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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

ELLIPTIC FUNCTIONS AND LAMBERT SERIES IN THE SUMMATION OF RECIPROCALS IN CERTAIN RECURRENCE-GENERATED SEQUENCES

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(Submitted July 1986)

1. INTRODUCTION

Consider the sequence of positive integers $\{w_n\}$ defined by the recurrence relation

$$w_{n+2} = pw_{n+1} - qw_n \tag{1.1}$$

with initial conditions

$$w_0 = a,\ w_1 = b, \tag{1.2}$$

where $a \geqslant 0$, $b \geqslant 1$, $p \geqslant 1$, $q \neq 0$ are integers with $p^2 \geqslant 4q$. We first consider the "nondegenerate" case: $p^2 > 4q$.

Roots of the characteristic equations of (1.1), namely,

$$\lambda^2 - p\lambda + q = 0 \tag{1.3}$$

are

$$\begin{cases} \alpha = (p + \sqrt{p^2 - 4q})/2, \\ \beta = (p - \sqrt{p^2 - 4q})/2. \end{cases} \tag{1.4}$$

Note $\alpha > 0$, $\beta \gtrless 0$ depending on $q \gtrless 0$. Then

$$\alpha + \beta = p,\ \alpha\beta = q,\ \alpha - \beta = \sqrt{p^2 - 4q} > 0. \tag{1.5}$$

The explicit *Binet form* for w_n is

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{1.6}$$

in which

$$\begin{cases} A = b - a\beta, \\ B = b - a\alpha. \end{cases} \tag{1.7}$$

It is the purpose of this paper to investigate the infinite sums

$$\sum_{n=1}^{\infty} \frac{1}{w_n} \tag{1.8}$$

$$\sum_{n=1}^{\infty} \frac{1}{w_{2n}} \tag{1.9}$$

$$\sum_{n=1}^{\infty} \frac{1}{w_{2n-1}}. \tag{1.10}$$

Special cases of $\{w_n\}$ which interest us here are:

the *Fibonacci sequence* $\{F_n\}$: $a = 0,\ b = 1,\ p = 1,\ q = -1$; (1.11)

the *Lucas sequence* $\{L_n\}$: $a = 2,\ b = 1,\ p = 1,\ q = -1$; (1.12)

the *Pell sequence* $\{P_n\}$: $a = 0,\ b = 1,\ p = 2,\ q = -1$; (1.13)

the *Pell-Lucas sequence* $\{Q_n\}$: $a = 2,\ b = 2,\ p = 2,\ q = -1$; (1.14)

the *Fermat sequence* $\{f_n\}$: $a = 0,\ b = 1,\ p = 3,\ q = 2$; (1.15)

the *"Fermat-Lucas" sequence* $\{g_n\}$: $a = 2,\ b = 3,\ p = 3,\ q = 2$; (1.16)

the *generalized Fibonacci sequence* $\{U_n\}$: $a = 0,\ b = 1$; (1.17)

the *generalized Lucas sequence* $\{V_n\}$: $a = 2,\ b = p$. (1.18)

The Fermat sequence (1.15) is also known as the *Mersenne sequence*.

Binet forms and related information are readily deduced for (1.11)-(1.18) from (1.4)-(1.7). Notice that $f_n = 2^n - 1$, $g_n = 2^n + 1$, and, for both (1.15) and (1.16), $\alpha = 2$, $\beta = 1$, in which case the roots of the characteristic equation are not irrational.

Sequences (1.11), (1.13), (1,15), and (1.17), in which $a = 0$, $b = 1$, may be alluded to as being of *Fibonacci type*. On the other hand, sequences (1.12), (1.14), (1.16), and (1.18), in which $a = 2$, $b = p$, may be said to be of *Lucas type*.

For Fibonacci-type sequences, we have $A = B = 1$, and the Binet form (1.6) reduces to

$$w_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.6$'$}$$

whereas for Lucas-type sequences, in which $A = -B = \alpha - \beta$, we have the simpler form

$$w_n = \alpha^n + \beta^n. \tag{1.6$''$}$$

From (1.6),

$$\lim_{n\to\infty}\left[\frac{\dfrac{1}{w_{n+1}}}{\dfrac{1}{w_n}}\right] = \lim_{n\to\infty}\frac{w_n}{w_{n+1}} = \lim_{n\to\infty}\left[\frac{A\alpha^n - B\beta^n}{A\alpha^{n+1} - B\beta^{n+1}}\right] \tag{1.19}$$

$$= \frac{1}{\alpha}\lim_{n\to\infty}\left[\frac{A - B\left(\frac{\beta}{\alpha}\right)^n}{A - B\left(\frac{\beta}{\alpha}\right)^{n+1}}\right] = \frac{1}{\alpha} \quad \text{since } \left|\frac{\beta}{\alpha}\right| < 1$$

$$< 1 \quad \text{since } \alpha > 1.$$

To prove this last assertion, we note that $2\alpha = p + \sqrt{p^2 - 4q} \geqslant 1 + 1 = 2$. If $p + \sqrt{p^2 - 4q} = 2$, then $q = p - 1$; but $q \neq 0$, so $p \neq 1 \Rightarrow p > 1 \Rightarrow \alpha > 1$.

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{w_n} \text{ converges absolutely.} \tag{1.20}$$

All the sequences (1.11)-(1.18) satisfy (1.20).

2. BACKGROUND

Historical

The desire to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{F_n} \tag{2.1}$$

seems to have been stated first by Laisant [21] in 1899 in these words:

> "*A-t-on déjà étudié la série*
>
> $$\frac{1}{1}\ \frac{1}{1}\ \frac{1}{2}\ \frac{1}{3}\ \frac{1}{5}\ \ldots,$$
>
> *que forment les inverses des termes de Fibonacci, et qui est évidemment convergente?*"

Barriol [3] responded to this challenge by approximating (2.1) to 10 decimal places:

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3.3598856662\ldots \tag{2.1$'$}$$

which concurs with that obtained by Brousseau ([6], p. 45) in calculating

$$\sum_{n=1}^{400} \frac{1}{F_n} \tag{2.1$''$}$$

to 400 decimal places. (Actually, in (2.1)′, the first decimal digit, 3, is misprinted in [3] as 2.) However, we find in Escott [11] the claim:

> "*J'ai calculé la valeur de cette somme avec quinze décimales et vérifié les résultats à l'aide de la formule*
>
> $$\frac{1}{p_{n+2}} = \frac{1}{p_n} - \frac{1}{p_{n+1}} - \frac{(-1)^n}{p_n p_{n+1} p_{n+2}}$$
>
> *où* p_n *est le* $n^{ième}$ *terme de la série de Fibonacci.*
>
> *J'obtiens* 3,3598856672—*qui diffère du résultat de M. Barriol par le* 10*e* *chiffre.*"

For the Lucas numbers, the approximation corresponding to (2.1)″ given by Brousseau ([6], p. 45) is

$$\sum_{n=1}^{400} \frac{1}{L_n} = 1.9628581732\ldots \tag{2.2}$$

Catalan [9] in 1883, and earlier Lucas [24] in 1878, had divided the problem of investigating $\sum_{n=1}^{\infty}(1/F_n)$ into two parts, namely,

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}}, \text{ expressible in terms of Jacobian elliptic functions,} \tag{2.3}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n}}, \text{ expressible in terms of Lambert series.} \tag{2.4}$$

Landau [23] in 1899 elaborated on Catalan's result in the case of (2.3) by expressing the answer in terms of *theta functions*.

Moreover, Catalan [9] also obtained an expression for

$$\sum_{n=1}^{\infty}\frac{1}{L_{2n}} \tag{2.5}$$

in terms of Jacobian elliptic functions. No mention in the literature available to me was made by Catalan for

$$\sum_{n=1}^{\infty}\frac{1}{L_{2n-1}}. \tag{2.6}$$

Results for Pell and Pell-Lucas numbers corresponding to those in (2.3)-(2.6) were obtained in [26] by Horadam and Mahon.

For a wealth of detailed, numerical information on the matters contained in, and related to, (2.3)-(2.6), one might consult Bruckman [7], who obtained closed forms for the expressions in (2.3) and (2.5), among others, in terms of certain constants defined by Jacobian elliptic functions.

Observe in passing that in (2.5) the value $n = 0$ is omitted in the summation even though $L_0 = 2$ $(\neq 0)$. We do this for consistency because, in the non-Lucas type sequences, $a = 0$ (i.e., $w_0 = 0$, so $1/w_0$ is infinite).

From (1.6),

$$\sum_{n=1}^{\infty}\frac{1}{w_n} = (\alpha - \beta)\sum_{n=1}^{\infty}\frac{1}{A\alpha^n - B\beta^n} = (\alpha - \beta)\sum_{n=1}^{\infty}\frac{\beta^n}{A\alpha^n\beta^n - B\beta^{2n}} \tag{2.7}$$

$$= (\alpha - \beta)\sum_{n=1}^{\infty}\frac{\beta^n}{Aq^n - B\beta^{2n}} \quad \text{by (1.5)}$$

$$= (\alpha - \beta)\sum_{n=1}^{\infty}\frac{(1/A)\beta^n}{q^n - (B/A)\beta^{2n}}.$$

At this stage, we must pause. The algebra, it appears, is too fragile to bear the burden of both q^n and B/A being simultaneously unrestricted, so some constraints must be imposed.

Clearly, the evenness or oddness of n is important since q^n will alternate in sign if $q < 0$. Following historical precedent as indicated earlier, we find it necessary to dichotomize w_n into the cases n even, n odd.

Furthermore, the outcome of the expression on the right-hand side of (2.7) depends on whether B/A (or A/B) is >0 or <0.

For our purposes, two specific values concern us, viz.,

$$\frac{A}{B} = \pm 1.$$

I. $\frac{A}{B} = 1$

From (1.7), $A/B = 1$ means that

$$b - a\alpha = b - a\beta \qquad (\alpha \neq \beta),$$

whence

$$a = 0$$

without any new restrictions on b, p, or q. Combining this fact with the criterion for (1.6)′ (i.e., $b = 1$), we have

$$a = 0,\ b = 1 \Rightarrow A = B = 1. \tag{2.8}$$

Sequences satisfying the criteria $a = 0$, $b = 1$ are the Fibonacci-type sequences.

II. $\frac{A}{B} = -1$

In this case, (1.7) gives

$$\begin{aligned} b - a\alpha &= -(b - a\beta) \\ b &= \frac{ap}{2} \qquad \text{by (1.5)} \\ &= p \qquad \text{if } a = 2. \end{aligned}$$

Relating these criteria to (1.6)″, we see that

$$a = 2,\ b = p \Rightarrow A = -B = \alpha - \beta. \tag{2.9}$$

Sequences which satisfy the criteria $a = 2$, $b = p$ are the Lucas-type sequences.

Having set down some necessary background information, we now proceed to the main objective of the paper, to wit, the application to our summation requirements of Jacobian elliptic functions and Lambert series.

3. JACOBIAN ELLIPTIC FUNCTIONS

In *Jacobian elliptic function* theory, the elliptic integral constants (see [7], [18])

$$K = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \tag{3.1}$$

and

$$K' = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k'^2 \sin^2 t}} \tag{3.2}$$

are related by

$$k^2 + k'^2 = 1, \tag{3.3}$$

k' being the *complement* of k.

Write

$$r = e^{-K'\pi/K} \qquad (0 < r < 1). \tag{3.4}$$

Jacobi's symbol q [17] is here replaced by r to avoid confusion with the use of q in the recurrence relation (1.1).

Two of Jacobi's summation formulas [18] required for our purposes are

$$\frac{2K}{\pi} = 1 + \frac{4r}{1 + r^2} + \frac{4r^2}{1 + r^4} + \frac{4r^3}{1 + r^6} + \cdots \tag{3.5}$$

and

$$\frac{2kK}{\pi} = \frac{4\sqrt{r}}{1 + r} + \frac{4\sqrt{r^3}}{1 + r^3} + \frac{4\sqrt{r^5}}{1 + r^5} + \cdots . \tag{3.6}$$

Now, from (1.6).

$$\begin{aligned} \frac{1}{w_{2n-1}} &= \frac{\alpha - \beta}{A(\alpha^{2n-1} - (B/A)\beta^{2n-1})} \qquad (3.7) \\ &= (\alpha - \beta) \cdot \frac{\beta^{2n-1}}{(\alpha\beta)^{2n-1} - \beta^{4n-2}} \qquad \text{if } A = B = 1 \\ &= (\alpha - \beta) \cdot \frac{\beta^{2n-2}\beta}{-1 - \beta^{4n-2}} \qquad \text{if } q = -1 \text{ in (1.5)} \\ &= (\alpha - \beta) \cdot \sqrt{r} \cdot \frac{r^{n-1}}{1 + r^{2n-1}} \qquad \text{with } \begin{cases} r = \beta^2 \ (\beta < 0) \\ \sqrt{r} = -\beta, \text{ so } 0 < \sqrt{r} < 1. \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{w_{2n-1}} &= (\alpha - \beta) \cdot \sqrt{r} \sum_{n=1}^{\infty} \frac{r^{n-1}}{1 + r^{2n-1}} \qquad (3.8) \\ &= (\alpha - \beta) \cdot \frac{1}{4} \cdot \frac{2kK}{\pi} \qquad \text{from (3.6)} \\ &= \sqrt{p^2 - 4q} \cdot \frac{kK}{2\pi}. \end{aligned}$$

Since the restrictions placed in w_{2n-1} in (3.7) are $A = B = 1$ and $q = -1$, formula (3.8) applies to sequences such as the odd-subscript Fibonacci (2.3) and Pell sequences. Accordingly,

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} = \frac{\sqrt{5}kK}{2\pi} \quad \text{by (1.11)} \tag{3.9}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{P_{2n-1}} = \frac{\sqrt{2}kK}{\pi} \quad \text{by (1.13).} \tag{3.10}$$

Because $r = \beta^2$ is different for $\{F_n\}$ and $\{P_n\}$, the term kK is different in (3.9) and (3.10).

Result (3.9) is not new and may be found in Catalan ([9], p. 13) while result (3.10), obtained by the author, appears in [26]. Bruckman ([7], p. 310) gave

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} = 1.82451515\ldots \tag{3.9'}$$

while Bowen [4] obtained

$$\sum_{n=1}^{\infty} \frac{1}{P_{2n-1}} = 1.24162540\ldots\ . \tag{3.10'}$$

Next, from (1.6) again

$$\begin{aligned} \frac{1}{w_{2n}} &= \frac{\alpha - \beta}{A(\alpha^{2n} - (B/A)\beta^{2n})} \\ &= \frac{\beta^{2n}}{(\alpha\beta)^{2n} + \beta^{4n}} && \text{if } A = -B = \alpha - \beta \text{ [cf. (2.9)]} \\ &= \frac{\beta^{2n}}{1 + \beta^{4n}} && \text{if } q = \pm 1 \text{ [cf. (1.5)]} \\ &= \frac{r^n}{1 + r^{2n}} && \text{where } \begin{cases} r = \beta^2 \ (\beta < 0 \text{ if } q = -1) \\ \sqrt{r} = |\beta|, \end{cases} \end{aligned} \tag{3.11}$$

whence

$$\sum_{n=1}^{\infty} \frac{1}{w_{2n}} = \frac{1}{4}\left(\frac{2K}{\pi} - 1\right) \quad \text{by (3.5).} \tag{3.12}$$

Under the constraints imposed on w_{2n} in (3.11), namely $A/B = -1$ and $q = \pm 1$, formula (3.12) applies to even-subscript Lucas (2.5) and Pell-Lucas sequences (with $q = -1$). Consequently,

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = \frac{1}{4}\left(\frac{2K}{\pi} - 1\right) \tag{3.13}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{Q_{2n}} = \frac{1}{4}\left(\frac{2K}{\pi} - 1\right), \tag{3.14}$$

the K being different in the two cases, since $r = \beta^2$ is different for $\{L_n\}$ and $\{Q_n\}$. However, notice that K in (3.9) [(3.10)] is the same as that in (3.13) [(3.14)]. Excluded from the summations are $1/L_0 = 1/Q_0 = 1/2$.

Result (3.13) occurs in Catalan ([9], p. 49) while (3.14) is given in [26]. Using essentially the same method, but checking results by a different method, Bruckman ([7], p. 310) has calculated

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = 0.56617767\ldots \tag{3.13$'$}$$

and Bowen [4] found

$$\sum_{n=1}^{\infty} \frac{1}{Q_{2n}} = 0.20217495\ldots\ . \tag{3.14$'$}$$

Microcomputer calculations recorded above, and subsequently, which are due to my colleague, Dr. E. W. Bowen, are acknowledged with appreciation. All his computations were obtained using the recurrence relations for the sequences. Some of the numerical summations were found manually, to a lesser degree of accuracy, by the author.

Further standard information on Jacobian elliptic function theory may be found in Abramowitz and Stegun [1] and in Whittaker and Watson [29].

4. LAMBERT SERIES

The first reference to the series known as the Lambert series occurs in Lambert [22]—hence the name.

A "Lambert series" is a series of the type

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1 - x^n}. \tag{4.1}$$

Detailed information about Lambert series is to be found in Knopp [19] and [20]. Interesting number-theoretic applications (to primeness and divisibility), depending on the value of a_n, and some basic theory, are given in Knopp [20].

More particularly, we speak of *the Lambert series*

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} \qquad |x| < 1. \tag{4.2}$$

A *generalized Lambert series* used in Arista [2] is

$$L(a,\ x) = \sum_{n=1}^{\infty} \frac{ax^n}{1 - ax^n} \qquad |x| < 1,\ |ax| < 1, \tag{4.3}$$

where the number a has nothing to do with the initial value in (1.2). The series in (4.2) and (4.3) may be shown to be absolutely convergent within the indicated intervals of convergence.

From (1.6), we have

$$\frac{1}{w_{2n}} = \frac{\alpha - \beta}{A(\alpha^{2n} - (B/A)\beta^{2n})} \tag{4.4}$$

$$= (\alpha - \beta)\cdot\frac{\beta^{2n}}{(\alpha\beta)^{2n} - \beta^{4n}} \qquad \text{if } A = B = 1$$

$$= (\alpha - \beta)\cdot\frac{\beta^{2n}}{1 - \beta^{4n}} \qquad \text{if } q = \pm 1$$

$$= (\alpha - \beta)\left(\frac{\beta^{2n}}{1 - \beta^{2n}} - \frac{\beta^{4n}}{1 - \beta^{4n}}\right),$$

so

$$\sum_{n=1}^{\infty}\frac{1}{w_{2n}} = (\alpha - \beta)\left\{\sum_{n=1}^{\infty}\frac{\beta^{2n}}{1 - \beta^{2n}} - \sum_{n=1}^{\infty}\frac{\beta^{4n}}{1 - \beta^{4n}}\right\} \tag{4.5}$$

$$= (\alpha - \beta)\{L(\beta^2) - L(\beta^4)\}.$$

To obtain (4.4) it was necessary to impose the conditions $A = B = 1$ and $q = \pm 1$. Accordingly, we can apply (4.5) to the even-subscript Fibonacci (2.4) and Pell sequences (where $q = -1$). It follows that

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n}} = \sqrt{5}\left[L\left(\frac{3 - \sqrt{5}}{2}\right) - L\left(\frac{7 - 3\sqrt{5}}{2}\right)\right] \tag{4.6}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{P_{2n}} = 2\sqrt{2}[L(3 - 2\sqrt{2}) - L(17 - 12\sqrt{2})]. \tag{4.7}$$

Formula (4.6) has been known for a long time (cf. Catalan [9]), while (4.7) appears in [26].

It is known [4] that

$$\sum_{n=1}^{\infty}\frac{1}{P_{2n}} = 0.60057764\ldots . \tag{4.7'}$$

Brady [5] extended (4.6) to the summation $\sum_{n=1}^{\infty}(1/F_{2kn})$ and exhibited the graph of the function $y = L(x)$ for $|x| < 1$.

Let us now take a special case of $\{w_n\}$ which generalizes the Fibonacci sequence. Suppose in (1.1) we have $p = 1$, $q = -1$, and retain the initial values to be a and b. Call this sequence $\{H_n\}$, i.e., $H_0 = a$, $H_1 = b$. We impose the further condition: $b > a\alpha$, where $\alpha = (1 + \sqrt{5})/2$.

Write

$$H = \frac{A}{B} = \frac{b - a\beta}{b - a\alpha} \qquad \left(\alpha = \frac{1 + \sqrt{5}}{2},\ \beta = \frac{1 - \sqrt{5}}{2}\right). \tag{4.8}$$

Paralleling the argument in (4.4), we have

$$\frac{1}{H_{2n}} = \frac{(\alpha - \beta)\beta^{2n}}{A[(\alpha\beta)^{2n} - (B/A)\beta^{4n}]} = \frac{\sqrt{5}}{A} \cdot \frac{\beta^{2n}}{1 - (1/H)\beta^{4n}} \tag{4.9}$$

$$= \frac{\sqrt{5}}{A(1/\sqrt{H})} \cdot \frac{(1/\sqrt{H})\beta^{2n}}{1 - (1/H)\beta^{4n}} = \frac{\sqrt{5}}{\sqrt{AB}}\left\{\frac{(1/\sqrt{H})\beta^{2n}}{1 - (1/\sqrt{H})\beta^{2n}} - \frac{(1/H)\beta^{4n}}{1 - (1/H)\beta^{4n}}\right\}$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{H_{2n}} = \frac{\sqrt{5}}{\sqrt{AB}}\left\{\sum_{n=1}^{\infty} \frac{(1/\sqrt{H})\beta^{2n}}{1 - (1/\sqrt{H})\beta^{2n}} - \sum_{n=1}^{\infty} \frac{(1/H)\beta^{4n}}{1 - (1/H)\beta^{4n}}\right\} \tag{4.10}$$

$$= \frac{\sqrt{5}}{\sqrt{b^2 - ab - a^2}}\left\{L\left(\frac{1}{\sqrt{H}},\ \beta^2\right) - L\left(\frac{1}{H},\ \beta^4\right)\right\} \qquad \text{by (4.3)},$$

wherein $1/H_0$ has been omitted from the summation because a may be zero.

In (4.10), the conditions imposed in (4.3) are met, since

$$|\beta^2| < 1 \qquad \left(\beta = \frac{1 - \sqrt{5}}{2} = -0.618\ldots\right)$$

and

$$\frac{1}{\sqrt{H}} = \sqrt{\frac{b - a\alpha}{b - a\beta}} < 1 \qquad (\alpha > 0,\ \beta < 0,\ b > a\alpha),$$

whence

$$\left|\frac{1}{\sqrt{H}}\ \beta^2\right| < 1; \text{ also, } |\beta^4| < 1,\ \left|\frac{1}{H}\ \beta^4\right| < 1.$$

Shannon and Horadam [28] obtained a variation of (4.10) by using a different pair of specially defined generalized Lambert series, whereas Arista's generalization (4.3) has been utilized in (4.10).

Observe that $\sqrt{AB}$ in (4.10) must be real, i.e., $AB > 0$. So (4.10) excludes Lucas-type sequences with $a = 2$, $b = 1$, 2, or 3, for which a Jacobian elliptic expression is required in the answer.

Suppose we introduce a generalized Pell sequence $\{K_n\}$ in which $p = 2$, $q = -1$, $b > a\alpha$, where $\alpha = 1 + \sqrt{2}$. Then, by reasoning similar to that used to establish (4.10), we can determine a resolution of $\sum_{n=1}^{\infty}(1/K_{2n})$ in terms of generalized Lambert series (4.3).

Let us now revert to the odd-subscript series contained in $\{L_n\}$ and $\{Q_n\}$. More generally, from (1.6)″, we have

$$\frac{1}{w_{2n-1}} = \frac{1}{\alpha^{2n-1} + \beta^{2n-1}} = \frac{\beta^{2n-1}}{(\alpha\beta)^{2n-1} + \beta^{4n-2}} \tag{4.11}$$

$$= -\frac{\beta^{2n-1}}{1 - \beta^{4n-2}} \qquad \text{for } q = -1 \text{ by (1.5)},$$

whence

$$\sum_{n=1}^{\infty} \frac{1}{w_{2n-1}} = -\sum_{n=1}^{\infty} \frac{\beta^{2n-1}}{(1-\beta^{4n-2})} = -L(\beta) + 2L(\beta^2) - L(\beta^4), \tag{4.12}$$

after some algebraic manipulation.

Thus, for appropriate β, expressions in terms of Lambert series as specializations of (4.12) are found for

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}} \qquad \left(\beta = \frac{1-\sqrt{5}}{2}\right), \tag{4.13}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{Q_{2n-1}} \qquad (\beta = 1 - \sqrt{2}). \tag{4.14}$$

Bowen [4] calculated

$$\sum_{n=1}^{\infty} \frac{1}{Q_{2n-1}} = 0.58614901952408\ldots \tag{4.14$'$}$$

Furthermore, it was computed in [4] that

$$\sum_{n=1}^{\infty} \frac{1}{P_n} = 1.8422030498275\ldots \tag{4.15}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{Q_n} = 0.7883239758197\ldots\ . \tag{4.16}$$

Addition of (4.7)′ and (3.10)′ verifies (4.15), while addition of (3.14)′ and (4.14)′ leads us to (4.16).

To complete this section, we revert to an extension of $\{U_n\}$ (1.17) which Arista [2] examined in some depth. In his investigation, Arista imposed no restriction on q other than that it is a positive or negative integer. To avoid confusion with our notation, we will designate the sequence studied by Arista as $\{u_n\}$, where $u_0 = 0$, $u_1 = 1$, q being a positive or negative integer. Further, we will retain the condition $p^2 > 4q$, to avoid complex expressions, along with $p \geqslant 1$.

Changing to our notation, we record Arista's conclusions.

$$\sum_{n=1}^{\infty} \frac{1}{u_n} = (\alpha - \beta) \sum_{h=0}^{\infty} \frac{\dfrac{\beta}{q}\left(\dfrac{\beta^2}{q}\right)^h}{1 - \dfrac{\beta}{q}\left(\dfrac{\beta^2}{q}\right)^h} = (\alpha - \beta)\left\{\frac{1}{\alpha - 1} + L\left(\frac{1}{\alpha}, \frac{\beta}{\alpha}\right)\right\}, \tag{4.17}$$

since $\left|\dfrac{\beta}{q}\right| < 1$, $\left|\dfrac{\beta}{q}\right| \left|\dfrac{\beta^2}{q}\right| < 1 \qquad [q = \alpha\beta\ (1.5)]$.

If $q > 0$, then $\beta/\alpha > 0$, and Arista showed that (4.17) is then expressible in terms of a complicated definite integral involving logarithmic and trigonometrical functions.

When $q < 0$,

$$\sum_{n=1}^{\infty} \frac{1}{u_n} = (\alpha - \beta)\left\{\frac{1}{\alpha - 1} + L\left(\frac{1}{\alpha}, \left(\frac{\beta}{\alpha}\right)^2\right) + L\left(\frac{1}{\beta}, \left(\frac{\beta}{\alpha}\right)^2\right)\right\}, \tag{4.18}$$

which again leads to a lengthy expression containing indefinite integrals of the kind mentioned above.

