The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

TABLE OF CONTENTS

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
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*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

TERMINATING DECIMALS IN THE CANTOR TERNARY SET

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(Submitted October 1987)

1. Introduction

The classical Cantor set is usually constructed by beginning with the interval [0, 1], deleting the middle third, and then continuing to delete the middle third of each interval remaining after the previous step. Another characterization is that the Cantor set consists of all numbers between 0 and 1 that can be written in base three using only 0 and 2 as digits. In this paper, we show that there are only 14 terminating decimals in the Cantor set, namely,

$$\frac{1}{4}, \frac{3}{4}, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}, \frac{1}{40}, \frac{3}{40}, \frac{9}{40}, \frac{13}{40}, \frac{27}{40}, \frac{31}{40}, \frac{37}{40}, \frac{39}{40}.$$

Clearly, we may restrict our attention to fractions N/M where $M = 2^a 5^b$ ($a \geq 0$, $b \geq 0$, $ab \neq 0$) and gcd $(N, M) = 1$. If N/M is a fraction in the Cantor set, then so is $1 - N/M$ and so is $3N/M$, provided $3N$ is reduced modulo M: the former is the 2's complement, and the latter is the fractional part after shifting the ternary point. In what follows, it will be convenient to write $M = \mu p$ and $\phi(M) = \gamma q$, where ϕ is Euler's function (the numbers μ and γ will be specified).

The claim above will be established by examining eight infinite classes of denominators and eight special cases. We will show that no fractions in the eight infinite classes are in the Cantor set; the eight special cases will yield the fourteen terminating decimal fractions listed above.

For j relatively prime to M, we will find it convenient to use the notation

$$[j] = \{j \cdot 3^k \pmod{M} : k = 0, 1, 2, \ldots\}.$$

If g is the smallest positive exponent for which $3^g \equiv 1 \pmod{M}$ and $(j, M) = 1$, then each set $[j]$ contains g elements, and there are $\phi(M)/g$ distinct sets $[j]$. Note that either all elements of $[j]$ are numerators of fractions in the Cantor set or none are, and that $[j]$ is eliminated if and only if $[-j]$ is.

Note that:

$$\begin{aligned} 3^{4k} &\equiv 1 \pmod{80} & 7^{4k} &\equiv 1 \pmod{80} \\ 3^{4k+1} &\equiv 3 \pmod{80} & 7^{4k+1} &\equiv 7 \pmod{80} \\ 3^{4k+2} &\equiv 9 \pmod{80} & 7^{4k+2} &\equiv 49 \pmod{80} \\ 3^{4k+3} &\equiv 27 \pmod{80} & 7^{4k+3} &\equiv 23 \pmod{80} \end{aligned}$$

Therefore,

Lemma 1: If $80|M$, then the sets $[j]$ are pairwise disjoint for $j = \pm 1, \pm 7, \pm 49$, and ± 343.

Reduction of the congruences yields

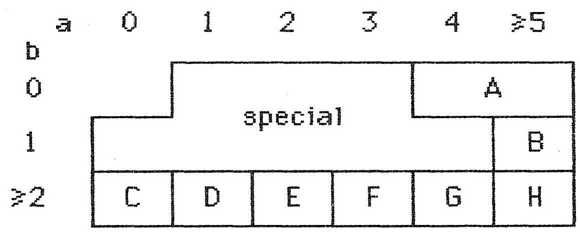
Lemma 2: If $40|M$, then the sets $[j]$ are pairwise disjoint for $j = \pm 1$ and ± 7.

Lemma 3: If $4|M$, then the sets $[j]$ are pairwise disjoint for $j = \pm 1$.

2. General Cases

In this section, we will examine eight infinite classes of denominators $M > 1$. For each class, we will describe the behavior of the numbers 3^k

(mod M). In each case, the congruence for 3^q may be proved by mathematical induction, and the others follow from it. No induction proofs will be presented because they are all easy (the hard part is spotting the patterns, not proving them). Then, we will show how each set $[j]$ with $(j, M) = 1$ contains an element N for which N/M is between 1/3 and 2/3, thus proving that the class contains no elements of the Cantor set. The scheme of proof is summarized by the following chart:



Class A: Suppose $M = 2^a$ with $a \geq 4$. Then $\phi(M) = 2^{a-1}$ and we write $M = 2p$ and $\phi(M) = 4q$. We observe that

$$3^q \equiv p + 1 \pmod{M} \qquad 3^{2q} \equiv 1 \pmod{M}$$

Then, by Lemma 3, the sets [1] and [-1] are disjoint, but [1] contains $p + 1$, which is obviously in the middle third. Details may be found in Reference 1, where it was proved that 1/4 and 3/4 are the only dyadic rationals in the Cantor set.

Class B: Suppose $M = 2^a 5$ with $a \geq 5$. We write $M = 2p$ and $\phi(M) = 2^{a+1} = 16q$. Then we may prove that:

$$3^q \equiv p + 1 \pmod{M} \qquad 3^{2q} \equiv 1 \pmod{M}$$

By Lemma 1, it suffices to examine the sets $[j]$ for $j = \pm 1, \pm 7, \pm 49$, and ± 343. Because $p + 1$ is in the middle third, sets [1] and [-1] do not qualify. Note that $7(p + 1) \equiv p + 7$, which is in the middle third, so [±7] is eliminated. Similarly, $49(p + 1) \equiv p + 49$ and $p + 49$ is in the middle third except for $p = 80$ ($M = 160$); but $243(49) \equiv 67 \pmod{160}$ and $67/160 = 0.41\ldots$, eliminating [±49]. Note that $343(p + 1) \equiv p + 343$ and $p + 343$ is in the middle third unless $p \leq 1029$, so [±343] is eliminated except possibly for $M = 160, 320, 640$, and 1280. But each of these possibilities includes an element of the middle third:

$M = 160$: $3 \cdot 343 \equiv 69 = 0.43\ldots M$
$M = 320$: $9 \cdot 343 \equiv 207 = 0.64\ldots M$
$M = 640$: $343 = 0.53\ldots M$
$M = 1280$: $9 \cdot 343 \equiv 527 = 0.41\ldots M$

Therefore, Class B is eliminated.

Class C: Suppose $M = 5^b$ with $b \geq 2$. We write $M = 5p$ and $\phi(M) = 4 \cdot 5^{b-1} = 10q$. Then:

$$3^q \equiv 2p - 1 \pmod{M} \qquad 3^{5q} \equiv -1 \pmod{M}$$
$$3^{2q} \equiv p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

But then the numbers 3^j for $0 < j \leq \phi(M)$ are distinct, so none of the numbers can be in the Cantor set, since $2p - 1$ obviously is not.

Class D: Suppose $M = 2 \cdot 5^b$ with $b \geq 2$. We write $M = 5p$ and $\phi(M) = 4 \cdot 5^{b-1} = 10q$. Then:

$$3^q \equiv p - 1 \pmod{M} \qquad 3^{5q} \equiv -1 \pmod{M}$$
$$3^{2q} \equiv 3p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

As in Class C, we cannot have all the numbers ($3p + 1$ in particular), so we have none of them.

Class E: Suppose $M = 2^2 5^b$ with $b \geq 2$. We write $M = 10p$ and $\phi(M) = 8 \cdot 5^{b-1} = 20q$. Then:

$$3^q \equiv p - 1 \pmod{M} \qquad 3^{5q} \equiv 5p - 1 \pmod{M}$$
$$3^{2q} \equiv 8p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

We have only the sets $[\pm 1]$ to check, but they are eliminated because $5p - 1$ is in the middle third.

Class F: Suppose $M = 2^3 5^b$ with $b \geq 2$. We write $M = 20p$ and $\phi(M) = 16 \cdot 5^{b-1} = 40q$. Then:

$$3^q \equiv p - 1 \pmod{M} \qquad 3^{5q} \equiv 5p - 1 \pmod{M}$$
$$3^{2q} \equiv 8p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

By Lemma 2, there are the four sets $[\pm 1]$ and $[\pm 7]$ to check. We quickly eliminate $[\pm 1]$ because $8p + 1$ is in the middle third. If $p > 21$, we eliminate $[\pm 7]$ because $7(p - 1)$ is in the middle third. If $p \leq 21$, then $p = 10$ and $M = 200$, but $3^5 7 \equiv 101 \pmod{200}$; thus, Class F yields no members of the Cantor set.

Class G: Suppose $M = 2^4 5^b$ with $b \geq 2$. We write $M = 80$ and $\phi(M) = 32 \cdot 5^{b-1} = 80q$. A "leapfrog" induction shows that

$$3^q \equiv 2p - 1 \pmod{M} \qquad 3^{5q} \equiv 50p - 1 \pmod{M}$$
$$3^{2q} \equiv 16p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

if b is even, while

$$3q \equiv 42p - 1 \pmod{M} \qquad 3^{5q} \equiv 10p - 1 \pmod{M}$$
$$3^{2q} \equiv 16p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

if b is odd. In any event, we have to examine the sets $[\pm 1]$, $[\pm 7]$, $[\pm 49]$, and $[\pm 343]$.

Suppose b is even. Because $50p - 1$ is in the middle third, we eliminate $[\pm 1]$. Note that $7(50p - 1) \equiv 30p - 7$, which is in the middle third, and that $49(50p - 1) \equiv 50p - 49$, which is also in the middle third, eliminating $[\pm 7]$ and $[\pm 49]$. Now, $343(2p - 1) \equiv 46p - 343$, which is in the middle third except when $p = 5$ and $M = 400$. Coupling this with the fact that $3(343) \equiv 229 \pmod{400}$, we eliminate $[\pm 343]$ and, therefore, all of Class G.

Class H: Suppose $M = 2^a 5^b$ with $a \geq 5$ and $b \geq 2$. We write $M = 10p$ and $\phi(M) = 2^{a+1} 5^{b-1} = 80q$. Then double induction shows that:

$$3^q \equiv p + 1 \pmod{M} \qquad 3^{5q} \equiv 5p + 1 \pmod{M}$$
$$3^{2q} \equiv 2p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

Once again, we must examine $[\pm 1]$, $[\pm 7]$, $[\pm 49]$, and $[\pm 343]$. But $5p + 1$ is in the middle third, eliminating $[\pm 1]$. Also, $7(2p + 1) \equiv 4p + 7$ and $49(5p + 1) \equiv 5p + 49$, so we may eliminate $[\pm 7]$ and $[\pm 49]$. Because $343(5p + 1) \equiv 5p + 343$, we may eliminate $[\pm 343]$ except possibly for $p = 80$ ($M = 800$) and $p = 160$ ($M = 1600$). But $343/800 = 0.42\ldots$ and $3(343)/1600 = 0.64\ldots$, so the exceptional cases present no problem.

3. Special Cases

Classes A through H yield no terminating decimals in the Cantor set, so the only possible denominators are 2, 4, 5, 8, 10, 20, 40, and 80. If $M = 2$, the only fraction possible is $1/2$, which is clearly in the middle third. For the other choices of M, we will simply list (in the order obtained) the elements of the sets $[j]$; an asterisk denotes a member of the middle third:

$$
\begin{array}{lrl}
M = 4 & [1] &= \{1,\ 3\} \\
M = 5 & [1] &= \{1,\ 3^*,\ 4,\ 2^*\} \\
M = 8 & [1] &= \{2,\ 3^*\} \\
 & [-1] &= \{7,\ 5^*\} \\
M = 10 & [1] &= \{1,\ 3,\ 9,\ 7\} \\
M = 20 & [1] &= \{1,\ 3,\ 9^*,\ 7^*\} \\
 & [-1] &= \{19,\ 17,\ 11^*,\ 13^*\} \\
M = 40 & [1] &= \{1,\ 3,\ 9,\ 27\} \\
 & [-1] &= \{39,\ 37,\ 31,\ 13\} \\
 & [7] &= \{7,\ 21^*,\ 23^*,\ 29\} \\
 & [-7] &= \{33,\ 19^*,\ 17^*,\ 11\} \\
M = 80 & [1] &= \{1,\ 3,\ 9,\ 27^*\} \\
 & [-1] &= \{79,\ 77,\ 71,\ 53^*\} \\
 & [7] &= \{7,\ 21,\ 63,\ 29^*\} \\
 & [-7] &= \{73,\ 59,\ 17,\ 51^*\} \\
 & [49] &= \{49^*,\ 67,\ 41^*,\ 43^*\} \\
 & [-49] &= \{31^*,\ 13,\ 39^*,\ 37^*\} \\
 & [343] &= \{23,\ 69,\ 47^*,\ 61\} \\
 & [-343] &= \{57,\ 11,\ 33^*,\ 19\}
\end{array}
$$

Thus, the terminating decimals in the Cantor set are precisely those claimed earlier.

Reference

1. C. R. Wall. Solution to Problem H-339. *Fibonacci Quarterly* 21.3 (1983):239.

*****

NOTE ON THE RESISTANCE THROUGH A STATIC CARRY LOOK-AHEAD GATE

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(Submitted March 1988)

In this paper, I show that a problem arising in hardware design has a solution that is the ratio of consecutive Fibonacci numbers.

One of the problems in VLSI designs of adders is to minimize the amount of time needed for addition [1]. A straightforward way of adding is to have a separate adder cell for each bit of the operands. The function to be performed by each one-bit adder cell is to take inputs A_i and B_i and a carry bit C_{i-1} from the previous stage, and compute

$$\begin{aligned}\text{SUM}_i &= A_iB_iC_{i-1} + A_i\overline{B}_i\overline{C}_{i-1} + \overline{A}_i\overline{B}_iC_{i-1} + \overline{A}_iB_i\overline{C}_{i-1}\\ &= A_i \oplus B_i \oplus C_{i-1}\end{aligned}$$

and

$$C_i = A_iB_i + A_iC_{i-1} + B_iC_{i-1},$$

where SUM_i is the i^{th} bit of the sum and C_i becomes the carry input to the next stage. Unfortunately, this scheme means that the i^{th} adder cannot compute its result until the $(i-1)^{\text{th}}$ adder has propagated its carry to it.

One way to get around this problem is to look ahead to compute the carry bit to be propagated to each stage. The idea is that each adder can make a quick decision whether to propagate or generate a carry by using the formulas:

$$\text{GEN} = A_iB_i \quad \text{and} \quad \text{PROP} = A_i \oplus B_i.$$

A carry from the previous stage will be propagated if either A_i or B_i is true, and one will be generated at this stage, regardless of the previous carry value, if both A_i and B_i are true. The pull-down transistor part of a 4-stage static carry look-ahead gate as it might be implemented in CMOS or nMOS is shown in Figure 1, where the output is the negation of the fourth carry bit value, the inputs on the left are the zeroth carry bit and the first four PROP values, and the inputs on the right are the first four GEN values.

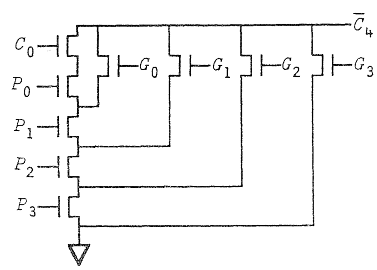

FIGURE 1. 4-stage static carry look-ahead gate

The circuit works by setting things so that the output $\overline{C}_4$ will be high (true) unless there is a path between it and ground. The overbar indicates a negated signal, that is, one which is true when it is at ground and false when it is at the power supply voltage. The transistors can be viewed as switches which allow current to flow if their inputs are high (true). In this circuit, there will be a path to ground if G_3 is true, which means that the fourth stage would generate a carry. If there is no carry generated in the fourth stage, the output can still be pulled low (true) if a carry was propagated through the fourth stage (P_3 is true) and a carry was somehow passed through the third stage. This analysis proceeds recursively, so that if, for example, all the generate bits were false, a carry would only be generated if all the propagate bits were true and the initial C_0 carry was true.

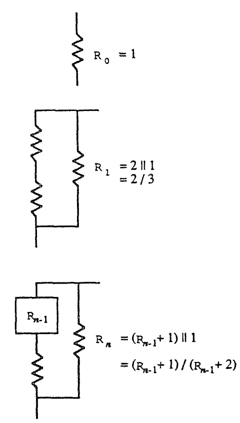

FIGURE 2. Source of the recurrence relation for resistance

In order to compute the delay through this circuit, it is necessary to compute the resistance and capacitance between ground and the output. This note concentrates on the resistance. The approximation made in computing resistance in this paper is that each transistor with a high input is in the conducting state and represents a unit of resistance. A generalized n-stage resistance network for this circuit has a very regular structure, as shown in Figure 2. A "zero-stage" look-ahead gate would comprise but a single resistor and thus have a resistance of one. A one-stage gate has a series of two resistors in parallel with a third; the composite resistance is computed by using the parallel resistance formula:

$$a \| b = \frac{ab}{a + b}.$$

In this case, $a = 2$, since resistors in series sum, and $b = 1$. Thus, $R_1 = 2/3$, and we get a general recurrence relation for R_n:

$$R_0 = 1,$$
$$R_n = \frac{R_{n-1} + 1}{R_{n-1} + 2}.$$

We can attack this recurrence by splitting R_n into its numerator and denominator:

$$R_n = \frac{N_n}{D_n} = \frac{N_{n-1}/D_{n-1} + 1}{N_{n-1}/D_{n-1} + 2} = \frac{N_{n-1} + D_{n-1}}{N_{n-1} + 2D_{n-1}}.$$

So we have a double recurrence:

$$N_0 = 1$$

$$N_n = N_{n-1} + D_{n-1}, \qquad n \geq 1$$

$$D_0 = 1$$

$$D_n = N_{n-1} + 2D_{n-1}. \qquad n \geq 1$$

So far, we have only demonstrated this as a formal solution because the fraction N_n/D_n may not be in lowest terms. The lemma below demonstrates that this is the actual lowest-term solution.

Lemma: N_n and D_n are relatively prime.

Proof: This is a proof by induction. This base case is easy:

$$\gcd(N_0, D_0) = \gcd(1, 1) = 1.$$

Assume that $\gcd(D_{n-1}, N_{n-1}) = 1$. We use a result by Euclid that if $n > m$, then $\gcd(n, m) = \gcd(m, n - m)$ (see [2]). Thus,

$$\begin{aligned}\gcd(D_n, N_n) &= \gcd(N_{n-1} + 2D_{n-1}, N_{n-1} + D_{n-1})\\ &= \gcd(N_{n-1} + D_{n-1}, D_{n-1})\\ &= \gcd(D_{n-1}, N_{n-1})\\ &= 1. \quad \square\end{aligned}$$

We can create ordinary generating functions $N(z)$ and $D(z)$ to find the closed-form solutions for the series. If we define $N_n = D_n = 0$ for $n < 0$ (the ratio R_n will thus be undefined in those cases), then we have formulas for them which are valid for all n:

$$N_n = N_{n-1} + D_{n-1} + \delta_{n0}$$

$$D_n = N_{n-1} + 2D_{n-1} + \delta_{n0}.$$

Multiplying both sides of these equations by z^n and summing over all n gives us the ordinary generating functions:

(1) $\quad N(z) = zN(z) + zD(z) + 1$

(2) $\quad D(z) = zN(z) + 2zD(z) + 1.$

Subtracting (2) - (1) and leaving off the (z)'s for clarity,

$$D - N = zD,$$

or

(3) $\quad N = D(1 - z).$

Plugging this back into (2) gives

$$D = \frac{1}{1 - 3z + z^2}.$$

Hence, by (3),

$$N = \frac{1 - z}{1 - 3z + z^2}.$$

We can get a closed-form expression for N_n from the generating function by factoring the denominator $(1 - 3z + z^2)$ into $(1 - az)(1 - bz)$ and expanding in terms of partial fractions. Using the quadratic formula, we get that

$$a = \frac{3 + \sqrt{5}}{2}, \quad b = \frac{3 - \sqrt{5}}{2}.$$

Here we make the observation that, if we let

$$\phi = \frac{1 + \sqrt{5}}{2}, \quad \hat{\phi} = \frac{1 - \sqrt{5}}{2},$$

then

$$a = \phi^2, \qquad b = \hat{\phi}^2.$$

We can also note that

(4) $\phi^2 - 1 = \phi, \quad \hat{\phi}^2 - 1 = \hat{\phi}$

and

(5) $\phi^2 - \hat{\phi}^2 = \sqrt{5}.$

Therefore, to expand the partial fraction

$$\frac{1 - z}{(1 - \phi^2 z)(1 - \hat{\phi}^2 z)} = \frac{\alpha}{1 - \phi^2 z} + \frac{\beta}{1 - \hat{\phi}^2 z},$$

we can find α by multiplying by $(1 - \phi^2 z)$ and setting z to $1/\phi^2$:

$$\alpha = \frac{\phi}{\sqrt{5}},$$

using identities (4) and (5).

Similarly,

$$\beta = -\frac{\hat{\phi}}{\sqrt{5}}.$$

This gives us a closed form for N_n:

$$N = \sum_n N_n z^n = \alpha \sum_n (\phi^2 z)^n + \beta \sum_n (\hat{\phi}^2 z)^n$$

by substituting the series for the partial fraction form. Equating coefficients of z^n:

$$N_n = \frac{1}{\sqrt{5}}(\phi\phi^{2n} - \hat{\phi}\hat{\phi}^{2n}) = \frac{1}{\sqrt{5}}(\phi^{2n+1} - \hat{\phi}^{2n+1}).$$

We can get D_n from N_n:

$$D_n = N_{n+1} - N_n = \frac{1}{\sqrt{5}}(\phi^{2n+3} - \hat{\phi}^{2n+1}) - \frac{1}{\sqrt{5}}(\phi^{2n+3} - \hat{\phi}^{2n+1})$$
$$= \frac{1}{\sqrt{5}}(\phi^{2n+2} - \hat{\phi}^{2n+2}) = F_{2n+2},$$

where F_i is the i^{th} Fibonacci number [2]. It seems there should have been an easier way to find the solution. We can rewrite the joint recurrences slightly to yield

$$N_0 = 1$$
$$N_n = N_{n-1} + D_{n-1}, \qquad n \geq 1$$
$$D_0 = 1$$
$$D_n = N_n + D_{n-1}. \qquad n \geq 1$$

Therefore, we can build the following table:

n	N_n	D_n	R_n
0	1	1	1.000000
1	2	3	0.666667
2	5	8	0.625000
3	13	21	0.619048
4	34	55	0.618182
5	89	144	0.618056

In other words, we have the Fibonacci numbers alternating between the N_n's and the D_n's. Thus,

$$R_n = \frac{F_{2n+1}}{F_{2n+2}} = \frac{\phi^{2n+1} - \hat{\phi}^{2n+1}}{\phi^{2n+2} - \hat{\phi}^{2n+2}}.$$

It is also possible to compute the asymptotic resistance, since as $n \to \infty$, $\hat{\phi}^n \to 0$ but $\phi^n \to \infty$. This gives

$$R_\infty = \frac{1}{\phi} = \frac{\sqrt{5} - 1}{2}.$$

The convergence, it can be seen, is quite rapid.

A similar result for the resistance through a ladder network was obtained by Basin [3] and independently by Manuel & Santiago [4]. The resistance of their circuit was also a ratio of consecutive Fibonacci numbers, but with the larger number in the numerator:

$$R_n = \frac{F_{2n}}{F_{2n-1}}.$$

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4. George Manuel & Amalia Santiago. "An Unexpected Appearance of the Golden Ratio." *College Math. J. 19.2* (1988):168-170.

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DIVISIBILITY PROPERTIES OF THE FIBONACCI NUMBERS MINUS ONE, GENERALIZED TO $C_n = C_{n-1} + C_{n-2} + k$

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1. Introduction

The numbers $\{C_n(a, b, k)\}$, defined by

$$C_n(a, b, k) = C_{n-1}(a, b, k) + C_{n-2}(a, b, k) + k,$$

with $C_1(a, b, k) = a$, $C_2(a, b, k) = b$, where k is a constant, have been studied in [1]. The Fibonacci sequence arises as the special case $F_n = C_n(1, 1, 0)$, while the Lucas sequence is $L_n = C_n(1, 3, 0)$. The sequence

$$\{C_n\} = \{\ldots, 0, 0, 1, 2, 4, 7, 12, 20, \ldots\},$$

where $C_n = C_n(0, 0, 1)$, has the property that $C_n = F_n - 1$, the sequence of Fibonacci numbers minus one.

The sequence $\{C_n\}$ has remarkable divisibility properties since almost every term is a composite number and at least one factor can always be named by examining the subscript of C_n. Further, $\{C_n\}$ contains exactly two prime terms, and two-thirds of its terms are even numbers. Analogous properties extend to the generalized sequence $\{C_n(a, b, k)\}$.

2. Prime Factors of C_n

First, since F_{3m} gives all the even Fibonacci numbers, C_{3m} is always odd, and $C_{3m\pm 1}$ is always even, so the probability of choosing an even term from $\{C_n\}$ at random is 2/3. Since $C_n = F_n - 1$, we can use [2] to prove some theorems in one step.

Theorem 1: For primes of the form $p = 5k \pm 2$, p divides both C_{p-1} and C_{2p+1}.

Proof: We have $F_p \equiv -1 \pmod p$ and $F_{p+1} \equiv 0 \pmod p$ from [2]. Then

$$C_{p-1} = F_{p-1} - 1 = F_{p+1} - (F_p + 1)$$

while

$$C_{2p+1} = F_{2p+1} - 1 = (F_{p+1})^2 + (F_p + 1)F_p - (F_p + 1),$$

where all terms on the right-hand side are divisible by p in both cases.

Theorem 2: For primes of the form $p = 5k \pm 1$, p divides C_p, C_{p+1}, C_{p-2}, C_{2p-1}, C_{2p}, and C_{2p-3}.

Proof: We have $F_p \equiv 1 \pmod p$ and $F_{p-1} \equiv 0 \pmod p$ from [2]. We write C_p, C_{p+1}, and C_{p-2} in forms in which p divides the terms on the right-hand side:

$$C_p = (F_p - 1),$$

$$C_{p+1} = F_{p+1} - 1 = F_{p-1} + (F_p - 1),$$

$$C_{p-2} = F_{p-2} - 1 = (F_p - 1) - F_{p-1}.$$

Since

$$C_{p+n-1} = F_{p+n-1} - 1 = F_p(F_n - 1) + F_{p-1}F_{n-1} + (F_p - 1),$$

where $p|F_{p-1}F_{n-1}$ and $p|(F_p - 1)$ but p does not divide F_p, observe that whenever $p|(F_n - 1)$, then $p|C_{p+n-1}$. Let $n = p$, $p + 1$, and $p - 2$ to write that

$$p|C_{2p-1},\ p|C_{2p},\ \text{and } p|C_{2p-3}.$$

Further, a little rewriting lets us prove the following corollary.

Corollary: If $p|C_n$, then $p|C_{n+m(p-1)}$, $m = 0, \pm 1, \pm 2, \ldots$, where p is a prime of the form $5k \pm 1$.

Proof: From the proof of Theorem 2, if $p|C_n$, then $p|C_{n+(p-1)}$. The corollary holds by the Axiom of Mathematical Induction, since whenever $p|C_{n+m(p-1)}$, then

$$p|C_{[n+m(p-1)]+(p-1)} = C_{n+(m+1)(p-1)}.$$

Theorem 3: If $\Pi(p)$ is the period of a prime p in the Fibonacci sequence modulo p, then

$$p|C_{k\Pi(p)-1},\quad p|C_{k\Pi(p)+1},\quad \text{and } p|C_{k\Pi(p)+2}.$$

Proof: Since

$$C_{k\Pi(p)+n} - C_n = F_{k\Pi(p)+n} - F_n,$$

and since p divides the right-hand side by definition of $\Pi(p)$, if $p|C_n$, then $p|C_{k\Pi(p)+n}$. Theorem 3 follows because $C_{-1} = C_1 = C_2 = 0$.

Corollary: The prime 5 divides C_{20k-1}, C_{20k+1}, C_{20k+2}, and C_{20k+8}.

Proof: $\Pi(5) = 20$, and 5 divides C_{-1}, C_1, C_2, and C_8.

Theorem 4: If p is a prime of the form $5k \pm 2$, then $p|C_{q(p+1)-2}$ if q is odd. If q is even, $p|C_{q(p+1)-1}$, $p|C_{q(p+1)+1}$, and $p|C_{q(p+1)+2}$.

Proof: If $p|C_n$, then $p|C_{n+m\Pi(p)}$ as in the proof of Theorem 3. From [3], if p is a prime of the form $5k \pm 2$, then $\Pi(p)|2(p + 1)$. Then, $p|C_{n+2m(p+1)}$, m any integer. Since

$$p|C_{p-1},\ p|C_{p-1+2m(p+1)} = C_{(2m+1)p+(2m-1)},$$

or, for q odd,

$$p|C_{qp+(q-2)} = C_{q(p+1)-2}.$$

If q is even, let $q(p + 1) = k\Pi(p)$ for some k, since $\Pi(p)|2(p + 1)$, and use Theorem 3.

Corollary: If $p = 5k \pm 2$, then

(i) p divides $C_{(p+2)(p-1)}$, $C_{p(p+3)}$, and $C_{p^s(p+1)-2}$;

(ii) p divides $C_{p(p+2)}$, C_{p^2-2}, C_{p^2}, and C_{p^2+1}.

Proof: (i) Take q odd, $q = p$, $q = p + 2$, and $q = p^s$, in Theorem 4. To show (ii), take q even, $q = p + 1$, $q = p - 1$.

Theorem 5: If p is a prime of the form $5k \pm 1$, then

$$p|C_{(m+1)p-(m+2)},\quad p|C_{(m+1)p-(m-1)},\quad \text{and } p|C_{(m+1)p-m} \text{ for any integer } m.$$

Proof: From the Corollary to Theorem 2, if $p|C_n$, then $p|C_{n+m(p-1)}$. From Theorem 2, take $n = p - 2$, $p + 1$, and $n = p$, and simplify.

Corollary: For any prime p, $p \neq 5$, $p|C_{p^2}$, $p|C_{p^2+1}$, and $p|C_{p^2-2}$.

Proof: If $p = 5k \pm 1$, let $m = p$ in Theorem 5. If $p = 5k \pm 2$, use the Corollary to Theorem 4.

Theorem 6: If $\Pi(j)$ is the period of any integer j, $j \neq 0$, in the Fibonacci sequence modulo j, then, for all integers k,

$$j|C_{k\Pi(j)-1}, \quad j|C_{k\Pi(j)+1}, \quad \text{and } j|C_{k\Pi(j)+2}.$$

Proof: See the proof of Theorem 3. Notice that any integer will eventually divide C_n for some n.

3. Fibonacci and Lucas Factors of C_n

Since $C_{m+n} - C_{m-n} = F_{m+n} - F_{m-n}$, we can write

$$\text{(3.1)} \qquad \begin{aligned} C_{m+n} - C_{m-n} &= F_m L_n, \quad \text{if } n \text{ is odd,} \\ C_{m+n} - C_{m-n} &= L_m F_n, \quad \text{if } n \text{ is even.} \end{aligned}$$

Observe that, if $L_n|C_{m-n}$, then $L_n|C_{m+n}$, and L_n has period $2n$ if n is odd. Similarly, F_n has period $2n$ if n is even. Putting these together with Theorem 6, we write

Theorem 7: If n is odd, L_n divides C_{2rn-1}, C_{2rn+1}, and C_{2rn+2}, while if n is even, F_n divides C_{2rn-1}, C_{2rn+1}, and C_{2rn+2} for any integer r.

Now things are getting exciting. Since we can take $n = 2k + 1$ to find that L_{2k+1} divides C_{4k+1}, C_{4k+3}, and C_{4k+4}, and $n = 2k$ to see that F_{2k} divides C_{4k-1}, C_{4k+1}, and C_{4k+2}, notice that C_n is always divisible either by L_{2k+1} or by F_{2k}. Now, if $k = 1$, $F_2 = 1$ divides any integer, so take $|k| \geq 2$. Thus, if $n \geq 7$, or if $n \leq -5$, then C_n always has at least one factor smaller than C_n and greater than 1 which we can write exactly, so C_n is not prime. We examine the sequence from C_{-4} through C_6: -4, 1, -2, 0, -1, 0, 0, 1, 2, 4, 7, and find that the only primes are 2 and 7.

Theorem 8: The sequence of Fibonacci numbers minus one, $C_n = F_n - 1$, contains only composite numbers for all $n \geq 7$ and all $n \leq -5$. The only primes which appear in $\{C_n\}$ are $C_4 = 2$, $C_6 = 7$, and $|C_{-2}| = 2$.

