The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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VOLUME 27 FEBRUARY 1989 NUMBER 1

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

ON PRIMITIVE PYTHAGOREAN TRIANGLES WITH EQUAL PERIMETERS

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(Submitted November 1983)

Dedicated to my wife Pesia and my son John

1. Introduction

A triple (x, y, z) of natural numbers is called a *Pythagorean Triangle* if x, y, z satisfy the Pythagorean equation

$$x^2 + y^2 = z^2.$$

The triple (x, y, z) is a *Primitive Pythagorean Triangle* (PPT) if x, y, z have no common factor greater than 1. If x is assumed to be odd, the set of PPT's can be generated by the set of pairs of natural numbers (u, v) satisfying

$$u > v > 0, \quad (u, v) = 1, \quad u + v \equiv 1 \pmod 2, \tag{1}$$

the well-known generating formula being

$$(x, y, z) = (u^2 - v^2, 2uv, u^2 + v^2).$$

The pair (u, v) is called the *generator* of the PPT (x, y, z).

In terms of the generator, the *perimeter* S of (x, y, z), $S = x + y + z$, may be expressed as

$$S = 2u(u + v).$$

Denote by H the set of all such perimeters. Let H_k be the subset of H defined by the relation: $S \in H_k$ if S is the perimeter of *exactly* k PPT's.

It is not difficult to show that H_1 is an infinite set, i.e., there is an infinite set of PPT's each one of which has a perimeter not shared by any other PPT. The surprising fact that H_2 is also an infinite set is proved in [1]. It is the main purpose of this paper to prove that H_k is an infinite set for any k, $k \geq 3$; see Proposition 3.3 below. The proof may appear to be constructive, but it is ultimately seen to depend on a known existential Theorem of analytic number theory, the so-called modern version of Bertrand's postulate.

Necessary conditions for the construction of k PPT's with equal perimeters are given in the next section. That the conditions can be met is shown in the proof of Proposition 3.3.

2. A Constructive Device

Let us first construct k different generators (u, v) of PPT's with equal perimeters.

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*This paper is the final version of two papers submitted for publication by Leon Bernstein before he died on March 12, 1984, of a cerebral hemorrhage. It benefitted from the advice of a number of anonymous referees.

Proposition 2.1: Let $B_1, B_2, \ldots, B_k$ be k ($k \geq 3$) odd positive integers, pairwise relatively prime, $B_1 < B_2 < \cdots < B_k$, and

$$B_k < B_1\sqrt{2}. \tag{2}$$

Let

$$A_k = \prod_{i=1}^{k} B_i \text{ and } u_t = A_k/B_t \text{ for } t \in T,\ T = \{1, 2, \ldots, k\}.$$

Assume there exists an odd positive integer P_k satisfying the two conditions

$$(P_k, u_t) = 1,\ t \in T, \tag{3}$$

$$\frac{B_2B_3 \ldots B_k}{B_1} < P_k < 2\,\frac{B_1B_2 \ldots B_{k-1}}{B_k}. \tag{4}$$

If $v_t = P_kB_t - u_t$, $t \in T$, then the pairs (u_t, v_t) are generators of k PPT's having equal perimeters S, $S = 2P_kA_k$.

Proof: We show first that (u_t, v_t) is the generator of a PPT for each $t \in T$, i.e., that (u_t, v_t) satisfies (1). From the definitions of u_t, v_t, it follows that

$$u_1 > u_2 > \cdots > u_k \quad \text{and} \quad v_1 < v_2 < \cdots < v_k. \tag{5}$$

Since by (4),

$$v_1 = P_kB_1 - u_1 = P_kB_1 - B_2B_3 \ldots B_k > 0,$$

it follows from (5) that $v_t > 0$ for $t \in T$. Moreover, it follows from (5) that $u_t > v_t$, $t \in T$, provided $u_k > v_k$. And this is a consequence of (4):

$$u_k - v_k = 2u_k - P_kB_k > (2A_k/B_k) - 2B_1B_2 \ldots B_{k-1} = 0.$$

Thus, $u_t > v_t$, $t \in T$.

Next, $(u_t, v_t) = 1$ if and only if $(u_t, u_t + v_t) = (A_k/B_t, P_kB_t) = 1$, which is true since, by assumption, $(u_t, P_k) = 1$ and the B_i's are pairwise relatively prime.

Since $u_t + v_t$ is odd, u_t and v_t must have opposite parity, i.e., $u_t + v_t \equiv 1 \pmod 2$. This concludes the proof that (u_t, v_t) satisfies (1) for each $t \in T$.

Finally, since $S = 2u_t(u_t + v_t) = 2P_kA_k$ is independent of t, the k PPT's generated by (u_t, v_t), $t \in T$, have equal perimeters.

3. Infinity of H_k

The main argument of this section rests on the following existential result; see [2], page 371.

Theorem 3.1: For every positive number ε there exists a number ξ such that for each x, $x > \xi$, there is a prime number between x and $(1 + \varepsilon)x$. (It will be used to prove the following proposition which has a certain interest in itself.)

Proposition 3.2: Let $k \geq 3$ and let $\delta > 0$. Then there is a number ξ such that for every y, $y > \xi$, there are k consecutive primes $B_1, B_2, \ldots, B_k$ and a prime P_k satisfying the inequalities

$$y < B_1 < B_2 < \cdots < B_k < \sqrt{1+\delta}\,y,$$

$$\frac{B_2B_3 \ldots B_k}{B_1} < P_k < (1+\delta)\frac{B_1B_2 \ldots B_{k-1}}{B_k}.$$

Proof: Let ε_1 be a given number such that $0 < \varepsilon_1 < \sqrt{1+\delta} - 1$. By Theorem 3.1, there is a number ξ_1 such that for every $x > \xi_1$, there are at least k consecutive primes $B_1, B_2, \ldots, B_k$ in the open interval $(x, (1+\varepsilon_1)x)$. Let

$$\varepsilon = \frac{1+\delta}{(1+\varepsilon_1)^2} - 1$$

and take ξ_2 so large that for each x, $x > \xi_2$, there is at least one prime number in the interval $(x, (1+\varepsilon)x)$.

Let $\xi = \max(\xi_1, \xi_2)$. Then for every y, $y > \xi$, we have that the interval $(y, (1+\varepsilon_1)y)$ contains k consecutive primes,

$$y < B_1 < B_2 < \cdots < B_k < (1+\varepsilon_1)y, \tag{6}$$

and the interval $(y, (1+\varepsilon)y)$ contains a prime number $\overline{P}_k$,

$$y < \overline{P}_k < (1+\varepsilon)y. \tag{7}$$

We show next that the interval

$$[X, Y] = \left[\frac{B_2B_3 \ldots B_k}{B_1},\ (1+\delta)\frac{B_1B_2 \ldots B_{k-1}}{B_k}\right]$$

contains $\overline{P}_k$. On the one hand, we know from (7) that $[X, (1+\varepsilon)X]$ contains at least the prime $\overline{P}_k$, since for $k \geq 3$, $X = B_2B_3 \ldots B_k/B_1 > B_2$ and $B_2 > y$ by (6). On the other hand, $[X, (1+\varepsilon)X]$ is a subinterval of $[X, Y]$ if we show $(1+\varepsilon)X < Y$. This last inequality is equivalent to

$$(1+\delta)\frac{B_1B_2 \ldots B_{k-1}}{B_k} > \frac{1+\delta}{(1+\varepsilon_1)^2}\cdot\frac{B_2B_3 \ldots B_k}{B_1},$$

which, in turn, is equivalent to

$$(1+\varepsilon_1)^2B_1^2 > B_k^2.$$

But $(1+\varepsilon_1)B_1 > (1+\varepsilon_1)y > B_k$ by (6). Thus $Y > (1+\varepsilon)X$. This concludes the proof.

We are now ready to prove the main proposition.

Proposition 3.3: Let H_k, $k \geq 3$, be the set of integers S such that S is the perimeter of exactly k PPT's. Then H_k is infinite.

Proof: Taking $\delta = 1$ in Proposition 3.2, we can count on k consecutive primes $B_1, B_2, \ldots, B_k$ such that

$$B_k < \sqrt{2}B_1,$$

so condition (2) is satisfied; moreover there is a prime P_k such that condition (4) is satisfied.

Defining A_k, u_t, and v_t as in Proposition 2.1, we see that (3) is also satisfied, so we may conclude that (u_t, v_t), $t \in T$, generate k PPT's having equal perimeter $S = 2P_k A_k$.

Since y in (6) may be taken to be any number larger than ξ, it is clear that the above process may be repeated infinitely often. Each time we obtain a new set of k PPT's having equal perimeters.

It remains to show that no PPT, other than the ones constructed, can have perimeter $S = 2P_k A_k$. To do so, assume (u, v) generates a PPT with perimeter $S = 2P_k A_k$. We will show that (u, v) is not a generator of a PPT unless (u, v) is one of the pairs (u_t, v_t) constructed above.

Since $S = 2u(u + v) = 2P_k A_k = 2B_1 B_2 \ldots B_k P_k$, there are but a finite number of possible values for u and $u + v$. We assume first that P_k is a factor of u and consider the three possibilities:

(i) $u = P_k$, $u + v = B_1 B_2 \ldots B_k$,

(ii) $u = B_1 B_2 \ldots B_k P_k$, $u + v = 1$,

(iii) $u = q_1 q_2 \ldots q_m P_k$, $u + v = q_{m+1} q_{m+2} \ldots q_k$,

where $q_1 q_2 \ldots q_m$, $m \in \{1, 2, \ldots, k - 1\}$, denotes any one of the products of m different primes from the set $\{B_1, B_2, \ldots, B_k\}$, and $q_{m+1} q_{m+2} \ldots q_k$ the product of the remaining primes.

In case (i), condition (4) implies

$$2u = 2P_k < 4B_1 B_2 \ldots B_{k-1}/B_k < B_1 B_2 \ldots B_k = u + v,$$

so that $u < v$, a contradiction of (1).

For case (ii), $v = 1 - u < 0$, contradicting (1).

For case (iii), using (4), we write

$$\begin{aligned}(q_1 q_2 \ldots q_m)(q_{m+1} q_{m+2} \ldots q_k) P_k &= A_k P_k > A_k^2 / B_1^2 \\ &= B_2^2 \ldots B_k^2 \geq (q_{m+1} q_{m+2} \ldots q_k)^2.\end{aligned}$$

Then

$$u = q_1 q_2 \ldots q_m P_k > q_{m+1} q_{m+2} \ldots q_k = u + v,$$

contradicting (1).

Next, we shall assume that P_k is not a factor of u. Then P_k must be a factor of $(u + v)$, and we consider the four possibilities:

(I) $u + v = P_k$, $u = B_1 B_2 \ldots B_k$,

(II) $u + v = B_1 B_2 \ldots B_k P_k$, $u = 1$,

(III) $u + v = q_{m+1} q_{m+2} \ldots q_k P_k$, $u = q_1 q_2 \ldots q_m$,

where $q_1 q_2 \ldots q_m$, $m \in \{1, 2, \ldots, k - 2\}$, denotes any one of the products of m different primes from the set $\{B_1, B_2, \ldots, B_k\}$, and $q_{m+1} q_{m+2} \ldots q_k$ the product of the remaining primes. Note that $u + v$ contains at least two of the primes B_i as factors.

(IV) $u + v = B_t P_k$, $u = B_1 B_2 \ldots B_{t-1} B_{t+1} \ldots B_k$, $t \in T$.

In case (I), using (4), we get

$$u + v = P_k < 2A_k / B_k^2 < B_1 B_2 \ldots B_k = u,$$

contradicting (1).

In case (II), $v = B_1 B_2 \ldots B_k P_k - 1 > 1 = u$, contradicting (1).

For case (III), using (4), we have

$$\begin{aligned} u + v &= q_{m+1} q_{m+2} \cdots q_k P_k > q_{m+1} q_{m+2} \cdots q_k A_k / B_1^2 \\ &= (q_{m+1} q_{m+2} \cdots q_k)^2 (q_1 q_2 \cdots q_m) / B_1^2 > 2 q_1 q_2 \cdots q_m = 2u, \end{aligned}$$

a contradiction of (1).

Case (IV) is seen to describe the k pairs (u_t, v_t) defined above. These k pairs then generate k PPT's with equal perimeters $S = 2P_k A_k$, and no other PPT can have this perimeter.

4. Examples

Let us conclude with a few examples.

(1) When $k = 3$, we have:

B_1	B_2	B_3	P_3	(u_1, v_1)	(u_2, v_2)	(u_3, v_3)	S
11	13	15	19	(195, 14)	(165, 82)	(143, 142)	81,510
31	37	43	53	(1591, 52)	(1333, 628)	(1147, 1132)	5,228,026
17	19	21	25	(399, 26)	(357, 118)	(323, 202)	339,150
17	19	21	29	(399, 94)	(357, 194)	(323, 286)	393,414
23	25	29	33	(725, 34)	(667, 158)	(575, 382)	1,110,550
23	29	31	41	(899, 44)	(713, 476)	(667, 604)	1,695,514
23	29	31	43	(899, 90)	(713, 534)	(667, 666)	1,778,222
29	31	37	41	(1147, 42)	(1073, 198)	(899, 618)	2,727,566

(2) Finally, let $k = 4$ and

$$B_1 = 17,\ B_2 = 19,\ B_3 = 21,\ B_4 = 23.$$

For the integer P_4 within the bounds in (4), we can select any prime P_4 in the set

$$\{541, 547, 557, 563, 569, 571, 577, 587\};$$

moreover, Proposition 2.1 allows us to take any nonprime P_4 in the set

$$\{545, 559, 565, 581, 583\}.$$

References

1. Leon Bernstein. "Primitive Pythagorean Triples." *Fibonacci Quarterly* 20.3 (1982):227-241.
2. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 5th ed. Oxford: Oxford University Press, 1900.

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ON A GENERALIZATION OF THE FIBONACCI SEQUENCE IN THE CASE OF THREE SEQUENCES

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A new direction for generalizing the Fibonacci sequence was introduced in [1], and [2]. In this paper, we shall continue that direction of research.

Let $C_1, C_2, \ldots, C_6$ be fixed real numbers. Using C_1 to C_6, we shall construct new schemes which are of the Fibonacci type and are called 3-F-sequences. Our analogy is of [1] and [2]; the form is

$$\begin{cases} a_0 = C_1,\ b_0 = C_2,\ c_0 = C_3,\ a_1 = C_4,\ b_1 = C_5,\ c_1 = C_6 \\ a_{n+2} = x^1_{n+1} + y^1_n & (n \geq 0), \\ b_{n+2} = x^2_{n+1} + y^2_n \\ c_{n+2} = x^3_{n+1} + y^3_n \end{cases}$$

where $\langle x^1_{n+1}, x^2_{n+1}, x^3_{n+1}\rangle$ is any permutation of $\langle a_{n+1}, b_{n+1}, c_{n+1}\rangle$ and $\langle y^1_n, y^2_n, y^3_n\rangle$ is any permutation of $\langle a_n, b_n, c_n\rangle$.

The number of different schemes is obviously 36.

In [3], the specific scheme

$$\begin{cases} a_0 = C_1,\ b_0 = C_2,\ c_0 = C_3,\ a_1 = C_4,\ b_1 = C_5,\ c_1 = C_6 \\ a_{n+2} = b_{n+1} + c_n & (n \geq 0), \\ b_{n+2} = c_{n+1} + a_n \\ c_{n+2} = a_{n+1} + b_n \end{cases}$$

is discussed in detail. For the sake of brevity, we devise the following representation for this scheme:

$$S = \begin{Bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{Bmatrix}. \tag{1}$$

Note that we have merely eliminated the subscripts and the equal and plus symbols so that our notation is similar to that used in representing a system of linear equations in matrix form. Using this notation, it is important to remember that the elements in their first column are always in the same order while the elements in the other column can be permuted within that column. Every element a, b, and c must be used in each column.

We now define an operation called substitution over these 3-F-sequences and adopt the notation $[p, q]S$, where $p, q \in \{a, b, c\}$, $p \neq q$. Applying the operation to S merely interchanges all occurrences of p and q in each column. For example, using (1), we have

$$[a, c]S = \begin{Bmatrix} c & b & a \\ b & a & c \\ a & c & b \end{Bmatrix}. \tag{2}$$

Note that in the result we do not maintain the order of the elements in the first column. To maintain this order we interchange the first and last rows of [2] to obtain

$$S' = \begin{Bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{Bmatrix},$$

which corresponds to the scheme

$$\begin{cases} a_0 = C'_1,\ b_0 = C'_2,\ c_0 = C'_3,\ a_1 = C'_4,\ b_1 = C'_5,\ c_1 = C'_6 \\ a_{n+2} = c_{n+1} + b_n \\ b_{n+2} = a_{n+1} + c_n \qquad (n \geq 0), \\ c_{n+2} = b_{n+1} + a_n \end{cases}$$

where $C'_1, C'_2, \ldots, C'_6$ are real numbers.

We shall say that the two schemes S and S' are equivalent under the operation of substitution and denote this by

$$S \leftrightarrow S'.$$

It is now obvious that for any two 3-F-sequences S and S', if $[p, q]S \leftrightarrow S'$, then $[p, q]S' \leftrightarrow S$. To investigate the concept of equivalence to a deeper extent, it is necessary to list all 36 schemes:

$$S_1 = \begin{Bmatrix} a & a & a \\ b & b & b \\ c & c & c \end{Bmatrix} \quad S_2 = \begin{Bmatrix} a & a & a \\ b & b & c \\ c & c & b \end{Bmatrix} \quad S_3 = \begin{Bmatrix} a & a & a \\ b & c & b \\ c & b & c \end{Bmatrix} \quad S_4 = \begin{Bmatrix} a & a & a \\ b & c & c \\ c & b & b \end{Bmatrix}$$

$$S_5 = \begin{Bmatrix} a & a & b \\ b & b & a \\ c & c & c \end{Bmatrix} \quad S_6 = \begin{Bmatrix} a & a & b \\ b & b & c \\ c & c & a \end{Bmatrix} \quad S_7 = \begin{Bmatrix} a & a & b \\ b & c & a \\ c & b & c \end{Bmatrix} \quad S_8 = \begin{Bmatrix} a & a & b \\ b & c & c \\ c & b & a \end{Bmatrix}$$

$$S_9 = \begin{Bmatrix} a & a & c \\ b & b & a \\ c & c & b \end{Bmatrix} \quad S_{10} = \begin{Bmatrix} a & a & c \\ b & b & b \\ c & c & a \end{Bmatrix} \quad S_{11} = \begin{Bmatrix} a & a & c \\ b & c & a \\ c & b & b \end{Bmatrix} \quad S_{12} = \begin{Bmatrix} a & a & c \\ b & c & b \\ c & b & a \end{Bmatrix}$$

$$S_{13} = \begin{Bmatrix} a & b & a \\ b & a & b \\ c & c & c \end{Bmatrix} \quad S_{14} = \begin{Bmatrix} a & b & a \\ b & a & c \\ c & c & b \end{Bmatrix} \quad S_{15} = \begin{Bmatrix} a & b & a \\ b & c & b \\ c & a & c \end{Bmatrix} \quad S_{16} = \begin{Bmatrix} a & b & a \\ b & c & c \\ c & a & b \end{Bmatrix}$$

$$S_{17} = \begin{Bmatrix} a & b & b \\ b & a & a \\ c & c & c \end{Bmatrix} \quad S_{18} = \begin{Bmatrix} a & b & b \\ b & a & c \\ c & c & a \end{Bmatrix} \quad S_{19} = \begin{Bmatrix} a & b & b \\ b & c & a \\ c & a & c \end{Bmatrix} \quad S_{20} = \begin{Bmatrix} a & b & b \\ b & c & c \\ c & a & a \end{Bmatrix}$$

$$S_{21} = \begin{Bmatrix} a & b & c \\ b & a & a \\ c & c & b \end{Bmatrix} \quad S_{22} = \begin{Bmatrix} a & b & c \\ b & a & b \\ c & c & a \end{Bmatrix} \quad S_{23} = \begin{Bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{Bmatrix} \quad S_{24} = \begin{Bmatrix} a & b & c \\ b & c & b \\ c & a & a \end{Bmatrix}$$

$$S_{25} = \begin{Bmatrix} a & c & a \\ b & a & b \\ c & b & c \end{Bmatrix} \quad S_{26} = \begin{Bmatrix} a & c & a \\ b & a & c \\ c & b & b \end{Bmatrix} \quad S_{27} = \begin{Bmatrix} a & c & a \\ b & b & b \\ c & a & c \end{Bmatrix} \quad S_{28} = \begin{Bmatrix} a & c & a \\ b & b & c \\ c & a & b \end{Bmatrix}$$

$$S_{29} = \begin{Bmatrix} a & c & b \\ b & a & a \\ c & b & c \end{Bmatrix} \quad S_{30} = \begin{Bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{Bmatrix} \quad S_{31} = \begin{Bmatrix} a & c & b \\ b & b & a \\ c & a & c \end{Bmatrix} \quad S_{32} = \begin{Bmatrix} a & c & b \\ b & b & c \\ c & a & a \end{Bmatrix}$$

$$S_{33} = \begin{Bmatrix} a & c & c \\ b & a & a \\ c & b & b \end{Bmatrix} \quad S_{34} = \begin{Bmatrix} a & c & c \\ b & a & b \\ c & b & a \end{Bmatrix} \quad S_{35} = \begin{Bmatrix} a & c & c \\ b & b & a \\ c & a & b \end{Bmatrix} \quad S_{36} = \begin{Bmatrix} a & c & c \\ b & b & b \\ c & a & a \end{Bmatrix}$$

Note that $S = S_{23}$ and $S' = S_{30}$, so that $S_{23} \leftrightarrow S_{30}$.

We say that a 3-F-sequence S is trivial if at least one of the resulting sequences is a Fibonacci sequence. Otherwise, S is said to be an Essential Generalization of the Fibonacci sequence.

Observe that there are ten trivial 3-F-sequences. They are

$$S_1,\ S_2,\ S_3,\ S_4,\ S_5,\ S_{10},\ S_{13},\ S_{17},\ S_{27},\ S_{36}.$$

These 10 schemes are easy to detect since they have at least one row all with the same letter. Furthermore, for these schemes one of the three possible substitutions returns the scheme itself. For example,

$$[b,\ c]S_i \leftrightarrow S_i,\ i = 1,\ 2,\ 3,\ 4$$

$$[a,\ b]S_i \leftrightarrow S_i,\ i = 1,\ 5,\ 13,\ 17$$

$$[a,\ c]S_i \leftrightarrow S_i,\ i = 1,\ 10,\ 27,\ 36.$$

The twenty-six remaining schemes are Essential Generalizations of the Fibonacci sequence. For eight of these schemes, the result is independent of the substitution made. That is,

$$[p,\ q]S_6 \leftrightarrow S_9 \qquad [p,\ q]S_{15} \leftrightarrow S_{25}$$

$$[p,\ q]S_{20} \leftrightarrow S_{33} \qquad [p,\ q]S_{23} \leftrightarrow S_{30}$$

for all p and q. This means the substitution operation for these schemes is cyclic of length 2.

For the other eighteen Essential Generalizations of the Fibonacci sequence schemes, all three possible substitutions generate three different schemes. For example,

$$[a,\ b]S_7 \leftrightarrow S_{31} \qquad [a,\ b]S_8 \leftrightarrow S_{35} \qquad [a,\ b]S_{16} \leftrightarrow S_{34}$$

$$[a,\ c]S_7 \leftrightarrow S_{14} \qquad [a,\ c]S_8 \leftrightarrow S_{21} \qquad [a,\ c]S_{16} \leftrightarrow S_{29}$$

$$[b,\ c]S_7 \leftrightarrow S_{12} \qquad [b,\ c]S_8 \leftrightarrow S_{11} \qquad [b,\ c]S_{16} \leftrightarrow S_{26}$$

All of the substitutions associated with the remaining eighteen schemes and their results are conveniently illustrated by the following three figures. That is, these pictures determine all possible cycles.

For example,

$$[a,\ b]S_{29} \leftrightarrow S_{19}$$

and

$$[a,\ b]([b,\ c]([a,\ c]S_{24})) \leftrightarrow S_{29}.$$

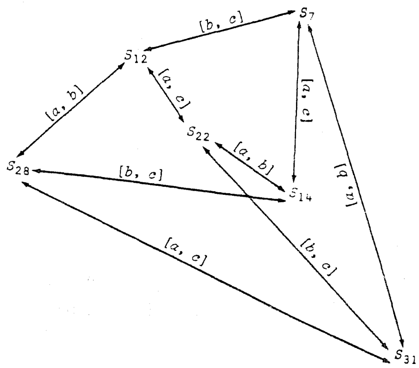

FIGURE 1

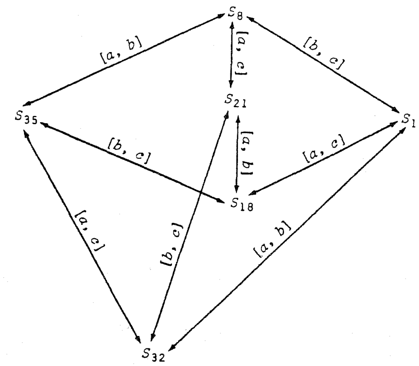

FIGURE 2

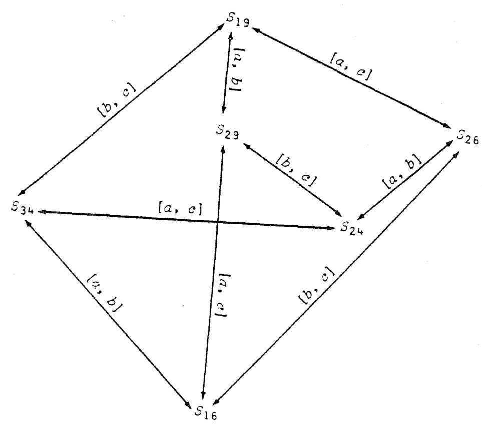

FIGURE 3

Note that the figures tell us that many of the schemes are independent. That is, S_{24} and S_{18} are independent. In fact, S_{18} is related only to the six schemes listed in Figure 3. Similar results can be found for the other schemes.

The closed form equation of the members for all three sequences of scheme S_{23} is given in [3]. By a method similar to that given in [3] or [1], the closed form equation of the members for all three sequences of the other schemes can be determined. We leave this task to the reader. Obviously, these results could be generalized to the case of four or more sequences with very little difficulty.

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ON THE PERIODS OF THE FIBONACCI SEQUENCE MODULO M

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This article deals with Fibonacci's sequence

$$u_0 = 0,\ u_1 = 1,\ u_{n+2} = u_{n+1} + u_n$$

and with the arithmetical function

$$K(m) = \text{length of the period of Fibonacci's sequence when reduced modulo } m.$$

In the last few years I had some occasions to guide activities in a mathematics-with-computer club for 15-year-olds, where we investigated the function $K(m)$. Theorems 1 and 2 of the present article were found (without proofs) by members of these clubs. To be more specific, these are those of the students' results, which I was not able to find in the literature either before or after they have emerged in the club. The rest of the students' discoveries can be found either in [1] or in [4]. One of these is the following lemma which was suggested by the student Oded Farago.

Lemma: For any m and n, $K([m, n]) = [K(m), K(n)]$.

Proof: Follows from [5], Lemma 13.

Theorem 2 in [1] says almost the same, but only for m and n that are relatively prime, so Oded's present version is more symmetric. (The lemma holds for every sequence that becomes periodical when reduced modulo a natural number.)

Theorem 1: For any fixed m let $\lambda_m(n) = K(m^{n+1})/K(m^n)$. Then:

I. $\lambda_m(n) | m$ for all n;

II. $\lambda_m(n) | \lambda_m(n + 1)$ for all n;

III. there exists t such that $\lambda_m(n) = m$ for all $n \geq t$.

This theorem emerged from the work of four girls: Shoshi Pashkes, Sigalit Teshuva, Mali Gana, and Chenit Lotan.

Proof:

(i) If p is prime and t is the largest integer such that $K(p^t) = K(p)$, then Theorem 5 in [1] implies

$$\lambda_p(n) = \begin{cases} 1 & \text{if } 1 \leq n \leq t - 1 \\ p & \text{if } n \geq t \end{cases},$$

from which I, II, and III immediately follow.

(ii) If $m = p^e$, then the conclusion follows from (i), since

$$\lambda_m(n) = \lambda_p(ne)\lambda_p(ne + 1) \ldots \lambda_p(ne + e - 1).$$

(iii) Now let $(a, b) = 1$ and assume that the theorem holds for $m = a$ and $m = b$. By hypothesis and the lemma,

$$\lambda_a(n) | a, \ \lambda_b(n) | b, \ K(a^n b^n) = [K(a^n), K(b^n)],$$

also

$$K(a^{n+1}b^{n+1}) = [\lambda_a(n)K(a^n), \lambda_b(n)K(b^n)].$$

Let $\lambda_{ab}(n) = K(a^{n+1}b^{n+1})/K(a^n b^n)$. Then $\lambda_{ab}(n) | \lambda_a(n)\lambda_b(n)$. Thus, $\lambda_{ab}(n) | ab$.

Let p be a prime such that $p^{z_n} \| \lambda_{ab}(n)$. Then $p | ab$; without loss of generality, let $p | a$.

Let $p^{x_n} \| \lambda_a(n)$, $p^{y_n} \| K(a^n)$, $p^c \| K(b)$.

Since $p \nmid b$, by part I we have $p \nmid \lambda_b(n)$, so $p^c \| K(b^n)$ for all n. Therefore,

$$z_n = \text{Max}\{x_n + y_n, c\} - \text{Max}\{y_n, c\},$$

that is, $z_n = x_n$, $x_n + y_n - c$, or 0. By hypothesis, $x_n \le x_{n+1}$ and $y_n \le y_{n+1}$. Therefore, $z_n \le z_{n+1}$, so $p^{z_n} | \lambda_{ab}(n + 1)$. Since p is arbitrary, we have

$$\lambda_{ab}(n) | \lambda_{ab}(n + 1).$$

By hypothesis, there exists t_a such that $\lambda_a(n) = a$ for all $n \ge t_a$. Since $\lambda_a(n) = a$ means that $K(a^{n+1}) = aK(a^n)$ this implies that $y_{n+1} \ge y_n + 1$. It follows that there exists a $T > t_a$ such that for all $n \ge T$ we have $y_n > c$ and thus $z_n = x_n$.

Since, for such an n, $\lambda_a(n) = a$, it follows from $z_n = x_n$ that $p^{z_n} \| a$.

Since $p \nmid b$, it follows that, for all $n \ge T$, $p^{z_n} \| ab$.

For n sufficiently large, this holds for every prime p that divides ab; for such an n, $\lambda_{ab}(n) = ab$.

Theorem 2: For any even $i > 3$, $K(u_i) = 2i$. For any odd $i > 4$, $K(u_i) = 4i$.

Remark 1: Amihai and Moshe, the boys who found this, used different words. They said that the elements of the sequence $K(u_4)$, $K(u_5)$, $K(u_6)$, ... are, alternatively, the elements of two arithmetical sequences, one with the difference 4 and one with the difference 8.

Remark 2: The second part of Theorem 2 follows from Theorem 3 in [3].

Proof: $K(m)$ is the first i after 0 such that $u_i \equiv 0$ and $u_{i+1} \equiv 1 \pmod m$.

Theorem 3 in [1] says: For every m there is a d such that $u_j \equiv 0 \pmod m$ if and only if $d | j$.

If $m = u_i > 1$ then $d = i$, since the elements before u_i are not changed when reduced modulo u_i. [This proves that $K(u_i)$ is a multiple of i.]

For the same reason, if $i > 3$ then $u_{i-1} \not\equiv 1 \pmod{u_i}$.

