25.48

The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

TABLE OF CONTENTS

VOLUME 28 FEBRUARY 1990 NUMBER 1

PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

EDITORIAL POLICY

THE FIBONACCI QUARTERLY seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

SUBMITTING AN ARTICLE

Articles should be submitted in the format of the current issues of **THE FIBONACCI QUARTERLY.** They should be typewritten or reproduced typewritten copies, that are clearly readable, double spaced with wide margins and on only one side of the paper. The full name and address of the author must appear at the beginning of the paper directly under the title. Illustrations should be carefully drawn in India ink on separate sheets of bond paper or vellum, approximately twice the size they are to appear in print. Since the Fibonacci Association has adopted $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \geq 2$ and $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$, $n \geq 2$ as the standard definitions for The Fibonacci and Lucas sequences, these definitions *should not* be a part of future papers. However, the notations *must* be used.

Two copies of the manuscript should be submitted to: **GERALD E. BERGUM, EDITOR, THE FIBONACCI QUARTERLY, DEPARTMENT OF COMPUTER SCIENCE, SOUTH DAKOTA STATE UNIVERSITY, BOX 2201, BROOKINGS, SD 57007-0194.**

Authors are encouraged to keep a copy of their manuscripts for their own files as protection against loss. The editor will give immediate acknowledgment of all manuscripts received.

SUBSCRIPTIONS, ADDRESS CHANGE, AND REPRINT INFORMATION

Address all subscription correspondence, including notification of address change, to: **RICHARD VINE, SUBSCRIPTION MANAGER, THE FIBONACCI ASSOCIATION, SANTA CLARA UNIVERSITY, SANTA CLARA, CA 95053.**

Requests for reprint permission should be directed to the editor. However, general permission is granted to members of The Fibonacci Association for noncommercial reproduction of a limited quantity of individual articles (in whole or in part) provided complete reference is made to the source.

Annual domestic Fibonacci Association membership dues, which include a subscription to **THE FIBONACCI QUARTERLY,** are $30 for Regular Membership, $40 for Sustaining Membership, and $67 for Institutional Membership; foreign rates, which are based on international mailing rates, are somewhat higher than domestic rates; please write for details. **THE FIBONACCI QUARTERLY** is published each February, May, August and November.

All back issues of **THE FIBONACCI QUARTERLY** are available in microfilm or hard copy format from **UNIVERSITY MICROFILMS INTERNATIONAL, 300 NORTH ZEEB ROAD, DEPT. P.R., ANN ARBOR, MI 48106.** Reprints can also be purchased from **UMI CLEARING HOUSE** at the same address.

The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau (1907-1988)

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

REFEREES

In addition to the members of the Board of Directors and our Assistant Editors, the following mathematicians, engineers, and physicists have assisted **THE FIBONACCI QUARTERLY** by refereeing papers during the past year. Their special efforts are sincerely appreciated, and we apologize for any names that have inadvertently been overlooked or misspelled.

AGARWAL, A.K.
Pennsylvania State University-Mont Alto
AKRITAS, A.G.
University of Kansas
ALMKVIST, G.
University of Lund
ANDERSON, P.G.
Rochester Institute of Technology
ANDERSON, S.
University of Minnesota-Duluth
ANDO, S.
Hosei University
ANDREWS, G.E.
Pennsylvania State University
ATANASSOV, K.T.
Sofia, Bulgaria
BERZSENYI, G.
Rose-Hulman University
BEZUSZKA, S.J.
Boston College
BOLLINGER, R.C.
Pennsylvania State University-Behrend
BRESSOUD, D.M.
Pennsylvania State University
BROWN, E.
Virginia Polytechnic Institute
BRUCE, I.
University of Adelaide
BUMBY, R.T.
Rutgers University
CAMPBELL, C.M.
University of St. Andrews
CANTOR, D.G.
University of California at LA
CAPOCELLI, R.M.
University of Rome
CASTELLANOS, D.
Valencia, Venezuela
CHANG, D.
University of California at LA
CHARALAMBIDES, C.A.
University of Athens
COHEN, D.I.A.
Hunter College
COHEN, M.E.
California State University, Fresno
COOPER, C.
Central Missouri State University
CREELY, J.W.
Vincentown, New Jersey
DAVIS, P.J.
Brown University
DEARDEN, B.
University of North Dakota
DE BRUIN, M.G.
University Delft
DENCE, T.P.
Ashland College
DEO, N.
University of Central Florida
DODD, F.
University of South Alabama
DOWNEY, P.J.
University of Arizona
D'SOUZA, H.
University of Michigan-Flint
DUDLEY, U.
DePauw University
ENNEKING, E.A.
Portland State University
ENTRINGER, R.C.
University of New Mexico
EVANS, R.J.
University of California-San Diego
EWELL, J.A.
Northern Illinois University
FARRELL, E.J.
University of the West Indies
FILASETA, M.
University of South Carolina
FISHBURN, P.C.
AT&T Bell Laboratories
FRAENKEL, A.S.
Weizmann Institute of Science
FUCHS, E.
University of J.E. Purkyne
GBUR, M.E.
Texas A and M University
GOOD, I.J.
Virginia Polytechnic Institute
GORDON, B.
University of California at LA
GRANVILLE, A.
Institute for Advanced Studies
HARBORTH, H.
Braunschweig, West Germany
HAUKKANEN, P.
University of Tampere
HENSLEY, D.
Texas A and M University
HOFT, M.
University of Michigan-Dearborn
HORAK, P.
J.E. Purkyne University
HORIBE, Y.
Shizuoka University
HSU, L.C.
Dalian Institute of Technology
JACOBSON, E.
Ohio University
JOHNSON, R.A.
Washington State University
JOYNER, R.N.
East Carolina University
KALMAN, D.
Rancho Palos Verdes, CA
KATZ, T.M.
Hunter College
KENNEDY, R.E.
Central Missouri State University
KILLGROVE, R.B.
Indiana State University
KIMBERLING, C.
University of Evansville
KISS, P.
Eger, Hungary
KNOEBEL, A.
New Mexico State University
KONHAUSER, J.D.E.
Macalester College
KUIPERS, L.
Suisse, Switzerland
LAHR, J.
Grand Duchy of Luxembourg
LENSTRA, H.W.
University of California
LEVESQUE, C.
Universite Laval
LIGH, S.
University of Southwestern Louisiana
LORD, G.
Princeton, New Jersey
McNEILL, R.B.
Northern Michigan University
METZGER, J.
University of North Dakota
MILOVANOVIC, G.V.
University of NIS
MIRON, D.B.
South Dakota State University
MONTGOMERY, P.
Los Angeles, CA
MONZINGO, M.G.
Southern Methodist University
MOON, J.W.
University of Alberta-Edmonton
NIEDERREITER, H.G.
Austrian Acadamy of Science
OWENS, M.A.
California State University-Chico
PHILIPPOU, A.N.
Nicosia, Cyprus
PIERCE, K.R.
University of Minnesota-Duluth
POPOV, B.S.
Macedonian Academy of Sciences & Arts
RAWSTHORNE, D.A.
Rockville, MD
ROBBINS, N.
San Francisco State University
ROBERTS, J.B.
Reed College
SANDER, J.W.
Universitat Hannover

Continued on page 59

THIRD-ORDER DIAGONAL FUNCTIONS OF PELL POLYNOMIALS

Br. J. M. Mahon
Benilde High School, Bankstown, N.S.W., 2200

A. F. Horadam
University of New England, Armidale, N.S.W., 2351

(Submitted December 1987)

1. Introduction

This paper is concerned with the study of some third-order sequences of polynomials. While it is only of an introductory nature, it does give something of the flavor of the research involved. In particular, we have found that an examination of the roots of the auxiliary equation to be a challenging and rewarding endeavor.

The first of these sequences is $\{r_n(x)\}$. It is defined thus:

$$(1.1)\quad \begin{cases} r_0(x) = 0,\ r_1(x) = 1,\ r_2(x) = 2x \\ r_{n+1}(x) = 2xr_n(x) + r_{n-2}(x) \quad (n \geq 2) \end{cases}$$

Two other sequences, namely $\{s_n(x)\}$ and $\{t_n(x)\}$, are also considered. They are defined thus:

$$(1.2)\quad \begin{cases} s_0(x) = 0,\ s_1(x) = 2,\ s_2(x) = 2x \\ s_{n+1}(x) = 2xs_n(x) + s_{n-2}(x) \quad (n \geq 2) \end{cases}$$

$$(1.3)\quad \begin{cases} t_0(x) = 3,\ t_1(x) = 2x,\ t_2(x) = 4x^2 \\ t_{n+1}(x) = 2xt_n(x) + t_{n-2}(x) \quad (n \geq 2) \end{cases}$$

These sequences are called *third-order diagonal functions of Pell polynomials* [5], or simply *Pell diagonal functions*, because the first two coincide with sequences derived by taking the "diagonals" of gradient 1 from the arrays produced by Pell and Pell-Lucas polynomials [10].

The three sequences can be considered to be constructed from the diagonals of gradient 2 from the arrays produced by expansions of

$$(2x + 1)^n, \quad (2x + 2)(2x + 1)^{n-1}, \quad (2x + 3)(2x + 1)^{n-1},$$

where $n \geq 1$.

Considered as a sequence of order three, $\{s_n(x)\}$ appears to be of little significance. The sequences $\{r_n(x)\}$ and $\{t_n(x)\}$ may be deemed to be the *fundamental* and *primordial* sequences, respectively, for those obeying the recurrence relation in (1.1)-(1.3) [9]. All of these sequences are too special to provide subject matter for the study of third-order sequences in general. In a later paper some generalizations of these polynomials may be considered and these are closer to typical third-order sequences.

Jaiswal [6] and Horadam [4] studied the diagonal functions of Chebyshev polynomials of the second and first kinds, respectively, $\{p_n(x)\}$ and $\{q_n(x)\}$. It may be shown that

$$(1.4)\quad \begin{cases} r_n(x) = (-1)^{n-1}p_n(-x), & r_n(-x) = (-1)^{n-1}p_n(x) \\ s_n(x) = (-1)^{n-1}q_n(-x), & s_n(-x) = (-1)^{n-1}q_n(x) \end{cases}$$

Simple relations such as these are to be expected as Pell and Pell-Lucas sequences are complex Chebyshev polynomials [5].

2. Roots of the Auxiliary Equation of the Pell Diagonal Functions

The auxiliary equation of the diagonal functions (1.1)-(1.3) is the cubic

$$(2.1)\quad f(y) \equiv y^3 - 2xy^2 - 1 = 0.$$

By Descartes' Rule, one of the roots is real and positive. Denote this by α. For $x \geq 0$, the other two roots, β and γ are conjugate complex numbers. It is noted that, from (2.1),

$$(2.1')\quad \begin{cases} \alpha + \beta + \gamma = 2x \\ \alpha\beta + \beta\gamma + \gamma\alpha = 0 \\ \alpha\beta\gamma = 1 \end{cases}$$

By using Cardano's procedure [3], it is found that

$$(2.2)\quad \begin{cases} \alpha = 2x/3 + \sqrt[3]{\{16x^3 + 27 + \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \\ \qquad + \sqrt[3]{\{16x^3 + 27 - \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \\ \beta = 2x/3 + \omega\sqrt[3]{\{16x^3 + 27 + \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \\ \qquad + \omega^2\sqrt[3]{\{16x^3 + 27 - \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \\ \gamma = 2x/3 + \omega^2\sqrt[3]{\{16x^3 + 27 + \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \\ \qquad + \omega\sqrt[3]{\{16x^3 + 27 - \sqrt{(864x^3 + 729)}\}}/3\sqrt[3]{2} \end{cases}$$

where ω and ω^2 are complex cube roots of unity; α, β, and γ are clearly algebraic functions of x. We use function notation with the roots where appropriate. From (2.2), it is seen that, for $x > -3/2\sqrt[3]{4}$, the quantities

$$(2.3)\quad \begin{cases} u = \sqrt[3]{\{16x^3 + 27 + \sqrt{(864x^3 + 729)}\}} \\ v = \sqrt[3]{\{16x^3 + 27 - \sqrt{(864x^3 + 729)}\}} \end{cases}$$

are real and so β and γ are conjugate complex numbers. If

$$(2.3')\quad x = -3/2\sqrt[3]{4} = d, \text{ then } \beta = \gamma.$$

Again from (2.2), it may be shown that

$$(2.4)\quad \alpha^2 - |\beta|^2 = 2x\alpha \text{ for } x > d.$$

Hence

$$(2.5)\quad \begin{cases} \alpha > |\beta| = |\gamma| \text{ for } x > 0 \\ \alpha = |\beta| = |\gamma| = 1 \text{ for } x = 0 \\ \alpha < |\beta| = |\gamma| \text{ for } d < x < 0. \end{cases}$$

For $x < d$, it is convenient to consider the roots to be given by

$$(2.6)\quad \begin{cases} \alpha = 2x/3 + 4x/3 \cos(4\pi + \theta)/3 \\ \beta = 2x/3 + 4x/3 \cos(\theta/3) \\ \gamma = 2x/3 + 4x/3 \cos(2\pi + \theta)/3 \end{cases}$$

where

$$(2.7)\quad \cos\theta = (16x^3 + 27)/16x^3, \quad \sin\theta = 3\sqrt{\{3(32x^3 + 27)\}}/16ix^3 = 3\sqrt{(3D)}/16x^3$$

D being the discriminant of (2.1), and thus

$$(2.8)\quad D = -(32x^3 + 27).$$

It may be shown that, for $x < d$,

$$(2.9)\quad \begin{cases} -\pi < \theta < 0 \\ \alpha > 0 \\ \beta,\ \gamma < 0 \\ |\beta| > |\gamma| > \alpha \\ |\beta| > 1 \\ |\gamma| > 1 \text{ for } -1 < x < d \\ |\gamma| < 1 \text{ for } x < -1 \\ \lim_{x \to -\infty} \theta = 0^- \\ \lim_{x \to -\infty} \alpha = 0^+ \\ \lim_{x \to -\infty} \beta = -\infty \\ \lim_{x \to -\infty} \gamma = 0^- \end{cases}$$

Some simple correspondences for x, θ, α, β, and γ are recorded in Table 2.1.

TABLE 2.1

x	θ	α	β	γ
d	$-\pi$	$1/\sqrt[3]{4}$	$-2/\sqrt[3]{4}$	$-2/\sqrt[3]{4}$
-1	$-\cos^{-1}(-11/16)$	$(\sqrt{5} - 1)/2$	$-(\sqrt{5} + 1)/2$	-1
$-3/2\sqrt[3]{2}$	$-\pi/2$	$(\sqrt{3} - 1)/\sqrt[3]{2}$	$-(\sqrt{3} + 1)/\sqrt[3]{2}$	$-1/\sqrt[3]{2}$
$-\infty$	0	0	$-\infty$	0

A computer investigation carried out by Br. V. Cotter indicates that, in the natural domain, α is an increasing function, that $|\beta|$ is a decreasing function, and that $|\gamma|$ increases, reaches a maximum near $x = d$ and then decreases to zero.

It is noted that $\alpha(-1)$ and $\beta(-1)$ are negatives of the roots of the auxiliary equation of the Fibonacci sequence. As a result, we would expect that there are simple relations between $\{r_n(-1)\}$, $\{s_n(-1)\}$, and $\{t_n(-1)\}$ and the Fibonacci and Lucas sequences. In fact, a study of the diagonal functions has resulted in obtaining what appears to be a large number of highly specific identities for these numbers. We were alerted to these possibilities by the work of Jaiswal [6] and Horadam [4] dealing with the diagonal functions of the Chebyshev polynomials.

3. Binet Formulas for the Diagonal Functions

A variety of procedures may be followed to give a number of formulas for the diagonal functions in terms of the roots, α, β, and γ. It may be shown that, for $\beta \neq \gamma$,

$$(3.1)\qquad r_n(x) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^{n+1} & \beta^{n+1} & \gamma^{n+1} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} = \Delta_{n+1}(x)/\Delta_2(x)$$

$$(3.2)\qquad r_n(x) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^{n+3} & \beta^{n+3} & \gamma^{n+3} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha & \beta & \gamma \end{vmatrix} = \delta_{n+3}(x)/\delta_1(x)$$

$$(3.3)\qquad r_n(x) = A_\alpha^n + B_\beta^n + C_\gamma^n$$

where

$$(3.4)\qquad A = \frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)},\ B = \frac{\beta}{(\beta - \gamma)(\beta - \alpha)},\ \text{and } C = \frac{\gamma}{(\gamma - \alpha)(\gamma - \beta)}$$

and

$$(3.5)\qquad r_n(x) = \frac{\alpha^{n+1}}{f'(\alpha)} + \frac{\beta^{n+1}}{f'(\beta)} + \frac{\gamma^{n+1}}{f'(\gamma)}$$

where $f(y)$ is as defined in (2.1).

The formula (3.1) may be considered to be the third-order analogue of the Binet formula for the Fibonacci numbers expressed as the quotient of two determinants. The third-order number sequence equivalents of (3.3) and (3.4) occur in Jarden [7] and Spickerman [11] and (3.5) may be compared to a formula of Levesque [8].

Starting with (3.1), we can deduce (1.1). Hence (3.1) could be taken as the definition of $\{r_n(x)\}$. This new definition would allow us to introduce negative subscripts.

Binet formulas for $\{s_n(x)\}$ include, for $\beta \neq \gamma$,

$$(3.6)\qquad s_n(x) = A'\alpha^n + B'\beta^n + C'\gamma^n$$

where

$$(3.7)\qquad A' = \frac{\alpha - \beta - \gamma}{(\alpha - \beta)(\alpha - \gamma)},\ B' = \frac{\beta - \alpha - \gamma}{(\beta - \gamma)(\beta - \alpha)},\ \text{and } C' = \frac{\gamma - \beta - \alpha}{(\gamma - \alpha)(\gamma - \beta)}$$

and

$$(3.8)\qquad s_n(x) = \frac{\alpha^{n+1} + \alpha^{n-2}}{f'(\alpha)} + \frac{\beta^{n+1} + \beta^{n-2}}{f'(\beta)} + \frac{\gamma^{n+1} + \gamma^{n-2}}{f'(\gamma)}.$$

The Binet formula for $t_n(x)$ is

$$(3.9)\qquad t_n(x) = \alpha^n + \beta^n + \gamma^n.$$

It may be shown that when $\beta = \gamma$, i.e., when $x = d$,

$$(3.10)\qquad r_n(d) = (-1)^{n-1}2^{(2-2n)/3}\{(3n + 1)2^n - (-1)^n\}/9.$$

The Binet formulas lead to some simple identities involving the diagonal functions, for example,

(3.11) $s_n(x) = r_n(x) + r_{n-3}(x) = 2xr_{n-1}(x) + 2r_{n-3}(x);$

(3.12) $t_{n-1}(x) = r_n(x) + 2r_{n-3}(x) = 2xr_{n-1}(x) + 3r_{n-3}(x) = s_n(x) + r_{n-3}(x).$

The formulas also give the following relations with the Fibonacci and Lucas numbers, $\{f_n\}$ and $\{\ell_n\}$:

$$(3.13)\quad \begin{cases} r_n(-1) = (-1)^{n+1}(f_{n+2} - 1) \\ r_{-n}(-1) = f_{n-2} + (-1)^n \\ s_n(-1) = 2(-1)^{n-1}f_n \\ s_{-n}(-1) = 2f_n \\ t_n(-1) = (-1)^n(\ell_n + 1) \\ t_{-n}(-1) = \ell_n + (-1)^n \end{cases}$$

4. Determinantal Generators for the Diagonal Functions

Let us now introduce a new sequence $\{\phi_n(x)\}$ of determinants of which the first few members are:

$$\phi_1(x) = |2x|,\ \phi_2(x) = \begin{vmatrix} 2x & 1 \\ 0 & 2x \end{vmatrix},\ \phi_3(x) = \begin{vmatrix} 2x & 1 & 0 \\ 0 & 2x & 1 \\ 1 & 0 & 2x \end{vmatrix}$$

The n^{th} term is defined thus:

$$(4.1)\quad \phi_n(x):\ \left\{\begin{array}{ll} d_{rr} = 2x & \text{for } r = 1, 2, \ldots, n \\ d_{r,r+1} = 1 & \text{for } r = 1, 2, \ldots, n-1 \\ d_{r,r-2} = 1 & \text{for } r = 3, 4, \ldots, n \\ d_{rc} = 0 & \text{otherwise} \end{array}\right\}$$

where d_{rc} is the entry in the r^{th} row and c^{th} column. It may be proved by induction that, for $n > 0$,

(4.2) $\phi_n(x) = r_{n+1}(x).$

The sequences $\{\phi_n^*(x)\}$, $\{\phi_n^{**}(x)\}$ are defined similarly, except that

$d_{12} = 2, 3$, respectively.

Induction shows that, for $n > 0$,

(4.3) $\phi_n^*(x) = s_{n+1}(x);$

(4.4) $\phi_n^{**}(x) = t_n(x).$

Next we introduce a further sequence $\{\eta_n(x)\}$ of which the first few members are:

$$\eta_1(x) = |0|,\ \eta_2(x) = \begin{vmatrix} 0 & 1 \\ 2x & 0 \end{vmatrix},\ \eta_3(x) = \begin{vmatrix} 0 & 1 & 0 \\ 2x & 0 & 1 \\ 1 & 2x & 0 \end{vmatrix}$$

The n^{th} term is specified thus:

$$(4.5)\qquad \eta_n(x):\ \left\{\begin{array}{lll} d_{r,r+1} = 1 & \text{for } r = 1, 2, \ldots, n-1 \\ d_{r,r-1} = 2x & \text{for } r = 2, 3, \ldots, n \\ d_{r,r-2} = 1 & \text{for } r = 3, 4, \ldots, n \\ d_{rc} = 0 & \text{otherwise} \end{array}\right\}$$

Induction may be employed to prove that

$$(4.6)\qquad \eta_n(x) = r_{-n-2}(x).$$

From these determinants, some new determinantal generators for Fibonacci and Lucas numbers may be derived, namely:

$$(4.7)\qquad \phi_n(-1) = (-1)^n(f_{n+3} - 1) \quad \text{from (3.13) and (4.2)}$$

$$(4.8)\qquad \phi_n^*(-1) = (-1)^n 2f_{n+1} \quad \text{from (3.13) and (4.3)}$$

$$(4.9)\qquad \phi_n^{**}(-1) = (-1)^n(\ell_n + 1) \quad \text{from (3.13) and (4.4)}$$

$$(4.10)\qquad \eta_n(-1) = f_n + (-1)^n \quad \text{from (3.13) and (4.6)}$$

5. Explicit Summation Expressions for Diagonal Functions

It is assumed in what follows that n is sufficiently large so that all the subscripts are greater than or equal to -1. Repeated application of the formula in (1.1) gives the lines below:

$$\begin{aligned}(5.1)\qquad r_n(x) &= 2xr_{n-1}(x) + r_{n-3}(x) \\ &= (2x)^2 r_{n-2}(x) + r_{n-3}(x) + (2x)r_{n-4}(x) \\ &= \{(2x)^3 + 1\}r_{n-3}(x) + (2x)r_{n-4}(x) + (2x)^2 r_{n-5}(x) \\ &= \{(2x)^4 + 2(2x)\}r_{n-4}(x) + (2x)^2 r_{n-5}(x) + \{(2x)^3 + 1\}r_{n-6}(x) \\ &= \{(2x)^5 + 3(2x)^2\}r_{n-5}(x) + \{(2x)^3 + 1\}r_{n-6}(x) \\ &\qquad + \{(2x)^4 + 2(2x)\}r_{n-7}(x) \\ &= \{(2x)^6 + 4(2x)^3 + 1\}r_{n-6}(x) + \{(2x)^4 + 2(2x)\}r_{n-7}(x) \\ &\qquad + \{(2x)^5 + 3(2x)^2\}r_{n-8}(x)\end{aligned}$$

One formula suggested by these lines is:

$$\begin{aligned}(5.2)\qquad r_n(x) &= \left\{\sum_{i=0}^{[j/3]}\binom{j-2i}{i}(2x)^{j-3i}\right\}r_{n-j}(x) \\ &\quad + \left\{\sum_{i=0}^{[(j-2)/3]}\binom{j-2-2i}{i}(2x)^{j-2-3i}\right\}r_{n-j-1}(x) \\ &\quad + \left\{\sum_{i=0}^{[(j-1)/3]}\binom{j-1-2i}{i}(2x)^{j-1-3i}\right\}r_{n-j-2}(x)\end{aligned}$$

This may be proved by induction. Put $j + 1 = n$ in (5.2) to get

$$(5.3)\qquad r_n(x) = \sum_{i=0}^{(n-1)/3}\binom{n-1-2i}{i}(2x)^{n-1-3i},$$

since $r_0(x) = r_{-1}(x) = 0$.

By substituting (5.3) in (5.2), it is found that

$$(5.4) \qquad r_n(x) = r_{j+1}(x)r_{n-j}(x) + r_{j-1}(x)r_{n-j-1}(x) + r_j(x)r_{j-n-2}(x)$$

or

$$(5.5) \qquad r_{m+n}(x) = r_{m+1}(x)r_n(x) + r_{m-1}(x)r_{n-1}(x) + r_m(x)r_{n-2}(x).$$

The identity (5.5) is similar to one found in Agronomoff [1] and Jarden [7] for third-order sequences of numbers.

Other explicit expressions for the diagonal functions include

$$(5.6) \qquad r_{-2n-1}(x) = \sum_{i=0}^{[(n-2)/3]} \binom{n-1-i}{2i+1}(-2x)^{n-2-3i}$$

$$(5.7) \qquad r_{-2n}(x) = \sum_{i=0}^{[(n-1)/3]} \binom{n-1-i}{2i}(-2x)^{n-1-3i}$$

$$(5.8) \qquad s_n(x) = (2x)^{n-1} + \sum_{i=1}^{[(n-1)/3]} \frac{n-1-i}{i}\binom{n-2-2i}{i-1}(2x)^{n-1-3i}$$

$$(5.9) \qquad s_{-2n}(x) = \sum_{i=0}^{[(n-1)/3]} \frac{n+1+i}{2i+1}\binom{n-1-i}{2i}(-2x)^{n-1-3i}$$

$$(5.10) \qquad s_{-2n-1}(x) = (-2x)^{n+1} + \sum_{i=1}^{[(n+1)/3]} \frac{n+1+i}{2i}\binom{n-i}{2i-1}(-2x)^{n+1-3i}$$

$$(5.11) \qquad t_n(x) = \sum_{i=0}^{[n/3]} \frac{n}{n-2i}\binom{n-2i}{i}(2x)^{n-3i}$$

$$(5.12) \qquad t_{-2n}(x) = \sum_{i=0}^{[n/3]} \frac{2n}{n-i}\binom{n-i}{2i}(-2x)^{n-3i}$$

$$(5.13) \qquad t_{-2n-1}(x) = \sum_{i=0}^{[(n-1)/3]} \frac{2n+1}{n-i}\binom{n-i}{2i+1}(-2x)^{n-1-3i}$$

If the method used to prove (5.3) is applied to the other sequences of Pell diagonal polynomials, then it is possible to prove that

$$(5.14) \qquad s_{m+n}(x) = r_{m+1}(x)s_n(x) + r_{m-1}(x)s_{n-1}(x) + r_m(x)s_{n-2}(x);$$

$$(5.15) \qquad t_{m+n}(x) = r_{m+1}(x)t_n(x) + r_{m-1}(x)t_{n-1}(x) + r_m(x)t_{n-2}(x).$$

The formulas (5.3) and (5.6)-(5.13) lead to some new explicit expressions for the Fibonacci and Lucas numbers:

$$(5.16) \qquad f_{n+2} - 1 = \sum_{i=0}^{[(n-1)/3]} (-1)^i \binom{n-1-2i}{i} 2^{n-1-3i}$$

$$(5.17) \qquad f_{2n-1} - 1 = \sum_{i=0}^{[(n-2)/3]} \binom{n-1-i}{2i+1} 2^{n-2-3i}$$

$$(5.18) \qquad f_{2n} + 1 = \sum_{i=0}^{[n/3]} \binom{n-i}{2i} 2^{n-3i}$$

$$(5.19) \qquad f_n = 2^{n-2} + \sum_{i=1}^{[(n-1)/3]} \frac{n-1-i}{i}\binom{n-2-2i}{i-1}(-1)^i 2^{n-2-3i}$$

$$(5.20)\quad f_{2n} = \sum_{i=0}^{[(n-1)/3]} \frac{n+1+i}{2i+1}\binom{n-1-i}{2i} 2^{n-2-3i}$$

$$(5.21)\quad f_{2n+1} = 2^n + \sum_{i=1}^{[(n+1)/3]} \frac{n+1+i}{2i}\binom{n-i}{2i-1} 2^{n-3i}$$

$$(5.22)\quad \ell_n + 1 = \sum_{i=0}^{[n/3]} \frac{n}{n-2i}\binom{n-2i}{i}(-1)^i 2^{n-3i}$$

$$(5.23)\quad \ell_{2n+1} - 1 = \sum_{i=0}^{[(n-1)/3]} \frac{2n+1}{n-i}\binom{n-i}{2i+1} 2^{n-1-3i}$$

$$(5.24)\quad \ell_{2n} + 1 = \sum_{i=0}^{[n/3]} \frac{2n}{n-i}\binom{n-i}{2i} 2^{n-3i}$$

By Descartes' Rule, $r_n(x)$ can have no positive roots and, at most, $[(n-1)/3]$ negative roots. It is believed that this maximal number of roots is, in fact, the actual number of roots. We shall attempt to prove this in some future paper.

References

1. M. Agronomoff. "Sur une suite récurrente." *Mathesis*, Ser. 4, *4* (1914): 125-126.
2. W. S. Burnside & A. W. Panton. *Theory of Equations*. New York: Dover, 1960.
3. F. Cajori. *An Introduction to the Theory of Equations*. New York: Dover, 1969.
4. A. F. Horadam. "Polynomials Associated with Chebyshev Polynomials of the First Kind." *Fibonacci Quarterly 15.3* (1977):255-257.
5. A. F. Horadam & Br. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *Fibonacci Quarterly 23.1* (1985):7-20.
6. D. V. Jaiswal. "Polynomials Related to Tchebichef Polynomials of the Second Kind." *Fibonacci Quarterly 12.3* (1974):263-265.
7. Dov Jarden. *Recurring Sequences*. Israel: Riveon Lematematika, 1966.
8. C. Levesque. "On m^{th}-Order Recurrences." *Fibonacci Quarterly 15.4* (1977): 290-293.
9. E. Lucas. *Théorie des nombres*. Paris: Blanchard, 1958.
10. Br. J. M. Mahon. "Pell Polynomials." M.A. Thesis presented to the University of New England, 1984.
11. W. R. Spickerman. "Binet's Formula for the Tribonacci Sequence." *Fibonacci Quarterly 20.2* (1982):118-122.

*****

ODD NONUNITARY PERFECT NUMBERS

Peter Hagis, Jr.
Temple University, Philadelphia, PA 19122
(Submitted December 1987)

1. Introduction

Throughout this paper lower-case letters will be used to denote natural numbers, with p and q always representing primes. As usual, (c, d) will symbolize the greatest common divisor of c and d. If $cd = n$ and $(c, d) = 1$, then d is said to be a unitary divisor of n and we write $d \| n$. $\sigma(n)$ and $\sigma^*(n)$ denote, respectively, the sum of the divisors and unitary divisors of n. Both σ and σ^* are multiplicative, and $\sigma(p^e) = 1 + p + \cdots + p^e$ while $\sigma^*(p^e) = 1 + p^e$.

In [1] Ligh & Wall have defined d to be a nonunitary divisor of n if $cd = n$ and $(c, d) > 1$. If $\sigma^\#(n)$ denotes the sum of the nonunitary divisors of n, it is immediate that $\sigma^\#(n) = \sigma(n) - \sigma^*(n)$. It is easy to see that $\sigma^\#$ is not multiplicative, and that $\sigma^\#(n) = 0$ if and only if n is squarefree. Now, n has a unique representation of the form $n = \overline{n} \cdot n^\#$ where $(\overline{n}, n^\#) = 1$, $\overline{n}$ is squarefree, and $n^\#$ is powerful. (The value of $\overline{n}$ is 1 if n is powerful, $n^\# = 1$ if n is squarefree, and $1 = 1 \cdot 1$.) It follows easily that

$$\sigma^\#(n) = \sigma(\overline{n}) \cdot \sigma^\#(n^\#)$$

so that

$$\sigma^\#(n) = \prod_{p \| n} (1 + p)\left\{\prod_{p^e \| n} (1 + p + \cdots + p^e) - \prod_{p^e \| n} (1 + p^e)\right\} \tag{1}$$

where $e > 1$.

Ligh & Wall [1] say that n is a k-fold nonunitary perfect number if $\sigma^\#(n) = kn$. In particular, if $\sigma^\#(n) = n$, then n is said to be a nonunitary perfect number. The integers m and n are nonunitary amicable numbers if $\sigma^\#(m) = n$ and $\sigma^\#(n) = m$. All known k-fold nonunitary perfect numbers and all known nonunitary amicable pairs are even. In the present paper we initiate the study of *odd* nonunitary perfect numbers. Nonunitary aliquot sequences will also be discussed.

