The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

TABLE OF CONTENTS

VOLUME 26 FEBRUARY 1988 NUMBER 1

PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

EDITORIAL POLICY

THE FIBONACCI QUARTERLY seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

SUBMITTING AN ARTICLE

Articles should be submitted in the format of the current issues of the **THE FIBONACCI QUARTERLY.** They should be typewritten or reproduced typewritten copies, that are clearly readable, double spaced with wide margins and on only one side of the paper. The full name and address of the author must appear at the beginning of the paper directly under the title. Illustrations should be carefully drawn in India ink on separate sheets of bond paper or vellum, approximately twice the size they are to appear in print.

Two copies of the manuscript should be submitted to: **GERALD E. BERGUM, EDITOR, THE FIBONACCI QUARTERLY, DEPARTMENT OF COMPUTER SCIENCE, SOUTH DAKOTA STATE UNIVERSITY, BOX 2201, BROOKINGS, SD 57007-0194.**

Authors are encouraged to keep a copy of their manuscripts for their own files as protection against loss. The editor will give immediate acknowledgment of all manuscripts received.

SUBSCRIPTIONS, ADDRESS CHANGE, AND REPRINT INFORMATION

Address all subscription correspondence, including notification of address change, to: **RICHARD VINE, SUBSCRIPTION MANAGER, THE FIBONACCI ASSOCIATION, SANTA CLARA UNIVERSITY, SANTA CLARA, CA 95053.**

Requests for reprint permission should be directed to the editor. However, general permission is granted to members of The Fibonacci Association for noncommercial reproduction of a limited quantity of individual articles (in whole or in part) provided complete reference is made to the source.

Annual domestic Fibonacci Association membership dues, which include a subscription to **THE FIBONACCI QUARTERLY,** are $27 for Regular Membership, $37 for Sustain Membership, and $67 for Institutional Membership; foreign rates, which are based on international mailing rates, are somewhat higher than domestic rates; please write for details. **THE FIBONACCI QUARTERLY** is published each February, May, August and November.

All back issues of **THE FIBONACCI QUARTERLY** are available in microfilm or hard copy format from **UNIVERSITY MICROFILMS INTERNATIONAL, 300 NORTH ZEEB ROAD, DEPT. P.R., ANN ARBOR, MI 48106.** Reprints can also be purchased from **UMI CLEARING HOUSE** at the same address.

The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

Announcement

THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 25-29, 1988
Department of Mathematics, University of Pisa
Pisa, Italy

International Committee

Horadam, A.F. (Australia), *Co-Chairman*
Philippou, A.N. (Greece), *Co-Chairman*
Ando, S. (Japan)
Bergum, G.E. (U.S.A.)
Johnson, M.D. (U.S.A.)
Kiss, P. (Hungary)
Schinzel, Andrzej (Poland)
Tijdeman, Robert (The Netherlands)
Tognetti, K. (Australia)

Local Committee

Robert Dvornicich, *Chairman*
Piero Filipponi
Alberto Perelli
Carlo Viola
Umberto Zannier

FIBONACCI'S STATUE

Have you ever seen Fibonacci's portrait? This photo is a close-up of the head of the statue of Leonardo Pisano in Pisa, Italy, taken by Frank Johnson in 1978.

Since Fibonacci's statue was difficult to find, here are the directions from the train station in Pisa (about 8 blocks to walk): Cross Piazza Vitt. Em. II, bearing right along Via V. Croce to Piazza Toniolo, and then walk through the Fortezza. The statue is found within Fortezza Campo Santo off Lungarno Fibonacci or Via Fibonacci along the Arno River at Giardino Scotto (Teatro Estivo).

CALL FOR PAPERS

The THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of Pisa, Pisa, Italy, from July 25-29, 1988. This conference is sponsored jointly by The Fibonacci Association and The University of Pisa.

Papers on all branches of mathematics and science related to the Fibonacci numbers and their generalizations are welcome. Abstracts are to be submitted by March 15, 1988. Manuscripts are requested by May 1, 1988. Abstracts and manuscripts should be sent to G.E. Bergum (address below). Invited and contributed papers will appear in the Conference Proceedings, which are expected to be published.

The program for the Conference will be mailed to all participants, and to those individuals who have indicated an interest in attending the conference, by June 15, 1988. All talks should be limited to one hour or less.

For further information concerning the conference, please contact Gerald Bergum, The Fibonacci Quarterly, Department of Computer Science, South Dakota State University, P.O. Box 2201, Brookings, South Dakota 57007-0194.

FIBONACCI AND LUCAS CURVES

A. F. HORADAM
University of New England, Armidale, Australia

A. G. SHANNON
N.S.W. Institute of Technology, Sydney, Australia

(*Submitted February 1986*)

1. INTRODUCTION

Define the recurrence-generated sequence $\{H_n\}$ for integers n by

$$H_{n+2} = H_{n+1} + H_n, \quad H_0 = 2b, \quad H_1 = a + b \quad (n \geqslant 0) \tag{1.1}$$

where a and b are arbitrary but are generally considered to be integers. Negative subscripts of H can be included in an extended definition if necessary.

Using [2], equation (δ), we have, for the Binet form of this generalized sequence, *mutatis mutandis*,

$$H_n = \frac{A\alpha^n - B\beta^n}{\sqrt{5}} \tag{1.2}$$

where

$$\begin{cases} \alpha = \dfrac{1+\sqrt{5}}{2} \\ \beta = \dfrac{1-\sqrt{5}}{2} = -1/\alpha \end{cases} \tag{1.3}$$

are the roots of

$$\lambda^2 - \lambda - 1 = 0 \tag{1.4}$$

and

$$\begin{cases} A = a + b\sqrt{5} \\ B = a - b\sqrt{5} \end{cases} \tag{1.5}$$

From (1.2), it follows readily that

$$H_n = aF_n + bL_n \tag{1.6}$$

where

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} \tag{1.7}$$

and

$$L_n = \alpha^n + \beta^n \tag{1.8}$$

are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively, occurring in (1.1), (1.2), and (1.6) when:

$$a = 1, \quad b = 0 \quad \text{for } F_n;$$
$$a = 0, \quad b = 1 \quad \text{for } L_n.$$

The explicit expressions (1.7) and (1.8) are the *Binet forms* of F_n and L_n.

Following an idea of Wilson [5], we set

$$x = \{A\alpha^{2n} + B\cos(n-1)\pi\}/\sqrt{5}\alpha^n \tag{1.9}$$

and

$$y = B\sin(n-1)\pi/\sqrt{5}\alpha^n \tag{1.10}$$

which we now regard as Cartesian coordinates in a plane (though Wilson [6] expressed his notion in terms of polar coordinates).

Certain geometrical features relating to circles and rectangular hyperbolas were shown [3] to be consequences of (1.9) and (1.10). These features were extended to Pell numbers and Pell-Lucas numbers in [4].

Here we examine (1.9) and (1.10) in a rather different geometrical context.

2. GENERALIZED BINET FORMS

First, we generalize (1.9) and (1.10) from an integer exponent n to a real exponent θ:

$$x = \{A\alpha^{2\theta} + B\cos(\theta-1)\pi\}/\sqrt{5}\alpha^\theta; \tag{2.1}$$

$$y = B\sin(\theta-1)\pi/\sqrt{5}\alpha^\theta. \tag{2.2}$$

Expanding the trigonometrical components of (2.1) and (2.2), we find

$$x = \{A\alpha^\theta - B\alpha^{-\theta}\cos\theta\pi\}/\sqrt{5} \tag{2.3}$$

and

$$y = -B\alpha^{-\theta}\sin\theta\pi/\sqrt{5}. \tag{2.4}$$

We will be particularly interested in the Fibonacci and Lucas aspects of (2.3). For the Fibonacci case $a = 1$, $b = 0$, so $A = B = 1$, and (2.3) becomes, with (1.3),

$$x = \frac{\alpha^\theta - \alpha^{-\theta}\cos\theta\pi}{\sqrt{5}} = \{\alpha^\theta - (-1)^\theta\beta^\theta\cos\theta\pi\}/\sqrt{5} \tag{2.5}$$

while for the Lucas case $a = 0$, $b = 1$, so $A = -B = \sqrt{5}$, and (2.3) reduces to

$$x = \alpha^\theta + (-1)^\theta\beta^\theta\cos\theta\pi. \tag{2.6}$$

When θ is an integer n, (2.5) and (2.6) simplify to the Binet forms (1.7) and (1.8), respectively. Therefore, we are justified in referring to (2.5) and (2.6) as the *generalized Binet forms* of F_n and L_n, i.e., the Binet forms of F_θ and L_θ.

It is the object of this paper to consider, *inter alia*, the locus generated by the parametric equations (2.3) and (2.4). Efforts to express the equation of this locus in Cartesian form, i.e., to eliminate the parameter θ, have not met with success.

From (2.4) we have

$$\frac{dy}{d\theta} = \frac{B\alpha^{-\theta}}{\sqrt{5}}(\log \alpha \sin \theta\pi - \pi \cos \theta\pi) \tag{2.7}$$

while from (2.3)

$$\frac{dx}{d\theta} = \frac{\alpha^{-\theta}}{\sqrt{5}}\{A\alpha^{2\theta} \log \alpha + B(\log \alpha \cos \theta\pi + \pi \sin \theta\pi)\} \tag{2.8}$$

whence

$$\frac{dy}{dx} = \frac{B(\log \alpha \sin \theta\pi - \pi \cos \theta\pi)}{A\alpha^{2\theta} \log \alpha + B(\log \alpha \cos \theta\pi + \pi \sin \theta\pi)} = 0 \tag{2.9}$$

when

$$\tan \theta\pi = \frac{\pi}{\log \alpha} \quad (\doteqdot 6.53 \text{ to two decimal places}) \tag{2.10}$$

yielding

$$\theta\pi \doteqdot 81°18' \text{ from tables,} \tag{2.11}$$

that is,

$$\theta \doteqdot 0.45 \quad (\doteqdot 26° \text{ in degree measure}). \tag{2.12}$$

Thus, the stationary points on the curve occur when

$$\tan(\theta - m)\pi = \frac{\pi}{\log \alpha} \quad (m \text{ an integer}), \tag{2.13}$$

that is,

$$\theta = \frac{1}{\pi} \tan^{-1}\left(\frac{\pi}{\log \alpha}\right) + m. \tag{2.14}$$

The nature of these stationary points, i.e., whether they yield maxima or minima, can be determined by the usual elementary methods.

Next, we discover the locus of the stationary points.

Write

$$\sin(\theta - m)\pi = k\pi \quad \text{i.e., } \sin \theta\pi = \pm k\pi \tag{2.15}$$

and

$$\cos(\theta - m)\pi = k \log \alpha \quad \text{i.e., } \cos \theta\pi = \pm k \log \alpha, \tag{2.16}$$

where

$$k = (\pi^2 + \log^2\alpha)^{-1/2} \qquad (\doteqdot 3.2). \tag{2.17}$$

Because $\sin\theta\pi$ and $\cos\theta\pi$ (and therefore θ) now have specified numerical values for the stationary points, we can eliminate α^θ from (2.3) and (2.4).

Substitute from (2.15) and (2.16) in (2.3) and (2.4) to obtain

$$\sqrt{5}x \cdot \mp \frac{Bk\pi}{\sqrt{5}y} = \frac{AB^2k^2\pi^2}{5y^2} \mp Bk \log\alpha$$

$$y^2 - \frac{\pi}{\log\alpha}xy = \frac{\pm ABk\pi^2}{5\log\alpha} \tag{2.18}$$

i.e., the branch in the first quadrant of the hyperbola,

$$y^2 - \frac{\pi}{\log\alpha}xy = \frac{ABk\pi^2}{5\log\alpha}, \tag{2.19}$$

and the branch in the fourth quadrant of the conjugate hyperbola,

$$y^2 - \frac{\pi}{\log\alpha}xy = -\frac{ABk\pi^2}{5\log\alpha}. \tag{2.20}$$

Common asymptotes of these two hyperbolas are

$$y = 0, \qquad y = \frac{\pi}{\log\alpha}x. \tag{2.21}$$

The oblique asymptote $y = \frac{\pi}{\log\alpha}x$ has gradient 81°18′ (approx.) by (2.10) and (2.11).

Of course, there are infinitely many points on (2.18) which do not satisfy (2.10), i.e., which are not stationary points. Therefore, the loci (2.18) are lacunary.

Inflections on the parametric curve (2.1) and (2.2) are given by the vanishing of $\frac{d^2y}{dx^2}$. Differentiating (2.9) a second time, we get

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} \tag{2.22}$$

$$= \frac{[A\alpha^{2\theta}\log\alpha(3\pi\log\alpha\cos\theta\pi + (\pi^2 - 2\log^2\alpha)\sin\theta\pi) + B\pi k^2]\sqrt{5}\alpha^\theta}{\{A\alpha^{2\theta}\log\alpha + B(\log\alpha\cos\theta\pi + \pi\sin\theta\pi)\}^3}$$

after some simplification.

Inflections are then given by those values of θ for which

$$A\alpha^{2\theta}\log\alpha(3\pi\log\alpha\cos\theta\pi + (\pi^2 - 2\log^2\alpha)\sin\theta\pi) + B\pi k^2 = 0. \tag{2.23}$$

To test for maxima and minima, use (2.15)-(2.17), keeping in mind that

$$\pi\cos\theta\pi = \log\alpha\sin\theta\pi.$$

Then, at the stationary points (letting the variable θ be replaced by constants Θ), we find that the left-hand side of (2.23) is, after tidying up,

$$k^2\pi\{A\alpha^{2\Theta}\log\alpha. \pm k^{-3} + B\} \tag{2.24}$$

which becomes

$$k^2\pi\{\pm k^{-3}\alpha^{2\Theta}\log\alpha + 1\} \tag{2.25}$$

in the Fibonacci case, and

$$\frac{k^2\pi}{\sqrt{5}}\{\pm k^{-3}\alpha^{2\Theta}\log\alpha - 1\} \tag{2.26}$$

in the Lucas case.

If the numerical values of Θ are known, the nature of the turning points may be determined from (2.25) and (2.26). Note that $k^{-3}\alpha^{\Theta}\log\alpha$ is always positive.

No obviously derived differential equation satisfies (3.3) and (3.4) for the curve.

Finally, if we rewrite (2.3) and (2.4) as

$$x(\theta) = (A\alpha^{\theta} + (-1)^{\theta-1}B\beta^{\theta}\cos\theta\pi)/\sqrt{5} \tag{2.3$'$}$$

and

$$y(\theta) = c(-1)^{\theta-1}\beta^{\theta}\sin\pi \tag{2.4$'$}$$

(on putting $c = B/\sqrt{5}$ temporarily), we can see from the tables that the recurrence relation (1.1) is, in effect, satisfied as

$$x(\theta) = x(\theta - 1) + x(\theta - 2) \tag{2.3$''$}$$

and

$$y(\theta) = y(\theta - 1) + y(\theta - 2). \tag{2.4$''$}$$

The proofs follow. We have

$$\begin{aligned}
x(\theta - 1) &= (A\alpha^{\theta-1} + (-1)^{\theta-2}B\beta^{\theta-1}\cos(\theta - 1)\pi)/\sqrt{5}\\
&= (A\alpha^{\theta-1} + (-1)^{\theta-1}B\beta^{\theta-1}\cos\theta\pi)/\sqrt{5}\\
x(\theta - 2) &= (A\alpha^{\theta-2} + (-1)^{\theta-3}B\beta^{\theta-2}\cos(\theta - 2)\pi)/\sqrt{5}\\
&= (A\alpha^{\theta-2} + (-1)^{\theta-1}B\beta^{\theta-2}\cos\theta\pi)/\sqrt{5}\\
x(\theta - 1) + x(\theta - 2) &= (A\alpha^{\theta-2}(\alpha + 1) + (-1)^{\theta-1}B\beta^{\theta-2}(\beta + 1)\cos\theta\pi)/\sqrt{5}\\
&= (A\alpha^{\theta} + (-1)^{\theta-1}B\beta^{\theta}\cos\theta\pi)/\sqrt{5} = x(\theta)
\end{aligned}$$

as required, since α, β satisfy (1.4).

Similarly,

$$\begin{aligned}
y(\theta - 1) &= c(-1)^{\theta-2}\beta^{\theta-1}\sin(\theta - 1)\pi = c(-1)^{\theta-1}\beta^{\theta-1}\sin\theta\pi\\
y(\theta - 2) &= c(-1)^{\theta-3}\beta^{\theta-2}\sin(\theta - 2)\pi = c(-1)^{\theta-1}\beta^{\theta-2}\sin\theta\pi
\end{aligned}$$

and $\qquad y(\theta - 1) + y(\theta - 2) = c(-1)^{\theta-1}\beta^{\theta-2}(\beta + 1)\sin \theta\pi$

$$= c(-1)^{\theta-1}\beta^{\theta}\sin \theta\pi \qquad \text{since } \beta \text{ satisfies (1.4)}$$

$$= y(\theta).$$

Thus, it has been demonstrated that the parametric forms (2.3)″ and (2.4)″ do indeed satisfy recurrence relation (1.1).

We need this assurance to preserve the continuity of our curves in Figures 1, 2, and 3, which we now examine.

3. THE FIBONACCI CURVE

Table 1 sets out the values of x in (2.5), and y in (2.2) where $B = 1$, for the Fibonacci case $a = 1$, $b = 0$, when we proceed to increase θ by multiples of 0.2.

Table 1. The Fibonacci Curve

θ	x	y	θ	x	y
1	1	0			
1.2	0.999799314	0.14755316	6	8.000000000	0
1.4	0.947653586	0.216839615	6.2	8.817334649	-0.013304890
1.6	0.901827097	0.196943249	6.4	9.721923304	-0.019552416
1.8	0.911232402	0.110549283	6.6	10.71685400	-9.96822E-03
2	1	0	6.8	11.80690074	-9.96822E-03
2.2	1.163587341	-0.091192868	7	13.00000000	0
2.4	1.375792509	-0.134014252	7.2	14.3076953	8.22286E-03
2.6	1.602274541	-0.121717622	7.4	15.744608	0.012084058
2.8	1.814640707	-0.068323214	7.6	17.32733182	0.010975271
3	2.000000000	0	7.8	19.07328767	6.16070E-03
3.2	2.163386655	0.056360292	8	21.00000000	0
3.4	2.323446095	0.082825363	8.2	23.12502995	-5.08200E-03
3.6	2.504101639	0.075225627	8.4	25.4665313	-7.46836E-03
3.8	2.725873109	0.042226069	8.6	28.04418582	-6.78309E-03
4	3.000000000	0	8.8	30.8801884	-3.80752E-03
4.2	3.326973997	-0.034832576	9	34.00000000	0
4.4	3.699238605	-0.051188889	9.2	37.43272525	3.14085E-03
4.6	4.10637618	-0.046491995	9.4	41.21113931	4.611570E-03
4.8	4.540513816	-0.026097146	9.6	45.37151764	4.19218E-03
5	5.000000000	0	9.8	49.953447608	2.35318E-03
5.2	5.490360652	0.021527716	10	55.00000000	0
5.4	6.022684699	0.031636473			
5.6	6.610477819	0.028733633			
5.8	7.266386925	0.016128923			

Figure 1 shows the computer-drawn graph corresponding to the data in Table 1. We may call it the *Fibonacci curve*.

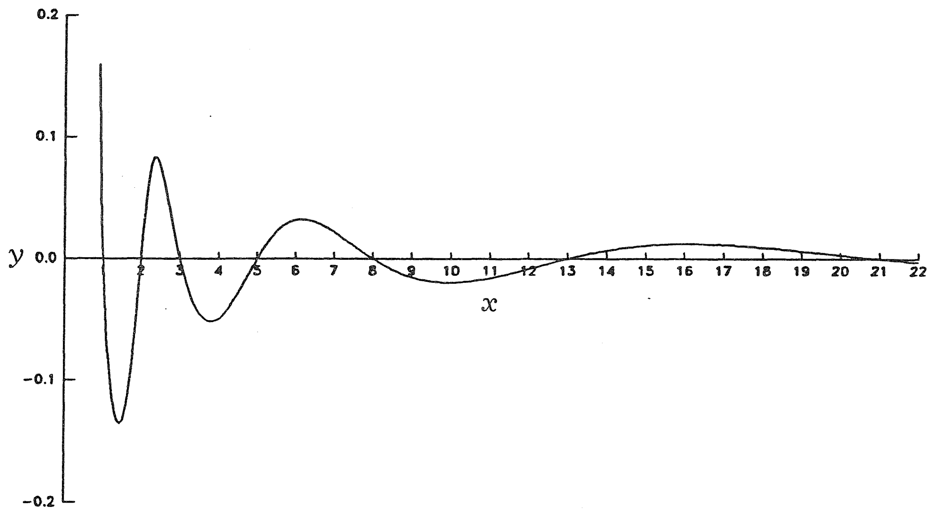

Figure 1. The Fibonacci Curve

Using (2.19) and (2.20) with $A = B = 1$ for the Fibonacci curve, we see that the locus of the stationary points is the appropriate branches of the hyperbolas

$$y^2 - \frac{\pi}{\log \alpha} xy = \pm \frac{k\pi^2}{5 \log \alpha}$$

From the observed stationary points on the plotted curve, one can visualize the need for a slight deviation (about 8.3°) from $x = 0$ of the "vertical" asymptote [refer to (2.11) and (2.21)]. The stationary points of the Fibonacci curve approach $y = 0$ asymptotically at a very quick rate (of necessity, since, in (2.2), $\alpha^\theta \to \infty$ rather rapidly as $\theta \to \infty$).

It is interesting to compare details of our Table 1 with similar figures given by Halsey [1]. See Table 2, in which the numbers in the first column for F_n are Halsey's and those in the second column for F_n are ours (to the same number of decimal places).

Starting from a quantity $n\not\Delta^m$ (read "*n delta-slash m*") which he defined for integers $m, n \geqslant 1$ and using the Pascal triangle generation of Fibonacci numbers (the elements of the Pascal triangle being expressed in terms of $n\not\Delta^m$ for various n and m), Halsey [1] established the following nice results:

$$F_n = \sum_{k=0}^{m} (n - 2k)\not\Delta^k \quad \left(\frac{n}{2} - 1 \leqslant m \leqslant \frac{n}{2}\right); \tag{3.1}$$

$$n\not\Delta^m = \binom{n + m - 1}{m}; \tag{3.2}$$

$$n\not\Delta^m = \left[(n + m)\int_0^1 x^{n-1}(1 - x)^m dx\right]^{-1}; \tag{3.3}$$

$$F_\theta = \sum_{k=0}^{m}\left[(\theta - k)\int_0^1 x^{\theta-2k-1}(1-x)^k\,dx\right]^{-1} \quad \left(\frac{\theta}{2} - 1 \leqslant m \leqslant \frac{\theta}{2}\right). \tag{3.4}$$

where θ is real.

Table 2

θ	F_θ	F_θ
2	1	1
2.2	1.2	1.2
2.4	1.4	1.4
2.6	1.6	1.6
2.8	1.8	1.8
3	2	2
3.2	2.2	2.2
3.4	2.4	2.3
3.6	2.6	2.5
3.8	2.8	2.7
4	3	3
4.2	3.32	3.33
4.4	3.68	3.70
4.6	4.08	4.11
4.8	4.52	4.54
5	5	5
5.2	5.52	5.49
5.4	6.08	6.02
5.6	6.68	6.61
5.8	7.32	7.27
6	8	8

To obtain the definite integral expressions, Halsey had recourse to basic properties of Beta functions and Gamma functions. It might be noted, as Halsey observed, that the Gamma function "extends the concept of factorials to numbers that are not integers," e.g., $\left(\frac{1}{2}\right)! = \sqrt{\pi}/2$. In this spirit, he extended the theory of Fibonacci numbers to noninteger values.

4. THE LUCAS CURVE

Table 3 lists the values of x in (2.5), and y in (2.2) where $B = -\sqrt{5}$, for the Lucas case $a = 0$, $b = 1$, when we increase θ by multiples of 0.2.

Figure 2 shows the computer-drawn graph corresponding to the data in Table 3. We may call it the *Lucas curve*.

As in the case of the Fibonacci curve, the locus of the stationary points on the Lucas curve, for which $A = -B = \sqrt{5}$, is the appropriate branches of the hyperbolas

$$y^2 - \frac{\pi}{\log \alpha}xy = \pm\frac{k\pi^2}{\log \alpha}.$$

Table 3. The Lucas Curve

θ	x	y
1	0	0
1.2	1.327375368	-0.329938896
1.4	1.803931433	-0.484868119
1.6	2.302721986	-0.440378493
1.8	2.718049012	-0.247195712
2	3	0
2.2	3.16318597	0.203913452
2.4	3.271099682	0.299664977
2.6	3.405928737	0.272168877
2.8	3.637105513	0.152775352
3	4.000000002	0
3.2	4.49056134	-0.126025444
3.4	5.075031117	-0.185203141
3.6	5.708650725	-0.168209616
3.8	6.355154527	-0.094420360
4	7.000000004	0
4.2	7.653747312	0.077888006
4.4	8.3461308	0.114461836
4.6	9.114579464	0.103959260
4.8	9.992260042	0.058354992
5	11	0
5.2	12.14430866	-0.048137436
5.4	13.42116192	-0.070741305
5.6	14.82323019	-0.064250356
5.8	16.34641457	-0.036065368

θ	x	y
6	18.00000002	0
6.2	19.79805597	0.029750572
6.4	21.76729273	0.043720531
6.6	23.93780966	0.039708904
6.8	26.33967462	0.022289623
7	29.00000003	0
7.2	31.94236463	-0.018386864
7.4	35.18845465	-0.027020774
7.6	38.76103987	-0.024541452
7.8	42.6870892	-0.013775745
8	47.00000006	0
8.2	51.74042062	0.011363707
8.4	56.95574739	0.016699757
8.6	62.69884954	0.015167452
8.8	69.02676384	8.51388E-03
9	76.0000001	0
9.2	83.68278528	-7.02316E-03
9.4	92.14420207	-0.010321017
9.6	101.4598894	-9.37400E-03
9.8	111.713853	-5.26187E-03
10	123.0000002	0

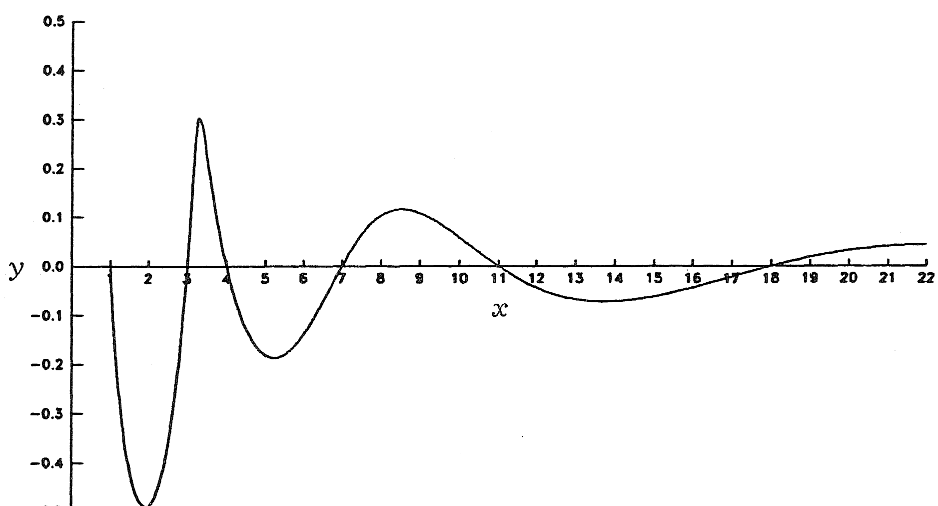

Figure 2. The Lucas Curve

Again, for the Lucas curve, the skewness (obliqueness) of the "vertical" asymptote is visually apparent.

Halsey [1] has no formulas for the Lucas numbers corresponding to those for the Fibonacci numbers, i.e., (3.1) and (3.4). This is because the Pascal triangle generates the Fibonacci numbers but not the Lucas numbers. However, as is well known,

$$L_n = F_{n+1} + F_{n-1} \tag{4.1}$$

for integers. This carries over to real number subscripts, e.g., from Tables 1 and 3,

$$\begin{aligned} F_{7.8} + F_{9.8} &= 69.026763\ldots \quad \text{(to 6 decimal places)} \\ &= L_{8.8}. \end{aligned}$$

On this basis, one could combine $F_{\theta+1}$ and $F_{\theta-1}$ from (3.4) to obtain an integral expression for L_θ.

5. THE *H* CURVES

Putting $a = b = 1$ (i.e., $A = 2\alpha$, $B = 2\beta$) in (1.5), we have, from (1.6),

$$\begin{aligned} H_n &= F_n + L_n \qquad\qquad (5.1) \\ &= F_{n+1} - F_{n-1} + F_{n+1} + F_{n-1} \quad \text{by definition of } F_n \text{ and (4.1)} \\ &= 2F_{n+1}. \end{aligned}$$

Hence, a composite curve for $F_\theta + L_\theta$ is equivalent to the Fibonacci curve for $2F_{\theta+1}$. This H-curve ($a = 1$, $b = 1$) is drawn in Figure 3, where it is to be compared with the Fibonacci and Lucas curves in Figures 1 and 2, respectively.

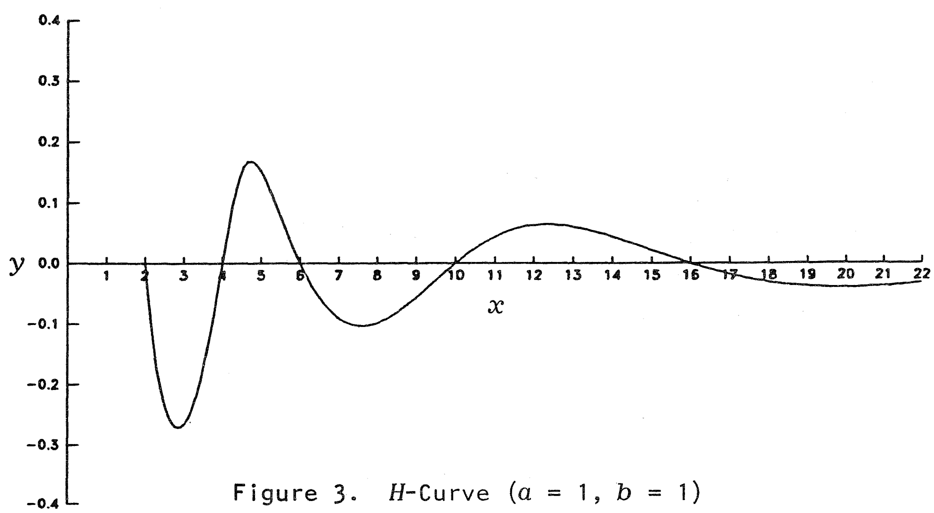

Figure 3. H-Curve ($a = 1$, $b = 1$)

Figure 3 might be taken as an illustration of the conclusion by Stein [5] regarding the intersection of Fibonacci sequences, e.g.,

$$\{F_n\} \cap \{L_n\} = 1,\ 3$$
$$\{F_n\} \cap \{F_n + L_n\} = 2$$
$$\{L_n\} \cap \{F_n + L_n\} = 4$$

Further, from (1.6),

$$\begin{aligned} H_n &= aF_n + bF_{n-1} + bF_{n+1} && \text{by (4.1)} \\ &= aF_n + bF_{n-1} + bF_n + bF_{n-1} && \text{by definition of } F_n \\ &= (a + b)F_n + 2bF_{n-1} \\ &= pF_n + qF_{n-1} \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} p &= a + b, & q &= 2b \\ &= H_1 & &= H_0 \quad \text{as in (1.1).} \end{aligned}$$

ACKNOWLEDGMENT

Acknowledgment is gratefully made to Dr. E. W. Bowen, University of New England, Australia, for assistance with the computer-drawn graphs in Figures 1, 2, and 3. (These graphs are drawn from the equations of the curves, and supersede graphs which were drawn from the tables.)

