n L(1)/M(1) - \sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\alpha n+\beta} \\ &= \frac{B^n L(1) - BW_{\alpha(n+1)+\beta} + (-A_1)^{\alpha} W_{\alpha n+\beta}}{B^2 - BS_{\alpha} + (-A_1)^{\alpha}}.\end{aligned}$$

These formulas are equal to Theorems 1-3 of [3].

Example 2: The general Tribonacci sequence. Take $m = 2$. Then we obtain

$$W_{n+3} = A_0 W_{n+2} + A_1 W_{n+1} + A_2 W_n,$$

$$M(2) = B^3 - B^2 S_{\alpha} + BA_2^{\alpha} S_{-\alpha} - A_2^{\alpha},$$

$$\sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\beta} = (B^2 W_{\alpha+\beta} + B(W_{2\alpha+\beta} - S_{\alpha} W_{\alpha+\beta}) + A_2^{\alpha} W_{\beta})/M(2),$$

$$\sum_{k=0}^{\infty} B^{-k} W_{\alpha k+\beta} = B(B^2 W_{\beta} + B(W_{\alpha+\beta} - S_{\alpha} W_{\beta}) + A_2^{\alpha} W_{\beta-\alpha})/M(2),$$

$$\sum_{k=0}^{n} B^{n-k} W_{\alpha k+\beta} = B^n L(2)/M(2) - \sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\alpha n+\beta}$$

(continued)

$$= \frac{B^n L(2) - B^2 W_{\alpha(n+1)+\beta} + B(S_\alpha W_{\alpha(n+1)+\beta} - W_{\alpha(n+2)+\beta}) - A_2^\alpha W_{\alpha n+\beta}}{B^3 - B^2 S_\alpha + BA_2^\alpha S_{-\alpha} - A_2^\alpha}.$$

These formulas are equal to (9) and Theorems 7 and 8 in [3], and a misprint in Theorem 7 in [3] is corrected.

Example 3: The general Tetranacci sequence. Take $m = 3$. Then we have

$$W_{n+4} = A_0 W_{n+3} + A_1 W_{n+2} + A_2 W_{n+1} + A_3 W_n,$$

$$M(3) = B^4 - B^3 S_\alpha + B^2(S_\alpha^2 - S_{2\alpha})/2 - B(-A_3)^\alpha S_{-\alpha} + (-A_3)^\alpha,$$

$$\sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\beta} = \{B^3 W_{\alpha+\beta} + B^2(W_{2\alpha+\beta} - S_\alpha W_{\alpha+\beta}) + B(-A_3)^\alpha (S_{-\alpha} W_\beta - W_{\beta-\alpha}) - (-A_3)^\alpha W_\beta\}/M(3),$$

$$\sum_{k=0}^{\infty} B^{-k} W_{\alpha k+\beta} = B\{B^3 W_\beta + B^2(W_{\alpha+\beta} - S_\alpha W_\beta) + B(2W_{2\alpha+\beta} - 2S_\alpha W_{\alpha+\beta} + (S_\alpha^2 - S_{2\alpha})W_\beta)/2 - (-A_3)^\alpha W_{\beta-\alpha}\}/M(3),$$

$$\sum_{k=0}^{n} B^{n-k} W_{\alpha k+\beta} = \{B^n L(3) - B^3 W_{\alpha(n+1)+\beta} - B^2(W_{\alpha(n+2)+\beta} - S_\alpha W_{\alpha(n+1)+\beta}) - B(-A_3)^\alpha (S_{-\alpha} W_{\alpha n+\beta} - W_{\alpha(n+1)+\beta}) + (-A_3)^\alpha W_{\alpha n+\beta}\}/M(3).$$

Formulas for $m \geqslant 4$ can be obtained in a similar manner.

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THE EXISTENCE OF INFINITELY MANY k-SMITH NUMBERS

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(Submitted May 1985)

1. INTRODUCTION

A Smith number has been defined by A. Wilansky [2] to be a composite number whose digit sum is equal to the sum of the digits of all its prime factors. Wilansky presents figures indicating that 360 Smith numbers occur among the first ten thousand positive integers, and asks whether there are infinitely many Smith numbers. Oltika and Wayland [1] have noted that relatively large Smith numbers are easily generated from primes whose digits are all 0's or 1's, but that only a small number of such primes are known.

We show in this paper that infinitely many Smith numbers do exist, using an approach that does not depend upon the primality of the integers used in the construction. This approach shows that, in fact, a much more general result holds.

Let m be a positive integer greater than 1. We denote the number of digits of m by $N(m)$, the sum of the digits of m by $S(m)$, and the sum of all the digits of all the prime factors of m by $S_p(m)$. It may be noted that $S_p(m) = S(m)$ if m is prime, and $S_p(m) = S_p(m_1) + S_p(m_2)$ if $m = m_1m_2$ $(m_1, m_2 > 1)$.

Definition: Let m be a composite integer and k be any positive integer. m is a k-Smith number if $S_p(m) = kS(m)$.

An example of a 2-Smith number is $m = 104 = 2^3 \cdot 13$:

$$S_p(m) = 2 + 2 + 2 + 1 + 3 = 10 = 2(1 + 0 + 4) = 2S(m).$$

An example of a 3-Smith number is $402 = 2 \cdot 3 \cdot 67$. Among the positive integers less than 1000, there are 47 k-Smith numbers for $k = 1$ (see [2] for additional information on the distribution of Smith numbers), twenty-one for $k = 2$, three for $k = 3$, and one k-Smith number for each of $k = 7$, 9, and 14.

The principal result of this paper is that infinitely many k-Smith numbers exist for every positive integer k.

2. SOME FUNDAMENTAL PROPERTIES

First, we obtain an upper bound on $S_p(m)$ which does not involve the specific prime factors of m.

Theorem 1: If $p_1, \ldots, p_r$ are prime numbers, not necessarily distinct, and if $m = p_1p_2 \cdots p_r$, then $S_p(m) < 9N(m) - .54r$.

