The Fibonacci Quarterly

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# The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
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*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

# THE ARITHMETIC OF POWERS AND ROOTS IN $Gl_2(C)$ AND $Sl_2(C)$

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(Submitted June 1986)

## Introduction

Let $A$ be in $\mathrm{Sl}_2(\mathbf{C})$ ("The special linear group of degree 2 over $\mathbf{C}$"; see [5]) and let $n$ be a positive integer. Let us look at all $B$'s in $\mathrm{Sl}_2(\mathbf{C})$ for which $B^n = A$.

If $\chi = \mathrm{Tr}\, A \neq \pm 2$, then $A$ is diagonalizable since it has two different eigenvalues, namely, $(\chi \pm \sqrt{\chi^2 - 4})/2$, and it is trivial to compute all $n^{\text{th}}$ roots of $A$.

If $A$ is the identity matrix and if $\delta$ is an eigenvalue of some $n^{\text{th}}$ root $B$ of $A$, then, unless $\delta = \pm 1$, the other eigenvalue of $B$ is different (as it is $1/\delta$, the determinant being 1) and therefore $B$ is diagonalizable, that is, $B$ is a conjugate of $\begin{pmatrix} \delta & 0 \\ 0 & 1/\delta \end{pmatrix}$, with $\delta$ an $n^{\text{th}}$ root of 1; note that when $\delta = \pm 1$, it is easy to check that $B$ is $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The case $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is similar.

Finally, if $\chi = \pm 2$, but $A \neq \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the problem is slightly more difficult; it turns out that there are either 0, 1, or 2 $n^{\text{th}}$ root(s) in $\mathrm{Sl}_2(\mathbf{C})$, depending on $n$ and $A$.

If $A \in \mathrm{Gl}_2(\mathbf{C})$ ("The general linear group of degree 2 over $\mathbf{C}$"; see [5]) is not a multiple of the identity, then $A$ has exactly $n$ $n^{\text{th}}$ roots. If $A$ is a multiple of the identity, then $A$ has infinitely many $n^{\text{th}}$ roots for any $n$.

Although we will compute roots in $\mathrm{Sl}_2(\mathbf{C})$ and $\mathrm{Gl}_2(\mathbf{C})$, the immediate purpose of this paper *is not* to compute roots in these groups. Our purpose is to give a *nonlinear-algebra* approach to computing roots which rests on the arithmetic involved in computing the powers of an element of $\mathrm{Sl}_2(\mathbf{C})$ or $\mathrm{Gl}_2(\mathbf{C})$. Computing these powers involves a finite number of multiplications and additions; this gives rise to polynomials and the arithmetic of these polynomials yields another method to compute roots in $\mathrm{Sl}_2(\mathbf{C})$ *without any linear-algebra concept*. We obtain a complete description of these roots in this way, with transcendental functions in expressions not naturally given by the linear-algebra approach [see, e.g., (1.14-C)]. We will explore this arithmetic and see how it connects most naturally with Chebyshev's polynomials. It also yields a *natural* meaning to *arbitrary complex powers* in $\mathrm{Sl}_2(\mathbf{C})$ and $\mathrm{Gl}_2(\mathbf{C})$, and we obtain an explicit formula allowing computations of $A$ for any $n$ in a time which theoretically does not depend on $n$ [see (1.6), (1.8), and (2.1)]. As far as computing roots is concerned, the arithmetic of these polynomials gives an elegant nonlinear-algebra solution which solves the problem of extracting roots in all cases in the same way, be the matrix diagonalizable or not.

Computing roots of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is achieved first through computing roots of

$A/\sqrt{ad - bc}$ [which is trivially in $\mathrm{Sl}_2(\mathbf{C})$];

therefore, we first study the arithmetic of powers and roots in $\mathrm{Sl}_2(\mathbf{C})$. It rests on two families of polynomials; if $\chi$ and $\chi_n$ are, respectively, the traces of $A$ and

$$A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix},$$

then $\chi_n$ is a polynomial in $\chi$ which depends only on $n$ and not on $A$. In addition, there is a polynomial $P_n$ which gives the values of $b_n$ and $c_n$ through

$$b_n = bP_n(\chi) \quad \text{and} \quad c_n = cP_n(\chi).$$

These polynomials are deeply related to Chebyshev's polynomials and a full description of their zeros yields a full description of all roots of any element of $\mathrm{Sl}_2(\mathbf{C})$. The $P_n$'s, which appear naturally in our problem have been considered more or less directly in some other contexts (see [1], [3], and [4]).

The $P_n$'s have received much attention, but as far as we know the $\chi_n$'s have received little; the computation of roots in $\mathrm{Sl}_2(\mathbf{C})$ has also received little attention because in most practical cases there is an obvious linear-algebra solution (which however masks the arithmetic behind the calculations). As far as the raw computation of roots is concerned, we found a vague and partial answer in [6] which triggered our investigation, and an exercise in [2] coming from [7] which concerns the sole case when $A$ is hermitian and $n = 2$. We are thankful to Professor G. Bergum for bringing to our attention references [3] and [4] regarding the $P_n$'s.

## Powers and Roots in $\mathrm{Sl}_2(\mathbf{C})$

The starting point of this paper is the following family of polynomials: for each $n \in \mathbf{Z}$, we define a polynomial $P_n$ by

$$\text{(a)} \quad P_0(t) = 0 \text{ and } P_1(t) = 1; \quad \text{(b)} \quad P_{n+1}(t) = tP_n(t) - P_{n-1}(t) \tag{1.1}$$

These polynomials have the easily verified properties:

$$\text{a)} \quad P_n(\pm 2) = n(\pm 1)^{n+1}; \quad \text{b)} \quad P_{-n} = -P_n. \tag{1.2}$$

Their roots are studied in [3] and [4], where $P_n = A_{2n}$ in their notation. The following proposition, the proof of which is an easy induction on $n$ using $ad - bc = 1$ [this matrix is in $\mathrm{Sl}_2(\mathbf{C})$], ignited our interest in this family of polynomials; we lately discovered a more general version of it in [1], but we state in Proposition 1 just the particular case we need.

*Proposition 1:* Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{Sl}_2(\mathbf{C})$ and let us set $\chi = a + d$. Then, for each $n \in \mathbf{Z}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} aP_n(\chi) - P_{n-1}(\chi) & bP_n(\chi) \\ cP_n(\chi) & dP_n(\chi) - P_{n-1}(\chi) \end{pmatrix} \tag{1.3}$$

*Corollary:* For $A \in \mathrm{Sl}_2(\mathbf{C})$ and $n \in \mathbf{Z}$, if $\chi$ and $\chi_n$ are, respectively, the trace of $A$ and $A^n$, then

$$\chi_n = P_{n+1}(\chi) - P_{n-1}(\chi). \tag{1.4}$$

Using (1.4) as a motivation, we introduce yet another family of polynomials: for each $n \in \mathbf{Z}$, we set

$$\chi_n(t) = P_{n+1}(t) - P_{n-1}(t);$$

each $\chi_n$ is a polynomial of degree $|n|$; moreover $\chi_n = \chi_{-n}$, as is easily checked from $P_n = -P_{-n}$. The table in the Appendix shows these polynomials for all values of $n$ in the range $2 \le n \le 20$.

We shall need the zeros of all polynomials of the form $\chi_n - \xi$, with $\xi \in \mathbf{C}$. Fortunately, these zeros are easy to describe and, surprisingly, this result seems to be new.

*Proposition 2:* Let $n > 0$ and $\xi$ be an arbitrary complex number, and let us set $\rho = \xi/2$. Then the $n$ complex numbers $\xi_0, \xi_1, \ldots, \xi_{n-1}$ defined by

$$\xi_k = 2 \cosh\left(\frac{\operatorname{argcosh} \rho + 2k\pi i}{n}\right) = 2 \cos\left(\frac{\arccos \rho + 2k\pi}{n}\right) \tag{1.5}$$

are the zeros of $\chi_n - \xi$ $(k = 0, \ldots, n - 1)$.

*Proof:* If $T_n$ is the $n^{\text{th}}$ Chebyshev polynomial of the first kind (see [8] or [9]), then one easily proves that the $T_n$'s are defined in terms of the $P_n$'s by

$$2T_n(t) = P_{n+1}(2t) - P_{n-1}(2t). \tag{1.6}$$

Since we look for the solutions of

$$\frac{\chi_n}{2} = \frac{\xi}{2}, \text{ or equivalently of } \frac{P_{n+1}(x) - P_{n-1}(x)}{2} = \rho,$$

when we set $x = 2s$, the problem reduces, using (1.6), to solving $T_n(s) = \rho$; using the identities $T_n(\cos\theta) = \cos(n\theta)$ and $T_n(\cosh\theta) = \cosh(n\theta)$, we see that $T_n(s) = \rho$ has $n$ solutions, which are given by

$$s_k = \cosh\left(\frac{\operatorname{argcosh} \rho + 2k\pi i}{n}\right) = \cos\left(\frac{\arccos \rho + 2k\pi}{n}\right), \tag{1.7}$$

where $k = 0, \ldots, n - 1$ (simply write $T_n(s)$ as $T_n(\cos \arccos s) = \rho \ldots$]. These solutions yield the solutions $\xi_k = 2s_k$. Q.E.D.

*Remark:* It follows from (1.6) that the value of the $n^{\text{th}}$ Chebyshev polynomial at any complex number $s$ is the half-trace of $A^n$, where $A$ is *any* element in $\mathbf{Sl}_2(\mathbf{C})$ with half-trace $s$. Therefore, if $s_n$ is the half-trace of the $n^{\text{th}}$ power of an element of $A$ in $\mathbf{Sl}_2(\mathbf{C})$ with half-trace $s$, we have

$$s_n = \cosh(n \operatorname{argcosh} s) = \cos(n \arccos s).$$

This is an easy exercise in linear algebra since given $A$ there exists an invertible matrix $X$ such that

$$XAX^{-1} = \begin{pmatrix} \delta & * \\ 0 & 1/\delta \end{pmatrix}.$$

Because the trace is invariant under conjugation, we have

$$s = \cosh(\ln \delta) \quad \text{and} \quad s_n = \cosh(\ln \delta^n) = \cosh(n \operatorname{argcosh} s).$$

Next we need an explicit description of the zeros of the $P_n$'s. These are known (see [3] and [4], where $A_{2n}$ in their notation is our $P_n$) but our proof is simpler and yields an explicit expression for the values of the $P_n$'s [see (1.8) below] which is used in proving Proposition 5.

*Proposition 3:* For each integer $n$, the zeros of $P_n$ are

$$S_k = 2\cos(k\pi/|n|), \quad (k = 1, \ldots, |n| - 1).$$

In particular, they are all real and distinct.

*Proof:* In view of (1.2)-(b), we will suppose, in full generality, that $n > 0$. Using the easily proved identity

$$(s^2 - 1)P_n(2s) = T_{n+1}(s) - sT_n(s),$$

which defines the $P_n$'s in terms of the $T_n$'s, and the trivial identities (see [8])

$$s = \cosh(\operatorname{argcosh} s) \quad \text{and} \quad T_k(\cosh x) = \cosh(kx),$$

we have

$$(s^2 - 1)P_n(2s) = \sinh(\operatorname{argcosh} s)\sinh(n \operatorname{argcosh} s).$$

Upon writing $s = \cosh(\operatorname{argcosh} s)$, using the standard identities for hyperbolic functions and using the relation (1.2)-(a) to take care of the case $s = \pm 1$, we obtain the following explicit formula for $P_n(2s)$, which one will observe gives the value of $P_n$ without any iteration, and hence of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n$$

without iteration:

$$P_n(2s) = \begin{cases} ns^{n+1} & \text{if } s = \pm 1; \\ \dfrac{\sinh(n \operatorname{argcosh} s)}{\sinh(\operatorname{argcosh} s)} & \text{otherwise.} \end{cases} \tag{1.8}$$

[Note that, with $s \neq \pm 1$, the denominator of the lower part of (1.8) cannot be 0, and that the value of the quotient does not depend on which value is chosen for argcosh $s$.] It follows from (1.8) that the solutions of $P_n(2s) = 0$ are the values of $s$ for which ($n$ argcosh $s$) is a multiple of $\pi i$; these values are given by

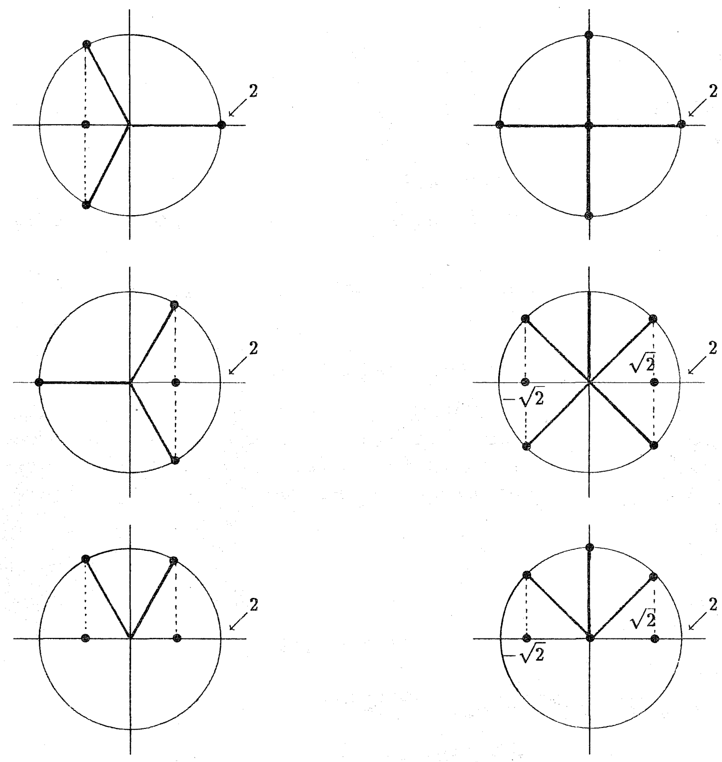
$$s_k = \cos\left(\frac{k\pi}{n}\right), \quad (k = 1, \ldots, n - 1),$$

whence the result

$$S_k = 2s_k = 2\cos\left(\frac{k\pi}{n}\right), \quad (k = 1, \ldots, n - 1). \quad \text{Q.E.D.}$$

It is interesting here to compare the zeros of $\chi_n \pm 2$ with those of $P_n$. From (1.5):

$$\left\{\begin{array}{ll} \text{The zeros of } \chi_n - 2\text{:} & \xi_0 = 2\cos 0,\ \xi_1 = 2\cos\frac{2\pi}{n},\ \ldots,\ \xi_{n-1} = 2\cos\frac{2(n-1)\pi}{n} \\ \text{The zeros of } \chi_n + 2\text{:} & \xi_0 = 2\cos\frac{\pi}{n},\ \xi_1 = 2\cos\frac{3\pi}{n},\ \ldots,\ \xi_{n-1} = 2\cos\frac{(2n-1)\pi}{n} \\ \text{The zeros of } P_n\text{:} & S_1 = 2\cos\frac{\pi}{n},\ S_2 = \cos\frac{2\pi}{n},\ \ldots,\ S_{n-1} = \cos\frac{(n-1)\pi}{n} \end{array}\right\}.$$




**Zeros**

$\chi_3 - 2: \; \xi_0 = 2, \; \xi_1 = -1, \; \xi_2 = -1$

$\chi_3 + 2: \; \xi_1 = 1, \; \xi_2 = -2, \; \xi_3 = 1$

$P_3: \; S_1 = 1, \; S_2 = -1$

**Zeros**

$\chi_4 - 2: \; \xi_0 = 2, \; \xi_1 = 0, \; \xi_2 = -2, \; \xi_3 = 0$

$\chi_4 + 2: \; \xi_0 = \sqrt{2}, \; \xi_1 = -\sqrt{2}, \; \xi_2 = -\sqrt{2}, \; \xi_3 = \sqrt{2}$

$P_4: \; S_1 = \sqrt{2}, \; S_2 = 0, \; S_3 = -\sqrt{2}$

FIGURE 1

The zeros of $\chi_n + 2$, $\chi_n - 2$, and $P_n$

Figure 1 shows the cases $n = 3$ and $n = 4$ and illustrates the essential content of Corollary 1 below; for convenience, let us call the zeros of $\chi_n + 2$ and $\chi_n - 2$ "small zeros" when they are *strictly less* than 2 in absolute value. Then, we clearly have the following:

*Corollary 1:*

1. The small zeros of $\chi_n + 2$ and $\chi_n - 2$ are each of multiplicity 2.
2. The small zeros of $\chi_n + 2$ and the small zeros of $\chi_n - 2$ form two disjoint subsets, the union of which is the set of zeros of $P_n$.

*Corollary 2:* For any $n \in \mathbb{Z}$ and for any $\xi \neq \pm 2$, $P_n$ does not vanish on a zero of $\chi_n - \xi$.

Corollary 1 states something on the values of $P_n$ at the zeros of $\chi_n \pm 2$, and Corollary 2 on the values of $P_n$ at the zeros of $\chi_n - \xi$, with $\xi \neq \pm 2$. If we agree to say that a function $f$ separates $n$ points $z_1, \ldots, z_n$ when $f$ takes $n$ different values on $\{z_1, \ldots, z_n\}$, then Proposition 4 below completes the information of Corollaries 1 and 2. This proposition will be responsible for the fact that the nonmultiple of the identity in $\mathrm{Gl}_2(\mathbf{C})$ has *exactly n distinct* $n^{\text{th}}$ roots.

*Proposition 4:* For all $n \in \mathbb{Z}$ and all $\xi \neq \pm 2$, $P_n$ separates the $|n|$ zeros of $\chi_n - \xi$.

*Proof:* Since $P_{-n} = -P_n$ and $\chi_{-n} = \chi_n$, we may suppose, in all generality, that $n \geq 0$. The cases $n = 0$ and $n = 1$ are vacuously true because $\chi_0$ and $\chi_1$ have, respectively, 0 and 1 zero [recall that $\chi_0(t) = 2$ and $\chi_1(t) = t$]. Therefore, we suppose that $n \geq 2$.

In order to consider the value of $P_n$ on each of the zeros of $\chi_n - \xi$, let us set

$$a + bi = \frac{\operatorname{argcosh} \xi/2}{n}.$$

Saying that $\xi \neq \pm 2$ means that $a + bi$ is not a multiple of $\pi i/n$. The roots of $\chi_n - \xi$ are, after (1.5),

$$\xi_k = 2 \cosh\left(a + bi + \frac{2k\pi i}{n}\right) \quad (k = 0, \ldots, n - 1),$$

and therefore,

$$P_n(\xi_k) = P_n\left(2 \cosh\left(a + bi + \frac{2k\pi i}{n}\right)\right).$$

Now,

$$\cosh\left(a + bi + \frac{2k\pi i}{n}\right) \neq \pm 1,$$

since the contrary would imply that $a + bi$ is a multiple of $\pi i/n$. It follows from (1.8) that, for $r = 0, \ldots, n - 1$,

$$P_n(\xi_r) = \frac{\sinh n(a + bi)}{\sinh a \cos\left(b + \frac{2\pi r}{n}\right) + i \cosh a \sin\left(b + \frac{2\pi r}{n}\right)}. \tag{1.9}$$

$\left[\text{The denominator is the expansion of } \sinh\left(a + bi + \frac{2\pi r i}{n}\right).\right]$

If $a \neq 0$, then, from (1.9), $P_n$ separates all $\xi_r$, for the denominator takes $n$ different values, which are $n$ different points on the ellipse with center 0 going through $\sinh a$ and $i \cosh a$. On the other hand, if $a = 0$, $P_n$ cannot identify two $\xi_r$'s, for, in the case $a = 0$, (1.9) becomes

$$P_n(\xi_r) = \frac{\sin nb}{\sin\left(b + \frac{2r\pi}{n}\right)},$$

and $P_n$ identifying two $\xi_r$'s, say $\xi_h$ and $\xi_k$ (with $h \neq k$), would imply that

$$\sin\left(b + \frac{2\pi k}{n}\right) = \sin\left(b + \frac{2\pi h}{n}\right)$$

(because $\sin nb \neq 0$ by Corollary 2 to Proposition 3), which would imply in turn that $b$ is a multiple of $\pi/n$, contradicting the hypothesis. Q.E.D.

*Proposition 5:*

(a) The set of $n^{\text{th}}$ roots of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is made of all diagonal matrices $\begin{pmatrix} \delta & 0 \\ 0 & 1/\delta \end{pmatrix}$, where $\delta$ is an $n^{\text{th}}$ root of 1, and of all matrices

$$\begin{pmatrix} \cos \frac{k\pi}{n} + T & Y \\ Z & \cos \frac{k\pi}{n} - T \end{pmatrix} \tag{1.10}$$

where $Y$, $Z$, $T$, and $k$ satisfy the following constraints:

(C1): $T$ is any complex number and $YZ = -\left(T^2 + \sin^2 \frac{k\pi}{n}\right)$
[This means exactly that the determinant of (1.10) is 1];

(C2): $k$ is even and $1 \le k \le n - 1$.

(b) The set of $n^{\text{th}}$ roots of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\mathbf{Sl}_2(\mathbf{C})$ is made of all diagonal matrices $\begin{pmatrix} \delta & 0 \\ 0 & 1/\delta \end{pmatrix}$, where $\delta$ is an $n^{\text{th}}$ root of $-1$, and of all matrices of the form (1.10) satisfying constraint (C1) above, constraint (C2) being replaced by constraint (C3):

(C3): $k$ is odd and $1 \le k \le n - 1$.

*Proof:*

(a) Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be an $n^{\text{th}}$ root of the identity in $\mathbf{Sl}_2(\mathbf{C})$, and let $Q = \alpha + \delta$. By (1.3), we have

$$\begin{pmatrix} \alpha P_n(Q) - P_{n-1}(Q) & \beta P_n(Q) \\ \gamma P_n(Q) & \delta P_n(Q) - P_{n-1}(Q) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.11}$$

If $P_r(Q) = 0$, then $\beta = \gamma = 0$ and $\alpha = 1/\delta$, which implies that $\alpha$ is an $n^{\text{th}}$ root of 1, and we have a root which is a diagonal matrix $\begin{pmatrix} \delta & 0 \\ 0 & 1/\delta \end{pmatrix}$ as desired. We will therefore suppose that $P_n(Q) = 0$. From Proposition 3, we know that

$$Q = 2 \cos \frac{k\pi}{n} \text{ for some } k \text{ in } \{1, 2, \ldots, n - 1\}.$$

It is clear from (1.11) that $\beta$ and $\gamma$ obey no other constraints than $\alpha\delta - \beta\gamma = 1$. On the other hand, $\alpha$ and $\delta$ are determined by:

$$\text{(A)} \quad \alpha + \delta = Q; \qquad \text{(B)} \quad P_{n-1}(Q) = -1. \tag{1.12}$$

Using (1.8) to work out the value of $P_{n-1}(Q) = P_{n-1}(2 \cos k\pi/n)$ we obtain

$$P_{n-1}\ 2\left(\cos \frac{k\pi}{n}\right) = \frac{\sinh \frac{(n-1)k\pi i}{n}}{\sinh \frac{k\pi i}{n}} = \frac{\sin \frac{(n-1)k\pi}{n}}{\sin \frac{k\pi}{n}} = (-1)^{k+1}. \tag{1.13}$$

It follows from (1.12)-(B) and (1.13) that $k$ must be even. [Remark: the constraint "$k$ is even and $1 \le k \le n - 1$" in (C2) implies that in (1.10) $n \ge 3$; therefore, $P_2(Q) \neq 0$ and the identity matrix has no square roots of the form (1.10)]. Finally, (1.12)-(A) implies that if $P_n(Q) = 0$, then the diagonal of

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is of the form of the diagonal of (1.10). Moreover, constraint (C1) is satisfied since $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is in $\mathbf{Sl}_2(\mathbf{C})$. As matrices of the form (1.10) are clearly $n^{\text{th}}$ roots of the identity matrix (to see this, apply Proposition 1), we have all $n^{\text{th}}$ roots of the identity with trace a zero of $P_n$. This completes the proof of (a).

(b) The proof runs parallel to the proof of (a). The constraint (1.12)-(B) is to be replaced by $P_{n-1}(Q) = 1$, which, by (1.13), implies that $k$ is odd. [Remark: The fact that $k$ is odd allows $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ to have infinitely many roots of any order in $\mathrm{Sl}_2(\mathrm{C})$, as opposed to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which has only two square roots in $\mathbf{Sl}_2(\mathbf{C})$.] Q.E.D.

We now hold all the necessary results to give a complete description of all $n^{\text{th}}$ roots of any element of $\mathbf{Sl}_2(\mathbf{C})$.

*Theorem A:* Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{Sl}_2(\mathbf{C}),$$

$n$ be any positive integer, $t = (a - d)/2$, and $\chi = a + d$. Then the set of all $n^{\text{th}}$ roots of $A$ in $\mathbf{Sl}_2(\mathbf{C})$ is described as follows:

<u>Case 1</u>. $A = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The $n^{\text{th}}$ roots of $A$ are exactly the conjugates of $\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}$, where $\mu$ is an $n^{\text{th}}$ root of $\pm 1$. [Remark: When $A$ is the identity and $n = 2$,

$$\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix} = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is in the center of $\mathbf{Sl}_2(\mathbf{C})$ and thus has no proper conjugates; this is why the identity has only two square roots. Apart from this case $A$ has infinitely many $n^{\text{th}}$ roots for each $n$.]

<u>Case 2</u>. $A$ is not the identity and $\chi = 2$.

There are only one or two root(s), depending on the parity of $n$; this (these) root(s) is (are)

$$(\sigma/n)\begin{pmatrix} a + (n-1) & b \\ c & d + (n-1) \end{pmatrix}, \tag{1.14-A}$$

where $\sigma$ is $\pm 1$ if $n$ is even and $+1$ if $n$ is odd.

<u>Case 3</u>. $A \neq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ but $\chi = -2$.

There are no roots in $\mathbf{Sl}_2(\mathbf{C})$ if $n$ is even and only one root if $n$ is odd, in which case this root is

$$(1/n)\begin{pmatrix} a - (n-1) & b \\ c & d - (n-1) \end{pmatrix}. \tag{1.14-B}$$

<u>Case 4</u>. $\chi \neq \pm 2$.

There are exactly $n$ distinct $n^{\text{th}}$ roots. If we set

$$\mu_k = \left(\operatorname{argcosh}\frac{x}{2}\right) + 2k\pi i \quad \text{and} \quad M_k = \frac{\sinh \mu_k/n}{\sinh \mu_k},$$

then these $n^{\text{th}}$ roots are $A_0, \ldots, A_{n-1}$, where

$$A_k = \begin{pmatrix} \cosh \frac{\mu_k}{n} + tM_k & bM_k \\ cM_k & \cosh \frac{\mu_k}{n} - tM_k \end{pmatrix}. \tag{1.14-C}$$

*Proof of Theorem A:*

Throughout, $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ will represent an $n^{\text{th}}$ root of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = x + w$.

Case 1. We will consider only the case of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, as the case $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ follows immediately from it. We must prove that the set of conjugates of $\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}$, where $\mu$ is an $n^{\text{th}}$ root of 1, is the set of roots described by Proposition 5(a). Let us write $\mathscr{R}$ (for $\mathscr{R}$oot) for the set described by (1.5)-(a) and $\mathscr{C}$ (for $\mathscr{C}$onjugate) for the set of conjugates of $\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}$.

First, we need a detailed description of $\mathscr{C}$; a direct calculation yields

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \mu\alpha\delta - \dfrac{\beta\gamma}{\mu} & -2\alpha\beta \sinh(\ln \mu) \\ 2\gamma\delta \sinh(\ln \mu) & -\left(\beta\gamma\mu - \dfrac{\alpha\delta}{\mu}\right) \end{pmatrix}, \tag{1.15}$$

where $\alpha\delta - \beta\gamma = 1$. If we use the identity

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} \dfrac{x + w}{2} + \dfrac{x - w}{2} & y \\ z & \dfrac{x + w}{2} - \dfrac{x - w}{2} \end{pmatrix}$$

to rewrite (1.15), we obtain

$$\begin{pmatrix} \cosh(\ln \mu) + \Gamma \sinh(\ln \mu) & -2\alpha\beta \sinh(\ln \mu) \\ 2\gamma\delta \sinh(\ln \mu) & \cosh(\ln \mu) - \Gamma \sinh(\ln \mu) \end{pmatrix} \tag{1.16}$$

where $\Gamma = \alpha\delta + \beta\gamma$. If $\mu = e^{2K\pi i/n}$, $K = 0, \ldots, n - 1$, then (1.15) becomes

$$\begin{pmatrix} \cos \dfrac{2K\pi}{n} + i\Gamma \sin \dfrac{2K\pi}{n} & -2\alpha\beta i \sin \dfrac{2K\pi}{n} \\ 2\gamma\delta i \sin \dfrac{2K\pi}{n} & \cos \dfrac{2K\pi}{n} - i\Gamma \sin \dfrac{2K\pi}{n} \end{pmatrix}. \tag{1.17}$$

Matrix (1.17) characterizes the elements of $\mathscr{C}$ and entails the detailed description of $\mathscr{C}$ that we now use.

We first show that $\mathscr{C} \subset \mathscr{R}$. If $K = 0$, (1.17) is the identity which is trivially in $\mathscr{R}$. If $1 \le 2K \le (n - 1)$, it is trivial to show that (1.17) has the form (1.10) (see Proposition 5) by solving

$$\begin{pmatrix} \cos \dfrac{2K\pi}{n} + i\Gamma \sin \dfrac{2K\pi}{n} & -2\alpha\beta i \sin \dfrac{2K\pi}{n} \\ 2\gamma\delta i \sin \dfrac{2K\pi}{n} & \cos \dfrac{2K\pi}{n} - i\Gamma \sin \dfrac{2K\pi}{n} \end{pmatrix}$$
$$= \begin{pmatrix} \cos \dfrac{k\pi}{n} + T & Y \\ Z & \cos \dfrac{k\pi}{n} - T \end{pmatrix} \tag{1.18}$$

with $k$, $T$, $Y$, $Z$ as unknowns. Finally, if $n \le 2K \le 2(n - 1)$, then (1.17) is a matrix with inverse of the form (1.17) for a value of $K$ for which $0 \le 2K \le (n - 1)$; since $\mathscr{R}$ is closed for inversion, (1.17) is in $\mathscr{R}$.

We next show that $\mathscr{R} \subset \mathscr{C}$. All matrices $\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}$, where $\mu$ is an $n^{\text{th}}$ root of 1, are trivially in $\mathscr{C}$. Let us consider the system (1.18) with left-hand side as unknown (that is, $K$, $\alpha$, $\beta$, $\gamma$, $\delta$ are unknown) and $\Gamma$ set to $\alpha\delta + \beta\gamma$. Let us set $K = k/2$. [Note that the left-hand side of (1.18) is a typical member of $\mathscr{C}$, and that the right-hand side is a typical member of $\mathscr{R}$. Moreover, the left-hand side of (1.18) is the left-hand side of (1.15) rearranged.]

If $\sin(2K\pi/n) = 0$, the left-hand side of (1.18) is $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, which is trivially in $\mathscr{C}$ $\left[\text{note that } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ occurs only if } n \text{ is even and } K = n/2\right]$. Therefore, we will suppose that $\sin(2K\pi/n) \neq 0$. We wish to show that the elements of $\mathscr{R}$ of the form of the right-hand side of (1.10) are in $\mathscr{C}$, that is, that (1.18), with the left-hand side as unknown, has a solution. This is achieved through showing that the following system has a solution, where (b) comes from $i\Gamma = T$ [see (1.18)], and (C) and (D) from the nondiagonal terms of (1.18):

$$\left\{\begin{array}{ll} a) & \alpha\delta - \beta\gamma = 1 \\ \\ b) & \alpha\delta + \beta\gamma = \dfrac{T}{i \sin \frac{k\pi}{n}} \end{array}\right\} \text{ or } \left\{\begin{array}{ll} A) & \alpha\delta = \dfrac{T + i \sin \frac{k\pi}{n}}{2i \sin \frac{k\pi}{n}} \\ \\ B) & \beta\gamma = \dfrac{T - i \sin \frac{k\pi}{n}}{2i \sin \frac{k\pi}{n}} \end{array}\right\}$$

$$\left\{\begin{array}{ll} C) & \alpha\beta = \dfrac{-Y}{2i \sin \frac{k\pi}{n}} \\ \\ D) & \gamma\delta = \dfrac{Z}{2i \sin \frac{k\pi}{n}} \end{array}\right\}$$

and where

$$YZ + T^2 + \sin^2 \frac{k\pi}{n} = 0 \text{ (Constraint C1, Proposition 5).}$$

The subsystem $(A, B, C)$ has the following solution in terms of $\alpha$:

$$\left\{\begin{array}{l} \beta = \dfrac{-Y}{2\alpha i \sin \frac{k\pi}{n}} \\ \\ \gamma = \dfrac{-\alpha\left(T - i \sin \frac{k\pi}{n}\right)}{Y} \\ \\ \alpha = \dfrac{T + i \sin \frac{k\pi}{n}}{2\alpha i \sin \frac{k\pi}{n}} \end{array}\right\} \tag{1.19}$$

[Note that, if $Y = 0$, we may use (D) to express $\gamma$ in terms of $\alpha$, since $Y = Z = 0$ is possibly only when (1.18) is $\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}$, a case which is trivially in $\mathscr{C}$; therefore, we assume that $|Y| + |Z| \neq 0$ and, without loss of generality, that $Y \neq 0$.] Constraint

$$YZ + T^2 + \sin^2 \frac{k\pi}{n} = 0$$

precisely means that the solutions (1.19) are compatible with ($D$). Case 1 has thus been established.

