The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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# The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau (1907-1988)

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION*
*DEVOTED TO THE STUDY*
*OF INTEGERS WITH SPECIAL PROPERTIES*

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## In Memoriam

# BROTHER ALFRED BROUSSEAU

**February 17, 1907–May 31, 1988**

Brother Alfred Brousseau, F.S.C., cofounder of The Fibonacci Association and Managing Editor of the *Fibonacci Quarterly* for its first thirteen years, died May 31 at the age of 81. He was associated with Saint Mary's College, Moraga, California, since the 1930s, where he was Chair of the School of Science for many years. He taught until he was 71, and then continued to attend mathematics meetings and give lectures to teachers.

In the early 1960s, Brother Alfred became interested in Fibonacci numbers and their applications. He and Verner E. Hoggatt, Jr., got a group of people together in 1963 and, as he said in *Time,* April 4, 1969, "just like a bunch of nuts, we started a mathematics magazine." Of course, twenty-five years later, the *Fibonacci Quarterly* continues to thrive. Brother Alfred's role in founding The Fibonacci Association in reported in our February 1987 issue.

Brother Alfred was an avid botanist, naturalist, and photographer, and he has donated his extensive collection of slides and wildflowers to Saint Mary's College. Also quite a hiker, he collected specimens of all twenty species of native California pine trees to study their growth patterns. He made a phyllotaxis exhibit, showing the spiral counts of the cones, to interest high school students in Fibonacci numbers. His exhibit was very popular at meetings of mathematics teachers as well.

Brother Alfred was always a dedicated teacher, and wanted to interest young people in mathematics. He wrote many articles especially for beginners in the *Fibonacci Quarterly* as well as six books still published by The Fibonacci Association. He gave countless lectures on Fibonacci Numbers and mathematical discovery to high school students and their teachers. He directed Saint Mary's College's joint program with the National Science Foundation which each year attracted hundreds of students to Saint Mary's College for a problem-solving competition.

Besides these serious pursuits, Brother Alfred always had time to play his accordion after teacher meetings, and to cheer patients at convalescent hospitals. He approached all of life with great enthusiasm and energy. He will be missed.

—*Marjorie Bicknell-Johnson*

# LENGTH OF THE 7-NUMBER GAME

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*(Submitted July 1986)*

## 1. INTRODUCTION

The $n$-number game is defined as follows. Let $S = (s_1, s_2, \ldots, s_n)$ be an $n$-tuple of nonnegative integers. Applying the difference operator $D$ we obtain a new $n$-tuple $D(S) = (s_1', s_2', \ldots, s_n')$ by taking numerical differences; that is, $s_i' = |s_i - s_{i+1}|$. Subscripts are reduced modulo $n$ so that $s_n' = |s_n - s_1|$. If this process is repeated, a sequence of tuples is generated; that is,

$$S,\ D^1(S),\ D^2(S),\ \ldots;$$

this sequence is referred to as *the $n$-number game generated by $S$*. The $n$-number game has been studied extensively, beginning with the 4-number game (see [5], [6], [8], [9], [13], and [19]).

As $k \Rightarrow \infty$, what happens to $D^k(S)$? Is it possible to generate an infinite, never repeating sequence? The answer is clearly no. For, let $|S| = \max(s_i)$. Then $|S| \geqslant |D(S)|$; since there are only a finite number of $n$-tuples with entries less than or equal to $|S|$, eventually the sequence $\{D^k(S)\}$ must repeat. When $n = 2^r$, it is well known that, for every $S$, the resulting sequence terminates with the zero-tuple $(0, 0, \ldots, 0)$. That this is not the case for other values of $n$ is easily seen by considering the triple $S = (4, 5, 3)$. Applying the difference operator to this tuple gives the following:

$$\begin{aligned}
S &= (4, 5, 3)\\
D^1(S) &= (1, 2, 1)\\
D^2(S) &= (1, 1, 0)\\
D^3(S) &= (0, 1, 1)\\
D^4(S) &= (1, 0, 1)\\
D^5(S) &= (1, 1, 0) = D^2(S)
\end{aligned}$$

We call $\{D^2(S), D^3(S), D^4(S)\}$ a cycle. When $n$ is not a power of 2, there are always tuples that appear in such cycles; indeed, for odd $n$, $(1, 1, 0, 0, \ldots, 0)$ is one such tuple. In an earlier paper this author characterized those tuples which can occur in a cycle. In particular, an $n$-tuple $S$ is in a cycle only if all the entries in $S$ are either 0 or $|S|$. Further, when $n$ is odd, such tuples are in a cycle if and only if the number of nonzero entries is even [11].

For any $n$-tuple $S$, we say the game generated by $S$ has *length* $\lambda$ if $D^{\lambda}(S)$ is in a cycle but $D^{\lambda-1}(S)$ is not. We will denote the length of this game by $L(S)$; thus, in the example above, $L(S) = 2$. Note that we consider the zero-tuple to be in a cycle, namely the trivial one. There is no bound on the length of an $n$-number game. That is, for any $\lambda$ there exists an $n$-tuple $S$ such that $L(S) > \lambda$. We seek to characterize the upper bound of $L(S)$ for all tuples $S$ for which $|S| \leqslant M$. Only when $n$ equals 4 or $2^r + 1$ has this question been answered [18], [12]. In all other cases, only partial results are known; a complete solution seems very difficult. In this paper we will resolve the question for $n = 7$.

## 2. GENERAL RESULTS

There are $N = (M + 1)^n - M^n$ $n$-tuples with the property that $|S| = M$. However, we may consider several of these tuples related to each other and hence, in actuality, the number of tuples that we need consider is far less than $N$. For $S = (s_1, s_2, \ldots, s_n)$ with $|S| = M$, define $\mathfrak{M}(S)$ as the $n$-tuple given by

$$\mathfrak{M}(S) = (M - s_1, M - s_2, \ldots, M - s_n).$$

Further, let $\mathfrak{D}_n$ denote the dihedral group of a regular $n$-gon. Then we say two $n$-tuples *$S$ and $R$ are related* if either $R = \sigma(S)$ or $R = \sigma(\mathfrak{M}(S))$ for some $\sigma \in \mathfrak{D}_n$. If $S$ and $R$ are related, then we write $S \approx R$. It is easily seen that $\approx$ is an equivalence relation. We now show that related tuples have the same length. Thus, in determining those tuples which give games of maximum length, we need only consider the question up to equivalence classes.

**Theorem 1:** Suppose $S \approx R$, then $L(R) = L(S)$.

**Proof:** We may think of the entries of an $n$-tuple as the vertices of a regular $n$-gon. Thus, if $R = \sigma(S)$, $R$ represents either a rotation, a flip, or a rotation followed by a flip of the $n$-gon whose vertices represent $S$. Clearly, the entries in $D(R)$ are the same as those in $D(S)$. It is only their order that is changed, and that change may be represented by a member of $\mathfrak{D}_n$. Since the entries are unchanged at each step, the length of $R$ is the same as that of $S$.

Now, if $R = \mathfrak{M}(S)$, then it is easily seen that $D(S) = D(\mathfrak{M}(S))$ and hence the two tuples have the same length. ■

We will now turn our attention to those tuples which give games of maximum length. We first prove a lemma; the corollary that follows is then an immediate consequence.

**Lemma 1:** Let $S$ be an $n$-tuple with $|S| = M$. Suppose that $S$ has a predecessor and that $S$ is not contained in a cycle. Then $|D^{n-2}(S)| \leqslant M - 1$.

**Proof:** If $S = (M, M, \ldots, M)$, then $|D(S)| = 0$ and thus the conclusion holds. Otherwise, since $S$ has a predecessor and is not in a cycle, there exists some $s_i$ that is different from 0 and $M$. Further, $S$ has a predecessor if and only if there exist values $\delta_i \in \{-1, 1\}$ such that $\sum \delta_i s_i = 0$ [11]. Thus, $S$ must have at least two entries that are different from 0 and $M$. This implies that $D(S)$ has at least three entries less than $M$; for, if $0 < s_i < M$, then $s'_{i-1} < M$ and $s'_i < M$, where $D(S) = (s'_1, s'_2, \ldots, s'_n)$. It follows that $D^{n-2}(S)$ has $n$ entries less than $M$; i.e., $|D^{n-2}(S)| \leqslant M - 1$. ■

**Corollary 1:** Let $S$ be an $n$-tuple, $|S| = M$, then $L(S) \leqslant (n - 2)(M - 1) + 1$. ■

This corollary gives an upper bound for $L(S)$ when $|S| \leqslant M$. For $n = 2^r + 1$, this upper bound is actually taken on by the $n$-tuple

$$R_M = (0, 0, \ldots, 0, M - 1, M) \text{ for } M \geqslant 1 \quad \text{(see [12])}.$$

For other values of $n$ the actual bound is less than that found in Corollary 1. In general, for a given $n$, the least upper bound for $L(S)$ with $|S| = M$ is not known. The main result of this paper will be to characterize $L(S)$ when $n = 7$. In particular we will show that if $S$ is a 7-tuple with $|S| \leqslant M$ and $M$ is sufficiently large, then

$$L(S) \leqslant \begin{cases} 7(M - 1)/2 & \text{if } M \text{ is odd,} \\ 7(M - 2)/2 + 4 & \text{if } M \text{ is even.} \end{cases}$$

This can be fairly easily proved for tuples $S$ for which $|D^7(S)| \leqslant M - 2$. Thus, we first determine the tuples for which $|D^7(S)| = M - 1$; note that by Lemma 1, $|D^7(S)| < M$.

## 3. TUPLES FOR WHICH $|D^7(S)| = M - 1$

In the following discussion we will only consider tuples up to the equivalence relation $\approx$. Thus, for example, when we state in Lemma 4 that

$$D^2(S) = (1, M, M, M, 1, 1, \cdot),$$

we really mean $D^2(S) \approx (1, M, M, M, 1, 1, \cdot)$.

By Lemma 1, if $|D^2(S)| = M - 1$, then $|D^7(S)| \leqslant M - 2$. Thus, in determining tuples for which $|D^7(S)| = M - 1$, we may restrict our attention to those with $|D^2(S)| = M$. First, consider an $n$-tuple $R = (r_1, r_2, \ldots, r_n)$ with $D(R) = (r'_1, r'_2, \ldots, r'_n)$. Suppose $D(R)$ has some entry $r'_i = M - 1$. Then either $r_i = M - 1$ and $r_{i+1} = 0$ or $r_i = M$ and $r_{i+1} = 1$. Thus, in the case under consideration, we must have, for some $k$, $2 \leqslant k \leqslant 6$,

$$|D^k(S)| = M, \quad D^k(S) \approx (M,\ 1,\ .,\ .,\ .,\ .,\ .),$$
$$D^{k+1}(S) \approx (M-1,\ .,\ .,\ .,\ .,\ .,\ .), \text{ and } |D^7(S)| = M - 1.$$

For the tuple $D^{k+1}(S)$, we may have either $|D^{k+1}(S)| = M$ or $|D^{k+1}(S)| = M - 1$. However, what must happen is that $D^k(S)$ have enough consecutive $M$'s and 1's to yield a sufficient number of $(M-1)$'s and 0's in $D^{k+1}(S)$ so that $D^7(S)$ contains at least one $M - 1$. Thus, it is the $M - 1$ and 0 terms in $D^{k+1}(S)$ that are the important ones. These possibilities are illustrated by the following two examples:

$$\begin{aligned}
S_1 &= (M,\ M,\ 0,\ M,\ 1,\ M,\ 2)\\
D^1(S_1) &= (0,\ M,\ M,\ M-1,\ M-1,\ M-2,\ M-2)\\
D^2(S_1) &= (M,\ 0,\ 1,\ 0,\ 1,\ 0,\ M-2)\\
D^3(S_1) &= (M,\ 1,\ 1,\ 1,\ 1,\ M-2,\ 2)\\
D^4(S_1) &= (M-1,\ 0,\ 0,\ 0,\ M-3,\ M-4,\ M-2)\\
D^5(S_1) &= (M-1,\ 0,\ 0,\ M-3,\ 1,\ 2,\ 1)\\
D^6(S_1) &= (M-1,\ 0,\ M-3,\ M-4,\ 1,\ 1,\ M-2)\\
D^7(S_1) &= (M-1,\ M-3,\ 1,\ M-5,\ 0,\ M-3,\ 1)
\end{aligned}$$

$$\begin{aligned}
S_2 &= (M,\ M,\ 0,\ M,\ 1,\ M-2,\ 0)\\
D^1(S_2) &= (0,\ M,\ M,\ M-1,\ M-3,\ M-2,\ M)\\
D^2(S_2) &= (M,\ 0,\ 1,\ 2,\ 1,\ 2,\ M)\\
D^3(S_2) &= (M,\ 1,\ 1,\ 1,\ 1,\ M-2,\ 0)\\
D^4(S_2) &= (M-1,\ 0,\ 0,\ 0,\ M-3,\ M-2,\ M)\\
D^5(S_2) &= (M-1,\ 0,\ 0,\ M-3,\ 1,\ 2,\ 1)\\
D^6(S_2) &= (M-1,\ 0,\ M-3,\ M-4,\ 1,\ 1,\ M-2)\\
D^7(S_2) &= (M-1,\ M-3,\ 1,\ M-5,\ 0,\ M-3,\ 1)
\end{aligned}$$

Note that in the examples above $|D^4(S_1)| = M - 1$, while $|D^4(S_2)| = M$. However, in both, it is the presence of five consecutive $M$'s and 1's in $D^3(S_i)$ that gives rise to $|D^7(S_i)| = M - 1$. In general, then, if for a tuple $S$, we have $|S| = M$ and $|D^7(S)| = M - 1$, we will denote by $\kappa(S)$ that step where the presence of consecutive $M$'s and 1's in $D^{\kappa(S)}(S)$ gives rise to $|D^7(S)| = M - 1$. Thus, in the examples above, $\kappa(S_1) = 3$ and $\kappa(S_2) = 3$. In general, we must have $2 \leqslant \kappa(S) \leqslant 6$. The following lemmas characterize $D^{\kappa(S)}(S)$ for various possibilities of $\kappa(S)$.

**Lemma 2:** Suppose $S$ is a 7-tuple with the properties that $|S| = M$ and $|D^7(S)| = M - 1$. Let $k = \kappa(S)$ and let $\ell$ be the number of consecutive 1's and $M$'s in $D^k(S)$, with $\ell$ as large as possible. Then $\ell \geqslant 8 - k$.

**Proof:** It is easily seen that the number of consecutive 0's and $(M - 1)$'s in $D^{k+1}(S)$ is $\ell - 1$ and, hence, $D^{k+1}(S)$ has at most $\ell - 1$ consecutive terms that equal $M - 1$. Similarly, the number of consecutive 0's and $(M - 1)$'s in $D^{k+t}(S)$ is $\ell - t$ and, hence, $D^{k+t}(S)$ has at most $\ell - t$ consecutive terms that equal

$M - 1$. Continuing, we find that $D^{k+\ell}(S)$ has no terms that equal $M - 1$. Since $|D^7(S)| = M - 1$, we must have $k + \ell \geqslant 8$ or $\ell \geqslant 8 - k$. ■

**Lemma 3**: Suppose $S$ is a 7-tuple with the properties that $|S| = M$ and $|D^7(S)| = M - 1$. Let $k = \kappa(S)$. Then $D^k(S) = (1, \ldots, 1, M, \ldots, M, 1, \ldots, 1, a, \ldots, b)$, where $a$ and $b$ are neither 1 nor $M$ and the number of consecutive 1's and $M$'s is at least $8 - k$.

**Proof**: By Lemma 2, all that we need show is that $D^k(S)$ cannot have the form $(M, 1, \ldots, 1, M, c, \ldots, d)$. Suppose $D^k(S) = (M, 1, M, \ldots)$. Then $D^{k-1}(S)$ must equal $(0, M, M - 1, 2M - 1, \ldots)$ or $(M, 0, 1, M + 1, \ldots)$, both of which are impossible because $|S| = M$. Similarly, we find that $D^k(S)$ cannot equal $(M, 1, 1, 1, M, \ldots)$ or $(M, 1, 1, 1, 1, 1, M, \cdot)$.

Now, if $D^k(S) = (M, 1, 1, M, \ldots)$, then $D^{k-1}(S)$ must equal $(0, M, M - 1, M, 0, \ldots)$ or $(M, 0, 1, 0, M, \ldots)$, neither of which has a predecessor that contradicts the hypothesis. ■

The above lemmas mean that it is possible that there is a 7-tuple $S$ for which $D^4(S) = (1, M, 1, 1, \ldots)$ and $|D^7(S)| = M - 1$. But there is no 7-tuple for which $D^3(S) = (1, M, 1, 1, a, \cdot, b)$, $a \neq 1$, $a \neq M$, $b \neq 1$, $b \neq M$, and $|D^7(S)| = M - 1$.

**Lemma 4**: Suppose $S$ is a 7-tuple with the properties that $|S| = M$, $|D^7(S)| = M - 1$, and $\kappa(S) = 2$. Then one of the following must hold:

$$D^2(S) = (M, M, M, M, M, 1, \cdot) \qquad D^2(S) = (M, 1, 1, 1, 1, 1, \cdot)$$
$$D^2(S) = (1, M, 1, 1, 1, 1, \cdot) \qquad D^2(S) = (1, M, M, 1, 1, 1, \cdot)$$
$$D^2(S) = (1, M, M, M, 1, 1, \cdot)$$

**Proof**: It is easily verified that these five tuples give $|D^7(S)| = M - 1$. On the other hand, suppose $R$ is a tuple such that $D^2(R) = (1, \ldots, 1, M, \ldots, M, 1, \ldots, 1, a)$. By direct computation, it can be shown that $|D^7(R)| < M - 1$; e.g., if $D^2(R) = (1, 1, M, 1, 1, 1, a)$, then $|D^7(R)| < M - 1$ unless $a = 1$. ■

**Theorem 2**: Suppose $M \geqslant 12$ and $S$ is a 7-tuple with the properties that $|S| = M$, $|D^7(S)| = M - 1$, and $\kappa(S) = 2$. Then $S$ is related to one of the following:

$$(1, 0, 0, M, M, 0, M - 1) \qquad (0, 13, 13, 12, 10, 7, 3)$$
$$(0, 15, 15, 14, 12, 9, 5)$$

**Proof**: We use Lemma 4, consider the various cases, and work our way backward to obtain $S$. We begin by assuming that $D^2(S) = (1, M, M, M, 1, 1, \cdot)$ and find possible tuples equal to $D(S)$ by first setting each of its elements equal, in turn, to $M$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | $M$ | $M-1$ | $-1$ | $M-1$ | $-1$ | $0$ | $1$ |
| 2. | $M-1$ | $M$ | $0$ | $M$ | $0$ | $1$ | $0/2$ |
| 3. | $1$ | $0$ | $M$ | $0$ | $M$ | $M-1$ | $M-2/M$ |
| 4. | same as Row 2 | | | | | | |
| 5. | same as Row 3 | | | | | | |
| 6. | $0$ | $-1$ | $M-1$ | $-1$ | $M-1$ | $M$ | $M-1$ |
| 7. | same as Row 3 | | | | | | |

When more than one number appears, such as "0/2" in Row 2, it means that either number is possible at that stage. Rows 1 and 6 are not possible because negative elements and present. We now treat Rows 2 and 3 in the above fashion. Starting with Row 3, we find possible tuples for $S$ when $D(S) = (1, 0, M, 0, M, M-1, M-2)$, as follows:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1a. | $M$ | $M+1$ | $M+1$ | $1$ | $1$ | $M+1$ | $2$ |
| 2a. | $M-1$ | $M$ | $M$ | $0$ | $0$ | $M$ | $1$ |
| 3a. | same as Row 2a | | | | | | |
| 4a. | $1$ | $0$ | $0$ | $M$ | $M$ | $0$ | $M-1$ |
| 5a. | same as Row 4a | | | | | | |
| 6a. | same as Row 2a | | | | | | |
| 7a. | $2$ | $1$ | $1$ | $M+1$ | $M+1$ | $1$ | $M$ |

Rows 1a and 7a are impossible because of the presence of elements greater than $M$. Rows 2a and 4a are possible, but they are related. Row 4a is the first tuple listed in the theorem. Continuing with Row 3, we find that there are no predecessors when $D(S) = (1, 0, M, 0, M, M-1, M)$. Similarly, there are no predecessors for Row 2 if $M < 12$.

We repeat this process for each of the other four conditions in Lemma 4 and find that $D^2(S) = (M, 1, 1, 1, 1, 1, \cdot)$ gives rise to the other tuples for $S$ listed in the theorem. ■

The other cases, with $2 < \kappa(S)$, proceed similarly.

**Lemma 5:** Suppose $S$ is a 7-tuple with the properties that $|S| = M$, $|D^7(S)| = M - 1$, and $2 < \kappa(S) \leqslant 6$. Then one of the following must hold:

$$\begin{array}{ll}
D^3(S) = (M, 1, 1, 1, 1, \cdot, \cdot) & D^3(S) = (M, M, 1, 1, 1, \cdot, \cdot) \\
D^3(S) = (M, M, M, 1, 1, \cdot, \cdot) & D^3(S) = (M, M, M, M, 1, \cdot, \cdot) \\
D^4(S) = (M, 1, 1, 1, \cdot, \cdot, \cdot) & D^4(S) = (M, M, M, 1, \cdot, \cdot, \cdot) \\
D^4(S) = (1, M, 1, 1, \cdot, \cdot, \cdot) & D^5(S) = (M, 1, 1, \cdot, \cdot, \cdot, \cdot) \\
D^5(S) = (M, M, 1, \cdot, \cdot, \cdot, \cdot) & D^6(S) = (M, 1, \cdot, \cdot, \cdot, \cdot, \cdot)
\end{array}$$

**Proof:** It is easily verified that the above ten tuples give $|D^7(S)| = M - 1$. On the other hand, suppose $R$ is a tuple not in the above list, such that $D^k(R) = (1, \ldots, 1, M, \ldots, M, 1, \ldots, 1, \ldots)$, where the number of consecutive 1's and $M$'s is at least $8 - k$ and there is at least one 1. Then it must be the case that

$D^3(R) = (1, M, 1, 1, 1, \cdot, \cdot)$ $\quad$ $D^3(R) = (1, M, M, 1, 1, \cdot, \cdot)$
$D^3(R) = (1, M, M, M, 1, \cdot, \cdot)$ $\quad$ $D^3(R) = (1, 1, M, 1, 1, \cdot, \cdot)$
$D^4(R) = (M, M, 1, 1, \cdot, \cdot, \cdot)$ $\quad$ $D^4(R) = (1, M, M, 1, \cdot, \cdot, \cdot)$
$D^5(R) = (1, M, 1, \cdot, \cdot, \cdot, \cdot)$

or, more precisely, $D^k(S)$ must be related to one of these. By direct computation, it can be shown that $|D^7(R)| < M - 1$. ■

**Theorem 3:** Suppose $M \geqslant 12$ and $S$ is a 7-tuple with the properties that $|S| = M$, $|D^7(S)| = M - 1$, and $2 < \kappa(S) < 6$. Then $S$ is related to one of the following tuples:

$D^3(S) = (M, 1, 1, 1, 1, \cdot, \cdot, \cdot)$:

$(0, M, 0, 0, M-1, M, 2)$ $\quad$ $(0, M, 0, 0, M-1, M-2, 0)$
$(0, 0, M, 0, M-1, 2, M-4)$ $\quad$ $(0, 0, M, 0, M-1, 2, M-2)$
$(0, 0, M, 0, M-1, 2, M)$ $\quad$ $(0, M, 0, 0, M-1, M, 0)$
$(0, 0, M, 0, M-1, 0, M-2)$ $\quad$ $(0, 0, M, 0, M-1, 0, M)$
$(0, 12, 12, 12, 11, 8, 2)$ $\quad$ $(0, 12, 12, 12, 11, 8, 4)$
$(0, 14, 14, 14, 13, 10, 4)$ $\quad$ $(0, 14, 14, 14, 13, 10, 6)$
$(0, 18, 18, 18, 17, 14, 8)$ $\quad$ $(0, 20, 20, 20, 19, 16, 10)$

$D^3(S) = (M, M, 1, 1, 1, \cdot, \cdot)$:

$(0, M, 0, 0, 0, 1, 2)$ $\quad$ $(0, M, 0, 0, 0, 1, 0)$
$(0, M, 0, 0, 1, 4)$

$D^3(S) = (M, M, M, 1, 1, \cdot, \cdot)$:

$(0, M, 0, 0, 0, M, M-1)$

$D^4(S) = (M, 1, 1, 1, \cdot, \cdot, \cdot)$:

$(0, 13, 13, 13, 13, 12, 8)$ $\quad$ $(0, 15, 15, 15, 15, 14, 10)$

$D^4(S) = (1, M, 1, 1, \cdot, \cdot, \cdot)$:

$(1, 0, 0, 0, 0, M, 1)$ $\quad$ $(M-1, 0, 0, M, M, M, 1)$
$(1, 0, M, 0, 0, M, M-1)$

$D^5(S) = (M, 1, 1, \cdot, \cdot, \cdot, \cdot)$:

$(0, M, 0, M, 0, 0, M-1)$ $\quad$ $(0, 0, 0, M, 0, M, M-1)$
$(0, M, M, M, 0, 0, 1)$ $\quad$ $(0, M, M, 0, M, M, M-1)$
$(0, 0, 0, 0, M, 0, 1)$ $\quad$ $(0, M, 0, 0, M, M, 1)$

**Proof:** The proof proceeds as in Theorem 2; due to the number of cases, the calculations are tedious. Although originally obtained by hand, these results were verified by computer. A copy of the program and/or output may be obtained from the author. ■

**Theorem 4:** Suppose $S$ is related to $(1, 0, M, M, M, 0, 0)$. Then, for $M \geqslant 6$, $|D^{14}(S)| \leqslant M - 4$.

**Proof:** By direct calculation, we find $D^{12}(S) = (M-5, M-6, 1, 1, 2, 0, 1)$, so $|D^{12}(S)| \leqslant M - 4$ for $M \geqslant 6$ and thus $|D^{14}(S)| \leqslant M - 4$. ■

**Theorem 5:** Suppose $S$ is a 7-tuple with the properties that $|S| = M$, $|D^7(S)| = M - 1$, and $S$ is *not* related to $(1, 0, M, M, M, 0, 0)$. Then $|D^{10}(S)| \leqslant M - 3$ whenever $M \geqslant 12$.

**Proof:** Suppose that $S$ is related to $(0, 1, M, M, 0, M, M)$. Computing $D^n(S)$ for $1 \leqslant n \leqslant 10$, we find $D^{10}(S) = (1, 1, M-6, M-4, 0, 1, 1)$ and thus the conclusion holds. Likewise, if $S$ is related to $(0, 2, 8, 11, 12, 12, 12)$, then $D^{10}(S) = (5, 3, 3, 3, 2, 2, 2)$ and thus $|D^{10}(S)| \leqslant 9$. In a similar manner, the theorem can be verified by calculating $D^{10}(S)$ for each of the other twenty-nine tuples found in Theorems 2 and 3. ■

## 4. TUPLES WHICH GIVE LONGEST GAMES

**Theorem 6:** Let $T = (1, M, 1, M, M, 0, M)$ for $M \geqslant 1$. Then, for $M \geqslant 3$,

$$L(T_M) = \begin{cases} 7(M-1)/2 & \text{if } M \text{ is odd,} \\ 7(M-2)/2 + 4 & \text{if } M \text{ is even.} \end{cases}$$

**Proof:** For $M \geqslant 3$, it is easily seen by direct calculation that $D^7(T_M) = T_{M-2}$. Since $L(T_1) = 0$ and $L(T_2) = 4$, the result follows. ■

We will show that for $M \geqslant 8$ the tuples $T_M$, as defined above, give the games of maximum length. The following lemma is essentially a corollary of the previous theorem.

**Lemma 6:** For the tuples $T_M$ defined as in Theorem 6, the following hold:

$$\begin{aligned} 7 + L(T_{M-2}) &= L(T_M) \\ 10 + L(T_{M-3}) &\leqslant L(T_M) \\ 14 + L(T_{M-4}) &= L(T_M) \end{aligned}$$

**Proof:** First suppose that $M$ is even. Then we have:

$$\begin{aligned} 7 + L(T_{M-2}) &= 7 + 7(M-4)/2 + 4 = 7(M-2)/2 + 4 = L(T_M) \\ 10 + L(T_{M-3}) &= 10 + 7(M-4)/2 \phantom{+ 4} = 7(M-2)/2 + 3 < L(T_M) \\ 14 + L(T_{M-4}) &= 14 + 7(M-6)/2 + 4 = 7(M-2)/2 + 4 = L(T_M) \end{aligned}$$

When $M$ is odd, the calculations are similar, except in that case

$$10 + L(T_{M-3}) = L(T_M). \blacksquare$$

**Theorem 7:** If $|S| = M$ and $M \geqslant 8$, then $L(S) \leqslant L(T_M)$.

**Proof:** It is easily verified by computer that the theorem holds for $M = 8$, 9, 10, and 11. This verification is not as lengthy as it might first appear. As noted above, we need only consider one member of each equivalence class. Further, note that if $S = (s_1, s_2, \ldots, s_n)$ with all $s_i \geqslant 1$, then $D(S) = D(T)$, where $T$ is defined by $T = (s_1 - 1, s_2 - 1, \ldots, s_n - 1)$. Thus, we need only consider those tuples which have at least one zero entry. This significantly reduces the number of tuples that need to be checked.

We have shown that the theorem is true for all tuples $S$ for which $|S| \leqslant m$ with $11 < m < M$. Consider a tuple $S$ for which $|S| = M$ and $|D^7(S)| \leqslant M - 2$. Then we have, using Lemma 6,

$$L(S) \leqslant 7 + L(D^7(S)) \leqslant 7 + L(T_{M-2}) = L(T_M).$$

[Note that $L(S) = 7 + L(D^7(S))$ so long as $L(S) > 7$.] If $|D^7(S)| = M - 1$, then by Theorems 4 and 5, either $|D^{10}(S)| \leqslant M - 3$ or $|D^{14}(S)| \leqslant M - 4$. Thus, by induction and Lemma 6, either

$$L(S) \leqslant 10 + L(D^{10}(S)) \leqslant 10 + L(T_{M-3}) \leqslant L(T_M)$$

or

$$L(S) \leqslant 14 + L(D^{14}(S)) \leqslant 14 + L(T_{M-4}) = L(T_M)$$

whenever $M \geqslant 12$. ■

## 5. FURTHER QUESTIONS

Although showing that $T_M$ gives a game of maximum length was not difficult, there were many details to consider. Additionally, there were many special cases for small values of $M$. This indicates that verifying an upper bound for the length of the general $n$-game is likely to be difficult. As stated above, for $n = 2^r + 1$, $r \geqslant 1$, games of maximum length are given by the tuples $(0, 0, \ldots, 0, M - 1, M)$ [12]. That is, games of maximum length arise from tuples with identical form. Whether this happens for other $n$ is not known. For example, when $n = 2^r - 1$, $r \geqslant 4$, are games of maximum length given by tuples that are in some way similar in form to $(1, M, 1, M, M, 0, M)$?