Proof: Let $b_i = N(p_i) - 1$, $i = 1, 2, \ldots, r$, and $b = b_1 + \cdots + b_r$. Now, the sum of the digits of a prime is not a multiple of 9, so

$$S(p_i) \leqslant 9N(p_i) - 1 = 9b_i + 8.$$

We partition the prime factors of m into 9 disjoint classes by means of the following: Let c_i be defined by $S(p_i) = 9b_i + c_i$, $c_i \leqslant 8$, $i = 1, 2, \ldots, r$, n_0 be the number of integers i $(1 \leqslant i \leqslant r)$ for which c_i is negative, and n_j be the number of integers i $(1 \leqslant i \leqslant r)$ such that $c_i = j$, for $1 \leqslant j \leqslant 8$. Then,

$$S_p(m) = \sum_{i=1}^{r} S(p_i) = \sum_{i=1}^{r} (9b_i + c_i) = 9b + \sum_{j=1}^{8} jn_j + \sum c_i,$$

where this last sum is over the n_0 values of i for which $c_i < 0$ (note that $c_i \neq 0$ for any i). Since the last sum is less than or equal to $-n_0$, we have

$$S_p(m) \leqslant 9b + \sum_{j=1}^{8} jn_j - n_0. \tag{1}$$

Now, $S(p_i) = 9b_i + c_i$ implies, for $c_i < 0$, that $p_i > 10^{b_i}$, and, for $1 \leqslant c_i \leqslant 8$, that

$$p_i \geqslant (c_i + 1) \cdot 10^{b_i} - 1 \geqslant (c_i + 9/10) \cdot 10^{b_i}, \text{ if } b_i > 0$$

(i.e., unless p_i is one of the primes 2, 3, 5, or 7), and

$$p_i = c_i 10^{b_i}, \text{ if } b_i = 0.$$

It follows that

$$\begin{aligned} m &= p_1 p_2 \cdots p_r \\ &\geqslant (1.9)^{n_1}(2)^{n_2}(3)^{n_3}(4.9)^{n_4}(5)^{n_5}(6.9)^{n_6}(7)^{n_7}(8.9)^{n_8} \cdot 10^b. \end{aligned}$$

Rewriting m as $a \cdot 10^{N(m)-1}$, for some rational number $1 \leqslant a < 10$, and taking logarithms, base 10, we have

$$\log a + N(m) - 1 \geqslant n_1 \log 1.9 + \cdots + n_8 \log 8.9 + b,$$

so

$$9N(m) \geqslant 9b + n_1(9 \log 1.9) + \cdots + n_8(9 \log 8.9) + 9(1 - \log a).$$

For each integer j $(1 \leqslant j \leqslant 8)$, we find that the coefficient of n_j is greater than $j + .54$. Hence,

$$9b + \sum_{j=1}^{8} n_j(j + .54) < 9N(m),$$

that is,

$$9b + \sum_{j=1}^{8} jn_j < 9N(m) - .54(n_1 + \cdots + n_8). \tag{2}$$

Combining (1) and (2), we have

$$S_p(m) < 9N(m) - .54(n_0 + n_1 + \cdots + n_8) - .46n_0 \leqslant 9N(m) - .54r. \text{ Q.E.D.}$$

We now state without proof a fact that is surely well known but which we have not found in the literature. The proof follows readily upon writing t as

$$\sum_{i=0}^{k} a_i 10^i \quad (k < n).$$

Theorem 2: If $10^n - 1$ is multiplied by a positive integer $t \leqslant 10^n - 1$, the digit sum of the product is $9n$ $(n \geqslant 1)$.

3. k-SMITH NUMBERS

Theorem 3: Let c be any nonnegative integer. There exist infinitely many integers M for which $S_p(M) = S(M) + c$.

Proof: Let $m = 10^n - 1$, $n \geqslant 2$. Since $3^2 | m$, m has at least three prime factors, so, by Theorem 1, $S_p(m) \leqslant 9N(m) - 2 = 9n - 2$. Let $h = 9n - S_p(m) \geqslant 2$. We define

$$T = \{2, 3, 4, 5, 8, 7, 15\},$$

making

$$\{S_p(t) | t \in T\} = \{2, 3, 4, 5, 6, 7, 8\}$$

a complete residue system (mod 7).

Since c nonnegative implies that $h + c \geqslant 2$, there exists an integer $t \in T$ such that $S_p(t) = h + c - 7b$ for some nonnegative integer b. We now consider the product $M = t(10^n - 1) \cdot 10^b$.

Noting that a power of 10 times a number has the same digit sum as the number, we have, by Theorem 2, $S(M) = 9$. Hence,

$$\begin{aligned} S_p(M) &= S_p(t) + S_p(10^n - 1) + S_p(10^b) \\ &= (h + c - 7b) + (9n - h) + 7b \\ &= 9n + c \\ &= S(M) + c. \end{aligned}$$

This secures the theorem, since each n determines a unique M.

Corollary: There exist infinitely many k-Smith numbers for each positive integer k.

Proof: Let k and n be positive integers, and M be defined as in Theorem 3. We need only choose c equal to $(k - 1) \cdot 9n = (k - 1)S(M)$; thus,

$$S_p(M) = S(M) + (k - 1)S(M) = kS(M).$$

When $k = 1$, we have, of course, a Smith number for each integer $n \geqslant 2$ [actually, for $n \geqslant 1$, since $S(t(10^1 - 1)) = 9$ for each $t \in T$].

The following algorithm for constructing k-Smith numbers is implicit in the proofs of Theorem 3 and the Corollary.

Algorithm:

1. Let $n \geqslant 2$ and factor $m = 10^n - 1$;
2. Compute $S_p(m)$ and set $h = 9n - S_p(m)$;
3. Solve $x = h + (k - 1)9n - 7b$, $2 \leqslant x \leqslant 8$, and find $t \in T$ such that $S_p(t) = x$.
4. $M = t(10^n - 1) \cdot 10^b$ is a k-Smith number.

Example 1: A Smith number ($k = 1$).

Let $m = 10^6 - 1 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$ ($n = 6$ has been chosen arbitrarily). $S_p(m) = 32$; $h = 54 - 32 = 22$. $x = 22 - 7b$ implies that $x = 8$ and $b = 2$. $S(t) = 8$ implies $t = 15$. Hence, $M = 15(10^6 - 1) \cdot 10^2 = 1{,}499{,}998{,}500$ is a Smith number. [$S_p(M) = S_p(3) + S_p(5) + 32 + 14 = 54 = S(M)$.]

Example 2: A 6-Smith number.

Let $m = 10^2 - 1 = 3^2 \cdot 11$. $S_p(m) = 8$; $h = 18 - 8 = 10$. $x = 100 - 7b$ implies that $x = 2$ and $b = 14$. $S(t) = 2$ implies $t = 2$. Hence, $M = 2(10^2 - 1) \cdot 10^{14} = 2^{15} \cdot 3^2 \cdot 11 \cdot 5^{14}$ is a 6-Smith number. [$S_p(M) = 30 + 6 + 2 + 70 = 108 = 6S(M)$.]