REFERENCES

1. E. Halsey. "The Fibonacci Number F_u Where u Is Not an Integer." *The Fibonacci Quarterly* 3, no. 2 (1965):147-52, 233.
2. A. F. Horadam. "A Generalized Fibonacci Sequence." *Amer. Math. Monthly* 68 (1961):455-59.
3. A. F. Horadam. "Coaxal Circles Associated With Recurrence-Generated Sequences." *The Fibonacci Quarterly* 22, no. 3 (1983):270-72.
4. A. F. Horadam. "Pell Numbers and Coaxal Circles." *The Fibonacci Quarterly* 22, no. 4 (1984):324-26.
5. S. K. Stein. "The Intersection of Fibonacci Sequences." *Michigan Mathematical Journal* 9 (1962):399-402.
6. L. G. Wilson. Private communication.

◆◇◆◇◆

A NOTE ON THE GENERALIZED FIBONACCI NUMBERS

JIN-ZAI LEE
Dept. of Computer Science, Soochow University, Taipei, Taiwan, R.O.C.

JIA-SHENG LEE
Graduate Institute of Management Sciences, Tamkang University
and
National Taipei Business College, Taipei, Taiwan, R.O.C.

(Submitted March 1986)

1. INTRODUCTION

This note is an extension of the results of L. Carlitz [1] concerning the problem of the multiple generating functions of F_k and L_k, where F_k and L_k are the k^{th} Fibonacci and Lucas numbers, respectively. Our proofs are very similar to those given by Carlitz [1]. Notation and content of [3] are assumed, when required.

Consider the sequence of numbers W_n defined by the second-order recurrence relation

$$W_{n+2} = pW_{n+1} - qW_n, \text{ with } W_0 = a \text{ and } W_1 = b, \tag{1}$$

i.e.,

$$W_n = W_n(a, b; p, q),$$

where a, b, p, and q are real numbers, usually integers.

From [2] and [3], we have

$$W_n = A\alpha^n + B\beta^n, \tag{2}$$

where

$$\begin{cases} \alpha = (p+d)/2,\ \beta = (p-d)/2,\ d = (p^2 - 4q)^{1/2}, \\ A = (b - a\beta)/d,\ B = (a\alpha - b)/d. \end{cases} \tag{3}$$

Standard methods enable us to derive the following generating function for $\{W_n\}$,

$$\sum_{n=0}^{\infty} W_n x^n = \{a + (b - ap)x\}/(1 - px + qx^2). \tag{4}$$

2. MAIN RESULTS

Define

$$\begin{cases} C_W(x_1, \ldots, x_m) = \prod_{j=1}^{m} (1 - qx_j)(1 - V_2 x_j + q^2 x_j^2), \\ W_1(x_1, \ldots, x_m; k) = A\alpha^k \cdot \prod_{j=1}^{m} (a + (b - ap)\alpha x_j)(1 - \beta^2 x_j) \\ \qquad + B\beta^k \cdot \prod_{j=1}^{m} (a + (b - ap)\beta x_j)(1 - \alpha^2 x_j), \\ W_2(x_1, \ldots, x_m; k) = \alpha^k \cdot \prod_{j=1}^{m} (a + (b - ap)\alpha x_j)(1 - \beta^2 x_j) \\ \qquad + \beta^k \cdot \prod_{j=1}^{m} (a + (b - ap)\beta x_j)(1 - \alpha^2 x_j), \end{cases}$$

where $V_n = W_n(2, p; p, q)$. That is, $V_0 = 2$, $V_1 = p$, $V_2 = p^2 - 2q$, ...

Theorem 1: $$\sum_{n_1, \ldots, n_m = 0}^{\infty} W_{n_1 + \cdots + n_m + k} W_{n_1} \ldots W_{n_m} x_1^{n_1} \ldots x_m^{n_m}$$

$$= W_1(x_1, \ldots, x_m; k)/C_W(x_1, \ldots, x_m).$$

Proof: $$\sum_{n_1, \ldots, n_m = 0}^{\infty} W_{n_1 + \cdots + n_m + k} W_{n_1} \ldots W_{n_m} x_1^{n_1} \ldots x_m^{n_m}$$

$$= \sum_{n_1, \ldots, n_m = 0}^{\infty} (A\alpha^{n_1 + \cdots + n_m + k} + B\beta^{n_1 + \cdots + n_m + k}) W_{n_1} \ldots W_{n_m} x_1^{n_1} \ldots x_m^{n_m}, \text{ by (2)}$$

$$= A\alpha^k \sum_{n_1, \ldots, n_m = 0}^{\infty} W_{n_1} \ldots W_{n_m} (\alpha x_1)^{n_1} \ldots (\alpha x_m)^{n_m}$$

$$+ B\beta^k \sum_{n_1, \ldots, n_m = 0}^{\infty} W_{n_1} \ldots W_{n_m} (\beta x_1)^{n_1} \ldots (\beta x_m)^{n_m}$$

$$= A\alpha^k \cdot \prod_{j=1}^{m} (a + (b - ap)\alpha x_j)/(1 - p\alpha x_j + q\alpha^2 x_j^2)$$

$$+ B\beta^k \cdot \prod_{j=1}^{m} (a + (b - ap)\beta x_j)/(1 - p\beta x_j + q\beta^2 x_j^2), \text{ by (4)}$$

$$= A\alpha^k \cdot \prod_{j=1}^{m} (a + (b - ap)\alpha x_j)/\{(1 - \alpha^2 x_j)(1 - \alpha\beta x_j)\}$$

$$+ B\beta^k \cdot \prod_{j=1}^{m} (a + (b - ap)\beta x_j)/\{(1 - \beta^2 x_j)(1 - \alpha\beta x_j)\}, \text{ by (3)}$$

$$= W_1(x_1, \ldots, x_m; k)/C_W(x_1, \ldots, x_m).$$

Using a method similar to that used for Theorem 1, we have

Theorem 2: $$\sum_{n_1, \ldots, n_m = 0}^{\infty} V_{n_1 + \cdots + n_m + k} W_{n_1} \ldots W_{n_m} x_1^{n_1} \ldots x_m^{n_m}$$

$$= W_2(x_1, \ldots, x_m; k)/C_W(x_1, \ldots, x_m).$$

Taking $m = 2$ in Theorems 1 and 2, we obtain

Corollary 1: $$\sum_{m,n=0}^{\infty} W_{m+n+k} W_m W_n x^m y^n = W_1(x, y; k)/C_W(x, y)$$

and $$\sum_{m,n=0}^{\infty} V_{m+n+k} W_m W_n x^m y^n = W_2(x, y; k)/C_W(x, y),$$

where

$$C_W(x, y) = (1 - qx)(1 - qy)(1 - V_2 x + q^2x^2)(1 - V_2 y + q^2 y^2),$$

$$\begin{aligned} W_1(x, y; k) = {} & a^2 W_k + a((b - ap)W_{k+1} - aq^2 W_{k-2})(x + y) \\ & - a(b - ap)q^2 W_{k-1}(x + y)^2 + ((b - ap)^2 W_{k+2} + a^2q^4 W_{k-4})xy \\ & + (b - ap)(aq^4 W_{k-3} - (b - ap)q^2 W_k)xy(x + y) \\ & + (b - ap)^2 q^4 W_{k-2} x^2 y^2, \end{aligned}$$

$$\begin{aligned} W_2(x, y; k) = {} & a^2 V_k + a((b - ap)V_{k+1} - aq^2 V_{k-2})(x + y) \\ & - a(b - ap)q^2 V_{k-1}(x + y)^2 + ((b - ap)^2 V_{k+2} + a^2q^4 V_{k-4})xy \\ & + (b - ap)(aq^4 V_{k-3} - (b - ap)q^2 V_k)xy(x + y) \\ & + (b - ap)^2 q^4 V_{k-2} x^2 y^2. \end{aligned}$$

Taking $k = 0$ in Corollary 1, we derive

Corollary 2: $$\sum_{m,n=0}^{\infty} W_{m+n} W_m W_n x^m y^n = W_1(x, y; 0)/C_W(x, y)$$

and $$\sum_{m,n=0}^{\infty} V_{m+n} W_m W_n x^m y^n = W_2(x, y; 0)/C_W(x, y),$$

where

$$\begin{aligned} W_1(x, y; 0) = {} & a^3 + a(b^2 - a^2(p^2 - q))(x + y) + a(b - ap)^2 q(x + y)^2 \\ & + ((b - ap)^2(bp - aq) + a^3(p^4 - 3p^2q + q^2) \\ & - a^2b(p^3 - 2pq))xy + aq(b - ap)(ap(p^2 - q) \\ & - b(p^2 - 2q))xy(x + y) + (b - ap)^2q^2(a(p^2 - q) - bp)x^2y^2, \end{aligned}$$

$$\begin{aligned} W_2(x, y; 0) = {} & 2a^2 + a(bp - 2a(p^2 - q))(x + y) - a(b - ap)pq(x + y)^2 \\ & + ((b - ap)^2(p^2 - 2q) + a^2(p^4 - 4p^2q + 2q^2))xy \\ & + q(b - ap)(a(p^3 - 3pq) - 2q(b - ap))xy(x + y) \\ & + (b - ap)^2q^2(p^2 - 2q)x^2y^2. \end{aligned}$$

Obviously, all formulas of §2 in [1] are special cases of Theorems 1 and 2 and Corollaries 1 and 2 since $F_n = W_n(0, 1; 1, -1)$ and $L_n = W_n(2, 1; 1, -1)$. Note that (2.2), (2.3), and (2.8) of [1] are misprinted.

Taking $m = 3$ and $k = 0$ in Theorems 1 and 2, we have

Corollary 3: $$\sum_{m,n,k=0}^{\infty} W_{m+n+k} W_m W_n W_k x^m y^n z^k = W_1(x, y, z; 0)/C_W(x, y, z)$$

and $$\sum_{m,n,k=0}^{\infty} V_{m+n+k} W_m W_n W_k x^m y^n z^k = W_2(x, y, z; 0)/C_W(x, y, z),$$

where

$$\begin{aligned} C_W(x, y, z) = {} & (1 - qx)(1 - qy)(1 - qz)(1 - V_2x + q^2x^2)(1 - V_2y \\ & + q^2y^2)(1 - V_2z + q^2z^2), \end{aligned}$$

$$\begin{aligned} W_1(x, y, z; 0) = {} & a^4 + a((b - ap)W_1 - aq^2W_{-2})(x + y + z) + a((b - ap)^2W_2 \\ & + a^2q^4W_{-4})(xy + yz + zx) + ((b - ap)^3W_3 - a^3q^6W_{-6})xyz \\ & - a^2q^2(b - ap)W_{-1}(x + y + z)^2 + a^2q(b - ap) \\ & \cdot (W_3 - (b - ap)q)(x + y + z)(xy + yz + zx) \\ & + a(b - ap)^2q^4W_{-2}(xy + yz + zx)^2 - q^2(b - ap) \\ & \cdot ((b - ap)^2W_1 + a^2q^4W_{-5})xyz(x + y + z) + (b - ap)^2 \\ & \cdot ((b - ap)W_{-1} + aq^2W_{-4})q^4xyz(xy + yz + zx) \\ & - (b - ap)^3q^6W_{-3}x^2y^2z^2, \end{aligned}$$

$$\begin{aligned} W_2(x, y, z; 0) = {} & 2a^3 + a((b - ap)p - aV_2)(x + y + z) + a((b - ap)^2V_2 \\ & + a^2V_4)(xy + yz + zx) + ((b - ap)^3V_3 - a^3V_6)\ xyz \\ & - a^2(b - ap)pq(x + y + z)^2 + a^2q(b - ap) \\ & \cdot (V_3 - (b - ap)q)(x + y + z)(xy + yz + zx) \\ & + aq^2(b - ap)^2V_2(xy + yz + zx)^2 - q(b - ap) \\ & \cdot ((b - ap)^2pq + a^2V_5)xyz(x + y + z) + q^2(b - ap)^2 \\ & \cdot ((b - ap)pq + aV_4)xyz(xy + yz + zx) \\ & - (b - ap)^3q^3V_3x^2y^2z^2. \end{aligned}$$

Obviously, all formulas of §3 in [1] are also special cases of Theorems 1 and 2 and Corollary 3. Note that (3.2)-(3.5) of [1] are misprinted.

Define

$$W(k, m) = A^k\alpha^m - (-B)^k\beta^m.$$

From (2), (3), and the binomial theorem, we have

Lemma 1: $$d^{k-1}W(k, m) = \sum_{r=0}^{k-1}\binom{k-1}{r}(-aq)^r b^{k-r-1}W_{m-r}.$$

Proof:
$$\begin{aligned}
d^{k-1}W(k,\, m) &= d^{k-1}(A^k\alpha^m - (-B)^k\beta^m), \text{ by (2)}\\
&= A\alpha^m(dA)^{k-1} + B\beta^m(-dB)^{k-1}\\
&= A\alpha^m(b - a\beta)^{k-1} + B\beta^m(b - a\alpha)^{k-1}, \text{ by (3)}\\
&= A\alpha^m\sum_{r=0}^{k-1}\binom{k-1}{r}(-a\beta)^r b^{k-r-1} + B\beta^m\sum_{r=0}^{k-1}\binom{k-1}{r}(-a\alpha)^r b^{k-r-1}\\
&= \sum_{r=0}^{k-1}\binom{k-1}{r}(-aq)^r b^{k-r-1}(A\alpha^{m-r} + B\beta^{m-r}), \text{ by (3)}\\
&= \sum_{r=0}^{k-1}\binom{k-1}{r}(-aq)^r b^{k-r-1}W_{m-r}, \text{ by (2)}.
\end{aligned}$$

Define
$$\begin{cases}
D_W(x,\, y,\, z) = d^2(1 - V_2x + q^2x^2)(1 - V_2y + q^2y^2)(1 - V_2z + q^2z^2)\\
W_3(x,\, y,\, z;\, k) = \displaystyle\sum_{j=0}^{3}(-q^2)^j h_j\left\{\sum_{r=0}^{2}\binom{2}{r}(-aq)^r b^{2-r}W_{3k-2j-r}\right\},
\end{cases}$$

where h_j is the j^{th} elementary symmetric function of x, y, and z. That is to say, $h_0 = 1$, $h_1 = x + y + z$, $h_2 = xy + yz + zx$, and $h_3 = xyz$.

Theorem 3:
$$\sum_{m,\, n,\, t=0}^{\infty} W_{m+n+k}W_{n+t+k}W_{t+m+k}x^my^nz^t = W_3(x,\, y,\, z;\, k)/D_W(x,\, y,\, z)$$
$$+ eq^kd^{-2}\Sigma(W_k - q^2W_{k-2}x)/\{(1 - V_2x + q^2x^2)(1 - qy)(1 - qz)\}.$$

Proof:
$$\begin{aligned}
&\sum_{m,\, n,\, t=0}^{\infty} W_{m+n+k}W_{n+t+k}W_{t+m+k}x^my^nz^t\\
&= \sum_{m,\, n,\, t=0}^{\infty}(A\alpha^{m+n+k} + B\beta^{m+n+k})(A\alpha^{n+t+k} + B\beta^{n+t+k})\\
&\qquad\cdot(A\alpha^{t+m+k} + B\beta^{t+m+k})x^my^nz^t, \text{ by (2)}\\
&= A^3\alpha^{3k}/\{(1 - \alpha^2x)(1 - \alpha^2y)(1 - \alpha^2z) + \Sigma A^2Bq^k\alpha^k/\{(1 - \alpha^2x)(1 - qy)\\
&\quad\cdot(1 - qz)\} + \Sigma AB^2q^k\beta^k/\{(1 - \beta^2x)(1 - qy)(1 - qz)\}\\
&\quad + B^3\beta^{3k}/\{(1 - \beta^2x)(1 - \beta^2y)(1 - \beta^2z)\}, \text{ by (4)}\\
&= f(x,\, y,\, z;\, k)/\ (1 - V_2x + q^2x^2)(1 - V_2y + q^2y^2)(1 - V_2z + q^2z^2)\\
&\quad + ABq^k\Sigma(A\alpha^k(1 - \beta^2x) + B\beta^k(1 - \alpha^2x))/\{(1 - V_2x + q^2x^2)\\
&\quad\cdot(1 - qy)(1 - qz)\}\\
&= d^2\cdot f(x,\, y,\, z;\, k)/D_W(x,\, y,\, z)\\
&\quad + eq^kd^{-2}\Sigma(W_k - q^2W_{k-2}x)/\{(1 - V_2x + q^2x^2)(1 - qy)(1 - qz)\},
\end{aligned}$$

where
$$\begin{aligned}
f(x,\, y,\, z;\, k) &= A^3\alpha^{3k}(1 - \beta^2x)(1 - \beta^2y)(1 - \beta^2z)\\
&\quad + B^3\beta^{3k}(1 - \alpha^2x)(1 - \alpha^2y)(1 - \alpha^2z)
\end{aligned}$$

$$= \sum_{j=0}^{3} (-q^2)^j h_j W(3, 3k - 2j).$$

From Lemma 1, we obtain

$$d^2 \cdot f(x, y, z; k) = \sum_{j=0}^{3} (-q^2)^j h_j \left\{ \sum_{r=0}^{2} \binom{2}{r} (-aq)^r b^{2-r} W_{3k-2j-r} \right\}$$

$$= W_3(x, y, z; k),$$

which proves Theorem 3.

Taking $k = 0$ in Theorem 3, we have

Corollary 4: $$\sum_{m,n,t=0}^{\infty} W_{m+n} W_{n+t} W_{t+m} x^m y^n z^t = W_3(x, y, z; 0)/D_W(x, y, z)$$
$$+ ed^{-2} \Sigma (a - q^2 W_{-2} x)/\{(1 - V_2 x + q^2 x^2)(1 - qy)(1 - qz)\}.$$

Obviously, all formulas of §4 in [1] are special cases of Theorem 3 and Corollary 4.

ACKNOWLEDGMENTS

We are deeply thankful to Professor Horng-Jinh Chang and to the referee for their helpful comments and thorough discussions regarding this note.

REFERENCES

1. L. Carlitz. "Fibonacci Notes 2: Multiple Generating Functions." *The Fibonacci Quarterly 12*, no. 3 (1974):179-85.
2. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* 3, no. 2 (1965):161-77.
3. J.-Z. Lee & J.-S. Lee. "Some Properties of the Sequence $\{W_n(a, b; p, q)\}$." *The Fibonacci Quarterly* 25, no. 3 (1987):268-78, 283.

◆◇◆◇◆

GENERALIZED GAUSSIAN LUCAS PRIMORDIAL FUNCTIONS

S. PETHE
University of Malaya, Kuala Lumpur, Malaysia

A. F. HORADAM
University of New England, N.S.W., Australia

(Submitted April 1986)

1. INTRODUCTION

The Fibonacci numbers F_n are defined as $F_0 = 0$, $F_1 = 1$ with the successive numbers given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$.

Horadam[6] extended these numbers to the complex number field by defining them as $F_n^* = F_n + iF_{n+1}$.

Taking a different approach, Berzsenyi [2] defined the set of complex numbers at the Gaussian integers and called them the Gaussian Fibonacci Numbers. He defined them as follows: Let $n \in \mathbb{Z}$ and m be a nonnegative integer. Then, the Gaussian Fibonacci numbers $F(n, m)$ are defined as

$$F(n, m) = \sum_{k=0}^{m} \binom{m}{k} i^k F_{n-k},$$

where F_j are the (real) Fibonacci numbers defined above. He proved that

$$F(n, m) = F(n - 1, m) + F(n - 2, m), \quad n \geqslant 2.$$

This relation implies that any adjacent triplets on the horizontal line possess a Fibonacci-type recurrence relation. In a paper in 1981, Harman (see [4]) elaborated Berzsenyi's idea and defined another set of complex numbers by directly using the Fibonacci recurrence relation. He defined them as follows: Let $(n, m) = n + im$, where $n, m \in \mathbb{Z}$. The complex Fibonacci numbers denoted by $G(n, m)$ are those which satisfy

$$G(0, 0) = 0, \; G(0, 1) = 1, \; G(1, 0) = i, \; G(1, 1) = 1 + i,$$

and

$$G(n + 2, m) = G(n + 1, m) + G(n, m),$$
$$G(n, m + 2) = G(n, m + 1) + G(n, m).$$

The initial values and the recurrence relations are sufficient to specify uniquely the value of $G(n, m)$ for each (n, m) in the plane. It is easy to see that

$$G(n, 0) = F_n \quad \text{and} \quad G(0, m) = iF_m.$$

The advantage of Harman's definition over Berzsenyi's is threefold:

1. While in Berzsenyi's definition, any adjacent horizontal triplets in the plane satisfy the Fibonacci recurrence relation, in Harman's definition, any adjacent horizontal *and* vertical triplets do the same.
2. Horadam's complex Fibonacci numbers F_n^* come as a special case for Harman's. Indeed, $F_m^* = G(1, m)$.
3. By obtaining a recurrence relation for $G(n, m)$ itself, Harman was able to prove some new summation identities for $\{F_n\}$.

Pethe, in collaboration with Horadam, extended Harman's idea to define Generalized Gaussian Fibonacci Numbers [10]. They again denoted these numbers by $G(n, m)$ and defined them at the Gaussian integers (n, m) as follows: Let p_1, p_2 be two fixed nonzero real numbers. Define

$$G(0, 0) = 0,\ G(1, 0) = 1,\ G(0, 1) = i,\ G(1, 1) = p_2 + ip_1,$$

with the conditions $G(n + 2, m) = p_1G(n + 1, m) - q_1G(n, m)$, and $G(n, m + 2) = p_2G(n, m + 1) - q_2G(n, m)$.

With the help of this extension of Harman's definition, the authors were able to obtain a wealth of summation identities involving the combinations of Fibonacci numbers and polynomials, Pell numbers and polynomials, and Chebyshev polynomials of the second kind. Observe that these numbers and polynomials all have the first two initial values as 0 and 1. Consequently, it is natural to ask, as in Remark 4 of [10], if a further extension that would include numbers and polynomials whose first two initial values were other than 0 and 1 is possible. The positive answer to this question is precisely the object of this paper.

Our main result is Theorem 6.1. With the help of a single equation, (6.1) of this theorem, various summation identities involving the product terms of Fermat's numbers, Fibonacci numbers and polynomials, Pell numbers and polynomials, Lucas numbers and polynomials, and Chebyshev polynomials of the first and second kinds are obtained. Besides these identities, (6.1) has the potential for obtaining many more by varying the values of m and n. The extension, first thought to be straightforward, did not turn out to be so. It still had to be formulated in terms of the Lucas fundamental sequence [9] whose first two terms are 0 and 1.

2. PRELIMINARIES

Let $\{U_n\}$ and $\{W_n\}$ denote the sequences defined as follows,

$$U_0 = 0,\ U_1 = 1,\ U_{n+2} = pU_{n+1} - qU_n,\quad n \geqslant 0,$$
$$W_0 = a,\ W_1 = b,\ W_{n+2} = pW_{n+1} - qW_n,\quad n \geqslant 0,$$

where a, b, p, and q are any real numbers, p, $q \neq 0$. The sequence $\{U_n\}$ is the fundamental sequence defined by Lucas and $\{W_n\}$ is the one defined and extensively studied by Horadam (see [9], [7], and [8]). Lucas's primordial function is the special case of $\{W_n\}$ with $W_0 = 2$ and $W_1 = p$. The relation between the terms of $\{W_n\}$ and $\{U_n\}$ is given by

$$W_n = bU_n - aqU_{n-1}. \tag{2.1}$$

Let $\{V_n\}$ be the complex-valued variant of Horadam's sequence defined by $V_0 = a$, $V_1 = ib$, with the recurrence relation $V_{n+2} = pV_{n+1} - qV_n$.

As above, it is clear that

$$V_n = ibU_n - aqU_{n-1}. \tag{2.2}$$

3. DEFINITION

Let (n, m), n, $m \in \mathbb{Z}$, denote the set of Gaussian integers $(n, m) = n + im$. Further, let

$$G: (n, m) \to \mathbb{C},$$

where $\mathbb{C}$ is the set of complex numbers, be the function defined as follows.

For fixed real numbers p and q, define

$$G(0, 0) = a,\ G(1, 0) = b,\ G(0, 1) = ib,\ G(1, 1) = pb(1 + i) \tag{3.1}$$

with the following conditions:

$$G(n + 2, m) = pG(n + 1, m) - qG(n, m), \tag{3.2}$$
and
$$G(n, m + 2) = pG(n, m + 1) - qG(n, m). \tag{3.3}$$

Conditions (3.2) and (3.3) with the initial values (3.1) are sufficient to obtain a unique value for every Gaussian integer.

4. EXPRESSION FOR $G(n, m)$

Lemma 4.1: We have

$$G(n, 0) = W_n, \quad G(0, m) = V_n. \tag{4.1}$$

Proof: The proof is simple and, therefore, omitted here.

Remark: Observe that if $a = 0$ and $b = 1$, the definition for $G(n, m)$ reduces to that of Pethe & Horadam's "Generalised Gaussian Fibonacci Numbers" [10], where $p_1 = p_2 = p$ and $q_1 = q_2 = q$. Further, if $a = 0$, $b = 1$, and $p = 1$, $q = 1$, this definition reduces to Harman's "Complex Fibonacci Numbers" [4].

Theorem 4.2: $G(n, m)$ is given by

$$G(n, m) = bU_nU_{m+1} + aq^2U_{n-1}U_{m-1} + ibU_{n+1}U_m. \tag{4.2}$$

Proof: We use induction for the proof. Suppose (4.2) holds for all integers $0, 1, \ldots, n$ for the first number in the ordered pair (n, m) and for all integers $0, 1, \ldots, m$ for the second number. By (3.2), we have

$$G(n + 1, m) = pG(n, m) - qG(n - 1, m). \tag{4.3}$$

Applying (4.2) to the right side of (4.3), we obtain

$$\begin{aligned} G(n + 1, m) &= p[bU_nU_{m+1} + aq^2U_{n-1}U_{m-1} + ibU_{n+1}U_m] \\ &\qquad - q[bU_{n-1}U_{m+1} + aq^2U_{n-2}U_{m-1} + ibU_nU_m] \\ &= b(pU_n - qU_{n-1})U_{m+1} + aq^2(pU_{n-1} - qU_{n-2})U_{m-1} \\ &\qquad + ib(pU_{n+1} - qU_n)U_m. \end{aligned}$$

Therefore, by the recurrence relation of $\{U_n\}$, we get

$$G(n + 1, m) = bU_{n+1}U_{m+1} + aq^2U_nU_{m-1} + ibU_{n+2}U_m. \tag{4.4}$$

The right side of (4.4) is exactly the right side of (4.2) with n replaced by $n + 1$. Similarly, we prove that

$$G(n, m + 1) = bU_nU_{m+2} + aq^2U_{n-1}U_m + ibU_{n+1}U_{m+1}. \tag{4.5}$$

By (4.4), (4.5), and the induction principle, (4.2) holds for all nonnegative integers.

5. RECURRENCE RELATION FOR $G(n, m)$

Theorem 5.1: For fixed n and m, the recurrence relation for $G(n, m)$ is given by

$$G(n+2k+s,\ m+2k+s) = bp(1+i)\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j+s}U_{m+j+s} \tag{5.1}$$
$$+\ apq^2\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j-2+s}U_{m+j-1+s} + q^{2k}G(n+s,\ m+s),$$

where $s = 0$ or 1.

Proof: For the proof, we again use induction on k. First we find the expressions for $G(n+2,\ m+2)$ and $G(n+3,\ m+3)$. By (4.2), we have

$$\begin{aligned}
G(n+2,\ m+2) &= bU_{n+2}U_{m+3} + aq^2U_{n+1}U_{m+1} + ibU_{n+3}U_{m+2}\\
&= bU_{n+2}(pU_{m+2} - qU_{m+1}) + aq^2(pU_n - qU_{n-1})(pU_m - qU_{m-1})\\
&\quad + ib(pU_{n+2} - qU_{n+1})U_{m+2}\\
&= bp(1+i)U_{n+2}U_{m+2} - bqU_{n+2}U_{m+1} - ibqU_{n+1}U_{m+2}\\
&\quad + aq^2(p^2U_nU_m - pqU_nU_{m-1} - pqU_{n-1}U_m + q^2U_{n-1}U_{m-1})\\
&= bp(1+i)U_{n+2}U_{m+2} - bq(pU_{n+1} - qU_n)U_{m+1} - ibqU_{n+1}(pU_{m+1} - qU_m)\\
&\quad + aq^2(p^2U_nU_m - pqU_nU_{m-1} - pqU_{n-1}U_m + q^2U_{n-1}U_{m-1})\\
&= bp(1+i)(U_{n+2}U_{m+2} - qU_{n+1}U_{m+1}) + ap^2q^2U_nU_m - apq^3U_{n-1}U_m\\
&\quad - apq^3U_nU_{m-1} + q^2(bU_nU_{m+1} + aq^2U_{n-1}U_{m-1} + ibU_{n+1}U_m)\\
&= bp(1+i)(U_{n+2}U_{m+2} - qU_{n+1}U_{m+1}) + apq^2U_n(pU_m - qU_{m-1})\\
&\quad - apq^3U_{n-1}U_m + q^2G(n,\ m).
\end{aligned}$$

Using the recurrence relation for $\{U_m\}$ once again, we finally obtain

$$\begin{aligned}
G(n+2,\ m+2) &= bp(1+i)(U_{n+2}U_{m+2} - qU_{n+1}U_{m+1})\\
&\quad + apq^2(U_nU_{m+1} - qU_{n-1}U_m) + q^2G(n,\ m),
\end{aligned} \tag{5.2}$$

which is the same as (5.1) when $k = 1$ and $s = 0$.

