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The Fibonacci Quarterly

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*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

EXPONENTIAL GENERATING FUNCTIONS FOR PELL POLYNOMIALS

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1. INTRODUCTION

Following our description [6] of the properties of the ordinary generating functions of Pell polynomials $P_n(x)$ and Pell-Lucas polynomials $Q_n(x)$ [3], we offer here a compact exposition of similar properties of the exponential generating functions of these polynomials.

Earlier authors have written about the exponential generating functions of the Fibonacci numbers [2] and of generalized Fibonacci numbers [7].

Details of the main properties of the Pell-type polynomials may be found in [3] and [4], and will be assumed, where necessary. For visual simplicity, we will abbreviate the functional notation thus: $P_n(x) \equiv P_n$, $Q_n(x) \equiv Q_n$.

Binet forms of P_n and Q_n are

$$P_n = (\alpha^n - \beta^n)/(\alpha - \beta) \tag{1.1}$$

and

$$Q_n = \alpha^n + \beta^n, \tag{1.2}$$

where

$$\begin{cases} \alpha = x + \sqrt{x^2 + 1} \\ \beta = x - \sqrt{x^2 + 1} \end{cases} \tag{1.3}$$

$$(\text{so } \alpha + \beta = 2x,\ \alpha\beta = -1,\ \alpha - \beta = 2\sqrt{x^2 + 1})$$

are the roots of

$$\lambda^2 - 2x\lambda - 1 = 0. \tag{1.4}$$

Some symbolism we shall employ include:

$$\nabla = (1 - 2xz - z^2)^{-1} \quad (= \Delta \text{ in [6] with } y \text{ replaced by } z) \tag{1.5}$$

$$\nabla_{(m)} = (1 - Q_m z + (-1)^m z^2)^{-1}, \text{ i.e., } \nabla_{(1)} \equiv \nabla \tag{1.6}$$

$$\nabla' = (1 + 2xz - z^2)^{-1}, \text{ i.e., replace } z \text{ by } -z \text{ in (1.5)} \tag{1.7}$$

$$\nabla^{(2)} \equiv \Delta^{(2)} \text{ in [6] with } y \text{ replaced by } z \tag{1.8}$$

$$P = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \tag{1.9}$$

$$P^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} \tag{1.10}$$

Usage of the matrix P (1.9) is to be found, for example, in [3], [4], [5], and [6]. Inevitably, some of the simpler results for Pell-type polynomials in the ensuing pages may have been obtained by other methods in our papers listed as references.

2. BASIC MATERIAL

Write

$$P(x, y, 0) = \frac{e^{\alpha y} - e^{\beta y}}{\alpha - \beta} = \sum_{r=0}^{\infty} \frac{P_r y^r}{r!} \tag{2.1}$$

and

$$Q(x, y, 0) = e^{\alpha y} + e^{\beta y} = \sum_{r=0}^{\infty} \frac{Q_r y^r}{r!}. \tag{2.2}$$

Both (2.1) and (2.2) satisfy

$$\frac{\partial^2 t}{\partial y^2} - 2x\frac{\partial t}{\partial y} - t = 0. \tag{2.3}$$

From (2.1)

$$P(x, y, k) = \frac{\partial^k}{\partial y^k} P(x, y, 0) = \sum_{r=0}^{\infty} \frac{P_{r+k} y^r}{r!}, \tag{2.4}$$

whence

$$P(x, y, n + 1) - 2xP(x, y, n) - P(x, y, n - 1) = 0. \tag{2.5}$$

Also

$$Q(x, y, k) = \frac{\partial^k}{\partial y^k} Q(x, y, 0) = \sum_{r=0}^{\infty} \frac{Q_{r+k} y^r}{r!}, \tag{2.6}$$

whence

$$Q(x, y, n + 1) - 2xQ(x, y, n) - Q(x, y, n - 1) = 0. \tag{2.7}$$

Formulas (2.5) and (2.7) suggest the matrix representations:

$$\begin{bmatrix} P(x, y, n) \\ P(x, y, n - 1) \end{bmatrix} = P^{n-1} \begin{bmatrix} P(x, y, 1) \\ P(x, y, 0) \end{bmatrix} \tag{2.8}$$

$$\begin{bmatrix} Q(x, y, n) \\ Q(x, y, n - 1) \end{bmatrix} = P^{n-1} \begin{bmatrix} Q(x, y, 1) \\ Q(x, y, 0) \end{bmatrix} \tag{2.9}$$

$$P(x, y, n) = [1 \quad 0]P^{n-1} \begin{bmatrix} P(x, y, 1) \\ P(x, y, 0) \end{bmatrix} \tag{2.10}$$

$$Q(x, y, n) = [1 \quad 0]P^{n-1} \begin{bmatrix} Q(x, y, 1) \\ Q(x, y, 0) \end{bmatrix} \tag{2.11}$$

3. PROPERTIES OF EXPONENTIAL GENERATING FUNCTIONS

First, from (2.4) and (2.1) or by matrices,

$$\begin{aligned} P(x, y, n + 1) + P(x, y, n - 1) &= \frac{\alpha^{n+1}e^{\alpha y} - \beta^{n+1}e^{\beta y} + \alpha^{n-1}e^{\alpha y} - \beta^{n-1}e^{\beta y}}{\alpha - \beta} \\ &= \alpha^n e^{\alpha y} + \beta^n e^{\beta y} \qquad (3.1) \\ &= Q(x, y, n) \quad \text{by (2.6)} \end{aligned}$$

while, similarly,

$$Q(x, y, n+1) + Q(x, y, n-1) = 4(x^2+1)P(x, y, n). \tag{3.2}$$

Generalizations, with variations, of (3.1) and (3.2) are:

$$P(x, y, n+r) + (-1)^r P(x, y, n-r) = Q_r P(x, y, n) \tag{3.3}$$

$$P(x, y, n+r) - (-1)^r P(x, y, n-r) = P_r Q(x, y, n) \tag{3.4}$$

$$Q(x, y, n+r) + (-1)^r Q(x, y, n-r) = Q_r Q(x, y, n) \tag{3.5}$$

$$Q(x, y, n+r) - (-1)^r Q(x, y, n-r) = 4(x^2+1)P_r P(x, y, n) \tag{3.6}$$

An elementary property is, by (2.1), (2.6), and (2.4),

$$P(x, y, n)Q(x, y, n) = P(x, 2y, 2n)/2^n. \tag{3.7}$$

Combining (3.3) and (3.4) with (3.7), we arrive at:

$$P^2(x, y, n+r) - P^2(x, y, n-r) = P_{2r}P(x, 2y, 2n)/2^n \tag{3.8}$$

$$Q^2(x, y, n+r) - Q^2(x, y, n-r) = 4(x^2+1)P_{2r}P(x, 2y, 2n)/2^n \tag{3.9}$$

For variety, we use matrices to demonstrate the *Simson formula* (3.10) for $P(x, y, n)$. Details are:

$$P(x, y, n+1)P(x, y, n-1) - P^2(x, y, n) \tag{3.10}$$

$$= \begin{vmatrix} P(x, y, n+1) & P(x, y, n) \\ P(x, y, n) & P(x, y, n-1) \end{vmatrix}$$

$$= \left| P^n \begin{bmatrix} P(x, y, 1) \\ P(x, y, 0) \end{bmatrix} \,\vdots\, P^{n-1} \begin{bmatrix} P(x, y, 1) \\ P(x, y, 0) \end{bmatrix} \right| \qquad \text{by } (2.8)$$

$$= (-1)^{n-1} \begin{vmatrix} P(x, y, 2) & P(x, y, 1) \\ P(x, y, 1) & P(x, y, 0) \end{vmatrix} \qquad \text{by } (2.8) \quad [|P^{n-1}| = (-1)^{n-1}]$$

$$= (-1)^{n-1}\{(\alpha^2 e^{\alpha y} - \beta^2 e^{\beta y})(e^{\alpha y} - e^{\beta y}) - (\alpha e^{\alpha y} - \beta e^{\beta y})^2\}/(\alpha - \beta)^2 \qquad \text{by (2.1) and (2.4)}$$

$$= (-1)^{n-1}\{-(\alpha^2 + \beta^2 - 2\alpha\beta)e^{(\alpha+\beta)y}\}/(\alpha - \beta)^2$$

$$= (-1)^n e^{2xy} \qquad \text{by } (1.3)$$

Likewise,

$$Q(x, y, n+1)Q(x, y, n-1) - Q^2(x, y, n)$$
$$= (-1)^{n-1}4(x^2+1)e^{2xy}. \tag{3.11}$$

The clear similarity of the results in this section with the corresponding formulas for P_n and Q_n is noticeable.

Obviously, the number of relationships involving exponential generating functions themselves alone is extensive. Three such are, for example:

$$P(x, y, n)P(x, y, r+1) + P(x, y, n-1)P(x, y, r)$$
$$= P(x, 2y, n+r)/2^{n+r}; \tag{3.12}$$

$$Q(x, y, n)Q(x, y, r+1) + Q(x, y, n-1)Q(x, y, r)$$
$$= 4(x^2+1)P(x, 2y, n+r)/2^{n+r}; \tag{3.13}$$

and

$$P(x, y, n)Q(x, y, r + 1) + P(x, y, n - 1)Q(x, y, r) = Q(x, 2y, n + r)/2^{n+r}. \tag{3.14}$$

Put $r = n - 1$ in (3.12) and (3.13) to get, in succession,

$$P^2(x, y, n) + P^2(x, y, n - 1) = P(x, 2y, 2n - 1)/2^{2n-1} \tag{3.15}$$

and

$$Q^2(x, y, n) + Q^2(x, y, n - 1) = 4(x^2 + 1)P(x, 2y, 2n - 1)/2^{2n-1}. \tag{3.16}$$

Finally,

$$P(x, y, m)Q(x, y, n) + P(x, y, n)Q(x, y, m) = P(x, 2y, m + n)/2^{m+n-1} \tag{3.17}$$

and

$$Q(x, y, m)Q(x, y, n) + 4(x^2 + 1)P(x, y, m)P(x, y, n) = Q(x, 2y, m + n)/2^{m+n-1} \tag{3.18}$$

Reverting now to the formulas relating exponential generating functions to Pell polynomials, we may establish, either by means of the definitions or by the matrix representations, the following:

$$P(x, y, n + r) = P_r P(x, y, n + 1) + P_{r-1}P(x, y, n) \tag{3.19}$$

$$\begin{aligned} Q(x, y, n + r) &= P_r Q(x, y, n + 1) + P_{r-1}Q(x, y, n) \\ &= Q_r P(x, y, n + 1) + Q_{r-1}P(x, y, n) \end{aligned} \tag{3.20}$$

$$4(x^2 + 1)P(x, y, n + r) = Q_r Q(x, y, n + 1) + Q_{r-1}Q(x, y, n) \tag{3.21}$$

Special cases of interest occur when $r = n$ in (3.19)-(3.21).
Also,

$$P(x, y, n + r) = \tfrac{1}{2}\{P_r Q(x, y, n) + Q_r P(x, y, n)\}, \tag{3.22}$$

$$Q(x, y, n + r) = \tfrac{1}{2}\{4(x^2 + 1)P_r P(x, y, n) + Q_r Q(x, y, n)\}, \tag{3.23}$$

$$P(x, y, n + r)P(x, y, n - r) - P^2(x, y, n) = (-1)^{n-r+1}P_r^2 e^{2xy}, \tag{3.24}$$

$$Q(x, y, n + r)Q(x, y, n - r) - Q^2(x, y, n) = (-1)^{n-r}4(x^2 + 1)P_r^2 e^{2xy}. \tag{3.25}$$

Results (3.24) and (3.25) are the *generalized Simson formulas*.
Lastly, in this section,

$$P(x, y, n)P(x, y, n + r + 1) - P(x, y, n - s)P(x, y, n + r + s + 1) = (-1)^{n-s}P_{r+s+1}P_s e^{2xy}, \tag{3.26}$$

and

$$Q(x, y, n)Q(x, y, n + r + 1) - Q(x, y, n - s)Q(x, y, n + r + s + 1) = (-1)^{n-s+1}4(x^2 + 1)P_{r+s+1}P_s e^{2xy}. \tag{3.27}$$

4. SERIES INVOLVING EXPONENTIAL GENERATING FUNCTIONS

Rearranging (2.5) and (2.7), and adding, we find

$$\sum_{r=1}^{n} P(x, y, r) = \{P(x, y, n + 1) + P(x, y, n) - P(x, y, 1) - P(x, y, 0)\}/2x \tag{4.1}$$

and

$$\sum_{r=1}^{n} Q(x, y, r) = \{Q(x, y, n + 1) + Q(x, y, n) - Q(x, y, 1) - Q(x, y, 0)\}/2x. \tag{4.2}$$

Binet forms give us the difference equations,

$$P(x,\ y,\ m(r+1)+k) - Q_m P(x,\ y,\ mr+k) + (-1)^m P(x,\ y,\ m(r-1)+k) = 0 \tag{4.3}$$

and

$$Q(x,\ y,\ m(r+1)+k) - Q_m Q(x,\ y,\ mr+k) + (-1)^m Q(x,\ y,\ m(r-1)+k) = 0. \tag{4.4}$$

Using (4.3) and (4.4), we may derive

$$\sum_{r=1}^{n} P(x,\ y,\ mr+k) \tag{4.5}$$

$$= \frac{P(x,\ y,\ m(n+1)+k) - P(x,\ y,\ m+k) - (-1)^m\{P(x,\ y,\ mn+k) - P(x,\ y,\ k)\}}{Q_m - 1 - (-1)^m}$$

and

$$\sum_{r=1}^{n} Q(x,\ y,\ mr+k) \tag{4.6}$$

$$= \frac{Q(x,\ y,\ m(n+1)+k) - Q(x,\ y,\ m+k) - (-1)^m\{Q(x,\ y,\ mn+k) - Q(x,\ y,\ k)\}}{Q_m - 1 - (-1)^m}.$$

Next, (2.8) and (3.19) used in conjunction with the matrix property

$$P^2 = 2xP + I$$

yield

$$P^{2n}\begin{bmatrix} P(x,\ y,\ 1) \\ P(x,\ y,\ 0) \end{bmatrix} = (2xP + I)^n \begin{bmatrix} P(x,\ y,\ 1) \\ P(x,\ y,\ 0) \end{bmatrix}. \tag{4.7}$$

Equating corresponding elements, we obtain

$$P(x,\ y,\ 2n) = \sum_{r=0}^{n} \binom{n}{r}(2x)^r P(x,\ y,\ r) \tag{4.8}$$

and

$$P(x,\ y,\ 2n+1) = \sum_{r=0}^{n} \binom{n}{r}(2x)^r P(x,\ y,\ r+1). \tag{4.9}$$

Similarly,

$$Q(x,\ y,\ 2n) = \sum_{r=0}^{n} \binom{n}{r}(2x)^r Q(x,\ y,\ r) \tag{4.10}$$

and

$$Q(x,\ y,\ 2n+1) = \sum_{r=0}^{n} \binom{n}{r}(2x)^r Q(x,\ y,\ r+1). \tag{4.11}$$

Extensions of (4.10) and (4.11) to $P(x,\ y,\ 2n+j)$ and $Q(x,\ y,\ 2n+j)$ readily follow.

Now let us consider a variation of the type of sequence being summed. Applying the Simson formula (3.10), simplifying, and summing, we derive

$$\sum_{r=1}^{n} \frac{(-1)^{r-1}}{P(x,\ y,\ r)P(x,\ y,\ r+1)} = \frac{1}{e^{2xy}}\left\{\frac{P(x,\ y,\ n)}{P(x,\ y,\ n+1)} - \frac{P(x,\ y,\ 0)}{P(x,\ y,\ 1)}\right\}. \tag{4.12}$$

Similarly,

$$\sum_{r=1}^{n} \frac{(-1)^r}{Q(x, y, r)Q(x, y, r+1)}$$

$$= \frac{1}{e^{2xy}}\left\{\frac{Q(x, y, n)}{Q(x, y, n+1)} - \frac{Q(x, y, 0)}{Q(x, y, 1)}\right\}\frac{1}{4(x^2+1)}. \tag{4.13}$$

5. ORDINARY GENERATING FUNCTIONS OF EXPONENTIAL GENERATING FUNCTIONS

Summing and using (2.5),

$$\sum_{r=0}^{\infty} P(x, y, r)z^r = (P(x, y, 0) + P(x, y, -1)z)\nabla \tag{5.1}$$

where $P(x, y, -1)$ is the *primitive function* of $P(x, y, 0)$ w.r.t. y.

Similarly,

$$\sum_{r=0}^{\infty} Q(x, y, r)z^r = (Q(x, y, 0) + Q(x, y, -1)z)\nabla, \tag{5.2}$$

$$\sum_{r=0}^{\infty} (-1)\ P(x, y, r)z^r = (P(x, y, 0) - P(x, y, -1)z)\nabla', \tag{5.3}$$

and

$$\sum_{r=0}^{\infty} (-1)^r Q(x, y, r)z^r = (Q(x, y, 0) - Q(x, y, -1)z)\nabla'. \tag{5.4}$$

More generally,

$$\sum_{r=0}^{\infty} P(x, y, mr+k)z^r = \{P(x, y, k) - (-1)^m P(x, y, -m+k)z\}\nabla_{(m)}, \tag{5.5}$$

and

$$\sum_{r=0}^{\infty} Q(x, y, mr+k)z^r = \{Q(x, y, k) - (-1)^m Q(x, y, -m+k)z\}\nabla_{(m)}. \tag{5.6}$$

Induction gives

$$\frac{\partial^n}{\partial z^n}\sum_{r=0}^{\infty} P(x, y, r)z^r = n!\left\{\sum_{r=0}^{n+1}\binom{n+1}{r}P(x, y, n-r)z^r\right\}\nabla^{n+1} \tag{5.7}$$

and

$$\frac{\partial^n}{\partial z^n}\sum_{r=0}^{\infty} Q(x, y, r)z^r = n!\left\{\sum_{r=0}^{n+1}\binom{n+1}{r}Q(x, y, n-r)z^r\right\}\nabla^{n+1} \tag{5.8}$$

with extensions when r is replaced by $r+m$.

Equating coefficients of z^r in (5.7) and (5.8) yields, in turn,

$$P(x, y, n+r) = \left\{\sum_{i=0}^{n+1}\binom{n+1}{i}P(x, y, n-i)P_{r+1-i}^{(n)}\right\}\Big/\binom{n+r}{r} \tag{5.9}$$

and

$$Q(x, y, n+r) = \left\{\sum_{i=0}^{n+1}\binom{n+1}{i}Q(x, y, n-i)P_{r+1-i}^{(n)}\right\}\Big/\binom{n+r}{r}, \tag{5.10}$$

since

$$\nabla^{n+1} = \sum_{t=0}^{\infty} P_{t+1}^{(n)}z^t,$$

where $\{P_i^{(n)}\}$, $i = 1, 2, 3, \ldots$ is the n^{th} *convolution sequence* for Pell polynomials [4].

Now, by (2.1) and (2.4), we can demonstrate that

$$P^2(x, y, r+1) - Q_2 P^2(x, y, r) + P^2(x, y, r-1) = 2(-1)^r e^{2xy}. \tag{5.11}$$

Using this as a difference equation, we obtain

$$\begin{aligned}\sum_{r=1}^{n} P^2(x, y, r) = [&P^2(x, y, n+1) - P^2(x, y, 1)\\ &- \{P^2(x, y, n) - P^2(x, y, 0)\} + 2(1-(-1)^n)e^{2xy}]/4x^2\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}\sum_{r=0}^{\infty} P^2(x, y, r)z^r = [&P^2(x, y, 0) + z\{P^2(x, y, 0) - P^2(x, y, -1)\}\\ &- P^2(x, y, -1)z^2 + 2ze^{2xy}]\nabla^{(2)}/(1+z)\end{aligned} \tag{5.13}$$

by (1.8).

Furthermore,

$$\begin{aligned}P^2(x, y, n+3) &- (4x^2+1)P^2(x, y, n+2)\\ &- (4x^2+1)P^2(x, y, n+1) + P^2(x, y, n) = 0,\end{aligned} \tag{5.14}$$

$$\sum_{r=0}^{\infty} \frac{P_{mr+k}y^r}{r!} = (\alpha^k e^{\alpha^m y} - \beta^k e^{\beta^m y})/(\alpha - \beta), \tag{5.15}$$

and

$$\sum_{r=0}^{\infty} \frac{P_r^2 y^r}{r!} = (e^{\alpha^2 y} + e^{\beta^2 y} - 2e^{-y})/(\alpha - \beta)^2. \tag{5.16}$$

6. FURTHER APPLICATIONS OF EXPONENTIAL GENERATING FUNCTIONS

Techniques employed for Fibonacci numbers in [1] are now cultivated for Pell polynomials.

To illustrate the method, we show that

$$P_{2n} = \sum_{r=0}^{n} \binom{n}{r}(2x)^r P_r. \tag{6.1}$$

Consider

$$\begin{aligned}A &= \{(e^{2\alpha xy} - e^{2\beta xy})e^y\}/(\alpha - \beta) && \\ &= \{e^{(2\alpha x+1)y} - e^{(2\beta x+1)y}\}/(\alpha - \beta) && \\ &= (e^{\alpha^2 y} - e^{\beta^2 y})/(\alpha - \beta) && \text{by (1.3)}\\ &= \sum_{n=0}^{\infty} \frac{P_{2n}y^n}{n!} && \text{by (1.1).}\end{aligned} \tag{6.2}$$

However, also,

$$\begin{aligned}A &= \left\{\sum_{n=0}^{\infty} \frac{(2x)^n P_n y^n}{n!}\right\}\left\{\sum_{n=0}^{\infty} \frac{y^n}{n!}\right\} \qquad \text{by (6.2) and (1.1)}\\ &= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n} \frac{(2x)^i P_i}{i!(n-i)!}\right\} y^n.\end{aligned} \tag{6.3}$$

By equating the coefficients of y^n in (6.2) and (6.3), we get

$$\frac{P_{2n}}{n!} = \sum_{i=0}^{n} \frac{(2x)^i P_i}{i!(n-1)!}, \tag{6.4}$$

which is equivalent to (6.1).

Observe that (6.2) and (6.3) lead to

$$\frac{\partial^r A}{\partial y^r} = \sum_{n=0}^{\infty} \frac{P_{2n+2r}y^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n+r} \frac{(n+1)_r(2x)^i P_i y^n}{i!(n+r-i)!}\right\}$$

where $(n)_r$ is the rising factorial.

Hence,

$$P_{2(n+r)} = \sum_{i=0}^{n+r}\binom{n+r}{i}(2x)^i P_i, \tag{6.5}$$

which is an extension of (6.4).

Turning our attention to

$$B = (e^{\alpha y} - e^{\beta y})e^{-2xy}/(\alpha - \beta), \tag{6.6}$$

we obtain, in a similar manner,

$$(-1)^{n+1}P_n = \sum_{i=0}^{n}\binom{n}{i}(-2x)^{n-i}P_i. \tag{6.7}$$

Likewise, from

$$C = (e^{\alpha^2 y} - e^{\beta^2 y})e^{-y}/(\alpha - \beta), \tag{6.8}$$

we derive

$$(2x)^n P_n = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}P_{2i}. \tag{6.9}$$

Next, consider

$$\begin{aligned} D &= (e^{\alpha^m y} - e^{\beta^m y})(e^{\alpha^m y} + e^{\beta^m y})/(\alpha - \beta) \qquad (6.10)\\ &= (e^{2\alpha^m y} - e^{2\beta^m y})/(\alpha - \beta)\\ &= \sum_{n=0}^{\infty}\frac{2^n P_{mn}y^n}{n!} \quad \text{by (1.1).}\end{aligned}$$

Now, also,

$$D = \sum_{n=0}^{\infty}\left\{\frac{P_{mn}y^n}{n!}\right\}\left\{\sum_{n=0}^{\infty}\frac{Q_{mn}y^n}{n!}\right\} = \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\frac{P_{mi}Q_{m(n-i)}}{i!(n-i)!}\right\}y^n. \tag{6.11}$$

So

$$2^n P_{mn} = \sum_{i=0}^{n}\binom{n}{i}P_{mi}Q_{m(n-i)}. \tag{6.12}$$

If we investigate

$$E = (e^{\alpha^m y} - e^{\beta^m y})(e^{\alpha^m y} - e^{\beta^m y})/(\alpha - \beta)^2, \tag{6.13}$$

we are led by the above process, eventually, to

$$2^n Q_{mn} - 2Q_m^n = 4(x^2 + 1)\sum_{r=0}^{n}\binom{n}{r}P_{mr}P_{m(n-r)}. \tag{6.14}$$

Similarly,

$$2\ Q_{mn} + 2Q_m^n = \sum_{r=0}^{n}\binom{n}{r}Q_{mr}Q_{m(n-r)}\ . \tag{6.15}$$

Suppose now that

$$\begin{aligned} F &= \{(e^{\alpha^{4m}y} - e^{\beta^{4m}y})e^y\}/(\alpha - \beta) \qquad (6.16)\\ &= \{e^{(\alpha^{4m}+1)y} - e^{(\beta^{4m}+1)y}\}/(\alpha - \beta)\\ &= \{e^{(\alpha^{4m}+\alpha^{2m}\beta^{2m})y} - e^{(\beta^{4m}+\alpha^{2m}\beta^{2m})y}\}/(\alpha - \beta)\\ &= \{e^{\alpha^{2m}(\alpha^{2m}+\beta^{2m})y} - e^{\beta^{2m}(\alpha^{2m}+\beta^{2m})y}\}/(\alpha - \beta)\\ &= \sum_{n=0}^{\infty}\frac{P_{2mn}Q_{2m}^n y^n}{n!} \qquad \text{by (1.1) and (1.2).} \end{aligned}$$

But, also,

$$\begin{aligned} F &= \left\{\sum_{n=0}^{\infty}\frac{P_{4mn}y^n}{n!}\right\}\left\{\sum_{n=0}^{\infty}\frac{y^n}{n!}\right\} \qquad \text{by (6.16) and (1.1)} \qquad (6.17)\\ &= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\frac{P_{4mi}}{i!(n-i)!}\right\}y^n. \end{aligned}$$

Consequently,

$$P_{2mn}Q_{2m}^n = \sum_{i=0}^{n}\binom{n}{i}P_{4mi}\ . \tag{6.18}$$

Differentiating r times partially w.r.t. y the two expressions (6.16) and (6.17) for F, as we did earlier for A [cf. (6.5)], we obtain the extension of (6.18), namely,

$$P_{2m(n+r)}Q_{2m}^{n+r} = \sum_{i=0}^{n+r}\binom{n+r}{i}P_{4mi}\ . \tag{6.19}$$

Finally, consider

$$\begin{aligned} G &= (e^{\alpha^m y} - e^{\beta^m y})/(\alpha - \beta) \qquad (6.20)\\ &= \left\{e^{(\alpha P_m + P_{m-1})y} - e^{(\beta P_m + P_{m-1})y}\right\}/(\alpha - \beta)\\ &= \left\{e^{P_{m-1}y}(e^{\alpha P_m y} - e^{\beta P_m y})\right\}/(\alpha - \beta)\\ &= \left\{\sum_{n=0}^{\infty}\frac{P_{m-1}^n y^n}{n!}\right\}\left\{\sum_{n=0}^{\infty}\frac{P_n P_m^n y^n}{n!}\right\} \qquad \text{by (1.1)}\\ &= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\frac{P_{m-i}^i P_{n-i} P_m^{n-i}}{i!(n-i)!}\right\}y^n. \end{aligned}$$

Also,

$$G = \sum_{n=0}^{\infty}\frac{P_{mn}y^n}{n!} \qquad \text{by (6.20) and (1.1).} \tag{6.21}$$

Then

$$P_{mn} = \sum_{i=0}^{n} \binom{n}{i} P_{m-1}^{i} P_{n-i} P_{m}^{n-i} = \sum_{i=0}^{n} \binom{n}{i} P_{m-1}^{n-i} P_{i} P_{m}^{i}, \tag{6.22}$$

whence

$$\frac{\partial^r G}{\partial y^r} = \sum_{n=0}^{\infty} \frac{P_{m(n+r)} y^n}{n!} = \sum_{n=0}^{\infty} \left\{ \sum_{i=0}^{n+r} \frac{(n+1)_r P_m^i P_{m-1}^{n+r-i} P_i}{i!(n+r-i)!} \right\} y^n \tag{6.23}$$

and

$$P_{m(n+r)} = \sum_{i=0}^{n+r} \binom{n+r}{i} P_m^i P_{m-1}^{n+r-i} P_i . \tag{6.24}$$

The presentation in this article of the properties of the exponential generating functions of Pell and Pell-Lucas polynomials suffices to give us something of their mathematical flavor.

Important special cases of the Pell polynomials and Pell-Lucas polynomials are noted in [3] and may, for variety and visual convenience, be tabulated as:

	P_n	Q_n
$x = 1$	Pell numbers	Pell-Lucas numbers
$x = \frac{1}{2}$	Fibonacci numbers	Lucas numbers
$x \to \frac{1}{2}x$	Fibonacci polynomials	Lucas polynomials

Results given in this paper for exponential generating functions, and in [6] for ordinary generating functions, of P_n and Q_n may clearly be specialized to corresponding results for the tabulated mathematical entities.

REFERENCES

1. C. A. Church & M. Bicknell. "Exponential Generating Functions for Fibonacci Identities." *The Fibonacci Quarterly* **11**, no. 3 (1973):275-81.
2. H. W. Gould. "Generating Function for the Products of Powers of Fibonacci Numbers." *The Fibonacci Quarterly* **1**, no. 1 (1963):1-16.
3. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23**, no. 1 (1985):7-20.
4. Bro. J. M. Mahon. "Pell Polynomials." M.A. (Hons.) Thesis, University of New England, 1984.
5. Bro. J. M. Mahon & A. F. Horadam. "Matrix and Other Summation Techniques for Pell Polynomials." *The Fibonacci Quarterly* **24**, no. 4 (1986):290-309.
6. Bro. J. M. Mahon & A. F. Horadam. "Ordinary Generating Functions for Pell Polynomials." *The Fibonacci Quarterly* **25**, no. 1 (1987):45-56.
7. J. E. Walton. "Properties of Second Order Recurrence Relations." M.Sc. Thesis, University of New England, 1968.

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ITERATING THE DIVISION ALGORITHM

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INTRODUCTION

The division algorithm guarantees that when an arbitrary integer b is divided by a positive integer a there is a unique quotient q and remainder r satisfying

$$0 \leqslant r < a$$

so that

$$b = qa + r.$$

We will assume that $0 < a \leqslant b$ in this paper.

Euclid's algorithm iterates this division as

$$\begin{aligned}
b &= q_1 a + r_1, \; 0 < r_1 < a\\
a &= q_2 r_1 + r_2, \; 0 < r_2 < r_1\\
r_1 &= q_3 r_2 + r_3, \; 0 < r_3 < r_2\\
&\vdots\\
r_{n-3} &= q_{n-1} r_{n-2} + r_{n-1}, \; 0 < r_{n-1} < r_{n-2}\\
r_{n-2} &= q_n r_{n-1} + 0.
\end{aligned}$$

Euclid's algorithm terminates when $r_n = 0$. What makes the algorithm useful is that r_{n-1} is then the greatest common divisor of a and b. The worst case, in the sense that the algorithm takes the longest possible number of iterations to terminate, is when the sequence

$$a > r_1 > r_2 > \cdots > r_n = 0$$

decreases to 0 as slowly as possible. The smallest pairs (b,a) for which this happens are found by choosing each quotient q_i to be 1 except the last one, where $r_{n-2} = 2$ and $r_{n-1} = 1$ forces $q_n = 2$. This makes $r_{n-3} = r_{n-2} + r_{n-1}$, $r_{n-4} = r_{n-3} + r_{n-2}$, and so on, back until we have that a and b are consecutive Fibonacci numbers. Lamé first noticed the connection between Fibonacci numbers and Euclid's algorithm in 1844 (see [3]).

General results based on this insight include:

1. If $a < F_n$, then Euclid's algorithm terminates in at most $n - 2$ steps, and the smallest pair (b,a) taking exactly $n - 2$ steps is (F_n, F_{n-1}).

