The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

TWENTY-FIFTH ANNIVERSARY YEAR

CONTENTS

VOLUME 25 MAY 1987 NUMBER 2

PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

EDITORIAL POLICY

THE FIBONACCI QUARTERLY seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

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All back issues of **THE FIBONACCI QUARTERLY** are available in microfilm or hard copy format from **UNIVERSITY MICROFILMS INTERNATIONAL, 300 NORTH ZEEB ROAD, DEPT. P.R., ANN ARBOR, MI 48106.** Reprints can also be purchased from **UMI CLEARING HOUSE** at the same address.

The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

THE SECOND INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

A MEMORY-LADEN EXPERIENCE

HERTA T. FREITAG

There I was—alone on a strange campus, at the University of California at Berkeley, where the startling number of 3,970 had gathered for ICM-86, The International Congress of Mathematicians. Did someone just call my name? He had done it again!—Professor A. N. Philippou, Chairman of our First International Conference on Fibonacci Numbers and Their Applications two years ago at The University of Patras, Greece, the man who at the time had "recognized" me without ever having seen me, now managed to "run into me" amidst this "almost nondenumerable" crowd.

To encounter—just before our Conference—Professor Philippou, the originator of the idea to set the stage for a meeting of "Fibonacci friends" on an international scale, was a very special omen to me. It was an appropriate and beautiful overture to our Second International Conference on Fibonacci Numbers and Their Applications, which was to begin two days later, and convened from August 13-16 at San Jose State University. This site was befittingly chosen, as it is the home of *The Fibonacci Quarterly*.

Professor Calvin Long, Chairman of the Board of The Fibonacci Association, and Professor Hugh Edgar, a member of the University's Mathematics Department, participated in the Conference. This gave us the opportunity to express our appreciation of the fact that our Conference was co-sponsored by The Fibonacci Association and San Jose State University.

Professor Gerald E. Bergum, Editor of *The Fibonacci Quarterly* and Chairman of the Local Committee, and Professor A. N. Philippou, who chaired the International Committee, immediately earned our admiration and praise. So did the Co-Chairmen—Professors A. F. Horadam and Hugh Edgar, and, indeed, Professor Calvin Long and all the other helpers "on the stage" and "in the wings."

The organization of our Conference was exemplary. And the atmosphere was charged with that most appealing blend of the seriousness and profundity of scholarliness and the enthusiasm and warmth of personal relationships. This seems to be the trademark of "Fibonaccians"—mathematicians who are dedicated to a common cause: a deep and abiding fascination with "Fibonacci-type" mathematics.

Approximately twenty-five papers were presented by a group which came from some ten countries. There were several joint authorships. Some had resulted from a cooperation between authors separated by oceans—a situation which, predictably, poses many obstacles: one just has to "hover by the mailbox until the anxiously awaited response can possibly arrive." Many of the papers exhibited the phenomenon that one mathematical idea begot another, and yet another, maybe a generalization, and yet a further one, etc., the very development mathematicians cherish so much. Our understanding of the goldmine that number sequences and the intricacies of their interrelationships constitute was enriched, and our appreciation of the value of such investigations was deepened. While the variety of topics was striking, dedication to the beauty of mathematical patterns and joy over the wealth of mathematical relationships provided the common bond. The Conference Proceedings will be published in the near future.

A small nucleus, just seven participants, were "second-timers," people who had previously experienced the unique pleasure of this kind of gathering on an international scale. Their friendships were welded together more meaningfully yet, and many newcomers were initiated. Many of us had accents but, in a very significant way, we all spoke the same language.

Professor Hoggatt's widow, Herta Hoggatt, most graciously invited our entire mathematical community to convene at her charming home—outdoors, amidst the beauty of flowers and trees. In a deeply touching way did the late Professor Verner E. Hoggatt, Jr., thus participate in our thoughts.

I believe that all of our Fibonacci friends—here and across the oceans—greatly valued the fact that the dream, first voiced in Greece, about continuation of our international gatherings had been realized. Now, we confidently rejoice over the prospect: "Until we meet again..., in two years, in Italy..., maybe in Pisa!"

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A. F. HORADAM—AD MULTOS ANNOS

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1. INTRODUCTION

"The sequence will involve a cumulative process of capital appreciation that will accrue to the benefit of the early rich and their heirs" [2]. Though very much out of context, the quotation came to mind when I sat down to respond to the Editor's invitation to write a paper about Alwyn Horadam, an elder statesman of The Fibonacci Association, on his retirement. As a former student, or intellectual heir, of "Horrie" (as he was affectionately known to thousands of students), my life has been enriched by the appreciation of the capital of his early generalizations of the Fibonacci sequence.

These attracted the attention of the Founding Editors of *The Fibonacci Quarterly*, Brother U. Alfred and Professor Verner E. Hoggatt, Jr. He accepted their invitation to become a foundation sustaining member and to join the initial board of Assistant Editors for Volume 1, Number 1, in February 1963.

Of that initial Board, Alwyn is still serving, together with Maxey Brooke, Paul Byrd, Leonard Carlitz, Henry Gould, and D. E. Thoro, but Alwyn is the only non-American still on it twenty-four years later. This is no mean feat and is indicative of his great virtue as a correspondent—the prompt reply—an asset which was to serve him well during his forty years on the staff of the University of New England where more than half the graduates have studied externally at a distance from UNE.

What follows is not a critical exposition of Alwyn's work—I have been too closely involved with him for twenty years as student, colleague, and friend. Nor is it an obituary—no one who knows Alwyn can expect an idle retirement. It is, in the words of the Editor's invitation, a list of his professional accomplishments and a summary of some of his work.

2. CURRICULUM VITAE

Alwyn was born in 1923, son of a dairy farmer at Singleton, a small town in the Hunter Valley about 230 km north of Sydney, the capital of New South Wales. The original Horadams came from Wiesbaden in Germany in 1846 as "vine tenderers." (The Hunter Valley is a renowned wine-producing area of Australia.)

It is of interest to note, in passing, that the other branch of the Horadam family emigrated to Texas, USA, and recently the two branches of the family have been in contact. A couple of years ago the Horadams had a family reunion on the original property, "Glendon," where there is a small church and cemetery.

Alwyn was educated at Maitland Boys' High School, which has a justifiably high reputation in Australia with many famous alumni. (Maitland is about 45 km southeast of Singleton, half way to the coastal city of Newcastle.) Alwyn distinguished himself at school, where he was Dux, Captain of the School, Captain of the Criket team, and a member of the Football team.

In 1939, he went to the New England University College in Armidale at the other end of the Hunter Valley in the Northern Tablelands of the State. The

NEUC had only been founded the year before. It "was affiliated with the University of Sydney, and was Australia's first experiment in the establishment of a university in a purely country area—an experiment that had been urged by some educationists since the closing years of the nineteenth century" [25].

The New England region is one of the finest sheep-raising areas of Australia, and it is fitting that a university which was to become a world leader in Rural Science and Agricultural Economics began with the munificence of local pastoralists who donated a mansion and farms to secure its foundation. NEUC obtained its full autonomy in 1954, and it now attracts students from overseas as well as from every State of Australia. Its rapid expansion in recent years has not been at the expense of its rural setting: its beautiful campus is an attraction for academics who visit Australia.

Alwyn graduated as BA in 1942 at NEUC with first class honors awarded by the parent University of Sydney. During World War II, the University worked a four-term year, so he graduated in three years instead of four.

After graduating, he served as a school teacher with the NSW Department of Education. His studies included Mathematics, Education, and English. These, together with his work in schools, helped to make him not only a gifted teacher of mathematics but also a practitioner skilled in writing and research.

With further academic work, he completed the requirements for his Master of Arts, Diploma in Education, Doctor of Philosophy of the University of Sydney, and Bachelor of Education of the University of Melbourne. His Ph.D. was done under the supervision of Professor T. G. Room, FRS, a world-renowned geometer who was Head of the Department of Pure Mathematics at the University of Sydney for about thirty years. Alwyn's life-time interest in geometry culminated in the publication of his book on projective geometry [13]. His Ph.D. involved work with Clifford Matrices and showed the wide range of algebraic skills that he was later able to apply to number theory.

Alwyn joined the staff of NEUC in 1947 as a lecturer, and his role in the development of the University can be seen from the following list of his contributions to the university community. From the date of its independence in 1954 until 1972, he served as a Member of the University of New England's Governing Council. He was elected to this position by the University Convocation, that is, by the full university community of graduates and staff.

He has been Captain and President of the University Cricket Club, Vice-President of the University Football Club, President of the Science Society, President of the University Union, Foundation Secretary of the UNE Teacher's Association, Foundation Chairman of the UNE Alumni Association, and Foundation Fellow of Robb College (one of the residential colleges of UNE).

He also has been a Governor of Robb, Duval, and Wright Colleges, and the University Esquire Bedell. More recently, he has been the University Ombudsman, a difficult role and the appointment to which is an indication of the esteem of the university community for his integrity. He was also the UNE Delegate for the 13th Quinquennial Congress of British Commonwealth Universities in Birmingham (UK).

Considering all this involvement, it is almost a surprise to learn that he had time to get married in 1950. He and Mollie have now been together for 37 years and have three lovely daughters and two grand-daughters. Mollie is an engineer and number theorist with degrees BSc(Eng) (London), MA (Cambridge), Ph.D. (UNE), and, for many years was on the staff of UNE, which she continues to serve as a Member of Council and an Honorary Fellow in Mathematics.

The three daughters are all married. Kathryn, with a Ph.D. in mathematics from the Australian National University, is now on the staff of the Royal Melbourne Institute of Technology. Readers of *Mathematical Reviews* will thus have seen references to A. F., E. M., and K. J. Horadam at times. Actually, there are five Dr. Horadams, because the other two daughters, Kerry and Alanna, have graduated as medical doctors!

Other academic appointments have included Dean of the Faculty of Science, and, in recent years, Head of the Department of Mathematics, Statistics and Computing Science, the position from which he retires. Is it any wonder that in correspondence, the Editor agreed that Alwyn is an excellent role model for an academic?

3. MATHEMATICS

In commenting on Alwyn's contributions to mathematics, I must first take account of his teaching ability, for that is how I first came to know him.

Since 1955, UNE has played a major role in distance-education through its Department of External Studies. Many school teachers of Mathematics took advantage of its facilities to upgrade their qualifications, and through them I had come to learn of Alwyn's expository skills. For this reason, I wrote to UNE in 1966 and I was fortunate to be assigned to Alwyn's care.

With undergraduates he aimed at the educational ideals of humane and liberal education through the medium of mathematics. With postgraduates, he encouraged optimism and a positive approach to research. I always found him cheerful but serious, able to ask the right questions and to resist the temptation to do too much for the novice researcher.

He encouraged his Master's and Doctoral research students, of whom he has successfully supervised 49, to correspond with mathematicians around the world to avoid insular or parochial frames of mind. His own research has been similarly stimulated with periods at the Universities of North Carolina, Cambridge, Leeds, Liverpool, East Anglia, Reading, York, Exeter, Iceland, and Malaya (where he was seconded to advise on their mathematics curriculum).

His influence on the teaching of mathematics at the high-school level in NSW has been threefold. First, as the NSW Universities' representative on the Board of Secondary Schools Studies Mathematics Syllabus Committee for the last 26 years. Second, as the co-author of a number of high school texts, of which [19] is an example. Finally, but not least significantly, through his help to high school mathematics teachers by his teaching of them, through his work for mathematics teaching associations, and through his writing of articles related to the teaching of mathematics.

His undergraduate expositions, which were clear and effective, would no doubt have, in turn, influenced the teaching styles of many of his proteges. Two of his short teaching texts for external students exemplify this. They also illustrate his interest in combinatorics. *Applied Combinatorics* [17] deals with graph theory, block designs, and enumeration techniques including recurrence relations and generating functions. Finite combinatorical structures and combinatorical circuits compose the two parts of his *Combinatorial Mathematics* [18]. He also co-authored a number of research papers which looked at combinatorial techniques for unravelling patterns (e.g., [21]).

To return briefly to his teaching texts. These are amply illustrated with worked examples and historical allusions. A more neutral observer, the eminent Oxford mathematician W. L. Ferrar, has noted in his review of Alwyn's *Outline Course of Pure Mathematics* [12]: "What a task!—and how well it has been carried out. The task? A unified treatment of the Algebra, Geometry and Calculus considered basic for the foundation of undergraduate mathematics.... Throughout, the author seizes every opportunity to interweave the variety of topics he is handling.... The range of knowledge and detailed reference displayed by the author is most striking.... That the author is an experienced teacher is everywhere apparent; he knows all the pitfalls" [1].

Alwyn's early research publications were in algebra and geometry [3-6, 8, 11] but the two which were seminal and continue to be cited frequently were in number theory. The second of these [10] generalized the Fibonacci numbers and introduced useful notation by considering the sequence

$$\{W_n\} \equiv \{W_n(a, b; p, q)\}$$

defined by the second-order linear homogeneous recurrence relation

$$W_n = pW_{n-1} - qW_{n-2}, \; n \geqslant 2,$$

with initial conditions $W_0 = a$, $W_1 = b$. Thus, the ordinary Fibonacci numbers $\{F_n\}$ are given by $\{W_n(0, 1; 1, -1)\}$. That paper, and a number which followed (e.g., [9]), developed the properties of this generalization, as he had done earlier for $\{H_n\}$ given by $\{W_n(a, b; 1, -1)\}$ [7]. These generalizations were not only elegant, they also clarified the roles of the fundamental and primordial sequences introduced by Lucas eighty odd years earlier [24].

By highlighting these two papers, I do not mean to do injustice to others, but they had a big influence on me, and they have been utilized by many others as well. The algebraic and geometric influences in Alwyn's research not surprisingly recur from time to time, as do techniques from the special functions of mathematical physics, especially the Chebyshev polynomials (e.g., [15]). Recently, he has co-authored material on the Gegenbauer polynomials and Gaussian Fibonacci numbers (e.g., [27]). There have also been numerous papers that deal with various properties of the Pell numbers $\{W_n(0, 1; 2, -1)\}$ and their generalizations and polynomials [22].

From time to time, too, there have been papers on other topics in number theory such as a proof of a problem posed by Morgan-Ward on the Staudt-Clausen theorem [23], Oresme numbers [14], and Wythoff pairs [16]. Surprisingly, since they were both number theorists, Mollie and Alwyn published only one paper together [20]. This dealt with finding the zeros of Fibonacci and Lucas polynomials and connections among them.

Among his many co-authors have been Stanley Collings (Open University, UK), Jamie Walton (Northern Rivers College), Brother Jim Mahon (Catholic College of Education, Sydney), Peter Sekhon (NSWIT), Sharad Pethe (University of Malaya), Merv Dunkley (Macquarie University), I. W. Stewart (Mitchell College), Carl Chiarella (University of NSW), and Phil Loh (University of Sydney). Jamie, Jim, Peter, and the present writer are among his former Ph.D. students.

As well as the references cited, he has published in journals in Portugal, India, Argentina, and Malaysia. He has also presented papers at Conferences of the Australian Mathematical Society, the New Zealand Mathematics Colloquium, the Australian and New Zealand Association for the Advancement of Science, and as a guest speaker at numerous universities. As readers will be aware, he has also served on the International Committee of the First (Patras) and Second (San Jose) International Conferences on Fibonacci Numbers and Their Applications. He was also Co-Chairman of the latter.

4. CONCLUSION

If this article has only skimmed Alwyn's work, it is because the job could not be readily tackled in a brief space because he is the author (sole, joint) of 8 books and more than 90 papers. The task would be more appropriate to a Master's dissertation.

Nor have I gone into Alwyn's personal characteristics, to avoid him embarrassment. One, however, deserves mention, and it is his loyalty to people and to causes. In any case, I associate much of what is best in Alwyn with team sports such as cricket—at which he excelled, and which is not now held in high regard by some educationists: "The old idea of a healthy mind in a healthy body, the benefit which physical training gives to the character by its discipline, the virtues of loyalty and self-abregation inculcated by team games, are all ideas which now evoke little but a faintly patronising tolerance or even contemptuous ridicule" [26].

I am very grateful for material supplied by Professor J. Hempel and Dr. E. M. Horadam of UNE to help me get started on this article. I apologize to readers and to Alwyn for any errors and inevitable omissions: it is difficult to write adequately about a living person without access to his files! I thank the Editor for the opportunity to honor a real university man, who has given much to mathematics internationally and, in a particular way, to the Fibonacci community.

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THIRD ANNUAL CONFERENCE IS IN PLANNING STAGES

The third International Conference on Fibonacci Numbers and Their Applications is in the planning stages. Currently, it looks like the place will be Pisa, Italy, from July 25 to July 29, 1988.

More details will follow in the August 1987 issue. Plan now for another great conference.

A CONSTELLATION OF SEQUENCES OF GENERALIZED PELL POLYNOMIALS

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(Submitted March 1985)

1. INTRODUCTION

In [1] and [2], Byrd introduced a sequence of polynomials which we call Pell. These polynomials may be defined, in the first instance, thus:

$$\begin{cases} p_0(x) = 0,\ p_1(x) = 1, \\ p_{n+1}(x) = 2xp_n(x) + p_{n-1}(x), \text{ for } n \geqslant 1. \end{cases} \tag{1.1}$$

The polynomials cognate to these, the Pell-Lucas, may be defined thus:

$$\begin{cases} q_0(x) = 2,\ q_1(x) = 2x, \\ q_{n+1}(x) = 2xq_n(x) + q_{n-1}(x), \text{ for } n \geqslant 1. \end{cases} \tag{1.2}$$

These two sequences have been studied in more detail in [5]-[10]. The Binet formulas for the two sequences of polynomials are

$$p_n(x) = \frac{\eta^n - \psi^n}{\eta - \psi} \tag{1.3}$$

and

$$q_n(x) = \eta^n + \psi^n \tag{1.4}$$

where η, ψ are roots of the equation

$$y^2 - 2xy - 1 = 0. \tag{1.5}$$

Hence, η, ψ are given by

$$\eta = x + \sqrt{(x^2 + 1)}, \quad \psi = x - \sqrt{(x^2 + 1)}. \tag{1.6}$$

In [12]-[14], Walton, and Walton & Horadam have studied a sequence of generalized Pell polynomials. They are defined thus:

$$\begin{cases} A_0(x) = q,\ A_1(x) = p, \\ A_{n+1}(x) = 2xA_n(x) + A_{n-1}(x), \text{ for } n \geqslant 1. \end{cases} \tag{1.7}$$

Another sequence of generalized Pell polynomials or, rather, a constellation of them is proposed here.

2. FIRST ENCOUNTER WITH THE CONSTELLATION OF SEQUENCES OF GENERALIZED PELL POLYNOMIALS

This constellation was first encountered in an effort to replicate for Pell polynomials what Gould [3] and others had done with a formula of Lucas.

An important identity for $p_n(x)$, easily proved from Binet formulas (1.3) and (1.4) is:

$$p_{n+m}(x) - q_m(x)p_n(x) + (-)^m p_{n-m}(x) = 0 \tag{2.1}$$

This may be regarded as a generalization for (1.1). By repeated applications of (2.1), we get:

$$\begin{cases} p_n(x) \\ = q_m(x)p_{n-m}(x) + (-)^{m-1}p_{n-2m}(x) \\ = (q_m^2(x) + (-)^{m-1})p_{n-2m}(x) + (-)^{m-1}q_m(x)p_{n-3m}(x) \\ = (q_m^3(x) + 2(-)^{m-1}q_m(x))p_{n-3m}(x) + (-)^{m-1}(q_m(x) + (-)^{m-1})p_{n-4m}(x) \\ = (q_m^4(x) + 3(-)^{m-1}q_m^2(x) + (-)^{2(m-1)})p_{n-4m}(x) + \\ \qquad + (-)^{m-1}(q_m^3(x) + 2(-)^{m-1}q_m(x))p_{n-5m}(x) \end{cases} \tag{2.2}$$

We may present these lines thus:

$$\begin{cases} p_n(x) \\ = p_{1,m}(x)p_n(x) + (-)^{m-1}p_{0,m}(x)p_{n-m}(x) \\ = p_{2,m}(x)p_{n-m}(x) + (-)^{m-1}p_{1,m}(x)p_{n-2m}(x) \\ = p_{3,m}(x)p_{n-2m}(x) + (-)^{m-1}p_{2,m}(x)p_{n-3m}(x) \\ = p_{4,m}(x)p_{n-3m}(x) + (-)^{m-1}p_{3,m}(x)p_{n-4m}(x) \\ = p_{5,m}(x)p_{n-4m}(x) + (-)^{m-1}p_{4,m}(x)p_{n-5m}(x) \end{cases} \tag{2.3}$$

where

$$\begin{cases} p_{0,m}(x) = 0 \\ p_{1,m}(x) = 1 \\ p_{2,m}(x) = q_m(x) \\ p_{3,m}(x) = q_m^2(x) + (-)^{m-1} \\ p_{4,m}(x) = q_m^3(x) + 2(-)^{m-1}q_m(x) \\ p_{5,m}(x) = q_m^4(x) + 3(-)^{m-1}q_m(x) + (-)^{2(m-1)} \end{cases} \tag{2.4}$$

The procedure followed in (2.2) and (2.3) may be continued indefinitely, when allowance is made for the first subscript to be negative. It is clear from (2.2) that

$$p_{n,m}(x) = q_m(x)p_{n-1,m}(x) + (-)^{m-1}p_{n-2,m}(x). \tag{2.5}$$

Starting again, we may define the sequence $\{p_{n,m}(x)\}$ thus:

$$\begin{cases} p_{0,m}(x) = 0, \; p_{1,m}(x) = 1, \\ p_{n+1,m}(x) = q_m(x)p_{n,m}(x) + (-)^{m-1}p_{n-1,m}(x), \text{ for } n \geqslant 1. \end{cases} \tag{2.6}$$

The defining equation gives rise to a constellation of sequences, one for each value of m.

3. SOME IDENTITIES AND GENERATORS FOR THE SEQUENCE $\{p_{n,m}(x)\}$

The results in (2.4) may be used as the basis for a proof by induction of an explicit formula for $p_{n,m}(x)$. It is:

$$p_{n,m}(x) = \sum_{i=0}^{[(n-1)/2]} (-)^{i(m-1)}\binom{n-1-i}{i}q_m^{n-1-2i}(x) \tag{3.1}$$

From this we may show that:

$$q_m^n(x) = \sum_{r=0}^{[n/2]} (-)^{rm} \binom{n}{r} \frac{n - 2r + 1}{n - r + 1} p_{n+1-2r,\,m}(x) \tag{3.2}$$

The Binet formula, also proved by induction, is:

$$p_{n,m}(x) = \frac{\eta^{nm} - \psi^{nm}}{\eta^m - \psi^m} \tag{3.3}$$

where η and ψ are as introduced in (1.6). If the Binet formula were used to define the sequence, negative integral values for n and m are easily introduced.

From (1.3) and (3.3), we have:

$$p_{nm}(x) = p_{n,m}(x)p_m(x) \tag{3.3'}$$

A determinantal generator for $p_{n,m}(x)$ is $\delta_{n,m}(x)$. The determinant is of order n and is defined thus:

$$\delta_{n,m}(x):\ \begin{cases} d_{rr} = q_m(x) & \text{for } r = 1, 2, \ldots, n \\ d_{r,r+1} = (-)^m & \text{for } r = 1, 2, \ldots, n - 1 \\ d_{r,r-1} = 1 & \text{for } r = 2, 3, \ldots, n \\ d_{rc} = 0 & \text{otherwise} \end{cases} \tag{3.4}$$

where d_{rc} is the entry in the r^{th} row and c^{th} column of $\delta_{n,m}(x)$. One may prove by induction that

$$\delta_{n,m}(x) = p_{n+1,m}(x) \text{ for } n \geqslant 1. \tag{3.5}$$

A matrix generator for $p_{n,m}(x)$ is:

$$\mathscr{P}_m = \begin{bmatrix} q_m(x) & (-)^{m-1} \\ 1 & 0 \end{bmatrix} \tag{3.6}$$

We can easily show, by induction again, that:

$$\mathscr{P}_m^n = \begin{bmatrix} p_{n+1,m}(x) & (-)^{m-1}p_{n,m}(x) \\ p_{n,m}(x) & (-)^{m-1}p_{n-1,m}(x) \end{bmatrix} \tag{3.7}$$

The matrix $\mathscr{P}_m$ has been employed to establish several identities. There are other matrix generators for the sequence.

An algebraic generator is

$$\sum_{n=0}^{\infty} p_{n+1,m}(x) = 1/(1 - q_m(x)y + (-)^m y^2), \tag{3.8}$$

and an exponential generator is:

$$\sum_{n=0}^{\infty} p_{n,m}(x)y^n/n! = \frac{e^{\eta^m y} - e^{\psi^m y}}{\eta^m - \psi^m}$$

The justification for regarding $\{p_{n,m}(x)\}$ as a generalization for $\{p_n(x)\}$ is that, when we put $m = 1$ in the results given above and in others, we obtain the corresponding formulas for the Pell polynomials. First and foremost, we have

$$p_{n,1}(x) = p_n(x). \tag{3.10}$$

We mention, finally, in this section two identities which have been proved by using the matrix $\mathscr{P}_m$. They are the Simson formula and its generalization for $p_{n,m}(x)$.

$$p_{n+1,m}(x)p_{n-1,m}(x) - p^2_{n,m}(x) = (-)^{m(n-1)+1} \tag{3.11}$$

$$p_{n+r,m}(x)p_{n-r,m}(x) - p^2_{n,m}(x) = (-)^{m(n-r)+1}p^2_{r,m}(x) \tag{3.12}$$

4. RELATIONS OF $\{p_{n,m}(x)\}$ WITH CHEBYSHEV POLYNOMIALS

In [1], [2], [5], [6], and [7] some relations of Pell and Pell-Lucas polynomials with Chebyshev polynomials were explored. If we regard $\{p_{n,m}(x)\}$ as a generalization of Pell polynomials, then we would also expect that it should have connections. However, we need to construct first a generalization for Chebyshev polynomials of the second kind [11]. These are $\{U_{n,m}(x)\}$ defined in the following manner:

$$\left.\begin{aligned} &U_{0,m}(x) = 1,\ U_{1,m}(x) = 2T_m(x), \\ &U_{n+1,m}(x) = 2T_m(x)U_{n,m}(x) - U_{n-1,m}(x), \text{ for } n \geqslant 1, \end{aligned}\right. \tag{4.1}$$

where $T_m(x)$ is the m^{th} Chebyshev polynomial of the first kind [11].

With this definition, it is possible to prove by induction that

$$U_{n,m}(x) = \sum_{j=0}^{[n/2]} (-)^j \binom{n-j}{j} (2T_m(x))^{n-2j}, \text{ for } n \geqslant 1. \tag{4.2}$$

Following from (4.2), we can prove that

$$p_{n,m}(x) = (-i)^{(n-1)m}U_{n-1,m}(ix). \tag{4.3}$$

A hypergeometric representation for $p_{n,m}(x)$ follows from (4.3). It is

$$p_{n,m}(x) = n\,{}_2F_1(n+1, -n+1; 3/2; Y_m)/i^{(n-1)m} \tag{4.4}$$

where

$$Y_m = (2 - i^m q_m(x))/4. \tag{4.5}$$

Another explicit expression for $p_{n,m}(x)$ may also be derived from (4.3), namely,

$$p_{n,m}(x) = \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} (q_m(x)/2)^{n-1-2k}(X_m/4)^k \tag{4.6}$$

where X_m is the discriminant of the auxiliary equation of $p_{n,m}(x)$, i.e.,

$$y^2 - q_m(x)y + (-)^m = 0. \tag{4.7}$$

This means that

$$X_m = q^2_m(x) + 4(-)^{m-1}. \tag{4.8}$$

Starting from (2.5) and the identity below, easily established from Binet formulas,

$$q_{(n+1)m}(x) - (q^2_m(x) + 4(-)^{m-1})p_{n,m}(x) + (-)^{m-1}q_{(n-1)m}(x) = 0, \tag{4.9}$$

we obtain other explicit expressions for $p_{n,m}(x)$. They are:

$$p_{2n+1,m}(x) = \sum_{k=0}^{n} (-)^{km} \frac{2n+1}{2n+1-k} \binom{2n+1-k}{k} X_m^{n-k}; \tag{4.10}$$

and

$$p_{2n,m}(x) = \left\{ \sum_{k=0}^{n-1} (-)^{km} \binom{2n-1-k}{k} X_m^{n-1-k} \right\} q_m(x). \tag{4.11}$$

These interesting and aesthetically appealing formulas for the constellation of sequences $\{p_{n,m}(x)\}$ are a sample of the large number that have been obtained.

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SOME PROPERTIES OF THE GENERALIZATION OF THE FIBONACCI SEQUENCE

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(Submitted April 1985)

1. INTRODUCTION

Let the arbitrary real numbers a, b, c, and d be given. Construct two sequences $\{X_n\}$ and $\{Y_n\}$ for which

$$\begin{cases} X_0 = a,\ X_1 = c,\ Y_0 = b,\ Y_1 = d, \\ X_{n+2} = Y_{n+1} + Y_n \\ Y_{n+2} = X_{n+1} + X_n \end{cases} \quad (n \geqslant 0). \tag{1}$$

Clearly, if we set $a = b$ and $c = d$, then the sequences $\{X_n\}$ and $\{Y_n\}$ will coincide with each other and with the sequence $\{F_n(a, c)\}$.