4. Divisibility of the Generalized Sequence $\{C_n(a, b, k)\}$

From [1], the sequence $\{C_n(a, b, k)\}$ with initial values $C_1 = a$ and $C_2 = b$ is given by

$$\text{(4.1)} \qquad \begin{aligned} C_n(a, b, k) &= C_{n-1}(a, b, k) + C_{n-2}(a, b, k) + k \\ &= aF_{n-2} + bF_{n-1} + kC_n(0, 0, 1) \\ &= H_n + kC_n \end{aligned}$$

for the generalized Fibonacci numbers H_n, $H_n = C_n(a, b, 0)$, and $C_n(0, 0, 1) = C_n$ of the earlier section.

As in Section 3,

$$C_{m+n}(a, b, k) - C_{m-n}(a, b, k) = (H_{m+n} - H_{m-n}) + k(C_{m+n} - C_{m-n}),$$

so that we can write

(4.2) $$C_{m+n}(a, b, k) - C_{m-n}(a, b, k) = L_n H_m + kF_m L_n, \quad \text{if } n \text{ is odd;}$$
$$C_{m+n}(a, b, k) - C_{m-n}(a, b, k) = F_n(H_{m+1} + H_{m-1}) + kL_m F_n, \text{ if } n \text{ is even.}$$

Thus, the periods of F_n and L_n are still $2n$, where we again distinguish n even and n odd. Also, since every nonzero integer eventually divides F_k for some k, every nonzero integer will divide $C_n(a, b, k)$ for some n if $\{C_n(a, b, k)\}$ contains a zero term. If $\{C_n(a, b, k)\}$ contains two zero terms, in some cases we will again have a finite number of primes occurring.

Theorem 9: If $C_q(a, b, k) = 0$, and if a nonzero integer j has period $\Pi(j)$ in the Fibonacci sequence, then $j \mid C_{q+m\Pi(j)}(a, b, k)$ for all integers m.

Theorem 10: If $F_{2m} \mid C_q(a, b, k)$, then

$$F_{2m} \mid C_{q+4m}(a, b, k),$$

and if $L_{2m+1} \mid C_q(a, b, k)$, then

$$L_{2m+1} \mid C_{q+4m+2}(a, b, k),$$

for any integer m.

Now, Theorem 10 gives us some interesting special cases. Notice that if $C_q(a, b, k) = 0$, and if $C_{q+r}(a, b, k) = 0$, where r is an odd number, then $\{C_n(a, b, k)\}$ will contain a finite number of primes, because for n larger than certain beginning values, $C_n(a, b, k)$ will always be divisible either by F_{2m} or L_{2m+1}, where $F_{2m} \neq 0, \pm 1$, and $L_{2m+1} \neq \pm 1$.

Without loss of generality, if $\{C_n(a, b, k)\}$ has a zero term, renumber the terms, taking new starting values, so that

$$a = 0 = C_1(0, b, k).$$

Then, if $C_{r+1}(0, b, k) = 0$ for some $r > 0$, from (4.1),

$$C_{r+1}(0, b, k) = 0 \cdot F_{r-1} + bF_r + kC_{r+1} = 0,$$

where we list some possibilities and special cases. Notice that $k = F_r$ and $b = -C_{r+1} = -F_{r+1} + 1$ always is a solution, and write the resulting

$$C_n(a, b, k) = C_n(0, -C_{r+1}, F_r).$$

For $r = 1$, we have $C_n(0, 0, 1) = C_n$; for $r = 2$, $C_n(0, -1, 1) = C_{n-2}$; and $r = 3$ gives $C_n(0, -2, 2) = 2C_{n-2}$, all the sequence of Fibonacci numbers minus one.

Consider $r = 4$ and $\{C_n(0, -4, 3)\} = \{\ldots, 0, -4, 1, -2, 0, 1, 4, 8, 15, 26, 44, 73, 120, \ldots\}$. We can show that

$$C_n(0, -4, 3) = -4F_{n-1} + 3C_n = L_{n-3} - 3.$$

From [2], we have $L_{2p} \equiv 3 \pmod p$ where p is any prime, so $p \mid L_{2p} - 3$, and we have

$$p \mid C_{2p+3}(0, -4, 3).$$

All odd-subscripted $C_n(0, -4, 3)$ have F_m or L_m for a divisor for some m, but we cannot easily say whether or not $\{C_n(0, -4, 3)\}$ contains a finite number of primes. However, any prime terms will have a subscript of the form $6m$. If r is even, we cannot determine whether or not $\{C_n(0, b, k)\}$ will contain a finite number of prime terms.

However, for $r = 5$, $\{C_n(0, -7, 5)\}$ contains only two primes, 2 and 7. We write $C_n(0, -7, 5)$ for $-3 \le n \le 10$: $-24, 7, -12, 0, -7, -2, -4, -1, 0, 4, 9, 18, 32$. We observe $|C_1| = 2$ and $|C_3| = 7 = C_{-2}$. From Theorem 10,

$$L_{2k+1}|C_{1+4k+2},\ L_{2k+1}|C_{6+4k+2},\ F_{2k}|C_{1+4k},\ \text{and}\ F_{2k}|C_{6+4k},$$

covering every possible subscript, so that $C_n(0, -7, 5)$ always has F_{2k} or L_{2k+1} for a divisor. But $F_{2k} = \pm 1$ for $k = \pm 1$, and $L_{2k+1} = \pm 1$ for $k = 0$ and $k = -1$. So terms $C_n(0, -7, 5)$ for $n > 10$ or $n < -3$ have a divisor greater than 1 and less than $C_n(0, -7, 5)$ and thus are not prime. For $r = 7$, in a similar fashion, we find only the three primes 7, 73, and 79 in $\{C_n(0, -20, 13)\}$. If $r = 9$, all the terms of $\{C_n(0, -54, 34)\}$ are even, but, if we instead consider $\{C_n(0, -27, 17)\}$, we find

$$|C_5| = 13 = C_{11},\ |C_8| = 11,\ \text{and}\ C_{14} = 107$$

as the only primes. Finally, $r = 11$ has only two primes

$$|C_5| = 73 \text{ and } |C_8| = 79,$$

but $r = 13$ is the best of all, containing no primes at all!

From the preceding discussion, we can write the following theorem.

Theorem 11: If $\{C_n(a, b, k)\}$ has $C_1(a, b, k) = 0$ and $C_{1+r}(a, b, k) = 0$ for r an odd integer, then $|C_n(a, b, k)|$ is prime for only a finite number of values for n.

Now, recall from above that the probability of choosing an even term from $\{C_n\} = \{C_n(0, 0, 1)\}$ is 2/3. $\{C_n(a, b, k)\}$ has the same property only when k is odd, and when at least one of a or b is even. These results can be verified by examining $C_n(a, b, k)$ from (4.1) for $n = 3m$, $3m + 1$, and $3m + 2$, where we always take k odd.

(i) $C_{3m}(a, b, k) = aF_{3m-2} + bF_{3m-1} + kC_{3m}$.

Note that kC_{3m}, F_{3m-1}, and F_{3m-2} are all odd. Then, if a and b have the same parity, $C_{3m}(a, b, k)$ is odd, while if a and b have opposite parity, $C_{3m}(a, b, k)$ is even.

(ii) $C_{3m+1}(a, b, k) = aF_{3m-1} + bF_{3m} + kC_{3m+1}$.

Here both bF_{3m} and kC_{3m+1} are always even, while F_{m-1} is odd, so $C_{3m+1}(a, b, k)$ is even or odd as a is even or odd.

(iii) $C_{3m+2}(a, b, k) = aF_{3m} + bF_{3m+1} + kC_{3m+2}$.

Now, aF_{3m} and kC_{3m+2} are always even, while F_{3m+1} is odd, so $C_{3m+2}(a, b, k)$ is even or odd as b is even or odd.

Putting the three cases together, first notice that, if all of a, b, and k are odd, $C_n(a, b, k)$ is always odd. If a and b are both even, $C_{3m}(a, b, k)$ is odd but $C_{3m+1}(a, b, k)$ and $C_{3m+2}(a, b, k)$ are both even. If a and b have opposite parity, $C_{3m}(a, b, k)$ is even, and either $C_{3m+1}(a, b, k)$ or $C_{3m+2}(a, b, k)$ is even, but not both. Then, if k is odd, and at least one of a or b is even, the probability that a term chosen at random from $\{C_n(a, b, k)\}$ will be even is 2/3.

Next, re-examine the three cases for k even. If a, b, and k are all even, $C_n(a, b, k)$ is always even, a trivial result. In (i), kC_{3m} is even, while F_{3m-2} and F_{3m-1} are odd, so that $C_{3m}(a, b, k)$ is odd if a and b have opposite parity, but even if a and b have the same parity. From (ii), both bF_{3m} and kC_{3m+1} are even, while F_{3m-1} is odd, so $C_{3m+1}(a, b, k)$ is even or odd as a is even or odd. From (iii), both aF_{3m} and kC_{3m+2} are even, while F_{3m+1} is odd, so $C_{3m+2}(a, b, k)$ is even or odd as b is even or odd. Putting these results together, if k is even, and a and b have opposite parity, then $C_{3m}(a, b, k)$ is odd while exactly one of $C_{3m+1}(a, b, k)$ or $C_{3m+2}(a, b, k)$ is odd. If k is even and both a and b are odd, $C_{3m}(a, b, k)$ is even but both $C_{3m+1}(a, b, k)$ and $C_{3m+2}(a, b, k)$ are odd. Thus, if k is even and at least one of a or b is odd, the probability of randomly choosing an even term from $\{C_n(a, b, k)\}$ is 1/3. We summarize in Theorem 12.

Theorem 12: If k is odd, and at least one of a or b is even, the probability that a term chosen at random from $\{C_n(a, b, k)\}$ will be even is 2/3. If k is even, and at least one of a or b is odd, the probability that a term chosen at random from $\{C_n(a, b, k)\}$ will be even is 1/3. If a, b, and k are all odd, $C_n(a, b, k)$ is always odd.

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PALINDROMIC DIFFERENCES

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Let x denote a positive integer, written in the ordinary denary form, and define its palindromic inverse x' to be the integer obtained from x by writing its digits in reverse order. We ignore leading zeros so that both 1234 and 12340 have palindromic inverse 4321. A number is called a palindrome if $x = x'$. Similar definitions apply to bases other than 10.

A notorious problem concerns palindromic sums [3]. From any starting point x_1 we form a sequence inductively by $x_{k+1} = x_k + x_k'$, and the question is whether one always arrives at a palindrome. A negative answer is conjectured, and specifically that for $x_1 = 196$ a palindrome is never reached. Although this problem is unsolved, the conjecture is known to be correct for base 2 [2]. The problem, however, is somewhat artificial since the property of being a palindrome will not persist throughout the iteration even if ever attained. We consider here the problem of taking palindromic differences; starting with x_1, define

$$x_{k+1} = |x_k - x_k'|$$

inductively. In this case, if x_k were a palindrome, all its successors would vanish, and the first question that arises is whether this always occurs. This problem has been considered previously (see [1], [4], [5]).

Clearly, if x_1 has only one digit, then $x_2 = 0$, and if x_1 has two digits, then x_2 will have at most two digits and be divisible by 9. If $x_2 = 9$ or 99, then $x_3 = 0$, whereas all other cases do eventually reach zero, as the sequence 90, 81, 63, 27, 45, 9, 0 shows, for this sequence together with all palindromic inverses contains all integers of no more than two digits divisible by 9. The same reasoning applies to three-digit numbers, for then x_2 will be divisible by 99, and the sequence 990, 891, 693, 297, 495, 99, 0 shows just as before that, for any x_1 under 1000, the process leads to zero in the end. As we shall see presently, the close connection between the behavior for two- and three-digit numbers is not mere coincidence.

Given an x_1 having n digits, it is not necessarily true that $x_2 < x_1$, but certainly x_2 has n or fewer digits. Accordingly, from any starting point x_1 of digit length n one of two things must happen; either in the sequence of iterates we find one with fewer than n digits, which property will then persist, or else the sequence becomes periodic eventually with all the numbers in the period having n digits. Within a period, the period-length p, is the number of iterations required to return to the starting point. We have already seen that there are no periods with $0 < n < 4$. However, there is a period with $n = 4$, $p = 2$, with $x_1 = 2178$, $x_2 = 6534$. So there are nontrivial periods. We seek to determine for each n, all possible periods; alternately, we might desire to find all possible p.

It is easily seen that $p = 1$ cannot occur except for $x_1 = 0$, for it would require $x_2 = x_1$ and so $x_1' = 2x_1$. Suppose then that the first and last digits of x_1 were a and b, respectively. Then we should find that $b = 2a$ or $2a + 1$ and also that $a \equiv 2b \pmod{10}$, which cannot hold simultaneously. [Incidentally, it can be shown that if instead of base 10 we consider base β the same result holds if $\beta = 2$ or if $\beta \equiv 1 \pmod 3$. However, in other cases, there are nontrivial periods with $p = 1$, e.g., $x = ab$ with

$$a = (\beta - 2)/3,\ b = (2\beta - 1)/3 \quad \text{if } \beta \equiv 2 \pmod 3,$$

and $x = abcd$ with

$$a = \beta/3,\ b = (\beta - 3)/3,\ c = (2\beta - 3)/3,\ d = 2\beta/3 \quad \text{if } \beta \equiv 0 \pmod 3.$$

We shall, however, concentrate on the denary case in the sequel.

We observed before a connection between the behavior of three-digit numbers and that of two-digit numbers, and we now use this to dispose of the case in which n is odd. Suppose that we have a period in which $n = 2m + 1$ is odd, and let $x_1 = a_0a_1 \ldots a_{2m-1}a_{2m}$ be any number in any period with digit length n. Then x_2 is the modulus of the difference

$$\begin{array}{l} a_0\ a_1 \quad \ldots\ a_m\ \ldots\ a_{2m-1}a_{2m}\ - \\ a_{2m}a_{2m-1}\ \ldots\ a_m\ \ldots\ a_1 \quad a_0 \end{array}$$

and since the two middle digits coincide, the middle digit of the difference will be 9 or 0 accordingly as there is or there is not a carry in the middle of the subtraction. Hence, for every number in such a period the middle digit will be 0 or 9, and moreover, were this digit to be removed in all cases, we should obtain a period with the same p but with n reduced by 1. Conversely, all periods with n odd can be obtained from exactly similar ones with n one less by the insertion of a suitable middle digit 0 or 9; thus, the period 2178, 6534, 2178 leads to 21978, 65934, 21978. In fact, we can produce a period with n one larger still by doubling this middle digit and, of course, the process can be carried on indefinitely. We call a period *old* if it is derived in this way from one with smaller n, and we shall from now onward concentrate on finding the new periods; since all new periods have n even, we shall write $n = 2m$.

Much of what follows was obtained by computation, and economy soon becomes a major consideration. At first sight, it might appear that to find all periods of digit length $2m$ it might be necessary to consider all $9 \cdot 10^{2m-1}$ possible n-digit numbers and their iterates to find all possible periods. Such a procedure would be extremely wasteful, for all the integers in a period are themselves iterates, and there are far fewer of these. For suppose that $x_1 = a_0a_1 \ldots a_{n-1}$ and without loss of generality that $x_1 < x_1'$. Then

$$x_2 = \sum_{r=0}^{m-1} A_r(10^{n-r-1} - 10^r),$$

where $A_r = a_{n-r-1} - a_r$. Since x_2 has n digits (and not less), it is easily seen that this requires

$$1 \le A_0 \le 9 \quad \text{and} \quad -9 \le A_r \le 9, \quad r = 1, 2, \ldots, m - 1.$$

Secondly, the observation that second iterates cannot have $A_0 = 9$ reduces the number of cases to be considered to $8 \cdot 19^{m-1}$. Despite this reduction and some other refinements, the number of cases still grows exponentially with n, which soon makes complete computation impossible.

We shall represent the iterate x, a number of $2m$ digits, by the corresponding A's in the canonical form $\{A_0, A_1, \ldots, A_{m-1}\}$ where it is to be understood that A_0 lies between 1 and 8 and the others between -9 and 9. From this, the denary form for x is found by writing

$$A_0A_1\ \ldots\ A_{m-1}\ -A_{m-1}\ \ldots\ -A_1\ -A_0$$

where, of course, some of the numbers will be negative. To deal with this, we start at the right, and whenever we encounter a negative number add 10 to it and subtract 1 from its predecessor in the usual "borrow and carry" fashion, familiar from elementary arithmetic. The successor is then easily calculated in the same canonical form and the process repeated, in a manner eminently suitable for computation.

It will be clear that if $A_{m-1} = 0$, then in the denary form the number will have its two middle digits both 0 or both 9, and its successor will also have $A_{m-1} = 0$; such a number cannot appear in a new period, and so can be ignored in a search for new periods. At first this appears to produce only a small saving in the computation, a factor of 18/19, but this is not so, for we can ignore any x_1 any of whose *iterates* has $A_{m-1} = 0$, and this observation saves a very large proportion of the time required to compute the periods.

Since we now assume that $A_{m-1} \neq 0$, we can associate with each number x of digit length $2m$ in a new period, the rational number $\mu = \sum A_r \cdot 19^{-r}$ whose denominator is precisely 19^{m-1}, and conversely, each such μ yields a unique x. Within each period we call that x the *first* in the period if the corresponding μ is the least μ of any x in the period. It clearly suffices to find all the first numbers in the periods.

For any r with $0 \le r < m - 1$, we write

$$x_1 = \{A_0, A_1, \ldots, A_r, +\}$$

or

$$x_1 = \{A_0, A_1, \ldots, A_r, -\}$$

according as the first nonvanishing integer in the sequence $A_{r+1}, \ldots, A_{m-1}$ is positive or negative. The utility of this lies in the fact that if

$$x_2 = \{B_0, B_1, \ldots, B_{m-1}\},$$

then $B_0, B_1, \ldots, B_r$ depend only upon $A_0, A_1, \ldots, A_r$ and the value + or - and not on the *actual* values of $A_{r+1}, \ldots, A_{m-1}$. Using this fact, we see that no period contains any element $\{5, +\}$, for the successor would have $B_0 = 0$. Furthermore, no period has $\{4, +\}$ as its first element, for the successor would have $B_0 = 2$, contradicting the assumption that $\{4, +\}$ came first in the period. In this way, we can write a program to determine whether any period could start with $(A_0, A_1, \ldots, A_r, \varepsilon)$, where $\varepsilon = +$ or $-$, for we can calculate the first $r + 1$ digits in the canonical form of its successor, then there would be two possible second successors, four possible third successors, and so on. At each stage, we can delete any suggested successor which comes before x_1 and so determine whether we could eventually return to x_1, and if so what is the minimum possible period. For $r = 0$, it is possible to show on the back of an envelope that, for the first element of any period $A_0 = 1$ or 2. For $r = 2$, about 3 minutes on a simple home computer suffice to prove

Result 1: The only period with $m = 2$ starts at $\{2, 2\}$ corresponding to 2178, and for $m > 2$, every new period must start at one of

$$\{1, 0, +\}, \{1, 1, \pm\}, \{1, 2, \pm\}, \{1, 3, \pm\}, \{2, -9, \pm\}, \{2, -8, \pm\},$$

$$\{2, -6, \pm\}, \{2, -5, \pm\}, \{2, -3, -\}, \{2, 0, -\}, \text{ or } \{2, 2, -\}.$$

The same program showed that the only periods with $p = 2$ are $\{2, 2\}$ and possibly more starting at $\{2, 2, -\}$. Use of this fact allows us to find all periods with $p = 2$. Let $\sigma(m)$ denote the number of periods both old and new with $p = 2$ and $n = 2m$. One such is, of course, $\{2, 2, 0, 0, \ldots, 0\}$, but this apart, we must have $x_1 = \{2, 2, -\}$ and so $x_2 = \{6, 6, \ldots\}$. If

$$x_2 = \{6, 6, +\} \quad \text{or} \quad \{6, 6, 0, 0, \ldots, 0\},$$

then

$$x_3 = \{2, 3, \pm\} \quad \text{or} \quad \{2, 2, 0, 0, \ldots, 0\},$$

respectively, and in either case $x_3 \neq x_1$. Thus,

$$x_2 = \{6, 6, -\}.$$

Now consider the number 2199 ... 9978 - x_1. It is easily seen that for some $k \geq 2$ this number has its first k digits zero, its last k digits zero, and a number y, which occupies the middle $2m - 2k$ digits; then

$$x_1 = \{2, 2, 0, \ldots, 0, A_k, \ldots, A_{m-1}\}$$

with $A_k < 0$. Then

$$y_1 = \{-A_k, \ldots, -A_{m-1}\}.$$

Also, $y_1 < y_1'$, otherwise we should not have $x_2 = \{6, 6, -\}$ and, moreover, y_1 must also be periodic with period dividing 2, and hence equal to 2. Therefore, $y_1 = \{2, 2, \ldots\}$, etc. Conversely, given such a y, we can find a corresponding x_1 of digit length $2m$. Hence,

$$\sigma(m) = 1 + \sigma(1) + \cdots + \sigma(m - 2)$$

and so

$$\sigma(m + 1) = \sigma(m) + \sigma(m - 1).$$

Since $\sigma(1) = 0$ and $\sigma(2) = 1$, it follows that $\sigma(m + 1) = F_m$, the m^{th} Fibonacci number. Also, the number of *old* periods with $p = 2$ and of digit length $2m$ equals $\sigma(m - 1)$; hence, for $m \geq 3$, the number of new periods of digit length $2m$ equals $F_{m-1} - F_{m-2} = F_{m-3}$.

We show next that all periods starting at $\{2, 2, -\}$ have $p = 2$. For, let x_1 be the first element in the period; then $x_2 = \{6, 6, \pm\}$. We cannot have $x_2 = \{6, 6, +\}$, otherwise $x_3 = \{2, 3, \pm\}$ or $\{2, 4, \pm\}$, whence $x_4 = \{2, -\}$ —impossible, since x_1 was assumed to be the first in the period. Thus,

$$x_2 = \{6, 6, -\} \quad \text{and} \quad x_3 = \{2, 2, \pm\}.$$

Again the + sign is impossible, since it would be found that x_5 came before x_1. Thus, we find that, for all k,

$$x_{2k+1} = \{2, 2, -\} \quad \text{and} \quad x_{2k} = \{6, 6, -\}$$

and, accordingly, p must be even. If we now subtract x_1 from 2199 ... 9978, we find that after deleting leading and trailing zeros we obtain either zero or else a number y_1 which also forms part of a periodic sequence with the properties that, for each k,

$$y_{2k+1} < y_{2k+1}' \quad \text{and} \quad y_{2k} > y_{2k}'.$$

It is not very difficult to establish that these conditions also require y_1 to start $\{2, 2, \pm\}$; we omit the details. Hence, all periods starting at $\{2, 2, -\}$ are obtained by the construction above; thus, by induction on m, all have period 2. Summing up, we have

Result 2: Every period with $p = 2$ starts with $\{2, 2, \ldots\}$ and conversely. For given digit-length $2m$ where $m \geq 3$, there are precisely F_{m-1} such distinct periods of which precisely F_{m-3} are new periods, F_k denoting the m^{th} Fibonacci number.

For other values of p, there does not seem to be such a neat description. We have carried out a complete search for $m \leq 8$ and obtained the following

Result 3: For $m \le 8$, the only periods are:

m	p	First x_1	Canonical Form
2	2	2178	2, 2
3	2	219978	2, 2, 0
4	2	21999978	2, 2, 0, 0
	2	21782178	2, 2, -2, -2
	14	11436678	2, -8, -6, 4
5	2	2199999978	2, 2, 0, 0, 0
	2	2178002178	2, 2, -2, -2, 0
	2	2197821978	2, 2, 0, -2, -2
	14	1143996678	2, -8, -6, 4, 0
6	2	219999999978	2, 2, 0, 0, 0, 0
	2	217800002178	2, 2, -2, -2, 0, 0
	2	217821782178	2, 2, -2, -2, 2, 2
	2	219780021978	2, 2, 0, -2, -2, 0
	2	219978219978	2, 2, 0, 0, -2, -2
	12	118722683079	1, 2, -1, -3, 2, 3
	14	114399996678	2, -8, -6, -4, 0, 0
	22	125520874479	1, 2, 5, 5, 2, 1
7	2	eight periods	
	12	one old period	
	14	one old period	
	22	one old period	
8	2	thirteen periods	
	12	one old period	
	14	one old period	
	14	1143667811436678	2, -8, -6, 4, -4, 6, 8, -2
	17	1186781188132188	2, -9, 9, -3, -3, 9, -9, 2
	22	one old period	

It will be observed in the above that certain of the canonical forms of new periods read the same left to right as right to left, e.g., {2, 2} and {1, 2, 5, 5, 2, 1 } and that others do so with a change of sign, e.g., {2, 2, -2, -2}. Consider any $x = \{A_0, \ldots, A_{m-1}\}$ in which $A_{m-1} \neq 0$ and define the *dual* of x, $z = \{C_0, \ldots, C_{m-1}\}$ where the A's have been written down back to front and the signs changed throughout if $A_{m-1} < 0$; formally

$$C_r = \operatorname{sgn}\{A_{m-1}\} \cdot A_{m-r-1}, \quad 0 \le r \le m-1.$$

Clearly, performing the operation twice will yield x again, justifying the name "dual." There is one difficulty that arises, for if $A_{m-1} = \pm 1$ and A_{m-2} has opposite sign to A_{m-1}, then $z = \{1, -\}$ and on expansion this fails to have $2m$ digits. We shall deal with this as it occurs. The utility of the definition lies in the following

Lemma: The iterate of the dual equals the dual of the iterate.

Proof; There are two cases depending on the sign of A_{m-1}. We give the proof for $A_{m-1} < 0$, the other case being less transparent but essentially similar. If $x = \{A_0, \ldots, A_{m-1}\}$, then $z = \{-A_{m-1}, \ldots, -A_0\}$. Thus, to find the denary representation for x, we have to perform the "borrow and carry" routine on the expression

$$A_0A_1 \ldots A_{m-1}(-A_{m-1}) \ldots (-A_1)(-A_0),$$

whereas for z we must do the same for

$$(-A_{m-1}) \ldots (-A_1)(-A_0)A_0A_1 \ldots A_{m-1}.$$

Now observing that both A_0 and $-A_{m-1}$ are positive, and the fact that the "first half" of the former expression is identical to the "second half" of the latter and vice-versa, it becomes clear that this property remains intact after the borrowing and carrying; recalling how the iterate is formed from the denary form proves the result.

Now consider any new period which guarantees that $A_{m-1} \neq 0$ for every x in the period. At first sight, the lemma would appear to give a new dual period, obtained by taking duals throughout. There are, however, three reasons why this need not be. In the first place, we might have a period in which x_1 is its own dual, and then by the lemma this property would persist throughout the period. Thus, the dual period does indeed exist, but is identical to the given one. This case can be further subdivided into two cases. If x_1 is its own dual, then we have either $A_r = A_{m-r-1}$ for each r, in which case we call x_1 symmetric, or else $A_r = -A_{m-r-1}$ for each r, in which case x_1 is said to be skew-symmetric. It is not difficult to see that the property of being symmetric or skew-symmetric also persists throughout the iterations and so we also call the respective periods symmetric or skew-symmetric. Both types do exist, as we see in Result 3. The symmetric cases are interesting, and can occur not only if m is even but also with m odd. The skew-symmetric cases, however, are all formed from periods with fewer digits in the following manner. Let

$$x_1 = \{A_0, \ldots, A_{m-1}\}$$

be the first member of any period whatsoever. Then we can obtain a skew-symmetric period with the same p starting at

$$y_1 = \{A_0, \ldots, A_{m-1}, 0, \ldots, 0, -A_{m-1}, \ldots, -A_0\}$$

where the number of zeros written in the middle is arbitrary and can be zero; conversely, any skew-symmetric period is of this form. The symmetric case is entirely different, and although $\{1, 2, 5, 5, 2, 1\}$ belongs to a period, neither $\{1, 2, 5\}$ nor $\{1, 2, 5, 0, 5, 2, 1\}$ does.

A second reason why the dual period may not be interesting is that although x_1 may not be self-dual, it may be the dual of one of its iterates. Thus, if

$$x_1 = \{2, -8, -6, 4\}$$

then

$$x_8 = \{4, -6, -8, 2\}.$$

In such cases it is reasonable to call the period self-dual although the elements themselves are not. It is plain that for all self-dual periods p must be even.

There is a third reason why the dual period may not yield anything interesting. It is possible that one x in a period is of the form we mentioned above with $A_{m-1} = \pm 1$ and A_{m-2} of opposite sign to A_{m-1}, in which case the dual "collapses," in not having the requisite number of digits. This does indeed occur; one example, which may well not be simplest, is the one given in Result 4 below for $p = 9$. It has

$$x_3 = \{4, 3, 4, 7, 0, -3, 9, 1, -6, 2, 2, 3, 2, 6, 7, -9, 6, 8, -4, 1, -4, 9, -2, 1\}.$$

There are some divisibility properties of the x which can occur in a period. Naturally, all are multiples of 9, but the observant reader may have noticed that all the x_1 with $m \le 8$, and indeed all those for $n \le 17$, including those with n odd are multiples of 11. If $n = 2m + 1$ is odd, then any iterate is a multiple of 11 since, if $x_1 = a_0a_1 \ldots a_{2m}$, then

$$\pm x_2 = \sum_{r=0}^{2m} (a_r - a_{2m-r})(10^{2m-r} - 10^r) \equiv 0 \pmod{11}.$$

If $n = 2m$ is even and $x = a_0a_1 \ldots a_{2m-1}$, then

$$x_1 + x_1' = \sum_{r=0}^{2m-r} (a_r + a_{2m-1-r})(10^{2m-1-r} + 10^r) \equiv 0 \pmod{11},$$

and so

$$x_2 = |x_1 - x_1'| \equiv \pm 2x_1 \pmod{11}.$$

Hence, x_1 and x_2 are either both divisible by 11 or neither is. Therefore, in any period either all or none of the numbers are multiples of 11. Let us consider how we might hope to discover periods consisting of nonmultiples of 11. In the first place, if $x_1 = \{A_0, \ldots, A_{m-1}\}$, then

$$x_1 = \sum_{r=0}^{m-1} A_r(10^{2m-r-1} - 10^r) \equiv 2\sum_{r=0}^{m-1}(-1)^{r-1}A_r \pmod{11}.$$

Thus, if x_1 is symmetric and m even, then $11|x_1$. Similarly, if x_1 is skew-symmetric and m odd, but this case is not really interesting, because whatever the parity of m, the property of being divisible by 11 or not is inherited from the shorter period from which x_1 can be formed.

We have seen that $x_2 \equiv \pm 2x_1 \pmod{11}$ and so, if x_1 is not divisible by 11, then

$$x_1 = x_{p+1} \equiv \pm 2^p x_1 \pmod{11}$$

which implies that

$$2^p \equiv \pm 1 \pmod{11},$$

i.e., that 5 divides p. It is not too difficult to show that $p = 5$ will not yield such a value, for if $p = 5$ it can be shown that

$$x_1 = x_6 \equiv 2^5 x_1 \equiv -x_1 \pmod{11}.$$

So in the search for possible periods not divisible by 11, it seems natural to look for numbers with period 10, which are not symmetric with m even nor skew-symmetric. In this way we have been able to find such a period, which is the one listed in Result 4 below; it is self-dual.

From the computational point of view, the existence of such numbers is rather a pity, for had we been able to show that all periods were divisible by 11, the necessary computation to exhaust all possibilities for a given n could have been reduced by a factor of 11.