2. Odd Nonunitary Perfect Numbers

We begin this section by proving the following

Theorem 1: The value of $\sigma^\#(n)$ is odd if and only if $n = 2^\alpha M^2$ where $(M, 2) = 1$, $M > 1$, $\alpha \ge 0$.

Proof: Suppose that $\sigma^\#(n)$ is odd and $n = 2^\alpha K$ where $(K, 2) = 1$ and $\alpha \ge 0$. Then $K \ge 3$ since $\sigma^\#(2^0) = \sigma^\#(2) = 0$ and $\sigma^\#(2^\alpha)$ is even if $\alpha \ge 2$. Since $2 \mid (1 + p^e)$ if p is odd, and since $2 \mid (1 + p + \cdots + p^e)$ if and only if e is odd, it follows easily from (1) [since $\sigma^\#(n)$ is odd] that $K = M^2$ *and* $n = 2^\alpha M^2$. Now suppose that $n = 2^\alpha M^2$ where $(M, 2) = 1$, $M > 1$, $\alpha \ge 0$. Since $(1 + p^e)$ is even and $(1 + p + \cdots + p^e)$ is odd if e is even and p is odd, it follows from (1) that $\sigma^\#(n) = \sigma^\#(2^\alpha M^2)$ is odd for $\alpha \ge 0$.

The following corollaries are immediate consequences of Theorem 1.

Corollary 1: If n is an odd nonunitary perfect number (or an odd k-fold nonunitary perfect number where k is odd), then $n = M^2$.

Corollary 2: If m and n are nonunitary odd amicable numbers, then $m = M^2$ and $n = N^2$.

Corollary 3: If m and n are nonunitary amicable numbers of opposite parity ($2|m$ and $2\nmid n$), then $m = 2^{\alpha}M^2$ where $(M,2) = 1$, $\alpha \geq 1$.

Now suppose that n is an odd nonunitary perfect number. From Corollary 1, $n = p_1^{e_1}p_2^{e_2} \ldots p_t^{e_t}$ where $2|e_i$ for $i = 1, 2, \ldots, t$. From (1), we have

$$n = \prod_{i=1}^{t}(1 + p_i + \cdots + p_i^{e_i}) - \prod_{i=1}^{t}(1 + p_i^{e_i}), \tag{2}$$

and it follows that

$$1 = \prod_{i=1}^{t}(1 + p_i^{-1} + \cdots + p_i^{-e_i}) - \prod_{i=1}^{t}(1 + p_i^{-e_i}). \tag{3}$$

Therefore,

$$1 < \prod_{i=1}^{t}(1 + p_i^{-1} + p_i^{-2} + \cdots) - \prod_{i=1}^{t}1$$

or

$$\prod_{p|n} p/(p-1) > 2. \tag{4}$$

It is well known that (4) holds for (ordinary) odd perfect numbers. Let $\omega(n)$ denote the number of distinct prime factors of n. From the table given by Norton in [2], we have

Proposition 1: Suppose that n is a nonunitary odd perfect number. Then $\omega(n) \geq 3$. If $3\nmid n$, then $\omega(n) \geq 7$ and

$$n \geq (5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23)^2 > 10^{15}.$$

If $(15,n) = 1$, then $\omega(n) \geq 15$ and

$$n \geq (7 \cdot 11 \cdot 13 \cdot \cdots \cdot 59 \cdot 61)^2 > 10^{43}.$$

A computer search was made for all odd nonunitary perfect numbers less than 10^{15}. None was found. Therefore, we have

Proposition 2: If n is an odd nonunitary perfect number, then $n > 10^{15}$.

If $n = p_1^{e_1}p_2^{e_2} \ldots p_t^{e_t}$ (where $2|e_i$) is an odd nonunitary perfect number and $1 \leq f_i \leq e_i$, then it follows easily from (3) that

$$\prod_{i=1}^{t}(1 + p_i^{-1} + \cdots + p_i^{-f_i}) - \prod_{i=1}^{t}(1 + p_i^{-f_i}) \leq 1. \tag{5}$$

In particular, if n is an odd nonunitary perfect number,

$$\prod_{p|n}(1 + p^{-1} + p^{-2}) - \prod_{p|n}(1 + p^{-2}) \leq 1. \tag{6}$$

Lemma 1: Suppose that $N = p_1^{a_1} \ldots p_r^{a_r}q_1^{b_1} \ldots q_s^{b_s} = RS$ where $(R,S) = 1$ and $S \geq 1$. If

$$(7)\qquad \sigma^{\#}(p_1^{c_1} \cdots p_r^{c_r})/(p_1^{c_1} \cdots p_r^{c_r})$$
$$= \prod_{i=1}^{r}(1 + p_i^{-1} + \cdots + p_i^{-c_i}) - \prod_{i=1}^{r}(1 + p_i^{-c_i}) > 1$$

where $2 \le c_i \le a_i$ for $i = 1, 2, \ldots, r$, then N is not an odd nonunitary perfect number.

Proof: If $W - V > 1$ and $V > 0$, it is easy to see that

$$W(1 + p^{-1} + \cdots + p^{-b}) - V(1 + p^{-b}) > W - V > 1.$$

It follows from (7) that N cannot satisfy the inequality (5). Therefore, N is not an odd nonunitary perfect number.

Now suppose that n is an odd nonunitary perfect number and $3|n$. Then

$$n = 3^{e_1}p_2^{e_2} \cdots p_t^{e_t} \text{ where } 2|e_i.$$

Since $1 + 3 + \cdots + 3^{e_1} \equiv 1 + 3^{e_1} \equiv 1 \pmod 3$ and since $1 + p^e \equiv -1 \pmod 3$ if $p > 3$ and $2|e$, it follows from (1) that

$$(8)\qquad \sigma^{\#}(n) = n \equiv 0 \equiv \prod_{p^e \| n}(1 + p + \cdots + p^e) + (-1)^t \pmod 3 \text{ where } p > 3.$$

If $p \equiv -1 \pmod 3$, then $1 + p + \cdots + p^e \equiv 1 \pmod 3$ if e is even; if $p \equiv 1 \pmod 3$, then $1 + p + \cdots + p^e \equiv 0, -1, 1 \pmod 3$ according as $e \equiv 2, 4, 6 \pmod 6$, respectively. The following lemma is an immediate consequence of (8) and the preceding remark.

Lemma 2: Suppose that n is an odd nonunitary perfect number such that $3|n$ and $\omega(n) = t$. If $p^e \| n$ and $p \equiv 1 \pmod 3$, then $e \ge 4$. [More precisely, $e \equiv 0, 4 \pmod 6$.] If $2|t$, then n has an odd number of components p^e such that $p \equiv 1 \pmod 3$ and $e \equiv 4 \pmod 6$. If $2\nmid t$, then n has an even number of components p^e such that $p \equiv 1 \pmod 3$ and $e \equiv 4 \pmod 6$.

Now assume that n is an odd nonunitary perfect number such that $3 \cdot 5 \cdot 7|n$. From Lemma 2, $7^4|n$. Suppose that $3^4|n$. Then, since $\sigma^{\#}(3^4 5^2 7^4)/3^4 5^2 7^4 > 1$, Lemma 1 yields a contradiction. Therefore, $3^2\|n$. Since $\sigma^{\#}(3^2 5^2 7^4 13^2)/3^2 5^2 7^4 13^2 > 1$, Lemma 1 shows that $13\nmid n$; and since $1 + 3 + 3^2 = 13$ and $1 + 5^2 = 2 \cdot 13$, we conclude from (2) that $5^2\nparallel n$ so that $5^4|n$.

If $p > 7$, let $F(p) = \sigma^{\#}(3^2 5^4 7^4 p^2)/3^2 5^4 7^4 p^2$. It is easy to verify that F is a monotonic decreasing function of p and that $F(271) > 1$. [($F(277) < 1$.] We have proved

Proposition 3: If n is an odd nonunitary perfect number and if $3 \cdot 5 \cdot 7|n$, then $3^2\|n$ and $5^4 7^4|n$. Also, $p\nmid n$ if $11 \le p \le 271$.

Theorem 2: If n is an odd nonunitary perfect number, then $\omega(n) \ge 4$.

Proof: Assume that $\omega(n) < 4$. Then from Proposition 1, $\omega(n) = 3$ and $3|n$. Since $(3/2)(7/6)(11/10) < 2$ and $x/(x - 1)$ is monotonic decreasing for $x > 1$, it follows from (4) that $5|n$. Since $(3/2)(5/4)(17/16) < 2$, $p\nmid n$ if $p \ge 17$.

Assume that $3 \cdot 5 \cdot 7|n$. From Lemma 2, $3^2\|n$ and it follows easily from (3) that $1 < (13/9)(5/4)(7/6) - (10/9)$. This is a contradiction.

Now suppose that $3 \cdot 5 \cdot 13|n$. If $3^2\|n$, then, from (3),

$$1 < (13/9)(5/4)(13/12) - (10/9).$$

If $5^2 \| n$, then

$$1 < (3/2)(31/25)(13/12) - (26/25).$$

In each case, we have a contradiction. Therefore, $3^4 5^4 13^2 | n$. But,

$$\sigma^{\#}(3^4 5^4 13^2)/3^4 5^4 13^2 > 1$$

and, from Lemma 1, n is not a nonunitary perfect number.

Finally, assume that $3 \cdot 5 \cdot 11 | n$. If $3^2 \| n$, then, from (3),

$$1 < (13/9)(5/4)(11/10) - (10/9)$$

and we have a contradiction. If $3^4 \| n$, then, since $1 + 3 + 3^2 + 3^3 + 3^4 = 11^2$ and $1 + 3^4 = 82$, it follows from (2) that $0 \equiv -5(1 + 5^e) \pmod{11}$. This is impossible since $11 \nmid (1 + 5^e)$ if $2 | e$. Therefore, $3^6 | n$. Now assume that $5^2 \| n$. If $11^4 | n$, then, since

$$\sigma^{\#}(3^6 5^2 11^4)/3^6 5^2 11^4 > 1,$$

we have a contradiction. Therefore, $11^2 \| n$. Since $n = 3^e 5^2 11^2$, it follows from (2) that

$$5^2 \cdot 11^2 \cdot 3^e = 31 \cdot 133 \cdot (3^{e+1} - 1)/2 - 26 \cdot 122 \cdot (1 + 3^e).$$

Therefore, $25 \cdot 3^e = 10467$ and we have a contradiction. We conclude that $5^4 | n$. But

$$\sigma^{\#}(3^6 5^4 11^2)/3^6 5^4 11^2 > 1$$

and, from Lemma 1, n is not a nonunitary perfect number.

3. Nonunitary Aliquot Sequences

A t-tuple of distinct natural numbers $(n_0; n_1; \ldots; n_{t-1})$ with $n_i = \sigma^{\#}(n_{i-1})$ for $i = 1, 2, \ldots, t - 1$ and $n_0 = \sigma^{\#}(n_{t-1})$ is called a *nonunitary t-cycle*. A nonunitary 1-cycle is a nonunitary perfect number; a nonunitary 2-cycle is a nonunitary amicable pair. A search was made for all nonunitary t-cycles with $t > 2$ and $n_0 \le 10^6$. One was found:

(619368; 627264; 1393551)

The *nonunitary aliquot sequence* $\{n_i\}$ with leader n is defined by

$$n_0 = n,\ n_1 = \sigma^{\#}(n_0),\ n_2 = \sigma^{\#}(n_1),\ \ldots,\ n_i = \sigma^{\#}(n_{i-1}),\ \ldots .$$

Such a sequence is said to be *terminating* if n_k is squarefree for some index k (so that $n_i = 0$ for $i > k$). [We define $\sigma^{\#}(0) = 0$.] A nonunitary aliquot sequence is said to be *periodic* if an index k exists such that $(n_k; n_{k+1}; \ldots; n_{k+t-1})$ is a nonunitary t-cycle. A nonunitary aliquot sequence which is neither terminating nor periodic is *unbounded*. Whether or not unbounded nonunitary aliquot sequences exist is an open question.

An investigation was made of all nonunitary aliquot sequences with leader $n \le 10^6$. About 40 minutes of computer time was required. 740671 sequences were found to be terminating; 1440 were periodic (194 ended in 1-cycles, 1195 in 2-cycles, and 51 in 3-cycles); and in 257889 cases an $n_k > 10^{12}$ was encountered and (for practical reasons) the sequence was terminated with its final behavior undetermined. As was pointed out by the referee, since there are 607926 squarefree numbers between 1 and 10^6, more than 82% of the 740671 terminating sequences were guaranteed to terminate *before* the investigation just described even began. From this perspective we see that the behavior of only about one-third of the "doubtful" sequences with leaders less than 10^6 has been determined. The first sequence with unknown behavior has leader $n_0 = 792$.

$n_{52} = 1{,}780{,}270{,}202{,}880$ is the first term of this sequence which exceeds 10^{12}.

References

1. S. Ligh & C. R. Wall. "Functions of Nonunitary Divisors." *Fibonacci Quarterly* 25.4 (1987):333-338.
2. K. K. Norton. "Remarks on the Number of Factors of an Odd Perfect Number." *Acta Arithmetica* 6 (1961):365-374.

*****

Announcement

FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 30-August 3, 1990

Department of Mathematics and Computer Science

Wake Forest University

Winston-Salem, North Carolina 27109

International Committee

Horadam, A.F. (Australia), *Co-Chairman*
Philippou, A.N. (Cyprus), *Co-Chairman*
Ando, S. (Japan)
Bergum, G. (U.S.A.)
Johnson, M. (U.S.A.)
Kiss, P. (Hungary)
Filipponi, Piero (Italy)
Campbell, Colin (Scotland)
Turner, John C. (New Zealand)

Local Committee

Fred T. Howard, *Co-Chairman*
Marcellus E. Waddill, *Co-Chairman*
Elmer K. Hayashi
Theresa Vaughan
Deborah Harrell

CALL FOR PAPERS

The FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Wake Forest University, Winston-Salem, N.C., from July 30 to August 3, 1990. This Conference is sponsored jointly by the Fibonacci Association and Wake Forest University.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts are to be submitted by March 15, 1990, while manuscripts are due by May 1, 1990. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

Professor Gerald E. Bergum
The Fibonacci Quarterly
Department of Computer Science, South Dakota State University
P.O. Box 2201, Brookings, South Dakota 57007-0194

A VON STAUDT-CLAUSEN THEOREM FOR CERTAIN BERNOULLIANLIKE NUMBERS AND REGULAR PRIMES OF THE FIRST AND SECOND KIND

Esayas George Kundert
University of Massachusetts, Amherst, MA 01003
(Submitted January 1988)

In a previous paper [6] we have shown that certain operators in a certain completion $\hat{A}$ of the s-d-ring A over the rational numbers determine a well-defined basis. One of the operators which we considered there was $H' = E - Q_1 D$ and we called its corresponding basis $\{u_n'\}$. It was shown in that paper that

$$(u_1')^2 = \sum_{n=0}^{\infty} b_n u_n',$$

where the coefficients b_n are the Bernoulli numbers. The partial fraction decomposition of these numbers is given by the von Staudt-Clausen Theorem (see, for example, [1]):

$$b_0 = 1,\ b_1 = 1/2,\ b_{2m+1} = 0,\ b_{2m} = (-1)^m \left(\text{integer} + \sum_i 1/p_i\right),\ m \geq 1,$$

where p_i is a prime number so that $(p_i - 1) | 2m$. (Note that p_i occurs in the first power only.)

Now, let

$$(u_1')^3 = \sum_{n=0}^{\infty} c_n u_n'.$$

In this paper we will give the partial fraction decomposition for the coefficients c_n. It will turn out for certain c_n that higher powers of primes in the partial fraction denominators will occur, namely, second and third powers of 2 and at most second powers of the other primes.

Definition: We will call a prime $p > 3$ regular of the first kind if a partial fraction belonging to p^1 does occur for all $n \equiv 2m \bmod p - 1$, $n \not\equiv 2m \bmod p$, $m = 1, 2, \ldots, (p - 3)/2$.

We will call a prime $p > 3$ regular of the second kind if a partial fraction belonging to p^1 does occur for all $n \equiv 0 \bmod p - 1$, $n \not\equiv 0 \bmod p$.

It will be seen that our definition of a regular prime of the first kind is equivalent to Kummer's definition of a regular prime [5]. It is not known whether there exist an infinite number of such primes. On the other hand, it is well known that there exist infinitely many irregular primes of the first kind. Robert Gonter from the Computer Center at the University of Massachusetts was kind enough to test all primes up to about 12×10^6 for regularity of the second kind and found that 5, 13, and 563 are the only irregular ones under those primes (see [8]).

Theorem: The partial fraction decomposition of the coefficients c_n with respect to the rest system $\{0, \pm 1, \pm 2, \ldots, (p - 1)/2\}$ is as follows for $n \geq 1$:

I. Partial fractions with 2, 2^2, 2^3 in the denominator. Let

$$r_{n1} = \begin{cases} 0 & \text{for } n = 2, 6, 7, 9, 10 \\ 1 & \text{for } n = 1, 3, 4, 5, 8 \end{cases}$$

$$r_{n2} = \begin{cases} 0 & \text{for } n = 4, 6, 7, 8, 9, 10 \\ 1 & \text{for } n = 1, 2, 3, 5 \end{cases}$$

$$r_{n3} = \begin{cases} 0 & \text{for } n = 1, 2, 4, 6, 8, 10 \\ 1 & \text{for } n = 3, 5, 7, 9 \end{cases}$$

then

$$s_n = \frac{r_{n1}}{2} + \frac{r_{n2}}{2^2} + \frac{r_{n3}}{2^3}$$

occur as partial fractions of c_n for $n = 1$ through 10 and when $n' \equiv n \bmod 8$ for $n, n' \geq 3$, then $s_{n'} = s_n$ occurs in $c_{n'}$.

II. Partial fractions with 3, 3^2 in the denominator. Let

$$\rho_{n1} = \begin{cases} -1 & \text{for } n = 4, 5, 11, 12, 17 \\ 0 & \text{for } n = 1, 6, 7, 8, 10, 13, 14, 19, 20 \\ 1 & \text{for } n = 2, 3, 9, 15, 16, 18 \end{cases}$$

$$\rho_{n2} = \begin{cases} -1 & \text{for } n = 2, 4, 6, 10, 12, 16, 18 \\ 0 & \text{for } n = 1, 3, 5, 7, 8, 9, 11, 13, 14, 15, 17, 19, 20 \end{cases}$$

then

$$\sigma_n = \frac{\rho_{n1}}{3} + \frac{\rho_{n2}}{3^2}$$

occur as partial fractions of c_n for $n = 1$ through 20 and when $n' \equiv n \bmod 18$ for $n, n' \geq 3$, then $\sigma_{n'} = \sigma_n$ occurs in $c_{n'}$.

III. Partial fractions with p or p^2 ($p \geq 5$) in the denominator.

(a) If $n \equiv 1 \bmod p - 1$ and $n \not\equiv p - 2 \bmod p$, $n > 1$, let

$$\rho \equiv -1 + (n - 1)[(p - 1)/2] + n[(p - 1)/2]^2 \bmod p \text{ in } R,$$

then ρ/p occurs as a partial fraction.

(b) Let b_{2m} be the $2m^{\text{th}}$ Bernoulli number, N_{2m} the numerator, and D_{2m} the denominator of b_{2m}, $n \equiv 2m \bmod p - 1$, $m = 1, 2, \ldots, (p - 3)/2$, and $p \nmid N_{2m}$ and $n \not\equiv 2m \bmod p$, $\rho \equiv (2m)^{-1}D_{2m}^{-1}N_{2m}(n - 2m) \bmod p$ in R, then ρ/p occurs as a partial fraction.

(c) By Wilson's theorem, we may write $1 + (p - 1)! = \alpha p$. If $n \equiv 0 \bmod p - 1$, $n \not\equiv 0 \bmod p$, $\alpha \not\equiv 0 \bmod p$, let $\rho \equiv -n\alpha \bmod p$ in R, then $-1/p^2 + \rho/p$ occurs in the decomposition.

Remark 1: Let

$$2m = \prod p_i^{s_i} \prod q_j^{r_j} \quad \text{(prime factorization!)}$$

so that $(p_i - 1) \mid 2m$. Let

$$\tau = N_{2m} / \prod q_j^{r_j}$$

which is an integer, then we may also use

$$\rho \equiv \tau \prod p_i^{-(s_i+1)} (n - 2m) \bmod p \text{ in } R \text{ in III(b).}$$

Remark 2: It can be shown that

$$1 + (p - 1)! \equiv pb_{p-1} - p + 1 \bmod p^2.$$

See [2] where this has been used to show that $1 + (p - 1)! \not\equiv 0 \bmod p^2$ for all $p < 114$ except for $p = 5$ and 13, but, as mentioned above, R. Gonter has shown, using the computer, that 563 is the only other irregular prime $< 12 \times 10^6$. See [8]. Other interpretations of α are given in [3] and [7].

Corollary 1: Let $m = 1, 2, \ldots, (p - 3)/2$, then

$$p \text{ is regular of the } 1^{\text{st}} \text{ kind} \iff p \nmid N_{2m} \iff p \text{ is Kummer regular.}$$

Corollary 2: If $\alpha = [1 + (p - 1)!]/p$, then

$$1 + (p - 1)! \not\equiv 0 \bmod p^2 \iff \alpha \not\equiv 0 \bmod p \iff p \text{ is regular of the } 2^{\text{nd}} \text{ kind.}$$

Proofs: From [6], we know that

$$u_1' = \sum_{k=1}^{\infty} (1/k)x_k',$$

where $\{x_k'\}$ is the basis belonging to the operator $D' = E - D$. For this basis, the multiplication in $\hat{A}$ is especially simple, namely component-wise, so that

$$(u_1')^3 = \sum_{k=1}^{\infty} (1/k^3)x_k'.$$

Also

$$x_k' = \sum_{n=0}^{\infty} B_n^k u_n',$$

where the B_n^k are defined as follows:

$$B_n^k = (-1)^{k+1}k!S_{n+1}^k \text{ where } S_{n+1}^k \text{ is determined by the iteration}$$
$$S_{n+1}^k = S_n^{k-1} + kS_n^k,\ S_n^1 = 1, \text{ and } S_1^k = 0 \text{ for } k > 1.$$

The reader should be warned that our definition of the B_n^k differs from the one in [6] by a factor of $(-1)^n$. If we now put

$$(u_1')^3 = \sum_{n=0}^{\infty} c_n u_n',$$

it follows that $c_0 = 1$ and

$$c_n = \sum_{k=1}^{n+1} B_n^k/k^3.$$

After this we do not have to refer to [6] anymore. In the following proofs, "~" always means "equal up to an added integer."

I. To prove the statements of the theorem in part I, we note first that powers of 2 in the prime factorization of k^3 divide into $k!$ unless $k = 2, 4$, or 8. For $k = 2$, $2!/2^3 = 1/4$; for $k = 4$, $4!/4^3 = 3/8$; for $k = 8$, $8!/8^3 \sim -1/4$. Using this and the iteration from above to calculate the reduced numerators of

B_n^k/k^3 for $k = 2, 4, 8 \bmod 4, 8, 4$, respectively, we see that they repeat periodically with increasing n with periods of length 1, 2, and 8, respectively. Computing next the partial fractions of the so reduced sums $B_n^2/2^3 + B_n^4/4^3 + B_n^8/8^3$ for $n = 1, 2, \ldots, 10$, we get the statements in part I of the theorem.

II. Similarly, one proves the the statements in part II of the theorem.

III. To prove the statements of part III, one uses the following formulas:

(1) $S_n^k = (-1)^k 1/k! \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^n$ (see, for example, [4]);

(2) $S_n^k = 0$ for $n < k$ and $S_n^n = 1$ [this follows readily from (1)];

(3) $B_n^k = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} j^{n+1}$ [follows from (1)];

(4) Let ${}_B\Delta_{(r,s)}^k = B_{s+(r+1)(p-1)}^k - B_{s+r(p-1)}^k$, then

$$ {}_B\Delta_{(r+1,s)}^k - {}_B\Delta_{(r,s)}^k = \sum_{j=1}^{k} (-1)^{j+1} j^{s+r(p-1)} (j^{p-1} - 1)^2 \equiv 0 \bmod p^2 $$

(since, by Fermat's theorem, $j^{p-1} - 1 \equiv 0 \bmod p$). It follows that ${}_B\Delta_{(r,s)}^k$ is independent with respect to $r \bmod p^2$.

(5) Wilson's theorem: $(p + 1)! + 1 \equiv 0 \bmod p$.

Now let $p \geq 5$. First we realize that $k!/k^3$ contains a power of p in the denominator (after cancellation) only if $k = p$ or $k = 2p$. For $k = p$ we have

$$ p!/p^3 = (p - 1)!/p^2 $$

and for $k = 2p$ we have

$$ (2p)!/(2p)^3 \sim [(p-1)/2]^2/p. $$

To compute $B_n^{2r}/(2p)^3$ that is $\sim -[(p - 1)/2]^2 S_{n+1}^{2p}/p$, one uses $S_{n+1}^{2p} \equiv S_n^{2p-1} \bmod p$ and

$$ S_{s+t(p-1)}^{2p-1} \equiv t\,{}_S\Delta_{(0,s)}^{2p-1} \equiv \begin{cases} 0 & \text{if } s < p \\ t & \text{if } s = p \end{cases} \bmod p $$

where ${}_S\Delta_{(r,s)}^k$ is defined as $S_{s+(r+1)(p-1)}^k - S_{s+r(p-1)}^k$ and shows independence with respect to $r \bmod p$ by using formula (4) from above. It follows that

$$ B_{s+t(p-1)}^{2p}/(2p)^3 \sim \begin{cases} 0 & \text{if } s < p \\ \rho_1/p & \text{if } s = p \end{cases} \quad \text{where } \rho_1 \equiv -t[(p-1)/2]^2 \bmod p. $$

To compute B_n^p/p^3 which is $\sim (p - 1)! S_{n+1}^p/p^2 \sim (p - 1)! S_{s+r(p-1)}^{p-1}/p^2$ if $s < p$ and $\sim (p - 1)! S_{s+r(p-1)}^{p-1}/p^2 - 1/p$ if $s = p$ where n has been replaced by $s + r(p - 1)$ but

$$ S_{s+r(p-1)}^{p-1} \equiv \begin{cases} r\,{}_S\Delta_{(0,s)}^{p-1} & \text{for } s < p - 1 \\ 1 + r\,{}_S\Delta_{(0,s)}^{p-1} & \text{for } s = p - 1 \end{cases} \bmod p^2. $$

The statements in III(a) can now be proved. Let $s = 1$ so that

$$ \Delta_{(0,1)}^{p-1} \equiv -p/2 \bmod p^2 $$

and, therefore,

$$B^p_{1+r(p-1)}/p^3 \sim \rho_2/p$$

where $\rho_2 \equiv -[r(p-1)/2+1] \bmod p$ and $B_1/p^3 \sim 0$. Putting

$$n = 1 + r(p-1) = p + t(p-1)$$

and

$$\rho = \rho_1 + \rho_2 \equiv -1 + (n-1)(p-1)/2 + n[(p-1)/2]^2 \bmod p,$$

then ρ/p occurs as a partial fraction of c_n if $n \neq 1$ and $n \not\equiv p-2 \bmod p$. It is clear that, if $n = 1$ and $n \equiv p-2 \bmod p$, then $\rho \equiv 0 \bmod p$ and p does not occur in a partial fraction.

III(b). Let $n = s + r(p-1)$ where $s = 2, 3, \ldots, p-2$.

$$B^p_n/p^3 \sim B^{p-1}_{n-1}/p^2 \sim -rS^{p-1}_{s+p-1}/p/p.$$

To compute S^{p-1}_{s+p-1}/p, we utilize the following Bernoulli numbers:

$$b_s = \sum_{k=1}^{s+1} B^k_s/k^2 = \sum_{k=1}^{s+1}(-1)^{k+1}[(k-1)!/k]S^k_{s+1},\ s = 2, 3, \ldots, p-2,$$

and

$$b_{s+p-1} \equiv \sum_{k=1}^{s+1}(-1)^{k+1}[(k-1)!/k]S^k_{s+1} + \sum_{k=s+2}^{p-1}(-1)^{k+1}[(k-1)!/p]S^k_{s+1}$$

$$-\frac{1}{p}S^p_{s+p} + \sum_{k=p+1}^{s+p}(-1)^{k+1}[(k-1)!/p]S^k_{s+p} \bmod p.$$

The first sum is equal to b_s, the second and third sums $\equiv 0 \bmod p$. Therefore, we have

$$-S^{p-1}_{s+p-1}/p \sim -S^p_{s+p}/p \equiv b_{s+p-1} - b_s \equiv -(1/s)b_s \bmod p.$$

The last congruence follows from a theorem of Kummer. (See, for example, Nr. 14 in [1].). Finally, we have

$$B^p_n/p^3 \sim \begin{cases} 0 & \text{for } s \text{ odd, since } b_s = 0 \\ \rho/p & \text{for } s = 2m,\ m = 1, \ldots, \dfrac{p-3}{2} \end{cases}$$

where

$$\rho \equiv -(r/s)b_s \equiv -(2m)^{-1}D^{-1}_{2m}N_{2m}r \bmod p$$

where D_{2m} and N_{2m} are the denominator and numerator of b_{2m}. Note that $(2m)^{-1}$ exists for our m's and that

$$D^{-1}_{2m} = \prod p_i^{-1} \text{ for } (p_i - 1)\,|\,2m \text{ (by the von Staudt-Clausen theorem)}$$

exists also for our m's. Furthermore, $n = 2m + r(p-1)$, so $-r \equiv n - 2m \bmod p$ and therefore

$$\rho \equiv (2m)^{-1}D^{-1}_{2m}N_{2m}(n-2m) \bmod p \text{ if } p \nmid N_{2m} \text{ and } p \nmid n - 2m$$

which proves III(b).

III(c). Here $n = p - 1 + r(p-1) \equiv 0 \bmod p-1$,

$$B^p_{p-1+r(p-1)}/p^3 \sim -B^{p-1}_{p-2+r(p-1)}/p^3$$

and

$$-B^{p-1}_{p-2+r(p-1)} \equiv -B^{p-1}_{p-2} - r\cdot {}_B\Delta^{p-1}_{(r,p-2)}\cdot$$

$$(p-1)! - r\cdot {}_B\Delta^{p-1}_{(p-1,p-2)} \bmod p^2,$$

but

$$-{}_B\Delta^{p-1}_{(p-1,p-2)} \equiv B^{p-1}_{p^2-p-1} - B^{p-1}_{p-2} \equiv 1 + (p+1)! \equiv \alpha\cdot p \bmod p^2$$

for some integer $\alpha = 0, 1, 2, \ldots, p - 1$. Therefore,

$$-B^{p-1}_{p-2+r(p-1)} \equiv -1 + (r + 1)\alpha p \bmod p^2.$$

Put $\rho - (r + 1)\alpha \equiv -n\alpha \bmod p$, then $\rho/p - 1/p^2$ occurs in the decomposition of c_n provided that $n \not\equiv 0 \bmod p$ and $\alpha \not\equiv 0 \bmod p$, which proves III(c).

Proof of Remark 1: We use a theorem of von Staudt (see, for example, [1], vol. 2, p. 55) which says that τ is an integer, then

$$(2m)^{-1}D^{-1}_{2m}N_{2m} = \tau \prod p_i^{-1-s_i}.$$

Proof of Remark 2:

$$b_{p-1} = \sum_{k=1}^{p-1} (-1)^{k+1}[(k - 1)!/k]S^k_p + (p - 1)!/p,$$

so

$$pb_{p-1} \equiv p + (p - 1)! \bmod p^2 \text{ (since } pS^k_p \equiv 0 \bmod p^2 \text{ for } 1 \le k \le p - 1),$$

so

$$1 + (p - 1)! \equiv pb_{p-1} - p + 1 \bmod p^2.$$

Proof of Corollary 1: The first equivalence follows at once from our definition of a regular prime of the first kind and from III(b). The second equivalence was proved by Kummer himself [5].

Proof of Corollary 2: The first equivalence follows from the proof of III(c) and the second equivalence from the definition of primes of the second kind.

References

1. P. Bachmann. *Niedere Zahlentheorie*, 2-ter Teil Nr. 14 and 15. New York: Chelsea, 1968.
2. N. G. W. H. Beeger. "Quelques remarques sur les congruences $r^{p-1} \equiv 1$ (mod p^2) et $(p - 1)! \equiv -1$ (mod p^2)." *Messenger Math.* *43* (1913):72-84.
3. Ch. Y. Chao. "Generalizations of Theorems of Wilson, Fermat and Euler." *J. Number Theory* *15* (1982):95-114.
4. Ch. Jordan. "On Stirling Numbers." *Tohoku Math. J.* *37* (1933):254-278.
5. E. E. Kummer. "Allgemeiner Beweis des Fermat's schen Satzes etc." *J. fuer Math.* (Crelle) *40* (1850):130-138.
6. E. G. Kundert. "Basis in a Certain Completion of the s-d-Ring over the Rational Numbers." Nota II, *Rendiconti della Academia dei Lincei*, Serie VIII, vol. LXIV, fasc. 6 (1979):543-547.
7. E. Lehmer. "On Congruences Involving Bernoulli Numbers and the Quotients of Fermat and Wilson." *Annals of Math.* *39* (1938):350-359.
8. R. Gonter & E. G. Kundert. "Wilson's Theorem $(p - 1)! + 1 \equiv 0 \bmod p^2$." *SIAM Conference on Discrete Mathematics in San Francisco, June 13-16, 1988.* Report pages 1-8.