2. If (b_n, a_n) denotes the pair (b,a) with smallest b for which Euclid's algorithm first takes n steps to terminate, then

$$\lim_{n\to\infty} b_n/a_n = \lim_{n\to\infty} F_{n+2}/F_{n+1} = (1 + 5^{1/2})/2.$$

The intermediate steps in Euclid's algorithm can be unwound to find integers x and y satisfying

$$d = ax + by,$$

where d is the greatest common divisor of a and b. A short BASIC program for iterating the division algorithm is given in Figure 1.

```
60 PRINT "WHAT TWO NUMBERS TO START WITH";:INPUT B,A
70 Q=INT(B/A):R=B-Q*A
80 PRINT B,"=";Q,"*";A,"+";R
90 B=A:A=R
100 IF A=0 THEN GOTO 120
110 GOTO 70
120 PRINT "ALGORITHM TERMINATES."
```

Figure 1. A BASIC Program for Euclid's Algorithm

The algorithm for radix conversion can also be written as a succession of divisions. Starting with b positive and $a \geqslant 2$, we can write

$$\begin{aligned} b &= q_1 a + r_1, \; 0 \leqslant r_1 < a \\ q_1 &= q_2 a + r_2, \; 0 \leqslant r_2 < a \\ &\vdots \\ q_{n-2} &= q_{n-1} a + r_{n-1}, \; 0 \leqslant r_{n-1} < a \\ q_{n-1} &= q_n a + r_n, \; 0 \leqslant r_n < a. \end{aligned}$$

In the i^{th} step, $q_i = [b/a^i]$, so, using the natural stopping place $q_n = 0$, the algorithm takes n steps to complete, where $a^{n-1} \leqslant b < a^n$. The value of this algorithm is that successive substitution gives

$$\begin{aligned} b &= r_1 + aq_1 = r_1 + a(r_2 + aq_2) = \cdots \\ &= r_1 + a(r_2 + a(r_3 + a(\ldots(r_{n-1} + ar_n)\ldots))) \\ &= r_1 + ar_2 + a^2 r_3 + \cdots + a^{n-1} r_n, \end{aligned}$$

which says that the remainders can be interpreted as successive digits (from right to left) in the expansion of b using the base a.

The BASIC program used for Euclid's algorithm works here as well with only minor modifications. Line 90 becomes

```
90 B=Q
```

and the test for completion in line 100 uses B instead of A.

Whatever number is used for b, it is clear there is no value for a that can make the algorithm take longer to terminate than $a = 2$. With this choice for a, the first b that makes the algorithm terminate in exactly n steps is 2^{n-1}.

In this paper we investigate ways in which the four numbers b, a, q, and r of the division algorithm can be rearranged to give a terminating sequence of quotients q_i and remainders r_i when the division algorithm is iterated. The combinatorial and number theoretic properties of some of the sequences so generated are of interest.

ALTERNATE ALGORITHMS

Line 90 of the BASIC program in Figure 1 provides the pattern for iterating the divisions in Euclid's algorithm. The substitution made is that the old A becomes the new B, and the old R becomes the new A. In the radix conversion algorithm, the old A never changes, and the new B is the old Q. We classify possible algorithms by analyzing possible replacement lines for line 90 in the BASIC program. Naively, there are sixteen possibilities, summarized in Figure 2, but ten of these are uninteresting in that their behavior is independent of the particular numbers a and b we start with. There is a single equation which repeats, a pair of equations which replace one another, or a sequence of equations that terminates to avoid a zero division. Of the six interesting cases, two are the radix conversion algorithm and Euclid's algorithm. The others are merely labelled in the table, and their analysis occupies the remainder of the paper.

\ B = A =\	B	A	Q	R
B	b = q a + r b = 1 b + 0 repeats	b = q a + r a = 0 b + a b = q a + r cycles	b = q a + r q = 0 b + q 0 = 0 q + 0 terminates	b = q a + r r = 0 b + r r = 1 r + 0 0 = 0 r + 0 terminates
A	b = q a + r repeats	b = q a + r a = 1 a + 0 repeats	Radix Conversion	b = q a + r r = 0 a + r repeats
Q	b = q a + r b = a q + r cycles once r < min(a,q)	Algorithm 5	b = q a + r q = 1 q + 0 1 = 1 1 + 0 repeats	Algorithm 4
R	Algorithm 6	Euclid's Algorithm	Algorithm 3	b = q a + r r = 1 r + 0 repeats

Figure 2. Possibilities for Line 90

ALGORITHM 3

Iterate the division algorithm as

$$\begin{aligned} b &= q_1 a + r_1, \quad 0 < r_1 < a \\ q_1 &= q_2 r_1 + r_2, \quad 0 < r_2 < r_1 \\ q_2 &= q_3 r_2 + r_3, \quad 0 < r_3 < r_2 \\ &\vdots \\ q_{n-2} &= q_{n-1} r_{n-2} + r_{n-1}, \quad 0 < r_{n-1} < r_{n-2} \\ q_{n-1} &= q_n r_{n-1} + 0, \quad r_n = 0. \end{aligned}$$

Stretching the algorithm out as long as possible is accomplished by taking $r_{n-1} = 1$, $r_{n-2} = 2$, ..., $r_1 = n - 1$. Then the smallest possible choices for the q_i would be given by

$$\begin{aligned}
q_{n-1} &= 0\ 1 + 0 = 0\\
q_{n-2} &= 0\ 2 + 1 = 1\\
q_{n-3} &= 1\ 3 + 2 = 5\\
q_{n-4} &= 5\ 4 + 3 = 23\\
&\vdots\\
q_{n-i} &= q_{n-i+1}i + i - 1\\
&\vdots
\end{aligned}$$

This implies that $q_{n-i} = i! - 1$, and hence $a = n$ and $b = n! - 1$. Thus, we obtain

Theorem 1: If $b < n! - 1$, then Algorithm 3 terminates in $< n$ steps. Algorithm 3 terminates in exactly n steps when $b = n! - 1$ and $a = n$.

Back substituting in Algorithm 3 gives an interesting pattern for the r's in terms of the q's. We have

$$\begin{aligned}
r_{n-1} &= q_{n-1}/q_n,\\
r_{n-2} &= (q_{n-2} - (q_{n-1}/q_n))/q_{n-1},\\
r_{n-3} &= (q_{n-3} - (q_{n-2} - (q_{n-1}/q_n)/q_{n-1})/q_{n-2},\\
&\vdots
\end{aligned}$$

and so on back in an inverted continued fraction expansion, to

$$a = (b - (q_1 - (q_2 - (\cdots - (q_{n-2} - (q_{n-1}/q_n)/q_{n-1})/\cdots/q_2)/q_1.$$

As a one-line summary of Algorithm 3 more in the spirit of radix conversion, we have

$$\begin{aligned}
b &= r_1 + aq_1 = r_1 + a(r_2 + r_1q_2) = \cdots\\
&= r_1 + a(r_2 + r_1(r_3 + r_2(\ldots(r_{n-2} + r_{n-3}(r_{n-2} + r_{n-1}q_n))\ldots))).
\end{aligned}$$

In the worst case $b = n! - 1$, $a = n$ of Theorem 1, we generate here a representation in the factorial number system (see [2]).

ALGORITHM 4

Here the division algorithm is iterated as

$$\begin{aligned}
b &= q_1a + r_1,\ 0 \leqslant r_1 < a\\
r_1 &= q_2q_1 + r_2,\ 0 \leqslant r_2 < q_1\\
r_2 &= q_3q_2 + r_3,\ 0 \leqslant r_3 < q_2\\
&\vdots\\
r_{n-2} &= q_{n-1}q_{n-2} + r_{n-1},\ 0 \leqslant r_{n-1} < q_{n-2}\\
r_{n-1} &= 0q_{n-1} + r_n,\ 0 \leqslant r_n < q_{n-1}.
\end{aligned}$$

This time the algorithm terminates just before the first zero division, i.e., when $q_n = 0$. It could be considered the dual of Algorithm 3 in that the roles of the A and B assignments in line 90 of the BASIC program are reversed.

We build backwards to see what the smallest possible values are for b and a to give a certain number of steps before the algorithm terminates. It is clear that the sequence of r's is strictly decreasing until the next to last term. If q_n is the first quotient that is 0, the smallest possible choice for q_{n-1} is 1. Since $r_n < q_{n-1}$, that forces $r_n = 0$. Then

$$r_{n-1} = q_n q_{n-1} + r_n = 0\,1 + 0 = 0,$$

and since $q_{n-2} > r_{n-1}$, $q_{n-2} = 1$ is the smallest possible choice. Then

$$r_{n-2} = q_{n-1} q_{n-2} + r_{n-1} = 1\,1 + 0 = 1,$$

and $q_{n-3} > r_{n-2}$ gives $q_{n-3} = 2$ as the smallest possible choice. We continue building the sequences of q's and r's backward from their n^{th} values by

$$r_{n-i} = q_{n-i+1} q_{n-i} + r_{n-i+1}$$
$$q_{n-i-1} = r_{n-i} + 1.$$

Writing $f(m) = r_{n-m}$, the sequence of r's is described by the recurrence

$$f(0) = f(1) = 0,$$
$$f(m) = (f(m-2)+1)(f(m-1)+1) + f(m-1) \text{ for } m > 1.$$

Writing $q_{n-m} = g(m) = f(m-1) + 1$, we obtain the neater recurrence

$$g(n+1) = g(n)(g(n-1)+1).$$

This is summarized in

Theorem 2: Define $g(n)$ for $n \geqslant 0$ by

$$g(0) = 0,\ g(1) = 1,$$
$$g(n+1) = g(n)(g(n-1)+1) \text{ for } n \geqslant 1.$$

Then the pair (b_n, a_n) for which Algorithm 4 first takes n steps to terminate is given by

$$b_n = g(n+2) - 1,\ a_n = g(n+1).$$

The sequence $b_1, b_2, b_3, \ldots$ begins

1, 3, 11, 59, 779, 47579, 37159979, ...

and the sequence $a_1, a_2, a_3, \ldots$ starts out

1, 2, 4, 12, 60, 780, 47580, ...

Neither of these sequences, nor any of their more obvious variants, seems to occur in Sloane's *Handbook* [5].

$\lim_{n\to\infty} b_n/a_n = \infty$ for Algorithm 4, but

$$\lim_{n\to\infty} \ln b_n/\ln a_n = (1 + 5^{1/2})/2.$$

This can be seen by noting that

$$\begin{aligned}\lim_{n\to\infty} \ln b_n/\ln a_n &= \lim_{n\to\infty} \ln(b_n+1)/\ln a_n = \lim_{n\to\infty} \ln g(n+2)/\ln g(n+1)\\ &= \lim_{n\to\infty}(\ln g(n+1) + \ln(g(n)+1))/\ln g(n+1)\\ &= 1 + 1/\lim_{n\to\infty}(\ln g(n+1)/\ln g(n)),\end{aligned}$$

and this process can be iterated to produce as many convergents to the continued fraction for $(1 + 5^{1/2})/2$ as desired. The limit has to be well behaved by the inequality

$$2^{F_{n-1}} \leqslant g(n) \leqslant 2^{F_n - 1},$$

which is easy to establish for $n \geqslant 1$ by induction.

ALGORITHM 5

Iterate the division algorithm as

$$\begin{aligned} b &= q_1 a + r_1, \; 0 \leqslant r_1 < a \\ a &= q_2 q_1 + r_2, \; 0 \leqslant r_2 < q_1 \\ q_1 &= q_3 q_2 + r_3, \; 0 \leqslant r_3 < q_2 \\ &\vdots \\ q_{n-3} &= q_{n-1} q_{n-2} + r_{n-1}, \; 0 \leqslant r_{n-1} < q_{n-2} \\ q_{n-2} &= 0 q_{n-1} + r_n, \; 0 \leqslant r_n < q_{n-1}. \end{aligned}$$

The iteration should end just before a zero division, i.e., when $q_n = 0$. $q_1, q_2, \ldots$ form a strictly decreasing sequence out to q_{n-2}, so the algorithm is guaranteed to terminate. Choosing r's and q's so as to build the longest possible algorithm for the smallest possible b and a, we find $q_n = 0$ and $q_{n-1} = 1$ forces $r_n = 0$, since $r_n < q_{n-1}$, and then $q_{n-2} = q_n q_{n-1} + r_n = 0$, which cannot happen. $q_n = 0$, $q_{n-1} = 2$, and $r_n = 1$ gives $q_{n-2} = 0\,2 + 1 = 1$. Now, $r_{n-1} = 0$ gives no trouble, and $q_{n-3} = q_{n-1}q_{n-2} + r_{n-3} = 2\,1 + 0 = 2$, and all the other r's $= 0$ give the q's satisfying the recurrence

$$q_{n-k} = q_{n-k+1} q_{n-k+2},$$

with $q_n = 0$, $q_{n-1} = 2$. Thus, we obtain, in general, that

$$q_{n-k} = 2^{F_{k-2}},$$

with the $(k - 2)^{\text{th}}$ Fibonacci number in the exponent. This is summarized in

Theorem 3: Writing (b_n, a_n) as the pair for which Algorithm 5 first takes n iterations to finish, we have, for $n \geqslant 2$,

$$b_n = 2^{F_{n-1}} \quad \text{and} \quad a_n = 2^{F_{n-2}}.$$

Thus,

$$\lim_{n \to \infty} \ln b_n / \ln a_n = (1 + 5^{1/2})/2.$$

Successive substitution provides a one-line summary of Algorithm 5:

$$\begin{aligned} b &= q_1 a + r_1 = r_1 + q_1(r_2 + q_2 q_1) = \cdots \\ &= r_1 + q_1(r_2 + q_2(r_3 + q_3(\ldots(r_{n-1} + q_{n-1}r_n)\ldots))). \end{aligned}$$

Multiply this out to obtain the "mixed radix expansion" of b relative to the sequence of quotients $q_1, q_2, q_3, \ldots$

$$b = r_1 + r_2(q_1) + r_3(q_1 q_2) + \cdots + r_n(q_1 q_2 \cdots q_{n-1}).$$

The relationship between systems of numeration and the division algorithm is explored by Fraenkel (see [2]).

ALGORITHM 6

The last variation we consider is

$$\begin{aligned} b &= q_1 a + r_1, \; 0 < r_1 < a \\ b &= q_2 r_1 + r_2, \; 0 < r_2 < r_1 \end{aligned}$$

$$b = q_3 r_2 + r_3, \; 0 < r_3 < r_2$$
$$\vdots$$
$$b = q_{n-1} r_{n-2} + r_{n-1}, \; 0 < r_{n-1} < r_{n-2}$$
$$b = q_n r_{n-1} + 0, \; r_n = 0.$$

If the sequence of r's is chosen to decrease as slowly as possible so that $r_n = 0$, $r_{n-1} = 1$, $r_{n-2} = 2, \ldots$, then b would satisfy the system of congruences

$$b \equiv 1 \pmod{2}$$
$$b \equiv 2 \pmod{3}$$
$$\vdots$$
$$b \equiv n - 1 \pmod{n}.$$

The smallest such b is clearly l.c.m. $(2, 3, \ldots, n) - 1$, with $a = n$. For $n \geqslant 4$, however, there are smaller values of b that provide an algorithm terminating after n steps. Table 1 summarizes "worst case" behavior up to $n = 16$.

Table 1. b_n, a_n that First Make Algorithm 6 Run for n Steps

n	b_n	a_n	n	b_n	a_n
1	1	1	9	53	32
2	3	2	10	95	61
3	5	3	11	103	65
4	11	4	12	179	115
5	11	7	13	251	161
6	19	12	14	299	189
7	35	22	15	503	316
8	47	30	16	743	470

We bound the number of steps that Algorithm 6 can take in the next result.

Theorem 4: Given b, no value for a makes Algorithm 6 take more than $2b^{1/2} + 2$ iterations to terminate.

Proof: Given b, form the sequence $R_1, R_2, \ldots, R_b$ of remainders associated with dividing b by each of the numbers $1, 2, \ldots, b$. Applying Algorithm 6 to a pair (b, a) is equivalent to picking out the increasing subsequence

$$0 = R_{n_1} < R_{n_2} < \cdots < R_{n_m} = R_a$$

satisfying

$$R_{n_{i+1}} = n_i.$$

The sequence $R_1, R_2, \ldots, R_b$ has its last $b - [b/2]$ elements decreasing by 1 (corresponding to quotients 1 in the divisions), preceded by $[b/2] - [b/3]$ elements decreasing by 2, preceded by $[b/3] - [b/4]$ elements decreasing by 3, and so on back. Most of the larger values for j have no elements between $[b/j]$ and $[b/j + 1]$. Choose $k = [b^{1/2}]$, and consider as a worst case that there could be an increasing subsequence with

$$R_{n_1} = 0,\ R_{n_2} = 1,\ \ldots,\ R_{n_{[b^{1/2}]+1}} = [b^{1/2}]$$

and working backward from the other end,

R_{n_m} one of the last $b - [b/2]$ elements

$R_{n_{m-1}}$ one of the next to last $[b/2] - [b/3]$ elements

$\vdots$

$R_{n_{m-[b^{1/2}]+1}}$ between $[b/[b^{1/2}]]$ and $[b/([b^{1/2}] + 1)]$.

This would yield an increasing subsequence of maximum length

$$[b^{1/2}] + 1 + [b/[b^{1/2}]] \leqslant 2b^{1/2} + 2.$$

One would expect that the longest sequences would be obtained from pairs (b, a) such that the sequence of quotients $q_1, q_2, q_3, \ldots$ grows as slowly as possible and the sequence of remainders $r_n, r_{n-1}, r_{n-2}, \ldots$ also stays as small as possible. Keeping the remainders small is achieved by choosing b to satisfy a number of low-order congruences. The quotients' size is controlled by the relative sizes of b and a.

Theorem 5: Let $\{(b_n, a_n)\}$ be any sequence of ordered pairs of integers with the property that for any positive integer m there exists an N such that, when Algorithm 6 is applied to (b_n, a_n) for $n > N$, $q_i = i$ for $i = 1, 2, \ldots, m$. Then

$$\lim_{n\to\infty} b_n/a_n = e/(e - 1).$$

Proof: A pair (b, a) with $q_1 = 1$ satisfies $b = 1a + r_1$, with $r_1 < a$. Hence, $b < 2a$, so $b/a < 2$;

$q_2 = 2$ implies $b = 2r_1 + r_2 < 3r_1 = 3(b - a)$, so $b/a > 3/2$;

$q_3 = 3$ implies $b = 3r_2 + r_3 < 4r_2 = 4(2a - b)$, so $b/a < 8/5$;

$q_4 = 4$ implies $b < 5(4b - 6a)$, so $b/a > 30/19$;

$q_5 = 5$ implies $b < 6(24a - 15b)$, so $b/a < 144/91$.

Continue this procedure to build a sequence of fractions

$$\{f(n)/g(n)\} = 2/1,\ 3/2,\ 8/5,\ 30/19,\ 144/91,\ 840/531,\ 5760/3641,\ \ldots$$

satisfying

$$f(2)/g(2) < f(4)/g(4) < \cdots < b/a < \cdots < f(3)/g(3) < f(1)/g(1).$$

It is easy to establish that $f(n) = (n + 1)(n - 1)!$.

$g(n)$ arises as the sum of coefficients of b in the inequalities generated from the assumptions

$$q_{n+1} = n + 1 \text{ and } q_n = n.$$

This sequence of coefficients,

$$\{c_n\} = 1,\ 1,\ 4,\ 15,\ 76,\ 455,\ \ldots,$$

has arisen in the literature before in an analysis of the game of Mousetrap [6], and satisfies the recurrence

$$c_n = nc_{n-1} + (-1)^{n+1}.$$

The analogy with subfactorials is compelling. See the note by Rumney and Primrose [4] for an analysis of the sequence $\{u_n\}$, which satisfies

$$u_{n-1} = f(n) - g(n).$$

A combinatorial interpretation of this sequence in terms of consecutive ascending pairs of numbers in permutation is given in [1]. Properties of $\{u_n\}$ can be used to establish the recurrences

$$\begin{aligned} g(n) &= ng(n-1) + \sum_{i=2}^{n-1} (-1)^{i+1} g(n-i) \\ &= (n-1)g(n-1) + (n-2)g(n-2) \end{aligned}$$

and the formula

$$g(n) = (n+1)(n-1)!(1 - 1/2! + 1/3! - \cdots + (-1)^{n+1}/(n+1)!).$$

Since the sum is a truncated series expansion for $1 - 1/e$, the theorem is established.

Examples of pairs (b, a) for which Algorithm 6 takes a relatively large number of iterations to terminate can be constructed by starting with two consecutive convergents a/b and c/d in the continued fraction expansion of

$$e/(e-1) = [1, 1, 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$$

and then choosing positive integers x and y so that the numerator of the intermediate fraction

$$(ax + cy)/(bx + dy)$$

satisfies a number of low-order congruences.

Algorithm 6 provides a weaker statement about divisibility than Euclid's Algorithm does. It is easy to show that, if Algorithm 6 ends at the n^{th} step with $b = q_n r_{n-1} + 0$, then $\gcd(b, a)$ divides r_{n-1}, which in turn divides b.

The k^{th} quotient q_k is given in terms of b, a, and earlier quotients by

$$q_k = [b/(b - q_{k-1}(b - q_{k-2}(b - \cdots q_2(b - q_1 a) \cdots)))].$$

$r_{n-1} = 1$ is a sufficient condition for $\gcd(b, a) = 1$. It is not necessary, because, for example, $b = 9999$ and $a = 343$ ends with $r_{n-1} = 9$.

The iterations in Algorithm 6 say that $b \equiv r_{k+1} \pmod{r_k}$. Thus, we are led to the following number theory problem: Given n, for each decreasing sequence of positive integers

$$x_1, x_2, x_3, \ldots, x_n$$

find the smallest positive number b satisfying

$$\begin{aligned} b &\equiv x_2 \pmod{x_1} \\ b &\equiv x_3 \pmod{x_2} \\ &\vdots \\ b &\equiv x_n \pmod{x_{n-1}}, \end{aligned}$$

if a solution exists. A solution is guaranteed to exist if, for example, the numbers $x_1, x_2, \ldots, x_{n-1}$ are pairwise relatively prime. If a solution does exist, it is unique (mod $\operatorname{lcm}(x_1, x_2, \ldots, x_{n-1})$). What is the smallest solution b among all possible decreasing sequences of n terms? It is the same b as first makes Algorithm 6 take exactly n steps to terminate.

REFERENCES

1. F. M. David, M. G. Kendall, & D. E. Barton. *Symmetric Functions and Allied Tables*. Cambridge: Cambridge University Press, 1966.
2. A. S. Fraenkel. "Systems of Numeration." *Amer. Math. Monthly 92* (1985): 105-14.
3. G. Lamé. "Note sur la limite du nombre des divisions dans la recherche du plus grand commun diviseur entre deux nombres entiers." *Compte Rendu 19* (Paris, 1844):867-69.
4. M. Rumney & E. J. F. Primrose. "A Sequence Connected with the Sub-Factorial Sequence." *Math. Gazette 52* (1968):381-82.
5. N. J. A. Sloane. *A Handbook of Integer Sequences*. New York and London: Academic Press, 1973.
6. A. Steen. "Some Formulae Respecting the Game of Mousetrap." *Q. J. Pure and App. Math. 15* (1878):230-41.

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A NOTE ON DIVISIBILITY SEQUENCES

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In [1], Marshall Hall defined U_n to be a *divisibility sequence* if $U_m | U_n$ whenever $m | n$. Well-known examples of such sequences include geometric sequences and the Fibonacci numbers and their various generalizations (see [2], [3], and the references therein). The purpose of this note is to prove the following theorem.

Theorem: Let U_n be the sequence generated by the recurrence relation

$$U_{n+2} = aU_{n+1} + bU_n,$$

with a, b nonzero integers satisfying $a^2 + 4b = 0$. Then U_n is a nongeometric divisibility sequence if and only if $U_0 = 0$.

Proof: The Binet formula for the sequence U_n is given by

$$U_n = \left(\frac{a}{2}\right)^n (c_1 + c_2 n).$$

If $U_0 = 0$, then $c_1 = U_0 = 0$, $U_n = (a/2)^n c_2 n$, and U_n is a (nongeometric) divisibility sequence.

Conversely, suppose $c_1 = U_0 \neq 0$ and that $U_m | U_n$ whenever $m | n$, i.e., suppose

$$c_1 + c_2 m \left| \left(\frac{a}{2}\right)^{n-m} (c_1 + c_2 n) \right. \text{ whenever } m | n.$$

Replace m by $c_1 a_0 m$, n by $c_1 a_0 n$, and let $a_0 = a/2$ and $e = c_1 a_0 n - c_1 a_0 m$. Then

$$c_1 + c_2 c_1 a_0 m | a_0^e (c_1 + c_1 c_2 a_0 n) \text{ whenever } m | n.$$

Therefore,

$$1 + c_2 a_0 m | a_0^e (1 + c_2 a_0 n) \text{ whenever } m | n.$$

If $e \leqslant 0$, then

$$1 + c_2 a_0 m | 1 + c_2 a_0 n$$

is immediate, while if $e > 0$, since $\gcd(1 + c_2 a_0 m, a_0) = 1$, we also have

$$1 + c_2 a_0 m | 1 + c_2 a_0 n \text{ whenever } m | n.$$

Letting $m = 1$, $n = 2$, gives

$$1 + c_2 a_0 | 1 + 2c_2 a_0 \text{ or } 1 + c_2 a_0 | c_2 a_0.$$

Since $\gcd(1 + c_2 a_0, c_2 a_0) = 1$, it follows that $1 + 2c_2 a_0 = \pm 1$. Hence, either $c_2 a_0 = 0$ or $c_2 a_0 = -2$. If $c_2 a_0 = 0$, then $c_2 = 0$, since $a_0 \neq 0$ by assumption, and we have the geometric sequence $c_1 (a/2)^n$. On the other hand, if $c_2 a_0 = -2$, then we have

$$1 - 2m | 1 - 2n \text{ whenever } m | n,$$

which is false for $m = 2$, $n = 4$.

REFERENCES

1. Marshall Hall. "Divisibility Sequences of 3rd Order." *Amer. J. Math.* 58 (1936):577-84.
2. Clark Kimberling. "Divisibility Properties of Recurrent Sequences." *The Fibonacci Quarterly* 14, no. 4 (1976):369-76.
3. Clark Kimberling. "Generating Functions of Linear Divisibility Sequences." *The Fibonacci Quarterly* 18, no. 3 (1980):193-208.

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A NOTE ON THE PELL EQUATION

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1. INTRODUCTION

The *Pellian sequence* $\{x_n,\ n = 1, 2, 3, \ldots\}$ is defined by the rule: x_n is the least positive integer x such that $nx^2 + 1$ is the square of an integer; if no such x exists, x_n is taken to be 0. Briefly, x_n is the least positive solution to the Pell equation $nx^2 + 1 = y^2$. The sequence behaves irregularly; the first few terms are

$$0,\ 2,\ 1,\ 0,\ 4,\ 2,\ 3,\ 1,\ 0,\ 6,\ 3,\ 2,\ 180,\ 4,$$

while $x_{61} = 1766319049$. It is easy to see that if n is a perfect square, then $x_n = 0$. The converse is also true: it is shown in [2] that for positive nonsquare n, if $\sqrt{n}$ has continued fraction expansion $[a_0, \overline{a_1, \ldots, a_k}]$, then the convergent p_{2k-1}/q_{2k-1} provides a solution $x = q_{2k-1}$, $y = p_{2k-1}$ to the Pell equation $nx^2 + 1 = y^2$ ([2] also serves as a good reference for terminology and facts about continued fractions used in Section 3 of this note). It is also easy to show that $x_n = 1$ if and only if n is one less than a square. In this note, a method will be described which produces all the occurrences of any integer $m > 1$ in the Pellian sequence.

2. POSSIBLE OCCURENCES OF m

It is not difficult to restrict the possible occurences of m in the Pellian sequence to a small list. The method as given in [1] is as follows:

Suppose m is an odd integer greater than 1 and that $x_n = m$. Say $nm^2 + 1 = y^2$ for a positive integer y. Since $nm^2 = (y - 1)(y + 1)$, and m is odd, while $y - 1$ and $y + 1$ share no common odd factors, there must be positive integers a, b with $(a, b) = 1$, $m = ab$, and such that $a^2 \mid (y + 1)$ and $b^2 \mid (y - 1)$. Hence,

$$n = (y^2 - 1)/m^2 = ((y + 1)/a^2)((y - 1)/b^2).$$

If m is even, write $m = 2^e M$ with M odd. In this case, if $nm^2 + 1 = y^2$, then y must be odd and so

$$n2^{2e-2}M^2 = ((y + 1)/2)((y - 1)/2).$$

The factors on the right are consecutive integers. It follows that

$$m/2 = 2^{e-1}M = ab$$

with $(a, b) = 1$ and such that $a^2 \mid (y + 1)/2$ and $b^2 \mid (y - 1)/2$. Thus,

$$n = ((y + 1)/2a^2)((y - 1)/2b^2).$$

So the only possible occurrences of m in the Pellian sequence are found as follows:

1. For odd m write m as a product ab with $(a, b) = 1$ in all possible ways. For even m write $m/2$ as a product ab with $(a, b) = 1$ in all possible ways.

2. For each such factorization ab find the positive solutions to

$y \equiv -1 \pmod{a^2}$

$y \equiv 1 \pmod{b^2}$

if m is odd, or to

$y \equiv -1 \pmod{2a^2}$

$y \equiv 1 \pmod{2b^2}$

if m is even.

Then m can occur in the Pellian sequence only for the numbers $n = (y^2 - 1)/m^2$.

For example, if $m = 35$, there are four systems to solve:

1. $y \equiv -1 \pmod{1^2}$, $y \equiv 1 \pmod{35^2}$
2. $y \equiv -1 \pmod{5^2}$, $y \equiv 1 \pmod{7^2}$
3. $y \equiv -1 \pmod{7^2}$, $y \equiv 1 \pmod{5^2}$
4. $y \equiv -1 \pmod{35^2}$, $y \equiv 1 \pmod{1^2}$

The solutions are, respectively,

1. $y = 1 + 35^2t$,
2. $y = 99 + 35^2t$,
3. $y = 1126 + 35^2t$,
4. $y = 1224 + 35^2t$,

each with $t \geqslant 0$.

Each solution y proivdes a candidate $n = (y^2 - 1)/35^2$, where $x_n = 35$ is possible. These candidates for the four solution sets are, respectively (with $t \geqslant 0$),

1. $(2 + 35^2t)t = 0, 1227, 4904, \ldots,$
2. $(4 + 7^2t)(2 + 5^2t) = 8, 1431, 5304, \ldots,$
3. $(23 + 5^2t)(45 + 7^2t) = 1035, 4512, 10439, \ldots,$
4. $(1 + t)(1224 + 35^2t) = 1224, 4896, 11019, \ldots .$

In fact, x_n is 35 for all the listed values of n except the 0 of solution 1 (x_0 is not even defined) and the 8 of solution 2 ($x_8 = 1$ since 8 is one less than a square). Thus, while the method produces all possible occurrences of m in the Pellian sequence, some exceptional values of n can creep into the lists.

3. EXCEPTIONAL VALUES

When m is odd, the two trivial factorizations of m,

$$m = (1)(m) \quad \text{and} \quad m = (m)(1),$$

give exceptional values of n which are easy to determine. For the first factorization, the system to solve is

$y \equiv -1 \pmod{1^2}$

$y \equiv 1 \pmod{m^2}$,

with solutions $y = 1 + m^2t$, $t \geqslant 0$, which yields candidates

$$n = (y^2 - 1)/m^2 = (2 + m^2t)t.$$

Of course $t = 0$ gives an exceptional value of n. However, all other values of t are good. To see that is so, it must be shown for each $t > 0$ that, if x is a positive integer and $(2 + m^2t)tx^2 + 1 = y^2$, then $x \geqslant m$. From $(2 + m^2t)tx^2 + 1 = y^2$, it follows that

$$2tx^2 + 1 = y^2 - (mtx)^2 \geqslant (mtx + 1)^2 - (mtx)^2 = 2mtx + 1,$$

which shows $x \geqslant m$.

