The Fibonacci Quarterly

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The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

EUCLID'S ALGORITHM AND LAMÉ'S THEOREM ON A MICROCOMPUTER

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To the memory of my friend and colleague Hugo D'Alarcao.

1. Introduction

We denote the greatest common divisor of two nonzero integers m and n by $\gcd(m, n)$. Since it is true that $\gcd(m, n) = \gcd(\pm m, \pm n)$, and since $\gcd(m, n) = \gcd(n, m)$, we may assume that both m and n are positive and $m \le n$.

The Euclidean algorithm for computing $\gcd(m, n)$ is a familiar process of iterated long division which can be written as follows:

$$\begin{aligned} n &= mq_1 + r_1;\ 0 < r_1 < m, \\ m &= r_1q_2 + r_2;\ 0 < r_2 < r_1, \\ r_1 &= r_2q_3 + r_3;\ 0 < r_3 < r_2, \\ &\vdots \\ r_{k-2} &= r_{k-1}q_k + r_k;\ 0 < r_k < r_{k-1}, \\ r_{k-1} &= r_kq_{k+1} + r_{k+1};\ r_{k+1} = 0. \end{aligned}$$

The process halts when a remainder 0 is obtained and then $\gcd(m, n)$ is the divisor r_k in the last step of division.

A theorem of Gabriel Lamé (1795-1870) asserts that the number of divisions required to find $\gcd(m, n)$ by Euclid's algorithm is no more than five times the number of digits (base 10) in the smaller of m and n. For proofs see [1] and [2].

Our idea is to keep a count of the number of divisions required to produce $\gcd(m, n)$ by Euclid's algorithm, for a range of values of m and n, and to study the distribution of these numbers.

2. Implementation

The actual computations were accomplished using a BASIC program (written for the APPLE II computers but easily modified for other equipment). The program is listed in Figure 1.

In this program, the variable DC represents a division count, that is, the number of steps of division in using Euclid's algorithm to obtain $\gcd(m, n)$.

The program actually calculates both $\gcd(m, n)$ and $\gcd(n, m)$ within the nested loops of lines 140-230 and, while the second computation is redundant, we have chosen to allow it because it gives us a program that should be easier to follow than otherwise. The program is also fairly slow to execute, and a compiled version of it is preferred.

```
100   REM  DYNAMIC VIEW OF LAME'S THM
110   PRINT "PLOT WHAT DIV. COUNT "
    : INPUT CH
    : REM  USER CHOICE
120   HGR2
130   H = 140
    : V = 95
140   FOR M = 1 TO H
150   FOR N = 1 TO V
160   DC = 0
170   IF M > N THEN DC = - 1
180   GOSUB 240
190   IF DC < > CH THEN 220
200   HCOLOR= 3
210   HPLOT M + H,V - N
    : HPLOT M + H,V + N
    : HPLOT H - M,V + N
    : HPLOT H - M,V - N
220   NEXT N
230   NEXT M
240   REM  SUBROUTINE FOR GCD VIA EUCLID
250   M1 = M
260   N1 = N
270   R = N1 - M1 * INT (N1 / M1)
280   DC = DC + 1
290   N1 = M1
300   M1 = R
310   IF R > 0 THEN 270
320   RETURN
330   END
```

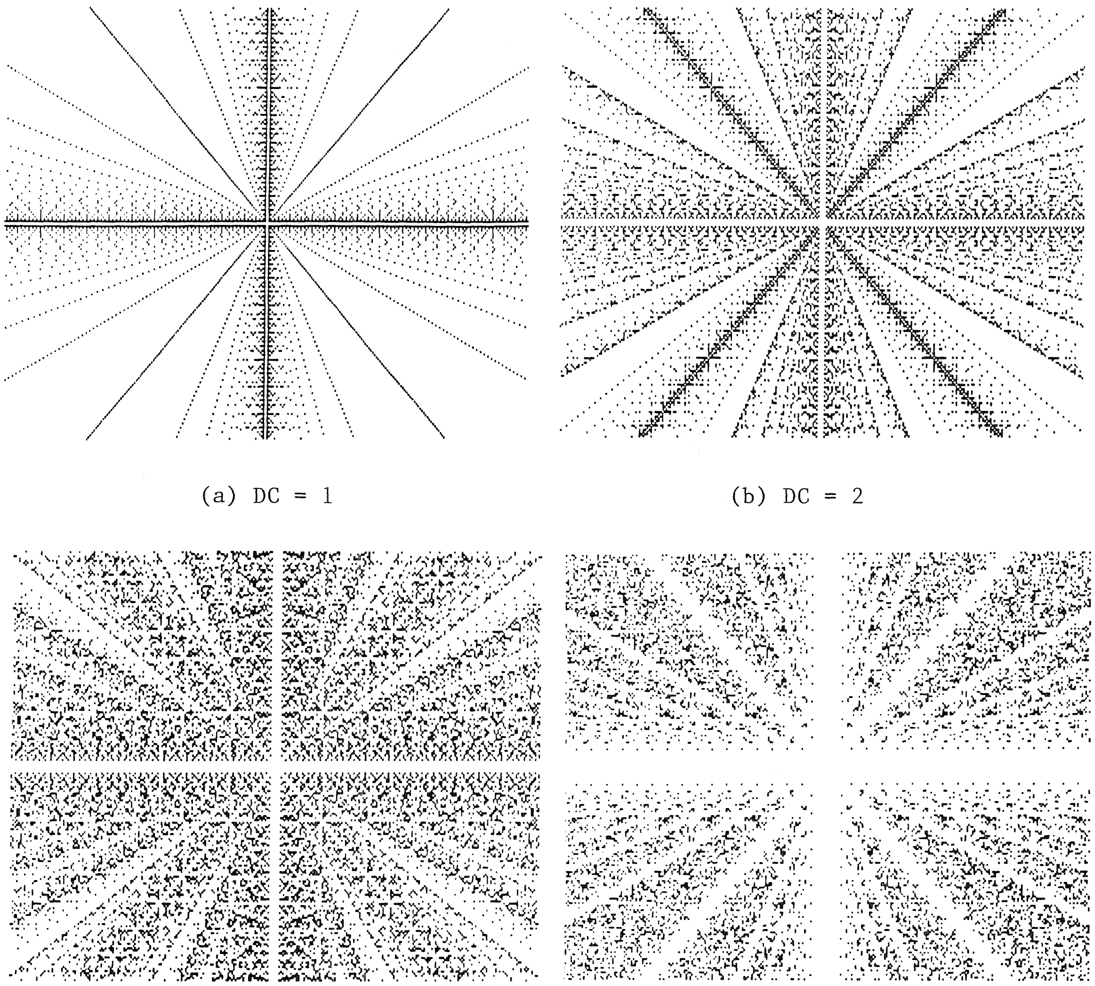
FIGURE 1

The graphics display capability of the computer with a monitor suggested that we interpret each pair of integers m and n as a lattice point (m, n) in the plane and that we plot or do not plot this point on the monitor screen according to the value DC obtained in finding gcd(m, n). Thus, the program asks the user to declare the value of DC in which he is interested.

From the observation that the values of $\gcd(\pm m, \pm n)$ are all equal, we note that a fourfold symmetry can be achieved if the display includes all four quadrants. Hence, the origin (0, 0) is translated to screen coordinates (140, 95) and all subsequently lit points are, similarly, translates of the actual $(\pm m, \pm n)$.

The screen images resulting from four different choices of division counts are shown in Figure 2. In each case, the range of positive integers for which $\gcd(m, n)$ is computed and a division count kept is $1 \le m \le 140$, $1 \le n \le 95$. (These bounds were determined by the graphics page of memory HGR2 in the APPLE II and by the decision to display four quadrants.)

Figure 2 not only illustrates the expected symmetry but also shows patterns of distribution that invite further investigation.



(a) DC = 1 (b) DC = 2

(c) DC = 3 (d) DC = 5

FIGURE 2

Screen dumps showing integer pairs (m, n) in the range $-140 \le m \le 140$, $-95 \le n \le 95$ whose gcd has been obtained by Euclid's algorithm in the same number of steps

3. Analysis

Consider the displays in Figure 2 and the striking fact that the plotted points arrange themselves along various lines. For example, in Figure 2(a) these are the lines in the x-y plane with the equations $y = kx$ and $y = (1/k)x$, for integers k: $k \neq 0$.

Indeed, if gcd(m, n) is found in one step, then m divides n (recall $m \le n$) and, if $n = mq$, then the point (m, n) is on $y = qx$. Since $x = (1/q)y$, then (n, m) is on $y = (1/q)x$.

Again in Figure 2(a), scanning it in the direction of increasing x, we can observe vertical segments at x-values that are multiples of 6 and still longer segments at multiples of 12. For example, at $x = 60$ (see Fig. 3), we note that the plotted points are $(60, \pm k)$ for $k = 1, 2, \ldots, 6$, and clearly $\gcd(60, \pm k)$ is accomplished in one step by Euclid's algorithm. We point out here that $60 = \mathrm{lcm}(1, 2, \ldots, 6)$ and that similar vertical segments will occur at all x such that $x = \mathrm{lcm}(1, 2, \ldots, m)$.

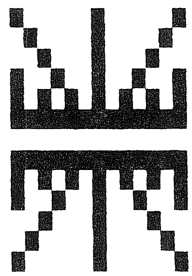


FIGURE 3

Still in Figure 2(a), there is also an X-shape of plotted points at the locations where x is a multiple of 6. The arms of the X-shape at $x = 60$ (see Fig. 3), for example, are just the pairs $(60 \pm k, \pm k)$, for $k = 1, 2, \ldots, 6$, whose greatest common divisor is obtained in one step.

In Figure 2(b), if we scan along the line $y = x$ in the first quadrant, then we can observe +-shapes centered on this line at x-values that are once more multiples of 6. For example, locating (60, 60) as the unplotted (white) center of one such shape (see Fig. 4), we find that this shape is the collection of points $(60, 60 \pm k)$ and $(60 \pm k, 60)$, for $k = 1, 2, \ldots, 6$. Each pair has corresponding $\gcd(m, n)$ obtained in two steps by Euclid's algorithm, as follows:

$$60 + k = (60)(1) + k;\ 0 < k < 60,$$
$$60 = (k)(60/k) + 0$$

or

$$60 = (60 - k)(1) + k;\ 0 < k < 60 - k,$$
$$60 - k = (k)((60 - k)/k) + 0.$$

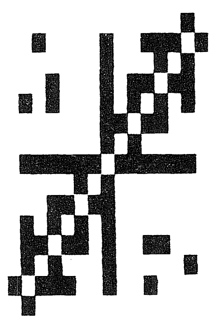


FIGURE 4

Another strongly recurring visual element in Figure 2(b) are blocks of four consecutively plotted horizontal or vertical points. In quadrant one, these occur at the points $(12a + k, 12)$ and $(12, 12a + k)$ for $a \geq 1$, $k = 1, 2, 3, 4$. For these integer pairs, Euclid's algorithm is done in two steps as follows:

$$12a + k = (12)(a) + k;\ 0 \le k < 12,$$
$$12 = (k)(12/k) + 0.$$

There are other discernible patterns in these figures, such as in Figure 2(c) where a pattern of mostly white lines parallel to the axes defines an irregular grid. What is behind it? What is the rule for spacing between successive lines? The interested reader may pursue this line of questioning.

4. Cyclic Behavior

In another direction, we study the distribution of the values DC for fixed $m \ge 1$ and $n \ge m$.

Example 1: $m = 4$.

n	4	5	6	7	8	9	10	11	12	13	14	15 ...
DC	1	2	2	3	1	2	2	3	1	2	2	3 ...

That is, the values of DC for consecutive $n \ge 4$ form the cycle (1223) of length 4.

Example 2: $m = 5, 6, 7$.

	n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20 ...
$(m = 5)$	DC	1	2	3	4	3	1	2	3	4	3	1	2	3	4	3	1 ...
$(m = 6)$	DC		1	2	2	2	3	3	1	2	2	2	3	3	1	2	2 ...
$(m = 7)$	DC			1	2	3	3	4	4	3	1	2	3	3	4	4	3 ...

In each case, the values DC form a cycle of length m. In fact, we find this is easy to prove generally.

Theorem: For fixed $m \ge 1$ and integers $n \ge m$, let DC be the number of steps required to find $\gcd(m, n)$ by Euclid's algorithm. Then the successive values of DC form a cycle of length m.

Proof: Let r be fixed, $0 \le r < m$. It is sufficient to prove that the values DC are the same for the computations of $\gcd(m, m + r)$ and $\gcd(m, km + r)$ for all integers $k \ge 1$. This follows at once from the initial division in each computation. The former begins

$$m + r = (m)(1) + r$$

and the latter begins

$$km + r = (m)(k) + r.$$

Thus, in each case, the second step of division and all succeeding steps are correspondingly equal.

Corollary: If $\gcd(m, n)$ is accomplished in s steps by Euclid's algorithm, then $\gcd(m, n + km)$ is accomplished in s steps for all integers $k \ge 1$.

Example 3: It is well known that $\gcd(89, 144)$ takes 10 steps of division, the maximum predicted by Lamé's theorem. It follows that infinitely many integers can be paired to 89 in this way, namely, the integers $144 + 89k$, and each gcd computation takes 10 steps.

5. Queries and Conclusion

The cycles for all $m \geq 3$ necessarily have the form (12...3), with the remaining DC values of the cycle showing considerable variety. We ask for a rule in terms of m and the position of a value within the cycle that will deliver this value. We have also observed that DC values can be consecutively repeated within a cycle. Is there a rule governing this? Specifically, for a given value DC and any positive integer k, is there a cycle such that DC is repeated consecutively k times?

The microcomputer has been used to gain insight into both the Euclidean algorithm and Lamé's theorem. More can be gained, and some directions to pursue have been given.

References

1. J. V. Uspensky & M. A. Heaslet. *Elementary Number Theory.* New York and London: McGraw-Hill, 1939.
2. D. E. Thoro. "The Euclidean Algorithm II." *Fibonacci Quarterly 2* (1964): 135-137.

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UNIQUE FIBONACCI FORMULAS

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I. Introduction

In this paper we consider the generating function

$$G(x)^{-k} = 1/(1 - a_1x - a_2x^2 - \cdots - a_mx_m)^k \quad \text{(where } m \geq 2 \text{ and } k \geq 1)$$

as a formal power series. Note that we can write the expression as

$$G(x)^{-k} = F_{m,k}(0) + F_{m,k}(1)x + F_{m,k}(2)x^2 + \cdots + F_{m,k}(n)x^n + \cdots \tag{1}$$

$$\text{(for } n \geq 0\text{, and where } F_{m,k}(0) = 1).$$

However, before considering equation (1), we shall develop certain identities by the use of partitions. Let $p(n)$ denote the number of partitions of n; that is, the number of solutions of the equation

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = n$$

in nonnegative integers. We state the following identity established in [1]:

$$p(n) = -\sum_{\substack{0 \leq i < m \\ m < j \leq n}} p(i)e(j - i)p(n - j) \tag{2}$$

$$\text{where } \begin{cases} e(k) = (-1)^k & \text{if } k = (1/2)(3h^2 \pm h)\text{, where } h \text{ is an integer,} \\ e(k) = 0 & \text{otherwise,} \end{cases}$$

and $p(0) = 1$.

The proof of (2) will be evident as a special case of a more general form to be given later. Let

$$g(x) = \sum_{n=0} a(n)x^n \tag{2a}$$

and

$$g(x)^{-1} = \sum_{n=0} b(n)x^n \tag{2b}$$

where, for convenience, $a(0) = b(0) = 1$. Then it can be shown that

$$\sum_{j=0}^{n} a(j)b(n - j) = 0 \quad \text{(for } n > 0). \tag{3}$$

For the sums

$$S = \sum_{\substack{0 \le i < m \\ m < j \le n}} a(i)b(j-i)a(n-j) \quad \text{and} \quad T = \sum_{0 \le i \le j \le m} a(i)b(j-i)a(n-j)$$

where $0 < m < n$, using (3) above, it can be shown that

$$T = \sum_{j=0}^{m} a(n-j) \sum_{i=0}^{j} a(i)b(j-i) = a(n). \tag{4}$$

Furthermore,

$$S + T = \sum_{0 \le i \le m} \sum_{i \le j \le n} a(i)b(j-i)a(n-j) = \sum_{0 \le i < m} a(i) \sum_{s=0}^{n-i} b(s)a(n-i-s).$$

Note also that the inner sum on the extreme right vanishes unless $n - 1 = 0$, because $m < n$. Hence, we have $S + T = 0$.

Combining this with (4), we get $S = -a(n)$, or, explicitly,

$$S = \sum_{\substack{0 \le i < m \\ m < j \le n}} a(i)b(j-i)a(n-j) = -a(n) \quad (0 < m < n). \tag{5}$$

Since we may equally well have started out with $g(x)^{-1}$, rather than $g(x)$, we also have

$$S = \sum_{\substack{0 \le i < m \\ m < j \le n}} b(i)a(j-i)b(n-j) = -b(n) \quad (0 < m < n).$$

II. Some Relations Involving Fibonacci and Tribonacci Numbers

Referring to (1), we first examine what happens when $k = 1$ and the $a_j = 1$, for $i \le j \le m$, where $m \ge 2$. For convenience, we let $F_{m,1}(n) = F_m(n)$, where $m \ge 2$ and $n \ge 0$. note that $F_m(0) = 1$, $F_2(n)$ denotes the n^{th} Fibonacci numbers, $F_3(n)$ denotes the n^{th} Tribonacci numbers, etc. Letting $n = 2Z$, for $Z \ge 1$ and $Z > m$, we have

$$F_m(2Z) = F_m(2Z-1) + F_m(2Z-2) + F_m(2Z-3) + \cdots + F_m(2Z-m). \tag{6}$$

Combining (6) with the results of (4) and (5) for the coefficients, we have the following table of values that are used to determine $F_m(2Z)$.

TABLE 1

a	b	1	2	3	...	m
Z	$F_m(Z)$	$F_m(Z-1)$	$F_m(Z-2)$	$F_m(Z-3)$	...	$F_m(Z-m)$
Z+1	$F_m(Z-1)$		$F_m(Z-1)$	$F_m(Z-2)$	...	$F_m(Z-m+1)$
Z+2	$F_m(Z-2)$			$F_m(Z-1)$	...	$F_m(Z-m+2)$
					...	
Z+m-1	$F_m(Z-m+1)$					$F_m(Z-m+(m-1))$

Multiplying the elements in column 2 by the sum of the corresponding elements in each row, we have:

$$\begin{aligned}F_m(2Z) = {} & F_m(Z)(F_m(Z-1) + F_m(Z-2) + F_m(Z-3) + \cdots + F_m(Z-m)) \\ & + F_m(Z-1)(F_m(Z-1) + F_m(Z-2) + \cdots + F_m(Z-m+1)) \\ & + F_m(Z-2)(F_m(Z-1) + F_m(Z-2) + \cdots + F_m(Z-m+2) \\ & \vdots \\ & + F_m(Z-m+1)(F_m(Z-m+(m-1))).\end{aligned} \tag{7}$$

When $m = 2$, we get the Fibonacci numbers, and (7) becomes

$$F_2(2Z) = F_2(Z)(F_2(Z-1) + F_2(Z-2)) + F_2(Z-1)(F_2(Z-1)), \tag{8}$$

or

$$F_2(2Z) = (F_2(Z))^2 + (F_2(Z-1))^2, \text{ for } Z \geq 1. \tag{9}$$

When $m = 3$, we get the Tribonacci numbers, and (7) becomes

$$\begin{aligned}F_3(2Z) = {} & F_3(Z)(F_3(Z-1) + F_3(Z-2) + F_3(Z-3)) \\ & + F_3(Z-1)(F_3(Z-1) + F_3(Z-2)) \\ & + F_3(Z-2)(F_3(Z-1)),\end{aligned} \tag{10}$$

so that

$$F_3(2Z) = (F_3(Z))^2 + (F_3(Z-1))^2 + 2F_3(Z-1)F_3(Z-2), \text{ for } Z \geq 1. \tag{11}$$

Continuing the process of (8)-(11), with $m = 2a$, we have:

$$\begin{aligned}F_{2a}(2Z) = {} & (F_{2a}(Z))^2 + (F_{2a}(Z-1))^2 + \cdots + (F_{2a}(Z-a))^2 \\ & + 2F_{2a}(Z-1)(F_{2a}(Z-2) + F_{2a}(Z-3) + \cdots + F_{2a}(Z-(2a-1))) \\ & + 2F_{2a}(Z-2)(F_{2a}(Z-3) + F_{2a}(Z-4) + \cdots + F_{2a}(Z-(2a-2))) \\ & \vdots \\ & + 2F_{2a}(Z-(a-1))(F_{2a}(Z-a) + F_{2a}(Z-(2a-(a-1)))),\end{aligned} \tag{12}$$

$a \geq 1$ and $Z \geq 1$.

Furthermore,

$$F_{2a}(0) = 1,\ F_{2a}(1) = 1,\ F_{2a}(2) = 2,\ \ldots,\ F_{2a}(2a) = 2^{2a-1}.$$

Continuing the process for $m = 2a + 1$, we have:

$$\begin{aligned}F_{2a+1}(2Z) = {} & (F_{2a+1}(Z))^2 + (F_{2a+1}(Z-1))^2 + \cdots + (F_{2a+1}(Z-a))^2 \\ & + 2F_{2a+1}(Z-1)(F_{2a+1}(Z-2) + F_{2a+1}(Z-3) + \cdots + F_{2a+1}(Z-2a)) \\ & + 2F_{2a+1}(Z-2)(F_{2a+1}(Z-3) + F_{2a+1}(Z-4) + \cdots \\ & \qquad + F_{2a+1}(Z-(2a-1))) \\ & \vdots \\ & + 2F_{2a+1}(Z-(a-1))(F_{2a+1}(Z-a) + F_{2a+1}(Z-(a+1)) \\ & \qquad + F_{2a+1}(Z-(a+2))) \\ & + 2F_{2a+1}(Z-a)(F_{2a+1}(Z-(a+1))),\end{aligned} \tag{13}$$

for $a \geq 1$, $Z \geq 1$. Here,

$$F_{2a+1}(0) = 1,\ F_{2a+1}(1) = 1,\ F_{2a+1}(2) = 2,\ F_{2a+1}(3) = 4,\ \ldots,$$

and $F_{2a+1}(2a+1) = 2^{2a}$.

For $n = 2Z + 1$, we also consider $F_{2a}(2Z+1)$ and $F_{2a+1}(2Z+1)$, where $Z \geq a$ and $Z \geq 1$. In the exact way we obtained (12) and (13) but with added induction, we now get

$$
\begin{aligned}
F_{2a}(2Z + 1) = {} & (F_{2a}(Z + 1))^2 - (F_{2a}(Z - a))^2 - (F_{2a}(Z - (a + 1)))^2 - \cdots \\
& \qquad - (F_{2a}(Z - (2a - 1)))^2 \\
& - [2F_{2a}(Z - (2a - 1))(F_{2a}(Z - 1) + F_{2a}(Z - 2) + F_2\ (Z - 3) + \cdots \\
& \qquad + F_{2a}(Z - (2a - 2)))] \\
& - [2F_{2a}(Z - (2a - 2))(F_{2a}(Z - 2) + F_{2a}(Z - 3) + \cdots \\
& \qquad + F_{2a}(Z - (2a - 3)))] \\
& \vdots \\
& - [2F_{2a}(Z - (a + 2))(F_{2a}(Z - (a - 2)) + F_{2a}(Z - (a - 1)) \\
& \qquad + F_{2a}(Z - a) + F_{2a}(Z - (a + 1)))] \\
& - [2F_{2a}(Z - (a + 1))(F_{2a}(Z - (a - 1)) + F_{2a}(Z - a))] \qquad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
F_{2a+1}(2Z + 1) = {} & (F_{2a+1}(Z + 1))^2 - (F_{2a+1}(Z - (a + 1)))^2 - \cdots \\
& \qquad - (F_{2a+1}(Z - 2a))^2 \\
& - [2F_{2a+1}(Z - 2a)(F_{2a+1}(z - 1) + F_{2a+1}(Z - 2) + F_{2a+1}(Z - 3) + \\
& \qquad \cdots + F_{2a+1}(Z - (a\ - 1)))] \\
& - [2F_{2a+1}(Z - (2a - 1))(F_{2a+1}(Z - 2) + F_{2a+1}(Z - 3) + \cdots \\
& \qquad + F_{2a+1}(Z - (2a - 2)))] \\
& \vdots \\
& - [2F_{2a+1}(Z - (a + 2))(F_{2a+1}(Z - (a - 1)) + F_{2a+1}(Z - a) \\
& \qquad + F_{2a+1}(Z - (a + 1)))] \\
& - [2F_{2a+1}(Z - (a + 1))(F_{2a+1}(Z - a))]. \qquad (15)
\end{aligned}
$$

In closing this section, we note that for $k = 1$, we can combine the coefficients in (1) to obtain

$$F_m(n) = \sum_{j=1}^{m} a_j F_m(n - j). \tag{16}$$

Hence, we can solve for $F_m(n)$ in terms of the a_j, where the a_j are arbitrary numbers, that is

$$F_m(0) = 1,\ F_m(1) = a_1,\ F_m(2) = (a_1)^2 + a\ ,\ \text{etc., where } m \geq 1.$$

It might also be noted here that all the numbers, the Fibonacci, Tribonacci, Quadranacci, ..., and Hoganacci numbers, are, respectively, the sums and differences of Fibonacci, Tribonacci, Quadranacci, ..., and Hoganacci *squares*.

III. A Congruence for $F_{m,k}(n)$ and Related Identities

We now return to (1) and consider the function $G(x)^{-k}$. Let

$$y = G(x). \tag{17}$$

Since

$$1/y = 1 + (1 - y)/y, \tag{18}$$

we see that

$$1/y = 1 + (a_1x/y) + (a_2x^2/y) + \cdots + (a_mx^m/y). \tag{19}$$

Multiplying (19) by $1/y^k$, we have

$$1/y^{k+1} = 1/y^k + (a_1x/y^{k+1}) + (a_2x\ /y^{k+1}) + \cdots + (a_mx^m/y^{k+1}). \tag{20}$$

Combining the coefficients in (20) leads to

$$F_{m,\,k+1}(n) = \sum_{j=1}^{m} (a_jF_{m,\,k+1}(n-j)) + F_{m,\,k}(n), \quad (n \geq 1). \tag{21}$$

Substitution of (17) into (1) yields

$$1/y^k = \sum_{n=0}^{\infty} F_{m,\,k}(n)x^n, \tag{22}$$

which, after differentiation and multiplying through by x, gives

$$-kx(y^{-k-1}dy/dx) = \sum_{n=1}^{\infty} nF_{m,\,k}(n)x^n. \tag{23}$$

Now, using the values of y in (18) and (1) and combining the coefficients of both sides of (23), we have

$$nF_{m,\,k}(n) = k\sum_{j=1}^{m} ja_jF_{m,\,k+1}(n-j). \tag{24}$$

We observe that (24) is a special case of (12) and (13). Hence, we get the following congruence:

$$F_{m,\,k}(n) \equiv 0 \ (\text{mod } k/(n,\,k), \text{ for } n \geq k. \tag{25}$$

Multiplying both sides of the equation in (21) by n, we have $nF_{m,\,k}(n)$ in both (21) and (24). Hence, combining (21) with (24) leads to

$$nF_{m,\,k+1}(n) = \sum_{j=1}^{m} (jka_j + na_j)F_{m,\,k+1}(n-j). \tag{26}$$

Replacing k with $k - 1$, we get

$$nF_{m,\,k}(n) = \sum_{j=1}^{m} (j(k-1)a_j + na_j)F_{m,\,k}(n-j), \tag{27}$$

where n, $k \geq 1$ and $m \geq 2$.

IV. A Table of Fibonacci Extensions

We now use equation (21) to make a table of Fibonacci extensions (see [5]) where, for convenience, we let $m = 2$. Of course, we could have considered any other value for m, where $m \geq 3$.

In Table 2, below, we consider the values of $F_{2,\,k}(n)$, where the $a_j = 1$ and $F_{2,\,k}(0) = 1$.

Table 2 was constructed by using the following rule:

> To get the k^{th} element in the n^{th} column, add the k^{th} element in the $(n-1)^{\text{st}}$ column and the k^{th} element in the $(n-2)^{\text{nd}}$ column together with the $(k-1)^{\text{st}}$ element in the n^{th} column. Note that the second row is the Fibonacci numbers.

When $m = 3$, we obtain the Tribonacci numbers. To get the Tribonacci extensions we merely proceed as in Table 2, except that we have one more term, that is, our rule is:

To get the k^{th} element in the n^{th} column of the Tribonacci extensions, we add the k^{th} element in the $(n - 1)^{\text{st}}$ column, the k^{th} element in the $(n - 2)^{\text{nd}}$ column and the k^{th} element in the $(n - 3)^{\text{rd}}$ column together with the $(k - 1)^{\text{st}}$ element in the n^{th} column.

We can construct a table for any $m \geq 4$ in the same way we found the table for $m = 3$ but with added induction.

TABLE 2

	0	1	2	3	4	5	6	...
0	0	0	0	0	0	0	0	...
1	1	1	2	3	5	8	13	...
2	1	2	5	10	20	38	71	...
3	1	3	9	22	51	111	233	...
...	...	...	...	...	...	...	...	...
k	1	k	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...

In order to construct Table 2 for the k^{th} powers, one might think we need to construct k lines, which is a great deal of work. However, this is really not necessary, since by equation (27) it is evident we need only find the numbers in line k.

The following is a table of the generalized Fibonacci numbers. For convenience, we have replaced a_1 with a and a_2 with b, where a and b are arbitrary numbers.