4. SOME REMAINING QUESTIONS

Thus far, it has become clear that there exist infinitely many integers m for which $S_p(m)$ far exceeds $S(m)$. Now, it is conceivable that the "opposite" relationship may also hold. If, in fact, one examines the composite integers $m < 1000$, one finds that $S_p(m) < S(m)$ for approximately 37% of these values. We make the following definition.

Definition: Let m be a composite integer and k be any positive integer. m is a k^{-1}-Smith number if $S_p(m) = k^{-1}S(m)$. [That is, if $kS_p(m) = S(m)$.]

There are nine k^{-1}-Smith numbers ($k > 1$) less than 1000—all 2^{-1}-Smith numbers. The smallest is 88:

$$S_p(88) = S_p(2^3 \cdot 11) = 8 - \tfrac{1}{2}S(88),$$

and an example of a 3^{-1}-Smith number is 19,998. The largest k for which we have found a k^{-1}-Smith number is 6:

$$3^2 \cdot 11 \cdot 101 \cdot (100003)^2 = 99{,}995{,}999{,}489{,}991$$

is a 6^{-1}-Smith number.

The following argument shows that it is possible that k^{-1}-Smith numbers exist for larger integers k.

Suppose that, for some integer $n > 2$, $10^n + 1$ is a prime (this implies that n is a power of 2 and $n \geqslant 1024$; see [3, p. 63]).

Let $\binom{t}{a}$ be the largest binomial coefficient in the expansion of $(10^n + 1)^t$, t any integer such that $\binom{t}{a} < 10^4 - 1$, and let $m = 9999(10^n + 1)^t$. The restriction on t assures that the coefficient of 10^{jn} ($0 \leqslant j \leqslant t$) in the expansion of m has digit sum 36, by Theorem 2. Since $9999\binom{t}{a}10^{jn} < 10^{(j+1)n}$, $S(m) = 36(t + 1)$, and it is clear that $S_p(m) = 10 + 2t$. Thus, for $t = 1, 3, 7$, and 13, m is a k^{-1}-Smith number for $k = 6, 9, 12$, and 14, respectively. However, at present no primes of the form $10^n + 1$, other than 11 and 101, are known.

Accordingly, we pose the following questions: Is there a k^{-1}-Smith number for every integer k? If not, what is the largest k for which k^{-1}-Smith numbers exist? Do there exist infinitely many 2^{-1}-Smith numbers? Do there exist infinitely many k^{-1}-Smith numbers for any $k > 2$?

We conjecture that the answer to each of the last two questions is "yes."

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A FAMILY OF FIBONACCI-LIKE SEQUENCES

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We consider the recurrence relation

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j,$$

where $G_0 = G_1 = 1$, and we express G_n in terms of the Fibonacci numbers F_n and F_{n-1}, and in the parameters $\alpha_0, \ldots, \alpha_k$.

For integer values of $k, \alpha_0, \ldots, \alpha_k$, the relation

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j, \tag{1}$$

where $G_0 = G_1 = 1$, forms a difference equation that can be solved by standard methods. In this note, we provide such a solution for equations of this type, in which we treat $\alpha_0, \ldots, \alpha_k$ as parameters.

First, the solution $G_n^{(h)}$ of the corresponding homogeneous equation equals

$$G_n^{(h)} = C_1\phi_1^n + C_2\phi_2^n,$$

where $\phi_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\phi_2 = \frac{1}{2}(1 - \sqrt{5})$; cf. e.g., [1] and [3].

Second, as a particular solution, we try

$$G_n^{(p)} = \sum_{i=0}^{k} A_i n^i,$$

which yields

$$\sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k} A_i (n-1)^i - \sum_{i=0}^{k} A_i (n-2)^i - \sum_{i=0}^{k} \alpha_i n^i = 0$$

or

$$\sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k} \left(\sum_{\ell=0}^{i} A_i \binom{i}{\ell} (-1)^{i-\ell} (1 + 2^{i-\ell}) n^\ell \right) - \sum_{i=0}^{k} \alpha_i n^i = 0.$$

For each i $(0 \leqslant i \leqslant k)$, we have

$$A_i - \sum_{m=i}^{k} \beta_{im} A_m - \alpha_i = 0, \tag{2}$$

where, for $m \geqslant i$,

$$\beta_{im} = \binom{m}{i} (-1)^{m-i} (1 + 2^{m-i}).$$

From the recurrence relation (2), $A_k, \ldots, A_0$ can be computed (in that order): A_i is a linear combination of $\alpha_i, \ldots, \alpha_k$. However, a more explicit expression for A_i can be obtained by setting

$$A_i = -\sum_{j=i}^{k} a_{ij}\alpha_j.$$

(The minus sign happens to be convenient in the sequel.) Then (2) implies

$$-\sum_{j=i}^{k} a_{ij}\alpha_j + \sum_{m=i}^{k} \beta_{im}\left(\sum_{\ell=m}^{k} a_{m\ell}\alpha_\ell\right) - \alpha_i = 0.$$

Since $\beta_{ii} = 2$, we have, for $0 \leqslant i \leqslant k$,

$$a_{ii} = 1$$
$$a_{ij} = -\sum_{m=i+1}^{j} \beta_{im}a_{mj}, \text{ if } j > i.$$

Hence,

$$G_n^{(p)} = -\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}\alpha_j n^i = -\sum_{j=0}^{k}\alpha_j\left(\sum_{i=0}^{j} a_{ij}n^i\right).$$

Finally, we ought to determine C_1 and C_2: $G_0 = G_1 = 1$ implies

$$C_1 + C_2 = 1 - G_0^{(p)},\ C_1\phi_1 + C_2\phi_2 = 1 - G_1^{(p)}.$$

These equalities yield

$$\begin{aligned} C_1 &= ((G_0^{(p)} - 1)\phi_2 + 1 - G_1^{(p)})(\sqrt{5})^{-1} \\ &= ((1 - G_0^{(p)})\phi_1 - G_1^{(p)} + G_0^{(p)})(\sqrt{5})^{-1}, \end{aligned}$$

$$\begin{aligned} C_2 &= ((G_0^{(p)} - 1)\phi_1 + G_1^{(p)} - 1)(\sqrt{5})^{-1} \\ &= -((1 - G_0^{(p)})\phi_2 - G_1^{(p)} + G_0^{(p)})(\sqrt{5})^{-1}, \end{aligned}$$

and

$$G_n = (1 - G_0^{(p)})F_n + (-G_1^{(p)} + G_0^{(p)})F_{n-1} + G_n^{(p)}.$$

Summarizing, we have the following proposition.

