The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

TWENTY-FIFTH ANNIVERSARY YEAR

CONTENTS

VOLUME 25 NOVEMBER 1987 NUMBER 4

PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

EDITORIAL POLICY

THE FIBONACCI QUARTERLY seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

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Articles should be submitted in the format of the current issues of the **THE FIBONACCI QUARTERLY.** They should be typewritten or reproduced typewritten copies, that are clearly readable, double spaced with wide margins and on only one side of the paper. The full name and address of the author must appear at the beginning of the paper directly under the title. Illustrations should be carefully drawn in India ink on separate sheets of bond paper or vellum, approximately twice the size they are to appear in print.

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All back issues of **THE FIBONACCI QUARTERLY** are available in microfilm or hard copy format from **UNIVERSITY MICROFILMS INTERNATIONAL, 300 NORTH ZEEB ROAD, DEPT. P.R., ANN ARBOR, MI 48106.** Reprints can also be purchased from **UMI CLEARING HOUSE** at the same address.

The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

Announcement

THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 25-29, 1988
Department of Mathematics, University of Pisa
Pisa, Italy

International Committee

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FIBONACCI'S STATUE

Have you ever seen Fibonacci's portrait? This photo is a close-up of the head of the statue of Leonardo Pisano in Pisa, Italy, taken by Frank Johnson in 1978.

Since Fibonacci's statue was difficult to find, here are the directions from the train station in Pisa (about 8 blocks to walk): Cross Piazza Vitt. Em. II, bearing right along Via V. Croce to Piazza Toniolo, and then walk through the Fortezza. The statue is found within Fortezza Campo Santo off Lungarno Fibonacci or Via Fibonacci along the Arno River at Giardino Scotto (Teatro Estivo).

CALL FOR PAPERS

The THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of Pisa, Pisa, Italy, from July 25-29, 1988. This conference is sponsored jointly by The Fibonacci Association and The University of Pisa.

Papers on all branches of mathematics and science related to the Fibonacci numbers and their generalizations are welcome. Abstracts are to be submitted by March 15, 1988. Manuscripts are requested by May 1, 1988. Abstracts and manuscripts should be sent to G.E. Bergum (address below). Invited and contributed papers will appear in the Conference Proceedings, which are expected to be published.

The program for the Conference will be mailed to all participants, and to those individuals who have indicated an interest in attending the conference, by June 15, 1988. All talks should be limited to one hour or less.

For further information concerning the conference, please contact Gerald Bergum, The Fibonacci Quarterly, Department of Computer Science, South Dakota State University, P.O. Box 2201, Brookings, South Dakota 57007-0199.

MIXED PELL POLYNOMIALS

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(Submitted April 1985)

1. INTRODUCTION

Pell polynomials $P_n(x)$ are defined ([8], [13]) by

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x) \qquad P_0(x) = 0,\ P_1(x) = 1. \tag{1.1}$$

Pell-Lucas polynomials $Q_n(x)$ are likewise defined ([8], [13]) by

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x) \qquad Q_0(x) = 2,\ Q_1(x) = 2x. \tag{1.2}$$

Properties of $P_n(x)$ and $Q_n(x)$ can be found in [8] and [13], while convolution polynomials for $P_n(x)$ and $Q_n(x)$ are investigated in detail in [9].

The k^{th} *convolution sequence for Pell polynomials* $\{P_n^{(k)}(x)\}$, $n = 1,2,3,\ldots,$ is defined in [9] by the equivalent expressions

$$P_n^{(k)}(x) = \begin{cases} \sum_{i=1}^{n} P_i(x)P_{n+1-i}^{(k-1)}(x) & k \geqslant 1 \\ \sum_{i=1}^{n} P_i^{(1)}(x)P_{n+1-i}^{(k-2)}(x) & \begin{cases} P_n^{(0)}(x) = P_n(x) \\ P_0^{(k)}(x) = 0 \end{cases} \\ \cdots & \\ \sum_{i=1}^{n} P_i^{(m)}(x)P_{n+1-i}^{(k-1-m)}(x) & 0 \leqslant m \leqslant k-1 \end{cases} \tag{1.3}$$

for which the generating function is

$$(1 - 2xy - y^2)^{-(k+1)} = \sum_{n=0}^{\infty} P_{n+1}^{(k)}(x)y^n. \tag{1.4}$$

The k^{th} *convolution sequence for Pell-Lucas polynomials* $\{Q_n^{(k)}(x)\}$, $n = 1,2,$ $3,\ldots,$ is defined in [9] by

$$Q_n^{(k)}(x) = \sum_{i=1}^{n} Q_i(x)Q_{n+1-i}^{(k-1)}(x),\ k \geqslant 1,\ Q_n^{(0)}(x) = Q_n(x) \tag{1.5}$$

with similar equivalent expressions in (1.5) for $Q_n^{(k)}(x)$ to those in (1.3) for $P_n^{(k)}(x)$. [$Q_0^{(k)}(x) = 0$ if $k \geqslant 1$; $Q_0^{(0)}(x) = 2$.]

The generating function for Pell-Lucas convolution polynomials is

$$\left\{\frac{2x + 2y}{1 - 2xy - y^2}\right\}^{k+1} = \sum_{n=0}^{\infty} Q_{n+1}^{(k)}(x)y^n. \tag{1.6}$$

Explicit summation formulas for the k^{th} convolutions are

$$P_n^{(k)}(x) = \sum_{r=0}^{[(n-1)/2]} \binom{k+n-1-r}{k}\binom{n-1-r}{r}(2x)^{n-2r-1} \tag{1.7}$$

and

$$Q_n^{(k)}(x) = 2^{k+1}\sum_{r=0}^{n-1}\binom{k+1}{r}x^{k+1-r}P_{n-r}^{(k)}(x) \tag{1.8}$$

where, in the latter case, the Pell-Lucas convolutions are expressed in terms of Pell convolutions.

A result needed subsequently is:

$$nP_{n+1}^{(k)}(x) = 2(k+1)\{xP_n^{(k+1)}(x) + P_{n-1}^{(k+1)}(x)\}. \tag{1.9}$$

Some of the simplest convolution polynomials are set out in Table 1.

Table 1. Convolutions for $P_n^{(k)}(x)$, $Q_n^{(k)}(x)$, $k = 1, 2$; $n = 1, 2, 3, 4, 5$

	$n = 1$	2	3	4	5
$P_n^{(1)}(x)$	1	$4x$	$12x^2 + 2$	$32x^3 + 12x$	$80x^4 + 48x^2 + 3$
$Q_n^{(1)}(x)$	$4x^2$	$16x^3 + 8x$	$48x^4 + 40x^2 + 4$	$128x^5 + 144x^3 + 32x$	$320x^6 + 448x^4 + 156x^2 + 8$
$P_n^{(2)}(x)$	1	$6x$	$24x^2 + 3$	$80x^3 + 24x$	$240x^4 + 120x^2 + 6$
$Q_n^{(2)}(x)$	$8x^3$	$48x^4 + 24x^2$	$192x^5 + 168x^3 + 24x$	$640x^6 + 768x^4 + 216x^2 + 8$	$1920x^7 + 2880x^5 + 1220x^3 + 120x$

Worth noting are the facts that

$$C_n^k(ix) = i^n P_{n+1}^{(k-1)}(x) \qquad (i = \sqrt{-1}), \tag{1.10}$$

where $C_n^k(x)$ is the *Gegenbauer polynomial* of degree n and order k [12], and

$$P_{n+1}^{(k)}(x) = P_n(2, x, -1, -(k+1), 1), \tag{1.11}$$

in which the right-hand side is a special case of the *generalized Humbert polynomial* $P_n(m, x, y, p, C)$ defined [3] by

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)t^n \qquad (m \geqslant 1). \tag{1.12}$$

Pell-Lucas convolution polynomials $Q_n^{(k)}(x)$ can be expressed in terms of the complex Gegenbauer polynomials by a complicated formula, but they are not expressible as specializations of generalized Humbert polynomials [cf. (1.6) and (1.12)].

Specializations of $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$ of interest to us occur when $x = 1$, giving the convolution sequences for *Pell numbers* and *Pell-Lucas numbers*. If x is replaced by $\frac{1}{2}x$, the sequence of *Fibonacci polynomial convolutions* and the sequence of *Lucas convolution polynomials* arise; in this case, putting $x = 1$ gives convolution sequences for *Fibonacci numbers* and for *Lucas numbers*.

The chief object of this paper is not to concentrate on $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$, but to examine convolution polynomials when $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$ are combined together. This will lead to the concept of "mixed Pell convolutions" and of a convolution of convolutions.

2. MIXED PELL CONVOLUTIONS

Let us introduce the *mixed Pell convolution* $\pi_n^{(a,b)}(x)$ in which

(i) $a + b \geqslant 1$

(ii) $\pi_n^{(0,0)}(x)$ is not defined.

Let

$$\sum_{n=0}^{\infty} \pi_{n+1}^{(a,b)}(x)y^n = \frac{(2x+2y)^b}{(1-2xy-y^2)^{a+b}} \tag{2.1}$$

$$= (2x+2y)^{b-j}\frac{1}{(1-2xy-y^2)^{a+b-j}}\left(\frac{2x+2y}{1-2xy-y^2}\right)^j$$

$$= (2x+2y)^{b-j}\left(\sum_{n=0}^{\infty}\pi_{n+1}^{(a+b-j,j)}(x)y^n\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{b-j}\binom{b-j}{i}(2x)^{b-j-i}2^i\pi_{n+1-i}^{(a+b-j,j)}(x)\right)y^n$$

whence

$$\pi_{n+1}^{(a,b)}(x) = 2^{b-j}\sum_{i=0}^{b-j}\binom{b-j}{i}x^{b-j-i}\pi_{n+1-i}^{(a+b-j,j)}(x). \tag{2.2}$$

Put $j = 1$ in (2.2). Then

$$\pi_{n+1}^{(a,b)}(x) = 2^{b-1}\sum_{i=0}^{b-1}\binom{b-1}{i}x^{b-1-i}\pi_{n+1-i}^{(a+b-1,1)}(x) \tag{2.3}$$

Special cases of (2.1) occur when $a = 0$, and when $b = 0$.

Thus, for $b = 0$, and $a = k$, (1.4) and (2.1) show that, with $n + 1$ replaced by n,

$$\pi_n^{(k,0)}(x) = P_n^{(k-1)}(x), \tag{2.4}$$

i.e.,

$$\pi_n^{(1,0)}(x) = P_n(x) \text{ by (1.3)}, \quad \pi_n^{(2,0)}(x) = P_n^{(1)}(x).$$

On the other hand, when $a = 0$ and $b = k$, (1.6) and (2.1) yield

$$\pi_n^{(0,k)}(x) = Q_n^{(k-1)}(x), \tag{2.5}$$

i.e.,

$$\pi_n^{(0,1)}(x) = Q_n(x) \text{ by (1.5)}, \quad \pi_n^{(0,2)}(x) = Q_n^{(1)}(x).$$

Now let $j = 0$ in (2.2). Hence, by (2.4), with $n + 1$ replaced by n,

$$\pi_n^{(a,b)}(x) = 2^b\sum_{i=0}^{b}\binom{b}{i}x^{b-i}P_{n-i}^{(a+b-1)}(x). \tag{2.6}$$

An explicit formulation for $\pi_n^{(a,b)}(x)$ could then be given by substituting for $P_{n-i}^{(a+b-1)}(x)$ from (1.7).

From (2.1), with (1.4) and (1.6), it is seen that

$$\pi_n^{(a,b)}(x) = \sum_{i=1}^{n}P_i^{(a-1)}(x)Q_{n+1-i}^{(b-1)}(x) \qquad (a \geqslant 1,\ b \geqslant 1). \tag{2.7}$$

Let us differentiate both sides of (2.1) w.r.t. y. Then

$$\sum_{n=0}^{\infty} n\pi_{n+1}^{(a,b)}(x)y^{n-1} = 2b\sum_{n=0}^{\infty}\pi_{n+1}^{(a+1,b-1)}(x)y^n + (a+b)\sum_{n=0}^{\infty}\pi_{n+1}^{(a,b+1)}(x)y^n,$$

whence

$$n\pi_{n+1}^{(a,b)}(x) = 2b\pi_n^{(a+1,b-1)}(x) + (a+b)\pi_n^{(a,b+1)}(x). \tag{2.8}$$

From the identity

$$\frac{(2x+2y)^b}{(1-2xy-y^2)^{a+b}} \cdot \frac{(2x+2y)^a}{(1-2xy-y^2)^{b+a}} = \frac{(2x+2y)^{a+b}}{(1-2xy-y^2)^{2a+2b}}$$

we derive a *convolution of convolutions*

$$\pi_n^{(a+b,a+b)}(x) = \sum_{i=1}^{n} \pi_i^{(a,b)}(x)\pi_{n+1-i}^{(b,a)}(x). \tag{2.9}$$

So, when $b = a$,

$$\pi_n^{(2a,2a)}(x) = \sum_{i=1}^{n} \pi_i^{(a,a)}(x)\pi_{n+1-i}^{(a,a)}(x). \tag{2.10}$$

From (2.9), when $b = 0$,

$$\pi_n^{(a,a)}(x) = \sum_{i=1}^{n} \pi_i^{(a,0)}(x)\pi_{n+1-i}^{(0,a)}(x) = \sum_{i=1}^{n} P_i^{(a-1)}(x)Q_{n+1-i}^{(a-1)}(x) \tag{2.11}$$

on using (2.4) and (2.5). [Cf. (2.7) also for $b = a$.]

Putting $b = a$ in (2.8) leads to

$$n\pi_{n+1}^{(a,a)}(x) = 2a\pi_n^{(a+1,a-1)}(x) + 2a\pi_n^{(a,a+1)}(x). \tag{2.12}$$

Combining (2.9) and (2.12), we have

$$2a\{\pi_n^{(a+1,a-1)}(x) + \pi_n^{(a,a+1)}(x)\} = n\sum_{i=1}^{n} \pi_i^{(a,0)}(x)\pi_{n+1-i}^{(0,a)}(x). \tag{2.13}$$

Equations (2.5) and (2.6), in which $a = 0$ and $b = k + 1$, give

$$\pi_n^{(0,k+1)}(x) = Q_n^{(k)}(x) = 2^{k+1}\sum_{i=0}^{k+1}\binom{k+1}{i}P_{n-i}^{(k)}(x)x^{k+1-i} \tag{2.14}$$

as in (1.8).

Next, put $b = 0$, $a = k$ in (2.8) to get

$$\begin{aligned}\pi_n^{(k,1)}(x) &= \frac{n}{k}\,\pi_{n+1}^{(k,0)}(x) = \sum_{i=1}^{n} P_i^{(k-1)}(x)Q_{n+1-i}(x) && \text{by (2.7)} \\ &= \frac{n}{k}\,P_{n+1}^{(k-1)}(x) && \text{by (2.4)} \\ &= 2xP_n^{(k)}(x) + 2P_{n-1}^{(k)}(x) && \text{by (1.9).}\end{aligned} \tag{2.15}$$

To exemplify some of the above results, we write down alternative expressions for $\pi_3^{(2,2)}(x)$.

We have

$$\begin{aligned}\pi_3^{(2,2)}(x) &= 4\{x^2P_3^{(3)}(x) + 2xP_2^{(3)}(x) + P_1^{(3)}(x)\} && \text{by (2.6)} \\ &= P_1^{(1)}(x)Q_3^{(1)}(x) + P_2^{(1)}(x)Q_2^{(1)}(x) + P_3^{(1)}(x)Q_1^{(1)}(x) && \text{by (2.7)} \\ &\left\{\begin{aligned}&= 2\{x\pi_3^{(3,1)}(x) + \pi_2^{(3,1)}(x)\} \\ &= 2\{x(3/3)P_4^{(2)}(x) + (2/3)P_3^{(2)}(x)\}\end{aligned}\right. && \begin{aligned}&\text{by (2.3)}\\ &\text{by (2.15)}\end{aligned} \\ &= \pi_1^{(2,0)}(x)\pi_3^{(0,2)}(x) + \pi_2^{(2,0)}(x)\pi_2^{(0,2)}(x) + \pi_3^{(2,0)}(x)\pi_1^{(0,2)}(x) && \text{... by (2.11)} \\ &= 160x^4 + 80x^2 + 4\end{aligned}$$

on using Table 1 and $P_1^{(3)}(x) = 1$, $P_2^{(3)}(x) = 8x$, and $P_3^{(3)}(x) = 40x^2 + 4$. Observe that the second and fifth lines of the chain of equalities above are the same, by virtue of (2.4) and (2.5).

Some interesting results for particular values of a and b may be found. For example, with $a = 0$, $b = 2$, we have, by (2.5) and (2.8),

$$nQ_{n+1}^{(1)}(x) = 4\pi_n^{(1,1)}(x) + 2Q_n^{(2)} = 4(1 + x^2)\pi_n^{(2,1)} + Q_n^{(2)}$$

on rearranging in another way the terms in the differentiation of (2.1). [For instance, when $n = 2$, the common value is $90x^4 + 80x^2 + 8$ on using

$$P_3^{(1)}(x) = \pi_2^{(2,1)}(x) \qquad \text{by (2.15)},$$

and Table 1.]

Thus,

$$Q_n^{(2)}(x) = 4(1 + x^2)\pi_n^{(2,1)}(x) - \pi_n^{(1,1)}(x).$$

Using

$$\pi_n^{(1,1)}(x) = nP_{n+1}(x) = \sum_{i=1}^{n} P_i(x)Q_{n+1-i}(x), \tag{2.16}$$

from (2.15) and (1.3), we find that the simplest values of $\pi_n^{(1,1)}(x)$ are:

$$\begin{cases} \pi_1^{(1,1)}(x) = 2x, \quad \pi_2^{(1,1)}(x) = 8x^2 + 2, \quad \pi_3^{(1,1)}(x) = 24x^3 + 12x \\ \pi_4^{(1,1)}(x) = 64x^4 + 48x^2 + 4, \quad \pi_5^{(1,1)}(x) = 160x^5 + 160x^3 + 30x \;\ldots \end{cases}$$

Theoretically, one may obtain a Simson-type analogue for the mixed convolution function $\pi_n^{(a,b)}(x)$. However, the task is rather daunting, so we content ourselves with the Simson formula in the simple instance when $a = b = 1$.

Computation, with the aid of (2.16) produces

$$\pi_{n+1}^{(1,1)}(x)\pi_{n-1}^{(1,1)}(x) - (\pi_n^{(1,1)}(x))^2 = (-1)^{n+1}(n^2 - 1) - P_{n+1}^2(x) \tag{2.17}$$

(both sides being equal to $-16x^4 - 8x^2 - 4$ when, say, $n = 2$).

3. MISCELLANEOUS RESULTS

A. Pell Convolutions

Two results given in [3] are worth relating to convolution polynomials.

First, apply (1.11) to [3, (3.10)]. Then

$$P_{n+1}^{(k)}(x) = \sum_{i_1+i_2+\cdots+i_j=n} P_{i_1+1}(x)P_{i_2+1}(x)\ldots P_{i_j+1}(x) \tag{3.1}$$

in our system of polynomials. Observe the restriction on the summation. Putting $k = 2$ and $n = 2$, say, gives, on applying (1.3) the appropriate number of times,

$$\begin{aligned} P_3^{(2)}(x) &= P_1(x)P_2(x)P_2(x) + P_2(x)P_1(x)P_2(x) + P_2(x)P_2(x)P_1(x) \\ &\quad + P_1(x)P_1(x)P_3(x) + P_1(x)P_3(x)P_1(x) + P_3(x)P_1(x)P_1(x) \\ &= 24x^2 + 3 \end{aligned}$$

which is precisely the summation expansion in (3.1). We may think of the ordered subscripts in each three-term product of the sum as a solution-set of $x + y + z = 5$ for nonnegative integers.

Second, suppose we wish to expand a given Fibonacci polynomial, say $F\ (x)$, in terms of Pell polynomials (an example of a well-known type of problem in classical analysis—see [2]).

Using notation in [3, (6.9), (6.10)], we have

$$F_5(x) = x^4 + 3x^2 + 1 = \sum_{n=0}^{4} A_n x^n \tag{3.2}$$

i.e.,

$$A_0 = 1,\ A_1 = 0,\ A_2 = 3,\ A_3 = 0,\ A_4 = 1, \tag{3.3}$$

whence

$$F_5(x) = \sum_{n=0}^{4} V_n P_{n+1}(x), \tag{3.4}$$

where

$$V_n = \sum_{j=0}^{[(4-n)/2]} (-1)^n \frac{\binom{-n-1-j}{j}}{\binom{-1}{n+2j}} \cdot \frac{n+1}{n+1+j} \cdot \frac{A_{n+2j}}{2^{n+2j}}. \tag{3.5}$$

Expanding (3.5) and using (3.3), we calculate

$$V_0 = A_0 - \frac{A_2}{4} + \frac{A_4}{8},\quad V_1 = -\left(\frac{A_1}{2} - \frac{A_3}{4}\right) = 0,\quad V_2 = \frac{A_2}{4} - \frac{3A_4}{16},$$

$$V_3 = -\frac{A_3}{8} = 0,\quad V_4 = \frac{A_4}{16}$$

whence the right-hand side of (3.4) simplifies to (3.2) on using (1.1) to obtain appropriate Pell polynomials. Thus,

$$F_5(x) = \frac{3}{8} P_1(x) + \frac{9}{16} P_3(x) + \frac{1}{16} P_5(x).$$

Again,

$$P_5^{(1)}(x) = P_1(x) - 3P_3(x) + 5P_5(x) \qquad (= 80x^4 + 48x^2 + 3)$$

on paralleling the calculations above.

Computations involving Pell convolution polynomials $P_n^{(k)}(x)$ for $k \geqslant 1$ could be effected in a similar manner.

B. Even and Odd Pell Convolutions

Let us now introduce $*P_n^{(1)}(x)$, the *first convolution of even Pell polynomials*, i.e., of Pell polynomials with even subscripts.

Consider

$$\sum_{n=0}^{\infty} P_{2n+2}(x) y^n = \frac{2x}{1 - Q_2(x)y + y^2}, \tag{3.6}$$

where $Q_2(x) = 4x^2 + 2$ [by (1.2)] and the nature of the generating function is determined by the recurrence relation for the Pell polynomials with even subscripts, which is obtained by a repeated application of (1.1), namely

$$P_n(x) = (4x^2 + 2)P_{n-2}(x) - P_{n-4}(x). \tag{3.7}$$

Then

$$\left(\sum_{n=0}^{\infty} P_{2n+2}(x) y^n\right)^2 = \frac{4x^2}{(1 - Q_2(x)y + y^2)^2} \tag{3.8}$$

that is,

$$\sum_{n=0}^{\infty} {}^*P_{n+1}^{(1)}(x)y^n = \frac{4x^2}{(1 - Q_2(x)y + y^2)^2}, \tag{3.9}$$

where

$$^*P_n^{(1)}(x) = \sum_{i=1}^{n} P_{2i}(x)P_{2n+2-2i}(x). \tag{3.10}$$

Some expressions for these convolutions are:

$$\begin{cases} {}^*P_1^{(1)}(x) = P_2(x)P_2(x) = 4x^2 \\ {}^*P_2^{(1)}(x) = P_2(x)P_4(x) + P_4(x)P_2(x) = 32x^4 + 16x^2 \\ {}^*P_3^{(1)}(x) = P_2(x)P_6(x) + P_4(x)P_4(x) + P_6(x)P_2(x) \\ \qquad\qquad = 192x^6 + 192x^4 + 40x^2 \end{cases}$$

Properties similar to those given in [9; (4.3), (4.4), (4.5), ...] may be obtained. Analogous to [9, (4.3)], for instance, we have the basic recursion-type relation

$$^*P_n^{(1)}(x) - Q_2(x)^*P_{n-1}^{(1)}(x) + {}^*P_{n-2}^{(1)}(x) = P_2(x)P_{2n}(x). \tag{3.11}$$

If we differentiate in (3.6) w.r.t. y and compare the result with (2.4), we deduce the analogue of [9, 4.4)]:

$$2nxP_{2n+2}(x) = Q_2(x)^*P_n^{(1)}(x) - 2^*P_{n-1}^{(1)}(x). \tag{3.12}$$

Experimentation has also been effected with convolutions of *odd* Pell polynomials (i.e., Pell polynomials with odd subscripts), with convolutions for Pell polynomials having subscripts, say, of the form $3m$, $3m + 1$, $3m + 2$, and generally with convolutions for Pell polynomials having subscripts of the form $rm + k$.

For the odd-subscript Pell polynomials, the recurrence relation is of the same form as that in (3.7). Indeed, $x = 1$ gives the recurrence

$$P_n = 6P_{n-2} - P_{n-4},$$

which is valid for sequences of Pell numbers with even subscripts or odd subscripts. Compare the situation for sequences of Fibonacci numbers with even subscripts or odd subscripts for which the recurrence is

$$F_n = 3F_{n-2} - F_{n-4}.$$

Other possibilities include convolving even and odd Pell polynomials, and powers of Pell polynomials.

Generalizing the above work to results for n^{th} convolutions is a natural extension.

Of course, investigations involving Pell polynomials automatically include considerations of cognate work on Pell-Lucas polynomials, and of a study of mixed convolutions of artibrary order, as for $\pi_n^{(a,b)}(x)$.

C. Further Developments

Among other possible developments of our ideas, we mention the generation of $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$ by rising diagonals of a Pascal-type array as was done in [8] for $P_n(x)$ and $Q_n(x)$. Work on this aspect is under way.

A variation of this approach is an examination of the polynomials produced by the rising (and descending) diagonals of arrays whose rows are the coefficients of powers of x in $P_n^{(k)}(x)$, where $n = 1, 2, 3, \ldots$, for a given k. Such a treatment as this has been done in [6], [7], and [10] for Chebyshev, Fermat, and Gegenbauer polynomials.

Another problem which presents itself is a discussion of the convolutions of Pell polynomials and *Pell-Jacobsthal polynomials* which might be defined by the recurrence relation

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x) \qquad J_0(x) = 0,\ J_1(x) = 1. \tag{3.13}$$

Evidently, one can proceed *ad infinitum, ad nauseam*! Convolution work on on Fibonacci polynomials and *Jacobsthal polynomials*, defined in [5] and [11], is summarized in [14]. The chapter on Convolutions in [14], a thesis dedicated to the mathematical research of the late Verner E. Hoggatt, Jr., contains much other information on convolution arrays for well-known sequences, such as the *Catalan sequence*, studied by Hoggatt and his associates.

D. Case $x = 1$

Following procedures established in [1] and [4] for Fibonacci number convolutions, we may demonstrate *inter alia* the results:

$$8P_n^{(1)} = (3n + 1)P_{n+1} - (n + 1)P_{n-1}; \tag{3.14}$$

$$8P_n^{(1)} = nQ_{n+1} + 2P_n; \tag{3.15}$$

$$P_{n+4}^{(1)} = 4P_{n+3}^{(1)} - 2P_{n+2}^{(1)} - 4P_{n+1}^{(1)} - P_n^{(1)}; \tag{3.16}$$

$$Q_{n-1}P_n^{(1)} - Q_{n+1}P_{n-2}^{(1)} = 2P_n^2; \tag{3.17}$$

$$\begin{vmatrix} P_{n+3}^{(1)} & P_{n+2}^{(1)} & P_{n+1}^{(1)} & P_n^{(1)} \\ P_{n+2}^{(1)} & P_{n+1}^{(1)} & P_n^{(1)} & P_{n-1}^{(1)} \\ P_{n+1}^{(1)} & P_n^{(1)} & P_{n-1}^{(1)} & P_{n-2}^{(1)} \\ P_n^{(1)} & P_{n-1}^{(1)} & P_{n-2}^{(1)} & P_{n-3}^{(1)} \end{vmatrix} = +1. \tag{3.18}$$

Clearly, all the work in this paper for k^{th} convolutions of the Pell and Pell-Lucas polynomials can be specialized for Pell and Pell-Lucas numbers.

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A GENERALIZATION OF FIBONACCI POLYNOMIALS AND A REPRESENTATION OF GEGENBAUER POLYNOMIALS OF INTEGER ORDER

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(Submitted September 1985)

1. INTRODUCTION

Various sequences of polynomials by the name of Fibonacci and Lucas polynomials occur in the literature. For example, Doman & Williams [2] introduced the polynomials

$$F_{n+1}(z) := \sum_{m=0}^{[n/2]} \binom{n-m}{m} z^m, \tag{1}$$

$$L_n(z) := \sum_{m=0}^{[n/2]} \frac{n}{n-m}\binom{n-m}{m} z^m, \tag{2}$$

for $n = 1, 2, 3, \ldots$, and $F_0(z) := 0$, $F_1(z) := 1$, $L_0(z) := 2$; $[n/2]$ denotes the integer part of $n/2$. Several properties of these polynomials were derived in [2] and, more recently, by Galvez & Dehesa [3].

The Fibonacci and Lucas polynomials which occur, for example, in [4], are different from but closely related to the $F_n(z)$ and $L_n(z)$. The properties derived in [4] and in the papers cited there can easily be adapted to the polynomials defined in (1) and (2); they mainly concern zeros and divisibility properties.

In [2], the connection to the Gegenbauer (or ultraspherical) and Chebyshev polynomials $C_n^{\alpha}(z)$ and $T_n(z)$ was given, namely

$$C_n^1(z) = (2z)^n F_{n+1}(-1/4z^2),$$

$$T_n(z) = \frac{1}{2}(2z)^n L_n(-1/4z^2).$$

We also note that $C_n^1(z) = U_n(z)$, the Chebyshev polynomial of the second kind. Because $2T_n(z) = nC_n^0(z)$ (see, e.g., [1], p. 779), we now have

$$F_{n+1}(z) = (-z)^{n/2}\, C_n^1(1/2\sqrt{-z}), \tag{3}$$

$$\frac{1}{n} L_n(z) = (-z)^{n/2}\, C_n^0(1/2\sqrt{-z}); \tag{4}$$

here and in the following the square root is to be considered as the principal branch.

The purpose of this note is to use these identities as a starting point to define a wider class of sequences of polynomials which contains (1) and (2) as special cases, and to derive some properties.

*Supported by a Killam Postdoctoral Fellowship.

2. THE POLYNOMIALS $F_n^{(k)}(z)$

For $k = -1, 0, 1, \ldots$, we introduce

$$F_n^{(k)}(z) := (-z)^{n/2} C_n^{k+1}(1/2\sqrt{-z}); \tag{5}$$

by (3) and (4), we have the special cases

$$F_n^{(0)}(z) = F_{n+1}(z) \quad \text{and} \quad F_n^{(-1)}(z) = L_n(z)/n.$$

We now use the explicit expressions for the Gegenbauer polynomials (see, e.g., [1], p. 775):

$$C_n^{\alpha}(x) = \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{[n/2]} (-1)^m \frac{\Gamma(\alpha + n - m)}{m!\,(n-2m)!}(2x)^{n-2m}, \tag{6}$$

for $\alpha > -1/2$, $\alpha \neq 0$, and

$$C_n^0(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!}(2x)^{n-2m}. \tag{7}$$

The connection between (7) and (2) is immediate and, for $\alpha = k + 1 \geqslant 1$, we have

$$\frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha + n - m)}{m!\,(n-2m)!} = \frac{(n+k-m)!}{k!\,m!\,(n-2m)!} = \binom{n+k-m}{m}\binom{n+k-2m}{k}$$

with (6) and (5), this yields the explicit expression

$$F_n^{(k)}(z) = \sum_{m=0}^{[n/2]} \binom{n+k-m}{m}\binom{n+k-2m}{k} z^m, \tag{8}$$

for $k \geqslant 0$. This could also serve as a definition of the $F_n^{(k)}(z)$, in analogy to (1).