Now $u_{i+1} \equiv u_{i-1} \pmod{u_i}$; therefore, if $i > 3$, then the i^{th} element of the Fibonacci sequence modulo u_i does not start a new period, instead, it starts a sequence of u_{i-1} multiples $\pmod{u_i}$ of the original sequence. Hence,

$$u_{2i+1} \equiv u_{2i-1} \equiv u_{i-1}^2 \pmod{u_i}.$$

For every i, $u_{i-1}^2 = u_{i-2}u_i + (-1)^i$; therefore, if i is even, then $u_{2i+1} \equiv 1 \pmod{u_i}$ and $K(u_i) = 2i$.

For odd i, $u_{2i+1} \equiv -1$; therefore, $u_{4i+1} \equiv 1$, so $u_{3i+1} \not\equiv 1$ and $K(u_i) = 4i$.

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HOW MANY 1'S ARE NEEDED?

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Let $f(n)$ denote the number of 1's necessary to express n, using the operations + and × (and parentheses). Determining $f(n)$ is an old problem, originally considered by Mahler and Popken in 1953 [3]. We have calculated $f(n)$ for $n \le 3^{10}$ and we present some statistics. (The reason for a power of 3 will be explained later.)

The Problem

With $f(n)$ defined as above, we have

$$f(n) = \begin{cases} 1, \text{ for } n = 1, \quad \text{and} \\ \min\limits_{\substack{ab=n \text{ or} \\ a+b=n}} \{f(a) + f(b)\}, \text{ for } n > 1. \end{cases} \tag{1}$$

This formula is very time consuming to use for large n, but we know of no other way to calculate $f(n)$.

The behavior of $f(n)$ is interesting. Selfridge has shown that $3^k + \Theta 3^{k-1}$ is the largest n for which $f(n) = 3k + \Theta$, for $\Theta = 0, \pm 1$. His proof is by induction, and the induction step is based on the following observation: If $b(m)$ is the largest n for which $f(n) = m$, then $b(m)$ is the largest element of the set

$$\bigcup_{r+s=m} \{b(r) + b(s),\ b(r)b(s)\}. \tag{2}$$

Using (2), it is fairly easy to show that $b(m)$ has the required form.

There are two competing conjectures about the behavior of $f(n)$ for large n. It has been conjectured that

$$f(n) < 3(1 + \varepsilon)\log_3 n, \text{ for large } n, \text{ and any } \varepsilon > 0. \tag{3}$$

It has also been conjectured that there is a set S (possibly of positive density) and a positive constant c so that

$$f(n) > 3(1 + c)\log_3 n, \text{ for all } n \text{ in } S. \tag{4}$$

The Results

We calculated $f(n)$, using equation (1), for $n \le 3^{10}$ (59,049). We chose 3^{10} because we would have all n for which $f(n) \le 30$, by Selfridge's results.

We broke the interval into 30 subintervals between the values of the form $3^k + \Theta 3^{k-1}$ for $\Theta = 0, \pm 1$ and we also looked at the sets $S(m) = \{n : f(n) = m\}$, for $m = 1, 2, \ldots, 37$, incomplete beyond $m = 30$.

Analysis of the thirty subintervals

One typical subinterval is the interval $18 \le n < 27$. The values at the "endpoints" differ by 1. If conjecture (3) were true, we would expect the values of $f(n)/\log_3 n$ in the interval to "flatten out" and approach the values at the endpoints, as n gets large. Table 1 gives the mean and standard deviation of $f(n)/\log_3 n$ in each interval.

While the analysis of such small values of $f(n)$ has very little to do with behavior at large n, it is clear that in this range conjecture (4) is strongly supported.

The single worst value of $f(n)/\log_3 n$ encountered was at $n = 1439$, with $f(n) = 26$, and $f(n)\log_3 n = 3.9281$.

TABLE 1

Mean and Standard Deviation of $f(n)/\log_3 n$, for $a \le n < b$

a	b	mean	std dev
1	2		0.0
2	3	3.1699	0.0
3	4	3.0	0.0
4	6	3.2915	0.1216
6	9	3.2077	0.1340
9	12	3.3350	0.2716
12	18	3.2928	0.1382
18	27	3.3613	0.2273
27	36	3.3430	0.1754
36	54	3.3653	0.1607
54	81	3.3726	0.1748
81	108	3.3959	0.1630
108	162	3.3743	0.1307
162	243	3.3973	0.1473
243	324	3.3988	0.1395
324	486	3.3996	0.1327
486	729	3.4031	0.1290
729	972	3.4031	0.1194
972	1458	3.4037	0.1191
1458	2187	3.4031	0.1130
2187	2916	3.4039	0.1043
2916	4374	3.4017	0.1040
4374	6561	3.4012	0.0995
6561	8748	3.4016	0.0945
8748	13122	3.3996	0.0931
13122	19683	3.3985	0.0893
19683	26244	3.3987	0.0860
26244	39366	3.3965	0.0840
39366	59049	3.3949	0.0806

Analysis of the sets $S(m)$

Let $S(m) = \{n : f(n) = m\}$. In Table 2 we consider the following questions about $S(m)$.

- What is its first element?
- How many elements are in $S(m)$?
- What is its last element?
- What is its average element?

One result about the sets $S(m)$ not captured in Table 2 is: If $b(m)$ and $b_1(m)$ are the largest and second largest elements of $S(m)$, then $b_1(m) = [(8/9)b(m)]$, where $[\cdot]$ is the greatest integer function.

Outline of proof: The proof is by induction.

The result is true by inspection for small values of m. For large values of m we have an equation similar to (2): $b_1(m)$ is the second-largest member of the set

$$\bigcup_{r+s=m} \{b(r) + b(s),\ b(r)b(s),\ b(r) + b_1(s),\ b(r)b_1(s)\}. \tag{5}$$

For $m \geq 9$, it is easy to show that both $b(m)$ and $(8/9)b(m)$ belong to this set. It only remains to show that there are no elements between these two values, and this can be done by a simple case-by-case examination using the results of Selfridge and the induction hypothesis.

TABLE 2

Analysis of the sets $S(m)$

m	first	last	|S(m)|	mean	median
1	1	1	1	1.0	1.0
2	2	2	1	2.0	2.0
3	3	3	1	3.0	3.0
4	4	4	1	4.0	4.0
5	5	6	2	5.5	5.5
6	7	9	3	8.0	8.0
7	10	12	2	11.0	11.0
8	11	18	6	14.5	14.5
9	17	27	6	21.3	20.5
10	22	36	7	28.4	28.0
11	23	54	14	37.7	37.5
12	41	81	16	55.2	53.5
13	47	108	20	73.3	73.5
14	59	162	34	100.4	98.5
15	89	243	42	141.9	137.0
16	107	324	56	191.7	185.5
17	167	486	84	266.0	257.5
18	179	729	108	371.8	362.5
19	263	972	152	501.3	482.5
20	347	1458	214	701.3	675.0
21	467	2187	295	966.1	931.0
22	683	2916	398	1335.4	1284.5
23	719	4374	569	1842.9	1783.0
24	1223	6561	763	2571.0	2478.0
25	1438	8748	1094	3513.8	3382.5
26	1439	13122	1475	4914.9	4734.0
27	2879	19683	2058	6792.4	6533.5
28	3767	26244	2878	9378.7	9020.0
29	4283	39366	3929	13061.5	12534.0
30	6299	59049	5493	18051.5	17315.0
31	10079	78732			
32	11807	118098			
33	15287	177147			
34	21599	236196			
35	33599	354294			
36	45197	531441			
37	56039	708588			

Comments

In his paper [2], Guy relays some questions about the function $f(n)$. We comment on three of these:

Q: For what values a and b does $f(2^a3^b) = 2a + 3b$?
A: $f(2^a3^b) = 2a + 3b$ for all $2^a3^b < 3^{10}$, at least.

Q: If $f(2^a3^b) = 2a + 3b$ and there is a larger n' so that $f(n') = 2a + 3b$ $(a,\ b \geq 0)$, must $n' = 2^a3^b$, for some r, s?
A: No. Two counterexamples are $2^7 < 3^35$, with $f(2^7) = f(3^35) = 14$, and $2^73^2 < 3^55$, with $f(2^73^2) = f(3^55) = 20$.

Q: When the value of $f(n)$ is of the form $f(a) = f(b)$, with $a + b = n$, and this minimum is not achieved as a product, is either a or b equal to 1?
A: Yes, at least for $n \leq 3^{10}$.

The calculation of $f(n)$ was performed on a Symbolics 3645 LISP machine using equation (1), and we used over 50 hours of CPU time.

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OPTIMAL SPACING OF POINTS ON A CIRCLE

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1. Introduction

Consider N points placed on the circle of unit circumference in the following way: begin by placing a point anywhere on the circle. Now place another point so that the angle (or circumferential distance) between the two points, measured clockwise from the first point, is equal to α. The third point is now placed at a clockwise angle of α from the second point. Thus, we successively place N points on the circle by our angle α.

Our problem is to find the value for α so that these points are spread about the circle in the most even (which we call optimal) fashion. We show that, in certain senses, the golden section ($\alpha = \tau = (\sqrt{5} - 1)/2$) provides the optimal spacing of points, where the number of points can assume any value.

This problem originally arose while investigating the phenomenon of phyllotaxis—regular leaf arrangement. Most higher-order plants exhibit a remarkable degree of regularity in the positioning of their leaves. In a sunflower, for instance, one can perceive two sets of opposed spirals which each partition the set of florets. Intriguingly, the number of spirals are almost certainly consecutive members of the Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2,\ F_0 = 0,\ F_1 = 1.$$

This pattern (which we call Fibonacci phyllotaxis) manifests itself in 95% of those plants which produce their leaves sequentially. In parallel to this observation, the divergence angle subtended by consecutively formed leaves is quite close in value to the ratio of these consecutive Fibonacci numbers. In the limit, F_{n-1}/F_n is equal to the golden section. To simplify the situation, we consider just the angular displacement of the leaves and thus we develop a simplistic model of plant growth with leaves appearing as points on a meristematic ring, successively placed at a constant angle.

What this paper shows is that the plant places its leaves in the optimal manner—in order to spread its leaves most evenly (and thus reduce leaf overlap) the optimal divergence angle is shown to be the golden section. The partition of the circle by the golden section is also examined in detail to reveal a rather self-similar structure.

We use results from The Three Gap Theorem (originally the Steinhaus Conjecture) which states that the above N points partition the circle into arcs, or gaps, of at most three and at least two different lengths! The result is all the more remarkable since it holds for all irrational α *and* for any number of points. It also holds for rational $\alpha = p/q$ with the number of points less than q. (For $N = q$ the circle is partitioned into q equal gaps.) Even though this has been proved by various mathematicians ([1], [2]-[7]), the result does not appear to be well known.

Note that, in order to conserve space, where complete proofs of results are not presented we either refer the reader to an existing proof or briefly outline a proof.

2. The Three Gap Theorem

Suppose that we have consecutively placed N points on a circle by the angle α. Let $(u_1(N), u_2(N), \ldots, u_N(N))$ be the sequence of points as they appear on the circle, ordered clockwise from the origin $u_1(N) = 0$. That is,

$$\{u_1(N), u_2(N), \ldots, u_N(N)\} = \{0, 1, 2, \ldots, N - 1\} \text{ where } \{u_j\alpha\} < \{u_{j+1}\alpha\}.$$

Thus, for example, with $\alpha = \sqrt{2}$ the first 12 points placed on the circle appear in the order (0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7). We call $u_{j+1}(N) = u_{j+1}$ the successor to u_j, or $u_{j+1} = \mathrm{Suc}(u_j)$. Equivalent to the original statement of The Three Gap Theorem is the fact that the difference between succeeding points assumes at most three, and at least two, different values. The following determines the ordering of points around the circle. (For a proof, see van Ravenstein [7, Theorem 2.2].).

Theorem 1:

$$\mathrm{Suc}(m) - m = \begin{cases} u_2, & 0 \le m < N - u_2, \\ u_2 - u_N, & N - u_2 \le m < u_N, \\ -u_N, & u_N \le m < N. \end{cases}$$

Thus, for our example with $\alpha = \sqrt{2}$ and $N = 12$,

$$\mathrm{Suc}(m) - m = \begin{cases} 5, & 0 \le m < 7, \\ -7, & 7 \le m < 12. \end{cases}$$

It is easily seen that $u_j = 5(j - 1) \bmod 12$, where $y \bmod x = y - x[y/x] = x\{y/x\}$. In general, if $N = u_2 + u_N$, the circle is partitioned into gaps of just two different lengths and then

$$u_j = ((j - 1)u_2) \bmod N, \; j = 1, 2, \ldots, N. \tag{1}$$

It is easy to see that the length of the gap formed by point m and $\mathrm{Suc}(m)$ is equal to $\{(\mathrm{Suc}(m) - m)\alpha\}$ where $\{x\}$ denotes the fractional part of x such that $x = [x] + \{x\}$ where $[x]$ is the largest integer not greater than x. In fact, for gap lengths less than $\frac{1}{2}$ this gap length is equal to

$$\|(\mathrm{Suc}(m) - m)\alpha\|, \quad \text{where } \|x\| = \min(\{x\}, 1 - \{x\}) = |x - [x + \tfrac{1}{2}]|,$$

the difference between x and its nearest integer. (This is always the case for $N > q_1$ [notation defined in Theorem 2]; in what follows, we will always make this assumption. Note that q_1 is the first point to replace 1 as the closest point to the origin.) Thus, Theorem 1 shows that the circle of N points is partitioned into $N - u_2$ gaps of length $\|u_2\alpha\|$, $N - u_N$ gaps of length $\|u_N\alpha\|$ and $u_2 + u_N - N$ gaps of length $\|u_2\alpha\| + \|u_N\alpha\|$. The same applies for rational α, say $\alpha = p/q$ in lowest terms, where $N < q$. In this paper, however, we will always assume that α is irrational.

Point u_2 is the successor to 0, while 0 is the successor to u_N; that is, u_2 and u_N are the points which neighbor the origin. We see that we need only know the values of these two points to determine the entire ordering.

We can characterize the angle α by the following. Let $V(\alpha)$ denote the *path* of α defined to be a sequence of pairs (u_2, u_N), the points which neighbor the origin as points are successively included on the circle. For example,

$$V(\sqrt{2}) = ((1, 2), (3, 2), (5, 2), (5, 7), (5, 12), \ldots).$$

It can be shown ([7, Proposition 4.2]) that each point always enters one of the larger gaps. Two gaps are formed, one equal in length to the smallest gap present. Thus, it is natural to define the ratio of gap division as the ratio of the smallest to the largest gap present. Hence, we let

$$r_N(\alpha) = \frac{\min(\|u_2\alpha\|, \|u_N\alpha\|)}{\|u_2\alpha\| + \|u_N\alpha\|}.$$

[This is in fact the ratio point $N - 1$ that divides some (large) gap.]

The path sequence $V(\alpha)$ and the ratio of gap division $r_N(\alpha)$ are quantities we will use in our analysis of the golden section's unique distribution properties. We can in fact determine explicitly their values in terms of the continued fraction expansion of α, which is expressed by

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

$$\{a_0; a_1, a_2, a_3, \ldots\}.$$

The n^{th} tail of α is

$$t_n = \{a_n; a_{n+1}, a_{n+2}, \ldots\}, \tag{2}$$

such that

$$\alpha = \{a_0; a_1, a_2, \ldots, a_{n-1}, t_n\}.$$

We say that α is *equivalent* to β if some tail in α is equal to some tail in β.

Partial convergents are defined by the (irreducible) fractions

$$\frac{p_{n,i}}{q_{n,i}} = \frac{p_{n-2} + ip_{n-1}}{q_{n-2} + iq_{n-1}} = \{a_0; a_1, a_2, \ldots, a_{n-1}, i\}, \; i = 1, 2, \ldots, a_n,$$

where

$$\frac{p_{n,a_n}}{q_{n,a_n}} = \frac{p_n}{q_n}, \; p_{-2} = q_{-1} = 0, \; q_{-2} = p_{-1} = 1.$$

We call p_n/q_n a *total* convergent to α.

The reader is referred to [7, Theorem 3.3] for a proof of the following.

Theorem 2:

$$u_2 = \begin{cases} q_{n-1}, & n \text{ odd}, \\ q_{n,i-1}, & n \text{ even}, \end{cases} \qquad u_N = \begin{cases} q_{n,i-1}, & n \text{ odd}, \\ q_{n-1}, & n \text{ even}, \end{cases}$$

where $q_{n,i-1} < N \le q_{n,i}$, $2 \le i \le a_n$ $(n \ge 2)$.

For $q_{n-1} < N \le q_{n,1}$ $(n \ge 2)$,

$$u_2 = \begin{cases} q_{n-1}, & n \text{ odd}, \\ q_{n-2}, & n \text{ even}, \end{cases} \qquad u_N = \begin{cases} q_{n-2}, & n \text{ odd}, \\ q_{n-1}, & n \text{ even}. \end{cases}$$

For $N \le q_1$, $u_j = j - 1$, $j = 1, 2, \ldots, N$.

The following proposition may be easily proved from the definition of $r_N(\alpha)$, Theorem 2, and the continued fraction theory.

Proposition 3:

$$r_N(\alpha) = \begin{cases} \dfrac{1}{1 + t_n}, & q_{n-1} < N \le q_{n,1}, \\[2ex] \dfrac{1}{2 - i + t_n}, & q_{n,i-1} < N \le q_{n,i}, \end{cases}$$

where $i = 2, 3, \ldots, a_n$, $(n \geq 2)$. [t_n is defined by (2).]

From Theorem 2,

$$V(\alpha) = ((1, q_1)', (q_{n,i}, q_{n-1})''; i = 1, 2, \ldots, a_n, n = 2, 3, \ldots), \quad (3)$$

where

$$(1, q_1)' = \begin{cases} (1, q_1), & 0 < \alpha < \frac{1}{2}, \\ (q_1, 1), & \frac{1}{2} < \alpha < 1, \end{cases}$$

$$(q_{n,i}, q_{n-1})'' = \begin{cases} (q_{n-1}, q_{n,i}), & n \text{ odd}, \\ (q_{n,i}, q_{n-1}), & n \text{ even}. \end{cases}$$

3. The Golden Section

For convenience, let the partition of the circle of unit circumference by the successive placement of points 0, 1, 2, ... by the golden section, τ, be denoted by G. The partition by $1 - \tau$ we denote by G'.

The continued fraction of τ is given by

$$\tau = \{0; 1 + \tau\} = \{0; 1, 1 + \tau\} = \{0; 1, 1, 1, \ldots\}.$$

All convergents to τ are total convengents and

$$p_n = q_{n-1} = F_n = F_{n-1} + F_{n-2}, \; n \geq 1, \; F_{-1} = 1, \; F_0 = 0.$$

That is, convergents to τ are equal to the ratio of consecutive Fibonacci numbers. From Theorems 1 and 2, for $F_n < N \leq F_{n+1}$,

$$\operatorname{Suc}(m) - m = \begin{cases} \left.\begin{array}{ll} F_n, & 0 \leq m \leq N - F_n, \\ F_{n-2}, & N - F_n < m < F_{n-1}, \\ -F_{n-1}, & F_{n-1} \leq m < N, \end{array}\right\} & n \text{ odd}, \\ \left.\begin{array}{ll} F_{n-1}, & 0 \leq m \leq N - F_{n-1}, \\ -F_{n-2}, & N - F_{n-1} < m < F_n, \\ -F_n, & F_n \leq m < N, \end{array}\right\} & n \text{ even}. \end{cases} \quad (4)$$

When $N = F_{n+1}$, from (1),

$$u_j = ((-1)^{n-1}(j - 1)F_n) \bmod F_{n+1}.$$

Since $F_{n-1} - F_n\tau = (-\tau)^n$ (by induction), (4) shows that N points ($F_n < N \leq F_{n+1}$) partition the circle into $N - F_{n-1}$ gaps of length τ^{n-1}, $N - F_n$ gaps of length τ^n and $F_{n+1} - N$ gaps of length τ^{n-2}.

From Proposition 3, since $t_n = 1 + \tau$, $n = 1, 2, \ldots$,

$$r_N(\tau) = \tau^2 = 1 - \tau, \quad F_n < N \leq F_{n+1}. \quad (5)$$

Theorem 4.1 from [7] describes the partition G by looking at the transformation of gap types as points are included on the circle. Gap types are either "large" or "small" when $N = u_2 + u_N$, that is, when N is the denominator of a convergent to α. For the golden section, this theorem describes the following: each large gap present when $N = F_n$ is divided by the addition of a further F_{n-1} points into two new gaps which can be labelled (in clockwise order) as small:large (n odd) or large:small (n even) when $N = F_{n+1}$. Those small gaps present when $N = F_n$ can then be labelled as large.

This fact and (5) can be used to prove the following self-similarity property of G, thus demonstrating the beautiful symmetry inherent in Fibonacci phyllotaxis.

Theorem 4: Consider any large gap present on the circle partitioned by the placement of F_n points by the golden section. Include further points on the circle and observe the resulting partitioning of this gap. If we pretend the gap is itself a circle of unit circumference (by lengthening it by the factor τ^{n-2} and identifying its endpoints as the same) then its partition is identical to G if n is odd, or equal to G' if n is even.

Let us interpret N to be a time variable and define the age of a gap to be the time it has survived without being divided. That is, the age of the gap with endpoints u_j, u_{j+1} is $N - 1 - \max(u_j, u_{j+1})$. From [7, Proposition 4.2] each point, for all α, divides the oldest of the larger gaps. Using [7, Theorem 4.1] it can be shown that only for the golden section does the formation of a large gap always coincide with that of a small gap. This proves the following. (Note that we assume that α is between 0 and 1. If $\alpha > 1$, the following results hold if α is replaced by its fractional part, $\{\alpha\}$.)

Theorem 5: For $\frac{1}{2} < \alpha < 1$, each point always enters the oldest gap if and only if $\alpha = \tau$. For $0 < \alpha < \frac{1}{2}$, each point enters the oldest gap if and only if $\alpha = \tau^2 = (3 - \sqrt{5})/2$.

Intuitively, in terms of phyllotaxis, it seems sensible that points be inserted in the oldest gap as the above result shows. This property must ensure an ideal distribution of points. In fact, the following theorem shows that the golden section provides the optimal value for gap division (our criteria for an optimal distribution) in the sense that the smallest value assumed by the ratio of gap division is *largest* for the golden section (where $\frac{1}{2} < \alpha < 1$). However, the golden section is somewhat of a compromise as Theorem 7a shows (that the ratio of gap division's maximum value is smallest for the golden section, where $\frac{1}{2} < \alpha < \frac{2}{3}$).

Theorem 6: $\max\limits_{\frac{1}{2}<\alpha<1} \min\limits_{N} r_N(\alpha) = \tau^2$, exclusively attained by $\alpha = \tau$.

$\max\limits_{0<\alpha<\frac{1}{2}} \min\limits_{N} r_N(\alpha) = \tau^2$, exclusively attained by $\alpha = \tau^2$.

Proof: We first consider the case where $\frac{1}{2} < \alpha < 1$. From Proposition 3,

$$\min_N r_N(\alpha) = \min_n \min_{q_{n-1}<N\le q_n} r_N(\alpha) = \min_n \frac{1}{1+t_n} = \frac{1}{1+\max\limits_n t_n},$$

where $n = 2, 3, \ldots$ ($q_1 = a_1 = 1$ since $\alpha > \frac{1}{2}$).

Consider $\alpha = \alpha' \neq \tau$, which has $a_k > 1$ for some integer k greater than 1. Then $\max\limits_n t_n \ge t_k > 2$, so $r_N(\alpha') < \frac{1}{3}$. The result follows since $\frac{1}{3} < r_N(\tau) = \tau^2$.

The second statement follows by symmetry (note that $r_N(1-\alpha) = r_N(\alpha)$).

Theorems 7a and 7b follow from Proposition 3 in a similar fashion.

Theorem 7a: $\min\limits_{\frac{1}{2}<\alpha<\frac{2}{3}} \max\limits_{N} r_N(\alpha) = \tau^2$, exclusively attained by $\alpha = \tau$.

$\min\limits_{\frac{1}{3}<\alpha<\frac{1}{2}} \max\limits_{N} r_N(\alpha) = \tau^2$, exclusively attained by $\alpha = \tau^2$.

Theorem 7b: $\min\limits_{\frac{2}{3}<\alpha<1} \max\limits_{N} r_N(\alpha) = \tau^2$, exclusively attained by

$$\alpha = \{0;\ 1,\ a,\ 1+\tau\} = \frac{\tau + a}{\tau + a + 1},$$ where a is any integer greater than 1.

$$\min_{0<\alpha<\frac{1}{3}} \max_N r_N(\alpha) = \tau^2, \text{ exclusively attained by}$$

$$\alpha = \{0;\ a+1,\ 1+\tau\} = \frac{1}{\tau + a + 1}, \text{ where } a \text{ is any integer greater than 1.}$$

We determine the value of α which ensures a path which consistently maximizes the length of the smallest gap on the circle. For each value of N, the points are generated by a constant angle α. This value of α may change with N but only in such a way that the path is retained: so that the addition of extra points does not alter the relative order of existing points. We show that $V(\tau)$ is the path which ensures that the smallest gaps are consistently as large as possible [where, initially, $(u_2(4),\ u_3(4)) = (2,\ 1)$, $\frac{1}{2} < \alpha < \frac{2}{3}$]. That is, as N increases, if the pair $(u_2(N),\ u_N(N))$ does not assume the value equal to the appropriate successive element of $V(\tau)$, then the smallest gap thus formed will not be as large.

Note that the golden section has path

$$\begin{aligned} V(\tau) &= ((1,\ 1),\ (2,\ 1),\ (2,\ 3),\ (5,\ 3),\ \ldots), \\ &= ((1,\ 1),\ (F_{n+1},\ F_n)',\ n = 2,\ 3,\ \ldots). \end{aligned}$$

Theorem 8: Suppose that $(u_2(4),\ u_3(4)) = (2,\ 1)$ generated by a constant angle α where $\frac{1}{2} < \alpha < \frac{2}{3}$. Then $V(\tau)$ is the path which consistently maximizes the length of the smallest gap.

Proof: We prove the result by induction. Initially, $\frac{1}{2} < \alpha < \frac{2}{3}$ or $\alpha = \{0;\ 1,\ 1,\ t_3\}$, $1 < t_3 < \infty$, such that point 2 is closest to the origin. The next element in the path must, from (3), be (2, 3). From Proposition 3, point 2 is furthest from the origin if $a_3 = 1$ than if $a_3 > 1$ since then it divides the gap bordered by the origin and the first point into a larger ratio. Hence, $\alpha = \{0;\ 1,\ 1,\ 1,\ t_4\}$. This ensures, from (3), that the next element in the path is (5, 3). Thus, the first three terms in the path belong to $V(\tau)$.

Now, assume that the terms in the path equal successive Fibonacci pairs and that $(u_2(N),\ u_N(N)) = (F_{n-1},\ F_n)$ where $F_n < N \le F_{n+1}$, n even. Then, $\alpha = \{0;\ 1,\ 1,\ 1,\ \ldots,\ 1,\ t_n\}$ $(n-1)$ ones, $1 < t_n < \infty$. The next element in the path must be $(F_{n+1},\ F_n)$ succeeded by $(F_{n+1},\ F_{n+2})$ if $a_n = 1$. From Proposition 3, the small gap bordered by origin and point F_n is larger if $a_n = 1$ than if $a_n > 1$. The case is similar for odd n. Thus, the path is equal to $V(\tau)$.

Note that the theorem shows that maximizing the length of the smallest gap ensures convergence to the golden section. Similarly, $V(\tau^2)$ consistently maximizes the length of the smallest gap where, initially, there are three points on the circle and $\frac{1}{3} < \alpha < \frac{1}{2}$. The following generalizes Theorem 8. Its proof is similar in manner and is omitted.

Theorem 9: Suppose that we have placed $q_n + 1$ points $(n \ge 2)$ generated by $\alpha = \{0;\ a_1,\ a_2,\ \ldots,\ a_n,\ t_{n+1}\}$. Then as more points are added, $V(\alpha')$, where

$$\alpha' = \{0;\ a_1,\ a_2,\ \ldots,\ a_n,\ 1+\tau\} = \frac{p_n + \tau p_{n-1}}{q_n + \tau q_{n-1}},$$

is the path which consistently maximizes the length of the smallest gap.

Acknowledgments

The author is grateful to Drs. Keith Tognetti and Graham Winley for their help.

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CONGRUENCE RELATIONS FOR k^{th}-ORDER LINEAR RECURRENCES

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(Submitted January 1987)

1. Introduction

Let k be a positive integer and let $\{T_n\}_{n=0}^{\infty}$ be a k^{th}-order integral linear recurrence defined by

$$T_{n+k} = a_1T_{n+k-1} + a_2T_{n+k-2} + \cdots + a_kT_n \tag{1}$$

with arbitrary initial terms $T_0, T_1, \ldots, T_{k-1}$. Associated with the recursion relation (1) is the characteristic polynomial

$$f(x) = x^k - a_1x^{k-1} - \cdots - a_{k-1}x - a_k \tag{2}$$

with characteristic roots $r_1, r_2, \ldots, r_k$. We will seek subsequences of $\{T_n\}$ such that the recursion relation (1) is also satisfied as a congruence modulo some integer m. Specifically, we will endeavor to find positive integers d and n such that

$$T_{n+kd} \equiv a_1T_{n+(k-1)d} + a_2T_{n+(k-2)d} + \cdots + a_{k-1}T_{n+d} + a_kT_n \pmod m \tag{3}$$

for all nonnegative integers n. This investigation was suggested by Freitag [2] and by Freitag and Phillips [3] and [4], and will generalize the results of these papers.

Two approaches will be taken in satisfying congruence (3). In the first approach, given a fixed modulus m we will seek to find integers d such that (3) is satisfied. Along these lines, Freitag [2] proved the following theorem:

Theorem 1: Let $\{F_n\}$ as usual denote the Fibonacci sequence. Then

$$F_{n+2d} \equiv F_{n+d} + F_n \pmod{10} \tag{4}$$

for all nonnegative integers n if and only if $d \equiv 1$ or $5 \pmod{12}$.□

The second approach will be to take the integer d from among the integers appearing in a specified sequence such as the sequence of primes and then find moduli m, depending on d, such that congruence (3) is satisfied. Corresponding to this approach, Freitag and Phillips proved Theorems 2 and 3 in [3] and [4], respectively.

Theorem 2: Let $\{T_n\}$ be a second-order recurrence defined by

$$T_{n+2} = a_1T_{n+1} + a_2T_n.$$

Then, if p is a prime greater than 3,

$$T_{n+2p} \equiv a_1T_{n+p} + a_2T_n \pmod{2p}$$

for all nonnegative integers n.□

Theorem 3: Let $\{T_n\}$ be a k^{th}-order recurrence with distinct characteristic roots satisfying

$$T_{n+k} = a_1T_{n+k-1} + a_2T_{n+k-2} + \cdots + a_kT_n.$$

Then, if p is a prime,

$$T_{n+kp} \equiv a_1 T_{n+(k-1)p} + a_2 T_{n+(k-2)p} + \cdots + a_{k-1} T_{n+p} + a_k T_n \pmod{p}$$

for all nonnegative integers n.□

2. Definitions and Known Results

We will need the following definitions and lemmas to continue.

Lemma 1: Let $\{T_n\}$ be a k^{th}-order linear recurrence with distinct characteristic roots $r_1, r_2, \ldots, r_m$. Then

$$T_n = \sum_{i=1}^{m} (c_i^{(0)} + c_i^{(1)} n + \cdots + c_i^{(s_i-1)} n^{s_i-1}) r_i^n,$$

where the $c_i^{(j)}$ are complex constants and s_i is the multiplicity of the root r_i.

Proof: This is a classical result in the theory of finite differences (see, for example, Milne-Thomson [5, Ch. XIII]).□

Definition 1: The primary linear recurrence $\{V_n\}_{n=0}^{\infty}$ is the recurrence satisfying (1) and defined by

$$V_n = r_1^n + r_2^n + \cdots + r_k^n,$$

where $r_1, r_2, \ldots, r_k$ are the zeros of the characteristic polynomial (2). If any characteristic root $r_i = 0$, we define r_i^0 to be 1.

