The Fibonacci Quarterly

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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

ON A NEW KIND OF NUMBERS

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1. Introduction

Recently, properties of the sequence $\{L_{2n+1}\}$, where L_{2n+1} is a Lucas number of order $2n + 1$, were studied [1]. In the present paper, we introduce a new class of numbers defined by

$$(1.1)\qquad f(n, k) = (-1)^{n-k}\binom{2n+1}{n-k}L_{2k+1},$$

where n is any nonnegative integer and $0 \le k \le n$.

These numbers have the interesting property (see [1], (1.5)):

$$(1.2)\qquad \sum_{k=0}^{n} f(n, k) = 1,$$

for every nonnegative integral value of n. Property (1.2) is very much analogous to the following property of Stirling numbers of the first kind (see [2], (6), p. 145):

$$(1.3)\qquad \sum_{k=1}^{n} S_n^k = 0.$$

Also, these new numbers generalize the Catalan numbers in a nontrivial way. First, we recall that the Catalan numbers c_n are defined by means of the generating relation ([5], p. 82)

$$(1.4)\qquad \sum_{n=0}^{\infty} c_n t^n = (2t)^{-1}(1 - \sqrt{1 - 4t}),$$

or by the explicit formula ([5], p. 101)

$$(1.5)\qquad c_n = (n + 1)^{-1}\binom{2n}{n}.$$

The following relationship is obvious:

$$(1.6)\qquad c_n = \frac{(-1)^n f(n, 0)}{(2n + 1)}.$$

Results obtained in this paper include a table, recurrence relations, generating functions, and summation formulas for these new numbers. In view of (1.6), many results reduce to their corresponding results for the Catalan numbers found in the literature. In our last section, we pose two significant open problems.

As usual $(\alpha)_n$ is Pochhammer's symbol and is defined by

$$(1.7)\qquad (\alpha)_n = \begin{cases} 1 & \text{if } n = 0 \\ \alpha(\alpha + 1) \ldots (\alpha + n - 1), & \text{for all } n \in \{1, 2, 3, \ldots\}, \end{cases}$$

${}_2F_1$ will denote the hypergeometric function defined by

$$(1.8)\qquad {}_2F_1\left[\begin{matrix} a, & b; & x \\ & c; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad c \ne 0, -1, -2, \ldots,$$

and the Jacobi polynomials are defined by

$$(1.9)\qquad P_n^{(\alpha,\beta)}(x) = \frac{(1+\beta)_n}{n!}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n, & -\alpha-n; & \dfrac{x+1}{x-1}\\ 1+\beta; & & \end{matrix}\right].$$

2. Table of $f(n, k)$

In this section, we give a table of $f(n, k)$ produced by SCRATCHPAD—IBM's symbolic manipulation language.

TABLE OF $f(n, k)$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	-3	4									
2	10	-20	11								
3	-35	84	-77	29							
4	126	-336	396	-261	76						
5	-462	1320	-1815	1595	-836	199					
6	1716	-5148	7865	-8294	5928	-2587	521				
7	-6435	20020	-33033	39585	-34580	20895	-7815	1364			
8	24310	-77792	136136	-179452	180880	-135320	70856	-23188	3571		
9	-92378	302328	-554268	786828	-883728	771324	-504849	233244	-67849	9349	
10	352716	-1175720	2238390	-3372120	4124064	-4049451	3118185	-1814120	749910	-196329	24476

3. Recurrence Relations

In equation (1.3) of [1], it is noted that

$$(3.1)\qquad 3L_{2n+1} - L_{2n-1} = L_{2n+3},\ n \ge 2.$$

Using (3.1) with n replaced by $n - 1$ and (1.1), we see that

$$(3.2)\qquad (n+k+1)(n+k)f(n, k) + 3(n+k)(n-k+1)f(n, k-1) + (n-k+1)(n-k+2)f(n, k-2) = 0,\ k \ge 2.$$

Furthermore, eliminating L_{2k+1} from

$$f(n+1, k) = (-1)^{n+1-k}\binom{2n+3}{n+1-k}L_{2k+1}$$

and

$$f(n, k) = (-1)^{n-k}\binom{2n+1}{n-k}L_{2k+1},$$

we obtain the formula

$$(3.3)\qquad f(n+1, k) = \frac{-(2n+3)(2n+2)}{(n-k+1)(n+k+2)}f(n, k).$$

Following the method of proof of formula (3.3) and using (1.7), we can obtain its straightforward generalization in the form

$$(3.4)\qquad f(n+m, k) = \frac{(-1)^m(2n+2)_{2m}}{(n-k+1)_m(n+k+2)_m}f(n, k),$$

where m is a nonnegative integer.

4. Generating Relations

We first obtain generating functions for $f(n, k)$ with respect to n. That is, we prove the following theorem.

Theorem 4.1: Let $f(n, k)$ be defined by (1.1). Then

$$(4.1)\qquad \sum_{n=0}^{\infty} \frac{f(n+k, k)}{2(k+n)+1} t^n = \frac{(1+v)^{2k+1}}{2k+1} L_{2k+1}$$

and

$$(4.2)\qquad \sum_{n=0}^{\infty} f(n+k, k) t^n = \frac{(1+v)^{2k+2}}{1-v} L_{2k+1},$$

where v is the function of t defined by

$$(4.3)\qquad v = -t(1+v)^2.$$

Remark: In view of (1.6), (4.1) yields (1.4) while (4.2) yields the following generating relation for the Catalan numbers:

$$(4.4)\qquad \sum_{n=0}^{\infty} (2n+1) c_n t^n = \frac{(1+v)^2}{1-v},\ v = t(1+v)^2.$$

Proof of (4.1): First, multiply both sides of (1.1) by $(2k+1)/[2(k+n)+1]$. Then sum over n from 0 to ∞. Finally, appeal to the well-known identity ([4], p. 348, Prob. 212),

$$(4.5)\qquad \sum_{n=0}^{\infty} \frac{\alpha}{\alpha+\beta n}\binom{\alpha+\beta n}{n} t^n = (1+u)^{\alpha},\ u = t(1+u)^{\beta}$$

to obtain (4.1).