Replacing n and m by $n + 1$ and $m + 1$, respectively, in (5.2) we have

$$\begin{aligned}
G(n+3,\ m+3) &= bp(1+i)(U_{n+3}U_{m+3} - qU_{n+2}U_{m+2})\\
&\quad + apq^2(U_{n+1}U_{m+2} - qU_nU_{m+1}) + q^2G(n+1,\ m+1).
\end{aligned} \tag{5.3}$$

Again, it is easily seen that (5.3) is exactly the same as (5.1) when $k = 1$ and $s = 1$. Thus, (5.1) holds for the initial values $k = 1$, $s = 0$, and $k = 1$, $s = 1$. Suppose next that (5.1) holds for, and up to, some positive integer k. We will show, then, that it also holds for $k + 1$. First let $s = 0$. Now, although n and m are assumed to be fixed in (5.2), it is clear that (5.2) is true for any positive integers n and m. Therefore, we can write the expression for $G(n + 2k + 2,\ m + 2k + 2)$ by replacing n and m in (5.2) by $n + 2k$ and $m + 2k$, respectively. Thus, we have

$$G(n+2k+2,\ m+2k+2) = bp(1+i)(U_{n+2k+2}U_{m+2k+2} - qU_{n+2k+1}U_{m+2k+1}) + apq^2(U_{n+2k}U_{m+2k+1} - qU_{n+2k-1}U_{m+2k}) + q^2G(n+2k,\ m+2k). \tag{5.4}$$

Using (5.1) for $s = 0$ in (5.4), we get

$$\begin{aligned} G(n+2k+2,\ m+2k+2) &= bp(1+i)(U_{n+2k+2}U_{m+2k+2} - qU_{n+2k+1}U_{m+2k+1}) \\ &\quad + apq^2(U_{n+2k}U_{m+2k+1} - qU_{n+2k-1}U_{m+2k}) \\ &\quad + q^2\Big\{bp(1+i)\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j}U_{m+j} \\ &\qquad + apq^2\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j-2}U_{m+j-1} + q^{2k}G(n,\ m)\Big\}. \end{aligned}$$

Combining the first four terms on the right with the corresponding terms in the braces, we have

$$\begin{aligned} G(n+2k+2,\ m+2k+2) &= bp(1+i)\sum_{j=1}^{2k+2}(-1)^j(q)^{2k+2-j}U_{n+j}U_{m+j} \\ &\quad + apq^2\sum_{j=1}^{2k+2}(-1)^j(q)^{2k+2-j}U_{n+j-2}U_{m+j-1} + q^{2k+2}G(n,\ m). \end{aligned} \tag{5.5}$$

Identity (5.5) shows that (5.1) with $s = 0$ is true if k is replaced by $k + 1$. Similarly, we can show that (5.5) with $s = 1$ also holds if k is replaced by $k + 1$. Induction on k then shows that (5.1) holds for all k when k is a positive integer.

6. IDENTITY FOR THE SEQUENCE $\{W_n\}$

Equation (5.1) enables us to prove an important identity involving the product terms of the sequences $\{W_n\}$ and $\{U_n\}$. We prove

Theorem 6.1:

$$\sum_{j=1}^{N} p(-1)^{j+1}(q)^{N-j}W_{n+j}U_{m+j} = q^N W_n U_{m+1} + \begin{cases} W_{n+N+1}U_{m+N}, & N \text{ odd}, \\ -W_{n+N}U_{m+N+1}, & N \text{ even}. \end{cases} \tag{6.1}$$

Proof: Equating the real and imaginary parts of (5.1), we get

$$\begin{aligned} bp\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j+s}U_{m+j+s} + apq^2\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j-2+s}U_{m+j-1+s} \\ = bU_{n+2k+s}U_{m+2k+1+s} + aq^2U_{n+2k-1+s}U_{m+2k-1+s} \\ - q^{2k}(bU_{n+s}U_{m+1+s} + aq^2U_{n-1+s}U_{m-1+s}) \end{aligned} \tag{6.2}$$

and

$$bp\sum_{j=1}^{2k}(-1)^j(q)^{2k-j}U_{n+j+s}U_{m+j+s} = bU_{n+2k+1+s}U_{m+2k+s} - q^{2k}bU_{n+1+s}U_{m+s}. \tag{6.3}$$

Note that if $a = 0$ and $b = 1$, (6.2) and (6.3) reduce, respectively, to (5.1) and (5.2) of [10], where $p_2 = p_1 = p$ and $q_2 = q_1 = q$.

To convert identity (6.2) to the one containing the terms of the sequence $\{W_n\}$, we proceed as follows.

The left-hand side of (6.2) equals

$$p\sum_{j=1}^{2k-1}(-1)^j(q)^{2k-j}(bU_{n+j+s} - aqU_{n+j-1+s})U_{m+j+s} \tag{6.4}$$
$$+ bpU_{n+2k+s}U_{m+2k+s} - apq^{2k+1}U_{n-1+s}U_{m+s}.$$

Using (2.1) in (6.4), we see that the left-hand side of (6.2) equals

$$p\sum_{j=1}^{2k-1}(-1)^j(q)^{2k-j}W_{n+j+s}U_{m+j+s}$$
$$+ bpU_{n+2k+s}U_{m+2k+s} - apq^{2k+1}U_{n-1+s}U_{m+s}.$$

Therefore, equation (6.2), after rearranging terms, becomes

$$\sum_{j=1}^{2k-1}p(-1)^j(q)^{2k-j}W_{n+j+s}U_{m+j+s}$$
$$= bU_{n+2k+s}U_{m+2k+1+s} - bpU_{n+2k+s}U_{m+2k+s} + aq^2U_{n+2k-1+s}U_{m+2k-1+s}$$
$$+ apq^{2k+1}U_{n-1+s}U_{m+s} - aq^{2k+2}U_{n-1+s}U_{m-1+s} - bq^{2k}U_{n+s}U_{m+1+s}$$
$$= bU_{n+2k+s}(U_{m+2k+1+s} - pU_{m+2k+s}) + aq^2U_{n+2k-1+s}U_{m+2k-1+s}$$
$$+ aq^{2k+1}U_{n-1+s}(pU_{m+s} - qU_{m-1+s}) - bq^{2k}U_{n+s}U_{m+1+s}$$
$$= bU_{n+2k+s}(-qU_{m+2k-1+s}) + aq^2U_{n+2k-1+s}U_{m+2k-1+s} + aq^{2k+1}U_{n-1+s}U_{m+1+s}$$
$$- bq^{2k}U_{n+s}U_{m+1+s}$$
$$= -q(bU_{n+2k+s} - aqU_{n+2k-1+s})U_{m+2k-1+s} - q^{2k}(bU_{n+s} - aqU_{n-1+s})U_{m+1+s}.$$

Therefore,

$$\sum_{j=1}^{2k-1}(-1)^{j+1}pq^{2k-j-1}W_{n+j+s}U_{m+j+s} \tag{6.5}$$
$$= W_{n+2k+s}U_{m+2k-1+s} + q^{2k-1}W_{n+s}U_{m+1+s}.$$

Putting $s = 0$ in (6.5), adding $-pW_{n+2k}U_{m+2k}$ to both sides of (6.5), and then using the recurrence relation for $\{U_n\}$, we get

$$\sum_{j=1}^{2k}(-1)^{j+1}pq^{2k-j}W_{n+j}U_{m+j} = -W_{n+2k}U_{m+2k+1} + q^{2k}W_nU_{m+1}. \tag{6.6}$$

Replacing $2k - 1$ and $2k$ in, respectively (6.5) with $s = 0$ and (6.6) by N, we finally obtain (6.1).

7. APPLICATION TO SOME SEQUENCES

7.1 Arithmetic Progression: Let $p = 2$ and $q = 1$. Taking $U_0 = 0$, $U_1 = 1$, and $W_0 = a$, $W_1 = a + d$, it is easily seen that $\{W_n\}$ becomes an arithmetic progression $\{A_n\}$ and $\{U_n\}$, the sequence of nonnegative integers, where $U_n = n$. Equation (6.1) reduces to

$$\sum_{j=1}^{N} 2(-1)^{j+1}(m+j)A_{n+j} = (m+1)A_n + \begin{cases} (m+N)A_{n+N+1}, & N \text{ odd}, \\ -(m+N+1)A_{n+N}, & N \text{ even}. \end{cases} \tag{7.1}$$

7.2 Geometric Progression: Let $p = q + 1$, $W_0 = a$, and $W_1 = aq$. Consequently, the sequence $\{W_n\}$ becomes the geometric progression with common ratio q and $W_n = aq^n$, and the sequence $\{U_n\}$ with $U_0 = 0$ and $U_1 = 1$ has the n^{th} term U_n given by

$$U_n = \sum_{j=0}^{n-1} q^j, \quad n = 1, 2, \ldots .$$

Let us denote the geometric sequence $\{W_n\}$ by $\{G_n^{(q)}\}$. Equation (6.1) reduces to

$$(q+1)\sum_{j=1}^{N} (-1)^{j+1} q^{N-j} G_{n+j}^{(q)} U_{m+j} = q^N G_n^{(q)} U_{m+1} + \begin{cases} G_{n+N+1}^{(q)} U_{m+N}, & N \text{ odd}, \\ -G_{n+N}^{(q)} U_{m+N+1}, & N \text{ even}. \end{cases} \tag{7.2}$$

7.3 Fermat's Sequence: Let $p = 3$, $q = 2$, $W_0 = 2$, and $W_1 = 3$. Then $\{W_n\}$ is Fermat's sequence (see [7]). Let us denote it by $\{M_n\}$. With these values of p and q, $\{U_n\}$ is easily seen to be the sequence given by $U_n = 2^n - 1$. Equation (6.1) reduces to

$$3\sum_{j=1}^{N} (-1)^{j+1} 2^{N-j} M_{n+j} U_{m+j} = 2^N M_n U_{m+1} + \begin{cases} M_{n+N+1} U_{m+N}, & N \text{ odd}, \\ -M_{n+N} U_{m+N+1}, & N \text{ even}. \end{cases} \tag{7.3}$$

Remark: In fact, $\{U_n\}$ is also known as Fermat's sequence. M_n and U_n are given by

$$M_n = 2^n + 1 \quad \text{and} \quad U_n = 2^n - 1.$$

7.4 Fibonacci and Pell Polynomials: Next, let $p = x$ and $q = -1$. Then, with $W_0 = 1$ and $W_1 = x$, $\{W_n\}$ reduces to the Fibonacci sequence $\{F_n(x)\}$, and with $U_0 = 0$ and $U_1 = 1$, $\{U_n\}$ becomes the Pell polynomial sequence $\{P_n(x)\}$, see [5]. It is easy to see that for $x = 1$ and $x = 2$, $\{U_n\}$ reduces to Fibonacci and Pell numbers, respectively, see [5]. Equation (6.1) becomes

$$\sum_{j=1}^{N} xF_{n+j}(x)P_{m+j}(x) = -F_n(x)P_{m+1}(x) + \begin{cases} F_{n+N+1}(x)P_{m+N}(x), & N \text{ odd}, \\ F_{n+N}(x)P_{m+N+1}(x), & \text{even}. \end{cases} \tag{7.4}$$

Remark on Lucas Polynomials: If $p = x$, $q = -1$, $W_0 = x$, and $W_1 = x^2 + 2$, $\{W_n\}$ reduces to the Lucas polynomial sequence $\{L_n(x)\}$ [5]. Since p, q and $\{U_n\}$ are the same as in section 7.4 above, equation (6.1) reduces to (7.4), where F_n is changed to L_n, that is

$$\sum_{j=1}^{N} xL_{n+j}(x)P_{m+j}(x) = -L_n(x)P_{m+1}(x) + \begin{cases} L_{n+N+1}(x)P_{m+N}(x), & N \text{ odd}, \\ L_{n+N}(x)P_{m+N+1}(x), & N \text{ even}. \end{cases} \tag{7.5}$$

7.5 Chebyshev Polynomials: Now let $p = 2x$, $q = 1$, $W_0 = 1$, and $W_1 = x$. Then $W_n(x)$ reduces to the n^{th} Chebyshev polynomial $T_n(x)$ of the first kind and $U_n(x)$ reduces to $S_n(x)$, that of the second kind [1], where

$$T_n(x) = \cos n\theta,\ S_n(x) = \frac{\sin n\theta}{\sin \theta},\ \text{and } \theta = \cos^{-1}x.$$

From (6.1), we obtain

$$\sum_{j=1}^{N} 2(-1)^{j+1}xT_{n+j}(x)S_{m+j}(x) = T_n(x)S_{m+1}(x) + \begin{cases} T_{n+N+1}(x)S_{m+N}(x), & N \text{ odd}, \\ -T_{n+N}(x)S_{m+N+1}(x), & N \text{ even}. \end{cases} \tag{7.6}$$

8. SPECIAL NUMERICAL CASES

Results of section 7 are more comprehensible and more interesting for some particular values of n and m. These are listed below. Some of these identities are known, and some appear to be new.

(A) $n = 0,\ m = 0$

$$\sum_{j=1}^{N} (-1)^{j+1}jA_j = \frac{a}{2} + \begin{cases} \frac{1}{2}NA_{N+1}, & N \text{ odd}, \\ -\frac{1}{2}(N+1)A_N, & N \text{ even}, \end{cases} \tag{7.1*}$$

where $A_0 = a$ is the first term of the arithmetic progression $\{A_n\}$.

$$\sum_{j=1}^{N} (-1)^{j+1}q^{N-j}(q+1)G_j^{(q)}U_j = aq^N + \begin{cases} G_{N+1}^{(q)}U_N, & N \text{ odd}, \\ -G_N^{(q)}U_{N+1}, & N \text{ even}, \end{cases} \tag{7.2*}$$

where a is the first term of the geometric progression $\{G_n^{(q)}\}$. Using the fact that $G_n^{(q)} = aq^n$, we find that (7.2)* reduces to

$$\sum_{j=1}^{N} (-1)^{j+1}(q+1)U_j = 1 + \begin{cases} qU_N, & N \text{ odd}, \\ -U_{N+1}, & N \text{ even}. \end{cases}$$

Observing that in (7.3) $U_n = 2^n - 1$, we see that (7.3), with $n = 0$, $m = 0$, reduces to

$$\sum_{j=1}^{N} (-1)^{j+1}2^{N-j}(2^j - 1)M_j = \frac{1}{3}\left[2^{N+1} + \begin{cases} (2^N - 1)M_{N+1}, & N \text{ odd}, \\ -(2^{N+1} - 1)M, & N \text{ even}; \end{cases}\right] \tag{7.3*}$$

$$\sum_{j=1}^{N} xF_j(x)P_j(x) = -1 + \begin{cases} F_{N+1}(x)P_N(x), & \text{odd}, \\ F_N(x)P_{N+1}(x), & \text{even}; \end{cases} \tag{7.4*}$$

$$\sum_{j=1}^{N} xL_j(x)P_j(x) = - \quad + \begin{cases} L_{N+1}(x)P_N(x), & N \text{ odd}, \\ L_N(x)P_{N+1}(x), & N \text{ even}; \end{cases} \tag{7.5)*$$

$$\sum_{j=1}^{N} (-1)^{j+1} xT_j(x)S_j(x) = \frac{1}{2} + \begin{cases} \dfrac{T_{N+1}S_N}{2}, & N \text{ odd}, \\ \dfrac{-T_N S_{N+1}}{2}, & N \text{ even}. \end{cases} \tag{7.6)*$$

(B) $n = 0,\ m = 1$

$$\sum_{j=1}^{N} (-1)^{j+1}(j+1)A_j = a + \begin{cases} \dfrac{(N+1)A_{N+1}}{2}, & N \text{ odd}, \\ \dfrac{-(N+2)A_N}{2}, & N \text{ even}; \end{cases} \tag{7.1)**$$

$$\sum_{j=1}^{N} (q+1)q^{N-j}(-1)^{j+1}G_j^{(q)}U_{j+1} = q^N a(q+1) + \begin{cases} G_{N+1}^{(q)}U_{N+1}, & N \text{ odd}, \\ -G_N^{(q)}U_{N+2}, & N \text{ even}; \end{cases} \tag{7.2)**$$

$$\sum_{j=1}^{N} (-1)^{j+1}2^{N-j}M_jU_{j+1} = 2^{N+1} + \begin{cases} \dfrac{M_{N+1}U_{N+1}}{3}, & N \text{ odd}, \\ \dfrac{-M_NU_{N+2}}{3}, & N \text{ even}; \end{cases} \tag{7.3)**$$

$$\sum_{j=1}^{N} xF_j(x)P_{j+1}(x) = -x + \begin{cases} F_{N+1}(x)P_{N+1}(x), & N \text{ odd}, \\ F_N(x)P_{N+2}(x), & N \text{ even}; \end{cases} \tag{7.4)**$$

$$\sum_{j=1}^{N} xL_j(x)P_{j+1}(x) = -x^2 + \begin{cases} L_{N+1}(x)P_{N+1}(x) & N \text{ odd}, \\ L_N(x)P_{N+2}(x), & N \text{ even}; \end{cases} \tag{7.5)**$$

$$\sum_{j=1}^{N} (-1)^{j+1}xT_j(x)S_{j+1}(x) = x + \begin{cases} \dfrac{T_{N+1}(x)S_{N+1}(x)}{2}, & N \text{ odd}, \\ \dfrac{-T_N(x)S_{N+2}(x)}{2}, & N \text{ even}. \end{cases} \tag{7.6)**$$

Remark: Obviously, various other identities may be obtained by other choices of n and m. This bears out the fact that this technique provides an abundance of identities by substituting suitable values for m, n, p, and q is just one identity (6.1)!

REFERENCES

1. W. W. Bell. *Special Functions for Scientists and Engineers*. London: D. Van Nostrand Co., Ltd.

2. G. Berzsenyi. "Gaussian Fibonacci Numbers." *The Fibonacci Quarterly* 15, no. 3 (1977):233-36.

3. Marjorie Bicknell. "A Primer for the Fibonacci Numbers: Part VII." *The Fibonacci Quarterly* 8, no. 4 (1970):407-20.

4. C. J. Harman. "Complex Fibonacci Numbers." *The Fibonacci Quarterly* 19, no. 1 (1981):82-86.

5. V. E. Hoggatt, Jr., & Marjorie Bicknell. "Roots of Fibonacci Polynomials." *The Fibonacci Quarterly* 11, no. 3 (1973):271-74.

6. A. F. Horadam. "Complex Fibonacci Numbers and Fibonacci Quaternions." *Amer. Math. Monthly* 70 (1963):289-91.

7. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." *Duke Math. J.* 32 (1965):437-46.

8. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* 3, no. 2 (1965):161-76.

9. E. Lucas. *Théorie des nombres*. Paris: Albert Blanchard, 1961, Ch. 18.

10. S. Pethe & A. F. Horadam. "Generalised Gaussian Fibonacci Numbers." *Bull. Australian Math. Soc.* (to appear).

◆◇◆◇◆

ON PRIME DIVISORS OF SEQUENCES OF INTEGERS INVOLVING SQUARES

M. G. MONZINGO
Southern Methodist University, Dallas, TX 75275

(Submitted May 1986)

The following problem appears on page 65 of *Elementary Number Theory* by David M. Burton:

> Show that 13 is the *largest* prime that can divide two successive integers of the form $n^2 + 3$.

In this note, it will be shown that 13 is the *only* prime that will divide two successive integers of the form $n^2 + 3$, and these pairs will be determined. In addition, the following questions will be investigated: Is the prime 13 unique? That is, if p is an odd prime, is there an integer a such that p is the *largest* prime that divides successive integers of the form $n^2 + a$? And, under what conditions will the prime p be the *only* divisor? Finally, precisely which pairs of successive integers are divisible by p?

The following theorem will answer these questions. The case $p = 13$ will be treated in a corollary following the theorem.

Theorem: Let p be an odd prime. If p is of the form $4k+1$, then p is the *only* prime that divides successive integers of the form $n^2 + k$, and p divides successive pairs precisely when n is of the form $bp + 2k$, for any integer b. If p is of the form $4k + 3$, then p is the *largest* prime that divides successive integers of the form $n^2 + (3k + 2)$, and p divides successive pairs precisely when n is of the form $bp + (2k + 1)$, for any integer b. Furthermore, p will be the *only* prime divisor if and only if $p = 3$.

Proof: In both cases, substitution can be used to show that the prescribed divisibility will hold; hence, only the necessity of the indicated forms will need to be shown.

Let p be of the form $4k + 1$, and suppose that q is any prime divisor of $n^2 + k$ and $(n + 1)^2 + k$. Since q divides the difference of these integers, q must divide $2n + 1$. Now,

$$4(n^2 + k) = (2n + 1)(2n - 1) + (4k + 1).$$

Since q divides both $n^2 + k$ and $2n + 1$, q divides $p = 4k + 1$. Hence, $q = p$, and p is the only such prime divisor. Since p must divide $2n + 1$, $2n + 1 \equiv 0 \pmod{p}$. This congruence has the unique solution, $n \equiv (p - 1)/2 \pmod{p}$; thus, n must be of the form $bp + 2k$, where b is any integer.

Let p be of the form $4k + 3$, and suppose that q is any prime divisor of $n^2 + (3k + 2)$ and $(n + 1)^2 + (3k + 2)$. As before, q must divide $2n + 1$. Now,

$$4(n^2 + (3k + 2)) = (2n + 1)(2n - 1) + 3(4k + 3).$$

As before, q must divide the last term $3(4k + 3)$, but in this case q can be 3 or p. If $p = 3$, then p is the only such prime divisor; if not, then p is simply the largest such prime divisor. (Of course, it should be noted that 3 does, in fact, divide some successive pairs in the case $k > 0$. This will be the case when n is of the form $3c + 1$, c any integer.) Finally, the same argument used previously can be used to show that n must be of the form $bp + (2k + 1)$, b any integer.

Corollary: The prime $p = 13$ is the only prime that divides successive terms of the form $n^2 + 3$ and does so precisely when n is of the form $13b + 6$, where b is any integer.

Proof: The first case of the Theorem applies with $k = 3$.

◆◇◆◇◆

AN EXPANSION OF x^m AND ITS COEFFICIENTS

YASUYUKI IMAI
4-3-14, Bingo-cho, Nada-ku, Kobe, 657 Japan

YASUO SETO
Hyogo Upper Secondary School, 1-4-1, Teraike-cho, Nagata-ku, Kobe, 653 Japan

SHOTARO TANAKA
Naruto University of Education, Takashima, Naruto-shi, 772 Japan

HIROSHI YUTANI
Naruto University of Education, Takashima, Naruto-shi, 772 Japan

(Submitted June 1986)

1. INTRODUCTION

This paper is concerned with an interesting expansion of x^m, where x and m are positive integers, and with the properties of its coefficients. One of the authors, Y. Imai, obtained expressions for 3^6 and 10^7 experimentally.

3^6 is systematically expressed by the sum of products below.

$$\begin{aligned}3^6 = {} & \frac{1}{6!}\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8 + \frac{57}{6!}\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 \\ & + \frac{302}{6!}\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 + \frac{302}{6!}\cdot 0\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5 \\ & + \frac{57}{6!}\cdot (-1)\cdot 0\cdot 1\cdot 2\cdot 3\cdot 4 + \frac{1}{6!}\cdot (-2)\cdot (-1)\cdot 0\cdot 1\cdot 2\cdot 3.\end{aligned}$$

10^7 is systematically expressed by the sum of products below.

$$\begin{aligned}10^7 = {} & \frac{1}{7!}\cdot 10\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15\cdot 16 + \frac{120}{7!}\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15 \\ & + \frac{1191}{7!}\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13\cdot 14 + \frac{2416}{7!}\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13 \\ & + \frac{1191}{7!}\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12 + \frac{120}{7!}\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11 \\ & + \frac{1}{7!}\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10.\end{aligned}$$

To generalize the above expressions, we introduce a notion called the Z-coefficient. We note that Z is a number-theoretic function. We also note the following. If m and x are positive integers, then x^m can be expanded as follows:

$$x^m = \sum_{r=1}^{m}\left(\frac{Z(m, r)}{m!} \cdot \prod_{i=1}^{m}(x + i - r)\right).$$

The numerator $Z(m, r)$ is a number-theoretic function (we call it the Z-coefficient) defined by

$$Z(m, r) = \sum_{k=1}^{r}(-1)^{r+k} \cdot \binom{m+1}{r-k} \cdot k^m, \quad (r = 1, \ldots, m).$$

Another construction method for Z-coefficients $Z(m, r)$, $r = 1, \ldots, m$, and their properties will be given. The Z-coefficients have properties similar to those of the Pascal triangle.

2. PROPERTIES OF EXPANSIONS

These expansions have the following four properties:

1. In each case, the sum of these coefficients is equal to 1. That is:

$$\frac{1}{6!} + \frac{57}{6!} + \frac{302}{6!} + \frac{302}{6!} + \frac{57}{6!} + \frac{1}{6!} = 1,$$

$$\frac{1}{7!} + \frac{120}{7!} + \frac{1191}{7!} + \frac{2416}{7!} + \frac{1191}{7!} + \frac{120}{7!} + \frac{1}{7!} = 1.$$

If we denote these coefficients by $I(6, r)$ and $I(7, r)$, then

$$\sum_{r=1}^{6} I(6, r) = 1 \quad \text{and} \quad \sum_{r=1}^{7} I(7, r) = 1.$$

2. The denominators of these coefficients are 6! and 7! in these cases, respectively. Denoting the numerators of these coefficients by $Z(6, r)$, $r = 1, \ldots, 6$, and $Z(7, r)$, $r = 1, \ldots, 7$, we have

$$I(6, r) = \frac{Z(6, r)}{6!} \quad (r = 1, \ldots, 6), \qquad \sum_{r=1}^{6} Z(6, r) = 6!.$$

$$I(7, r) = \frac{Z(7, r)}{7!} \quad (r = 1, \ldots, 7), \qquad \sum_{r=1}^{7} Z(7, r) = 7!.$$

$Z(6, r)$ and $Z(7, r)$ are called Z-coefficients.

3. In both cases, Z-coefficients systematically distribute, i.e.,

   1, 57, 302, 302, 57, 1 and 1, 120, 1191, 2416, 2291, 120, 1.

4. In the expressions for 3^6 and 10^7, the first members of each product except their coefficients are, respectively,

   3, 2, 1, 0, -1, -2 and 10, 9, 8, 7, 6, 5, 4.

As is easily seen, the first integers of these descending sequences are 3 (the base of 3^6) and 10 (the base of 10^7).

The question now arises: Can we generalize the above properties?

3. THE COEFFICIENTS $Z(m, r)$ AND THE THEOREM

The answer to the question above is affirmative. We now have the following definition and theorem.

Definition: Let m and r be integers. $Z(m, r)$ is defined by

$$Z(m, r) = \sum_{k=1}^{r} (-1)^{r+k} \cdot \binom{m+1}{r-k} \cdot k^m, \quad (m \geqslant 1,\ r = 1, \ldots, m), \tag{1}$$

$$Z(m, r) = 0 \text{ for } m \leqslant 0 \text{ or } r \leqslant 0 \text{ or } m < r.$$

Theorem: Let x and m be positive integers. Then

$$\begin{aligned} x^m &= \frac{Z(m, 1)}{m!} \cdot x \cdot (x+1) \cdot (x+2) \cdot \cdots \cdot (x + (m-1)) \\ &\quad + \frac{Z(m, 2)}{m!} \cdot (x-1) \cdot x \cdot \cdots \cdot (x + (m-2)) \\ &\quad + \cdots + \frac{Z(m, m)}{m!} \cdot (x - (m-1)) \cdot \cdots \cdot x \\ &= \sum_{r=1}^{m} \left(\frac{Z(m, r)}{m!} \cdot \prod_{i=1}^{m} (x + i - r) \right). \end{aligned} \tag{2}$$

In order to prove the Theorem, we need the following Lemmas concerning the Z-coefficients.

Lemma 1: Let $Z(m, r)$ be Z-coefficients. Then:

$$Z(m+1, r) = (m - r + 2) \cdot Z(m, r-1) + r \cdot Z(m, r); \tag{3}$$

$$Z(m, r) = (m - r + 1) \cdot Z(m-1, r-1) + r \cdot Z(m-1, r); \tag{4}$$

$$Z(m+1, r+1) = (m - r + 1) \cdot Z(m, r) + (r+1) \cdot Z(m, r+1). \tag{5}$$

Proof of Lemma 1: It is clear that (3), (4), and (5) are equivalent to each other. We prove (5). By the definition of $Z(m, r)$, the right-hand side of (5) is written in the form

$$\sum_{k=1}^{r} \left((-1)^{r+k} \cdot (m - r + 1) \cdot \binom{m+1}{r-k} \cdot k^m \right) + \sum_{k=1}^{r} \left((-1)^{r+1+k} \cdot (r+1) \right.$$
$$\left. \cdot \binom{m+1}{r+1-k} \cdot k^m \right) + (-1)^{2r+2} \cdot (r+1) \cdot \binom{m+1}{0} \cdot (r+1)^m.$$

A general term is expressed by the following:

$$(-1)^{r+k} \cdot (m - r + 1) \cdot \binom{m+1}{r-k} \cdot k^m + (-1)^{r+1+k} \cdot (r + 1) \cdot \binom{m+1}{r+1-k} \cdot k^m$$

$$= (-1)^{r+k} \cdot k^m \cdot \frac{(m+1)!}{(m + 1 - r + k)! \cdot (r + 1 - k)!} \cdot (-k) \cdot (m + 2)$$

$$= (-1)^{r+k+1} \cdot k^{m+1} \cdot \binom{m+2}{r-k+1}.$$

Therefore, the right-hand side of (5) is equal to

$$\sum_{k=1}^{r}\left((-1)^{r+k+1} \cdot \binom{m+2}{r-k+1} \cdot k^{m+1}\right) + (-1)^{2r+2} \cdot (r + 1) \cdot \binom{m+1}{0} \cdot (r + 1)^m,$$

which is

$$\sum_{k=1}^{r+1}\left((-1)^{r+k+1} \cdot \binom{m+2}{r-k+1} \cdot k^{m+1}\right).$$

By the definition of $Z(m, r)$, the last expression is equal to $Z(m + 1, r + 1)$. Hence, the proof is complete.

Lemma 2: Let $Z(m, r)$ be Z-coefficients. Then:

$$\sum_{r=1}^{m} Z(m, r) = m!, \quad (m \geqslant 1,\ r = 1, \ldots, m); \tag{6}$$

$$Z(m, r) = Z(m, m + 1 - r). \tag{7}$$

Equation (7) shows that Z-coefficients distribute symmetrically.