Proposition: The solution of (1) can be expressed as

$$G_n = (1 + \Lambda_k)F_n + \lambda_k F_{n-1} - \sum_{j=0}^{k}\alpha_j p_j(n),$$

where Λ_k is a linear combination of $\alpha_0, \ldots, \alpha_k$, λ_k is a linear combination of $\alpha_1, \ldots, \alpha_k$, and for each j $(0 \leqslant j \leqslant k)$, $p_j(n)$ is a polynomial of degree j:

$$\Lambda_k = \sum_{j=0}^{k} a_{0j}\alpha_j,\quad \lambda_k = \sum_{j=1}^{k}\left(\sum_{i=1}^{j} a_{ij}\right)\alpha_j,\quad p_j(n) = \sum_{i=0}^{j} a_{ij}n^i.$$

Remarks:

(1) For $j = 0, 1, \ldots, 8$, the polynomials $p_j(n)$ are given in Table 1.

(2) No assumptions on $\alpha_0, \ldots, \alpha_k$ have been made; thus, they may be rational or real numbers as well.

(3) Changing $G_1 = 1$ into $G_1 = c$ only affects λ_k; it has to be increased with $c - 1$.

Table 1

j	$p_j(n)$
0	1
1	$n + 3$
2	$n^2 + 6n + 13$
3	$n^3 + 9n^2 + 39n + 81$
4	$n^4 + 12n^3 + 78n^2 + 324n + 673$
5	$n^5 + 15n^4 + 130n^3 + 810n^2 + 3365n + 6993$
6	$n^6 + 18n^5 + 195n^4 + 1620n^3 + 10095n^2 + 41958n + 87193$
7	$n^7 + 21n^6 + 273n^5 + 2835n^4 + 23555n^3 + 146853n^2 + 610351n + 1268361$
8	$n^8 + 24n^7 + 364n^6 + 4536n^5 + 47110n^4 + 391608n^3 + 2441404n^2 + 10146888n + 21086113$

(4) The coefficients of $\alpha_0, \alpha_1, \alpha_2, \ldots$ in Λ_k and of $\alpha_1, \alpha_2, \ldots$ in λ_k are independent of k. Thus, they give rise to two infinite sequences Λ and λ of natural numbers, as k tends to infinity, of which the first few elements are

Λ: 1, 3, 13, 81, 673, 6993, 87193, 1268361, 21086113, ...,

λ: 1, 7, 49, 415, 4321, 53887, 783889, 13031935, ...

Neither of these sequences is included in [2].

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-586 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Show that $5\sum_{k=0}^{n} F_{k+1}F_{n+1-k} = (n+1)F_{n+3} + (n+3)F_{n+1}$.

B-587 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Let $y = \sum_{n=0}^{\infty} F_n x^n/n!$ and $z = \sum_{n=0}^{\infty} L_n x^n/n!$.

Show that $y'' = y' + y$ and $z'' = z' + z$.

B-588 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Find the y and z of Problem B-587 in closed form.

B-589 *Proposed by Herta T. Freitag, Roanoke, VA*

The number $N = 0434782608695652173913$ has the property that the digits of KN are a permutation of the digits of N for $K = 1, 2, \ldots, m$. Determine the largest such m.

B-590 *Proposed by Herta T. Frietag, Roanoke, VA*

Generalize on Problem B-589 and describe a method for predicting the leftmost digit of KN.

B-591 *Proposed by Mihaly Bencze, Jud. Brasa, Romania*

Let $F(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ with each a_n in $\{0, 1\}$.

Prove that $f(x) \neq 0$ for all x in $-1/a < x < 1/a$, where $a = (1 + \sqrt{5})/2$.

SOLUTIONS

Constant Modulo 5

B-562 *Proposed by Herta T. Freitag, Roanoke, VA*

Let c_n be the integer in $\{0, 1, 2, 3, 4\}$ such that

$$c_n \equiv L_{2n} + [n/2] - [(n - 1)/2] \pmod 5,$$

where $[x]$ is the greatest integer in x. Determine c_n as a function of n.

Solution by J. Suck, Essen, Germany

$c_n = 3$ for all $n \in Z$. From the very definition, we see that $L_n \equiv 2, 1, 3, 4 \pmod 5$ for $n \equiv 0, 1, 2, 3$, respectively, (mod 4). Hence

$$L_{2n} \equiv \begin{cases} 2 & \text{for } n \text{ even} \\ 3 & \text{for } n \text{ odd.} \end{cases}$$

But for n even,

$$\left[\frac{n}{2}\right] - \left[\frac{n}{2} - \frac{1}{2}\right] = \frac{n}{2} - \left(\frac{n}{2} - 1\right) = 1,$$

and for n odd,

$$\left[\frac{n-1}{2} + \frac{1}{2}\right] - \left[\frac{n-1}{2}\right] = \frac{n-1}{2} - \frac{n-1}{2} = 0.$$

So,

$$L_{2n} + \left[\frac{n}{2}\right] - \left[\frac{n-1}{2}\right] \equiv \begin{cases} 2 + 1, & n \text{ even} \\ 3 + 0, & n \text{ odd} \end{cases} = 3 \pmod 5.$$

Also solved by Paul S. Bruckman, Lászlô Cseh, L. A. G. Dresel, Piero Filipponi, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Imre Merényi, Bob Prielipp, Heinz-Jürgen Seiffert, and the proposer.

2 of 3 Are Multiples of 4

B-563 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_n = \sum_{i=1}^{n} L_{2i+1} L_{2i-2}$. For which values of n is S_n exactly divisible by 4?

Solution by J. Suck, Essen, Germany

From the definition of the Lucas numbers we see that if $k \equiv 0, 1, 2, 3, 4, 5 \pmod 6$, then $L_k \equiv 2, 1, 3, 0, 3, 3 \pmod 4$, respectively. Hence, if $i \equiv 1$, 2, 0 (mod 3), then $L_{2i+1}L_{2i-2} \equiv 0 \cdot 2 \equiv 0,\ 3 \cdot 3 \equiv 1,\ 1 \cdot 3 \equiv 3 \pmod 4$, respectively. This, of course, implies that $S_n \equiv 0 \pmod 4$ if and only if $n \equiv 1$ or $0 \pmod 3$ and $S_n \equiv 1$ otherwise.