3. SOME PROPERTIES

With (5) and the recurrence relation for Gegenbauer polynomials (see, e.g., [1], p. 782), we obtain

$$(n+1)F_{n+1}^{(k)}(z) = (n+k+1)F_n^{(k)}(z) + (n+2k+1)zF_{n-1}^{(k)}(z). \tag{9}$$

More properties of the $F_n^{(k)}(z)$ can be derived, with (5), from the corresponding properties of the Gegenbauer polynomials. This includes generating functions, differential relations, and more recurrence relations; we just mention

$$\frac{d}{dz} F_{n+1}^{(k)}(z) = (k+1)F_{n-1}^{(k+1)}(z) \qquad (\text{for } k \geqslant 0),$$

and

$$\frac{d}{dz} L_n(z) = nF_{n-1}(z), \tag{10}$$

which can also be verified directly using (8), (1), and (2). If we differentiate the recurrence

$$P_{n+1}(z) = P_n(z) + zP_{n-1}(z) \tag{11}$$

which, by (9), holds for $L_n(z)$ and $F_n(z)$, we get, with (10),

$$(n+1)F_n(z) = nF_{n-1}(z) + L_{n-1}(z) + (n-1)zF_{n-2}(z);$$

this, combined with (11), for $F_n(z)$, yields

$$L_{n-1}(z) = 2F_n(z) - F_{n-1}(z).$$

This last equation can also be derived from the corresponding well-known identity connecting the Chebyshev polynomials of the first and second kind.

The following recurrence relation involves polynomials $F_n^{(k)}(z)$ of different orders $k \geqslant 1$.

$$F_{n+2}^{(k)}(z) - F_{n+1}^{(k)}(z) - zF_n^{(k)}(z) = F_{n+2}^{(k-1)}(z),$$

which can be verified by elementary manipulations, using (8).

4. THE $F_n^{(k)}(z)$ AS ELEMENTARY SYMMETRIC FUNCTIONS

We begin with the following

Lemma: (a) For integers $n \geqslant 0$ and for complex $z \neq 1$ and x, we have

$$\sum_{j=0}^{n} (-1)^j F_j^{(n-j)}(x) z^{n-j} = (z-1)^n F_{n+1}\left(\frac{x}{(z-1)^2}\right) \tag{12}$$

(b) $$\sum_{j=0}^{n} (-1)^j F_j^{(n-j)}(x) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ x^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Proof: Let $f_n(x, z)$ denote the left-hand side of (12). With (8), we have

$$\begin{aligned} f_n(x, z) &= \sum_{j=0}^{n} (-1)^j \sum_{m=0}^{[j/2]} \binom{n-m}{m}\binom{n-2m}{n-j} x^m z^{n-j} \\ &= \sum_{m=0}^{[n/2]} x^m \binom{n-m}{m} \sum_{j=2m}^{n} (-1) \binom{n-2m}{j-2m} z^{n-j} \\ &= \sum_{m=0}^{[n/2]} x^m \binom{n-m}{m} \sum_{j=0}^{n-2m} (-1)^j \binom{n-2m}{j} z^{n-2m-j}, \end{aligned}$$

which yields assertion (b) if we put $z = 1$. For $z \neq 1$, we have

$$f_n(x, z) = \sum_{m=0}^{[n/2]} x^m \binom{n-m}{m} (z-1)^{n-2m} = (z-1)^n \sum_{m=0}^{[n/2]} \binom{n-m}{m}\left(\frac{x}{(z-1)^2}\right)^m,$$

which proves (a).

Proposition: For $k = 1, 2, \ldots, n$, we have

$$F_k^{(n-k)}(x) = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n} A_{j_1}^{(n)}(x) \ldots A_{j_k}^{(n)}(x),$$

where

$$A_j^{(n)}(x) := 1 + 2\sqrt{-x} \cos \frac{j\pi}{n+1}.$$

Proof: Because $C_n^1(z) = U_n(z)$, we have, with (3) and the definition of $A_j^{(n)}(x)$,

$$F_{n+1}(x(A_j^{(n)}(x) - 1)^{-2}) = F_{n+1}\left(-1/4 \cos^2 \frac{j\pi}{n+1}\right)$$

$$= \left(2 \cos \frac{j\pi}{n+1}\right)^{-n} U_n\left(\cos \frac{j\pi}{n+1}\right).$$

Now $\cos(j\pi/(n+1))$, for $j = 1, 2, \ldots, n$, are known to be the zeros of the Chebyshev polynomials of the second kind $U_n(z)$. Furthermore, if n is odd, then $\cos(j\pi/(n+1)) = 0$ for $j = (n+1)/2$, in which case $A_j^{(n)}(x) = 1$ for all x. So we have, by both parts of the Lemma,

$$\sum_{k=0}^{n} (-1)^k F_k^{(n-k)}(x)(A_j^{(n)}(x))^{n-k} = 0$$

for all $j = 1, 2, \ldots, n$. But this means that the $F_k^{(n-k)}(x)$, $k = 0, 1, \ldots, n$, with x held constant, are the elementary symmetric functions of the n roots $A_j^{(n)}(x)$ of $f(x, z) = 0$. This proves the Proposition.

Finally, if we let $x = 1/2\sqrt{-z}$, the proposition together with (5) yields the following representation of the ultraspherical polynomials of integer order.

Corollary: If $k \geqslant 1$ is an integer, then

$$C_n^k(x) = 2^n \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n+k-1} \left(x + \cos \frac{j_1\pi}{n+k}\right) \cdot \cdots \cdot \left(x + \cos \frac{j_n\pi}{n+k}\right).$$

In closing, we note that [5] and [6] deal with Gegenbauer polynomials from another (related) point of view.

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THE RECIPROCAL OF THE BESSEL FUNCTION $J_k(z)$

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(Submitted September 1985)

1. INTRODUCTION

For $k = 0, 1, 2, \ldots$, let $J_k(z)$ be the Bessel function of the first kind. Put

$$f_k(z) = J_k(2\sqrt{z})/z^{k/2} = \sum_{m=0}^{\infty} \frac{(-1)^m z^m}{m!(m+k)!} \tag{1.1}$$

and define the polynomial $u_m(k; x)$ by means of

$$k!f_k(xz)/f_k(z) = \sum_{m=0}^{\infty} u_m(k; x)\frac{z^m}{m!(m+k)!}, \tag{1.2}$$

Certain congruences for $w_m(x) = u_m(0; x)$ and the integers $w_m = w_m(0)$ were derived by Carlitz [3] in 1955, and an interesting application was presented.

The purpose of the present paper is to extend Carlitz's results to the polynomials $u_m(k; x)$ and the rational numbers $u_m(k) = u_m(k; 0)$.

In particular, we show in §§3 and 4 that, if p is a prime number, $p > 2k$, and

$$m = c_0 + c_1 p + c_2 p^2 + \cdots \quad (0 \leqslant c_0 < p - 2k)$$
$$(0 \leqslant c_i < p \text{ for } i > 0), \tag{1.3}$$

then

$$u_m(k) \equiv u_{c_0}(k) \cdot w_{c_1} w_{c_2} \ \ldots \pmod{p}, \tag{1.4}$$

$$u_m(k; x) \equiv u_{c_0}(k; x) \cdot w_{c_1}^{p}(x) \cdot w_{c_2}^{p^2}(x) \ \ldots \pmod{p}. \tag{1.5}$$

In §5, we prove more general congruences of this type. In §6, applications of these general results are given. Finally, in §7, we examine in more detail the positive integers $u_n(1)$.

2. PRELIMINARIES

Throughout the paper, we use the notation $w_m(x) = u_m(0; x)$ and $w_m = w_m(0)$.

In the proofs of Theorems 1-6, we use the divisibility properties of binomial coefficients given in the lemmas below. These lemmas follow from well-known theorems of Kummer [4] and Lucas [5].

Lemma 1: If p is a prime number, then

$$\binom{mp}{rp} \equiv \binom{m}{r} \pmod{p}.$$

Also, if $p - 2k > s \geqslant 0$, then, for $j = s + 1, s + 2, \ldots, p - 1$,

$$\binom{np + s + k}{rp + j + k}\binom{np + s + k}{rp + j} \equiv 0 \pmod{p}.$$

Lemma 2: Suppose p is a prime number and

$$n = n_0 + n_1 p + \cdots + n_j p^j \quad (0 \leqslant n_i < p),$$
$$r = r_0 + r_1 p + \cdots + r_j p^j \quad (0 \leqslant r_i < p),$$

If, for some fixed i, we have $r_i > n_i$ and $r_{i+v} \geqslant n_{i+v}$ for $v = 1, \ldots, t-1$, then

$$\binom{n}{r} \equiv 0 \pmod{p^t}.$$

Lemma 3: Let p be a prime number, $p > 2k$. Then

$$\binom{n+k}{r+k}\binom{n+k}{r}\Big/\binom{n+k}{k}$$

is integral (mod p) for $r = 0, 1, \ldots, n$. Also

$$\binom{mp}{rp+k}\Big/\binom{mp}{k} \equiv \binom{m-1}{r} \pmod{p},$$

$$\binom{mp}{rp-k}\Big/\binom{mp}{k} \equiv \binom{m-1}{r-1} \pmod{p}.$$

3. THE NUMBERS $u_m(k)$

We first note that the numbers $u_m(k)$ were introduced in [2], where Carlitz showed they cannot satisfy a certain type of recurrence formula.

It follows from (1.2) that

$$\{f_k(z)\}^{-1} = \sum_{m=0}^{\infty} u_m(k) \frac{z^m}{m!(m+k)!}. \tag{3.1}$$

Thus, we have

$$u_0(k) = u_1(k) = (k!)^2,$$
$$u_2(k) = (k!)^2(k+3)/(k+1),$$
$$u_3(k) = (k!)^2(k^2+8k+19)/(k+1)^2,$$

and

$$\sum_{r=0}^{m} (-1)^r \binom{m+k}{r+k}\binom{m+k}{r} u_r(k) = 0 \quad (m > 0). \tag{3.2}$$

It follows from (3.2) and Lemma 3 that if p is a prime number, $p \geqslant 2k$, then the numbers $u_m(k)$ are integral (mod p); in particular, $u_n(0)$ and $u_n(1)$ are positive integers for $n = 0, 1, 2, \ldots$.

Theorem 1: If p is a prime number and if $0 \leqslant s < p - 2k$, then

$$u_{np+s}(k) \equiv u_s(k) \cdot w_n \pmod{p}. \tag{3.3}$$

Proof: We use induction on the total index $np + s$. If $np + s = 0$, (3.3) holds since $w_0 = 1$. Assume (3.3) holds for all $rp + j < np + s$, with $j < p - 2k$. We then have, by (3.2),

$$(-1)^{n+s+1}\binom{s+k}{s} u_{np+s}(k) \equiv \sum_{r=0}^{n-1} \sum_{j=0}^{s} (-1)^{j+r} \binom{s+k}{j+k}\binom{s+k}{j}\binom{n}{r}^2 u_{rp+j}(k)$$
$$+ (-1)^n \sum_{j=0}^{s-1} (-1)^j \binom{s+k}{j+k}\binom{s+k}{j} u_{np+j}(k)$$

$$\equiv \sum_{r=0}^{n-1} (-1)^r \binom{n}{r}^2 w_r \cdot \sum_{j=0}^{s} (-1)^j \binom{s+k}{j+k}\binom{s+k}{j} u_j(k)$$

$$+ (-1)^n w_n \sum_{j=0}^{s-1} (-1)^j \binom{s+k}{j+k}\binom{s+k}{j} u_j(k)$$

$$\equiv \begin{cases} 0 + (-1)^{n+s+1}\binom{s+k}{s} w_n u_s(k) \pmod{p} & \text{if } s > 0, \\ (-1)^{n+1} w_n u_0(k) \pmod{p} & \text{if } s = 0. \end{cases}$$

We see that (3.3) follows, and the proof is complete.

Corollary (Carlitz): With the hypotheses of Theorem 1 and with m defined by (1.3) with $k = 0$,

$$w_m \equiv w_{c_0} w_{c_1} w_{c_2} \cdots \pmod{p}.$$

Corollary: With the hypotheses of Theorem 1 and with m defined by (1.3),

$$u_m(k) \equiv u_{c_0}(k) \cdot w_{c_1} w_{c_2} \cdots \pmod{p}.$$

Theorem 2: If p is a prime number, $p > 2k$, then

$$u_{np-k}(k) \equiv (-1)^k u_0(k) \cdot w_n \pmod{p}.$$

Proof: The proof is by induction on n. For $n = 1$ we have, by (3.1),

$$(-1)^k u_{p-k}(k) \equiv \sum_{r=0}^{p-k-1} (-1)^r \binom{p}{r+k}\binom{p}{r} u_r(k) \Big/ \binom{p}{k}$$

$$\equiv u_0(k) \equiv u_0(k) \cdot w_1 \pmod{p}.$$

Theorem 2 is therefore true for $n = 1$; assume it is true for $n = 1, \ldots, s - 1$. Then

$$(-1)^{s+k+1} u_{sp-k}(k) \equiv \sum_{r=0}^{sp+k-1} (-1)^r \binom{sp}{r+k}\binom{sp}{r} u_r(k) \Big/ \binom{sp}{k}$$

$$\equiv \sum_{r=0}^{s-1} (-1)^r \binom{sp}{rp+k}\binom{sp}{rp} u_{rp}(k) \Big/ \binom{sp}{k}$$

$$+ \sum_{r=1}^{s-1} (-1)^{r-k} \binom{sp}{rp}\binom{sp}{rp-k} u_{rp-k}(k) \Big/ \binom{sp}{k}$$

$$\equiv \sum_{r=0}^{s-1} (-1)^r \binom{s}{r}\binom{s-1}{r} u_0(k) w_r + \sum_{r=1}^{s-1} (-1)^r \binom{s}{r}\binom{s-1}{r-1} u_0(k) w_r$$

$$\equiv u_0(k) \sum_{r=0}^{s-1} (-1)^r \binom{s}{r}^2 w_r \equiv (-1)^{s-1} u_0(k) w_s \pmod{p}.$$

This completes the proof of Theorem 2.

If m is defined by (1.3) with $c_0 = p - k$, and if $c_i = p - 1$ for $1 \leqslant i \leqslant j - 1$ with $c_j < p - 1$, then Theorem 2 says

$$u_m(k) \equiv u_{c_0}(k) \cdot w_{1+c_j} w_{c_{j+1}} w_{c_{j+2}} \cdots \pmod{p}.$$

In particular, if $p > 2k$, and $n = p^t - k$,

$$u_n(k) \equiv u_{p-k}(k) \equiv (-1)^k u_0(k) \equiv (-1)^k (k!)^2 \pmod{p}.$$

4. THE POLYNOMIALS $u_m(k;\, x)$

We now consider the polynomials $u_m(k;\, x)$ defined by (1.2). It is clear that

$$u_m(k;\, 0) = u_m(k), \quad u_m(k,\, 1) = 0 \quad (m > 0).$$

Also, it follows from (1.1) and (1.2) that

$$\binom{m+k}{k} u_m(k;\, x) = \sum_{r=0}^{m} (-1)^{m-r} \binom{m+k}{r+k} \binom{m+k}{r} u_r(k) x^{m-r}. \tag{4.1}$$

Theorem 3: If p is a prime number and if $0 \leqslant s < p - 2k$, then

$$u_{np+s}(k;\, x) \equiv u_s(k;\, x) \cdot w_{np}(x) \pmod{p}. \tag{4.2}$$

Proof: The proof is by induction on the total index $np + s$. We first note that

$$u_0(k;\, x) \equiv u_0(k;\, x) \cdot w_0(x) \pmod{p},$$

since $w_0(x) = 1$.

Assume (4.2) is true for all $rp + j < np + s$ with $0 \leqslant j < p - 2k$. Then, by (4.1) and (3.3),

$$\binom{s+k}{s} u_{np+s}(k;\, x) \equiv \sum_{r=0}^{np+s} (-1)^{n-s-r} \binom{np+s+k}{r} \binom{np+s+k}{r+k} u_r(k) x^{np+s-r}$$

$$\equiv \sum_{j=0}^{s} \sum_{r=0}^{n} \binom{np+s+k}{rp+j} \binom{np+s+k}{rp+j+k} (-1)^{n+s+j+r} u_{rp+j}(k) x^{np-rp+s-j}$$

$$\equiv \sum_{j=0}^{s} \sum_{r=0}^{n} \binom{n}{r}^2 \binom{s+k}{j} \binom{s+k}{j+k} (-1)^{n+s+j+r} w_r u_j(k) x^{np-rp+s-j}$$

$$\equiv \sum_{j=0}^{s} \binom{s+k}{j} \binom{s+k}{j+k} (-1)^{s+j} u_j(k) x^{s-j} \cdot \sum_{r=0}^{n} \binom{n}{r}^2 (-1)^{n-r} w_r x^{np-rp}$$

$$\equiv \binom{s+k}{k} u_s(k;\, x) \cdot w_n(x^p) \equiv \binom{s+k}{s} u_s(k;\, x) \cdot w_{np}(x) \pmod{p}.$$

This completes the proof of Theorem 3. We note that Theorem 1 was used in the proof.

Corollary (Carlitz): With the hypotheses of Theorem 3 and with m defined by (1.3) with $k = 0$,

$$w_m(x) \equiv w_{c_0}(x) \cdot w_{c_1}^{p}(x) \cdot w_{c_2}^{p^2}(x) \ \ldots \pmod{p}.$$

Corollary: With the hypotheses of Theorem 3 and with m defined by (1.3),

$$u_m(k;\, x) \equiv u_{c_0}(k;\, x) \cdot w_{c_1}^{p}(x) \cdot w_{c_2}^{p^2}(x) \ \ldots \pmod{p}.$$

5. GENERAL RESULTS

For each integer $k \geqslant 0$, let $\{F_n(k)\}$ and $\{G_n(k)\}$, $n = 0, 1, 2, \ldots$, be polynomials in an arbitrary number of indeterminates with coefficients that are integral (mod p) for $p > 2k$. We use the notation $F_n(0) = F_n$ and $G_n(0) = G_n$, and we assume $F_0 = G_0 = 1$. For each m of the form (1.3), suppose

$$F_m(k) \equiv F_{c_0}(k) \cdot F_{c_1}^{p} \cdot F_{c_2}^{p^2} \ldots \pmod{p}, \tag{5.1}$$

$$G_m(k) \equiv G_{c_0}(k) \cdot G_{c_1}^p \cdot G_{c_2}^{p^2} \cdots \pmod{p}. \tag{5.2}$$

For each integer $k \geqslant 0$, define $H_n(k)$ and $Q_n(k)$ by means of

$$\binom{n+k}{n} H_n(k) = \sum_{r=0}^{n} (-1)^{n-r} \binom{n+k}{r} \binom{n+k}{r+k} F_r(k) G_{n-r}(k) \tag{5.3}$$

and

$$\binom{n+k}{k} F_n(k) = \sum_{r=0}^{n} (-1)^{n-r} \binom{n+k}{r} \binom{n+k}{r+k} Q_r(k) G_{n-r}(k). \tag{5.4}$$

Theorem 4: Let the sequences $\{H_n(k)\}$ and $\{Q_n(k)\}$ be defined by (5.3) and (5.4), respectively, and let $H_j = H_j(0)$, $Q_j = Q_j(0)$. If p is a prime, $0 \leqslant s \leqslant p - 2k$, then

$$H_{np+s}(k) \equiv H_s(k) \cdot H_{np} \pmod{p}. \tag{5.5}$$

If $G_0(k) \not\equiv 0 \pmod{p}$, we also have

$$Q_{np+s}(k) \equiv Q_s(k) \cdot Q_{np} \pmod{p}. \tag{5.6}$$

Proof: From (5.3), we have

$$\begin{aligned}
\binom{s+k}{s} H_{np+s}(k) &\equiv \sum_{j=0}^{s} \sum_{r=0}^{n} (-1)^{n+s+r+j} \binom{np}{rp}^2 \binom{s+k}{j} \binom{s+k}{j+k} F_{rp+j}(k) G_{np-rp+s-j}(k) \\
&\equiv \sum_{j=0}^{s} (-1)^{s+j} \binom{s+k}{j} \binom{s+k}{j+k} F_j(k) G_{s-j}(k) \cdot \sum_{r=0}^{n} (-1)^{n+r} \binom{n}{r}^2 F_r^p G_{n-r}^p \\
&\equiv \binom{s+k}{s} H_s(k) \cdot H_n^p \equiv \binom{s+k}{s} H_s(k) \cdot H_{np} \pmod{p}.
\end{aligned}$$

This completes the proof of (5.5).

As for (5.6), we first observe that for $n = 0$ and $0 \leqslant s < p - 2k$, congruence (5.6) is valid. Assume that (5.6) is true for all $rp + j < np + s$ with $0 \leqslant j < p - 2k$. Then, from (5.4), we have

$$\begin{aligned}
\binom{s+k}{s} F_{np+s}(k) &\equiv \sum_{j=0}^{s} \sum_{r=0}^{n} (-1)^{n+s+r+j} \binom{np}{rp}^2 \binom{s+k}{j} \binom{s+k}{j+k} Q_{rp+j}(k) G_{np-rp+s-j}(k) \\
&\equiv \sum_{j=0}^{s} (-1)^{s-j} \binom{s+k}{j} \binom{s+k}{j+k} Q_j(k) G_{s-j}(k) \cdot \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r}^2 Q_r^p G_{n-r}^p \\
&\quad - \binom{s+k}{s} Q_s(k) G_0(k) Q_n^p + \binom{s+k}{s} Q_{np+s}(k) G_0(k) \\
&\equiv \binom{s+k}{s} F_s(k) \cdot F_n^p - \binom{s+k}{s} Q_s(k) G_0(k) Q_n^p \\
&\quad + \binom{s+k}{s} Q_{np+s}(k) G_0(k) \pmod{p}.
\end{aligned}$$

Now, since $F_{np+s}(k) \equiv F_s(k) \cdot F_{np} \pmod{p}$, we have

$$Q_{np+s}(k) \equiv Q_s(k) \cdot Q_n^p \equiv Q_s(k) \cdot Q_{np} \pmod{p},$$

and the proof is complete.

<u>Corollary (Carlitz)</u>: Using the hypotheses of Theorem 4 with m defined by (1.3) and $k = 0$,

$$H_m \equiv H_{c_0} \cdot H_{c_1}^p \cdot H_{c_2}^{p^2} \cdots \pmod{p},$$

$$Q_m \equiv Q_{c_0} \cdot Q_{c_1}^p \cdot Q_{c_2}^{p^2} \cdots \pmod{p}.$$

Corollary: Using the hypotheses of Theorem 4 with m defined by (1.3),

$$H_m(k) \equiv H_{c_0}(k) \cdot H_{c_1}^p \cdot H_{c_2}^{p^2} \cdots \pmod{p}.$$

If $G_0(k) \not\equiv 0 \pmod{p}$, we also have

$$Q_m(k) \equiv Q_{c_0}(k) \cdot Q_{c_1}^p \cdot Q_{c_2}^{p^2} \cdots \pmod{p}.$$

6. APPLICATIONS

As an application of Theorem 4, for each integer $k \geqslant 0$ consider the expansion

$$(k!)^{r+s-1} \frac{f_k(x_1 z) \cdots f_k(x_r z)}{f_k(y_1 z) \cdots f_k(y_s z)} = \sum_{n=0}^{\infty} F_n(k) \frac{z^m}{n!(n+k)!}, \tag{6.1}$$

where $f_k(z)$ is defined by (1.1), r, s are arbitrary nonnegative integers, and the x_i, y_i are indeterminates (not necessarily distinct). By (1.1) and (3.1), $F_n(k)$ is a polynomial in $x_1, \ldots, x_r$, and $y_1, \ldots, y_s$ with coefficients that are integral (mod p) if $p > 2k$. The following result may be stated.

Theorem 5: If m is of the form (1.3), then the polynonial $F_m(k)$ defined by (6.1) satisfies

$$F_m(k) \equiv F_{c_0}(k) \cdot F_{c_1}^p \cdot F_{c_2}^{p^2} \cdots \pmod{p},$$

where $F_j = F_j(0)$. In particular, if the x_i, y_i are replaced by rational numbers that are integral (mod p), then

$$F_m(k) \equiv F_{c_0}(k) \cdot F_{c_1} F_{c_2} \cdots \pmod{p}.$$

As a special case of (6.1), we may take

$$(k!)^{r-1}\{f_k(z)\}^{-r} = \sum_{n=0}^{\infty} u_n^{(r)}(k) \frac{z^n}{n!(n+k)!}.$$

Then the $u_n^{(r)}(k)$ are integral (mod p) if $p > 2k$, and they satisfy

$$u_m^{(r)}(k) \equiv u_{c_0}^{(r)}(k) \cdot u_{c_1}^{(r)}(0) \cdot u_{c_2}^{(r)}(0) \cdots \pmod{p}$$

for all r (positive or negative).

7. THE NUMBERS $u_n(1)$

For $n = 0, 1, 2, \ldots$, let $w_n = u_n(0)$ and let $u_n = u_n(1)$. The positive integers w_n were studied by Carlitz [3] and were shown to satisfy (1.4) (with $k = 0$). Since the u_n are also positive integers, it may be of interest to examine their properties in more detail. The generating function and recurrence formula are given by (1.1), (3.1), and (3.2) with $k = 1$. From them we can compute the following values:

$u_0 = u_1 = 1$	$u_5 = 321$
$u_2 = 2$	$u_6 = 3681$
$u_3 = 7$	$u_7 = 56197$
$u_4 = 39$	$u_8 = 1102571$

Suppose that p is an odd prime number and that m is defined by (1.3) with $0 \leqslant c_0 \leqslant p - 3$. Then, by Theorems 1 and 2, we have

$$u_m \equiv u_{c_0} w_{c_1} w_{c_2} \ \cdots \pmod{p}, \tag{7.1}$$

$$u_{np+(p-1)} \equiv -w_{n+1} \pmod{p}. \tag{7.2}$$

The case $c_0 = p - 2$ is considered in the next theorem. This theorem makes use of the positive integers h_n defined by means of

$$\{J_1(z)\}^2/\{J_0(z)\}^3 = \sum_{n=0}^{\infty} h_n \frac{(z/2)^{2n}}{n!n!} \tag{7.3}$$

These numbers are related to the integers a_n defined by Carlitz [1]:

$$a_n = 2^{2n} n!(n-1)!\sigma_{2n}(0),$$

where $\sigma_{2n}(0)$ is the Rayleigh function. It can be determined from properties of a_n that a generating function is

$$J_1(z)/J_0(z) = \sum_{n=1}^{\infty} a_n \frac{(z/2)^{2n-1}}{n!(n-1)!} \tag{7.4}$$

as well as

$$\{J_1(z)/J_0(z)\}^2 = \sum_{n=1}^{\infty} a_{n+1} \frac{(z/2)^{2n}}{n!n!}. \tag{7.5}$$

Now it follows from (3.1), (7.3), and (7.5) that

$$h_n = \sum_{r=0}^{n-1} \binom{n}{r}^2 w_r a_{n+1-r} \quad (n > 0), \tag{7.6}$$

$$(-1)^n a_{n+1} = \sum_{r=0}^{n} (-1)^r \binom{n}{r}^2 h_r \quad (n > 0). \tag{7.7}$$

The first few values of h_n are $h_0 = 0$, $h_1 = 1$, $h_2 = 8$, $h_3 = 96$, $h_4 = 1720$.

In the proof of Theorem 6, we use the relationship

$$\sum_{r=0}^{n-1} (-1)^r \binom{n}{r}\binom{n}{r+1} w_{r+1} = (-1)^{n+1} a_{n+1}, \tag{7.8}$$

which follows from (7.4).

Theorem 6: If p is an odd prime number, then

$$u_{np+(p-2)} \equiv u_{p-2} w_n - h_n \pmod{p},$$

where h_n is defined by (7.3).

Proof: The proof is by induction on n. The theorem is true for $n = 0$, since $h_0 = 0$ and $w_0 = 1$. Assume that Theorem 6 is true for $n = 0, \ldots, s - 1$. Then by (3.2), (7.1), (7.2), and (7.8) we have

$$\begin{aligned}(-1)^{s-1} u_{sp+(p-2)} \equiv &\sum_{r=0}^{s} \sum_{j=0}^{p-3} (-1)^{r+j} \binom{sp+p-1}{rp+j}\binom{sp+p-1}{rp+j+1} u_{rp+j} \\ &+ \sum_{r=0}^{s-1} \sum_{j=p-2}^{p-1} (-1)^{r+j} \binom{sp+p-1}{rp+j}\binom{sp+p-1}{rp+j+1} u_{rp+j}\end{aligned}$$

$$\equiv \sum_{r=0}^{s}(-1)^r\binom{s}{r}^2 w_r \cdot \sum_{j=0}^{p-3}(-1)^j\binom{p-1}{j}\binom{p-1}{j+1}u_j + u_{p-2}\sum_{r=0}^{s-1}(-1)^r\binom{s}{r}^2 w_r$$

$$+ \sum_{r=0}^{s-1}(-1)^{r+1}\binom{s}{r}^2 h_r + \sum_{r=0}^{s-1}(-1)^{r+1}\binom{s}{r}\binom{s}{r+1}w_{r+1}$$

$$\equiv (-1)^{s-1}u_{p-2}w_s + (-1)^s h_s + (-1)^{s-1}a_{s+1} + (-1)^s a_{s+1}$$

$$\equiv (-1)^{s-1}(u_{p-2}w_s - h_s) \pmod{p}.$$

This completes the proof of Theorem 6.

Using (7.7) we can prove, for $p > 2$,

$$h_{np+s} \equiv h_s w_n \pmod{p} \quad (0 \leqslant s \leqslant p-2),$$

$$h_{np+(p-1)} \equiv h_{p-1}w_n + h_n \pmod{p}.$$

Theorem 6 can be refined by means of these congruences. For example, if m is defined by (1.3) with $c_0 = p - 2$ and $c_1 = 0$, we have

$$u_m \equiv u_{c_0}w_{c_1}w_{c_2} \cdots \pmod{p}.$$

The proofs in this section are not valid for $p = 2$. However, it is not difficult to show by induction that if $m \not\equiv 2 \pmod 4$ then u_m is odd. The proof is similar to the proofs of Theorems 1-6. If $m \equiv 2 \pmod 4$, we can write

$$m = 4n + 2 = 2^{v+1}j + 2^v - 2$$

for some $v > 1$. Using (3.2) and induction on n, we can prove

$$u_m \equiv \begin{cases} 0 \pmod 2 & \text{if } v \text{ is even,} \\ 1 \pmod 2 & \text{if } v \text{ is odd.} \end{cases}$$

Thus, for $p = 2$, we have the following theorem.

Theorem 7: If $m = c_0 + c_1 2 + c_2 2^2 + \cdots$, with each $c_i = 0$ or 1, then

$$u_m \equiv u_{c_0}u_{c_1}u_{c_2} \cdots \pmod 2,$$

unless $m = 2^{v+1}j + 2^v - 2$ with v even, $v \geqslant 2$.

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ON THE LARGEST ODD COMPONENT OF A UNITARY PERFECT NUMBER*

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(Submitted September 1985)

1. INTRODUCTION

A divisor d of an integer n is a *unitary divisor* if gcd $(d, n/d) = 1$. If d is a unitary divisor of n we write $d \| n$, a natural extension of the customary notation for the case in which d is a prime power. Let $\sigma^*(n)$ denote the sum of the unitary divisors of n:

$$\sigma^*(n) = \sum_{d \| n} d.$$

Then σ^* is a multiplicative function and $\sigma^*(p^e) = 1 + p^e$ for p prime and $e > 0$.