TABLE 3

Values of $F_{2,k}(n, a, b)$

		1	2	3	4
0	0	0	0	0	0
1	1	a	a^2+b	a^3+2ab	$a^4+3a^2b+b^2$
2	1	$2a$	$3a^2+2b$	$4a^3+6ab$	$5a^4+12a^2b+3b^2$
3	1	$3a$	$6a^2+2b$	$10a^3+12ab$	$15a^4+30a^2b+6b^2$
...	...	...	...	...	...
k	1	ka	...	...	...
...	...	...	...	...	...

Table 3 was constructed using the rule:

To get the k^{th} element in the n^{th} column, we add the product of a multiplied by the k^{th} element in the $(n - 1)^{\text{st}}$ column and the product of b multiplied by the k^{th} element in the $(n - 2)^{\text{nd}}$ column together with the $(k - 1)^{\text{st}}$ element in the n^{th} column.

To obtain the table for $m = 3$ that gives us the generalized Tribonacci numbers, we use the rule:

> To get the k^{th} element in the n^{th} column, we add the product of a_1 multiplied by the k^{th} element in the $(n - 1)^{\text{st}}$ column to the product of a_2 multiplied by the k^{th} element in the $(n - 2)^{\text{nd}}$ column and we add those two products together with the third product of a_3 multiplied by the k^{th} element in the $(n - 3)^{\text{rd}}$ column. We then add the sum of the three products together with the $(k - 1)^{\text{st}}$ element in the n^{th} column.

To obtain the table for $m \geq 4$, we do exactly as we did for $m = 3$ but with added induction.

We conclude this paper by noting that in exactly the way we found Table 2 (with the $a_j = 1$) we may also construct Table 3 with the a_j equal to arbitrary numbers.

Using step-by-step induction, it is easy to show that, by equation (27), we can find any element on line k using only the numbers found on line k and a_j.

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FIBONACCI-LIKE DIFFERENTIAL EQUATIONS WITH A POLYNOMIAL NONHOMOGENEOUS PART

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1. Introduction

In [1] and [2] we studied difference equations of the form

$$G_n = G_{n-1} + G_{n-2} + p(n) \tag{1}$$

where $G_0 = G_1 = 1$ and $p(n)$ is either a (ordinary or power) polynomial [1] or a factorial polynomial [2], i.e.,

$$p(n) = \sum_{i=0}^{k} \alpha_i n^i \quad \text{or} \quad p(n) = \sum_{i=0}^{k} \alpha_i n^{(i)}, \tag{2}$$

respectively, where

$$n^{(i)} = n(n-1)(n-2)\ \ldots\ (n-i+1) \text{ for } i \geq 1 \text{ and } n^{(0)} = 1.$$

The main results established in [1] and [2] provide expressions for the solution of (1) in terms of the coefficients $\alpha_1, \ldots, \alpha_k$ of (2) and in the Fibonacci numbers F_n, i.e., in the solution of the homogeneous difference equation

$$F_n = F_{n-1} + F_{n-2}, \tag{3}$$

where $F_0 = F_1 = 1$; cf. also [5].

In this note we derive similar expressions for the family of differential equations corresponding to (1) and (2), viz. we consider differential equations of the form

$$x''(t) + x'(t) - x(t) = p(t), \tag{4}$$

where $x(0) = c$, $x'(0) = d$,

$$p(t) = \sum_{i=0}^{k} \alpha_i t^i \quad \text{or} \quad p(t) = \sum_{i=0}^{k} \alpha_i t^{(i)},$$

and we express the solution of (4) in terms of the coefficients $\alpha_1, \ldots, \alpha_k$ and in the solution of the homogeneous differential equation corresponding to (3), i.e., the solution of

$$y''(t) + y'(t) - y(t) = 0 \tag{5}$$

where $y(0) = y'(0) = 1$.

Essential in our approach is the following proposition in which $p(t)$ now need not be a (factorial) polynomial at all; it may be an arbitrary function which, however, gives rise to a particular solution $x_p(t)$ of (4).

Let $F_{-1} = 0$ and $F_{-n} = (-1)^n F_{n-2}$ for each $n \geq 2$.

Proposition 1.1: Let $x_p(t)$ be a particular solution of (4). If $x(0) = c$ and $x'(0) = d$, then the solution of (4) can be expressed as

$$x(t) = (c - x_p(0))\left(\sum_{n=0}^{\infty} F_{-n}\frac{t^n}{n!}\right) + (d - x_p'(0))\left(\sum_{n=0}^{\infty} F_{-n-1}\frac{t^n}{n!}\right) + x_p(t). \tag{6}$$

Proof: Using standard methods (cf. e.g., [3]), we first determine the solution $x_h(t)$ of the homogeneous equation corresponding to (4). To this end, we solve (5) with $y(0) = y'(0) = 1$:

$$y(t) = -(1 + \phi_2)(\sqrt{5})^{-1} \exp(-\phi_1 t) + (1 + \phi_1)(\sqrt{5})^{-1} \exp(-\phi_2 t),$$

where $\phi_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\phi_2 = \frac{1}{2}(1 - \sqrt{5})$. Then we obtain

$$\begin{aligned} y(t) &= -(1 + \phi_2)(\sqrt{5})^{-1}\left(\sum_{n=0}^{\infty} \frac{(-\phi_1 t)^n}{n!}\right) + (1 + \phi_1)(\sqrt{5})^{-1}\left(\sum_{n=0}^{\infty} \frac{(-\phi_2 t)^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{\sqrt{5}}(\phi_1^{n-2} - \phi_2^{n-2})\frac{t^n}{n!} = \sum_{n=0}^{\infty} F_{-n+1}\frac{t^n}{n!}, \end{aligned}$$

since $(1 + \phi_2)\phi_1^2 = 1$ and $(1 + \phi_1)\phi_2^2 = 1$. Notice that

$$y'(t) = \sum_{n=0}^{\infty} F_{-n}\frac{t^n}{n!} \quad \text{and} \quad y''(t) = \sum_{n=0}^{\infty} F_{-n-1}\frac{t^n}{n!}. \tag{7}$$

Now it is straightforward to show that for the solution $x(t)$ of (4) we have

$$x(t) = x_h(t) + x_p(t) = (c - x_p(0))y'(t) + (d - x_p'(0))y''(t) + x_p(t),$$

which yields together with (7) the desired equality (6). □

From Proposition 1.1 it is clear that we now need a particular solution of (4). As in [1] and [2] we distinguish two cases, viz. $p(t)$ is a polynomial (Section 2) and $p(t)$ is a factorial polynomial (Section 3).

2. Polynomials

Throughout this section, we assume that $p(t)$ is an ordinary or power polynomial

$$p(t) = \sum_{i=0}^{k} \alpha_i t^i.$$

As a particular solution of (4) we try

$$x_p(t) = \sum_{i=0}^{k} A_i t^i.$$

For $p(t)$ and $x_p(t)$, we write

$$p(t) = \sum_{i=0}^{k} \beta_i \frac{t^i}{i!} \quad \text{and} \quad x_p(t) = \sum_{i=0}^{k} B_i \frac{t^i}{i!},$$

respectively, where $\beta_i = i!\alpha_i$ and $B_i = i!A_i$ for each i $(0 \le i \le k)$. Hence, (4) yields

$$\sum_{i=0}^{k-2} B_{i+2} \frac{t^i}{i!} + \sum_{i=0}^{k-1} B_{i+1} \frac{t^i}{i!} - \sum_{i=0}^{k} B_i \frac{t^i}{i!} = \sum_{i=0}^{k} \beta_i \frac{t^i}{i!}.$$

From a comparison of the coefficients of $t^i/i!$, it follows that

$$B_k = -\beta_k,$$
$$B_{k-1} = -\beta_{k-1} - \beta_k,$$
$$B_i = -\beta_i + B_{i+2} + B_{i+1}, \quad \text{for } 0 \le i \le k-2.$$

Thus, we can successively compute B_k, B_{k-1}, ..., B_0; B_i is a linear combination of β_i, ..., β_k. Therefore, we write

$$B_i = -\sum_{j=i}^{k} a_{ij}\beta_j$$

(cf. [1] and [2]), which gives

$$-\sum_{j=i}^{k} a_{ij}\beta_j = -\beta_i - \sum_{j=i+2}^{k} a_{i+2,j}\beta_j - \sum_{j=i+1}^{k} a_{i+1,j}\beta_j.$$

Comparing the coefficients of β_j yields the following difference equation for each j $(1 \le j \le k)$:

$$a_{ij} = a_{i+2,j} + a_{i+1,j}, \quad \text{for } j - i \ge 2,$$

where $a_{jj} = a_{j-1,j} = 1$. But this means that

$$a_{ij} = F_{j-i}, \quad \text{for } 0 \le i \le j,$$

and hence

$$x_p(t) = \sum_{i=0}^{k} B_i \frac{t^i}{i!} = -\sum_{i=0}^{k}\sum_{j=i}^{k} F_{j-i}j!\,\alpha_j \frac{t^i}{i!} = -\sum_{j=0}^{k}\alpha_j\left(\sum_{i=0}^{j} j^{(j-i)}F_{j-i}t^i\right)$$

which implies

$$x_p(0) = B_0 = -\sum_{j=0}^{k} j!F_j\alpha_j \quad \text{and} \quad x_p'(0) = B_1 = -\sum_{j=1}^{k} j!F_{j-1}\alpha_j.$$

These equalities together with Proposition 1.1 yield the following proposition.

TABLE 1

j	$p_j(t)$
0	1
1	$t + 1$
2	$t^2 + 2t + 4$
3	$t^3 + 3t^2 + 12t + 18$
4	$t^4 + 4t^3 + 24t^2 + 72t + 120$
5	$t^5 + 5t^4 + 40t^3 + 180t^2 + 600t + 960$
6	$t^6 + 6t^5 + 60t^4 + 360t^3 + 1800t^2 + 5760t + 9360$
7	$t^7 + 7t^6 + 84t^5 + 630t^4 + 4200t^3 + 20160t^2 + 65520t + 105840$
8	$t^8 + 8t^7 + 112t^6 + 1008t^5 + 8400t^4 + 53760t^3 + 262080t^2 + 846720t + 1370880$
9	$t^9 + 9t^8 + 144t^7 + 1512t^6 + 15120t^5 + 120960t^4 + 786240t^3 + 3810240t^2 + 12337920t + 19958400$

Proposition 2.1: The solution of (4) with $x(0) = c$, $x'(0) = d$, and

$$p(t) = \sum_{i=0}^{k} \alpha_i t^i$$

can be expressed as

$$x(t) = (c + L_k)\left(\sum_{n=0}^{\infty} F_{-n} \frac{t^n}{n!}\right) + (d + \ell_k)\left(\sum_{n=0}^{\infty} F_{-n-1} \frac{t^n}{n!}\right) - \sum_{j=0}^{k} \alpha_j p_j(t),$$

where L_k and ℓ_k are linear combinations of $\alpha_0, \ldots, \alpha_k$, and for each j $(0 \le j \le k)$, $p_j(t)$ is a polynomial of degree j:

$$L_k = \sum_{j=0}^{k} j!F_j\alpha_j; \quad \ell_k = \sum_{j=1}^{k} j!F_{j-1}\alpha_j; \quad p_j(t) = \sum_{i=0}^{j} j^{(j-i)}F_{j-i}t^i. \quad \square$$

The polynomials $p_j(t)$ are given in Table 1 above for $j = 0, 1, 2, \ldots, 9$.

The coefficients of α_j in L_k and ℓ_k are independent of k; cf. [1] and [2]. They give rise to two infinite sequences L and ℓ of natural numbers (not mentioned in [4]) as k tends to infinity. The first few elements of these new sequences are

L: 1,1,4,18,120,960,9360,105840,1370880,19958400,...,
ℓ: 0,1,2,12,72,600,5760,65520,846720,12337920,... .

3. Factorial Polynomials

This section is devoted to the case in which $p(t)$ is a factorial polynomial

$$p(t) = \sum_{i=0}^{k} \alpha_i t^{(i)}.$$

In order to try

$$x_p(t) = \sum_{i=0}^{k} A_i t^{(i)} \tag{8}$$

as a particular solution of (4), we first ought to determine the derivative of $t^{(n)}$.

Lemma 3.1: $\dfrac{dt^{(n)}}{dt} = \displaystyle\sum_{k=0}^{n-1} \binom{n}{k} t^{(k)}(-1)^{(n-k-1)}$.

Proof: The argument is by induction on n. The basis of which $(n = 1)$ is trivial. Suppose the equality holds for $n - 1$:

$$\frac{dt^{(n-1)}}{dt} = \sum_{k=0}^{n-2} \binom{n-1}{k} t^{(k)}(-1)^{(n-k-2)}. \tag{9}$$

To perform the induction step, consider

$$dt^{(n)}/dt = d(t(t-1)^{(n-1)})/dt = (t-1)^{(n-1)} + td((t-1)^{(n-1)})/dt.$$

Now, by the Chain Rule, we have

$$d((t-1)^{(n-1)})/dt = d((t-1)^{(n-1)})/d(t-1).$$

Applying the Binomial Theorem from [2] to $(t-1)^{(n-1)}$ and the induction hypothesis (9) yields:

$$\frac{dt^{(n)}}{dt} = \sum_{k=0}^{n-1}\binom{n-1}{k}t^{(k)}(-1)^{(n-k-1)} + t\sum_{k=0}^{n-2}\binom{n-1}{k}(t-1)^{(k)}(-1)^{(n-k-2)}$$

$$= (-1)^{(n-1)} + \sum_{k=1}^{n-1}\binom{n-1}{k}t^{(k)}(-1)^{(n-k-1)} + \sum_{k=1}^{n-1}\binom{n-1}{k-1}t^{(k)}(-1)^{(n-k-1)}$$

$$= (-1)^{(n-1)} + \sum_{k=1}^{n-1}\binom{n}{k}t^{(k)}(-1)^{(n-k-1)} = \sum_{k=0}^{n-1}\binom{n}{k}t^{(k)}(-1)^{(n-k-1)}$$

which completes the induction. □

From Lemma 3.1, (4), and (8), we obtain

$$\sum_{i=2}^{k} A_i\left(\sum_{m=1}^{i-1}\binom{i}{m}\left(\sum_{\ell=0}^{m-1}\binom{m}{\ell}t^{(\ell)}(-1)^{(m-\ell-1)}\right)(-1)^{(i-m-1)}\right)$$

$$+ \sum_{i=1}^{k} A_i\left(\sum_{m=0}^{i-1}\binom{i}{m}t^{(m)}(-1)^{(i-m-1)}\right) - \sum_{i=0}^{k} A_i t^{(i)} = \sum_{i=0}^{k}\alpha_i t^{(i)}.$$

Comparing the coefficients of $t^{(i)}$ yields

$$A_k = -\alpha_k,$$

$$A_{k-1} = -\alpha_{k-1} + k\alpha_k,$$

$$A_i = -\alpha_i + \sum_{n=i+1}^{k} A_n\binom{n}{i}(-1)^{(n-i-1)}$$

$$+ \sum_{n=i+2}^{k} A_n\left(\sum_{m=i+1}^{n-1}\binom{n}{m}\binom{m}{i}(-1)^{(m-i-1)}(-1)^{(n-m-1)}\right)$$

for each i $(0 \le i \le k-2)$. As $(-x)^{(n)} = (-1)^n(x+n-1)^{(n)}$ and $n^{(n)} = n!$, this latter recurrence can be rewritten as

$$A_i = -\alpha_i + \sum_{n=i+1}^{k} A_n(-1)^{n-i-1}\frac{n^{(n-i)}}{n-i}$$

$$+ \sum_{n=i+2}^{k} A_n\left(\sum_{m=i+1}^{n-1}(-1)^{n-i-2}\frac{n^{(n-i)}}{(n-m)(m-i)}\right)$$

or

$$A_i = -\alpha_i + (i+1)A_{i+1} + \sum_{n=i+2}^{k}\zeta_{in}A_n, \tag{10}$$

where

$$\zeta_{in} = (-1)^{n-i-1}n^{(n-i)}\left((n-i)^{-1} - \sum_{m=i+1}^{n-1}(n-m)^{-1}(m-i)^{-1}\right).$$

Now (10) enables us to compute $A_k, \ldots, A_0$: A_i is a linear combination of $\alpha_i, \ldots, \alpha_k$. Thus

$$A_i = -\sum_{j=i}^{k} b_{ij}\alpha_j$$

and (10) becomes

$$\sum_{j=i}^{k} b_{ij}\alpha_j = \alpha_i + (i+1)\sum_{j=i+1}^{k} b_{i+1,j}\alpha_j + \sum_{n=i+2}^{k}\zeta_{in}\sum_{j=n}^{k} b_{nj}\alpha_j.$$

From the coefficients of α_j, it follows that

$$b_{ii} = 1,$$

$$b_{i,\,i+1} = i + 1,$$

$$b_{ij} = (i + 1)b_{i+1,\,j} + \sum_{n=i+2}^{j} \zeta_{in} b_{nj} \quad \text{for } j \geq i + 2.$$

Hence,

$$x_p(t) = -\sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}\alpha_j t^{(i)} = -\sum_{j=0}^{k} \alpha_j \left(\sum_{i=0}^{j} b_{ij} t^{(i)} \right)$$

and

$$x_p'(t) = -\sum_{j=1}^{k} \alpha_j \left(\sum_{i=1}^{j} b_{ij} \left(\sum_{\ell=0}^{i-1} \binom{i}{\ell} t^{(\ell)} (-1)^{(i-\ell-1)} \right) \right).$$

Since

$$x_p(0) = -\sum_{j=0}^{k} b_{0j}\alpha_j \quad \text{and} \quad x_p'(0) = -\sum_{j=1}^{k} \left(\sum_{i=1}^{j} (-1)^{(i-1)} b_{ij} \right) \alpha_j,$$

we have the following result.

Proposition 3.2: The solution of (4) with $x(0) = c$, $x'(0) = d$, and

$$p(t) = \sum_{i=0}^{k} \alpha_i t^{(i)}$$

can be expressed as

$$x(t) = (c + M_k)\left(\sum_{n=0}^{\infty} F_{-n} \frac{t^n}{n!} \right) + (d + m_k)\left(\sum_{n=0}^{\infty} F_{-n-1} \frac{t^n}{n!} \right) - \sum_{j=0}^{k} \alpha_j \pi_j(t),$$

where M_k and m_k are linear combinations of $\alpha_0, \ldots, \alpha_k$, and for each j ($0 \leq j \leq k$), $\pi_j(t)$ is a factorial polynomial of degree j:

$$M_k = \sum_{j=0}^{k} b_{0j}\alpha_j; \quad m_k = \sum_{j=1}^{k} \left(\sum_{i=1}^{j} (-1)^{(i-1)} b_{ij} \right) \alpha_j; \quad \pi_j(t) = \sum_{i=0}^{j} b_{ij} t^{(i)}. \quad \square$$

For $j = 0, 1, \ldots, 9$, the factorial polynomials $\pi_j(t)$ are given in Table 2.

TABLE 2

j	$\pi_j(t)$
0	1
1	$t^{(1)} + 1$
2	$t^{(2)} + 2t^{(1)} + 3$
3	$t^{(3)} + 3t^{(2)} + 9t^{(1)} + 8$
4	$t^{(4)} + 4t^{(3)} + 18t^{(2)} + 32t^{(1)} + 50$
5	$t^{(5)} + 5t^{(4)} + 30t^{(3)} + 80t^{(2)} + 250t^{(1)} + 214$
6	$t^{(6)} + 6t^{(5)} + 45t^{(4)} + 160t^{(3)} + 750t^{(2)} + 1284t^{(1)} + 2086$
7	$t^{(7)} + 7t^{(6)} + 63t^{(5)} + 280t^{(4)} + 1750t^{(3)} + 4494t^{(2)} + 14602t^{(1)} + 11976$
8	$t^{(8)} + 8t^{(7)} + 84t^{(6)} + 448t^{(5)} + 3500t^{(4)} + 11984t^{(3)} + 58408t^{(2)} + 95808t^{(1)} + 162816$
9	$t^{(9)} + 9t^{(8)} + 108t^{(7)} + 672t^{(6)} + 6300t^{(5)} + 26964t^{(4)} + 175224t^{(3)} + 431136t^{(2)} + 1465344t^{(1)} + 1143576$

As in the previous section and [1] and [2], the coefficients of α_j in M_k and m_k are independent of k. The first few elements of the limit sequences (not mentioned in [4]) M and m (obtained from M_k and m_k for $k \to \infty$) are

M: 1,1,3,8,50,214,2086,11976,162816,1143576,...,
m: 0,1,1,8,16,224,608,13320,41760,1366152,... .

Finally, we remark that the coefficients b_{ij} (and hence the elements of the sequences M and m) can also be computed from a_{ij} by means of

$$b_{ij} = \sum_{m=i}^{j} S(i, m)\left(\sum_{\ell=m}^{j} a_{m\ell}\, s(\ell, j)\right) \qquad (i \le j),$$

where $s(\ell, j)$ and $S(i, m)$ are Stirling numbers of the first and of the second kind, respectively.

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INTEGRAL TRIANGLES AND CIRCLES

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(Submitted July 1987)

Having noticed that Pythagorean triangles have integral diameter circumcircles and integral diameter incircles, A. Hunter was prompted to inquire as to whether there were any integer-sided, non-right-angled triangles having integral diameter incircles or integral diameter circumcircles. After a couple of weeks of Diophantine analysis, the answer to these questions was found to be in the affirmative in both cases.

The first solutions found in this way were:

SIDES			DIAMETER
Incircle Problem			
7	15	20	4
Circumcircle Problem			
182	560	630	650

On investigating the circumcircle problem, M. Kovarik devised the construction in Figure 1 which shows how that problem may be solved by means of Pythagorean triangles.

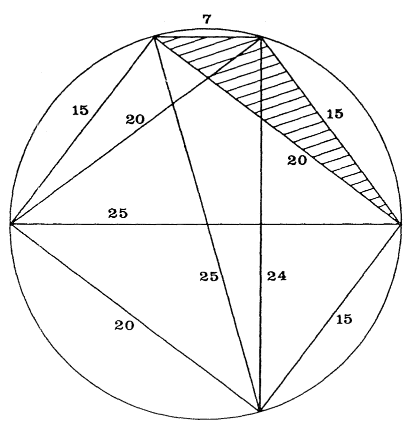

FIGURE 1

Kovarik's construction for the integer-sided scalene triangle 7,15,20 having an integral diameter circumcircle

The first integer-sided scalane triangle produced by Kovarik's method was the 7,15,20 triangle, which happened to coincide with the first solution to the integral diameter incircle problem found above. This prompted Hunter to inquire whether other triangles constructed by Kovarik's method had integral diameter incircles.

Generalizations of Kovarik's construction are shown in Figures 2-5. The two Pythagorean triplets a,b,c and r,s,t are scaled to have the common hypotenuse, ct.

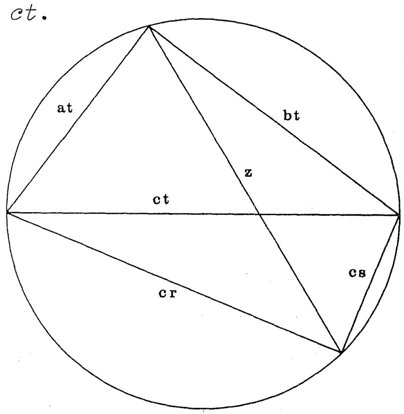

FIGURE 2

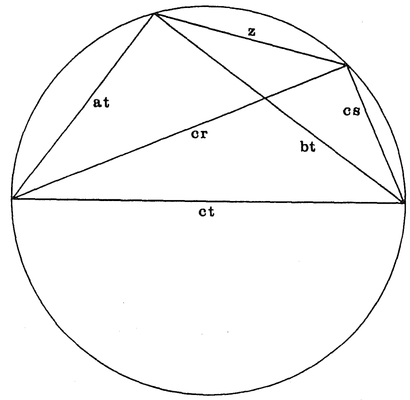

FIGURE 3

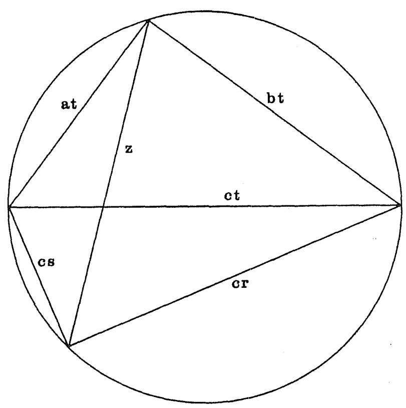

FIGURE 4

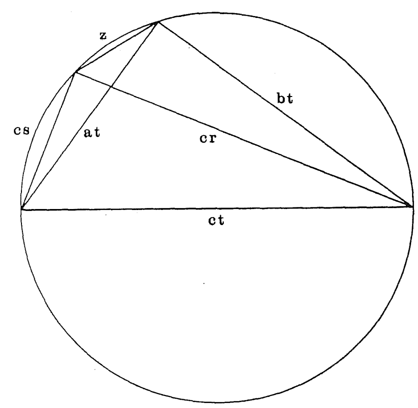

FIGURE 5

FIGURES 2-5

Four constructions for integer-sided, non-right-angled triangles having an integral diameter circumcircle, based on the pair of Pythagorean triplets a,b,c and r,s,t

For a cyclic quadrilateral, Ptolemy's theorem states that the sum of the products of the opposite sides is equal to the product of the diagonals. It follows that the values of z corresponding to Figures 2, 3, 4, and 5 are

$$as + br,\quad |as - br|,\quad ar + bs,\quad \text{and}\quad |ar - bs|,$$

respectively, and are clearly integers. There are two non-right-angled, integer-sided triangles x,y,z for each of Figures 2-5 as given in Table 1.

TABLE 1

Triangle No.	x	y	z
1	at	cr	$as + br$
2	bt	cs	$as + br$
3	at	cr	$\|as - br\|$
4	bt	cs	$\|as - br\|$
5	at	cs	$ar + bs$
6	bt	cr	$ar + bs$
7	at	cs	$\|ar - bs\|$
8	bt	cr	$\|ar - bs\|$

It remained to investigate the diameters of the incircles of these triangles.

The diameter of the incircle of a triangle whose sides are x,y,z is given by

$$d = \sqrt{\frac{(x + y - z)(y + z - x)(z + x - y)}{x + y + z}}$$

Substitution of the values of x, y, and z given in Table 1 yields the incircle diameters given in Table 2.

TABLE 2

Triangle No.	Incircle Diameter
1	$ar - (c - b)(t - s)$
2	$bs - (c - a)(t - r)$
3	$ar - (c + b)(t - s)$
4	$bs - (c - a)(t + r)$
5	$as - (c - b)(t - r)$
6	$br - (c - a)(t - s)$
7	$as - (c - b)(t + r)$
8	$br - (c + a)(t - s)$

Clearly, the diameters of all incircles are integers. The integral triangles and circles (itacs) generated from the Pythagorean triplets

$$\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

are shown in Figure 6. When the two Pythagorean triplets are equal, triangles numbered 1 and 2 become isosceles, numbers 3 and 4 diminish to a point, 5 and 6 become right-angled, and 7 and 8 are scalene. The itacs generated from

$$\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 17 \end{bmatrix}$$

are shown in Figure 7.

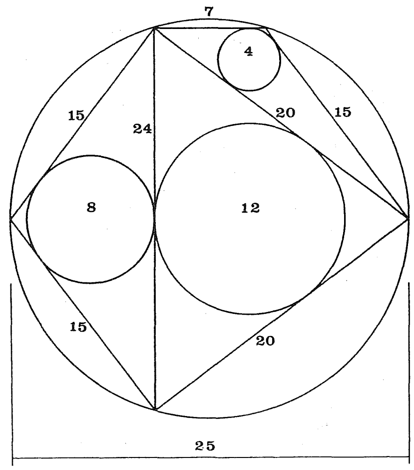

FIGURE 6

Itacs based on a,b,c equal to r,s,t equal to 3,4,5

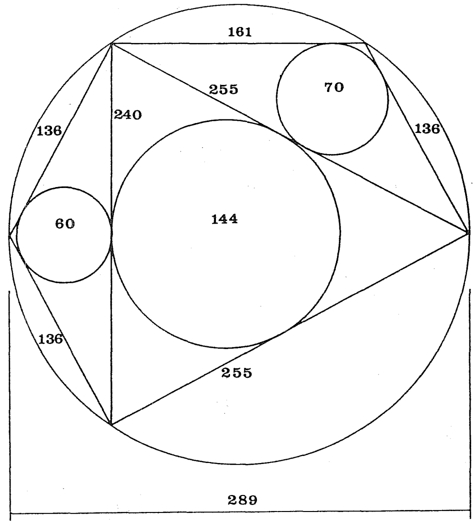

FIGURE 7

Itacs based on a,b,c equal to r,s,t equal to 8,15,17

Where a,b,c and r,s,t are independent Pythagorean triplets, all eight triangles are, in general scalene. Itacs generated from

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix}$$

are shown in Figures 8 and 9 and itacs generated from

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 17 \end{bmatrix}$$

are shown in Figures 10 and 11.