Also solved by Paul S. Bruckman, Lászlő Cseh, L. A. G. Dresel, Piero Filipponi, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Bob Prielipp, Heinz-Jürgen Seiffert, and the proposer.

Summing $[aF_k]$

B-564 *Proposed by Lászlő Cseh, Cluj, Romania*

Let $a = (1 + \sqrt{5})/2$ and $[x]$ be the greatest integer in x. Prove that

$$[aF_1] + [aF_2] + \cdots + [aF_n] = F_{n+3} - [(n + 4)/2].$$

Solution by Paul S. Bruckman, Fair Oaks, CA

First we note that $aF_k = 5^{-1/2}(a^{k+1} - b^{k+1} + b^k(b - a)) = F_{k+1} - b^k$. Since $-1 < b < 0$, thus $[aF_{2k}] = F_{2k+1} - 1$, $[aF_{2k+1}] = F_{2k+2}$, or $[aF_k] = F_{k+1} - e_k$, where e_k is the characteristic function of the even integers.

Let $S_n \equiv \sum_{k=1}^{n} [aF_k]$. Then

$$S_n = \sum_{k=1}^{n} (F_{k+1} - e_k) = \sum_{k=1}^{n} (F_{k+3} - F_{k+2}) - \left[\frac{n}{2}\right] = F_{n+3} - F_3 - \left[\frac{n}{2}\right]$$
$$= F_{n+3} - \left[\frac{n+4}{2}\right]. \quad \text{Q.E.D.}$$

Also solved by Piero Filipponi, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Imre Merényi, Bob Prielipp, Heinz-Jürgen Seiffert, J. Suck, and the proposer.

Fibonacci-Pell Products Summed

B-565 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Let $P_0, P_1, \ldots$ be the sequence of Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \in \{2, 3, \ldots\}$. Show that

$$9\sum_{k=0}^{n} P_k F_k = P_{n+2}F_n + P_{n+1}F_{n+2} + P_n F_{n-1} - P_{n-1}F_{n+1}.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

Let R_n denote the right member in the statement of the problem. Then

$$R_n = (2P_{n+1} + P_n)F_n + P_{n+1}(F_{n+1} + F_n) + P_n(F_{n+1} - F_n)$$
$$- (P_{n+1} - 2P_n)F_{n+1};$$

after simplification, this reduces to

$$R_n = 3(P_{n+1}F_n + P_n F_{n+1}). \tag{1}$$

Therefore,

$$\Delta R_n \equiv R_{n+1} - R_n = 3(P_{n+2}F_{n+1} - P_{n+1}F_n + P_{n+1}F_{n+2} - P_nF_{n+1})$$
$$= 3\{(2P_{n+1} + P_n)F_{n+1} - P_{n+1}F_n + P_{n+1}(F_{n+1} + F_n) - P_nF_{n+1}\},$$

which reduces to

$$\Delta R_n = 9P_{n+1}F_{n+1}. \tag{2}$$

On the other hand, let S_n denote the left member in the statement of the problem. Clearly,

$$\Delta S_n = 9P_{n+1}F_{n+1}. \tag{3}$$

Since $\Delta R_n = \Delta S_n$, this implies that

$$R_n = S_n + c,\ n = 0, 1, 2, \ldots, \tag{4}$$

for some constant c (independent of n). Since $P_0 = F_0 = 0$, thus

$$R_0 = 0 \quad \text{and} \quad S_0 = 9P_0F_0 = 0.$$

Setting $n = 0$ in (4), we find that $0 = R_0 = S_0 + c = c$, i.e., $c = 0$. Therefore,

$$R_n = S_n \text{ for all } n. \quad \text{Q.E.D.} \tag{5}$$

Also solved by L. A. G. Dresel, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Heinz-Jürgen Seiffert, and the proposer.

Lucas-Pell Products Summed

B-566 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*

Let P_n be as in B-565. Show that

$$9\sum_{k=0}^{n} P_kL_k = P_{n+2}L_n + P_{n+1}L_{n+2} + P_nL_{n-1} - P_{n-1}L_{n+1} - 6.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

The proof is similar to that of B-565. Using the same notation, we find, as before, that

$$\Delta R_n = 9P_{n+1}L_{n+1} = \Delta S_n, \tag{1}$$

and

$$R_n = S_n + c,\ n = 0, 1, 2, \ldots, \tag{2}$$

for some constant c (independent of n).

Also, however, we have the following relation, which differs from (1) in the solution of B-565:

$$R_n = 3(P_{n+1}L_n + P_nL_{n+1}) - 6. \tag{3}$$

As before, $S_0 = 9P_0L_0 = 0$; also, using (3), $R_0 = 3(1 \cdot 2 + 0 \cdot 1) - 6 = 0$. Setting $n = 0$ in (2), as before, we find that $c = 0$. Thus,

$$R_n = S_n \text{ for all } n. \quad \text{Q.E.D.} \tag{4}$$

Also solved by L. A. G. Dresel, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, J. Suck, and the proposer.

Relatives of Hermite Polynomials

B-567 *Proposed by P. Rubio, Dragados Y Construcciones, Madrid, Spain*

Let $a_0 = a_1 = 1$ and $a_{n+1} = a_n + na_{n-1}$ for n in $Z^+ = \{1, 2, \ldots\}$. Find a simple formula for

$$G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k.$$

Solution by L. A. G. Dresel, Reading. England

Putting $A_k = a_k/k!$, we have

$$G(x) = \sum_{k=0}^{\infty} A_k x^k,$$

where $A_0 = A_1 = 1$ and $(n + 1)A_{n+1} = A_n + A_{n-1}$ for $n = 1, 2, \ldots$. It follows that the series for $G(x)$ is convergent and differentiable, and

$$\frac{dG}{dx} = \sum_{k=0}^{\infty} (k + 1)A_{k+1}x^k = A_1 + \sum_{k=1}^{\infty} (A_k + A_{k-1})x^k = \sum_{k=0}^{\infty} (A_k x^k + A_k x^{k+1})$$
$$= (1 + x)G.$$

Since $G(0) = 1$, we can integrate the differential equation for G to obtain

$$G(x) = e^{x + \frac{1}{2}x^2}.$$

Also solved by Duane Broline, Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, Dario Castellanos, László Cseh, Alberto Facchini, J. Foster, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Imre Merényi, Heinz-Jürgen Seiffert, J. Suck, David Zeitlin, and the proposer.