*****

FIBONACCI HYPERBOLAS

Clark Kimberling
Mathematics Software Co., 419 S. Boeke Rd., Evansville, IN 47714
(Submitted January 1988)

1. Introduction

Is it possible for a hyperbola $h(x, y) = 0$ to pass through infinitely many points of the form (F_m, F_n), whose coordinates are distinct Fibonacci numbers? The answer to this question is yes. For example, the hyperbola $x^2 + xy - y^2 + 1 = 0$ passes through the points (1, 2), (3, 5), (8, 13), (21, 34), (55, 89), ...

It is not difficult to discover other hyperbolas

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

that pass through infinitely many (F_m, F_n). We shall call such a hyperbola a *Fibonacci hyperbola*. Bergum [1] and Horadam [2] have discussed classes of conic sections that include Fibonacci hyperbolas. In particular, formulas (1) and (1′) below occur, after substitutions, among those discussed by Bergum and Horadam. The purpose of this note is to prove that these formulas account for *all* the Fibonacci hyperbolas. There are no others.

2. Formula, Examples, and Graphs

As usual, let F_0, F_1, F_2, F_3, F_4, F_5, F_6, ... denote the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, ..., and let L_0, L_1, L_2, L_3, L_4, L_5, L_6, ... denote the Lucas sequence 2, 1, 3, 4, 7, 11, 18, ... We extend these sequences in the usual way:

$$F_n = (-1)^{n+1}F_{-n} \quad \text{and} \quad L_n = (-1)^n L_{-n}, \quad \text{for } n = -1, -2, -3, \ldots .$$

It will be helpful to list the first few hyperbolas of the form

(1) $$p_n(x, y) = x^2 + (-1)^{n+1}L_n xy + (-1)^n y^2 + F_n^2 = 0, \quad \text{for } n = 1, 2, 3, \ldots,$$

along with representative points that lie on each hyperbola:

TABLE 1

Hyperbola	Representative Points
$p_1(x, y) = x^2 + xy - y^2 + 1 = 0$	(1, 2), (3, 5), (8, 13), (21, 34), (55, 89)
$p_2(x, y) = x^2 - 3xy + y^2 + 1 = 0$	(1, 2), (2, 5), (5, 13), (13, 34), (34, 89)
$p_3(x, y) = x^2 + 4xy - y^2 + 4 = 0$	(1, 5), (3, 13), (8, 34), (21, 89), (55, 233)
$p_4(x, y) = x^2 - 7xy + y^2 + 9 = 0$	(1, 5), (2, 13), (5, 34), (13, 89), (34, 233)
$p_5(x, y) = x^2 + 11xy - y^2 + 25 = 0$	(1, 13), (3, 34), (8, 89), (21, 233), (55, 610)
$p_6(x, y) = x^2 - 18xy + y^2 + 64 = 0$	(1, 13), (2, 34), (5, 89), (13, 233), (34, 610)

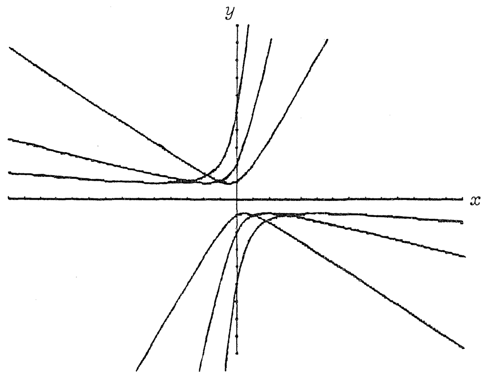

FIGURE 1

The Fibonacci hyperbolas
$p_1(x, y)$, $p_3(x, y)$, and $p_5(x, y)$

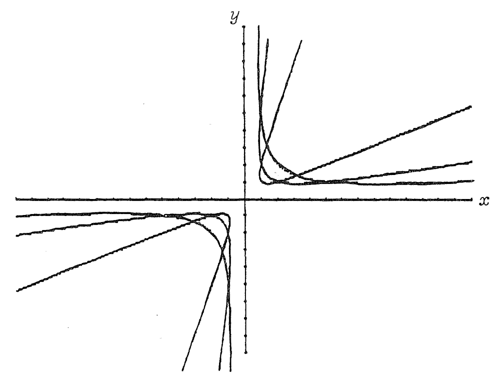

FIGURE 2

The Fibonacci hyperbolas
$p_2(x, y)$, $p_4(x, y)$, and $p_6(x, y)$

Theorem 1: Each hyperbola of the form

(1) $\quad p_n(x, y) = x^2 + (-1)^{n+1}L_n xy + (-1)^n y^2 + F_n^2 = 0$, for $n = 1, 2, 3, \ldots$,

is a Fibonacci hyperbola.

Proof: Well-known identities, given as I_{24} and I_{19} in Hoggatt [3], show that for odd n and even m,

$$\begin{aligned} F_m^2 + (L_nF_m - F_{n+m})F_{n+m} + F_n^2 &= F_m - F_{n-m}F_{n+m} + F_n^2 \\ &= F_m^2 + F_n^2 - [F_n^2 + (-1)^{m+n+1}F_m^2] \\ &= 0. \end{aligned}$$

Similarly, identities I_{22} and I_{19} yield analogous results for even n and odd m. Thus, for any positive integer n, positive even integer h, and integer k for which $k + n$ is odd, all the points

$$(F_k, F_{k+n}), (F_{k+h}, F_{k+n+h}), (F_{k+2h}, F_{k+n+2h}), \ldots$$

lie on hyperbola (1).

Theorem 2: Each hyperbola of the form

(1′) $\quad q_n(x, y) = x^2 + (-1)^{n+1}L_n xy + (-1)^n y^2 - F_n^2 = 0$, for $n = 1, 2, 3, \ldots$,

is a Fibonacci hyperbola.

Proof: For odd n and odd m, identities I_{22} and I_{19} yield

$$\begin{aligned} F_m^2 + (L_nF_m - F_{n+m})F_{n+m} - F_n^2 &= F_m^2 + F_{n-m}F_{n+m} - F_n^2 \\ &= F_m^2 - F_n^2 + (F_n^2 - F_m^2) \\ &= 0. \end{aligned}$$

Similarly, $q_n(F_m, F_{n+m}) = 0$ for even n and even m. As in the proof of Theorem 1, it now follows that for any positive integer n, positive even integer h, and integer k for which $k + n$ is even, all the points

$$(F_k, F_{k+n}), (F_{k+h}, F_{k+n+h}), (F_{k+2h}, F_{k+n+2h}), \ldots$$

lie on hyperbola (1′).

TABLE 2

Hyperbola	Representative Points
$q_1(x, y) = x^2 + xy - y^2 - 1 = 0$	(1, 1) (2, 3), (5, 8), (13, 21), (34, 55)
$q_2(x, y) = x^2 - 3xy + y^2 - 1 = 0$	(1, 3), (3, 8), (8, 21), (21, 55), (55, 144)
$q_3(x, y) = x^2 + 4xy - y^2 - 4 = 0$	(1, 3), (2, 8), (5, 21), (13, 55), (34, 144)
$q_4(x, y) = x^2 - 7xy + y^2 - 9 = 0$	(1, 8), (3, 21), (8, 55), (21, 144), (55, 377)
$q_5(x, y) = x^2 + 11xy - y^2 - 25 = 0$	(1, 8), (2, 21), (5, 55), (13, 144), (34, 377)
$q_6(x, y) = x^2 - 18xy + y^2 - 64 = 0$	(1, 21), (3, 55), (8, 144), (21, 377), (55, 987)

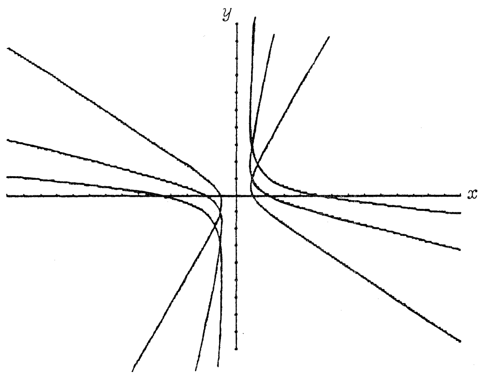

FIGURE 3

The Fibonacci polynomials $q_1(x, y)$, $q_3(x, y)$, and $q_5(x, y)$

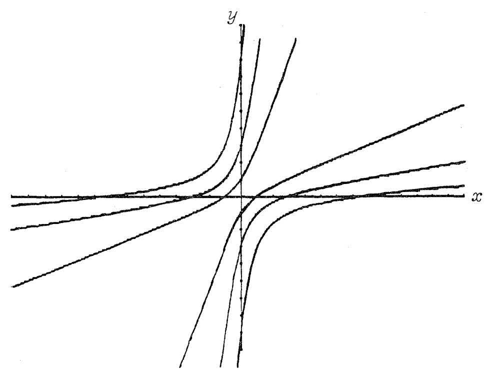

FIGURE 4

The Fibonacci polynomials $q_2(x, y)$, $q_4(x, y)$, and $q_6(x, y)$

3. The Main Theorem

In this section we shall state and prove the main theorem of this paper, which expresses every Fibonacci hyperbola in terms of the polynomials $p_n(x, y)$ and $q_n(x, y)$.

Lemma 3.1: The coefficients a, b, c, d, e, f in the equation

(2) $$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

of a Fibonacci hyperbola can be chosen to be integers. (Following the proof of this lemma, these coefficients will be understood to be integers except where stated otherwise.)

Proof: Divide both sides of (2) by one of the nonzero coefficients, and then substitute for (x, y) any five distinct (F_m, F_n) that lie on the hyperbola. Cramer's Rule applied to the resulting five equations shows that each coefficient is a rational number. Let D be the least common multiple of the five denominators. Write (2) using the five rational numbers and 1 as coefficients, and then multiply both sides by D. The resulting coefficients are integers.

Lemma 3.2: Suppose (2) is a hyperbola that passes through the points (F_{s_n}, F_{t_n}) for some pair $s_1, s_2, s_3, \ldots$ and $t_1, t_2, t_3, \ldots$ of nondecreasing sequences of integers. Then there exist constants m and N such that $t_n - s_n = m$ for all $n > N$.

Proof: The proof will be in three cases.

Case 1. Suppose $c = 0$. Then $b \neq 0$, else (2) would not represent a hyperbola. Divide both sides of (2) by x^2 to find

$$-a/b = \lim_{x\to\infty} y/x = \lim_{n\to\infty} F_{t_n}/F_{s_n}$$
$$= \lim_{n\to\infty}(\alpha^{t_n} - \beta^{t_n})/(\alpha^{s_n} - \beta^{s_n}) = \lim_{n\to\infty} \alpha^{t_n - s_n},$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.

If $a = 0$, then $\lim\limits_{x\to\infty} u = -d/b$, so that $\lim\limits_{n\to\infty} F_{t_n} = -d/b$, which is impossible, and so $a \neq 0$. Consequently $\lim\limits_{n\to\infty} \alpha^{t_n - s_n}$ is a nonzero constant. The exponent $t_n - s_n$ is an integer for all n, so that $t_n - s_n$ is a constant for all sufficiently large n.

Case 2. If $c \neq 0$ and $a = 0$, then $b \neq 0$, else (2) would not represent a hyperbola. Divide both sides of (2) by y^2 to find

$$-c/b = \lim_{y\to\infty} x/y = \lim_{n\to\infty} F_{s_n}/F_{t_n} = \lim_{n\to\infty} \alpha^{s_n - t_n},$$

so that $s_n - t_n$, and hence $t_n - s_n$ is a constant for all sufficiently large n.

Case 3. If $c \neq 0$ and $a \neq 0$, then divide both sides of (2) by x^2 and solve the resulting equation for y/x to obtain the slopes of the asymptotes:

$$(-b \pm \sqrt{b^2 - 4ac})/2c = \lim_{x\to\infty} y/x = \lim_{n\to\infty} F_{t_n}/F_{s_n} = \lim_{n\to\infty} \alpha^{t_n - s_n},$$

so that $t_n - s_n$ must be a constant for all sufficiently large n.

Theorem 3: For $n = 1, 2, \ldots$, let

$$p_n(x, y) = x^2 + (-1)^{n+1}L_n xy + (-1)^n y^2 + F_n^2 = 0,$$

and let $q_n(x, y) = p_n(x, t) - 2F_n^2$. Every Fibonacci polynomial is one of the following forms:

$$p_n(x, y) = 0, \quad q_n(x, y) = 0, \quad p_n(-x, y) = 0, \quad \text{or} \quad q_n(-x, y) = 0.$$

Proof: Suppose

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{2}$$

is a hyperbola that passes through the points (F_{s_n}, F_{t_n}) for some pair $s_1, s_2, s_3, \ldots$ and $t_1, t_2, t_3, \ldots$ of nondecreasing sequences of integers. We refer to the three cases of Lemma 3.2 and show first that Case 1 and Case 2 cannot occur. Let m be as in Lemma 3.2; note that m can be negative.

In Case 1, if $m = 0$, then infinitely many (F_{s_n}, F_{s_n}) lie on the conic section (2), and so (2) represents the line $y = x$, not a hyperbola. If $m = 0$, then the equation

$$-a/b = \alpha^m = [(1 + \sqrt{5})/2]^m = (L_m + \sqrt{5}F_m)/2$$

shows that a/b is not a rational number, contrary to Lemma 3.1. We conclude that $c \neq 0$.

In Case 2, $-c/b = \alpha^{t_n - s_n}$, an irrational constant for all sufficiently large n, contrary to Lemma 3.1. Consequently, $c \neq 0$ and $a \neq 0$, which is Case 3.

In Case 3, $(-b \pm \sqrt{b^2 - 4ac})/2c = \alpha^m = (L_m + \sqrt{5}F_m)/2$. Separating rational and irrational parts yields

(3) $\quad -b/c = L_m \quad$ and $\quad \pm(\sqrt{b^2 - 4ac})/c = \sqrt{5}F_m.$

Substitute $-cL_m$ for b into the second equation and obtain

$$c = 4a/(L_m^2 - 5F_m^2) = (-1)^m a.$$

We may and do assume that $a = 1$ (allowing d, e, f to be rational numbers), so that (2) takes the form

(4) $\quad x^2 - (-1)^m L_m xy + (-1)^m y^2 + dx + ey + f = 0, \; m = \pm 1, \pm 2, \pm 3, \ldots .$

Now, substitute (F_{s_n}, F_{s_n+m}) for (x, y) into (4):

$$F_{s_n}^2 - (-1)^m L_m F_{s_n} F_{s_n+m} + (-1)^m F_{s_n+m}^2 + dF_{s_n} + eF_{s_n+m} + f = 0.$$

Using identities I_{21} (if m is even) and I_{23} (if m is odd) from [3] gives

$$(-1)^m [F_{s_n+m}^2 - F_{s_n} F_{s_n+2m}] + dF_{s_n} + eF_{s_n+m} + f = 0.$$

Identity I_{19} from [3] then gives

$$(-1)^{s_n+m} F_m^2 + dF_{s_n} + eF_{s_n+m} + f = 0.$$

Let n_1, n_2, n_3 be any three integers, exceeding N, for which the three integers s_{n_1}, s_{n_2}, s_{n_3} are either all odd or all even. Then the system

(5) $\quad dF_{s_{n_i}} + eF_{s_{n_i}+m} + f = (-1)^{s_{n_i}+m+1} F_m^2$, for $i = 1, 2, 3,$

has the unique solution

$$d = 0, \; e = 0, \; f = (-1)^{s_{n_i}+m+1} F_m^2.$$

Clearly, the s_{n_i}, for $i = 1, 2, 3, \ldots,$ must all be odd or must all be even, else the infinite system (5) has no solution.

Case 1. Suppose the s_{n_i} are all odd. Rewrite (4) as

(6) $\quad x^2 - (-1)^m L_m xy + (-1)^m y^2 + (-1)^m F_m = 0, \; m = \pm 1, \pm 2, \pm 3, \ldots .$

If $m < 0$, then $L_m = (-1)^m L_{-m}$ and $F_m = (-1)^{m+1} F_{-m}$, so that

$$x^2 - L_{-m} xy + (-1)^m y^2 + (-1)^m F_{-m}^2 = 0.$$

Substitute n for $-m$ to obtain

(7) $\quad x^2 - L_n xy + (-1)^n y^2 + (-1)^n F_n^2 = 0.$

If n is even, (7) is $p_n(x, y) = 0$; if n is odd, (7) is $q_n(-x, y) = 0$.

If $n = m > 0$ and is even, then (6) is $p_n(x, y) = 0$. If $n = m > 0$ is odd, then (6) is $q_n(x, y) = 0$.

Case 2. Suppose the s_{n_i} are all even. Rewrite (4) as

(6′) $\quad x^2 - (-1)^m L_m xy + (-1)^m y^2 - (-1)^m F_m^2 = 0, \; m = \pm 1, \pm 2, \pm 3, \ldots .$

If $m < 0$, then write $n = -m$, so that

(7′) $\quad x^2 - L_n xy + (-1)^n y^2 - (-1)^n F_n^2 = 0.$

If n is even, (7′) is $q_n(x, y) = 0$; if n is odd, (7′) is $q\ (-x, y) = 0$.

If $n = m > 0$ and is even, then (6′) is $q_n(x, y) = 0$. If $n = m > 0$ is odd, then (6′) is $p_n(x, y) = 0$.

4. Concluding Remarks

Theorem 3 establishes the following representation for all Fibonacci hyperbolas:

$$y^2 + bxy + (-1)^n x^2 + f = 0, \quad \text{where } |b| = L_n \text{ and } |f| = F_n^2.$$

Each of these hyperbolas consists of two branches:

$$y = (-bx + \sqrt{[b^2 - 4(-1)^n]x^2 - 4f})/2$$

and

$$y = (-bx - \sqrt{[b^2 - 4(-1)^n]x^2 - 4f})/2.$$

The representative points listed in Tables 1 and 2 lie on the *upper* branch of their respective hyperbolas. Does the *lower* branch also pass through points that are closely associated with Fibonacci numbers? The affirmative answer to this question follows from Bergum [1, pp. 27-28].

References

1. Gerald E. Bergum. "Addenda to Geometry of a Generalized Simson's Formula." *Fibonacci Quarterly 22.1* (1984):22-28.
2. A. F. Horadam. "Geometry of a Generalized Simson's Formula." *Fibonacci Quarterly 20.2* (1982):164-168.
3. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.

*****

ON CIRCULAR FIBONACCI BINARY SEQUENCES

Derek K. Chang
California State University, Los Angeles, CA 90032
(Submitted January 1988)

The number of combinations of n elements taken k at a time is given by the binomial coefficient $\binom{n}{k}$. If the n elements are arranged in a circle, any two circular combinations are said to be indistinguishable if one can be obtained by a cyclic rotation of the other. Let $C(n, k)$ denote the number of distinguishable circular combinations of n elements taken k at a time. Using a formula for $C(n, k)$, we consider a problem on circular Fibonacci binary sequences.

We recall that a Fibonacci binary sequence is a $\{0, 1\}$-sequence with no two 1's adjacent. Similarly, a circular Fibonacci sequence is a circular $\{0, 1\}$-sequence with no two 1's adjacent. Let $H(n)$ denote the number of distinguishable circular Fibonacci binary sequences of length n, and let $W(n)$ denote the total number of 1's in all such sequences. The ratio $Q(n) = W(n)/nH(n)$ gives the proportion of 1's in all the distinguishable circular Fibonacci binary sequences of length n. In the case of ordinary Fibonacci binary sequences, this ratio tends to the limit $(5 - \sqrt{5})/10$ as $n \to \infty$ [2]. In the case of circular Fibonacci binary sequences, a similar result can be proved.

For any integer

$$m = p_1^{r_1} p_2^{r_2} \ldots p_j^{r_j} \geq 2,$$

where p_i's are distinct prime numbers and $r_i \geq 1$, let $\phi(m)$ be defined by

$$\phi(m) = m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_j}\right);$$

for $m = 1$, let $\phi(m) = 1$. Thus, ϕ is the Euler totient function. The number $C(n, k)$ of all distinguishable circular combinations of n elements taken k at a time is given by the following formula.

$$C(n, k) = \frac{1}{n} \sum_{1 \leq m | (n, k)} \phi(m)\binom{n/m}{k/m}.$$

(See [1], p. 208.)

Now let $g(n, k)$ denote the number of distinguishable circular Fibonacci binary sequences of length n which contain a total of k 1's. Since each 1 must be followed by a 0 in the sequence,

$$g(n, k) = C(n - k, k).$$

If n is a prime number, the ratio

$$Q(n) = \frac{W(n)}{nH(n)} = \frac{1}{n} \frac{C(n - 1, 1) + 2C(n - 2, 2) + 3C(n - 3, 3) + \cdots}{1 + C(n - 1, 1) + C(n - 2, 2) + C(n - 3, 3) + \cdots}$$

$$= \frac{1 + \binom{n-3}{1} + \binom{n-4}{2} + \cdots}{n\left[1 + 1 + \binom{n-3}{1}/2 + \binom{n-4}{2}/3 + \cdots\right]}$$

Using the following formula (see [3], p. 76),

$$\sum_{k \geq 0} \binom{n-k}{k} x^k = \frac{1}{2^{n+1}s}[(1 + s)^{n+1} - (1 - s)^{n+1}],$$

where $s = \sqrt{1 + 4x}$, one has

$$W(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^{n-1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n-1}\right],$$

$$nH(n) = n - 1 + \left[\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots\right]$$

$$+ \left[\binom{n-2}{0} + \binom{n-3}{1} + \binom{n-4}{2} + \cdots\right]$$

$$= n - 1 + \left(\frac{1 + \sqrt{5}}{2}\right)^{n} + \left(\frac{1 - \sqrt{5}}{2}\right)^{n}.$$

Thus, the limit through prime numbers is

$$\lim_{\substack{n \to \infty \\ n \text{ is prime}}} Q(n) = (5 - \sqrt{5})/10.$$

In general, for any positive integer $n = p_1^{r_1} p_2^{r_2} \ldots p_j^{r_j}$, one has

$$nH(n) = n\left[1 + 1 + \binom{n-3}{1}/2 + \binom{n-4}{2}/3 + \cdots\right]$$

$$+ \sum_{i=1}^{j} \frac{n}{p_i} \phi(p_i) \sum_{r \geq 1} \frac{1}{r}\left[\binom{n/p_i - r - 1}{r - 1} + \binom{n/p_i^2 - r - 1}{r - 1}\right.$$

$$\left. + \cdots + \binom{n/p_i^{r_i} - r - 1}{r - 1}\right]$$

$$+ \sum_{\substack{i, m = 1 \\ i \neq m}}^{j} \frac{n}{p_i p_m} \phi(p_i p_m) \sum_{r \geq 1} \frac{1}{r}\left[\binom{n/p_i p_m - r - 1}{r - 1}\right.$$

$$\left. + \binom{n/p_i^2 p_m - r - 1}{r - 1} + \cdots + \binom{n/p_i^{r_i} p_m^{r_m} - r - 1}{r - 1}\right] + \cdots$$

where the successive terms enumerate sequences having patterns of increasing multiplicity.

Let $y = (1 + \sqrt{5})/2$, $z = (1 - \sqrt{5})/2$. Then

$$nH(n) = (y^n + z^n + n - 1) + \sum_{i=1}^{j} \phi(p_i) \frac{n}{p_i}\left[\frac{p_i}{n}(y^{n/p_i} + z^{n/p_i} - 1)\right.$$

$$\left. + \frac{p_i^2}{n}(y^{n/p_i^2} + z^{n/p_i^2} - 1) + \cdots + \frac{p_i^{r_i}}{n}(y^{n/p_i^{r_i}} + z^{n/p_i^{r_i}} - 1)\right]$$

$$+ \sum_{\substack{i, m = 1 \\ i \neq m}}^{j} \phi(p_i p_m) \frac{n}{p_i p_m}\left[\frac{p_i p_m}{n}(y^{n/p_i p_m} + z^{n/p_i p_m} - 1)\right.$$

$$+ \frac{p_i^2 p_m}{n}(y^{n/p_i^2 p_m} + z^{n/p_i^2 p_m} - 1)$$

$$\left. + \cdots + \frac{p_i^{r_i} p_m^{r_m}}{n}(y^{n/p_i^{r_i} p_m^{r_m}} + z^{n/p_i^{r_i} p_m^{r_m}} - 1)\right] + \cdots$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III} + \cdots$$

Since $\phi(r)/r \le 1$ for any positive integer r, and $|z| < 1$, we have:

$$\text{II} \le \sum_{i=1}^{j} y^{n/p_i}(p_i + p_i^2 + \cdots + p_i^{r_i}) < \sum_{i=1}^{j} y^{n/2} 2p_i^{r_i}$$

$$\le \sum_{i=1}^{j} y^{n/2} 2n = \binom{j}{1} 2ny^{n/2};$$

$$\text{III} \le \sum_{\substack{i,m=1 \\ i \ne m}}^{j} y^{n/p_i p_m}(p_i + p_i^2 + \cdots + p_i^{r_i})(p_m + p_m^2 + \cdots + p_m^{r_m})$$

$$< \sum_{\substack{i,m=1 \\ i \ne m}}^{j} y^{n/2} 2p_i^{r_i} 2p_m^{r_m} \le \sum_{\substack{i,m=1 \\ i \ne m}}^{j} y^{n/2} 4n = \binom{j}{2} 4ny^{n/2}.$$

But for large n,

$$\sum_{i=0}^{j} \binom{j}{i} 2^i ny^{n/2} \le \sum_{i=0}^{j} \binom{j}{i} n^2 y^{n/2} = 2^j n^2 y^{n/2} \le n^3 y^{n/2} = o(y^n).$$

So

$$nH(n) = y^n + o(y^n).$$

Similarly,

$$W(n) = \frac{1}{\sqrt{5}}(y^{n-1} - z^{n-1}) + \frac{1}{\sqrt{5}} \sum_{i=1}^{j} \phi(p_i)[(y^{n/p_i - 1} - z^{n/p_i - 1})$$

$$+ (y^{n/p_i^2 - 1} - z^{n/p_i^2 - 1}) + \cdots + (y^{n/p_i^{r_i} - 1} - z^{n/p_i^{r_i} - 1})]$$

$$+ \frac{1}{\sqrt{5}} \sum_{\substack{i,m=1 \\ i \ne m}}^{j} \phi(p_i p_m)[(y^{n/p_i p_m - 1} - z^{n/p_i p_m - 1})$$

$$+ \cdots + (y^{n/p_i^{r_i} p_m^{r_m} - 1} - z^{n/p_i^{r_i} p_m^{r_m} - 1})] + \cdots = \frac{y^{n-1}}{\sqrt{5}} + o(y^n).$$

Thus, we have the following result on the asymptotic proportions of 1's in circular Fibonacci binary sequences.

$$\lim_{n \to \infty} Q(n) = (5 - \sqrt{5})/10.$$

The author wishes to thank the referee for his comments and suggestions.

References

1. M. Eisen. *Elementary Combinatorial Analysis*. New York: Gordon and Breach, 1969.
2. P. H. St. John. "On the Asymptotic Proportions of Zeros and Ones in Fibonacci Sequences." *Fibonacci Quarterly* *22.2* (1984):144-145.
3. J. Riordan. *Combinatorial Identities*. New York: Wiley, 1968.

*****

PYTHAGOREAN NUMBERS

Supriya Mohanty and S. P. Mohanty
Bowling Green State University, Bowling Green, OH 43403-0221
(Submitted February 1988)

Let M be a right angled triangle with legs x and y and hypotenuse z. Then x, y, and z satisfy $x^2 + y^2 = z^2$, and conversely. If x, y, and z are natural numbers, then M is called a *Pythagorean triangle* and (x, y, z) a *Pythagorean triple*. If the natural numbers x, y, and z further satisfy $(x, y) = 1$ or $(y, z) = 1$ or $(z, x) = 1$ (if one of these three holds, then all three hold), then M is called a *primitive Pythagorean triangle* and (x, y, z) a *primitive Pythagorean triple*. It is well known [4] that all primitive Pythagorean triangles or triples (x, y, z) are given, without duplication, by:

(1) $x = 2uv,\ y = u^2 - v^2,\ z = u^2 + v^2$ or

$x = u^2 - v^2,\ y = 2uv,\ z = u^2 + v^2,$

where u and v are relatively prime natural numbers of opposite parity and satisfy $u > v$. Conversely, if u and v $(u > v)$ are relatively prime natural numbers of opposite parity, then they generate a Pythagorean triangle according to (1). Every primitive Pythagorean triangle (x, y, z) generates an infinite number of primitive Pythagorean triangles, namely (tx, ty, tz) where t is a natural number. Conversely, if (x, y, z) is a Pythagorean triangle, then $(x/t, y/t, z/t)$ is a primitive Pythagorean triangle provided $(x, y) = t$.

We see that the area of a primitive Pythagorean triangle

$$(2uv,\ u^2 - v^2,\ u^2 + v^2),$$

where $u > v$, $(u, v) = 1$, and u and v are of opposite parity is

$$uv(u^2 - v^2).$$

Conversely, a natural number n of the form $uv(u^2 - v^2)$ with $u > v$, $(u, v) = 1$, and u and v of opposite parity is the area of the primitive Pythagorean triangle $(2uv, u^2 - v^2, u^2 + v^2)$.

Definition 1: The area of a Pythagorean triangle is called a *Pythagorean number* and that of a primitive Pythagorean triangle a *primitive Pythagorean number*.

From the discussion above, it is clear that if n is a Pythagorean number then t^2n is also a Pythagorean number for every natural number t. But, if t^2n is a Pythagorean number, it does not imply that n is a Pythagorean number. For example, $84 = 2^2 \cdot 21$ is a Pythagorean number but we shall see shortly that 21 is not.

The following is a list of Pythagorean numbers below 10,000. There are 150 in all, out of which 43 are primitive. The primitive ones are underlined.

6, 24, 30, 54, 60, 84, 96, 120, 150, 180, 210, 216, 240, 270, 294, 330, 336, 384, 480, 486, 504, 540, 546, 600, 630, 720, 726, 750, 756, 840, 864, 924, 960, 990, 1014, 1080, 1176, 1224, 1320, 1344, 1350, 1386, 1470, 1500, 1536, 1560, 1620, 1710, 1716, 1734, 1890, 1920, 1944, 2016, 2100, 2160, 2166, 2184, 2310, 2340, 2400, 2430, 2520, 2574, 2646, 2730, 2880, 2904, 2940, 2970, 3000, 3024, 3036, 3174, 3360, 3456, 3570, 3630, 3696, 3750, 3840, 3900, 3960, 4056, 4080, 4116, 4290, 4320, 4374, 4500, 4536, 4620, 4704, 4860, 4896, 4914, 5016, 5046, 5070, 5250, 5280, 5376, 5400, 5544, 5610, 5670, 5766, 5814, 5880, 6000, 6090, 6144, 6240, 6480, 6534, 6630, 6750, 6804, 6840, 6864, 6936, 7140, 7260, 7350, 7440, 7560, 7680, 7776, 7854, 7956, 7980, 8064, 8214, 8250, 8316, 8400, 8640, 8664, 8670, 8736, 8820, 8910, 8970, 8976, 9126, 9240, 9360, 9600, 9690, 9720

If $P.P_i$ and P_i stand, respectively for the number of primitive Pythagorean numbers and number of Pythagorean numbers in the i^{th} thousand, then we have:

$$(P.P_i,\ P_i) = (13, 34), (6, 19), (34, 17), (3, 13), (4, 13), (3, 13), (2, 12), (5, 10), (2, 13), \text{ and } (1, 6)$$

for $i = 1, 2, \ldots, 10$ in order.

This shows that the distribution of Pythagorean numbers is very irregular.

From the above table, we see that

(i) every Pythagorean number is divisible by 6.

(ii) the unit's place of a Pythagorean number is 0, 4, or 6.

(iii) out of the first 150 Pythagorean numbers there are 86 with 0, 31 with 4, and 33 with 6 in their unit's places. Thus, there are more Pythagorean numbers with 0 in their unit's places than with 4 or 6. Pythagorean numbers with 4 or 6 in their unit's places occur almost the same number of times when we consider all Pythagorean numbers up to a given integer.

We shall see that (i), (ii), and (iii) are facts not accidents.

We can construct as many primitive Pythagorean or Pythagorean numbers as we like. But given a Pythagorean number, we cannot tell or construct the next Pythagorean number. We shall give some necessary and sufficient conditions for an integer n to be Pythagorean or primitive Pythagorean, but they are not very useful for practical purposes when n is very large.