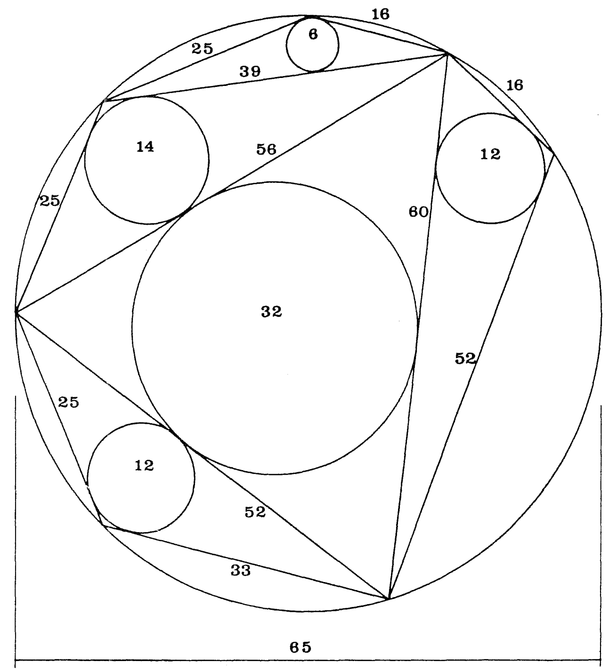

FIGURE 8

Itacs based on a,b,c equal to 3,4,5 and r,s,t equal to 5,12, 13 (5 incircles)

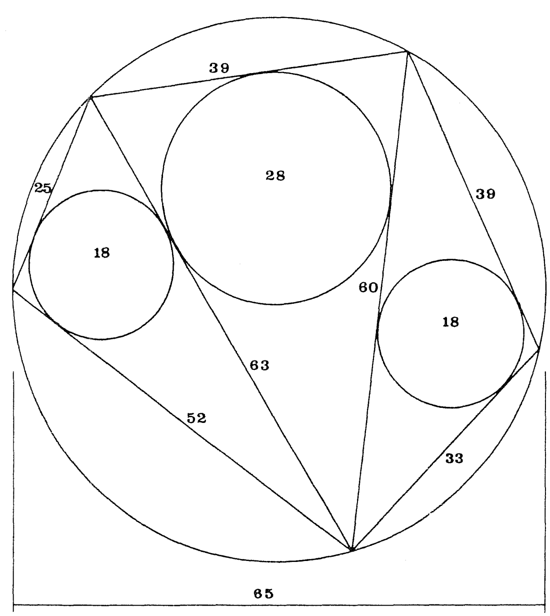

FIGURE 9

Itacs based on a,b,c equal to 3,4,5 and r,s,t equal to 5,12,13 (3 incircles)

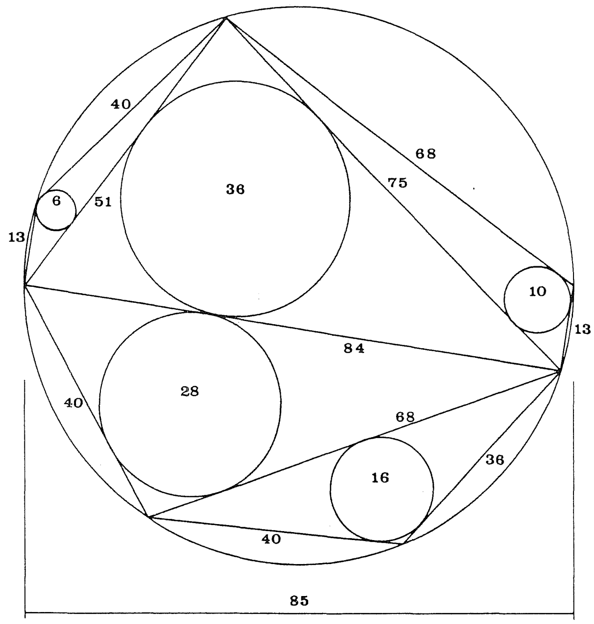

FIGURE 10

Itacs based on a,b,c equal to 3,4,5 and r,s,t equal to 8,15,17 (5 incircles)

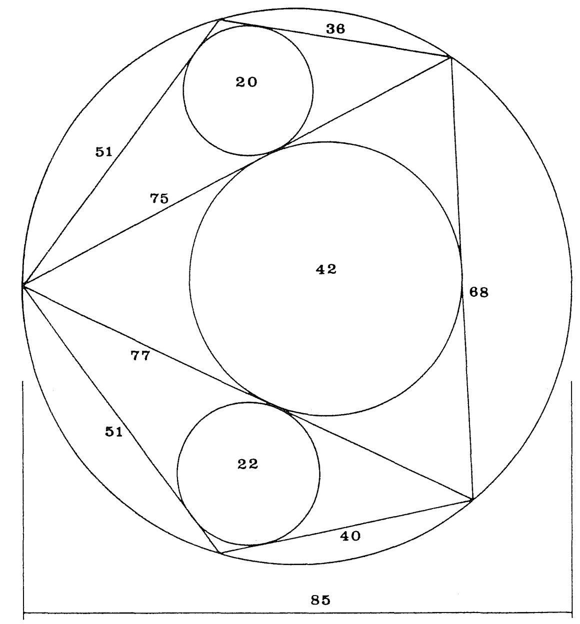

FIGURE 11

Itacs based on a,b,c equal to 3,4,5 and r,s,t equal to 8,15,17 (3 incircles)

Triangles arising in itacs are also Heronian; that is, have integral areas. The area of a triangle is given by

$$A = \frac{xyz}{2d},$$

where d is the diameter of the circumcircle, and x, y, and z are the sides. Substituting from Table 1, we obtain expressions such as

$$A = \frac{ar(as + br)}{2}.$$

For Pythagorean triplets a,b,c and r,s,t, ab and rs are both even, so A is always an integer.

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Announcement

FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 30-August 3, 1990

Department of Mathematics and Computer Science

Wake Forest University

Winston-Salem, North Carolina 27109

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CALL FOR PAPERS

The FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Wake Forest University, Winston-Salem, N.C., from July 30 to August 3, 1990. This Conference is sponsored jointly by the Fibonacci Association and Wake Forest University.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts are to be submitted by March 15, 1990, while manuscripts are due by May 1, 1990. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

Professor Gerald E. Bergum
The Fibonacci Quarterly
Department of Computer Science
South Dakota State University
P.O. Box 2201
Brookings, South Dakota 57007-0194

UNITARY PERFECT NUMBERS WITH SQUAREFREE ODD PART

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(Submitted July 1987)

1. Introduction

A divisor d of a natural number n is said to be *unitary* if and only if

$$(d, n/d) = 1.$$

The sum of the unitary divisors of n is denoted $\sigma^*(n)$. It is straightforward to show that if

$$n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k},$$

then

$$\sigma^*(n) = (p_1^{a_1} + 1)(p_2^{a_2} + 1) \cdots (p_k^{a_k} + 1).$$

A natural number n is said to be *unitary perfect* if $\sigma^*(n) = 2n$.

Subbarao and Warren [2] discovered the first four unitary perfect numbers:

$$6 = 2 \cdot 3,\ 60 = 2^2 \cdot 3 \cdot 5,\ 90 = 2 \cdot 3^2 \cdot 5,\ 87360 = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13.$$

Wall [3] discovered another such number,

$$146361946186458562560000 = 2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313,$$

and he later showed [4] that this is the fifth unitary perfect number. No other unitary perfect numbers are known, and Wall [5] has shown that any other such number must have an odd prime divisor exceeding 2^{15}.

In this paper, we consider the existence of unitary perfect numbers of the form $2^m s$, where s is a squarefree odd integer. We shall prove that there are only three such numbers.

Theorem: If $2^m s$ is a unitary perfect number and s is squarefree, then either $m = 1$ and $s = 3$, $m = 2$ and $s = 3 \cdot 5$, or $m = 6$ and $s = 3 \cdot 5 \cdot 7 \cdot 13$.

2. Preliminaries

Throughout this paper, the letter s shall be used to denote an odd squarefree number. The letter p, with or without a subscript, shall denote an odd prime. The letter q, with or without a subscript, shall denote a Mersenne prime.

Our starting point is the observation that, for any fixed m, it is easy to determine all unitary perfect numbers of the form $2^m s$. From the previously stated formula for $\sigma^*(n)$, we see that if $s = p_1 p_2 \ldots p_r$, then $2^m s$ is unitary perfect if and only if

$$2 = \frac{\sigma^*(2^m s)}{2^m s} = \frac{2^m + 1}{2^m} \cdot \frac{p_1 + 1}{p_1} \cdot \frac{p_2 + 2}{p_2} \cdot \ldots \cdot \frac{p_r + 1}{p_r}. \tag{1}$$

Any odd prime dividing $2^m + 1$ must appear as a denominator on the right-hand side. If p is such a prime, then all odd prime divisors of $p + 1$ must also appear as denominators on the right-hand side. If we can force a prime to appear more than once, then we can conclude that there is no unitary perfect number of the form $2^m s$.

For example, suppose $m = 7$. Since $2^7 + 1 = 3 \cdot 43$, 3 and 43 must appear as denominators on the right-hand side of (1). Since $11 \mid (43 + 1)$, 11 must also appear. But $3 \mid (11 + 1)$, so 3 must appear twice. Therefore, there is no unitary perfect number of the form $2^7 s$.

On the other hand, suppose $m = 6$. Since $2^6 + 1 = 5 \cdot 13$, both 5 and 13 must be prime divisors of s. Since $3 \mid (5 + 1)$ and $7 \mid (13 + 1)$, 7 and 13 must be prime divisors of s. If any other p divides s, then

$$\frac{\sigma^*(2^m s)}{2^m s} \geq \frac{2^6 + 1}{2^6} \cdot \frac{3 + 1}{3} \cdot \frac{5 + 1}{5} \cdot \frac{7 + 1}{7} \cdot \frac{13 + 1}{13} \cdot \frac{p + 1}{p} > 2.$$

Therefore, the only unitary perfect number of the form $2^6 s$ is $2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$.

Proceeding in this fashion, it is easy to show that the only unitary perfect numbers of the form $2^m s$ with $m < 10$ are those listed in the theorem. Thus, we may assume henceforth that $m \geq 10$. (Alternatively, we could reduce to the case $m \geq 10$ by quoting a result of Subbarao [1].)

The method of the preceding paragraphs is "top-down": we start with divisors of $2^m + 1$ and work down. While this procedure works well for specific m, it does not lend itself well to a proof in the general case. We therefore introduce an alternative "bottom-up" procedure. This procedure starts with the Mersenne primes dividing s and works up to the divisors of $2^m + 1$. (A Mersenne prime is a prime of the form $2^k - 1$; the first few such primes are

$$3 = 2^2 - 1,\ 7 = 2^3 - 1,\ 31 = 2^5 - 1,\ 127 = 2^7 - 1,\ 8191 = 2^{13} - 1.)$$

First we note that s does have Mersenne prime divisors. For in equation (1), all odd prime divisors of $\sigma^*(s) = p_1 p_2 \ldots p_r$ must appear in the denominator of the right-hand side. But some of the p_i's divide $2^m + 1$, so at least one of the terms $p_i + 1$ must be free of any odd prime factors. It follows that p_i is a Mersenne prime.

Suppose q is a Mersenne prime dividing s. Renumber the primes in (1) so that $q = p_1$. There is some (necessarily unique) prime p_2 dividing s such that $p_1 \mid (p_2 + 1)$. Note that $p_2 \geq 2p_1 - 1$. Either $p_2 \mid (2^m + 1)$ or there is some p_3 such that $p_2 \mid (p_3 + 1)$. Continuing in this way, we obtain a sequence of primes

$$p_1 < p_2 < \cdots < p_k, \tag{2}$$

where p_1 is a Mersenne prime, $p_k \mid (2^m + 1)$, and $p_{i+1} \geq 2p_i - 1$.

To formalize the ideas of the preceding paragraph, we introduce the following function f. Let p be an odd prime in the denominator of the right-hand side of (1). We define $f(p)$ to be 1 if $p \mid (2^m + 1)$. Otherwise, we define $f(p)$ to be the unique prime p' such that $p' \mid s$ and $p \mid (p' + 1)$. We define

$$f_0(p) = p,\ f_1(p) = f(p),\ \text{and}\ f_{k+1}(p) = f(f_k(p)).$$

We also define

$$f(1) = 1 \quad \text{and} \quad f_\infty(p) = \prod_{i=0}^{\infty} f_i(p).$$

For example, if $m = 6$ and $s = 3 \cdot 5 \cdot 7 \cdot 13$, then

$$f_1(3) = 5,\ f_2(3) = 1,\ \text{and}\ f_\infty(3) = 3 \cdot 5.$$

Similarly,

$$f_\infty(7) = 7 \cdot 13.$$

Let $q_1, q_2, \ldots, q_\ell$ be the Mersenne primes dividing s. Then all odd primes dividing s occur in the product

$$f_\infty(q_1)f_\infty(q_2) \cdots f_\infty(q_\ell). \tag{3}$$

At this point, we cannot rule out the possibility that this product contains repeated prime factors. For example, if $41|s$, then $41 | f_\infty(3)$ and $41|f_\infty(7)$. Accordingly, for each Mersenne prime q , we define $F(q_i)$ to be the product of all primes that divide $f_\infty(q_i)$ but do not divide any of $f_\infty(q_1), f_\infty(q_2), \ldots, f_\infty(q_{i-1})$. With this definition, we have

$$\frac{2^m + 1}{2^m} \cdot \frac{\sigma^*(F(q_1))}{F(q_1)} \cdot \ldots \cdot \frac{\sigma^*(F(q_\ell))}{F(q_\ell)} = 2.$$

If we write

$$G(q) = \frac{\sigma^*(F(q))}{F(q)},$$

then the above may be rewritten as

$$\frac{2^m + 1}{2^m} G(q_1) \cdots G(q_\ell) = 2. \tag{4}$$

The idea behind the proof is to obtain upper bounds for $G(q)$ that make (4) untenable. The crucial point here is that, if $p_1, p_2, \ldots, p_k$ are the primes described in (2), then $p_2 \geq 2p_1$, $p_3 \geq 4p_1 - 3$, etc. It follows that

$$G(q) \leq \prod_{i=0}^{\infty} \frac{2^i p_i - 2^i + 2}{2^i p_i - 2^i + 1}.$$

As we shall show in Lemmas 1 and 2, this product converges. This bound for G is sufficient for the larger Mersenne primes. A more elaborate analysis is needed for the smaller primes.

3. Lemmas

Lemma 1: If ρ and δ are real numbers with $\rho > 1$, then

$$\prod_{i=0}^{\infty} \frac{\delta\rho^i - (\rho + \rho^2 + \cdots + \rho^i)}{\delta\rho^i - (1 + \rho + \rho^2 + \cdots + \rho^i)} = \frac{(\rho - 1)\delta}{(\rho - 1)\delta - \rho}.$$

Proof: The K^{th} partial product is

$$\frac{\delta}{\delta - 1} \cdot \frac{\rho\delta - \rho}{\rho\delta - \rho - 1} \cdot \ldots \cdot \frac{\rho^K\delta - \rho^K - \cdots - \rho}{\rho^K\delta - \rho^K - \cdots - \rho - 1}.$$

Note that the numerator of each term after the first is ρ times the denominator of the previous term. Therefore, the K^{th} partial product is

$$\frac{\delta\rho^K}{\rho^K\delta - \rho^K - \cdots - \rho - 1} = \frac{(\rho - 1)\delta}{(\rho - 1)\delta - \rho + \rho^{-K}}.$$

The result follows by letting K tend to infinity.

Lemma 2: If $q = 2^m - 1$ is a Mersenne prime, then $G(q) \leq \dfrac{2^{m-1}}{2^{m-1} - 1}$.

Proof: Let $p_1, p_2, \ldots, p_k$ be the primes dividing $G(q)$. Since $p_1 = q = 2^m$ and and $p_{i+1} \geq 2p_i$, we see that

$$p_i \geq 2^{m+i-1} - 2^i + 1.$$

Therefore,

$$G(q) \leq \frac{2^m}{2^m - 1} \cdot \frac{2^{m+1} - 2}{2^{m+1} - 3} \cdots .$$

The result now follows by applying Lemma 1 with $\delta = 2^m$ and $\rho = 2$.

Lemma 3: Let $q_j, \ldots, q_\ell$ be the Mersenne primes that divide s and are at least 8191. Then

$$G(q_j) \cdots G(q_\ell) \leq \frac{3072}{3071}.$$

Proof: It is well known that, if $2^m - 1$ is prime, then m must be prime. Thus, $m = 2$ or m is odd. Consequently

$$G(q_j) \cdots G(q_\ell) \leq \prod_{i=0}^{\infty} \frac{2^{12+2i}}{2^{12+1i} - 1}$$

We bound this by observing that

$$\prod_{i=0}^{\infty} \frac{2^{12+2i}}{2^{12+2i} - 1} \leq \prod_{i=0}^{\infty} \frac{2^{12+2i} - 4 - 4^2 - \cdots - 4^i}{2^{12+2i} - 1 - 4 - 4^2 - \cdots - 4^i}.$$

The result now follows from Lemma 1 with $\delta = 2^{12}$ and $\rho = 4$.

Lemma 4: Let $q_j, \ldots, q_\ell$ be the Mersenne primes that divide s and are at least 127. Then

$$G(q_j) \cdots G(q_\ell) \leq \frac{122}{121}.$$

Proof: We first get a bound on $G(127)$. Let $p_1, \ldots, p_r$ be the primes that divide $F(127)$. If $r \leq 1$, then $G(127) \leq 128/127$. Assume that $r \geq 2$. Then $p_1 = 127$ and p_2 is a prime of the form $127h - 1$, where all the odd prime divisors of h are at least 8191. Now $127 \cdot 2^i - 1$ is composite for $1 \leq i \leq 7$, so $p_2 \geq 127 \cdot 2^8 - 1 = 32511$. Therefore,

$$G(127) \leq \frac{128}{127} \prod_{i=0}^{\infty} \frac{32511 \cdot 2^i - 2 - 2^2 - \cdots - 2^i}{32511 \cdot 2^i - 1 - 2 - \cdots - 2^i} = \frac{128}{127} \cdot \frac{16256}{16255}.$$

From this and Lemma 3, we see that

$$G(q_j) \cdots G(q_\ell) \leq G(127)G(8191) \cdots \leq \frac{128}{127} \cdot \frac{16256}{16255} \cdot \frac{3072}{3071} \leq \frac{122}{121}.$$

4. Proof of the Theorem

As stated in Section 2, we may assume that $m \geq 10$.

The proof breaks into three cases: (1) m odd, (2) $m \equiv 0 \bmod 4$, and (3) $m \equiv 2 \bmod 4$.

Case 1: Assume that m is odd. Then $3 \mid 2^m + 1$, and $G(3) = 4/3$. It follows that the left-hand side of (4) is

$$\frac{2^m + 1}{2^m} \frac{4}{3} G(7)G(31) \cdots \leq \frac{1025}{1024} \frac{4}{3} \frac{4}{3} \frac{16}{15} \frac{122}{121} < 2.$$

Case 2: Assume that $m \equiv 0 \bmod 4$. Then $2^m + 1 \equiv 2 \bmod 3$ and $2^m + 1 \equiv 2 \bmod 5$. It follows that there is some prime p such that $p \mid 2^m + 1$, $p \equiv 2 \bmod 3$, and $p > 5$. Moreover, the congruence $x^4 \equiv -1 \bmod p$ has the solution $x \equiv 2^{m/4}$, so we have $p \equiv 1 \bmod 8$. By the Chinese Remainder Theorem, $p \equiv 17 \bmod 24$. We cannot have $p = 17$ since $3^2 \mid \sigma^*(17)$. Therefore, $p \geq 41$, and the left-hand side of (4) is

$$\frac{2^m+1}{2^m}\,\frac{4}{3}\,\frac{p+1}{p}\,G(7)G(31)\,\cdots \leq \frac{1025}{1024}\,\frac{4}{3}\,\frac{42}{41}\,\frac{4}{3}\,\frac{16}{15}\,\frac{122}{121} < 2.$$

Case 3: Assume that $m \equiv 2 \bmod 4$. Then $5 \mid 2^m + 1$, and

$$G(3) = \frac{4}{3}\,\frac{6}{5}.$$

This case breaks into four subcases: (i) $7 \nmid s$; (ii) $7 \mid s$ and $13 \nmid s$; (iii) $7 \mid s$, $13 \mid s$, and $103 \nmid s$; (iv) $7 \mid s$, $13 \mid s$, and $103 \mid s$.

Subcase 3(i): Assume that $7 \nmid s$. Then the left-hand side of (4) is

$$\frac{2^m+1}{2^m}\,G(3)G(31)G(127)\,\cdots \leq \frac{1025}{1024}\,\frac{4}{3}\,\frac{6}{5}\,\frac{16}{15}\,\frac{122}{121} < 2.$$

Subcase 3(ii): Assume that $7 \mid s$ and $13 \nmid s$. Other than 13, the least prime of the form $7h - 1$ with all odd prime divisors of h greater than or equal to 31 is $7 \cdot 32 - 1 = 223$. Therefore,

$$G(7) \leq \frac{8}{7}\prod_{i=0}^{\infty}\frac{224 \cdot 2^i - (2 + 2^2 + \cdots + 2^i)}{224 \cdot 2^i - (1 + 2 + \cdots + 2^i)} = \frac{8}{7}\,\frac{112}{111}.$$

Therefore, the left-hand side of (4) is

$$\leq \frac{1025}{1024}\,\frac{4}{3}\,\frac{6}{5}\,\frac{8}{7}\,\frac{112}{111}\,\frac{16}{15}\,\frac{122}{121} < 2.$$

Subcase 3(iii): Assume that $7 \mid s$, $13 \mid s$, and $103 \mid s$. Then $31 \mid s$ since

$$\frac{\sigma^*(3 \cdot 5 \cdot 7 \cdot 13 \cdot 31)}{3 \cdot 5 \cdot 7 \cdot 13 \cdot 31} > 2.$$

If $F(7)$ contains any prime factors other than 7 or 13, then the least such factor is of the form $13h - 1$, where all odd prime factors of h are ≥ 127. Other than 103, the least prime of this form is $13 \cdot 2^7 - 1 = 1663$. Therefore,

$$G(7) \leq \frac{8}{7}\,\frac{14}{13}\,\frac{832}{831},$$

and the left-hand side of (4) is

$$\leq \frac{1025}{1024}\,\frac{4}{3}\,\frac{6}{5}\,\frac{8}{7}\,\frac{14}{13}\,\frac{832}{831}\,\frac{122}{121} < 2.$$

Subcase 3(iv): Assume that $7 \mid s$, $13 \mid s$, and $103 \mid s$. Then $127 \nmid s$ since

$$\frac{\sigma^*(3 \cdot 5 \cdot 7 \cdot 13 \cdot 103 \cdot 127)}{3 \cdot 5 \cdot 7 \cdot 13 \cdot 103 \cdot 127} > 2$$

The least prime of the form $103h - 1$ is $103 \cdot 8 - 1 = 823$. Therefore,

$$G(7) \leq \frac{8}{7}\,\frac{14}{13}\,\frac{104}{103}\,\frac{412}{411},$$

and the right-hand side of (4) is

$$\leq \frac{1025}{1024}\,\frac{4}{3}\,\frac{6}{5}\,\frac{8}{7}\,\frac{14}{13}\,\frac{104}{103}\,\frac{412}{411}\,\frac{3072}{3071} < 2.$$

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RECURRING-SEQUENCE TILING

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In this paper we consider the problem of dividing a rectangle into nonoverlapping squares and rectangles using recurring-sequence tiling. The results obtained herein are illustrated with appropriate figures. These results, with the exception of basic introductory material, are believed to be new. There seem to be no such results in the literature.

Among the many generating functions possible, we choose the following:

$$G(x)^{-k} = 1/(1 - x - x^2 - \cdots - x^m)^k \tag{1}$$

(where $m = 2, 3, 4, \ldots$ and $k = 1, 2, 3, \ldots$).

Note that

$$G(x)^{-k} = F_{m,k}(0) + F_{m,k}(1)x + F_{m,k}(2)x^2 + \cdots + F_{m,k}(n)x^n + \cdots \tag{2}$$

[where $F_{m,k}(0) = 1$, for all m and k].

In this paper, we limit k to the value of 1. Therefore, with $k = 1$, for convenience, we can write

$$F_{m,k}(n) = F_m(n).$$

When $m = 2$, the above formulas will result in the well-known Fibonacci numbers. When $m = 3$, one will obtain the Tribonacci numbers.

A tile representing a number in one of these sequences will be a square whose sides are of a length equal to that number. As we examine various versions of equation (2) above, we will attempt to combine tiles so that rectangular regions are formed. When this is not possible, we will identify the gaps left in the almost-rectangular region, and attempt to generalize the sizes of those gaps.

First let $m = 2$, then for $n = 0, 1, 2, \ldots$, we have the Fibonacci numbers and the following consecutive values for $F_2(n)$:

$$F_2(0), F_2(1), F_2(2), \ldots, F_2(n), \ldots$$
$$= 1, 1, 2, 3, 5, 8, 13, 21, \ldots . \tag{3}$$

When tiles are fashioned from these numbers, they may be arranged as in Figure 1. *Note that each tile is a square.* There are, of course, other ways to arrange these tiles. This method of arrangement is simple to follow, and facilitates the arguments below. In this case, as each new tile is added to the region, a full rectangle results.

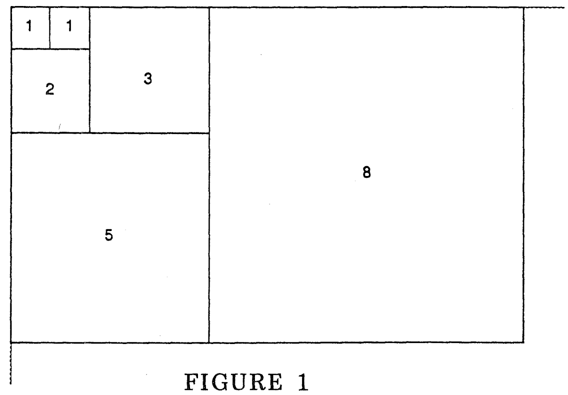

FIGURE 1

When $m = 3$, then, for $n = 0, 1, 2, 3, \ldots$, we have the Tribonacci numbers consisting of the following consecutive values for $F_3(n)$:

$$F_3(0),\ F_3(1),\ F_3(2),\ \ldots,\ F_3(n),\ \ldots$$
$$= 1, 1, 2, 4, 7, 13, 24, 81, 149, 274, \ldots . \tag{4}$$

Tiles representing these numbers may be arranged as in Figure 2. Again, though distorted because of space, each tile is a square. Note that in this case the addition of a new tile results in a region of irregular shape. To form a full rectangle, smaller rectangles (not necessarily squares) must be added to fill in the "gaps." We shall examine these gaps further below.

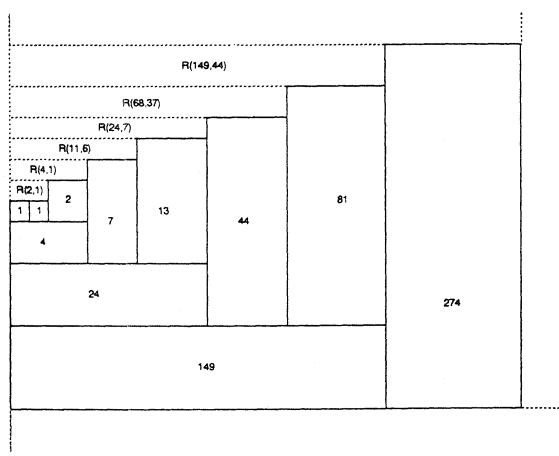

FIGURE 2

We will later continue in this way for larger values of m, and step by step, and with added induction we will develop a systematic way of placing tiles using the recurring sequence in equation (2) above.

Let us now examine the gaps in the rectangle we are attempting to tile. Refer to the upper left corner of Figure 2 to see these gaps. Our approach will be to find a recurrence relation which might be used to construct smaller rectangles, which will in turn fill in the gaps. We shall label these smaller rectangles $R_m(x, y)$, where x and y are the horizontal and vertical components of the rectangles, respectively.

In Table 3 below, for $m = 3$, we list the consecutive values of the horizontal component x (where $x = 2, 4, 11, 24, 68, 149, \ldots$) in the consecutive rectangles $R_m(x, y)$.

TABLE 3

Horizontal Components of Rectangles, for $m = 3$

Rectangle	x	Formula
R(2,1)	2	$F_3(0) + F_3(1)$
R(4,1)	4	$F_3(3)$
R(11,6)	11	$F_3(3) + F_3(4)$
R(24,7)	24	$F_3(6)$
R(68,37)	68	$F_3(6) + F_3(7)$
R(149,44)	149	$F_3(9)$
...	...	...

Now, using Table 3 and by induction, we obtain the following general values of x in $R_3(x, y)$:

$$x = F_3(3n) + F_3(3n + 1) \quad \text{or} \quad x = F_3(3n + 3), \text{ where } n \geq 0. \tag{5}$$

In Table 4 below, for $m = 3$, we list the consecutive values of the vertical component y (for $y = 1, 1, 6, 7, 37, 44, \ldots$) of each $R_3(x, y)$ from Figure 2:

TABLE 4

Vertical Components of Rectangles, for $m = 3$

Rectangle	y	Formula
R(2,1)	1	$F_3(0)$
R(4,1)	1	$F_3(1)$
R(11,6)	6	$F_3(2) + F_3(3)$
R(24,7)	7	$F_3(4)$
R(68,37)	37	$F_3(5) + F_3(6)$
R(149,44)	44	$F_3(7)$
...	...	...

Now, using Table 4 step by step and with added induction, we obtain the following values of y in $R_3(x, y)$:

$$y = F_3(0) = 1 \quad \text{or} \quad y = F_3(3n + 1) \quad \text{or}$$
$$F_3(3n + 2) + F_3(3n + 3), \text{ where } n \geq 0. \tag{6}$$

Combining (5) and (6) above, we observe that the general form of each gap-filling rectangle $R_3(x, y)$ may be written as

$$R_3(2, 1) \quad \text{or} \quad R_3(F_3(3n + 3), F_3(3n + 1)) \quad \text{or}$$
$$R_3(F_3(3n + 3) + F_3(3n + 4), F_3(3n + 2) + F_3(3n + 3)), \text{ where } n \geq 0. \tag{7}$$

We now consider the rectangles that we are attempting to tile. The notation we use will be $A_m(x, y)$, where once again x and y are the horizontal and vertical lengths of the tiled rectangle.

For $m = 3$, we have $A_3(x, y)$. As we add the tiles one by one, certain of the rectangles $A_3(x, y)$ tiled, in order of their construction, are shown in Table 5. Also shown are the components of each tiled rectangle, with each square tile followed by an S.