Proof of (4.2): Starting with the definition (1.1) of $f(n, k)$, we have

$$\sum_{n=0}^{\infty} f(n+k, k) t^n = L_{2k+1} \sum_{n=0}^{\infty} \binom{2n+2k+1}{n} (-t)^n = L_{2k+1} \frac{(1+v)^{2k+2}}{1-v},$$

by virtue of the identity (see [4], p. 349, Prob. 216), which is

$$(4.6)\qquad \sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} t^n = \frac{(1+u)^{\alpha+1}}{1-\beta u},$$
$$u = t(1+u)^{\beta+1}.$$

Next, we prove the following theorem on generating functions involving double series:

Theorem 4.2: Let $f(n, k)$ be defined by (1.1). Then

$$(4.7)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(n+k, k) t^{n+k} = (1-t)^{-1},\ |t| < 1,$$

and

$$(4.8)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{f(n+k, k)}{2(k+n)+1} t^{n+k} = (1+v) F_4\left[1, \frac{1}{2}; \frac{3}{2}, \frac{1}{2}; \frac{-v}{4}, \frac{-5v}{4}\right],$$

where F_4 is Appell's double hypergeometric function of the fourth kind defined by ([6], p. 14), that is,

$$(4.9)\qquad F_4[a, b; c, c'; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m (c')_m} \frac{x^m}{m!} \frac{y^n}{n!},\ \sqrt{|x|} + \sqrt{|y|} < 1.$$

Proof of (4.7): By making use of (1.1), we observe that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(n+k, k) t^{n+k} = \sum_{k=0}^{\infty} L_{2k+1} t^k \sum_{n=0}^{\infty} \binom{2n+2k+1}{n} (-t)^n.$$

Summing the inner series with the help of (4.6) and then interpreting the resulting expression by means of the generating relation (1.4) of [1], which is,

$$(4.10)\qquad \sum_{n=0}^{\infty} L_{2n+1} t^n = (1+t)(1-3t+t^2)^{-1},\ |t| < 1,$$

we are led to (4.7). Alternatively, (4.7) can be obtained by using (1.2).

Proof of (4.8): Comparing (4.10) with the following generating function for Jacobi polynomials (see [3], Eq. 10, p. 256),

$$(4.11)\quad \sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n} t^n = (1-t)^{-1-\alpha-\beta}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}(1+\alpha+\beta),\ \frac{1}{2}(2+\alpha+\beta); \\ 1+\alpha;\end{matrix}\ \frac{2t(x-1)}{(1-t)^2}\right]$$

we obtain the formula

$$(4.12)\quad L_{2n+1} = \frac{n!}{(1/2)_n} P_n^{(1/2,-1/2)}(3/2).$$

If we use (4.12) in the following generating function (see [3], p. 271),

$$(4.13)\quad \sum_{n=0}^{\infty} \frac{(\gamma)_n (\delta)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n (1+\beta)_n} t^n = F_4[\gamma, \delta; 1+\alpha, 1+\beta; \tfrac{1}{2}t(x-1), \tfrac{1}{2}t(x+1)],$$

remembering that

$$(4.14)\quad (3/2)_n = (1/2)_n (2n+1),$$

we find that L_{2n+1} satisfies the generating relation

$$(4.15)\quad \sum_{n=0}^{\infty} \frac{L_{2n+1}}{2n+1} t^n = F_4\left(1, \frac{1}{2}; \frac{3}{2}, \frac{1}{2}; \frac{t}{4}, \frac{5t}{4}\right).$$

If we now start with the left-hand side of (4.8), we have

$$\sum_{n,k=0}^{\infty} \frac{f(n+k,k)}{2(n+k)+1} t^{n+k} = \sum_{k=0}^{\infty} \frac{L_{2k+1}}{2k+1} t^k \sum_{n=0}^{\infty} \frac{2k+1}{2(k+n)+1} \binom{2n+2k+1}{n} (-t)^n.$$

Summing the inner series by using (4.5), we get

$$\sum_{n,k=0}^{\infty} \frac{f(n+k,k) t^{n+k}}{2(n+k)+1} = (1+v) \sum_{k=0}^{\infty} \frac{L_{2k+1}}{2k+1} (-v)^k.$$

Interpreting the last infinite series by means of (4.15) along with the second member of the generating function, (4.8) follows at once. This concludes the proof of Theorem 4.2.

5. Summation Formulas

In this section we propose to prove the following summation formulas:

$$(5.1)\quad \sum_{m=0}^{n-1} \left\{ f(n+k-m-1, k) + \frac{L_{2k+1}}{L_{2k-1}} f(n+k-m-1, k-1) \right\} f(m, 0) = f(n+k, k) - f(n, 0) L_{2n+1},\ k \ge 1.$$

$$(5.2)\quad \sum_{m=0}^{n-1} \frac{f(m,0) f(n-m-1, 0)}{(2m+1)\{2(n-m)-1\}} = -\frac{f(n,0)}{2n+1}.$$

$$(5.3)\quad \sum_{m=0}^{\infty} \frac{\{n-2m(1+k)\}}{(2m+1)\{2(k+n-m)+1\}} f(n+k-m, k) f(m, 0) = 0.$$

$$(5.4)\quad \sum_{m=0}^{n-1} f(m,0) \left[\frac{f(n-m+k-1, k)}{2(k+n-m)-1} + \frac{L_{2k+1}}{L_{2k-1}} \frac{f(n+k-m-1, k-1)}{(2m+1)(2k+1)} \right] = \frac{f(n+k,k)}{2(k+n)+1} - \frac{L_{2k+1}}{(2k+1)} \frac{f(n,0)}{(2n+1)},\ k \ge 1.$$

$$(5.5)\quad \sum_{k=0}^{n}(-1)^k\left[\frac{f(n,\ n-k)}{L_{2(n-k)+1}}+\frac{f(n,\ n-k+1)}{L_{2(n-k)+3}}\right]=(2n+1)c_n.$$

$$(5.6)\quad \sum_{k=0}^{n}\frac{f(n,\ k)f(n,\ n-k)}{L_{2k+1}L_{2(n-k)+1}}=\frac{f(2n,\ n)}{L_{2n+1}}-\frac{f(2n,\ n+1)}{L_{2n+3}}.$$