In 1985, K. T. Atanassov, L. C. Atanassova, & D. D. Sasselov [1] showed that

$$X_{n+2} = \frac{1}{2}\left\{\left(F_{n+1} + 3\left[\frac{n+2}{3}\right] - n - 1\right)a + \left(F_{n+1} - 3\left[\frac{n+2}{3}\right] + n + 1\right)b \right.$$
$$\left. + \left(F_{n+2} - 3\left[\frac{n}{3}\right] + n - 1\right)c + \left(F_{n+2} + 3\left[\frac{n}{3}\right] - n + 1\right)d\right\}$$

and $Y_n(a, b, c, d) = X_n(b, a, d, c)$, for $n \geqslant 0$. (2)

2. THE GENERALIZATION OF THE FIBONACCI SEQUENCE

Consider the generalized recursive form of (1), as follows:

$$\begin{cases} X_0 = a,\ X_1 = c,\ Y_0 = b,\ Y_1 = d, \\ X_{n+2} = r_1X_{n+1} + r_2X_n + r_3Y_{n+1} + r_4Y_n \\ Y_{n+2} = r_1Y_{n+1} + r_2Y_n + r_3X_{n+1} + r_4X_n \end{cases} \quad (n \geqslant 0), \tag{3}$$

where r_i is real.

Define

$$\begin{cases} X_n = X_{n,1}a + X_{n,2}b + X_{n,3}c + X_{n,4}d \\ Y_n = Y_{n,1}a + Y_{n,2}b + Y_{n,3}c + Y_{n,4}d \\ U_n = X_n + Y_n = U_{n,1}a + U_{n,2}b + U_{n,3}c + U_{n,4}d \\ V_n = X_n - Y_n = V_{n,1}a + V_{n,2}b + V_{n,3}c + V_{n,4}d \end{cases}$$

then $\{U_n\}$ and $\{V_n\}$ can be defined by the recursions:

$$\begin{cases} U_0 = a + b,\ U_1 = c + d \ \text{ and } \ U_{n+2} = (r_1 + r_3)U_{n+1} + (r_2 + r_4)U_n \\ V_0 = a - b,\ V_1 = c - d \ \text{ and } \ V_{n+2} = (r_1 - r_3)V_{n+1} + (r_2 - r_4)V_n \end{cases}$$

i.e.,

$$\begin{cases} U_n = W_n(a + b,\ c + d;\ r_1 + r_3,\ -r_2 - r_4) \\ V_n = W_n(a - b,\ c - d;\ r_1 - r_3,\ r_4 - r_2) \end{cases} \qquad \text{(See [2, 3].)}$$

Since

$$\begin{aligned} U_n(a, b, c, d) &= X_n(a, b, c, d) + Y_n(a, b, c, d) \\ &= X_n(a, b, c, d) + X_n(b, a, d, c), \text{ by symmetrical property,} \\ &= X_n(a + b,\ a + b,\ c + d,\ c + d) \\ &= X_{n,1}(a + b) + X_{n,2}(a + b) + X_{n,3}(c + d) + X_{n,4}(c + d) \\ &= (X_{n,1} + X_{n,2})a + (X_{n,1} + X_{n,2})b + (X_{n,3} + X_{n,4})c \\ &\quad + (X_{n,3} + X_{n,4})d \end{aligned}$$

and

$$\begin{aligned} V_n(a, b, c, d) = (X_{n,1} - X_{n,2})a + (X_{n,2} - X_{n,1})b + (X_{n,3} - X_{n,4})c \\ + (X_{n,4} - X_{n,3})d, \end{aligned}$$

compare with the coefficients of a, b, c, and d, we obtain:

$$\begin{cases} U_{n,1} = U_{n,2} = W_n(1,\ 0;\ r_1 + r_3,\ -r_2 - r_4) \\ \qquad\qquad = \displaystyle\sum_{k=1}^{[n/2]} \binom{n - k - 1}{k - 1}(r_1 + r_3)^{n-2k}(r_2 + r_4)^k \\ U_{n,3} = U_{n,4} = W_n(0,\ 1;\ r_1 + r_3,\ -r_2 - r_4) \\ \qquad\qquad = \displaystyle\sum_{k=1}^{[(n+1)/2]} \binom{n - k}{k - 1}(r_1 + r_3)^{n-2k+1}(r_2 + r_4)^{k-1} \\ V_{n,1} = -V_{n,2} = W_n(1,\ 0;\ r_1 - r_3,\ -r_2 + r_4) \\ \qquad\qquad = \displaystyle\sum_{k=1}^{[n/2]} \binom{n - k - 1}{k - 1}(r_1 - r_3)^{n-2k}(r_2 - r_4)^k \\ V_{n,3} = -V_{n,4} = W_n(0,\ 1;\ r_1 - r_3,\ -r_2 + r_4) \\ \qquad\qquad = \displaystyle\sum_{k=1}^{[(n+1)/2]} \binom{n - k}{k - 1}(r_1 - r_3)^{n-2k+1}(r_2 - r_4)^{k-1} \end{cases} \tag{4}$$

for $n \geqslant 2$.

Hence,

$$\begin{cases} U_n = (a + b)U_{n,1} + (c + d)U_{n,3} \\ V_n = (a - b)V_{n,1} + (c - d)V_{n,3}. \end{cases}$$

Since $U_n = X_n + Y_n$ and $V_n = X_n - Y_n$, thus,

$$\begin{cases} X_n = (U_n + V_n)/2 \\ Y_n = (U_n - V_n)/2 \end{cases}$$

is the solution of (3).

Example 1: Let $r_1 = r_2 = 0$ and $r_3 = r_4 = 1$. Then, we have:

$$\begin{cases} U_{n,1} = W_n(1, 0; 1, -1) = F_{n-1} \\ U_{n,3} = W_n(0, 1; 1, -1) = F_n \\ V_{n,1} = W_n(1, 0; -1, 1) = [1, 0, -1] \\ V_{n,3} = W_n(0, 1; -1, 1) = [0, 1, -1] \end{cases}$$

where $[t_1, t_2, \ldots, t_k] = t_j$ if $n \equiv j \pmod{k}$.

Hence,

$$X_n = \{(F_{n-1} + [1, 0, -1])a + (F_{n-1} + [-1, 0, 1])b + (F_n + [0, 1, -1])c + (F_n + [0, -1, 1])d\}/2 \tag{5}$$

and $Y_n(a, b, c, d) = X_n(b, a, d, c)$ is the solution of (1), where F_i is the i^{th} Fibonacci number. Note that (5) is the simple form of (2).

Example 2: Let $r_3 = r_4 = 0$ and $r_1 = r_2 = 1$. Then, we have:

$$U_{n,1} = V_{n,1} = W_n(1, 0; 1, -1) = F_{n-1}$$

and

$$U_{n,3} = -V_{n,3} = W_n(0, 1; 1, -1) = F_n.$$

Thus,

$$\begin{cases} X_n = F_{n-1}a + F_n c \\ Y_n = F_{n-1}b + F_n d \end{cases}$$

is the solution of (3) in [1].

Example 3: Let $r_1 = r_4 = 0$ and $r_2 = r_3 = 1$. Then, we have:

$$U_{n,1} = W_n(1, 0; 1, -1) = (-1)^n V_{n,1} = F_{n-1}$$

and

$$U_{n,3} = W_n(0, 1; 1, -1) = (-1)^n V_{n,3} = F_n.$$

Thus,

$$\begin{aligned} X_n &= \{(1 + (-1)^n)F_{n-1}a + (1 - (-1)^n)F_{n-1}b + (1 - (-1)^n)F_n c + (1 + (-1)^n)F_n d\}/2 \\ &= \begin{cases} F_{n-1}a + F_n d, & n \text{ even} \\ F_{n-1}b + F_n c, & n \text{ odd} \end{cases} \end{aligned}$$

and $Y_n(a, b, c, d) = X_n(b, a, d, c)$ is the solution of (4) in [1].

Example 4: Let $r_2 = r_3 = 0$ and $r_1 = r_4 = 1$. Then, we have:

$$U_{n,1} = W_n(1, 0; 1, -1) = F_{n-1}$$

$$U_{n,3} = W_n(0, 1; 1, -1) = F_n$$

$$V_{n,1} = W_n(1, 0; 1, 1) = [1, 0, -1, -1, 0, 1]$$

$$V_{n,3} = W_n(0, 1; 1, 1) = [0, 1, 1, 0, -1, -1]$$

Thus,

$$X_n = \{(F_{n-1} + [1, 0, -1, -1, 0, 1])a + (F_{n-1} + [-1, 0, 1, 1, 0, -1])c + (F_n + [0, 1, 1, 0, -1, -1])c + (F_n + [0, -1, -1, 0, 1, 1])d\}/2$$

and $Y_n(a, b, c, d) = X_n(b, a, d, c)$ is the solution of (5) in [1].

<u>Example 5</u>: Let $r_1 = r_2 = 0$, $r_3 = 2$, and $r_4 = 1$. Then, we have:

$$U_{n,1} = W_n(1,\ 0;\ 2,\ -1) = \{(2 - \sqrt{2})(1 + \sqrt{2})^n + (2 + \sqrt{2})(1 - \sqrt{2})^n\}/4$$
$$U_{n,3} = W_n(0,\ 1;\ 2,\ -1) = \sqrt{2}\{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n\}/4$$
$$V_{n,1} = W_n(1,\ 0;\ -2,\ 1) = (n - 1)(-1)^{n-1}$$
$$V_{n,3} = W_n(0,\ 1;\ -2,\ 1) = n(-1)^{n-1}$$

Thus,

$$X_{n,1} = \{(2 - \sqrt{2})(1 + \sqrt{2})^n + (2 + \sqrt{2})(1 - \sqrt{2})^n + 4(n - 1)(-1)^{n-1}\}/8$$
$$X_{n,2} = \{(2 + \sqrt{2})(1 + \sqrt{2})^n + (2 + \sqrt{2})(1 - \sqrt{2})^n + 4(n - 1)(-1)^{n}\}/8$$
$$X_{n,3} = X_{n+1,\,2} = \{\sqrt{2}(1 + \sqrt{2})^n - \sqrt{2}(1 - \sqrt{2})^n + 4n(-1)^n\}/8$$
$$X_{n,4} = X_{n+1,\,1} = \{\sqrt{2}(1 + \sqrt{2})^n - \sqrt{2}(1 - \sqrt{2})^n + 4n(-1)^{n-1}\}/8$$

Hence,

$$X_n = X_{n,1}a + X_{n,2}b + X_{n,3}c + X_{n,4}d$$

and $Y_n(a,\ b,\ c,\ d) = X_n(b,\ a,\ d,\ c)$ is the solution of the following system:

$$\begin{cases} X_0 = a,\ X_1 = c,\ Y_0 = b,\ Y_1 = d, \\ X_{n+2} = 2Y_{n+1} + Y_n & (n \geqslant 0) \\ Y_{n+2} = 2X_{n+1} + X_n \end{cases}$$

By the five examples above and (4), we obtain the following formulas:

$$\sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1} = F_{n-1}$$
$$\sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1}(-1)^k = [1,\ 0,\ -1,\ -1,\ 0,\ 1] = (-1)^n[1,\ 0,\ -1]$$
$$\sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1}(-1)^{k-1}2^{n-2k} = n - 1.$$

<u>3. THE TRIBONACCI SEQUENCE</u>

Let the arbitrary real numbers a, b, c, d, e, and h be given. Construct two sequences $\{X_n\}$ and $\{Y_n\}$ for which

$$\begin{cases} X_0 = a,\ X_1 = b,\ X_2 = c,\ Y_0 = d,\ Y_1 = e,\ Y_2 = h, \\ X_{n+3} = Y_{n+2} + Y_{n+1} + Y_n & (n \geqslant 0). \\ Y_{n+3} = X_{n+2} + X_{n+1} + X_n \end{cases} \tag{6}$$

Define:

$$X_n = X_{n,1}a + X_{n,2}b + X_{n,3}c + X_{n,4}d + X_{n,5}e + X_{n,6}h$$
$$Y_n = Y_{n,1}a + Y_{n,2}b + Y_{n,3}c + Y_{n,4}d + Y_{n,5}e + Y_{n,6}h$$
$$U_n = X_n + Y_n = U_{n,1}a + U_{n,2}b + U_{n,3}c + U_{n,4}d + U_{n,5}e + U_{n,6}h$$
$$V_n = X_n - Y_n = V_{n,1}a + V_{n,2}b + V_{n,3}c + V_{n,4}d + V_{n,5}e + V_{n,6}h$$

Then, we have:

$$U_{n,1} = U_{n,4} \qquad V_{n,1} = -V_{n,4} = [1, 0, 0, -1]$$
$$U_{n,2} = U_{n,5} = U_{n+2,1} - U_{n+1,1} \qquad V_{n,2} = -V_{n,5} = [0, 1, 0, -1]$$
$$U_{n,3} = U_{n,6} = U_{n+1,1} \qquad V_{n,3} = -V_{n,6} = [0, 0, 1, -1]$$

where $\{U_{n,1}\}$ can be defined by the recursions (cf. the definition of the Tribonacci numbers in [4] and [5]):

$$U_{0,1} = 1,\ U_{1,1} = U_{2,1} = 0,\ \text{and } U_{n+3,1} = U_{n+2,1} + U_{n+1,1} + U_{n,1},$$

for $n \geqslant 0$. That is to say,

$$U_{0,1} = 1,\ U_{1,1} = U_{2,1} = 0,\ U_{3,1} = 1,$$

and

$$U_{n+4,1} = 2U_{n+3,1} - U_{n,1} \quad (n \geqslant 0),$$

since $x^3 - x^2 - x - 1$ is a factor of $x^4 - 2x^3 + 1$. Thus, we have:

$$X_{n,1} = (U_{n,1} + V_{n,1})/2 = (U_{n,1} + [1, 0, 0, -1])/2$$
$$X_{n,2} = (U_{n,2} + V_{n,2})/2 = (U_{n+2,1} - U_{n+1,1} + [0, 1, 0, -1])/2$$
$$X_{n,3} = (U_{n,3} + V_{n,3})/2 = (U_{n+1,1} + [0, 0, 1, -1])/2$$
$$X_{n,4} = (U_{n,4} + V_{n,4})/2 = (U_{n,1} + [-1, 0, 0, 1])/2$$
$$X_{n,5} = (U_{n,5} + V_{n,5})/2 = (U_{n+2,1} - U_{n+1,1} + [0, -1, 0, 1])/2$$
$$X_{n,6} = (U_{n,6} + V_{n,6})/2 = (U_{n+1,1} + [0, 0, -1, 1])/2$$

Hence,

$$X_n = \{(a + d)U_{n,1} + (c + h - b - d)U_{n+1,1} + (b + d)U_{n+2,1} + [a - d, b - e, c - h, d + e + h - a - b - c]\}/2$$

and $Y_n(a, b, c, d, e, h) = X_n(d, e, h, a, b, c)$ is the solution of (6).

4. THE FIBONACCI-TRIPLES SEQUENCE

Let the arbitrary real numbers a, b, c, d, e, and h be given. Construct three sequences $\{X_n\}$, $\{Y_n\}$, and $\{Z_n\}$ for which

$$\begin{cases} X_0 = a,\ X_1 = b,\ Y_0 = c,\ Y_1 = d,\ Z_0 = e,\ Z_1 = h, \\ X_{n+2} = Y_{n+1} + Z_n \\ Y_{n+2} = Z_{n+1} + X_n \quad (n \geqslant 0). \\ Z_{n+2} = X_{n+1} + Y_n \end{cases} \tag{7}$$

The first ten terms of the sequences defined in (7) are shown below:

n	X_n	Y_n	Z_n	n	X_n	Y_n	Z_n
0	a	c	e				
1	b	d	h	6	$8h + 5a$	$8b + 5c$	$8d + 5e$
2	$d + e$	$h + a$	$b + c$	7	$13b + 8c$	$13d + 8e$	$13h + 8a$
3	$2h + a$	$2b + c$	$2d + e$	8	$21d + 13e$	$21h + 13a$	$21b + 13c$
4	$3b + 2c$	$3d + 2e$	$3h + 2a$	9	$34h + 21a$	$34b + 21c$	$34d + 21e$
5	$5d + 3e$	$5h + 3a$	$5b + 3c$	10	$55b + 34c$	$55d + 34e$	$55h + 34a$

Define

$$U_n = X_n + Y_n + Z_n = U_{n,1}a + U_{n,2}b + U_{n,3}c + U_{n,4}d + U_{n,5}e + U_{n,6}h,$$

then $\{U_n\}$ can be defined by the recursion

$$U_0 = a + c + e,\ U_1 = b + d + h, \text{ and } U_{n+2} = U_{n+1} + U_n,$$

i.e., $U_n = W_n(a + c + e,\ b + d + h;\ 1, -1)$.

Compare with the coefficients of a, b, c, d, e, and h. We have:

$$U_{n,1} = U_{n,3} = U_{n,5} = W_n(1, 0; 1, -1) = F_{n-1}$$

$$U_{n,2} = U_{n,4} = U_{n,6} = W_n(0, 1; 1, -1) = F_n$$

Thus,

$$U_n = (a + c + e)F_{n-1} + (b + d + h)F_n.$$

Since $X_n = Y_{n+2} - Z_{n+1}$ and $X_n = Z_{n+1} - Y_{n-1}$, we have:

$$\begin{cases} X_n = (Y_{n+2} - Y_{n-1})/2 \\ Y_n = (Z_{n+2} - Z_{n-1})/2 \\ Z_n = (X_{n+2} - X_{n-1})/2 \end{cases}$$

Since $X_n = Y_{n-1} + Z_{n-2}$ and $X_n = Z_{n+1} - Y_{n-1}$, we obtain:

$$\begin{cases} X_n = (Z_{n+1} + Z_{n-2})/2 \\ Y_n = (X_{n+1} + X_{n-2})/2 \\ Z_n = (Y_{n+1} + Y_{n-2})/2 \end{cases}$$

Since $4X_{n+3} = 2(Y_{n+5} - Y_{n+2}) = (X_{n+6} + X_{n+3}) - (X_{n+3} + X_n) = X_{n+6} - X_n$, we have:

$$\begin{cases} X_{n+6} = 4X_{n+3} + X_n \\ Y_{n+6} = 4Y_{n+3} + Y_n \\ Z_{n+6} = 4Z_{n+3} + Z_n \end{cases}$$

When $n \equiv 0 \pmod 3$, taking $n = 3m$, we have:

$$X_{3(m+2)} = 4X_{3(m+1)} + X_{3m} \text{ with } X_0 = a \text{ and } X_3 = 2h + a.$$

Letting $V_m = X_{3m}$, we have:

$$V_{m+2} = 4V_{m+1} + V_m \text{ with } V_0 = a \text{ and } V_1 = 2h + a.$$

Therefore, we get:

$$\begin{aligned} V_m &= \frac{2h + (\sqrt{5} - 1)a}{2\sqrt{5}}(2 + \sqrt{5})^m + \frac{(\sqrt{5} + 1)a - 2h}{2\sqrt{5}}(2 - \sqrt{5})^m \\ &= \frac{1}{\sqrt{5}}\left\{\left(\left(\frac{1 + \sqrt{5}}{2}\right)^{3m-1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{3m-1}\right)a + \left(\left(\frac{1 - \sqrt{5}}{2}\right)^{3m} - \left(\frac{1 - \sqrt{5}}{2}\right)^{3m}\right)h\right\}, \\ &= F_{3m-1}a + F_{3m}h \quad \text{by } \left(\frac{1 \pm \sqrt{5}}{2}\right)^3 = 2 \pm \sqrt{5}, \end{aligned}$$

i.e., $X_n = F_{n-1}a + F_nh$.

Using a similar method, we have: $X_n = \begin{cases} F_{n-1}a + F_nh, & \text{if } n \equiv 0 \pmod 3 \\ F_{n-1}c + F_nb, & \text{if } n \equiv 1 \pmod 3 \\ F_{n-1}e + F_nd, & \text{if } n \equiv 2 \pmod 3 \end{cases}$

$$Y_n(a, b, c, d, e, h) = X_n(e, h, a, b, c, d)$$

and

$$Z_n(a, b, c, d, e, h) = X_n(c, d, e, h, a, b)$$

as the solution of (7).

Numerous similar pairs of sequences can be constructed. However, the ones introduced here stand most closely to the very spirit of the Tribonacci sequence (or the Fibonacci-triples sequence) and its generalization rules.

We would like to thank Professor Horng-Jinh Chang for his helpful comments and the referee for his thorough discussions.

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SECTIONS, GOLDEN AND NOT SO GOLDEN

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(Submitted May 1985)

INTRODUCTION

The idea of the golden section is familiar to a wide audience. While many of the properties that arise from consideration of the golden section seem to be unique to it, they may belong to a much wider class of "sections." This paper presents the golden section and certain related ideas as special cases of such a wider class.

To provide the context for what follows and to introduce some notation, we include here a quick reference to the golden ratio, ϕ. Let a line segment $\overline{AB}$ be given and, for convenience, let its length $AB = 1$. If we determine a point C between A and B and such that $AB/AC = AC/CB$, we say the point C divides AB in golden section (see Fig. 1).

Figure 1

It is a simple matter to find the ratio $\phi = AB/AC$ that belongs to the golden section. If we set $x = AC$, then $CB = 1 - x$ and we have the requirement

$$\phi = 1/x = x/(1 - x), \; 0 < x < 1.$$

From this equation, we infer that $1/\phi = \phi - 1$, or

$$\phi^2 - \phi - 1 = 0. \tag{1}$$

From the quadratic equation, and since $x > 0$, we have

$$\phi = (\sqrt{5} + 1)/2 \doteq 1.61803.$$

The number ϕ has the interesting property that if we subtract the value 1 from it we obtain its reciprocal.

As readers of this journal know well, the golden ratio bears a relation to the Pentagon of Pythagoras, so much admired by the Greeks, to the golden rectangle where it gives the ratio of adjacent sides, to the logarithmic spiral, and to the Fibonacci numbers. Specifically, if F_k denotes the k^{th} Fibonacci number, then as $n \to \infty$, $F_{n+1}/F_n \to \phi$.

THE MODIFIED GOLDEN SECTION AND GOLDEN RATIO

In what follows we will develop some ideas similar to those alluded to above and growing out of a generalization of the definition of the golden section. So consider again a line segment $\overline{AB}$ of length $AB = 1$ and let C be a point between A and B and such that $AB/AC = \alpha^2 AC/CB$ with $\alpha > 0$. We can write this relationship as

$$\frac{AB}{\alpha AC} = \frac{\alpha AC}{CB}, \quad \alpha > 0.$$

Set $\psi_\alpha = AB/\alpha AC$ and let $AC = x$. Then $CB = 1 - x$, $\psi_\alpha = 1/\alpha x = \alpha x/(1 - x)$, and

$$\psi_\alpha^2 - \frac{1}{\alpha}\psi_\alpha - 1 = 0, \tag{2}$$

which is the analogue of equation (1). For convenience, we let β represent the reciprocal of ψ_α:

$$\beta = 1/\psi_\alpha = \alpha x = \frac{\sqrt{1 + 4\alpha^2} - 1}{2\alpha}.$$

Suppose now that we let $\alpha > 0$ be chosen and construct a rectangle $ABDF$ (see Fig. 2) whose sides are in the ratio $\psi_\alpha = 1/\beta = AB/BD$. A few simple calculations show that if from such a rectangle we remove the rectangle $ACEF$ whose sides AC and CE are in the ratio $1/\alpha$, i.e., $AC = x = \beta/\alpha$ and $CE = \beta$, then the remaining rectangle $BDEC$ has sides also in the ratio

$$\alpha x/(1 - x) = \beta/(1 - x) = \psi_\alpha.$$

Thus, as in the case of the golden rectangle, the two rectangles $ABDF$ and $BDEC$ are similar and, by continuation of the process described here, we can construct an infinite nested sequence $R_1, R_2, R_3, \ldots, R_n, \ldots$ of rectangles, all of which are mutually similar.

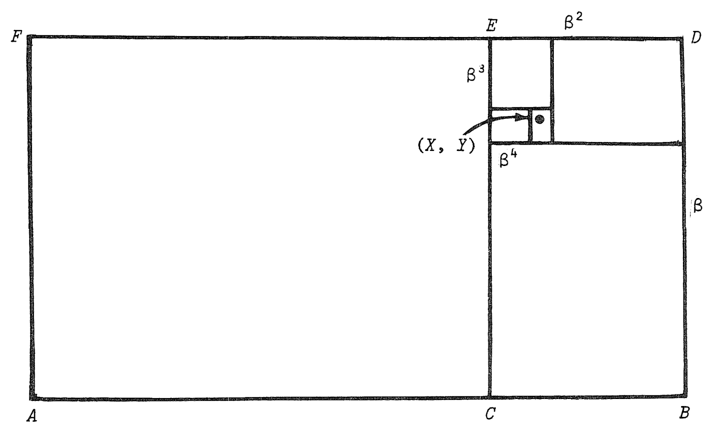

Figure 2

By varying α, of course, we vary the value of β and so also the shape of the rectangles. Since

$$\lim_{\alpha \to 0} \beta = \lim_{\alpha \to 0} \frac{\sqrt{1 + 4\alpha^2} - 1}{2\alpha} = 0,$$

It is clear that to small α there correspond small values of β. Thus, as $\alpha \to 0$ the rectangles tend toward "degenerate" rectangles, i.e., toward line segments.

From

$$\lim_{\alpha \to \infty} \beta = 1,$$

we infer that as α increases without bound the rectangles approach squares. We note also that for $0 < \alpha < \infty$, $0 < \beta < 1$.

Suppose that, for some value of α, we let $\{R_n\}$ be the associated sequence of rectangles obtained by the construction described above. Recall $R_{n-1} \supset R$. Take the vertex A (see Fig. 2) to be the origin of a rectangular coordinate system with the side $\overline{AB}$ lying on the positive x axis. Cantor's nested set theorem then assures us that there is a point (X, Y) which lies in each rectangle R_n. Now each rectangle R_n has sides of length β^{n-1} and β^n, with β^{n-1} being the longer side. It is clear from Figure 2 that

$$\begin{aligned} X &= 1 - \beta^2 + \beta^4 - \beta^6 + \cdots \\ &= 1/(1 - \beta^2) \qquad [= \alpha/(2\alpha - \beta)] \end{aligned}$$

and

$$\begin{aligned} Y &= \beta - \beta^3 + \beta^5 - \beta^7 + \cdots \\ &= \beta X \qquad [= \alpha\beta/(2\alpha - \beta)]. \end{aligned}$$

If we eliminate β from these equations, we find that

$$Y^2 = X - X^2 \quad \text{or} \quad \left(X - \frac{1}{2}\right)^2 + Y^2 = \left(\frac{1}{2}\right)^2.$$

Thus, the points (X, Y) lie on a circle of radius 1/2 and having its center at (1/2, 0). As $\alpha \to 0$, $(X, Y) \to (1, 0)$ along the circle, and as $\alpha \to \infty$, $(X, Y) \to (1/2, 1/2)$. Specifically, the points (X, Y) lie on the quarter-circle shown in Figure 3.

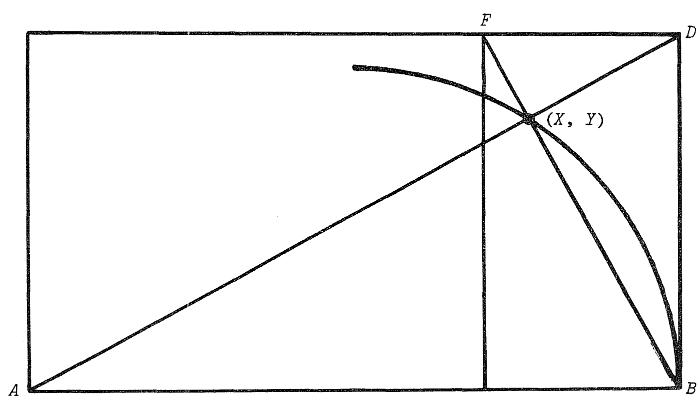

Figure 3

The point (X, Y) can be found by a very simple geometrical construction. If R_n and R_{n+1} are any two consecutive rectangles, then (X, Y) lies at the intersection of corresponding (and orthogonal) diagonals of these rectangles. Figure 3 above illustrates the case when R_1 and R_2 are the given rectangles. The diagonals here are $\overline{AD}$ and $\overline{BF}$.