Case 2. $A$ is not the identity but $\chi = 2$.

Then, by (1.2),

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} xP_n(X) - P_{n-1}(X) & yP_n(X) \\ zP_n(X) & wP_n(X) - P_{n-1}(X) \end{pmatrix}. \tag{1.20}$$

The Möbius transformation defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{PSl}_2(\mathbf{C})$ ("The projective special linear group of degree 2 over $\mathbf{C}$") has a unique fixed point as $\chi = 2$ (see [5]); therefore, the one defined by $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$, the $n^{\text{th}}$ iteration of which is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, has also a unique fixed point; thus, we have $X = \pm 2$. On the other hand,

$$\chi_n(-2) = -2 \text{ if } n \text{ is odd,}$$

as is easily checked. From $\chi_n(X) = \chi = 2$, we see that in the case where $n$ is odd we must have $X = 2$. Therefore, from (1.20) and

$$P_n(\pm 2) = n(\pm 1)^{n+1} = \begin{cases} n & \text{if } n \text{ is odd} \\ \pm n & \text{if } n \text{ is even} \end{cases}$$

we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \begin{pmatrix} nx - (n-1) & ny \\ nz & nw - (n-1) \end{pmatrix} & \text{if } n \text{ if odd,} \\[2ex] \begin{pmatrix} \pm nx - (n-1) & \pm ny \\ \pm nz & \pm nw - (n-1) \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

Solving then for $x$, $y$, $z$, $w$ in terms of $a$, $b$, $c$, $d$ yields

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{cases} (1/n)\begin{pmatrix} a + (n-1) & b \\ c & d + (n-1) \end{pmatrix} & \text{if } n \text{ is odd,} \\ (\pm 1/n)\begin{pmatrix} a + (n-1) & b \\ c & d + (n-1) \end{pmatrix} & \text{if } n \text{ is even,} \end{cases}$$

which is exactly (1.14-A).

Case 3. $A \neq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ but $\chi - 2$.

As in Case 2, we must have $X = \pm 2$; however, since

$$\chi_n(\pm 2) = 2(\pm 1)^{n+1} = \chi,$$

we must have $X = -2$ and $n$ odd. Moreover, we then have

$$P_n(X) = n \quad \text{and} \quad P_{n-1}(X) = -(n-1).$$

The result follows immediately from (1.20).

Case 4. $\chi \neq \pm 2$.

$X$ is a zero of $\chi_n - \chi$, say (see Proposition 2),

$$X = \xi_k = 2\cosh\frac{\operatorname{argcosh}(\chi/2) + 2k\pi i}{n} = 2\cos\frac{\arccos(\chi/2) + 2k\pi}{n}; \tag{1.21}$$

consequently, from

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}^n = \begin{pmatrix} xP_n(X) - P_{n-1}(X) & yP_n(X) \\ zP_n(X) & wP_n(X) - P_{n-1}(X) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we obtain the following possibilities for $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$:

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix} = \begin{pmatrix} \dfrac{a + P_{n-1}(\xi_k)}{P_n(\xi\ )} & \dfrac{b}{P_n(\xi_k)} \\ \dfrac{c}{P_n(\xi_k)} & \dfrac{d + P_{n-1}(\xi_k)}{P_n(\xi_k)} \end{pmatrix} \tag{1.22}$$

[(1.22) uses tacitly Corollary 2 of Proposition 3 in using $P\ (\xi\ )$ in the denominator.] We first show that each of the $n$ matrices defined by (1.22), ($k = 0, \ldots, n - 1$), is an $n^{\text{th}}$ root of $A$ in $\mathbf{Sl}_2(\mathbf{C})$ (see Lemma 1 below). Then we show that these matrices are all different (see Lemma 4 below, which requires Lemmas 2 and 3).

*Lemma 1:* $x_k w_k - y_k z_k = 1$ $\left[\text{the possible values of } \begin{pmatrix} x & y \\ z & w \end{pmatrix} \text{ obtained from Proposition 2 are all in } \mathbf{Sl}_2(\mathbf{C})\right]$.

*Proof:* Let us set

$$u = \frac{\operatorname{argcosh}(\chi/2) + 2k\pi i}{n}.$$

Then

$$\frac{1}{P_n(\xi_k)} = \frac{\sinh u}{\sinh nu} \text{ by (1.8) and (1.22), and } \frac{\xi_k}{2} = \cosh u \text{ by (1.22),}$$

which gives

$$\begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix} = \begin{pmatrix} \cosh u + t\dfrac{\sinh u}{\sinh nu} & b\dfrac{\sinh u}{\sinh nu} \\ c\dfrac{\sinh u}{\sinh nu} & \cosh u - t\dfrac{\sinh u}{\sinh nu} \end{pmatrix}.$$

Therefore,

$$x_k w_k - y_k z_k = \cosh^2 u - (bc + t^2)\frac{\sinh^2 u}{\sinh^2 nu},$$

but

$$bc + t^2 = bc + \frac{\chi^2 - 4ad}{4} = \left(\frac{\chi}{2}\right)^2 - 1 = \cosh^2 nu - 1 = \sinh^2 nu,$$

whence the result. This completes the proof of Lemma 1.

*Lemma 2:* $x_k + y_k = \xi_k$.

*Proof:* From (1.22), we have

$$x_k + y_k = \frac{\chi + 2P_{n-1}(\xi_k)}{P_n(\xi_k)} \tag{1.23}$$

But (see the Remark following Proposition 2),

$$\chi = 2T_n\left(\cosh\left(\frac{\operatorname{argcosh}(\chi/2) + 2k\pi i}{n}\right)\right) = 2T_n\left(\frac{\xi_k}{2}\right);$$

therefore, from (1.6), we have

$$\chi = P_{n+1}(\xi_k) - P_{n-1}(\xi_k),$$

which, by definition (1.1), gives

$$\chi = \xi_k P_n(\xi_k) - 2P_{n-1}(\xi_k).$$

Substituting this value of $X$ into (1.23) yields the result and completes the proof of Lemma 2.

*Lemma 3:* $\begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix} = \begin{pmatrix} \xi_k/2 + t/P_n(\xi_k) & b/P_n(\xi_k) \\ c/P_n(\xi_k) & \xi_k/2 - t/P_n(\xi_k) \end{pmatrix}$ $\left(\text{Recall: } t = \dfrac{a-d}{2}\right)$.

*Proof:* From (1.22) and Lemma 2, we have the linear system

$$\begin{cases} x_k + w_k = \xi_k \\ x_k - w_k = \dfrac{2t}{P_n(\xi_k)}, \end{cases}$$

the solution of which is the required result; thus, Lemma 3 is proved.

*Lemma 4:* The matrices $\begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix}$ $(k = 0, \ldots, n-1)$ are all different.

*Proof:* This is simply a consequence of Lemma 3 and Proposition 4, since $P_n$ separates the $\xi_k$'s. This completes the proof of Lemma 4, and Theorem 4 has thus been proved. Q.E.D.

*Remark:* The denominator of $M_k$ [see (1.14-C)], which is sinh $\mu_k$, with

$$\mu_k = \operatorname{argcosh} \frac{\chi}{2} + 2k\pi i,$$

does not depend on $k$ because, if we set $s = \chi/2$, we have

$$\sinh \mu_k = \pm\sqrt{s^2 - 1},$$

where the sign is chosen so as to agree with the principal value of argcosh $s$; note that $M_k$ does not depend on the choice of this principal value.

In the same fashion, we have

$$\sinh \frac{\mu_k}{n} = \pm\sqrt{s_k^2 - 1}$$

for the numerator of $M_k$ when we set $s_k = \cosh(\mu_k/n)$. Thus, we have

$$M_k = \pm\sqrt{s_k^2 - 1}/\sqrt{s^2 - 1},$$

and, clearly, only the numerator of this expression depends on $k$.

## Roots in $\mathbf{Gl}_2(\mathbf{C})$

Let us conclude with the computation of roots in $\mathbf{Gl}_2(\mathbf{C})$. For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{Gl}_2(\mathbf{C}),$$

let $\delta$ be one of the two square roots of det $A$; we will write $\delta_+$ for $\delta$, $\delta_-$ for $-\delta$, $A_+$ for $A/\delta_+$ and $A_-$ for $A/\delta_-$. Clearly, $A_+$ and $A_-$ are in $\mathbf{Sl}_2(\mathbf{C})$.

We first observe that the $n^{\text{th}}$ roots of $A$ in $\mathbf{Gl}_2(\mathbf{C})$ are elements of $\Phi B$ with:

$$\left\{\begin{matrix} \Phi \text{ an } n^{\text{th}} \text{ root of } \delta_+ \text{ and } B \text{ an } n^{\text{th}} \text{ root of } A_+ \\ \text{or} \\ \Phi \text{ an } n^{\text{th}} \text{ root of } \delta_- \text{ and } B \text{ an } n^{\text{th}} \text{ root of } A_- \end{matrix}\right\}. \tag{2.1}$$

It is clear that an element $\Phi B$ is an $n^{\text{th}}$ root of $A$, for

$$(\Phi B)^n = \Phi^n B^n = \left\{\begin{matrix} \delta_+ A_+ \\ \text{or} \\ \delta_- A_- \end{matrix}\right\} = A.$$

Conversely, all $n^{\text{th}}$ roots of $A$ are of this form, for let $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ be an $n^{\text{th}}$ root of $A$ and $\tau$ be one of the two square roots of $(xw - yz)$; then

$$A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}^n = \tau^n \begin{pmatrix} x/\tau & y/\tau \\ z/\tau & w/\tau \end{pmatrix}^n$$

from which we get

$$(1/\tau)^n A = \begin{pmatrix} x/\tau & y/\tau \\ z/\tau & w/\tau \end{pmatrix}^n.$$

The determinant of the right-hand side being

$$\left(\frac{xw - yz}{\tau^2}\right)^n = 1,$$

we have that $\tau^n = \delta_\pm$; thus,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \tau \begin{pmatrix} x/\tau & y/\tau \\ z/\tau & w/\tau \end{pmatrix}$$

is of the form $\Phi B$.

To obtain all $n^{\text{th}}$ roots of $A$, we shall compute all products $\Phi B$ with $\Phi$ and $B$ satisfying (2.1); note that, since $A_+$ and $A_-$ are in $\mathrm{Sl}_2(\mathbf{C})$, Theorem A gives all possible $B$'s. Let us agree that $\delta$ is one of the square roots of $(ad - bc)$ for which $(\Re \operatorname{Tr} A_+) \geq 0$.

We first suppose that $A$ *is not a multiple of the identity*. We consider separately three cases:

<u>Case A</u>. $\operatorname{Tr} A_+ = 2$ and $n$ is even (say $n = 2k$). By Case 2 of Theorem A, $A_+$ has two roots in $\mathrm{Sl}_2(\mathbf{C})$ which are of opposite signs [see (1.14-A)]; on the other hand, the roots $\Phi$ of $\delta_+$ come in pairs with opposite signs and there are $2k$ of them. If $\Phi_1, \ldots, \Phi_k, -\Phi_1, \ldots, -\Phi_k$ are the $n$ possible values for $\Phi$ and $A_0$ and $-A_0$ are two roots of $A_+$, then the set

$$\{\Phi_1, \ldots, \Phi_k, -\Phi_1, \ldots, -\Phi_k\}\{A_0, -A_0\} \tag{2.2}$$

contains $n$ elements.

On the other hand, $A_-$ has no $n^{\text{th}}$ root (see Case 3 of Theorem A); thus, in this case the products of the form

$$(\text{a root of } \delta_-)(\text{a root of } A_-) \tag{2.3}$$

contribute nothing. $A_-$ has therefore altogether $n$ distinct $n^{\text{th}}$ roots and these are the elements of the set (2.2).

<u>Case B</u>. $\operatorname{Tr} A_+ = 2$ and $n$ is odd.

Each of $A_+$ and $A_-$ has exactly one $n^{\text{th}}$ root in $\mathrm{Sl}_2(\mathbf{C})$ (Cases 2 and 3 of Theorem A), namely:

$$\text{The root of } A_+\text{:} \quad A_0 = \frac{1}{n}\begin{pmatrix} a/\delta_+ + (n-1) & b/\delta_+ \\ c/\delta_+ & d/\delta_+ + (n-1) \end{pmatrix}$$

$$\text{The root of } A_-\text{:} \quad -A_0 = \frac{1}{n}\begin{pmatrix} a/\delta_- - (n-1) & b/\delta_- \\ c/\delta_- & d/\delta_- - (n-1) \end{pmatrix}$$

(since $\delta_+ = -\delta_-$, these two roots are of opposite signs). If $r = |\delta|$ and $\theta$ is the argument of $\delta_+$, then the $n^{\text{th}}$ roots of $\delta_+$ and $\delta_-$ are

for $\delta_+$: $re^{i\theta/n}\{\sigma_0, \ldots, \sigma_{n-1}\}$,

for $\delta_-$: $re^{i(\theta+\pi)/n}\{\sigma_0, \ldots, \sigma_{n-1}\}$,

where $\sigma_0, \ldots, \sigma_{-1}$ are the $n$ $n^{\text{th}}$ roots of 1. Note that the second set is the first set multiplied by $-1$. Therefore, the $n^{\text{th}}$ roots of $A$ form the union of the following two sets:

$$X_1 = re^{i\theta/n}\{\sigma_0, \ldots, \sigma_{n-1}\}A_0$$
$$X_2 = re^{i\theta/n}\{-\sigma_0, \ldots, -\sigma_{n-1}\}(-A_0)$$

Clearly, $X_1 = X_2$ and their union contains exactly $n$ elements.

Case C. Tr $A_+ \neq 2$.

Let $\sigma_0, \ldots, \sigma_{-1}$ be the $n$ $n^{\text{th}}$ roots of 1, and let $B$ be one of the $n$ $n^{\text{th}}$ roots of $A_+$ [see Case 4 of Theorem A, (1.14)-C)]. Then $\sigma_0 B, \ldots, \sigma_{n-1}B$ are all distinct and each of them is an $n^{\text{th}}$ root of $A_+$ since $(\sigma_k B)^n = B^n = A_+$. It follows from Theorem A, Case 4, that $\sigma_0 B, \ldots, \sigma_n B$ are the $n$ roots of $A_+$, and therefore that the set of elements of the form

$$(\text{a root of } \delta_+)(\text{a root of } A_+) \tag{2.4}$$

is, using the notation of Case B,

$$re^{i\theta/n}\{\sigma_0, \ldots, \sigma_{n-1}\}\{\sigma_0 B, \ldots, \sigma_{n-1}B\},$$

which is the set

$$re^{i\theta/n}\{\sigma_0 B, \ldots, \sigma_{n-1}B\}; \tag{2.5}$$

this set contains $n$ elements.

If $\sigma$ is any $n^{\text{th}}$ root of -1, a similar argument yields

$$re^{i\theta/n}\{\sigma\sigma_0, \ldots, \sigma\sigma_{n-1}\}\{\sigma\sigma_0 B, \ldots, \sigma\sigma_{n-1}B\}$$

for the set of elements of the form (2.3). This is

$$re^{i\theta/n}\sigma^2\{\sigma_0, \ldots, \sigma_{n-1}\}\{\sigma_0 B, \ldots, \sigma_{n-1}B\},$$

which contains exactly $n$ distinct elements. Now, since $\sigma$ is an $n^{\text{th}}$ root of -1, $\sigma^2$ is an $n^{\text{th}}$ root of 1; then $\sigma^2$ is one of $\sigma_0, \ldots, \sigma_{n-1}$, which implies that the set of elements of the form (2.3) is described by (2.5), which is already the set of elements of the form (2.4). Therefore, $A$ has exactly $n$ distinct $n^{\text{th}}$ roots in $\mathrm{Gl}_2(\mathbf{C})$.

The case when $A$ *is a (nonzero) multiple of the identity* is immediate; $A$ has infinitely many $n^{\text{th}}$ roots, for if $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, then $A = -a\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ has infinitely many $n^{\text{th}}$ roots for each $n$ (see Theorem A, Case 1). Hence, we have proved the following theorem, which is our conclusion.

*Theorem B:* Let $A$ be in $\mathrm{Gl}_2(\mathbf{C})$.

a) If $A$ is a nonzero multiple of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $A$ has infinitely many $n^{\text{th}}$ roots;

b) If $A$ is not a multiple of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $A$ has exactly $n$ distinct $n^{\text{th}}$ roots. They are of the form $\Phi B$ satisfying (2.1).

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## Appendix

Polynomials $P_n$ and $\chi_n$ for $2 \le n \le 20$

| $n$ | $P_n$<br>$\chi_n$ |
|---|---|
| 2 | $x$<br>$x^2 - 2$ |
| 3 | $x^2 - 1$<br>$x^3 - 3x$ |
| 4 | $x^3 - 2x$<br>$x^4 - 4x^2 + 2$ |
| 5 | $x^4 - 3x^2 + 1$<br>$x^5 - 5x^3 + 5x$ |
| 6 | $x^5 - 4x^3 + 3x$<br>$x^6 - 6x^4 + 9x^2 - 2$ |
| 7 | $x^6 - 5x^4 + 6x^2 - 1$<br>$x^7 - 7x^5 + 14x^3 - 7x$ |
| 8 | $x^7 - 6x^5 + 10x^3 - 4x$<br>$x^8 - 8x^6 + 20x^4 - 16x^2 + 2$ |
| 9 | $x^8 - 7x^6 + 15x^4 - 10x^2 + 1$<br>$x^9 - 9x^7 + 27x^5 - 30x^3 + 9x$ |
| 10 | $x^9 - 8x^7 + 21x^5 - 20x^3 + 5x$<br>$x^{10} - 10x^8 + 35x^6 - 50x^4 + 25x^2 - 2$ |
| 11 | $x^{10} - 9x^8 + 28x^6 - 35x^4 + 15x^2 - 1$<br>$x^{11} - 11x^9 + 44x^7 - 77x^5 + 55x^3 - 11x$ |
| 12 | $x^{11} - 10x^9 + 36x^7 - 56x^5 + 35x^3 - 6x$<br>$x^{12} - 12x^{10} + 54x^8 - 112x^6 + 105x^4 - 36x^2 + 2$ |
| 13 | $x^{12} - 11x^{10} + 45x^8 - 84x^6 + 70x^4 - 21x^2 + 1$<br>$x^{13} - 13x^{11} + 65x^9 - 156x^7 + 182x^5 - 91x^3 + 13x$ |
| 14 | $x^{13} - 12x^{11} + 55x^9 - 120x^7 + 126x^5 - 56x^3 + 7x$<br>$x^{14} - 14x^{12} + 77x^{10} - 210x^8 + 294x^6 - 196x^4 + 49x^2 - 2$ |
| 15 | $x^{14} - 13x^{12} + 66x^{10} - 165x^8 + 210x^6 - 126x^4 + 28x^2 - 1$<br>$x^{15} - 15x^{13} + 90x^{11} - 275x^9 + 450x^7 - 378x^5 + 140x^3 - 15x$ |
| 16 | $x^{15} - 14x^{13} + 78x^{11} - 220x^9 + 330x^7 - 252x^5 + 84x^3 - 8x$<br>$x^{16} - 16x^{14} + 104x^{12} - 352x^{10} + 660x^8 - 672x^6 + 336x^4 - 64x^2 + 2$ |
| 17 | $x^{16} - 15x^{14} + 91x^{12} - 286x^{10} + 495x^8 - 462x^6 + 210x^4 - 36x^2 + 1$<br>$x^{17} - 17x^{15} + 119x^{13} - 442x^{11} + 935x^9 - 1122x^7 + 714x^5 - 204x^3 + 17x$ |
| 18 | $x^{17} - 16x^{15} + 105x^{13} - 364x^{11} + 715x^9 - 792x^7 + 462x^5 - 120x^3 + 9x$<br>$x^{18} - 18x^{16} + 135x^{14} - 546x^{12} + 1287x^{10} - 1782x^8 + 1386x^6 - 540x^4 + 81x^2 - 2$ |
| 19 | $x^{18} - 17x^{16} + 120x^{14} - 455x^{12} + 1001x^{10} - 1287x^8 - 924x^6 - 330x^4 + 45x^2 - 1$<br>$x^{19} - 19x^{17} + 152x^{15} - 665x^{13} + 1729x^{11} - 2717x^9 + 2508x^7 - 1254x^5 + 285x^3 - 19x$ |
| 20 | $x^{19} - 18x^{17} + 136x^{15} - 560x^{13} + 1365x^{11} - 2002x^9 + 1716x^7 - 792x^5 + 165x^3 - 10x$<br>$x^{20} - 20x^{18} + 170x^{16} - 800x^{14} + 2275x^{12} - 4004x^{10} + 4290x^8 - 2640x^6 + 825x^4 - 100x^2 + 2$ |

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# DERIVATION OF A FORMULA FOR $\sum r^k x^r$

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## 1. Introduction

Many different approaches have been proposed to evaluate the sums of powers of consecutive integers,

$$\sum_{r=0}^{n} r^k.$$

Interest in these sums is very old: the Greeks, the Hindus, and the Arabs had rules for the first few cases. Modern interest in these sums goes back more than 350 years to Faulhaber's (1631) "Academia algebrae." Fermat (1636), Pascal (1654), Bernoulli (1713), Jacobi (1834), and many others have also considered this question. Recent contributions are due to Sullivan [1], Edwards [2], Scott [3], and Khan [4]. Sullivan uses a simple and elegant recursion formula to study this problem. Edwards and Scott make use of a matrix formulation which is very intimately connected to Pascal's triangle and the binomial theorem. Khan introduces a simple integral approach that can be presented in all generality with just a basic knowledge of calculus. The interested reader will find a textbook account in Jordan [5], for example.

The purpose of the present note is to study sums of the type

$$\sum_{r=0}^{n} r^k x^r,$$

where $n$, $k \geq 0$ are integers and $x$ is an arbitrary parameter (real or complex). The sums of powers of consecutive integers can be obtained from our results, as a special case, by letting $x \to 1$. But since the latter sums ($x = 1$) have been studied extensively in the literature, the main emphasis of the present note will be on the former sums ($x \neq 1$).

## 2. A Method for Evaluating $\sum r^k x^r$

In this section, we present a calculus-based method for evaluating $\sum r^k x^r$. To our knowledge, this approach has not been discussed before. An alternative approach is to use Sullivan's technique [1] by setting $a_r = x^r$, instead of $a_r = 1$, in his expressions. However, after examination, it was found that this approach is not analytically as transparent as the present approach; thus, the details are not reported here.

Let $x \neq 1$ be an arbitrary real or complex parameter, and note the following identity,

$$\sum_{r=0}^{n} x^r = (1 - x^{n+1})/(1 - x). \tag{1}$$

By $k$ successive applications of the differential operator $D = xd/dx$ to both sides of (1), we immediately obtain

$$\sum_{r=0}^{n} r^k x^r = D^k(1 - x^{n+1})/(1 - x). \tag{2}$$

For $k = 0$, (2) is to give back (1) and so we adopt the convention that $r^0 = 1$ for all $r$, including the case $r = 0$. The above formula provides a compact analytic expression for the desired sums.

By observing that $k$ applications of $D$ on the right-hand side of (2) produces a result with a common denominator of $(1- x)^{k+1}$, we define a set of polynomials of degree $n + k + 1$, $Q_{n+1}(x; k)$; thus:

$$\sum_{0}^{n} r^k x^r = Q_{n+1}(x; k)/(1 - x)^{k+1}, \tag{3}$$

with

$$Q_{n+1}(x; 0) \equiv 1 - x^{n+1}, \tag{4}$$

from (2) and (1). From this point on, the summation index will be $r$, unless otherwise specified. A recursion formula in $k$ is obtained by noting that

$$\sum_{0}^{n} r^{k+1} x^r = D \sum_{0}^{n} r^k x^r. \tag{5}$$

Identifying each side of (5) with a $Q$-polynomial as given in (3) we get

$$Q_{n+1}(x; k + 1) = x[(1 - x)Q'_{n+1}(x; k) + (k + 1)Q_{n+1}(x; k)] \tag{6}$$

for $k$ integer $\geq 1$; $Q_{n+1}(x; 0)$ is defined by (4), and a prime denotes differentiation with respect to $x$.

The first few $Q$-polynomials are:

$$\begin{aligned} Q_{n+1}(x; 1) &= x - (n + 1)x^{n+1} + nx^{n+2}; \\ Q_{n+1}(x; 2) &= x + x^2 - (n + 1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}; \\ Q_{n+1}(x; 3) &= x + 4x^2 + x^3 - (n + 1)^3 x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} \\ &\quad - (3n^3 + 3n^2 - 3n + 1)x^{n+3} + n^3 x^{n+4}. \end{aligned} \tag{7}$$

## 3. General Properties of $\sum r^k x^r$

An inspection of (7) suggests that the $Q$-polynomials may be written as $x^n$ times a polynomial of degree $k$ in $n$, plus a term which is $n$-independent. Consequently, this property also holds for $\sum r^k x^r$, by (3). To see this more clearly, rewrite (2) as follows:

$$\sum_{0}^{n} r^k x^r = D^k \frac{x^{n+1}}{x - 1} - D^k \frac{1}{x - 1}. \tag{8}$$

The first term on the right-hand side generates $x^n$ times a polynomial of degree $k$ in $n$ and the second term generates a term which is independent of $n$. As a result, in an effort to display the $n$-dependence of the right-hand side as explicitly as possible, we rewrite (8) in the form

$$\sum_{0}^{n} r^k x^r = x^n P_k(x; n) + R_k(x), \tag{9}$$

where

$$P_k(x; n) = \sum_{r=0}^{k} a_r^{(k)}(x) n^r \tag{10}$$

is a polynomial of degree $k$ in $n$, with coefficients $a_r^{(k)}$ which depend on $x$. The term $R$ is independent of $n$ and so, by setting $n = 0$ in (9), we find that $R_k = -a_0^{(k)}$, except when $k = 0$. Indeed, because of our earlier convention that $r^0$ be equal to 1 for all $r \geq 0$, the case $k = 0$ has to be handled differently. From (2), with $k = 0$, we find that

$$R_0(x) = -1/(x - 1) \quad \text{and} \quad a_0^{(0)}(x) = x/(x - 1).$$

Finally, with this restriction in mind, we rewrite (9) in the form

$$\sum_0^n r^k x^r = x^n \sum_{r=0}^k a_r^{(k)}(x) n^r - a_0^{(k)}(x) \tag{11}$$

and establish rules to obtain the coefficients $a_r^{(k)}$. To obtain these coefficients, we will use two different methods: A) a method of recursion on $k$; and B) a method of recursion on $n$.

A) $k$-Recursive Method: This method consists in assuming that the $a_r^{(k)}$'s are known for some $k$. Then, by using (5), the next set of coefficients, $a_r^{(k+1)}$, is determined. By (5) and (9)-(11), we get

$$x^n P_{k+1}(x;\, n) - a_0^{(k+1)} = D[x^n P_k(x;\, n) - a_0^{(k)}]. \tag{12}$$

To reduce this expression, perform the derivative and get

$$x^n[P_{k+1}(x;\, n) - nP_k(x;\, n) - DP_k(x;\, n)] = a_0^{(k+1)}(x) - Da_0^{(k)}(x). \tag{13}$$

The right-hand side of (13) is independent of $n$ but the left-hand side has a factor which grows exponentially with $n$. Consequently, for (13) to hold for all values of $n$, with $x$ fixed but arbitrary, we must have

$$a_0^{(k+1)} = Da_0^{(k)}, \tag{14}$$

$$P_{k+1} = nP_k + DP_k. \tag{15}$$

To reduce (15) further, define

$$a_{k+1}^{(k)} \equiv 0, \quad a_{-1}^{(k)} \equiv 0, \tag{16}$$

and use (10) to get

$$\sum_0^{k+1} [a_r^{(k+1)}(x) - a_{r-1}^{(k)}(x) - Da_r^{(k)}(x)] n^r = 0. \tag{17}$$

In order for this expression to hold for all $n$, with $x$ fixed but arbitrary, we must have

$$a_r^{(k+1)} = a_{r-1}^{(k)} + Da_r^{(k)}. \tag{18}$$

Because of (16), the case $r = 0$ is consistent with (14) above; similarly, for $r = k + 1$, we get

$$a_{k+1}^{(k+1)} = a_k^{(k)}, \tag{19}$$

and so we conclude, from (3), that

$$a_k^{(k)}(x) = x/(x - 1) \tag{20}$$

for all $k$, including $k = 0$. One significant drawback of this $k$-recursive approach is that all previous sums must be known in order to determine the $k^{\text{th}}$ one. Fortunately, however, using method B, it is possible to determine the $k^{\text{th}}$ sum independently from the others.

B) <u>Induction on $n$</u>: By induction on $n$, (10) and (11) give

$$\sum_0^{n+1} r^k x^r = x^{n+1} P_k(x;\, n+1) - a_0^{(k)}(x) \tag{21}$$

or, equivalently,

$$(n+1)^k x^{n+1} = x^{n+1} P_k(x;\, n+1) - x^n P_k(x;\, n). \tag{22}$$

With (10), this gives

$$x(n+1)^k = x\sum_0^k a_r^{(k)}(n+1)^r - \sum_0^k a_r^{(k)} n^r. \tag{23}$$

To simplify the notation in what follows, we will write $a_r$ for $a_r^{(k)}$ because the upper index $k$ is kept fixed.

Using the binomial expansion, (23) becomes

$$x\sum_{j=0}^k a_j \sum_{r=0}^j \binom{j}{r} n^r - \sum_{j=0}^k a_j n^j = x\sum_{j=0}^k \binom{k}{j} n^j. \tag{24}$$

For this equation to hold for all $n$, we must have equality of the coefficients of like powers of $n$ on both sides; hence,

$$a_k = x/(x-1), \tag{25}$$

as observed previously, and

$$a_r = \frac{x}{x-1}\left[\binom{k}{r} - \sum_{j=r+1}^k \binom{j}{r} a_j\right], \tag{26}$$

for $0 \le r \le k-1$. We give here the first few $a_r$'s, for arbitrary $k$; $a_k$ is given by (25), and

$$\begin{aligned} a_{k-1} &= -kx/(x-1)^2, \\ a_{k-2} &= k(k-1)x(x+1)/2(x-1)^3, \\ a_{k-3} &= -k(k-1)(k-2)x(x^2+4x+1)/6(x-1)^4. \end{aligned} \tag{27}$$

Others are determined readily using (26).

To conclude this section, we extend (2) to negative values of $n$. To do so, first note that the right-hand side of (2) is well defined for all values of $n$, with $k$ an integer $\ge 0$. For $n = -1$, the right-hand member of (2) is zero, so we adopt the convention that

$$\sum_0^{-1} r^k x^r = 0 \text{ for all } x \ne 1 \text{ or } 0.$$

For $n$ an integer $\ge 2$, we let

$$\begin{aligned} \sum_0^{-n} r^k x^r \equiv D^k(x^{-n+1} - 1)/(x-1) &= -D^k\left(\frac{1}{x^{n-1}} - 1\right)\Big/\left(x\left(\frac{1}{x} - 1\right)\right) \\ &= -D^k \sum_1^{n-1} x^{-r}, \end{aligned} \tag{28}$$

i.e.,

$$\sum_0^{-n} r^k x^r \equiv -\sum_1^{n-1} (-r)^k x^{-r}, \tag{29}$$

with $\sum_1^0 \equiv 0$ on the right-hand side.