Remark: In view of relationship (1.6), (5.2) and (5.3) yield the following formulas for the Catalan numbers:

$$(5.7)\quad \sum_{m=0}^{n}c_mc_{n-m}=c_{n+1},$$

and

$$(5.8)\quad n\sum_{m=0}^{\infty}c_mc_{n-m}=2\sum_{m=0}^{\infty}mc_mc_{n-m},$$

respectively.

Proof of (5.1): Changing the dummy index k to $k-1$ in (4.2), we get

$$(5.9)\quad \sum_{n=0}^{\infty}f(n+k-1,\ k-1)t^n=L_{2k-1}\frac{(1+v)^{2k}}{1-v}.$$

On the other hand, for $k=0$, (4.2) reduces to

$$(5.10)\quad \sum_{n=0}^{\infty}f(n,\ 0)t^n=\frac{(1+v)^2}{1-v}.$$

In view of (5.9), (4.2) can be written as

$$\sum_{n=0}^{\infty}f(n+k,\ k)t^n=\frac{L_{2k+1}}{L_{2k-1}}(1+v)^2\sum_{n=0}^{\infty}f(n+k-1,\ k-1),$$

which can be rewritten in the form

$$\left[1-t\sum_{m=0}^{\infty}f(m,\ 0)t^m\right]\sum_{n=0}^{\infty}f(n+k,\ k)t^n$$
$$=\frac{L_{2k+1}}{L_{2k-1}}\sum_{m,n=0}^{\infty}f(m,\ 0)f(n+k-1,\ k-1)t^{n+m}$$

by virtue of (5.10).

Equating the coefficients of t^n and using (2.4), we arrive at (5.1).

Proof of (5.2): Setting $k=0$ in (4.1), we get

$$(5.11)\quad \sum_{n=0}^{\infty}\frac{f(n,\ 0)}{2n+1}t^n=1+v.$$

In view of the definition of v in (4.3), (5.11) can be written as

$$\sum_{n=0}^{\infty}\frac{f(n,\ 0)}{2n+1}t^n=1-t(1+v)^2=1-t\sum_{n,m=0}^{\infty}\frac{f(n,\ 0)f(m,\ 0)}{(2n+1)(2m+1)}t^{n+m}.$$

Comparing coefficients of t^n, we get (5.2).

Proof of (5.3): Combining (4.1) and (4.2), we have

$$\sum_{n=0}^{\infty}f(n+k,\ k)t^n=\frac{(2k+1)(1+v)}{1-v}\sum_{n=0}^{\infty}\frac{f(n+k,\ k)}{2(k+n)+1}t^n.$$

Multiplying both sides by $(1+v)$ and then using (5.10) and (5.11), we obtain

$$\sum_{n,m=0}^{\infty}\frac{f(m,\ 0)f(n+k,\ k)}{2m+1}t^{n+m}=(2k+1)\sum_{n,m=0}^{\infty}\frac{f(m,\ 0)f(n+k,\ k)}{2(k+n)+1}t^{n+m}.$$

By equating the coefficients of t^n, we get (5.3).

Proof of (5.4): This proof is similar to that of (5.1) and, hence, will be omitted.

Proof of (5.5); Equation (5.5) is an immediate consequence of (1.1), the following identity (see [5], p. 65),

$$(5.12)\quad \binom{2n+1}{k} = \sum_{k=0}^{n}\left[\binom{2n+1}{k} - \binom{2n+1}{k-1}\right],$$

and (1.5).

Proof of (5.6): Using (1.1), we have

$$\sum_{k=0}^{n}\frac{f(n,k)f(n,n-k)}{L_{2k+1}L_{2(n-k)+1}} = (-1)^k\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1}{n-k} = (-1)^k\binom{4n+2}{n},$$

where we obtained the last equation by using the Vandermonde addition formula, which is

$$(5.13)\quad \sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}.$$

Appealing to the binomial identity

$$(5.14)\quad \binom{x+1}{n} = \binom{x}{n} + \binom{x}{n-1},$$

we have

$$\sum_{k=0}^{n}\frac{f(n,k)f(n,n-k)}{L_{2k+1}L_{2(n-k)+1}} = (-1)^n\left\{\binom{4n+1}{n} + \binom{4n+1}{n-1}\right\}.$$

Using (1.1) to interpret the right-hand side, we arrive at (5.6).

6. Questions

The most obvious questions arising from this work are:

(i) Do the numbers $f(n, k)$ have a nice combinatorial meaning?

(ii) We have seen that $f(n, k)$ has a property analogous to the Stirling numbers of the first kind and that they also generalize the Catalan numbers. Is it possible to associate $f(n, k)$ with some other known mathematical objects?