The next question is, determine for which p periods exist. We have seen that there are none with $p = 1$, but some with $p = 2$, 10, 12, 14, 17, and 22. There is in principle no difficulty, given a suggested p, to search for periods in a systematic way. Suppose that we have reason to think that there might be a period starting at $x_1 = \{A_0, \ldots, A_r, \pm\}$ of period-length p. Then, as mentioned above, we can calculate the 2^{p-1} possible p^{th} successors of x_1 and check whether any one can be $\{A_0, \ldots, A_r, \pm\}$. If not, we can discard this starting point; if yes, then we can increase r by one and look at the 19 possible starting points with the first $r + 1$ entries and the sign given, etc., inductively. Although the task sounds quite formidable, it is actually very efficient at least for small p, apparently more so than a complete search for a given m. In this way, we have been able to show

Result 4: For $p \le 14$, there are no periods with $p = 1, 3, 6$, or 13. For the other ten values of p, one example each is provided by:

p	Canonical form for x_1
2	{2, 2}
4	{2, -3, 0, -9, 5, -9, 0, -3, 2}
5	{1, 0, 5, 9, 1, 3, -4, 6, 6, -4, 3, 1, 9, 5, 0, 1}
7	{2, -6, 2, 8, -9, 1, -7, 5, 4, 3, 5, 3, 4, 5, -7, 1, -9, 8, 2, -6, 2}
8	{2, -3, 0, -9, 5, -9, -2, 0, -5, 0, 4, 1, 8, 2, -2, -1, 7, 1, -4, -6, -7, -3}
9	{2, -8, -8, -4, 0, 3, 5, 2, -1, -3, 2, 2, -8, -4, 6, -1, 6, 0, 3, 7, 3, 0, 3, -3}
10	{1, 0, 6, -7, 0, -7, -8, 6, -6, -8, 1, 1}
11	{2, -3, -4, 5, -7, -3, 5, 5, -6, 5, -1, 3, -5, -5, 3, -1, 5, -6, 5, 5, -3, -7, 5, -4, -3, 2}
12	{1, 2, -1, -3, 2, 3}
14	{2, -8, -6, 4}

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A HYPERCUBE PROBLEM

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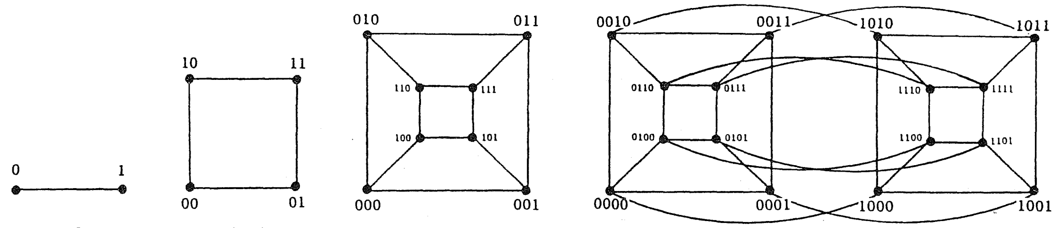
1. Introduction

The n-dimensional hypercube, Q_n, is the graph whose vertex set, $V(Q_n)$, is the set of all n-bit strings, any two of which are adjacent iff they differ in exactly one bit. We refer to Q_n as the n-cube. The 1-, 2-, 3-, and 4-cubes are illustrated in Figure 1.



(a) Q_1 (b) Q_2 (c) Q_3 (d) Q_4

FIGURE 1

Sometime in the early 1980s, Paul Erdös asked for the largest order of an induced subgraph of Q_n which contains no 4-cycle. This question has been answered and extremal graphs characterized [1]. Since a 4-cycle in Q_n can be interpreted as a sub-Q_2, it is natural to generalize and ask for the order of a largest induced subgraph of Q_n which contains no sub-Q_k, $k \in \{1, 2, 3, \ldots\}$. It is also natural to ask for the order of a largest induced subgraph of Q_n which contains no $2k$-cycle, $k \in \{2, 3, \ldots\}$, but this question seems far more difficult. Partial results in this direction appear in [2].

With the advent of the hypercube computer, these questions assume a new significance. An n-dimensional hypercube computer is a multicomputer with 2^n processors, possessing the network topology of an n-dimensional hypercube; i.e., each vertex of the cube is associated with a processor and each edge represents a direct communication link between the two processors incident with that edge. A question that has generated some interest recently ([3], [4]) is *how does the hypercube computer behave in the presence of faulty nodes (or links)?* In particular, given a set of faulty nodes (links), what is the largest subcube that remains? The question is pertinent because there are algorithms which are designed to run on a cube structure, and in the presence of faulty nodes (links) will run on the largest remaining subcube [3].

In the following, F_n and L_n will denote the n^{th} Fibonacci and Lucas numbers, respectively, having the initial conditions $F_0 = 0$, $F_1 = 1$ and $L_1 = 1$, $L_2 = 3$. We use $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the greatest integer less than or equal to x and the least integer greater than or equal to x, respectively. Now, let $f(n, k)$ denote the largest order of an induced subgraph of Q_n that contains no sub-Q_k. It is known that

$$f(n,\ 2) = \left\lceil \frac{2}{3} \cdot 2^n \right\rceil \quad [1].$$

A good lower bound for $f(n,\ 3)$ is known, namely,

$$f(n,\ 3) \geq \frac{3}{4} \cdot 2^n + 2^{\left\lfloor \frac{n-2}{2} \right\rfloor} \quad [5].$$

In general, it is easy to show [3] that

$$f(n,\ k) \geq \frac{k}{k+1} \cdot 2^n. \tag{1}$$

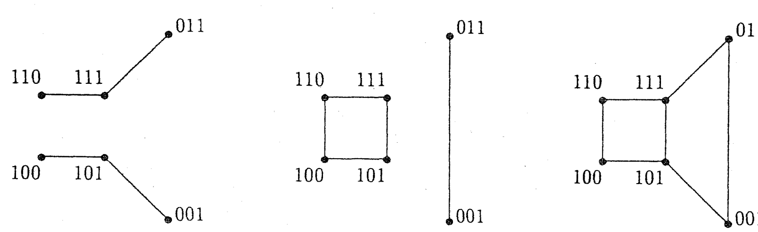
In this paper we prove, in Theorem 2 and its corollary, a result which enables us to improve on the inequality in (1) for the special case $k = 4$. We obtain

$$f(n,\ 4) \geq \begin{cases} \frac{4}{5} \cdot 2^n + \frac{1}{5} L_{n+1}, & n \text{ even}, \\ \frac{4}{5} \cdot 2^n + \frac{2}{5} L_n, & n \text{ odd}. \end{cases} \tag{2}$$

2. The Hypercube Problem

The *order* of a graph is the size of its vertex set. Given a graph G with vertex set $V(G)$ and edge set $E(G)$, a *subgraph of* G is a graph whose vertex and edge sets are subsets of $V(G)$ and $E(G)$, respectively. If H is a subgraph of Q_n and there is a subgraph of H isomorphic to some Q_k, $1 \leq k \leq n$, then H is said to contain a sub-Q_k. Given any graph G with vertex set $V(G)$ and $S \subseteq V(G)$, the subgraph of G which is *induced* by S, denoted $\langle S \rangle$, is the graph with vertex set S and two vertices of $\langle S \rangle$ are adjacent iff they are adjacent in G.

In Figure 2, G_1, G_2, and G_3 are all subgraphs of Q_3. The graphs G_1 and G_2 are not induced subgraphs of Q_3, while G_3 is. G_2 and G_3 both contain a sub-Q_2.



(a) G_1 (b) G_2 (c) G_3

FIGURE 2

Example: Let W be the set of 16 vertices listed in Figure 3. For each $v = v_1v_2v_3v_4v_5v_6v_7$ in W, we have $v_5 = v_6 = v_7 = 1$, while the first four bits range from 0000 to 1111. Hence $\langle W \rangle$, the subgraph of Q_7 induced by W, contains a sub-Q_4. (In fact, $\langle W \rangle$ is isomorphic to Q_4.)

For $v \in V(Q_n)$, the *weight* of v, denoted wgt(v), is defined to be the number of 1's in v. Observe that the vertices of W have weights ranging from 0 to 4 (mod 5). In fact, for all n, any sub-Q_4 in Q_n contains vertices with weights of 0, 1, 2, 3, and 4 (mod 5). For $n \in Z^+$, $k \in \{0, 1, 2, 3, 4\}$, let

$$V_k^n = \{v \in V(Q_n) : \text{wgt}(v) \equiv k \pmod 5\}.$$

If $V \subseteq V(Q_n)$ and $\langle V \rangle$ contains a sub-Q_4, then

$$V \cap V_k^n \neq \emptyset \text{ for all } k \in \{0, 1, 2, 3, 4\}.$$

0	0	0	0	1	1	1
0	0	0	1	1	1	1
0	0	1	0	1	1	1
0	0	1	1	1	1	1
0	1	0	0	1	1	1
0	1	0	1	1	1	1
0	1	1	0	1	1	1
0	1	1	1	1	1	1
1	0	0	0	1	1	1
1	0	0	1	1	1	1
1	0	1	0	1	1	1
1	0	1	1	1	1	1
1	1	0	0	1	1	1
1	1	0	1	1	1	1
1	1	1	0	1	1	1
1	1	1	1	1	1	1

FIGURE 3. The vertex set W

Hence for any k, $\langle V(Q_n) - V_k^n \rangle$ contains no sub-Q_4. This implies (1). To obtain the inequality in (2), we first let $V_k^n = \# V_k^n$. Clearly,

$$V_k^n = \sum_{\substack{j \equiv k \\ \bmod 5}} \binom{n}{j},$$

and if we define

$$V(n) = \min_{0 \le k \le 4} V_k^n,$$

then we obtain $f(n, k) \ge 2^n - V(n)$. Determination of a formula for $V(n)$ is the content of the next two sections.

3. Properties of the V_k^n

We begin with an example. By definition,

$$V_0^7 = \binom{7}{0} + \binom{7}{5} = 1 + 21 = 22 = V_2^7 = \binom{7}{2} + \binom{7}{7}.$$

Similarly,

$$V_1^7 = \binom{7}{1} + \binom{7}{6} = 14 \quad \text{and} \quad V_3^7 = \binom{7}{3} = \binom{7}{4} = V_4^7 = 35.$$

Hence, $V(7) = 14 = V_1^7$. On the other hand, if we compute values of V_k^6, we find that $V(6) = V_0^6 = V_1^6$. In Theorem 1 we will show that, if we define

$$k(n) = \left\lfloor \frac{n}{2} \right\rfloor - 2 \pmod 5,$$

then $V(n) = V_{k(n)}^n$.

Because the terms V_k^n are computed in terms of binomial coefficients, we would expect the V_k^n to reflect some of the properties of binomial coefficients. That this is the case is illustrated in the following lemma.

Lemma: For $n \in Z^+$, $k \in \{0, 1, 2, 3, 4\}$,

(1) *(Recursion Formula)*

$$V_k^n = V_k^{n-1} + V_{k-1}^{n-1}, \text{ where } k - 1 \text{ is computed modulo 5.}$$

(2) *(Symmetry Formula)*

$$V_k^n = V_j^n, \text{ where } k + j \equiv n \pmod 5.$$

(3) *(Initial Conditions)*

(i) For $n < 5$, $V_k^n = \binom{n}{k}$,

(ii) $V_0^5 = 2$, $V_k^5 = \binom{5}{k}$ for $k \in \{1, 2, 3, 4\}$.

Proof: To prove (1) let W^n be a set of size n and let W_k^n denote the collection of all subsets of W^n of size congruent to k (mod 5), $k \in \{0, 1, 2, 3, 4\}$. Then clearly $\#W_k^n = V_k^n$. Now let $w \in W^n$, $W \in W_k^n$. If $w \in W$, the remaining elements of W can be chosen from the $n - 1$ elements of $W^n - \{w\}$ in V_{k-1}^{n-1} ways. Otherwise, if $w \notin W$, the elements of W can be chosen from the $n - 1$ elements of $W^n - \{w\}$ in V_k^{n-1} ways.

To prove (2) let $n \in Z^+$, $k \in \{0, 1, 2, 3, 4\}$. The division algorithm yields integers m and j such that

$$n - k = 5m + j \quad \text{where } j \in \{0, 1, 2, 3, 4\},$$

and hence $k + j \equiv n$ (mod 5). Using this we can relate V_k^n and V_j^n as follows:

$$\begin{aligned} V_k^n &= \binom{n}{k} + \binom{n}{k+5} + \cdots + \binom{n}{k+5m} \\ &= \binom{n}{k} + \binom{n}{k+5} + \cdots + \binom{n}{n-j} \\ &= \binom{n}{n-k} + \quad \cdots \quad + \binom{n}{j} \\ &= \binom{n}{j+5m} + \quad \cdots \quad + \binom{n}{j} = V_j^n. \end{aligned}$$

The proof of (3) is trivial and so omitted. □

Using the initial conditions and the recursion for the V_k^n, we can build a table of values for the V_k^n similar to Pascal's triangle. Since the V_k^n are computed mod 5, there will be 5 entries in each row of our *Pascalian Rectangle*. In Figure 4 we illustrate the general form of the table and in Figure 5 we fill in specific values.

$$\begin{array}{ccccccccccc|c}
 & & & & & & & & & & & \text{Row} \\
V_3^1 & & V_4^1 & & V_0^1 & & V_1^1 & & V_2^1 & & & 1 \\
 & V_4^2 & & V_0^2 & & V_1^2 & & V_2^2 & & V_3^2 & & 2 \\
V_4^3 & & V_0^3 & & V_1^3 & & V_2^3 & & V_3^3 & & & 3 \\
 & V_0^4 & & V_1^4 & & V_2^4 & & V_3^4 & & V_4^4 & & 4 \\
V_0^5 & & V_1^5 & & V_2^5 & & V_3^5 & & V_4^5 & & & 5 \\
 & & & & \vdots & & & & & & & \vdots
\end{array}$$

FIGURE 4

Remark: Notice the wrap-around property of the table. The right-most entry in an even row (or the left-most entry in an odd row) is the sum of the left-most and right-most entries of the previous row, e.g.,

$$V_0^5 = V_4^4 + V_0^4 \quad \text{and} \quad V_4^4 = V_4^3 + V_3^3.$$

If the table is constructed as in Figure 4 above and Figure 5 below, then the left-most entry of the n^{th} row is next seen to be a smallest entry of the n^{th} row. Recalling the definition of $V(n)$, we state the following theorem.

										Row
0		0		1		1		0		1
	0		1		2		1		0	2
0		1		3		3		1		3
	1		4		6		4		1	4
2		5		10		10		5		5
	7		15		20		15		7	6
14		22		35		35		22		7
	36		57		70		57		36	8
72		93		127		127		93		9
					⋮					⋮

FIGURE 5

Theorem 1: For $n \in Z^+$,

$$V(n) = V^n_{k(n)}, \quad \text{with } k(n) = \left\lfloor \frac{n}{2} \right\rfloor - 2 \pmod 5.$$

Proof: That the left-most entry of the n^{th} row is of the form $[n/2] - 2$ follows from the recursion formula and induction. Next, we must show that the left-most entry of each row in Figures 4 and 5 is also a smallest entry of that row. This follows easily by induction once we verify that the symmetry of each row is maintained. But this is immediate from the symmetry formula of the Lemma. If n is even, then

$$\left\lfloor \frac{n}{2} \right\rfloor - 2 = \frac{n}{2} - 2.$$

If the left-most entry of the n^{th} row is $V^n_{(n/2)-2}$, then the right-most entry is of the form

$$V^n_{(n/2)-2+4} = V^n_{(n/2)+2}.$$

Since

$$\left(\frac{n}{2} - 2\right) + \left(\frac{n}{2} + 2\right) \equiv n \pmod 5$$

we have, by the Lemma, that

$$V^n_{(n/2)-2} = V^n_{(n/2)+2}.$$

Similarly,

$$\left(\frac{n}{2} - 2 + 1\right) + \left(\frac{n}{2} - 2 + 3\right) \equiv n \pmod 5$$

so that the second and fourth entries of the row are equal. Similar reasoning verifies the shifted row symmetry for n odd. An easy induction completes the proof. □

4. A Recursion for $V^n_{k(n)}$

Our next theorem provides a recursion and closed formula for $V(n)$.

Theorem 2: For any integer $n \in Z^+$,

$$\text{(i)} \quad V(n) = \begin{cases} 2V(n-1), & n \text{ odd}, \\ 2V(n-1) + F_{n-2}, & n \text{ even}. \end{cases}$$

$$\text{(ii)} \quad V(n) = \begin{cases} \frac{1}{5}\cdot 2^n - \frac{2}{5}L_n, & n \text{ odd}, \\ \frac{1}{5}\cdot 2^n - \frac{1}{5}L_{n+1}, & n \text{ even}. \end{cases}$$

Proof: By the established symmetry of the table in Figure 5, the first and last entries in an even row are identical. Also, for n odd, we have

$$k(n) = \left\lfloor \frac{n}{2} \right\rfloor - 2 = \left\lfloor \frac{n-1}{2} \right\rfloor - 2 = k(n-1).$$

Therefore, we have, for n odd

$$(3) \qquad V^n_{k(n)} = V^{n-1}_{k(n)} + V^{n-1}_{k(n)-1} = V^{n-1}_{k(n-1)} + V^{n-1}_{k(n-1)-1} = 2V^{n-1}_{k(n-1)}.$$

For n even, in $V^n_{k(n)}$, we need to take a somewhat less direct approach. To this end, we define $D(n)$, for all n, as follows

$$(4) \qquad D(n) = \begin{cases} V^n_{k(n)+2} - V^n_{k(n)+1} & n \text{ odd}. \\ V^n_{k(n)+1} - V^n_{k(n)} & n \text{ even}. \end{cases}$$

We will show that $D(n) = F_n$. To begin with, consultation of Figure 5 verifies that

$$D(1) = 1 - 0 = 1,\ D(2) = 1 - 0 = 2,\ D(3) = 3 - 1 = 2,\ D(4) = 4 - 1 = 3.$$

Now, for n even, we have

$$\begin{aligned} D(n) &= V^n_{k(n)+1} - V^n_{k(n)} \\ &= [V^{n-1}_{k(n)+1} + V^{n-1}_{k(n)}] - [V^{n-1}_{k(n)} + V^{n-1}_{k(n)-1}] \\ &= [V^{n-1}_{k(n-1)+2} + V^{n-1}_{k(n-1)+1}] - [V^{n-1}_{k(n-1)+1} + V^{n-1}_{k(n-1)}] \\ &= [V^{n-1}_{k(n-1)+2} - V^{n-1}_{k(n-1)+1}] + [V^{n-1}_{k(n-1)+1} - V^{n-1}_{k(n-1)}] \\ &= D(n-1) + [V^{n-2}_{k(n-2)+1} - V^{n-2}_{k(n-2)}] \\ &= D(n-1) + D(n-2). \end{aligned}$$

A similar argument shows that the recursion holds for n odd. Since $D(n)$ satisfies the same recursion as F_n and the initial conditions are the same, we have that $D(n) = F_n$.

We return now to $V^n_{k(n)}$. For n even, we have

$$\begin{aligned} (5) \qquad V^n_{k(n)} &= V^{n-1}_{k(n)} + V^{n-1}_{k(n)-1} \\ &= V^{n-1}_{k(n-1)+1} + V^{n-1}_{k(n-1)} \\ &= 2V^{n-1}_{k(n-1)} + [V^{n-1}_{k(n-1)+1} - V^{n-1}_{k(n-1)}] \\ &= 2V^{n-1}_{k(n-1)} + [V^{n-2}_{k(n-2)+1} - V^{n-2}_{k(n-2)}] \\ &= 2V^{n-1}_{k(n-1)} + D(n-2) = 2V^{n-1}_{k(n-1)} + F_{n-2}. \end{aligned}$$

Combining the results in (3) and (5) yields

$$(6) \qquad V^n_{k(n)} = \begin{cases} 2V^{n-1}_{k(n-1)}, & n \text{ odd}, \\ 2V^{n-1}_{k(n-1)} + F_{n-2}, & n \text{ even}. \end{cases}$$

To solve this recursion we note that $[x/(1-x-x^2)]$ is the generating function for the sequence F_0, F_1, F_2, ..., so that $[-x/(1+x-x^2)]$ is the generating function for the sequence F_0, $-F_1$, F_2, $-F_3$, ..., and therefore

$$\frac{1}{2}\left[\frac{x}{1-x-x^2} - \frac{x}{1+x-x^2}\right]$$

is the generating function for the sequence F_0, 0, F_2, 0, F_4, ... Let

$$V(x) = \sum_{n \geq 1} V_{k(n)}^{n} x^n.$$

then (6) gives

$$V(x) = 2xV(x) + x - x^2 - x^3 - 2x^4 + \frac{1}{2}\left[\frac{x^2}{1-x-x^2} - \frac{x^2}{1+x-x^2}\right].$$

A partial fraction expansion of the rational function $V(x)$ leads, after some calculation, to the closed form:

$$(7) \qquad V_{k(n)}^{n} = \begin{cases} \frac{1}{5}\cdot 2^n - \frac{1}{5}L_{n+1}, & n \text{ even,} \\ \frac{1}{5}\cdot 2^n - \frac{2}{5}L_n, & n \text{ odd.} \end{cases}$$

Combining the results of (6) and (7) with the definition of $V(n)$ completes the proof. □

Corollary: Let $f(n, k)$ denote the largest order of an induced subgraph of Q_n that contains no sub-Q_4. Then

$$f(n, 4) \geq \begin{cases} \frac{4}{5}\cdot 2^n + \frac{1}{5}L_{n+1}, & n \text{ even.} \\ \frac{4}{5}\cdot 2^n + \frac{2}{5}L_n, & n \text{ odd.} \end{cases}$$

Proof: This follows from Theorem 2 and the fact that $f(n, 4) \geq 2^n - V(n)$. □

Remarks: (1) Recalling that $V_{k(n)}^{n}$ is a sum of binomial coefficients, it is interesting to observe the locations of these binomial coefficients in Pascal's triangle. In Figure 6, the circled entries in the n^{th} row of Pascal's triangle are the binomial coefficients that sum to $V_{k(n)}^{n}$. Observe that the circled entries are "as far as possible" from the binomial coefficients of the form $\binom{n}{n/2}$ [6].

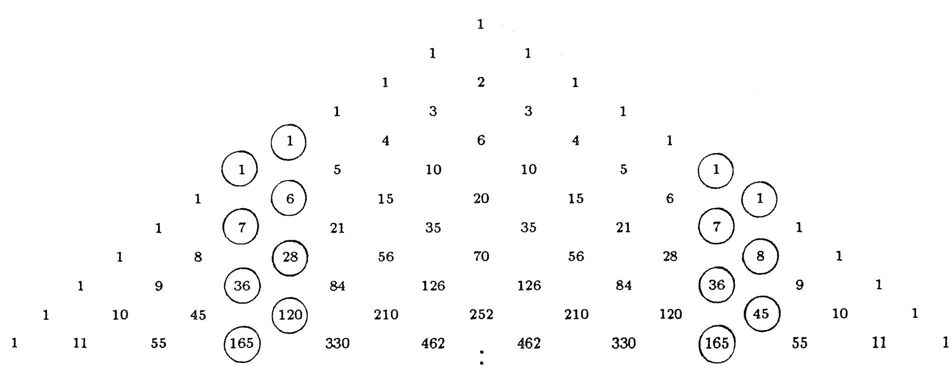


FIGURE 6

(2) A related problem appeared in the 35th W. L. Putnam Intercollegiate Mathematical Competition [7]; that problem asked for a calculation of S_k^n, where

$$S_k^n = \sum_{\substack{j \equiv k \\ \text{mod } 3}} \binom{n}{j}, \quad k = 0, 1, 2.$$

5. Conclusion

By defining the terms V_k^n and $V(n)$ modulo 5, we were able to obtain an improved lower bound for $f(n, 4)$, the largest order of an induced subgraph of Q_n that contains no sub-Q_4. In general, by working modulo m, we can improve on the inequality (1) for $k = m - 1$; for $k \in \{0, 1, \ldots, m - 1\}$, let

$$V_{k,m}^n = \sum_{\substack{j \equiv k \\ \text{mod } m}} \binom{n}{j} \quad \text{and} \quad V(n, m) = \min_{0 \le k \le m-1} V_{k,m}^n.$$

Then $f(n, m - 1) \ge 2^n - V(n, m)$. Work on determination of $V(n, m)$, for all $m \le \{0, 1, \ldots, n\}$ is in progress by this author. It was originally conjectured that $f(n, m - 1) = 2^n - V(n, m)$ but this is now known to be true only for $m \in \{0, 1, 2\}$ [8].

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REPEATING DECIMALS REPRESENTED BY TRIBONACCI SEQUENCES APPEARING FROM LEFT TO RIGHT OR FROM RIGHT TO LEFT

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1. Introduction

In 1953 Fenton Stancliff [1] noted that

$$\sum 10^{-(i+1)} F_i = \frac{1}{89},$$

where F_i denotes the i^{th} Fibonacci number. This curious property of Fibonacci numbers attracts many Fibonacci fanciers. Afterward, Long [2], Hudson & Winans [3], Winans [4], and Lin [5] discussed this Fibonacci phenomenon from different viewpoints. Köhler [6] and Hudson [7] then discussed Tribonacci series decimal expansions. In Lin [8], the characteristics of four types of Tribonacci series

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \text{ where } T_1 = 1,\ T_2 = 1,\ T_3 = 2,$$

$$R_n = R_{n-1} + R_{n-2} + R_{n-3}, \text{ where } R_1 = 1,\ R_2 = 3,\ R_3 = 7,$$

$$S_n = S_{n-1} + S_{n-2} + S_{n-3}, \text{ where } S_1 = 2,\ S_2 = 5,\ S_3 = 10,$$

$$U_n = U_{n-1} + U_{n-2} + U_{n-3}, \text{ where } U_1 = 1,\ U_2 = 2,\ U_3 = 3,$$

are further explored in their $X^3 - X^2 - X - 1 = 0$ format. But, in Lin [8], there was a question left open, which is whether T_n, R_n, S_n, and U_n could be described as one of the four different types of decimal expansions represented by sequential Tribonacci series of the form:

A. $0.\,T_{n1}T_{n2}T_{n3}T_{n4}T_{n5}T_{n6}T_{n7}\ldots = N_a/M_a$,

B. $0.\,T_{n1}\overline{T}_{n2}T_{n3}\overline{T}_{n4}T_{n5}\overline{T}_{n6}T_{n7}\ldots = N_b/M_b$,

C. N_c/M_c ends in $\ldots T_{n7}T_{n6}T_{n5}T_{n4}T_{n3}T_{n2}T_{n1}$,

D. for $N_d/M_d > 0$, N_d/M_d ends in $\ldots T_{n7}\overline{T}_{n6}T_{n5}\overline{T}_{n4}T_{n3}\overline{T}_{n2}T_{n1}$,

for $N_d/M_d < 0$, N_d/M_d ends in $\ldots \overline{T}_{n7}T_{n6}\overline{T}_{n5}T_{n4}\overline{T}_{n3}T_{n2}\overline{T}_{n1}$,

where $\overline{T}_{nm} = -T_{nm}$.

The terms of decimal expansion A are all positive, and those of decimal expansion B appear positive and negative alternately. The repetends of C and D are viewed in retrograde fashion, reading from the rightmost digit of the repeating cycle toward the left. The terms of repetend C are all positive, and those of repetend D appear positive and negative alternately. This question has been given a positive answer in this article. In the following, each of those four types of decimal expansions will be explored.

2. Decimal Fractions That Can Be Represented in Terms of Tribonacci Series Reading from Left to Right

Summing the geometric progressions using the same method described in Lin [5], Köhler [6], and Hudson [7], we can easily obtain the decimal fractions of the Tribonacci series T_{nm+p} as equation (1).

Theorem 1:

$$(1)\qquad \sum_{m=1}^{\infty} \frac{T_{nm+p}}{10^{km}} = \frac{T_{n+p} \cdot 10^{2k} + (T_{2n+p} - R_n \cdot T_{n+p}) \cdot 10^k + T_p}{10^{3k} - R_n \cdot 10^{2k} + R_{-n} \cdot 10^k - 1}.$$

R_{mn+p}, S_{mn+p}, and U_{mn+p} have the same representation if we change T into R, S, and U, respectively.

When $p = 0$, they become

$$(2)\qquad \sum_{m=1}^{\infty} \frac{T_{nm}}{10^{km}} = \frac{T_n \cdot 10^{2k} + (T_{2n} - T_n \cdot R_n) \cdot 10^k + T_0}{10^{3k} - R_n \cdot 10^{2k} + R_{-n} \cdot 10^k - 1},$$

$$(3)\qquad \sum_{m=1}^{\infty} \frac{R_{nm}}{10^{km}} = \frac{R_n \cdot 10^{2k} + (R_{2n} - R_n^2) \cdot 10^k + R_0}{10^{3k} - R_n \cdot 10^{2k} + R_{-n} \cdot 10^k - 1},$$

$$(4)\qquad \sum_{m=1}^{\infty} \frac{S_{nm}}{10^{km}} = \frac{S_n \cdot 10^{2k} + (S_{2n} - S_n \cdot R_n) \cdot 10^k + S_0}{10^{3k} - R_n \cdot 10^{2k} + R_{-n} \cdot 10^k - 1},$$

$$(5)\qquad \sum_{m=1}^{\infty} \frac{U_{nm}}{10^{km}} = \frac{U_n \cdot 10^{2k} + (U_{2n} - U_n \cdot R_n) \cdot 10^k + U_0}{10^{3k} - R_n \cdot 10^{2k} + R_{-n} \cdot 10^k - 1},$$

where n and k must satisfy

$$(6)\qquad \frac{1}{3 \cdot 10^k}\left[R_n + \frac{S_{n-1}}{3}(X + Y) + \frac{T_{n-2}}{3}(X^2 + Y^2)\right] < 1,$$

where $X = \sqrt[3]{19 + 3\sqrt{33}}$ and $Y = \sqrt[3]{19 - 3\sqrt{33}}$. Also,

$$(7)\qquad R_{-n} = R_{-n+3} - R_{-n+2} - R_{-n+1}.$$

Some particular values for the above series are summarized in Tables 1-4.

TABLE 1. Some values of $\sum_{m=1}^{\infty} \frac{R_{nm}}{10^{km}}$

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{123}{889}$	$\frac{323}{689}$	$\frac{603}{349}$				
2	$\frac{10203}{989899}$	$\frac{30203}{969899}$	$\frac{69003}{930499}$	$\frac{111003}{889499}$	$\frac{210203}{789899}$	$\frac{387803}{611099}$	$\frac{713003}{288499}$
3	$\frac{1002003}{998998999}$	$\frac{3002003}{996998999}$	$\frac{6990003}{993004999}$	$\frac{11010003}{988994999}$	$\frac{21002003}{978998999}$	$\frac{38978003}{961010999}$	$\frac{71030003}{928984999}$

TABLE 2. Some values of $\sum_{m=1}^{\infty} \frac{S_{nm}}{10^{km}}$

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{233}{889}$	$\frac{523}{689}$	$\frac{893}{349}$				
2	$\frac{20303}{989899}$	$\frac{50203}{969899}$	$\frac{98903}{930499}$	$\frac{171203}{889499}$	$\frac{320103}{789899}$	$\frac{587603}{611099}$	$\frac{1083503}{288499}$
3	$\frac{2003003}{998998999}$	$\frac{5002003}{996998999}$	$\frac{9989003}{993004999}$	$\frac{17012003}{988994999}$	$\frac{32001003}{978998999}$	$\frac{58976003}{961010999}$	$\frac{108035003}{928984999}$

TABLE 3. Some values of $\sum_{m=1}^{\infty} \frac{T_{nm}}{10^{km}}$

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{100}{889}$	$\frac{110}{689}$	$\frac{190}{349}$				
2	$\frac{10000}{989899}$	$\frac{10100}{969899}$	$\frac{19900}{930499}$	$\frac{40000}{889499}$	$\frac{70200}{789899}$	$\frac{129700}{611099}$	$\frac{240100}{288499}$
3	$\frac{1000000}{998998999}$	$\frac{1001000}{996998999}$	$\frac{1999000}{993004999}$	$\frac{4000000}{988994999}$	$\frac{7002000}{978998999}$	$\frac{12997000}{961010999}$	$\frac{24001000}{928984999}$

TABLE 4. Some values of $\sum_{m=1}^{\infty} \frac{U_{nm}}{10^{km}}$

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{110}{889}$	$\frac{200}{689}$	$\frac{290}{349}$				
2	$\frac{10100}{989899}$	$\frac{20000}{969899}$	$\frac{29900}{930499}$	$\frac{60200}{889499}$	$\frac{109900}{789899}$	$\frac{199800}{611099}$	$\frac{370500}{288499}$
3	$\frac{1001000}{998998999}$	$\frac{2000000}{996998999}$	$\frac{2999000}{993004999}$	$\frac{6002000}{988994999}$	$\frac{10999000}{978998999}$	$\frac{19998000}{961010999}$	$\frac{37005000}{928984999}$

Using (6) and $k = 1, 2, 3$, $n = 4, 8, 12$, respectively, we obtain:

$$[11 + 10 \cdot 4.51786.../3 + 2 \cdot 12.41106.../3]/30 = 1.14445... > 1;$$

$$[131 + 108 \cdot 4.51786.../3 + 24 \cdot 12.41106.../3]/300 = 1.30977... > 1;$$

$$[1499 + 1238 \cdot 4.51786.../3 + 274 \cdot 12.41106.../3]/3000 = 1.49897... > 1.$$

These indicate that the ratios of geometric progressions are greater than 1; thus, the sums are divergent. This explains all the blanks in Tables 1-4.