We turn next to the logarithmic spiral associated with the rectangles R_n. Before doing so, however, we mention briefly the so-called rectangular spiral constructed from the longer sides of the rectangles R_n (see Fig. 4). This spiral "terminates," of course, at the point (X, Y) and has length (measured from the origin)

$$\begin{aligned} L &= 1 + \beta + \beta^2 + \cdots \\ &= 1/(1 - \beta) \\ &= \frac{2\alpha}{2\alpha - \sqrt{1 + 4\alpha^2} + 1} \\ &= \alpha(1 + \psi_\alpha). \end{aligned}$$

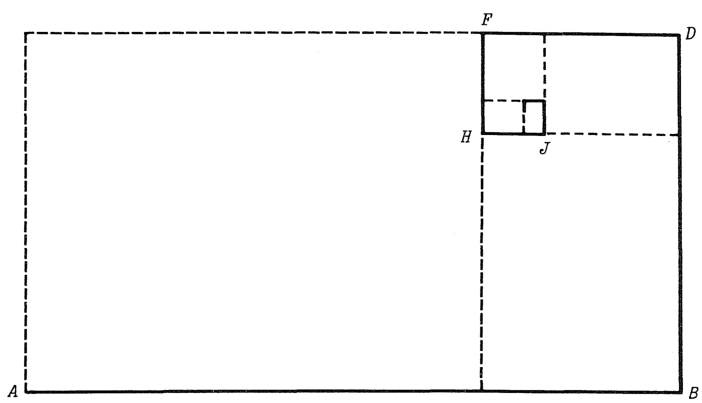

Figure 4

By a translation, we can place the origin of our coordinate system at the point (X, Y). Let P_n, $n = 1, 2, 3, \ldots$, be corresponding corners of the rectangles R_n and so also corners of the rectangular spiral. The points P_n can be shown, after some calculation, to have the following representations in terms of the new coordinate system:

$$\begin{aligned} P_1 &= (-X, -Y) \\ P_2 &= (1 - X, -Y) \\ P_3 &= (1 - X, \beta - Y) \\ P_4 &= (1 - \beta^2 - X, \beta - \beta^3 - Y) \end{aligned}$$

$$P_n = \begin{cases} \left(\dfrac{\beta^n}{1 + \beta^2}, \dfrac{-\beta^{n-1}}{1 + \beta^2}\right) & \text{if } n = 4k - 2 \\ \left(\dfrac{\beta^{n-1}}{1 + \beta^2}, \dfrac{\beta^n}{1 + \beta^2}\right) & \text{if } n = 4k - 1 \\ \left(\dfrac{-\beta^n}{1 + \beta^2}, \dfrac{\beta^{n+1}}{1 + \beta^2}\right) & \text{if } n = 4k \\ \left(\dfrac{-\beta^{n-1}}{1 + \beta^2}, \dfrac{-\beta^n}{1 + \beta^2}\right) & \text{if } n = 4k + 1. \end{cases}$$

If we express these points in terms of the polar coordinates r and θ, we obtain the cleaner expressions:

$$P_n = \left(\frac{\beta^{n-1}}{\sqrt{1+\beta^2}},\ \text{Arctan}\,\beta + ([n/2]-1)\pi\right) \qquad \text{if } n \text{ is odd,}$$

or

$$P_n = \left(\frac{\beta^{n-1}}{\sqrt{1+\beta^2}},\ \text{Arctan}(-1/\beta) + ([n/2]-1)\pi\right) \quad \text{if } n \text{ is even.}$$

Here $[\cdot]$ denotes the greatest integer function.

A few additional calculations convince us that each of the points P_n lies on a logarithmic spiral $r = ae^{b\theta}$. The constants a and b are easily determined from the requirement that the spiral pass through, let us say, P_2 and P_3. That it passes through P_2 implies that

$$r = \frac{\beta}{\sqrt{1+\beta^2}} \quad \text{and} \quad \theta = \text{Arctan}(-1/\beta) = -\text{Arctan}(1/\beta).$$

Thus,

$$\frac{\beta}{\sqrt{1+\beta^2}} = ae^{-b\ \text{Arctan}(1/\beta)}. \tag{3}$$

That the spiral passes through P_3 implies

$$r = \frac{\beta^2}{\sqrt{1+\beta^2}} \quad \text{and} \quad \theta = \text{Arctan}\,\beta.$$

Thus,

$$\frac{\beta^2}{\sqrt{1+\beta^2}} = ae^{b\ \text{Arctan}\,\beta}. \tag{4}$$

Combining equations (3) and (4) yields

$$\beta = e^{b(\text{Arctan}\,\beta + \text{Arctan}(1/\beta))}.$$

But the exponent here reduces to $b\pi/2$. So $\beta = e^{b\pi/2}$ and $b = 2\ln\beta/\pi < 0$. Now from (4) we can conclude that

$$a = \frac{\beta^2}{\sqrt{1+\beta^2}\ \exp(2\ \ln\beta\ \text{Arctan}\,\beta/\pi)}$$

and that

$$r = \frac{\beta^2}{\sqrt{1+\beta^2}\ \exp(2\ \ln\beta\ \text{Arctan}\,\beta/\pi)}\ \exp\left(\frac{2\ \ln\beta}{\pi}\theta\right).$$

Alternately,

$$r = \frac{1}{\sqrt{1+\beta^2}}\ \beta^{\frac{2(\theta+\pi-\text{Arctan}\,\beta)}{\pi}} \tag{5}$$

Figure 5 shows the spiral when $\alpha = 2$.

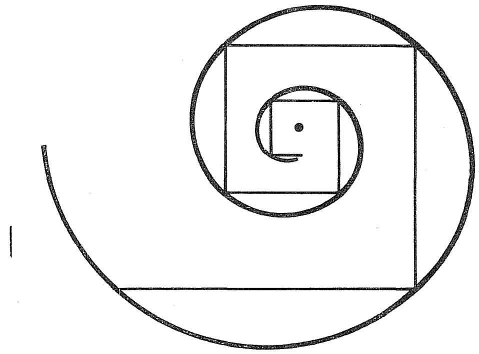

Figure 5

In the construction above, we have taken the points P_n to be at corresponding corners of the rectangles R_n. The decision to use these corners was arbitrary. If P_1 is chosen to be *any* point within or on R_1 and $P_2, P_3, \ldots$ to be the corresponding points of $R_2, R_3, \ldots$, then the spiral passing through all of the points P_n would again be logarithmic.

OTHER RELATED IDEAS

We consider next some relationships which are analogous to those between the golden section, golden rectangles, and the Fibonacci sequence. Let α be given and consider the sequence $\{u_n\}$, where

$$u_n = \frac{1}{\sqrt{1 + 4\alpha^2}}\left[\left(\frac{1 + \sqrt{1 + 4\alpha^2}}{2}\right)^n - \left(\frac{1 - \sqrt{1 + 4\alpha^2}}{2}\right)^n\right].$$

Readers familiar with the Fibonacci sequence will recognize that if $\alpha = 1$, then the last expression is the Binet formula and $\{u_n\}$ is nothing more than the Fibonacci sequence. To simplify calculations for the moment, set

$$z = \sqrt{1 + 4\alpha^2},\ a = (1 + z)/2, \text{ and } b = (1 - z)/2$$

so that $u_n = (a^n - b^n)/z$. Then it easily follows that

$$\begin{aligned} u_{n-1} + \alpha^2 u_{n-2} &= (1/z)[(a^{n-1} - b^{n-1}) + \alpha^2(a^{n-2} - b^{n-2})] \\ &= (1/z)[a^{n-2}(a + \alpha^2) - b^{n-2}(b + \alpha^2)]. \end{aligned}$$

But $a + \alpha^2 = a^2$ and $b + \alpha^2 = b^2$. Hence,

$$u_n = u_{n-1} + \alpha^2 u_{n-2}. \tag{6}$$

This serves as the law of generation for the sequence $\{u_n\}$. If $\alpha = 1$, this reduces to the familiar law of generation for the Fibonacci sequence.

Although we will not prove their validity, we list here a few of the relationships which are analogous to the relationships between terms of the Fibonacci sequence.

1. $u_n = (1 + 2\alpha^2)u_{n-2} - \alpha^4 u_{n-4}$

2. $u_1 + u_2 + u_3 + \cdots + u_n = (u_{n+2} - 1)/\alpha^2$

3. $u_1^2 + u_2^2 + \cdots + u_n^2 = (1 - \alpha^2)[u_1u_2 + u_2u_3 + \cdots + u_{n-1}u_n] + u_nu_{n+1}$

4. $u_n^2 - u_{n-1}u_{n+1} = (-\alpha^2)^{n-1}$

5. For any positive integer k, $u_n | u_{kn}$.

The first four of these relationships can be shown by an appeal to (6) and/or an induction proof. The fifth follows directly from the definition of u_n. From the second of these relationships, we can infer that when α is an integer, then $u_n \equiv 1 \bmod \alpha^2$. The table below gives values of u_n for some choices of α and of n.

$n =$	$\alpha =$ 1/3	1/2	2	3	4
1	1	1	1	1	1
2	1	1	1	1	1
3	1.11111	1.25	5	10	17
4	1.22222	1.50	9	19	33
5	1.345679	1.8125	29	109	205
6	1.4814814	2.1875	65	280	833
7	1.6310013	2.640625	181	1261	5713
8	1.7956104	3.18145	441	3781	19041
9	1.9768328	3.84765	1165	15130	110449
10	2.1763450	4.64453	2929	49159	415105
⋮			⋮	⋮	⋮
15			325525	28607050	884773585

Our next question is the obvious one:

How does $\lim_{n\to\infty} u_{n+1}/u_n$ relate to the ratio ψ_α?

From the definition of u_n, we can write

$$\frac{u_{n+1}}{u_n} = \frac{a^{n+1} - b^{n+1}}{a^n - b^n} = a\,\frac{1 - (b/a)^{n+1}}{1 - (b/a)^n}.$$

But $|b/a| = |(1 - z)/(1 + z)| < 1$, since $z > 0$ for all α. Thus,

$$\lim_{n\to\infty} u_{n+1}/u_n = a = (1 + z)/2 = \alpha\psi_\alpha.$$

This relationship is the analogue of

$$\lim_{n\to\infty} F_{n+1}/F_n = \phi,$$

where $\{F_n\}$ is the Fibonacci sequence and ϕ is the golden ratio.

Two further interesting properties which belong to the Fibonacci numbers also belong to the "modified Fibonacci numbers" u_n. If we form a matrix M of order $m \geqslant 3$ and whose entries, row by row, are m^2 successive terms u_k, u_{k+1}, u_{k+2}, ..., u_{k+m^2-1}, then det $M = 0$. So, for example, if $m = 3$ and $\alpha = 2$, and if we choose the nine successive terms of $\{u_n\}$ to be 1, 5, 9, 29, 65, 181, 441, 1165, and 2929, then

$$M = \begin{bmatrix} 1 & 5 & 9 \\ 29 & 65 & 181 \\ 441 & 1165 & 2929 \end{bmatrix}.$$

That the determinant of M is zero follows from the fact that, in any matrix M constructed as above from the successive terms u_n, the third column, U_3 (regarded here as a column vector), is equal to $U_2 + \alpha^2 U_1$, where U_1 and U_2 are the first and second column vectors belonging to M.

Another interesting property relates to magic squares of order 3. We illustrate this with an example. Thus, consider the magic square

8	1	6
3	5	7
4	9	2

Summing along rows, columns, or diagonals yields the same result, 15. Now let α be given and determine the terms u_1, u_2, u_3, ..., u_9 of the sequence $\{u_n\}$. In the magic square above, replace the number k with the term u_k to obtain the square

u_8	u_1	u_6
u_3	u_5	u_7
u_4	u_9	u_2

Then

$$u_8 u_1 u_6 + u_3 u_5 u_7 + u_4 u_9 u_2 = u_8 u_3 u_4 + u_1 u_5 u_9 + u_6 u_7 u_2.$$

For $\alpha = 3$, the above square with the associated products and sums is:

3781	1	280	1058680
10	109	1261	1374490
19	15130	1	287470
718390	1649170	35380	2720640

More generally, let the magic square

h	i	j
k	ℓ	m
p	q	r

be given, where

$$h + i + j = k + \ell + m = p + q + r = h + k + p = i + \ell + q = j + m + r.$$

Now, construct the corresponding square

u_h	u_i	u_j
u_k	u_ℓ	u_m
u_p	u_q	u_r

whose entries are the modified Fibonacci numbers.

Employing the notation of page 123 $[u_n = (a^n - b^n)/z]$, it is a simple matter to show that

$$u_h u_i u_j + u_k u_\ell u_m + u_p u_q u_r = u_h u_k u_p + u_i u_\ell u_q + u_j u_m u_r.$$

The reader will quickly observe, for example, that the expansion of the expression $u_h u_i u_j$ contains the term $(1/z^3)a^{h+i+j}$, while the expansion of $u_h u_k u_p$ contains the term $(1/z^3)a^{h+k+p}$. But $h+i+j = h+k+p$. Similarly, the expansion of $u_k u_\ell u_m$ contains the term $(-1/z^3)a^{k+\ell}b^m$, while the expansion of $u_j u_m u_r$ contains the term $(-1/z^3)a^{j+r}b^m$. But $j + r = k + \ell$ so that the terms in question are equal.

While the property alluded to here holds for any 3×3 magic square, it does not hold generally for larger magic squares. The reader may verify this by considering the 4×4 magic square:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

CONCLUSION

We have provided here only a few of the most significant relationships and properties arising from consideration of the ratio ψ_α. Many others analogous to those arising from the golden ratio may be found. Indeed, what we have shown here places the golden section, the golden rectangles, the Fibonacci sequence, and the properties pertaining to them within a continuum in which they appear as a part of the special case $\alpha = 1$.

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ON ASSOCIATED AND GENERALIZED LAH NUMBERS AND APPLICATIONS TO DISCRETE DISTRIBUTIONS

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(*Submitted May 1985*)

1. INTRODUCTION

First, we consider some definitions and preliminary results needed in this study. Ahuja & Enneking [1] have defined the associated Lah numbers $B(n, r, k)$ by

$$B(n, r, k) = (n!/k!)\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}\binom{n + ri - 1}{n}, \tag{1}$$

where

$B(n, r, k) = 0$ for $k > n$, $B(n, r, 0) = 0$,
$B(n, r, 1) = r(r + 1)\ldots(r + n - 1)$, $B(n, r, n) = r^n$
and $B(n, 1, k) = |L(n, k)|$,

the signless Lah numbers (see Riordan [12], p. 44).

Ahuja & Enneking have also obtained (see [2]) the following relations for the $B(n, r, k)$'s:

$$B(n + 1, r, k) = (n + rk)B(n, r, k) + rB(n, r, k - 1), \tag{2}$$

and

$$[B(n, r, k)]^2 > B(n, r, k + 1)B(n, r, k - 1) \text{ for } k = 2, 3, \ldots, n - 1. \tag{3}$$

We now introduce two other equivalent definitions of $B(n, r, k)$. First, we write

$$B(n, r, k) = [(E^r - I)^k y^{[n]}]_{y=0}/k! \quad (k = 1, \ldots, n), \tag{4}$$

where $Ef(x) = f(x + 1)$ and I is the unit operator.

Second, we have

$$B(n, r, k) = (n!/k!)\sum_{k}\prod_{i=1}^{k}\binom{n_i + r - 1}{n_i}, \tag{5}$$

where $\sum_k$ denotes the sum over all positive integral values of the n_i's such that $n_1 + \cdots + n_k = n$ and $n = k, k + 1, \ldots$.

Equation (5) follows from the following combinatorial identity:

$$\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}\binom{n + ri - 1}{n} = \sum_{k}\prod_{i=1}^{k}\binom{n_i + r - 1}{n_i}, \tag{6}$$

where the summation in the right-hand member is extended over integral values of each $n_i \geqslant 1$ such that $n_1 + \cdots + n_k = n$ and $n = k, k + 1, \ldots$.

Further, let $R(n, r, k)$ be a sequence of real numbers defined by

$$R(n, r, k) = B(n + 1, r, k)/B(n, r, k), \; k = 1, 2, \ldots, n, \tag{7}$$

for given n. These numbers are useful in calculating probability functions independent of rapidly growing associated Lah numbers.

Ahuja & Enneking [1] have introduced the generalized Lah numbers $L_{c,r}(n, k)$ defined by:

$$L_{c,r}(n, k) = (n!/k!) \sum (-1)^{k-r_1} \frac{k!}{r_1!r_2!\ldots r_{c+2}!} \times \prod_{j=0}^{c} \left[\binom{j+r-1}{j}\right]^{r_{j+2}} \binom{n - \sum_{j=0}^{c} jr_{j+2} + rr_1 - 1}{rr_1 - 1} \tag{8}$$

for integral $c \geqslant 0$, and $n = k(c + 1)$, $k(c + 1) + 1$, ..., where the summation extends over all $r_j > 0$ such that $\sum_{j=1}^{c+2} r_j = k$.

Using the combinatorial identity

$$\sum_J (-1)^{k-r_1} \frac{k!}{r_1!r_2!\ldots r_{c+2}!} \prod_{j=0}^{c} \left[\binom{j+r-1}{j}\right]^{r_{j+2}} \binom{n - \sum_{j=0}^{c} jr_{j+2} + rr_1 - 1}{rr_1 - 1}$$
$$= \sum_K \prod_{i=1}^{k} \binom{x_i + r - 1}{x_i}, \tag{9}$$

for $c > 0$, and $n = k(c+1)$, $k(c+1) + 1, \ldots$, where $\sum_J$ extends over all $r_j > 0$ such that $\sum_{j=1}^{c+2} r_j = k$ and $\sum_k$ extends over all $x_i > c$ such that $\sum_{i=1}^{k} x_i = n$, we find an alternative representation of the generalized Lah number as

$$L_{c,r}(n, k) = (n!/k!) \sum_K \prod_{i=1}^{k} \binom{x_i + r - 1}{x_i}, \tag{10}$$

where $\sum_K$ is extended over all ordered k-tuples $(x_1, x_2, \ldots, x_k)$ of integers $x_i > c$, $i = 1, 2, \ldots, k$ with $x_1 + x_2 + \cdots + x_k = n$.

Section 2 is devoted to the study of properties of associated Lah numbers. Section 3 is concerned with the properties of ratios of associated Lah numbers. Section 4 deals with a discrete probability distribution involving associated Lah numbers via a generalized occupancy problem. Section 5 contains the problem of estimating a parameter of the population discussed in the preceding section. Section 6 discusses limiting forms of the discrete distribution studied in Section 4. Section 7 introduces an inverse probability distribution involving associated Lah numbers. Section 8 considers the definitions and properties of a conditional multivariate distribution involving associated Lah numbers. The last two sections deal with some applications of generalized Lah numbers.

2. SOME PROPERTIES OF $B(n, r, k)$

We now investigate properties of $B(n, r, k)$ and their limiting forms.

Property 1:

$$(rx)^{[n]} = \sum_{k=1}^{\infty} B(n, r, k)(x)_k, \tag{11}$$

where $(rx)^{[n]} = rx(rx + 1) \ldots (rx + n - 1)$ and $(x)_k = x(x - 1) \ldots (x - k + 1)$, x being any real number and r a positive integer.

Proof: $(rx)^{[n]} = [E^{rx}y^{[n]}]_{y=0} = [\{I + (E^r - I)\}^x y^{[n]}]_{y=0}$

$$= \sum_{k=0}^{\infty} \binom{x}{k} [(E^r - I)^k y^{[n]}]_{y=0} = \sum_{k=1}^{\infty} B(n, r, k)(x)_k \text{ from (4).}$$

However, if x is a positive integer, then

$$(rx)^{[n]} = \sum_{k=1}^{\min(x,n)} B(n, r, k)(x)_k. \tag{12}$$

Property 2:

$$1/(x-1)_k = \sum_{n=k}^{\infty} B(n, r, k)/(rx+1)^{[n]}. \tag{13}$$

This can be proved by induction on k.

Property 3:

$$B(n, r, k) = (1/k)\sum_{x=1}^{n-k+1} (n)_x \binom{x+r-1}{x} B(n-x, r, k-1). \tag{14}$$

Property 4:

$$\lim_{r\to 0} B(n, r, k)/r^k = |s(n, k)|, \tag{15}$$

where $|s(n, k)|$ is the signless Stirling number of the first kind.

Property 5:

$$\lim_{r\to\infty} B(n, r, k)/r^n = S(n, k), \tag{16}$$

where $S(n, k)$ denotes the Stirling number of the second kind.

3. SOME PROPERTIES OF $R(n, r, k)$

In this section we study the following properties of $R(n, r, k)$.

Property 1: The sequence (7) satisfies the recurrence relation

$$R(n, r, k) - (n + rk)$$
$$= R(n-1, r, k-1) - [(n + rk - 1)/R(n-1, r, k)] \tag{17}$$

for $1 < k < n$ and for all n, where

$R(n, r, 1) = (n + r)$ and $R(n, r, n) = [n(n+1)(r+1)]/2$.

The relation (17) follows directly from (2).

Property 2: The sequence (7) increases with k for given n and satisfies the inequality

$$R(n, r, k+1) > R(n, r, k), \text{ for } k = 2, 3, \ldots, n-1. \tag{18}$$

This follows immediately from (3).

Property 3: The sequence (7) satisfies the inequality

$$R(n-1, r, k) + 1 \geqslant R(n, r, k) \quad (n = k+1, k+2, \ldots) \tag{19}$$

with equality only for $k = 1$.

Relation (19) is observed from (17). It shows that the ratio $R(n, r, k)$ grows very slowly with n.

4. A DISCRETE PROBABILITY DISTRIBUTION INVOLVING ASSOCIATED LAH NUMBERS

This section is devoted to the study of a discrete probability distribution involving the associated Lah numbers derived via the following generalized occupancy problem.

Suppose n indistinguishable balls are distributed in $r\theta$ cells constituting θ groups of r cells each. Then the probability that k groups are occupied with n_1 balls in one group, n_2 balls in the second group, ..., n_k balls in the k^{th} group, and the remaining $(\theta - k)$ groups are empty is

$$Pr\{K = k \cap N_1 = n_1, \ldots, N_{k-1} = n_{k-1} | n, r, \theta\}$$
$$= n!(\theta)_k \prod_{i=1}^{k} \binom{n_i + r - 1}{n_i} \Big/ \{(r\theta)^{[n]} k!\}, \tag{20}$$

where $(r\theta)^{[n]} = (r\theta)(r\theta + 1) \ldots (r\theta + n - 1)$ and $n_k = n - n_1 - \cdots - n_{k-1}$.

From (20), the probability that k different groups are occupied out of θ groups (without regard to frequencies) is

$$Pr\{K = k|n, r, \theta\} = f_K(k|n, r, \theta)$$
$$= [(\theta)_k/(r\theta)^{[n]}](n!/k!) \sum \prod_{i=1}^{k} \binom{n_i + r - 1}{n_i}, \tag{21}$$

where the summation extends over all positive integral values of $n_1, \ldots, n_{k-1}$ subject to $n > n_1 + \cdots + n_{k-1}$.

Now, using the definition of associated Lah numbers in (5), the probability function (pf) of the random variable K is

$$f_K(k|n, r, \theta) = B(n, r, k)(\theta)_k/(r\theta)^{[n]}, \quad k = 1, \ldots, n. \tag{22}$$

From (11), it follows that

$$\sum_{k=1}^{n} f_K(k|n, r, \theta) = 1,$$

which verifies that $f_K(k|n, r, \theta)$ is a proper pf.

In particular, if $r = 1$ in (22),

$$f_K(k|n, \theta) = |L(n, k)|(\theta)_k/\theta^{[n]}, \quad (k = 1, \ldots, n), \tag{23}$$

where the $|L(n, k)|$'s are the signless Lah numbers.

The probability model (23) describes the distribution of K, the number of occupied cells, when n indistinguishable balls are assigned to θ cells. Analogously, it gives the distribution of K, the number of occupied energy levels, if n like particles (e.g., protons, nuclei, or atoms containing an even number of elementary particles for the Bose-Einstein system of physical statistics) are assigned to θ energy levels.

The pf (22) satisfies the recurrence relation

$$f_K(k|n, r, \theta) = r(\theta - k + 1)f_K(k - 1|n, r, \theta)/[R(n, r, k) - (n + rk)] \tag{24}$$

for $k = 2, 3, \ldots, n$, where $f_K(1|n, r, \theta) = \theta r^{[n]}/(r\theta)^{[n]}$.

Relation (17) seems to be quite useful in preparing a table for $R(n, r, k)$. The values of $R(n, r, k)$ are necessary in computing the pf from (24).

The mean and variance of K are given by:

$$E(K) = \theta[(r\theta)^{[n]} - (r\theta - r)^{[n]}]/(r\theta)^{[n]}; \tag{25}$$

$$E(K(K - 1)) = (\theta)_2[(r\theta)^{[n]} - 2(r\theta - r)^{[n]} + (r\theta - 2r)^{[n]}]/(r\theta)^{[n]}; \tag{26}$$

$$\mathrm{Var}(K) = E(K(K - 1)) + E(K) - [E(K)]^2. \tag{27}$$

5. ESTIMATION OF THE PARAMETER θ OF THE PROBABILITY DISTRIBUTION OF THE PREVIOUS SECTION

Suppose we have a population of θr cells consisting of θ groups of r cells each, in which r is known but θ is unknown. Suppose n indistinguishable balls are randomly distributed in these cells and k groups are found to be occupied. Here K, the number of occupied groups, has probability function (22). We wish to estimate the underlying parameter θ based upon the observed k.

First, following the arguments of Patil [10], we shall show that a uniformly minimum variance unbiased (UMVU) estimator of θ based on the complete sufficient statistic K does not exist. Second, we shall show that, in some special case, a suitable estimator of θ is obtainable. Suppose we proceed heuristically to construct an unbiased estimator $t(K|n, r)$ of θ based on K. Then the condition of unbiasedness

$$E[t(K|n, r)] = \theta \tag{28}$$

yields

$$t(k|n, r) = [R(n, r, k) - n]/r \quad (k = 1, \ldots, n - 1) \tag{29}$$

and

$$B(n, r, n) = 0. \tag{30}$$

But, by definition, $B(n, r, n) = r^n$, and we arrive at a contradiction. Hence, there is no unbiased estimator of θ.

Here the relative bias of $t(K|n, r)$ satisfies

$$E[t(K|n, r)/\theta] - 1 = -[r^{n+1}(\theta)_{n+1}/\{(r\theta)(r\theta)^{[n]}\}]. \tag{31}$$

We observe that

$$[r^{n+1}(\theta)_{n+1}/\{(r\theta)(r\theta)^{[n]}\}] < 1,$$

thus the relative bias approaches zero for moderately large value of n. Further, in practice, the probability of the maximum outcome may be negligibly small. So the use of (29) may often be justified in a special case where the bias of the estimator is not serious, and in such a case the estimate (29) of the parameter θ is obtainable from the recurrence relation

$$\begin{aligned} &t(k|n, r) - k \\ &= [t(k|n - 1, r) - k][rt(k - 1|n - 1, r) + n - 1]/[rt(k|n - 1, r) + n - 1] \end{aligned} \tag{32}$$

where $1 < k < n$ with

$$t(1|n, r) = n + r + 1 \quad \text{and} \quad t(n|n, r) = [n(n - 1)(r + 1)/2r] + n.$$

The above relation follows from (17).

6. TWO LIMITING DISTRIBUTIONS

We now consider two limiting forms of the distribution (22) which are of much practical use.

First, if $r\theta = \phi$ is constant and $r \to 0$ in (22), then $f_K(k|n, r, \theta)$ becomes the limiting distribution

$$f_K(k|n, \phi) = |s(n, k)|\phi^k/\phi^{[n]} \quad (k = 1, \ldots, n), \tag{33}$$

which has application in genetic studies (see Johnson & Kotz, [8], p. 246) and the distribution of the number of hearers directly from a source (see Bartholomew, [4], p. 317). We observe that (33) is a special case of the power series distribution (see [8], p. 85). When $\phi = 1$, (33) reduces to

$$f_K(k|n) = |s(n, k)|/n! \quad (k = 1, \ldots, n) \tag{34}$$

which has been used by Barlow *et al.* ([3], p. 143) in connection with some problems of testing statistical hypotheses under order restrictions. Equation (34) gives the probability that a permutation of n elements picked at random has k cycles.

Second, if $r \to \infty$ in (22), we find

$$f_K(k|n, \theta) = S(n, k)(\theta)_k/\theta^n \quad (k = 1, \ldots, n). \tag{35}$$

This is known as Steven-Craig's distribution (see Patil & Joshi, [11], p. 56) and sometimes called Arfwedson's distribution (see Johnson & Kotz, [7], p. 251). It is a particular case of the factorial series distribution introduced by Berg [5]. It is also useful in the study of the ecology of plants and animals (see Lewontin & Prout, [9] and Watterson, [13]) and in some problems of sample surveys (see Des Raj & Khamis, [6]). In addition, it can be applied to finding the critical values of the empty cell test (see, e.g., Wilks, [14], pp. 433-37].

7. A PROBABILITY MODEL UNDER AN INVERSE SAMPLING SCHEME

We introduce a probability model involving associated Lah numbers under an inverse sampling scheme.

Suppose that, instead of n being fixed and k variable, random distribution of like balls, one at a time, is continued until a predetermined number k, say, of groups have been occupied. Let the required size be n. Then we have a probability model under the inverse sampling scheme having the pf

$$\begin{aligned} h_N(n|k, r, \theta) &= Pr\{N = n|k, r, \theta\} \\ &= rB(n - 1, r, k - 1)(\theta)_k/(r\theta)^{[n]}, \quad n = k, k + 1, \ldots . \end{aligned} \tag{36}$$

It is seen from (13) that

$$\sum_{n=k}^{\infty} h_N(n|k, r, \theta) = 1.$$

The pf (36) is recognized as a special case of inverse factorial series distribution (see [8], p. 88). It satisfies the following recurrence relation:

$$h_N(n|k, r, \theta) = [R(n - 2, r, k - 1)/(r\theta + n - 1)]h_N(n - 1|k, r, \theta), \tag{37}$$

where the $R(n, r, k)$ satisfy (17).