Theorem 1: A natural number n is Pythagorean if and only if it has at least four different positive factors a, b, c, and d such that

$$ab = cd = n \quad \text{and} \quad a + b = c - d.$$

Proof: Let n be a Pythagorean number. Then

$$n = m^2uv(u^2 - v^2)$$

where u and v $(u > v)$ are of different parity with $(u, v) = 1$. Clearly, n has four different factors,

$$a = mv(u + v),\ b = mu(u - v),\ c = mu(u + v), \text{ and } d = mv(u - v),$$

and they satisfy $ab = cd = n$ and $a + b = m(u^2 + v^2) = c - d$. Conversely, let n be a natural number with four different positive factors a, b, c, and d such that $ab = cd = n$ and $a + b = c - d$. From $ab = cd$ and $a + b = c - d$, we eliminate d and get

$$c^2 - c(a + b) - ab = 0.$$

Since the discriminant $(a + b)^2 + 4ab > (a + b)^2$ and c is a positive integer, we take

$$c = \frac{1}{2}\{a + b + \sqrt{(a + b)^2 + 4ab}\}.$$

For c to be an integer, we must have

$$(a + b)^2 + 4ab = t^2$$

where t is a positive integer. The necessary condition is also sufficient. Now

$$(a + b)^2 + 4ab = t^2 \quad \text{or} \quad 2(a + b)^2 = t^2 + (a - b)^2$$

can be rewritten as

$$4(a + b)^2 = (t + a - b)^2 + (t - a + b)^2.$$

Clearly, $t + a - b$ and $t - a + b$ are both even integers. Therefore,

$$(a + b)^2 = \left(\frac{t + a - b}{2}\right)^2 + \left(\frac{t - a + b}{2}\right)^2.$$

If

$$\left(\frac{t + a - b}{2}, \frac{t - a + b}{2}\right) = m$$

then m divides $a + b$. Hence,

$$\left(\frac{a + b}{m}\right)^2 = \left(\frac{t + a - b}{2m}\right)^2 + \left(\frac{t - a + b}{2m}\right)^2.$$

Now

$$\left(\frac{t + a - b}{2m}, \frac{t - a + b}{2m}, \frac{a + b}{m}\right)$$

is a primitive Pythagorean triple. Taking

$$\frac{t + a - b}{2m} = 2uv, \quad \frac{t - a + b}{2m} = u^2 - v^2, \quad \text{and} \quad \frac{a + b}{m} = u^2 + v^2,$$

where $u > v$, $(u, v) = 1$, and u and v are of opposite parity, we get

$$a = m(v^2 + uv),\ b = m(u^2 - uv),$$
$$c = m(u^2 + uv),\ d = m(uv - v^2).$$

If we take

$$\frac{t + a - b}{2m} = u^2 - v^2, \quad \frac{t - a + b}{2m} = 2uv, \quad \text{and} \quad \frac{a + b}{m} = u^2 + v^2,$$

then

$$a = m(u^2 - uv),\ b = m(v^2 + uv),$$
$$c = m(u^2 + uv),\ d = m(uv - v^2),$$

then

$$n = ab = m^2uv(u^2 - v^2),$$

which is the area of the Pythagorean triangle

$$(2muv,\ m(u^2 - v^2),\ m(u^2 + v^2)).$$

Hence, n is a Pythagorean number. We note that

$$a + b = c - d = m(u^2 + v^2)$$

is the hypotenuse of the Pythagorean triangle with area n.

Bert Miller [6] defines a nasty number n as a positive integer n with at least four different factors a, b, c, and d such that

$$a + b = c - d \quad \text{and} \quad ab = cd = n.$$

By Theorem 1, n is nasty if and only if it is Pythagorean. "Pythagorean number" is a better name for "nasty number."

Theorem 2: If four positive integers r, s, t, and are such that r, s, and t are in arithmetic progression with m as their common difference, then $n = rstm$ is a Pythagorean number. If s and m are relatively prime and of opposite parity, then n is a primitive Pythagorean number.

Proof: As r, s, and t are in arithmetic progression with m as their common difference,

$$n = rstm = r(r + m)(r + 2m)m.$$

Taking

$$a = r(r + m),\ b = (r + 2m)m,\ c = (r + m)(r + 2m),\ d = rm,$$

we have four different positive integers a, b, c, and d such that

$$ab = cd = n \quad \text{and} \quad a + b = r^2 + 2rm + 2m^2 = c - d.$$

Therefore, by Theorem 1, $n = rstm$ is a Pythagorean number. If s and m, i.e., $r + m$ and m are relatively prime and of different parity, we take $r + m = u$, $m = v$ and get

$$n = uv(u^2 - v^2)$$

where $(u, v) = 1$, $u > v$, and u and v are of different parity. Hence, n is a primitive Pythagorean number.

Corollary 2.1: The product of three consecutive integers n, $(n + 1)$, $(n + 2)$ is a Pythagorean number. It is primitive only if n is odd.

Proof: Since $n(n + 1)(n + 2) = n(n + 1)(n + 2) \cdot 1$ is the product of three integers n, $n + 1$, $n + 2$ that are in arithmetic progression with common difference 1, $n(n + 1)(n + 2)$ is a Pythagorean number. The triangle is

$$(2n + 2,\ n^2 + 2n,\ n^2 + 2n + 2).$$

The numbers $n + 1$ and 1 are always relatively prime. They will be of different parity if and only if n is odd. Hence, $n(n + 1)(n + 2)$ is a primitive Pythagorean number if and only if n is odd.

Corollary 2.2: The number $6\sum_{k=1}^{n} k^2$ is a primitive Pythagorean number.

Proof: The number

$$6\sum_{k=1}^{n} k \ = n(n + 1)(2n + 1).$$

Since $(n + 1)$ and n are relatively prime and of opposite parity

$$1 \cdot (n + 1)(2n + 1) \cdot n$$

is a primitive Pythagorean number, by Theorem 2.

Corollary 2.3: $F_{2n}F_{2n+2}F_{2n+4}$ is a Pythagorean number where F_n is the n^{th} Fibonacci number. It is primitive if and only if F_{2n+2} is even.

Proof: The Fibonacci numbers are defined by

$$F_1 = 1,\ F_2 = 1,\ F_{n+1} = F_n + F_{n-1},\ n \geq 2.$$

It is well known that

$$F_{2n}F_{2n+4} = (F_{2n+2})^2 - 1 = (F_{2n+2} + 1)(F_{2n+2} - 1).$$

Therefore,

$$\begin{aligned} F_{2n}F_{2n+2}F_{2n+4} &= (F_{2n+2} - 1)F_{2n+2}(F_{2n+2} + 1) \\ &= \text{product of three consecutive integers.} \end{aligned}$$

Hence, by Corollary 2.1, it is a Pythagorean number. It is primitive if and only if $F_{2n+2} - 1$ is odd, i.e., F_{2n+2} is even.

Corollary 2.4: The product of three consecutive Fibonacci numbers F_{2n}, F_{2n+1}, and F_{2n+2} is a Pythagorean number. It is primitive if and only if F_{2n+1} is even.

Proof: Use $F_{2n+2} \cdot F_{2n} = (F_{2n+1})^2 - 1$.

Corollary 2.5: The product of four consecutive Fibonacci numbers F_n, F_{n+1}, F_{n+2}, and F_{n+3} is a Pythagorean number. It is primitive if and only if F_{n+1} and F_{n+2} are of different parity.

Proof: We have

$$F_nF_{n+1}F_{n+2}F_{n+3} = (F_{n+2} - F_{n-1})F_{n+2}(F_{n+2} + F_{n+1})F_{n+1}.$$

Since

$$F_{n+2} - F_{n+1},\ F_{n+2},\ \text{and}\ F_{n+2} + F_{n+1}$$

are in arithmetic progression with common difference F_{n+1}, by Theorem 2

$$F_nF_{n+1}F_{n+2}F_{n+3}$$

is a Pythagorean number. Since

$$(F_{n+1},\ F_{n+2}) = 1,$$

$F_nF_{n+1}F_{n+2}F_{n+3}$ is primitive if and only if F_{n+1} and F_{n+2} are of different parity.

Corollary 2.6: The product of four consecutive Lucas numbers L_n, L_{n+1}, L_{n+2}, L_{n+3} is a Pythagorean number. It is primitive if and only if L_{n+1}, L_{n+2} are of opposite parity.

Proof: The Lucas sequence is defined by

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1},\ n \geq 1.$$

Since

$$L_nL_{n+1}L_{n+2}L_{n+3} = (L_{n+2} - L_{n+1})L_{n+2}(L_{n+2} + L_{n+1}) \cdot L_{n+1},$$

it is a Pythagorean number, by Theorem 2. Since

$$(L_{n+1}, L_{n+2}) = 1,$$

it is primitive if and only if L_{n+1} and L_{n+2} are of different parity.

We have already seen that there are infinitely many Pythagorean and primitive Pythagorean numbers which are products of three consecutive integers. Since

$$x(x + 1)(x + 2)(x + 3)$$

is always Pythagorean if either x or $x + 3$ is a square, we have an infinite number of Pythagorean numbers which are products of four consecutive integers. Now a natural question is:

Do we have infinitely many Pythagorean and primitive Pythagorean numbers which are products of two consecutive integers?

The following theorems give affirmative answers to our question.

Theorem 3: There are infinitely many Pythagorean numbers which are products of two consecutive integers.

Proof: Let

$$n = a^2(a^2 - 1)a^2(a^2 + 1), \quad a > 1.$$

Since $(a^2 - 1)a^2(a^2 + 1)$ is a product of three consecutive integers, it is a Pythagorean number, by Corollary 2.1. The product of a Pythagorean number and a square number is always Pythagorean. Thus,

$$n = a^2(a^2 - 1)a^2(a^2 + 1)$$

is Pythagorean. Since $n = a^4(a^4 - 1)$, it is a product of two consecutive integers.

Again, let

$$n = a^2\left(\frac{a^2 - 3}{2}\right)\left(\frac{a^2 - 1}{2}\right)(a^2 - 2)$$

where a is an odd natural number > 1. Since 1, $(a^2 - 1)/2$, $a^2 - 2$ form an arithmetic progression with common difference $(a^2 - 3)/2$ and a is odd,

$$\left(\frac{a^2 - 3}{2}\right)\left(\frac{a^2 - 1}{2}\right)(a^2 - 2)$$

is Pythagorean, whence,

$$n = a^2\left(\frac{a^2 - 3}{2}\right)\left(\frac{a^2 - 1}{2}\right)(a^2 - 2)$$

is Pythagorean. But

$$n = \left(\frac{a^4 - 3a^2}{2}\right)\left(\frac{a^4 - 3a^2}{2} + 1\right)$$

is a product of two consecutive integers.

Theorem 4: There are infinitely many primitive Pythagorean numbers which are products of two consecutive integers.

Proof: Consider the product number $F_nF_{n+1}F_{n+2}F_{n+3}$ where F_n is the n^{th} Fibonacci number and F_{n+1}, F_{n+2} are of opposite parity. By Corollary 2.5, $F_nF_{n+1}F_{n+2}F_{n+3}$ is a primitive Pythagorean number. Since $F_nF_{n+3} = F_{n+1}F_{n+2} + (-1)^n$,

$$F_n F_{n+1} F_{n+2} F_{n+3} = F_{n+1} F_{n+2} (F_{n+1} F_{n+2} + (-1)^n)$$

is a product of two consecutive integers.

Although there are infinitely many Pythagorean numbers which are products of (a) three consecutive integers, (b) two consecutive integers, there are only two Pythagorean numbers 6 and 210 which are simultaneously products of two as well as three consecutive integers [7].

Theorem 5: Every Pythagorean number is divisible by 6.

Proof: Every Pythagorean number n is of the form $m^2uv(u^2 - v^2)$ where $u > v$, $(u, v) = 1$ and u and v are of opposite parity. Since u and v are of opposite parity, n is already divisible by 2. We show that $n \equiv 0 \pmod 3$. Since, by Fermat's little theorem $u^3 \equiv u \pmod 3$ and $v^3 \equiv v \pmod 3$,

$$n = m^2uv(u^2 - v^2) = m^2(u^3v - uv^3) \equiv m^2(uv - uv) \equiv 0 \pmod 3.$$

Corollary 5.1: No Pythagorean number except 6 is perfect.

Proof: By Theorem 5 every Pythagorean number n is divisible by 6. So

$$n \equiv 0,\ 3,\text{ or } 6 \pmod 9.$$

As every Pythagorean number is even, no odd perfect number (the existence or nonexistence of which is an open problem) can be a Pythagorean number. The number $2^{n-1}(2^n - 1)$ when n and $2^n - 1$ are primes is an even perfect number and every even perfect number is of this form [4]. It is an easy exercise to see that every even perfect number except 6 is congruent to 1 (mod 9). Therefore, no even perfect number > 6 can be Pythagorean. Thus, 6 is the only number that is both Pythagorean and perfect.

By Bertrand's postulate [4] there is a prime number between n and $2n$ for every integer $n > 1$. The following theorem shows that we can have a similar result for Pythagorean numbers.

Theorem 6: For every integer $n > 12$ there is a Pythagorean number between n and $2n$.

Proof: The number 24 does the job for $13 \le n \le 23$, 30 does the job for $24 \le n \le 29$, and 54 does the job for $30 \le n \le 53$. We see that

$$6(t + 1)^2 < 12t^2 \text{ for } t \ge 3.$$

Therefore, the Pythagorean number $6(t + 1)^2$ lies between $6t^2$ and $12t^2$. Thus, $6(t + 1)^2$ does the job for

$$6t^2 \le n \le 6(t + 1)^2 - 1 \text{ for } t \ge 3.$$

Since $6t^2$ is Pythagorean for every positive integer t, there is a Pythagorean number between n and $2n$ for every $n > 12$.

We know that if n is Pythagorean then t^2n is Pythagorean for every natural number t. If n and tn are both Pythagorean, then it follows easily that $t^m n$ is Pythagorean for every positive integral exponent m. Thus, $5^m \cdot 6$, $2^m \cdot 30$, $7^m \cdot 30$ are Pythagorean for every positive integral exponent m. Hence, there are an infinite number of Pythagorean numbers of the form $10k$. If $t = 10s + 2$ or $10s + 3$, then $6t^2 \equiv 4 \pmod{10}$. Since $6t^2$ is Pythagorean for every positive integer t, we have an infinite number of Pythagorean numbers of the form

$10k + 4$. Similarly, for $t = 10s + 4$ and $t = 10s + 6$, we have $6t^2 \equiv 6 \pmod{10}$, whence there are an infinite number of Pythagorean numbers of the form $10k + 6$. Thus, we have

Theorem 7: There are infinitely many Pythagorean numbers of the form (i) $10k$, (ii) $10k + 4$, and (iii) $10k + 6$.

The next theorem shows that every Pythagorean number is of the form $10k$, $10k + 4$, or $10k + 6$.

Theorem 8: No Pythagorean number can have 2 or 8 in its unit's place.

Proof: As every Pythagorean number is divisible by 6, it can have 0, 2, 4, 6, or 8 in its unit's place. We shall show that it can have only 0, 4, or 6 in its unit's place. Every Pythagorean number is of the form $t^2uv(u^2 - v^2)$ where t, u, and v are natural numbers with $(u, v) = 1$, $u > v$, and u and v are of opposite parity. It is an easy exercise that number n is the area of the Pythagorean triangle

$$(2tuv,\ t(u^2 - v^2),\ t(u^2 + v^2)).$$

A Pythagorean triangle has one of its sides divisible by 5. If one of the legs or t is divisible by 5, then n is divisible by 10 and, hence, has 0 in its unit's place. Now suppose that neither t nor one of the legs is divisible by 5. Then $u \not\equiv 0 \pmod 5$, $v \not\equiv 0 \pmod 5$, and $u^2 - v^2 \not\equiv 0 \pmod 5$, but then $u^2 + v^2 \equiv 0 \pmod 5$. As $u^2 + v^2$ is odd, we have $u^2 + v^2 \equiv 5 \pmod{10}$. Now, considering modulo 10, we have

$$\begin{aligned}(u, v) \equiv\ & (1, 2), (1, 8), (2, 1), (2, 9), (3, 4), (3, 6), (4, 3),\\ & (4, 7), (6, 3), (7, 4), (7, 60, (8, 1), (8, 9), (9, 2),\\ & \text{and } (9, 8).\end{aligned}$$

For every (u, v) written above, $uv(u^2 - v^2) \equiv 4$ or $6 \pmod{10}$. If $t \not\equiv 0 \pmod 5$, then $t^2 \equiv 1, 4, 6, 9 \pmod{10}$ and $t^2uv(u^2 - v^2)$ can have only 4 or 6 in its unit's place. Thus, every Pythagorean number can have 0, 4, or 6 in its unit's place.

Corollary 8.1: No four Pythagorean numbers can form an arithmetic progression with common difference 6 or 24.

Proof: We shall prove the corollary for the common difference 6. The proof for the common difference 24 is analogous. We show that n, $n + 6$, $n + 12$, and $n + 18$ cannot be simultaneously Pythagorean. The number n being Pythagorean, it must have 0, 4, or 6 in its unit's place (Theorem 8). If n has 0 in its unit's place, then $n + 12$ will have 2 in its unit's place. So $n + 12$ cannot be Pythagorean. If n has 4 in its unit's place, then $n + 18$ cannot be Pythagorean. If n has 4 in its unit's place, then $n + 18$ cannot be Pythagorean by the same argument. If n has 6 in its unit's place, then $n + 6$ will have 2 in its unit's place. So $n + 6$ cannot be Pythagorean. Therefore, n, $n + 6$, $n + 12$, and $n + 18$ cannot be simultaneously Pythagorean.

Arguing as above, we have

Corollary 8.2: No three Pythagorean numbers can form an arithmetic progression with common difference 12 or 18.

It is clear that for any a.p. series of Pythagorean numbers with common difference d and of length L we have an a.p. series of Pythagorean numbers of length at least L with common difference dt^2, t an integer.

Conjecture 1: The numbers n, $n + 6$, and $n + 12$ cannot be simultaneously Pythagorean.

Conjecture 2: The numbers n, $n + 24$, and $n + 48$ are simultaneously Pythagorean if and only if $n = 6$.

We note that if Conjecture 2 is true then Conjecture 1 is true. Suppose n, $n + 6$, and $n + 12$ are simultaneously Pythagorean, then $4n$, $4n + 24$, and $4n + 48$ are Pythagorean. If Conjecture 2 is true, then $4n = 6$, which is nonsense. So Conjecture 1 is true. We see from our list of Pythagorean numbers that 120, 150, 180, 210, 240, 270 form an a.p. series with common difference 30. It has length 6. From this a.p. series, we can construct an a.p. series of length at least 6 with common difference $30t^2$, t a positive integer. For example, 480, 840, 960, 2080 is an a.p. series with common difference 120.

Problem 1: What can be the maximum length of an a.p. series all of whose terms are Pythagorean numbers?

If two Pythagorean numbers are 6 apart, then we call them *twin Pythagorean numbers* like twin primes. For example, twin Pythagorean numbers below 10,000 are:

(24, 30), (54, 60), (210, 216), (330, 336), (480, 486), (540, 546),
(720, 726), (750, 756), (1710, 1716), (2160, 2166), (8664, 8670),
(8970, 8976).

Although we do not know whether the number of twin primes is finite or infinite we do have a definite answer for the twin Pythagorean numbers.

Theorem 9: The number of twin Pythagoreans is infinite.

Proof: Since 6 and 30 are Pythagorean numbers, $6x^2$ and $30y^2$ are Pythagorean for all integral values of x and y. $6x^2$ and $30y^2$ are twin if

$$6x^2 - 30y^2 = \pm 6 \quad \text{or} \quad x^2 - 5y^2 = \pm 1.$$

The pellian equation $x^2 - 5y^2 = -1$ has fundamental solution

$$u_1 + v_1\sqrt{5} = 2 + \sqrt{5}.$$

All solutions of $x^2 - 5y^2 = -1$ are given by

$$(2 + \sqrt{5})^{2\quad 1} = u_{2\quad 1} + v_{2\quad 1}\sqrt{5}.$$

Again, all solutions of $x^2 - 5y^2 = 1$ are given by

$$(2 + \sqrt{5})^{2} = u_{2} + v_{2}\sqrt{5}.$$

We have the recurrence relation

$$u_{n+2} = 4u_{n+1} + u_n,\ v_{n+2} = 4v_{n+1} + v_n \text{ with}$$
$$u_1 = 1,\ v_1 = 0,\ u_2 = 2,\ v_2 = 1.$$

The first few solutions for $x^2 - 5y^2 = \pm 1$ are (1, 0), (2, 1), (9, 4), (38, 17), (161, 72) etc. They give us, respectively, twin Pythagorean numbers (6, 0), (24, 30), (486, 480), (8664, 8670), (155526, 155520).

Since we have an infinite number of solutions for each of the equations

$$x^2 - 5y^2 = -1 \quad \text{and} \quad x^2 - 5y^2 = 1,$$

we have an infinite number of twin Pythagorean numbers. For Pell's equation, one can refer to [8].

Since 6 and 60 are Pythagorean, $6x^2$ and $60y^2$ will be twin Pythagorean if

$$6x^2 - 60y^2 = \pm 6 \quad \text{or} \quad x^2 - 10y^2 = \pm 1.$$

All solutions of $x^2 - 10y^2 = \pm 1$ are given by

$$u_n + \sqrt{10}v_n = (3 + \sqrt{10})^n;$$

n is even for $x^2 - 10y^2 = 1$ and odd for $x^2 - 10y^2 = -1$. The solutions satisfy the recurrence relation

$$u_{n+2} = 6u_{n+1} + u_n \quad \text{and} \quad v_{n+2} = 6v_{n+1} + v_n \text{ with}$$
$$u_1 = 1,\ v_1 = 0,\ u_2 = 3,\ v_2 = 1.$$

The first solutions are: (1, 0), (3, 1), (19, 6), (117, 37). They give us, respectively, (6, 0), (54, 60), (2166, 2160), (82134, 82140). We again have an infinite number of Pythagorean twins from the solutions of the equations $6x^2 - 60y^2 = \pm 6$.

Definition 2: A Pythagorean number n is called independent if it cannot be obtained from another Pythagorean number m by multiplying it by t^2, where t is a natural number. For example, 6 is independent, while 24 is not.

It follows from Theorem 1 that for an integer to be an independent Pythagorean number, it is necessary that it should be primitive. The following example shows that the necessary condition is not sufficient, and hence C. K. Brown's statement [2] is incorrect.

Consider the number 840. It is primitive because it is the area of a primitive triangle (112, 15, 113). It is also four times the area of another primitive triangle (20, 21, 29). Hence, 840 is primitive but not independent.

Theorem 10: There are an infinite number of primitive Pythagorean numbers which are not independent.

Proof: Consider the number n given by

$$n = (18k^2 + 12k + 2)(6k^2 + 4k + 1)(24k^2 + 16k + 3)(12k^2 + 8k + 1),$$

where $k \geq 1$. Let

$$u = 18k^2 + 12k + 2 \quad \text{and} \quad v = 6k^2 + 4k + 1.$$

Now u is even, v is odd, and $(u, v) = 1$ with $u > v$. So $n = uv(u + v)(u - v)$ is the area of a primitive Pythagorean triangle, and hence n is a primitive Pythagorean number. Again n can be written as

$$\begin{aligned} n &= (3k + 1)^2(12k^2 + 8k + 2)(24k^2 + 16k + 3)(12k^2 + 8k + 1) \\ &= (3k + 1)^2 n', \end{aligned}$$

where n' is of the form $a(a + 1)(2a + 1)$ where $a = 12k^2 + 8k + 1$. So n' is primitive Pythagorean by Corollary 2.2. If $k \geq 1$, n is not independent.

We give two more examples for the above fact.

Example 1:

$$\begin{aligned} n &= (18k^2 + 24k + 8)(6k^2 + 8k + 3)(24k^2 + 32k + 11)(12k^2 + 16k + 5),\ k \geq 1, \\ &= (3k + 2)^2(12k^2 + 16k + 6)(24k^2 + 32k + 11)(12k^2 + 16k + 5) \end{aligned}$$

Example 2:

$$n = (6k + 2)(2k + 1)(8k + 3)(4k + 1) \text{ with } k \geq 1 \text{ and } 3k + 1 = s^2,$$
$$= (3k + 1)(4k + 2)(8k + 3)(4k + 1).$$

Problem 2: Find a sufficient condition for an integer n to be an independent Pythagorean number.

Definition 3: A natural number n is called a *twice* (*thrice*) *Pythagorean number* if it can be the area of two (three) different Pythagorean triangles.

Since by Theorem 10 we have infinitely many primitive Pythagorean numbers which are not independent we have an infinite number of twice Pythagorean numbers. The number $n = 840$ is a thrice Pythagorean number because n is the area of three Pythagorean triangles (40, 42, 58), (70, 24, 74), and (112, 15, 113). Hence, $840t^2$ is triply Pythagorean for every natural number t.

Some positive integers are twice primitive Pythagorean. There are three such numbers below 10,000. They are 210, 2730, and 7980. For example, (i) $n = 210$ is the area of two primitive Pythagorean triangles (12, 35, 37) and (20, 21, 29), (ii) $n = 2730$ is the area of two primitive Pythagorean triangles (28, 195, 197) and (60, 91, 109), and (iii) $n = 7980$ is the area of two primitive Pythagorean triangles (40, 399, 401) and (168, 95, 193).

To find all positive integers n which can be the area of two primitive Pythagorean triangles is an interesting problem which, to the best of our knowledge, has escaped the notice of mathematicians so far.

Problem 3: Find all positive integers n which are twice primitive Pythagorean.

Problem 4: Is there an integer n which is thrice primitive Pythagorean?

Problem 5: Let m be the maximum number of Pythagorean triangles having the same area. Can we say something about m?

Definition 4: A powerful number [3] is a positive integer n satisfying the property that p^2 divides n whenever the prime p divides n, i.e., in the canonical prime decomposition of n, no prime appears with exponent 1.

Definition 5: A number is powerful Pythagorean if it is powerful and Pythagorean.

Theorem 11: A Pythagorean number is never a square.

Proof: If possible, let $m^2uv(u^2 - v^2) = s^2$ where $(u, v) = 1$, $u > v$, and u and v are of opposite parity. Then

$$uv(u - v)(u + v) = \frac{s^2}{m^2} = s'^2$$

yields

$$u = a^2,\ v = b^2,\ u - v = c^2, \text{ and } u + v = d^2,$$

where a, b, c, and d are natural numbers. Now we have

$$a^2 - b^2 = c^2 \quad \text{and} \quad a^2 + b^2 = d^2,$$

which is impossible [4]. If $(u, v) = 1$ and $uv(u - v)(u + v) = s^k$, then there exist natural numbers a, b, c, and d such that

$$u = a^k,\ v = b^k,\ u - v = c^k, \text{ and } u + v = d^k,$$

whence $a^k + b^k = d^k$. A primitive Pythagorean number is never a k^{th} power of an integer if Fermat's last theorem is true for the exponent k (i.e., $x^k + y^k = z^k$ has no nontrivial solution).

Theorem 12: There are infinitely many powerful Pythagorean numbers.

Proof: If n is Pythagorean, then $t^2 n^{2m+1}$ is powerful Pythagorean for every positive integer t and m.

The smallest powerful Pythagorean number is $6^3 = 216$. Some other powerful Pythagorean numbers are $t^2 \cdot 6^3$, $t^2 \cdot 2^m \cdot 30^3$, $t^2 \cdot 5^m \cdot 6^3$, $t^2 \cdot 7^m \cdot 30^3$.

Theorem 13: There is no Pythagorean number in the Lucas sequence.

Proof: A Pythagorean number is divisible by 6 and has 0, 4, or 6 in its unit's place. For the n^{th} Lucas number to be Pythagorean, it is necessary that $L_n \equiv 0 \pmod 6$ and $L_n \equiv 0, 4,$ or $6 \pmod{10}$. We consider the Lucas sequence modulo 6 and modulo 10 separately.

Modulo 6 the Lucas sequence is

$$\langle 2,1,3,4,1,5,0,5,5,4,3,1,4,5,3,2,5,1,0,1,1,2,3,5 \rangle 2,1,3,\ldots .$$

Its period is 24. We have

$$L_{24k+6} \equiv 0 \pmod 6 \quad \text{and} \quad L_{24k+18} \equiv 0 \pmod 6.$$

Modulo 10 the Lucas sequence is

$$\langle 2,1,3,4,7,1,8,9,7,6,3,9 \rangle 2,1,\ldots .$$

Its period is 12 and

$$L_{24k+6} = L_{12k'+6} \equiv 8 \pmod{10},$$
$$L_{24k+18} = L_{12(2k+1)+6} \equiv 8 \pmod{10}.$$

The Lucas numbers that are divisible by 6 have 8 in their unit's place; therefore, they cannot be Pythagorean.

Conjecture 4: There is no Pythagorean number in the Fibonacci sequence.

We shall discuss the problems and conjectures in this paper and other interesting questions on Pythagorean numbers in a future paper.

References

1. B. L. Bhatia & Supriya Mohanty. "Nasty Numbers and Their Characterizations." *Mathematical Education 2.1* (1985):34-37.
2. K. Charles Brown. "Nasties are Primitive." *The Mathematics Teacher 74.7* (1981):502-504.
3. S. W. Golomb. "Powerful Numbers." *Amer. Math. Monthly 77* (1970):848-852.
4. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 4th ed. Oxford: Clarendon Press, 1960.
5. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.
6. Bert Miller. "Nasty Numbers." *The Mathematics Teacher 73.9* (1980):649.
7. L. J. Mordell. "On Integer Solutions of $y(y + 1) = x(x + 1)(x + 2)$." *Pacific J. Math. 13* (1963):1347-1351.
8. Trygve Nagell. *Introduction to Number Theory*. New York: Chelsea, 1964.

*****

ON FRIENDLY-PAIRS OF ARITHMETIC FUNCTIONS

N. Balasubramanian
(Former Director, Joint Cipher Bureau, Govt. of India)
c/o CMC Ltd., 1 Ring Road, Kilokri, New Delhi 110014 India
(Submitted February 1988)

1. In [1] it is shown that there exists a "friendly-pair" of multiplicative functions $\{f, g\}$ such that

(1.1) $\quad f(n^\alpha) = g(n), \; g(n^\alpha) = f(n), \; f(n)g(n) = 1$

for a fixed integer $\alpha \geq 2$. It is clear that f and g must satisfy the functional relation,

(1.2) $\quad F(n^{\alpha^2}) = F(n)$ for all natural numbers n.

Hence, it is natural to examine whether pairs of functions $\{f, g\}$, not necessarily multiplicative, exist so that

(1.3) $\quad f(n^\alpha) = g(n), \; g(n^\beta) = f(n)$

for a given pair $\alpha, \beta \geq 1$. Relation (1.3) implies that f and g must both satisfy the following functional equation where $r = \alpha \cdot \beta$.

(1.4) $\quad F(n^r) = F(n) \; \forall \; n \in \mathbb{N}$ (the set of all natural numbers).

Conversely, if F satisfying (1.4) for some r exists, then for any factorization of r as $\alpha \cdot \beta$ we could define

(1.5) $\quad f(n) = F(n), \; g(n) = F(n^\alpha)$ so that $g(n^\beta) = f(n)$

and so f and g satisfy (1.3). N.B. If r is prime, then both f and g are the same as F defined by (1.4).

Thus, it suffices to look for arithmetic functions F that satisfy what may be called the "power-periodicity" expressed in (1.4).

2. A complete characterization of such a power-periodic function F is more straightforward than when F is required to be multiplicative: Given a natural number $r > 1$, define $F(m)$ arbitrarily for every m that is not an r^{th} power of a natural number. Every natural number n that is an r^{th} power is uniquely expressible as

(2.1) $\quad n = m^{r^i}$, m a non-r^{th} power and i a natural number.

So $F(n)$ with power-period r is easily characterized by its values at non-r^{th} powers.

3. Suppose F is required to be multiplicative. Then (1.4) implies:

(3.1) $\quad \prod_{p|n} F(p^{ra}) = \prod_{p|n} F(p^a)$ where $n = \prod p^a$

in the standard form of unique factorization into powers of primes. Writing $F(p^a)$ as $G_p(a)$ and considering G as an arithmetic function of a, we are led to the following property of G that would suffice to ensure the power-periodicity of F.

Define a "multiplicatory-periodic" arithmetic function with period r by the relation

(3.2) $G(rn) = G(n)$ for all n and a given integer $r > 1$.

An infinity of such functions G exists. For we can define $G(m)$ arbitrarily for every m that is not a multiple of r, and then every n that is a multiple of r can be uniquely expressed as

(3.3) $n = m \cdot r^i$ where $r \nmid m$ and $i \geq 1$.

Taking a countable infinity of such functions G and labelling each of them with a unique prime number suffix p, set up a function $F(n)$ defined as

(3.4) $F(n) = \prod_{p|n} G\,(a)$ when $n = \prod p$ in the standard form.