We say that an integer N is *unitary perfect* if $\sigma^*(N) = 2N$. In 1966, Subbaro and Warren [2] found the first four unitary perfect numbers:

$$6 = 2 \cdot 3;\ 60 = 2^2 3 \cdot 5;\ 90 = 2 \cdot 3^2 5;\ 87{,}360 = 2^6 3 \cdot 5 \cdot 7 \cdot 13.$$

In 1969, I announced [3] the discovery of another such number,

$$146{,}361{,}936{,}186{,}458{,}562{,}560{,}000 = 2^{18} 3 \cdot 5^4 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313,$$

which I later proved [4] to be the fifth unitary perfect number. No other unitary perfect numbers are known.

Throughout what follows, let $N = 2^a m$ (with m odd) be unitary perfect and suppose that K is the largest odd component (i.e., prime power unitary divisor) of N. In this paper we outline a proof that, except for the five known unitary perfect numbers, $K > 2^{15}$.

2. TECHNIQUES

In light of the fact that $\sigma^*(p^e) = 1 + p^e$ for p prime, the problem of finding a unitary perfect number is equivalent to that of expressing 2 as a product of fractions, with each numerator being 1 more than its denominator, and with the denominators being powers of distinct primes. If such an expression for 2 exists, then the denominator of the unreduced product of fractions is unitary perfect. The main tool is the epitome of simplicity: we must eventually divide out any odd prime that appears in either a numerator or a denominator.

If p is an odd prime, then $\sigma^*(p^e) = 1 + p^e$ is even. Thus, if some of the odd components of a unitary perfect number N are known or assumed, there is an implied lower bound for a, where $2^a \| N$, since all but one of the 2's in the numerator of $\sigma^*(N)/N$ must divide out. Another lower bound, useful in many cases, is Subbarao's result [1] that $a > 10$ except for the first four unitary perfect numbers.

This paper was written while the author was Visiting Professor at The University of Southwestern Louisiana, Lafayette, LA.

A simple program was run on a microcomputer to find, for each odd prime $p < 2^{15}$, the smallest A for which $2^A \equiv \pm 1 \pmod{p}$. If $2^A \equiv 1 \pmod{p}$, then p never divides $1 + 2^a$. If $2^A \equiv -1 \pmod{p}$, then p divides $1 + 2^a$ if and only if a is an odd integer times A, and we refer to A as the *entry point* of p.

If an odd prime p has entry point A and $p^2 | (1 + 2^A)$, it is easy to see that $2^{p-1} \equiv 1 \pmod{p^2}$. There are only two primes less than $3 \cdot 10^9$ for which this this phenomenon occurs, and they are 1093 and 3511. Then $1 + 2^A$ would have a component larger than 10^6. Thus, for the primes $p < 2^{15}$ under consideration here, either p never divides $1 + 2^a$ or $p \| (1 + 2^A)$ or $1 + 2^a$ has a component larger than 2^{15}.

The odd primes less than 2^{15} having entry points were ordered by entry point. Then it was a fairly easy procedure to consider algebraic factors and conclude that $1 + 2^a$ has all components less than 2^{15} for only $a < 11$ and the a shown in Table 1.

Table 1

2^a	$1 + 2^a$	2^a	$1 + 2^a$
2^{11}	3*683	2^{24}	97*257*673
2^{12}	17*241	2^{25}	3*11*251*4051
2^{13}	3*2731	2^{26}	5*53*157*1613
2^{14}	5*29*113	2^{30}	5^2*13*41*61*1321
2^{15}	3^2*11*331	2^{33}	3^2*67*683*20857
2^{18}	5*13*37*109	2^{34}	5*137*953*26317
2^{21}	3^2*43*5419	2^{42}	5*13*29*113*1429*14449
2^{22}	5*397*2113	2^{46}	5*277*1013*1657*30269
		2^{78}	5*13^2*53*157*313*1249*1613*3121*21841

In many of the proofs, cases are eliminated because under the stated conditions $\sigma^*(N)/N$ would be less than 2. A number n for which $\sigma^*(n) < 2n$ is called *unitary deficient* (abbreviated "u-def"). Finally, we will write $a = A \cdot \text{odd}$ to indicate that a is an odd integer times A.

3. PRELIMINARY CASES

If $K = 3$, we have $3 | \sigma^*(2^a)$, so a is odd. But N is u-def if $a \geqslant 3$, so $a = 1$; hence, $N = 2 \cdot 3 = 6$, the first unitary perfect number.

If $K = 5$, we immediately have $3 \| N$ and $a = 2 \cdot \text{odd}$. But N is u-def if $a \geqslant 6$, so $a = 2$; therefore, $N = 2^2 3 \cdot 5 = 60$, the second unitary perfect number.

Note that $K = 7$ is impossible, because 7 never divides $1 + 2^a$. In general, the largest component cannot be the first power of a prime that has no entry point.

If $K = 3^2 = 9$, then $5 \| N$, and $\sigma^*(5)$ uses one of the two 3's. To use the other 3, we must have $3 | \sigma^*(2^a)$, so a is odd. Now, $7 \nmid N$ or else $7 | \sigma^*(2^a)$, which is impossible. Then N is u-def if $a \geqslant 3$, so $a = 1$; hence, $N = 2 \cdot 3^2 5 = 90$, the third unitary perfect number.

If $K = 11$, then $11 | \sigma^*(2^a)$, so $a = 5$ odd; hence, $3 | \sigma^*(2^a)$. But $3 | \sigma^*(11)$ as well, so $3^2 \| N$. Then $5 | \sigma^*(3^2)$, so $5 \| N$, and since $3 | \sigma^*(5)$ we have $3^3 | N$, contradicting the maximality of K.

If $K = 13$, we have $13 | \sigma^*(2^a)$, so $a = 6$ odd; hence, $5 | \sigma^*(2^a)$. Then $5 \| N$, so $3 \| N$ because $3^2 \| N$ would imply $5^2 | N$, a contradiction. Because $13 \| N$, we have $7 \| N$, but we cannot have $11 \| N$ or else $3^2 | N$. But N is u-def if $\alpha \geqslant 18$, so $\alpha = 6$, from which it follows that $N = 2^6 3 \cdot 5 \cdot 7 \cdot 13 = 87{,}360$, the fourth unitary perfect number.

We have now accounted for the first four unitary perfect numbers. In light of Subbarao's results [1], we may assume that $\alpha > 10$ from now on.

Now suppose $\alpha = 78$. Because $313 \cdot 1249 | \sigma^*(2^{78})$ and the squares of these primes exceed 2^{15}, we have $313 \cdot 1249 \| N$. But $157^2 | \sigma^*(2^{78} 313)$, so $157^2 \| N$. However, $5^7 | \sigma^*(2^{78} 157^2 1249)$, so $5^7 | N$. But $5^7 > 2^{15}$, so $\alpha = 78$ is impossible.

At this stage, a table was constructed to list all odd prime powers which might be components in the remaining cases. For the sake of brevity, the table and most of the remaining proofs are omitted here. However, the table may be obtained from the author. The table was constructed to include: (1) the odd primes that appear in Table 1 (except for $\alpha = 78$); (2) all odd primes dividing $\sigma^*(q)$, where q is any other number also in Table 2 below; and (3) all allowable powers of primes also in Table 2. A "possible sources" column listed all components of unreduced denominators in $\sigma^*(N)/N$ for which a particular prime might appear in a numerator; multiple appearances were also indicated.

Insufficient entries in the "possible sources" column allow us to eliminate some possible components. For example, there are only two possible sources for 23, so 23^3 cannot occur. We eliminate: 23^3; 31^2; 31^3; 67^2 and hence 449; 71^2 and hence 2521; 73^2; in succession, 79^2, 3121, and 223; successively, 101^2, 5101, and 2551; successively, 131^2, 8581, 613, and 307; successively, 139^2, 9661, 4831, 151^2, 877, and 439; successively, 149^2, 653, 109^2, 457, 229, and 23^2; and successively, 181^2, 16381, and 8191.

4. REMAINING CASES

We have $11 \leqslant \alpha \leqslant 46$, so there can be no more than 47 odd components. The smallest odd component must be smaller than 17 because a $\sigma^*(N)/N$ ratio of 1.926... occurs if N is the product of 2^{11} and the following 47 prime powers:

17, 19, 23, 25, 27, 29, 31, 37, 41, 43, 47, 49, 53, 61, 67, 73,
79, 83, 97, 101, 109, 113, 121, 131, 137, 139, 149, 151, 157,
169, 181, 191, 193, 199, 211, 241, 251, 257, 269, 271, 277,
281, 313, 331, 337, 397, 421

If $83^2 \| N$, then $331 \cdot 829 \| N$. If 829 is a component, then 1657 is also, and hence $\alpha = 46$. Now, 331 is a component only if $\alpha = 15$ or $661 \| N$, and since $\alpha = 46$, $661 \| N$. But then $1321 \| N$, so $\alpha = 30$, a contradiction. Therefore, 83^2 cannot be a component.

Suppose $\alpha = 46$. Then $277 \cdot 1013 \cdot 1657 \cdot 30269 \| N$, so $139 \cdot 829 \cdot 1009 \| N$, hence $83 \cdot 101 \| N$. Therefore, $3^4 5^5 7^2 13^2 | N$, so $11 \| N$, because there must be a component smaller than 17, and $\sigma^*(11)$ contributes another 3 to the numerator of $\sigma^*(N)/N$. Now, either $5^5 \| N$ or $5^6 \| N$. If $5^5 \| N$, then $521 \| N$ and we have, successively, 29^2, 421, 211, and 53 as components; but then $3^{10} | N$, which is impossible. Thus, $5^6 \| N$, so $601 \| N$, hence $43 | N$. But $43 \| N$ or else there are too many 5's. Now, $7^3 \| N$ would force $43^2 | N$, and $7^4 \| N$ would force $1201 \| N$, hence $601^2 \| N$, so $7^5 \| N$; however, then $11^2 | N$, a contradiction. Therefore, we may eliminate $\alpha = 46$. As a result, we may eliminate 277, 1657, 829, and 30269 as components, then 139 and 1009, and then 101.

For the sake of brevity, the other cases, except $\alpha = 24$, are summarized in Table 2.

Table 2

CASE	ELIMINATED	CASE	ELIMINATED
$2^{42}*17^3$	$2^{42}*17^3$	$2^{12}*11^4$	$2^{12}*11^4$
$2^{42}*7$	$2^{42}*7$	$2^{12}*11^3$	$2^{12}*11^3$
2^{42}	2^{42}; 113^2; 1277; 71	2^{12}	2^{12}
2^{26}	2^{26}	$2^{21}*43^2$	$2^{21}*43^2$
53^2	53^2; 281; 47^2	$2^{21}*5$	$2^{21}*5$
2^{34}	2^{34}; 26317; 13159; 47	2^{21}	2^{21}
2^{33}	2^{33}; 67	61^2	61^2; 1861
41^2	41^2	19^3	19^3
$2^{30}*61$	$2^{30}*61$	$2^{22}*19^2$	$2^{22}*19^2$
2^{30}	2^{30}	2^{22}	2^{22}
2^{25}	2^{25}	$2^{18}*37^2$	$2^{18}*37^2$
2^{11}	2^{11}	$2^{18}*19^2$	$2^{18}*19^2$
2^{13}	2^{13}	$2^{18}*5^3$	$2^{18}*5^3$
2^{15}	2^{15}; 441; 83	$2^{18}*5^5$	$2^{18}*5^5$
$2^{14}*29^2$	$2^{14}*29^2$	$2^{18}*5^6$	$2^{18}*5^6$
2^{14}	2^{14}; 113	2^{18}	2^{18} unless $N = W$; 109

The ordering of cases presented in Table 2 works fairly efficiently. The reader should rest assured that sudden departures from an orderly flow are deliberate and needed. The case $\alpha = 24$ is especially difficult, and so is presented here.

Suppose $\alpha = 24$. We immediately have $257 \cdot 673 \| N$, hence $337 \| N$, so $13^2 | N$. To avoid having N u-def, the smallest component must be 3, 5, or 7.

If the smallest component is 7, then $97^2 \| N$ or else $97 \| N$ and $7^2 | N$. Therefore, $941 \| N$, so $193 \| N$. Then $3^2 11 \cdot 17 \| N$ or N is u-def. But $3^3 | \sigma^*(17 \cdot 257)$, so $3^3 | N$, a contradiction. Thus, the smallest component is not 7.

If the smallest component is 3, there are no more components $\equiv -1 \pmod 3$ as $3 | \sigma^*(257)$. Then we must have 7, 19, 25, and 31 as components or N is u-def. But then, no more than nine more odd components are allowable, and N is u-def. Therefore, the smallest component must be 5.

Because $5 \| N$, we must have $43 \| N$, since $5^2 | \sigma^*(43^2)$. We know that $13^2 | N$, so either $13^2 \| N$ or $13^3 \| N$ or $13^4 \| N$.

Suppose $13^4 \| N$. We cannot have 5^2 or 5^6 as components, so we must have 181 and 17^3. Starting with $2^{24} 5 \| N$, we have, successively, as unitary divisors, $257 \cdot 673$, $337 \cdot 43$, 13^4, $17^3 181 \cdot 14281$, and $19^2 193$. Because $19^2 \| N$, we must have $3^9 37 \| N$. But $37^2 | \sigma^*(3^9 13281)$, contradicting $37 \| N$. Therefore, 13^4 is not a component.

Suppose $13^3 \| N$. Then $157 \| N$ or else $157^2 \| N$, hence $5^2 | N$. Consequently, $79 \| N$ and no more components $\equiv -1 \pmod 5$ are allowable. Then $97 \| N$ or else $97^2 \| N$, hence $5^2 | N$. If $7^3 \| N$, then $43^2 | N$, which cannot be, and if $7^4 \| N$, then $1201 \| N$, so $601 \| N$, and again $43^2 | N$. Therefore, $7^5 \| N$, so $191 \| N$. But then N is u-def.

Hence, $13^2 \| N$, so no more components $\equiv -1 \pmod 5$ are allowable. In particular, we must have $97 \| N$ to avoid $97^2 \| N$, and then we must have $3^3 7^3 | N$. But then N is u-def, so $\alpha = 24$ is impossible.

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A NOTE ON $n(x, y)$-REFLECTED LATTICE PATHS

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(Submitted October 1985)

INTRODUCTION

A natural bijection between the class of lattice paths from (0, 0) to ($2m$, $2m$) having the property that, for each (x, y) in the path, ($2m - x$, $2m - y$) is also on the path and the class of partitions of $2m^2$ into at most $2m$ parts, each part $\leqslant 2m$ and the parts which are strictly less than $2m$ can be paired such that the sum of each pair is $2m$, is shown.

1. DEFINITION AND THE MAIN RESULT

Describing the n-reflected lattice paths [paths from (0, 0) to (n, n) having the property that, for each (x, y) in the path, ($n - y$, $n - x$) is also on the path] of the paper "Hook Differences and Lattice Paths" [1] as $n(y, x)$-reflected, we define here $n(x, y)$-reflected lattice paths as follows:

Definition: A lattice path from (0, 0) to (n, n) is said to be $n(x, y)$-reflected if, for each (x, y) in the path, ($n - x$, $n - y$) is also on the path.

Example: The two $2(x, y)$-reflected lattice paths are:

In the present note we propose to prove the following.

Theorem: The number of partitions of $2m^2$ into at most $2m$ parts each $\leqslant 2m$ and the parts which are strictly less than $2m$ can be paired such that the sum of each pair is $2m$ equals $\binom{2m}{m}$.

2. PROOF OF THE THEOREM

We describe a partition of $2m^2$ as a multiset

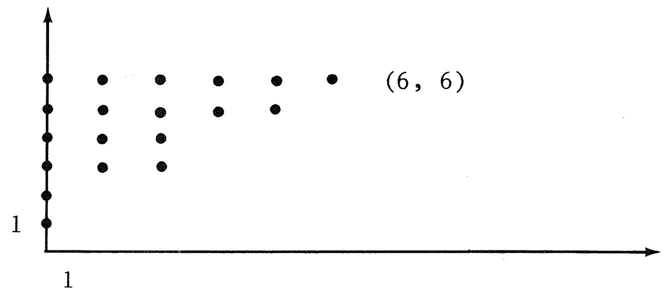
$$\mu = \mu(m) : = [a_1, \ldots, a_s]$$

of $s(\in \{1, 2, \ldots, 2m^2\})$ positive integers a_i ($i = 1, 2, \ldots, s$) such that

$$\sum_{i=1}^{s} a_i = 2m^2 \quad (\text{conventionally, } a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s).$$

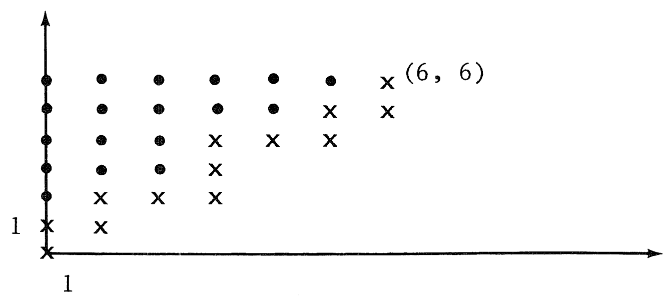
In this notation let $\mathcal{S}(m)$ denote the set of all partitions $\mu = [a_1, a_2, \ldots, a_s]$ of $2m^2$ such that $s \leqslant 2m$, $2m \geqslant a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s$; and, all of the a_j for which $a_j < 2m$ can be paired such that the sum of each pair equals $2m$. Further, let $\mathfrak{J}(m)$ denote the set of all $2m(x, y)$-reflected lattice paths. To establish a one-to-one correspondence from $\mathcal{S}(m)$ onto $\mathfrak{J}(m)$, we represent any $\mu = [a_1, a_2, \ldots, a_s] \in \mathcal{S}(m)$ by its Ferrers graph in the coordinate plane as follows:

We fit the leftmost node of the i^{th} row of nodes (counted by a_i) over the point $(0, 2m - i + 1)$ as shown in Graph A (in the graph, $m = 3$ and $\mu = [6, 5, 3, 3, 1]$).



Graph A

We now place crosses at one unit of length below every free horizontal node and at one unit of length to the right of every free vertical node. Through these crosses, we then complete the lattice path from (0, 0) to $(2m, 2m)$, as shown in Graph B.



Graph B

We observe that each partition μ corresponds uniquely to a $2m(x, y)$-reflected lattice path. It may be noted here that the corresponding path will not be $2m(x, y)$-reflected if

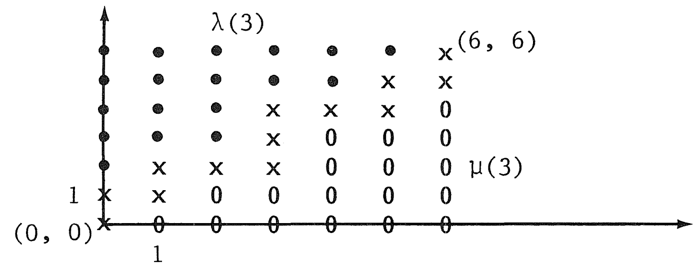
$$s = 2m = a_1. \tag{1}$$

For, in this case, $(2m, 2m - 1)$ belongs to the path, but $(0, 1) = (2m - 2m, 2m - (2m - 1))$ does not. Therefore, in order to prove that the correspondence is one-to-one and onto, we first rule out the possibility (1) under the conditions of the theorem. There are only three possible cases: (i) $a_1 > a_2$. In this case, if (1) is true, then there are $2m - 1$ parts, viz. $a_2, \ldots, a_s$, that are strictly less than $2m$. Being odd in number, these parts cannot be paired; hence, (1) is false. (ii) $a_1 = a_2 = \cdots = a_r$, where r ($\geqslant 1$) is odd. In this case, if (1) is true, then the number of parts that are $< 2m$ is $2m - r$. Again, since $2m - r$ is odd, the parts that are $< 2m$ cannot be paired; hence, (1) is not possible. (iii) $a_1 = a_2 = \cdots = a_r$, where r ($\geqslant 2$) is even. As in the previous case, if (1) is true, then the number of parts that are $< 2m$ is $2m - r$. But in this case, $2m - r$ is even. So the parts that are $< 2m$ can be paired. However, since the sum of each pair is $2m$, the number being partitioned is:

$$2m \cdot r + \frac{(2m - r)}{2} \cdot 2m = 2m^2 + mr > 2m^2.$$

This is a contradiction since we are considering the partitions of $2m^2$. Thus, (1) does not hold true.

We also note that each $2m(x, y)$-reflected lattice path uniquely splits $4m^2$ into two identical partitions of $2m^2$, say, $\lambda(m)$ and $\mu(m)$. (See Graph C, where $m = 3$ and $\lambda(3) = [6, 5, 3, 3, 1]$).



Graph C

Now if a_i $(i = 1, 2, \ldots, s) \in \lambda$, and $a_i < 2m$, there must exist b_j $(j = 1, 2, \ldots, s) \in \mu$, where $b_j < 2m$, such that $a_i + b_j = 2m$. But since λ and μ are identical, $b_j = a_k$ for some $k \in \{1, 2, \ldots, s\}$. Thus, $a_i + a_k = 2m$. This is how the restriction "all of the a_j for which $a_j < 2m$ can be paired such that the sum of each pair equals $2m$" enters into the argument. After establishing a one-to-one correspondence from $\mathcal{S}(m)$ onto $\mathcal{J}(m)$, we use the fact that each $2m(x, y)$-reflected lattice path determines and is determined uniquely by its first half, i.e., the nondecreasing path between (0, 0) to (m, m). Hence, the number of $2m(x, y)$-reflected lattice paths or the number of relevant partitions equals the number of paths between (0, 0) to (m, m), i.e., $\binom{2m}{m}$. This completes the proof of the theorem.

As an example, let us consider the case in which $m = 3$. We get the following relevant partitions:

$$3^6,\ 43^4 2,\ 4^2 3^2 2^2,\ 4^3 2^3,\ 53^4 1,\ 54^2 2^2 1,\ 543^2 21,\ 5^2 3^2 1^2,\ 5^2 421^2,\ 5^3 1^3,$$
$$63^4,\ 64^2 2^2,\ 643^2 2,\ 653^2 1,\ 65^2 1^2,\ 6^2 3^2,\ 6^2 42,\ 6^2 51,\ 6^3,\ 65421.$$

We remark here that in all there are 58 partitions of 18 into at most 6 parts and each part $\leqslant 6$ $\left(\text{see [2], p. 243, coefficient of } q^{18} \text{ in the expansion of } \begin{bmatrix} 12 \\ 8 \end{bmatrix}\right)$. But 38 partitions, such as 6543, $5^3 3$, 543^3, $4^3 3^2$, $6^2 2^3$, $5^3 21$, etc., do not satisfy the condition "the parts which are < 6 can be paired such that the sum of each pair is 6."

ACKNOWLEDGMENT

Thanks are due to the referee for his suggestions, which led to a better presentation of the paper.

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FRIENDLY-PAIRS OF MULTIPLICATIVE FUNCTIONS

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(Submitted October 1985)

1. INTRODUCTION

An arithmetic function f is said to be multiplicative if

$$f(m)f(n) = f(mn) \text{ whenever } (m, n) = 1. \tag{1.1}$$

It is a consequence of (1.1) that f is known if $f(p^r)$ is known for every prime p and $r \geqslant 1$.

Definition: A pair $\{f, g\}$ of multiplicative functions is called a "friendly-pair" of the type α ($\alpha \geqslant 2$) if, for $n \geqslant 1$,

$$f(n^\alpha) = g(n), \quad g(n^\alpha) = f(n) \tag{1.2}$$

and

$$f(n)g(n) = 1. \tag{1.3}$$

Question: Do friendly-pairs of multiplicative functions exist?

We answer this question in the affirmative.

2. A FRIENDLY-PAIR

We exhibit a friendly-pair of multiplicative functions by actual construction. As f, g are multiplicative, it is enough if we work with prime-powers.

Let p be a prime and $r \geqslant 1$.

We define f and g by the expressions:

$$f(p^r) = \exp\left(\frac{2\pi i k}{\alpha + 1}\right) \text{ if } r \equiv k \pmod{\alpha + 1} \tag{2.1}$$

$$g(p^r) = \exp\left(\frac{-2\pi i k}{\alpha + 1}\right) \text{ if } r \equiv k \pmod{\alpha + 1} \tag{2.2}$$

We immediately deduce that

$$f(p^{r\alpha}) = \exp\left(\frac{2\pi i k\alpha}{\alpha + 1}\right) = \exp\left(\frac{-2\pi i k}{\alpha + 1}\right) = g(p^r).$$

Similarly, we obtain

$$g(p^{r\alpha}) = f(p^r).$$

Therefore, we get

$$f(n^\alpha) = g(n) \quad \text{and} \quad g(n^\alpha) = f(n).$$

Also, $f(p^{\alpha+1}) = g(p^{\alpha+1}) = 1$. Thus, from (2.1) and (2.2), we obtain

$$f(p^r)g(p^r) = 1,\ r \geqslant 1.$$

Or, $f(n)$ and $g(n)$ are such that $f(n)g(n) = 1$.

Example: For $\alpha = 2$, we note that f, g would form a friendly-pair satisfying

$$f(n^2) = g(n),\ g(n^2) = f(n), \text{ and } f(n)g(n) = 1,\ n \geqslant 1.$$

In this case, f and g are given by:

$$f(p^r) = \begin{cases} \exp(2\pi i/3) & \text{if } r \equiv 1 \pmod 3 \\ \exp(4\pi i/3) & \text{if } r \equiv 2 \pmod 3 \\ 1 & \text{if } r \equiv 0 \ (\text{mod} \quad) \end{cases} \tag{2.3}$$

$$g(p^r) = \begin{cases} \exp(-2\pi i/3) & \text{if } r \equiv 1 \pmod 3 \\ \exp(-4\pi i/3) & \text{if } r \equiv 2 \pmod 3 \\ 1 & \text{if } r \equiv 0 \pmod 3 \end{cases} \tag{2.4}$$

Before concluding, we remark that there exist pairs $\{f, g\}$ which satisfy (1.2) but not (1.3). This point is elucidated for the case $\alpha = 2$.

Let $\mu(n)$ be the Möbius function. We define $f(n)$ and $g(n)$ as follows:

$$f(n) = \sum_{n = dt^3} \mu(d), \tag{2.5}$$

where the summation is over the divisors d of n for which the complementary divisor n/d is a perfect cube.

$$g(n) = \sum_{n = d^2t^3} \mu(d),$$

where the summation is over the square divisors d^2 of n for which the complementary divisor n/d^2 is a perfect cube.

We observe that f and g are multiplicative. Further,

$$f(p^r) = \begin{cases} -1 & \text{if } r \equiv 1 \pmod 3 \\ 0 & \text{if } r \equiv 2 \pmod 3 \\ 1 & \text{if } r \equiv 0 \pmod 3 \end{cases} \tag{2.7}$$

$$g(p^r) = \begin{cases} 0 & \text{if } r \equiv 1 \pmod 3 \\ -1 & \text{if } r \equiv 2 \pmod 3 \\ 1 & \text{if } r \equiv 0 \pmod 3 \end{cases} \tag{2.8}$$

It is easy to check that $f(n^2) = g(n)$ and $g(n^2) = f(n)$ for $n \geqslant 1$. However,

$$f(n)g(n) = \begin{cases} 1 & \text{if } n \text{ is a perfect cube,} \\ 0 & \text{otherwise.} \end{cases}$$

This pair $\{f, g\}$ is not a friendly-pair.

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HOGGATT SEQUENCES AND LEXICOGRAPHIC ORDERING

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(Submitted October 1985)

DEDICATION

While I was a graduate student at San Jose State University, Vern Hoggatt and I worked with sequences of positive integers which we were calling "generalized r-nacci numbers." In this paper, I have gathered some of our results concerning these sequences which I have renamed "the Hoggatt sequences." I would like to dedicate this paper to the memory of Professor Hoggatt.

M. A. O.

INTRODUCTION

The Zeckendorf Theorem states that every positive integer can be represented as a sum of distinct Fibonacci numbers and that this representation is unique, provided no two consecutive Fibonacci numbers appear in any sum.

In [2] the Zeckendorf Theorem is extended to a class of sequences obtained from the generalized Fibonacci polynomials; in particular, an analogous theorem holds for the generalized Fibonacci sequences. In Section 1, a collection of sequences called the Hoggatt sequences is introduced, and it is shown that these sequences also enjoy a "Zeckendorf Theorem"; in fact, the Hoggatt sequences share many of the representation and ordering properties of the generalized Fibonacci sequences discussed in [2] and [3].

1. HOGGATT SEQUENCES AND ZECKENDORF REPRESENTATIONS

For each fixed integer r with $r \geqslant 2$, the generalized Fibonacci polynomials yield a generalized Fibonacci sequence [2] which will be denoted $\{R_n\}_{n=1}^{\infty}$. The *generalized Fibonacci sequence* associated with the integer r can be defined as follows [3]:

$$R_1 = 1;$$

$$R_j = 2^{j-2} \text{ for } j = 2, 3, \ldots, r;$$

$$R_{k+r} = R_{k+r-1} + R_{k+r-2} + \cdots + R_k \text{ for all positive integers } k.$$

Note that with $r = 2$, 3, 4, and 5 we obtain, respectively, the Fibonacci numbers $\{F_n\}$, the Tribonacci numbers $\{T_n\}$, the Quadranacci numbers $\{Q_n\}$, and the Pentanacci numbers $\{P_n\}$.

The *Hoggatt sequence of degree* r, where r is once again a fixed integer greater than 1, will be denoted $\{\mathcal{R}_n\}$ and can be obtained by taking differences of adjacent generalized Fibonacci numbers; more precisely, $\mathcal{R}_n = R_{n+2} - R_{n+1}$ for all positive integers n. The defining properties of the sequences $\{R_n\}$ and $\{\mathcal{R}_n\}$ give rise to the following recursive description of the Hoggatt sequence of degree r:

$$\mathcal{R}_j = 2^{j-1} \text{ for } j = 1, 2, \ldots, r-1;$$
$$\mathcal{R}_r = 2^{r-1} - 1 = \mathcal{R}_1 + \mathcal{R}_2 + \cdots + \mathcal{R}_{r-1};$$
$$\mathcal{R}_{k+r} = \mathcal{R}_{k+r-1} + \mathcal{R}_{k+r-2} + \cdots + \mathcal{R}_k \text{ for all positive integers } k.$$

Note that the second-degree Hoggatt sequence coincides with the Fibonacci sequence; moreover, for $r > 2$, the sequences $\{R_n\}$ and $\{\mathcal{R}_n\}$ differ in their initial (and subsequent) entries but share a common recursion relation.

Identities similar (but not identical) to those developed for the generalized Fibonacci sequences in [3] can be obtained for the Hoggatt sequences.

For $r = 2$ the Hoggatt sequence is the Fibonacci sequence $\{F_n\}$, and we have the two identities

$$F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$$

and

$$F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2} - 1.$$

Let the third-degree Hoggatt sequence be denoted $\{\mathcal{T}_n\}$. Three identities arise in this case:

$$(\mathcal{T}_2 + \mathcal{T}_3) + (\mathcal{T}_5 + \mathcal{T}_6) + \cdots (\mathcal{T}_{3n-1} + \mathcal{T}_{3n}) = \mathcal{T}_{3n+1} - 1;$$
$$\mathcal{T}_1 + (\mathcal{T}_3 + \mathcal{T}_4) + (\mathcal{T}_6 + \mathcal{T}_7) + \cdots + (\mathcal{T}_{3n} + \mathcal{T}_{3n+1}) = \mathcal{T}_{3n+2} - 1;$$
$$\mathcal{T}_2 + (\mathcal{T}_4 + \mathcal{T}_5) + (\mathcal{T}_7 + \mathcal{T}_8) + \cdots + (\mathcal{T}_{3n+1} + \mathcal{T}_{3n+2}) = \mathcal{T}_{3n+3} - 1.$$

In general, we have the following lemma.