Proof of (6): By (4), the following equalities hold:

$$\begin{aligned}
Z(m, 1) &= m \cdot Z(m - 1, 0) + 1 \cdot Z(m - 1, 1),\\
Z(m, 2) &= (m - 1) \cdot Z(m - 1, 1) + 2 \cdot Z(m - 1, 2),\\
Z(m, 3) &= (m - 2) \cdot Z(m - 1, 2) + 3 \cdot Z(m - 1, 3),\\
\vdots\quad & \quad\ \vdots\\
Z(m, m) &= 1 \cdot Z(m - 1, m - 1) + m \cdot Z(m - 1, m).
\end{aligned}$$

From these equalities with $Z(m - 1, 0) = 0$ and $Z(m - 1, m) = 0$, we have

$$\begin{aligned}
\sum_{r=1}^{m} Z(m, r) &= m \cdot (Z(m - 1, 1) + \cdots + Z(m - 1, m - 1))\\
&= m \cdot \sum_{r=1}^{m-1} Z(m - 1, r).
\end{aligned}$$

Hence, by the definition of $Z(1, 1)$,

$$\sum_{r=1}^{m} Z(m, r) = m \cdot (m - 1) \cdot \cdots \cdot 2 \cdot Z(1, 1) = m!.$$

Proof of (7): We prove (7) by induction on m. It is clear that (7) holds for $m = 1$. We assume that (7) holds for the positive integers not greater than m. We now show that (7) holds for $m + 1$, i.e.,

$$Z(m + 1, r) = Z(m + 1, m + 2 - r). \tag{8}$$

By (3), we have

$$Z(m + 1, m + 2 - r) = r \cdot Z(m, m - r + 1) + (m + 2 - r) \cdot Z(m, m + 2 - r).$$

By the induction hypothesis,

$$Z(m, m - r + 1) = Z(m, r), \qquad Z(m, m + 2 - r) = Z(m, r - 1).$$

Hence, by (3),

$$\begin{aligned} Z(m + 1, m + 2 - r) &= r \cdot Z(m, r) + (m - r + 2) \cdot Z(m, r - 1) \\ &= Z(m + 1, r). \end{aligned}$$

Therefore, (8) holds, as required.

Now, we return to the proof of the Theorem.

Proof of Theorem: We shall prove the Theorem by induction on m. It is clear that the Theorem holds for $m = 1$. We assume that (2) holds for positive integers not greater than m. We shall prove that (2) holds for $m + 1$, i.e.,

$$x^{m+1} = \frac{1}{(m + 1)!} \cdot \sum_{r=1}^{m+1} \left(Z(m + 1, r) \cdot \prod_{i=1}^{m+1} (x + i - r) \right). \tag{9}$$

By (3), we have

$$\begin{aligned} &\sum_{r=1}^{m+1} \left(Z(m + 1, r) \cdot \prod_{i=1}^{m+1} (x + i - r) \right) \\ &\quad = \sum_{r=1}^{m+1} \left((m - r + 2) \cdot Z(m, r - 1) \cdot \prod_{i=1}^{m+1} (x + i - r) \right) \\ &\qquad + \sum_{r=1}^{m+1} \left(r \cdot Z(m, r) \cdot \prod_{i=1}^{m+1} (x + i - r) \right). \end{aligned}$$

Since $Z(m, r - 1) = 0$ for $r = 1$ and $Z(m, r) = 0$ for $r = m + 1$, the right-hand side of the above is equal to

$$\begin{aligned} &\sum_{r=2}^{m+1} \left((m - r + 2) \cdot Z(m, r - 1) \cdot \prod_{i=1}^{m+1} (x + i - r) \right) \\ &\qquad + \sum_{r=1}^{m} \left((r \cdot Z(m, r) \cdot \prod_{i=1}^{m+1} (x + i - r) \right). \end{aligned}$$

Changing $r - 1$ to r in the first term, we have

$$\sum_{r=1}^{m}\left((m+1-r)\cdot Z(m, r)\cdot \prod_{i=1}^{m+1}(x+i-r-1)\right)$$

$$+\sum_{r=1}^{m}\left(r\cdot Z(m, r)\cdot \prod_{i=1}^{m+1}(x+i-r)\right)$$

$$= (m+1)\cdot \sum_{r=1}^{m}\left(Z(m, r)\cdot \prod_{i=1}^{m+1}(x+i-r-1)\right)$$

$$+\sum_{r=1}^{m}\left(r\cdot Z(m, r)\cdot\left(\prod_{i=1}^{m+1}(x+i-r) - \prod_{i=1}^{m+1}(x-1+i-r)\right)\right)$$

$$= (m+1)\cdot \sum_{r=1}^{m}\left(Z(m, r)\cdot \prod_{i=1}^{m+1}(x+i-r-1)\right)$$

$$+ (m+1)\cdot \sum_{r=1}^{m}\left(r\cdot Z(m, r)\cdot \prod_{i=1}^{m}(x+i-r)\right)$$

$$= (m+1)\cdot x\cdot \sum_{r=1}^{m}\left(Z(m, r)\cdot \prod_{i=1}^{m}(x+i-r)\right).$$

By the induction hypothesis, the last expression is equal to

$$(m+1)!\cdot x^{m+1}.$$

Hence, (9) holds, as required.

4. REMARKS

4.1 If x and m $(x < m)$ are positive integers, then (2) is reduced as follows:

$$x^m = \frac{Z(m, 1)}{m!}\cdot x\cdot (x+1)\cdot \cdots \cdot (x+(m-1)) + \frac{Z(m, 2)}{m!}\cdot (x-1)\cdot x\cdot \cdots$$

$$\cdot (x+(m-2)) + \cdots + \frac{Z(m, x)}{m!}\cdot 1\cdot 2\cdot \cdots \cdot m$$

$$= \sum_{r=1}^{x}\left(\frac{Z(m, r)}{m!}\cdot \prod_{i=1}^{m}(x+i-r)\right).$$

4.2 Calculating $Z(m, r)$ for $1 \leqslant m \leqslant 6$, $r = 1, \ldots, m$, the following triangle is obtained:

$m = 1$						1						
$m = 2$					1		1					
$m = 3$				1		4		1				
$m = 4$			1		11		11		1			
$m = 5$		1		26		66		26		1		
$m = 6$	1		57		302		302		57		1	

Clearly, this triangle is obtained by simple calculation. For example, to get

$Z(6, 3) = 302$, write all the values of $Z(5, r)$ $(r = 1, 2, 3, 4, 5)$ in one line from left to right (see the line for $m = 5$). Next, write r as a left subscript for $Z(5, r)$, i.e., ${}_rZ(5, r)$. Finally, write $5 - (r - 1)$ as the right subscript for $Z(5, r)$, i.e., $Z(5, r)_{5-(r-1)}$. Then, we obtain

$$\begin{array}{ccccc} {}_1 1_5 & {}_2 26_4 & {}_3 66_3 & {}_4 26_2 & {}_5 1_1 \\ & \multicolumn{2}{c}{\searrow\ \swarrow} & & \\ & \multicolumn{2}{c}{302} & & \end{array}$$

$Z(6, 3) = 302 = 26 \cdot 4 + 3 \cdot 66$, which gives equation (4):

$$Z(m, r) = (m - r + 1) \cdot Z(m - 1, r - 1) + r \cdot Z(m - 1, r).$$

The symmetry of Z-coefficients is clear from the viewpoint of this construction method. The Pascal triangle is a special case of our triangle, i.e., the Pascal triangle is obtained by using 1 for all right- and left-hand subscripts. Let us call our triangle the "I-triangle."

4.3 By (6), it is clear that

$$\sum_{r=1}^{m} I(m, r) = \sum_{r=1}^{m} \frac{Z(m, r)}{m!} = 1.$$

4.4 It is an interesting problem to find the relation between Z-coefficients and Stirling numbers of the second kind (see [1]).

REFERENCE

1. M. Aigner. *Combinatorial Theory*. New York: Springer-Verlag, 1979.

◆◇◆◇◆

A NEW GENERALIZATION OF DAVISON'S THEOREM

DOUGLAS BOWMAN
P.O. Box 91592, Pasadena, CA 91109

(Submitted June 1986)

1. INTRODUCTION

In [3] Davison proved that

$$\sum_{n \geqslant 1} \frac{1}{2^{\lfloor n\alpha \rfloor}} = \frac{1}{2^{F_0} +} \frac{1}{2^{F_1} +} \frac{1}{2^{F_2} + \cdots}, \text{ with } \alpha = \frac{1 + \sqrt{5}}{2},$$

where $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, for $n \geqslant 0$, and $\lfloor x \rfloor$ is the greatest integer $\leqslant x$. In [1] the authors found the simple continued fraction for

$$T(x, C) = (C - 1) \sum_{n \geqslant 1} \frac{1}{C^{\lfloor nx \rfloor}}, \text{ with real } x > 1 \text{ and } C > 1.$$

In this paper, we shall prove a new generalization of Davison's Theorem (see Theorem 1).

2. CONVENTIONS AND USEFUL THEOREMS

Throughout this paper, make the following conventions:

$$\alpha = \frac{1 + \sqrt{5}}{2}.$$

Let F_n be defined for negative n by $F_{n+2} = F_{n+1} + F_n$.

Define Y_n by: Y_0 and Y_1 are given real numbers such that $Y_0 + Y_1\alpha > 0$, and all other values of Y_n are defined by $Y_{n+2} = Y_{n+1} + Y_n$, n any integer.

Also, throughout, let the Fibonacci representation of an integer $K \geqslant 1$ be written as

$$K = F_{V_1} + F_{V_2} + \cdots + F_{V_n}, \tag{1}$$

where $2 \leqslant V_1 <_2 V_2 <_2 \ldots <_2 V$ and $a <_2 b$ means that $a + 2 \leqslant b$.

Define the function $e(K)$, for K an integer $\geqslant 0$, by

$$e(K) = 0 \text{ if } K = 0;$$

otherwise,

$$e(K) = F_{V_1 - 1} + F_{V_2 - 1} + \cdots + F_{V_n - 1}, \text{ where } K \text{ has the representation (1).}$$

In the paper [4], setting $\alpha = \dfrac{\sqrt{5}+1}{2}$ gives

$$e(k) = \lfloor (k+1)\alpha^{-1} \rfloor, \text{ for } k \geqslant 0. \tag{2}$$

The convergence ranges for the series in this paper can easily be justified by comparing the series to geometric series. Because of the limit passing below, the convergence ranges for the continued fractions are also justified.

From [6], we will use the Euler-Minding Theorem:

If $\dfrac{A_p}{B_p} = 1 + \dfrac{C_1}{1+}\,\dfrac{C_2}{1+}\,\dfrac{C_3}{1+\cdots}\,\dfrac{C_p}{1}$, where $\{C_k\}$ is a sequence of nonzero real numbers for $k \geqslant 1$, then,

$$A_p = 1 + \sum_{n \geqslant 1,\ 1 \leqslant V_1 <_2 \cdots <_2 V_n \leqslant P} C_{V_1} C_{V_2} \cdots C_{V_n},$$

and

$$B_p = 1 + \sum_{n \geqslant 1,\ 2 \leqslant V_1 <_2 \cdots <_2 V_n \leqslant P} C_{V_1} C_{V_2} \cdots C_{V_n}.$$

Actually, all that is needed is the following corollary:

Write $A\ (C_1, C_2, \ldots, C_p) = A_p$, then notice $B_p = A_{p-1}(C_2, C_3, \ldots, C_p)$. Now, let $P \to \infty$ and we have:

$$1 + \frac{C_1}{1+}\,\frac{C_2}{1+}\,\frac{C_3}{1+\cdots} = \frac{A_\infty(C_1, C_2, \ldots)}{A_\infty(C_2, C_3, \ldots)}. \tag{3}$$

Notice that the indices on the summation for A_∞ will be:

$$n \geqslant 1,\ 1 \leqslant V_1 <_2 V_2 <_2 \cdots <_2 V_n.$$

3. THE MAIN THEOREMS

Theorem 1:
$$\frac{\displaystyle\sum_{n \geqslant 1} \left(\frac{1}{C}\right)^{Y_0 n + Y_{-1} \lfloor n\alpha^{-1} \rfloor}}{\displaystyle\sum_{n \geqslant 1} \left(\frac{1}{C}\right)^{Y_1 n + Y_0 \lfloor n\alpha^{-1} \rfloor}} = C^{Y_{-1}} + \frac{1}{C^{Y_0}+}\,\frac{1}{C^{Y_1}+}\,\frac{1}{C^{Y_2}+}\,\frac{1}{C^{Y_3}+\cdots},$$
where $C > 1$.

Proof: Set $C_n = a^{F_{n-1}} b^{F_n}$ in (3), with $|a|, |b| \leqslant 1$, not both 1, to get

$$1 + \frac{a^{F_0} b^{F_1}}{1+}\,\frac{a^{F_1} b^{F_2}}{1+\cdots} = \frac{1 + \displaystyle\sum_{n \geqslant 1,\ 1 \leqslant V_1 <_2 \cdots <_2 V_n} a^{F_{v_1-1}+\cdots+F_{v_n-1}} b^{F_{v_1}+\cdots+F_{v_n}}}{1 + \displaystyle\sum_{n \geqslant 1,\ 1 \leqslant V_1 <_2 \cdots <_2 V_n} a^{F_{v_1}+\cdots+F_{v_n}} b^{F_{v_1+1}+\cdots+F_{v_n+1}}}$$

Denote the numerator by $F(a, b)$ and the denominator by $G(a, b)$.

Now,

$$F(b, ab) = 1 + \sum_{n \geqslant 1,\ 1 \leqslant V_1 <_2 \cdots <_2 V_n} a^{F_{v_1}+\cdots+F_{v_n}} b^{F_{v_1+1}+\cdots+F_{v_n+1}} = G(a, b). \tag{4}$$

Hence, we have

$$\frac{F(a,\ b)}{F(b,\ ab)} = 1 + \frac{a^{F_0}b^{F_1}}{1 +}\ \frac{a^{F_1}b^{F_2}}{1 + \cdots}. \tag{5}$$

From this, it follows that

$$\frac{F(b,\ ab)}{F(ab,\ ab^2)} = 1 + \frac{a^{F_1}b^{F_2}}{1 +}\ \frac{a^{F_2}b^{F_3}}{1 + \cdots},$$

so we find that

$$F(a,\ b) = F(b,\ ab) + bF(ab,\ ab^2), \tag{6}$$

with $|a|,\ |b| \leqslant 1$, and not both 1.

An expansion for $F(a,\ b)$ could now be reached by setting

$$F(a,\ b) = \sum k_{n,m}a^n b^m, \text{ with } n,\ m \geqslant 0,$$

and equating coefficients in (6), but this route is tedious. Instead, notice that if in (4) the exponent of b is k, then the exponent of a will be $e(k)$ and because of Zeckendorf's Theorem (see [2]), k will range over the integers > 0. Hence,

$$F(b,\ ab) = 1 + \sum_{n \geqslant 1} a^{e(n)}b^n = \sum_{n \geqslant 0} a^{e(n)}b^n.$$

Thus, we also get

$$F(a,\ b) = \sum_{n \geqslant 0} a^{n - e(n)}b^{e(n)}.$$

Using (2), we have

$$F(a,\ b) = \sum_{n \geqslant 0} a^{n - \lfloor (n+1)\alpha^{-1} \rfloor} b^{\lfloor (n+1)\alpha^{-1} \rfloor}, \tag{7}$$

and

$$F(b,\ ab) = \sum_{n \geqslant 0} a^{\lfloor (n+1)\alpha^{-1} \rfloor} b^n. \tag{8}$$

Let $a = C^A$ and $b = C^B$ in (7) and (8) to get

$$F(C^A,\ C^B) = \sum_{n \geqslant 1} C^{A(n-1) + (B-A)\lfloor n\alpha^{-1} \rfloor}, \tag{9}$$

and

$$F(C^B,\ C^{A+B}) = \sum_{n \geqslant 1} C^{B(n-1) + A\lfloor n\alpha^{-1} \rfloor}. \tag{10}$$

Set $A = Y_0 - Y_1$ and $B = -Y_0$ in (10) to get

$$F\left(\left(\frac{1}{C}\right)^{Y_0},\ \left(\frac{1}{C}\right)^{Y_1}\right) = \sum_{n \geqslant 1} \left(\frac{1}{C}\right)^{Y_0(n-1) + (Y_1 - Y_0)\lfloor n\alpha^{-1} \rfloor},\quad |C| > 1, \tag{11}$$

or set $A = -Y_0$ and $B = -Y_1$ in (10) to get

$$F\left(\left(\frac{1}{C}\right)^{Y_1},\ \left(\frac{1}{C}\right)^{Y_0 + Y_1}\right) = \sum_{n \geqslant 1} \left(\frac{1}{C}\right)^{Y_1(n-1) + Y_0\lfloor n\alpha^{-1} \rfloor},\quad |C| > 1. \tag{12}$$

From (5), we see that

$$\frac{F(C^{Y_0}, C^{Y_1})}{F(C^{Y_1}, C^{Y_0+Y_1})} = 1 + \frac{C^{Y_0F_0+Y_1F_1}}{1+}\;\frac{C^{Y_0F_1+Y_1F_2}}{1+}\;\frac{C^{Y_0F_2+Y_1F_3}}{1+\cdots}, \; 0 < C < 1.$$

It is easy to show by induction that $Y_n = Y_0F_{n-1} + Y_1F_n$, for integer n; hence,

$$\frac{F(C^{Y_0}, C^{Y_1})}{F(C^{Y_1}, C^{Y_0+Y_1})} = 1 + \frac{C^{Y_1}}{1+}\;\frac{C^{Y_2}}{1+}\;\frac{C^{Y_3}}{1+\cdots}, \; 0 < C < 1.$$

Replacing C with its reciproval variable,

$$\frac{F\left(\left(\frac{1}{C}\right)^{Y_0}, \left(\frac{1}{C}\right)^{Y_1}\right)}{F\left(\left(\frac{1}{C}\right)^{Y_1}, \left(\frac{1}{C}\right)^{Y_0+Y_1}\right)} = 1 + \frac{C^{-Y_1}}{1+}\;\frac{C^{-Y_2}}{1+}\;\frac{C^{-Y_3}}{1+\cdots}, \; C > 1$$

$$= 1 + \frac{C^{Y_0}C^{-Y_1}}{C^{Y_0}+}\;\frac{C^{Y_0}C^{Y_1}C^{-Y_2}}{C^{Y_1}+}\;\frac{C^{Y_1}C^{Y_2}C^{-Y_3}}{C^{Y_2}+}\;\frac{C^{Y_2}C^{Y_3}C^{-Y_4}}{C^{Y_3}+\cdots}, \; C > 1,$$

(by the equivalence relation (3.1) of [7])

$$= 1 + \frac{C^{Y_0-Y_1}}{C^{Y_0}+}\;\frac{1}{C^{Y_1}+}\;\frac{1}{C^{Y_2}+}\;\frac{1}{C^{Y_3}+\cdots}, \; C > 1.$$

Hence,

$$\frac{F\left(\left(\frac{1}{C}\right)^{Y_0}, \left(\frac{1}{C}\right)^{Y_1}\right)C^{-Y_0}}{F\left(\left(\frac{1}{C}\right)^{Y_1}, \left(\frac{1}{C}\right)^{Y_0+Y_1}\right)C^{-Y_1}} = C^{Y_{-1}} + \frac{1}{C^{Y_0}+}\;\frac{1}{C^{Y_1}+}\;\frac{1}{C^{Y_2}+}\;\frac{1}{C^{Y_3}+\cdots}, \; C > 1.$$

Substituting in (11) and (12) and simplifying yields the theorem.

Theorem 2: $\sum_{n\geqslant 1} C^{A(n-1)+(B-A)\lfloor n\alpha^{-1}\rfloor} = \sum_{n\geqslant 1} C^{B(n-1)+A\lfloor n\alpha^{-1}\rfloor} + C^{A(n-1)+B\lfloor n\alpha\rfloor}$, for $|C| < 1$.

Proof: Let $a = C^{A+B}$ and $b = C^{A+2B}$ in (7) and simplify to get

$$F(C^{A+B}, C^{A+2B}) = \sum_{n\geqslant 1} C^{(A+B)(n-1)+B\lfloor n\alpha^{-1}\rfloor}. \tag{13}$$

Let $a = C^A$ and $b = C^B$ in (6) to get

$$F(C^A, C^B) = F(C^B, C^{A+B}) + C^B F(C^{A+B}, C^{A+2B}). \tag{14}$$

Now substitute (9), (10), and (13) into (14) and simplify to get the theorem.

Corollary 1: If $T = \sum_{n\geqslant 1} C^{Y_k n + Y_{k+1}\lfloor n\alpha\rfloor}$, for $C < 1$, then $T_{k+2} = T_k - C^{-Y_{k+1}}T_{k+1}$, where k is any integer.

Proof: Let $A = Y_{k+2}$ and $B = Y_{k+3}$ in Theorem 2 and simplify.

Corollary 2: $\sum_{n\geqslant 1} C^{F_k n + F_{k+1}\lfloor n\alpha\rfloor}$, for $C < 1$, can be evaluated in terms of $\sum_{n\geqslant 1} C^{\lfloor n\alpha\rfloor}$ and rational functions of C for any integer k. For example,

$$\sum_{n\geqslant 1} C^{n+2\lfloor n\alpha\rfloor} = (1 + C^{-1})\sum_{n\geqslant 1} C^{\lfloor n\alpha\rfloor} - (1 + C)^{-1}. \tag{15}$$

Proof: Put $Y_k = F_k$ in Corollary 1. Notice that

$$T_{-1} = \sum_{n\geqslant 1} C^n = \frac{C}{C-1} \quad \text{and} \quad T_0 = \sum_{n\geqslant 1} C^{\lfloor n\alpha\rfloor}.$$

Now Corollary 2 follows by induction using Corollary 1. For example, we find

$$T_1 = \frac{C}{C-1} - T_0 \quad \text{or} \quad \sum_{n\geqslant 1} C^{\lfloor n\alpha^2\rfloor} = \frac{C}{C-1} - \sum_{n\geqslant 1} C^{\lfloor n\alpha\rfloor},$$

which is easily verified by Beatty's Theorem (see [5]). Applying Corollary 1 another time gives (15).

Corollary 3: $\sum_{n\geqslant 1} \left(\frac{1}{C}\right)^{F_k n + F_{k+1}\lfloor n\alpha\rfloor}$ is trancendental for integer $k \neq -1$ and integer $C > 1$.

Proof: From Corollary 2 we can see that the sum for $k \neq -1$ and rational function of C added to a rational function of C multiplied by $\sum_{n\geqslant 1}\left(\frac{1}{C}\right)^{\lfloor n\alpha\rfloor}$ which is transcendental by setting $\alpha = (1 + \sqrt{5})/2$ in [1]. We can show by induction that the rational function which multiplies $\sum_{n\geqslant 1}\left(\frac{1}{C}\right)^{\lfloor n\alpha\rfloor}$ is nonzero; hence, the corollary follows.

Corollary 4: If A and B are integers not both zero, then the number of times that any integer occurs in the sequence

$$A(n - 1) + (B - A)\lfloor n\alpha^{-1}\rfloor, \text{ for } n \geqslant 1,$$

is equal to the total number of times that integer occurs in the following sequences:

$$B(n - 1) + A\lfloor n\alpha^{-1}\rfloor, \text{ for } n \geqslant 1, \text{ and } A(n - 1) + B\lfloor n\alpha\rfloor, \text{ for } n \geqslant 1.$$

Proof: The proof follows immediately from Theorem 2.

REFERENCES

1. W. W. Adams & J. L. Davison. "A Remarkable Class of Continued Fractions." *Proceedings of the American Mathematical Society* 65, no. 2 (1977):194-98.
2. L. Carlitz. "Fibonacci Representations." *The Fibonacci Quarterly* 6, no. 2 (1968):193-220.

3. J. L. Davison. "A Series and Its Associated Continued Fraction." *Proceedings of the American Mathematical Society 63* (1977):29-32.

4. A. S. Fraenkel, J. Levitt, & M. Shimshoni. "Characterization of the Set of Values $f(n) = [n\alpha]$." *Discrete Mathematics 2* (1977):335-45.

5. J. Lambek & L. Moser. "Inverse and Complementary Sequences of Natural Numbers." *American Mathematical Monthly 61* (1954):454-58.

6. O. Perron. *Kettenbruchen.* New York: Chelsea, 1954.

7. H. S. Wall. *Analytic Theory of Continued Fractions.* New York: Chelsea, 1973, p. 15.

◆◇◆◇◆

GENERALIZED FIBONACCI PRIMITIVE ROOTS, AND CLASS NUMBERS OF REAL QUADRATIC FIELDS*

R. A. MOLLIN
Mathematics Department, University of Calgary
2500 University Drive N.W., Calgary, Alberta, Canada, T2N 1N4

(Submitted June 1986)

1. INTRODUCTION

It is the purpose of this paper to generalize the concept of Fibonacci primitive roots introduced by Shanks in [22]. This work was motivated by attempts to prove a conjecture of S. Chowla on class numbers of certain real quadratic fields. The generalized Fibonacci primitive roots which we introduce are interesting in their own right. Moreover, it turns out that Chowla's conjecture is more closely related to a generalized sequence of Fibonacci numbers which we introduce in §2 as a precursor to the generalized Fibonacci primitive roots. Thus, we first establish the generalized Fibonacci primitive roots and several of their properties in §2 before displaying the connection with the motivating work on Chowla's conjecture, at the end of the paper in §3.

2. GENERALIZED FIBONACCI PRIMITIVE ROOTS

Linear recurring sequences of the second order have been extensively explored since the last century. We have such sequences of integers $\{G_i\}$ defined by $G_i = mG_{i-1} + nG_{i-2}$ for $i > 1$, where G_0, G_1, m and n are given integers. There has more recently been a plethora of papers dealing with these sequences as generalized Fibonacci numbers. As evidence, the reader may consult any of [1]-[4], [6]-[17], [20]-[21], and [26]-[31]. However, heretofore, there has been no generalization of Fibonacci primitive roots in the literature.

We consider the particular case of the G_i where $m = 1$ and $n > 0$. Set $G_i = F_i(n)$ and let $F_0(n) = 1$ and $F_1(n) = g$, a positive integer. Thus,

$$F_i(n) = F_{i-1}(n) + nF_{i-2}(n), \text{ for } i > 1.$$

The author's research is supported by N.S.E.R.C. Canada, grant #A8484.

Call $\{F_i(n)\}$ the n^{th}-*Fibonacci sequence with base* g (or simply the n^{th}-FS base g). The first Fibonacci sequence with base 1 is the ordinary Fibonacci sequence. Now let p be a prime and let g be a primitive root modulo p. We call g an n^{th}-*Fibonacci primitive root modulo* p (or simply an n^{th}-FPR mod p) if satisfies:

$$x^2 \equiv x + n \pmod{p}, \tag{1}$$

where g.c.d. $(p, n) = 1$. The $n = 1$ case yields the ordinary Fibonacci primitive roots introduced by Shanks [22] and for which properties were developed in [23] and [24] which, among others, we will have occasion to generalize later.

For the remainder of the paper we assume that p is an odd prime and n is a positive integer.

Lemma 1: If the positive integer g is a solution of (1), then

$$F_i(n) \equiv gF_{i-1}(n) \equiv g^i \pmod{p}$$

for all positive integers i.

Proof: We use induction on i. If $i = 1$, then $F_1(n) = g = gF_0(n)$. By definition of the n^{th}-FS base g, we have that $F_i(n) = F_{i-1}(n) + nF_{i-2}(n)$ for $i > 1$. By induction hypothesis:

$$F_{i-1}(n) \equiv gF_{i-2}(n) \equiv g^{i-1} \pmod{p}. \tag{2}$$

Therefore, $F_i(n) \equiv (g + n)F_{i-2}(n) \pmod{p}$. Thus, from (1), we obtain:

$$F_i(n) \equiv g^2F_{i-2}(n) \pmod{p}.$$

Hence, from (2) again, we get:

$$F_i(n) \equiv gF_{i-1}(n) \equiv g^i \pmod{p}. \quad \text{Q.E.D.}$$

As an illustration of Lemma 1, we have:

Example 1: Let $n = 5$, $p = 101$, and $g = 42$; 42 is a 5^{th}-FPR mod 101. Moreover:

$$F_0(5) = 1,\ F_1(5) = 42,\ F_2(5) = 47 = 42 + 5 \equiv 42^2,$$
$$F_3(5) = 257 = 47 + 5 \cdot 42 \equiv 42 \cdot 47 \equiv 42^3,$$
$$F_4(5) = 492 = 257 + 5 \cdot 47 \equiv 42 \cdot 257 \equiv 42^4,$$
$$F_5(5) = 1777 = 492 + 5 \cdot 257 \equiv 42 \cdot 492 \equiv 42^5,$$

etc. (where $\equiv$ denotes congruence modulo 101).

The following observations will prove to be useful, and they generalize Shanks [23, A-D, p. 164].

Remark 1: If g is an n^{th}-FPR mod p, then either

$$4n \equiv -1 \pmod{p} \quad \text{or} \quad ((4n+1)/p) = 1,$$

where $(*/*)$ denotes the Legendre symbol. This is verified from the observation that $(2g-1)^2 \equiv 4n+1 \pmod{p}$ if (1) is satisfied by g.

Remark 2: If $(-n/p) = -1$, then there exists at most one n^{th}-FPR mod p. To see this, we observe that the two solutions of (1) are

$$g_1 = (1 + \sqrt{4n+1})/2 \quad \text{and} \quad g_2 = (1 - \sqrt{4n+1})/2;$$

whence, $g_1 g_2 \equiv -n \pmod{p}$. Therefore, one of g_1 or g_2 is a quadratic residue and the other is not. Hence, there is at most one n^{th}-FPR mod p. We now give examples of each case.

Example 2: If $n = 4$ and $p = 19$, $g = 13$ is a 4^{th}-FPR mod 19. Since $(-4/19) = -1$, $g = 13$ is the only 4^{th}-FPR mod 19 by Remark 2.

Example 3: If $n = 1$ and $p = 3$, then $((4n+1)/p) = (5/3) = -1$, whence 3 has no 1^{st}-FPR by Remark 1.

Remark 3: If $(-n/p) = 1$, there may be two, one, or no n^{th}-FPR's mod p. The following examples illustrate the three cases.

Example 4: If $n = 2$ and $p = 41$, the solutions of (1) are $g_1 = 2$ and $g_2 = 40$, both of which are quadratic residues modulo 41. Hence, 41 has no 2^{nd}-FPR's.

Example 5: If $n = 3$ and $p = 13$, $g = 7$ is a 3^{rd}-FPR mod 13. However, $7^2 \equiv -3 \pmod{13}$ and $7x \equiv -3 \pmod{13}$ has only one solution. Hence, there is exactly one 3^{rd}-FPR mod 13.

Example 6: If $n = 6$ and $p = 7$, then $g_1 = 3$ and $g_2 = 5$ are 6^{th}-FPR's mod 7.

Remark 4: If two n^{th}-FPR's mod p exist, say g_1 and g_2 with $0 < g_i < p$ for $i = 1, 2$, then $g_1 + g_2 = 1 + p$. This follows from Remark 2. As an instance of this, see Example 6, where $g_1 + g_2 = 8 = p + 1$.