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## REFERENCES

1. E. R. Berlekamp. "The Design of Slowly Shrinking Labelled Squares." *Math. Comp.* *29* (1975):25-27.
2. K. D. Boklan. "The $n$-Number Game." *The Fibonacci Quarterly* *22* (1984):152-155.
3. M. Burmeister, R. Forcade, & E. Jacobs. "Circles of Numbers." *Glasgow Math. J.* *19* (1978):115-119.

4. C. Ciamberlini & A. Marengoni. "Su una Interessante Curiosita Numerica." *Period. Mat.* Ser. 4 **17** (1937):25-30.

5. M. Dumont & Jean Meeus. "The Four-Number Game." *J. Rec. Math.* **13** (1980-1981):89-96.

6. A. Ehrlich. "Columns of Differences." *Math Teaching* (1977):42-45.

7. B. Freedman. "The Four Number Game." *Scripta Math.* **14** (1948):35-47.

8. R. Honsberger. *Ingenuity in Mathematics.* New York: Random House, 1970.

9. H. H. Lammerich. "Quadrupelfolgen." *Praxis der Mathematik* **17** (1975):219-223.

10. M. Lotan. "A Problem in Difference Sets." *Amer. Math. Monthly* **56** (1949): 219-223.

11. A. L. Ludington. "Cycles of Differences of Integers." *J. Number Theory* **13** (1981):155-261.

12. A. L. Ludington. "Length of the $n$-Number Game." (Submitted.)

13. L. Meyers. "Ducci's Four-Number Problem: A Short Bibliography." *Crux Mathematicorum* **8** (1982):262-266.

14. R. Miller. "A Game with $n$ Numbers." *Amer. Math. Monthly* **85** (1978):183-185.

15. S. P. Mohanty. "On Cyclic Difference of Pairs of Integers." *Math. Stud.* **49** (1981):96-102.

16. D. Richman. "Iterated Absolute Differences." *Discrete Math.* (In press.)

17. W. Webb. "A Mathematical Curiosity." *Math. Notes from Wash. State Univ.* **20** (1980).

18. W. Webb. "The Length of the Four-Number Game." *The Fibonacci Quarterly* **20** (1982):33-35.

19. F. Wong. "Ducci Processes." *The Fibonacci Quarterly* **20** (1982):97-105.

20. P. Zvengrowski. "Iterated Absolute Differences." *Math. Magazine* **52** (1979): 36-37.


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# ON FOLYOMINOES AND FEUDOMINOES

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*(Submitted August 1986)*

## 1. INTRODUCTION

In 1953 Solomon Golomb [1] "invented" polyominoes and gave them to the world in a talk to the Harvard Mathematics Club. Since then polyominoes have given pleasure to tens of thousands, not only through puzzle- and game-type activities carried out with them but also as a source of problems amenable to mathematical study.

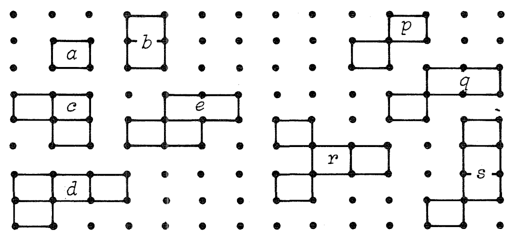
This year we contrived a creative project in combinatorics for a first-year University class. We took the polyominoes and added to them the integers of the Fibonacci sequence in a way to be described below. We christened the resulting objects *folyominoes* and *feudominoes*. In the notes for the project, we wrote: "Thus we have acted as midwife to the birth of twins Folyomino and Feudomino, born of two venerable and well-loved parents, viz. Polyomino and Fibonacci-sequence. We offer the twins to you, to rear, to nourish, and to study; to play with; to build ideas with; to create mathematics with."

In this paper we define the objects of study and describe some of their properties. The linking of the two fields of mathematics will be seen to have given rise to a wealth of new problems, the solution of which can provide the basis for a new field of study. This field might be named *integer sequence geometry*.

## 2. FOLYOMINOES AND FEUDOMINOES

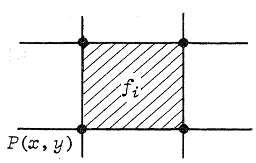
Polyominoes dwell amid the integer points of the Cartesian plane (see [1], [2]). They are formed by connecting unit squares into shapes, by 'glueing' one or more pairs of sides together. Thus, an *n-omino* is a shape consisting of $n$ squares of a large chessboard, connected in such a way that a rook (a chess piece) could be moved from any square of it to any other square of it, in one or more valid rook moves. On the other hand, a *pseudo n-omino* has $n$ unit squares joined together, but this time connection by 'glueing' two vertices is allowed as well as by 'glueing' two sides. In order to traverse all pseudo-polyominoes with a chess piece, we would need to use a king (or a queen) which can make diagonal moves as well as row and column moves.

Examples of both types of $n$-omino are given below for $n = 1, 2, 3$, and 4.




$a$ is a 1-omino (monomino)
$b$ is a 2-omino (domino)
$c$ is a 3-omino (tromino)
$d$ and $e$ are 4-ominoes (tetromino)
$p$ is a pseudo-domino
$q$ is a pseudo-tromino
$r$ and $s$ are pseudo-tetrominoes

In order to derive folyominoes from polyominoes, we first place a pair of rectangular axes on the lattice and then assign Fibonacci integers to the unit squares of the positive quadrant by the following *rule*:




The square having the point $P(x, y)$ at its bottom left-hand corner receives the Fibonacci integer $f_i$, where $i = x + y + 1$ and $f_{i+2} = f_{i+1} + f_i$, with $f_1 = 1 = f_2$.

We may call the result the *Fibonacci lattice*.

Now if we construct a polyomino on this lattice, we may add up the integers in its cells. Let us call the total of the integers in a polyomino $p$ the *value* $v(p)$ of the polyomino.

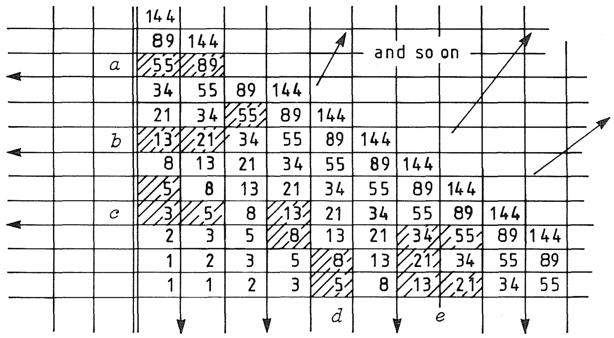
**Definitions:**

(i) If the value of a polyomino is a Fibonacci integer, the numbered polyomino is a *folyomino*.

(ii) If the value of a pseudo-polyomino is a Fibonacci integer, the numbered pseudo-polyomino is a *feudomino*.

In the following diagram, we show the positive quadrant of the Fibonacci lattice, with three example folyominoes marked on it.

The numbering of the lattice could be extended into the other three quadrants. Here, however, all our problems and discoveries will be confined to the positive quadrant.

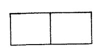



$a$ is a 2-folyomino (total 55 + 89 = 144 = $f_{12}$)
$b$ is a 3-feudomino (total 13 + 21 + 55 = 89 = $f_{11}$)
$c$ is a 3-folyomino
$d$ is a 4-feudomino
$e$ is a 5-folyomino

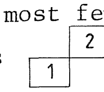
Note that a folyomino is also a feudomino (since a king as well as a rook can traverse a folyomino); but a feudomino with at least one vertex connection cannot also be called a folyomino, since it cannot be traversed by a rook.

## 3. FIRST CLASSIFICATION

Tables 1 and 2 show all the folyominoes having $n$ = 1, 2, 3, 4, or 5 cells, and the feudominoes with $n$ = 1, 2, 3, or 4 cells.

It should be noted that each folyomino is a representative of an infinite class. with any class, the members all have the same shape but differ in their values. For example, the 2-folyominoes  form the class

$$\{\boxed{1\,|\,1},\ \boxed{1\,|\,2},\ \boxed{2\,|\,3},\ \ldots,\ \boxed{f_i\,|\,f_{i+1}},\ \ldots\};$$

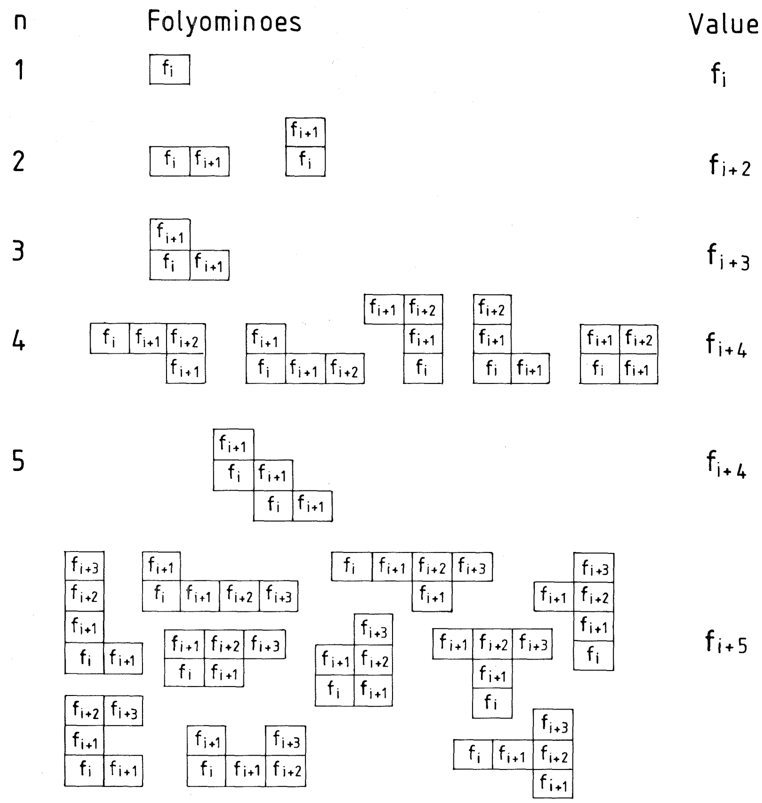
their values form the set $\{f_{i+2} : i = 1, 2, \ldots\}$. The same is true of most feudominoes; however, there are some unique feudominoes. One example is ; we give other examples in Table 2.

Notes:

(i) A polyomino has *size* (i.e., the number of cells in it) and *orientation* in the plane. One can translate it from one part of the plane to another; one can rotate it through 90°, 180°, or 270°; one can flip it over; and it still remains the same polyomino unless one expressly forbids one or the other of these transformations.

A folyomino, on the other hand, also has a *value* (i.e., the total of its cell values); so, under any of the above transformations its value may change. Let us agree that, if the value remains the same after *some* rotations and/or flippings, then the differently oriented folyominoes are *equivalent*. Otherwise, they are inequivalent. (Recall that when we speak of a folyomino, we refer to a representative of an infinite class of folyominoes having the same shape and orientation. We define all members of such a class to be equivalent, too.)

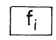
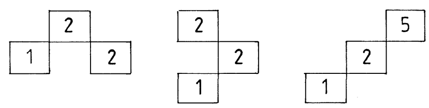
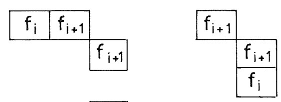
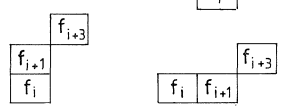
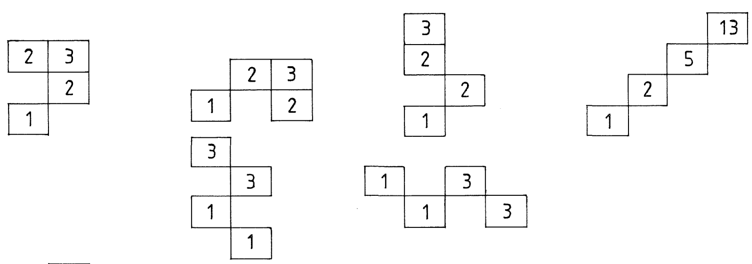
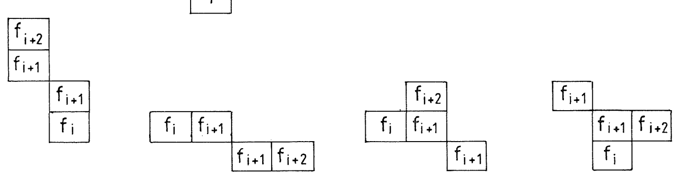
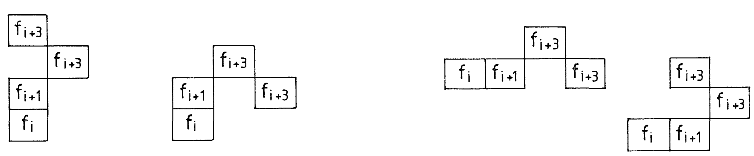
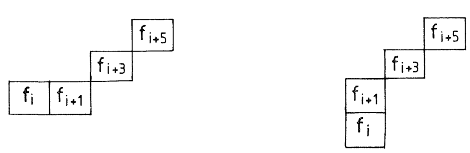
TABLE 1. FOLYOMINOES UP TO n = 5




(ii) Referring to Table 1, we see that the numbers of inequivalent folyominoes, for $n = 1, \ldots, 5$, is as given in the table below. We give also the number of different $n$-polyominoes, for comparison.

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| # folyominoes | 1 | 1 | 1 | 2 | 6 |
| # polyominoes | 1 | 1 | 2 | 5 | 12 |

## TABLE 2. FEUDOMINOES UP TO n=4

| n | Feudominoes (N.B. The Folyominoes in Table 1 are also Feudominoes) | Values |
|---|---|---|
| 1 |  | $f_i$ |
| 2 |  | 2,3 (unique) |
| 3 |  | 5,5,8 (unique) |
| |  | $f_{i+3}$ |
| |  | $f_{i+4}$ |
| 4 |  | 8,8,8,8,8,21 (unique |
| |  | $f_{i+4}$ |
| |  | $f_{i+5}$ |
| |  | $f_{i+6}$ |

(iii) Combining the information of Tables 1 and 2 for $n = 1, \ldots, 4$, and leaving aside the unique feudominoes, we get the following table:

| $n$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| # feudominoes | 1 | 1 | 3 | 7 |
| # pseudo-ominoes | 1 | 2 | 5 | 22 |

(iv) Note that usually if a folyomino is not a square, it has one equivalent folyomino (it is always the case with the feudominoes in Table 2); there are 5 exceptions in Table 1, for $n = 3$ and for $n = 5$.

(v) All four 3-cell feudominoes have the same shape; but two different values occur. It never happens, among the folyominoes of Table 1, that two folyominoes have the same shape and have different values. We ask whether it is possible to construct such a pair of folyominoes.

## 4. TILING PROBLEMS

Many of the attractive problems concerning polyominoes involve finding how to use certain sets of them in order to fill a given shape exactly. For example, there are just 12 different pentominoes, and one problem is to use a set of these to fill (i.e., to *tile*) a $6 \times 10$ rectangle. It has been shown that there are 2339 different ways of doing this (although it is surprisingly difficult to find even one of these, if one cuts the pentominoes out of cardboard and attempts a jig-saw approach to the problem!).

With folyominoes, the number and types of possible tiling problem multiply, because not only can one aim to tile a given shape with them, but also one can aim to achieve certain kinds of total value for the shape (e.g., a Fibonacci number of a particular kind.) Further, one can aim to produce a sequence of shapes that have a given sequence of integer values; we discuss below, in Sections 5 and 6, two problems of this kind.

First we discuss problems of tiling (i) squares, (ii) rectangles, and (iii) the quarter-plane.

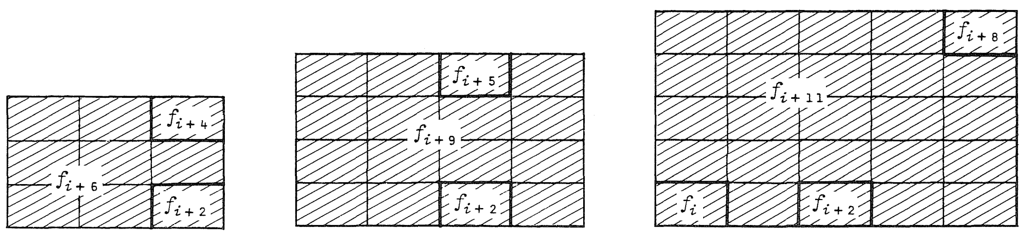
(i) Tiling an $n \times n$ square: Every $1 \times 1$ square is, of course, a folyomino. So, too, is every $2 \times 2$ square, since each has the following arrangement:

| | |
|---|---|
| $f_{i+1}$ | $f_{i+2}$ |
| $f_i$ | $f_{i+1}$ |

which has value $f_{i+4}$. It is worth noting here that if we were to create an $r$-bonacci lattice, assigning integers from an $r$-bonacci sequence to the cells, then every $r \times r$ square would have a value that was an integer of the sequence. We give a tribonacci example of this in Section 7.

Therefore, when $n$ is 1 or 2, the $n \times n$ square can be tiled with a single folyomino. A natural question to ask is: What is the minimal number, say $\phi$, of feudominoes required to tile a given square? We have not yet found a general answer to this question; however, the answers for small $n$ may be found by inspection. A table, and example minimal tilings for $n$ = 3, 4, 5 follow:

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\phi$ | 1 | 2 | 3 | 3 | 4 |




Using the fact that $f_r + \cdots + f_s = f_{s+2} - f_{r+1}$, we easily find the following formula for the value of an $n \times n$ square which has $f_i$ in its least-value cell:

$$V_{n \times n} = f_{i+2n+2} - 2f_{i+n+2} + f_{i+2}.$$

To find a solution for $\phi$ for a given value of $n$, we have to find a minimal partition of $V_{n \times n}$ using Fibonacci integers as addends.

<u>(ii) Tiling an $m \times n$ rectangle</u>: Rather than ask for the minimum number of feudominoes required to tile a given shape, as in (i), we ask what is the *total* number that can be found in the shape, differing in any way.

Let $\Phi_{mn}$ be the total required for an $m \times n$ rectangle. It is easy to show that $\Phi_{m1} = 2m - 1$; but we have not yet found a formula for $\Phi_{m2}$, even when adding the restriction that only folyominoes be counted.

<u>(iii) Tiling the quarter-plane</u>: Referring to the 'positive', or 'north-east', portion of the plane only, simple tiling problems are: Tile the quarter-plane using only

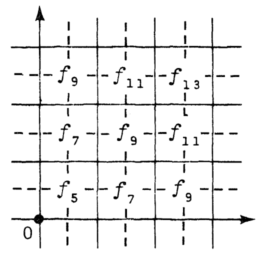
(a) even-valued folyominoes;

(b) odd-valued folyominoes;

(c) folyominoes with even-subscripted $F$-values;

(d) folyominoes with odd-subscripted $F$-values.

We easily found solutions for (a), (b), and (d); but for (c) our only solution so far uses a 5-feudomino of value $f_6$. The simplest solution for (d) uses the 2 × 2 squares, thus:




Incidentally, this solution with odd-subscripted folyominoes suggests the following generalization. Defining a *lolyomino* to be a polyomino whose value is an integer of the Lucas sequence, $\{L_i\} = 1, 3, 4, 7, \ldots$, the diagram above immediately gives a tiling in terms of even-subscripted lolyominoes. This follows from the fact that $L_i = f_{i-1} + f_{i+1}$; so placing two 2 × 2 squares side by side gives a lolyomino. Thus, $\boxed{f_5}\boxed{f_7} = \boxed{\ \ L_6\ \ }$; and the required type of quarter-plane tiling, using 2 × 4 lolyominoes of even-subscript values, is immediately evident.

We turn now to a new kind of tiling problem: Given any integer, does a shape (i.e., a combination of cells) exist whose total value equals the integer, and which can be tiled by distinct folyominoes? We shall call this the *integer tiling problem*; and, in view of Zeckendorf's theorem on Fibonacci partitioning of the integers, it is easy to arrive at a solution.

## 5. ZECKENDORF INTEGER TILINGS

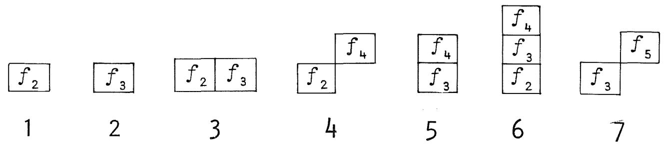
Zeckendorf's theorem (see [4] for details) tells us that any integer can be partitioned into distinct Fibonacci integers in such a way that there is no gap larger than one in the sequence of $f_i$-values used in the partition, with all sequences beginning with $f_2 = 1$ or $f_3 = 2$.

We construct the required partitions recursively as follows: Let the partition of 1, namely $f_2$, be written as a set $P_1 = \{f_2\}$; and the partitions of 2 and 3 be written as $P_2 = \{f_3\}$, $P_3 = \{f_1, f_2\}$, respectively. Then the partitions of the next three (= $f_4$) integers are given by:

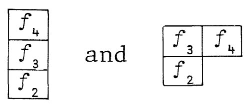
$$P_4 = P_1 \cup \{f_4\}; \quad P_5 = P_2 \cup \{f_4\}; \quad P_6 = P_3 \cup \{f_4\}.$$

The partitions of the next five (= $f_5$) integers are given by taking the union of $\{f_5\}$ with each of $P_2, P_3, P_4, P_5$, and $P_6$, in turn. For the next eight (= $f_6$), we take the union $\{f_6\}$ of $P_4, P_5, \ldots, P_{11}$, in turn. And so on.

Using the same recurrence procedure, and with each union taking the corresponding cells from the Fibonacci lattice, we can construct shapes which constitute Zeckendorf tilings for each integer. The tilings for $n = 1, \ldots, 7$ are shown below:




Note that, for $n = 6$, two types of tile arise, viz:




Therefore, the answer to the integer tiling problem is: for each integer, a Zeckendorf tiling can be constructed. Some integers have more than one type of Zeckendorf tiling (Z-tiling).

Now that we have shown how to construct Zeckendorf integer tilings, we can classify the integers according to defined properties of their respective tilings. Four interesting properties are:

$\phi$ = minimal number of folyominoes in a Z-tiling;
$\delta$ = number of diagonal connections in a Z-tiling;
$\tau$ = number of types of Z-tiling (different up to rotations and flippings of the shape only) of a given integer;
$\sigma$ = size (i.e., number of cells used) of a Z-tiling.

**Remark:** $\phi$, $\delta$, and $\sigma$ are invariant over tiling types, and $\delta = 0$ for $n = f_i - 2$ and $f_i - 3$, $i \geqslant 5$.

We will conclude this section by tabulating the four properties for the Z-tilings of $n = 1, \ldots, 19$. A recurrence formula can be written down to generate the sequence of $\sigma$ values.

**Table 3.** Properties of Z-Tilings

| $n$: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\phi$: | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 2 | 3 | 2 | 3 |
| $\delta$: | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 2 | 1 | 1 | 2 | 1 | 1 | 0 | 0 |
| $\tau$: | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 2 | 3 | 1 | 1 | 2 | 1 | 1 | 2 | 3 | 6 |
| $\sigma$: | 1 | 1 | 2 | 2 | 2 | 3 | 2 | 3 | 3 | 3 | 4 | 3 | 3 | 4 | 3 | 4 | 4 | 4 | 5 |

## 6. LATTICE PATHS AND $F$-CHAINS

There is a large literature on the combinatoric theory of paths defined on rectangular lattices, and [3] gives a good review of this. It is natural that we should now combine the notion of folyomino with that of paths on a Fibonacci lattice.

**Definitions:**

(i) A *simple path* on a Fibonacci lattice is a sequence of distinct cells on the lattice, each cell arrived at being adjacent (horizontally, vertically, or diagonally) to the previous cell. In chess terms, then, a simple path is a king's tour with no repetitions of cells. Let us use the symbols $c_1, c_2, \ldots, c_n$ to describe a simple path starting at $c_1$ and ending at cell $c_n$; the sequence of cell values will be described by $v_1, v_2, \ldots, v_n$. The *length* of a simple path is the number of cells in it. The *value* of the path is

$$V_n = \sum_{i=1}^{n} v_i .$$

The $r^{\text{th}}$ *partial path* is $c_1, c_2, \ldots, c_r$, with $1 \leqslant r \leqslant n$, having value

$$V_r = \sum_{i=1}^{r} v_i .$$

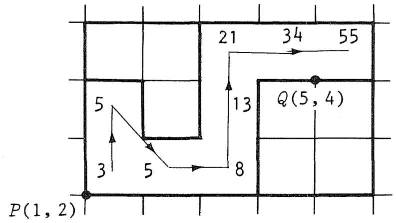
(ii) An *F-chain* is a simple path on a Fibonacci lattice such that all its partial paths are feudominoes; that is, all the partial path values $V_1, V_2, \ldots, V_n$ are Fibonacci integers.

**Counting the $F$-chains**

We will address the basic problem only, namely that of counting the number of $F$-chains that start at $P(i, j)$, in the cell $c_1$ having value $f_{i+j+1}$, and end at $Q(r, s)$, in the cell $c_n$ having value $f_{r+s+1}$. We assume that $0 \leqslant i \leqslant r$ and $0 \leqslant j \leqslant s$. There are many cases to consider, if one looks at the different possible steps from cell to cell; if one does or does not allow unique steps [e.g., $P(1, 1)$ to $0(0, 0)$, involving the value sum $f_3 + f_1$]; if one imposes boundaries that a path cannot cross. To keep this introduction short, we give solutions for just two cases.

**Case 1:** Only steps in one of four directions $\uparrow$, $\rightarrow$, $\nwarrow$, $\searrow$ (i.e., N, E, NW, SE) are allowed; and all the paths are to lie entirely within the boundary of the rectangle determined by the diagonal $PQ$.

**Solution:** We refer to the example in which $i = 1$, $j = 2$, $r = 5$, and $s = 4$; the inference to the general solution given at the end is elementary.




The $F$-chain shown in the diagram has partial path values:

3, 8, 13, 21, 34, 55, 89, 144

Note that the first two steps of all $F$-chains from $P$ are forced to be either N, SE (giving partial value $3+5+5 = 13$) or E, NW (again giving partial value 13). From there on, all paths can proceed by only N or E steps. To get from lower 5-cell to the 55-cell, starting with value 13, required two N-steps and three E-steps. The number of different ways of doing this is equal to the number of different arrangements of the symbols N N E E E, which is $\binom{5}{2}$. Similarly, to get from the upper 5-cell to the 55-cell requires one N-step and four E-steps; the number of ways of doing this is $\binom{5}{1}$. Hence, the total number of $F$-chains from $P$ to $Q$ is $\binom{5}{1} + \binom{5}{2} = 15$.

Generalizing, the number of $F$-chains from $P(i, j)$ to $Q(r, s)$ is given by:

$$\binom{r+s-i-j-1}{s-j-1} + \binom{r+s-i-j-1}{s-j} = \binom{m}{n},$$

where $m = (r+s) - (i+j)$ and $n = s - j$, $m, n > 0$.

The value of each $F$-chain is the same, namely $f_{r+s+3}$. This is remarkable, in that the value is independent of $i$ and $j$. Thus, we can state the following proposition regarding $F$-chains.

**Proposition**: Given $Q(r, s)$, and any other point $P(i, j)$ with $0 \leqslant i < r$ and $0 \leqslant j < s$. All $F$-chains from $P$ to $Q$, with the conditions of Case 1, have the same value $f_{r+s+3}$.

**Case 2**: Only steps in one of the five directions $\uparrow$, $\rightarrow$, $\nwarrow$, $\searrow$, $\nearrow$ (i.e., N, E, NW, SE, NE) are allowed; $i \geqslant 1$ and $j \geqslant 1$; and no boundary conditions imposed.

**Solution**: Allowing for the NE steps (which were not allowed in Case 1) and removing boundary conditions leads to many more possibilities for constructing $F$-chains from $P(i, j)$ to $Q(r, s)$. We give the solution in terms of two coupled partial recurrence equations. To explain them, we must first define the following three counting functions.

(i) $A(i, j)$ is the number of $F$-chains from $P(i, j)$ to $Q(r, s)$, with all cells having their usual assigned $F$-values.

(ii) $B(i, j)$ is the number of $F$-chains from $P$ to $Q$, with the first cell in each chain having value $f_{i+j+2}$ and the others having their usual values.

(iii) $C(i, j)$ is the number of $F$-chains from $P$ to $Q$, with the first cell having value $f_{i+j+3}$ and the others having their usual values.

Now let us consider the first steps of $F$-chains from $P(i, j)$, beginning with the first cell-value $f_{i+j+1}$. There are two possibilities; namely, either a step $N$ leading to cell $(i, j+1)$ and partial value $V_2 = f_{i+j+2}$, or else a step $E$ leading to cell $(i+1, j)$, again with partial value $V_2 = f_{i+j+2}$. We can, therefore, write down the equation:

$$A(i, j) = B(i, j+1) + B(i+1, j) \tag{1}$$

Considering $F$-chains starting from $P(i, j)$ with the first cell having value $f_{i+j+2}$, we see that three different first steps are possible, the first two being to cells $(i-1, j+1)$ or $(i+1, j-1)$ in which cases the partial values $V_2 = f_{i+j+3}$ are achieved; the third is to cell $(i+1, j+1)$, achieving $V_2 = f_{i+j+4}$. From this information we can write down the equation:

$$B(i, j) = C(i-1, j+1) + C(i+1, j-1) + B(i+1, j+1) \tag{2}$$

Finally, we need an equation for $C(i, j)$. In fact, as explained above in Case 1, we can obtain a formula for it, thus,

$$C(i, j) = \binom{m}{n}, \tag{3}$$

where $m = (r+s) - (i+j)$ and $n = s - j$. (N.B. It is no accident that this number is precisely the same as the total for Case 1, as a moment's reflection on the two cases will show.)

Putting formula (3) into equation (2) gives:

$$B(i, j) = B(i+1, j+1) + \binom{m}{n-1} + \binom{m}{n+1} \tag{4}$$

For any given pair of values of $(r, s)$, we can use equation (4) to compute a table of values $B(i, j)$; then, finally, using equation (1) with a particular pair $(i, j)$ will give us the total $A(i, j)$, which is the object of the study.

As mentioned earlier, there are many other problems we could pose about $F$-chains, the solutions of which we could seek by means of lattice-path counting methods; but we must leave them here.

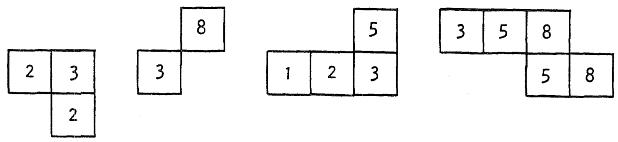
## 7. SUMMARY AND EXTENSIONS

We have shown how an integer sequence can be assigned to a lattice, and be used to give values to polyominoes constructed on the lattice. We chose to use the Fibonacci sequence, and studied tiling and path problems related to the folyominoes which resulted.

Many interesting possibilities suggest themselves for varying and extending our studies. We end by briefly indicating some of these.