Editorial Note: Castellanos and Zeitlin pointed out that $a_n = 2^{-n/2}i^n H_n(-i/\sqrt{2})$, where the H_n are the Hermite polynomials. Bruckman, Seiffert, and Zeitlin gave the explicit formula:

$$a_n = n! \sum_{k=0}^{[n/2]} (1/2^k(n - 2k)!k!).$$

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-406 *Proposed by R. A. Melter, Long Island University, Southampton, NY and I. Tomescu, University of Bucharest, Romania*

Let A_n denote the set of points on the real line with coordinates 1, 2, ..., n. If $F(n)$ denotes the number of pairwise noncongruent subsets of A_n, then prove

$$F(n) = \begin{cases} 2^{n-2} + 2^{n/2} - 1 & \text{for } n \text{ even,} \\ 2^{n-2} + 3 \cdot 2^{(n-3)/2} - 1 & \text{for } n \text{ odd.} \end{cases}$$

H-407 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Find a closed form for the infinite product

$$\prod_{n=0}^{\infty} \frac{(5n+2)(5n+3)}{(5n+1)(5n+4)}. \tag{1}$$

H-408 *Proposed by Robert Shafer, Berkeley, CA*

a) Define $u_0 = 3$, $u_1 = 0$, $u_2 = 2$, and $u_{n+1} = u_{n-1} + u_{n-2}$ for all integers n.

b) In addition, let $w_0 = 3$, $w_1 = 0$, $w_2 = -2$, and $w_{n+1} = -w_{n-1} + w_{n-2}$ for all integers n.

Prove: $u_p \equiv w_p \equiv 0 \pmod{p}$ and $u_{-p} \equiv -w_{-p} \equiv -1 \pmod{p}$, where p is a prime number.

SOLUTIONS

Here's the Limit!

H-383 *Proposed by Clark Kimberling, Evansville, IN (Vol. 23, no. 1, February 1985)*

For any $x > 0$, let

$$c_1 = 1, \quad c_2 = x, \quad \text{and} \quad c_n = \frac{1}{n}\sum_{i=1}^{n} c_i c_{n-i} \quad \text{for } n = 3, 4, \ldots .$$

Prove or disprove that there exists $y > 0$ such that $\lim_{n\to\infty} y^n c_n = 1$.

Solution by Paul S. Bruckman, Fair Oaks, CA

We form the generating function

$$u = f(z, x) = \sum_{n=1}^{\infty} c_n z^n, \tag{1}$$

assumed valid for some disk of convergence $C: |z| < r$ (z complex). Under this assumption,

$$\lim_{n\to\infty} c_{n+1}/c_n = 1/r. \tag{2}$$

Note that from the defining recurrence and the condition $x > 0$ it follows that all c_n's are positive. Within C, the series defining f represents an analytic function of z, hence may be differentiated term by term. Thus,

$$\begin{aligned} u' &= \sum_{n=1}^{\infty} nc_n z^{n-1} = 1 + 2xz + \sum_{n=3}^{\infty} nc_n z^{n-1} = 1 + 2xz + \sum_{n=3}^{\infty} z^{n-1}\sum_{k=1}^{n-1} c_k c_{n-k} \\ &= 1 + 2xz - z + \sum_{n=2}^{\infty} z^{n-1}\sum_{k=1}^{n-1} c_k c_{n-k} \\ &= 1 - (1 - 2x)z + \sum_{k=1}^{\infty} c_k z^{k-1}\sum_{n=1}^{\infty} c_n z^n, \end{aligned}$$

or

$$u' = u^2/z + 1 - az, \tag{3}$$

where

$$a = 1 - 2x. \tag{4}$$

Note also the conditions

$$f(0, x) = 0, \quad f'(0, x) = 1. \tag{5}$$

For reasons which will become clear subsequently, we make the initial restriction $x \neq 1/2$, i.e., $a \neq 0$. We make the fortuitous substitutions:

$$u = -zv'/v, \text{ where } v = g(z, x), \tag{6}$$

$$v = w/\sqrt{\theta}, \tag{7}$$

$$w = h(\theta, x), \tag{8}$$

$$\theta = 2bz, \tag{9}$$

$$b = \sqrt{a}. \tag{10}$$

Upon transformation, we obtain the following differential equation:

$$w'' + (-1/4 + 1/2b\theta + 1/4\theta^2)w = 0, \tag{11}$$

where differentiation in (11) is with respect to θ.

Equation (11) is a special case of Whittaker's Equation, given in 13.1.31 of [1] in the following (paraphrased) modified form:

$$w'' + \left(-\frac{1}{4} + \frac{k}{\theta} + \frac{1/4 - \mu^2}{\theta^2}\right)w = 0. \tag{12}$$

Equation (12) possesses multiple-valued solutions, but we are not concerned with these; there exists a single-valued solution of (12) which meets all the necessary criteria. This is given by Whittaker's Function (13.1.32 [1]):

$$M_{k,\mu}(\theta) = e^{-1/2\theta}\,\theta^{1/2+\mu}M(1/2 + \mu - k,\ 1 + 2\mu,\ \theta), \tag{13}$$

where $M(A, B, Z)$ is the Kummer (or Confluent Hypergeometric) function defined by

$$M(A, B, Z) = \sum_{n=0}^{\infty} \frac{(A)_n}{(B)_n}\cdot\frac{z^n}{n!}, \tag{14}$$

using Pochhammer's notation: $(s)_n = s(s + 1)(s + 2)\ldots(s + n - 1)$.