The same reasoning shows that the system

$$\begin{aligned} y &\equiv -1 \pmod{m^2} \\ y &\equiv 1 \pmod{1^2} \end{aligned}$$

yields no exceptional values of n.

Similarly, for even m, the factorization $(1)(m/2)$ of $m/2$ yields one exceptional value of n (namely, $n = 0$), while the factorization $(m/2)(1)$ gives no exceptional values.

For the nontrivial factorizations of m, the exceptional values will be determined by noting a peculiar feature of the continued fraction expansions of $\sqrt{n}$ for the candidate n values produced by each of the systems: the expansions all share common "middle terms." For example, looking at the solutions to system 2 in the example above, the following CFEs are found:

$$\sqrt{8} = [2, \overline{1, 4}] = [2, \overline{1, 4, 1, 4, 1, 4}];$$

$$\sqrt{1431} = [37, \overline{1, 4, 1, 4, 74}];$$

$$\sqrt{5304} = [72, \overline{1, 4, 1, 4, 1, 144}].$$

To see why this is so, let us suppose m is odd and $m = ab$, with $a, b > 1$, $(a, b) = 1$. Let Y be the least positive solution of

$$\begin{aligned} y &\equiv -1 \pmod{a^2} \\ y &\equiv 1 \pmod{b^2}, \end{aligned}$$

so that all positive solutions are given by $y = Y + m^2t$, $t \geqslant 0$. For each $t \geqslant 0$, put

$$n_t = ((Y + m^2t)^2 - 1)/m^2,$$

the t^{th} candidate n. If it is observed that

$$\begin{aligned} [\sqrt{n_t}] &= [\sqrt{(Y + m^2t)^2 - 1}/m] = [[\sqrt{(Y + m^2t)^2 - 1}]/m] \\ &= [(Y + m^2t - 1)/m] = [Y/m] + mt, \end{aligned}$$

where $[\cdot]$ denotes the greatest integer function, it is not difficult to verify that the sequence $\sqrt{n_t} - [\sqrt{n_t}]$, $t = 0, 1, \ldots$ is monotone increasing and converges to $Y/m - [Y/m]$. Thus, for all $t \geqslant 1$, we have

$$\sqrt{n_0} - [\sqrt{n_0}] < \sqrt{n_t} - [\sqrt{n_t}] < Y/m - [Y/m].$$

Now, $x = m$, $y = Y$ is certainly a solution to the Pell equation $n_0x^2 + 1 = y^2$, and, consequently, Y/m must be a convergent of the CFE of $\sqrt{n_0}$; in fact, it can be said that

$$\sqrt{n_0} = [q_0, \overline{q_1, \ldots, q_k, 2q_0}],$$

where k is odd, and $q_0 = [Y/m]$, since $[Y/m]$ is the greatest integer in $\sqrt{n_0}$ and, finally, Y/m has CFE $[q_0, q_1, \ldots, q_k]$. The period of the expansion of $\sqrt{n_0}$ is not necessarily $k + 1$, but must be some divisor of $k + 1$. In addition, it is known that $2q_0$ is the largest integer appearing in the CFE of $\sqrt{n_0}$.

So the CFEs of

$$\sqrt{n_0} - [\sqrt{n_0}] = [0, q_1, \ldots, q_k, \ldots]$$

and

$$Y/m - [Y/m] = [0, q_1, \ldots, q_k]$$

are identical out to the entry q_k. Since the numbers $\sqrt{n_t} - [\sqrt{n_t}]$ are trapped between these two values, they also must have continued fraction expansions which begin with $[0, q_1, q_2, \ldots, q_k]$. Furthermore, since $x = m$ certainly provides a solution to the Pell equation $n_t x^2 + 1 = y^2$, it follows that the CFE of $\sqrt{n_t}$ has the form

$$[Q, \overline{q_1, \ldots, q_k, 2Q}], \text{ where } Q = [\sqrt{n_t}].$$

Since the values $q_1, q_2, \ldots, q_k$ are all less than $2q_0$, and so certainly less than $2Q$, it must be that the period of the CFE of $\sqrt{n_t}$ is exactly $k + 1$; hence, m is the least positive x that satisfies the Pell equation $n_t x^2 + 1 = y^2$, which proves that m occurs in the Pellian sequence at every n_t except, possibly, the value n_0.

In a similar fashion, it is found for even m that each nontrivial factorization of m yields at most one exceptional value of n, namely the value

$$n_0 = (Y^2 - 1)/m^2,$$

where Y is the least positive solution for the system.

Thus, the following theorem has been established.

Theorem 1: For $m > 1$ odd, write $m = ab$ with $(a, b) = 1$, and let Y be the least positive solution of the system

$$\begin{aligned} y &\equiv -1 \pmod{a^2} \\ y &\equiv 1 \pmod{b^2}. \end{aligned} \tag{1}$$

Then $m = x_n$, the n^{th} term of the Pellian sequence, where n is given by

$$n = ((Y + m^2 t)^2 - 1)/m^2, \text{ for all } t \geqslant 1,$$

and possibly for $t = 0$ as well. This accounts for all occurrences of m.

For $m > 1$ even, write $m/2 = ab$ with $(a, b) = 1$, and let Y be the least positive solution of the system

$$\begin{aligned} y &\equiv -1 \pmod{2a^2} \\ y &\equiv 1 \pmod{2b^2}. \end{aligned} \tag{2}$$

Then $m = x_n$, the n^{th} term of the Pellian sequence, where n is given by

$$n = ((Y + m^2 t)^2 - 1)/m^2, \text{ for all } t \geqslant 1,$$

and possibly for $t = 0$ as well. This accounts for all occurrences of m.

It is natural to ask exactly when $t = 0$ will yield an exceptional n. While a general solution of this problem appears to be difficult, for some particular nontrivial facotrizations ab of m (or $m/2$), the answer can be provided. For example, when m is odd, a factorization of the form $a(a + 2)$ always gives an exceptional value of n (as was seen for the case $35 = 5 \cdot 7$ in the earlier example). To see why this is true, suppose $a = 2k + 1$ and $b = 2k + 3$. The least positive solution to the system

$$\begin{aligned} y &\equiv -1 \pmod{a^2} \\ y &\equiv 1 \pmod{b^2} \end{aligned}$$

is

$$Y = (k + 2)(2k + 1)^2 - 1 = k(2k + 3)^2 + 1,$$

which provides us with

$$n = k(k + 2) = (k + 1)^2 - 1,$$

always one less than a square. Hence, $x_n = 1$, and this n is exceptional. However, such factorizations do not account for all exceptional values of n. For

$m = 1197 = 19 \cdot 63$, the least positive solution to

$$y \equiv -1 \pmod{19^2}$$
$$y \equiv 1 \pmod{63^2}$$

is $Y = 3970$, which yields $n = 11$. But $x_{11} = 3$ and not 1197. Likewise, it can be shown that if m is even and $m/2$ is factored as $(m/4)(2)$ (assuming m is a multiple of 4), then for the n produced, $x_n = 2$, and not m. Again there are other factorizations which yield exceptional values of n.

REFERENCES

1. S. P. Kaler. Properties of the Pellian Sequence." Masters Thesis. University of North Dakota, 1985.
2. W. J. LeVeque. *Fundamentals of Number Theory*. Reading, Mass.: Addison-Wesley, 1977.

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ON r^{th}-ORDER RECURRENCES*

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This note will generalize results obtained by Wyler [5] concerning periods of second-order recurrences.

Let $r \geqslant 2$ and let (u) be an r^{th}-order linear recurrence over the rational integers satisfying the recursion relation

$$u_{n+r} = a_1 u_{n+r-1} - a_2 u_{n+r-2} + \cdots + (-1)^{r+1} a_r u_n \tag{1}$$

with initial terms $u_0 = u_1 = \cdots = u_{r-2} = 0$, $u_{r-1} = 1$. Then (u) is called a unit sequence with coefficients $a_1, a_2, \ldots, a_r$. For a positive integer M, the primitive period of (u) modulo M, denoted by $K(M)$, is the least positive integer m such that $u_{n+m} \equiv u_n \pmod{M}$ for all nonnegative integers n greater than or equal to some fixed integer n_0. It is known that the primitive period modulo M of a unit sequence (u) is a period modulo M of any other recurrence satisfying the same recursion relation (see [4], pp. 603-04). The rank of (u) modulo M, denoted by $k(M)$, is the least integer m such that $u_{n+m} \equiv su_n \pmod{M}$ for some residue s and for all integers n greater than or equal to some fixed nonnegative integer n_0. We call s the principal multiplier of (u) modulo M. If $(a_r, M) = 1$, then it is known from [1] that (u) is purely periodic modulo M and $K(M) \mid k(M)$. Furthermore, if $(a_r, M) = 1$, Carmichael [1] has shown that the principal multiplier s is a unit modulo M and $K(M)/k(M) = E(M)$ is the exponent of the multiplier s modulo M. In this paper, we will put constraints on $K(M)$ given $k(M)$ and the exponent of a_r modulo M.

Our two main results are Theorems 1 and 2. Theorem 2 is a refinement of Theorem 1.

Theorem 1: Let (u) be a unit sequence with coefficients $a_1, a_2, \ldots, a_r$. Let $M \geqslant 2$ be a positive integer such that $(a_r, M) = 1$. Let h be the exponent of a_r modulo M. Let $k = k(M)$ and $K = K(M)$. Let H be the least common multiple of h and k. Then $H \mid K$ and $K \mid rH$.

Theorem 2: Let (u) be a unit sequence with coefficients $a_1, a_2, \ldots, a_r$. Let $M \geqslant 2$ be a positive integer such that $(a_r, M) = 1$. Let h, k, K, and H be defined as in Theorem 1. Let

$$r = \prod_{i=1}^{n} p_i^{\alpha_i},$$

where the p_i are distinct primes and $a_i \geqslant 1$. Let

$$h = \left(\prod_{i=1}^{n} p_i^{\beta_i}\right) h', \quad k = \left(\prod_{i=1}^{n} p_i^{\gamma_i}\right) k',$$

This note is based partly on results in the author's Ph.D. Dissertation, The University of Illinois at Urbana-Champaign, 1985.

where $\beta_i \geqslant 0$, $\gamma_i \geqslant 0$, and $(h', r) = (k', r) = 1$. Let j vary over all the indices i, $1 \leqslant i \leqslant n$, such that $\beta_i > \gamma_i$. Let $c = 1$ if there is no subscript i such that $\beta_i > \gamma_i$. Otherwise, let

$$c = \prod_j p_j^{\alpha_j}.$$

Then

$$cH \mid K$$

and

$$K \mid k(rH/k, \phi(M)),$$

where $\phi(M)$ denotes Euler's totient function.

To prove Theorems 1 and 2, we will need the following lemmas.

Lemma 1: For the unit sequence (u) given in (1), define the persymmetric determinant

$$D_n^{(r)}(u) = \begin{vmatrix} u_n & u_{n+1} & \cdots & u_{n+r-1} \\ u_{n+1} & u_{n+2} & \cdots & u_{n+r} \\ \cdots & \cdots & \cdots & \cdots \\ u_{n+r-1} & u_{n+r} & & u_{n+2r-2} \end{vmatrix}$$

Then

$$D_{n+1}^{(r)}(u) = a_r D_n^{(r)}(u).$$

Proof: This is Heymann's Theorem and a proof is given in [2, ch. 12.12]. ∎

Lemma 2: Let $k = k(M)$. Suppose

$$u_m \equiv u_{m+1} \equiv \cdots \equiv u_{m+r-2} \equiv 0 \pmod{M}$$

and $(a_r, M) = 1$. Then $k \mid m$. Furthermore,

$$u_{mi+n} \equiv u_{m+r-1}^i u_n \pmod{M} \tag{2}$$

and for all non-negative integers n,

$$u_{m+r-1}^r \equiv a_r^m \pmod{M}. \tag{3}$$

In particular, if s is the principal multiplier of (u), then

$$s^r \equiv a_r^k \pmod{M}.$$

Proof: Suppose $m = tk + d$, where $0 \leqslant d < k$. Since (u) is purely periodic modulo M, it follows that, for $0 \leqslant n \leqslant r - 2$,

$$0 \equiv u_{m+n} \equiv su_{m+n-k} \equiv s^2 u_{m+n-2k} \equiv \cdots \equiv s^t u_{m+n-tk} = s^t u_{d+n} \pmod{M},$$

where s is the principal multiplier of (u) modulo M. However, if $d > 0$, this is impossible since s is a unit modulo M and, by definition, k is the smallest positive integer j such that $u_{j+n} \equiv 0 \pmod{M}$ for $0 \leqslant n \leqslant r - 2$. Thus, $d = 0$ and $k \mid m$.

We now note that

$$u_{m+n} \equiv u_{m+r-1} u_n \pmod{M} \tag{4}$$

for $0 \leqslant n \leqslant r - 1$. It follows from the linearity of the r^{th}-order recursion relation defining (u) that (4) holds for all nonnegative integers n, and u_{m+r-1}

is a multiplier modulo M, though not necessarily principal, of (u). By applying congruence (4) repeatedly, we obtain

$$\begin{aligned} u_{mi+n} &= u_{m+(m(i-1)+n)} \equiv u_{m+r-1}u_{m(i-1)+n} = u_{m+r-1}u_{m+(m(i-2)+n)} \\ &\equiv u^2_{m+r-1}u_{m(i-2)+n} \equiv \cdots \equiv u^i_{m+r-1}u_n \pmod{M}, \end{aligned}$$

and congruence (2) holds.

To prove (3), we note that since $u_m \equiv u_{m+1} \equiv \cdots \equiv u_{m+r-2} \equiv 0 \pmod{M}$, one easily calculates that

$$D_m^{(r)}(u) \equiv (-1)^{r(r-1)/2}u^r_{m+r-1} \pmod{M}.$$

Moreover, since $u_0 = u_1 = \cdots = u_{r-2} = 0$ and $u_{r-1} = 1$,

$$D_0^{(r)}(u) = (-1)^{r(r-1)/2}.$$

By applying Lemma 1 m times, we now obtain

$$D_m^{(r)}(u) \equiv (-1)^{r(r-1)/2}u^r_{m+r-1} \equiv a^m D_0^{(r)}(u) = a_r^m(-1)^{r(r-1)/2} \pmod{M},$$

and congruence (3) is seen to hold. Finally, noting that $s \equiv u_{k+r-1} \pmod{M}$, the lemma now follows. ■

We are now ready for the proofs of Theorems 1 and 2.

Proof of Theorem 1: Note that $u_{K+r-1} \equiv u_{r-1} = 1 \pmod{M}$. By Lemma 2,

$$u^r_{K+r-1} \equiv a_r^K \equiv 1 \pmod{M}.$$

Thus, K is a multiple of h. Since $k|K$, K is also a multiple of H. On the other hand, by Lemma 2,

$$u_{rH} \equiv u_{rH+1} \equiv \cdots \equiv u_{rH+r-2} \equiv 0 \pmod{M}$$

and

$$u_{rH+r-1} \equiv u^r_{H+r-1} \equiv a_r^H \equiv 1 \pmod{M}.$$

Hence, rH is a multiple of K and we are done. ■

Proof of Theorem 2: By Theorem 1, $K|rH$. Since $K = kE(M)$ and $E(M)|\phi(M)$, it follows that

$$K|k(rH/k,\ \phi(M)).$$

For a given index j, let $\delta_j = \alpha_j + \beta_j$. Then it follows from the definitions of c and H that

$$p_j^{\delta_j} \,\|\, cH \quad \text{and} \quad p_j^{\delta_j} \,\|\, rH,$$

where $p_j^x \,\|\, N$ means x is the highest power of p_j dividing N. Since $H|K$ by Theorem 1 and $cH|rH$, it suffices to prove that if p_j is a prime dividing c, then

$$K \nmid (rH/p_j).$$

By Lemma 2, we thus need to show that

$$u_{(rH/p_j)+r-1} \not\equiv 1 \pmod{M}.$$

Note that $p_jk|H$ since $\beta_j > \gamma_j$. Thus, $rH/p_j = kN$ for some integer N. Moreover, $r|N$ since $k|H/p_j$. By Lemma 2,

$$\begin{aligned} u_{(rH/p_j)+r-1} &= u_{kN+r-1} \equiv u^N_{k+r-1}u_{r-1} = (u^r_{k+r-1})^{N/r} \\ &\equiv (s^r)^{N/r} \equiv (a_r^k)^{N/r} = a_r^{H/p_j} \pmod{M}. \end{aligned}$$

Now,

$$p_j^{\beta_j-1} \,\|\, (H/p_j),\ p_j^{\beta_j} \,\|\, h.$$

Thus,

$$u_{(rH/p_j)+r-1} \equiv a_r^{H/p_j} \not\equiv 1 \pmod{M}.$$

Consequently, $K \nmid (rH/p_j)$ and we are done. ■

REFERENCES

1. R. D. Carmichael. "On Sequences of Integers Defined by Recurrence Relations." *Quart. J. Pure Appl. Math.* 48 (1920):343-72.
2. L. M. Milne-Thomson. *The Calculus of Finite Differences*. London: Macmillan, 1960.
3. L. Somer. "The Divisibility and Modular Properties of k^{th}-Order Linear Recurrences Over the Ring of Integers of an Algebraic Number Field with Respect to Prime Ideals." Ph.D. dissertation, The University of Illinois at Urbana-Champaign, 1985.
4. M. Ward. "The Arithmetical Theory of Linear Recurring Series." *Trans. Amer. Math. Soc.* 35 (1933):600-28.
5. O. Wyler. "On Second-Order Recurrences." *Amer. Math. Monthly* 72 (1965): 500-06.

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POWERFUL k-SMITH NUMBERS

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1. INTRODUCTION

Let $S(m)$ denote the sum of the digits of the positive integer $m > 1$, and $S_p(m)$ denote the sum of all the digits of all the prime factors of m. If k is a positive integer such that $S_p(m) = kS(m)$, m is called a k-Smith number, and when $k = 1$, simply, a Smith number [5].

A powerful number is an integer m with the property that if $p|m$ then $p^2|m$. The number of positive powerful numbers less than $x > 0$ is between $cx^{1/2} - 3x^{1/3}$ and $cx^{1/2}$, where $c \approx 2.173$ (see [1]). By actual count, for example, there are 997 powerful numbers less than 250,000.

Precious little is known about the frequency of occurrence of Smith numbers or of their distribution. Wilansky [5] has found 360 Smith numbers among the integers less than 10,000, and we have shown [2] that infinitely many k-Smith numbers exist ($k \geqslant 1$). In this paper, we investigate the existence of k-Smith numbers in two complementary sets: the set of powerful numbers and its complement. A basic relationship between $S_p(m)$ and the number $N(m)$ of digits of m is first obtained. We then show (not surprisingly) that there exist infinitely many k-Smith numbers ($k \geqslant 1$) which are not powerful numbers. Finally, we use the basic relationship to show that there exist infinitely many k-Smith numbers ($k > 1$) among the integers in each of the two categories of powerful numbers: square and nonsquare.

2. TWO LEMMAS

Lemma 1: If b, k, and n are positive integers, $k < n$, and

$$t = a_k 10^k + \cdots + a_1 10 + a_0$$

is an integer with $0 < a_0 \leqslant 5$ and $0 \leqslant a_i < 5$ for $1 \leqslant i \leqslant k$, then

$$S(t(10^n - 1)^2 \cdot 10^b) = 9n.$$

Proof: If in the product of t and $10^{2n} - 2 \cdot 10^n + 1$ we replace $a_0 10^{2n}$ by

$$(a_0 - 1)10^{2n} + 9 \cdot 10^{2n-1} + \cdots + 9 \cdot 10^{n+1} + 10 \cdot 10^n,$$

we obtain

$$\begin{aligned} t(10^n - 1)^2 \cdot 10^b = [&a_k 10^{2n+k} + \cdots + a_1 10^{2n+1} + (a_0 - 1)10^{2n} \\ &+ 9 \cdot 10^{2n-1} + \cdots + 9 \cdot 10^{n+k+1} \\ &+ (9 - 2a_k)10^{n+k} + \cdots + (9 - 2a_1)10^{n+1} \\ &+ (10 - 2a_0)10^n + a_k 10^k + \cdots + a_0] \cdot 10^b. \end{aligned}$$

Each coefficient is a nonnegative integer less than 10; hence the digit sum of the product is

$$\begin{aligned} &(a_k + \cdots + a_1 + a_0 - 1) + 9(n - 1) \\ &\qquad + 10 - 2(a_k + \cdots + a_0) + (a_k + \cdots + a_0) = 9n. \end{aligned}$$

Let $m = p_1 p_2 \ldots p_r$ with $p_1, \ldots, p_r$ primes not necessarily distinct. We define

$$c_i = 9N(p_i) - S(p_i) - 9, \text{ for } 1 \leqslant i \leqslant r,$$

let

$$A = \{c_i \mid c_i > 0,\ 1 \leqslant i \leqslant r\},$$

and let n_0 be the number of integers in A.

Lemma 2: $S_p(m) < 9N(m) - \sum_A c_i - .54(r - n_0)$.

The proof involves partitioning the prime factors of m in accordance with their digit sums. Since the result is essentially a refinement of Theorem 1 in [2] (replacing c_i by the number 1 yields that theorem), and the proof is similar, we omit it here.

The above lemma is useful only if some, but not all, of the prime factors of m are known, or, if a lower bound (the higher, the better) on the number of factors of m is known.

3. POWERFUL AND k-SMITH NUMBERS

Theorem 1: There exist infinitely many k-Smith numbers which are not powerful numbers, for each positive integer k.

Proof: Let $n = 2u \not\equiv 0 \pmod{11}$. We have shown in [2] that there exists an integer $b \geqslant 1$ and an integer t belonging to the set $\{2, 3, 4, 5, 7, 8, 15\}$ such that $m = t(10^n - 1) \cdot 10^b$ is a Smith number. Since

$$\begin{aligned} 10^{2u} - 1 &= (10^2 - 1)(10^{2(u-1)} + \cdots + 10^2 + 1) \\ &\equiv 9 \cdot 11 \cdot u \pmod{11}, \end{aligned}$$

it is clear that $11 \mid m$ and $11^2 \nmid m$; hence, m is not a powerful number.

Theorem 2: These exist infinitely many square k-Smith numbers and infinitely many nonsquare powerful k-Smith numbers, for $k > 1$.

Proof: Let $m = (10^n - 1)^2$ and $n = 4u$, u any positive integer. Since $10^4 - 1$ divides $10^{4u} - 1$, $11^2 \cdot 101^2 \mid m$. Setting $p_1 = p_2 = 11$ and $p_3 = p_4 = 101$, we have

$$c_1 = c_2 = 9 \cdot 2 - 2 - 9 = 7 \quad \text{and} \quad c_3 = c_4 = 9 \cdot 3 - 2 - 8 = 16;$$

thus, by Lemma 2, $S_p(m) < 18n - 46$. Let $h = 18n - S_p(m) > 46$. We define

$$T_1 = \{5^3, 2, 2^5, 5^5, 5, 11^3, 2^3 \cdot 5^3\}$$

and

$$T_2 = \{3^4 \cdot 5^2, 15^2, 5^2, 2^2, 2^2 \cdot 3^2 \cdot 17^2, 3^2 \cdot 7^2, 2^4 \cdot 3^2\},$$

and observe that

$$\{S_p(t) \mid t \in T_1\} = \{15, 2, 10, 25, 5, 6, 21\}$$

and

$$\{S_p(t) \mid t \in T_2 = \{22, 16, 10, 4, 26, 20, 14\}$$

are complete residue systems (mod 7).

It follows that there exists an element t in either of T_1 and T_2 such that

$$S_p(t) \equiv h + (k - 2) \cdot 9n \pmod 7,\ k \geqslant 2.$$

Since $h + (k - 2) \cdot 9n > 46$ and $S_p(t) \leqslant 26$, we have

$$S_p(t) = h + (k - 2) \cdot 9n - 7b, \text{ for } b > 2.$$

Let $M = t(10^n - 1)^2 \cdot 10^b$; M is clearly a powerful number. Noting that the hypotheses of Lemma 1 are satisfied, we have $S(M) = 9n$. Thus,

$$\begin{aligned} S_p(M) &= S_p(t) + S_p((10^n - 1)^2) + S_p(10^b) \\ &= [h + (k - 2) \cdot 9n - 7b] + (18n - h) + 7b \\ &= 9kn = kS(M). \end{aligned}$$

This shows that M is a powerful k-Smith number. Now, $m = (10^n - 1)^2$ implies that

$$S_p(m) = 2S_p((10^n - 1))$$

is an even integer. We observe that this implies that h is even, and, since $n = 4u$, that b is even. Since each element of T_1 contains an odd power of a prime, and each element of T_2 is a square, it follows that M is a square if $t \in T_2$, and a nonsquare if $t \in T_1$. Q.E.D.

4. SOME OPEN QUESTIONS

It seems very likely that there exist infinitely many powerful Smith numbers, both squares and nonsquares, i.e., that Theorem 2 is true also when $k = 1$. It would be interesting to know, too, whether there are infinitely many k-Smith numbers which are n^{th} powers of integers for n greater than 2.

Several questions whose answers would provide additional insight into the distribution of k-Smith numbers, but which would appear to be more difficult to answer are also readily suggested: Are there infinitely many consecutive k-Smith numbers for any k (or for every k)? Or, more generally, do infinitely many representations of any integer n exist as the difference of k-Smith numbers for any k? Does every integer have at least one such representation? Although we have not examined an extensive list of Smith numbers, we have found among the composite integers less than 1000, for example, representations of n as the difference of Smith numbers for $n = 2, 3, 4, 5, 6, 7$, and, of course, many larger values of n. We conjecture that every integer is so representable.

Powerful k-Smith numbers occur, of course, much less frequently. Among the integers less than 1000, there are ten: 4, 27, 121, 576, 648, and 729 are powerful Smith numbers, and 32, 361, 200, and 100 are powerful k-Smith numbers for $k = 2, 2, 9$, and 14, respectively. Unexpectedly, however, the frequency with which Smith numbers occur among the powerful numbers less than 1000 is nearly five times as great as the frequency of occurrence among the composite integers less than 1000 which are not powerful. Is this related to the smallness of our sample, or is there another explanation? Finally, in view of the fact that there exist infinitely many representations of every integer as the difference of two powerful numbers [3], we ask: "Which integers are representable as the difference of powerful k-Smith numbers?"

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REFERENCES

1. S. W. Golomb. "Powerful Numbers." *Amer. Math. Monthly* 77 (1970):848-52.
2. W. L. McDaniel. "The Existence of Infinitely Many k-Smith Numbers." *The Fibonacci Quarterly* 25, no. 1 (1987):76-80.
3. W. L. McDaniel. "Representations of Every Integer as the Difference of Powerful Numbers." *The Fibonacci Quarterly* 20, no. 1 (1982):85-87.

4. S. Oltikar & K. Wayland. "Construction of Smith Numbers." *Math. Magazine* *56* (1983):36-37.
5. A. Wilansky. "Smith Numbers." *Two-Year College Math. J.* *13* (1982):21.
6. S. Yates. "Special Sets of Smith Numbers." *Math. Magazine* *59* (1986):293-296.

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ON THE DERIVATIVES OF COMPOSITE FUNCTIONS

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1. INTRODUCTION AND STATEMENT OF RESULTS

1.1 Let f, g be functions sufficiently differentiable. Put $G(z) = f(z^z)$, where $z^z := \exp(z \ln z)$ ($\exp t := e^t$, $\ln 1 = 0$). If f is the identity function, i.e., if $G(z) = z^z$, then (see [7], p. 110)

$$G^{(m)}(1) = \begin{vmatrix} 1 & -1 & 0 & \dots & 0 \\ (-1)^2 0! & \binom{1}{1} & -1 & \dots & 0 \\ (-1)^3 1! & (-1)^2 0!\binom{2}{1} & \binom{2}{2} & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{m-1}(m-3)! & (-1)^{m-2}(m-4)!\binom{m-2}{1} & (-1)^{m-3}(m-5)!\binom{m-2}{2} & \dots & -1 \\ (-1)^m (m-2)! & (-1)^{m-1}(m-3)!\binom{m-1}{1} & (-1)^{m-2}(m-4)!\binom{m-1}{2} & \dots & \binom{m-1}{m-1} \end{vmatrix} \tag{1}$$

for $m = 1, 2, 3, \ldots$. A particular case of a result obtained in this article shows that (1) may be replaced by

$$G^{(m)}(1) = \sum_{k=1}^{m} \sum_{\ell=1}^{k} (-1)^{k+m} S_1(m, k) \ell^{k-\ell} \binom{k}{\ell}, \tag{2}$$

where $S_1(m, k)$ is the sequence of Stirling numbers of the first kind, which may be defined by

$$S_1(m, 1) = (m-1)!,$$

$$S_1(m, m) = 1,$$

and

$$S_1(m, k) = (m-1)S_1(m-1, k) + S_1(m-1, k-1), \quad 1 < k < m.$$

Let us consider the sequence $\omega(m, k, j)$ defined, for $0 \leqslant j \leqslant k$, $1 \leqslant k \leqslant m$, in the following way:

$$j!\omega(m, k, j) := \binom{m}{k-j} \sum_{s=0}^{j} (-1)^s \binom{j}{s} (k-s)^{m-k+j}. \tag{3}$$

We have

$$\omega(m, k, 0) = \binom{m}{k} k^{m-k},$$

$$\omega(m, m, j) = \binom{m}{j}$$

$$\left(\text{since } \sum_{s=0}^{j} (-1) \binom{j}{s} (m-s) = j!; \text{ note that } s\binom{j+1}{s} = (j+1)\binom{j}{s-1}\right)$$

and (see [3], II, p. 38) $\omega(m, k, k) = S(m, k)$, the sequence of Stirling numbers of the second kind, which may be defined by

$$S(m,\ 1) = S(m,\ m) = 1$$

and

$$S(m,\ k) = kS(m-1,\ k) + S(m-1,\ k-1),\ 1 < k < m.$$

That kind of generalization of Stirling numbers has already been considered by Carlitz ([1]; see also [2] and [4]). In fact, we have (see [1], II, p. 243)

$$\omega(m,\ k,\ j) = (-1)^{k+m}\binom{m}{k-j}R(m-k+j,\ j,\ -k),$$

where

$$\sum_{m=0}^{\infty}\sum_{j=0}^{m}R(m,\ j,\ \lambda)\frac{x^m y^j}{m!} = \exp(\lambda x + y(e^x - 1)),\ \lambda \in \mathbb{R}.$$

The combinatorial aspect of the sequence $R(m,\ j,\ \lambda)$ and other related numbers have been studied in the aforesaid articles. We want, here, to give some complements. To begin, we state the following theorem.

Theorem 1: Suppose that $G(z)$ is defined as above; we have

$$G^{(m)}(z) = \sum_{k=1}^{m}\sum_{\ell=1}^{k}\sum_{r=1}^{\ell}\sum_{s=0}^{\ell}(-1)^{k+m}S_1(m,\ k)S(\ell,\ r)\omega(k,\ \ell,\ s)z^{rz+\ell-m}(\ln z)^s f^{(r)}(z^z). \tag{4}$$

If $f(z) \equiv z$, then $G(z) = z^z$ and (4) becomes

$$G^{(m)}(z) = \sum_{k=1}^{m}\sum_{\ell=1}^{k}\sum_{s=0}^{\ell}(-1)^{k+m}S_1(m,\ k)\omega(k,\ \ell,\ s)z^{z+\ell-m}(\ln z)^s; \tag{5}$$

we obtain (2) with $z = 1$.