Lucas polyominoes (lolyominoes)

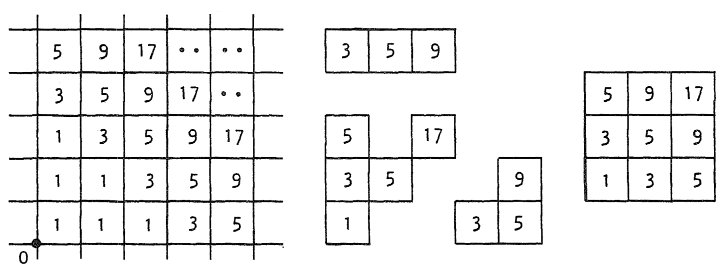
We have defined a *lolyomino* to be a polyomino whose value is a member of the Lucas sequence 1, 3, 4, 7, 11, 18, ... . Examples of lolyominoes found on the Fibonacci lattice are:




We can study lolyominoes on the Fibonacci lattice. Likewise, we can use the Lucas sequence to produce a Lucas lattice: then we can study folyominoes on the Lucas lattice. It is clear that interesting comparisons and dual relations between the two systems will abound.

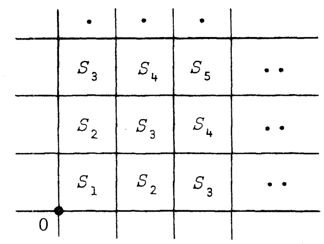
Integer sequence geometry

In Section 4(i), we noted a result concerning polyominoes defined on $r$-bonacci lattices. To give one example of such a lattice, with $r = 3$ and the sequence 1, 1, 1, 3, 5, 9, 17, 31, ..., we show a portion of the lattice, and a few small-size *trolyominoes*.



Note that the $3 \times 3$ square is a trolyomino, as claimed in 4(i). Note also that only odd-sized trolyominoes are possible: this is easily proved true, for all single cells have an odd value, and any combination of an even number of them would have an even total value. Since all members of this tribonacci sequence are odd, an even-valued combination of cells cannot be a trolyomino.

Finally, we do not have to stay with $r$-bonacci sequences. Generally, we can use the sequence $s_1$, $s_2$, $s_3$, ... to define an $S$-lattice thus: Our definition of what constitutes an $S$-polyomino (see the figure on the following page) will depend on whatever property or properties of the sequence $\{S_i\}$ we wish to highlight. Then our discoveries concerning the $S$-polyominoes (or *solyominoes*) will constitute results in *integer sequence geometry*.




Generalizations will take place when we compare results on solyominoes drawn from a class of $S$-lattices, defined using a class of related integer sequences. An obvious candidate for such studies is a class of $F$-lattices, using the sequences $F(a, b)$ defined by

$$F_1 = a,\ F_2 = b,\ F_{i+2} = F_{i+1} + F_i, \quad (a, b) \in Z \times Z.$$

## REFERENCES

1. S. W. Golomb. *Polyominoes*. New York: Scribner, 1965.
2. D. A. Klarner, ed. *The Mathematical Gardner*. New York: Prindle, Weber & Schmidt, 1981.
3. S. G. Mohanty. *Lattice Path Counting and Applications*. New York: Academic Press, 1979.
4. J. L. Brown, Jr. "Zeckendorf's Theorem and Some Applications." *The Fibonacci Quarterly* 2, no. 3 (1964):163-168.

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# GENERALIZED FIBONACCI CONTINUED FRACTIONS

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## 1. INTRODUCTION

Eisenstein [3] proposed and Lord [8] solved elegant problems to the effect that the infinite continued fractions (in the preferred notation of Khovanskii [7])

$$L_n - \frac{(-1)^n}{L_n} - \frac{(-1)^n}{L_n} - \cdots = \alpha^n, \tag{1.1}$$

where $L_n$ is the $n^{\text{th}}$ lucas number and $\alpha$ is the positive root of $x^2 - x - 1 = 0$.

The purpose of this note is to generalize (1.1), which we do in (4.2) for the sequence $\{w_n\} \equiv \{w_n(a, b; p, q)\}$ (see Horadam [5]). This is defined by the initial conditions $w_0 = a$, $w_1 = b$, and the recurrence relation

$$w_n = pw_{n-1} - qw_{n-2},\ n \geqslant 2, \tag{1.2}$$

where $p$ and $q$ are arbitrary integers.

## 2. NOTATION

Following Horadam, we let $\alpha = (p + \sqrt{(p^2 - 4q)})/2$, $\beta = (p - \sqrt{(p^2 - 4q)})/2$, with $|\beta| < 1$, be the roots of

$$x^2 - px + q = 0, \tag{2.1}$$

so that $\{w_n\}$ has the general term

$$w_n = A\alpha^n + B\beta^n, \tag{2.2}$$

where $A = (b - a\beta)/d$, $B = (a\alpha - b)/d$, and $AB = e/d^2$ in which $e = pab - qa^2 - b^2$, $d = \alpha - \beta$, $p = \alpha + \beta$, and $q = \alpha\beta$. Furthermore, for notational convenience, let

$$Q_n = ABq^n. \tag{2.3}$$

For example, for the sequence of Fibonacci numbers $\{F_n\} \equiv \{w_n(0, 1; 1, -1)\}$, $Q_n = (-1)^{n+1}/5$; for the Lucas numbers $\{L_n\} \equiv \{w_n(2, 1; 1, -1)\}$, $Q_n = (-1)^n$; and for the Pell numbers $\{P_n\} \equiv \{w_n(0, 1; 2, -1)\}$, $Q_n = (-1)^n/8$.

## 3. THE CONVERGENTS

Let $x_k = p_k/q_k$ be the $k^{\text{th}}$ convergent of the continued fraction (CF)

$$\mathrm{CF}(w_n) = w_n - \frac{Q_n}{w_n -}\,\frac{Q_n}{w_n -}\,\cdots\,. \tag{3.1}$$

$$x_k - x_{k+1} = \frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} = (p_k^2 - p_{k+1}p_{k-1})/q_kq_{k+1},$$

since $p_k = q_{k+1}$ (Khinchin [6]). So, from equations (1.9) and (4.3) of [5],

$$x_k - x_{k+1} = Q_n^k/q_kq_{k+1}. \tag{3.2}$$

For further notational convenience, suppose we write

$$X_k = x_{k+1} \tag{3.3}$$

so that (3.2) has the form

$$X_k - X_{k-1} = -Q_n^k/q_kq_{k-1}. \tag{3.4}$$

Replace $k$ by $k + 1$ in (3.4) to get

$$X_{k+1} - X_k = -Q_n^{k+1}/q_{k+1}q_k. \tag{3.5}$$

If we add (3.4) and (3.5), then

$$\begin{aligned} X_{k+1} - X_{k-1} &= -\frac{Q_n^k}{q_k}\left(\frac{Q_n}{q_{k+1}} + \frac{1}{q_{k-1}}\right) = -\frac{Q_n^k}{q_k}\left(\frac{q_{k-1}Q_n + q_{k+1}}{q_{k+1}q_{k-1}}\right) \\ &= -\frac{Q_n^k}{q_k}\left(\frac{q_{k-1}Q_n + w_nq_k - Q_nq_{k-1}}{q_{k+1}q_{k-1}}\right) \quad \text{[from (4.3)]} \\ &= -w_nQ_n^k/q_{k+1}q_{k-1}. \end{aligned}$$

Replace $k$ by $2K$, so that

$$X_{2K+1} - X_{2K-1} = -w_nQ_n^{2K}/q_{2K-1}q_{2K+1}. \tag{3.6}$$

Now, by (3.6),

$$\begin{aligned} X_3 - X_1 &= -w_nQ_n^2/q_1q_3 \\ X_5 - X_3 &= -w_nQ_n^4/q_3q_5 \\ &\vdots \\ X_{2K+1} - X_{2K-1} &= -w_nQ_n^{2K}/q_{2K-1}q_{2K+1}. \end{aligned}$$

On adding, we get

$$X_{2K+1} = w_n\left(1 - \frac{Q_n^2}{q_1q_3} - \frac{Q_n^4}{q_3q_5} - \cdots - \frac{Q_n^2}{q_{2K-1}q_{2K+1}}\right) - \frac{Q_n}{w_n} \tag{3.7}$$

since $X_1 = w_n - Q_n/w_n$.

Similarly, on replacing $k$ by $2K - 1$, we obtain

$$X_{2K} = w_n\left(1 - \frac{Q_n}{q_0 q_2} - \frac{Q_n^3}{q_2 q_4} - \cdots - \frac{Q_n^{2K-1}}{q_{2K-2}q_{2K}}\right) \tag{3.8}$$

since $X_0 = w_n$.

With our notation adapted to Khovanskii's treatment, he established that when all the coefficients $w_n$ and $-Q_n$ are positive:

(i) the convergents of odd order generate a monotonically increasing sequence with upper bound the even convergent $w_n - Q_n/w_n$, that is, $\lim_{K\to\infty} X_{2K}$ exists and is smaller than each even convergent; and

(ii) the convergents of even order generate a monotonically decreasing sequence with lower bound the odd convergent $w_n$, that is, $\lim_{K\to\infty} X_{2K-1}$ exists and is greater than each odd convergent.

## 4. THE CONTINUED FRACTION

In either case, the value of the limit is a root of the equation

$$x = w_n - Q_n/x, \tag{4.1}$$

which can be rewritten as

$$\begin{aligned} 0 &= x^2 - w_n x + Q_n \\ &= x^2 - (A\alpha^n + B\beta^n)x - AB\alpha^n\beta^n = (x - A\alpha^n)(x - B\beta^n). \end{aligned}$$

Since $x_k > w_n - 1$, we have

$$(w_n) = A\alpha^n. \tag{4.2}$$

For example, $\mathrm{CF}(L_n) = \alpha^n$, $\mathrm{CF}(F_n) = \alpha^n/d$, $\mathrm{CF}(P_n) = \alpha_1^n\sqrt{2}/4$, and $\mathrm{CF}(L_1) = \alpha$ (see Vorob'ev [9]), where $\alpha = (1 + \sqrt{5})/2$ and $\alpha_1 = 1 + \sqrt{2}$. This is consistent with $|w_n - A\alpha^n| = B|\beta|^n < B$ if $|\beta| < 1$, or $|F_n - \alpha^n/d| < 1/2$ and $|L_n - \alpha^n| < 1/2$ as in Hoggatt [4].

Since $x_k = w_n - Q_n/x_{k-1}$, we have $\dfrac{p_k}{q_k} = w_n - \dfrac{Q_n}{p_{k-1}/q_{k-1}}$ or

$$p_k = w_n p_{k-1} - Q_n p_{k-2}, \; k \geqslant 2, \tag{4.3}$$

with $p_0 = 1$ and $p_1 = w_n$ since $p_k = q_{k+1}$. Note that (3.2) can also be expressed as $p_{k+1}q_k - p_k q_{k+1} = -Q_n^k$ or, in determinantal form, as

$$\begin{vmatrix} p_k & p_{k+1} \\ q_k & q_{k+1} \end{vmatrix} = Q_n^k. \tag{4.4}$$

## 5. CONCLUDING COMMENTS

It can be seen from the recurrence relation (4.3) and the initial conditions for $p_k$, that the numerators of the convergents, $\{p_k\} \equiv \{p_k(1, w_n; w_n, Q_n)\}$, form a generalized Fibonacci sequence. The first few terms can be constructed as follows:

$$\begin{aligned} p_0 &= 1 & p_4 &= w_n^4 - 3Q_n w_n^2 + Q_n^2 \\ p_1 &= w_n & p_5 &= w_n^5 - 4Q_n w_n^3 + 3Q_n^2 w_n \\ p_2 &= w_n^2 - Q_n & p_6 &= w_n^6 - 5Q_n w_n^4 + 6Q_n^2 w_n^2 - Q_n^3 \\ p_3 &= w_n^3 - 2Q_n w_n & p_7 &= w_n^7 - 6Q_n w_n^5 + 10Q_n^2 w_n^3 - 4Q_n^3 w_n \end{aligned}$$

It can be seen that the values of the numerical coefficients seem to satisfy the partial recurrence relation

$$a_{ij} = a_{i-1,j} - a_{i-2,j-1}, \quad i, j \geqslant 0, \tag{5.1}$$

with boundary conditions given by $a_{i0} = 1$ and $a_{ij} = 0$ if $i < 0$ or $j < 0$ or $j > [i/2]$ (the integer part of $i/2$). (Note: $i$ and $j$ refer to row and column numbers, respectively.)

To establish that the numerical coefficients satisfy (5.1), we first solve (5.1) and then show that the solutions in (5.3) can be used to generate $p_k$ in (5.4). Following Carlitz [2], we set (formally)

$$F(x, y) = \sum_{i=1}^{\infty} \sum_{j=0}^{[i/2]} a_{ij} x^i y^j, \tag{5.2}$$

and rewrite (5.1) using the boundary conditions on $a_{ij}$:

$$\begin{aligned} F(x, y) &= x \sum_{i=1}^{\infty} \sum_{j=0}^{[i/2]} a_{i-1,j} x^{i-1} y^j - x^2 y \sum_{i=1}^{\infty} \sum_{j=0}^{[i/2]} a_{i-2,j-1} x^{i-2} y^{j-1} \\ &= x \sum_{i=0}^{\infty} \sum_{j=0}^{[i/2]} a_{ij} x^i y^j - x^2 y \sum_{i=0}^{\infty} \sum_{j=0}^{[i/2]} a_{ij} x^i y^j \\ &= x + x \sum_{i=1}^{\infty} \sum_{j=0}^{[i/2]} a_{ij} x^i y^j - x^2 y - x^2 y \sum_{i=1}^{\infty} \sum_{j=0}^{[i/2]} a_{ij} x^i y^j \\ &= x + xF(x, y) - x^2 y - x^2 y F(x, y) \\ &= (x - x^2 y)/(1 - x + x^2 y) \\ &= x(1 - xy)(1 - x(1 - xy))^{-1} \\ &= \sum_{i=0}^{\infty} x^{i+1}(1 - xy)^{i+1} = \sum_{i=1}^{\infty} x^i (1 - xy)^i \\ &= \sum_{i=1}^{\infty} \sum_{j=0}^{i} (-1)^j \binom{i-j}{j} x^i y^j, \end{aligned}$$

whence, on equating coefficients of $xy$,

$$a_{ij} = (-1)^j \binom{i-j}{j}. \tag{5.3}$$

So, from equation (2.8) of Barakat [1],

$$p_k = \sum_{j=0}^{[k/2]} (-1)^j \binom{k-j}{j} Q_n^j w^{k-2j}, \tag{5.4}$$

and it can be confirmed by induction on $k$ that (5.4) satisfies the recurrence relation (4.3).

## REFERENCES


1. R. Barakat. "The Matrix Operator $e^x$ and the Lucas Polynomials." *Journal of Mathematics and Physics* **43** (1964):332-335.
2. L. Carlitz. "Some Difference Equations." *Duke mathematical Journal* **33** (1966):27-32.
3. M. Eisenstein. B-530, B-531, Problems Proposed. *The Fibonacci Quarterly* **22** (1984):274.
4. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.
5. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Integers." *The Fibonacci Quarterly* **3** (1965):161-176.
6. A. Ya. Khinchin. *Continued Fractions*, 3rd ed. (trans. from the Russian by Scripta Technica, Inc.). Chicago: University of Chicago Press, 1964.
7. A. N. Khovanskii. *The Application of Continued Fractions* (trans. from the Russian by Peter Wynn). Gronigen: Noordhoff, 1963.
8. G. Lord. B-530, B-531, Problems Solved. *The Fibonacci Quarterly* **23** (1985): 280-281.
9. N. N. Vorob'ev. *Fibonacci Numbers* (trans. from the Russian by Halina Moss). Oxford: Pergamon, 1961.


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# CARLITZ FOUR-TUPLES

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**Definition 1:** We say that $\langle a, b, c, d\rangle$ is a *Carlitz four-tuple* iff $a, b, c, d$ are integers such that $ab \equiv 1 \pmod{c}$, $ab \equiv 1 \pmod{d}$, $cd \equiv 1 \pmod{a}$, and $cd \equiv 1 \pmod{b}$. For convenience, we shall often write CFT instead of Carlitz four-tuple.

As can easily be verified, $\langle 1, 6, 1, 1\rangle$, $\langle 4, 24, 5, 5\rangle$, and $\langle 15, 90, 19, 19\rangle$ are Carlitz four-tuples. More generally, for every integer $a$, $\langle 1, a, 1, 1\rangle$, $\langle a, a(a+2), a+1, a+1\rangle$, and $\langle a(a+2), a(a+2)(a+3), a(a+3)+1, a(a+3)+1\rangle$ are CFTs. The latter two of these are, in some sense (see the comments between Theorem 17 and Proposition 18), generated by $\langle 1, a, 1, 1\rangle$ and, in fact, $\langle 1, a, 1, 1\rangle$ generates not just these two CFTs but infinitely many CFTs.

Both $\langle 4, 16, 7, 7\rangle$ and $\langle 5, 20, 11, 11\rangle$ are CFTs; more generally, for every integer $a$, $\langle a, 4a, 2a-1, 2a-1\rangle$ is a CFT.

Carlitz proved in [1] that, if $\langle a, b, c, d\rangle$ is a Carlitz four-tuple, then either $a = b$ or $c = d$. Thus, in the sequel, we shall only consider CFTs of the form $\langle a, b, c, c\rangle$.

There are CFTs $\langle a, b, c, c\rangle$ for which $a = b$ and for which $a = -b$. Some examples are:

$$\langle a, a, a+1, a+1\rangle;\ \langle a, a, a^2-1, a^2-1\rangle;\ \langle a, -a, a^2+1, a^2+1\rangle.$$

Notice also that, if $\langle a, b, c, c\rangle$ is a CFT, then so are $\langle b, a, c, c\rangle$, $\langle -a, -b, c, c\rangle$, $\langle a, b, -c, -c\rangle$, and $\langle -a, -b, -c, -c\rangle$.

**Definition 2:** The Carlitz four-tuple $\langle a, b, c, c\rangle$ is *primitive* iff there does not exist an integer $m > 1$ such that $\left\langle \frac{a}{m}, bm, c, c\right\rangle$ is a CFT.

The CFTs $\langle 8, 12, 5, 5\rangle$ and $\langle 30, 45, 19, 19\rangle$ are not primitive; for each of these we could choose $m = 2$.

The following result shows that CFTs occur in pairs.

**Proposition 3:** If $\langle a, b, c, c\rangle$ is a Carlitz four-tuple, then so is

$$\left\langle a, b, \frac{ab-1}{c}, \frac{ab-1}{c}\right\rangle.$$

**Proof:** Let $d = \frac{ab - 1}{c}$, which is an integer, since $ab \equiv 1 \pmod{c}$. Since $cd = ab - 1 \equiv -1 \pmod{[a, b]}$,

$$1 \equiv c^2d^2 \equiv d^2 \pmod{[a, b]}.$$

Since we also have that $ab = cd + 1 \equiv 1 \pmod{d}$, we see that

$$\left\langle a, b, \frac{ab - 1}{c}, \frac{ab - 1}{c}\right\rangle$$

is also a CFT. ■

Given a Carlitz four-tuple $\langle a, b, c, c\rangle$, we shall prove, after a lemma, a necessary and sufficient condition for this CFT to be primitive. Then, after another lemma, we shall prove that $a|b$ for any primitive CFT $\langle a, b, c, c\rangle$.

**Lemma 4:** Let $\langle a, b, c, c\rangle$ be a Carlitz four-tuple and let $m$ be an integer. We have that $\left\langle \frac{a}{m}, bm, c, c\right\rangle$ is a Carlitz four-tuple iff $m$ divides $\left(a, \frac{c^2 - 1}{b}\right)$.

**Proof:** First, assume that $\left\langle \frac{a}{m}, bm, c, c\right\rangle$ is a CFT. Thus, $m|a$ and $c^2 \equiv 1 \pmod{bm}$. Since $m|a$ and $m$ divides $\frac{c^2 - 1}{b}$, $m$ divides $\left(a, \frac{c^2 - 1}{b}\right)$.

Conversely, assume that $m$ divides $\left(a, \frac{c^2 - 1}{b}\right)$. Thus, $\frac{a}{m}$ is an integer. Now, since $\langle a, b, c, c\rangle$ is a CFT,

$$\frac{a}{m}bm = ab \equiv 1 \pmod{c}$$

and $c^2 \equiv 1 \pmod{a}$. Hence, $c^2 \equiv 1 \left(\bmod \frac{a}{m}\right)$. Also, since $m$ divides $\frac{c^2 - 1}{b}$, $c^2 \equiv 1 \pmod{bm}$. Therefore, $\left\langle \frac{a}{m}, bm, c, c\right\rangle$ is a CFT. ■

Following directly from Lemma 4 is

**Theorem 5:** Let $\langle a, b, c, c\rangle$ be a Carlitz four-tuple. We have that $\langle a, b, c, c\rangle$ is a primitive Carlitz four-tuple iff $\left(a, \frac{c^2 - 1}{b}\right) = 1$.

**Lemma 6:** If $\langle a, b, c, c\rangle$ is a Carlitz four-tuple, then $a$ divides $b\left(a, \frac{c^2 - 1}{b}\right)$.

**Proof:** Let $e = \frac{c^2 - 1}{b}$. Since $a$ divides $c^2 - 1 = be$ and $a$ divides $ab$, $a$ divides $(ab, be) = |b|(a, e)$. ■

**Proposition 7:** If $\langle a, b, c, c\rangle$ is a primitive Carlitz four-tuple, then $a|b$.

**Proof:** By Lemma 6 and Theorem 5, $a$ divides $b\left(a, \frac{c^2 - 1}{b}\right) = b$. ■

The converse of Proposition 7 is not true. A counterexample is

$$\langle 12, 24, 7, 7\rangle,$$

which is a nonprimitive CFT.

We shall now prove two propositions. Given a CFT, the first proposition will enable us to find the primitive CFT that, in some sense, generates the given CFT. The second proposition does the opposite, i.e., given a primitive CFT, this proposition will enable us to find all CFTs that are generated by the given primitive CFT.

**Proposition 8:** If $\langle a, b, c, c\rangle$ is a Carlitz four-tuple and $e = \frac{c^2 - 1}{b}$, then

$$\left\langle\frac{a}{(a, e)}, b(a, e), c, c\right\rangle$$

is a primitive Carlitz four-tuple.

**Proof:** By Lemma 4, $\left\langle\frac{a}{(a, e)}, b(a, e), c, c\right\rangle$ is a CFT. Since

$$\left(\frac{a}{(a, e)}, \frac{c^2 - 1}{b(a, e)}\right) = \left(\frac{a}{(a, e)}, \frac{e}{(a, e)}\right) = 1,$$

by Theorem 5, $\left\langle\frac{a}{(a, e)}, b(a, e), c, c\right\rangle$ is a primitive CFT. ∎

The converse of this result is false. For example, choose $a = 75$, $b = 18$, and $c = 19$. Thus, $e = \frac{c^2 - 1}{b} = 20$ and $(a, e) = 5$. Now,

$$\left\langle\frac{a}{(a, e)}, b(a, e), c, c\right\rangle = \langle 15, 90, 19, 19\rangle$$

is a primitive CFT but $\langle a, b, c, c\rangle = \langle 75, 18, 19, 19\rangle$ is not a CFT.

**Proposition 9:** Let $\langle a, b, c, c\rangle$ be a primitive Carlitz four-tuple. We have that $\left\langle aj, \frac{b}{j}, c, c\right\rangle$ is a Carlitz four-tuple iff $j \mid \frac{b}{a}$.

**Proof:** First, assume that $\left\langle aj, \frac{b}{j}, c, c\right\rangle$ is a CFT. By Lemma 6, Theorem 5, and without loss of generality, assuming $j > 0$, we see that $aj$ divides

$$\frac{b}{j}\left(aj, \frac{c^2 - 1}{b/j}\right) = \frac{b}{j}\left(aj, \frac{j(c^2 - 1)}{b}\right) = b\left(a, \frac{c^2 - 1}{b}\right) = b.$$

Conversely, assume that $aj \mid b$. First, notice that

$$aj\frac{b}{j} = ab \equiv 1 \pmod{c}.$$

Since we have that $c^2 \equiv 1 \pmod{b}$, $aj \mid b$, and $\frac{b}{j} \mid b$,

$$c^2 \equiv 1 \pmod{aj} \quad \text{and} \quad c^2 \equiv 1 \left(\text{mod } \frac{b}{j}\right). \blacksquare$$

The next two theorems (Theorems 10 and 13) consider the cojnection between a CFT $\langle a, b, c, c\rangle$ and the equation $ab + c^2 - 1 = bck$.

**Theorem 10:** Let $a$, $b$, $c$ be integers. We have that $\langle a, b, c, c\rangle$ is a Carlitz four-tuple iff there is an integer $k$ such that $a \mid bk$ and $ab + c^2 - 1 = bck$.

**Proof:** First, assume that $\langle a, b, c, c\rangle$ is a CFT. Thus, $b$ divides $ab + c^2 - 1$ and $c$ divides $ab + c^2 - 1$. Hence, since $(b, c) = 1$, there is an integer $k$ such that $ab + c^2 - 1 = bck$. Furthermore, since $a$ divides $ab + c^2 - 1 = bck$ and $(a, c) = 1$, $a$ divides $bk$.

Conversely, assume that there is an integer $k$ such that $a|bk$ and $ab + c^2 - 1 = bck$. Clearly, $ab \equiv 1 \pmod{c}$ and $c^2 \equiv 1 \pmod{b}$. Also, since $a$ divides $bck - ab = c^2 - 1$, $c^2 \equiv 1 \pmod{a}$.

The condition $a|bk$ in Theorem 10 cannot be deleted. For example, let $a = 5$, $b = 8$, $c = 3$, and $k = 2$. Now

$$ab + c^2 - 1 = 48 = bck,$$

but $\langle a, b, c, c\rangle = \langle 5, 8, 3, 3\rangle$ is not a CFT. ■

**Lemma 11:** If $a$, $b$, $c$, and $k$ are integers such that $ab + c^2 - 1 = bck$, then $\left(a, \frac{c^2 - 1}{b}\right)$ divides $k$.

**Proof:** Let $d = \left(a, \frac{c^2 - 1}{b}\right)$. Since $d$ divides $a + \frac{c^2 - 1}{b} = ck$ and $(d, c) = 1$, $d$ divides $k$. ■

**Proposition 12:** Let $a$, $b$, $c$, and $k$ be integers. If $k|a$, $a|bk$, and $ab + c^2 - 1 = bck$, then $\left\langle \frac{a}{k}, bk, c, c\right\rangle$ is a primitive Carlitz four-tuple and

$$|k| = \left(a, \frac{c^2 - 1}{b}\right).$$

**Proof:** By Theorem 10, $\langle a, b, c, c\rangle$ is a CFT. Since $k$ divides both $a$ and $ck - a = \frac{c^2 - 1}{b}$, $k$ divides $\left(a, \frac{c^2 - 1}{b}\right)$. This implies, by Lemma 11, that

$$|k| = \left(a, \frac{c^2 - 1}{b}\right).$$

By Proposition 8, $\left\langle \frac{a}{|k|}, b|k|, c, c\right\rangle$ is a primitive CFT. Hence, $\left\langle \frac{a}{k}, bk, c, c\right\rangle$ is a primitive CFT. ■

The converse of this result is false. For example, choose $a = 75$, $b = 18$, $c = 19$, and $k = 5$.

As a special case of Proposition 12, we have

**Theorem 13:** Let $a$, $b$, $c$ be integers. If $a|b$ and $ab + c^2 - 1 = bc$, then both $\langle a, b, c, c\rangle$ and $\langle a, b, b-c, b-c\rangle$ are primitive Carlitz four-tuples, and if $a|b$ and $ab + c^2 - 1 = -bc$, then both $\langle a, b, c, c\rangle$ and $\langle a, b, b+c, b+c\rangle$ are primitive Carlitz four-tuples.

**Proof:** We shall just prove this result for $ab + c^2 - 1 = -bc$; the proof for $ab + c^2 - 1 = bc$ is similar. Since $ab + c^2 - 1 = -bc$ and

$$ab + (b + c)^2 - 1 = ab + c^2 - 1 + b^2 + 2bc = b(b + c),$$

by Proposition 12, $\langle -a, -b, c, c\rangle$ and $\langle a, b, b+c, b+c\rangle$ are primitive CFTs. Since $\langle -a, -b, c, c\rangle$ is a primitive CFT, so is $\langle a, b, c, c\rangle$. ■

The condition $a|b$ cannot be deleted in Theorem 13. For example, for $a = 7$, $b = 30$, and $c = 11$, we see that

$$ab + c^2 - 1 = 330 = bc,$$

but $\langle a, b, c, c\rangle = \langle 7, 30, 11, 11\rangle$ is not even a CFT.

**Corollary 14:** If $a$ and $b$ are integers greater than 1 such that $a|b$ and $b^2 - 4ab + 4$ is a perfect square, then there is an integer $c$ such that $\langle a, b, c, c\rangle$ is a primitive Carlitz four-tuple and $1 < c < \frac{b}{2}$.

**Proof:** If we let $c = \frac{b - \sqrt{b^2 - 4ab + 4}}{2}$, then it can easily be shown that $ab + c^2 - 1 = bc$. Therefore, by Theorem 13, $\langle a, b, c, c\rangle$ is a primitive CFT. Also $1 < c < \frac{b}{2}$. ■

In the preceding corollary, we do need $a|b$; this is shown by considering $a = 7$ and $b = 30$. For assume there is an integer $c$ such that $\langle a, b, c, c\rangle$ is a CFT. Thus, by Theorem 10, there is an integer $k$ such that

$$7(30) + c^2 - 1 = 30ck.$$

This implies that $c|209$, so $c = 11$ or $c = 19$. Neither of these is possible, since we must have $c^2 \equiv 1 \pmod 7$.

Using the following lemma, we shall find a connection between a diophantine equation and primitive CFTs.

**Lemma 15:** For $a$, $b$, $c$, $q$ complex numbers with $q = b/a$, we have that

$$ab + c^2 - 1 = bc$$

iff

$$(b - 2c)^2 - (q^2 - 4q)a^2 = 4.$$

**Proof:** Since $b = aq$, this result follows from the identity

$$\begin{aligned}(b - 2c)^2 - (q^2 - 4q)a^2 &= b^2 - 4bc + 4c^2 - (qa)^2 + 4a(qa)\\ &= b^2 - 4bc + 4c^2 - b^2 + 4ab = 4(ab + c^2 - bc).\end{aligned}$$

**Theorem 16:** If $q$, $u$, $v$ are integers such that $u^2 - (q^2 - 4q)v^2 = 4$, then both

$$\left\langle v, qv, \frac{qv-u}{2}, \frac{qv-u}{2}\right\rangle \quad \text{and} \quad \left\langle v, qv, \frac{qv+u}{2}, \frac{qv+u}{2}\right\rangle$$

are primitive Carlitz four-tuples.

**Proof:** Let $a = v$, $b = qv$, and $c = \frac{qv - u}{2}$. Thus, $a|b$ and

$$(b - 2c)^2 - (q^2 - 4q)a^2 = u^2 - (q^2 - 4q)v^2 = 4.$$

Therefore, by Lemma 15 and Theorem 13, the proof is complete. ∎

As we saw in the preceding theorem, there is a strong connection between primitive CFTs and the diophantine equation

$$u^2 - (q^2 - 4q)v^2 = 4.$$

For this reason, we shall now consider the diophantine equation

$$u^2 - Dv^2 = 4, \tag{1}$$

where $D$ is a natural number that is not a perfect square. Our discussion will be based on work by Trygve Nagell [2, pp. 3-4].