Now set $n = -1$ in (11) to obtain

$$x^{-1}\sum_{0}^{k} a_r^{(k)}\cdot(-1)^r - a_0^{(k)} = 0, \tag{30}$$

i.e.,

$$(x - 1)a_0^{(k)} = \sum_{1}^{k}(-1)^r a_r^{(k)} \text{ for all } x \neq 1. \tag{31}$$

This interesting property can be observed in the special cases that follow.

## 4. Interesting Special Cases

In this section, results for $k = 1, 2, 3, 4$, and 5 are presented.

To begin with, we let $x = 2$ and find the following sums:

$$\begin{aligned}
&\sum_{0}^{n} r\cdot 2^r = 2[2^n(n - 1) + 1];\\
&\sum_{0}^{n} r^2\cdot 2^r = 2[2^n(n^2 - 2n + 3) - 3];\\
&\sum_{0}^{n} r^3\cdot 2^r = 2[2^n(n^3 - 3n^2 + 9n - 13) + 13];\\
&\sum_{0}^{n} r^4\cdot 2^r = 2[2^n(n^4 - 4n^3 + 18n^2 - 52n - 75) + 75];\\
&\sum_{0}^{n} r^5\cdot 2^r = 2[2^n(n^5 - 5n^4 + 30n^3 - 130n^2 + 375n - 541) + 541].
\end{aligned} \tag{32}$$

There is an interesting regularity in the coefficients of $n$ in the parentheses; for example, the absolute value of the coefficient of $n^0$ is equal to the sum of the absolute values of the coefficients of all the higher-order terms.

The second sum in (32) belongs to a class of sums where the summand $r^k x^r$ is symmetric under the interchange of $r$ and $k$: $r^k\cdot k^r$. Such sums have an intrinsic appeal and we give a few examples below:

$$\begin{aligned}
&\sum_{0}^{n} r^2\cdot 2^r = \frac{2}{1^2}[2^n(n^2 - 2n + 3) - 3];\\
&\sum_{0}^{n} r^3\cdot 3^r = \frac{3}{2^3}[3^n(4n^3 - 6n^2 + 12n - 11) + 11];\\
&\sum_{0}^{n} r^4\cdot 4^r = \frac{4}{3^4}[4^n(27n^4 - 36n^3 + 90n^2 - 132n + 95) - 95].
\end{aligned} \tag{33}$$

The case $\sum_0^n r^1\cdot 1^r$ has to be handled differently because (1) does not hold for $x = 1$; we shall discuss this type of situation in C) below.

Other interesting results are now given in A)-C).

A) For $x = -1$:

$$\sum_{0}^{n}(-1)^r r = \frac{1}{4}[(-1)^n(2n + 1) - 1];$$

$$\sum_0^n (-1)^r r^2 = \frac{1}{2}[(-1)^n(n^2+n)] = (-1)^n \sum_0^n r; \tag{34}$$

$$\sum_0^n (-1)^r r^3 = \frac{1}{8}[(-1)^n(4n^3+6n^2-1)+1].$$

B) For $x = i$ imaginary, we get, for example,

$$\sum_0^n i^r r^2 = \frac{1}{2}[i^n(n^2+2n+i(1-n^2)) - i]. \tag{35}$$

If the real and imaginary terms are gathered separately, for $n$ even, two identities are obtained. The identity for the real terms gives back the second equation of (34) and that for the imaginary terms gives the new identity,

$$\sum_0^{n/2-1} (-1)^r (2r+1)^2 = [(-1)^{n/2}(1-n^2) - 1]. \tag{36}$$

C) For $x = 1$: In order to obtain the sums of powers of consecutive integers, take the limit $x \to 1$ in (3) and get

$$\sum_0^n r^k = \lim_{x\to 1} \frac{Q_{k+1}(x;\,k)}{(1-x)^{k+1}} = \frac{(-1)^{k+1}}{(k+1)!} \lim_{x\to 1} \frac{d^{k+1}}{dx^{k+1}} Q_{n+1}(x;\,k) \tag{37}$$

after $k+1$ applications of l'Hôpital's rule. For $k = 0, 1, 2, 3$, equations (7) give, respectively:

$$\begin{aligned}
&\lim_{x\to 1} \frac{d}{dx} Q_{n+1}(x;\,0) = -(n+1);\\
&\lim_{x\to 1} \frac{d^2}{dx^2} Q_{n+1}(x;\,1) = n(n+1);\\
&\lim_{x\to 1} \frac{d^3}{dx^3} Q_{n+1}(x;\,2) = -n(n+1)(2n+1);\\
&\lim_{x\to 1} \frac{d^4}{dx^4} Q_{n+1}(x;\,3) = 6n^2(n+1)^2.
\end{aligned} \tag{38}$$

Insertion of these results in (37) gives the expected results for the appropriate sums. The present technique is, however, somewhat cumbersome to handle. Indeed, $k$ derivatives are first required to find $Q_{n+1}(x;\,k)$ followed by $k+1$ additional ones in order to compute the limit. Cases with $x = 1$ can be handled easily with Khan's technique or by the method of induction on $n$ presented earlier. Indeed, by observing, from (8), that

$$\lim_{x\to 1} D^k(x^{n+1}-1)/(x-1)$$

is a polynomial of degree $k+1$ in $n$, we may write

$$\sum_0^n r^k = \sum_0^{k+1} a_r^{(k)} n^r - a_0 \tag{39}$$

and proceed as before.

## Acknowledgment

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# *Announcement*

# FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

## Monday through Friday, July 30-August 3, 1990

**Department of Mathematics and Computer Science**

## Wake Forest University

**Winston-Salem, North Carolina 27109**

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### CALL FOR PAPERS

The FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Wake Forest University, Winston-Salem, N.C., from July 30 to August 3, 1990. This Conference is sponsored jointly by the Fibonacci Association and Wake Forest University.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts are to be submitted by March 15, 1990, while manuscripts are due by May 1, 1990. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

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# CHARACTERIZATIONS OF THREE TYPES OF COMPLETENESS

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## Introduction

A sequence is *complete* if every positive integer is a sum of distinct terms of the sequence [1, 3]. In this paper I discuss and characterize this definition and two definitions that generalize it.

In Section 1, I give several examples of complete sequences. Section 2 describes how a theorem due to Brown & Weiss [1] can be used to characterize the complete sequences. In Section 3, *weak completeness* [3] is defined, a sufficient condition for a sequence to be weakly complete is given and, finally, a condition equivalent to weak completeness is presented.

In Section 4, the concept of *completability* is introduced. Several conditions which imply the completability of a sequence are described. A theorem characterizing the completable sequences is proved, and it is used to find an infinite noncompletable sequence. The relations between the concepts of "completeness" discussed are described.

## 1. Sequences and Completeness

A sequence is a collection of numbers in one-to-one correspondence with the positive integers. Since only sequences of nonnegative integers are considered in this paper, the word "number" will be understood to refer to a nonnegative integer, and the word "sequence" will refer only to sequences of such numbers.

*Definition 1:* A sequence $f$ is *complete* [3] if every natural number is a sum of one or more distinct terms of the sequence.

Erdös & Graham [2] mean, by a *complete* sequence, a sequence such that every sufficiently large natural number is a sum of distinct terms of the sequence. We will not use "complete" in this sense.

Clearly, the sequence $\{n\} = \{1, 2, 3, 4, 5, \ldots\}$ is complete. However, there exist infinitely many other complete sequences. For example, the sequence $\{1, 2, 3, 4, 8, 12, 16, 20, 24, 28, \ldots\}$ is complete. This follows from our ability to represent each positive integer in mod 4. A similar sequence may be obtained from any number $m > 1$, by appending the numbers from 1 to $m - 1$ to the multiples of $m$. As we see in the following example, any sequence constructed in a similar manner is complete.

*Example 1:* Let $m$ be a natural number. Then the sequence $f$, where

$$f(n) = \begin{cases} (n - m + 1)m, & \text{if } n > m, \\ n, & \text{if } 1 \le n \le m \end{cases}$$

is complete.

*Proof:* Let $n$ be a natural number, and let $r$ be its least residue mod $m$. If $r = 0$, then $n$ is a term of $f$. If $r \neq 0$, then $n - r$ is a multiple of $m$. If $n - r = 0$, then, once again, $n$ is a term of $f$; otherwise, $n$ is a sum of distinct terms of $f$, namely $n - r$ and $r$.

The Fibonacci sequence $\{1, 1, 2, 3, 5, 8, \ldots\} = f_{1,1}$ is an example of a complete sequence [3]. In this sequence,

$$f_{1,1}(1) = 1, \quad f_{1,1}(2) = 1, \quad \text{and}$$

$$f_{1,1}(n) = f_{1,1}(n - 1) + f_{1,1}(n - 2) \quad \text{if } n \geq 3.$$

Consider the class of all sequences $f$ that satisfy the recurrence relation

$$f(n) = f(n - 1) + f(n - 2) \quad \text{if } n \geq 3. \tag{R}$$

The sequences in this class have only two degrees of freedom, since, given the first two terms, the recurrence relation (R) determines all remaining terms. Any ordered pair of whole numbers can be the first two terms of a sequence satisfying (R). The class of these sequences is countably infinite, but any illusions we might have that an infinite number of them are complete are shattered by Proposition 1 which follows. But first, a definition:

*Definition 2:* Suppose $f_{i,j}(1) = i$, $f_{i,j}(2) = j$, and that $f_{i,j}$ satisfies (R). Then $f_{i,j}$ is called *the Fibonacci sequence beginning with $i$ and $j$.*

*Proposition 1:* The Fibonacci sequence beginning with $i$ and $j$ is complete if and only if $(i, j)$ is one of the pairs $(0, 1)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 1)$.

*Proof:* ("If" part.) Parallels exactly the proof that $f_{1,1}$ is complete.

("Only if" part.) Let $f$ be the Fibonacci sequence beginning with $i$ and $j$. Suppose $f$ is complete. It is easily seen that 1 must be one of the first two terms of $f$. If $i = 1$ and $j > 2$, then 2 is not a sum of distinct terms of $f$. If $j = 1$ and $i > 2$, then 2 is, again, not a sum. So if $f$ is complete, then $(i, j)$ is one of the pairs $(0, 1)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 1)$.

In the next section, we shall derive a characterization of the complete sequences.

## 2. Brown's Criterion and Its Use in Characterizing All Complete Sequences

Of the three sequences $\{1^{n-1}\}$, $\{2^{n-1}\}$, and $\{3^{n-1}\}$, the first two are complete, and the third is not. the following relations are true for all natural numbers $n$:

$$1^n < 1 + \sum_{i=1}^{n} 1^{i-1}, \quad 2^n = 1 + \sum_{i=1}^{n} 2^{i-1}, \quad \text{and} \quad 3^n > 1 + \sum_{i=1}^{n} 3^{i-1}.$$

These data suggest that a sequence $f$ may be complete iff, for all $p \geq 1$,

$$f(p + 1) \leq 1 + \sum_{i=1}^{p} f(i).$$

A counterexample shows that this is not so: If $f$ is the complete sequence $\{8, 4, 2, 1, 16, 32, 64, 128, \ldots, 2^{n-1}, \ldots\}$, then the inequality is false for some $p$. For instance, $f(2) = 4$, even though $1 + f(1) = 2$. The important difference between the sequences $\{1^{n-1}\}$, $\{2^{n-1}\}$, and $\{3^{n-1}\}$, and the sequence $\{8, 4, 2, 1, 16, \ldots\}$ is that the first three sequences are nondecreasing, while the fourth is not. The following theorem about nondecreasing sequences with first term 1 can be used to characterize all complete sequences. The theorem is known as "Brown's criterion" since it was first proved by Brown & Weiss [1].

*Brown's Criterion:* If $f$ is a nondecreasing sequence, and if $f(1) = 1$, then $f$ is complete iff, for all $p \geq 1$,

$$f(p + 1) \leq 1 + \sum_{i=1}^{p} f(i).$$

Let $f$ be any sequence. If $f$ is finite, then $f$ is not complete. If $f$ is infinite but contains no 1, then it is not complete, since 1 is not representable. If $f$ is infinite and contains a 1, then it is either nondecreasing or not. Suppose $f$ is not nondecreasing. Either there is a term that occurs infinitely often in the sequence, or there is not. If there is not, then, without affecting its completeness, the terms of the sequence can be rearranged so that it is nondecreasing. Suppose there is a term of the sequence $f$ that repeats infinitely often. The following theorem will show that there is a nondecreasing sequence $g$ that is complete if and only if $f$ is.

*Theorem 1:* Let the sequence $f$ contain a term which is repeated infinitely often. Then there is a sequence $g$ that is nondecreasing and which is complete if and only if $f$ is.

*Proof (and construction):* Suppose the value of the least term, in magnitude, that repeats infinitely often is $k$. If there is no term of $f$ greater than $k$, then the terms of $f$ less than $k$ can be reordered, and the term $k$ (infinitely repeated) tacked on to the end, to obtain the sequence $g$. By this procedure, $\{5, 4, 3, 2, 6, 6, 1, 6, \ldots\}$ can be turned into $\{1, 2, 3, 4, 5, 6, 6, \ldots\}$.

If there are terms greater than $k$, then we show that the removal of all terms of $f$ that are greater than $k$ will not affect its completeness. First, note that the removal of terms from a sequence that is not complete cannot render the sequence complete; so all that must be proved is that, if the sequence $f$ is complete prior to the removal of all terms greater than $k$, it will remain complete.

Suppose $f$ is complete and all such terms are removed. Let $n$ be a natural number. If $n \leq k$, then $n$ is a sum of distinct terms of the original sequence none of which is greater than $k$, so it is a sum of distinct terms of the new sequence. If $n$ is greater than $k$, then $n$ is the sum of a multiple of $k$ and a nonnegative integer less than $k$, that is,

$$n = ak + r, \text{ where } 0 \leq r < k.$$

If $r = 0$, then, since $k$ is infinitely repeated, $n$ is a sum of distinct terms of the new sequence. If $r \neq 0$, then $ak$ is the sum of distinct terms $k$, while $r$ is a sum of distinct terms all less than $k$. So $n$ is the sum of distinct terms of the new sequence. The cases have been exhausted; thus, the new sequence is complete if the original sequence is complete.

Hence, there is no loss of generality in assuming that no term of $f$ is greater than $k$, since all such terms can be dropped, and the resulting sequence can be reordered into a nondecreasing sequence, as described above.

So we may assume, without loss of generality, that $f$ is nondecreasing. If $f$ contains zeros, they can be removed, again without affecting completeness, so assume $f$ contains no zeros. Brown's criterion may immediately be applied to decide whether $f$ is complete—for, since $f$ contains a 1 but no zeros, $f(1)$ must be 1.

Briefly, then, the procedure for testing a sequence $f$ for completeness is as follows:

i) If $f$ is finite, or if $f$ contains no 1, then $f$ is not complete.

ii) If some number occurs infinitely often in the sequence $f$, then remove all terms of $f$ that are greater than the least term so repeated, if any terms greater than the least term do exist.

iii) If $f$ is not nondecreasing, then reorder it so that it is. Do not remove any nonzero terms of the sequence to accomplish this.

iv) If $f$ contains any zeros, remove them, since a sum of distinct integers containing zeros clearly is still a sum of distinct integers.

v) Prove or disprove that the inequality

$$f(p+1) \leq 1 + \sum_{i=1}^{p} f(i)$$

holds for all $p \geq 1$.

The complete sequences have been characterized!

The limitation of completeness, as a mathematical statement of the intuitive idea of the "richness" of a sequence, is not one of undue generality but, rather, is a failure to include sequences which are so "nearly complete," or which are so easily "turned into complete sequences," that to call them "incomplete" seems little more than nitpicking. For example, the sequences $\{2, 3, 4, 5, 6, \ldots\}$ and $\{2, 2, 4, 6, 10, 16, \ldots\}$ are not complete, although every integer $\geq 2$ is a sum of distinct terms of the first sequence, and although the sequence $\{1, 2, 2, 4, 6, 10, \ldots\}$, obtained by appending a 1 to the second sequence, is complete.

## 3. Weak Completeness

*Definition 3:* A sequence $f$ is *weakly complete* [3] if a positive integer $n$ exists such that every integer greater than $n$ is a sum of distinct terms of $f$. Erdös & Graham [2] call such sequences *complete*.

A complete sequence is weakly complete. The sequence $f(n) = n + 1$, to give a trivial example, is weakly complete but not complete. The following theorem specifies a condition implying weak completeness.

*Theorem 2:* A sequence $f$ is weakly complete if a positive integer $n$ and a real number $s > 2$ exist such that:

i) If $x > n$, then there is a term of the sequence strictly between $x$ and $(2 - 2/s)x$, and

ii) every integer between $n$ and $sn$ (inclusive) is a sum of distinct terms of the sequence.

*Proof:* By strong induction. Given an integer $w > sn$, we must show that $w$ is a sum of distinct terms of the sequence. Let our induction hypothesis be that every integer inclusively between $n$ and $w - 1$ is a sum of distinct terms of $f$.

There exists a term of the sequence, $f(t)$, strictly between $w/2$ and $(1 - 1/s)w$, by hypothesis i). Let

$$m = w - f(t).$$

Then $m < w/2$, and $m > w/s > n$. Since $n < m < w/2$, $m$ is a sum of distinct terms of $f$; and, since $m < w/2 < f(t)$, none of these distinct terms equals $f(t)$. Since $w = m + f(t)$, $w$ is a sum of distinct terms of $f$. By strong induction, the theorem is proved.

The two properties i) and ii) are not necessary for weak completeness. In particular, the function $f(n) = 2^{n-1}$ fails condition i) for all positive $n$ and real $s > 2$. The sequence $f(n)$ is nevertheless complete. (I am obliged to the referee for this example.) The sequence $f(n) = n + 1$, on the other hand, satisfies i) and ii) for suitable $s$ and $n$, and yet is incomplete. Thus, conditions i) and ii) are sufficient, but not necessary, for weak completeness, and are neither sufficient nor necessary for completeness.

The following examples of sequences which fail to be weakly complete show that this concept is not too broad.

*Example 2:* The Fibonacci sequence beginning with 2 and 2 is not weakly complete; neither is $\{2n\}$.

*Proof:* Let $f$ be either of these sequences. Any term $f(n)$ is even, so any sum of distinct terms of $f$ is even. No matter how large $n > 0$ is chosen, $2n + 1$ is greater than $n$ and is not a sum of distinct terms of $f$.

If any two terms of the Fibonacci sequence $f_{1,1}$ are replaced by zeros, the resulting sequence is not weakly complete. A proof of this can be found in [3]. Thus, the Fibonacci sequence beginning with 2 and 3 is not weakly complete.

*Definition 4:* A sequence $f$ is *finite* if a number $n$ exists such that, for all natural numbers $m > n$, $f(n) = 0$. A sequence is *infinite* iff it is not finite. An infinite sequence $f$ is *increasing* if, for any two natural numbers $m$ and $n$ such that $m > n$, $f(m) > f(n)$.

*Definition 5:* Let $f$ be weakly complete. Then the greatest integer which is not a sum of distinct terms of $f$ is called the *threshold* (of completeness) of $f$. Erdös & Graham [2] use the term "threshold" as well, but may not mean the same thing by it.

*Theorem 3:* The following conditions on a sequence $f$ are equivalent.

a) Every infinite increasing sequence contains a term that is a sum of distinct terms of $f$.

b) Every infinite increasing sequence contains a subsequence each of whose terms is a sum of distinct terms of $f$.

c) $f$ is weakly complete.

*Proof:* c) → b). Suppose c) holds. Let an infinite increasing sequence $h$ be given. Then, if $h(m)$ is the least term of $h$ greater than $T$, the threshold of $f$, then $g(n) = h(n + m)$ defines a subsequence of $h$ each of whose terms is a sum of distinct terms of $f$.

b) → a). Obvious.

a) → c). If $f$ were not weakly complete, then the sequence of numbers that are not sums of distinct terms of $f$ would form an infinite increasing sequence containing no sum of distinct terms of $f$, so a) would not be true. So, if a) holds, then c) holds.

(I am obliged to the referee for suggestions which shortened this proof.)

## 4. Completability

The sequence {2, 2, 4, 6, 10, 16, 26, 42, 68, ...} is not weakly complete, even though it is "sufficiently rich" that the mere attachment of a 1 to this sequence renders it complete. This suggests the definition of a third, very general sort of completeness, called *completability*, such that a completable sequence becomes complete after a suitable finite sequence is prefixed to it, that is, attached to it at its beginning.

*Definition 6:* Suppose $f$ is a sequence, and $I$ is a finite sequence. If $I(n) = 0$ for all $n$, then define *the result of prefixing $I$ to $f$* to be $f$. Otherwise, if $m$ is a natural number such that $I(m)$ is nonzero and, if $n > m$, $I(n) = 0$, then define *the result of prefixing $I$ to $f$* as the sequence $h$ such that $h(n) = I(n)$ if $n \le m$, and $h(n) = f(n - m)$ if $n > m$.

The formal tools are now available with which to define completability:

*Definition 7:* A sequence $f$ is *completable* if there exists a finite sequence $I$ such that the result of prefixing $I$ to $f$ is complete.

Note that the completability of a sequence is not affected by the removal or prefixing of a finite number of terms from or to the sequence.

*Theorem 4:* A weakly complete sequence is completable.

*Proof:* Let $f$ be weakly complete, and let $T$ be its threshold (see Definition 5). Define the sequence $I$ by letting

$$I(n) = n \text{ if } n \le T \quad \text{and} \quad I(n) = 0 \text{ if } n > T.$$

Then $I$ is finite, and the result of prefixing $I$ to $f$ is complete.

The following two theorems derive sufficient conditions that a sequence be completable.

*Theorem 5:* Let $f$ be a sequence. If a positive integer $n$ and a real number $v$ strictly between 1 and 2 exist such that, if $x > n$, there is a term of $f$ strictly between $x$ and $vx$, then $f$ is completable.

*Proof:* Let $s = 2/(2 - v)$. Then $v = 2 - 2/s$, and $s > 2$. Define the sequence $I$ to contain the integers between $n$ and $sn$, inclusive, in numerical order, followed by zeros. The sequence $I$ is finite. Let $h$ be the result of prefixing $I$ to $f$. If $h$ is weakly complete, then it is completable by Theorem 4; hence, $f$ is completable. Theorem 2 now applies: Our $s$ is the $s$ of that theorem.

The preceding theorem can be used to show that if $f$ is a sequence, and if there exists a real number $v$ strictly between 1 and 2 such that, for all sufficiently large $n$,

$$f(n) < f(n + 1) < vf(n),$$

then $f$ is completable. If $v$ is greater than 2 and the right-hand inequality is reversed, i.e., if

$$f(n+1) > vf(n)$$

for all sufficiently large $n$, then $f$ is not completable. This will be shown in Theorem 9.

*Theorem 6:* Let $f$ be a sequence. Suppose there is a natural number $m > 1$ such that all but a finite number (possibly zero) of terms of $f$ are divisible by $m$. Suppose, in addition, that the sequence $I$ defined by

$$I(n) = f(n+s)/m,$$

where $f(s)$ is the last term of $f$ that is not divisible by $m$, is complete. Then $f$ is completable.

*Proof:* If there is a term of $f$ not divisible by $m$, let $r$ be the largest; if every term of $f$ is divisible by $m$, let $r = 0$. Let

$$\begin{cases} h(n) = n & \text{if } n \le m, \\ h(n) = 0 & \text{otherwise,} \end{cases}$$

and let the sequence $j$ be the result of prefixing the finite sequence $h$ to $f$. We obtain that $j$ is complete by a similar argument to that of the proof of Theorem 1.

*Counterexample:* The converse to Theorem 6 is false: there exists a completable sequence any two consecutive terms of which are relatively prime.

Let $f$ be the Fibonacci sequence whose first two terms are 2 and 3. Then $f$ is completable because the result of prefixing the finite sequence $h$ defined by $h(1) = 1$, $h(2) = 1$, and $h(n) = 0$ if $n > 2$, is complete.

This shows that Theorem 6 does not characterize the completable sequences.

Theorem 6 proves that completable sequences can be obtained by multiplying every term of a complete sequence by a constant and prefixing some finite number, possibly zero, of terms. It is also true that, if every term of a weakly complete sequence is multiplied by a constant and a finite sequence is then prefixed, the result is completable (replace $m$ by $m + T$, where $T$ is the threshold of the weakly complete sequence, in the proof of Theorem 6). The concept of completability is certainly not restrictive. There is now the problem of proving that it is not too general—that the class of completable sequences does not coincide with the class of sequences. This will be done in the next four theorems.

*Definition 8:* Let $f$ be a sequence. Then $P(f)$ is the set of all natural numbers that are sums of distinct terms of $f$. This notation is due to Erdös & Graham [2].

It follows from this definition that a sequence $f$ is complete iff $P(f) = N$. Similarly, a sequence $f$ is weakly complete iff $P(f)$ is cofinite (i.e., iff its complement in $N$ is a finite set).

*Theorem 7:* A sequence $f$ is completable iff there exists a positive integer $c$ such that, if $q$ is greater than $c$ and is not in $P(f)$, then there exists a number $m$ in $P(f)$ such that $0 < q - m \le c$.

*Proof:* ("Only if" part.) If $f$ is weakly complete, then upon choosing $c$ to be the threshold of $f$, the theorem follows trivially. Suppose $f$ is completable but not weakly complete. If $I$ is a finite sequence such that the result of prefixing $I$ to $f$ is complete, then let $c$ be the maximum element of $P(I)$. Suppose $q$ is greater than $c$ and not in $P(f)$. Then $q$ is the sum of distinct terms of $I$ and distinct terms of $f$. Let the distinct terms in this sum from $f$, taken by themselves, have the sum $m$. The distinct terms in this sum from $I$ are greater than zero, but cannot exceed $c$. However, $q$ is the sum of $m$ and these distinct terms of $I$, so $m < q \le m + c$. This implies $0 < q - m \le c$.

("If" part.) If $I$ is the finite sequence consisting of $c$ ones followed by zeros, then prefixing $I$ to $f$ we obtain a sequence $g$. Let $q$ be a natural number. If $q \le c$, then $q$ is the sum of $q$ ones from $I$. If $q > c$, then either $q$ is in $P(f)$ or is not. If $q$ is not in $P(f)$, then there is $m$ in $P(f)$ with $0 < q - m \le c$, and $q$ is the sum of $m$ in $P(f)$ and $q - m$ in $P(I)$. The terms whose sums are $m$ and $q - m$, respectively, do not overlap because the terms of $I$ precede the terms of $f$ in the sequence. So every natural number is a sum of distinct terms of $g$. Thus, $f$ is completable.

It follows from Theorem 7 that an infinite sequence $f$ is complete iff the sequence $h$, defined by $h(n) = n^{\text{th}}$ term of $P(f)$ in order of magnitude, has the property that the difference between consecutive terms of $h$, $h(n + 1) - h(n)$, is a function of $n$ that is bounded from above.

Theorem 7 is a necessary and sufficient condition that a sequence $f$ be completable. The following theorem applies the contrapositive of the "only if" part of Theorem 7 to obtain a condition that a sequence not be completable.

*Definition 9:* A sequence $f$ is *superincreasing* if the quantity

$$f(n) - \sum_{i=1}^{n-1} f(i)$$

is positive for all sufficiently large $n$. [Note that superincreasing sequences are increasing for n sufficiently large.]

*Theorem 8:* Let $f$ be a superincreasing sequence. Suppose

$$f(n) - \sum_{i=1}^{n-1} f(i)$$

is unbounded from above. Then $f$ is not completable.

*Proof:* Suppose the condition of Theorem 7 held. Then there would exist a number $c$ such that, if $n$ were greater than $c$ and not in $P(f)$, there would exist $m$ in $P(f)$ such that $0 < n - m \le c$. For all positive integers $c$, we will exhibit $t > c$ which is not in $P(f)$ and such that, if $m$ is in $P(f)$ and is less than $t$, $c + m$ is also less than $t$, so that the sequence $f$ cannot satisfy the necessary condition of Theorem 7.

Let $c > 0$ be given. By hypothesis, there are infinitely many $n$ such that

$$f(n) - \sum_{i=1}^{n-1} f(i) > c + 1.$$

Choose any such $n$, and define $t = f(n) - 1$. Then:

a) $t$ is not in $P(f)$.

For suppose $t$ were in $P(f)$. Then $t_i$ and $r$ would exist such that

$$t = \sum_{i=1}^{r} f(t_i), \text{ for all } i,\ t_i < n,$$

since $t < f(n)$ and $f$ is increasing beyond the $n^{\text{th}}$ term. This implies that $t$, the sum of the terms $f(t_i)$ can be no greater than the sum of all terms up to the $(n-1)^{\text{th}}$, that is,

$$\sum_{i=1}^{n-1} f(i);$$

and so

$$t = \sum_{i=1}^{r} f(t_i) \le \sum_{i=1}^{n-1} f(i) < f(n) - c - 1 = t - c < t,$$

which is impossible.

b) $t > c$.

Since

$$f(n) - \sum_{i=1}^{n-1} f(i) - 1 > c,$$

and since $t = f(n) - 1$,

$$t > \sum_{i=1}^{n-1} f(i) + c \ge c.$$

c) If $m$ is in $P(f)$ and $m < t$, then $m + c < t$.

Since $m < t$, $m < f(n)$. Since $m$ is in $P(f)$, $m$ is a sum of distinct terms of $f$, and since $m < f(n)$ this sum can be no greater than

$$\sum_{i=1}^{n-1} f(i).$$

So

$$c + 1 < f(n) - \sum_{i=1}^{n-1} f(i) \le f(n) - m;$$

hence,

$$c < t - \sum_{i=1}^{n-1} f(i) \le t - m \quad \text{and} \quad m + c < t.$$

*Theorem 9:* Let $f$ be a sequence. Suppose there exists a real number $v > 2$ such that, for all sufficiently large $n$,

$$f(n+1) > vf(n).$$

Then $f$ is not completable.

*Proof:* Let

$$h(n) = f(n) - \sum_{i=1}^{n-1} f(i) \quad (n \ge 2).$$

Suppose $n$ is sufficiently large such that, for all $r \ge n$,

$$f(r+1) > vf(r).$$

Since,

$$f(r+1) > 2f(r),$$
$$f(r+1) - f(r) > f(r);$$

thus, subtracting

$$\sum_{i=1}^{r-1} f(i)$$

from both sides, we obtain

$$h(r+1) > h(r) \text{ for all } r \geq n.$$

We will show that the function $h$ satisfies the condition of Theorem 8; that is, $h(r)$ is positive for sufficiently large $r$ and unbounded from above.

Since

$$h(r+1) \geq h(r) + 1$$

for all $r \geq n$, it is true by induction that

$$h(r+m) \geq h(r) + m \text{ for all } m \geq 1 \text{ and } r \geq n.$$

Let $z = h(n)$. If $z \geq 0$, then $h(r)$ is positive for all $r > n$; so suppose $z < 0$. Then, if $m > -z$,

$$h(n+m) \geq h(n) + m > h(n) - z = 0,$$

so, if $r > n - z$, $h(r) > 0$. Thus, in any case, $h(r)$ is positive for all sufficiently large $r$.

Suppose $h(r)$ is bounded above by $w$, for all $r$. Again, let $z = h(n)$. Then $z < w$; let $m = w - z$. Then

$$h(n+m+1) \geq h(n) + m + 1 = z + m + 1 = w + 1,$$

a contradiction. So $h(r)$ is unbounded from above.

This theorem, and Theorem 5, relate completability to the rate of growth of a sequence. However, there are infinite sequences whose completability neither theorem can decide. For example, let $f$ be the sequence defined by

$$f(n) = \begin{cases} 1, \text{ if } n = 1, \\ f(n-1)^2, \text{ if } n \text{ is even}, \\ f(n-1) + 1, \text{ if } n \geq 3 \text{ is odd}. \end{cases}$$

If $n$ is sufficiently large, then

$$f(n+1) < 2f(n), \text{ if } n \text{ is even},$$
$$f(n+1) > 2f(n), \text{ if } n \text{ is odd}.$$

$f$ satisfies neither the hypothesis of Theorem 5 nor that of Theorem 9.

Theorem 9 yields infinite noncompletable sequences, for example, the sequence $f(n) = 3^n$.