While proving (4), we shall obtain some identities relating two differential operators, denoted by $f_m^{(3)}$, $f_m^{(4)}$, and defined by

$$f_0^{(3)} := f,\ f_1^{(3)}(z) := \exp\left(\frac{f'(z)}{f(z)}\right),\ f_m^{(3)} := (f_{m-1}^{(3)})_1^{(3)},\ m > 1, \tag{6}$$

and

$$f_0^{(4)} := f,\ f_1^{(4)}(z) := \exp\left(\frac{zf'(z)}{f(z)}\right),\ f_m^{(4)} := (f_{m-1}^{(4)})_1^{(4)},\ m > 1. \tag{7}$$

We shall in fact consider two well-known operators, denoted here by $f_m^{(1)}$, $f_m^{(2)}$, and defined by

$$f_0^{(1)} := f,\ f_1^{(1)}(z) := f'(z),\ f_m^{(1)} := (f_{m-1}^{(1)})_1^{(1)},\ m > 1, \tag{6'}$$

and

$$f_0^{(2)} := f,\ f_1^{(2)}(z) := zf'(z),\ f_m^{(2)} := (f_{m-1}^{(2)})_1^{(2)},\ m > 1. \tag{7'}$$

Those operators have been studied for a very long time. The operator f_1 is the ordinary derivative of f; it is easy to verify that

$$f_m^{(2)}(z) = \sum_{k=1}^{m}S(m,\ k)z^k f^{(k)}(z).$$

Of course $\ln f_1^{(3)}$ is nothing but the logarithmic derivative of f. The operator $\ln f_1^{(4)}$ is useful in geometric function theory; for example, a function $f(z)$, holomorphic in the unit disk, is called starlike (see [6], p. 46) if

$$|f_1^{(4)}(z)| \geqslant 1$$

in that disk.

1.2 A classical formula of Faa Di Bruno ([3], I, p. 148; [5], p. 177) says that if $h(z) := f(g(z))$ then

$$h^{(m)}(z) = \sum_{k=1}^{m} \sum_{\pi(m,k)} c(k_1, \ldots, k_m) \prod_{j=1}^{m} (g^{(j)}(z))^{k_j} \cdot f^{(k)}(g(z)) \tag{8}$$

where $\pi(m, k)$ means that the summation is extended over all nonnegative integers $k_1, \ldots, k_m$ such that $k_1 + 2k_2 + \cdots + mk_m = m$ and $k_1 + k_2 + \cdots + k_m = k$; we have put

$$c(k_1, \ldots, k_m) := \frac{m!}{k_1! \ldots k_m!(1!)^{k_1} \ldots (m!)^{k_m}}.$$

Formula (8) is equivalent to

$$\ln h_m^{(3)}(z) = \sum_{k=1}^{m} \sum_{\pi(m,k)} c(k_1, \ldots, k_m) \prod_{j=1}^{m} (g^{(j)}(z))^{k_j} \cdot \ln f_k^{(3)}(g(z)). \tag{8'}$$

It can be proved in several ways; a simple proof is contained in [8]. We can prove the next theorem using only the principle of mathematical induction.

Theorem 2: If $h(z) := f(g(z))$, then we have the identities

$$h_m^{(2)}(z) = \sum_{k=1}^{m} \sum_{\pi(m,k)} c(k_1, \ldots, k_m) \prod_{j=1}^{m} (g_j^{(2)}(z))^{k_j} \cdot f_k^{(1)}(g(z)) \tag{9}$$

and

$$\ln h_m^{(4)}(z) = \sum_{k=1}^{m} \sum_{\pi(m,k)} c(k_1, \ldots, k_m) \prod_{j=1}^{m} (\ln g_j^{(4)}(z))^{k_j} \cdot \ln f_k^{(4)}(g(z)). \tag{9'}$$

Formula (9′) may also be written in the form

$$H_m^{(2)}(z) = \sum_{k=1}^{m} \sum_{\pi(m,k)} c(k_1, \ldots, k_m) \prod_{j=1}^{m} (g_j^{(2)}(z))^{k_j} \cdot f_k^{(2)}(e^{g(z)}), \tag{9''}$$

where $H(z) := f(\exp(g(z)))$.

1.3 If f^{-1} denotes the inverse function of f [i.e.,

$$f(f^{-1}(z)) \equiv f^{-1}(f(z)) \equiv z],$$

then (see [3], I, p. 161), for $m = 2, 3, 4, \ldots,$

$$(f^{-1})_m^{(1)}(z) \tag{10}$$

$$= \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} \frac{(-1)^k (m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \prod_{j=2}^{m} (f^{(j)}(f^{-1}(z)))^{k_j} \cdot (f'(f^{-1}(z)))^{-m-k},$$

where $\pi_1(m, k)$ means that the summation is extended over all nonnegative integers $k_2, \ldots, k_m$ such that $2k_2 + \cdots + mk_m = m + k - 1$ and $k_2 + \cdots + k_m = k$. Here,

$$c_1(k_1, \ldots, k_m) := c(0, k_2, \ldots, k_m).$$

The same kind of reasoning which could be used to prove (9) or (9′) will help us to verify the following theorem.

Theorem 3: If f^{-1} denotes the inverse function of f, then the following identities are valid for $m = 2, 3, 4, \ldots$:

$$(f^{-1})_m^{(2)} = \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} \frac{(-1)^k(m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \tag{11}$$

$$\cdot \prod_{j=2}^{m} (\ln f_j^{(3)}(f^{-1}(z)))^{k_j} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k};$$

$$\ln(f^{-1})_m^{(3)}(z) = \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} \frac{(-1)^k(m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \tag{11$'$}$$

$$\cdot \prod_{j=2}^{m} (f_j^{(2)}(f^{-1}(z)))^{k_j} \cdot (f_1^{(2)}(f^{-1}(z)))^{-m-k};$$

$$\ln(f^{-1})_m^{(4)}(z) = \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} \frac{(-1)^k(m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \tag{11$''$}$$

$$\cdot \prod_{j=2}^{m} (\ln f_j^{(4)}(f^{-1}(z)))^{k_j} \cdot (\ln f_1^{(4)}(f^{-1}(z)))^{-m-k}.$$

It is to be noted that (11′) may be obtained from (11″) by replacing $f(z)$ by $\exp f(z)$: also, if we replace $f(z)$ by $f(e^z)$ in (11) , then we obtain (11″). The distinction between formulas (8) and (9) and formulas (10) and (11) is also to be observed. Finally, while the identity

$$\ln\left(f(z)^{g(z)}\right)_m^{(3)} = \sum_{k=0}^{m} \binom{m}{k} g^{(m-k)}(z) \ln f_k^{(3)}(z)$$

is nothing but the Leibnitz formula, we have

$$\ln\left(f(z)^{g(z)}\right)_m^{(4)} = \sum_{k=0}^{m} \binom{m}{k} g_{m-k}^{(2)}(z) \ln f_k^{(4)}(z)$$

or, what is the same thing (see [5], p. 222):

$$(f(z)g(z))_m^{(2)} = \sum_{k=0}^{m} \binom{m}{k} f_k^{(2)}(z) g_{m-k}^{(2)}(z).$$

2. COMPLEMENTARY RESULTS

It follows from the recurrence relations for Stirling's numbers that:

Lemma 1: We have, for $m = 1, 2, 3, \ldots,$

$$f_m^{(2)}(z) = \sum_{k=1}^{m} S(m, k) z^k \cdot f_k^{(1)}(z) \tag{12}$$

and

$$z^m f_m^{(1)}(z) = \sum_{k=1}^{m} (-1)^{k+m} S_1(m, k) \cdot f_k^{(2)}(z). \tag{12$'$}$$

To obtain (4), we shall also need the following lemma.

Lemma 2: The sequence $\omega(m, k, j)$, defined by (3), satisfies the following recurrence relation:

$$\omega(m, 1, 0) = m, \ \omega(m, m, j) = \binom{m}{j} \quad (0 \leqslant j \leqslant m),$$
$$\omega(m, k, k) = S(m, k) \quad (1 \leqslant k \leqslant m),$$

and
$$\omega(m+1, k, 0) = k\omega(m, k, 0) + \omega(m, k-1), 0) + \omega(m, k, 1), \ 1 < k \leqslant m;$$

$$\omega(m+1, k, j) = k\omega(m, k, j) + (j+1)\omega(m, k, j+1) + \omega(m, k-1, j-1) + \omega(m, k-1, j), \ 1 \leqslant j < k \leqslant m. \tag{13}$$

Proof: If $m = 1$, then $k = 1$ and $j = 0$ or 1; in that case the relation (13) is trivial. Also, since

$$\omega(m, k, 0) = \binom{m}{k}k^{m-k}$$

and
$$\omega(m, k, 1) = (k^{m-k+1} - (k-1)^{m-k+1})\binom{m}{k-1},$$

we have immediately

$$k\omega(m, k, 0) + \omega(m, k-1, 0) + \omega(m, k, 1) = \omega(m+1, k, 0), \ 1 < k \leqslant m.$$

Now, for $1 \leqslant j < k$,

$$j![k\omega(m, k, j) + (j+1)\omega(m, k, j+1) + \omega(m, k-1, j-1) + \omega(m, k-1, j)]$$
$$= k\binom{m}{k-j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-s)^{m-k+j} + \binom{m}{k-j-1}\sum_{s=0}^{j+1}(-1)^s\binom{j+1}{s}(k-s)^{m-k+j+1}$$
$$+ j\binom{m}{k-j}\sum_{s=0}^{j-1}(-1)^s\binom{j-1}{s}(k-1-s)^{m-k+j} + \binom{m}{k-j-1}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-1-s)^{m-k+j+1}$$
$$= \binom{m}{k-j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-s)^{m+1-k+j} + \binom{m}{k-j-1}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-s)^{m+1-k+j}$$
$$= \binom{m+1}{k-j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-s)^{m+1-k+j} = j!\omega(m+1, k, j).$$

This completes the proof of Lemma 2.

3. PROOFS OF THE THEOREMS

The proof of Theorem 2 is similar to that of Theorem 3; it suffices to define the sequence corresponding to (11*) below in an appropriate manner.

Proof of Theorem 1: Let us verify that if $G(z) := f(z^z)$ then

$$G_m^{(2)}(z) = \sum_{k=1}^{m}\sum_{j=0}^{k}\omega(m, k, j)z^k(\ln z)^j f_k^{(2)}(z^z). \tag{14}$$

It is sufficient to show that if we write

$$G_m^{(2)}(z) = \sum_{k=1}^{m}\sum_{j=0}^{k}w(m, k, j)z^k(\ln z)^j f_k^{(2)}(z^z)$$

then the sequence $w(m, k, j)$ satisfies the same recurrence relation (13) as $\omega(m, k, j)$ with the same initial conditions. Observe that

$$(f + g)_1^{(2)}(z) \equiv f_1^{(2)}(z) + g_1^{(2)}(z);$$

it follows from (7′) that

$$\begin{aligned} G_{m+1}^{(2)}(z) &= \sum_{k=1}^{m} \sum_{j=0}^{k} kw(m, k, j)z^k(\ln z)^j f_k^{(2)}(z^z) \\ &+ \sum_{k=1}^{m} \sum_{j=0}^{k} jw(m, k, j)z^k(\ln z)^{j-1} f_k^{(2)}(z^z) \\ &+ \sum_{k=1}^{m} \sum_{j=0}^{k} w(m, k, j)z^{k+1}(\ln z)^{j+1} f_{k+1}^{(2)}(z^z) \\ &+ \sum_{k=1}^{m} \sum_{j=0}^{k} w(m, k, j)z^{k+1}(\ln z)^{j} f_{k+1}^{(2)}(z^z). \end{aligned} \tag{15}$$

Relation (13) then follows immediately if we change, respectively, j to $j + 1$, j to $j - 1$ and k to $k - 1$, and k to $k - 1$ in the second, third, and fourth double summation of the right-hand side of (15). To see that $w(m, k, j)$ satisfies the same initial conditions as $\omega(m, k, j)$, we may use the observations made after the definition (3).

Now, using (12′) and (14), then (12), we obtain

$$\begin{aligned} G_m^{(1)}(z) &= \sum_{k=1}^{m} (-1)^{k+m} S_1(m, k) z^{-m} G_k^{(2)}(z) \\ &= \sum_{k=1}^{m} \sum_{\ell=1}^{k} \sum_{s=0}^{\ell} (-1)^{k+m} S_1(m, k)\omega(k, \ell, s) z^{\ell-m}(\ln z)^s \cdot f_\ell^{(2)}(z^z) \\ &= \sum_{k=1}^{m} \sum_{\ell=1}^{k} \sum_{s=0}^{\ell} \sum_{r=1}^{\ell} (-1)^{k+m} S_1(m, k) S(\ell, r)\omega(k, \ell, s) z^{rz+\ell-m}(\ln z)^s f_r^{(1)}(z^z). \end{aligned}$$

Proof of Theorem 3: It remains only to prove (11). That formula is clear for $m = 2$. Suppose that it is satisfied for a given $m > 2$. Then

$$\begin{aligned} (f^{-1})_{m+1}^{(2)}(z) &= \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} (-1)^k \frac{(m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \\ &\quad \cdot \prod_{i=2}^{m} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i} \\ &\quad \cdot \sum_{j=2}^{m} k_j \frac{\ln f_{j+1}^{(3)}(f^{-1}(z))}{\ln f_j^{(3)}(f^{-1}(z))} (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1} \\ &\quad - \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} (-1)^k \frac{(m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \\ &\quad \cdot \prod_{i=2}^{m} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i} \cdot \ln f_2^{(3)}(f^{-1}(z)) \\ &\quad \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-2}. \end{aligned} \tag{16}$$

Let us put

$$k_i^{(1)} = \begin{cases} k_2 + 1, & i = 2 \\ k_i, & 2 < i \leqslant m \\ 0, & i = m + 1, \end{cases}$$

$$k_i^{(j)} = \begin{cases} k_i, & 2 \leqslant i < j \\ k_j - 1, & i = j \\ k_{j+1} + 1, & i = j + 1 \\ k_i, & j + 1 < i \leqslant m \\ 0, & i = m + 1,\ 2 \leqslant j < m, \end{cases} \tag{11*}$$

and

$$k_i^{(m)} = \begin{cases} k_i, & 2 \leqslant i < m \\ k_m - 1, & i = m \\ 1, & i = m + 1. \end{cases}$$

We have

$$\sum_{i=2}^{m+1} ik_i^{(1)} = m + k + 1, \quad \sum_{i=2}^{m+1} k_i^{(1)} = k + 1,$$

and

$$\sum_{i=2}^{m+1} ik_i^{(j)} = m + k, \quad \sum_{i=2}^{m+1} k_i^{(j)} = k, \quad 1 < j \leqslant m.$$

Identity (16) may thus be written in the form

$$\begin{aligned}(f^{-1})_{m+1}^{(2)}(z) = &\sum_{j=2}^{m} \sum_{k=1}^{m-1} \sum_{\pi_1^{(j)}(m+1,\, k)} (-1)^k \frac{(m+k-1)!}{m!} c_1(k_1^{(j)}, \ldots, k_m^{(j)})(j+1)k_{j+1}^{(j)} \\ &\cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^{(j)}} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1} \\ &- \sum_{k=1}^{m-1} \sum_{\pi_1^{(1)}(m+1,\, k+1)} (-1)^k \frac{(m+k)!}{m!} c_1(k_1^{(1)}, \ldots, k_m^{(1)}) \cdot 2k_2^{(1)} \\ &\cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^{(1)}} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-2},\end{aligned} \tag{17}$$

where $\pi_1^{(j)}(m + 1, k)$ means that the summation is extended over the numbers $k_2^{(j)}, \ldots, k_m^{(j)}$, related to the numbers $k_2, \ldots, k_m$ by (11*), satisfying

$$2k_2^{(j)} + \cdots + mk_m^{(j)} = m + k, \quad k_2^{(j)} + \cdots + k_m^{(j)} = k, \quad 1 < j \leqslant m;$$

$\pi_1^{(1)}(m + 1, k + 1)$ means that

$$2k_2^{(1)} + \cdots + mk_m^{(1)} = m + k + 1, \quad k_2^{(1)} + \cdots + k_m^{(1)} = k + 1.$$

We have put

$$c_1(k_1^{(j)}, \ldots, k^{(j)}) := \frac{m!}{k_2^{(j)}! \ldots k_m^{(j)}!(2!)^{k_2^{(j)}} \ldots (m!)^{k_m^{(j)}}}, \quad 1 \leqslant j \leqslant m.$$

Replacing k by $k - 1$ in the last summation of (17), we readily obtain

$$(f^{-1})^{(2)}_{m+1}(z) = \sum_{j=2}^{m} \sum_{k=1}^{m-1} \sum_{\pi_1^{(j)}(m+1,k)} (-1)^k \frac{(m+k-1)!}{m!} c_1(k_1^{(j)}, \ldots, k_m^{(j)})(j+1)k_{j+1}^{(j)}$$
$$\cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^{(j)}} \cdot \ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1}$$
$$+ \sum_{k=2}^{m} \sum_{\pi_1^{(1)}(m+1,k)} (-1)^k \frac{(m+k-1)!}{m!} c_1(k_1^{(1)}, \ldots, k_m^{(1)}) \cdot 2k_2^{(1)}$$
$$\cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^{(1)}} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1}. \tag{18}$$

Now, let $(k_2^*, \ldots, k_{m+1}^*)$ be a solution of the system

$$2k_2^* + \cdots + (m+1)k_{m+1}^* = m + k,$$
$$k_2^* + \cdots + k_{m+1}^* = k,$$
$$k_j^* \geqslant 0, \; 1 < j \leqslant m+1, \; (1 \leqslant k \leqslant m).$$

(i) If $k_2^* \neq 0$, then $k_{m+1}^* = 0$ (otherwise, $k_{m+1}^* = 1$ and $2K_2^* + \cdots + mk_m^* = k - 1 = k_2^* + \cdots + k_m^*$, which implies that $k_2^* = \cdots = k_m^* = 0$); in that case, to each solution $(k_2^*, \ldots, k_m^*, 0)$ there corresponds a solution $(k_2^{(1)}, \ldots, k_m^{(1)}, 0)$; it is possible, since the hypothesis $k_2^* \neq 0$ implies that $k_2 = k_2^{(1)} - 1 = k_2^* - 1 \geqslant 0$. Conversely, to each solution $(k_2^{(1)}, \ldots, k_{m+1}^{(1)})$, there corresponds a solution $(k_2^*, \ldots, k_m^*, k_{m+1}^* = 0)$.

(ii) Suppose that $1 < j < m$. If $k_{j+1}^* \neq 0$ then $k_{m+1}^* = 0$; in that case, to each solution $(k_2^*, \ldots, k_{m+1}^*)$, there corresponds a solution $(k_2^{(j)}, \ldots, k_{m+1}^{(j)} = 0)$; it is possible, since $k_{j+1} = k_{j+1}^{(j)} - 1 = k_{j+1}^* - 1 \geqslant 0$.

(iii) If $k_{m+1}^* \neq 0$, then $k_{m+1}^* = 1$ and $k_2^* = \cdots = k_m^* = 0$, $k = 1$. In that case, to the solution $(0, \ldots, 0, k_{m+1}^* = 1)$, there corresponds the solution $(0, \ldots, 0, k_{m+1}^{(m)} = 1)$.

Rearranging the terms in the summations of (18), we may thus write

$$(f^{-1})^{(2)}_{m+1}(z) = \sum_{j=2}^{m} \sum_{k=1}^{m-1} \sum_{\pi_1^*(m+1,k)} (-1)^k \frac{(m+k-1)!}{(m+1)!} c_1(k_1^*, \ldots, k_{m+1}^*)(j+1)k_{j+1}^*$$
$$\cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^*} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1}$$
$$+ \sum_{k=2}^{m} \sum_{\pi_1^*(m+1,k)} (-1)^k \frac{(m+k-1)!}{(m+1)!} c_1(k_1^*, \ldots, k_{m+1}^*) \cdot 2k_2^*$$
$$\cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^*} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1}, \tag{19}$$

where

$$2k_2^* + \cdots + (m+1)k_{m+1}^* = m + k, \; k_2^* + \cdots + k_{m+1}^* = k,$$

and

$$c_1(k_1^*, \ldots, k_{m+1}^*) := \frac{(m+1)!}{k_1^* \ldots k_{m+1}^*!(1!)^{k_1^*} \ldots ((m+1)!)^{k_{m+1}^*}}.$$

In the first summation of (19) we may add the terms corresponding to $k = m$ since $2k_2^* + \cdots + (m+1)k_{m+1}^* = 2m$, $k_2^* + \cdots + k_{m+1}^* = m$ imply

$$(m-1)k_{m+1}^* + \cdots + 2k_4^* + k_3^* = 0,$$

i.e., $k_3^* = \cdots = k_{m+1}^* = 0$. Similarly, we may add, in the second summation of (19), the terms corresponding to $k = 1$. Writing

$$\sum_{j=2}^{m} (j+1)k_{j+1}^* = m + k - 2k_2^*,$$

we obtain

$$(f^{-1})_{m+1}^{(2)} = \sum_{k=1}^{m} \sum_{\pi_1^*(m+1,k)} (-1)^k \frac{(m+k)!}{(m+1)!} c_1(k_1^*, \ldots, k_{m+1}^*) \tag{20}$$

$$\cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^*} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-1-k}.$$

This completes the proof of Theorem 3.

4. SOME REMARKS AND EXAMPLES

4.1 Remark on Taylor's formula: Let us write

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!}(g(z - z_0))^k, \; a_0 := f(z_0). \tag{21}$$

We have, in a neighborhood of $z = z_0$, $(g(0) = 0)$,

$$a_k = (f(z_0 + g^{-1}(z))^{(k)}(z = 0).$$

Put

$$f_1(z_0) := a_1 = \frac{f'(z_0 + g^{-1}(0))}{g'(g^{-1}(0))} \quad \text{and } f_k := (f_{k-1})_1, \; k > 1. \tag{22}$$

In order that $a_k \equiv f_k(z_0)$, we must have

$$(f(z_0 + g^{-1}(z)))^{(k)}(z = 0) \equiv \frac{f^{(k)}(z_0 + g^{-1}(0))}{(g'(g^{-1}(0))^k},$$

whence

$$f(z_0 + g^{-1}(z)) \equiv \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0 + g^{-1}(0))}{k!}\left(\frac{z}{g'(g^{-1}(0))}\right)^k$$

$$= f\left(\frac{z}{g'(g^{-1}(0))} + z_0 + g^{-1}(0)\right),$$

in a neighborhood of $z = 0$. It follows that if g is normalized by the conditions

$$g(0) = 0,\ g'(0) = 1 \tag{24}$$

then $g(z) \equiv z$. The unique function g, normalized by (24), for which the expansion (21) is valid, where a_k is the k^{th} iteration of the operator induced by $f_1 := a_1$, is the identity function $g(z) = z$; in that case, $f_1 = f'$. A similar argument may be made for expansions of the form

$$\sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\ln \frac{z}{z_0}\right)^k,\quad \sum_{k=0}^{\infty} \frac{\ln a_k}{k!}(z - z_0)^k,\quad \sum_{k=0}^{\infty} \frac{\ln a_k}{k!}\left(\ln \frac{z}{z_0}\right)^k. \tag{25}$$

It is in fact easy to come down to the previous case. For the expansions (25) we have, respectively, $f_1 = f_1^{(2)}$, $f_1 = f_1^{(3)}$, $f_1 = f_1^{(4)}$ [see (6), (7), and (7')].

It is of interest to observe here that for expansions of the form

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!}(g(z) - g(z_0))^k,\quad a_0 := f(z_0), \tag{21'}$$

we have always that a_k is the k^{th} iteration of the operator induced by

$$f_1(z_0) := \frac{f'(z_0)}{g'(z_0)}.$$

To see this, we may easily show that

$$f_k(z_0) = \left.\frac{\partial^k f(g^{-1}(z + g(z_0)))}{\partial z^k}\right|_{z=0},\quad k = 1, 2, 3, \ldots .$$

4.2 (i) Let us take $f(z) = e^z$, then $z = 1$, in (4); we obtain:

$$(\exp(z^z))_m^{(1)}(z = 1) = e\sum_{k=1}^{m}\sum_{\ell=1}^{k}\sum_{r=1}^{\ell}(-1)^{k+m}S_1(m, k)S(\ell, r)\cdot\binom{k}{\ell}\ell^{k-\ell}. \tag{26}$$

(ii) If $g(z) = z^z$ in (9'), then we obtain, using (14) and $g_j^{(4)}(z) = z^z e^{jz}$, $j = 0, 1, 2, \ldots$, the identity

$$\sum_{\pi(m,k)} c(k_1, \ldots, k_m)\prod_{j=1}^{m}(z + j)^{k_j} = \sum_{j=0}^{k}\omega(m, k, j)z^j,\quad z \in \mathbb{C}. \tag{27}$$

Note that we can deduce from (8) (see [5], p. 191) the relation

$$\sum_{\pi(m,k)}\frac{k!}{k_1!\ \cdots\ k_m!}\prod_{j=1}^{m} j^{k_j} = \binom{m + k - 1}{m - k},\quad 1 \leqslant k \leqslant m.$$

(iii) Lagrange expansion [concerning a root of equations of the form $z = a + \xi\phi(z)$, $\xi \to 0$] in conjunction with (8) may be used to prove the formula

$$\sum_{\pi(m,k)} c(k_1, \ldots, k_m)\prod_{j=1}^{m}((\phi^j(a))^{(j-1)})^{k_j} \equiv \binom{m-1}{k-1}(\phi^m(a))^{(m-k)}, \tag{28}$$

$$1 \leqslant k \leqslant m,$$

which implies that

$$\sum_{\pi(m)} c(k_1, \ldots, k_m)\prod_{j=1}^{m}((\phi^j(a))^{(j-1)})^{k_j} \equiv e^{-a}(\phi^m(a)e^a)^{(m-1)}, \tag{29}$$

where $\pi(m)$ means that the summation is extended over all nonnegative integers $k_1, \ldots, k_m$ such that $k_1 + 2k_2 + \cdots + mk_m = m$.

REFERENCES

1. L. Carlitz. "Weighted Stirling Numbers of the First and Second Kind, I," and "Weighted Stirling Numbers of the First and Second Kind, II." *The Fibonacci Quarterly 18*, no. 2 (1980):147-62, and no. 3 (1980):242-57.
2. Ch. A. Charalambides. "On Weighted Stirling and Other Related Numbers and Some Combinatorial Applications." *The Fibonacci Quarterly 22*, no. 4 (1984): 296-309.
3. L. Comtet. *Analyse Combinatoire*, vols. I and II. Paris: Presses Universitaires de France, 1970.
4. F. T. Howard. "Weighted Associated Stirling Numbers." *The Fibonacci Quarterly 22*, no. 2 (1984):156-65.
5. J. Riordan. *Combinatorial Identities*. New York: Wiley & Sons, 1968.
6. St. Ruscheweyh. *Convolutions in Geometric Function Theory*. Séminaire de Mathématiques Supérieurs. Montréal: Les Presses de l'Université de Montréal, 1982.
7. I. J. Schwatt. *An Introduction to the Operations with Series*. New York: Chelsea Publishing Company, 1924.
8. R. Steven. "The Formula of Faa Di Bruno." *Amer. Math. Monthly 87*, no. 10 (1980):805-09.

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LETTER TO THE EDITOR

NOTE ON "REPRESENTING $\binom{2n}{n}$ AS A SUM OF SQUARES"

[Neville Robbins, *The Fibonacci Quarterly* 25, no. 1 (1987):29]

In addition to the theorems Dr. Robbins presented, it is the case that

$$\binom{2n}{n} = \sum_{i=\emptyset}^{n} \binom{n}{i}^2. \tag{1}$$

Proof: In general, the coefficients of terms in a polynomial that is the product of two other polynomials is the convolution of the terms of the two-factor polynomials. In particular, the coefficients of the terms in the binomial expansion can be expressed by such a convolution:

$$\binom{p}{q} = \sum_{i=\emptyset}^{n} \binom{p-r}{i}\binom{r}{q-i}. \tag{2}$$

If we chose $r = q = n$, then $p = 2n$, and we get

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}, \tag{3}$$

which is obviously equivalent to (1).

Equation (2) is a rendering of the first form of the Vandermonde convolution (see [1]), with the term $\binom{n}{p}$ replaced by 1. Equation (3) is a particular case of that, with the substitutions noted.

Corollary: $n!$ can be written recursively not only as $n(n-1)!$, but also (for even n) as

$$n! = (n/2)!^2 \sum_{i=\emptyset}^{n/2} \binom{n/2}{i}^2. \tag{4}$$

Proof: This is made clear by rewriting the summation according to (1) above:

$$n! = (n/2)!^2 \binom{n}{n/2}. \tag{5}$$

We then expand the combination $\binom{n}{n/2}$ to give,

$$n! = (n/2)!^2 \frac{n!}{(n/2)!(n-n/2)!}, \tag{6}$$

which is fairly obviously an identity.

Reference

1. John Riordan. *Combinatorial Identities*. New York: Wiley & Sons, 1968, p. 15, Eq. (9), form 1.

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A NOTE ON A GENERALIZATION OF EULER'S ϕ FUNCTION

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(Submitted August 1985)

P. G. Garcia and Steve Ligh [3] introduced the following generalization of the Euler function $\phi(n)$: For an arithmetic progression

$$D(s, d, n) = \{s, s + d, \ldots, s + (n - 1)d\},$$

where $(s, d) = 1$, let $\phi(s, d, n)$ denote the number of elements in $D(s, d, n)$ that are relatively prime to n. Observe that $\phi(1, 1, n) \equiv \phi(n)$.

Garcia and Ligh showed that $\phi(s, d, n)$ is multiplicative in n, i.e., for $(m, n) = 1$, we have

$$\phi(s, d, mn) = \phi(s, d, m)\phi(s, d, n)$$

(cf. [3], Theorem 1), and deduced the formula:

$$\phi(s, d, p^k) = \begin{cases} p^k\left(1 - \dfrac{1}{p}\right), & \text{if } p \nmid d, \\ p^k, & \text{if } p \mid d, \end{cases} \tag{1}$$

(cf. [3], Lemma 2).

The aim of this note is to establish an asymptotic formula for the summatory function of $\phi(s, d, n)$ using an elementary method.

Let μ denote the Möbius function, I the Dirac function, for which

$$I(n) = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases}$$

and let I_d be the arithmetic function defined by $I_d(n) = I((n, d))$. We need the following result, which is the generalization of the familiar Dedekind-Liouville evaluation of $\phi(n)$:

$$\phi(n) = \sum_{er = n} \mu(e)r.$$

Lemma 1: $\phi(s, d, n) = \sum_{er = n} \mu(e)I_d(e)r \equiv \sum_{\substack{er = n \\ (e, d) = 1}} \mu(e)r.$ (2)

Proof: The functions μ, I_d, and $\mu \cdot I_d$ are multiplicative [moreover, I_d is totally multiplicative, i.e., $I_d(mn) = I_d(m)I_d(n)$ for arbitrary m and n] and so the right-hand sum, being the Dirichlet convolution of two multiplicative functions, is also multiplicative. It has been noted that $\phi(s, d, n)$ is multiplicative; thus, it is enough to verify the above identity for $n = p^k$. We have:

$$\sum_{er = p^k} \mu(e)I_d(e)r = \begin{cases} p^k - p^{k-1} = p^k\left(1 - \dfrac{1}{p}\right), & \text{if } p \nmid d \\ p^k, & \text{if } p \mid d \end{cases}$$

$$= \phi(s, d, p^k) \text{ by (1).}$$

Corollary 1: $\sum_{er=n} I_d(e)\phi(s, d, r) \equiv \sum_{\substack{er=n\\(e, d)=1}} \phi(s, d, r) = n.$ (3)

Proof: By Lemma 1 we have $\phi(s, d, n) = \mu \cdot I_d * E$, where $E(n) = n$ and $*$ denotes the Dirichlet convolution. Thus,

$$I_d * \phi(s, d, n) = I_d * \mu \cdot I_d * E,$$

and, using the distributivity property of the totally multiplicative functions (see, for example, [4], Theorem 1):

$$I_d * \phi(s, d, n) = I_d(U * \mu) * E,$$

where $U(n) = 1$ and $(U * \mu) * E = I * E = E$. Hence,

$$I_d * \phi(s, d, n) = E$$

and the proof is complete.