If $u = u^*$ and $v = v^*$ are integers which satisfy (1), then we say, for simplicity, that the number $u^* + v^*\sqrt{D}$ is a solution of (1). From among all solutions in positive integers to (1), there is a solution in whch both $u$ and $v$ have their least positive values; this solution is called the *fundamental solution of* (1). The following theorem [2, Theorem 1] states that from the fundamental solution of (1), one can generate all solutions in positive integers to (1).

**Theorem 17:** We have that $u + v\sqrt{D}$ is a solution in positive integers to (1) iff there is a positive integer $n$ such that

$$\frac{1}{2}(u + v\sqrt{D}) = \left[\frac{1}{2}(u_1 + v_1\ D)\right]^n,$$

where $u_1 + v_1\sqrt{D}$ is the fundamental solution to (1).

For $D = a^2 - 4a$, we can easily see that $(a - 2) + \sqrt{D}$ is a fundamental solution to (1). Thus, by Theorem 16, $\langle 1, a, 1, 1\rangle$ is a primitive CFT. In some sense, from a trivial solution to (1), we obtained a trivial CFT. It turns out though that from this trivial fundamental solution to (1), we can get some distinctly nontrivial primitive CFTs.

Using Theorem 17 and doing some calculations, we see that two more solutions to (1) are

$$u_2 + v_2\sqrt{D} = (a^2 - 4a + 2) + (a - 2)\sqrt{D}$$

and

$$u_3 + v_3\sqrt{D} = (a - 2)(a^2 - 4a + 1) + (a - 3)(a - 1)\sqrt{D},$$

where $D = a^2 - 4a$. Using Theorem 16 and, for convenience, replacing $a$ by $a + 2$, we see that $u_2 + v_2\sqrt{D}$ gives rise to

$$\langle a,\ a(a + 2),\ a + 1,\ a + 1\rangle \text{ and } \langle a,\ a(a + 2),\ a^2 + a - 1,\ a^2 + a - 1\rangle,$$

and $u_3 + v_3\sqrt{D}$, with $a$ replaced by $a + 3$, gives rise to

$$\langle a(a + 2),\ a(a + 2)(a + 3),\ a(a + 3) + 1,\ a(a + 3) + 1\rangle$$

and

$$\langle a(a + 2),\ a(a + 2)(a + 3),\ a(a + 1)(a + 3) - 1,\ a(a + 1)(a + 3) - 1\rangle.$$

Of course, using Theorems 17 and 16, we could continue to get infinitely many primitive CFTs from the fundamental solution $(a - 2) + \sqrt{D}$, where $D = a^2 - 4a$.

Notice also that, for any integer $a$, $u = 2$ and $v = a$ is a solution to (1), where $D = 4^2 - 4 \cdot 4 = 0$. This gives rise to the primitive CFTs

$$\langle a,\ 4a,\ 2a - 1,\ 2a - 1\rangle \text{ and } \langle a,\ 4a,\ 2a + 1,\ 2a + 1\rangle.$$

The preceding discussions gives

**Proposition 18:** For all integers $a$, the following are primitive CFTs:

$$\langle 1,\ a,\ 1,\ 1\rangle \text{ and } \langle 1,\ a,\ a - 1,\ a - 1\rangle;$$
$$\langle a,\ 4a,\ 2a - 1,\ 2a - 1\rangle \text{ and } \langle a,\ 4a,\ 2a + 1,\ 2a + 1\rangle;$$
$$\langle a,\ a(a + 2),\ a + 1,\ a + 1\rangle \text{ and } \langle a,\ a(a + 2),\ a^2 + a - 1,\ a^2 + a - 1\rangle;$$
$$\langle a(a + 2),\ a(a + 2)(a + 3),\ a(a + 3) + 1,\ a(a + 3) + 1\rangle \text{ and}$$
$$\langle a(a + 2),\ a(a + 2)(a + 3),\ a(a + 1)(a + 3) - 1,\ a(a + 1)(a + 3) - 1\rangle.$$

The next result relates CFTs to another diophantine equation.

**Proposition 19:** For $a$, $b$, $c$ integers, we have that $ab + c^2 - 1 = bc$ iff

$$a^2 + c^2 + (b - c)^2 - (b - a)^2 = 2.$$

**Proof:** This result follows from the identity

$$\begin{aligned}&a^2 + c^2 + (b - c)^2 - (b - a)^2\\ &= a^2 + c^2 + b^2 - 2bc + c^2 - b^2 + 2ab - a^2\\ &= 2ab + 2c^2 - 2bc = 2(ab + c^2 - bc). \blacksquare\end{aligned}$$

The following two results concern the relative size of $a$, $b$, and $c$, where $\langle a, b, c, c\rangle$ is a Carlitz four-tuple.

**Lemma 20:** Let $\langle a, b, c, c\rangle$ be a Carlitz four-tuple. If $0 < a < c < b$, then $ab + c^2 - 1 = bc$.

**Proof:** Since $0 < a < c < b$,

$$0 < ab + c^2 - 1 < bc + bc - 1 < 2bc.$$

Furthermore, by Theorem 10, $ab + c^2 - 1 = bc$. $\blacksquare$

**Theorem 21:** Let $\langle a, b, c, c\rangle$ be a Carlitz four-tuple. If $0 < a < c < b$, then $\langle a, b, c, c\rangle$ is a primitive Carlitz four-tuple.

**Proof:** By the preceding lemma, $ab + c^2 - 1 = bc$. Thus,

$$\left(a, \frac{c^2 - 1}{b}\right) = (a, c - a) = (a, c) = 1.$$

Thus, by Theorem 5, $\langle a, b, c, c\rangle$ is a primitive Carlitz four-tuple. ■

**Theorem 22:** Let $a$, $b$, and $c$ be positive integers such that $a \neq b$, $c > 1$, and $c \neq ab - 1$. The following six conditions are equivalent.

(i) If, for some integer $k$, $ab + c^2 - 1 = bck$ and $a$ divides $c^2 - 1$, then $k|a$.

(ii) If $\langle a, b, c, c\rangle$ is a Carlitz four-tuple, then $ab + c^2 - 1 = bc(a, e)$, where $e = \dfrac{c^2 - 1}{b}$.

(iii) If $\langle a, b, c, c\rangle$ is a primitive Carlitz four-tuple, then
$$ab + c^2 - 1 = bc.$$

(iv) If $\langle a, b, c, c\rangle$ is a primitive Carlitz four-tuple, then
$$u^2 - (q^2 - 4q)v^2 = 4,$$
where $u = b - 2c$, $v = a$, and $q = b/a$.

(v) If $\langle a, b, c, c\rangle$ is a primitive Carlitz four-tuple, then
$$0 < a < c < b.$$

(vi) If $\langle a, b, c, c\rangle$ is a primitive Carlitz four-tuple, then
$$b^2 - 4ab + 4 = (b - 2c)^2.$$

We see that statements (ii)-(vi) in Theorem 22 are related to Theorem 10 and also to the converses of Theorems 13, 16, and 21, and Corollary 14, respectively.

**Proof:** First, we show that (i), (ii), and (iii) are equivalent. We then show that (iii) is equivalent to each of (iv), (v), and (vi).

**Proof that (i) implies (iii):** Assume that $\langle a, b, c, c\rangle$ is a primitive CFT. Thus, by Theorem 10, for some integer $k$, $ab + c^2 - 1 = bck$. Hence, by (i), we have $k|a$. Thus, $k$ divides $ck - a = \dfrac{c^2 - 1}{b}$. Hence, $k$ divides $\left(a, \dfrac{c^2 - 1}{b}\right) = 1$ by Theorem 5. Therefore, $ab + c^2 - 1 = bck = bc$.

**Proof that (iii) implies (ii):** Assume that $\langle a, b, c, c\rangle$ is a Carlitz four-tuple. By Proposition 8, $\left\langle \dfrac{a}{(a, e)}, b(a, e), c, c\right\rangle$ is a primitive CFT. Thus, by (iii), $ab + c^2 - 1 = bc(a, e)$.

**Proof that (ii) implies (i):** Assume that $a$ divides $c^2 - 1$ and, for some integer $k$, $ab + c^2 - 1 = bck$. Since $a|bck$ and $(a, c) = 1$, $a|bk$. Thus, by

Theorem 10, $\langle a, b, c, c\rangle$ is a CFT. Hence, by (ii), $ab + c^2 - 1 = bc(a, e)$. Therefore, $k = (a, e)$, so $k|a$.

**Proof that (iii) and (iv) are equivalent:** This follows from Lemma 15.

**Proof that (iii) implies (v):** Assume that $\langle a, b, c, c\rangle$ is a primitive CFT. Thus, $ab + c^2 - 1 = bc$.

First, assume $c \geqslant b$. Since $c^2 \geqslant bc = ab + c^2 - 1$, we have the contradiction that $1 \geqslant ab$.

Second, assume that $a \geqslant c$. Since $ab \geqslant bc = ab + c^2 - 1$, we have the contradiction that $1 \geqslant c^2$.

**Proof that (v) implies (iii):** This follows from Lemma 20.

**Proof that (iii) and (vi) are equivalent:** This follows from the identity

$$(b - 2c)^2 - b^2 + 4ab = 4(ab + c^2 - bc). \blacksquare$$

Based on some computer-genreated data, it seems reasonable to believe that Theorem 22(iii) is true. Hence, we make the following conjecture.

**Conjecture 23:** The six statements of Theorem 22 are true.

## REFERENCES


1. L. Carlitz. "An Application of the Reciprocity Theorem for Dedekind Sums." *The Fibonacci Quarterly* **22** (1984):266-270.
2. Trygve Nagell. "Contributions to the Theory of a Category of Diophantine Equations of the Second Degree with Two Unknowns." *Nova Acta Soc. Sci. Upsal.* (4) **16**, no. 2 (1955), 38 pp.


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# FIBONACCI WORD PATTERNS AND BINARY SEQUENCES

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## 1. INTRODUCTION

In this paper, we introduce the notion of Fibonacci word patterns, and use these to construct Fibonacci binary sequences. Spaces of the binary sequences are defined, and many properties of the spaces and sequences are obtained. Suggestions are given for using word patterns to generate other types of number sequences.

## 2. DEFINITIONS

Suppose we are given a *character set* $\mathbf{c} = \{c_1, \ldots, c_k\}$, whose members may be letters or digits. For example, if $k = 2$, and $c_1 = 0$, $c_2 = 1$, the character set is $\mathbf{c} = \{0, 1\}$, which is the binary set usually denoted by $\mathscr{B}$.

Using the characters of $\mathbf{c}$ we can, by juxtaposing characters, form *words*. Then, by juxtaposing words, we can form a pattern of words. A finite pattern of words we shall call a *sentence*.

**Definitions:**

(i) Given two initial words $W_1$ and $W_2$ (called *seed words*), the following recurrence defines an infinite sequence of words:

$$W_{n+2} = W_n W_{n+1},\ n = 1, 2, \ldots . \tag{1}$$

(ii) The juxtaposition of the first $i$ words generated by recurrence (1) is called a *Fibonacci sentence of length* $i$.

(iii) The name *Fibonacci word pattern* (or *word sequence*) will be used to denote the infinite juxtaposition $W_1 W_2 W_3 \ldots W_i \ldots$ . We shall often use letters $A$, $B$ for the seed words, and write $F(A, B) \equiv F(W_1, W_2)$ for the Fibonacci pattern. With this notation, the first part of the pattern is $ABABBABABBAB\ldots$, with $W_3 = AB$, $W_4 = BAB$, $W_5 = ABBAB$, and so on. The first four Fibonacci sentences in the pattern are:

$A$, $AB$, $ABAB$, and $ABABBAB$.

(iv) If the character set used for the seed words $W_1$ and $W_2$ is

$\mathscr{B} = \{0, 1\}$,

the resulting word pattern is a (0, 1)-sequence which we call a *Fibonacci binary pattern* (an FBP).

## 3. A FIBONACCI BINARY PATTERN

The following example of a Fibonacci binary pattern is the one whose discovery motivated our development of a theory of such patterns.

With $\mathscr{B} = \{0, 1\}$ as the character set, and seed words $A = 0$ and $B = 10$, we obtain the pattern:

$$F(0, 10) = 0100101001001010010\ldots .$$

This particular FBP we have given the symbol $\omega$, after Wythoff. Its interest and importance arise from the following facts.

(i) The positions of the 0's in the sequence are

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, \ldots,$$

which is the sequence $\{a_n\} = \{[n\alpha]\}$, where $n = 1, 2, 3, \ldots$, and where $\alpha = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio.

(ii) The positions of the 1's in the sequence are

$$2, 5, 7, 10, 13, 15, 18, 20, 23, \ldots,$$

which is the sequence $\{b_n\} = \{[n\alpha^2]\}$.

It is well known (see [1], for example) that $(a_n, b_n)$ are the *Wythoff pairs*, much studied in the literature on Fibonacci sequences.

## 4. SPACES OF FIBONACCI BINARY PATTERNS (FBPs)

Any FBP is determined by choosing two binary words $W_1$ and $W_2$ as seeds, and applying the recurrence (1). Let $\mathscr{B}^i$ be the set of all binary words of length $i$ (i.e., words having $i$ characters, each character being either 0 or 1). The number of words in $\mathscr{B}^i$, which we shall denote by $|\mathscr{B}^i|$, is $2^i$. Thus, for examples, $\mathscr{B}^1 = \mathscr{B} = \{0, 1\}$ has the two words 0 and 1, and $\mathscr{B}^2 = \{00, 01, 10, 11\}$ has the four words shown.

Suppose that we choose the seed $W_1$ from $\mathscr{B}^m$, and the second seed $W_2$ from $\mathscr{B}^n$. There are $2^m \times 2^n = 2^{m+n}$ ways of making this double choice; each choice determines an FBP, which we denote by $F(W_1, W_2)$. We shall use the symbols $\mathscr{F}^{mn}$ to denote the set of all the possible $2^{m+n}$ FBPs obtained in this way, and call the set the *mn-FBP-space*. Using set notation, the space is defined thus:

$$\mathscr{F}^{mn} = \left\{F(W_1, W_2);\ W_1 \in B^m,\ W_2 \in B^n\right\}, \tag{2}$$

with

$$|\mathscr{F}^{mn}| = 2^{m+n}. \tag{3}$$

## 5. PROPERTIES OF FBP-SPACES

The FBP-space with the fewest elements is $\mathscr{F}^{11}$. We can list this space completely as follows (we give names to the members in the right-hand column):

**Table 1.** The First FBP-Space

| FBP | First 13 Characters ... | | Name(s) |
|---|---|---|---|
| $F(0, 0)$ | 0000000000000 | ... | $0$, $z$ |
| $F(1, 0)$ | 1010010100100 | ... | $\alpha$ |
| $F(0, 1)$ | 0101101011011 | ... | $\overline{\alpha}$ (complement of $\alpha$) |
| $F(1, 1)$ | 1111111111111 | ... | $1$, $u$, $\overline{z}$ |

Note that the space contains the *zeros sequence* 0 (or $z$), and its [0, 1]-component, the *units sequence* 1 (or $u$). It is clear that every $mn$-FBP-space will contain 0 and 1. It is also clear that whenever an FBP-space contains an element $F(A, B)$, it also contains the complement $F(\overline{A}, \overline{B})$, since, if $(A, B)$ belongs to $\mathscr{B}^m \times \mathscr{B}^n$, so does $(\overline{A}, \overline{B})$. Thus, in $\mathscr{F}^{11}$ we find 0 and $\alpha$, together with their complements 1 and $\overline{\alpha}$.

We now define equality of two FBPs as follows:

Let $F_1 = \{b_i\}_{i=1}^{\infty}$ and $F_2 = \{c_i\}_{i=1}^{\infty}$.
Then $F_1 = F_2$ if and only if $b_i = c_i \ \forall\, i$.

**Proposition 5.1:** Let $F_1, F_2 \in \mathscr{F}^{mn}$; then $F_1 = F_2$ iff they have the same seed words.

**Proof:** Trivial. ■

Thus, there are $2^{m+n}$ different FBPs in the space $F^{mn}$; up to complementation, however, there are $2^{m+n-1}$ different FBPs.

One may note that, if we define addition of two FBPs by

$$F_1 \oplus F_2 \equiv \{b_i + c_i\}_{i=1}^{\infty},$$

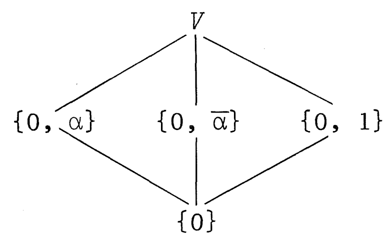
where the binary operation is addition modulus 2 [also known as "exclusive or (XOR)" or "ring sum" addition], the set of elements in any FBP-space form a group under $\oplus$. The details of this group for $\mathscr{F}^{11}$ are shown in the table and graph on the following page.

All the properties noted so far are possessed by pairs of finite binary words of lengths $m$ and $n$, respectively. To determine something new, which is a property of infinite FBPs and which warrants further study, we ask whether an FBP (other than 0 or 1) occurring in one $\mathscr{F}^{mn}$ space also occurs in another $\mathscr{F}^{mn}$

Group $G(\mathscr{F}^{11})$

| $\oplus$ | 0 | $\alpha$ | $\overline{\alpha}$ | 1 |
|---|---|---|---|---|
| 0 | 0 | $\alpha$ | $\overline{\alpha}$ | 1 |
| $\alpha$ | $\alpha$ | 0 | 1 | |
| $\overline{\alpha}$ | $\overline{\alpha}$ | 1 | 0 | |
| 1 | 1 | $\overline{\alpha}$ | $\alpha$ | 0 |

Group Lattice




The Viergruppe (Klein's 4-Group)

space. The answer is "Yes"; every FBP occurs in an infinity of $\mathscr{F}^{mn}$ spaces, as stated in the following theorem.

**Theorem 5.1:** Let $F(W_1, W_2) \in \mathscr{F}^{mn}$. Then $F(W_1, W_2)$ is also a member of spaces $\mathscr{F}^{rs}$, where

$$(r, s) \in \{(m+n, m+2n), (2m+3n, 3m+5n), \ldots, (p_1m+q_1n, p_2m+q_2n), \ldots\},$$

with the coefficients $\{p_1, q_1, p_2, q_2\}$ being ordered sets of Fibonacci numbers of type $\{f_i, f_{i+1}, f_{i+1}, f_{i+2}\}$.

**Proof:** We shall write $A$, $B$ for $W_1$, $W_2$, to avoid subscripts, and begin by proving a lemma.

**Lemma:** $F(A, B) = ABF(AB, BAB)$. (4)

This follows immediately from (1), since the recurrence generation of words produces $W_3 = AB$, and then $W_4 = BAB$; thus, $F(W_3, W_4)$ is the continuation of $F(A, B)$ after words $A$ and $B$ are juxtaposed.

We shall now prove that

$$F(A, B) = F(AB, ABB). \tag{5}$$

Using (4) on the left-hand side, we obtain

$$F(A, B) = ABF(AB, BAB) = ABx_1, x_2, x_3, \ldots, \text{ say;}$$

and the right-hand side of (5) is

$$F(AB, ABB) = y_1, y_2, y_3, \ldots, \text{ say; where each } x_i, y_j \in \{A, B\}.$$

We have to show that $y_1 = A$, $y_2 = B$, $y_3 = x_1, \ldots, y_i = x_{i-2}, \ldots$ . To show that this is so, we shall replace the two $AB$s in the $x$ seed words by $C$ and $C^*$, respectively, and those in the $y$ seed words by $D$ and $D^*$, respectively. Then the expanded sequences are

$$x\text{:}\quad F(AB, BAB) = F(C, BC^*) = C, BC^*, CBC^*, BC^*CBC^*, \ldots;$$

$$y:\ F(AB,\ ABB) = F(D,\ D^*B) = D,\ D^*B,\ DD^*B,\ D^*BDD^*B,\ \ldots\ .$$

Comparison of the elements of these two expansions completes the proof of (5).

Note now that, if $A \in \mathscr{B}^m$, and $B \in \mathscr{B}^n$, $F(AB,\ ABB) \in \mathscr{F}^{(m+n)(m+2n)}$; and if we replace $AB$ by $A'$ and $ABB$ by $B'$, we can use the same proof to show that

$$F(AB,\ ABB) = F(A'B',\ A'B'B') = F(ABABB,\ ABABBABB) \in \mathscr{F}^{rs},$$

with $r = 2m + 3n$ and $s = 3m + 5n$.

Inductive argument establishes that this process can be continued indefinitely, with $r$ and $s$ being Fibonacci integers as claimed. ■

**Corollaries:**

(i) From (5) we see that we can write $F(A,\ B) = F(S_i,\ T_i)$, where $(S_i,\ T_i)$ are obtainable from the following double recurrence system:

$$S_{i+1} = S_i T_i,\ \text{with } S_1 = A,\ T_1 = B,\ \text{and } T_{i+1} = S_{i+1}T_i. \tag{6}$$

Let us denote the *length of a word* $W$ (i.e., the number of characters it contains) by $\ell(W)$. Then, if $\ell(A) = m$ and $\ell(B) = n$, by Theorem 5.1 we have

$$\ell(S_i) = f_i m + f_{i+1}n \quad \text{and} \quad \ell(T_i) = f_{i+1}m + f_{i+2}n. \tag{7}$$

Thus, since $S_i$ is repeated infinitely often, the first $(f_i m + f_{i+1}n)$ characters of the FBP, for $i = 1, 2, \ldots$, occur together infinitely often later in the sequence.

Indeed, if we take *any* subsequence $\{b_j,\ b_{j+1},\ \ldots,\ b_k\}$ of an FBP, and if we choose $i$ large enough, the subsequence will be included in $S_i$, and hence will be repeated infinitely often. We call this property of FBPs the *strong recurrence property*.

(ii) Let us define scalar multiplication of a sequence of words thus: If $\alpha$ is a scalar, then $\alpha(W_1,\ W_2,\ \ldots) = W_1W_1 \ldots W_1W_2W_2 \ldots W_2\ \ldots$, each word being taken $\alpha$ times before continuing the sequence with the next word.

With this notation, repeated application of the lemma in Theorem 5.1 shows that

$$F(W_1,\ W_2) = 2(W_3,\ W_6,\ \ldots,\ W_{3i},\ \ldots), \tag{8}$$

where $W_3 = W_1W_2$, etc.

We may say that any FBP has a *scalar factor of 2*, with a meaning which is clear from (8).

Now that we know any given FBP occurs in an infinite number of $\mathscr{F}^{mn}$ spaces, we may ask how many new FBPs can be found in a given space $\mathscr{F}^{mn}$, new in the sense that they have not already occurred in an *earlier* space. To give meaning to "earlier," we define an ordering of the FBP spaces by the following ordering of $(m, n)$ pairs:

| $m \backslash n$ | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|
| 1 | (1, 1) → | (1, 2) | (1, 3) → | (1, 4) | ... |
| 2 | (2, 1) | (2, 2) | (2, 3) | ⋮ | ... |
| 3 | (3, 1) | ⋮ | ⋮ | | ... |
| ⋮ | ⋮ | | | | ... |

Using the symbol < for the order relation, we can now write

$$\mathscr{F}^{11} < \mathscr{F}^{12} < \mathscr{F}^{21} < \mathscr{F}^{31} < \mathscr{F}^{22} < \cdots . \tag{9}$$

At this point, we will add to the difficulty of determining how many new FBPs occur in a given $\mathscr{F}^{mn}$ by defining "new" in a broader sense than "not equal to an earlier one." To do this, however, we need to introduce the concept of *eventual equality*.

Consider the two sequences

$$F_1 = c_1c_2c_3c_4\ldots, \quad \text{and} \quad F_2 = xyzc_1c_2c_3c_4\ldots,$$

where after $xyz$ the sequence for $F_2$ continues exactly as for $F_1$. We shall say that $F_1$ and $F_2$ are "eventually equal." In general, we define eventually-equal sequences thus:

Let $F_1$, $F_2$ be any two FBPs; if $F_1 = B_1F$ and $F_2 = B_2F$, where $F$ is a FBP and $B_1$, $B_2$ are binary words (possibly empty), then $F_1$ and $F_2$ are *eventually equal*. We shall write this as

$$F_1 \overset{\text{ev}}{=} F_2.$$

We now define an *equivalence relation* for FBPs thus:

Let $F_1$, $F_2$ be any two FBPs; then $F_1 \equiv F_2$ if either $F_1 = F_2$ or $F_1 \overset{\text{ev}}{=} F_2$. Otherwise, $F_1 \not\equiv F_2$.

With this notion of equivalence and inequivalence of FBPs, we can sort members of FBP spaces into equivalence classes and attempt to count the classes.

**Examples:**

(1) $F(1, 0) = 1, 0, 10, 010, 10010, \ldots$
$= 1F(0, 10)$
$= 10F(10, 010)$ etc. (by lemma, Theorem 5.1),
$= F(10, 100)$
$= F(10100, 10100100)$ etc, (by Theorem 5.1);

therefore,

$$F(1, 0) \equiv F(0, 10) \equiv F(10, 010) \equiv \cdots$$
$$\equiv F(10, 100) \equiv F(10100, 10100100) \equiv \cdots .$$

(ii) The following table lists the FBPs in the first five spaces, showing only the new ones that appear in each space. The [0, 1]-complements of the sequences are listed in bar-notation at the end of each space. Thus, $\alpha = F(1, 0)$ is described in full, but $\overline{\alpha} = F(0, 1)$ is merely listed with all other complements at the end of the $\mathscr{F}^{11}$ section.

**Table 2.** Inequivalent FBPs in the First Five Spaces

$$N = |\mathscr{F}^{mn}| = 2^{m+n}$$

| Space $\mathscr{F}^{mn}$ $(m, n)$ | Sequence $F(A, B)$ (first thirteen characters) | Descriptive Names |
|---|---|---|
| (1, 1)<br><br>$N = 4$ | $F(0, 0)$ = 0000000000000<br>$F(1, 0)$ = 1010010100100<br>0, $z$ = 1111111111111<br>$\overline{\alpha}$ | 0, $z$, zero<br>$\alpha$, alpha<br>1, $u$, unity |
| (1, 2)<br><br>$N = 8$ | $F(1, 00)$ = 1001000010010<br>$\overline{\beta}$<br>The other six in this space are 0,<br>$F(0, 10) \overset{\text{ev}}{=} \alpha$, $F(0, 01) \overset{\text{ev}}{=} \alpha$, and<br>their complements. | $\beta$, beta |
| (2, 1)<br><br>$N = 8$ | $F(01, 0)$ = 0100100010010<br>$F(11, 0)$ = 1101100110110<br>$\overline{\gamma}$, $\overline{\varepsilon}$<br>The other four in this space are 0,<br>$F(10, 0) = \gamma$, 1, $\overline{\gamma}$. | $\gamma$, gamma<br>$\varepsilon$, epsilon |
| (3, 1)<br><br>$N = 16$ | $F(100, 0)$ = 1000010000100<br>$F(011, 0)$ = 0110011000110<br>$F(101, 0)$ = 1010101001010<br>$F(111, 0)$ = 1110111001110<br>$\overline{\zeta}$, $\overline{\eta}$, $\overline{\mu}$, $\overline{\nu}$<br>The other eight in this space are 0,<br>$F(010, 0) \overset{\text{ev}}{=} F(001, 0) \overset{\text{ev}}{=} \zeta$,<br>$F(110, 0) = \eta$, and their complements. | $\zeta$, zeta<br>$\eta$, eta<br>$\mu$, mu<br>$\nu$, nu |
| (2, 2)<br><br>$N = 16$ | $F(00, 01)$ = 0001000101000<br>$F(00, 11)$ = 0011001111001<br>$F(01, 10)$ = 0110011010011<br>$F(01, 11)$ = 0111011111011<br>$F(01, 01)$ = 0101010101010<br>$\overline{\pi}$, $\overline{\rho}$, $\overline{\sigma}$, $\overline{\tau}$, $\overline{c}_1$, 0, 1<br>$F(00, 10) \overset{\text{ev}}{=} \pi$, $F(10, 11) \overset{\text{ev}}{=} \tau$,<br>and their complements. | $\pi$, phi<br>$\rho$, rho<br>$\sigma$, sigma<br>$\tau$, tau<br>$c_1$, first cyclic |

**Notes:**

(i) The first cyclic FBPs are 0, 1, $c_1$, $\overline{c}_1$. We show later that cyclic sequences can only occur when $m = n$.

(ii) The list count of "esentially new" (i.e., up to complementation) FBPs that are noncyclic grows by the following increments: 1, 1, 2, 4, 4, ... as we proceed through the ordered FBP spaces.

We have not yet found a general formula for these increments. However, we have a useful sequence parameter for determining whether or not two FBPs may be equivalent, namely the limit density of the words of the binary sequences. We describe this parameter next.

## 6. THE DENSITY OF AN FBP

Consider the FBP given by $F(A, B)$, where $A$, $B$ are binary seed words having weights (numbers of 1's) $\omega(A) = a$ and $\omega(B) = b$, respectively. Let the lengths (numbers of characters) of $A$, $B$ be $\ell(A) = m$ and $\ell(B) = n$, respectively. Let $F(A, B) = W_1W_2W_3, \ldots, W_i, \ldots$, the $W_i$ being the words generated by the Fibonacci recurrence.

**Definitions:**

(i) The *density* of word $W_i$ is $\delta_i \equiv \dfrac{\omega(W_i)}{\ell(W_i)}$.

(ii) The *density* of $F(A, B)$ is $\delta \equiv \lim\limits_{i\to\infty} \delta_i$, assuming such a limit exists.

**Theorem 6.1:** The density of $F(A, B)$ is

$$\delta = \frac{a + b}{m + n} = c + d\alpha,$$

where

$$\alpha = \frac{1}{2}(1 + \sqrt{5}), \quad c = \frac{1}{\Delta}\begin{vmatrix} a & n \\ b & m + n \end{vmatrix}, \quad d = \frac{1}{\Delta}\begin{vmatrix} m & a \\ n & b \end{vmatrix},$$

and

$$\Delta = \begin{vmatrix} m & n \\ n & m + n \end{vmatrix} = m^2 - n^2 + mn.$$

**Proof:** The $i^{\text{th}}$ word $W_i$ of the Fibonacci word pattern $F(A, B)$ contains $f_{i-2}$ $A$'s and $f_{i-1}$ $B$'s; this follows by induction from the recurrence construction of the pattern. Therefore,

$$\delta_i = \frac{\omega(W_i)}{\ell(W_i)} = \frac{f_{i-2}a + f_{i-1}b}{f_{i-2}m + f_{i-1}n}.$$

Dividing numerator and denominator by $f_{i-2}$ and taking limits gives

$$\delta = \lim_{i\to\infty} \delta_i = \frac{a + b\alpha}{m + n\alpha},$$

with $\alpha$ the golden ratio $\frac{1}{2}(1 + \sqrt{5})$.

Setting $\frac{a + b\alpha}{m + n\alpha} = c + d\alpha$, algebra gives

$$a + b\alpha = (cm + dn) + \alpha[cn + d(m + n)],$$

using the fact that $\alpha^2 = \alpha + 1$.