3. Decimal Fractions That Can Be Represented in Terms of Alternating Positive and Negative Tribonacci Series Reading from Left to Right

Long [2] gave a proof for

$$\sum_{m=1}^{\infty} \frac{F_{m-1}}{(-10)^m} = 1/109;$$

Lin [5] proved

$$\sum_{m=1}^{\infty} \frac{F_{nm}}{(-10^k)^{m+1}} = \frac{F_n}{10^{2k} + 10^k \cdot L_n + (-1)^n}$$

and

$$\sum_{m=1}^{\infty} \frac{L_{nm}}{(-10^k)^{m+1}} = \frac{L_n}{10^{2k} + 10^k \cdot L_n + (-1)^n},$$

where L_m is the m^{th} Lucas number. These equations show that Fibonacci and Lucas numbers appear as the positive and negative terms of alternated Fibonacci and Lucas series, viz.,

$$N/M = 0.\ F_1\overline{F}_2F_3\overline{F}_4F_5\overline{F}_6\ldots,$$

where $\overline{F}_m = -F_m$, and the F_m appears successively in the repetend in blocks of k digits. In this case of Tribonacci sequences, if we substitute (-10^k) for 10^k in equation (2), it will appear as:

$$(8) \qquad \sum_{m=1}^{\infty} \frac{-T_{nm}}{(-10^k)^m} = \frac{T_n \cdot 10^{2k} + (T_nR_n - T_{2n}) \cdot 10^k + T_0}{10^{3k} + R_n \cdot 10^{2k} + R_{-n} \cdot 10^k + 1}$$

Changing T into R, S, and U, it will still be true.

TABLE 5. Some particular values for the T_{mn} series

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{100}{1091}$	$\frac{90}{1291}$	$\frac{210}{1751}$				
2	$\frac{1000}{1009901}$	$\frac{9900}{1029901}$	$\frac{20100}{1070501}$	$\frac{40000}{1109501}$	$\frac{69800}{1209901}$	$\frac{130300}{1391101}$	$\frac{239900}{1708501}$
3	$\frac{1000000}{1000999001}$	$\frac{999000}{1002999001}$	$\frac{2001000}{1007005001}$	$\frac{4000000}{1010995001}$	$\frac{6998000}{1020999001}$	$\frac{13003000}{1039011001}$	$\frac{23999000}{1070985001}$

4. Decimal Fractions That Can Be Represented in Terms of Tribonacci Series Reading from Right to Left

Winans [4] pointed out that 1/109, 9/71, and 1/10099 can be expressed as a reverse diagonalization of sums of Fibonacci numbers reading from the far right on the repeating cycle, where 1/109 ends in

```
   13853211
  21
 34
55
...
...8623853211
```

Johnson [9] gave a short solution to this kind of problem. Summing from the rightmost digit of the repeating cycle toward the left, she got the result:

$$(9)\qquad \frac{F_n}{10^{2k} + L_n \cdot 10^k - 1}, \ n \text{ is odd.}$$

Summing the geometric progressions by using the Binet form for Tribonacci T_n as Lin did in [8], and using the method indicated in Johnson [9], for $k > 0$, we can derive:

$$(10)\qquad \sum_{m=1}^{L} 10^{k(m-1)} T_{nm} = \frac{[T_{n(L-1)} \cdot 10^{k(L+1)} + (T_{n(L+1)} - R_n T_{nL}) \cdot 10^{kL} +}{10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1}$$

$$\underline{T_{nL} \cdot 10^{k(L-1)} - T_0 \cdot 10^{2k} - (T_{2n} - R_n T_n) \cdot 10^k - T_n]}$$

Let the denominator be acronymed as M, and $L(M)$ be the length of the period of M. We add

$$[-T_0 \cdot 10^{2k} - (T_{2n} - R_n T_n) \cdot 10^k - T_n] \cdot 10^{L(M)}$$
$$+ [T_0 \cdot 10^{2k} + (T_{2n} - R_n T_n) \cdot 10^k + T_n] \cdot 10^{L(M)}$$

to the numerator and divide both sides of (10) by $10^{k(L(M))}$; then it becomes

$$\sum_{m=1}^{L} 10^{k(m-1-L(M))} T_{nm}$$
$$= \frac{T_{n(L-1)} \cdot 10^{k(L+1-L(M))} + (T_{n(L+1)} - R_n T_{nL}) \cdot 10^{k(L-L(M))}}{M}$$
$$+ \frac{T_{nL} \cdot 10^{k(L-1-L(M))} - T_0 \cdot 10^{2k} - (T_{2n} - R_n T_n) \cdot 10^k - T_n}{M}$$
$$+ \frac{(T_0 \cdot 10^{2k} + (T_{2n} - R_n T_n) \cdot 10^k + T_n)(10^{L(M)} - 1)}{M \cdot 10^{L(M)}}$$

and, we get

Theorem 2: The decimal representation of

$$(11)\qquad \frac{N}{M} = \frac{T_0 \cdot 10^{2k} + (T_{2n} - R_n T_n) \cdot 10^k + T_n}{10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1}, \ N > 0,$$

ends in successive terms of T_{mn}, $m = 1, 2, 3, \ldots$, reading from the right end of the repeating cycle and appearing in groups of k digits.

If $N < 0$, then we have

Theorem 3: The decimal representation of

$$(12)\qquad \frac{M + N}{M} = \frac{M + T_0 \cdot 10^{2k} + (T_{2n} - R_n T_n) \cdot 10^k + T_n}{10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1}$$

ends in successive terms of T_{mn}, $m = 1, 2, 3, \ldots$, reading from the right end of the repeating cycle and appearing in groups of k digits, if 1 is added to the rightmost digit.

Proof: If N is negative, the N/M still has a positive term there. The numerator needs to be adjusted as below:

$$\frac{(T_0 \cdot 10^{2k} + (T_{2n} - R_nT_n) \cdot 10^k + T_n)(10^{L(M)} - 1)}{10^{L(M)} \cdot M}$$

$$= \frac{(T_0 \cdot 10^{2k} + (T_{2n} - R_nT_n) \cdot 10^k + T_n)(10^{L(M)} - 1) + (10^{L(M)} - 1)M - (10^{L(M)} - 1)M}{10^{L(M)} \cdot M}$$

$$= \frac{(M + T_0 \cdot 10^{2k} + (T_{2n} - R_nT_n) \cdot 10^k + T_n)(10^{L(M)} - 1)}{10^{L(M)} \cdot M} + \frac{1}{10^{L(M)}} - 1$$

The fractional part represents $(M + N)/M$ times one cycle of the repetend of $1/M$, when 1 is added to the rightmost digit.

Using the same method, we derive (11) and (12), and we can further generalize them to

Theorem 4:

$$\frac{N}{M} = \frac{T_p \cdot 10^{2k} + (T_{2n+p} - R_nT_{n+p}) \cdot 10^k + T_{n+p}}{10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1}, \quad N > 0, \tag{13}$$

$$\frac{M + N}{M} = \frac{M + T_p \cdot 10^{2k} + (T_{2n+p} - R_nT_{n+p}) \cdot 10^k + T_{n+p}}{10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1}, \quad N < 0, \tag{14}$$

ends in T_{mn+p}, reading from the right end of the repeating cycle and appearing in groups of k digits. If $N < 0$, 1 is added to the rightmost digit.

From the above method, we can easily obtain the decimal fractions that end in successive terms of R_{mn+p}, S_{mn+p}, and U_{mn+p} by changing T into R, S, and U, respectively.

Tables 6-9 show some values of T_{mn+p}, R_{mn+p}, S_{mn+p}, and U_{mn+p}, for $p = -3, -2, -1, 0, 1, 2, 3$, and $n = 1, 2, 3, 4, 5$.

TABLE 6. Fractions whose repetends end with successive terms of $T_{mn \pm p}$, occurring in repeating blocks of one digit

p \ n	1	2	3	4	5
−3	$\frac{1000}{1109}$	$\frac{1039}{1129}$ ▲	$\frac{489}{569}$ ▲	$\frac{1470}{1609}$ ▲	$\frac{1240}{1309}$ ▲
−2	$\frac{100}{1109}$	$\frac{110}{1129}$	$\frac{71}{569}$	$\frac{121}{1609}$	$\frac{122}{1309}$
−1	$\frac{10}{1109}$	$\frac{1120}{1129}$ ▲	$\frac{1}{569}$ ▲	$\frac{22}{1609}$	$\frac{1283}{1309}$ ▲
0	$\frac{1}{1109}$	$\frac{11}{1129}$	$\frac{561}{569}$ ▲	$\frac{4}{1609}$	$\frac{27}{1309}$
1	$\frac{111}{1109}$	$\frac{112}{1129}$	$\frac{64}{569}$	$\frac{147}{1609}$	$\frac{123}{1309}$
2	$\frac{122}{1109}$	$\frac{114}{1129}$	$\frac{57}{569}$	$\frac{173}{1609}$	$\frac{124}{1309}$
3	$\frac{234}{1109}$	$\frac{237}{1129}$	$\frac{113}{569}$	$\frac{324}{1609}$	$\frac{274}{1309}$

Note▲: 1 is added to the rightmost digit.

TABLE 7. Fractions whose repetends end with successive terms of $R_{mn \pm p}$, occurring in repeating blocks of one digit

p \ n	1	2	3	4	5
−3	$\frac{499}{1109}$	$\frac{539}{1129}$	$\frac{363}{569}$	$\frac{601}{1609}$	$\frac{583}{1309}$
−2	$\frac{1048}{1109}$ ▲	$\frac{972}{1129}$ ▲	$\frac{510}{569}$ ▲	$\frac{1572}{1609}$ ▲	$\frac{1056}{1309}$ ▲
−1	$\frac{992}{1109}$ ▲	$\frac{1070}{1129}$ ▲	$\frac{472}{569}$ ▲	$\frac{1456}{1609}$ ▲	$\frac{11}{1309}$
0	$\frac{321}{1109}$	$\frac{323}{1129}$	$\frac{207}{569}$	$\frac{411}{1609}$	$\frac{341}{1309}$
1	$\frac{143}{1109}$	$\frac{107}{1129}$	$\frac{51}{569}$	$\frac{221}{1609}$	$\frac{93}{1309}$
2	$\frac{347}{1109}$	$\frac{371}{1129}$	$\frac{161}{569}$	$\frac{479}{1609}$	$\frac{451}{1309}$
3	$\frac{811}{1109}$	$\frac{801}{1129}$	$\frac{419}{569}$	$\frac{1111}{1609}$	$\frac{891}{1309}$

Note ▲: 1 is added to the rightmost digit.

TABLE 8. Fractions whose repetends end with successive terms of $S_{mn \pm p}$, occurring in repeating blocks of one digit

p \ n	1	2	3	4	5
−3	$\frac{409}{1109}$	$\frac{420}{1129}$	$\frac{293}{569}$	$\frac{502}{1609}$	$\frac{698}{1309}$
−2	$\frac{1039}{1109}$	$\frac{992}{1129}$ ▲	$\frac{501}{569}$ ▲	$\frac{1554}{1609}$ ▲	$\frac{1109}{1309}$ ▲
−1	$\frac{1102}{1109}$ ▲	$\frac{42}{1129}$	$\frac{544}{569}$ ▲	$\frac{990}{1609}$ ▲	$\frac{107}{1309}$
0	$\frac{332}{1109}$	$\frac{325}{1129}$	$\frac{200}{569}$	$\frac{437}{1609}$	$\frac{342}{1309}$
1	$\frac{255}{1109}$	$\frac{230}{1129}$	$\frac{107}{569}$	$\frac{372}{1609}$	$\frac{249}{1309}$
2	$\frac{580}{1109}$	$\frac{597}{1129}$	$\frac{282}{569}$	$\frac{799}{1609}$	$\frac{1117}{1309}$
3	$\frac{58}{1109}$ *	$\frac{23}{1129}$ *	$\frac{20}{569}$ *	$\frac{1608}{1609}$	$\frac{1289}{1309}$

Note ▲: 1 is added to the rightmost digit.
*: −1 is added to the rightmost digit.

TABLE 9. Fractions whose repetends end with successive terms of $U_{mn \pm p}$, occurring in repeating blocks of one digit

p \ n	1	2	3	4	5
−3	$\frac{1019}{1109}$ ▲	$\frac{1010}{1129}$ *	$\frac{499}{569}$ ▲	$\frac{1510}{1609}$ ▲	$\frac{1161}{1309}$ ▲
−2	$\frac{1100}{1109}$ ▲	$\frac{20}{1129}$	$\frac{560}{569}$ ▲	$\frac{1591}{1609}$ ▲	$\frac{53}{1309}$
−1	$\frac{110}{1109}$	$\frac{101}{1129}$	$\frac{72}{569}$	$\frac{1598}{1609}$	$\frac{96}{1309}$
0	$\frac{11}{1109}$	$\frac{2}{1129}$	$\frac{562}{569}$ ▲	$\frac{26}{1609}$	$\frac{1}{1309}$
1	$\frac{112}{1109}$	$\frac{123}{1129}$	$\frac{56}{569}$	$\frac{151}{1609}$	$\frac{150}{1309}$
2	$\frac{233}{1109}$	$\frac{226}{1129}$	$\frac{121}{569}$	$\frac{320}{1609}$	$\frac{247}{1309}$
3	$\frac{356}{1109}$	$\frac{351}{1129}$	$\frac{170}{569}$	$\frac{497}{1609}$	$\frac{398}{1309}$

Note ▲: 1 is added to the rightmost digit.
*: −1 is added to the rightmost digit.

5. Decimal Fractions That Can Be Represented in Terms of Alternating Positive and Negative Tribonacci Series Reading from Right to Left

Starting from Theorem 4 of Johnson [9], we rewrite it as:

The repeating cycle of $\dfrac{(-1)^n \cdot F_p \cdot 10^k - F_{n+p}}{(-1)^n \cdot 10^{2k} - L_n \cdot 10^k + 1}$ ends in F_{mn+p},

and the repeating cycle of $\dfrac{(-1)^n \cdot L_p \cdot 10^k - L_{n+p}}{(-1)^n \cdot 10^{2k} - L_n \cdot 10^k + 1}$ ends in L_{mn+p},

for $m = 1, 2, 3, 4, \ldots$, occurring in blocks of k digits. Substituing (-10^k) for (10^k), we get

Theorem 5:

(15) The repeating cycle of $\dfrac{N}{M} = \dfrac{(-1)^{n+1} \cdot F_p \cdot 10^k - F_{n+p}}{(-1)^n \cdot 10^{2k} + L_n \cdot 10^k + 1}$ ends in F_{mn+p},

(16) and the repeating cycle of $\dfrac{N}{M} = \dfrac{(-1)^{n+1} \cdot L_p \cdot 10^k - L_{n+p}}{(-1)^n \cdot 10^{2k} + L_n \cdot 10^k + 1}$ ends in L_{mn+p},

for $m = 1, 2, 3, 4, \ldots$, occurring in blocks of k digits. If $N/M > 0$, all even terms are negative, if $N/M < 0$, all odd terms are negative. For example,

for $k = 1$, $n = 1$,

$$N/M = 1/89 = 0 \cdot \ldots 38202247191 = 0 \cdot \ldots\ldots\ldots \bar{8}5\bar{3}2\bar{1}1$$

...893413 5 2 1	positive
....5521 8 3 1	negative
....38202247191	summation

for $k = 1$, $n = 12$,

$$N/M = -16/369 = -0.\overline{04336}$$

```
.....40
...323072
...11879264
     4807526976
            46368   positive
-----------------------------
                144  negative
      14930352
...008755920
...6367088
.....096
.....4
-----------------------------
......60433604336 summation
```

Using (15) and (16), we can derive Tables 10 and 11 for $k = 1, 2, 3$, and n from 1 to 7.

TABLE 10. Fractions whose repetends end in F_{nm} with positive and negative terms alternated, positive fractions begin with positive F_{mn}, negative fractions opposite

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{1}{89}$	$\frac{-1}{131}$	$\frac{2}{59}$	$\frac{-3}{171}$	$\frac{-5}{11}$	$\frac{-8}{281}$	$\frac{-13}{191}$
2	$\frac{1}{9899}$	$\frac{-1}{10301}$	$\frac{2}{9599}$	$\frac{-3}{10701}$	$\frac{5}{8899}$	$\frac{-8}{11801}$	$\frac{13}{7099}$
3	$\frac{1}{998999}$	$\frac{-1}{1003001}$	$\frac{2}{995999}$	$\frac{-3}{1007001}$	$\frac{5}{988999}$	$\frac{-8}{1018001}$	$\frac{13}{970999}$

TABLE 11. Fractions whose repetends end in L_{nm} with positive and negative terms alternated, positive fractions begin with positive L_{mn}, negative fractions opposite

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{-19}{89}$	$\frac{-23}{131}$	$\frac{-16}{59}$	$\frac{-27}{171}$	$\frac{9}{11}$	$\frac{-38}{281}$	$\frac{-9}{191}$
2	$\frac{-199}{9899}$	$\frac{-203}{10301}$	$\frac{-196}{9599}$	$\frac{-207}{10701}$	$\frac{-189}{8899}$	$\frac{-218}{11801}$	$\frac{-171}{7099}$
3	$\frac{-1999}{998999}$	$\frac{-2003}{1003001}$	$\frac{-1996}{995999}$	$\frac{-2007}{1007001}$	$\frac{-1980}{988999}$	$\frac{-2018}{1018001}$	$\frac{-1991}{970999}$

Because

$$0.\ldots\overline{F}_7F_6\overline{F}_5F_4\overline{F}_3F_2\overline{F}_1 + 0.\ldots F_7\overline{F}_6F_5\overline{F}_4F_3\overline{F}_2F_1$$
$$-0.\ 0000\ldots0001 = 0.\ 9999\ldots999,$$

$0.\ldots\overline{F}_7F_6\overline{F}_5F_4\overline{F}_3F_2\overline{F}_1$ and $0.\ldots F_7\overline{F}_6F_5\overline{F}_4F_3\overline{F}_2F_1$

are complementary numbers. This result can be described in another way:

If $N/M > 0$, N/M ends in $0.\ldots F_7\overline{F}_6F_5\overline{F}_4F_3\overline{F}_2F_1$

then $N/M - 1$ ends in $0.\ldots \overline{F}_7F_6\overline{F}_5F_4\overline{F}_3F_2\overline{F}_1$;*

if $N/M < 0$, N/M ends in $0.\ldots \overline{F}_7F_6\overline{F}_5F_4\overline{F}_3F_2\overline{F}_1$

then $1 + N/M$ ends in $0.\ldots F_7\overline{F}_6F_5\overline{F}_4F_3\overline{F}_2F_1$.*

*: -1 is added to the rightmost digit.

So, Tables 10 and 11 have their complementary tables.

From Theorem 4, if we use (-10^k) instead of (10^k), then we will have

Theorem 6: For $N/M > 0$, the repeating cycle of

$$(17)\qquad \frac{N}{M} = \frac{T_p \cdot 10^{2k} - (T_{2n+p} - R_nT_{n+p}) \cdot 10^k + T_{n+p}}{-10^{3k} - R_{-n} \cdot 10^{2k} - R_n \cdot 10^k - 1}$$

ends with T_{nm+p}, even terms are negative; for $N/M < 0$, the repeating cycle of

$$(18)\qquad \frac{N}{M} = \frac{T_p \cdot 10^{2k} - (T_{2n+p} - R_nT_{n+p}) \cdot 10^k + T_{n+p}}{-10^{3k} - R_{-n} \cdot 10^{2k} - R_n \cdot 10^k - 1}$$

ends with T_{nm+p}, odd terms are negative, both appearing in blocks of k digits. Table 12 shows some illustrations of (17) and (18).

As before, the above results can be developed as follows:

If $N/M > 0$, N/M ends in $0.\ldots T_{7n}\overline{T}_{6n}T_{5n}\overline{T}_{4n}T_{3n}\overline{T}_{2n}T_n$,

then $N/M - 1$ ends in $0.\ldots \overline{T}_{7n}T_{6n}\overline{T}_{5n}T_{4n}\overline{T}_{3n}T_{2n}\overline{T}_n$;*

if $N/M < 0$, N/M ends in $0.\ldots \overline{T}_{7n}T_{6n}\overline{T}_{5n}T_{4n}\overline{T}_{3n}T_{2n}\overline{T}_n$,

then $1 + N/M$ ends in $0.\ldots T_{7n}\overline{T}_{6n}T_{5n}\overline{T}_{4n}T_{3n}\overline{T}_{2n}T_n$.*

*: -1 is added to the rightmost digit.

So, Table 12 has its complementary table, too.

TABLE 12. Fractions whose repetends end in T_{nm} appearing with positive and negative terms alternated

$k \backslash n$	1	2	3	4	5	6	7
1	$\frac{-1}{911}$	$\frac{9}{931}$	$\frac{-12}{1571}$	$\frac{-4}{611}$	$\frac{13}{1111}$	$\frac{-43}{2491}$	$\frac{-14}{211}$
2	$\frac{-1}{990101}$	$\frac{99}{990301}$	$\frac{-102}{1050701}$	$\frac{-4}{951101}$	$\frac{193}{992101}$	$\frac{-313}{1113901}$	$\frac{76}{857101}$
3	$\frac{-1}{999001001}$	$\frac{999}{999003001}$	$\frac{-1002}{1005007001}$	$\frac{-4}{995011001}$	$\frac{1993}{999021001}$	$\frac{-3013}{1011039001}$	$\frac{976}{985071001}$

6. Conclusion

Tables 1-5 and Tables 6-12 have a great difference, the former tables contain blanks, the latter do not. Examining $M = 10^{3k} - R_{-n} \cdot 10^{2k} + R_n \cdot 10^k - 1$, R_n is always greater then R_{-n}, so we can calculate M whenever we wish.

From the above discussion, we can find the following interesting results:

$1/89 = 0.0112358\ldots = 0.\ F_0F_1F_2F_3F_4F_5F_6\ldots,$

$10/89 = 0.112358\ldots = 0.\ F_1F_2F_3F_4F_5F_6\ldots,$

$10/109 = 0.1\bar{1}2\bar{3}5\bar{8}\ldots = 0.\ F_1\bar{F}_2F_3\bar{F}_4F_5\bar{F}_6\ldots,$

$1/109$ ends in $\ldots 853211$ or $\ldots F_6F_5F_4F_3F_2F_1,$

$1/89$ ends in $\ldots \bar{8}5\bar{3}2\bar{1}1$ or $\ldots \bar{F}_6F_5\bar{F}_4F_3\bar{F}_2F_1,$

$88/89$ ends in $\ldots 8\bar{5}3\bar{2}1\bar{1}$ or $\ldots F_6\bar{F}_5F_4\bar{F}_3F_2\bar{F}_1,$*

$100/889 = 0.112485939\ldots = 0.\ T_1T_2T_3T_4T_5T_6T_7\ldots,$

$100/1091 = 0.1\bar{1}2\bar{4}7\ldots = 0.\ T_1\bar{T}_2T_3\bar{T}_4T_5\bar{T}_6T_7\ldots,$

$1/1109$ ends in $\ldots 374211$ or $\ldots T_7T_6T_5T_4T_3T_2T_1,$

$1/911$ ends in $\ldots 3\bar{7}4\bar{2}1\bar{1}$ or $\ldots \bar{T}_7T_6\bar{T}_5T_4\bar{T}_3T_2\bar{T}_1,$

$910/911$ ends in $\ldots \bar{3}7\bar{4}2\bar{1}1$ or $\ldots T_7\bar{T}_6T_5\bar{T}_4T_3\bar{T}_2T_1,$*

*: -1 is added to the rightmost digit.

One of the above,

$1/1109 = 0.00\ldots 862385374211,$

can not only end in T_m, $m = 1, 2, 3, 4, 5, \ldots$, but can also end in T_{9m}, $m = 1, 2, 3, 4, 5, \ldots$. Summing up, we may find different forms of the decimal expansion for a particular fraction. Perhaps, they could be explored on another occasion.

In another article written by this author (unpublished), even Tetrabonacci series can also be divided into four types, as above.

Acknowledgment

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BOOK REVIEW

A New Chapter for Pythagorean Triples by A. G. Schaake and J. C. Turner

In this book, the authors develop a new method for generating all Pythagorean triples. They also illustrate that their new method can be used to find solutions to the Pellian equations $x^2 - Ny^2 = \pm 1$ where N is square-free. Since the book contains only accusations and examples, it is impossible to verify that their method is mathematically correct even though the numerous examples found in the book seem to imply that it does work. The authors have published a Departmental Research Report, with proofs of their methods, which may be had, on request, with the book. The reviewer has not read the Research Report.

The method, at least to this reviewer, appears to be new. Furthermore, the method is *a very neat way of relating Pythagorean triples to continued fractions* via what is called a "decision tree." However, the reviewer does not accept the new method with the enthusiasm of the authors because they make claims which, in the opinion of the reviewer, may not be true. Several of these claims will be discussed later in this report.

The basic claim of the authors is essentially that (x, y, z) is a Pythagorean triple if and only if

$$x = \frac{q - r}{2n}, \quad y = \frac{p + s}{2n}, \quad z = \frac{q + r}{2n}$$

where r/s and p/q are, respectively, the last two convergents of a continued fraction of the form

$$[0; u_1, u_2, \ldots, u_i, v, 1, j, (v + 1), u_i, \ldots, u_2, u_1].$$

Using the parity of v, a nice contraction method developed by the authors for the set of values $u_1, u_2, \ldots, u_i$ and the size of j, the authors illustrate that there are five families which predict the value of n.

Most of the book is spent on the development of the techniques used and examples which show how the techniques work. The explanations are clear and the examples are well done. Actually, there are far more examples than are probably needed. The book is very easy to read. In fact, several chapters could be reduced in size or eliminated since anyone with a background in number theory would know most if not all of the material in Chapters 1 and 2. Other parts of the book could also be left out. For example, the tables on pages 127 to 137 were of no value to the reviewer. To be fair to the authors on this point, however, in the Foreword they do state that the material is intended to be accessible to teachers and college students, as well as to number theorists and professional mathematicians. *(Please turn to page 155)*

EQUAL SUMS OF UNLIKE POWERS

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1. Introduction

Solutions are given for the Diophantine equation

$$x_1^p + x_2^p + \cdots + x_m^p = y_1^q + y_2^q + \cdots + y_n^q, \quad p > 0,\ q > 0,\ m + n > 2,$$

for which we use the notation $(p.q.m.n)$. In a previous paper [1] we surveyed solutions of this equation for $p = q$ with p and $q \le 10$. We now show that $(p.q.m.n)$ has nontrivial parametric solutions in which the number of terms m, n on both sides of the equation depend on p and q. Some of these solutions will be valid when $p = q$ as a special case, but in general we assume that $p > q$. That is, we always write the equation with the higher exponent on the left-hand side. We assume that none of the x_i or y_j is zero, and that $x_i^p \ne y_j^q$, i.e., that equal individual terms on both sides of the equation have been removed. Rarely does this condition invalidate one of the many solutions available by our algorithms.

Related work includes a number of parametric solutions and also numerical solutions, usually involving low values of either p or q or both. Uspenski [2] gives a general solution in relatively prime integers of $z^n = x^2 + y^2$ for $n > 1$. Various solutions of the equation $z^2 = x^3 + y^3$ by Euler, Hoppe, Thue, and Schwering are given in Dickson [3]. The equation $(3.2.n.1)$ was solved for various values of n by a number of investigators [4], [5]. Cunningham gave a procedure for solving $(2n.4.2.3)$ in [6]. Several writers solved $(4.2.m.n)$ for various values of m and n [7]. Some numerical examples of biquadrates as the sum of several cubes or squares are given in [8]. A parametric solution of (5.2.3.1) was obtained by Bouniakowsky [9]. Cunningham solved (8.2.6.1) in [10] and both (4.2.3.3) and (8.4.3.3) in [11]. Rignaux solved (6.2.2.2) in [12]. Killgrove [13] discussed the equation $x^n + y^m = z^k$ and gave a proof for a theorem of Lebesgue [19] which states that if $x^{2t} + y^{2t} = z^2$ has a nontrivial solution, then t is odd and $u^t + v^t = w^t$ has a nontrivial solution. Beerensson [14] proved that $x^n + y^n = z^m$ has infinitely many integer solutions if m, n are relatively prime, but did not present explicit solutions. In [20], Kelemen proved two theorems on conditions for the solvability and form of solutions of the general equation

$$a_1x_1^{k1} + a_2x_2^{k2} + \cdots + a_nx_n^{kn} = 0,$$

and gave examples.

2. Solution for all Positive Values of p, q

Theorem 1: The Diophantine equation

$$x_1^p + x_2^p + \cdots + x_m^p = y_1^q + y_2^q + \cdots + y_n^q, \tag{1}$$

where $p > 0$, $q > 0$, $m > 0$, $n > 0$, and $m + n > 2$, has a nontrivial parametric integer solution, as follows. If d is the greatest common divisor of p and q, this solution exists for all m, n such that

$$m = \sum_{k=2}^{r} (u_k + v_k k^d), \quad n = \sum_{k=2}^{r} (v_k + u_k k^d),$$

where r is any integer > 1 and, for $k = 2, 3, \ldots, r$, the u_k and v_k are arbitrary nonnegative integers not all zero.

Proof: Since d is the greatest common divisor of p and q, there exist positive integers A, B, C, D such that

(2) $$Ap - Bq = Cq - Dp = d.$$

Let $a_1, a_2, \ldots, a_s$ and $b_1, b_2, \ldots, b_t$ be arbitrary nonzero integers where $s > 1$ and $t > 1$, and let

(3) $$u = \sum_{k=1}^{s} a_k^p, \quad v = \sum_{k=1}^{t} b_k^q.$$

Then u^d, when expanded by multiplication, is the sum of s^d terms, each of which is the product of d numbers of the form a_k^p. Therefore, each term of u^d is of the form y^p, where y is an integer. Thus, we have

(4) $$u^d = \sum_{j=1}^{s^d} y_j^p,$$

where the y_j are all integers. Similarly, we have

(5) $$v^d = \sum_{j=1}^{t^d} z_j^q,$$

where the z_j are all integers. Then, from (2) and (4),

$$u^{Cq} = u^{Dp}u^{Cq-Dp} = u^{Dp}u^d = \sum_{j=1}^{s^d} u^{Dp}y_j^p,$$

so that

(6) $$(u^C)^q = \sum_{j=1}^{s^d} (y_j u^D)^p$$

is a nontrivial parametric solution of (1) with $m = s^d$, $n = 1$, and having $s > 1$ arbitrary nonzero integer parameters $a_1, a_2, \ldots, a_s$. Similarly,

$$v^{Ap} = v^{Bq}v^{Ap-Bq} = v^{Bq}v^d = \sum_{j=1}^{t^d} v^{Bq}z_j^q,$$

or

(7) $$(v^A)^p = \sum_{j=1}^{t^d} (z_j v^B)^q,$$

which is a nontrivial parametric solution of (1) with $m = 1$, $n = t^d$, and having $t > 1$ arbitrary nonzero integer parameters $b_1, b_2, \ldots, b_t$.

Next, we may "add" two or more solutions of (1) by summing the terms with exponent p to form the left-hand side of the new solution and summing the terms with exponent q to form the right-hand side. Therefore, a valid nontrivial parametric solution of (1) may be obtained by summing u_k solutions of the form given by (7) for $t = k$, together with v_k solutions of the form given by (6) with $s = k$, where k takes on the values $2, 3, \ldots, r$ for any arbitrary integer $r > 1$. The numbers of solutions to be "added" in this way, u_k and v_k, may be any nonnegative integers not all zero. Then m, n, the number of terms in the resultant equation having exponents p, q, respectively, will be as given in the theorem.

Example 1: Let $p = 4$ and $q = 3$ so that $d = 1$. Take $A = B = 1$, $C = 3$, $D = 2$. Let $r = 2$ so that $s = 2$ and $t = 2$. We have

$$u = a_1^4 + a_2^4,\ v = b_1^3 + b_2^3,\ y_1 = a_1,\ y_2 = a_2,\ z_1 = b_1,\ z_2 = b_2.$$

The solution (6) becomes

$$(6.1) \qquad [(a_1^4 + a_2^4)^3]^3 = [a_1(a_1^4 + a_2^4)^2]^4 + [a_2(a_1^4 + a_2^4)^2]^4$$

and the solution (7) becomes

$$(7.1) \qquad (b_1^3 + b_2^3)^4 = [b_1(b_1^3 + b_2^3)]^3 + [b_2(b_1^3 + b_2^3)]^3.$$

Two numerical examples of (6.1) for $(a_1, a_2) = (1, 1)$ and $(2, 1)$ are

$$8^3 = 4^4 + 4^4;\ 4913^3 = 578^4 + 289^4.$$

Two numerical examples of (7.1) for $(b_1, b_2) = (2, 1)$ and $(3, 2)$ are

$$9^4 = 18^3 + 9^3;\ 35^4 = 105^3 + 70^3.$$

We may obtain further solutions by combining (through "addition") any number of the individual solutions. For example, from those given, we get

$$9^4 + 4^4 + 4^4 = 18^3 + 9^3 + 8^3;\ 578^4 + 289^4 + 35^4 = 4913^3 + 105^3 + 70^3,$$

and so on.