*Remark:* Those results of the past two sections which relate the three definitions of completeness may be summarized as follows:

> Let $J$ be the class of complete sequences, let $K$ be the class of weakly complete sequences, let $L$ be the class of completable sequences, and let $M$ be the class of all infinite sequences (see Definition 4). Then $J \subset K \subset L \subset M$, and all containments are proper. The Remark after Definition 3, Theorem 4, and the Remark after Theorem 9 prove that $J \subseteq K$, $K \subseteq L$, $L \subset M$. The relations $J \subset K$ and $K \subset L$ are true because $f(n) = n + 1$ is in $K$ but not in $J$, and because $f(n) = 2n$ is in $L$ but not in $K$.

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3. Ross Honsberger. "A Second Look at the Fibonacci and Lucas Numbers." *Math. Gems III*, pp. 102-138. Mathematical Association of America Inc., 1985.

4. Torleiv, Kløve. "Sums of Distinct Elements from a Fixed Set." *Math. Comp.* (October 1975):1144-1149.

The following two articles are not utilized in my paper, but contain further information about complete sequences.

5. J. L. Brown, Jr. "A Characterization of 0-Sequences." *Math. Mag.* (September 1972):209-213.

A 0-sequence is a complete sequence which is rendered incomplete by the deletion of any term from the sequence. This article derives a necessary and sufficient condition that a nondecreasing sequence beginning with 1 be a 0-sequence.

6. J. L. Brown, Jr. "Integer Representations and Complete Sequences." *Math. Mag.* (January 1976):30-32.

This article derives a theorem which gives a necessary and sufficient condition that a nondecreasing sequence beginning with 1 be complete.

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# HURWITZ'S THEOREM AND THE CONTINUED FRACTION WITH CONSTANT TERMS

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## Introduction

We are concerned with finding the convergents

$$C_j(\alpha) = \frac{p_j}{q_j},$$

in lowest terms, to the positive real number $\alpha$ which satisfy the inequality relating to Hurwitz's theorem,

$$|\alpha - C_j(\alpha)| < \frac{\beta}{\sqrt{5}q_j^2}, \quad 0 < \beta < 1, \tag{1}$$

where $\alpha$ has a simple continued fraction expansion $\{i; i, i, \ldots\}$ and $i$ is a positive integer.

Van Ravenstein, Winley, & Tognetti [5] have solved this problem for the case where $i = 1$, which means $\alpha$ is the Golden Mean, and extended that result in [6] to the case where $\alpha$ is a Noble Number that is a number equivalent to the Golden Mean.

The Markov constant for $\alpha$, $M(\alpha)$, is defined at the upper limit on $\sqrt{5}/\beta$ such that (1) has infinitely many solutions $p_j$, $q_j$ (see Le Veque [4]). Thus, in order to determine $M(\alpha)$, we require the lower limit on values of $\beta$ such that there are infinitely many solutions.

Using the notation of [6] and the well-known facts concerning simple continued fractions (see Chrystal [2], Khintchine [3]), we have:

(i) If $\alpha = \{i; i, i, \ldots\}$ where $i$ is an integer and $i \geq 1$, then

$$\alpha = \frac{i + \sqrt{i^2 + 4}}{2},$$

which is the positive root of the equation $x^2 - ix - 1 = 0$;

(ii) $$p_j = \frac{\left(\alpha^{j+2} - \left(-\frac{1}{\alpha}\right)^{j+2}\right)}{\left(\alpha + \frac{1}{\alpha}\right)}, \qquad q_j = \frac{\left(\alpha^{j+1} - \left(-\frac{1}{\alpha}\right)^{j+1}\right)}{\left(\alpha + \frac{1}{\alpha}\right)} = p_{j-1} \tag{2}$$

where $j = 0, 1, 2, \ldots$ .

Hence, $$C_j(\alpha) = \frac{p_j}{q_j} = \frac{\left(\alpha^{j+2} - \left(-\frac{1}{\alpha}\right)^{j+2}\right)}{\left(\alpha^{j+1} - \left(-\frac{1}{\alpha}\right)^{j+1}\right)}.$$

The numbers $p_j$ have been studied extensively by Bong [1] where their relationship with Fibonacci and Pell numbers is described in detail.

## Solutions to (1)

**Case 1.** If $j$ is odd ($j = 2k + 1$, $k = 0, 1, 2, \ldots$), then (1) becomes

$$q_j(p_j - \alpha q_j) < \frac{\beta}{\sqrt{5}}$$

which, using (2)(ii), finally reduces to

$$\left(\frac{1}{\alpha^4}\right)^k > \alpha^4\left(1 - \frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right)\right). \tag{3}$$

From (3), we see that;

(i) there are no solutions for $k$ if

$$0 < \beta \le \frac{\sqrt{5}(\alpha^2 - 1)}{\alpha^3}; \tag{4}$$

(ii) there is a nonzero finite number of solutions for $k$ if

$$0 < \alpha^4\left(1 - \frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right)\right) < 1,$$

which simplifies to

$$0 < \frac{\sqrt{5}(\alpha^2 - 1)}{\alpha^3} < \beta < \frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)} \le 1. \tag{5}$$

We note that equality holds on the right in (5) only when $\alpha$ is the Golden Mean.

(iii) All nonnegative integers are solutions for $k$ if

$$\frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)} \le \beta < 1. \tag{6}$$

**Case 2.** If $j$ is even ($j = 2k$, $k = 0, 1, 2, \ldots$), then (1) becomes

$$q_j(\alpha q_j - p_j) < \frac{\beta}{\sqrt{5}}$$

and again using (2)(ii), this reduces to

$$\left(\frac{1}{\alpha^4}\right)^k < \alpha^2\left(\frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right) - 1\right). \tag{7}$$

From (7), we see that:

(i) there are no solutions for $k$ if

$$0 < \beta \le \frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)}; \tag{8}$$

(ii) there is a nonzero finite number of nonsolutions for $k$ if

$$0 < \alpha^2\left(\frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right) - 1\right) < 1,$$

which simplifies to

$$0 < \frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)} < \beta < \frac{\sqrt{5}}{\alpha}; \tag{9}$$

(iii) all nonnegative integers are solutions for $k$ if

$$\frac{\sqrt{5}}{\alpha} \le \beta < 1. \tag{10}$$

In the particular case $i = 1$, $\alpha$ is the Golden Mean, $\alpha + (1/\alpha) = \sqrt{5}$, and there will be no convergents $C_j(\alpha)$ that satisfy (1) when $j$ is even. However, if $i \ge 2$, then $(\sqrt{5}/\alpha) < 1$ and there are convergents that satisfy (1) when $j$ is even.

## Summary

Define

$$\beta_L = \frac{\sqrt{5}(\alpha^2 - 1)}{\alpha^3}, \quad \beta_M = \frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)}, \quad \beta_U = \frac{\sqrt{5}}{\alpha}$$

Using (4)-(10), we see that:

(i) If $i \ge 2$, then $\beta_L < \beta_M < \beta_U < 1$ and there are no convergents that satisfy (1) when $0 < \beta \le \beta_L$.

If $\beta_L < \beta < \beta_M$, there are a finite number of convergents $C_j(\alpha)$ that satisfy (1) with $j = 1, 3, 5, \ldots, 2[R] + 1$ and

$$R = \frac{\ln\left\{\alpha^4\left(1 - \frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right)\right)\right\}}{\ln\left(\frac{1}{\alpha^4}\right)}. \tag{11}$$

If $\beta = \beta_M$, there are an infinite number of convergents that satisfy (1) given by all $C_j(\alpha)$ where $j$ is odd.

If $\beta_M < \beta < \beta_U$, there are an infinite number of solutions to (1). These are given by all $C_j(\alpha)$ for $j$ odd and all but a finite number of $C_j(\alpha)$ when $j = 0, 2, 4, \ldots, 2[S]$ where

$$S = \frac{\ln\left\{\alpha^2\left(\frac{\beta}{\sqrt{5}}\left(\alpha + \frac{1}{\alpha}\right)\right) - 1\right\}}{\ln\left(\frac{1}{\alpha^4}\right)}. \tag{12}$$

If $\beta_U \le \beta < 1$, there are an infinite number of solutions to (1) given by $C_j(\alpha)$ for $j = 0, 1, 2, \ldots$ .

(ii) If $i = 1$, then $\beta_L < \beta_M = 1 < \beta_U$ and there are no convergents that satisfy (1) unless $\beta_L < \beta < 1$. In this case, the only convergents that are solutions to (1) are given by

$$C_j(\alpha) = \frac{F_{j+1}}{F_j}, \; j = 1, 3, 5, \ldots, 2[R] + 1,$$

where

$$R = \ln \frac{(1 - \beta)(7 + 3\sqrt{5})}{2} \Big/ \ln \frac{(7 - 3\sqrt{5})}{2} \text{ as specified in [5].} \tag{13}$$

(iii) The lower limit on numbers $\beta$ such that (1) has infinitely many solutions is given by

$$\beta_M = \frac{\sqrt{5}}{\left(\alpha + \frac{1}{\alpha}\right)},$$

and in this case the Markov constant for $\alpha$ is given by

$$M(\alpha) = \frac{\sqrt{5}}{\beta_M} = \alpha + \frac{1}{\alpha} = \sqrt{i^2 + 4}. \tag{14}$$

## Examples

1. If $i = 2$, then $\alpha = 1 + \sqrt{2} = \{2; 2, 2, \ldots\}$, $\beta_L \simeq 0.77$, $\beta_M \simeq 0.79$, $\beta_U \simeq 0.93$. Hence, we see that for:

   (i) $\beta \in (0, 0.77]$, there are no convergents satisfying (1);

   (ii) $\beta \in (0.77, 0.79)$, there are a finite number of convergents satisfying (1) and these are specified by (11);

   (iii) $\beta = 0.79$, there are an infinite number of convergents satisfying (1) given by all $C_j(\alpha)$ where $j = 1, 3, 5, \ldots$;

   (iv) $\beta \in (0.79, 0.93)$, all the convergents $C_j(\alpha)$ satisfy (1) for $j$ odd, whereas all but those specified by (12) satisfy (1) for $j$ even;

   (v) $\beta \in (0.93, 1)$, all convergents satisfy (1).

   In particular, it is seen from (14) that $M(1 + \sqrt{2}) = 2\sqrt{2}$.

2. If $\alpha = \{1; 1, 1, 1, \ldots\} = \dfrac{1 + \sqrt{5}}{2}$, then $\beta_L \simeq 0.85$, $\beta_M = 1$, $\beta_U \simeq 1.38$.

   Consequently, if $\beta \in (0, 0.85]$, there are no convergents that satisfy (1), whereas, if $\beta \in (0.85, 1)$, there are a finite number of solutions to (1) specified by (13). If $\beta = 1$, there are an infinite number of solutions given by all $C_j(\alpha)$ where $j$ is odd and we see from (14) that

$$M\left(\frac{1 + \sqrt{5}}{2}\right) = \sqrt{5}.$$

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# A GENERALIZATION OF BINET'S FORMULA AND SOME OF ITS CONSEQUENCES

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## 1. A Generalization of Binet's Formula

We derive a simple generalization of Binet's formula for Fibonacci and Lucas numbers. From the equations

$$F_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right], \tag{1.1}$$

and

$$L_m = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m, \tag{1.2}$$

we have at once,

$$\frac{L_m + \sqrt{5}F_m}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^m,$$

and

$$\frac{L_m - \sqrt{5}F_m}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^m.$$

Raising both sides to the $n^{\text{th}}$ power, and combining the results by means of (1.1) and (1.2), we find

$$F_{nm} = \frac{1}{\sqrt{5}}\left[\left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n - \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n\right], \tag{1.3}$$

and

$$L_{nm} = \left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n + \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n, \tag{1.4}$$

which are the desired generalizations. Equations (1.3) and (1.4) reduce to equations (1.1) and (1.2), respectively, when $m = 1$. Note that, in the right-hand sides of equations (1.3) and (1.4), $m$ and $n$ can be interchanged.

A number of interesting results can be obtained from (1.3) and (1.4). Note, for instance, that one has

$$(L_m + \sqrt{5}F_m)^n = L_m^n + \binom{n}{1}L_m^{n-1}\sqrt{5}F + \binom{n}{2}L_m^{n-2}5F_m^2 + \cdots + (\sqrt{5})^n F_m^n, \tag{1.5}$$

and

$$(L_m - \sqrt{5}F_m)^n = L_m^n - \binom{n}{1}L_m^{n-1}\sqrt{5}F_m + \binom{n}{2}L_m^{n-2}5F_m^2 - \cdots + (-1)^n(\sqrt{5})^n F_m^n. \tag{1.6}$$

If these results are substituted into (1.3), we see that $L_m^n$ cancels out. The remaining terms all have a nonzero power of $F_m$, and we have found a simple proof of the known result that $F_{nm}$ is divisible by $F_m$ and $F_n$. For Lucas numbers, we observe that cancellation of the last term in (1.5) and (1.6) will take place only if $n$ is odd. Hence, $L_{mn}$ is divisible by $L_m$ only if $n$ is odd.

With the aid of (1.3) and (1.4), it is possible to obtain some appealing generating functions for Fibonacci and Lucas numbers. We proceed as follows:

$$\sum_{n=0}^{\infty} \frac{F_{nm}t^n}{n!} = \frac{1}{\sqrt{5}}\left[\sum_{n=0}^{\infty} \frac{\left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n t^n}{n!} - \sum_{n=0}^{\infty} \frac{\left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n t^n}{n!}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\exp\left\{\left(\frac{L_m + \sqrt{5}F_m}{2}\right)t - \exp\left(\frac{L_m - \sqrt{5}F_m}{2}\right)t\right\}\right]$$

$$= \frac{2}{\sqrt{5}} \exp\left(\frac{L_m t}{2}\right) \sinh\left(\frac{\sqrt{5}F_m}{2} t\right) \tag{1.7}$$

An identical procedure gives

$$2 \exp\left(\frac{L_m}{2} t\right) \cosh\left(\frac{\sqrt{5}F_m}{2} t\right) = \sum_{n=0}^{\infty} \frac{L_{mn}t^n}{n!}. \tag{1.8}$$

Some curious formulas may be obtained from (1.7) and (1.8). From (1.7), for example, one has

$$F_m t \exp\left(\frac{L_m}{2} t\right) = \frac{\sqrt{5}F_m}{2} t \;\operatorname{csch}\left(\frac{\sqrt{5}F_m}{2} t\right) \sum_{n=0}^{\infty} \frac{F_{nm}t^n}{n!}. \tag{1.9}$$

Using the expansion [1],

$$z \operatorname{csch} z = \sum_{k=0}^{\infty} -\frac{2(2^{2k-1} - 1)B_{2k}}{(2k)!} z^{2k}, \quad |z| < \pi, \tag{1.10}$$

where the $B_{2k}$ are Bernoulli numbers, and forming the Cauchy product, we have

$$F_m t \exp\left(\frac{L_m}{2} t\right) = -\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{F_{nm}\, 2(2^{2k-1} - 1)B_{2k}5^k F_m^{2k} t^{n+2k}}{n!(2k)!2^{2k}}$$

$$= -\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{2(2^{2k-1} - 1)5^k F_m^{2k} F_{(n-2k)m} B_{2k} t^n}{(n - 2k)!(2k)!2^{2k}},$$

where $[n/2]$ designates the greatest integer in $n/2$.

Expanding the exponential and equating corresponding powers of $t$, we get

$$L_m^{n-1} = -\frac{1}{nF_m}\sum_{k=0}^{[n/2]} \binom{n}{2k} 2^{n-2k}(2^{2k-1} - 1)5^k F_m^{2k} F_{(n-2k)m} B_{2k}, \tag{1.11}$$

which gives powers of Lucas numbers in terms of Fibonacci and Bernoulli numbers.

From (1.8), one has

$$2 \cosh\left(\frac{\sqrt{5}F_m}{2} t\right) = \exp\left(-\frac{L_m}{2} t\right) \sum_{n=0}^{\infty} \frac{L_{mn}t^n}{n!}.$$

Expanding the exponential term, forming the Cauchy product, and separating the even part, since the left-hand side is even, one finds

$$F_m^{2n} = \frac{2^{2n-1}}{5^n}\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k 2^{-k} L_m^k L_{m(2n-k)}, \tag{1.12}$$

which gives even powers of Fibonacci numbers in terms of Lucas numbers.

In (1.8), change $t$ to $-t$ and add the result to (1.8) to obtain

$$2\cosh\left(\frac{L_m}{2}t\right)\cosh\left(\frac{\sqrt{5}F_m}{2}t\right) = \sum_{n=0}^{\infty}\frac{L_{2nm}t^{2n}}{(2n)!},$$

which may be written

$$2\cosh\left(\frac{\sqrt{5}F_m}{2}t\right) = \operatorname{sech}\left(\frac{L_m}{2}t\right)\sum_{n=0}^{\infty}\frac{L_{2nm}t^{2n}}{(2n)!}.$$

Using the expansion [1],

$$\operatorname{sech} z = \sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}z^{2n}, \qquad |z| < \tfrac{1}{2}\pi$$

where the $E_{2n}$ are Euler numbers, we find

$$F^{2n} = \frac{1}{5^n}\sum_{k=0}^{n}\binom{2n}{2k}2^{2k-1}L_{2km}L_m^{2n-2k}E_{2n-2k}, \tag{1.13}$$

which gives even powers of Fibonacci numbers in terms of Lucas and Euler numbers.

Byrd [4], [5] has obtained expressions for Fibonacci and Lucas numbers which bear some resemblance to the expressions obtained by the author.

Now, observe that

$$\left(\frac{L_m + \sqrt{5}F_m}{2}\right)\left(\frac{L_m - \sqrt{5}F_m}{2}\right) = (-1)^m. \tag{1.14}$$

This relation can be used to advantage to obtain sums of reciprocals of Fibonacci and Lucas numbers. For this purpose, it is convenient to introduce the abbreviations:

$$a_m = \frac{L_m + \sqrt{5}F_m}{2}, \quad a_1 = a = \frac{1+\sqrt{5}}{2}, \tag{1.15}$$

$$b_m = \frac{L_m - \sqrt{5}F_m}{2}, \quad b_1 = b = \frac{1-\sqrt{5}}{2}. \tag{1.16}$$

We define the *Lambert series* as

$$L(\beta) = \sum_{n=1}^{\infty}\frac{\beta^n}{1-\beta^n}, \qquad |\beta| < 1, \tag{1.17}$$

and note that

$$L(\beta) - L(\beta^2) = \sum_{n=1}^{\infty}\frac{\beta^n}{1-\beta^n} - \sum_{n=1}^{\infty}\frac{\beta^{2n}}{1-\beta^{2n}} = \sum_{n=1}^{\infty}\frac{\beta^{2n}}{1-\beta^{2n}}. \tag{1.18}$$

We will make use of *Jacobi's identity* [2]

$$\theta_3^2(q) = 1 + 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}, \qquad |q| < 1, \tag{1.19}$$

where

$$\theta_3(q) = \sum_{n=-\infty}^{\infty}q^{n^2},$$

is a special case of the *third theta function* of Jacobi. Jacobi's *second theta function*:

$$\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2},$$

is related to the third theta function through the identity

$$\theta_3^2(q^2) + \theta_2^2(q^2) = \theta_3^2(q). \tag{1.20}$$

Recalling (1.14), observe that we have

$$\sum_{n=1}^{\infty} \frac{1}{a_m^n - b_m^n} = \sum_{n=1}^{\infty} \frac{b_m^n}{(-1)^{nm} - b_m^{2n}},$$

which, for even $m$, gives at once

$$\sum_{n=1}^{\infty} \frac{1}{F_{2nm}} = \sqrt{5}[L(b_{2m}) - L(b_{2m}^2)], \tag{1.21}$$

while, for odd $m$, remembering that $b_{2m+1}$ is negative, we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{(2n+1)(2m+1)}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{(-b_{2m+1})^{2n+1}}{1 + (-b_{2m+1})^{4n+2}}$$

$$= \frac{\sqrt{5}}{4}[\theta_3^2(-b_{2m+1}) - \theta_3^2(b_{2m+1}^2)] = \frac{\sqrt{5}}{4}\,\theta_2^2(b_{2m+1}^2). \tag{1.22}$$

Equations (1.21) and (1.22) are generalizations of results obtained by Landau [8].

For Lucas numbers, one has

$$\sum_{n=1}^{\infty} \frac{1}{a_m^n + b_m^n} = \sum_{n=1}^{\infty} \frac{b_m^n}{(-1)^{nm} + b_m^{2n}},$$

which, for even $m$, gives

$$\sum_{n=1}^{\infty} \frac{1}{L_{2nm}} = \tfrac{1}{4}[\theta_3^2(b_{2m}) - 1], \tag{1.23}$$

while, for odd $m$, we get

$$\sum_{n=0}^{\infty} \frac{1}{L_{(2n+1)(2m+1)}} = \sum_{n=0}^{\infty} \frac{(-b_{2m+1})^{2n+1}}{1 - (-b_{2m+1})^{4n+2}}$$

$$= L(-b_{2m+1}) - 2L(b_{2m+1}^2) + L(b_{2m+1}^4). \tag{1.24}$$

The last equality above is established in a manner wholly analogous to equation (1.18).

Many more relations can be established by simply imitating the procedures used for ordinary Fibonacci and Lucas numbers. The only change is to replace $a$ and $b$ by $a_m$ and $b_m$. In particular, Borwein & Borwein [2], and Bruckman [3] give a host of such relations.

## 2. A Class of Series for the Arc Tangent

In reference [6], we made use of Chebyshev polynomials of the first and second kinds

$$T_n(x) = \frac{1}{2}[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n], \tag{2.1}$$

$$U_n(x) = \frac{1}{2}\left[\frac{(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}}{\sqrt{x^2 - 1}}\right] \tag{2.2}$$

to establish, with $x = \sqrt{5}/2$, the relations

$$F_{2n} = \frac{1}{\sqrt{5}} U_{2n-1}\left(\frac{\sqrt{5}}{2}\right), \ n \geq 1, \tag{2.3}$$

$$F_{2n+1} = \frac{2}{\sqrt{5}} T_{2n+1}\left(\frac{\sqrt{5}}{2}\right). \tag{2.4}$$

$$L_{2n} = 2T_{2n}\left(\frac{\sqrt{5}}{2}\right), \tag{2.5}$$

$$L_{2n+1} = U_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{2.6}$$

Equations (2.5) and (2.6) were given in a different guise.

In reference [6], we also established the two series for the arc tangent:

$$\tan^{-1}\alpha = \frac{2}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{(-1)^n T_{2n+1}(x) t^{2n+1}}{5^n(2n + 1)}, \tag{2.7}$$

with

$$t = \frac{\sqrt{5}\alpha}{x + \sqrt{x^2 + \alpha^2}}, \tag{2.8}$$

and

$$\tan^{-1}\alpha = 4 \sum_{n=0}^{\infty} \frac{(-1)^n T^2_{2n+1}(x)}{(2n + 1)(t + \sqrt{t^2 + 1})^{2n+1}}, \tag{2.9}$$

with

$$t = \frac{x^2}{\alpha}(1 + \sqrt{1 + [\alpha^2(2x^2 - 1)/x^4]}). \tag{2.10}$$

These series give, with $x = \sqrt{5}/2$, the results

$$\tan^{-1}\alpha = \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1} t^{2n+1}}{5^n(2n + 1)}, \tag{2.11}$$

with

$$t = \frac{2\alpha}{1 + \sqrt{1 + (4\alpha^2/5)}}, \tag{2.12}$$

and

$$\tan^{-1}\alpha = 5 \sum_{n=0}^{\infty} \frac{(-1)^n F^2_{2n+1}}{(2n + 1)(t + \sqrt{t^2 + 1})^{2n+1}}, \tag{2.13}$$

with

$$t = \frac{5}{4\alpha}(1 + \sqrt{1 + (24\alpha^2/25)}). \tag{2.14}$$

To generalize these equations, we need an analogue of (1.3) for Chebyshev polynomials of the first kind.

We know that

$$T_m(\cos\theta) = \cos m\theta. \tag{2.15}$$

Let $m\theta = \phi$, and we have

$$T_n[T_m(\cos\theta)] = T_n(\cos\phi) = \cos n\phi = \cos nm\theta,$$

or

$$T_n[T_m(z)] = T_{nm}(z), \tag{2.16}$$

which is the desired relation.

For $n$ and $m$ odd and $z = \sqrt{5}/2$, (2.16) gives

$$T_{2n+1}\left(\frac{\sqrt{5}}{2}F_{2m+1}\right) = \frac{\sqrt{5}}{2}F_{(2n+1)(2m+1)}, \tag{2.17}$$

while, for even $m$, we get

$$T_n(\tfrac{1}{2}L_{2m}) = \tfrac{1}{2}L_{2nm}. \tag{2.18}$$

Similar relations may be obtained directly from (2.1) and (2.2):

$$T_{2n+1}\left(i\frac{\sqrt{5}}{2}F_{2m}\right) = \frac{(-1)^n i\sqrt{5}}{2}F_{(2n+1)2m}, \tag{2.19}$$

$$T_n(\tfrac{1}{2}iL_{2m+1}) = \tfrac{1}{2}i^n L_{n(2m+1)}, \tag{2.20}$$

$$T_{2n}\left(\frac{\sqrt{5}}{2}F_{2m+1}\right) = \tfrac{1}{2}L_{2n(2m+1)}, \tag{2.21}$$

$$U_{2n-1}\left(\frac{\sqrt{5}}{2}F_{2m+1}\right) = \sqrt{5}\,\frac{F_{2n(2m+1)}}{L_{2m+1}}, \tag{2.22}$$

$$U_{2n-1}\left(i\frac{\sqrt{5}}{2}F_{2m}\right) = (-1)^{n+1}i\sqrt{5}\,\frac{F_{4nm}}{L_{2m}}, \tag{2.23}$$

$$U_{n-1}(\tfrac{1}{2}L_{2m}) = \frac{F_{2nm}}{F_{2m}}, \tag{2.24}$$

$$U_{n-1}(\tfrac{1}{2}iL_{2m+1}) = i^{n-1}\frac{F_{n(2m+1)}}{F_{2m+1}}, \tag{2.25}$$

$$U_{2n}\left(\frac{\sqrt{5}}{2}F_{2m+1}\right) = \frac{L_{(2n+1)(2m+1)}}{L_{2m+1}}. \tag{2.26}$$

In equations (2.19), (2.20), (2.23), and (2.25), $i$ is the imaginary unit.

Changing $x$ to $T_{2m+1}(x)$ in (2.7) through (2.10), letting $x = \sqrt{5}/2$, and using (2.16), (2.4), and (2.17), we find

$$\tan^{-1}\alpha = \sum_{n=0}^{\infty}\frac{(-1)^n F_{(2n+1)(2m+1)}t^{2n+1}}{5^n(2n+1)}, \tag{2.27}$$

with

$$t = \frac{2\alpha}{F_{2m+1} + \sqrt{F_{2m+1}^2 + (4/5)\alpha^2}}, \tag{2.28}$$

and

$$\tan^{-1}\alpha = 5\sum_{n=0}^{\infty}\frac{(-1)^n F_{(2n+1)(2m+1)}^2}{(2n+1)(t+\sqrt{t^2+1})^{2n+1}}, \tag{2.29}$$

with

$$t = \frac{5F_{2m+1}^2}{4\alpha}(1 + \sqrt{1 + [(4\alpha/5)^2((5/2)F_{2m+1}^2 - 1)/F_{2m+1}^4]}). \tag{2.30}$$

Observe that the right-hand sides of (2.27) and (2.29) are independent of $m$. Equations (2.27) through (2.30) reduce to (2.11) through (2.14) when $m = 0$.

Series (2.27) and (2.29) converge most rapidly for $m = 0$. The reader should have no trouble showing that, as m increases without bound, (2.27) and (2.29) degenerate into Gregory's series:

$$\tan^{-1}\alpha = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{2n+1}, \quad |\alpha| \le 1. \tag{2.13}$$

Equations (2.27) through (2.30) provide a class of series for the arc tangent whose convergence lies between those of series (2.11) and (2.13) and that of series (2.31).

## 3. Some Series for $\pi$

The series we obtained in the previous section can be used to obtain some curious expressions for $\pi$.

For instance, (2.27) through (2.30) give, with $\alpha = 1$,

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n F_{(2n+1)(2m+1)}}{5^n(2n+1)} \frac{2^{2n+1}}{(F_{2m+1} + \sqrt{F_{2m+1}^2 + (4/5)})^{2n+1}}, \tag{3.1}$$

and

$$\frac{\pi}{4} = 5\sum_{n=0}^{\infty} \frac{(-1)^n F_{(2n+1)(2m+1)}^2}{(2n+1)(t + \sqrt{t^2+1})^{2n+1}}, \tag{3.2}$$

with

$$t = \frac{5F_{2m+1}^2}{4}(1 + \sqrt{1 + [(4/5)^2((5/2)F_{2m+1}^2 - 1)/F_{2m+1}^4]}). \tag{3.3}$$

For $m = 0$, (3.1) and (3.2) become

$$\frac{\pi}{4} = \sqrt{5}\sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1} 2^{2n+1}}{(2n+1)(3+\sqrt{5})^{2n+1}}, \tag{3.4}$$

and

$$\frac{\pi}{4} = 5\sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1}^2}{(2n+1)(3+\sqrt{10})^{2n+1}} \tag{3.5}$$

Series (3.4) and (3.5) were published by the author in [6].

Note that, as $m$ increases, series (3.1) and (3.20 will go from equations (3.4) and (3.5) to the limiting case of Leibniz's series

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots . \tag{3.6}$$

An explicit evaluation of series (3.4) and (3.5) requires a rapid algorithm for the numerical determination of $\sqrt{5}$ and $\sqrt{10}$. The interested reader may use the series

$$\sqrt{5} = \frac{1117229}{499640}\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k 9559^k}{20^{3k} 12491^{2k} k!}, \tag{3.7}$$

and

$$\sqrt{10} = \frac{790269}{249905}\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (-1)^k 8444^k}{10^{3k} 49981^{2k} k!}, \tag{3.8}$$

where

$(\alpha)_k = \alpha(\alpha+1)(\alpha+2) \cdots (\alpha+k-1)$, $\alpha \ne 0$,

$(\alpha)_0 = 1$ is Pochhammer's symbol.

Either series taken to $k = 12$ gives one hundred decimal places of the corresponding root.

Series (3.7) and (3.8) are special cases of the following general expression

$$\sqrt[N]{s/t} = \frac{n}{smt^{r-1}} \sum_{k=0}^{\infty} \frac{(1/N)_k \alpha^k}{s^{(N+1)k} t^{[(r-1)N-1]k} m^{Nk} k!}, \tag{3.9}$$

where $s$, $t$, and $N$ are positive integers, $n$ is the positive integer nearest

$$\frac{s}{t} t^r m \left(\frac{s}{t}\right)^{1/N}, \quad r > 1, \tag{3.10}$$

determined with a calculator, and $r$ and $m$ are arbitrary positive integers ($m$ may even be a positive rational). $\alpha$ is an integer, positive or negative, that satisfies the equation

$$\frac{s^{N+1}}{t^{N+1}} t^{rN} m^N - \alpha = n^N. \tag{3.11}$$

Equation (3.9) is simply an identity found by expanding the expression

$$\left(1 - \frac{\alpha}{[(s^{N+1})/(t^{N+1})] t^{rN} m^N}\right)^{-1/N} \tag{3.12}$$

in two different ways: (1) by putting the quantity inside parentheses under a common denominator and using (3.11) and (2) by expanding (3.12) by Newton's binomial theorem:

$$(1 - z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!}, \quad |z| < 1. \tag{3.13}$$

Generally, the larger the $m$ and $r$ are, the more rapidly converging the series is.

For series (3.7), we searched for a value of $m$ in the neighborhood of 100,000 for which $n$ would differ from an integer by not more than ±0.01. This makes $\alpha$ small and improves convergence. The parameter $r$, of course, plays no part when $t = 1$.

For $m = 99928$, we found

$$5 \cdot 99928 \cdot \sqrt{5} = 1117229.00427,$$

so we take $n = 1117229$ and find, using (3.11),

$$5^3 \cdot 99928^2 - \alpha = 1117229^2,$$

which gives $\alpha = 9559$.

For the series (3.8), we found $m = 99962$, and

$$10 \cdot 99962 \cdot \sqrt{10} = 3161075.99464,$$

which gives $n = 3161076$, and

$$10^3 \cdot 99962^2 - \alpha = 3161076^2,$$

which gives $\alpha = -33776$.

These sets of values, when substituted in (3.9), give series (3.7) and (3.8).