Lemma 1.1: For each integer r greater than 1, there arise r identities involving groupings of $(r-1)$ consecutive terms of the Hoggatt sequence of degree r.

$$(\mathcal{R}_2 + \mathcal{R}_3 + \cdots + \mathcal{R}_r) + (\mathcal{R}_{r+2} + \mathcal{R}_{r+3} + \cdots + \mathcal{R}_{2r}) + \cdots$$
$$+ (\mathcal{R}_{r(n-1)+2} + \mathcal{R}_{r(n-1)+3} + \cdots + \mathcal{R}_{rn}) = \mathcal{R}_{rn+1} - 1;$$
$$\mathcal{R}_1 + (\mathcal{R}_3 + \mathcal{R}_4 + \cdots + \mathcal{R}_{r+1}) + (\mathcal{R}_{r+3} + \mathcal{R}_{r+4} + \cdots + \mathcal{R}_{2r+1}) + \cdots$$
$$+ (\mathcal{R}_{r(n-1)+3} + \mathcal{R}_{r(n-1)+4} + \cdots + \mathcal{R}_{rn+1}) = \mathcal{R}_{rn+2} - 1;$$
$$\mathcal{R}_1 + \mathcal{R}_2 + (\mathcal{R}_4 + \mathcal{R}_5 + \cdots + \mathcal{R}_{r+2}) + (\mathcal{R}_{r+4} + \mathcal{R}_{r+5} + \cdots + \mathcal{R}_{2r+2}) + \cdots$$
$$+ (\mathcal{R}_{r(n-1)+4} + \mathcal{R}_{r(n-1)+5} + \cdots + \mathcal{R}_{rn+2}) = \mathcal{R}_{rn+3} - 1;$$
$$\vdots$$
$$\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \cdots + \mathcal{R}_{r-2} + (\mathcal{R}_r + \mathcal{R}_{r+1} + \cdots + \mathcal{R}_{2r-2}) + \cdots$$
$$+ (\mathcal{R}_{rn} + \mathcal{R}_{rn+1} + \cdots + \mathcal{R}_{rn+r-2}) = \mathcal{R}_{rn+r-1} - 1;$$
$$\mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4 + \cdots + \mathcal{R}_{r-1} + (\mathcal{R}_{r+1} + \mathcal{R}_{r+2} + \cdots + \mathcal{R}_{2r-1}) + \cdots$$
$$+ (\mathcal{R}_{rn+1} + \mathcal{R}_{rn+2} + \cdots + \mathcal{R}_{rn+r-1}) = \mathcal{R}_{rn+r} - 1.$$

Proof: For a fixed r, each of the identities can be verified by adding 1 to the expression on the left and applying the appropriate recursion relation.

In the first equation, note that

$$1 + \mathcal{R}_2 + \mathcal{R}_3 + \cdots + \mathcal{R}_r = \mathcal{R}_{r+1}.$$

When the term $\mathcal{R}_{r+1}$ is added to the next $(r-1)$ consecutive terms the result is $\mathcal{R}_{2r+1}$, which can be added to the next $(r-1)$ consecutive terms; this process can be repeated until addition yields $\mathcal{R}_{rn+1}$.

In general for the i^{th} equation, where $2 \leqslant i \leqslant r-1$, note that

$$1 + \mathcal{R}_1 + \mathcal{R}_2 + \cdots + \mathcal{R}_{i-1} = 1 + 1 + 2 + \cdots + 2^{i-2} = 2^{i-1} = \mathcal{R}_i.$$

Since the next parenthetic expression is

$$\mathcal{R}_{i+1} + \mathcal{R}_{i+2} + \cdots + \mathcal{R}_{r+i-1},$$

the addition process described for the first equation can now be applied.

The final identity follows by recalling that $1 + \mathcal{R}_2 + \mathcal{R}_3 + \cdots + \mathcal{R}_{r-1} = \mathcal{R}_r$ and applying the addition process.

In [1] a proof of a Zeckendorf Theorem for the generalized Fibonacci polynomials is given; a consequence of this theorem is the existence and uniqueness of the Zeckendorf representation for positive integers in terms of the generalized Fibonacci numbers. A generalized Zeckendorf Theorem also holds for the Hoggatt numbers of degree r. That is, for a given r, every positive integer can be represented as the sum of distinct terms of the sequence $\{\mathcal{R}_n\}$ provided no r consecutive terms of the sequence are used in the representation; however, since the sum of the first $(r-1)$ terms of the sequence is $\mathcal{R}_r$, in order to ensure uniqueness of the representation, we must also require that no representation use the first $(r-1)$ consecutive terms of $\{\mathcal{R}_n\}$.

Theorem 1.2: For each fixed integer $r \geqslant 2$, every positive integer N has a unique representation in terms of $\{\mathcal{R}_n\}$ of the form

$$N = N_1\mathcal{R}_1 + N_2\mathcal{R}_2 + \cdots + N_i\mathcal{R}_i, \text{ where } N_j \in \{0, 1\} \text{ for } j = 1, 2, \ldots, i,$$

$$N_1 N_2 \cdot \cdots \cdot N_{r-1} = 0,$$

and

$$N_k N_{k+1} \cdot \cdots \cdot N_{k+r-1} = 0 \text{ for all positive integers } k;$$

i.e., every integer has a unique Zeckendorf representation in terms of $\{\mathcal{R}_n\}$.

Proof: Note that for $r = 2$, the Hoggatt sequence in question is the Fibonacci sequence and the Zeckendorf Theorem holds.

The nature of the inductive proof of the theorem can best be seen by considering a particular small value of r. We concentrate our efforts on the case in which $r = 3$. Suppose for some n every positive integer $N \leqslant \mathfrak{I}_{3n+2} - 1$ has a unique Zeckendorf representation; it suffices to prove that every positive integer $N \leqslant \mathfrak{I}_{3n+3} - 1$ has a unique Zeckendorf representation.

It follows from Lemma 1.1 that

$$\mathfrak{I}_{3n+2} - 1 = \mathfrak{I}_1 + (\mathfrak{I}_3 + \mathfrak{I}_4) + (\mathfrak{I}_6 + \mathfrak{I}_7) + \cdots + (\mathfrak{I}_{3n} + \mathfrak{I}_{3n+1}),$$

and this equation must give the unique Zeckendorf representation for $\mathfrak{I}_{3n+2} - 1$. Next, we note that the representation for $\mathfrak{I}_{3n+2} - 1$ implies that the largest integer which can be represented without using $\mathfrak{I}_{3n+2}$ or any succeeding term of $\{\mathfrak{I}_n\}$ is $\mathfrak{I}_{3n+2} - 1$; therefore, the term $\mathfrak{I}_{3n+2}$ is itself the unique Zeckendorf representation for $\mathfrak{I}_{3n+2}$.

Since $\mathfrak{I}_{3n+1} - 1 < \mathfrak{I}_{3n+2} - 1$, the integer $\mathfrak{I}_{3n+1} - 1$ has a unique Zeckendorf representation. Moreover, this unique representation is given by the following identity from Lemma 1.1:

$$\mathfrak{I}_{3n+1} - 1 = (\mathfrak{I}_2 + \mathfrak{I}_3) + (\mathfrak{I}_5 + \mathfrak{I}_6) + \cdots + (\mathfrak{I}_{3n-1} + \mathfrak{I}_{3n}).$$

An immediate consequence of the preceding observations is that

$$\mathfrak{I}_{3n+2} + \mathfrak{I}_{3n+1} - 1$$

is uniquely representable by

$$\mathfrak{J}_{3n+2} + (\mathfrak{J}_2 + \mathfrak{J}_3) + (\mathfrak{J}_5 + \mathfrak{J}_6) + \cdots + (\mathfrak{J}_{3n-1} + \mathfrak{J}_{3n}).$$

It also follows that, for any positive integer M less than $\mathfrak{J}_{3n+1}$, there is a unique Zeckendorf representation for $\mathfrak{J}_{3n+2} + M$ consisting of adding $\mathfrak{J}_{3n+2}$ to the unique Zeckendorf representation for M.

Finally, we apply the only remaining third-degree identity in Lemma 1.1. Since $\mathfrak{J}_{3n} - 1 < \mathfrak{J}_{3n+2} - 1$, the integer $\mathfrak{J}_{3n} - 1$ has a unique Zeckendorf representation, and this representation is given by the identity

$$\mathfrak{J}_{3n} - 1 = \mathfrak{J}_2 + (\mathfrak{J}_4 + \mathfrak{J}_5) + (\mathfrak{J}_7 + \mathfrak{J}_8) + \cdots + (\mathfrak{J}_{3n-2} + \mathfrak{J}_{3n-1}).$$

It follows immediately that

$$\mathfrak{J}_{3n+2} + \mathfrak{J}_{3n+1} + \mathfrak{J}_{3n} - 1$$

has the unique Zeckendorf representation

$$\mathfrak{J}_{3n+2} + \mathfrak{J}_{3n+1} + [\mathfrak{J}_2 + (\mathfrak{J}_4 + \mathfrak{J}_5) + (\mathfrak{J}_7 + \mathfrak{J}_8) + \cdots + (\mathfrak{J}_{3n-2} + \mathfrak{J}_{3n-1})].$$

It is also apparent that $\mathfrak{J}_{3n+2} + M$ has a unique Zeckendorf representation for every positive integer M less than $\mathfrak{J}_{3n+1} + \mathfrak{J}_{3n}$.

Noting that

$$\mathfrak{J}_{3n+2} + \mathfrak{J}_{3n+1} + \mathfrak{J}_{3n} - 1 = \mathfrak{J}_{3n+3} - 1$$

concludes the proof of the theorem in the case $r = 3$.

The only major difference between the proof for $r = 3$ and the proof for an arbitrary value of r is that in the general case all r identities appearing in Lemma 1.1 must be used.

2. THE HOGGATT SEQUENCE OF DEGREE 3

If $r = 3$, the associated Hoggatt sequence $\{\mathfrak{J}_n\}$ is defined by taking

$$\mathfrak{J}_1 = 1,\ \mathfrak{J}_2 = 2,\ \mathfrak{J}_3 = \mathfrak{J}_1 + \mathfrak{J}_2 = 1 + 2 = 3$$

and

$$\mathfrak{J}_i = \mathfrak{J}_{i-1} + \mathfrak{J}_{i-2} + \mathfrak{J}_{i-3} \text{ for } i \geqslant 4;$$

the first seven terms of the resulting sequence are:

$\mathfrak{J}_1$	$\mathfrak{J}_2$	$\mathfrak{J}_3$	$\mathfrak{J}_4$	$\mathfrak{J}_5$	$\mathfrak{J}_6$	$\mathfrak{J}_7$
1	2	3	6	11	20	37

By Theorem 1.2, every positive integer has a unique Zeckendorf representation in terms of the third-degree Hoggatt numbers. In the next theorem, we prove that the terms used in the Zeckendorf representation of integers give information about the natural ordering of the integers being represented; in particular, we investigate lexicographic orderings which were defined and examined in [3] and [5]. We now define this kind of ordering as in [3].

Let the positive integers be represented in terms of a strictly increasing sequence of integers, $\{A_n\}$, so that for integers M and N,

$$M = \sum_{i=1}^{k} M_i A_i \quad \text{and} \quad N = \sum_{i=1}^{k} N_i A_i,$$

where the coefficients M_i and N_i lie in the set $\{0, 1, 2, \ldots, q\}$ for some fixed integer q; moreover, suppose m is an integer such that $M_i = N_i$ for all $i > m$.

If, for every pair of integers M and N, $M_m > N_m$ implies $M > N$, then the representation is a *lexicographic ordering*.

In [3], identities analogous to those in Lemma 1.1 are used to show that the Zeckendorf representation of the positive integers in terms of the Tribonacci numbers is a lexicographic ordering; a similar proof is used in the following theorem.

Theorem 2.1: The Zeckendorf representation of the positive integers in terms of the third-degree Hoggatt sequence $\{\mathfrak{I}_n\}$ is a lexicographic ordering.

Proof: Let M and N be two positive integers expressed in Zeckendorf form in terms of the third-degree Hoggatt numbers; then, for some positive integer t,

$$M = \sum_{i=1}^{t} M_i \mathfrak{I}_i \quad \text{and} \quad N = \sum_{i=1}^{t} N_i \mathfrak{I}_i,$$

where M_i, $N_i \in \{0, 1\}$, $M_1 M_2 = N_1 N_2 = 0$ and, for all i,

$$M_i M_{i+1} M_{i+2} = N_i N_{i+1} N_{i+2} = 0.$$

Let m be a positive integer such that $M_i = N_i$ for all $i > m$, and suppose that $M_m > N_m$. Then $M_m = 1$ and $N_m = 0$. In order to prove that $M > N$, we consider the following truncated portions of M and N:

$$M^* = M_1 \mathfrak{I}_1 + M_2 \mathfrak{I}_2 + \cdots + M_{m-1} \mathfrak{I}_{m-1} + \mathfrak{I}_m \geqslant \mathfrak{I}_m$$

and

$$N^* = N_1 \mathfrak{I}_1 + N_2 \mathfrak{I}_2 + \cdots + N_{m-1} \mathfrak{I}_{m-1}.$$

It is clear from the nature of the Zeckendorf representation and the recursion relation for members of $\{\mathfrak{I}_n\}$ that in order to maximize N^* we must have $N_{m-1} = N_{m-2} = 1$. Let k be a positive integer so that $m = 3k + j$, where $j = 1$, 2, or 3. The three pertinent identities in Lemma 1.1 imply that, for any of the three possible values of j, the maximal possible value of N^* is $\mathfrak{I}_m - 1$. Consequently, $N^* < \mathfrak{I}_m \leqslant M^*$, and it follows that $N < M$.

In [3], it was demonstrated that the positive integers can be represented in terms of the Tribonacci numbers by means of a "second canonical form," and it was proved that this new representation also gives rise to a lexicographic ordering. Analogous results hold for the sequence $\{\mathfrak{I}_n\}$. We begin by developing the second canonical form for a representation.

For each positive integer N, let $\mathfrak{I}_k$ be the least term of $\{\mathfrak{I}_n\}$ used in the Zeckendorf representation for N; of course, the subscript k depends on the particular integer N being examined. The uniqueness of the Zeckendorf representation implies it is possible to partition the positive integers into two sets as follows:

S_1 is the set of all positive integers N such that
$k \equiv 0 \pmod 3$ or $k \equiv 1 \pmod 3$,

and

S_2 is the set of all positive integers N such that
$k \equiv 2 \pmod 3$.

Suppose the elements of the sets S_1 and S_2 are written in natural order, and let $S_{i,n}$ denote the n^{th} element in the set S_i for $i = 1$ or 2. We list the first ten entries in each set.

Table 1

n	$S_{1,n}$	$S_{2,n}$
1	$1 = \mathfrak{J}_1$	$2 = \mathfrak{J}_2$
2	$3 = \mathfrak{J}_3$	$5 = \mathfrak{J}_3 + \mathfrak{J}_2$
3	$4 = \mathfrak{J}_3 + \mathfrak{J}_1$	$8 = \mathfrak{J}_4 + \mathfrak{J}_2$
4	$6 = \mathfrak{J}_4$	$11 = \mathfrak{J}_5$
5	$7 = \mathfrak{J}_4 + \mathfrak{J}_1$	$13 = \mathfrak{J}_5 + \mathfrak{J}_2$
6	$9 = \mathfrak{J}_4 + \mathfrak{J}_3$	$16 = \mathfrak{J}_5 + \mathfrak{J}_3 + \mathfrak{J}_2$
7	$10 = \mathfrak{J}_4 + \mathfrak{J}_3 + \mathfrak{J}_1$	$19 = \mathfrak{J}_5 + \mathfrak{J}_4 + \mathfrak{J}_2$
8	$12 = \mathfrak{J}_5 + \mathfrak{J}_1$	$22 = \mathfrak{J}_6 + \mathfrak{J}_2$
9	$14 = \mathfrak{J}_5 + \mathfrak{J}_3$	$25 = \mathfrak{J}_6 + \mathfrak{J}_3 + \mathfrak{J}_2$
10	$15 = \mathfrak{J}_5 + \mathfrak{J}_3 + \mathfrak{J}_1$	$28 = \mathfrak{J}_6 + \mathfrak{J}_4 + \mathfrak{J}_2$

Theorem 2.2: The sets S_1 and S_2 can be characterized as follows:

> S_1 is the set of all positive integers N which can be represented in the form $\mathfrak{J}_1 + N_2\mathfrak{J}_2 + N_3\mathfrak{J}_3 + \ldots$, where each $N_i \in \{0, 1\}$ and $N_iN_{i+1}N_{i+2} = 0$ if $i > 1$;
>
> S_2 is the set of all positive integers N which can be represented in the form $\mathfrak{J}_2 + N_3\mathfrak{J}_3 + N_4\mathfrak{J}_4 + \cdots$, where each $N_i \in \{0, 1\}$ and $N_iN_{i+1}N_{i+2} = 0$ if $i > 2$.

Moreover, every positive integer has a unique representation in one of the above two forms.

Proof: Let N be a positive integer and let $\mathfrak{J}_k$ be the least member of $\{\mathfrak{J}_n\}$ used in the Zeckendorf representation of N in terms of $\{\mathfrak{J}_n\}$. There are three cases to consider depending on whether k is congruent to 0, 1, or 2 modulo 3.

If $k \equiv 0 \pmod 3$, then N is an element of S_1 and, for some nonnegative integer m, $k = 3m + 3$. Using the identities in Lemma 1.1 and the Zeckendorf representation for N, the term $\mathfrak{J}_k$ can be replaced by

$$(\mathfrak{J}_1 + \mathfrak{J}_2) + (\mathfrak{J}_4 + \mathfrak{J}_5) + \cdots + (\mathfrak{J}_{3m+1} + \mathfrak{J}_{3m+2});$$

moreover, this is the only admissible representation for $\mathfrak{J}_k$. These observations and the uniqueness of the Zeckendorf representation imply the uniqueness of this new representation for N.

If $k \equiv 1 \pmod 3$, again N lies in S_1 and, for some nonnegative integer m, $k = 3m + 1$. In this case, $\mathfrak{J}_k$ must be replaced by

$$\mathfrak{J}_1 + (\mathfrak{J}_2 + \mathfrak{J}_3) + (\mathfrak{J}_5 + \mathfrak{J}_6) + \cdots + (\mathfrak{J}_{3m-1} + \mathfrak{J}_{3m}).$$

This illustrates the reason for permitting $N_1N_2N_3 = 1$. Again, this new representation for N is the unique allowable representation.

Finally, if $k \equiv 2 \pmod 3$, then N lies in S_2 and, for some nonnegative integer m, $k = 3m + 2$. From Lemma 1.1, we have

$$\mathfrak{J}_k = 1 + \mathfrak{J}_1 + (\mathfrak{J}_3 + \mathfrak{J}_4) + (\mathfrak{J}_6 + \mathfrak{J}_7) + \cdots + (\mathfrak{J}_{3m} + \mathfrak{J}_{3m+1})$$

$$\mathfrak{J}_k = \mathfrak{J}_2 + (\mathfrak{J}_3 + \mathfrak{J}_4) + (\mathfrak{J}_6 + \mathfrak{J}_7) + \cdots + (\mathfrak{J}_{3m} + \mathfrak{J}_{3m+1}).$$

In this case, we see that $N_1N_2N_3 = 1$ may be necessary in representing some integers. The uniqueness of this new representation for N follows as in the previous cases.

The preceding theorem suggests a definition for a second canonical representation with respect to $\{\mathfrak{I}_n\}$: a positive integer N is being represented in *second canonical form* in terms of the sequence $\{\mathfrak{I}_n\}$ if, for some m,

$$N = N_1\mathfrak{I}_1 + N_2\mathfrak{I}_2 + N_3\mathfrak{I}_3 + \cdots + N_m\mathfrak{I}_m,$$

where (1) each $N_i \in \{0, 1\}$,
(2) at least one of N_1 and N_2 is nonzero,
(3) if $N_1 = 1$, then $N_iN_{i+1}N_{i+2} = 0$ for all $i > 1$,
and (4) if $N_2 = 1$, then $N_iN_{i+1}N_{i+2} = 0$ for all $i > 2$.

The following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.3: Every positive integer can be uniquely represented in second canonical form in terms of the Hoggatt sequence of degree 3.

In [3], it is noted that the representation of the positive integers in second canonical form with respect to the Tribonacci numbers is a lexocigraphic ordering. Although the second canonical form of a representation with respect to $\{\mathfrak{I}_n\}$ is not defined in the same way as the second canonical form with respect to $\{T_n\}$, the two forms are similar and an analogous theorem holds for the third-degree Hoggatt numbers.

Theorem 2.4: The second canonical representation of the positive integers in terms of the sequence $\{\mathfrak{I}_n\}$ is a lexicographic ordering.

Proof: We begin as in the proof of Theorem 2.1.

Let M and N be two positive integers expressed in second canonical form in terms of $\{\mathfrak{I}_n\}$. There is some positive integer t such that, in second canonical form,

$$M = \sum_{i=1}^{t} M_i\mathfrak{I}_i \quad \text{and} \quad N = \sum_{i=1}^{t} N_i\mathfrak{I}_i.$$

Let m be a positive integer such that $M_i = N_i$ for all $i > m$; further, suppose $M_m = 1$ and $N_m = 0$. Consider the following truncations of M and N:

$$M^* = M_1\mathfrak{I}_1 + M_2\mathfrak{I}_2 + \cdots + M_{m-1}\mathfrak{I}_{m-1} + \mathfrak{I}_m$$

and

$$N^* = N_1\mathfrak{I}_1 + N_2\mathfrak{I}_2 + \cdots + N_{m-1}\mathfrak{I}_{m-1}.$$

Since M has been represented in second canonical form, either M_1 or M_2 is nonzero; therefore, $M^* \geqslant \mathfrak{I}_1 + \mathfrak{I}_m > \mathfrak{I}_m$. Again, in order or maximize N^*, we must have $N_{m-1} = N_{m-2} = 1$. Let K be a positive integer such that $m = 3k + j$ for some $j = 1, 2,$ or 3. Conisder the three appropriate identities in Lemma 1.1, and the three possible values of j.

If $m = 3k + 1$, then the maximum possible value of N^* is

$$\mathfrak{I}_{3k+1} - 1 + \mathfrak{I}_1 = \mathfrak{I}_{3k+1} = \mathfrak{I}_m.$$

If $m = 3k + 2$, then the maximum value for N^* is

$$\mathfrak{I}_{3k+2} - 1 = \mathfrak{I}_m - 1.$$

Finally, if $m = 3k + 3$, then the maximum possible N^* is

$$\mathfrak{I}_{3k+3} - 1 + \mathfrak{I}_1 = \mathfrak{I}_{3k+3} = \mathfrak{I}_m.$$

In any case, N^* does not exceed $\mathfrak{I}_m$ in value, and we have $N^* \leqslant \mathfrak{I}_m < M^*$; consequently, $N < M$.

Before proceeding to the generalizations of the preceding theorems in this section to degree r, we note a special property of the third-degree Hoggatt sequence.

Let $S_1, S_2, \ldots, S_n$ be nonempty sequences of positive integers such that every positive integer appears exactly once in exactly one of the sequences; in [1], such sequences are called *complementary* or a *complementary system*. In [3], properties of $\{T_n\}$ and a theorem of Lamdek and Moser [4] are used to demonstrate the existence of a pair of complementary sequences $\{X_n\}$ and $\{Y_n\}$ in natural order with the property that $\{X_n + Y_n\}$ and $\{Y_n - X_n\}$ is another pair of complementary sequences of positive integers in natural order. In the next theorem, we prove the existence and uniqueness of $\{X_n\}$ and $\{Y_n\}$.

Theorem 2.5: There exist exactly two sequences, $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$, of positive integers in natural order such that $\{X_n\}$ and $\{Y_n\}$ are complementary sequences and the sequences $\{X_n + Y_n\}$ and $\{Y_n - X_n\}$ are also complementary sequences in natural order.

Proof: We develop four sequences $\{X_n\}$, $\{Y_n\}$, $\{P_n\}$, and $\{Q_n\}$ as follows: let $X_1 = 1$, $P_1 = 1$, $Y_1 = X_1 + P_1 = 2$, and $Q_1 = X_1 + Y_1 = 3$; in general, to find X_n, P_n, Y_n, and Q_n, let

(1) X_n = the first positive integer not yet appearing as an X_i or a Y_i,
(2) P_n = the first positive integer not yet appearing as a P_i or a Q_i,
(3) $Y_n = X_n + P_n$, and
(4) $Q_n = X_n + Y_n$.

The following array arises.

Table 2

n	X_n	P_n	Y_n	Q_n
1	1	1	2	3
2	3	2	5	8
3	4	4	8	12
4	6	5	11	17
5	7	6	13	20
6	9	7	16	25
7	10	9	19	29
8	12	10	22	34
9	14	11	25	39
10	15	13	28	43
⋮	⋮	⋮	⋮	⋮

Note that (1)-(4) guarantee that $\{X_n\}$ and $\{Y_n\}$ are complementary sequences in natural order, as are $\{P_n\}$ and $\{Q_n\}$. From (3) and (4) it follows that

$$\{P_n\} = \{Y_n - X_n\} \quad \text{and} \quad \{Q_n\} = \{X_n + Y_n\},$$

as desired. Hence, the exixtence of the sequences $\{X_n\}$ and $\{Y_n\}$ has been established.

To verify the uniqueness of the sequences $\{X_n\}$ and $\{Y_n\}$, we note that the method of generating the four sequences yields exactly one pair of sequences satisfying the conditions in the statement of the theorem; therefore, any other pair of sequences satisfying these conditions must be obtained by some method other than that used to generate $\{X_n\}$ and $\{Y_n\}$.

Suppose there is another pair of sequences, denoted $\{\overline{X}_n\}$ and $\{\overline{Y}_n\}$, satisfying the conditions of the theorem. Let $\{\overline{Q}_n\}$ and $\{\overline{P}_n\}$ represent, respectively, the sum and difference sequences $\{\overline{X}_n + \overline{Y}_n\}$ and $\{\overline{Y}_n - \overline{X}_n\}$; it follows that properties (3) and (4) hold for the four new sequences. Suppose property (1) does not hold. Then, for some n, $\overline{X}_n$ is not the first positive integer not yet appearing as an $\overline{X}_i$ or a $\overline{Y}_i$; since $\{\overline{X}_n\}$ and $\{\overline{Y}_n\}$ are complementary and in natural order, $\overline{X}_n > \overline{Y}_n$. Consequently, $\overline{Y}_n - \overline{X}_n < 0$ and $\overline{P}_n$ is not a positive integer, a contradiction. Therefore, property (1) is necessary to the solution of the problem; similarly, property (2) must hold. Hence, the method used to generate $\{X_n\}$ and $\{Y_n\}$ provides the only pair of sequences satisfying the conditions of the theorem.

Consider the sets S_1 and S_2 defined earlier in this section. Recall that S_1 and S_2 are written in natural order, and $S_{i,n}$ denotes the n^{th} element of S_i for $i = 1$ or 2. We have seen that $\{S_{1,n}\}$ and $\{S_{2,n}\}$ are complementary sequences of positive integers in natural order. It has also been shown in [3] that

$$\{S_{1,n} + S_{2,n}\} \quad \text{and} \quad \{S_{2,n} - S_{1,n}\}$$

are complementary sequences in natural order. It follows that $\{S_{1,n}\}$ and $\{S_{2,n}\}$ are the sequences $\{X_n\}$ and $\{Y_n\}$ of Theorem 2.5. Therefore, the sets S_1 and S_2 can be generated by the method described in the proof of Theorem 2.5; no appeal to representations in terms of $\{\mathfrak{T}_n\}$ is necessary.

3. THE HOGGATT SEQUENCE OF DEGREE r

In this section, we note that the theorems of Section 2 involving lexicographic ordering have analogs for the r^{th}-degree Hoggatt sequence. Since the theorems of this section can be proved by using the same techniques as in Section 2, only sketches of proofs are given. Recall that from Section 1 we have r identities involving the sequence $\{\mathfrak{R}_n\}$ and a unique Zeckendorf representation for every positive integer in terms of $\{\mathfrak{R}_n\}$.

Theorem 3.1: The Zeckendorf representation of the positive integers in terms of the r^{th}-degree Hoggatt sequence $\{\mathfrak{R}_n\}$ is a lexicographic ordering.

Proof: Let M and N be two positive integers expressed in Zeckendorf form:

$$M = \sum_{i=1}^{t} M_i \mathfrak{R}_i \quad \text{and} \quad N = \sum_{i=1}^{t} N_i \mathfrak{R}_i,$$

where $M_i,\ N_i \in \{0,\ 1\}$,

$$M_1 M_2 \cdot \cdots \cdot M_{r-1} = N_1 N_2 \cdot \cdots \cdot N_{r-1} = 0,$$

and $\quad M_i M_{i+1} \cdot \cdots \cdot M_{i+r-1} = N_i N_{i+1} \cdot \cdots \cdot N_{i+r-1} = 0$ for all i.

Let m be a positive integer such that $M_i = N_i$ for all $i > m$, let $M_m = 1$, and let $N_m = 0$. Consider the truncations M^* and N^* as in the proof of Theorem 2.1, and note that $M^* \geqslant \mathfrak{R}_m$. In order to maximize N^*, we must let

$$N_{m-1} = N_{m-2} = \cdots = N_{m-(r-1)} = 1.$$

From the r identities in Lemma 1.1, it follows that $N^* < \mathfrak{R}_m \leqslant M^*$, and consequently, $N < M$.

We next develop the second canonical form for a representation in terms of $\{\mathcal{R}_n\}$.

For a particular positive integer N, let $\mathcal{R}_k$ be the smallest term of $\{\mathcal{R}_n\}$ used in the Zeckendorf representation for N. Using the uniqueness of the Zeckendorf representation, the positive integers can be partitioned into $(r - 1)$ sets as follows:

> S_1 is the set of all positive integers N such that $k \equiv 0 \pmod{r}$ or $k \equiv 1 \pmod{r}$,

and for integers i such that $2 \leqslant i \leqslant r - 1$,

> S_i is the set of all positive integers N such that $k \equiv i \pmod{r}$.

Let the elements of the sets $S_1, S_2, \ldots, S_{r-1}$ be written in natural order.

Theorem 3.2: The sets $S_1, S_2, \ldots, S_{r-1}$ can be characterized as follows: for $j = 1, 2, \ldots, r - 1$,

> S_j is the set of all positive integers which can be represented in the form $N = \mathcal{R}_j + N_{j+1}\mathcal{R}_{j+1} + N_{j+2}\mathcal{R}_{j+2} + \cdots$, where each $N_i \in \{0, 1\}$ and $N_i N_{i+1} \cdot \cdots \cdot N_{i+r-1} = 0$ if $i > j$.

Moreover, every positive integer has a unique representation in terms of $\{\mathcal{R}_n\}$ in one of these $(r - 1)$ forms.

Proof: Let N be a positive integer and let $\mathcal{R}_k$ be the least term of $\{\mathcal{R}_n\}$ used in the Zeckendorf representation of N. There are r cases to consider depending on whether k is congruent to $0, 1, 2, \ldots,$ or $(r - 1)$ modulo r. In each of these cases, the uniqueness of Zeckendorf representations and one of the identities in Lemma 1.1 yield the desired representation for N; moreover, the new representation is unique.