Acknowledgment

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SOME SEQUENCES OF LARGE INTEGERS

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1. Introduction

One of the many interesting problems posed in the book *Unsolved Problems in Number Theory* [1] concerns the sequence

$$nx_n = x_{n-1}^m(x_{n-1} + n - 1),\ x_1 \in N.$$

It was introduced by Fritz Göbel and has been studied by Lenstra [1] for $m = 1$ and $x_1 = 2$. Lenstra states that x_n is an integer for all $n \le 42$, but x_{43} is not. For $m = 2$ and $x_1 = 2$, David Boyd and Alf van der Poorten state that for $n \le 88$ the only possible denominators in x_n are products of powers of 2, 3, 5, and 7. Why do these denominators cause a problem? Is it possible to find even longer sequences of integers by choosing different values of x_1 and m? These questions were posed by M. Mudge [2].

The terms in these sequences grow fast. For $m = 1$, $x_1 = 2$, the first ten terms are:

3, 5, 10, 28, 154, 3520, 15518880, 267593772160, 160642690122633501504.

If the number of digits in x_n is denoted $N(n)$, then $N(11) = 43$, $N(12) = 85$, $N(13) = 168$, $N(14) = 334$, $N(15) = 667$, $N(16) = 1332$, and $N(17) = 2661$. The last integer in this sequence, x_{42}, has approximately 89288343500 digits.

The purpose of this study is to find a method of determining the number of integers in the sequence and apply the method for the parameters $1 \le m \le 10$ and $2 \le x_1 \le 11$. In particular, the problem of Boyd and van der Poorten will be solved. Some explanations will be given to why some of these sequences are so long. It will be observed and explained why the integer sequences are in general longer for even than for odd values of m.

2. Method

For given values of x_1 and m consider the equation

$$kx_k = x_{k-1}(x_{k-1}^m + k - 1) \tag{1}$$

where the prime factorization of k is given by

$$k = \prod_{i=1}^{\ell} p_i^{n_i}. \tag{2}$$

Let us assume that x_{k-1} is an integer and expand x_{k-1} and $x_{k-1}^m + k - 1$ in a number system with $G_i = p_i^{t_i}$, $(t_i > n_i)$ as base.

$$x_{k-1} = \sum_j a_j G_i^j \quad (0 \le a_j < G_i) \tag{3a}$$

and

$$x_{k-1}^m + k - 1 = \sum_j b_j G_i^j \quad (0 \le b_j < G_i). \tag{3b}$$

Since $x_{k-1} \ne 0$, it is always possible to choose t_i so that $a_0 \ne 0$ and $b_0 \ne 0$. With this t_i we have

$$x_{k-1}(x_{k-1}^m + k - 1) = \sum_{j,\ell} a_j b_\ell G_i^{j+\ell} \equiv a_0 b_0 \pmod{G_i}. \tag{4}$$

The congruence

(5) $kx_k \equiv a_0b_0 \pmod{G_i}$

is soluble iff $(k, G_i)|a_0b_0$, or, in this case, iff $p_i^{n_i}|a_0b_0$. But, if $p_i^{n_i}|a_0b_0$, then by (4) we also have

$$p_i^{n_i}|x_{k-1}(x_{k-1}^m + k - 1).$$

Furthermore, if (5) is soluble for all expansions originating from (2), then it follows that

$$k|x_{k-1}(x_{k-1}^m + k - 1)$$

and, consequently, that x_k is an integer. The solution $x_k \pmod{G_i}$ to $kx_k \equiv a_0b_0 \pmod{G_i}$ is equal to the first term in the expansion of x_k using the equivalent of (3a). The previous procedure is repeated using (3b), (4), and (5) to examine if $x_{k+1} \pmod{G_i}$ is an integer.

From the computational point of view, the testing is done up to a certain pre-set limit $k = k_{\max}$ for consecutive primes $p = 2, 3, 5, 7, \ldots$ to $p \le k_{\max}$. One of three things will happen:

1. All congruences are soluble modulus G_i for $k \le k_{\max}$ for all $p_i \le k_{\max}$.
2. $a_0b_0 = 0$ for a certain set of values $k \le k_{\max}$, $p_i \le k_{\max}$.
3. The congruence $kx_k \equiv a_0b_0 \pmod{G_i}$ is soluble for all $k < n \le k_{\max}$, but not soluble for $k = n$ and $p = p_i$.

In cases 1 and 2 increase $k_{\max}$, respectively, t_i in $G_i = p_i^{t_i}$ (if computer facilities permit) and recalculate. In case 3, x_n is not an integer, viz. n has been found so that x_k is an integer for $k < n$ but not for $k = n$.

3. Results

The results from using this method in the 100 cases $1 \le m \le 10$, $2 \le x_1 \le 11$ are shown in Table 1. In particular, it shows that the integer sequence holds up to $n = 88$ for $m = 2$, $x_1 = 2$ which corresponds to the problem of Boyd and van der Poorten. The longest sequence of integers was found for $x_1 = 11$, $m = 2$. For these parameters, the 600 first terms are integers, but x_{601} is not. In the 100 cases studied, only 32 different primes occur in the terminating values n. In 7 cases, the integer sequences are broken by values of n which are not primes. In 6 of these, the value of n is 2 times a prime which had terminated other sequences. For $x_1 = 3$, $m = 10$, the sequence is terminated by $n = 2 * 13^2$. The prime 239 is involved in terminating 10 of the 100 sequences studied. It occurs 3 times for $m = 6$ and 7 times for $m = 10$. It is seen from the table that integer sequences are in general longer for even than for odd values of m.