Remark 1: The author thanks the referee for the following direct proof of (3):

We write n as $n = PQ$, $(P, Q) = 1$, where $(P, d) = 1$ and $(Q, d) > 1$ or $Q = 1$. By the multiplicative property of $\phi(s, d, n)$,

$$\phi(s, d, n) = \phi(s, d, P)\phi(s, d, Q) = \phi(P)Q$$

(cf. [5], Lemma 2). Thus,

$$\sum_{\substack{er=n\\(e, d)=1}} \phi(s, d, r) = \sum_{J|P} \phi(s, d, jQ) = \sum_{J|P} \phi(s, d, j)\phi(s, d, Q)$$
$$= Q \sum_{J|P} \phi(j) = PQ = n.$$

Remark 2: Ligh and Garcia have obtained a formula for $\sum_{r|n} \phi(s, d, r)$ (see [5], Theorem 2).

Let $J(n)$ denote the Jordan totient function of second order,

$$J(n) = n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) \quad \text{(see [2], p. 147).}$$

Lemma 2 (cf. [1], Lemma 5.1):

$$\sum_{n=1}^{\infty} \frac{\mu(n)I_d(n)}{n^2} = \frac{6d^2}{\pi^2 J(d)} \tag{4}$$

Proof: The series is absolutely convergent and the general term is a multiplicative function of n; thus, it can be expanded into an infinite product of the Euler type (see [2], § 17.4):

$$\sum_{n=1}^{\infty} \frac{\mu(n)I_d(n)}{n^2} = \prod_p \left(\sum_{i=0}^{\infty} \frac{\mu(p^i)I_d(p^i)}{p^{2i}}\right) = \prod_{p \nmid d} \left(1 - \frac{1}{p^2}\right)$$
$$= \frac{\prod_p \left(1 - \frac{1}{p^2}\right)}{\prod_{p|d} \left(1 - \frac{1}{p^2}\right)} = \frac{d^2}{\zeta(s)J(d)} = \frac{6d^2}{\pi^2 J(d)},$$

where $\zeta(s)$ is the Riemann Zeta function.

We shall use the following well-known estimates.

Lemma 3: $$\sum_{n \leqslant x} n = \frac{x^2}{2} + 0(x) \tag{5}$$

$$\sum_{n \leqslant x} \frac{1}{n} = 0(\log x) \tag{6}$$

$$\sum_{n > x} \frac{1}{n^2} = 0\left(\frac{1}{x}\right) \tag{7}$$

Theorem: $$\sum_{n \leqslant x} \phi(s, d, n) = \frac{3d^2}{\pi^2 J(d)} x^2 + 0(x \log x). \tag{8}$$

Proof: Using (2) and (5), we have:

$$\begin{aligned}\sum_{n \leqslant x} \phi(s, d, n) &= \sum_{er \leqslant x} \mu(e) I_d(e) r = \sum_{e \leqslant x} \mu(e) I_d(e) \sum_{r \leqslant x/e} r \\ &= \sum_{e \leqslant x} \mu(e) I_d(e) \left\{\frac{x^2}{2e^2} + 0\left(\frac{x}{e}\right)\right\} = \frac{x^2}{2} \sum_{e \leqslant x} \frac{\mu(e) I_d(e)}{e^2} + 0\left(x \sum_{e \leqslant x} \frac{1}{e}\right) \\ &= \frac{x^2}{2} \sum_{e=1}^{\infty} \frac{\mu(e) I_d(e)}{e^2} + 0\left(x^2 \sum_{e > x} \frac{1}{e^2}\right) + 0\left(x \sum_{e \leqslant x} \frac{1}{e}\right).\end{aligned}$$

And now, by (4), (7), and (6),

$$\begin{aligned}\sum_{n \leqslant x} \phi(s, d, n) &= \frac{x^2}{2} \cdot \frac{6d^2}{\pi^2 J(d)} + 0(x) + 0(x \log x) \\ &= \frac{3d^2}{\pi^2 J(d)} x^2 + 0(x \log x).\end{aligned}$$

Corollary 2: The average order of $\phi(s, d, n)$ is $\dfrac{6d^2}{\pi^2 J(d)} n$.

Proof: From (8), we have

$$\frac{1}{x} \sum_{n \leqslant x} \phi(s, d, n) \sim \frac{1}{x} \sum_{n \leqslant x} f_d(n), \text{ where } f_d(n) = \frac{6d^2}{\pi^2 J(d)} n.$$

For $d = 1$, we reobtain Mertens' formula:

Corollary 3: $$\sum_{n \leqslant x} \phi(n) = \frac{3}{\pi^2} x^2 + 0(x \log x).$$

REFERENCES

1. E. Cohen. "Arithmetical Functions Associated with the Unitary Divisors of an Integer." *Math z.* *74* (1960):66-80.
2. L. E. Dickson. *History of the Theory of numbers.* Vol. I. New York: Chelsea, 1952.
3. P. G. Garcia & Steve Ligh. "A Generalization of Euler's ϕ Function." *The Fibonacci Quarterly 21*, no. 1 (1983):26-28.
4. J. Lambek. "Arithmetical Functions and Distributivity." *Amer. Math. Monthly 73* (1966):969-73.

5. Steve Ligh & P. G. Garcia. "A Generalization of Euler's ϕ Function, II." *Jath. Japonica* 30 (1985):519-22.

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Announcement

THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 25-29, 1988
Department of Mathematics, University of Pisa
Pisa, Italy

International Committee

Horadam, A.F. (Australia), *Co-Chairman*
Philippou, A.N. (Greece), *Co-Chairman*
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Robert Dvornicich, *Chairman*
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FIBONACCI'S STATUE

Have you ever seen Fibonacci's portrait? This photo is a close-up of the head of the statue of Leonardo Pisano in Pisa, Italy, taken by Frank Johnson in 1978.

Since Fibonacci's statue was difficult to find, here are the directions from the train station in Pisa (about 8 blocks to walk): Cross Piazza Vitt. Em. II, bearing right along Via V. Croce to Piazza Toniolo, and then walk through the Fortessa. The statue is found within Fortezza Campo Santo off Lungarno Fibonacci or Via Fibonacci along the Arno River at Giardino Scotto (Teatro Estivo).

CALL FOR PAPERS

The THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of Pisa, Pisa, Italy, from July 25-29, 1988. This conference is sponsored jointly by The Fibonacci Association and The University of Pisa.

Papers on all branches of mathematics and science related to the Fibonacci numbers and their generalizations are welcome. Abstracts are to be submitted by March 15, 1988. Manuscripts are requested by May 1, 1988. Abstracts and manuscripts should be sent to G.E. Bergum (address below). Invited and contributed papers will appear in the Conference Proceedings, which are expected to be published.

The program for the Conference will be mailed to all participants, and to those individuals who have indicated an interest in attending the conference, by June 15, 1988. All talks should be limited to one hour or less.

For further information concerning the conference, please contact Gerald Bergum, The Fibonacci Quarterly, Department of Mathematics, South Dakota State University, P.O. Box 2220, Brookings, South Dakota 57007-1297.

SOLUTION OF THE SYSTEM
$a^2 \equiv -1 \pmod{b}$, $b^2 \equiv -1 \pmod{a}$

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University of Maryland, College Park, MD 20742

(Submitted August 1985)

INTRODUCTION

On page 64 of *Introduction to Number Theory* by Adams and Goldstein [1], problem number 7 asks: "Does $x^2 \equiv -1 \pmod{65}$ have a solution?" An obvious solution is $x = 8$, but if one first solves the congruences $x^2 \equiv -1 \pmod{5}$ and $x^2 \equiv -1 \pmod{13}$ and then applies the Chinese Remainder Theorem, one finds that $x^2 \equiv -1 \bmod (5 \cdot 13) \Longleftrightarrow x \equiv \pm 5 \pm \pm 3 \bmod (5 \cdot 13)$. This leads to the following obvious question. For which pairs of numbers a, b do we have $(\pm a \pm b)^2 \equiv -1 \pmod{ab}$? This is equivalent to $ab | a^2 + b^2 + 1$ which, in turn, is equivalent to the pair of conditions $a | b^2 + 1$ & $b | a^2 + 1$ (if the latter conditions hold, it is clear that a and b are relatively prime).

Let

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_n + F_{n+1},\ F_{n-2} = F_n - F_{n-1}$$

so that F_n, the n^{th} Fibonacci number, is defined for all integers n. Clearly $(\pm a \pm b)^2 \equiv -1 \pmod{ab}$ is equivalent to $(a - b)^2 \equiv -1 \pmod{ab}$. We will show that $(a - b)^2 \equiv -1 \pmod{ab}$, where $1 \leqslant a \leqslant b$, iff for some $n \geqslant 0$, $a = F_{2n-1}$ & $b = F_{2n+1}$. Thus, the solutions are (1, 1), (1, 2), (2, 5), (5, 13), (13, 34), (34, 89), (89, 233), (233, 610), ... Since we are also interested in the equation $(a - b)^2 \equiv +1 \pmod{ab}$, we shall carry out many of our calculations with ± 1 in place of -1.

1. EQUIVALENCE TO THE DIOPHANTINE EQUATION $z^2 - (x^2 - 4)y^2 = \pm 4$

Since $(a - b)^2 \equiv \pm 1 \pmod{ab}$, we write $(a - b)^2 \mp 1 = rab$, that is,

$$a^2 - (2 + r)ab + b^2 \mp 1 = 0.$$

Let $k = 2 + r$. If b and k are given, then there will exist an a satisfying $a^2 - kab + b^2 \pm 1 = 0$ iff

$$\frac{1}{2}(kb \pm \sqrt{(k^2 - 4)b^2 \pm 4})$$

is an integer. By examining the cases k even, b even, k and b both odd, we see that this is equivalent to $(k^2 - 4)b^2 \pm 4 = z^2$, for some z. We let $x = k$, $y = b$, and obtain the Diophantine equation

$$z^2 - (x^2 - 4)y^2 = \pm 4.$$

Every solution of this equation except for $(z, x, y) = (0, 0, \pm 1)$ corresponds to two solutions of $(a - b)^2 = \pm 1 + (x - 2)ab$, namely,

$$b = y,\ a = \frac{xy \pm z}{2}.$$

Here 4 corresponds to +1 and -4 corresponds to -1.

2. THE EQUATION $z^2 - (x^2 - 4)y^2 = -4$

We now concentrate on the -1 case. First, we prove a useful lemma.

Lemma 1: $z^2 - (x^2 - 4)y^2 = -4$ is solvable in integers iff $z^2 - (x^2 - 4)y^2 = -1$ is solvable in integers. One direction is easy, since $z^2 - (x^2 - 4)y^2 = -1$ implies $(2z^2) - (x^2 - 4)(2y)^2 = -4$. So suppose $z^2 - (x^2 - 4)y^2 = -4$ is solvable. If x were even, then 4 would divide $x^2 - 4$, so 2 would divide z, and we would obtain

$$\left(\frac{z}{2}\right)^2 - \left(\left(\frac{x}{2}\right)^2 - 1\right)y^2 = -1.$$

Since -1 is not a square (mod 4), y is odd. Thus,

$$\left(\frac{z}{2}\right)^2 = -1 + \left(\left(\frac{x}{2}\right)^2 - 1\right)y^2 \equiv \left(\frac{x}{2}\right)^2 - 2 \equiv 2,\ 3 \pmod{4},$$

which is impossible. Therefore, x is odd.

Let (z_0, y_0) be a solution of $z^2 - (x^2 - 4)y^2 = -4$. Then $z_0 \equiv y_0 \pmod{2}$. If z_0 and y_0 are both even, then

$$\left(\frac{z_0}{2}\right)^2 - (x^2 - 4)\left(\frac{y_0}{2}\right)^2 = -1$$

and we are done. Therefore, we assume that z_0, y_0 are odd. We now quote the following easy and well-known result.

Multiplication Principle: If $u_0^2 - Dv_0^2 = A$ and $u_1^2 - Dv_1^2 = B$, then $u_2^2 - Dv_2^2 = AB$ where

$$u_2 + \sqrt{D}v_2 = (u_0 + \sqrt{D}v_0)(u_1 + \sqrt{D}v_1) = (u_0u_1 + Dv_0v_1) + \sqrt{D}(u_0v_1 + u_1v_0).$$

$(x, \pm 1)$ are solutions of $z^2 - (x^2 - 4)y^2 = 4$; so, by the Multiplication Principle with $D = x^2 - 4$, (z_i, y_i), $i = 1, 2$, are solutions of

$$z^2 - (x^2 - 4)y^2 = (-4)(4) = -16,$$

where

$$(z_i, y_i) = (z_0x + (-1)^iDy_0,\ xy_0 + (-1)^iz_0).$$

Since $4^2 | 16$, it is clear that $4 | z_i$ iff $4 | y_i$. Also, since x, z_0, y_0, and D are all odd, z_1, y_1, z_2, and y_2 are even. Also,

$$z_2 - z_1 = 2Dy_0 \equiv 2 \pmod{4} \quad \text{and} \quad y_2 - y_1 = 2z_0 \equiv 2 \pmod{4}.$$

So, for some i, $z_i \equiv y_i \equiv 0 \pmod{4}$. Hence,

$$\left(\frac{z_i}{4}\right)^2 - (x^2 - 4)\left(\frac{y_i}{4}\right)^2 = -1$$

and Lemma 1 is proved.

Lemma 2: $z^2 - (x^2 - 4)y^2 = -1$ is solvable only when $x = \pm 3$.

When $x = \pm 3$, we may take $z = 2$, $y = 1$. Suppose $z^2 - (x^2 - 4)y^2 = -1$ is solvable. Then x is odd. Suppose $x > 0$ and $x \neq 3$. Then $x > 3$ since, otherwise,

$$z^2 - (x^2 - 4)y^2 \geqslant 0.$$

Let (z^*, y^*) be that solution characterized by $z^* > 0$, $y^* > 0$, and y^* is minimal (the so-called *fundamental* solution). Since $x > 3$, $x^2 - 4$ is not a perfect square; so, by the general theory of Pell equations (see [1], p. 201, Theorem

106), if (z, y) is any solution of $z^2 - (x^2 - 4)y^2 = +1$ with $z > 0$, $y > 0$, then

$$z + \sqrt{x^2 - 4}y = (z^* + \sqrt{x^2 - 4}y^*)^n,$$

where n is an even positive integer.

In order to arrive at a contradiction, we need to find a small solution of $z^2 - (x^2 - 4)y^2 = 1$ with x odd. We have two obvious solutions of

$$z^2 - (x^2 - 4)y^2 = 4,$$

namely, $(x, 1)$ and $(x^2 - 2, x)$. Therefore,

$$(x(x^2 - 2) + (x^2 - 4)x,\ x^2 + (x^2 - 2)) = (2(x^3 - 3x),\ 2(x^2 - 1))$$

is a solution of $z^2 - (x^2 - 4)y^2 = 16$, by the Multiplication Principle. Since x is odd, $x^3 - 3x$ and $x^2 - 1$ are even. Hence,

$$\left(\frac{x^3 - 3x}{2}\right)^2 - (x^2 - 4)\left(\frac{x^2 - 1}{2}\right)^2 = 1.$$

Let

$$(A,\ B) = \left(\frac{x^3 - 3x}{2},\ \frac{x^2 - 1}{2}\right).$$

(A, B) is probably the fundamental solution of $z^2 - (x^2 - 4)y^2 = 1$, but we do not have a proof [William Adams has shown, using the theory of continued fractions, that (A, B) is the fundamental solution]. In any case,

$$A + \sqrt{x^2 - 4}B = (z^* + \sqrt{x^2 - 4}y^*)^n, \text{ where } n \text{ is even.}$$

Therefore, there exist positive numbers U and V such that

$$A + \sqrt{x^2 - 4}B = (U + \sqrt{x^2 - 4}V)^2.$$

Let $D = \sqrt{x^2 - 4}$. Then $A = U^2 + DV^2$, $B = 2UV$. Hence,

$$A = U^2 + D\left(\frac{B}{2U}\right)^2.$$

Let $W = U^2$. Then $4W^2 - 4AW + DB^2 = 0$. So $(2W - A)^2 = A^2 - DB^2 = 1$, and

$$U^2 = W = \frac{A \pm 1}{2} = \frac{x^3 - 3x \pm 2}{4}.$$

Thus,

$$U = \frac{1}{2}\sqrt{x^3 - 3x \pm 2} = \frac{1}{2}\sqrt{(x \mp 1)^2(x \pm 2)} = \frac{x \mp 1}{2}\sqrt{x \pm 2}$$

and

$$V = \frac{B}{2U} = \frac{x^2 - 1}{2(x \mp 1)\sqrt{x \pm 2}} = \frac{x \pm 1}{2\sqrt{x \pm 2}}.$$

It turns out that if $2W = A - 1$, then $U^2 - DV^2 = -1$, while if $2W = A + 1$, then $U^2 - DV^2 = +1$. We do not, however, need this information. We have shown

Proposition: If $z^2 - (x^2 - 4)y^2 = -1$ is solvable in integers, then either

$$x - 2 \text{ is a perfect square and } \sqrt{x - 2}\,|\,x - 1$$

or

$$x + 2 \text{ is a perfect square and } \sqrt{x + 2}\,|\,x + 1.$$

Suppose that $x - 2 = t^2$ and $t | x - 1$. Then $t | t^2 + 1$. So $t = 1$. Therefore $x = 3$, a contradiction. Suppose that $x + 2 = t^2$ and $t | x + 1$. Then $t | t^2 - 1$. So $t = 1$. Thus $x = -1$, a contradiction. This completes the proof of Lemma 2.

Putting Lemmas 1 and 2 together, we see that $z^2 - (x^2 - 4)y^2 = -4$ is solvable in integers iff $x = \pm 3$.

3. SOLUTION OF $a^2 - 3ab + b^2 = \pm 1$

In solving the congruence $(\pm a \pm b)^2 \equiv -1 \pmod{ab}$, it clearly suffices to find all solutions (a, b) with $a, b \geqslant 1$. Also, the equation is equivalent to $(a - b)^2 \equiv -1 \pmod{ab}$, i.e., $(a - b)^2 + 1 = rab$, where, because $a, b > 0$, we know $r > 0$. By §2, $2 + r = k = \pm 3$. Therefore, $k = 3$ and $r = 1$. So, if $a, b \geqslant 1$, the congruence $(\pm a \pm b)^2 \equiv -1 \pmod{ab}$ is equivalent to the equation

$$a^2 - 3ab + b^2 = -1.$$

Theorem: Let a and b be any two integers. Then

1) $a^2 - 3ab + b^2 = -1$ iff $(a, b) = \pm(F_n, F_{n\pm 2})$ where n is odd, and
2) $a^2 - 3ab + b^2 = 1$ iff $(a, b) = \pm(F_n, F_{n\pm 2})$ where n is even.

Proof: We could reduce our equations to the Pell equation $u^2 - 5v^2 = 1$ using well-known methods. However, it is easier to apply the methods developed in [3]. Consider the equation $a^2 - 3ab + b^2 = -1$. The idea is that any solution (a, b) generates two other solutions (a, b') and (a', b), where a' and b' are determined by the recurrences $a' = 3b - a$, $b' = 3a - b$. If we apply these recurrences over and over, we develop a two-way infinite chain $\ldots b'\,a\,b\,a' \ldots$ of integers in which any adjacent pair represents a solution. According to ([3], p. 56), every chain of solutions to our equation must contain an a-value in the set $\{0, \pm 1\}$ or a b-value in the set $\{0, \pm 1\}$. The only solutions (a, b) having this property are $\pm(1, 1)$, $\pm(1, 2)$, and $\pm(2, 1)$. So, except for changes of sign, every solution lies in the single chain

$$\ldots \underline{34}\ 13\ \underline{5}\ 2\ \underline{1}\ 1\ \underline{2}\ 5\ \underline{13}\ 34 \ldots,$$

where we have underlined the a-values. Since $F_{-1} = 1$ and $F_1 \equiv 1$, and since

$$3F_n - F_{n-2} = 2F_n + F_{n-1} = F_n + F_{n+1} = F_{n+2}$$

holds for every integer n, we see that this sequence of numbers is

$$\ldots F_{-5}\ F_{-3}\ F_{-1}\ F_1\ F_3\ F_5 \ldots .$$

Therefore $a^2 - 3ab + b^2 = -1$ iff, for some odd number n, $(a, b) = \pm(F_n, F_{n\pm 2})$. The equation $a^2 - 3ab + b^2 = +1$ is handled in a similar fashion.

Corollary: If $0 \leqslant a \leqslant b$, then $(\pm a \pm b)^2 \equiv -1 \pmod{ab}$ iff, for some $n \geqslant 0$,

$$(a, b) = (F_{2n-1}, F_{2n+1}).$$

4. DISCUSSION OF $(\pm a \pm b)^2 \equiv 1 \pmod{ab}$

We shall briefly discuss the equation $(\pm a \pm b)^2 \equiv 1 \pmod{ab}$, equivalent to $(a - b)^2 \equiv 1 \pmod{ab}$, which we rewrite as $a^2 - kab + b^2 = 1$. In §1 we showed that this equation is solvable iff $z^2 - (k^2 - 4)y^2 = +4$ is solvable. The latter equation has an obvious solution, namely $(z, y) = (k, 1)$. So we have solutions of $a^2 - kab + b^2 = 1$ for every k, not just $k = 3$. When $k = 3$, we have only the solutions given by the Theorem of §3, but when $k = 4$ we have, for example, $(a, b) = (1, 4)$, and when $k = 5$ we have, for example, $(a, b) = (5, 24)$. When $k = 2$, we get the infinite class $(a, b) = (n, n \pm 1)$. Clearly,

$$n^2 - 2n(n \pm 1) + (n \pm 1)^2 = 1,$$

and if $a^2 - 2ab + b^2 = 1$, then $b = a \pm 1$. A complete classification for all k would be an interesting project.

5. WHEN a AND b ARE PRIMES

If a and b are distinct primes, or if one is an odd prime and the other is twice another odd prime, the congruence $x^2 \equiv -1 \pmod{ab}$, if solvable, will have precisely four solutions. Therefore,

$$x^2 \equiv -1 \pmod{ab} \iff x = \pm a \pm b$$

holds for the following pairs (a, b):

$$(2, 5), (5, 13), (13, 34), (34, 89), (89, 233).$$

However, it does not hold for the pair (233, 610). There are eight solutions of $x^2 \equiv -1 \bmod (233 \cdot 610)$, four of which are $\pm 233 \pm 610 = \pm 377, \pm 843$. The other four are $\pm 121 \cdot 233 \pm 610 = \pm 27583, \pm 28803$. Thus, the question arises: How many pairs of primes a, b are there satisfying $(\pm a \pm b)^2 = -1 \pmod{ab}$? Since n is prime whenever F_n is prime, if there are finitely many twin primes, there are only finitely many such pairs. However, it is generally believed that the set of twin primes is infinite. Nevertheless, based on separate probabilistic considerations, Daniel Shanks has conjectured that (89, 233) is the last such pair.

ACKNOWLEDGMENT

We should like to acknowledge the role of Daniel Shanks in the development of this paper. It was he who first noticed that the sequence

$$2, 5, 13, 34, 89, 233, \ldots$$

provides infinitely many solutions to the congruence $(\pm a \pm b)^2 \equiv -1 \pmod{ab}$.

REFERENCES

1. William W. Adams & Larry J. Goldstein. *Introduction to Number Theory*. Englewood Cliffs, NJ.: Prentice-Hall, 1976.
2. Trygve Nagell. *Introduction to Number Theory*. New York: Chelsea Publishing Co., 1964.
3. James C. Owings, Jr. "Diophantine Chains." *Rocky Mountain Journal of Mathematics 13*, no. 1 (1983):55-60.

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RENCONTRES GRAPHS: A FAMILY OF BIPARTITE GRAPHS*

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(Submitted August 1985)

I. INTRODUCTION

A number of different families of graphs have recently been proposed as possible interconnection models for computer networks. A tree is the cheapest interconnection, but has unacceptably poor connectivity properties. On the other hand, the complete graphs K_n, although most reliable and best connected, is prohibitively expensive (too many edges). A number of other graph families that lie between these two extremes have been proposed and analyzed for relevant properties such as path lengths, connectivities, cost, reliability, potential congestions, throughput, etc. The search for "good" interconnection graphs for various situations continues. This paper is an outcome of our attempt to find a class of graphs which satisfy certain desired properties.

In Section II, we derive a family of adjacency matrices from Rencontres numbers, and call the corresponding graphs Rencontres graphs, which are connected, undirected, bipartite graphs. In Section III, the connectivity of Rencontres graphs is explored. In that section, we also prove that the complete bipartite graph $K_{t,t}$ is a subgraph of the Rencontres graph of 2^t vertices. An expression for the number of edges in a Rencontres graph in terms of the number of vertices is developed in Section IV. In Section V, it is shown that all Rencontres matrices of order other than 2 are singular.

We have used standard graph theoretic terms, for which readers may refer to [3] or [4]. All logarithms are with respect to base 2.

II. BASIC CONCEPTS AND DEFINITIONS

A classical combinatorial problem, known generally by its French name, "le problème des rencontres," is to find the number of permutations of n distinct elements (say, 1, 2, ..., n) such that no element is in its own position, or element k is not in the k^{th} position, $k = 1, 2, \ldots, n$. It is also known as the derangement problem. Its solution by Montmort (1713) effectively uses the principle of inclusion and exclusion [1]. More generally, the derangement problem enumerates permutations of n distinct elements according to the number of elements in "their own positions."

Let $D_{n,k}$ be the number of permutations of n elements with exactly k of them not displaced. In particular, $D_{n,0}$ is the number of permutations of n elements with all of them displaced, and $D_{n,n}$ is the number of permutations of n elements with none of them displaced. It has been shown in [1] that

$$D_{n,k} = \binom{n}{k} D_{n-k,0}.$$

The numbers $D_{n,k}$ for given n and k, $0 \leqslant k \leqslant n$, are called *Rencontres numbers.*

This work was supported by the United States Army Research Office under grant DAAG29-82-K-0107.

For $n = 0, 1, \ldots, 10$ and $k = 0, 1, \ldots, 10$, the numbers $D_{n,k}$ are given in Table 1, henceforth referred to as the *Rencontres table*.

Table 1. Rencontres Numbers $D_{n,k}$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	1	0	1								
3	2	3	0	1							
4	9	8	6	0	1						
5	44	45	20	10	0	1					
6	265	264	135	40	15	0	1				
7	1854	1855	924	315	70	21	0	1			
8	14833	14832	7420	2464	630	112	28	0	1		
9	133496	133497	66744	22260	5544	1134	168	36	0	1	
10	1334961	1334960	667485	222480	55650	11088	1890	240	45	0	1

The following results can be derived easily.

$$D_{0,0} = 1$$

$$D_{n,n} = \binom{n}{n} D_{0,0} = 1 \text{ for all } n \geqslant 0$$

$$D_{n,0} = nD_{n-1,0} + (-1)^n \text{ for all } n \geqslant 1$$

$$D_{n+1,n} = 0 \text{ for all } n \geqslant 0$$

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k,0} \text{ for all } n \geqslant 0$$

$$D_{n,k} = D_{n-1,k-1} + \binom{n-1}{k} D_{n-k,0} \text{ for all } n \geqslant 1 \text{ and } 1 \leqslant k \leqslant n$$

$$D_{i,j} = 0 \text{ if either or both } i \text{ and } j \text{ are negative integers.}$$

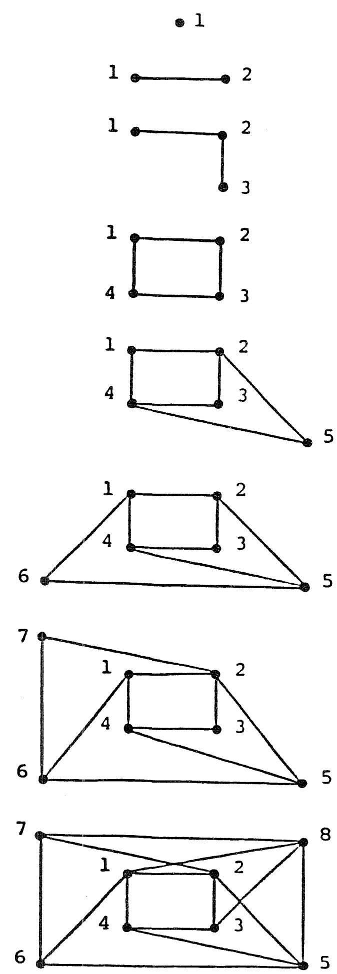
Let us define a few terms used in this paper.

Definition 1: An $n \times n$ symmetric binary matrix is called the *Rencontres matrix* $RM(n)$ of order n if its principal diagonal entries are all 0's and its lower triangle (and therefore the upper also) consists of the first $n-1$ rows of the Rencontres table modulo 2. Let $rm_{i,j}$ denote the element in the i^{th} row and the j^{th} column of the Rencontres matrix.

Definition 2: The simple, undirected graph with n vertices corresponding to $RM(n)$ as its adjacency matrix is called the *Rencontres graph* $RG(n)$ of order n.

The matrix $RM(10)$ is shown below followed (in Figure 1) by the first eight Rencontres graphs.

$$\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}$$



RG(1)

RG(2)

RG(3)

RG(4)

RG(5)

RG(6)

RG(7)

RG(8)

Figure 1. Rencontres Graphs $RG(n)$, $1 \leqslant n \leqslant 8$

Definition 3: Let $rt_{i,j}$ be the j^{th} element in the i^{th} row of the Rencontres table, where rows and their elements are numbered beginning with 0.

Thus, by the definition of the Rencontres matrix,

$$\begin{aligned} rm_{i,j} &= rt_{i-2,j-1} \pmod 2 \text{ for } i > j \geqslant 1 \\ &= \binom{i-2}{j-1} rt_{i-j-1,0} \pmod 2 \\ &= \binom{i-2}{j-1} rm_{i-j+1,1} \pmod 2. \end{aligned}$$

Definitions 1-3 are similar to those in [5], in the context of Pascal graphs.

Definition 4: Let $BS(M)$ denote the *binary representation* of a nonnegative integer M; if q is the smallest integer such that $2^{q+1} > M$, then q will be called the *length* of $BS(M)$. The p^{th} bit of $BS(M)$ will be denoted as $BS_p(M)$, where the bits are counted from right to left and the rightmost bit is the 0^{th} bit.

Definition 5: The *B-sequence* of a positive integer N is defined as the strictly decreasing sequence $B(N) = (p_1, p_2, \ldots, p_\ell)$ of ℓ nonnegative integers such that

$$N = \sum_{i=1}^{\ell} 2^{p_i}.$$

Note that the B-sequence of any positive integer N gives the positions of 1's in the binary representation of N in decreasing order. Also, the B-sequence of zero is defined to be a null sequence. This definition is the same as in [6].

III. CONNECTIVITY PROPERTIES OF THE RENCONTRES GRAPHS

Lemma 1: Graph $RG(n)$ is a subgraph of $RG(n+1)$ for all $n \geqslant 1$.

Proof: This property is a direct consequence of the definition of the Rencontres matrix.

Theorem 1: All graphs $RG(i)$, $1 \leqslant i \leqslant 7$, are planar; all Rencontres graphs of higher order are nonplanar.

Proof: Figure 1 clearly shows that all graphs $RG(i)$ for $1 \leqslant i \leqslant 7$ are planar. It is easy to see that Kuratowski's second graph $K_{3,3}$ is a subgraph of $RG(8)$. Thus, by Lemma 1, all graphs of order 8 and higher are nonplanar.

Theorem 2: (a) Vertex v_i is adjacent to v_{i+1} in the Rencontres graph for every $i \geqslant 1$.

(b) Vertex v_1 is adjacent only to all even-numbered vertices in the Rencontres graph.

(c) Vertex v_2 is adjacent only to all odd-numbered vertices in the Rencontres graph.

Proof: (a) By the definition of the Rencontres matrix,

$$rm_{i,j} = rt_{i-2,j-1} \pmod 2, \; i > j \geqslant 1.$$

For all $i \geqslant 1$, $rm_{i+1,i} = rt_{i-1,i-1} \pmod 2 = 1$. Thus, vertex v_i is adjacent to v_{i+1} for all $i \geqslant 1$.