It can be shown [7] that, if $p_n/q_n$ is a convergent in the expansion of a real number $x$ as a continued fraction, then there does not exist any rational number $a/b$ with $b \le q_n$ that approximates $x$ better than $p_n/q_n$. Hence, a sensible way to make (3.10) nearly an integer is to choose $m$ as the denominator of a high enough convergent in the expansion of the $N^{\text{th}}$ root of $s/t$ as a continued fraction.

For the case of square roots, the identity

$$\sqrt{a} - b = (a - b^2)/(2b + (\sqrt{a} - b)),$$

due to Michel Rolle (1652-1719), *Mémoires de mathématiques et de physiques*, vol. 3, p. 24 (Paris, 1692), gives at once the continued fraction

$$\sqrt{a} = b + \frac{a - b^2}{2b} + \frac{a - b^2}{2b} + \frac{a - b^2}{2b} + \cdots \tag{3.14}$$

from which we can obtain suitable continued fraction expansions by giving appropriate values to $a$ and $b$.

For instance, $a = 5$, $b = 2$ gives

$$\sqrt{5} = 2 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots \tag{3.15}$$

and $a = 10$, $b = 3$ gives

$$\sqrt{10} = 3 + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \cdots. \tag{3.16}$$

Note that $a = 5/4$, $b = 1/2$, gives the well-known result

$$\frac{\sqrt{5} + 1}{2} = 1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \cdots.$$

Let $p_n/q_n$ be the $n^{\text{th}}$ convergent in the expansion of a real number $\sqrt{D}$ in a continued fraction. Consider the following identity

$$\sqrt{D} = \frac{p_n}{q_n}\left(\frac{p_n^2}{Dq_n^2}\right)^{-\frac{1}{2}} \quad \frac{p_n}{q_n}\left(1 + \frac{p_n^2 - Dq_n^2}{Dq_n^2}\right)^{-\frac{1}{2}} \tag{3.17}$$

Now, it is known that, for an appropriate value of $n$, the expression $p_n^2 - Dq_n^2$ will be either +1 or -1, a fact intimately bound up with the properties of Pell's equation. These $n$'s occur in cycles; hence, we can make the second term in parentheses in (3.17) as small as we please by choosing a sufficiently large value of $n$.

For the continued fraction (3.16), we choose the convergent

9238605483/2921503573,

and from (3.17) find the series

$$\sqrt{10} = \frac{9238605483}{2921503573}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k}{10^k 2921503573^{2k} k!}, \tag{3.18}$$

which picks up about twenty decimals per term, i.e., series (3.18) carried to $k = 5$ gives one hundred decimal places of the square root of ten.

For the square root of five, we can use (3.15), but we can do better if we remember that $L_n/F_n \to \sqrt{5}$ as $n$ increases. Using the same idea exemplified in (3.17), we obtain

$$\sqrt{5} = \frac{L_n}{F_n}\left(1 + \frac{L_n^2 - 5F_n^2}{5F_n^2}\right)^{-\frac{1}{2}}. \tag{3.19}$$

Since $L_n^2 - 5F_n^2 = 4(-1)^n$, we see that the numerator in the fraction inside parentheses is $4(-1)^n$ and the corresponding series will give any number of decimal places per term by choosing $n$ large enough. It is desirable to choose $n$ as a multiple of 3, because then $F_n$ is even and the factor of 4 cancels out. In that case, (3.19) becomes identical to (3.17) with $D = 5$.

Choosing $n = 48$, we have $F_{48} = 4807526976$ and

$$\sqrt{5} = \frac{5374978561}{2403763488} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (-1)^k}{5^k 2403763488^{2k} k!}, \tag{3.20}$$

which also picks up about twenty decimal places per term.

Needless to say, one may also use the computer to search for a good value of $m$, and then use this value to construct an appropriate series. Note, for instance, the values found in this manner:

$$5 \cdot \sqrt{5} \cdot 83204 = 930248.9999994625, \tag{3.21}$$

$$10 \cdot \sqrt{10} \cdot 777526 = 24587531.00000079. \tag{3.22}$$

For (3.21), we searched for a value of $m$ in the neighborhood of 100,000. The corresponding series gives about twelve decimal places per term. For (3.22), we searched for a value in the neighborhood of 750,000. The corresponding series picks up about seventeen decimal places per term.

By way of comparison, the values we used in series (3.18) and (3.20) give

$$10 \cdot \sqrt{10} \cdot 2921503573 = 92386054830.00000000057353236,$$

$$5 \cdot \sqrt{5} \cdot 2403763488 = 26874892804.999999999624625216.$$

## 4. Some Identities for Fibonacci Numbers

Equations (2.17) through (2.26) provide many interesting relations for Fibonacci and Lucas numbers.

The identity [1]

$$2(x^2 - 1) \sum_{m=0}^{n} U_{2m}(x) = T_{2n+2}(x) - 1 \tag{4.1}$$

gives, with $x = \frac{1}{2}L_{2k}$ and use of (2.24) and (2.18), the result

$$\sum_{m=0}^{n} F_{(2m+1)2k} = \frac{L_{(2n+2)2k} - 2}{5F_{2k}} \tag{4.2}$$

The identity [1]

$$2(1 - x^2) \sum_{m=1}^{n} U_{2m-1}(x) = x - T_{2n+1}(x) \tag{4.3}$$

gives, with $x = \frac{1}{2}L_{2k}$ and use of (2.24) and (2.18), the result

$$\sum_{m=1}^{n} F_{(2m)2k} = \frac{L_{(2n+1)2k} - L_{2k}}{5F_{2k}}. \tag{4.4}$$

Equations (4.2) and (4.4) can be combined to give

$$\sum_{m=1}^{n} F_{m2k} = \frac{L_{(n+1)2k} + L_{n2k} - L_{2k} - 2}{5F_{2k}}. \tag{4.5}$$

Equation (4.3), with $x = (\sqrt{5}/2)F_{2k+1}$ and use of (2.17) and (2.22) gives

$$\sum_{m=1}^{n} F_{2m(2k+1)} = \frac{F_{(2n+1)(2k+1)} - F_{2k+1}}{L_{2k+1}}. \tag{4.6}$$

The identity [1]

$$\sum_{m=0}^{n-1} T_{2m+1}(x) = \tfrac{1}{2}U_{2n-1}(x) \tag{4.7}$$

gives, with $x = (\sqrt{5}/2)F_{2k+1}$ and the use of (2.17) and (2.22), the result

$$\sum_{m=0}^{n-1} F_{(2m+1)(2k+1)} = \frac{F_{2n(2k+1)}}{L_{2k+1}}. \tag{4.8}$$

Equations (4.6) and (4.8) combine to give the expression

$$\sum_{m=1}^{n} F_{m(2k+1)} = \frac{F_{(n+1)(2k+1)} + F_{n(2k+1)} - F_{2k+1}}{L_{2k+1}}. \tag{4.9}$$

Equations (4.5) and (4.9) are generalizations of well-known results.

The reader should note that these formulas, once established, may be verified by induction.

## 5. Other Numerical Sequences Associated with Classical Polynomials

Much of the success we have had in obtaining properties of Fibonacci and Lucas numbers has depended largely on our being able to associate a recurrent sequence of numbers with the set of Chebyshev polynomials. The question naturally arises as to whether other such sequences of positive integers exist associated with other classical polynomials. Surprisingly, such sequences do exist in a number of important cases.

For example, if $P_n(x)$ designates Legendre polynomials, the expression

$$b_n = 2^n i^{-n} P_n(i) \tag{5.1}$$

gives a recurrent sequence of positive integers associated with Legendre polynomials. We have, explicitly,

$$b_n = \sum_{k=0}^{[n/2]} \frac{(2n-2k)!}{k!(n-k)!(n-2k)!}. \tag{5.2}$$

The pure recurrence relation for Legendre polynomials

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x),\ n \geq 2, \tag{5.3}$$

together with (5.1) gives

$$nb_n = 2(2n-1)b_{n-1} + 4(n-1)b_{n-2},\ n \geq 2,\ b_0 = 1,\ b_1 = 2, \tag{5.4}$$

which defines the $b_n$ recurrently. The first few are

$$b_0 = 1,\ b_1 = 2,\ b_2 = 8,\ b_3 = 32,\ b_4 = 136,\ b_5 = 592, \text{ etc.}$$

Similarly, if $L_n(x)$ designates the simple Laguerre polynomials, the expression

$$c_n = n!L_n(-1) \tag{5.5}$$

gives a recurrent sequence of positive integers associated with simple Laguerre polynomials. We have, explicitly,

$$c_n = n! \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!}. \tag{5.6}$$

The pure recurrence relation for simple Laguerre polynomials

$$nL_n(x) = (2n - 1 - x)L_{n-1}(x) - (n - 1)L_{n-2}(x),\ n \ge 2, \tag{5.7}$$

together with (5.5) gives

$$c_n = 2nc_{n-1} - (n - 1)^2 c_{n-2},\ n \ge 2,\ c_0 = 1,\ c_1 = 2, \tag{5.8}$$

which defines the $c_n$ recurrently. The first few are

$$c_0 = 1,\ c_1 = 2,\ c_2 = 7,\ c_3 = 34,\ c_4 = 209,\ c_5 = 1546, \text{ etc.}$$

Using the known generating function for simple Laguerre polynomials

$$(1 - t)^{-1}\exp\left(\frac{-xt}{1 - t}\right) = \sum_{n=0}^{\infty} L_n(x)t^n,$$

we obtain at once

$$(1 - t)^{-1}\exp\left(\frac{t}{1 - t}\right) = \sum_{n=0}^{\infty} \frac{c_n t^n}{n!}.$$

Now, replacing $t/(1 - t)$ by $x$, we find the interesting expansion

$$e^x = \sum_{n=0}^{\infty} \frac{c_n}{n!} \frac{x^n}{(1 + x)^{n+1}},\ x > -\tfrac{1}{2}. \tag{5.9}$$

Another curious series for the exponential is found from the expression

$$e^{xt} = \left(\frac{t}{2}\right)^{-\nu}\Gamma(\nu)\sum_{n=0}^{\infty}(\nu + n)I_{\nu+n}(t)C_n^{\nu}(x), \tag{5.10}$$

due to Gegenbauer, where $I_k(t)$ are modified Bessel functions of the first kind [9], given by

$$I_k(t) = \frac{(\tfrac{1}{2}t)^k}{\Gamma(k + 1)}\ {}_0F_1(-;\ 1 + k;\ \tfrac{1}{4}t^2),$$

and $C_n^{\nu}(x)$ are ultraspherical polynomials [9] defined by

$$C_n^{\nu}(x) = \frac{(2\nu)_n P_n^{(\nu-\frac{1}{2},\ \nu-\frac{1}{2})}(x)}{(\nu + \tfrac{1}{2})_n},$$

where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials.

In terms of ultraspherical polynomials, Chebyshev polynomials are given by

$$U_n(x) = C_n^1(x), \tag{5.11}$$

$$T_n(x) = \lim_{\nu\to 0}\frac{C_n^{\nu}(x)}{C_n^{\nu}(1)}. \tag{5.12}$$

With appropriate substitutions in (5.10) and making use of (2.3) and (2.6), we have

$$e^x = \frac{\sqrt{5}}{x}\sum_{n=1}^{\infty} n\left[\frac{1 + (-1)^n}{2}\sqrt{5}F_n + \frac{1 - (-1)^n}{2}L_n\right]I_n(2x/\sqrt{5}). \tag{5.13}$$

If $H_n(x)$ designates Hermite polynomials, then the expression

$$d_n = 2^{-n/2}\, i^n H_n(-i/\sqrt{2}), \tag{5.14}$$

gives a recurrent sequence of positive integers.

The pure recurrence relation for Hermite polynomials

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x),$$

together with (5.14) gives

$$d_{n+1} = d_n + nd_{n-1},\ n \geq 1,\ d_0 = 1,\ d_1 = 1. \tag{5.15}$$

Sequence (5.15) has been studied by P. Rubio, *Dragados y Construcciones* (Madrid, Spain), although (5.14) was, to my knowledge, discovered by me (see [10]).

The first few $d_n$'s are

$$d_0 = 1,\ d_1 = 1,\ d_2 = 2,\ d_3 = 4,\ d_4 = 10,\ d_5 = 26,\ \text{etc.}$$

Known relations for Hermite polynomials provide interesting expansions with the $d_n$'s as coefficients. For instance, the generating relation [9]

$$(1 - 4t^2)^{-\frac{1}{2}}\exp\left[y^2 - \frac{(y - 2xt)^2}{1 - 4t^2}\right] = \sum_{k=0}^{\infty} \frac{H_k(x)H_k(y)t^k}{k!}$$

gives, on changing both $x$ and $y$ to $-i/\sqrt{2}$, and $t$ to $-t/2$, the interesting relation

$$(1 - t^2)^{-\frac{1}{2}}\exp\left(\frac{t}{1 - t}\right) = \sum_{k=0}^{\infty} \frac{d_k^2 t^k}{k!}. \tag{5.16}$$

Changing $\frac{t}{1 - t}$ to $x$, we find

$$e^x = (2x + 1)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{d_k^2}{k!} \frac{x^k}{(x + 1)^{k+1}},\ x > -\tfrac{1}{2}. \tag{5.17}$$

Series (5.9), (5.13), and (5.17) are offered only as mathematical curiosities. None of them converges faster than Euler's exponential series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Series (5.9) and (5.17), in particular, converge very slowly.

These recurrent sequences of positive integers associated with classical polynomials seem not to have been studied in the existing literature, in spite of the fact that they may well be used to advantage in numerical work.

## 6. Continued Fraction Expansions for Fibonacci and Lucas Numbers

We will close this paper by showing how to expand Fibonacci and Lucas numbers in nontrivial finite continued fractions. This result is rather surprising inasmuch as Fibonacci and Lucas numbers are integers.

The expression

$$S = \alpha_0 + \alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_3\alpha_4 + \cdots, \tag{6.1}$$

is easily seen to be equivalent to the infinite continued fraction

$$S = \alpha_0 + \frac{\alpha_1}{1} - \frac{\alpha_2}{(1 + \alpha_2)} - \frac{\alpha_3}{(1 + \alpha_3)} - \cdots. \tag{6.2}$$

If we let

$$\alpha_0 = \beta(z), \tag{6.3}$$

where $\beta(z)$ is an arbitrary function of $z$,

$$\alpha_1 = \beta(z)\frac{a_1a_2 \ldots a_p}{b_1b_2 \ldots b_q}z, \tag{6.4}$$

$$\alpha_k = \frac{(a_1 + k - 1)(a_2 + k - 1) \ldots (a_p + k - 1)}{(b_1 + k - 1)(b_2 + k - 1) \ldots (b_q + k - 1)}\frac{z}{k}, \; k > 1, \tag{6.5}$$

where none of the $b$'s is zero or a negative integer, then (6.1) becomes

$$\beta(z)\left[1 + \frac{a_1a_2 \ldots a_p}{b_1b_2 \ldots b_q}\frac{z}{1!} + \frac{(a_1)_2(a_2)_2 \ldots (a_p)_2}{(b_1)_2(b_2)_2 \ldots (b_q)_2}\frac{z^2}{2!} + \ldots\right]$$

$$= \beta(z)\,{}_pF_q\left[\begin{matrix}a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q;\end{matrix}\; z\right]. \tag{6.6}$$

Use of (6.2) with the values (6.3), (6.4), and (6.5) gives a continued fraction expansion for the generalized hypergeometric function ${}_pF_q(z)$ times an arbitrary function of $z$, $\beta(z)$. The continued fraction expansion converges, of course, whenever the infinite series defining the hypergeometric function converges. The continued fraction and the series converge and diverge together.

One of the known expressions for Jacobi polynomials is

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n(1+\beta)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n, 1+\alpha+\beta+n; \\ 1+\beta;\end{matrix}\; \frac{1+z}{2}\right]. \tag{6.7}$$

In terms of Jacobi polynomials, Chebyshev polynomials are given by

$$T_n(x) = \frac{n!}{(\frac{1}{2})_n}P_n^{(-\frac{1}{2},-\frac{1}{2})}(x), \tag{6.8}$$

and

$$U_n(x) = \frac{(n+1)!}{(\frac{3}{2})_n}P_n^{(\frac{1}{2},\frac{1}{2})}(x). \tag{6.9}$$

Simple substitutions, and use of (2.3) through (2.6), gives continued fraction expansions for Fibonacci and Lucas numbers.

Let us illustrate this by finding a continued fraction expansion for $L_4$. One has

$$L_{2n} = 2\,{}_2F_1\left[\begin{matrix}-2n, 2n; \\ \frac{1}{2};\end{matrix}\; \frac{2+\sqrt{5}}{4}\right],$$

from which one gets, for $n = 2$,

$$\beta(z) = 2,$$

$$\alpha_0 = 2,$$

$$\alpha_1 = -16(2+\sqrt{5}),$$

$$\alpha_2 = -\frac{5}{4}(2+\sqrt{5}),$$

$$\alpha_3 = -\frac{2}{5}(2+\sqrt{5}),$$

$$\alpha_4 = -\frac{1}{8}(2+\sqrt{5}),$$

$\alpha_k = 0, \; k > 4.$

From these, it follows that

$$L_4 = 2 - \frac{16(2+\sqrt{5})}{1} \; \frac{5(2+\sqrt{5})}{+\, -6 - 5\sqrt{5}} \; \frac{8(2+\sqrt{5})}{+\; 1 - 2\sqrt{5}} \; \frac{5(2+\sqrt{5})}{+\; 6 - \sqrt{5}} = 7,$$

as the reader can verify easily.

Putting the Jacobi polynomial into its several equivalent forms [9] gives different, but equivalent, continued fractions for Fibonacci and Lucas numbers.

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# ON $K^{th}$-ORDER COLORED CONVOLUTION TREES AND A GENERALIZED ZECKENDORF INTEGER REPRESENTATION THEOREM

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## 1. Introduction

In Turner [6], a sequence of trees was defined such that the $n^{\text{th}}$ tree $T_n$ had $F_n$ leaf-nodes, where $F_n$ is the $n^{\text{th}}$ element of the Fibonacci sequence. It was shown how to construct the trees so that the nodes were weighted with integers from a general sequence $\{C_n\}$ using a sequential method described in Section 2.

This produced a sequence of Fibonacci convolution trees $\{T_n\}$, so called because the sum of the weights assigned to the nodes of $T_n$ was equal to the nth term of the convolution product of $\{F_n\}$ and $\{C_n\}$. That is, the $\Omega$ meaning the sum of weights:

$$\Omega(T_n) = (F * C)_n = \sum_{i=1}^{n} F_i C_{n-i+1}.$$

This result is illustrated in Section 3.

With this construction, a graphical proof of a dual of Zeckendorf's theorem was given, namely that every positive integer can be represented as the sum of distinct Fibonacci numbers, with no gap greater than one in any representation, and that such a representation is unique [2].

To develop this procedure further, we give a construction for $k^{\text{th}}$-order colored trees, and for colors consider generalized Fibonacci numbers of order 2 and greater. To this end, we define the recurring sequence

$$\{W_n\} = \{W_n(a, b; p, q)\}$$

as in Horadam [4] by the homogeneous linear recurrence relation

$$W_n = pW_{n-1} - qW_{n-2}, \; n > 2,$$

with initial conditions $W_1 = a$, $W_2 = b$. The ordinary Fibonacci numbers are then

$$\{F_n\} \equiv \{W_n(1, 1; 1, -1)\}.$$

## 2. Construction of $K^{th}$-Order Colored Trees

Given a sequence of colors, $C = \{C_1, C_2, C_3, \ldots\}$, we construct $k^{\text{th}}$-order colored trees, $T_n$, as follows:

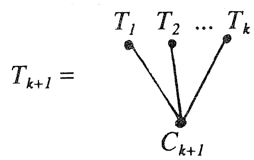
$T_1 = C_1^{\bullet}$

$T_n = T_{n-1}^{\bullet}\!\!-\!\!\!-\!\!\!-\!\!{}^{\bullet}\, C_n$, with $C_1^{\bullet}$ being the root node in each case,

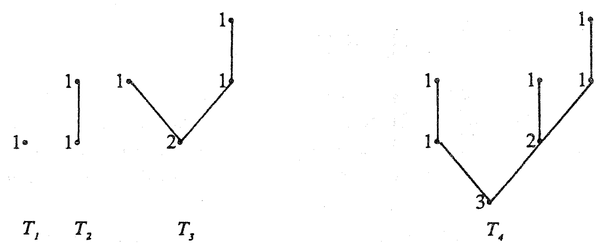
$n = 2, 3, \ldots, k;$

$$T_{k+m} = C_{k+m} \mathop{\mathbf{V}}_{i=0}^{k-1} T_{m+i},\ m = 1, 2, \ldots$$

in the "drip-feed" construction, in which the $k^{\text{th}}$-order fork operation $V$ is to mount trees $T_m$, $T_{m+1}$, ..., $T_{m+k-1}$ on separate branches of a new tree with root node at $C_{m+k}$ for $m \geq 1$. Thus, when $m = 1$, we get:




For example, when $k = 2$ and $C = \{F_n\}$, the sequence of Fibonacci convolution trees is

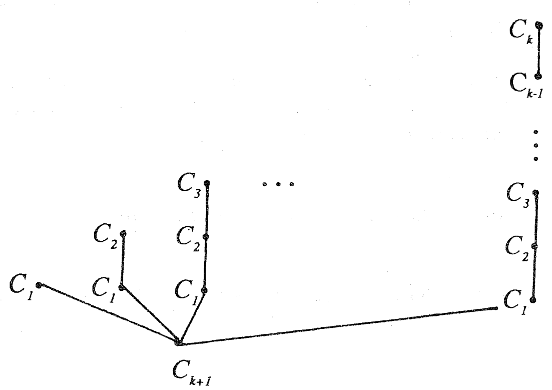



where $T_1$, $T_2$, are the initial trees, and subsequent ones are:

$$T_3 = F_3 \mathop{\mathbf{V}}_{i=0}^{1} T_{i+1} = F_3 V(T_1,\ T_2),$$

$$T_4 = F_4 V(T_2,\ T_3),\ \text{and so on (see Turner [8]).}$$

The tree $T_{k+1}$ for the general case is:




In Section 4, we take the colors from the $k^{\text{th}}$-order Fibonacci sequence (cf. Shannon [5]) given by

$$C_{n+k} = \sum_{i=0}^{k-1} C_{n+i},\ n \geq 1,\ \text{and with initial elements } C_1,\ C_2,\ \ldots,\ C_k.$$

The colors on the leaves taken from all the trees in the sequence, from left to right, form the Fibonacci word pattern (Turner [8]):

$$f(C_1, C_2, \ldots, C_k) = C_1, C_2, \ldots, C_k, C_1C_2 \ldots C_k, C_2 \ldots C_kC_1 \ldots C_k, \ldots$$

There is a remarkable relationship between the leaf-word pattern and the tree shade sets, which we shall discuss in Section 4.

Examples of Fibonacci word patterns, showing how they are formed by $r^{\text{th}}$-order word-juxtaposition recurrences are,

$$r = 2: \quad f(a, b) = a, b, ab, bab, abbab, \ldots$$
$$r = 3: \quad f(a, b, c) = a, b, c, abc, bcabc, cabcbcabc, \ldots$$
$$r = 4: \quad f(a, b, c, d) = a, b, c, d, abcd, bcdabcd, \ldots .$$

It is of interest to note that

$$f(a, b) = \{W_n(a, b; 1, -10 \uparrow F_{n-1})\}$$

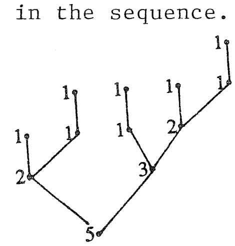
where $10 \uparrow m$ represents $10^m$ for notational convenience, $\{W_n\}$ is Horadam's generalized Fibonacci sequence, and $F_n$ are the ordinary Fibonacci numbers. Thus,

$$W_1 = a$$
$$W_2 = b,$$
$$\begin{aligned} W_3 &= W_2 + 10^1 W_1 \\ &= b + 10a \\ &= ab, \text{ in the above notation, and so on.} \end{aligned}$$

## 3. Number Properties of the Trees

### (i) Node weight totals

As stated in the Introduction, when $k = 2$ the sum of all node weights of $T_n$ is equal to the convolution term $(F * C)_n$. We illustrate this for the case $C = \{F_n\}$ and with the fifth tree in the sequence.

$T_5 =$ 


From observation,

$$\Omega(T_5) = 1 + (1 + 1 + 1 + 1) + (1 + 1 + 1 + 2) + (2 + 3) + 5 = 20.$$

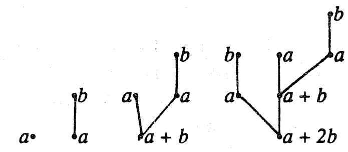
Using the formula, we get

$$\begin{aligned} \Omega(T_5) &= F_1F_5 + F_2F_4 + F_3F_3 + F_4F_2 + F_5F_1 \\ &= 2F_1F_5 + 2F_2F_4 + F_3^2 \\ &= 10 + 6 + 4 = 20. \end{aligned}$$

### (ii) Number of nodes with colors $C_1$, $C_2$, $\ldots$, $C_k$ at different nodes

Now consider the first four trees associated with the color sequence

$$C = F(a, b) = a, b, a + b, a + 2b, \ldots$$




$T_1$ $T_2$ $T_3$ $T_4$

Note that we use $f(a, b)$ to denote the word pattern and $F(a, b)$ for the color sequence.

Let $(n_a, n_b)$ represent the number of $a$'s and the number of $b$'s at any level in the tree; we get the following table for this pair.

| Tree $T_n$ | m = (level + 1) | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $T_1$ | | (1, 0) | | | | | |
| $T_2$ | | (1, 0) | (0, 1) | | | | |
| $T_3$ | | (1, 1) | (2, 0) | (0, 1) | | | |
| $T_4$ | | (1, 2) | (2, 1) | (2, 1) | (0, 1) | | |
| $T_5$ | | (2, 3) | (2, 3) | (4, 1) | (2, 2) | (0, 1) | |
| $T_6$ | | (3, 5) | (3, 5) | (4, 4) | (6, 2) | (2, 3) | (0, 1) |

If we represent the element in the $n^{\text{th}}$ row and $m^{\text{th}}$ column of this array by $x_{nm}$, then $x_{nm}$ satisfies the partial recurrence relation

$$x_{nm} = x_{n-1,m-1} + x_{n-2,m-1}, \quad 1 < m < n, \; n > 2,$$

with boundary conditions

$$x_{11} = x_{21} = (1, 0);$$

$$x_{22} = (0, 1);$$

$$x_{n1} = (F_{n-2}, F_{n-1}), \; n > 2;$$

$$x_{nm} = (0, 0), \; m > n.$$

The proof follows from the construction of the trees and the fact that the root color for $T_n$, after $n = 2$, is the $n^{\text{th}}$ term of $f(a, b)$, which is $aF_{n-2} + bF_{n-1}$ (see Horadam [3]).

## 4. Shades and Leaf Patterns

Consider the set of all leaf-to-root paths in a given convolution tree. Each leaf node determines a unique path, say $P_i$. We can label the paths $P_1$, $P_2$, ..., $P_g$ according to their position (taken from left to right) on the tree diagram. If we add up the node weights on path $P_i$ and denote this path weight by $W_i$, we obtain a sequence $\{W_1, W_2, \ldots, W_g\}$, called the shade of the tree (Turner [6]). This is denoted by $Z(T_n)$ and is described in more detail in Section 5.

Recall from Section 2 that the colors on the leaves of the trees in the sequence form a Fibonacci word pattern. For example, the pattern $f(1, 4)$ of leaf nodes for $k = 2$ can be seen in Figure 1.

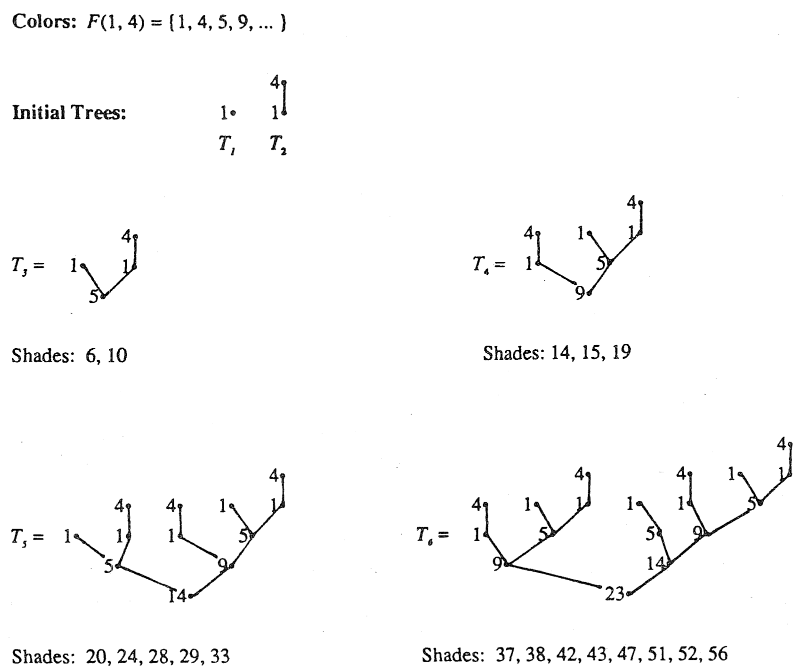



FIGURE 1

The shades can be generated by the $\phi$ function of Atanassov [1] defined by $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi(0) = 0$ and

$$\phi(n) \equiv \phi(a_1 a_2 \ldots a_k) = \sum_{i=1}^{k} a_i,$$

where

$$n = \sum_{i=1}^{k} 10^{k-i} a_i.$$

The shades and leaf numbers of the first four trees for $f(1, 4)$ are as follows:

| Tree | $T_1$ | $T_2$ | $T_3$ | $T_4$ | ... |
|---|---|---|---|---|---|
| Leaf numbers | 1 | 4 | 14 | 414 | ... |
| Shade | 1 | 5 | 6, 10 | 14, 15, 19 | |
| | $\phi(1) = 1$ | $\phi(14) = 5$ | $\phi(51) = 6$ | $\phi(10, 4) = 14$ | |
| | | | $\phi(64) = 10$ | $\phi(14, 1) = 15$ | |
| | | | | $\phi(15, 4) = 19$ | |

Note that $\phi(n)$ just accumulates the elements of the total leaf number pattern

$$f(1, 4) = 1, 4, 14, 414, 14414, \ldots,$$

from the left. Thus, the shade set for the tree sequence is

$$\begin{aligned} &1 \\ 1 + 4 &= 5 \\ 1 + 4 + 1 &= 6 \\ 1 + 4 + 1 + 4 &= 10\ldots \end{aligned}$$

In the general second-order case,

$$f(a, b) = a, b, ab, bab, \ldots,$$

and the shade sequence $r_i a + s_i b$ is, in turn,

$$\begin{array}{ccccccc} 1a, & 1a, & 2a, & 2a, & 2a, & 3a, & 3a, \\ & + & + & + & + & + & + \\ & 1b & 1b & 2b & 3b & 3b & 4b. \end{array}$$

An example can be seen in Figure 1 for $f(1, 4)$.

Various results can be developed from the coefficients of a and b in this sequence. For example, if we write them as two-component vectors, we get

$$\left\{\binom{r_i}{s_i}\right\} = \left\{\binom{1}{0}, \binom{1}{1}, \binom{2}{1}, \binom{2}{2}, \binom{2}{3}, \binom{3}{3}, \binom{3}{4}, \binom{4}{4}, \binom{4}{5}, \ldots\right\}$$

then the first differences are:

$$\begin{aligned} \{\Delta_i\} &= \left\{\binom{r_{i+1}}{s_{i+1}} - \binom{r_i}{s_i}\right\} \\ &= \left\{\binom{0}{1}, \binom{1}{0}, \binom{0}{1}, \binom{0}{1}, \binom{1}{0}, \binom{0}{1}, \binom{1}{0}, \binom{0}{1}, \binom{0}{1}, \ldots\right\} \\ &= f\left(\binom{0}{1}, \binom{1}{0}, \binom{0}{1}\right). \end{aligned}$$

Note that the elements of $\{\Delta_i\}$ determine the Wythoff pairs, much studied in the Fibonacci literature [7]. To see this, consider the positions of the 1's in the upper elements of $\{\Delta_i\}$, and likewise in the lower elements: the upper 1's indicate the sequence $\{[n\alpha^2]\}$, and the lower 1's the sequence $\{[n\alpha]\}$.