Equating coefficients of $\alpha^0$ and $\alpha^1$ gives

$$a = cm + dn \quad \text{and} \quad b = cn + d(m + n).$$

Solving for $c$, $d$ by the method of determinants gives the formulas required. ■

Before presenting a table of densities for the first fifteen FBPs, we make three remarks and state a proposition on the density of a complement sequence.

**Remarks:**

(i) It is clear that if $F_1 \overset{\text{ev}}{=} F_2$, the densities of $F_1$ and $F_2$ are equal, because the limit is applied to $W_i$, and beyond certain points in both sequences all characters correspond.

(ii) It might seem a better procedure to define density by

$$\delta(F) = \lim_{i \to \infty} \frac{\omega(S_i)}{\ell(S_i)},$$

where $S_i$ is the Fibonacci sentence $W_1 W_2 \ldots W_i$. In fact, perhaps surprisingly, this limit is the same as the one derived above, which can be proved using the identity

$$\sum_{r=1}^{i} f_r = f_{i+2} - 1.$$

(iii) From the definition of $\delta$ it is evident that $0 \leqslant \delta(F) \leqslant 1$ for all $F$.

**Proposition:** Let $F \equiv F(A, B)$ have density $\delta(F) = c + d\alpha$ as in Theorem 6.1. Then the [0, 1]-complement sequence $\overline{F} \equiv F(\overline{A}, \overline{B})$ has density

$$\delta(\overline{F}) = 1 - \delta(F) = \frac{(m - a) + (n - b)\alpha}{m + n\alpha} = (1 - c) - d\alpha. \tag{10}$$

**Proof:** The proof follows immediately from consideration of the composition of $W_i$. ■

We could say that $\delta(F)$ is a measure of the density of 1's in the sequence $F$, and $\delta(\overline{F})$ is a measure of the density of 0's in $F$. (See Table 3.)

We have used the density parameter in two ways. First, when we checked for equivalence of two FBPs to produce Table 2. From Remark (i) above we know that two FBPs are *inequivalent* if they have different densities. However, the converse is not true, as can be seen by scanning Table 3; $\eta$ and $\mu$ have equal densities, as have $\sigma$ and $c_1$. To distinguish between equal density pairs, one must compare their patterns of 0's and 1's. Thus:

$$\eta = 0110011000110\ldots \quad \text{and} \quad \mu = 1010101001010\ldots$$

are clearly distinct, since the former contains pairs of 1's while the latter does not.

Similarly for $\sigma = 01100110\ldots$ and $c_1 = 01010101\ldots$ .

**Table 3.** Densities of the First Fifteen FBPs

| Sequence | $m, n$ | Parameter Values $a, b$ | $c, d$ | $\delta$ (to 3 d.p.) |
|---|---|---|---|---|
| $0 = F(0, 0)$ | 1, 1 | 0, 0 | 0, 0 | 0 |
| $\alpha = F(1, 0)$ | 1, 1 | 1, 0 | 2, −1 | 0.382 |
| $1 = F(1, 1)$ | 1, 1 | 1, 1 | 1, 0 | 1 |
| $\beta = F(1, 00)$ | 1, 2 | 1, 0 | −3, 2 | 0.236 |
| $\gamma = F(01, 0)$ | 2, 1 | 1, 0 | $\frac{1}{5}(3, -1)$ | 0.276 |
| $\varepsilon = F(11, 0)$ | 2, 1 | 2, 0 | $\frac{2}{5}(3, -1)$ | 0.553 |
| $\zeta = F(100, 0)$ | 3, 1 | 1, 0 | $\frac{1}{11}(4, -1)$ | 0.217 |
| $\eta = F(011, 0)$ | 3, 1 | 2, 0 | $\frac{2}{11}(4, -1)$ | 0.433 |
| $\mu = F(101, 0)$ | 3, 1 | 2, 0 | $\frac{2}{11}(4, -1)$ | 0.433 |
| $\nu = F(111, 0)$ | 3, 1 | 3, 0 | $\frac{3}{11}(4, -1)$ | 0.650 |
| $\pi = F(00, 01)$ | 2, 2 | 0, 1 | $\frac{-1}{2}, \frac{-1}{2}$ | 0.309 |
| $\rho = F(00, 11)$ | 2, 2 | 0, 2 | −1, −1 | 0.618 |
| $\sigma = F(01, 10)$ | 2, 2 | 1, 1 | $\frac{1}{2}, 0$ | 0.5 |
| $\tau = F(01, 11)$ | 2, 2 | 1, 2 | $0, \frac{1}{2}$ | 0.809 |
| $c_1 = F(01, 01)$ | 2, 2 | 1, 1 | $\frac{1}{2}, 0$ | 0.5 |

Our second use of $\delta$ was to study the question: "Given an FBP, how many equivalent forms has it for a fixed $m$, and for a fixed $n$ (we have already seen that it has an infinite number of equivalent forms when $m$ and $n$ are allowed to vary)?". Again, we have no general answer to this question, but examining the density of an FBP provides a useful start. We give one example.

**Example:** Find all the equivalents of $\alpha = F(1, 0)$ in spaces $\mathscr{F}^{mn}$, for $1 \leqslant m \leqslant 2$, $1 \leqslant n \leqslant 4$.

**Solution:** The parameters of $\alpha$ and $\delta = 0.382$ with $(m, n) = (1, 1)$, and $(a, b) = (1, 0)$; therefore, any other FBP is a candidate for equivalence if

$$0.382 = \frac{1}{1 + \alpha} = \frac{a + b\alpha}{m + n\alpha}.$$

Equating coefficients of $\alpha^0$ and $\alpha^1$ gives conditions for $a$ and $b$ as follows:

$$\left.\begin{aligned} a &= 2m - n \\ b &= m - n \end{aligned}\right\} \text{ with } 0 < m, \quad 0 < n. \tag{11}$$

Thus, feasible solutions for $(m, n)$ are the lattice points on and between lines $m = n$ and $m = \frac{1}{2}n$. For fixed $m$, the values for $n$ are $m, m + 1, \ldots, 2m$. To solve our problem, we need only look at the following $(m, n)$-points:

$(1, 1)$, $(1, 2)$, $(2, 2)$, $(2, 3)$, and $(2, 4)$.

From (11) we compute the corresponding $(a, b)$-values; then we can write out all possible FBPs having the same density as $\alpha$. Finally, we can check these for equivalences. Table 4 shows the FBPs with $\delta = \delta(\alpha)$.

**Table 4.** The FBPs with Density Equal to $\delta(\alpha) = 0.382$

| $(m, n)$ | Parameter Values $a = 2m - n$ | $b = n - m$ | Fibonacci Binary Patterns (FBPs) |
|---|---|---|---|
| 1, 1 | 1 | 0 | $F(1, 0) = \alpha$ |
| 1, 2 | 0 | 1 | $F(0, 10)$, $F(0, 01)$ (both are $\overset{\text{ev}}{=} \alpha$) |
| 2, 2 | 2 | 0 | $F(11, 00) = 2\alpha$ |
| 2, 3 | 1 | 1 | $F(10, 100)$, $F(01, 100)$<br>$F(10, 010)$, $F(01, 010)$<br>$F(10, 001)$, $F(01, 001)$ |
| 2, 4 | 0 | 2 | $F(00, 1100)$, $F(00, 0110)$<br>$F(00, 0011)$, $F(00, 1001)$<br>$F(00, 1010)$, $F(00, 0101)$ |

**Combinatoric Formula:** The total number of FBPs with density $\delta(\alpha)$ is given by the formula

$$\sum_{m=1}^{m^*} \sum_{n=m}^{2m} \binom{m}{2m - n}\binom{n}{n - m} = \sum_{m=1}^{m^*} \sum_{n=m}^{2m} \frac{[n]_m}{(2m - n)!\,(n - m)!}, \tag{12}$$

where $m^*$ is a given upper limit for $m$, and $[n]_m$ is the falling factorial

$n(n - 1) \cdots (n - m + 1)$.

**Proof:** We obtained the limits for $m$ and $n$ above. The binomial coefficients count the numbers of ways in which the $a$ 1's and $b$ 1's can be placed in the seed words $A$ and $B$, respectively. ∎

To complete the solution to our problem, we have to examine all the FBPs found, to check for equivalences. By inspection, we find the following set of inequivalent sequences which have $\delta = \delta(\alpha) = 0.382$, for $m = 1, 2$, $n = 1, 2, 3$ 4.

$$\{F(1, 0) = \alpha, \quad F(11, 00) = 2\alpha, \quad F(01, 100), \quad F(10, 001),$$
$$F(01, 001), \quad F(00, 1100), \quad F(00, 1001), \quad F(00, 1010)\}.$$

The cardinal number of this set is 8, which is half the total number of equal-density FBPs found.

## 7. GENERALIZATIONS, FURTHER PROPERTIES OF $F(A, B)$; APPLICATIONS

In this final section, we give density formulas, without proofs, for two new kinds of binary pattern; then we list propositions concerning run-lengths of $A$ and $B$ in the pattern $F(A, B)$. Details of these results may be found in [2] and [3]. We also indicate briefly how word patterns can be used to generate number sequences. Two ways of doing this are given; we are investigating others. We believe that studies of number sequences derived from word patterns will be very fruitful, in that they will provide classes of sequences with interesting properties related to those of word patterns. Developing links between theories of word patterns and theories of number sequences will prove beneficial to both topics.

(1) The density of an FBP with $W_1 = rA$ and $W_2 = sB$

Let $W_1 = AA \ldots A$ (with $A$ taken $r$ times) and $W_2 = BB \ldots B$ (with $B$ taken $s$ times), with $A$, $B$ being binary words. Then

$$\delta(F(rA, sB)) = \frac{ra + sb\alpha}{rm + sn\alpha}. \tag{13}$$

(2) Tribonacci binary patterns

$T(W_1, W_2, W_3)$ is the tribonacci word pattern

$$W_1W_2W_3 \ldots W_n \ldots, \text{ where } W_n \equiv W_{n-3}W_{n-2}W_{n-1}.$$

If the seed words $W_1$, $W_2$, and $W_3$ have the binary character set, we have a tribonacci binary pattern (a TBP) whose density is given by

$$\delta(T) = \frac{\tau\omega_1 + (\tau + 1)\omega_2 + \tau^2\omega_3}{\tau\ell_1 + (\tau + 1)\ell_2 + \tau^2\ell_3}, \tag{14}$$

where $\tau = 1.839$ is the positive root of $x^3 - x^2 - x - 1 = 0$. It is clear that we can extend these definitions and formulas to give $n$-bonacci patterns and their densities.

(3) Further properties of $F(A, B) = ABABBABBBBAB\ldots$

The following propositions concerning runs within the pattern are easily proved:

All $A$-runs have length 1; all $B$-runs have length 1 or 2. (15)

The number of $A$'s in the $i^{\text{th}}$ word of the pattern is $f_{i-2}$; the number of $B$'s is $f_{i-1}$, with $f_{-1} = 1$, $f_0 = 0$. (16)

The number of $B$-runs of length 2 in $W_{2i+1}$ is $f_{2(i-1)}$, and in $W_{2i+2}$ is $f_{2i-1} - 1$, $i = 1, 2, \ldots$ . (17)

The number of $B$-runs of length 1 in $W_i$ can be determined using (16) and (17).

Consider the $i^{\text{th}}$ Fibonacci sentence $S_i = W_1W_2 \ldots W_i$. The number of $B$-runs of length 1 in $S_i$ is $f_{i-2}+1$, of length 2 is $f_{i-1}-1$, and of either length is $f_i$, for $i > 1$. (18)

Define the *chaos* $\chi_i$ of $W_i$ to be the number of transpositions of adjacent letters required to set the word into the form $AA\ldots ABB\ldots B$. Then $\chi_i$ satisfies the recurrence $\chi_i - \chi_{i-2} - \chi_{i-1} = f_{i-3}^2$, $i \geqslant 4$, with $\chi_1 = \chi_2 = \chi_3 = 0$.

(4) Two applications in number theory

(i) Generation of $r$-tuple integer sequences

In [2] we show generally how FBPs may be used to generate sequences of $r$-tuples of integers, whose properties we have only begun to study. One simple example must suffice here, with $r = 2$.

Suppose we use seed words $W_1 = a$, $W_2 = ba$, then consider the *positions* of $a$ and $b$, respectively, in the resulting Fibonacci word pattern. Thus, the word pattern is

$$F(a, ba) = abaababaaba\ldots,$$

and the $a$-positions are 1, 3, 4, 6, 8, 9, 11, ... with the $b$-positions being 2, 5, 7, 10, etc. Taking these in pairs, we get the 2-tuple sequence

$$(1, 2), (3, 5), (4, 7), (6, 10), \text{etc.}$$

We see that $F(a, ba)$ in this manner generates the Wythoff-pairs sequence.

It is clear how we can generate 3-tuple sequences if we use character set $\{a, b, c\}$; and so on.

(ii) The Fibonacci reals

If we take any Fibonacci binary pattern and place a decimal point in front of it, we obtain a binary representation of a real number in the interval (0, 1). We believe the class of all such numbers, namely the *Fibonacci reals* to be worthy of study.

## REFERENCES

1. V. E. Hoggatt, Jr., & M. Bicknell-Johnson. "Additive Partition of the Positive Integers and Generalized Fibonacci Representations." *The Fibonacci Quarterly* 22, no. 1 (1984):2-21.
2. J. C. Turner. "Fibonacci Word Patterns and Derived Number Sequences." Research Report No. 138. University of Waikato, N.Z., 1985.
3. W. Klitscher. "Fibonacci Structures and Representation Sets." M. Sc. Thesis, University of Waikato, 1986.
4. M. Lothaire, ed. *Combinatorics on Words: Encyclopaedia of Mathematics and Its Applications*. Section: Algebra. Vol. 17, p. 236. New York: Addison-Wesley.

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# A NEW EXTREMAL PROPERTY OF THE FIBONACCI RATIO

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## 0. INTRODUCTION

In some problems in the geometry of numbers and in the theory of diophanting approximation, sequences of lattices play an important role. Especially, it sometimes is very useful to consider the sequence of lattices $(\Gamma_N(\alpha))$, $N \in \mathbb{N}$, where $\alpha$ is a real number and $\Gamma_N(\alpha)$ is the two-dimensional lattice spanned by the vectors $\binom{1/N}{\alpha}$ and $\binom{0}{1}$. See, for example, [2], [9], [10].

It is easy to see that, if $\alpha$ is irrational, then the set of points of $\Gamma_N(\alpha)$ in $\mathbb{R}^2$ will become more and more dense in $\mathbb{R}^2$. We will explain this more exactly and define

$$d(\Gamma) := \sup_{x \in \mathbb{R}^2} \inf_{y \in \Gamma} d(x, y),$$

where $d(x, y)$ denotes the euclidean metric, the "dispersion" of the lattice $\Gamma$. (We do this in analogy to the notion of the dispersion of a point-sequence in a metric space; see [4], [5].) Since, by Kronecker's theorem, the sequence $k\alpha$ is dense modulo one, if and only if $\alpha$ is irrational, it is easy to see that

$$\lim_{N \to \infty} d(\Gamma\ (\alpha)) = 0$$

if and only if $\alpha$ is irrational. An obvious question is, what can be said about the speed of convergence of $d(\Gamma_N(\alpha))$ for given $\alpha$. (Similar questions regarding the dispersion of a sequence have been considered, e.g., in [1], [3], [6], and [7].)

It can be shown that the speed of convergence never can be faster than $O(1/\sqrt{N})$, and that $d(\Gamma_N(\alpha)) = O(1/\sqrt{N})$ if and only if $\alpha$ has bounded continued fraction coefficients. This follows directly from obvious connections of our dispersion $d$ with the dispersion of the sequence $(k/N, \{k\alpha\})$, $k = 1, 2, \ldots, N$, in the unit square and from results on this dispersion in [1] and [3], for example. Thus, it is obvious to ask for which $\alpha$ the value

$$D(\alpha) := \limsup_{N \to \infty} \sqrt{N} \cdot d(\Gamma_N(\alpha))$$

is minimal.

We will find that this problem is quite interesting because it provides a new sort of something like a Markov-spectrum (compare especially with [5]) and a new extremal property of the Fibonacci ratio $\xi = (1 + \sqrt{5})/2$.

**Theorem:** $\inf D(\alpha) = 1/\sqrt{2}$

$D(\alpha) = 1/\sqrt{2}$ if and only if $\alpha$ is equivalent to $\frac{1 + \sqrt{5}}{2}$

$D(\alpha) \geqslant \sqrt{\frac{3\sqrt{3} - 1}{8}} = \frac{1.024\ldots}{\sqrt{2}}$ if $\alpha$ is not equivalent to $\frac{1 + \sqrt{5}}{2}$.

(Here, "equivalent" is used in the sense of the theory of continued fractions. See Perron [8].)

## 1. NOTATIONS

The irrational number $\alpha$ is represented by the infinite continued fraction $\alpha := [a_0; a_1, a_2, \ldots]$ and has best approximation denominators $1 = q_0 \leqslant q_1 \leqslant q_2 \leqslant \cdots$ with $q_{\ell+1} = a_{\ell+1}q_\ell + q_{\ell-1}$;

$$\frac{p_i}{q_i} + \frac{\phi_i}{q_i \cdot (q_i + q_{i+1})} = \alpha = \frac{p_i}{q_i} + \frac{\theta_i}{q_i \cdot q_{i+1}} \text{ with } |\theta_i| \leqslant 1 \text{ and } |\phi_i| \geqslant 1.$$

(see [8].) Further we denote

$$s_i := \frac{q_i}{q_{i-1}}; \quad \alpha_i := [a_i; a_{i+1}, a_{i+2}, \ldots]$$

and, for a given fixed $N$, the index $\ell := \ell(N)$, such that $q^2_{\ell(N)} \leqslant N \leqslant q^2_{\ell(N)+1}$.

We denote the distance of $x$ to the nearest integer by $\|x\|$. For given $N$ and for $r \in \mathbb{N}$, we define

$$M(r) := \left(\left(\frac{r}{N}\right)^2 + \|r\alpha\|^2\right)^{1/2},$$

and again, for given $N$, we denote by $\lambda_1$ and $\lambda_2$ the successive minima of $\Gamma_N$ with respect to the euclidean norm, and also two linearly independent vectors in $\Gamma_N$ with length $\lambda_1$ and $\lambda_2$.

$F$ is the parallelogram built by $\lambda_1$ and $\lambda_2$, and $\mu$ is the shorter diagonal, and also its length.

$\xi = (1 + \sqrt{5})/2$ always is the positive Fibonacci ratio.

$\alpha \cong \beta$ means that $\alpha$ is equivalent to $\beta$.

## 2. GENERAL RESULTS

**Lemma 1:** $d(\Gamma_N) = \frac{1}{2} \cdot \lambda_1 \cdot \lambda_2 \cdot \mu \cdot N$

**Proof:** Every $x \in \mathbb{R}^2$ lies in one fundamental parallelogram $F_x$ of $\Gamma$. Let

$$d_x := \min_{y \in \Gamma} d(x, y),$$

then, by using the triangle inequality, it is easy to see that $d_x$ will be attained for a vertex $y$ of $F_x$. In the triangle built by the vectors $\lambda_1$ and $\lambda_2$ and by $\mu$, the angle between $\lambda_1$ and $\lambda_2$ is between $\pi/3$ and $\pi/2$.

The two other angles in the triangle are less than or equal to this angle. Therefore, the center of the circle through the vertices of this triangle is in the interior or on the boundary of the triangle and $d(\Gamma)$ is equal to the radius of the circle. Thus,

$$d(\Gamma_N) = r = \frac{\lambda_1 \cdot \lambda_2 \cdot \mu}{2 \cdot |F|} = \frac{\lambda_1 \cdot \lambda_2 \cdot \mu \cdot N}{2}. \blacksquare$$

**Remark 1:** Because of the approximation properties of the $q_i$, we obviously have for given $N$: $\lambda_1 = \min_{k \in \mathbb{Z}} M(k) = \min_i M(q_i)$.

**Remark 2:** For $i \neq j$, the two vectors $[(q_i/N), q_i\alpha - p_i]$ and $[(q_j/N), q_j\alpha - p_j]$ are always linearly independent.

**Lemma 2:** If $\lambda_1 = M(q_j)$, then $\lambda_2 = M(k)$ with $k = q_{j+1} - c \cdot q_j$ and $0 \leqslant c \leqslant a_{j+1}$.

**Proof:** If $q_m \leqslant k \leqslant q_{m+1}$ with $m \neq j$, then $M(q_m) \leqslant M(k)$ and, therefore, $k = q_m$. Further, we have

$$\frac{1}{N} = |\det(\lambda_1, \lambda_2)| = \frac{1}{N} \cdot |p_j q_m - p_m q_j|;$$

thus, $m = j + 1$ or $m = j - 1$ (see [8], p. 14).

If $q_{j-1} < k < q_{j+1}$ and if $t$ is the largest intermediate convergent's denominator less than or equal to $k$, or if $t$ is $q_{j-1}$ if $q_{i-1} \leqslant k < q_{i-1} + q_i$ (i.e., if $t = q_{j+1} - c \cdot q_j$ with a $c$ with $0 < c \leqslant a_{j+1}$), then $M(t) \leqslant M(k)$ and, therefore, $k = t$. $\blacksquare$

**Lemma 3:** If $\limsup_{i \to \infty} a_i \geqslant 4$, then $D(\alpha) > \dfrac{1.025\ldots}{\sqrt{2}}$.

**Proof:** In the following, we will write $d_N$ instead of $d(\Gamma_N)$. Then we have

$$\sqrt{N} \cdot d_N = \frac{N^{3/2}\,\lambda_1\lambda_2\mu}{2} \geqslant \frac{1}{2\sqrt{N}\lambda_1} \quad \text{because } \lambda_1 \cdot \lambda_2 \text{ and } \lambda_1 \cdot \mu \text{ and } \geqslant \frac{1}{N}.$$

Further,

$$N\lambda_1^2 = N \cdot \min_i M(q\ )^2 \leqslant 2 \cdot \min_i \max\left(\frac{q_i^2}{N}, \frac{N}{q_{i+1}^2}\right) = 2 \cdot \max\left(\frac{q_\ell^2}{N}, \frac{N}{q_{\ell+1}^2}\right).$$

If we choose $N = q_\ell \cdot q_{\ell+1}$, then $N \cdot \lambda_1^2 \leqslant 2(q_\ell/q_{\ell+1})$, and so

$$D(\alpha) \geqslant \limsup_{\ell \to \infty} \frac{1}{2\sqrt{2}} \cdot \sqrt{\frac{q_{\ell+1}}{q_\ell}} \geqslant \frac{1}{2\sqrt{2}} \cdot \sqrt{[4;\ 4,\ 1,\ 4,\ 1,\ \ldots]} = \frac{\sqrt{7} + \sqrt{2}}{4}$$

$$= \frac{1.025\ldots}{\sqrt{2}}. \blacksquare$$

**Lemma 4:** If $\limsup_{i\to\infty} a_i \leqslant 3$, then, for every $k \geqslant 2$ and all $\ell$ large enough, we have:

a) $M(q_{\ell-k}) > M(q_\ell)$;

b) $M(q_{\ell+k}) > M(q_\ell)$.

**Proof of a):** It is sufficient to show that $NM^2(q_{\ell-k}) > NM^2(q_\ell)$ with $N = q_\ell^2$, since $NM^2(q_{\ell-k}) - NM^2(q_\ell)$ is monotonically increasing in $N$ for $q_\ell^2 \leqslant N \leqslant q_{\ell+1}^2$.

We always have $q_\ell^2 \cdot \|q_\ell \alpha\|^2 \leqslant 1$, and so

$$q_\ell^2 \cdot M^2(q_{\ell-k}) \geqslant q_\ell^2 \cdot \|q_{\ell-k}\alpha\|^2 \geqslant \left(\frac{q_\ell}{q_{\ell-k} + q_{\ell-k+1}}\right)^2 = \left(\frac{a_\ell q_{\ell-1} + q_{\ell-2}}{q_{\ell-k+1} + q_{\ell-k}}\right)^2$$
$$\geqslant \left(\frac{3}{2}\right)^2 > 2 \geqslant 1 + q_\ell^2 \cdot \|q_\ell \alpha\|^2 = q_\ell^2 \cdot M^2(q_\ell)$$

if $a_\ell \geqslant 2$ or if $k > 2$.

If $a_\ell = 1$ and $k = 2$, then we have to show that

$$\frac{q_{\ell-2}^2}{q_\ell^2} + \frac{q_\ell^2}{q_{\ell-2}\left(s_{\ell-1} + \dfrac{1}{\alpha_\ell}\right)^2} > 1 + \frac{1}{\left(s_{\ell+1} + \dfrac{1}{\alpha_{\ell+2}}\right)^2},$$

because $\|q_\ell \alpha\| = \dfrac{1}{q_\ell\left(s_{\ell+1} + \dfrac{1}{\alpha_{\ell+2}}\right)}$ (see [8], p. 36).

If we write $A := \alpha_{\ell+1}$ and $s := s_{\ell-1}$, then

$$\frac{q_\ell}{q_{\ell-2}} = s + 1,\ s_{\ell+1} + \frac{1}{\alpha_{\ell+2}} = \frac{s}{s+1} + A,\ \frac{1}{\alpha_\ell} = \frac{A}{A+1},$$

and

$$\frac{1}{2} + \sqrt{\frac{7}{12}} - \varepsilon_\ell = [1;\ 3,\ 1,\ 3,\ \ldots] - \varepsilon_\ell \leqslant s,\ A \leqslant [3;\ 1,\ 3,\ 1,\ \ldots] + \varepsilon_\ell$$
$$= \frac{3}{2}\left(\sqrt{\frac{7}{3}} + 1\right) + \varepsilon_\ell,$$

with an $\varepsilon_\ell$ with $\lim_{\ell\to\infty} \varepsilon_\ell = 0$.

So it remains to show that, for all $A$ and $s$ in the above region and all $\ell$ large enough, we have

$$\frac{1}{(s+1)^2} + \frac{(s+1)^2 \cdot (A+1)^2}{(sA + s + A)^2} > 1 + \frac{(s+1)^2}{(s + sA + A)^2},$$

and with $r := s + 1$ and $b := A + 1$, this is equivalent to

$$b^2 - 2b\left(\frac{1}{r} - r\right) - \left(r^2 + 1 - \frac{1}{r^2}\right) > 0,$$

which is true for all

$$b > \frac{1}{r} - r + \sqrt{2r^2 - 1} =: f(r).$$

$f$ is monotonically increasing for $r \geqslant 1/\sqrt{2}$; therefore,

$$f(r) \leqslant f\left(1 + \frac{3}{2} \cdot \left(\sqrt{\frac{7}{3}} + 1\right) + \varepsilon_\ell\right) < \frac{3}{2} + \sqrt{\frac{7}{12}} - \varepsilon_\ell \leqslant b$$

for $\ell$ large enough, and thus the inequality holds. ■

**Proof of b):** Analogous to a), it is sufficient to show that, with $N = q_{\ell+1}^2$, we have $NM^2(q_{\ell+k}) > NM^2(q_\ell)$. We can write

$$q_{\ell+1}^2 \cdot \|q_\ell \alpha\|^2 \leqslant 1 \quad \text{and} \quad \frac{1}{s_{\ell+1}} = s_{\ell+2} - a_{\ell+2};$$

therefore,

$$q_{\ell+1}^2 \cdot M^2(q_{\ell+k}) > \frac{q_{\ell+k}^2}{q_{\ell+1}^2} \geqslant s_{\ell+2}^2 \geqslant (s_{\ell+2} - a_{\ell+2})^2 + 1 = \frac{q_\ell^2}{q_{\ell+1}^2} + 1$$

$$\geqslant q_{\ell+1}^2 \cdot M^2(q_\ell). \text{ ■}$$

## 3. THE CASE $\alpha \cong \xi$

**Lemma 5:** If $\alpha \cong \xi$, then, for every $\ell$ large enough, we have

$$M(q_{\ell+2}) > M(q_{\ell-1}).$$

**Proof:** It is sufficient to show $NM^2(q_{\ell+2}) > NM^2(q_{\ell-1})$ with $N = q_{\ell+1}^2$. In all that follows, $\varepsilon_\ell(i)$ are reals with $\lim\limits_{\ell \to \infty} \varepsilon_\ell(i) = 0$.

For every $\alpha \cong \xi$, we have

$$\frac{q_{\ell+1}}{q_\ell} = \xi + \varepsilon_\ell(1) \quad \text{and} \quad q_\ell \cdot \|q_\ell \alpha\| = 1/\sqrt{5} + \varepsilon_\ell(2).$$

So, $q_{\ell+1}^2 \cdot M^2(q_{\ell+2}) \geqslant \xi^2 + 1/5\xi^2 - \varepsilon_\ell(3) > 1/\xi^4 + \xi^4/5 + \varepsilon_\ell(3) \geqslant q_{\ell+1}^2 \cdot M^2(q_{\ell-1})$ for $\ell$ large enough. ■

**Remark:** By Lemmas 2 and 4, $\lambda_1$ and $\lambda_2$ (not necessarily in this order) will be attained by $M(q_\ell)$ and $M(q_{\ell-1})$ or by $M(q_\ell)$ and $M(q_{\ell+1})$.