A positive integer N is represented in *second canonical form* in terms of the sequence $\{\mathcal{R}_n\}$ if, for some m,

$$N = N_1\mathcal{R}_1 + N_2\mathcal{R}_2 + \cdots + N_m\mathcal{R}_m,$$

where

(1) each $N_i \in \{0, 1\}$,
(2) at least one of the coefficients $N_1, N_2, \ldots, N_{r-1}$ is nonzero, and
(3) if $N_j = 1$, then $N_i N_{i+1} \cdot \cdots \cdot N_{i+r-1} = 0$ for all $i > j$.

We immediately have the following corollary to Theorem 3.2.

Corollary 3.3: Every positive integer can be uniquely represented in second canonical form in terms of the sequence $\{\mathcal{R}_n\}$.

Finally, we have the analog to Theorem 2.4.

Theorem 3.4: The second canonical representation of the positive integers in terms of the sequence $\{\mathcal{R}_n\}$ is a lexicographic ordering.

Proof: With notation as in the proof of Theorem 3.1, but with the representation in second canonical form, consider the truncations of M and N:

$$M^* = M_1\mathcal{R}_1 + M_2\mathcal{R}_2 + \cdots + M_{m-1}\mathcal{R}_{m-1} + \mathcal{R}_m$$

and

$$N^* = N_1\mathcal{R}_1 + N_2\mathcal{R}_2 + \cdots + N_{m-1}\mathcal{R}_{m-1}.$$

One of the coefficients M_1, M_2, ..., M_{r-1} is nonzero; therefore,

$$M^* \geqslant \mathcal{R}_1 + \mathcal{R}_m > \mathcal{R}_m.$$

In order to maximize N^*, we let

$$N_{m-1} = N_{m-2} = \cdots = N_{m-(r-1)} = 1$$

and note that the identities in Lemma 1.1 imply that the maximum possible value for N^* is $\mathcal{R}_m$; therefore, $N^* \leqslant \mathcal{R}_m < M^*$ and $N < M$.

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FUNCTIONS OF NON-UNITARY DIVISORS

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(Submitted November 1985)

1. INTRODUCTION

A divisor d of n is a *unitary divisor* if gcd $(d, n/d) = 1$; in such a case, we write $d \| n$. There is a considerable body of results on functions of unitary divisors (see [2]-[7]). Let $\tau^*(n)$ and $\sigma^*(n)$ denote, respectively, the number and sum of the unitary divisors of n.

We say that a divisor d of n is a *non-unitary divisor* if $(d, n/d) > 1$. If d is a non-unitary divisor of n, we write $d |^{\#} n$. In this paper, we examine some functions of non-unitary divisors.

We will find it convenient to write

$$n = \overline{n} \cdot n^{\#},$$

where $\overline{n}$ is the largest squarefree unitary divisor of n. We call $\overline{n}$ the *square-free part* of n and $n^{\#}$ the *powerful part* of n. Then, if p is prime, $p | \overline{n}$ implies $p \| n$, while $p | n^{\#}$ implies $p^2 | n$. Naturally, either $\overline{n}$ or $n^{\#}$ can be 1 if required (if n is powerful or squarefree, respectively).

2. THE SUM OF NON-UNITARY DIVISORS FUNCTION

Let $\sigma^{\#}(n)$ be the sum of the non-unitary divisors of n:

$$\sigma^{\#}(n) = \sum_{d |^{\#} n} d.$$

Now, every divisor is either unitary or non-unitary. Because $\overline{n}$ and $n^{\#}$ are relatively prime and the σ and σ^* functions are multiplicative, we have

$$\sigma^{\#}(n) = \sigma(n) - \sigma^*(n) = \sigma(\overline{n})\sigma(n^{\#}) - \sigma^*(\overline{n})\sigma^*(n^{\#}).$$

But $\sigma(\overline{n}) = \sigma^*(\overline{n})$, so

$$\sigma^{\#}(n) = \sigma(\overline{n})\{\sigma(n^{\#}) - \sigma^*(n^{\#})\}.$$

Therefore,

$$\sigma^{\#}(n) = \left\{\prod_{p \| n} (p+1)\right\} \cdot \left\{\prod_{\substack{p^e \| n \\ e>1}} \frac{p^{e+1}-1}{p-1} - \prod_{\substack{p^e \| n \\ e>1}} (p^e+1)\right\}.$$

Note that $\sigma^{\#}(n) = 0$ if and only if n is squarefree, and that $\sigma^{\#}$ is *not* multiplicative.

Recall that an integer n is perfect [unitary perfect] if it equals the sum of its proper divisors [unitary divisors]. This is usually stated as $\sigma(n) = 2n$ [$\sigma^*(n) = 2n$] in order to be dealing with multiplicative functions. But all non-unitary divisors are proper divisors, so the analogous definition here is that n is *non-unitary perfect* if $\sigma^{\#}(n) = n$.

Theorem 1: If $2^p - 1$ is prime, so that $2^{p-1}(2^p - 1)$ is an even perfect number, then $2^{p+1}(2^p - 1)$ is non-unitary perfect.

Proof: Suppose $n = 2^{p+1}(2^p - 1)$, where p is prime. Then

$$\begin{aligned}\sigma^{\#}(n) &= \sigma(2^p - 1)\{\sigma(2^{p+1}) - \sigma^*(2^{p+1})\}\\ &= 2^p[(2^{p+2} - 1) - (2^{p+2} + 1)]\\ &= 2^p(2^{p+1} - 2) = 2^{p+1}(2^p - 1) = n.\end{aligned}$$

A computer search written under our direction by Abdul-Nasser El-Kassar found no other non-unitary perfect numbers less than one million. Accordingly, we venture the following:

Conjecture 1: An integer is non-unitary perfect if and only if it is 4 times an even perfect number.

If $n^{\#}$ is known or assumed, it is relatively easy to search for $\overline{n}$ to see if n is non-unitary perfect. Many cases are eliminated because of having $\sigma^{\#}(n^{\#}) > n^{\#}$. In most other cases, the search fails because $\overline{n}$ would have to contain a repeated factor. For example, if $n^{\#} = 2^2 5^2$, then no $\overline{n}$ will work, for

$$\sigma^{\#}(2^2 5^2) = 7 \cdot 31 - 5 \cdot 26 = 87 = 3 \cdot 29,$$

so $3 \cdot 29 | \overline{n}$; but $2^2 5^2 29 \| n$ implies $3^2 | n$, so $3 | \overline{n}$ is impossible.

The second author generated by computer all powerful numbers not exceeding 2^{15}. Examination of the various cases verified that there is no non-unitary perfect number n with $n^{\#} \leqslant 2^{15}$ except when n satisfies Theorem 1 [i.e., $n = 2^{p+1}(2^p - 1)$, where $2^p - 1$ is prime].

More generally, we say that n is *k-fold non-unitary perfect* if $\sigma^{\#}(n) = kn$, where $k \geqslant 1$ is an integer. We examined all $n^{\#} \leqslant 2^{15}$ and all $n \leqslant 10^6$ and found the k-fold non-unitary perfect numbers ($k > 1$) listed in Table 1. Based on the profusion of examples and the relative ease of finding them, we hazard the following (admittedly shaky) guess:

Conjecture 2: There are infinitely many k-fold non-unitary perfect numbers.

Table 1. k-fold Non-Unitary Perfect Numbers ($k > 1$)

k	n
2	$2^3 3^2 5 \cdot 7 = 2520$
2	$2^3 3^3 5 \cdot 29 = 31\ 320$
2	$2^3 3^4 5 \cdot 359 = 1\ 163\ 160$
2	$2^7 3^5 71 = 2\ 208\ 384$
2	$2^4 3^2 7 \cdot 13 \cdot 233 = 3\ 053\ 232$
2	$2^7 3^3 31 \cdot 61 = 6\ 535\ 296$
2	$2^5 3^2 7 \cdot 41 \cdot 163 = 13\ 472\ 928$
2	$2^5 5^2 3 \cdot 19 \cdot 37 \cdot 73 = 123\ 165\ 600$
2	$2^7 3^4 47 \cdot 751 = 365\ 959\ 296$
2	$2^4 3^4 11 \cdot 131 \cdot 2357 = 4\ 401\ 782\ 352$
2	$2^{10} 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 = 5\ 517\ 818\ 880$
3	$2^7 3^2 5^2 \cdot 7 \cdot 13 \cdot 71 = 186\ 076\ 800$
3	$2^8 3^4 5 \cdot 7 \cdot 11 \cdot 53 \cdot 769 = 325\ 377\ 803\ 520$
3	$2^6 3^2 7^2 5 \cdot 13 \cdot 19 \cdot 113 \cdot 677 = 2\ 666\ 567\ 816\ 640$

We say that n is *non-unitary subperfect* if $\sigma^{\#}(n)$ is a proper divisor of n. Because $\sigma^{\#}(18) = 9$ and $\sigma^{\#}(p^2) = p$ if p is prime, we have the following:

Theorem 2: If $n = 18$ or $n = p^2$, where p is prime, then n is non-unitary subperfect.

An examination of all $n^{\#} \leqslant 2^{15}$ and all $n \leqslant 10^6$ found no other non-unitary subperfect numbers, so we are willing to risk the following:

Conjecture 3: An integer n is non-unitary subperfect if and only if $n = 18$ or $n = p^2$, where p is prime.

It is possible to define non-unitary harmonic numbers by requiring that the harmonic mean of the non-unitary divisors be integral. If $\tau^{\#}(n) = \tau(n) - \tau^*(n)$ counts the number of non-unitary divisors, the requirement is that $n\tau^{\#}(n)/\sigma^{\#}(n)$ be integral. We found several dozen examples less than 10^6, including all k-fold non-unitary perfect numbers, as well as numbers of the forms

$$2 \cdot 3p^2,\ p^2(2p - 1),\ 2 \cdot 3p^2(2p - 1),\ 2^{p+1}3(2^p - 1),\ 2^{p+1}3 \cdot 5(2^p - 1),$$
$$\text{and } 2^{p+1}(2p - 1)(2^p - 1),$$

where p, $2p - 1$, and $2^p - 1$ are distinct primes. Many other examples seemed to fit no general pattern.

Recall that integers n and m are amicable [unitary amicable] if each is the sum of the proper divisors [unitary divisors] of the other. Similarly, we say that n and m are *non-unitary amicable* if

$$\sigma^{\#}(n) = m \quad \text{and} \quad \sigma^{\#}(m) = n.$$

Theorem 3: If $2^p - 1$ and $2^q - 1$ are prime, then $2^{p+1}(2^q - 1)$ and $2^{q+1}(2^p - 1)$ are non-unitary amicable.

Proof: Trivial verification.

Thus, there are at least as many non-unitary amicable pairs as there are pairs of Mersenne primes. Our computer search for $n < m$ and $n \leqslant 10^6$ revealed only four non-unitary amicable pairs that are not characterized by Theorem 3:

$$n = 252 = 2^2 3^2 7 \qquad m = 328 = 2^3 41$$
$$n = 3240 = 2^3 3^4 5 \qquad m = 6462 = 2 \cdot 3^2 359$$
$$n = 11616 = 2^5 3 \cdot 11^2 \qquad m = 17412 = 2^2 \cdot 3 \cdot 1451$$
$$n = 11808 = 2^5 3^2 41 \qquad m = 20538 = 2 \cdot 3^2 \cdot 7 \cdot 163$$

3. THE NON-UNITARY ANALOG OF EULER'S FUNCTION

Euler's function

$$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right) = \prod_{p^e \| n} (p^e - p^{e-1})$$

is usually defined as the number of positive integers not exceeding n that are relatively prime to n. The unitary analog is

$$\varphi^*(n) = n \prod_{p^e \| n} \left(1 - \frac{1}{p^e}\right) = \prod_{p^e \| n} (p^e - 1).$$

Our first task here is to give equivalent alternative definitions for φ and φ^* which will suggest a non-unitary analog. In particular, we may define $\varphi(n)$ as the number of positive integers not exceeding n that are not divisible by any of the divisors $d > 1$ of n. Similarly, $\varphi^*(n)$ may be defined as the number of positive integers not exceeding n that are not divisible by any of the unitary divisors $d > 1$ of n.

Recalling that 1 is never a non-unitary divisor of n, it is natural in light of the alternative definitions of φ and φ^* to define $\varphi^{\#}(n)$ as the number of positive integers not exceeding n that are not divisible by any of the non-unitary divisors of n. By imitating the usual proofs for φ and φ^*, it is easy to show that $\varphi^{\#}$ is multiplicative, and that

$$\varphi^{\#}(n) = \bar{n}\varphi(n^{\#}). \tag{1}$$

The following result neatly connects divisors, unitary divisors, and non-unitary divisors in a, perhaps, unexpected way:

Theorem 4: $\sum_{d|n} \varphi^{\#}(d) = \sigma^*(n)$.

Proof: The Dirichlet convolution preserves multiplicativity, and $\varphi^{\#}$ is multiplicative, so we need only check the assertion for prime powers. In light of (1), doing so is easy, because the sum telescopes:

$$\begin{aligned}\sum_{d|p^e} \varphi^{\#}(d) &= \varphi^{\#}(1) + \varphi^{\#}(p) + \varphi^{\#}(p^2) + \cdots + \varphi^{\#}(p^e)\\ &= 1 + p + (p^2 - p) + \cdots + (p^e - p^{e-1})\\ &= 1 + p^e = \sigma^*(p^e).\end{aligned}$$

It is well known that

$$\sum_{d|n} \varphi(d) = n \quad \text{and} \quad \sum_{d\|n} \varphi^*(d) = n,$$

and one might anticipate a similar result involving $\varphi^{\#}$. However, the situation is a bit complicated. We write

$$\sum_{d|^{\#}n} \varphi^{\#}(d) = \sum_{d|n} \varphi^{\#}(d) - \sum_{d\|n} \varphi^{\#}(d). \tag{2}$$

Now, both convolutions on the right side of (2) preserve multiplicativity and, as a result, it is possible to obtain the following:

Theorem 5: $\sum_{d|^{\#}n} \varphi^{\#}(d) = \sigma(\bar{n})\left\{\sigma^*(n^{\#}) - \prod_{p^e\|n^{\#}} (p^e - p^{e-1} + 1)\right\}$

Theorem 5 was first obtained by Scott Beslin in his Master's thesis [1], written under the direction of the first author of this paper.

Two questions arise in connection with Theorem 5. First, is it possible to find a subset $S(n)$ of the divisors of n for which

$$\sum_{d\in S(n)} \varphi^{\#}(d) = n?$$

It is indeed possible to do so. Let $\omega(n)$ be the number of distinct primes that divide n. We say that d is an ω-*divisor* of n if $d|n$ and $\omega(d) = \omega(n)$, i.e., if every prime that divides n also divides d. Let $\Omega(n)$ denote the set of all ω-divisors of n.

Theorem 6: $\sum_{d \in \Omega(n)} \varphi^{\#}(d) = n.$

Proof: Trivial if $\omega(n) = 0$. But if $\omega(n) = 1$, the sum is that in the proof of Theorem 4 except that the term "$\varphi^{\#}(1) = 1$" is missing. Easy induction on $\omega(n)$, using the multiplicativity of $\varphi^{\#}$, completes the proof.

The other question that arises from Theorem 5 is whether it is possible to have

$$\sum_{d|n} \varphi^{\#}(d) = n, \quad n > 1. \tag{3}$$

We know of ten solutions to (3), and they are given in Table 2. By Theorem 5, if n satisfies (3), then

$$\sigma(\overline{n})/\overline{n} = n^{\#}/\left\{\sigma^*(n^{\#}) - \prod_{p^e \| n^{\#}} (p^e - p^{e-1} + 1)\right\}. \tag{4}$$

This observation makes it easy to search for $\overline{n}$ if $n^{\#}$ is known. The first eight numbers in Table 2 are the only solutions to (3) with $1 < n \leqslant 2^{15}$.

Table 2. Solutions to (3), Ordered by $n^{\#}$

n	$n^{\#}$	$\overline{n}$
5 220	$2^2 3^2$	$5 \cdot 29$
3 960	$2^3 3^2$	$5 \cdot 11$
8 447 040	$2^6 3^2$	$5 \cdot 7 \cdot 419$
6 773 440	$2^7 3^2$	$5 \cdot 7 \cdot 167$
18 685 336 320	$2^8 3^2$	$5 \cdot 7 \cdot 139 \cdot 1667$
341 863 562 880	$2^7 3^3$	$5 \cdot 7 \cdot 29 \cdot 41 \cdot 2377$
1 873 080	$2^3 3^2 11^2$	$5 \cdot 43$
1 018 887 932 160	$2^8 3^4$	$5 \cdot 7 \cdot 19 \cdot 37 \cdot 1997$
20 993 596 382 889 043 200	$2^8 3^2 5^2$	$7 \cdot 19 \cdot 2393 \cdot 23929 \cdot 47857$
357 174 165 248	$2^{13} 3^2$	$7 \cdot 11 \cdot 13 \cdot 47 \cdot 103$

It seems unlikely that one could completely characterize the solutions to (3). However, we do know the following:

Theorem 7: If $n > 1$ satisties (3), then $n^{\#}$ is divisible by at least two distinct primes.

Proof: We must have $n^{\#} > 1$ because $\sigma(\overline{n}) \geqslant \overline{n}$ with equality only if $\overline{n} = 1$. Suppose $n^{\#} = p^e$, where p is prime and $e \geqslant 2$. Then, from (4), we have $\sigma(\overline{n})/\overline{n} = p$. If $p = 2$, then $\overline{n}$ is an odd squarefree perfect number, which is impossible. Now, $\overline{n}$ is squarefree, and any odd prime that divides $\overline{n}$ contributes at least one factor 2 to $\sigma(\overline{n})$, and since $p \neq 2$, we have $2 \| \overline{n}$. Then $\overline{n} = 2q$, where q is prime, and the requirement $\sigma(\overline{n})/\overline{n} = p$ forces $q = 3/(2p - 3)$, which is impossible if $p > 2$.

We strongly suspect the following is true:

Conjecture 4: If n satisfies (3), then $n^{\#}$ is even.

If the right side of (4) does not reduce, then Conjecture 4 is true: If we suppose that $n^{\#}$ is odd, then $4 | \sigma^*(n^{\#})$, as $n^{\#}$ has at least two distinct prime divisors by Theorem 7. Then, it is easy to see that the denominator of the right side of (4) is of the form $4k - 1$, and if the right side of (4) does not reduce, then $\overline{n}$ is of the form $4k - 1$, whence $4 | \sigma(\overline{n})$, making (4) impossible. Thus, any counterexample to Conjecture 4 requires that the fraction on the right side of (4) reduce.

ACKNOWLEDGMENTS

The authors express their gratitude to fellow participants in a problems seminar at the University of Southwestern Louisiana. The contributions of Abdul-Nasser El-Kassar and Scott Beslin have already been noted. P. G. Garcia and Pat Jones also contributed to our investigation of functions of non-unitary divisors.

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SOME PROPERTIES OF BINOMIAL COEFFICIENTS

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(Submitted November 1985)

1. INTRODUCTION

In 1982, M. Boscarol [1] gave a demonstration of the following property of binomial coefficients:

$$\sum_{i=0}^{m} 2^{-(n+i)}\binom{n+i}{i} + \sum_{j=0}^{n} 2^{-(n+m-j)}\binom{n+m-j}{m} = 2 \tag{1}$$

for each pair of integers $n,\ m \geqslant 0$. For instance, let $m = 4$ and $n = 3$, then we have

$$\sum_{i=0}^{4} 2^{-i-3}\binom{3+i}{i} + \sum_{j=0}^{3} 2^{j-7}\binom{7-j}{4} = 2,$$

i.e.,

$$2^{-3} + 4 \cdot 2^{-4} + 10 \cdot 2^{-5} + 20 \cdot 2^{-6} + 35 \cdot 2^{-7} + 35 \cdot 2^{-7} + 15 \cdot 2^{-6} + 5 \cdot 2^{-5} + 2^{-4} = 2.$$

The purpose of this note is to present a generalization of (1).

2. MAIN RESULTS

Theorem 1: For each pair of integers $n,\ m \geqslant 0$ and $r > 0$, the following identity holds:

$$\sum_{i=0}^{m} r^{m-i}\binom{n+i}{i} = \binom{n+m+1}{m} + (r-1)\sum_{i=0}^{m-1} r^{i}\binom{n+m-i}{n+1}. \tag{2}$$

Proof: For $m = 0$, we have

$$\binom{n}{0} = 1 = \binom{n+1}{0}$$

from the definition. We now show that the formula for $m + 1$ follows from the formula for m.

$$\sum_{i=0}^{m+1} r^{(m+1)-i}\binom{n+i}{i} = \binom{n+m+1}{m+1} + r\sum_{i=0}^{m} r^{m-i}\binom{n+i}{i}$$

$$= \binom{n+m+1}{m+1} + r\left\{\binom{n+m+1}{m} + (r-1)\sum_{i=0}^{m-1} r^{i}\binom{n+m-i}{n+1}\right\}, \text{ by assumption}$$

$$= \left\{\binom{n+m+1}{m+1} + \binom{n+m+1}{m}\right\}$$

$$+ (r-1)\left\{\binom{n+m+1}{m} + \sum_{i=0}^{m-1} r^{i+1}\binom{n+m-i}{n+1}\right\}$$

$$= \binom{n+m+2}{m+1} + (r-1)\sum_{i=-1}^{m-1} r^{i+1}\binom{n+m-i}{n+1}$$

$$= \binom{n+(m+1)+1}{m+1} + (r-1)\sum_{j=0}^{m} r^{j}\binom{n+(m+1)-j}{n+1},$$

completing our proof.

Theorem 2: For each pair of integers $n, m \geqslant 0$ and $r > 0$, define

$$L(n, m; r) = \sum_{i=0}^{m} r^{m-i}\binom{n+i}{i} + \sum_{j=0}^{n} r^{j}\binom{n+m-j}{m}, \tag{3}$$

then $L(n, m; r)$ satisfies the following recursive form:

$$L(n+1, m+1; r) = L(n, m+1; r) + L(n+1, m; r)$$

and

$$L(0, n; r) = L(n, 0; r) = \sum_{j=0}^{n} r^{j} + 1.$$

Proof: By (3), we have

$$L(0, n; r) = L(n, 0; r) = \sum_{j=0}^{n} r^{j} + 1.$$

Using a dummy variable, we obtain

$$L(n, m; r) = \sum_{i=0}^{m} r^{m-i}\binom{n+i}{i} + \sum_{j=0}^{n} r^{n-j}\binom{m+j}{j}$$

or (4)

$$L(n, m; r) = \sum_{i=0}^{m} r^{i}\binom{n+m-i}{n} + \sum_{j=0}^{n} r^{j}\binom{n+m-j}{m}.$$

Since

$$\sum_{i=0}^{m+1} r^{i}\binom{n+m+1-i}{n} + \sum_{i=0}^{m} r^{i}\binom{n+m+1-i}{n+1}$$

$$= \sum_{i=0}^{m+1} r^{i}\left\{\binom{n+m+1-i}{n} + \binom{n+m+1-i}{n+1}\right\} = \sum_{i=0}^{m+1} r^{i}\binom{n+m+2-i}{n+1}$$

and

$$\sum_{j=0}^{n} r^{j}\binom{n+m+1-j}{m+1} + \sum_{j=0}^{n+1} r^{j}\binom{n+m+1-j}{m} = \sum_{j=0}^{n+1} r^{j}\binom{n+m+2-j}{m+1},$$

we have

$$L(n+1, m; r) + L(n, m+1; r)$$

$$= \left\{\sum_{i=0}^{m} r^{i}\binom{n+m+1-i}{n+1} + \sum_{j=0}^{n+1} r^{j}\binom{n+m+1-j}{m}\right\}$$

$$+ \left\{\sum_{i=0}^{m+1} r^{i}\binom{n+m+1-i}{n} + \sum_{j=0}^{n} r^{j}\binom{n+m+1-j}{m+1}\right\}$$

$$= \sum_{i=0}^{m-1} r^i \binom{n+m+2-i}{n+1} + \sum_{j=0}^{n+1} r^j \binom{n+m+2-j}{m+1} = L(n+1, m+1; r).$$

In fact, the reverse of this theorem is also true by the generating function method.

Theorem 3: For each pair of integers $n, m \geqslant 0$ and $r > 0$, we have

$$L(n, m; r) = \sum_{i=0}^{m} (r-1)^i \binom{n+m+1}{m-i} + \sum_{j=0}^{n} (r-1)^j \binom{n+m+1}{n-j}. \tag{5}$$

Proof: By (2) and the dummy variable, we have

$$\sum_{i=0}^{m} r^{m-i} \binom{n+i}{i} = \binom{n+m+1}{m} + (r-1) \sum_{i=0}^{m-1} r^i \binom{n+m-i}{n+1}$$

$$= \binom{n+m+1}{m} + (r-1) \sum_{j=0}^{m-1} r^{(m-1)-j} \binom{n+1+j}{j}.$$

Repeating the above procedure, we obtain

$$\sum_{i=0}^{m} r^{m-i} \binom{n+i}{i} = \sum_{i=0}^{m} (r-1)^i \binom{n+m+1}{m-i}, \tag{6}$$

completing our proof.

Corollary 1: For each pair of integers $n, m \geqslant 0$, the following identity holds:

$$\sum_{i=0}^{m} 2^{m-i} \binom{n+i}{i} = \sum_{i=0}^{m} \binom{n+m+1}{i}. \tag{7}$$

Proof: Taking $r = 2$ in (6), we have

$$\sum_{i=0}^{m} 2^{m-i} \binom{n+i}{i} = \sum_{i=0}^{m} \binom{n+m+1}{m-i} = \sum_{j=0}^{m} \binom{n+m+1}{j}, \text{ by } j = m - i.$$

Corollary 2: For each pair of integers $n, m \geqslant 0$, we have

$$L(n, m; 2) = 2^{n+m+1}. \tag{8}$$

Proof: $L(n, m; 2) = \sum_{i=0}^{m} 2^{m-i} \binom{n+i}{i} + \sum_{j=0}^{m} 2^j \binom{n+m-j}{m}$

$$= \sum_{i=0}^{m} 2^{m-i} \binom{n+i}{i} + \sum_{j=0}^{n} 2^{n-j} \binom{m+j}{j}$$

$$= \sum_{i=0}^{m} \binom{n+m+1}{i} + \sum_{j=0}^{n} \binom{n+m+1}{j}$$

$$= \sum_{i=0}^{m} \binom{n+m+1}{i} + \sum_{k=m+1}^{n+m+1} \binom{n+m+1}{k} = 2^{n+m+1}.$$

Dividing identity (8) by 2^{n+m}, we obtain (1).

3. EXAMPLES

Example 1: Take $r = 3$. We have the values of $L(n, m; 3)$ as follows:

$m \backslash n$	0	1	2	3	4	5	6	7
0	2	5	14	41	122	365	1094	3281
1	5	10	24	65	187	552	1646	4927
2	14	24	48	113	300	852	2498	7425
3	41	65	113	226	526	1378	3876	11301
4	122	187	300	526	1052	2430	6306	17607
5	365	552	852	1378	2430	4860	11166	28773
6	1094	1646	2498	3876	6306	11166	22332	51105
7	3281	4927	7425	11301	17607	28773	51105	102210

Example 2: Take $r = 4$. We obtain the values of $L(n, m; 4)$ as follows:

$m \backslash n$	0	1	2	3	4	5	6	7
0	2	6	22	86	342	1366	5462	21846
1	6	12	34	120	462	1828	7290	29136
2	22	34	68	188	650	2478	9768	38904
3	86	120	188	376	1026	3504	13272	52176
4	342	462	650	1026	2052	5556	18828	71004
5	1366	1828	2478	3504	5556	11112	29940	100944
6	5462	7290	9768	13272	18828	29940	59880	160824
7	21846	29136	38904	52176	71004	100944	160824	321648

Example 3: Take $r = 5$. We have the values of $L(n, m; 5)$ as follows:

$m \backslash n$	0	1	2	3	4	5	6	7
0	2	7	32	157	782	3907	19532	97657
1	7	14	46	203	985	4892	24424	122081
2	32	46	92	295	1280	6172	30596	152677
3	157	203	295	590	1870	8042	38638	191315
4	782	985	1280	1870	3740	11782	50420	241735
5	3907	4892	6172	8024	11782	23564	73984	315719
6	19532	24424	30596	38638	50420	73984	147968	463687
7	97657	122081	152677	191315	241735	315719	463687	927374

ACKNOWLEDGMENTS

We would like to thank Professor Horng-Jinh Chang for his helpful comments and the referee for his thorough discussions.

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ANALOGS OF SMITH'S DETERMINANT*

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(Submitted December 1985)

Over a century ago, according to Dickson [1], H. J. S. Smith [3] showed that

$$\begin{vmatrix} (1, 1) & \cdots & (1, j) & \cdots & (1, n) \\ \vdots & & \vdots & & \vdots \\ (i, 1) & \cdots & (i, j) & \cdots & (i, n) \\ \vdots & & \vdots & & \vdots \\ (n, 1) & \cdots & (n, j) & \cdots & (n, n) \end{vmatrix} = \varphi(1)\ \varphi(2)\ \ldots\ \varphi(n),$$

where (i, j) is the greatest common divisor of i and j, and φ is Euler's function. P. Mansion [2] proved a generalization of Smith's result: If

$$f(m) = \sum_{d|m} g(d),$$

and we write $f(i, j)$ for $f(\gcd(i, j))$, then

$$\begin{vmatrix} f(1, 1) & \cdots & f(1, j) & \cdots & f(1, n) \\ \vdots & & \vdots & & \vdots \\ f(i, 1) & & f(i, j) & & f(i, n) \\ \vdots & & \vdots & & \vdots \\ f(n, i) & \cdots & f(n, j) & \cdots & f(n, n) \end{vmatrix} = g(1)\ g(2)\ \ldots\ g(n).$$

Note that Mansion's result becomes Smith's when $f(m) = m$, because

$$m = \sum_{d|m} \varphi(d).$$

In this paper, we present an extension of Mansion's result to a wide class of arithmetic convolutions.

Suppose $S(m)$ defines some set of divisors of m for each m. If $d|m$, we say that d is an S-divisor of m if $d \in S(m)$. We will denote by $(i, j)_S$ the largest common S-divisor of i and j.

Now m might or might not be an element of $S(m)$, as can be seen if we let $S(m)$ be the largest squarefree divisor of m. Also, the property

$$d \in S(i) \cap S(j) \text{ if and only if } d \in S((i, j)_S)$$

might or might not be true. It is true if $S(m)$ consists of all the divisors of

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m, but not if $S(m)$ consists of all divisors d of m for which $(d, m/d) > 1$, for then 6 is the largest common S-divisor of 12 and 24, and 2 is an S-divisor of 12 and 24, but not of 6.