TABLE 1. x_n is the first noninteger term in the sequence defined by $nx_n = x_{n-1}(x_{n-1}^m + n - 1)$. The table gives n for parameters x_1 and m.

m	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$x_1 = 5$	$x_1 = 6$	$x_1 = 7$	$x_1 = 8$	$x_1 = 9$	$x_1 = 10$	$x_1 = 11$
1	43	7	17	34	17	17	51	17	7	34
2	89	89	89	89	31	151	79	89	79	601
3	97	17	23	97	149	13	13	83	23	13
4	214	43	139	107	269	107	214	139	251	107
5	19	83	13	19	13	37	13	37	347	19
6	239	191	359	419	127	127	239	191	239	461
7	37	7	23	37	23	37	17	23	7	37
8	79	127	158	79	103	103	163	103	163	79
9	83	31	41	83	71	83	71	23	41	31
10	239	338	139	137	239	239	239	239	239	389

4. A Model To Explain Some Features of the Sequence

The congruence

$$x(k) \equiv \alpha(k) \pmod{p}, \ \alpha(k) \in \{-1, 0, 1, \ldots, p - 2\}$$

studied in a number system with a sufficiently large base p^t, is of particular interest when looking at the integer properties of the sequence. Five cases will be studied. These are:

1. $\alpha(k)$ does not belong to cases 2, 3, 4, or 5 below
2. $\alpha(k) = -1$, $p \neq 2$
3. $\alpha(k) = 0$
4. $\alpha(k) = 1$
5. $\alpha(k) = \alpha(k + 1)$ and/or $\alpha(k) = \alpha(k - 1)$, $\alpha(k) \neq -1, 0, 1$

These cases are mutually exclusive; however, in case 5 there may be more than one sequence of the described type for a given p, for example, for $m = 10$, $x_1 = 7$, and $p = 11$, we have $\alpha(k) = 7$ for $k = 1, 2, \ldots, 10$ and $\alpha(k) = 4$ for $k = 11, 12, \ldots, 15$. Therefore, when running through the values of k for a given p, it is possible to classify $\alpha(k)$ into states corresponding to cases 1, 2, 3, 4 or into one of several possible states corresponding to case 5. In this model, $\alpha(1)$ appears as a result of creation rather than transition from one state to another but, formally, it will be considered as resulting from transition from a state 0 ($k = 0$) to the state corresponding to $\alpha(1)$.

The study of transitions from one state to another in the above model is useful in explaining why there are such long sequences of integers and why they are in general longer for even than for odd m. Table 2 shows the number of transitions of each kind in the 100 cases studied. Let a_r be the number of transitions from state r to state s:

$$A_r = \sum_r a_{rs}, \ B_s = \sum_s a_{rs}, \ Q_s = 100A_s/B_s.$$

(Note that r and s refer to states not rows and columns in Table 2.) The transitions for odd and even values of m are treated separately. It is seen that transitions from states 4, 5, and 2 (for even m) are rare. Only between 5% and 14% of all such states "created" are "destroyed," while the corresponding percentage for other transitions range between 85% and 99%. It is the fact that transitions from certain states are rare, which makes some of these integer sequences so long. That transitions from state 2 are rare for even m (11%) and frequent for odd m (99%) make the integer sequences in general longer for even than for odd m. In all the many transitions observed, it was noted that certain types (underscored in Table 2) only occurred for values of k divisible by p, while other types never occurred for k divisible by p. Transitions from state 3 all occur for k divisible by p but, unlike the other transitions which occur for k divisible by p, they have a high frequency. Some of the observations made on the model are explained in the remainder of this paper.

TABLE 2. The number of transitions of each type for odd and even m

From state	To state 1		To state 2		To state 3		To state 4		To state 5		A_r	
	$2\nmid m$	$2\mid m$	$2\nmid m$	$2\mid m$	$2\nmid m$	$2\mid m$	$2\nmid m$	$2\mid m$	$2\nmid m$	$2\mid m$	$2\nmid m$	$2\mid m$
0	467	1847	38	40	60	60	55	55	32	69	652	2071
1			220	701	252	791	247	642	75	307	794	2241
2	181	$\underline{55}$			71	$\underline{21}$	39	$\underline{7}$	2	0	293	83
3	$\underline{202}$	$\underline{634}$	$\underline{36}$	$\underline{30}$			$\underline{111}$	$\underline{80}$	$\underline{9}$	$\underline{16}$	358	760
4	$\underline{20}$	$\underline{35}$	$\underline{2}$	$\underline{6}$	$\underline{39}$	$\underline{12}$				$\underline{3}$	61	56
5	$\underline{2}$	$\underline{2}$	$\underline{1}$	$\underline{2}$	0	$\underline{3}$	0	$\underline{2}$	$\underline{2}$	$\underline{11}$	5	20
B_s	872	2573	297	779	422	887	452	786	120	406	2163	5431
Q_s %	92	95	99	11	85	86	14	8	5	5		

Transitions from state 4 and, for even m *only, from state 2*

It is evident from $kx_k = x_{k-1}(x_{k-1}^m + k - 1)$ that, if $x_{k-1} \equiv \pm 1 \pmod p$ and $(k, p) = 1$, then $x_k = \pm 1 \pmod p$. Assume that we arrive at $x_{k-1} \equiv \pm 1 \pmod p$ for $k < p - m$ and $m < p$. We can then write

(6) $$x_{p-m-1} \equiv \pm 1 + \alpha p \pmod{p^2},\ 0 \le \alpha < p$$

and

(7) $$x_{p-m-1}^m \equiv (\pm 1 + \alpha p)^m \equiv 1 \pm m\alpha p \pmod{p^2}\ (m \text{ even}).$$

Equations (6) and (7) give

$$(p - m)x_{p-m} \equiv \pm(p - m) \pmod{p^2}$$

or, since $(p - m, p) = 1$,

$$x_{p-m} \equiv \pm 1 \pmod{p^2} \quad \text{or} \quad x_k \equiv \pm 1 \pmod{p^2} \text{ for } p - m \le k \le p - 1.$$

For $k = p$, we have

$$px_p \equiv \pm 1(1 + p - 1) \pmod{p^2}$$

or, after division by p throughout

$$x_p \equiv \pm 1 \pmod p.$$

It is now easy to see that $x_k \equiv 1 \pmod p$ continues to hold also for $k > p$. The integer sequence may, however, be broken for $k = p^2$.