In Remarks 2 and 3, we saw that it is possible that no n^{th}-FPR's mod p exist. We now provide a class of primes p for which an n^{th}-FPR mod p always exists. First we need a preliminary result that generalizes an idea of Shanks and Taylor [24].

Lemma 2: Suppose that either $n = 1$ or $p > n > 2$ and $p = 1 + 2q$ where q is prime and n has order q modulo p. If g is a solution of (1), then g is a primitive root modulo p if and only if $g - 1$ is one.

Proof: If $n = 1$, then $g(g - 1) \equiv 1 \pmod{p}$ implies that g and $g - 1$ have the same order modulo p. Now we assume $p > n > 2$, $p = 2q + 1$, and n has order q modulo p. Since $g(g - 1) \equiv n \pmod{p}$ from (1), we get $g^q \equiv (g - 1)^{-q} \pmod{p}$. If g is a primitive root modulo p, then $(g - 1)^q \equiv -1 \pmod{p}$. We cannot have $g - 1 \equiv -1 \pmod{p}$, whence $g - 1$ is a primitive root modulo p. Conversely, if $g - 1$ is a primitive root modulo p, then $g^q \equiv -1 \pmod{p}$. If $g \equiv -1 \pmod{p}$, then from (1) we get that $n \equiv 2 \pmod{p}$, contradicting the hypothesis. Q.E.D.

The following example illustrates the above.

Example 7: Let $p = 47$, $g = 20$, and $n = 4$. 4 has order 23 modulo 47, 20 is a primitive root mod 47, and $g = 20$ is a solution of (1), whence 19 is a primitive root mod 47.

Now, we provide a sufficient condition for the existence of an n^{th}-FPR mod p. The following generalizes Mays's [18, Theorem, p. 111]. We follow Mays's reasoning in the initial part of the proof.

Theorem 1: Suppose that $n = 1$ or $p > n > 2$, and $((4n + 1)/p) = 1$ where $p = 1 + 2q$ is a prime with q an odd prime. Furthermore, suppose that either $n = 1$ or n has order q modulo p. Then p has an n^{th}-FPR.

Proof: Since $p \equiv 3 \pmod{4}$, at most one of α or $-\alpha$ is a primitive root modulo p for any α in the range $2 \leqslant \alpha \leqslant (p - 1)/2 = q$. But there are exactly

$$\phi(p - 1) = q - 1 = (p - 3)/2$$

primitive roots modulo p, so exactly one of α or $-\alpha$ is a primitive root modulo p. Since $((4n + 1)/p) = 1$, there are two distinct solutions of (1), namely, g and $1 - g$ (see Remarks 1 and 2). It suffices to show that either g or $1 - g$ is a primitive root modulo p. Suppose that g is not a primitive root modulo p. Then, by Lemma 2, $g - 1$ is not a primitive root modulo p. Also, $g - 1 \not\equiv 0, \pm 1 \pmod{p}$ because g satisfies (1) and $n \neq 0, 2$. Consequently, $g - 1 \equiv \pm\beta \pmod{p}$ for some β satisfying $2 \leqslant \beta \leqslant (p - 1)/2 = q$; and so, $1 - g$ is a primitive root modulo p. Q.E.D.

The following generalizes Shanks-Taylor [24, Theorem, p. 159].

Theorem 2: Suppose that either $n = 1$ or $p > n > 2$, and $p = 1 + 2q$, where q is an odd prime and n has order q modulo p. If g is an n^{th}-FPR mod p, then $g - 1$ and $g - (n + 1)$ are primitive roots modulo p.

Proof: By Lemma 2, $g - 1$ is a primitive root modulo p. Therefore, since

$$(g - 1)^2 \equiv 1 - g + n \pmod{p},$$

we get

$$(g - 1)^{2+q} \equiv g - (n + 1) \pmod{p}.$$

Since g.c.d. $(2q, 2 + q) = 1$, we see that $g - (n + 1)$ is a primitive root modulo p. Q.E.D.

Corollary 1: Suppose that n is a positive integer such that $((4n + 1)/p) = 1$, where $p = 1 + 2q$ is prime, with q an odd prime. Further, suppose that either $n = 1$ or $p > n > 2$, where n has order q modulo p. Then, there is an n^{th}-FPR mod p. If g is such an FPR, then $g - 1$ and $g - (n + 1)$ are primitive roots modulo p.

Proof: The proof follows immediately from Theorems 1 and 2.

The following illustrates Corollary 1.

Example 8: Let $n = 3$ and $p = 23$. Then,

$$((4n + 1)/p) = (13/23) = 1 \equiv 3^{11} \pmod{23}.$$

Thus, the hypothesis of Corollary 1 is satisfied and 15 is the 3^{rd}-FPR mod 23. Moreover, $14^{11} \equiv -1 \pmod{23}$ and $11^{11} \equiv -1 \pmod{23}$.

We close this section with the observation that it is possible to give a more restrictive generalization of Fibonacci primitive roots, albeit a natural one.

Let n be a positive integer and p a prime with $p \equiv 1 \pmod{n}$. Define g to be an n^{th}-FPR modulo p whenever g has order $(p - 1)/n$ modulo p and (1) is satisfied by g.

Example 9: If $n = 3$, $p = 103$, and $g = 31$, then 31 satisfies (1) and has order 34 modulo 103. Hence, under the preceding definition, 31 is a 3^{th}-FPR mod 103, but it is not one under the earlier definition.

Example 10: If $n = 2$ and $p = 5$, then 2 satisfies (1) but 2 is a primitive root modulo 5, so 2 is not a 2^{nd}-FPR mod 5 under the preceding definition but it is one under the earlier definition.

It would be of interest to see what developments would come out of a study of the latter definition.

3. CLASS NUMBERS OF REAL QUADRATIC FIELDS

In [5] S. Chowla conjectured that, if $p = m^2 + 1$ is prime and $m > 26$, then $h(p) > 1$ where $h(p)$ is the class number of $Q(\sqrt{p})$. In [19] we established that, if $r = m^2 + 1 > 17$ is square free where either r is composite or $m \neq 2q$ for an odd prime q, then $h(r) > 1$. Furthermore, we showed that in the remaining case, $h(r) = 1$ for at most finitely many q. Also we established

Theorem 3: Let $r = 4m^2 + 1$ be square free where m is a positive integer. Then the following are equivalent.

(a) $h(r) = 1$.

(b) p is inert in $Q(\sqrt{r})$ for all primes $p < m$.

(c) $f(x) = -x^2 + x + m^2 \not\equiv 0 \pmod{p}$ for all integers x and primes p satisfying $0 < x < p < m$.

(d) $f(x)$ is equal to a prime for all integers x satisfying $1 < x < m$.

The following links §2 and §3 and provides a criterion for the solvability of (1). For convenience, we let $F_i(n) = F_i$ in what follows.

Theorem 4: If n is a positive integer relatively prime to p, then g is a solution of (1) if and only if the n^{th}-FS base g satisfies $F_{i+1}F_{i-1} \equiv F_i^2 \pmod{p}$ for some $i > 1$. Moreover, if g is a solution of (1), then $F_{i+1}F_{i-1} \equiv F_i^2$ (mod p) for all $i > 0$.

Proof: By Horadam [12, (27), p. 440]: $F_{i+1}F_{i-1} - F_i^2 = (-n)^{i-1}(g + n - g^2)$ for all $i > 0$. The result follows. Q.E.D.

Therefore, we have the following conjecture based on the preceding data.

Conjecture: If $n = q^2$, where $q > 13$ is an odd prime and $4q^2 + 1$ is prime, then there is an n^{th}-FS base g, $\{F_i(n)\}$, for some g satisfying $F_{i+1}F_{i-1} \equiv F_i^2 \pmod{p}$ for a prime p with $0 < g < p < q$.

ACKNOWLEDGMENT

The author welcomes the opportunity to thank the referee for many valuable comments and suggestions.

REFERENCES

1. R. V. Anderson. "Generalized Fibonacci Sequences as Powers of a Square Matrix." *Univ. Lisboa Revista Fac. Ci A 14* (1972-1973):113-21.
2. G. E. Bergum & V. E. Hoggatt, Jr. "An Application of the Characteristic of the Generalized Fibonacci Sequence." *The Fibonacci Quarterly 15*, no. 3 (1977):215-20.
3. G. Berzsenyi. "Sums of Products of Generalized Fibonacci Numbers." *The Fibonacci Quarterly 13*, no. 4 (1975):343-44.
4. M. W. Bunder. "A Special Case of the Generalized Fibonacci Sequence over an Arbitrary Ring with Identity." *The Fibonacci Quarterly 13*, no. 3 (1975): 280.
5. S. Chowla & J. Friedlander. "Class Numbers and Quadratic Residues." *Glasgow Math. J. 17* (1976):47-52.
6. D. E. Daykin, L. A. Presel, & A. J. W. Hilton. "The Structure of Second Order Sequences in a Finite Field." *J. Reine Angew. Math. 270* (1974):77-96.
7. L. E. Fuller. "Geometric Recurrence Relation." *The Fibonacci Quarterly 18*, no. 2 (1980):126-29.
8. L. E. Fuller. "Representations for r, s Recurrence Relations." *The Fibonacci Quarterly 18*, no. 2 (1980):129-35.
9. W. Gerdes. "Generalized Fibonacci Numbers and Their Convergent Sequences." *The Fibonacci Quarterly 16*, no. 3 (1978):269-75.
10. A. F. Horadam. "On Generating Functions for Powers of a Generalized Sequence of Numbers." *The Fibonacci Quarterly 12*, no. 4 (1974):348, 350, 353, 362.
11. A. F. Horadam. "Extensions of a Paper on Diagonal Functions." *The Fibonacci Quarterly 18*, no. 1 (1980):1, 3-8.
12. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." *Duke Math. J. 32* (1965):437-46.
13. D. V. Jaisival. "Some Geometrical Properties of the Generalized Fibonacci Sequence." *The Fibonacci Quarterly 12*, no. 1 (1974):67-70.
14. B. W. King. "A Polynomial with Generalized Fibonacci Coefficients." *The Fibonacci Quarterly 11*, no. 6 (1973):527-32.
15. H. V. Krishna. "Identities of a Generalized Fibonacci Sequence." *A Collection of Manuscripts Related to the Fibonacci Sequence.* Santa Clara, Calif.: The Fibonacci Association, 1980, pp. 65-66.
16. H. V. Krishna. "Divisibility Properties of a Generalized Fibonacci Sequence." *Ibid.*, pp. 66-67.
17. C. L. Lau. "The Periodic Generating Sequence." *The Fibonacci Quarterly 15*, no. 2 (1977):178-82.

18. M. E. Mays. "A Note on Fibonacci Primitive Roots." *The Fibonacci Quarterly* *20*, no. 2 (1982):111.

19. R. A. Mollin. "Necessary and Sufficient Conditions for the Class Number of a Real Quadratic Field To Be One, and a Conjecture of S. Chowla." (To appear in *Proc. Amer. Math. Soc.*)

20. J. C. Parnami & T. N. Shorey. "Subsequences of Binary Recursive Sequences." *Acta Arith.* *40* (1981-1982):193-96.

21. H. J. A. Sallé. "A Maximum Value for the Rank of Apparition of Integers in Recursive Sequences." *The Fibonacci Quarterly* *13*, no. 2 (1975):159-61.

22. D. Shanks. *Solved and Unsolved Problems in Number Theory*. 2nd ed. New York: Chelsea, 1978.

23. D. Shanks. "Fibonacci Primitive Roots." *The Fibonacci Quarterly* *10*, no. 2 (1972):163-68, 181.

24. D. Shanks & L. Taylor. "An Observation on Fibonacci Primitive Roots." *The Fibonacci Quarterly* *11*, no. 2 (1973):159-60.

25. A. G. Shannon & A. F. Horadam. "Reciprocals of Generalized Fibonacci Numbers." *The Fibonacci Quarterly* *9*, no. 4 (1971):299-306, 312.

26. C. Smith & V. E. Hoggatt, Jr. "Primitive Periods of Generalized Fibonacci Sequences." *The Fibonacci Quarterly* *14*, no. 4 (1976):343-47.

27. L. Somer. "Fibonacci-Like Groups and Periods of Fibonacci-Like Sequences." *The Fibonacci Quarterly* *15*, no. 1 (1977):35-41.

28. M. N. S. Swamy. "On Generalized Fibonacci Quaternions." *The Fibonacci Quarterly* *11*, no. 6 (1973):547-49.

29. M. E. Waddill. "Matrices and Generalized Fibonacci Sequences." *The Fibonacci Quarterly* *12*, no. 4 (1974):381-86.

30. J. E. Walton & A. F. Horadam. "Some Aspects of Generalized Fibonacci Numbers." *The Fibonacci Quarterly* *12*, no. 3 (1974):241-50.

31. J. E. Walton & A. F. Horadam. "Some Further Identities for the Generalized Fibonacci Sequence $\{H_n\}$." *The Fibonacci Quarterly* *12*, no. 3 (1974):272-80.

Note added in proof: Since the writing of this paper, substantial progress has been made. The author and H. C. Williams have used a suitable Riemann hypothesis to prove the Chowla conjecture. In fact, we have found *all* real quadratic fields of Richaud-Degert-type having class number one.

◆◇◆◇◆

ON CERTAIN SEMI-PERFECT CUBOIDS

W. J. A. COLMAN
North East London Polytechnic, Degenham, Essex RM8 2AS, England

(Submitted June 1986)

1. The classical cuboid is a rectangular block with integral edges and face diagonals. If we consider the internal diagonal as well, then there are seven lengths in all. It is known [3] that any six of the seven lengths can be integral. We can call such cuboids semi-perfect. Semi-perfect cuboids fall into three categories such that there is no integral specification for:

(1) the internal diagonal,
(2) one face diagonal,
(3) one edge.

If all seven lengths were integral, then we would have what is known as a perfect cuboid. No such perfect cuboids are known; indeed, their existence is a classical open question. It is known [3] that there are an infinity of semi-perfect cuboids in all categories, as certain parametric solutions are known. Unfortunately, none of these solutions is complete. Clearly, if perfect cuboids exist, they must fall into all three categories and so the complete determination of all semi-perfect cuboids in any one category would reduce the problem of perfect cuboids to the consideration of the seventh nonspecified length. It has been shown that some of these partial parametric solutions cannot be perfect (see [2], [3], and [4]). In this paper we shall determine a two-parameter solution for category (3) which is the generalization of a solution first given by Bromhead [1], and then show in a simple manner that this too can never give a perfect cuboid.

2. It is instructive first to consider the smallest real solutions (with $c > 0$) in category (3). If we measure the size of the cuboid by the length of the internal diagonal d (say) with edges a, b, and $\sqrt{c}$, then Leech [3] has given the smallest solutions. The first four being

a	b	c	d
520	576	618849	1105
1800	1443	461776	2405
1480	969	6761664	3145
124	957	13852800	3845

where $a^2 + b^2$, $a^2 + c$, $b^2 + c$, and $a^2 + b^2 + c$ are all square. If $\sqrt{c}$ were itself integral, then of course the cuboid would be perfect.

3. Following Bromhead's solution [1], we have $a^2 + b^2$ square and, hence,

$$a = k(2uv) \quad \text{and} \quad b = k(u^2 - v^2).$$

If we write $a^2 + c = p^2$ and $b^2 + c = q^2$, then $p^2 + b^2 = q^2 + a^2$, and each is a square. Therefore, $p = k_1(2u_1v_1)$ or $k_1(u_1^2 - v_1^2)$ with $k_1(u_1^2 - v_1^2) = k(u^2 - v^2)$ or $k_1(2u_1v_1) = k(u^2 - v^2)$. Similarly, $q = k_2(u_2^2 - v_2^2)$ or $k_2(2u_2v_2)$; hence,

$$k_2(2u_2v_2) = k(2uv) \quad \text{or} \quad k_2(u_2^2 - v_2^2) = k(u^2 - v^2).$$

Finally, $k_1(u_1^2 + v_1^2) = k_2(u_2^2 + v_2^2)$ in all cases.

Since we need only consider cuboids with $(a^2, b^2, c) = 1$, we can reduce the problem to solving the systems:

3.1 $$(1) \begin{cases} k_1(u_1^2 + v_1^2) = k_2(u_2^2 + v_2^2) \\ k_1(u_1^2 - v_1^2) = k(u^2 - v^2) \\ k_2(2u_2v_2) = k(2uv) \end{cases} \quad \text{or} \quad (2) \begin{cases} k_1(u_1^2 + v_1^2) = k_2(u_2^2 + v_2^2) \\ k_1(2u_1v_1) = k(u^2 - v^2) \\ k_2(2u_2v_2) = k(2uv) \end{cases}$$

in integers. Thus, we can say that all primitive semi-perfect cuboids in category (3) must satisfy either system (1) or system (2). Of the four "smallest" real solutions listed above the smallest satisfies system (2) and the next three satisfy system (1). Bromhead's one-parameter solution satisfies system (1) when $k = k_1 = k_2$, and the smallest solution with this condition is the fourth.

4. We shall now determine a two-parameter solution of system (1) when $k = k_1 = k_2$. We have:

4.1 $$u_1^2 + v_1^2 = u_2^2 + v_2^2;$$

4.2 $$u_1^2 - v_1^2 = u^2 - v^2;$$

4.3 $$u_2v_2 = uv.$$

The general solution of 4.1 is

4.4 $$(mp + nq)^2 + (mq - np)^2 = (mq + np)^2 + (mp - nq)^2.$$

Writing 4.2 as $u_1^2 + v^2 = u^2 + v_1^2$, its solution is:

4.5 $$(m_1p_1 + n_1q_1)^2 + (m_1q_1 - n_1p_1)^2 = (m_1p_1 - n_1q_1)^2 + (m_1q_1 + n_1p_1)^2.$$

Putting $\quad mp + nq = m_1p_1 + n_1q_1$

and $\quad mq - np = m_1q_1 + n_1p_1,$

a rational solution is given by:

4.6 $\quad p_1 = q,\ q_1 = p,\ m_1 = \dfrac{n(q^2 + p^2)}{q^2 - p^2},\ n_1 = \dfrac{m(q^2 - p^2) - 2npq}{q^2 - p^2}$

Therefore,

4.7 $\quad u = (nq(q^2 + 3p^2) - mp(q^2 - p^2))/q^2 - p^2,$

4.8 $\quad v = (np(3q^2 + p^2) - mq(q^2 - p^2))/q^2 - p^2.$

Finally, we require, from 4.3, that

4.9 $\quad (mq + np)(mp - nq) = \left(mp - \dfrac{nq(q^2 + 3p^2)}{q^2 - p^2}\right)\left(mq - \dfrac{np(3q^2 + p^2)}{q^2 - p^2}\right)$

Let $n = \lambda(q^2 - p^2)$, then

4.10 $\quad (mq + \lambda p(q^2 - p^2))(mp - \lambda q(q^2 - p^2))$
$\quad = (mp - \lambda q(q^2 + 3p^2))(mq - \lambda p(3q^2 + p^2)).$

Multiplying in (4.10) and simplifying, we have,

$$2mpq = \lambda(q^2 + p^2)^2;$$

therefore,

$$m = \frac{\lambda(q^2 + p^2)^2}{2pq}.$$

Let $\lambda = 2pq$, then

$$n = 2pq(q^2 - p^2) \quad \text{and} \quad m = (q^2 + p^2)^2.$$

Hence, we have a solution where:

4.11
$$\begin{aligned}
u_1 &= p(q^2 + p^2)^2 + 2pq^2(q^2 - p^2) = p(p^4 + 3q^4);\\
v_1 &= q(q^2 + p^2)^2 - 2p^2q(q^2 - p^2) = q(3p^4 + q^4);\\
u_2 &= q(q^2 + p^2)^2 + 2p^2q(q^2 - p^2) = q(q^4 + 4q^2p^2 - p^4);\\
v_2 &= p(q^2 + p^2)^2 - 2pq^2(q^2 - p^2) = p(p^4 + 4p^2q^2 - q^4);\\
u &= 2pq^2(q^2 + 3p^2) - p(q^2 + p^2)^2 = p(q^4 + 4p^2q^2 - p^4);\\
v &= 2p^2q(3q^2 + p^2) - q(q^2 + p^2)^2 = q(p^4 + 4p^2q^2 - q^4).
\end{aligned}$$

This gives the solution:

4.12
$$\begin{aligned}
a &= 2pq(p^4 + 4p^2q^2 - q^4)(q^4 + 4p^2q^2 - p^4);\\
b &= p^2(q^4 + 4p^2q^2 - p^4)^2 - q^2(p^4 + 4p^2q^2 - q^4)^2;\\
c &= 32p^2q^2(p^4 - q^4)^2(p^8 + 14p^4q^4 + q^8).
\end{aligned}$$

Bromhead's solution corresponds to $p = t + 1$, $q = t$. We need only consider values of p and q such that $(p, q) = 1$. If $p = 2$ and $q = 1$, we have:

$$a = 124; \quad b = 957; \quad c = 13852800.$$

The signs of a and b are, of course, irrelevant and so we will always take the absolute value.

5. We know that if c itself is square, then the cuboid will be perfect. Looking at the form given for c, this requires that $p^8 + 14p^4q^4 + q^8$ is twice a square. We shall now prove that this is not possible. Set

5.1 $$p^8 + 14p^4q^4 + q^8 = 2w^2, \text{ where } (p, q) = 1,$$

then p and q must both be odd, and

5.2 $$(p^4 - q^4)^2 + (4p^2q^2)^2 = 2w^2.$$

The general solution of 5.2 is known to be

5.3 $$p^4 - q^4 = k(m^2 - 2mn - n^2) \quad \text{or} \quad k(m^2 + 2mn - n^2),$$

5.4 $$4p^2q^2 = k(m^2 + 2mn - n^2) \quad \text{or} \quad k(m^2 - 2mn - n^2)$$
$$w = k(m^2 + n^2).$$

From 5.3 and 5.4,

$$p^4 - q^4 + 4p^2q^2 = 2k(m^2 - n^2).$$

If m and n have the same parity, then $8 | 2k(m^2 - n^2)$. However, $8 \nmid p^4 - q^4 + 4p^2q^2$ since $8 | p^4 - q^4$ but not $4p^2q^2$. Therefore, m and n must have opposite parities, in which case $m^2 \pm 2mn - n^2$ is odd. Hence, from 5.3, we have that $8 | k$. From 5.4, it follows that $8 | 4p^2q^2$, which is impossible because p and q are both odd. It also follows that c can never be square, so the semi-perfect cuboids generated by 4.12 can never be perfect.

REFERENCES

1. T. Bromhead. "On Square Sums of Squares." *Math. Gazette 44* (1960):219-20.
2. E. Z. Chein. "On the Derived Cuboid of an Eulerian Triple." *Can. Math. Bull. 20* (1977):509-10.
3. J. Leech. "The Rational Cuboid Revisited." *Amer. Math. Monthly 84* (1977): 518-33.
4. W. G. Spohn. "On the Integral Cuboid." *Amer. Math. Monthly 79* (1972):57-59.

◆◇◆◇◆

GENERALIZED TRANSPOSABLE INTEGERS

ANNE L. LUDINGTON
Loyola College, Baltimore, MD 21210

(Submitted July 1986)

1. INTRODUCTION

Let x be an n-digit number expressed in base g; thus,

$$x = \sum_{i=0}^{n-1} a_i g^i \text{ with } 0 \leqslant a_i < g \text{ and } a_{n-1} \neq 0.$$

Let k be a positive integer. Then x is called k-transposable if and only if

$$kx = \sum_{i=0}^{n-2} a_i g^{i+1} + a_{n-1}. \tag{1}$$

Clearly, x is 1-transposable if and only if all of its digits are equal. Thus, we assume $k > 1$.

Kahan [2] studied decadic k-transposable integers. He showed that k must equal 3, that $x_1 = 142857$ and $x_2 = 285714$ are 3-transposable, and that all other 3-transposable integers are obtained by concatenating x_1 or x_2 m times, $m \geqslant 1$.

In [1], this author studied k-transposable integers for an arbitrary base g. Necessary and sufficient conditions were given for an n-digit, g-adic number to be k-transposable.

When a k-transposable integer is multiplied by k, its digits are shifted one place to the left with the leading digit moving to the units place. In this paper, we will generalize this shift of one place to a shift of j places, $1 \leqslant j < n$.

2. TRANSPOSABLE INTEGERS WITH ARBITRARY SHIFTS

We say that the n-digit number $x = \sum_{i=0}^{n-1} a_i g^i$ is a k-transposable, j-shift integer, or a (k, j)-integer for short, if and only if

$$kx = \sum_{i=0}^{n-1-j} a_i g^{i+j} + \sum_{i=n-j}^{n-1} a_i g^{i-(n-j)}, \text{ for } 1 \leqslant j < n \text{ and } 1 < k < g. \tag{2}$$

For example, again consider the decadic integers 142857 and 285714. Since

$$6(142857) = 857142,$$
$$2(285714) = 571428,$$

142857 is a (6, 3)-integer, while 285714 is a (2, 2)-integer.

We shall study (k, j)-integers for an arbitrary base g. Kahan [3] has determined all decadic n-digit $(k, n - 1)$-integers. He called these k-reverse transposable integers.

Rearranging the terms in (2), we get

$$(kg^{n-j} - 1)\sum_{i=n-j}^{n-1} a_i g^{i-(n-j)} = (g^j - k)\sum_{i=0}^{n-1-j} a_i g^i. \tag{3}$$

Let d be the greatest common divisor of $kg^{n-j} - 1$ and $g^j - k$. Then the following lemma is immediate.

Lemma 1: Let x be an n-digit, (k, j)-integer and let $d = (kg^{n-j} - 1, g^j - k)$. Then d satisfies the following:

(i) $(g, d) = 1$
(ii) $(k, d) = 1$
(iii) $k < d$
(iv) $g^n \equiv 1 \pmod{d}$

The following theorem gives necessary and sufficient conditions for the existence of (k, j)-integers.

Theorem 1: There exists an n-digit, (k, j)-integer if and only if there is an integer d with the following properties:

(i) $(k, d) = 1$
(ii) $k < d$
(iii) $d | g^j - k$
(iv) $g^n \equiv 1 \pmod{d}$

Proof: Lemma 1 shows that (i)-(iv) are necessary with $d = (kg^{n-j} - 1, g^j - k)$.

Now, suppose there exists a d satisfying (i)-(iv). Note that d divides $kg^{n-j} - 1$ since

$$kg^{n-j} - 1 \equiv g^j g^{n-j} - 1 \equiv g^n - 1 \equiv 0 \pmod{d}.$$

We now construct $\left[\frac{d}{k}\right]$ (k, j)-integers x_t. Let

$$x_t = \sum_{i=0}^{n-1} b_{t,i} g^i, \text{ with } t = 1, \ldots, \left[\frac{d}{k}\right].$$

The coefficients $b_{t,n-1}, \ldots, b_{t,n-j}$ are given by

$$\sum_{i=n-j}^{n-1} b_{t,i}\, g^{i-(n-j)} = \frac{g^j - k}{d}\, t. \tag{4}$$

We obtain (4) by dividing (3) by $g^j - k$ and requiring that $\sum_{n-j}^{n-1} b_{t,i}\, g^{i-(n-j)}$ be a multiple of $\frac{g^j - k}{d}$, since d divides $kg^{n-j} - 1$. Note that the highest power of g which occurs on each side of (4) is $j - 1$, so the coefficients $b_{t,i}$ are well defined. Using (3) we find that $b_{t,0}, \ldots, b_{t,n-j-1}$ are to be defined by

$$\sum_{i=0}^{n-1-j} b_{t,i}\, g^i = \frac{kg^{n-j} - 1}{d}\, t. \tag{5}$$

Equation (5) is also well defined, since $kt \leqslant d$.

We note here that the proof of Theorem 1 is a constructive one. The digits of k-transposable integers are found using (4) and (5). We now show that all g have (k, j)-integers.

Theorem 2: If $g = 5$ or $g \geqslant 7$, then g has a (k, j)-integer for all $j \geqslant 1$. If $g = 3$, 4, or 6, then g has a (k, j)-integer for $j \geqslant 2$.

Proof: If $g = 5$ or $g \geqslant 7$, choose k satisfying the following:

$$2 \leqslant k \leqslant g/2 \quad \text{and} \quad (k, g) = 1.$$

Then $d = g^j - k$, $j \geqslant 1$, satisfies (i)-(iii) of Theorem 1; further, $(d, g) = 1$. Hence, there exists n such that $g^n \equiv 1 \pmod{d}$. By Theorem 1, g has a (k, j)-integer.

For $g = 3$, 4, or 6, choose k such that

$$2 \leqslant k < g \quad \text{and} \quad (k, g) = 1.$$

Again, let $d = g^j - k$, $j \geqslant 2$, and apply Theorem 1. For these g, no $(k, 1)$-integers exist.

For j fixed, we now show that up to concatenation there are only a finite number of (k, j)-integers.

Theorem 3: Suppose $x = \sum_{i=0}^{n-1} a_i g^i$ is a (k, j)-integer. Let $d = (kg^{n-j} - 1, g^j - k)$ and let N be the order of g in U_d, the group of units of Z_d. Then x equals some (k, j)-integer concatenated n/N times.

Proof: Since $g^n \equiv 1 \pmod{d}$, n is a multiple of N. Let

$$x_t = \sum_{i=0}^{N-1} b_{t,i}\, g^i,\quad t = 1, \ldots, \left[\frac{d}{k}\right],$$

be the N-digit integers given by equations (4) and (5).

In (3), $\sum_{i=n-j}^{n-1} a_i g^{i-(n-j)}$ must be a multiple of $\frac{g^j - k}{d}$. Thus, for some t,

$$\sum_{i=n-j}^{n-1} a_i g^{i-(n-j)} = \left(\frac{g^j - k}{d}\right)t = \sum_{i=N-j}^{N-1} b_{t,i} g^{i-(N-j)}$$

so

$$a_{n-i} = b_{t,N-i}, \text{ for } i = 1, \ldots, j.$$

Thus,

$$\sum_{i=0}^{n-1-j} a_i g^i = \left(\frac{kg^{n-j} - 1}{d}\right)t = g^{n-N}\left(\frac{kg^{N-j} - 1}{d}\right)t + \left(\frac{g^{n-N} - 1}{d}\right)t.$$

Note that $kt \leqslant d$. Now, since

$$\sum_{i=0}^{N-1-j} b_{t,i} g^i = \left(\frac{kg^{N-j} - 1}{d}\right)t,$$

we must have

$$a_{n-i} = b_{t,N-i}, \; i = j + 1, \ldots, N.$$

Further,

$$\left(\frac{g^{n-N} - 1}{d}\right)t = \left(\frac{g^j - k}{d}\right)tg^{n-N-j} + \left(\frac{kg^{n-N-j} - 1}{d}\right)t.$$

Hence,

$$\sum_{i=n-N-j}^{n-N-1} a_i g^i = \left(\frac{g^j - k}{d}\right)tg^{n-N-j}$$

or

$$\sum_{i=n-N-j}^{n-N-1} a_i g^{i-(n-N-j)} = \left(\frac{g^j - k}{d}\right)t = \sum_{i=N-j}^{N-1} b_{t,i} g^{i-(N-j)}.$$

Thus, $a_{n-N-i} = b_{t,N-i}$, $i = 1, \ldots, j$, and $a_{n-N-i} = b_{t,N-i}$, $i = j - 1, \ldots, N$. Continuing, we find that x equals x_t concatenated n/N times.