Note (11) is obtained from (12) by setting $k = 1/2b$, $\mu = 0$; the restriction $b \neq 0$ now becomes evident. We therefore obtain the solution of (11):

$$g(\theta) = e^{-1/2\theta}\,\theta^{1/2}\,M(c,\ 1,\ \theta), \tag{15}$$

where

$$c = \frac{b - 1}{2b}. \tag{16}$$

Thus, using (6) and (7),

$$v = g(z, x) = e^{-bz}M(c,\ 1,\ 2bz). \tag{17}$$

To check that the boundary conditions in (5) are satisfied, we note that

$$\frac{d}{dZ}M(A, B, Z) = \frac{A}{B}M(A + 1,\ B + 1,\ Z);$$

hence,

$$v' = e^{-bz}\{2bcM(C + 1,\ 2,\ 2bz) - bM(c,\ 1,\ 2bz)\},$$

$$v'' = e^{-bz}\{2b^2c(c + 1)M(c + 2,\ 3,\ 2bz) - 4b^2cM(c + 1,\ 2,\ 2bz) + b^2M(c,\ 1,\ 2bz)\}.$$

Since $M(A, B, 0) = 1$, thus,

$$g(0, x) = 1,\quad g'(0, x) = 2bc - b = -1),$$

$$g''(0, x) = 2b^2c(c + 1) - 4b^2c + b^2 = b^2(2c \quad - 2c + 1)$$

$$= 1/2(b - 1)^2 - b(b - 1) + b^2 = 1/2(b^2 + 1) = 1 - x.$$

Using (6), $f(0, x) = \dfrac{0(1)}{1} = 0$. Also,

$$u' = \frac{-v(zv'' + v') + z(v')^2}{v^2} = \frac{-zv''}{v} - \frac{v'}{v} + z(v'/v)^2;$$

hence,

$$f'(0, x) = 0(1 - x)/1 + 1/1 + 0(-1/1)^2 = 1.$$

Thus, the boundary conditions are satisfied.

Using the divverential expression for v' obtained above and simplifying, we obtain as the (single-valued) solution of (3):

$$u = f(z, x) = bz - (b - 1)z \cdot \frac{M(c + 1, 2, 2bz)}{M(c, 1, 2bz)}, \tag{18}$$

provided $x \neq 0$.

If $0 < x < 1/2$, then $0 < a < 1$, $0 < b < 1$, $c < 0$. In this situation, it is known that $M(c, 1, 2bz)$ has a zero z_0 of minimum modulus $|z_0| > 0$. Since e^{-bz} cannot vanish for any values of z, thus z_0 is also a zero of g. Hence, from (6), z_0 is a simple pole of f; moreover, there are no other singularities of f with smaller modulus than z_0. It follows from (2) that

$$\lim_{n\to\infty} c_{n+1}/c_n = |z_0|^{-1}. \tag{19}$$

From a known result in analysis (Ex. 68.1 [2]):

$$\lim_{n\to\infty} c_n^{1/n} = |z_0|^{-1}. \tag{20}$$

Now (20) implies

$$\lim_{n\to\infty} |z_0|^n c_n = 1. \tag{21}$$

Hence, if $0 < x < 1/2$, the original conjecture is true, with $y = |z_0|$. If $x > 1/2$, then $a < 0$, $b = ik = i\sqrt{-a}$, say, and $c = 1/2 + i/2k$. Less seems to be known about the zeros of $M(A, B, Z)$ when A, B, and Z are complex, in particular of the function

$$M\left(\frac{1}{2} + \frac{i}{2k}, 1, 2ikz\right);$$

it seems likely, however, that, in this case as well, there exists a nonzero zero of this function, which leads to (21), as before. Certainly, the numerical evidence suggests this conclusion; namely, that the conjecture is true for $x > 1/2$.

Only the case $x = 1/2$ remains to be investigated. In this case, $a = b = 0$, but as $x \to 1/2^-$, $c \to -\infty$. To handle this case, we return to (3), which now becomes

$$u' = u^2/z + 1. \tag{22}$$

Making the substitution in (6), but with

$$g(z, 1/2) = v = w = H(\theta, 1/2), \text{ where } \theta = 2\sqrt{z}, \tag{23}$$

we obtain the differential equation

$$\theta w'' + w' + \theta w = 0. \tag{24}$$

The single-valued solution of (24) that satisfies the appropriate boundary conditions is the Bessel function of order zero:

$$H(\theta, 1/2) = J_0(\theta). \tag{25}$$

Therefore, $v = g(z, 1/2) = J_0(2\sqrt{z})$. Since $\frac{d}{d\theta} J_0(\theta) = -J_1(\theta)$, thus,

$$\frac{d}{dz} J_0(2\sqrt{z}) = -z^{-1/2} J_1(2\sqrt{z}).$$

Hence

$$f(z, 1/2) = \frac{\sqrt{z}J_1(2\sqrt{z})}{J_0(2\sqrt{z})}. \tag{26}$$

The function $J_0(\theta)$ has a simple zero at $\theta_0 \doteq 2.4048255577$ (viz. 9.5 of [1]), which has the smallest modulus of any other zero. Therefore, reasoning as before, $y = z_0 = (\theta_0/2)^2 \doteq 1.4457964906$, and in this case also,

$$\lim_{n\to\infty} y^n c_n = 1. \tag{27}$$

Hence, the conjecture is certainly true for $0 < x \leqslant 1/2$, and *appears* true for $x > 1/2$ as well.

Note: The function $M(-k, B, Z)$, where k is a positive integer, is a polynomial in Z; this leads to *rational* solutions of (3), when c is a negative integer. This occurs when $x = x_k = 2k(k+1)/(2k+1)^2$, $k = 1, 2, \ldots$; letting u_k, v_k, $c_n^{(k)}$, and y_k denote the appropriate quantities (previously denoted by u, v, c_n, and y), we find:

$$v_k(z) = \exp(-z/(2k+1)) \quad M(-k, 1, 2z/(2k+1)),$$

and

$$u_k(z) = \frac{z}{2k+1}\left\{\frac{2kM(-k+1, 2, 2z/(2k+1)) + M(-k, 1, 2z/(2k+1))}{M(-k, 1, 2z/(2k+1))}\right\}.$$

This leads to *algebraic* values for $c_n^{(k)}$, which facilitate the task of finding the appropriate value of y_k satisfying $\lim_{n\to\infty} y_k^n c_n^{(k)} = 1$.

For example, $x_1 = 4/9$ yields:

$$M(-1, 1, 2z/3) = 1 - 2z/3, \; v_1(z) = e^{-z/3}(1 - 2z/3),$$

$$u_1(z) = z(1 - 2z/9)(1 - 2z/3)^{-1} = \frac{z}{3} + \frac{2z/3}{1 - 2z/3} = \frac{z}{3} + \sum_{n=1}^{\infty}(2z/3)^n$$

$$= z + \sum_{n=2}^{\infty}(2z/3)^n.$$

Thus, $c_n^{(1)} = (2/3)^n$, $n \geqslant 2$, which implies $y_1 = 1.5$. Also,

$$v_2(z) = e^{-z/5}M(-2, 1, 2z/5) = e^{-z/5}(1 - 4z/5 + 2z^2/25),$$

and

$$u_2(z) = \frac{z(1 - 8z/25 + 2z^2/125)}{1 - 4z/5 + 2z^2/25} = z + \sum_{n=2}^{\infty}(2z/25)^n(p^n + q^n),$$

where $p = 5(1 + 2^{-1/2})$, $q = 5(1 - 2^{-1/2})$. Hence, $c_n^{(2)} = (2/25)^n(p^n + q^n)$, $n \geqslant 2$. Since $0 < q < p$, thus $y_2 = 25/2p = q \doteq 1.4644661$. Note that $\lim_{k\to\infty} x_k = 1/2$, so

$$\lim_{k\to\infty} y_k \doteq 1.4457964906,$$

the value obtained previously in connection with J_0.