TABLE 5

Construction of Tiled Rectangles, for $m = 3$

Tiled Rectangle	Component Squares and Rectangles
$A_3(2,1)$	1S + 1S
$A_3(4,6)$	$A_3(2,1)$ + 2S + 4S + $R_3(2,1)$
$A_3(11,7)$	$A_3(4,6)$ + 7S + $R_3(4,1)$
$A_3(24,37)$	$A_3(11,7)$ + 13S + 24S + $R_3(11,6)$
$A_3(68,44)$	$A_3(24,37)$ + 44S + $R_3(24,7)$
...	...

Continuing in this way, by induction, we conclude that the general formulas for the areas $A_3(x, y)$ are

$$A_3(F_3(3n) + F_3(3n + 1), F_3(3n + 1)) \quad \text{or}$$
$$A_3(F_3(3n + 3), F_3(3n + 2) + F_3(3n + 3)), \text{ where } n = 0, 1, 2, \ldots . \tag{8}$$

Now, let $m = 4$ in equation (2). Then, for $n = 0, 1, 2, \ldots$, we have the following consecutive values for $F_4(n)$:

$$F_4(0), F_4(1), F_4(2), \ldots, F_4(n), \ldots$$
$$= 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, \ldots . \tag{9}$$

An arrangement of the tiles corresponding to these values is shown in Figure 6. Again, note that the arrangement of square tiles results in an irregular shape. Once again, "filler" rectangles must be generated to fill in the gaps, to construct a fully tiled rectangle.

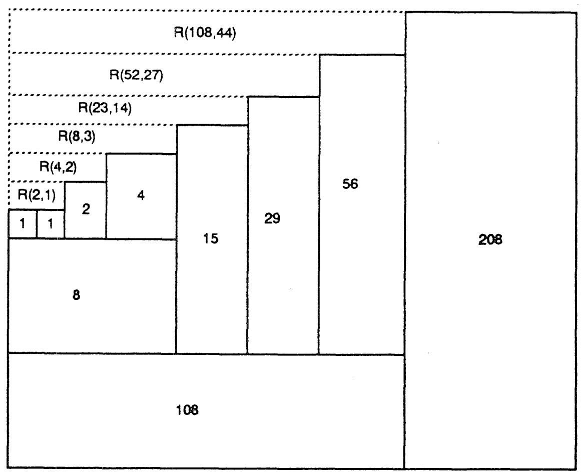

FIGURE 6

In Table 7 below, for $m = 4$, we list the consecutive values of the horizontal component x (where $x = 2, 4, 8, 23, 52, 108, 316, 717, 1490, \ldots$) in the consecutive rectangles $R_4(x, y)$.

TABLE 7

Horizontal Components of Rectangles, for $m = 4$

Rectangle	x	Formula
R(2,1)	2	$F_4(0) + F_4(1)$
R(4,2)	4	$F_4(0) + F_4(1) + F_4(2)$
R(8,3)	8	$F_4(4)$
R(23,14)	23	$F_4(4) + F_4(5)$
R(52,27)	52	$F_4(4) + F_4(5) + F_4(6)$
R(108,44)	108	$F_4(8)$
...	...	...

Now, using Table 7 with added induction, we find the following general values of x in $R_4(x, y)$:

$$x = F_4(4n) + F_4(4n + 1) \quad \text{or}$$
$$x = F_4(4n) + F_4(4n + 1) + F_4(4n + 2) \quad \text{or}$$
$$x = F_4(4n + 4), \text{ where } n \geq 0. \tag{10}$$

In Table 8 below, for $m = 4$, we list the consecutive values of the vertical component y ($y = 1, 2, 3, 14, 27, 44, 193, 372, 609, \ldots$) of each $R_4(x, y)$:

TABLE 8

Vertical Components of Rectangles, for $m = 4$

Rectangle	y	Formula
R(2,1)	1	$F_4(1)$
R(4,2)	2	$F_4(2)$
R(8,3)	3	$F_4(1) + F_4(2)$
R(23,14)	14	$F_4(2) + F_4(3) + F_4(4)$
R(52,27)	27	$F_4(3) + F_4(4) + F_4(5)$
R(108,44)	44	$F_4(5) + F_4(6)$
...	...	...

Now, using Table 8 step by step and with added induction, we find the following general values of y in $R_4(x, y)$:

$$
\begin{aligned}
&y = F_4(1) = 1 \quad \text{or} \quad y = F_4(2) \quad \text{or} \qquad (11)\\
&y = F_4(4n + 1) + F_4(4n + 2) \quad \text{or}\\
&y = F_4(4n + 2) + F_4(4n + 3) + F_4(4n + 4) \quad \text{or}\\
&y = F_4(4n + 3) + F_4(4n + 4) + F_4(4n + 5), \text{ where } n = 0, 1, 2, \ldots .
\end{aligned}
$$

Combining equations (10) and (11) above, we observe that the general formulas of the $R_4(x, y)$ may be written as

$$
\begin{aligned}
&R_4(2, 1), \quad R_4(4, 2) \qquad (12)\\
&R_4(F_4(4n + 4), F_4(4n + 1) + F_4(4n + 2)),\\
&R_4(F_4(4n + 4) + F_4(4n + 5), F_4(4n + 2) + F_4(4n + 3) + F_4(4n + 4)),\\
&R_4(F_4(4n + 4) + F_4(4n + 5) + F_4(4n + 6),\\
&\qquad F_4(4n + 3) + F_4(4n + 4) + F_4(4n + 5)), \text{ where } n \geq 0.
\end{aligned}
$$

We now consider selected tiled rectangles $A_4(x, y)$ in the order of their construction:

TABLE 9

Construction of Tiled Rectangles, for $m = 4$

Tiled Rectangle	Component Squares and Rectangles
$A_4(2,1)$	1S + 1S
$A_4(4,2)$	$A_4(2,1)$ + 2S + $R_4(2,1)$
$A_4(8,12)$	$A_4(4,2)$ + 4S + 8S + $R_4(4,2)$
$A_4(23,15)$	$A_4(8,12)$ + 15S + $R_4(8,3)$
$A_4(52,29)$	$A_4(23,15)$ + 29S + $R_4(23,14)$
$A_4(108,164)$	$A_4(52,29)$ + 56S + 108S + $R_4(52,27)$
...	...

Continuing in this way, we conclude that the general formulas for the areas $A_4(x, y)$ are

$$A_4(F_4(4n) + F_4(4n + 1), F_4(4n + 1)),$$

$$A_4(F_4(4n) + F_4(4n + 1) + F_4(4n + 2), F_4(4n + 2)),$$

$$A_4(F_4(4n + 4), F_4(4n + 3) + F_4(4n + 4)), \text{ where } n \geq 0. \tag{13}$$

Now, by induction, we tile the functions in equation (2) for all $m = 5, 6, 7, \ldots$. We are concerned with the sequence of values for $F_m(n)$:

$$F_m(0), F_m(1), F_m(2), \ldots, F_m(n), \ldots$$

$$= 1, 1, 2, 4, 8, 16, \ldots, 2^{m-1}, 2^m - 1, 2^{m+1} - 3,$$

$$2^{m+2} - 8, 2^{m+3} - 20, \ldots, \text{ where } n \geq 0. \tag{14}$$

An arrangement of the tiles corresponding to these values is shown in Figure 10.

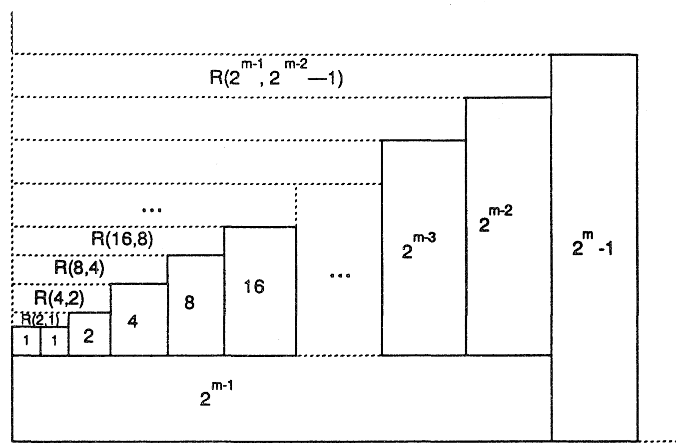

FIGURE 10

Now, by induction, we may systematically generalize the values of the horizontal component x in each rectangle $F_m(x, y)$:

For $n = 0$,

$$F_m(m \cdot 0 + m)$$

$$F_m(m \cdot 0 + m) + F_m(m \cdot 0 + m + 1)$$

$$F_m(m \cdot 0 + m) + F_m(m \cdot 0 + m + 1) + F_m(m \cdot 0 + m + 2)$$

$$\vdots$$

$$F_m(m \cdot 0 + m) + F_m(m \cdot 0 + m + 1) + F_m(m \cdot 0 + m + 2)$$
$$+ \cdots + F_m(m \cdot 0 + 2m - 2).$$

For $n = 1$,

$$F_m(m \cdot 1 + m)$$

$$F_m(m \cdot 1 + m) + F_m(m \cdot 1 + m + 1)$$

$$F_m(m \cdot 1 + m) + F_m(m \cdot 1 + m + 1) + F_m(m \cdot 1 + m + 2)$$

$$\vdots$$

$$F_m(m \cdot 1 + m) + F_m(m \cdot 1 + m + 1) + F_m(m \cdot 1 + m + 2)$$
$$+ \cdots + F_m(m \cdot 1 + 2m - 2).$$

Then, in general, for n (where $n \geq 0$),

$$F_m(m \cdot n + m) \tag{15}$$

$$F_m(m \cdot n + m) + F_m(m \cdot n + m + 1)$$

$$F_m(m \cdot n + m) + F_m(m \cdot n + m + 1) + F_m(m \cdot n + m + 2)$$

$$\vdots$$

$$F_m(m \cdot n + m) + F_m(m \cdot n + m + 1) + F_m(m \cdot n + m + 2)$$
$$+ \cdots + F_m(m \cdot n + 2m - 2).$$

Now, step by step and with added induction, we obtain the generalized values of the vertical component y in each rectangle $R_m(x, y)$:

For $n = 0$,

$$F_m(m \cdot 0 + 1) + F_m(m \cdot 0 + 2) + \cdots + F_m(m \cdot 0 + m - 2)$$

$$F_m(m \cdot 0 + 2) + F_m(m \cdot 0 + 3) + \cdots + F_m(m \cdot 0 + m)$$

$$F_m(m \cdot 0 + 3) + F_m(m \cdot 0 + 4) + \cdots + F_m(m \cdot 0 + m + 1)$$

$$\vdots$$

$$F_m(m \cdot 0 + m - 1) + F_m(m \cdot 0 + m) + \cdots + F_m(m \cdot 0 + 2m - 3).$$

For $n = 1$,

$$F_m(m \cdot 1 + 1) + F_m(m \cdot 1 + 2) + \cdots + F_m(m \cdot 1 + m - 2)$$

$$F_m(m \cdot 1 + 2) + F_m(m \cdot 1 + 3) + \cdots + F_m(m \cdot 1 + m)$$

$$F_m(m \cdot 1 + 3) + F_m(m \cdot 1 + 4) + \cdots + F_m(m \cdot 1 + m + 1)$$

$$\vdots$$

$$F_m(m \cdot 1 + m - 1) + F_m(m \cdot 1 + m) + \cdots + F_m(m \cdot 1 + 2m - 3).$$

For n is general, where $n = 0, 1, 2, \ldots,$

$$F_m(m \cdot n + 1) + F_m(m \cdot n + 2) + \cdots + F_m(m \cdot n + m - 2) \tag{16}$$

$$F_m(m \cdot n + 2) + F_m(m \cdot n + 3) + \cdots + F_m(m \cdot n + m)$$

$$F_m(m \cdot n + 3) + F_m(m \cdot n + 4) + \cdots + F_m(m \cdot n + m + 1)$$

$$\vdots$$

$$F_m(m \cdot n + m - 1) + F_m(m \cdot n + m) + \cdots + F_m(m \cdot n + 2m - 3).$$

One should note that

$$F_m(m \cdot 0 + m) = 2^{m-1}$$

and

$$F_m(m \cdot 0 + 1) + F_m(m \cdot 0 + 2) + \cdots + F_m(m \cdot 0 + m - 2) = 2^{m-2} - 1.$$

Combining the generalized formulas for the x and y components of each rectangle, equations (15) and (16) above, we observe that the general formulas of each rectangle $R_m(x, y)$ may be written as

$$R_m(2, 1) \tag{17}$$

$$R_m(4, 2)$$

$$R_m(8, 4)$$

$$R_m(16, 8)$$

$$\vdots$$

$$R_m(2^{m-2}, 2^{m-3})$$

$R_m(F_m(m \cdot 0 + m),\ F_m(m \cdot 0 + 1) + F_m(m \cdot 0 + 2) + \cdots + F_m(m \cdot 0 + m - 2))$

$R_m(F_m(m \cdot 0 + m) + F_m(m \cdot 0 + m + 1),$

$\quad F_m(m \cdot 0 + 2) + F_m(m \cdot 0 + 3) + \cdots + F_m(m \cdot 0 + m))$

$R_m(F_m(m \cdot 0 + m) + F_m(m \cdot 0 + m + 1) + F_m(m \cdot 0 + m + 2),$

$\quad F_m(m \cdot 0 + 3) + F_m(m \cdot 0 + 4) + \cdots + F_m(m \cdot 0 + m + 1))$

$\vdots$

$R_m(F_m(m \cdot 0 + m) + F_m(m \cdot 0 + m + 1) + F_m(m \cdot 0 + m + 2)$

$\quad + \cdots + F_m(m \cdot 0 + 2m - 2),\ F_m(m \cdot 0 + m - 1) + F_m(m \cdot 0 + m)$

$\quad + \cdots + F_m(m \cdot 0 + 2m - 3))$

$R_m(F_m(m \cdot 1 + m),\ F_m(m \cdot 1 + 1) + F_m(m \cdot 1 + 2) + \cdots + F_m(m \cdot 1 + m - 2))$

$R_m(F_m(m \cdot 1 + m) + F_m(m \cdot 1 + m + 1)$

$\quad F_m(m \cdot 1 + 2) + F_m(m \cdot 1 + 3) + \cdots + F_m(m \cdot 1 + m))$

$R_m(F_m(m \cdot 1 + m) + F_m(m \cdot 1 + m + 1) + F_m(m \cdot 1 + m + 2),$

$\quad F_m(m \cdot 1 + 3) + F_m(m \cdot 1 + 4) + \cdots + F_m(m \cdot 1 + m + 1))$

$\vdots$

$R_m(F\ (m \cdot 1 + m) + F_m(m \cdot 1 + m + 1) + F_m(m \cdot 1 + m + 2)$

$\quad + \cdots + F_m(m \cdot 1 + 2m - 2),\ F_m(m \cdot 1 + m - 1) + F_m(m \cdot 1 + m)$

$\quad + \cdots + F_m(m \cdot 1 + 2m - 3))$

$R_m(F_m(m \cdot n + m),\ F_m(m \cdot n + 1) + F_m(m \cdot n + 2) + \cdots + F_m(m \cdot n + m - 2))$

$R_m(F_m(m \cdot n + m) + F_m(m \cdot n + m + 1),$

$\quad F_m(m \cdot n + 2) + F_m(m \cdot n + 3) + \cdots + F_m(m \cdot n + m))$

$R_m(F_m(m \cdot n + m) + F_m(m \cdot n + m + 1) + F_m(m \cdot n + m + 2),$

$\quad F_m(m \cdot n + 3) + F_m(m \cdot n + 4) + \cdots + F_m(m \cdot n + m + 1))$

$\vdots$

$R_m(F_m(m \cdot n + m) + F_m(m \cdot n + m + 1) + F_m(m \cdot n + m + 2)$

$\quad + \cdots + F_m(m \cdot n + 2m - 2),$

$\quad F_m(m \cdot n + m - 1) + F_m(m \cdot n + m) + \cdots + F_m(m \cdot n + 2m - 3)),$

where $n = 0, 1, 2, \ldots$.

We now consider the tiled rectangles $A_m(x, y)$ in order of their construction:

$$A_m(F_m(m \cdot 0) + F_m(m \cdot 0 + 1),\ F_m(m \cdot 0 + 1)) \tag{18}$$

$A_m(F_m(m \cdot 0) + F_m(m \cdot 0 + 1) + F_m(m \cdot 0 + 2),\ F_m(m \cdot 0 + 2))$

$A_m(F_m(m \cdot 0) + F_m(m \cdot 0 + 1) + F_m(m \cdot 0 + 2) + F_m(m \cdot 0 + 3),\ F_m(m \cdot 0 + 3))$

$\vdots$

$A_m(F_m(m \cdot 0) + F_m(m \cdot 0 + 1) + F_m(m \cdot 0 + 2)$

$\quad + \cdots + F_m(m \cdot 0 + m - 2),\ F_m(m \cdot 0 + m - 2))$

$A_m(F_m(m \cdot 0 + m),\ F_m(m \cdot 0 + m - 1) + F_m(m \cdot 0 + m))$

$A_m(F_m(m \cdot 1) + F_m(m \cdot 1 + 1),\ F_m(m \cdot 1 + 1))$

$A_m(F_m(m \cdot 1) + F_m(m \cdot 1 + 1) + F_m(m \cdot 1 + 2),\ F_m(m \cdot 1 + 2))$

$$A_m(F\ (m\cdot 1) + F_m(m\cdot 1+1) + F_m(m\cdot 1+2) + F_m(m\cdot 1+3),\ F_m(m\cdot 1+3))$$
$$\vdots$$
$$A_m(F_m(m\cdot 1) + F_m(m\cdot 1+1) + F_m(m\cdot 1+2) + \cdots + F_m(m\cdot 1+m-2),\ F_m(m\cdot 1+m-2))$$
$$A_m(F_m(m\cdot 1+m),\ F_m(m\cdot 1+m-1) + F_m(m\cdot 1+m))$$

$$A_m(F_m(mn)+F_m(mn+1),\ F_m(mn+1))$$
$$A_m(F_m(mn)+F_m(mn+1)+F_m(mn+2),\ F_m(mn+2))$$
$$A_m(F_m(mn)+F_m(mn+1)+F_m(mn+2)+F_m(mn+3),\ F_m(mn+3))$$
$$\vdots$$
$$A_m(F_m(mn)+F_m(mn+1)+F_m(mn+2) + \cdots + F_m(mn+m-2),\ F_m(mn+m-2))$$
$$A_m(F_m(mn+m),\ F_m(mn+m-1)+F_m(mn+m)),$$

where $n = 0, 1, 2, \ldots$.

This establishes the recurring sequences for the tiling with $k = 1$ and $m = 2, 3, 4, \ldots$ for equation (1), using the construction of Figure 10. We intend to generalize this procedure for $k > 1$ in later work.

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A DIOPHANTINE EQUATION WITH GENERALIZATION

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In [1] the authors showed that the diophantine equation $Nb^2 = c^2 + N + 1$ does not admit any integral solution except for the trivial case $N = -1$ and $b = c = 0$. At the end of the proof, a conjecture about its generalization was made, namely that

$$Nb^2 = c^2 + N(4k + 1) + 1 \tag{1}$$

would not yield any nontrivial solutions.

In this paper we give a new proof of the original equation. We also prove that (1) does not have a solution when N is a positive integer. A counter-example is given to show that there may exist infinitely many solutions of the general equation when N takes negative values, so the conjecture in (1) was not correct.

Omitting the trivial case when $b = c = 0$, we show that

$$N = \frac{c^2 + 1}{b^2 - 1}$$

is not an integer for all integral values of b and c. Suppose that N is an integer. We consider two cases. Suppose b is even. This means that $b^2 - 1 \equiv 3 \pmod 4$, which implies that there exists at least one prime $p \equiv 3 \pmod 4$ such that p divides $b^2 - 1$. This in turn leads to $c^2 + 1 \equiv 0 \pmod p$, which is impossible since -1 is a quadratic nonresidue $\pmod p$. If b is odd, then $b^2 - 1 \equiv 0 \pmod 8$, so $c^2 + 1 = 0 \pmod 8$, which is also impossible.

To show that (1) has solutions when N is negative, take $N = -2$. The equation becomes $8k - 2b^2 = c^2 - 1$, which has infinitely many solutions given by

$$b = 2m, \quad c = 2n + 1, \quad \text{and } k = m^2 + \frac{n(n + 1)}{2},$$

where m, n are arbitrary integers. The reader can easily generate infinitely many solutions by selecting other specific negative values of N.

Theorem: The diophantine equation $Nb^2 = c^2 + N(4k + 1) + 1$ does not admit any solution when $N > 0$.

Proof: We consider five cases:

1. Let $N \equiv 3 \pmod 4$. There is a prime factor p of N such that $p \equiv 3 \pmod 4$. This implies $c^2 + 1 \equiv 0 \pmod p$, which is impossible.

2. $N \equiv 1 \pmod 4$. Let $N = 4t + 1$ with $t \geq 0$. The equation becomes

$$(4t + 1)b^2 = c^2 + 4M + 2,$$

where $M = 4tk + t + k$. This equation is solvable only if the congruence $b^2 - c^2 \equiv 2 \pmod 4$ is solvable. But since $b^2 - c^2 \equiv 0, 1, 3 \pmod 4$ for all possible choices of b and c, $b^2 - c^2 \equiv 2 \pmod 4$ is not solvable.

3. $N \equiv 0 \pmod 4$. This implies $c^2 + 1 \equiv 0 \pmod 4$, which is impossible.

4. $N \equiv 2 \pmod 8$. Let $N = 8t + 2$ with $t \geq 0$. The equation becomes

$$(8t + 2)b^2 = c^2 + (8t + 2)(4k + 1) + 1,$$

which implies $2b^2 - c^2 \equiv 3 \pmod 8$ is solvable. Since $x^2 \equiv 0, 1, 4 \pmod 8$ for all integers x, $2b^2 - c^2 \equiv 0, 1, 2, 4, 6, 7 \pmod 8$; thus, $2b^2 - c^2 \equiv 3 \pmod 8$ is not solvable.

5. $N \equiv 6 \pmod 8$. Let $N = 8t + 6 = 2(4t + 3)$. Then N contains a prime factor p, where $p \equiv 3 \pmod 4$. Thus, the solution of the equation is not possible for the reason discussed in Case 1 above, and the proof is complete.

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ALMOST UNIFORM DISTRIBUTION OF THE FIBONACCI SEQUENCE

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Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ denote the sequence of Fibonacci numbers. For an integer $m > 1$, recall that (F_n) is *uniformly distributed* modulo m if all residues modulo m occur with the same frequency in any period (see [2], [4]). This happens precisely when $m = 5^k$ with $k > 0$, in which case (F_n) has (shortest) period of length $4 \cdot 5^k$, and each residue occurs four times (see [1], [3]). In this paper we study moduli with more complex distributions.

For any r, $0 \leq r < m$, denote by $\nu(r)$ the number of times r occurs as a residue in one (shortest) period of $F_n \pmod{m}$. If m is a power of 5, then $\nu(r) = 4$ for all r. However, if $m = 11$, then the period of $F_n \pmod{11}$ is 0, 1, 1, 2, 3, 5, 8, 2, 10, 1, so that $\nu(r)$ takes on four different values.

Definition: For an integer $m > 1$, (F_n) is *almost uniformly distributed* modulo m [notation: (F_n) AUD $\pmod{m}$] if $\nu(r)$ assumes exactly two values for $0 \leq r < m$.

In this paper we describe four infinite sequences of AUD moduli, along with describing the function ν precisely for these moduli. Our proof makes use of a recent result of Velez [2], which we state here for the reader's convenience.

Lemma: For any integer $s \geq 0$, the sequence

$$F_{s+4q},\ q = 0, 1, \ldots, 5^k - 1,$$

consists of a complete residue system modulo 5^k.

Main Theorem: (F_n) is AUD $\pmod{m}$ for $m \in \{2 \cdot 5^k, 4 \cdot 5^k, 3 \cdot 5^k, 9 \cdot 5^k : k \geq 0\}$. For these moduli, the following data appertain:

Modulus	Period	Distribution
2	3	$\nu(0) = 1,\ \nu(1) = 2$
4	6	$\nu(0) = \nu(2) = \nu(3) = 1,\ \nu(1) = 3$
$2 \cdot 5^k,\ k > 0$	$3 \cdot 4 \cdot 5^k$	$\nu(r) = \begin{cases} 4 & r \text{ is even} \\ 8 & r \text{ is odd} \end{cases}$
$4 \cdot 5^k,\ k > 0$	$3 \cdot 4 \cdot 5^k$	$\nu(r) = \begin{cases} 2 & r \not\equiv 1 \pmod{4} \\ 6 & r \equiv 1 \pmod{4} \end{cases}$
$3 \cdot 5^k,\ k \geq 0$	$8 \cdot 5^k$	$\nu(r) = \begin{cases} 2 & r \equiv 0 \pmod{3} \\ 3 & r \not\equiv 0 \pmod{3} \end{cases}$
$9 \cdot 5^k,\ k \geq 0$	$3 \cdot 8 \cdot 5^k$	$\nu(r) = \begin{cases} 2 & r \not\equiv 1, 8 \pmod{9} \\ 5 & r \equiv 1, 8 \pmod{9} \end{cases}$

Proof: The cases $m = 2, 3, 4, 9$ can be checked directly. Assume that $k \geq 1$. Because of the similarity of the proofs of the four cases, we only prove the cases $m = 2 \cdot 5^k$ and $m = 9 \cdot 5^k$, leaving the proofs of the remaining cases to the reader.

Case 1. $m = 2 \cdot 5^k$. As the period of $F_n \pmod{m}$ is the least common multiple of its periods modulo 2 and 5^k, it is clear that the period is $3 \cdot 4 \cdot 5^k$.

To compute $\nu(r)$, it suffices, by the Chinese Remainder Theorem, to compute the number of simultaneous solutions to the system

$$\begin{cases} F_n \equiv r_1 \pmod{2} \\ F_n \equiv r_2 \pmod{5^k} \end{cases}$$

with $0 \le n < 3 \cdot 4 \cdot 5^k$, for ordered pairs of residues (r_1, r_2) with $0 \le r_1 < 2$ and $0 \le r_2 < 5^k$. Fix r_2.

For n in the indicated range, n can be expressed uniquely in the form $n = s + 4q$, with $0 \le s < 4$ and $0 \le q \le 3 \cdot 5^k - 1$. By the lemma, for fixed s, there is a unique q_1 with $0 \le q_1 \le 5^k - 1$ such that

$$F_{s+4q_1} \equiv r_2 \pmod{5^k}.$$

Then, also,

$$F_{s+4(q_1+5^k)} \equiv r_2 \pmod{5^k}$$

and

$$F_{s+4(q_1+2\cdot 5^k)} \equiv r_2 \pmod{5^k},$$

because F_n has period $4 \cdot 5^k$ modulo 5^k. Now observe that

$$s + 4q_1 \equiv s + q_1 \pmod{3},$$

$$s + 4(q_1 + 5^k) \equiv s + q_1 + (-1)^k \pmod{3},$$

$$s + 4(q_1 + 2 \cdot 5^k) \equiv s + q_1 + (-1)^{k+1} \pmod{3},$$

and these are incongruent modulo 3. Thus, for fixed s, there are exactly two solutions q to the system

$$\begin{cases} F_{s+4q} \equiv 1 \pmod{2} \\ F_{s+4q} \equiv r_2 \pmod{5^k} \end{cases}$$

and exactly one solution q of the system

$$\begin{cases} F_{s+4q} \equiv 0 \pmod{2} \\ F_{s+4q} \equiv r_2 \pmod{5^k} \end{cases}$$

with $0 \le q \le 3 \cdot 5^k - 1$.

Now s has four possible values, so that there are exactly eight solutions of

$$\begin{cases} F_n \equiv 1 \pmod{2} \\ F_n \equiv r_2 \pmod{5^k} \end{cases}$$

and exactly four solutions of

$$\begin{cases} F_n \equiv 0 \pmod{2} \\ F_n \equiv r_2 \pmod{5^k} \end{cases}$$

with $0 \le n \le 3 \cdot 4 \cdot 5^k - 1$. This translates via the Chinese Remainder Theorem to the stated distribution.

The method of proof is now clear, and we provide few details in Case 2.

Case 2. $m = 9 \cdot 5^k$. The period is $\operatorname{lcm}(24, 4 \cdot 5^k) = 8 \cdot 3 \cdot 5^k$. Express $n = s + 4q$, where $0 \le s \le 3$, $0 \le q \le 6 \cdot 5^k - 1$. For fixed s and residue $r_2 \pmod{5^k}$, there is a unique q_1 such that $F_{s+4q_1} \equiv r_2 \pmod{5^k}$ with $0 \le q_1 \le 5^k - 1$. Now the Fibonacci numbers have period 0, 1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 0, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1 (mod 9) of length 24, so we consider the subscripts $s + 4(q_1 + t \cdot 5^k) \pmod{24}$ for $t = 0, 1, 2, 3, 4, 5$. A straightforward calculation yields that these are congruent (in some order) to s, $s + 4$, $s + 8$, $s + 12$, $s + 16$, $s + 20 \pmod{24}$. Thus, for fixed s, r_2 there are 6 values of q, $0 \le q \le 6 \cdot 5^k - 1$, with $F_{s+4q} \equiv r_2 \pmod{5^k}$, (namely, $q = q_1 + t \cdot 5^k$, $0 \le t \le 5$). Now, for this sequence of q's, we have that:

$$\underline{s = 0} \Rightarrow F_{s+4q} \equiv 0, 3, 3, 0, 6, 6 \pmod{9}$$
$$\underline{s = 1} \Rightarrow F_{s+4q} \equiv 1, 5, 7, 8, 4, 2 \pmod{9}$$
$$\underline{s = 2} \Rightarrow F_{s+4q} \equiv 1, 8, 1, 8, 1, 8 \pmod{9}$$
$$\underline{s = 3} \Rightarrow F_{s+4q} \equiv 2, 4, 8, 7, 5, 1 \pmod{9}$$

Again, the stated distribution follows from the Chinese Remainder Theorem. □

Remarks: It is clear from the proof that the given method will decide the distribution of any family of the form $m \cdot 5^k$, where $5 \nmid m$, once it is known explicitly modulo m. However, there does not appear to be a general theorem valid for all m that will let one forgo this tedium.