Example 2: Let $p = 6$ and $q = 4$ so that $d = 2$. Take $A = B = 1$. Set $r = 2$ so that $t = 2$. Then we have $v^2 = (b_1^4 + b_2^4)^2$, so that

$$z_1 = b_1^2,\ z_2 = z_3 = b_1b_2,\ z_4 = b_2^2.$$

Solution (7) becomes)

$$(7.2) \qquad (b_1^4 + b_2^4)^6 = [b_1^2(b_1^4 + b_2^4)]^4 + 2[b_1b_2(b_1^4 + b_2^4)]^4 + [b_2^2(b_1^4 + b_2^4)]^4.$$

Two numerical examples of (7.2) for $(b_1, b_2) = (1, 1)$ and $(2, 1)$ are

$$2^6 = 2^4 + 2^4 + 2^4 + 2^4;\ 17^6 = 68^4 + 34^4 + 34^4 + 17^4.$$

Note that the terms in each equation of the type (6) and (7) are not relatively prime. However, since the exponents p and q are different, it is not usually possible to remove a common factor and still have an equation remaining with the same exponents p and q. This would be possible if in equation (1) there is a divisor F of all the terms $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$, where F is of the form z^f and f is divisible by p and q, and $z > 1$. When solutions involving different sets of parameters a_i and b_j are combined by "addition," the resultant solution will not in general have such a common divisor (as in the examples given above).

3. Solution for p and q Relatively Prime

Theorem 2: Whenever p and q are relatively prime, equation (1) of Theorem 1 has a nontrivial parametric integer solution for all positive values of m, n such that $m + n > 2$.

Proof: In Theorem 1, let $d = 1$. We use the notation $(p.q.m.n)$ to denote equation (1). Then (6) gives a solution of $(p.q.s.1)$ for arbitrary $s > 1$, which we denote by (S). If $n = 1$, set $s = m$ to solve $(p.q.m.n)$ with m integer parameters. Similarly, (7) gives a solution of $(p.q.1.t)$ for arbitrary $t > 1$, which we denote by (T). If $m = 1$, set $t = n$ to solve $(p.q.m.n)$ with n integer parameters. Next, assume that $m > 2$ and $n > 2$. Now set $s = m - 1$ and $t = n - 1$. Then "add" the two solutions (S), (T) to obtain a new solution of $(p.q.s + 1.t + 1) = (p.q.m.n)$. This solution will have $s + t = m + n - 2$ arbitrary integer parameters. Next, if $m = 2$ and $n > 3$, add solution (T) with $t = 2$ to solution (T) with $t = n - 2$ to obtain a solution of $(p.q.2.n)$ having n integer parameters. Similarly, if $n = 2$ and $m > 3$, add solution (S) with $s = 2$ to solution (S) with $s = m - 2$ to obtain a solution of $(p.q.m.2)$ having m integer parameters.

There remain only three cases, namely, $(p.q.2.2)$, $(p.q.2.3)$, and $(p.q.3.2)$. For the case $m = n = 2$, let a, b be distinct positive integers, arbitrary except that both are even or both are odd. Then $a^q + b^q = 2w$, where w is an integer. Then, since p and q are relatively prime, we have $Ap - Bq = 1$ for integers A, B and

$$w^{Bq}(a^q + b^q) = w^{Bq}(2w) = 2w^{Bq+Ap-Bq} = 2w^{Ap}.$$

Then

$$(aw^B)^q + (bw^B)^q = (w^A)^p + (w^A)^p$$

is a solution of $(p.q.2.2)$ having two integer parameters a, b of equal parity but otherwise arbitrary. For the case $m = 2$, $n = 3$, let a, b, and c be distinct positive integers, arbitrary except that the sum $a^q + b^q + c^q = 2w$, where w is an integer. This can be achieved by selecting a, b, and c to all be even, or by choosing one of a, b, or c to be even and the others odd.

Then, as before, we have $Ap - Bq = 1$ for integers A, B, and

$$w^{Bq}(a^q + b^q + c^q) = w^{Bq}(2w) = 2w^{Bq+Ap-Bq} = 2w^{Ap}.$$

Therefore,

$$(aw^B)^q + (bw^B)^q + (cw^B)^q = (w^A)^p + (w^A)^p$$

is a solution of $(p.q.2.3)$ having three integer parameters. In a similar manner, we can generate a three-parameter solution of $(p.q.3.2)$. This completes the proof.

Example 3: Let $p = 8$ and $q = 5$. First, to solve $(8.5.2.2)$, take $a = 3$, $b = 1$ so that $3^5 + 1^5 = 244 = 2(122)$ and $w = 122$. Then, since $2(8) - 3(5) = 1$, we may take $A = 2$, $B = 3$, and $122^{15}(3^5 + 1^5) = 122^{16}(2)$, or

$$[3(122)^3]^5 + [(122^3)]^5 = [(122^2)]^8 + [(122^2)]^8.$$

To solve $(8.5.2.3)$, take $a = 2$, $b = c = 1$, so that $2^5 + 1^5 + 1^5 = 34 = 2(17)$ and $w = 17$. Then, $17^{15}(2^5 + 1^5 + 1^5) = 17^{16}(2)$, or

$$[2(17^3)]^5 + (17^3)^5 + (17^3)^5 = (17^2)^8 + (17^2)^8.$$

4. Derived Solutions

Theorem 3: If a specific nontrivial solution of equation $(p.q.m.n)$ exists for which all of the n terms y_j^q in equation (1) are equal, then a nontrivial solution exists for the equation $(q + pr.p.n.m)$, where r is any nonnegative integer.

Proof: If

$$nb^q = \sum_{i=1}^{m} a_i^p$$

is the specific nontrivial solution of $(p.q.m.n)$, then

$$nb^q b^{pr} = b^{pr}\sum_{i=1}^{m} a_i^p = \sum_{i=1}^{m}(a_i b^r)^p = nb^{q+pr}$$

is a solution of the equation $(q + pr.p.n.m)$.

Example 4: A computer search by the author yielded the smallest nontrivial solution of (6.2.3.1) as $100^6 + 81^6 + 42^6 = 1134865^2$. If we set $b = 1134865$, we have

$$(100b^r)^6 + (81b^r)^6 + (42b^r)^6 = b^{6r+2}$$

as a solution of equation $(6r + 2.6.1.3)$ for $r \geq 0$.

Theorem 3 can also be applied when $p = q$. The solutions recently found by Eklies [15] and Frye [16] to the equation $x^4 + y^4 + z^4 = t^4$ allows us to solve the equation $(4r + 4.4.1.3)$, for any integer $r \geq 0$. In particular, for $r = 1$, we have

$$(tx)^4 + (ty)^4 + (tz)^4 = t^8$$

as a solution of (8.4.1.3), where $x = 95800$, $y = 217519$, $z = 414560$, and $t = 422481$. Other solutions to the equation $(p.p.m.n)$ can be found in [1].

5. Incompleteness of the Theorems

The solutions to (1) produced by the algorithms of Theorems 1, 2, and 3 are not complete. The smallest nontrivial solution of (4.2.3.1) is

$$20^4 + 15^4 + 12^4 = 481^2,$$

which cannot be produced by Theorem 1, since 481 is prime to 20, 15, and 12. The smallest nontrivial solution of (4.3.2.2) is

$$11^4 + 8^4 = 24^3 + 17^3.$$

This solution cannot be produced by Theorem 2, which yields only solutions of the form

$$x_1^p + x_2^p = 2y_1^q,$$

or by Theorem 3, which yields only solutions of the form

$$x_1^p + x_2^p + \cdots + x_m^p = ny_1^q.$$

6. Table of Solutions

We supplement the discussion by presenting in Table 1 a list of solutions to equation $(p.q.m.n)$ for p and $q < 10$ and m and $n < 4$. The solutions were obtained by a combination of methods, including the use of Theorems 1, 2, and 3, computer search, and reference to the literature. As illustrated in the table, the solutions produced by use of Theorems 1, 2, and 3 are incomplete, since solutions exist for which the terms in (1) have no common divisor > 1. Table 1 lists the solutions in smallest integers known to the author. Some equations have no nontrivial solutions. The equations (6.3.1.2), (6.3.2.1), (9.3.1.2), (9.3.2.1), (9.6.1.2), and (9.6.2.1) have no nontrivial solution because, as Euler proved [17], the equation $x^3 + y^3 = z^3$ has no solution with $xy \neq 0$; similarly, equations (4.2.2.1), (6.4.1.2), (8.2.2.1) and (8.6.2.1) cannot be solved nontrivially because Euler showed that the equation $x^4 + y^4 = z^2$ has no solution with $xy \neq 0$ [18]. The equations (6.2.2.1), (6.4.2.1), and (8.6.1.2) are impossible (because $x^3 + y^3 = z^3$ is impossible) by a theorem of Lebesgue [19]. As shown in Table 1, the equations for small values of p, q, m, and n which appear to be the most difficult to solve in small integers are (6.3.3.2), (6.3.3.1), (6.4.2.2), (6.4.3.1), (6.4.3.2), (8.2.3.1), (8.4.2.2), (8.4.3.1), (8.4.3.2); $(8.6.m.n)$ for $m < 4$, $n < 4$ except (8.6.1.3); (9.3.3.1); and $(9.6.m.n)$ with $m < 4$, $n < 4$. Although solutions were not found for these specific values of p, q, m, n, we can obtain solutions for the same values of p, q with larger values of m, n by applying Theorem 1. For example, solutions for (9.6.1.8) and (9.6.8.1) are

$$u^9 = (a^3u)^6 + (b^3u)^6 + 3[(a^2bu)^6 + (ab^2u)^6], \; u = a^6 + b^6;$$

$$(v^2)^6 = (a^3v)^9 + (b^3v)^9 + 3[(a^2bv)^9 + (ab^2v)^9], \; v = a^9 + b^9,$$

where a and b are arbitrary integers. If $a = 2$ and $b = 1$, then $u = 65$ and $v = 513$ and these solutions become

$$65^9 = 520^6 + 3(260^6) + 3(130^6) + 65^6;$$

$$263169^6 = 4104^9 + 3(2052^9) + 3(1026^9) + 513^9.$$

The author would be pleased to receive correspondence concerning any new solutions to the equations discussed above.

TABLE 1. Solutions of $\sum_{i=1}^{m} x_i^p = \sum_{j=1}^{n} y_j^q$

Legend: The entry $x_1, x_2, \ldots, x_m = y_1, y_2, \ldots, y_n$ denotes the solution.

p.q \ m.n	1.2	1.3	2.1	2.2
3.2	2=2,2 5=10,5 5=11,2	3=3,3,3 3=5,1,1	2,1=3 2,2=4 8,4=24	4,1=7,4 4,2=6,6
4.2	5=24,7 5=20,15	3=6,6,3 3=7,4,4	Impossible	5,5=35,5
4.3	2=2,2 9=18,9	3=3,3,3	4,4=8 32,32=128 108,108=648	11,8=24,17 14,14=42,14
5.2	2=4,4 5=41,38	3=9,9,9 3=11,11,1	2,2=8 8,8=256	3,1=12,10 4,1=31,8
5.3	3=6,3 4=8,8	6=18,12,6 9=27,27,27	2,2=4	6,6=24,12 12,10=70,18
5.4	2=2,2	3=3,3,3	8,8=16	41,41=123,41
6.2	5=100,75 5=117,44 5=120,35	3=18,18,9 3=26,7,2	Impossible	2,1=7,4 3,1=21,17
6.3	Impossible	3=8,6,1 5=22,17,4 6=30,24,18	Impossible	18,12=330,102 172,86=27778,16942
6.4	Impossible	481=20(481), 15(481),12(481)	Impossible	Unknown
6.5	2=2,2 33=66,33	3=3,3,3 34=68,34,34	16,16=32	122,122=366,122
7.2	2=8,8 5=205,190 5=250,125 5=278,29	3=45,9,9 3=43,17,17	2,2=16 8,8=2048	4,1=127,16 4,1=103,76 4,1=92,89
7.3	2=4,4 9=162,81	3=9,9,9 6=64,26,6	4,4=32 32,32=4096	14,14=588,196 16,12=620,404
7.4	8=32,32	11=55,55,33 27=243,243,243	2,2=4	$41^3,41^3=3(41^5),41^5$
7.5	8=16,16	27=81,81,81	4,4=8	$122^3,122^3=3(122^4),122^4$
7.6	2=2,2 65=130,65	3=3,3,3 66=132,66,66	32,32=64 $2a^5,a^5=a^6$ a=129	365,365=1095,365

TABLE 1 (continued)

p.q \ m.n	1.2	1.3	2.1	2.2
8.2	5=500,375 5=585,220 5=600,165	3=54,54,27 3=63,36,36 3=79,16,8	Impossible	3,1=71,39 3,2=79,24 4,3=264,49
8.3	3=18,9 4=32,32	6=108,72,36 8=255,57,22 9=243,243,243	2,2=8	6,6=144,72
8.4	Impossible	See Text, Section 4.	Impossible	Unknown
8.5	8,8=32 $33^2=2(33^2)$, 33^2	9=27,27,27	8,8=32	$a^2,a^2=3a^3,a^3$ a=122
8.6	Impossible	a=100a,81a, 42a a=1134865	Impossible	Unknown
8.7	2=2,2	3=3,3,3	64,64=128	a,a=3a,a a=1094
9.2	2=16,16	3=81,81,81	2,2=32	5,5=1875,625
9.3	Impossible	3=24,18,3 5=110,85,20	Impossible	4,2=60,36 7,2=322,191 8,4=480,288
9.4	2=4,4 17=578,289	3=9,9,9	8,8=128	$a,a=3a^2,a^2$ a=41
9.5	16=128,128	$81=3^7,3^7,3^7$	2,2=4	$a^4,a^4=3a^7,a^7$ a=122
9.6	Impossible	Unknown	Impossible	Unknown
9.7	16=32,32	$81=3^5,3^5,3^5$	8,8=16 $2a^3,a^3=a^4$ a=513	$a^4,a^4=3a^5,a^5$ a=1094
9.8	2=2,2 257=514,257	3=3,3,3	128,128=256	a,a=3a,a a=3281

TABLE 1 (continued)

p.q \ m.n	2.3	3.1	3.2	3.3
3.2	3,2=5,3,1 3,3=5,5,2	3,2,1=6 3,3,3=9 6,2,1=15	3,1,1=5,2 4,2,1=8,3	2,2,1=3,2,2
4.2	2,1=3,2,2 3,1=8,3,3	20,15,12=481	2,1,1=3,3 3,2,1=7,7	2,2,1=5,2,2 3,1,1=7,5,3
4.3	5,4=9,5,3	5,5,3=11 9,9,9=27	3,3,3=6,3 8,5,4=17,4	4,1,1=5,5,2
5.2	2,1=5,2,2 3,1=12,8,6 3,3=22,1,1	3,3,3=27 12,12,12=864 15,5,5=875	2,1,1=5,3 2,2,1=7,4	2,1,1=4,3,3 2,2,1=6,5,2
5.3	6,4=17,15,8 8,3=32,6,3 9,3=36,21,15	3,3,3=9 24,24,24=288 68,34,34=1156	9,3,3=39,6 9,9,9=54,27	3,2,2=6,4,3 3,3,3=8,6,1
5.4	9,9=18,9,9	27,27,27=81	17,4,1=37,17	6,4,3=9,7,3 6,6,6=12,6,6
6.2	2,1=6,5,2 5,3=127,12,9 5,5=176,15,7	100,81,42= 1134865	3,2,1=25,13 3,2,2=29,4 3,3,2=39,1	2,1,1=5,5,4 2,2,1=10,5,2 2,2,1=11,2,2
6.3	7,5=46,33,1 7,6=50,34,1	Unknown	Unknown	3,3,1=11,4,4 6,2,1=30,25,16
6.4	3,3=6,3,3 7,7=19,18,1 7,7=21,14,7	Unknown	Unknown	10,6,1=30,22,7 10,9,1=34,21,5
6.5	17,17=34,17, 17	81,81, 81,=243	11,11,11=22,11	16,16,2=32,2,2
7.2	2,1=11,2,2 2,1=10,5,2 2,1=8,7,4	3,3,3=81 12,12,12= 10368	2,1,1=11,3 2,1,1=9,7	2,2,1=15,4,4 2,2,1=12,8,7 2,2,2=16,8,8
7.3	4,2=20,20,8 5,4=44,21,4	9,9,9=243	3,3,1=15,10 3,3,3=18,9	4,4,2=32,4,4 6,6,2=76,49,15

TABLE 1 (continued)

p.q \ m.n	2.3	3.1	3.2	3.3
7.4	9,9=54,27,27	3,3,3=9	$a^3,a^3,a^3=6a^5,3a^5$ $a=459$	6,1,1=23,3,2 8,2,2=32,32,4
7.5	$a^3,a^3=2a^4,$ a^4,a^4 $a=17$	9,9,9=27	$2a^2,a^2,a^2=a^3,a^3$ $a=65$	8,4,4=16,16,8
7.6	33,33=66, 33,33	243,243,243 =729	$a,a,a=6a,3a$ $a=15795$	32,32,2=64,2,2
8.2	2,1=11,10,6 2,1=12,8,7	Unknown	3,3,2=97,63	2,1,1=11,11,4 2,1,1=13,18,5
8.3	4,2=40,12,4 4,2=33,31,4	3,3,3=27	9,9,9=486,243	2,1,1=5,5,2 3,2,2=18,9,8
8.4	7,7=56,35,21 7,7=55,39,16	Unknown	Unknown	4,4,3=18,13,8 5,4,3=24,19,5
8.5	$a^2,a^2=2a^3,$ a^3,a^3 $a=17$	27,27,27=243	$11^2,11^2,11^2=$ $2(11^3),11^3$	$a^2,a^2,a^2=3a^3,$ $2a^3,a^3$ $a=92$
8.6	Unknown	Unknown	Unknown	Unknown
8.7	65,65=130, 65,65	$3^6,3^6,3^6=3^7$	43,43,43=86,43	$a,a,a=3a,2a,a$ $a=772$
9.2	3,3=162, 81,81	3,3,3=243	2,1,1=17,15 $2a,a,a=a^5,a^5$ $a=257$	2,2,2=32,16,16
9.3	4,2=57,42,15 4,3=65,19,7	Unknown	12,8,8=1808,-784	3,2,1=23,18,3 3,3,2=32,17,13
9.4	9,9=162,81, 81	$27,27,27=3^7$	$a,a,a=6a^2,3a^2$ $a=459$	8,8,2=128,4,4
9.5	$a^4,a^4=2a^7,$ a^7,a^7 $a=17$	3,3,3=9	$2a,a,a=a^2,a^2$ $a=257$	$3a,2a,a=a^2,a^2,a^2$ $a=6732$
9.6	Unknown	Unknown	Unknown	Unknown
9.7	$a^4,a^4=2a^5,$ a^5,a^5 $a=65$	27,27,27=81	$2a^3,a^3,a^3=a^4,a^4$ $a=257$	$2a^3,a^3,16=a^4,32,$ 32 $a=513$
9.8	$a,a=2a,a,a$ $a=129$	$3^7,3^7,3^7=3^8$	$2a^7,a^7,a^7=a^8,a^8$ $a=257$	128,128,2=256, 2,2

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A POLYNOMIAL FORMULA FOR FIBONACCI NUMBERS

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1. Introduction

A Fibonacci sequence is defined by two initial terms, $F(1)$ and $F(2)$, together with the recursion equation

(1) $F(n+1) = F(n) + F(n-1),\ n = 2, 3, 4, \ldots .$

A closed form expression for the n^{th} Fibonacci number is given by

(2) $$F(n) = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n,\ n = 1, 2, 3, \ldots .$$

If we let $F(1) = F(2) = 1$ in equation (1), then we get the well-known sequence of Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots .$$

Because $F(n)$ is defined recursively in (1), we must know $F(n)$ and $F(n-1)$ in order to find $F(n+1)$. Therefore, to find $F(100)$ for example, we must first compute $F(3)$, $F(4)$, ..., $F(98)$, $F(99)$. This becomes a formidable computing task as n gets large. Finding $F(n)$ for large values of n from equation (2) is also a laborious task. Computing time, machine limits, and round-off error are problems that must be considered.

In this paper we assume that m terms of the Fibonacci sequence are known. To construct a formula that generates the m terms, one can use the Lagrangian approach to obtain the collocation polynomial. This method is based on the following theorem from [3].

Theorem: Let (x_k, f_k), $k = 0, 1, 2, \ldots, n$, denote $(n+1)$ points that would lie on the graph of a function. Then there exists a unique collocation polynomial $p(x) = \sum_{j=0}^{n} a_j x_j$ whose graph passes through the given $(n+1)$ points.

The Lagrangian method may require sophisticated numerical techniques in order to produce the collocation polynomial. However, the finite differences procedure and the examples presented here are at a level that can appeal to high school teachers with a desire to add interesting exercises involving Fibonacci numbers (or any sequence). Therefore, the emphasis in this paper is not on the derivation of the formula, but on the application of the formula to reproduce the given m Fibonacci numbers. In addition, the formula presented is in a more directly useable form than is usually available, and its purpose is different from equations (1) and (2). In some applications, such a formula may prove to be quite useful.

2. A Polynomial Formula Using Finite Differences

In this section we describe a general method for constructing a polynomial that generates the terms of a sequence. Let $s_1, s_2, \ldots, s_m$ be the terms of a sequence. Form the successive order differences as shown in Table 1.

TABLE 1

n	Sequence Terms	Difference 1st	2nd	3rd	4th	5th $\cdots$
1	s_1					
		D_1^1				
2	s_2		D_1^2			
		D_2^1		D_1^3		
3	s_3		D_2^2		D_1^4	
		D_3^1		D_2^3		D_1^5
4	s_4		D_3^2		D_2^4	
		D_4^1		D_3^3		
5	s_5		D_4^2			
		D_5^1				
6	s_6					
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

where

$$D_1^1 = s_2 - s_1 \qquad D_1^2 = D_2^1 - D_1^1 \qquad D_1^3 = D_2^2 - D_1^2$$

$$D_2^1 = s_3 - s_2 \qquad D_2^2 = D_3^1 - D_2^1 \qquad D_2^3 = D_3^2 - D_2^2$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$D_{m-1}^1 = s_m - s_{m-1} \qquad D_{m-2}^2 = D_{m-1}^1 - D_{m-2}^1 \qquad D_{m-3}^3 = D_{m-2}^2 - D_{m-3}^2 \quad \cdots$$

We assume that some order difference becomes constant. That is, $D_j^i = c$, $j = 1, 2, 3, \ldots, m - i$, for some $i = 1, 2, \ldots, m - 2$. Thus, the next order difference D_j^{i+1} is zero for all j.

Let $k \le m - 1$ be a positive integer such that D_j^k is zero for all $j = 1, 2, \ldots, m - k$. The general term of the original sequence can now be expressed by a polynomial in n. The polynomial formula that generates the sequence is based on the above finite difference table and is given by

$$s_n = s_1 + (n-1)D_1^1 + \frac{(n-1)(n-2)}{2!}D_1^2 + \frac{(n-1)(n-2)(n-3)}{3!}D_1^3 + \cdots + \frac{(n-1)(n-2)\cdots(n-(k-1))}{(k-1)!}D_1^{k-1} \tag{3}$$

Equation (3) is in terms of s_1, the first term of the sequence, and D_1^1, D_1^2, $\ldots$, D_1^{k-1}, the leading first terms of the various order differences. The complete derivation of (3) is given in [1] and [2].

Equation (3) assumes that the order differences, D_j^i, $j = 1, 2, \ldots, m - i$, are zero for some $i = 1, 2, \ldots, m - 1$. However, we have found that this condition is not necessary for the derivation of a generating polynomial. Equation (3) can be extended in order to construct a polynomial that generates the terms of any sequence whether or not the order differences, D_j^i, $j = 1, 2, \ldots, m - 1$, are zero for some $i = 1, 2, \ldots, m - 1$. We use the first term of the sequence, s_1, and the differences D_1^1, D_1^2, $\ldots$, D_1^{m-1}. The general term of the sequence if given by

$$s_n = s_1 + (n-1)D_1^1 + \frac{(n-1)(n-2)}{2!}D_1^2 + \frac{(n-1)(n-2)(n-3)}{3!}D_1^3 + \cdots + \frac{(n-1)(n-2)\cdots 2\ 1}{(m-1)!}D_1^{m-1} \tag{4}$$

3. Examples

In this section we apply equation (4) to several sequences. Consider the first four terms of the Fibonacci sequence, 1, 1, 2, 3. Form the order differences as shown in Table 2.

TABLE 2

n	Sequence	Differences		
1	1			
		0		
2	1		1	
		1		-1
3	2		0	
		1		
4	3			

Thus, $s_1 = 1$, $D_1^1 = 0$, $D_1^2 = 1$, and $D_1^3 = -1$. Substituting these values into (4) yields

$$(5)\qquad s_n = 1 + (n - 1)(0) + \frac{(n - 1)(n - 2)}{2}(1) + \frac{(n - 1)(n - 2)(n - 3)}{6}(-1)$$
$$= \frac{1}{6}(-n^3 + 9n^2 - 20n + 18).$$

For $n = 1, 2, 3, 4$, equation (5) yields the Fibonacci numbers 1, 1, 2, 3. Using (5), it is possible to generate $F(4)$ without having to compute $F(1)$, $F(2)$, $F(3)$ as in the recursion equation (1). Note that (5) does not generate the correct term $F(5) = 5$ for $n = 5$. This procedure produces a polynomial that generates only the terms of the initial sequence.

We do not have to begin the sequence of terms with $F(1)$ in order to apply (4). For example, consider $F(10)$, $F(11)$, $F(12)$, $F(13)$, $F(14)$, namely, 55, 89, 144, 233, 377. Table 3 contains the order differences.

TABLE 3

n	Sequence	Differences			
1	55				
		34			
2	89		21		
		55		13	
3	144		34		8
		89		21	
4	233		55		
		144			
5	377				

Here, $s_1 = 55$, $D_1^1 = 34$, $D_1^2 = 21$, $D_1^3 = 13$, $D_1^4 = 8$, $s_1 = F(10)$, $s_2 = F(11)$, $s_3 = F(12)$, $s_4 = F(13)$, $s_5 = F(14)$. Using (4), we obtain a polynomial that generates the sequence:

$$(6)\qquad s_n = 55 + (n - 1)(34) + \frac{(n - 1)(n - 2)}{2}(21) + \frac{(n - 1)(n - 2)(n - 3)}{6}(13)$$
$$+ \frac{(n - 1)(n - 2)(n - 3)(n - 4)}{24}(8)$$
$$= \frac{1}{6}(2n^4 - 7n^3 + 55n^2 + 58n + 222)$$

For $n = 1, 2, 3, 4, 5$, equation (6) yields the Fibonacci numbers

$$s_1 = F(10) = 55, \ldots, s_5 = F(14) = 377.$$

Once again we can generate any single term of the sequence without computing previous terms. For example, in order to generate $F(14) = 377$, we let $n = 5$ in (6). As in the previous example, we do not obtain $F(15) = 610$ by letting $n = 6$ in (6).

Suppose we are given a longer sequence of Fibonacci numbers. To obtain the generating polynomial, the above procedure suggests we must calculate *all* the order differences. Fortunately, this is not the case.

Consider the sequence consisting of the first ten Fibonacci numbers and the order differences given in Table 4.

TABLE 4

Fibonacci Numbers

$F(1)$	$F(2)$	$F(3)$	$F(4)$	$F(5)$	$F(6)$	$F(7)$	$F(8)$	$F(9)$	$F(10)$
1	1	2	3	5	8	13	21	34	55

Differences for Equation (4)

s_1	D_1^1	D_1^2	D_1^3	D_1^4	D_1^5	D_1^6	D_1^7	D_1^8	D_1^9
1	0	1	-1	2	-3	5	-8	13	-21

There is a definite pattern in the differences given in Table 4. The leading differences alternate in sign beginning with D and the absolute value of these differences yields the first eight Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21. The following examples further illustrate the pattern in the leading differences.

Consider the sixteen Fibonacci numbers beginning with $F(5) = 5$ through $F(2) = 6765$. The Fibonacci numbers and the leading differences are given in Table 5.

TABLE 5

Fibonacci Numbers

$F(5)$	$F(6)$	$F(7)$	$F(8)$	$F(9)$	$F(10)$	$F(11)$	$F(12)$
5	8	13	21	34	55	89	144
$F(13)$	$F(14)$	$F(15)$	$F(16)$	$F(17)$	$F(18)$	$F(19)$	$F(20)$
233	377	610	987	1597	2584	4181	6765

Differences for Equation (4)

s_1	D_1^1	D_1^2	D_1^3	D_1^4	D_1^5	D_1^6	D_1^7	D_1^8	D_1^9	D_1^{10}	D_1^{11}	D_1^{12}	D_1^{13}	D_1^{14}	D_1^{15}
5	3	2	1	1	0	1	-1	2	-3	5	-8	13	-21	34	-55

From Table 5, we see that

$$D_1^1 = F(4),\ D_1^2 = F(3),\ D_1^3 = F(2),\ D_1^4 = F(1).$$

After D_1^5, the differences follow the same pattern of differences as in the previous example. That is, the differences alternate in sign, and the absolute value of the differences yields the first ten Fibonacci numbers.

Therefore, suppose we consider a sequence of sixteen Fibonacci numbers beginning with $F(10) = 55$. Then the differences are found quickly and simply without computation from the patterns in the above examples. The differences for (4) are:

D_1^1	D_1^2	D_1^3	D_1^4	D_1^5	D_1^6	D_1^7	D_1^8	D_1^9	D_1^{10}	D_1^{11}	D_1^{12}	D_1^{13}	D_1^{14}	D_1^{15}
34	21	13	8	5	3	2	1	1	0	1	-1	2	-3	5

Substituting these values into (4), we obtain a polynomial in n which generates the sixteen Fibonacci numbers $F(10) = 55$ through $F(25) = 75025$.

These examples demonstrate a technique for obtaining a polynomial that generates any finite sequence of Fibonacci numbers. The leading order differences must be calculated in order to determine the polynomial, but they follow a discernible pattern. The resulting polynomial generates only those terms in the initial sequence and is useful in some applications.

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(continued from page 140)

The reviewer has some problems with comments made by the authors. First, the authors could, I believe, have misinterpreted the quote by Schalau and Opolka which is given in the Foreword. The Pythagorean triple problem was completely solved in antiquity if by this statement Schalau and Opolka meant that a method had been developed which totally solved the problem of finding all Pythagorean triples. If Schalau and Opolka meant that no new results could be found, then the authors are correct. I believe that the former is the case.

The authors also claim that there is no technique for systematically generating all Pythagorean triples by the old method. This is, I believe, a matter of opinion. The reviewer happens to believe that the original technique developed by Diophantus is very systematic. That is, (x, y, z) is a Pythagorean triple if and only if $x = u^2 - v^2$, $y = 2uv$, and $z = u^2 + v^2$, where $u > v$. The problem here is the meaning of "systematic." The authors also feel that their method is more time efficient. The reviewer has a problem with this. Finding the greatest common divisor of two integers, even when large, is not a problem for the computer. It does take time but would it take any more time than is needed to go through the contraction method developed by the authors or to find the convergents needed for the continued fraction or to pick and implement the method (class) that gives the correct value of n? I think not.

Overall, I would recommend the book and suggest that those interested in Pythagorean triples or Pellian equations read it.

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SPRINGS OF THE HERMITE POLYNOMIALS

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1. Introduction

The Hermite polynomials, Legendre polynomials, Laguerre polynomials, Gegenbauer polynomials, and Jacobi polynomials belong to the system of classical orthogonal polynomials (see, e.g., [4]). For each class of these polynomials, it is well known that the orthogonal property, differential equation (generalized), Rodrigues representation, and three-term recurrence relation are all equivalent (see, e.g., [4]) in the sense that any one of the above four properties implies the other three.

Throughout this paper we concentrate exclusively on the Hermite polynomials $H_n(x)$. There exist in the literature (see, e.g., [1]-[3], [5], [6], [8]) many starting points for developing the properties of the Hermite polynomials: (i) Hermite differential equation (see, e.g., [6]), (ii) Rodrigues' representation [8], (iii) the simple but beautiful relation [9], given in Arfken ([2], Prob. 13.1.5, p. 718),

(1) $$H_n(x) = (2x - D)^n 1, \; D \equiv d/dx, \; n \geq 0,$$

and (iv) the following generating function (see, e.g., [1]-[3], [5])

(2) $$\exp(2tx - t^2) = \sum_{n=0}^{\infty} H_n(x) t^n/n!$$

Many generating functions exist for the Hermite polynomials (see, e.g., [5]). However, throughout this paper by generating function for $H_n(x)$ we only mean the more familiar one defined by (2). Moreover, we follow the convention that $W^0 = I$, the unit operator, for any operator W. The purpose of this paper is to present the following relation

(3) $$H_n(x) = g^{-1}[2x - D + g^{-1}\{Dg\}]^n g, \; D \equiv d/dx, \; n \geq 0,$$

where $g(x)$ is *any* differentiable function not identically zero, as the spring (starting point) for the starting points. We begin with a derivation of (3) and then show that all properties of the Hermite polynomials and many a beautiful relation follow from it.