It is now clear that for the leaf number pattern

$$f(1, 1) = 1, 11, 111, 11111, \ldots$$

the shade is

$$1, 23, 456, 7891011, \ldots,$$

as Figure 2 so graphically illustrates: for we merely have to accumulate the sequence of 1's, from the left, to get the shade, which is the sequence at the base of the straight lines from the trees to the horizontal axis.

Thus, each natural number $n$ corresponds to a leaf-to-root path; and the path's color-sum provides a representation of $n$ as a sum of distinct Fibonacci numbers:

$$n = \sum e_i F_i, \qquad e_i \in \{0, 1\}.$$

Furthermore, $e_i + e_{i+1} > 0$ for each $i$, which means that there is never a gap greater than one among the Fibonacci numbers constituting any representation, which is evident from the "drip-feed" tree-coloring procedure. Deleting the 1 from each leaf node, in each representation, one obtains integer representations with the same properties but in terms of distinct members of the sequence $\{u_n\} = \{F_{n+1}\}$. This integer-representation result has come to be known as Zeckendorf's dual theorem [2].

We now present two general results about the leaf patterns and shade sets.

*Theorem 1:* For the $k^{th}$-order tree sequence defined in Section 2, the colors on the leaves, from left to right, form the Fibonacci word-pattern

$$f(C_1, C_2, \ldots, C_k).$$

*Proof:* The colors on the leaves, from left to right, are initially by construction $C_1, C_2, \ldots, C_k$ in turn, and then for $T_{k+1}$ they are $C_2 \ldots C_k C_1$, and so on, as in the recurrence that produces the $k^{th}$-order Fibonacci word-pattern.

*Corollary:* It follows that if $C$ is the $k^{\text{th}}$-order Fibonacci sequence with initial elements $C_1, C_2, \ldots, C_k$ as in Section 2, then

$$\sum_{T_n} (\text{leaf colors}) = (\text{root color } C_n),\ n > k.$$

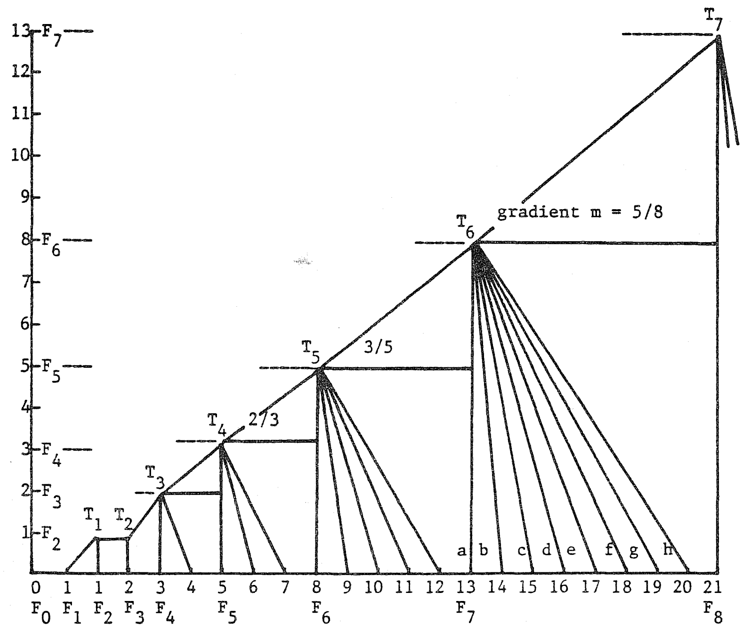



FIGURE 2(a)

Weights of leaf-to-root paths versus max. node weight

| | |
|---|---|
| a:1 1 3 8 | = 13 |
| b:1 2 3 8 | = 14 |
| c:1 1 2 3 8 | = 15 |
| d:1 2 5 8 | = 16 |
| e:1 1 2 5 8 | = 17 |
| f:1 1 3 5 8 | = 18 |
| g:1 2 3 5 8 | = 19 |
| h:1 1 2 3 5 8 | = 20 |

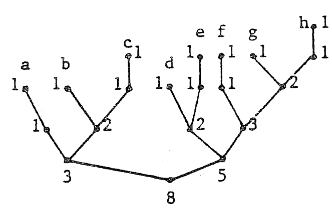



FIGURE 2(b)

Leaf-to-root paths for $T_6$

*Theorem 2:* The shade set of $T_n$ (from the sequence of Theorem 1 and its corollary) is given by adding leaf-colors from the left, that is, by computing the partial sums of the leaf-pattern. Thus, if the leaf nodes of $T_n$ have the color pattern $L_1L_2 \ldots L_r$ with each $L_i \in \{C_1, C_2, \ldots, C_k\}$, then the shade

$$Z(T_n) = \{S_1 + R_{n-1}, S_2 + R_{n-1}, \ldots, S_r + R_{n-1}\} \text{ for } n > k,$$

where $L_m$ is the $m^{\text{th}}$ partial sum of the leaf color pattern (left to right) of $T_n$, and

$$R_{n-1} = \sum_{i=1}^{n-1} C_i$$

is the sum of the root colors for the previous $n - 1$ trees.

*Proof:* An inductive proof is easily established.

*Corollary:* If $C_i = 1$ for $i = 1, \ldots, k$,

$$\lim_{n\to\infty} \bigcup_n Z(T_n) = Z^+, \text{ for any } k \geq 2.$$

This corollary provides the integer representations which are the subject of the next section.

We can also represent the shades in terms of $\{W_n\}$, as defined in Section 1. For $k = 2$ and $f(1, 1)$, we can define the sequence $\{S_{nm}\}$ by

$$S_{nm} \equiv 10W_n + 1 \pmod{10 \uparrow m},\ 1 \leq m \leq n.$$

Then, for example, for $\{W_n\} \equiv \{W_n(1, 4; 1, -10 \uparrow F_n)\}$, we have

| | | | | | |
|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 |
| $W_n$ | 1 | 4 | 14 | 414 | 14414 |

and

| | | | | | |
|---|---|---|---|---|---|
| $m$ | 1 | 2 | 3 | 4 | 5 |
| $S_{5m}$ | 1 | 41 | 141 | 4141 | 44141 |
| $\phi(S_{5m})$ | 1 | 5 | 6 | 10 | 14 |

which is the shade sequence we found in Section 3.

## 5. Integer Representation Theorem

A family of integer representations using the $k^{\text{th}}$-order Fibonacci sequence with 1's for the first $k$ elements is given by the following theorem.

*Theorem 3:* Any integer $n \in Z^+$ has a representation of the form

$$n = \sum e_i C_i, \quad e_i \in \{0, 1\}.$$

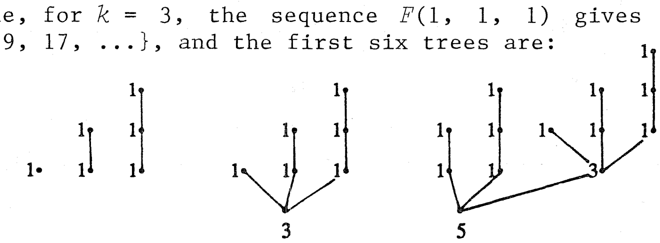
where the $C_i$ are distinct elements of the $k^{\text{th}}$-order Fibonacci sequence $F(1, 1, \ldots, 1)$, and

$$\sum_{j=0}^{k-1} e_{i+j} > 0 \text{ for all } i,\ k \geq 1.$$

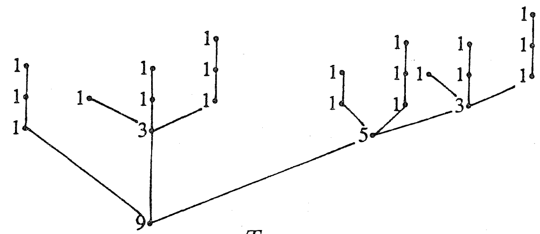
*Proof:* The proof follows immediately from Theorem 2 and its corollary and the manner of construction of the trees. (The Zeckendorf dual occurs when $k = 2$.)

*Corollary:* We can use the initial 1's in each representation in a manner which provides representations for all integers in terms of distinct elements of the sequence whose first elements are $1, 2, 3, \ldots, k$, and whose subsequent elements are the corresponding 1's of $F(1, 1, 1, \ldots, 1)$.

As an example, for $k = 3$, the sequence $F(1, 1, 1)$ gives the color set $\{1, 1, 1, 3, 5, 9, 17, \ldots\}$, and the first six trees are:




$T_1$ $T_2$ $T_3$ $T_4$ $T_5$



$T_6$

The following table shows the shades and the corresponding integer representations for integers $N$ = 1, ..., 15 when the initial 1's are replaced by a 2 when (1, 1) occurs and by 1, 2 when (1, 1, 1) occurs in a representation.

| $T_n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $Z(T_n)$ | 1 | 2 | 3 | 4 5 6 | 7 8 9 10 11 | 12 13 14 15 ... |
| Integers in Representation | 1 | 2 | 3 | 1 2 1<br>3 3 2<br>3 | 2 3 1 2 1<br>5 5 3 3 2<br>5 5 3<br>5 | 1 1 2 1<br>2 3 3 2<br>9 9 9 3<br>9 |
| Maximum gap | - | - | - | 1 0 0 | 1 0 1 0 0 | 2 1 1 1 |

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# RETROGRADE RENEGADES AND THE PASCAL CONNECTION: REPEATING DECIMALS REPRESENTED BY FIBONACCI AND OTHER SEQUENCES APPEARING FROM RIGHT TO LEFT

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Repeating decimals show a surprisingly rich variety of number sequence patterns when their repetends are viewed in retrograde fashion, reading from the rightmost digit of the repeating cycle towards the left. They contain geometric sequences as well as Fibonacci numbers generated by an application of Pascal's triangle. Further, fractions whose repetends end with successive terms of $F_{nm}$, $m = 1, 2, \ldots$, occurring in repeating blocks of $k$ digits, are completely characterized, as well as fractions ending with $F_{nm+p}$ or $L_{nm+p}$, where $F_n$ is the $n^{\text{th}}$ Fibonacci number,

$$F_1 = 1,\ F_2 = 1,\ F_{n+1} = F_n + F_{n-1},$$

and $L_n$ is the $n^{\text{th}}$ Lucas number,

$$L_1 = 1,\ L_2 = 3,\ L_{n+1} = L_n + L_{n-1}.$$

## 1. The Pascal Connection

It is no surprise that 1/89 contains the sum of successive Fibonacci numbers in its decimal expansion [2], [3], [4], [5], as

```
1/89 = .012358
            13
             21
              34
               ...
```

However, 1/89 can also be expressed as the sum of successive powers of 11, as

```
1/89 = .01
       .0011
       .000121
       .00001331
       .0000014641
       ...
```

where

$$1/89 = 1/10^2 + 11/10^4 + 11^2/10^6 + \ldots,$$

which is easily shown by summing the geometric progression. If the array above had the leading zeros removed and was left-justified, we would have Pascal's triangle in a form where the Fibonacci numbers arise as the sum of the rising diagonals. Notice that $11^k$ generates rows of Pascal's triangle, and that the columns of the array expressing 1/89 are the diagonals of Pascal's triangle. We call this relationship "the Pascal connection."

Now, 1/89 leads to the Fibonacci numbers by summing diagonals of Pascal's triangle. Since $89 = 10^2 - 11$, consider $9899 = 10^4 - 101$. By summing the geometric series,

$$1/9899 = 1/10^4 + 101/10^8 + 101^2/10^{12} + \ldots .$$

However, $101^k$ generates rows of Pascal's triangle where the columns are interspersed with columns of zeros. By the Pascal connection, we obtain Fibonacci numbers in every second place, as

$$1/9899 = .0001010203050813\ldots .$$

The Pascal connection also gives us

$$1/998999 = .000001001002003005008013\ldots,$$

since $998999 = 10^6 - 1001$. In general,

$$1/(10^{2k} - 100\ldots01),$$

where $(k - 1)$ zeros appear between the two 1's, gives successive Fibonacci numbers at every $k^{\text{th}}$ place by the Pascal connection.

Looking again at $89 = 10^2 - 11$, observe that $889 = 10^3 - 111$, and summing a geometric series gives

$$1/889 = 1/10^3 + 111/10^6 + 111^2/10^9 + \ldots,$$

where

$$\begin{array}{ll} 1/889 = .001 & = .001124859\ldots \\ \quad\quad\quad\; .000111 & \\ \quad\quad\quad\; .000012321 & \\ \quad\quad\quad\; .000001367631 & \\ \quad\quad\quad\; \ldots & \end{array}$$

and we generate the Tribonacci numbers

$$0,\ 1,\ 1,\ 2,\ \ldots,\ T_{n+1} = T_n + T_{n-1} + T_{n-2},$$

by the Pascal connection, since $111^k$ generates rows of the trinomial triangle, and the sums of the rising diagonals of the trinomial triangle yield the Tribonacci numbers [1].

The results of expressing 1/89, 1/9899, 1/998999 in terms of Fibonacci numbers have been developed by other methods by Long [2] and by Hudson & Winans [3]. Winans [4] also gives 1/109 and 1/10099 as a reverse diagonalization of sums of Fibonacci numbers, reading from the far right of the repeating cycle, where 1/109 ends in

```
      13853211
     21
    34
   55
  89
...
```

We next apply the Pascal connection to repeating decimals, looking out to the far end of the repetend and reading from right to left. 1/109 has a period length of 108, and 1/109 ends in powers of 11, as

$$1/10^{108} + 11/10^{107} + 11^2/10^{106} + \ldots,$$

or, a reverse diagonalization of powers of 11,

```
        1
        11
      121
    1331
  14641
...
```

Summing the geometric progression,

$$\sum_{i=1}^{108} \frac{11^{i-1}}{10^{109-i}} = \frac{11^{108} \cdot 10^{108} - 1}{10^{108} \cdot (110 - 1)} = \frac{11^{108} - 1}{109} + \frac{10^{108} - 1}{10^{108} \cdot 109}$$

Now, $109 \mid (11^{108} - 1)$ because 109 is prime, so that the left term is an integer. The rightmost term represents one cycle of the repetend of 1/109, since 109 has period length 108. Thus, 1/109 gives $F_n$, $n = 1, 2, \ldots$, reading from the right, by the Pascal connection.

Notice that $109 = 11(10) - 1$, and 109 is prime with period 108. Now,

$$1109 = 111(10) - 1,$$

where 1109 is prime with period 1108. We can generate the last digits of the repeating cycle for 1/1109 in exactly the same way by writing

$$1/10^{1108} + 111/10^{1107} + 111^2/10^{1106} + \ldots .$$

By the Pascal connection, 1/1109 ends in the Tribonacci sequence, ...74211.

Generalizing 109 in another way, 10099 is a prime with period length 3366, where $10099 = 101(10^2) - 1$, so that 1/10099 can be expressed in terms of powers of 101 from the far right. As before, $101^k$ generates rows of Pascal's triangle where the columns are interspersed by zeros, so that the Pascal connection shows 1/10099 ending in ...0503020101. Similarly, $1000999 = 1001(10^3) - 1$, and 1000999 is prime with period length 500499 [6], so that, by the Pascal connection, 1/1000999 must end in $F_n$ appearing as every third entry, as ..005003002001001.

We can immediately write fractions which generate the Lucas numbers $L_n$ from the right. Since 1/109 ends in $F_n$, $n = 1, 2, \ldots$, reading from the right, and $L_n = 2F_{n-1} + F_n$, multiplying 1/109 by 21 in effect adds $2F_{n-1} + F_n$ in the expansion except for the rightmost digit. But because the digit on the right of $F_1$ is indeed 0, the last digit also fits the pattern, so that 21/109 ends in $L_{n-1}$ from the right. Also, multiplying 1/109 by 101 in effect adds $F_{n-1} + F_{n+1}$ to make $L_n$ except for the rightmost digit. Thus, 101/109 ends in $L_n$ except for the rightmost digit. That is, 101/109 ends in ...74311, and $L_n$ reads from the right to left beginning at the $107^{\text{th}}$ digit. Since 1/10099 gives $F_n$, $n = 1, 2, \ldots$, reading from the right with every second digit, 201/10099 ends in $L_n$ from the right as ...181107040301. Similarly, 2001/1000999 ends in $L_n$ as every third digit. Finally, 10001/10099 ends in ...18110704030101 while 1000001/1000999 ends in ...018011007004003001001.

We will eventually prove these notions, but to enjoy these relationships one needs an easy way to write the far right-hand digits in these long repeating cycles. If $(A, 10) = 1$, $A > 1$, then $A \cdot 1/A = 1 = .99999\ldots$ . To generate 1/109 from the right, simply fill in the digits to make a product of ...9999999:

```
       109
  ...53211
  --------
       109
      109
     218
    327
   545
 ---------
 ...999999
```

The last digit of the next partial product must be 2 to make the next digit in the product be 9. So the digit preceding 5 in the multiplier must be an 8. One proceeds thusly, filling in the digits of the multiplier one at a time. The multiplier gives successive digits of 1/109 as read from the right.

## 2. Retrograde Renegades: Repeating Decimals that Contain Geometric Series

Any repeating decimal can itself be considered as a geometric series, but here we want to study repeating decimals which contain geometric series within their repetends. First, we list some general known results in Lemma 1 [7], [8].

*Lemma 1:* Let $n$ be an integer, $(n, 10) = 1$, $n > 1$. Then $L(n)$, the length of the period of $n$, is given by

(i) $\quad 10^{L(n)} \equiv 1 \pmod{n}$,

where $L(n)$ is the smallest exponent possible to solve the congruence; if $R(n)$ denotes the repetend of $n$, then $R(n)$ has $L(n)$ digits and

(ii) $\quad R(n) = (10^{L(n)} - 1)/n$;

the remainder $B$ after $A$ divisions by $n$ in finding $1/n$ is given by

(iii) $\quad 10^A \equiv B \pmod{n}$,

and

(iv) $\quad m^{L(n)} \equiv 1 \pmod{n}$, $(m, n) = 1$.

While $L(n)$ can be calculated as in Lemma 1(i), Yates [6] has calculated period lengths for all primes through 1370471.

We first look at repetends which contain powers of numbers reading left to right, or right to left, such as $1/97 = .01030927\ldots$ and $1/29$, which ends in $\ldots 931$, both of which seem to involve powers of 3.

*Lemma 2:* The decimal expansion of $1/(100 - k)$, $(100, k) = 1$, contains powers of $k$ from left to right, $k < 100$.

*Proof:* Summing the geometric series,

$$1/10^2 + k/10^4 + k^2/10^6 + \cdots = 1/(100 - k).$$

*Lemma 3:* The repetend of $1/(10k - 1)$ contains powers of $k$ as seen from the right.

*Proof:* Let $n = 10k - 1$. Then the sum after $L(n)$ terms of

$$S = \frac{1}{10^{L(n)}} + \frac{k}{10^{L(n)-1}} + \frac{k^2}{10^{L(n)-2}} +$$

is given by summing the geometric progression for $L(n)$ terms as

$$S = \frac{1}{10^{L(n)}} \cdot \frac{(10^{L(n)}k^{L(n)} - 1)}{(10k - 1)}$$
$$= \frac{1}{10^{L(n)}} \cdot \frac{[10^{L(n)}k^{L(n)} - 10^{L(n)} + 10^{L(n)} - 1]}{(10k - 1)} = \frac{k^{L(n)} - 1}{n} + \frac{10^{L(n)} - 1}{10^{L(n)}n},$$

where the left-hand term is an integer and the right-hand term gives one cycle of $1/n$ following the decimal point, both by Lemma 1.

Notice that $1/89$ has powers of 11 or Fibonacci numbers as seen from the left and powers of 9 from the right, while $1/109$ has powers of $(-0.09)$ from the left (where the initial term is 0.01), and powers of 11 or Fibonacci numbers as seen from the right. Also, $1/889$ has Tribonacci numbers as seen from the left, and powers of 89 on the right, since $889 = 10 \cdot 89 - 1$.

Next, consider pairs of fractions whose repeating decimal representations end in each other. For example, 31 appears as the rightmost two digits of 1/29 (period length 28), and 29 is the last pair of digits of 1/31 (period length 15). Now, $29 \cdot 31 = 9 \cdot 10^2 - 1$, and the digits in the two cycles, reading from the right, can be represented as

$$1/29: \quad 31/10^{28} + 9 \cdot 31/10^{26} + 9^2 \cdot 31/10^{24} + \ldots;$$

$$1/31: \quad 29/10^{15} + 9 \cdot 29/10^{13} + 9^2 \cdot 29/10^{11} + \ldots .$$

Further, 1/29 ends in ...137931, and 1/31 ends in ...29, 1/931 in ...029, 1/7931 in ...0029, 1/37931 in ...00029, and, finally, $1/B = 0.000\ldots29$ (26 zeros in the repetend), where $B$ is the entire repetend of 1/29. Also, there are many representations of a fraction reading from the right, such as, for 1/59 with its 58-digit period length, ending in ...779661, we have

$$1/10^{58} + 6/10^{57} + 6^2/10^{56} + \ldots,$$

$$61/10^{58} + 36 \cdot 61/10^{56} + 36^2 \cdot 61/10^{54} + \ldots,$$

$$661/10^{58} + 39 \cdot 661/10^{55} + 39^2 \cdot 661/10^{52} + \ldots,$$

$$9661/10^{58} + 57 \cdot 9661/10^{54} + 57^2 \cdot 9661/10^{50} + \ldots,$$

where

$$10^{58} \equiv 1 \pmod{59}, \quad 10^{57} \equiv 6 \pmod{59}, \quad 10^{56} \equiv 36 \pmod{59},$$

$$10^{55} \equiv 39 \pmod{59}, \text{ and } 10^{54} \equiv 57 \pmod{59}.$$

Notice that the multipliers are the remainders in reverse order in the division to obtain 1/59.

Both of these examples of retrograde renegades are explained by Theorem 1.

Let $A$ and $B$ be integers, $(A, 10) = 1$, $(B, 10) = 1$, $A > 1$. Let $L(A)$ be the number of digits in the period of $A$. If $1/A$ ends in $B$, then the end of $1/A$ can be expressed as

$$B/10^{L(A)} + KB/10^{L(A)-k} + K^2B/10^{L(A)-2k} + \ldots,$$

where

$$AB + 1 = K \cdot 10^k,$$

and the number of terms is $L(A)/k$ if $k$ divides $L(A)$, or $[L(A)/k] + 1$ otherwise, where $[x]$ is the greatest integer in $x$.

*Proof:* $AB + 1 = K \cdot 10^k$ because $K \cdot 10^k$ is a partial dividend where $A$ is the divisor, $B$ is the quotient, and 1 is the remainder, in the long division process to find $1/A$. By Lemma 1,

$$10^{L(A)} \equiv 1 \pmod{A} \quad \text{and} \quad 10^{L(A)-k} \equiv K \pmod{A}.$$

Case 1. Let $k|L(A)$. Sum the geometric progression with $L(A)/k$ terms to obtain

$$S = \frac{B}{10^{L(A)}} \cdot \frac{(K^{L(A)/k} \cdot 10^{L(A)} - 1)}{(K \cdot 10^{k-1})}$$

$$= \frac{B}{10^{L(A)}} \cdot \frac{(K^{L(A)/k} \cdot 10^{L(A)} - 10^{L(A)} + 10^{L(A)} - 1)}{AB}$$

$$= \frac{K^{L(A)/k} - 1}{A} + \frac{10^{L(A)} - 1}{A \cdot 10^{L(A)}}$$

Now, the right-hand term represents one cycle of the repetend of $1/A$ following the decimal point, by Lemma 1. Next, if the left-hand term is an integer, we are done. By Lemma 1,

$$10^{L(A)-k} \equiv K \pmod{A},$$

so

$$K^{L(A)/k} \equiv (10^{L(A)-k})^{L(A)/k} \equiv (10^{L(A)})^{(L(A)-k)/k} \equiv 1 \pmod{A},$$

which means that the left-hand term is an integer.

Case 2. If $k$ does not divide $L(A)$, then $L(A) = km + r$, $0 < r < m$, and there are $(m + 1)$ terms. Then, summing as before,

$$S = \frac{B}{10^{L(A)}} \cdot \frac{K^{m+1} \cdot 10^{k(m+1)} - 1}{K \cdot 10^k - 1}$$

$$= \frac{K^{m+1} \cdot 10^{k(m+1)} - 10^{L(A)}}{10^{L(A)} \cdot A} + \frac{10^{L(A)} - 1}{10^{L(A)} \cdot A}$$

Notice that the right-hand term is the same as in Case 1. If the left-hand term is an integer, then Case 2 is done. The left-hand term is equivalent to

$$(K^{m+1} \cdot 10^{k(m-1)-L(A)} - 1)/A$$

so we have an integer if

$$K^{m+1} \cdot 10^{k(m+1)-L(A)} \equiv 1 \pmod{A}.$$

But $K \equiv 10^{L(A)-k} \pmod{A}$, and substituting above,

$$(10^{L(A)-k})^{m+1} \cdot 10^{k(m+1)-L(A)} = 10^{L(A)(m+1)-k(m+1)+k(m+1)-L(A)}$$

$$= 10^{mL(A)} = (10^{L(A)})^m \equiv 1 \pmod{A}$$

and we are done.

*Corollary* (due to G. E. Bergum): Let $A$ be a prime with $k$ digits. If $B$ is the integer formed by writing the last $i$ digits of the repetend of $1/A$, $L(A) \geq 1 \geq k$, then $1/B$ ends in $\ldots 000 \ldots A$, where $A$ is preceded by $(i - k)$ zeros.

## 3. Fractions that Contain $F_{nm}$ in Their Decimal Representations

Hudson & Winans [3] completely characterized decimal fractions which can be represented in terms of $F_{nm}$, reading from the left. In particular, they give

$$1/71 = \sum_{i=1}^{\infty} F_{2i}/10^{i+1}.$$

Winans [4] gives 9/71 as ending in Fibonacci numbers with odd subscripts. Since 9/71 also begins with $F_{2m-1}$ reading from the left and

$$L_{2m} = F_{2m-1} + F_{2m+1},$$

we write $11 \cdot 9 = 99 \equiv 28 \pmod{71}$ and 28/71 begins with $L_{2m}$, $m = 1, 2, \ldots$ . Since we find that 19/71 ends in $F_{2m-3}$, and

$$L_{2m-2} = F_{2m-1} + F_{2m-3},$$

$19/71 + 9/71 = 28/71$ ends in $L_{2m-2}$, $m = 1, 2, \ldots$, reading from the right. Further, Hudson & Winans [3] give

$$1/9701 = .000103082156\ldots,$$

where $F_{2m}$ appears in groups of two digits. We note that $9701 = 89 \cdot 109$, with 1188 digits in its repeating cycle. It turns out that

$$99/9701 = .0102051334\ldots$$

and that 99/9701 ends in ...893413050201, where $F_{2m-1}$ appears in groups of two digits, reading either from the left or from the right. Since

$$L_{2m} = F_{2m-1} + F_{2m+1},$$

and

$$101 \cdot 99 = 9999 \equiv 298 \pmod{9701},$$

we should have 298/9701 both beginning and ending in Lucas numbers with even subscripts. In fact,

$$298/9701 = .03071848\ldots$$

and ends with ...4718070302, or begins with $L_{2m}$ and ends with $L_{2m-2}$, $m = 1, 2, \ldots$, moving in blocks of two.

Next, we give a description of fractions with a decimal representation using $F_{nm}$, reading from right to left.

*Theorem 2:* The decimal representation of

$$\frac{F_n}{10^{2k} + L_n \cdot 10^k - 1}, \quad n \text{ odd},$$

ends in successive terms of $F_{nm}$, $m = 1, 2, \ldots$, reading from the right end of the repeating cycle, and appearing in groups of $k$ digits.

*Proof:* Change the sum written in (i) to geometric progressions by using the Binet form for $F_n$,

$$F_n = (\alpha^n - \beta^n)/\sqrt{5}, \text{ where } \alpha = (1 + \sqrt{5})/2 \text{ and } \beta = (1 - \sqrt{5})/2.$$

Then sum the geometric progressions, making use of $\alpha\beta = -1$ and $L_n = \alpha^n + \beta^n$. After sufficient algebraic patience, one can write, for $k > 0$,

$$\text{(i)} \qquad \sum_{i=1}^{L} 10^{k(i-1)} F_{ni} = \frac{(-1)^{n+1} 10^{k(L+1)} F_{nL} + 10^{kL} F_{n(L+1)} - F_n}{(-1)^{n+1} 10^{2k} + L_n \cdot 10^k - 1}.$$

Notice that the sum is a positive integer at this point, and dividing by $10^y$, $y > 0$, will move the decimal point $y$ places to the left. Let

$$M = (-1)^{n+1} 10^{2k} + L_n \cdot 10^k - 1,$$

where $M > 0$ when $n$ is odd, and let $L(M)$ be the length of the period of $M$. The number of terms $L$ in the sum must be chosen so that $L \geq L(M)/k$. We divide both sides of (i) by $10^{L(M)}$, and add $(F_n - F_n)$ to the numerator on the right-hand side, making

$$\text{(ii)} \qquad \sum_{i=1}^{L} 10^{k(i-1) - L(M)} F_{ni} = \frac{10^{kL - L(M)} \left( (-1)^{n+1} 10^k F_{nL} + F_{n(L+1)} \right) - F_n}{M} + \frac{F\ (10^{L(M)} - 1)}{10^{L(M)}\, M}.$$

Since $kL \geq L(M)$, $10^{kL - L(M)} \geq 1$, and the decimal point has been shifted $L(M)$ places left. Now, the rightmost term is $F_n$ times one cycle of the repetend of $1/M$. Thus, when $n$ is odd,

$$M = 10^{2k} + L_n \cdot 10^k - 1, \text{ and } F_n/M \text{ has the needed form.}$$

Now, if $n$ is even,

$$M = (-1)^{n+1}10^{2k} + L_n \cdot 10^k - 1$$

is negative, and we have to modify Theorem 2.

*Theorem 3:* The decimal representation of

$$\frac{M - F_n}{M}, \ M = 10^{2k} - L_n \cdot 10^k + 1, \ n \text{ is even,}$$

ends in successive terms of $F_{nm}$, $m = 1, 2, \ldots$, reading from the right end of the repeating cycle and appearing in groups of $k$ digits, if 1 is added to the rightmost digit.

*Proof:* Return to (ii) in the proof of Theorem 2. When $n$ is even, both numerator and denominator of the left-hand term are negative, so we still have a positive term there. Since $M$ is negative when $n$ is even, rewrite the right-hand term as

$$-F_n(10^{L(M)} - 1)/10^{L(M)}M$$

for adjusted $M$,

$$M = 10^{2k} - L_n \cdot 10^k + 1.$$

Then write

$$\frac{-F_n(10^{L(M)} - 1)}{10^{L(M)}M} = \frac{-F_n(10^{L(M)} - 1) + (M(10^{L(M)} - 1)) - (M(10^{L(M)} - 1))}{10^{L(M)}M}$$
$$= \frac{(M - F_n)(10^{L(M)} - 1)}{10^{L(M)}M} + \frac{1}{10^{L(M)}} - 1$$

The fractional part represents $(M - F_n)$ times one cycle of the repetend of $1/M$, with 1 added to the rightmost digit, which finishes Theorem 3.

Further, notice that if $F_n/M$ is represented in terms of $F_{nm}$, then other fractions with the same denominator will have representations in terms of $F_{nm+r}$ and $L_{nm+r}$, $r = 0, 1, \ldots, n - 1$. For example, for $n = 2$, $k = 1$ and $m = 1, 2, \ldots$.

$2/139$ ends in $F_{3m}$, $20/139$ in $F_{3m-3}$, $11/139$ in $F_{3m-1}$, $13/139$ in $F_{3m+1}$;

$24/139$ ends in $L_{3m}$, $31/139$ in $L_{3m-2}$, $41/139$ in $L_{3m+2}$.

In general, for $n = 3$, $m = 1, 2, \ldots$, and $M = 10^{2k} + 4 \cdot 10^k - 1$, we have

$2/M$ ends in $F_{3m}$; $\quad 2 \cdot 10^k/M$ ends in $F_{3m-3}$.

Since $F_{3m} + F_{3m-3} = 2F_{3m-1}$, and $F_{3m+1} = F_{3m} + F_{3m-1}$, we find that

$(10^k + 1)/M$ ends in $F_{3m-1}$; $\quad (10^k + 3)/M$ ends in $F_{3m+1}$.