Lemma 2: Suppose $\{T_n\}$ is a k^{th}-order linear recurrence satisfying

$$T_{n+k} = a_1 T_{n+k-1} + a_2 T_{n+k-2} + \cdots + a_k T_n.$$

Suppose m is a positive integer such that $(a_k, m) = 1$. Then $\{T_n\}$ is purely periodic modulo m.

Proof: This is proved by Carmichael [1, p. 344].□

Lemma 3: Let $\{T_n\}$ be a k^{th}-order integral linear recurrence with characteristic roots $r_1, r_2, \ldots, r_k$. Let h be a fixed positive integer, and let q be a fixed nonnegative integer. Then the sequence

$$\{S_n\}_{n=0}^{\infty} = \{T_{hn+q}\}_{n=0}^{\infty}$$

also satisfies a linear integral recursion relation

$$S_{n+k} = a_1^{(h)} S_{n+k-1} + a_2^{(h)} S_{n+k-2} + \cdots + a_k^{(h)} S_n, \tag{5}$$

where $a_1^{(h)}, a_2^{(h)}, \ldots, a_k^{(h)}$ are integral constants dependent on h but not on q. Further, if j is a fixed integer such that $1 \le j \le k$, then

$$a_j^{(h)} = \sum (-1)^j r_{i_1}^h r_{i_2}^h \cdots r_{i_j}^h, \tag{6}$$

where one sums over all indices $i_1, i_2, \ldots, i_j$ such that

$$1 \le i_1 < i_2 < \cdots < i_j \le k.$$

Proof: This is proved in [6].□

3. Main Results

We now present our principal theorems.

Theorem 4: Let $\{T_n\}$ be a k^{th}-order recurrence defined by

$$T_{n+k} = a_1T_{n+k-1} + a_2T_{n+k-2} + \cdots + a_kT_n.$$

Let p be a prime. Then for all nonnegative integers b,

$$T_{n+kp^b} \equiv a_1T_{n+(k-1)p^b} + a_2T_{n+(k-2)p^b} + \cdots + a_{k-1}T_{n+p^b} + a_kT_n \pmod{p},$$

where n is any nonnegative integer.

Proof: Let $r_1, r_2, \ldots, r_m$ be the distinct characteristic roots of $\{T_n\}$. Let R denote the integers of the algebraic number field $Q(r_1, r_2, \ldots, r_m)$, where Q denotes the rational numbers. Let Z denote the rational integers. Let P be a prime ideal of R dividing p. Let σ be the Frobenius automorphism of the finite field R/P having Z/p as a fixed field. Then σ is defined by $\sigma(x) = x^p$. Then, for any nonnegative integer b, σ^b, defined by $\sigma^b(x) = x^{p^b}$, is also an automorphism of R/P fixing Z/p.

Now, for $1 \le i \le m$,

$$r_i^k = a_1r_i^{k-1} + a_2r_i^{k-2} + \cdots + a_{k-1}r_i + a_k. \tag{7}$$

Applying σ^b to equation (7), we have, for $1 \le i \le m$,

$$\begin{aligned}\sigma^b(r_i^k) &\equiv r_i^{kp^b} \equiv \sigma^b(a_1r_i^{k-1} + a_2r_i^{k-2} + \cdots + a_k) \equiv \sum_{j=1}^{k} a_j\sigma^b(r_i^{k-j}) \\ &\equiv \sum_{j=1}^{k} a_jr_i^{(k-j)p^b} \pmod{P}.\end{aligned} \tag{8}$$

By (8), (1), and Lemma 1, we have

$$\begin{aligned}T_{n+kp^b} &= \sum_{i=1}^{m}\left[\left(c_i^{(0)} + c_i^{(1)}n + \cdots + c_i^{(m_i-1)}n^{m_i-1}\right)r_i^n\right]r_i^{kp^b} \\ &\equiv \sum_{i=1}^{m}\left[\left(c_i^{(0)} + c_i^{(1)}n + \cdots + c_i^{(m_i-1)}n^{m_i-1}\right)r_i^n\right]\sum_{j=1}^{k}a_jr_i^{(k-j)p^b} \\ &\equiv \sum_{j=1}^{k} a_j \sum_{i=1}^{m}\left(c_i^{(0)} + c_i^{(1)}n + \cdots + c_i^{(m_i-1)}n^{m_i-1}\right)r_i^{n+(k-j)p^b} \\ &\equiv \sum_{j=1}^{k} a_jT_{n+(k-j)p^b} \pmod{P}.\end{aligned} \tag{9}$$

Since the first and last terms of (9) are rational integers, we have

$$T_{n+kp^b} \equiv \sum_{j=1}^{k} a_jT_{n+(k-j)p^b} \pmod{p}. \square$$

Remark: We note that Theorem 4 is a generalization of Theorem 3.

Theorem 5: Let $\{T_n\}$ be a k^{th}-order recurrence defined by

$$T_{n+k} = a_1T_{n+k-1} + a_2T_{n+k-2} + \cdots + a_kT_n.$$

Let c be a fixed positive integer such that $(c, a_k) = 1$. Then there exists a fixed modulus g such that if $h \equiv 1 \pmod{g}$, then

$$T_{n+kh} \equiv a_1T_{n+(k-1)h} + a_2T_{n+(k-2)h} + \cdots + a_{k-1}T_{n+h} + a_kT_n \pmod{c},$$

where n is any nonnegative integer.

Proof: If h is any positive integer, then by (5) and (6),

$$T_{n+kh} = a_1^{(h)} T_{n+(k-1)h} + a_2^{(h)} T_{n+(k-2)h} + \cdots + a_k^{(h)} T_n, \tag{10}$$

where, for $1 \le j \le k$,

$$a_j^{(h)} = \sum (-1)^{j+1} r_{i_1}^h r_{i_2}^h \ldots r_{i_j}^h, \tag{11}$$

where one sums over all indices $i_1, i_2, \ldots, i_j$ such that

$$1 \le i_1 < i_2 < \cdots < i_j \le k.$$

Let $n_j = \binom{k}{j}$. Let $1 \le j \le k$ be a fixed integer and let $t_1^{(j)}, t_2^{(j)}, \ldots, t_{n_j}^{(j)}$ denote the $\binom{k}{j}$ algebraic integers $(-1)^{j+1} r_{i_1} r_{i_2} \ldots r_{i_j}$, where these represent all the $\binom{k}{j}$ products taken j at a time of the characteristic roots $r_1, r_2, \ldots, r_k$ of $\{T_n\}$. By the theory of symmetric polynomials, for a fixed integer j such that $1 \le j \le k$, the n_j algebraic integers $t_1^{(j)}, t_2^{(j)}, \ldots, t_{n_j}^{(j)}$ are the roots, possibly with repetitions, of a monic polynomial of degree n_j with rational integral coefficients.

Let $\{V_n^{(j)}\}$, defined by

$$V_n^{(j)} = (t_1^{(j)})^n + (t_2^{(j)})^n + \cdots + (t_{n_j}^{(j)})^n$$

be the primary linear recurrence with characteristic roots $t_1^{(j)}, t_2^{(j)}, \ldots, t_{n_j}^{(j)}$. Since $(a_k, c) = 1$, it follows by Lemma 2 that $\{V_n^{(j)}\}$ is purely periodic modulo c. Let d_j denote the period modulo c of $\{V_n^{(j)}\}$ for $1 \le j \le k$. Let g be the least common multiple of $d_1, d_2, \ldots, d_k$. Since by (11),

$$V_1^{(j)} = a_j^{(1)} = a_j,$$

it follows that if $h \equiv 1 \pmod{g}$, then

$$a_j^{(h)} = V_h^{(j)} \equiv V_1^{(j)} = a_j \pmod{c}. \tag{12}$$

The result now follows by (10).□

Corollary: Let $\{T_n\}$ be a k^{th}-order linear recurrence defined by

$$T_{n+k} = a_1 T_{n+k-1} + a_2 T_{n+k-2} + \cdots + a_k T_n.$$

Let p be a fixed prime such that $p \nmid a_k$. Then there exists a fixed modulus g such that if $h \equiv p^b \pmod{g}$, where b is any nonnegative integer, then

$$T_{n+kh} \equiv a_1 T_{n+(k-1)h} + a_2 T_{n+(k-2)h} + \cdots + a_k T_n \pmod{p},$$

where n is any nonnegative integer.

Proof: Let $\{V_n\}$ be any primary linear recurrence with characteristic roots $r_1, r_2, \ldots, r_k$. Then

$$V_{p^b} = r_1^{p^b} + r_2^{p^b} + \cdots + r_k^{p^b} \equiv (r_1 + r_2 + \cdots + r_k)^{p^b} = (V_1)^{p^b} \equiv V_1 \pmod{p}.$$

Let the primary linear recurrences $\{V_n^{(j)}\}$ and the integers $a_j^{(h)}$, where $1 \le j \le k$, be defined as in the proof of Theorem 5. Choose the modulus g in the same manner as in the proof of Theorem 5, letting $p = c$. Then

$$V_{p^b}^{(j)} \equiv V_h^{(j)} \pmod{g}$$

and

$$a_j^{(h)} = V_h^{(j)} \equiv V_{p^b}^{(j)} \equiv V_1^{(j)} = a_j \pmod{p}$$

for all j such that $1 \le j \le k$. The proof now follows by (10).□

Remark 1: Note that if p is a fixed prime, the corollary to Theorem 5 is a strengthening of Theorem 4.

Remark 2: Theorem 1 follows from the corollary to Theorem 5. By the proof of this corollary, it can be shown that if $d \equiv 1$ or 5 (mod 12), then

$$F_{n+2d} \equiv F_{n+d} + F_n \pmod{5}. \tag{13}$$

Similarly, it can be shown that if $d \equiv 1$ or 2 (mod 3), then

$$F_{n+2d} \equiv F_{n+d} + F_n \pmod{2}. \tag{14}$$

It thus follows that if $d \equiv 1$ or 5 (mod 12), then (14) holds. Since 2 and 5 are relatively prime, it follows from (13)-(14) that if $d \equiv 1$ or 5 (mod 12), then congruence (4) holds. This proves the necessity of Theorem 1. The sufficiency of Theorem 1 follows from the fact that $\{F_n\}$ has a period modulo 10 equal to 60. Examining (4) for all integral values of d between 1 and 60 establishes the result.

Theorem 6: Let $\{T_n\}$ be a k^{th}-order linear recurrence defined by

$$T_{n+k} = a_1T_{n+k-1} + a_2T_{n+k-2} + \cdots + a_kT_n.$$

Let c be a fixed positive integer such that $(c, a_k) = 1$. Then for all nonnegative integers b, there exists an infinite number of primes p of positive density in the set of primes such that

$$T_{n+kp^b} \equiv a_1T_{n+(k-1)p^b} + a_2T_{n+(k-2)p^b} + \cdots + a_{k-1}T_{n+p^b} + a_kT_n \pmod{cp}, \tag{15}$$

where n is any nonnegative integer. Furthermore, there exists a fixed modulus g such that if $p \equiv 1 \pmod{g}$, then congruence (15) is satisfied.

Proof: By Theorem 4, the congruence (15) is satisfied modulo p for any prime p. Given the integer c, we choose the modulus g in the same manner as in the proof of Theorem 5. By Dirichlet's theorem on the infinitude of primes in arithmetic progressions, there exists an infinite number of primes p such that $p \equiv 1$ (mod g). Further, the density of such primes is $1/\phi(g)$, where ϕ denotes Euler's totient function. By Theorem 5, congruence (15) is also satisfied modulo c, since p^b is also congruent to 1 modulo g for any nonnegative integer b. Since we can also assume that $(p, c) = 1$, it follows that (15) is satisfied modulo cp.□

Corollary 1: Let $\{T_n\}$ be a k^{th}-order linear recurrence defined by

$$T_{n+k} = a_1T_{n+k-1} + a_2T_{n+k-2} + \cdots + a_kT_n.$$

Let c be a fixed prime such that $c \nmid a_k$. Then for all nonnegative integers b, there exists an infinite number of primes p of positive density in the set of primes such that

$$T_{n+kp^b} \equiv a_1T_{n+(k-1)p^b} + a_2T_{n+(k-2)p^b} + \cdots + a_{k-1}T_{n+p^b} + a_kT_n \pmod{cp}, \tag{16}$$

where n is any nonnegative integer. Furthermore, there exists a fixed modulus g such that if the prime $p \equiv c^b$ (mod g), where b is any nonnegative integer, then congruence (16) is satisfied.

Proof: This follows by the corollary to Theorem 5 and the proof of Theorem 6.

Corollary 2: Let $\{T_n\}$ be a second-order linear recurrence defined by

$$T_{n+2} = a_1 T_{n+1} + a_2 T_n.$$

Then for all primes $p > 3$ and for all nonnegative integers b,

$$T_{n+2p^b} \equiv a_1 T_{n+p^b} + a_2 T_n \pmod{2p}, \tag{17}$$

where n is any nonnegative integer.

Proof: Let $p > 3$ be a prime. By Theorem 4, congruence (17) holds modulo p for all n. We will show that (17) also holds modulo 2 for all n. The corollary will then follow since $(2, p) = 1$.

First, suppose that $2|a_2$. Considering the characteristic polynomial $f(x)$ of $\{T_n\}$ modulo 2, we have

$$f(x) = x^2 - a_1 x - a_2 \equiv x(x - a_1) \pmod 2.$$

Hence, the characteristic roots of $\{T_n\}$ modulo 2 are $r_1 \equiv a_1 \pmod 2$ and $r_2 \equiv 0 \pmod 2$. As in the proof of Theorem 5, we have that if h is any nonnegative integer, then

$$T_{n+2h} = a_1^{(h)} T_{n+h} + a_2^{(h)} T_n, \tag{18}$$

where $a_1^{(h)}$ and $a_2^{(h)}$ are defined as in equation (11). Constructing the primary linear recurrences $\{V_n^{(1)}\}$ and $\{V_n^{(2)}\}$ as in the proof of Theorem 5, we observe that

$$V_n^{(1)} \equiv a_1 \pmod 2 \tag{19}$$

for all $n \geq 1$ and

$$V_n^{(2)} \equiv a_2 \equiv 0 \pmod 2 \tag{20}$$

for all $n \geq 1$. By (12) and (18)-(20), we see that for $j = 1$ or 2,

$$a_j^{(h)} = V_h^{(j)} \equiv a_j \pmod 2 \tag{21}$$

for all positive integers h. Letting $h = p^b$, equation (18) and congruence (21) lead to the congruence

$$T_{n+2p^b} \equiv a_1 T_{n+p^b} + a_2 T_n \pmod 2,$$

which is what we wanted to show.

Now, suppose that $2 \nmid a_2$. Constructing the primary recurrences $\{V_n^{(1)}\}$ and $\{V_n^{(2)}\}$ as in the proof of Theorem 5, we see that $\{V_n^{(1)}\}$ and $\{V_n^{(2)}\}$ are each purely periodic modulo 2 by Lemma 2. Further, one can easily determine that the period of the second-order recurrence $\{V_n^{(1)}\}$ modulo 2 is either 2 or 3, and the period of the first-order recurrence $\{V_n^{(2)}\}$ modulo 2 is 1. It thus follows that if we determine the modulus g, as in the proof of Theorem 5, then $g = 2$ or 3. By Theorem 5, if $g = 2$ and p is a prime such that $p \equiv 1 \pmod 2$, then congruence (17) holds modulo 2. By the corollary to Theorem 5, if $g = 3$ and p is a prime such that $p \equiv 1$ or $2 \pmod 3$, then the congruence (17) again holds modulo 2. Since for any $p > 3$, $p \equiv 1 \pmod 2$ and $p \equiv 1$ or $2 \pmod 3$, the result now follows. □

Remark: Note that Corollary 2 to Theorem 6 generalizes Theorem 2.

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In addition to the members of the Board of Directors and our Assistant Editors, the following mathematicians, engineers, and physicists have assisted **THE FIBONACCI QUARTERLY** by refereeing papers during the past year. Their special efforts are sincerely appreciated, and we apologize for any names that have inadvertently been overlooked or misspelled.

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A RECURRENCE RESTRICTED BY A DIAGONAL CONDITION: GENERALIZED CATALAN ARRAYS

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1. Introduction

For a real sequence $\{a_k\}_{k\geq 0}$, $a_0 \neq 0$, and $\mu \in \mathbb{N}$, consider the real array $A(k, n, \mu)$, $(k, n) \in \mathbb{Z} \times \mathbb{Z}$, which satisfies the recurrence

$$A(k, n, \mu) = \sum_{j\geq 0} a_j A(k - j, n - 1, \mu) \tag{1.1}$$

subject to the *diagonal* condition

$$A(k, \mu k, \mu) = 0 \text{ for } k > 0, \tag{1.2}$$

and the conditions

$$A(0, 0, \mu) = 1 \quad \text{and} \quad A(k, n, \mu) = 0 \text{ for } k < 0. \tag{1.3}$$

We wish to use lattice path combinatorics to obtain known formulas for $A(k, n, \mu)$. Collectively these constitute a Lagrange inversion formula. Others have made similar studies; our explanations are influenced by those of Raney [18] and Gessel [9]. We examine specific examples of recurrences and their solutions, the generalized Catalan arrays. We illustrate our approach by enumerating certain plane trees.

For the given sequence $\{a_k\}_{k\geq 0}$, let $\alpha(x; \mu)$ denote

$$\sum_{k\geq 0} A(k, \mu k + 1, \mu)x^k,$$

which we view as a *diagonal series*. In particular, let

$$a(x) = \alpha(x; 0) = \sum_{k\geq 0} a_k x^k \text{ (the } \textit{initial series}\text{)},$$

and let

$$\alpha(x) = \alpha(x; 1) \text{ (the } \textit{principal} \text{ diagonal series)}.$$

For any power series, let $[x^k]\Sigma f_j x^j$ denote the coefficient f_k. Let

$$\alpha_k = [x^k]\alpha(x) = A(k, k + 1, 1) \text{ [the } \textit{principal diagonal} \text{ of } A(k, n, 1)\text{]}.$$

It is immediate from (1.1) and elementary properties of formal power series (see [3], [12]) that

$$A(k, n, 0) = [x^k]a^n(x). \tag{1.4}$$

The following record solutions to (1.1, 1.2, and 1.3).

Propositions: For $m, n \in \mathbb{Z}$ and $k, \lambda \in \mathbb{N}$:

1. $$A(k, n, \mu) = \frac{n - \mu k}{n} A(k, n, 0),\ n \neq 0. \tag{1.5}$$

2. $$A(k, n, \mu) = \sum_{j\geq 0} (1 - \mu j)a_j A(k - j, n - 1, 0);\ A(k, 1, \mu) = (1 - \mu k)a_k. \tag{1.6}$$

3. $A(k, m + n + \mu k, \lambda) = \sum_{j \geq 0} A(j, m + \mu j, \lambda) A(k - j, n + \mu(k - j), \mu)$. (1.7)

4. $A(k, n + \mu k, \mu) = [x^k]\alpha^n(x; \mu)$. (1.8)

5. $\phi(x) = \alpha(x; \mu)$ is a unique series satisfying $\phi(x) = a(x\phi^{\mu}(x))$. (1.9)

These are proven in Sections 3 and 4. In proving (1.5), we interpret the factor $(n - \mu k)/n$. For (1.6) we interpret $\{A(k, 1, \mu)\}_{k \geq 0}$. Proposition (1.7) is a Vandermonde-type convolution; (1.8) shows that $A(k, n, \mu)$ is a convolution array; (1.9) gives a functional relationship between $\alpha(x; \mu)$ and $a(x)$ which immediately yields $x\alpha(x)$ as the compositional inverse of $x/a(x)$. Correspondingly, (1.9) with (1.4) and (1.5) yields a Lagrange inversion formula; another is given in Section 5.

A lattice path is a directed path in the Cartesian plane with vertices the lattice points (integer pairs) and with steps (directed edges) of the form $((x, y), (x + u, y + v))$. There will be various restrictions on (u, v); the set of permitted (u, v)'s is called the *step set*. A lattice path from $(0, 0)$ to (k, n) which lies strictly above the line $y = \mu x$ for $0 < x \leq k$ is called a (k, n, μ)-*path*. If we restrict to steps of the form $((x, y), (x + j, y + 1))$ with weight a_j, and if the weight of a path is the product of the weights of its steps, then we shall show that $A(k, n, \mu)$ is the sum of the weights of the (k, n, μ)-paths for $n \geq \mu k$.

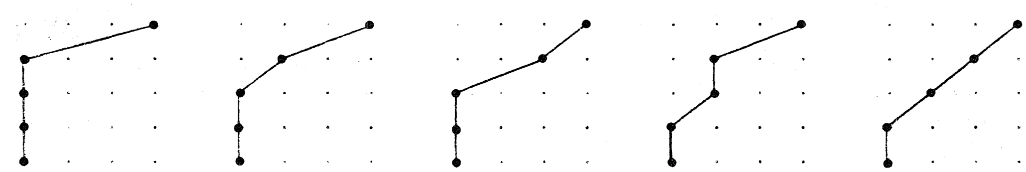

FIGURE 1
$A(3, 4, 1)$ counts the $(3, 4, 1)$-paths with step set $\{(j, 1) : j \in \mathbb{N}\}$ and $a_j = 1$ for $j \in \mathbb{N}$. [$A(3, 4, 1) = C(3, 4, 1) = \gamma_3$ of Example 2B).]

2. Examples of Recurrences and Their Solutions, the Catalan Arrays

The recurrences are defined by their initial series. $A(k, n, 0)$ and $\alpha = A(k, k + 1, 1)$ (often in [25]) are found from (1.4) and (1.5). For reference $b(x)$, $c(x)$, etc., $B(k, n, \mu)$, $C(k, n, \mu)$, etc., and β, γ, etc. denote the specific $a(x)$, $A(k, n, \mu)$, and α. Here "PA," "CA," and "CN" abbreviate Pascal's array, Catalan's array (see [21], [24]) and the Catalan numbers [11]: 1, 1, 2, 5, 14, 42, ... These examples are unnecessary for Sections 3 and 4.

Example 2A: $b(x) = 1 + x$. $B(k, n, 0) = \binom{n}{k}$, PA, where, for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$\binom{n}{k} = (n)(n - 1) \cdots (n - k + 1)/k! \text{ if } k > 0 \text{ and } \binom{n}{0} = 1.$$

$$B(k, n, 1) = \binom{n - 1}{k}, \text{ another PA, and } \beta_k = 1 \text{ for } k \geq 0.$$

$$B(k, n, 2) = \frac{n - 2}{n}\binom{n}{k}, \text{ CA (see Table 1).}$$

$$[x^k]\beta(x; 2) = B(k, 2k + 1, 2) = \frac{1}{2k + 1}\binom{2k + 1}{k}, \text{ CN (marked} \leftarrow \text{in Table 1).}$$

$\beta(x; 2) = 1 + x\beta^2(x; 2)$ by (1.9). The step set $\{(0, 1), (1, 1)\}$ yields $\binom{n}{k}$ as the number of $(k, n, 0)$-paths.

$n\backslash k$	0	1	2	3	4
5	1	3	2←	−2	−3
4	1	2	0	−2	−1
3	1	1←	−1	−1	0
2	1	0	−1	0	0
1	1←	−1	0	0	0
0	1	−2	2	−2	2
−1	1	−3	5	−7	9
−2	1	−4	9	−16	25

TABLE 1
A section of $B(k, n, 2)$

$n\backslash k$	0	1	2	3	4
5	1	4	9	14	14←
4	1	3	5	5←	0
3	1	2	2←	0	−5
2	1	1←	0	−2	−5
1	1←	0	−1	−2	−3
0	1	−1	−1	−1	−1
−1	1	−2	0	0	0
−2	1	−3	2	0	0

TABLE 2
A section of $C(k, n, 1)$

Example 2B: $c(x) = \sum_{k \geq 0} x^k = (1 - x)^{-1}$.

$$C(k, n, 0) = \binom{n + k - 1}{k}, \text{ PA.}$$

$C(k, n, 1)$, CA, are the ballot numbers [3], [16], see Table 2.

$$\gamma_k = \frac{1}{k + 1}\binom{2k}{k}, \text{ CN (marked ← in Table 2).}$$

$x\gamma^2(x) - \gamma(x) + 1 = 0$ by (1.9). $C(k, n, 0)$ counts the $(k, n, 0)$-paths with step set $\{(0, 1), (1, 0)\}$ with $(0, 1)$ as the initial step. $C(k, n, 0)$ also counts the $(k, n, 0)$-paths with step set $\{(j, 1) : j \in \mathbb{N}\}$; see Figure 1.

Example 2C: $\hat{b}(x) = 1 + x^\nu = b(x^\nu)$, where $\nu \in \mathbb{N}$, $\nu > 0$.

$$\hat{B}(k, n, 0) = \binom{n}{K} \text{ if } k = \nu K \text{ and } = 0 \text{ otherwise, a variant PA.}$$

$$\hat{B}(k, n, 1) = \frac{n - k}{n}\binom{n}{K} = \frac{n - \nu K}{n}\binom{n}{K} = B(K, n, \nu) \text{ if } k = \nu K \text{ and } = 0 \text{ otherwise.}$$

Example 2D: $\check{c}(x) = 1 + x^2$.

$$\check{C}(k, n, 0) = \binom{n}{k/2} \text{ for } k \text{ even and } = 0 \text{ otherwise, a variant PA.}$$

$$\check{\gamma}_k = \frac{1}{k + 1}\binom{k + 1}{k/2} = 1, 0, 1, 0, 2, \ldots, \text{ zero-interspersed CN.}$$

Example 2E: $\tilde{c}(x) = 1 + 2x + x^2$.

$$\tilde{C}(k, n, 0) = \binom{2n}{k}, \text{ a PA with every other row missing.}$$

$$\tilde{C}(k, n, 1) = B(k, 2n, 2).$$

$$\tilde{\gamma}_k = B(k, 2k + 2, 2) = \frac{1}{k + 1}\binom{2k + 2}{k}, \text{ CN with first entry missing.}$$

Note that $[x^k]\tilde{\gamma}(x) = [x^k]\beta^2(x; 2) = [x^k]x^{-1}(\beta(x; 2) - 1)$.

Example 2F: $\tilde{m}(x) = 1 + x + x^2$.

$$\tilde{M}(k, n, 0) = [x^k](1 - x^3)(1 - x)^{-1} = \sum_{i, j: 3j + i = k} (-1)^j \binom{n}{j} \binom{n + i - 1}{i}.$$

Also

$$\tilde{M}(k, n, 0) = [x^k]((1 + x^2) + x)^n = \sum_{i, j: n - j + 2i = k} \binom{n}{j} \binom{j}{i}.$$

$\tilde{\mu}_k = \tilde{M}(k, k + 1, 1) = 1, 1, 2, 4, 9, 21, \ldots$, named for Motzkin [17], who found them to count the ways of placing nonintersecting cords between k points on a circle. Note

$$\tilde{\mu}_k = \frac{1}{k + 1}[x^k]((1 + x^2) + x)^{k+1} = \frac{1}{k + 1} \sum_{j \geq 0} \binom{k + 1}{j} [x^{j-1}](1 + x^2)^j$$

$$= \sum_{j \geq 0} \binom{k}{j - 1} \check{\gamma}_{j-1}$$

and

$$\tilde{\gamma}_k = \frac{1}{k + 1}[x^k]((1 + x + x^2) + x)^{k+1} = \frac{1}{k + 1} \sum_{j \geq 0} \binom{k + 1}{j} [x^{j-1}](1 + x + x^2)^j$$

$$= \sum_{j \geq 0} \binom{k}{j - 1} \tilde{\mu}_{j-1}.$$

See Example 7A and [4], [5], and [14].

3. Lattice Path Analysis for Propositions 1 and 2

We use weighted paths with steps of the form $((x, y), (x + j, y + 1))$, denoted by $\langle j \rangle$ and assigned the weight a_j, $j \geq 0$. $\langle j_{1:n} \rangle$ denotes an arbitrary path $\langle j_1 \rangle \langle j_2 \rangle \cdots \langle j_n \rangle$ and $\prod_{i=1}^n a_{j_i}$ denotes its weight. $P(k, n, \mu)$ denotes the set of all (k, n, μ)-paths and $|P(k, n, \mu)|$ denotes the sum of the weights of the paths in $P(k, n, \mu)$. When appropriate, $|A|$ denotes the cardinality of A.

Since all (k, n, μ)-paths pass through $\{(k - j, n - 1) : 0 \leq j \leq k\}$ exactly once,

$$|P(k, n, \mu)| = \sum_{\langle j_{1:n} \rangle \in P(k, n, \mu)} \prod_{i=1}^{n} a_{j_i}$$

$$= \sum_{0 \leq j \leq k} \sum_{\langle j_{1:n-1} \rangle \in P(k - j, n - 1, \mu)} \prod_{i=1}^{n-1} a_{j_i} a_j$$

$$= \sum_{0 \leq j \leq k} a_j |P(k - j, n - 1, \mu)|.$$

$|P(0, 0, \mu)| = 1$ and $|P(k, \mu k, \mu)| = 0$. Hence, $|P(k, n, \mu)|$ satisfies (1.1), (1.2), and (1.3) for $n \geq \mu k$. Thus,

Proposition 6: $|P(k, n, \mu)| = A(n, k)$ for $n \geq \mu k$. (3.1)

We next determine $|P(k, n, \mu)|$ by a "radiation" scheme, which extends the method used by Dvoretzky and Motzkin [7] on Barbier's ballot problem of counting the (k, n, μ)-paths with a two element step set. Grossman [13], [16] reformulated their technique as "penetrating analysis." See also [18].

Each path $\langle j_{1:n} \rangle \in P(k, n, 0)$ determines a sequence of cyclic permutations, each being a path in $P(k, n, 0)$:

$$\langle j_{1:n} \rangle, \ \langle j_{2:1} \rangle = \langle j_2 \rangle \langle j_3 \rangle \cdots \langle j_n \rangle \langle j_1 \rangle, \ \langle j_{3:2} \rangle = \langle j_3 \rangle \langle j_4 \rangle \cdots \langle j_1 \rangle \langle j_2 \rangle,$$
$$\ldots, \ \langle j_{n:n-1} \rangle = \langle j_n \rangle \langle j_1 \rangle \cdots \langle j_{n-2} \rangle \langle j_{n-1} \rangle. \quad (3.2)$$

Let p be the period of $\langle j_{1:n}\rangle$. $[\langle j_{1:n}\rangle]$ denotes the cyclic permutation class $\{\langle j_{1:n}\rangle, \langle j_{2:1}\rangle, \ldots, \langle j_{p:p-1}\rangle\}$, the set of distinct paths in (3.2).

Let $\langle j_{1:n}\rangle$ be a fixed [fixed until (3.5)] path in $P(k, n, 0)$ of period p.

$$\langle j_{1:n}\rangle = \langle j_1\rangle\langle j_2\rangle \cdots \langle j_p\rangle \in P(kp/n, p, 0),$$

is the initial subpath of $\langle j_{1:n}\rangle$. Each path in $[\langle j_{1:n}\rangle]$ is the concatenation of n/p copies of a cyclic permutation of $\langle j_{1:p}\rangle$. Distinguish the steps in $\langle j_{1:p}\rangle$ by their index. Thus, each step in $\langle j_{1:p}\rangle$ initiates a unique path in $[\langle j_{1:n}\rangle]$. A step $\langle j\rangle$ is called a zero step if $j = 0$; otherwise it is called positive. Let $J_+ = \{i : j_i > 0 \text{ and } i \le p\}$, the index set of the positive distinguished steps of $\langle j_{1:p}\rangle$. See Figure 2.