Finally, the "degenerate" case in which the roots α, β are equal is considered as a limiting process to produce

$$\lim_{\alpha \to \beta} \sum_{n=1}^{\infty} \frac{1}{u_n} = \alpha \log\left(\frac{\alpha}{\alpha - 1}\right). \tag{4.19}$$

In the nondegenerate case ($\alpha \neq \beta$) Arista [2] also studied the consequences of $x \to 1$, and of $|a| < 1$. It is interesting to discern the usage made by him of the relevant researches of earlier and contemporary mathematicians, e.g., Cesàro [10], Schlömilch [27], and Catalan, *inter alia.*

Lucas [25] undertook to give *plus tard* (analogous) formulas deduced from the theory of elliptic functions, "*et, en particulier, les sommes des inverses des termes* U_n *et de leurs puissances semblables*". Writing a quarter of a century afterwards, Arista [2] remarked *à propos* this undertaking: "... *ma non esiste alcuna sua pubblicazione su questo argomento*".

5. APPLICATION OF METHODS OF GOOD AND GREIG

In this section we wish to develop some interesting techniques for summing reciprocals when the subscript of w (and of its specialized sequences) is not n, $2n$, or $2n - 1$, but is some related number.

Following an approach for Fibonacci numbers due to Good [12], we establish the corresponding result for Pell numbers:

$$\sum_{m=0}^{n} \frac{1}{P_{2^n}} = 2 - P_{2^n - 1}/P_{2^n}. \tag{5.1}$$

Proof of (5.1): The proof is by induction.

When $n = 1$, the result is obviously true, since

$$\frac{1}{P_1} + \frac{1}{P_2} \left(= 1 + \frac{1}{2}\right) = 2 - \frac{P_1}{P_2} \left(= 2 - \frac{1}{2}\right).$$

Assume it is true for $n = k$. Then the validity of (5.1) for $n = k + 1$ requires that

$$P_{2^k - 1}/P_{2^k} - P_{2^{k+1} - 1}/P_{2^{k+1}} = \frac{1}{P_{2^{k+1}}}.$$

This is readily demonstrated by using the Binet form for P_n [cf. (1.6)′ and (1.13)]. Thus, (5.1) is proved.

Now let $n \to \infty$. If, temporarily, $N = 2^n$, then $\lim_{n\to\infty}(P_{N-1}/P_N) = 1/\alpha = \sqrt{2} - 1$. Hence, (5.1) yields

$$\sum_{m=0}^{\infty} \frac{1}{P_{2^m}} = 3 - \sqrt{2}. \tag{5.2}$$

This might be compared with the corresponding value for Fibonacci numbers (Good [12]—see also Gould's reference [13], p. 67, to Millin):

$$\sum_{m=0}^{\infty} \frac{1}{F_{2^m}} = \frac{7 - \sqrt{5}}{2}. \tag{5.3}$$

Next, following the method and notation of Greig [14] for Fibonacci numbers, adapted for Pell numbers, let us write $b = 2^m$, $B = 2^n$. Then we may show that

$$\sum_{m=0}^{n} \frac{1}{P_{kb}} = C_k - P_{kB-1}/P_{kB} \qquad (n, k \geqslant 1), \tag{5.4}$$

where

$$C_k = \begin{cases} (1 + P_{k-1})/P_k & \text{for } k \text{ even,} \\ (1 + P_{k-1})/P_k + 2/P_{2k} & \text{for } k \text{ odd,} \end{cases} \tag{5.5}$$

i.e., C_k is independent of n.

Proof of (5.4): Again, the proof is by induction.

Assume (5.4) holds for a given n. Then its validity for $n + 1$ requires us to show that

$$P_{2kB}P_{kB-1} - P_{kB}P_{2kB-1} = P_{kB} \tag{5.6}$$

or, more succinctly, on writing $j = kB$,

$$P_{2j}P_{j-1} - P_jP_{2j-1} = (-1)^jP_j. \tag{5.6$'$}$$

This may be demonstrated by appealing to the Binet form for P_n.

[Alternatively, we may use

$$P_{h+1}P_j + P_hP_{j-1} = P_{h+j} \qquad (h = -2j,\ P_{-n} = (-1)^{n+1}P_n).] \tag{5.6$''$}$$

Put $n = 1$ in (5.4). Then

$$C_k = \frac{1}{P_k} + \frac{1 + P_{2k-1}}{P_{2k}} \tag{5.7}$$

$$= \begin{cases} (1 + P_{k-1})/P_k & \text{when } k \text{ is even,} \\ (1 + P_{k-1})/P_k + 2/P_{2k} & \text{when } k \text{ is odd.} \end{cases}$$

To obtain (5.7), we employ the Binet form in

$$\frac{1}{P_{2k}} + \frac{P_{2k-1}}{P_{2k}} - \frac{P_{k-1}}{P_k} = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \dfrac{2}{P_{2k}} & \text{if } k \text{ is odd.} \end{cases} \tag{5.8}$$

Our proof of (5.4) is now complete.

The first few values of C_k are calculated from (5.7):

$$C_1 = 2,\ C_2 = 1,\ C_3 = \frac{22}{35},\ C_4 = \frac{1}{2},\ C_5 = \frac{534}{1189},\ \ldots . \tag{5.9}$$

Let $n \to \infty$. Then (5.4) becomes

$$\sum_{m=0}^{\infty} \frac{1}{P_{k \cdot 2^m}} = C_k - \frac{1}{\alpha}, \tag{5.10}$$

since $\displaystyle\lim_{n \to \infty}\left(\frac{P_{j-1}}{P_j}\right) = \frac{1}{\alpha} \qquad (j = kB = k \cdot 2^n;\ \alpha = 1 + \sqrt{2}).$

Observing from Gould [13] and Greig [14] that for $k \geqslant 0$, $m \geqslant 0$, $(2k + 1)2^m$ generates each positive integer just once, we have (cf. [14]) that

$$\sum_{n=1}^{\infty} \frac{1}{P_n} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \sum_{m=0}^{\infty} \frac{1}{P_{kb}} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \left(C_k - \frac{1}{\alpha}\right) \quad \left(\frac{1}{\alpha} = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1\right). \tag{5.11}$$

Summing the right-hand side of (5.11) as far as $k = 15$ (at which stage $C_{15} - 1/\alpha = 0.000005\ldots$), we find the value to six decimal places to be 1.842202... which concurs with the summation of $\sum_{n=1}^{20} (1/P_n)$. From these computations, we can state that

$$\sum_{n=1}^{\infty} \frac{1}{P_n} = 1.842202\ldots \tag{5.12}$$

approximately to six decimal places. See (4.15) for a slightly more accurate value.

One may observe that $C_k \to 1/\alpha$ as $k \to \infty$ on using the Binet form in (5.7), whence it follows that $C_{k+2}/C_k \to 1/\alpha^2$ as $k \to \infty$. This gives us an estimate for C_{k+2} when C_k is known, which increases in accuracy as k increases in value.

If one tries to parallel the above work for $\{Q_n\}$, one finds that the presence of the plus sign (rather than a minus sign) in the Binet form [cf. (1.6)″ and (1.14)] causes the straightforwardness of the treatment, e.g., at the stage (5.6), to collapse. A similar remark in relation to $\{L_n\}$ is made by Gould in [13], p. 68 (wherein the relation to the Riemann zeta function and to sine and cosine expressions is discussed).

Nevertheless, if we simply take a summation of reciprocals as far as $n = 20$, we obtain $\sum_{n=1}^{\infty}(1/Q_n)$ correct to six decimal places, namely, 0.7883239, as in (4.16).

Generalizing the results produced above for the Fibonacci-type sequences $\{F_n\}$ and $\{P_n\}$ to results for $\{w_n\}$ can be accomplished without too much effort.

Induction (details of which are available on request) can be applied to generate the following chain of formulas:

$$\sum_{m=0}^{n} \frac{1}{w_{2^m}} = C - w_{2^n - 1}/w_{2^n} \tag{5.13}$$

in which

$$C = \frac{1}{w_1} + \frac{1 + w_1}{w_2}; \tag{5.14}$$

$$\sum_{m=0}^{n} \frac{1}{w_{k \cdot 2^m}} = C_k - P_{k \cdot 2^n - 1}/P_{k \cdot 2^n} \qquad (n,\ k \geqslant 1) \tag{5.15}$$

where

$$C_k = \frac{1}{w_k} + \frac{1 + w_{2k-1}}{w_{2k}} = \begin{cases} (1 + w_{k-1})/w_k & \text{when } k \text{ is even,} \\ (1 + w_{k-1})/w_k + 2/w_{2k} & \text{when } k \text{ is odd;} \end{cases} \tag{5.16}$$

and

$$\sum_{m=0}^{\infty} \frac{1}{w_{k \cdot 2^m}} = C_k - \frac{1}{\alpha}, \tag{5.17}$$

where α is given by (1.4) ($q = -1$).

Note that, in (5.14),

$C = 3$ for Fibonacci numbers, $C = 2$ for Pell numbers.

For a generalization of (5.14) and (5.11), the reader might consult Greig [15]. Entries in row 2 of his table ([15], p. 257) give ratios of Pell numbers which are our C_1, C_2, C_3, ... in (5.9).

6. GENERALIZED BERNOULLI AND EULER POLYNOMIALS

In this final section, it is desired to find a suitable form for the expression of w_n^{-t} and for the generating function of $\{w_n^{-t}\}$. The results generalize material in [26] which itself extends the work in [28].

First, we define the *generalized Bernoulli polynomial* $B_r^{(t)}(x)$ by

$$\sum_{r=0}^{\infty} B_r^{(t)}(x)\frac{m^r}{r!} = \frac{m^t e^{mx}}{(e^m - 1)^t} \tag{6.1}$$

and the *generalized Euler polynomial* $E_r^{(t)}(x)$ by

$$\sum_{r=0}^{\infty} E_r^{(t)}(x)\frac{n^r}{r!} = \frac{2^t e^{nx}}{(e^n + 1)^t}. \tag{6.2}$$

When $t = 1$, $B_r^{(1)}(x) = B_r(x)$ and $E_r^{(1)}(x) = E_r(x)$ are the ordinary Bernoulli polynomial and Euler polynomial, respectively. Let

$$c = \frac{\beta}{\alpha}. \tag{6.3}$$

Temporarily write

$$m = n \log C \qquad (\text{i.e., } C^n = e^m). \tag{6.4}$$

From (1.6)′, for Fibonacci-type sequences,

$$\frac{1}{w_n^t} = (\beta - \alpha)^t \cdot \frac{1}{\alpha^{nt}(C^n - 1)^t} \tag{6.5}$$

$$= \frac{(\beta - \alpha)^t \cdot C^{nx}}{(C^x\alpha^t)^n (C^n - 1)^t} \qquad \text{introducing the variable } x$$

$$= \frac{(\beta - \alpha)^t}{m^t (C^x\alpha^t)^n} \cdot \frac{m^t e^{mx}}{(e^m - 1)^t} \qquad \text{by (6.4)}$$

$$= \frac{(\beta - \alpha)^t}{m\ (C^x\alpha\ )} \sum_{r=0}^{\infty} B_r^{(t)}(x)\frac{m^r}{r!} \qquad \text{by (6.1),}$$

whence arises the generating function

$$\sum_{n=1}^{\infty} \frac{1}{w_n^t} y^n = (\beta - \alpha)^t \sum_{r=0}^{\infty} B_r^{(t)}(x)\left(\frac{\log C}{r!}\right)^{r-t} \sum_{n=1}^{\infty} n^{r-t}\left(\frac{y}{\alpha^{t-x}\beta^x}\right)^n. \tag{6.6}$$

Putting $t = 1$ in (6.5) gives

$$\frac{1}{w_n} = \frac{(\beta - \alpha)}{(\alpha^{1-x}\beta^x)^n} \sum_{r=0}^{\infty} B_r(x)\ \frac{\left(\log\left(\frac{\beta}{\alpha}\right)\right)^{r-1}}{r!}\ n^{r-1}. \tag{6.7}$$

This expresses the reciprocal of appropriate w_n in terms of the Bernoulli polynomial.

A chain of results similar to (6.5)-(6.7) may be obtained from (1.6) and (6.2) for Lucas-type sequences. We then obtain an expression for the reciprocal of appropriate w_n in terms of the Euler polynomial.

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A MATRIX APPROACH TO CERTAIN IDENTITIES

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1. INTRODUCTION AND GENERALITIES

In the theory of functions of matrices [3], the domain of an analytic function f is extended to include a square matrix M of arbitrary order k by defining $f(M)$ as a polynomial in M of degree less than or equal to $k - 1$ provided f is defined on the spectrum of M. Then, if f is represented by a power series expansion in a circle containing the eigenvalues of M, this expansion remains valid when the scalar argument is replaced by the matrix M. Moreover, we point out that identities between functions of a scalar variable extend to matrix values of the argument. Thus, for example, the sum $(\sin M)^2 + (\cos M)^2$ equals the identity matrix of order k.

The purpose of this article is to use functions of two-by-two matrices Q to obtain a large number of Fibonacci-type identities, most of which we believe to be new.

To achieve this objective we generally proceed in the following way:

First we determine a closed form expression of the entries a_{ij} of any function $f(Q) = A = [a_{ij}]$ based on a polynomial representation of the function itself.

Then we consider a set of functions f such that $f(Q)$ can be found by means of a power series expansion $\hat{A} = [\hat{a}_{ij}] = f(Q)$ and equate a_{ij} and $\hat{a}_{ij}$ for some i and j, thus getting one or more Fibonacci-type identities.

We shall only be concerned with some of the elementary functions, namely, the square root function, the inverse function, and the exponential, circular, hyperbolic, and logarithm functions.

To illustrate the principles being used, we choose to proceed from the particular to the general, i.e., from use of the matrix Q defined in (1.3) to use of the more general matrix P defined in (2.7).

Throughout, we shall follow the usual notational convention that F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively.

First we recall ([2], [3]) that, if M has m distinct eigenvalues μ_k $(k = 1, 2, \ldots, m)$, the coefficients c_i of the polynomial representation

$$f(M) = \sum_{i=0}^{m-1} c_i M^i \tag{1.1}$$

of any analytic function f defined on the spectrum of M are given by the solution of the following system of m equations and m unknowns

$$\sum_{i=0}^{m-1} c_i \mu_k^i = f(\mu_k) \qquad (k = 1, 2, \ldots, m). \tag{1.2}$$

Then we consider the well-known matrix (e.g., see [4])

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1.3}$$

Since the distinct eigenvalues of Q are $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$, it follows from (1.1) and (1.2) that the coefficients c_0 and c_1 of the polynomial representation

$$f(Q) = c_0 I + c_1 Q \tag{1.4}$$

(where I denotes the two-by-two identity matrix)

of any function f defined on the spectrum of Q are given by the solution of the system

$$\begin{cases} c_0 + c_1\alpha = f(\alpha) \\ c_0 + c_1\beta = f(\beta). \end{cases} \tag{1.5}$$

In fact, from (1.5), we obtain

$$\begin{cases} c_0 = (\alpha f(\beta) - \beta f(\alpha))/\sqrt{5} \\ c_1 = (f(\alpha) - f(\beta))/\sqrt{5}. \end{cases} \tag{1.6}$$

Therefore, from (1.4) and (1.6), we can write

$$f(Q) = A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha f(\alpha) - \beta f(\beta) & f(\alpha) - f(\beta) \\ f(\alpha) - f(\beta) & \alpha f(\beta) - \beta f(\alpha) \end{bmatrix}. \tag{1.7}$$

It can be noted that the main property of the matrix Q, that is,

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \tag{1.8}$$

can be derived immediately from (1.7) by specializing f to the integral n^{th} power.

2. THE SQUARE ROOT MATRIX

In general, a two-by-two matrix possesses at least two square roots [3]. In the case of Q, the existence of a negative eigenvalue (β) implies that the entries a_{ij} of any square root A will be complex. Specializing f to the square root, from (1.7) we obtain the following equations defining one square root of Q,

$$\begin{cases} a_{11} = (\alpha\sqrt{\alpha} + i\sqrt{1/\alpha^3})/\sqrt{5} \\ a_{12} = a_{21} = (\sqrt{\alpha} - i\sqrt{1/\alpha})/\sqrt{5} \\ a_{22} = (\sqrt{1/\alpha} + i\sqrt{\alpha})/\sqrt{5}, \end{cases} \tag{2.1}$$

where $i = \sqrt{-1}$.

An alternative way to obtain a square root of Q is to solve the matrix equation $\hat{A}^2 = Q$, that is,

$$\begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.2}$$

from which the following system can be written:

$$\begin{cases} \hat{a}_{11}^2 + \hat{a}_{12}\hat{a}_{21} = 1 \\ \hat{a}_{11}\hat{a}_{12} + \hat{a}_{12}\hat{a}_{22} = 1 \\ \hat{a}_{21}\hat{a}_{11} + \hat{a}_{22}\hat{a}_{21} = 1 \\ \hat{a}_{21}\hat{a}_{12} + \hat{a}_{22}^2 = 0. \end{cases} \tag{2.3}$$

From the second and third equations we can write

$$\hat{a}_{12}(\hat{a}_{11} + \hat{a}_{22}) = \hat{a}_{21}(\hat{a}_{11} + \hat{a}_{22}),$$

from which the equality $\hat{a}_{12} = \hat{a}_{21}$ is obtained (i.e., as expected, $\sqrt{Q}$ is a symmetric matrix). Therefore, from the fourth equation we get $\hat{a}_{12} = \hat{a}_{21} = \pm i\hat{a}_{22}$. Substituting these values in the first and second equations and dividing the corresponding sides one by the other, we obtain $\hat{a}_{11} = (1 \pm i)\hat{a}_{22}$. Hence, the solutions of the system (2.3) are:

$$\begin{cases} \hat{a}_{11} = (1 \pm i)\hat{a}_{22} \\ \hat{a}_{12} = \hat{a}_{21} = \pm i\hat{a}_{22} \\ \hat{a}_{22} = \pm\sqrt{(-1 \mp 2i)/5}. \end{cases} \tag{2.4}$$

Since

$$-1 \mp 2i = \sqrt{5}\,e^{i(\pi \pm \arctan 2)},$$

the complex entry $\hat{a}_{22}$ can be written as

$$\hat{a}_{22} = (1/5)^{1/4}\,e^{i(\pi \pm \arctan 2)/2 + ik\pi} \quad (k = 0, 1).$$

The real part of $\hat{a}_{22}$ is

$$\mathrm{Re}(\hat{a}_{22}) = (-1)^k (1/5)^{1/4} \cos \frac{\pi \pm \arctan 2}{2} \quad (k = 0, 1). \tag{2.5}$$

Since every square root of Q must satisfy (2.3), the matrix A defined by (2.1) does. Equating the real parts of a_{22} and $\hat{a}_{22}$, and squaring both sides of this equation, from (2.1) and (2.5) we have

$$1/(5\alpha) = \sqrt{1/5} \sin^2 \frac{\arctan 2}{2},$$

thus obtaining the trigonometrical identity

$$\alpha = 1/\left(\sqrt{5} \sin^2 \frac{\arctan 2}{2}\right). \tag{2.6}$$

Equating the imaginary parts of a_{22} and $\hat{a}_{22}$, we obtain the equivalent identity

$$\alpha = \sqrt{5} \cos^2 \frac{\arctan 2}{2}. \tag{2.6$'$}$$

The preceding treatment may be generalized in the following way:

Let

$$P = \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix} \tag{2.7}$$

whence, by induction

$$P^n = \begin{bmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{bmatrix} \tag{2.8}$$

where U_n $(n = 0, 1, 2, \ldots)$ is defined by the recurrence relation

$$U_{n+2} = pU_{n+1} + U_n; \; U_0 = 0, \; U_1 = 1. \tag{2.9}$$

When $p = 1$, we get the Fibonacci numbers F_n. When $p = 2$, the Pell numbers P_n result.

Writing

$$\Delta = \sqrt{p^2 + 4}, \tag{2.10}$$

we find that the eigenvalues of P in (2.7) are

$$\alpha_p = (p + \Delta)/2, \; \beta_p = (p - \Delta)/2. \tag{2.11}$$

From (2.11) and (2.10), it can be noted that $\alpha_p\beta_p = -1$, i.e., $\beta_p = -1/\alpha_p$.

When $p = 1$, these eigenvalues are $(1 \pm \sqrt{5})/2$ as given earlier (namely, the values of $\alpha = \alpha_1$ and $\beta = \beta_1$). If $p = 2$, these eigenvalues reduce to

$$\alpha_2 = 1 + \sqrt{2} \quad \text{and} \quad \beta_2 = 1 - \sqrt{2}.$$

Paralleling the argument for Fibonacci numbers outlined above, we may derive the identity corresponding to (2.6):

$$\alpha_p = 1/\left(\Delta \sin^2 \frac{\arctan(2/p)}{2}\right). \tag{2.12}$$

Taking $p = 2$, we have the identity for Pell numbers:

$$\alpha_2 = 1/\left(2\sqrt{2}\ \sin^2 \frac{\arctan 1}{2}\right). \tag{2.13}$$

It must be noted that identity (2.12) may be verified directly. In fact, the identity $\sin^2(x/2) = (1 - \cos x)/2$ implies

$$\sin^2 \frac{\arctan(2/p)}{2} = \frac{1 - \cos(\arctan(2/p))}{2} = (1 - p/\sqrt{p^2 + 4})/2$$
$$= (1 - p/\Delta)/2 = (\Delta - p)/(2\Delta) = -\beta_p/\Delta = 1/(\alpha_p\Delta).$$

3. THE EXPONENTIAL FUNCTION MATRIX

The previous results follow for $f(x) = \sqrt{x}$. Other particular identities emerge for other choices of f. Specializing f to the exponential function, from (1.7) we obtain:

$$\begin{cases} a_{11} = (\alpha e^{\alpha} - \beta e^{\beta})/\sqrt{5} \\ a_{12} = a_{21} = (e^{\alpha} - e^{\beta})/\sqrt{5} \\ a_{22} = (\alpha e^{\beta} - \beta e^{\alpha})/\sqrt{5}. \end{cases} \tag{3.1}$$

An alternative way of obtaining $\hat{A} = [\hat{a}_{ij}] = \exp Q$ is (see [1], [5], [6]) to use the power series expansion

$$\exp Q = \sum_{n=0}^{\infty} \frac{Q^n}{n!}. \tag{3.2}$$

From (1.8), it is easily seen that:

$$\begin{cases} \hat{a}_{11} = \sum_{n=0}^{\infty} \frac{F_{n+1}}{n!} \\ \hat{a}_{12} = \hat{a}_{21} = \sum_{n=0}^{\infty} \frac{F_n}{n!} \\ \hat{a}_{22} = \sum_{n=0}^{\infty} \frac{F_{n-1}}{n!}. \end{cases} \tag{3.3}$$

Therefore, equating the corresponding entries of $\hat{A}$ and A, from (3.1) and (3.3) we obtain the following known Fibonacci identities (see [4]):

$$\sum_{n=0}^{\infty} \frac{F_n}{n!} = (e^{\alpha} - e^{\beta})/\sqrt{5} \tag{3.4}$$

$$\sum_{n=0}^{\infty} \frac{F_{n+1}}{n!} = (\alpha e^{\alpha} - \beta e^{\beta})/\sqrt{5} \tag{3.5}$$

$$\sum_{n=0}^{\infty} \frac{F_{n-1}}{n!} = (\alpha e^{\beta} - \beta e^{\alpha})/\sqrt{5}. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\sum_{n=0}^{\infty} \frac{L_n}{n!} = e^{\alpha} + e^{\beta}. \tag{3.7}$$

It is evident that the above results may be generalized by using the exponential of the matrix P. As an example, for $p = 2$, the following identity involving Pell numbers,

$$\sum_{n=0}^{\infty} \frac{P_n}{n!} = e(e^{\sqrt{2}} - e^{-\sqrt{2}})/(2\sqrt{2}), \tag{3.8}$$

is obtained. Similar results to those in (3.5)-(3.7) readily follow.

4. OTHER FUNCTIONAL MATRICES

Let us consider the following power series expansions ([3], [6]):

$$\sin Q = \sum_{n=0}^{\infty} (-1)^n \frac{Q^{2n+1}}{(2n+1)!} \tag{4.1}$$

$$\cos Q = \sum_{n=0}^{\infty} (-1)^n \frac{Q^{2n}}{(2n)!} \tag{4.2}$$

$$\sinh Q = \sum_{n=0}^{\infty} \frac{Q^{2n+1}}{(2n+1)!} \tag{4.3}$$

$$\cosh Q = \sum_{n=0}^{\infty} \frac{Q^{2n}}{(2n)!}. \tag{4.4}$$

Using reasoning similar to the preceding, we may obtain a large number of Fibonacci identities, some of which are well known [6]. These identities have the following general forms,

$$\sum_{n=0}^{\infty} c_n F_n = (f(\alpha) - f(\beta))/\sqrt{5}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} c_n F_{n+1} = (\alpha f(\alpha) - \beta f(\beta))/\sqrt{5}, \tag{4.6}$$

$$\sum_{n=0}^{\infty} c_n F_{n-1} = (\alpha f(\beta) - \beta f(\alpha))/\sqrt{5}, \tag{4.7}$$

where

$$f(y) = \sum_{n=0}^{\infty} c_n y^n.$$

A brief selection of particular cases is shown below:

$$\sum_{n=0}^{\infty} (-1)^n \frac{F_{2n+1}}{(2n+1)!} = (\sin \alpha - \sin \beta)/\sqrt{5} \tag{4.8}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{F_{2n}}{(2n)!} = (\cos \alpha - \cos \beta)/\sqrt{5} \tag{4.9}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n+1}}{(2n+1)!} = (\sinh \alpha - \sinh \beta)/\sqrt{5} \tag{4.10}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n}}{(2n)!} = (\cosh \alpha - \cosh \beta)/\sqrt{5} \tag{4.11}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n+1}}{(2n)!} = (\alpha \cosh \alpha - \beta \cosh \beta)/\sqrt{5} \tag{4.12}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n-1}}{(2n)!} = (\alpha \cosh \beta - \beta \cosh \alpha)/\sqrt{5}. \tag{4.13}$$

Combining some of the above-mentioned results, we may obtain analogous identities involving Lucas numbers. For example, combining (4.12) and (4.13) gives

$$\sum_{n=0}^{\infty} \frac{L_{2n}}{(2n)!} = \cosh \alpha + \cosh \beta. \tag{4.14}$$

Again, we point out that these identities may be generalized by using circular and hyperbolic functions of the matrix P. In particular, we may obtain results for Pell numbers similar to these listed for Fibonacci and Lucas numbers.