The mean and variance of N are obtained as follows:

$$E(N) = -(r\theta - 1)(\theta)_k \underset{1/r}{\Delta}[1/(\theta - 1/r)_k] \tag{38}$$

and

$$E(N(N + 1)) = (r\theta - 1)(r\theta - 2)(\theta)_k \underset{1/r}{\Delta^2}[1/(\theta - 2/r)_k], \tag{39}$$

where $\underset{1/r}{\Delta} f(\theta) = f(\theta + 1/r) - f(\theta)$.

From (38) and (39), Var(N) can be obtained easily.

Here we note that N is a complete, sufficient statistic for θ. Making use of this statistic, we now consider the problem of estimation.

Arguing as in Section 5, we can show that the UMVU estimator of θ based on N does not exist. However, if we assume $g(N)$ to be an unbiased estimator of θ, then we find that the relative bias of $g(N)$ is:

$$E[g(N)/\theta] - 1 = r^{k-1}(\theta - 1)_{k-2}/(r\theta)^{[k-1]}. \tag{40}$$

This relative bias does not depend upon n. Thus, it cannot be reduced by taking a large sample. Therefore, it is not possible to provide any usable estimate of θ.

8. A CONDITIONAL MULTIVARIATE DISTRIBUTION INVOLVING ASSOCIATED LAH NUMBERS

We now investigate the properties of a conditional multivariate distribution whose pf can be obtained readily from the associated Lah numbers.

From (5), the joint distribution of $\overline{N} = (N_1, \ldots, N_k)$ (given $N_1 + \cdots + N_k + N_{k+1} = n$) is:

$$Pr\left\{\overline{N} = \overline{n} \,\middle|\, \text{each } n_i > 0,\ i = 1, \ldots, k,\ n > \sum_{i=1}^{k} n_i,\ k \text{ and } r \text{ are positive integers}\right\}$$

$$= (n!/(k+1)!)\prod_{i=1}^{k+1}\binom{n_i + r - 1}{n_i}\Big/B(n, r, k+1), \tag{41}$$

where the mass points (the sample points) of $\overline{n}$ are defined by the set:

$$\left\{\overline{n} \,\middle|\, \text{each } n_i > 0,\ n > \sum_{i=1}^{k} n_i,\ k \text{ and } r \text{ are fixed positive integers}\right\}.$$

It represents the pf of $\overline{N}$ (the group frequencies), if $n > r(k+1)$ indistinguishable balls are put into $r(k+1)$ cells constituting $k+1$ groups of r cells each with no group empty.

To find the mean and variance of N_i, we put, for convenience,

$$A(n, r, k+1) = [r/B(n, r, k+1)]\sum_{j=1}^{n-k}\left[\binom{j+r-1}{j}B(n-j, r, k)/(n-j-1)!\right]. \tag{42}$$

Then

$$E(N_i) = n/(k+1) \tag{43}$$

and

$$\mathrm{Var}(N_i) = [n^2k/(k+1)^2 - (n!/(k+1)!)A(n, r, k+1)]. \tag{44}$$

Further,

$$\mathrm{Cov}(N_i, N_j) = -(1/k)\mathrm{Var}(N_i) \qquad (i \neq j) \tag{45}$$

and

$$\mathrm{Corr}(N_i, N_j) = -(1/k). \tag{46}$$

The marginal distribution of N_1 is:

$$Pr\left\{N_1 = n_1 \,\middle|\, \sum_{i=1}^{k+1} N_i = n, k, r\right\}$$

$$= (n)_{n_1}\binom{n_1 + r - 1}{n_1}B(n - n_1, r, k)/[(k+1)B(n, r, k+1)], \qquad n_1 = 1, \ldots, n-k. \tag{47}$$

The joint distribution of the subset $(N_1, \ldots, N_m)$ of the N_i's is:

$$Pr\left\{N_1 = n_1, \ldots, N_m = n_m \,\middle|\, \sum_{i=1}^{k+1} N_i = n, k, r\right\}$$

$$= (n)_{n_0}\prod_{i=1}^{m}\binom{n_i + r - 1}{n_i}B(n - n_0, r, k - m + 1)/[(k+1)_m B(n, r, k+1)], \tag{48}$$

where $n_0 = n_1 + \cdots + n_m$, each n_i being a positive integer.

The conditional distribution of N_j, where $N_1, \ldots, N_{j-1}$ are fixed, is:

$$Pr\left\{N_j = n_j \middle| N_1 = n_1, \ldots, N_{j-1} = n_{j-1}, \sum_{i=1}^{k+1} N_i = n,\ k,\ r \text{ being positive integers}\right\}$$

$$= (n - n_0 + n_j)!\binom{n_j + r - 1}{n_j} B(n - n_0, r, k - j + 1)/\{(n - n_0)!(k - j + 2)$$

$$\times B(n - n_0 + n_j, r, k - j + 2)\}, \tag{49}$$

where $n_0 = n_1 + \cdots + n_j$ and $n_j = 1, \ldots, n - n_0 + n_j - k + j - 1$.

It is interesting to note that the distribution of the vector $\overline{N}$ in (41) is the same as that of the joint distribution of the independent random variables $N_1, \ldots, N_{k+1}$, each following a zero truncated negative binomial distribution with arbitrary parameters θ $(0 < \theta < 1)$ and r (a positive integer), subject to the condition $N_1 + \cdots + N_{k+1} = n$.

9. AN APPLICATION OF $L_{c,r}(n, k)$

Let $n > ck$ indistinguishable balls be distributed in rk cells constituting k groups of r cells each. Then the probability that j groups of cells are occupied with each group containing at least $c + 1$ balls is given by

$$P_{c,r}(j \mid n) = (k)_j L_{c,r}(n, j)/(rk)^{[n]}, \tag{50}$$

where $(k)_j = k(k - 1) \ldots (k - j + 1)$ and

$$(rk)^{[n]} = (rk)(rk + 1) \ldots (rk + n - 1).$$

Proof: The probability that j groups $g_1, \ldots, g_j$ contain $x_1, \ldots, x_j$ balls, respectively, with $x_1 + \cdots + x_j = n$ is given by

$$\binom{x_1 + r - 1}{x_1} \ldots \binom{x_j + r - 1}{x_j} \bigg/ \binom{n + rk - 1}{n}. \tag{51}$$

Therefore, the probability that the groups $g_1, \ldots, g_j$ are occupied each containing at least $c + 1$ balls is given by

$$\sum \binom{x_1 + r - 1}{x_1} \ldots \binom{x_j + r - 1}{x_j} \bigg/ \binom{n + rk - 1}{n} \tag{52}$$

where the summation is extended over all ordered j-tuples $(x_1, \ldots, x_j)$ of integers $x_i > c$, $i = 1, \ldots, j$ with $x_1 + \cdots + x_j = n$.

Now, from (10), (52), and noting that j groups out of k can be selected in $\binom{k}{j}$ ways, we obtain (50).

10. A CONDITIONAL MULTIVARIATE DISTRIBUTION INVOLVING GENERALIZED LAH NUMBERS

From (10), the joint distribution of $\overline{N} = (N_1, \ldots, N_k)$ (given $N_1 + \cdots + N_k + N_{k+1} = n$) is

$$Pr\left\{\overline{N} = \overline{n} \middle| \text{each } n_i > c,\ i = 1, \ldots, k,\ n > \sum_{i=1}^{k} n_i,\ k,\ c, \text{ and } r \text{ are positive integers}\right\}$$

$$= (n!/(k + 1)!)\prod_{i=1}^{k+1}\binom{n_i + r - 1}{n_i} \bigg/ L_{c,r}(n, k), \tag{53}$$

where the mass points of $\overline{n}$ are given by the set

$$\left\{\overline{n}\,\middle|\,\text{each } n_i > c,\ n > \sum_{i=1}^{k} n_i,\ k,\ c, \text{ and } r \text{ are fixed positive integers}\right\}.$$

We note that (53) represents the joint distribution of $k + 1$ independent random variables $N_1, \ldots, N_{k+1}$ each following a c-truncated negative binomial distribution with arbitrary parameters θ $(0 < \theta < 1)$, r and c subject to the condition $N_1 + \cdots + N_{k+1} = n$.

Distribution (53) has properties analogous to those of distribution (41).

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RATIOS OF GENERALIZED FIBONACCI SEQUENCES

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(Submitted May 1985)

INTRODUCTION

In addition to the well-known Fibonacci sequence $F(n)$, recursively defined by

$$F(1) = 1,\ F(2) = 1,\ F(n+1) = F(n) + F(n-1),\ \text{for } n > 2,$$

is the Lucas sequence $L(n)$, similarly defined by

$$L(1) = 1,\ L(2) = 3,\ L(n+1) = L(n) + L(n-1).$$

Although the difference $L(n) - F(n)$ increases without bound, the ratio $L(n)/F(n)$ tends to a limiting value of $\sqrt{5}$. This result follows from the two representations:

$$F(n) = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n;\ L(n) = \left[\frac{1+\sqrt{5}}{2}\right]^n + \left[\frac{1-\sqrt{5}}{2}\right]^n.$$

For a given integer $m \geqslant 3$, we now consider the sequence $G_m(n)$ defined by

$$G_m(1) = 1,\ G_m(2) = m,\ G_m(n+1) = G_m(n) + G_m(n-1).$$

From this we have $G_m(n) = L(n) + (m-3)F(n-1)$ and, consequently, the ratio $G_m(n)/F(n)$ has a limiting value of $\sqrt{5} + (m-3)(\sqrt{5}-1)/2$. This relationship also holds for any integral m since the inequality $m \geqslant 3$ was not crucial to the validity of the statement. Indeed, the result is valid for all real m.

For Fibonacci-type sequences that begin with a nonzero first term other than one, say, for example, the sequence $H_{a,b}(n)$ defined by

$$H_{a,b}(1) = a,\ H_{a,b}(2) = b,\ H_{a,b}(n+1) = H_{a,b}(n) + H_{a,b}(n-1),$$

each term of which is merely a constant multiple of a G_m sequence, namely,

$$H_{a,b}(n) = aG_{b/a}(n).$$

This means that the ratio $H_{a,b}(n)/F(n)$ has a limiting value of

$$a[\sqrt{5} + (b/a - 3)(\sqrt{5} - 1)/2].$$

Finally, for real numbers a, b, c, and d with $ac \neq 0$, the ratio of $H_{a,b}(n)$ to $H_{c,d}(n)$ has a limiting value shown by

$$\frac{H_{a,b}(n)}{H_{c,d}(n)} = \frac{H_{a,b}(n)/F(n)}{H_{c,d}(n)/F(n)} \to \frac{2a\sqrt{5} + (b-3a)(\sqrt{5}-1)}{2c\sqrt{5} + (d-3c)(\sqrt{5}-1)}. \tag{1}$$

GENERALIZED SEQUENCES

Let us consider the more general case where a Fibonacci-type sequence F_k is recursively defined by the sum of the previous k terms. The first k terms are arbitrarily defined by $F_k(0) = a_1$, $F_k(1) = a_2$, ..., $F_k(k-1) = a_k$, and then

$$F_k(i) = \sum_{j=i-k}^{i-1} F_k(j),\ \text{for } i \geqslant k.$$

From the theory of recursion we know that F_k is generated by a finite k-sum of powers by

$$F_k(n) = f_1 r_1^n + f_2 r_2^n + \cdots + f_k r_k^n,$$

where f_i are constants (real or complex), and the r_i are the zeros of the polynomial

$$p(x) = x^k - x^{k-1} - \cdots - x - 1. \tag{2}$$

It is shown in [3] that the roots of p are all distinct, and all lie within the unit circle in the complex plane except one root which is real and lies between 1 and 2. For simplicity this real root will be labeled r_k, and the others will be denoted by $r_1, r_2, \ldots, r_{k-1}$. This means $|r_i| < 1$ for $i < k$, and $1 < r_k < 2$.

The graphs of p for various k help to illustrate the location of the roots, as well as the additional fact that $r_k \to 2$ as k increases [2]. It is important to realize that these roots are determined as soon as k is known, and that they have nothing to do with the initial values given to $F_k(0), F_k(1), \ldots, F_k(k-1)$.

The constants f_i can be determined from the side conditions $a_i = F_k(i-1)$, and by applying Cramer's rule we get:

$$f_i = \frac{\begin{vmatrix} 1 & 1 & & 1 & a_1 & 1 & & 1 \\ r_1 & r_2 & & r_{i-1} & a_2 & r_{i+1} & & r_k \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ r_1^{k-1} & r_2^{k-1} & & r_{i-1}^{k-1} & a_k & r_{i+1}^{k-1} & & r_k^{k-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & & 1 \\ r_1 & r_2 & & r_k \\ \vdots & \vdots & \cdots & \vdots \\ r_1^{k-1} & r_2^{k-1} & & r_k^{k-1} \end{vmatrix}}. \tag{3}$$

Since the denominator of this expression is the $k \times k$ Vandermonde determinant, its value is given by

$$\prod_{\substack{i=2 \\ i>j}}^{k} (r_i - r_j). \tag{4}$$

Suppose we have two such Fibonacci sequences of the same type, say F_k and G_k, where

$$F_k(i) = a_{i+1} \quad \text{and} \quad G_k(i) = b_{i+1} \text{ for } 0 \leqslant i \leqslant k-1.$$

Then there exist constants $f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_k$ such that

$$F_k(n) = \sum_{i=1}^{k} f_i r_i^n \quad \text{and} \quad G_k(n) = \sum_{i=1}^{k} g_i r_i^n \tag{5}$$

and the r_i are the roots to (2). The ratio $F_k(n)/G_k(n)$ must then approach f_k/g_k as n increases. The problem then becomes one to evaluate f_k and g_k which, in turn, reduces to solving $p(x) = 0$.

TWO SPECIFIC CASES

When $k = 2$, we have $F_2(n) = f_1r_1^n + f_2r_2^n$ and $G_2(n) = g_1r_1^n + g_2r_2^n$, and the ratio $F_2(n)/G_2(n) \to f_2/g_2$ where, from (3), we get

$$f_2 = \frac{a_2 - a_1r_1}{r_2 - r_1} \quad \text{and} \quad g_2 = \frac{b_2 - b_1r_1}{r_2 - r_1}. \tag{6}$$

Since r_1, r_2 are the roots to $x^2 - x - 1 = 0$, with r_2 being the root of modulus between 1 and 2, then $r_1 = (1 - \sqrt{5})/2$. Thus, the ratio f_2/g_2 reduces to

$$\frac{2a_2 - a_1(1 - \sqrt{5})}{2b_2 - b_1(1 - \sqrt{5})}, \tag{7}$$

and this agrees with our earlier result from (1).

For $k = 3$, $F_3(n) = f_1r_1^n + f_2r_2^n + f_3r_3^n$ and $G_3(n) = g_1r_1^n + g_2r_2^n + g_3r_3^n$, and then

$$f_3 = \frac{a_1r_1r_2 - a_2(r_1 + r_2) + a_3}{(r_3 - r_2)(r_3 - r_1)} \quad \text{and} \quad g_3 = \frac{b_1r_1r_2 - b_2(r_1 + r_2) + b_3}{(r_3 - r_2)(r_3 - r_1)}$$

so then

$$f_3/g_3 = \frac{a_1r_1r_2 - a_2(r_1 + r_2) + a_3}{b_1r_1r_2 - b_2(r_1 + r_2) + b_3}. \tag{8}$$

The values for r_1, r_2 are determined by using Cardano's formula:

$$r_1 = \frac{1}{6}\left[2 - \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}} + 3i\left\{\sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}\right\}\right]$$

$$r_2 = \frac{1}{6}\left[2 - \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}} - 3i\left\{\sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}\right\}\right].$$

This gives the approximate values $r_1 = -.4196433 + .6062906i$ and $r_2 = \overline{r}_1$. Consequently, the ratio f_3/g_3 is real (since r_1r_2 and $r_1 + r_2$ are real) with the approximate value

$$f_3/g_3 = \frac{.5436888a_1 + .8392866a_2 + a_3}{.5436888b_1 + .8392966b_2 + b_3}. \tag{9}$$

Evaluation of f_k/g_k for $k > 3$ ultimately rests on effectively computing the complex roots to $p(x) = 0$.

APPROXIMATING COMPLEX ROOTS

Among the many iterative numerical methods available for locating roots to polynomial equations, probably the best known is Newton's method. Typically, this method is employed to find real roots, but it can be generalized to the complex plane [5]. To this end, we begin with a complex seed z_0, and consider the sequence $\{z_n\}$ of iterates, $z_{n+1} = z_n - p(z_n)/p'(z_n)$. It appears, from data gathered, that every complex seed generates a sequence that eventually converges to a root of $p(z)$ with, of course, varying rates of convergence. But an interesting question, and one that was posed as far back as 1879 by Arthur Cayley [4], is to determine the regions of the plane whose members generate sequences that converge to identical roots of $p(z)$. The readers may wish to determine the corresponding regions for a specific polynomial. The author gathered data on $z^3 - z^2 - z - 1 = 0$ and approximated the partitions of

$\{(x, y) : |x| \leqslant 1, |y| \leqslant 1\}$.

The shaded regions in Figure 1 consist of those "seeds" that generate sequences that converge to the root r_2 with $r_2 = -.4196 - .6063i$. Obviously there is no reason to suspect that the points in the plane that generate sequences that converge to the same root form a connected set. Likewise, statements concerning symmetry of regions are not obvious to formulate. Instead, there is some considerable disconnectedness to the regions, especially for this one in the near vicinity of the x-axis, where one can find seeds that generate sequences that converge to each of the three roots to the polynomial.

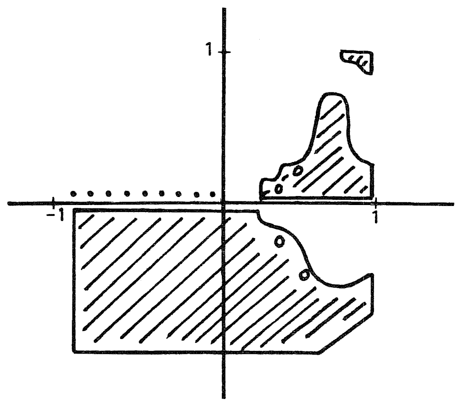

FIGURE 1. A region whose members generate the same polynomial root

It is of interest to point out that the associated notion of Julia sets (a concept developed by Julia and Fatou at the turn of the century in regard to the iteration of rational functions in the plane) is discussed in [4] and accompanied by some excellent color computer graphics.

CONSECUTIVE FIBONACCI NUMBERS

Suppose we take a more careful look at the sequence of ratios of consecutive Fibonacci numbers. For the standard Fibonacci sequence $F(n)$, the sequence of ratios $F(n)/F(n-1)$ alternates monotonically. Thus, setting

$$r(n) = F(n)/F(n-1),$$

we have

$$r(2i) < r(2i+2),\ r(2i+1) > r(2i+3),\ r(2i) < r(2i+1), \text{ for all } i.$$

But what happens if $F(n)$ is replaced by the more general Fibonacci sequence $F_k(n)$, where

$$F_k(i) = a_i, \qquad \text{for } 1 \leqslant i \leqslant k,$$

$$F_k(i) = \sum_{j=i-k}^{i-1} F_k(j), \text{ for } i \geqslant k+1.$$

In this general setting the sequence of ratios $r_k(n) = F_k(n)/F_k(n-1)$ does not alternate monotonically, nor does it alternate in k-tuples. Patterns seem to be haphazard at best. But one can make a statement about the maximum number of ratios that form a consecutive monotone string. More specifically, this means (monotone increasing is sufficient)

$$\max\{j : \exists i,\ i > k \text{ and } r_k(i+1) < r_k(i+2) < \cdots < r_k(i+j)\} \leqslant k. \tag{10}$$

This inequality will be established if we show that whenever

$$\frac{F_k(i+1)}{F_k(i)} < \frac{F_k(i+2)}{F_k(i+1)} < \cdots < \frac{F_k(i+k)}{F_k(i+k-1)} \tag{11}$$

it follows that

$$\frac{F_k(i+k)}{F_k(i+k-1)} > \frac{F_k(i+k+1)}{F_k(i+k)}. \tag{12}$$

Setting $f_j = F_k(i+j)$ to simplify notation, it follows that

$$f_1 f_{k-1} < f_0 f_k,\ f_2 f_{k-1} < f_1 f_k,\ \ldots,\ f_{k-1} f_{k-1} < f_{k-2} f_k,$$

so summing gives

$$\sum_{i=1}^{k-1} f_i f_{k-1} < \sum_{i=0}^{k-2} f_i f_k \tag{13}$$

and then adding $f_{k-1} f_k$ to both summations yields

$$f_{k+1} f_{k-1} = \sum_{i=1}^{k} f_i f_{k-1} < \sum_{i=0}^{k-1} f_i f_k = f_k f_k, \tag{14}$$

which establishes the desired result.

So for each given choice of k, each string of ratios of consecutive k-generalized Fibonacci numbers $F_k(n)/F_k(n-1)$ will contain a maximum of k consecutive monotone terms. Consider the following example with $k = 3$.

TABLE 1. Generalized Fibonacci Numbers and Their Ratios

n	$F_3(n)$	$F_3(n)/F_3(n-1)$
1	1	
2	1	
3	2	
4	4	2.00
5	7	1.75
6	13	1.85714
7	24	1.84615
8	44	1.83333
9	81	1.84091

Here we have the three consecutive monotone terms,

$$\frac{F_3(6)}{F_3(5)} > \frac{F_3(7)}{F_3(6)} > \frac{F_3(8)}{F_3(7)} \tag{15}$$

and, of course, the next ratio reverses the monotonicity,

$$\frac{F_3(8)}{F_3(7)} < \frac{F_3(9)}{F_3(8)}. \tag{16}$$

Data seem to indicate that this result is best, in the sense that every Fibonacci sequence F_k contains a string of exactly k consecutive monotone ratios. What we can prove here is the existence of a sequence, for each k, which satisfies this conjecture. Thus, for $k \geqslant 2$, we define the sequence F_k by $F_k(n) = 1$ for $n < k$ and $F_k(k) = k$. Then

$$F_k(k+1) = 2k - 1,\ F_k(k+2) = 4k - 3,$$

$$F_k(k+3) = 8k - 7,\ F_k(k+4) = 16k - 15,$$

and the pattern continues up to

$$F_k(2k-1) = 2^{k-1}k - (2^{k-1} - 1) \quad \text{and} \quad F_k(2k) = 2^k k - (2^k - 1).$$

The pattern breaks with the next term for

$$F_k(2k+1) = (2^{k+1} - 2)k - (2^{k+1} - 2 - k).$$

The ratios $F_k(k+i)/F_k(k+i-1)$ form an increasing sequence for $i = 1, 2, \ldots, k$ because the inequality

$$\frac{2^n k - (2^n - 1)}{2^{n-1}k - (2^{n-1} - 1)} < \frac{2^{n+1}k - (2^{n+1} - 1)}{2^n k - (2^n - 1)}$$

holds for all $n \geqslant 1$. Furthermore, the string of increasing ratios is then reversed with the next ratio because

$$\frac{F_k(2k)}{F_k(2k-1)} > \frac{F_k(2k+1)}{F_k(2k)}. \tag{18}$$

It is interesting to look at the similar question of finding a Fibonacci sequence with k consecutive decreasing ratios. Unlike the previous example, such a solution cannot be found by defining the initial k terms in the sequence from among the elements $1, 2, \ldots, k$. We need to choose from a larger set of positive integers. Thus, for $k \geqslant 2$, we define the sequence F_k by

$$F_k(1) = 1,\ F_k(2) = 2,\ F_k(3) = 4,\ \ldots,\ F_k(k-1) = 2^{k-2},\ \text{and}\ F_k(k) = 1.$$

For values of i with $1 \leqslant i \leqslant k$, the term $F_k(k+i)$ has the value

$$F_k(k+i) = 2^{k+(i-2)} - (i-1)2^{i-2} \tag{19}$$

and, consequently,

$$\frac{F_k(k+i)}{F_k(k+i-1)} > \frac{F_k(k+i+1)}{F_k(k+i)},\ \text{for } i = 1, 2, \ldots, k-1. \tag{20}$$

Many other questions remain for the interested reader to investigate. Can one predict when these monotone strings of ratios of length k will occur, or how often they will occur? Are there strings of length i for each i less than k for each given sequence? Are there as many increasing strings as decreasing strings?

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BI-UNITARY AMICABLE AND MULTIPERFECT NUMBERS

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(Submitted May 1985)

1. INTRODUCTION

In what follows, lower-case letters will be used to denote natural numbers, with p and q always representing primes. As usual, (c, d) will symbolize the greatest common divisor of c and d. If $cd = n$ and $(c, d) = 1$, then d is said to be a unitary divisor of n. If $(c, d)^*$ denotes the greatest common unitary divisor of c and d, then d is said to be a bi-unitary divisor of n if $cd = n$ and $(c, d)^* = 1$. The notion of a bi-unitary divisor was first introduced by Subbarao & Suryanarayana in 1971 (see [6]).

We shall symbolize by $\sigma(n)$, $\sigma^*(n)$, and $\sigma^{**}(n)$, respectively, the sums of the (positive) divisors, unitary divisors, and bi-unitary divisors of n. It is well known that $\sigma(p^a) = (p^{a+1} - 1)/(p - 1)$ and $\sigma^*(p^a) = (p^a + 1)$ *and* that both σ and σ^* are multiplicative functions. It is not difficult to verify that $\sigma^{**}(p^a) = \sigma(p^a)$ if a is odd and $\sigma^{**}(p^a) = \sigma(p^a) - p^{a/2}$ if a is even *and* that σ^{**} is multiplicative. It follows that $\sigma^{**}(n) = \sigma(n)$ if every exponent in the prime-power decomposition of n is odd and that $\sigma^{**}(n) = \sigma^*(n)$ if n is cube-free. It is also immediate that $\sigma^{**}(n)$ is even unless $n = 2^a$ or $n = 1$.

2. BI-UNITARY MULTIPERFECT NUMBERS

A number n is said to be perfect if $\sigma(n) = 2n$ and to be multiperfect if $\sigma(n) = kn$, where $k \geqslant 3$. Perfect and multiperfect numbers have been studied extensively. Subbarao & Warren [7] have defined n to be a unitary perfect number if $\sigma^*(n) = 2n$, and Wall [11] has defined n to be a bi-unitary perfect number if $\sigma^{**}(n) = 2n$. Five unitary perfect numbers have been found to date (see [10]), while Wall [11] has proved that 6, 60, and 90 are the *only* bi-unitary perfect numbers.

If $\sigma^*(n) = kn$, where $k \geqslant 3$, n is said to be a unitary multiperfect number. The properties of such numbers have been studied by Harris & Subbaro [4] and by Hagis [3]. It is known that, if n is a unitary multiperfect number, then $n > 10^{102}$ and n has at least 46 distinct prime factors (including 2). No unitary multiperfect numbers have, as yet, been found.

We shall state that n is a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$, where $k \geqslant 3$. It is easy to show that every such number is even.

Theorem 1: There are no odd bi-unitary multiperfect numbers.

Proof: Suppose that $\sigma^{**}(n) = kn$, where $k \geqslant 3$, and

$$n = p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}, \text{ with } 3 \leqslant p_1 < p_2 < \cdots < p_s.$$

Suppose, also, that $k = 2^c M$, where $2 \nmid M$ and $c \geqslant 0$. Since

$$\sigma^{**}(n) = \prod_{i=1}^{s} \sigma^{**}(p_i^{a_i})$$

and since $2 \mid \sigma^{**}(p_i^{a_i})$ for $i = 1, 2, \ldots, s$, we see that $s \leqslant c$. Also,

$$2^c \leqslant 2^c M = k = \sigma^{**}(n)/n \leqslant \sigma(n)/n$$

$$= \prod_{i=1}^{s} \sigma(p_i^{a_i})/p_i^{a_i} < \prod_{i=1}^{s} p_i/(p_i - 1) < 2^s \leqslant 2^c.$$

This contradiction completes the proof.

Using the CDC CYBER 750 at the Temple University Computing Center a search was made for all bi-unitary multiperfect numbers less than 10^7. The search required about 1.5 hours of computer time, and thirteen numbers were found, nine with $k = 3$ and four with $k = 4$. They, along with the three bi-unitary perfect numbers, are listed in Table 1.

TABLE 1

The Bi-Unitary Perfect and Multiperfect Numbers Less than 10**7

1.	6 = 2.3	k = 2
2.	60 = 2**2.3.5	k = 2
3.	90 = 2.3**2.5	k = 2
4.	120 = 2**3.3.5	k = 3
5.	672 = 2**5.3.7	k = 3
6.	2160 = 2**4.3**3.5	k = 3
7.	10080 = 2**5.3**2.5.7	k = 3
8.	22848 = 2**6.3.7.17	k = 3
9.	30240 = 2**5.3**3.5.7	k = 4
10.	342720 = 2**6.3**2.5.7.17	k = 3
11.	523776 = 2**9.3.11.31	k = 3
12.	1028160 = 2**6.3**3.5.7.17	k = 4
13.	1528800 = 2**5.3.5**2.7**2.13	k = 3
14.	6168960 = 2**7.3**4.5.7.17	k = 4
15.	7856640 = 2**9.3**2.5.11.31	k = 3
16.	7983360 = 2**8.3**4.5.7.11	k = 4

3. BI-UNITARY AMICABLE NUMBERS

m and n are said to be amicable numbers if $\sigma(m) = \sigma(n) = m + n$. A history of these numbers may be found in [5]. If $\sigma^*(m) = \sigma^*(n) = m + n$, then m and n are said to be unitary amicable numbers (see [2]). Similarly, we shall say that m and n are bi-unitary amicable numbers if $\sigma^{**}(m) = \sigma^{**}(n) = m + n$.