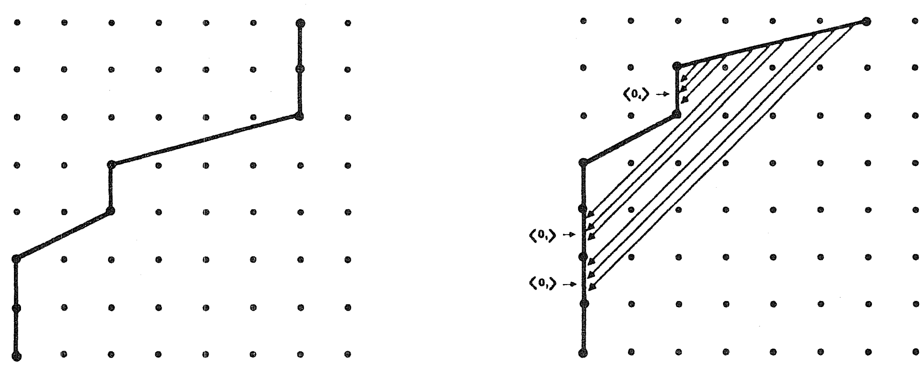

FIGURE 2

If $\mu = 1$ and $\langle j_{1:7}\rangle = \langle 0_1\rangle\langle 0_2\rangle\langle 2_3\rangle\langle 0_4\rangle\langle 4_5\rangle\langle 0_6\rangle\langle 0_7\rangle$ (the subscripts distinguish the steps), then $J_+ = \{3, 5\}$. $Z_3 = \{2\}$, $Z_5 = \{1, 4, 7\}$, $Z_c = \{6\}$. $\langle j_{1:7}\rangle$ and $\langle j_{6:5}\rangle$ are shown with 3 subswaths of rays indicated on $\langle j_{6:5}\rangle$.

In the following $n \ge \mu k$. Fix $i \in J_+$ and consider the geometrical configuration of $\langle j_{i+1:i}\rangle \in [\langle j_{1:p}\rangle]$ where each step is a line segment ($i + 1$ is replaced by 1 if $i = p$). From the points on the last step of $\langle j_{i+1:i}\rangle$, namely $\langle j_i\rangle$, draw rays in the direction of the ray from (0, 0) through $(-1, -\mu)$. See Figure 2. Since the terminal vertex of $\langle j_{i+1:i}\rangle$ is above or on the line $y = \mu x$, all rays must strike and be absorbed on the right side of $\langle j_{i+1:i}\rangle$ by the zero steps, positive steps being too inclined to be hit. By examining a triangle with vertices (0, 0), $(0, \mu j - 1)$, and $(j, \mu j)$, we see that the vertical width of the swath of rays from $\langle j_i\rangle$ is $\mu j_i - 1$. This swath can be partitioned into $\mu j_i - 1$ equal parallel subswaths. Since each subswath passes between vertically adjacent lattice points, each subswath must irradiate the entire interior of a zero step. If Z_i denotes the set of indices with respect to $\langle j_{1:p}\rangle$ of the zero steps which are irradiated by the rays from $\langle j_i\rangle$, then

$$|Z_i| = \mu j_i - 1. \tag{3.3}$$

We claim that the Z_i, $i \in J_+$, are disjoint from one another. Suppose there is a zero step that is irradiated by both $\langle j_i\rangle$ and $\langle j_{i'}\rangle$, where the zero step appears earlier in, say, $\langle j_{i'+1:i'}\rangle$. But the configuration of $\langle j_{i'+1:i'}\rangle$ shows that the step $\langle j_i\rangle$ will shield this zero step from the irradiation of $\langle j_{i'}\rangle$.

Let Z_c be the index set of *clean* (nonirradiated) zero steps in $\langle j_{1:p}\rangle$. By considering the vertical and the horizontal dimensions of $\langle j_{1:p}\rangle$,

$$p = \sum_{i \in J_+} |Z_i| + |J_+| + |Z_c|$$

and

$$\mu kp/n = \mu \sum_{i \in J_+} j_i = \sum_{i \in J_+} (\mu j_i - 1) + |J_+| = \sum_{i \in J_+} |Z_i| + |J_+|,$$

and thus,

$$|Z_c| = \frac{p(n - \mu k)}{n}. \tag{3.4}$$

As noted, each path in $[\langle j_{1:n}\rangle]$ is uniquely determined by its initial step which is a distinguished step of $\langle j_{1:p}\rangle$. A path beginning with a positive step touches or is below $y = \mu x$ by the first step. A path beginning with a zero step touches $y = \mu x$ for the first time on its radiating positive step. Thus, the paths beginning with a clean zero step are precisely those belonging to $P(k, n, \mu)$. By (3.4), we have

Lemma: The number of paths in $[\langle j_{1:n}\rangle] \cap P(k, n, \mu)$ is

$$|Z_c| = \frac{p(n - \mu k)}{n} = \frac{n - \mu k}{n}|[\langle j_{1:n}\rangle]|. \tag{3.5}$$

Since every path in a cyclic permutation class has the same weight and since the classes are disjoint with union $P(k, n, 0)$,

$$A(k, n, \mu) = |P(k, n, \mu)| = \sum \frac{n - \mu k}{n}|[\langle j_{1:n}\rangle]| \prod_{i=1}^{n} a_{j_i} \quad \text{(sum over all c.p. classes)}$$

$$= \frac{n - \mu k}{n} \sum |[\langle j_{1:n}\rangle]| \prod_{i=1}^{n} a_{j_i} = \frac{n - \mu k}{n}|P(k, n, 0)|$$

$$= \frac{n - \mu k}{n} A(k, n, 0).$$

Thus, a formula for $A(k, n, \mu)$ has been constructed for $n \geq \mu k$. Simple arithmetic shows that this formula satisfies (1.1), (1.2), and (1.3) for $n \neq 0$ and $k \geq 0$; hence (1.5) is valid.

A second realization of the contribution of each cyclic permutation class to $|P(k, n, \mu)|$ establishes (1.6). By (3.3) and (3.5), the weight contributed by $[\langle j_{1:n}\rangle]$ for $n \geq \mu k$ is

$$|Z_c| \prod_{i=1}^{n} a_{j_i} = \sum_{t \in Z_c} (1 - \mu j_t) a_{j_t} \prod_{i \neq t} a_{j_i} + \sum_{t \in J_+} \left\{ \left[\sum_{s \in Z_t} (1 - \mu j_s) a_{j_s} \prod_{i \neq s} a_{j_i} \right] \right.$$

$$\left. + (1 - \mu j_t) a_{j_t} \prod_{i \neq t} a_{j_i} \right\} \quad \begin{pmatrix}\text{since } j_t = j_s = 0 \text{ for zero steps and} \\ \text{since the term in } \{\ \} \text{ is } 0\end{pmatrix}$$

$$= \sum_{t=1}^{p} (1 - \mu j_t) a_{j_t} \prod_{i \neq t} a_{j_i}.$$

Summing over all cyclic permutation classes yields

$$A(k, n, \mu) = |P(k, n, \mu)| = \sum_{\langle j_{1:n}\rangle \in P(k, n, 0)} (1 - \mu j_1) a_{j_1} \prod_{i=2}^{n} a_{j_i} \quad \text{(sum over all paths)}$$

$$= \sum_{j \ge 0} (1 - \mu j) a_j \sum_{\langle j_{2:n} \rangle} \prod_{i=2}^{n} a_{j_i}$$ [Since all paths pass through the line $y = 1$, the second sum is over all paths from $(j, 1)$ to (k, n).]

$$= \sum_{j \ge 0} (1 - \mu j) a_j |P(k - j, n - 1, 0)|$$ (weights are transition invariant)

$$= \sum_{j \ge 0} (1 - \mu j) a_j A(k - j, n - 1, 0).$$

Thus we have constructed the formula of (1.6) for $n \ge \mu k$. If $\bar{A}(k, n)$ momentarily denotes this formula, then it is easily shown that $\bar{A}(k, n)$ satisfies (1.1) for $(k, n) \in \mathbb{N} \times \mathbb{Z}$. Since $\bar{A}(k, n)$ and $A(k, n, \mu)$ agree when $n \ge \mu k$, they agree for all $(k, n) \in \mathbb{N} \times \mathbb{Z}$, yielding (1.6).

Equation (1.6) yields a nice interpretation for $A(k, n, \mu)$ on both sides of $y = \mu x$ and $n \ge 1$. Retaining the definitions of this section, reassign the weight of $(1 - \mu j)a_j$ to the initial steps $((0, 0), (j, 1))$, $j \ge 0$. Then $A(k, n, \mu)$ is the sum of the modified weights of *all unrestricted* paths from $(0, 0)$ to (n, k).

With $a'(x)$ denoting the usual formal derivative, immediately (1.6) is equivalent to (similar to a result in [1])

$$A(k, n, \mu) = [x^k](a(x) - \mu x a'(x)) a^{n-1}(x) \text{ for } n \in \mathbb{Z}. \tag{3.6}$$

4. The Proofs of Propositions 3, 4, and 5

We establish (1.7), a useful generalized Vandermonde-type convolution [10], [16]. Then, using the tractable notation of series, we reformulate both the convolution and (1.1) in terms of diagonal series.

First we give a lattice path proof of (1.7) for $m, n \ge 0$ and $m + \mu k \ge \lambda k$. Since $((x, y), (x + j, y + 1))$, $j \ge 0$, is the form of the lattice steps, any path in $P(k, m + n + \mu k, \lambda)$ must intersect the line $M = \{(j, m + \mu j) : 0 \le j \le k\}$. Since the weight of a path is invariant under translation, the sum of the weights of the paths from $(j, m + \mu j)$ to $(k, m + n + \mu k)$ which remain above M is $|P(k - j, n + \mu(k - j), \mu)|$. Hence the sum of the weights of the paths in $P(k, m + n + \mu k, \lambda)$ that pass through M *for a last time* at $(j, m + \mu j)$ is the product

$$|P(j, m + \mu j, \lambda)| |P(k - j, n + \mu(k - j), \mu)|.$$

Summing over M and putting $A(x, y, \mu) = |P(x, y, \mu)|$ yields (1.7) in this case.

Now for $m \in \mathbb{Z}$ and $n > 0$, (1.7) can be proved by induction on the value of $n + \mu k$ by observing that, for $n + \mu i - 1 < n + \mu k$,

$$\sum_{i \ge 0} a_i A(k - i, m + n + \mu i - 1 + \mu(k - i), \lambda)$$

$$= \sum_{i \ge 0} \sum_{j \ge 0} a_i A(j, m + \mu j, \lambda) A(k - i - j, n + \mu i - 1 + \mu(k - i - j), \mu)$$

$$= \sum_{j \ge 0} A(j, m + \mu j, \lambda) \sum_{i \ge 0} a_i A(k - j - i, n - 1 + \mu(k - j), \mu).$$

The case for $m \in \mathbb{Z}$ and $n \le 0$ can be proved by induction with respect to $-n$ upon noting that (1.1) yields

$$A(k, m + (n - 1) + \mu k, \mu)$$

$$= A(k, m + n + \mu k, \mu) - \sum_{j \ge 0} a_j A(k - j, m + (n - 1 + \mu j) + \mu(k - j), \mu)$$

and that $-n + 1 - \mu j \le -n$.

Equivalent to (1.8) is

$$\alpha^n(x;\ \mu) = \sum_{k\ge 0} A(k,\ \mu k + n,\ \mu)x^k \text{ for } n \in \mathbb{Z}.$$

For $n \ge 0$, this can be proved inductively since, by (1.7),

$$\alpha^{n+1}(x;\ \mu) = \alpha(x;\ \mu)\alpha^n(x;\ \mu) = \sum_{k\ge 0} A(k,\ \mu k + 1,\ \mu)x^k \sum_{k\ge 0} A(k,\ \mu k + n,\ \mu)x^k$$
$$= \sum_{k\ge 0} A(k,\ \mu k + n + 1,\ \mu)x^k.$$

The case for $n \ge 0$ and (1.7) yields

$$\alpha^{-n}(x) = \alpha^{-n}(x)\sum_{k\ge 0} A(k,\ \mu k + n,\ \mu)x^k \sum_{k\ge 0} A(k,\ \mu k - n,\ \mu)x^k$$
$$= \alpha^{-n}(x)\alpha^n(x)\sum_{k\ge 0} A(k,\ \mu k - n,\ \mu)x^k = \sum_{k\ge 0} A(k,\ \mu k - n,\ \mu)x^k.$$

As in [9], equation (1.8) has the following meaning for $n \ge \mu k$: Since each $(k,\ \mu k + n,\ \mu)$-path must sequentially intersect *for a last time* each of the lines $y = \mu x + i$ for $1 \le i \le n$, each $(k,\ \mu k + n,\ \mu)$-path is an n-fold concatenation of $(k,\ \mu j + 1,\ \mu)$-paths for various j. Correspondingly, the total weight of the $(k,\ \mu k + n,\ \mu)$-paths is a coefficient of an n-fold convolution of $\alpha(x;\ \mu)$.

Moreover, since each $(k,\ \mu k + 1,\ \mu)$-path intersects the line $y = \mu k$ only preceding its last step, the set of $(k,\ \mu k + 1,\ \mu)$-paths is the disjoint union

$$P(k,\ \mu k + 1,\ \mu) = \bigcup_{j=0}^{k} \{\langle j_{1:\mu k}\rangle\langle j\rangle : \langle j_{1:\mu k}\rangle \text{ is a } (k - j,\ \mu(k - j) + \mu j,\ \mu)\text{-path}\}$$
$$= \bigcup_{j=0}^{k} \{\langle j_{1:\mu k}\rangle\langle j\rangle : \langle j_{1:\mu k}\rangle \text{ is a } \mu j\text{-fold concatenation of various } (j,\ \mu j + 1,\ \mu)\text{-paths}\}.$$

More precisely, we have that

$$\alpha(x;\ \mu) = \sum_{k\ge 0} A(k,\ \mu k + 1,\ \mu)x^k$$
$$= \sum_{k\ge 0}\sum_{j\ge 0} a_j x^j A(k - j,\ \mu(k - j) + \mu j,\ \mu)x^{k-j}$$
$$= \sum_{j\ge 0} a_j x^j \sum_{k\ge 0} A(k - j,\ \mu(k - j) + \mu j,\ \mu)x^{k-j}$$
$$= \sum_{j\ge 0} a_j x^j \alpha^{\mu j}(x;\ \mu) = \sum_{j\ge 0} a_j (x\alpha^{\mu}(x;\ \mu))^j.$$

This establishes (1.9) since comparing coefficients shows the uniqueness.

As a consequence of (1.9), we have

Proposition 7: For each $\mu \in \mathbb{N}$, if $\alpha(x;\ \mu)$ is taken as the initial series, then $\alpha(x;\ \mu + 1)$ is the corresponding principal diagonal series.

Proof: If $\overline{\alpha}(x)$ denotes the principal diagonal for $\alpha(x;\ \mu)$, then by (1.9),

$$\overline{\alpha}(x) = \alpha(x\overline{\alpha}(x);\ \mu) = a(x\overline{\alpha}(x)[\alpha(x\overline{\alpha}(x);\ \mu)]^{\mu})$$
$$= a(x\overline{\alpha}(x)[\overline{\alpha}(x)]^{\mu}) = a(x[\overline{\alpha}(x)]^{\mu+1}).$$

But $\overline{\alpha}(x)$ must be $\alpha(x;\ \mu + 1)$ by the uniqueness in (1.9).

5. A Lagrange Inversion Formula

A common Lagrange inversion formula [3], [12], is included as it easily follows (1.4), (1.5), (1.6), and (1.9). See [3], [8], [9], [12], and [18] for more general formulas.

Proposition 8: For any initial series $a(x)$, there exists a unique series

$$\omega(x) = \sum_{k \geq 1} \omega_k x^k$$

such that $\omega(x) = xa(\omega(x))$. Moreover, if $f(x)$ is a formal Laurent series so that

$$f(x) = \sum_{k \geq t} f_k x^k \text{ for some } t \in \mathbb{Z},$$

$$[x^n]f(\omega(x)) = \begin{cases} \frac{1}{n}[x^{n-1}]f'(x)a^n(x) & \text{for } n \neq 0, \\ [x^0]f(x) + [x^{-1}]f'(x)\log(a(x)a^{-1}(0)) & \text{for } n = 0. \end{cases}$$

Proof: By (1.9), $\omega(x) = x\alpha(x)$ is the unique solution. It suffices to show the second part for $f(x) = x^k$, $k \in \mathbb{Z}$. For $n \neq 0$,

$$[x^n](x\alpha(x))^k = [x^{n-k}]\alpha^k(x) = A(n-k, n, 1) = [x^{n-k}]\frac{k}{n} a^n(x)$$
$$= \frac{1}{n}[x^{n-1}]kx^{k-1}a^n(x).$$

As noted in [12],

$$0 = [x^{-1}]\frac{d}{dx}(x^k \log(a(x)a^{-1}(0)))$$
$$= [x^{-1}]kx^{k-1}\log(a(x)a^{-1}(0)) + [x^{-1}]x^k a'(x)a^{-1}(x).$$

For $n = 0$, it follows from (3.6) that

$$[x^0]\omega^k(x) = [x^0]x^k\alpha^k(x) = [x^{-1}]\alpha^k(x) = A(-k, 0, 1)$$
$$= [x^{-k}](1 - xa'(x)a^{-1}(x)) = [x^{-1}]kx^{k-1}\log(a(x)a^{-1}(0)).$$

6. More Examples of Recurrences

Example 6A: $r(x) = 1 + (w+1)\sum_{k \geq 1} x^k = (1 + wx)(1 - x)^{-1}$.

$$R(k, n, 0) = [x^k] \sum_{j \geq 0} \binom{n}{j} w^j x^j \sum_{i \geq 0} \binom{n+i-1}{i} x^i$$
$$= \sum_{i \geq 0} w^{k-i} \binom{n}{k-i} \binom{n+i-1}{i}.$$

Note how r_k relates to a_k of Section 3. Also,

$$R(k, n, \mu) = R(k, n-1, \mu) + wR(k-1, n-1, \mu) + R(k-1, n, \mu),$$

and $R(k, n, \mu)$ is the sum of the weights of the (k, n, μ)-paths with the step set $\{(0, 1), (1, 1), (1, 0)\}$ where $(1,1)$ has weight w. As shown in [21], [22], or from $R(k, n, 0)$, we have

$$R(k, n, 1) = \sum_{j \geq 0} w^j \binom{n+k-j-1}{j} C(k-j, n-j, 1).$$

For $w = 1$, see Table 3, where

$$\rho_k = \sum_{j \geq 0} \binom{2k - j}{j} \gamma_{k-j},$$

the r-Schröder numbers, are marked. See [19], [21], and [22]. Note that

$$\rho_k = 2\sigma_k \quad (k > 0);$$

see Example 6E. $x\rho^2(x) + (wx - 1)\rho(x) + 1 = 0$ by (1.9).

$n\backslash k$	0	1	2	3	4
4	1	6	16	22←	0
3	1	4	6←	0	−22
2	1	2←	0	−6	−16
1	1←	0	−2	−4	−6
0	1	−2	0	−2	0
−1	1	−4	6	−8	10

TABLE 3
$R(k, n, 1)$ for $w = 1$

$n\backslash k$	0	1	2	3	4
4	1	3	7	11←	0
3	1	2	3←	0	−22
2	1	1←	0	−6	−28
1	1←	0	−2	−8	−24
0	1	−1	−3	−7	−15
−1	1	−2	−3	−4	−5

TABLE 4
$S(k, n, 1)$ for $w = 1$

General Example 6B: Given any sequence $w_1, w_2, w_3, \ldots$, consider the step set $\{(0, 1)\} \cup \{(j, 0) : j > 0\}$ where $(0, 1)$ has weight 1 and $(j, 0)$ has weight w_j. If $A(k, n, \mu)$ is the sum of the weights of the (k, n, μ)-paths (the initial step must be vertical), we have

$$A(k, n, \mu) = A(k, n - 1, \mu) + \sum_{j > 0} w_j A(k - j, n, \mu).$$

It follows inductively that this $A(k, n, \mu)$ satisfies (1.1) for $\{a_k\}_{k \geq 0}$ defined by $a_0 = 1$, $a_1 = w_1$, $a_2 = w_2 + w_1w_1$, $a_3 = w_3 + w_2w_1 + w_1w_2 + w_1w_1w_1$, and in general

$$a_k = \sum_{i_1 + i_2 + \cdots + i_m = k} \prod_{1 \leq t \leq m} w_{i_t}.$$

Hence,

$$a(x) = \sum_{k \geq 0} a_k x^k = 1 + \sum_{k \geq 1} \Big(\sum_{i \geq 1} w_i x^i \Big)^k = \Big(1 - \sum_{i \geq 1} w_i x^i \Big)^{-1}. \quad \text{See [26].}$$

Example 6C: In Example 6B put $w_1 = w_2 = 1$ and $w_i = 0$ for $i > 2$.

$$a(x) = (1 - x - x^2)^{-1}.$$

Thus, $a_k = 1, 1, 2, 3, 5, \ldots$, the Fibonacci numbers. See 7B.

Example 6D: In Example 6B put $w_i = 1$ for $i = \nu$ and $= 0$ otherwise.

$$\hat{c}(x) = (1 + x^\nu)^{-1} = c(x^\nu) \text{ of (2.2).}$$

$$\hat{\gamma}_k = \hat{C}(k, k + 1, 1) = C(K, k + 1, \nu) = \frac{1}{\nu K + 1} \binom{(\nu + 1)K}{K}$$

for $k = \nu K$ and $= 0$ otherwise.

Example 6E: In Example 6B put $w_i = w$ for $i > 0$:

$$s(x) = 1 + \sum_{k > 0} (w + 1)^{k-1} x^k = (1 - wx)(1 - (w + 1)x)^{-1}.$$

$$S(k,\ n,\ 0) = \sum_{j \geq 0} (-1)^j w^j (w + 1)^{k-j} \binom{n}{j} \binom{n + k - j - 1}{k - j}.$$

When $w = 1$, see Table 4 for $S(k,\ n,\ 1)$ where $\sigma_k = S(k,\ k + 1,\ 1)$, the s-Schröder numbers [23], are marked; see Example 7B.

From $(w + 1)x\sigma^2(x) - (1 + wx)\sigma(x) + 1 = 0$ [by (1.9)] and the last identity of Example 6A, one can show:

(i) $(w + 1)(\sigma(x) - 1) = \rho(x) - 1$,

(ii) $(1 + wx\rho(x)) = (1 - wx\sigma(x))^{-1}$, and

(iii) $\sigma(x) = (1 - x\rho(x))^{-1}$.

These are illustrated in Tables 3 and 4: (i) relates the principal diagonals. (ii) and (iii) relate the partial row sums in the triangle above the zeros in one array to the principal diagonal in the other, as generalized in the following:

Proposition 9: If $t_n = \sum_{k=0}^{n-1} w^{n-k} A(k,\ n,\ 1) = \sum_{j \geq 0} w^j A(n - j,\ n,\ 1)$, a weighted partial row sum, then $t(x) = \sum_{n \geq 0} t_n x^n = (1 - wx\alpha(x))^{-1}$.

Proof: $t(x) = \sum_{n \geq 0} \sum_{j \geq 0} w^j A(n - j,\ n,\ 1)x^n = \sum_{j \geq 0} w^j x^j \sum_{n \geq 0} A(n - j,\ n - j + j,\ 1)^{n-j}$

$$= \sum_{j \geq 0} w^j x^j \alpha^j(x).$$

This extends a result in [20].

7. Enumerating Plane Trees

Informally, a rooted plane tree is an unlabeled tree which is oriented in the plane so that it branches upward from a root (a distinguished vertex which need not be univalent) to the leaves. Two plane trees are equal if one can be continuously transformed into the other in the plane so that the nonroot vertices remain above the level of the root. A more formal definition is given by Klarner [14] and [16]. A planted plane tree is a plane tree with univalent root. See Figure 3.

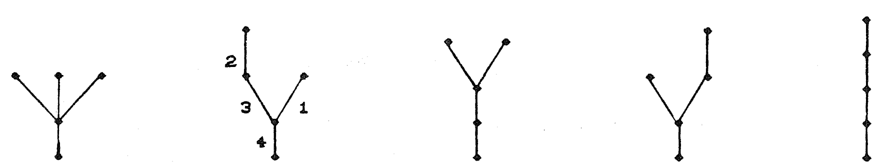

FIGURE 3
This illustrates the planted plane trees with 4 edges and no degree restriction. These trees are listed as they correspond to the paths of Figure 1 under Bijection A. The numbers indicate the order of growth.

Here we enumerate rooted plane trees with vertex degree restrictions by establishing bijections between the trees and previously counted lattice paths. Equivalently, one can establish directly a recurrence for the tree counts in the form of (1.1). The following examples are enumerated by other methods in [4], [5], [7], [12], [14], [15], [16], and [22]. One common method for planted plane trees is to establish a functional equation for a generating function and then solve the equation perhaps, and not surprisingly, by the Lagrange inversion formula as in [12].

Bijection A: Counting rooted plane trees with respect to the number of edges

Let $T(n, \rho)$ be the set of such trees with <u>n edges</u> and root degree ρ. <u>*Let D*</u> be a specified set of permitted degrees for <u>the nonroot vertices</u>. Let $P(n - \rho, n; D)$ be the set of $(n - \rho, n, 1)$-paths with step set

<u>$\{(j, 1) : j + 1 \in D\}$</u>.

A bijection from $P(n - \rho, n; D)$ to $T(n, \rho)$ is defined inductively. The trivial zero-length path in $P(0, 0; D)$ corresponds to the tree consisting of just a root. A path in $P(n - \rho, n; D)$ and the corresponding tree in $T(n, \rho)$ can be extended in two ways. (i) The path can be extended to a path in $P(n - \rho, n + 1; D)$ by attaching a new step $(0, 1)$, while the corresponding tree is extended to a tree in $T(n + 1, \rho + 1)$ by grafting a new left-most edge to the root. Such a new step corresponds to a new leaf. (ii) For <u>$j + 1 \in D$ and $j \leq \rho$</u>, the path can be extended to a path in <u>$P(n - \rho + j, n + 1; D)$</u> by attaching the <u>*step* $(j, 1)$</u> while the corresponding tree is extended to a tree in <u>$T(n + 1, \rho - j + 1)$</u> by cutting at the root *the* j left-most incident edges and then grafting the lower vertices of these edges to the upper vertex of a new left-most edge incident to the root. Thus, a <u>$(j, 1)$ *step*</u> corresponds to a new vertex of <u>*degree* $j + 1$</u>. Hence,

$$|T(n, \rho)| = A(n - \rho, n, 1) = \frac{\rho}{n}[x^{n-\rho}]\Big(\sum_{j+1 \in D} x^j\Big)^n.$$

Bijection B: Counting rooted plane trees with respect to the number of leaves

Modify the scheme of Bijection A by replacing the underlined phrases sequentially by: n leaves; let D $(2 \notin D)$; $\{(0, 1)\} \cup \{(j, 0) : j + 2 \in D\}$; $j + 2 \in D$ and $j \leq \rho - 1$; $P(n - \rho + j, n; D)$; step $(j, 0)$; $T(n, \rho - j)$; the $j + 1$; $(j, 0)$ step; degree $j + 2$. Thus, by Example 6B with $w_j = 1$ if $j + 2 \in D$ and $= 0$ otherwise,

$$|T(n, \rho)| = A(n - \rho, n, 1) = \frac{\rho}{n}[x^{n-\rho}]\Big(1 - \sum_{j+2 \in D} x^j\Big)^{-n}.$$

Example 7A: Applications of Bijection A

If $D = \mathbb{N} - \{0\}$, no degree restriction, $|T(n, 1)| = C(n - 1, n, 1) = \gamma_{n-1}$; see Example 2B and Figure 3. For $D = \{1, 3\}$, trivalent planted trees, $|T(n, 1)| = \check{C}(n - 1, n, 1) = \check{\gamma}_{n-1}$ of 2D. For $D = \{1, \nu + 1\}$, use 2C. For $D = \{1, 2, 3\}$, no vertex has degree greater than 3, $|T(n, 1)| = \tilde{M}_1(n - 1, n, 1) = \tilde{\mu}_{n-1}$; see 2F.

If $D = \mathbb{N} - \{0, 2\}$, no bivalent nonroot vertices, let $d(x) = (1 - x)^{-1} - x$.

$$|T(n, \rho)| = \frac{\rho}{n}[x^{n-\rho}]d^n(x) = \frac{\rho}{n}\sum_{i \geq 0}(-1)^{n-i}\binom{n}{i}\binom{2i - \rho - 1}{i - \rho}.$$

$$|T(n, 1)| = \sum_{i \geq 0}(-1)^{n-i}\binom{n - 1}{i - 1}\delta_{i-1}.$$

$(x + 1)\delta(x) = (1 - x\delta(x))^{-1}$ by (1.9). One can show

$$(x + 1)(\delta(x) - 1) = x(\tilde{\mu}(x) - 1) \quad (\text{thus, } \delta_{n-1} + \delta_n = \tilde{\mu}_{n-1},\ n > 0).$$

Therefore, by Proposition 9,

$$\sum_{k=0}^{n-1} D(k, n, 1) = \tilde{\mu}_{n-1}.$$

Hence

$$\sum_{\rho \geq 1} |T(n, \rho)| = \tilde{\mu}_{n-1},$$

as in [4].

Example 7B: Applications of Bijection B

An immediate source for such trees is the problem [3] of counting the ways to bracket n nonassociative, noncommutative factors so that the number of factors associated by a pair of brackets is restricted to some set B. If $B = \{2\}$, we have the problem of Catalan [2], 1838. There is a simple bijection between the usual pairwise bracketings on n factors and the planted plane trees with $D = \{1, 3\}$ and n leaves. For $D = \{1, 3\}$,

$$|T(n, \rho)| = \frac{\rho}{n}[x^{n-\rho}](1 - x)^{-n}.$$

$|T(n, 1)| = \gamma_{n-1}$, the appropriately named sequence of Example 2B.

If $B = \mathbb{N} - \{0, 1\}$, we have the problem of Schröder [23], 1870. If $n = 4$, the bracketings are

$$(a(b(cd))),\ (a((bc)d)),\ (a(bcd)),\ (((ab)c)d),\ ((a(bc))d),$$

$$((abc)d),\ ((ab)(cd)),\ (a(bc)d),\ (ab(cd)),\ ((ab)cd),\ (abcd).$$

There is a simple bijection between the unrestricted bracketings on n factors and the planted trees with n leaves and no bivalent vertices. For $D = \mathbb{N} - \{0, 2\}$, refer to Example 6E with $w = 1$:

$$|T(n, \rho)| = S(n - \rho, n, 1) \quad \text{and} \quad |T(n, 1)| = \sigma_{n-1}.$$

If $D = \{1, \nu + 2\}$,

$$|T(n, 1)| = \hat{C}(n - 1, n, 1) = \hat{\gamma}_{n-1};$$

refer to Example 6D. If $D = \{1, 3, 4\}$, refer to Example 6C.

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FIBONACCI-LIKE MATRICES

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It is well known that the powers of the matrix

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

are matrices of the form

$$\begin{bmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{bmatrix}$$

whose entries are the Fibonacci numbers. If these matrices are normalized by dividing by F, so that the first entry is always 1, then the resulting sequence converges to

$$\begin{bmatrix} 1 & f \\ f & f^2 \end{bmatrix} \tag{1}$$

where $f = (1 + \sqrt{5})/2$ is the golden ratio.

Moore [2] noticed that if the 2 in M is replaced by $1 + x$ and the same procedure (taking powers of M and normalizing to obtain a 1 in the first entry) is performed then the resulting sequence seems to converge to a matrix of the same form with $f = (x + \sqrt{x^2 + 4})/2$. These observations naturally suggest the following questions.

1. If we start with any symmetric 2×2 matrix M, with positive integral entries, does a similar phenomenon occur and, if so, what is the corresponding value of f?

2. If there is convergence with $f = (a + \sqrt{d})/b$, what are the values of d that can occur, i.e., in what quadratic number fields do we find such f?