We come now to the promised generalization:

Theorem: Let $S(m)$ and $(i, j)_S$ be defined as above. If

(1) $m \in S(m)$ for each m,

(2) $d \in S(i) \cap S(j)$ if and only if $d \in S((i, j)_S)$, and

(3) $f(m) = \sum_{d \in S(m)} g(d)$,

then

$$\begin{vmatrix} f((1, 1)_S) & \cdots & f((1, j)_S) & \cdots & f((1, n)_S) \\ \vdots & & \vdots & & \vdots \\ f((i, 1)_S) & \cdots & f((i, j)_S) & \cdots & f((i, n)_S) \\ \vdots & & \vdots & & \vdots \\ f((n, 1)_S) & \cdots & f((n, j)_S) & \cdots & f((n, n)_S) \end{vmatrix} = g(1) \ \cdots \ g(n).$$

Proof: Assume the hypotheses, and define

$$S(a, b) = \begin{cases} 1 & \text{if } b \in S(a), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $S(a, b) = 0$ if $b > a$, and by (1) we have $S(a, a) = 1$ for each a. Now, $S(i, d)S(j, d)$ is 0 unless d is an S-divisor of both i and j, in which case the product is 1, and by (2) and (3) it is easy to see that

$$f((i, j)_S) = S(i, 1)S(j, 1)g(1) + S(i, 2)S(j, 2)g(2) + \cdots + S(i, n)S(j, n)g(n)$$

for each i and j. Then

$$[f((i, j)_S)] = A \cdot B,$$

where

$$A = \begin{bmatrix} S(1, 1) & S(1, 2) & \cdots & S(1, i) & \cdots & S(1, n) \\ S(2, 1) & S(2, 2) & \cdots & S(2, i) & \cdots & S(2, n) \\ \vdots & \vdots & & \vdots & & \vdots \\ S(i, 1) & S(i, 2) & \cdots & S(i, i) & \cdots & S(i, n) \\ \vdots & \vdots & & \vdots & & \vdots \\ S(n, 1) & S(n, 2) & \cdots & S(n, i) & \cdots & S(n, n) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ S(2, 1) & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ S(i, 1) & S(i, 2) & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ S(n, 1) & S(n, 2) & \cdots & S(n, i) & \cdots & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} S(1, 1)g(1) & S(2, 1)g(1) & \ldots & S(j, 1)g(1) & \ldots & S(n, 1)g(1) \\ S(1, 2)g(2) & S(2, 2)g(2) & \ldots & S(j, 2)g(2) & \ldots & S(n, 2)g(2) \\ \vdots & \vdots & & \vdots & & \vdots \\ S(1, j)g(j) & S(2, j)g(j) & \ldots & S(j, j)g(j) & \ldots & S(n, j)g(j) \\ \vdots & \vdots & & \vdots & & \vdots \\ S(1, n)g(n) & S(2, n)g(n) & \ldots & S(j, n)g(n) & \ldots & S(n, n)g(n) \end{bmatrix}$$

$$= \begin{bmatrix} g(1) & S(2, 1)g(1) & \ldots & S(j, 1)g(1) & \ldots & S(n, 1)g(1) \\ 0 & g(2) & \ldots & S(j, 2)g(2) & \ldots & S(n, 2)g(2) \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \ldots & g(j) & \ldots & S(n, j)g(j) \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & g(n) \end{bmatrix}$$

The theorem then follows from the observations

$$\det A = 1 \quad \text{and} \quad \det B = g(1)\, g(2)\, \ldots\, g(n). \quad \blacksquare$$

In particular, if $S(m)$ consists of all divisors of m, the theorem yields Mansion's result. Another special case of some interest arises if we let $S(m)$ consist of the unitary divisors of m: We say that d is a unitary divisor of m if gcd $(d, m/d) = 1$. Let $(i, j)^*$ be the largest common unitary divisor of i and j. Also, let $\tau^*(m)$ and $\sigma^*(m)$ be the number and sum, respectively, of the unitary divisors of m. Then $g(d) = 1$ and $g(d) = d$, respectively, yield

$$|\tau^*((i, j)^*)| = 1 \quad \text{and} \quad |\sigma^*((i, j)^*)| = n!$$

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GENERALIZED STIRLING NUMBER PAIRS ASSOCIATED WITH INVERSE RELATIONS

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(Submitted December 1985)

1. INTRODUCTION

Stirling numbers and some of their generalizations have been investigated Intensively during the past several decades. Useful references for various results may be found in [1], [2, ch. 5], [3], [6], [7], etc.

The main object of this note is to show that the concept of a generalized Stirling number pair can be characterized by a pair of inverse relations. Our basic idea is suggested by the well-known inverse relations as stated explicitly in Riordan's classic book [7], namely

$$a_n = \sum_{k=0}^{n} S_1(n, k) b_k, \qquad b_n = \sum_{k=0}^{n} S_2(n, k) a_k,$$

where $S_1(n, k)$ and $S_2(n, k)$ are Stirling numbers of the first and second kind, respectively. Recall that $S_1(n, k)$ and $S_2(n, k)$ may be defined by the exponential generating functions

$$(\log(1 + t))^k/k! \quad \text{and} \quad (e^t - 1)^k/k!,$$

respectively, where

$$f(t) = \log(1 + t) \quad \text{and} \quad g(t) = e^t - 1$$

are just reciprocal functions of each other, namely $f(g(t)) = g(f(t)) = t$ with $f(0) = g(0) = 0$. What we wish to elaborate is a comprehensive generalization of the known relations mentioned above.

2. A BASIC DEFINITION AND A THEOREM

Denote by $\Gamma \equiv (\Gamma, +, \cdot)$ the commutative ring of formal power series with real or complex coefficients, in which the ordinary addition and Cauchy multiplication are defined. Substitution of formal power series is defined as usual (cf. Comtet [2]).

Two elements f and g of Γ are said to be reciprocal (inverse) of each other if and only if $f(g(t)) = g(f(t)) = t$ with $f(0) = g(0) = 0$.

Definition: Let f and g belong to Γ, and let

$$\frac{1}{k!}(f(t))^k = \sum_{n \geqslant 0} A_1(n, k)\frac{t^n}{n!}, \tag{2.1}$$

$$\frac{1}{k!}(g(t))^k = \sum_{n \geqslant 0} A_2(n, k)\frac{t^n}{n!}. \tag{2.2}$$

Then $A_1(n, k)$ and $A_2(n, k)$ are called a generalized Stirling number pair, or a GSN pair if and only if f and g are reciprocal of each other.

From (2.1) and (2.2), one may see that every GSN pair has the property

$$A_1(n, k) = A_2(n, k) = 0 \text{ for } n < k.$$

Moreover, one may define

$$A_1(0, 0) = A_2(0, 0) = 1.$$

Let us now state and prove the following:

Theorem: Numbers $A_1(n, k)$ and $A_2(n, k)$ defined by (2.1) and (2.2) just form a GSN pair when and only when there hold the inverse relations

$$a_n = \sum_{k=0}^{n} A_1(n, k)b_k, \qquad b_n = \sum_{k=0}^{n} A_2(n, k)a_k, \tag{2.3}$$

where $n = 0, 1, 2, \ldots,$ and either $\{a_k\}$ or $\{b_k\}$ is given arbitrarily.

Proof: We have to show that $(2.3) \Longleftrightarrow f(g(t)) = g(f(t)) = t$ with $f(0) = g(0) = 0$. As may easily be verified, the necessary and sufficient condition for (2.3) to hold is that the orthogonality relations

$$\sum_{n \geq 0} A_1(m, n)A_2(n, k) = \sum_{n \geq 0} A_2(m, n)A_1(n, k) = \delta_{mk}, \tag{2.4}$$

hold, where δ_{mk} is the Kronecker symbol. Clearly, both summations contained in (2.4) consist of only a finite number of terms inasmuch as

$$A_1(m, n) = A_2(m, n) = 0 \text{ for } n > m.$$

Let us prove $\Longrightarrow$. Since (2.4) is now valid, we may substitute (2.1) into (2.2), and by the rule of function composition we obtain

$$\frac{1}{k!}(g(f(t)))^k = \sum_{n \geq 0} A_2(n, k) \sum_{m \geq 0} A_1(m, n)\frac{t^m}{m!}$$
$$= \sum_{m \geq 0} \frac{t^m}{m!}\left(\sum_{n \geq 0} A_1(m, n)A_2(n, k)\right) = \sum_{m \geq 0} \frac{t^m}{m!}\delta_{mk} = \frac{t^k}{k!}.$$

Thus, it follows that $g(f(t)) = t$. Similarly, we have $f(g(t)) = t$. This proves $\Longrightarrow$.

To prove $\Longleftarrow$, suppose that $f(g(t)) = g(f(t)) = t$, $f(0) = g(0) = 0$. Substituting (2.2) into (2.1), we obtain

$$\frac{1}{k!}t^k = \frac{1}{k!}(f(g(t)))^k = \sum_{m \geq 0} \frac{t^m}{m!}\left(\sum_{n \geq 0} A_2(m, n)A_1(n, k)\right).$$

Comparing the coefficients of t on both sides, we get

$$\sum_{n \geq 0} A_2(m, n)A_1(n, k) = \delta_{mk}.$$

In a similar manner, the first equation contained in (2.4) can be deduced. Recalling that (2.4) is precisely equivalent to (2.3), the inverse implication $\Longleftarrow$ is also verified; hence, the theorem.

Evidently, the theorem just proved may be restated as follows:

Equivalence Proposition: The following three assertions are equivalent to each other.

(i) $\{A_1(n, k), A_2(n, k)\}$ is a GSN pair.

(ii) Inverse relations (2.3) hold.

(iii) $\{f, g\}$ is a pair of reciprocal functions of Γ.

3. EXAMPLES AND REMARKS

Examples: Some special GSN pairs may be displayed as shown below.

$f(t)$	$g(t)$	$A_1(n, k)$	$A_2(n, k)$
$\log(1 + t)$	$e^t - 1$	$S_1(n, k)$	$S_2(n, k)$
$\tan t$	$\arctan t$	$T_1(n, k)$	$T_2(n, k)$
$\sin t$	$\arcsin t$	$\bar{s}_1(n, k)$	$\bar{s}_2(n, k)$
$\sinh t$	$\operatorname{arc\,sinh} t$	$\sigma_1(n, k)$	$\sigma_2(n, k)$
$\tanh t$	$\operatorname{arc\,tanh} t$	$\tau_1(n, k)$	$\tau_2(n, k)$
$t/(t - 1)$	$t/(t - 1)$	$(-1)^{n-k}L(n, k)$	$(-1)^{n-k}L(n, k)$

Note that $L(n, k)$ is known as Lah's number, which has the expression

$$L(n, k) = (-1)^n \frac{n!}{k!}\binom{n - 1}{k - 1}.$$

In what follows, we will give a few brief remarks that follow easily from the ordinary theory about exponential generating functions.

Remark 1: For a pair of reciprocal elements $f, g \in \Gamma$, write:

$$f(t) = \sum_1^\infty \alpha_k t^k/k!, \qquad g(t) = \sum_1^\infty \beta_k t^k/k! \tag{3.1}$$

Making use of the definition of Bell polynomials (cf. Riordan [7]),

$$Y_n(gf_1, \ldots, gf_n) = \sum_{(J)} \frac{n!g_k}{j_1! \cdots j_n!}\left(\frac{f_1}{1!}\right)^{j_1} \cdots \left(\frac{f_n}{n!}\right)^{j_n},$$

where (J) indicates the summation condition $j_1 + \cdots + j_n = k$, $1j_1 + 2j_2 + \cdots + nj_n = n$, $k = 1, 2, \ldots, n$, one may obtain

$$A_1(n, k) = Y_n(f\alpha_1, \ldots, f\alpha_n), \qquad A_2(n, k) = Y_n(f\beta_1, \ldots, f\beta_n),$$

where $f_i = \delta_{ki}$ $(i = 1, \ldots, n)$ and δ_{ki} is the Kronecker symbol. Consequently, certain combinatorial probabilistic interpretation may be given of $A_i(n, k)$ $(i = 1, 2)$. Moreover, for any given $\{\alpha_k\}$, the sequence $\{\beta_k\}$ can be determined by the system of linear equations

$$Y_n(\beta\alpha_1, \ldots, \beta\alpha_n) = \delta_{n1} \qquad (n = 1, 2, \ldots). \tag{3.2}$$

Remark 2: It is easy to write down double generating functions for $A_i(n, k)$, viz.,

$$\Phi(t, u) := \sum_{n, k \geqslant 0} A_1(n, k)\frac{t^n u^k}{n!} = \exp[uf(t)],$$

$$\Psi(t, u) := \sum_{n, k \geq 0} A_2(n, k)\frac{t^n u^k}{n!} = \exp[ug(t)].$$

Moreover, for each $A_i(n, k)$ $(i = 1, 2)$, we have the convolution formula

$$\binom{k_1 + k_2}{k_1} A_i(n, k_1 + k_2) = \sum_{j=0}^{n} \binom{n}{j} A_i(j, k_1) A_i(n - j, k_2), \tag{3.3}$$

and, consequently, there is a vertical recurrence relation for $A_i(n, k)$, viz.,

$$kA_i(n, k) = \sum_{j=0}^{n-1} \binom{n}{j} A_i(j, k - 1) A_i(n - j, 1), \tag{3.4}$$

where $A_1(j, 1) = \alpha_j$ and $A_2(j, 1) = \beta_j$. A similar recurrence relation takes the form

$$A_i(n + 1, k) = \sum_{j=0}^{n} \binom{n}{j} A_i(j, k - 1) A_i(n - j + 1, 1). \tag{3.5}$$

However, we have not yet found any useful horizontal recurrence relations for $A_i(n, k)$ $(i = 1, 2)$. Also unsolved are the following:

Problems: How to determine some general asymptotic expansions for $A_i(n, k)$ as $k \to \infty$ with $k = o(n)$ or $k = O(n)$? Is it true that the asymptotic normality of $A_1(n, k)$ implies that of $A_2(n, k)$? Is it possible to extend the concept of a GSN pair to a case involving multiparameters?

4. A CONTINUOUS ANALOGUE

We are now going to extend, in a similar manner, the reciprocity of the relations (2.3) to the case of reciprocal integral transforms so that a kind of GSN pair containing continuous parameters can be introduced.

Let $\phi(x)$ and $\psi(x)$ be real-valued reciprocal functions decreasing on $[0, 1]$ with $\phi(0) = \psi(0) = 1$ and $\phi(1) = \psi(1) = 0$, such that

$$\phi(\psi(x)) = \psi(\phi(x)) = x \qquad (0 \leq x \leq 1).$$

Moreover, $\phi(x)$ and $\psi(x)$ are assumed to be infinitely differentiable in $(0, 1)$. Introduce the substitution $x = e^{-t}$, so that we may write

$$e^{-u} = \phi(e^{-t}), \quad e^{-t} = \psi(e^{-u}), \quad t, u \in [0, \infty). \tag{4.1}$$

For given measurable functions $f(s) \in L(0, \infty)$, consider the integral equation

$$F(u) := \int_0^\infty f(s) e^{-us}\, ds = \int_0^\infty g(s)(\psi(e^{-u})^s)\, ds, \tag{4.2}$$

where $g(s)$ is to be determined. Evidently, (4.2) is equivalent to the following:

$$G(t) := \int_0^\infty f(s)(\phi(e^{-t}))^s\, ds = \int_0^\infty g(s) e^{-ts}\, ds. \tag{4.3}$$

Denote $G(t) = F(u) = F(-\log \phi(e^{-t}))$. Suppose that $G(t)$ satisfies the Widder condition D (cf. [8], ch. 7, §6, §17):

(i) $G(t)$ is infinitely differentiable in $(0, \infty)$ with $G(\infty) = 0$.

(ii) For every integer $m \geq 1$, $L_{m,x}[G] \equiv L_{m,x}[G(\cdot)]$ is Lebesgue integrable on $(0, \infty)$, where $L_{m,x}[G]$ is the Post-Widder operator defined by

$$L_{m,x}[G] := \frac{(-1)^m}{m!}\left(\frac{m}{x}\right)^{m+1}\left(\frac{d}{dt}\right)^m G(t)\bigg|_{t=(m/x)}. \tag{4.4}$$

(iii) The sequence $\{L_{m,x}[G]\}$ converges in mean of index unity, namely

$$\lim_{m,n\to\infty}\int_0^\infty |L_{m,x}[G] - L_{n,x}[G]|dx = 0.$$

Then by the representation theorem of Widder (cf. [8], Theorem 17, p. 318) one may assert the existence of $g(s) \in L(0, \infty)$ such that (4.3) holds. Consequently, the well-known inversion theorem of Post-Widder (*loc. cit.*) is applicable to both (4.3) and (4.2), yielding

$$g(x) = \lim_{m\to\infty}\int_0^\infty f(s)L_{m,x}[(\phi(e^{-(\cdot)}))^s]ds, \tag{4.5}$$

$$f(x) = \lim_{m\to\infty}\int_0^\infty g(s)L_{m,x}[(\psi(e^{-(\cdot)}))^s]ds, \tag{4.6}$$

whenever $x > 0$ belongs to the Lebesgue sets of g and f, respectively.

In fact, the reciprocity (4.5) $\Longleftrightarrow$ (4.6) so obtained is just a generalization of the inverse relations for self-reciprocal integral transforms (in the case $\phi \equiv \psi$) discussed previously (cf. [4], Theorem 8).

Notice that $A_i(n, k)$ $(i = 1, 2)$ may be expressed by using formal derivatives:

$$A_1(n, k) = \frac{1}{k!}\left(\frac{d}{dt}\right)^n (f(t))^k\bigg|_{t=0}, \qquad A_2(n, k) = \frac{1}{k!}\left(\frac{d}{dt}\right)^n (g(t))^k\bigg|_{t=0}.$$

Thus, recalling (4.4) and comparing (4.5) and (4.6) with (2.3), it seems to be reasonable to consider the following two sequences of numbers:

$$A_1^*(x, y; m) = L_{m,x}[(\phi(e^{-(\cdot)}))^y],$$
$$A_2^*(x, y; m) = L_{m,x}[(\psi(e^{-(\cdot)}))^y] \qquad (m = 1, 2, \ldots),$$

as a kind of GSN pair involving continuous parameters $x, y \in (0, \infty)$.

In conclusion, all we have shown is that the continuous analogue of the concept for a GSN pair is naturally connected to a general class of reciprocal integral transforms. Surely, special reciprocal functions $\phi(x)$ and $\psi(x)$ $(0 \leqslant x \leqslant 1)$ may be found—as many as one likes. For instance, if one takes

$$\phi_1(x) = 1 - x, \quad \phi_2(x) = \cos\frac{\pi x}{2}, \quad \phi_3(x) = \log(e - (e - 1)x),$$

their corresponding inverse functions are given by

$$\psi_1(x) = 1 - x, \quad \psi_2(x) = \frac{2}{\pi}\text{ arc cos } x, \quad \psi_3(x) = (e - e^x)/(e - 1),$$

respectively. Monotone and boundary conditions

$$\phi_i(0) = \psi_i(0) = 1 \quad\text{and}\quad \phi_i(1) = \psi_i(1) = 0 \qquad (i = 1, 2, 3)$$

are obviously satisfied.

ACKNOWLEDGMENT

The author wishes to thank the referee for giving useful suggestions that led to an improvement and revision of this note.

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A CURIOUS SET OF NUMBERS

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(Submitted February 1986)

1. INTRODUCTION

Recently, these two authors proved a theorem involving necessary and sufficient conditions on when a real ordinary differential expression can be made formally self-adjoint [1]. A differential expression

$$L(y) = \sum_{k=0}^{r} a_k(x)y^{(k)}(x)$$

is said to be *symmetric* or *formally self-adjoint* if $L(y) = L^+(y)$, where L^+ is the Lagrange adjoint of L defined by

$$L^+(y) = \sum_{k=0}^{r} (-1)^k (a_k(x)y(x))^{(k)}.$$

It is easy to see that if $L = L^+$ then it is necessary that r be even. If $L(y)$ is a differential expression and $f(x)$ is a function such that $f(x)L(y)$ is symmetric, then $f(x)$ is called a symmetry factor for $L(y)$. In [2], Littlejohn proved the following theorem.

Theorem: Suppose $a_k(x) \in C^k(I)$, $a_k(x)$ is real valued, $k = 0, 1, \ldots, 2n$, $a_{2n}(x) \neq 0$, where I is some interval of the real line. Then there exists a symmetry factor $f(x)$ for the expression

$$L(y) = \sum_{k=0}^{2n} a_k(x)y^{(k)}(x)$$

if and only if $f(x)$ simultaneously satisfies the n differential equations

$$\sum_{s=k}^{n} \sum_{j=0}^{2s-2k+1} \binom{2s}{2k-1}\binom{2s-2k+1}{j} \frac{2^{2s-2k+2}-1}{s-k+1} B_{2s-2k+2}a_{2s}^{(2s-2k+1-j)}f^{(j)} - a_{2k-1}f = 0, \tag{1}$$

$k = 1, 2, \ldots, n$, where B_{2i} is the Bernoulli number defined by

$$\frac{x}{e^x - 1} = 2 - \frac{x}{2} + \sum_{k=1}^{\infty} \frac{B_{2i}x^{2i}}{(2i)!}. \quad \blacksquare$$

However, these two authors have significantly improved the n equations that the symmetry factor must satisfy [1]. Directly from the definition of symmetry it is easy to see that $f(x)$ is a symmetry factor for (1) if and only if $A_{k+1} = 0$, $k = 0, 1, \ldots (2n - 1)$, where

$$A_{k+1} = \sum_{j=0}^{2n-k} (-1)^{k+j}\binom{k+j}{j}(f(x)a_{k+j}(x))^{(j)} - f(x)a_k(x).$$

Littlejohn & Krall show that $A_{k+1} = 0$, $k = 0, 1, \ldots (2n - 1)$ if and only if

$$C_{k+1} = \sum_{i=2k+1}^{2n} (-1)^i \binom{i-k-1}{k} (a_i(x)f(x))^{(i-2k-1)} = 0, \tag{2}$$

$k = 0, 1, \ldots (n - 1)$. If we express the C_k's in terms of the A_k's, we see that:

$$C_1' = A_1,$$

$$C_2'' + 2C_1 = A_2,$$

and, for $3 \leqslant k \leqslant 2n - 1$,

$$1C_k + kC_{k-1}^{(k-2)} + \sum_{j=3}^{\left[\frac{k+3}{2}\right]} \frac{k(k-j)(k-j-1)(k-j-2)\ldots(k-2j+3)}{(j-1)!} C_{k-j+1}^{(k-2j+2)} = A_k,$$

where $C_k = 0$ if $k > n$ and $[\cdot]$ denotes the greatest integer function.

From the coefficients of these equations, we get the following array:

1st row	1	0						
2nd row	1	2						
3rd row	1	3	0					
4th row	1	4	2					
5th row	1	5	5	0				
6th row	1	6	9	2				
7th row	1	7	14	7	0			
8th row	1	8	20	16	2			
9th row	1	9	27	30	9	0		
10th row	1	10	35	50	25	2		
11th row	1	11	44	77	55	11	0	
12th row	1	12	54	112	105	36	2	
13th row	1	13	65	156	182	91	13	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

This array has many interesting properties, some of which we shall discuss in this note.

2. PROPERTIES OF THE ARRAY

If we add all of the entries in each row, we arrive at the sequence

1, 3, 4, 7, 11, 18, 29, 47, 76, ...

A Fibonacci sequence! (Actually, this sequence is called the Lucas sequence.) From this, we can desily derive

Theorem 1: For $n \geqslant 3$,

$$1 + n + \sum_{j=3}^{\left[\frac{n+3}{2}\right]} \frac{n(n-j)(n-j-1)(n-j-2)\ldots(n-2j+3)}{(j-1)!}$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \quad ■$$

For $n \geqslant 3$ and $j \geqslant 3$, the number

$$A_{n,j} = \frac{n(n-j)(n-j-1)\ldots(n-2j+3)}{(j-1)!}$$

is the entry in the n^{th} row and j^{th} column. Alternatively, $A_{n,j}$ is the r^{th} element in the j^{th} column where $r = n - 2j + 4$. We now show how to obtain any element in the j^{th} column by looking at the $(j - 1)^{\text{st}}$ column. Consider, for example, $A_{11,4} = 77$, which is the *seventh entry* in the fourth column. Observe that we can also obtain 77 by adding the first *seven* entries in the third column:

$$77 = 0 + 2 + 5 + 9 + 14 + 20 + 27.$$

As another example, $A_{13,6} = 91$, the *fifth* number in the sixth column can also be obtained by adding the first *five* numbers in the fifth column:

$$91 = 0 + 2 + 9 + 25 + 55.$$

From this, we get

Theorem 2: For $n \geqslant 4$ and $3 \leqslant j \leqslant \left[\frac{n+3}{2}\right]$,

$$\sum_{i=2j-4}^{n-2} i(i-j+1)(i-j)\ldots(i-2j+5) = \frac{n(n-j)(n-j-1)\ldots(n-2j+3)}{j-1}. \blacksquare$$

Of course, this process can also be reversed; that is, we can obtain the entries in the $(j - 1)^{\text{st}}$ column by looking at the j^{th} column. More specifically, by taking differences of successive elements in the j^{th} column, we obtain the entries in the $(j - 1)^{\text{st}}$ column. The reason for this is the identity

$$A_{n,j} - A_{n-1,j} = A_{n-2,j-1}.$$

There are probably many other patterns appearing in this array; we list a few more:

$$1 + n + A_{n+1,3} + A_{n+2,4} + A_{n+3,5} + \cdots + A_{2n-1,n+1}$$
$$= 3 \cdot 2^{n-2},\ n \geqslant 2, \tag{3}$$
$$A_{2n,n} = n^2. \tag{4}$$

How many new patterns can you find?

The first set of necessary and sufficient conditions for the existence of a symmetry factor [i.e., equation (1)] involve the Bernoulli numbers. We have shown that the second set of conditions [equation (2)], which are equivalent to the first set, involve the Fibonacci numbers. What is the connection between these two sets of numbers?

ACKNOWLEDGMENTS

The authors wish to thank the referee for many helpful suggestions as well as the referee's student who found a few more patterns in the array. If we add the entries on the (main) diagonals, we obtain the sequence

$$1, 1, 3, 4, 5, 8, 12, 17, 25, 37, 54, 79, \ldots$$

or $a_n = a_{n-1} + a_{n-3}$, $n \geqslant 4$. Another interesting pattern is the following:

$$A_{n,j} = \sum_{k=1}^{j} A_{n-2j+2k-1,k}, \tag{5}$$

where we adopt the notation that $A_{0,1} = 1$, $A_{n,k} = 0$ when $n < 0$ and $A_{n,k} = 0$ if

$$k > \left[\frac{n+2}{2}\right].$$

For example,

$$A_{11,5} = A_{2,1} + A_{4,2} + A_{6,3} + A_{8,4} + A_{10,5} = 1 + 4 + 9 + 16 + 25 = 55.$$

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ON SOME PROPERTIES OF HUMBERT'S POLYNOMIALS

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(Submitted February 1986)

1. INTRODUCTION

In 1921, Humbert [8] defined a class of polynomials $\{\Pi_{n,m}^{\lambda}\}_{n=0}^{\infty}$ by the generating function

$$(1 - mxt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} \Pi_{n,m}^{\lambda}(x)t^n. \tag{1}$$

These satisfy the recurrence relation

$$(n + 1)\Pi_{n+1,m}^{\lambda}(x) - mx(n + \lambda)\Pi_{n,m}^{\lambda}(x) - (n + m\lambda - m + 1)\Pi_{n-m+1,m}^{\lambda}(x) = 0.$$

Particular cases of these polynomials are Gegenbauer polynomials [1]

$$C_n^{\lambda}(x) = \Pi_{n,2}^{\lambda}(x)$$

and Pincherle polynomials (see [8])

$$\mathcal{P}_n(x) = \Pi_{n,3}^{-1/2}(x).$$

Later, Gould [2] studied a class of generalized Humbert polynomials

$$P_n(m, x, y, p, C)$$

defined by

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)t^n, \tag{2}$$

where $m \geqslant 1$ is an integer and the other parameters are unrestricted in general. The recurrence relation for the generalized Humbert polynomials is

$$CnP_n - m(n - 1 - p)xP_{n-1} + (n - m - mp)yP_{n-m} = 0, \quad n \geqslant m \geqslant 1, \tag{3}$$

where we put $P_n = P_n(m, x, y, p, C)$.

In [6], Horadam and Pethe investigated the polynomials associated with the Gegenbauer polynomials

$$C^{\lambda}(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(\lambda)_{n-k}}{k!(n - 2k)!}(2x)^{n-2k}, \tag{4}$$

where $(\lambda)_0 = 1$, $(\lambda)_n = \lambda(\lambda + 1) \ldots (\lambda + n - 1)$, $n = 1, 2, \ldots$. Listing the polynomials of (4) horizontally and taking sums along the rising diagonals, Horadam and Pethe obtained the polynomials denoted by $p_n^{\lambda}(x)$. For these polynomials, they proved that the generating function $G^{\lambda}(x, t)$ is given by

$$G^{\lambda}(x, t) = \sum_{n=1}^{\infty} p_n^{\lambda}(x)t^{n-1} = (1 - 2xt + t^3)^{-\lambda}. \tag{5}$$

Some special cases of these polynomials were considered in several papers (see [3], [4], and [7], for example).

Comparing (5) to (1), we see that their polynomials are Humbert polynomials for $m = 3$, with x replaced by $2x/3$, i.e., $p_{n+1}^{\lambda}(x) = \Pi_{n,3}^{\lambda}(2x/3)$.

2. THE POLYNOMIALS $p_{n,m}^{\lambda}(x)$

In this paper, we consider the polynomials $\{p_{n,m}^{\lambda}\}_{n=0}^{\infty}$ defined by

$$p_{n,m}^{\lambda}(x) = \Pi_{n,m}^{\lambda}(2x/m).$$

Their generating function is given by

$$G_m^{\lambda}(x, t) = (1 - 2xt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} p_{n,m}^{\lambda}(x)t^n. \tag{6}$$

Note that

$$p_{n,2}^{\lambda}(x) = C_n^{\lambda}(x) \qquad \text{(Gegenbauer polynomials)}$$

and

$$p_{n,3}^{\lambda}(x) = p_{n+1}^{\lambda}(x) \qquad \text{(Horadam-Pethe polynomials).}$$

For $m = 1$, we have

$$G_1^{\lambda}(x, t) = (1 - (2x - 1)t)^{-\lambda} = \sum_{n=0}^{\infty} p_{n,1}^{\lambda}(x)t^n$$

and

$$p_{n,1}^{\lambda}(x) = (-1)^n \binom{-\lambda}{n}(2x - 1)^n = \frac{(\lambda)_n}{n!}(2x - 1)^n.$$

These polynomials can be obtained from descending diagonals in the Pascal-type array for Gegenbauer polynomials (see Horadam [5]).

Expanding the left-hand side of (6), we obtain the explicit formula

$$p_{n,m}^{\lambda}(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{(\lambda)_{n-(m-1)k}}{k!(n - mk)!}(2x)^{n-mk}. \tag{7}$$

These polynomials can be obtained from (2) by putting $C = y = 1$, $p = -\lambda$, and $x := 2x/m$. Then we have

$$p_{n,m}^{\lambda}(x) = P_n(m, 2x/m, 1, -\lambda, 1).$$

Also, if we put $C = y = m/2$ and $p = -\lambda$, we obtain

$$p_{n,m}^{\lambda}(x) = \left(\frac{2}{m}\right)^{\lambda} P_n(m, x, m/2, -\lambda, m/2).$$

Then, from (3), we get the following recurrence relation

$$np_{n,m}^{\lambda}(x) = (\lambda + n - 1)2xp_{n-1,m}(x) - (n + m(\lambda - 1))p_{n-m,m}(x), \tag{8}$$

for $n \geqslant m \geqslant 1$.