Theorem 2: If $(m; n)$ is a bi-unitary amicable pair, then m and n have the same parity.

Proof: Assume that $m + n$ is odd. Then $\sigma^{**}(m)$ is odd, and it follows that $m = 2^a$. Similarly, $n = 2^a$, and we have a contradiction.

Theorem 3: Suppose that $(m; n)$ is a bi-unitary amicable pair *and* that $m = 2^a M$ and $n = 2^b N$ where $M \equiv N \equiv 1 \pmod 2$ and $a < b$. If $\omega(M) = s$ and $\omega(M) = t$ [where $\omega(L)$ denotes the number of distinct prime factors of L], then $s \leqslant a$ and $t \leqslant a$.

Proof: If $p^c \| M$, then $2 | \sigma^{**}(p^c)$, and we see that $2^s | \sigma^{**}(m)$. But,

$$\sigma^{**}(m) = m + n = 2^a(M + 2^{b-a}N) = 2^a K \text{ where } K \text{ is odd,}$$

and it follows that $s \leqslant a$. Similarly, $t \leqslant a$.

Corollary 3.1: If $(2M; 2^b N)$, where $b > 1$ *and* M and N are odd, is a bi-unitary amicable pair, then $M = p^c$ and $N = q^d$.

Theorem 4.1: Suppose that $(m; n)$ is a bi-unitary amicable pair such that $m = aM$ and $n = aN$ where $(a, M) = (a, N) = 1$. If b is a natural number such that $\sigma^{**}(b)/b = \sigma^{**}(a)/a$ *and* $(b, M) = (b, N) = 1$, then $(bM; bN)$ is a bi-unitary amicable pair.

Proof: $\sigma^{**}(bM) = \sigma^{**}(b)\sigma^{**}(M) = a^{-1}b\sigma^{**}(a)\sigma^{**}(M) = a^{-1}b\sigma^{**}(aM) = a^{-1}b(aM + aN) = bM + bN$. Similarly, $\sigma^{**}(bN) = bM + bN$.

The proofs of the next two theorems are similar to that of Theorem 4.1 and are, therefore, omitted.

Theorem 4.2: Suppose that $(m; n)$ is a unitary amicable pair such that $m = aM$ and $n = aN$ where $(a, M) = (a, N) = 1$ *and* where M and N are cube-free. If

$$\sigma^{**}(b)/b = \sigma^{*}(a)/a \text{ and } (b, M) = (b, N) = 1,$$

then $(bM; bN)$ is a bi-unitary amicable pair.

Theorem 4.3: Suppose that $(m; n)$ is an amicable pair such that $m = aM$ and $n = aN$ where $(a, M) = (a, N) = 1$ *and* where every exponent in the prime-power decomposition of M and N is odd. If

$$\sigma^{**}(b)/b = \sigma(a)/a \text{ and } (b, M) = (b, N) = 1,$$

then $(bM; bN)$ is a bi-unitary amicable pair.

A computer search among distinct natural numbers a and b such that $2 \leqslant a, b \leqslant 10^4$ yielded 667 cases where $\sigma^{**}(b)/b = \sigma^{**}(a)/a$, 1325 cases where $\sigma^{**}(b)/b = \sigma^{*}(a)/a$, and 673 cases where $\sigma^{**}(b)/b = \sigma(a)/a$.

Example 1: Since $(8 \cdot 17 \cdot 41 \cdot 179; 8 \cdot 23 \cdot 5669)$ is a bi-unitary amicable pair, and since

$$\sigma^{**}(144)/144 = \sigma^{**}(8)/8,$$

it follows from Theorem 4.1 that $(144 \cdot 17 \cdot 41 \cdot 179; 144 \cdot 23 \cdot 5669)$ is also a bi-unitary amicable pair.

Example 2: Since $(135 \cdot 2 \cdot 19 \cdot 47; 135 \cdot 2 \cdot 29 \cdot 31)$ is a unitary amicable pair, and since

$$\sigma^{**}(2925)/2925 = \sigma^{*}(135)/135,$$

it follows from Theorem 4.2 that $(2925 \cdot 2 \cdot 19 \cdot 47; 2925 \cdot 2 \cdot 29 \cdot 31)$ is a bi-unitary amicable pair.

Example 3: Since (47 • 7 • 19 • 2663; 45 • 11 • 73 • 479) is an amicable pair, and since

$$\sigma^{**}(450)/450 = \sigma(45)/45,$$

it follows from Theorem 4.3 that (450 • 7 • 19 • 2663; 450 • 11 • 73 • 479) is a bi-unitary amicable pair.

A search was made for all bi-unitary amicable pairs $(m; n)$ such that $m < n$ and $m \leqslant 10^6$. The search required about five minutes on the CDC CYBER 750 and sixty pairs were found. These are listed in Table 2.

TABLE 2

The Bi-Unitary Amicable Pairs with Smallest Member Less than 10**6

1.	114 = 2.3.19	126 = 2.3**2.7
2.	594 = 2.3**3.11	846 = 2.3**2.47
3.	1140 = 2**2.3.5.19	1260 = 2**2.3**2.5.7
4.	3608 = 2**3.11.41	3952 = 2**4.13.19
5.	4698 = 2.3**4.29	5382 = 2.3**2.13.23
6.	5940 = 2**2.3**3.5.11	8460 = 2**2.3**2.5.47
7.	6232 = 2**3.19.41	6368 = 2**5.199
8.	7704 = 2**3.3**2.107	8496 = 2**4.3**2.59
9.	9520 = 2**4.5.7.17	13808 = 2**4.863
10.	10744 = 2**3.17.79	10856 = 2**3.23.59
11.	12285 = 3**3.5.7.13	14595 = 3.5.7.139
12.	13500 = 2**2.3**3.5**3	17700 = 2**2.3.5**2.59
13.	41360 = 2**4.5.11.47	51952 = 2**4.17.191
14.	44772 = 2**2.3.7.13.41	49308 = 2**2.3.7.587
15.	46980 = 2**2.3**4.5.29	53820 = 2**2.3**2.5.13.23
16.	60858 = 2.3**3.7**2.23	83142 = 2.3**2.31.149
17.	62100 = 2**2.3**3.5**2.23	62700 = 2**2.3.5**2.11.19
18.	67095 = 3**3.5.7.71	71145 = 3**3.5.17.31
19.	67158 = 2.3**2.7.13.41	73962 = 2.3**2.7.587
20.	73360 = 2**4.5.7.131	97712 = 2**4.31.197
21.	79650 = 2.3**3.5**2.59	107550 = 2.3**2.5**2.239
22.	79750 = 2.5**3.11.29	88730 = 2.5.19.467
23.	105976 = 2**3.13.1019	108224 = 2**6.19,89
24.	118500 = 2**2.3.5**3.79	131100 = 2**2.3.5**2.19.23
25.	141664 = 2**5.19.233	153176 = 2**3.41.467
26.	142310 = 2.5.7.19.107	168730 = 2.5.47.359
27.	177750 = 2.3**2.5**3.79	196650 = 2.3**2.5**2.19.23

TABLE 2—continued

28. 185368 = 2**3.17.29.47	203432 = 2**3.59.431
29. 193392 = 2**4.3**2.17.79	195408 = 2**4.3**2.23.59
30. 217840 = 2**4.5.7.389	287600 = 2**4.5**2.719
31. 241024 = 2**7.7.269	309776 = 2**4.19.1019
32. 298188 = 2**2.3**3.11.251	306612 = 2**2.3**3.17.167
33. 308220 = 2**2.3.5.11.467	365700 = 2**2.3.5**2.23.53
34. 308992 = 2**8.17.71	332528 = 2**4.7.2969
35. 356408 = 2**3.13.23.149	399592 = 2**3.199.251
36. 399200 = 2**5.5**2.499	419800 = 2**3.5**2.2099
37. 415264 = 2**5.19.683	446576 = 2**4.13.19.113
38. 415944 = 2**3.3**2.53.109	475056 = 2**4.3**2.3299
39. 462330 = 2.3**2.5.11.467	548550 = 2.3**2.5**2.23.53
40. 545238 = 2.3**3.23.439	721962 = 2.3**2.19.2111
41. 600392 = 2**3.13.23.251	669688 = 2**3.97.863
42. 608580 = 2**2.3**3.5.7**2.23	831420 = 2**2.3**2.5.31.149
43. 609928 = 2**3.11.29.239	686072 = 2**3.191.449
44. 624184 = 2**3.11.41.173	691256 = 2**3.71.1217
45. 627440 = 2**4.5.11.23.31	865552 = 2**4.47.1151
46. 635624 = 2**3.11.31.233	712216 = 2**3.127.701
47. 643336 = 2**3.29.47.59	652664 = 2**3.17.4799
48. 669900 = 2**2.3.5**2.7.11.29	827700 = 2**2.3.5**2.31.89
49. 671580 = 2**2.3**2.5.7.13.41	739620 = 2**2.3**2.5.7.587
50. 699400 = 2**3.5**2.13.269	774800 = 2**4.5**2.13.149
51. 726104 = 2**3.17.19.281	796696 = 2**3.53.1879
52. 785148 = 2**2.3.7.13.719	827652 = 2**2.3.7.59.167
53. 796500 = 2**2.3**3.5**3.59	1075500 = 2**2.3**2.5**3.239
54. 815100 = 2**2.3.5**2.11.13.19	932100 = 2**2.3.5**2.13.239
55. 818432 = 2**8.23.139	844768 = 2**5.26399
56. 839296 = 2**7.79.83	874304 = 2**6.19.719
57. 898216 = 2**3.11.59.173	980984 = 2**3.47.2609
58. 930560 = 2**8.5.727	1231600 = 2**4.5**2.3079
59. 947835 = 3**3.5.7.17.59	1125765 = 3**3.5.31.269
60. 998104 = 2**3.17.41.179	1043096 = 2**3.23.5669

4. BI-UNITARY ALIQUOT SEQUENCES

The function s^{**} is defined by $s^{**}(n) = \sigma^{**}(n) - n$, the sum of the bi-unitary *aliquot* divisors of n. $s^{**}(1) = 0$ and we define $s^{**}(0) = 0$. A t-tuple of distinct natural numbers $(n_0;\ n_1;\ \ldots;\ n_{t-1})$ with $n_i = s^{**}(n_{i-1})$ for $i = 1, 2, \ldots, t-1$ and $s^{**}(n_{t-1}) = n_0$ is called a *bi-unitary t-cycle*. A bi-unitary 1-cycle is a bi-unitary perfect number; a bi-unitary 2-cycle is a bi-unitary amicable pair. All of the bi-unitary t-cycles with $t > 2$ and smallest member less than 10^5 are listed in Table 3.

TABLE 3

The Bi-Unitary t-Cycles with $t > 2$ and First Member Less than 10**5

t = 4

(162;174;186;198), (1026;1374;1386;1494), (1620;1740;1860;1980),

(10098;15822;19458;15102), (10260;13740;13860;14940),

(41800;51800;66760;83540), (51282;58158;62802;76878)

t = 6

(12420;16380;17220;23100;26820;18180)

t = 13

(6534;8106;10518;10530;17694;11826;13038;14178;16062;16074;12726;

11754;7866)

It is not difficult to modify Theorems 4.1, 4.2, 4.3 so that one can obtain "new" bi-unitary t-cycles from known t-cycles (see [1]), unitary t-cycles (see [8] and [9]), and bi-unitary t-cycles. For example, since

$$\sigma^{**}(20)/20 = \sigma^{**}(2)/2,$$

it follows from Table 3 that

(100980; 158220; 194580; 151020) and (512820; 581580; 628020; 768780)

are bi-unitary 4-cycles.

The *bi-unitary aliquot sequence* $\{n_i\}$ with leader n is defined by

$$n_0 = n,\ n_1 = s^{**}(n_0),\ n_2 = s^{**}(n_1),\ \ldots,\ n_i = s^{**}(n_{i-1}),\ \ldots .$$

Such a sequence is said to be *terminating* if $n_k = 1$ for some index k (so that $n_i = 0$ for $i > k$). This will occur if $n_{k-1} = p$ or p^2. A bi-unitary aliquot sequence is said to be *periodic* if there is an index k such that $(n_k;\ n_{k+1};\ \ldots;\ n_{k+t-1})$ is a bi-unitary t-cycle. A bi-unitary aliquot sequence which is neither terminating nor periodic is (obviously) *unbounded*. Whether or not unbounded bi-unitary aliquot sequences exist is an open question. I would conjecture that such sequences *do* exist.

An investigation was made of all bi-unitary aliquot sequences with leader $n \leqslant 10^5$. About 2.5 hours of computer time was required. 69045 sequences were found to be terminating; 15560 were periodic (6477 ended in 1-cycles, 5556 in

2-cycles and 3527 in t-cycles with $t > 2$); and in 15395 cases an $n_k > 10^{12}$ was encountered and (for practical reasons) the sequence was terminated with its behavior undetermined. The "first" sequence with unknown behavior has leader $n_0 = 2160$. $n_{306} = 1{,}301{,}270{,}618{,}226$ is the first term of this sequence which exceeds 10^{12}.

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ON SOME MIXTURES OF DISTRIBUTIONS OF ORDER k

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(Submitted May 1985)

1. INTRODUCTION

Problems associated with the frequency of occurrence of runs of like elements in a series of Bernoulli trials have recently attracted quite a lot of attention. The reasons may possibly be traced not only to the theoretical interest they present as generalizing the usual binomial set up, but also to the practical value that any theoretical results in this direction would have with regard to statistical hypothesis testing. Feller [3] considered a series of Bernoulli trials and concentrated on the relationship between the probability distributions of the number of runs of k successes in n trials ($N_{k,n}$) and the number of trials needed to get r runs of k successes ($T_{k,r}$). He showed that

$$P(N_{k,n} \geqslant r) = P(T_{k,r} \leqslant n),\ r = 0,\ 1,\ \ldots,\ \left[\frac{n}{k}\right]$$

and examined the asymptotic behavior of the distributions of $N_{k,n}$ and $T_{k,r}$. Fréchet [4] led the way in considering the problem of deriving the exact distribution of $N_{k,n}$ and $T_{k,1}$ using his theory on the probability of the conjunction of events. More recently, Shane [21] and Turner [23] obtained expressions for the probability distribution of $T_{k,1}$ using the polynacci polynomials of order k and the entries of the Pascal triangle, respectively. Philippou & Muwafi [19] provided an alternative formula for this probability distribution in terms of the multinomial coefficients. Also, Uppuluri & Patil [24] gave an explicit expression in terms of weighted binomial coefficients that was implicit in the work of Philippou *et al.* [16]. Philippou *et al.* [17] obtained the exact distribution of $T_{k,r}$ ($r \geqslant 1$) by pointing out that $T_{k,r}$ can be represented by the sum of r independent and identically distributed random variables whose distribution coincides with that of $T_{k,1}$ (see also Philippou [15]). The exact distributions of $T_{k,r}$ and $T_{k,1}$ are called the "negative binomial distribution of order k" and "geometric distribution of order k," respectively. Hirano [6] and Philippou & Makri [18] employed the combinatorial argument of Philippou & Muwafi [19] to derive the exact distribution of $N_{k,n}$ which they named "the binomial distribution of order k." Certain limiting cases and/or mixtures of the above distributions have also been examined. Philippou *et al.* [17] showed that the distribution of $T_{k,r} - kr$ as $r \to +\infty$ reduces to a certain form of generalized Poisson distribution examined in further detail by Philippou [14], who

A preliminary version of this paper was presented at the First International Conference on Fibonacci Numbers and Their Applications, Patras, Greece, August 1984.

names it "the Poisson distribution of order k" (as being the limit of an "order k" distribution). In addition, Philippou (in [14]) discussed a gamma compound (mixed) Poisson distribution of order k. Aki *et al.* [1] derived a logarithmic distribution of order k as the limiting distribution of the random variable

$$T_{k,r} \mid (T_{k,r} > kr) \text{ as } r \to 0$$

(see also the work of Hirano *et al.* [7] who gave figures of distributions of order k). Finally, Panaretos & Xekalaki [13] defined and studied some other distributions of order k. These are the hypergeometric and the negative hypergeometric distributions of order k, a limiting case of the zero-truncated compound Poisson distribution of order k (the logarithmic series distribution of order k) as well as the Polya, the inverse Polya, and the generalized Waring distributions of order k.

As is well known, the number of applications of the above-mentioned distributions when $k = 1$ (ordinary binomial, geometric, or negative binomial distributions) is vast. However, applying these distributions presupposes a constant probability of success p which is a requirement that can hardly hold in practice. So, in many instances, combinations of different binomial, geometric, or negative binomial distributions have been considered. That is, p is allowed to vary from trial to trial according to some probability law thus giving rise to compound (mixed) forms of these distributions. The particular case of a beta distributed p gives rise to distributions belonging to the class of inverse factorial series distributions that have played an important role in the medical and biological fields. Two such distributions are the beta-compound geometric, also known as the Yule distribution (see [32]), and the beta-compound negative binomial distribution, also known as the generalized Waring distribution (see Xekalaki [25]). Their applications, however, are not confined to these fields. They have also been applied to fields such as accident, income, or geographical analysis, linguistics, bibliographic research, and reliability. A selection of their contribution to these fields can be found in Dacey [2], Haight [5], Irwin [8, 9, 10], Kendall [11], Krishnaji [12], Schubert & Glänzel [20], Simon [22], Xekalaki [26-30], and Xekalaki & Panaretos [31].

In this paper we consider generalizations of beta-geometric and beta-negative binomial distribution. These are obtained in Sections 2 and 3 as mixtures of the Poisson distribution of order k, in a manner similar to the derivation of the geometric and the negative binomial distributions as mixtures of the ordinary Poisson distribution. Expressions for their probabilities and the first two moments are given. In Section 4 it is shown that the Poisson and the gamma-compound Poisson distributions of order k are limiting cases of the generalized beta-negative binomial so that the theory of those distributions that are of negative binomial form is a particular case of that shown in Section 4.

Before providing the main results, let us introduce some notation and terminology.

A nonnegative, integer-valued random variable (r.v.) X is said to have the beta-geometric (Yule) distribution with parameter c if its probability function (p.f.) is given by

$$P(X = x) = \frac{cx!}{(c+1)_{(x+1)}}, \; c > 0, \; x = 0, 1, 2, \ldots, \tag{1.1}$$

where

$$a_{(\beta)} = \Gamma(a + \beta)/\Gamma(a), \; a > 0, \; \beta \in R.$$

A nonnegative, integer-valued r.v. X is said to have the beta-negative binomial (generalized Waring) distribution with parameters a, b, c if its p.f. is

$$P(X = x) = \frac{c_{(b)}}{(a + c)_{(b)}} \frac{a_{(x)}b_{(x)}}{(a + b + c)_{(x)}} \frac{1}{x!} \tag{1.2}$$

$$a, b, c > 0, \; x = 0, 1, 2, \ldots .$$

Their probability generating functions (p.g.f.) are of the form

$${}_2F_1(a, \beta; \gamma; s)/{}_2F_1(a, \beta; \gamma; 1),$$

where ${}_2F_1$ is the Gauss hypergeometric function defined by the series

$${}_2F_1(a, \beta; \gamma; s) = \sum_{r=0}^{\infty} \frac{a_{(r)}\beta_{(r)}}{\gamma_{(r)}} \frac{s^r}{r!}, \tag{1.3}$$

which is convergent for all $|s| \leqslant 1$ provided that $\gamma > a + \beta$.

In the sequel, we will refer to the distribution with p.g.f.

$$H_{(s)} = p^r(1 - qs)^{-r}, \; r \geqslant 1, \tag{1.4}$$

as the negative binomial distribution. For $r = 1$ the resulting distribution will be termed "the geometric distribution."

A continuous r.v. X will be said to have the beta distribution of the first kind with parameters a, b [beta I (a, b)] if its probability density function (p.d.f.) is given by

$$f(x) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1 - x)^{b-1}, \; a > 0, \; b > 0, \; 0 < x < 1. \tag{1.5}$$

Finally, a continuous r.v. X will be said to have the beta distribution of the second kind with parameters a, b [beta II (a, b)] if its p.d.f. is given by

$$h(x) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1 + x)^{-(a+b)}, \; a, b > 0, \; x > 0. \tag{1.6}$$

2. THE BETA-GEOMETRIC DISTRIBUTION OF ORDER k

As implied by its name, the beta-geometric (Yule) distribution defined by (1.1) is obtained as a mixture on p of the geometric distribution when p is a beta r.v. In fact, that was the theoretical model on which Yule [32] derived this distribution. In particular, if $\hat{\theta}$ denotes the mixing with respect to a parameter θ and ~ denotes equivalence, then

$$\text{beta-geometric } (c) \sim \text{geometric } (p) \underset{\hat{p}}{} \text{ beta I } (c, 1).$$

Since the geometric distribution arises as an exponential mixture of the Poisson distribution, this model is equivalent to

$$\text{beta-geometric } (c) \sim \text{Poisson } (\lambda) \underset{\hat{\lambda}}{} \text{ exponential } (1/b) \underset{\hat{b}}{} \text{ beta II } (1, c).$$

The structure of the latter model reveals the possibility of extending the beta-geometric distribution by replacing the Poisson distribution by a generalized Poisson distribution.

Consider a r.v. X which, conditional on some other r.v. λ $(\lambda > 0)$, has a generalized Poisson distribution. Then its p.g.f. is of the form

$$G_{X|\lambda}(s) = \exp\{\lambda(g(s) - 1)\}$$

where $g(s)$ is a valid p.g.f., or, equivalently (see Feller [3], p. 291) of the form

$$G_{X|\lambda}(s) = \exp\left\{\sum_{i=1}^{k} \lambda_i(s^i - 1)\right\}, \tag{2.1}$$

$$\lambda_i = \lambda g^{(i)}(0)/i!, \; k \in \mathrm{I}^+ \cup \{+\infty\}, \; \sum_{i=1}^{k} \lambda_i < +\infty.$$

Assume that $k < +\infty$ and that $\lambda_i = \lambda_j$, $i \neq j$, $i, j = 1, 2, \ldots, k$, i.e., that $g(s)$ is the p.g.f. of the discrete uniform distribution on $\{1, 2, \ldots, k\}$. Then $G_{X|\lambda}(s)$ is the p.g.f. of the Poisson distribution with parameter λ/k generalized by the uniform distribution on $\{1, 2, \ldots, k\}$, i.e.,

$$G_{X|\lambda}(x) = \exp\left\{\frac{\lambda}{k}\sum_{i=1}^{k}(s^i - 1)\right\}. \tag{2.2}$$

The probability distribution defined by (2.2) is known in the literature as the Poisson distribution of order k (Philippou *et al.* [17]). Thus, we have shown that the Poisson distribution of order k with parameter λ can be viewed as the distribution of $X_1 + X_2 + \cdots + X_N$, where N is a Poisson (λk) r.v. and X_1, X_2, ... are independent r.v.'s that are distributed on $\{1, 2, \ldots, k\}$ uniformly and independently of N.

Suppose now that λ has an exponential distribution whose parameter is itself a r.v. having a beta II $(1, c)$ distribution, i.e., the p.d.f. of λ is of the form

$$f(\lambda) = \int_0^{+\infty}\frac{c}{m}e^{-(1/m)\lambda}(1 + m)^{-(c+1)}dm.$$

Then the unconditional distribution of X has p.g.f.

$$\begin{aligned}G_X(s) &= c\int_0^{+\infty}\int_0^{+\infty}m^{-1}(1 + m)^{-(c+1)}\exp\left\{-\lambda\left(\frac{1}{m} + k - \sum_{i=1}^{k}s^i\right)\right\}dm\,d\lambda\\ &= c\int_0^{+\infty}(1 + m)^{-(c+1)}\left(1 + m\left(k - \sum_{i=1}^{k}s^i\right)\right)^{-1}dm,\end{aligned}$$

i.e.,

$$G_X(s) = \frac{c}{c + 1}\,{}_2F_1\left(1, 1; c + 2; \sum_{i=1}^{k}s^i - k + 1\right). \tag{2.3}$$

For $k = 1$, (2.3) reduces to

$$G_X(s) = \frac{c}{c + 1}\,{}_2F_1(1, 1; c + 2; s)$$

which is the p.g.f. of the beta-geometric distribution. Hence, (2.3) is a generalized form of the beta-geometric distribution. In the sequel, we will refer to this distribution as the beta-geometric distribution of order k with parameter c.

The first two factorial moments of the beta-geometric distribution of order k can be obtained using (2.3); thus,

$$\begin{aligned}E(X) &= \frac{c}{c + 1}\frac{1}{c + 2}\,{}_2F_1(2, 2; c + 3; 1)\sum_{i=1}^{k}i\\ &= \frac{c}{(c + 1)}\frac{1}{(c + 2)}\frac{(c + 2)_{(2)}}{(c - 1)_{(2)}}\frac{k(k + 1)}{2} = \frac{k(k + 1)}{2(c - 1)};\end{aligned} \tag{2.4}$$

$$\begin{aligned}E(X(X - 1)) &= \frac{c}{c + 1}\frac{4}{(c + 2)_{(2)}}\,{}_2F_1(3, 3; c + 4; 1)\left(\sum_{i=1}^{k}i\right)^2\\ &\quad + \frac{c}{c + 1}\frac{1}{(c + 2)}\,{}_2F_1(2, 2; c + 3; 1)\sum_{i=2}^{k}i(i - 1)\\ &= \frac{k^2(k + 1)^2}{(c - 1)(c - 2)} + \frac{k(k + 1)(k - 1)}{3(c - 1)}.\end{aligned}$$

Hence, the variance is

$$V(X) = \frac{k^2(k+1)^2c^2}{4(c-1)^2(c-2)} + \frac{k(k^2-1)(3k+2)}{36(c-1)}. \tag{2.5}$$

Note that both the mean and the variance of the beta-geometric distribution are greater than or equal to the corresponding mean and variance of the ordinary beta-geometric distribution and do not exist when $c \leqslant 1$ and $c \leqslant 2$, respectively.

Because of their simplicity, relationships (2.4) and (2.5) can be of great practical value as far as moment estimation of the parameter c is concerned, especially because of the complexity of the maximum likelihood method for generalized hypergeometric-type distributions. Thus, based on a random sample of size n, the moment estimator of c is

$$\hat{c} = \frac{k(k+1)}{2\overline{X}} + 1 \tag{2.6}$$

with variance

$$V(\hat{c}) = \frac{c^2(c-1)^2}{n(c-2)} + \frac{(c-1)(k-1)(3k+2)}{9nk(k+1)}, \tag{2.7}$$

where $\overline{X}$ is the sample mean.

Now, we shall show that if X is a r.v. having the beta-geometric distribution of order k with parameter $c > 0$, its p.f. is given by

$$P(X = x) = c\sum_{\ell=0}^{\infty}\frac{(1-k)^{\ell}}{\ell!}\sum_{\Sigma i x_i = x}\frac{((\Sigma x_j + \ell)!)^2}{(c+1)_{(\Sigma x_j + \ell + 1)}}\,\frac{1}{\prod_{j=1}^{k} x_j!}, \tag{2.8}$$
$$x = 0, 1, 2, \ldots .$$

From (2.3), we have that

$$G_X(s) = \frac{c}{c+1}\,{}_2F_1\Big(1, 1; c+2; \sum_{i=1}^{k} s^i + 1 - k\Big),$$

i.e.,

$$\begin{aligned}
G_X(s) &= \frac{c}{c+1}\sum_{r=0}^{\infty}\frac{r!}{(c+2)_{(r)}}\left(\sum_{i=1}^{k} s^i + 1 - k\right)^r \\
&= \frac{c}{c+1}\sum_{r=0}^{\infty}\frac{(r!)^2}{(c+2)_{(r)}}\sum_{\ell + \Sigma r_i = r}\binom{r}{r_1, r_2, \ldots, r_k, \ell}\frac{(1-k)^{\ell}\prod_{i=1}^{k} s^{jr_j}}{r!} \\
&= \frac{c}{c+1}\sum_{r=0}^{\infty}\sum_{\ell=0}^{r}\sum_{\Sigma r_i = r-\ell}\frac{((\Sigma r_i + \ell)!)^2(1-k)^{\ell}s^{\Sigma i r_i}}{(c+2)_{(\Sigma r_i + \ell)}\,\ell!\prod_{j=1}^{k} r_j!}.
\end{aligned}$$

Setting

$$r_i = x_i,\ i = 1, 2, \ldots, k, \text{ and } r + \sum_{i=1}^{k}(i-1)r_i = x,$$

we obtain

$$G_X(s) = \frac{c}{c+1}\sum_{x=0}^{\infty} s^x \sum_{\ell=0}^{\infty}\frac{(1-k)^{\ell}}{\ell!}\sum_{\Sigma i x_i = x}\frac{\left(\left(\sum_{j=1}^{k} x_j + \ell\right)!\right)^2}{(\rho+2)_{(\Sigma x_j + \ell)}\prod_{j=1}^{k} x_j!}$$

from which (2.8) follows.