It is easily found that this F satisfies (1.4).

4. We are, in turn, led to finding multiplicative functions that have a multiplicatory-period as defined in (3.2). In such a case

(4.1) $\prod_p G(p^{a+i}) = \prod_p G(p^a),\ n = \prod_p p^a,\ r = \prod_p p^i,$

where p runs through all the primes so that $a, i \geq 0$. Writing $G(p^a)$ as $H_p(a)$, we see that a sufficient condition for (4.1) to hold is that H_p be periodic in a with period i (in the normal sense of periodicity). That is, for every prime p and the corresponding i such that

(4.2) $p^i | r,\ p^{i+1} \nmid r$

we should have

(4.3) $H_p(a + i) = H_p(a)\ \ \forall\, a \in \mathbb{N}$.

A function $H_p(a)$ satisfying (4.2) and (4.3) can be easily constructed by (i) defining $H_p(o)$ as an arbitrary function of the prime argument p and (ii) further defining arbitrary values for $H(a)$ for the values of a in the interval $0 < a < i$, where i is the unique integer corresponding to p given by (4.2). These arbitrary values completely determine the values of $H_p(a)$ for every prime p and every nonnegative integer a, in order that (4.2) and (4.3) hold. Hence, a function G defined by

(4.4) $G(n) = \prod_p H\,(a),\ n = \prod_p p^a,$

where p is a variable prime, is multiplicative and multiplicatory-periodic with n as that period.

5. Special Solutions

The preceding general solution notwithstanding, the particular pairs of functions given in [1] are still of interest. They show how certain simple expressions of known arithmetic functions exhibit the power-periodic relation (1.4), and hence generate friendly-pairs.

The two instances given in [1] actually can be shown to be representatives of two classes of such arithmetic functions.

Write P-periodic for power-periodic, which is the property expressed by (1.4) and M-periodic for the multiplicatory-periodic property expressed in (3.2).

Class I:

Consider the m^{th} root of unity, $\omega = \exp(2\pi i/m)$ for a given $m > 1$. Obviously

(5.1) $k \equiv 1 \pmod{m} \Rightarrow \omega^{kr} = \omega^r$.

That is, ω^r as a function of r is M-periodic with k as an M-period. Construct the multiplicative $f(n)$ defined by its values for powers of primes as $f(p^r) = \omega^r$. Clearly, $f(n) = \omega^{\Omega(n)}$ where $\Omega(n)$ is the total number of prime divisors, repetition reckoned, in the factorization of n. It is also clear that $f(n)$ is P-periodic, with P-period k, i.e., $f(n^k) = f(n)\ \forall\, n \in \mathbb{N}$.

When k happens to be a square, say $k = \alpha^2$, we have

$$f(n^\alpha) = g(n),\ g(n^\alpha) = f(n).$$

In the first friendly-pair given in [1], m is taken as $\alpha + 1$ so that $\alpha \equiv -1 \pmod{m}$, so $\omega^{\alpha r} = \omega^r$ and hence $f(n)g(n) = 1$.

Class II:

The concluding pair of functions given in [1], "friendly" except for the fact that they are not reciprocals of each other, is

$$(5.2)\qquad f(n) = \sum_{dt^3 = n} \mu(d); \quad g(n) = \sum_{d^2t^3 = n} \mu(d)$$

so that

$$(5.3)\qquad f(n^2) = g(n); \quad g(n^2) = f(n); \quad f(n)g(n) = \begin{cases} 1 & \text{if } n \text{ is a cube} \\ 0 & \text{if not.} \end{cases}$$

The summand μ is the Möbius function. The first summation is over divisors d of n such that n/d is a perfect cube. The other summation is over the divisors d of n such that $d^2|n$ and n/d^2 is a perfect cube.

The general class, of which the given example turns out to be representative, is given below.

Take a multiplicative function $c(n)$ that vanishes when n is divisible by an r^{th} power (for a fixed r). There are infinitely many such functions, since $c(p^\lambda)$ can be defined arbitrarily for every prime p and $1 \le \lambda \le r - 1$. Set

$$(5.4)\qquad F(n) = \sum_{dt^r = n} c(d) \quad \text{and} \quad G(n) = \sum_{d^\ell t^r = n} c(d)$$

where r and c are as just assumed and ℓ is any integer such that

$$\exists\, k\colon k\ell \equiv 1 \pmod{r}.$$

The summations are over divisors d of n such that n/d is an r^{th} power in the first case and $d^\ell|n$ and n/d^ℓ is an r^{th} power in the second case.

F and G can be proved to be multiplicative. Define

$$(5.5)\qquad T_r(n) = \begin{cases} 1 & \text{if } n \text{ is an } r^{\text{th}} \text{ power} \\ 0 & \text{if not.} \end{cases}$$

Observe that $T_r(n)$ is multiplicative. F and G can now be written as divisor-convolution products.

$$(5.6)\qquad F(n) = \sum_{d|n} c(d)T_r(n/d) \quad \text{and} \quad G(n) = \sum_{d|n} c(d^{1/\ell})T_\ell(d)T_r(n/d),$$

where, in the second summation, c is understood to be zero when $d^{1/\ell}$ is not an integer. Such convolution products of multiplicative functions are multiplicative. Hence, F and G are multiplicative and are consequently characterized by their values for powers of primes. For every prime $p \ge 2$ and $\alpha \ge 1$, we have, by virtue of (5.6),

$$(5.7)\qquad F(p^\alpha) = \sum_{i=0}^{\mathrm{Min}(r-1,\ \alpha)} c(p^i)T\ (p^{\alpha - i}),$$

where "Min" denotes the minimum value from among the arguments within the parentheses. By the nature of the function T_r, it is clear that all the terms but one on the right-hand side of (5.6) have to be zero. The result is that

$$F(p^{\alpha}) = c(p^{\alpha \bmod r}), \tag{5.8}$$

where "α mod r" stands for the remainder left when α is divided by r.

If k and ℓ are two integers such that $k\ell \equiv 1 \pmod r$, then

$$F(p^{k\ell\alpha}) = c(p^{k\ell\alpha \bmod r}) = c(p^{\alpha \bmod r}) \tag{5.9}$$

which, by (5.8) $= F(p^{\alpha})$.

Hence

$$\begin{aligned} F(n^{k\ell}) &= \prod_{p|n} F(p^{k\ell\alpha}) \quad \left[\text{where } n = \prod p^{\alpha}\right] \\ &= \prod F(p^{\alpha}) = F(n). \end{aligned} \tag{5.10}$$

That is, F defined in (5.4) is P-periodic, with $k\ell$ for a P-period. So, if we set $F(n^k) = G^*(n)$, then $G^*(n^{\ell}) = F(n)$. We prove below that G^* is the same as G defined in (5.4).

$$F(p^{k\alpha}) = \sum_{i=0}^{\mathrm{Min}(r-1,\,k\alpha)} c(p^i)T_r(p^{k\alpha - i}). \tag{5.11}$$

Now note that

$$T_r(p^{k\alpha - i}) = T_r(p^{k(\alpha - \ell i)}) \tag{5.12}$$

since the indices on both of the sides differ by a multiple of r and T_r is not affected thereby.

Using (5.11) and (5.12), we deduce

$$\begin{aligned} F(n^k) &= \prod_p F(p^{k\alpha}) \quad \text{where } n = \prod p^{\alpha} \text{ in the standard form} \\ &= \prod_p [T_r(p^{k\alpha}) + c(p)T_r(p^{k(\alpha-\ell)}) + c(p^2)T_r(p^{k(\alpha-2\ell)}) \\ &\quad + \cdots \text{ until the index on } p \text{ becomes negative}] \\ &= \sum_{d^{\ell}t^{r} = n} c(d) \quad \text{(multiplied out)} \end{aligned} \tag{5.13}$$

which $= G(n)$ as defined.

6. Three Points and an Open Problem

Before concluding, we make three observations and indicate a promising problem.

Note (i): Pair-wise "friendliness" being found only on off-shoots of power-periodicity, one could study friendly-pairs defined on the basis of M-periodicity and normal periodicity also: Say

$$\begin{aligned} &f(kn) = g(n),\ g(\ell n) = f(n), \text{ so that} \\ &f(k\ell n) = f(n) \quad \text{and} \quad g(k\ell n) = g(n); \end{aligned} \tag{6.1}$$

$$\begin{aligned} &f(n+k) = g(n),\ g(n+\ell) = f(n), \text{ so that} \\ &f(n+k+\ell) = f(n) \quad \text{and} \quad g(n+k+\ell) = g(n). \end{aligned} \tag{6.2}$$

The former of these cases does not appear to be as trivial as the latter, as seen from the construction of M-periodic functions given earlier.

Note (ii): The definitions of P- and M-periodicities, leading to interesting consequences in the case of arithmetic functions, would seem to degenerate into trivialities in the case of functions of a continuous variable.

For instance, defining $f(kx) = f(x)$ for all real x or $f(x^k) = f(x)$ for all real x leads only to f being a constant, if f is to be continuous at zero in the first case and at one in the second case.

Note (iii): Why pairs only? one could ask for r-tuples of functions f_i, $0 \le i \le n - 1$, satisfying the mutual relation.

(6.3) $\quad f_i(n\ (\cdot)\ k_i) = f_{i+1 \bmod r}(n)$,

where $(\cdot)$ stands for multiplication or "to the power of." Obviously, every f_i is "$(\cdot)$"-periodic; with $\prod_i k_i$ for a "$(\cdot)$"-period.

Note (iv): In the case of normal periodicity it is well known that if k is a period then there is a divisor of k that is the minimal period (considering arithmetic functions), and a function cannot have more than one fundamental period. That is not true for M- and P-periodic arithmetic functions. It appears promising to study the set of integers

$$\{k^r \ell^s : r, s \in \mathbb{N} + \{0\}\}$$

for a given pair of natural numbers k and ℓ.

Acknowledgment

Thankful acknowledgment is due to Dr. R. Sivaramakrishnan of the University of Calicut, India, currently at the University of Kansas, U.S.A., for many fruitful discussions on the subject of this paper.

Reference

1. N. Balasubramanian & R. Sivaramakrishnan. "Friendly-Pairs of Multiplicative Functions." *Fibonacci Quarterly* 25.4 (1987):320-321.

*****

A PROOF FROM GRAPH THEORY FOR A FIBONACCI IDENTITY

Lee Knisley Sanders
Miami University-Hamilton, 1601 Peck Boulevard, Hamilton, Ohio 45011
(Submitted February 1988)

One of the beauties of mathematics is its consistency. To find, serendipitously, a verification of a result from an area other than the one being studied is an unexpected bonus. One such bonus is a proof of the Fibonacci identity

(1) $$f_{n+2} = f_{i+1}f_{n-i} + f_{i+2}f_{n-i+1}, \quad 1 \le i \le n-2$$

which arose during a count of maximal independent sets in trees.

First, we need some definitions from graph theory [1].

A graph G is a nonempty finite set of points, or vertices, V, along with a prescribed set E of unordered pairs of distinct points of V, called edges. We write $G = (V, E)$.

If two distinct points, x and y, of a graph are joined by an edge, they are said to be adjacent, and we write x adj y.

A walk of a graph G is a finite sequence of points such that each point of the walk is adjacent to the point of the walk immediately preceding it and to the point immediately following it. If the last vertex of the walk is the same as the first, the walk is closed. If a closed walk contains at least three distinct points and all are distinct except the first and last, then we have a cycle. A graph is acyclic if it contains no cycles. A walk is a path if it contains no cycles. A walk is a path if all the points are distinct.

A graph is connected if every pair of points is joined by a path.

A tree is a connected, acyclic graph.

The degree of a point v in G, denoted deg v, is the number of edges incident with v.

An endpoint or end vertex or leaf of a tree is a vertex of degree one. (Every tree has at least two endpoints.) An interior point of a tree is any vertex with a degree greater than one.

An independent set for graph G is a set of vertices with the property that no two vertices in the set are adjacent.

A maximal independent set (MIS) of G is an independent set which is contained in no other independent set of G.

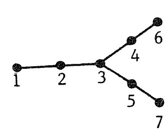

FIGURE 1

For the tree in Figure 1, {1, 3, 6} is an independent set; {1, 3, 6, 7} and {2, 4, 5} are maximal independent sets. Note that not all maximal independent sets are the same size. Also note that any vertex v is either included in a given maximal independent set or adjacent to a vertex in that maximal independent set.

It was hoped that the number and sizes of maximal independent sets would be a key to the structure of a tree. Although that was not the case, it was while counting the maximal independent sets of a narrow class of trees that the counterexample was found, along with the Fibonacci identity (1).

Let T be a tree with n vertices. Let $p(T)$ be the tree obtained by adding one edge and one end vertex to each vertex of T. Then $p(T)$ has $2n$ vertices, and is called the expanded tree of T, and T is the reduced or core tree of $p(T)$. The expanded tree has exactly n end vertices. If T is a tree with $2n$ vertices and exactly n end vertices, then each of the end vertices (with its adjoining edge) can be removed to obtain the core tree, which we call $p^{-1}(T)$. Figure 2 shows a core tree with its expanded tree. The added vertices are circled.

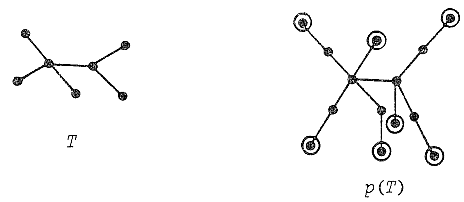

FIGURE 2

If e is an endpoint of tree T which is adjacent to a vertex x of degree 2, call e a remote end vertex.

Now consider only the set of trees that are expanded trees of n-paths, n = 1, 2, 3, ... Let us count the number of maximal independent sets for each of these trees.

Let M_T = the number of maximal independent sets of T.

Let T be the expanded tree of an n-path. For each vertex v in T, define $\lambda(v)$ to be the number of maximal independent sets of T that contain v. If v is an interior point (i.e., not an endpoint) and if w is the endpoint adjacent to v, then

$$\lambda(v) + \lambda(w) = M_T,$$

since every maximal independent set must contain either v or w. In particular, if e is a remote end vertex and e adj x, then

$$\lambda(e) + \lambda(x) = M_T.$$

If x adj y, $y \neq e$, then

$$\lambda(y) + \lambda(x) = \lambda(e),$$

since e belongs to every MIS containing y, and if a MIS S contains x, then $(S - \{x\}) \cup \{e\}$ is also a MIS of the proper size, and these are the only MIS's that could possibly contain e.

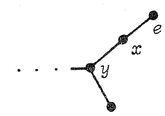

FIGURE 3

By combining these two equalities, we find that $M_T = \lambda(y) + 2\lambda(x)$.

The following proposition states some more facts about relationships among λ-numbers.

Proposition 1: Let T be an expanded tree that looks like the tree in figure 4; that is, e is a remote end vertex, e adj x, x adj y, $y \neq e$, z_1 is the end vertex adjacent to y, z_2 adj y, $z_2 \neq z_1$, $z_2 \neq x$, and z_3 is the end vertex adjacent to z_2. The structure of the rest of T (which is attached at z_2) does not matter.

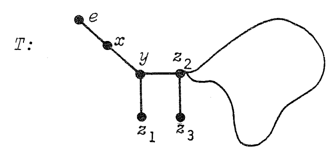

FIGURE 4

Then:

(i) $\lambda(z_3) = 3\lambda(y)$, so that $\lambda(z_3)$ is divisible by 3;

(ii) $\lambda(z_1) = 2\lambda(y) + \lambda(z_2)$;

(iii) $\lambda(z_2)$ is even;

(iv) $\lambda(z_1)$ is even;

(v) $\lambda(y)$ and M_T have the same parity;

(vi) $\lambda(v)$ is independent of the number of remote end vertices attached to v for any $v \in T$ that is an interior point.

In addition,

(vii) $\lambda(e) = M_{T-\{e,x\}}$.

Proof:

(i) For every MIS S containing y, $z_3 \in S$, $S - \{y\} \cup \{z_1\}$,

$S - \{y, e\} \cup \{z_1, x\}$,

so $\lambda(z_3) = 3\lambda(y)$, and 3 divides $\lambda(z_3)$.

(ii) This can be proved in two ways:

(a) $M_T = \lambda(z_2) + \lambda(z_3) = \lambda(z_2) + 3\lambda(y)$ by (i);

$M_T = \lambda(y) + \lambda(z_1)$.

The difference of these two equations is

$$0 = 2\lambda(y) + \lambda(z_2) - \lambda(z_1)$$

or

$$\lambda(z_1) = 2\lambda(y) + \lambda(z_2).$$

(b) For every MIS S containing y,

$$z_1 \in S - \{y, e\} \cup \{x, z_1\}$$

and

$$z_1 \in S - \{y\} \cup \{z_1\}.$$

z_1 is also in every MIS containing z_2.

(iii) Let T_1 be the part of T containing e, x, y, z_1, z_2, z_3, and

$$T_2 = (T - T_1) \cup \{z_2\}.$$

$\lambda(z_2)$ = (number of MIS's in T_1 containing z_2) $\times$ (number of MIS's in T_2 containing z_2) = 2 $\times$ (number of MIS's in T_2 containing z_2). The two sets in T_1 are $\{z_1, z_2, e\}$ and $\{z_1, z_2, x\}$. Thus, 2 divides $\lambda(z_2)$.

(iv) follows immediately from (ii) and (iii); and (v) follows from (iv) and

$$M_T = \lambda(y) + \lambda(z_1).$$

(vi) Let $e_1, e_2, \ldots, e_k$ be the remote end vertices attached to some interior point v in T, with e_1 adj x_1, x_i adj v, $i = 1, 2, \ldots, k$. Then every MIS containing v must also contain all the e_i's, and if a MIS contains even one x_i, then v is *not* a member of that set. Thus, $\lambda(v)$ is not affected by the size of k.

(vii) Let $\lambda'(v)$ be the number of MIS's containing v in $T - \{e, x\}$, for any $v \in T - \{e, x\}$.

$$\lambda(e) + \lambda(x) = M_T$$

and

$$\lambda(x) + \lambda(y) = \lambda(e)$$

so that

$$2\lambda(e) = M_T + \lambda(y)$$

so

$$\lambda(e) = \frac{M_T + \lambda(y)}{2}; \qquad (2)$$

also

$$M_T = 2M_{T-\{e,x\}} - \lambda'(y)$$

since for every MIS S in $T - \{e, x\}$ we have, in T, the MIS $S \cup \{e\}$ and the MIS $S \cup \{x\}$, except when S contains y.

But (vi) implies that $\lambda(y) = \lambda'(y)$, so

$$\lambda(y) = \lambda'(y) = 2M_{T-\{e,x\}} - M_T. \qquad (3)$$

(2) and (3) lead to

$$\lambda(e) = \frac{M_T + 2M_{T-\{e,x\}} - M_T}{2} = M_{T-\{e,x\}}.$$

Now to determine M_T: The expanded trees of n-paths have exactly two remote end vertices.

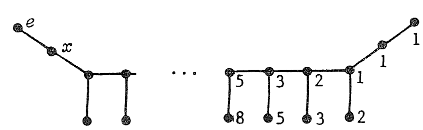

FIGURE 5

We will call the central $n - 2$ points of the core tree the <u>central path</u>, and will find the λ-numbers for all points of the central path, as well as for the nonremote end vertices.

Starting at the right-hand end of the central path, we label each vertex of the central path and each corresponding end vertex with the number of maximal independent sets containing the point that include only points that have been previously labeled, or points only "to the right" of the given point. Points "to the right" of an end vertex shall include the point in the core tree to which it is connected. These labels will be elements of the Fibonacci sequence (1, 1, 2, 3, 5, 8, ... is the Fibonacci sequence, where the n^{th} term, f_n, equals the sum of the two previous terms: $f_n = f_{n-1} + f_{n-2}$).

Since a vertex in the central path is contained in exactly the same number of maximal independent sets to its right as the most recently labeled end vertex, its λ-number will be the same as the label of that end vertex. (Note that because of the order of labeling described above, these two points are *not* adjacent.)

Since any end vertex can be added to a MIS containing the most recently labeled end vertex (to the right) or to a MIS containing the vertex in the central path adjacent to the most recently labeled end vertex, the label of the end vertex in question will be the sum of the labels of those two previously labeled vertices.

Then since the farthest remote end vertex to the right and the point adjacent to it can each only be in one MIS to the right, we see (refer to Fig. 5) that the labels of the interior points are indeed a Fibonacci sequence. The labels of the end vertices form a Fibonacci sequence but start with the second term.

A portion of the tree in Figure 5 with labels that will correspond to the following discussion is shown in Figure 6.

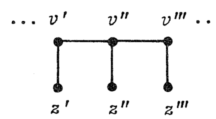

FIGURE 6

Define $r(w)$ to be the number of MIS's containing vertex w and points "to the right" of w.

For a vertex v' in the central path, v' is in the same number of maximal independent sets to its right as z'' is in, where z'' is the endpoint adjacent to v'', v'' adj v' and v'' to the right of v'. $r(z') = r(v'') + r(z'')$ since, if z' is in a MIS S (containing points only "to the right" of z'), then either $z'' \in S$ or $v'' \in S$.

Therefore,

$$r(v') = r(z'') = r(z''') + r(v''') = r(v'') + r(v''').$$

Thus, if we number the vertices of the central path from left to right by $v_{n-2}, v_{n-3}, \ldots, v_2, v_1$, and the end vertices by $z_{n-2}, z_{n-3}, \ldots, z_2, z_1$, where z_i adj v_i, then

$$r(v_i) = f_{i+1} \quad \text{and} \quad r(z_i) = f_{i+2}.$$

Then,

$$\lambda(v_{n-2}) = f_{n-1} \quad \text{and} \quad \lambda(z_{n-2}) = 2f_n$$

(since for every MIS "to the right" we could add either the remote end vertex e or the adjacent point x) and hence,

$$M_T = \lambda(v_{n-2}) + \lambda(z_{n-2}) = f_{n-1} + 2f_n$$
$$= f_{n-1} + f_n + f_n = f_{n+1} + f_n = f_{n+2}.$$

Now label each point v_i and end point z_i in a similar manner from the left by $\ell(v_i)$ and $\ell(z_i)$.

$$\lambda(v_i) = r(v_i) \cdot \ell(v_i) \quad \text{and} \quad \lambda(z_i) = r(z_i) \cdot \ell(z_i).$$

Note that $\ell(v_i) = f_{n-i}$ since v_i is the $n - i - 1^{\text{st}}$ point from the left, and

$$\ell(z_i) = f_{n-i+1}.$$

Since

$$\lambda(v_i) + \lambda(z_i) = M_T, \ 1 \le i \le n - 2,$$

we have the following well-known [2] number theoretic result.

Theorem 3.24: $f_{n+2} = f_{i+1}f_{n-i} + f_{i+2}f_{n-i+1}$ for $1 \le i \le n - 2$.

For more general expanded trees, we follow a similar procedure. If v is a member of the core tree and deg $v = k$, then $\lambda(v)$ is the product of $k - 1$ labels—one from each of the $k - 1$ branches incident with v. If v adj z, z an end vertex, then $\lambda(z)$ is also the product of $k - 1$ labels. In this general case, the labels will not always be elements of the Fibonacci sequence, but each individual label will be obtained as the sum of the two previous labels in the same branch. It is not necessary to find all labels for every point in order to find M_T. Only the λ-numbers for one end vertex and its adjacent point are needed.

As an example, in Figure 7 is a tree with 20 vertices. The points e and y are the ones for which the λ-numbers are being found. We are labeling from the endpoints of the separate branches *toward* the vertices e and y, in the order of the indices on the v's.

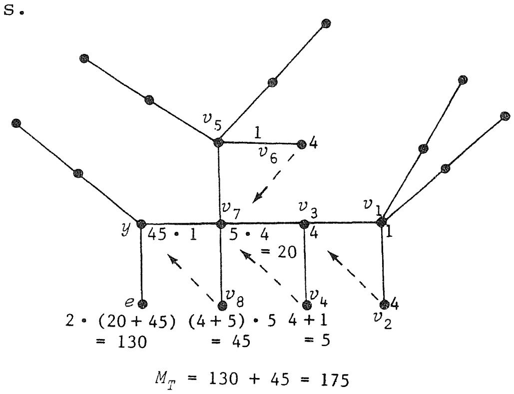

FIGURE 7

v_1 is in only one MIS to the right.

v_2 is in 4 MIS's to the right—there are two choices on each of the paths leading to the remote end vertices for $2 \cdot 2$ MIS's.

v_3 is in the same MIS's to the right as v_2.

v_4 is in the same MIS's as v_1 or the same MIS's as v_2, for a total of $1 + 4$, or 5.

v_5 and v_6 are like v_1 and v_2, respectively, when labeling from the end of their branch, i.e., "from above."

v_7 gets a label of 4 from above (the same as v_6) and a label of 5 from the right (the same as v_4) for a total of $5 \cdot 4$ MIS's.

v_8 is in 4 + 5 sets to the right (the sum of labels from v_3 and v_4) and in 1 + 4 sets from above (the sum of labels from v_5 and v_6) for a total of $9 \cdot 5$, or 45.

$\lambda(y) = 1 \cdot 45$, the product of the number of MIS's to the left and the number of MIS's to the right (from the v_8 label).

$\lambda(e) = 2 \cdot (20 + 45)$, with 2 being the number of MIS's to the left and 20 + 45 being the sum of the labels from v_7 and v_8.

$M_T = \lambda(y) + \lambda(e) = 45 + 130 = 175.$

If we look at the triangular array of λ-numbers for the central $n - 2$ vertices of the core tree of the expanded tree of an n-path, we see a triangle whose entries grow along diagonals in a Fibonacci-like manner. Figure 8 shows the first 3 trees and gives the λ-numbers of the circled vertices. In the following chart, $2n$ is the number of vertices of the expanded n-path.

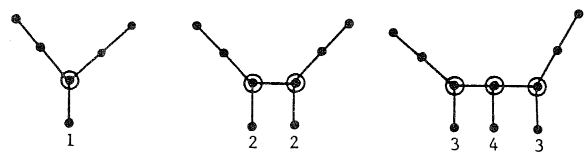

FIGURE 8

													$2n$
						1							6
					2		2						8
				3		4		3					10
			5		6		6		5				12
		8		10		9		10		8			14
	13		$\boxed{16}$		15		15		16		13		16
12		26		24		25		24		26		21	18
						⋮							⋮

16 = 2 × 8

Notice that the triangle is symmetric about a vertical line through its center. The two outer diagonals are the Fibonacci sequence without the first term. All other diagonals are Fibonacci-like in that each term, starting with the third is the sum of the two terms immediately preceding it in the diagonal. Also, each diagonal is a set of multiples of the first element, and the members correspond to multiples of the shortened Fibonacci sequence seen in the outer diagonals.

Also notice that if there are $2n$ vertices in the tree, there are n vertices in its core tree and $n - 2$ vertices of that core tree will not be adjacent to a remote end vertex in the original tree. Therefore, there will be $n - 2$ vertices to label and $n - 2$ numbers in the row of the triangle associated with $2n$ (see Fig. 8).

Now, the remarkable coincidences of the triangle can be understood if we recall the way in which the vertices of the central path are labeled. Each label is the product of the number of MIS's to the right and the number of MIS's to the left. However, the numbers of MIS's to the right for the central path are just the Fibonacci numbers, starting with the second term. Likewise, because of symmetry, the numbers of MIS's to the left are also the Fibonacci numbers, starting with the second term. So for the $n - 2$ elements of the triangle row associated with $2n$, we have $f_{n-1-i} \cdot f_i$, $2 \le i \le n - 1$.

For example, if we let the factor on the left represent the number of MIS's to the left and the factor on the right represent the number of MIS's to the right, we see that the row associated with $2n = 14$ and $n - 2 = 5$ is:

$$1 \cdot 8 \quad 2 \cdot 5 \quad 3 \cdot 3 \quad 5 \cdot 2 \quad 8 \cdot 1$$

Certainly, the growth of the numbers related to maximal independent sets in this special class of trees is related to Fibonacci numbers and patterns, and the study of one enhances the other.

References

1. F. Harary. *Graph Theory*. Reading: Addison-Wesley, 1969.
2. I. Niven & H. S. Zuckerman. *An Introduction to the Theory of Numbers*. New York: Wiley & Sons, 1960, pp. 98-99.

*****

ON THE SUM $\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)a$

M. G. Monzingo
Southern Methodist University, Dallas, TX 75275
(Submitted March 1988)

In this note, the sum

$$\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)a, \text{ where } p \text{ is an odd prime and } \left(\frac{a}{p}\right) \text{ is the Legendre symbol,}$$

will be written in an expanded form. Special cases of this form yield the results that, for $p \equiv 7 \pmod 8$,

$$\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right)a = 0,$$

and for $p \equiv 3 \pmod 8$,

$$\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right) \text{ is an odd multiple of 3.}$$

This latter result implies that for such primes the difference in the number of quadratic residues and quadratic nonresidues in the first half of the interval $1 \le a \le p - 1$ must be an odd multiple of three.

Let p be an odd prime, and let $\left(\frac{a}{p}\right)$ denote the Legendre symbol.

Theorem 1: Let q, $1 \le q \le p - 1$, be a divisor of $p - 1$ and k such that $p - 1 = kq$; then,

$$S = \sum_{a=1}^{p-1}\left(\frac{a}{p}\right)a$$

satisfies

$$(*) \qquad \left\{\left(\frac{q}{p}\right)q - 1\right\}S = p\left\{(q-1)\sum_{t=0}^{k-1}\left(\frac{tq+1}{p}\right) + (q-2)\sum_{t=0}^{k-1}\left(\frac{tq+2}{p}\right) + \cdots + \sum_{t=0}^{k-1}\left(\frac{tq+(q-1)}{p}\right)\right\}.$$

Proof: $\left(\frac{q}{p}\right)qS = \sum_{a=1}^{p-1}\left(\frac{qa}{p}\right)qa.$

This sum can be expanded as follows:

$$(1) \qquad \sum_{s=1}^{q}\left\{\sum_{a=(s-1)k+1}^{sk}\left(\frac{qa}{p}\right)qa\right\} = \sum_{s=1}^{q}\left\{\sum_{t=1}^{k}\left(\frac{((s-1)k+t)q}{p}\right)((s-1)k+t)q\right\}.$$

Next,
$$\begin{aligned}((s-1)k+t)q &= (s-1)kq + tq \\ &= (s-1)(p-1) + tq \\ &= (s-1)p + tq - (s-1) \\ &= (s-1)p + (t-1)q + (q-(s-1)).\end{aligned}$$

Substitution into the right-hand side of (1) and noting that $(s-1)p \equiv 0 \pmod p$ yields

$$\sum_{s=1}^{q}\left\{\sum_{t=1}^{k}\left(\frac{(t-1)q+q-(s-1)}{p}\right)\{(s-1)p+(t-1)q+(q-(s-1))\}\right\}. \tag{2}$$

In (2), letting $v = q - (s-1)$, splitting the sum, and summing on v yields

$$\sum_{v=1}^{q}\left\{\sum_{t=1}^{k}\left(\frac{(t-1)q+v}{p}\right)((t-1)q+v)\right\} + \sum_{v=1}^{q}\left\{\sum_{t=1}^{k}\left(\frac{(t-1)q+v}{p}\right)(q-v)p\right\}. \tag{3}$$

Note that the first sum in (3) is S. In the second sum, replace $t-1$ with t; then, the second sum can be written

$$p\sum_{v=1}^{q-1}\left\{(q-v)\sum_{t=0}^{k-1}\left(\frac{tq+v}{p}\right)\right\}.$$

Putting the pieces together, we have

$$\left(\frac{q}{p}\right)qS = S + p\sum_{v=1}^{q-1}\left\{(q-v)\sum_{t=0}^{k-1}\left(\frac{tq+v}{p}\right)\right\},$$

from which the conclusion follows.

Corollary 1: If q, $1 < q \le p-1$, is a quadratic residue modulo p, then $q-1$ divides

$$(q-2)\sum_{t=0}^{k-1}\left(\frac{tq+2}{p}\right) + \cdots + \sum_{t=0}^{k-1}\left(\frac{tq+(q-1)}{p}\right).$$

And, if q, $1 \le q < p-1$, is a quadratic nonresidue modulo p, then $q+1$ divides

$$(q-1)\sum_{t=0}^{k-1}\left(\frac{tq+1}{p}\right) + \cdots + \sum_{t=0}^{k-1}\left(\frac{tq+(q-1)}{p}\right).$$

Proof: In the second case, $(q/p) = -1$, and so $q+1$ divides the left side of (*) and, consequently, the right side of (*). The conclusion follows by noting that $(q+1, p) = 1$. The first case follows in a similar fashion with $q-1$ replacing $q+1$, and by noting that the first sum on the right side of (*) is multiplied by $q-1$.

Example: Let $p = 17$ and $q = 4$; then $k = 4$. Since 4 is a quadratic residue, the conclusion from Corollary 1 is that 3 divides

$$2\sum_{t=0}^{3}\left(\frac{4t+2}{17}\right) + \sum_{t=0}^{3}\left(\frac{4t+3}{17}\right).$$

Corollary 2: If $p \equiv 7 \pmod 8$, then $S = -p\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right)$.