3. $(k, 1)$-INTEGERS ARE ALSO (ℓ, j)-INTEGERS

In some cases $(k, 1)$-integers are also (ℓ, j)-integers. Consider the multiples of the decadic $(3, 1)$-integer $y = 142857$:

$$2y = 285714; \quad 4y = 571428; \quad 5y = 714285; \quad 6y = 857142.$$

Thus, y is also a $(2, 2)$, $(4, 4)$, $(5, 5)$, and $(6, 3)$-integer. We observe that y is an (ℓ, j)-integer when $\ell \equiv 3^j \pmod 7$. Here $7 = d = (g - k, kg^{n-1} - 1)$, with $g = 10$, $k = 3$, and $n = 6$. We will show that this is always the case when ℓy is an n-digit number. The following lemmas will be useful.

Lemma 2: Suppose $x = \sum_{i=0}^{n-1} a_i g^i$ is a $(k, 1)$-integer. Let $d = (g - k, kg^{n-1} - 1)$.

Then

$$dx = \frac{d}{g - k} a_{n-1}(g^n - 1).$$

Proof: Since d divides $g - k$, $d = \frac{g - k}{r}$ for some r. Thus, we have:

$$d\sum_{i=0}^{n-1} a_i g^i = \frac{1}{r}(g - k)\sum_{i=0}^{n-1} a_i g^i = \frac{1}{r}\left[\sum_{i=0}^{n-1} a_i g^{i+1} - \sum_{i=0}^{n-2} a_i g^{i+1} - a_{n-1}\right]$$

$$= \frac{1}{r} a_{n-1}(g^n - 1) = \frac{d}{g - k} a_{n-1}(g^n - 1).$$

Lemma 3: Suppose $x = \sum_{i=0}^{n-1} a_i g^i$ is a $(k, 1)$-integer. Then, for $j \geqslant 2$, we have

$$k^j x = \sum_{i=0}^{n-j-1} a_i g^{i+j} + \sum_{i=n-j}^{n-1} a_i g^{i-(n-j)} + r_j (g^n - 1),$$

where

$$r_j = \sum_{i=2}^{j} (a_{n-i} - k^{i-1} a_{n-1}) g^{j-i}.$$

Proof: The proof is by induction. Since the initial step with $j = 2$ is similar to the induction step, we will do only the latter. Consider

$$k^{j+1}x = k^j\left(\sum_{i=0}^{n-2} a_i g^{i+1} + a_{n-1}\right) = gk^j\sum_{i=0}^{n-1} a_i g^i - k^j a_{n-1}(g^n - 1)$$

$$= g\left[\sum_{i=0}^{n-j-1} a_i g^{i+j} + \sum_{i=n-j}^{n-1} a_i g^{i-(n-j)} + r_j (g^n - 1)\right] - k^j a_{n-1}(g^n - 1)$$

$$= \sum_{i=0}^{n-j-2} a_i g^{i+j+1} + \sum_{i=n-j-1}^{n-1} a_i g^{i-(n-j-1)}$$

$$+ (a_{n-j-1} - k^j a_{n-1})(g^n - 1) + r_j g(g^n - 1)$$

$$= \sum_{i=0}^{n-j-2} a_i g^{i+j+1} + \sum_{i=n-j-1}^{n-1} a_i g^{i-(n-j-1)} + r_{j+1}(g^n - 1).$$

Theorem 4: Suppose that $x = \sum_{i=0}^{n-1} a_i g^i$ is a $(k, 1)$-integer. Let $d = (g - k, kg^{n-1} - 1)$. Suppose ℓx is an n-digit number with $\ell < d$. Then x is an (ℓ, j)-integer if $\ell \equiv k^j \pmod{d}$.

Proof: Since $\ell \equiv k^j \pmod{d}$, $\ell = k^j - sd$ for some nonnegative integer s. Then by Lemmas 2 and 3,

$$\ell x = \sum_{i=0}^{n-j-1} a_i g^{i+j} + \sum_{i=n-j}^{n-1} a_i g^{i-(n-j)} + \left(r_j - s\frac{d}{g - k} a_{n-1}\right)(g^n - 1).$$

Since ℓx is an n-digit number, $r_j - s\frac{d}{g - k} a_{n-1}$ must equal zero. Hence, x is an (ℓ, j)-integer.

While $(k, 1)$-integers give rise to (ℓ, j)-integers, an (ℓ, j)-integer need not be a $(k, 1)$-integer. For example, the decadic number 153846 is a (4, 5)-integer, but it is not a $(k, 1)$-integer for any k.

ACKNOWLEDGMENT

Work on this paper was done while the author was a faculty member at Hamilton College, Clinton, New York. She is grateful for the support and encouragement she received during her eleven-year association with Hamilton College.

REFERENCES

1. Anne Ludington. "Transposable Integers in Arbitrary Bases." *The Fibonacci Quarterly* 25, no. 3 (1987):263-67.
2. Steven Kahan. "*K*-Transposable Integers." *Math. Magazine* 49, no. 1 (1976): 27-28.
3. Steven Kahan. "*K*-Reverse-Transposable Integers." *J. Recreational Math.* 9 (1976-1977):11-18.

◆◇◆◇◆

LIMITS OF q-POLYNOMIAL COEFFICIENTS

KUNG-WEI YANG

Western Michigan University, Kalamazoo, MI 49008

(Submitted July 1986)

INTRODUCTION

It is well known that the q-binomial (Gaussian) coefficients $\begin{bmatrix} n \\ r \end{bmatrix}$ satisfy the "finite" Euler identity ([2], p. 101):

$$\prod_{n-1 \geqslant i \geqslant 0} (1 + q^i x) = 1 + \sum_{n \geqslant r \geqslant 1} \begin{bmatrix} n \\ r \end{bmatrix} q^{\binom{r}{2}} x^r,$$

and that their q-adic limits

$$\lim_{n \to \infty} \begin{bmatrix} n \\ r \end{bmatrix} = \prod_{r \geqslant i \geqslant 1} (1 - q^i)^{-1}$$

satisfy the "infinite" Euler identity ([1], p. 254; [2], p. 105):

$$\prod_{i \geqslant 0} (1 + q^i x) = 1 + \sum_{r \geqslant 1} \prod_{r \geqslant i \geqslant 1} (1 - q^i)^{-1} q^{\binom{r}{2}} x^r.$$

In [5], we showed that the q-polynomial coefficients $\begin{bmatrix} n.m \\ r \end{bmatrix}$ satisfy the generalized "finite" Euler identity:

$$\prod_{n-1 \geqslant i \geqslant 0} \left(\sum_{m \geqslant j \geqslant 0} q^{ijm + \binom{j}{2}} x^j \right) = 1 + \sum_{nm \geqslant r \geqslant 1} \begin{bmatrix} n.m \\ r \end{bmatrix} q^{\binom{r}{2}} x^r.$$

We now complete the analogy by showing that the q-adic limits of these q-polynomial coefficients $G_r^{(m)}$ (for each $m \geqslant 1$) satisfy a recurrence relation which generalizes that satisfied by

$$\prod_{r \geqslant i \geqslant 1} (1 - q^i)^{-1},$$

and the generalized infinite Euler identity:

$$\prod_{i \geqslant 0} \left(\sum_{m \geqslant j \geqslant 0} q^{ijm + \binom{j}{2}} x^j \right) = 1 + \sum_{r \geqslant 1} G_r^{(m)} q^{\binom{r}{2}} x^r.$$

This paper is organized as follows. We begin in Section 1 by defining the basic graphical terms. We then make the first of two valuations of the digraph in Section 2. In Section 3, the recurrence formula for $G_r^{(m)}$ is proved. The generalized infinite Euler identity is proved in Section 4, and Section 5 contains a short discussion of the special cases $m = 1$ and $m = 2$.

We recall here the definition of the q-polynomial coefficients (see [4], [5], and [6]). Let $(m_1, \ldots, m_n)$ denote the *multiset* on $\{1, \ldots, n\}$ in which the multiplicity of i is m_n. The number of elements in $(m_1, \ldots, m_n)$ is $m_1 + \cdots + m_n$ and is denoted by $|(m_1, \ldots, m_n)|$. We abbreviate the multiset $(m_1, \ldots, m_n)$ in which $m_1 = \cdots = m_n = m$ to $(n.m)$. A *multisubset* $(a_1, \ldots, a_n)$ of $(n.m)$ satisfies $a_i \leqslant m$, for $i = 1, \ldots, n$, and it uniquely determines a *complementary multisubset* $(a_1', \ldots, a_n')$ satisfying $a_i + a_i' = m$ $(i = 1, \ldots, n)$. An *inversion* between the multisets $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, in that order, is a pair (i, j), where i is an element of the multiset $(a_1, \ldots, a_n)$ and j is an element of $(b_1, \ldots, b_n)$, and $i > j$. Let $I(a_1, \ldots, a_n)$ denote the number of inversions between $(a_1, \ldots, a_n)$ and $(a_1', \ldots, a_n')$, where $(a_1, \ldots, a_n)$ is a multisubset of $(n.m)$. The *q-polynomial coefficient* $\begin{bmatrix} n.m \\ r \end{bmatrix}$ is defined to be the generating function

$$\begin{bmatrix} n.m \\ r \end{bmatrix} = \sum_{|(a_1, \ldots, a_n)| = r} q^{I(a_1, \ldots, a_n)}.$$

1. GRAPHS

Let m be a fixed positive integer. We consider the digraph with vertices all the lattice points in the first quadrant of the plane

$$\{(i, j) \mid i, j \geqslant 0\}$$

and directed edges

$$(i, j) \to (i + 1, j), \ (i, j) \to (i, j + 1)(i, j \geqslant 0).$$

We will call a vertex an *m-vertex* if there is a nonnegative integer k such that $i + j = km$. We will call a path of the form

$$\begin{aligned}(i, j) &\to (i + 1, j) \to \cdots \to (i + a, j) \\ &\to (i + a, j + 1) \to \cdots \to (i + a, j + b),\end{aligned}$$

where (i, j) is an m-vertex and $a + b = m$, an *m-arc*, and we will denote it by

$$(i, j) \to\to (i + a, j + b).$$

An m-arc of the form $(i, j) \to\to (i, j + m)$ will be called a *vertical m-arc*.

A finite sequence of consecutive m-arcs beginning with the origin followed by an infinite sequence of consecutive vertical m-arcs is called an *m-path*. In an m-path, if $(r - a, s - b) \to\to (r, s)$, where $a + b = m$, is the *last* nonvertical m-arc, (r, s) will be called the *terminal m-vertex* of the m-path. The part of an m-path between $(0, 0)$ and its terminal m-vertex will be called the *valuable part* of the m-path.

2. VALUATION

Until Section 4, we will assign to all directed edges of the form $(i, j) \to (i + 1, j)$ the monomial $q^j x$ and directed edges of the form $(i, j) \to (i, j + 1)$ the trivial monomial 1 $(i, j \geqslant 0)$.

The product of all the monomials on the m-path p (m-arc) is then called the *value* of the m-path p (m-arc) and is denoted by $v(p; q, x)$. Clearly, the value of an m-path is completely determined by its valuable part. In fact, if (r, s) is the terminal m-vertex, and if

$$(0, 0) \to\to (a_1, a_1') \to\to (a_1 + a_2, a_1' + a_2') \to\to \cdots$$
$$\to\to (a_1 + \cdots + a_n, a_1' + \cdots + a_n') = (r, s)$$

is the valuable part of the m-path, the value of the m-path p is

$$v(p; q, x) = q^{a_2a_1' + a_3(a_1' + a_2') + \cdots + a_n(a_1' + \cdots + a_{n-1}')}x^r.$$

Observe that

$$I(a_1, \ldots, a_n) = a_2a_1' + a_3(a_1' + a_2') + \cdots + a_n(a_1' + \cdots + a_{n-1}').$$

This shows $v(p; q, x) = q^{I(a_1, \ldots, a_n)}x^r$. Hence,

Lemma 1: $\begin{bmatrix} n.m \\ r \end{bmatrix} = \sum v(p; q, 1)$, where the sum is over all m-paths from $(0, 0)$ to $(r, nm - r)$.

We note that $I(a_1, \ldots, a_n)$ is also equal to the number of unit squares (area) under the m-path p ([3], p. 13).

Theorem 1: Keeping the above notation, we have

$$I(a_1, \ldots, a_n) = I(a_n', \ldots, a_1').$$

Proof:
$$\begin{aligned} I(a_n', \ldots, a_1') &= a_{n-1}'a_n + a_{n-2}'(a_n + a_{n-1}) + \cdots + a_1'(a_n + \cdots + a_2) \\ &= a_2a_1' + a_3(a_1' + a_2') + \cdots + a_n(a_1' + \cdots + a_{n-1}') \\ &= I(a_1, \ldots, a_n). \quad \text{Q.E.D.} \end{aligned}$$

3. RECURRENCE RELATIONS

Let $G^{(m)}(q, x)$ denote the power series obtained from summing the value of all the m-paths. Writing in the ascending powers of x,

$$G^{(m)}(q, x) = 1 + \sum_{r\geqslant 1} G_r^{(m)} x^r,$$

we see that $G_r^{(m)} = \sum v(p; q, 1)$, where the sum is over the set of m-paths with terminal m-vertex on the line $x = r$. Lemma 1 now implies

Corollary 2: $\begin{bmatrix} n.m \\ r \end{bmatrix} \to G_r^{(m)}$, as $n \to \infty$.

Theorem 3: Let $G_0^{(m)} = 1$, $G_r^{(m)} = 0$, if $r < 0$. Then, for all $r > 1$,

$$G_r^{(m)} = (1 - q^{rm})^{-1}\left(\sum_{m\geqslant i\geqslant 1} q^{(r-i)(m-i)} G_{r-i}^{(m)}\right).$$

Proof: Let p be an m-path with terminal m-vertex on the line $x = r$. Choose the largest k such that $(0, km)$ is an m-vertex of p and let $(i, (k + 1)m - i)$ be the next m-vertex, $1 \leqslant i \leqslant m$. Then

$$v(p; q, 1) = q^{rkm + (r-i)(m-i)} v(p'; q, 1),$$

where p' is the m-path obtained by deleting the part from $(0, 0)$ to $(i, (k + 1)m - i)$ from p and then translating so that the starting point is at the origin. The sum of $v(p'; q, 1)$ for all such p' is $G_{r-i}^{(m)}$. Thus,

$$\begin{aligned} G_r^{(m)} &= \sum_{k\geqslant 0} q^{rkm}\left(\sum_{m\geqslant k\geqslant 1} q^{(r-i)(m-i)} G_{r-i}^{(m)}\right) \\ &= (1 - q^{rm})^{-1}\left(\sum_{m\geqslant i\geqslant 1} q^{(r-i)(m-i)} G_{r-i}^{(m)}\right). \quad \text{Q.E.D.} \end{aligned}$$

4. IDENTITIES

Now, we multiply an additional factor of q^i to each monomial $q^j x$ already assigned to the directed edges between the lines $x = i$ and $x = i + 1$. Thus, the total sum of the values of all the m-paths is clearly changed from

$$1 + \sum_{r\geqslant 1} G_r^{(m)} x^r$$

to

$$1 + \sum_{r\geqslant 1} G_r^{(m)} q^{\binom{r}{2}} x^r.$$

On the other hand, the sum of the values of the m-arcs emanating from *each* m-vertex (r, s) satisfying $r + s = im$ is now uniformly equal to

$$\sum_{m\geqslant j\geqslant 0} q^{ijm+\binom{j}{2}} x^j.$$

Since each m-path consists of a valuable part followed by an infinite sequence of consecutive vertical m-arcs the value of which is 1, and since the valuable part consists of a finite sequence of consecutive m-arcs starting with (0, 0) and ending at its terminal m-vertex, the total sum of the values of the m-paths is equal to

$$\prod_{i\geqslant 0}\left(\sum_{m\geqslant j\geqslant 0} q^{ijm+\binom{j}{2}}x^j\right).$$

Equating these two formal power series and invoking Corollary 2, we obtain

Theorem 4: Let $G_r^{(m)}$ be the q-adic limit of $\begin{bmatrix} n.m \\ r \end{bmatrix}$ as $n \to \infty$. Then they satisfy

$$\prod_{i\geqslant 0}\left(\sum_{m\geqslant j\geqslant 0} q^{ijm+\binom{j}{2}}x^j\right) = 1 + \sum_{r\geqslant 1} G_r^{(m)} q^{\binom{r}{2}} x^r.$$

It should be noted that Theorem 4 also follows directly from Theorem 3.

5. SPECIAL CASES

The case $m = 1$ is, of course, the Euler identity:

$$\prod_{i\geqslant 0}(1 + q^i x) = 1 + \sum_{r\geqslant 1} G_r^{(1)} q^{\binom{r}{2}} x^r,$$

where $G_0^{(1)} = 1$, and $G_r^{(1)} = \prod_{r\geqslant i\geqslant 1}(1 - q^i)^{-1}$, if $r \geqslant 1$.

When $m = 2$, the recurrence for $G_r^{(2)}$ is

$$G_r^{(2)} = (1 - q^{2r})^{-1} q^{r-1} G_{r-1}^{(2)} + (1 - q^{2r})^{-1} G_{r-2}^{(2)},$$

where $G_0^{(2)} = 1$, $G_{-1}^{(2)} = 0$. If we let r be $\geqslant 1$, $a_{r-1} = (1 - q^{2r})^{-1} q^{r-1}$, and $b_{r-2} = (1 - q^{2r})^{-1}$, the recurrence can be written as

$$G_r^{(2)} = a_{r-1} G_{r-1}^{(2)} + b_{r-2} G_{r-2}^{(2)}.$$

Using this notation, we may write the infinite product identity for the case $m = 2$ as

$$\begin{aligned}
&(1 + x + qx^2)(1 + q^2x + q^5x^2)\ \ldots\ (1 + q^2\ x + q^{4r+1}x^2)\ \ldots \\
&= 1 + a_0 q^{\binom{1}{2}} x + (a_0 a_1 + b_0) q^{\binom{2}{2}} x^2 + (a_0 a_1 a_2 + b_0 a_2 + a_0 b_1) q^{\binom{3}{2}} x^3 \\
&\quad + (a_0 a_1 a_2 a_3 + b_0 a_2 a_3 + a_0 b_1 a_3 + a_0 a_1 b_2 + b_0 b_1) q^{\binom{4}{2}} x^4 \\
&\quad + \cdots + \left(\sum_{a_i a_{i+1} | \to b_i} a_0 a_1 \ldots a_{r-1}\right) q^{\binom{r}{2}} x^r + \cdots \\
&= 1 + (1 - q^2)^{-1} q^{\binom{1}{2}} x + \{(1 - q^2)^{-1} q (1 - q^4)^{-1} + (1 - q^4)^{-1}\} q^{\binom{2}{2}} x^2
\end{aligned}$$

(continued)

$$+ \{(1 - q^2)^{-1}q(1 - q^4)^{-1}q^2(1 - q^6)^{-1} + (1 - q^4)^{-1}q^2(1 - q^6)^{-1} + (1 - q^2)^{-1}(1 - q^6)^{-1}\}q^{\binom{3}{2}}x^3 + \cdots .$$

Here, by the notation,

$$\sum_{a_i a_{i+1} | \to b_i} a_0 a_1 \ldots a_{r-1}$$

we mean that the sum is over all possible products obtainable from $a_0 a_1 \ldots a_{r-1}$ by replacing in it blocks of two consecutive $a_i a_{i+1}$ by b_i. There are F_r (Fibonacci number) such formal terms in $G_r^{(2)}$. This can be seen, by induction, from

$$\begin{aligned} G_r^{(2)} &= a_{r-1}G_{r-1}^{(2)} + b_{r-2}G_{r-2}^{(2)} \\ &= \left(\sum_{a_i a_{i+1} | \to b_i} a_0 a_1 \ldots a_{r-2}\right)a_{r-1} + \left(\sum_{a_i a_{i+1} | \to b_i} a_0 a_1 \ldots a_{r-3}\right)b_{r-2} \\ &= \sum_{a_i a_{i+1} | \to b_i} a_0 a_1 \ldots a_{r-1}. \end{aligned}$$

ACKNOWLEDGMENT

I would like to thank the referee for suggestions which greatly simplified the proof of Theorem 3.

REFERENCES

1. J. Goldman & G.-C. Rota. "On the Foundations of Combinatorial Theory. IV: Finite Vector Spaces and Eulerian Generating Functions." *Studies in Applied Math.* 49 (1970):239-58.
2. I. P. Goulden & D. M. Jackson. *Combinatorial Enumeration.* New York: John Wiley & Sons, 1983.
3. G. Polya & G. Szego. *Problems and Theorems in Analysis.* New York: Springer Verlag, 1972.
4. K.-W. Yang. "A q-Polynomial Identity." *Proc. Amer. Math. Soc.* 95 (1985): 152-54.
5. K.-W. Yang. "Generalized Euler and Chu-Vandermonde Identities." *J. Comb. Theory (A)* (to appear).
6. K.-W. Yang. "q-Polynomial Coefficients." *Ars. Comb.* (to appear).

◆◇◆◇◆

PERIODICITY OF A COMBINATORIAL SEQUENCE

BJORN POONEN

Student, Harvard College, Cambridge, MA 02138

(Submitted July 1986)

In June 1985, twenty-three other high school students and I trained for the International Mathematical Olympiad in a three-week session hosted by the U.S. Military Academy. There I, along with three classmates (John Dalbec, Jeremy Kahn, and Joseph Keane) and the two coaches (Professor Cecil Rousseau and Gregg Patruno) considered $\omega(n)$, defined as the number of possible outcomes in a race among n horses with multiple ties permitted. This sequence was first studied by A. Cayley [1] as the number of a certain type of tree having $n + 1$ terminal nodes. His results have been extended by the more recent papers of Gross [3] and Good [2].

Before uncovering these three papers, we independently proved eleven results which can be found in [1], page 113, [2], pages 11-14, and [3], pages 5-8. Although we found that Good's statement (p. 13),

$$\left|\omega(n) - \frac{n!}{2(\ln 2)^{n+1}}\right| < \frac{1}{2} \text{ for all } n < 16, \tag{1}$$

could be extended to $n < 17$, the only important new results were my proofs of Good's Conjectures 1-5. These conjectures are concerned with the behavior of the sequence modulo r. To prove these, we need the following lemmas.

Lemma 1: If $n, k \geqslant 1$, and we define $\omega(0) = 1$, then

$$2^k\omega(n) = \sum_{j=1}^{k-1} 2^{k-j-1}j^n + \sum_{j=0}^{n} k^j\binom{n}{j}\omega(n-j). \tag{2}$$

Proof of Lemma 1: We have, by equation (4) of [2],

$$\begin{aligned} 2^k\omega(n) &= 2^k\sum_{j=1}^{k-1}\frac{j^n}{2^{j+1}} + 2^k\sum_{i=0}^{\infty}\frac{(i+k)^n}{2^{i+k+1}} \\ &= \sum_{j=1}^{k-1} 2^{k-j-1}j^n + \sum_{i=0}^{\infty}\sum_{j=0}^{n}\frac{\binom{n}{j}i^{n-j}k^j}{2^{i+1}} \\ &= \sum_{j=1}^{k-1} 2^{k-j-1}j^n + \sum_{j=0}^{n} k^j\binom{n}{j}\omega(n-j). \end{aligned}$$

Note that, we we let $k = 1$ in Lemma 1, we obtain a relation derived by Cayley ([1], p. 113). Similarly, we can prove

$$2^{-k}\omega(n) = -\sum_{j=0}^{k} 2^{j-k-1}(-j)^n + \sum_{j=1}^{n} (-k)^j \binom{n}{j}\omega(n - j). \tag{3}$$

From Lemma 1, we have the following useful result.

Corollary; If $n, k \geqslant 1$, then

$$(2^k - 1)\omega(n) \equiv \sum_{j=1}^{k-1} 2^{k-j-1} j^n \pmod{k}. \tag{4}$$

It is interesting to note that the corollary, along with Fermat's Theorem, provides a simple proof of Theorem 5 in [2]. Now we shall use the corollary to prove another lemma.

Lemma 2: For an odd prime p, let $q = p^m$ and $r = p^{m+1}$ be consecutive powers of p. Suppose the sequence $\omega(a)$, $\omega(a + 1)$, ... modulo r has period c, where c is a multiple of $\phi(q)$. Then

$$0 \equiv \sum_{k=0}^{q-1} 2^{q-k-1}[(g + kp)^c - 1] \pmod{r} \tag{5}$$

for $g = 1, 2, \ldots, p - 1$.

Proof: From the corollary to Lemma 1, we find that, for all $n \geqslant a$,

$$0 \equiv (2^r - 1)[\omega(n + c) - \omega(n)] \equiv \sum_{j=1}^{r-1} 2^{r-j-1} j^n (j^c - 1) \pmod{r}.$$

It follows that for any polynomial $P(j)$ with integral coefficients,

$$\sum_{j=1}^{r-1} 2^{r-j-1} j^n P(j)(j^c - 1) \equiv 0 \pmod{r}.$$

Let $P(j) = 1 - (j - g)^{p-1}$ and let n be a multiple of $\phi(r)$ greater than a. By repeated use of theorems of Fermat and Euler, we make the following sequence of observations concerning the terms of the sum that are nonvanishing (mod r):

$$j \not\equiv 0 \pmod{p},\ j^n \equiv 1 \pmod{r},\ j^c - 1 \equiv 0 \pmod{q},$$
$$j \equiv g \pmod{p},\ P(j) \equiv 1 \pmod{p},\ P(j)(j^c - 1) \equiv j^c - 1 \pmod{r}.$$

Thus, the sum reduces to

$$\sum_{k=0}^{q-1} 2^{r-(g+kp)-1}[(g + kp)^c - 1] \equiv 0 \pmod{r}.$$

Now $r - (g + kp) - 1 \equiv q - g - k - 1 \pmod{p - 1}$. Also, since c is a multiple of $\phi(q)$, we have $[(g + kp)^c - 1] \equiv 0 \pmod{q}$. Thus, by Fermat's Theorem, we may substitute $2^{q-g-k-1}$ for $2^{r-(g+kp)-1}$ in the last equation. Finally, multiplying by 2^g, we obtain

$$\sum_{k=0}^{q-1} 2^{q-k-1}[(g+kp)^c - 1] \equiv 0 \pmod{r}.$$

Now we are ready to prove the theorems.

Theorem 1: Modulo a prime p, the period of the sequence $[\omega(n)]$ is at least $p - 1$. This, along with Good's Theorem 5, implies that the period is exactly $p - 1$.

Proof of Theorem 1: For $p = 2$, the result is clear. If $p \geqslant 3$, let c be the minimum period. Applying Lemma 2 with $a = 1$ and $q = 0$, and with g a primitive root modulo p, we have

$$0 \equiv 2^{p-1-g}(g^c - 1) \pmod{p}.$$

However, 2^{p-1-g} is not divisible by p, so $g^c - 1$ must be. Since we chose g as a primitive root modulo p, we must have $c \geqslant p - 1$.

Theorem 1 does *not* imply that, if $\omega(n) \equiv 0 \pmod{p}$, then $n \equiv 0 \pmod{p-1}$. [A counterexample is $\omega(3) \equiv 0$ (mod 13.] Proofs of three of Good's conjectures in [1] depended on this result:

$$\mathrm{GCF}(\omega(n),\ \omega(n+1)) = 1,\ \mathrm{GCF}(\omega(n) - 1,\ \omega(n+1) - 1) = 2,\ \text{and}\ n \mid \omega(n),$$

for all n. The first is false because $\omega(1090)$, $\omega(1091)$, and $\omega(1092)$ are all divisible by 1093. The second and third are still open.

Theorem 2: If $q = p^m$ with p prime, then for all $n \geqslant m$,

$$\omega(n + \phi(q)) \equiv \omega(n) \pmod{q}, \tag{6}$$

where ϕ is Euler's totient function.

Proof of Theorem 2: Since $n \geqslant m$, the terms in the sum given by (4) with j divisible by p will drop out. The result then follows from $j^{n+\phi(q)} \equiv j^n \pmod{q}$, which is Euler's Theorem.

Theorem 2 does not tell us that the period of the sequence $\{\omega(n)\}$ modulo q is exactly $\phi(q)$ for q a power of a prime. We know only that the minimum period must be a factor of $\phi(q)$. Theorem 3 shows that, when q is the power of an odd prime, this fundamental period is no less than $\phi(q)$. To prove this, we need one more lemma.

Lemma 3: For an odd prime p, let $q = p^m$ and $r = p^{m+1}$. Then, for any integer k, $(1 + kp)^{\phi(q)} - 1 = -kq \pmod{r}$.

Proof: By the binomial theorem,

$$(1 + kp)^{\phi(q)} = \sum_{i=0}^{\phi(q)} \binom{\phi(q)}{i} (kp)^i .$$

Let $f(n)$ denote the greatest integer d such that p^d divides n. Then

$$f\left(\binom{\phi(q)}{i} p^i\right) = \sum_{j=\phi(q)-i+1}^{\phi(q)} f(j) - \sum_{j=1}^{i} f(j) + i = f(\phi(q)) - f(i) + i,$$

Since $f(\phi(q) - j) = f(j)$ for any j with $0 < j < \phi(q)$. But if $f(i) > 0$, then

$$i \geqslant p^{f(i)} \geqslant 3^{f(i)} \geqslant f(i) + 2,$$

so $i - f(i) \geqslant 2$ for all $i \geqslant 2$. Also, $f(\phi(q)) = m - 1$, so if we look at the binomial expansion modulo r, all but the first two terms drop out:

$$(1 + kp)^{\phi(q)} - 1 \equiv 1 + \phi(q)(kp) - 1 \equiv -kq \pmod{r}.$$

Theorem 3: Let p be an odd prime. Then, modulo p^m, the sequence

$$\omega(m),\ \omega(m + 1),\ \omega(m + 1),\ \ldots$$

has period exactly $\phi(p^m)$.

Proof of Theorem 3: Theorem 1 proved the case $m = 1$. Now suppose that Theorem 3 holds for a certain m. We shall prove that it must also hold for $m + 1$. Let $q = p^m$, let $r = p^{m+1}$, and let c be the minimum period of the sequence $\{\omega(n)\}$ modulo r. By the inductive hypothesis, $\phi(q)$ is the period modulo q, so c must be a multiple of $\phi(q)$. By Theorem 2, c must be a factor of $\phi(r)$. But $\phi(r) = p\phi(q)$, so c is either $\phi(q)$ or $\phi(r)$.