(b) Since $rm_{2,1} = rt_{0,0} \pmod 2 = 1$, so vertex v_1 is adjacent to v_2.

For $i \geqslant 3$, $rm_{i,1} = rt_{i-2,0} \pmod 2$

$$= (i-2)rt_{i-3,0} + (-1)^{i-2} \pmod 2$$

$$= (i-2)rm_{i-1,1} \pmod 2 + (-1)^{i-2} \pmod 2 \pmod 2.$$

Now, if i is even,

$$(i-2) \pmod 2 = 0 \quad \text{and} \quad (-1)^{i-2} = 1,$$

so that $rm_{i,1} = 1$ for all even $i \geqslant 2$. On the other hand, if i is odd,

$$(i-2) \pmod 2 = 1 \quad \text{and} \quad (-1)^{i-2} = -1;$$

also, since $i - 1$ is even, $rm_{i-1,1} = 1$. Hence, $rm_{i,1} = 0$ for all odd $i \geqslant 3$. Thus, vertex v_1 is adjacent to all even-numbered vertices and to no others in the Rencontres graph.

(c) Vertex v_2 is obviously adjacent to v_1.

For $i \geqslant 3$, $rm_{i,2} = \binom{i-2}{1} rm_{i-1,1} \pmod 2$

$$= (i-2)rm_{i-1,1} \pmod 2.$$

Clearly, when i is even, $rm_{i,2} = 0$. But, when i is odd, $rm_{i,2} = 1$, since $rm_{i-1,1} = 1$ by Theorem 2(b). Therefore, vertex v_2 is adjacent only to all odd-numbered vertices in the Rencontres graph.

Corollary 1: Graph $RG(n)$, for all $n \geqslant 2$, is connected, and contains a Hamiltonian path $[1, 2, 3, \ldots, n]$. Moreover, for all even $n \geqslant 4$, graph $RG(n)$ contains a Hamiltonian circuit $[1, 2, \ldots, n-1, n, 1]$.

Corollary 2:* In graph $RG(n)$, degree $(v_1) = \left\lfloor \frac{n}{2} \right\rfloor$, and degree $(v_2) = \left\lceil \frac{n}{2} \right\rceil$.

Themrem 3: $RG(n)$ is bipartite for $n \geqslant 2$.

Proof: The proof consists of showing that neither two even-numbered nor two odd-numbered vertices in a Rencontres graph are adjacent. Let both i and j be even integers, $i > j$. Then,

$$rm_{i,j} = \binom{i-2}{j-1} rm_{i-j+1,1} \pmod 2.$$

Since the integer $i - j$ is even, by Theorem 2(b) $rm_{i-j+1,1} = 0$, and therefore, $rm_{i,j} = 0$. Thus, no two even-numbered vertices in a Rencontres graph are adjacent. Similar argument shows that no two odd-numbered vertices in a Rencontres graph are adjacent.

Corollary 3: Since $RG(4)$ is a 4-cycle, the girth of the Rencontres graph $RG(n)$ is 4 for all $n > 3$.

Theorem 4: Vertex v_i is adjacent to v_{i+3} in the Rencontres graph iff i is 1 or 2 (mod 4).

---

*$\lceil a \rceil$ *is the least integer greater than or equal to* a. $\lfloor a \rfloor$ *is the greatest integer less than or equal to* a.

Proof: $rm_{i+3,i} = \binom{i+1}{i-1} rm_{4,1} \pmod 2$

$$= \binom{i+1}{i-1} \pmod 2, \text{ by Theorem 2(b)}$$

$$= \frac{i(i+1)}{2} \pmod 2$$

$$= 1, \text{ iff } i \text{ is 1 or 2} \pmod 4.$$

The following theorem gives a necessary and sufficient condition for any two vertices to be adjacent in a Rencontres graph.

Theorem 5: Vertex v_i is adjacent to v_j, where $i > j$ and one is odd and the other even, iff there does not exist an integer p, $0 \leqslant p \leqslant k$, such that

$$BS_p(i-2) = 0 \quad \text{and} \quad BS_p(j-1) = 1,$$

where k is the length of $BS(j-1)$.

Proof: We have

$$rm_{i,j} = \binom{i-2}{j-1} rm_{i-j+1,1} \pmod 2.$$

If one of i and j is odd and the other even, by Theorem 2(b) $rm_{i-j+1,1} = 1$. Thus, we have to determine the condition under which

$$\binom{i-2}{j-1} \pmod 2 = 1$$

so that vertex v_i is adjacent to v_j. Let

$$BS(i-2) = m_q m_{q-1} \ldots m_1 m_0 \quad \text{and} \quad BS(j-1) = n_k n_{k-1} \ldots n_1 n_0,$$

where $q \geqslant k$. Following [2], we can write:

$$\binom{i-2}{j-1} \pmod 2 = \binom{m_k}{n_k}\binom{m_{k-1}}{n_{k-1}} \cdots \binom{m_1}{n_1}\binom{m_0}{n_0} \pmod 2$$

$$= \begin{cases} 1 & \text{iff } m_i \geqslant n_i,\ 0 \leqslant i \leqslant k \\ 0 & \text{iff } \exists p,\ 0 \leqslant p \leqslant k \ni m_p < n_p, \\ & \text{i.e., } m_p = 0 \text{ and } n_p = 1. \end{cases}$$

Thus, $rm_{i,j} = 1$ iff there does not exist an integer p, $0 \leqslant p \leqslant k$, such that

$$BS_p(i-2) = 0 \quad \text{and} \quad BS_p(j-1) = 1,$$

where k is the length of $BS(j-1)$, and in that case vertex v_i is adjacent to v_j.

Theorem 6: If $i = 2^k + 1$, where $k \geqslant 1$, then vertex v_i is adjacent to all even-numbered vertices v_j, $2 \leqslant j < 2i$, $j \neq i$.

Proof: Let $i = 2^k + 1$, $k \geqslant 1$. Since i is odd, j must be even, if vertex v_i is adjacent to v_j.

Case 1. $2 \leqslant j < i$

$$rm_{i,j} = \binom{2^k - 1}{j-1} rm_{i-j+1,1} \pmod 2 = 1, \text{ by Theorems 2(b) and 5.}$$

Case 2. $i < j < 2i$

$$rm_{i,j} = rm_{j,i} = \binom{j-2}{2^k} rm_{i-j+1,1} \pmod 2 = 1, \text{ by Theorems 2(b) and 5.}$$

Since, for all even j, $2 \leqslant j < 2i$ and $j \neq i$, $rm_{i,j} = 1$, vertex v_i is adjacent to all such v_j.

Corollary 4: If $i = 2^k + 1$, $k \geqslant 1$, then degree $(v_i) = 2^{k-1}$ in graph $RG(i)$, and degree $(v_i) = 2^k$ in graph $RG(2^{k+1})$.

Theorem 7: If $i = 2^k$, where k is a positive integer, then vertex v_i is adjacent to all odd-numbered vertices in the Rencontres graph.

Proof: Let $i = 2^k$, where $k \geqslant 1$. Since i is even, j must be odd for adjacency. We have

$$rm_{i,j} = \binom{2^k - 2}{j-1} rm_{i-j+1,1} \pmod 2 = 1, \text{ by Theorems 2(b) and 5.}$$

Since, for all odd j, $1 \leqslant j < i$, $rm_{i,j} = 1$, vertex v_i is adjacent to all such v_j.

Corollary 5: If $i = 2^k$, $k \geqslant 1$, then

(a) degree $(v_i) = 2^{k-1}$ in graph $RG(i)$,

(b) degree $(v_i) = 2^{k-1} + 1$ in graph $RG(2i)$.

Proof: (a) Follows directly from Theorem 7.

(b) Theorem 7 considers the adjacency of vertex v_i with v_j, $1 \leqslant j < i$. Here we also need to consider odd j such that $i < j \leqslant 2i$. In this case,

$$rm_{i,j} = \binom{j-1}{2^k - 1} rm_{j-i+1,1} \pmod 2 = 0 \text{ except when } j = 2^k + 1,$$

by Theorem 5. That is, for $i < j \leqslant 2i$, vertex v_i is adjacent to v_{i+1} only. Hence, degree $(v_i) = 2^{k-1} + 1$ in graph $RG(2i)$.

Theorem 8: If $i = 2^k + 2$, $k \geqslant 1$, then vertex v_i is adjacent to v_1, v_{i-1}, and all odd-numbered vertices v_j, $i < j < 2^{k+1}$.

Proof: Let $i = 2^k + 2$, where k is a positive integer. That v_i is adjacent to v_1 and v_{i-1} is evident by Theorems 2(a) and 2(b).

Case 1. $1 < j < i - 1$, and j is odd.

$$rm_{i,j} = \binom{2^k}{j-1} rm_{i-j+1,1} \pmod 2 = 0, \text{ by Theorem 5.}$$

Thus, v_i is not adjacent to any odd-numbered vertex v_j, $1 < j < i - 1$.

Case 2. $i < j < 2^{k+1}$, and j is odd.

$$rm_{i,j} = \binom{j-2}{2^k + 1} rm_{i-j+1,1} \pmod 2 = 1, \text{ by Theorem 5.}$$

Hence the theorem.

Corollary 6: If $i = 2^k + 2$, $k \geqslant 1$, then

(a) degree $(v_i) = 2$ in graph $RG(i)$,

(b) degree $(v_i) = 2^{k-1} + 1$ in graph $RG(2^{k+1})$.

Proof: (a) Follows from Theorems 2(a), 2(b), and Case 1 of Theorem 8.

(b) By Theorem 8, in graph $RG(2^{k+1})$, vertex v_i is adjacent to v_1, v_{i-1}, and $2^{k-1} - 1$ even-numbered vertices v_j, $i < j < 2^{k+1}$. Therefore, degree $(v_i) = 2^{k-1} + 1$ in $RG(2^{k+1})$.

The following theorem identifies the subset of Rencontres graphs which contain complete bipartite graphs as subgraphs.

Theorem 9: Complete bipartite graph $K_{t,t}$ is a subgraph of $RG(2^t)$ for all $t \geqslant 1$.

Proof: By Theorem 3, $RG(2^t)$ is a bipartite graph with the following partitioning of its vertex set,

$$V_1 = \{v_{2m+1} \mid 0 \leqslant m < 2^{t-1}\} \quad \text{and} \quad V_2 = \{v_{2m} \mid 1 \leqslant m \leqslant 2^{t-1}\}.$$

Now, choose $V'_{t1} \subset V_1$, and $V'_{t2} \subset V_2$ such that

$$V'_{t1} = \{v_1\} \cup \{v_{2^i+1} \mid 0 < i < t\} \quad \text{and} \quad V'_{t2} = \{v_{2^i} \mid 1 \leqslant i \leqslant t\}.$$

We shall prove by induction that $K_{t,t}$ is a subgraph of $RG(2^t)$, and consists of sets V'_{t1} and V'_{t2}.

Basis. Graph $K_{1,1}$ is identical to $RG(2)$. Thus, the theorem is true for $t = 1$.

Induction Hypothesis. Let the theorem be true for $t = j \geqslant 1$, i.e., $k_{j,j}$ is a subgraph of $RG(2^j)$, and the vertex sets V'_{j1} and V'_{j2} are well defined.

Induction Step. To prove it to be true for $t = j + 1$, define

$$V'_{j+1,1} = V'_{j1} \cup \{v_{2^j+1}\} \quad \text{and} \quad V'_{j+1,2} = V'_{j2} \cup \{v_{2^{j+1}}\}.$$

Then, by Theorem 6, the vertex v_{2^j+1} is adjacent to all even-numbered vertices and, by Theorem 7, the vertex $v_{2^{j+1}}$ is adjacent to all odd-numbered vertices in $K_{j,j}$. Hence, we obtain the graph $K_{j+1,j+1}$, which is a subgraph of $RG(2^{j+1})$.

The following connectivity properties are useful in the design of reliable communication and computer networks. From Theorems 2(b), 2(c), 6, and 7, we conclude that vertices v_1 and $v_{2^{\lceil \log n \rceil - 1}+1}$ always serve as two central vertices adjacent to all even-numbered vertices in graph $RG(n)$; and v_2 is always the central vertex adjacent to all odd-numbered vertices in $RG(n)$. Moreover, when $n = 2^k$, $k \geqslant 1$, vertices v_2 and v_n are centrally adjacent to all odd-numbered vertices in $RG(n)$.

Theorem 10: There are at least two edge-disjoint paths of length $\leqslant 3$ between any two distinct vertices in graph $RG(n)$, $n \geqslant 4$.

Proof: Let v_i and v_j be two vertices of graph $RG(n)$, $n \geqslant 4$, $i \neq j$.

Case 1. $i = 1$ and $j = 2$

Two edge-disjoint paths are $[v_1, v_2]$ and $[v_1, v_4, v_3, v_2]$.

Case 2. $i = 1$ and $j > 2$

Two edge-disjoint paths are $[v_1, v_j]$ and $[v_1, v_{j+2}, v_{j+1}, v_j]$ for j even; and $[v_1, v_2, v_j]$ and $[v_1, v_{j-1}, v_j]$ for j odd.

Case 3. $i > 2$ and $j > 2$

If there is an edge between v_i and v_j, then it constitutes one path. Even if there is no such edge, we have the following two edge-disjoint paths in different subcases.

(i) i even and j odd

$[v_i, v_{i-1}, v_2, v_j]$ and $[v_i, v_1, v_{j-1}, v_j]$

(ii) i odd and j even

$[v_i, v_{i-1}, v_1, v_j]$ and $[v_i, v_2, v_{j-1}, v_j]$

(iii) i even and j even

$[v_i, v_1, v_j]$ and $[v_i, v_{2^{\lceil \log n \rceil - 1}+1}, v_j]$

(iv) i odd and j odd

$[v_i, v_2, v_j]$ and $[v_i, v_{2^{\lceil \log n \rceil}}, v_j]$ if i and $j \leqslant 2^{\lceil \log n \rceil} + 1$

or

$[v_i, v_2, v_j]$ and $[v_i, v_{2^{\lceil \log n \rceil}+2}, v_j]$ if i and $j \geqslant 2^{\lceil \log n \rceil} + 3$

Theorem 10 implies that the edge-connectivity $\geqslant 2$ and that the diameter is 3 for all $RG(n)$, $n \geqslant 4$.

IV. NUMBER OF EDGES IN RENCONTRES GRAPHS

Since the cost of a communication network is proportional to the number of edges in the graph (these edges represent the full duplex communication lines among processors), an estimation of the number of edges in graph $RG(n)$ is important. In the following, we derive an expression for the number of edges in $RG(n)$ in terms of n, the number of vertices in the graph. Before doing this, we need some lemmas.

Lemma 2: If $n = 2^k + i$, $k \geqslant 1$ and $1 < i \leqslant 2^k$, then $d(n) = 2 \cdot d(i)$, where $d(n)$ is the degree of vertex v_n in $RG(n)$ and $d(i)$ is the degree of vertex v_i in $RG(i)$.

Proof: Let i and j have different parity. For $1 \leqslant j < i$, we have

$$rm_{i,j} = \binom{i-2}{j-1} rm_{i-j+1,1} \pmod 2$$

$$= \binom{i-2}{j-1} \pmod 2, \text{ by Theorem 2(b).}$$

Let q be the length of $BS(j - 1)$. Then, by Theorem 5,

$$d(i) = \sum_{1 \leqslant j < i} \left[\binom{i-2}{j-1} \pmod 2 \right]$$

= the number of j's, $1 \leqslant j < i$, for which

$BS_p(i - 2) \geqslant BS_p(j - 1)$, for $0 \leqslant p \leqslant q$.

Now, let $n = 2^k + i$, $k \geqslant 1$ and $1 < i \leqslant 2^k$. Let $2^k + i$ and r have different parity. Then, for $1 \leqslant r < n$, we have

$$rm_{n,r} = \binom{n-2}{r-1} rm_{n-r+1,1} \pmod 2 = \binom{2^k + i - 2}{r-1} \pmod 2.$$

Clearly,

$$d(n) = \sum_{1 \leqslant r < n} \left[\binom{2^k + i - 2}{r-1} \pmod 2 \right]$$

$= 2$ times the number of j's, $1 \leqslant j < i$, for which $BS_p(i-2) \geqslant BS_p(j-1)$ for each p, $0 \leqslant p \leqslant q$.

This is because $BS_k(2^k + i - 2) = 1$ and $BS_k(r-1)$ can be 0 or 1, while

$$BS_k(i-2) = BS_k(j-1) = 0 \text{ (always)}.$$

Thus, $d(n) = 2 \cdot d(i)$ for all i, $1 < i \leqslant 2^k$ and $k \geqslant 1$.

Corollary 7: If $n = 2^k + 1 + i$, for $k \geqslant 1$ and $1 \leqslant i \leqslant 2^k$, then the degree $d(n)$ of vertex v_n in $RG(n)$ is given by

$$d(n) = 2 \cdot d(i+1),$$

where $d(i+1)$ is the degree of vertex v_{i+1} in $RG(i+1)$.

Proof: This corollary is identical to Lemma 2 for all i, $1 \leqslant i < 2^k$. Hence, to prove this corollary, we need to consider another case where $i = 2^k$. In that case, $n = 2^{k+1} + 1$, and by Corollary 4, $d(n) = 2^k$ and $d(i+1) = 2^{k-1}$. Thus, $d(n) = 2 \cdot d(i+1)$ for all i such that $1 \leqslant i \leqslant 2^k$ and $k \geqslant 1$.

Lemma 3: Define $e(n)$ to be the number of edges in the bipartite graph $RG(n)$. Then

$$e(2^k) = \begin{cases} 3 \cdot e(2^{k-1}) + 2^{k-2}, & k > 1 \\ 1, & k = 1 \end{cases} \tag{1}$$

Proof: When $k = 1$, $e(2) = 1$ is obviously true. Let $n = 2^k$, $k > 1$. Then,

$$\begin{aligned} e(2^k) &= e(2^{k-1}) + \text{the number of edges added because of the addition of extra } 2^{k-1} \text{ vertices, e.g., } v_{(n/2)+1}, v_{(n/2)+2}, \ldots, v_n \\ &= e(2^{k-1}) + \text{the number of edges added because of the addition of vertices } v_{(n/2)+2}, v_{(n/2)+3}, \ldots, v_n \\ &\quad + \text{the number of edges added because of the addition of vertex } v_{2^{k-1}+1} \\ &= e(2^{k-1}) + 2 \cdot e(2^{k-1}) + 2^{k-2}, \text{ by Lemma 2 and Corollary 4.} \end{aligned}$$

Therefore, $e(2^k) = 3 \cdot e(2^{k-1}) + 2^{k-2}$, for $k > 1$.

Theorem 11: If $n = 2^k$, $k \geqslant 1$, then $e(n) = 2 \cdot 3^{k-1} - 2^{k-1} = \frac{2}{3} \cdot n^{\log 3} - \frac{n}{2}$.

Proof: We shall prove this theorem by solving the recurrence equation (1). Let $n = 2^k$, i.e., $k = \log n \geqslant 1$. The homogeneous solution of (1) is $e(n) = A \cdot 3^k$, where the arbitrary constant A is to be evaluated from $e(2)$. The particular solution of (1) is $e(n) = -2^{k-1}$, so the general solution for $e(n)$ is given by

$$e(n) = A \cdot 3^k - 2^{k-1}.$$

Since $e(2) = 1$ yields $A = 2/3$, we have

$$e(n) = 2 \cdot 3^{k-1} - 2^{k-1} = \frac{2}{3} \cdot n^{\log 3} - \frac{n}{2}.$$

Corollary 8: The number of edges in graph $RG(2^k - 1)$ is

$$e(2^k - 1) = e(2^k) - 2^{k-1} = 2 \cdot 3^{k-1} - 2^k, \text{ for all } k \geqslant 1.$$

Proof: Follows from Corollary 5 and Theorem 11.

Corollary 9: The number of edges in graph $RG(2^k + 1)$ is given by

$$e(2^k + 1) = e(2^k) + 2^{k-1} = 2 \cdot 3^{k-1}, \text{ for } k \geqslant 1.$$

Proof: Corollary 9 can be proved easily using Corollary 4 and Theorem 11.

Another proof can be given as follows:

$$\begin{aligned} e(2^k + 1) &= e(2^{k-1} + 1) + \text{the number of edges addes owing to the addition of extra } 2^{k-1} \text{ vertices} \\ &= e(2^{k-1} + 1) + 2 \cdot e(2^{k-1} + 1), \text{ by Corollary 7} \\ &= 3 \cdot e(2^{k-1} + 1) \\ &\;\;\vdots \\ &= 3^{k-1} \cdot e(3). \end{aligned}$$

Now, $e(3)$ corresponds to the number of edges in graph $RG(3)$, which is 2; thus, $e(2^k + 1) = 2 \cdot 3^{k-1}$.

The expression for $e(n)$, the number of edges in graph $RG(n)$, is different for even and odd n. We prove this in the following theorem.

Theorem 12: The number of edges in graph $RG(n)$ is given by

$$e(n) = \begin{cases} \displaystyle\sum_{i=1}^{\ell} 2^i \cdot 3^{p_i - 1}, & \text{if } n \geqslant 3 \text{ is odd} \\ \displaystyle\sum_{i=1}^{\ell - 1} 2^i \cdot 3^{p_i - 1} + 2^{\ell - 1}, & \text{if } n \text{ is even,} \end{cases}$$

where $B(n - 1) = (p_1, p_2, \ldots, p_\ell)$ is the B-sequence of $n - 1$.

Proof:

Case 1. Let $n \geqslant 3$ be odd. Then $n - 1 = n_1 + n_2 + \cdots + n_\ell$, where $n_i = 2^{p_i}$ with $p_i \geqslant 1$, $1 \leqslant i \leqslant \ell$. Thus,

$$\begin{aligned} e(n) &= e(n_1 + n_2 + n_3 + \cdots + n_\ell) \\ &= e(n_1 + 1) + \text{the number of edges because of the addition of vertices } v_{n_1 + 2}, \ldots, v_n \text{ to } RG(n_1 + 1) \\ &= 2 \cdot 3^{p_1 - 1} + 2 \cdot e(n_2 + 1 + n_3 + \cdots + n_\ell), \end{aligned}$$

by Corollaries 7 and 9.

Repeating the process, we get

$$\begin{aligned} e(n) &= 2 \cdot 3^{p_1 - 1} + 2^2 \cdot 3^{p_2 - 1} + 2^2 \cdot e(n_3 + 1 + n_4 + \cdots + n_\ell) \\ &= 2 \cdot 3^{p_1 - 1} + 2^2 \cdot 3^{p_2 - 1} + \cdots + 2^{\ell - 1} \cdot 3^{p_{\ell-1} - 1} + 2^\ell \cdot 3^{p_\ell - 1} \\ &= \sum_{i=1}^{\ell} 2^i \cdot 3^{p_i - 1}. \end{aligned}$$

Case 2. Let n be even. Then, $n - 1 = n_1 + n_2 + \cdots + n_{\ell-1} + n_\ell$, where $n_i = 2^{p_i}$ with $p_i \geqslant 1$ for $1 \leqslant i \leqslant \ell - 1$, $p_\ell = 0$, and $n_\ell = 1$. Following the same procedure as in the proof of Case 1 of this theorem, we get

$$\begin{aligned} e(n) &= 2 \cdot 3^{p_1 - 1} + 2^2 \cdot 3^{p_2 - 1} + \cdots + 2^{\ell - 1} \cdot 3^{p_{\ell-1} - 1} \\ &\qquad + 2^{\ell-1} \cdot e(n_\ell + 1) \\ &= \sum_{i=1}^{\ell - 1} 2^i \cdot 3^{p_i - 1} + 2^{\ell - 1}, \text{ since } e(n_\ell + 1) = e(2) = 1. \end{aligned}$$

In Section V we shall investigate the determinants of Rencontres matrices.

V. DETERMINANTS OF RENCONTRES MATRICES

Theorem 13: Let $\det(RM(n))$ be the determinant of the Rencontres matrix $RM(n)$ of order n. Then $\det(RM(n)) = 0$ for all $n \geqslant 1$ except for $n = 2$ and $\det(RM(2)) = -1$.

Proof: $\det(RM(1))$ is obviously zero, and

$$\det(RM(2)) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

For $n > 2$, there always exists $k \geqslant 1$ such that $k = [\log n] - 1$ and row $2^k + 1$ is identical to row 1 in matrix $RM(n)$ by Theorem 6. Therefore, $\det(RM(n)) = 0$ for all $n > 2$.

VI. CONCLUSION

We have defined Rencontres matrices, a new class of adjacency matrices constructed from the Rencontres number table modulo 2. The corresponding graphs are connected and bipartite with edge connectivity $\geqslant 2$, diameter 3, and girth 4. The number of edges $\leqslant (2/3) \cdot n^{\log 3} - (n/2)$. Since the binary representation of a vertex number provides a great deal of information on its adjacencies, the situation may be exploited (1) in economic storage of these graphs and (2) in designing a routing algorithm between a pair of communicating vertices. These are some of the desirable properties; additional properties need to be studied to determine how well these graphs are suited for computer interconnection networks.

ACKNOWLEDGMENTS

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REFERENCES

1. J. Riordan. *An Introduction to Combinatorial Analysis.* New York: Wiley & Sons, 1958.
2. N. J. Fine. "Binomial Coefficients Modulo a Prime." *Amer. Math. Monthly* *54* (1947):89-92.
3. N. Deo. *Graph Theory with Applications to Engineering and Computer Science.* Englewood Cliffs, NJ: Prentice-Hall, 1974.
4. F. Harary. *Graph Theory.* Reading, MA: Addison-Wesley, 1969.
5. N. Deo & M. J. Quinn. "Pascal Graphs and Their Properties." *The Fibonacci Quarterly 21*, no. 3 (1983):203-14.
6. B. P. Sinha, S. Ghose, B. B. Bhattacharya, & P. K. Srimani. "A Further Note to Pascal Graphs." *The Fibonacci Quarterly 24*, no. 3 (1986):251-57.

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TRANSPOSABLE INTEGERS IN ARBITRARY BASES*

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1. INTRODUCTION

Let k be a positive integer. The n-digit number $x = a_{n-1}a_{n-2} \dots a_1a_0$ is called k-transposable if and only if

$$kx = a_{n-2}a_{n-3} \dots a_0a_{n-1}. \tag{1}$$

Clearly x is 1-transposable if and only if all of its digits are equal. Thus, we assume $k > 1$.

Kahan has studied decadic k-transposable integers (see [1]); that is, numbers expressed in base 10. The numbers $x_1 = 142857$ and $x_2 = 285714$ are both 3-transposable:

$$3(142857) = 428571$$
$$3(285714) = 857142$$

Kahan has shown that decadic k-transposable numbers exist only when $k = 3$. Further, all 3-transposable integers are obtained by concatenating x_1 or x_2 m times, $m \geqslant 1$ [1]. In this paper we will study k-transposable integers for an arbitrary base g.

2. TRANSPOSABLE INTEGERS IN BASE g

Let x be an n-digit number expressed in base g; that is,

$$x = \sum_{i=0}^{n-1} a_i g^i$$

with $0 \leqslant a_i < g$ and $a_{n-1} \neq 0$. Then x will be k-transposable if and only if

$$kx = \sum_{i=0}^{n-2} a_i g^{i+1} + a_{n-1}. \tag{2}$$

Again we assume $k > 1$; further, we can assume that $k < g$, since $k \geqslant g$ would imply that kx has more digits than x. By rewriting (2), we see that the digits of x must satisfy the following equation:

$$(kg^{n-1} - 1)a_{n-1} = (g - k)\sum_{i=0}^{n-2} a_i g^i. \tag{3}$$

Let d be the greatest common divisor of $g - k$ and $kg^{n-1} - 1$, written

$$d = (g - k,\ kg^{n-1} - 1).$$

Work on this paper was done while the author was a faculty member at Hamilton College, Clinton, NY. She is grateful for the support and encouragement given her during the eleven years she was associated with Hamilton College.

Then the following lemma gives information about d.

Lemma: Let x be an n-digit k-transposable g-adic integer and let

$$d = (g - k,\ kg^{n-1} - 1).$$

Then d must satisfy the following:

(i) $(d, k) = 1$
(ii) $k < d$
(iii) $k^n \equiv 1 \pmod{d}$

Proof: Properties (i) and (iii) follow immediately from the definition of d.

To show (ii), suppose $d \leqslant k - 1$. Then, in (3), $(g - k)$ divides the left-hand side (LHS) as follows:

$$d \text{ divides } kg^{n-1} - 1 \quad \text{and} \quad \frac{g - k}{d} \text{ divides } a_{n-1}.$$

Thus,

$$\frac{kg^{n-1} - 1}{d} > \frac{(k - 1)g^{n-1}}{d} \geqslant g^{n-1} \text{ by the assumption.}$$

But, then, the LHS divided by $g - k$ has a g^{n-1} term, while the right-hand side (RHS) does not. Since $(d, k) = 1$, $k < d$.

We are now able to determine those g-adic numbers which are k-transposable for some k.

Theorem 1: There exists an n-digit g-adic k-transposable integer if and only if there exists an integer d which satisfies the following properties:

(i) $(d, k) = 1$
(ii) $k < d$
(iii) $d | g - k$
(iv) $k^n \equiv 1 \pmod{d}$

Proof: If x is k-transposable then, by the lemma, $d = (g - k, kg^{n-1} - 1)$ satisfies (i)-(iv).

To show the converse, we first observe that d divides $kg^{n-1} - 1$:

$$kg^{n-1} - 1 \equiv kk^{n-1} - 1 \equiv k^n - 1 \equiv 0 \pmod{d}.$$

We now define $x = \sum_{i=0}^{n-1} a_i g^i$ which satisfies (3). Let

$$a_{n-1} = \frac{g - k}{d}. \tag{4}$$

Since $k < d$, $(kg^{n-1} - 1)/d$ has no g^{n-1} term. Thus, $a_{n-2}, \ldots, a_0$ are well defined by the following equation:

$$\sum_{i=0}^{n-2} a_i g^i = \frac{kg^{n-1} - 1}{d}. \tag{5}$$

Note that (5) is obtained by dividing (3) by $g - k = d((g - k)/d)$.

For d satisfying (i)-(iv), we can actually find $[d/k]$ k-transposable integers. We will define

$$x_t = \sum_{i=0}^{n-1} b_{t,i} g^i, \text{ where } t = 1, \ldots, \left[\frac{d}{k}\right].$$

Let $b_{t,i}$ be given by

$$b_{t,n-1} = \left(\frac{g-k}{d}\right)t \tag{6}$$

and

$$\sum_{i=0}^{n-2} b_{t,i}g^i = \left(\frac{kg^{n-1}-1}{d}\right)t. \tag{7}$$

Note that in (7) the RHS has no g^{n-1} term since $kt \leqslant d$; thus, the $b_{t,i}$ are well defined.

We will shortly give an example to show how Theorem 1 is used to determine all k-transposable integers for a given g. We note here that the proof of Theorem 2 is a constructive one. The digits of k-transposable numbers are found using (6) and (7). We now show that almost all g have k-transposable integers.

Theorem 2: If $g = 5$ or $g \geqslant 7$, then there exists a k-transposable integer for some k. No k-transposable numbers exist for $g = 2, 3, 4, 6$.

Proof: Recall that $k > 1$. For the first part we must find k with the following properties:

$$2 \leqslant k < \frac{g}{2}$$

$$(k, g) = 1$$

If g is odd, let $k = 2$. Otherwise, if $g = 2h$, $h \geqslant 4$, choose

$$k = \begin{cases} h - 1 & \text{if } h \text{ is even,} \\ h - 2 & \text{if } h \text{ is odd.} \end{cases}$$

Now let $d = g - k$. Then, clearly, d satisfies (i)-(iii) of Theorem 1. Since $(d, k) = 1$ and $k < d$, there exists n with $k^n \equiv 1 \pmod{d}$. Hence, by Theorem 1, there is an n-digit g-adic k-transposable integer.

It is a straightforward matter to check that there are no k-transposable integers when $g = 2, 3, 4, 6$.

We now show that up to concatenation there are only a finite number of k-transposable integers for a given k, and hence a finite number for a given g.