5. EXTENSIONS

The results obtained *primo impetu* in Sections 3 and 4 may be extended using functions of the matrix

$$Q_{k,x} = xQ^k = \begin{bmatrix} xF_{k+1} & xF_k \\ xF_k & xF_{k-1} \end{bmatrix}, \tag{5.1}$$

where x is an arbitrary real quantity and k is a nonnegative integer. Since $Q_{k,x}$ is a polynomial $r(Q)$ in Q, it follows that its eigenvalues are

$$\begin{cases} \chi_1(k, x) = r(\alpha) = x\alpha^k \\ \chi_2(k, x) = r(\beta) = x\beta^k, \end{cases} \tag{5.2}$$

and $f(Q_{k,x}) = f(r(Q))$ derives values in terms of $f(r(\alpha))$ and $f(r(\beta))$. Thus, any function f defined on the spectrum of $Q_{k,x}$ can be obtained from (1.7) by replacing $f(\alpha)$ and $f(\beta)$ with $f(\chi_1(k, x))$ and $f(\chi_2(k, x))$, respectively. Moreover, from (5.1) and (1.8), it is easily seen that $Q_{k,x}$ enjoys the property

$$Q_{k,x}^n = (xQ^k)^n = x^n Q^{kn} = \begin{bmatrix} x^n F_{kn+1} & x^n F_{kn} \\ x^n F_{kn} & x^n F_{kn-1} \end{bmatrix}. \tag{5.3}$$

5.1 The Exponential Function of $Q_{k,x}$

Specializing f to the exponential function, from (1.7) and (5.2) we obtain the following values of the entries of the polynomial representation $A_{k,x} = [a_{ij}(k, x)]$ of $\exp Q_{k,x}$:

$$\begin{cases} a_{11}(k, x) = (\alpha e^{x\alpha^k} - \beta e^{x\beta^k})/\sqrt{5} \\ a_{12}(k, x) = a_{21}(k, x) = (e^{x\alpha^k} - e^{x\beta^k})/\sqrt{5} \\ a_{22}(k, x) = (\alpha e^{x\beta^k} - \beta e^{x\alpha^k})/\sqrt{5}. \end{cases} \tag{5.4}$$

Calculating $\exp Q_{k,x}$ by means of (3.2), we have

$$\exp Q_{k,x} = \sum_{n=0}^{\infty} \frac{Q_{k,x}^n}{n!} = \hat{A}_{k,x} = [\hat{a}_{ij}(k, x)]. \tag{5.5}$$

Equating $\hat{a}_{ij}(k, x)$ and $a_{ij}(k, x)$, from (5.5), (5.3), and (5.4) we obtain:

$$\sum_{n=0}^{\infty} \frac{x^n F_{kn+1}}{n!} = (\alpha e^{x\alpha^k} - \beta e^{x\beta^k})/\sqrt{5} \tag{5.6}$$

$$\sum_{n=0}^{\infty} \frac{x^n F_{kn}}{n!} = (e^{x\alpha^k} - e^{x\beta^k})/\sqrt{5} \tag{5.7}$$

$$\sum_{n=0}^{\infty} \frac{x^n F_{kn-1}}{n!} = (\alpha e^{x\beta^k} - \beta e^{x\alpha^k})/\sqrt{5}. \tag{5.8}$$

Combining (5.6) and (5.8), we get

$$\sum_{n=0}^{\infty} \frac{x^n L_{kn}}{n!} = e^{x\alpha^k} + e^{x\beta^k}. \tag{5.9}$$

The above results (5.6)-(5.9) may be generalized using the exponential of the matrix xP^k [refer to (2.8)].

5.2 Circular and Hyperbolic Functions of $Q_{k,x}$

By means of a procedure similar to the preceding one, the use of $\sin Q_{k,x}$, $\cos Q_{k,x}$, $\sinh Q_{k,x}$, and $\cosh Q_{k,x}$ yields a set of identities having the following general forms,

$$\sum_{n=0}^{\infty} c_n x^n F_{kn} = (f(x\alpha^k) - f(x\beta^k))/\sqrt{5}, \tag{5.10}$$

$$\sum_{n=0}^{\infty} c_n x^n F_{kn+1} = (\alpha f(x\alpha^k) - \beta f(x\beta^k))/\sqrt{5}, \tag{5.11}$$

$$\sum_{n=0}^{\infty} c_n x^n F_{kn-1} = (\alpha f(x\beta^k) - \beta f(x\alpha^k))/\sqrt{5}, \tag{5.12}$$

where

$$f(y) = \sum_{n=0}^{\infty} c_n y^n.$$

A brief selection of particular cases is shown below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} F_{k(2n+1)}}{(2n+1)!} = (\sin(x\alpha^k) - \sin(x\beta^k))/\sqrt{5} \tag{5.13}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} F_{2kn}}{(2n)!} = (\cos(x\alpha^k) - \cos(x\beta^k))/\sqrt{5} \tag{5.14}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} F_{k(2n+1)}}{(2n+1)!} = (\sinh(x\alpha^k) - \sinh(x\beta^k))/\sqrt{5} \tag{5.15}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} F_{2kn}}{(2n)!} = (\cosh(x\alpha^k) - \cosh(x\beta^k))/\sqrt{5} \tag{5.16}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} L_{2kn}}{(2n)!} = \cosh(x\alpha^k) + \cosh(x\beta^k). \tag{5.17}$$

The above-mentioned identities may be generalized using circular and hyperbolic functions of the matrix xP^k [refer to (2.8)].

5.3 The Logarithm of $Q_{k,x}$ for k Even and Particular Values of x

The principal value of the function $\ln Q$ can be calculated by (1.7), thus getting a complex matrix A. Unfortunately, since Q has a negative eigenvalue, the power series expansion of the matrix logarithm (see [3])

$$\ln Q = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(Q - I)^n \tag{5.18}$$

does not converge and a matrix $\hat{A}$ cannot be obtained in this way. On the other hand, the use of $Q_{k,x}$, with k even, allows us to utilize this function. We will show how, setting x equal to the reciprocal of the k^{th} Lucas number, some interesting results can be worked out.

First we define the two-by-two matrix

$$R_{k,x} = Q_{k,x} - I = xQ^k - I \tag{5.19}$$

whence, using induction, it can be proved that, if n is a nonnegative integer, then

$$R^n_{k,1/L_k} = \frac{1}{L_k^n}\begin{bmatrix} (-1)^n F_{kn-1} & (-1)^{n+1} F_{kn} \\ (-1)^{n+1} F_{kn} & (-1)^n F_{kn+1} \end{bmatrix}. \tag{5.20}$$

Incidentally, it can also be proved that

$$R^n_{2,1/2} = \frac{1}{2^n}\begin{bmatrix} (-1)^n F_{n-1} & (-1)^{n+1} F_n \\ (-1)^{n+1} F_n & (-1)^n F_{n+1} \end{bmatrix} \tag{5.21}$$

Then replacing f in (1.7) with the function $f(y) = \ln(xy^k)$, we have $f(\alpha) = \ln(x\alpha^k)$, $f(\beta) = \ln(x\beta^k)$, and we calculate the matrix

$$\ln Q_{k,x} = A_{k,x} = [a_{ij}(k, x)]$$

which is real if and only if k is even and $x > 0$. In fact, we obtain

$$\begin{cases} a_{11}(k, x) = \dfrac{k}{\sqrt{5}} \ln \alpha + \ln x \\[2ex] a_{12}(k, x) = a_{21}(k, x) = \dfrac{2k}{\sqrt{5}} \ln \alpha \\[2ex] a_{22}(k, x) = -\dfrac{k}{\sqrt{5}} \ln \alpha + \ln x \end{cases} \tag{5.22}$$

where it can be noted that $a_{12}(k, x) = a_{21}(k, x)$ is independent of x.

Finally, since for k even the inequality

$$|\chi_i(k, 1/L_k) - 1| < 1 \qquad (i = 1, 2)$$

holds [see (5.2)], we can calculate the function $\ln Q_{k,1/L_k}$ by means of the power series expansion (5.18):

$$\ln Q_{k,1/L_k} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} R^n_{k,1/L_k} = \hat{A}_{k,1/L_k} = [\hat{a}_{ij}(k, 1/L_k)]. \tag{5.23}$$

Replacing x by $1/L_k$ in (5.22) and equating $\hat{a}_{ij}(k, 1/L_k)$ and $a_{ij}(k, 1/L_k)$, from (5.23), (5.20), and (5.22), we obtain:

$$\sum_{n=1}^{\infty} \frac{F_{kn-1}}{nL_k^n} = \ln L_k - \frac{k}{\sqrt{5}} \ln \alpha \qquad (k = 0, 2, 4, \ldots) \tag{5.24}$$

$$\sum_{n=1}^{\infty} \frac{F_{kn}}{nL_k^n} = \frac{2k}{\sqrt{5}} \ln \alpha \qquad (k = 0, 2, 4, \ldots) \tag{5.25}$$

$$\sum_{n=1}^{\infty} \frac{F_{kn+1}}{nL_k^n} = \ln L_k + \frac{k}{\sqrt{5}} \ln \alpha \qquad (k = 0, 2, 4, \ldots). \tag{5.26}$$

Combining (5.24) and (5.26), we have

$$\sum_{n=1}^{\infty} \frac{L_{kn}}{nL_k^n} = \ln L_k^2 \qquad (k = 0, 2, 4, \ldots). \tag{5.27}$$

Using the matrix $Q_{2,1/2}$ [see (5.21)], by means of the same procedure we obtain

$$\sum_{n=1}^{\infty} \frac{F_n}{n2^n} = \frac{4}{\sqrt{5}} \ln \alpha = \sum_{n=1}^{\infty} \frac{F_{2n}}{n3^n} \tag{5.28}$$

and

$$\sum_{n=1}^{\infty} \frac{L_n}{n2^n} = \ln 4, \tag{5.29}$$

where the right-hand side of (5.28) was derived by setting $k = 2$ in (5.25).

We conclude this subsection by pointing out that, from the equality

$$(Q^k/L_k - I)^n = R^n_{k,\,1/L_k}$$

[directly derived from (5.19)] and from (5.20), the following identities can be obtained:

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \frac{F_{ki\pm 1}}{L_k^i} = (-1)^n \frac{F_{kn\mp 1}}{L_k^n} \tag{5.30}$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \frac{F_{ki}}{L_k^i} = (-1)^{n+1} \frac{F_{kn}}{L_k^n} \tag{5.31}$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \frac{L_{ki}}{L_k^i} = (-1)^n \frac{L_{kn}}{L_k^n}. \tag{5.32}$$

5.4 The Inverse of $I - Q_{k,x}$

Let us consider the two-by-two matrix

$$S_{k,x} = -R_{k,x} = I - Q_{k,x} = I - xQ^k. \tag{5.33}$$

For

$$x \neq \begin{cases} \alpha^k,\ \beta^k & (k \text{ even}) \\ -\alpha^k,\ -\beta^k & (k \text{ odd}), \end{cases} \tag{5.34}$$

$S_{k,x}$ admits its inverse

$$S^{-1}_{k,x} = \frac{1}{D}\begin{bmatrix} 1 - xF_{k-1} & xF_k \\ xF_k & 1 - xF_{k+1} \end{bmatrix} = A_{k,x} = [a_{ij}(k,\,x)], \tag{5.35}$$

where

$$D = (-1)^k x^2 - xL_k + 1.$$

The inverse of $S_{k,x}$ can be obtained from (1.7) by replacing $f(\alpha)$ and $f(\beta)$ with $1/(1 - x\alpha^k)$ and $1/(1 - x\beta^k)$, respectively.

It is apparent that the inequality

$$|\chi_i(k,\,x)| < 1 \quad (i = 1,\,2)$$

holds for $-\alpha^{-k} < x < \alpha^{-k}$ [see (5.2)]. Under this restriction, we can calculate $S^{-1}_{k,x}$ by means of the power series expansion [3]:

$$S^{-1}_{k,x} = \sum_{n=0}^{\infty} Q^n_{k,x} = \hat{A}_{k,x} = [\hat{a}_{ij}(k,\,x)] \tag{5.36}$$

Equating $\hat{a}_{ij}(k,\,x)$ and $a_{ij}(k,\,x)$, from (5.36), (5.3), and (5.35), we obtain:

$$\sum_{n=0}^{\infty} x^n F_{kn+1} = (1 - xF_{k-1})/D \quad (-\alpha^{-k} < x < \alpha^{-k}) \tag{5.37}$$

$$\sum_{n=0}^{\infty} x^n F_{kn} = xF_k/D \quad (-\alpha^{-k} < x < \alpha^{-k}) \tag{5.38}$$

$$\sum_{n=0}^{\infty} x^n F_{kn-1} = (1 - xF_{k+1})/D \quad (-\alpha^{-k} < x < \alpha^{-k}). \tag{5.39}$$

Combining (5.37) and (5.39), we have

$$\sum_{n=0}^{\infty} x^n L_{kn} = (2 - xL_k)/D \quad (-\alpha^{-k} < x < \alpha^{-k}). \tag{5.40}$$

Setting $k = 1$ and $x = 1/2$ in (5.38), we obtain, as a particular case,

$$\sum_{n=0}^{\infty} \frac{F_n}{2^n} = 2. \tag{5.41}$$

Setting $k = 1$ and $x = 1/2$, $1/3$ in (5.40), we have

$$\sum_{n=0}^{\infty} \frac{L_n}{2^n} = 6, \tag{5.42}$$

and

$$\sum_{n=0}^{\infty} \frac{L_n}{3^n} = 3, \tag{5.43}$$

respectively.

6. CONCLUDING REMARKS

While the authors know that a few of the results presented in this article have been established by others (e.g., [1], [5], [6]), they believe that most of them are original. Certainly, more possibilities exist than those developed here.

It is possible that some of the work presented above could be extended to simple cases of three-by-three matrices.

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ON THE NUMBER OF SOLUTIONS OF THE DIOPHANTINE EQUATION $\binom{x}{p} = \binom{y}{2}$

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It is well known that the binomial coefficients are equal in the trivial cases

$$1 = \binom{n}{0} = \binom{m}{0} = \binom{k}{k}, \; \binom{n}{k} = \binom{n}{n-k}, \text{ and } N = \binom{n}{k} = \binom{N}{1}$$

for any positive integers n, m, and k ($\leqslant n$). Apart from these cases, it is more difficult to decide whether there are infinitely many pairs of equal binomial coefficients or not.

The problem of equal binomial coefficients was studies by several authors (e.g., Singmaster [6], [7]; Lind [4]; Abbot, Erdös, & Hanson [1]). Recently, in an article in this *Quarterly*, Tovey [8] showed that the equation

$$\binom{n}{k} = \binom{n-1}{k+1} \tag{1}$$

has infinitely many solutions; furthermore, (1) holds if and only if

$$n = F_{2i}(F_{2i} + F_{2i-1}) \text{ and } k = F_{2i}F_{2i-1} - 1 \quad (i = 1, 2, \ldots),$$

where F_j denotes the j^{th} Fibonacci number. Another type of result was conjectured by W. Sierpinski and solved by Avanesov [2]: There are only finitely many pairs $(x; y)$ of natural numbers such that $\binom{x}{3} = \binom{y}{2}$. Avanesov proved that this holds only in the cases $(x; y) = (3; 2), (5; 5), (10; 16), (22; 56)$, and $(36; 120)$.

The purpose of this paper is to prove an extension of Sierpinski's conjecture. We shall show that the conjecture is true even if we exchange 3 for any odd prime.

Theorem: Let p ($\geqslant 3$) be a fixed prime. Then the Diophantine equation

$$\binom{x}{p} = \binom{y}{2} \tag{2}$$

has only finitely many positive integer x, y solutions.

We need the following lemmas for the proof of our theorem.

Lemma 1: Let $m \geqslant 2$ and $n \geqslant 3$ be rational integers and let $a_n \neq 0, a_{n-1}, \ldots, a_0$ and b be rational numbers. If the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

has at least 3 simple roots, then all integer solutions x, z of the Diophantine equation

$$f(x) = b \cdot z^m$$

satisfy $\max(|x|, |z|) < C$, where C is a number which is effectively computable in terms of $a_0, \ldots, a_{n-1}, a_n$, and b.

Proof: The lemma is known if the coefficients of $f(x)$ are integers and $b = 1$ (see, e.g., Baker [3]). If b and the coefficients are rational numbers, then there is an integer d ($\neq 0$) such that $d \cdot f(x)$ is a polynomial with integer coefficients and $d \cdot b$ is an integer. Thus, our equation can be written in the form

$$(bd)^{m-1} d \cdot f(x) = (bdz)^m$$

which, by the result mentioned above, has only finitely many integer solutions.

Lemma 2: Let $p \geqslant 3$ be a fixed prime number. Then all the roots of the polynomial

$$f(x) = x(x - 1)(x - 2) \cdots (x - (p - 1)) + \frac{p!}{8}$$

are simple.

Proof: First, we assume that $p > 3$. We only have to prove that $f(x)$ and its derivative $f'(x)$ are relatively prime, since that implies the lemma.

Let us consider the polynomial

$$f_1(x) = x(x - 1)(x - 2) \cdots (x - (p - 1)). \tag{3}$$

It is a polynomial of degree p with leading coefficient 1; furthermore, the number of the solutions of the congruence

$$f_1(x) \equiv 0 \pmod{p}$$

is p ($x \equiv 0, 1, \ldots, p - 1$). So, as is well known,

$$f_1(x) \equiv x^p - x \pmod{p},$$

that is, $f_1(x)$ has the form

$$f_1(x) = x^p - x + p \cdot g_1(x), \tag{4}$$

where $g_1(x)$ is a polynomial of degree less than p and has integer coefficients (see, e.g., Theorem 2.22 in [5]).

Since $p \geqslant 5$, $p!/8$ is an integer and $p \mid (p!/8)$; so, by (3) and (4), the polynomial $f(x)$ and its derivative $f'(x)$ are of the form

$$f(x) = x^p - x + p \cdot g(x)$$

and

$$f'(x) = -1 + p \cdot h(x),$$

respectively, for some polynomials $g(x)$ and $h(x)$ with integer coefficients. It follows that

$$f(x) - x \cdot f'(x) = x^p + p \cdot (g(x) - x \cdot h(x))$$
$$= b_p x^p + b_{p-1}x^{p-1} + \cdots + b_0,$$

where the b_i's are integers. It can be easily checked that $b_p = 1 - p$ and that $b_0 = p!/8$. Furthermore, $p \nmid b_p$, $p | b_i$ for $i = 0, 1, \ldots, p - 1$ and $p^2 \nmid b_0$. So, by Eisenstein's irreducibility criterion, $f(x) - x \cdot f'(x)$ is an irreducible polynomial over the rational number field. Hence, $f(x)$ and $f'(x)$ are relatively prime. This proves the lemma in the case in which $p > 3$.

When $p = 3$, one can directly show that the roots of $f(x)$ are simply, which completes the proof of Lemma 2.

Proof of the Theorem: Let x and y be integers for which (2) holds. Then

$$\frac{y(y-1)}{2} = \binom{x}{p};$$

thus, the equation

$$y^2 - y - 2\binom{x}{p} = 0$$

has a positive integer solution y. From this it follows that there is an integer z such that

$$8\binom{x}{p} + 1 = z^2.$$

Consequently, x and z satisfy the Diophantine equation

$$f(x) = x(x-1)(x-2) \cdots (x-(p-1)) + \frac{p!}{8} = \frac{p!}{8} \cdot z^2. \tag{5}$$

However, by Lemma 2, the roots of the polynomial $f(x)$ are simple; therefore, by Lemma 1, (5) has only finitely many integer solutions x, z, and the Theorem is proved.

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Announcement

THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 25-29, 1988
Department of Mathematics, University of Pisa
Pisa, Italy

International Committee

Horadam, A.F. (Australia), *Co-Chairman*
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FIBONACCI'S STATUE

Have you ever seen Fibonacci's portrait? This photo is a close-up of the head of the statue of Leonardo Pisano in Pisa, Italy, taken by Frank Johnson in 1978.

Since Fibonacci's statue was difficult to find, here are the directions from the train station in Pisa (about 8 blocks to walk): Cross Piazza Vitt. Em. II, bearing right along Via V. Croce to Piazza Toniolo, and then walk through the Fortezza. The statue is found within Fortezza Campo Santo off Lungarno Fibonacci or Via Fibonacci along the Arno River at Giardino Scotto (Teatro Estivo).

The THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of Pisa, Pisa, Italy, from July 25-29, 1988. This conference is sponsored jointly by The Fibonacci Association and The University of Pisa.

Forty-two abstracts on all branches of mathematics and science related to the Fibonacci numbers and their generalizations have been received. All contributed papers will appear subject to approval by a referee in the Conference Proceedings, which are expected to be published in 1989.

The program for the Conference will be mailed to all participants, and to those individuals who have indicated an interest in attending the conference, by June 15, 1988. All talks have been limited to one hour or less.

For further information concerning the conference, please contact Gerald Bergum, The Fibonacci Quarterly, Department of Computer Science, South Dakota State University, P.O. Box 2201, Brookings, South Dakota 57007-0194.

DE MOIVRE-TYPE IDENTITIES FOR THE TRIBONACCI NUMBERS

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(Submitted February 1986)

1. INTRODUCTION

Recently *The Fibonacci Quarterly* has published a number of articles establishing for the Tribonacci sequence some analogs of properties of the Fibonacci sequence.

It is well known that, for $x^2 - x - 1 = 0$, the two roots are $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, and that

$$\left(\frac{1 \pm \sqrt{5}}{2}\right)^n = \frac{L_n \pm \sqrt{5}F_n}{2} \tag{1}$$

as well as

$$\left(\frac{L_n \pm \sqrt{5}F_n}{2}\right)^m = \frac{L_{mn} \pm \sqrt{5}F_{mn}}{2}, \tag{2}$$

where L_n are the Lucas numbers and F_n are the Fibonacci numbers with m and n integers. Identities (1) and (2) are called "de Moivre-type" identities [9]. The purpose of this article is to establish de Moivre-type identities for the Tribonacci numbers.

2. DE MOIVRE-TYPE IDENTITIES FOR THE TRIBONACCI NUMBERS

From references [1] and [2], we get the three roots of $x^3 - x^2 - x - 1 = 0$. They are

$$r_1 = \frac{1}{3}(1 + X + Y), \tag{3}$$

$$r_2 = \frac{1}{3}\left[1 - \frac{3}{6}(X + Y) + \frac{3\sqrt{3}}{6}i\,(X - Y)\right], \tag{4}$$

and

$$r_3 = \frac{1}{3}\left[1 - \frac{3}{6}(X + Y) - \frac{3\sqrt{3}}{6}i\,(X - Y)\right], \tag{5}$$

where $X = \sqrt[3]{19 + 3\sqrt{33}}$ and $Y = \sqrt[3]{19 - 3\sqrt{33}}$. Using $X \cdot Y = 4$, and $X^3 + Y^3 = 38$, we have

$$r_1^2 = \frac{1}{3}\left[3 + \frac{2}{3}(X + Y) + \frac{1}{3}(X^2 + Y^2)\right],$$

$$r_1^3 = \frac{1}{3}\left[7 + \frac{5}{3}(X + Y) + \frac{1}{3}(X^2 + Y^2)\right],$$

$$r_1^4 = \frac{1}{3}\left[11 + \frac{10}{3}(X + Y) + \frac{2}{3}(X^2 + Y^2)\right],$$

$$r_1^5 = \frac{1}{3}\left[21 + \frac{17}{3}(X + Y) + \frac{4}{3}(X^2 + Y^2)\right],$$

and

$$r_1^6 = \frac{1}{3}\left[39 + \frac{32}{3}(X + Y) + \frac{7}{3}(X^2 + Y^2)\right].$$

The coefficients of the above equations are three Tribonacci sequences, which we denote by R_n, S_n, and T_n, respectively. The first ten numbers of these sequences are shown in the following table.

n	0	1	2	3	4	5	6	7	8	9	10
R_n	3	1	3	7	11	21	39	71	131	241	443
S_n	3	2	5	10	17	32	59	108	199	366	673
T_n	1	1	2	4	7	13	24	44	81	149	274
U_n	0	1	2	3	6	11	20	37	68	125	230

By induction we establish that

$$r_1^n = \frac{1}{3}\left[R_n + \frac{S_{n-1}}{3}(X + Y) + \frac{T_{n-2}}{3}(X^2 + Y^2)\right]. \tag{6}$$

Using the same method, we obtain

$$r_2^n = \frac{1}{3}\left\{R_n - \frac{1}{6}[S_{n-1}(X + Y) + T_{n-2}(X^2 + Y^2)] + \frac{\sqrt{3}}{6}i\,[S_{n-1}(X - Y) + T_{n-2}(X^2 - Y^2)]\right\} \tag{7}$$

and

$$r_3^n = \frac{1}{3}\left\{R_n - \frac{1}{6}[S_{n-1}(X + Y) + T_{n-2}(X^2 + Y^2)] - \frac{\sqrt{3}}{6}i\,[S_{n-1}(X - Y) + T_{n-2}(X^2 - Y^2)]\right\}. \tag{8}$$

Hence, we find that r_1^n, r_2^n, and r_3^n can be expressed in terms of R_n, S_{n-1}, and T_{n-2}, so we have formulas equivalent to (1) for the Tribonacci numbers.

3. BINET'S FORMULA FOR R_n, S_n, AND T_n

From Spickerman [2] and Köhler [3], we can obtain Binet's formula for R_n, S_n, and T_n. That is,

$$R_n = r_1^n + r_2^n + r_3^n \tag{9}$$

and

$$S_n = d_1 r_1^n + d_2 r_2^n + d_3 r_3^n, \tag{10}$$

where $S_0 = 3$, $S_1 = 2$, and $S_2 = 5$.

From (10), it follows that

$$d_1 = \frac{3r_2r_3 + 2r_1 + 3}{(r_1 - r_2)(r_1 - r_3)} = \frac{r_1(3r_1 - 1)}{(r_1 - r_2)(r_1 - r_3)},$$

$$d_2 = \frac{3r_3r_1 + 2r_2 + 3}{(r_2 - r_3)(r_2 - r_1)} = \frac{r_2(3r_2 - 1)}{(r_2 - r_3)(r_2 - r_1)},$$

$$d_3 = \frac{3r_1r_2 + 2r_3 + 3}{(r_3 - r_1)(r_3 - r_2)} = \frac{r_3(3r_3 - 1)}{(r_3 - r_1)(r_3 - r_2)},$$

and

$$T_n = \frac{r_1^{n+2}}{(r_1 - r_2)(r_1 - r_3)} + \frac{r_2^{n+2}}{(r_2 - r_3)(r_2 - r_1)} + \frac{r_3^{n+2}}{(r_3 - r_1)(r_3 - r_2)}. \tag{11}$$

T_n and R_n were originally discussed by Mark Feinberg [1] and Günter Köhler [3]. Equation (11) was derived by Spickerman [2].

4. SOME PROPERTIES OF R_n, S_n, AND T_n

As Ian Bruce shows in [6], using the Tribonacci sequence definition, some interesting results can be derived. We have also found the following:

$$R_n = R_{n-1} + R_{n-2} + R_{n-3} \tag{12}$$

$$S_n = S_{n-1} + S_{n-2} + S_{n-3} \tag{13}$$

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \tag{14}$$

$$U_n = U_{n-1} + U_{n-2} + U_{n-3} \tag{15}$$

$$U_n = T_{n-1} + T_{n-2} \tag{16}$$

$$R_n = T_{n-1} + 2T_{n-2} + 3T_{n-3} \tag{17}$$

$$S_n = 3T_n - T_{n-1} \tag{18}$$

$$\sum_{i=1}^{n} U_i = T_{n+1} - 1 \tag{19}$$

$$\sum_{i=1}^{n} R_i = 2U_{n+2} + U_n - 3 \tag{20}$$

$$\sum_{i=1}^{n} S_i = \frac{3U_{n+1} + 2U_n - U_{n-1} - 2}{2} \tag{21}$$

$$\sum_{i=0}^{n} T_i = \frac{U_{n+2} + U_{n+1} - 1}{2} \tag{22}$$

$$T_0T_1 + T_1T_2 + T_2T_3 + T_3T_4 + \cdots + T_{n-1}T_n = \frac{U_n^2 + U_{n-1}^2 - 1}{4} \tag{23}$$

and

$$U_{4n+1}U_{4n+3} + U_{4n+2}U_{4n+4} = T_{4n+3}^2 - T_{4n+1}^2 \tag{24}$$

$$U_{n+1}^2 + U_{n-1}^2 = 2(T_{n-1}^2 + T_n^2) \tag{25}$$

$$T_n^2 - T_{n-1}^2 = U_{n+1} \cdot U_{n-1}. \tag{26}$$

ACKNOWLEDGMENT

The author is extremely grateful to the referee and to Mr. Hwang Kae Shyuan for their helpful comments and suggestions.