It is natural to ask if the list in the Theorem is complete. A computer search of moduli $m \le 1000$ indicates this is so. However, the converse proof quickly reduces to showing that a modulus m where ν takes on only the values 0 and f for that m does not exist. The question of whether there exists a prime $p > 7$ such that only the frequencies 0 and f occur mod p is a well-known open problem.

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THE GENERALIZED ZECKENDORF THEOREMS

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We recall the Zeckendorf Theorem and its dual, credited to E. Zeckendorf, which deals with the representation of integers as sums of distinct Fibonacci numbers. These theorems were restated and proved by J. L. Brown, Jr., in [1] and [2]. Throughout this paper, we let N denote the set of positive integers.

Zeckendorf Theorem: If $n \in N$, n may be uniquely expressed in the following form:

$$n = \sum_{k=1}^{r} \theta_k F_{k+1}, \tag{1}$$

where

$$\theta_k \in \{0, 1\},\ \theta_k = 0 \text{ if } k > r, \text{ and } \theta_k + \theta_{k+1} < 2,\ k = 1, 2, \ldots . \tag{2}$$

Dual Zeckendorf Theorem: If $n \in N$, n may be uniquely expressed in the form shown in (1), but with the conditions:

$$\theta_k \in \{0, 1\},\ \theta_k = 0 \text{ if } k > r, \text{ and } \theta_k + \theta_{k+1} > 0,\ k = 1, 2, \ldots, r. \tag{3}$$

[Note: The usual statement of the condition on the θ_k's in (2) is, $\theta_k\theta_{k+1} = 0$, which is equivalent. The condition as stated in (2) is more amenable to the proper generalization.]

Before stating and proving the appropriate generalizations of the above theorems, we introduce some useful definitions.

Given integers b and t with $b \geq 2$, $t \geq 2$, we say that a given integer $n \in N$ is b, *t-upper representable* iff there exists an increasing sequence

$$H = H_k(b, t)_{k=1}^{\infty}$$

of positive integers such that n may be uniquely expressed in the following form:

$$n = \sum_{k=1}^{r} \theta_k(b, t) H_k(b, t), \tag{4}$$

where

$$\theta_k(b, t) \in \{0, 1, \ldots, b-1\},\quad \theta_k(b, t) = 0 \text{ if } k > r, \tag{5}$$

and

$$\theta_k + \theta_{k+1} + \cdots + \theta_{k+t-1} < (b-1)t,\ k = 1, 2, \ldots . \tag{6}$$

We say that $n \in N$ is b, *t-lower representable* iff the same conditions hold as in (4) and (5), but (6) is replaced by:

$$\theta_k + \theta_{k+1} + \cdots + \theta_{k+t-1} > 0,\ k = 1, 2, \ldots, r. \tag{7}$$

Let $S(H)$ and $T(H)$ denote the sets of b, t-upper representable and b, t-lower representable numbers, respectively. For brevity, we may write the sum in (4) in the form:

$$n = (\theta_r\theta_{r-1} \ldots \theta_2\theta_1)_H, \tag{8}$$

omitting the arguments "b, t" where no confusion is likely to arise. We may let the notation in (8) represent the b, t-representation of n [an element of $\overline{S}(H)$ or $\overline{T}(H)$] as well as the value of the sum indicated in (4) [an element of $S(H)$ or $T(H)$]. Here, $\overline{S}(H)$ and $\overline{T}(H)$ denote the sets of b, t-upper and -lower representations, respectively, of the form given in (8). Note that condition (6) for b, t-upper representations states that no representation in $\overline{S}(H)$ is to contain t consecutive digits equal to $(b - 1)$; similarly, condition (7) requires that no element of $\overline{T}(H)$ is to contain t consecutive digits equal to zero.

Let $\overline{S}_r(H)$ and $\overline{T}_r(H)$ denote the subsets of $\overline{S}(H)$ and $\overline{T}(H)$, respectively, which contain r digits in the representation (that is, with $\theta_r > 0$, $\theta_k = 0$, if $k > r \geq 1$). Let the corresponding integers represented by $\overline{S}_r(H)$ and $\overline{T}_r(H)$ be arranged in nondecreasing order (as yet, we do not know if any duplication occurs), and call these ordered sets $S_r(H)$ and $T_r(H)$, respectively. Let $U_r(H)$ and $V_r(H)$ denote the sizes of $\overline{S}_r(H)$ and $\overline{T}_r(H)$, respectively, that is,

$$U_r(H) = |\overline{S}_r(H)|, \quad V_r(H) = |\overline{T}_r(H)|. \tag{9}$$

Let $A_r(H)$ and $B_r(H)$ denote the smallest and largest values, respectively, of $S_r(H)$; let $C_r(H)$ and $D_r(H)$ denote the smallest and largest values, respectively of $T_r(H)$. Finally, we observe that:

$$S(H) = \bigcup_{r=1}^{\infty} S_r(H), \quad T(H) = \bigcup_{r=1}^{\infty} T_r(H). \tag{10}$$

We may now express and prove the following theorems.

Theorem 1 (Generalized Zeckendorf): We define the sequence $G = (G_k(b, t))_{k=1}^{\infty}$ as follows:

$$G_k = b^{k-1}, \; k = 1, 2, \ldots, t; \tag{11}$$

$$G_{k+t} = (b - 1)(G_{k+t-1} + G_{k+t-2} + \cdots + G_{k+1} + G_k), \; k = 1, 2, \ldots . \tag{12}$$

Then

$$N = S(G). \tag{13}$$

Moreover, if $N = S(H)$ for some sequence $H = (H_k(b, t))_{k=1}^{\infty}$, then $H = G$.

Theorem 2 (Generalized Dual Zeckendorf): If G is as defined in (11) and (12), then $N = T(G)$. Moreover, if $N = T(H)$ for some sequence $H = (H_k(b, t))_{k=1}^{\infty}$, then $H = G$.

Proof of Theorem 1: We begin by deriving the values of $U_r(H)$. Since

$$\theta_1 \in \{1, 2, \ldots, b - 1\} \text{ if } r = 1,$$

we have

$$U_1(H) = b - 1 = G_2 - G_1.$$

If $r = 2$ (with $t > 2$), then

$$\theta_1 \in \{0, 1, 2, \ldots, b - 1\} \text{ and } \theta_2 \in \{1, 2, \ldots, b - 1\},$$

independently, so

$$U_2(H) = b(b - 1) = G_3 - G_2.$$

Continuing in this fashion, we see that

$$U_r(H) = b^{r-1}(b - 1) = G_{r+1} - G_r, \; r = 1, 2, \ldots, t - 1.$$

Setting $k = 1$ in (12) yields:

$$G_{t+1} = (b - 1)(b^{t-1} + b^{t-2} + \cdots + 1) = b^t - 1.$$

Also, note that $\overline{S}_t(H)$ may be generated by $(b-1)$ choices for θ_t and b choices for each of $\theta_{t-1}, \theta_{t-2}, \ldots, \theta_1$; however, we must subtract from this composition the (one) choice where all digits are equal to $(b-1)$. Therefore,

$$U_t(H) = b^{t-1}(b-1) - 1 = b^t - 1 - b^{t-1} = G_{t+1} - G_t.$$

So far, we have shown:

$$U_r(H) = G_{r+1} - G_r, \; r = 1, 2, \ldots, t. \tag{14}$$

Next (for brevity, omitting the argument "H"), assuming $m \geq t$, we let $\overline{S}'_m$ and $\overline{S}''_m$ denote the subsets of $\overline{S}_m$ with initial digit in $\{1, 2, \ldots, b-2\}$ and equal to $(b-1)$, respectively. Let U'_m and U''_m denote the sizes of $\overline{S}'_m$ and $\overline{S}''_m$, respectively. Also, let

$$W_m = U_1 + U_2 + \cdots + U_m, \quad W'_m = U'_1 + U'_2 + \cdots + U'_m.$$

Now $\overline{S}_m = \overline{S}'_m \cup \overline{S}''_m$; thus, $U_m = U'_m + U''_m$. In what follows, we let x represent any of the digits in $\{1, 2, \ldots, b-2\}$, $y = (b-1)$, and 0 the zero digit; also, z represents either x or y. We note that $\overline{S}''_m$ may be formed in any of the following (mutually exclusive and exhaustive) ways:

$$\begin{array}{ccccc}
y\overline{S}'_{m-1} & y0\overline{S}_{m-2} & y00\overline{S}_{m-3} & \cdots \; y00\ldots00\overline{S}_{t-1} & y000\ldots0 \\
yy\overline{S}'_{m-2} & yy0\overline{S}_{m-3} & yy00\overline{S}_{m-4} & \cdots \; yy00\ldots0\overline{S}_{t-2} & yy00\ldots0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\underbrace{yy\ldots y}_{t-1}\overline{S}'_{m-t+1} & \underbrace{yy\ldots y}_{t-1}0\overline{S}_{m-t} & \underbrace{yy\ldots y}_{t-1}00\overline{S}_{m-t-1} & \cdots \; \underbrace{yy\ldots y}_{t-1}00\ldots00\overline{S}_1 & \underbrace{yy\ldots y}_{t-1}00\ldots0
\end{array}$$

Therefore,

$$\begin{aligned}
U''_m &= (U'_{m-1} + U'_{m-2} + \cdots + U'_{m-t+1}) + (U_{m-2} + U_{m-3} + \cdots + U_{m-t}) \\
&\quad + (U_{m-3} + U_{m-4} + \cdots + U_{m-t-1}) + \cdots + (U_{t-1} + U_{t-2} + \cdots + U_1) + t - 1 \\
&= (W'_{m-1} - W'_{m-t}) + (W_{m-2} - W_{m-t-1}) + (W_{m-3} - W_{m-t-2}) + \cdots + W_{t-1} + t - 1.
\end{aligned}$$

Taking the first difference, we obtain:

$$U''_{m+1} - U''_m = U'_m - U'_{m-t+1} + W_{m-1} - W_{m-t}. \tag{15}$$

Next, we consider the possible ways to generate S'_m, namely, as follows:

$$x\overline{S}_{m-1},\; x0\overline{S}_{m-2},\; x00\overline{S}_{m-3},\; \ldots,\; x00\ldots00\overline{S}_1, \text{ or } x00\ldots00.$$

Since x may be chosen in $b-2$ ways, we have:

$$U'_m = (b-2)(U_{m-1} + U_{m-2} + \cdots + U_1 + 1) = (b-2)(W_{m-1} + 1).$$

Taking first differences in the last expression, we have:

$$U'_{m+1} - U'_m = (b-2)U_m. \tag{16}$$

Now, adding the expressions in (15) and (16), we obtain:

$$\begin{aligned}
U_{m+1} - U_m &= U'_m - U'_{m-t+1} + W_{m-1} - W_{m-t} + (b-2)U_m \\
&= (b-2)(W_{m-1} + 1 - W_{m-t} - 1) + W_{m-1} - W_{m-t} + (b-2)U_m;
\end{aligned}$$

hence,

$$U_{m+1} = (b-1)(U_m + W_{m-1} - W_{m-t}) = (b-1)(W_m - W_{m-t}).$$

Equivalently,

$$U_{m+1} = (b-1)(U_m + U_{m-1} + \cdots + U_{m-t+1}), \quad m = t, t+1, t+2, \ldots . \tag{17}$$

Note that (17) is the same recursion satisfied by the G_m's in (12). Since G_{m+1} and G_m satisfy this recursion, so does $G_{m+1} - G_m$. It follows from (14) and (17) that we have:

$$U_r(H) = G_{r+1} - G_r,\ r = 1, 2, \ldots, \text{ for all } H. \tag{18}$$

Next, we derive expressions for $A_r(H)$ and $B_r(H)$ [recalling that these are the smallest and largest values, respectively, of $S_r(H)$]. For any admissible H, we see that

$$A_r(H) = (1\underbrace{00\ldots0}_{r-1})_H,$$

or, equivalently,

$$A_r(H) = H_r. \tag{19}$$

In particular,

$$A_r(G) = G_r. \tag{20}$$

Also, using the notation introduced earlier, we see that

$$B_r(H) = (\underbrace{yy\ldots y}_{t-1}\ y-1\quad \underbrace{yy\ldots y}_{t-1}\ y-1\ \ldots\ \underbrace{yy\ldots y}_{t-1}\ y-1\quad \underbrace{yy\ldots y}_{v})_H,$$

where $r = ut + v,\ 0 \le v < t$,

and in the above representation there are u blocks of length t of the type:

$$yy\ldots y\ y-1.$$

Therefore,

$$B_r(H) = (b-1)(H_r + H_{r-1} + \cdots + H_1) - (H_{v+1+(u-1)t} + \cdots + H_{v+1}).$$

In particular,

$$\begin{aligned} B_r(G) &= (b-1)\sum_{k=1}^{v} G_k + (b-1)\sum_{j=0}^{u-1}\sum_{k=1}^{t} G_{v+jt+k} - \sum_{j=0}^{u-1} G_{v+1+jt} \\ &= \sum_{k=1}^{v}(b-1)b^{k-1} + \sum_{j=0}^{u-1} G_{v+1+(j+1)t} - \sum_{j=0}^{u-1} G_{v+1+jt} \\ &= b^v - 1 + G_{v+1+ut} - G_{v+1} = b^v - 1 + G_{r+1} - b^v, \end{aligned}$$

or

$$B_r(G) = G_{r+1} - 1. \tag{21}$$

By definition of $A_r(G)$ and $B_r(G)$, we see from (21) that the $S_r(G)$ are disjoint. Moreover, from (20), (21), and (18), we have:

$$B_r(G) - A_r(G) = G_{r+1} - G_r - 1 = U_r(G) - 1. \tag{22}$$

Thus, the difference between the largest and smallest elements of $S_r(G)$ is one less than the number of elements in $\overline{S}_r(G)$. If we can prove that $N \subset S(G)$ (i.e., that all positive integers have a b, t-upper representation, with G the underlying sequence), this in turn will imply that $N = S(G)$. We will need a lemma.

Lemma: $(b-1)G_m < G_{m+1} \le bG_m$, $m = 1, 2, \ldots$.

Proof: The left inequality is clearly true, from (11) and (12). If $1 \le m \le t$, $G_m = b^{m-1}$, so $G_{m+1} = bG_m$ in the range $1 \le m < t$. Also $G_{t+1} = b^t - 1 < bG_t$. Replacing $k + t$ by $m + 1$ and m, respectively, in (12), and subtracting the results, we obtain:

$$G_{m+1} - G_m = (b - 1)(G_m - G_{m-t})$$

or

$$bG_m - G_{m+1} = (b - 1)G_{m-t}, \text{ if } m > t.$$

Therefore, if $m > t$, $bG_m > G_{m+1}$, which yields the right inequality in the statement of the lemma.

Let J_r denote the set $\{1, 2, \ldots, G_r - 1\}$, $r = 2, 3, \ldots$. Assuming $2 \le r \le t$, $G_r = b^{r-1}$, so if $n \in J_r$, n may be uniquely represented as a b-adic number with digits in $\{0, 1, \ldots, b - 1\}$; this representation is also a b, t-upper representation, as well as a b, t-lower representation. Hence,

$$J_r \subset S(G), \quad J_r \subset T(G), \quad \text{if } 2 \le r \le t. \tag{23}$$

Note that $J_1 = \emptyset$, $J_2 = \{1, 2, \ldots, b - 1\}$.

Suppose next that $r \ge t$, and assume $J_r \subset S(G)$; this inductive hypothesis is seen to be true for $r = t$. Given an integer n' with $G_r \le n' < G_{r+1}$, then

$$pG_r \le n' < (p + 1)G_r, \text{ where } 1 \le p \le b - 1.$$

Then $0 \le n' - pG_r < G_r$, so $(n' - pG_r) \in J_r$. Hence, by (23),

$$(n' - pG_r) \in S(G),$$

which implies that

$$n' - pG_r = (\theta_{r-1}\theta_{r-2} \ldots \theta_2\theta_1)_G,$$

which is an element of $T_{r-1}(G)$ (note that $\theta_r = 0$, otherwise $n' - pG_r \ge G_r$, a contradiction). Therefore,

$$n' = (p\theta_{r-1}\theta_{r-2} \ldots \theta_1)_G.$$

A priori, we could have

$$p = \theta_{r-1} = \theta_{r-2} = \cdots = \theta_{r-t+1} = b - 1;$$

if so,

$$n' \ge (b - 1)(G_r + G_{r-1} + \cdots + G_{r-t+1}) = G_{r+1},$$

which would be a contradiction. Hence, $n' \in S(G)$. Therefore, if $r \ge t$ and $J_r \subset S(G)$, we must have the set

$$\{G_r, G_r + 1, G_r + 2, \ldots, bG_r - 1\} \subset S(G).$$

However, by the Lemma, $G_{r+1} \le bG_r$. Therefore, $J_r \subset S(G)$ implies $J_{r+1} \subset S(G)$. Due to (23), it follows by induction that

$$\bigcup_{r=2}^{\infty} J_r \subset S(G).$$

But G is an increasing sequence, so

$$\bigcup_{r=2}^{\infty} J_r = N.$$

Thus, $N \subset S(G)$. By our previous comments, it follows that $N = S(G)$; in other words, there is a 1-to-1 correspondence between N and $S(G)$.

The final part of Theorem 1 states that G is the only sequence generating b, t-upper representations. To prove this, we will assume $N = S(H)$ for some sequence $H = (H_k(b, t))_{k=1}^{\infty}$. Since H must be increasing, and since 1 must have a (unique) representation, it is apparent that $H_1 = 1$. Then, by (18) and (19),

$$U_r(H) = G_{r+1} - G_r \quad \text{and} \quad A_r(H) = H_r.$$

Also, since the $S_r(H)$ must be disjoint, and since all representations must be unique, we must have

$$B_r(H) = A_{r+1}(H) - 1;$$

therefore, by (19), $B_r(H) = H_{r+1} - 1$. Also, however, we see that

$$B_r(H) = U_r(H) + U_{r-1}(H) + \cdots + U_1(H),$$

so

$$B_r(H) = \sum_{k=1}^{r}(G_{k+1} - G_k) = G_{r+1} - G_1 = G_{r+1} - 1.$$

Therefore, $B_r(H) = H_{r+1} - 1 = G_{r+1} - 1$, so $H_{r+1} = G_{r+1}$ for all $r \ge 1$. It follows that $H = G$, which completes the proof of Theorem 1.

Proof of Theorem 2: The proof follows that of Theorem 1. We begin by deriving the values of $V_r(H)$. The initial values of $V_r(H)$ are derived by reasoning identical to that used in the derivation of the initial values of $U_r(H)$, with the exception of $V_t(H)$. Thus,

$$V_r(H) = (b - 1)b^{r-1}, \; r = 1, 2, \ldots, t - 1,$$

i.e., in this range, $V_r(H) = (b - 1)G_r$. For $\overline{T}_t(H)$, we must avoid t consecutive zero digits; this will automatically be satisfied if $\theta_t > 0$. Hence,

$$V_t(H) = (b - 1)b^{t-1} = (b - 1)G_t.$$

Thus,

$$V_r(H) = (b - 1)G_r, \; r = 1, 2, \ldots, t. \tag{24}$$

Next, we observe that if $m \ge t$, $\overline{T}_{m+1}(H)$ may be formed in the following mutually exclusive and exhaustive ways (using the same notation as before):

$$z\overline{T}_m, \; z0\overline{T}_{m-1}, \; z00\overline{T}_{m-2}, \; \ldots, \; z\underbrace{00\ldots0}_{t-1}\overline{T}_{m-t+1}.$$

Since z may be chosen in $(b - 1)$ ways, we have:

$$V_{m+1} = (b - 1)(V_m + V_{m-1} + \cdots + V_{m-t+1}), \tag{25}$$
$$m = t, t + 1, t + 2, \ldots .$$

Note that (25) is the same recursion as satisfied by the G_m's (and the U_m's). We conclude from (24) that

$$V_r(H) = (b - 1)G_r, \; r = 1, 2, \ldots, \text{ for all } H. \tag{26}$$

Next, we derive expressions for $C_r(H)$ and $D_r(H)$, the smallest and largest values, respectively, of $T_r(H)$. We see that, for any admissible H,

$$C_r(H) = (1\underbrace{00\ldots0}_{t-1} \; 1\underbrace{00\ldots0}_{t-1} \; \ldots \; 1\underbrace{00\ldots0}_{t-1} \; 1\underbrace{00\ldots0}_{v-1})_H,$$
$$\text{where } r = ut + v, \; 1 \le v \le t,$$

and the representation above contains u blocks of t digits, of the type

$$1\underbrace{00\ldots0}_{t-1}.$$

Hence,

$$C_r(H) = \sum_{j=0}^{u} H_{v+jt}. \tag{27}$$

Also, it is clear that $D_r(H) = (\underbrace{yy\ldots y}_{r})_H$, or

$$D_r(H) = (b - 1)\sum_{k=1}^{r} H_k. \tag{28}$$

In particular, $D_r(G) = (b - 1)(G_1 + G_2 + \cdots + G_r)$. If $1 \le v \le t - 1$, then

$$\begin{aligned} D_r(G) &= (b - 1)\sum_{k=1}^{v} G_k + (b - 1)\sum_{j=0}^{u-1}\sum_{k=1}^{t} G_{v+k+jt} \\ &= (b - 1)\sum_{k=1}^{v} b^{k-1} + \sum_{j=0}^{u-1} G_{v+1+(j+1)t} \\ &= b^v - 1 + \sum_{j=1}^{u} G_{v+1+jt} = b^v - 1 + \sum_{j=0}^{u} G_{v+1+jt} - G_{v+1} \\ &= b^v - 1 + C_{r+1}(G) - b^v, \end{aligned}$$

or

$$D_r(G) = C_{r+1}(G) - 1, \text{ where } r = ut + v,\ v = 1, 2, \ldots, t - 1. \tag{29}$$

Also, if $v = t$, then $r = (u + 1)t$, so

$$D_r(G) = (b - 1)\sum_{k=1}^{(u+1)t} G_k = \sum_{j=1}^{u+1} G_{1+jt};$$

note that in this case

$$\begin{aligned} C_{r+1}(G) &= (\overbrace{1\underbrace{00\ldots0}_{t-1}\ \ 1\underbrace{00\ldots0}_{t-1}\ \ldots\ 1\underbrace{00\ldots0}_{t-1}}^{(u+1) \text{ blocks of } t \text{ digits}}\ 1)_G = \sum_{j=0}^{u+1} G_{1+jt} \\ &= D_r(G) + (G_1 = 1), \end{aligned}$$

which shows that (29) holds also for $v = t$. We may therefore conclude:

$$D_r(G) = C_{r+1}(G) - 1,\ r = 1, 2, \ldots . \tag{30}$$

Note, from (28), that

$$D_r(H) - D_{r-1}(H) = (b - 1)H_r,$$

so

$$D_r(G) - D_{r-1}(G) = (b - 1)G_r = V_r(G).$$

Using (30):

$$D_r(G) - C_r(G) = V_r(G) - 1. \tag{31}$$

We see from (30) that the $T_r(G)$'s are disjoint, by definition of the $C_r(G)$ and $D_r(G)$. Thus, as before, if we can establish that $N \subset T(G)$, (30) and (31) would imply that $N = T(G)$.

Recall that $J_r \subset T(G)$ for $2 \le r \le t$. Suppose next that $r \ge t$, and assume $J_r \subset T(G)$. Given an integer n'' with $G_r \le n'' < G_{r+1}$, it must satisfy

$$pG_r \le n'' < (p + 1)G_r, \text{ where } 1 \le p \le b - 1;$$

then $0 \le n'' - pG_r < G_r$, so $(n'' - pG_r) \in T(G)$, by the inductive hypothesis. Now

$$n'' - pG_r = (\theta_{r-1}\theta_{r-2} \ldots \theta_1)_G,$$

which is an element of $T_{r-1}(G)$ [for, if $\theta_r > 0$, then $(n'' - pG_r) \ge G_r$, a contradiction). Thus,

$$n'' = (p\theta_{r-1}\theta_{r-2} \ldots \theta_1)_G,$$

so $n'' \in T(G)$. Hence, if $r \ge t$ and $J_r \subset T(G)$, we have that

$$\{G_r, G_r + 1, \ldots, bG_r - 1\} \text{ is a subset of } T(G).$$

Since $G_{r+1} \le bG_r$, by the Lemma, $J_r \subset T(G)$ implies $J_{r+1} \subset T(G)$. So, as before, $N \subset T(G)$. By our previous remarks, $N = T(G)$.

To prove that G is the only sequence allowing b, t-lower representations, we suppose that $N = T(H)$ for some sequence H. Then

$$V_r(H) = (b - 1)G_r, \text{ from (26).}$$

Since $N = T(G) = T(H)$, it follows that

$$D_r(H) = C_{r+1}(H) - 1.$$

Also,

$$D_r(H) - D_{r-1}(H) = (b - 1)H_r, \text{ from (28).}$$

But

$$D_r(H) = V_1(H) + V_2(H) + \cdots + V_r(H),$$

so

$$D_r(H) - D_{r-1}(H) = V_r(H) = (b - 1)G_r.$$

From this, it follows that $H_r = G_r$ for all $r \ge 1$, so $H = G$. Q.E.D.

We now illustrate these two theorems with two examples. For $b = t = 2$, we have the "ordinary" Zeckendorf Theorem and its dual, and the appropriate sequence G is the sequence of distinct Fibonacci numbers:

$$\{1, 2, 3, 5, 8, \ldots\} = (F_{k+1})_{k=1}^{\infty}.$$

For $b = 3$, $t = 2$,

$$G = \{1, 3, 8, 22, 60, \ldots\}$$

and we have the following representations:

n	$\overline{S}(G(3, 2))$	$\overline{T}(G(3, 2))$	n	$\overline{S}(G(3, 2))$	$\overline{T}(G(3, 2))$
1	1	1	25	1010	1010
2	2	2	26	1011	1011
3	10	10	27	1012	1012
4	11	11	28	1020	1020
5	12	12	29	1021	1021
6	20	20	30	1100	1022
7	21	21	31	1101	1101
8	100	22	32	1102	1102
9	101	101	33	1110	1110
10	102	102	34	1111	1111
11	110	110	35	1112	1112
12	111	111	36	1120	1120
13	112	112	37	1121	1121
14	120	120	38	1200	1122
15	121	121	39	1201	1201
16	200	122	40	1202	1202
17	201	201	41	1210	1210
18	202	202	42	1211	1211
19	210	210	43	1212	1212
20	211	211	44	2000	1220
21	212	212	45	2001	1221
22	1000	220	46	2002	1222
23	1001	221	47	2010	2010
24	1002	222	48	2011	2011 etc.

For $b = 2$, $t = 3$,

$$G = \{1,\ 2,\ 4,\ 7,\ 13,\ 24,\ 44,\ \ldots\},$$

which is the sequence of distinct Tribonacci numbers, and we have the following representations:

n	$\overline{S}(G(2,\ 3))$	$\overline{T}(G(2,\ 3))$	n	$\overline{S}(G(2,\ 3))$	$\overline{T}(G(2,\ 3))$
1	1	1	26	100010	11110
2	10	10	27	100011	11111
3	11	11	28	100100	100100
4	100	100	29	100101	100101
5	101	101	30	100110	100110
6	110	110	31	101000	100111
7	1000	111	32	101001	101001
8	1001	1001	33	101010	101010
9	1010	1010	34	101011	101011
10	1011	1011	35	101100	101100
11	1100	1100	36	101101	101101
12	1101	1101	37	110000	101110
13	10000	1110	38	110001	101111
14	10001	1111	39	110010	110010
15	10010	10010	40	110011	110011
16	10011	10011	41	110100	110100
17	10100	10100	42	110101	110101
18	10101	10101	43	110110	110110
19	10110	10110	44	1000000	110111
20	11000	10111	45	1000001	111001
21	11001	11001	46	1000010	111010
22	11010	11010	47	1000011	111011
23	11011	11011	48	1000100	111100
24	100000	11100	49	1000101	111101
25	100001	11101	50	1000110	111110 etc.

It is of interest to indicate a generating function for the $G_n(b,\ t)$'s, namely:

$$F(z;\ b,\ t) = \frac{z + z^2 + \cdots + z^t}{1 - (b - 1)(z + z^2 + \cdots + z^t)} = \sum_{n=1}^{\infty} G_n(b,\ t)z^n. \tag{32}$$

This may be verified by multiplying each side of the last equation by the denominator of the fraction, then applying the relations in (11) and (12) defining $G_n(b,\ t)$. By multinomial expansion, we may derive the following explicit expression for $G_n(b,\ t)$ from (32):

$$G_n(b,\ t) = \sum_{m=1}^{n} (b - 1)^{m-1} \sum_{S} \binom{x_1 + \cdots + x_t}{x_1,\ \ldots,\ x_t}, \tag{33}$$

where S is the set of t-ples of nonnegative integers $x_1,\ x_2,\ \ldots,\ x_t$ satisfying

$$x_1 + x_2 + \cdots + x_t = m, \quad x_1 + 2x_2 + \cdots + tx_t = n.$$

We may also show the following result, expressed as a divided difference:

$$G_n(b,\ t) = (b - 1)^{-1} \Delta^{t-1} z^{n+t-1}(z_1,\ z_2,\ \ldots,\ z_t), \tag{34}$$

where $z_1,\ z_2,\ \ldots,\ z_t$ are the (distinct) roots of the equation:

$$p(z) = p(z;\ b,\ t) = z^t - (b - 1)(z^{t-1} + z^{t-2} + \cdots + 1) = 0. \tag{35}$$

This may be simplified to the following sum:

$$G_n(b,\ t) = (b - 1)^{-1}\sum_{k=1}^{t} z_k^{n+t-1}/p'(z_k). \tag{36}$$

An alternative expression, in terms of a contour integral, is given by:

$$G_n(b,\ t) = (b - 1)^{-1}\ \frac{1}{2i\pi}\oint_C \frac{z^{n+t-1}}{p(z)}\, dz, \tag{37}$$

where C is any simple closed contour in the complex plane, with positive direction and surrounding $z_1,\ z_2,\ \ldots,\ z_t$ within its interior.