Theorem 3: Suppose $x = \sum_{i=0}^{n-1} a_i g^i$ is a k-transposable integer. Let

$$d = (g - k, kg^{n-1} - 1)$$

and let N be the order of k in U_d, the group of units of Z_d. Then x equals some N-digit k-transposable integer concatenated n/N times.

Proof: Since $k^n \equiv 1 \pmod{d}$, n is a multiple of N. Let

$$x_t = \sum_{i=0}^{N-1} b_{t,i}g^i,\ t = 1, \ldots, \left[\frac{d}{k}\right],$$

be the N-digit integers given by equations (6) and (7).

As shown in the proof of Theorem 1, $(g - k)/d$ divides a_{n-1} while d divides $kg^{n-1} - 1$. Thus,

$$a_{n-1} = \frac{g-k}{d} \cdot t = b_{t,N-i} \text{ for some } t \leqslant \left[\frac{d}{k}\right].$$

Now,

$$\sum_{i=0}^{n-2} a_i g^i = \left(\frac{kg^{n-1} - 1}{d}\right)t = g^{n-N}\left(\frac{kg^{N-1} - 1}{d}\right)t + \left(\frac{g^{n-N} - 1}{d}\right)t.$$

Hence,

$$a_{n-i} = b_{t,N-i}, \; i = 2, \ldots, N,$$

since

$$\sum_{i=0}^{N-2} b_{t,i} g^i = \left(\frac{kg^{N-1} - 1}{d}\right)t.$$

But now we have

$$\left(\frac{g^{n-N} - 1}{d}\right)t = \left(\frac{g - k}{d}\right)tg^{n-N-1} + \left(\frac{kg^{n-N-1} - 1}{d}\right)t.$$

Thus,

$$a_{n-N-1} = \left(\frac{g - k}{d}\right)t = b_{t,N-1}$$

and

$$a_{n-N-i} = b_{t,N-i}, \; i = 2, \ldots, N.$$

Continuing, we see that x equals x_t concatenated n/N times.

The N-digit numbers x_t are called basic k-transposable integers, since all others are obtained by concatenating these.

3. SOME EXAMPLES

We show how to determine all k-transposable integers for a given g by considering an example. By Theorem 3, we need only determine the basic k-transposable numbers.

Before beginning the example, we note that we need only consider $k < g/2$. By Theorem 1, $k < d$ and $d | g - k$; thus, $k \leqslant g/2$. Since $(d, k) = 1$, $k \neq g/2$.

Let $g = 9$: the possibilities for k, d, and N are given in the table.

k	$g - k$	d	N
2	7	7	3
3	6	–	–
4	5	5	2

When $k = 2$, there are $\left[\frac{d}{k}\right] = 3$, 2-transposable integers. These are found using (6) and (7):

$$b_{t,2} = t;$$

$$b_{t,1} \cdot 9 + b_{t,0} = \left(\frac{2 \cdot 9^2 - 1}{7}\right)t = 23t, \; t = 1, 2, 3.$$

Thus, the basic 2-transposable integers are 125, 251, 376. (Note that these numbers are expressed in base 9.) When $k = 4$, there is one 4-transposable integer, namely, 17.

It is possible that, for a given g and k, there will be more than one d which satisfies (i)-(iii) of Theorem 1. We illustrate this with an example. Suppose $g = 17$ and $k = 2$. Since $g - k = 15$, d can equal 3, 5, or 15. The 2-transposable integers for each case are given in the following table.

d	N	$\left[\frac{d}{k}\right]$	x
3	2	1	$5\ \overline{11}$
5	4	2	$3\ 6\ \overline{13}\ \overline{10}$ $6\ \overline{13}\ \overline{10}\ 3$
15	4	7	$1\ 2\ 4\ 9$ $4\ 9\ 1\ 2$ $2\ 4\ 9\ 1$ $5\ \overline{11}\ 5\ \overline{11}$ $7\ \overline{15}\ \overline{14}\ \overline{12}$ $3\ 6\ \overline{13}\ \overline{10}$ $6\ \overline{13}\ \overline{10}\ 3$

Note that the 2-transposable integers corresponding to $d = 3, 5$ are included among those for $d = 15$, except that $5\ \overline{11}\ 5\ \overline{11}$ is not basic.

REFERENCE

1. Steven Kahan. "k-Transposable Integers." *Math. Magazine* 49, no. 1 (1976): 27-28.

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SOME PROPERTIES OF THE SEQUENCE $\{W_n(a, b; p, q)\}$

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1. INTRODUCTION

Elsewhere in this journal [5], the sequence $\{W_n(a, b; p, q)\}$ has been introduced and its basic properties exhibited. Here, we investigate the finite sum of W_k^t (k from 0 to $n - 1$) and the properties of W_{mn}. Notation and content of [5] are assumed, when required.

Particular cases of $\{W_n\}$ are the sequences $\{U_n\}$, $\{V_n\}$, $\{H_n\}$, $\{F_n\}$, and $\{L_n\}$ given by:

$$U_n(p, q) = W_n(1, p; p, q) \tag{1}$$

$$V_n(p, q) = W_n(2, p; p, q) = pU_{n-1}(p, q) - 2qU_{n-2}(p, q) \tag{2}$$

$$H_n(r, s) = W_n(r, r + s; 1, -1) = rF_{n+1} + sF_n \tag{3}$$

$$F_n = W_n(0, 1; 1, -1) = H_n(0, 1) = U_{n-1}(1, -1) \tag{4}$$

$$L_n = W_n(2, 1; 1, -1) = H_n(2, -1) = V_n(1, -1) \tag{5}$$

Historical information about these second-order recurrence sequences can be found in L. Dickson [3]. Of course, $\{F_n\}$ is the famous Fibonacci sequence, $\{L_n\}$ is the Lucas sequence, $\{U_n\}$ and $\{V_n\}$ are generalizations of these, and $\{H_n\}$, discussed in [4], is a different generalization of them, while $\{W_n\}$ is the complete generalization of them. Chief properties of $\{W_n\}$, $\{U_n\}$, $\{V_n\}$, $\{H_n\}$, $\{F_n\}$, and $\{L_n\}$ can be found, for example, in V. E. Hoggatt, Jr. [3], A. F. Horadam [4], [5], [6], D. Jarden [7], E, Lucas [8], K. Subba Rao [9], A. Tagiuri [10], [11], and N. N. Vorobév [12].

Two interesting specializations of (1) and (2) are the Fermat sequences

$$\{U_n(3, 2)\} = \{2^{n+1} - 1\} \quad \text{and} \quad \{V_n(3, 2)\} = \{2^n + 1\}$$

and the Pell sequences

$$\{U_n(2, -1)\} \quad \text{and} \quad \{V_n(2, -1)\}$$

(see [1], [6], [8]).

From (1)-(5), it follows (See [4], [5], [6]) that ($p^2 \neq 4q$),

$$\begin{cases} W_n = \{(b - a\beta)\alpha^n + (a\alpha - b)\beta^n\}/(\alpha - \beta) & (6) \\ U_n = (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta) & (7) \\ V_n = \alpha^n + \beta^n & (8) \\ H_n = \{(r + s - r\beta_0)\alpha_0^n - (r + s - r\alpha_0)\beta_0^n\}/\sqrt{5} & (9) \\ F_n = (\alpha_0^n - \beta_0^n)/\sqrt{5} & (10) \\ L_n = \alpha_0^n + \beta_0^n & (11) \end{cases}$$

where

$$\alpha = (p + \sqrt{p^2 - 4q})/2, \ \beta = (p - \sqrt{p^2 - 4q})/2$$
$$\alpha_0 = (1 + \sqrt{5})/2, \text{ and } \beta_0 = (1 - \sqrt{5})/2.$$

In the meantime, from [4], [5], and [6], we have:

$$\begin{cases} W_{k+1} = pW_k - qW_{k-1} & (12) \\ W_{k+1}W_{k-1} = W_k^2 + eq^{k-1}, \text{ where } e = abp - a^2q - b^2 & (13) \\ W_{n+k} = W_nV_k - q^kW_{n-k} & (14) \\ W_{m+r}U_{n-r-1} - q^kW_{m+r-k}U_{n-r-k-1} = W_{m+n-k}U_{k-1} & (15) \\ W_{m+r}W_{n-r} - q^kW_{m+r-k}W_{n-r-k} = (bW_{m+n-k} - aqW_{m+n-k-1})U_{k-1} & (16) \end{cases}$$

2. THE FINITE SUM $\sum_{k=0}^{n-1} W_k^t$

Define

$$G_k(m, j) = \sum_{i=0}^{m} \binom{m}{i} W_{k+1}^{m+j-i+1}(qW_{k-1})^{j+i+1}; \tag{17}$$

we have

Lemma 1: $G_k(m, j) = q^{j+1}(pW_k)^m(W_k^2 + eq^{k-1})^{j+1}$ (18)

$$= p^m\left\{\sum_{i=0}^{j+1}\binom{j+1}{i} e^{j-i+1}q^{k(j-i+1)+i}W_k^{m+2i}\right\}, \tag{19}$$

where $e = abp - a^2q - b^2$.

Proof: $G_k(m, j) = \sum_{i=0}^{m}\binom{m}{i} W_{k+1}^{m+j-i+1}(qW_{k-1})^{j+i+1}$, by (17)

$= (qW_{k+1}W_{k-1})^{j+1}\left\{\sum_{i=0}^{m}\binom{m}{i}W_{k+1}^{m-i}(qW_{k-1})^i\right\}$

$= (qW_{k+1}W_{k-1})^{j+1}(W_{k+1} + qW_{k-1})^m$, by the binomial theorem

$= q^{j+1}(W_k^2 + eq^{k-1})^{j+1}(W_{k+1} + qW_{k-1})^n$, by (13)

$= q^{j+1}(W_k^2 + eq^{k-1})^{j+1}(pW_k)^m$, by (12)

$= q^{j+1}(pW_k)^m\left\{\sum_{i=0}^{j+1}\binom{j+1}{i}W_k^{2i}(eq^{k-1})^{j-i+1}\right\}$, by the binomial theorem

$= p^m\left\{\sum_{i=0}^{j+1}\binom{j+1}{i}e^{j-i+1}q^{k(j-i+1)+i}W_k^{m+2i}\right\}.$

Consider $a_j(t)$ satisfying the following recurrence,

$$a_{j+1}(t + 2) = a_{j+1}(t + 1) + a_j(t), \tag{20}$$

subject to the initial conditions $a_0(t) = 1$ for $t \geqslant 1$, $a_j(2j) = 2$ for $j \geqslant 0$, with $a_j(t) = 0$ for $j < 0$ and $j > [t/2]$. It is easy to prove directly from (20) that

$$a_j(t) = \binom{t - j}{j} + \binom{t - j - 1}{j - 1}. \tag{21}$$

The first few value of $a_j(t)$ are shown in Table 1.

Table 1. The Values of $a_j(t)$

$j \backslash t$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	0	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	0	0	0	2	5	9	14	20	27	35	44	54	65	77	90
3	0	0	0	0	0	0	2	7	16	30	50	77	112	156	210	275
4	0	0	0	0	0	0	0	0	2	9	25	55	105	182	294	450
5	0	0	0	0	0	0	0	0	0	0	2	11	36	91	196	378
6	0	0	0	0	0	0	0	0	0	0	0	0	2	13	49	140
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	15

Lemma 2: $$\sum_{i=1}^{t-1} \binom{t}{i} W_{k+1}^{t-i}(qW_{k-1})^i = \sum_{j=1}^{[t/2]} (-1)^{j+1} a_j(t) G_k(t - 2j, j - 1). \tag{22}$$

Proof: $$\sum_{i=1}^{t-1} \binom{t}{i} W_{k+1}^{t-i}(qW_{k-1})^i = \sum_{i=0}^{t-2} \binom{t}{i + 1} W_{k+1}^{t-i-1}(qW_{k-1})^{i+1}, \text{ by a dummy variable}$$

$$= t\sum_{i=0}^{t-2}\binom{t - 2}{i} W_{k+1}^{t-i-1}(qW_{k-1})^{i+1} - \frac{t}{2}\binom{t - 3}{1}\sum_{i=0}^{t-4}\binom{t - 4}{i} W_{k+1}^{t-i-2}(qW_{k-1})^{i+2}$$

$$+ \frac{t}{3}\binom{t - 4}{2}\sum_{i=0}^{t-6}\binom{t - 6}{i} W_{k+1}^{t-i-3}(qW_{k-1})^{i+3} - \cdots, \text{ by expansion}$$

$$= \sum_{j=1}^{[t/2]} (-1)^{j+1} a_j(t) \left\{\sum_{i=0}^{t-2j} \binom{t - 2j}{i} W_{k+1}^{t-j-i}(qW_{k-1})^{j+i}\right\}, \text{ by summation}$$

$$= \sum_{j=1}^{[t/2]} (-1)^{j+1} a_j(t) G_k(t - 2j, j - 1), \text{ by (17)}.$$

Consider $A(j, t; p, q) \equiv A(j, t)$ satisfying the following recurrence,

$$A(j + 1, t + 2) = pA(j + 1, t + 1) - qA(j + 1, t) + A(j, t) \tag{23}$$

subject to the initial conditions $A(j, 2j) = 2$ for $j \geqslant 0$, $A(0, 1) = p$, with $A(j, t) = 0$ for $j < 0$ and $j > [t/2]$. It is easy to prove directly from (23) that

$$A(j, t) = p^{t-2j}\left\{\sum_{i=0}^{[t/2]-j} \binom{i + j}{j} (-p^{-2}q)^i a_{i+j}(t)\right\}. \tag{24}$$

The first few values of $A(j, t)$ are shown in Table 2. Note that

$$A(j, t) = \frac{(-1)^j}{j!} V_t^{(j)}, \text{ where } V_t^{(j)} = \frac{\partial^j V_t}{\partial q^j}.$$

Table 2. The Values of $A(j,\ t)$

t \ j	0	1	2	3	4
0	2	0	0	0	0
1	p	0	0	0	0
2	$p^2 - 2q$	2	0	0	0
3	$p^3 - 3pq$	$3p$	0	0	0
4	$p^4 - 4p^2q + 2q^2$	$4p^2 - 4q$	2	0	0
5	$p^5 - 5p^3q + 5pq^2$	$5p^3 - 10pq$	$5q$	0	0
6	$p^6 - 6p^4q + 9p^2q^2 - 2q^3$	$6p^4 - 18p^2q + 6q^2$	$9p^2 - 6q$	2	0
7	$p^7 - 7p^5q + 14p^3q^2 - 7pq^3$	$7p^5 - 28p^3q + 21pq^2$	$14p^3 - 21pq$	$7p$	0

Now, define

$$L_W(r,\ t) = \sum_{k=0}^{n-1} q^{kr} W_k^t \tag{25}$$

and

$$W(t) = \sum_{k=0}^{n-1} W_k^t = L_W(0,\ t), \tag{26}$$

where r and t are nonnegative integers; then we have the following lemmas and theorem.

Lemma 3:
$$\sum_{j=1}^{[t/2]} (-1)^{j+1} a_j(t) p^{t-2j} \left\{ \sum_{i=1}^{j} \binom{j}{i} e^i q^{j-i} L_W(r+i,\ t-2i) \right\} \tag{27}$$
$$= -\sum_{j=1}^{[t/2]} (-e)^j A(j,\ t) L_W(r+j,\ t-2j).$$

Proof:
$$\sum_{j=1}^{[t/2]} (-1)^{j+1} a_j(t) p^{t-2j} \left\{ \sum_{i=1}^{j} \binom{j}{i} e^i q^{j-i} L_W(r+i,\ t-2i) \right\}$$
$$= a_1(t) p^{t-2} e L_W(r+1,\ t-2) - a_2(t) p^{t-4} \left\{ \sum_{i=1}^{2} \binom{2}{i} e^i q^{2-i} L_W(r+i,\ t-2i) \right\}$$
$$+ a_3(t) p^{t-6} \left\{ \sum_{i=1}^{3} \binom{3}{i} e^i q^{3-i} L_W(r+i,\ t-2i) \right\} - \cdots, \text{ by expansion}$$
$$= eA(1,\ t)L\ (r+1,\ t-2) - e^2 A(2,\ t) L_W(r+2,\ t-4)$$
$$+ e^3 A(3,\ t) L_W(r+3,\ t-6) - \cdots, \text{ by collecting terms in } L_W(r+i,\ t-2i) \text{ for all positive integers } i$$
$$= -\sum_{j=1}^{[t/2]} (-e)^j A(j,\ t) L_W(r+j,\ t-2j), \text{ by summation.}$$

Lemma 4:
$$\sum_{k=0}^{n-1} q^{kr} G_k(t-2j,\ j-1) = p^{t-2j} \left\{ \sum_{i=0}^{j} \binom{j}{i} e^i q^{j-i} L_W(r+i,\ t-2i) \right\}. \tag{28}$$

Proof: $$\sum_{k=0}^{n-1} q^{kr} G_k(t - 2j,\ j - 1) = \sum_{k=0}^{n-1} q^{kr}\{q^j (pW_k)^{t-2j}(W_k^2 + eq^{k-1})^j\},\ \text{by (18)}$$

$$= \sum_{k=0}^{n-1} q^{kr+j}(pW_k)^{t-2j}\left\{\sum_{i=0}^{j}\binom{j}{i} W^{2j-2i}(eq^{k-1})^i\right\},\ \text{by the binomial theorem}$$

$$= p^{t-2j}\left\{\sum_{i=0}^{j}\binom{j}{i} e^i q^{j-i}\left\{\sum_{k=0}^{n-1} q^{k(r+i)} W_k^{t-2i}\right\}\right\}$$

$$= p^{t-2j}\left\{\sum_{i=0}^{j}\binom{j}{i} e^i q^{j-i} L_W(r + i,\ t - 2i)\right\},\ \text{by (25).}$$

Consider $B(t;\ p,\ q) \equiv B(t)$ satisfying the following recurrence,

$$B(t + 2) = pB(t + 1) - qB(t) + a_0(t)p^t q, \tag{29}$$

subject to the initial conditions $B(0) = B(1) = 0$.

Let $C(t) = B(t) - a_0(t)p^t$; then $C(t)$ satisfies the following recurrence,

$$C(t + 2) = pC(t + 1) - qC(t) \text{ with } C(0) = -2,\ C(1) = -p, \tag{30}$$

i.e.,

$$C(t) = -p^t\left\{\sum_{j=0}^{[t/2]}(-p^{-2}q)^j a_j(t)\right\}, \tag{31}$$

$$B(t) = -p^t\left\{\sum_{j=1}^{[t/2]}(-p^{-2}q)^j a_j(t)\right\}. \tag{32}$$

Table 3. The Values of $B(t)$ and $C(t)$

t	0	1	2	3	4	5
$B(t)$	0	0	$2q$	$3pq$	$4p^2q - 2q^2$	$5p^3q - 5pq^2$
$C(t)$	-2	$-p$	$-p^2 + 2q$	$-p^3 + 3pq$	$-p^4 + 4p^2q - 2q^2$	$-p^5 + 5p^3q - 5pq^2$

Lemma 5: $$\sum_{k=0}^{n-1} q^{kr}\left\{\sum_{i=1}^{t-1}\binom{t}{i} W_{k+1}^{t-i}(qW_{k-1})^i\right\}$$

$$= B(t)L_W(r,\ t) - \sum_{j=1}^{[t/2]}(-e)^j A(j,\ t)L_W(r + j,\ t - 2j). \tag{33}$$

Proof: $$\sum_{k=0}^{n-1} q^{kr}\left\{\sum_{i=1}^{t-1}\binom{t}{i} W_{k+1}^{t-i}(qW_{k-1})^i\right\}$$

$$= \sum_{k=0}^{n-1} q^{kr}\left\{\sum_{j=1}^{[t/2]}(-1)^{j+1} a_j(t) G_k(t - 2j,\ j - 1)\right\},\ \text{by (22)}$$

$$= \sum_{j=1}^{[t/2]}(-1)^{j+1} a_j(t)\left\{\sum_{k=0}^{n-1} q^{kr} G_k(t - 2j,\ j - 1)\right\}$$

$$= \sum_{j=1}^{[t/2]}(-1)^{j+1} a_j(t) p^{t-2j}\left\{\sum_{i=0}^{j}\binom{j}{i} e^i q^{j-i} L_W(r + i,\ t - 2i)\right\},\ \text{by (28)}$$

(continued)

$$= B(t)L_W(r,\ t) - \sum_{j=1}^{[t/2]}(-e)^j A(j,\ t)L_W(r+j,\ t-2j),\ \text{by (27) and (32).}$$

Theorem 1: $L_W(r,\ t)$ satisfies the following recursion,

$$\begin{aligned}&\{1 + q^{2r+t} - a_0(t)p^t q^r + q^r B(t)\}L_W(r,\ t)\\ &= q^{nr}(q^{r+t}W_{n-1}^t - W_n^t) - (q^{r+t}W_{-1}^t - W_0^t)\\ &\quad + q^r\left\{\sum_{j=1}^{[t/2]}(-e)^j A(j,\ t)L_W(r+j,\ t-2j)\right\},\end{aligned} \tag{34}$$

for $t \geqslant 1$ or ($t = 0$ and $r \geqslant 1$).

Proof: (1) When $t = 0$ and $r \geqslant 1$:

$$L_W(r,\ 0) = \sum_{k=0}^{n-1} q^{kr},\ \text{from (25).}$$

Hence, $L_W(r,\ 0)$ satisfies (34).

(2) When $t \geqslant 1$: Since

$$\begin{aligned}p^t L_W(r,\ t) &= \sum_{k=0}^{n-1} q^{kr}(pW_k)^t,\ \text{by (25)}\\
&= \sum_{k=0}^{n-1} q^{kr}(W_{k+1} + qW_{k-1})^t,\ \text{by (12)}\\
&= \sum_{k=0}^{n-1} q^{kr}\left\{\sum_{i=0}^{t}\binom{t}{i}W_{k+1}^{t-i}(qW_{k-1})^i\right\},\ \text{by the binomial theorem}\\
&= \sum_{k=0}^{n-1} q^{kr}\left\{W_{k+1}^t + q^t W_{k-1}^t + \sum_{i=1}^{t-1}\binom{t}{i}W_{k+1}^{t-i}(qW_{k-1})^i\right\},\ \text{by expansion}\\
&= \{q^{-r}L_W(r,\ t) + q^{(n-1)r}W_n^t - q^{-r}W_0^t\} + q^t\{q^r L_W(r,\ t) - q^{nr}W_{n-1}^t + W_{-1}^t\}\\
&\quad + B(t)L_W(r,\ t) - \sum_{j=1}^{[t/2]}(-e)^j A(j,\ t)L_W(r+j,\ t-2j),\ \text{by (33),}\end{aligned}$$

we have

$$\begin{aligned}&\{q^{-r} + q^{r+t} - p^t + B(t)\}L_W(r,\ t)\\ &= q^{(n-1)r}(q^{r+t}W_{n-1}^t - W_n^t) - q^{-r}(q^{r+t}W_{-1}^t - W_0^t)\\ &\quad + \sum_{j=1}^{[t/2]}(-e)^j A(j,\ t)L_W(r+j,\ t-2j).\end{aligned}$$

Hence,

$$\begin{aligned}&\{1 + q^{2r+t} - p^t q^r + q^r B(t)\}L_W(r,\ t)\\ &= q^{nr}(q^{r+t}W_{n-1}^t - W_n^t) - (q^{r+t}W_{-1}^t - W_0^t)\\ &\quad + q^r\left\{\sum_{j=1}^{[t/2]}(-e)^j A(j,\ t)L_W(r+j,\ t-2j)\right\}.\end{aligned}$$

This completes the proof of Theorem 1, since $a_0(t) = 1$ for $t \geqslant 1$.

Setting $t = 0, 1, 2,$ and 3 in Theorem 1, we have the following four corollaries.

Corollary 1: $(1 - q^r)L_W(r, 0) = 1 - q^{nr}$, for all $r \geqslant 1$ [cf. (25)].

Proof: Setting $t = 0$ in Theorem 1, we have

$$(1 + q^{2r} - a_0(0)q^r + q^rB(0))L_W(r, 0) = q^{nr}(q^r - 1) - (q^r - 1),$$

i.e.,

$$(1 - q^r)L_W(r, 0) = 1 - q^{nr}, \text{ since } a_0(0) = 2.$$

See also Proof (1) of Theorem 1.

Corollary 2: $(1 + q^{2r+1} - pq^r)L_W(r, 1) = q^{nr}(q^{r+1}W_{n-1} - W_n) - (q^{r+1}W_{-1} - W_0)$.

Proof: Setting $t = 1$ in Theorem 1, we have

$$(1 + q^{2r+1} - a_0(1)pq^r + q^rB(1))L_W(r, 1)$$
$$= q^{nr}(q^{r+1}W_{n-1} - W_n) - (q^{r+1}W_{-1} - W_0),$$

completing the proof of Corollary 2.

Corollary 3: $(1 + q^{2r+2} - p^2q^r + 2q^{r+1})L_W(r, 2)$

$$= q^{nr}(q^{r+2}W_{n-1} - W_n^2) - (q^{r+2}W_{-1}^2 - W_0^2) - 2eq^rL_W(r + 1, 0).$$

Proof: Setting $t = 2$ in Theorem 1, we have

$$(1 + q^{2r+2} - a_0(2)p^2q^r + q^rB(2))L_W(r, 2)$$
$$= q^{nr}(q^{r+2}W_{n-1}^2 - W_n^2) - (q^{r+2}W_{-1}^2 - W_0^2) - eq^rA(1, 2)L_W(r + 1, 0),$$

completing the proof of Corollary 3.

Corollary 4: $(1 + q^{2r+3} - p^3q^r + 3pq^{r+1})L_W(r, 3)$

$$= q^{nr}(q^{r+3}W_{n-1}^3 - W_n^3) - (q^{r+3}W_{-1}^3 - W_0^3) - 3epq^rL_W(r + 1, 1).$$

Proof: Setting $t = 3$ in Theorem 1, we have

$$(1 + q^{2r+3} - a_0(3)p^3q^r + q^rB(3))L_W(r, 3)$$
$$= q^{nr}(q^{r+3}W_{n-1}^3 - W_n^3) - (q^{r+3}W_{-1}^3 - W_0^3) - eq^rA(1, 3)L_W(r + 1, 1),$$

completing the proof of Corollary 4.

Since $C(t) = B(t) - a_0(t)p^t$, we have

Theorem 1′: $L_W(r, t)$ satisfies the following recursion,

$$\{1 + q^{2r+t} + q^rC(t)\}L_W(r, t)$$
$$= q^{nr}(q^{r+t}W_{n-1}^t - W_n^t) - (q^{r+t}W_{-1}^t - W_0^t)$$
$$+ q^r\left\{\sum_{j=1}^{[t/2]}(-e)^jA(j, t)L_W(r + j, t - 2j)\right\}, \tag{35}$$

for $t \geqslant 1$ or ($t = 0$ and $r \geqslant 1$).

Setting $r = 0$ in Theorem 1′, we have

Theorem 2: $W(t)$ satisfies the following recursion,

$$\{1 + q^t + C(t)\}W(t)$$
$$= (q^t W_{n-1}^t - W_n^t) - (q^t W_{-1}^t - W_0^t) + \sum_{j=1}^{[t/2]} (-e)^j A(j, t) L_W(j, t - 2j), \tag{36}$$

for $t \geqslant 1$.

Now, we have the following five formulas about $W(t)$ for t, respectively, 1 to 5:

$$(1 + q - p)W(1) = (qW_{n-1} - W_n) - (qW_{-1} - W_0); \tag{37}$$
$$(1 + q^2 - p^2 + 2q)W(2) = (q^2W_{n-1}^2 - W_n^2) - (q^2W_{-1}^2 - W_0^2) - 2eL_W(1, 0); \tag{38}$$
$$(1 + q^3 - p^3 + 3pq)W(3) = (q^3W_{n-1}^3 - W_n^3) - (q^3W_{-1}^3 - W_0^3) - 3epL_W(1, 1); \tag{39}$$
$$(1 + q^4 - p^4 + 4p^2q - 2q^2)W(4)$$
$$= (q^4W_{n-1}^4 - W_n^4) - (q^4W_{-1}^4 - W_0^4) - 4e(p^2 - q)L_W(1, 2) + 2e^2L_W(2, 0); \tag{40}$$
$$(1 + q^5 - p^5 + 5p^3q - 5pq^2)W(5)$$
$$= (q^5W_{n-1}^5 - W_n^5) - (q^5W_{-1}^5 - W_0^5) - 5ep(p^2 - 2q)L_W(1, 3) + 5e^2pL_W(2, 1). \tag{41}$$

We note that (37) is the equivalent form of (3.5) in [5], (38) is the simple form of (4.16) in [5], and (39) is the simple form of (4.28), misprinted, in [5].

Finally, we consider the corresponding special cases of $W(t)$:

(1) When $a = r$, $b = r + s$, $p = 1$, and $q = -1$, then $H(t) = \sum_{k=0}^{n-1} H_k^t(r, s)$ has the following properties:

$H(1) = H_n + H_{n-1} - r - s = H_{n+1} - r - s$, by (37);

$H(2) = H_n^2 - H_{n-1}^2 - r^2 + s^2 + (1 - (-1)^n)(r^2 - rs - s^2)$, by (38) and Cor. 1;

$4H(3) = H_n^3 + H_{n-1}^3 - r^3 - s^3 + 3(r^2 - rs - s^2)\{(-1)^{n+1}H_{n-2} + r - s\}$, by (39) and Cor. 2;

$5H(4) = H_n^4 - H_{n-1}^4 - r^4 + s^4 + 6n(r^2 - rs - s^2)^2/5 + 8(r^2 - rs - s^2)\{(-1)^{n+1}(H_n^2 + H_{n-1}^2) + r^2 + s^2\}/5$, by (40) and Cors. 1, 3;

$11H(5) = H_n^5 + H_{n-1}^5 - r^5 - s^5 + 25(r^2 - rs - s^2)^2(H_{n+1} - r - s)/4 + 15(r^2 - rs - s^2)\{(-1)^{n+1}(H_n^3 - H_{n-1}^3) + r^3 - s^3\}/4$, by (41) and Cors. 2, 4.

(2) When $a = 0$, $b = p = 1$, and $q = -1$, then $F(t) = \sum_{k=0}^{n-1} F_k^t$ has the following properties:

$F(1) = F_{n+1} - 1$

$F(2) = F_n^2 - F_{n-1}^2 + (-1)^n = (F_{2n} - F_n^2)/2$

$4F(3) = F_n^3 + F_{n-1}^3 + 3(-1)^nF_{n-2} + 2$

$5F(4) = F_n^4 - F_{n-1}^4 + 8(-1)^n(F_n^2 + F_{n-1}^2)/5 + 6n/5 - 3/5$

$11F(5) = F_n^5 + F_{n-1}^5 + 15(-1)^n(F_n^3 - F_{n-1}^3)/4 + 25F_{n+1}/4 - 7/2$

(3) When $a = 2$, $b = p = 1$, and $q = -1$, then $L(t) = \sum_{k=0}^{n-1} L_k^t$ has the following properties:

$L(1) = L_{n+1} - 1$

$$L(2) = L_n^2 - L_{n-1}^2 + 5(-1)^{n+1} + 2$$
$$4L(3) = L_n^3 + L_{n-1}^3 + 15(-1)^{n+1}L_{n-2} + 38$$
$$5L(4) = L_n^4 - L_{n-1}^4 + 8(-1)^{n+1}(L_n^2 + L_{n-1}^2) + 30n + 25$$
$$11L(5) = L_n^5 + L_{n-1}^5 + 75(-1)^{n+1}(L_n^3 - L_{n-1}^3)/4 + 625L_{n+1}/4 - 37/2$$

3. THE PROPERTIES OF W_{mn}

Define

$$\tilde{L}_m(q) \equiv \tilde{L}_m = \sum_{k=0}^{[(m-1)/2]} \binom{m-k-1}{k}(-q^n)^k V_n^{m-2k-1}, \quad \text{with } \tilde{L}_0 = 0,$$

where m and n are nonnegative integers. Then we obtain the following lemma.