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ENTROPY OF TERMINAL DISTRIBUTIONS AND THE FIBONACCI TREES

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(Submitted March 1986)

Continuing the previous papers (see [1] and [2]), several new properties of binary trees, especially Fibonacci trees, have been found and will be shown in this note. For this, we shall occasionally need to refer to some of the notations, definitions, and results given in those papers.

1. BINARY TREES WITH BRANCH COST

Consider a binary tree with $n-1$ internal nodes 1, 2, ..., $n-1$ and n terminal nodes (leaves) 1, 2, ..., n. An internal node has two sons, while a terminal node has none. A node is at level ℓ if the path from the root to this node has ℓ branches. When, as in [1] and [2], unit cost 1 is assigned to each left branch and cost $c > 0$ to each right, we say the tree is "$(1, c)$-assigned." The cost of a node is then defined as follows: The cost of the root node is 0, and the cost of the left [right] son of a node of cost b is $b + 1$ [$b + c$]. Denoting by a_i [b_j] the cost of terminal node i [internal node j], we have the relation:

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n-1} b_j + (n - 1)(1 + c). \tag{1}$$

This is proved easily by induction on n (see [1]).

The sum on the left-hand side of (1) is called the *total cost* of the tree. Let us say that a binary tree is *c-minimal* (or *c-optimal* [2]) if, when $(1, c)$-assigned, it has the minimum total cost of all the $(1, c)$-assigned binary trees having the same number of terminal nodes.

2. BINARY TREES WITH BRANCH PROBABILITY

We may also assign, instead of cost, probability p $(0 < p < 1)$ to each left branch and $\overline{p} = 1 - p$ to each right. We then say the tree is "$(p, \overline{p})$-assigned." The probability of a node is defined as follows: The probability of the root is 1 and the probability of the left [right] son of a node of probability q is pq [$\overline{p}q$]. Let p_i [q_j] be the probability of terminal node i [internal node j].

The $(p, \overline{p})$-assignment may be interpreted as a transportation of "nourishment" of unit amount, along paths from the root to leaves, with rates p and $\overline{p}$ to the left and right branches at each internal node. The probabilities p_i of terminal nodes, whose sum is of course 1, show the distribution of the nourishment among leaves, and will be called the *terminal distribution*.

We are especially interested in such trees that have terminal distribution as uniform as possible, given p and the number of terminal nodes. For fixed n, the uniformity of a probability distribution $p_1, \ldots, p_n$ can be measured appropriately by the entropy function

$$H(p_1, \ldots, p_n) = -\sum p_i \log p_i \quad \text{(log-base = 2)},$$

as will be seen in the following sections.

A binary tree is called *p-maximal* if, when $(p, \overline{p})$-assigned, it has the maximum entropy of all $(p, \overline{p})$-assigned binary trees having the same number of terminal nodes.

3. ENTROPY FUNCTION

The entropy function measures the uniformity or the uncertainty of the probability distribution (see [3], and also [1]). It is well known that $H(p_1, \ldots, p_n)$ attains its maximum value $\log n$ only in the case of the complete uniformness:

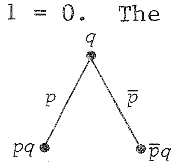
$$p_1 = \cdots = p_n = 1/n.$$

The following lemma is a variant of the so-called "branching property" of the entropy.

Lemma 1: Given a $(p, \overline{p})$-assigned tree, the entropy of the terminal distribution is given by

$$H(p_1, \ldots, p_n) = H(p, \overline{p}) \sum_{j=1}^{n-1} q_j. \tag{2}$$

Proof: Our binary tree can be viewed as grown by $n - 1$ successive branchings, starting with the branching of the root node. The entropy is initially zero: $H(1) = -1 \log 1 = 0$. The entropy increase due to the branching of a node of probability q:



is readily seen to be

$$-(pq)\log(pq) - (\overline{p}q)\log(\overline{p}q) - (-q \log q) = (-p \log p - \overline{p} \log \overline{p})q, \tag{3}$$

hence completing the proof by induction. ■

It should be noted that the sum of the internal node probabilities is equal to the *average path length* for the terminal nodes:

$$\sum_{j=1}^{n-1} q_j = \sum_{i=1}^{n} p_i \ell_i.$$

Here, ℓ_i is the level where terminal node i exists. This equality holds because each p_i contributes to both sides exactly ℓ_i times.

Lemma 1 can, therefore, be interpreted as follows: "A terminal node can be reached from the root with Σq_j branchings on the average, and the uncertainty produced per branching is $H(p, \overline{p})$, so the uncertainty of the terminal distribution should be $H(p, \overline{p})\Sigma q_j$."

Let us digress here to consider the following question: Suppose, conversely, that the following functional equation in the same form as (3) is given for some nonnegative function $f(t)$ defined on $0 < t \leqslant 1$:

$$f(pq) + f(\overline{p}q) - f(q) = (f(p) + f(\overline{p}))q, \quad 0 < p < 1, \; 0 < q \leqslant 1. \tag{4}$$

Then, how well will f be characterized?

Theorem 0: If $f(t)$ is defined on $0 < t \leqslant 1$ and satisfies (4), then $f(t) = -ct \log t$ for some constant $c \geqslant 0$.

Proof: Take $q = 1$. Then $f(1) = 0$. Let us put $g(t) = f(t)/t$. We have $g(1) = 0$, and (4) becomes

$$pg(pq) + \overline{p}g(\overline{p}q) - g(q) = pg(p) + \overline{p}g(\overline{p}). \tag{5}$$

Taking $p = 2^{-1}$ gives $g(2^{-1}q) = g(2^{-1}) + g(q)$. Repeating this gives $g(2^{-N}q) = Ng(2^{-1}) + g(q)$; hence,

$$2^{-N}g(2^{-N}q) \to 0, \; N \to \infty. \tag{6}$$

Rearrange terms in (5) to obtain:

$$p^2 \frac{g(p) - g(pq)}{p - pq} + \overline{p}^2 \frac{g(\overline{p}) - g(\overline{p}q)}{\overline{p} - \overline{p}q} = \frac{f(1) - f(q)}{1 - q} \cdot \frac{1}{q}, \text{ for } q < 1. \tag{7}$$

Letting $q \to 1$ in (7) yields

$$p^2 g'(p) = \overline{p}^2 g'(\overline{p}) = -c_1 \text{ (constant)}. \tag{8}$$

Next, we take the integral $\int_{2^{-N}}^{1} dq$ to both sides of (5). Then,

$$\int_{2^{-N}p}^{p} g(t)dt + \int_{2^{-N}\overline{p}}^{\overline{p}} g(t)dt - \int_{2^{-N}}^{1} g(t)dt = (pg(p) + \overline{p}g(\overline{p}))(1 - 2^{-N}). \tag{9}$$

Differentiating (9) with respect to p, and then letting N go to infinity, we have, using (6),

$$pg'(p) = \overline{p}g'(\overline{p}). \tag{10}$$

From (8) and (10), we have $pg'(p) = -c_1$. Hence, $g(p) = -c_1 \ln p + d$. We must have $d = 0$ and $c_1 \geqslant 0$, because $g(1) = 0$ and $g(p) \geqslant 0$. Consequently,

$$g(t) = -c \log t \text{ on } 0 < t \leqslant 1, \text{ for some constant } c \geqslant 0. \blacksquare$$

(For a derivation of the entropy function under a more general condition, see [4].)

4. DUALITY

In this section, we present and prove the following theorem.

Theorem 1: Let $c > 0$ and $0 < p < 1$ satisfy $p^c = \overline{p}$. Then a binary tree is c-minimal if and only if it is p-maximal.

Proof: Consider the infinite complete binary tree T_∞. Because of (1), a c-minimal tree having n terminal nodes can be found in T_∞ by picking the $n - 1$ cheapest nodes $1, 2, \ldots, n - 1$ to be internal, if the nodes of the $(1, c)$-assigned T_∞ are numbered $1, 2, \ldots$ such that

$$b_1 \leqslant b_2 \leqslant \cdots . \tag{11}$$

(Also see [1] in this respect.) The ordering (11) is equivalent to the ordering

$$p^{b_1} \geqslant p^{b_2} \geqslant \cdots . \tag{12}$$

If node j is reached from the root by r left branches and s right branches, we have $b_j = r + sc$. Now, from the assumption $p^c = \overline{p}$, we have $q_j = p^r(\overline{p})^s = p^{b_j}$ in the $(p, \overline{p})$-assigned T_∞. Hence, because of Lemma 1, the tree thus found must be p-maximal. $\blacksquare$

A most interesting c, p satisfying $p^c = \overline{p}$ is $c = 2$,

$$p = \psi = (\sqrt{5} - 1)/2 \quad (\overline{\psi} = \psi^2).$$

5. FIBONACCI TREES

We can now apply Theorem 1 to the Fibonacci trees (see [2]). The Fibonacci tree of order k, denoted by T_k, is a binary tree having $n = F_k$ terminal nodes, and defined inductively as follows: T_1 and T_2 are simply the root nodes only. The left subtree of T_k $(k \geqslant 3)$ is T_{k-1} and the right is T_{k-2}. Here, F_k is the k^{th} Fibonacci number: $F_0 = 0$, $F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$.

It was shown in [1] that T_k is 2-minimal for every k. Hence, by Theorem 1, T_k is ψ-maximal for every k.

The following theorem was proved in [2].

Theorem 2: When $1 \leqslant c < 2$, T_k $(k \geqslant 3)$ is c-minimal if and only if

$$k \leqslant 2\left\lfloor \frac{1}{2 - c} \right\rfloor + 3.$$

When $c > 2$, T_k $(k \geqslant 3)$ is c-minimal if and only if

$$k \leqslant 2\left\lfloor \frac{1}{c-2} \right\rfloor + 4.$$

($\lfloor x \rfloor$ is the largest integer $\leqslant x$.)

Translating this into its dual form by using Theorem 1, we have

Theorem 3: For even $k \geqslant 6$, T_k is p-maximal if and only if

$$p^{2\left(1+\frac{1}{k-4}\right)} \leqslant \overline{p} \leqslant p^{2\left(1-\frac{1}{k-2}\right)}.$$

For odd $k \geqslant 5$, T_k is p-maximal if and only if

$$p^{2\left(1+\frac{1}{k-3}\right)} \leqslant \overline{p} \leqslant p^{2\left(1-\frac{1}{k-3}\right)}.$$

In [1] it was shown that the (1, 2)-assigned T_k has F_{k-1} terminal nodes (called α-nodes in [2]) of cost $k - 2$ and F_{k-2} terminal nodes (β-nodes) of cost $k - 1$. Since each α-node [β-node] has probability ψ^{k-2} [ψ^{k-1}] in the $(\psi, \overline{\psi})$-assigned T_k, we have the following terminal distribution:

$$\underbrace{\psi^{k-2}, \ldots, \psi^{k-2}}_{F_{k-1}}, \underbrace{\psi^{k-1}, \ldots, \psi^{k-1}}_{F_{k-2}}.$$

Hence, we have

$$F_{k-1}\psi^{k-2} + F_{k-2}\psi^{k-1} = \frac{1}{\sqrt{5}}\psi^{-1} + \frac{1}{\sqrt{5}}\psi = 1.$$

$$\left(F_{k-1}\psi^{k-2} \sim \frac{1}{\sqrt{5}}\psi^{-1} = 0.724,\ F_{k-2}\psi^{k-1} \sim \frac{1}{\sqrt{5}}\psi = 0.276.\right)$$

The entropy of the above terminal distribution is computed as

$$\begin{aligned} &-F_{k-1}\psi^{k-2}\log\psi^{k-2} - F_{k-2}\psi^{k-1}\log\psi^{k-1} \\ &= (-\log\psi)\{(k-2) + F_{k-2}\psi^{k-1}\}. \end{aligned} \tag{13}$$

By a numerical computation, the ratio of this entropy and the entropy $\log F_k$ of the completely uniform distribution is approximately $1 - (0.05)/(k - 1.67)$.

Finally, let us compute the entropy of the terminal distribution of the $(p, \overline{p})$-assigned T_k. Denote the entropy by H_k for simplicity. Then, trivially, $H_1 = H_2 = 0$ and $H_3 = H(p, \overline{p})$. By Lemma 1 and by the recursive structure of the Fibonacci tree, the sum of the internal node probabilities of T_k is given by

$$1 + p\frac{H_{k-1}}{H_3} + \overline{p}\frac{H_{k-2}}{H_3} \quad (k \geqslant 3).$$

Hence, we have the "Fibonacci branching of the entropy":

$$H_k = H_3 + pH_{k-1} + \overline{p}H_{k-2}.$$

Putting $\Delta H_k = H_k - H_{k-1}$, we have $\Delta H_k = -\overline{p}\Delta H_{k-1} + H_3$; therefore,

$$\Delta H_k = \frac{H_3}{2 - p}\{1 - (-\overline{p})^{k-2}\}, \ k \geqslant 3,$$

$$H_k = \sum_{m=3}^{k} \Delta H_m = \frac{H(p, \overline{p})}{2 - p}\left\{(k - 2) + \frac{\overline{p} + (-\overline{p})^{k-1}}{1 + \overline{p}}\right\}, \ k \geqslant 3.$$

When $p = \psi$, H_k becomes (13), as can be checked. The p that maximizes H_k approaches ψ as k becomes large. This is because the maximization then almost becomes the maximization of the function

$$F(p) = \frac{H(p, \overline{p})}{2 - p},$$

and the maximum ($= -\log \psi$) of $F(p)$ is attained only when $p = \psi$. The maximization of $F(p)$ has already appeared in [1] in a closely related context.

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THE LENGTH OF A THREE-NUMBER GAME

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1. THE THREE-NUMBER PROBLEM

Let $B = (b_1, b_2, b_3)$ represent a column vector of three elements and define the operator D_3 on B as

$$D_3(b_1, b_2, b_3) = (|b_1 - b_3|, |b_1 - b_2|, |b_2 - b_3|).$$

Given any initial vector B_0, we obtain a sequence $\{B_n\}$ with $B_n = D_3 B_{n-1}$. This sequence is called the "three-number game" because of its similarity to the four-number game studied by Webb [2].

Define $rB = \max(|b_1|, |b_2|, |b_3|)$. Then, $rB \geqslant rD_3 B$ with equality only if $D_3 B$ is of the form B', where

$$B' \in [(b', b', 0), (0, b', b'), (b', 0, b')], \quad b' \geqslant 0.$$

Definition 1.1: The length of the sequence $\{B_n\}$, denoted $L(B)$, is the smallest n such that B_n takes the form B'.

The three-number problem is to determine $L(B)$ given B. Note that, if $b_1 = b_2 = b_3$, $B' = 0$ and $L(B) = 1$.

Definition 1.2: If $L(B) = L(C)$, B and C are said to be virtually equivalent, $B \simeq C$.

Let $C_0 = P_0 B_0$, a vector in which the elements of B_0 are rearranged, then $C_i = P_i B_i$, $i = 1, 2, \ldots, n$, where P_i is some permutation matrix. Therefore, $C_0 \simeq B_0$ and

$$B_0 \simeq P_0 B_0. \tag{1.1}$$

Definition 1.3: The vector B is said to be *proper* if $B = (a, b, 0) + cU$, where $a > b \geqslant 0$, c is arbitrary, and $U = (1, 1, 1)$.

Note that either $L(B) = 1$ or B is virtually equivalent to a proper vector. If B is proper, then

$$D_3 B \simeq \begin{cases} (b, 2b - a, 0) + (a - b)U & \text{if } 2b \geqslant a > b > 0, \\ (a - b, a - 2b, 0) + bU & \text{if } a \geqslant 2b. \end{cases} \tag{1.2}$$

In either case, D_3B is virtually equivalent to a proper vector of the form $(a', b', 0) + c_1U$, where $c_1 = a' - b'$ and is independent of c.

If c' is arbitrary and B is proper, then

$$B + c'U \simeq B. \tag{1.3}$$

If k is an integer and B is proper, $D_3kB = |k|D_3B$; hence,

$$kB \simeq B. \tag{1.4}$$

The three-number problem can be solved, in general, by use of the above equations. If B is proper, it reduces to a solution of the two-number problem as shown below.

2. THE TWO-NUMBER PROBLEM

The two-number game has been studied by the author (see [1]). Let D_2 represent an operator defined on a vector $A = (a, b)$, $a \geqslant b > 0$, by

$$D_2A = \begin{cases} (b,\ a - b) & 2b \geqslant a, \\ (a - b,\ b) & a \geqslant 2b. \end{cases} \tag{2.1}$$

Definition 2.1: The complement of A is defined as $C(a, b) = (a, a - b)$. Then, $C = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ and if $a > b > 0$,

$$D_2CA = D_2A. \tag{2.2}$$

Given any initial vector A_0, we obtain a sequence $\{A_n\}$ with $A_n = D_2A_{n-1}$. This sequence is called the "two-number game."

Definition 2.2: The length of the sequence $\{A_n\}$, denoted $L_2(A)$ or $L_2(a, b)$ is the smallest n such that $A_n = (a', 0)$ for some integer $a' > 0$.

It follows that $L_2(n, 1) = n$ and that

$$L_2(a, b) = [a/b] + L_2(b, a \pmod{b}), \tag{2.3}$$

where $[x]$ represents the greatest integer in the number x.

The two-number problem has been solved for $a \geqslant b > 0$ as the result of repeated applications of this formula.

3. THE MAIN RESULT

Theorem 3.1: If $B = (a, b, 0) + cU$ is proper, then $L(B) = L_2(a, b)$.

Proof: Comparing equations (2.1) and (1.2), we see that

$$D_3B_0 \simeq (CD_2A_0, 0) + c_1U \quad \text{or}$$
$$B_1 \simeq (CA_1, 0) + c_1U,$$
$$B_2 \simeq (CA_2, 0) + c_2U, \text{ etc., where } c_i \text{ is an integer.}$$

For some n, $B_n = (b', b', 0)$, $c_n = 0$, $B_{n-1} \neq B_n$, and $L(B_0) = n$, but $B_n \simeq (CA_n, 0)$, so $A_n = (b', 0)$. Since $D_2(b', 0)$ does not exist, there is only one n such that $A_n = (b', 0)$. It follows that, if $B = (A, 0) + cU$ is proper, then

$$L(B) = L_2(A). \blacksquare$$

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SOME IDENTITIES FOR TRIBONACCI SEQUENCES

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1. INTRODUCTION

The sequence $\{F_n\}$ of Fibonacci numbers is defined by

$$F_0 = 0,\ F_1 = 1,$$

with the recurrence relation

$$F_{n+2} = F_{n+1} + F_n.$$

A number of identities for $\{F_n\}$ are well known. Among them are

$$F_{N-1}F_{N+1} - F_N^2 = (-1)^N \quad \text{and} \quad F_{N-1}F_{N+1} - F_{N-2}F_{N+2} = 2(-1)^N.$$

These identities were generalized by Harman in [1] by introducing the complex Fibonacci numbers. Similar generalized identities involving the combinations of the Fibonacci, Lucas, Pell, and Chebyshev sequences were obtained by this author (see [2]) by introducing the Generalized Gaussian Fibonacci Numbers defined using Harman's technique.

This gave rise to a natural question: Is it possible to achieve similar results for the Tribonacci numbers? This paper gives the answer in the affirmative. To achieve this, we define in Section 3 the complex Tribonacci numbers at the Gaussian integers. Our main result is equation (5.1).

2. TRIBONACCI NUMBER SEQUENCES

Denote by $\{S_n\}$ a sequence defined by the third-order recurrence relation given by

$$S_{n+3} = PS_{n+2} + QS_{n+1} + RS_n.$$

We consider the following particular cases of $\{S_n\}$ and call them the fundamental sequences of third order.

a. $\{J_n\}$ where $J_0 = 0$, $J_1 = 1$, and $J_2 = P$,

b. $\{K_n\}$ where $K_0 = 1$, $K_1 = 0$, and $K_2 = Q$,

c. $\{L_n\}$ where $L_0 = 0$, $L_1 = 0$, and $L_2 = R$.

If $P = Q = R = 1$, then $\{J_n\}$, $\{K_n\}$, and $\{L_n\}$ will be called the special fundamental sequences and will be denoted by $\{J_n^*\}$, $\{K_n^*\}$, and $\{L_n^*\}$, respectively.

The following relations are easily proved:

$$H_{n+1} = PJ_n + K, \quad n \geqslant 0; \tag{2.1}$$

$$K_{n+1} = QJ_n + RJ_{n-1}, \quad n \geqslant 1; \tag{2.2}$$

$$L_{n+1} = RJ_n, \quad n \geqslant 0. \tag{2.3}$$

By (2.3), (2.2) can also be written as

$$K_{n+1} = QJ_n + L_n. \tag{2.4}$$

It is helpful to know the first few terms of the above sequences. We present them in Table 2.1. These sequences have been studied by many researchers (see, e.g., Shannon [3], Shannon & Horadam [4], and Waddill & Sacks [5]).

Table 2.1

$\{S_n\}$ \ n	0	1	2	3	4	5	6
$\{J_n\}$	0	1	P	P^2+Q	$P^3+2PQ+R$	$P^4+3P^2Q+2PR+Q^2$	$P^5+4P^3Q+3P^2R+3PQ^2+2QR$
$\{K_n\}$	1	0	Q	$PQ+R$	$P^2Q+PR+Q^2$	$P^3Q+P^2R+2PQ^2+2QR$	$P^4Q+P^3R+3P^2Q^2+4PQR+Q^3+R^2$
$\{L_n\}$	0	0	R	PR	P^2R+QR	$P^3R+2PQR+R^2$	$P^4R+3P^2QR+2PR^2+Q^2R$

3. DEFINITION

Let (n, m), $n, m \in \mathbb{Z}$, denote the set of Gaussian integers $(n, m) = n + im$. Let $G: (n, m) \to \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers, be a function defined as follows:

For fixed real numbers P, Q, and R, define

$$\begin{cases} G(0, 0) = 0, \ G(1, 0) = 1, \ G(2, 0) = P \\ G(0, 1) = i, \ G(1, 1) = P + iP, \ G(2, 1) = P^2 + i(P^2 + Q) \\ G(0, 2) = iP, \ G(1, 2) = P^2 + Q + iP^2, \ G(2, 2) = P^3 + PQ + i(P^3 + PQ) \end{cases} \tag{3.1}$$

with the following conditions:

$$G(n + 3, m) = PG(n + 2, m) + QG(n + 1, m) + RG(n, m), \tag{3.2}$$

and

$$G(n, m + 3) = PG(n, m + 2) + QG(n, m + 1) + RG(n, m). \tag{3.3}$$

The conditions (3.2) and (3.3) with the initial values (3.1) are sufficient to obtain a unique value for every Gaussian integer with nonnegative values for n and m.

4. RESULTS CONCERNING $G(n, m)$

Lemma 4.1: $G(n, 0)$ and $G(0, m)$ are given by

$$G(n, 0) = J_n, \text{ and } G(0, m) = iJ_m. \tag{4.1}$$

Proof: The proof is simple and hence omitted.

Theorem 4.2: $G(n, m)$ is given by

$$G(n, m) = J_n J_{m+1} + iJ_{n+1}J_m. \tag{4.2}$$

Proof: Although an elegant proof can be given by using the technique of mathematical induction, we give another below, which although not so elegant brings out more clearly the interaction. We have:

$$\begin{aligned} G(n, m) &= PG(n-1, m) + QG(n-2, m) + RG(n-3, m) \\ &= P\{PG(n-2, m) + QG(n-3, m) + RG(n-4, m)\} \\ &\qquad + QG(n-2, m) + RG(n-3, m) \\ &= (P^2 + Q)G(n-2, m) + (PQ + R)G(n-3, m) + PRG(n-4, m) \\ &= J_3G(n-2, m) + K_3G(n-3, m) + L_3G(n-4, m) \\ &= J_3[PG(n-3, m) + QG(n-4, m) + RG(n-5, m)] \\ &\qquad + K_3G(n-3, m) + L_3G(n-4, m) \\ &= (PJ_3 + K_3)G(n-3, m) + (QJ_3 + L_3)G(n-4, m) + RJ_3G(n-5, m) \end{aligned}$$

Now we make use of (2.1), (2.4), and (2.3) to set

$$G(n, m) = J_4G(n-3, m) + K_4G(n-4, m) + L_4G(n-5, m).$$

Continuing this process, we finally get

$$G(n, m) = J_{n-1}G(2, m) + K_{n-1}G(1, m) + L_{n-1}G(0, m). \tag{4.3}$$

We apply the same technique for $G(2, m)$, $G(1, m)$, and $G(0, m)$ to get

$$G(2, m) = J_{m-1}G(2, 2) + K_{m-1}G(2, 1) + L_{m-1}G(2, 0),$$

$$G(1, m) = J_{m-1}G(1, 2) + K_{m-1}G(1, 1) + L_{m-1}G(1, 0),$$

and $$G(0, m) = J_{m-1}G(0, 2) + K_{m-1}G(0, 1) + L_{m-1}G(0, 0).$$

Then (3.1) gives

$$\begin{cases} G(2, m) = \{P^3 + PQ + i(P^3 + PQ)\}J_{m-1} + [P^2 + i(P^2 + Q)]K_{m-1} + PL_{m-1}, \\ G(1, m) = (P^2 + Q + iP^2)J_{m-1} + (P + iP)K_{m-1} + L_{m-1}, \text{ and} \\ G(0, m) = iPJ_{m-1} + iK_{m-1}. \end{cases} \tag{4.4}$$

Substituting the values of $G(2, m)$, $G(1, m)$, and $G(0, m)$ from (4.4) into (4.3) and simplifying, we get:

$$\begin{aligned} G(n, m) &= \{(P^3 + PQ)J_{n-1} + (P^2 + Q)K_{n-1} + i[(P^3 + PQ)J_{n-1} \\ &\qquad + P^2K_{n-1} + PL_{n-1}]\}J_{m-1} \\ &\quad + \{P^2J_{n-1} + PK_{n-1} + i[(P^2 + Q)J_{n-1} + PK_{n-1} + L_{n-1}]\}K_{m-1} \\ &\quad + \{PJ_{n-1} + K_{n-1}\}L_{m-1} \end{aligned}$$

Using equations (2.1)-(2.4), we obtain:

$$\begin{aligned} G(n, m) &= J_{m-1}\{P^2J_n + QJ_n + i(P^2J_n + PK_n)\} \\ &\qquad + K_{m-1}\{PJ_n + i[PJ_n + K_n]\} + L_{m-1}J_n \\ &= J_n\{P^2J_{m-1} + QJ_{m-1} + PK_{m-1} + L_{m-1} + i[P^2J_{m-1} + PK_{m-1}]\} \\ &\qquad + iK_n\{PJ_{m-1} + K_{m-1}\} \\ &= J_nJ_{m+1} + iJ_{n+1}J_m \end{aligned}$$

Theorem 4.3: For fixed n, m (n, m = 0, 1, ...), the recurrence relation for $G(n, m)$ is given by the following:

$$\begin{aligned}G(n + k, m + k) = {} & (P + iP)\sum_{j=1}^{k} Q^{k-j} J_{n+j}J_{m+j} \qquad (4.5)\\ & + R\left[\sum_{j=1}^{[k/2]+s} Q^{k-2j+s} J_{n+2j-s}J_{m+2j-2-s}\right.\\ & \left. + \sum_{j=1}^{[k/2]} Q^{k-2j+1-s} J_{n+2j-3+s}J_{m+2j-1+s}\right]\\ & + iR\left[\sum_{j=1}^{[k/2]+s} Q^{k-2j+s} J_{n+2j-2-s}J_{m+2j-s}\right.\\ & \left. + \sum_{j=1}^{[k/2]} Q^{k-2j+1-s} J_{n+2j-1+s}J_{m+2j-3+s}\right]\\ & + Q^k\begin{cases} G(n, m), & \text{if } k \text{ is even}\\ G(m, n), & \text{if } k \text{ is odd,}\end{cases}\end{aligned}$$

where $s = \begin{cases} 0, & \text{if } k \text{ is even}\\ 1, & \text{if } k \text{ is odd}\end{cases}$ and $[k/2]$ denotes the greatest integer function.