Other expressions may be derived which can be shown to be equivalent, namely:

$$G_n(b,\ t) = \sum_{m=1}^{n} \frac{(b - 1)^{m-1}}{(n - m)!}\cdot\frac{d^{n-m}}{dz^{n-m}}(1 + z + z^2 + \cdots + z^{t-1})^m\bigg|_{z=0}, \tag{38}$$

and

$$G_n(b,\ t) = \sum_{m=1}^{n}(b - 1)^{m-1}\sum_{k=0}^{[(n-m)/t]}(-1)^k\binom{m}{k}\binom{n - 1 - kt}{m - 1}. \tag{39}$$

Undoubtedly, further analysis of such relations should lead to additional interesting results.

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MINIMUM PERIODS OF BINOMIAL COEFFICIENTS MODULO M

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1. Introduction

The minimum period of the sequence

$$\left\{\binom{n}{k} \pmod{M}\right\}_{n \ge k}$$

was discovered by Zabek [6] by investigating the Pascal triangle. Applying Vandermonde's convolution to $\binom{n+N}{k}$, Trench [5] obtained identical periods. By studying the highest prime power dividing $q^n - 1$, Fray [2] extended the results to q-binomial coefficients. All these approaches depend directly on the properties of the binomial coefficients. It is difficult to apply these techniques to other infinite integer sequences. In this paper, we will look at the problem from another perspective. In particular, we will consider the generating function of

$$\left\{\binom{n}{k}\right\}_{n \ge k}: \quad \frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n.$$

The problem can then be solved by studying divisibility of polynomials over $\mathbb{Z}_M$. This approach relies on the generating functions only, so it can also be applied to other sequences with similar generating functions. Thus, in this paper, we will assume all sequences to be infinite integer sequences.

2. Preliminaries

A sequence $\{a_n\}_{n \ge 0}$ is said to be *periodic modulo* M, with *period* π, if there is an integer $n_0 \ge 0$ such that, for $n \ge n_0$,

$$a_{n+\pi} \equiv a_n \pmod{M}.$$

If $n_0 = 0$, $\{a_n\}_{n \ge 0}$ is said to be *purely periodic modulo* M. If

$$A(x) = \sum_{n \ge 0} a_n x^n$$

generates $\{a_n\}_{n \ge 0}$, we also call π a period of $A(x)$ modulo M. Clearly, any period always divides the *minimum period*, which is, by definition, the smallest period. The next two theorems are obvious.

Theorem 2.1: If $\{a_n\}_{n \ge 0}$ is generated by $A(x)$, then π is a period of $\{a_n \pmod{M}\}_{n \ge 0}$ iff

$$(1 - x^{\pi})A(x) \in \mathbb{Z}_M[x].$$

Theorem 2.2: If $\{a_n\}_{n \ge 0}$ is generated by $A(x)$ and periodic modulo M with period π, then it is purely periodic modulo M iff the degree of $(1 - x^{\pi})A(x)$ is at most $\pi - 1$ in $\mathbb{Z}_M[x]$.

We will study generating functions of the form $1/f(x)$, where $f(x) \in \mathbb{Z}[x]$ and $f(0) = 1$. Under these conditions, π is a period of $1/f(x)$ modulo M iff $f(x)$ divides $1 - x^{\pi}$ (mod M).

Theorem 2.3: Given $f(x)$, $u(x) \in \mathbb{Z}[x]$, where $f(0) = u(0) = 1$, let μ and μ' be the minimum periods of $1/f(x)$ and $1/f(x)u(x)$ modulo M, respectively. Then μ divides μ'.

Proof: It suffices to show that μ' is a period of $1/f(x)$ modulo M. Equivalently, it suffices to show that $f(x)$ divides $1 - x^{\mu'}$ (mod M), which follows from the fact that $f(x)u(x)$ divides $1 - x^{\mu'}$ (mod M). □

The next result, which is again obvious, allows us to assume, for the rest of this paper, that M is a prime power.

Theorem 2.4: Let $p_1^{e_1} \ldots p_s^{e_s}$ be the prime factorization of M, and $\mu(p_i^{e_i})$ be the minimum period of $\{a_n \text{ (mod } p_i^{e_i})\}_{n \geq 0}$; then the minimum period of

$$\{a_n \text{ (mod } M)\}_{n \geq 0}$$

is the least common multiple of $\mu(p_i^{e_i})$, where $1 \leq i \leq s$.

Finally, if we know a period of $\{a_n \text{ (mod } p)\}_{n \geq 0}$, we have an upper bound for the period of $\{a_n \text{ (mod } p^N)\}_{n \geq 0}$.

Theorem 2.5: If π is a period of $1/f(x)$ modulo p^N, then $p\pi$ is a period of $1/f(x)$ modulo p^{N+1}.

Proof: From Theorem 2.1,

$$x^{\pi} = 1 - f(x)h(x) + p^N g(x), \text{ for some } h(x),\ g(x) \in \mathbb{Z}[x].$$

Then, for some $H(x)$, $G(x) \in \mathbb{Z}[x]$,

$$x^{p\pi} = \{1 - f(x)h(x)\}^p + p^{N+1}G(x) = 1 - f(x)H(x) + p^{N+1}G(x).$$

Thus, $f(x)$ divides $1 - x^{p\pi}$ (mod p^{N+1}). □

Corollary 2.6: If π is a period of $1/f(x)$ modulo p, then πp^{N-1} is a period of $1/f(x)$ modulo p^N for $N \geq 1$.

3. Binomial Coefficients

Let $\mu(t;\ p^N)$ be the minimum period of $1/(1 - x)^t$ modulo p^N. Since $\mu(1;\ p^N) = 1$ for $N \geq 1$, we may assume that $t > 1$. From Theorem 2.3, $\mu(t;\ p^N)$ always divides $\mu(t + 1;\ p^N)$ for $N \geq 1$. What we would like to know is, when will

$$\mu(t;\ p^N) \neq \mu(t + 1;\ p^N);$$

which would imply that $\mu(t;\ p^N)$ divides $\mu(t + 1;\ p^N)$ *properly*. The following theorem provides one such criterion.

Theorem 3.1: Let π be a period of $1/(1 - x)^t$ modulo M. Then π is also a period of $1/(1 - x)^{t+1}$ modulo M iff $h(1) \equiv 0$ (mod M), where $h(x)$ is the polynomial $(1 - x^{\pi})/(1 - x)^t$ in $\mathbb{Z}_M[x]$.

Proof: Let $h(x) = \sum_{n=0}^{D} a_n x^n \in \mathbb{Z}_M[x]$. Then

$$\begin{aligned}\frac{1-x^{\pi}}{(1-x)^{t+1}} &\equiv \frac{h(x)}{1-x} \pmod{M}\\ &= \left(\sum_{m=0}^{\infty} x^m\right)\left(\sum_{n=0}^{D} a_n x^n\right)\\ &= \sum_{m=0}^{D-1}\left(\sum_{n=0}^{m} a_n\right)x^m + h(1)\sum_{m=D}^{\infty} x^m\end{aligned}$$

is a polynomial modulo M iff $h(1) \equiv 0 \pmod{M}$. □

For $b \geq 0$, $(1-x)^{p^b} \equiv 1 - x^{p^b} \pmod{p}$; thus, $\mu(p^b; p) = p^b$. Consequently, Corollary 2.6 implies that $\mu(p^b; p^N) \mid p^{N+b-1}$. But, from Theorem 2.3,

$$\mu(t; p^N) \mid \mu(p^b; p^N) \quad \text{if } t \leq p^b.$$

Hence, for $p^{b-1} < t \leq p^b$, $b \geq 1$, we have

$$G(x) = \frac{1 - x^{N+b-1}}{(1-x)^t} \in \mathbb{Z}_M[x].$$

Since the leading coefficient of $(1-x)^t$ is ± 1, the degree of $G(x)$ is at most $p^{N+b-1} - 1$. We conclude from Theorem 2.2 that $1/(1-x)^t$ is purely periodic modulo M. In other words,

Theorem 3.2: $\left\{\binom{n}{k} \pmod{M}\right\}_{n \geq k}$ is purely periodic, for $k \geq 0$.

In particular,

$$H(x) = (1 - x^{p^{N+b-2}})/(1-x)^{b-1}$$

is a polynomial modulo p^N:

$$H(x) \equiv \sum_{j=0}^{p^{N+b-2}-1} \binom{p^{b-1}+j-1}{j} x^j \pmod{p^N}.$$

We want to know if p^{N+b-2} is still a period of $1/(1-x)^{p^{b-1}+1}$ modulo p^N. In order to apply Theorem 3.1, we evaluate

$$H(1) \equiv \sum_{j=0}^{p^{N+b-2}-1} \binom{p^{b-1}+j-1}{j} = \binom{p^{N+b-2}+p^{b-1}-1}{p^{N+b-2}-1} \pmod{p^N}.$$

The highest power of p which divides $\binom{A+B}{A}$ is the number of carries in the p-ary addition of $A + B$. (See, for example, [1], pp. 270-271.) Thus,

$$p^{N-1} \| H(1) \quad \text{and} \quad H(1) \not\equiv 0 \pmod{p^N}.$$

It now follows from Theorem 3.1 that

$$p^{N+b-2} \text{ divides } \mu(p^{b-1}+1; p^N) \text{ properly.}$$

So, p^{N+b-2} is a *proper divisor* of $\mu(t; p^N)$ for all $t > p^{b-1}$. On the other hand, for $t \leq p^b$,

$$\mu(t; p^N) \mid \mu(p^b; p^N) \quad \text{and} \quad \mu(p^b; p^N) \mid p^{N+b-1}.$$

Therefore, we have just proved

Theorem 3.3: The minimum period for $1/(1-x)^t$ modulo p^N is 1 if $t = 1$, and p^{N+b-1} if $p^{b-1} < t \leq p^b$, $b \geq 1$.

Corollary 3.4: The minimum period of

$$\left\{\binom{n}{k} \pmod{p^N}\right\}_{n \ge k}$$

is 1 if $k = 0$, and p^{N+b-1} if $p^{b-1} \le k < p^b$, $b \ge 1$.

Corollary 3.5: If $p_1^{e_1} \ldots p_s^{e_s}$ is the prime factorization of M, then the minimum period of

$$\left\{\binom{n}{k} \pmod{M}\right\}_{n \ge k}$$

is 1 if $k = 0$, and

$$\prod_{i=1}^{s} p_i^{e_i + b_i - 1}$$

if $p_i^{b_i - 1} \le k < p_i^{b_i}$, $b_i \ge 1$ for $1 \le i \le s$.

4. Final Remarks

Our approach can be used to determine minimum periods of many other infinite integer sequences. For example, the minimum periods of the Stirling numbers of the second kind are determined in [3]. In particular, we found the minimum periods of $1/f(x)$ modulo M, where the factors of $f(x)$ are all linear (in the form of $1 - rx$), or are all binomials of the form $1 - x^r$. These allow us to extend the results in [4] to any prime power modulus, and hence to any modulus. These results will appear in a forthcoming article elsewhere.

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TWO-SIDED GENERALIZED FIBONACCI SEQUENCES

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1. Introduction

This paper investigates a concept called a *two-sided generalized Fibonacci sequence* (TGF) that was motivated by problems of uniqueness in measurement representations [2-4, 6-8]. The particular context that gives rise to TGFs is finite algebraic difference measurement [2, 6-8]. For simplicity, suppose that $n+1$ objects $a_1, \ldots, a_{n+1}$ are linearly ordered by a real-valued function u as

$$u(a_1) < u(a_2) < \cdots < u(a_{n+1})$$

and that comparisons can be made between positive differences $u(a_j) - u(a_i)$, $i < j$. In measurement theory, we are sometimes concerned with conditions which guarantee that the u values are unique up to a positive affine transformation

$$u \to \alpha u + \beta, \ \alpha > 0.$$

Let $d_i > 0$ be defined by

$$d_i = u(a_{i+1}) - u(a_i).$$

Then we search for conditions which guarantee that the d_i are unique up to multiplication by a positive constant α. Each *equality*-of-differences comparison yields an equation of the form

$$d_i + d_{i+1} + \cdots + d_j = d_k + d_{k+1} + \cdots + d_\ell, \ 1 \le i \le j < k \le \ell \le n,$$

in the variables d_i. If there are $n - 1$ linearly independent equations of this type that have a strictly positive solution, then their solution by positive d_i is unique up to multiplication of every d_i by the same positive constant. For example, the three equations

$$d_1 = d_2, \quad d_2 + d_3 = d_4, \quad d_1 + d_2 = d_3 + d_4$$

have solution $d_1^* \ldots d_4^* = 2213$, and if $d_1' \ldots d_4'$ is any other positive solution then there is a $\lambda > 0$ such that $d_i' = \lambda d_i^*$ for each i. We refer the interested readers to [2] for additional discussion of this type of uniqueness in the general algebraic difference setting.

A *TGF* is a finite sequence of positive integers constructed by starting with a 1 and adding terms one by one at either end of the sequence S constructed thus far so that each new term equals the sum of one or more contiguous terms on the *end* of S at which the new term is placed. A new term v added to $S = x_1 \ldots x_m$ produces either $vx_1 \ldots x_m$ with

$$v \in \{x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_m\}$$

or $x_1 \ldots x_m v$ with

$$v \in \{x_m, x_m + x_{m-1}, \ldots, x_m + \cdots + x_1\}.$$

TGFs arise from specialized sets of equations of the type described in the preceding paragraph. One example for $n = 4$ is 2114, which is the unique positive solution (up to multiplication by a positive constant) to

$$d_2 = d_3, \quad d_1 = d_2 + d_3, \quad d_4 = d_1 + d_2 + d_3.$$

Although many unique solutions to equations for the general algebraic difference setting do not correspond to TGFs, as is true for

$$d_1^* \ldots d_4^* = 2213,$$

two-sided generalized Fibonacci sequences constitute an important subset of all such unique solutions, and it is this subset that we study here.

Let T_n denote the set of all n-term TGFs, and let $t_n = |T_n|$. Then

$$T_1 = \{1\}, \quad T_2 = \{(1, 1)\} = \{11\}, \quad T_3 = \{111, 112, 211\},$$

$$T_4 = \{1111, 1112, 1113, 1122, 1123, 1124, 2111, 2112, 2114, 2211, 3111, 3211, 4112, 4211\},$$

and so forth, with $t_1 = t_2 = 1$, $t_3 = 3$, $t_4 = 14$, and, as we shall see, $t_5 = 85$, $t_6 = 626, \ldots$. We note that every TGF for $n \geq 2$ has the *monotonicity property*, which means that there is a subsequence of two or more contiguous 1's and the sequence is nondecreasing in both directions away from that subsequence. Given any finite integer sequence

$$b_j \ldots b_2 b_1 1 \ldots 1 a_1 a_2 \ldots a_k$$

with the monotonicity property, a simple outside-in algorithm identifies whether it is a TGF. At each step of the algorithm, we ask whether a largest end term is the sum of a contiguous block of terms next to it. If not, the sequence is not a TGF; else delete that end term and repeat the question. If deletions leave only 1's, the sequence is a TGF.

We close this section by summarizing our main results. Our first main counting result is the nonlinear recurrence

$$t_{n+1} = 2nt_n - (n - 1)^2 t_{n-1} \text{ for } n \geq 2,$$

which has the Fibonacci feature that each new term in

$$(t_1, t_2, \ldots) = (1, 1, \ldots)$$

is determined from its two immediate predecessors. Since the t_n sequence is not in Sloane's book [10] and has not been brought to that author's attention by others (N. J. A. Sloane, personal communication), it may not have been studied previously.

The recurrence implies that

$$(\sqrt{n} + 1/2)^2 - 1/\sqrt{n} < \frac{t_{n+1}}{t_n} < (\sqrt{n} + 1/2)^2 \text{ for } n \geq 2.$$

This gives nice bounds on the ratio of successive t_n and indicates the growth rate of the t_n sequence. We omit the proof of these bounds, which follow without great difficulty from the recurrence by induction, algebraic manipulation, and subsidiary inequalities such as

$$1/2 < \sqrt{n}(\sqrt{n} - \sqrt{n - 1}).$$

Our other main result for t_n is an asymptotic estimate obtained from the exponential generating function

$$F(x) = \sum_{n=1}^{\infty} \frac{t_n x^{n-1}}{(n - 1)!}.$$

We prove that

$$F(x) = \frac{e^{\frac{1}{1-x}}}{1-x}\left[\frac{1}{e} - \int_{y=0}^{x} \frac{e^{-\frac{1}{1-y}}}{1-y}\,dy\right]$$

and use this to obtain

$$t_n \sim K(n-1)!e^{2\sqrt{n}}/n^{1/4},$$

where $K = K_1\sqrt{e/\pi}/2$ and

$$K_1 = \frac{1}{e} - \int_{y=0}^{1} \frac{e^{-\frac{1}{1-y}}}{1-y}\,dy = 0.148495\ldots .$$

The ratios of successive values of this approximation of t_n lie well within the bounds of the preceding paragraph. The generating function can also be used to obtain a fuller asymptotic approximation to t_n.

The results for t_n are proved in the next section. Section 3 examines $f(k_1, \ldots, k_m)$, the length of a shortest TGF that contains at least one permutation of the positive integer sequence $(k_1, \ldots, k_m)$ as a (not necessarily contiguous) subsequence. We note first that $f(k_1, \ldots, k_m)$ is always defined for $m \leq 4$ but can be undefined for $m \geq 5$ because no TGF has a permutation of $k_1, \ldots, k_m$ as a subsequence. We then show for a single integer $k \geq 2$ that

$$f(k) = \lceil \log_2 k \rceil + 2,$$

where $\lceil x \rceil$ is the smallest integer at least as great as x. This result is followed by a proof that, when $k_1 \leq k_2 \leq k_3 \leq k_4$, $f(k_1, k_2, k_3, k_4) - f(k_2, k_3, k_4)$ can be arbitrarily large. We do not know whether the same thing holds for $f(k_1, k_2, k_3) - f(k_2, k_3)$ or for $f(k_1, k_2) - f(k_2)$ when $k_1 \leq k_2 \leq k_3$, but conjecture that $f(k_1, k_2) \leq f(k_2) + 1$.

The paper concludes with remarks on open problems and generalizations.

2. Counting TGFs

Theorem 1: $t_1 = 1$, $t_2 = 1$, and $t_{n+1} = 2nt_n - (n-1)^2 t_{n-1}$ for $n \geq 2$.

Proof: Each TGF $x_1 \ldots x_n$ in T_n yields n left extensions $vx_1 \ldots x_n$ in T_{n+1} for the n different values in

$$\{x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_n\}.$$

It also yields n right extensions $x_1 x_2 \ldots x_n v$ in T_{n+1} for the n different values in

$$\{x_n, x_n + x_{n-1}, \ldots, x_n + \cdots + x_1\}.$$

Thus, T_n induces nt_n distinct members of T_{n+1} by left extension and nt_n distinct members of T_{n+1} by right extension. But the $2nt_n$ total can contain duplications between left and right extensions.

Call a sequence in T_{n+1} a *sequence of duplication* if it arises from *both* a left extension and a right extension of sequences in T_n. Consider the condition

$$z_2 \ldots z_n \in T_{n-1},\ z_1 = z_2 + \cdots + z_j \text{ for some } 2 \leq j \leq n, \tag{A}$$
$$\text{and } z_{n+1} = z_n + \cdots + z_k \text{ for some } 2 \leq k \leq n.$$

If (A) holds, then $z_1 z_2 \ldots z_n z_{n+1}$ is clearly a sequence of duplication, since $z_1 \ldots z_n$ and $z_2 \ldots z_{n+1}$ are in T_n.

Conversely, if $z_1 \dots z_{n+1}$ is a sequence of duplication, then (A) holds. To see this, suppose

$$z_1 \dots z_{n+1} = ax_1 \dots x_n = y_1 \dots y_n b$$

with $x_1 \dots x_n$ and $y_1 \dots y_n$ in T_n,

$$a = x_1 + \cdots + x_j \text{ for some } 1 \le j \le n, \text{ and}$$

$$b = y_n + \cdots + y_k \text{ for some } 1 \le k \le n.$$

We cannot have $a = x_1 + \dots + x_n$, since otherwise $y_1 > y_2 + \dots + y_n$, contradicting $y_1 \dots y_n \in T_n$. Similarly, b cannot equal $y_n + \dots + y_1$. We can conclude that (A) holds for $z_1 = a$ and $z_{n+1} = b$, provided that we can show that $S = z_2 \dots z_n$ is in T_{n-1}. Suppose, to the contrary, that $S \notin T_{n-1}$. Then

$$x_k = x_{k+1} + \cdots + x_n \text{ for some } k \le n - 1.$$

If this is true only for $k = n - 1$, then x_n can be the last term added in the construction of $x_1 \dots x_n$ so that its deletion leaves member $x_1 \dots x_{n-1} = S$ of T_{n-1}. Hence, we suppose that

$$x_k = x_{k+1} + \cdots + x_n \text{ for some } k \le n - 2.$$

By a symmetric argument for $y_1 \dots y_n$, $S \notin T_{n-1}$ implies that

$$y_j = y_1 + \cdots + y_{j-1} \text{ for some } j \ge 3.$$

With k and j as just noted, $x_k = z_{k+1}$, $y_j = z_j$, and the monotonicity property for $z_1 \dots z_{n+1}$ requires that there be some 1's to the left of z_j and some 1's to the right of z_{k+1}. Therefore, $k + 1 < j$. But then $z_{k+1} > z_j$ ($x_k > y_j$), since z_{k+1} is a sum of terms that include z_j, and $z_j > z_{k+1}$ ($y_j > x_k$), since z_j is a sum of terms that include z_{k+1}. We therefore have a contradiction and conclude that $S \in T_{n-1}$.

We have shown that (A) holds if and only if $z_1 \dots z_{n+1}$ is a sequence of duplication. Since for every member of T_{n-1} each of z_1 and z_{n+1} can be chosen independently in $n - 1$ ways to satisfy (A), there are precisely $(n - 1)^2 t_{n-1}$ sequences of duplication. Each of these corresponds to one left extension and one right extension from T_n. Therefore,

$$t_{n+1} = 2nt_n - (n - 1)^2 t_{n-1}. \quad \square$$

A simple application of Theorem 1 shows that

$$t_5 = 85,\ t_6 = 626,\ t_7 = 5387,\ t_8 = 52{,}882,$$

$$t_9 = 582{,}149,\ t_{10} = 7{,}094{,}234,\ t_{11} = 94{,}730{,}611,\ \dots .$$

Theorem 2: $t_n \sim (n - 1)!K_1\sqrt{e/\pi}\, e^{2\sqrt{n}}/(2n^{1/4})$, where

$$K_1 = \frac{1}{e} - \int_{y=0}^{1} \frac{e^{-\frac{1}{1-y}}}{1 - y}\, dy = 0.148495\dots .$$

Proof: The proof is based on the saddle point method of asymptotic analysis described, for example, in de Bruijn [1]. As we note shortly, the main step in the proof is covered by results of Hayman [5].

We begin with the recurrence of Theorem 1 and form the exponential generating function

$$F(x) = \sum_{n=1}^{\infty} \frac{t_n x^{n-1}}{(n - 1)!}.$$

Using the recurrence, we get

$$F'(x)(1-x)^2 - F(x)(2-x) = -1.$$

We solve this linear differential equation by a standard method to obtain

$$F(x) = \frac{e^{\frac{1}{1-x}}}{1-x}\left[K_1 + \int_x^1 \frac{e^{-\frac{1}{1-y}}}{1-y}\,dy\right],$$

where K_1 is as defined in Theorem 2.

Ignoring $\int_x^1 \ldots dy$ for the moment, we use the saddle point method to obtain the asymptotic estimate of the coefficient c_n of x^n in the power series expansion of $e^{1/(1-x)}/(1-x)$. It follows from Hayman [5] (and by our independent verification) that

$$c_n \sim \frac{1}{2}\sqrt{e/\pi}\,e^{2\sqrt{n}}/n^{1/4}.$$

Since $c_n/c_{n-1} \to 1$ and (see below) $\int_x^1 \ldots dy$ is insignificant compared to K_1, we conclude that

$$t_n/(n-1)! \sim K_1\frac{1}{2}\sqrt{e/\pi}\,e^{2\sqrt{n}}/n^{1/4}$$

as claimed in Theorem 2.

To show that the $\int_x^1 \ldots dy$ part of $F(x)$ can be ignored asymptotically, we first extend this part of $F(x)$ to the complex plane by defining

$$g(z) = \frac{e^{\frac{1}{1-z}}}{1-z}\int_z^1 \frac{e^{-\frac{1}{1-u}}}{1-u}\,du = \frac{1}{1-z}\int_z^1 \frac{e^{\frac{z-u}{(1-z)(1-u)}}}{1-u}\,du = \sum_{n=0}^{\infty} d_n z^n.$$

By Cauchy's integral equation,

$$d_n = \frac{1}{2\pi i}\oint_{|z|=r}\frac{g(z)}{z^n}\,\frac{dz}{z},$$

and therefore,

$$|d_n| \le \frac{\max|g(z)|}{|z|^n} = \frac{\max|g(z)|}{r^n},$$

where $r = 1 - 1/\sqrt{n}$ and the max is taken on the circle $|z| = r$. We shall show that

$$|g(z)| = O(\sqrt{n}) \text{ for all } z \text{ with } |z| = r.$$

It then follows that

$$|d_n| = O(\sqrt{n}\,e^{\sqrt{n}})$$

and hence that

$$\frac{|d_n|}{K_1 c_n} = O(n^{3/4}/e^{\sqrt{n}}) \to 0.$$

Therefore, the total coefficient of x^n in the power series expansion of $F(x)$ is $\sim K_1 c_n$.

To obtain

$$|g(z)| = O(\sqrt{n}) \text{ on } |z| = r,$$

we begin with the second integral expression of $g(z)$ in the preceding paragraph and define α by

$$u = 1 - \alpha(1-z)$$

to obtain

$$g(z) = \frac{1}{1-z}\int_{\alpha=0}^{1} e^{(1-1/\alpha)/(1-z)}\frac{d\alpha}{\alpha}.$$

Since $\mathrm{Re}(1/(1-z)) = (1-\mathrm{Re}(z))/|1-z|^2$ and $1 - 1/\alpha < 0$, this yields

$$|g(z)| \le \frac{1}{|1-z|}\int_{\alpha=0}^{1} e^{(1-1/\alpha)(1-\mathrm{Re}(z))/|1-z|^2}\frac{d\alpha}{\alpha}.$$

With $z = r(\cos\theta + i\sin\theta)$ in polar coordinates,

$$|1-z| = (1 - 2r\cos\theta + r^2)^{1/2}.$$

This is minimized at $\theta = 0$, so

$$\min|1-z| = 1 - r = 1/\sqrt{n}.$$

Therefore,

$$\max(1/|1-z|) = \sqrt{n}.$$

Moreover, $\mathrm{Re}(1/(1-z))$ is easily seen to be maximized at $\theta = \pi$, where it equals $1/(1+r)$, or about $1/2$. Let $\beta > 0$ be a constant less than $\mathrm{Re}(1/(1-z))$ for all $|z| = r$. Since $1 - 1/\alpha$ in the exponent of the preceding integral is negative, it follows that

$$|g(z)| = O\left(\sqrt{n}\int_{\alpha=0}^{1} e^{(1-1/\alpha)\beta}d\alpha/\alpha\right).$$

We break the range of integration for α into $[0, 1/10]$ and $[1/10, 1]$. Since

$$\int_{\alpha=1/10}^{1} e^{(1-1/\alpha)\beta}d\alpha/\alpha = O(1),$$

$$O\left(\sqrt{n}\int_{\alpha=1/10}^{1} \ldots\ d\alpha/\alpha\right) = O(\sqrt{n}).$$

On $[0, 1/10]$, $1 - 1/\alpha < -1/2\alpha$, so

$$\sqrt{n}\int_{\alpha=0}^{1/10} e^{(1-1/\alpha)\beta}d\alpha/\alpha = O\left(\sqrt{n}\int_{0}^{1/10} e^{-\beta/2\alpha}\,d\alpha/\alpha\right).$$

Let $\gamma = \beta/(2\alpha)$, so $d\alpha/\alpha = -d\gamma/\gamma$ and

$$\int_{0}^{1/10} e^{-\beta/2\alpha}\,d\alpha/\alpha = \int_{\gamma=5\beta}^{\infty} e^{-\gamma}\,d\gamma/\gamma.$$

Since β is only required to be less than $1/(1+r)$, and $0 < r < 1$, we can presume that $5\beta > 1$. Then, since

$$\int_{1}^{\infty}(e^{-x}/x)dx = O(1),$$

we get

$$\sqrt{n}\int_{\alpha=0}^{1/10} e^{(1-1/\alpha)\beta}d\alpha/\alpha = O(\sqrt{n}).$$

Hence $|g(z)| = O(\sqrt{n})$ regardless of where z lies on $|z| = r$. □

3. Inclusion of Specific Terms in TGFs

Recall that $f(k_1, \ldots, k_m)$ is the length of a shortest TGF which contains at least one permutation of the positive integer sequence $(k_1, \ldots, k_m)$. If there is no such TGF, we say that $f(k_1, \ldots, k_m)$ is undefined.

Theorem 3: $f(k_1, \ldots, k_m)$ is always defined for $m \le 4$ but can be undefined for $m \ge 5$.

Proof: Let $k = \max\{k_1, k_2, k_3, k_4\}$ and assume with no loss in generality that $k_1 \le k_2$ and $k_3 \le k_4$. Then $k_2k_11 \ldots 1k_3k_4$ with k 1's in the middle is a TGF. However, $f(4, 5, 6, 7, 8)$ is undefined since, according to the monotonicity property, at least three numbers from $\{4, 5, 6, 7, 8\}$ must appear in increasing order (away from the 1's) on the same side of the block of 1's, and this is clearly impossible for a TGF. □

Theorem 4: $f(k) = \lceil \log_2 k \rceil + 2$ for $k \ge 2$.