Transitions from state 3

Let us assume that $x_j \equiv 0 \pmod p$ for some $j < p$. If $(j + 1, p) = 1$, it follows that $x_{j+1} \equiv 0 \pmod p$ or, generally, $x_k \equiv 0 \pmod p$ for $j \le k \le p - 1$. For $k = p - 1$, we can write $x_{p-1} \equiv pa \pmod{p^2}$, $0 \le a < p - 1$. We then have

$$px_p \equiv pa(p^m a^m + p - 1) \pmod{p^2},$$

from which follows $x_p \equiv -a \pmod p$, viz. x_p is an integer; however, if $a \nmid 0$, the state is changed.

Transitions from states of type 5

When, for some $j < p - 1$, it happens that $x_j^m \equiv 1 \pmod p$, it is easily seen that $x_k \equiv x_j \pmod p$ for $j \le k < p$. This implies

$$px_p \equiv x_j(1 + p - 1) \pmod p,$$

from which it is seen that x_p may not be congruent to $x_j \pmod p$ but also that x_p is an integer.

References

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2. *Personal Computer World*, December 1987, p. 213.

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CHARACTERISTICS AND THE THREE GAP THEOREM

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1. Introduction

In order to determine the sequence $v = ([m\alpha],\ m = 1, 2, 3, \ldots)$, for irrational α (where $[x]$ denotes the largest integer not exceeding x), Bernoulli [1] considered the sequence of differences $d_1, d_2, d_3, \ldots$, where

(1) $\quad d_m = [(m + 1)\alpha] - [m\alpha],\ m = 1, 2, 3, \ldots .$

Clearly then,

$$[m\alpha] = \sum_{i=1}^{m-1} d_i + [\alpha],\ m = 3, 4, 5, \ldots .$$

Thus, knowing the first two terms of v, one can then determine the entire sequence from (1). For example, with $\alpha = \sqrt{2}$, we have the following.

m	d_m	$[m\alpha]$
1	1	1
2	2	2
3	1	4
4	2	5
5	1	7
6	1	8
7	2	9
8	1	11
9	2	12
10	1	14

It may be shown that d_m may only equal $[\alpha]$ or $[\alpha] + 1$ (that is, 0 or 1 when $0 < \alpha < 1$). If we replace $[\alpha]$ by s (small) and $[\alpha] + 1$ by l (large), then we obtain a string of such characters. This we will refer to as the *characteristic* of α. For example, the characteristic of $\alpha = \sqrt{2}$ is $slslsslsl\ldots$.

String operations may be used to generate the characteristic from its first few terms, by utilizing the continued fraction expansion of α. Bernoulli was the first to guess the rules which were the basis of these string operations. These were reformulated in a more attractive form by Christoffel [2]. However, it had to wait until Markoff [9] before the first proofs were offered. In Section 4 we show how the characteristic is generated.

In this paper we demonstrate a rather intriguing connection between the characteristic of α and the sequence of arcs or gaps formed by the partition of the circle by the successive placement of points by the angle α revolutions. The connection is not immediately obvious and does not hold for all values of α. We use results from the Three Gap Theorem, a result first conjectured by Steinhaus (see [6, 10, 11, 13-15, 18, 19]) which states that N points placed on the circle as above partition it into gaps of either three or two different lengths.

Consider such a circle when N is equal to the denominator of a convergent [see (2)] to α. Only in this case is the circle partitioned into gaps of exactly two different lengths. We can label these gaps as large or small, assigning l or s where appropriate and thus we have a string of gap types, ordered clockwise about the circle, with the first element describing the gap adjacent to the origin.

We show that when N is the denominator of a *total* convergent [see (3)] this string, after a trivial permutation, forms the first few terms of the characteristic, but *only* for special values of α (for example, those numbers with identical terms in their continued fraction expansion). One such value is the golden number, $\alpha = \tau = (\sqrt{5} - 1)/2$. The golden number's characteristic has interesting properties (see [16]) and we give it a special name—the *Golden Sequence*.

In order to state Christoffel's rule for generating the characteristic, we introduce in Section 2 some aspects from the theory of continued fractions. The Three Gap Theorem is later described in more detail in Section 3 before we prove our main result (in Section 4.2).

2. Continued Fractions

Write $t_0 = \alpha$ and express (for $n = 0, 1, 2, \ldots$),

$$a_n = [t_n],$$

$$t_{n+1} = \frac{1}{\{t_n\}},$$

where $\{x\} = x - [x]$ is the fractional part of x. Thus, we can generate the simple continued fraction expansion of α, namely,

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

$$= \{a_0;\ a_1,\ a_2,\ a_3,\ \ldots\}.$$

The *partial* convergents to α are defined as

(2) $$\frac{p_{n,i}}{q_{n,i}} = \{a_0;\ a_1,\ a_2,\ \ldots,\ a_{n-1},\ i\},\quad i = 1, 2, \ldots, a_n - 1,$$

while

(3) $$\frac{p_{n,a_n}}{q_{n,a_n}} = \frac{p_n}{q_n} = \{a_0;\ a_1,\ a_2,\ \ldots,\ a_{n-1},\ a_n\},$$

defines the *total* convergents.