Since we normalize at each step, we can assume that

$$M = \begin{bmatrix} 1 & y \\ y & 1 + x \end{bmatrix}.$$

Diagonalizing M gives $M = PDP^{-1}$ with

$$P = \begin{bmatrix} 1 & 1 \\ (x + \sqrt{d})/2y & (x - \sqrt{d})/2y \end{bmatrix}, \quad D = \begin{bmatrix} (x + 2 + \sqrt{d})/2 & 0 \\ 0 & (x + 2 - \sqrt{d})/2 \end{bmatrix},$$

where $d = x^2 + 4y^2$. Thus we have $M^n = PD^nP^{-1}$. When we normalize D to make the leading entry 1, the second diagonal entry is less than one and so the sequence of its powers converges to

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and so $PD\,P^{-1}$, when normalized, can be seen to converge to a matrix of the form (1) with $f = (x + \sqrt{d})/2y$, where $d = x^2 + 4y^2$.

The set of d that can be written as $x^2 + 4y^2$ can be easily found, since we know what numbers can be written as the sum of two squares. (A positive integer is the sum of two squares if, when factored, all its prime factors congruent to 3 modulo 4 occur with even exponent, see, e.g., [1].) If d is odd, then d is the sum of two squares if and only if it is of the form $x^2 + (2y)^2 = x^2 + 4y^2$, since one of the terms must be even. If d is even, then $d = x^2 + 4y^2$ if and only if $d = 4m$, where m can be written as the sum of two squares. Since d is even, x is even, and so d is divisible by 4. The 4 can then be factored out giving m as the sum of two squares. The converse is also easy. Thus, d is of the form $x^2 + 4y^2$ exactly if, when factored, all its odd prime factors congruent to 3 modulo 4 occur with even exponent and 2 does not occur with exponent 1.

Acknowledgment

The author would like to thank the editor for pointing out references [3], [4], and [5] which are also concerned with Moore's conjecture.

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REPRESENTATIONS FOR REAL NUMBERS VIA k^{th} POWERS OF INTEGERS

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Introduction

The three related classical series for representing real numbers as the sums of reciprocals of integers were all studied toward the end of the nineteenth century. These are, respectively, the series of Sylvester, Engel, and Lüroth (see Perron [1]). More precisely, *given any real number A there exist three (different) sequences of integers $\{a_i\}$ such that*

$$\text{(i)} \quad A = a_1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

where $a_1 \geq 2$, $a_{i+1} \geq a_i(a_i - 1) + 1$ for $i \geq 1$,

$$\text{(ii)} \quad A = a_0 + \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots,$$

where $a_1 \geq 2$, $a_{i+1} \geq a_i$ for $i \geq 1$,

$$\text{(iii)} \quad A = a_0 + \frac{1}{a_1} + \frac{1}{(a_1 - 1)a_1} \cdot \frac{1}{a_2} + \frac{1}{(a_1 - 1)a_1(a_2 - 1)a_2} \cdot \frac{1}{a_3} + \cdots,$$

where $a_i \geq 2$ for $i \geq 1$.

Observe that as we move from the Sylvester series (i) to the Lüroth series (iii), the denominators in the expansion become increasingly more complex while at the same time the growth conditions on the digits a_i become simpler. We now generalize the expansions in (i) and (ii) above, to obtain new representations for real numbers that depend on a power $k > 0$. These new representations have the desirable property of having terms only slightly more complex than in (i) and (ii) above, yet their digits need satisfy only mild growth conditions. Two different sets of algorithms leading to results of the types mentioned are considered. We state the main results in the case where the digits a_i grow least.

Given any fixed real $k \geq 1$ and any real number A, there exist sequences of integers $\{a_i\}$ such that

$$\text{(i)} \quad A = a_0 + \frac{1}{a_1^k} + \frac{1}{a_2^k} + \frac{1}{a_3^k} + \cdots,$$

where $a_{i+1} \geq a_i \geq 2$ for $i \geq 1$, and for i sufficiently large,

$$a_i + 1 \leq a_{i+1} \leq 2^{1/k} a_i + 1,$$

$$\text{(ii)} \quad A = a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1 a_2)^k} + \frac{1}{(a_1 a_2 a_3)^k} + \cdots,$$

where $a_1 = 2$, $1 \leq a_i \leq 2$ for $i \geq 2$, and $a_i = 2$ infinitely often,

(iii) $$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1^k a_2} + \frac{1}{(a_1 a_2)^k a_3} + \cdots,$$

where $a_1 = 2$, $1 \le a_i \le 2$ for $i \ge 2$, and $a_i = 2$ infinitely often.

Since only the digits 1 and 2 are used, the representations (ii) and (iii) above could be regarded as being analogous in some fashion to the binary representation for real numbers. Expansions of the above form where the digits have no upper bounds are also considered. We note in particular that, by setting $k = 1$ in the above results, we obtain expansions for real numbers with the same form as the Sylvester and Engel series but whose digits are considerably smaller. In addition, when k is a positive integer, *rational* numbers have representations of types (ii) and (iii) above for which the digits a_i become *periodic*. This condition is analogous to that of the Lüroth series when A is rational.

The paper is set out as follows. In Section 2, we consider k^{th} power analogues of the Sylvester series. In Section 3, we consider k^{th} power analogues of the Engel series. Finally, in Section 4, we consider k^{th} power expansions that are related to a simplified version of the Lüroth series.

For convenience we introduce the following notational conventions. The lower case letters a_i and a_n denote *integers* throughout the paper. Furthermore, unless otherwise stated, the lower case letter k represents a positive real number.

2. Generalizations of Sylvester Series

We introduce two different algorithms that lead to a k^{th} power generalization of the series of Sylvester. The first coincides with the ordinary Sylvester algorithm for $k = 1$. The second leads to a restricted growth of the digits in all cases, including $k = 1$.

Theorem 2.1: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1^k} + \frac{1}{a_2^k} + \frac{1}{a_3^k} + \cdots,$$

where:

if $k > 1$, then $a_{i+1} \ge a_i \ge 2$ for $i \ge 1$, and for i sufficiently large, $a_{i+1} \ge a_i + 1$,

if $0 < k \le 1$, then $a_{i+1} \ge a_i(a_i - 1) + 1$ for $i \ge 1$, $a_1 \ge 2$.

Proof: In order to obtain this result, we introduce the following algorithm:

Given any real number A, let $A_1 = A - a_0$, $0 < A_1 \le 1$.

Then we recursively define

$$a_n = \left[\frac{1}{A_n^{1/k}}\right] + 1 \quad \text{for } n \ge 1,\ A_n > 0,$$

where

$$A_{n+1} = A_n - \frac{1}{a_n^k} \quad \text{for } A_n > 0.$$

First, repeated application of the algorithm yields

$$A = a_0 + A_1 = a_0 + \frac{1}{a_1^k} + A_2 = \cdots = a_0 + \frac{1}{a_1^k} + \frac{1}{a_2^k} + \cdots + \frac{1}{a_n^k} + A_{n+1}.$$

Now $a_n = [1/A_n^{1/k}] + 1$ implies that, for $0 < A_n$, $(a_n - 1)^k \le 1/A_n < a_n^k$. Thus,

$$A_n > \frac{1}{a_n^k},$$

and provided $a_n \ge 2$ $(0 < A_n \le 1)$

$$A_n \le \frac{1}{(a_n - 1)^k}.$$

Now $0 < A_1 \le 1$ implies $a_1 \ge 2$ and

$$A_2 = A_1 - \frac{1}{a_1^k} > \frac{1}{a_1^k} - \frac{1}{a_1^k} = 0.$$

Continuing this process inductively we see that $A_n > 0$ for all n. Furthermore, since $\{A_n\}$ is a strictly decreasing sequence of positive values, we deduce that $a_{n+1} \ge a_n \ge 2$ for $n \ge 1$. Therefore,

$$A_{n+1} = A_n - \frac{1}{a_n^k} \le \frac{1}{(a_n - 1)^k} - \frac{1}{a_n^k} = \frac{a_n^k - (a_n - 1)^k}{(a_n - 1)^k a_n^k}.$$

Thus,

$$a_{n+1}^k > \frac{1}{A_{n+1}} \ge \frac{(a_n - 1)^k a_n^k}{a_n^k - (a_n - 1)^k}.$$

In the case $0 < k \le 1$, $a_n^k - (a_n - 1)^k \le 1$; so

$$a_{n+1} \ge (a_n - 1)a_n + 1, \; n \ge 1.$$

In the case $k > 1$, we have

$$a_{n+1} \ge a_n + 1$$

provided

$$\frac{(a_n - 1)^k}{a_n^k - (a_n - 1)^k} \ge 1.$$

This is true if $a_n \ge 2^{1/k}/(2^{1/k} - 1)$. and if $A_n \le (2^{1/k} - 1)^k/2$. On the contrary, suppose that

$$2 \le a_n \le \left[\frac{2^{1/k}}{2^{1/k} - 1}\right] = c(k),$$

say. Then

$$A_{n+1} = A_n - \frac{1}{a_n^k} \le A_n - \frac{1}{(c(k))^k}.$$

Now, either $A_{n+1} \le (2^{1/k} - 1)^k/2$ or

$$A_{n+2} = A_{n+1} - \frac{1}{a_{n+1}^k} \le A_n - \frac{2}{(c(k))^k} \quad \text{if } a_{n+1} \le c(k).$$

Thus, at each stage, A_{n+i} is decreasing by at least a fixed constant, so after a finite number of steps we must reach a stage at which $A_j \le (2^{1/k} - 1)^k/2$. The result for $k > 1$ now follows, since

$$A_{j+n} \le \frac{1}{(a_{j+n} - 1)^k} \le \frac{1}{(c(k) + n - 1)^k} \to 0 \text{ as } n \to \infty.$$

For $0 < k < 1$, $a_{n+1} \ge (a_n - 1)a_n + 1 \ge a_n + 1$ as $a_n \ge 2$; hence,

$$A_n \leq \frac{1}{(n+1)^k} \to 0 \text{ as } n \to \infty.$$

A slight modification to the algorithm leads to the following results.

Theorem 2.2: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1^k} + \frac{1}{a_2^k} + \frac{1}{a_3^k} + \cdots,$$

where

$$a_n + 1 \geq a_n \geq 2 \text{ for } n \geq 1,$$

and for n sufficiently large,

$$a_n + 1 \leq a_{n+1} \leq 2^{1/k}a_n + 1.$$

Proof: We use the same algorithm as previously, except that now we let

$$a_n = \left[\left(\frac{2}{A_n}\right)^{1/k}\right] + 1 \text{ for } n \geq 1,\ A_n > 0.$$

As before, $a_{n+1} \geq a_n \geq 2$ for $n \geq 1$, but now

$$\frac{2}{a_n^k} < A_n \leq \frac{2}{(a_n - 1)^k} \text{ if } 0 < A_n \leq 1.$$

Therefore, for $n \geq 1$,

$$A_{n+1} = A_n - \frac{1}{a_n^k} > \frac{1}{a_n^k}$$

and

$$a_{n+1} = 1 + \left[\left(\frac{2}{A_{n+1}}\right)^{1/k}\right] < 1 + 2^{1/k}a_n.$$

Also,

$$A_{n+1} \leq \frac{2}{(a_n - 1)^k} - \frac{1}{a_n^k} = \frac{2a_n^k - (a_n - 1)^k}{a_n^k(a_n - 1)^k}.$$

So

$$a_{n+1} > \frac{2}{A_{n+1}} \geq \frac{2a_n^k(a_n - 1)^k}{2a_n^k - (a_n - 1)^k},$$

and we have $a_{n+1} \geq a_n + 1$, provided that

$$\frac{2(a_n - 1)^k}{2a_n^k - (a_n - 1)^k} \geq 1.$$

This is easily seen to be true if $a_n \geq 3^{1/k}/(3^{1/k} - 2^{1/k})$ and if

$$A_n < \frac{2(3^{1/k} - 2^{1/k})^k}{3}.$$

An argument similar to that used in the previous proof shows that these conditions must hold after at most a finite number of steps. Thereafter, as before, $A_n \to 0$ as $n \to \infty$, and the result follows.

We note in particular that by setting $k = 1$ in Theorem 2.2 we get an analogue of the Sylvester series

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots,$$

where $a_{n+1} \geq a_n \geq 3$ for $n \geq 1$ and for n sufficiently large

$$a_n + 1 \leq a_{n+1} \leq 2a_n + 1.$$

This is a much milder growth condition than the condition $a_{n+1} \geq a_n(a_n - 1) + 1$ for $n \geq 1$ of Sylvester. However, under these weaker conditions we no longer obtain uniqueness for the expansions. For example, if we let instead

$$a_n = \left[\left(\frac{m}{A_n}\right)^{1/k}\right] + 1.$$

where $m > 1$ is a fixed constant, we obtain a new expansion for A where the digits satisfy very similar growth conditions.

As a particular case of these expansions, we note that, by definition, the Riemann zeta function $\zeta(k)$ for $k > 1$ has expansion

$$\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots .$$

Euler's well-known formula for $\zeta(2m)$, $m = 1, 2, 3, \ldots$, then yields

$$2^{2m-1}B_m \frac{\pi^{2m}}{(2m)!} = 1 + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \frac{1}{4^{2m}} + \cdots,$$

where B_m is a Bernoulli number.

3. A Generalization of the Engel Series

Using algorithms essentially similar to those introduced in Section 2, we obtain k^{th} power analogues of the Engel series.

Theorem 3.1: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1a_2)^k} + \frac{1}{(a_1a_2a_3)^k} + \cdots,$$

where:

if $k > 1$, then $a_1 \geq 2$, $a_i \geq 1$ for $i \geq 1$, and $a_i \geq 2$ infinitely often,

if $0 < k \leq 1$, then $a_{i+1} \geq a_i \geq 2$ for $i \geq 1$.

Proof: We make use of the following algorithm. Given any real number A, let $A_1 = A - a_0$, $0 < A_1 \leq 1$. Then we recursively define

$$a_n = \left[\frac{1}{A_n^{1/k}}\right] + 1 \text{ for } n \geq 1,\ A_n > 0,$$

where

$$A_{n+1} = a_n^k A_n - 1 \text{ for } A_n > 0.$$

First, repeated application of the above algorithm yields

$$\begin{aligned} A &= a_0 + A_1 = a_0 + \frac{1}{a_1^k} + \frac{A_2}{a_1^k} = \cdots \\ &= a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1a_2)^k} + \cdots + \frac{1}{(a_1a_2 \ldots a_n)^k} + \frac{A_{n+1}}{(a_1a_2 \ldots a_n)^k}. \end{aligned}$$

Again, if $A_n > 0$, we have $A_n^k > 1/a_n$, and if also $a_n \geq 2$, then

$$A_n \leq \frac{1}{(a_n - 1)^k}$$

Now $0 < A_1 \le 1$ implies $a_1 \ge 2$ and $A_2 = a_1^k A - 1 > 0$, thus $a_2 \ge 1$. Continuing the process inductively we see that $A_n > 0$ and hence $a_n \ge 1$ for all n. We consider now the case $k > 1$. Suppose $a_n \ge 2$, then

$$A_{n+1} = a_n^k A_n - 1 \le \frac{a_n^k}{(a_n - 1)^k} - 1 = \left(1 + \frac{1}{a_n - 1}\right)^k - 1 \le 2^k - 1,$$

since we have assumed $a_n \ge 2$. Now, if $A_{n+1} \le 1$, then $a_{n+1} \ge 2$. Otherwise $a_{n+1} = 1$ and $A_{n+2} = A_{n+1} - 1$. Continuing this process, we see that after at most $[2^k - 1]$ steps with

$$a_{n+i} = 1,\ A_{n+i+1} = A_{n+i} - 1 = A_{n+1} - i,$$

we must reach a stage at which

$$A_{n+j} \le 1 \quad \text{and} \quad a_{n+j} \ge 2.$$

We deduce that the sequence $\{A_n\}$ is bounded above by $2^k - 1$ for all n. Furthermore, there exists a sequence of integers $n_1 = 0 < n_2 < n_3 < \ldots$ such that

$$0 < A_{n_i+1} \le 1,\ a_{n_i+1} \ge 2,$$

and $a_n = 1$ for all other $n > 1$. Then

$$0 < \frac{A_{n_i+1}}{(a_1 a_2 \ldots a_{n_i})^k} \le \frac{1}{2^{k(i-1)}},$$

and so $S_{n_i} \to A$ as $i \to \infty$, where

$$S_n = a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1 a_2)^k} + \cdots + \frac{1}{(a_1 \ldots a_n)^k} = A - \frac{A_{n+1}}{(a_1 \ldots a_n)^k}.$$

Now let $n_{i-1} \le n < n_i$. Then $S_{n_{i-1}} \le S_n < S_{n_i}$, and $n \to \infty$ iff $i \to \infty$. So $S_n \to A$ as $n \to \infty$, i.e., the series converges. For the case $0 < k \le 1$, if $a_n \ge 2$, then

$$a_{n+1}^k > \frac{1}{A_{n+1}} \ge \frac{(a_n - 1)^k}{a_n^k - (a_n - 1)^k}, \text{ since } A_{n+1} \le \frac{a_n^k}{(a_n - 1)^k} - 1.$$

Now for $k \le 1$, $a_n^k - (a_n - 1)^k \le 1$ and since $a_1 \ge 2$ we deduce that $a_{n+1} \ge a_n \ge 2$ for all $n \ge 1$. Thus,

$$\frac{A_{n+1}}{(a_1 \ldots a_n)^k} \le \frac{1}{2^{kn}} \to 0 \text{ as } n \to \infty,$$

and again the series converges.

This result gives the ordinary Engel series for $k = 1$.

It is possible to further restrict the growth of the digits a_i, so that for $i \ge 1$ they need only take on the values $a_i = 1$ and $a_i = 2$, for any $k \ge 1$.

Theorem 3.2: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1 a_2)^k} + \frac{1}{(a_1 a_2 a_3)^k} + \cdots,$$

where:

If $k \ge 1$, then $a_1 = 2$, $1 \le a_i \le 2$ for $i \ge 2$, and $a_i = 2$ infinitely often, If $0 < k < 1$, then $a_1 \ge 2$, $1 \le a_i \le 1 + [2^{1/k}]$ for $i \ge 1$, and $a_i \ge 2$ infinitely often.

Proof: We use the same algorithm as in Theorem 3.1 except that now we let

$$A_1 = A - a_0,\ 1 < A_1 \le 2,$$

and

$$a_n = \left[\left(\frac{2}{A_n}\right)^{1/k}\right] + 1 \text{ for } n \ge 1,\ A_n > 0.$$

As in the previous result, $A_n > 0$ and $a_n \ge 1$ for all $n \ge 1$. Also,

$$A_n > \frac{2}{a_n^k}$$

which implies

$$A_{n+1} = a_n^k A_n - 1 > 1$$

and, in the case $k \ge 1$,

$$a_{n+1} = 1 + \left[\left(\frac{2}{A_{n+1}}\right)^{1/k}\right] < 1 + 2^{1/k} \le 3.$$

Thus, $1 \le a_n \le 2$ for $n \ge 2$, and (since $1 < A_1 \le 2$) $a_1 \ge 2$. Also provided $a_n = 2$ (the case $a_n > 2$ cannot occur for $n \ge 1$, by the preceding inequalities) we get

$$A_n \le \frac{2}{(a_n - 1)^k} = 2 \quad \text{and} \quad A_{n+1} \le \frac{2a_n^k}{(a_n - 1)^k} - 1 = 2^{k+1} - 1,$$

since we assumed $a_n = 2$. Now, in the same way as in the previous theorem, after at most $[2^{k+1} - 1]$ steps of $a_{n+i} = 1$, we must reach a stage at which $A_{n+j} \le 2$ and $a_{n+j} = 2$. Therefore, the sequence $\{A_n\}$ is bounded above by $2^{k+1} - 1$ for all $n \ge 1$. The convergence of the series for $k \ge 1$ is now shown in exactly the same way as in the previous theorem. The proof for $0 < k < 1$ is exactly the same except that $a_1 \ge 2$ and for $n \ge 1$,

$$1 \le a_n \le 1 + [2^{1/k}].$$

In particular, by setting $k = 1$ in Theorem 3.2, we get an analogue of the Engel series

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots,$$

where $a_1 = 2$, $1 \le a_i \le 2$ for $i \ge 2$ and $a_i = 2$ infinitely often. Compare this to the growth condition $a_{i+1} \ge a_i \ge 2$ of Engel. Again under these weaker conditions the expansion is not unique. For example, in Section 4 we consider a different algorithm which for $k = 1$ gives another series with the same form and conditions on the digits, as the series noted here.

We note as well that, if we had defined $A_1 = A - a_0$ with $0 < A_1 \le 1$, as we did in Theorem 3.1, the digits obtained would have satisfied the same conditions as above for $i \ge 2$, but would have had $a_1 > 2$ if $0 < A_1 < 1/2^{k-1}$. The representation thus obtained would no longer be entirely in a "binary" form.

The representation of rational numbers when k takes on integer values 1, 2, 3, ... is also of interest. The condition that holds, i.e., that A is rational if and only if the digits in the expansion eventually become periodic, corresponds to the criterion for the representation of rational numbers via the Lüroth series. The result below applies to both the algorithms of Theorem 3.1 and Theorem 3.2.

Proposition 3.3: Let $k = 1, 2, 3, \ldots$. The digits in the k^{th} power expansions of Theorem 3.1 (or Theorem 3.2) become periodic if and only if A is rational.

Proof: Suppose firstly that $A_1 = p/q$ is rational (with $p, q \in \mathbb{N}$). Then, since $k \in \mathbb{N}$, each A_n is also rational, with

$$\begin{aligned} A_n &= a_{n-1}^k A_{n-1} - 1 = a_{n-1}^k(a_{n-2}^k A_{n-2} - 1) - 1 \\ &= \cdots = a^k A_1 + b = \frac{p_n}{q}, \end{aligned}$$

where $a \in \mathbb{N}$, $b \in \mathbb{Z}$. Now, for the first algorithm (Theorem 3.1) we have

$$0 < A_n = \frac{p_n}{q} \le 2^k - 1.$$

Thus, every

$$A_n \in \left\{\frac{1}{q}, \frac{2}{q}, \frac{3}{q}, \ldots, \frac{(2^k - 1)q}{q}\right\},$$

and so there exist $m, n \in \mathbb{N}$ such that $A_n = A_{n+m}$. Then the algorithm applied to A_{n+m} gives the same successive digits as when applied to A_n, i.e., the digits become periodic. The same argument applies in the case of the second algorithm except that now $0 < A_n = p_n/q \le 2^{k+1} - 1$.

Conversely, suppose that eventually $a_n = a_{n+m}$. If we use the notation

$$X_n = a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1a_2)^k} + \cdots + \frac{1}{(a_1 \ldots a_{n-1})^k}$$

and let $\alpha_r = (a_1a_2 \ldots a_r)^k$, and $\alpha_* = \alpha_{n+m-1}/\alpha_{n-1}$, we have

$$\begin{aligned} A &= X_n + \frac{1}{\alpha_{n-1}}\left\{\left(\frac{1}{a_n^k} + \frac{1}{a_n^k a_{n+1}^k} + \cdots + \frac{1}{a_n^k a_{n+1}^k \ldots a_{n+m-1}^k}\right)\right. \\ &\quad + \left(\frac{1}{\alpha_*^k a_n} + \frac{1}{\alpha_* a_n^k a_{n+1}^k} + \cdots + \frac{1}{\alpha_* a_n^k a_{n+1}^k \ldots a_{n+m-1}^k}\right) \\ &\quad \left. + \left(\frac{1}{\alpha_*^2 a_n^k} + \frac{1}{\alpha_*^2 a_n^k a_{n+1}^k} + \cdots\right) + \cdots\right\} \\ &= X_n + \frac{1}{\alpha_{n-1}}\left(\frac{1}{a_n^k} + \frac{1}{a_n^k a_{n+1}^k} + \cdots + \frac{1}{a_n^k a_{n+1}^k \ldots a_{n+m-1}^k}\right)\left(1 + \frac{1}{\alpha_*} + \frac{1}{\alpha^2} + \cdots\right) \\ &= \text{a rational number.} \end{aligned}$$

Note that for the ordinary Engel series the condition $a_{n+1} \ge a_n$ implies that for some n sufficiently large $a_{n+i} = a_n$ for all $i \ge 1$.

4. k^{th} Power Series Related to the Lüroth Series

We could at this stage investigate expansions for real numbers whose terms take the form of the terms of the Lüroth series raised to a power. However, we consider instead a similar type of algorithm which leads to an expansion of simpler form, yet where the digits satisfy similar conditions. In particular, by setting $k = 2$ in the results below, we obtain a series expansion for real numbers with the appearance of a simplified Lüroth series.

Theorem 4.1: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1^k a_2} + \frac{1}{(a_1 a_2)^k a_3} + \frac{1}{(a_1 a_2 a_3)^k a_4} + \cdots,$$

where:

if $k > 1$, then $a_1 \geq 2$, $a_i \geq 1$ for $i \geq 1$, and $a_i \geq 2$ infinitely often,

if $0 < k \leq 1$, then $a_{i+1} \geq a_i \geq 2$ for $i \geq 1$.

Proof: We derive this result from the following algorithm. Given any real number A, let $A_1 = A - a$, $0 < A_1 \leq 1$. Then we recursively define

$$a_n = 1 + \left[\frac{1}{A_n}\right] \text{ for } n \geq 1,\ A_n > 0,$$

where

$$A_{n+1} = a_n^k A_n - a_n^{k-1} \text{ for } A_n > 0.$$

Applying this algorithm repeatedly, we obtain

$$A = a_0 + A_1 = a_0 + \frac{1}{a_1} + \frac{A_2}{a_1^k} = \cdots$$
$$= a_0 + \frac{1}{a_1} + \frac{1}{a_1^k a_2} + \cdots + \frac{1}{(a_1 \ldots a_{n-1})^k a_n} + \frac{A_{n+1}}{(a_1 \ldots a_n)^k}.$$

Now $a_n = 1 + [1/A_n]$ implies that for $A_n > 0$,

$$A_n > \frac{1}{a_n},$$

and provided $a_n \geq 2$

$$A_n \leq \frac{1}{a_n - 1}.$$

Now $0 < A_1 \leq 1$ implies that $a_1 \geq 2$ and $A_2 = a_1^k A_1 - a_1^{k-1} > 0$; thus $a_2 \geq 1$. Continuing this process inductively, we see that $A_n > 0$; hence, $a_n \geq 1$ for all n. Consider the case $k > 1$. Suppose now that $a_n \geq 2$; then

$$A_{n+1} = a_n^k A_n - a_n^{k-1} \leq \frac{a_n^k}{a_n - 1} - a^{k-1} = \frac{a_n^{k-1}}{a_n - 1}.$$

Now if $A_{n+1} \leq 1$, then $a_{n+1} \geq 2$. Otherwise, $a_{n+1} = 1$ and $A_{n+2} = A_{n+1} - 1$. Continuing this process, we see that after at most $[a_n^{k-1}/(a_n - 1)]$ steps with

$$a_{n+i} = 1,\ A_{n+i+1} = A_{n+i} - 1 = A_{n+1} - 1,$$

we must reach a stage at which $A_{n+j} \leq 1$ and $a_{n+j} \geq 2$. Hence, there exists a sequence of integers $n_1 = 0 < n_2 < n_3 < \ldots$ such that

$$0 < A_{n_i+1} \leq 1,\ a_{n_i+1} \geq 2, \text{ and } a_n = 1$$

for all other $n > 1$. Then

$$0 < \frac{A_{n_i+1}}{(a_1 a_2 \ldots a_{n_i})^k} \leq \frac{1}{2^{k(i-1)}},$$

and so $S_{n_i} \to A$ as $i \to \infty$, where

$$S_n = a_0 + \frac{1}{a_1} + \cdots + \frac{1}{(a_1 \ldots a_{n-1})^k a_n} = A - \frac{A_{n+1}}{(a_1 \ldots a_n)^k}.$$

Now let $n_{i-1} \le n < n_i$. Then

$$S_{n_{i-1}} \le S_n < S_{n_i}, \text{ and } n \to \infty \text{ iff } i \to \infty.$$

So $S_n \to A$ as $n \to \infty$, and the series converges. For the case $0 < k \le 1$, if $a_n \ge 2$ then

$$a_{n+1} > \frac{1}{A_{n+1}} \ge \frac{a_n - 1}{a_n^{k-1}}.$$

Since $k \le 1$, $a_n^{k-1} \le 1$, and as $a_1 \ge 2$ we deduce that $a_{n+1} \ge a_n \ge 2$ for all $n \ge 1$. Thus,

$$\frac{A_{n+1}}{(a_1 \ldots a_n)^k} \le \frac{1}{2^{kn}} \to 0 \text{ as } n \to \infty,$$

and again the series converges to A.

We note that by setting $k = 2$ in Theorem 4.1 we obtain the expansion

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1^2 a_2} + \frac{1}{(a_1 a_2)^2 a_3} + \cdots,$$

where $a_1 \ge 2$, $a_i \ge 1$ for $i \ge 1$, and $a_i \ge 2$ infinitely often. In many ways this could be regarded as a simplified version of the Lüroth series. In addition, we shall show shortly that, as in the Lüroth case, A is rational if and only if the digits in the expansion become periodic.

A second algorithm for $k \ge 1$ leads to a "binary" series of this type where the digits a_i are equal to 1 or 2, for $i \ge 1$.

Theorem 4.2: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1^k a_2} + \frac{1}{(a_1 a_2)^k a_3} + \cdots,$$

where:

If $k \ge 1$, then $a_1 = 2$, $1 \le a_i \le 2$ for $i \ge 2$, and $a_i = 2$ infinitely often,

if $0 < k < 1$, then $a_1 = 2$, $1 \le a_{i+1} < 1 + 2a_i^{1-k}$ for $i \ge 1$, and $a_i \ge 2$ infinitely often.

Proof: We use the algorithm of Theorem 4.1 except that now we let $A_1 = A - a_0$, $1 < A_1 \le 2$, and

$$a_n = 1 + \left[\frac{2}{A_n}\right] \text{ for } n \ge 1,\ A_n > 0.$$

In the same way as before, we can show $A_n > 0$ and $a_n \ge 1$ for all $n \ge 1$. Also in this case

$$A_n > \frac{2}{a_n}$$

which implies

$$A_{n+1} = a_n^k A_n - a_n^{k-1} > a_n^{k-1}.$$

It follows that for $k \ge 1$

$$a_{n+1} = 1 + \left[\frac{2}{A_{n+1}}\right] < 1 + \frac{2}{a_n^{k-1}} \le 3.$$

Thus, $1 \le a_n \le 2$ for $n \ge 2$ and, as $1 < A_1 \le 2$, $a_1 = 2$. Also, provided $a_n = 2$ (the case $a_n > 2$ cannot occur from the above), we get

$$A_n \le \frac{2}{a_n - 1} = 2,$$

and

$$A_{n+1} \le \frac{2a_n^k}{a_n - 1} - a_n^{k-1} = 3.2^{k-1},$$

since we assumed $a_n = 2$. Now in the same way as in the previous theorem, after at most $[3.2^{k-1}]$ steps of $a_{n+i} = 1$, we reach a stage at which $A_{n+j} \le 2$ and $a_{n+j} = 2$. The convergence of the series for $k \ge 1$ is now shown in exactly the same way as in the previous theorem. However here, unlike that case, the sequence $\{A_n\}$ is bounded above for all n by a fixed constant as well. The proof for $0 < k < 1$ is the same except that we now have, for $n \ge 1$,

$$A_{n+1} \le 3.2^{k-1} < 3, \text{ and } a_{n+1} < 1 + 2a_n^{1-k}.$$

We consider now the expansion of rational numbers via these algorithms when k is a positive integer. We show that as in the previous section A is rational if and only if A has an expansion in which the digits become periodic.