Proof: Since the largest possible term in a sequence in T_n is 2^{n-2} (from 11248 ... 2^{n-2}, for example), $f(k) \ge \lceil \log_2 k \rceil + 2$ for $k \ge 2$. Conversely, given $k \ge 2$ and its expansion as a sum of powers of 2, say,

$$k = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_p} \text{ with } 0 \le k_1 < k_2 < \cdots < k_p,$$

let $u_1 < u_2 < \cdots < u_q$ be all integers in $\{0, 1, \ldots, k_p\}\backslash\{k_1, \ldots, k_p\}$. Then the $(k_p + 2)$-term sequence

$$2^{k_p}, \ldots, 2^{k_2}, 2^{k_1}, 1, 2^{u_1}, 2^{u_2}, \ldots, 2^{u_q}$$

is a TGF since each 2^x equals 1 plus all terms 2^y with $y < x$. If $k = 2^{k_p}$, then it follows that

$$f(k) \le k_p + 2 = \log_2 k + 2;$$

if $k > 2^{k_p}$, then the addition of k to the left end of the sequence gives another TGF, from which

$$f(k) \le k_p + 3 \le \lceil \log_2 k \rceil + 2$$

follows. Hence,

$$f(k) = \lceil \log_2 k \rceil + 2 \text{ for } k \ge 2. \quad \square$$

The next steps beyond Theorem 4 are to consider $f(k_1, k_2)$ and $f(k_1, k_2) - f(k_2)$ when $k_1 \le k_2$. We have systematically verified

$$f(k_1, k_2) \le f(k_2) + 1 \quad (k_1 \le k_2)$$

for all $k_2 \le 16$, but do not know if this holds for larger k_2. Similarly, we do not know if there is a fixed c such that

$$f(k_1, k_2, k_3) \le f(k_2, k_3) + c \text{ whenever } k_1 \le k_2 \le k_3.$$

However, we do have the following result.

Theorem 5: If $k_1 \le k_2 \le k_3 \le k_4$, then $f(k_1, k_2, k_3, k_4) - f(k_2, k_3, k_4)$ can be arbitrarily large.

The following lemma is used in the proof of Theorem 5. We will prove the lemma shortly. Here, $\lfloor x \rfloor$ is the integer part of x.

Lemma 1: $f(k, k + 1, k + 2, k + 3) \ge \lfloor k/3 \rfloor + 6$ for $k > 3$.

Proof of Theorem 5: Let

$$(k_1, k_2, k_3, k_4) = (k, k+1, k+2, k+3)$$

with $k + 1 = 2^p$ and $p \geq 3$. Then

$$f(k+1, k+2, k+3) \leq p + 5 = \log_2(k+1) + 5$$

since

$$2^p + 2, 2^p + 1, 1, 1, 1, 2, 4, 8, \ldots, 2^p$$

is a TGF in T_{p+5}. When this is combined with the conclusion of Lemma 1, we have

$$f(k_1, k_2, k_3, k_4) - f(k_2, k_3, k_4) \geq \lfloor k/3 \rfloor + 1 - \log_2(k+1),$$

and the right-hand side can be made arbitrarily large. □

Proof of Lemma 1: Let $S = x_1 \ldots x_n$ be a *shortest* TGF that contains the integers in

$$K = \{k, k+1, k+2, k+3\}, \; k > 3.$$

By the monotonicity property, $x_i \leq k + 3$ for all i.

CLAIM: $K = \{x_1, x_2, x_{n-1}, x_n\}$.

To prove the Claim, note first that since $k > 3$, it is impossible for more than two elements of K to appear in increasing order away from the center on the same side of the sequence 1, 1. Thus, there must be two elements of K on each side of the block of 1's. Since S is a shortest TGF, elements of K should be at the beginning and end of S, and there are no repetitions of elements of K. Thus, x_1 and x_n are in K. The Claim follows by monotonicity of S.

We now use the Claim to analyze the following three cases:

Case 1: $x_1, x_2 = k+1, k;\; x_{n-1}, x_n = k+2, k+3.$

Case 2: $x_1, x_2 = k+2, k;\; x_{n-1}, x_n = k+1, k+3.$

Case 3: $x_1, x_2 = k+3, k;\; x_{n-1}, x_n = k+1, k+2.$

The other three possible cases are symmetric to these.

Case 1: By the construction process, this case forces S to be

$$k+1, k, 1, \ldots, 1, k+2, k+3.$$

By monotonicity, all remaining terms are 1's and so there are $k + 2$ 1's. It follows that $n = (k+2) + 4 = k + 6$, and $k + 6 \geq \lfloor k/3 \rfloor + 6$.

Case 2: For this case, let

$$S = k+2, k, p, \ldots, q, k+1, k+3.$$

To obtain $k + 2$ by the construction process, we must have $p \leq 2$, and similarly, $q \leq 2$. Hence, all terms from p through q are ≤ 2. Since there must be at least two 1's, and since $p + \cdots + q \geq k + 1$, we note that to obtain $k + 1$ by construction, we must have

$$n \geq 2 + \left\lceil \frac{k-1}{2} \right\rceil + 4 = \left\lceil \frac{k-1}{2} \right\rceil + 6 \geq \lfloor k/3 \rfloor + 6.$$

Case 3: Let

$$S = k + 3,\ k,\ p,\ \ldots,\ q,\ k + 1,\ k + 2$$

which forces $q = 1$ and $p \le 3$. Since the p through q part must end in 111 or 211, and since every other term in this part is ≤ 3 by the monotonicity property,

$$n \ge 3 + \left\lceil \frac{k + 1 - 4}{3} \right\rceil + 4 = \lceil k/3 \rceil + 6 \ge \lfloor k/3 \rfloor + 6. \ \Box$$

4. Remarks

Questions of uniqueness in finite measurement structures are proving to be a rich source of interesting combinatorial and number-theoretic problems, as shown in [2, 3] and the present paper, and summarized in [4, 8]. Our story here is the familiar one of encountering Fibonacci-like structures in an area where none was visible at the start. Not only are TGFs natural generalizations of the basic Fibonacci sequence in their two-sidedness and their relaxation of the requirement that a new addition be the sum of exactly two neighbors, but the sequence $t_1, t_2, t_3, \ldots$ that counts the number of TGFs has a recurrence in which the next term is determined by precisely its two immediate predecessors.

The most obvious problems left open in the paper concern boundedness, or better, of $f(k_1, k_2, k_3) - f(k_2, k_3)$ and $f(k_1, k_2) - f(k_2)$ when $k_1 \le k_2 \le k_3$. A further possibility for investigation is $f^*(k_1, \ldots, k_m)$, the length of the shortest TGF, if any, that has $k_1, \ldots, k_m$ as a subsequence.

We mention two generalizations of two-sided generalized Fibonacci sequences. The first is also two-sided and is constructed like a TGF except that the value of a new term at either end can equal the sum of one or more contiguous terms (including a single 1) located anywhere in the sequence constructed thus far. Some results for this generalization are included in [2].

The other generalization is one of a large number of things that might be referred to as generalized Fibonacci trees. The tree we have in mind is constructed like a TGF except that it has N rather than 2 branches extending away from a root that consists of two 1's. The value of a new term added to a branch is the sum of one or more extant terms consisting of either (a) immediate predecessors on that branch, or (b) all those predecessors plus one or both root 1's, or (c) all its branch predecessors plus both root 1's plus terms contiguous to the root along some other branch. We are not aware of results for this generalization.

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EQUIVALENCE OF PIZA'S PRIMALITY CRITERION WITH THAT OF GOULD-GREIG AND ITS DUAL RELATIONSHIP TO THE MANN-SHANKS CRITERION

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1. Introduction

The famous amateur mathematician Pedro A. Piza (1896-1956), who spent most of his life in business in Puerto Rico, discovered many interesting things in geometry and number theory. In his paper on Fermat coefficients [5], he discovered a criterion for primality which runs as follows:

$$2n + 1 \text{ is prime iff } k \,\Big|\, \binom{2n-k}{k-1} \text{ for all } k,\ 1 \le k \le n. \tag{1.1}$$

Actually, he left an ellipsis in the proof. He said that when k is composite the proof that k divides the binomial coefficient when $2n + 1$ is prime is somewhat more complicated but not difficult and he left this to the interested reader.

Eighteen years later, Henry Mann and Daniel Shanks [4] discovered another attractive primality criterion which may be stated as follows:

$$C \ge 2 \text{ is prime iff } R \,\Big|\, \binom{R}{C-2R} \text{ for every } R \text{ such that}$$
$$C/3 \le R \le C/2,\ R \ge 1. \tag{1.2}$$

In their criterion, the binomial coefficients are arranged in an array where each row is shifted two units over from the preceding. Then the criterion may be stated more pithily in the following way: A column number C is prime if and only if every binomial coefficient in that column is divisible by its corresponding row number R. (See Table 3 below.)

Then Gould and Greig [3] obtained a primality criterion using a "Lucas" triangle. Diagonals in this triangle sum to Lucas numbers, whereas in the usual Pascal triangle they sum to Fibonacci numbers. The criterion runs as follows:

$$D \ge 2 \text{ is prime iff } D \,|\, A(D-j,\ j) \text{ for all } j,\ 1 \le j \le D/2, \tag{1.3}$$

where

$$A(n,\ k) = \binom{n}{k} + \binom{n-1}{k-1}.$$

It was shown in [3] that this can be reformulated in the equivalent form:

$$C \ge 2 \text{ is prime iff } R \,\Big|\, \binom{-R}{C-2R} \text{ for all } R,\ 1 \le R \le C/2. \tag{1.4}$$

This made the criterion dual to that of Mann and Shanks, since only a minus sign is different in comparison to (1.2), but of course the coefficients differ.

We shall show here that Piza's criterion may be reformulated as follows:

$$C \geq 2 \text{ is prime iff } R \,\Big|\, (-1)^C \binom{-R}{C - 2R} \text{ for all } R,\ 1 \leq R \leq C/2. \tag{1.5}$$

Since the sign $(-1)^C$ does not affect divisibility, it follows that Piza's criterion is equivalent to that of Gould-Greig. Table 4 below shows Piza's coefficients.

All of these criteria are susceptible to extensions to generalized binomial coefficients—such as Gaussian or q-coefficients, Fibonomial coefficients, s-Fibonomial coefficients, etc.—as was shown for the Mann-Shanks criterion in [1] and [2]. The Mann-Shanks criterion, requiring fewer divisibility tests, is more efficient than the criteria of Piza or Gould-Greig.

2. Proofs and Discussion

Let us first examine Piza's array from (1.1). We have the following display of $\binom{2n-k}{k-1}$:

TABLE 1. Piza Array

n	$2n+1$	1	2	3	4	5	6	7 ... k
1	3	1						
2	5	1	2					
3	7	1	4	3				
4	9	1	6	10	4			
5	11	1	8	21	20	5		
6	13	1	10	36	56	35	6	
7	15	1	12	55	120	126	56	7

Thus 9 is not prime since $3 \nmid 10$; 15 is not prime since $3 \nmid 55$, $5 \nmid 126$, $6 \nmid 56$. We may rearrange the table to make it look more like the usual Pascal array of $\binom{n}{k}$:

TABLE 2. Modified Piza Array

n	$n+2$	0	1	2	3	4	5	6 ... k
1	3	1	1					
2	4	1	2	1				
3	5	1	3	3	1			
4	6	1	4	6	4	1		
5	7	1	5	10	10	5	1	
6	8	1	6	15	20	15	6	1
7	9	1	7	21	35	35	21	7
8	10	1	8	28	56	70	56	28
9	11	1	9	36	84	126	126	84
10	12	1	10	45	120	210	252	210
11	13	1	11	55	165	330	462	462
12	14	1	12	66	220	495	792	924

Examination of these arrays shows that Piza's criterion may be reformulated as follows:

$$n + 2 \text{ is prime iff } k + 1 \,\Big|\, \binom{n-k}{k} \text{ for all } k,\ 0 \le k \le n/2. \tag{2.1}$$

Make the replacement $n \leftarrow n - 2$ and this becomes

$$n \text{ is prime iff } k + 1 \,\Big|\, \binom{n-2-k}{k} \text{ for all } k,\ 0 \le k \le n/2 - 1. \tag{2.2}$$

Then make the replacement $k \leftarrow k - 1$ and this, in turn, becomes

$$n \text{ is prime iff } k \,\Big|\, \binom{n-k-1}{k-1} \text{ for all } k,\ 1 \le k \le n/2. \tag{2.3}$$

However,

$$\binom{n-k-1}{k-1} = \binom{n-k-1}{n-2k} = (-1)^n \binom{-k}{n-2k},$$

so that Piza's criterion takes the form

$$n \text{ is prime iff } k \,\Big|\, (-1)^n \binom{-k}{n-2k} \text{ for all } k,\ 1 \le k \le n/2, \tag{2.4}$$

which is what we asserted in (1.5). The $(-1)^n$ may be dropped, as we said, so that Piza's criterion is equivalent to the Gould-Greig criterion.

Now let us return to the original (1.1) and make the replacement $k \leftarrow n - k$. We obtain the equivalent form of Piza's criterion:

$$2n + 1 \text{ is prime iff } n - k \,\Big|\, \binom{n+k}{n-k-1} \text{ for all } k,\ 0 \le k \le n - 1. \tag{2.5}$$

Since $\binom{n+k}{n-k-1} = \binom{n+k}{2k+1}$, we may restate (2.5) as:

$$2n + 1 \text{ is prime iff } n - k \,\Big|\, \binom{n+k}{2k+1} \text{ for all } k,\ 0 \le k \le n - 1. \tag{2.6}$$

This is an interesting variant because in [1] it was shown that the Mann-Shanks criterion could be rephrased in the form:

$$2n + 1 \text{ is prime iff } n - k \,\Big|\, \binom{n-k}{2k+1} \text{ for all } k,\ 0 \le k \le \frac{n-1}{3}. \tag{2.7}$$

Thus, the Piza criterion is a kind of dual to that of Mann-Shanks in that one has $n + k$ and the other has $n - k$.

We close by setting down the original Mann-Shanks array (1.2) followed by the array of Piza-Gould-Greig in the form (1.5):

TABLE 3. Mann-Shanks Array

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	... C
1	1	1																	
2			1	2	1														
3					1	3	3	1											
4							1	4	6	4	1								
5									1	5	10	10	5	1					
6											1	6	15	20	15	6	1		
7													1	7	21	35	35	21	
8															1	8	28	56	
9																	1	9	
⋮																			
R																			

TABLE 4. Piza-Gould-Greig Array

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	... C
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
2			1	2	3	4	5	6	7	8	9	10	11	12	
3					1	3	6	10	15	21	28	36	45	55	
4							1	4	10	20	35	56	84	120	
5									1	5	15	35	70	126	
6											1	6	21	56	
7													1	7	
⋮															
R															

Mann-Shanks is far more efficient, requiring fewer divisibility tests in a column. In Table 4 we must test each R for $1 \le R \le C/2$, whereas in Table 3 we test only values of R with $C/3 \le R \le C/2$.

The multiple charms of Pascal's triangle are far from exhausted.

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ON THE SCHNIRELMANN DENSITY OF M-FREE INTEGERS

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It is well known that a positive integer is said to be r-free ($r \geq 2$) if it contains no r^{th} power factor greater than 1. Let Q_r denote the set of all r-free integers. If the integers r and k are such that $2 \leq r < k$, an integer of the form $a^k b$, where a is any natural number and b is r-free is called a (k, r)-integer. The set of all (k, r)-integers is denoted by $Q_{k,r}$. The (k, r)-integers were introduced by Cohen [1] and by Subbarao & Harris [6], independently, under different notations. Observe that (∞, r)-integers are the r-free integers; therefore, the (k, r)-integers can be considered as generalized r-free integers.

The Schnirelmann density for a set, S, of positive integers is denoted by $D(S)$. That is,

$$D(S) = \inf_{n \geq 1} \frac{S(n)}{n},$$

where $S(n)$ is the number of integers in S not exceeding n.

Using computational methods, Rogers [5] proved that $D(Q_2) = 53/88$. Duncan [2] showed, by elementary methods, that

$$D(Q_r) > 1 - \sum_p \frac{1}{p^r}, \tag{1}$$

in which the summation is over all primes p. Later, Feng & Subbarao [3] established

$$D(Q_{k,r}) \geq \alpha_{k,r}, \tag{2}$$

where

$$\alpha_{k,r} = \zeta(k)\left(1 - \sum_p \frac{1}{p^r}\right) - \frac{1}{k}\left(1 - \frac{1}{k}\right)^{k-1}, \tag{3}$$

in which $\zeta(k)$ is the Riemann zeta function.

Rieger [4] introduced M-free integers as follows: Suppose M is a set of positive integers with minimal element $r > 1$. A positive integer $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$, where $p_1, p_2, \ldots, p_t$ are distinct primes, is said to be M-free if $\alpha_i \notin M$ for $i = 1, 2, \ldots, t$. The set of all M-free integers is denoted by Q_M.

If r, k are integers such that $2 \leq r < k$, write

$$A = \{r, r+1, r+2, \ldots\},$$

$$B = \{n: n \geq r, n \equiv j \pmod{k} \text{ for some } j\ (r \leq j \leq k-1)\},$$

$$C = \{r\},$$

$$D = \{r, 2r, 3r, \ldots\}.$$

Then observe that $Q_A = Q_r$; $Q_B = Q_{k,r}$, the set of all (k, r)-integers; $Q_C = S_r$, the set of all semi-r-free integers introduced by Suryanarayana [7]; and $Q_D = U_r$, the set of all unitarily r-free integers given by Cohen [1].

The object of this note is to obtain a lower bound for $D(Q_M)$. This bound improves (2) in the case $M = B$. In fact, we prove the following:

Theorem: $D(Q_M) \geq 1 - 2\sum_p (p-1)\sum_{a \in M} p^{-a-1}$.

Proof: If $Q_M(n)$ is the number of integers in Q_M not exceeding n, then

$$Q_M(n) \geq n - \sum_p \alpha_{M,n}(p), \tag{4}$$

where $\alpha_{M,n}(p)$ is the number of integers $m \leq n$ such that $p^a \| m$ for some $a \in M$. To count $\alpha_{M,n}(p)$, for each fixed $a \in M$, we find the number of integers $m \leq n$ with $p^a | m$ and $p^{a+1} \nmid m$, and the latter number is

$$[n/p^a] - [n/p^{a+1}]$$

so that

$$\alpha_{M,n}(p) = \sum_{a \in M}\left(\left[\frac{n}{p^a}\right] - \left[\frac{n}{p^{a+1}}\right]\right) \leq \sum_{a \in M}\left(1 - \frac{1}{p}\right)\left(\left[\frac{n}{p^a}\right] + 1\right). \tag{5}$$

Now, from (4) and (5), we obtain

$$Q_M(n) \geq n - \sum_p \sum_{a \in M}\left(1 - \frac{1}{p}\right)\left(\left[\frac{n}{p^a}\right] + 1\right) \geq n - 2\sum_p (p-1)\sum_{a \in M} n \cdot p^{-a-1},$$

where the sum on the right side is over primes p with $p^a \leq n$ for some $a \in M$, which gives

$$\frac{Q_M(n)}{n} \geq 1 - 2\sum_p (p-1)\sum_{a \in M} p^{-a-1}.$$

Since this is also true when summed over all primes, the theorem follows.

Corollary: For $k > r \geq 2$, $D(Q_{k,r}) \geq b_{k,r}$, where

$$b_{k,r} = 1 - 2\sum_p \frac{p^{k-r} - 1}{p^k - 1}.$$

Proof: Since

$$\sum_{a \in B} p^{-a-1} = \sum_{m=0}^{\infty}\sum_{j=0}^{k-1}\frac{1}{p^{mk+j+1}} = \frac{p^{k-r} - 1}{(p-1)(p^k - 1)}$$

and $Q_B = Q_{k,r}$, the Corollary follows from the Theorem.

Remark 1: For any $k > r \geq 2$, $a_{k,r} < b_{k,r}$. In fact, since

$$\begin{aligned}
b_{k,r} &= 1 - 2\sum_p\left(\frac{1}{p^r} - \frac{1}{p^k}\right)\left(1 - \frac{1}{p^k}\right)^{-1} \\
&= 1 - 2\sum_p \frac{1}{p^r}\left(1 + \frac{1}{p^k} + \frac{1}{p^{2k}} + \cdots\right) + 2\sum_p \frac{1}{p^k}\left(1 - \frac{1}{p^k}\right)^{-1} \\
&= 1 - 2\sum_p \frac{1}{p^r} - 2\sum_p \frac{1}{p^{r+k}}\left(1 - \frac{1}{p^k}\right)^{-1} + 2\sum_p \frac{1}{p^k}\left(1 - \frac{1}{p^k}\right)^{-1} \\
&= \left(1 - \sum_p \frac{1}{p^r}\right) - \sum_p \frac{1}{p^r} + 2\sum_p\left(1 - \frac{1}{p^r}\right)\frac{1}{p^k - 1}.
\end{aligned}$$

In view of (3), it suffices to show that

$$2\sum_p\left(1-\frac{1}{p^r}\right)\frac{1}{p^k-1} > \sum_p \frac{1}{p^r} + \left(1-\sum_p \frac{1}{p^r}\right)\left(\sum_{n=2}^{\infty}\frac{1}{n^k}\right)$$
$$= \sum_{n=2}^{\infty}\frac{1}{n^k} + \left(\sum_p \frac{1}{p^r}\right)\left(1-\sum_{n=2}^{\infty}\frac{1}{n^k}\right),$$

and this follows if we prove that

$$\sum_{n=2}^{\infty}\frac{1}{n^k} - 2\sum_p\left(1-\frac{1}{p^r}\right)\frac{1}{p^k-1} < \left(\sum_{n=2}^{\infty}\frac{1}{n^k}-1\right)\left(\sum_p\frac{1}{p^r}\right). \tag{6}$$

If $a_n = -1$ or 1, according as $n = 1$ or $n > 1$, then $b_n = n^{k-r}$ or 0, according as n is a prime or not and $c_n = [(n^r - 1)/(n^k - 1)]b_n$, so the inequality in (6) can be written as

$$\sum_{n=2}^{\infty}\frac{a_n}{n^k} - 2\sum_{n=2}^{\infty}\frac{c_n}{n^k} < \left(\sum_{n=1}^{\infty}\frac{a_n}{n^k}\right)\left(\sum_{n=1}^{\infty}\frac{b_n}{n^k}\right). \tag{7}$$

But, by the multiplication of Dirichlet series, the right side of (7) is:

$$\sum_{n=1}^{\infty}\frac{d_n}{n^k},\text{ where } d_n = \begin{cases} 0 & \text{if } n = 1,\\ -p^{k-r} & \text{if } n = p, \text{ a prime},\\ \displaystyle\sum_{\substack{p\mid n\\ p<n}} p^{k-r} & \text{otherwise.}\end{cases}$$

Since $d_n > a_n - 2c_n$ for all n, the inequality (7) holds; hence

$a_{k,r} < b_{k,r}$.

Thus, the Corollary improves (2). However, the inequality (1) gives a better lower bound for $D(Q_r)$ than the one obtained from the Theorem.

Remark 2: In the special cases of $Q_C = S_r$ and $Q_D = U_r$, defined earlier, the Theorem gives

$$D(S_r) \geq 1 - 2\sum_p \frac{p-1}{p^{r+1}} \quad\text{and}\quad D(U_r) \geq 1 - 2\sum_p \frac{p-1}{p(p^r-1)}.$$

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TRAPPING A REAL NUMBER BETWEEN ADJACENT RATIONALS

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Introduction

In this article we ask the following question: Given any real number σ can one find a rational number p/q such that $(p + 1)/(q + 1) < \sigma < p/q$.? Clearly, one of the necessary conditions of this problem is that $\sigma > 1$. But this condition is not sufficient. Interestingly enough, the question came up as a result in algebraic geometry in [2], where Sommese essentially proves the sufficiency of $\sigma > 2$ in the first theorem.

We give explicit conditions under which the above question is true using a somewhat stronger hypothesis: Given any real number $\sigma > 1$ and $N > 0$, can one find positive integers r and s such that $r > s > N$, and s divisible by some fixed integer m, and the denominator of a *fixed* rational number t and satisfying $r - ts > M$, *for any* M, where

$$1 < t < \sigma \quad \text{and} \quad \frac{r+1}{s+1} < \sigma < \frac{r}{s}?$$

The answer depends on whether σ is rational or irrational. We have the following two theorems:

Theorem 1: Let $\sigma = p/q$ be a positive rational number. Then the following are equivalent:

i) $\sigma > 2$

ii) Given any positive integers m, M, N and a rational number $t = a/b$ such that $0 < t < \sigma$, then one can find r and s such that $r > s > N$, s is divisible by mb, and

$$r - ts > M \quad \text{and} \quad \frac{r+1}{s+1} < \sigma < \frac{r}{s}.$$

Proof: First we prove that ii) $\Rightarrow$ i). Since mb divides s, write $s = nmb$, where n is a positive integer. Since $r - ts > M$, we must have $r = ts + M + u$ for some integer $u \geq 1$. Hence, $r = nma + M + u$. Thus,

$$\begin{aligned}\frac{p}{q} > \frac{r+1}{s+1} &\Rightarrow p(s+1) > q(r+1)\\ &\Rightarrow sp + p - q > qr\\ & nmbp + p - q > qnma + qM + qu\\ &\Rightarrow p - q(u+1) > qnma - nmbp + qM\\ &\Rightarrow \frac{p - q(u+1)}{nqb} > m\left(\frac{a}{b} - \frac{p}{q}\right) + \frac{M}{nb}.\end{aligned}$$

Now choose M sufficiently large so that

$$m\left(\frac{a}{b} - \frac{p}{q}\right) + \frac{M}{nb} \geq 0.$$

Hence, we conclude that

$$\frac{p - q(u + 1)}{nqb} > 0.$$

Thus, $p > q(u + 1)$, from which it follows that

$$\sigma = \frac{p}{q} > 2.$$

Next we show that i) $\Rightarrow$ ii). Let $r = np + 1$, $s = nq$. Choose $n = kmb$. Then $r - ts = n(p - tq) + 1 \to \infty$, as $k \to \infty$, since $p - tq > 0$. It is also easily seen that

$$\sigma > 2 \Rightarrow \frac{r + 1}{s + 1} < \sigma. \quad \square$$

Before discussing the next theorem, we need a few results.

Let $a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$ denote a continued fraction.

Let $a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = \frac{p_n}{q_n}$, then $p_n = Q(a_0, a_1, \ldots, a_n)$ and $q_n = Q(a_1, \ldots, a_n)$.

Unlike [1], we use $Q(a_0, a_1, \ldots, a_n)$ to denote Euler continuants, where each of p_n and q_n are expanded using Euler's rule ([1], p. 82). Also well known is that (see [1], p. 83),

$$Q(a_0, \ldots, a_n) = a_0 Q(a_1, \ldots, a_n) + Q(a_2, \ldots, a_n). \tag{*}$$

Remark 1: By Euler's rule, as $n \to \infty$, $p_n \to \infty$, $q_n \to \infty$, and $Q(a_2, \ldots, a_n) \to \infty$.

$$p_n - q_n = (a_0 - 1)q_n + Q(a_2, \ldots, a_n), \text{ by } (*).$$

We also know (see [1], p. 84) that

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}. \tag{**}$$

Let α be an irrational number.

Let $a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{1}{\alpha_{n+1}} = \alpha.$

Then $\alpha_{n+1} > 1$, and is irrational. Moreover (see [1], p. 89),

$$\alpha = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}}.$$

And, by (**), it follows that, if *n is even*, then

$$\frac{p_n}{q_n} < \alpha < \frac{p_{n-1}}{q_{n-1}}. \tag{***}$$

This brings us to Theorem 2.

Theorem 2: Suppose that σ is irrational, and $\sigma > 1$. Let $t = a/b$ be a fixed rational number and m a fixed positive integer. Given any $N > 0$, one can find positive integers r and s, with $r > s > N$, s is divisible by mb, satisfying $r - ts > M$, for any given M, where

$$0 < t < \frac{r + 1}{s + 1} < \sigma < \frac{r}{s}.$$

Proof: Let σ have a continued fraction representation as α above. By (***), we see that, for n even,

$$\frac{p_n}{q_n} < \alpha < \frac{p_{n-1}}{q_{n-1}}.$$

Let $r = mabMp_{n-1}$ and $s = mabMq_{n-1}$, then

$$r - ts = maM(bp_{n-1} - aq_{n-1}) > M, \text{ since } \frac{a}{b} < \sigma < \frac{p_{n-1}}{q_{n-1}}.$$

$$\frac{p_n}{q_n} - \frac{r+1}{s+1} = \frac{p_n - q_n - mabM}{(mabMq_{n-1} + 1)q_n} > 0 \text{ if } n >> 0, \text{ and } n \text{ is even.}$$

This follows from (**) and Remark 1 above, noting that m, a, b, and M are given and n is arbitrary. Also

$$br - as = mabM(bp_{n-1} - aq_{n-1}) > a - b.$$

The last inequality holds since $bp_{n-1} - aq_{n-1} \geq 1$ and $ab > a - b$; hence,

$$t < \frac{r+1}{s+1}.$$

This proves the theorem. □

Example 1: The following example shows that if the conditions in part ii) of Theorem 1 are relaxed, then the implication is false. Let $\sigma = 8/5$, $r = 5$, and $s = 3$, then $6/4 < \sigma < 5/3$.

Example 2: If $\sigma = (n+1)/n$, then it is easy to see that it is impossible to find r and s in Theorem 1, even under relaxed conditions. If $\sigma = p/q$, a careful examination of the proof shows that $p - q \geq 2$ is a necessary condition.