For example, the continued fraction of τ is given by

$$\tau = \{0;\ 1 + \tau\} = \{0;\ 1,\ 1 + \tau\} = \{0;\ 1,\ 1,\ 1,\ \ldots\}.$$

All convergents to τ are total convergents and

$$p_n = q_{n-1} = F_n = F_{n-1} + F_{n-2},\ n \ge 1,\ F_{-1} = 1,\ F_0 = 0.$$

We quote some results from the theory of continued fractions (see Khintchine [7]);

(4) $$\frac{p_{n,i}}{q_{n,i}} = \frac{p_{n-2} + ip_{n-1}}{q_{n-2} + iq_{n-1}},\ p_{-2} = q_{-1} = 0,\ q_{-2} = p_{-1} = 1,$$

(5) $$p_{n-1}q_{n,i} - q_{n-1}p_{n,i} = (-1)^n,$$

(6) $$q_n\alpha - p_n = \frac{(-1)^n}{t_{n+1}p_n + p_{n+1}},$$

(7) $$q_{n,i}\|q_{n-1}\alpha\| + q_{n-1}\|q_{n,i}\alpha\| = 1,$$

(8) $$p_{n,i}\|q_{n-1}\alpha\| + p_{n-1}\|q_{n,i}\alpha\| = \alpha,$$

(9) $$\min_{0<q<q_{n+1}} \|q\alpha\| = \|q_n\alpha\|,$$

(10) $$\|q_n\alpha\| = \begin{cases} 1 - \{q_n\alpha\}, & n \text{ odd}, \\ \{q_n\alpha\}, & n \text{ even}, \end{cases}$$

where $\|q\alpha\| = |q\alpha - p|$, $p = [q\alpha + 1/2]$. That is, $\|q\alpha\|$ is equal to the absolute difference between $q\alpha$ and its nearest integer. Note that $p_n = [q_n\alpha + 1/2]$.

Also, if $\alpha = \{0; a, a, \ldots\} = (\sqrt{a^2 + 4} - a)/2$, then

(11) $$p_n = \frac{(1/\alpha)^n - (-\alpha)^n}{\alpha + 1/\alpha} = q_{n-1}.$$

3. The Three Gap Theorem

The reader is referred to van Ravenstein [18] for an account of the Three Gap Theorem as well as the proofs of many of the results used in this section. Alternatively, the reader may see van Ravenstein [19] where the theorem is also discussed with special reference to the golden number.

3.1 Order of Points

Consider N points placed in succession on a circle at an angle of α. We are interested in determining the order of the points as they appear in clockwise order on the circle. This is equivalent to ordering ($\{n\alpha\} = n\alpha \bmod 1$, $n = 0, 1, 2, \ldots, N - 1$) into an ascending sequence. ($y \bmod x = y - x[y/x] = x\{y/x\}$.) Let ($\{u_j\alpha\}$), $j = 1, 2, \ldots, N$ be that ordered sequence. That is,

$$\{u_1, u_2, \ldots, u_N\} = \{0, 1, \ldots, N - 1\},$$

where $\{u_j\alpha\} < \{u_{j+1}\alpha\}$. It is shown in Slater [11] and Sós [14] (or see [18], Th. 2.2) that the elements u_j are obtained by the following relation,

(12) $$u_{j+1} - u_j = \begin{cases} u_2, & 0 \le u_j < N - u_2, \\ u_2 - u_N, & N - u_2 \le u_j < u_N, \\ -u_N, & u_N \le u_j < N, \end{cases}$$

for $j = 1, 2, \ldots, N$, $u_1 = u_{N+1} = 0$. Points u_j and u_{j+1} delimit the j^{th} gap, which is of length $\{(u_{j+1} - u_j)\alpha\}$.

Here, we will only be concerned with the case where the circle is partitioned into gaps of just two different lengths. This occurs when $N = u_2 + u_N$ or, equivalently, when N is the denominator of a convergent to α.

It may be shown (from [18], Lemma 2.1) that, for $N = u_2 + u_N = q_{n,i}$ ($i = 1, 2, \ldots, a_n$, $n \ge 2$),

(13) $$u_j = ((-1)^{n-1}(j - 1)q_{n-1}) \bmod q_{n,i},\quad j = 1, 2, \ldots, q_{n,i}.$$

For any other value of N, the circle is composed of gaps of three different lengths.

3.2 The String of Gap Types

Now let us consider the more dynamic situation—we will describe the change in gap structure induced by the addition of extra points. In particular, we are interested in the transition from a circle of q_{n-1} gaps to one of q_n gaps. Notation is needed.

Suppose the circle is partitioned into gaps of only two different lengths, say large and small. We label a large gap l and a small gap s. Let

$$\Phi_n = \phi_{n,1}\phi_{n,2} \cdots \phi_{n,q_n}$$

denote the string of gap types when $N = q_n$, ordered clockwise from the origin around the circle so that $\phi_{n,j}$ denotes the gap type (either s or l) of the j^{th} gap formed by points u_j and u_{j+1}. Assume that $\Phi_0 = s$.

For any string S and nonnegative integer t, denote by S^t the concatenation of S with itself t times, where S^0 is the empty string. For any strings S_1, S_2, we write S_1S_2 for the concatenation of S_1 followed by S_2.

Define P_n such that

$$P_n(l) = \begin{cases} s^{a_n}l, & n \text{ odd}, \\ ls^{a_n}, & n \text{ even}, \end{cases} \qquad P_n(s) = \begin{cases} s^{a_n-1}l, & n \text{ odd}, \\ ls^{a_n-1}, & n \text{ even}. \end{cases}$$

The following theorem shows that P_n is the *production rule* which describes the manner in which the string of gap types develops as more points are included on the circle. The result may, after a little effort, be derived from (12). We omit the proof, and refer the reader to Theorem 4.1 in [18].

Theorem 1:

$$\Phi_n = P_n(\Phi_{n-1}) = P_n(\phi_{n-1,1})P_n(\phi_{n-1,2}) \ldots P_n(\phi_{n-1,q_{n-1}}). \ \square$$

Example: For the golden number, $\tau = (\sqrt{5} - 1)/2$,

$$P_n(l) = \begin{cases} sl, & n \text{ odd}, \\ ls, & n \text{ even}, \end{cases} \qquad P_n(s) = l.$$

Hence,

$$\begin{aligned} \Phi_0 &= s, \\ \Phi_1 &= l, \\ \Phi_2 &= ls, \\ \Phi_3 &= sll, \\ \Phi_4 &= llsls, \\ \Phi_5 &= slsllsll. \end{aligned}$$

We now introduce the following two results which we will need to prove our main result in Section 4.2. Proposition 2 demonstrates a simple property of the production rule P_n, while Proposition 3 shows that a component of the string Φ_n is symmetric.