Suppose $c = \phi(q)$. Applying Lemma 2 with $a = m$ and $g = 1$ yields

$$0 \equiv \sum_{k=0}^{q-1} 2^{q-k-1}[(1 + kp)^c - 1] \equiv \sum_{k=0}^{q-1} 2^{q-k-1}(-kq) \pmod{r},$$

by Lemma 3. Evaluating this sum, we obtain

$$0 \equiv (2^q - q - 1)q \equiv -q \pmod{r} \quad \text{(by Fermat's Theorem)},$$

a contradiction. Thus, $c = \phi(r)$, and the induction is complete.

We now proceed to consider the sequence modulo a power of 2.

Theorem 4: If $1 \leqslant m \leqslant n - 4$, then

$$\omega(n + 2^m) \equiv \omega(n) + 2^{m+4} \pmod{2^{m+5}}. \tag{7}$$

Proof of Theorem 4: Set $k = 2^{m+5}$ in the corollary of Lemma 1. Then $2^k - 1 \equiv -1 \pmod{k}$, so

$$\omega(n + c) - \omega(n) \equiv -\sum_{j=1}^{k-1} 2^{k-j-1} j^n (j^c - 1) \pmod{k}.$$

Now set $c = 2^m$. The terms with even j drop out because 2^{k-j-1} is even and j^n is divisible by 2^{m+4}. The terms with odd $j \leqslant k - 5$ also drop out since 2^{k-j-1} is divisible by 2^4 and $j^c - 1$ is divisible by 2^{m+1} (by Euler's Theorem). Thus, our sum reduces to

$$\begin{aligned} &\omega(n + c) - \omega(n) \\ &\equiv -2^2(k - 3)^n[(k - 3)^c - 1] - (k - 1)^n[(k - 1)^c - 1] \pmod{k} \\ &\equiv -4(-3)^n(3^c - 1) \pmod{k}. \end{aligned}$$

To show that this is congruent to 2^{m+4} modulo 2^{m+5}, it suffices to prove that 2^{m+2} is the highest power of 2 dividing

$$3^{2^m} - 1 = (3^{2^{m-1}} + 1)(3^{2^{m-2}} + 1) \ldots (3 + 1)(3 - 1).$$

This is true since the second-to-last factor is 4 and each of the other m factors is congruent to 2 modulo 4.

Theorem 5: If $\omega(n)$ is expressed in binary notation as

$$a_{n0} + 2a_{n1} + 2^2a_{n2} + 2^3a_{n3} + \ldots,$$

then the sequence a_{mm}, $a_{(m+1)m}$, $a_{(m+2)m}$, ... runs into a cycle whose lengths for $m = 0, 1, 2, 3, \ldots$ are, respectively, 1, 2, 2, 1, 2, 4, 8, ... From this, it follows that, modulo 2^m, the sequence $\omega(m - 1)$, $\omega(m)$, $\omega(m + 1)$, ... has period 1 when $m = 1$, period 2 when $2 \leqslant m \leqslant 4$, and period 2^{m-4} when $m \geqslant 5$. [We define $\omega(0) = 1$.]

Proof of Theorem 5: By Theorem 4 with $m = 1$, if $n \geqslant 5$, then

$$\omega(n + 1) \equiv \omega(n) \pmod{32},$$

so for $m < 5$, the sequence a_{5m}, a_{6m}, a_{7m}, ... is periodic with period dividing 2. [The period is 1 iff $a_{5m} = a_{6m}$, which we see holds iff $m = 3$, by observing the five least significant binary digits of $\omega(5)$ and $\omega(6)$.] Also, by observing the five least significant binary digits of $\omega(0)$, $\omega(1)$, ..., $\omega(4)$, we see that the periodicity begins with a_{mm} instead of a_{5m} for $m < 5$.

If $m \geqslant 5$, then in the sequence a_{mm}, $a_{(m+1)m}$, $a_{(m+2)m}$, ... of zeros and ones, the terms are the opposite of what they were, after every 2^{m-4} terms, by Theorem 4. This implies that, after 2^{m-3} terms, the sequence repeats. Hence, the sequence runs into a cycle whose length is a factor of 2^{m-3} but not of 2^{m-4}. Thus, the period is exactly 2^{m-3}.

Finally, to summarize and extend our results, we have the following:

Theorem 6: Let the prime factorization of $r > 1$ be $2^m p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$. If

$$a = \max\{m - 1, m_1, m_2, m_3, \ldots, m_k\}$$

and

$$b = \mathrm{LCM}(\phi(p_1^{m_1}), \phi(p_2^{m_2}), \ldots, \phi(p_k^{m_k})),$$

then the period of the sequence

$$\omega(a), \omega(a + 1), \omega(a + 2), \ldots \text{ modulo } r$$

is exactly

$$\begin{array}{ll} b, & \text{if } m = 0 \text{ or } 1, \\ \mathrm{LCM}(2, b), & \text{if } 2 \leqslant m \leqslant 4, \\ \mathrm{LCM}(2^{m-4}, b), & \text{if } m \geqslant 5. \end{array}$$

Note that the period of $\{\omega(n)\}$ modulo r is not the product of the periods modulo its prime power factors, but is, rather, their lowest common multiple. This implies that even when r is odd, the period modulo r is not necessarily $\phi(r)$, although it must be a factor of $\phi(r)$. The smallest example of this is $r = 15$, in which case the period is $\mathrm{LCM}(\phi(3), \phi(5)) = 4$ instead of $\phi(15) = 8$.

Proof of Theorem 6: Let c be the claimed period. If $n \geqslant m - 1$, then

$$\omega(n + c) \equiv \omega(n) \pmod{2^m}$$

by Theorem 5, since c is a multiple of the period of $\{\omega(n)\}$ modulo 2^m. Also, if $n \geqslant m_i$, then

$$\omega(n + c) \equiv \omega(n) \pmod{p_i^{m_i}}$$

by Theorem 3, since c is a multiple of $\phi(p_i^{m_i})$, for $i = 1, 2, \ldots, k$. Hence, if $n \geqslant a$,

$$\omega(n + c) \equiv \omega(n) \pmod{r}.$$

If the actual period d of $\{\omega(n)\}$ modulo r were any smaller than c, then it could not be a multiple of all the necessary periods modulo 2^m and $p_i^{m_i}$, since c is their LCM. Suppose d is not a multiple of the necessary period modulo p^q. Then, for some $n \geqslant a$, $\omega(n + d) \not\equiv \omega(n) \pmod{p^q}$, so

$$\omega(n + d) \not\equiv \omega(n) \pmod{r},$$

a contradiction. Hence, the period given is minimum.

Now that we have finished proving the main theorems, we will conclude with a few applications of Theorem 6 and other miscellaneous results:

(a) $\omega(12k) \equiv \omega(12k + 3) \equiv 0 \pmod{13}$.

(b) $59|\omega(11)$, so $59|\omega(58k + 11)$. Dirichlet's Theorem implies that there are infinitely many primes of the form $58k + 11$, so there are infinitely many primes p for which $\omega(p)$ is composite.

(c) $9\nmid\omega(n)$ for any n, so there seems to be no generalization of $p|\omega(p - 1)$ ([12], p. 23) to powers of odd primes.

(d) For any prime p and any $m \geqslant 1$, $\omega(p^m) \equiv 1 \pmod{p}$, so if $n|\omega(n)$, n has at least two distinct prime factors.

(e) For odd primes p and q, $pq|\omega(pq)$ iff $p|\omega(q)$ and $q|\omega(p)$. There are no such primes less than 1700, but I conjecture on probabilistic grounds that such primes do exist.

(f) For all n, $\mathrm{GCF}(\omega(n) - 1, \omega(n + 1) - 1)$ has no divisor less than 1700 except 2. Yet, again on probabilistic grounds, I conjecture that there exists n for which $\mathrm{GCF}(\omega(n) - 1, \omega(n + 1) - 1) \neq 2$.

(g) The only r for which the period of $\{\omega(n)\}$ modulo r is exactly $\phi(r)$ are the numbers of the form p^m and $2p^m$, where p is an odd prime, and 4.

ACKNOWLEDGMENTS

I would like to thank the two coaches at the training session, Professor Cecil Rousseau and Gregg Patruno, who suggested that I write this paper, and who provided many helpful comments.

REFERENCES

1. A. Cayley. *Collected Works*, *4*:112-15. Cambridge, Mass.: Cambridge University Press, 1891.
2. I. J. Good. "The Number of Orderings of n Candidates When Ties Are Permitted." *The Fibonacci Quarterly* *13*, no. 1 (1975):11-18.
3. O. A. Gross. "Preferential Arrangements." *Amer. Math. Monthly* *69* (1962): 4-8.

◆◇◆◇◆

JACOBSTHAL AND PELL CURVES

A. F. HORADAM
University of New England, Armidale, NSW, 2351, Australia

(Submitted August 1986)

1. INTRODUCTION

In an earlier paper [2], a study was made of Fibonacci and Lucas curves in the plane, and their Laser-printed graphs were exhibited. These graphs were drawn from the equations of the curves, rather than from the tabulated lists of values of the Cartesian coordinates x and y, which also served a purpose of their own. It is desirable to extend the work in [2] and so produce a more complete theory.

Here, we present basic information about the corresponding curves associated with (i) Pell and Pell-Lucas numbers, and (ii) Jacobsthal and *Jacobsthal-Lucas* numbers, in our nomenclature.

Curves associated with (i) will carry the generic name of *Pell curves* while those connected with (ii) will be designated *Jacobsthal curves*. There seems to be no theory related to (i) and (ii) which corresponds to the result of Halsey [1] for Fibonacci numbers.

To avoid unnecessary duplication in our discussion, we will consider the numbers in (i) and (ii) (as well as the Fibonacci and Lucas numbers) to be special instances of a general sequence $\{w_n\}$ whose relevant properties will be investigated.

Thus, the Pell and Jacobsthal curves, as well as the Fibonacci and Lucas curves, may be thought of as members of a family of curves which we shall designate as *w-curves*.

The two Pell curves and the two Jacobsthal curves resemble the Fibonacci and Lucas curves, so we will not reproduce them here. Instead, the reader is invited to compare them in the mind's eye with the curves exhibited in [2].

2. GENERALITIES

Let a, b, p, and q be real numbers, usually integers.

Define the sequence $\{w_n\}$ by

$$w_{n+2} = pw_{n+1} - qw_n,\ w_0 = 2b,\ w_1 = a + b \quad (n \geqslant 0). \tag{2.1}$$

Extension to negative values of n may be made, but we do not require it here.

The explicit Binet form for w_n is

$$w_n = (A\alpha^n - B\beta^n)/(\alpha - \beta), \tag{2.2}$$

where α, β are the roots of the characteristic equation

$$\lambda^2 - p\lambda + q = 0, \tag{2.3}$$

so that
$$\begin{cases} \alpha = (p + \sqrt{p^2 - 4q})/2 \\ \beta = (p - \sqrt{p^2 - 4q})/2 \\ \alpha + \beta = p,\ \alpha\beta = q,\ \alpha - \beta = \sqrt{p^2 - 4q} \end{cases} \tag{2.4}$$

and

$$\begin{cases} A = a + b - 2b\beta, \\ B = a + b - 2b\alpha. \end{cases} \tag{2.5}$$

Special cases of $\{w_n\}$ are:

SEQUENCE	p	q	a	b	α	β	
$\{F_n\}$: Fibonacci	1	-1	1	0	$(1 + \sqrt{5})/2$	$(1 - \sqrt{5})/2$	(2.5)
$\{L_n\}$: Lucas	1	-1	0	1	$(1 + \sqrt{5})/2$	$(1 - \sqrt{5})/2$	(2.6)
$\{P_n\}$: Pell	2	-1	1	0	$1 + \sqrt{2}$	$1 - \sqrt{2}$	(2.7)
$\{Q_n\}$: Pell-Lucas	2	-1	1	1	$1 + \sqrt{2}$	$1 - \sqrt{2}$	(2.8)
$\{J_n\}$: Jacobsthal	1	-2	1	0	2	-1	(2.9)
$\{j_n\}$:Jacobsthal-Lucas	1	-2	0	1	2	-1	(2.10)

The Fibonacci and Lucas sequences are well known.

Some values for the other sequences are:

	$n = 0$	1	2	3	4	5	6	7	...	
$\{P_n\}$	0	1	2	5	12	29	70	169	...	(2.11)
$\{Q_n\}$	2	2	6	14	34	82	198	478	...	(2.12)
$\{J_n\}$	0	1	1	3	5	11	21	43	...	(2.13)
$\{j_n\}$	2	1	5	7	17	31	65	127	...	(2.14)

In what follows, $q < 0$.

Write

$$q = -1 \cdot r \quad (r > 0), \tag{2.15}$$

so, by (2.4)

$$\beta = -1 \cdot \frac{r}{\alpha}. \tag{2.16}$$

Only the cases

$$q = -1, \text{ i.e., } r = 1 \quad [\text{cf. } (2.5)\text{-}(2.8)] \tag{2.17}$$

and

$$\beta = -1, \text{ i.e., } r = \alpha \ (= 2) \quad [\text{cf. } (2.9)\text{-}(2.10)] \tag{2.18}$$

will concern us.

3. THE w-CURVES

For Cartesian coordintes x, y of a point in the plane, let

$$x = \left(A\alpha^{\theta} - B\left(\frac{\alpha}{r}\right)^{-\theta} \cos \theta\pi\right) \Big/ (\alpha - \beta) \tag{3.1}$$

and

$$y = -B\alpha^{-\theta} \sin \theta\pi / (\alpha - \beta), \tag{3.2}$$

where θ is real.

Comparing (3.1) with (2.2), we may refer to (3.1) as the *generalized Binet form* of w_n. When $\theta = n$, integer, we have $x = w_n$ by (2.2) and (3.1).

As θ varies in (3.1) and (3.2), we obtain the *w-curves*.

Now, from (3.1) and (3.2), we have

$$\frac{dy}{d\theta} = B\alpha^{-\theta}(\log \alpha \sin \theta\pi - \pi \cos \theta\pi)/(\alpha - \beta) \tag{3.3}$$

and

$$\frac{dx}{d\theta} = \left[A\alpha^{\theta} \log \alpha + B\left(\frac{\alpha}{r}\right)^{-\theta}\left\{\log\left(\frac{\alpha}{r}\right)\cos \theta\pi + \pi \sin \theta\pi\right\}\right] \Big/ (\alpha - \beta), \tag{3.4}$$

whence $\dfrac{dy}{dx} = 0$ if

$$\tan \theta\pi = \frac{\pi}{\log \alpha} \doteqdot \begin{cases} 6.528 & \text{for } (2.5), (2.6)\text{—see [2]} \\ 3.565 & \text{for } (2.7), (2.8) \\ 4.538 & \text{for } (2.9), (2.10) \end{cases} \tag{3.5}$$

yielding

$$\theta\pi \doteqdot 81°16', \ 74°21', \ 77°34',$$

respectively, i.e.,

$$\theta \doteqdot \begin{cases} 0.45 \\ 0.41 \\ 0.43, \end{cases} \tag{3.6}$$

respectively, for the three cases in (3.5).

Write (3.5) as

$$\begin{cases} \sin \theta\pi = \pm k\pi \\ \cos \theta\pi = \pm k \log \alpha, \end{cases} \tag{3.7}$$

i.e.,

$$k = [\pi^2 + (\log \alpha)^2]^{-1/2} \tag{3.8}$$

giving

$$k \doteqdot \begin{cases} 0.31 \\ 0.30 \\ 0.299, \end{cases} \tag{3.8$'$}$$

respectively, for the three cases in (3.5).

Eliminate θ from (3.1) and (3.2) for the specific values of θ covered in (3.5) for stationary points, i.e., for

$$\tan(\theta - m)\pi = \frac{\pi}{\log \alpha} \qquad (m \text{ an integer}). \tag{3.5)$'$}$$

With the aid of (3.7), we find that the locus of the stationary points is generally given by

$$y^2 - \frac{\pi}{\log \alpha} xy = \pm \frac{ABk\pi^2}{\Delta^2 \log \alpha} \qquad (\Delta = \alpha - \beta), \tag{3.9}$$

which represents the branches of two hyperbolas (a hyperbola and its conjugate hyperbola) in the first and fourth quadrants.

Common asymptotes of these hyperbolas have equations

$$y = 0 \quad \text{and} \quad y = \frac{\pi}{\log \alpha}x, \tag{3.10}$$

the gradients of the oblique asymptote being given in (3.5).

Inflexions on these curves are established in the usual way. When x and y are replaced by the functional notation $x(\theta)$ and $y(\theta)$, it may be demonstrated as in [2] that (3.1) and (3.2) do reproduce the *w-type* recurrence relations, namely,

$$x(\theta) = px(\theta - 1) - qx(\theta - 2) \tag{3.1)$'$}$$

and

$$y(\theta) = py(\theta - 1) - qy(\theta - 2). \tag{3.2)$'$}$$

4. PELL CURVES

Consider the *generalized Pell sequence* $\{R_n\}$ defined by (2.1) in which $p = 2$ and $q = -1$, namely,

$$R_{n+2} = 2R_{n+1} + R_n, \quad R_0 = 2b, \quad R_1 = a + b \quad (n \geqslant 0). \tag{4.1}$$

From (2.2), we have the Binet form

$$R_n = (A\alpha^n - B\beta^n)/2\sqrt{2}, \tag{4.2}$$

where α, β are given in (2.7) [and (2.8)] and A and B in (2.5).

For the Pell numbers P_n given in (2.11), and for the Pell-Lucas numbers Q_n given in (2.12), we have

$$P_n : a = 1,\ b = 0,\ A = B = 1 \tag{4.3}$$

and

$$Q_n : a = 1,\ b = 1,\ A = -B = 2\sqrt{2}. \tag{4.4}$$

Binet forms for P_n and Q_n are then readily obtained from (4.2).

Substituting appropriately in (3.1) and (3.2), we derive

$$x = (\alpha^{\theta} - \alpha^{-\theta} \cos \theta\pi)/2\sqrt{2} \tag{4.5}$$

and

$$y = -\alpha^{-\theta} \sin \theta\pi/2\sqrt{2} \tag{4.6}$$

for the Pell case, and

$$x = \alpha^{\theta} + \alpha^{-\theta} \cos \theta\pi \tag{4.7}$$

and

$$y = \alpha^{-\theta} \sin \theta\pi \tag{4.8}$$

for the Pell-Lucas case.

Equations (4.5) and (4.7) are the *modified Binet forms* for P_n and Q_n, respectively. When $\theta = n$, integer, we get the usual Binet forms for P_n and Q_n, i.e., $x = P_n$ and $x = Q_n$, respectively.

The locus given by (4.5) and (4.6) is the *Pell curve*. Its stationary points lie on the two appropriate branches of the hyperbolas

$$y^2 - \frac{\pi}{\log \alpha} xy = \pm \frac{k\pi^2}{8 \log \alpha} \qquad (\alpha = 1 + \sqrt{2}). \tag{4.9}$$

Equations (4.7) and (4.8) yield the *Pell-Lucas curve*. Stationary points of this curve lie on the hyperbolic curves

$$y^2 - \frac{\pi}{\log \alpha} xy = \pm \frac{k\pi^2}{\log \alpha} \qquad (\alpha = 1 + \sqrt{2}). \tag{4.10}$$

In both (4.9) and (4.10), the value of k is given in (3.8)'. Asymptotes common to the curves in (4.9) and (4.10) are $y = 0$ and $y = (\pi/\log \alpha)x$, whose gradient is given in (3.5).

Suppose we put $a = 3$, $b = 1$ in (4.1) so that $A = 2\alpha$, $B = 2\beta$.

Then (4.1) or the Binet forms for P_n, Q_n, and R_n yield

$$\begin{aligned} R_n^* &= 2P_n + Q_n \\ &= 2P_{n+1} \qquad \text{since } Q_n = P_{n+1} + P_{n-1}. \end{aligned} \tag{4.11}$$

Thus, a composite curve for $2P_\theta + Q_\theta$ is equivalent to the Pell curve for $2P_{\theta+1}$. Furthermore, from (4.11) or the Binet forms, we deduce that

$$\begin{aligned} R_n &= (a - b)P_n + bQ_n \\ &= (a - b)P_n + b(P_{n+1} + P_{n-1}) \\ &= (a + b)P_n + 2bP_{n-1} \\ &= R_1P_n + R_0P_{n-1} \qquad \text{by (4.1)} \\ &[= R_n^* = 2P_{n+1} \text{ when } a = 3,\ b = 1, \text{ as in (4.11)}]. \end{aligned} \tag{4.12}$$

5. JACOBSTHAL CURVES

Next, consider the *generalized Jacobsthal sequence* $\{\mathcal{J}_n\}$ given by

$$\mathcal{J}_{n+2} = \mathcal{J}_{n+1} + 2\mathcal{J}_n, \quad \mathcal{J}_0 = 2b, \quad \mathcal{J}_1 = a + b \quad (n \geqslant 0). \tag{5.1}$$

From (2.2), we have the Binet form

$$\mathcal{J}_n = \frac{A\alpha^n - B\beta^n}{3} \tag{5.2}$$

in which α (= 2), β (= -1) are already given in (2.9) and (2.10), and A, B are given in (2.5).

Particular cases of (5.1) are the Jacobsthal numbers J_n given in (2.13) and the Jacobsthal-Lucas numbers j_n given in (2.14) for which

$$J_n : a = 1,\ b = 0,\ A = B = 1 \tag{5.3}$$

and

$$j_n : a = 0,\ b = 1,\ A = -B = 3. \tag{5.4}$$

Binet forms for J_n and j_n then readily follow from (5.2).

Appropriate substitution in (3.1) and (3.2) produces

$$x = (2^{\theta} - \cos\theta\pi)/3 \tag{5.5}$$

and

$$y = -2^{-\theta}\sin\theta\pi/3 \tag{5.6}$$

for J_n, and

$$x = 2^{\theta} + \cos\theta\pi \tag{5.7}$$

and

$$y = 2^{-\theta}\sin\theta\pi \tag{5.8}$$

for j_n. Note the effect of (2.18) on (3.1) in (5.5) and (5.7).

Equations (5.5) and (5.7) are the *modified Binet forms* for J_n and j_n, respectively. Setting $\theta = n$, integer, we have the ordinary Binet forms for J_n and j_n, i.e., $x = J_n$ and $x = j_n$, respectively.

The locus given by (5.5) and (5.6) is the *Jacobsthal curve*. Its stationary points lie on the appropriate branches of the rectangular hyperbolas

$$y\left(x \pm \frac{k\log 2}{3}\right) = \mp\frac{k\pi}{9}. \tag{5.9}$$

Equations (5.7) and (5.8) yield the *Jacobsthal-Lucas curve*. Its stationary points lie on the rectangular hyperbolic branches

$$y(x \pm k\log 2) = \mp k\pi. \tag{5.10}$$

In both (5.9) and (5.10), the value of k is given in (3.8)′.

Put $a = 1$, $b = 1$ in (5.1) so that $A = -B = 4$.

Hence, as for the Pell case,

$$\begin{aligned}\mathcal{J}_n^* &= J_n + j_n \\ &= 2J_{n+1}.\end{aligned} \tag{5.11}$$

Thus, a composite curve for $J_\theta + j_\theta$ is equivalent to the Jacobsthal curve $2J_{\theta+1}$.

Finally,

$$\begin{aligned}\mathcal{J}_n &= aJ_n + bj_n \qquad (5.12)\\ &= aJ_n + b(2J_{n-1} + J_{n+1}) \qquad \text{since } j_n = J_{n+1} + 2J_{n-1}\\ &= (a+b)J_n + 4bJ_{n-1}\\ &= \mathcal{J}_1 J_n + 2\mathcal{J}_0 J_{n-1} \qquad \text{by (5.1)}\\ [&= \mathcal{J}_n^* = 2J_{n+1} \quad \text{when } a = 1,\ b = 1 \text{ as in (5.11)}].\end{aligned}$$

Notice the formal similarity of the right-hand sides of (4.12) and (5.12).

In choosing the expression for y in (3.2), we could have opted to pick α/r instead of α, both of which seem to be permissible as extensions of Wilson's original idea [2] for Fibonacci curves. Choice of α/r, however, appears to be the less appropriate.

Some obvious limits might be noted, and compared with similar limits for $\{F_n\}$ and $\{L_n\}$. These are:

$$\lim_{n\to\infty}\left(\frac{Q_n}{P_n}\right) = 2\sqrt{2} \qquad \lim_{n\to\infty}\left(\frac{P_{n+1}}{P_n}\right) = \lim_{n\to\infty}\left(\frac{Q_{n+1}}{Q_n}\right) = 1 + \sqrt{2}; \qquad (5.13)$$

$$\lim_{n\to\infty}\left(\frac{j_n}{J_n}\right) = 3 \qquad \lim_{n\to\infty}\left(\frac{J_{n+1}}{J_n}\right) = \lim_{n\to\infty}\left(\frac{j_{n+1}}{j_n}\right) = 2. \qquad (5.14)$$

Our concluding comment is of a geometrical nature. If we consider sequences in which two terms have the same value, e.g., $J_1 = J_2 = 1$ and $F_1 = F_2 = 1$, we observe that, as θ passes through the set of values giving these coincident numbers, the curve will necessarily have a node (i.e., a loop at a double-point) there. Thus, in (5.5), the Jacobsthal curve has a node at $x = 1$, as θ lies in the range $2 \leqslant \theta \leqslant 3$. Similarly, the Fibonacci curve has a node occurring when $x = 1$ and $1 \leqslant \theta \leqslant 2$.

REFERENCES

1. E. Halsey. "The Fibonacci Number F_u Where u Is Not an Integer." *The Fibonacci Quarterly* 3, no. 2 (1965):147-52, 233.
2. A. F. Horadam & A. G. Shannon. "Fibonacci and Lucas Curves." *The Fibonacci Quarterly* 26, no. 1 (1988):3-13.
3. E. Jacobsthal. "Fibonaccische Polynome und Kreisteilungsgleichungen." *Sitzungsberichte der Berliner Mathematischen Gesellschaft* 17 (1919-1920):43-57.

◆◇◆◇◆

REFEREES

In addition to the members of the Board of Directors and our Assistant Editors, the following mathematicians, engineers, and physicists have assisted **THE FIBONACCI QUARTERLY** by refereeing papers during the past year. Their special efforts are sincerely appreciated, and we apologize for any names that have inadvertently been overlooked.

ANDO, Shiro
Hosie University
ANDREWS, George E.
Pennsylvania State University
BACKSTROM, Robert P.
Gymea, New South Wales
BENNETT, Larry F.
South Dakota State University
BERZSENYI, George
Lamar University
BOLLINGER, Richard C.
Pennsylvania State University-Erie
BRESSOUD, David M.
Pennsylvania State University
BRUCKMAN, Paul S.
Fair Oaks, CA
BURTON, David M.
University of New Hampshire
CAMPBELL, Colin
University of St. Andrews
CANTOR, David G.
Malibu, CA
CHARALAMBIDES, Ch. A.
University of Athens
CHUNG, Fan R. K.
Morristown, NJ
CHURCH, C.A., Jr.
University of North Carolina
COHEN, M. E.
California State University
CREELY, Joseph W.
Vincentown, NJ
DE BRUIN, Marcel G.
University of Amsterdam
DELEON, M. J.
Florida Atlantic University
DENCE, Thomas P.
Ashland College
DEO, Naisingh
Washington State University
DESMOND, James E.
Pensacola Junior College
DODD, Fred
University of South Alabama
DUDLEY, Underwood
DePauw University
ECKERT, Ernest J.
University of South Carolina
EVANS, Ronald J.
University of California
FILASETA, Michael
University of South Carolina
FILIPPONI, Piero
Roma, Italy
FINE, Nathan J.
Dearfield Beach, FL
GAAL, Lisl
University of Minnesota
GALLIAN, Joseph A.
University of Minnesota
GESSEL, Ira M.
Brandeis University
GOLOMB, Solomon W.
University of Southern California
GORDON, Basil
University of California
GUY, Richard
University of Calgary
HARBORTH, Heiko
West Germany
HARRIS, V.C.
San Diego, CA
HAYES, David F.
San Jose State University
HEUER, Gerald A.
Concordia College
HONSBERGER, Ross A.
University of Waterloo
HORIBE, Yasuichi
Shizuoka University
HUDSON, Richard H.
University of South Carolina
IBARRA, Oscar H.
University of Minnesota
JOICHI, James T.
University of Minnesota
JOYNER, R. N.
East Carolina University
KALMAN, Dan
Rancho Palos Verdes, CA
KATZ, Talbot M.
Hunter College
KENNEDY, Robert E.
Central Missouri State University
KNOPFMACHER, John
University of Witwatersrand
KNOEBEL, Arthur R.
University of Illinois
KNOSHAUG, Clayton
Bemidji State University
KONHAUSER, Joseph D. E.
Macalester College
KUIPERS, L.
Suisse, Switzerland
LANGE, L. J.
University of Missouri
LEHMER, D. H.
University of California
LENSTRA, H. W., Jr.
University of California
LEVESQUE, Claude
Universite Laval
LIGH, Steve
University of SW Louisiana
LIN, Pin-Yen
Republic of China
LONG, Calvin
Washington State University
LOUD, Warren S.
Minneapolis, MN
MAHON, J. M., Br.
Bankstown, Australia
MANN, H. B.
Tucson, AZ
MERCIER, Armel N.
University of Quebec
METZGER, Jerry
University of North Dakota
MEYEROWITZ, Aaron
Florida Atlantic University
NAGASAKA, Kenji
Chiba, Japan
NAJAR, R. M.
University of Wisconsin
NIEDERREITER, Harold G.
Vienna, Austria
OWENS, Margaret A.
California State University
PHILIPPOU, Andreas N.
University of Patras
PHILLIPS, George M.
University of St. Andrews
POLLIN, Jack M.
Tucson, AZ
POMERANCE, Carl
University of Georgia
PRODINGER, Helmut
Wien, Austria
RAWSTHORNE, Daniel A.
Silver Springs, MD
RICHERT, Norman J.
University of Houston
ROBBINS, Neville
San Francisco State University
ROBERTS, Joseph B.
Reed College
SELFRIDGE, John L.
Northern Illinois University
SHALLIT, J. O.
University of Chicago
SIVARAMAKRISHNAN, R.
University of Calicut
SNOW, Donald R.
Brigham Young University
SPICKERMAN, William R.
East Carolina University
STROEKER, R. J.
The Netherlands
SUBBARAO, M. V.
University of Alberta
TURNER, Stephen John
Babson College
WADDILL, Marcellus E.
Wake Forest University
WARREN, L. J.
San Diego State University
WEST, Douglas B.
Princeton, NJ
WILF, H. S.
University of Pennsylvania
YOKOTA, Hisashi
Hiroshima, Japan

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-610 *Proposed by L. Kuipers, Serre, Switzerland*

Prove that there are no positive integers r, s, t such that (F_r, F_s, F_t) is a Pythagorean triple (that is, such that $F_r^2 + F_s^2 = F_t^2$).