Remark 2: If $\sigma = 2$, then we can easily see that, for any $r/s > 2$, we must have $(r+1)/(s+1) \geq 2$. Hence, Theorem 1 fails in that case even in the relaxed form.

Acknowledgments

I would like to thank D. J. Lewis for many useful conversations. I also sincerely thank the referee for many suggestions that helped to improve the style of exposition.

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2. A. J. Sommese. "On the Adjunction Theoretic Structure of Projective Varieties; Proceedings of Conference in Gottingen, 1985. Springer Lecture Notes #1194.

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A COMBINATORIAL PROBLEM THAT AROSE IN BIOPHYSICS

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(Submitted December 1987)

The purpose of this note is to prove the following result that was conjectured by T. L. Hill ([1], [2], p. 148) in the course of his investigations of the "surface" properties of some long multi-stranded polymers.

Theorem: Let s be a positive integer, and for any nonnegative integer m, let $R(m)$ be the number of solutions, in *integers* $(m_1, \ldots, m_s)$ of the system

$$m_1 + \cdots + m_s = 0, \tag{1a}$$

$$|m_1| + \cdots + |m_s| = 2m. \tag{1b}$$

Then,

$$Q(\rho) := \sum_{m=0}^{\infty} R(m)\rho^m = (1 - \rho)^{-(s-1)} \sum_{k=0}^{s-1} \binom{s-1}{k}^2 \rho^k.$$

Proof: It is readily seen that $R(m)$ is the coefficient of $\rho^m t^0$ in

$$\left[\sum_{k=-\infty}^{\infty} t^k \rho^{|k|/2}\right]^s = [\rho^{1/2}t^{-1}/(1 - \rho^{1/2}t^{-1}) + 1 + \rho^{1/2}t/(1 - \rho^{1/2}t)]^s \tag{2}$$

$$= (1 - \rho)^s(1 - \rho^{1/2}t)^{-s}(1 - \rho^{1/2}t^{-1})^{-s}.$$

Thus, $Q(\rho)$ is the coefficient of t^0 in the right side of (2). Expanding the last two terms in the right side of (2) by Newton's binomial formula, and collecting the coefficient of t^0, we get

$$Q(\rho) = (1 - \rho)^s \sum_{k=0}^{\infty} \binom{s + k - 1}{s - 1}^2 \rho^k. \tag{3}$$

Using Euler's transformation for hypergeometric series (e.g., [3], Th. 21, p. 60), (3) can be expressed as the right-hand side of the Theorem. □

The same method of proof can be applied to treat the more general problem where the 0 at the left side of (1a) is replaced by a general integer i.

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1. T. L. Hill. "Effect of Fluctuating Surface Structure and Free Energy on the Growth of Linear Tubular Aggregates." *Biophysical J.* *49* (1986):1017-1031.
2. T. L. Hill. *Linear Aggregation Theory in Cell Biology.* New York: Springer-Verlag, 1987.
3. Earl D. Rainville. *Special Functions.* New York: Chelsea, 1971.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. Hillman

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-646 *Proposed by A. P. Hillman in memory of Gloria C. Padilla*

We know that $F_{2n} = F_n L_n = F_n(F_{n-1} + F_{n+1})$. Find m as a function of n so as to have the analogous formula $T_m = T_n(T_{n-1} + T_{n+1})$, where T_n is the triangular number $n(n + 1)/2$.

B-647 *Proposed by L. Kuipers, Serre, Switzerland*

Simplify

$$[L_{2n} + 7(-1)^n][L_{3n+3} - 2(-1)^n L_n] - 3(-1)^n L_{n-2}L_{n+2}^2 - L_{n-2}L_{n-1}L_{n+2}^3.$$

B-648 *Proposed by M. Wachtel, Zurich, Switzerland*

The Pell numbers P_n and Q_n are defined by

$$P_{n+2} = 2P_{n+1} + P_n,\ P_0 = 0,\ P_1 = 1;\ Q_{n+2} = 2Q_{n+1} + Q_n,\ Q_0 = 1 = Q_1.$$

Show that $(P_{4n}, P_{2n}^2 + 1, 3P_{2n}^2 + 1)$ is a primitive Pythagorean triple for n in $\{1, 2, \ldots\}$.

B-649 *Proposed by M. Wachtel, Zurich, Switzerland*

Give a rule for constructing a sequence of primitive Pythagorean triples (a_n, b_n, c_n) whose first few triples are in the table

n	1	2	3	4	5	6	7	8
a_n	24	28	88	224	572	1248	3276	7332
b_n	7	45	105	207	555	1265	3293	7315
c_n	25	53	137	305	797	1777	4645	10357

and which satisfy

$$|a_n - b_n| = 17,$$

$$a_{2n-1} + a_{2n} = 26P_{2n} = b_{2n-1} + b_{2n},$$

and $$c_{2n-1} + c_{2n} = 26Q_{2n}.$$

[P_n and Q_n are the Pell numbers of B-648.]

B-650 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome Italy & David Singmaster, Polytechnic of the South Bank, London, UK*

Let us introduce a pair of 1-month-old rabbits into an enclosure on the first day of a certain month. At the end of one month, rabbits are mature and each pair produces $k - 1$ pairs of offspring. Thus, at the beginning of the second month there is 1 pair of 2-month-old rabbits and $k - 1$ pairs of 0-month-olds. At the beginning of the third month, there is 1 pair of 3-month-olds, $k - 1$ pairs of 1-month-olds, and $k(k - 1)$ pairs of 0-month-olds. Assuming that the rabbits are immortal, what is their average age A_n at the end of the n^{th} month? Specialize to the first few values of k. What happens as $n \to \infty$?

B-651 *Proposed by L. Van Hamme, Vrije Universiteit, Brussels, Belgium*

Let $u_0, u_1, \ldots$ be defined by $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = u_{n+1} - u_n$. Also let p be a prime greater than 3, and for n in $X = \{1, 2, \ldots, p - 1\}$, let n^{-1} denote the v in X with $nv \equiv 1 \pmod{p}$. Prove that

$$\sum_{n=1}^{p-1} (n^{-1}u_{n+k}) \equiv 0 \pmod{p}$$

for all nonnegative integers k.

SOLUTIONS

Relationship between Variables

B-622 *Proposed by Philip L. Mana, Albuquerque, NM*

For fixed n, find all m such that $L_nF_m - F_{m+n} = (-1)^n$.

Solution by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

Using the Binet forms for L_n and F_m, after some simple manipulations, it can be shown that

$$S_{n,m} = L_nF_m - F_{m+n} = (-1)^n F_{m-n}.$$

It follows that $S_{n,m} = (-1)^n$ iff $F_{m-n} = 1$, that is $m = n - 1, n + 1, n + 2$.

Also solved by Paul S. Bruckman, Herta T. Freitag, C. Georghiou, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Amitabha Tripathi, and the proposer.

Multiple of L_n

B-623 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S(n) = \sum_{k=1}^{2n-1} L_{n+k}L_k.$$

Prove that $S(n)$ is an integral multiple of L_n for all positive integers n.

Solution by Sahib Singh, Clarion Univ. of Pennsylvania, Clarion, PA

Using the Binet form, $L_{n+k}L_k = L_{n+2k} + (-1)^k L_n$. Thus,

$$\begin{aligned}\sum_{k=1}^{2n-1} L_{n+k}L_k &= (L_{n+2} + L_{n+4} + \cdots + L_{5n-2}) - L_n\\ &= L_{5n-1} - L_{n+1} - L_n\\ &= L_{5n-1} - L_{n-1} - 2L_n.\end{aligned}$$

Since $L_{5n-1} - L_{n-1} = 5F_{2n}F_{3n-1} = 5L_nF_nF_{3n-1}$, $S(n) \equiv 0 \pmod{L_n}$ is true.

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, Amitabha Tripathi, and the proposer.

Multiple of F_n^2 or L_n^2

B-624 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$T_n = \sum_{i=1}^{n} L_{2(n+i)-1}.$$

For every positive integer n, prove that either $F_n | T_n$ or $L_n | T_n$.

Solution by Lawrence Somer, Washington, D.C.

We will prove the stronger result that either $F_n^2 | T_n$ or $L_n^2 | T_n$. By the solution to Problem B-605 on page 374 of the November 1988 issue of *The Fibonacci Quarterly*,

$$T_n = (L_{2n} - 2)(L_{2n} + 1).$$

We will show that either $F_n | (L_{2n} - 2)$ or $L_n | (L_{2n} - 2)$. The result will then follow.

It is well known that

$$L_{2n} = L_n^2 - 2(-1)^n \tag{1}$$

and

$$L_n^2 - 5F_n^2 = 4(-1)^n. \tag{2}$$

First, suppose that n is even. Then, by (1) and (2),

$$L_{2n} - 2 = L_n^2 - 4 = 5F_n^2.$$

Thus, $F_n^2 | (L_n - 2)$ if n is even.

Now, suppose that n is odd. Then, by (1),

$$L_{2n} - 2 = (L_n^2 + 2) - 2 = L_n^2,$$

and $L_n^2 \mid (L_{2n} - 2)$. Q.E.D.

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, and the proposer.

Recurrences for F_nP_n and L_nP_n

B-625 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let P_0, P_1, ... be the Pell numbers defined by

$$P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2} \text{ for } n \ge 2.$$

Let $G_n = F_nP_n$ and $H_n = L_nP_n$. Show that (G_n) and (H_n) satisfy

$$K_{n+4} - 2K_{n+3} - 7K_{n+2} - 2K_{n+1} + K_n = 0.$$

Solution by Amitabha Tripathi, SUNY, Buffalo, NY

Let us consider two second-order linear recurrence relations given by

$$x_{n+2} = ax_{n+1} + bx_n,\quad y_{n+2} = cy_{n+1} + dy_n,\quad n \ge 0,$$

with a, b, c, and d complex numbers with at least one of a, c nonzero. Then the sequence $\{z_n\} = \{x_ny_n\}$, $n \ge 0$, is also a linearly recurrent sequence of order at most four. In fact, for $n \ge 0$, we have

$$\begin{aligned}
z_{n+4} &= x_{n+4}y_{n+4} = (ax_{n+3} + bx_{n+2})(cy_{n+3} + dy_{n+2})\\
&= acz_{n+3} + bdz_{n+2} + ady_{n+2}(ax_{n+2} + bx_{n+1}) + bcx_{n+2}(cy_{n+2} + dy_{n+1})\\
&= acz_{n+3} + (bd + a^2d + bc^2)z_{n+2} + abdx_{n+1}(cy_{n+1} + dy_n)\\
&\qquad + bcdx_{n+2}\,\frac{y_{n+2} - dy_n}{c}\\
&= acz_{n+3} + (a^2d + 2bd + bc^2)z_{n+2} + abcdz_{n+1} - bd^2y_n(x_{n+2} - ax_{n+1})\\
&= acz_{n+3} + (a^2d + 2bd + bc^2)z_{n+2} + abcdz_{n+1} - b^2d^2z_n.
\end{aligned}$$

The result now follows with $a = b = d = 1$, $c = 2$ for each of the sequences $\{G_n\}$ and $\{H_n\}$.

Also solved by Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, C. Georghiou, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, Sahib Singh, and the proposer.

Generating Functions for F_nP_n and L_nP_n

B-626 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let G_n and H_n be as in B-625. Express the generating functions

$$G(z) = \sum_{n=0}^{\infty} G_nz^n \quad\text{and}\quad H(z) = \sum_{n=0}^{\infty} H_nz^n$$

as rational functions of z.

Solution by Amitabha Tripathi, SUNY, Buffalo, NY

It is well known (and follows easily from a Binet-type formula for the nth term of a linearly recurrent sequence) that, if

$$f_{n+k} + a_1 f_{n+k-1} + a_2 f_{n+k-2} + \cdots + a_k f_n = 0,$$

then the denominator of the rational expression for the generating function for the sequence f_n is given by the polynomial $(1 + a_1 z + a_2 z^2 + \cdots + a_k z^k)$. Thus,

$$(1 - 2z - 7z^2 - 2z^3 + z^4)K(z)$$

$$= K_0 + (K_1 - 2K_0)z + (K_2 - 2K_1 - 7K_0)z^2 + (K_3 - 2K_2 - 7K_1 - 2K_0)z^3,$$

where $K_{n+4} - 2K_{n+3} - 7K_{n+2} - 2K_{n+1} + K_n = 0$ $(n \geq 0)$. Hence,

$$G(z) = \frac{z - z^3}{1 - 2z - 7z^2 - 2z^3 + z^4} \quad \text{and} \quad H(z) = \frac{z + 4z^2 + z^3}{1 - 2z - 7z^2 - 2z^3 + z^4}.$$

Also solved by Paul S. Bruckman, Odoardo Brogia & Piero Filipponi, C. Georghiou, L. Kuipers, Y. H. Harris Kwong, Sahib Singh, and the proposer.

Integral Mean of Consecutive Cubes

B-627 *Proposed by Piero Filipponi, Fond U. Bordoni, Rome, Italy*

Let

$$C_{n,k} = (F_n^3 + F_{n+1}^3 + \cdots + F_{n+k-1}^3)/k.$$

Find the smallest k in $\{2, 3, 4, \ldots\}$ such that $C_{n,k}$ is an integer for every n in $\{0, 1, 2, \ldots\}$.

Solution by C. Georghiou, University of Patras, Greece

We find that

$$C_{n,k} = [F_{3n+3k-1} - F_{3n-1} + 6(-1)^{n+k}F_{n+k-2} - 6(-1)^n F_{n-2}]/10k.$$

Those k in $\{2, 3, 4, \ldots, 24\}$ such that $k|C_{0,k}$ are in the set $\{6, 9, 11, 19, 24\}$. The only k in the last set such that $k|C_{1,k}$ is $k = 24$. Therefore, the required smallest k is $k \geq 24$. From

$$C_{n+1,k} = C_{n,k} + (F_{n+k}^3 - F_n^3)/k,$$

we get

$$\begin{aligned} C_{n+1,24} &= C_{n,24} + (F_{n+24}^3 - F_n^3)/24 \\ &= C_{n,24} + (F_{n+24}^2 + F_{n+24}F_n + F_n^2)(F_{n+24} - F_n)/24 \\ &= C_{n,24} + 6L_{n+12}(F_{n+24}^2 + F_{n+24}F_n + F_n^2), \end{aligned}$$

from which it follows that the answer to the problem is $k = 24$.

Also solved by Paul S. Bruckman, L. Kuipers, Bob Prielipp, Sahib Singh, Amitabha Tripathi, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-430 *Proposed by Larry Taylor, Rego Park, NY (Corrected Version)*

Find integers j, k ($\neq 0, \pm 1, \pm 2$), m_i and n_i such that:

(A) $5F_{m_i}F_{n_i} = L_k + L_{j+i}$, for $i = 1, 5, 9, 13, 17, 21$;

(B) $5F_{m_i}F_{n_i} = L_k - L_{j+i}$, for $i = 3, 7, 11, 15, 19, 23$;

(C) $F_{m_i}L_{n_i} = F_k + F_{j+i}$, for $i = 1, 2, \ldots, 22, 23$;

(D) $L_{m_i}F_{n_i} = F_k - F_{j+i}$, for $i = 1, 3, \ldots, 21, 23$;

(E) $L_{m_i}L_{n_i} = L_k - L_{j+i}$, for $i = 1, 5, 9, 13, 17, 21$;

(F) $L_{m_i}L_{n_i} = L_{-k} + L_{j+i}$, for $i = 2, 4, 6, 8$;

(G) $L_{m_i}L_{n_i} = L_k + L_{j+i}$, for $i = 3, 7, 11, 15, 16, 18, 19, 20, 22, 23$;

(H) $L_{m_i}L_{n_i} = L_k + F_{j+i}$, for $i = 10$;

(I) $L_{m_i}F_{n_i} = L_k + F_{j+i}$, for $i = 12$;

(J) $5F_{m_i}F_{n_i} = L_k + F_{j+i}$, for $i = 14$.

H-433 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let $P_0, P_1, \ldots$ be the Pell numbers defined by

$$P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2.$$

Show that, for $n = 1, 2, \ldots,$

$$6(n+1)P_{n-1} + P_{n+1} \equiv (-1)^{n+1}(9n^2 - 7)F_{n+1} \pmod{27}.$$

H-434 *Proposed by Piero Filipponi & Odoardo Brugia, Rome, Italy*

Strange creatures live on a planet orbiting around a star in a remote galaxy. Such beings have three sexes (namely, sex A, sex B, and sex C) and are reproduced as follows:

(i) An individual of sex A (or simply A) generates individuals of sex C by parthenogenesis.

(ii) If A is fertilized by an individual of sex B, then A generates individuals of sex B.

(iii) In order to generate individuals of sex A, A must be fertilized by both an individual of sex B and an individual of sex C.

Find a closed form expression for the number T_n of ancestors of an individual of sex A in the n^{th} generation. Note that, according to (i), (ii), and (iii), A has three parents ($T_1 = 3$) and six grandparents ($T_2 = 6$).

SOLUTIONS

A Prize Problem

H-409 *Proposed by John Turner, University of Waikato, New Zealand (Vol. 25, no. 2, May 1987)*

The following arithmetic triangle has many properties of special interest to Fibonacci enthusiasts.

							1							
						1	1	1						
					1	2	2	2	1					
				1	3	4	5	4	3	1				
			1	4	7	10	11	10	7	4	1			
		1	5	11	18	24	26	24	18	11	5	1		
	1	6	16	30	46	58	63	58	46	30	16	6	1	
1	7	22	47	81	116	143	153	143	116	18	47	22	7	1

....

Denote the triangle by T, the i^{th} element in the n^{th} row by t_i^n, and the sum of elements in the n^{th} row by σ_n.

(i) Discover a rule to generate the next row from the previous rows.

(ii) Given your rule, prove the Fibonacci row-sum property, viz:

$$\sigma_n = 2\sum_{i=1}^{n-1} t_i^n + t_n^n = F_{2n}, \text{ for } n = 1, 2, \ldots,$$

where F_{2n} is a Fibonacci integer.

(iii) Discover and prove a remarkable functional property of the sequence of diagonal sequences, $\{d_i\}$:

$d_1 = 1 \quad 1 \quad 1 \quad 1 \quad 1 \ldots$

$d_2 = 1 \quad 2 \quad 3 \quad 4 \quad 5 \ldots$

$d_3 = 1 \quad 2 \quad 4 \quad 7 \quad 11 \ldots$

$d_4 = 2 \quad 5 \quad 10 \quad 18 \quad 30 \ldots$

$d_5 = 1 \quad 4 \quad 11 \quad 24 \quad 46 \ldots$

(iv) Discover another Fibonacci arithmetic triangle which has the same generating rule and other properties but with row-sums equal to F_{2n-1}, $n = 1, 2, \ldots$.

(v) Show how the numbers in the triangle are related to the dual-Zeckendorf theorem on integer representations, which states (see [1]) that every positive integer N has one and only one representation in the form

$$N = \sum_{i}^{k} e_i u_i,$$

where the e_i are binary digits, $e_i + e_{i+1} \neq 0$ for $1 \leq i < k$, and $\{u_i\} = 1, 2, 3, 5, \ldots$, the Fibonacci integers.

There are many interesting identities derivable from the triangle, relating the t_i^n with themselves, with the natural numbers and Fibonacci integers, and with the binomial coefficients. The proposer offers a prize of US$25 for the best list of identities submitted.

A final remark is that Pascal-T and Fibonacci-T triangles (see [2] and [3]) can curiously be linked to a common source. They both may be derived from studies of binary words whose digits have the properties of the e_i in (v) above.

References

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2. S. J. Turner. "Probability via the N^{th} Order Fibonacci-T Sequence." *Fibonacci Quarterly 17.1* (1979):23-28.
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Solution by Karl Dilcher, Halifax, Nova Scotia

(i) Claim: Each element in the n^{th} row of T is the sum of the three closest elements in the $(n-1)^{\text{th}}$ row minus the closest element in the $(n-2)^{\text{th}}$ row.

Proof: Let

$$G(z, t) := \frac{1}{1 - t(1 + z + z^2) + z^2 t^2} = \sum_{n=0}^{\infty} f_n(z) t^n. \tag{1}$$

The $f_n(z)$ are polynomials of degree $2n$, and we have the recursion

$$\begin{aligned} &f_0(z) = 1,\ f_1(z) = 1 + z + z^2, \text{ and} \\ &f_{n+1}(z) = (1 + z + z^2) f_n(z) - z^2 f_{n-1}(z). \end{aligned} \tag{2}$$

The $f_n(z)$ are self-inverse polynomials, i.e., $f_n(z) = z^{2n} f_n(1/z)$; hence, we can write

$$\begin{aligned} f_n(z) = t_1^{n+1} + t_2^{n+1} z + \cdots + t_n^{n+1} z^{n-1} + t_{n+1}^{n+1} z^n + t_n^{n+1} z^{n+1} + \cdots \\ + t_2^{n+1} z^{2n-1} + t_1^{n+1} z^{2n}. \end{aligned} \tag{3}$$

This, with (2), proves the claim.

(ii) The row-sums σ_n are obviously given by

$$\sigma_n = f_{n-1}(1),\ n = 1, 2, \ldots;$$

hence, by (2), the σ_n satisfy the recursion

$$\sigma_1 = 1,\ \sigma_2 = 3,\ \sigma_{n+1} = 3\sigma_n - \sigma_{n-1};$$

but this is the well-known recursion for the even-indexed Fibonacci numbers F_{2n}; hence, $\sigma_n = F_{2n}$ for $n = 1, 2, \ldots$.

(iii) Claim: The k^{th} differences of the sequence d_{k+1} are eventually all 1.

Proof: Obviously, the numbers $d_{k+1}(n)$ in the sequence d_{k+1} are the $(k+1)^{\text{th}}$ coefficients (counting from the constant coefficient upward) of the polynomials $f_n(z)$, as defined by (1). They can be found by taking the k^{th} derivative of $f_n(z)$:

$$d_{k+1}(n) = \frac{1}{k!} f_n^{(k)}(0). \tag{4}$$

We consider the generating function, see (1):

$$\frac{d^k}{dz^k} G(z, t)\Big|_{z=0} = \sum_{n=0}^{\infty} f_n^{(k)}(0) t^n. \tag{5}$$

To evaluate the left-hand side of (5), we use the partial fraction expansion

$$G(z, t) = \frac{1}{\sqrt{t^2 + 4t(1-t)^2}}\left[\frac{1}{z + \alpha(t)} - \frac{1}{z + \beta(t)}\right],$$

where

$$\alpha(t) := \frac{1}{2(1-t)} + \sqrt{\frac{1}{4(1-t)^2} + \frac{1}{t}},$$

$$\beta(t) := \frac{1}{2(1-t)} - \sqrt{\frac{1}{4(1-t)^2} + \frac{1}{t}},$$

which is easy to verify. Hence,

$$\frac{d^k}{dz^k} G(z, t) = \frac{(-1)^k k!}{\sqrt{t^2 + 4t(1-t)^2}}[(z + \alpha(t))^{-k-1} - (z + \beta(t))^{-k-1}],$$

$$\frac{d^k}{dz^k} G(z, t)\Big|_{z=0} = \frac{(-1)^k k!}{\sqrt{t^2 + 4t(1-t)^2}} \; \frac{(\beta(t))^{k+1} - (\alpha(t))^{k+1}}{(\alpha(t)\beta(t))^{k+1}}$$

$$= \frac{k!}{(1-t)^{k+1}} g_k(t),$$

where

$$g_k(t) = \frac{1}{2^{k+1}\sqrt{t^2 + 4t(1-t)^2}}[(t + \sqrt{t^2 + 4t(1-t)^2})^{k+1} - (t - \sqrt{t^2 + 4t(1-t)^2})^{k+1}]. \tag{6}$$

Hence, with (4) and (5), we have

$$\frac{1}{(1-t)^{k+1}} g_k(t) = \sum_{n=0}^{\infty} d_{k+1}(n) t^n. \tag{7}$$

It is easy to see from (6) that

$$g_k(1) = 1. \tag{8}$$

Using the binomial theorem, we can rewrite (6) as

$$g_k(t) = 2^{-k} \sum_{j=0}^{[k/2]} \binom{k+1}{2j+1} t^{k-j}(t + 4(1-t)^2)^j; \tag{9}$$

this shows that $g_k(t)$ is a polynomial of degree $k + [k/2]$.

The k^{th} difference $\Delta_k(n)$ of the sequence $\{d_{k+1}(n)\}_n$ is

$$\Delta_k(n) := \sum_{j=0}^{k} \binom{k}{j} (-1)^j d_{k+1}(n-j).$$

To evaluate it, we consider the generating function

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_k(n) t^n &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^j d_{k+1}(n-j) \right) t^n \\ &= \sum_{j=0}^{k} \binom{k}{j} (-1)^j \sum_{n=0}^{\infty} d_{k+1}(n-j) t^n \\ &= \sum_{j=0}^{k} \binom{k}{j} (-1)^j t^j \sum_{n=0}^{\infty} d_{k+1}(n-j) t^{n-j} \\ &= (1-t)^k g_k(t) \frac{1}{(1-t)^{k+1}} \quad \text{[by (7)]}.\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} \Delta_k(n) t^n = \frac{1}{1-t} g_k(t). \tag{10}$$

Finally, if we denote the coefficients of $g_k(t)$ by $a_j^{(k)}$, we get

$$\frac{1}{1-t} g_k(t) = \left(\sum_{n=0}^{\infty} t^n \right) \left(\sum_{j=0}^{k+[k/2]} a_j^{(k)} t^j \right) = \sum_{n=0}^{\infty} t^n \left(\sum_{j=0}^{n} a_j^{(k)} \right),$$

where we set $a_j^{(k)} := 0$ for $j > k + [k/2]$. Hence, by (8), the coefficients in the Taylor series for $g_k(t)/(1-t)$ are all 1 for $n \geq k + [k/2]$. Comparing coefficients on both sides of (10), we get

$$\Delta_k(n) = 1 \text{ for } n \geq k + [k/2],$$

which completes the proof.

(iv) Consider the generating function

$$\frac{1 - zt}{1 - t(1 + z + z^2) + z^2 t^2} = \sum_{n=0}^{\infty} h_n(z) t^n.$$

By multiplying both sides by the denominator of the left-hand side and comparing coefficients, we get the recursion

$$\begin{aligned}&h_0(z) = 1,\ h_1(z) = 1 + z^2, \text{ and} \\ &h_{n+1}(z) = (1 + z + z^2) h_n(z) - z^2 h_{n-1}(z).\end{aligned} \tag{11}$$

Hence, the coefficients of the $h_n(z)$ satisfy the same generating rule as those of $f_n(z)$, and we obtain the triangle

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 0 & 1 & & & \\
 & & 1 & 1 & 1 & 1 & 1 & & \\
 & 1 & 2 & 2 & 3 & 2 & 2 & 1 & \\
1 & 3 & 4 & 6 & 6 & 6 & 4 & 3 & 1 \\
 & & & & \vdots & & & &
\end{array}$$

If τ_n denotes the sum of the elements in the n^{th} row, we have, as in (i),

$$\tau_n = h_{n-1}(1),$$

so, by (11), the τ_n satisfy the recursion

$$\tau_1 = 1,\ \tau_2 = 2,\ \tau_{n+1} = 3\tau_n - \tau_{n-1},$$

and this is the recursion for the odd-indexed Fibonacci numbers F_{2n-1}; hence, $\tau_n = F_{2n-1}$ for $n = 1, 2, \ldots$.

Remark: K. B. Stolarsky considered the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}$$

and its difference analogue

$$2u(x, t+1) = u(x-1, t) + u(x, t) + u(x+1, t) - u(x, t-1)$$

which, after normalizing, leads to the triangle

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 1 & 2 & 1 & 2 & 1 & & \\
 & 1 & 3 & 2 & 3 & 2 & 3 & 1 & \\
1 & 4 & 4 & 4 & 5 & 4 & 4 & 4 & 1 \\
 & & & & \vdots & & & &
\end{array}$$

The generating rule for this triangle is very similar to that of T, namely, each element in the n^{th} row is the sum of the three closest elements in the $(n-1)^{\text{th}}$ row minus *twice* the closest element in the $(n-2)^{\text{th}}$ row. The generating function in this case is

$$[1 - t(1 + z + z^2) + 2z^2t^2]^{-1},$$

and the sum of the entries in the n^{th} row is $2^n - 1$.

Stolarsky also suggested to study the general case

$$[1 - t(1 + z + z^2) + \lambda z^2 t^2]^{-1},$$

where the corresponding triangle is generated as before, with the difference that λ *times* the closest element in the $(n-2)^{\text{th}}$ row is subtracted. This was carried out in [1], in a slightly more general setting. The sum of the entries of the n^{th} row turns out to be

$$\frac{\lambda^{n/2}}{\sqrt{9-4\lambda}}\left\{\left(\frac{3+\sqrt{9-4\lambda}}{2\sqrt{\lambda}}\right)^n - \left(\frac{3-\sqrt{9-4\lambda}}{2\sqrt{\lambda}}\right)^n\right\}.$$

For $\lambda = 1$, the Fibonacci connection becomes apparent again, since

$$(3 \pm \sqrt{5})/2 = ((1 \pm \sqrt{5})/2)^2.$$

Asymptotic formulas for the elements in the columns of the triangle are also given in [1]. For example, for $\lambda < 9/4$, the column elements in the (general) triangle are asymptotically

$$\frac{1}{2\sqrt{\pi(n-1)}}(9-4\lambda)^{-1/4}\left(\frac{3+\sqrt{9-4\lambda}}{2}\right)^n.$$

regardless of which column is considered; n denotes the row, numbered as in the problem. In particular, for $\lambda = 1$, this is

$$\frac{1}{2\sqrt{\pi(n-1)}}\;5^{-1/4}\left(\frac{3+\sqrt{5}}{2}\right)^n \sim \frac{5^{1/4}}{2\sqrt{\pi(n-1)}}\;F_{2n}.$$

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Also solved by J.-Z. Lee & J.-S. Lee and the proposer.

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