2. Spring of Springs

Actually, (3) is a combination of the pure recurrence relation (see, e.g., [5])

(4) $$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \; n \geq 1,$$

and the differential recurrence relation (see, e.g., [5])

(5) $$DH_n(x) = 2nH_{n-1}(x), \; n \geq 1,$$

and the results (see, e.g., [5])

(6) $$H_0(x) = 1,$$

(7) $$H_1(x) = 2x.$$

The proof is as follows. Using (4) and (5), we have

(8) $\qquad H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x) = (2x - D)H_m(x), \ m \geq 1.$

Moreover, in view of (6) and (7), $H_1(x) = (2x - D)H_0(x)$. Thus,

(9) $\qquad H_n(x) = (2x - D)H_{n-1}(x), \ n \geq 1.$

If $g(x)$ is any differentiable function not identically zero, then

(10)
$$\begin{aligned} gH_n(x) &= g(2x - D)H_{n-1}(x) \\ &= [2x - D + g^{-1}\{Dg\}]\{gH_{n-1}(x)\}, \ n \geq 1. \end{aligned}$$

Iteration of (10) yields

(11) $\qquad gH_n(x) = [2x - D + g^{-1}\{Dg\}]^n g, \ n \geq 1,$

since $H_0(x) = 1$. However, (11) is also true for $n = 0$. Relation (3) now follows immediately.

The interesting point about (3) is that one need not specify what $g(x)$ is. *Any* differentiable function not identically zero will suffice. Thus, for example, when $g = 1$, we obtain the beautiful relation given in Arfken ([2], Prob. 13.1.5, p. 718):

(1) $\qquad H_n(x) = (2x - D)^n 1, \ D \equiv d/dx, \ n \geq 0.$

When $g = \exp(-x^2/2)$, we derive the relation

(12) $\qquad H_n(x) = \exp(x^2/2)(x - D)^n \exp(-x^2/2), \ n \geq 0,$

a result that is very useful in the quantum mechanical treatment of a simple harmonic oscillator (see, e.g., [2]). When $g = \exp(-x^2)$, we deduce from (3) the Rodrigues' representation (see, e.g., [5])

(13) $\qquad H_n(x) = (-1)^n \exp(x^2) D^n\{\exp(-x^2)\}, \ n \geq 0.$

It is now clear that the spring of springs [i.e., (3)], the Rodrigues' representation [i.e., (13)], Arfken's formula [i.e., (1)] and (12) are all equivalent.

Relation (3) has been obtained as a natural consequence of the standard properties of the Hermite polynomials. We shall now show that (3) is a spring for developing the properties of $H_n(x)$. First we prove (9) starting from (3):

$$\begin{aligned} H_n(x) &= g^{-1}[2x - D + g^{-1}\{Dg\}]^n g \\ &= g^{-1}[2x - D + g^{-1}\{Dg\}]\{gH_{n-1}(x)\} \\ &= (2x - D)H_{n-1}(x), \ n \geq 1. \end{aligned}$$

Relation (9) plays a crucial role in establishing the results that (1) and (3) are springs of the Hermite polynomials. For example, the differential recurrence relation can be obtained from (9). If $DH_M(x) = 2MH_{M-1}(x)$ for some $M \geq 1$, then

(14)
$$\begin{aligned} DH_{M+1}(x) &= D\{(2x - D)H_M(x)\} \\ &= 2H_M(x) + (2x - D)DH_M(x) \\ &= 2H_M(x) + (2x - D)\{2MH_{M-1}(x)\} \\ &= 2H_M(x) + 2MH_M(x) \\ &= 2(M + 1)H_M(x). \end{aligned}$$

By using induction, we now obtain the differential recurrence relation, (5). The three-term recurrence relation, (4), then follows from (9) and (5). The differential equation satisfied by $H_M(x)$ can be obtained from (14), since

$$2H_M(x) + (2x - D)DH_M(x) = 2(M + 1)H_M(x),$$

so that

(15) $\quad (D^2 - 2xD + 2M)H_M(x) = 0, \ M \geq 0.$

From (9), one can obtain the power series expansion (see, e.g., [5]) using induction:

$$(16) \quad H_n(x) = \sum_{s=0}^{[n/2]} \frac{(-1)^s n!(2x)^{n-2s}}{s!(n-2s)!}, \ n \geq 0,$$

where $[r]$ is the greatest integer $\leq r$. Though tedious, the method is straightforward. For an alternative method of arriving at the power series expansion from (1), see also [8]. Following Simmons ([6], p. 191), we can obtain the generating function [see (2)] from the power series expansion. We show that (2) can also be derived from the pure recurrence relation as follows: (i) Assume the existence of a generating function of the form

$$(17) \quad G(x, t) = \sum_{n=0}^{\infty} H_n(x)t^n/n!$$

(ii) Differentiate $G(x, t)$ partially with respect to t and use the three-term recurrence relation *and* (6) and (7) to develop the following first-order differential equation for $G(x, t)$:

$$(18) \quad G^{-1}(\partial G/\partial t) = 2x - 2t.$$

(iii) Holding x fixed, integrate both sides of (18) with respect to t, from 0 to t, to obtain

$$(19) \quad G(x, t) = G(x, 0)\exp(2xt - t^2).$$

(iv) Since $G(x, 0) = H_0(x) = 1$, by (6), it follows that

$$(20) \quad G(x, t) = \exp(2xt - t^2).$$

Our procedure outlined above is just similar to the one used by Arfken ([2], Prob. 13.1.1, p. 717) to arrive at the generating function from the differential recurrence relation, (5), supplemented with the results

$$(21) \quad H_{2m+1}(0) = 0, \ m \geq 0,$$

$$(22) \quad H_{2m}(0) = (-1)^m(2m)!/m!, \ m \geq 0.$$

Rodrigues' representation is a simple corollary of (3) and the orthonormal property,

$$(23) \quad \int_{-\infty}^{\infty} \exp(-x^2)H_m(x)H_n(x)\,dx = 2^n n!\sqrt{\pi}\,\delta_{mn},$$

can be proved using it (see, e.g., [8]). Szegö [10] has elegantly shown that real orthogonal polynomials associated with an *even* weight function and an interval of orthogonality *symmetric* with respect to the *origin* have a definite parity. Hence,

$$(24) \quad H_n(-x) = (-1)^n H_n(x), \ n \geq 0.$$

In other words, $H_n(x)$ can contain only those powers of x that are congruent to n (mod 2). Using this result, Descartes's rule of signs, and the properties of the zeros of $H_n(x)$ (see, e.g., [5], [10]), it has been proved in [7] that $H_n(x)$ *does contain* only those and all those powers of x that are congruent to n (mod 2). Moreover, the adjacent coefficients of $H_n(x)$, $n \geq 2$, alternate in sign [7]. See also (16). Thus, starting from (3), one can obtain the differential recurrence relation, pure (i.e., without derivative) recurrence relation, differential equation, and orthonormal property satisfied by the Hermite polynomials in addition to their Rodrigues representation, power series expansion, and generating function.

3. The Relation $H_n(x) = 2^n\{\exp(-D^2/4)\}x^n$

We now prove the following interesting relation from Bell ([3], Th. 5.3, p. 159):

(25) $$H_n(x) = 2^n\{\exp(-D^2/4)\}x^n.$$

Here $\exp(-D^2/4)$ is formally expanded as

(26) $$\exp(-D^2/4) = \sum_{s=0}^{\infty} \{(-1/4)^s/s!\}D^{2s}.$$

Since

(27) $$D^{2s}x^n = \begin{cases} \{n!/(n-2s)!\}x^{n-2s}, & 2s \le n, \\ 0, & 2s > n, \end{cases}$$

one can obtain (25) directly from the power series expansion, (16), using (26) and (27). Our proof of (25) is an alternative to that given in Bell ([3], p. 159). By retracing the steps for obtaining (25) from (16), one can show that (25) implies (16). Thus, the power series expansion and Bell's formula [i.e., (25)] are equivalent.

4. Status of the Springs

We can clearly classify the starting points into two distinct groups: (a) full/complete/self-contained springs and (b) associate (incomplete or partial) springs. To the first category belong the generating function, the Rodrigues representation, the power series expansion, relations (1), (3), and (25), and the orthonormal property. These springs specify the Hermite polynomials completely. The differential equation, the pure and differential recurrence relations, the orthogonal property, and (9) belong to the second category because they require supplementary conditions to specify the Hermite polynomials fully. The constant term of any $H_n(x)$, $n \ge 1$, cannot be found from the differential recurrence relation, (5), since the operator D simply swallows it. In the case of the orthogonal property, we require the value of the right-hand side of (23) when $m = n$, for all $n \ge 0$ (the square root of the reciprocal of this quantity is the so-called normalization constant), and to make (9) a complete spring we require the result $H_0(x) = 1$.

An outline of the development of the various properties from the springs is shown schematically in Figure 1. (Of course, not all the paths are shown.) Certain properties can be more easily obtained from a given spring, while it may be tedious to derive another property from the same spring. For example, in view of (26), we have

$$[D\exp(-D^2/4)]f(x) \equiv \{\exp(-D^2/4)\}(Df),$$

where $f(x)$ is any differentiable function of x. Hence, from (25) and (26), we have

$$\begin{aligned} DH_n(x) &= D[2^n\{\exp(-D^2/4)\}x^n] \\ &= 2^n\{\exp(-D^2/4)\}(Dx^n) \\ &= 2n[2^{n-1}\{\exp(-D^2/4)\}x^{n-1}] \\ &= 2nH_{n-1}(x), \quad n \ge 1. \end{aligned}$$

Probably this is the simplest proof of the differential recurrence relation. The method of induction plays an elegant role in developing certain properties from a given starting point. Some properties can be independently obtained from a given spring without going either via the generating function or via the Rodrigues representation.

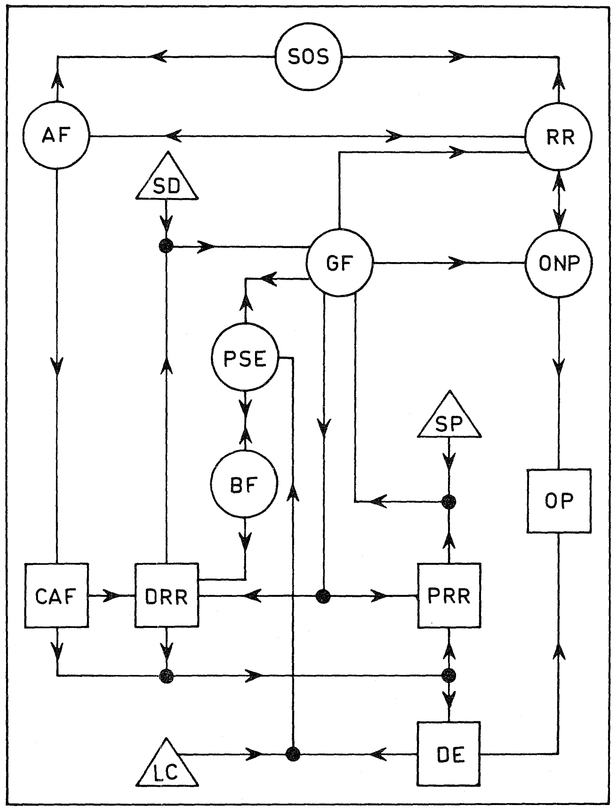

FIGURE 1

Schematic diagram showing the development of the various properties of the Hermite polynomials. Full springs are shown inside the circles. Squares enclose the associate starting points. Triangles stand for the supplementary conditions necessary to make the incomplete springs complete ones. We have not given the complete paths to arrive at all the properties, starting from a given spring. The following abbreviations have been used: (a) AF: Arfken's formula, (1) of text. (b) BF: Bell's formula, (25) of text. (c) CAF: Corollary to Arfken's formula, (9) of text. (d) DE: Differential equation. (e) DRR: Differential recurrence relation. (f) GF: Generating function. (g) LC: Leading coefficient of each and every $H_n(x)$, $n \geq 0$ $(= 2^n)$; supplement to the differential equation. (h) ONP: Orthonormal property. (i) OP: Orthogonal property. (A knowledge of the leading coefficient or the normalization constant for every $H_n(x)$ makes it a complete spring.) (j) PRR: Pure (three-term) recurrence relation. (k) PSE: Power series expansion. (l) RR: Rodrigues' representation. (m) SD: Supplement to the differential recurrence relation, (21) and (22) of text. (n) SOS: Spring of springs, (3) of text. (o) SP: Supplement to the pure recurrence relation, (6) and (7) of text.

5. Conclusions

Any relation or a set of relations that can specify all the Hermite polynomials completely should be a full starting point. One can level criticisms against any spring. For Simmons ([6], p. 189), the generating function method is totally unmotivated, though it has the advantage of efficiency for deducing the properties of the Hermite polynomials. While he prefers to develop the properties from the differential equation, Andrews ([1], p. vii) introduces the classical orthogonal polynomials by the generating function method and Rainville [5] revels in the generating function approach. Relation (1) is simple and handy, but may have the obvious weakness of being completely unmotivated.

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ON THE EQUATION $\phi(x) + \phi(k) = \phi(x + k)$

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Solutions of the equation

$$\phi(x) + \phi(k) = \phi(x + k)$$

(where ϕ is Euler's totient function) were considered by Makowski [3]. He showed that at least one solution exists if k is even, or k is not divisible by 3, or

$$k = mF_0^{a_0}F_1^{a_1} \ldots F_s^{a_s},$$

where $F_i = 2^{2^i} + 1$ is the i^{th} Fermat number, $a_i > 1$ for $0 \le i \le s$, F_{s+1} is prime, and $(m, 2F_0F_1 \ldots F_sF_{s+1}) = 1$. He did not determine whether solutions exist for other odd numbers that are divisible by 3. Makowski also raised the question whether there are positive integers for which no solution exists. In particular, he noted that it is not known whether there is a solution for $k = 3$.

This paper provides very severe necessity conditions for x when $k = 3$, and significantly enlarges the set of integers for which at least one solution is known to exist.

Throughout this paper, p, q, and r will denote distinct odd prime numbers.

Lemma 1: If $\phi(n) = 2j$ for $j > 1$ and odd, then $n = p^\alpha$ or $n = 2p^\alpha$.

The proof is given in [1].

Lemma 2: If $\phi(n) = 4j$ for some odd $j > 1$, then n is one of the following: p^α, $2p^\alpha$, $4p^\alpha$, $p^\alpha q^\beta$, or $2p^\alpha q^\beta$.

Proof: Clearly n cannot be divisible by 8 and cannot have more than two distinct odd prime factors.

Theorem I: If $\phi(x) + \phi(3) = \phi(x + 3)$, then

(i) $x = 2p^\alpha$ or $x = 2p^\alpha - 3$, and
(ii) $p > 3$.

Proof: (i) Let $\phi(x) = 2^v j$ and $\phi(x + 3) = 2^m k$ for j, k odd. Then the hypothesis gives us $2^v j + 2 = 2^m k$. Hence, $v = 1$ iff $m \ne 1$.

Case 1. Let $v = 1$. Then $x = p^\alpha$ or $x = 2p^\alpha$ by Lemma 1. $x = p^\alpha$ implies

$$p^\alpha - p^{\alpha-1} + 2 = \phi(p^\alpha + 3),$$

and since $p^\alpha + 3$ is even,

$$\phi(p^\alpha + 3) \le \frac{p^\alpha + 3}{2}.$$

Thus, $p^\alpha + 1 \le 2p^{\alpha-1}$, which is impossible.

Case 2. Let $m = 1$. Then $x = p^\alpha - 3$ or $x = 2p^\alpha - 3$ (Lemma 1). Since $p^\alpha - 3$ is even,

$$\phi(p^\alpha - 3) \le \frac{p^\alpha - 3}{2}.$$

However,

$$\phi(p^\alpha) \ge \frac{2}{3}p^\alpha;$$

so if $x = p^\alpha - 3$, we have

$$\frac{p^\alpha - 3}{2} + 2 \geq \phi(p^\alpha - 3) + \phi(3) \geq \frac{2}{3}p^\alpha,$$

which gives the contradiction $3 \geq p^\alpha$.

(ii) Suppose $p = 3$.

Case 1. Let $x = 2 \cdot 3^\alpha$ for $\alpha > 1$. Then

$$\phi(2 \cdot 3^\alpha) + \phi(3) = \phi(2 \cdot 3^\alpha + 3),$$

so that

$$3^{\alpha - 1} + 1 = \phi(2 \cdot 3^{\alpha - 1} + 1).$$

Notice that this implies that $2 \cdot 3^{\alpha-1} + 1$ and $\phi(2 \cdot 3^{\alpha-1} + 1)$ are relatively prime; hence, $2 \cdot 3^{\alpha-1} + 1$ is square-free. And since $8 \nmid (3^{\alpha-1} + 1)$, Lemma 2 gives us

$$2 \cdot 3^{\alpha-1} + 1 = q \quad \text{or} \quad 2 \cdot 3^{\alpha-1} + 1 = qr.$$

The supposition $2 \cdot 3^{\alpha-1} + 1 = q$ leads to the contradiction

$$\phi(2 \cdot 3^{\alpha-1} + 1) = 2 \cdot 3^{\alpha-1} = 3^{\alpha-1} + 1.$$

Hence, $2 \cdot 3^{\alpha-1} + 1 = qr$.

Assume $q > r$. Since

$$2\phi(qr) = 2(3^{\alpha-1} + 1) = qr + 1 = 2(qr - q - r + 1),$$

we get $qr = 2q + 2r - 1$. But $r \geq 5$, so $qr > 4q$. Therefore, $2r - 1 > 2q$, which contradicts $q > r$.

Case 2. Let $x = 2 \cdot 3^\alpha - 3$ for $\alpha > 1$. Then

$$2\phi(2 \cdot 3^{\alpha-1} - 1) + 2 = 2 \cdot 3^{\alpha-1} \quad \text{and} \quad \phi(2 \cdot 3^{\alpha-1} - 1) = 3^{\alpha-1} - 1.$$

Hence $2 \cdot 3^{\alpha-1} - 1$ and $\phi(2 \cdot 3^{\alpha-1} - 1)$ are relatively prime, which implies that $2 \cdot 3^{\alpha-1} - 1$ is square-free. Also, since $3 \nmid (3^{\alpha-1} - 1)$, we have $3 \nmid \phi(2 \cdot 3^{\alpha-1} - 1)$. So, if $q \mid (2 \cdot 3^{\alpha-1} - 1)$, then $q \not\equiv 1 \pmod 3$. Thus, $q \equiv 2 \pmod 3$. So,

$$\phi(2 \cdot 3^{\alpha-1} - 1) = (q_1 - 1)(q_2 - 1) \cdots (q_i - 1) \equiv 1 \pmod 3.$$

But $(3^{\alpha-1} - 1) \equiv 2 \pmod 3$. This contradiction completes the proof.

Lemma 3: If $\phi(2p^\alpha) + \phi(3) = \phi(2p^\alpha + 3)$, then $\dfrac{\phi(2p^\alpha + 3)}{2p^\alpha + 3} < \dfrac{1}{2}$.

Proof: $\phi(2p^\alpha + 3) = \phi(2p^\alpha) + \phi(3) = \left(\dfrac{p - 1}{p}\right)p^\alpha + 2 < \dfrac{2p^\alpha + 3}{2}$.

Lemma 4: If $\phi(2p^\alpha - 3) + \phi(3) = \phi(2p^\alpha)$, then $\dfrac{\phi(2p^\alpha - 3)}{2p^\alpha - 3} < \dfrac{1}{2}$.

Proof: $\phi(2p^\alpha - 3) = \phi(2p^\alpha) - \phi(3) = \left(\dfrac{p - 1}{p}\right)p^\alpha - 2 < \dfrac{2p^\alpha - 3}{2}$.

Lemma 5: Let $S = \{q \mid q \equiv 2 \pmod 3\}$. If n is a positive integer such that every prime factor of n belongs to S and $\phi(n)/n < 1/2$, then n has more than 32 distinct prime factors.

Proof: Calculations show that even if the 32 smallest primes in S all divide n, $\phi(n)/n$ is still greater than 1/2.

Theorem II: If $\phi(x) + \phi(3) = \phi(x + 3)$, then:

(i) x or $x + 3$ has at least 33 distinct prime factors, or
(ii) $x = 2p^\alpha$ for α odd, $p \equiv 2 \pmod 3$, $x > 10^{11}$, and $x + 3$ has at least 9 distinct prime factors.

Proof:

Case 1. Let $x = 2p^{\alpha} - 3$, α even. Suppose $q|x$. Then $2p^{\alpha} - 3 = qv$ for some integer v, and $4p^{\alpha} = 2qv + 6$. And since a is even, 6 is a quadratic residue mod q. Hence, the thirteen smallest primes that can divide x are 5, 19, 23, 29, 43, 47, 53, 67, 71, 73, 97, 101, and 139. Let $x = q_1^{m_1}q_2^{m_2} \ldots q_i^{m_i}$. Calculations show that

$$\frac{1}{2} < \frac{4}{5}\cdot\frac{18}{19}\cdot\frac{22}{23}\cdot\frac{28}{29}\cdot\frac{42}{43}\cdot\frac{46}{47}\cdot\frac{52}{53}\cdot\frac{66}{67}\cdot\frac{70}{71}\cdot\frac{72}{73}\cdot\frac{96}{97}\cdot\frac{100}{101}\cdot\left(\frac{138}{139}\right)^{28}.$$

So if $i \le 40$, then $\phi(x)/x > 1/2$. But $\phi(x)/x < 1/2$ by Lemma 4. Hence, $i > 40$.

Case 2. Let $x = 2p^{\alpha} - 3$, α odd. Suppose $q|x$ and $q \equiv 1 \pmod 3$. Then we have $\phi(x) \equiv 0 \pmod 3$. So

$$[\phi(x) + \phi(3)] \equiv 2 \pmod 3.$$

But $\phi(x) + \phi(3) = \phi(x + 3)$; hence,

$$\phi(x + 3) = \phi(2p^{\alpha}) = p^{\alpha-1}(p - 1) \equiv 2 \pmod 3.$$

And since α is odd, this is impossible. Thus, if $q|x$, then $q \equiv 2 \pmod 3$. So by Lemmas 4 and 5, x has at least 33 distinct prime factors.

Case 3. Let $x = 2p^{\alpha}$, α even. Suppose $q|(x + 3)$ and $q \equiv 1 \pmod 3$. Then $\phi(x + 3) \equiv 0 \pmod 3$. But

$$\phi(x + 3) = \phi(2p^{\alpha}) + \phi(3) = p^{\alpha-1}(p - 1) + 2.$$

So $p^{\alpha-1}(p - 1) + 2 \equiv 0 \pmod 3$, which implies

$$p^{\alpha-1}(p - 1) \equiv 1 \pmod 3.$$

And since α is even, this is impossible. Hence, if $q|(x + 3)$, then $q \equiv 2 \pmod 3$. Thus, by Lemmas 3 and 5, $x + 3$ has at least 33 distinct prime factors.

Case 4. Let $x = 2p^{\alpha}$, α odd, and $p \equiv 1 \pmod 3$. Suppose $q|x + 3$ and $q \equiv 1 \pmod 3$. Then $\phi(x + 3) \equiv 0 \pmod 3$. But

$$\phi(x + 3) = p^{\alpha-1}(p - 1) + 2 \equiv 2 \pmod 3.$$

Hence, every prime divisor of $x + 3$ belongs to $S = \{q|q = 2 \pmod 3\}$. Therefore, by Lemmas 3 and 5, $x + 3$ has at least 33 distinct prime factors.

Case 5. Let $x = 2p^{\alpha}$, α odd, and $p \equiv 2 \pmod 3$. Suppose that $5|(x + 3)$, $q|(x + 3)$, and $q \equiv 1 \pmod 5$. Then $\phi(x + 3) \equiv 0 \pmod 5$, $p^{\alpha} \equiv 1 \pmod 5$, and, since α is odd, $p^{\alpha-1} \equiv \pm 1 \pmod 5$. Therefore,

$$\phi(x + 3) = p^{\alpha} - p^{\alpha-1} + 2 \not\equiv 0 \pmod 5.$$

Hence, the prime factors of $x + 3$ all belong to $S_1 = \{q|q \ge 7\}$ or $5|(x + 3)$ and every other prime divisor of $x + 3$ belongs to $S_2 = \{q|q > 5 \text{ and } q \equiv 1 \pmod 5\}$. Let

$$x + 3 = q_1^{m_1}q_2^{m_2} \ldots q_i^{m_i}.$$

Calculations show that if all q_j belong to S_1 or $q_1 = 5$, and all other q_j belong to S_2, then $\phi(x + 3)/(x + 3) > 1/2$ whenever $i \le 8$. Therefore, by Lemma 3, $x + 3$ has at least 9 distinct prime factors. Calculations also show that in either case, $x > 10^{11}$.

Makowski did not determine whether solutions exist for $k = 18t \pm 3$ or for $k = 45m$, where $5\nmid m$. The following theorems not only prove that solutions exist for many of these integers, they characterize x for each k.

Theorem III: $\phi(x) + \phi(k) = \phi(x + k)$ has a solution if $k = 3m$ is odd and satisfies any of these conditions:

(i) $p^\alpha \| k$, $p^\beta = q - 2$, $\alpha > \beta$, and $q \nmid k$;

(ii) $p \| k$, $p = 3q - 4$, and $q \nmid k$;

(iii) $p \| k$, $p = 9q - 16$, and $q \nmid k$;

(iv) $p \| k$, $p = 3^\alpha q - 2^\alpha r$, $3^\alpha - 1 = 2^{\alpha - 1}(r + 1)$, $q \nmid k$ and $r \nmid k$.

Proof:

(i) Let $k = p^\alpha j$. Then $\phi(2q^{\alpha-\beta}j) + \phi(p^\alpha j) = \phi(qp^{\alpha-\beta}j)$.

(ii) Let $k = 3^\alpha pj$. Then $\phi(2^2 \cdot 3^\alpha j) + \phi(3^\alpha pj) = \phi(q \cdot 3^{\alpha+1}j)$.

(iii) Let $k = 3^\alpha pj$. Then $\phi(2^4 \cdot 3^\alpha j) + \phi(3^\alpha pj) = \phi(q \cdot 3^{\alpha+2}j)$.

(iv) Let $k = 3pj$. Then $\phi(3 \cdot 2^\alpha rj) + \phi(3pj) = \phi(3^{\alpha+1}qj)$.

Theorem IV: Let $2^m + 1 = 3^\alpha n$ where $(3, n) = 1$ and $\alpha \geq 0$; and suppose there exists a positive integer j such that $j - \phi(j) = n$ and $3^\alpha j - 2^{m+1} = p$. Then, if $k = 3pv$ where $(3v, 2pj) = 1$, the equation $\phi(x) + \phi(k) = \phi(x + k)$ has a solution.

Proof: $\phi(2^{m+1} \cdot 3v) + \phi(3pv) = \phi(3^{\alpha+1} \cdot jv)$.

Theorems III and IV provide a solution for 51 of the 91 positive odd integers that are less than 10,000, divisible by 45, and not divisible by 25. They also give solutions for 50 of the 112 k such that $k = 18t \pm 1$ and $k < 1000$. Since the solutions produced by these theorems depend on k being divisible by certain kinds of primes, it seems reasonable to expect that numbers with many prime divisors are much more likely to satisfy the hypotheses of the theorems than the relatively small numbers considered above.

It is not known whether there are solutions for $k = 3p$ where p = 5, 7, 13, 19, 23, 59, 67, 71, 73, 97, 113, 127, 131, 151, 163, 167, 181, or 199. For all other $p < 200$, $k = 3$ has a solution defined in Theorems III and IV.

Theorem IV raises the question: for which n does the equation $n = x - \phi(x)$ have at least one solution? This equation was considered by Erdös [2], but a characterization of all such n has not been found.

The calculations in part (i) of Theorem II could probably be refined to show that x or $x + 3$ must have 40 or more distinct prime divisors. But such a refinement would not be significant, since we have already shown that any solution for $k = 3$ must be very large. Now the real challenge is to prove that $\phi(x) + \phi(3) = \phi(x + 3)$ has no solution.

Finally, we mention two other related, unanswered questions:

1. For which positive integers n does $\phi(x) + \phi(n - x) = \phi(n)$ have at least one solution?
2. For which pairs of positive integers a, b does $\phi(a) + \phi(b) = \phi(a + b)$?

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SERIES TRANSFORMATIONS FOR FINDING RECURRENCES FOR SEQUENCES

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There is no *really* new theoretical result below. However, our paper will show how to use an old and clever idea in order to discover recurrences. Such an expository paper surveying these techniques may be of interest. A few specific books or papers are needed, but for general background as to notations and definitions for Fibonacci, Bernoulli, Bell, and Stirling numbers, etc., the reader may consult papers in the *Fibonacci Quarterly* or Riordan's books [6], [7]. Niven [5] has given a good, readable account of formal power series. It is shown there when and why convergence questions may be ignored. Finally, four papers of the author, [1], [2], [3], and [4], may be consulted for other background information. Reference [1] is especially useful for an abundance of intricate generating functions for powers of Fibonacci numbers.

We begin with a small theorem about *formal power series*.

Theorem 1. Exponential Series Transformation: Define

(1) $$S(n) = \sum_{k=0}^{n} \binom{n}{k} A_k,$$

(2) $$\mathscr{A}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} A_n,$$

and

(3) $$\mathscr{G}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} S(n).$$

Then

(4) $$\mathscr{G}(x) = e^x \mathscr{A}(x).$$

The proof is simple and runs as follows. We have

$$\begin{aligned}\mathscr{G}(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} \binom{n}{k} A_k = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{A_k}{(n-k)!k!} \\ &= \sum_{k=0}^{\infty} \frac{A_k}{k!} \sum_{n=k}^{\infty} \frac{x^n}{(n-k)!} = \sum_{k=0}^{\infty} \frac{A_k}{k!} \sum_{n=0}^{\infty} \frac{x^{n+k}}{n!} \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} A_k \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \mathscr{A}(x).\end{aligned}$$

What we wish to show here is that by clever manipulation, especially if e^x combines in a *novel* way with $\mathscr{A}$, we may often use (4) to find a *different* way of writing expansion (3) that does not use $S(n)$ again directly. Then, by equating coefficients, we get a *new* recurrence. This is a common piece of psychological trickery used in research. We say the same thing but in a seemingly different manner.

Relation (1) may easily be inverted to give

(5) $$A_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} S(k),$$

which is a well-known result [7] which follows readily from the Kronecker delta

$$(6) \qquad \sum_{k=j}^{n}(-1)^{n-k}\binom{n}{k}\binom{k}{j} = \binom{0}{n-j} = \begin{cases}1 & \text{if } j = n,\\ 0 & \text{if } j \neq n.\end{cases}$$

As a consequence of this inversion, we may also state Theorem 1 in a dual form.

Theorem 1': Define

$$(1') \qquad A_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}S(k),$$

$$(2') \qquad \mathscr{G}(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}S(n),$$

$$(3') \qquad \mathscr{A}(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}A_n.$$

Then

$$(4') \qquad \mathscr{A}(x) = e^{-x}\mathscr{G}(x).$$

We will now concentrate on applications of Theorem 1.

Application 1. Let $A_n = (-1)^n F_n$, where F_n is the n^{th} Fibonacci number defined by

$$F_{n+1} = F_n + F_{n-1},\ F_0 = 0,\ F_1 = 1.$$

We must recall that the exponential generating function for the Fibonacci numbers is

$$(7) \qquad \sum_{n=0}^{\infty}\frac{x^n}{n!}F_n = \frac{e^{ax} - e^{bx}}{a - b},$$

where $a + b = 1$, $ab = -1$. These are the roots of the characteristic equation associated with the recurrence relation. In fact, $a, b = (1 \pm \sqrt{5})/2$.