3. THE BETA-NEGATIVE BINOMIAL DISTRIBUTION OF ORDER k

The beta-negative binomial distribution (or generalized Waring distribution) was considered by Irwin [9] in the context of problems in accident analysis. It was obtained from the theoretical model

beta-negative binomial $(a, b; c) \sim$ negative binomial $(b, p) \underset{p}{\wedge}$ beta I (c, a)

which is equivalent to

beta-negative binomial $(a, b; c) \sim$ Poisson $(\lambda) \underset{\lambda}{\wedge}$ gamma $\left(\frac{1}{m}, b\right) \underset{m}{\wedge}$ beta II (a, c)

Then, an extension of the beta-negative binomial distribution can be defined by a slight modification of the latter mechanism.

Let X be a r.v. such that, conditional on another nonnegative r.v., λ has a Poisson distribution of order k with parameter λ and p.g.f. given by (2.3). Assume now that λ has a gamma distribution whose scale parameter is a beta II (a, c) r.v., i.e., assume that λ has a p.d.f. of the form

$$f(\lambda) = \frac{\Gamma(a+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \lambda^{b-1} \int_0^{+\infty} m^{a-b-1}(1+m)^{-(a+c)} e^{-\lambda/m}\, dm.$$

Then the final resulting distribution of X will have p.g.f.

$$\begin{aligned} G_X(s) &= \frac{\Gamma(a+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^{+\infty}\int_0^{+\infty} m^{a-b-1}(1+m)^{-(a+c)}\lambda^{b-1} \times \\ &\qquad \exp\left\{-\lambda\left(\frac{1}{m} + k - \sum_{i=1}^{k} s^i\right)\right\} d\lambda\, dm \\ &= \frac{\Gamma(a+c)}{\Gamma(a)\Gamma(c)} \int_0^{+\infty} m^{a-1}(1+m)^{-(a+c)}\left(1 + m\left(k - \sum_{i=1}^{k} s^i\right)\right)^{-b} dm, \end{aligned}$$

i.e.,

$$G_X(s) = \frac{c_{(b)}}{(a+c)_{(b)}}\ {}_2F_1\left(a, b; a+b+c; \sum_{i=1}^{k} s^i - k + 1\right). \tag{3.1}$$

The above relationship reduces, for $k = 1$, to

$$G_X(s) = \frac{c_{(b)}}{(a+c)_{(b)}}\ {}_2F_1(a, b; a+b+\rho; s),$$

i.e., it coincides with the p.g.f. of the usual beta-negative binomial distribution. Thus (3.1) defines a more general form of beta-negative binomial distribution in the framework of distributions of order k. We will refer to this distribution as the beta-negative binomial distribution of order k with parameters a, b, and c; $a, b, c > 0$.

The mean of this distribution can be obtained from (3.1) by differentiation at $s = 1$.

$$\begin{aligned} E(X) &= \frac{c_{(b)}}{(a+c)_{(b)}} \frac{ab}{a+b+c}\ {}_2F_1(a+1, b+1; a+b+c+1; 1) \sum_{i=1}^{k} i \\ &= \frac{c_{(b)}}{(a+c)_{(b)}} \frac{ab}{a+b+c} \frac{(a+c)_{(b+1)}}{(c-1)_{(b)}} \frac{k(k+1)}{2} \end{aligned}$$

i.e.,

$$E(X) = \frac{abk(k+1)}{2(c-1)}. \tag{3.2}$$

The second factorial moment is

$$\mu_{[2]} \equiv E(X(X-1))$$
$$= \frac{c_{(b)}}{(a+c)_{(b)}} \frac{a_{(2)}b_{(2)}}{(a+b+c)_{(2)}} \, {}_2F_1(a+2,\ b+2;\ a+b+c+2;\ 1)\left(\sum_{i=1}^{k} i\right)^2$$
$$+ \frac{c_{(b)}}{(a+c)_{(b)}} \frac{ab}{a+b+c} \, {}_2F_1(a+1,\ b+1;\ a+b+c+1;\ 1)\sum_{i=1}^{k} i(i-1),$$

i.e.,

$$\mu_{[2]} = \frac{ab(a+1)(b+1)k^2(k+1)^2}{4(c-1)(c-2)} + \frac{abk(k^2-1)}{3(c-1)}. \tag{3.3}$$

Hence, we have, for the variance,

$$V(X) = \frac{k^2(k+1)^2 ab(c+a-1)(c+b-1)}{4(c-1)^2(c-2)} + \frac{abk(k^2-1)(3k+2)}{36(c-1)}. \tag{3.4}$$

Because application of the distribution will require estimation of three parameters (a, b, and c), we also provide the third factorial moment.

$$\mu_{[3]} \equiv E(X(X-1)(X-2)) = \frac{a(a+1)(a+2)b(b+1)(b+2)k^3(k+1)^3}{8(c-1)(c-2)(c-3)}$$
$$+ \frac{a(a+1)b(b+1)k^2(k^2-1)(k+1)}{2(c-1)(c-2)}$$
$$+ \frac{abk(k^2-1)(k-2)}{4(c-1)}. \tag{3.5}$$

Equations (3.2), (3,3), and (3.5) can be used to develop estimators of the parameters a, b, and c if a moment method of estimation is to be considered.

Note that for $k = 1$ we obtain from equations (3.2)-(3.5) the corresponding moments of the usual beta-negative binomial distribution. Inspection of these formulas shows that $\mu_{[i]}$ is expressed in terms of the first i factorial moments of the beta-negative binomial distribution, $i = 1, 2, 3$. Hence $\mu_{[i]}$ exists only if $c > i$, $i = 1, 2, 3$.

Let us now consider a nonnegative, integer-valued r.v. X whose probability distribution is the beta-negative binomial distribution of order k. We will show that the p.f. of X is given by

$$P(X = x) = \frac{c_{(b)}}{(a+c)_{(b)}} \sum_{\ell=0}^{+\infty} \frac{(1-k)^{\ell}}{\ell!} \sum_{\Sigma i x_i = x} \frac{a_{(\Sigma x_i + \ell)} b_{(\Sigma x_i + \ell)}}{(a+b+c)_{(\Sigma x_i + \ell)} \prod_{j=1}^{k} x_j!}, \tag{3.6}$$
$$x = 0,\ 1,\ 2,\ \ldots\ .$$

Setting $c^* = \dfrac{c_{(b)}}{(a+c)_{(b)}}$ we have, from (3.1),

$$G_X(s) = c^* {}_2F_1\left(a,\ b;\ a+b+c;\ \sum_{i=1}^{k} s^i + 1 - k\right)$$
$$= c^* \sum_{r=0}^{\infty} \frac{a_{(r)}b_{(r)}}{(a+b+c)_{(r)}} \frac{\left(\sum_{i=1}^{k} s^i + 1 - k\right)^r}{r!}$$

$$= c^* \sum_{r=0}^{\infty} \frac{a_{(r)} b_{(r)}}{(a+b+c)_{(r)}} \sum_{\ell + \Sigma r_i = r} \binom{r}{\ell, r_1, \ldots, r_k} \frac{(1-k)^{\ell} \prod_{j=1}^{k} s^{j r_j}}{r!}$$

$$= c^* \sum_{r=0}^{\infty} \sum_{\ell=0}^{r} \sum_{\Sigma r_i = r - \ell} \frac{a_{(\Sigma r_i + \ell)} b_{(\Sigma r_i + \ell)} (1-k)^{\ell} s^{\Sigma i r_i}}{(a+b+c)_{(\Sigma r_i + \ell)} \ell! \prod_{j=1}^{k} r_j!}.$$

Let $x_i = r_i$, $i = 1, 2, \ldots, k$, and $x = r + \sum_{i=1}^{k} (i-1) r_i$. Then the p.g.f. of X becomes

$$G_X(x) = c^* \sum_{x=0}^{+\infty} s^x \sum_{\ell=0}^{\infty} \frac{(1-k)^{\ell}}{\ell!} \sum_{\Sigma i x_i = x} \frac{a_{(\Sigma x_i + \ell)} b_{(\Sigma x_i + \ell)}}{(a+b+c)_{(\Sigma x_i + \ell)} \prod_{j=1}^{k} x_j!}$$

which leads to (3.6).

It is interesting to observe that the beta-geometric distribution of order k defined in Section 2 and the beta-negative binomial distribution of order k defined in Section 3 are related in the same manner in which the ordinary beta-geometric and beta-negative binomial distributions are related. In particular, the beta-geometric distribution of order k can be thought of as a special case of the beta-negative binomial distribution of order k for $a = b = 1$.

4. SOME LIMITING CASES OF THE BETA-NEGATIVE BINOMIAL DISTRIBUTION OF ORDER k

It is known (see Irwin [8]) that the beta-negative binomial distribution can take a negative binomial or a Poisson form for certain limiting values of its parameters. So, naturally one would inquire whether its generalization as defined in Section 3, i.e., the beta-negative binomial distribution of order k tends to a negative binomial or Poisson type of distribution of the same order. It can be shown that, indeed, this is the case.

The Poisson distribution of order k and the gamma-compound Poisson distribution of order k are obtained as limiting cases of the beta-negative binomial distribution of order k as indicated by the following theorems.

Theorem 4.1: Let X be a nonnegative, integer-valued r.v. whose probability distribution is the beta-negative binomial of order k with parameters a, b, c. Then

$$\lim_{H} G_X(s) = \exp\left\{\frac{ab}{a+c}\left(\sum_{i=1}^{k} s^i - k\right)\right\}, \tag{4.1}$$

where $\lim_{H}$ stands for limit as $a \to +\infty$, $b \to +\infty$, $c \to +\infty$ so that $ab/(a+c) < +\infty$ and $a/(a+c) \to 0$.

The result of this theorem was not unexpected since, by its derivation, the beta-negative binomial distribution of order k can be regarded as a beta mixture of the gamma-compound Poisson distribution of order k (studied by Philippou [14]) with p.g.f.

$$G(s) = \left(1 + m\left(k - \sum_{i=1}^{k} s^i\right)\right)^{-b}, \quad m > 0, \; b > 0, \tag{4.2}$$

which converges to a Poisson distribution of order k as demonstrated by the following theorem.

Theorem 4.2: Let X be a r.v. having the gamma-compound Poisson distribution of order k with p.g.f. $G(s)$ given by (4.2). Then,

$$G(s) \to \exp\left\{mb\sum_{i=1}^{k}(s^i - 1)\right\} \tag{4.3}$$

as $m \to 0$, $b \to +\infty$ so that $mb < +\infty$.

Theorem 4.3: Let X be defined as in Theorem 4.1. Then,

$$\lim_{H'} G_X(s) = \left(1 + \frac{a}{c}\left(k - \sum_{i=1}^{k} s^i\right)\right)^{-b}, \tag{4.4}$$

where $\lim_{H'}$ stands for limit as $a \to +\infty$ and $c \to +\infty$ so that $a/(a + c) < +\infty$.

Note that, for $k = 1$, relationships (4.1) and (4.4) yield the Poisson and negative binomial limit of the ordinary beta-geometric distribution, respectively, while (4.3) yields the Poisson limit of the ordinary negative binomial distribution.

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COUNTING THE "GOOD" SEQUENCES

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(Submitted June 1985)

For a finite sequence of nonnegative integers, $A = \{a_{1j}\}$, $j = 1, 2, 3, \ldots, n$, define its set of absolute differences by the recursion relation

$$a_{ij} = |a_{i-1,\,j} - a_{i-1,\,j+1}|, \text{ for } i + j \leqslant n + 1.$$

We write A along with its set of absolute differences in the natural way indicated in the following table and call the resulting triangular array $T(A)$.

$$\begin{array}{ccccccccccccc}
1 & & 2 & & 4 & & 4 & & 8 & & 10 & & 6 \\
& 1 & & 2 & & 0 & & 4 & & 2 & & 4 & \\
& & 1 & & 2 & & 4 & & 2 & & 2 & & \\
& & & 1 & & 2 & & 2 & & 0 & & & \\
& & & & 1 & & 0 & & 2 & & & & \\
& & & & & 1 & & 2 & & & & & \\
& & & & & & 1 & & & & & &
\end{array}$$

If the left "column" of $T(A)$ consists totally of 1's, we say that A is a *good sequence*. There are a great many good sequences of length n, ranging from the "smallest," $\{1, 0, 0, \ldots, 0\}$, to the "largest," $\{1, 2, 4, \ldots, 2^{n-1}\}$. Galbreath conjectured that the sequence $\{p_i - 1\}$, where p_i is the i^{th} prime, is an infinite good sequence (see [1]). A natural question to ask is: *How many good sequences are there of length n?* In this paper, we shall answer this question for small n, and present a heuristic recursion relation.

Let $G(n)$ be the set of good sequences of length n, with $g(n) = \#G(n)$. If $g \in G(n)$, we note that each row of $T(g)$ is a good sequence. This observation, along with the obvious one that any initial subsequence of a good sequence is also good, leads to the following definitions.

For $g \in G(n-1)$, let $e(g) = \#\{g^* \in G(n)$, with g an initial subsequence of $g^*\}$, and $e^*(g) = \#\{g^* \in G(n)$, with g the second row of $T(g^*)\}$. We say that $e(g)$ is the number of ways to extend $T(g)$ to the right, and $e^*(g)$ is the number of ways to extend it upward.

Now, assume $g \in G(n-2)$, and extend $T(g)$ both to the right and upward, as in Figure 1. If we choose c so that $|b - c| = a$, we will have a triangular array that is $T(g^*)$ for some $g^* \in G(n)$. Since c can be chosen in either 1 or 2 ways for a given a and b, based on their relative magnitudes, we have the following equality.

$$g(n) = \sum_{g \in G(n-2)} e(g)e^*(g)\beta(g),$$

where $1 \leqslant \beta(g) \leqslant 2$.

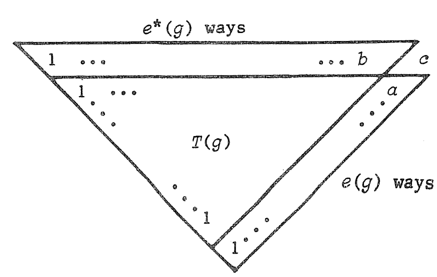

FIGURE 1

The average value of both $e(g)$ and $e^*(g)$ is $g(n-1)/g(n-2)$. Also, since a and b are each the last elements of members of $G(n-1)$, we expect $a \leqslant b$ about half the time, and vice versa. In other words, we expect $\beta(g) \sim 3/2$ on average. By replacing $e(g)$, $e^*(g)$, and $\beta(g)$ with these "averages" in the previous sum, we have an "expected" asymptotic recursion relation,

$$g(n) \sim \frac{3}{2}\frac{(g(n-1))^2}{g(n-2)}, \text{ as } n \to \infty.$$

To test this relation, $g(n)$ was calculated for $n \leqslant 10$. Its values, along with the values for $\beta(n) = g(n)g(n-2)/(g(n-1))^2$, are presented in Table 1.

TABLE 1

n	$g(n)$	$\beta(n)$
1	1	-
2	2	-
3	5	1.250
4	17	1.360
5	82	1.419
6	573	1.449
7	5,839	1.458
8	86,921	1.461
9	1,890,317	1.461
10	60,013,894	1.460

The following questions naturally arise:

Is there a formula for $g(n)$?

Does $\lim_{n\to\infty} g(n)g(n-2)/(g(n-1))^2$ exist? If so, what is it?

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CONVERGENCE OF TRIBONACCI DECIMAL EXPANSIONS

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1. INTRODUCTION

Let F_i denote the Fibonacci sequence defined by

$F_1 = F_2 = 1$, $F_i = F_{i-2} + F_{i-1}$, for $i \geqslant 3$;

that is, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... In 1953 Fenton Stancliff [5] observed that

$$\sum_{i=1}^{\infty} 10^{-(i+1)} F_i = \frac{1}{89}. \tag{1}$$

Since 1953 a number of authors including Wlodarski [8], Brousseau [1], Kohler [3], Winans [7], Long [5], Hudson and Winans [2], and Pin-Yen Lin [4] have investigated the convergence of Fibonacci decimal expansions,

$$\sum_{i=1}^{\infty} 10^{-k(i+1)} F_{\alpha i}, \ \alpha \geqslant 1.$$

C. F. Winans first observed that

$$\sum_{i=1}^{\infty} 10^{-(i+1)} F_{2i}$$

appears to converge to 1/71 employing decimal approximation, since

$$\frac{1}{71} = .014084507\ldots$$

and

$$\begin{aligned} \sum_{i=1}^{10} 10^{-(i+1)} F_{2i} = \ & .01 \\ + \ & .0003 \\ + \ & .00008 \\ + \ & .000021 \\ + \ & .0000055 \\ + \ & .00000144 \\ + \ & .000000377 \\ + \ & .0000000987 \\ + \ & .00000002584 \\ + \ & \underline{.000000006765} \\ & .010408448305 \end{aligned} \tag{2}$$

Convergence of (2) to 1/71 was proved in [2], as were

$$\sum_{i=1}^{\infty} 10^{-(i+1)} F_{2i} = \frac{2}{59} \quad \text{and} \quad \sum_{i=1}^{\infty} 10^{-(i+1)} F_{3i} = \frac{3}{31}.$$

The purpose of this paper is to prove an analogous conjecture of Winans for tribonacci decimal expansions and to generalize this result to obtain convergents in cases where Winans found that decimal approximation failed to give even a clue to the correct convergent. As in the Fibonacci case, the convergents include coefficients that involve a fascinating, though more complicated, tribonacci-like recurrence relation; see Theorem 2 in Section 3.

2. PROOF OF WINAN'S CONJECTURE

Let T_i denote the tribonacci sequence defined by $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, and

$$T_i = T_{i-3} + T_{i-2} + T_{i-1}, \; i \geqslant 3; \tag{3}$$

that is, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, ... Employing decimal approximation, Winans conjectured the following theorem which we now prove.

<u>Theorem 1</u>: Let T_i be defined as in (3). Then

$$\sum_{i=1}^{\infty} 10^{-k(i+2)} T_i = \frac{1}{10^{3k} - 10^{2k} - 10^k - 1}. \tag{4}$$

Proof: Define $f(z)$ by

$$f(z) = \sum_{i=1}^{\infty} T_i z^i \tag{5}$$

and note that since $T_1 = T_2$, $T_3 = T_1 + T_2$, and $T_i = T_{i-1} + T_{i-2} + T_{i-3}$ for $i \geqslant 4$, we have

$$\begin{aligned}(1 - z - z^2 - z^3)f(z) &= (1 - z - z^2 - z^3)(T_1 z + T_2 z^2 + \cdots)\\ &= T_1 z + (T_2 - T_1)z^2 + (T_3 - T_2 - T_1)z^3 + (T_4 - T_3 - T_2 - T_1)z^4 + \cdots\\ &\qquad + (T_n - T_{n-1} - T_{n-2} - T_{n-3})z^n + \cdots\\ &= T_1 z + (T_2 - T_1)z^2 + (T_3 - T_2 - T_1)z^3 = z.\end{aligned}$$

Therefore,

$$f(z) = \sum_{i=1}^{\infty} T_i z^i = \frac{z}{1 - z - z^2 - z^3}. \tag{6}$$

Since $|1 - z - z^2 - z^3| \geqslant 1 - |z| - |z^2| - |z^3| > 0$ if $|z| \leqslant 1/2$, the function $f(z)$ is analytic in the disc $\{z \in C : |z| \leqslant 1/2\}$. Consequently, its power series expansion is absolutely convergent for all z with $|z| \leqslant 1/2$ and (6) holds if we replace z by any complex number with modulus less than or equal to 1/2.

In particular, if we let $z = 10^{-k}$ with $k \geqslant 1$, we obtain

$$\sum_{i=1}^{\infty} T_i 10^{-ki} = \frac{10^{-k}}{1 - 10^{-k} - 10^{-2k} - 10^{-3k}} = \frac{10^{2k}}{10^{3k} - 10^{2k} - 10^k - 1}, \tag{7}$$

completing the proof of the conjecture of Winans.

<u>Remark</u>: Define an n-ary Fibonacci sequence by the recurrence relation

$$T_{i,n} = T_{i-n-1,n} + T_{i-n-2,n} + \cdots + T_{i-1,n} > n \geqslant 2, \; i \geqslant n. \tag{8}$$

Using the same method given in the proof of Theorem 1, one obtains:

$$\sum_{i=1}^{\infty} 10^{-k(i+n-1)} T_{i,n} = \frac{1}{10^{nk} - 10^{(n-1)k} - \cdots - 1}. \tag{9}$$

This result was conjectured by Winans for tetrabonacci and pentabonacci expansions.

Numerical Examples: Analogous to (1), observed by Stancliff, we have from (7),

$$\sum_{i=1}^{\infty} 10^{-(i+1)} T_i = \frac{1}{1000 - 100 - 10 - 1} = \frac{1}{889}.$$

Moreover, by (9), we have

$$\sum_{i=1}^{\infty} 10^{-(i+1)} T_{i,4} = \frac{1}{10000 - 1000 - 100 - 10 - 1} = \frac{1}{8889},$$

and, in general (with the dots denoting $n - 1$ eights),

$$\sum_{i=1}^{\infty} 10^{-(i+1)} T_{i,n} = 888\ldots 89$$

for an n-ary Fibonacci decimal expansion.

3. GENERALIZATION OF WINAN'S CONJECTURES

For $\alpha \geqslant 2$, Winans was unable to formulate a conjecture for the correct convergents for $\Sigma 10^{-ki} T_{\alpha i}$ even for $k = 1$, $\alpha = 2$. Once one establishes the correct convergent as we will in Theorem 3 of this section, one observes that $\Sigma 10^{-i} T_{2i}$ does converge fairly rapidly to 110/689. Indeed,

$$\begin{aligned} \sum_{i=1}^{10} 10^{-i} T_{\alpha i} = \ & .1 \\ + \ & .04 \\ + \ & .013 \\ + \ & .0044 \\ + \ & .00149 \\ + \ & .000504 \\ + \ & .0001705 \\ + \ & .00005768 \\ + \ & \underline{.000019513} \\ & .159641693 \end{aligned}$$

and $\dfrac{110}{689} = .159651699\ldots$.

First, we require a theorem involving a recurrence relation for tribonacci numbers which is interesting in itself and essential to the goal of determining all convergents of

$$\sum_{i=1}^{\infty} 10^{-ki} T_{\alpha i}, \ k \geqslant 1, \ \alpha \geqslant 1.$$

Theorem 2: Let $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, and let $T_i = T_{i-3} + T_{i-2} + T_{i-1}$ for $i \geqslant 3$.

Define sequences $\{a_i\}$ and $\{b_i\}$ by

$$a_i = a_{i-1} + a_{i-2} + a_{i-3} \text{ for } i \geqslant 4; \ a_1 = 1, \ a_2 = 3, \ a_3 = 7, \tag{10}$$

and

$$b_i = b_{i-1} + b_{i-2} + b_{i-3} \text{ for } i \geq 4;\ b_1 = b_2 = 1,\ b_3 = -5. \tag{11}$$

For every positive integer $\alpha \geq 1$,

$$T_{3\alpha+i} = a_\alpha T_{2\alpha+i} + b_\alpha T_{\alpha+i} + T_i;\ i \geq 0. \tag{12}$$

Proof: Let β_1, β_2, and β_3 be the distinct complex roots of $z^3 - z^2 - z - 1 = 0$ so that

$$(z - \beta_1)(z - \beta_2)(z - \beta_3) = z^3 - z^2 - z - 1. \tag{13}$$

Then there are constants u_1, u_2, and u_3 such that

$$T_i = u_1\beta_1^i + u_2\beta_2^i + u_3\beta_3^i \text{ for every } i \geq 0. \tag{14}$$

Define

$$A_\alpha = \beta_1^\alpha + \beta_2^\alpha + \beta_3^\alpha,\ B_\alpha = -[\beta_1^\alpha\beta_2^\alpha + \beta_1^\alpha\beta_3^\alpha + \beta_2^\alpha\beta_3^\alpha],\ C_\alpha = (\beta_1\beta_2\beta_3)^\alpha.$$

Now, it is easily checked that

$$(\beta_1^\alpha)^3 - (\beta_1^\alpha + \beta_2^\alpha + \beta_3^\alpha)(\beta_1^\alpha)^2 + [(\beta_1\beta_2)^\alpha + (\beta_1\beta_3)^\alpha + (\beta_2\beta_3)^\alpha]\beta_1^\alpha - (\beta_1\beta_2\beta_3)^\alpha$$
$$= \beta_1^{3\alpha} - \beta_1^{3\alpha} - \beta_2^\alpha\beta_1^{2\alpha} - \beta_3^\alpha\beta_1^{2\alpha} + \beta_2^\alpha\beta_1^{2\alpha} + \beta_3^\alpha\beta_1^{2\alpha} + (\beta_2\beta_3\beta_1)^\alpha - (\beta_1\beta_2\beta_3)^\alpha = 0,$$

and similarly for β_2^α and β_3^α, so that β_1^α, β_2^α, and β_3^α are the roots of the equation

$$z^3 - A_\alpha z^2 - B_\alpha z - C_\alpha = 0.$$

Using (14), we obtain

$$T_{i+3\alpha} - A_\alpha T_{i+2\alpha} - B_\alpha T_{i+\alpha} - C_\alpha T_i = 0. \tag{15}$$

From (13), it follows that

$$A_i = A_{i-1} + A_{i-2} + A_{i-3} \text{ for every } i \geq 1 \tag{16}$$

[Since, for $j = 1, 2, 3$, $\beta_j^i = \beta_j^{i-1} + \beta_j^{i-2} + \beta_j^{i-3} \Longleftrightarrow \beta_j^{i-3}(\beta_j^3 - \beta_j^2 - \beta_j - 1) = 0$] and clearly

$$C_i = 1 \text{ for every } i \geq 0. \tag{17}$$

In particular, (13) implies that $\beta_1\beta_2\beta_3 = 1$, so that

$$B_\alpha = -[\beta_1^{-\alpha} + \beta_2^{-\alpha} + \beta_3^{-\alpha}]. \tag{18}$$

Replacing z by z^{-1} in (13), we obtain

$$\left(\frac{1}{z} - \beta_1\right)\left(\frac{1}{z} - \beta_2\right)\left(\frac{1}{z} - \beta_3\right) = \frac{1}{z^3} - (\beta_1 + \beta_2 + \beta_3)\left(\frac{1}{z^2}\right)$$
$$+ (\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)\left(\frac{1}{z}\right) - \beta_1\beta_2\beta_3,$$

so that $\beta_1 + \beta_2 + \beta_3 = 1$ and $\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3 = -1$.

On the other hand, we have, as $\beta_1\beta_2\beta_3 = 1$,

$$\left(\frac{1}{z} - \frac{1}{\beta_1}\right)\left(\frac{1}{z} - \frac{1}{\beta_2}\right)\left(\frac{1}{z} - \frac{1}{\beta_3}\right) = \frac{1}{z^3} - \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3}\right)\left(\frac{1}{z^2}\right)$$
$$+ \left(\frac{1}{\beta_3}\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right) + \frac{1}{\beta_1\beta_2}\right)\left(\frac{1}{z}\right) - 1$$
$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} - 1,$$

so that $1/\beta_1$, $1/\beta_2$, and $1/\beta_3$ are roots of $(1/z)^3 + (1/z)^2 + (1/z) - 1 = 0$. However, $\beta_2\beta_3 = 1/\beta_1$, $\beta_1\beta_3 = 1/\beta_2$, and $\beta_1\beta_2 = 1/\beta_3$, so we have

$$B_i = -\left[\left(\frac{1}{\beta_3}\right)^i + \left(\frac{1}{\beta_2}\right)^i + \left(\frac{1}{\beta_1}\right)^i\right].$$

Consequently,

$$B_i = -B_{i-1} - B_{i-2} + B_{i-3} \text{ for } i \geqslant 3 \tag{19}$$

if and only if

$$-\left[\left(\frac{1}{\beta_3}\right)^i + \left(\frac{1}{\beta_2}\right)^i + \left(\frac{1}{\beta_1}\right)^i\right]$$
$$= \left[\left(\frac{1}{\beta_3}\right)^{i-1} + \left(\frac{1}{\beta_2}\right)^{i-1} + \left(\frac{1}{\beta_1}\right)^{i-1}\right] + \left[\left(\frac{1}{\beta_3}\right)^{i-2} + \left(\frac{1}{\beta_2}\right)^{i-2} + \left(\frac{1}{\beta_1}\right)^{i-2}\right]$$
$$- \left[\left(\frac{1}{\beta_3}\right)^{i-3} + \left(\frac{1}{\beta_2}\right)^{i-3} + \left(\frac{1}{\beta_1}\right)^{i-3}\right].$$

But this is equivalent, since $\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3 = -1$, to

$$-\left(\left(\left(\frac{1}{\beta_3}\right)^{i-3} + \left(\frac{1}{\beta_2}\right)^{i-3} + \left(\frac{1}{\beta_1}\right)^{i-3}\right)\left(\sum_{j=1}^{3}\left(\left(\frac{1}{\beta_j}\right)^3 + \left(\frac{1}{\beta_j}\right)^2 + \left(\frac{1}{\beta_j}\right) - 1\right)\right)\right) = 0,$$

which is true in view of the fact that $1/\beta_1$, $1/\beta_2$, $1/\beta_3$ are roots of

$$\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right) - 1 = 0.$$

Finally, checking initial values, we observe from (16), (17), and (19) that $a_\alpha = A_\alpha$, $b_\alpha = B_\alpha$, and $C_\alpha = 1$ for every $\alpha \geqslant 1$ completing the proof of Theorem 2.