In the first case, then, we have for $\alpha \cong \xi$ and for $\ell$ large enough (because $q_\ell + q_{\ell-1} = q_{\ell+1}$ and $q_\ell - q_{\ell-1} = q_{\ell-2}$):

$$d_N = \frac{N}{2} \cdot \min(M(q_{\ell-1}) \cdot M(q_\ell) \cdot M(q_{\ell+1}),\ M(q_{\ell-2}) \cdot M(q_{\ell-1}) \cdot M(q_\ell)).$$

In the second case, by Lemma 5:

$$d_N = \frac{N}{2} \cdot (M(q_{\ell-1}) \cdot M(q_\ell) \cdot M(q_{\ell+1})).$$

Further, $M(q_{\ell-1}) \cdot M(q_\ell) \cdot M(q_{\ell+1}) \leqslant M(q_{\ell-2}) \cdot M(q_{\ell-1}) \cdot M(q_\ell)$ and, therefore, in any case:

$$d_N = \frac{N}{2} \cdot \min(M(q_{\ell-1}) \cdot M(q_\ell) \cdot M(q_{\ell+1}),\ M(q_{\ell-2}) \cdot M(q_{\ell-1}) \cdot M(q_\ell)).$$

**Lemma 6:** If $\alpha \cong \xi$, then $D(\alpha) = 1/\sqrt{2}$.

**Proof:** We write $M_k$ for $M(q_k)$ and set

$$g(N) := N^3 \cdot \min(M_{\ell-2}^2 \cdot M_{\ell-1}^2 \cdot M_\ell^2,\ M_{\ell-1}^2 \cdot M_\ell^2 \cdot M_{\ell+1}^2) = 4Nd_N^2.$$

We have $\lim\limits_{\ell\to\infty}\left| \max\limits_{q_\ell^2 \leqslant N \leqslant q_{\ell+1}^2} g(N) - \max\limits_{1 \leqslant \sigma \leqslant \left(\frac{q_{\ell+1}}{q_\ell}\right)^2} g(\sigma \cdot q_\ell^2)\right| = 0$ and, therefore,

$$4 \cdot (D(\alpha))^2 = \limsup_{\ell\to\infty} \max_{1 \leqslant \sigma \leqslant \left(\frac{q_{\ell+1}}{q_\ell}\right)^2} g(\sigma \cdot q_\ell^2).$$

Now

$$\sigma \cdot q_\ell^2 \cdot M_{\ell-2}^2 \cdot M_{\ell-1}^2 \cdot M_\ell^2 = \left(\frac{q_{\ell-2}^2}{q_\ell^2 \cdot} + \frac{q_\ell^2 \cdot}{5q_{\ell-2}^2}\right) \cdot \left(\frac{q_{\ell-1}^2}{q_\ell^2 \cdot} + \frac{q_\ell^2 \cdot}{5q_{\ell-1}^2}\right) \cdot \left(\frac{q_\ell^2}{q_\ell^2 \cdot} + \frac{q_\ell^2 \cdot}{5q_\ell^2}\right)$$

$$= \left(\frac{1}{\xi^4\sigma} + \frac{\xi^4\sigma}{5}\right) \cdot \left(\frac{1}{\xi^2\sigma} + \frac{\xi^2\sigma}{5}\right) \cdot \left(\frac{1}{\sigma} + \frac{\sigma}{5}\right) + \varepsilon_\ell(4)$$

$$= \left(\frac{1}{\xi^2\sigma} + \frac{\xi^2\sigma}{5}\right) \cdot \left(\frac{1}{\xi^4\sigma^2} + \frac{\xi^4\sigma^2}{25} + \frac{1}{5\xi^4} + \frac{\xi^2}{5}\right) + \varepsilon_\ell(4)$$

$$= x^3 + x + \varepsilon_\ell(4),$$

with $x = x(\sigma) = 1/\xi^2\sigma + \xi^2\sigma/5$, and quite analogously we get:

$$\sigma \cdot q_\ell^2 \cdot M_{\ell-1}^2 \cdot M_\ell^2 \cdot M_{\ell+1}^2 = y^3 + y + \varepsilon_\ell(5) \text{ with } y = y(\sigma) = \frac{1}{\sigma} + \frac{\sigma}{5}.$$

Consequently, we have (with $\xi_\ell := q_{\ell+1}/q_\ell$):

$$4D^2(\alpha) = \limsup_{\ell\to\infty} \max_{1 \leqslant \sigma \leqslant \xi_\ell^2} \min(x^3 + x + \varepsilon_\ell(4),\ y^3 + y + \varepsilon_\ell(5))$$

$$= \max_{1 \leqslant \sigma \leqslant \xi^2} \min(x^3 + x,\ y^3 + y) = z^3 + z,$$

with $z = \max\limits_{1 \leqslant \sigma \leqslant \xi^2} \min(x(\sigma), y(\sigma))$.

We have $x(\sigma) \geqslant y(\sigma)$ if and only if $\sigma \geqslant \sqrt{5}/\xi$ and, therefore,

$$z = \max\left(\max_{1 \leqslant \sigma \leqslant \sqrt{5}/\xi} x(\sigma),\ \max_{\sqrt{5}/\xi \leqslant \sigma \leqslant \xi^2} y(\sigma)\right) = x(\sqrt{5}/\xi) = 1,$$

and so $D(\alpha) = 1/\sqrt{2}$. ■

## 4. THE CASE $\alpha \not\equiv \xi$

**Lemma 7:** If a) $m = q_{\ell+2} - c \cdot q_{\ell+1}$ with $0 < c < a_{\ell+2}$
or b) $m = q_\ell - c \cdot q_{\ell-1}$ with $0 < c < a_\ell$,
then $M(m) > M(q_\ell)$.

**Proof of a):** It is sufficient to show $NM^2(m) > NM^2(q_\ell)$ for $N = q_{\ell+1}^2$.

$$q_{\ell+1}^2 M^2(q_\ell) \geqslant \frac{(q_{\ell-2} - cq_{\ell+1})^2}{q_{\ell+1}^2} = \left(a_{\ell+2} + \frac{q_\ell}{q_{\ell+1}} - c\right)^2 \geqslant \left(\frac{q_\ell}{q_{\ell+1}} + 1\right)^2$$

(continued)

$$> \left(\frac{q_\ell}{q_{\ell+1}}\right)^2 + 1 \geqslant \frac{q_\ell^2}{q_{\ell+1}^2} + q_{\ell+1}^2 \cdot \|q_\ell \alpha\|^2 = q_{\ell+1}^2 \cdot M^2(q_\ell). \blacksquare$$

**Proof of b):** It is sufficient to show the assertion for $N = q_\ell^2$. We have:

$$\|m\alpha\| = \frac{c}{a_\ell} \cdot \|q_{\ell-2}\alpha\| + \frac{a_\ell - c}{a_\ell} \cdot \|q_\ell \alpha\|.$$

We set $a_\ell = a$, $s_{\ell-1} = s$, $\alpha_{\ell+1} = A$, and we get:

$$q_\ell^2 \cdot M^2(m) > q_\ell^2 \cdot \|m\alpha\|^2 = \left(\frac{c}{a} \cdot \frac{as + 1}{s + \dfrac{A}{aA + 1}} + \frac{a - c}{a} \cdot \frac{1}{A + \dfrac{s}{as + 1}}\right)^2$$

$$> c^2 \geqslant 4 > q_\ell^2 \cdot M^2(q_\ell) \quad \text{if} \quad c \geqslant 2.$$

If $c = 1$, then

$$q_\ell^2 \cdot \|m\alpha\|^2 = \left(\frac{1}{a} \cdot \frac{(as + 1) \cdot (Aa + 1)}{(Aas + a + s)} + \frac{a - 1}{a} \cdot \frac{(as + 1)}{(Aas + a + s)}\right)^2,$$

$$q_\ell^2 \cdot M^2(q_\ell) = 1 + \frac{(as + 1)^2}{(Aas + a + s)^2},$$

and the inequality $q_\ell^2 \cdot \|m\alpha\|^2 \geqslant q_\ell^2 \cdot M^2(q_\ell)$ is, therefore, equivalent to

$$(2Aa^2 - 2Aa - 1) \cdot s^2 + (4Aa - 2A) \cdot s + 2A \geqslant 0.$$

Since $1 = c < a_\ell$, we have $a_\ell \geqslant 2$ and, therefore, because of $A \geqslant 1$, the last inequality holds, and the result is proved. ■

**Remark:** From all this we have that $\lambda_1$ and $\lambda_2$ will be attained (not necessarily in this order) by $q_\ell$ and $q_{\ell-1} + cq$ with $0 \leqslant c \leqslant a_{\ell+1}$.

**Lemma 8:** $\inf\limits_{\alpha \not\equiv \xi} D(\alpha) \geqslant \sqrt{\dfrac{3\sqrt{3} - 1}{8}} = \dfrac{1.024\ldots}{\sqrt{2}}.$

**Proof:** $\alpha \not\equiv \xi$ iff $\limsup\limits_{i \to \infty} a_i > 1$. For $\limsup\limits_{i \to \infty} a_i \geqslant 4$, the result follows from Lemma 3.

First, let $\limsup\limits_{i \to \infty} a_i = 2$ and $a := a_{i+1} = 2$. $\lambda_1$ and $\lambda_2$ will be attained by $q_\ell$ and $q_{\ell-1} + cq_\ell$ with $c = 0$, 1, or 2. Therefore, we have

$$4Nd_N^2 \geqslant \min(T_1, T_2, T_3, T_4) =: g(N)$$

with

$$T_1 = N^3 M_{\ell-1}^2 M^2(q_\ell - q_{\ell-1}) M_\ell^2, \quad T_2 = N^3 M_{\ell-1}^2 M_\ell^2 M^2(q_\ell + q_{\ell-1}),$$
$$T_3 = N^3 M_\ell^2 M^2(q_\ell + q_{\ell-1}) M_{\ell+1}^2, \quad T_4 = N^3 M_\ell^2 M_{\ell+1}^2 M^2(q_\ell + q_{\ell+1}).$$

We write $x := q_{\ell-1}/q_\ell$, $A = \alpha_{\ell+1}$, $N = \sigma q_\ell^2$, then again we have

$$\frac{1}{\sqrt{3} + 1} - \varepsilon_\ell(6) \leqslant x \leqslant \sqrt{3} - 1 + \varepsilon_\ell(6)$$

and

$$2 + \frac{1}{\sqrt{3} + 1} - \varepsilon_\ell(7) \leqslant A \leqslant \sqrt{3} + 1 + \varepsilon_\ell(7) \text{ with } \lim_{\ell \to \infty} \varepsilon_\ell(6),\ \varepsilon_\ell(7) = 0$$

because, for example,

$$[0; 2, 1, 2, 1, \ldots] - \varepsilon_\ell(6) \leqslant x \leqslant [0; 1, 2, 1, 2, \ldots] + \varepsilon_\ell(6).$$

Further,

$$1 \leqslant \sigma \leqslant \left(\frac{q_{\ell+1}}{q_\ell}\right)^2 = (2 + x)^2,$$

and, for every $\sigma$,

$$4D^2(\alpha) \geqslant \limsup_{\substack{\ell \to \infty \\ a_{\ell+1} = 2}} g(\sigma q^2).$$

For every $\ell$, we choose $\sigma^2 := \dfrac{(1 + x)(x + A)^2}{(A - 1)}$; thus, we have

$$1 < 2.7\ldots \leqslant \sigma \leqslant 3.47\ldots < (2 + x)^2$$

and

$$\begin{aligned}
T_1 &= \left(\frac{x^2}{\sigma} + \frac{\sigma A^2}{(A + x)^2}\right)\left(\frac{(1 + x)^2}{\sigma} + \frac{\sigma(A + 1)^2}{(A + x)^2}\right)\left(\frac{1}{\sigma} + \frac{\sigma}{(x + A)^2}\right) \\
&= \left(\left(\frac{x}{\sigma} - \frac{\sigma A}{(x + A)^2}\right)^2 + 1\right) \cdot \left(\frac{(1 - x)^2}{\sigma} + \frac{\sigma(A + 1)^2}{(A + x)^2}\right) \\
&= \frac{1}{x + A}\left((1 - x)^2 \cdot \sqrt{\frac{A - 1}{x + 1}} + (A + 1)^2 \cdot \sqrt{\frac{x + 1}{A - 1}}\right) \\
&\qquad \cdot \left(\frac{1}{(1 + x)(A - 1)} + 1\right) \\
&\geqslant \frac{1}{x + A} \cdot \left((1 + x)^2 \cdot \sqrt{\frac{A - 1}{x + 1}} + (A - 1)^2 \cdot \sqrt{\frac{x + 1}{A - 1}}\right) \\
&\qquad \cdot \left(\frac{1}{(1 + x)(A - 1)} + 1\right) \\
&= \frac{1}{\sqrt{(1 + x)(A - 1)}} + \sqrt{(1 + x)(A - 1)} \\
&\geqslant \frac{1}{-\dfrac{1}{\sqrt{3} + 1} + 1} + -\frac{1}{\sqrt{3} + 1} + 1 - \varepsilon_\ell(8) = \frac{3\sqrt{3} - 1}{2} - \varepsilon_\ell(8).
\end{aligned}$$

And, quite analogously, we get $T_2, T_3, T_4 \geqslant \dfrac{3\sqrt{3} - 1}{2} - \varepsilon_\ell(9)$; therefore,

$$D(\alpha) \geqslant \frac{3\sqrt{3} - 1}{8}.$$

If $\limsup_{i \to \infty} a_j = 3$ and $a_{\ell+1} = 3$, then $4Nd_N^2 \geqslant \min(T_1, T_2, T_4, T_5, T_6)$ with

$$T_5 = N^3 M_\ell^2 M^2 (q_\ell + q_{\ell-1}) M^2 (2q_\ell + q_{\ell-1}) = T_3$$

and

$$T_6 = N^3 M_\ell^2 M^2 (2q_\ell + q_{\ell-1}) M_{\ell+1}^2 = T_2.$$

Now:

$$\sqrt{\frac{7}{12}} - \frac{1}{2} - \varepsilon_\ell(12) \leqslant x \leqslant \frac{1}{\frac{1}{2} + \sqrt{\frac{7}{12}}} + \varepsilon_\ell(10)$$

and

$$\frac{5}{2} + \sqrt{\frac{7}{12}} - \varepsilon_\ell(11) \leqslant A \leqslant 3 + \frac{1}{\frac{1}{2} + \sqrt{\frac{7}{12}}} + \varepsilon_\ell(11).$$

Therefore, in this case, $D(\alpha) \geqslant \frac{1.068\ldots}{\sqrt{2}}$, and the Lemma is proved. ■

Finally, the Theorem follows from Lemma 6 and Lemma 8.

## REFERENCES


1. W. Bayrhamer. "Quasi-zufällige Suchmethoden der Globalen Optimierung." Dissertation, Universität Salzburg, 1986.
2. H. Davenport & W. M. Schmidt. "A Theorem on Linear Forms." *Acta Arith.* *14* (1968):209-223.
3. G. Larcher. "On the Dispersion of a Special Sequence." *Arch. Math.* Vol. *47* (1986):347-352.
4. H. Niederreiter. "A Quasi-Monte Carlo Method for the Approximate Computation of the Extreme Values of a Function." In *Studies in Pure Mathematics (To the Memory of Paul Turan)*, pp. 523-529. Basel: Akadémiai Kiadó, Birkhäuser Verlag, 1983.
5. H. Niederreiter. "On a Measure of Denseness for Sequences." In *Topics in Classical Number Theory (Budapest 1981)*, pp. 1163-1208. Amsterdam: North-Holland Publishing Co., 1984.
6. H. Niederreiter & P. Peart. "A Comparative Study of Quasi-Monte Carlo Methods for Optimization of Functions of Several Variables." *Caribbean J. Math.* *1* (1982):27-44.
7. P. Peart. "The Dispersion of the Hammersley Sequence in the Unit Square." *Monatsh. Math.* *94* (1982):249-261.
8. O. Perron. *Die Lehre von den Kettenbrüchen I.* Stuttgart: Teubner, 1954.
9. W. M. Schmidt. "Diophantine Approximation and Certain Sequences of Lattices." *Acta Arith.* *18* (1971):168-178.
10. W. M. Schmidt. "Open Problems in Diophantine Approximation." In *Approximations Diophantiennes et Nombres Transcendants*, pp. 271-288. D. Bertrand and M. Waldschmidt, eds. Basel: Birkhäuser Verlag, 1983.


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# ON FIBONACCI AND LUCAS REPRESENTATIONS AND A THEOREM OF LEKKERKERKER

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## 1. INTRODUCTION

Let $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, $n = 1, 2, \ldots$, be the Fibonacci numbers and let $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$, $n = 0, 1, \ldots$, be the Lucas numbers. According to the Theorem of Zeckendorf (see, for example, [5, p. 74], [6], [1], [8]), every positive integer $m$ has a unique "minimal" representation as a sum of distinct Fibonacci numbers $F_2, F_3, \ldots$ such that no two consecutive Fibonacci numbers are used. If we denote by $f(m)$ the number of Fibonacci numbers in the representation of $m$, then Lekkerkerker [6] defined the average value

$$\psi_n = \left(\sum_{i=F_{n+1}}^{F_{n+2}-1} f(i)\right) \Big/ F_n$$

and proved that

$$\lim_{n \to \infty} \frac{\psi_n}{n} = \frac{5 - \sqrt{5}}{10}. \tag{1}$$

In [7] we gave a very simple proof of (1) and also proved a certain generalization of this result. In order to state this generalization, we introduce some notations and terminology from [7]. Let $1 = a_1 < a_2 < \cdots$ be a strictly increasing sequence of positive integers with the first element equal to 1. We call this an *A-sequence*. Suppose that $m$ is a positive integer. We write

$$m = a_{(1)} + a_{(2)} + \cdots + a_{(s)}, \tag{2}$$

where $a_{(1)}$ is the greatest element of the $A$-sequence $\leqslant m$, $a_{(2)}$ is the greatest element of the $A$-sequence $\leqslant m - a_{(1)}$, and, generally, $a_{(i)}$ is the greatest element of the $A$-sequence $\leqslant m - a_{(1)} - a_{(2)} - \cdots - a_{(i-1)}$. We denote by $h(m)$ the number $s$ in (2), that is, the number of terms in the representation of $m$.

Suppose that $k \geqslant 2$ is a positive integer and define an $A$-sequence by $a_1 = 1$, $a_2 = k$, and $a_{n+2} = a_{n+1} + a_n$, $n = 1, 2, \ldots$. We call this a *recursive A=sequence*. If $a_2 = k = 3$, then $a_n = L_n$, $n = 1, 2, \ldots$. If $a_2 = k = 2$, then

$a_n = F_{n+1}$, $n = 1, 2, \ldots$, and (2) is the Zeckendorf representation [7, Lemma 5.12, p. 45], so that $h(m) = f(m)$.

Consider now a recursive $A$-sequence. If $a_2 = k$, we defined

$$\psi_k(n) = \left(\sum_{i=a_n}^{a_{n+1}-1} h(i)\right) \bigg/ a_{n-1},$$

so that $\psi_2(n) = \psi_n$, and proved [7, Theorem 5.15, p. 46] that for all $a_2 = k \geqslant 2$,

$$\lim_{n\to\infty} \frac{\psi_k(n)}{n} = \frac{5 - \sqrt{5}}{10} \tag{3}$$

a generalization of (1).

In [4] Daykin has given a different generalization of (1). If $h$ and $k$ are positive integers such that $h \leqslant k \leqslant h + 1$, then he defined [4, p. 144] the $(h, k)^{\text{th}}$ *Fibonacci sequence* $(v_i)$ in the following way:

$$\begin{aligned} v_i &= i && \text{for } 1 \leqslant i \leqslant k, \\ v_i &= v_{i-1} + v_{i-h} && \text{for } k < i < h + k, \\ v_i &= v_{i-1} + v_{i-k} + (k - h) && \text{for } i \geqslant h + k. \end{aligned} \tag{4}$$

Clearly, the Fibonacci numbers $F_2$, $F_3$, ... are given by the $(2, 2)^{\text{th}}$ Fibonacci sequence.

Daykin generalizes the Theorem of Zeckendorf by proving [4, Theorem C, p. 144] that, if $(v_i)$ is the $(h, k)^{\text{th}}$ Fibonacci sequence, then for each positive integer $m$ there is one, and only one, system of positive integers $i_1, i_2, \ldots, i_d$ such that

$$m = v_{i_1} + v_{i_2} + \cdots + v_{i_d}, \tag{5}$$

where $i_2 \geqslant i_1 + h$ if $d > 1$, and $i_{\nu+1} \geqslant i_\nu + k$ for $2 \leqslant \nu < d$. [We note that the $(h, k)^{\text{th}}$ Fibonacci sequence is an $A$-sequence, and it is easy to see that the representation (5) is the same as (2).]

Let $\psi_n$ denote the average number of summands required in (5) for all those positive integers $m$ such that $v_n \leqslant m < v_{n+1}$. Then [4, Theorem E, p. 144] for $k \geqslant 2$,

$$\lim_{n\to\infty} \frac{\psi_n}{n} = \frac{\theta - 1}{1 + k(\theta - 1)}, \tag{6}$$

where $\theta = \theta(k)$ is the positive real solution of the equation $z - 1 = z^{1-k}$.

In this paper we consider three other kinds of unique representations using Fibonacci and Lucas numbers and prove the following corresponding results. Let $f'(m)$ denote the number of elements in the "maximal" representation (see [5, p. 74], [2]) of $m$ using Fibonacci numbers $F_2$, $F_3$, ... (where no "gaps" formed by two consecutive Fibonacci numbers are allowed). Let $g(m)$ denote the number of elements in the "minimal" representation (see [5, p. 76], [3], [8]) of $m$ and let $g'(m)$ denote the number of elements in the "maximal" representation (see [5, p. 77], [3]) of $m$ using Lucas numbers. These are similar to the corresponding Fibonacci representations but with certain additional restrictions to ensure uniqueness. We define

$$\psi'_n = \left(\sum_{i=F_{n+1}}^{F_{n+2}-1} f'(i)\right)\Big/F_n, \quad \lambda_n = \left(\sum_{i=L_{n+1}}^{L_{n+2}-1} g(i)\right)\Big/L_n,$$

and

$$\lambda'_n = \left(\sum_{i=L_{n+1}}^{L_{n+2}-1} g'(i)\right)\Big/L_n.$$

Then we have

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{5-\sqrt{5}}{10}, \tag{7}$$

$$\lim_{n\to\infty} \frac{\lambda'_n}{n} = \frac{5+\sqrt{5}}{10}, \tag{8}$$

and

$$\lim_{n\to\infty} \frac{\psi'_n}{n} = \frac{5+\sqrt{5}}{10}. \tag{9}$$

## 2. "MINIMAL" LUCAS REPRESENTATIONS

Let $a_1 = 1 = L_1$, $a_2 = L_0$, and $a_n = L_{n-1}$, $n = 3, 4, \ldots$, so that

$$a_{n+2} = a_{n+1} + a_n,\ n = 3, 4, \ldots . \tag{10}$$

**Lemma 1:** The representation of a positive integer $m$ corresponding to this $A$-sequence is the "minimal" Lucas representation.

**Proof:** Similar to that of Lemma 5.12 in [7, p. 45]. ■

It follows that $g(m) = h(m)$ for every positive integer $m$. Let

$$S(n) = \sum_{i=1}^{n} h(i) \quad \text{and} \quad S'(n) = S(a_{n+1} - 1) \quad [7,\text{ p. } 7].$$

Then it follows from (10) that we have (compare with Theorem 5.4 in [7, p. 41])

$$S'(n+2) = S'(n+1) + S'(n) + L_n,\ n = 2, 3, \ldots . \tag{11}$$

**Lemma 2:** $S'(n) = n \cdot F_{n-1}$, $n = 2, 3, \ldots$ .

**Proof:** Easily by induction, using (11) and [5, ($I_8$), p. 56]. ■

It follows that

$$\sum_{i=L_{n+1}}^{L_{n+2}-1} g(i) = \sum_{i=a_{n+2}}^{a_{n+3}-1} h(i) = S'(n+2) - S'(n+1) \tag{12}$$
$$= (n+2)F_{n+1} - (n+1)F_n = n \cdot F_{n-1} + L_n,\ n = 2, 3, \ldots .$$

(This holds also for $n = 1$, if we define $F_0 = 0$ as usual.) From (12), it follows that (7) holds, because

$$\lim_{n\to\infty} \frac{F_{n-1}}{L_n} = \frac{5-\sqrt{5}}{10} \quad \text{(see, for example, [7, (5.36), p. 47]).} \tag{13}$$

## 3. "MAXIMAL" LUCAS REPRESENTATIONS

**Lemma 3:** Suppose that $m$ is a positive integer such that $L_{n+1} \leqslant m \leqslant L_{n+2} - 1$, where $n \geqslant 1$. Then the greatest-indexed Lucas number in the "maximal" Lucas representation of $m$ is $L_n$.

**Proof:** This follows from Theorem 2 in [3, p. 250]. ■

Let

$$a(n) = \sum_{i=L_{n+1}}^{L_{n+2}-1} g'(i),\ n \geqslant 0,$$

so that $\lambda'_n = a(n)/L_n$.

**Lemma 4:** $a(n) = a(n-1) + a(n-2) + L_n,\ n = 2, 3, \ldots .$

**Proof:** Since $a(0) = a(1) = 2$, $a(2) = 7$, and $L_2 = 3$, the equation clearly holds for $n = 2$. Let $L_{n+1} \leqslant m \leqslant L_{n+2} - 1$, where $n \geqslant 3$. According to Lemma 3, the greatest-indexed Lucas number in the representation of $m$ is $L_n$. Let $m' = m - L_n$. Then

$$L_{n-1} \leqslant m' \leqslant L_{n+1} - 1. \tag{14}$$

According to Lemma 3, if $L_n \leqslant m' \leqslant L_{n+1} - 1$, then the greatest-indexed Lucas number in the representation of $m'$ is $L_{n-1}$, and if $L_{n-1} \leqslant m' \leqslant L_n - 1$, then it is $L_{n-2}$. It follows that in both cases we get the representation of $m$ by adding $L_n$ to the representation of $m'$. It follows that

$$g'(m) = g'(m') + 1, \tag{15}$$

which, together with (14), clearly completes the proof. ■

**Lemma 5:** $a(n) = n \cdot F_{n+1} + L_n,\ n = 0, 1, \ldots .$

**Proof:** Easily by induction using Lemma 4. ■

It follows that (8) holds because

$$\lim_{n\to\infty} \frac{F_{n+1}}{L_n} = \frac{5+\sqrt{5}}{10} \quad \text{(see, for example, [7, (5.34), p. 47]).} \tag{16}$$

## 4. "MAXIMAL" FIBONACCI REPRESENTATIONS

Let

$$b(n) = \sum_{i=F_{n+1}}^{F_{n+2}-1} f'(i), \; n \geqslant 1,$$

so that $\psi'_n = b(n)/F_n$. Let

$$c(n) = \sum_{i=F_{n+1}-1}^{F_{n+2}-2} f'(i), \; n \geqslant 2.$$

Then we have

**Lemma 6:** $b(n) = c(n) + \dfrac{1+(-1)^{n+1}}{2}$, $n = 2, 3, \ldots$ .

**Proof:** $b(n) - c(n) = f'(F_{n+2} - 1) - f'(F_{n+1} - 1)$. We use the formulas (see, for example, [5, $(I_5)$, $(I_6)$, p. 56])

$$F_{2k} - 1 = F_{2k-1} + F_{2k-3} + \cdots + F_5 + F_3, \; k = 2, 3, \ldots, \tag{17}$$

and

$$F_{2k+1} - 1 = F_{2k} + F_{2k-2} + \cdots + F_4 + F_2, \; k = 1, 2, \ldots . \tag{18}$$

If $n$ is even, $n = 2k$, we get $f'(F_{n+2} - 1) - f'(F_{n+1} - 1) = k - k = 0$ and if $n$ is odd, $n = 2k + 1$, we get $f'(F_{n+2} - 1) - f'(F_{n+1} - 1) = (k + 1) - k = 1$. ■

**Lemma 7:** Let $F_{n+1} - 1 \leqslant m \leqslant F_{n+2} - 2$, where $n \geqslant 2$. Then the greatest Fibonacci number in the "maximal" representation of $m$ is $F_n$.

**Proof:** [2, Theorem 1, p. 2]. ■

In a similar fashion as in the case of "maximal" Lucas representations, it follows now that

$$c(n) = c(n-1) + c(n-2) + F_n, \; n = 4, 5, \ldots . \tag{19}$$

**Lemma 8:** $c(n) = (1/5)(n \cdot L_{n+1} - 3F_n)$, $n = 2, 3, \ldots$ .

**Proof:** Easily by induction using (19) and [5, $(I_9)$, p. 56]. ■

Formula (9) now follows from Lemma 8 and Lemma 6 using the fact that

$$\lim_{n\to\infty} \frac{L_{n+1}}{F_n} = \frac{5+\sqrt{5}}{2} \quad \text{(see, for example, [7, (5.36), p. 47]).} \tag{20}$$

## REFERENCES

1. J. L. Brown, Jr. "Zeckendorf's Theorem and Some Applications." *The Fibonacci Quarterly 2* (1964):162-168.

2. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *The Fibonacci Quarterly* **3** (1965):1-8.

3. J. L. Brown, Jr. "Unique Representation of Integers as Sums of Distinct Lucas Numbers." *The Fibonacci Quarterly* **7** (1969):243-252.

4. D. E. Daykin. "Representation of Natural Numbers as Sums of Generalised Fibonacci Numbers." *J. London Math. Soc.* **35** (1960):143-160.

5. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.

6. C. G. Lekkerkerker. "Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci." *Simon Stevin* **29** (1951-1952):190-195.

7. J. Pihko. "An Algorithm for Additive Representation of Positive Integers." *Ann. Acad. Sci. Fenn., Ser. A I Math. Dissertations No.* **46** (1983):1-54.

8. E. Zeckendorf. "Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas." *Bull. Soc. Royale Sci. Liège* **41** (1972):179-182.

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# A WINNING STRATEGY AT TAXMAN®

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Taxman® is an educational computer game, brought out by the Minnesota Educational Consortium. Starting from an initial set, which in the standard game is

$$S_1 = \{1, 2, \ldots, n\},$$

the player chooses successive integers $k_1, k_2, \ldots$ . After each choice $k_j$, $k_j$ and its divisors in $S_j$ are deleted to form $S_{j+1}$. The player's score is increased by $k_j$ and the computer's by the sum of all the deleted proper divisors. It is illegal to choose $k \in S_j$ if $k$ has no proper divisor in $S_j$. Initially, any $k$ except 1 may be chosen in the standard game, since that $k$ has at least the proper divisor $1 \in S_1$. As play continues, the number of legal choices dwindles. Whenever the player has no legal move, the computer scores the sum of the remaining elements and the game is over. The objective is to have a higher score than the computer at the end.

Play can be described by listing the integers chosen in the order they were picked. For instance, with $n = 10$, we might play (10, 9, 8). The monitor would show, successively,

| | | YOU | ME |
|---|---|---|---|
| {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} | , | 0, | 0 |
| {3, 4, 6, 7, 8, 9} | , | 10, | 8 |
| {4, 6, 7, 8} | , | 19, | 11 |
| {6, 7} | , | 27, | 15 |
| GAME OVER | , | 27, | 28. |

We lost. We could have won if we had picked 7 first. The computer would have deleted 7 (for us) and 1 (for itself) to give $S_2 = \{2, 3, 4, 5, 6, 8, 9, 10\}$. After that we could still have chosen 10, 9, and 8, or better still, 9, 6, 8, and 10. In general, we should begin play by choosing the largest prime $p \leqslant n$. Aside from our choice, only 1 will be deleted, and it is deleted on any first move. However, for large $n$ there are $\approx (1/2)n^2$ points at stake, and this tactic makes at most an $n$ point difference. Let $f(n)$ denote the best possible score for the player on $\{1, 2, \ldots, n\}$. It is natural to conjecture that

$$\lim_{n\to\infty} f(n)\Big/\left(\frac{1}{2}n^2\right) = C$$

exists. If so, and if $C > 1/2$, then the player can win for all sufficiently large $n$.

In fact, we have the following theorem.

**Theorem:** $\lim_{n\to\infty} f(n)\Big/\left(\frac{1}{2}n^2\right) = C$ exists, and $\frac{1}{2} < C < \frac{3}{4}$.

From this it follows that the player can win for $n$ sufficiently large. On the basis of the proof we give, "sufficiently large" may be very large indeed. Yet a little experimentation strongly suggests that in fact the player can win for $n \geqslant 4$. Resolving the question of how large $n$ has to be is simple, in principle. Suppose our theoretical argument shows $f(n) > (1/4)(n^2 + n)$ for $n > N$. We have only to exhibit a winning line of play for all $n$, $4 \leqslant n \leqslant N$, to show the player wins for any $n \geqslant 4$. Unfortunately, the calculations will be lengthy unless the theoretical argument is greatly sharpened, reducing $N$ to tractable size. (I obtained $N = 6{,}000{,}000$.)