Proof: In Theorem 1, let $q = 2$ and, hence, 2 is a quadratic residue. Thus,

$$S = p\sum_{t=0}^{k-1}\left(\frac{2t+1}{p}\right);$$

that is,

$$S = p\sum_{\substack{a=1\\ a \text{ odd}}}^{p-1}\left(\frac{a}{p}\right).$$

The desired conclusion is obtained from the following:

$$\sum_{\substack{a=1\\ a \text{ odd}}}^{p-1}\left(\frac{a}{p}\right) = -\sum_{\substack{a=2\\ a \text{ even}}}^{p-1}\left(\frac{a}{p}\right) = -\sum_{a=1}^{(p-1)/2}\left(\frac{2a}{p}\right) = -\left(\frac{2}{p}\right)\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right) = -\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right).$$

Note that the conclusion in Corollary 2 also holds with $p \equiv 1 \pmod 8$, but trivially; both S and the sum are zero.

Theorem 2: If $p \equiv 3 \pmod 8$, $p > 3$, then 3 divides $\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right)$.

Proof: Let $q = 2$; then q is a quadratic nonresidue, and so Corollary 1 implies that 3 divides

$$\sum_{t=0}^{k-1}\left(\frac{2t+1}{p}\right),$$

that is,

$$\sum_{\substack{a=1\\ a \text{ odd}}}^{p-1}\left(\frac{a}{p}\right).$$

Now, by an argument similar to that used in the proof of Corollary 2; the conclusion follows.

Example: Let $p = 11$; then the quadratic residues are 1, 3, 4, 5, and 9, while the quadratic nonresidues are 2, 6, 7, 8, and 10. Hence, the sum in Theorem 2 is

$$\left(\frac{1}{11}\right) + \left(\frac{2}{11}\right) + \left(\frac{3}{11}\right) + \left(\frac{4}{11}\right) + \left(\frac{5}{11}\right) = 1 - 1 + 1 + 1 + 1 = 3.$$

Note that the conclusion in Theorem 2 also holds for $p \equiv 5 \pmod 8$, but trivially; the sum in question is zero.

Also note that in Theorem 2 with $p \equiv 3 \pmod 8$, $(p - 1)/2$ is odd. Therefore, the sum in Theorem 2 has an odd number of terms, each one equal to ± 1. It follows, then, that the number of quadratic residues and quadratic nonresidues are opposite in parity. Hence, from Theorem 2, the difference in the numbers of quadratic residues and quadratic nonresidues in the interval from 1 to $(p - 1)/2$ must be an odd multiple of three.

Theorem 3: If $p \equiv 7 \pmod 8$, then $\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right)a = 0$.

Proof: $S = \sum_{a=1}^{p-1}\left(\frac{a}{p}\right)a = \sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right)a + \sum_{b=(p+1)/2}^{p-1}\left(\frac{b}{p}\right)b.$

In the last sum, let $b = p - a$; then this sum can be rewritten as

$$\sum_{a=1}^{(p-1)/2}\left(\frac{p-a}{p}\right)(p - a) = \sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right)a - p\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right),$$

since $\left(\frac{p-a}{p}\right) = -\left(\frac{a}{p}\right)$. Hence,

$$S = 2\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right)a - p\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right).$$

[Note that this equation also holds for $p \equiv 3 \pmod 8$ since, in this case, it is also true that

$$\left(\frac{p-a}{p}\right) = -\left(\frac{a}{p}\right).]$$

Now, from Corollary 2,

$$S = -p\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right),$$

and so

$$\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right)a = 0.$$

*****

REFEREES Continued from page 2

SUMS OF POWERS BY MATRIX METHODS

Dan Kalman
The Aerospace Corporation, P.O. Box 92957, Los Angeles, CA 90009-2957
(Submitted March 1988)

Let $\{s_n^r\}$ be the sequence of sums of r^{th} powers given by

$$(1) \qquad s_n^r = \sum_{k=0}^{n} k^r.$$

These familiar sequences are the subject of an extensive literature, a few recent samples of which may be found among the references. The present note has two objectives:

- To illustrate the application of matrix methods in the context of finite difference equations; and
- To publicize the following beautiful matrix formula for s_n^r.

$$(2) \qquad s_n^r = \left[\binom{n}{1} \ \binom{n}{2} \ \binom{n}{3} \ \cdots \ \binom{n}{r+1}\right] \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \pm 1 & \mp\binom{r}{1} & \pm\binom{r}{2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1^r \\ 2^r \\ 3^r \\ \vdots \\ (r+1)^r \end{bmatrix}$$

This equation may be viewed as reducing the sum of n terms of the power sequence to a linear combination of the first $r + 1$ terms. Accordingly, there is an implicit assumption that $n > r$. Note that the matrix appearing as the middle factor on the right side of this equation is lower triangular. The zeros that should appear above the main diagonal have been omitted. The nonzero entries constitute a version of Pascal's triangle with alternating signs.

The scalar equivalent of equation (2) has appeared previously ([4], eq. 57, p. 33) and can be derived by standard elementary manipulations of series expansions for exponential functions. The main virtue of the matrix form is esthetic: it reveals a nice connection between s_n^r and Pascal's triangle, and is easily remembered.

The main idea we wish to present regarding the application of matrix analysis to difference equations may be summarized as follows. In general, an n^{th}-order difference equation with constant coefficients is expressible as a first-order vector equation. The solution of this first-order vector equation is given in terms of powers of the coefficient matrix. By reducing the coefficient matrix to its Jordan canonical form, the powers can be explicitly calculated, finally leading to a formula for a solution to the original difference equation. This approach was discussed previously [5] for the case in which the matrix is diagonalizable. In applying this method to the derivation of (2), the matrix is not diagonalizable. Another example with a nondiagonalizable matrix will also be presented, connected with reference [3].

In the interest of completeness, a few results about linear difference equations will be presented. These can also be found in any introductory text on the subject, for example [8].

1. Elementary Results about Difference Equations

The sequence s_n^r satisfies the recursive relation

$$s_{n+1}^r = s_n^r + (n + 1)^r.$$

This is an example of a difference equation. It expresses one term of a sequence as a function of the preceding term, and the sequence index n. More generally, a difference equation of order k specifies a term of a sequence $\{a_n\}$ as a function of the preceding k terms and n. We shall be especially concerned with linear, constant coefficient, homogeneous difference equations. Any equation of this type can be cast in the form

$$(3) \qquad a_{n+k} + c_{k-1}a_{n+k-1} + \cdots + c_0 a_n = 0; \quad n \geq 0,$$

where the coefficients c_j are constants. Hereafter, we assume all difference equations are of this type. Clearly, given initial terms a_0 through a_{k-1}, the remaining terms of the sequence are uniquely determined by equation (3). The main objective of the next section is to develop techniques to express these terms as a function of n.

The analysis of difference equations is expedited by reformulating equation (3) in terms of linear operators. Accordingly, we focus for the present on the linear space of sequences $\{a_k\}_{k=0}^{\infty}$ of complex numbers, and state

Definition 1: The linear operator $\mathbf{L}$, called the *lag operator*, is defined by the relation

$$(4) \qquad \mathbf{L}\{a_n\}_{n=0}^{\infty} = \{a_{n+1}\}_{n=0}^{\infty}.$$

$\mathbf{L}$ has the effect of *shifting* the terms of a sequence by one position. Thus, it is often convenient to write

$$\mathbf{L}a_n = a_{n+1}.$$

Now (3) may be expressed in the form

$$p(\mathbf{L})\{a_n\} = 0$$

where $p(t) = t^k + c_{k-1}t^{k-1} + \cdots + c_0$ is called the characteristic polynomial of the equation. We follow the usual convention that the constant term of the polynomial p operates on the sequence $\{a_n\}$ by scalar multiplication. Since $p(\mathbf{L})$ is a linear operator, solving (3) amounts to determining the null space.

Now we turn our attention to the application involving s_n^r. As a first step, we use the operator approach to characterize polynomials in n as solutions to a specific class of difference equations. The statement and proof of this result will be simplified by the following notation.

Definition 2: $\mathbf{D}$ is the operator $\mathbf{L} - 1$. N_k is the null space of $\mathbf{D}^k$. Note that $\mathrm{N}_1 \subseteq \mathrm{N}_2 \subseteq \mathrm{N}_3 \ldots$.

Theorem 1: For any k, N_k consists of the sequences $\{a_n\}$ such that $a_n = p(n)$ for some polynomial p of degree less than k.

Proof: We show first that polynomials of degree less than k are contained in N_k. For $k = 1$, with a_n a polynomial of degree 0, and thus constant, it is clear that $\mathbf{D}\{a_n\} = 0$. Proceeding by induction, assume that polynomials of degree less than $k - 1$ are in N_{k-1}, and hence in N_k. Showing that $\{n^{k-1}\}$ is in N_k then assures that all polynomials of degree less than or equal to $k - 1$ are contained in N_k. One application of $\mathbf{D}$ to $\{n^{k-1}\}$ produces the sequence

$$\{(n + 1)^{k-1} - n^{k-1}\}.$$

This result is a polynomial of degree $k - 2$ and so is annihilated by $\mathbf{D}^{k-1}$, by the induction hypothesis. This shows that $\mathbf{D}^k\{n^{k-1}\} = 0$, and completes the first part of the proof.

For the converse, we must show that the polynomials of degree less than k exhaust N_k. Since these polynomials comprise a subspace of dimension k, it will suffice to show that N_k has dimension no more than k. This statement is clearly true for the case that $k = 1$. As before, the general case shall be established by induction.

Assume that N_j has dimension j for all j less than k, and suppose that a_n and b_n are in N_k but not in N_{k-1}. Then $\mathbf{D}^{k-1}a_n$ and $\mathbf{D}^{k-1}b_n$ are nonzero elements of N_1, which is one dimensional. This implies that, for some scalar c,

$$\mathbf{D}^{k-1}a_n = c\mathbf{D}^{k-1}b_n.$$

Hence, $a_n - cb_n$ lies in N_{k-1}. We conclude that the dimension of N_k can exceed that of N_{k-1} by at most 1. Finally, by the induction hypothesis, the dimension of N_k is no more than k, completing the proof. $\square$

This result may be immediately applied to the analysis of s_n^r. As observed previously,

$$s_{n+1}^r - s_n^r = (n + 1)^r$$

which is, in operator notation,

$$\mathbf{D}s_n^r = (n + 1)^r.$$

Now the right side is a polynomial in n of degree r, so is annihilated by $\mathbf{D}^{r+1}$. Thus, applying $\mathbf{D}^{r+1}$ to both sides yields

$$\mathbf{D}^{r+2}s_n^r = 0$$

and hence, s_n^r is in N_{r+2}. Moreover,

$$\mathbf{D}^{r+1}s_n^r = \mathbf{D}^r(n + 1)^r,$$

which is not zero. It follows that

$$s_n^r \in \mathrm{N}_{r+2}\backslash\mathrm{N}_{r+1},$$

and that s_n^r is a polynomial in n of degree $r + 1$.

The realization of s_n^r as a solution to the equation

$$\mathbf{D}^{r+2}a_n = 0$$

is more significant for our purposes than is the characterization of s_n^r as a polynomial. For future reference, it is convenient to express this equation in the form

$$(\mathbf{L} - 1)^{r+2}a_n = 0. \tag{5}$$

We show next how matrix methods can be employed to solve difference equations. Then, as a particular example, we apply the method to (5) to derive (2).

2. Matrix Methods for Difference Equations

Matrices appear as the result of a standard device for transforming a k^{th}-order scalar equation into a linear vector equation. The transformation is perfectly analogous to one used in the analysis of differential equations ([1], p. 192), and was used in the form presented below in [7].

Suppose a_n satisfies a difference equation of order k, as in equation (3). For each $n \geq 0$, define the k-dimensional vector $\mathbf{v}_n$ according to

$$\mathbf{v}_n = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

The vector $\mathbf{v}_n$ may be visualized as a window displaying k entries in the infinite column

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}$$

Now the transformation from $\mathbf{v}_n$ to $\mathbf{v}_{n+1}$ can be formulated as multiplication by the $k \cdot k$ matrix $\mathbf{C}$ given by

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{k-1} \end{bmatrix}$$

This matrix is the companion matrix for the characteristic polynomial of the original difference equation (3). It can also be understood as a combination of row operations. In this view, $\mathbf{C}$ has the effect of rolling rows 2 through k up one position, and creating in place of row k the linear combination

$$-c_0(\text{row } 1) - c_1(\text{row } 2) - \cdots - c_{k-1}(\text{row } k).$$

These operations correspond exactly to the transformation from $\mathbf{v}_n$ to $\mathbf{v}_{n+1}$.

Visually, multiplying $\mathbf{v}_n$ by $\mathbf{C}$ has the effect of moving the window described earlier down one position. Algebraically, $\mathbf{v}_n$ satisfies the *vector* difference equation

$$(6) \qquad \mathbf{v}_{n+1} = \mathbf{C}\mathbf{v}_n.$$

Evidently, a solution $\mathbf{v}_n$ of (6) may be characterized by

$$(7) \qquad \mathbf{v}_n = \mathbf{C}^n\mathbf{v}_0,$$

and so, the solution a_n of (3) is given as the first component of the right side of (7). These remarks may be summarized by expressing a_n in the equation

$$(8) \qquad a_n = [1 \quad 0 \quad 0 \cdots 0] \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{k-1} \end{bmatrix}^n \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{k-1} \end{bmatrix}$$

This formula is not really useful as a functional representation of a_n because the powers of $\mathbf{C}$ must be computed by what is essentially a recursive procedure (although the computation can be made very efficient by exploiting the special structure of $\mathbf{C}$, see [2]). However, if the roots of the characteristic polynomial are known, the Jordan canonical form of $\mathbf{C}$ can be explicitly formulated as described in [7]. Thus, if the reduction of $\mathbf{C}$ to its Jordan form

J is expressed by $C = SJS^{-1}$, then the matrix C^n may be replaced by SJ^nS^{-1} in (8). The special case in which the roots are distinct features a diagonal J and so the powers of J are simply expressed. This case is discussed in [5]. In the sequel, we shall focus on the application of matrix methods to the analysis of s_n^r, based on equation (5). Observe that the characteristic polynomial is given by $(t - 1)^{r+2}$, hence, rather than distinct roots, we have a single root of multiplicity $r + 2$. The next section will discuss the properties of the Jordan form for this case, and derive equation (2).

3. Analysis of s_n^r

As observed in the preceding section, s_n^r satisfies the difference equation (5) with the characteristic polynomial $(t - 1)^{r+2}$, and with k therefore equal to $r + 2$. Using this information, the general equation (8) may be particularized to give

$$(9) \qquad s_n^r = [1 \quad 0 \quad 0 \cdots 0]\, \mathbf{C}^n \begin{bmatrix} s_0^r \\ s_1^r \\ s_2^r \\ \vdots \\ s_{r+1}^r \end{bmatrix}$$

C is the companion matrix for $(t - 1)^{r+2}$. It can be shown that the Jordan canonical form for the companion matrix of a polynomial has one Jordan block for each distinct root. (A simple proof of this assertion may be constructed using Theorems 4.5 and 8.5 of [9].) In the present case, the Jordan form J is therefore a single Jordan block corresponding to the root 1. That is, J is a square matrix of dimension $r + 2$ with entries of 1 along the main diagonal and first superdiagonal, and all other entries zero. It will be convenient to write $J = I + N$, where I is the identity matrix. The matrix N is familiar as a nilpotent matrix whose j^{th} power has 1's on the j^{th} superdiagonal, and 0's elsewhere, for $0 \le j \le r + 1$. Accordingly, J^n may be computed as

$$(I + N)^n = \sum_{j=0}^{r+1} \binom{n}{j} N^j,$$

and we observe that this result has constants along each diagonal. Specifically, it is an upper-triangular matrix with 1's on the main diagonal, $\binom{n}{1}$'s on the super diagonal, $\binom{n}{2}$'s on the next diagonal, and so on.

The matrix S is also described in [7], and is given by

$$S = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{r+1}{0} & \binom{r+1}{1} & \binom{r+1}{2} & \cdots & 1 \end{bmatrix}$$

This matrix is a special case of a more general form

$$M(\lambda) = \left(\binom{i-1}{j-1}\lambda^{i-j}\right)_{ij}.$$

In fact, $M(\lambda)$ plays the role of S when the characteristic polynomial is $(t - \lambda)^{r+2}$, and the specific instance of S above is $M(1)$. There are several interesting properties of $M(\lambda)$ described in [6]. Of special interest here is that

$$M(\lambda)^{-1} = M(-\lambda),$$

and in particular,

$$S^{-1} = M(-1).$$

This shows that S^{-1} has the same form as S, but with a minus sign introduced before each entry of the odd numbered subdiagonals. Note that the square matrix which appears in (2) has exactly this form, but with one less row and column. Put another way, the matrix in (2) is the $(r + 1)$-dimensional principal submatrix of S^{-1}. For future reference, we shall denote this matrix by S^*.

Combining the results presented so far, we have

$$s_n^r = [1 \quad 0 \quad 0 \quad \cdots \quad 0]SJ^nS^{-1}[s_0^r \quad s_1^r \quad s_2^r \quad \cdots \quad s_{r+1}^r]^T. \tag{10}$$

This equation can be simplified by observing that premultiplication of a square matrix by the row $[1 \quad 0 \quad 0 \cdots 0]$ results in just the first row of the matrix. The first row of S is again $[1 \quad 0 \quad 0 \cdots 0]$ so that the product of the first three factors on the right side of (10) is simply the first row of J^n, or

$$\left[\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \cdots \binom{n}{r+1}\right].$$

Therefore, we may write

$$s_n^r = \left[\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \cdots \quad \binom{n}{r+1}\right] S^{-1}[s_0^r \quad s_1^r \quad s_2^r \quad \cdots \quad s_{r+1}^r]^T \tag{11}$$

This is similar to (2), and is interesting in its own right.

Next, to replace the initial terms of the sequence $\{s_n^r\}$ with initial terms of $\{n^r\}$, we observe that

$$\begin{bmatrix} s_0^r \\ s_1^r \\ s_2^r \\ \vdots \\ s_{r+1}^r \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 0^r \\ 1^r \\ 2^r \\ \vdots \\ (r+1)^r \end{bmatrix}. \tag{12}$$

When the right side of (12) is substituted in (11), the product $S^{-1}T$ appears, where T is the triangular matrix in (12). A straightforward computation reveals that $S^{-1}T$ may be expressed as the partitioned matrix

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S^* \end{array}\right].$$

Thus, the combination of (11) and (12) results in the partitioned matrix equation

$$s_n^r = \left[\binom{n}{0} \middle| \binom{n}{1} \quad \binom{n}{2} \quad \cdots \binom{n}{r+1}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S^* \end{array}\right] [0|1^r \quad 2^r \quad \cdots \quad (r+1)^r]^T. \tag{13}$$

Carrying out the partitioned multiplication completes the derivation of (2).

It is instructive to use (2) to derive a formula for s_n^r in a particular case. For example, with $r = 2$, we have

$$s_n^2 = \left[\binom{n}{1} \ \binom{n}{2} \ \binom{n}{3}\right]\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3}.$$

This gives s_n^2 in terms of binomial coefficients, and simplification produces the well-known equation

$$s_n^2 = \frac{n(n+1)(2n+1)}{6}.$$

The derivation of (2) generalizes immediately. Let p be a polynomial of degree r, and define $s_n = \sum_{k=0}^{n} p(k)$. All of the analysis through equation (13) remains valid when k^r is replaced by $p(k)$. This leads to the following analog of equation (13).

$$(14) \qquad s_n = \left[\binom{n}{0}\middle|\binom{n}{1} \ \binom{n}{2} \ \cdots \ \binom{n}{r+1}\right]\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathbf{S}^* \end{array}\right][p(0)\,|\,p(1) \quad p(2) \ \cdots \ p(r+1)]^{\mathrm{T}}$$

Carrying out the partitioned product now yields the identity

$$(15) \qquad s_n - p(0) = \left[\binom{n}{1} \ \binom{n}{2} \ \binom{n}{3} \ \cdots \ \binom{n}{r+1}\right]\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & & \\ \vdots & \vdots & \vdots & & \\ \pm 1 & \mp\binom{r}{1} & \pm\binom{r}{2} & \cdots & 1 \end{bmatrix}\begin{bmatrix} p(1) \\ p(2) \\ p(3) \\ \vdots \\ p(r+1) \end{bmatrix}.$$

This equation may be used for adding up the first n terms of the sequence $\{p(k)\}$ starting from $k = 1$ instead of $k = 0$.

An interesting class of examples involves summing the r^{th} powers of the first n integers equivalent to b modulo a. In these cases, the polynomial has the form $p(k) = (ak + b)^r$. With $a = 4$ and $b = -3$, for example, the left-hand side of (15) is the sum of the first n terms of the progression $1^r, 5^r, 9^r, \ldots$. For an even more specific example, let $r = 2$. Then (15) reduces to

$$1^2 + 5^2 + 9^2 + \cdots + (4n-3)^2 = \left[\binom{n}{1} \ \binom{n}{2} \ \binom{n}{3}\right]\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1^2 \\ 5^2 \\ 9^2 \end{bmatrix}$$

$$= \binom{n}{1} + 24\binom{n}{2} + 32\binom{n}{3}.$$

A review of the derivation of (2) and (15) reveals a natural division into two parts. In the first, culminating in equation (11), the analysis has general validity. Any difference equation for which the Jordan canonical form can be calculated can be subjected to a similar analysis, resulting in an analogous identity. The second part depends on the fact that the characteristic polynomial for the difference operator is a power of $t - 1$. Therefore, the final result (15) should not be expected to generalize in any obvious fashion to a larger class of difference equations. In the final section, another example is considered. As expected, a result analogous to (11) is obtained, but no analog for (2) appears.

4. Geometrically Weighted Power Sums

In [3], recursive procedures are presented for expressing formulas for the geometrically weighted power sum

$$s_n^r(x) = \sum_{k=0}^{n} k^r x^k.$$

This is a generalization of s_n^r in the sense that $s_n^r(1) = s_n^r$. The sequence $\{s_n^r(x)\}$ (indexed by n) can be analyzed by matrix difference equation methods. As a first step, we have the following simple generalizations of earlier material.

Definition 3: $\mathrm{D_x}$ is the operator $\mathrm{L} - x$, where x acts as a scalar multiplier. $\mathrm{N}_k(x)$ is the null space of D_x^k. Note that $\mathrm{N}_1(x) \subseteq \mathrm{N}_2(x) \subseteq \mathrm{N}_3(x) \ldots$.

Theorem 2: For any k, $\mathrm{N}_k(x)$ consists of the sequences $\{a_n\}$ defined as the termwise product of the exponential sequence x^n with a polynomial in n of degree less than k.

We omit a proof for this theorem; one can be obtained by modifying the proof of the earlier theorem in an obvious way. The main significance for the present discussion is as follows. Since

$$\mathrm{D}s_n^r(x) = (n+1)^r x^{n+1},$$

it must be annihilated by $\mathrm{D}_\mathrm{x}^{r+1}$. Therefore, $s_n^r(x)$ is a solution to the difference equation

$$(\mathrm{L} - x)^{r+1}(\mathrm{L} - 1)a_n = 0.$$

The characteristic polynomial for this equation is $(t - x)^{r+1}(t - 1)$. We represent the reduction of its companion matrix to Jordan form in the usual way as $\mathrm{C} = \mathrm{SJS}^{-1}$. Once again, the analysis of [7] is directly applicable. It tells us that J has one Jordan block of dimension $r + 1$ for the root x, and a 1×1 block for the simple root 1. The matrix S is closely related to $\mathrm{M}(x)$ defined above. In fact, the first $r + 1$ columns of S are identical to the corresponding columns of $\mathrm{M}(x)$, but the final column consists of all 1's. [Indeed, this final column is really the first column of M(1). In general, the matrix S is a combination of $\mathrm{M}(x)$'s for the various roots of the characteristic polynomial, with the number of columns for each x given by its multiplicity as a root.] With these definitions for J and S, and with $s_n^r(x)$ in place of s_n^r, we may calculate $s_n^r(x)$ using (11).

Unfortunately, there is a bit more work required to determine an explicit representation for the inverse of S in this example. For simplicity, shorten $\mathrm{M}(x)$ to M, and define E to be the difference S − M. Thus, E is given by

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix} [0 \quad 0 \quad \cdots \quad 0 \quad 1].$$

Let the column and row matrices in this factorization be called $\mathrm{E_c}$ and $\mathrm{E_r}$, respectively. Now we claim that

$$\mathrm{S}^{-1} = (\mathrm{I} - \mathrm{S}^{-1}\mathrm{E})\mathrm{M}^{-1}.$$

This can be verified by premultiplying by S. The product $S^{-1}E$ can be computed using the factorization of E as soon as $S^{-1}E_c$ is determined. Thus, the problem of inverting S is reduced to finding the inverse image of a single vector. This is not surprising: since S was obtained by making a rank 1 modification to M, it is reasonable to expect a corresponding rank 1 modification to link the inverse matrices.

Proceeding with this approach, $S^{-1}E_c$ is computed by solving the equation $S\mathbf{v} = E_c$ for $\mathbf{v}$. Again using $S = M + E_cE_r$, write the equation as

$$M\mathbf{v} + E_cE_r\mathbf{v} = E_c.$$

Since $E_r\mathbf{v}$ is a scalar, namely v, the last entry of $\mathbf{v}$, the equation may now be rearranged as

$$M\mathbf{v} = (1 - v)E_c.$$

This leads to

$$\mathbf{v} = (1 - v)M^{-1}E_c,$$

and by equating the final entries of the vectors on either side, to an equation for v. Once v is found, $\mathbf{v}$ simply requires the computation shown at the right side of the previous equation. Carrying out these steps produces

$$\mathbf{v} = \frac{1}{(1-x)^{r+1}}\begin{bmatrix} 1 \\ 1 - x \\ (1-x)^2 \\ \vdots \\ (1-x)^r \\ (1-x)^{r+1} - 1 \end{bmatrix}$$

With this result, it is now possible to express S^{-1} as $M^{-1} - \mathbf{v}E_rM^{-1}$. Let $\mathbf{w} = E_rM^{-1}$, which is simply the last row of M^{-1}. This gives $S^{-1} = M^{-1} - \mathbf{v}\mathbf{w}$.

At this point, the factors appearing at right in (11) cannot be simplified much further. As before, the first two factors yield the first row of S. However, this row has a 1 in the last position as well as the first, so multiplying by J^n results in the sum of first and last rows of that matrix. Meanwhile, J is a block diagonal matrix. The first block is $(r + 1)$-dimensional and of the form $xI + N$. Its powers are computed just as before, exploiting the properties of N. Specifically, the first row of the n^{th} power is

$$\left[\binom{n}{0}x^n \quad \binom{n}{1}x^{n-1} \quad \cdots \quad \binom{n}{r}x^{n-r}\right]$$

and contributes all but the last entry of the first row of J^n. The second block is just the scalar 1 at the end of the diagonal. It contributes the only nonzero entry in the last row of J^n. When the first and last row are added, the result is

$$\left[\binom{n}{0}x^n \quad \binom{n}{1}x^{n-1} \quad \cdots \quad \binom{n}{r}x^{n-r} \quad 1\right].$$

When all of the foregoing calculations and reductions are combined into a single equation, the result is

$$s_n^r(x) = \left[\binom{n}{0}x^n \quad \binom{n}{1}x^{n-1} \quad \cdots \quad \binom{n}{r}x^{n-r} \quad 1\right]S^{-1}\begin{bmatrix} s_1^r(x) \\ s_1^r(x) \\ s_2^r(x) \\ \vdots \\ s_{r+1}^r(x) \end{bmatrix} \tag{16}$$

where

$$S^{-1} = \begin{bmatrix} 1 & & & & \\ -x & 1 & & & \\ x^2 & -2x & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ (-x)^{r+1} & \binom{r+1}{1}(-x)^r & \binom{r+1}{2}(-x)^{r-1} & \cdots & 1 \end{bmatrix}$$

$$- \frac{1}{(1-x)^{r+1}} \begin{bmatrix} 1 \\ 1-x \\ \vdots \\ (1-x)^r \\ (1-x)^{r+1} - 1 \end{bmatrix} \begin{bmatrix} (-x)^{r+1} & \binom{r+1}{1}(-x)^r & \binom{r+1}{2}(-x)^{r-1} & \cdots & 1 \end{bmatrix}$$

This formulation is not as compact as (2) but is sufficiently orderly to permit convenient calculation for specific values of r and x. The following formulas were obtained by writing a short computer program to define and calculate the product of the last two matrices on the right side of (16), then running it with x set to 2 and r set to 1, 2, 3, 4, and 5.

$$s_n^1(2) = 2^n\left[-2 + 2\binom{n}{1}\right] + 2$$
$$s_n^2(2) = 2^n\left[6 - 2\binom{n}{1} + 4\binom{n}{2}\right] - 6$$
$$s_n^3(2) = 2^n\left[-26 + 14\binom{n}{1} + 24\binom{n}{3}\right] + 26$$
$$s_n^4(2) = 2^n\left[150 - 74\binom{n}{1} + 52\binom{n}{2} + 24\binom{n}{3} + 48\binom{n}{4}\right] - 150$$
$$s_n^5(2) = 2^n\left[-1082 + 542\binom{n}{1} - 240\binom{n}{2} + 300\binom{n}{3} + 250\binom{n}{4} + 240\binom{n}{5}\right] + 1082$$

These equations are similar to the ones derived by Gauthier ([3], eqs. 31), but express $s_n^r(2)$ in terms of binomial coefficients instead of as polynomials in n.

It is also feasible to use (16) symbolically for small values of r. As an example, we carry through the matrix multiplication for $r = 2$.

The algebra will be simplified if the factors of $(1 - x)$ appearing in the denominator of entries of S^{-1} are transferred to the corresponding entries of the first matrix factor. In pursuit of this goal, rewrite (16) in the form

$$s_n^2(x) = RS^{-1}C,$$

where R and C are the row and column vectors, respectively, appearing in (16). Next, define the diagonal matrix D with entries

$$(1-x)^{-3},\ (1-x)^{-2},\ (1-x)^{-1},\ \text{and}\ (1-x)^{-3}.$$

Then we may write (17) as

$$s_n^2(x) = (RD)(D^{-1}S^{-1})C.$$

Focusing separately on each factor in (18), observe that

$$RD = \begin{bmatrix} \binom{n}{0}\frac{x^n}{(1-x)^3} & \binom{n}{1}\frac{x^{n-1}}{(1-x)^2} & \binom{n}{2}\frac{x^{n-2}}{1-x} & \frac{1}{(1-x)^3} \end{bmatrix}$$
$$C^T = \begin{bmatrix} 0 & x & 4x^2 + x & 9x^3 + 4x^2 + x \end{bmatrix}$$

and

$$
D^{-1}S^{-1} = \begin{bmatrix} (1-x)^3 & & & \\ -x(1-x)^2 & (1-x)^2 & & \\ x^2(1-x) & -2x(1-x) & 1-x & \\ -x^3(1-x)^3 & 3x^2(1-x)^3 & -3x(1-x)^3 & (1-x)^3 \end{bmatrix}
$$

$$
- \begin{bmatrix} 1 \\ 1 \\ 1 \\ (1-x)^3 - 1 \end{bmatrix} [-x^3 \quad 3x^2 \quad -3x \quad 1].
$$

Expressing the right side of this equation as a single matrix produces

$$
D^{-1}S^{-1} = \begin{bmatrix} * & -3x^2 & 3x & -1 \\ * & -2x^2 - 2x + 1 & 3x & -1 \\ * & -x^2 - 2x & 2x + 1 & -1 \\ * & 3x^2 & -3x^2 & 1 \end{bmatrix}.
$$

The entries in the first column have not been explicitly presented because they have no effect on the final formula for $s^2(x)$; these entries are each multiplied by the zero in the first position of **C**. Indeed, multiplying this last expression by **C** now yields

$$
D^{-1}S^{-1}C = \begin{bmatrix} -x(x+1) \\ x^2(x-3) \\ -2x^3 \\ x(x+1) \end{bmatrix}.
$$

Finally, after multiplying by **RD**, the following formula is obtained:

$$
s_n^2(x) = x^{n+1}\left[\frac{-(x+1)}{(1-x)^3} + \binom{n}{1}\frac{x-3}{(1-x)^2} - \binom{n}{2}\frac{2}{1-x}\right] + \frac{x^2+x}{(1-x)^3}
$$

As before, this result is consistent with the analysis presented in [3].

5. Summary

In this paper, matrix methods have been used to derive closed form expressions for the solutions of difference equations. The general tool of analysis involves expressing a scalar difference equation of order k as a first-order vector equation, then using the Jordan canonical form to express powers of the system matrix, thus describing the solution to the equation. Two specific examples of the method have been presented, differing from previous work in that neither example features a diagonalizable system matrix. In the first example, an esthetically appealing formula for the sum $\sum_{k=0}^{n} k^r$ was derived. In the second example, the more general sum $\sum_{k=0}^{n} k^r x^k$ was analyzed. In each case, the results have been derived previously using other methods. However, the main point of the article has been to show that the methods of matrix algebra can be a powerful tool, and provide a distinct heuristic insight, for the study of difference equations.