The starting polynomials are

$$p_{n,m}^{\lambda}(x) = \frac{(\lambda)_n}{n!}(2x)^n, \; n = 0, 1, \ldots, m - 1.$$

Remark: For corresponding monic polynomials $\hat{p}^{\lambda}_{n,m}$, we have

$$\hat{p}^{\lambda}_{n,m}(x) = x\hat{p}^{\lambda}_{n-1,m}(x) - b_n\hat{p}^{\lambda}_{n-m,m}(x), \; n \geqslant m \geqslant 1,$$

$$\hat{p}^{\lambda}_{n,m}(x) = x^n, \; 0 \leqslant n \leqslant m - 1,$$

where

$$b_n = \frac{(n-1)!}{(m-1)!} \cdot \frac{n + m(\lambda - 1)}{2^m(\lambda + n - m)_m}.$$

The classes of polynomials $\mathbb{P}_{m,\lambda} = \{p^{\lambda}_{n,m}\}^{\infty}_{n=0}$, $m = 2, 3, \ldots$, can be found by repeating the "diagonal functions process," starting from $p^{\lambda}_{n,1}(x)$. Listing the terms of polynomials horizontally,

$$p^{\lambda}_{n,m}(x) = \sum_{k=0}^{[n/m]} a^{\lambda}_{n,m}(k)(2x)^{n-mk}, \quad a^{\lambda}_{n,m}(k) = \frac{(-1)^k(\lambda)_{n-(m-1)k}}{k!(n-mk)!},$$

and taking sums along the rising diagonals, we obtain $p^{\lambda}_{n,m+1}(x)$, because

$$a^{\lambda}_{n-k,m}(k) = (-1)^k \frac{(\lambda)_{n-k-(m-1)k}}{k!(n-k-mk)!} = a^{\lambda}_{n,m+1}(k).$$

3. SOME DIFFERENTIAL RELATIONS

In this section we shall give some differential equalities for the polynomials $p^{\lambda}_{n,m}$. Here, D is the differentiation operator and $p^{\lambda}_{k,m}(x) \equiv 0$ when $k \leqslant 0$.

Theorem 1: The following equalities hold:

$$D^k p^{\lambda}_{n+k,m}(x) = 2^k(\lambda)_k p^{\lambda+k}_{n,m}(x), \tag{9}$$

$$2np^{\lambda}_{n,m}(x) = 2xDp^{\lambda}_{n,m}(x) - mDp^{\lambda}_{n-m+1,m}(x), \tag{10}$$

$$mDp^{\lambda}_{n+1,m}(x) = 2(n + m\lambda)p^{\lambda}_{n,m}(x) + 2x(m-1)Dp^{\lambda}_{n,m}(x), \tag{11}$$

$$2\lambda p^{\lambda}_{n,m}(x) = Dp^{\lambda}_{n+1,m}(x) - 2xDp^{\lambda}_{n,m}(x) + Dp^{\lambda}_{n-m+1,m}(x). \tag{12}$$

Proof: Using the differentiation formula (cf. [2, Eq. (3.5)])

$$D^k_x P_{n+k}(m, x, y, p, C) = (-m)^k k!\binom{p}{k} P_n(m, x, y, p-k, C)$$

we obtain (9).

To prove (10), we differentiate the generating function (6) w.r.t. x and t. Then, elimination $(1 - 2xt + t^m)^{-\lambda-1}$ from the expressions, we find

$$\sum_{n=1}^{\infty} 2np^{\lambda}_{n,m}(x)t^n = (2x - mt^{m-1})\sum_{n=0}^{\infty} Dp^{\lambda}_{n,m}(x)t^n.$$

Equating coefficients of t^n in this identity, we get (10).

By differentiating the recurrence relation (8), with $n + 1$ substituted for n, and using (10), we obtain (11).

Finally, by differentiating the generating function (6) w.r.t. x, replacing $G^{\lambda}_m(x, t)$ by its series expansion in powers of t, and equating coefficients of t^{n+1}, we obtain the relation (12).

4. THE DIFFERENTIAL EQUATION

Let the sequence $(f_r)_{r=0}^n$ be given by $f_r = f(r)$, where

$$f(t) = (n - t)\left(\frac{n - t + m(\lambda + t)}{m}\right)_{m-1}.$$

Also, we introduce two standard difference operators, the forward difference operator Δ and the displacement (or shift) operator E, by

$$\Delta f_r = f_{r+1} - f_r \quad \text{and} \quad Ef_r = f_{r+1},$$

and their powers by

$$\Delta^0 f_r = f_r, \quad \Delta^k f_r = \Delta(\Delta^{k-1} f_r), \quad E^k f_r = f_{r+k}.$$

Theorem 2: The polynomial $x \mapsto p_{n,m}^{\lambda}(x)$ is a particular solution of the following m-order differential equation

$$y^{(m)} + \sum_{s=0}^{m} a_s x^s y^{(s)} = 0, \tag{13}$$

where the coefficients a_s are given by

$$a_s = \frac{2^m}{s!m}\Delta^s f_0 \qquad (s = 0, 1, \ldots, m). \tag{14}$$

Proof: Let $n = pm + q$, where $p = [n/m]$ and $0 \leqslant q \leqslant m - 1$. By differentiating (7), we find

$$x^s D^s p_{n,m}^{\lambda}(x) = \sum_{k=0}^{\left[\frac{n-s}{m}\right]} (-1)^k \frac{(\lambda)_{n-(m-1)k}}{k!(n - mk - s)!}(2x)^{n-mk}$$

and

$$D^m p_{n,m}^{\lambda}(x) = \sum_{k=0}^{p-1} (-1)^k \frac{(\lambda)_{n-(m-1)k}\, 2^m}{k!(n - m(k + 1))!}(2x)^{n-m(k+1)},$$

where $\left[\frac{n - s}{m}\right] = p$ when $s \leqslant q$, or $= p - 1$ when $s > q$.

If we substitute these expressions in the differential equation (13) and compare the corresponding coefficients, we obtain the following relations:

$$\sum_{s=0}^{m}\binom{n - mk}{s}s!a_s = 2^m k(\lambda + n - (m - 1)k)_{m-1} \qquad (k = 0, 1, \ldots, p - 1) \tag{15}$$

and

$$\sum_{s=0}^{q}\binom{n - mp}{s}s!a_s = 2^m p(\lambda + n - (m - 1)p)_{m-1}.$$

First, we consider the second equality, i.e.,

$$\sum_{s=0}^{q}\binom{q}{s}\frac{2^m}{m}\Delta^s f_0 = 2^m\frac{n - q}{m}\left(\lambda + q + \frac{n - q}{m}\right)_{m-1}.$$

This equality is correct, because it is equivalent to

$$(1 + \Delta)^q f_0 = E^q f_0 = f_q = f(q).$$

Equality (15) can be written in the form

$$\sum_{s=0}^{m}\binom{n - mk}{s}\Delta^s f_0 = f_{n-mk} \qquad (k = 0, 1, \ldots, p - 1). \tag{16}$$

Since $t \mapsto f(t)$ is a polynomial of degree m, the last equalities are correct; (16) is a forward-difference formula for f at the point $t = n - mk$.

Thus, the proof is completed.

From (14), we have

$$a_0 = \frac{2^m n}{m}\left(\frac{n + m\lambda}{m}\right)_{m-1} = \frac{2^m n}{m^m}\prod_{i=1}^{m-1}(n + m(\lambda + i - 1)),$$

$$a_1 = \frac{2^m}{m}\left\{(n - 1)\left(\frac{n - 1 + m(\lambda + 1)}{m}\right)_{m-1} - n\left(\frac{n + m\lambda}{m}\right)_{m-1}\right\}, \text{ etc.}$$

Since

$$f(t) = -\left(\frac{m - 1}{m}\right)^{m-1} t^m + \text{terms of lower degree},$$

we find

$$a_m = -\frac{2^m}{m}\left(\frac{m - 1}{m}\right)^{m-1}.$$

For $m = 1, 2, 3$, we have the following differential equations:

$$(1 - 2x)y' + 2ny = 0,$$

$$(1 - x^2)y'' - (2\lambda + 1)xy' + n(n + 2\lambda)y = 0,$$

$$\begin{aligned}\left(1 - \frac{32}{27}x^3\right)y''' &- \frac{16}{9}(2\lambda + 3)x^2y'' \\ &- \frac{8}{27}(3n(n + 2\lambda + 1) - (3\lambda + 2)(3\lambda + 5))xy' \\ &+ \frac{8}{27}n(n + 3\lambda)(n + 3(\lambda + 1))y = 0.\end{aligned}$$

Note that the second equation is the Gegenbauer equation.

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ANOTHER FAMILY OF FIBONACCI-LIKE SEQUENCES

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In [1] we studied the class of recurrence relations

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j \tag{1}$$

with $G_0 = G_1 = 1$. The main result of [1] consists of an expression for G_n in terms of the Fibonacci numbers F_n and F_{n-1}, and in the parameters $\alpha_0, \ldots, \alpha_n$.

The present note is devoted to the related family of recurrences that is obtained by replacing the (ordinary or power) polynomial in (1) by a factorial polynomial; viz.

$$H_n = H_{n-1} + H_{n-2} + \sum_{j=0}^{k} \gamma_j n^{(j)} \tag{2}$$

with $H_0 = H_1 = 1$, $n^{(j)} = n(n-1)(n-2) \ldots (n-j+1)$ for $j \geqslant 1$, and $n^{(0)} = 1$. The structure of this note resembles the one of [1] to a large extent.

As usual (cf. e.g., [2] and [4]) the solution $H_n^{(h)}$ of the homogeneous equation corresponding to (2) is

$$H_n^{(h)} = C_1\phi_1^n + C_2\phi_2^n$$

with $\phi_1 = \frac{1}{2}(1+\sqrt{5})$ and $\phi_2 = \frac{1}{2}(1-\sqrt{5})$.

Next we try as a particular solution

$$H_n^{(p)} = \sum_{i=0}^{k} B_i n^{(i)},$$

which yields

$$\sum_{i=0}^{k} B_i n^{(i)} - \sum_{i=0}^{k} B_i (n-1)^{(i)} - \sum_{i=0}^{k} B_i (n-2)^{(i)} - \sum_{i=0}^{k} \gamma_i n^{(i)} = 0.$$

In order to rewrite this equality, we need the following *Binomial Theorem for Factorial Polynomials.*

Lemma 1: $(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}$.

Proof (A. A. Jagers):

$$(x+y)^{(n)} t^{x+y} = t^n \frac{d^n t^{x+y}}{dt^n}$$

(Leibniz's formula)

$$= t^n \sum_{k=0}^{n} \binom{n}{k} x^{(k)} t^{x-k} y^{(n-k)} t^{y-n+k}.$$

Cancellation of t^{x+y} yields the desired equality. ■

Thus, we have

$$\sum_{i=0}^{k} B_i n^{(i)} - \sum_{\ell=0}^{k}\left(\sum_{i=0}^{i} B_i \binom{i}{\ell}\left((-1)^{(i-\ell)} + (-2)^{(i-\ell)}\right)n^{(\ell)}\right) - \sum_{i=0}^{k} \gamma_i n^{(i)} = 0;$$

hence, for each i $(0 \leqslant i \leqslant k)$,

$$B_i - \sum_{m=i}^{k} \delta_{im} B_m - \gamma_i = 0 \tag{3}$$

with, for $m \geqslant i$,

$$\delta_{im} = \binom{m}{i}((-1)^{(m-i)} + (-2)^{(m-i)}).$$

Since $(-x)^{(n)} = (-1)^n(x + n - 1)^{(n)}$ and $n^{(n)} = n!$, we have

$$\begin{aligned}\delta_{im} &= \binom{m}{i}(-1)^{m-i}((m - i)! + (m - i + 1)!)\\ &= \binom{m}{i}(-1)^{m-i}(m - i + 2)(m - i)!\\ &= (-1)^{m-i}(m - i + 2)m^{(m-i)}.\end{aligned}$$

From the family of recurrences (3), we can successively determine $B_k, \ldots, B_0$: the coefficient B_i is a linear combination of $\gamma_i, \ldots, \gamma_k$. Therefore, we set

$$B_i = -\sum_{j=i}^{k} b_{ij}\gamma_j$$

(cf. [1]) which yields, together with (3),

$$-\sum_{j=i}^{k} b_{ij}\gamma_j + \sum_{m=i}^{k} \delta_{im}\left(\sum_{\ell=m}^{k} b_{m\ell}\gamma_\ell\right) - \gamma_i = 0.$$

Thus, for $0 \leqslant i \leqslant j \leqslant k$, we have

$$\begin{aligned}b_{jj} &= 1\\ b_{ij} &= -\sum_{m=i+1}^{j} \delta_{im} b_{mj}, \text{ if } i < j.\end{aligned}$$

Hence, for the particular solution $H_n^{(p)}$ of (2), we obtain

$$H_n^{(p)} = -\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}\gamma_j n^{(i)} = -\sum_{j=0}^{k}\gamma_j\left(\sum_{i=0}^{j} b_{ij} n^{(i)}\right).$$

As in [1] the determination of C_1 and C_2 from $H_0 = H_1 = 1$ yields

$$H_n = (1 - H_0^{(p)})F_n + (-H_1^{(p)} + H_0^{(p)})F_{n-1} + H_n^{(p)}.$$

Therefore, we have

Proposition 2: The solution of (2) can be expressed as

$$H_n = (1 + M_k)F_n + \mu_k F_{n-1} - \sum_{j=0}^{k}\gamma_j \pi_j(n),$$

where M_k is a linear combination of $\gamma_0, \ldots, \gamma_k$, μ_k is a linear combination of $\gamma_1, \ldots, \gamma_k$, and for each j $(0 \leqslant j \leqslant k)$, $\pi_j(n)$ is a factorial polynomial of degree j:

$$M_k = \sum_{j=0}^{k} b_{0j}\gamma_j, \quad \mu_k = \sum_{j=1}^{k} b_{1j}\gamma_j, \quad \pi_j(n) = \sum_{i=0}^{j} b_{ij}n^{(i)}. \quad \blacksquare$$

Table 1

j	$\pi_j(n)$
0	1
1	$n^{(1)} + 3$
2	$n^{(2)} + 6n^{(1)} + 10$
3	$n^{(3)} + 9n^{(2)} + 30n^{(1)} + 48$
4	$n^{(4)} + 12n^{(3)} + 60n^{(2)} + 192n^{(1)} + 312$
5	$n^{(5)} + 15n^{(4)} + 100n^{(3)} + 480n^{(2)} + 1560n^{(1)} + 2520$
6	$n^{(6)} + 18n^{(5)} + 150n^{(4)} + 960n^{(3)} + 4680n^{(2)} + 15120n^{(1)} + 24480$
7	$n^{(7)} + 21n^{(6)} + 210n^{(5)} + 1680n^{(4)} + 10920n^{(3)} + 52920n^{(2)} + 171360n^{(1)} + 277200$
8	$n^{(8)} + 24n^{(7)} + 280n^{(6)} + 2688n^{(5)} + 21840n^{(4)} + 141120n^{(3)} + 685440n^{(2)} +$ $+ 2217600n^{(1)} + 3588480$
9	$n^{(9)} + 27n^{(8)} + 360n^{(7)} + 4032n^{(6)} + 39312n^{(5)} + 317520n^{(4)} + 2056320n^{(3)} +$ $+ 9979200n^{(2)} + 32296320n^{(1)} + 52254720$

Table 1 displays the factorial polynomials $\pi_j(n)$ for $j = 0, 1, \ldots, 9$.

The coefficients of $\gamma_0, \gamma_1, \gamma_2, \ldots$ in M_k and of $\gamma_1, \gamma_2, \ldots$ in μ_k are independent of k; cf. [1]. As k tends to infinity they give rise to two infinite sequences M and μ of natural numbers (not mentioned in [3]) of which the first few elements are

M: 1, 3, 10, 48, 312, 2520, 24480, 277200, 3588480, 52254720, ...

μ: 1, 6, 30, 192, 1560, 15120, 171360, 2217600, 322963 , ...

Contrary to the corresponding sequences Λ and λ in [1], M and μ obviously show more regularity. Formally, this is expressed in

Proposition 3: For each i and j with $0 \leqslant i \leqslant j \leqslant k$,

$$b_{jj} = 1$$
$$b_{ij} = j^{(j-i)}F_{j-i+2}, \text{ if } i < j.$$

Consequently,

$$M_k = \gamma_0 + \sum_{j=1}^{k} j!F_{j+2}\gamma_j \quad \text{and} \quad \mu_k = \gamma_1 + \sum_{j=2}^{k} j!F_{j+1}\gamma_j.$$

Proof: The argument proceeds by induction on $j - i$.

Initial step $(j - i = 1)$: $b_{j-1,j} = -\delta_{j-1,j}b_{jj} = -(-1)^1 \cdot 3j \cdot 1 = j^{(1)}F_3$.

Induction hypothesis: For all m with $i < m < j$, $b_{mj} = j^{(j-m)}F_{j-m+2}$.

Induction step: $b_{ij} = -\sum_{m=i+1}^{j} \delta_{im} b_{mj} = -\delta_{ij} b_{jj} - \sum_{m=i+1}^{j-1} \delta_{im} b_{mj}$

$$= (-1)^{j-i+1}(j - i + 2)j^{(j-i)} + \sum_{m=i+1}^{j-1} (-1)^{m-i+1}(m - i + 2)m^{(m-i)} b_{mj}.$$

From the induction hypothesis, it follows that

$$b_{ij} = j^{(j-i)}\left((-1)^{j-i+1}(j - i + 2) + \sum_{m=i+1}^{j-1} (-1)^{m-i+1}(m - i + 2)F_{j-m+2}\right).$$

As $F_0 = F_1 = 1$, we may replace $j - i + 2$ by $F_0 + (j - i + 1)F_1$. Adding

$$j^{(j-i)}\left((-1)^{j-i}(F_0 + F_1 - F_2) + \sum_{m=i+1}^{j-1} (-1)^{m-i+1}(m - i + 1)(F_{j-m} + F_{j-m+1} - F_{j-m+2})\right) = 0$$

yields, after rearranging,

$$b_{ij} = j^{(j-i)}(F_{j-i} + F_{j-i+1}) = j^{(j-i)}F_{j-i+2},$$

which completes the induction. ■

Clearly, Proposition 3 provides a different way of computing the coefficients a_{ij} (and hence the elements of the sequences Λ and λ) from [1]; viz. by

$$a_{ij} = \sum_{m=i}^{j} s(i, m) \left(\sum_{\ell=m}^{j} b_{m\ell} S(\ell, j)\right) \qquad (i \leqslant j),$$

where $s(i,m)$ and $S(\ell, j)$ are the Stirling numbers of the first and second kind, respectively.

ACKNOWLEDGMENTS

For some useful discussions, I am indebted to Frits Göbel and particularly to Bert Jagers who brought factorial polynomials to my notice and provided the proof of Lemma 1.

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$N!$ HAS THE FIRST DIGIT PROPERTY

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(Submitted March 1986)

Observation of extensive collections of numerical data shows that the distribution of first digits is not equally likely. Frank Benford, a General Electric Company physicist hypothesized in 1938 that for any extensive collection of real numbers expressed in decimal form $Pr(j = p) = \log_{10}(1 + 1/p)$ or, equivalently, $Pr(j < p) = \log_{10} p$, where j is the first significant digit and p is an integer $1 \leqslant p \leqslant 9$. Benford presented extensive data to back up his claim. Sequences that have this property are said to obey Benford's law or to have the first digit property.

One can certainly create data which does not obey Benford's law. However, many "natural" collections do behave in this manner. It has been shown that the geometric sequence ar^n is a Benford sequence as long as r is not a rational power of 10, as is any sequence which is asymptotically geometric (see, e.g., [7]). The Fibonacci numbers F_k are asymptotic to $(\sqrt{5}/5)[(1 + \sqrt{5})/2]^k$, so they have the first digit phenomenon.

R. A. Raimi [7] gives an extensive bibliography of work done in the field until 1976. More recently, others have considered the distribution of first digits in specific sequences of mathematical interest using both the natural density

$$n(S) = \lim_{m \to \infty} \frac{(\text{the number of elements in } S < m)}{m}$$

and other density functions (see, e.g., [1], [2], [6]). In this paper, I show that $N!$ obeys Benford's law using the natural density.

Let D_p be the set of all members of R^+ written with standard expansion in terms of some positive integer base b whose most significant digit is an integer $\leqslant p$. Then,

$$D_p = \bigcup_{n=-\infty}^{\infty} [b^n, (p + 1)b^n).$$

This set maps into $E_p = [0, \log_b(p + 1))$ if we take $\log_b D_p$ (mod 1). Using the notation of [4] let (x_n), $n = 1, 2, \ldots$, be a sequence of positive integers in R^+ written in base b and let $((\log_b x_n))$ be the sequence of fractional parts of $(\log_b x_n)$. Note that $b^{(\log_b x_n)}$ has the same first digit as x_n. Let

$$A[S; N; (x_n)]$$

be the number of terms of (x_n), $1 \leqslant n \leqslant N$, for which $x_n \in S$. Then

$$A[D_p; N; (x_n)] = A[E_p; N; ((\log_b x_n))].$$

A sequence (x_n) is said to be uniformly distributed modulo 1 (written u.d. mod 1) if, for every pair of real numbers with $0 \leqslant a \leqslant b \leqslant 1$, we have

$$\lim_{N \to \infty} \frac{A[[a, b); N; ((x_n))]}{N} = b - a.$$

Recall that E_p is simply $[0, (p+1))$ so that if $((\log_b x_n))$ is u.d. mod 1, then (x_n) is Benford under the natural density. Hence, the problem is reduced to considering the sequence $((\log_b x_n))$ for any sequence (x_n), where b is the base in which the sequence is expanded.

For convenience I will consider sequences written in decimal form and will write $\log x$ for $\log_{10} x$.

Theorem: Let $F = \{N! | N = 1, 2, 3, \ldots\}$ and let

$F_k = \{n | n \in F$ and the first digit of n is $K\}$.

Then $N!$ is Benford; that is,

$$\lim_{m\to\infty} \frac{(\text{the number of elements in } F_k < m)}{(\text{the number of elements in } F < m)} = \log \frac{k+1}{k}.$$

This can be proven utilizing the following theorems from [4]:

(a) If the sequence (x_n), $n = 1, 2, \ldots$, is u.d. mod 1, and if (y_n) is a sequence with the property

$$\lim_{n\to\infty}(x_n - y_n) = a,$$

a real constant, then (y_n) is u.d. mod 1.

(b) The Weyl Criterion: A sequence (x_n), $n = 1, 2, \ldots$, is u.d. mod 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \text{ for all integers } h \neq 0.$$

(c) Let a and b be integers with $a < b$, and let f be twice differentiable on $[a, b)$ with $f'' \geqslant p > 0$ or $f'' \leqslant -p$ for $x \in [a, b)$. Then,

$$\left|\sum_{n=a}^{b} e^{2\pi i f(n)}\right| \leqslant (|f'(b) - f'(a)| + 2)\left(\frac{4}{\sqrt{p}} + 3\right).$$

We observe that

$$\lim_{n\to\infty}|\log[(n/e)^n\sqrt{2\pi n}] - \log n!| = 0$$

since

$$n! = \sqrt{2\pi n}(n/e)^n e^{r(n)/12n} \text{ with } 1 - 1/(12n+1) < r(n) < 1,$$

so that

$$\lim_{n\to\infty} \log[(\sqrt{2\pi n}(n/e)^n]/n!)] = 0.$$

Thus, if $\sqrt{2\pi n}(n/e)^n$ is Benford, so is $n!$. This is convenient for a statistical analysis because it is much simpler and faster to obtain the first digit of $\sqrt{2\pi n}(n/e)^n$ than that of $n!$ despite the fact that, today, programs are available to compute $n!$ for very large n (see, e.g., [3]). Moreover, using (b) and (c), we can show that $\log(\sqrt{2\pi n}(n/e)^n)$ is u.d. mod 1 so that $(\log n!)$ is also, which means $n!$ is Benford. Define $f(x) = h(\log[\sqrt{2\pi x}(x/e)^x])$. Then

$$f''(x) = h(\log_e 10)^{-1}[(2x-1)/x^2] > h(N \log_e 10)^{-1} \geqslant h/3N \text{ for } 1 \leqslant x \leqslant N.$$

Substituting into Theorem (c) with $p = h/3N$ yields:

$$\left|\sum_{n=1}^{N} e^{2\pi i f(n)}\right| \leqslant \left(\left|\frac{h(1-N)}{6N} + \log N\right| + 2\right)\left(4\sqrt{\frac{3N}{h}} + 3\right).$$

Thus,

$$\lim_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i f(n)} \right| = 0$$

and $f(n)$ is u.d. mod 1, which implies $\sqrt{2\pi n}(n/e)^n$ is Benford and, therefore, as indicated previously, so is $n!$.

Another interesting sequence to consider is a^{p_k} where p_k is the k^{th} prime. It has been shown that the primes themselves do have the first digit phenomenon under some non-standard densities (see, e.g., [1]). In a chi-squared analysis at the 95% level for 8 degrees of freedom we would reject the Benford hypothesis if chi-squared is greater than 15.5. Tallying the first digit of the sequence 2^{p_k} for the first 65 primes gives a value of chi-squared of 9.8, while in an analysis of a random sequence of 56 primes less than 10000 a chi-squared value of 12.74 was obtained. Using a Kolmogorov-Smirnov analysis at the 95% level, in the first case, $K = .072$ compared to the table value of .16, and for the random sample, a K value of .14 was obtained, compared to .18 (for table values see, e.g., [5]). These results seem to indicate that 2^{p_k} or, more generally, a^{p_k} may be Benford under other than the natural density. However, this remains an open question.

ACKNOWLEDGMENT

This research was partially supported by a CW Post Campus LIU faculty research grant.

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THE K^{th}-ORDER ANALOG OF A RESULT OF L. CARLITZ

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(Submitted March 1986)

This note is an extension of the work of Carlitz [1] and of Laohakosol and Roenrom [2]. The proofs given here are very similar to those of Laohakosol and Roenrom as presented in [2].

Consider the k^{th}-order difference equation

$$\sum_{j=0}^{k} \sum_{m=0}^{j} (-1)^{m+k-j}\binom{j}{m}p^m n^{(m)} a_{k-j} f_{n+j-m}(x) = x^{k-1} f_{n+k-1}(x) \tag{1}$$

for all $n = 0, 1, 2, \ldots$, with initial conditions

$$f_0(x) = f_1(x) = \cdots = f_{k-2}(x) = 0, \; f_{k-1}(x) = 1, \tag{2}$$

and $a_0 = 1$; a_i $(i = 1, 2, \ldots, k)$ are arbitrary parameters, where

$$n^{(m)} = n(n-1) \ldots (n-m+1)$$

subject to the following three restrictions:

I. $p \neq 0$.

II. All k roots α_i $(i = 1, 2, \ldots, k)$ of the equation $G(0, \alpha, k) = 0$ are distinct and none is a nonpositive integer, where

$$G(r, \alpha, k) = \sum_{j=0}^{k} (-1)^{k-j} a_{k-j} p^j (\alpha + r + j - 1)^{(j)}.$$

III. All $k - 1$ roots r_i $(i = 1, 2, \ldots, k-1)$ of the equation $L(r, \alpha, k) = 0$ are nonpositive integers, where α denotes any one of $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ or α_k from II and

$$L(r, \alpha, k) = \{G(r, \alpha, k) - G(0, \alpha, k)\}/r.$$

Let

$$F(t) := F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n/(n!) \tag{3}$$

be the exponential generating function for $f_n(x)$. From (1)-(3), we have

$$\sum_{j=0}^{k} (-1)^{k-j}(1-pt)^j a_{k-j} F^{(j)}(t) = x^{k-1}F^{(k-1)}(t).$$

Next, we define an operator

$$\Delta := \sum_{j=0}^{k} (-1)^{k-j}(1-pt)^j a_{k-j} D^j \quad (\text{where } D = d/dt).$$

Then our differential equation becomes

$$\Delta F(t) = x^{k-1}F^{(k-1)}(t).$$

We expect k independent solutions of this differential equation to be of the form

$$\phi(t, \alpha) := \phi(t, \alpha, x) = \sum_{m=0}^{\infty} T_m x^m (1 - pt)^{-\alpha - m},$$

where α is any one of $\alpha_1, \alpha_2, \ldots, \alpha_k$. Thus, we must compute $T_m = T_m(\alpha)$.

Using a method similar to that given in [2], we derive

$$T_{j(k-1)+i} = \frac{(\alpha + jk - j + i - 1)^{j(k-1)}}{p^j \left\{ \prod_{m=1}^{j} (mk - m + i) \left[\prod_{s=1}^{k-1} (mk - m + i - r) \right] \right\}} T_i$$

for all $i = 0, 1, \ldots, k - 2$.

Let $C_n(\alpha) := C_n(x, \alpha)$ be the coefficient of $t^n/(n!)$ in $\phi(t, \alpha)$, then

$$C_n(\alpha) = \sum_{j=0}^{\infty} T_{j(k-1)} (\alpha + jk - j + n - 1)^{(n)} p^n x^{j(k-1)}.$$

Hence, we have the general solution of (1) as

$$f_n(x) = \sum_{i=0}^{k} w_i C_n(x, \alpha_i)$$

where

$$w_i = w_i(\alpha_1, \alpha_2, \ldots, \alpha_k) \qquad (i = 1, 2, \ldots, k)$$

are to be chosen so that the initial conditions (2) are fulfilled, namely

$$C \cdot W = E$$

where

$$C = \begin{bmatrix} C_0(\alpha_1) & C_0(\alpha_2) & \cdots & C_0(\alpha_k) \\ C_1(\alpha_1) & C_1(\alpha_2) & \cdots & C_1(\alpha_k) \\ \vdots & \vdots & & \vdots \\ C_{k-1}(\alpha_1) & C_{k-1}(\alpha_2) & \cdots & C_{k-1}(\alpha_k) \end{bmatrix}_{k*k}, \quad W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix}_{k*1}, \quad E = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{k*1},$$

and det $C \neq 0$. Using Cramer's rule, we obtain the solution of W. With these values, we have completely solved (1). Obviously, the difference equations of [1] and [2] are the special cases of (1).

ACKNOWLEDGMENTS

I am deeply grateful to Professors Horng-Jinh Chang, Jin-Zai Lee, and the referees for their helpful comments and thorough discussions.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-604 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*

Let c be a fixed number and $u_{n+2} = cu_{n+1} + u_n$ for n in $N = \{0, 1, 2, \ldots\}$. Show that there exists a number h such that

$$u_{n+4}^2 = hu_{n+3}^2 - hu_{n+1}^2 + u_n \text{ for } n \text{ in } N.$$

B-605 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S(n) = \sum_{i=1}^{n} L_{2n+2i-1}.$$

Determine the positive integers n, if any, for which $S(n)$ is prime.

B-606 *Proposed by L. Kuipers, Sierre, Switzerland*

Simplify the expression

$$L_{n+1}^2 + 2L_{n-1}L_{n+1} - 25F_n^2 + L_{n-1}^2.$$

B-607 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Let

$$C_n = \sum_{k=0}^{n} \binom{n}{k} F_k L_{n-k}.$$

Show that $C_n/2^n$ is an integer for n in $\{0, 1, 2, \ldots\}$.

B-608 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

For $k = \{2, 3, \ldots\}$ and n in $N = \{0, 1, 2, \ldots\}$, let

$$S_{n,k} = \frac{1}{k}\sum_{j=n}^{n+k-1} F_j^2$$

denote the quadratic mean taken over k consecutive Fibonacci numbers of which the first is F_n. Find the smallest such $k \geqslant 2$ for which $S_{n,k}$ is an integer for all n in N.

B-609 *Proposed by Adina DiPorto & Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Find a closed form expression for

$$S = \sum_{k=1}^{n} (kF_k)^2$$

and show that $S_n \equiv n(-1)^n \pmod{F_n}$.