Then $L_{3m} = F_{3m+1} + F_{3m-1}$ and $L_{3m-2} = F_{3m-3} + F_{3m-1}$, give us that

$(2 \cdot 10^k + 4)/M$ ends in $L_{3m}$; $\quad (3 \cdot 10^k + 1)/M$ ends in $L_{3m-2}$.

Lastly, $L_{3m+2} = F_{3m} + 3F_{3m+1}$ means that

$(3 \cdot 10^k + 11)/M$ ends in $L_{3m+2}$,

where all of the above occur in groups of $k$ digits.

The even examples are both more difficult and more entertaining. For $n = 2$, $m = 1, 2, \ldots$, $M = 10^{2k} - 3 \cdot 10^k + 1$, the following occur in blocks of $k$ digits from the right:

$(10^k - 3)/M$ ends in $F_{2m+2}$, $\quad (10^k - 2)/M$ ends in $F_{2m+1}$;

$(2 \cdot 10^k - 3)/M$ ends in $L_{2m}$, $\quad (10^k - 4)/M$ ends in $L_{2m+1}$.

For $n = 4$, $m = 1, 2, \ldots$, $M = 10^{2k} - 7 \cdot 10^k + 1$, the following occur in blocks of $k$ digits from the right:

$(10^k - 5)/M$ ends in $F_{4m+1}$, $\quad (10^k - 8)/M$ ends in $F_{4m+2}$;

$(3 \cdot 10^k - 18)/M$ ends in $L_{4m+2}$, $\quad (4 \cdot 10^k - 29)/M$ ends in $L_{4m+3}$.

These are by no means exhaustive. Fibonacci and Lucas numbers abound but encountering negative numerators causes addition of multiples of $M$ to write a fraction with a positive numerator and the same repetend, and there will be adjustments to the last digit in the representation.

When $n$ is even, Theorem 3 gives the same denominators as found by Hudson & Winans [3] for the even case, in representations using $F_{nm}$ from left to right. We find examples such as 9/71 and 99/9701, which both begin and end in $F_{2m-1}$, and 98/9301, which has $F_{4m-3}$ from the left and $F_{4m-1}$ from the right. We can write a corollary to Theorem 3.

*Corollary:* (i) $\dfrac{10^k - 1}{10^{2k} - 3 \cdot 10^k + 1}$ begins and ends with $F_{2m-1}$,

(ii) $\dfrac{10^k - 2}{10^{2k} - 7 \cdot 10^k + 1}$ begins with $F_{4m-3}$ and ends with $F_{4m-1}$,

both appearing in blocks of $k$ digits.

*Proof:* Case (i), where $n = 2$. From left to right, $1/M$ begins with $F_{2m-2}$ and $10^k/M$ begins with $F_{2m}$, so subtracting gives $(10^k - 1)/M$ for $F_{2m-1}$. From right to left,

$$(M - 1)/M = (10^{2k} - 3 \cdot 10^k)/M \text{ ends in } F_{2m}$$

except for the last digit, so moving one block left,

$$(10^k - 3)/M \text{ ends in } F_{2m+2}.$$

Using $F_{2m-1} = F_{2m+2} - 2F_{2m}$, compute

$$(10^k - 3 - 2(-1))/M = (10^k - 1)/M,$$

where the numerator is positive, ending in $F_{2m-1}$.

Case (ii), where $n = 4$. From left to right, $3/M$ begins with $F_{4m-4}$, so $3 \cdot 10^k/M$ begins with $F_{4m}$. Since $3F_{4m-3} = F_{4m} - 2F_{4m-4}$, we find that

$$(10^k - 2)/M \text{ begins with } F_{4m-3}.$$

From right to left, except for the last digit,

$$(M - 3)/M \text{ ends in } F_{4m},$$

so that $F_{4m+4}$ ends in

$$(M - 3)/10^k M \equiv (3M - 3)/10^k M = (3 \cdot 10^k - 21)/M.$$

Now, $3F_{4m-1} = F_{4m+4} - 5F_{4m}$ allows us to compute

$$(3 \cdot 10^k - 21 - 5(-3))/3M = (10^k - 2)/M,$$

where the numerator is positive, ending in $F_{4m-1}$.

Examining the proof of the corollary, we have seen several examples for $n = 2$ and $n = 4$ where

$$\frac{F_p \cdot 10^k - F_{p+n}}{10^{2k} - L_n \cdot 10^k + 1} \text{ ends in } F_{nm+p}$$

and some earlier examples for $n = 3$ and $n = 1$, where

$$\frac{F_p \cdot 10^k + F_{p+n}}{10^{2k} + L_n \cdot 10^k - 1} \text{ ends in } F_{nm+p}.$$

We write our final generalization as Theorem 4.

*Theorem 4:* The repeating cycle of

$$\frac{F_p \cdot 10^k + (-1)^{n+1}F_{p+n}}{10^{2k} + (-1)^{n+1}(L_n \cdot 10^k - 1)} \text{ ends in } F_{nm+p}$$

and the repeating cycle of

$$\frac{L_p \cdot 10^k + (-1)^{n+1}L_{p+n}}{10^{2k} + (-1)^{n+1}(L_n \cdot 10^k - 1)} \text{ ends in } L_{nm+p},$$

for $m = 1, 2, \ldots,$ occurring in blocks of $k$ digits, for positive integers $k$ and $n$ such that

$$10^{2k} + (-1)^{n+1}(L_n \cdot 10^k - 1) > F_p \cdot 10^k + (-1)^{n+1}F_{p+n} > 0.$$

The proof of the Fibonacci case follows from summing

$$\sum_{i=1}^{L} 10^{k(i-1)-L(M)} F_{ni+p},$$

using the techniques of Theorems 2 and 3. Since we force cases where the numerator and denominator are both positive, we can do the proof as one case, and the proof is fairly straightforward but very long and tedious. The Lucas case follows by adding the fractions which represent $F_{nm+(p-1)}$ and $F_{nm+(p+1)}$.

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# GROUPS OF INTEGRAL TRIANGLES

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**The Group $H_\gamma$.** An *integral* (*rational*) triangle $T = (a, b, c)$ is a triangle having integral (rational) side lengths $a$, $b$, $c$. Two rational triangles

$$T_1 = (a_1, b_1, c_1) \quad \text{and} \quad T_2 = (a_2, b_2, c_2)$$

are *equivalent* if one is a rational multiple of the other:

$$(a_2, b_2, c_2) = (ra_1, rb_1, rc_1) \text{ for rational } r.$$

As our favorite representative for a class of equivalent rational triangles, we take the *primitive* integral triangle $(a, b, c)$ in which $a$, $b$, $c$ have no common factor greater than 1. That is, for any positive rational number $r$ we identify $(ra, rb, rc)$ with $(a, b, c)$.

A *pythagorean* triangle is an integral triangle $(a, b, c)$ in which the angle $\gamma$, opposite side $c$, is a right angle. Equivalently, $(a, b, c)$ is a pythagorean triangle if $a$, $b$, $c$ satisfy the pythagorean equation

$$a^2 + b^2 = c^2. \tag{1}$$

An angle $\beta$, $0 < \beta < \pi$, is said to be *pythagorean* if $\beta$, or $\pi - \beta$, is an angle in some pythagorean triangle or, equivalently, if it has rational sine and cosine. A *heronian* triangle is an integral triangle with rational area. Clearly, an integral triangle is heronian if and only if each of its angles $\alpha$, $\beta$, $\gamma$, is pythagorean.

In [1] the set of primitive pythagorean triangles is made into a group $H_{\pi/2}$. The group operation is, basically, *addition of angles modulo* $\pi/2$. When placed in *standard position* (Fig. 1), a primitive pythagorean triangle $T = (a, b, c)$ is uniquely determined by the point $P$ on the unit circle,

$$P = (\cos\beta, \sin\beta) = (a/c, b/c).$$

Geometrically, the sum of two such triangles, $(a, b, c)$ and $(A, B, C)$, is obtained by adding their central angles $\beta_1$ and $\beta_2$. If $\beta_1 + \beta_2$ equals or exceeds $\pi/2$, then the angle sum is reduced modulo $\pi/2$. The identity element is the (degenerate) triangle $(1, 0, 1)$ with $\beta = 0$. The inverse of $T = (a, b, c)$ is $-T = (b, a, c)$. Thus, in $H_{\pi/2}$ we must distinguish between $(a, b, c)$ and $(b, a, c)$, even though they are congruent triangles. Analytically, the sum of $(a, b, c)$ and $(A, B, C)$ may be expressed

$$(a, b, c) + (A, B, C) = \begin{cases} (aA - bB,\ bA + aB,\ cC) & \text{when } aA - bB > 0, \\ (bA + aB,\ bB - aA,\ cC) & \text{when } aA - bB \le 0. \end{cases} \tag{2}$$

With this definition of the sum of two triangles, $H_{\pi/2}$ becomes a *free abelian group*. The set of generators of $H_{\pi/2}$ may be taken to be the set of triangles $T_p = (r, s, p)$ with $p$ prime, $p \equiv 1 \pmod 4$, and $r > s$ ([1, p. 25]). Thus, any primitive pythagorean triangle can be written as a unique linear combination of the generators with integral coefficients.

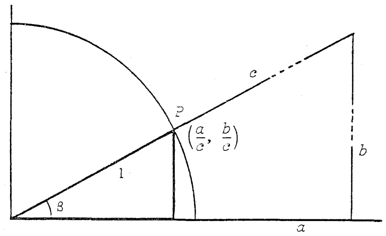


FIGURE 1

The set of pythagorean triangles may be characterized as that subset of the integral triangles for which $\gamma = \pi/2$. One may ask if the requirement that $\gamma$ be a right angle is essential for defining a group structure. The answer is no. Let us fix an angle $\gamma$, $0 < \gamma < \pi$, with rational cosine, $\cos\gamma = u/w$, and denote by $H_\gamma$ that subset of the integral triangles having one fixed angle $\gamma$. The condition for a triangle $T = (a, b, c)$ to belong to $H_\gamma$ is that $a$, $b$, and $c$ satisfy the generalized pythagorean theorem (the law of cosine)

$$c^2 = a^2 + b^2 - 2abu/w. \tag{3}$$

To define addition of two triangles in $H_\gamma$, we proceed almost as we did for the case of $H_{\pi/2}$ above, guided by our geometric intuition. When placed in *standard position* (Fig. 2), a primitive $\gamma$-angled triangle $T = (a, b, c)$ is uniquely determined by its central angle $\beta$ or, equivalently, by the point $P$ on the unit circle. Geometrically, the sum of two triangles $(a, b, c)$ and $(A, B, C)$ in $H_\gamma$ is obtained by adding their two central angles $\beta_1$ and $\beta_2$. If $\beta_1 + \beta_2$ equals or exceeds $(\pi - \gamma)$, then the angle sum is reduced modulo $(\pi - \gamma)$. The identity element is the (degenerate) triangle $(1, 0, 1)$ with $\beta = 0$. The inverse of $T = (a, b, c)$ is $-T = (b, a, c)$. Thus, in $H_\gamma$ we must distinguish between $(a, b, c)$ and $(b, a, c)$ even though they are congruent triangles. Analytically, the sum of $(a, b, c)$ and $(A, B, C)$ may be expressed

$$(a, b, c) + (A, B, C) = \begin{cases} (aA - bB,\ aB + bA - 2bBu/w,\ cC) & \text{if } aA - bB > 0, \\ (aB + bA - 2aAu/w,\ bB - aA,\ cC) & \text{if } aA - bB \le 0. \end{cases} \tag{4}$$

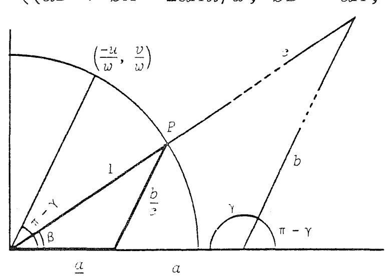


FIGURE 2

To see that (4) defines a group operation on $H_\gamma$, we must show that

$$T = (a, b, c) + (A, B, C)$$

satisfies (3). This is not very difficult but somewhat tedious. Since addition of $\gamma$-angled triangles corresponds to addition of their central angles, the operation is associative and commutative. Simple computations will show that $(1, 0, 1)$ is the identity element, and that $-(a, b, c) = (b, a, c)$. We note that (4) reduces to (2) when $\gamma = \pi/2$.

Using the angle $\gamma$ for which $\cos\gamma = -5/13$, we give a few examples of addition of triangles as defined by (4):

$(13, 11, 20) + (119, 65, 156) = (4, 13, 15)$,
$(13, 11, 20) + (65, 119, 156) = (182, 29, 195)$,
$2(11, 13, 20) = (11, 13, 20) + (11, 13, 20) = (308, 39, 325)$,
$3(11, 13, 20) = (2881, 4823, 6500)$.

The sum of two triangles is required to be a primitive member of $H_\gamma$, so cancellation of common factors of the three coordinates on the right of (4) may be necessary. To obtain integral components, multiplication of the three coordinates by $w = 13$ may be necessary. For computational purposes, it is worth noting, from (4), that if a triple $T = (a, b, c)$ appears with $a \le 0$, then

$$T = (b - 2au/w, -a, c)$$

after the required reduction of the central angle modulo $(\pi - \gamma)$. We summarize in Proposition 1.

*Proposition 1:* The set $H_\gamma$ of primitive $\gamma$-angled triangles is an abelian group under the operation, called addition, defined by (4). The identity element in $H_\gamma$ is $(1, 0, 1)$, and the inverse of $(a, b, c)$ is $(b, a, c)$.

The group $H_{\pi/2}$ is a free abelian group. For values of $\gamma$ other than $\pi/2$, a characterization of the group $H_\gamma$ is not so simple. One difficulty is that we have no easy way, so far, of identifying the members of $H_\gamma$. For $H_{\pi/2}$, it is well known [2] that a member $(r, s, t)$, i.e., a primitive pythagorean triangle, is generated by a pair of positive integers $(m, n)$, $m > n$ that are relatively prime and have $m + n \equiv 1 \pmod 2$. The generation process is:

$$r = m^2 - n^2,\quad s = 2mn,\quad t = m^2 + n^2 \qquad \text{or} \tag{5}$$

$$r = 2mn,\quad s = m^2 - n^2,\quad t = m^2 + n^2. \tag{5'}$$

In fact, (5) establishes a one-to-one correspondence between the set of all such pairs $(m, n)$ and the set of all primitive pythagorean triangles $(r, s, t)$ with odd $r$. The pair $(m, n)$ is called the *generator* for $(r, s, t)$.

To obtain a generating process for $\gamma$-angled triangles, akin to (5) and (5′) for pythagorean triangles, we shall make the restriction that $\cos\gamma = u/w$, $\sin\gamma = v/w$ are both rational numbers. To simplify the derivation of the process, which is geometrically inspired, we also make the assumption that $\pi > \gamma > \pi/2$, so that $u/w < 0$, i.e., $u < 0$. Thus, $(|u|, v, w)$ is a pythagorean triangle, and $(\pi - \gamma)$ is a pythagorean angle, $(\pi - \gamma) < \pi/2$. The central angle of any triangle $T$ in $H_\gamma$ is then also pythagorean, and $T$ is a heronian triangle.

With each pythagorean angle $\beta$, $0 < \beta < (\pi - \gamma)$, we can associate a unique primitive pythagorean triangle $(r, s, t)$ having central angle $\beta$, and also a unique primitive $\gamma$-angled triangle $(a, b, c)$ having central angle $\beta$ (Fig. 3). It follows that there is a one-to-one mapping $\phi$ from the subset of $H_{\pi/2}$ having central angle $\beta < (\pi - \gamma)$ onto $H$ . Using only elementary geometry (Fig. 3), one may show that $\phi$ and $\phi^{-1}$ can be represented by matrices in the following way.

$$(a, b, c) = \phi(r, s, t) = \begin{bmatrix} v & u & 0 \\ 0 & w & 0 \\ 0 & 0 & v \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix} = (vr + us, ws, vt), \tag{6}$$

$$(r, s, t) = \phi^{-1}(a, b, c) = \begin{bmatrix} w & -u & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (wa - ub, vb, wc). \tag{7}$$

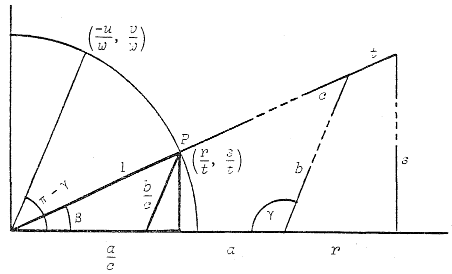


FIGURE 3

$$\frac{a}{c} = \frac{r}{t} - \frac{s}{t}\cos(\pi - \gamma), \quad \frac{b}{c} = \frac{s}{t}\csc(\pi - \gamma),$$
$$(a,\ b,\ c) - (vr + us,\ ws,\ vt)$$

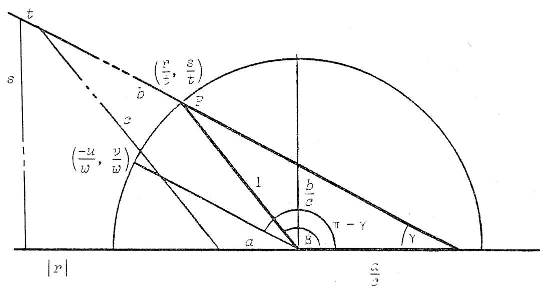


FIGURE 4

$$\frac{a}{c} = \frac{r}{t} - \frac{s}{t}\cot(\pi - \gamma), \quad \frac{b}{c} = \frac{s}{t}\csc(\pi - \gamma).$$
$$(a,\ b,\ c) = (vr + us,\ ws,\ vt)$$

Cancellations of common factors of the three components on the right of (6) and (7) may be necessary in order to arrive at primitive triangles. We give a few examples to illustrate the use of (6). As above, $\gamma$ is defined by

$$\cos\gamma = -5/13 = u/w, \ \sin\gamma = 12/13 = v/w.$$

$$\phi(4,\ 3,\ 5) = \begin{bmatrix} 12 & -5 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 12 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = (33,\ 39,\ 60) = (11,\ 13,\ 20).$$

$$\phi(3,\ 4,\ 5) = (4,\ 13,\ 15). \qquad \phi(35,\ 12,\ 37) = (30,\ 13,\ 37).$$

Equation (6) makes it possible to construct members of $H_\gamma$ from members of $H_{\pi/2}$. The generating process for $\gamma$-angled triangles is at hand: substitute (5) and (5′) into (6) and remember that the central angle $\beta$ must satisfy $\beta < (\pi - \gamma)$. Thus, when using (5), we must have

$$\tan\frac{\beta}{2} = \frac{n}{m} < \tan\frac{\pi - \gamma}{2} = \frac{w + u}{v},$$

and, when using (5′), we must have

$$\tan\frac{\beta}{2} = \frac{m - n}{m + n} < \frac{w + u}{v}, \text{ so that } \frac{n}{m} > \frac{w - u - v}{w - u + v} = \frac{v - w}{u}.$$

Summarizing, we formulate the generating process for $\gamma$-angled triangles in Proposition 2.

*Proposition 2:* For a given angle $\gamma$, $\pi/2 < \gamma < \pi$, $(\cos\gamma, \sin\gamma) = (u/w, v/w)$, a $\gamma$-angled triangle $(a, b, c)$ is generated by means of:

$$a = v(m^2 - n^2) + 2umn, \quad b = 2wmn, \quad c = v(m^2 + n^2) \quad \text{or} \tag{8}$$

$$a = 2vmn + u(m^2 - n^2), \quad b = w(m^2 - n^2), \quad c = v(m^2 + n^2). \tag{8'}$$

Here $m$ and $n$ are relatively prime positive integers, $m > n$ and $m + n = 1$ (mod 2). Furthermore, for (8), $(m, n)$ must satisfy $n/m < (w + u)/v$, and for (8′), $n/m > u/(w + v)$. Each $\gamma$-angled triangle is obtained in this way.

We illustrate the use of (8) by a few examples. As before, $\gamma$ is given by $(\cos\gamma, \sin\gamma) = (-5/13, 12/13) = (u/w, v/w)$.

$(m, n) = (2, 1)$. $\quad (a, b, c) = (16, 52, 60) = (4, 13, 15)$.
$(m, n) = (3, 2)$. $\quad (a, b, c) = (0, 1, 1) = (1, 0, 1)$.
$(m, n) = (4, 1)$. $\quad (a, b, c) = (35, 26, 51)$.
$(m, n) = (5, 2)$. $\quad (a, b, c) = (38, 65, 77)$.

If $(u, v, w) = (0, 1, 1)$, so that $\gamma = \pi/2$, then (8) and (8′) reduce to (5) and (5′). Thus, Proposition 2 is a generalization of the euclidean process of generating pythagorean triangles.

In the case of $0 < \gamma < \pi/2$, not covered by Proposition 2, the derivation of the generation process of $\gamma$-angled triangles is only slightly more complicated. The key step, however, is still the formal substitution of (5), (5′) into (6) with a slight modification. If $0 < \beta < \pi/2$, so that $P = (r/t, s/t)$ is in the first quadrant, we simply substitute (5) or (5′) into (6) depending on whether $r$ is odd or even. For $\beta = \pi/2$, $P = (0/1, 1/1)$, we use $(m, n) = (1, 1)$ to generate the pythagorean triple $(0, 1, 1)$, from which we obtain

$$\phi(0, 1, 1) = (u, w, v).$$

If $\pi/2 < \beta < (\pi - \gamma)$, so that $P$ is in the second quadrant (Fig. 4), we write

$$P = \left(\frac{-r}{t}, \frac{s}{t}\right), \quad r > 0,$$

and consider the point $P' = (r/t, s/t)$ in the first quadrant. The corresponding angle $\beta'$, $(\cos\beta', \sin\beta') = (r/t, s/t)$, must satisfy $\gamma < \beta' < \pi/2$. We then apply $\phi$ to $(-r, s, t)$ to arrive at the member of $H_\gamma$ that corresponds to $P$. We summarize in Proposition 2′.

*Proposition 2′:* For a given angle $\gamma$, $0 < \gamma < \pi/2$, $(\cos\gamma, \sin\gamma) = (u/w, v/w)$, a $\gamma$-angled triangle $(a, b, c)$ is generated by means of:

$$a = v(m^2 - n^2) + 2umn, \quad b = 2wmn, \quad c = v(m^2 + n^2) \quad \text{or} \tag{8}$$

$$a = 2vmn + u(m^2 - n^2), \quad b = w(m^2 - n^2), \quad c = v(m^2 + n^2) \quad \text{or} \tag{8'}$$

$$a = -v(m^2 - n^2) + 2umn, \quad b = 2wmn, \quad c = v(m^2 + n^2) \quad \text{or} \tag{9}$$

$$a = -2vmn + u(m^2 - n^2), \quad b = w(m^2 - n^2), \quad c = v(m^2 + n^2). \tag{9'}$$

Here $m$ and $n$ are relatively prime positive integers, $m > n$ and $m + n \equiv 1$ (mod 2). Furthermore, for (9), $(m, n)$ must satisfy

$$\frac{n}{m} > \frac{v}{w + u},$$

and for (9′),

$$\frac{n}{m} < \frac{u}{w + v}.$$

Each $\gamma$-angled triangle is obtained in this way except $(u, w, v)$, which is obtained from (8) by taking $(m, n) = (1, 1)$.

**$H_0$ and $H_\pi$.** It is tempting to consider the two limiting cases: $\gamma = 0, \pi$. Since then the triangles collapse, we shall call $(a, b, c)$ a *triple*. But the rule (4) for adding two such triples makes sense and, in fact, defines a group operation on $H_\gamma$, $\gamma = 0, \pi$.

For $\gamma = 0$, we have $\cos\gamma = 1 = u/w$, $\sin\gamma = 0 = v/w$, and we may take

$$(u, v, w) = (1, 0, 1).$$

The condition for the triple $(a, b, c)$ to belong to $H_0$ is:

$$c^2 = a^2 + b^2 - 2ab, \text{ i.e., } c = |a - b|. \tag{10}$$

Similarly, the condition for the triple $(a, b, c)$ to belong to $H_\pi$ is:

$$c^2 = a^2 + b^2 + 2ab, \text{ i.e., } c = a + b. \tag{11}$$

Addition of two triples in $H_0$ and $H_\pi$ is defined by rewriting (4) with $\cos\gamma = 1$ and $\cos\gamma = -1$, respectively.

For $(a, b, c)$ and $(A, B, C)$ in $H_0$:

$$(a, b, c) + (A, B, C) = \begin{cases} (aA - bB,\ aB + bA - 2bB,\ cC) & \text{when } aA - bB > 0, \\ (aB + bA - 2aA,\ bB - aA,\ cC) & \text{when } aA - bB \le 0. \end{cases}$$

For $(a, b, c)$ and $(A, B, C)$ in $H_\pi$:

$$(a, b, c) + (A, B, C) = \begin{cases} (aA - bB,\ aB + bA + 2bB,\ cC) & \text{when } aA - bB > 0, \\ (aB + bA + 2aA,\ bB - aA,\ cC) & \text{when } aA - bB \le 0. \end{cases}$$

It is straightforward to verify that $H_0$ and $H_\pi$ become groups under these operations. Note that the generation process described above makes no sense for triples in $H_0$ and $H_\pi$. Fortunately, (10) and (11) already provide easy methods of constructing members of $H_0$ and $H_\pi$.

**Open Problems.** How does the group structure of $H_\gamma$ vary as the parameter $\gamma$ runs through the pythagorean angles from 0 to $\pi$? For each such $\gamma$, the group operation is defined by (4), but the group structures are, in general, different. For example, $H_{\pi/2}$ has no nontrivial element of finite order, whereas $H_\pi$ has no element of infinite order. For other values of $\gamma$, $H_\gamma$ has nontrivial elements of finite order as well as elements of infinite order. A description of the isomorphism classes of the family of groups $H_\gamma$, $0 \le \gamma \le \pi$, is desirable. Another question is: For which angles $\gamma$ is $H_\gamma$ isomorphic with $H_{\pi/2}$? More generally, for which angles $\gamma_1$ and $\gamma_2$ are $H_{\gamma_1}$ and $H_{\gamma_2}$ isomorphic? If $H_{\pi/2}$ and $H_\gamma$ are isomorphic, which properties of pythagorean triangles can be transferred to $\gamma$-angled triangles? For example, the number of primitive pythagorean triangles $(r, s, t)$ having the same hypotenuse $t$ may be determined by using the group structure of the group $H_{\pi/2}$, see [1]. If $(a, b, c)$ is in $H_\gamma$ and $H_\gamma$ is isomorphic with $H_{\pi/2}$, can one determine the number of $\gamma$-angled triangles having the same "hypotenuse" $c$?

A great many papers about pythagorean triangles have appeared in the literature presenting various properties of these beautiful triangles. See, for example [2]. Very possibly some properties of pythagorean triangles can be generalized so as to be applicable to $\gamma$-angled triangles. How exclusive is the requirement that $\gamma$ be a right angle?

## Acknowledgment

The authors wish to express their gratitude to the referee for making corrections, suggesting improvements in style, and finding a counterexample to a conjecture we had made.

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2. Waclaw Sierpinski. *Pythagorean Triangles*. The Scripta Mathematica Studies, No. 9. New York: Yeshiva University, 1962.

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# A BOX FILLING PROBLEM

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(Submitted January 1988)

## 1. Introduction

For an arbitrary but fixed integer $b > 1$, consider the set of ordered pairs $S_b = \{(i, a_i)\colon 0 \le i \le b - 1$, $a_i$ equals the number of occurrences of $i$ in the sequence $a_0, a_1, \ldots, a_{b-1}\}$. A complete solution for $S_b$ is given explicitly in terms of $b$. It is shown that there is a unique solution for each $b > 6$ and for $b = 5$, that there are two solutions for $b = 4$, and that there is none for $b = 2$, 3, or 6.

Let $b$ be an arbitrary but fixed integer, $b > 1$. We wish to determine, whenever possible, the integers $a_i$ $(0 \le i \le b - 1)$, where $a_i$ denotes the number of occurrences of $i$ in the lower row of boxes in the table below.

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | $\cdots$ | $i$ | $\cdots$ | $b - 1$ |
| $a_0$ | $a_1$ | $a_2$ | $\cdots$ | $a_i$ | $\cdots$ | $a_{b-1}$ |

This may be viewed as a problem in determining all possible sets whose members are functions that satisfy a special property. It is easy to see that the case $b = 2$ gives no solution; henceforth, we shall assume that $b \ge 3$. It is convenient to consider the cases $b > 6$ and $3 \le b \le 6$ separately.

## 2. The Case $b > 6$

It is clear from the definition of each $a_i$, that $a_0 \ne 0$. Thus, the set $T_b = \{a_i\colon a_i \ne 0\}$ is nonempty. In fact, $|T_b| = b - a_0$. Since $a_i$ boxes are filled by $i$ and since each box is necessarily filled by an integer at most $b - 1$, we have

$$\sum_{0 \le i \le b-1} a_i = b.$$

Define the set $T_{0,b} = T_b - \{a_0\}$. Clearly,

$$|T_{0,b}| = b - a_0 - 1 \quad \text{and} \quad \sum a_i = b - a_0.$$

Since each member of $T_{0,b}$ is at least 1, it follows that $T_{0,b}$ consists of $(b - a_0 - 2)$ 1's and one 2. (*)

If $a_0 = 1$, $T_{0,b}$ would consist of $(b - 3)$ 1's and one 2, and $T_b$ would consist of $(b - 2)$ 1's and one 2. This is impossible since the boxes are being filled by 0, 1, and 2, while $a_1 = b - 2 > 4$.

If $a_0 = 2$, $T_{0,b}$ would consist of $(b - 4)$ 1's and one 2, and $T_b$ would consist of $(b - 4)$ 1's and two 2's. This, too, is impossible since the boxes are being occupied by 0, 1, and 2, while $a_1 = b - 4 > 2$.

Thus, $a_0 \geq 3$ and $a_A = 1$ where $A = a_0$. Hence,

$$T_b = \{a_0,\ a_1 = b - a_0 - 2,\ a_2 = 1,\ a_A = 1\}.$$

But $|T_b| = b - a_0 = 4$ implies that $a_0 = b - 4$ and the unique solution in this case is given in the table below.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | ... | $b - 5$ | $b - 4$ | $b - 3$ | $b - 2$ | $b - 1$ |
| $b - 4$ | 2 | 1 | 0 | ... | 0 | 1 | 0 | 0 | 0 |

## 3. The Case $b \leq 6$

By repeating the argument in the case $b > 6$ until (*), if $a_0 \neq 1$ or 2, we would have $|T_b| = b - a_0 = 4$ and so $b = a_0 + 4 \geq 7$. Hence, $a_0 = 1$ or 2.

If $a_0 = 1$, $T_b$ would consist of $(b - 2)$ 1's and one 2. Since all the boxes are being occupied by 0, 1, and 2, we must have $b - 2 \leq 2$. If $b = 3$, we have $a_0 = a_1 = a_2 = 1$, which does not give a solution. If $b = 4$, we have $a_0 = 1$, $a_1 = 2$, $a_2 = 1$, which does give a solution.

If $a_0 = 2$, $T_b$ would consist of $(b - 4)$ 1's and two 2's. Since all of the boxes are filled by 0, 1, and 2, we must have $b - 4 \leq 2$. If $b = 4$, we have $a_0 = 2$, $a_1 = 0$, $a_2 = 2$, which gives a solution. If $b = 5$, we have $a_0 = a_1 = a_2 = 2$, which does not give a solution.

We thus have two solutions if $b = 4$, one solution if $b = 5$, and no solution if $b = 2$, 3, or 6, and these are listed in the tables below.

| 0 | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 2 | 1 | 0 |
| 2 | 0 | 2 | 0 |

$b = 4$

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 2 | 1 | 2 | 0 | 0 |

$b = 5$

## Acknowledgment

The author wishes to thank Michael Esser for having suggested the problem for the case $b = 10$.

## Reference

1. H. J. Ryser. *Combinatorial Mathematics*. The Carus Mathematical Monographs #14. New York: The Mathematical Association of America, 1965.

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# ELEMENTARY PROBLEMS AND SOLUTIONS

*Edited by*

A. P. Hillman

*Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.*

*Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.*

*Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).*

## BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

## PROBLEMS PROPOSED IN THIS ISSUE

**B-652** *Proposed by Herta T. Freitag, Roanoke, VA*

Let $\alpha = (1 + \sqrt{5})/2$,

$$S_1(n) = \sum_{k=1}^{n} \alpha^k \quad \text{and} \quad S_2(n) = \sum_{k=1}^{n} \alpha^{-k}.$$

Determine $m$ as a function of $n$ such that $\dfrac{S_1(n)}{S_2(n)} - \alpha F_m$ is a Fibonacci number.