The idea of the asymptotically winning strategy is to divide and conquer, by partitioning the game into subgames playable separately on certain nonstandard initial sets. We select a prime $p$ and let

$D_p$ denote $\{d\text{: if } q|d \text{ and } q \text{ is prime, then } q \leqslant p\}$,

$$A_p = \prod_{\substack{q \leqslant p \\ q \text{ prime}}} q \quad \text{and} \quad B_p = \prod_{\substack{q \leqslant p \\ q \text{ prime}}} (1 - 1/q).$$

Thus, $D_3 = \{1, 2, 3, 4, 6, 8, 9, 12, 16, \ldots\}$, $A_3 = 6$, and $B_3 = 1/3$. Next, for the chosen $p$, we partition $\{1, 2, \ldots, n\}$ into sets

$$N_{k,p}(n) = \{kd\text{: } d \in D_p \text{ and } kd \leqslant n\}$$

for $k$ relatively prime to $A_p$. Thus, with $p = 3$, $\{1, 2, \ldots, 40\}$ partitions as

$\{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36\}$, $\{5, 10, 15, 20, 30, 40\}$
$\{7, 14, 21, 28\}$, $\{11, 22, 33\}$, $\{13, 26, 39\}$, $\{17, 34\}$, $\{19, 38\}$

and some singletons. Any time we choose a number in $N_{1,p}(n)$, only elements of $N_{1,p}(n)$ are deleted. In general,

*However we play* on $N_{k,p}(n)$, *the sets* $N_{k',p}(n)$ for $k' > k$ *are undisturbed.* (1)

Let $f_p(x)$ denote the best score possible for the player if the (nonstandard) initial set is $\{d \in D_p\text{: } 1 \leqslant d \leqslant x\}$. Thus, $f_p(x)$ is defined for real $x$, but only changes at elements of $D_p$; e.g., $f_3(5) = 7$, because on $[1, 5] \cap D_3 =$

$\{1, 2, 3, 4\}$, the best play is to take 3 and then 4. Similarly, $f_3(36) = 144$, taking 3, 4, 27, 18, 36, 24, and 32, starting from $[1, 36] \cap D_3$.

The best score possible on $N_{k,p}(n)$ is clearly $kf_p(n/k)$. In view of (1), then, if we play on $N_{k,p}(n)$ in order of increasing $k$, we get

$$\sum_{k \leqslant n}^{*} kf_p(n/k),$$

where $\Sigma^*$ denotes summation only over $k$ relatively prime to $A_p$. This score is a lower bound for $f(n)$, that is,

$$f(n) \geqslant \sum_{k \leqslant n}^{*} kf_p(n/k). \tag{2}$$

In our example $n = 40$, the same line of play is applied to $\{11, 22, 33\}$ and $\{13, 26, 39\}$, and from these we score $11f_3(3)$ and $13f_3(3)$, respectively. In general, grouping partition pieces having the same number of elements puts (2) into the form

$$f(n) \geqslant \sum_{\substack{j \leqslant n \\ j \in D_p}} f_p(j) \sum_{\frac{n}{j'} < k \leqslant \frac{n}{j}}^{*} k, \tag{3}$$

where $j'$ denotes the next element of $D_p$ after $j$.

Let us now temporarily put aside rigor and look ahead to the answer. If $B_p = b/A_p$, then $b$ is an integer, and of any $A_p$ consecutive integers, $b$ of them are relatively prime to $A_p$. Thus, the inner sum in (3) is the sum of, roughly, $B_p\left(\frac{n}{j} - \frac{n}{j'}\right)$ integers, with an average value of about $\frac{1}{2}\left(\frac{n}{j} + \frac{n}{j'}\right)$. This suggests something like

$$f(n) \geqslant \frac{1}{2}n^2 B_p \sum_{\substack{j \leqslant n \\ j \in D_p}} f_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right). \tag{3'}$$

Happily, essentially the same sum as in (3) provides an upper bound for $f(n)$.

Suppose we choose $p$ prime and then play a game on $\{1, 2, \ldots, n\}$. For each integer $m \leqslant n$ we pick, we note which is the largest proper divisor of $m$ in play at the time, and call it $t(m)$. Distinct $m$'s have distinct $t(m)$'s, since $t(m)$ is deleted when we pick $m$. We separate the $m$ we pick into two sets:

$$M_1 = \{m\colon t(m) < n/p\} \quad \text{and} \quad M_2 = \{m\colon t(m) \geqslant n/p\}.$$

Clearly, $M_1$ has fewer than $n/p$ elements, so our score from $M_1$ is less than $n^2/p$. To bound from above our score from $M_2$, we need a lemma.

**Lemma 1:** Suppose $k$ is relatively prime to $A_p$, $d \in D_p$, and $kd \in M_2$. Then

$$k \mid t(kd).$$

**Proof:** If $k \nmid t(kd)$, then some prime $q > p$ divides $k$ to a higher power than it does $t(kd)$. But then

$$kd/t(kd) \geqslant q > p,$$

in contradiction to the assumption $t(kd) \geqslant n/p$. ∎

From Lemma 1, we claim that, for $k \leqslant n$ and relatively prime to $A_p$,

$$\sum_{\substack{d \in D_p \\ kd \in M_2}} kd \leqslant kf_p(n/k). \tag{4}$$

For consider the sequence of moves in the standard game we just played, but restricted to those moves which chose a number of the form $kd$, with $d \in D_p$ and $kd \in M_2$. We can map this sequence of moves onto a shadow game played on the initial set $D_p \cap \{1, 2, \ldots, [n/k]\}$. The image of a choice of $kd$ in the real game is the choice of $d$ in the shadow game. This $d$ will be a legal move. First, $k^{-1}t(kd)$ is a proper divisor of $d$, since $t(kd)$ was a proper divisor of $kd$, and since, from Lemma 1, $k^{-1}t(kd)$ is an integer. Second, since $t(kd)$ had not yet been deleted at the time $kd$ was chosen in the standard game, no multiple $kd'$ of $t(kd)$ had yet been chosen in that game. Thus, in the shadow game, no multiple $d'$ of $k^{-1}t(kd)$ can yet have been chosen. Therefore, $k^{-1}t(kd)$ must still be in play in the shadow game and available as an as yet undeleted proper divisor of $d$. By its definition, the sum of the numbers $d$ so chosen in the shadow game is less than or equal to $f_p(n/k)$, and (4) follows on multiplication by $k$.

Summing (4) over $k$ and using our observation about $M_1$ now gives

$$\sum_{m \in M_1 \cup M_2} m \leqslant n^2/p + \sum_{k \leqslant n}{}^{*} kf(n/k), \tag{5}$$

and since this holds even for best play, we can group $k$'s as before and get

$$f(n) \leqslant n^2/p + \sum_{\substack{j \leqslant n \\ j \in D_p}} f_p(j) \sum_{\frac{n}{j'} < k \leqslant \frac{n}{j}}{}^{*} k. \tag{6}$$

The analog of (3′) is then

$$f(n) \leqslant \frac{1}{2}n^2\left(\frac{2}{p} + B_p \sum_{\substack{j \leqslant n \\ j \in D_p}} f_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right)\right). \tag{6′}$$

Now assuming that the sum here is convergent (and it is, as we shall prove later), (3′) and (6′) converge to give

$$\lim_{n \to \infty} \frac{f(n)}{(1/2)n^2} = \lim_{p \to \infty} B_p \sum_{j \in D_p} f_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right) = C. \tag{7}$$

The path now splits. We should like to have some notion of the value of $C$, and the demands of rigor must be met. First, let us work on $C$.

In principle we have only to pick $p$ large, calculate $f_p(j)$ for enough terms that the "tail" of

$$\sum_{D_p} f_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right)$$

is less than $1/p$, and we shall get $C$ to within an error on the order of $1/p$. The catch is that it is hard to find $f_p(j)$ for large $j$ and large $p$.

A crude upper bound is not so hard. Any odd numbers $m$ between $n/2$ and $n$ that are picked have odd $t(m)$ between 1 and $n/3$. There are, thus, $\leqslant (n + 3)/6$ such $m$. We can pick in all no more than $(1/2)n$ numbers. The sum of a set of $\leqslant n/2$ numbers, all $\leqslant n$ and containing at most $(n + 3)/6$ odd numbers between $n/2$ and $n$, is at most $(35/96)n^2 + 0(n)$.

Thus, $C \leqslant 35/48 < 3/4$. The proof that $C > 1/2$ is more difficult.

We choose $p = 5$ and calculate $f_5(j)$ for $1 \leqslant j \leqslant 36$, and then a lower bound for $f_5(j)$ for $40 \leqslant j \leqslant 200$. It turns out that

$$\sum_{\substack{j \in D_5 \\ j \leqslant 200}} f_5(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right) > 1.9 \quad \text{and} \quad B_5 = \frac{4}{15}.$$

Now, $$C \geqslant B_5 \sum_{j \in D_5} f_5(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right) > \frac{4}{15} \cdot \frac{19}{10} > \frac{1}{2}.$$

[See the table of $f_5(j)$, 1 to 36, and the lower bound, 40 to 200.] More extensive calculations with $p = 7$ suggest that in fact $C > .56$. Before proceeding to the problem of justifying (3′) and (6′) (which are not claimed to hold verbatim), it would be well to spell out the winning strategy.

(A) Partition $\{1, 2, \ldots, n\}$ into sets of the form $N_{k,5}(n) = \{kd\colon d \in D_5, d \leqslant n/k\}$ with $k$ relatively prime to 30.

(B) Discard $N_{k,5}(n)$ if $n/k > 200$. Make no attempt to score from these $k$.

(C) For all $k$ relatively prime to 30 and satisfying $(n/200) \leqslant k \leqslant n$, play $N_{k,5}(n)$ as instructed by the table. Start with smaller values of $k$ and work up.

This will win if $n$ is large enough. For lesser $n$, we might do well to go ahead and play the $N_{k,5}(n)$ for small $k$ by ear, starting with $k = 1$. And, of course, first pick the largest prime.

We now justify (3′) and (6′) and show that

$$\sum_{D_p} f_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right)$$

is convergent. The "0" notation will be helpful from this point on. We say

$\phi_1(n) = O(\phi_2(n))$ if there exists $C_1 > 0$ such that $|\phi_1(n)| \leqslant C_1\phi_2(n)$ for all $n$. A subscript $O_p$ denotes that for each $p$ such a constant $C_p$ exists.

**Table A.** $f_5(n)$ for $n \in D_5$, $n \leqslant 36$

| $n$ | $f_5(n)$ | Moves |
|---|---|---|
| 1 | 0 | none |
| 2 | 2 | (2) |
| 3 | 3 | (3) |
| 4 | 7 | (3, 4) |
| 5 | 9 | (5, 4) |
| 6 | 15 | (5, 4, 6) |
| 8 | 19 | (5, 6, 8) |
| 9 | 28 | (5, 9, 6, 8) |
| 10 | 33 | (9, 6, 10, 8) |
| 12 | 44 | (5, 9, 10, 8, 12) |
| 15 | 54 | (9, 15, 10, 8, 12) |
| 16 | 62 | (9, 15, 10, 12, 16) |
| 18 | 80 | (9, 15, 20, 18, 12, 16) |
| 20 | 96 | (5, 15, 10, 20, 12, 18, 16) |
| 24 | 112 | (9, 15, 10, 18, 20, 16, 24) |
| 25 | 128 | (25, 15, 10, 20, 16, 18, 24) |
| 27 | 155 | (25, 15, 27, 10, 18, 20, 16, 24) |
| 30 | 177 | (2, 25, 15, 27, 18, 30, 20, 16, 24) |
| 32 | 193 | (2, 25,15, 27, 18, 30, 20, 24, 32) |
| 36 | 219 | (3, 4, 25, 27, 18, 36, 24, 20, 30, 32) |

**Table B.** The lower bound for $f_5(n)$ given here comes from first playing the odd numbers by hand, then taking $2f_5(n/2)$ for our score on the evens.

| $n$, | $f_5(n) \geqslant$, | Moves | $n$, | $f_5(n) \geqslant$, | Moves |
|---|---|---|---|---|---|
| 40, | 259, | (25, 15, 27) | 100, | 941 | |
| 45, | 292, | (3, 25, 27, 45) | 108, | 1049 | |
| 48, | 324 | | 120, | 1137 | |
| 50, | 356 | | 125, | 1239, | (5, 125, 75, 45, 81) |
| 54, | 410 | | 128, | 1303 | |
| 60, | 454 | | 135, | 1402, | (5, 9, 81, 125, 75, 135) |
| 64, | 486 | | 144, | 1506 | |
| 72, | 538 | | 150, | 1610 | |
| 75, | 590, | (5, 27, 45, 75) | 160, | 1770 | |
| 80, | 670 | | 162, | 1924 | |
| 81, | 747, | (3, 25, 75, 45, 81) | 180, | 2056 | |
| 90, | 813 | | 192, | 2184 | |
| 96, | 877 | | 200, | 2312 | |

**Lemma 2:** For $0 < x < y$,

$$\sum_{x<k\leqslant y}^{*} k = \frac{1}{2}B_p(y^2 - x^2) + O(A_p^2) + O(yA_p).$$

**Proof:** Consider the set $R_p$ of reduced residues mod $A_p$ that are relatively prime to $A_p$. $R_p$ has $A_pB_p$ elements. For each $r \in R_p$, the arithmetic progression $(r, r + A_p, r + 2A_p, \ldots)$ intersects the interval $(x, y)$ in either $[(y - x)/A_p]$ or $1 + [(y - x)/A_p]$ points, whose average lies between $\frac{1}{2}(x + y - A_p)$ and $\frac{1}{2}(x + y + A_p)$ if there are any. Thus, for $r \in R_p$,

$$\sum_{\substack{k = r + jA_p \\ x < k < y}} k = \left(\frac{x + y}{2} + 0(A_p)\right)\left(\frac{y - x}{A_p} + 0(1)\right) = \frac{y^2 - x^2}{2A_p} + 0(y) + 0(A_p). \quad (8)$$

Now, summing over the $A_pB_p$ elements of $R_p$ gives

$$\sum_{x<k<y}^{*} k = \frac{1}{2}B_p(y^2 - x^2) + 0(yA_p) + 0(A_p^2). \blacksquare \quad (9)$$

**Remark:** We could get much sharper estimates here from the literature on sieves. The quantity estimated in (9) is a weighted count of how many numbers survive sifting by the small primes $q \leqslant p$. See [2] for a readable introduction to sieves.

From Lemma 2, and from (3),

$$\sum_{k \leqslant n} kf_p(n/k) = \frac{1}{2}n^2B_p \sum_{\substack{j \in D_p \\ j \leqslant n}} f_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right) + 0_p\left(n \sum_{\substack{j \in D_p \\ j \leqslant n}} \frac{1}{j} f_p(j)\right). \quad (10)$$

Now, let $g_p(j)$ be the total number of points at stake in $D_p \cap [1, j]$, that is,

$$g_p(j) = \sum_{\substack{d \leqslant j \\ d \in D_p}} d. \quad (11)$$

Then $f_p(j) < g_p(j)$, so (10) holds with $g_p$ in place of $f_p$ in the error term. Now,

$$\frac{1}{j}g_p(j) \leqslant \sum_{\substack{d \in D_p \\ d \leqslant j}} 1 = \Psi(j, p).$$

The counting function $\Psi(x, y)$ of integers $\leqslant x$ composed exclusively of primes $\leqslant y$ has been the topic of numerous studies over the past fifty years. For an elementary but surprisingly good estimate, see [1].

Here, because we are not trying to see how small we can take $n$ with a given $p$, a simple estimate will do.

**Lemma 3:** $\Psi(x, p) = 0(\log x)^p$.

**Proof:** There are $\leqslant \left[\frac{\log x}{\log 2}\right] + 1$ possible values for the number of powers of 2 in a number $\leqslant x$, $\left[\frac{\log x}{\log 3}\right] + 1$ possibilities for the number of powers of 3, ..., and there are clearly fewer than $p$ primes $\leqslant p$. $\blacksquare$

Thus, from Lemma 3,

$$\sum_{\substack{j \in D_p \\ j \leqslant n}} \frac{1}{j} g_p(j) = O(\log n)^{2p}, \tag{12}$$

since the sum in (12) has $\Psi(n, p)$ terms. Thus,

$$\sum_{k \leqslant n}{}^{*} k f_p(n/k) = \frac{1}{2} n^2 B_p \sum_{\substack{j \in D_p \\ j \leqslant n}} f_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right) + O_p(n(\log n)^{2p}). \tag{13}$$

Our other unfinished business is to show that

$$\sum_{\substack{j \in D_p \\ j \leqslant n}} f_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right)$$

is convergent. For purposes of computation, some estimate of the *rate* of convergence would also be helpful—how many terms must we take to bring the partial sum to within $\varepsilon$ of its limit?

Convergence of

$$\sum_{\substack{j \in D_p \\ j \leqslant n}} f_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right)$$

is simple. Since $f_p(j) < g_p(j)$, we need only prove

$$\sum_{D_p} g_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right)$$

convergent. It is, to $1/B_p$. ■

**Proof:**
$$\sum_{D_p} g_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right) = \sum_{j \in D_p}\left(\frac{1}{j^2} - \frac{1}{j'^2}\right)\sum_{\substack{d \in D_p \\ d \leqslant j}} d = \sum_{d \in D_p} d \sum_{\substack{j \in D_p \\ j \geqslant d}}\left(\frac{1}{j^2} - \frac{1}{j'^2}\right)$$
$$= \sum_{d \in D_p} d/d^2 = \sum_{d \in D_p} 1/d = \prod_{\substack{q \leqslant p \\ q \text{ prime}}} (1 - 1/q)^{-1} = 1/B_p.$$

Now, for any fixed $p$, if

$$\sum_{\substack{j \in D_p \\ j \leqslant n}} g_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right)$$

is within $\varepsilon$ of $1/B_p$, then

$$\sum_{\substack{j \in D_p \\ j > n}} f_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right) < \varepsilon. \tag{14}$$

But how does $n$ in (14) depend on $\varepsilon$ and $p$? Here is an estimate—the technique is taken from probabilistic number theory, and we omit the proof.

$$\sum_{\substack{j \in D_p \\ j \geqslant x}} g_p(j)\left(\frac{1}{j^2} - \frac{1}{j'^2}\right) = O\left(\frac{1}{x}(\log x)^p\right) + O\left(\frac{(\log p)^2}{\log x}\right).$$

## REFERENCES


1. N.G. de Bruijn. "On the Number of Positive Integers $\leqslant x$ and Free of Prime Factors $> y$ II. *Nederl. Akad. Wetensch. Proc.* Ser. A 69 = *Indag. Math. 288* (1966):239-247.
2. H. Halberstam & K. Roth. *Sequences* (Chapter 4). Oxford: Oxford University Press, 1966.
3. *MECC Elementary Volume 1 Mathematics (Games and Drills)*. Minnesota Educational Computing Consortium, 3490 Lexington Ave. North, St. Paul MN 55112, 1983, pp. 47-52.


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# ON PRIME NUMBERS

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(Submitted October 1986)

## I. RAREFACTION OF PRIMES IN THE SERIES OF INTEGERS

Let $p$ be a prime number and $\pi(n)$ the number of primes up to $n$, inclusive. In the capricious succession of primes in the series of integers, the inequality

$$\pi(2n) < 2\pi(n) \qquad (n > 10) \tag{1}$$

shows a certain regularity. It is equivalent to the following proposition.

**Theorem 1:** The first $n$ integers contain more primes than the $n$ following, for $n$ greater than 10.

I submitted this proposition as a conjecture to Professor G. Robin of the University of Limoges, with two remarks:

—It is true for $n < 10{,}000$, as I have verified it on the computer.

—The inequalities

$$\frac{n}{\log n}\left(1 + \frac{1}{2 \log n}\right) < \pi(n) < \frac{n}{\log n}\left(1 + \frac{3}{2 \log n}\right) \qquad (n \geqslant 52)$$

established by Rosser and Schoenfeld [1] are not sufficient to prove (1), for they give

$$2\pi(n) - \pi(2n) > \frac{2n}{\log n \log 2n}\left[\log 2 - \frac{3(\log n)^2 - (\log 2n)^2}{2 \log n \log 2n}\right],$$

where the expression in brackets is negative.

Robin sent me the following ingenious demonstration of Theorem 1: Suppose that for $n \geqslant n_0$

$$\frac{n}{\log n}\left(1 + \frac{a}{\log n}\right) \leqslant \pi(n) \leqslant \frac{n}{\log n}\left(1 + \frac{b}{\log n}\right). \tag{2}$$

Then for $n \geqslant n_0$

$$\Delta_n = 2\pi(n) - \pi(2n) \geqslant \frac{2n}{\log n \log 2n}\left[\log 2 + \frac{a(\log 2n)^2 - b(\log n)^2}{\log 2n \log n}\right]$$

$$\geqslant \frac{2n}{\log n \log 2n}[\log 2 + (a - b)].$$

Therefore, if $b - a < \log 2$, $\Delta_n > 0$ for $n \geqslant n_0$.

We shall see that we can choose $a = 5/6$ for $n \geqslant 10{,}000$, and verify directly afterward that we can also do this for $n \geqslant 227$. We take $b = 3/2$. We write, as usual, $\theta(n) = \sum_{p \leqslant n} \log p$. Then (see [1], p. 359),

$$\theta(n) = n + R(n),$$

where $R(n) < \dfrac{an}{\log n}$ for $a = \dfrac{5}{6}$ and $n > 10{,}000$.

**Lemma:** We have

$$\pi(n) \geqslant \frac{n}{\log n} + \frac{5}{6}\frac{n}{(\log n)^2} \qquad (n \geqslant 227).$$

For $n > n_0 = 10{,}000$, with $\log_i(n) = \displaystyle\int_2^n \frac{dt}{\log t}$,

$$\pi(n) - \pi(n_0) = \int_{n_{\bar{0}}}^{n} \frac{d\theta(t)}{\log t}$$

$$= \log_i(n) - \log_i(n_0) + \frac{R(n)}{\log n} - \frac{R(n_{\bar{0}})}{\log n_0} + \int_{n_0}^{n} \frac{R(t)dt}{t(\log t)^2}$$

$$\geqslant \log_i(n) - \log_i(n_0) - \frac{an}{(\log n)^2} - a\int_{n_0}^{n} \frac{dt}{(\log t)^3}$$

since $R(n_{\bar{0}}) < 0$.

Let

$$f(n) = \log_i(n) - \frac{n}{\log n} - \frac{n}{(\log n)^2}.$$

Then, $f'(n) = 2/(\log t)^3$; hence,

$$f(n) - f(n_0) = 2\int_{n_0}^{n} \frac{dt}{(\log t)^3}$$

and

$$\pi(n) - \frac{n}{\log n} - \frac{n}{(\log n)^2} \geqslant \frac{-an}{(\log n)^2} + \left(1 - \frac{a}{2}\right)[f(n) - f(n_0)] + \pi(n_0) - \frac{n_0}{\log n_0} - \frac{n_0}{(\log n_0)^2}.$$

So, for $n_0 = 10{,}000$,

$$\pi(n_0) = 1229 > \frac{n_0}{\log n_0} + \frac{n_0}{(\log n_0)^2}.$$

Since $f(n) > f(n_0)$, we have

$$\pi(n) \geqslant \frac{n}{\log n} + \frac{5}{6}\frac{n}{(\log n)^2} \qquad (n \geqslant 10{,}000).$$

**Remark:** G. Robin adds: "In [2] the authors claim to have proved (1) and state their demonstration will be published at a later date. As far as I know, it has not yet appeared, but as we saw, the proof is an easy consequence of the results from Rosser and Schoenfeld."

Professor Robin also sent me a demonstration of a more general proposition, which I submitted to him again as a conjecture.

**Theorem 2:** For all integers $k$ and $n$ greater than 2,

$$\pi(kn) < k\pi(n). \tag{2}$$

Suppose (3) is verified for $n \geqslant n_0$. Then,

$$k\pi(n) - \pi(kn) \geqslant \frac{kn}{\log n \log kn}\left[\log k + \frac{a(\log kn)^2 - b(\log n)^2}{\log kn \log n}\right]$$

$$\geqslant \frac{kn}{\log n \log kn}[\log k + (a - b)].$$

We have seen that with $n_0 \geqslant 10{,}000$, we can take $a = 5/6$, $b = 3/2$. So

$$\log k + (a - b) > 0 \quad \text{for } k \geqslant 2.$$

For $n \geqslant 59$, we can choose $a = \frac{1}{2}$ and $b = \frac{3}{2}$; consequently, $\log k + (a - b) > 0$, for $k \geqslant 3$. We verify directly afterward that (3) holds for $k = 3$ and $n < 59$. For $n \geqslant 17$, we can choose $a = 0$ and $b = \frac{3}{3}$; consequently, $\log k + (a - b) > 0$, for $k \geqslant 5$.

The case $k = 4$ being treated as $k = 2$, it remains to study (3) for $n \leqslant 16$. We did this successively for

$$13 \leqslant n \leqslant 16,\ 11 \leqslant n \leqslant 12,\ 7 \leqslant n \leqslant 10,\ 5 \leqslant n \leqslant 6,\ 3 \leqslant n \leqslant 4,$$

using for $\pi(km)$ the majoration

$$\pi(n) < \frac{5}{4}\frac{n}{\log n} \quad (n \geqslant 2).$$

## II. RAREFACTION OF TWINS IN THE SERIES OF PRIMES

Twins are two primes with a difference of 2.

**Theorem 3:** In the infinite series of great primes, twins are extremely rare.

Let $p_n$ be the $n^{\text{th}}$ prime number. The probability that 7, for instance, does not divide $p_n + 2$ is 5/6, for the equiprobable remainders in the division of $p_n$ by 7 are 2, 3, 4, 5, and 6. The probability $P_n$ that $p_n + 2$ will be a prime is also

$$P_n = \prod_{2 < p < \sqrt{p_n}} \frac{p - 2}{p - 1}.$$

Therefore, if $n$ tends to infinity, $P_n$ tends to zero, like

$$\prod_{p < \sqrt{p_n}} \frac{p - 1}{p},$$

which is greater.

**Remark**: Our reasoning is not quite rigorous, because we utilize implicitly the independence of prime numbers. A conjecture in [3] substitutes an approached value of $P_n$,

$$\prod_{2<p<p_n\cdot 0,5615\ldots} \frac{p-2}{p-1} \quad \text{to} \quad \prod_{2<p<p_n\cdot 0,5} \frac{p-2}{p-1},$$

where $0,5615\ldots = e^{-\gamma}$, with Euler's constant $\gamma = 0,5772\ldots$ . Mertens proved that

$$\log P_m \prod_{i \leqslant m} \frac{p_i - 1}{p_i}$$

tends to $e^{-\gamma}$, if $m$ tends to infinity.

Yet our reasoning carries away, I think, any doubt that the probability $P_n$ tends to zero. If you disagree, consider Theorems 3 and 4 as conjectures.

**Theorem 4**: The series of primes presents arbitrarily great intervals without twins.

Indeed, if every interval of $k$ consecutive primes presented at least one pair of twins, the probability $P_n$ would be at least $1/k$ and could not tend to zero.

Remarks (see [4]):

1. The table below gives an idea of the rarefaction of twins; the 150,000 first integers after $10^n$ present $t$ pairs of twins:

| $n$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|
| $t$ | 584 | 461 | 314 | 309 | 259 | 211 | 191 | 166 |

2. One estimates empirically that the number of pairs of twins up to $n$ has the order of

$$132032 \frac{n}{(\log n)^2}.$$

The estimate leads one to believe there exist an infinity of twin primes.

## REFERENCES


1. L. Schoenfeld. "Shaper Bounds for the Chebychev Functions $\theta(x)$ and $\psi(x)$." *Mathematics of Computation* **30** (1976):337-360.
2. J. Rosser & L. Schoenfeld. "Approximate Formulas for Some Functions of Prime Numbers." *Illinois Journal of Mathematics* **6** (1962):64-94.
3. H. Riesel. *Prime Numbers and Computer Methods for Factorisation* (Chap. 3). Basel: Birkhäuser, 1935.
4. P. Davis & R. Hersch. *The Mathematical Experiences*, pp. 215-216. Boston, 1982.


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# A GENERALIZATION OF METROD'S IDENTITY

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*(Submitted November 1986)*

In 1913, G. Métrod published an arithmetical identity involving Euler's function $\phi$ and the Jordan function $J_2$, and asked if a similar identity holds for other Jordan functions [2, p. 155]. The Jordan functions $J_k$, for $k = 1, 2, \ldots$, are the Dirichlet convolutions of the functions $\zeta_k$ and the Möbius function $\mu$, where $\zeta_k(n) = n^k$ for all $n$, i.e., $J_k = \zeta_k * \mu$. Cohen [1] answered Métrod's question by showing that, for all $n$,

$$\sum_{d|n} \sum_{e|d} J_k(n/d) J_s(n/e) e^k = n^{k+s}. \tag{1}$$

Métrod's identity is the special case $k = 1$, $s = 2$.

H. Stevens [5] defined a class of arithmetical functions which includes the Jordan functions and he showed that a suitable identity analogous to (1) holds for any two functions in the class. All of Stevens' functions can be written in the form $g * h^{-1}$, where $g$ and $h$ are completely multiplicative, i.e., $g(mn) = g(m)g(n)$ for all $m$ and $n$, and similarly for $h$. The function $h^{-1}$ is the inverse of $h$ with respect to Dirichlet convolution.

In this note, we point out that there is an identity which extends (1) and Stevens' identity in several ways. It involves an arbitrary finite number of functions, and the functions are not restricted as severely as those described above. Furthermore, it holds for an arbitrary regular arithmetical convolution. We shall derive the identity for the Dirichlet convolutions, and restate it in the more general setting at the end of the note. Our terminology and notation will be that used in [3].

For $i = 1, \ldots, k$, let $f_i = g_i * h_i^{-1}$, and assume that $g_i$ is completely multiplicative for $i = 1, \ldots, k - 1$. Then, for all $n$,

$$\begin{aligned}\sum_{d_1|n} \sum_{d_2|d_1} \cdots \sum_{d_k|d_{k-1}} g_1(d_2)\ldots g_{k-1}(d_k) h_1(d_1/d_2)\ldots h_{k-1}(d_{k-1}/d_k) h_k(d_k) \\ \cdot f_1(n/d_1)\ldots f_k(n/d_k) \\ = g_1(n)\ldots g_k(n).\end{aligned} \tag{2}$$

When $k = 1$, (2) is simply the expression of the fact that $g_1 = f_1 * h_1$. We shall complete the proof of (2) by induction on $k$. Assume $k > 1$, and that (2) holds when $k$ is replaced by $k - 1$. Then, for all $n$,

$$g_1(n)\ldots g_k(n)$$
$$= g_1(n) \sum_{d_2|n} \cdots \sum_{d_k|d_{k-1}} g_2(d_3)\ldots g_{k-1}(d_k)h_2(d_2/d_3)\ldots h_{k-1}(d_{k-1}/d_k)h_k(d_k) \cdot f_2(n/d_2)\ldots f_k(n/d_k).$$

Since $g_1$ is completely multiplicative,

$$g_1(n) = g_1(d_2)g_1(n/d_2) = g_1(d_2) \sum_{e|(n/d_2)} h_1(e)f_1(n/d_2e)$$

for every choice of $d_2$. Hence,

$$g_1(n)\ldots g_k(n)$$
$$= \sum_{d_2|n} \cdots \sum_{d_k|d_{k-1}} g_1(d_2)g_2(d_3)\ldots g_{k-1}(d_k)h_2(d_2/d_3)\ldots h_{k-1}(d_{k-1}/d_k)h_k(d_k) \cdot f_2(n/d_2)\ldots f_k(n/d_k) \sum_{d_2e|n} h_1(e)f_1(n/d_2e).$$

If we write $d_1$ for $d_2e$, then $d_1$ runs over all divisors of $n$, and for each $d_1$, $d_2$ runs over all divisors of $d_1$, and $e = d_1/d_2$. Hence, we obtain the left-hand side of (2).