References

1. Tom M. Apostol. *Calculus*, Vol. II, 2nd ed. Waltham, Mass.: Blaisdell, 1969.
2. M. C. Er. "Matrices of Fibonacci Numbers," *Fibonacci Quarterly* 22.2 (1984): 134-139.

3. N. Gauthier. "Derivation of a Formula for $\sum r^k x^r$." *Fibonacci Quarterly* *27.5* (1989):402-408.
4. H. W. Gould. *Sums of Powers of Integers*, volume Part I of *Number Theory Class Notes*. West Virginia University, 1974-1975.
5. Dan Kalman. "Generalized Fibonacci Numbers by Matrix Methods." *Fibonacci Quarterly* *20.1* (1982):73-76.
6. Dan Kalman. "Polynomial Translation Groups." *Math. Mag.* *56.1* (1983):23-25.
7. Dan Kalman. "The Generalized Vandermonde Matrix." *Math. Mag.* *57.1* (1984): 15-21.
8. Kenneth A. Miller. *Linear Difference Equations*. New York: Benjamin, 1968.
9. Evar D. Nering. *Linear Algebra and Matrix Theory*, 2nd ed. New York: Wiley, 1970.
10. Jeffrey Nunemacher & Robert M. Young. "On the Sum of Consecutive k^{th} Powers." *Math. Mag.* *60.4* (1987):237-238.
11. J. L. Paul. "On the Sum of the k^{th} Powers of the First n Integers." *Amer. Monthly* *78.3* (1971):271-272.
12. Henry J. Schultz. "The Sum of the k^{th} Powers of the First n Integers." *Amer. Math. Monthly* *87.6* (1980):478-481.
13. Barbara Turner. "Sums of Powers of Integers via the Binomial Theorem." *Math. Mag.* *53.2* (1980):92-96.

*****

ON THE FIBONACCI NUMBER OF AN $M \times N$ LATTICE

Konrad Engel
Wilhelm-Pieck-Universität, Sektion Mathematik, 2500 Rostock, German Democratic Republic
(Submitted March 1988)

1. Introduction

Let

$$Z_{m,n} := \{(i, j): 1 \le i \le m,\ 1 \le j \le n\},$$

$$\mathscr{A}_{m,n} := \{A \subseteq Z_{m,n}: \text{ there are no } (i_1, j_1), (i_2, j_2) \in A \text{ with } |i_1 - i_2| + |j_1 - j_2| = 1\}$$

and

$$\kappa_{m,n} := |\mathscr{A}_{m,n}|.$$

So $\kappa_{m,n}$ equals the number of independent (vertex) sets in the Hasse graph of a product of two chains with m resp. n elements, i.e., in the $m \times n$ lattice. Following Prodinger & Tichy [11], we call $\kappa_{m,n}$ the Fibonacci number of the $m \times n$ lattice.

In this paper we study the numbers $\kappa_{m,n}$ using linear algebraic techniques. We prove several inequalities for these numbers and show that

$$1.503 \le \lim_{n \to \infty} \kappa_{n,n}^{1/n^2} \le 1.514.$$

We conjecture that this limit equals 1.50304808... .

The problem of the determination of the number of independent sets in graphs goes back to Kaplansky [6] who determined in his well-known lemma the number of k-element independent sets in the $1 \times n$ lattice, i.e., in a path on n vertices. Burosch suggested to consider other graphs, and some results were obtained in [3].

Answering a question of Weber, the number of independent sets in the Hasse graph of the Boolean lattice was determined asymptotically by Korshunov & Saposhenko [9]. Prodinger & Tichy [11] and later together with Kirschenhofer [7], [8] considered that problem in particular for trees. They introduced the notion of the Fibonacci number of a graph for the number of independent sets in it because the case of paths yields the Fibonacci numbers. We will see that the numbers $\kappa_{m,n}$ preserve many properties of the classical Fibonacci numbers, i.e., the results do not hold only for $m = 1$ but for all positive integers m. The first results on the numbers $\kappa_{m,n}$ have been obtained by Weber [12]. Among other things he proved the inequality

$$1.45^{mn} < \kappa_{m,n} < 1.74^{mn} \text{ if } mn > 1,$$

the existence of

$$\lim_{n \to \infty} \kappa_{m,n}^{1/n} \quad \text{and} \quad \lim_{n \to \infty} \kappa_{n,n}^{1/n^2}$$

as well as the inequality

$$1.45 \le \lim_{n \to \infty} \kappa_{n,n}^{1/n^2} \le 1.554.$$

2. Inequalities and Eigenvalues

Let

$$\varphi_m := \{S \subseteq \{1, \ldots, m\}\colon \text{there are no } i, j \in S \text{ with } |i - j| = 1\},$$
$$\mathscr{A}_{m,n,S} := \{A \in \mathscr{A}_{m,n}\colon (i, n) \in A \text{ iff } i \in S\},$$
$$x_{m,n,S} := |\mathscr{A}_{m,n,S}|.$$

So $x_{m,n,S}$ counts those sets of $\mathscr{A}_{m,n}$ for which the elements in the top line (with second coordinate n) are fixed by S. Obviously, $|\varphi_m| = \kappa_{m,1}$. Briefly, we set $z_m := \kappa_{m,1}$.

Throughout this section we consider m to be fixed. To avoid too many indices, we omit the index m everywhere. Obviously,

(1) $\quad \kappa_n = x_{n+1,\phi}.$

Moreover,

(2) $\quad x_{n+1,S} = \sum_{\substack{T \in \varphi \\ T \cap S = \phi}} x_{n,T}$ for all $S \in \varphi$, $n = 1, 2, \ldots$.

Let $\mathbf{x}_n$ be the vector whose coordinates are the numbers $x_{n,S}$ $(S \in \varphi)$ and $A = (a_{S,T})_{S,T \in \varphi}$ that $z \times z$-matrix for which

$$a_{S,T} := \begin{cases} 1 \text{ if } S \cap T = \phi, \\ 0 \text{ otherwise.} \end{cases}$$

Because of (2), we have

(3) $\quad \mathbf{x}_{n+1} = A\mathbf{x}_n$, $n = 1, 2, \ldots$.

Let the vector $\mathbf{e}$ with coordinates e_S, $S \in \varphi$, be defined by

$$e_S := \begin{cases} 1 \text{ if } S = \phi, \\ 0 \text{ otherwise,} \end{cases}$$

and let, for an integer k, the vector $\mathbf{k}$ be composed only of k's. Then we have

(4) $\quad \mathbf{x}_1 = A\mathbf{e} = \mathbf{1},$

and because of (3),

(5) $\quad \mathbf{x}_n = A^n\mathbf{e}.$

Finally, if (,) denotes the inner product, then

(6) $\quad \kappa_n = x_{n+1,\phi} = (A^{n+1}\mathbf{e}, \mathbf{e}).$

In our proofs, we often use the fact that A is symmetric. In particular, we have, for all vectors $\mathbf{x}$, $\mathbf{y}$,

(7) $\quad (A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A\mathbf{y}).$

Theorem 1: For all positive integers k and ℓ,

(8) $\quad \kappa_{k+1}^2 \le \kappa_{2k-1}\kappa_{2\ell+1}.$

Proof: By (6), (7), and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\kappa_{k+1}^2 &= (A^{k+\ell+1}\mathbf{e}, \mathbf{e})^2 = (A^k\mathbf{e}, A^{\ell+1}\mathbf{e})^2 \le (A^k\mathbf{e}, A^k\mathbf{e})(A^{\ell+1}\mathbf{e}, A^{\ell+1}\mathbf{e}) \\ &= (A^{2k}\mathbf{e}, \mathbf{e})(A^{2\ell+2}\mathbf{e}, \mathbf{e}) = \kappa_{2k-1}\kappa_{2\ell+1}. \quad \square\end{aligned}$$

Corollary 2: $\frac{\kappa_3}{\kappa_1} \le \frac{\kappa_5}{\kappa_3} \le \frac{\kappa_7}{\kappa_5} \le \cdots$. □

Since A is symmetric, all eigenvalues of A are real numbers. Let λ be the largest eigenvalue of A.

Proposition 3: λ has multiplicity 1, to λ belongs an eigenvector $\mathbf{u}$ with coordinates $u_S > 0$ for all $S \in \varphi$, and $|\lambda| > |\mu|$ for all eigenvalues μ of A.

Proof: The column and row of A which correspond to the empty set ϕ contain only ones; hence, the matrix A is irreducible and A^2 is positive (see [4], p. 395). Now the statements in the proposition are direct consequences of two theorems of Frobenius (see [4], pp. 398, 422). □

Theorem 4: Let $\mathbf{u}$ be that eigenvector of A to the largest eigenvalue λ for which $u_S > 0$ for all $S \in \varphi$ and $\sum_{S \in \varphi} u_S^2 = 1$ holds. Then

$$x_{n,S} \sim u_\phi u_S \lambda^n \text{ as } n \to \infty.$$

Proof: We use standard techniques. Let U be the orthogonal matrix whose columns are normed, pairwise orthogonal eigenvectors of A and let D be the diagonal matrix of the corresponding eigenvalues. Then

$$U^T A U = D \quad \text{and} \quad A^n = U D^n U^T.$$

Consequently,

$$\mathbf{x}_n = A^n \mathbf{e} = U D^n U^T \mathbf{e} \text{ [note (5)]}.$$

Because of Proposition 3, the asymptotic behavior of the components of $\mathbf{x}_n$ is determined by the terms containing λ^n which yields the formula in the theorem. □

Noting (1) and Corollary 2, we derive immediately

Corollary 5:

(a) $\lim_{n \to \infty} \kappa^{1/n} = \lambda$, (b) $\lim_{n \to \infty} \frac{\kappa_{n+1}}{\kappa_n} = \lambda$,

(c) $\lim_{n \to \infty} \frac{\kappa_{n+2}}{\kappa_n} = \lambda^2$, (d) $\frac{\kappa_{2k+1}}{\kappa_{2k-1}} \le \lambda^2$ for all $k = 1, 2, \ldots$. □

Remarks:

(a) If $p(\mu) = \det(\mu E - A) = \mu^z + a_{z-1}\mu^{z-1} + \cdots + a_0$ is the characteristic polynomial of A, we have $a_{z-1} = -\text{trace } A = -1$ and (by induction)

$$|a_0| = |\det A| = 1.$$

From the Cayley-Hamilton relation, it follows that

$$\mathbf{x}_{n+z} = -a_{z-1}\mathbf{x}_{n+z-1} - \cdots - a_0 \mathbf{x}_n$$

and, in particular, the recursion

$$\kappa_{n+z} = -a_{z-1}\kappa_{n+z-1} - \cdots - a_0 \kappa_n.$$

(b) Corollary 5(b) contains in effect the crucial point of the well-known power method of *v.* Mises for the determination of the absolute maximal eigenvalue of a matrix.

Theorem 6: For all positive integers h, k, ℓ,

$$(\kappa_{h+2\ell-1}/\kappa_{2\ell-1})^{1/h} \le \lambda \le \kappa_k^{1/k}.$$

Proof: It is well known that the Rayleigh-Quotient does not exceed the largest eigenvalue. Hence, by (6) and (7),

$$\begin{aligned}\kappa_{h+2\ell-1}/\kappa_{2\ell-1} &= (A^{h+2\ell}\mathrm{e}, \mathrm{e})/(A^{2\ell}\mathrm{e}, \mathrm{e}) \\ &= (A^h(A^\ell \mathrm{e}), A^\ell \mathrm{e})/(A^\ell \mathrm{e}, A^\ell \mathrm{e}) \\ &\le \text{largest eigenvalue of } A^h = \lambda^h.\end{aligned}$$

This proves the left inequality.

To show the right inequality, we use a standard technique for the estimation of the largest eigenvalue of nonnegative matrices (see, e.g., [10], 11.14). Let $\mathbf{u}$ be the eigenvector of A to λ with $u_S > 0$ for all $S \in \varphi$ and with $u_\phi = 1$. Then $A\mathbf{u} = \lambda\mathbf{u}$ implies

$$\sum_{T\in\varphi} u_T = \lambda u_\phi = \lambda, \qquad \sum_{\substack{T\in\varphi \\ T\cap S=\phi}} u_T = \lambda u_S,\ S \in \varphi.$$

Hence, $1 \ge u_S$ for all $S \in \varphi$ and, consequently,

$$\mathbf{u} \le \mathbf{1}.$$

It follows [note (4)] that

$$\lambda^k \mathbf{u} = A^k \mathbf{u} \le A^k \mathbf{1} = A^{k+1}\mathrm{e},$$

which gives [note (6)]

$$\lambda^k = (\lambda^k \mathbf{u}, \mathrm{e}) \le (A^{k+1}\mathrm{e}, \mathrm{e}) = \kappa_k,$$

i.e., the right inequality. □

Corollary 7: For all positive integers ℓ, k with $k > 2\ell - 1$,

$$\kappa_k^{1/k} \le \kappa_{2\ell-1}^{1/(2\ell-1)}.$$

Proof: We choose $h := k - (2\ell - 1)$ in Theorem 6. Then

$$(\kappa_k/\kappa_{2\ell-1})^{1/h} \le \kappa_k^{1/k}$$

and, equivalently,

$$\kappa_k^k/\kappa_{2\ell-1}^k \le \kappa_k^h, \qquad \kappa_k^{2\ell-1} \le \kappa_{2\ell-1}^k. \ \square$$

3. Limits

Now we consider the dependence of m and introduce again everywhere the index m. We will study the sequence $\{\lambda_m^{1/m}\}$, where

$$\lambda_m = \text{largest eigenvalue of } A_m = \lim_{n\to\infty} \kappa_{m,n}^{1/n} \text{ [see Corollary 5(a)].}$$

In the following, we often use the obvious fact that

(9) $\qquad \kappa_{m,n} = \kappa_{n,m}$ for all n, m.

Proposition 8: For all integers ℓ, k with $k > 2\ell - 1$,

$$\lambda_k^{1/k} \le \lambda_{2\ell-1}^{1/(2\ell-1)}.$$

Proof: By Corollary 7 and (9), we have

$$\kappa_{k,m}^{1/k} \le \kappa_{2\ell-1,m}^{1/(2\ell-1)}$$

and further

$$(\kappa_{k,m}^{1/m})^{1/k} \le (\kappa_{2\ell-1,m}^{1/m})^{1/(2\ell-1)}.$$

Now, if m tends to infinity, we obtain

$$\lambda_k^{1/k} \le \lambda_{2\ell-1}^{1/(2\ell-1)}. \quad \square$$

Proposition 9: The limit $g := \lim_{m \to \infty} \lambda_m^{1/m}$ exists, and

$$\lambda_{2\ell}/\lambda_{2\ell-1} \le g \le \lambda_k^{1/k}$$

holds for all positive integers ℓ and k.

Proof: First we note that the existence of the limit is trivial if Conjectures 2 and 3 are true, because then the sequence $\{\lambda_m^{1/m}\}$ is monotoniously decreasing. Let

$$\gamma := \lim_{n \to \infty} \inf \lambda_m^{1/m} \quad \text{(note Proposition 8).}$$

Now choose $\varepsilon > 0$ and let M be a number that satisfies

$$\text{(10)} \qquad \lambda_M^{1/M} < \gamma + \varepsilon/2.$$

Because M is fixed, by Corollary 5(a) there is a number m_0 such that, for all $m > m_0$,

$$\text{(11)} \qquad (\kappa_{M,m}^{1/m})^{1/M} < \lambda_M^{1/M} + \varepsilon/2.$$

Finally, by Theorem 6,

$$\text{(12)} \qquad \lambda_m \le \kappa_{m,M}^{1/M} \text{ for all } m = 1, 2, \ldots .$$

From (10), (11), and (12), we derive, for all $m > m_0$,

$$\lambda_m^{1/m} \le (\kappa_{m,M}^{1/m})^{1/M} < \lambda_M^{1/M} + \varepsilon/2 < \gamma + \varepsilon.$$

Consequently,

$$g = \lim_{m \to \infty} \lambda_m^{1/m} = \gamma.$$

Last, but not least, again by Theorem 6 (with $h = 1$),

$$\begin{aligned}(\kappa_{2\ell,m}/\kappa_{2\ell-1,m})^{1/m} &= (\kappa_{m,2\ell}/\kappa_{m,2\ell-1})^{1/m} \le \lambda_m^{1/m} \le (\kappa_{m,k}^{1/k})^{1/m} \\ &= (\kappa_{k,m}^{1/m})^{1/k},\end{aligned}$$

and with $m \to \infty$, we obtain

$$\lambda_{2\ell}/\lambda_{2\ell-1} \le g \le \lambda_k^{1/k}. \quad \square$$

Theorem 10: The limit $\lim_{n \to \infty} \kappa_{n,n}^{1/n^2}$ exists, and it is equal to g. In particular,

$$\lambda_{2\ell}/\lambda_{2\ell-1} \le \lim_{n \to \infty} \kappa_{n,n}^{1/n^2} \le \lambda_k^{1/k}$$

for all positive integers k and ℓ.

Proof: By Theorem 6 (with $h = k = m = n$ and $\ell = 1$) and using the obvious fact that $\kappa_{n,n} \le \kappa_{n,n+1}$,

$$(\kappa_{n,n}/\kappa_{n,1})^{1/n} \le (\kappa_{n,n+1}/\kappa_{n,1})^{1/n} \le \lambda_n \le \kappa_{n,n}^{1/n}.$$

Hence,

$$\lambda_n^{1/n} \le \kappa_{n,n}^{1/n^2} \le (\kappa_{n,1}^{1/n})^{1/n} \lambda_n^{1/n}.$$

If $n \to \infty$, then the lower and upper bounds tend to g, by Corollary 5(a) and Proposition 9; hence, $\kappa_{n,n}^{1/n^2}$ also tends to g. The inequality in this theorem is a reformulation of the inequality in Proposition 9. $\square$

We note here that the existence of the limit in Theorem 10 was previously proved by Weber (see [12]).

To find bounds for $\lim_{n\to\infty} \kappa_{n,n}^{1/n^2}$, we used a computer (see Table 1).

TABLE 1

m	λ_m/λ_{m-1}	$\lambda_m^{1/m}$
2	1.49206604	1.55377397
3	1.50416737	1.53705928
4	1.50292823	1.52845453
5	1.50306010	1.52334155
6	1.50304676	1.51994015
7	1.50304821	1.51751544
8	1.50304807	1.51569943
9	1.50304808	1.51428849
10	1.50304808	1.51316067

Because of Theorem 10 and the numerical results, we have the following estimation.

Corollary 11: $1.50304808 \le \lim_{n\to\infty} \kappa_{n,n}^{1/n^2} \le 1.51316067$. □

Conjecture 1: For all positive integers m and ℓ,

$$\kappa_{m,\,2\ell+1}/\kappa_{m,\,2\ell} \ge \lambda_m.$$

If this conjecture is true, then it would follow, as above, that

$$\lim_{n\to\infty} \kappa_{n,n}^{1/n^2} \le \lambda_{2\ell+1}/\lambda_{2\ell};$$

hence (with $\ell = 4$),

$$\lim_{n\to\infty} \kappa_{n,n}^{1/n^2} = 1.50304808\ldots\ .$$

Let us note that, for numerical purposes, the bound $\lambda_m^{1/m}$ is weak, because $\lambda_m^{1/m}$ decreases slowly whereas the size of the matrix A_m increases exponentially with m (like the Fibonacci numbers). The following conjecture is stronger.

Conjecture 2: For all positive integers m and k,

$$\kappa_{m,\,2k}^2 \ge \kappa_{m,\,2k-2}\kappa_{m,\,2k+2}.$$

If this Conjecture is true, then, together with Theorem 1 and Corollary 5, it would follow (we omit again the index m)

$$\frac{\kappa_3}{\kappa_1} \le \frac{\kappa_5}{\kappa_3} \le \frac{\kappa_7}{\kappa_5} \le \cdots \le \lambda^2 \le \cdots \le \frac{\kappa_6}{\kappa_4} \le \frac{\kappa_4}{\kappa_2} \le \frac{\kappa_2}{\kappa_0}$$

and, further,

$$\frac{\kappa_2}{\kappa_1} \le \frac{\kappa_4}{\kappa_3} \le \frac{\kappa_6}{\kappa_5} \le \cdots \le \lambda \le \cdots \le \frac{\kappa_5}{\kappa_4} \le \frac{\kappa_3}{\kappa_2} \le \frac{\kappa_1}{\kappa_0}.$$

Conjecture 3: For all positive integers m and k,

$$(\kappa_{m,2k+1}/\kappa_{m,2k})^2 \le \kappa_{m,2k}/\kappa_{m,2k-2}.$$

If Conjectures 2 and 3 are true, then one can derive (again without index m)

$$(\kappa_{2k+1}/\kappa_{2k})^{2k} \le (\kappa_{2k}/\kappa_{2k-2})^k \le \frac{\kappa_{2k}}{\kappa_{2k-2}} \frac{\kappa_{2k-2}}{\kappa_{2k-4}} \cdots \frac{\kappa_2}{\kappa_0} = x_{2k},$$

i.e.,

$$\kappa_{2k+1}^{2k} \le \kappa_{2k}^{2k+1}, \quad \kappa_{2k+1}^{1/(2k+1)} \le \kappa_{2k}^{1/2k},$$

and, together with Corollary 7, this means that the sequence $\{\kappa_{m,n}^{1/n}\}$ decreases monotoniously in n. Finally, as in the proof of Proposition 8, one can conclude that $\{\lambda_m^{1/m}\}$ decreases monotoniously in m.

Because of the recursions

$$\kappa_{1,n+2} = \kappa_{1,n+1} + \kappa_{1,n} \quad \text{and} \quad \kappa_{2,n+2} = 2\kappa_{2,n+1} + \kappa_{2,n},$$

one can easily verify these conjectures for $m = 1, 2$ (see also [2]). Using a computer, we verified them also for the numbers $\kappa_{m,n}$ for which $3 \le m \le 10$ and $1 \le n \le 20$.

References

1. D. M. Cvetković, M. Doob, & H. Sachs. *Spectra of Graphs*. Berlin: VEB Deutscher Verlag der Wissenschaften, 1980.
2. T. P. Dence. "Ratios of Generalized Fibonacci Sequences." *Fibonacci Quarterly 25.2* (1987):137-143.
3. K. Engel. "Über zwei Lemmata von Kaplansky." *Rostock. Math. Kolloq. 9* (1978):5-26.
4. F. R. Gantmacher. *Matrizentheorie*. Berlin: VEB Deutscher Verlag der Wissenschaften, 1980.
5. G. Hopkins & W. Staton. "Some Identities Arising from the Fibonacci Numbers of Certain Graphs." *Fibonacci Quarterly 22.3* (1984):255-258.
6. I. Kaplansky. "Solution of the 'Problème des ménages.'" *Bull. Amer. Math. Soc. 49* (1943):784-785.
7. P. Kirschenhofer, H. Prodinger, & R. F. Tichy. "Fibonacci Numbers of Graphs II." *Fibonacci Quarterly 21.3* (1983):219-229.
8. P. Kirschenhofer, H. Prodinger, & R. F. Tichy. "Fibonacci Numbers and Their Applications." In *Fibonacci Numbers and Their Applications* (Proc. 1st Int. Conf., Patras, Greece, 1984). Dordrecht, Holland: D. Reidel, 1986, pp. 105-120.
9. A. D. Korshunov & A. A. Saposhenko. "On the Number of Codes with Distance *2T*." *Problemy Kibernet. 40* (1983):111-130.
10. L. Lovász. *Combinatorial Problems and Exercises*. Budapest: Akadémiai Kiadó, 1979.
11. H. Prodinger & R. F. Tichy. "Fibonacci Numbers of Graphs." *Fibonacci Quarterly 20.1* (1982):16-21.
12. K. Weber. "On the Number of Stable Sets in an $m \times n$ Lattice." *Rostock. Math. Kolloq. 34* (1988):28-36.

*****

ON FIBONACCI PRIMITIVE ROOTS

J. W. Sander
Universität Hannover, Welfengarten 1, 3000 Hannover 1, Fed. Rep. of Germany
(Submitted March 1988)

1. Introduction

In [6] D. Shanks introduced the concept of a Fibonacci Primitive Root (FPR) mod p, i.e., an integer g which is a primitive root mod p and satisfies the congruence $g^2 \equiv g + 1 \bmod p$. He proved some properties of FPR's, for instance: If for a prime p, $p \neq 5$, there is an FPR mod p, then $p \equiv \pm 1 \bmod 10$. He also made the following conjecture:

Let

$$F(x) = \text{card}\{p \leq x\colon p \in \mathbb{P},\ \exists_g\ g \text{ is FPR mod } p\},$$

and

$$\pi(x) = \text{card}\{p \leq x\colon p \in \mathbb{P}\}.$$

Conjecture: As $x \to \infty$,

$$\frac{F(x)}{\pi(x)} \sim C,$$

where $C = \dfrac{27}{38} \displaystyle\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p(p-1)}\right)$.

Note that

$$\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p(p-1)}\right) = 0.3739558136\ldots$$

is Artin's constant.

By a theorem of DeLeon [3] and deep-lying work of Göttsch [4] using methods of Hooley [5] on Artin's conjecture, we will prove the Conjecture above on the assumption of a certain Riemann hypothesis, namely,

Theorem: Let $\rho = (1 + \sqrt{5})/2$, ζ be a primitive $2n^{\text{th}}$ root of unity, where n is a positive integer, and C be defined as in the Conjecture. If the Riemann hypothesis holds for all fields $\mathbb{Q}(\sqrt[n]{\rho}, \zeta)$, then

$$\frac{F(x)}{\pi(x)} = C + O\left(\frac{\log \log x}{\log x}\right).$$

2. Preliminaries

Let (f_n) be the classical Fibonacci sequence, i.e.,

$$f_0 = 0,\ f_1 = 1,\ f_{n+2} = f_{n+1} + f_n \quad (n \geq 0).$$

An easy pigeon-hole principle argument yields the periodicity of (f_n) mod m for any integer $m > 1$. Let $\lambda(m)$ be the length of the smallest period mod m.

Lemma 1: ([3], Theorem 1) Let $p \neq 5$ be a prime. Then there exists an FPR mod p iff $p \equiv \pm 1 \bmod 10$ and $\lambda(p) = p - 1$.

The following lemma has been proved by Göttsch [4]. A rather obvious generalization which, however, is more accessible has been given by Antoniadis [1].

Lemma 2: ([4], Kor. 2.10; [1], Satz 2 and Kor. 4) Let

$$A(x) = \text{card}\{p \le x \colon p \equiv \pm 1 \bmod 10,\ \lambda(p) = p - 1\}.$$

Under the assumption made in the Theorem, we have

$$A(x) = C\frac{x}{\log x} + O\left(\frac{x \log\log x}{(\log x)^2}\right),$$

where C is defined in the Conjecture.

It should be remarked that, without assuming the Riemann hypothesis, the applied methods only give upper bounds for $A(x)$ (see [4]). These are useless with regard to the Conjecture.

3. Proof of the Theorem

Since there is an FPR mod 5, we have, by Lemma 1, for $x \ge 5$,

$$F(x) = 1 + A(x).$$

Applying Lemma 2, we get

$$F(x) = C\frac{x}{\log x} + O\left(\frac{x \log\log x}{(\log x)^2}\right).$$

By the Prime Number Theorem (see, e.g., [2]),

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

Thus,

$$F(x) = C\pi(x) + O\left(\frac{x \log\log x}{(\log x)^2}\right),$$

which implies the Theorem.

References

1. J. A. Antoniadis. "Über die Periodenlänge mod p einer Klasse rekursiver Folgen." *Arch. Math.* *42* (1984):242-252.
2. H. Davenport. *Multiplicative Number Theory*. 2nd ed. New York-Heidelberg-Berlin: Springer-Verlag, 1980.
3. M. J. DeLeon. "Fibonacci Primitive Roots and the Period of the Fibonacci Numbers modulo p." *Fibonacci Quarterly* *15* (1977):353-355.
4. G. Göttsch. "Über die mittlere Periodenlänge der Fibonacci-Folgen modulo p." Dissertation, Hannover, 1982.
5. Ch. Hooley. "On Artin's Conjecture." *J. Reine Angew. Math.* *225* (1967):209-220.
6. D. Shanks. "Fibonacci Primitive Roots." *Fibonacci Quarterly* *10* (1972):163-168.

*****

A RESULT ON 1-FACTORS RELATED TO FIBONACCI NUMBERS

Ivan Gutman
University of Kragujevac, P.O. Box 60, YU-34000 Kragujevac, Yugoslavia
Sven J. Cyvin
The University of Trondheim, N-7034 Trondheim-NTH, Norway
(Submitted March 1988)

1. Introduction

The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$. It is well known [3] that the "ladder" composed of n squares (Fig. 1) has F_{n+2} 1-factors.

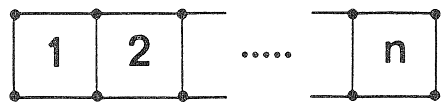

FIGURE 1

A 1-factor of a graph G with $2n$ vertices is a set of n independent edges of G, where independent means that two edges do not have a common endpoint. In the present paper, we investigate the number of 1-factors in a graph $Q_{p,q}$, composed of $p + q + 1$ squares, whose structure is depicted in Figure 2.

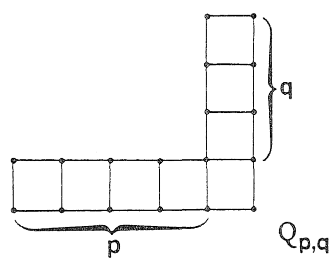

FIGURE 2

Throughout this paper, we assume that the number of squares in $Q_{p,q}$ is fixed and is equal to $n + 1$.

The number of 1-factors of a graph G is denoted by $K\{G\}$.

Lemma 1: $K\{Q_{p,q}\} = F_{n+2} + F_{p+1}F_{q+1}$ where $n = p + q$.

Before proceeding with the proof of Lemma 1 we recall an elementary property of the Fibonacci numbers, which is frequently employed in the present paper:

(1) $\quad F_m = F_k F_{m-k+1} + F_{k-1}F_{m-k}, \quad 1 \leq k \leq m.$

Proof: Let the edges of $Q_{p,q}$ be labeled as indicated in Figure 3.

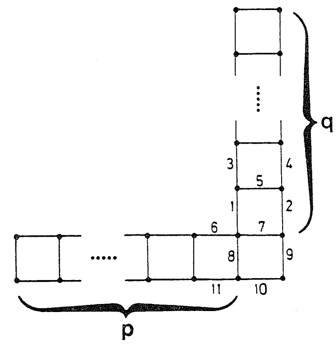

FIGURE 3

First observe that above and below the edges 1 and 2 there is an even number of vertices. Therefore, a 1-factor of $Q_{p,q}$ either contains both the edges 1 and 2 or none of them.

A 1-factor of $Q_{p,q}$ containing the edges 1 and 2 must not contain the edges 3, 4, ..., 9 because they have common endpoints with 1 and/or 2. Then, however, the edge 10 must and the edge 11 must not belong to this 1-factor. The remaining edges of $Q_{p,q}$ form two disconnected ladders with $p - 1$ and $q - 2$ squares, respectively, whose number of 1-factors is evidently $F_{p+1}F_q$. Therefore, there are $F_{p+1}F_q$ 1-factors of $Q_{p,q}$ containing the edges 1 and 2.

The edges of $Q_{p,q}$ without 1 and 2 form two disconnected ladders with $p + 1$ and $q - 1$ squares, respectively. Consequently, there are $F_{p+3}F_{q+1}$ 1-factors of $Q_{p,q}$ which do not contain the edges 1 and 2.

This gives

$$\begin{aligned} K\{Q_{p,q}\} &= F_{p+3}F_{q+1} + F_{p+1}F_q = F_{p+2}F_{q+1} + F_{p+1}F_{q+1} + F_{p+1}F_q \\ &= F_{p+q+2} + F_{p+1}F_{q+1}, \end{aligned}$$

where the identity (1) was used. Lemma 1 follows from the fact that $p + q = n$. □

2. Minimum and Maximum Values of $K\{Q_{p,q}\}$

Theorem 1: The minimum value of $K\{Q_{p,q}\}$, $p + q = n$, is achieved for $p = 1$ or $q = 1$.

Proof: Bearing in mind Lemma 1, it is sufficient to demonstrate that for $0 \le p \le n$,

$$F_2F_n \le F_{p+1}F_{n-p+1},$$

with equality if and only if $p = 1$ or $p = n - 1$.

Now, using (1),

$$\begin{aligned} F_{p+1}F_{n-p+1} &= F_{n+1} - F_pF_{n-p} = F_n + F_{n-1} - F_pF_{n-p} \\ &= F_n + F_pF_{n-p} + F_{p-1}F_{n-p-1} - F_pF_{n-p} \\ &= F_n + F_{p-1}F_{n-p-1} \ge F_n = F_2F_n. \end{aligned}$$

Because of $F_0 = 0$, equality in the above relation occurs if and only if $p - 1 = 0$ or $n - p - 1 = 0$. □

Theorem 2: The maximum value of $K\{Q_{p,q}\}$, $p + q = n$, is achieved for $p = 0$ or $q = 0$.