**B-653** *Proposed by Herta T. Freitag, Roanoke, VA*

The sides of a triangle are $a = F_{2n+3}$, $b = F_{n+3}F_n$, and $c = F_3F_{n+2}F_{n+1}$, with $n$ a positive integer.

(i) Is the triangle acute, right, or obtuse?
(ii) Express the area as a product of Fibonacci numbers.

**B-654** *Proposed by Alejandro Necochea, Pan American U., Edinburgh, TX*

Sum the infinite series

$$\sum_{k=1}^{\infty} \frac{1 + 2^k}{2^{2k}} F_k.$$

B-655 *Proposed by L. Kuipers, Sierre, Switzerland*

Prove that the ratio of integers $x/y$ such that

$$\frac{F_{2n}}{F_{2n+2}} < \frac{x}{y} < \frac{F_{2n+1}}{F_{2n+3}}$$

and with smallest denominator $y$ is $(F_{2n} + F_{2n+1})/(F_{2n+2} + F_{2n+3})$.

B-656 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Find a closed form for the sum

$$S_n = \sum_{k=0}^{n} w_k p^{n-k},$$

where $w_n$ satisfies $w_n = pw_{n-1} - qw_{n-2}$ for $n$ in $\{2, 3, \ldots\}$, with $p$ and $q$ nonzero constants.

B-657 *Proposed by Clark Kimberling, U. of Evansville, Evansville, IN*

Let $m$ be an integer and $m \geq 3$. Prove that no two of the integers

$$k(mF_n + F_{n-1}) \text{ for } k = 1, 2, \ldots, m - 1 \text{ and } n = 0, 1, 2, \ldots$$

are equal. Here $F_{-1} = 1$.

## SOLUTIONS

### Average Age of Fibonacci's Rabbits

B-628 *Proposed by David Singmaster, Polytechnic of the South Bank, London, England*

What is the present average age of Fibonacci's rabbits? (Recall that he introduced a pair of mature rabbits at the beginning of his year and that rabbits mature in their second month. Further, no rabbits died. Let us say that he did this at the beginning of 1202 and that he introduced a pair of 1-month-old rabbits. At the end of the first month, this pair would have matured and produced a new pair, giving us a pair of 2-month-old rabbits and a pair of 0-month-old rabbits. At the end of the second month we have a pair of 3-month-old rabbits and pairs of 1-month-old and of 0-month-old rabbits.) Before solving the problem, make a guess at the answer.

*Solution by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

The solution to this problem is rather amazing. If $n$ is the number of months within the interval (1st Jan. 1202-1st Nov. 1988), then the number of pairs of Fibonacci rabbits in the enclosure is currently $F_{n+1}$. On the basis of the growth rule, their average age $A_n$ (in months) is

$$(1) \qquad A_n = \left(n + \sum_{i=1}^{n-2} iF_{n-1-i}\right) F_{n+1}.$$

By using the identity

$$(2) \qquad \sum_{i=1}^{N} iF_{k-i} = F_{k+3} - (N + 2)F_{k-N+1} - F_{k-N},$$

the proof of which is omitted in this context, we can evaluate (1)

(3) $A_n = (n + F_{n+2} - nF_2 - F_1)/F_{n+1} = (F_{n+2} - 1)/F_{n+1}.$

Since $n$ is sufficiently large (> 9000), we have

$$A_n \approx \alpha = (1 + \sqrt{5})/2 \text{ months}.$$

*Also solved by Charles Ashbacher, Paul Bruckman, John Cannell, Carl Libis, and the proposer.*

## Always at Least One Solution

B-629 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

For which integers $a$, $b$, and $c$ is it possible to find integers $x$ and $y$ satisfying

$$(x + y)^2 - cx^2 + 2(b - a + ac)x - 2(a - b)y + (a - b)^2 - ca^2 = 0?$$

*Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA*

We prove more than what is asked; namely, for every possible choice of the ordered triple $(a, b, c)$, we find the corresponding solution $(x, y)$ satisfying the given equation.

The given equation can be written as:

$$(x + y)^2 - 2(a - b)(x + y) + (a - b)^2 = c(x - a)^2$$

or
$$[(x + y) - (a - b)]^2 = c(x - a)^2$$

or
$$[(x - a) + (y + b)]^2 = c(x - a)^2$$

The following cases are possible:

(i) If $c = 0$, then $(x, y)$ takes infinitely many integral values, namely,

$$(x,\ a - b - x) \text{ where } a,\ b,\ x \text{ are arbitrary integers.}$$

Thus, with $a$, $b$ as arbitrary integers and $c = 0$ the corresponding solution is

$$(x,\ a - b - x) \text{ where } x \text{ is any integer.}$$

(ii) If $x = a$, then $y = -b$ and $c$ can be any arbitrary integer. Hence, with any choice of integral values of $a$, $b$ and arbitrary integer $c$, we have $(a, -b)$ as the solution for $(x, y)$.

(111) If $x \neq a$, $c \neq 0$, then it follows that $c$ must be the square of an integer and $(x - a)$ must divide $(y + b)$. Consequently, if $c = n^2$ where $n$ is a positive integer, then with $a$, $b$ as arbitrary integers and $c = n^2$, we get two possible integral solutions:

$$[x,\ (n - 1)(x - a) - b] \quad \text{and} \quad [x,\ -(n + 1)(x - a) - b]$$

for $(x, y)$ where $x$ is an arbitrary integer.

*Also solved by Paul Bruckman, L. Kuipers, Amitabha Tripathi, and the proposer.*

## Golden Geometric Progression

B-630 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $a$ and $b$ be constants and define the sequences

$$\{A_n\}_{n=1}^{\infty} \quad \text{and} \quad \{B_n\}_{n=1}^{\infty}$$

by $A_1 = a$, $A_2 = b$, $B_1 = 2b - a$, $B_2 = 2a + b$, and $A_n = A_{n-1} + A_{n-2}$ and $B_n = B_{n-1} + B_{n-2}$ for $n \geq 3$.

(i) Determine $a$ and $b$ so that $(A_n + B_n)/2 = [(1 + \sqrt{5})/2]^n$.

(ii) For these $a$ and $b$, obtain $(B_n + A_n)/(B_n - A_n)$.

*Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY*

(i) Since $C_n = (A_n + B_n)/2$ satisfies the second-order recurrence relation

$$C_n = C_{n-1} + C_{n-2} \text{ for } n \geq 3,$$

$C_1$ and $C_2$ determine the sequence $\{C_n\}_{n=1}^{\infty}$. Solving the system of equations

$$b = C_1 = (1 + \sqrt{5})/2$$

$$a + b = C_2 = [(1 + \sqrt{5})/2]^2 = (3 + \sqrt{5})/2$$

we obtain $a = 1$ and $b = (1 + \sqrt{5})/2$.

(ii) For these $a$ and $b$, we have

$$B_1 = \sqrt{5} = \sqrt{5}A_1 \quad \text{and} \quad B_2 = (5 + \sqrt{5})/2 = \sqrt{5}A_2.$$

So

$$B_n = \sqrt{5}A_n \quad \text{and} \quad (B_n - A_n)/2 = [(\sqrt{5} - 1)/2]A_n \text{ for all } n \geq 1.$$

Thus,

$$(B_n + A_n)/2 = [(\sqrt{5} + 1)/2]A_n$$

and

$$\frac{B_n + A_n}{B_n - A_n} = \frac{\sqrt{5} + 1}{\sqrt{5} - 1} = \left(\frac{\sqrt{5} + 1}{2}\right)^2.$$

*Also solved by Charles Ashbacher, Paul Bruckman, Russell Euler, Piero Filipponi, L. Kuipers, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Amitabha Tripathi, and the proposer.*

## Closed Form

B-631 *Proposed by L. Kuipers, Sierre, Switzerland*

For $N$ in $\{1, 2, \ldots\}$ and $N \geq m + 1$, obtain, in closed form,

$$u_N = \sum_{k=m+1}^{m+N} k(k-1)\cdots(k-m)\binom{n+k}{k}.$$

*Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY*

$$u_N = \frac{1}{n!}\sum_{k=m+1}^{m+N}\frac{(n+k)!}{(k-m-1)!} = \frac{(n+m+1)!}{n!}\sum_{k=m+1}^{m+N}\binom{n+k}{n+m+1}$$

$$= \frac{(n+m+1)!}{n!}\binom{n+m+N+1}{n+m+2}$$

*Also solved by Paul Bruckman, Odoardo Brugia & Piero Filipponi, and the proposer.*

## Golden Determinant

B-632 *Proposed by H.-J. Seiffert, Berlin, Germany*

Find the determinant of the $n$ by $n$ matrix $(x_{ij})$ with $x_{ij} = (1 + \sqrt{5})/2$ for $j > i$, $x_{ij} = (1 - \sqrt{5})/2$ for $j < i$, and $x_{ij} = 1$ for $j = i$.

*Solution by Hans Kappus, Rodersdorf, Switzerland*

More generally, let us determine the characteristic polynomial

$$f_n(t) = \det(x_{ij} - t\delta_{ij}).$$

Subtracting the $(n - i)^{\text{th}}$ line from the $(n - i + 1)^{\text{th}}$ line $(i = 1, \ldots, n - 1)$ we obtain the determinant

$$f_n(t) = \begin{vmatrix} 1 - t & \alpha & \ldots & \alpha & \alpha \\ \beta - 1 + t & 1 - \alpha - t & \ldots & 0 & 0 \\ 0 & \beta - 1 + t & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & \beta - 1 + t & 1 - \alpha - t \end{vmatrix}$$

which, after expanding with respect to the $n^{\text{th}}$ column, may be written as

$$\begin{aligned} f_n(t) &= \alpha(1 - \beta - t)^{n-1} + (1 - \alpha - t)f_{n-1}(t) \\ &= \alpha(\alpha - t)^{n-1} + (\beta - t)f_{n-1}(t). \end{aligned}$$

Because of symmetry, we may interchange $\alpha$ and $\beta$ and eliminate $f_{n-1}(t)$. Thus, we arrive at

$$\begin{aligned} f_n(t) &= (1/\sqrt{5})\{\alpha(\alpha - t)^n - \beta(\beta - t)^n\} \\ &= \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\frac{\alpha^{k+1} - \beta^{k+1}}{\sqrt{5}} t^{n-k} = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}F_{k+1}t^{n-k}. \end{aligned}$$

Therefore, the solution to the original problem is given by $f_n(0) = F_{n+1}$.

Editor's Note: Bob Prielipp pointed out that B-632 is a special case of the determinant of Problem A-2 of the 1978 W. L. Putnam Mathematical Competition. (The solution is in *American Mathematical Monthly*, Nov. 1979, p. 753.)

*Also solved by Paul Bruckman, Odoardo Brugia & Piero Filipponi, Russell Euler, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Bob Prielipp, Sahib Singh, Amitabha Tripathi, and the proposer.*

## Ratio of Series

B-633 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $n \geq 2$ be an integer and define

$$A_n = \sum_{k=0}^{\infty} \frac{F_k}{n^k}, \quad B_n = \sum_{k=0}^{\infty} \frac{L_k}{n^k}.$$

Prove that $B_n/A_n = 2n - 1$.

*Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI*

$$F_k = (\alpha^k - \beta^k)/\sqrt{5}$$

and

$$L_k = \alpha^k + \beta^k \text{ where } \alpha = (1 + \sqrt{5})/2 \text{ and } \beta = (1 - \sqrt{5})/2.$$

The infinite geometric series

$$C_n = \sum_{k=0}^{\infty}\left(\frac{\alpha}{n}\right)^k \quad \text{and} \quad D_n = \sum_{k=0}^{\infty}\left(\frac{\beta}{n}\right)^k$$

both converge since the absolute value of each of their common ratios is less than 1. (Notice that the condition $n \geq 2$ is needed to insure the convergence of the first series.) Thus,

$$(1) \qquad A_n = \frac{1}{\sqrt{5}}(C_n - D_n) = \frac{1}{\sqrt{5}}\left(\frac{n}{n - \alpha} - \frac{n}{n - \beta}\right) = \frac{n}{\sqrt{5}}\,\frac{\alpha - \beta}{(n - \alpha)(n - \beta)}$$

$$= \frac{n}{(n - \alpha)(n - \beta)} \quad \text{since } \alpha - \beta = \sqrt{5}$$

and

$$(2) \qquad B_n = C_n + D_n = \frac{n}{n - \alpha} + \frac{n}{n - \beta} = \frac{n(2n - (\alpha + \beta))}{(n - \alpha)(n - \beta)}$$

$$= \frac{n(2n - 1)}{(n - \alpha)(n - \beta)} \quad \text{because } \alpha + \beta = 1.$$

Therefore, $B_n/A_n = 2n - 1$.

*Also solved by Paul Bruckman, Russell Euler, Herta T. Freitag, Jay Hendel, Hans Kappus, L. Kuipers, Carl Libis, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, and the proposer.*

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## ADVANCED PROBLEMS AND SOLUTIONS

*Edited by*
Raymond E. Whitney

*Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.*

### PROBLEMS PROPOSED IN THIS ISSUE

<u>H-435</u> *Proposed by Ratko Tošič, University of Novi Sad, Yugoslavia*

(a) Prove that, for $n \geq 1$,

$$F_{n+1} + \sum_{\substack{0 < i_1 < \cdots < i_k \leq n \\ 1 \leq k \leq n}} F_{n+1-i_k} F_{i_k - i_{k-1}} \cdots F_{i_2 - i_1} F_{i_1} = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n+1}{2k+1} \cdot 2^k,$$

where $\lfloor x \rfloor$ is the greatest integer $\leq x$.

(b) Prove that, for $n \geq 3$,

$$\sum_{\substack{0 < i_1 < \cdots < i_k \leq n \\ 1 \leq k \leq n}} (-1)^{n-k} F_{n-1-i_k} F_{i_k - i_{k-1}} \cdots F_{i_2 - i_1} F_{i_1 - 2} \cdot 2^k = F_{n+3} + (-1)^{n+1} F_{n-3}.$$

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(<u>Comment</u>: The identity is valid for $n \geq 0$, if we define
$F_{-3} = 2$, $F_{-2} = -1$; $F_i = F_{i-1} + F_{i-2}$, for $i \geq -1$.)

<u>H-436</u> *Proposed by Piero Filipponi, Rome, Italy*

For $p$ an arbitrary prime number, it is known that

$$(p-1)! \equiv p-1 \pmod{p}, \quad (p-2)! \equiv 1 \pmod{p},$$

and

$$(p-3)! \equiv (p-1)/2 \pmod{p}.$$

Let $k_0$ be the smallest value of an integer $k$ for which $k! > p$.

The numerical evidence turning out from computer experiments suggests that the probability that, for $k$ varying within the interval $[k_0, p-3]$, $k!$ reduced modulo $p$ is either even or odd is 1/2. Can this conjecture be proved?

## SOLUTIONS

### Integrate Your Results

H-410 *Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 25, no. 2, May 1987)*

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ \text{for } n \ge 2.$$

Prove or disprove that, for $n \ge 1$,

$$\int_0^1 F_n(x)dx = \frac{1}{n}(L_n - (-1)^n - 1).$$

*Solution by Paul Bruckman, (formerly) Fair Oaks, CA*

The conjecture is true.

*Proof:* The characteristic equation of the $F_n(x)$ is given by:

(1) $$z^2 - xz - 1 = 0,$$

which has solutions

(2) $$u \equiv u(x) = \tfrac{1}{2}(x + \theta),\ v \equiv v(x) = \tfrac{1}{2}(x - \theta),$$

where $\theta \equiv \theta(x) = (x^2 + 4)^{\frac{1}{2}} = u - v$.

From the initial conditions on the $F_n(x)$, we readily find:

(3) $$F_n(x) = \frac{u^n - v^n}{u - v},\ n = 0, 1, 2, \ldots .$$

Also, we define $L_n(x)$ as follows:

(4) $$L_n(x) = u^n + v^n,\ n = 0, 1, 2, \ldots .$$

We may differentiate the quantities in (2) with respect to $x$; denoting such derivatives by prime symbols, we readily obtain:

(5) $$\theta' = x/\theta;\quad u' = u/\theta;\quad v' = -v/\theta.$$

From (4) and (5), we find:

$$L_n'(x) = nu^{n-1}\cdot u/\theta - nv^{n-1}\cdot v/\theta = n(u^n - v^n)/\theta,$$

or

(6) $$L_n'(x) = nF_n(x).$$

It follows from (6) that

$$\int_0^1 F_n(x)dx = L_n(x)/n\Big]_0^1, \text{ or}$$

(7) $$\int_0^1 F_n(x)dx = \frac{1}{n}(L_n(1) - L_n(0)),\ n = 1, 2, \ldots .$$

Now $\theta(1) = 5^{\frac{1}{2}}$, so $u(1) = \alpha$, $v(1) = \beta$ (the usual Fibonacci constants), and $L_n(1) = L_n$. Also, $\theta(0) = 2$, so $u(0) = 1$, $v(0) = -1$, and $L(0) = 1 + (-1)^n$. Thus,

(8) $$\int_0^1 F_n(x)dx = \frac{1}{n}(L_n - 1 - (-1)^n),\ n = 1, 2, \ldots .\quad \text{Q.E.D.}$$

*Also solved by O. Brugia & P. Filipponi, C. Georghiou, L. Kuipers, J.-Z. Lee & J.-S. Lee, B. Prielipp, and the proposer.*

## Close Ranks

H-411 *Proposed by Paul S. Bruckman, Fair Oaks, CA (Vol. 25, no. 2, May 1987)*

Define the simple continued fraction $\theta(a, d)$ as follows:

$$\theta(a, d) \equiv [a, a + d, a + 2d, a + 3d, \ldots], \; a \text{ and } d \text{ real}, \; d \neq 0.$$

Find a closed form for $\theta(a, d)$.

*Solution by C. Georghiou, University of Patras, Patras, Greece*

Take the differential equation

$$(*) \qquad zw'' + bw' - w = 0.$$

Then, for $b \neq 0, -1, -2, \ldots$, we have

$$\frac{w}{w'} = b + \frac{z}{w'/w''}.$$

By differentiating (*), we get

$$\frac{w'}{w''} = b + 1 + \frac{z}{w''/w'''},$$

and by repeated differentiation of (*), we get the continued fraction

$$(**) \qquad f(z) = \frac{w}{w'} = b + \cfrac{z}{b + 1 + \cfrac{z}{b + 2 + \cfrac{z}{b + 3 + \cdots}}}$$

Now it is shown in W. B. Jones & W. J. Thron, *Continued Fractions* (New York: Addison-Wesley, 1980), pp. 209-210, that the above continued fraction converges to the meromorphic function

$$f(z) = \frac{b\,{}_0F_1(b;\, z)}{{}_0F_1(b+1;\, z)}$$

for all complex numbers $z$ and, moreover, the convergence is uniform on every compact subset of $\mathbb{C}$ that contains no poles of $f(z)$.

From the theory of continued fractions, we know that

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots = b_0 + \frac{c_1a_1}{c_1b_1} + \frac{c_1c_2a_2}{c_2b_2} + \frac{c_2c_3a_3}{c_3b_3} + \cdots$$

Where $c_n \neq 0$. Then, if we take $b = a/d$ and $z = 1/d^2$ in (**) and use the above identity, we find

$$\theta(a, d) = \frac{a\,{}_0F_1(a/d);\, 1/d^2)}{{}_0F_1(a/d + 1;\, 1/d^2)}$$

valid for $a/d \neq 0, -1, -2, \ldots$ and $d \neq 0$. Since

$${}_0F_1(b + 1; \tfrac{1}{4}z^2) = e^{-z}M(b + \tfrac{1}{2}, 2b + 1, 2z) = \Gamma(b + 1)(\tfrac{1}{2}z)^{-b}I_b(z)$$

where $M(a, b, z)$ and $I_b(z)$ are the Confluent Hypergeometric function and the Modified Bessel function of the first kind, respectively, $\theta(a, d)$ can be expressed in terms of these functions as

$$\theta(a, d) = \frac{I_{a/d-1}(2/d)}{I_{a/d}(2/d)} = \frac{aM(a/d - \frac{1}{2}, 2a/d - 1, 4/d)}{M(a/d + \frac{1}{2}, 2a/d + 1, 4/d)}.$$

When $a/d = -k$, $k = 0, 1, 2, \ldots$, we have

$$\theta(-kd, d) = -kd + \frac{1}{\theta(-(k-1)d, d)}$$

and since

$$\theta(0, d) = \frac{1}{\theta(d, d)} = \frac{I_1(2/d)}{I_0(2/d)} = \frac{I_{-1}(2/d)}{I_0(2/d)},$$

it is easily shown by induction that

$$\theta(-kd, d) = \frac{I_{-k-1}(2/d)}{I_{-k}(2/d)}.$$

Therefore,

$$\theta(a, d) = \frac{I_{a/d-1}(2/d)}{I_{a/d}(2/d)}$$

for all (complex) $a$ and $d$, $d \neq 0$ and $I_{a/d}(2/d)$ does not vanish. Since the Modified Bessel functions have no real zeros, the above expression is valid for all real $a$ and $d$, $d \neq 0$.

Finally, for (real) $a$ and $d = 0$, we have the simple periodic continued fraction

$$\theta(a, 0) = (a + \sqrt{a^2 + 4})/2 \text{ for } a > 0,$$

$\theta(-a, 0) = -\theta(a, 0)$, and $\theta(0, 0)$ diverges.

*Also solved by the proposer who noted the following interesting result:*

(*) $\theta(1, 2) = \coth 1.$

## It Adds Up!

H-412 *Proposed by Andreas N. Philippou & Frosso S. Makri, University of Patras, Patras, Greece (Vol. 25, no. 3, August, 1987)*

Show that

$$\sum_{i=0}^{k-1} \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} = \binom{n}{r}, \; k \geq 1, \; 0 \leq r \leq k - 1 \leq n,$$

where the inner summation is over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n - i$ and $n_1 + \cdots + n_k = n - r$.

*Solution by W. Moser, McGill University, Montreal, Canada*

The number of solutions $(x_1, x_2, \ldots, x_{n-r}, i)$ of

(1) $x_1 + x_2 + \cdots + x_{n-r} + i = r$

(where $x_1, x_2, \ldots, x_{n-r}, i$ are nonnegative integers)—or, equivalently, the number of ways of distributing $r$ like objects into $n - r - 1$ unlike boxes—is $\binom{n}{r}$. This is well known and easy to prove. Let

(2) $n_q = \#\{j \mid x_j = q - 1, \; j = 1, 2, \ldots, n - r\}, \; q = 1, 2, \ldots, k,$

i.e., $n_q$ is the number of $x_j$'s in (1) equal to $q - 1$. Since

$$x_1, x_2, \ldots, x_{n-r}, \; i \in \{0, 1, \ldots, r\} \text{ and } k - 1 \geq r,$$

every $x_j$ is counted once in the sum

(3) $\quad n_1 + n_2 + \cdots + n_k = n - r,$

while $x_1 + x_2 + \cdots + x_{n-r}$ is equal to

(4) $\quad n_2 + 2n_3 + \cdots + (k - 1)n_k = r - i.$

[Note that (3) and (4) are together equivalent to (3) and $n_1 + 2n_2 + \cdots + kn_k = n - i$.] Thus, every solution of (1) yields a solution of (3) satisfying (4). For each $i$ ($i = 0, 1, \ldots, r$), how many solutions of (1) yield the same solution of (3)? Corresponding to a solution of (3) satisfying (4) there are

$$\binom{n - r}{n_1, n_2, \ldots, n_k}$$

linear displays of $n - r$ integers--$n_1$ 0's, $n_2$ 1's, ..., $n_k$ $k - 1$'s--and these integers named from left to right $x_1, x_2, \ldots, x_{n-r}$ have sum $r - i$. The identity follows.

*Also solved by P. Bruckman, G. Dinside, and the proposers & D. Antzoulakos.*

## Generally True!

H-413 *Proposed by Gregory Wulczyn, Bucknell U. (retired), Lewisburg, PA (Vol. 25, no. 3, August 1987)*

Let $m$, $n$ be integers. If $m$ and $n$ have the same parity, show that

(1) $\quad (2m + 1)F_{2n+1} - (2n + 1)F_{2m+1} \equiv 0 \pmod 5;$

(2) $\quad (2m + 1)F_{2n+1} - (2n + 1)F_{2m+1} \equiv 0 \pmod{25}$ if either

(a) $2m + 1$ or $2n + 1$ is a multiple of 5, or
(b) $m \equiv n \equiv 0$ or $m \equiv n \equiv -1 \pmod 5$.

If $m$ and $n$ have the opposite parity, show that

(3) $\quad (2m + 1)F_{2n+1} + (2n + 1)F_{2m+1} \equiv 0 \pmod 5;$

(4) $\quad (2m + 1)F_{2n+1} + (2n + 1)F_{2m+1} \equiv 0 \pmod{25}$ if either

(a) $2m + 1$ or $2n + 1$ is a multiple of 5, or
(b) $m \equiv n \equiv 0$ or $m \equiv n \equiv -1 \pmod 5$.

*Solution by Paul S. Bruckman, (formerly) Fair Oaks, CA*

The indicated results are true, but under more general conditions. We prove the more general result. We define $D(m, n)$ for all integers $m$ and $n$ as follows:

(1) $\quad D(m, n) = (2m + 1)F_{2n+1} - (-1)^{m+n}(2n + 1)F_{2m+1}.$

Also, for all integers $k$, we define $\theta_k$ as follows:

(2) $\quad \theta_k = \dfrac{(-1)^k F_{2k+1}}{2k + 1}.$

Note:
(3) $\quad D(m, n) = (-1)^n(2m + 1)(2n + 1)(\theta_n - \theta_m).$

We now investigate the values of $\theta_k \pmod{25}$. Clearly, if $k \equiv 2 \pmod 5$, then $2k + 1 \equiv 0 \pmod 5$, so $\theta_k \pmod{25}$ and $\theta_k \pmod 5$ are not defined in this case. We find that $\theta_k \pmod{25}$ (as defined) is periodic, with period 50, and we may form the following table (mod 25), omitting values of $k$ with $k \equiv 2 \pmod 5$:

| $k$ | $(2k+1)^{-1}$ | $(-1)$ | $F_{2k+1}$ | $\theta_k$ | $k$ | $(2k+1)^{-1}$ | $(-1)$ | $F_{2k+1}$ | $\theta_k$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 17 | -1 | 2 | 16 | 28 | 18 | 1 | 12 | 16 |
| 3 | 18 | -1 | 13 | 16 | 29 | 14 | -1 | 16 | 1 |
| 4 | 14 | 1 | 9 | 1 | 30 | 16 | 1 | 11 | 1 |
| 5 | 16 | -1 | 14 | 1 | 31 | 2 | -1 | 17 | 16 |
| 6 | 2 | 1 | 8 | 16 | 33 | 3 | -1 | 3 | 16 |
| 8 | 3 | 1 | 22 | 16 | 34 | 4 | 1 | 19 | 1 |
| 9 | 4 | -1 | 6 | 1 | 35 | 6 | -1 | 4 | 1 |
| 10 | 6 | 1 | 21 | 1 | 36 | 12 | 1 | 18 | 16 |
| 11 | 12 | -1 | 7 | 16 | 38 | 13 | 1 | 7 | 16 |
| 13 | 13 | -1 | 18 | 16 | 39 | 19 | -1 | 21 | 1 |
| 14 | 19 | 1 | 4 | 1 | 40 | 21 | 1 | 6 | 1 |
| 15 | 21 | -1 | 19 | 1 | 41 | 22 | -1 | 22 | 16 |
| 16 | 22 | 1 | 3 | 16 | 43 | 23 | -1 | 8 | 16 |
| 18 | 23 | 1 | 17 | 16 | 44 | 9 | 1 | 14 | 1 |
| 19 | 9 | -1 | 11 | 1 | 45 | 11 | -1 | 9 | 1 |
| 20 | 11 | 1 | 16 | 1 | 46 | 7 | 1 | 13 | 16 |
| 21 | 7 | -1 | 12 | 16 | 48 | 8 | 1 | 2 | 16 |
| 23 | 8 | -1 | 23 | 16 | 49 | 24 | -1 | 1 | 1 |
| 24 | 24 | 1 | 24 | 1 | 50 | 1 | 1 | 1 | 1 |
| 25 | 1 | -1 | 24 | 1 | 51 | 17 | -1 | 2 | 16 |
| 26 | 17 | 1 | 23 | 16 | etc. | | | | |

Inspection of the foregoing table yields the following result:

(4) $\theta_k \equiv 1 \pmod{25}$ iff $k \equiv 0$ or $4 \pmod 5$;

$\theta_k \equiv 16 \pmod{25}$ iff $k \equiv 1$ or $3 \pmod 5$.

It follows from (3) that $D(m, n) \equiv 0 \pmod{25}$ if any of the following conditions on $m$ and $n$ (mod 5) hold:

$$(m, n) = (0, 0), (0, 4), (4, 0), (4, 4), (1, 1), (1, 3), (3, 1), \text{ or } (3, 3).$$

This proves parts (2)(b) and (4)(b) of the problem, but gives more general conditions for which $D(m, n) \equiv 0 \pmod{25}$.

Now, if $m \equiv 2 \pmod 5$, then $2m + 1 \equiv 0 \pmod 5$ and $F_{2m+1} = 0 \pmod 5$. Letting $U_n = F_{2n+1} - (-1)^n(2n + 1)$ and $V_n = F_{2n+1} + (-1)^n(2n + 1)$, we may form the following table (mod 25), which is periodic with period 50:

| $m$ | $2m+1$ | $F_{2m+1}$ | $D(m, n)$ |
|---|---|---|---|
| 2 | 5 | 5 | $5U_n$ |
| 7 | -10 | 10 | $-10U_n$ |
| 12 | 0 | 0 | 0 |
| 17 | 10 | -10 | $10U_n$ |
| 22 | -5 | -5 | $-5U_n$ |
| 27 | 5 | 5 | $5V_n$ |
| 32 | -10 | 10 | $-10V_n$ |
| 37 | 0 | 0 | 0 |
| 42 | 10 | -10 | $10V_n$ |
| 47 | -5 | -5 | $-5V_n$ |
| 52 | 5 | 5 | $5U_n$ |

From the table, we see that if $m \equiv 2 \pmod 5$, then $D(m, n) \equiv 0 \pmod{25}$ for all $n$ only if $5 \mid U_n$ or $5 \mid V_n$, i.e., $F_{2n+1} \equiv \pm(2n + 1) \pmod 5$ for all $n$. To test this, we prepare the following table (mod 5), which has period 20:

| $k$ | $F_k$ | $F_k + k$ or $F_k - k$* |
|---|---|---|
| 1 | 1 | $1 - 1 \equiv 0$ |
| 3 | 2 | $3 + 2 \equiv 0$ |
| 5 | 0 | $0 + 0 \equiv 0$ |
| 7 | 3 | $7 + 3 \equiv 0$ |
| 9 | 4 | $9 - 4 \equiv 0$ |
| 11 | 4 | $11 + 4 \equiv 0$ |
| 13 | 3 | $13 - 3 \equiv 0$ |
| 15 | 0 | $15 + 0 \equiv 0$ |
| 17 | 2 | $17 - 2 \equiv 0$ |
| 19 | 1 | $19 + 1 \equiv 0$ |
| 21 | 1 | $21 - 1 \equiv 0$ |

*whichever is applicable

Thus, $5 \mid U_n$ or $5 \mid V_n$ for all $n$, which proves that $D(m, n) \equiv 0 \pmod{25}$ if $m \equiv 2 \pmod 5$. Similarly, $D(m, n) \equiv 0 \pmod{25}$ if $n \equiv 2 \pmod 5$. This proves parts (2)(a) and (4)(a) of the problem. Thus, if $m \equiv 2$ or $n \equiv 2 \pmod 5$, $D(m, n) \equiv 0 \pmod 5$. On the other hand, if $m \not\equiv 2$ and $n \not\equiv 2 \pmod 5$, then $\theta_m \equiv \theta_n \equiv 1 \pmod 5$ (from the first table); in the latter case, therefore, $D(m, n) \equiv 0 \pmod 5$ also [using (3)]. This proves parts (1) and (3).

*Also solved by L. Kuipers, L. Sohmer, and the proposer.*

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*A Primer for the Fibonacci Numbers.* Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

*Fibonacci's Problem Book.* Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

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