Proof: Fix n and m. From (4.2), we have:

$$\begin{aligned}G(n + 1, m + 1) &= J_{n+1}J_{m+2} + iJ_{n+2}J_{m+1}\\ &= J_{n+1}[PJ_{m+1} + QJ_m + RJ_{m-1}] + i[PJ_{n+1} + QJ_n + RJ_{n-1}]J_{m+1}\end{aligned}$$

By algebraic manipulation and interchanging n and m in (4.2), we get

$$\begin{aligned}G(n + 1, m + 1) = {} & (P + iP)J_{n+1}J_{m+1} + RJ_{n+1}J_{m-1}\\ & + iRJ_{n-1}J_{m+1} + QG(m, n). \qquad (4.6)\end{aligned}$$

Similarly, we have

$$\begin{aligned}G(n + 2, m + 2) = {} & (P + iP)[J_{n+2}J_{m+2} + QJ_{n+1}J_{m+1}]\\ & + R[J_{n+2}J_m + QJ_{n-1}J_{m+1}]\\ & + iR[J_nJ_{m+2} + QJ_{n+1}J_{m-1}] + Q^2G(n, m). \qquad (4.7)\end{aligned}$$

(4.6) and (4.7) show that (4.5) holds for $k = 1$ and $k = 2$. Now, suppose (4.5) holds for the first k positive integers. We prove that then it also holds for the integer $k + 2$. Now, although n and m are assumed to be fixed in (4.7), it is clear that (4.7), in fact, is true for any positive integers n and m. Thus, replacing n and m by $n + k$ and $m + k$, respectively, in (4.7), we get:

$$\begin{aligned}G(n + k + 2, m + k + 2) = {} & (P + iP)[J_{n+k+2}J_{m+k+2} + QJ_{n+k+1}J_{m+k+1}]\\ & + R[J_{n+k+2}J_{m+k} + QJ_{n+k-1}J_{m+k+1}]\\ & + iR[J_{n+k}J_{m+k+2} + QJ_{n+k+1}J_{m+k-1}]\\ & + Q^2G(n + k, m + k)\end{aligned}$$

Substituting for $G(n + k, m + k)$ from (4.5), we get:

$$\begin{aligned} G(n+k+2,\, m+k+2) = (P + iP)&[J_{n+k+2}J_{m+k+2} + QJ_{n+k+1}J_{m+k+1}] \\ &+ R[J_{n+k+2}J_{m+k} + QJ_{n+k-1}J_{m+k+1}] \\ &+ iR[J_{n+k}J_{m+k+2} + QJ_{n+k+1}J_{m+k-1}] \\ &+ Q^2\left\{(P + iP)\sum_{j=1}^{k} Q^{k-j}J_{n+j}J_{m+j}\right. \\ &+ R\left[\sum_{j=1}^{[k/2]+s} Q^{k-2j+s}J_{n+2j-s}J_{m+2j-2-s}\right. \\ &+ \left.\sum_{j=1}^{[k/2]} Q^{k-2j+1-s}J_{n+2j-3+s}J_{m+2j-1+s}\right] \\ &+ iR\left[\sum_{j=1}^{[k/2]+s} Q^{k-2j+s}J_{n+2j-2-s}J_{m+2j-s}\right. \\ &+ \left.\sum_{j=1}^{[k/2]} Q^{k-2j+1-s}J_{n+2j-1+s}J_{m+2j-3+s}\right] \\ &+ Q^k\left\{\begin{matrix} G(n,\, m), & k \text{ even} \\ G(m,\, n), & k \text{ odd}\end{matrix}\right\} \end{aligned} \tag{4.8}$$

We observe the following on the right-hand side of (4.8):

The coefficient of $P + iP$ is

$$J_{n+k+2}J_{m+k+2} + QJ_{n+k+1}J_{m+k+1} + \sum_{j=1}^{k} Q^{k+2-j}J_{n+j}J_{m+j}$$

$$= \sum_{j=1}^{k+2} Q^{k+2-j}J_{n+j}J_{m+j}.$$

The coefficient of R is

$$J_{n+k+2}J_{m+k} + QJ_{n+k-1}J_{m+k+1} + \sum_{j=1}^{[k/2]+s} Q^{k+2-2j+s}J_{n+2j-s}J_{m+2j-2-s}$$
$$+ \sum_{j=1}^{[k/2]} Q^{k+3-2j-s}J_{n+2j-3+s}J_{m+2j-1+s}.$$

Observing that, if $j = [k/2] + 1 + s$ and $j = [k/2] + 1$, $2j = k + 2 + s$ and $k + 2 - s$, respectively, where s is as defined before, we see that:

The coefficient of R is

$$\sum_{j=1}^{[k/2]+1+s} Q^{k+2-2j+s}J_{n+2j-s}J_{m+2j-2-s}$$
$$+ \sum_{j=1}^{[k/2]+1} Q^{k+3-2j-s}J_{n+2j-3+s}J_{m+2j-1+s}.$$

Similarly:

The coefficient of iR is

$$\sum_{j=1}^{[k/2]+1+s} Q^{k+2-2j+s} J_{n+2j-2-s} J_{m+2j-s} + \sum_{j=1}^{[k/2]+1} Q^{k+3-2j-s} J_{n+2j-1+s} J_{m+2j-3+s}.$$

The last term is

$$Q^{k+2}\begin{cases} G(n, m), & k \text{ even}, \\ G(m, n), & k \text{ odd}. \end{cases}$$

These coefficients are exactly the same as those, respectively, on the right-hand side of (4.5) with k replaced by $k + 2$. This completes the proof.

5. IDENTITIES FOR $\{J_n\}$

Equating the real parts of (4.5), and making use of (4.2), we get:

$$P\sum_{j=1}^{k} Q^{k-j} J_{n+j} J_{m+j} + R\left[\sum_{j=1}^{[k/2]+s} Q^{k-2j+s} J_{n+2j-s} J_{m+2j-2-s} + \sum_{j=1}^{[k/2]} Q^{k-2j+1-s} J_{n+2j-3+s} J_{m+2j-1+s}\right] = J_{n+k} J_{m+k+1} - Q^k J_{n+s} J_{m+1-s} \tag{5.1}$$

Remark 1: Equation (5.1) gives the sum of $2k$ terms as that of just two terms. Note that equating the imaginary parts of (4.5) gives (5.1) with m and n interchanged and, therefore, effectively the same equation.

We now consider some special cases.

6. SPECIAL CASES

(A) $m = n$

Putting $s = 0$ and $s = 1$, in turn, for k even and k odd, respectively, we readily observe that, for both even and odd k, (5.1) reduces to a single equation given by

$$P\sum_{j=1}^{k} Q^{k-j} J_{n+j}^2 + R\sum_{j=1}^{k} Q^{k-j} J_{n+j-2} J_{n+j} = J_{n+k} J_{n+k+1} - Q^k J_n J_{n+1}. \tag{6.1}$$

(B) $m = n = 0$

With these values of m and n, (6.1) reduces to

$$P\sum_{j=1}^{k} Q^{k-j} J_j^2 + R\sum_{j=1}^{k} Q^{k-j} J_{j-2} J_j = J_k J_{k+1}. \tag{6.2}$$

(C) $n = 1$, $m = 0$

Equation (5.1) takes the following form:

$$P\sum_{j=1}^{k} Q^{k-j}J_j J_{j+1} + R\left\{\sum_{j=1}^{[k/2]} Q^{k-2j}J_{2j-2+2s}\,[QJ_{2j-1-2s} + J_{2j+1-2s}]\right\} \tag{6.3}$$
$$= \begin{cases} J_{k+1}^2 - Q^k & \text{if } k \text{ is even,} \\ J_{k+1}^2 - RJ_{k+1}J_{k-2} & \text{if } k \text{ is odd.} \end{cases}$$

Remark 2: Various other identities may be obtained for other choices of m and n. Thus, equation (5.1) provides an abundance of identities.

Remark 3: If $P = Q = R = 1$, the identities in Sections 5 and 6 reduce to those for the "special fundamental sequences." It is interesting to compare these identities with similar ones for Fibonacci sequences. For example, for $n = 0$, $m = 0$, (6.2) becomes

$$\sum_{j=1}^{k} J_j^{*2} + \sum_{j=1}^{k} J_{j-2}^* J_j^* = J_k^* J_{k+1}^*,$$

and for $n = 1$, $m = 0$, (6.3) reduces to

$$\sum_{j=1}^{k} J_j^* J_{j+1}^* + \sum_{j=1}^{[k/2]} J_{2j-2+2s}^*\,[J_{2j-1-2s}^* + J_{2j+1-2s}^*]$$
$$= \begin{cases} J_{k+1}^{*2} - 1, & \text{if } k \text{ is even,} \\ J_{k+1}^{*2} - J_{k+1}^* J_{k-2}^*, & \text{if } k \text{ is odd.} \end{cases}$$

Similar identities for the Fibonacci sequence are

$$\sum_{j=1}^{k} F_j^2 = F_k F_{k+1},$$

and $$\sum_{j=1}^{k} F_j F_{j+1} = \begin{cases} F_{k+1}^2 - 1, & \text{if } k \text{ is even,} \\ F_{k+1}^2, & \text{if } k \text{ is odd.} \end{cases}$$

(See [1].)

Remark 4: If $R = 0$, the sequence $\{J_n\}$ reduces to the sequence with second-order recurrence relation. If, in addition, $P = p$ and $Q = -q$, $\{J_n\}$ becomes Lucas's fundamental sequence [2]. If $P = 1$ and $Q = 1$, $\{J_n\}$ reduces to the Fibonacci sequence. In these cases, equation (5.1) and the rest of the equations reduce to equation (5.1) and the others, respectively, of [2].

Remark 5: Define the initial terms as follows:

$G(0, 0) = 0$, $G(1, 0) = iQ$, $G(2, 0) = i(PQ + R)$

$G(0, 1) = Q$, $G(1, 1) = 0$, $G(2, 1) = Q^2$

$G(0, 2) = PQ + R$, $G(1, 2) = iQ^2$, $G(2, 2) = Q(PQ + R) + iQ(PQ + R)$

Then, following a technique similar to that used in Theorem 4.2, we prove that

$$G(n, m) = K_n K_{m+1} + iK_{n+1}K_m. \tag{6.4}$$

Since (6.4) is exactly the same as (4.2) with J_i replaced by K_i, it can be readily seen that with such a replacement all identities proved in Sections 5 and 6 can be transformed into ones with $\{K_n\}$ and $\{K_n^*\}$. The same is true for $\{L_n\}$ and $\{L_n^*\}$.

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FIBONACCI SEQUENCES OF SETS AND THEIR DUALS

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In this paper, Fibonacci sequences of sets and their duals are defined and used first to obtain short proofs of two well-known theorems on the representation of integers as sums of Fibonacci numbers, and second to produce two sets of binary numbers that resemble Cantor's ternary set. It is also shown how Fibonacci sequences of sets and their duals can be represented by sequences of trees.

Given any sequence $C = (c_1, c_2, \ldots)$ of real numbers, let the corresponding Fibonacci sequence of sets and its dual be defined by

$$S_0 = \{0\},\ S_1 = \{c_1\},\ S_n = \{x : (x - c_n) \in (S_{n-1} \cup S_{n-2})\}, \tag{1}$$

and

$$S'_0 = \{0\},\ S'_1 = \{c_1\},\ S'_n = \{x : (x - c_n) \in (S_0 \cup S_1 \cup \cdots \cup S_{n-2})\}. \tag{1'}$$

These definitions resemble the recurrence relations that may be used to define the sequence $F = (u_1, u_2, \ldots)$ of distinct positive Fibonacci numbers, namely,

$$u_0 = u_1 = 1,\quad u_n = u_{n-1} + u_{n-2}; \tag{2}$$

$$u_0 = u_1 = 1,\quad u_n = 1 + u_0 + u_1 + \cdots + u_{n-2}. \tag{2'}$$

The following lemmas are easily proved by induction.

Lemma 1: $x \in S_n$ if and only if x is of the form

$$x = \sum_{j=1}^{n} e_j c_j,\quad n \geqslant 1, \tag{3}$$

where

$$e_j \in \{0, 1\},\quad e_n = 1,\quad e_j + e_{j+1} \neq 0 \text{ if } 1 \leqslant j < n. \tag{4}$$

There are exactly u_n distinct n-tuples $(e_1, \ldots, e_n)$ satisfying (4).

Lemma 1′: $x \in S'$ if and only if x is of the form (3), where

$$e_j \in \{0, 1\},\quad e_n = 1,\quad e_j e_{j+1} = 0 \text{ if } 1 \leqslant j < n. \tag{4'}$$

There are exactly u_{n-1} distinct n-tuples $(e_1, \ldots, e_n)$ satisfying (4′) if $n \geqslant 1$.

Two special choices of C are of interest. The first choice, $C = F$, yields short proofs of two well-known theorems.

Theorem 1 (Brown [1]): Every positive integer has one and only one representation (the so-called Dual of the Zeckendorf representation) in the form

$$x = \sum_{j=1}^{n} e_j u_j, \quad n \geqslant 1, \tag{5}$$

where $(e_1, \ldots, e_n)$ satisfies (4).

Theorem 1′ (Lekkerkerker [2]): Every positive integer has one and only one representation (the so-called Zeckendorf representation) in the form (5), where $(e_1, \ldots, e_n)$ satisfies (4′).

Proofs: Let $C = F$ and let S_n and S_n' be defined by (1) and (1′). It is seen, by induction on n, that

$$S_n = \{u_{n+1} - 1, u_{n+1}, u_{n+1} + 1, \ldots, u_{n+2} - 2\},$$

and

$$S_n' = \{u_n, u_n + 1, u_n + 2, \ldots, u_{n+1} - 1\},$$

for $n = 1, 2, 3, \ldots$. Theorems 1 and 1′ now follow from Lemmas 1 and 1′.

The second choice of C is $C = B$, where

$$B = \left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\right). \tag{6}$$

We now show that this choice leads to two binary sets that resemble Cantor's ternary set.

Theorem 2: Let $\overline{S}$ be the set of all real numbers x whose binary expansion is $x = 0 \cdot e_1 e_2 \ldots$, where $e_j + e_{j+1} \neq 0$ for all $j \geqslant 1$ if the expansion does not terminate, for $1 \leqslant j < n$ if the expansion terminates with the digit $e_n = 1$. Then $\overline{S}$ is an uncountable closed set of measure 0.

Theorem 2′: Let $\overline{S'}$ be the set of all real numbers x whose binary expansion is $x = 0 \cdot e_1 e_2 \ldots$, where $e_j e_{j+1} = 0$ for all $j \geqslant 1$. Then $\overline{S'}$ is an uncountable closed set of measure 0.

Proofs: Let $C = B$, defined by (6). Let S_n and S_n' be defined by (1) and (1′). Let

$$S = \bigcup_{n=1}^{\infty} S_n, \quad S' = \bigcup_{n=1}^{\infty} S_n'.$$

By Lemma 1 (Lemma 1′), S_n (S_n') contains exactly the binary fractions in $\overline{S}$ ($\overline{S'}$) that terminate with the digit $e_n = 1$, and it is clear that $\overline{S}$ ($\overline{S'}$) is the closure of S (S'). Also, it is easily seen that

$$\overline{S} \subseteq \left[\frac{1}{4}, 1\right] \quad \text{and} \quad \overline{S'} \subseteq \left[0, \frac{2}{3}\right].$$

Now $z \in [1/4, 1] - \overline{S}$ if and only if z is a binary fraction of the form

$$z = 0 \cdot e_1 e_2 \ldots e_n 0\, 0 e_{n+3} \ldots,$$

where $0 \cdot e_1 e_2 \ldots e_n \in S_n$, $n \geqslant 1$, and $e_m = 1$ for at least one $m \geqslant n + 3$. It follows that the complement of S in [1/4, 1] is the open set

$$C = \left[\frac{1}{4}, 1\right] - \overline{S} = \bigcup_{n=1}^{\infty} \bigcup_{x \in S_n} (x, x + 2^{-n-2}). \tag{7}$$

The intervals on the right of (7) are disjoint because their end-points belong to $\overline{S}$, and their total length is

$$\sum_{n=1}^{\infty} \frac{1}{2^{n+2}} u_n = \frac{3}{4}$$

by Lemma 1 and the well-known result

$$\sum_{n=0}^{\infty} u_n x^n = \frac{1}{1 - x - x^2} \text{ if } |x| < \frac{\sqrt{5} - 1}{2}.$$

It follows readily that $\overline{S} = [1/4, 1] - C$ has measure 0.

While S is clearly countable, $\overline{S}$ is not. For, if

$$0 \cdot e_{k,1} e_{k,2} \ldots \quad (k = 1, 2, \ldots)$$

is any countable list of elements of $\overline{S}$ in binary notation, consider

$$x = 0 \cdot e_1 e_2 \ldots,$$

where $(e_{3k-2}, e_{3k-1}, e_{3k}) = (1, 0, 1)$ or $(1, 1, 1)$ according as

$$(e_{k,3k-2}, e_{k,3k-1}, e_{k,3k}) = (1, 1, 1)$$

or not; clearly, x belongs to $\overline{S}$ but does not occur in the list.

Before proceeding to the proof of Theorem 2′, note that $\overline{S}$ can be written as the disjoint union

$$\overline{S} = \overline{S}^* \cup \overline{S}^{**},$$

where $\overline{S}^{**}$ is the set consisting of all elements of $\overline{S}$ whose binary expansion terminates with 01, and where $\overline{S}^* = \overline{S} - \overline{S}^{**}$. Clearly, $\overline{S}^{**}$ is countable, and it is easily seen that $\overline{S}^{**}$ consists of all the isolated points of $\overline{S}$, while $\overline{S}^*$ consists of all the limit points of $\overline{S}$. Like $\overline{S}$, $\overline{S}^*$ is, therefore, an uncountable, closed set of measure 0. Thus, Theorem 2′ follows from Theorem 2, since $x \in \overline{S'}$ if and only if $1 - x \in \overline{S}^*$. However, it is interesting to note that, if $x \in S_n'$, $n \geqslant 1$, then

$$\left(x - \frac{1}{3} 2^{-n}, x\right) \subseteq C', \text{ where } C' = \left[0, \frac{2}{3}\right] - \overline{S'}.$$

Conversely, suppose that $z \in C'$. Then z must be a binary fraction of the form

$$z = e_0 \cdot e_1 e_2 \ldots e_m e_{m+1} \ldots,$$

where $e_0 \cdot e_1 e_2 \ldots e_m \in S_m'$ and $e_{m+1} = 1$. Let n be the largest subscript such that $1 \leqslant n < m$, $e_{n-1} = e_n = 0$, and $e_{n+1} = 1$. This n exists because

$$e_0 \cdot e_1 e_2 \ \ldots \ e_m < z < \frac{2}{3}.$$

Put $x = e_0 \cdot e_1 e_2 \ \ldots \ e_{n-1} 1$. Then $x \in S'_n$, $n \geqslant 1$, and

$$z \in \left(x - \frac{1}{3}\, 2^{-n},\ x\right).$$

It follows that the complement of $\overline{S'}$ in [0, 2/3] is the open set

$$C' = \left[0, \frac{2}{3}\right] - \overline{S'} = \bigcup_{n=1}^{\infty} \bigcup_{x \in S'_n} \left(x - \frac{1}{3}\, 2^{-n},\ x\right). \tag{7'}$$

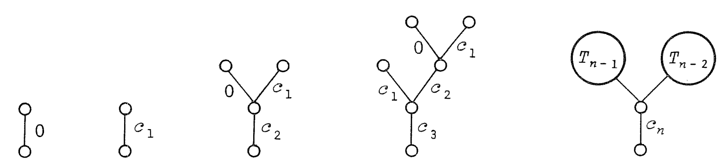
The intervals on the right of (7′) are disjoint because their endpoints belong to $\overline{S'}$, and their total length is

$$\sum_{n=1}^{\infty} \frac{1}{3}\, 2^{-n} u_{n-1} = \frac{2}{3},$$

which proves that $\overline{S'} = [0, 2/3] - C'$ has measure 0.

Equations (7) and (7′) emphasize the similarity between the constructions of $\overline{S}$ and $\overline{S'}$ and the construction of Cantor's ternary set. There are further similarities: $\overline{S}$ and $\overline{S'}$ are nowhere dense, $\overline{S'}$ is a perfect set, and the derived set $\overline{S}^*$ of $\overline{S}$ is also perfect.

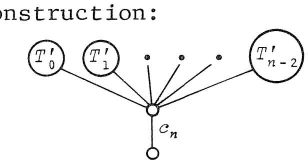
The Fibonacci sequence of sets $(S_0, S_1, S_2, \ldots)$ may be represented graphically by a sequence of weighted, rooted trees $(T_0, T_1, T_2, \ldots)$ as follows:



T_0 $\quad T_1$ $\quad T_2$ $\quad T_3$ $\quad T_n$ $(n \geqslant 2)$

For each of the u_n leaf-nodes of T_n, we may compute the total weight of the path to it from the root of T_n. The set of these u_n total weights is called "the shade of T_n" (cf. Turner [3]). The shade of T_n is obviously equal to the set S_n. A similar representation can be obtained for the dual of the Fibonacci sequence of sets by using the tree construction:

$T'_0 = T_0$, $\quad T'_1 = T_1$, $\quad T'_n =$



$(n \geqslant 2)$

In particular, very pretty graphical illustrations of Theorems 1 and 1′ can be obtained (cf. Turner [3]).

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ON THE L^p-DISCREPANCY OF CERTAIN SEQUENCES

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1. INTRODUCTION

Let (x_n), $n = 1, 2, \ldots$ be a sequence of real numbers contained in $[0, 1)$. Let $A([0, x); N)$ be the number of x_n, $1 \leqslant n \leqslant N$, that lie in the subinterval $[0, x)$ of the unitinterval. The number

$$D_N^{(p)} = \left(\int_0^1 \left|\frac{A([0, x); N)}{N} - x\right|^p dx\right)^{1/p}, \ldots, \tag{1}$$

where $1 \leqslant p < \infty$, is called the L^p-discrepancy of the given sequence ([2], p. 97).

As is well known, the notion of discrepancy is at the center of most theories in the area of uniform distribution (and other types of distributions as well) and quantitative aspects of certain limit passages are expressed by estimates of the discrepancy.

The following relation was given by Koksma [1] and by Niederreiter [3]:

$$(D_N^{(2)})^2 = \frac{1}{12N^2} + \frac{1}{N}\sum_{n=1}^{N}(x_n - s_n)^2, \ldots, \tag{2}$$

where $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant 1$ and $s_n - (2n - 1)/2N$.

In the following, we show (2) (for the sake of completeness) and consider the case $p = 4$. The proofs are given by elementary methods. Some sum formulas are established and only integration results are used.

2. THE CASE $p = 2$

To prove (2), the following lemma is useful.

Lemma 1: $\displaystyle\sum_{n=1}^{N}\sum_{m=1}^{N}\max(x_n, x_m) = \sum_{n=1}^{N}(2n - 1)x_n$, where $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant 1$.

Proof: $\displaystyle\sum_{n=1}^{N}\sum_{m=1}^{N}\max(x_n, x_m) = \sum_{n=1}^{N}\left(\sum_{m=1}^{N}\max(x_n, x_m)\right) = \sum_{n=1}^{N}(2n - 1)x_n$,

since for any n there are n values of m satisfying $m \leqslant n$ taking care of the $2n$ pairs $(x_n, x_1), \ldots, (x_n, x_n), (x_1, x_n), \ldots, (x_n, x_n)$. But (x_n, x_n) is counted twice, so for any n there are $2n - 1$ values of x_n.

Let $c(t, x_n) = \begin{cases} 0 (x_n \geqslant t) \\ 1 (x_n < t) \end{cases}$, $0 \leqslant t \leqslant 1$. Then

$$\int_0^1 \left(\sum_{n=1}^N c(t, x_n)\right)^2 dt = \int_0^1 \sum_{n=1}^N \sum_{m=1}^N c(t, x_n)c(t, x_m)dt$$
$$= \sum_{n=1}^N \sum_{m=1}^N \int_0^1 c(t, \max(x_n, x_m))dt.$$

Now we show (2). We have:

$$(ND_N^{(2)})^2 = \int_0^1 \left|\sum_{n=1}^N c(t, x_n) - Nt\right|^2 dt$$
$$= \int_0^1 \left(\sum_{n=1}^N c(t, x_n)\right)^2 dt - 2N\int_0^1 t \sum_{n=1}^N c(t, x_n)dt + N^2 \int_0^1 t^2\, dt$$
$$= \sum_{n=1}^N \sum_{m=1}^N \int_0^1 c(t, \max(x_n, x_m))dt - 2N\int_0^1 t \sum_{n=1}^N c(t, x_n)dt + \frac{1}{3}N^2$$
$$= \sum_{n=1}^N \sum_{m=1}^N (1 - \max(x_n, x_m)) - 2N \sum_{n=1}^N \frac{1}{2}(1 - x_n^2) + \frac{1}{3}N^2$$
$$= N^2 - \sum_{n=1}^N \sum_{m=1}^N \max(x_n, x_m) - N\left(N - \sum_{n=1}^N x_n^2\right) + \frac{1}{3}N^2$$
$$= \frac{1}{3}N^2 - \sum_{n=1}^N (2n - 1)x_n + N\sum_{n=1}^N x_n^2 \quad \text{(by Lemma 1)}.$$

Hence,

$$(D_N^{(2)})^2 = \frac{1}{3} + \frac{1}{N}\sum_{n=1}^N \left(x_n^2 - \frac{2n - 1}{N} x_n\right) = \frac{1}{3} + \frac{1}{N}\sum_{n=1}^N (x_n^2 - 2s_n x_n)$$
$$= \frac{1}{3} + \frac{1}{N}\sum_{n=1}^N (x_n - s_n)^2 - \frac{1}{N}\sum_{n=1}^N s_n^2.$$

Since

$$\sum_{n=1}^N s_n^2 = \sum_{n=1}^N \left(\frac{2n - 1}{2N}\right)^2 = \frac{1}{4N^2}\sum_{n=1}^N (4n^2 - 4n + 1) = \frac{1}{4N^2}\left(\frac{4N^3}{3} - \frac{N}{3}\right),$$

we have

$$(D_N^{(2)})^2 = \frac{1}{N}\sum_{n=1}^N (x_n - s_n)^2 + \frac{1}{12N^2},$$

and this proves (2).