It then follows in this special Fibonacci case that

$$\mathscr{G}(x) = -\mathscr{A}(-x).$$

To show this, we have

$$\mathscr{G}(x) = e^x\mathscr{A}(x) = e^x\frac{e^{-ax} - e^{-bx}}{a - b} = \frac{e^{(1-a)x} - e^{(1-b)x}}{a - b} = \frac{e^{bx} - e^{ax}}{a - b}$$

$$= -\frac{e^{ax} - e^{bx}}{a - b} = -\mathscr{A}(-x) = -\sum_{n=0}^{\infty}\frac{x^n}{n!}F_n.$$

Recalling (1) and (3), we have, upon equating coefficients, the new recurrence relation $S(n) = -F_n$, i.e.,

$$(8) \qquad \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}F_k = F_n.$$

The reader may find it interesting to try to provide a *simple inductive proof* of relation (8) using the binomial and Fibonacci recurrences

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},\quad F_{n+1} = F_n + F_{n-1}.$$

Such a proof requires a certain algebraic skill.

Application 2. Let $A_n = B_n$, the n^{th} Bernoulli number, whose exponential generating function is known to be

$$(9) \qquad \mathscr{A}(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}B_n = \frac{x}{e^x - 1}.$$

It can then easily be seen that

$$\mathscr{G}(x) = e^x \frac{x}{e^x - 1} = \frac{-x}{e^{-x} - 1} = \mathscr{A}(-x),$$

and it thus follows from Theorem 1 that $S(n) = (-1)^n B_n$, i.e.,

(10) $$\sum_{k=0}^{n} \binom{n}{k} B_k = (-1)^n B_n, \text{ valid for all } n \geq 0.$$

Remark: Because $B_n = 0$ for all odd $n \geq 3$, this familiar recurrence may be modified to read as

(11) $$\sum_{k=0}^{n} \binom{n}{k} B_k = B_n, \text{ valid for all } n \geq 2.$$

Symbolically, in the umbral notation of Blissard, this is often written in the compact form $(B + 1)^n = B^n$ (expand and demote powers to subscripts).

Application 3. Let $A_n = B(n)$, the n^{th} Bell, or exponential number. These numbers have the well-known exponential generating function

(12) $$e^{e^x - 1} = \exp(e^x - 1) = \sum_{n=0}^{\infty} \frac{x^n}{n!} B(n),$$

so this is our $\mathscr{A}(x)$.

By out theorem, using relation (4), we find that

$$\mathscr{G}(x) = e^x \exp(e^x - 1) = D_x \exp(e^x - 1) = D_x \mathscr{A}(x),$$

$$= \sum_{n=0}^{\infty} \frac{nx^{n-1}}{n!} B(n) = \sum_{n=0}^{\infty} \frac{x^n}{n!} B(n + 1),$$

whence by our theorem we find the recurrence relation $S(n) = B(n + 1)$, i.e.,

(13) $$\sum_{k=0}^{n} \binom{n}{k} B(k) = B(n + 1), \text{ valid for all } n \geq 0.$$

By the inversion (5), this yields

(14) $$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} B(k + 1) = B(n),$$

which, in terms of the finite difference quotient operator, says that

(15) $$\mathop{\Delta}_{k,1}^{n} B(k + 1) = B(n),$$

which is the analogue of the differential calculus formula

(16) $$(D_x)^n e^x = e^x.$$

This parallel of (15) with (16) is a further reason why the Bell numbers are reasonably called "exponential" numbers.

The reader may look for other examples where a generating function has some nice relation to the exponential function, which is part of the secret of success. Such research requires an artistic touch of intuition.

It is possible to set down a parallel theorem for binomial generating functions. We offer the following.

Theorem 2. Binomial Series Transformation: Define as before in (1),

(17) $$S(n) = \sum_{k=0}^{n} \binom{n}{k} A_k,$$

(18) $$\mathscr{B}(x) = \sum_{n=0}^{\infty} x^n A_n,$$

and

(19) $$\mathscr{H}(x) = \sum_{n=0}^{\infty} x^n S(n).$$

Then

(20) $$\mathscr{H}(x) = \sum_{n=0}^{\infty} A_n \frac{x^n}{(1-x)^{n+1}}.$$

and the best we can do to parallel (4) is to write this as

(21) $$\mathscr{H}(x) = \frac{1}{1-x}\mathscr{B}(z), \text{ where } z = \frac{x}{1-x}.$$

The proof is easy and runs as follows. We have

$$\begin{aligned}\mathscr{H}(x) &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n}\binom{n}{k}A_k = \sum_{k=0}^{\infty} A_k \sum_{n=k}^{\infty}\binom{n}{k}x^n \\ &= \sum_{k=0}^{\infty} A_k \sum_{n=0}^{\infty}\binom{n+k}{k}x^{n+k} = \sum_{k=0}^{\infty} x^k A_k \sum_{n=0}^{\infty}\binom{n+k}{k}x^n \\ &= \sum_{k=0}^{\infty} x^k A_k (1-x)^{-k-1} = \frac{1}{1-x}\sum_{k=0}^{\infty} A_k z^k = \frac{1}{1-x}\mathscr{B}(z).\end{aligned}$$

This result is useful in a different way than Theorem 1. We give as an example,

Application 4. Let $A_n = (-1)^n F_n$ as in Application 1. Then

$$\mathscr{B}(x) = \sum_{n=0}^{\infty} (-x)^n F_n = \frac{-x}{1+x-x^2}$$

and

$$\begin{aligned}\mathscr{H}(x) &= \frac{1}{1-x}\mathscr{B}(z) = \frac{1}{1-x}\,\frac{-z}{1+z-z^2} = \frac{-x}{1-x-x^2} \\ &= -\mathscr{B}(-x) = -\sum_{n=0}^{\infty} F_n x^n,\end{aligned}$$

so that by Theorem 2 we have the recurrence $S(n) = -F_n$, i.e.,

(22) $$\sum_{k=0}^{n} (-1)^{k+1}\binom{n}{k}F_k = F_n,$$

which is precisely result (8) again, but it required a bit more work to obtain it by use of Theorem 2. This gives some feeling for the elegance of the exponential generating function when it can be used.

Application 5. In Theorem 2, let $A_n = F_n$ using the Fibonacci numbers again. Then

$$\mathscr{B}(x) = \frac{x}{1-x-x^2}$$

and the reader may verify that a bit of algebra using $A_n = 1$ and $m = 2$ in equation (2.11) in [1] yields

(23) $$\mathscr{H}(x) = \frac{x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n}x^n,$$

so that we have the recurrence $S(n) = F_{2n}$, i.e.,

(24) $$\sum_{k=0}^{n}\binom{n}{k}F_k = F_{2n}.$$

Application 6. Let us apply Theorem 1 to a generating function studied by Euler (cf. [2], p. 48, and [4], Sect. 6). Euler used the generating function

(25) $\mathscr{A}(x) = \mathscr{A}(x, p) = (e^x - 1)^p$

to evaluate the series

(26) $S(n, p) = \dfrac{1}{p!}\displaystyle\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}j^n,$

which we have designated here by the "Stirling Number of Second Kind" notation of Riordan. It is known (see [4], Sect. 6) that

(27) $\mathscr{A}(x, p) = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{k=0}^{p}(-1)^k\binom{p}{k}k^n.$

In Theorem 1 then, with this for $\mathscr{A}(x)$, and taking $S(n)$ to be given by

(28) $S(n) = \displaystyle\sum_{i=0}^{n}\binom{n}{i}\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}k^i,$

(29) $A_n(p) = \displaystyle\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}k^n,$

we then find by Theorem 1 that

$$\begin{aligned}\mathscr{G}(x) &= e^x\mathscr{A}(x, p) = e^x(e^x - 1)^p = (e^x - 1 + 1)(e^x - 1)^p\\ &= (e^x - 1)(e^x - 1)^p + (e^x - 1)^p = (e^x - 1)^{p+1} + (e^x - 1)^p\end{aligned}$$

or, more simply,

(30) $\mathscr{G}(x) = \mathscr{A}(x, p + 1) + \mathscr{A}(x, p).$

Therefore,

(31) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}S(n) = \sum_{n=0}^{\infty}\frac{x^n}{n!}[A_n(p + 1) + A_n(p)],$

so that we find the recurrence

(32) $S(n) = A_n(p + 1) + A_n(p),$

which, in view of (28) and (29), says

(33) $\displaystyle\sum_{j=0}^{n}\binom{n}{j}A_j(p) = A_n(p + 1) + A_n(p).$

Comparing (26) and (29), we have the correspondence

(34) $A_n(p) = p!S(n, p)$

for translating our results into Riordan's "Stirling Number" notation. Thus, we find

(35) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}S(k, p) = (p + 1)S(n, p + 1) + S(n, p)$

or

$$\sum_{n=0}^{n-1}\binom{n}{k}S(k, p) = (p + 1)S(n, p + 1).$$

However, $S(k, p) = 0$ whenever $0 \le j < p$, so we finally get the recurrence formula for the Stirling Numbers of the Second Kind, i.e.,

(36) $\displaystyle\sum_{k=p}^{n-1}\binom{n}{k}S(k, p) = (p + 1)S(n, p + 1).$

Conclusion. The work we have presented here was based on the use of the binomial coefficient $\binom{n}{k}$ in the defining relationships (1) and (17). It is easy to replace this by other functions $g(n, k)$ and obtain parallel theorems. We just have to impose interesting properties on $g(n, k)$ in order to get interesting theorems. In later papers we will exhibit such results for q-analogs, Fibonomial coefficients, and the bracket function.

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PARTIAL ORDERS AND THE FIBONACCI NUMBERS

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Introduction

We present an approach to the Fibonacci numbers by considering finite partially ordered sets (posets). The n^{th} Fibonacci number, F_n can be interpreted as the number of ideals in a very simple poset, usually called a *fence*.

The purpose of this note is *not* to prove new theorems about the sequence $\{F_n\}$. However, we wish to demonstrate that the approach has several advantages. By attaching to each Fibonacci number a geometrical object, the number gets an additional dimension, that might be of value in proving identities for the Fibonacci numbers.

While, in general, it may be difficult to count the number of ideals in a poset, the simple structure of a fence enables one to calculate the number of ideals in several different ways.

Even the simple partition of the ideals in a fence into two classes, those that contain a given element x, and those that do not contain x, can be used to show properties of the Fibonacci numbers that usually are verified by an inductive proof. This may, in some cases, add to our understanding of "why" the proof is valid.

Another advantage is that, after having established that F_n is the number of ideals in a fence with n elements, we have at our disposal theorems from the general theory of posets, see for instance [2].

Preliminaries

Our terminology on posets is, with a few exceptions, standard, and we refer to for instance Birkhoff [1], but for the convenience of the reader, we define briefly the basic concepts.

We let $[n]$ denote the set $\{1, \ldots, n\}$.

In this paper a partially ordered set (*poset*) is a *finite* set equipped with a relation $\geq$ that is reflexive, antisymmetric, and transitive.

An *ideal* in a poset P is a subset I of P such that, for any $x \in P$ and any $y \in I$, if $x \geq y$ then $x \in I$. Both $\emptyset$ and P are ideals in P. Actually, an ideal in the present paper is usually called an *upper* ideal, *dual* ideal, or *filter*.

For any poset P, $Id(P)$ denotes the number of ideals in P. Moreover, $Id(x)$, $Id(x \,\&\, y)$, and $Id(x \,\&\, \neg y)$ denote the number of ideals (in P) that contain x, contain x and y, contain x but not y, respectively.

Given a subset A of a poset P, let A^* denote the set of elements $x \in P$ such that $x \geq a$ for some $a \in A$, and A_* denotes the elements $x \in P$ such that $a \geq x$ for some $a \in A$.

Any subset A of a poset P, may be considered as a poset in itself with the inherited relations from the set P. Hence, $Id(A)$ denotes the number of ideals in the *poset* A. This should not be confused with the earlier definitions of $Id(x)$, $Id(x \,\&\, y)$, etc.

The elements x and y in a poset P are *path connected* if there exists a sequence of elements $x_1, \ldots, x_n$ in P such that $x_1 = x$, $x_n = y$, and x_i and x_{i+1} are comparable for each $1 \leq i \leq n - 1$. Two subsets A and B of a poset are *separated* if x and y are *not* path connected for any $x \in A$ and $y \in B$.

The *fence* Γ_n with n elements is the poset

$$\Gamma_n = \{x_1 \geq x_2 \leq x_3 \geq \cdots \leq (\text{or} \geq)\ x_n\}.$$

Let Γ_0 refer to the empty fence, with one ideal only.

A fence can be pictured as a lattice path; we show Γ_5 in Figure 1.

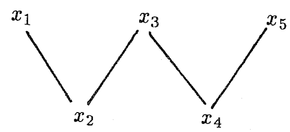

FIGURE 1: Γ_5

The following observation, whose simple proof is omitted, will be found to be very useful.

Lemma 1: Let A be a subset of the poset P. Then:

1. The number of ideals in P that contain A equals $Id(P - A^*)$.
2. The number of ideals in P that are disjoint with A equals $Id(P - A_*)$.
3. If $P = A \cup B$, where A and B are separated subsets of P, then $Id(P) = Id(A)Id(B)$.

As an illustration of Lemma 1, we shall find $Id(x_3)$ and $Id(\neg x_3)$ for the fence Γ_5. In order to find $Id(x_3)$, Lemma 1.1 says that one shall erase all y such that $y \geq x_3$, and find the number of ideals in the remaining poset. In this case, we only erase x_3 itself, and are left with a poset consisting of two separated parts, each being isomorphic to Γ_2. Hence, by Lemma 1.3 it follows that $Id(x_3) = Id^2(\Gamma_2)$.

In order to find $Id(\neg x_3)$, one must erase $\{x_3\}_* = \{x_2, x_3, x_4\}$. One is left with two separated copies of Γ_1; thus, $Id(\neg x_3) = Id^2(\Gamma_1)$. Hence,

$$Id(\Gamma_5) = Id^2(\Gamma_2) + Id^2(\Gamma_1).$$

Ideals in a Fence

Let $F_0 = 1$, $F_1 = 2$, $F_2 = 3$, etc., refer to the Fibonacci numbers, and Γ_n to the fence of cardinality n.

Theorem 1: $Id(\Gamma_n) = F_n$ for $n = 0, 1, 2, \ldots$.

Proof: By definition $Id(\Gamma_0) = 1$, and trivially $Id(\Gamma_1) = 2$. We shall show that

$$Id(\Gamma_n) = Id(\Gamma_{n-1}) + Id(\Gamma_{n-2}) \text{ for } n \geq 2.$$

In general,

$$Id(\Gamma_n) = Id(x_n) + Id(\neg x_n).$$

If n is even, it follows from Lemma 1 that

$$Id(x_n) = Id(\Gamma_{n-2}) \quad \text{and} \quad Id(\neg x_n) = Id(\Gamma_{n-1}),$$

and if n is odd, Lemma 1 yields that

$$Id(x_n) = Id(\Gamma_{n-1}) \quad \text{and} \quad Id(\neg x_n) = Id(\Gamma_{n-2}).$$

This proves Theorem 1.

We shall consider a few simple applications of Theorem 1.

Corollary 1: $F_n = F_{i-1}F_{n-i} + F_{i-2}F_{n-i-1}$ for $2 \leq i < n$.

Proof: Follows from Theorem 1, Lemma 1, and the identity

$$Id(\Gamma_n) = Id(x_i) + Id(\neg x_i).$$

In the remainder of this note we simplify our notation by letting the nodes of Γ_n be denoted by 1, ..., n instead of $x_1, \ldots, x_n$.

Corollary 2:

$$F_{2n-1} = \#\{(a_1, \ldots, a_k) \mid a_i \text{ is odd and } a_i \geq 1 \text{ and } a_1 + \cdots + a_k = 2n + 1\}.$$

Proof: A subset X of $[n]$ can uniquely be given by an odd (i.e., k = odd) tuple $(a_1, \ldots, a_k)$ of positive integers whose sum equals $n + 2$. To such a tuple we assign the set X defined by: a_1 is the smallest number belonging to X, $a_1 + a_2$ is the smallest number greater than a_1 that does not belong to X, $a_1 + a_2 + a_3$ is the smallest number after $a_1 + a_2$ that belongs to X, etc.

The following example illustrates the correspondence. Let $n = 11$ and let $(a_1, \ldots, a_5) = (2, 3, 2, 2, 4)$. This vector corresponds with the set {2, 3, 4, 7, 8}.

It is easily seen that by this correspondence, the set corresponding to a vector $(a_1, \ldots, a_k)$ is an ideal in Γ_{2n-1} iff each a_i is an odd integer.

This proves Corollary 2.

Corollary 3: $F_{2n-1} = \sum_{i=0}^{n} \binom{n+i}{2i}$

Proof: By Corollary 2, F_{2n-1} equals the number of tuples $(a_1, \ldots, a_k)$ of odd positive integers whose sum is $2n + 1$. Put $a_j = 2b_j - 1$, and since k is odd, there exists an integer i such that $k = 2i + 1$. One derives the condition

$$b_1 + \cdots + b_{2i+1} = n + i + 1$$

and since

$$\#\{(c_1, \ldots, c_i) \mid c_i \geq 1 \text{ and } c_1 + \cdots + c_i = m\} = \binom{m-1}{i-1}.$$

Corollary 3 follows.

Finally, let us add that many more identities can be shown in this simple manner.

A slightly more complicated application is achieved by defining an equivalence relation on Γ_{2n-1} by declaring two ideals to be equivalent if they contain the same *odd* numbers in $[2n - 1]$. Counting the number of ideals in each equivalence class leads to the following identity, whose proof is left to the reader.

$$F_{2n-1} = 1 + \sum \binom{s-1}{k-1}\binom{n+1-s}{k} 2^{s-k},$$

where the sum is over all (s, k) such that $s \geq k \geq 1$ and $s + k \leq n + 1$.

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GENERALIZED FIBONACCI POLYNOMIALS AND THE FUNCTIONAL ITERATION OF RATIONAL FUNCTIONS OF DEGREE ONE

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1. Introduction

One of the great advances in mathematics recently has been in the analysis of nonlinear dynamical systems. In this paper we will study the properties of a set of polynomials in two variables using techinques from nonlinear dynamic theory. These polynomials are variants of the class of generalized Fibonacci polynomials (see, for example, [7]) defined by

$$P_0(z_1, z_2) = 0, \quad P_1(z_1, z_2) = 1,$$
$$P_{n+1}(z_1, z_2) = (1 - z_1)P_n(z_1, z_2) - (z_2 - z_1)P_{n-1}(z_1, z_2), \; n \geq 1.$$

The results derived here are not new in the sense that they can be proven from existing work on generalized Fibonacci polynomials but the approach is entirely novel in that it provides a link between the analysis of generalized Fibonacci numbers and the theory of dynamical systems via the iteration of rational functions of degree one.

Fundamental to the concept of the analysis of nonlinear dynamical systems is the functional iteration of the form

$$(1) \qquad x_{n+1} = f(\lambda, x_n),$$

where λ is a parameter that can be varied. In this paper we will consider the iterative behavior of the general rational function of degree one given by

$$(2) \qquad f(k, \lambda_1, \lambda_2, x) = k\,\frac{1 - \lambda_1 x}{1 - \lambda_2 x},$$

where k, λ_1, and λ_2 can be complex, and relate these iterations to a family of polynomials, defined in two variables by

$$(3) \qquad P_0(z_1, z_2) = 0, \quad P_1(z_1, z_2) = 1,$$
$$P_{n+1}(z_1, z_2) = (1 - z_1)P_n(z_1, z_2) - (z_2 - z_1)P_{n-1}(z_1, z_2), \; n \geq 1.$$

We will also consider as a special example the case when $k = 1$ and $\lambda_1 = 0$, so that

$$(4) \qquad f(\lambda, x) = \frac{1}{1 - \lambda x},$$

and relate the iterations of this class of functions to a family of polynomials defined by

$$(5) \qquad P_0(z) = 0, \; P_1(z) = 1, \; P_{n+1}(z) = P_n(z) - zP_{n-1}(z), \; n \geq 1.$$

We note that in our terminology $P_n(0, z) = P_n(z)$. The polynomials presented in (3) and (5) are in fact variants of two well-known classes of polynomials known as generalized Fibonacci polynomials and Fibonacci polynomials, respectively.

In Section 2 we will present a review of some of the known results concerning generalized Fibonacci polynomials and show that they can be generalized to the polynomials defined in (3) and (5). The analysis in Section 3 will prove some of these results anew but using a completely different approach. This approach is based on the concept of topological conjugacy. Two maps $f: A \to A$

and $g:B \to B$ are said to be topologically conjugate if there exists a homeomorphism $h:A \to B$ such that

(6) $$h \circ f = g \circ h.$$

Topologically conjugate maps are equivalent in terms of their dynamics (see, for example [4]). Now, if g is the function μz, then (6) is called the Schröder Functional Equation (SFE). It is well known (see, for example, [1]) that, if f is a rational function of degree two or more, then the SFE does not have a solution if μ is a root of unity. On the other hand, Siegel [11] has shown that, if $\mu = e^{2\pi i\alpha}$, where α is irrational, then the SFE has a solution if there exist a, $b > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{a}{q^b}, \quad \forall p,\ q \in \mathbf{Z}.$$

This condition is satisfied for a set of μ of full measure on the unit circle. In this paper we will make use of the well-known fact that f, given by (2), is topologically conjugate to μx. Hence, the dynamics of f and μx are equivalent and the zeros of the generalized Fibonacci polynomials can be related to the roots of unity.

2. Generalized Fibonacci Polynomials

Although Fibonacci polynomials have been studied for well over a century, there was initially no common agreement on how to define this class of polynomials. For example, Catalan [3] defined them by

$$F_0(z) = 0,\ F_1(z) = 1,\ F_{n+1}(z) = zF_n(z) + F_{n-1}(z),\ n \ge 1,$$

while Jacobsthal [9] defined them by

$$F_0(z) = 0,\ F_1(z) = 1,\ F_{n+1}(z) = F_n(z) + zF_{n-1}(z),\ n \ge 1,$$

and Byrd [2] by

$$F_0(z) = 0,\ F_1(z) = 1,\ F_{n+1}(z) = 2zF_n(z) + F_{n-1}(z),\ n \ge 1.$$

However, the general consensus (see [6], for example) is that the class of Fibonacci polynomials is defined by

(7) $$F_0(z) = 0,\ F_1(z) = 1,\ F_{n+1}(z) = zF_n(z) + F_{n-1}(z),\ n \ge 1.$$

It is easy to obtain a simple closed expression for these polynomials in terms of trigonometric functions (see [6], for example) and hence show that the zeros of F_n are given by

$$2i \cos \frac{k\pi}{n},\ k = 1,\ \dots,\ n - 1.$$

In addition, it is easy to show

(8) $$F_n(z) = \sum_{j=0}^{p} \binom{n - 1 - j}{j} z^{n-2j-1},\ p = \left[\frac{n - 1}{2}\right].$$

Horadam [8] has considered generalized sequences of Fibonacci numbers given by

$$w_0 = a,\ w_1 = b,\ w_{n+1} = pw_n - qw_{n-1},\ n \ge 1,$$

where w_n is a function of a, b, p, and q, and obtained closed expressions for many special classes of w_n. The case in which $a = 0$, $b = 1$ so that

(9) $$F_0(z_1,\ z_2) = 0,\quad F_1(z_1,\ z_2) = 1,$$
$$F_{n+1}(z_1,\ z_2) = z_1F_n(z_1,\ z_2) + z_2F_{n-1}(z_1,\ z_2),\ n \ge 1$$

is now known as the family of generalized Fibonacci polynomials. The properties of these polynomials have been studied extensively by Hoggatt & Long [7], which builds on the earlier work of Webb & Parberry [12] who consider the divisibility properties of Fibonacci polynomials.

In particular, Hoggatt & Long [7] show that

(10) $$F_n(z_1, z_2) = \sum_{j=0}^{p} \binom{n-1-j}{j} z_1^{n-2j-1} z_2^j, \quad p = \left[\frac{n-1}{2}\right],$$

and that $F_n(z_1, z_2) = 0$ iff

$$z_1 = 2i\sqrt{z_2} \cos \frac{k\pi}{n}, \quad k = 1, \ldots, n-1.$$

Furthermore, they show that, for $m \geq 2$, $F_m | F_n$ iff $m|n$ and that F_n is irreducible over the rationals iff A is prime. A consequence of this is, if $n_1, \ldots, n_\ell$ are the factors of n, then all the zeros of $F_{n_1}, \ldots, F_{n_\ell}$ are zeros of F_n.

This work has been generalized by Kimberling [10] who shows that each generalized Fibonacci polynomial F_n has one and only one irreducible factor that is not a factor of F_k for any $k < n$, which is called the n^{th} Fibonacci cyclotomic polynomial $G_n(z_1, z_2)$. Kimberling shows

$$F_n(z_1, z_2) = \prod_{d|n} G_n(z_1, z_2).$$

The polynomials defined in (3) and (5), which will prove significant when analyzing the behavior of the iteration of rational functions of degree one, can easily be related to generalized Fibonacci polynomials and Fibonacci polynomials. In fact, comparing (3) and (9), we see

(11) $$P_n(z_1, z_2) = F_n(1 - z_1, z_1 - z_2),$$

while

(12) $$P_n\left(\frac{-1}{x^2}\right) = \frac{F_n(x)}{x^{n-1}}$$

or

(13) $$P_n(z) = \frac{F_n\left(\frac{i}{\sqrt{z}}\right)}{\left(\frac{i}{\sqrt{z}}\right)^{n-1}}.$$

This can be seen by substituting (12) into (5) and noting that (7) results. Consequently, it is trivial to show

$$P_n(z) = \sum_{j=0}^{p} (-1)^j \binom{n-1-j}{j} z^j, \quad p = \left[\frac{n-1}{2}\right],$$

while

(14) $$P_n(z_1, z_2) = \sum_{j=0}^{p} \binom{n-1-j}{j} (1 - z_1)^{n-2j-1}(z_1 - z_2)^j, \quad p = \left[\frac{n-1}{2}\right].$$

In addition, the zeros of $P_n(z_1, z_2)$ and P_n can be found from (11) and (13). Thus, the zeros of P_n are simple and given by

$$\frac{1}{4}\sec^2 \frac{k\pi}{n}, \quad k = 1, \ldots, n-1,$$

so that all zeros are real distinct and lie in the interval $(1/4, \infty)$. Similarly, if $z_1 \neq 1$, then $P(z_1, z_2) = 0$ iff

(15) $$z_2 = z_1 + (1 - z_1)^2 \frac{1}{4} \sec^2 \frac{k\pi}{n}, \quad k = 1, \ldots, n-1,$$

where all the roots in this set are simple, so that if $n = 2p + 1$ there are p distinct zeros while if $n = 2p$ there are $p - 1$. On the other hand if $z_1 = 1$, then (14) implies

(16) $\quad P_{2n}(1,\ z_2) = 0,\ n = 1,\ 2,\ \ldots,$

while it can easily be seen from (14) that

(17) $$P_n\left(z_1,\ \frac{(1 + z_1)^2}{8}\right) = n\left(\frac{1 - z_1}{2}\right)^{n-1}.$$

We also note that a formula for P_n can be derived by considering the characteristic polynomial associated with (3) given by

$$x^2 - (1 - z_1)x + z_2 - z_1 = 0.$$

The roots of this equation are

$$\theta_{\pm} = \frac{1 - z_1 \pm \sqrt{(1 + z_1)^2 - 4z_2}}{2}$$

and so it is easily seen that

(18) $$P_n(z_1,\ z_2) = \left(\frac{\theta_+^n - \theta_-^n}{\theta_+ - \theta_-}\right).$$

In the next section we will show that some of the above results can be proved by noting the behavior of the iterations of rational functions of degree one. For ease of notation we will henceforth refer to the polynomials $P_n(z_1,\ z_2)$ as the Shifted Generalized Fibonacci Polynomials (SGFP).

3. Functional Iteration

Consider the iteration scheme given by (1) where f is as in (2). We will denote the iterations of $\{x,\ x_1,\ x_2,\ \ldots,\ x_n,\ \ldots\}$ by

$$\{f^{(k)}(x);\ k = 0,\ 1,\ \ldots\}.$$

The following result gives the value of x_n after n iterations.

Lemma 1: Let $z_1 = k\lambda_1$, $z_2 = k\lambda_2$, and $P_n(z_1,\ z_2)$ represent the n^{th} shifted generalized Fibonacci polynomial then

$$f^{(n)}(x) = \frac{kP_n(z_1,\ z_2) - x(P_n(z_1,\ z_2) - P_{n+1}(z_1,\ z_2))}{P_{n+1}(z_1,\ z_2) + z_1P_n(z_1,\ z_2) - x\lambda_2P_n(z_1,\ z_2)}.$$

Proof: The proof is by induction. From (2),

$$f^{(2)}(x) = k\,\frac{1 - \lambda_1k\dfrac{1 - \lambda_1x}{1 - \lambda_2x}}{1 - \lambda_2k\dfrac{1 - \lambda_1x}{1 - \lambda_2x}} = \frac{k(1 - z_1) - x(z_2 - z_1^2)}{1 - z_2 - \lambda_2x(1 - z_1)}$$

$$= \frac{kP_2(z_1,\ z_2) - x(P_2(z_1,\ z_2) - P_3(z_1,\ z_2))}{P_3(z_1,\ z_2) + z_1P_2(z_1,\ z_2) - \lambda_2xP_2(z_1,\ z_2)},$$

where $z_1 = k\lambda_1$, $z_2 = k\lambda_2$. Now,

$$f^{(n+1)}(x) = f^{(n)}(f(x)) = \frac{kP_n - k\dfrac{1 - \lambda_1x}{1 - \lambda_2x}(P_n - P_{n+1})}{P_{n+1} + z_1P_n - z_2\dfrac{1 - \lambda_1x}{1 - \lambda_2x}P_n}$$

$$= \frac{kP_{n+1} - x(z_2P_n - z_1(P_n - P_{n+1}))}{P_{n+1} + z_1P_n - z_2P_n - \lambda_2xP_{n+1}} = \frac{kP_{n+1} - x(P_{n+1} - P_{n+2})}{P_{n+2} + z_1P_{n+1} - \lambda_2xP_{n+1}},$$

by (3), and the lemma is proved.

From Lemma 1, it can be seen that

(19) $$f^{(n)}(x) = x + P_n(z_1, z_2)\frac{\lambda_2 x^2 - (1 + z_1)x + k}{P_{n+1}(z_1, z_2) + z_1 P_n(z_1, z_2) - \lambda_2 x P_n(z_1, z_2)},$$

so that x is a fixed point of $f^{(n)}$ iff

(20) $$P_n(z_1, z_2) = 0, \quad \text{or} \quad \lambda_2 x^2 - (1 + z_1)x + k = 0.$$

Thus, it can be seen that, if

$$P_n(z_1, z_2) = 0,$$

then f is periodic of order n no matter what the starting value [or, equivalently, $f^{(n)}(x)$ is the identity function]. From this, we deduce that the result in [7] about the common zeros of generalized Fibonacci polynomials is a direct consequence of (20). For, if N is a multiple of n, and z_1 and z_2 are chosen so that $P_n(z_1, z_2) = 0$, then f will be periodic of order n for any starting value. But f will also be periodic of order N, and so from (19), $P_N(z_1, z_2) = 0$. Thus, $P_n | P_N$ iff $n | N$.

The above property is due to the well-known fact that the map given by (2) is topologically conjugate to the map μz by a Möbius transformation (see, for example, [4]). Consequently, if the function $g(z) = \mu z$ is iterated, then g will be periodic of order n for any initial guess if $\mu^n - 1 = 0$; hence, the zeros of the shifted generalized Fibonacci polynomials are related to the n^{th} roots of unity.

Some simple analysis gives the relationship between μ and (2) as

(21) $$\mu = \frac{1 - 2z_2 + z_1^2 \pm (1 - z_1)\sqrt{(1 + z_1)^2 - 4z_2}}{2(z_2 - z_1)} = \frac{\theta_\pm}{\theta_\mp},$$

where $z_1 = k\lambda_1$, $z_2 = k\lambda_2$. This can also be written as

(22) $$\mu^2 - \mu\left(\frac{1 - 2z_2 + z_1^2}{z_2 - z_1}\right) + 1 = 0.$$

Hence, from (18) and (21), we have

(23) $$\mu^n - 1 = \frac{\theta_\pm^n - \theta_\mp^n}{\theta_\mp^n} = \frac{\theta_\pm - \theta_\mp}{\theta_\mp^n} P_n(z_1, z_2) = \frac{\sqrt{(1 + z_1)^2 - 4z_2}}{\theta_\mp^n} P_n(z_1, z_2).$$

Now the dynamics of g and f are equivalent (see, for example, [4]). If $|\mu| < 1$, then the iterations of g converge to 0 for any starting value while, if $|\mu| > 1$, the iterations converge to infinity for any starting value apart from 0. On the other hand, if $|\mu| = 1$, there are two possibilities: if μ is an n^{th} root of unity, the iterations of g are periodic of order n for any starting value, so that $g^{(n)}$ is the identity function while, if $\mu^n \neq 1$, then the iterations of $g(x)$ wander chaotically on the unit disk of radius x taking on all possible values. Thus, the relationship between the zeros of unity and the zeros of P_n are obtained from (22) and (23) by noting the following:

(i) $\mu = 1$ corresponds to $(1 + z_1)^2 = 4z_2$, so that from (17) and Lemma 1,

$$f^{(n)}(x) = \frac{k - x\left(1 - \frac{1}{2}\left(1 + \frac{1}{n}\right)(1 - z_1)\right)}{\frac{1}{2}\left(1 + \frac{1}{n}\right)(1 - z_1) + z_1 - \frac{x}{4k}(1 + z_1)^2} \to \frac{2k}{1 + z_1} \text{ as } n \to \infty.$$

(ii) $\mu = -1$, which is equivalent to $\mu^n = 1$ for n even, corresponds [by (16) and (22)] to $z_1 = 1$. In this case f is periodic of order 2 for any starting value.