Using Theorem 2, we can now easily establish our main result, from which convergents of all tribonacci expansions of the form $\sum 10^{-ki} T_{\alpha i}$, $\alpha \geqslant 1$, $k \geqslant 1$, may be calculated. Clearly, this contains Theorem 1 as the special case $\alpha = k = 1$. However, we note that the proof of the following theorem does not appear to generalize trivially to n-ary Fibonacci expansions, $n > 3$, because of its dependence on Theorem 2.

Theorem 3: Let $\{T_i\}$, $\{a_i\}$, and $\{b_i\}$ be defined as in Theorem 2. Then

$$\sum_{i=1}^{\infty} 10^{-ki} T_{\alpha i} = \frac{T_\alpha \cdot 10^{2k} + (T_{2\alpha} - a_\alpha T_\alpha) \cdot 10^k}{10^{3k} - a_\alpha \cdot 10^{2k} - b_\alpha \cdot 10^k - 1} \tag{20}$$

iff the denominator is nonnegative.

Proof: Define $F(z)$ by

$$F(z) = \sum_{i=1}^{\infty} T_{\alpha i} z^i, \ \alpha \geqslant 1, \tag{21}$$

and observe that

$$(1 - a_\alpha z - b_\alpha z^2 - z^3)(T_\alpha z + T_{2\alpha} z^2 + T_{3\alpha} z^3 + \cdots)$$
$$= T_\alpha z + (T_{2\alpha} - a_\alpha T_\alpha) z^2 + (T_{3\alpha} - T_{2\alpha} a_\alpha - T_\alpha b_\alpha) z^3 + \text{terms of higher degree.}$$

Using (12), it is easily seen that the coefficients of all powers of z greater than 2 vanish. Hence,

$$F(z) = \frac{T_\alpha z + (T_{2\alpha} - a_\alpha T_\alpha)z^2}{1 - a_\alpha z - b_\alpha z^2 - z^3}.$$

Let γ_1^α, γ_2^α, and γ_3^α be the roots of

$$1 - a_\alpha z - b_\alpha z^2 - z^3 = 0. \tag{22}$$

We begin by showing that exactly one of the roots of (22) is real. Indeed, it suffices to consider the case $\alpha = 1$. For, assume that one of γ_1^α, γ_2^α, γ_3^α, say γ_1, is nonreal and that all of γ_1^α, γ_2^α, γ_3^α are real. Clearly, $Q(\gamma_1^\alpha)$ is a proper subfield of $Q(\gamma_1)$, so that $\deg(Q(\gamma_1^\alpha/Q)) < 3$ and divides 3; that is, it is 1. Consequently, γ_1^α is an algebraic integer lying in Q. Indeed, it is a unit because it is a root of $1 - a_\alpha z - b_\alpha z^2 - z^3 = 0$, so that $\gamma_1^\alpha = \pm 1$. Thus, γ_1 is a root of unity, which is clearly impossible.

It is now easy to show that (22) has exactly one positive real root when $\alpha = 1$. Let $f(z) = 1 - z - z^2 - z^3$ and observe that $f'(z) = -1 - 2z - 3z^2 < 0$ for all real z since $f''(z) = -2 - 6z = 0$ only if $z = -1/3$ and $f'''(-1/3) < 0$ so that $f'(z)$ has a maximum at $z = -1/3$. However, $f'(-1/3) < 0$ so that $f(z)$ is decreasing for all real z and, since $f(1/2) > 0$ and $f(1) < 0$, it is clear that $f(z) = 0$ has one real root z, $(1/2) < z < 1$, and so must have two nonreal roots which are conjugate pairs.

Now, let $h(z)$ be the polynomial defined by

$$h(z) = T_\alpha z + (T_{2\alpha} - a_\alpha T_\alpha)z^2.$$

Then, applying partial fractions, we have, as $\alpha = 1$,

$$\begin{aligned} F(z) &= \frac{u_1 z}{\gamma_1 - z} + \frac{u_2 z}{\gamma_2 - z} + \frac{u_3 z}{\gamma_3 - z} \\ &= z\left(\frac{u_1}{\gamma_1}\sum_{n=0}^{\infty}\left(\frac{z}{\gamma_1}\right)^n + \frac{u_2}{\gamma_2}\sum_{n=0}^{\infty}\left(\frac{z}{\gamma_2}\right)^n + \frac{u_3}{\gamma_3}\sum_{n=0}^{\infty}\left(\frac{z}{\gamma_3}\right)^n\right). \end{aligned}$$

This converges if $\left|\frac{z}{\gamma_1}\right| < 1$, $\left|\frac{z}{\gamma_2}\right| < 1$, and $\left|\frac{z}{\gamma_3}\right| < 1$.

Now the denominator of $F(z)$ can be written as

$$(\gamma_1 - z)(\gamma_2 - z)(\gamma_3 - z)$$

and if we let γ_1 be the real root between 1/2 and 1 and note that

$$(\gamma_2 - z)(\gamma_3 - z) > 0,$$

since γ_2 and γ_3 are complex conjugates, we see that, for real z,

$$1 - z - z^2 - z^3 > 0 \text{ if and only if } \gamma_1 - z > 0 \text{ or } z < \gamma_1. \tag{23}$$

Clearly, then, as $\gamma_1\gamma_2\gamma_3 = 1$ and $|\gamma_2| = |\gamma_3| = (1/\sqrt{\gamma_1}) > 1$, we also have

$$|z| < |\gamma_2| \quad \text{and} \quad |z| < |\gamma_3|,$$

completing the proof.

Example 1: Let $k = 3$ and let $\alpha = 8$. Then, by Theorem 3,

$$\sum_{i=1}^{\infty} 10^{-3i} T_{8i} = \frac{44 \cdot 10^6 + 4 \cdot 10^3}{10^9 - 131 \cdot 10^6 + 3 \cdot 10^3 - 1} = \frac{44{,}004{,}000}{869{,}002{,}999}.$$

Note that this fraction is approximately equal to .050637... and that with

$T_8 = 44$, $T_{16} = 5768$, $T_{24} = 755476$,

we have

$$\begin{aligned}\sum_{i=1}^{\infty} 10^{-3i} T_{8i} = &\quad .044\\ &+ .005768\\ &+ .000755476\\ \hline &\quad .050523476\end{aligned}$$

so that the series converges quite rapidly for $k = 3$ although it does not converge at all for $k = 2$.

Example 2: Listed in the table below are the convergents of

$$\sum_{i=1}^{\infty} 10^{-ki} T_{\alpha i} \text{ for } k = 1, 2, 3 \text{ and } \alpha \leqslant 4.$$

	$k = 1$	$k = 2$	$k = 3$
$\alpha = 1$	$\frac{100}{889}$	$\frac{10{,}000}{989{,}899}$	$\frac{1{,}000{,}000}{998{,}998{,}999}$
$\alpha = 2$	$\frac{110}{689}$	$\frac{10{,}100}{969{,}899}$	$\frac{1{,}001{,}000}{996{,}998{,}999}$
$\alpha = 3$	$\frac{190}{349}$	$\frac{19{,}900}{930{,}499}$	$\frac{1{,}999{,}000}{993{,}004{,}999}$
$\alpha = 4$	None	$\frac{40{,}000}{889{,}499}$	$\frac{4{,}000{,}000}{988{,}994{,}999}$

ACKNOWLEDGMENT

I would like to express my deep thanks to Stephen Eure and David Richman for their encouragement and assistance in writing this paper. In particular, Professor Richman proved Theorem 2 in this paper.

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A PROPERTY OF NUMBERS EQUIVALENT TO THE GOLDEN MEAN

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(Submitted June 1985)

We are concerned with finding the convergents $C_j(\alpha) = \dfrac{p_j}{q_j}$, in lowest terms, to a positive real number α that satisfy the inequality,

$$|\alpha - C_j(\alpha)| < \frac{\beta}{\sqrt{5}q_j^2}, \quad 0 < \beta < 1. \tag{1}$$

From Le Veque [3] or Roberts [4], we have the following theorems.

Hurwitz's theorem states that, if α is irrational and $\beta = 1$, there are infinitely many irreducible rational solutions to (1).

Dirichlet's theorem states that, if $\beta = \sqrt{5}/2$, then all rational solutions to (1) are convergents to α.

Since $1/\sqrt{5} < 1/2$, we note that the expression "irreducible rational solutions" in Hurwitz's theorem may always be replaced by "convergents."

It is readily shown (see [4]) that if $\alpha = \tau = (1 + \sqrt{5})/2$ (the Golden Mean) then there are only finitely many convergents to τ which satisfy (1). In [5], van Ravenstein, Winley, & Tognetti have determined the convergents explicitly.

We now extend [5] by determining the solutions to (1) when α is equivalent to τ, which means the Noble Number α has a simple continued fraction expansion $(a_0; a_1, a_2, \ldots, a_n, 1, 1, 1, \ldots)$ where the terms $a_1, a_2, \ldots, a_n$ are positive integers, $a_n \geqslant 2$ and a_0 is a nonnegative integer.

Using the notation of [5], with C_j replaced by $C_j(\alpha)$, and well-known facts [see Chrystal [1] and Khintchine [2]):

$$\left.\begin{array}{ll}
\text{(i)} & p_j = p_{j-2} + a_j p_{j-1}, \\
 & q_j = q_{j-2} + a_j q_{j-1}, \\
 & \text{for } j \geqslant 0,\ p_{-2} = q_{-1} = 0 \text{ and } q_{-2} = p_{-1} = 1; \\
\text{(ii)} & q_{j+1} > q_j > q_{j-1} > \cdots > q_0 = 1; \\
\text{(iii)} & p_{j-1}q_j - p_j q_{j-1} = (-1)^j; \\
\text{(iv)} & C_j(\tau) = \dfrac{F_{j+1}}{F_j}, \text{ where } F_j \text{ is the } j^{\text{th}} \text{ term of the} \\
 & \text{Fibonacci sequence } \{1, 1, 2, 3, 5, \ldots\}; \\
\text{(v)} & F_j = \dfrac{\tau^{j+1} - (1-\tau)^{j+1}}{\sqrt{5}}.
\end{array}\right\} \tag{2}$$

It follows from (2(1)) that

$$C_j(\alpha) = \frac{p_j}{q_j} = \begin{cases} \dfrac{p_{j-2} + a_j p_{j-1}}{q_{j-2} + a_j q_{j-1}}, & j = 0, 1, 2, \ldots, n \\[2ex] \dfrac{F_{j-n} p_n + F_{j-n-1} p_{n-1}}{F_{j-n} q_n + F_{j-n-1} q_{n-1}}, & j = n+1, n+2, \ldots, \end{cases} \tag{3}$$

and

$$\alpha = \lim_{j \to \infty} C_j(\alpha) = \frac{p_{n-1} + \tau p_n}{q_{n-1} + \tau q_n} = C_n(\alpha) + \frac{(-1)^n}{q_n(\tau q_n + q_{n-1})}.$$

Using (2(111)), and (2(iv)) in (3), we see that, for $j \geqslant n + 1$,

$$C_j(\alpha) = \frac{C_{j-n-1}(\tau) p_n + p_{n-1}}{C_{j-n-1}(\tau) q_n + q_{n-1}},$$

$$C_{j-n-1}(\tau) = \frac{F_{j-n}}{F_{j-n-1}}, \tag{4}$$

and

$$|\alpha - C_j(\alpha)| = \frac{|\tau - C_{j-n-1}(\tau)|}{(q_{n-1} + q_n\tau)(C_{j-n-1}(\tau) q_n + q_{n-1})}$$

Hence, for $j \geqslant n + 1$, (1) reduces to

$$|\tau - C_{j-n-1}(\tau)| < \frac{\beta(q_{n-1} + q_n\tau)}{\sqrt{5}F^2_{j-n-1}(C_{j-n-1}(\tau)q_n + q_{n-1})} \tag{5}$$

If $j - n - 1$ is even ($j = n + 1 + 2k$, $k = 0, 1, 2, \ldots$), then using (4) and $\tau^2 = 1 + \tau$ in (5) we seek nonnegative values of k such that

$$(\tau F_{2k} - F_{2k+1})(F_{2k+1}q_n + F_{2k}q_{n-1}) < \frac{\beta}{\sqrt{5}}(q_{n-1} + \tau q_n).$$

Using (2(v)), this reduces to

$$k < \ln\left(\frac{q_n - \tau q_{n-1}}{\tau^3(1 - \beta)(\tau q_n + q_{n-1})}\right) \Big/ 4 \ln \tau. \tag{6}$$

Now nonnegative values of k in (6) exist only if

$$\ln\left(\frac{q_n - \tau q_{n-1}}{\tau^3(1 - \beta)(\tau q_n + q_{n-1})}\right) > 0,$$

which means that

$$\beta_u < \beta < 1, \text{ where } \beta_u = \frac{\sqrt{5}}{\tau}\left[\frac{q_n + q_{n-1}}{\tau q_n + q_{n-1}}\right].$$

If $j - n - 1$ is odd ($j = n + 2 + 2k$, $k = 0, 1, 2, \ldots$), then (5) reduces to

$$(F_{2k+2} - \tau F_{2k+1})(F_{2k+2}q_n + F_{2k+1}q_{n-1}) < \frac{\beta}{\sqrt{5}}(q_{n-1} + q_n\tau).$$

Using (2(v)), this further reduces to

$$\tau^{4k+6}(1 - \beta) < \frac{\tau(\tau q_{n-1} - q_n)}{\tau q_n + q_{n-1}}. \tag{7}$$

Since the left side is positive and the right side is negative,

$$\tau - \frac{q_n}{q_{n-1}} < \tau - a_n < 0,$$

there are no nonnegative values of k which satisfy (7).

This completes the solutions to (1) for $j \geqslant n + 1$.

If $j = n$, then from (3) we have

$$|\alpha - C_n(\alpha)| = \frac{1}{q_n(\tau q_n + q_{n-1})},$$

and so (1) becomes

$$\frac{1}{q_n(\tau q_n + q_{n-1})} < \frac{\beta}{\sqrt{5}q_n^2},$$

which means $\beta > \dfrac{\sqrt{5}q_n}{\tau q_n + q_{n-1}}$.

However, since $\tau - (q_n/q_{n-1}) < 0$, we have $q_n > \tau q_{n-1}$, and this gives

$$\beta > \frac{\sqrt{5}q_n}{\tau q_n + q_{n-1}} > 1,$$

which is not possible. Hence, $C_n(\alpha)$ does not satisfy (1).

Consequently, there are no convergents that satisfy (1) if $\beta \leqslant \beta_u$ and $j \geqslant n$.

On the other hand, if $\beta > \beta_u$, then there are $[S] + 1$ convergents that satisfy (1). They are given by

$$C_j(\alpha) = \frac{F_{j-n}p_n + F_{j-n-1}p_{n-1}}{F_{j-n}q_n + F_{j-n-1}q_{n-1}}, \; j = n+1,\ n+3,\ \ldots,\ n+1+2[S], \tag{8}$$

where

$$S = \ln\left(\frac{q_n - \tau q_{n-1}}{\tau^3(1-\beta)(\tau q_n + q_{n-1})}\right)\bigg/ 4 \ln \tau,$$

and $[S]$ denotes the integer part of S.

We note that if $n = 0$, then $\alpha = (a_0; 1, 1, 1, \ldots)$, $a_0 \geqslant 2$, and the result (8) reduces to that given in [5].

It does not appear to be possible to make a precise statement as to which of the convergents $C_j(\alpha)$ for $j = 0, 1, 2, \ldots, n - 1$ will satisfy (1) without knowing the values of $a_0, a_1, \ldots, a_{n-1}$. However, we have shown that, if $0 < \beta < 1$, then there are only finitely many convergents to α which satisfy (1).

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THE LENGTH OF A TWO-NUMBER GAME

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(Submitted July 1985)

1. INTRODUCTION

Let D be an operator defined on a pair of integers

$A = (a_1, a_2),\ a_1 \geqslant a_2 > 0,$

by

$$D(a_1, a_2) = \begin{cases} (a_2, a_1 - a_2), & 2a_2 \geqslant a_1, \\ (a_1 - a_2, a_2), & a_1 \geqslant 2a_2. \end{cases} \tag{1.1}$$

Given any initial pair A_0, we obtain a sequence $\{A_n\}$ with $A_n = DA_{n-1}$, $n > 0$. This sequence is called the "two-number game."

Definition 1.1: The length of the sequence $\{A_n\}$, denoted $L(A)$, is n such that $A_n = (a', 0)$ for some integer $a' > 0$.

Definition 1.2: The complement of A is $CA = (a_1, a_1 - a_2)$.

It follows that $C^2A = A$ and

$$DCA = DA. \tag{1.2}$$

The effect of D on (a_1, a_2) is to reduce a_1 by a_2 and then arrange $a_1 - a_2$ and a_2 in order of decreasing magnitude to form $D(a_1, a_2)$.

The number pair (a_1, a_2) may be replaced by a rectangle $(a_1 . a_2)$ of sides a_1 and a_2. In such a case, $D(a_1 . a_2)$, $C(a_1 . a_2)$, and $L(a_1 . a_2)$ may be defined as above, but by replacing the comma with a dot. $D(a_1 . a_2)$ and $C(a_1 . a_2)$ are then rectangles. The length $L(a_1 . a_2)$ is equal to the number of squares obtained by removing the largest square $(a_1 . a_2)$ from an end of $(a_1 . a_2)$, then the largest square from an end of the remaining rectangle, and so on, until no squares remain. Therefore,

$$L(a_1, a_2) = L(a_1 . a_2). \tag{1.3}$$

For example,

$$\begin{aligned}(5 . 3) &= (3 . 3) + (3 . 2) = (3 . 3) + (2 . 2) + (2 . 1) \\ &= (3 . 3) + (2 . 2) + (1 . 1) + (1 . 1)\end{aligned}$$

from which $L(5 . 3) = 4$. See Figure 1 on page 175.

Replace (a_1, a_2) by the vector $A = \binom{a_1}{a_2}$, and write D in matrix form:

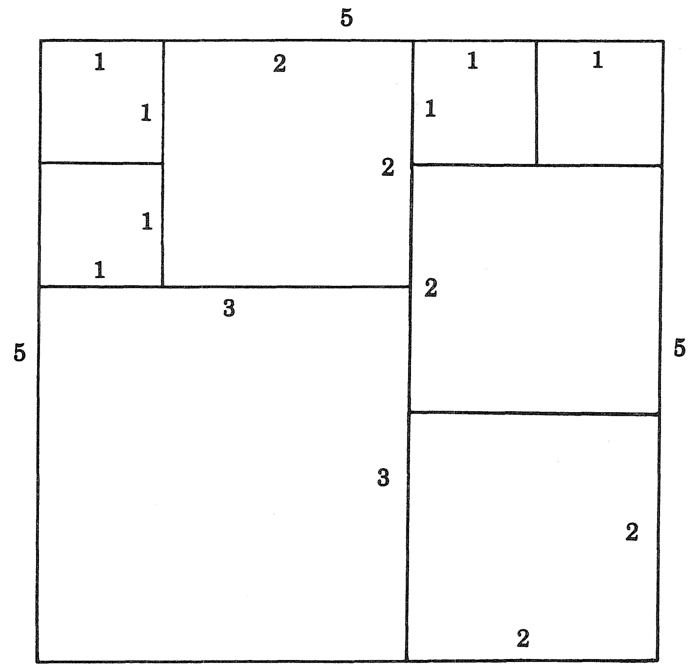

FIGURE 1. $L(5\,.\,3) = L(5\,.\,2) = 4$, $C(5\,.\,3) = (5\,.\,2)$

$$DA = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} A, & 2a_2 \geqslant a_1, \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} A, & a_1 \geqslant 2a_2. \end{cases} \tag{1.4}$$

Then $DkA = kDA$ for $k > 0$, and

$$L(kA) = L(A). \tag{1.5}$$

It follows from the definition that

$$L\begin{pmatrix} a_1 + na_2 \\ a_2 \end{pmatrix} = n + L\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \; n > 0. \tag{1.6}$$

Choose c such that $a_2 | (a_1 - c)$ and $a_1 > a_2 > c > 0$. Then,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \dfrac{(a_1 - c)}{a_2} a_2 + c \\ a_2 \end{pmatrix},$$

and from (1.6),

$$L\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{a_1 - c}{a_2} + L\begin{pmatrix} a_2 \\ c \end{pmatrix}. \tag{1.7}$$

Now, $(a_1 - c)/a_2$ is the greatest integer in a_1/a_2, since a_2 divides $a_1 - c$ and $a_2 > c > 0$, so

$$\frac{a_1 - c}{a_2} = \left[\frac{a_1}{a_2}\right],$$

where $[x]$ represents the greatest integer function of x. Since c represents the quantity $a_1 \pmod{a_2}$, Equation (1.7) may be written

$$L\binom{a_1}{a_2} = \left[\frac{a_1}{a_2}\right] + L\binom{a_2}{a_1 \pmod{a_2}}. \tag{1.8}$$

This relation may be iterated as in the following example:

$$L\binom{23}{5} = \left[\frac{23}{5}\right] + \left[\frac{5}{3}\right] + \left[\frac{3}{2}\right] + \left[\frac{2}{1}\right] = 8.$$

Table 1 exhibits $L\binom{a_1}{a_2}$ for a_1, a_2 equal to 1, 2, ..., 15.

TABLE 1. $L\binom{a_1}{a_2}$

a_2 \ a_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2		1	3	2	4	3	5	4	6	5	7	6	8	7	9
3			1	4	4	2	5	5	3	6	6	4	7	7	5
4				1	5	3	5	2	6	4	6	3	7	5	7
5					1	6	5	5	6	2	7	6	6	7	3
6						1	7	4	3	4	7	2	8	5	4
7							1	8	6	6	6	6	8	2	9
8								1	9	5	6	3	6	5	9
9									1	10	7	4	7	7	4
10										1	11	6	7	5	3
11											1	12	8	7	7
12												1	13	7	5
13													1	14	9
14														1	15
15															1

Let

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\quad C = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},\quad \text{and } P = CQ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

From (1.4), we have two forms of D^{-1}: $D_0^{-1} = Q$ and $D_1^{-1} = P$. D^{-2} has 2^2 forms, namely Q^2, QP, PQ, and P^2. D^{-n} has 2^n forms called D_j^{-n} which are the terms in the expansion of $(Q + P)^n$, where P and Q do not commute. The 2^n numbers $j = 0, 1, 2, \ldots, 2^n - 1$ may be expressed uniquely in binary form using n digits so that each D_j^{-n} may be paired with a distinct binary number.

Definition 1.3: We choose to define D_j^{-n} as the product derived from the binary number j of n digits in which 0 is replaced by Q and 1 by P.

For example, if $j = 3$, $n = 4$, the binary form of j is 0 0 1 1, so that $D_3^{-4} = Q^2P^2$.

It follows that $D^{-1}D^{-n} = D^{-n-1}$ and

$$D_i^{-m}D_j^{-n} = D_k^{-m-n}, \text{ where } k = 2^n i + j. \tag{1.9}$$

Note that D_i^{-m} and D_j^{-n} do not commute.

2. SEQUENCES OF VECTORS

Definition 2.1: If $a_1 \geqslant a_2$, A is said to be *proper*, and if a_1 and a_2 are relatively prime, then A is said to be *prime*.

We will assume henceforth that A is a proper prime vector. It follows that PA and QA are proper prime vectors, and hence $D^{-n}A$ in any of its forms is proper and prime.

Definition 2.2: Let $A(i, j)$ represent the vector A of length $i = L(A)$ as follows:

$$A(1, 0) = DA(2, 0) = \binom{1}{1},$$

$$A(2, 0) = D^0A(2, 0) = \binom{2}{1},$$

$$A(3, 0) = D_0^{-1}A(2, 0) = \binom{3}{2},$$

$A(3, 1) = D_1^{-1}A(2, 0) = \binom{3}{1}$ and if $i > 2$, $j = 0, 1, 2, \ldots, 2^{i-2} - 1$,

$$A(i, j) = D_j^{-i+2}A(2, 0).$$

Consider the sequence $\{X_n = A(n + 2, j), n = 1, 2, \ldots\}$, where

$$X_n = D_j^{-n}\binom{2}{1} \text{ and } L(X_n) = n + 2.$$

If $j = 0$, then

$$X_n = Q^n\binom{2}{1} \text{ and } X_{n+2} - X_{n+1} - X_n = 0 \text{ from the identity } Q^2 - Q - I = 0.$$

This identity may also be applied to cases where $j = 2^{n-1}$, 1, and $2^{n-1} + 1$ to yield the same recurrence relation. If $j = 2^n - 1$,

$$X_n = P^n\binom{2}{1} \text{ and } X_{n+2} - 2X_{n+1} + X_n = 0 \text{ from the identity } P^2 - 2P + I = 0.$$

This relation also holds for $j = 2^{n-1} - 1$, where $X_n = QP^{n-1}\binom{2}{1}$.

Note that X is represented as a product of elements selected from the set (P, Q) and a vector $\binom{2}{1}$. Then CX_n is X_n in which its first matrix (P or Q) is replaced by its complement (Q or P). X_n and CX_n have the same recurrence relations. See Table 2.

TABLE 2. Sequences $\{X_n = A(n + 2, j)\}$

j	X_n	Recurrence
0	$Q^n\binom{2}{1} = \binom{F_{n+3}}{F_n}$	$X_{n+2} - X_{n+1} - X_n = 0$
2^{n-1}	$PQ^{n-1}\binom{2}{1} = \binom{F_{n+3}}{F_{n+1}}$	$X_{n+2} - X_{n+1} - X_n = 0$
1	$Q^{n-1}P\binom{2}{1} = \binom{L_{n+1}}{L_n}$	$X_{n+2} - X_{n+1} - X_n = 0$
$2^{n-1} + 1$	$PQ^{n-2}P\binom{2}{1} = \binom{L_{n+1}}{L_{n-1}}$ if $n > 1$	$X_{n+2} - X_{n+1} - X_n = 0$
$2^n - 1$	$P^n\binom{2}{1} = \binom{n+2}{1}$	$X_{n+2} - 2X_{n+1} + X_n = 0$
$2^{n-1} - 1$	$QP^{n-1}\binom{2}{1} = \binom{n+2}{n+1}$	$X_{n+2} - 2X_{n+1} + X_n = 0$

Let $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$, $X_i = \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix}$, and $KX_i = X_{i+1}$, $i = 0, 1, 2, \ldots$, so that

$$K^n X_0 = X_n. \tag{2.1}$$

The characteristic equation for k is $|yI - K| = 0$ or

$$y^2 - (k_{11} + k_{22})y + |K| = 0.$$

By the Cayley-Hamilton theorem,

$$K^2 - (k_{11} + k_{22})K + |K|I = 0.$$

Multiply both sides of this equation on the right by $K^{n-2}X_0$, then

$$K^n X_0 - (k_{11} + k_{22})K^{n-1}X_0 + |K|K^{n-2}X_0 = 0.$$

From Equation (2,1),

$$X_n = (K_{11} + K_{22})X_{n-1} - |K|X_{n-2}, \tag{2.2}$$

a recurrence relation for X_n. We will assume here that

$$X_0 = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

The sequences $\{x_{n1}\}$ and $\{x_{n2}\}$ have been described by Horadam [1] as

$$\{w_n\} = \{w_n(a, b; p, q)\} : w_0 = a,\ w_1 = b,\ w_n = pw_{n-1} - qw_{n-2},\ n \geqslant 2.$$

In either sequence, $p = \operatorname{tr}(K)$, the trace of K, and $q = |K|$.

We may substitute D_j^{-r} for K and $A(2, 0)$ for X_0 in (2.1) to yield a sequence with the property $L(X_n) = rn + 2$. Let $D_j^{-r} = S_1 S_2 \ldots S_r$, where $S_i \in (P, Q)$.

Note that any 2×2 matrices A and B have the property $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, so

$$\operatorname{tr}(S_1 S_2 \ldots S_r) = \operatorname{tr}(S_2 S_3 \ldots S_r S_1).$$

Therefore, p is the same for K equal to any cyclic product of the S_i. Since

$$|P| = 1 \quad \text{and} \quad |Q| = -1,$$

$q = |K| = (-1)^s$, where s represents the number of S_i equal to Q. Consider the example:

$$D_{10}^{-5} = QPQPQ = \begin{pmatrix} 7 & 3 \\ 5 & 2 \end{pmatrix}.$$

There are five different cyclic products of the S_i: $j = 5, 9, 10, 18, 20$. These form the sequences

$$\{D_j^{-5n} A(2,\ 0) = X_n : n = 0,\ 1,\ 2,\ \ldots\}$$

having the recurrence relation

$$X_n = 9X_{n-1} + X_{n-2}$$

and satisfying $L(X_n) = 5n + 2$. These sequences are exhibited in Table 3.