Corollary 1: $D_N^{(2)} \geqslant \dfrac{1}{2N\sqrt{3}}$; $D_N^{(2)} = \dfrac{1}{2N\sqrt{3}}$ iff $x_n = s_n$ $(n = 1, 2, \ldots, N)$.

Corollary 2: $D_N^{(2)} \leqslant \dfrac{1}{\sqrt{3}}$ if $x_n \leqslant \dfrac{2n - 1}{N}$ $(n = 1, 2, \ldots, N)$.

Proof: $(D_N^{(2)})^2 = \frac{1}{3} + \frac{1}{N}\sum_{n=1}^{N} x_n\left(x_n - \frac{2n-1}{N}\right)$; hence,

$(D_N^{(2)}) \leqslant \frac{1}{3}$ if $x_n \leqslant \frac{2n-1}{N}$ $(n = 1, 2, \ldots, N)$.

3. THE CASE $p = 4$

We shall use the following lemma.

Lemma 2: Let $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant 1$. Then,

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max^2(x_n, x_m, x_\ell) = \sum_{n=1}^{N}(3n^2 - 3n + 1)x_n^2.$$

Proof: $\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max^2(x_n, x_m, x_\ell) = \sum_{n=1}^{N}\sum_{m=1}^{N}(2m-1)\max^2(x_n, x_m)$

$$= 2\sum_{n=1}^{N}\sum_{m=1}^{N} m \max^2(x_n, m) - \sum_{n=1}^{N}\sum_{m=1}^{N}\max^2(x_n, x_m).$$

Now, $\sum_{n=1}^{N}\sum_{m=1}^{N}\max^2(x_n, x_m) = \sum_{n=1}^{N}(2n-1)x_n^2$ (by Lemma 1).

$$\sum_{n=1}^{N}\sum_{m=1}^{N} m \max^2(x_n, x_m) = \sum_{n=1}^{N}[2\{1 + 2 + \cdots + (n-1)\} + n]x_n^2$$

$$= \frac{1}{2}\sum_{n=1}^{N} n(3n-1)x_n^2.$$

Hence, we have

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max^2(x_n, x_m, x_\ell) = \sum_{n=1}^{N}(3n^2 - 3n + 1)x_n^2.$$

Corollary of Lemma 2:

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max(x_n, x_m, x_\ell) = \sum_{n=1}^{N}(3n^2 - 3n + 1)x_n.$$

Lemma 3: Let $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant 1$. Then,

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\sum_{u=1}^{N}\max(x_n, x_m, x_\ell, x_u) = \sum_{n=1}^{N}(4n^3 - 6n^2 + 4n - 1)x_n.$$

Proof: First we consider

$$\sum_{n=1}^{N}\sum_{m=1}^{N} m^2 \max(x_n, x_m) = \sum_{n=1}^{N}\left(\sum_{m=1}^{N} m^2 \max(x_n, x_m)\right).$$

Keeping n fixed, we have to take the pairs $(x_1, x_n), \ldots, (x_n, x_n), (x_n, x_1), \ldots, (x_n, x_{n-1})$ into consideration; the maximums of the first set must be multiplied by n^2, those of the right set by $1^2, 2^2, \ldots, (n-1)^2$, respectively. Hence,

$$\sum_{n=1}^{N}\left(\sum_{m=1}^{N} m^2 \max(x_n, x_m)\right) = \sum_{n=1}^{N}\left\{\frac{1}{6}(n-1)n(2n-1) + n^3\right\}x_n.$$

So we have

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\sum_{u=1}^{N}\max(x_n, x_m, x_\ell, x_u)$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N}(3m^2 - 3m + 1)\max(x_n, x_m) \quad \text{(by the Corollary of Lemma 2)}$$

$$= 3\sum_{n=1}^{N}\sum_{m=1}^{N} m^2 \max(x_n, x_m) - 3\sum_{n=1}^{N}\sum_{m=1}^{N} m \max(x_n, x_m) + \sum_{n=1}^{N}\sum_{m=1}^{N}\max(x_n, x_m)$$

$$= 3\sum_{n=1}^{N}\left\{\frac{1}{6}n(n-1)(2n-1) + n^3\right\}x_n - 3\sum_{n=1}^{N}\frac{1}{2}n(3n-1)x_n + \sum_{n=1}^{N}(2n-1)x_n$$

$$= \sum_{n=1}^{N}(4n^3 - 6n^2 + 4n - 1)x_n.$$

Lemma 4: $1^3 + 2^3 + \cdots + N^3 = \frac{1}{4}N^2(N+1)^2,$

$$1^4 + 2^4 + \cdots + N^4 = \frac{1}{30}N(N+1)(2N+1)(3N^2 + 3N - 1).$$

Theorem: Let $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant 1$. Then,

$$(D_N^{(4)})^4 = \frac{1}{N}\sum_{n=1}^{N}\left\{(x_n - s_n)^4 + \frac{1}{2N^2}(x_n - s_n)^2\right\} + \frac{1}{80N^4}, \text{ where } s_n = \frac{2n-1}{2N}.$$

Proof: First, we have

$$-4N^3\sum_{n=1}^{N}\int_0^1 c(t, x_n)t^3\,dt = -4N^3\int_0^1\sum_{n=1}^{N}c(t, x_n)t^3\,dt$$

$$= -4N^3\sum_{n=1}^{N}\int_{x_n}^1 t^3\,dt = -N^3\sum_{n=1}^{N}(1 - x_n^4)$$

$$= -N^4 + N^3\sum_{n=1}^{N}x_n^4.$$

Second, we have

$$6N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\int_0^1 t^2 c(t, \max(x_n, x_m))dt = 6N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{1}{3}t^3\Big]_{\max(x_n, x_m)}^1$$

$$= 6N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\left(\frac{1}{3} - \frac{1}{3}(\max(x_n, x_m))^3\right) = 6N^2\left(\frac{N^2}{3} - \frac{1}{3}\sum_{n=1}^{N}\sum_{m=1}^{N}(\max(x_n, x_m))^3\right)$$

$$= 2N^4 - 2N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\max{}^3(x_n, x_m).$$

Hence,

$$(ND_N^{(4)})^4 = \int_0^1\left\{\sum_{n=1}^{N}c(t, x_n) - Nt\right\}^4 dt$$

(continued)

$$= \int_0^1 \left(\sum_{n=1}^{N} c(t,\ x_n) \right)^4 dt \ - \ 4N\int_0^1 t\left(\sum_{n=1}^{N} c(t,\ x_n) \right)^3 dt \ - \ 4N^3\int_0^1 t^3 \sum_{n=1}^{N} c(t,\ x_n) dt$$

$$+ \ 6N^2\int_0^1 t^2 \left(\sum_{n=1}^{N} c(t,\ x_n) \right)^2 dt \ + \ N^4\int_0^1 t^4\, dt$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\sum_{u=1}^{N}\int_0^1 c(t,\ \max(x_n,\ x_m,\ x_\ell,\ x_u))dt \ - \ 4N^3\sum_{n=1}^{N}\int_0^1 c(t,\ x_n)t^3\, dt$$

$$+ \ 6N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\int_0^1 c(t,\ \max(x_n,\ x_m))t^2\, dt$$

$$- \ 4N\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\int_0^1 tc(t,\ \max(x_n,\ x_m,\ x_\ell))dt + \frac{1}{5}N^4$$

$$= \frac{1}{5}N^4 \ - \ N^4 \ + \ N^3\sum_{n=1}^{N} x_n^4 \ + \ 2N^4 \ - \ 2N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\max{}^3(x_n,\ x_m)$$

$$- \ 2N^4 \ + \ 2N\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max{}^2(x_n,\ x_m,\ x_\ell)$$

$$+ \ N^4 \ - \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\sum_{u=1}^{N}\max(x_n,\ x_m,\ x_\ell,\ x_u)$$

$$= \frac{1}{5}N^4 \ + \ N^3\sum_{n=1}^{N} x_n^4 \ - \ 2N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\max{}^3(x_n,\ x_m)$$

$$+ \ 2N\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max{}^2(x_n,\ x_m,\ x_\ell)$$

$$- \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\sum_{u=1}^{N}\max(x_n,\ x_m,\ x_\ell,\ x_u)$$

$$= \frac{1}{5}N^4 \ + \ N^3\sum_{n=1}^{N} x_n^4 \ - \ 2N^2\sum_{n=1}^{N}(2n\ -\ 1)x_n^3$$

$$+ \ 2N\sum_{n=1}^{N}(3n^2\ -\ 3n\ +\ 1)x_n^2 \ - \ \sum_{n=1}^{N}(4n^3\ -\ 6n^2\ +\ 4n\ -\ 1)x_n.$$

Hence,

$$(D_N^{(4)})^4 \ = \ \frac{1}{5} + \frac{1}{N}\sum_{n=1}^{N}\left(x_n^4 \ - \ \frac{2(2n\ -\ 1)}{N}x_n^3 \ + \ \frac{2(3n^2\ -\ 3n\ +\ 1)}{N^2}x_n^2 \right.$$

$$\left. - \ \frac{4n^3\ -\ 6n^2\ +\ 4n\ -\ 1}{N^3}x_n \right)$$

$$= \frac{1}{5} + \frac{1}{N}\sum_{n=1}^{N}\left((x_n\ -\ s_n)^4 \ + \ \frac{1}{2N^2}(x_n\ -\ s_n)^2 \right) - \frac{1}{N}\sum_{n=1}^{N}\left(s_n^4 \ + \ \frac{s_n^2}{2N^2} \right).$$

Now,
$$\sum_{n=1}^{N}\left(s_n^4 \ + \ \frac{s_n^2}{2N^2} \right) = \sum_{n=1}^{N}\left(\left(\frac{2n\ -\ 1}{2N} \right)^4 \ + \ \frac{(2n\ -\ 1)^2}{8N^4} \right)$$

$$= \frac{1}{16N^4}\sum_{n=1}^{N}(16n^4\ -\ 32n^3\ +\ 32n^2\ -\ 16n\ +\ 3)$$

$$= \frac{1}{16N^4}\left(16\sum_{n=1}^{N} n^4 \ - \ 32\sum_{n=1}^{N} n^3 \ + \ 32\sum_{n=1}^{N} n^2 \ - \ 16\sum_{n=1}^{N} n \ + \ \sum_{n=1}^{N} 3 \right)$$

$$= \frac{1}{16N}\left(\frac{16}{5} N^5 - \frac{1}{5} N\right).$$

Finally,

$$(D_N^{(4)})^4 = \frac{1}{5} + \frac{1}{N}\sum_{n=1}^{N}\left((x_n - s_n)^4 + \frac{1}{2N^2}(x_n - s_n)^2\right) - \frac{1}{N}\cdot\frac{1}{16N^4}\left(\frac{16N^5}{5} - \frac{N}{5}\right)$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left((x_n - s_n)^4 + \frac{1}{2N^2}(x_n - s_n)^2\right) + \frac{1}{80N^4}.$$

Corollary 1: $D_N^{(4)} \geqslant \dfrac{1}{2N\sqrt[4]{5}}$; $D_N^{(4)} = \dfrac{1}{2N\sqrt[4]{5}}$ iff $x_n = s_n$ $(n = 1, 2, \ldots, N)$.

Corollary 2: $D_N^{(4)} \leqslant \dfrac{1}{\sqrt[4]{5}}$ if $x_n \leqslant \dfrac{2n-1}{N}$.

Proof: $$(D_4^{(4)})^4 = \frac{1}{5} + \frac{1}{N}\sum_{n=1}^{N}\left\{x_n\left(x_n - \frac{2n-1}{N}\right)\left(x_n^2 - \frac{2n-1}{N}x_n + \frac{2n^2 - 2n + 1}{N^2}\right)\right\}.$$

Now, $$x_n - \frac{2n-1}{N}x_n + \frac{2n^2-2n+1}{N^2} = \left(x_n - \frac{2n-1}{2N}\right)^2 + \frac{4n^2 - 4n + 3}{4N^2} > 0.$$

Hence, $D_4^{(4)} \leqslant \dfrac{1}{\sqrt[4]{5}}$.

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A PARTIAL ASYMPTOTIC FORMULA FOR THE NIVEN NUMBERS

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(Submitted April 1986)

A Niven number is a positive integer that is divisible by its digital sum. That is, if n is an integer and $s(n)$ denotes the digital sum of n, then n is a Niven number if and only if $s(n)$ is a factor of n. This idea was introduced in [1] and investigated further in [2], [3], and [4].

One of the questions about the set N of Niven numbers was the status of

$$\lim_{x\to\infty}\frac{N(x)}{x},$$

where $N(x)$ denotes the number of Niven numbers less than x. This limit, if it exists, is called the "natural density" of N.

It was proven in [3] that the natural density of the set of Niven numbers is zero, and in [4] a search for an asymptotic formula for $N(x)$ was undertaken. That is, does there exist a function $f(x)$ such that

$$\lim_{x\to\infty}\frac{N(x)}{f(x)} = 1?$$

If such an $f(x)$ exists, then this would be indicated by the notation

$$N(x) \sim f(x).$$

Let k be a positive integer. Then k may be written in the form

$$k = 2^a 5^b t,$$

where $(t, 10) = 1$. In [4] the following notation was used.

$$\begin{aligned} N_k &= \text{The set of Niven numbers with digital sum } k.\\ \overline{e}(k) &= \text{The maximum of } a \text{ and } b. \\ e(k) &= \text{The order of 10 mod } t.\end{aligned} \tag{1}$$

With this notation, it was then proven [4; Corollary 4.1] that

$$N_k(x) \sim c(\log x)^k, \tag{2}$$

where c depends on k.

Thus, a partial answer concerning an asymptotic formula for $N(x)$ was found in [4]. Exact values of the constant c can be calculated for a given k. But, as noted in [4], this would involve an investigation of the partitions of k and solutions to certain Diophantine congruences. In what follows, we give the exact value of the constant c for a given integer k.

Let k be a positive integer such that $(k, 10) = 1$. We define the sets S and $\overline{S}$ as

$$S = \left\{\langle x_i \rangle : \sum_{i=1}^{e(k)} x_i = k\right\},$$

and

$$\overline{S} = \left\{\langle x_i \rangle : \sum_{i=1}^{e(k)} x_i = k \text{ and } \sum_{i=1}^{e(k)} 10^{i-1}x_i \equiv 0 \pmod{k}\right\},$$

where $\langle x_i \rangle$ is an $e(k)$-tuple of nonnegative integers. Since $(k, 10) = 1$, it follows that, for a positive integer n,

$$N_k(10^{e(k)n}) = \sum_{\langle x_i \rangle \in \overline{S}} \prod_{i=1}^{e(k)} \binom{n}{x_i}_{10},$$

where $\binom{n}{t}_{10}$ denotes the t^{th} coefficient in the expansion of

$$G(x) = (1 + x + x^2 + \cdots + x^9)^n.$$

That is,

$$\frac{G^{(t)}(0)}{t!} = \binom{n}{t}_{10}, \tag{4}$$

where $G^{(t)}(0)$ is the t^{th} derivative of $G(x)$ at $x = 0$.

The expression given in (3) can be realized by noting that, for each

$$\langle x_i \rangle \in \overline{S},$$

the product

$$\prod_{i=1}^{e(k)} \binom{n}{x_i}_{10}$$

is the number of Niven numbers y less than $10^{e(k)n}$ with decimal representation

$$y = \sum_{j=1}^{ne(k)} y_j 10^{j-1}$$

such that

$$x_i = \sum_{j \equiv i \pmod{e(k)}} y_j.$$

Noting that $G^{(t)}(0) \sim n^t$, and using (4), we have that

$$\binom{n}{t}_{10} \sim \frac{n^t}{t!}.$$

Hence, for a positive k such that $(k, 10) = 1$, it follows from (3) that

$$N_k(10^{ne(k)}) \sim n^k \sum_{\langle x_i \rangle \in \overline{S}} \frac{1}{x_1!x_2! \ldots x_{e(k)}!}.$$

Therefore,

$$N_k(10^{ne(k)}) \sim \frac{n^k}{k!} \sum_{\langle x_i \rangle \in \overline{S}} \frac{k!}{x_1!x_2! \ldots x_{e(k)}!},$$

which may be rewritten in terms of multinomial coefficients as:

$$N_k(10^{ne(k)}) \sim \frac{n^k}{k!} \sum_{\langle x_i \rangle \in \overline{S}} \binom{k}{x_1,\ x_2,\ \ldots,\ x_{e(k)}}. \tag{5}$$

Let w be the k^{th} root of unity $\exp(2\pi i/k)$, and consider the sum

$$\sum_{g=0}^{k-1} f(w^g),$$

where f is the function given by

$$f(u) = (u + u^{10} + u^{10^2} + \cdots + u^{10^{e(k)-1}})^k. \tag{6}$$

Then

$$\begin{aligned} \sum_{g=0}^{k-1} f(w^g) &= \sum_{g=0}^{k-1} \left(\sum_{i=0}^{e(k)-1} (w^g)^{10^i} \right)^k \\ &= \sum_{g=0}^{k-1} \sum_{\langle x_i \rangle \in S} \binom{k}{x_1,\ \ldots,\ x_{e(k)}} (w^g)^{x_1 + 10x_2 + \cdots + 10^{e(k)-1}x_{e(k)}} \end{aligned} \tag{7}$$

In order to make the notation more compact, we will let

$$W(g,\ \langle x_i \rangle) = (w^g)^{x_1 + 10x_2 + \cdots + 10^{e(k)-1}x_{e(k)}}$$

Thus, after interchanging the order of summation, (7) becomes:

$$\begin{aligned} &\sum_{\langle x_i \rangle \in S} \sum_{g=0}^{k-1} \binom{k}{x_1,\ \ldots,\ x_{e(k)}} W(g,\ \langle x_i \rangle) \\ &= \sum_{\langle x_i \rangle \in \overline{S}} \sum_{g=0}^{k-1} \binom{k}{x_1,\ \ldots,\ x_{e(k)}} W(g,\ \langle x_i \rangle) \\ &\quad + \sum_{\langle x_i \rangle \in S - \overline{S}} \sum_{g=0}^{k-1} \binom{k}{x_1,\ \ldots,\ x_{e(k)}} W(g,\ \langle x_i \rangle) \\ &= \sum_{\langle x_i \rangle \in \overline{S}} \binom{k}{x_1,\ \ldots,\ x_{e(k)}} \sum_{g=0}^{k-1} W(g,\ \langle x_i \rangle) \\ &\quad + \sum_{\langle x_i \rangle \in S - \overline{S}} \binom{k}{x_1,\ \ldots,\ x_{e(k)}} \sum_{g=0}^{k-1} W(g,\ \langle x_i \rangle). \end{aligned}$$

But noting that $W(g,\ \langle x_i \rangle)$ is equal to 1 when $\langle x_i \rangle \in \overline{S}$ and $\sum_{g=0}^{k-1} W(g,\ \langle x_i \rangle) = 0$ when $\langle x_i \rangle \in S - \overline{S}$, we conclude that

$$\sum_{g=0}^{k-1} f(w^g) = k \sum_{\langle x_i \rangle \in \overline{S}} \binom{k}{x_1,\ \ldots,\ x_{e(k)}}.$$

Hence, from (5), the following theorem is immediate.

Theorem 1: For any positive integer k, relatively prime to 10, let f, w, and $e(k)$ be given as above. Then

$$N_k(10^{ne(k)}) \sim \frac{n^k}{k!k} \sum_{g=0}^{k-1} f(w^g),$$

where n is any positive integer.

Some specific examples using Theorem 1 are:

$$N_3(10^n) \sim \frac{n^3}{6},$$

$$N_7(10^{6n}) \sim \frac{n^7}{7!7}(6^7 - 6),$$

$$N_{49}(10^{42n}) \sim \frac{n^{49}}{49!49}(42^{49} - 6(7^{49})),$$

and

$$N_{31}(10^{15n}) \sim \frac{n^{31}}{31!31}\left[15^{31} + 15\left(\left(\frac{-1 + (31)^{1/2}i}{2}\right)^{31} + \left(\frac{-1 - (31)^{1/2}i}{2}\right)^{31}\right)\right],$$

where $e(k) = 1, 6, 42$, and 15 when $k = 3, 7, 49$, and 31, respectively. Note that i denotes the square root of -1 in the last formula.

It is perhaps clear that the determination of such asymptotic formulas involves sums of complex expressions dependent on the orbit of 10 modulo k, and might be difficult to generalize.

Finally, we can use the above development as a model to generalize to the case where k is any positive integer, not necessarily relatively prime to 10. Recalling (1), we see that, if $(k, 10) \neq 1$, then it follows that $\bar{e}(k) \neq 0$. So $\bar{S}$ would be replaced by

$$\bar{S} = \left\{\langle x_i;\, y_i\rangle : \sum_{i=1}^{e(k)} x_i + \sum_{i=1}^{\bar{e}(k)} y_i = k \right.$$

and

$$\left. \sum_{i=1}^{e(k)} x_i 10^{i+\bar{e}(k)-1} + \sum_{i=1}^{\bar{e}(k)} y_i 10^{i-1} \equiv 0 \pmod{k}\right\},$$

where y_i is a decimal digit for each i and where $\langle x_i;\, y_i\rangle$ is the $(e(k)+\bar{e}(k))$-tuple

$$(x_1, x_2, \ldots, x_{e(k)}, y_1, \ldots, y_{\bar{e}(k)}).$$

Thus, similarly to (3), it follows that

$$N_k(10^{ne(k)+\bar{e}(k)}) = \sum_{\langle x_i;\, y_i\rangle \in \bar{S}} \prod_{i=1}^{e(k)} \binom{n}{x_i}_{10} \prod_{i=1}^{\bar{e}(k)} \binom{1}{y_i}_{10}. \tag{8}$$

But $\binom{1}{y_i}_{10} = 1$ for each $1 \leqslant i \leqslant \bar{e}(k)$, so (8) may be rewritten as

$$N_k(10^{ne(k)+\bar{e}(k)}) = \sum_{\langle x_i;\, y_i\rangle \in \bar{S}} \prod_{i=1}^{e(k)} \binom{n}{x_i}_{10}.$$

Therefore,

$$N_k(10^{ne(k)+\bar{e}(k)}) \sim \sum_{\langle x_i;\, 0\rangle \in \bar{S}} \prod_{i=1}^{e(k)} \binom{n}{x_i}_{10},$$

and replacing f as given in (6) by

$$f(u) = (u^{\bar{e}(k)} + \cdots + u^{\bar{e}(k)+e(k)-1})^k,$$

we are able to state the following theorem.

Theorem 2: For any positive integer k, let f, w, $e(k)$, and $\overline{e}(k)$ be given as above. Then

$$N_k(10^{ne(k)+\overline{e}(k)}) \sim \frac{n^k}{k!k}\sum_{g=0}^{k-1} f(w^g),$$

where n is any positive integer.

If $e(k) = 1$, the following corollary is also immediate since $f(w^g) = 1$ for each $0 \leqslant g \leqslant k - 1$.

Corollary: If k is a positive integer such that $e(k) = 1$, then, for any positive integer n,

$$N_k(10^{n+\overline{e}(k)}) \sim \frac{n^k}{k!}.$$

Using Theorem 2, we can determine an asymptotic formula for $N_k(x)$ for any positive real number x. This follows since there exists an integer n such that

$$10^{ne(k)+\overline{e}(k)} \leqslant x < 10^{(n+1)e(k)+\overline{e}(k)}. \tag{9}$$

But, by Theorem 2, we have that

$$N_k(10^{ne(k)+\overline{e}(k)}) \sim N_k(10^{(n+1)e(k)+\overline{e}(k)})$$

since $n_k \sim (n + 1)^k$. Hence,

$$N_k(x) \sim \frac{n^k}{k!k}\sum_{g=0}^{k-1} f(w^g),$$

and because (9) implies that

$$n \sim \frac{[\log x] - \overline{e}(k)}{e(k)} \sim \frac{\log x}{e(k)},$$

we have, in conclusion, Theorem 3.

Theorem 3: For any positive real number x and any positive integer k, let f, w, and $e(k)$ be given as above. Then

$$N_k(x) \sim \frac{(\log x)^k}{k!k(e(k))^k}\sum_{g=0}^{k-1} f(w^g).$$

Thus, an explicit formula for the constant c referred to in (2) has been given. The determination of an asymptotic formula for $N(x)$, however, is left as an open problem.

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ON CERTAIN DIVISIBILITY SEQUENCES

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In [1], Marshall Hall defined U_n to be a *divisibility sequence* if $U_m | U_n$ whenever $m|n$. If we let $U_n = A^n(c_0 + c_1 n)$ for integers A, c_0, and c_1, then a corollary to the theorem in [2] is that U_n is a divisibility sequence if and only if exactly one of the coefficients c_0 or c_1 equals 0. The purpose of this paper is to establish a similar result for $U_n = A^n(c_0 + c_1 n + c_2 n^2)$.

Theorem: Let $U_n = A^n(c_0 + c_1 n + c_2 n^2)$ for integers A, c_0, c_1, and c_2. U_n is a divisibility sequence if and only if exactly two of the coefficients c_0, c_1, and c_2 are 0.

Proof: It is easy to see that, if exactly two of the coefficients c_0, c_1, and c_2 are 0, then U_n is a divisibility sequence. Consequently, in what follows, we assume that $A^m(c_0 + c_1 m + c_2 m^2) | A^n(c_0 + c_1 n + c_2 n^2)$ if $m|n$, and, without loss of generality, that $A > 0$.

Case 1: $c_0 = 0$

Assume $c_1 \neq 0$, for, otherwise, we have $c_0 = c_1 = 0$ and $c_2 m^2 A | c_2 n^2 A$ if $m|n$, and we are finished. Replace m by $c_1 mA$, n by $c_1 nA$, and let $e = c_1 A(n - m)$. Then we have $(c_1^2 mA + c_2 c_1^2 m^2 A^2) | A^e(c_1^2 nA + c_2 c_1^2 n^2 A^2)$ if $m|n$. Consequently,

$$(m + c_2 m^2 A) | A^e(n + c_2 n^2 A) \text{ if } m|n.$$

In particular,

$$(1 + c_2 A) | A^e(n + c_2 n^2 A).$$

If $e \leqslant 0$, then $(1 + c_2 A) | (n + c_2 n^2 A)$ is immediate, while if $e > 0$, since

$$\gcd(1 + c_2 A, A^e) = 1,$$

we also have $(1 + c_2 A) | (n + c_2 n^2 A)$.

Set $n = 2$. $(1 + c_2 A) | (2 + 4c_2 A)$. Since $2 + 4c_2 A = 2(1 + c_2 A) + 2c_2 A$, we have $(1 + c_2 A) | 2c_2 A$, which implies that $(1 + c_2 A) | 2$; hence, $1 + c_2 A = \pm 1$ or ± 2.