Lemma 6: $\tilde{L}_m$ satisfies the following recursion,

$$\tilde{L}_{m+2} = V_n\tilde{L}_{m+1} - q^n\tilde{L}_m, \text{ with } \tilde{L}_0 = 0 \text{ and } \tilde{L}_1 = 1.$$

Using Lemma 6 and mathematical induction, we have

Theorem 3: $W_{mn} = \tilde{L}_m W_n - aq^n\tilde{L}_{m-1}$.

Proof: For $m = 1$, we have $W_n = \tilde{L}_1 W_n - aq^n\tilde{L}_0$ from the definition and from the formula. Similarly, the theorem is true if $m = 2$. We now show that the formula for $m + 1$ follows from the formula for m and $m - 1$.

$$\begin{aligned} W_{(m+1)n} &= V_n W_{mn} - q^n W_{(m-1)n}, \text{ by (14)} \\ &= V_n(\tilde{L}_m W_n - aq^n\tilde{L}_{m-1}) - q^n(\tilde{L}_{m-1}W_n - aq^n\tilde{L}_{m-2}) \\ &= (V_n\tilde{L}_m - q^n\tilde{L}_{m-1})W_n - aq^n(V_n\tilde{L}_{m-1} - q^n\tilde{L}_{m-2}) \\ &= \tilde{L}_{m+1}W_n - aq^n\tilde{L}_m, \text{ by Lemma 6,} \end{aligned}$$

completing the proof.

In particular, we have the following six corollaries.

Corollary 5: $U_{mn-1} = \tilde{L}_m U_{n-1}$, i.e., $U_{n-1} \mid U_{mn-1}$.

Corollary 6: $U_{mn} = \tilde{L}_m U_n - q^n\tilde{L}_{m-1}$

$$= \sum_{k=0}^{\infty} (-q^n)^k V_n^{m-2k-2}\left\{\binom{m-k-1}{k}U_n V_n - \binom{m-k-2}{k}q^n\right\}.$$

Corollary 7: $V_{mn} = \tilde{L}_m V_n - 2q^n\tilde{L}_{m-1} = V_n^m + \sum_{k=1}^{\infty} (-q^n)^k V_n^{m-2k} a_k(m)$

$$= \sum_{k=0}^{\infty} (-q^n)^k V_n^{m-2k-2}\left\{\binom{m-k-1}{k}V_n^2 - 2\binom{m-k-2}{k}q^n\right\}.$$

That is to say, $V_n \mid V_{mn}$ if m is odd.

Corollary 8: $H_{mn}(r, s) = \tilde{L}_m(-1)H_n(r, s) - r(-1)^n\tilde{L}_{m-1}(-1)$

$$\begin{aligned} = \sum_{k=0}^{\infty} (-1)^{(n+1)k} L_n^{m-2k-2}\Bigg\{&\binom{m-k-1}{k}L_n H_n(r, s) \\ &+ r(-1)^{n+1}\binom{m-k-2}{k}\Bigg\}. \end{aligned}$$

Corollary 9: $F_{mn} = \tilde{L}_m(-1)F_n = \sum_{k=0}^{\infty} (-1)^{(n+1)k}\binom{m-k-1}{k} L_n^{m-2k-1} F_n$, i.e., $F_n | F_{mn}$.

Corollary 10:
$$L_{mn} = \tilde{L}_m(-1)L_n - 2(-1)^n \tilde{L}_{m-1}(-1)$$
$$= L_n^m + \sum_{k=1}^{\infty} (-1)^{(n+1)k} L_n^{m-2k} a_k(m)$$
$$= \sum_{k=0}^{\infty} (-1)^{(n+1)k} L_n^{m-2k-2} \left\{\binom{m-k-1}{k} L_n^2 + 2(-1)^{n+1}\binom{m-k-2}{k}\right\}.$$

That is to say, $L_n | L_{mn}$ if m is odd.

Example 1: Setting $m = 2$, we have the following seven properties:

$W_{2n} = V_n W_n - aq^n$

$U_{2n-1} = V_n U_{n-1}$ (see [5]; [8])

$U_{2n} = V_n U_n - q^n$

$V_{2n} = V_n^2 - 2q^n$ (see [5]; [8])

$H_{2n}(r, s) = L_n H_n(r, s) - r(-1)^n$

$F_{2n} = L_n F_n$

$L_{2n} = L_n^2 - 2(-1)^n$

Example 2: Setting $m = 3$, we obtain the following seven properties:

$W_{3n} = (V_n^2 - q^n)W_n - aq^n V_n$

$U_{3n-1} = (V_n^2 - q^n)U_{n-1}$ (see [5]; [8])

$U_{3n} = (V_n^2 - q^n)U_n - q^n V_n$

$V_{3n} = (V_n^2 - 3q^n)V_n$ (see [5]; [8])

$H_{3n}(r, s) = (L_n^2 - (-1)^n)H_n(r, s) - r(-1)^n L_n$

$F_{3n} = (L_n^2 - (-1)^n)F_n$

$L_{3n} = (L_n^2 - 3(-1)^n)L_n$

Example 3: Setting $m = 4$, we have the following seven properties:

$W_{4n} = (V_n^2 - 2q^n)V_n W_n - aq^n(V_n^2 - q^n)$

$U_{4n-1} = (V_n^2 - 2q^n)V_n U_{n-1}$

$U_{4n} = (V_n^2 - 2q^n)V_n U_n - q^n(V_n^2 - q^n)$

$V_{4n} = V_n^4 - 4q^n V_n^2 + 2q^{2n}$

$H_{4n}(r, s) = (L_n^2 - 2(-1)^n)L_n H_n(r, s) - r(-1)^n L_n^2 + r$

$F_{4n} = (L_n^2 - 2(-1)^n)L_n F_n = (L_n^2 - 2(-1)^n)F_{2n}$

$L_{4n} = L_n^4 - 4(-1)^n L_n^2 + 2$

4. THE POWER EXPANSION OF W_n

Since
$$\begin{cases} W_n(1, 0; p, q) = \sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1} p^{n-2k}(-q)^k \\ W_n(0, 1; p, q) = \sum_{k=1}^{[(n+1)/2]} \binom{n-k}{k-1} p^{n-2k+1}(-q)^{k-1}, \end{cases}$$
we have
$$W_n(a, b; p, q) = \sum_{k=1}^{\infty} p^{n-2k}(-q)^{k-1}\left\{bp\binom{n-k}{k-1} - aq\binom{n-k-1}{k-1}\right\}.$$

Now, we consider the special cases of $W_n(a, b; p, q)$:

$$U_n(p, q) = \sum_{k=1}^{\infty} p^{n-2k}(-q)^{k-1}\left\{p^2\binom{n-k}{k-1} - q\binom{n-k-1}{k-1}\right\}$$
$$= \sum_{j=0}^{\infty} (-1)^j \binom{n-j}{j} p^{n-2j} q^j$$
$$V_n(p, q) = \sum_{k=1}^{\infty} p^{n-2k}(-q)^{k-1}\left\{p^2\binom{n-k}{k-1} - 2q\binom{n-k-1}{k-1}\right\}$$
$$H_n(r, s) = \sum_{k=0}^{\infty} \left\{r\binom{n-k}{k} + s\binom{n-k-1}{k}\right\} = rF_{n+1} + sF_n$$
$$F_n = \sum_{k=0}^{\infty} \binom{n-k-1}{k}$$
$$L_n = \sum_{k=0}^{\infty} \left\{2\binom{n-k}{k} - \binom{n-k-1}{k}\right\} = 2F_{n+1} - F_n$$

Remark: $W_{mn+k} = \sum_{i=0}^{m} \binom{m}{i} U_{n-1}^i (-qU_{n-2})^{m-i} W_{k+i}$.

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REFERENCES

1. E. Bessel-Hagen. *Repertorium der höheren Mathematic.* Leipzig, 1929, p. 1563.
2. L. Dickson. *History of the Theory of Numbers.* New York, 1952, Vol. I, ch. 17.
3. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1980.
4. A. F. Horadam. "A Generalized Fibonacci Sequence." *Amer. Math. Monthly 68*, no. 5 (1961):455-59.
5. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* 3, no. 2 (1965):161-77.

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to DR. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-598 *Proposed by Herta T. Freitag, Roanoke VA*

For which positive integers n is $(2L_n,\ L_{2n} - 3,\ L_{2n} - 1)$ a Pythagorean triple? For which of these n's is the triple primitive?

B-599 *Proposed by Herta T. Freitag, Roanoke, VA*

Do B-598 with the triple now $(2L_n,\ L_{2n} + 1,\ L_{2n} + 3)$.

B-600 *Proposed by Philip L. Mana, Albuquerque, NM*

Let n be any positive integer and $m = n^{13} - n$. Prove that F_n is an integral multiple of 30290.

B-601 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $A_{n,k} = (F_n + F_{n+1} + \cdots + F_{n+k-1})/k$. Find the smallest k in $\{2, 3, 4, \ldots\}$ such that $A_{n,k}$ is an integer for every n in $\{0, 1, 2, \ldots\}$.

B-602 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Let H_n represent either F_n or L_n.

(a) Find a simplified expression for $\dfrac{1}{H_n} - \dfrac{1}{H_{n+1}} - \dfrac{1}{H_{n+2}}$.

(b) Use the result of (a) to prove that

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + 2\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}F_{2n+1}F_{2n+2}}.$$

B-603 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Do the Lucas analogue of B-602(b).

SOLUTIONS

Downrounded Square Roots

B-574 *Proposed by Valentina Bakinova, Rondout Valley, NY*

Let $a_1, a_2, \ldots$ be defined by $a_1 = 1$ and $a_{n+1} = [\sqrt{s_n}]$, where $s_n = a_1 + a_2 + \cdots + a_n$ and $[x]$ is the integer with $x - 1 < [x] \leqslant x$. Find a_{100}, s_{100}, a_{1000}, and s_{1000}.

Solution by L. A. G. Dressel, University of Reading, England

Starting with $s_1 = 1$, we have $a_2 = a_3 = a_4 = 1$ and $s_4 = 4$. Suppose now that, for some integer h, $h \geqslant 2$, we have $s_t = h^2$. Then, since $(h+1)^2 = h^2 + 2h + 1$, we obtain

$$a_{t+1} = a_{t+2} = a_{t+3} = h \quad \text{and} \quad s_{t+3} = (h+1)^2 + h - 1;$$

further,

$$a_{t+4} = a_{t+5} = h + 1 \quad \text{and} \quad s_{t+5} = (h+2)^2 + h - 2,$$

and continuing as long as $j \leqslant h$, $s_{t+2j+1} = (h+j)^2 + h - j$, so that for $j = k$ we obtain $s_{t+2h+1} = (2h)^2$.

Since $s_4 = 2^2$, it follows that whenever s_n is a perfect square it is of the form 2^{2i} $(i = 0, 1, 2, \ldots)$, and that if

$$s_{t_i} = 2^{2i} \quad \text{and} \quad s_{t_{i+1}} = 2^{2(i+1)},$$

then $t_{i+1} = t_i + 2^{i+1} + 1$.

Since $s_1 = 1$, $t_0 = 1$, and we can show that

$$t_i = 2^{i+1} + i - 1, \text{ for } i = 0, 1, 2, \ldots .$$

To find a_{100} and s_{100}: we have $t_5 = 64 + 4 = 68$, so that $s_{68} = (32)^2$,

$$s_{99} = (32 + 15)^2 + 32 - 15,\ a_{100} = 47,\ s_{100} = (47)^2 + 64 = 2273.$$

To find a_{1000} and s_{1000}: $t_8 = 2^9 + 7 = 519$ and $s_{519} = (256)^2$,

$$s_{998} = (256 + 239)^2 + 256 - 239,\ a_{999} = a_{1000} = 495$$

and

$$s_{1000} = (256 + 240)^2 + 256 - 240 = (496)^2 + 16 = 246032.$$

Also solved by Charles Ashbacher, Paul S. Bruckman, Piero Filipponi, L. Kuipers, J. Suck, M. Wachtel, and the proposer.

Summing Products

B-575 *Proposed by L. A. G. Dresel, Reading, England*

Let R_n and S_n be sequences defined by given values R_0, R_1, S_0, S_1 and the recurrence relations $R_{n+1} = rR_n + tR_{n-1}$ and $S_{n+1} = sS_n + tS_{n-1}$, where r, s, t are constants and $n = 1, 2, 3, \ldots$. Show that

$$(r + s)\sum_{k=1}^{n} R_k S_k t^{n-k} = (R_{n+1}S_n + R_n S_{n+1}) - t^n(R_1 S_0 + R_0 S_1).$$

Solution by J. Suck, Essen, Germany

This identity may be hard to dream up but is easy to prove by induction: For $n = 1$, the left-hand side is $(r + s)R_1S_1$, and the right-hand side is

$$(rR_1 + tR_0)S_1 + R_1(sS_1 + tS_0) - t(R_1S_0 + R_0S_1),$$

i.e., both are the same.

For the step from n to $n + 1$, we have to show that

$$t(R_{n+1}S_n + R_nS_{n+1}) + (r + s)R_{n+1}S_{n+1}$$
$$= (rR_{n+1} + tR_n)S_{n+1} + R_{n+1}(sS_{n+1} + tS_n),$$

which, after a little sorting, is seen to be true.

Also solved by Paul S. Bruckman, L. Cseh, Piero Filipponi & Adina Di Porto, L. Kuipers, Andreas N. Philippou & Demetris Antzoulakos, George Philippou, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

Product of Three Fibonacci Numbers

B-576 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A = L_{2m+3(4n+1)} + (-1)^m$. Show that A is a product of three Fibonacci numbers for all positive integers m and n.

Solution by Lawrence Somer, Washington, D.C.

We prove the more general result that, if $r \geqslant 1$, then

$$L_{2r+1} + (-1)^{r+1} = 5F_rF_{r+1} = F_5F_rF_{r+1}.$$

Note that, if $2r + 1 = 2m + 3(4n + 1)$, then

$$m \equiv r + 1 \pmod 2 \quad \text{and} \quad (-1)^m = (-1)^{r+1}.$$

By the Binet formulas and using the fact that $\alpha\beta = -1$,

$$5F_rF_{r+1} = 5[(\alpha^r - \beta^r)/\sqrt{5}][(\alpha^{r+1} - \beta^{r+1})/\sqrt{5}]$$
$$= \alpha^{2r+1} + \beta^{2r+1} - (\alpha\beta)^r(\alpha + \beta)$$
$$= L_{2r+1} - (-1)^rL_1 = L_{2r+1} + (-1)^{r+1},$$

and we are done.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, George Koutsoukellis, L. Kuipers, Andreas N. Philippou & Demetris Antzoulakos, Bob Prielipp, H.-J. Seiffert, Sahib Singh, J. Suck, and the proposer.

Difference of Squares

B-577 *Proposed by Herta T. Freitag, Roanoke, VA*

Let A be as in B-575. Show that $4A/5$ is a difference of squares of Fibonacci numbers.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

Let m and n be arbitrary positive integers. We shall show that

$$4A/5 = F^2_{m+6n+3} - F^2_{m+6n}. \qquad (*)$$

In our solution to B-576, we establish that

$$A = 5F_{m+6n+2}F_{m+6n+1}.$$

Thus,

$$4A/5 = 4F_{m+6n+2}F_{m+6n+1}.$$

But it is known that $4F_kF_{k-1} = F^2_{k+1} - F^2_{k-2}$ [see (I_{36}) on p. 59 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969], so (*) follows.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, George Koutsoukellis, Andreas N. Philippou & Demetris Antzoulakos, H.-J. Seiffert, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Zeckendorf Representation for $[aF_n]$

B-578 *Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy*

It is known (Zeckendorf's theorem) that every positive integer N can be represented as a finite sum of distinct nonconsecutive Fibonacci numbers and that this representation is unique. Let $a = (1 + \sqrt{5})/2$ and $[x]$ denote the greatest integer not exceeding x. Denote by $f(N)$ the number of F-addends in the Zeckendorf representation for N. For positive integers n, prove that $f([aF_n]) = 1$ if n is odd.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

It suffices to show that, for each positive integer n, $[aF_{2n-1}]$ is a Fibonacci number. We shall show that,

for each positive integer n, $[aF_{2n-1}] = F_{2n}$.

Let n be an arbitrary positive integer, and let $b = (1 - \sqrt{5})/2$. It is known that, for each positive integer k, $aF_k = F_{k+1} - b^k$ [see p. 34 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969]. So $aF_{2n-1} = F_{2n} - b^{2n-1} = F_{2n} + (-b)^{2n-1}$. Since $0 < -b < 1$, $0 < (-b)^{2n-1} < 1$. It follows that $[aF_{2n-1}] = F_{2n}$.

Also solved by Paul S. Bruckman, L. Cseh, L. A. G. Dresel, Herta T. Freitag, L. Kuipers, Imre Merenyi, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Zeckendorf Representation, Even Case

B-579 *Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy*

Using the notation of B-578, prove that $f([aF_n]) = n/2$ when n is even.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

Let n be an arbitrary positive integer. We shall show that the Zeckendorf representation for $[aF_{2n}]$ is $F_2 + F_4 + F_6 + \cdots + F_{2n}$, which implies the required result.

Let $b = (1 - \sqrt{5})/2$. It is known that

$$aF_{2n} = F_{2n+1} - b^{2n}$$

[see p. 34 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969]. Since $0 < b^2 < 1$, $0 < b^{2n} < 1$. It follows that

$$[aF_{2n}] = F_{2n+1} - 1.$$

But

$$F_{2n+1} - 1 = F_2 + F_4 + F_6 + \cdots + F_{2n}$$

by (I_6) (*Ibid.*, p. 56). Hence, the Zeckendorf representation for $[aF_{2n}]$ is

$$F_2 + F_4 + F_6 + \cdots + F_{2n}$$

completing our solution.

Also solved by Paul S. Bruckman, L. Cseh, L.A.G. Dresel, Herta T. Freitag, L. Kuipers, Imre Merenyi, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

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6. A. F. Horadam. "Special Properties of the Sequence $W_n(a, b; p, q)$." *The Fibonacci Quarterly* 5, no. 5 (1967):424-34.
7. D. Jarden. *Recurring Sequences*. Jerusalem: Riveon Lematematika, 1958.
8. E. Lucas. *Théorie des nombres*. Paris: Blanchard, 1961, ch. 18.
9. K. Subba Rao. "Some Properties of Fibonacci Numbers." *Amer. Math. Monthly* 60, no. 10 (1953):680-84.
10. A. Tagiuri. "Recurrence Sequences of Positive Integral Terms." (Italian) *Period. di Mat.*, serie 2, no 3 (1901):1-12.
11. A. Tagiuri. "Sequences of Positive Integers." (Italiam) *Period. di Mat.*, serie 2, no. 3 (1901):97-114.
12. N. N. Vorobév. *The Fibonacci Numbers* (tr. from Russian). New York, 1961.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-412 *Proposed by Andreas N. Philippou and Frosso S. Makri, University of Patras, Patras, Greece*

Show that

$$\sum_{i=0}^{k-1} \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} = \binom{n}{r}, \; k \geqslant 1, \; 0 \leqslant r \leqslant k - 1 \leqslant n,$$

where the inner summation is over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n - i$ and $n_1 + \cdots + n_k = n - r$.

H-413 *Proposed by Gregory Wulczyn, Bucknell University (retired), Lewisburg, PA*

Let m, n be integers. If m and n have the same parity, show that

(1) $(2m + 1)F_{2n+1} - (2n + 1)F_{2m+1} \equiv 0 \pmod 5$;

(2) $(2m + 1)F_{2n+1} - (2n + 1)F_{2m+1} \equiv 0 \pmod{25}$ if either

(a) $2m + 1$ or $2n + 1$ is a multiple of 5, or
(b) $m \equiv n \equiv 0$ or $m \equiv n \equiv -1 \pmod 5$.

If m and n have the opposite parity, show that

(3) $(2m + 1)F_{2n+1} + (2n + 1)F_{2m+1} \equiv 0 \pmod 5$;

(4) $(2m + 1)F_{2n+1} + (2n + 1)F_{2m+1} \equiv 0 \pmod{25}$ if either

(a) $2m + 1$ or $2n + 1$ is a multiple of 5, or
(b) $m \equiv n \equiv 0$ or $m \equiv n \equiv -1 \pmod 5$.

H-414 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, m, and n be integers. Prove that

$$F_{m+j}F_{n+k} = F_{m+k}F_{n+j} - F_{k-j}F_{n-m}(-1)^{m+j}.$$

SOLUTIONS

A Wind from the Past

H-307 *Proposed by Larry Taylor, Briarwood, NY*
(Vol. 17, no. 4, December 1979)

(A) If $p \equiv \pm 1 \pmod{10}$ is prime, $x \equiv \sqrt{5}$ and

$$a \equiv \frac{2(x-5)}{x+7} \pmod{p},$$

prove that a, $a+1$, $a+2$, $a+3$, and $a+4$ have the same quadratic character modulo p if and only if $11 < p \equiv 1$ or $11 \pmod{60}$ and $(-2x/p) = 1$.

(B) If $p \equiv 1 \pmod{60}$, $(2x/p) = 1$, and

$$b \equiv \frac{-2(x+5)}{7-x} \pmod{p},$$

then b, $b+1$, $b+2$, $b+3$, and $b+4$ have the same quadratic character modulo p. Prove that $(11ab/p) = 1$.

Solution by the proposer

(A) Let $f \equiv (x+1)/2 \pmod{p}$. Then

$$\begin{aligned}
(x+7)a &\equiv 2x - 10 \equiv -4xf^{-1}\\
(x+7)(a+1) &\equiv 3x - 3 \equiv 6f^{-1},\\
(x+7)(a+2) &\equiv 4x + 4 \equiv 8f,\\
(x+7)(a+3) &\equiv 5x + 11 \equiv 2f^5,\\
(x+7)(a+4) &\equiv 6x + 18 \equiv 12f^2 \pmod{p}.
\end{aligned}$$

But $(f^{-1}/p) = (f/p) = (f^5/p)$ and $(4/p) = (f^2/p) = 1$. Therefore,

$$\left(\frac{-4xf^{-1}}{p}\right) = \left(\frac{6f^{-1}}{p}\right) \text{ if and only if } (-2x/p) = (3/p);$$

$$\left(\frac{6f^{-1}}{p}\right) = \left(\frac{8f}{p}\right) \text{ if and only if } (3/p) = 1;$$

$$\left(\frac{8f}{p}\right) = \left(\frac{2f^5}{p}\right) \text{ unconditionally;}$$

$$\left(\frac{2f^5}{p}\right) = \left(\frac{12f^2}{p}\right) \begin{array}{l}\text{if and only if } (6f/p) = 1,\\ \text{if and only if } (3(x+1)/p) = 1.\end{array}$$

Then, the five consecutive residues have the same quadratic character modulo p if and only if

$$(-2x/p) = ((x+1)/p) = (3/p) = 1.$$

The following result is given in [1], page 24:

$$\left(\frac{\sqrt{p}}{5}\right) = \left(\frac{-2x(x+1)}{p}\right).$$

Then $(-2x/p) = ((x+1)/p)$ if and only if $(\sqrt{p}/5) = 1$. But $(\sqrt{p}/5) = (3/p) = 1$ is equivalent to $p \equiv 1$ or $11 \pmod{60}$. Since

$$(\sqrt{p}/5) = 1 \text{ if } (-2x/p) = ((x+1)/p) = 1$$

and

$$(\sqrt{p}/5) = 1 \text{ if } (-2x/p) = ((x+1)/p) = -1,$$

it is necessary to include either $(-2x/p) = 1$ or $((x + 1)/p) = 1$ in the statement of the criterion.

Finally, if $p = 11$ and $(-2x/p) = 1$, then $x \equiv 4$ and $x + 7 \equiv 0 \pmod{11}$, so this result is not valid for $p = 11$.

(B) The second part of this problem should have been stated more generally as follows: If $p \neq 11$ and

$$b \equiv \frac{-2(x + 5)}{7 - x} \pmod{p},$$

prove that $(11ab/p) = 1$.

Then

$$ab \equiv \frac{(2(x - 5))(-2(x + 5))}{(x + 7)(7 - x)} \equiv 20/11 \pmod{p}$$

and the result follows.

Comment: There is a five-term arithmetic progression of Fibonacci-Lucas identities corresponding to this set of five consecutive residues having the same quadratic character modulo p, as follows:

$$-2L_{n-1};\ 3F_{n-1};\ 4F_{n+1};\ F_{n+5};\ 6F_{n+2}.$$

The common difference is $F_n + L_{n+1}$ (i.e., $-2L_{n-1} + F_n + L_{n+1} = 3F_{n-1}$, etc.).

Reference

1. Emma Lehmer. "Criteria for Cubic and Quartic Residuacity." *Mathematika* **5** (1958):20-29.

Somethings Are Constant

H-390 *Proposed by M. Wachtel, Zurich, Switzerland (Vol. 23, no. 3, August 1985)*

For every m,

$$2F_{2-m}F_{5+m} + (-1)^m(F_mF_{m+1} + F_{m+2}^2) \text{ has the unique value } 11.$$

Find a general formula for analogous constant values, which should represent the terms of an infinite sequence.

Prove that no divisor of any of these terms is congruent to 3 or 7 modulo 10.

Solution by Bjorn Poonen, Harvard College, Cambridge, MA

Since $F_n = \dfrac{a^n - b^n}{a - b}$, where $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$, we have:

$$\begin{aligned}
&(a - b)^2[2F_{k-m}F_{k+3+m} + (-1)^{m+k}(F_mF_{m+1} + F_{m+2}^2)]\\
&\quad = 2(a^{k-m} - b^{k-m})(a^{k+3+m} - b^{k+3+m}) + (-1)^{m+k}[(a^m - b^m)(a^{m+1} - b^{m+1})\\
&\qquad + (a^{m+2} - b^{m+2})^2]\\
&\quad = 2(a^{2k+3} + b^{2k+3} - a^{k-m}b^{k+3+m} - a^{k+3+m}b^{k-m})\\
&\qquad + (-1)^{m+k}[a^{2m+1} + b^{2m+1} - a^mb^{m+1} - a^{m+1}b^m\\
&\qquad + a^{2m+4} + b^{2m+4} - 2(ab)^{m+2}]\\
&\quad = 2[a^{2k+3} + b^{2k+3} - (ab)^{k-m}(a^{2m+3} + b^{2m+3})]\\
&\qquad + (-1)^{k-m}[a^{2m+1} + b^{2m+1} - (ab)^m(a + b) + a^{2m+4} + b^{2m+4} - 2(-1)^{m+2}]
\end{aligned}$$

(continued)

$$\begin{aligned}
&= 2(a^{2k+3} + b^{2k+3}) - 2(-1)^{k-m}(a^{2m+3} + b^{2m+3}) \\
&\quad + (-1)^{k-m}(a^{2m+1} + b^{2m+1}) - (-1)^{k-m}(-1)^m(1) \\
&\quad + (-1)^{k-m}(a^{2m+4} + b^{2m+4}) - 2(-1)^{k-m}(-1)^m \\
&= 2(a^{2k+3} + b^{2k+3}) - (-1)^k - 2(-1)^k + (-1)^{k-m}(a^{2m+4} - 2a^{2m+3} + a^{2m+1}) \\
&\quad + (-1)^{k-m}(b^{2m+4} - 2b^{2m+3} + b^{2m+1}) \\
&= [a^{2k+3} + b^{2k+3} - 4(-1)^k] + [a^{2k+3} + b^{2k+3} + (-1)^k] \\
&\quad + (-1)^{k-m}a^{2m+1}(a - 1)(a^2 - a - 1) + (-1)^{k-m}b^{2m+1}(b - 1)(b^2 - b - 1) \\
&= [a^{2k+3} + b^{2k+3} - (ab)^k(a^3 + b^3)] + [a^{2k+3} + b^{2k+3} - (ab)^{k+1}(a + b)] \\
&= (a^{k+3} - b^{k+3})(a^k - b^k) + (a^{k+2} - b^{k+2})(a^{k+1} - b^{k+1}) \\
&= (a - b)^2(F_{k+3}F_k + F_{k+2}F_{k+1}).
\end{aligned}$$

Thus,

$$2F_{k-m}F_{k+3+m} + (-1)^{m+k}(F_mF_{m+1} + F_{m+2}^2) = F_{k+3}F_k + F_{k+2}F_{k+1},$$

which yields the result given in the problem when $k = 2$. Now, we wish to show that no divisor of $F_{k+3}F_k + F_{k+2}F_{k+1}$ is congruent to 3 or 7 modulo 10. Let $x = F_k$ and $y = F_{k+1}$. Then

$$F_{k+3}F_k + F_{k+2}F_{k+1} = [(x + y) + y]x + (x + y)y = x^2 + 3xy + y^2.$$

Suppose that $x^2 + 3xy + y^2 \equiv 0 \pmod p$ for some prime p. Now, x and y could not both be divisible by p because then all the Fibonacci numbers would be divisible by p. Then, since the discriminant of the quadratic form $x^2 + 3xy + y^2$ is 5, if p is not 2 or 5, we must have $(5/p) = 1$, but by the Law of Quadratic Reciprocity, this is true iff $(p/5) = 1$, which holds iff $p \equiv \pm 1 \pmod 5$. Now, suppose there were a factor d of $F_{k+3}F_k + F_{k+2}F_{k+1}$ congruent to 3 or 7 modulo 10. Clearly, d has no factors of 2 or 5, so, by the above arguments, d is a product of primes congruent to ± 1 modulo 5. But any product of this sort is itself congruent to ± 1 modulo 5. Thus, d could not be congruent to 3 or 7 modulo 10.

Also solved by P. Bruckman, L. A. G. Dresel, and L. Kuipers.

The Law of Exclusion

H-391 *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 23, no. 3, August 1985)

For every n, show that no integral divisor of L_{2n} is congruent to 11, 13, 17, or 19 modulo 20. (This problem was suggested by Problem H-364 on p. 313 of the November 1983 issue of *The Fibonacci Quarterly*.)

Solution by L. A. G. Dresel, Reading, England

Let N_0 be the set of integers congruent to 1, 3, 7, or 9 modulo 20, and let N_1 be the set of integers congruent to 11, 13, 17, or 19 modulo 20. Then, since the product of any two integers in N_0 also belongs to N_0, it follows that any integer in N_1 is either a prime or divisible by at least one prime belonging to N_1. Hence, it is sufficient to show that, for all n, L_{2n} is not divisible by any prime belonging to N_1.

For the case of primes congruent to 13 or 17 (mod 20), this has been proved by Paul Bruckman in his solution to H-364, this journal Vol. 23, no. 4 (1985): 283-84.

Thus, there remains the case of primes p congruent to 11 or 19 (mod 20).

For these primes, we have $L_{p-1} \equiv 2 \pmod p$ and $\frac{1}{2}(p - 1)$ is odd. We also have the identity

$$L_t^2 = L_{2t} + 2(-1)^t,$$

so that putting $t = \frac{1}{2}(p - 1)$, we have

$$L^2_{\frac{1}{2}(p-1)} \equiv 0 \pmod{p}$$

and, therefore,

$$L_{\frac{1}{2}(p-1)} \equiv 0 \pmod{p}.$$

Then, if e denotes the entry point of p in the Lucas sequence, we have that e divides $\frac{1}{2}(p - 1)$ and, therefore, e is odd. Furthermore, L_k will be divisible by p only when k is an odd multiple of the entry point e, and any such k is also odd.

Hence, L_{2n} is not divisible by any prime congruent to 11 or 19 (mod 20). ∎

Also solved by P. Bruckman, B. Poonen, and the proposer.

Editorial Note: The following problems are as yet unsolved:

H-146, H-148, H-152, H-170, H-179, H-203, H-204, H-211, H-212, H-213, H-214, H-215, H-222, H-260, H-271, H-287, H-300, H-304, H-306, H-307, H-309, H-357, H-365.

LET'S CLEAN UP SOME OF THESE OLDIES!

ADDITIONAL PROBLEM PROPOSALS ARE NEEDED—PITCH IN AND HELP!!

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.

Recurring Sequences by Dov Jarden. Third and enlarged edition. Riveon Lematematika, Israel, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

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