Proposition 4.3: Let $k = 1, 2, 3, \ldots$. The digits in the k^{th} power expansions of Theorem 4.1 (or Theorem 4.2) become periodic if and only if A is rational.

Proof: First suppose that the expansion is periodic, that is, eventually

$$a_n = a_{n+m}.$$

Then with the notation

$$X_n = a_0 + \frac{1}{a_1} + \frac{1}{a_1^k a_2} + \cdots + \frac{1}{(a_1 \ldots a_{n-2})^k a_{n-1}}$$

and $\alpha_r = (a_1 a_2 \ldots a_r)^k$, $\alpha_* = \alpha_{n+m-1}/\alpha_{n-1}$, we have

$$\begin{aligned} A = X_n &+ \frac{1}{\alpha_{n-1}}\left\{\frac{1}{a_n} + \frac{1}{a_n^k a_{n+1}} + \cdots + \frac{1}{(a_n \ldots a_{n+m-2})^k a_{n+m-1}}\right. \\ &+ \frac{1}{\alpha_* a_n} + \frac{1}{\alpha_* a_n^k a_{n+1}} + \cdots + \frac{1}{\alpha_*(a_n \ldots a_{n+m-2})^k a_{n+m-1}} \\ &+ \left.\frac{1}{\alpha_*^2 a_n} \quad \frac{1}{\alpha_*^2 a_n^k a_{n+1}} + \cdots\right\} \\ = X_n &+ \frac{1}{\alpha_{n-1}}\left(\frac{1}{a_n} + \frac{1}{a_n^k a_{n+1}} + \cdots \right. \\ &+ \left.\frac{1}{(a_n \ldots a_{n+m-2})^k a_{n+m-1}}\right)\left(1 + \frac{1}{\alpha_*} + \frac{1}{\alpha_*^2} + \cdots\right) \\ = {} & \text{a rational.} \end{aligned}$$

Conversely, suppose $A_1 = p/q$ is rational (with $p, q \in \mathbb{N}$). Then, since $k \in \mathbb{N}$, each A_n is also rational, with

$$\begin{aligned} A_n &= a_{n-1}^k A_{n-1} - a_{n-1}^{k-1} = a_{n-1}^{k-1}\left(a_{n-2}^k A_{n-2} - a_{n-2}^{k-1}\right) - a_{n-1}^{k-1} \\ &= \cdots = a^k A_1 + b = p_n/q. \end{aligned}$$

where $b \in \mathbb{Z}$, a, $p_n \in \mathbb{N}$. Now in the case of the second algorithm (Theorem 4.2)

$$0 < A_n = p_n/q \leq 3.2^{k-1}.$$

Thus, every

$$A_n \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{3.2^{k-1}q}{q}\right\}$$

and we deduce that the expansion becomes periodic in the same way as in Proposition 3.3. In the case of the algorithm of Theorem 4.1, we do not have a fixed bound for A_n. However, when A is rational,

$$a_n = 1 + \left[\frac{1}{A_n}\right] = 1 + \left[\frac{q}{p_n}\right] \leq q + 1$$

as $p_n \geq 1$ for $A_n > 0$. Using the fact that any A_n for which $a_n = 1$ is bounded above by an A_m for which $a_m \geq 2$, and that (for $a_m \geq 2$)

$$A_m \leq \frac{a_m^{k-1}}{a_m - 1},$$

it follows that, for all $n \geq 1$,

$$A_n \leq (q + 1)^{k-1}.$$

Thus, every

$$A_n \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{(q + 1)^{k-1}q}{q}\right\},$$

and again we can deduce that the expansion must eventually become periodic.

In summary, we have found new classes of representations for real numbers that are related to the classical series of Sylvester, Engel, and Lüroth. In many cases, the expansions require very mild growth conditions on the digits and share with the Lüroth series the property of begin periodic when a number is rational. Unlike the classical series, however, the expansions for real numbers with $k \neq 1$ are not unique, slightly different algorithms yielding series with the same properties, but with different digits for the same real number A.

Reference

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CONGRUENCES FOR NUMBERS OF RAMANUJAN

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1. Introduction

In Chapter 3 of his second notebook [1, p. 164], Ramanujan defined polynomials

$$A_r(x) = \sum_{k=0}^{r-2} a(r, k)x^{2r-k-1} \quad (r \geq 2) \tag{1.1}$$

with $A_1(x) = x$. The numbers $a(r, k)$ are integers such that $a(2, 0) = 1$ and, for $r \geq 2$,

$$a(r + 1, k) = (r - 1)a(r, k - 1) + (2r - k - 1)a(r, k). \tag{1.2}$$

Also, $a(r, k) = 0$ when $k < 0$ or $k > r - 2$. Properties of $A_r(x)$, and the motivation for defining them, are discussed in [1, pp. 163-166]. Included in that reference is a list of the polynomials $A_r(x)$, $1 \leq r \leq 7$, and the following theorem:

$$\sum_{k=0}^{r-2} a(r, k) = A_r(1) = (r - 1)^{r-1}. \tag{1.3}$$

In [3] it was shown how $a(r, r - k)$ can be expressed in terms of Stirling numbers of the first kind, and the following special cases were worked out:

$$a(r, 0) = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2r - 3), \tag{1.4}$$

$$a(r, 1) = [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2r - 3)](r - 2)/3, \tag{1.5}$$

$$a(r, r - 2) = (r - 2)! \tag{1.6}$$

We note here that it is easy to prove by induction that

$$a(r, 2) = (r - 3)(r - 2)(r - 1)5 \cdot 7 \cdot \cdots \cdot (2r - 5)/3.$$

The main purpose of the present paper is to prove congruences for $a(r, k)$ (mod p), where p is a prime number. As an application of some of these congruences we prove $A_p(x)/x^{p+1}$ and $A_{p-1}(x)/x^p$ are irreducible over the rational field. We also determine, for all r, the least residues of $a(r, k)$ (mod 2), (mod 3), and (mod 4). For each r we find the largest k such that $a(r, k) \not\equiv 0$ (mod p), and we make a conjecture, based on computer evidence, about the smallest k such that $a(r, k) \not\equiv 0$ (mod p). We also conjecture the following periodicity property:

$$a(r + (p - 1)p, k + (p - 2)p) \equiv a(r, k) \pmod{p}.$$

This has been verified for all primes $p \leq 251$. A few other results and conjectures are given for moduli not necessarily prime.

2. Congruences (Mod P)

Theorem 2.1: For any prime number p,

$$a(p, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, p - 3),$$

$$a(p,\ p-2) \equiv 1 \pmod{p}.$$

Proof: In [1, p. 164] we have

$$A_r(x) = x(r-2)A_{r-1}(x) + x\sum_{k=1}^{r-1}\binom{r-1}{k}A_k(x)A_{r-k}(x), \tag{2.1}$$

and hence

$$A_{p+1}(x) \equiv -xA_p(x) + xA_p(x)A_1(x) \equiv (x^2 - x)A_p(x) \pmod{p}.$$

Comparing coefficients of x^{2p-k+1}, we have

$$a(p+1,\ k) \equiv a(p,\ k) - a(p,\ k-1) \pmod{p}. \tag{2.2}$$

From (1.2) we have

$$a(p+1,\ k) \equiv -a(p,\ k-1) - (k+1)a(p,\ k) \pmod{p}. \tag{2.3}$$

Combining (2.2) and (2.3), we see that

$$(k+2)a(p,\ k) \equiv 0 \pmod{p} \quad (k = 0,\ \ldots,\ p-2). \tag{2.4}$$

The theorem now follows from (2.4) and (1.6). We note that results similar to Theorem 2.1 have been proved for the Stirling numbers [2, pp. 218-219].

Theorem 2.2: For any odd prime number p,

$$a(p-1,\ k) \equiv 0 \pmod{p} \quad (k = 0,\ 1,\ \ldots,\ p-4),$$

$$a(p-1,\ p-3) \equiv (p-3)! \pmod{p}.$$

Proof: From (1.2) we have

$$a(p,\ k) \equiv -2a(p-1,\ k-1) - (k+3)a(p-1,\ k) \pmod{p}.$$

Thus, by Theorem 2.1,

$$(k+3)a(p-1,\ k) \equiv -2a(p-1,\ k-1) \pmod{p} \quad (k = 1,\ \ldots,\ p-3). \tag{2.5}$$

Since

$$a(p-1,\ 0) = 1 \cdot 3 \cdot \cdots \cdot (2p-5) \equiv 0 \pmod{p} \text{ for } p > 3,$$

the theorem follows from (2.5) and (1.6).

Theorem 2.3: For any odd prime number p, the polynomials

$$A_p(x)/x^{p+1} \quad \text{and} \quad A_{p-1}(x)/x^p$$

are irreducible over the rational field.

Proof: Assume $p > 2$. We know

$$a(p,\ k) \equiv 0 \pmod{p} \quad (k = 0,\ 1,\ \ldots,\ p-3),$$

$$a(p,\ 0) = 1 \cdot 3 \cdot \cdots \cdot (2p-3) \not\equiv 0 \pmod{p^2},$$

$$a(p,\ p-2) \equiv 1 \not\equiv 0 \pmod{p}.$$

Thus, $A_p(x)/x^{p+1}$ is irreducible by Eisenstein's Criteria. The proof is similar for $A_{p-1}(x)/x^p$.

We note here that Theorem 2.1 could be generalized by using p^j, $j \geq 1$, instead of p. Replacing p by p^j in the proof, we have

$$a(p^j,\ k) \equiv 0 \pmod{p} \quad (k \not\equiv -2 \pmod{p}).$$

Theorem 2.4: If p is an odd prime and if $m \geq p$, then

$$a(m, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, p - 3).$$

If $m > p$, then

$$a(m, p - 2) = a(p + t, p - 2) \equiv 1 \cdot 3 \cdot \cdots \cdot (2t - 1) \pmod{p}.$$

Proof: We use induction on m. The first part of the theorem is true for $m = p$. Assume it is true for $m = p, p + 1, \ldots, r$. Then, by (1.2) we have, for $k = 0, 1, \ldots, p - 3$,

$$a(r + 1, k) \equiv 0 \pmod{p};$$

therefore, the first part of the theorem is true for all $m \geq p$. Now, by (1.2), if $t > 0$, then

$$\begin{aligned} a(p + t, p - 2) &\equiv (2t - 1)a(p + t - 1, p - 2) \\ &\equiv 1 \cdot 3 \cdot \cdots \cdot (2t - 1)a(p, p - 2) \\ &\equiv 1 \cdot 3 \cdot \cdots \cdot (2t - 1) \pmod{p}. \end{aligned}$$

This completes the proof.

We note that when $t \geq (p + 1)/2$,

$$a(p + t, p - 2) \equiv 0 \pmod{p} \quad (p > 2).$$

We also note the following special cases of Theorem 2.4. For $k = 0, 1, 2, \ldots, p - 3$:

$$\begin{aligned} &a(p + 1, k) \equiv 0 \pmod{p}; && a(p + 2, k) \equiv 0 \pmod{p}: \\ &a(p + 1, p - 2) \equiv 1 \pmod{p}; && a(p + 2, p - 2) \equiv 3 \pmod{p}; \\ &a(p + 1, p - 1) \equiv -1 \pmod{p}; && a(p + 2, p - 1) \equiv -2 \pmod{p}; \\ & && a(p + 2, p) \equiv 0 \pmod{p}. \end{aligned}$$

Theorem 2.5: Let p be an odd prime. Then, for $k = 0, 1, \ldots, 2p - 5$:

$$\begin{aligned} &a(2p, k) \equiv 0 \pmod{p}; && (2p - 1, k) \equiv 0 \pmod{p}; \\ &a(2p, 2p - 4) \equiv 1 \pmod{p}; && a(2p - 1, 2p - 4) \equiv 1 \pmod{p}; \\ &a(2p, 2p - 3) \equiv -2 \pmod{p}; && a(2p - 1, 2p - 3) \equiv 0 \pmod{p}; \\ &a(2p, 2p - 2) \equiv 0 \pmod{p}. && \end{aligned}$$

Proof: We know by (1.6) and Theorem 2.4 that

$$a(2p, 2p - 2) \equiv 0 \equiv a(2p, p - 2) \pmod{p}.$$

From (2.1) we have

$$A_{2p+1}(x) \equiv (-x + x^2)A_{2p}(x) + 2xA_p(x)A_{p+1}(x) \pmod{p}.$$

Thus, by Theorem 2.1 and Theorem 2.4 (with $m = p + 1$),

$$A_{2p+1}(x) \equiv (-x + x^2)A_{2p}(x) + 2x^{2p+5} - 2x^{2p+4} \pmod{p}. \tag{2.6}$$

Congruence (2.6) gives, for $k \neq 2p - 3, 2p - 4$,

$$a(2p + 1, k) \equiv a(2p, k) - a(2p, k - 1) \pmod{p}, \tag{2.7}$$

and from (1.2) we have

$$a(2p + 1, k) \equiv -(k + 1)a(2p, k) - a(2p, k - 1) \pmod{p}. \tag{2.8}$$

Combining (2.7) and (2.8), we have, for $k \neq 2p - 3, 2p - 4$,

$$(k + 2)a(2p, k) \equiv 0 \pmod{p}.$$

For $k = 2p - 3$ and $k = 2p - 4$, (2.6) and (2.8) give

$$(2p - 1)a(2p, 2p - 3) \equiv 2 \pmod{p},$$
$$(2p - 2)a(2p, 2p - 4) \equiv -2 \pmod{p},$$

and we see that the congruences for $a(2p, k)$ in Theorem 2.5 are valid. Now, by (1.2) and (1.4), we have

$$a(2p, k) \equiv -2a(2p - 1, k - 1) - (k + 3)a(2p - 1, k) \pmod{p},$$
$$a(2p - 1, 0) \equiv 0 \pmod{p}. \tag{2.9}$$

Thus, $a(2p - 1, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, p - 4)$,

and by Theorem 2.4,

$$a(2p - 1, p - 3) \equiv 0 \pmod{p}.$$

It is now clear that the congruences for $a(2p - 1, k)$ follow from the congruences for $a(2p, k)$ and (2.9). This completes the proof.

Theorem 2.6: If p is prime and $m \geq 2p$, then

$$a(m, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, 2p - 5),$$
$$a(m, 2p - 4) = a(2p + t, 2p - 4) \equiv 1 \cdot 3 \cdot \cdots \cdot (2t + 1) \pmod{p}.$$

Proof: We use induction on m. The theorem is true for $m = 2p$. Assume it is true for $m = 2p, 2p + 1, \ldots, r$. Then, by (1.2), we have

$$a(r + 1, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, 2p - 5);$$

therefore, the first part of the theorem is true for all $m \geq 2p$. By (1.2), we have, for $t > 0$,

$$\begin{aligned} a(2p + t, 2p - 4) &\equiv (2t + 1)a(2p + t - 1, 2p - 4) \\ &\equiv 3 \cdot 5 \cdot \cdots \cdot (2t + 1)a(2p, 2p - 4) \\ &\equiv 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2t + 1) \pmod{p}. \end{aligned}$$

This completes the proof.

Using the same sort of proof as the proof of the first part of Theorem 2.5, we can show, for $p > 2$,

$$a(3p, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, 3p - 7;\ k \neq 2p - 2).$$

The case $a(3p, 2p - 2)$ has not been resolved. We indicate with Conjecture 1 in Section 4 what the general situation appears to be. The next theorem deals with a related problem, namely, the problem of finding the largest k such that $a(r, k) \not\equiv 0 \pmod{p}$.

Define $g(p, r)$ to be the largest k such that $a(r, k) \not\equiv 0 \pmod{p}$.

Theorem 2.7: Let r be a positive integer, $r \geq 2$. Write

$$r = 2 + (s(p - 1) + t)p + u$$

with $s \geq 0$, $0 \leq t \leq p - 2$, and $0 \leq u \leq p - 1$. Then

$$g(p, r) = m = sp(p - 2) + t(p - 1) + u.$$

Furthermore,

$$a(r, m) \equiv u!\ (p - 2)!/(p - 2 - t)! \pmod{p}.$$

Proof: We give a brief outline of the proof by induction on s, t, and u. Note that showing $a(r, k) \equiv 0 \pmod{p}$ for all $k > m$ is simple, and we omit the details. The recurrence relation (1.2) is the main tool in all of the following. The theorem is certainly true for $s = t = u = 0$. For fixed s and t, induction on u is straightforward. Note that the induction applies to arbitrarily large u; the statement of the theorem restricts u to the nonzero values of $a(r, m)$. If the theorem is true for some fixed value of s, $u = p - 1$, and some value of t, then it is not hard to show that the theorem must be true for the same s, $u = 0$, and the successor of t. By induction, if this theorem is true for some s and for $t = u = 0$, then it is also true for that s and all $0 \le t \le p - 2$ and $0 \le u \le p - 1$.

Now suppose the theorem is true for some s and all t and u such that $0 \le t \le p - 2$ and $0 \le u \le p - 1$. Let

$$r_0 = 2 + (s(p - 1) + (p - 2))p$$

and let

$$m_0 = sp(p - 2) + (p - 2)(p - 1).$$

Then, putting $t = p - 2$ in the induction hypothesis, we have

$$a(r_0 + u, m_0 + u) \equiv u! \pmod{p} \text{ for } 0 \le u \le p - 1.$$

Since $r_0 - 2 \equiv 0 \pmod{p}$ and $2r_0 - (m_0 + 1) - 3 \equiv 0 \pmod{p}$, we must have

$$a(r_0, m_0 - 1) \equiv 0 \equiv 0 \cdot 0! \pmod{p}.$$

Now induct on u to show that

$$a(r_0 + u, m_0 + u - 1) \equiv u \cdot u! \pmod{p}.$$

Finally, we can conclude that:

$$a(r_0 + p, m_0 + p) \equiv p! \equiv 0 \pmod{p};$$

$$a(r_0 + p, m_0 + p - 1) \equiv p \cdot p! \equiv 0 \pmod{p};$$

$$\begin{aligned} a(2 + (s + 1)(p - 1)p, (s + 1)p(p - 2)) &= a(r_0 + p, m_0 + p - 2) \\ &\equiv 0 \cdot a(r_0 + p - 1, m_0 + p - 3) + (4 - 0 - 3) \cdot a(r_0 + p - 1, m_0 + p - 2) \\ &\equiv (p - 1) \cdot (p - 1)! \equiv 1 \pmod{p}. \end{aligned}$$

It follows that the theorem is true for $t = u = 0$ and $s + 1$. By induction, the theorem is true for all $s \ge 0$, $0 \le t \le p - 2$, and $0 \le u \le p - 1$.

The proof of the following theorem follows the same lines as the proof of Theorem 2.2.

Theorem 2.8: Let p be prime, $p > 3$. Then, for $1 \le t \le (p - 3)/2$,

$$a(p - t, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, p - 2t - 2).$$

For $2 \le t \le p - 1$,

$$a(2p - t, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, 2p - 2t - 2).$$

For example, using Theorem 2.8, Theorem 2.2, and (1.2), we have, for $p > 5$,

$$a(p - 2, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, p - 6),$$

$$a(p - 2, p - 5) \equiv -(p - 4)!/3 \pmod{p},$$

$$a(p - 2, p - 4) \equiv (p - 4)! \pmod{p}.$$

3. Congruences (Mod 2), (Mod 3), and (Mod 4)

In this section we first determine when $a(r, k)$ is even and when it is odd.

Theorem 3.1:

$$a(r,\ 0) \equiv 1 \pmod 2 \quad (r \geq 2),$$
$$a(r,\ 1) \equiv r \pmod 2 \quad (r \geq 3),$$
$$a(r,\ k) \equiv 0 \pmod 2 \quad (k > 1).$$

Proof: The congruences for $a(r,\ 0)$ and $a(r,\ 1)$ are clear from (1.4) and (1.5). By (1.2) we have, for $k > 1$,

$$a(2r,\ k) \equiv (k + 1)a(2r - 1,\ k) \pmod 2.$$

If k is odd, we clearly have $a(2r,\ k)$ is even. If k is even, then

$$a(2r,\ k) \equiv a(2r - 1,\ k) \pmod 2.$$

And by (1.2), since $k - 1$ is odd,

$$a(2r - 1,\ k) \equiv a(2r - 2,\ k) \pmod 2.$$

Thus,

$$a(2r,\ k) \equiv a(2r - 2,\ k) \equiv \cdots \equiv a(k + 2,\ k) \equiv k! \equiv 0 \pmod 2.$$

Now since

$$a(2r + 1,\ k) \equiv a(2r,\ k - 1) + (k + 1)a(2r,\ k),$$

we have $a(2r + 1,\ k)$ is even if $k > 1$. This completes the proof.

The patterns (mod 4) and (mod 8) are suggested by the computer and can be proved by induction on r. For (mod 4) we have the following congruences.

Theorem 3.2: $a(r,\ k) \equiv 0 \pmod 4$ for all k except:

$$a(r,\ 0) \equiv \begin{cases} 1 \pmod 4 \text{ if } r \equiv 1 \text{ or } 2 \pmod 4, \\ 3 \pmod 4 \text{ if } r \equiv 0 \text{ or } 3 \pmod 4, \end{cases}$$

$$a(r,\ 1) \equiv \begin{cases} 1 \pmod 4 \text{ if } r \equiv 1 \text{ or } 3 \pmod 4, \\ 2 \pmod 4 \text{ if } r \equiv 0 \pmod 4, \end{cases}$$

$$a(r,\ 2) \equiv 2 \pmod 4 \text{ if } r \equiv 0 \pmod 4,$$
$$a(r,\ 3) \equiv 2 \pmod 4 \text{ if } r \equiv 1 \pmod 4.$$

Theorems 3.1 and 3.2 suggest the following, which can be proved by means of (1.2) and induction on n.

Theorem 3.3: If $k \geq 2n$, then $a(r,\ k) \equiv 0 \pmod{2^n}$.

To prove congruences (mod 3) we need the following lemma, which is a special case of Conjecture 4 of Section 4.

Lemma: For $r \geq 2$, $a(r,\ k) \equiv a(r + 6,\ k + 3) \pmod 3$.

Proof: The lemma is true for $r = 2$, since

$$a(8,\ 3) \equiv 1 \pmod 3,$$
$$a(8,\ k) \equiv 0 \pmod 3 \text{ if } k \neq 3.$$

Assume it is true for $r = m - 1$. Then, by (1.2),

$$a(m + 6,\ k + 3) \equiv (m - 2)a((m - 1) + 6,\ k + 2) + (2(m - 1) - 1 - k)a(m - 1,\ k)$$

(continued)

$$\equiv (m-2)a(m-1, k-1) + (2(m-1)-1-k)a(m-1, k)$$
$$\equiv a(m, k) \pmod 3.$$

Theorem 3.4: $a(r, k) \equiv 0 \pmod 3$ for all k except:

$$a\left(r, \left[\frac{r-1}{2}\right]\right) \equiv 1 \pmod 3, \quad r \geq 2,$$

$$a\left(r, \left[\frac{r+1}{2}\right]\right) \equiv \begin{array}{ll} r(r+1) & \pmod 3 \text{ if } r \not\equiv 0 \pmod 6, \\ 1 & \pmod 3 \text{ if } r \equiv 0 \pmod 6. \end{array}$$

Proof: Suppose $r \equiv 2 \pmod 6$, i.e., $r = 6j + 2$. Then, by the lemma,

$$a(r, k) \equiv a(6(j-1)+2, k-3) \equiv \cdots \equiv a(2, k-3j) \pmod 3.$$

Thus,

$$a(r, k) \equiv \begin{cases} 0 \pmod 3 \text{ if } k \neq 3j, \\ 1 \pmod 3 \text{ if } k = 3j = (r-2)/2. \end{cases}$$

The other cases of $r \pmod 6$ are handled in exactly the same way.

4. Conjectures

Theorem 2.4, Theorem 2.5, and information given by the computer suggest the following conjectures.

Conjecture 1: For all integers t and positive integers h such that $h + t \geq 1$,

$$a(hp+t, k) \equiv 0 \pmod p, \; k = 0, 1, \ldots, h(p-2)-1,$$

$$a(hp+t, h(p-2)) \equiv 1 \cdot 3 \cdot \cdots \cdot (2t+2h-3) \pmod p.$$

For $t \geq 0$, Conjecture 1 has already been proved in Section 2 of this paper for $h = 1$, $h = 2$. If we try induction and assume true for $h = m - 1$, we can show, as in Theorem 2.1 and Theorem 2.5,

$$(k+2)a(mp, k) \equiv 0 \pmod p \quad (k = 0, \ldots, m(p-2)-2).$$

Thus, the proof depends on showing

$$a(mp, k) \equiv 0 \pmod p \text{ if } k \equiv -2 \pmod p.$$

The rest of the proof, for $t > 0$, would then follow. The cases $t \geq 0$ have been verified by the computer for all primes less than or equal to 251. The case $h + t = 0$ leads to the next conjecture.

Conjecture 2: Let p be any prime.

(i) Let h be any nonnegative integer. Then

$$a(2+hp(p-1), m) \equiv 0 \pmod p \text{ if } m \neq hp(p-2).$$

(ii) Let h be a nonnegative integer, $h \not\equiv 0 \pmod p$. Then

$$a(1+h(p-1), m) \equiv 0 \pmod p \text{ if } m \neq h(p-2).$$

(iii) Let h be a nonnegative integer, $h \not\equiv 0$ or $p - 1 \pmod p$. Then

$$a(h(p-1), m) \equiv 0 \pmod p \text{ if } m \neq h(p-2)-1.$$

By Theorem 2.7, we know:

(i) $a(2+hp(p-1), hp(p-2)) \equiv 1 \pmod p$.

(ii) Let $h = sp + t$, $1 \leq t \leq p - 1$, $s \geq 0$, then

$$\begin{aligned} &a(1 + h(p - 1),\ h(p - 2)) \\ &= a(2 + (s(p - 1) + (t - 1))p + (p - 1) - t), \\ &\quad sp(p - 2) + (t - 1)(p - 1) + (p - 1) - t) \\ &\equiv (p - 1 - t)! \cdot (p - 2)!/(p - 2 - (t - 1))! \equiv 1 \pmod{p}. \end{aligned}$$

(iii) Let $h = sp + t$, $1 \le t \le p - 2$, $s \le 0$, then

$$\begin{aligned} &a(h(p - 1),\ h(p - 2) - 1) \\ &= a(2 + (s(p - 1) + (t - 1))p + (p - 2) - t), \\ &\quad sp(p - 2) + (t - 1)(p - 1) + (p - 2) - t) \\ &\equiv (p - 2 - t)! \cdot (p - 2)!/(p - 2 - (t - 1))! \equiv (p - 2)!/(p - 1 - t). \end{aligned}$$

The authors are grateful to the referee for suggesting the next conjecture. Part of this conjecture would follow from Conjectures 1 and 2.

Define $f(p, r)$ to be the smallest k such that $a(r, k) \not\equiv 0 \pmod{p}$.

Conjecture 3: Clearly $f(p, r) = 0$ if $r \le (p + 1)/2$. Thus, for $r \ge 2$:

(i) $f(p, r) = (p - 2)\left[\dfrac{r}{p - 1}\right] - 1$ if $(p - 1) | r$;

(ii) $f(p, r) = (p - 2)\left[\dfrac{r}{p - 1}\right]$ if $r \equiv t \pmod{p - 1}$, $1 \le t \le (p + 1)/2$;

(iii) $f(p, r) \ge (p - 2)\left[\dfrac{r}{p - 1}\right] + 2t$ if $r \equiv t + (p + 1)/2 \pmod{p - 1}$, $1 \le t \le (p - 5)/2$.

In some cases, $f(p, r)$ is larger than the formula given in (iii) above. For example, $f(11, 17) = 13$, $f(11, 48) = 42$, $f(13, 22) = 19$, and $f(13, 68) = 59$ are larger by 2, and $f(41, 350) = 334$, $f(43, 1743) = 1703$, $f(61, 2152) = 2111$, and $f(67, 2038) = 2002$ are larger by 4. It appears to be difficult to predict when $f(p, r)$ will be larger than the formula or by how much it will be bigger. There are many cases where $f(p, r)$ is larger by 2 or 4, and we suspect the formula could be off by even more for very large primes.

Conjecture 4: If p is any prime, then

$$a(r + p(p - 1),\ m + p(p - 2)) \equiv a(r, m) \pmod{p} \text{ for any } r \ge 2,\ m \ge 0.$$

Because of the recursion formula, (1.2), it suffices to show that

$$a(2 + p(p - 1),\ m + p(p - 2)) \equiv a(2, m) \pmod{p}$$

for all integers m. In this manner, Conjecture 4 has been proved on the computer for any prime $p \le 251$.

Conjecture 5: If p is any prime and $m \ge 0$, then

$$a(r + p^n(p - 1),\ m + p^n(p - 2)) \equiv a(r, m) \pmod{p^n}$$

for all sufficiently large r.

Conjecture 5 has been proved on the computer for p^n up to 2^{13}, 3^7, 5^5, 7^4, 11^3, 13^3, 17^2, 19^2, 23^2, 29^2, 31^2, 37^2, 41^2.

Conjecture 6: If p is an odd prime, then

(i) $a(r, 0) \equiv p \pmod{2p}$ for $r \ge (p + 3)/2$;

(ii) $a(r, 1) \equiv \begin{cases} 0 \pmod{2p} & \text{if } r \text{ is even} \\ p \pmod{2p} & \text{if } r \text{ is odd} \end{cases}$ and $r \geq (p + 3)/2$;

(iii) if $m \geq 2$ and $r \geq (p + 3)/2$, then

$$a(r + p(p - 1), m + p(p - 2)) \equiv a(r, m) \pmod{2p}.$$

Conjecture 6 can be proved to be true if Conjecture 4 is assumed to be true. Similar conjectures for other composite moduli also seem to hold, but are more complicated to state.

5. Concluding Remarks

Apparently not much is known about the numbers $a(r, k)$. It would be useful if a generating function and a combinatorial interpretation were found. Also, it appears difficult to find values of $A_r(x)$ for $x \neq 0$, $x \neq 1$. We remark that it is easy to find derivative formulas for $A_r(x)$, however. It follows from (1.2) and the definition of $A_r(x)$ that

$$x^3 A_r'(x) = A_{r+1}(x) - (r - 1)xA_r(x),$$

and thus it is easy to find a general formula for $A_r^{(j)}(x)$. For example, we have by (1.3) and the above comments,

$$A_r'(1) = r^r - (r - 1)^r$$

$$A_r''(1) = (r + 1)[(r + 1)^r - 2r^r + (r - 1)^r].$$

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ON A CERTAIN SEQUENCE OF QUOTIENTS OF A SEQUENCE

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Let $\{x_n\}_{n\geq 1}$ be a sequence of positive real numbers. The q-sequence corresponding to $\{x_n\}$ is defined as having as its n^{th} term

$$q_{n+1} = \frac{x_{n+1}}{x_n}$$

for all integers $n \geq 1$. One of the purposes of this note is to compare the sequence $\{x_n\}$ with its corresponding q-sequence $\{q_n\}$ so that conditions imposed on one of them will yield results concerning the other.

1. *Example:* Consider the Fibonacci sequence $\{x_n\}$ defined recursively by

$$x_1 = 1,\ x_2 = 1,\ x_{n+1} = x_n + x_{n-1} \text{ if } n \geq 2.$$

It is well known that the q-sequence corresponding to $\{x_n\}$ converges to the real number $(1 + \sqrt{5})/2$. This example shows that divergent sequences $\{x_n\}$ can have corresponding q-sequences that converge. On the other hand, examples can be found of convergent sequences of quotients of convergent sequences. See Theorem 5 below.