$1 + c_2 A = 1 \Rightarrow c_2 = 0$, and we are finished.

$1 + c_2 A = -1 \Rightarrow (m - 2m^2) | (n - 2n^2)$ if $m|n$ and m is odd, which is false for $m = 3$, $n = 6$.

$1 + c_2A = 2 \Rightarrow c_2A = 1 \Rightarrow (m + m^2) | (n + n^2)$ if $m|n$, which is false for $m = 2$, $n = 4$.

$1 + c_2A = -2 \Rightarrow A = 1$ or $A = 3 \Rightarrow (m - 3m^2) | A^e(n - 3n^2)$ if $m|n$, which is false for $m = 5$, $n = 10$.

Case 2: $c_0 \neq 0$

Replace m by c_0mA, n by c_0nA, and let $e = c_0A(n - m)$. This gives

$$(c_0 + c_0c_1mA + c_2c_0^2m^2A^2) | A^e(c_0 + c_0c_1nA + c_2c_0^2n^2A^2),$$

which implies that

$$(1 + c_1mA + c_2c_0m^2A^2) | A^e(1 + c_1nA + c_2c_0n^2A^2).$$

As in Case 1, this leads to

$$(1 + c_1mA + c_2c_0m^2A^2) | (1 + c_1nA + c_2c_0n^2A^2) \text{ if } m|n.$$

Select $m = 1$, $n = 1 + c_1A + c_2c_0A^2$. Then $(1 + c_1A + c_2c_0A^2) | 1$, i.e.,

$$1 + c_1A + c_2c_0A^2 = \pm 1.$$

Case a: $1 + c_1A + c_2c_0A^2 = 1$

$1 + c_1A + c_2c_0A^2 = 1 \Rightarrow A(c_1 + c_2c_0A) = 0 \Rightarrow c_2c_0A = -c_1$. Thus,

$$(1 + c_1mA - c_1m^2A) | (1 + c_1nA - c_1n^2A) \text{ if } m|n.$$

Set $n = 2m$. $(1 + c_1mA - c_1m^2A) | (1 + 2c_1mA - 4c_1m^2A)$ if $m|n$, or

$$(1 + c_1mA - c_1m^2A) | (1 + c_1mA - c_1m^2A + (c_1mA - 3c_1m^2A)).$$

Hence,

$$(1 + c_1mA - c_1m^2A) | 2(c_1mA - 3c_1m^2A). \tag{1}$$

Set $n = 3m$. In a similar manner to the above, we get

$$(1 + c_1mA - c_1m^2A) | (2c_1mA - 8c_1m^2A). \tag{2}$$

Together, (1) and (2) imply that $(1 + c_1mA - c_1m^2A) | (2c_1m^2A)$.

Set $m = 2$. We obtain $(1 - 2c_1A) | 8c_1A$. But $8c_1A = 4 - 4(1 - 2c_1A)$, so that $(1 - 2c_1A) | 4$, i.e., $1 - 2c_1A = \pm 1$.

$1 - 2c_1A = 1 \Rightarrow c_1 = 0$. Since $c_2c_0A = -c_1$, either $c_0 = 0$ or $c_2 = 0$, and we are finished.

$1 - 2c_1A = -1 \Rightarrow c_2A = 1 \Rightarrow (1 + m - m^2) | (1 + n - n^2)$ if $m|n$, which is false for $m = 3$, $n = 6$.

Case b: $1 + c_1A + c_2c_0A^2 = -1$

$1 + c_1A + c_2c_0A^2 = -1 \Rightarrow A(c_1 + c_2c_0A) = -2 \Rightarrow A = 1$ or 2.

Case i: $A = 1$, $c_1 + c_2c_0 = -2$

If $A = 1$, then

$$(1 + c_1m + c_2c_0m^2) \mid (1 + c_1n + c_2c_0n^2) \text{ if } m \mid n.$$

Let $m = 2$ and replace n by $2n$. Then

$$(1 + 2c_1 + 4c_2c_0) \mid (1 + 2c_1n + 4c_2c_0n^2).$$

Since $c_1 + c_2c_0 = -2$, we have

$$(2c_2c_0 - 3) \mid (1 + 2c_1n + 4c_2c_0n^2).$$

Let $n = 2c_2c_0 - 3$. Then $(2c_2c_0 - 3) \mid 1 \Rightarrow 2c_2c_0 - 3 = \pm 1$.

$2c_2c_0 - 3 = 1 \Rightarrow c_2c_0 = 4 \Rightarrow c_1 = -4 \Rightarrow (1 - 4m + 2m^2) \mid (1 - 4n + 2n^2)$ if $m \mid n$, which is false for $m = 4$, $n = 8$.

$2c_2c_0 - 3 = -1 \Rightarrow c_2c_0 = 1 \Rightarrow c_1 = -3 \Rightarrow (1 - 3m + m^2) \mid (1 - 3n + n^2)$ if $m \mid n$, which is false for $m = 4$, $n = 8$.

Case ii: $A = 2$, $c_1 + 2c_2c_0 = -1$

If $A = 2$, then

$$(1 + 2c_1m + 4c_2c_0m^2) \mid (1 + 2c_1n + 4c_2c_0n^2) \text{ if } m \mid n.$$

Let $m = 2$, and replace n by $2n$. Consequently,

$$(1 + 4c_1 + 16c_2c_0) \mid (1 + 4c_1n + 16c_2c_0n^2).$$

Since $c_1 + 2c_2c_0 = -1$, we have

$$(8c_2c_0 - 3) \mid (1 + 4c_1n + 16c_2c_0n^2).$$

Let $n = 8c_2c_0 - 3$. Then $(8c_2c_0 - 3) \mid 1$, which is impossible.

Remark: It is reasonable to conjecture that

$$U_n = A^n \sum_{i=0}^{k} c_i n^i$$

is a divisibility sequence if and only if exactly k of the c_i's are 0. It appears that this general case cannot be proved using the methods in this paper.

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INTEGRAL 4 BY 4 SKEW CIRCULANTS*

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(Submitted June 1986)

1. INTRODUCTION

A 4 by 4 skew circulant matrix is a matrix of the form

$$\begin{bmatrix} a & b & c & d \\ -d & a & b & c \\ -c & -d & a & b \\ -b & -c & -d & a \end{bmatrix}$$

and the determinant of such a matrix is called a "skew circulant." A pleasant article by I. J. Good [2] devoted to skew circulants contains, in particular, a study of the values such a determinant could take for integer entries a, b, c, and d. The numerical evidence led him to two conjectures:

Conjecture I. An odd prime p occurs as a value if and only if $p \equiv 1 \pmod 8$.

Conjecture II. A positive integer in general occurs as a value if and only if it is a power of 2 times a square times primes $\equiv 1 \pmod 8$.

In this note I shall prove that both conjectures are correct. This is not altogether a new result, for (as Good later pointed out in [3]) there is work on the topic going back to Jacobi; as we shall note at the end of the paper, much more general results have been obtained using advanced methods of algebraic number theory. But it is possible to prove the two conjectures by elementary means, using hardly anything beyond the material available (for instance) in Hardy and Wright [4].

2. REFORMULATION IN TERMS OF ROOTS OF UNITY

Following Good's paper, we begin by reformulating the question in terms of roots of unity. The point is that the particular matrix J with $a = c = d = 0$ and $b = 1$ generates the skew circulant matrices, in the sense that an arbitrary

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one can be expressed as

$$aI + bJ + cJ^2 + dJ^3 \text{ (with } J^4 = -I).$$

Thus, if $j = \exp(\pi i/4) = (1 + i)/\sqrt{2}$ is a primitive 8^{th} root of unity, then the map sending J to j induces an isomorphism (bijection preserving both sums and products) from the family of integral skew circulant matrices to the subring A of the complex numbers consisting of integral combinations of powers of j. The same would be true if we sent J to any one of the other primitive 8^{th} roots of unity, which are j^3, j^5, and $j^7 = j^{-1}$. When we deal with elements of A, we call these other values (obtained by replacing j by an appropriate power) the "conjugates" of the original element. Straightforward computation shows that the determinant is simply then the product of the element and its three conjugates, which in rings like this is usually called the "norm." Thus, our question is concerned with possible norms of elements. Worked out as a polynomial in a, b, c, and d, the norm $N(a + bj + cj^2 + dj^3)$ can be written as

$$(a^2 - c^2 + 2bd)^2 + (b^2 - d^2 - 2ac)^2, \text{ or as}$$
$$(a^2 + b^2 + c^2 + d^2)^2 - 2(ad - ab - bc - ac)^2, \text{ or as}$$
$$(a^2 - b^2 + c^2 - d^2)^2 + 2(ad + ab - bc + cd)^2.$$

In particular, of course, the first expression shows that the norm is positive for nonzero elements of A. Furthermore, these three factorizations (arising originally from different ways of grouping the conjugates in the product into pairs) reflect three subrings that will play a role in our analysis:

$$A_1 = \text{combinations of 1 and } j^2 = i,$$
$$A_2 = \text{combinations of 1 and } \sqrt{2} = j + j^7, \text{ and}$$
$$A_3 = \text{combinations of 1 and } i\sqrt{2} = j + j^3.$$

Note, at once, that a conjugate of a product of elements is the corresponding product of conjugates and, hence, the norm of a product is the product of the norms. Also note that $a = b = 1$, $c = d = 0$ gives $N = 2$. Hence, 2 and all its powers occur as norms; and if an odd number q occurs as a norm, so does every product $2^r q$. Thus, our main concern is with possible odd norms.

3. BASIC FACTS ABOUT FACTORIZATION IN A

The basic idea that we need was already suggested by the expression of the norm as a product: it is factorization. The facts involved are available in several texts, such as [4], and I shall state some of them here without proof.

The most important [4, p. 230] is that *unique factorization holds* for our ring A. That is, every element that is not a unit can be written as a product of primes, and this product is unique except for multiplication by units. Here a unit is an element of A that has an inverse in A, and a prime is an element that cannot be factored except by allowing one of the factors to be a unit.

Now, if an element x is a unit, then we have $xy = 1$ for some y in A. It follows that $N(x)N(y) = N(1) = 1$ and, hence, $N(x) = \pm 1$. But the first of the formulas for the norm above shows that norms are nonnegative; thus, any unit in A has norm 1. Conversely, whenever $N(x) = 1$, the product of x by its other conjugates is 1, and, of course, this shows that x has an inverse in A. Thus, we have the following lemma.

Lemma 1: An element of A is a unit if and only if its norm is 1.

The units of A have, in fact, been known at least since the time of Kronecker [5] and are listed in Good's paper [3]: they are powers of j times $(1+\sqrt{2})^r$ for integral r.

Furthermore, since every (nonunit) element in A is a product of prime elements, every norm except 0 and 1 will be a product of norms of prime elements.

Lemma 2: An integer larger than 1 occurs as a norm from A if and only if it is a product of integers that occur as norms of prime elements in A.

We already know that $2 = N(1 + j)$ occurs as a norm. Incidentally, this shows that $1 + j$ is a prime in A; for, if we have a factorization $1 + j = yz$, then

$$2 = N(1 + j) = N(y)N(z),$$

and, hence, either $N(y) = 1$ or $N(z) = 1$. Observe now that every prime element π in A divides an ordinary integer, namely $N(\pi)$. But we can write this positive integer as the product of its ordinary integer prime factors. Since π is prime in A and divides this product, unique factorization shows that π must divide one of the factors. Therefore, we have the following lemma.

Lemma 3: Every prime of A divides some ordinary prime integer.

Thus, we can determine the possible norms if only we can determine enough about how ordinary integer primes factor in A.

4. PROOF OF THE CONJECTURES

The next information we need [4, pp. 212-13] is that the rings A_1, A_2, and A_3 also have unique factorization (though, of course, the elements that are "prime" in them may factor when we allow the larger range of possible factors available in A). Furthermore, we know in detail just how the different odd integer primes p factor in these quadratic fields. (The integer 2 factors as a unit times a square of a prime in each of them, but we do not need that information.) The factorizations of p are essentially equivalent to information on the representability of the prime p by suitable quadratic forms; thus, for instance [4, p. 219], we can factor p nontrivially in A_1 iff it can be written as $(a + bi)(a - bi)$, which happens iff we can express p as $a^2 + b^2$. It is well known that this is possible iff p is congruent to 1 mod 4. Similar statements are true in the other two A_i: either p remains a prime in A_i or it factors into two primes, and the different behaviors depend only on p mod 8. (The result for A_2 is worked out in [4, p. 221], where it is remarked that A_3 can be treated similarly.) In A_2, the primes congruent to 1 or 7 mod 8 can be factored into two prime factors, while those congruent to 3 or 5 remain prime; and in A_3, those congruent to 1 or 3 mod 8 can be factored, while the others remain prime.

Now, first of all, this tells us at once that all squares of odd primes are norms from A. For, if (for instance) we have p congruent to 5 mod 8, then p factors at least as $(a + bi)(a - bi)$. We then have

$$p^4 = N(p) = N(a + bi)N(a - bi).$$

Furthermore, $a + bi$ and $a - bi$ are conjugates. Thus, they both must have the same norm, namely p^2. A simple congruence argument given by Good [2, pp. 55-56] shows that p cannot itself be a norm, and an argument like that after Lemma 2 shows then that $a \pm bi$ here are prime elements in A. Similarly, if p is congruent to 7 mod 8, then it factors as $(a + b\sqrt{2})(a - b\sqrt{2})$, and the factors have norm $= p^2$ and are prime in A; while, if p is congruent to 3 mod 8, then it factors as $(a + bi\sqrt{2})(a - bi\sqrt{2})$, and again the factors have norm $= p^2$ and are prime in A.

Of course, the primes p congruent to 1 mod 8 are the ones that deserve special attention. We know that such a p factors into two factors in each of the rings A , and hence, as before, p^2 occurs as a norm. But the existence of these different factorizations should lead us to suspect that we have not actually found the prime factors of p in A, and that is exactly what is true. We can, e.g., write p as $(a + bi)(a - bi)$; we can also write p as $(c + d\sqrt{2})(c - d\sqrt{2})$. If (say) $c + d\sqrt{2}$ is prime in A, then its conjugate $c - d\sqrt{2}$ is also prime, since the conjugations are isomorphisms. By unique factorization, the two nonunit factors $a \pm bi$ must be units times $c \pm d\sqrt{2}$. But since we know the units in A, this gives

$$a \pm bi = j^k(1+\sqrt{2})^r(c \pm d\sqrt{2}).$$

Thus, $a \pm bi$ would have to be j^k times a real number. Such an equality can occur only when $a = 0$ or $b = 0$ or $a = \pm b$, all of which are impossible when $a^2 + b^2 = p$. Thus, the element $c + d\sqrt{2}$ (of norm p^2) must have nontrivial factors, and they can only have norm p. Hence, we have proved both conjectures.

5. A SUBSIDIARY CONJECTURE

There is one other conjecture made in Good's paper [2], but it is closer to familiar results and we can dispose of it quickly; it is worth noting, however, that unique factorization is again the main idea. We already know that there exists a solution of the equation $p = a^2 - 2b^2$ when p is congruent to 1 or 7 mod 8, and the problem is then to determine all solutions. But one solution corresponds to a factorization $p = (a + b\sqrt{2})(a - b\sqrt{2})$ in A_2, and, hence, unique factorization shows that all other solutions must differ by units; and since we know the units (solutions of Pell's equation!), any other solution α, β must satisfy $\alpha + \beta\sqrt{2} = \pm(1+\sqrt{2})^r(a \pm b\sqrt{2})$. By proper choice of signs for α and β, we can assume that $\alpha + \beta\sqrt{2} = (1+\sqrt{2})^r(a + b\sqrt{2})$. To get the product to come out equal to p rather than $-p$, we must have r even, or, in other terms,

$$\alpha + \beta\sqrt{2} = (3 + 2\sqrt{2})^s(a + b\sqrt{2}).$$

Thus, the solutions are exactly those given by the recurrences in [2, p. 57].

6. GENERALIZATIONS

We have shown that in the ring A generated by 8^{th} roots of units, an odd prime p occurs as a norm iff p is congruent to 1 mod 8; along the way, we were reminded also that an odd prime p occurs as a norm from the ring A_1 generated by 4^{th} roots of unity iff p is congruent to 1 mod 4. The general fact is that essentially the same result holds in general, *but* the statement has to be modified because unique factorization usually fails to be true in the rings generated by higher roots of unity. This was the famous discovery of Kummer that set modern algebraic number theory on its way (cf. Edwards [1]). He introduced certain objects called "ideal prime factors" and he could prove that there was a unique factorization into them. Furthermore, when we take the ring generated over the integers by the n^{th} roots of unity, an odd prime p (relatively prime to n) will be a norm of one of these "ideal" factors iff it is congruent to 1 mod n. But these ideal primes correspond to actual single elements of the ring only when we have unique factorization, which holds in only finitely many cases (which are all known; see [6] or [7, Chap. 11]). In particular, it holds for $n = 16$ and for $n = 32$, but not for any higher powers of 2.

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ON A RESULT INVOLVING ITERATED EXPONENTIATION

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(Submitted July 1986)

In connection with recent work by M. Creutz and myself involving iterated exponentiation [1], [2], [3], e.g., the function

$$f(x) = x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}, \tag{1}$$

with an infinite number of x's, I have noticed an interesting property when only a finite number n of x's is considered.

I will now consider the bracketing a for $n = 4$. This is defined as

$$F_{4,a}(x) \equiv x^{[x^{(x^x)}]} = {}^4x. \tag{2}$$

In a Brookhaven National Laboratory Report [4], I have given a more extensive discussion of the present results (see, in particular, Table 1 of [4]). Obviously, when $x > 2$, the function $F_{4,a}(x)$ has a large numerical value. As an example, we consider

$$F_{4,a}(5) = 5^{[5^{(5^5)}]} = 5^{(5^{3125})}. \tag{3}$$

Now we find

$$5^{3125} \simeq 10^{2184.281} = 1.910 \times 10^{2184}, \tag{4}$$

where

$$2184.281 = 5^5 \log_{10} 5 = (3125)(0.69897). \tag{5}$$

From equations (3)-(5), one obtains

$$F_{4,a}(5) = 5^{(10^{2184.281})} = 5^{1.910 \times 10^{2184}} \tag{6}$$

A seemingly paradoxical result is obtained if we express $F_{4,a}(5)$ as a power of 10. Thus, we find the exponent

$$\log_{10}[5^{(10^{2184.281})}] = 10^{2184.281} \log_{10} 5 = 0.69897 \times 1.910 \times 10^{2184}$$
$$\doteq 1.335 \times 10^{2184} = 10^{2184.125}, \tag{7}$$

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which leads to the result

$$F_{4,a}(5) = 10^{(10^{2184.125})}, \tag{8}$$

showing [by comparison with (6)] that the exponent in the parentheses is hardly changed in going from a power of 5 to a power of 10.

To clarify this result, we consider the equation

$$x^{(10^y)} = 10^{(10^{y'})}, \tag{9}$$

which defines y', where in the present case $x = 5$ and $y = 2184.281$. To derive the relationship between y' and y, we take the logarithms of both sides of (9). This gives

$$10^y \log_{10} x = 10^{y'}, \tag{10}$$

By taking the logarithms of both sides of this equation, we obtain

$$y' = y + \log_{10}\log_{10} x. \tag{11}$$

For the case discussed above, it can be readily verified that $\log_{10}\log_{10} 5 = -0.1555$, leading to the results in (6) and (8), since $0.281 - 0.125 = 0.156$, which is clearly consistent with the value of $\log_{10}\log_{10} 5 = -0.1555$ obtained above. It is of interest that the correction to y, namely $\log_{10}\log_{10} x$, is independent of the value of y.

To make the above results more believable, note that the *ratio* of the two powers of 10 involved in (6) and (7) above is given by

$$R = 10^{2184.281} / 10^{2184.125} = 10^{0.156} = 1.432. \tag{12}$$

Thus, the very large exponent $10^{2184.125}$ is multiplied by 1.432 in going from $x = 10$ to $x = 5$. This is a very considerable increase. As a result, we write

$$5^{1.432 \times 10^{2184.125}} = 10^{10^{2184.125}}, \tag{13}$$

which is essentially correct because $5^{1.432} = 10.02$. (The small apparent discrepancy of 0.02 is due to rounding errors.)

As a final comment, I note that, if I had used $x = 1.1$ (instead of 5.0), with the correction $\log_{10}\log_{10} 1.1 = -1.383$, and $y' = 2184.125 + 1.383 = 2185.508$, I would have obtained

$$10^{10^{2184.125}} = 1.1^{10^{2185.508}}, \tag{14}$$

since $10^{1.383} = 24.15$ and $1.1^{24.15} \simeq 10$.

ACKNOWLEDGMENT

I would like to thank Dr. Michael Creutz for helpful discussions and for a critical reading of the manuscript.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-616 *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, MA*

(a) Find the smallest positive integer a such that

$$L_n \equiv F_{n+a} \pmod 6 \text{ for } n = 0, 1, \ldots .$$

(b) Find the smallest positive integer b such that

$$L_n \equiv F_{5n+b} \pmod 5 \text{ for } n = 0, 1, \ldots .$$

B-617 *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, MA*

Let R be a rectangle each of whose vertices has Fibonacci numbers as its coordinates x and y. Prove that the sides of R must be parallel to the coordinate axes.

B-618 *Proposed by Herta T. Treitag, Roanoke, VA*

Let $S(n) = L_{2n+1} + L_{2n+3} + L_{2n+5} + \cdots + L_{4n-1}$. Prove that $S(n)$ is an integral multiple of 10 for all even positive integers n.

B-619 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $T(n) = F_{2n+1} + F_{2n+3} + F_{2n+5} + \cdots + F_{4n-1}$. For which positive integers n is $T(n)$ an integral multiple of 10?

B-620 *Proposed by Philip L. Mana, Albuquerque, NM*

Prove that $F^n_{24k+3} + F^n_{24k+5} \equiv 2F^n_{24k+6} \pmod 9$ for all n and k in $N = \{0, 1, 2, \ldots\}$.

B-621 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $n = 2h - 1$ with h a positive integer. Also, let $K(n) = F_h L_{h-1}$. Find sufficient conditions on F_n to establish the congruence

$$F_{n+1}^{K(n)} \equiv 1 \pmod{F_n}.$$

SOLUTIONS

No Such Constants

B-592 *Proposed by Herta T. Freitag, Roanoke, VA*

Find all integers a and b, if any, such that $F_a L_b + F_{a-1} L_{b-1}$ is an integral multiple of 5.

Solution by J.-Z. Lee, Chinese Culture University and J.-S. Lee, National Taipei Business College, Taipei, Taiwan, R.O.C.

Since $F_a L_b + F_{a-1} L_{b-1} = L_{a+b-1}$ and $L_n \equiv [2, 1, 3, 4] \pmod 5$, i.e., $L_n \not\equiv 0 \pmod 5$, $F_a L_b + F_{a-1} L_{b-1}$ is not an integral multiple of 5 (for all integers a and b).

Also solved by P. S. Bruckman, F. H. Cunliffe, P. Filipponi, G. Koutsoukellis, L. Kuipers, B. Prielipp, H.-J. Seiffert, S. Singh, L. Somer, G. Wulczyn, and the proposer.

Multiple of 1220

B-593 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A(n) = F_{n+1} L_n + F_n L_{n+1}$. Prove that $A(15n - 8)$ is an integral multiple of 1220 for all positive integers n.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

By Problem B-294 on p. 375 of the December 1975 issue of this journal,

$$F_n L_k + F_k L_n = 2F_{n+k}.$$

Thus, $A(n) = 2F_{2n+1}$, so

$$A(15n - 8) = 2F_{30n-15} = 2F_{15(2n-1)}.$$

Because 15 divides $15(2n - 1)$, $610 = F_{15}$ divides $F_{15(2n-1)}$. Thus, $2(610) = 1220$ divides $A(15n - 8)$.

Also solved by P. S. Bruckman, F. H. Cunliffe, P. Filipponi, G. Koutsoukellis, L. Kuipers, J.-Z. Lee & J.-S. Lee, H.-J. Seiffert, S. Singh, L. Somer, G. Wulczyn, and the proposer.

Congruence Mod 60

B-594 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A(n) = F_{n+1}L_n + F_nL_{n+1}$ and $B(n) = \sum_{j=1}^{n}\sum_{k=1}^{j} A(k)$.

Prove that $B(n) \equiv 0 \pmod{20}$ when $n \equiv 19$ or $29 \pmod{60}$.

Solution by Paul S. Bruckman, Fair Oaks, CA

Using the expression derived in the solution to B-593, we have:

$$\begin{aligned}B(n) &= \sum_{j=1}^{n}\sum_{k=1}^{j} 2F_{2k+1} = 2\sum_{j=1}^{n}\sum_{k=1}^{j}(F_{2k+2} - F_{2k}) = 2\sum_{j=1}^{n}(F_{2j+2} - F_2)\\ &= 2\sum_{j=2}^{n+1}F_{2j} - 2n = 2\sum_{j=2}^{n+1}(F_{2j+1} - F_{2j-1}) - 2n = 2(F_{2n+3} - F_3) - 2n,\end{aligned}$$

or

$$B(n) = 2F_{2n+3} - (2n + 4). \tag{1}$$

Now $(F_n \pmod 4)_{n=1}^{\infty}$ and $(F_n \pmod 5)_{n=1}^{\infty}$ are periodic sequences of periods 6 and 20, respectively. Thus, $(F_n \pmod{20})_{n=1}^{\infty}$ has period equal to L.C.M.(6, 20) = 60, from which it follows that $(F_{2n+3} \pmod{20})_{n=1}^{\infty}$ has period 30, as well as the sequence $(2F_{2n+3} \pmod{20})_{n=1}^{\infty}$. Also, $((2n + 4) \pmod{20})^{\infty}$ has period 10, clearly. Therefore, $(B(n) \pmod{20})_{n=1}^{\infty} \equiv ((2F_{2n+3} - (2n + 4)) \pmod{20})_{n=1}^{\infty}$ has period 30. Inspecting the 30 possible values of this sequence, we find that

$$B(n) \equiv 0 \pmod{20} \text{ iff } n \equiv 0, 19, \text{ or } 29 \pmod{30}.$$

This is a stronger result than sought in the problem.

Also solved by P. Filipponi, L. Kuipers, J.-Z. Lee & J.-S. Lee, B. Prielipp, S. Singh, G. Wulczyn, and the proposer.

Convolution Congruence

B-595 *Proposed by Philip L. Mana, Albuquerque, NM*

Prove that $\sum_{k=0}^{n} k^3(n - k)^2 \equiv \binom{n+4}{6} + \binom{n+1}{6} \pmod 5$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

It is known that

$$\sum_{k=0}^{n} k^3(n - k)^2 = \binom{n+1}{6} + 5\binom{n+2}{6} + 5\binom{n+3}{6} + \binom{n+4}{6}.$$

(See p. 57 of "A Symmetric Substitute for Sterling Numbers" by A. P. Hillman, P. L. Mana, and C. T. McAbee in the February 1971 issue of this journal.) The desired result follows immediately.

Also solved by P. S. Bruckman, P. Filipponi, H. T. Freitag, L. Kuipers, J.-Z. Lee & J.-S. Lee, S. Singh, G. Wulczyn, and the proposer.