TABLE 3. Related Sequences

j	D_j^{-5}	$\{X_n : n = 0, 1, 2, \ldots\}$
5	$\begin{pmatrix} 7 & 5 \\ 3 & 2 \end{pmatrix}$	$\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 19 \\ 8 \end{pmatrix}, \begin{pmatrix} 173 \\ 73 \end{pmatrix}, \ldots\right\}$
9	$\begin{pmatrix} 8 & 3 \\ 3 & 1 \end{pmatrix}$	$\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 19 \\ 7 \end{pmatrix}, \begin{pmatrix} 173 \\ 64 \end{pmatrix}, \ldots\right\}$
10	$\begin{pmatrix} 7 & 3 \\ 5 & 2 \end{pmatrix}$	$\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ 12 \end{pmatrix}, \begin{pmatrix} 155 \\ 109 \end{pmatrix}, \ldots\right\}$
18	$\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}$	$\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 15 \\ 11 \end{pmatrix}, \begin{pmatrix} 137 \\ 100 \end{pmatrix}, \ldots\right\}$
20	$\begin{pmatrix} 5 & 7 \\ 3 & 4 \end{pmatrix}$	$\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ 10 \end{pmatrix}, \begin{pmatrix} 155 \\ 91 \end{pmatrix}, \ldots\right\}$

REFERENCE

1. A. F. Horadam. "Basic Properties of a Certain Sequence of Numbers." *The Fibonacci Quarterly* 3, no. 3 (1965):161-76.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-592 *Proposed by Herta T. Freitag, Roanoke, VA*

Find all integers a and b, if any, such that $F_a L_b + F_{a-1}L_{b-1}$ is an integral multiple of 5.

B-593 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A(n) = F_{n+1}L_n + F_nL_{n+1}$. Prove that $A(15n - 8)$ is an integral multiple of 1220 for all positive integers n.

B-594 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$A(n) = F_{n+1}L_n + F_nL_{n+1} \quad \text{and} \quad B(n) = \sum_{j=1}^{n}\sum_{k=1}^{j} A(k).$$

Prove that $B(n) \equiv 0 \pmod{20}$ when $n \equiv 19$ or $29 \pmod{60}$.

B-595 *Proposed by Philip L. Mana, Albuquerque, NM*

Prove that

$$\sum_{k=0}^{n} k^3(n-k)^2 \equiv \binom{n+4}{6} + \binom{n+1}{6} \pmod{5}.$$

B-596 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let

$$S(n, k, m) = \sum_{i=1}^{m} F_{ni+k}.$$

For positive integers a, m, and k, find an expression of the form XY/Z for $S(4a, k, m)$, where X, Y, and Z are Fibonacci or Lucas numbers.

B-597 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Do as in Problem B-596 for $S(4a + 2, k, 2b)$ and for $S(4a + 2, k, 2b - 1)$, where a and b are positive integers.

SOLUTIONS

Fibonacci-Lucas Hyperbola for Odd n

B-568 *Proposed by Wray G. Brady, Slippery Rock University, Slippery Rock, PA*

Find a simple curve passing through all of the points

$$(F_1, L_1), (F_3, L_3), \ldots, (F_{2n+1}, L_{2n+1}), \ldots .$$

Solution by C. Georghiou, University of Patras, Greece

It is easy to show that the given points do not lie on a straight line. However,

$$L_{2n+1}/F_{2n+1} \to \sqrt{5} \text{ as } n \to \infty,$$

and it is also known that

$$5F_{2n+1}^2 - L_{2n+1}^2 = 4.$$

Therefore, the given points lie on a branch of the hyperbola with equation

$$5x^2 - y^2 = 4.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Herta T. Freitag, L. Kuipers, J. Z. Lee & J. S. Lee, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, Tad P. White, and the proposer.

Fibonacci-Lucas Hyperbola for Even n

B-569 *Proposed by Wray G. Brady, Slippery Rock University, Slippery Rock, PA*

Find a simple curve passing through all of the points

$$(F_0, L_0), (F_2, L_2), \ldots, (F_{2n}, L_{2n}), \ldots .$$

Solution by J. Z. Lee, Chinese Culture University, Taipei, Taiwan, R.O.C. & J. S. Lee, National Taipei Business College, Taipei, Taiwan, R.O.C.

A simple curve passing through all of the points (F_2, L_2), (F_4, L_4), ..., (F_{2n}, L_{2n}), ... is $y^2 - 5x^2 = 4$, since $L_n^2 - 5F_n^2 = 4(-1)^n$.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Herta T. Freitag, C. Georghiou, L. Kuipers, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, Tad P. White, and the proposer.

Fibonacci Squareroot Triangle with Fixed Area

B-570 *Proposed by Herta T. Freitag, Roanoke, VA*

Let a, b, and c be the positive square roots of F_{2n-1}, F_{2n+1}, and F_{2n+3}, respectively. For $n = 1, 2, \ldots$, show that

$$(a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 4.$$

Solution by L. A. G. Dresel, University of Reading, England

Let

$$P = (a + b + c)(-a + b + c)(a - b + c)(a + b - c).$$

Then, since

$$(a + b + c)(a - b + c) = (a + c)^2 - b^2,$$

and

$$(a + b - c)(-a + b + c) = b^2 - (a - c)^2,$$

we have

$$\begin{aligned} P &= (2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2) \\ &= 4a^2c^2 - (a^2 + c^2 - b^2)^2 \\ &= 4F_{2n-1}F_{2n+3} - (F_{2n-1} + F_{2n+3} - F_{2n+1})^2 \\ &= 4F_{2n-1}F_{2n+3} - 4F_{2n+1}^2, \end{aligned}$$

since

$$F_{2n+3} = 3F_{2n+1} - F_{2n-1}.$$

Using the Binet forms, we find that $F_{2n-1}F_{2n+3} = F_{2n+1}^2 + 1$; hence, $P = 4$.

We note in passing that Heron's formula gives the area of a triangle of sides a, b, c as $\frac{1}{4}\sqrt{P}$, and therefore the area of a triangle whose sides are the positive square roots of F_{2n-1}, F_{2n+1}, and F_{2n+3} will be $\frac{1}{2}$ for $n = 1, 2, 3, \ldots$.

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Weighted Rising Diagonal Sum

B-571 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Conjecture and prove a simple expression for

$$\sum_{r=0}^{[n/2]} \frac{n}{n-r}\binom{n-r}{r}$$

where $[n/2]$ is the largest integer m with $2m \leqslant n$.

Solution by Philip L. Mana, Albuquerque, NM

Let S be the given sum and $q = [n/2]$. Then

$$S = \sum_{r=0}^{q} \binom{n-r}{r} + \sum_{r=0}^{q} \frac{r}{n-r}\binom{n-r}{r} = F_{n+1} + \sum_{n=1}^{q} \binom{n-r-1}{r-1}$$

$$= F_{n+1} + F_{n-1} = L_n,$$

using the rising diagonal formula

$$\sum_{r=0}^{q} \binom{n-r}{r} = F_{n+1}.$$

Also solved by Paul S. Bruckman, Oroardo Brugia & Piero Filipponi, L. A. G. Dresel, Herta T. Freitag, C. Goerghiou, L. Kuipers, J. Z. Lee & J. S. Lee, F. S. Makri & D. Antzoulakos, Bob Prielipp, J. Suck, Tad P. White, and the proposer.

Continued Fraction

B-572 *Proposed by Ambati Jaya Krishna, Student, Johns Hopkins University, Baltimore, MD, and Gomathi S. Rao, Orangeburg, SC*

Evaluate the continued fraction:

$$1 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7 + \cdots}}}$$

Solution by C. Georghiou, University of Patras, Greece

This is the same as Problem H-394 in this *Quarterly* (Vol. 24, no. 1 [1986], p. 88] proposed by the same authors. Its solution is as follows:

From the theory of continued fractions, it is known that (See, for example, M. Abramowitz & A. Stegun, *Handbook of Mathematical Functions* [New York: Dover, 1970], p. 19):

$$g_n(x) = \frac{1}{a_0} - \frac{x}{a_0a_1} + \frac{x^2}{a_0a_1a_2} - \cdots + (-1)^n \frac{x^n}{a_0a_1a_2 \cdots a_n}$$

$$= \frac{1}{a_0} + \frac{a_0x}{a_1 - x} + \frac{a_1x}{a_2 - x} + \cdots + \frac{a_{n-1}x}{a_n - x}.$$

Take $a_n = 2n + 2$ and $x = 1$. Then, the n^{th} convergent of the given continued fraction, f_n, is given by

$$f_n = \frac{1}{g_n} - 2.$$

Since $\lim\limits_{n\to\infty} g_n = 1 - e^{-1/2}$, we get $f = (e^{1/2} - 1)^{-1}$.

Also solved by Paul S. Bruckman, L. Kuipers & Peter S. J. Shiue, J. Z. Lee & J. S. Lee, and the proposer.

B-573 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

For all nonnegative integers n, prove that

$$\sum_{k=0}^{n}\binom{n}{k}L_kL_{n-k} = 4 + 5\sum_{k=0}^{n}\binom{n}{k}F_kF_{n-k}.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall show that

$$S = \sum_{k=0}^{n}\binom{n}{k}(L_kL_{n-k} - 5F_kF_{n-k}) = 4,$$

which is equivalent to the required result.

$$\begin{aligned} L_kL_{n-k} - 5F_kF_{n-k} &= (\alpha^k + \beta^k)(\alpha^{n-k} + \beta^{n-k}) - (\alpha^k - \beta^k)(\alpha^{n-k} - \beta^{n-k}) \\ &= 2\alpha^k\beta^{n-k} + 2\beta^k\alpha^{n-k}. \end{aligned}$$

Thus,

$$\begin{aligned} S = 2\sum_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k} + 2\sum_{k=0}^{n}\binom{n}{k}\beta^k\alpha^{n-k} &= 4(\alpha + \beta)^n \quad \text{[by the Binomial Theorem]} \\ &= 4 \cdot 1^n = 4. \end{aligned}$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, Herta T. Freitag, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Sahib Singh, J. Suck, Tad P. White, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-409 *Proposed by John Turner, University of Waikato, New Zealand*

Fibonacci-T Arithmetic Triangles

The following arithmetic triangle has many properties of special interest to Fibonacci enthusiasts.

							1							
						1	1	1						
					1	2	2	2	1					
				1	3	4	5	4	3	1				
			1	4	7	10	11	10	7	4	1			
		1	5	11	18	24	26	24	18	11	5	1		
	1	6	16	30	46	58	63	58	46	30	16	6	1	
1	7	22	47	81	116	143	153	143	116	18	47	22	7	1

....

Denote the triangle by T, the i^{th} element in the n^{th} row by t_i^n, and the sum of elements in the n^{th} row by σ_n.

(i) Discover a rule to generate the next row from the previous rows.

(ii) Given your rule, prove the Fibonacci row-sum property, viz:

$$\sigma_n = 2\sum_{i=1}^{n-1} t_i^n + t_n^n = F_{2n}, \text{ for } n = 1, 2, \ldots,$$

where F_{2n} is a Fibonacci integer.

(iii) Discover and prove a remarkable functional property of the sequence of diagonal sequences, $\{d_i\}$:

$$\begin{array}{lllllll} d_1 & = 1 & 1 & 1 & 1 & 1 & \ldots \\ d_2 & = 1 & 2 & 3 & 4 & 5 & \ldots \\ d_3 & = 1 & 2 & 4 & 7 & 11 & \ldots \\ d_4 & = 2 & 5 & 10 & 18 & 30 & \ldots \\ d_5 & = 1 & 4 & 11 & 24 & 46 & \ldots \end{array}$$

(iv) Discover another Fibonacci arithmetic triangle which has the same generating rule and other properties but with row-sums equal to F_{2n-1}, $n = 1, 2, \ldots$.

(v) Show how the numbers in the triangle are related to the dual-Zeckendorf theorem on integer representations, which states (see [1]) that every positive integer N has one and only one representation in the form

$$N = \sum_{1}^{k} e_i u_i,$$

where the e_i are binary digits and $e_i + e_{i+1} \neq 0$ for $1 \leqslant i < k$, and $\{u_i\} = 1, 2, 3, 5, \ldots$, the Fibonacci integers.

There are many interesting identities derivable from the triangle relating the t_i^n with themselves, with the natural numbers and Fibonacci integers, and with the binomial coefficients. The proposer offers a prize of US $25 for the best list of identities submitted.

A final remark is that Pascal-T (see [2] and [3]) and Fibonacci-T triangles can curiously be linked to a common source. They both may be derived from studies of binary words whose digits have the properties of the e_i in part (v) above.

References

1. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *The Fibonacci Quarterly* 3, no. 1 (1965):1-8.
2. S. J. Turner. "Probability via the Nth Order Fibonacci-T Sequence." *The Fibonacci Quarterly* 17, no. 1 (1979):23-28.
3. J. C. Turner. "Convolution Trees and Pascal-T Triangles." (Submitted to *The Fibonacci Quarterly*, 1986.)

H-410 *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ \text{for } n \geqslant 2.$$

Prove or disprove that, for $n \geqslant 1$,

$$\int_0^1 F_n(x)dx = \frac{1}{n}(L_n - (-1)^n - 1).$$

H-411 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Define the simple continued fraction $\theta(a, d)$ as follows:

$$\theta(a, d) \equiv [a, a + d, a + 2d, a + 3d, \ldots],\ a \text{ and } d \text{ real},\ d \neq 0. \quad (1)$$

Find a closed form for $\theta(a, d)$.

SOLUTIONS

Acknowledgment Correction:

H-377, H-379, and H-382 were solved by S. Papastavridis, P. Siafarikas, and P. Sypsas; H-381 was not solved by P. Siafarikas or P. Sypsas.

A Complex Problem

H-386 *Proposed by Paul S. Bruckman, Fair Oaks, CA*
(Vol. 23, no. 2, May 1985)

Define the multiple-valued *Fibonacci function* ${}^mF : \mathbb{C} \to \mathbb{C}$ as follows:

1. $${}^mF(z) = \frac{1}{\sqrt{5}}(\exp Lz - \exp L'z),\ z \in \mathbb{C},\ m \in \mathbb{Z},$$

where $L = \log \alpha$, $\alpha = \frac{1}{2}(1 + \sqrt{5})$, $L' = (2m + 1)i\pi - L$, and "log" denotes the principal logarithm.

a. Show that ${}^mF(n) = F_n$ for all integers m and n.

b. Prove the multiplication formula

2. $$\prod_{m=0}^{n-1} {}^mF\left(k + \frac{r}{n}\right) = 5^{-\frac{1}{2}(n-1)}F_{nk+r},$$ where n, k, r are integers with $0 < r < n$.

c. With m fixed, find the zeros of ${}^mF$.

Solution by the propeser

Proof of (a):
$$\begin{aligned}{}^mF(n) &= 5^{-\frac{1}{2}}\{\exp(nL) - \exp[(2m+1)ni\pi - nL]\}\\ &= 5^{-\frac{1}{2}}(\alpha^n - (-1)^{(2m+1)n}\alpha^{-n})\\ &= 5^{-\frac{1}{2}}(\alpha^n - (-\alpha)^{-n}) = 5^{-\frac{1}{2}}(\alpha^n - \beta^n),\end{aligned}$$
where $\beta = \frac{1}{2}(1 - \sqrt{5})$; hence, ${}^mF(n) = F_n$. Q.E.D.

Proof of (b): Let $\omega = \exp(i\pi r/n)$. Then

$$\begin{aligned}\prod_{m=0}^{n-1} {}^mF(k + r/n) &= \prod_{m=0}^{n-1} 5^{-\frac{1}{2}}\{\exp(k + r/n)L\\ &\qquad - \exp[(2m+1)(k + r/n)i\pi - (k + r/n)L]\}\\ &= 5^{-\frac{1}{2}n}\prod_{m=0}^{n-1}\{\alpha^{k+r/n} - (-1)^k\omega^{2m+1}\alpha^{-k-r/n}\}\\ &= 5^{-\frac{1}{2}n}\alpha^{nk+r}\prod_{m=0}^{n-1}\{1 - (-1)^k\omega^{2m+1}\alpha^{-2k-2r/n}\}.\end{aligned}$$

Since the solutions of the equation: $z^n = (-1)^r$ are given by ω, ω^3, ω^5, $\ldots$, ω^{2n-1}, it follows that, for all x,

$$(1 - x\omega)(1 - x\omega^3) \cdots (1 - x\omega^{2n-1}) = 1 - (x\omega)^n = 1 - (-1)^r x^n.$$

Therefore, setting $x = (-1)^k\alpha^{-2k-2r/n}$, we see that

$$\begin{aligned}\prod_{m=0}^{n-1} {}^mF(k + r/n) &= 5^{-\frac{1}{2}n}\alpha^{nk+r}\{1 - (-1)^{nk+r}\alpha^{-2nk-2r}\}\\ &= 5^{-\frac{1}{2}n}(\alpha^{nk+r} - \beta^{nk+r}) = 5^{-\frac{1}{2}(n-1)}F_{nk+r}.\quad \text{Q.E.D.}\end{aligned}$$

Note that setting $r = 0$ in (2) yields F_k^n [using (a)].

Solution of (c): ${}^mF(z) = (2/\sqrt{5})\exp(m + \tfrac{1}{2})i\pi z \quad \sinh z\theta_m$, where

$$\theta_m = \tfrac{1}{2}(L - L') = \log\alpha - (m + \tfrac{1}{2})i\pi.$$

Since $\exp uz$ vanishes for *no* complex u and z, the zeros of ${}^mF$ are precisely the zeros of $\sinh z\theta_m$, namely, ${}^mF(z_{r,m}) = 0$, where

$$z_{r,m} = ri\pi/\theta_m = r \cdot z_{1,m} = \frac{-r(m + \tfrac{1}{2})\pi^2 + \pi rLi}{L^2 + (m + \tfrac{1}{2})^2\pi^2}. \quad \text{Q.E.D.}$$

NOTE: Given m, ${}^mF$ is one of the Riemann sheets which extend the Fibonacci numbers to the complex domain.

Also solved by L. Kuipers.

Non Residual

H-387 *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 23, no. 2, May 1985)

Let $\{w_n\}_{n=0}^{\infty}$ be a second-order linear integral recurrence defined by the recursion relation

$$w_{n+2} = aw_{n+1} + bw_n,$$

where $b \neq 0$. Show the following:

(i) If p is an odd prime such that $p \nmid b$ and $w_1^2 - w_0w_2$ is a quadratic nonresidue of p, then

$$p \nmid w_{2n} \quad \text{for any } n \geqslant 0.$$

(ii) If p is an odd prime such that $(-b)(w_1^2 - w_0w_2)$ is a quadratic nonresidue of p, then

$$p \nmid w_{2n+1} \quad \text{for any } n \geqslant 0.$$

(iii) If p is an odd prime such that $-b$ is a nonzero quadratic residue of p and $w_1^2 - w_0w_2$ is a quadratic nonresidue of p, then

$$p \nmid w_n \quad \text{for any } n \geqslant 0.$$

Solution by the proposer

We first note that

$$w_n^2 - w_{n-1}w_{n+1} = (-b)^{n-1}(w_1^2 - w_0w_2) \tag{1}$$

for $n \geqslant 1$. This identity can be proven by induction using the recursion relation defining $\{w_n\}$. We now prove parts (i), (ii), and (iii).

(i) Suppose $p|w_{2n}$ for some $n \geqslant 0$. Then by (1),

$$w_{2n+1}^2 - w_{2n}w_{2n+1} \equiv w_{2n+1}^2 - 0 \equiv (-b)^{2n}(w_1^2 - w_0w_2) \pmod{p}.$$

However, this is contradicted by the fact that $w_1^2 - w_0w_2$ is a quadratic nonresidue of p and $-b$ is a nonzero residue of p. The result follows.

(ii) Suppose $p|w_{2n+1}$ for some $n \geqslant 0$. Then by (1),

$$\begin{aligned} w_{2n+1}^2 - w_{2n+1}w_{2n+3} &\equiv w_{2n+2}^2 - 0 \equiv (-b)^{2n+1}(w_1^2 - w_0w_2) \\ &\equiv (-b)^{2n}[(-b)(w_1^2 - w_0w_2)] \pmod{p}. \end{aligned}$$

This is a contradiction, since $(-b)(w_1^2 - w_0w_2)$ is a quadratic nonresidue of p and the product of a nonzero quadratic residue and a quadratic nonresidue is a quadratic nonresidue. Hence, assertion (ii) must hold.

(iii) This follows immediately from parts (i) and (ii). First, by (i), p cannot divide w_{2n} for any $n \geqslant 0$, since $p \nmid b$ and $w_1^2 - w_0w_2$ is a quadratic nonresidue of p. Also, by (ii), p cannot divide w_{2n+1} for any $n \geqslant 0$, since $(-b)(w_1^2 - w_0w_2)$ is a quadratic nonresidue of p. This again follows, because the product of a nonzero quadratic residue and a quadratic nonresidue is a quadratic nonresidue. Thus, $p \nmid w_n$ for any $n \geqslant 0$, and we are done.

Also solved by P. Bruckman, L. Kuipers, and T. White.

Across the Digraph!

H-388 *Proposed by Piero Filipponi, Rome, Italy*
(Vol. 23, no. 2, May 1985)

This problem arose in the determination of the diameter of a class of locally restricted digraphs [1].

For a given integer $n \geqslant 2$, let $P_1 = \{p_{1,1}, p_{1,2}, \ldots, p_{1,k_1}\}$ be a nonempty (i.e., $k_1 \geqslant 1$) increasing sequence of positive integers such that $p_{1,k_1} \leqslant n - 1$. Let $P_2 = \{p_{2,1}, p_{2,2}, \ldots, p_{2,k_2}\}$ be the increasing sequence containing all nonzero distinct values given by $p_{1,i} + p_{1,j} \pmod{n}$ $(i, j = 1, 2, \ldots, k_1)$. In general let $P_h = \{p_{h,1}, p_{h,2}, \ldots, p_{h,k_h}\}$ be the increasing sequence containing all nonzero distinct values given by $p_{h-1,i} + p_{1,j} \pmod{n}$ $(i = 1, 2, \ldots, k_{h-1}, j = 1, 2, \ldots, k_1)$. Furthermore, let B_m $(m = 1, 2, \ldots)$ be the increasing sequence containing all values given by

$$\bigcup_{j=1}^{m} P_j.$$

Find, in terms of $n, p_{1,1}, \ldots, p_{1,k_1}$, the smallest integer t such that

$$B_t = \{1, 2, \ldots, n - 1\}.$$

Remark: The necessary and sufficient condition for t to exist (i.e., to be finite) is given in [1]:

$$\gcd(n, p_{1,1}, \ldots, p_{1,k_1}) = 1.$$

In such a case we have $1 \leqslant t \leqslant n - 1$. It is easily seen that

$$k_1 = 1 \Longleftrightarrow t = n - 1$$

$$k_1 = n - 1 \Longleftrightarrow t = 1;$$

furthermore, it can be conjectured that either $t = n - 1$ or $1 \leqslant t \leqslant [n/2]$.

Reference

1. P. Filipponi. "Digraphs and Circulant Matrices." *Ricerca Operativa*, no. 17 (1981):41-62.

An Example

$$\begin{aligned}
n = 8 \quad & P_1 = \{3, 5\} && \to B_1 = \{3, 5\} \\
& P_2 = \{2, 6\} && \to B_2 = \{2, 3, 5, 6\} \\
& P_3 = \{1, 3, 5, 7\} && \to B_3 = \{1, 2, 3, 5, 6, 7\} \\
& P_4 = \{2, 4, 6\} && \to B_4 = \{1, 2, 3, 4, 5, 6, 7\};
\end{aligned}$$

hence, we have $t = 4$.

Comments (not a solution) by Paul S. Bruckman, Fair Oaks, CA

The proposer's conjecture may be refined to the following conjecture:

$$1 \leqslant t \leqslant \left[\frac{n-2}{k_1}\right] + 1. \tag{1}$$

This *seems* to be true, but a more exact expression eluded me, as did the proof of (1). I first conjectured that $t = t(n, k_1)$ was a function solely of n and k_1, given by:

$$t = \left[\frac{n-2}{k_1}\right] + 1. \tag{2}$$

Unfortunately, (2) is false; the first counter-example occurs with $n = 8$, $k_1 = 2$. If we take $P_1 = (5, 7)$, then $P_2 = (2, 4, 6)$, $P_3 = (1, 3, 5, 7)$, so $t = 3$ in this case. On the other hand, if $P_1 = (1, 2)$, then $P_2 = (2, 3, 4)$, $P_3 = (3, 4, 5, 6)$, $P_4 = (4, 5, 6, 7)$, so $t = 4$ in this case.

Thus t, in general, depends on P_1 as well as on n and k_1. It is conceivable, however, that (2) would hold, provided some additional constraints on P_1 are specified. Note that the expression given in (2) produces the correct values of t (for $n \geqslant 2$) if $k_1 = 1$ or $k_1 = n - 1$. It seems likely that the problem is more difficult than the proposer originally intended, at least in its general form, and that the true formula for $t = t(n, k_1, P_1)$ is more complicated than some concise expression such as indicated in (2).

Waiting for Success

H-389 *Proposed by Andreas N. Philippou, University of Patras, Greece (Vol. 23, no. 3, August 1985)*

Show that

$$F_{n+2}^{(n-i)} = 2^n - 2^i(1 + i/2) \qquad (n \geqslant 2i + 1)$$

for each nonnegative integer i, where $F_{n+2}^{(n-i)}$ is the $n + 2$ Fibonacci number of order $n - i$ [1] and $F_3^{(1)} = 1$.

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order k." *The Fibonacci Quarterly* 20, no. 1 (1982):28-32.

Solution by S. Papastavridis, P. Siafarikas, & P. Sypsas, U. of Patras, Greece

Setting $(n - i) = k$ and $(n + 2) = m$, the problem becomes

$$F_m^{(k)} = 2^{m-2} - 2^{m-k-2}\left(1 + \frac{m-2-k}{2}\right)$$

or

$$F_m^{(k)} = 2^{m-2} - 2^{m-k-3}(m - k) \tag{1}$$

for $k + 2 \leqslant m \leqslant 2k + 2$.

We shall prove (1) for $k + 2 \leqslant m \leqslant 2k + 2$. From here on, we suppress (k), since it is the same throughout. So we write F_m instead of $F_m^{(k)}$.

In the paper of Philippou and Muwafi ([1], p. 29, Lemma 2.1), it is proved that the sequence F_m satisfies the following recursion:

$$F_m = \begin{cases} 0 & \text{if } m = 0 \\ 1 & \text{if } m = 1,\ 2 \\ 2F_{m-1} & \text{if } k + 1m \geqslant 3 \\ 2F_{m-1} - F_{m-k-1} & \text{if } m \geqslant k + 2 \end{cases}$$

This clearly implies that the generating function $\sum_{m=0}^{\infty} F_m t^m$ equals

$$F(t) = \sum_{m=0}^{\infty} F_m t^m = \frac{t - t^2}{1 - 2t + t^{k+1}}$$

We are going to expand this generating function. Binomial expansion is all we need. Thus, we have:

$$F(t) = \frac{t - t^2}{1 - 2t + t^{k+1}} = \frac{t - t^2}{1 - t(2 - t^k)} \doteq (t - t^2) \sum_{i=0}^{\infty} t^i (2 - t^k)^i$$

$$= (t - t^2) \sum_{i=0}^{\infty} \sum_{j=0}^{i} (-1)^j \binom{i}{j} 2^{i-j} t^{kj+i}$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{i} (-1)^j \binom{i}{j} 2^{i-j} t^{kj+i+1} - \sum_{i=0}^{\infty} \sum_{j=0}^{i} (-1)^j \binom{i}{j} 2^{i-j} t^{kj+i+2}$$

(in the first summation we set $m = kj + i + 1$, and in the second summation we set $m = kj + i + 2$)

$$= \sum_{m=1}^{\infty} \left(\sum_{j=0}^{(m-1)/(k+1)} (-1)^j \binom{m - 1 - kj}{j} 2^{m-1-(k+1)j} \right) t^m$$

$$= \sum_{m=2}^{\infty} \left(\sum_{j=0}^{(m-2)/(k+1)} (-1)^j \binom{m - 2 - kj}{j} 2^{m-2-(k+1)j} \right) t^m.$$

Thus, since F_m is the coefficient of t^m in the expansion of $F(t)$, we get:

$$F_m = \sum_{j=0}^{(m-1)/(k+1)} (-1)^j \binom{m - 1 - kj}{j} 2^{m-(k+1)j-1} - \sum_{j=0}^{(m-2)/(k+1)} (-1)^j \binom{m - 2 - kj}{j} 2^{m-(k+1)j-2}, \text{ for } m \geqslant 2. \tag{2}$$

Formula (2) is a general closed expression of F_m. Let us look at the special case that we have with the conditions

$(m - 1)/(k + 1) < 2$ and $(m - 2)/(k + 1) \geqslant 1$,

which is equivalent to

$k + 3 \leqslant m \leqslant 2k + 2$.

In this case, the index j in the summations of (2) takes only the values $j = 0$ and $j = 1$. So, we obtain (for this case)

$$F_m = 2^{m-1} - (m - 1 - k)2^{m-k-2} - 2^{m-2} + (m - k - 2)2^{m-k-3}$$

$$= 2^{m-2} - 2^{m-k-2}\left(\frac{m - k}{2}\right)$$

$$= 2^{m-2} - 2^{m-k-3}(m - k),$$

which is exactly what we had to prove. The remaining case of $k + 2 = m$ is deduced similarly. The case of $F_3^{(1)}$ is obvious.

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order k." *The Fibonacci Quarterly* 20, no. 1 (1982):28-32.

Also solved by P. Bruckman, B. Poonen, and the proposer.

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Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

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