B-611 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S(n) = \sum_{k=1}^{n} L_{4k+2}.$$

For which positive integers n is $S(n)$ an integral multiple of 3?

B-612 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$T(n) = \sum_{k=1}^{n} F_{4k+2}.$$

For which positive integers n is $T(n)$ an integral multiple of 7?

B-613 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Show that there exist integers a, b, and c such that

$$F_{n+p}^2 + F_{n-p}^2 = aF_n^2F_p^2 + b(-1)^pF_n^2 + c(-1)^nF_p^2.$$

B-614 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $L(n) = L_{n-2}L_{n-1}L_{n+1}L_{n+2}$ and $F(n) = F_{n-2}F_{n-1}F_{n+1}F_{n+2}$. Show that

$$L(n) \equiv F(n) \pmod 8$$

and express $[L(n) - F(n)]/8$ as a polynomial in F_n.

B-615 *Proposed by Michael Eisenstein, San Antonio, TX*

Let $C(n) = L_n$ and $a_n = C(C(n))$. For $n = 0, 1, \ldots$, prove that

$$a_{n+3} = a_{n+2}a_{n+1} \pm a_n.$$

SOLUTIONS

Fibonacci Convolution

B-586 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Show that $5\sum_{k=0}^{n} F_{k+1}F_{n+1-k} = (n+1)F_{n+3} + (n+3)F_{n+1}$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

It is known that

$$\begin{aligned}&5(F_1F_{t-1} + F_2F_{t-2} + \cdots + F_{t-2}F_2 + F_{t-1}F_1)\\ &= (t-1)F_{t+1} + (t+1)F_{t-1}.\end{aligned}$$

[For a proof of this result, see (1.12) on p. 118 of "Fibonacci Convolution Sequences" by V. E. Hoggatt, Jr., and Marjorie Bicknell-Johnson, which appears in the April 1977 issue of this journal.] Thus,

$$\begin{aligned}5\sum_{k=0}^{n} F_{k+1}F_{n+1-k} &= 5(F_1F_{n+1} + F_2F_n + \cdots + F_nF_2 + F_{n+1}F_1)\\ &= [(n+2)-1]F_{(n+2)+1} + [(n+2)+1]F_{(n+2)-1}\\ &= (n+1)F_{n+3} + (n+3)F_{n+1}.\end{aligned}$$

Also solved by Demetris Antzoulakos, Paul S. Bruckman, Lászlő Cseh, Russell Euler, Piero Filipponi & Odoardo Brugia, Herta T. Freitag, George Koutsoukellis, L. Kuipers, Jia-Sheng Lee, Carl Libis, Sahib Singh, J. Suck, Nico Trutzenberg, Gregory Wulczyn, and the proposer.

D. E. for Fibonacci Generating Function

B-587 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Let $y = \sum_{n=0}^{\infty} F_n x^n/n!$ and $z = \sum_{n=0}^{\infty} L_n x^n/n!$.

Show that $y'' = y' + y$ and $z'' = z' + z$.

Solution by Alberto Facchini, Università di Udine, Udine, Italy

Since

$$y' = \sum_{n=0}^{\infty} F_{n+1}x^n/n!, \quad y'' = \sum_{n=0}^{\infty} F_{n+2}x^n/n! \text{ and } F_{n+2} = F_{n+1} + F_n,$$

the desired result follows. The proof for z is similar.

Also solved by Demetris Antzoulakos, Charles Ashbacher, Paul S. Bruckman, Gabriel B. Costa, Lászlő Cseh, Russell Euler, Piero Filipponi, L. Kuipers, J.-S. Lee, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Closed Form Exponential Generating Function

B-588 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Find the y and z of Problem B-587 in closed form.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

Let $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$. Then,

$$y = \sum_{n=0}^{\infty} F_n x^n/n! = \sum_{n=0}^{\infty} \frac{a^n - b^n}{\sqrt{5}}(x^n/n!) = \frac{1}{\sqrt{5}}(e^{ax} - e^{bx})$$

and

$$z = \sum_{n=0}^{\infty} L_n x^n/n! = \sum_{n=0}^{\infty} (a^n + b^n)(x^n/n!) = e^{ax} + e^{bx}.$$

Also solved by Demetris Antzoulakos, Paul S. Bruckman, Lászlő Cseh, Russell Euler, Alberto Facchini, Piero Filipponi, Jia-Sheng Lee, Carl Libis, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Periodic Decimal Expansions

B-589 *Proposed by Herta T. Freitag, Roanoke, VA*

The number $N = 0434782608695652173913$ has the property that the digits of KN are a permutation of the digits of N for $K = 1, 2, \ldots, m$. Determine the largest such m.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

The largest such m is 22.

N consists of the 22 digits in the period (in base 10) for 1/23. As is easily checked, $23N$ is the 22-digit numeral each of whose digits is a 9.

Dickson reports: "J. W. L. Glaisher . . . noted that if q is a prime such that the period for $1/q$ has $q - 1$ digits, the products of the period for $1/q$ by 1, 2, ..., $q - 1$ have the same digits in the same cyclic order. This property, well known for $q = 7$, holds also for $q = 17, 19, 23, 29, 47, 59, 61, 97$, and for $q = 7^2$." [See Dickson, *History of the Theory of Numbers*, Vol. I, p. 171 (New York: Chelsea Publishing Company, 1966).]

Also solved by Charles Ashbacher, Paul S. Bruckman, Piero Filipponi, L. Kuipers, Marjorie Johnson, Jia-Sheng Lee, Sahib Singh, Nico Trutzenberg, and the proposer.

Leftmost Digit

B-590 *Proposed by Herta T. Frietag, Roanoke, VA*

Generalize on Problem B-589 and describe a method for predicting the leftmost digit of KN.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

For the generalization, see the solution to B-589.

Let q be a prime number such that the period for $1/q$ has $q - 1$ digits. Also, let M consist of the $q - 1$ digits in the period (in base 10) for $1/q$. To predict the leftmost digit of KM, $K = 1, 2, \ldots, q - 1$, write the digits of M in increasing order with each digit appearing in the sequence S_M exactly as many times as it appears in M. Then the leftmost digit of KM is the K^{th} entry in the sequence S_M. This follows from the fact that the products KM have the same digits as M in the same cyclic order and increase as K increases.

Example: For N, $S_N = 0, 0, 1, 1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 6, 6, 6, 7, 7, 8, 8, 9, 9$. Thus, the leftmost digit of $6N$ is 2 and the leftmost digit of $12N$ is 5.

Editorial Note: Paul S. Bruckman gave the formula $[10K/q]$ for the leftmost digit of KN.

Also solved by Paul S. Bruckman, Piero Filipponi, Marjorie Johnson, Jia-Sheng Lee, Sahib Singh, and the proposer.

Interval With No Zeros

B-591 *Proposed by Mihaly Bencze, Jud. Brasa, Romania*

Let $F(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ with each a_n in $\{0, 1\}$.

Prove that $f(x) \neq 0$ for all x in $-1/a < x < 1/a$, where $a = (1 + \sqrt{5})/2$.

Solution by H.-J. Seiffert, Berlin, Germany

If $0 \leqslant x$, then, of course, $F(x) > 0$. Now assume that $-1/a < x < 0$. Then

$$F(x) = 1 + \sum_{n=1}^{\infty} a_n x^n \geqslant 1 + \sum_{k=1}^{\infty} a_{2k-1} x^{2k-1}$$

$$\geqslant 1 + \sum_{k=1}^{\infty} x^{2k-1} > 1 - \sum_{k=1}^{\infty} (1/a)^{2k-1} = \frac{a^2 - a - 1}{a^2 - 1} = 0.$$

Also solved by Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, L. Kuipers, and the proposer.

◆◇◆◇◆

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-418 *Proposed by Lawrence Somer, Washington, D.C.*

Let $m > 1$ be a positive integer. Suppose that m itself is a general period of the Fibonacci sequence modulo m; that is,

$$F_{n+m} \equiv F_n \pmod{m}$$

for all nonnegative integers n. Show that $24|m$.

H-419 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let $P_0, P_1, \ldots$ be the sequence of Pell numbers defined by

$$P_0 = 0,\ P_1 = 1, \text{ and } P_n = 2P_{n-1} + P_{n-2} \text{ for } n \in \{2, 3, \ldots\}.$$

Show that

(a) $9\sum_{k=0}^{n} kF_kP_k = 3(n+1)(F_nP_{n+1} + F_{n+1}P_n) - F_{n+2}P_{n+2} - F_nP_n + 2,$

(b) $9\sum_{k=0}^{n} kL_kP_k = 3(n+1)(L_nP_{n+1} + L_{n+1}P_n) - L_{n+2}P_{n+2} - L_nP_n,$

(c) $F_{m+n+2}P_{n+2} + F_{m+n}P_n \equiv 3(n+1)F_m + L_m \pmod{9}.$

(d) $L_{m+n+2}P_{n+2} + L_{m+n}P_n \equiv 3(n+1)L_m + 5F_m \pmod{9},$

where n is a nonnegative integer and m any integer.

H-420 *Proposed by Peter Kiss, Teachers Training College, Eger, Hungary and Andreas N. Philippou, University of Patras, Patras, Greece*

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2^{n-1}}}{2^{2^n} - 1} = 1.$

SOLUTIONS

Editorial Note: H-307 was listed as an unsolved problem. However, the solution for H-307 was inadvertently placed in the same issue as the problem.

Return from the Dead

H-211 *Proposed by S. Krishman, Orissa, India*
(Vol. 11, no. 1, February 1973)

A. Show that $\binom{2n}{n}$ is of the form $2n^3k + 2$ when n is prime and $n > 3$.

B. Show that $\binom{2n-2}{n-1}$ is of the form $n^3k - 2n^2 - n$ when n is prime.

$\binom{m}{j}$ represents the binomial coefficient, $m!/(j!(m-j)!)$.

Solution by Paul S. Bruckman, Fair Oaks, CA

Consider the expansion:

(1) $$(x+1)(x+2)\cdots(x+n-1) = x^{n-1} + A\, x^{n-2} + \cdots + A_{n-2} + A_{n-1},$$

where A_k is the sum of the products of the k different members of the set $1, 2, \ldots, n-1$.

If $n \geqslant 3$ is prime, Theorem 113 in [1] states:

(2) $$A_k \equiv 0 \pmod{n},\ k = 1, 2, \ldots, n-2.$$

Moreover, Wolstenholme's Theorem (Theorem 115 in [1]) states:

(3) $$A_{n-2} \equiv 0 \pmod{n^2}, \text{ provided } n > 3.$$

Also, Wilson's Theorem states:

(4) $$A_{n-1} = (n-1)! \equiv -1 \pmod{n}.$$

If $n > 3$ (and prime), then, by (4):

$$(n!)^2 = n^2(an-1)^2 \text{ for some integer } a.$$

Also, setting $x = n$ in (1) and applying (2), (3), and (4), we obtain

$$(2n-1)!/n! = an - 1 + n\cdot bn^2 + n^2\cdot cn + dn^3 = an - 1 + fn^3$$

(for integers b, c, d, and f; here "a" is the same integer as in the previous statement). Therefore,

$$\binom{2n}{n} = \frac{2n}{n!}\,\frac{(2n-1)!}{n!} = \frac{2}{an-1}(an - 1 + fn^3)$$
$$= 2 + \frac{2fn^3}{an-1} \equiv 2 \pmod{2n^3};$$

this proves part (A) of the problem.

Now,

$$\binom{2n-2}{n-1} = \frac{n}{2(2n-1)}\binom{2n}{n};$$

hence, again, if n is a prime greater than 3,

$$\binom{2n-2}{n-1} = \frac{n(2+2kn^3)}{2(2n-1)} \quad \text{(for some integer } k\text{), so}$$

$$\binom{2n-2}{n-1} = \frac{n(kn^3+1)}{2n-1} \equiv (kn^3+1)(-n)(1+2n) \pmod{n^3}$$

$$= -n - 2n^2 \pmod{n^3};$$

this proves part (B).

Reference

1. G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed. (Oxford: Clarendon Press, 1960), pp. 86-88.

Another Ancient One

H-213 *Proposed by V. E. Hoggatt, Jr., San Jose State University, (deceased) (Vol. 11, no. 1, February 1973)*

A. Let A_n be the left adjusted Pascal triangle, with n rows and columns and 0's above the main diagonal. Thus,

$$A_n = \begin{bmatrix} 1 & 0 & & & \cdots & 0 \\ 1 & 1 & 0 & & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n}$$

Find $A_n \cdot A_n^T$, where A_n^T represents the transpose of matrix A_n.

B. Let

$$C_n = \begin{bmatrix} 1 & 0 & 0 & & & \cdots & 0 \\ 0 & 1 & 0 & & & \cdots & 0 \\ 0 & 1 & 1 & 0 & & \cdots & 0 \\ 0 & 0 & 2 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n}$$

where the i^{th} column of the matrix C_n is the i^{th} row of Pascal's triangle adjusted to the main diagonal and the other entries are zeros. Find $C_n \cdot A_n^T$.

Solution by Paul S. Bruckman, Fair Oaks, CA

Part (A): We see that

$$a_{ij} = \binom{i}{j}, \quad 0 \leqslant i, j \leqslant n-1,$$

with the convention that $a_{ij} = 0$ outside this range. Hence, if $B = AA^T$, and

$m = \min(i, j)$, then

$$b_{ij} = \sum_{k=0}^{m} a_{ik}a_{jk} = \sum_{k=0}^{m} \binom{i}{k}\binom{j}{k} = \sum_{k=0}^{m} \binom{i}{k}\binom{j}{j-k} = \binom{i+j}{j}$$

(using Vandermonde's convolution), provided $0 \leqslant i, j \leqslant n - 1$.

This is a symmetric matrix, whose rows (and columns) are the coefficients of powers of $(1 - x)^{-1}$.

Part (B): We see that

$$c_{ij} = \binom{j}{i-j}, \text{ where } 0 \leqslant j \leqslant i \leqslant 2j \leqslant 2n - 2, \; c_{ij} = 0 \text{ elsewhere.}$$

If $D = CA^T$, and if $u = [\tfrac{1}{2}(i + 1)]$, then

$$d_{ij} = \sum_{k=u}^{m} c_{ik}a_{jk} = \sum_{k=u}^{m} \binom{k}{i-k}\binom{j}{k}.$$

Note that

$$F(x) = \sum_{i=0}^{\infty} d_{ij}x^i = \sum_{i=0}^{m} x^i \sum_{k=u}^{m} \binom{k}{i-k}\binom{j}{k} = \sum_{k=0}^{j} \binom{j}{k} \sum_{i=k}^{2k} \binom{k}{i-k} x^i$$

$$= \sum_{k=0}^{j} \binom{j}{k} x^k \sum_{i=0}^{k} \binom{k}{i} x^i = \sum_{k=0}^{j} \binom{j}{k} x^k (1 + x)^k$$

$$= (1 + x + x^2)^j = \sum \binom{n_1 + n_2 + n_3}{n_1, n_2, n_3} x^i,$$

where the last sum is over nonnegative integers n_1, n_2, n_3, such that

$$n_1 + n_2 + n_3 = j, \; n_1 + 2n_2 + 3n_3 = i + j.$$

Thus,

$$d_{ij} = \sum \binom{n_1 + n_2 + n_3}{n_1, n_2, n_3},$$

over the range indicated, for $0 \leqslant i, j \leqslant n - 1$ ($d_{ij} = 0$ if $i > 2j$). Hence, the columns of $D = CA^T$ are the rows of the Pascal *trinomial* triangle truncated after n terms.

Some Operator

H-397 *Proposed by Paul S. Bruckman, Fair Oaks, CA (Vol. 24, no. 2, May 1986)*

For any positive integer n, define the function F_n on $\mathbf{C}$ as follows:

$$F_n(x) \equiv (g^n - 1)(x), \tag{1}$$

where g is the operator

$$g(x) \equiv x^2 - 2. \tag{2}$$

[Thus, $F_3(x) = \{(x^2 - 2)^2 - 2\}^2 - 2 - x = x^8 - 8x^6 + 20x^4 - 16x^2 - x + 2$.] Find all 2^n zeros of F_n.

Solution by the proposer

We find that the following substitution yields fruitful results:

$$x = 2\cos\theta. \tag{3}$$

For then

$$g(x) = 4\cos^2\theta - 2 = 2\cos 2\theta,\ g^2(x) = 2\cos(2^2\theta),\ \text{etc.},$$

$$g^n(x) = 2^n\cos(2^n\theta).$$

Hence, $F_n(x) = 2\cos(2^n\theta) - 2\cos\theta$. Setting $F_n(x) = 0$ yields:

$$2^n\theta = \pm\theta + 2k\pi \text{ for all integers } k;$$

since

$$\theta = 2k\pi/(2^n \pm 1),$$

we may restrict k to the values $0, 1, \ldots, (2^n \pm 1) - 1$.

We consider the two cases implied by the $\pm$ sign above separately. If $\theta = 2k\pi/(2^n - 1)$, we may further restrict k to the values $0, 1, \ldots, 2^{n-1} - 1$; for if $2^{n-1} \leqslant k \leqslant 2^n - 2$, then $k' \equiv 2^n - 1 - k$ satisfies $1 \leqslant k' \leqslant 2^{n-1} - 1$, i.e., k' repeats the same values previously assumed by k, except for zero. Moreover,

$$\cos(2k'\pi/(2^n - 1) = \cos(2\pi - 2k\pi/(2^n - 1)) = \cos(2k\pi/(2^n - 1));$$

thus, $k \in [0, 2^{n-1} - 1]$ generates all zeros of F_n under this case.

If $\theta = 2k\pi/(2^n + 1)$, we may restrict k to the values $1, 2, \ldots, 2^{n-1}$; for if $2^{n-1} + 1 \leqslant k \leqslant 2^n$, then $k' \equiv 2^n + 1 - k$ satisfies $1 \leqslant k' \leqslant 2^{n-1}$, i.e., k' repeats the same values previously assumed by k. Moreover, as before,

$$\cos(2k'\pi/(2^n + 1)) = \cos(2k\pi/(2^n + 1)).$$

Thus, all zeros of F_n are generated in this case by the values $k \in [1, 2^{n-1}]$.

The zeros of F_n found above are 2^n in number, which is expected in an equation of degree 2^n. Further, they are distinct, since $(2^n - 1)$ and $(2^n + 1)$ are relatively prime, and all zeros in each of the two cases considered above are distinct. Thus, the zeros of F_n are as follows:

$$2\cos\{2(k - 1)\pi/(2^n - 1)\}$$

or

$$2\cos\{2k\pi/(2^n + 1)\},\ k = 1, 2, \ldots, 2^{n-1}.$$

For example,

$$F_3(x) = \prod_{k=1}^{4}\{x - 2\cos(2(k - 1)\pi/7)\}\{x - 2\cos(2k\pi/9)\}.$$

A Piece of Pie

H-398 *Proposed by Ambati Jaya Krishna, Freshman, Johns Hopkins University (Vol. 24, no. 2, May 1986)*

Let

$$a + b + c + d + e = \left(\sum_1^\infty\left(\frac{(-1)^{n+1}}{2n - 1}\,\frac{2}{3}\cdot 9^{1-n} + 7^{1-2n}\right)\right)^2$$

and

$$a^2 + b^2 + c^2 + d^2 + e^2 = \frac{45}{512}\sum_1^\infty n^{-4},$$

$a, b, c, d, e \in \mathbb{R}$. What are the values of a, b, c, d, and e if e is to attain its maximum value?

Solution by Paul S. Bruckman, Fair Oaks, CA

$$\left(\sum_0^\infty \frac{(-1)^n}{2n+1}\left(2\left(\frac{1}{3}\right)^{2n+1} + \left(\frac{1}{7}\right)^{2n+1}\right)\right)^2 = \left(2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right)^2$$

$$= \left(\tan^{-1}\left(\frac{2/3}{1 - 1/9}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right)^2 = \left(\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right)^2$$

$$= \left(\tan^{-1}\left(\frac{3/4 + 1/7}{1 - 3/28}\right)\right)^2 = (\tan^{-1} 1)^2 = (\pi/4)^2,$$

or

$$a + b + c + d + e = \pi^2/16. \tag{1}$$

Also,

$$\sum_1^\infty n^{-4} = \xi(4) = \pi^4/90,$$

so

$$a^2 + b^2 + c^2 + d^2 + e^2 = \pi^4/1024 = (\pi^2/32)^2. \tag{2}$$

To simplify the computations, we make the following substitutions:

$$a = \pi^2/32x_1,\ b = \pi^2/32x_2,\ \ldots,\ e = \pi^2/32x_5. \tag{3}$$

We observe that e is maximized iff x_5 is. Then, the equivalents of (1) and (2) are:

$$S = \sum_1^5 x_k = 2; \tag{4}$$

$$Q = \sum_1^5 x_k^2 = 1. \tag{5}$$

This is an extremal problem with constraints. Such problems may be solved by using Lagrange's method of multipliers (see Angus E. Taylor, *Advanced Calculus* [Ginn & Co., 1955], pp. 198-204). We form the function

$$u = u(x_1, x_2, x_3, x_4, x_5; \lambda_1, \lambda_2) = x_5 + \lambda_1 S + \lambda_2 Q, \tag{6}$$

where λ_1 and λ_2 are indeterminate "Multipliers." According to Lagrange's method, *all* extremal values of x_5, subject to the constraints given by (4) and (5), are provided as solutions of the equations

$$\frac{\partial u}{\partial x_k} = 0,\ k = 1, 2, 3, 4, 5, \text{ together with (4) and (5).} \tag{7}$$

We then obtain:

$$\lambda_1 + 2x_k\lambda_2 = 0,\ k = 1, 2, 3, 4; \tag{8}$$

$$1 + \lambda_1 + 2x_5\lambda_2 = 0. \tag{9}$$

We observe that we cannot have $\lambda_2 = 0$; for, if $\lambda_2 = 0$, then (8) implies $\lambda_1 = 0$. But then (9) would imply $1 = 0$, clearly impossible. Since $\lambda_2 \neq 0$, it follows from (8) that for any extremal solutions of the problem, we must have $x_1 = x_2 = x_3 = x_4$. Let x denote the common value of the x_k's ($k = 1, 2, 3, 4$), y the corresponding extremal value(s) of x_5. We then obtain, from (4) and (5):

$$4x + y = 2; \tag{10}$$

$$4x^2 + y^2 = 1. \tag{11}$$

We may readily solve (1) and (11), obtaining the two solutions

$$(x, y) = (\tfrac{1}{2}, 0), \text{ or } (3/10, 4/5). \tag{12}$$

Since this provides *all* extremal values y, we see that x_5 is maximized at $y = 4/5$ iff $x = 3/10$. Returning to our original notation [i.e., using (3)], it follows that e assumes its maximum value of $\pi^2/40$ iff

$$a = b = c = d = 3\pi^2/320.$$

Also solved by C. Georghiou, L. Kuipers, J.-Z. Lee & J.-S. Lee, and the proposer.

Rules, Rules, Rules

H-399 *Proposed by M. Wachtel, Zurich, Switzerland*
(Vol. 24, no. 2, May 1986)

The twin sequences: $$\frac{L_{1+6n} - 1}{2} = 0, 14, 260, 4674, 83880, \ldots$$

and $$\frac{L_{5+6n} - 1}{2} = 5, 99, 1785, 32039, \ldots$$

are representable by infinitely many identities, partitioned into several groups of similar structure (see *The Fibonacci Quarterly* 24, no. 2 [May 1986], p. 186 for details). Find the construction rules for S_n for each group.

Solution by Paul S. Bruckman, Fair Oaks, CA

The Group I formulas for S_m are as follows:

$$\tfrac{1}{2}(L_{6n+3+2k} - 1) = \tfrac{1}{2}(L_{6m-2} - 1)L_{6n-6m+5+2k} + \tfrac{1}{2}\{(\alpha^{3m-1} - \alpha^{-(3m-1)})(\alpha^{6n-9m+6+2k} + \alpha^{-(6n-9m+6+2k)}) - 1\}, \tag{1}$$

where $k = -1$ or $+1$, $m = 1, 2, 3, \ldots$.

Depending on whether m is odd or even, these may be expressed as follows:

$$\begin{aligned}\tfrac{1}{2}(L_{6n+3+2k} - 1) &= \tfrac{1}{2}(L_{6m-2} - 1)L_{6n-6m+5+2k} + \tfrac{1}{2}(5F_{3m-1}F_{6n-9m+6+2k} - 1), \ m \text{ odd}; \\ &= \tfrac{1}{2}(L_{6m-2} - 1)L_{6n-6m+5+2k} + \tfrac{1}{2}(L_{3m-1}L_{6n-9m+6+2k} - 1), \ m \text{ even}.\end{aligned} \tag{1a}$$

The corresponding Group II and III formulas are as follows, with a similar dichotomy as indicated below:

$$\tfrac{1}{2}(L_{6n+3+2k} - 1) = \tfrac{1}{2}(L_{6m-4} - 1)L_{6n-6m+7+2k} + \tfrac{1}{2}\{(\alpha^{3m-2} - \alpha^{-(3m-2)})(\alpha^{6n-9m+9+2k} + \alpha^{-(6n-9m+9+2k)}) - 1\}; \tag{2}$$

$$= \tfrac{1}{2}(L_{6m-4} - 1)L_{6n-6m+7+2k} + \tfrac{1}{2}(L_{3m-2}L_{6n-9m+9+2k} - 1),\ m \text{ odd}; \quad (2a)$$
$$= \qquad '' \qquad + \tfrac{1}{2}(5F_{3m-2}F_{6n-9m+9+2k} - 1),\ m \text{ even};$$

$$= \tfrac{1}{2}(L_{6m-4} + 1)L_{6n-6m+7+2k} \quad (3)$$
$$- \tfrac{1}{2}\{(\alpha^{3m-2} + \alpha^{-(3m-2)})(\alpha^{6n-9m+9+2k} - \alpha^{-(6n-9m+9+2k)}) + 1\};$$

$$= \tfrac{1}{2}(L_{6m-4} + 1)L_{6n-6m+7+2k} - \tfrac{1}{2}(5F_{3m-2}F_{6n-9m+9+2k} + 1),\ m \text{ odd}; \quad (3a)$$
$$= \qquad '' \qquad - \tfrac{1}{2}(L_{3m-2}L_{6n-9m+9+2k} + 1),\ m \text{ even}.$$

Proof of (1): The right member of (1) simplifies as follows:

$$\tfrac{1}{2}(L_{6n+3+2k} + L_{6n-12m+7+2k} - L_{6n-6m+5+2k} + L_{6n-6m+5+2k} - L_{6n-12m+7+2k} - 1)$$
$$= \tfrac{1}{2}(L_{6n+3+2k} - 1). \quad \text{Q.E.D.}$$

Proof of (2): The right member of (2) simplifies as follows:

$$\tfrac{1}{2}(L_{6n+3+2k} + L_{6n-12m+11+2k} - L_{6n-6m+7+2k} + L_{6n-6m+7+2k} - L_{6n-12m+11+2k} - 1)$$
$$= \tfrac{1}{2}(L_{6n+3+2k} - 1). \quad \text{Q.E.D.}$$

Proof of (3): The right member of (3) simplifies as follows:

$$\tfrac{1}{2}(L_{6n+3+2k} + L_{6n-12m+11+2k} + L_{6n-6m+7+2k} - L_{6n-6m+7+2k} - L_{6n-12m+11+2k} - 1)$$
$$= \tfrac{1}{2}(L_{6n+3+2k} - 1). \quad \text{Q.E.D.}$$

Also solved by J.-Z. Lee & J.-S. Lee as well as the proposer.

Editorial Note: Might as well dedicate this issue to Paul S. Bruckman.

◆◇◆◇◆

SUSTAINING MEMBERS

*Charter Members

*A.L. Alder
S. Ando
R. Andre-Jeannin
*J. Arkin
L. Bankoff
F. Bell
M. Berg
J.G. Bergart
G. Bergum
G. Berzsenyi
*M. Bicknell-Johnson
*Br. A. Brousseau
P.S. Bruckman
M.F. Bryn
P.F. Byrd
G.D. Chakerian
J.W. Creely
P.A. DeCaux
M.J. DeLeon
J. Desmond
H. Diehl
T.H. Engel
D.R. Farmer
P. Flanagan
F.F. Frey, Jr.
Emerson Frost
C.L. Gardner
R.M. Giuli
I.J. Good
*H.W. Gould
H.E. Heatherly
A.P. Hillman
*A.F. Hordam
F.T. Howard
R.J. Howell
R.P. Kelisky
C.H. Kimberling
J. Lahr
N. Levine
C.T. Long
Br. J.M. Mahon
*J. Maxwell
F.U. Mendizabal
L. Miller
M.G. Monzingo
S.D. Moore, Jr.
J.F. Morrison
K. Nagasaka
F.G. Ossiander
A. Prince
S. Rabinowitz
E.D. Robinson
S.E. Schloth
J.A. Schumaker
A.G. Shannon
D. Singmaster
J. Sjoberg
L. Somer
M.N.S. Swamy
*D. Thoro
J.C. Turner
R. Vogel
M. Waddill
J.E. Walton
G. Weekly
R.E. Whitney
B.E. Williams
A.C. Yanoviak

INSTITUTIONAL MEMBERS

THE BAKER STORE EQUIPMENT COMPANY
Cleveland, Ohio

BOSTON COLLEGE
Chestnut Hill, Massachusetts

BUCKNELL UNIVERSITY
Lewisburg, Pennsylvania

CALIFORNIA STATE UNIVERSITY, SACRAMENTO
Sacramento, California

ETH-BIBLIOTHEK
Zurich, Switzerland

FERNUNIVERSITAET HAGEN
Hagen, West Germany

HOWELL ENGINEERING COMPANY
Bryn Mawr, California

NEW YORK PUBLIC LIBRARY
GRAND CENTRAL STATION
New York, New York

NORTHWESTERN UNIVERSITY
Evanston, Illinois

PRINCETON UNIVERSITY
Princeton, New Jersey

SAN JOSE STATE UNIVERSITY
San Jose, California

SANTA CLARA UNIVERSITY
San Jose, California

KLEPKO, INC.
Sparks, Nevada

SYRACUSE UNIVERSITY
Syracuse, New York

TECHNOLOGICAL EDUCATION INSTITUTE
Larissa, Greece

UNIVERSITY OF CALIFORNIA, SANTA CRUZ
Santa Cruz, California

UNIVERSITY OF GEORGIA
Athens, Georgia

UNIVERSITY OF NEW ENGLAND
Armidale, N.S.W. Australia

WASHINGTON STATE UNIVERSITY
Pullman, Washington

BOOKS AVAILABLE THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.

Recurring Sequences by Dov Jarden. Third and enlarged edition. Riveon Lematematika, Israel, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

Please write to the Fibonacci Association, Santa Clara University, Santa Clara, CA 95053, U.S.A., for current prices.

VOLUME 26 FEBRUARY 1988 NUMBER 1