Proof:

$$F_{p+1}F_{n-p+1} = F_{n+1} - F_pF_{n-p} \leq F_{n+1} = F_1F_{n+1}$$

with equality if and only if $p = 0$ or $p = n$. Theorem 2 follows now from Lemma 1. □

Theorem 3: If $p \neq 0$, $q \neq 0$, then the maximum value of $K\{Q_{p,q}\}$, $p + q = n$, is achieved for $p = 2$ or $q = 2$.

Proof: From the proof of Theorem 1 we know that for $0 < p < n$,

$$F_2F_{n-2} \leq F_pF_{n-p}$$

with equality for $p = 2$ or $p = n - 2$. This inequality implies

$$F_{n+1} - F_2F_{n-2} \geq F_{n+1} - F_pF_{n-p},$$

i.e.,

$$F_3F_{n-1} + F_2F_{n-2} - F_2F_{n-2} \geq F_{p+1}F_{n-p+1} + F_pF_{n-p} - F_pF_{n-p},$$

i.e.,

$$F_3F_{n-1} \geq F_{p+1}F_{n-p+1},$$

from which Theorem 3 follows immediately. □

Theorem 4: If $p \neq 1$, $q \neq 1$, then the minimum value of $K\{Q_{p,q}\}$, $p + q = n$, is achieved for $p = 3$ or $q = 3$.

Proof: We start with the inequality

$$F_3F_{n-3} \geq F_pF_{n-p}$$

which was deduced within the proof of Theorem 3 and in a fully analogous manner obtain

$$F_4F_{n-2} \leq F_{p+1}F_{n-p+1}$$

with equality for $p + 1 = 4$ or $p + 1 = n - 2$. □

3. The Main Result

The reasoning employed to prove Theorems 3 and 4 can be further continued, leading ultimately to the main result of the present paper.

Theorem 5:

(a) If n is odd, then

$$K\{Q_{0,n}\} > K\{Q_{2,n-2}\} > K\{Q_{4,n-4}\} > \cdots > K\{Q_{n-3,3}\} > K\{Q_{n-1,1}\}.$$

(b) If n is even and divisible by four, then

$$K\{Q_{0,n}\} > K\{Q_{2,n-2}\} > \cdots > K\{Q_{n/2,\,n/2}\} > K\{Q_{n/2+1,\,n/2-1}\}$$
$$> K\{Q_{n/2+3,\,n/2-3}\} > \cdots > K\{Q_{n-3,3}\} > K\{Q_{n-1,1}\}.$$

(c) If n is even, but not divisible by four, then

$$K\{Q_{0,n}\} > K\{Q_{2,n-2}\} > \cdots > K\{Q_{n/2-1,\,n/2+1}\} > K\{Q_{n/2,\,n/2}\}$$
$$> K\{Q_{n/2+2,\,n/2-2}\} > \cdots > K\{Q_{n-3,3}\} > K\{Q_{n-1,1}\}.$$

All the above inequalities are strict.

4. Discussion and Applications

There seem to be many ways by which the present results can be extended and generalized. It is easy to see that if in the graph $Q_{p,q}$ some (or all) structural details of the type A and B are replaced by A^* and B^*, respectively (see Fig. 4), the number of 1-factors will remain the same. This means that our results hold also for chains of hexagons. In particular, it is long known

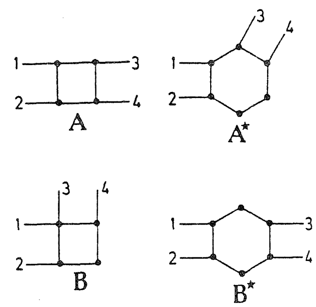

FIGURE 4

[2] that the zig-zag chain of n hexagons (Fig. 5) has F_{n+2} 1-factors. As a matter of fact, the number of 1-factors of chains of hexagons are of some importance in theoretical chemistry [1] and quite a few results connected with Fibonacci numbers have been obtained in this field (see [1] and the references cited therein).

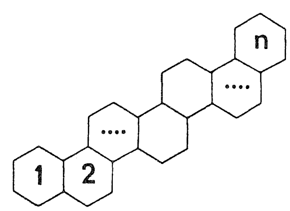

FIGURE 5

References

1. S. J. Cyvin & I. Gutman. *Kekulé Structures in Benzenoid Hydrocarbons*. Berlin: Springer-Verlag, 1988.
2. M. Gordon & W. H. T. Davison. "Theory of Resonance Topology of Fully Aromatic Hydrocarbons." *J. Chem. Phys.* *20* (1952):428-435.
3. L. Lovász. *Combinatorial Problems and Exercises*, pp. 32, 234. Amsterdam: North-Holland, 1979.

*****

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-658 *Proposed by Joseph J. Kostal, U. of Illinois at Chicago*

Prove that $Q_1^2 + Q_2^2 + \cdots + Q_n^2 \equiv P_n^2 \pmod 2$, where the P_n and Q_n are the Pell numbers defined by

$$P_{n+2} = 2P_{n+1} + P_n,\ P_0 = 0,\ P_1 = 1;$$

$$Q_{n+2} = 2Q_{n+1} + Q_n,\ Q_0 = 1,\ Q_1 = 1.$$

B-659 *Proposed by Richard Andre-Jeannin, Sfax, Tunisia*

For $n \geq 3$, what is the nearest integer to $F_n\sqrt{5}$?

B-660 *Proposed by Herta T. Freitag, Roanoke, VA*

Find closed forms for:

$$\text{(i)}\quad 2^{1-n}\sum_{i=0}^{[n/2]}\binom{n}{2i}5^i, \qquad \text{(ii)}\quad 2^{1-n}\sum_{i=1}^{[(n+1)/2]}\binom{n}{2i-1}5^{i-1},$$

where $[t]$ is the greatest integer in t.

B-661 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $T(n) = n(n + 1)/2$. In B-646, it was seen that $T(n)$ is an integral divisor of $T(2T(n))$ for all n in $Z^+ = \{1, 2, \ldots\}$. Find the n in Z^+ such that $T(n)$ is an integral divisor of

$$\sum_{i=1}^{n} T(2T(i)).$$

B-662 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let $H_n = L_n P_n$, where the L_n and P_n are the Lucas and Pell numbers, respectively. Prove the following congruences modulo 9:

(1) $H_{4n} \equiv 3n$, (2) $H_{4n+1} \equiv 3n + 1$,

(3) $H_{4n+2} \equiv 3n + 6$, (4) $H_{4n+3} \equiv 3n + 2$.

B-663 *Proposed by Clark Kimberling, U. of Evansville, Evansville, IN*

Let $t_1 = 1$, $t_2 = 2$, and $t_n = (3/2)t_{n-1} - t_{n-2}$ for $n = 3, 4, \ldots$. Determine lim sup t_n.

SOLUTIONS

When Is $2^n \equiv n \pmod 5$?

B-634 *Proposed by P. L. Mana, Albuquerque, NM*

For how many integers n with $1 \le n \le 10^6$ is $2^n \equiv n \pmod 5$?

Solution by Hans Kappus, Rodersdorf, Switzerland

More generally, we show that the number of solutions of

$$2^n \equiv n \pmod 5 \tag{*}$$

with $1 \le n \le 10^r$ is $2 \cdot 10^{r-1}$. In fact, it is easily checked that $(2^n - n)$ mod 5 is periodic with period $p = 20$ since $p = 20$ is the smallest number such that $2^n(2^p - 1) \equiv p \pmod 5$ for all $n \in N$. Now the only solutions of (*) with $1 \le n \le 20$ are n = 3, 14, 16, 17. Hence, the number of solutions of (*) in the interval $[1, 10^r]$ is $4 \cdot 10^r/20 = 2 \cdot 10^{r-1}$.

Also solved by R. Andre-Jeannin, Charles Ashbacher, Paul S. Bruckman, John Cannell, Nickolas D. Diamantis, Alberto Facchini, Piero Filipponi, Russell Jay Hendel, H. Klauser & M. Wachtel, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Sahib Singh, Lawrence Somer, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Application of the Inequality on the Means

B-635 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

For all positive integers n, prove that

$$2^{n+1}\left[1 + \sum_{k=1}^{n} (k!k)\right] < (n + 2)^{n+1}.$$

Solution by Bob Prielipp, U. of Wisconsin-Oshkosh

$$\sum_{k=1}^{n}(k!k) = \sum_{k=1}^{n}((k+1)! - k!) = (n+1)! - 1$$

Thus, the required inequality is equivalent to

$$(n+1)! < \left(\frac{n+2}{2}\right)^{n+1}.$$

This inequality follows immediately from the Arithmetic Mean-Geometric Mean Inequality, since

$$\frac{1 + 2 + \cdots + n + (n+1)}{n+1} = \frac{(n+1)(n+2)}{2(n+1)} = \frac{n+2}{2}.$$

Also solved by R. Andre-Jeannin, Charles Ashbacher, Paul S. Bruckman, J. E. Chance, Nicholas D. Diamantis, Russell Euler, Piero Filipponi, Hans Kappus, Y. H. Harris Kwong, Carl Libis, Alejandro Necochea, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Difference Equation

B-636 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

Solve the difference equation

$$x_{n+1} = (n+1)x_n + \lambda(n+1)^3[n!(n! - 1)]$$

for x_n in terms of λ, x_0, and n.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Divide the recurrence relation by $(n+1)!$:

$$\frac{x_{n+1}}{(n+1)!} = \frac{x_n}{n!} + \lambda[(n+1)(n+1)! - (n+1)^2].$$

Let $a_n = x_n/n!$. We then have

$$a_{n+1} = a_n + \lambda[(n+1)(n+1)! - (n+1)^2] \qquad \text{for } n \geq 0,$$

from which it follows immediately that

$$a_n = a_0 + \lambda\sum_{k=1}^{n}(k!k - k^2).$$

Since $\sum_{k=0}^{n} k!k = (n+1)! - 1$, $\sum_{k=0}^{n} k^2 = n(n+1)(2n+1)/6$, and $a_0 = x_0$, we obtain

$$x_n = n!\{x_0 + \lambda[(n+1)! - 1 - n(n+1)(2n+1)/6]\}.$$

Also solved by R. Andre-Jeannin, Paul S. Bruckman, Nicholas D. Diamantis, Guo-Gang Gao, Hans Kappus, L. Kuipers, H.-J. Seiffert, Amitabha Tripathi, and the proposer.

Golden Geometric Series

B-637 *Proposed by John Turner, U. of Waikato, Hamilton, New Zealand*

Show that

$$\sum_{n=1}^{\infty} \frac{1}{F_n + \alpha F_{n+1}} = 1,$$

where α is the golden mean $(1 + \sqrt{5})/2$.

Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA

By induction, $F_n + \alpha F_{n+1} = \alpha^{n+1}$. Thus, the given sum equals

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{n+1}}.$$

Since $1/\alpha < 1$, the sum of this geometric series is

$$\frac{1/\alpha^2}{1 - (1/\alpha)} = \frac{1}{\alpha(\alpha - 1)} = \frac{1}{1} = 1$$

Also solved by R. Andre-Jeannin, Paul S. Bruckman, John Cannell, J. E. Chance, Nickolas D. Diamantis, Russell Euler, Piero Filipponi, Herta T. Freitag, Guo-Gang Gao, Russell Jay Hendel, Hans Kappus, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Alejandro Necochea, Oxford Running Club (U. of Mississippi), Bob Prielipp, Elmer D. Robinson, H.-J. Seiffert, A. G. Shannon, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Summing Every Fourth Fibonacci Number

B-638 *Proposed by Herta T. Freitag, Roanoke, VA*

Find s and t as function of k and n such that

$$\sum_{i=1}^{k} F_{n-4k+4i-2} = F_s F_t.$$

Solution by Paul S. Bruckman, Edmonds, WA

$$\sum_{i=1}^{k} F_{n-4k+4i-2} = \sum_{i=0}^{k-1} F_{n-4i-2} = \frac{1}{5}\sum_{i=0}^{k-1}(L_{n-4i} - L_{n-4i-4})$$
$$= \frac{1}{5}(L_n - L_{n-4k}) = F_{2k}F_{n-2k}.$$

Hence, we may take $s = 2k$, $t = n - 2k$ (or $s = n - 2k$, $t = 2k$).

Also solved by R. Andre-Jeannin, Piero Filipponi, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Lucas Analogue

B-639 *Proposed by Herta T. Freitag, Roanoke, VA*

Find s and t as functions of k and n such that

$$\sum_{i=1}^{k} L_{n-4k+4i-2} = F_s L_t.$$

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

It is well known that $L_n = a^n + b^n$, where a and b are the zeros of $x^2 - x - 1$; so we can employ the same technique used in solving B-638. Alternately, using the result (from B-638)

$$\sum_{i=1}^{k} F_{n-4k+4i-2} = F_{2k}F_{n-2k},$$

and the fact that $L_n = F_{n+1} - F_{n-1}$, a solution follows immediately:

$$\begin{aligned}\sum_{i=1}^{k} L_{n-4k+4i-2} &= \sum_{i=1}^{k} F_{n+1-4k+4i-2} + \sum_{i=1}^{k} F_{n-1-4k+4i-2} \\ &= F_{2k}F_{n+1-2k} + F_{2k}F_{n-1-2k} \\ &= F_{2k}L_{n-2k}.\end{aligned}$$

Also solved by Paul S. Bruckman, R. Andre-Jeannin, Piero Filipponi, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

*****

****

LETTER TO THE EDITOR

Dear Professor Bergum,

On Tuesday 26th July, 1988, at the Pisa conference on Fibonacci Numbers and Their Applications, a specially convened meeting considered the problem of keeping up-to-date with new information relating to translations of Fibonacci's writings and to any matters concerning his life and works.

To some extent, the stimulus for that meeting was the recent appearance of L. E. Sigler's translation, with commentary, of Fibonacci's *Liber quadratorum* ("The Book of Squares").

Through the medium of "The Fibonacci Quarterly" we are appealing for any new details concerning Fibonacci's life and works—particularly translations of his works—which could be of interest to the international Fibonacci community.

Anyone possessing such knowledge could contact one of us. We would appreciate this assistance and cooperation very much.

Yours sincerely,

A. F. Horadam

A. F. Horadam
Department of Mathematics,
Statistics and Computing Science
University of New England
ARMIDALE N.S.W. 2351 AUSTRALIA

Josué Lahr

J. Lahr
14, Rue des Sept Arpents
L-1139 Luxembourg
GRAND DUCHY OF LUXEMBOURG

****

ADVANCED PROBLEMS AND SOLUTIONS

Edited by

Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-437 *Proposed by L. Kuipers, Sierre, Switzerland*

Let x, y, n be Natural numbers, where n is odd. If

$$L_n/L_{n+2} < x/y < L_{n+1}/L_{n+3}, \text{ show that } y > 1/5L_{n+4}. \qquad (*)$$

Are there fractions, x/y, satisfying (*) for which $y < L_{n+4}$?

H-438 *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \text{ for } n \geq 2.$$

Show that, for all odd integers $n \geq 3$,

$$\int_{-\infty}^{+\infty} \frac{dx}{F_n(x)} = \frac{\pi}{n}\left(1 + 1/\cos\frac{\pi}{n}\right).$$

H-439 *Proposed by Richard Andre-Jeannin, ENIS BP.W, Tunisia*

Let p be a prime number ($p \neq 2$) and m a Natural number. Show that

$$L_{2p^m} + L_{4p^m} + \cdots + L_{(p-1)p^m} \equiv 0 \pmod{p^{m+1}}.$$

SOLUTIONS

Some Difference

H-414 *Proposed by Larry Taylor, Rego Park, New York (Vol. 25, no. 3, August, 1987)*

Let j, k, m, and n be integers. Prove that

$$F_{m+j}F_{n+k} = F_{m+k}F_{n+j} - F_{k-j}F_{n-m}(-1)^{m+j}.$$

Solution by Tad White, UCLA, Los Angeles, CA

The proof is by induction on each of the four variables. Let $P(j, k, m, n)$ denote the above equality. It is trivial to verify this equality for $j, k, m, n \in \{0, 1\}$. We thus need only show that

(i) $P(j-2, k, m, n)$ and $P(j-1, k, m, n) \Rightarrow P(j, k, m, n)$, and

(ii) $P(j+2, k, m, n)$ and $P(j+1, k, m, n) \Rightarrow P(j, k, m, n)$,

and similarly for the other three variables. The proofs are essentially identical for each variable, so we will present only the induction on j here for brevity.

Notice that the equality $P(j, k, m, n)$ can be written in determinant form:

$$\begin{vmatrix} F_{m+k} & F_{m+j} \\ F_{n+k} & F_{n+j} \end{vmatrix} = F_{k-j}F_{n-m}(-1)^{m+j}.$$

Using the Fibonacci recursion relation, the determinant on the left can be rewritten as

$$\begin{vmatrix} F_{m+k} & F_{m+j-1} + F_{m+j-2} \\ F_{n+k} & F_{n+j-1} + F_{n+j-2} \end{vmatrix}.$$

By linearity of the determinant in the second column, this is

$$\begin{vmatrix} F_{m+k} & F_{m+j-1} \\ F_{n+k} & F_{n+j-1} \end{vmatrix} + \begin{vmatrix} F_{m+k} & F_{m+j-2} \\ F_{n+k} & F_{n+j-2} \end{vmatrix}$$

which, by the induction hypothesis, equals

$$\begin{aligned} &F_{k-j+1}F_{n-m}(-1)^{m+j-1} + F_{k-j+2}F_{n-m}(-1)^{m+j-2} \\ &= (F_{k-j+2} - F_{k-j+1})F_{n-m}(-1)^{m+j} \\ &= F_{k-j}F_{n-m}(-1)^{m+j}, \end{aligned}$$

as required. The induction in the negative direction is the same, except that one uses the Fibonacci recursion relation in the reverse direction.

Also solved by P. Bruckman, P. Filipponi, L. Kuipers, J. Mahon, F. Makri & D. Antzoulakos, B. Prielipp, H.-J. Seiffert, and the proposer.

A Little Reciprocity

H-416 *Proposed by Gregory Wulczyn, Bucknell U. (retired), Lewisburg, PA (Vol. 25, no. 4, November, 1987)*

(1) If $\left(\frac{p}{5}\right) = 1$, show that $\begin{cases} .5(L_{p-1} + F_{p-1}) \equiv 1 \pmod{p}, \\ .5(L_{p+1} - F_{p+1}) \equiv 1 \pmod{p}. \end{cases}$

(2) If $\left(\frac{p}{5}\right) = -1$, show that $\begin{cases} .5(L_{p-1} + F_{p-1}) \equiv -1 \pmod{p}, \\ .5(L_{p+1} - F_{p+1}) \equiv -1 \pmod{p}. \end{cases}$

Solution by Lawrence Somer, Washington, D.C.

It is well known that

$$F_{p-(5/p)} \equiv 0 \pmod{p} \quad \text{and} \quad F_p \equiv (5/p) \pmod{p}.$$

It is also known that

$$L_n = F_{n-1} + F_{n+1}.$$

It follows from the law of quadratic reciprocity that $(p/5) = (5/p)$ if p is a prime greater than 2.

(1) Suppose $(p/5) = 1$. Then $p \neq 2$. It follows that

$$F_{p-(5/p)} = F_{p-1} \equiv 0 \pmod{p}$$

and

$$F_p \equiv (5/p) \equiv 1 \pmod{p}.$$

Then

$$F_{p+1} = F_{p-1} + F_p \equiv 0 + 1 \equiv 1 \pmod{p}.$$

Thus,

$$L_{p-1} = F_{p-2} + F_p = (F_p - F_{p-1}) + F_p \equiv 1 - 0 + 1 \equiv 2 \pmod{p}$$

and

$$L_{p+1} = F_p + F_{p+2} = F_p + (F_p + F_{p+1}) \equiv 1 + 1 + 1 \equiv 3 \pmod{p}.$$

Hence,

$$.5(L_{p-1} + F_{p-1}) \equiv .5(2 + 0) \equiv 1 \pmod{p}$$

and

$$.5(L_{p+1} - F_{p+1}) \equiv .5(3 - 1) \equiv 1 \pmod{p}.$$

(2) Assume that $(p/5) = -1$. First suppose that $p = 2$. Then

$$L_{p-1} = L_1 = 1,\ L_{p+1} = L_3 = 4,\ F_{p-1} = F_1 = 1,\ F_{p+1} = F_3 = 2.$$

Then

$$.5(L_{p-1} + F_{p-1}) = .5(1 + 1) \equiv -1 \pmod{2}$$

and

$$.5(L_{p+1} - F_{p+1}) = .5(4 - 2) \equiv -1 \pmod{2}.$$

Now suppose that $p \neq 2$. It follows that

$$F_{p-(5/p)} = F_{p+1} \equiv 0 \pmod{p}$$

and

$$F_p \equiv (5/p) \equiv -1 \pmod{p}.$$

Then

$$F_{p-1} = F_{p+1} - F_p \equiv 0 - (-1) \equiv 1 \pmod{p}.$$

Hence,

$$L_{p-1} = F_{p-2} + F_p = (F_p - F_{p-1}) + F_p \equiv -1 - 1 - 1 \equiv -3 \pmod{p}$$

and

$$L_{p+1} = F_p + F_{p+2} = F_p + (F_p + F_{p+1}) \equiv -1 + (-1) + 0 \equiv -2 \pmod{p}.$$

Thus,

$$.5(L_{p-1} + F_{p-1}) \equiv .5(-3 + 1) \equiv -1 \pmod{p}$$

and

$$.5(L_{p+1} - F_{p+1}) \equiv .5(-2 - 0) \equiv -1 \pmod{p}.$$

Also solved by P. Bruckman, P. Filipponi, C. Georghiou, L. Kuipers, T. White, and the proposer.

A Mean Problem

H-417 *Proposed by Piero Filipponi, Rome, Italy*
(Vol. 25, no. 4, November, 1987)

Let $G(n, m)$ denote the geometric mean taken over m consecutive Fibonacci numbers of which the smallest is F_n. It can be readily proved that

$$G(n, 2k+1) \qquad (k = 1, 2, \ldots)$$

is not integral and is asymptotic to F_{n+k} (as n tends to infinity).

Show that if n is odd (even), then $G(n, 2k+1)$ is greater (smaller) than F_{n+k}, except for the case $k = 2$, where $G(n, 5) < F_{n+2}$ for every n.

Solution by Paul Bruckman, Edmonds, WA

$$(1) \qquad G(n, 2k+1) = \left(\prod_{j=0}^{2k} F_{n+j}\right)^{\frac{1}{2k+1}}.$$

Hence,

$$\begin{aligned}\log G(n, 2k+1) &= \frac{1}{2k+1}\sum_{j=0}^{2k} \log F_{n+j}\\ &= \frac{1}{2k+1}\sum_{j=0}^{2k} (\log a^{n+j} - \tfrac{1}{2}\log 5 + \log(1 - (b/a)^{n+j})\\ &= \frac{1}{2k+1}\sum_{j=0}^{2k} ((n+j)\log a - \tfrac{1}{2}\log 5 + \log(1 - x^{n+j})),\end{aligned}$$

where a and b are the usual Fibonacci constants and $x = b/a = -b^2$. (Note that $-1 < x < 0$.) Thus,

$$(2) \qquad \log G(n, 2k+1) = (n+k)\log a - \tfrac{1}{2}\log 5 + \frac{1}{2k+1}\sum_{j=0}^{2k}\log(1 - x^{n+j}).$$

Likewise,

$$(3) \qquad \log F_{n+k} = (n+k)\log a - \tfrac{1}{2}\log 5 + \log(1 - x^{n+k}).$$

We now make the definition:

$$(4) \qquad D(n, k) = \log\left(\frac{G(n, 2k+1)}{F_{n+k}}\right).$$

Thus, it suffices to prove that $D(n, k) > 0$ if n is odd and $D(n, k) < 0$ if n is even, unless $k = 2$, in which case $D(n, 2) < 0$ for all n.

Now, from (2) and (3), we have

$$(5) \qquad D(n, k) = \frac{1}{2k+1}\sum_{j=0}^{2k}\log(1 - x^{n+j}) - \log(1 - x^{n+k}).$$

Expanding into Maclaurin series, we obtain:

$$\begin{aligned}D(n, k) &= \frac{-1}{2k+1}\sum_{j=0}^{2k}\sum_{i=1}^{\infty} i^{-1}(x^{n+j})^i + \sum_{i=1}^{\infty} i^{-1}(x^{n+k})^i\\ &= \sum_{i=1}^{\infty} i^{-1}x^{ni}\left(x^{ki} - \frac{1}{2k+1}\left(\frac{1 - x^{(2k+1)i}}{1 - x^i}\right)\right),\end{aligned}$$

or, after some simplification,

$$(6) \qquad D(n,\ k) = \sum_{i=1}^{\infty} i^{-1} x^{(n+k)i}\left(1 - \frac{(-1)^{ki}}{2k+1} \cdot \frac{F_{(2k+1)i}}{F_i}\right).$$

We consider the various possibilities:

Case 1. k is even, $k \ge 2$. Then

$$D(n,\ k) = \sum_{i=1}^{\infty} \frac{x^{(n+k)i}}{i}\left(1 - \frac{F_{(2k+1)i}}{(2k+1)F_i}\right);$$

if, moreover, n is even, $(n + k)$ is even, and the last expression is clearly negative (the first term vanishing if $k = 2$). If n is odd, then

$$D(n,\ k) > b^{2n+2k}\left(\frac{F_{2k+1}}{2k+1} - 1\right) - \tfrac{1}{2}b^{4n+4k}\left(\frac{F_{4k+2}}{2k+1} - 1\right)$$

so

$$\begin{aligned} D(n,\ k) &> b^{2n+2k}\left(\frac{a^{2k+1}}{(2k+1)\sqrt{5}} - 1\right) - \tfrac{1}{2}b^{4n+4k}\left(\frac{a^{4k+2}}{((2k+1)\sqrt{5} - 1)}\right) \\ &\ge \frac{a^{-(2n-1)}}{2(2k+1)\sqrt{5}}(2 - a^{-(2n-1)}) - \tfrac{1}{2}b^{2n+2k}(2 - b^{2n+2k}) \\ &\ge \frac{a^{-(2n-1)}}{2(2k+1)\sqrt{5}}(2 - a^{-1}) - b^{2n+2k} \ge \frac{a^{-(2n-1)}\ a^{-1}}{2(2k+1)} - b^{2n+2k} \\ &= b^{2n}\left(\frac{1}{4k+2} - b^{2k}\right) > 0 \text{ if } k \ge 4 \text{ (since } a^{2k} > 4k + 2 \text{ if } k \ge 3). \end{aligned}$$

Thus far, we have shown that

$$(7) \qquad \begin{aligned} &D(n,\ k) < 0 \text{ if } k \text{ and } n \text{ are even;} \\ &D(n,\ k) > 0 \text{ if } k \ge 4 \text{ is even, } n \text{ is odd.} \end{aligned}$$

Also, if n is odd,

$$\begin{aligned} D(n,\ 2) &= \sum_{i=2}^{\infty} i^{-1}x^{(n+2)i}\left(1 - \frac{F_{5i}}{5F_i}\right) < -5x^{2n+4} - 20x^{3n+6} \\ &= -5x^{2n+4}(1 + 4x^{n+2}) \le -5x^{2n+4}(1 - 4b^6) < 0. \end{aligned}$$

Thus,

$$(8) \qquad D(n,\ 2) < 0 \text{ for all } n.$$

Case 2. k is odd, $k \ge 3$. Then

$$D(n,\ k) = \sum_{i=1}^{\infty} i^{-1}x^{(n+k)i}\left(1 - \frac{(-1)^i}{2k+1} \cdot \frac{F_{(2k+1)i}}{F_i}\right).$$

If n is odd,

$$D(n,\ k) > b^{2n+2k}\left(1 + \frac{a^{2k+1}}{(2k+1)\sqrt{5}}\right) - \tfrac{1}{2}b^{4(n+k)}\left(\frac{a^{4k+2}}{(2k+1)\sqrt{5}} - 1\right)$$

(continued)

$$\geq \tfrac{1}{2}b^{2(n+k)}(2 + b^{2(n+k)}) + \frac{a^{-(2n-1)}}{2(2k+1)\sqrt{5}}(2 - a^{-(2n-1)})$$

$$> b^{2(n+k)} + \frac{a^{-(2n-1)}}{2(2k+1)\ 5}(2 - a^{-1}) = b^{2n}\left(b^{2k} + \frac{1}{4k+2}\right) > 0.$$

If n is even,

$$D(n,\ k) = -b^{2(n+k)}\left(1 + \frac{F_{2k+1}}{2k+1}\right) - \tfrac{1}{2}b^{4(n+k)}\left(\frac{F_{4k+2}}{2k+1} - 1\right) - \cdots;$$

clearly, $D(n,\ k) < 0$ in this case. Therefore,

(9) $D(n,\ k) > 0$ if $k \geq 3$ and n are odd;
$D(n,\ k) < 0$ if $k \geq 3$ is odd, and n is even.

Combining (7), (8), and (9) yields the desired conclusion:

(10) $D(n,\ k) < 0$, if n is even, $k \neq 2$;
$D(n,\ k) > 0$, if n is odd, $k \neq 2$;
$D(n,\ 2) < 0$ for all n. Q.E.D.

Also solved by the proposer.

*****

SUSTAINING MEMBERS

M.H. Ahmadi
A.F. Alameddine
*A.L. Alder
G.L. Alexanderson
S. Ando
R. Andre-Jeannin
*J. Arkin
M.K. Azarian
L. Bankoff
F. Bell
M. Berg
J.G. Bergart
G. Bergum
G. Berzsenyi
*M. Bicknell-Johnson
P.S. Bruckman
M.F. Bryn
P.F. Byrd
G.D. Chakerian
J.W. Creely
P.A. DeCaux
M.J. DeLeon
J. Desmond
H. Diehl
T.H. Engel
D.R. Farmer
D.C. Fielder
P. Flanagan
F.F. Frey, Jr.
Emerson Frost
Anthony Gioia
R.M. Giuli
I.J. Good
*H.W. Gould
P. Hagis, Jr.
H. Harborth
H.E. Heatherly
A.P. Hillman
*A.F. Horadam
F.T. Howard
R.J. Howell
R.P. Kelisky
C.H. Kimberling
R.P. Kovach
J. Lahr
J.C. Lagarias
C.T. Long
Br. J.M. Mahon
*J. Maxwell
F.U. Mendizabal
L. Miller
M.G. Monzingo
S.D. Moore, Jr.
J.F. Morrison
K. Nagasaka
F.G. Ossiander
A. Prince
S. Rabinowitz
E.D. Robinson
S.E. Schloth
J.A. Schumaker
A.G. Shannon
D. Singmaster
J. Sjoberg
B. Sohmer
L. Somer
M.N.S. Swamy
*D. Thoro
J.C. Turner
T.P. Vaughan
R. Vogel
M. Waddill
J.E. Walton
G. Weekly
R.E. Whitney
B.E. Williams
A.C. Yanoviak

*Charter Members

INSTITUTIONAL MEMBERS

ACADIA UNIVERSITY LIBRARY
Wolfville, Nova Scotia

THE BAKER STORE EQUIPMENT COMPANY
Cleveland, Ohio

BUCKNELL UNIVERSITY
Lewisburg, Pennsylvania

CALIFORNIA STATE UNIVERSITY SACRAMENTO
Sacramento, California

ETH-BIBLIOTHEK
Zurich, Switzerland

FERNUNIVERSITAET HAGEN
Hagen, West Germany

HOWELL ENGINEERING COMPANY
Bryn Mawr, California

MATHEMATICS SOFTWARE COMPANY
Evansville, Indiana

NEW YORK PUBLIC LIBRARY GRAND CENTRAL STATION
New York, New York

PRINCETON UNIVERSITY
Princeton, New Jersey

SAN JOSE STATE UNIVERSITY
San Jose, California

SANTA CLARA UNIVERSITY
Santa Clara, California

KLEPCO, INC.
Sparks, Nevada

TECHNOLOGICAL EDUCATION INSTITUTE
Larissa, Greece

UNIVERSITY OF SALERNO
Salerno, Italy

UNIVERSITY OF CALIFORNIA, SANTA CRUZ
Santa Cruz, California

UNIVERSITY OF GEORGIA
Athens, Georgia

UNIVERSITY OF NEW ENGLAND
Armidale, N.S.W. Australia

UNIVERSITY OF NEW ORLEANS
New Orleans, Louisiana

WAKE FOREST UNIVERSITY
Winston-Salem, North Carolina

WASHINGTON STATE UNIVERSITY
Pullman, Washington

BOOKS AVAILABLE
THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

Applications of Fibonacci Numbers. Edited by A.N. Philippou, A.F. Horadam and G.E. Bergum.

Please write to the Fibonacci Association, Santa Clara University, Santa Clara CA 95053, U.S.A., for current prices.