(iii) $\mu^n = 1$, with $\mu \notin \{1, -1\}$, implies [from (22) and (23)] that the zeros of P_n are

$$z_2 = \frac{\mu}{(\mu + 1)^2}(1 - z_1)^2 + z_1. \tag{24}$$

For these values of z_1, $f^{(n)}$ is the identity function.

Thus, in conclusion, we have seen that by iterating the general rational function of degree one and noting that the dynamics of this function are the same as that of the function μz, we have obtained relationships between the zeros of generalized Fibonacci polynomials and the n^{th} roots of unity. These results are not new but the proofs are and they rely upon obtaining a general formula for the n^{th} iteration of a rational function of degree one in terms of a set of polynomials called Shifted Generalized Fibonacci Polynomials. Thus, we have related the study of Fibonacci theory to the iteration of the general rational function of degree one.

With respect to the mathematics of the iteration of nonlinear functions, since it is known that the Schröder Functional Equation has no solution for rational functions of degree 2 or more when μ is an n^{th} root of unity, we have, in this paper, essentially characterized the dynamics of all rational functions that satisfy the SFE when μ is a root of unity. Finally, in this paper we have obtained results about the nature of the zeros of a new class of polynomials by iterating an appropriate class of functions and this technique may well be generalizable.

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ON CERTAIN DIVISIBILITY SEQUENCES

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(Submitted June 1988)

In [1], U_n is defined to be a divisibility sequence if $U_m | U_n$ whenever $m|n$. It is conjectured that

$$U_n = A^n \sum_{i=0}^{k} c_i n^i,$$

A, c_i integers, is a divisibility sequence if and only if exactly k of the c_i are 0. In this note, the conjecture will be shown to be true.

Since the A^n factor offers no difficulty, it will be ignored. Furthermore, the sufficiency can be demonstrated easily; therefore, only the necessity will be proven in the following theorem.

Theorem: Let

$$U_n = \sum_{i=0}^{k} c_i n^i,$$

where the c_i are integers and $c_k \neq 0$, be a divisibility sequence; then, $c_i = 0$ for $0 \leq i \leq k - 1$. (Note that there is no loss of generality in assuming that U_n has this form.)

Proof: Let $n = mt$, n, m, t positive integers. Then,

$$U_n = U_{mt} = \sum_{i=0}^{k} c_i (mt)^i = \sum_{i=0}^{k} c_i m^i t^i = \left(\sum_{i=0}^{k} c_i m^i\right) t^k - \sum_{i=0}^{k-1} c^i (t^k - t^i) m^i.$$

Since $U_m | U_n$ for all t, U_m must divide the second sum on the right-hand side. (Note that the first sum is U_m.)

Now, fix $t > 1$ and let $d_i = c_i(t^k - t^i)$ for $0 \leq i \leq k - 1$; note that $t^k - t^i \neq 0$ for all i. Thus,

$$U_m \left| \sum_{i=0}^{k-1} d_i m^i \right. \text{ for all } m.$$

However, U_m is a polynomial in m of degree $k(c_k \neq 0)$; thus, for sufficiently large m,

$$|U_m| > \left| \sum_{i=0}^{k-1} d_i m^i \right|.$$

Hence,

$$\sum_{i=0}^{k-1} d_i m^i = 0 \text{ for all } m.$$

This implies that $d_i = 0$ for all i, and, consequently, $c_i = 0$, $0 \leq i \leq k - 1$.

Reference

1. R. B. McNeill. "On Certain Divisibility Sequences." *Fibonacci Quarterly 26.2* (1988):169-71.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n, satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-664 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

Let $a_0 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for n in $\{0, 1, \ldots\}$. Show that

$$\lim_{n \to \infty} a_n = \sum_{i=0}^{\infty} \left[\sum_{j=0}^{i} \binom{i}{j} \right]^{-1}.$$

B-665 *Proposed by Christopher C. Street, Morris Plains, NJ*

Show that $AB = 9$, where

$$A = (19 + 3\sqrt{33})^{1/3} + (19 - 3\sqrt{33})^{1/3} + 1,$$

$$B = (17 + 3\sqrt{33})^{1/3} + (17 - 3\sqrt{33})^{1/3} - 1.$$

B-666 *Taken from solutions to B-643 by Russell Jay Hendel, Dowling College, Oakdale, NY, and by Lawrence Somer, Washington, D.C.*

For primes p, prove that

$$\binom{n}{p} \equiv [n/p] \pmod{p},$$

where $[x]$ is the greatest integer in x.

B-667 *Proposed by Herta T. Freitag, Roanoke, VA*

Let p be a prime, $p \neq 2$, $p \neq 5$, and m be the least positive integer such that $10^m \equiv 1 \pmod{p}$. Prove that each m-digit (integral) multiple of p remains a multiple of p when its digits are permuted cyclically.

B-668 *Proposed by A. P. Hillman in memory of Gloria C. Padilla*

Let h be the positive integer whose base 9 numeral

100101102...887888

is obtained by placing all the 3-digit base 9 numerals end-to-end as indicated.

(a) What is the remainder when h is divided by the base 9 integer 14?
(b) What is the remainder when h is divided by the base 9 integer 81?

B-669 *Proposed by Gregory Wulczyn, Lewisburg, PA*

Do the equations

$$25F_{a+b+c}F_{a+b-c}F_{b+c-a}F_{c+a-b} = 4 - L_{2a}^2 - L_{2b}^2 - L_{2c}^2 + L_{2a}L_{2b}L_{2c},$$
$$L_{a+b+c}L_{a+b-c}L_{b+c-a}L_{c+a-b} = -4 + L_{2a}^2 + L_{2b}^2 + L_{2c}^2 + L_{2a}L_{2b}L_{2c},$$

hold for all even integers a, b, c?

SOLUTIONS

Circulant Determinant for F_{n+1}

B-640 *Proposed by Russell Euler, Northwest Missouri State U., Marysville, MO*

Find the determinant of the $n \times n$ matrix (x_{ij}) with $x_{ij} = 1$ for $j = i$ and for $j = i - 1$, $x_{ij} = -1$ for $j = i + 1$, and $x_{ij} = 0$, otherwise.

Solution by Paul S. Bruckman, Edmonds, WA

Let A_n denote the given matrix and D_n its determinant. Clearly, $D_1 = 1$, and $D_2 = 2$. We may expand D_n along its first row; doing so, we see that $D_n = D_{n-1} + B_{n-1}$, where B_n is the determinant of the $n \times n$ matrix obtained by replacing $x_{21} = 1$ by 0 in A_n, all other entries unchanged. Expanding B_{n-1} along its first column, we see that $B_{n-1} = D_{n-2}$. Therefore, we obtain the recurrence relation:

(1) $\quad D_n = D_{n-1} + D_{n-2}$, $n = 3, 4, \ldots$.

Together with the initial values of D_n, we see that

(2) $\quad D_n = F_{n+1}$ $(n = 1, 2, \ldots)$.

Also solved by R. André-Jeannin, C. Ashbacher, Piero Filipponi, Russell Jay Hendel, Hans Kappus, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Alex Necochea, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

F_{mn} and L_{mn} as Polynomials in F_m and L_m

B-641 *Proposed by Dario Castellanos, U. de Carabobo, Valencia, Venezuela*

Prove that

$$F_{mn} = \frac{1}{\sqrt{5}}\left[\left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n - \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n\right],$$

$$L_{mn} = \left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n + \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n.$$

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, NY

Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. It is known that

$$L_m = \alpha^m + \beta^m \quad \text{and} \quad \sqrt{5}F_m = \alpha^m - \beta^m.$$

Solving for α^m and β^m, we have

$$\alpha^m = \frac{L_m + \sqrt{5}F_m}{2} \quad \text{and} \quad \beta^m = \frac{L_m - \sqrt{5}F_m}{2}.$$

Therefore,

$$F_{mn} = \frac{1}{\sqrt{5}}[\alpha^{mn} - \beta^{mn}] = \frac{1}{\sqrt{5}}\left[\left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n - \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n\right],$$

$$L_{mn} = \alpha^{mn} + \beta^{mn} = \left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n + \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n.$$

Editor's note: The proposer asked for a proof that

$$F_{nm} = \frac{1}{\sqrt{5}}\left[\left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n - \left(\frac{L_n - \sqrt{5}F_n}{2}\right)^m\right]$$

and

$$L_{nm} = \left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n + \left(\frac{L_n - \sqrt{5}F_n}{2}\right)^m$$

and the Elementary Problems editor inadvertently interchanged some (but not all) m's and n's.

Also solved by R. André-Jeannin, Paul S. Bruckman, James E. Desmond, Russell Euler, Piero Filipponi, Herta T. Freitag, Guo-Gang Gao, Russell Jay Hendel, Hans Kappus, L. Kuipers, Alex Necochea, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

$L_{k(2n+1)}$ as a Polynomial in L_{2n+1}

B-642 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

It is known that

$$L_{2(2n+1)} = L_{2n+1}^2 + 2,$$

and it can readily be proven that

$$L_{3(2n+1)} = L_{2n+1}^3 + 3L_{2n+1}.$$

Generalize these identities by expressing $L_{k(2n+1)}$, for integers $k \geq 2$, as a polynomial in L_{2n+1}.

Solution by H.-J. Seiffert, Berlin, Germany

Define the Pell-Lucas polynomials $Q_k(x)$ as in [1], p. 7, (1.2), by

$$Q_0(x) = 2,\ Q_1(x) = 2x,\ Q_{k+2}(x) = 2xQ_{k+1}(x) + Q_k(x).$$

First, we show that

(1) $\qquad Q_k(L_{2n+1}/2) = L_{k(2n+1)}$

is true for $k = 0, 1$. Assuming (1) holds for all $j = 0, \ldots, k$, we get

$$\begin{aligned} Q_{k+1}(L_{2n+1}/2) &= L_{2n+1}Q_k(L_{2n+1}/2) + Q_{k-1}(L_{2n+1}/2) \\ &= L_{2n+1}L_{k(2n+1)} + L_{(k-1)(2n+1)} = L_{(k+1)(2n+1)}, \end{aligned}$$

where the last equality can easily be proven by using the known Binet form of the Lucas numbers. Thus (1) is established by induction on k. In [1], p. 9, (2.16), it is shown that, for $k > 0$,

$$(2) \qquad Q_k(x) = \sum_{j=0}^{[k/2]} \frac{k}{k-j}\binom{k-j}{j}(2x)^{k-2j},$$

where [ ] denotes the greatest integer function. From (1) and (2), we obtain

$$L_{k(2n+1)} = \sum_{j=0}^{[k/2]} \frac{k}{k-j}\binom{k-j}{j}L_{2n+1}^{k-2j}.$$

1. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *Fibonacci Quarterly* 23.1 (1985).

Also solved by R. André-Jeannin, Paul S. Bruckman, Herta T. Freitag, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, Sahib Singh, Paul Smith, and the proposer.

Binomial Coefficient Congruence

B-643 *Proposed by T. V. Padnakumar, Trivandrum, South India*

For positive integers a, n, and p, with p prime, prove that

$$\binom{n+ap}{p} - \binom{n}{p} \equiv a \pmod{p}.$$

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, NY

A well known result of E. Lucas [2] states that if the p-ary expansions of n and k are $\sum_{i\ge 0} n_i p^i$ and $\sum_{i\ge 0} k_i p^i$, respectively, then

$$\binom{n}{k} \equiv \prod_{i\ge 0}\binom{n_i}{k_i} \pmod{p}.$$

(For a short and simple proof, consult [1].) Suppose the p-ary expansions of a and $m = n + ap$ are $\sum_{i\ge 0} a_i p^i$ and $\sum_{i\ge 0} m_i p^i$, respectively. We have to show that

$$\binom{m}{p} - \binom{n}{p} \equiv \binom{m_1}{1} - \binom{n_1}{1} = m_1 - n_1 \equiv a \equiv a_0 \pmod{p}.$$

But it is clear from $m = n + ap$ that $m_1 \equiv n_1 + a_0 \pmod{p}$, so the proof is completed.

1. N. J. Fine. "Binomial Coefficients Modulo a Prime." *Amer. Math. Monthly* 54 (1947):589-92.
2. E. Lucas. *Théorie des nombres*. Vol. I. Paris: Librairie Scientifique et Technique Albert Blanchard, 1961. (Original printing, 1891.)

Also solved by R. André-Jeannin, Paul S. Bruckman, Piero Filipponi, Russell Jay Hendel, Joseph J. Kostal & Subramanyam Durbha, L. Kuipers, Bob Prielipp, Lawrence Somer, and the proposer.

Markov Chain

B-644 *Proposed by H. W. Corley, U. of Texas at Arlington, TX*

Consider three children playing catch as follows. They stand at the vertices of an equilateral triangle, each facing its center. When any child has the ball, it is thrown to the child on her or his left with probability 1/3 and to the child on the right with probability 2/3. Show that the probability that the initial holder has the ball after n tosses is

$$\frac{2}{3}\left(\frac{\sqrt{3}}{3}\right)^n \cos\left(\frac{5n\pi}{6}\right) + \frac{1}{3} \text{ for } n = 0, 1, 2, \ldots .$$

Solution by Hans Kappus, Rodersdorf, Switzerland

More generally, let us assign probabilities p, q ($p + q = 1$) for throws to the left and right, respectively. Denote by $p_i(n)$ the probability that child i has the ball after n tosses ($i = 1, 2, 3$) and suppose that child 1 is the initial holder, i.e., impose the initial conditions

$$(1) \qquad p_1(0) = 1,\ p_1(1) = 0.$$

Applying the rule of conditional probability and noting that

$$p_1(n) + p_2(n) + p_3(n) = 1,$$

we have the recursion

$$(2) \qquad \begin{cases} p_1(n+1) = q \cdot p_2(n) + p \cdot p_3(n) = -p \cdot p_1(n) + (q - p) \cdot p_2(n) + p \\ p_2(n+1) = p \cdot p_1(n) + q \cdot p_3(n) = (p - q) \cdot p_1(n) - q \cdot p_2(n) + q \end{cases}$$

Eliminating $p_2(n)$ we arrive at the inhomogeneous second-order difference equation

$$(3) \qquad p_1(n+2) + p_1(n+1) + (1 - 3pq) \cdot p_1(n) = 1 - pq,$$

which may be solved by standard methods. The solution turns out to be

$$(4) \qquad p_1(n) = \frac{2}{3} \cdot (1 - 3pq)^{n/2} \cos n\phi + \frac{1}{3},$$

where ϕ is given by

$$(5) \qquad \cos\phi = -\frac{1}{2} \cdot (1 - 3pq)^{-1/2},\ \sin\phi = \frac{1}{2}\left(\frac{3 - 12pq}{1 - 3pq}\right)^{1/2}.$$

For the special case $p = 1/3$, $q = 2/3$; this is the result of the proposer.

Remark: The process described in the problem is a Markov chain with transition matrix

$$P = \begin{bmatrix} 0 & p & q \\ q & 0 & p \\ p & q & 0 \end{bmatrix}.$$

Also solved by Paul S. Bruckman, Piero Filipponi, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

<u>H-440</u> *Proposed by T. V. Padmakumar, Trivandrum, India*

If $a_1, a_2, \ldots, a_m, n$ are positive integers such that $n > a_1, a_2, \ldots, a_m$ and $\phi(n) = m$ and a_i is relatively prime to n for $i = 1, 2, 3, \ldots, m$, prove

$$\left(\prod_{i=1}^{m} a_i\right)^2 \equiv 1 \pmod{n}.$$

<u>H-441</u> *Proposed by Albert A. Mullin, Huntsville, AL*

By analogy with palindrome, a *validrome* is a sentence, formula, relation, or verse that remains valid whether read forward or backward. For example, relative to *prime* factorization, 341 is a factorably validromic number since $341 = 11 \cdot 31$, and when backward gives $13 \cdot 11 = 143$, which is also correct. (1) What is the largest factorably validromic square you can find? (2) What is the largest factorably validromic square, *avoiding* palindromic numbers, you can find? Here are three examples of factorably validromic squares:

$13 \cdot 13, \quad 101 \cdot 101, \quad 311 \cdot 311.$

<u>H-442</u> *Proposed by Piero Filipponi, Rome, Italy*

Prove that the congruence

$$\prod_{i=1}^{(d-3)/2} (2i+1)^2 \equiv \begin{cases} 1 \pmod{d} \text{ if } (d+1)/2 \text{ is even} \\ -1 \pmod{d} \text{ if } (d+1)/2 \text{ is odd} \end{cases}$$

holds if and only if d is an odd prime.

SOLUTIONS

<u>A Fifth</u>

<u>H-365</u> *Proposed by Larry Taylor, Rego Park, NY*
(Vol. 22, no. 1, February 1984)

Call a Fibonacci-Lucas identity divisible by 5 if every term of the identity is divisible by 5. Prove that, for every Fibonacci-Lucas identity not divisible by 5, there exists another Fibonacci-Lucas identity not divisible by 5 that can be derived from the original identity in the following way:

1) If necessary, restate the original identity in such a way that a derivation is possible.

2) Change one factor in every term of the original identity from F_n to L_n or from L_n to $5F_n$ in such a way that the result is also an identity. If the resulting identity is not divisible by 5, it is the derived identity.

3) If the resulting identity is divisible by 5, change one factor in every term of the original identity from L_n to F_n or from $5F_n$ to L_n in such a way that the result is also an identity. This is equivalent to dividing every term of the first resulting identity by 5. Then, the second resulting identity is the derived identity.

For example, $F_nL_n = F_{2n}$ can be restated as $F_nL_n = F_{2n} \pm F_0(-1)^n$. This is actually two distinct identities, of which the derived identities are

$$L_n^2 = L_{2n} + L_0(-1)^n \quad \text{and} \quad 5F_n^2 = L_{2n} - L_0(-1)^n.$$

Partial solution (Outline) by the proposer

Define a Fibonacci-Lucas equation as an algebraic equation in one unknown in which one of the roots is equal to $(1 + \sqrt{5})/2$. Call a Fibonacci-Lucas equation divisible by $\sqrt{5}$ if every term of the equation is of the form $(5a + b\sqrt{5})/2$ where a and b are integers.

Define a Fibonacci-Lucas identity as the sum of a finite number of terms equated to zero, each of which terms is the product of a finite number of factors, one of which factors is either a Fibonacci or a Lucas number. Call a Fibonacci-Lucas identity divisible by 5 if every term of the identity is of the form $5a$ where a is an integer.

Theorem 1: There are only eight three-term Fibonacci-Lucas identities not divisible by 5.

Theorem 2: Every Fibonacci-Lucas identity can be derived from a three-term Fibonacci-Lucas identity by algebraic manipulation.

Theorem 3: From every Fibonacci-Lucas equation not divisible by $\sqrt{5}$ it is possible to derive two Fibonacci-Lucas identities not divisible by 5.

Theorem 4: There are only four three-term Fibonacci-Lucas equations not divisible by $\sqrt{5}$.

Theorem 5: Every Fibonacci-Lucas equation can be derived from a three-term Fibonacci-Lucas equation by algebraic manipulation.

Theorem 6: From every Fibonacci-Lucas identity not divisible by 5 it is possible to derive another Fibonacci-Lucas identity not divisible by 5 and a Fibonacci-Lucas equation not divisible by $\sqrt{5}$.

Comment: Theorem 6 uses Theorems 1 through 5 as lemmas; the proof of Theorem 6 is the complete solution of this problem.

Reference: L. Taylor. Partial solution of Problem H-365 (first segment). *Fibonacci Quarterly* 27.2 (1989):188-89.

Divide and Conquer

H-418 *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 26, no. 1, February, 1988)

Let $m > 1$ be a positive integer. Suppose that m itself is a general period of the Fibonacci sequence modulo m; that is $F_{n+m} \equiv F_n \pmod{m}$ for all nonnegative integers n. Show that $24|m$.

Solution by Paul Bruckman, Edmonds, WA

Let a and b denote the usual Fibonacci constants; we deal with congruences in $F(\sqrt{5})$, modulo some integer, in the normal way. Given m as defined, we may suppose that

(1) $a^m \equiv c, \quad b^m \equiv d \pmod{m}$.

Setting $n = 0$ in the original congruence, we have

(2) $m | F_m$.

Thus, (1) and (2) imply that $c \equiv d \pmod{m}$. Also $a^{m+1} \equiv ca$, $b^{m+1} \equiv cb \pmod{m}$, so $F_{m+1} \equiv c \pmod{m}$. However, setting $n = 1$ in the original congruence, we have

(3) $F_{m+1} \equiv 1 \pmod{m}$.

Therefore, $c = d = 1$, i.e.,

(4) $a^m \equiv b^m \equiv 1 \pmod{m}$.

Now, a result of Jarden [1] states that

(5) $m | F_m$, $m > 1$ implies either $5 | m$ or $12 | m$.

Note that $a = \frac{1}{2}(1 + \sqrt{5}) \equiv 2^{-1} \equiv 3 \pmod{5}$; also, $a^2 \equiv 4$, $a^3 \equiv 2$, and $a^4 \equiv 1 \pmod{5}$. Hence,

(6) $a^r \equiv 1 \pmod{5}$ iff $4 | r$.

Thus, $a^r \equiv 1 \pmod{20}$ only if $4 | r$. But $a^4 = 2 + 3a = 2^{-1}(7 + 3\sqrt{5})$, and $a^8 = 13 + 21a = 2^{-1}(47 + 21\sqrt{5})$, neither of which expression is defined (mod 20); on the other hand, $a^{12} \equiv 89 + 144a = 2^{-1}(322 + 144\sqrt{5}) = 161 + 36\sqrt{20} \equiv 1 \pmod{20}$. Hence,

(7) $a^r \equiv 1 \pmod{20}$ iff $12 | r$.

Suppose now that $5 | m$. Then $a^m \equiv 1 \pmod{m}$, by (4), so $a^m \equiv 1 \pmod{5}$, which implies $4 | m$, by (6); hence $20 | m$. Then $a^m \equiv 1 \pmod{20}$, so $12 | m$, by (7). Therefore, for m as defined,

(8) $5 | m$ implies $60 | m$.

Therefore, by Jarden's result in (5), we see that $3 | m$ in any event.

Next, we observe that

$$a^2 = 1 + a = 2^{-1}(3 + \sqrt{5}) \equiv 2\sqrt{5} \equiv -\sqrt{5} \pmod{3};$$
$$a^3 = 1 + 2a \equiv 1 - a = b \pmod{3};\ a^4 \equiv ab \equiv -1 \pmod{3};$$
$$a^5 \equiv -a \pmod{3};\ a^6 \equiv \sqrt{5} \pmod{3};$$
$$a^7 \equiv -b \pmod{3};\ a^8 \equiv 1 \pmod{3}.$$

Therefore

(9) $a^s \equiv 1 \pmod{3}$ iff $8 | s$.

Since $3 | m$, $a^m \equiv 1 \pmod{3}$, which implies $8 | m$, by (9); hence, $24 | m$. Q.E.D.

1. Dov Jarden. "Recurring Sequences." *Riveon Lematematika*, 3rd ed. (1973), Theorem F, p. 72.

Also solved by R. Jeannin, L. Kuipers, C. Long, P. Tzermias, and the proposer.

Pell-Mell

H-419 *Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 26, no. 1, February 1988)*

Let $P_0, P_1, \ldots$ be the sequence of Pell numbers defined by

$$P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2} \text{ for } n \in \{2, 3, \ldots\}.$$

Show that

(a) $9\sum_{k=0}^{n} kF_kP_k = 3(n+1)(F_nP_{n+1} + F_{n+1}P_n) - F_{n+2}P_{n+2} - F_nP_n + 2,$

(b) $9\sum_{k=0}^{n} kL_kP_k = 3(n+1)(L_nP_{n+1} + L_{n+1}P_n) - L_{n+2}P_{n+2} - L_nP_n,$

(c) $F_{m+n+2}P_{n+2} + F_{m+n}P_n \equiv 3(n+1)F_m + L_m \pmod 9,$

(d) $L_{m+n+2}P_{n+2} + L_{m+n}P_n \equiv 3(n+1)L_m + 5F_m \pmod 9,$

where n is a nonnegative integer and m any integer.

Solution by the proposer

Remark: (c) and (d) contain interesting special cases.

1) Taking $m = -n$ and using $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^nL_n$ in (c) yields

$$P_{n+2} \equiv (-1)^{n+1}(3(n+1)F_n - L_n) \pmod 9.$$

2) Taking $m = -(n+1)$ and using $P_{n+2} - P_n = 2P_{n+1}$ in (d) yields

$$2P_{n+1} \equiv (-1)^{n+1}(3(n+1)L_{n+1} - 5F_{n+1}) \pmod 9$$

or, after replacing n by $n - 1$

$$2P_n \equiv (-1)^n(3nL_n - 5F_n) \pmod 9.$$

3) Taking $m = -(n+1)$ in (c) and then replacing n by $n - 1$ yields

$$P_{n+1} + P_{n-1} \equiv (-1)^{n+1}(3nF_n - L_n) \pmod 9.$$

4) Taking $m = -n$ in (d) yields

$$3P_{n+2} + 2P_n \equiv (-1)^n(3(n+1)L_n - 5F_n) \pmod 9.$$

Let (G_n) denote either the sequence of Fibonacci or Lucas numbers. Then

$$\begin{aligned}
G_{n+3}P_{n+3} &= (G_{n+2} + G_{n+1})(2P_{n+2} + P_{n+1})\\
&= 2G_{n+2}P_{n+2} + G_{n+2}P_{n+1} + 2G_{n+1}P_{n+2} + G_{n+1}P_{n+1}\\
&= G_{n+2}P_{n+2} + G_{n+2}(P_{n+2} + P_{n+1}) + 2G_{n+1}P_{n+2} + G_{n+1}P_{n+1}\\
&= G_{n+2}P_{n+2} + G_{n+2}(3P_{n+1} + P_n) + 2G_{n+1}P_{n+2} + G_{n+1}P_{n+1}\\
&= G_{n+2}P_{n+2} + 3G_{n+2}P_{n+1} + G_{n+2}P_n + 2G_{n+1}P_{n+2} + G_{n+1}P_{n+1}\\
&= G_{n+2}P_{n+2} + 3(G_{n+2}P_{n+1} + G_{n+1}P_{n+2}) - G_{n+1}P_{n+1} + G_{n+2}P_n\\
&\qquad - G_{n+1}(P_{n+2} - 2P_{n+1})\\
&= G_{n+2}P_{n+2} + 3(G_{n+2}P_{n+1} + G_{n+1}P_{n+2}) - G_{n+1}P_{n+1} + G_nP_n\\
&\qquad + G_{n+1}(P_n - P_{n+2} + 2P_{n+1})
\end{aligned}$$

which yields

$$G_{n+2}P_{n+2} + G_nP_n + 3(G_{n+1}P_{n+2} + G_{n+2}P_{n+1}) = G_{n+3}P_{n+3} + G_{n+1}P_{n+1}. \tag{1}$$

Now we are able to prove (a) and (b) by induction on n.

Proof of (a) and (b): Obviously (a) and (b) hold for $n = 0$. To show that both hold for $n + 1$ if they hold for n, we have to prove the equation

$$\begin{aligned}
(*)\qquad &3(n+1)(G_nP_{n+1} + G_{n+1}P_n) - G_{n+2}P_{n+2} - G_nP_n + 9(n+1)G_{n+1}P_{n+1}\\
&= 3(n+2)(G_{n+1}P_{n+2} + G_{n+2}P_{n+1}) - G_{n+3}P_{n+3} - G_{n+1}P_{n+1}.
\end{aligned}$$

Using

$$\begin{aligned}
G_nP_{n+1} + G_{n+1}P_n + 3G_{n+1}P_{n+1} &= G_nP_{n+1} + G_{n+1}P_n + 2G_{n+1}P_{n+1} + G_{n+1}P_{n+1}\\
&= (G_n + G_{n+1})P_{n+1} + G_{n+1}(2P_{n+1} + P_n) = G_{n+1}P_{n+2} + G_{n+2}P_{n+1}
\end{aligned}$$

and (1), we get (*).

Proof of (c) and (d): In [1] it is proved that

(2) $$3\sum_{k=0}^{n} F_k P_k = F_n P_{n+1} + F_{n+1} P_n$$

and

(3) $$3\sum_{k=0}^{n} L_k P_k = L_n P_{n+1} + L_{n+1} P_n - 2,$$

which shows that 3 divides the right side of (2) and (3). Thus, from (a) and (b) we easily obtain

(4) $$F_{n+2}P_{n+2} + F_n P_n \equiv 2 \text{ (mod } 9,$$

(5) $$L_{n+2}P_{n+2} + L_n P_n \equiv 6(n+1) \quad (\text{mod } 9).$$

Now, if m is any integer, then we multiply (4) by L_m, (5) by F_m, and add the obtained congruences by using the formula $F_k L_m + L_k F_m = 2F_{m+k}$. Then we divide the obtained congruence by 2 [note that GCD(2, 9) = 1] to get (c).

To obtain (d) we multiply (4) by $5F_m$, (5) by L_m and add the obtained congruences by using the formula $5F_k F_m + L_k L_m = 2L_{m+k}$. Now, we again divide the obtained congruence to get (d). This completes the solution.

1. P. S. Bruckman. Solution of B565-B566. *Fibonacci Quarterly* 25.1 (1987):87-88.

Also solved by P. Bruckman, C. Georghiou, R. Andre-Jeannin, L. Kuipers, and G. Wulczyn.

Two Two Much

H-420 *Proposed by Peter Kiss, Eger, Hungary, and Andreas N. Philippou, Patras, Greece (Vol. 26, no. 1, February 1988)*

Show that

(1) $$\sum_{n=1}^{\infty} \frac{2^{2^{n-1}}}{2^{2^n} - 1} = 1.$$

Solution (and Generalization) by H. M. Srivastava, Victoria, Canada

It can easily be seen, by mathematical induction, that (see [1], Example 15, p. 24)

(2) $$\sum_{n=1}^{N} \frac{x^{2^{n-1}}}{x^{2^n} - 1} = \frac{1}{x-1} - \frac{1}{x^{2^N} - 1} \quad (x \neq 1).$$

Now let $N \to \infty$ in cases when $|x| > 1$ and $|x| < 1$, separately, and (2) leads us immediately to the sum

(3) $$\sum_{n=1}^{\infty} \frac{x^{2^{n-1}}}{x^{2^n} - 1} = \begin{cases} 1/(x-1), & \text{if } |x| > 1, \\ x/(x-1), & \text{if } |x| < 1. \end{cases}$$

Equation (1) follows at once from (3) in the *special* case when $x = 2$.

Remark: The general summation formulas (2) and (3) are attributed to De Morgan (1806-1871) and Tannery (1848-1910), respectively, by Bromwich (see [1], Example 15, p. 24; Example 24, p. 273). In fact, (3) has appeared in *numerous* books and tables.

1. T. J. I'A. Bromwich. *An Introduction to the Theory of Infinite Series.* 2nd ed. London: Macmillan, 1926.

Also solved by P. Bruckman, D. Carothers, C. Georghiou, W. Janous, R. Andre-Jaennin, C. Long, H.-J. Seiffert, P. Tzermias, and the proposer.

Editorial Note: The editor wishes to apologize to Paul Bruckman for the omission of his name in the solution of H-409. The editor would like anyone with identities relating to H-409 to submit them to John Turner, University of Waikato, New Zealand, for his judgment as to the awarding of the $25 prize.

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Announcement

FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

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CALL FOR PAPERS

The FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Wake Forest University, Winston-Salem, N.C., from July 30 to August 3, 1990. This Conference is sponsored jointly by the Fibonacci Association and Wake Forest University.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts were to be submitted by March 15, 1990. However, there is still some room on the schedule for speakers. Submit abstracts as soon as possible. Manuscripts are due by May 30, 1990. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

Applications of Fibonacci Numbers. Edited by A.N. Philippou, A.F. Horadam and G.E. Bergum.

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