Whenever a sequence $\{x_n\}$ is defined recursively, say

$$x_1 = a,\ x_2 = b,\ x_{n+1} = f(x_n,\ x_{n-1}) \text{ for } n \geq 2$$

and positive numbers a and b, let $S(a, b, f)$ denote the corresponding q-sequence, where f is a nonnegative function of two real variables which is defined and positive in the first quadrant and defined on the positive y-axis. If $\{q_n\}$ converges, let z be its limit

$$z = \lim_{n\to\infty} q_n.$$

2. *Theorem:* Let $\{q_n\} = S(a, b, f)$ be the q-sequence corresponding to a sequence $\{x_n\}$. If $\{q_n\}$ converges and f is continuous and positively homogeneous of degree 1 $[f(\lambda x, \lambda y) = \lambda f(x, y)$ for $\lambda > 0]$, then z satisfies the equation

$$w^2 = f(w, 1).$$

Proof: Since f is positively homogeneous of degree 1, it follows that

$$q_{n+1}q_n = \frac{x_{n+1}}{x_{n-1}} = \frac{f(x_n,\ x_{n-1})}{x_{n-1}} = f\left(\frac{x_n}{x_{n-1}},\ 1\right) = f(q_n,\ 1).$$

Consequently, $z^2 = f(z, 1)$ must hold because of the continuity of f.

3. *Examples:* (1) For the Fibonacci sequence, one has

$$f(x_n,\ x_{n-1}) = x_n + x_{n-1}$$

and the limit $z = (1 + \sqrt{5})/2$ satisfies $f(z, 1) = ((1 + \sqrt{5})/2)^2$, in agreement with the theorem.

(2) Consider $\{q_n\} = S(1, 2, x_n + 2x_{n-1})$. To find z, one might want first to solve the quadratic equation $z^2 = z + 2$, whose positive solution is $z = 2$. Unfortunately, Theorem 2 as stated does not guarantee that $\lim_{n \to \infty} q_n = 2$. If the limit exists, then

$$q_{n+1} = \frac{x_{n+1}}{x_n} = \frac{x_n + 2x_{n-1}}{x_n} = 1 + 2\bigg/\left(\frac{x_n}{x_{n-1}}\right) = 1 + 2/q_n$$

implies $z = 2$. A procedure for finding the limit is presented in the next result.

4. *Theorem:* Let $b > 0$ and $c \geq 0$ be real numbers. If $f(x, y) = bx + cy$, let $\{x_n\}$ be the sequence defined recursively by

$$x_1 = p > 0,\ x_2 = q > 0, \text{ and } x_{n+1} = f(x_n, x_{n-1}) \text{ for } n \geq 2.$$

Then the q-sequence $S(p, q, f)$ converges to

$$z = (b + \sqrt{b^2 + 4c})/2$$

independent of the initial values p and q. Moreover, the sequence $\{q_{n+1} - q_n\}$ is either the constant sequence $\{0\}$ or oscillates between positive and negative values.

Proof: Note that for $n \geq 2$,

$$q_{n+1} = \frac{x_{n+1}}{x_n} = \frac{bx_n + cx_{n-1}}{x_n} = b + \frac{c}{q_n},$$

and hence, for $n \geq 3$,

$$q_{n+1} - q_n = c\,\frac{q_{n-1} - q_n}{q_n q_{n-1}}.$$

Consequently, $\{q_{n+1} - q_n\}$ is either the sequence $\{0\}$ or oscillates between positive and negative values. Also,

$$q_n = b + \frac{c}{q_{n-1}} \text{ for } n \geq 3$$

implies that $q_n q_{n-1} = bq_{n-1} + c > b^2 + c$ for $n \geq 4$, and hence

$$|q_{n+1} - q_n| = \left|c\,\frac{q_{n-1} - q_n}{q_n q_{n-1}}\right| \leq \frac{c}{b^2 + c}|q_{n-1} - q_n|.$$

If $d = c/(b^2 + c)$, then $0 \leq d < 1$ and

$$|q_{n+1} - q_n| \leq d^{n-3}|q_4 - q_3| \text{ for } n \geq 3.$$

Since

$$\sum_{n=3}^{\infty} d^{n-3}|q_4 - q_3|$$

converges, it follows that $\{q_n\}$ is a Cauchy sequence, thus it converges to some number $z \geq 0$. Theorem 2 shows that $z^2 = bz + c$ must hold; therefore,

$$z = \frac{b + \sqrt{b^2 + 4c}}{2}. \ \square$$

One can, in some cases, compare the behavior of a given sequence $\{x_n\}$ with that of its corresponding q-sequence. It is to be pointed out here that the sequences referred to in the following result are not necessarily generated by recursion

5. *Theorem:* Let $\{x_n\}$ be a sequence of positive numbers and let $\{q_n\}$ denote its corresponding q-sequence. Then

(1) If $\{q_n\} \in c_0$, then $\{x_n\} \in \ell^{(1)}$; hence $\{x_n\} \in c_0$.

(2) If $\{x_n\} \in c - c_0$, then $\lim_{n \to \infty} q_n = 1$.

($\ell^{(1)}$, c_0, and c denote the spaces of summable, convergent to zero, and convergent sequences, respectively.)

6. *Example:* Consider the sequence defined by

$$y_n = \frac{1}{4n^2 - 1}.$$

Since $4n^2 - 1 > n^2$, then $\{y_n\} \in \ell^{(1)}$. Define a sequence $\{x_n\}$ by

$$x_n = \frac{t}{\prod_{k \ge n} (1 + y_k)},$$

where $t > 0$ is a real parameter. This sequence is well defined, because the infinite product

$$\prod_{k \ge n} (1 + y_k)$$

converges. It follows that $\{1 + y_n\}$ is a q-sequence. Indeed,

$$\frac{x_{n+1}}{x_n} = \frac{\prod_{k \ge n} (1 + y_k)}{\prod_{k \ge n+1} (1 + y_k)}$$

for $n \ge 1$. A simple computation shows that

$$\prod_{k \ge 1} (1 + y_k) = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\,\cdots,$$

where the product on the right was shown by John Wallis (1616-1703) to have the value $\pi/2$.

The next result is an attempt to answer a question suggested by the previous example: What sequences are q-sequences?

7. *Theorem:* Let $\{y_n\}$ be a sequence of positive terms in $\ell^{(1)}$. Then, there exists a 1-parameter family of sequences $\{x_n(t)\}$ for $t > 0$ such that

$$\frac{x_{n+1}(t)}{x_n(t)} = 1 + y_n$$

for all $n \ge 1$.

Proof: In order to have that $x_{n+1}/x_n = 1 + y_n$, one must solve the infinite linear system

$$\begin{cases} x_1(1 + y_1) - x_2 = 0 \\ x_2(1 + y_2) - x_3 = 0 \\ \qquad\qquad \vdots \end{cases}$$

Set $x_1 = t$, an arbitrary positive real number. Then $x_2 = t(1 + y_1)$, $x_3 = t(1 + y_1)(1 + y_2)$, and by induction,

$$x_{n+1} = t \prod_{k \le n} (1 + y_k).$$

Therefore, the q-sequence corresponding to $\{x_n(t)\}$ is given by

$$\frac{x_{n+1}(t)}{x_n(t)} = \frac{t \prod_{k \le n} (1 + y_k)}{t \prod_{k \le n-1} (1 + y_k)} = 1 + y_n.$$

This establishes the result.

In Theorem 2 above, the limit z of a convergent q-sequence corresponding to a recursively generated sequence was shown to satisfy a functional equation involving the generating function for the original sequence. This generating function was required to be positively homogeneous of degree 1, continuous and nonnegative in the first quadrant. According to Theorem 4, if the generating function is the restriction of a linear form, then the limit of the q-sequence can be explicitly calculated and does not depend on the initial two terms of the original sequence.

The result that follows explores the nature of the functional equation by characterizing the class of functions to which Theorem 2 applies and provides examples to show that the independence of the limit z of a q-sequence with respect to the initial terms p and q of the original sequence, which was one of the conclusions obtained in Theorem 4, no longer holds in the general case.

8. *Theorem:* A function $f: [0, \infty) \times (0, \infty) \to [0, \infty)$ is continuous, positive on $(0, \infty)^2$, and positively homogeneous of degree 1 if and only if there is a continuous function $\gamma: (0, \infty) \to (0, \infty)$ which is such that

(i) $\gamma(t) = f(1, t)$ for all $t \in (0, \infty)$

and

(ii) $\lim_{t \to \infty} \frac{\gamma(t)}{t}$ exists and is finite.

Proof: Suppose $f: [0, \infty) \times (0, \infty) \to [0, \infty)$ is continuous, positive on $(0, \infty)^2$ and positively homogeneous of degree 1. Set

$$\gamma(t) = f(1, t) \text{ for } t \in (0, \infty).$$

Then γ is continuous and positive, and

$$\lim_{t \to \infty} \frac{\gamma(t)}{t} = \lim_{t \to \infty} f\left(\frac{1}{t}, 1\right) = f(0, 1)$$

exists and is finite, due to the continuity of f and the fact that f is positively homogeneous of degree 1.

Conversely, suppose that $\gamma: (0, \infty) \to (0, \infty)$ is continuous and that

$$\lim_{t \to \infty} \frac{\gamma(t)}{t}$$

exists and is finite. Set

$$\alpha = \lim_{t \to \infty} \frac{\gamma(t)}{t}$$

and define f on $[0, \infty) \times (0, \infty)$ by setting

$$f(x, y) = \begin{cases} x\gamma(y/x) & \text{if } x \neq 0 \\ y\alpha & \text{if } x = 0. \end{cases}$$

f is continuous for $x > 0$ and $y > 0$ since is continuous on $(0, \infty)$, $(x, y) \to y/x$ is continuous for $x > 0$ and $y > 0$ and the projection $(x, y) \to x$ is continuous everywhere.

If $S_1 = \{(x, y) : x > 0, y > 0\}$, $S_2 = \{(0, y) : y > 0\}$ and if y_0 is a fixed positive number, then

$$\lim_{\substack{(x, y) \to (0, y_0) \\ (x, y) \in S_1}} f(x, y) = \lim_{\substack{(x, y) \to (0, y_0) \\ (x, y) \in S_1}} x\gamma\left(\frac{y}{x}\right) = \lim_{\substack{(x, y) \to (0, y_0) \\ (x, y) \in S_1}} y \left[\frac{\gamma\left(\frac{y}{x}\right)}{\frac{y}{x}}\right] = y_0\alpha$$
$$= f(0, y_0),$$

and

$$\lim_{\substack{(x, y) \to (0, y_0) \\ (x, y) \in S_2}} f(x, y) = \lim_{y \to y_0} \alpha y = y_0\alpha = f(0, y_0).$$

Thus,

$$\lim_{(x, y) \to (0, y_0)} f(x, y) = f(0, y_0),$$

and f is continuous at $(0, y_0)$. It has now been demonstrated that f is continuous on $[0, \infty) \times (0, \infty)$.

The function f is also positively homogeneous of degree 1. For, if $\lambda > 0$,

$$f(\lambda x, \lambda y) = \begin{cases} (\lambda x)\gamma\left(\frac{\lambda y}{\lambda x}\right) & \text{if } x \neq 0 \\ (\lambda y)\alpha & \text{if } x = 0 \end{cases} = \begin{cases} \lambda\, x\gamma\left(\frac{y}{x}\right) & \text{if } x \neq 0 \\ \lambda(y\alpha) & \text{if } x = 0. \end{cases}$$

Therefore, $f(\lambda x, \lambda y) = \lambda f(x, y)$. □

Since $x_{n+1} = f(x_n, x_{n-1})$ implies

$$q_{n+1} = f\left(1, \frac{1}{q_n}\right) = \gamma\left(\frac{1}{q_n}\right),$$

the question of the convergence of q-sequences is equivalent to examining the convergence of sequences $\{q_n\}$ generated by choosing $q_2 > 0$ and defining q_n for $n \geq 3$ by

$$q_n = \gamma\left(\frac{1}{q_{n-1}}\right)$$

for some positive continuous function γ on $(0, \infty)$ having the property that

$$\lim_{t \to \infty} \frac{\gamma(t)}{t}$$

exists and is finite. Using this fact, examples can be constructed quite easily. The following examples, which were constructed in this way, show that limits of q-sequences depend in general on the starting values x_1 and x_2.

9. *Example:* Let $\gamma(t) = 1/t^2$. Starting with $q_2 = 1$, it follows that $q_n = 1$ for all n and $\lim_{t \to \infty} q_n = 1$. However, if $q_2 = 2$, then it is easy to show that

$$q = 2^{2^{n-2}}$$

for $n \geq 2$, so that $\{q_n\}$ diverges. This shows convergence is dependent on the starting values. In this example,

$$f(x, y) = x\gamma(y/x) = x^3/y^2.$$

x_1 and x_2 could be taken as $x_1 = x_2 = 1$ and $x_1 = 1$, $x_2 = 2$, respectively.

10. Example: Let $\gamma(t) = 1/t$ and let $q_2 = r > 0$ be arbitrarily chosen. Then

$$q_3 = \gamma\left(\frac{1}{q_2}\right) = r.$$

Similarly, $q_n = r$ for all $n \geq 4$. Therefore, $\{q_n\}$ is the constant sequence

$$\{r, r, \ldots, r, \ldots\},$$

which converges to r. In this example, it is seen that each positive real number is the limit of some q-sequence for the same generating function. Here,

$$f(x, y) = x\gamma\left(\frac{y}{x}\right) = \frac{x^2}{y}.$$

Starting values may be taken as $x_1 = 1$ and $x_2 = r > 0$.

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THE ALPHA AND THE OMEGA OF THE WYTHOFF PAIRS

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(Submitted February 1987)

1. Introduction

The Wythoff number pairs have been much discussed in the literature on Fibonacci integers (see [1] for references up to 1978). And in [2] and [3] M. Bicknell-Johnson treats generalizations of Wythoff numbers which provide number triples with many interesting properties. In this paper we present three different ways to generate the Wythoff pairs, and, with some trepidation in view of the extent of the literature on them, claim that these are "new." We emphasize the notion of "generation" (in contrast to "giving a formula"), and introduce Fibonacci word patterns [3] as a tool to define n-tuple generating processes.

A determinantal relation for the Wythoff pairs is described, which makes further use of the word-pattern tools.

In the final section we show how similar methods can be used to generate and study sequences of integer triples. Three examples are given, and each is an attempt to generalize aspects of the Wythoff pairs-sequence.

It is clear to us that these tools and methods hold much promise for developing a general theory of sequences of integer n-tuples which have structures related to Fibonacci word patterns.

2. Notation and Definitions

The main tool to be used below is the Fibonacci word pattern, which we developed in [3]. We shall also use an operation of merging two integer sequences, and its inverse; we shall use the terms *addmerge* and *submerge* for these two operations.

Definitions and examples

To keep the exposition brief and readable, we now give somewhat informal definitions of the operations and concepts we wish to use. The examples will make the intended operations perfectly clear.

(i) *Fibonacci word patterns* $(W_1W_2W_3 \ldots W_n \ldots)$
A *word pattern* is a concatenation of a sequence of words W_1, W_2, W_3, ..., W_n, ... The words are formed using characters from a given *letter set* such as $\{0, 1\}$ or $\{a, b, c\}$. The basic word pattern is obtained by repeatedly using the concatenation recurrence

$$W_{n+2} = W_nW_{n+1}, \text{ with } W_1 = A,\ W_2 = B.$$

We shall denote the resulting pattern by $F(A, B)$. Then:

$$F(A, B) = A, B, AB, BAB, ABBAB, \ldots .$$

(N.B. The commas on the right should be removed; they are inserted to show the boundaries of successive words in the pattern.)

Examples to be used below:

$$\alpha = (1, 0) = 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ \ldots \text{ (using } A = 1, B = 0);$$

$$\omega = (1, 01) = 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ \ldots \text{ (using } A = 1, B = 01).$$

These two binary word patterns are, respectively, the *alpha* and the *omega* referred to in the title of this article. The ω pattern is named after Wythoff, for reasons which will become abundantly clear as the paper develops.

We shall also use the tribonacci word pattern (with $W_1 = A$, $W_2 = B$, $W_3 = C$)

$$F(A, B, C) = A, B, C, ABC, BCABC, \ldots \quad (W_{n+3} = W_n W_{n+1} W_{n+2}).$$

(ii) *Set-sequences*

In [4] we introduced the following construction (though with a slightly different notation). Let $\{S_n\}$ be a sequence of sets, and let $\{a_n\}$ be a sequence of integers. The set-sequence is formed using the following recurrence

$$S_{n+2} = S_n \cup S_{n+1} + a_n,$$

with S_1, S_2 being any given sets. The + operation is to be carried out as indicated by

$$\{s_1, s_2, \ldots, s_i, \ldots\} + a = \{s_1 + a, s_2 + a, \ldots, s_i + a, \ldots\}.$$

(iii) *Addmerging and submerging*

A merging operation, and its inverse, should be clear from the following definition and example.

Let S and T be any monotone increasing sequences. Then the *addmerge* of S and T (written $S \frown T$) is obtained by taking the multi-union of the two sequences and sorting them into monotonic increasing order. By "multi-union" we mean that integer repetitions are to be allowed.

Example

Let $S = \{1, 3, 5, 7, 9, \ldots\}$ and $T = \{2, 5, 8, 11, \ldots\}$. Then

$$S \frown T = \{1, 2, 3, 5, 5, 7, 8, 9, 11, \ldots\}.$$

The inverse of addmerge is *submerge*, which we shall write $S \smile T$. This operation simply removes the sequence S from the sequence S (all elements of T, that is, which happen to be in S).

(iv) *Sequence notations*

We shall use either Greek letters or underlined, lowercase Roman letters to denote sequences; and will use or omit subscripts on individual sequence elements as is appropriate. The following examples illustrate our notation, and will be needed below.

$\underline{n} = \{1, 2, 3, \ldots\}$ the natural numbers

$\underline{n}^+ = \{0, 1, 2, \ldots\}$ the natural numbers with zero

$\underline{f} = \{1, 1, 2, 3, 5, \ldots\}$ the Fibonacci integers $\{F_n\}$

$\underline{f}' = \{1, 2, 3, 5, \ldots\} = \{F_{n+1}\}$, $n = 1, 2, \ldots$

$\underline{f}'' = \{2, 3, 5, 8, \ldots\} = \{F_{n+2}\}$, $n = 1, 2, \ldots$

$\underline{\omega}_1 = \{1, 3, 4, 6, 8, \ldots\}$ first members of Wythoff pairs, equals $\{[n\alpha]\}$ where $\alpha = \frac{1}{2}(1 + \sqrt{5})$.

$\underline{\omega}_2 = \{2, 5, 7, 10, 13, \ldots\}$ second members of Wythoff pairs, equals $\{[n\alpha^2]\}$.

$$\underline{\omega} = \binom{\underline{\omega}_1}{\underline{\omega}_2} = \binom{1}{2}\binom{3}{5}\binom{4}{7}\binom{6}{10}\binom{8}{13} \ldots \text{ Wythoff pair-sequence.}$$

(v) *Binary word pattern representations*

Let B be a sequence of 0's and 1's, say,

$$B = b_1, b_2, b_3, \ldots, b_n, \ldots, \text{ all } b_i \in \{0, 1\}.$$

And let $\underline{s} = s_1, s_2, s_3, \ldots, s_n, \ldots$ be an integer sequence. Then,

$\underline{s} * B \equiv$ the subsequence from $\underline{s}$ whose elements are in the positions where the 1's occur in B.

For example, if $B = 0, 1, 0, 1, 0, 1, \ldots$, then,

$$\underline{n} * B = 2, 4, 6, 8, \ldots .$$

3. Three Ways of Generating the Wythoff Pairs

We now use our word patterns and sequence notations to give three different ways to generate the Wythoff pairs.

(i) *Use of the omega sequence*

Recall from Section 2 that the binary word pattern omega is

$$\omega = F(1, 01) = 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1 \ldots$$

It may be observed that the 1's occur in positions $\underline{\omega}_1 = 1, 3, 4, 6, 8, \ldots$; and the 0's occur in positions $\underline{\omega}_2 = 2, 5, 7, 10, 13, \ldots$. Thus, ω contains all the information needed for producing the Wythoff pairs. Using the notations of 2(iv) and 2(v), we can write:

$$\underline{\omega} = \binom{\underline{\omega}_1}{\underline{\omega}_2} = \binom{\underline{n} * F(1, 01)}{\underline{n} * F(0, 10)}$$

Note that we can do certain algebraic operations with sequences and our new notation.

Thus, for example:

$$\underline{n} = \underline{\omega}_1 \frown\!\frown \underline{\omega}_2 = \underline{n} * [F(1, 01) + F(0, 10)]$$

where + is mod 2 addition of elements.

In [6] two methods of proof are given to demonstrate that omega [i.e., $F(1, 01)$] does in fact generate the Wythoff sequences

$$\underline{\omega}_1 = \{[n\alpha]\} \quad \text{and} \quad \underline{\omega}_2 = \{[n\alpha^2]\}$$

as claimed.

(ii) *Use of Fibonacci set-sequences*

In 2(ii) above, we explained the recurrence for generating Fibonacci set-sequences, viz.,

$$S_{n+2} = S_n \cup S_{n+1} + a_n.$$

Let $S_{-1} = \{0\}$ and $S_0 = \{0\}$, and $\{a_n\} \equiv \underline{f}' = 1, 2, 3, 5, \ldots$. Then,

$$S_1 = \{1\},\ S_2 = \{3\},\ S_3 = \{4, 6\}, \text{ etc.};$$

and it soon becomes clear that

$$\bigcup_{n=1}^{\infty} S_n = \underline{\omega}_1.$$

Similarly, if $\{a_n\} = \underline{f}'' = 2, 3, 5, 8, \ldots$ and the same S_{-1}, S_0 are used, the infinite union

$$\bigcup_{n=1}^{\infty} S_n = \underline{\omega}_2.$$

Proofs of these assertions are given in [6].

(iii) *Use of Fibonacci magic matrices*

In [5] we decided that square matrices all of whose elements were Fibonacci integers, whose diagonal, row, and column sums were Fibonacci integers and, moreover, whose powers also possessed these properties, deserved to be called *magic*. The spectral radius of these matrices is α, the golden mean. The simplest such matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}. \quad \text{Note that } A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

The characteristic polynomial of A is $\alpha^2 - \alpha - 1$; and of A^2 is $\lambda^2 - 3\lambda + 1$, which has maximum root $\lambda = \alpha^2$ with $\alpha = \frac{1}{2}(1 + \sqrt{5})$.

Many properties of A have been noted in the literature, but the following relationships with the Wythoff pairs may possibly be new. We give them without full proof.

Proposition (generation of $\underline{\omega}$, the Wythoff pairs sequence):

(A) $\underline{\omega} = \left\{[nA + m_n I]\binom{1}{1}\right\}$,

where I is the 2×2 identity matrix, $\underline{n}$ is the natural number sequence, and

$$\underline{m} = \{m_n\} = 0, 1, 1, 2, 3, 3, 4, 4, 5, 6, 6, 7, \ldots .$$

(N.B. The generation of $\underline{m}$ by a Fibonacci word pattern is given below.)

(B) $\underline{\omega} = \left\{[r_n A + m_n A^2]\binom{1}{1}\right\}$, where

$$\underline{r} = \{r_n\} = 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, \ldots .$$

Note that (B) follows from (A) since $\underline{m} + \underline{r} = \underline{n}$ and $A^2 = A + I$.

Generation of $\underline{m}$ *and* $\underline{r}$

The sequence $\underline{m}$ is generated as follows. Take the Fibonacci word pattern

$$F(1, 2) = 1, 2, 12, 212, 12212, \ldots .$$

We can use the elements of this sequence as *frequencies*, drawing elements from the sequence $\underline{n} = 0, 1, 2, 3, 4, \ldots$ with these frequencies. This gives a natural extension to the star operation which we defined in 2(v), in connection with binary words. Thus, we get

$$\underline{m} = \underline{n}^+ * F(1, 2) = 0, 11, 2, 33, 44, 5, 66, \text{etc.}$$

as required. With this very useful extended operation (which includes the earlier one, if 0 frequency is interpreted as "leave out"), we see that the sequence $\underline{r}$ in proposition (B) for $\underline{\omega}$ is:

$$\underline{r} = 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, \ldots = \underline{n} * F(2, 3).$$

Corollary: $\underline{n}^+ * F(1, 2) + \underline{n} * F(2, 3) = \underline{n}$, since $\underline{m} + \underline{r} = \underline{n}$.

The attractiveness of the method of generation of the Wythoff pairs just given lies in comparisons that can be made with the classical generation of the individual sequences. To spell these out, we note that $\underline{\omega}_1 = 1, 3, 4, 6, 8, \ldots$ is generated by $[n\alpha]$, and $\underline{\omega}_2 = 2, 5, 7, 10, 13, \ldots$ is generated by $[n\alpha^2]$, where α is the golden mean. By comparison, the generation formula given in (A) above for

$$\underline{\omega} = \binom{1}{2}\binom{3}{5}\binom{4}{7}\binom{6}{10}\binom{8}{13} \ldots$$

uses only nA, where A is a matrix having α as spectral radius, and a sequence $\underline{n}^+ * F(1, 2)$: and the sequence $F(1, 2)$ has the same pattern as that other "α," the basic Fibonacci word pattern referred to in the title of our paper.

4. A Determinantal Relation for the Wythoff Pairs

The following interesting relationship is reminiscent of the well-known Fibonacci relation

$$\begin{vmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{vmatrix} = (-1)^{n+1}.$$

Consider the Wythoff pair-sequence

$$\underline{\omega} = \begin{pmatrix} \underline{\omega}_1 \\ \underline{\omega}_2 \end{pmatrix}.$$

To simplify the notation, we write $\underline{\omega}_1 = \underline{a} = \{a_i\}$ and $\underline{\omega}_2 = \underline{b} = \{b_i\}$. Then,

$$\underline{\omega}_1 = \binom{a_1}{b_1}\binom{a_2}{b_2}\binom{a_3}{b_3} \cdots = \binom{1}{2}\binom{3}{5}\binom{4}{7} \cdots$$

Let us pass a 2×2 window along this sequence and compute determinants as we go. Thus,

$$\left\{\begin{vmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{vmatrix}\right\} = \begin{array}{l} -1, 1, -2, -2, 3, -3, 4, -4, -4, \\ 6, -5, -5, 8, -6, 9, -7, -7, 11, -8, \ldots . \end{array}$$

There is clearly an interesting pattern to the sequence, but how can we capture it in a formula? It is here that our word pattern notation becomes really useful. Let us *submerge* the negative and the positive elements, to find:

$$-1, -2, -2, -3, -4, -4, -5, -5, -6, -7, -7, -8, \ldots$$

and

$$1, 3, 4, 6, 8, 9, 11, \ldots .$$

Now we see that the negative sequence can be written as $(-\underline{n}) * F(1, 2)$. And the positive sequence is just $\underline{\omega}_1$.

Thus, we state finally

Proposition:

(i) The determinants of successive Wythoff pairs are given by

$$\Delta\underline{\omega} \equiv \left\{ \begin{vmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{vmatrix} \right\} = \underline{\omega}_1 \frown [(-\underline{n}) * F(1,\ 2)],$$

(with the addmerge ignoring minus signs). □

It might be said that to complete the above proposition we must give a precise formula for the n^{th} term of $\Delta\underline{\omega}$, whereas we have given only a sequence generator. We shall do this later. As we said in the Introduction, we wish first to emphasize ways in which our notation can describe the generation of interesting sequences. Picking out particular values of a sequence is always harder to do. In [3] and [6] are given many formulas for making that task easier, being results concerning counts of runs and runlengths of particular letters or integers in given Fibonacci word patterns.

(ii) Formulas for the n^{th} term in the sequence of determinants are:

$$(\Delta\underline{\omega})_n = \begin{cases} \underline{\omega}_{1i}, & \text{when } n \in \underline{\omega}_2 \text{ with } n = [i\alpha^2], \\ & \text{where } \alpha \text{ is the golden ratio;} \\ -i, & \text{when } [(i-1)\alpha^2] < n < [i\alpha^2]. \end{cases}$$

Proof: It may be seen that the positive terms in the sequence of determinants, namely,

$$\underline{\omega}_1 = 1, 3, 4, 6, 8, 9, 11, \ldots,$$

occur at positions

$$\underline{\omega}_2 = 2, 5, 7, 10, 13, 15, 18, \ldots .$$

This is fascinating in itself, and immediately explains the given formulas, because the positive terms occur when $n = [i\alpha^2]$. □

[N.B. Because of the fact just noted in the proof, we could give the determinant sequence as

$$\Delta\underline{\omega} = \underline{n} * F(0,\ 10) \frown (-\underline{n}) * F(1,\ 2).$$

An immediate corollary of the fact that $\Delta\underline{\omega}$ includes the sequence

$$(-\underline{n}) * F(1,\ 2)$$

is the following proposition on the representation of the natural numbers in terms of the Wythoff numbers.

(iii) In terms of the Wythoff numbers, every integer N can be represented as follows:

either uniquely as $N = a_{i+1}b_i - b_{i+1}a_i$ using Wythoff pairs;

or in two ways using a run of three consecutive Wythoff pairs thus: $N = a_{i+1}b_i - b_{i+1}a_i = a_{i+2}b_{i+1} - b_{i+2}a_{i+1}$.

There any many other interesting things that could be said about the sequence $\Delta\underline{\omega}$. One more will have to suffice. Suppose we mark the sequence into words, each of which ends at a positive integer thus:

$$(-1,\ 1)\ (-2,\ -2,\ 3)\ (-3,\ 4)\ (-4,\ -4,\ 6)\ (-5,\ -5,\ 8)\ (-6,\ 9)$$
$$(-7,\ -7,\ 11)\ (-8,\ 12)\ \text{etc.}$$

The *lengths* of these words have the pattern $F(2,\ 3)$. And their *totals* follow the pattern

$$0,\ -1,\ 1,\ 2,\ -2,\ -2,\ 3,\ -3,\ 4,\ -4,\ -4,\ 6,\ -5,\ -5,\ 8,\ \ldots\ .$$

The first 0 indicates that the sum of the first two determinants is:

$$(\Delta\underline{\omega})_1 + (\Delta\underline{\omega})_2 = a_1b_2 + a_2(b_3 - b_1) - a_3b_2 = 0.$$

If we consider the sequence of word totals, it appears that it will oscillate with increasing amplitude; and that the sum

$$\sum_{i=1}^{n}(\Delta\underline{\omega})_i$$

will equal zero infinitely often; but we have not established proofs of these observations.

5. Generation of Other Pair-Sequences

Any Fibonacci word-pattern which uses a binary letter-set can be used to generate a pair-sequence. For example, the *alpha* sequence

$$\alpha = F(1,\ 0) = 1,\ 0,\ 10,\ 010,\ 10010,\ 01010010,\ \ldots$$

generates the following:

$$\underline{\alpha} = \begin{pmatrix}\underline{\alpha}_1\\ \underline{\alpha}_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}\begin{pmatrix}6\\5\end{pmatrix}\begin{pmatrix}8\\7\end{pmatrix}\ \begin{matrix}\ldots\text{ the 1 positions }\ldots\\ \ldots\text{ the 0 positions }\ldots\ .\end{matrix}$$

A question of interest now is whether $\underline{\alpha}$ can be expressed in terms of the Wythoff pairs, and vice-versa. Using our word-pattern tools, we find as follows:

$$\begin{aligned}\underline{\alpha}_1 &= 1\ 3\ 6\ 8\ 11\ 14\ 16\ 19\ \ldots\\ &= (1\ 3\ 4\ 6\ 8\ 9\ 11\ 12\ 14\ 16\ 17\ 19) \sim (4\ 9\ 12\ 17\ \ldots)\\ &= \underline{\omega}_1 \sim [\underline{\omega}_1 * F(0,\ 01)] = \underline{\omega}_1 * F(1,\ 10);\end{aligned}$$

and

$$\begin{aligned}\underline{\alpha}_2 &= 2\ 4\ 5\ 7\ 9\ 10\ 12\ 13\ 15\ 17\ 18\ \ldots\\ &= (2\ 5\ 7\ 10\ 13\ 15\ \ldots) \sim (4\ 9\ 12\ 17\ \ldots)\\ &= \underline{\omega}_2 \sim [\underline{\omega}_1 * F(0,\ 01)].\end{aligned}$$

Thus, we have:

$$\underline{\alpha} = \begin{pmatrix}\underline{\alpha}_1\\ \underline{\alpha}_2\end{pmatrix} = \begin{pmatrix}\underline{\omega}_1 \sim [\underline{\omega}_1 * F(0,\ 01)]\\ \underline{\omega}_2 \sim [\underline{\omega}_1 * F(0,\ 01)]\end{pmatrix}.$$

By similar methods we can invert these equations thus:

$$\left.\begin{aligned}\underline{\omega}_1 &= \underline{\alpha}_1 \sim [\underline{\alpha}_2 * F(0,\ 10)]\\ \underline{\omega}_2 &= \underline{\alpha}_2 \sim [\underline{\alpha}_2 * F(0,\ 10)]\end{aligned}\right\}.$$

And so the *alpha* pair-sequence can be expressed in terms of the *omega* (Wythoff) pair-sequence; and vice-versa.