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1. M. Abramowitz & I. A. Stegun, Eds. *Handbook of Mathematical Functions*. 9th printing. Washington, D.C.: National Bureau of Standards AMS 55, 1970.
2. G. Pólya & G. Szegö. *Problems and Theorems in Analysis*, Vol. I. New York, Heidelberg, Berlin: Springer-Verlag, 1972.

Sum Product!

H-384 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany (Vol. 23, no. 1, February 1985)*

Show that for $n = 0, 1, 2, \ldots$,

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} \prod_{j=0}^{k-1} \left[\left(n + \frac{1}{2}\right)^2 - j^2\right] = \frac{\sqrt{5}}{2} F_{2n+1}$$

Solution by the proposer

Let $F(a, b; c; z)$ denote the hypergeometric function defined by

$$F(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{c_k}{k!} z^k,$$

where

$$c_k = \frac{a(a+1) \ldots (a+k-1)b(b+1) \ldots (b+k-1)}{c(c+1) \ldots (c+k-1)}$$

We take $a = -n$, $b = n + 1$, $c = 1/2$, and $z = -1/4$. Then

$$c_k = (-1)^k 4^k \cdot \frac{k!(n+k)!}{(2k)!(n-k)!},$$

so that

$$F(-n, n+1; 1/2; -1/4) = 1 + \sum_{k=1}^{n} \frac{1}{(2k)!} \frac{(n+k)!}{(n-k)!}$$

$$= \sum_{k=0}^{n} \binom{n+k}{n-k} = \sum_{r=0}^{n} \binom{2n-r}{r} = F_{2n+1}. \qquad (1)$$

The hypergeometric function satisfies the following identity [see F. G. Tricomi, *Vorlesungen uber Orthogonalreihen*, Springer Verlag, p. 151, formula (2.7)]:

$$F\left(-n, n+1; \frac{1}{2}; -\frac{1}{4}\right) = \frac{2}{\sqrt{5}} F\left(n + \frac{1}{2}, -n - \frac{1}{2}; \frac{1}{2}; -\frac{1}{4}\right). \qquad (2)$$

Again, by using the above definition, we obtain

$$F\left(n + \frac{1}{2}, -n - \frac{1}{2}; \frac{1}{2}; -\frac{1}{4}\right) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{d_k}{k!} \cdot \frac{1}{4^k}$$

where

$$d_k = \left[\prod_{j=0}^{k-1} \left(\frac{1}{2} + j\right)\right]^{-1} \left[\prod_{j=0}^{k-1} \left(n + \frac{1}{2} + j\right)\right] \left[\prod_{j=0}^{k-1} \left(-n - \frac{1}{2} + j\right)\right]$$

$$= \frac{4^k k!}{(2k)!} (-1)^k \cdot \prod_{j=0}^{k-1} \left[\left(n + \frac{1}{2}\right)^2 - j^2\right].$$

Now the statement easily follows from (1) and (2). Q.E.D.

Also solved by P. Bruckman.

Gotta Have a System

H-385 *Proposed by M. Wachtel, Zurich, Switzerland*
(Vol. 23, no. 2, May 1985)

Solve the following system of equations:

I. $U^2_{f(n)} + V^2_{g(n)} - 3 \cdot U_{f(n)}V_{g(n)} = 1;$

II. $3 \cdot U_{h(n)}V_{i(n)} - (U^2_{h(n)} + V^2_{i(n)}) = 1.$

Solution by the proposer

I. 1) Let $U_{f(n)} = a$, $V_{g(n)} = b$. Then, we have $a^2 + b^2 - 3ab = 1$. It follows:

$$b = \frac{3a \pm \sqrt{5a^2 + 4}}{2}.$$

2) Now let $a = F_{2n}$, which leads to $\dfrac{3F_{2n} \pm \sqrt{5F^2_{2n} + 4}}{2}$.

3) Using the identity, Hoggatt I_{12}, $5F^2_{2n} + 4 = L^2_{2n}$, we obtain:

$$b_{1,2} = \frac{3F_{2n} \pm L_{2n}}{2} = F_{2n \pm 2}.$$

4) Hence, $F^2_{2n} + F^2_{2n \pm 2} - 3F_{2n}F_{2n \pm 2} = 1$, which is one of the solutions of the more generalized identity

$$F^2_{2n} + F^2_{2n \pm 2m} - L_{2m}F_{2n}F_{2n \pm 2m} = F^2_{2m},\ m = 1, 2, 3, \ldots, \text{ if } m = 1.$$

II. 1) Let $U_{h(n)} = a$, $V_{i(n)} = b$. Then, we have $3ab - (a^2 + b^2) = 1$. Thus:

$$b = \frac{3a \pm \sqrt{5a^2 - 4}}{2}.$$

2) Now let $a = F_{1+2n}$, which leads to $\dfrac{3F_{1+2n} \pm \sqrt{5F^2_{1+2n} - 4}}{2}$.

3) Using the identity, Hoggatt I_{12}, $5F^2_{1+2n} - 4 = L^2_{1+2n}$, we obtain:

$$b_{1,2} = \frac{3F_{1+2n} \pm L_{1+2n}}{2} = F_{1+2n \pm 2}.$$

4) Hence, $3F_{1+2n}F_{1+2n \pm 2} - (F^2_{1+2n} + F^2_{1+2n \pm 2}) = 1$,

which is one of the solutions of the more generalized identity

$$L_{2m}F_{1+2n}F_{1+2n \pm 2m} - (F^2_{1+2n} + F^2_{1+2n \pm 2m}) = F^2_{2m},\ m = 1, 2, 3, \ldots,$$
if $m = 1$.

Also solved by P. Bruckman.

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BOOKS AVAILABLE THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.

Recurring Sequences by Dov Jarden. Third and enlarged edition. Riveon Lematematika, Israel, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

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