SOLUTIONS

Nondivisors of the L_n

B-580 *Proposed by Valentina Bakinova, Rondout Valley, NY*

What are the three smallest positive integers d such that no Lucas number L_n is an integral multiple of d?

Solution by J. Suck, Essen, Germany

They are 5, 8, 10. Since $1|L_n$, $2|L_0$, $3|L_2$, $4|L_3$, $6|L_6$, $7|L_4$, $9|L_6$, it remains to show that $5\nmid L_n$ and $8\nmid L_n$ for all $n = 0, 1, 2, \ldots$. This follows from the fact that the Lucas sequence modulo 5 or 8 is periodic with period 2, 1, 3, 4 or 2, 1, 3, 4, 7, 3, 2, 5, 7, 4, 3, 7, respectively.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, Herta T. Freitag, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, and the proposer.

Third Degree Representations for F

B-581 *Proposed by Antal Bege, University of Cluj, Romania*

Prove that, for every positive integer n, there are at least $[n/2]$ ordered 6-tuples (a, b, c, x, y, z) such that

$$F_n = ax^2 + by^2 - cz^2$$

and each of a, b, c, x, y, z is a Fibonacci number. Here $[t]$ is the greatest integer in t.

Solution by Paul S. Bruckman, Fair Oaks, CA

We first prove the following relations:

$$F_{2n} = F_{2s+1}F_{n-s+1}^2 + F_{2s}F_{n-s}^2 - F_{2s-1}F_{n-s-1}^2; \tag{1}$$

$$F_{2n+1} = F_{2s+2}F_{n-s+1}^2 + F_{2s+1}F_{n-s}^2 - F_{2s}F_{n-s-1}^2, \tag{2}$$

valid for all integers s and n.

Proof of (1) and (2): We use the following relations repeatedly:

$$F_u F_v^2 = \frac{1}{5}(F_{2v+u} - (-1)^u F_{2v-u} - 2(-1)^v F_u), \tag{3}$$

which is readily proven from the Binet formulas and is given without proof.

Multiplying the right member of (1) by 5, we apply (3) to transform the result as follows:

$$\begin{aligned}
&(F_{2n+3} + F_{2n-4s+1} + 2(-1)^{n-s}F_{2s+1}) + (F_{2n} - F_{2n-4s} - 2(-1)^{n-s}F_{2s}) \\
&\quad - (F_{2n-3} + F_{2n-4s-1} + 2(-1)^{n-s}F_{2s-1}) \\
&= (F_{2n+3} - F_{2n-3} + F_{2n}) + (F_{2n-4s+1} - F_{2n-4s} - F_{2n-4s-1}) \\
&\quad + 2(-1)^{n-s}(F_{2s+1} - F_{2s} - F_{2s-1}) \\
&= (L_3F_{2n} + F_{2n}) + 0 + 0 = 5F_{2n}.
\end{aligned}$$

This proves (1).

Likewise, multiplying the right member of (2) by 5 yields:

$$\begin{aligned}
&(F_{2n+4} - F_{2n-4s} + 2(-1)^{n-s}F_{2s+2}) + (F_{2n+1} + F_{2n-4s-1} - 2(-1)^{n-s}F_{2s+1}) \\
&\quad - (F_{2n-2} - F_{2n-4s-2} + 2(-1)^{n-s}F_{2s}) \\
&= (F_{2n+4} - F_{2n-2} + F_{2n+1}) - (F_{2n-4s} - F_{2n-4s-1} - F_{2n-4s-2}) \\
&\quad + 2(-1)^{n-s}(F_{2s+2} - F_{2s+1} - F_{2s}) \\
&= (L_3F_{2n+1} + F_{2n+1}) - 0 + 0 = 5F_{2n+1}.
\end{aligned}$$

This proves (2).

We may combine (1) and (2) into the single formula:

$$F_n = F_{2s+1+o_n}F_{m-s+1}^2 + F_{2s+o_n}F_{m-s}^2 - F_{2s-1+o_n}F_{m-s-1}^2, \tag{4}$$

where

$$m \equiv [n/2], \quad o_n \equiv (1 - (-1)^n)/2 = \begin{cases} 1, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

We see that the 6-tuples

$$\begin{aligned}
&(a, b, c, x, y, z) \\
&= (F_{2s+1+o_n}, F_{2s+o_n}, F_{2s-1+o_n}, F_{m-s+1}, F_{m-s}, F_{m-s-1})
\end{aligned} \tag{5}$$

are solutions of the problem, as s is allowed to vary. For at least the values $s = 0, 1, \ldots, m - 1$, different 6-tuples are produced in (5). Hence, there are at least $m = [n/2]$ distinct 6-tuples solving the problem.

Also solved by the proposer.

Zeckendorf Representations

B-582 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

It is known that every positive integer N can be represented uniquely as a sum of distinct nonconsecutive positive Fibonacci numbers. Let $f(N)$ be the number of Fibonacci addends in this representation, $a = (1 + \sqrt{5})/2$, and $[x]$ be the greatest integer in x. Prove that

$$f([aF_n^2]) = [(n + 1)/2] \text{ for } n = 1, 2, \ldots .$$

Solution by L. A. G. Dresel, University of Reading, England

Since

$$F_r^2 - F_{r-2}^2 = (F_r - F_{r-2})(F_r + F_{r-2}) = F_{r-1}L_{r-1} = F_{2(r-1)},$$

we have, summing for *even* values $r = 2t$, $t = 1, 2, \ldots, m$,

$$F_{2m}^2 - 0 = F_{4m-2} + F_{4m-6} + \cdots + F_2,$$

and summing for *odd* values $r = 2t + 1$, $t = 1, 2, \ldots, m$,

$$F_{2m+1}^2 - 1 = F_{4m} + F_{4m-4} + \cdots + F_4.$$

Let $a = \frac{1}{2}(1 + \sqrt{5})$ and $b = \frac{1}{2}(1 - \sqrt{5})$, then

$$aF_{2s} = (a^{2s+1} - ab^{2s})/\sqrt{5} = F_{2s+1} + (b - a)b^{2s}/\sqrt{5} = F_{2s+1} - b^{2s}.$$

Applying the formula for F_{2m}^2, we obtain

$$aF_{2m}^2 = F_{4m-1} + F_{4m-5} + \cdots + F_3 - (b^{4m-2} + b^{4m-6} + \cdots + b^2)$$

and since $0 < (b^2 + b^6 + \cdots + b^{4m-2}) < b^2/(1 - b^4) < 1$, we have

$$[aF_{2m}^2] = F_{4m-1} + F_{4m-5} + \cdots + F_3 - 1.$$

Putting $F_3 - 1 = F_2$, we have a sum of m nonconsecutive Fibonacci numbers. Similarly,

$$aF_{2m+1}^2 = F_{4m+1} + F_{4m-3} + \cdots + F_5 + a - (b^{4m} + \cdots + b^8 + b^4),$$
$$0 < (b^4 + b^8 + \cdots + b^{4m}) < b^4/(1 - b^4) < b^2,$$

and $\quad 1 < a - b^2 < 2,$

so that

$$[aF_{2m+1}^2] = F_{4m+1} + F_{4m-3} + \cdots + F_5 + F_1,$$

which is the sum of $(m+1)$ nonconsecutive Fibonacci numbers. Finally, for $n = 1$, we have

$$[aF_1^2] = 1 = F_1.$$

Thus, in all cases, we have

$$f([aF_n^2]) = [(n + 1)/2], \; n = 1, 2, \ldots .$$

Also solved by Paul S. Bruckman, L. Kuipers, J. Suck, and the proposer.

Recursion for a Triangle of Sums

B-583 *Proposed by Dorin Andrica, University of Cluj-Napoca, Romania*

For positive integers n and s, let

$$S_{n,s} = \sum_{k=1}^{n} \binom{n}{k} k^s.$$

Prove that $S_{n,s+1} = n(S_{n,s} - S_{n-1,s})$.

Solution by J.-J. Seiffert, Berlin, Germany

With $\binom{n-1}{n} := 0$ and $\binom{n}{k} - \binom{n-1}{k} = \frac{k}{n}\binom{n}{k}$, we obtain

$$S_{n,s} - S_{n-1,s} = \sum_{k=1}^{n} \left(\binom{n}{k} - \binom{n-1}{k}\right) k^s = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k} k^{s+1} = \frac{1}{n} S_{n,s+1}.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Russell Euler, Piero Filipponi & Odoardo Brugia, Herta T. Freitag, Fuchin He, Joseph J. Kostal, L. Kuipers, Carl Libis, Bob Prielipp, J. Suck, Nicola Treitzenberg, Paul Tzermias, Tad P. White, and the proposer.

Product of Exponential Generating Functions

B-584 *Proposed by Dorin Andrica, University of Cluj-Napoca, Romania*

Using the notation of B-583, prove that

$$S_{m+n,s} = \sum_{k=0}^{s} \binom{s}{k} S_{m,k} S_{n,s-k}.$$

Solution by Heinz-Jürgen Seiffert, Berlin, Germany

The stated equation is not meaningful if one uses the notation of B-583. (To see this, put $s = 0$.) But such an equation can be proved for

$$S_{n,s} := \sum_{k=0}^{n} \binom{n}{k} k^s, \tag{1}$$

with the usual convention $0^0 := 1$. Consider the function

$$F(x, n) := \sum_{s=0}^{\infty} S_{n,s} \frac{x^s}{s!}. \tag{2}$$

Since $0 \leqslant S_{n,s} \leqslant 2^n n^s$, the above series converges for all real x. Using (1), one obtains

$$F(x, n) = \sum_{s=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(kx)^s}{s!} = \sum_{k=0}^{n} \binom{n}{k} \sum_{s=0}^{\infty} \frac{(kx)^s}{s!} = \sum_{k=0}^{n} \binom{n}{k} e^{kx}$$

or

$$F(x, n) = (e^x + 1)^n, \tag{3}$$

which yields

$$F(x, m+n) = F(x, m)F(x, n). \tag{4}$$

Cauchy's product leads to

$$F(x, m)F(x, n) = \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{S_{m,k}}{k!} \frac{S_{n,s-k}}{(s-k)!} x^s \tag{5}$$

From (2), (4), and (5), and by comparing coefficients, one obtains the equation as stated in the proposal for the $S_{n,s}$ defined in (1).

Also solved by Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, L. A. G. Dresel, L. Kuipers, Fuchin He, J. Suck, Nicola Treitzenberg, Paul Tzermias, Tad P. White, and the proposer.

Combinatorial Interpretation of the F_n

B-585 *Proposed by Constantin Gonciulea & Nicolae Gonciulea, Trian College, Drobeta Turnu-Severin, Romania*

For each subset A of $X = \{1, 2, \ldots, n\}$, let $r(A)$ be the number of j such that $\{j, j+1\} \subseteq A$. Show that

$$\sum_{A \subseteq X} 2^{r(A)} = F_{2n+1}.$$

Solution by J. Suck, Essen, Germany

Let us supplement the proposal by

$$\text{"and } \sum_{n \in A \subseteq X} 2^{r(A)} = F_{2n}.\text{"}$$

We now have a beautiful combinatorial interpretation of the Fibonacci sequence. The two identities help each other in the following induction proof.

For $n = 1$, $A = \emptyset$ or X, $r(A) = 0$. Thus, both identities hold here. Suppose they hold for $k = 1, \ldots, n$. Consider $Y := \{1, \ldots, n, n+1\}$. If $\{n, n+1\} \subseteq B \subseteq Y$, $r(B) = r(B\backslash\{n+1\}) + 1$. If $n \notin B \subseteq Y$, $r(B) = r(B\backslash\{n+1\})$. Thus,

$$\sum_{n+1 \in B \subseteq Y} 2^{r(B)} = \sum_{n \in A \subseteq X} 2^{r(A)+1} + \sum_{A \subseteq X\backslash\{n\}} 2^{r(A)} \quad \text{(the last sum is 1 for the step } 1 \to 1+1)$$

$$= 2F_{2n} + F_{2(n-1)+1} = F_{2n} + F_{2n+1} = F_{2(n+1)},$$

and

$$\sum_{B \subseteq Y} 2^{r(B)} = \sum_{n+1 \in B \subseteq Y} 2^{r(B)} + \sum_{A \subseteq X} 2^{r(A)} = F_{2(n+1)} + F_{2n+1} = F_{2(n+1)\ 1}.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, N. J. Kuenzi & Bob Prielipp, Paul Tzermias, Tad P. White, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-415 *Proposed by Larry Taylor, Rego Park, N.Y.*

Let n and w be integers with w odd. From the following Fibonacci-Lucas identity (Elementary Problem B-464, *The Fibonacci Quarterly*, December 1981, p. 466), derive another Fibonacci-Lucas identity using the method given in Problem 1:

$$F_{n+2w}F_{n+w} - 2L_wF_{n+w}F_{n-w} - F_{n-w}F_{n-2w} = (L_{3w} - 2L_w)F_n^2.$$

H-416 *Proposed by Gregory Wulczyn, Bucknell University (Ret.), Lewisburg, PA*

(1) If $\left(\frac{p}{5}\right) = 1$, show that $\begin{cases} .5(L_{p-1} + F_{p-1}) \equiv 1 \pmod{p}, \\ .5(L_{p+1} - F_{p+1}) \equiv 1 \pmod{p}. \end{cases}$

(2) If $\left(\frac{p}{5}\right) = -1$, show that $\begin{cases} .5(L_{p-1} + F_{p-1}) \equiv -1 \pmod{p}, \\ .5(L_{p+1} - F_{p+1}) \equiv -1 \pmod{p}. \end{cases}$

H-417 *Proposed by Piero Filipponi, Rome, Italy*

Let $G(n, m)$ denote the geometric mean taken over m consecutive Fibonacci numbers of which the smallest is F_n. It can be readily proved that

$$G(n, 2k + 1) \qquad (k = 1, 2, \ldots)$$

is not integral and is asymptotic to F_{n+k} (as n tends to infinity).

Show that if n is odd (even), then $G(n, 2k + 1)$ is greater (smaller) than F_{n+k}, except for the case $k = 2$, where $G(n, 5) < F_{n+2}$ for every n.

SOLUTIONS

Bracket Some Sums

H-392 *Proposed by Piero Filipponi, Rome, Italy [Vol. 23(4), Nov. 1985]*

It is known [1], [2], [3], [4] that every positive integer n can be represented uniquely as a finite sum of *F-addends* (distinct nonconsecutive Fibonacci numbers). Denoting by $f(n)$ the number of F-addends the sum of which represents the integer n and denoting by $[x]$ the greatest integer not exceeding x, prove that:

(i) $f([F_k/2]) = [k/3]$, $(k = 3, 4, \ldots)$;

(ii) $f([F_k/3]) = \begin{cases} [k/4] + 1, & \text{for } [k/4] \equiv 1 \pmod 2 \text{ and } k \equiv 3 \pmod 4 \\ [k/4], & \text{otherwise.} \end{cases}$ $(k = 4, 5, \ldots)$

Find (if any) a closed expression for $f([F_k/p])$ with p a prime and k such that $F_k \equiv 0 \pmod p$.

References

1. J. L. Brown, Jr. "Zeckendorf's Theorem and Some Applications." *The Fibonacci Quarterly* 2, no. 4 (1964):163-168.
2. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *The Fibonacci Quarterly* 3, no. 1 (1965):1-8.
3. D. E. Daykin. "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers." *J. London Math. Soc.* 35 (1960):143-160.
4. D. A. Klarner. "Partitions of N into Distinct Fibonacci Numbers." *The Fibonacci Quarterly* 6, no. 4 (1968):235-244.

Solution (partial) by the proposer

Proof (i): Let us put $k = 3h + v$ $(v = 0, 1, 2;\ h = 1, 2, \ldots)$. On the basis of the equalities

$$[F_{3h+v}/2] = \begin{cases} F_{3h}/2, & \text{for } v = 0 \\ (F_{3h+v} - 1)/2, & \text{for } v = 1, 2 \end{cases}$$

the relations

$$[F_{3h+v}/2] = \sum_{i=1}^{h} F_{3i+v-2} \quad (v = 0, 1, 2)$$

can be proven by induction on h. Therefore $[F_k/2]$ can be represented as a sum of $h = [k/3]$ F-addends.

Proof (ii): Let us put $k = 4h + v$ $(v = 0, 1, 2, 3;\ h = 1, 2, \ldots)$. By virtue of the identity

$$F_{t+s} = F_{t+1}F_s + F_tF_{s-1} \tag{1}$$

and of the congruence

$$F_{4h} \equiv 0 \pmod 3, \tag{2}$$

the congruences

$$F_{4h+1} \equiv \begin{cases} 1 \pmod 3, & \text{for } h \text{ even,} \\ 2 \pmod 3, & \text{for } h \text{ odd,} \end{cases} \tag{3}$$

can be readily proven by induction on h. From (1) and (2), we can write:

$$[F_{4h+v}/3] = \begin{cases} F_{4h}/3, & \text{for } v = 0, \\ [F_{4h+1}/3], & \text{for } v = 1, \\ F_{4h}/3 + [F_{4h+1}/3], & \text{for } v = 2, \\ F_{4h}/3 + [2F_{4h+1}/3], & \text{for } v = 3; \end{cases} \tag{4}$$

therefore, from (3) and (4), we obtain:

$$[F_{4h}/3] = F_{4h}/3, \forall h; \tag{5}$$

$$[F_{4h+1}/3] = \begin{cases}(F_{4h+1} - 1)/3, & \text{for } h \text{ even,}\\ (F_{4h+1} - 2)/3, & \text{for } h \text{ odd;}\end{cases} \tag{5$'$}$$

$$[F_{4h+2}/3] = \begin{cases}(F_{4h+2} - 1)/3, & \text{for } h \text{ even,}\\ (F_{4h+2} - 2)/3, & \text{for } h \text{ odd;}\end{cases} \tag{5$''$}$$

$$[F_{4h+3}/3] = \begin{cases}(F_{4h+3} - 2)/3, & \text{for } h \text{ even,}\\ (F_{4h+3} - 1)/3, & \text{for } h \text{ odd.}\end{cases} \tag{5$'''$}$$

From (5), (5′), (5″), (5‴), and on the basis of (1) and of the identity

$$L_n = F_{n-1} + F_{n+1}, \tag{6}$$

the relations

$$[F_{4h+v}/3] = \sum_{i=1}^{h/2} L_{8i+v-4} \quad (v = 0,\ 1,\ 2,\ 3;\ h \text{ even}) \tag{7}$$

$$[F_{4h+v}/3] = F_{v+1} + \sum_{i=1}^{(h-1)/2} L_{8i+v} \quad (v = 0,\ 1,\ 2;\ h \text{ odd}) \tag{7$'$}$$

$$[F_{4h+3}/3] = \sum_{i=1}^{(h+1)/2} L_{8i-5} \quad (h \text{ odd}) \tag{7$''$}$$

can be proven by induction on h. As an example, we consider the case h even and $v = 1$, and prove that

$$(F_{4h+1} - 1)/3 = \sum_{i=1}^{h/2} L_{8i-3}.$$

Setting $h = 2$, we obtain $(F_9 - 1)/3 = L_5$. Supposing the statement true for h, we have

$$\begin{aligned}\sum_{i=1}^{(h+2)/2} L_{8i-3} &= \sum_{i=1}^{h/2+1} L_{8i-3} = (F_{4h+1} - 1)/3 + L_{4h+5}\\ &= (F_{4h+1} - 1)/3 + F_{4h+4} + F_{4h+6}\\ &= (F_{4h+1} - 1)/3 + 18F_{4h} + 11F_{4h-1}\\ &= (34F_{4h+1} + 21F_{4h} - 1)/3\\ &= (F_{4h+9} - 1)/3 = (F_{4(h+2)+1} - 1)/3.\end{aligned}$$

From (7), (7′), (7″), and (6), it is seen that $[F_k/3]$ can be represented as a sum of $h + 1 = [k/4] + 1$ F-addends in the case $[k/4]$ odd and $k \equiv 3 \pmod 4$, and as a sum of $h = [k/4]$ F-addends otherwise.

Also solved (minus a closed form) by L. Kuipers and B. Poonen.

E Gads

H-394 *Proposed by Ambati Jaya Krishna, Baltimore, MD, and Gomathi S. Rao, Orangeburg, SC [Vol. 24(1), Feb. 1986]*

Find the value of the continued fraction $1 + \frac{2}{3+}\frac{4}{5+}\frac{6}{7+}\cdots$.

Solution by Paul S. Bruckman, Fair Oaks, CA

Define c_n, the n^{th} convergent of the indicated continued fraction, as follows:

(1) $\quad c_n \equiv u_n/v_n \equiv 1 + 2/3 + 4/5 + \cdots + 2n/(2n+1)$, $n = 1, 2, \ldots;$
$\quad c_0 \equiv 1 = 1/1.$

After a moment's reflection, it is seen that u_n and v_n satisfy the common recurrence relation:

(2) $\quad w_n = (2n+1)w_{n-1} + 2nw_{n-2}$, $n \geqslant 2$, where w_n denotes either u_n or v_n, and

(3) $\quad u_0 = v_0 = 1;\ u_1 = 5,\ v_1 = 3.$

We now define the generating functions:

(4) $\quad u(x) = \sum_{n=0}^{\infty} u_n \dfrac{x^n}{n!}$, $v(x) = \sum_{n=0}^{\infty} v_n \dfrac{x^n}{n!}$, $w(x)$ denoting either $u(x)$ or $v(x)$.

The initial conditions in (3) become:

(5) $\quad u(0) = v(0) = 1;\ u'(0) = 5,\ v'(0) = 3.$

The recurrence in (2) translates to the following differential equation:

(6) $\quad (2x - 1)w'' + (2x + 5)w' + 4w = 0.$

To solve (6), we find the following transformation useful:

(7) $\quad g(x) = (2x - 1)w'(x) + 4w(x).$

Then, we find (6) is equivalent to the first-order homogeneous equation:

(8) $\quad g' + g = 0,$

from which

(9) $\quad g(x) = ae^{-x}$, for an unspecified constant a.

Substituting this last result into (7), after first making the transformation:

(10) $\quad w(x) = h(x) \cdot (1 - 2x)^{-2},$

we find that $h'(x) = -a(1 - 2x)e^{-x}$, so

(11) $\quad h(x) = -a(1 + 2x)e^{-x} + b$, where b is another unspecified constant.

Thus,

(12) $\quad w(x) = (1 - 2x)^{-2}\{b - a(1 + 2x)e^{-x}\},$

where a and b are to be determined from (5), by appropriate differentiation in (12). Note that $w(0) = b - a = 1$. Also,

$$w'(x) = 4b(1-2x)^{-3} - 2ae^{-x}(1-2x)^{-3}(3+2x) + ae^{-x}(1+2x)(1-2x)^{-2},$$

so $w'(0) = 4b - 5a = 4 - a$. If $w(x) = u(x)$, then $a = -1$ and $b = 0$, while if $w(x) = v(x)$, then $a = 1$ and $b = 2$. Hence,

(13) $\quad u(x) = (1 + 2x)(1 - 2x)^{-2}e^{-x}$, $\quad v(x) = 2(1 - 2x)^{-2} - u(x).$

Next, we use (13) to obtain expansions for $u(x)$ and $v(x)$ and, therefore, explicit expressions for the u_n and v_n originally defined in (1). We start with

$$(1 + 2x)(1 - 2x)^{-2} = (1 + 2x)\sum_{n=0}^{\infty}(n + 1)2^n x^n$$

$$= \sum_{n=0}^{\infty}(n + 1)2^n x^n + \sum_{n=0}^{\infty} n2^n x^n = \sum_{n=0}^{\infty}(2n + 1)2^n x^n;$$

thus,

$$u(x) = \sum_{n=0}^{\infty}(2n + 1)2^n x^n \cdot \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{(-1)^k}{k!}(2n - 2k + 1)2^{n-k}$$

$$= \sum_{n=0}^{\infty}(2n + 1)(2x)^n \sum_{k=0}^{n}\frac{(-\frac{1}{2})^k}{k!} - 2\sum_{n=1}^{\infty}(2x)^n \sum_{k=1}^{n}\frac{(-\frac{1}{2})^k}{(k - 1)!};$$

letting

(14) $$r_n = \sum_{k=0}^{n}\frac{(-\frac{1}{2})^k}{k!}, \quad n = 0, 1, 2, \ldots,$$

we obtain

$$u(x) = \sum_{n=0}^{\infty}(2n + 1)(2x)^n r_n + \sum_{n=1}^{\infty}(2x)^n\left(r_n - \frac{(-\frac{1}{2})^n}{n!}\right)$$

$$= 1 + \sum_{n=1}^{\infty}\left\{2(n + 1)r_n - \frac{(-\frac{1}{2})^n}{n!}\right\}(2x)^n,$$

or

(15) $$u(x) = \sum_{n=0}^{\infty}\left\{2^{n+1}(n + 1)!r_n - (-1)^n\right\}\frac{x^n}{n!}.$$

It follows from comparison of coefficients in (4) and (15) that

(16) $$u_n = 2^{n+1}(n + 1)!r_n - (-1)^n, \quad n = 0, 1, 2, \ldots .$$

Likewise, since $v(x) = 2(1 - 2x)^{-2} - u(x)$, we find

$$v(x) = 2\sum_{n=0}^{\infty}(n + 1)2^n x^n - \sum_{n=0}^{\infty} u_n \frac{x^n}{n!} = \sum_{n=0}^{\infty}(n + 1)!2^{n+1}\frac{x^n}{n!} - \sum_{n=0}^{\infty} u_n \frac{x^n}{n!},$$

so

(17) $$v_n = 2^{n+1}(n + 1)! - u_n,$$

or

(18) $$v_n = 2^{n+1}(n + 1)!(1 - r_n) + (-1)^n, \quad n = 0, 1, 2, \ldots .$$

We note that

(19) $$\lim_{n\to\infty} r_n = e^{-\frac{1}{2}}.$$

Therefore,

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty}(u_n/v_n) = \lim_{n\to\infty}\left\{\frac{2^{n+1}(n + 1)!r_n - (-1)^n}{2^{n+1}(n + 1)!(1 - r_n) + (-1)^n}\right\} = \lim_{n\to\infty}\left(\frac{r_n}{1 - r_n}\right)$$

$$= e^{-\frac{1}{2}}/(1 - e^{-\frac{1}{2}}),$$

or

(20) $$\lim_{n\to\infty} c_n = (e^{\frac{1}{2}} - 1)^{-1} \doteq 1.541494083.$$

Also solved by W. Janous, A. Krishna & G. Rao, L. Kuipers & P. Shieu, J.-S. Lee & J.-Z. Lee, F. Steutel, and the proposer.

Easy Induction

H-395 *Proposed by Heinz-Jürgen Seiffert, Berlin Germany*
[Vol. 24(1), Feb. 1986]

Show that for all positive integers m and k,

$$\sum_{n=0}^{m-1} \frac{F_{2k(2n+1)}}{L_{2n+1}} = \sum_{j=0}^{k-1} \frac{F_{2m(2j+1)}}{L_{2j+1}}$$

Solution by J.-Z. Lee & J.-S. Lee, Soochow University, Taipei, Taiwan, R.O.C.

Define

$$S_1(m, k) = \sum_{n=0}^{m-1} (F_{2k(2n+1)}/L_{2n+1}),$$

$$S_2(m, k) = \sum_{j=0}^{k-1} (F_{2m(2j+1)}/L_{2j+1}).$$

From the definitions of F_n and L_n, we have

Lemma 1: $F_{(m+2k)(2n+1)} - F_{m(2n+1)} = F_{(m+k)(2n+1)}L_{k(2n+1)}$,

Lemma 2: $\sum_{n=0}^{m-1} F_{(2k-1)(2n+1)} = F_{2m(2k-1)}/L_{2k-1}$.

We will prove, using the induction hypothesis, that

$$S_1(m, k) = S_2(m, k) \tag{*}$$

for all positive integers m and k.

For $k = 1$, we obtain

$$S_1(m, 1) = \sum_{n=0}^{m-1} (F_{2(2n+1)}/L_{2n+1}) = \sum_{n=0}^{m-1} F_{2n+1} = F_{2m} = S_2(m, 1),$$

so (*) is true for $k = 1$. Suppose that (*) is true for all positive integers less than k, then

$$\begin{aligned} S_1(m, k) &= \sum_{n=0}^{m-1} (F_{2k(2n+1)}/L_{2n+1}) \\ &= \sum_{n=0}^{m-1} ((F_{2(k-1)(2n+1)} + F_{(2k-1)(2n+1)}L_{2n+1})/L_{2n+1}), \text{ by Lemma 1,} \\ &= \sum_{n=0}^{m-1} (F_{2(k-1)(2n+1)}/L_{2n+1}) + \sum_{n=0}^{m-1} F_{(2k-1)(2n+1)} \\ &= \sum_{j=0}^{k-2} (F_{2m(2j+1)}/L_{2j+1}) + F_{2m(2k-1)}/L_{2k-1}, \\ &\qquad \text{by the induction hypothesis and Lemma 2,} \\ &= \sum_{j=0}^{k-1} (F_{2m(2j+1)}/L_{2j+1}) = S_2(m, k); \end{aligned}$$

therefore, (*) is true for all positive integers k.

Also solved by P. Bruckman, L. A. G. Dresel, C. Georghiou, W. Janous, L. Kuipers, and the proposer.

Another Easy One

H-396 *Proposed by M. Wachtel, Zürich, Switzerland [Vol. 24(1), Feb. 1986]*

Establish the identity:

$$\sum_{i=1}^{\infty} \frac{F_{i+n}}{a^i} + \sum_{i=1}^{\infty} \frac{F_{i+n+1}}{a^i} = \sum_{i=1}^{\infty} \frac{F_{i+n+2}}{a^i}$$

$$a = 2, 3, 4, \ldots, n = 0, 1, 2, 3, \ldots .$$

Solution by Paul S. Bruckman, Fair Oaks, CA

The series defined as follows,

$$f(x, m) \equiv \sum_{i=1}^{\infty} F_{i+m} x^i, \quad m \in Z, \tag{1}$$

is absolutely convergent, with radius of convergence $\theta \equiv \frac{1}{2}(\sqrt{5} - 1) \doteq .618$. In fact, the sum of the series is readily found to be equal to

$$f(x, m) = \frac{xF_{m+1} + x^2 F_m}{1 - x - x^2}, \quad |x| < \theta. \tag{2}$$

Since $a^{-1} < \theta$ for $a = 2, 3, 4, \ldots$, each of the series indicated in the statement of the problem is absolutely convergent. Hence,

$$\sum_{i=1}^{\infty} F_{i+n} a^{-i} + \sum_{i=1}^{\infty} F_{i+n+1} a^{-i} = \sum_{i=1}^{\infty} (F_{i+n} + F_{i+n+1}) a^{-i} = \sum_{i=1}^{\infty} F_{i+n+2} a^{-i}.$$

This may also be demonstrated from (2), setting $x = a^{-1}$:

$$\begin{aligned} f(a^{-1}, n) + f(a^{-1}, n + 1) &= \frac{aF_{n+1} + F_n}{a^2 - a - 1} + \frac{aF_{n+2} + F_{n+1}}{a^2 - a - 1} \\ &= \frac{aF_{n+3} + F_{n+2}}{a^2 - a - 1} = f(a^{-1}, n + 2). \end{aligned}$$

Also solved by L. A. G. Dresel, P. Filipponi, C. Georghiou, W. Janous, L. Kuipers, J.-Z. Lee & J.-S. Lee, R. Whitney, and the proposer.

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

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Recurring Sequences by Dov Jarden. Third and enlarged edition. Riveon Lematematika, Israel, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

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A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

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