It is evident that by such means an infinite number of pair-sequences can be generated, and their properties studied by establishing relationships between them and the fundamental Wythoff pairs. A new kind of number theory could be developed, based upon the sequences $\underline{\omega}_1$ and $\underline{\omega}_2$, and related to the "ordinary" number theory based on $\underline{n}$ through the functions $[n\alpha]$ and $[n\alpha^2]$.

Finally, we give an indication of how these methods can be extended to study sequences of triples.

6. Generation of Triple-Sequences

We shall show, proceeding by examples and comments upon them, how to generate *triple-sequences* in three different ways. The first uses Fibonacci word-pattern with letter-set $\{a, b, c\}$; the second uses a tribonacci word pattern with letter-set $\{a, b, c\}$; and the third uses a "magic" tribonacci matrix.

(i) *Use of a Fibonacci word pattern*

Consider the following word pattern:

$$F(a, bc) = a,\ bc,\ abc,\ bcabc,\ abcbcabc,\ \dots\ .$$

Listing the *positions* of a, b, and c, respectively, produces the following triple-sequence:

$$\begin{bmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \\ 8 \end{bmatrix}\begin{bmatrix} 12 \\ 10 \\ 11 \end{bmatrix}\begin{bmatrix} 17 \\ 13 \\ 14 \end{bmatrix}\begin{bmatrix} 22 \\ 15 \\ 16 \end{bmatrix}\begin{bmatrix} 25 \\ 18 \\ 19 \end{bmatrix}\begin{bmatrix} 30 \\ 20 \\ 21 \end{bmatrix} \dots$$

It will be noted that, as might be expected since the word pattern is Fibonacci, the three component sequences can each be expressed in terms of the Wythoff numbers. Thus:

$$\begin{bmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \end{bmatrix} = \begin{bmatrix} \underline{\omega}_1 * \alpha \\ \underline{\omega}_2 \\ \underline{\omega}_2 + 1 \end{bmatrix}, \text{ where } \alpha = F(1, 0).$$

Then $\underline{a}_n = (\underline{\omega}_1)_i$, $\underline{b}_n = (\underline{\omega}_2)_n$, and $\underline{c}_n = (\underline{\omega}_2)_n + 1$, where

$$i = \underline{c}_{n-1} = (\underline{\omega}_2)_{n-1} + 1.$$

Note also that $\underline{a}$, $\underline{b}$, $\underline{c}$ are each strictly increasing sequences, they are non-intersecting, and their union equals $\mathbb{Z}^+$: all properties of the Wythoff pairs-sequence. Their proof is immediate from the way in which $F(a, bc)$ is expanded.

Another interesting point is that the parity of the terms in $\underline{a}$ is alternately odd and even. And then, since the sum $(b_n + c_n)$ is always odd, we have the sum $(a_n + b_n + c_n)$ also of alternating parity.

The parity patterns, and more generally mod 3, mod 4, etc., patterns of elements of multi-sequences generated from word patterns would seem to be worthy of study.

(ii) *Use of a tribonacci pattern*

Consider next the tribonacci expansion of $F(a, b, c)$ and the triple-sequence it generates through the positions of a, b, c in the resulting pattern.

$$F(a, b, c) = a,\ b,\ c,\ abc,\ bcabc,\ cabcbcabc,\ \ldots$$

$$\begin{bmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \\ 8 \end{bmatrix}\begin{bmatrix} 13 \\ 10 \\ 11 \end{bmatrix}\begin{bmatrix} 18 \\ 14 \\ 12 \end{bmatrix}\begin{bmatrix} 21 \\ 16 \\ 15 \end{bmatrix}\begin{bmatrix} 26 \\ 19 \\ 17 \end{bmatrix} \ldots$$

This triple-sequence again clearly has the property that each element sequence is monotone increasing, and the three sequences partition $\mathbb{Z}^+$.

When we first studied this sequence, we hoped that we would find simple relationships between $\underline{a}$, $\underline{b}$, and $\underline{c}$, respectively,

$$\{[n\tau]\},\ \{[n\tau^2]\},\ \text{and}\ \{[n\tau^3]\},$$

where τ is the positive root of the tribonacci equation

$$x^3 - x^2 - x - 1 = 0.$$

This would have been an excellent generalization of the Wythoff pairs property whereby $\underline{a} = \{[n\alpha]\}$ and $\underline{b} = \{[n\alpha^2]\}$. Unfortunately, we have not been able to find such relationships, although there seems to be hope for relating Fibonacci word patterns to the sequences of first differences $\{\Delta[n\tau]\}$, etc. To encourage the reader to search for these, we show the first few tribonacci triples ($\tau \doteq 1.839$):

$$\begin{bmatrix} [n\tau] \\ [n\tau^2] \\ [n\tau^3] \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \\ 18 \end{bmatrix}\begin{bmatrix} 7 \\ 13 \\ 24 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \\ 31 \end{bmatrix}\begin{bmatrix} 11 \\ 20 \\ 37 \end{bmatrix}\begin{bmatrix} 12 \\ 23 \\ 43 \end{bmatrix} \ldots$$

(iii) *Use of a tribonacci magic matrix*

Our third attempt to generalize the Wythoff pairs is to take a 3×3 matrix which generalizes the "magic" properties of the 2×2 matrix used in Section 3(iii), and attempt to generate with it a unique sequence of triples whose members partition $\mathbb{Z}^+$. Once again we must confess that we have not found a fully satisfactory way of defining such a unique sequence; but in the spirit of the aims of this paper we believe it is worth presenting our attempt.

The tribonacci magic matrix we shall use is:

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Note that the characteristic polynomial of T is $-(\lambda^3 - \lambda^2 - \lambda - 1)$ so its spectral radius is $\tau \doteq 1.839$.

Note further that the row sums of powers of T are

$$T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ T^2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \ldots,\ \text{which give} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \\ 17 \end{bmatrix}\begin{bmatrix} 9 \\ 17 \\ 31 \end{bmatrix},\ \ldots,$$

with each element of the triples being in tribonacci sequence. These are the magic properties of T.

The question we ask now is the following. In 3(iii)(B) we generated the Wythoff pairs using only the 2×2 matrices A and A^2 and coefficients from the sequences $\underline{n} * F(2, 3)$ and $\underline{n}^+ * F(1, 2)$. Can we generate a unique sequence of *triples*, a *T-sequence*, using only the 3×3 matrices T, T^2, and T^3, together with coefficients from sequences which can be defined in terms of Fibonacci word patterns? Furthermore, can we require the three member sequences of the triple sequence to be strictly increasing, and to partition $\mathbb{Z}^+$? If we can find such a T-sequence uniquely, it will constitute an excellent generalization of the Wythoff pairs sequence.

Out attempt, down through the first twenty triples, is tabulated below, showing the triples horizontally for convenience.

Triples (a, b, c) Elements			Row sums of $(pT + qT^2 + rT^3)$ Coefficients		
a	b	c	p	q	r
1	3	5	0	1	0
2	4	8	1	1	0
6	10	20	2	1	1
7	11	23	3	1	1
9	15	31	4	2	1
12	22	31	5	4	1
13	25	49	5	5	1
14	26	52	6	5	1
16	30	60	7	6	1
17	33	65	7	7	1
18	34	68	8	7	1
19	35	71	9	7	1
21	39	79	10	8	1
23	43	87	11	9	1
24	46	92	11	10	1
27	51	101	11	10	2
28	54	106	11	11	2
29	55	109	12	11	2
32	62	122	13	13	2
36	70	136	13	14	3
37	73	141	13	15	3

It will be noted that we have succeeded in advancing ($\underline{a}$, $\underline{b}$, $\underline{c}$) thus far without increasing p, q, and r by more than 1 at each step. But, as we confessed above, we have not determined a formula for advancing the triple sequence indefinitely while satisfying all the requirements for generalizing the Wythoff pairs to triples.

7. Summary

In this paper we have defined word patterns, and various tools derived from them, in order to generate and study increasing sequences of integer pairs and integer triples.

A particular outcome of our study of pair-sequences as derived from Fibonacci binary word patterns was to show how all such sequences (and there is an infinite class of them) might be related to the Wythoff pairs.

It is hoped that we have convinced the reader that there is much scope for developing a number theory of integer pairs defined by binary sequence representations and using tools such as we have described. The title of our paper, namely, "The Alpha and the Omega of Wythoff Pairs," might suggest that all has now been said upon the pairs. In fact we claim the opposite—that this paper can mark a beginning of a broad development in their study and application.

The path to a general study of triple sequences would seem to be a much harder (but nevertheless a most interesting) one to seek.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. Hillman

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

<u>B-634</u> *Proposed by P. L. Mana, Albuquerque, NM*

For how many integers n with $1 \le n \le 10^6$ is $2^n \equiv n \pmod 5$?

<u>B-635</u> *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

For all positive integers n, prove that

$$2^{n+1}\left[1 + \sum_{k=1}^{n}(k!k)\right] < (n + 2)^{n+1}.$$

<u>B-636</u> *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

Solve the difference equation

$$x_{n+1} = (n + 1)x_n + \lambda(n + 1)^3[n!(n! - 1)]$$

for x_n in terms of λ, x_0, and n.

<u>B-637</u> *Proposed by John Turner, U. of Waikato, Hamilton, New Zealand*

Show that

$$\sum_{n=1}^{\infty} \frac{1}{F_n + \alpha F_{n+1}} = 1,$$

where α is the golden mean $(1 + \sqrt{5})/2$.

B-638 *Proposed by Herta T. Freitag, Roanoke, VA*

Find s and t as function of k and n such that

$$\sum_{i=1}^{k} F_{n-4k+4i-2} = F_s F_t.$$

B-639 *Proposed by Herta T. Freitag, Roanoke, VA*

Find s and t as function of k and n such that

$$\sum_{i=1}^{k} L_{n-4k+4i-2} = F_s L_t.$$

SOLUTIONS

No Fibonacci Pythagorean Triples

B-610 *Proposed by L. Kuipers, Serre, Switzerland*

Prove that there are no positive integers r, s, and t such that (F_r, F_s, F_t) is a Pythagorean triple (that is, such that $F_r^2 + F_s^2 = F_t^2$).

Solution by Marjorie Bicknell-Johnson, Santa Clara, CA

V. E. Hoggatt, Jr., proved that no three distinct Fibonacci numbers can be the lengths of the three sides of a triangle. (See page 85 of *Fibonacci and Lucas Numbers*, Houghton Mifflin Mathematics Enrichment Series, Houghton Mifflin, Boston, 1969.) Since a Pythagorean triple gives integral lengths for the sides of a right triangle, his result is more general. Hoggatt's elegant proof follows, where a, b, and c are the sides of the triangle:

> In any triangle, we must have $a + b > c$, $b + c > a$, and $c + a > b$. For any three consecutive Fibonacci numbers, $F_n + F_{n+1} = F_{n+2}$, and so there can be no triangle with sides having measures F_n, F_{n+1}, F_{n+2}. In general, consider Fibonacci numbers, F_r, F_s, F_t, where $F_r \le F_{s-1}$ and $F_{s+1} \le F_t$. Since $F_{s-1} + F_s = F_{s+1}$ and $F_r \le F_{s-1}$, we have $F_r + F_s \le F_{s+1}$, and since $F_{s+1} \le F_t$, we have $F_r + F_s \le F_t$. Therefore, there can be no triangle with sides having measure F_r, F_s, and F_t.

Also solved by Charles Ashbacher, Paul S. Bruckman, Piero Filipponi, C. Georghiou, Sahib Singh, Lawrence Somer, and the proposer.

Each Term a Multiple of 3

B-611 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S(n) = \sum_{k=1}^{n} L_{4k+2}.$$

For which positive integers n is $S(n)$ an integral multiple of 3?

Solution by Bob Prielipp, U. of Wisconsin-Oshkosh

We shall show that $S(n)$ is an integral multkple of 3 for each positive integer n.

The claimed result is an immediate consequence of the following lemma.

Lemma: 3 divides L_{4k+2} for each nonnegative integer k.

Proof: Because $L_2 = 3$, the specified result holds when $k = 0$. Let j be a nonnegative integer. Then

$$\begin{aligned} L_{4(j+1)+2} &= L_{4j+6} = L_{4j+4} + L_{4j+5} \\ &= (L_{4j+2} + L_{4j+3}) + (L_{4j+2} + 2L_{4j+3}) = 2L_{4j+2} + 3L_{4j+3}. \end{aligned}$$

Hence, if 3 divides L_{4j+2}, then 3 divides $L_{4(j+1)+2}$. The required result now follows by mathematical induction.

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, L. Kuipers, Chris Long, Br. J. M. Mahon, H.-J. Seiffert, Sahib Singh, Lawrence Somer, H. J. M. Wijers, Gregory Wulczyn, and the proposer.

When the Sum Is a Multiple of 7

B-612 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$T(n) = \sum_{k=1}^{n} F_{4k+2}.$$

For which positive integers n is $T(n)$ an integral multiple of 7?

Solution by Lawrence Somer, Washington, D.C.

By inspection, we observe that the period of $\{F_n\}$ modulo 7 is 16. Now,

$$\begin{aligned} &F_2 = 1 \equiv 1 \pmod 7, \quad F_6 = 8 \equiv 1 \pmod 7, \\ &F_{10} = 55 \equiv -1 \pmod 7, \quad F_{14} = 377 \equiv -1 \pmod 7. \end{aligned}$$

It thus follows that

$$F_{4k+2} \equiv 1 \pmod 7 \text{ if } k \equiv 0 \text{ or } 1 \pmod 4$$

and

$$F_{4k+2} \equiv -1 \pmod 7 \text{ if } k \equiv 2 \text{ or } 3 \pmod 4.$$

Consequently, it follows that $T(n)$ is an integral multiple of 7 for a positive integer n if and only if n is an even integer.

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, Br. J. M. Mahon, Bob Prielipp, H.-J. Seiffert, Sahib Singh, H. J. M. Wijers, Gregory Wulczyn, and the proposer.

Finding the Constants

B-613 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Show that there exist integers a, b, and c such that

$$F_{n+p}^2 + F_{n-p}^2 = aF_n^2F_p^2 + b(-1)^pF_n^2 + c(-1)^nF_p^2.$$

Solution by C. Georghiou, University of Patras, Greece

We will show that $a = 5$ and $b = c = 2$. Indeed, from the identity

$$5F_n^2 = L_{2n} - 2(-1)^n,$$

we find

$$5F_{n+p}^2 + 5F_{n-p}^2 = L_{2n+2p} + L_{2n-2p} - 4(-1)^{n+p} = L_{2n}L_{2p} - 4(-1)^{n+p}$$

and

$$25F_n^2F_p^2 = L_{2n}L_{2p} - 2(-1)^pL_{2n} - 2(-1)^nL_{2p} + 4(-1)^{n+p}.$$

It follows, therefore, that

$$\begin{aligned} F_{n+p}^2 + F_{n-p}^2 - 5F_n^2F_p^2 &= (2(-1)^pL_{2n} + 2(-1)^nL_{2p} - 8(-1)^{n+p})/5 \\ &= 2(-1)^pF_n^2 + 2(-1)^nF_p^2. \end{aligned}$$

Also solved by Paul S. Bruckman, Herta T. Freitag, L. Kuipers, Br. J. M. Mahon, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

Quadruple Products Mod 8

B-614 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $L(n) = L_{n-2}L_{n-1}L_{n+1}L_{n+2}$ and $F(n) = F_{n-2}F_{n-1}F_{n+1}F_{n+2}$. Show that

$$L(n) \equiv F(n) \pmod 8$$

and express $[L(n) - F(n)]/8$ as a polynomial in F_n.

Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA

Using I_{20} and I_{29} in Hoggatt's *Fibonacci and Lucas Numbers*, we get:

$$L(n) = L_n^4 - 25 \quad \text{and} \quad F(n) = F_n^4 - 1.$$

Replacing L_n^2 by $5F_n^2 + 4(-1)^n$, we get

$$L(n) - F(n) = 24F_n^4 + 40(-1)^nF_n^2 - 8 \equiv 0 \pmod 8.$$

Hence,

$$\frac{L(n) - F(n)}{8} = 3F_n^4 + 5(-1)^nF_n^2 - 1.$$

Also solved by Paul S. Bruckman, Herta T. Freitag, C. Georghiou, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Gregory Wulczyn, David Zeitlin, and the proposer.

Identity for Iterated Lucas Numbers

B-615 *Proposed by Michael Eisenstein, San Antonio, TX*

Let $C(n) = L_n$ and $a_n = C(C(n))$. For $n = 0, 1, \ldots,$ prove that

$$a_{n+3} = a_{n+2}a_{n+1} \pm a_n.$$

Solution by C. Georghiou, University of Patras, Greece

It is easy to see that $a_n = \alpha^{L(n)} + \beta^{L(n)}$. Therefore,

$$\begin{aligned} a_{n+2}a_{n+1} &= (\alpha^{L(n+2)} + \beta^{L(n+2)})(\alpha^{L(n+1)} + \beta^{L(n+1)}) \\ &= \alpha^{L(n+3)} + \beta^{L(n+3)} + (-1)^{L(n+1)}(\alpha^{L(n)} + \beta^{L(n)}) \end{aligned}$$

from which the assertion follows.

Also solved by Paul S. Bruckman, Piero Filipponi, Herta T. Freitag, L. Kuipers, Br. J. M. Mahon, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, David Zeitlin, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

<u>H-428</u> *Proposed by Larry Taylor, Rego Park, NY*

Let j, m, and n be integers. Let a and b be relatively prime even-odd integers with b not divisible by 5. Let $A_n = aL_n + bF_n$. Then $A_n = A_{n+1} - A_{n-1}$ with initial values $A_1 = b + a$, $A_{-1} = b - a$.

Prove that the following three numbers

$$(2F_{n-j}A_{m-j},\ F_{n+j}A_{m+j},\ 2F_{2j}A_{n+m})$$

are in arithmetic progression.

<u>H-429</u> *Proposed by John Turner, Hamilton, New Zealand*

Fibonacci enthusiasts know what happens when they add two adjacent numbers of a sequence and put the result next in line.

Have they considered what happens if they put the results *in the middle*?

They will get the following increasing sequence of T-sets (multi-sets):

$T_1 = \{1\}$
given initial sets
$T_2 = \{1, 2\}$

$T_3 = \{1, 3, 2\}$,
$T_4 = \{1, 4, 3, 5, 2\}$,
$T_5 = \{1, 5, 4, 7, 3, 8, 5, 7, 2\}$,
$T_6 = \{1, 6, 5, 9, 4, 11, 7, 10, 3, 11, 8, 13, 5, 12, 7, 9, 2\}$,
etc.

Prove that for $3 \le i \le n$ the multiplicity of i in multi-set T_n is $\frac{1}{2}\phi(i)$, where ϕ is Euler's function.

SOLUTIONS

<u>What's the Point?</u>

<u>H-406</u> *Proposed by R. A. Melter, Long Island U., Southampton, NY and I. Tomescu, U. of Bucharest, Romania (Vol. 25, no. 1, February 1987)*

Let A_n denote the set of points on the real line with coordinates 1, 2, ..., n. If $F(n)$ denotes the number of pairwise noncongruent subsets of A_n, then prove

$$F(n) = \begin{cases} 2^{n-2} + 2^{n/2} - 1 & \text{for even } n, \\ 2^{n-2} + 3 \cdot 2^{(n-3)/2} - 1 & \text{for odd } n. \end{cases}$$

Solution by the proposers

Let $n = n_1 + \cdots + n_k$ be a decomposition of n into k nonnegative parts. It is well known that the number of such decompositions is equal to

$$\binom{n-1}{k-1}$$

The decompositions $n_1 + \cdots + n_k$ and $n_k + \cdots + n_1$ will be said to be conjugate.

It follows that

$$F(n) = 1 + \sum_{m=1}^{n-1} \alpha(m)$$

where $\alpha(m)$ is the number of pairwise nonconjugate decompositions of m.

Denote by $\alpha(m, k)$ the number of pairwise nonconjugate decompositions of m into k parts:

We shall consider four cases:

(i) $k = 2\ell$ and $m = 2p$.

In this case the number of self-conjugate decompositions of m with k parts is equal to

$$\binom{p-1}{\ell-1}$$

and hence,

$$\binom{p-1}{\ell-1} + \binom{p-2}{\ell-1} + \cdots + \binom{\ell-1}{\ell-1} + \binom{p}{\ell}.$$

Thus $\alpha(m, k) = \binom{m-1}{k-1} - \frac{1}{2}\left[\binom{m-1}{k-1} - \binom{p}{\ell}\right] = \frac{1}{2}\binom{2p}{2\ell} + \frac{1}{2}\binom{p}{\ell}.$

In order to calculate $\alpha(m)$, consider two subcases.

I. Let $m = 2p$. It follows that

$$\alpha(m) = \frac{1}{2}\left[\sum_{\ell \geq 0}\binom{2p-1}{2\ell} + \sum_{\ell \geq 0}\binom{p-1}{\ell}\right] + \frac{1}{2}\left[\sum_{\ell \geq 1}\binom{2p-1}{2\ell-1} + \sum_{\ell \geq 1}\binom{p-1}{\ell-1}\right]$$

$$= \frac{1}{2}\left[\sum_{\ell \geq 0}\binom{2p}{2\ell} + \sum_{\ell \geq 0}\binom{p}{\ell}\right] = \frac{1}{2}(2^{2p-1} + 2^p) = 2^{2p-2} + 2^{p-1}$$

$$= 2^{m-2} + 2^{(m-2)/2}.$$

II. Let $m = 2p + 1$. One can write

$$\alpha(m) = \frac{1}{2}\sum_{\ell \geq 1}\binom{2p}{2\ell-1} + \frac{1}{2}\left[\sum_{\ell \geq 0}\binom{2p}{2\ell} + \sum_{\ell \geq 0}\binom{p}{\ell}\right] = \frac{1}{2}\sum_{\ell \geq 0}\binom{2p}{\ell} + \frac{1}{2}\sum_{\ell \geq 0}\binom{p}{\ell}$$

$$= 2^{m-2} + 2^{(m-3)/2}.$$

$$\alpha(m, k) = \binom{m-1}{k-1} - \frac{1}{2}\left[\binom{m-1}{k-1} - \binom{p-1}{\ell-1}\right]$$

$$= \frac{1}{2}\binom{2p-1}{2\ell-1} + \frac{1}{2}\binom{p-1}{\ell-1}.$$

(ii) $k = 2\ell$, $m = 2p + 1$.

Here there are no self-conjugate decompositions; hence,

$$\alpha(m, k) = \frac{1}{2}\binom{m-1}{k-1}.$$

(iii) $k = 2\ell + 1$, $m = 2p$.

In order to count the number of self-conjugate decompositions, observe that the central position (m_ℓ) of $m_1 + \cdots + m_{2\ell+1}$ must be an even integer.

Thus, the number of self-conjugate decompositions is equal to

$$\binom{p-2}{\ell-1} + \binom{p-3}{\ell-1} + \cdots + \binom{\ell-1}{\ell-1} = \binom{p-1}{\ell}.$$

It follows that, in this case

$$\alpha(m, k) = \binom{m-1}{k-1} - \frac{1}{2}\left[\binom{m-1}{k-1} - \binom{p-1}{\ell}\right]$$

$$= \frac{1}{2}\binom{2p-1}{2\ell} + \frac{1}{2}\binom{p-1}{\ell}.$$

(iv) $k = 2\ell + 1$, $m = 2p + 1$.

It can be seen that the central position of a self-conjugate decomposition must be an odd number.

Finally, for odd n,

$$\begin{aligned} F(n) &= 1 + 2^{-1} + 2^{-1} + 2^0 + 2^0 + 2^1 + 2^0 + 2^2 + 2^1 + \cdots + 2^{n-3} + 2^{(n-3)/2} \\ &= 2 + (2^0 + \cdots + 2^{n-3}) + (2^0 + 2^0 + 2^1 + 2^1 + \cdots \\ &\qquad + 2^{(n-5)/2} + 2^{(n-5)/2}) + 2^{(n-3)/2} \\ &= 2^{n-2} + 3 \cdot 2^{(n-3)/2} - 1. \end{aligned}$$

For even n, one obtains

$$\begin{aligned} F(n) &= 1 + 2^{-1} + 2^{-1} + 2^0 + 2^0 + 2^1 + 2^0 + 2^2 + 2^1 + \cdots \\ &\qquad + 2^{n-4} + 2^{(n-4)/2} + 2^{(n-3)} + 2^{(n-4)/2} \\ &= 2 + (2^0 + \cdots + 2^{n-3}) + (2^0 + 2^0 + \cdots + 2^{(n-4)/2} + 2^{(n-4)/2}) \\ &= 2^{n-2} + 2^{n/2} - 1. \end{aligned}$$

Also solved by Paul Bruckman.

Nice End Product

H-407 *Proposed by Paul S. Bruckman, Lynwood, WA*
(Vol. 25, no. 1, February 1987)

Find a closed form for the infinite product:

$$\prod_{n=0}^{\infty} \frac{(5n+2)(5n+3)}{(5n+1)(5n+4)}. \tag{1}$$

Solution by Carl Libis, student, Tempe, AZ

Use Theorem 5, p. 14, in Rainville's *Special Functions*, which says:

"If $\sum_{k=1}^{s} a_k = \sum_{k=1}^{s} b_k$, and if no a_k or b_k is a negative integer,

$$\prod_{n=1}^{\infty} \frac{(n+a_1)(n+a_2)\ \dots\ (n+a_s)}{(n+b_1)(n+b_2)\ \dots\ (n+b_s)} = \frac{\Gamma(1+b_1)\Gamma(1+b_2)\ \dots\ \Gamma(1+b_s)}{\Gamma(1+a_1)\Gamma(1+a_2)\ \dots\ \Gamma(1+a_s)}."$$

Thus,

$$\prod_{n=0}^{\infty} \frac{(5n+2)(5n+3)}{(5n+1)(5n+4)} = \frac{\frac{2}{5}\cdot\frac{3}{5}}{\frac{1}{5}\cdot\frac{4}{5}} \prod_{n=1}^{\infty} \frac{\left(n+\frac{2}{5}\right)\left(n+\frac{3}{5}\right)}{\left(n+\frac{1}{5}\right)\left(n+\frac{4}{5}\right)}$$

$$= \frac{\frac{2}{5}\cdot\frac{3}{5}}{\frac{1}{5}\cdot\frac{4}{5}} \cdot \frac{\Gamma\left(\frac{6}{5}\right)\Gamma\left(\frac{9}{5}\right)}{\Gamma\left(\frac{7}{5}\right)\Gamma\left(\frac{8}{5}\right)}$$

$$= \frac{\frac{2}{5}\cdot\frac{3}{5}}{\frac{1}{5}\cdot\frac{4}{5}} \cdot \frac{\left(\frac{1}{5}\right)\Gamma\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\left(\frac{2}{5}\right)\Gamma\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)\Gamma\left(\frac{3}{5}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)} = \frac{\sin\left(\frac{2\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)} = 2\cos\left(\frac{\pi}{5}\right).$$

Also solved by D. Antzoulakos, O. Brugia & P. Filipponi, C. Georghiou, W. Janous, B. Prielipp, J. Shallit, and the proposer.

Ghost from the Past

H-125 *Proposed by Stanley Rabinowitz, Far Rockaway, NY (Vol. 5, no. 5, December 1967)*

Define a sequence of positive integers to be *left-normal* if given any string of digits, there exists a member of the given beginning with this string of digits, and define the sequence to be *right-normal* if there exists a member of the sequence ending with the string of digits.

Show that the sequences whose n^{th} terms are given by the following are left-normal but not right-normal.

a. $P(n)$, where $P(x)$ is a polynomial function with integral coefficients.
b. P_n, where P_n is the n^{th} prime.
c. $n!$
d. F_n, where F_n is the n^{th} Fibonacci number.

Comment by Chris Long, student, Rutgers U., New Brunswisk, NJ

Left-normality for all of the above was established by Raymond E. Whitney in this journal, vol. 11, no. 1, p. 77, and vol. 11, no. 2, pp. 186-187; he also established that (b) and (c) are not right-normal. For (d), note that the Fibonacci sequence is defective mod 8, and hence is defective mod 1000; this shows that the Fibonacci sequence is not right-normal. However, the statement that no polynomials with integer coefficients are right-normal is false, as the example $P(n) \equiv n$ demonstrates. Indeed, David Moews, a student at Harvard College, came up with the following characterization of right-normal polynomials with integer coefficients.

Theorem (Moews): If Q is a polynomial with integer coefficients, then Q is right-normal iff

(1) for all m there exists n with $Q(n) \equiv m \pmod{10}$,

(2) for all n, $(Q'(n), 10) = 1$, where $Q'(x)$ is the formal derivative of $Q(x)$.

Proof (Moews): Note that for all $m \geq 1$, Q can be viewed as a function from $\mathbb{Z}/10^m\mathbb{Z}$ into $\mathbb{Z}/10^m\mathbb{Z}$. Q will be right-normal just when this function is surjective for all m; since $\mathbb{Z}/10^m\mathbb{Z}$ is finite, this will be the case just when this function is injective for all m. We induce on m to show that (1) and (2) imply this.

If $m = 1$, this is clear; otherwise, let $m > 1$. Suppose we have x, y with $Q(x) \equiv Q(y) \pmod{10^m}$. Then $Q(x) \equiv Q(y) \pmod{10^{m-1}}$, so by the induction hypothesis, $x \equiv y \pmod{10^{m-1}}$. Let $x = y + k10^{m-1}$. Then, since $m > 1$, $2(m-1) \geq m$, so $Q(x) \equiv Q(y) + k10^{m-1}Q'(y) \pmod{10^m}$, and we must have $k10^{m-1}Q'(y) \equiv 0 \pmod{10^m}$, i.e., $kQ'(y) \equiv 0 \pmod{10}$, which gives $k \equiv 0 \pmod{10}$ since $Q'(y)$ is relatively prime to 10. Hence, $x \equiv y \pmod{10^m}$, which completes the induction.

For the other implication, it is clear that Q cannot be right-normal if (1) fails. If (2) fails, let n have $(Q'(n), 10) = a$, $a > 1$. Then, if $b = 10/a$,

$$Q(n + 10b) \equiv Q(n) + 10bQ'(n) \pmod{100},$$

and a divides $Q'(n)$ so 100 divides $10bQ'(n)$, which means that

$$Q(n + 10b) \equiv Q(n) \pmod{100}.$$

This proves that Q is not injective as a function from $\mathbb{Z}/100\mathbb{Z}$ to $\mathbb{Z}/100\mathbb{Z}$, so Q cannot be right normal. Q.E.D.

Examples that David Moews came up with of $Q(x)$'s which satisfy conditions (1) and (2), and that are therefore right-normal, include

$$Q(x) = ax + b \text{ for } (a, 10) = 1$$

and higher degree polynomials such as

$$Q(x) = 2x^5 + 5x^4 + 5x^2 + 9x.$$

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BOOKS AVAILABLE THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

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Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

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A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

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