X, Y, Z Affair

B-596 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let

$$S(n, k, m) = \sum_{i=1}^{m} F_{ni+k}.$$

For positive integers a, m, and k, find an expression of the form XY/Z for $S(4a, k, m)$, where X, Y, and Z are Fibonacci or Lucas numbers.

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

Let $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$. Using the Binet form for Fibonacci numbers,

$$S(n, k, m) = \frac{1}{\alpha - \beta}\left[\sum_{i=1}^{m} \alpha^{ni+k} - \sum_{i=1}^{m} \beta^{ni+k}\right]$$

$$= \frac{F_{(m+1)n+k} - F_{n+k} - (\alpha\beta)^n\{F_{mn+k} - F_k\}}{L_n - 1 - (\alpha\beta)^n}.$$

Thus,

$$S(4a, k, m) = \frac{F_{4a(m+1)+k} - F_{4a+k} - \{F_{4am+k} - F_k\}}{L_{4a} - 2}$$

$$= \frac{F_{2am}L_{2am+4a+k} - F_{2am}L_{2am+k}}{5F_{2a}^2}$$ by I_{16} and I_{24} of Hoggatt's *Fibonacci and Lucas Numbers*

$$= \frac{F_{2am}(5F_{2a} \cdot F_{2am+2a+k})}{5F_{2a}^2} = \frac{F_{2am} \cdot F_{2am+2a+k}}{F_{2a}} = \frac{XY}{Z},$$

where X, Y, and Z are all Fibonacci numbers.

Also solved by P. S. Bruckman, H. T. Freitag, J.-Z. Lee & J.-S. Lee, H.-J. Seiffert, G. Wulczyn, and the proposer.

More X, Y, Z Relations

B-597 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Do as in Problem B-596 for $S(4a + 2, k, 2b)$ and for $S(4a + 2, k, 2b - 1)$, where a and b are positive integers.

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

Using the result in B-596, we obtain:

Case I

$$S(4a + 2, k, 2b) = \frac{F_{2(2a+1)(2b+1)+k} - F_{2(2a+1)+k} - \{F_{4b(2a+1)+k} - F_k\}}{L_{4a+2} - 2}$$

$$= \frac{(F_{2(2a+1)(2b+1)+k} - F_{4b(2a+1)+k}) - \{F_{2(2a+1)+k} - F_k\}}{L_{2a+1}^2}$$

$$= \frac{F_{(2a+1)(4b+1)+k}L_{2a+1} - F_{(2a+1+k)}L_{2a+1}}{L^2_{2a+1}}$$

$$= \frac{F_{(2a+1)(4b+1)+k} - F_{2a+1+k}}{L_{2a+1}}$$

$$= \frac{F_{2(2a+1)b}L_{(2a+1)(2b+1)+k}}{L_{2a+1}},$$

by using I_{18}, I_{23}, and I_{24} in Hoggatt's *Fibonacci and Lucas Numbers.*

Case 2

$$S(4a+2, k, 2b-1) = \frac{F_{4(2a+1)b+k} - F_{2(2a+1)+k} - \{F_{2(2a+1)(2b-1)+k} - F_k\}}{L_{4a+2} - 2}$$

$$= \frac{F_{4(2a+1)b+k} - F_{2(2a+1)(2b-1)+k} - \{F_{2(2a+1)+k} - F_k\}}{L^2_{2a+1}}$$

$$= \frac{F_{(2a+1)(4b-1)+k}L_{2a+1} - F_{2a+1+k}L_{2a+1}}{L^2_{2a+1}}$$

$$= \frac{F_{(2a+1)(4b-1)+k} - F_{2a+1+k}}{L_{2a+1}}$$

$$= \frac{F_{2(2a+1)b+k}L_{(2a+1)(2b-1)}}{L_{2a+1}}.$$

Also solved by P. S. Bruckman, H. T. Freitag, L. Kuipers, J.-Z. Lee & J.-S. Lee, H.-J. Seiffert, G. Wulczyn, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-421 *Proposed by Piero Filipponi, Rome, Italy*

Let the numbers $U_n(m)$ (or merely U_n) be defined by the recurrence relation [1]

$$U_{n+2} = mU_{n+1} + U_n; \quad U_0 = 0, \ U_1 = 1,$$

where $m \in N = \{1, 2, \ldots\}$.

Find a compact form for

$$S(k, h, n) = \sum_{j=0}^{n-1} U_{k+jh} U_{k+(n-1-j)h} \quad (k, h, n \in N).$$

Note that, in the particular case $m = 1$, $S(1, 1, n) = F_n^{(1)}$ is the n^{th} term of the Fibonacci first convolution sequence [2].

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1. M. Bicknell. "A Primer on the Pell Sequence and Related Sequences." *The Fibonacci Quarterly* 13, no. 4 (1975):345-349.
2. V. E. Hoggatt, Jr. "Convolution Triangles for Generalized Fibonacci Numbers." *The Fibonacci Quarterly* 8, no. 2 (1970):158-171.

H-422 *Proposed by Larry Taylor, Rego Park, NY*

(A1) Generalize the numbers (2, 2, 2, 2, 2, 2, 2) to form a seven-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference F_n.

(A2) Generalize the numbers (1, 1, 1, 1, 1, 1) to form a six-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference F_n.

(A3) Generalize the numbers (4, 4, 4, 4, 4) to form a five-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference $5F_n$.

(A4) Generalize the numbers (3, 3, 3, 3), (3, 3, 3, 3), (3, 3, 3, 3) to form three four-term arithmetic progressions of integral multiples of Fibonacci and/or Lucas numbers with common differences F_n, $5F_n$, F_n, respectively.

(B) Generalize the Fibonacci and Lucas numbers in such a way that, if the Fibonacci numbers are replaced by the generalized Fibonacci numbers and the Lucas numbers are replaced by the generalized Lucas numbers, the arithmetic progressions still hold.

SOLUTIONS

Late Acknowledgment: *C. Georghiou solved H-394.*

A Simple Sequence

H-400 *Proposed by Arne Fransen, Stockholm, Sweden*
(Vol. 24, no. 3, August 1986)

For natural numbers h, k, with k odd, and an irrational a in the Lucasian sequence $V_{kh} = a^{kh} + a^{-kh}$, define $y_k \equiv V_{kh}$. Put

$$y_k = \sum_{r=0}^{n} c_r^{(2n+1)} y_1^{(2r+1)}, \text{ with } k = 2n + 1.$$

Prove that the coefficients are given by

$$c_r^{(2n+1)} \begin{cases} \equiv 1 & \text{for } r = n, \\ = (-1)^{n-r}(2n+1)\displaystyle\sum_{j=1}^{J} \frac{1}{2j-1}\binom{n-j}{2(j-1)}\binom{n-1-3(j-1)}{r-(j-1)} & \text{for } 0 \leqslant r < n, \end{cases}$$

where $J = \min\left(\left[\dfrac{n+2}{3}\right], \left[\dfrac{n+1-r}{2}\right], r+1\right)$.

Also, is there a simpler expression for $c_r^{(2n+1)}$?

Solution by Paul Bruckman, Fair Oaks, CA

Let $a^h = e^{i\theta}$, so that

$$y_k = 2 \cos k\theta. \tag{1}$$

Examining the Chebyshev polynomials of the first kind (viz. 22.3.15 of [1]), we find the following relation:

$$T_m(\cos\theta) = \cos m\theta, \quad m = 1, 2, 3, \ldots, \tag{2}$$

where (22.3.6, *ibid.*)

$$T_m(x) = \sum_{r=0}^{[\frac{1}{2}m]} \tfrac{1}{2}m(-1)^r \frac{\binom{m-r}{r}}{m-r}(2x)^{m-2r}. \tag{3}$$

Substitute $x = \cos\theta$, $m = k = 2n + 1$ in (3). Then, from (2) and (1),

$$\cos k\theta = \tfrac{1}{2}y_k = \sum_{r=0}^{n} \tfrac{1}{2}k(-1)^r \frac{\binom{k-r}{r}}{k-r} y_1^{k-2r};$$

further substituting $n - r$ for r gives

$$y_k = k\sum_{r=0}^{n}(-1)^{n-r}\frac{\binom{n+r}{2r}}{2r+1}y_1^{2r+1}. \tag{4}$$

It follows that we have obtained the desired simple expression:

$$c_r^{(k)} = \frac{k}{2r+1}(-1)^{n-r}\binom{n+r}{2r}. \tag{5}$$

Note the following:

$$c_n^{(k)} = 1. \tag{6}$$

Let the given alleged expression for $c_r^{(k)}$ be denoted by $b_r^{(k)}$. Thus,

$$b_r^{(k)} \equiv (-1)^{n-r}k\sum_{j=0}^{J-1}\frac{1}{2j+1}\binom{n-1-j}{2j}\binom{n-1-3j}{r-j},\ 0 \leqslant r < n. \tag{7}$$

Note that the conditions $2j \leqslant n-1-r$, $j \leqslant r$ imply $3j \leqslant n-1$; hence,

$$J - 1 = \min([\tfrac{1}{2}(n-1-r)],\ r).$$

After some manipulation, we obtain

$$b_r^{(k)} = (-1)^{n-r}\frac{k}{n-r}\sum_{j=0}^{J-1}\binom{n-r}{2j+1}\binom{n-1-j}{r-j}. \tag{8}$$

To sum (8), we use the following combinatorial identity (viz. 3.25 in [2]):

$$\sum_{j=0}^{r}\binom{x}{2j+1}\binom{x+r-j-1}{r-j} = \binom{x+2r}{2r+1}. \tag{9}$$

Let $x = n - r$ in (9). Note that terms for which $n - r < 2j + 1$ vanish, so $j \leqslant [\frac{1}{2}(n-1-r)]$; also, $j \leqslant r$. Thus, (9) becomes

$$\sum_{j=0}^{J-1}\binom{n-r}{2j+1}\binom{n-1-j}{r-j} = \binom{n+r}{2r+1}. \tag{10}$$

Comparison with (8) yields $b_r^{(k)} = \frac{k}{n-r}(-1)^{n-r}\binom{n+r}{2r+1}$, or

$$b_r^{(k)} = \frac{k}{2r+1}(-1)^{n-r}\binom{n+r}{2r},\ 0 \leqslant r < n. \tag{11}$$

Comparison of (5) and (11) yields the desired relation:

$$b_r^{(k)} \equiv c_r^{(k)},\ 0 \leqslant r < n.\quad \text{Q.E.D.} \tag{12}$$

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1. M. Abramowitz & I. A. Stegun, eds. *Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables.* 9th printing. National Bureau of Standards, 1970.
2. H. W. Gould. *Combinatorial Identities.* Morgantown, West Virginia, 1972.

Fibonacci in His Prime

H-401 *Proposed by Albert A. Mullin, Huntsville, AL*
(Vol. 24, no. 3, August 1986)

It is well known that, if $n \neq 4$ and the Fibonacci number F_n is prime, then n is prime.

(1) Prove or disprove the complementary result: If $n \neq 8$ and the Fibonacci number F_n is the product of two *distinct* primes then n is either prime or the product of two primes, in which case at least one prime factor of F_n is Fibonacci.

(2) Define the recursions $u_{n+1} = F_{u_n}$, $u_1 = F_m$, $m \geqslant 6$. Prove or disprove that each sequence $\{u_n\}$ represents only finitely many primes and finitely many products of two distinct primes.

Solution by Lawrence Somer, Washington, D.C.

(1) The result is true. It was proved in both [3] and [4] that F_n is the product of two distinct primes only if $n = 8$ or n is of the form p, $2p$, or p^2, where p is an odd prime. It is well known that if $m|n$, then $F_m|F_n$. A prime p is called a primitive divisor of F_n if $p|F_n$, but $p \nmid F_m$ for $0 < m < n$. In [1], R. Carmichael proved that F_n has a primitive prime divisor for every n except $n = 1, 2, 6$, or 12. If $n = 1, 2, 6$, or 12, then F_n is not the product of two distinct primes. It thus follows that if $n > 6$ and n is of the form $2p$ or p^2, then F_n has at least two distinct prime divisors—one of the primitive prime divisors of F_p and one of the primitive prime divisors of F_n. Clearly, every prime divisor of F_p is a primitive divisor. Thus, if F_n is the product of two distinct primes and $n = 2p$ or $n = p^2$, then F_p must be a prime divisor of F_n. The result now follows.

(2) As stated by the proposer, if $n \neq 4$, then F_n can be prime only if n is prime. Thus, it is conceivable that if $p > 6$, p is a prime, and $u_1 = F_p$ is prime, then u_n is prime for all n, and $\{u_n\}$ represents infinitely many primes. However, if u_n is not prime for some n, then we claim that, for any fixed positive integer k, there exist only finitely many positive integers n such that u_n has exactly k distinct prime divisors. In particular, $\{u_n\}$ represents only finitely many products of two distinct primes no matter what u_1 is. In fact, the following theorem and corollary are true.

Theorem: Let $\{u_n\}$ be defined by $u_{n+1} = F_{u_n}$, $u_1 = F_m$, $m \geqslant 6$. Let $d(u_n)$ denote the number of distinct prime divisors of u_n, then $d(u_{n+1}) \geqslant d(u_n)$. If $d(u_n) = r \geqslant 3$, then

$$d(u_{n+1}) \geqslant 2^r - 3 > d(u_n).$$

If $d(u_n) = 2$ and if it is not the case that both $n = 1$ and $u_n = F_9 = 34$, then $d(u_{n+1}) \geqslant 3 > d(u_n)$. If $u_n = F_9 = 34$, then $n = 1$ and $d(u_{n+1}) = 2 = d(u_n)$. If $d(u_n) = 1$ and $u_n = p^s$, where p is an odd prime and $s \geqslant 1$, then $d(u_{n+1}) \geqslant s$. If $d(u_n) = 1$ and $u_n = 2^s$, where $s \geqslant 2$, then $d(u_{n+1}) \geqslant s - 1$.

Corollary: Let t be the least positive integer, if it exists, such that u_t is not a prime. Then $\{u_n\}$ represents exactly $t - 1$ primes and at most t integers that are prime powers. If such a positive integer t does not exist, then $\{u_n\}$ represents infinitely many primes and only primes. For a fixed integer $k \geqslant 3$, $\{u_n\}$ represents at most one integer having exactly k distinct prime divisors. If $u_1 \neq 34 = F_9$, then $\{u_n\}$ represents at most one integer having exactly two prime divisors. If $u_1 = 34 = F_9$, then $\{u_n\}$ represents exactly two integers having exactly two distinct prime divisors.

Proof of the Theorem: By Carmichael's result in [2] stated earlier, F_n has a primitive prime divisor if $n \neq 1, 2, 6$, or 12. Suppose $d(u_n) = r \geqslant 3$. Then u_n has 2^r distinct divisors that are products of distinct primes or equal to 1. If k is a divisor of u_n which is the product of distinct primes and if $k \neq 1$, 2, or 6, then $F_k | F_{u_n}$ and F_k has at least one primitive prime divisor. It thus follows that $d(u_{n+1}) \geqslant 2^r - 3 > d(u_n) = r$.

Now suppose $d(u_n) = 2$ and $u_n \neq F_9 = 34$. We claim that $d(u_{n+1}) \geqslant 3$. First we prove that if $d(u_n) = 2$, $u_n \neq F_9 = 34$, and $u_n \neq F_{12} = 144$, then $2 \nmid u_n$. If $2 | F_j$, then it is known that $3 | j$. If $j = 3i$, where $i \geqslant 5$, then F_j is divisible by F_3, F_i, and F_{3i}, each of which has a primitive prime divisor. Thus, F_{3i}, $i \geqslant 5$, has at least three distinct prime divisors. The result now follows because F_3 and F_6 do not have exactly two distinct prime divisors. Thus, u_n has exactly two distinct odd prime divisors p and q. Then u_{n+1} is divisible by F_p, F_q, and F_{pq}, each of which has a primitive prime divisor. Hence, we have

$$d(u_{n+1}) \geqslant 3 > d(u_n) = 2.$$

If $u_n = F_{12} = 144$, then $u_{n+1} = F_{144}$. By the table given in [1, p. 8], $d(F_{144}) = 11$, and the claim follows. Now suppose $u_n = F_9 = 34$. Since 9 is not a Fibonacci number, we must have that $n = 1$. By the table given in [1, p. 2],

$$u_{n+1} = F_{34} = 5702887 = 1597.3571,$$

and $d(u_{n+1}) = 2 = d(u_n)$.

Now consider the case in which $d(u_n) = 1$ and $u_n = p^s$, where p is an odd prime and $s \geqslant 1$. Then u_{n+1} is divisible by F_{p^i} for $1 \leqslant i \leqslant s$, each of which has a primitive prime divisor. Hence, $d(u_{n+1}) \geqslant s$. Finally, suppose $d(u_n) = 1$ and $u_n = 2^s$, where $s \geqslant 2$. Then u_{n+1} is divisible by F_{2^i} for $2 \leqslant i \leqslant s$, each of which has a primitive prime divisor. Consequently,

$$d(u_{n+1}) \geqslant s - 1. \blacksquare$$

Proof of the Corollary: This follows immediately from the proof of the Theorem above upon noting that $u_{n+1} > u_n$ and that F_n is a power of 2 only in the cases $F_3 = 2$ and $F_6 = 8 = 2^3$. ■

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3. L. Somer. Solution to Problem B-456, proposed by A. A. Mullin. *The Fibonacci Quarterly 20*, no. 3 (1982):283.
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Also solved or partially solved by P. Bruckman, J. Desmond, and L. Kuipers.

Just a Game

H-402 *Proposed by Piero Filipponi, Rome, Italy*
(Vol. 24, no. 3, August 1986)

A MATRIX GAME (from the Italian TV serial *Pentathlon*).

For complete details of this very interesting problem, see pages 283-84 of *The Fibonacci Quarterly 24*, no. 3 (August 1986).

Solution by Paul S. Bruckman, Fair Oaks, CA

Given $n \geqslant 1$, let χ_n denote the set of $1 \times n$ vectors $(\theta_1, \theta_2, \ldots, \theta_n)$, χ_n' the set of $n \times 1$ vectors $(\theta_1, \theta_2, \ldots, \theta_n)'$ with $\theta_i = 0$ or 1 (chosen randomly). Let $T_n = \chi_n^n = \chi_n'^n$ denote the set of all $n \times n$ matrices with entries either 0 or 1. Let $\underline{\delta}_n \equiv (0, 0, \ldots, 0) \in \chi_n$, $\underline{\varepsilon}_n \equiv (1, 1, \ldots, 1) \in \chi_n$; likewise, $\underline{\delta}_n' \equiv (0, 0, \ldots, 0)' \in \chi_n'$, $\underline{\varepsilon}_n' \equiv (1, 1, \ldots, 1)' \in \chi_n'$. Let $\rho_n \equiv \{\underline{\delta}_n, \underline{\varepsilon}_n\}$, $\rho_n' \equiv \{\underline{\delta}_n', \underline{\varepsilon}_n'\}$; $\sigma_n \equiv \{\underline{\delta}_n, \underline{\delta}_n'\}$, $\tau_n \equiv \{\underline{\varepsilon}_n, \underline{\varepsilon}_n'\}$. We say a matrix *contains* a vector if the vector is either a row or a column, as appropriate, of the matrix.

Let A_n denote the subset of T_n containing at least one element of $\rho_n \cup \rho_n'$;
Let B_n denote the subset of T_n containing at least one element of ρ_n;
Let C_n denote the subset of T_n containing at least one element of ρ_n';
Let D_n denote the subset of T_n containing at least one element of ρ_n, ρ_n'.

We first observe that $|T_n| = 2^{n^2}$. Moreover,

$$P_n = |A_n|/|T_n| = 2^{-n^2}|A_n|. \tag{1}$$

By symmetry, we see that $|B_n| = |C_n|$. Also, $|A_n| = |B_n| + |C_n| - |D_n|$, so

$$|A_n| = 2|B_n| - |D_n|. \tag{2}$$

To evaluate $|B_n|$, we note that B_n^* is the subset of T_n containing no elements of ρ_n. Since each such (row) element of B_n^* may be chosen in $2^n - 2$ ways, thus, $|B_n^*| = (2^n - 2)^n$. Hence,

$$|B_n| = 2^{n^2} - (2^n - 2)^n. \tag{3}$$

To evaluate $|D_n|$, we first partition D_n into the two (disjoint) sets $D_n^{(0)}$ and $D_n^{(1)}$, defined as follows: $D_n^{(0)}$ is the subset of T_n containing σ_n, $D_n^{(1)}$ is the subset of T_n containing τ_n. Note that no element of T_n can contain $\{\underline{\delta}_n, \underline{\varepsilon}_n'\}$ or $\{\underline{\delta}_n', \underline{\varepsilon}_n\}$. By symmetry, $|D_n^{(0)}| = |D_n^{(1)}|$. Therefore,

$$|D_n| = 2|D_n^{(0)}|. \tag{4}$$

To evaluate $|D_n^{(0)}|$, we further partition $D_n^{(0)}$ into the (disjoint) sets $D_{n,k}^{(0)}$, $k = 1, 2, \ldots, n$, where $D_{n,k}^{(0)}$ is the subset of $D_n^{(0)}$ with at least one $\underline{\delta}_n$, with $\underline{\delta}_n'$ in the k^{th} column, but with no $\underline{\delta}_n'$ in any of the preceding columns. Thus,

$$|D_n^{(0)}| = \sum_{k=1}^{n} |D_{n,k}^{(0)}|. \tag{5}$$

Now, $D_{n,1}^{(0)}$ is the subset of T_n with at least one $\underline{\delta}_n$ and with first column $\underline{\delta}_n'$; this is equivalent to the set difference $E - F$, where E is the subset of T_n with first column $\underline{\delta}_n'$, F is the subset of E containing no $\underline{\delta}_n$. We enumerate E by considering the rows of any matrix in E. Each such row must have 0 as its first element, with the other elements random. This involves 2^{n-1} choices for each such row; hence, $|E| = 2^{(n-1)n}$. $|F|$ is enumerated similarly, except that each row of any matrix in F must also not be $\underline{\delta}_n$. This involves $2^{n-1} - 1$ choices for each row of any matrix in F; hence, $|F| = (2^{n-1} - 1)^n$. Therefore,

$$|D_{n,1}^{(0)}| = 2^{(n-1)n} - (2^{n-1} - 1)^n. \tag{6}$$

Next, we evaluate $|D_{n,2}^{(0)}|$. $D_{n,2}^{(0)}$ is the subset of T_n with at least one $\underline{\delta}_n$, with the first column *not* $\underline{\delta}_n'$ and with second column $\underline{\delta}_n'$. Thus, $D_{n,2}^{(0)}$ is equivalent to the set difference $G - H$, where G is the subset of T_n with at least one $\underline{\delta}_n$ and second column $\underline{\delta}_n'$, H is the subset of G where both first and second columns are $\underline{\delta}_n'$. By symmetry, we see that $|G| = |D_{n,1}^{(0)}|$. To evaluate $|H|$, we see that H is the set difference $J - K$, where J is the subset of T_n with both first and second columns $\underline{\delta}_n'$, and K is the subset of J containing no $\underline{\delta}_n$. By similar reasoning, $|J| = 2^{(n-2)n}$, $|K| = (2^{n-2} - 1)^n$. Hence, $|H| = 2^{(n-2)n} - (2^{n-2} - 1)^n$, and so

$$|D_{n,2}^{(0)}| = 2^{(n-1)n} - (2^{n-1} - 1)^n - \{2^{(n-2)n} - (2^{n-2} - 1)^n\}. \tag{7}$$

A moment's reflection shows us where this general process leads us; first, however, we make the following convenient definition:

$$a_k = 2^{(n-k)n} - (2^{n-k} - 1)^n, \; k = 0, 1, 2, \ldots, n. \tag{8}$$

We then find: $|D_{n,1}^{(0)}| = a_1$, $|D_{n,2}^{(0)}| = a_1 - a_2 = -\Delta a_1$, $|D_{n,3}^{(0)}| = a_1 - 2a_2 + a_3 = \Delta^2 a_1$, etc.; in general, we find

$$|D_{n,k}^{(0)}| = (-1)^{k-1}\Delta^{k-1}a_1, \; k = 1, 2, \ldots, n. \tag{9}$$

Therefore, by (5), $|D_n^{(0)}| = \sum_{k=1}^n (-1)^{k-1}\Delta^{k-1}a_1$. This expression can be slightly simplified as follows:

$$\sum_{k=1}^{n} (-1)^{k-1}\Delta^{k-1}a_1 = \sum_{k=1}^{n} (-1)^{k-1}\Delta^{k-1}(1 + \Delta)a_0$$

$$= \sum_{k=1}^{n} (-1)^{k-1}\Delta^{k-1}a_0 - \sum_{k=1}^{n} (-1)^{k}\Delta^{k}a_0 = -(-1)^{k-1}\Delta^{k-1}a_0\Big|_1^{n+1} = a_0 - (-1)^n\Delta^n a_0.$$

In terms of the binomial expansion,

$$|D_n^{(0)}| = \sum_{k=1}^{n} \binom{n}{k}(-1)^{k-1}a_k. \tag{10}$$

We may also express $|B_n|$ in (3) in terms of a_1, since we see from (3) that $|B_n| = 2^n(2^{(n-1)n} - (2^{n-1} - 1)^n)$, i.e.,

$$|B_n| = 2^n a_1. \tag{11}$$

Using (2), (4), (10), and (11), we therefore obtain:

$$|A_n| = 2\left(2^n a_1 - \sum_{k=1}^{n} \binom{n}{k}(-1)^{k-1}a_k\right). \tag{12}$$

Finally, from (1), we obtain the desired exact expression:

$$P_n = 2^{1-n^2}\left(2^n a_1 - \sum_{k=1}^{n} \binom{n}{k}(-1)^{k-1}a_k\right), \tag{13}$$

where the a_k's are given by (8).

After some computations, we obtain the following values from (13): $P_1 = 1$, $P_2 = .875$, $P_3 = 205/256 \doteq .8008$, as discovered by the proposer. However, we further obtain: $P_4 = 21{,}331/32{,}768 \doteq .6510$, $P_5 = 7{,}961{,}061/16{,}777{,}216 \doteq .4745$, $P_6 = 10{,}879{,}771{,}387/34{,}559{,}738{,}368 \doteq .3166$, $P_7 \doteq .1978$, $P_8 \doteq .1215$, $P_9 \doteq .0680$, and $P_{10} \doteq .0383$, all of which values are different from those published in the statement of the problem.

Nevertheless, the proposer's conjecture is correct, and is easily proved. Note, from (13), that $P_n < 2^{1-n^2}2^n a_1$. Also,

$$a_1 = 2^{n^2-n} - (2^{n-1} - 1)^n = 2^{n^2-n}\{1 - (1 - 2^{1-n})^n\}$$

$$= 2^{n^2-n}\{1 - 1 + n \cdot 2^{1-n} - \cdots\} < n \cdot 2^{n^2-2n+1}.$$

Hence, $P_n < 2^{1+n-n^2} \cdot n \cdot 2^{n^2-2n+1}$, or

$$P_n < \frac{4n}{2^n}. \tag{14}$$

Clearly, $\lim_{n\to\infty} 4n \cdot 2^{-n} = 0$. Hence, $\lim_{n\to\infty} P_n = 0$. Q.E.D.

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

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