Let us look at several examples. If $f_1 = J_k$, $f_2 = J_s$, and $f_3 = J_t$, then, for all $n$,

$$\sum_{c|n} \sum_{d|c} \sum_{e|d} J_k(n/c)J_s(n/d)J_t(n/e)d^k e^s = n^{k+s+t},$$

the three-function analogue of (1).

If we denote $\zeta_0$ by $\zeta$, so that $\zeta(n) = 1$ for all $n$, then $\zeta = \mu^{-1}$ and the divisor sum functions $\sigma_k$ are given by $\sigma_k = \zeta_k * \mu^{-1}$. Thus, for all $n$,

$$\sum_{d|n} \sum_{e|d} e^k \mu(d/e)\mu(e)\sigma_k(n/d)\sigma_s(n/d) = n^{k+s}.$$

This identity is not included in Stevens' extension of (1). If $\beta(n)$ = the number of integers $x$ such that $1 \leqslant x \leqslant n$ and $(x, n)$ is a square, then $\beta = \zeta_1 * h^{-1}$, where $h(n) = |\mu(n)|$ for all $n$ [3, p. 26]. Hence, for all $n$,

$$\sum_{d|n} \sum_{e|d} e|\mu(d/e)|\mu(e)\beta(n/d)\sigma(n/e) = n^2.$$

An identity exactly similar to (2) holds in the setting of an arbitrary regular arithmetical convolution. A discussion of these convolutions can be found in W. Narkiewicz's paper [4] or in Chapter 4 of [3]. Let $A$ be a regular arithmetical convolution. An arithmetical function $f$ is called *A-multiplicative* if $f(n) = f(d)f(n/d)$ for all $n$ and all $d \in A(n)$. This generalization of the notion of completely multiplicative function was introduced by K. L. Yocom [6], who obtained several characterizations of such functions.

For $i = 1. \ldots, k$, let $f_i = g_i *_A h_i^{-1}$, where $h_i^{-1}$ is the inverse of $h_i$ with respect to the regular arithmetical convolution $A$, and assume that $g_i$ is $A$-multiplicative for $i = 1, \ldots, k - 1$. Then, for all $n$,

$$\sum_{d_1 \in A(n)} \sum_{d_2 \in A(d_1)} \cdots \sum_{d_k \in A(d_{k-1})} g_1(d_2) \ldots g_{k-1}(d_k) h_1(d_1/d_2) \ldots h_{k-1}(d_{k-1}/d_k) \cdot h_k(d_k) f_1(n/d_1) \ldots f_k(n/d_k) = g_1(n) \ldots g_k(n). \tag{3}$$

As an example, consider the unitary convolution $U$, where $d \in U(n)$ means $d|n$ and $(d, n/d) = 1$. Usually, we write $d\|n$ rather than $d \in U(n)$. The unitary Jordan function is $J_k^* = \zeta_k *_U \zeta^{-1}$, where now $\zeta^{-1}$ is the inverse of $\zeta$ with respect to the unitary convolution. Then, for all $n$,

$$\sum_{d\|n} \sum_{e\|d} J_k^*(n/d) J_s^*(n/e) e^k = n^{k+s},$$

the unitary analogue of Cohen's identity (1).

The proof of (3) is exactly similar to the proof of (2).

## REFERENCES


1. E. Cohen. "Some Totient Functions." *Duke Math. J.* **23** (1956):515-522.
2. L. E. Dickson. *History of the Theory of Numbers*. Vol. I. New York: Chelsea Publishing Co., 1959.
3. P. J. McCarthy. *Introduction to Arithmetical Functions*. New York: Springer-Verlag, 1986.
4. W. Narkiewicz. "On a Class of Arithmetical Convolutions." *Colloq. Math.* **10** (1963):81-94.
5. H. Stevens. "Generalizations of the Euler $\phi$-Function." *Duke Math. J.* **38** (1971):181-186.
6. K. L. Yocom. "Totally Multiplicative Functions in Regular Convolution Rings." *Canad. Math. Bull.* **16** (1973):119-128.


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# ELEMENTARY PROBLEMS AND SOLUTIONS

*Edited by*
A. P. HILLMAN

*Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.*

## DEFINITIONS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

## PROBLEMS PROPOSED IN THIS ISSUE

B-662 *Proposed by Philip L. Mana, Albuquerque, NM*

For fixed $n$, find all $m$ such that $L_nF_m - F_{m+n} = (-1)^n$.

B-623 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S(n) = \sum_{k=1}^{2n-1} L_{n+k}L_k.$$

Prove that $S(n)$ is an integral multiple of $L_n$ for all positive integers $n$.

B-624 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$T_n = \sum_{i=1}^{n} L_{2(n+i)-1}.$$

For every positive integer $n$, prove that either $F_n | T_n$ or $L_n | T_n$.

B-625 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let $P_0, P_1, \ldots$ be the Pell numbers defined by

$$P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geqq 2.$$

Let $G_n = F_nP_n$ and $H_n = L_nP_n$. Show that $(G_n)$ and $(H_n)$ satisfy

$$K_{n+4} - 2K_{n+3} - 7K_{n+2} - 2K_{n+1} + K_n = 0.$$

B-626 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let $G_n$ and $H_n$ be as in B-625. Express the generating functions

$$G(z) = \sum_{n=0}^{\infty} G_n z^n \quad \text{and} \quad H(z) = \sum_{n=0}^{\infty} H_n z^n$$

as rational functions of $z$.

B-627 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let

$$C_{n,k} = (F_n^3 + F_{n+1}^3 + \cdots + F_{n+k-1}^3)/k.$$

Find the smallest $k$ in $\{2, 3, 4, \ldots\}$ such that $C_{n,k}$ is an integer for every $n$ in $\{0, 1, 2, \ldots\}$.

## SOLUTIONS

### 2 Problems on Pythagorean Triples

B-598 *Proposed by Herta T. Freitag, Roanoke, VA*

For which positive integers $n$ is $(2L_n, L_{2n} - 3, L_{2n} - 1)$ a Pythagorean triple? For which of these $n$'s is the triple primitive?

B-599 *Proposed by Herta T. Freitag, Roanoke, VA*

Do B-598 with the triple now $(2L_n, L_{2n} + 1, L_{2n} + 3)$.

*Solutions by Thomas M. Green, Contra Costa College, San Pablo, CA*

It is known that $L_{2n} = L_n^2 + 2(-1)^{n+1}$.

For $n$ odd, we have $L_{2n} = L_n^2 + 2$ and the triple

$$(2L_n, L_{2n} - 3, L_{2n} - 1) = (2L_n, L_n^2 - 1, L_n^2 + 1)$$

which is a Pythagorean triple. Furthermore, a Pythagorean triple of the type $(2m, m^2 - 1, m^2 + 1)$ is primitive if $m$ is even. Thus, if $L_n = m$, an even number, then $(2L_n, L_n^2 - 1, L_n^2 + 1)$ is primitive. But, if $n$ is odd, $L_n$ is even only when $n$ is an odd multiple of three.

Similarly, for $n$ even (B-599), the triple

$$(2L_n, L_{2n} + 1, L_{2n} + 3) = (2L_n, L_n^2 - 1, L_n^2 + 1)$$

is Pythagorean and will be primitive if $L_n$ is even. In this case, however, if $n$ is even, $L_n$ is even only when $n$ is an even multiple of three.

*Also solved by Paul S. Bruckman, Frank Conliffe, Richard Dry, Piero Filipponi & Adina Di Porto, C. Georghiou, L. Kuipers, Bob Prielip, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Paul Tzermias, and the proposer.*

## Fibonacci Multiples of 121160

B-600 *Proposed by Philip L. Mana, Albuquerque, NM*

Let $n$ be any positive integer and $m = n^{13} - n$. Prove that $F_m$ is an integral multiple of 30290.

*Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA*

We prove a more general result, namely: $F_m$ is an integral multiple of 121,160, where $m = n^{13} - n$; $n$ being a positive integer.

We can express

$$\begin{aligned} n^{13} - n &= (n^7 - n)(n^6 + 1) = (n^5 - n)(n^8 + n^4 + 1) \\ &= (n^3 - n)(n^{10} + n^8 + n^6 + n^4 + n^2 + 1). \end{aligned}$$

By Fermat's theorem: $n^p - n \equiv 0 \pmod{p}$, where $p$ is prime and $n$ is a positive integer.

Thus, we conclude that:

$$n^{13} - n \equiv 0 \pmod{13};\ n^{13} - n \equiv 0 \pmod{7};\ n^{13} - n \equiv 0 \pmod{5}.$$

Since $n^3 - n$ is a factor of $n^{13} - n$ and $n^3 - n$ is a product of three consecutive integers, $n - 1$, $n$, $n + 1$, we have:

$$n^3 - n \equiv 0 \pmod{6} \Rightarrow n^{13} - n \equiv 0 \pmod{6}$$

$$\Rightarrow F_5 \cdot F_6 \cdot F_7 \cdot F_{13} \text{ divides } F_m$$

(by the fact that $r$ divides $s$ implies $F_r$ divides $F_s$)

$$\Rightarrow 5 \cdot 8 \cdot 13 \cdot 233 \text{ is a factor of } F_m.$$

Thus, we are done.

*Also solved by Paul S. Bruckman, David M. Burton, Frank H. Conliffe, Piero Filipponi, C. Georghiou, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, and the proposer.*

## Integral Arithmetic Means

B-601 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $A_{n,k} = (F_n + F_{n+1} + \cdots + F_{n+k-1})/k$. Find the smallest $k$ in $\{2, 3, 4, \ldots\}$ such that $A_{n,k}$ is an integer for every $n$ in $\{0, 1, 2, \ldots\}$.

*Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI*

We shall show that 24 is the value of $k$ that is being sought.

Our solution will use the following known information:

(1) $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$, $n \geqq 1$, and

(2) $F_{n+t} - F_{n-t} = L_n F_t$, $t$ even.

[(1) is $(I_1)$ on p. 52 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr., Houghton Mifflin, Boston, 1969, and (2) is $(I_{24})$ on p. 59, ibid.]

Since

$$\begin{aligned}
&F_n + F_{n+1} + \cdots + F_{n+k-1} \\
&= (F_1 + F_2 + \cdots + F_{n+k-1}) - (F_1 + F_2 + \cdots + F_{n-1}) \\
&= (F_{n+k+1} - 1) - (F_{n+1} - 1) \quad \text{[by (1)]} \\
&= F_{n+k+1} - F_{n+1},
\end{aligned}$$

$$A_{n,k} = (F_{n+k+1} - F_{n+1})/k.$$

Let $n$ be an arbitrary nonnegative integer. If $k = 24$,

$$\begin{aligned}
F_{n+k+1} - F_{n+1} &= F_{(n+13)+12} - F_{(n+13)-12} = L_{n+13}F_{12} \quad \text{[by (2)]} \\
&= L_{n+13} \cdot 144 \equiv 0 \pmod{24}.
\end{aligned}$$

Thus, $A_{n,24}$ is an integer for each nonnegative integer $n$.

$A_{0,2} = (F_3 - F_1)/2 = (2 - 1)/2 = 1/2$. Proceeding in this same manner, it can be shown that $A_{0,k}$ is NOT an integer for $k = 2, 3, 5, 7, 8, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22$, and $23$ and that $A_{1,k}$ is NOT an integer for $k = 4, 6, 9, 11$, and $19$. Therefore, 24 is the smallest $k$ in $\{2, 3, 4, \ldots\}$ such that $A_{n,k}$ is an integer for every nonnegative integer $n$.

*Also solved by David M. Burton, C. Georghiou, L. Kuipers, H.-J. Seiffert, Sahib Singh, Lawrence Somer, David Zeitlin, and the proposer.*

## Fibonacci Infinite Series

B-602 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Let $H_n$ represent either $F_n$ or $L_n$.

(a) Find a simplified expression for $\dfrac{1}{H_n} - \dfrac{1}{H_{n+1}} - \dfrac{1}{H_{n+2}}$.

(b) Use the result of (a) to prove that

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + 2\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}F_{2n+1}F_{2n+2}}.$$

*Solution by C. Georghiou, University of Patras, Greece*

(a) After some simple algebra it is easy to see that

$$\frac{1}{H_n} - \frac{1}{H_{n+1}} - \frac{1}{H_{n+2}} = \frac{H_{n+1}^2 - H_nH_{n+2}}{H_nH_{n+1}H_{n+2}}$$

(b) For $H_n = F_n$, we have $F_{n+1}^2 - F_nF_{n+2} = (-1)^n$, and since $F_n = 0(\alpha^n)$ it follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{F_nF_{n+1}F_{n+2}} &= \sum_{n=1}^{\infty}\left(\frac{1}{F_{2n}F_{2n+1}F_{2n+2}} - \frac{1}{F_{2n-1}F_{2n}F_{2n+1}}\right) \\
&= -2\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}F_{2n+1}F_{2n+2}}.
\end{aligned}$$

On the other hand, we have

$$\sum_{i=1}^{\infty}\left(\frac{1}{F_n} - \frac{1}{F_{n+1}} - \frac{1}{F_{n+2}}\right) = -\sum_{i=1}^{\infty} \frac{1}{F_n} + \frac{2}{F_1} + \frac{1}{F_2}.$$

By equating the two sums we get the given expression.

*Also solved by Piero Filipponi, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Tzermias, and the proposer.*

## Lucas Analogue

B-603 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Do the Lucas analogue of B-602(b).

*Solution by C. Georguiou, University of Patras, Greece*

For $H_n = L_n$, we have $L_{n+1}^2 - L_nL_{n+2} = 5(-1)^{n+1}$, and since $L_n = 0(\alpha^n)$ it follows that

$$\sum_{n=1}^{\infty} \frac{5(-1)^{n+1}}{L_nL_{n+1}L_{n+2}} = 10 \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}L_{2n+1}L_{2n+2}}$$

On the other hand, we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{L_n} - \frac{1}{L_{n+1}} - \frac{1}{L_{n+2}}\right) = -\sum_{n=1}^{\infty}\frac{1}{L_n} + \frac{2}{L_1} + \frac{1}{L_2}.$$

By equating the two sums, we get

$$\sum_{n=1}^{\infty}\frac{1}{L_n} = \frac{7}{3} - 10\sum_{n=1}^{\infty}\frac{1}{L_{2n-1}L_{2n+1}L_{2n+2}}.$$

*Also solved by Piero Filipponi, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Tzermias, and the proposer.*

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# ADVANCED PROBLEMS AND SOLUTIONS

*Edited by*
RAYMOND E. WHITNEY

*Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.*

## PROBLEMS PROPOSED IN THIS ISSUE

H-423 *Proposed by Stanley Rabinowitz, Littleton, MA*

Prove that each root of the equation

$$F_n x^n + F_{n+1}x^{n-1} + F_{n+2}x^{n-2} + \cdots + F_{2n-1}x + F_{2n} = 0$$

has absolute value near $\phi$, the golden ratio.

H-424 *Proposed by Piero Filipponi & Adina Di Porto, Rome, Italy*

Let $F_n$ and $P_n$ denote the Fibonacci and Pell numbers, respectively.

Prove that, if $F_p$ is a prime ($p > 3$), then either $F_p | P_H$ or $F_p | P_{H+1}$, where $H = (F_p - 1)/2$.

## SOLUTIONS

Editorial Notes: Andrzej Makowski has pointed out that H-287 was published in the *American Mathematical Monthly* as Problem S 3 [1979, 55] and the solution appeared in [1980, 136].

Chris Long solved H-211 by using a Lemma of Wolstenholme [*Quart. Jour. Math.* 5 (1862), 35-39].

### Brush the Dust Off

H-152 *Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, CA (deceased) (Vol. 7, no. 1, February 1969)*

Let $m$ denote a positive integer and $F_n$ the $n^{\text{th}}$ Fibonacci number. Further, let $\{c_k\}$, $k = 1$ to $\infty$, be the sequence defined by

$$\{c_k\} \equiv \{\underbrace{(F_n)^m, (F_n)^m, \ldots, (F_n)^m}_{2^{m-1} \text{ copies}}\};\ m,\ k = 1 \text{ to } \infty,$$

Prove that $\{c_k\}$ is complete; i.e., show that every positive integer $n$ has at least one representation of the form

$$n = \sum_{k=1}^{p} \alpha_k c_k,$$

where $p$ is a positive integer and

$$\alpha_i = 0 \text{ or } 1 \text{ if } k = 1, 2, \ldots, p - 1,$$
$$\alpha_p = 1.$$

*Solution by Chris Long, student, Rutgers University, New Brunswick, NJ*

First some preliminaries.

Lemma 1: Let $\{x_i\}$, $i = 1$ to $\infty$, be a nondecreasing sequence of positive integers with $x_1 = 1$. Then $\{x_i\}$ is complete if and only if

$$x_{p+1} \leqq 1 + \sum_{1}^{p} x_i, \text{ for } p = 1, 2, \ldots .$$

Proof: This is proven in J. L. Brown, Jr., "Note on Complete Sequences of Integers," *Amer. Math. Monthly* 67 (1960):557-560.

Lemma 2: $(f_n)^m f_{n-1} + f_n(f_{n-1})^m \leqq (f_{n-1})^{m+1} + (f_n)^{m+1}$ for all $m$, $n \geqq 1$.

Proof: Since for $m$, $n \geqq 1$,

$$(f_{n-1})^m(f_n - f_{n-1}) \leqq (f_n)^m(f_n - f_{n-1}) \Rightarrow (f_n)^m f_{n-1} + f_n(f_{n-1})^m$$
$$\leqq (f_{n-1})^{m+1} + (f_n)^{m+1}.$$

Lemma 3: $(f_{n+1})^m \leqq 2^{m-1}((f_{n-1})^m + (f_n)^m)$ for all $m$, $n \geqq 1$.

Proof: We have $f_{n+1} \leqq f_{n-1} + f_n$ for all $n \geqq 1$. If

$$(f_{n+1})^m \leqq 2^{m-1}((f_{n-1})^m + (f_n)^m),$$

then, since $f_{n+1} = f_{n-1} + f_n$ and $f_{n+1} > 0$ for all $n \geqq 1$,

$$(f_{n+1})^m f_{n+1} \leqq 2^{m-1}((f_{n-1})^{m+1} + (f_n)^{m+1} + (f_n)^m f_{n-1} + f_n(f_{n-1})^m)$$
$$\leqq 2^m((f_{n-1})^{m+1} + (f_n)^{m+1}) \quad \text{by Lemma 2.}$$

Hence, by induction, $(f_{n+1})^m \leqq 2^{m-1}((f_{n-1})^m + (f_n)^m)$ for all $m$, $n \geqq 1$.

Since $c_1 = 1$, $\{c_k\}$ is complete if $c_{k+1} \leqq 1 + c_1 + \cdots + c_k$ for all $k \geqq 1$. Now, if $2 \leqq \alpha \leqq 2^{m-1}$, then we have, for $k = n2^{m-1} + \alpha$, that

$$c_k = (f_{n+1})^m \leqq 1 + 2^{m-1}((f_1)^m + \cdots + (f_n)^m) + (\alpha - 1)(f_{n+1})^m$$
$$= 1 + c_1 + \cdots + c_k;$$

therefore, we need only prove the case for $\alpha = 1$, and this is equivalent to

$$(f_{n+1})^m \leqq 1 + 2^{m-1}((f_1)^m + \cdots + (f_n)^m).$$

But by Lemma 3,

$$(f_{n+1})^m \leqq 2^{m-1}((f_{n-1})^m + (f_n)^m) \leqq 1 + 2^{m-1}((f_0) + \cdots + (f_n)^m)$$
$$= 1 + 2^{m-1}((f_1)^m + \cdots + (f_n)^m). \quad \text{Q.E.D.}$$

## At Last

H-215 *Proposed by Ralph Fecke, North Texas State University, Denton, TX (Vol. 11, no. 2, April 1973)*

a. Prove

$$\sum_{i=n}^{n+2} 2^i P_i \equiv 0 \pmod 5$$

for all positive integers $n$; $P_i$ is the $i^{\text{th}}$ term of the Pell sequence,

$$P_i = 1,\ P_2 = 2,\ P_{n+1} = 2P_n + P_{n-1} \quad (n \geqq 2).$$

b. Prove $2^n L_n \equiv 2 \pmod{10}$ for all positive integers $n$; $L_n$ is the $n^{\text{th}}$ term of the Lucas sequence.

*Solution by Chris Long, student, Rutgers University, New Brunswick, NJ*

a. Note that $2P_1 + 4P_2 + 8P_3 \equiv 4P_2 + 8P_3 + 16P_4 \equiv 0 \pmod 5$ and that

$$\begin{aligned} 2^{i+2}P_{i+2} + 2^{i+1}P_{i+1} + 2^i P_i &= 4(2^{i+1}P_{i+1} + 2^i P_i + 2^{i-1}P_{i-1}) \\ &\quad + 4(2^i P_i + 2^{i-1}P_{i-1} + 2^{i-2}P_{i-2}); \end{aligned}$$

hence, by induction,

$$2^{i+2}P_{i+2} + 2^{i+1}P_{i+1} + 2^i P_i \equiv 0 \pmod 5$$

for all positive integers $n$.

b. We have that $2L_1 \equiv 4L_2 \equiv 2 \pmod{10}$ and that

$$2^{n+2}L_{n+2} = 2(2^{n+1}L_{n+1} + 2(2^n L_n));$$

hence, by induction,

$$2^n L_n \equiv 2 \pmod{10}$$

for all positive integers $n$.

## Middle Aged

H-306 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA (deceased) (Vol. 17, no. 3, October 1979)*

(a) Prove that the system $S$,

$$a + b = F_p,\ b + c = F_q,\ c + a = F_r,$$

cannot be solved in positive integers if $F_p$, $F_q$, $F_r$, are positive Fibonacci numbers.

(b) Likewise, show that the system $T$,

$$a + b = F_p,\ b + c = F_q,\ c + d = F_r,\ d + e = F_s,\ e + a = F_t,$$

has no solution under the same conditions.

(c) Show that if $F_p$ is replaced by any positive non-Fibonacci integer, then $S$ and $T$ have solutions.

If possible, find necessary and sufficient conditions for the system $U$,

$$a + b = F_p,\ b + c = F_q,\ c + d = F_r,\ d + a = F_s,$$

to be solvable in positive integers.

*Solution by Chris Long, student, Rutgers University, New Brunswick, NJ*

It is unclear whether the $F$'s are meant to be distinct or not; I will consider both possibilities in the following.

(a) Assume WLOG that $F_q$ is the maximum of $F_p$, $F_q$, $F_r$. We have that $2a = F_p - F_q + F_r$. If the $F$'s are distinct, we then have that $F_q \geqq F_p + F_r$; hence, $2a \leqq 0$. Therefore, $S$ cannot be solved in positive integers if the $F$'s are distinct. If the $F$'s are not distinct, then this is false; e.g., take $a = b = c = 1$.

(b) This is similar to (a). Assume that $F_q$ is the maximum of the $F$'s. We have that

$$2a = F_p - F_q + F_r - F_s + F_t \text{ and } 2d = F_s - F_t + F_p - F_q + F_r;$$

if the $F$'s are distinct, then $F_q \geqq F_p + F_r$, which gives us that

$$2a \leqq F_t - F_s \text{ and } 2d \leqq F_s - F_t.$$

Adding gives the contradiction that $2(a + d) \leqq 0$; therefore, $T$ cannot be solved in positive integers if the $F$'s are distinct. Again, if the $F$'s are not distinct, this is false; e.g., take $a = b = c = d = e = 1$.

(c) This is false for both (a) and (b). Indeed, for system $S$ replace $F_p$ with 4 and let $F_q = 1$ and $F_r = 2$; these values imply that $2a = 5$. Similarly, for system $T$ replace $F_p$ with 4 and let $F_q = 1$, $F_r = 2$, $F_s = 3$, and $F_t = 5$; these values imply that $2a = 7$.

For system $U$, I claim that it is solvable in positive integers if and only if $F_p + F_r = F_q + F_s$ and $F_p, F_q, F_s, F_t \geqq 2$. Indeed, the necessity of the statement is obvious. For sufficiency, note that all possible solutions must be of the form

$$(a, b, c, d) = (t, F_p + t, F_r - F_s + t, F_s - t);$$

hence, all solutions with $a$, $b$, $c$, $d$ positive integers are given by

$$\{(t, F_p + t, F_r - F_s + t, F_s - t) \mid \max(1, F_s - F_r + 1) \leqq t \leqq F_s - 1\}.$$

In particular, $t = F_s - 1$ yields a solution under the given conditions. It is also interesting to note that the $F$'s cannot all be distinct, as this would imply that one of the $F$'s was $\leqq 0$.

## Close Ranks

H-403 *Proposed by Paul S. Bruckman, Fair Oaks, CA*
*(Vol. 24, no. 4, November 1986)*

Given $p$, $q$ real with $p \neq -1 - 2qk$, $k = 0, 1, 2, \ldots$, find a closed form expression for the continued fraction

$$\theta(p, q) \equiv p + \cfrac{p + q}{p + 2q + \cfrac{p + 3q}{p + 4q + \cdots}}. \tag{1}$$

HINT: Consider the *Confluent Hypergeometric* (or *Kummer*) function defined as follows:

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \cdot \frac{z^n}{n!}, \quad b \neq 0, -1, -2, \ldots . \tag{2}$$

NOTE: $\theta(1, 1) = 1 + \cfrac{2}{3 + \cfrac{4}{5 + \cdots}}$, which was Problem H-394.

*Solution by C. Georghiou, University of Patras, Greece*

Take the confluent hypergeometric differential equation

$$zw'' + (b - z)w' - aw = 0 \tag{*}$$

Then, for $a \neq 0, -1, -2, \ldots$ and $b \neq 0, -1, -2, \ldots$, we have that

$$\frac{w}{w'} = \frac{b - z}{a} + \frac{z/a}{w'/w''}.$$

By differentiating (*), we get

$$\frac{w'}{w''} = \frac{b + 1 - z}{a + 1} + \frac{z/(a + 1)}{w''/w'''}$$

and by repeated differentiation of (*), we get the continued fraction

$$\frac{bw'}{aw} \equiv f(z) = \cfrac{b}{b - z + \cfrac{z}{\cfrac{b + 1 - z}{a + 1} + \cfrac{z/(a + 1)}{\cfrac{b + 2 - z}{a + 2} + \cdots}}}$$

From the theory of continued fractions, we know that

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots = b_0 + \frac{c_1 a_1}{c_1 b_1} + \frac{c_1 c_2 a_2}{c_2 b_2} + \frac{c_2 c_3 a_3}{c_3 b_3} + \cdots \tag{**}$$

where $c_n \neq 0$, and setting $c_1 = 1$, $c_2 = a + 1$, $\ldots$, $c_n = a + n - 1$, $\ldots$, we get

$$f(z) = \frac{b}{b - z +} \frac{(a+1)z}{b + 1 - z +} \frac{(a+2)z}{b + 2 - z +} \cdots . \tag{***}$$

Now it is shown in W. B. Jones & W. J. Thron, "Continued Fractions," in G.-C. Rota, ed., *Encyclopedia of Mathematics and Its Applications*, Addison-Wesley, 1980, pp. 276-282, that the above continued fraction converges to the meromorphic function

$$f(z) = \frac{M(a + 1, b + 1, z)}{M(a, b, z)}$$

for all complex numbers $z$ and, moreover, the convergence is uniform on every compact subset of $\mathbb{C}$ which contains no poles of $f(z)$.

Before we proceed further, we note that the restriction $a \neq 0, -1, -2, \ldots$ can be removed by a limiting argument (see also the above-mentioned reference).

Now, for $b \neq 0, -1, -2, \ldots$, and $q \neq 0$ and $c_n = 2q$, $n = 1, 2, 3, \ldots$, (**) and (***) give

$$\frac{M(a+1,\ b+1,\ z)}{M(a,\ b,\ z)} = \frac{2qb}{2q(b-z)+}\ \frac{4q^2(a+1)z}{2q(b+1-z)+}\ \frac{4q^2(a+2)z}{2q(b+2-z)+}\ \cdots .$$

Finally, take $a = (p-q)/2q$, $b = (p+1)/2q$, and $z = 1/2q$. Then,

$$\frac{M\left(\dfrac{p+q}{2q},\ \dfrac{p+1+2q}{2q},\ \dfrac{1}{2q}\right)}{M\left(\dfrac{p-q}{2q},\ \dfrac{p+1}{2q},\ \dfrac{1}{2q}\right)} = \cfrac{p+1}{p+\cfrac{p+q}{p+2q+\cfrac{p+3q}{p+4q+\cdots}}} = \frac{p+1}{\theta(p,\ q)},$$

and the final result is

$$\theta(p,\ q) = (p+1)\,\frac{M\left(\dfrac{p-q}{2q},\ \dfrac{p+1}{2q},\ \dfrac{1}{2q}\right)}{M\left(\dfrac{p+q}{2q},\ \dfrac{p+1+2q}{2q},\ \dfrac{1}{2q}\right)}$$

valid for $p$, $q$ such that $q \neq 0$ and $p \neq -1-2qk$, $k = 0, 1, 2, \ldots$ .

Again the restriction $p \neq -1-2qk$ can be removed since it is easy to see that

$$\theta(-1-2qk,\ q)$$
$$= -1-2qk + \frac{-1-(2k-1)q}{-1-(2k-2)q+}\ \frac{-1-(2k-3)q}{-1-(2k-4)q+}\ \cdots\ \frac{-1-q}{+\ -1}\ \frac{-1+q}{+\ \theta(-1+2q,\ q)}.$$

For example, for $k = 0$ (and $q = 0$), we have

$$\theta(-1,\ q) = -1 + \frac{-1+q}{\theta(-1+2q,\ q)} = -1 + \frac{q-1}{2q}\,\frac{M\left(\dfrac{3q-1}{2q},\ 2,\ \dfrac{1}{2q}\right)}{M\left(\dfrac{q-1}{2q},\ 1,\ \dfrac{1}{2q}\right)}$$

and the same result is obtained from the given expression of $\theta(p,\ q)$ by a limiting argument when $p \to -1$. The same is true for $k > 0$.

Finally, when $q = 0$, we have a periodic continued fraction and

$$\theta(p,\ 0) = p + \cfrac{p}{p+\cfrac{p}{p+\cdots}} = p + \frac{p}{\theta(p,\ 0)}$$

which gives for $p > 0$ or $p \leqq -4$

$$\theta(p,\ 0) = (p + \sqrt{p^2+4p})/2.$$

For $-4 < p \leqq 0$, $\theta(p,\ 0)$ diverges.

*Also solved by the proposer.*

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*Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr. FA, 1972.

*A Primer for the Fibonacci Numbers.* Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

*Fibonacci's Problem Book.* Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

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