Let $\Theta = \theta_1\theta_2 \ldots \theta_k$ denote a string of k letters, where $\theta_i = s$ or l, $i = 1, 2, \ldots, k$. For any string S, let S^* denote the string S in reverse order. We write $P_n(\Theta)^*$ and $P_n(\theta_k)^*$ for $(P_n(\Theta))^*$ and $(P_n(\theta_k))^*$, respectively.

Proposition 2: $P_n(\Theta)^* = P_{n-1}(\Theta^*)$.

Proof:
$$\begin{aligned} P_n(\Theta)^* &= (P_n(\theta_1)P_n(\theta_2) \ldots P_n(\theta_k))^*, \\ &= P_n(\theta_k)^*P_n(\theta_{k-1})^* \ldots P_n(\theta_1)^*, \\ &= P_{n-1}(\theta_k)P_{n-1}(\theta_{k-1}) \ldots P_{n-1}(\theta_1), \\ &= P_{n-1}(\theta_k\theta_{k-1} \ldots \theta_1), \\ &= P_{n-1}(\Theta^*), \end{aligned}$$

where we have used the fact that $P_n(s)^* = P_{n-1}(s)$ and $P_n(l)^* = P_{n-1}(l)$. $\square$

Let $B_n = \phi_{n,2}\phi_{n,3} \ldots \phi_{n,q_n-1}$.

Proposition 3: $B_n^* = B_n$ $(n \geq 1)$.

Proof: For $n = 1$, the result is trivial since from Theorem 1, $\Phi_1 = s^{a_1-1}l$. Now consider the case $n \geq 2$. It is necessary to show that

$$\phi_{n,j} = \phi_{n,q_n-j+1}, \quad j = 2, 3, \ldots, q_n - 1.$$

From (13), with $i = a_n$ (since $N = q_n$),

(14) $\quad u_j = ((-1)^{n-1}(j - 1)q_{n-1}) \bmod q_n, \quad j = 1, 2, \ldots, q_n.$

Thus,

$$u_{q_n-j+2} = ((-1)^{n-1}(q_n - (j-1))q_{n-1}) \bmod q_n,$$
$$= ((-1)^n(j-1)q_{n-1}) \bmod q_n \quad (j = 2, 3, \ldots, q_n).$$

Hence,

$$u_j + u_{q_n-j+2} = q_n \quad (j = 2, 3, \ldots, q_n).$$

Thus,

$$u_{j+1} - u_j = q_n - u_{q_n-j+1} - (q_n - u_{q_n-j+2}) \quad (j = 2, 3, \ldots, q_n - 1),$$
$$= u_{q_n-j+2} - u_{q_n-j+1},$$

from which the result follows. □

4. The Characteristic of α

4.1 General α

The following method of constructing the characteristic is described in Venkov ([20], pp. 65-68). Markoff first showed that the characteristic of α is equal to $\beta_1\beta_2\beta_3\ldots$, where

$$\beta_n = \beta_{n-1}^{a_n-1}\beta_{n-2}\beta_{n-1}, \; \beta_0 = s, \; \beta_1 = s^{a_1-1}l.$$

We mention that if α is rational, say $\alpha = \{a_0; a_1, a_2, \ldots, a_N\}$, then $\beta_1\beta_2 \ldots \beta_{N-1}(\beta_N)^\infty$ is the characteristic where N is even (so that the number of terms is odd). If N is odd, the number of terms can be made odd, as $\ldots a_{N-1}, a_N\}$ can be replaced by $\ldots a_{N-1}, a_N - 1, 1\}$, if $a_N > 1$. If $a_N = 1$ (and $\alpha \neq 1$), then $\ldots a_{N-2}, a_{N-1}, a_N\}$ can be replaced by $\ldots a_{N-2}, a_{N-1} + 1\}$.

Let $\alpha = \{0; 1, 2, 3\} = \{0; 1, 2, 2, 1\} = 7/10$. Then

$$\beta_0 = s,$$
$$\beta_1 = l,$$
$$\beta_2 = \beta_1\beta_0\beta_1 = lsl,$$
$$\beta_3 = \beta_2\beta_1\beta_2 = lsllsl,$$
$$\beta_4 = \beta_2\beta_3 = lsllsllsl.$$

The characteristic is then given by $\beta_1\beta_2\beta_3(\beta_4)^\infty$, that is,

$$llsllsllsl(lsllsllsl)^\infty.$$

Fraenkel et al. ([4], Theorem 1) offer an alternative method of construction: they show that the characteristic is equal to $\lim_{n\to\infty} \delta_n$, where

$$\delta_n = \delta_{n-1}^{a_n}\delta_{n-2}, \; \delta_0 = s, \; \delta_1 = s^{a_1-1}l. \tag{16}$$

They actually form the characteristic by means of "shift operators." It may be shown, however, that the recurrence relation (16) is an equivalent means of formulating the characteristic, in terms of the actual operations required.

Note that if $\alpha = \{a_0; a_1, a_2, \ldots, a_N\}$, then δ_N^∞ is the characteristic (if N is even).

Example: As in the above example, consider $\alpha = 7/10$. Then

$$\delta_0 = s,$$
$$\delta_1 = l,$$
$$\delta_2 = lls,$$
$$\delta_3 = llsllsl,$$
$$\delta_4 = llsllsllls.$$

Thus, δ_4^∞ is the characteristic.

The method of Fraenkel et al. [4] generalizes the work done by Stolarsky ([12], Theorem 2), who shows how to generate the characteristic for the particular case where $\alpha = \{1; a, a, \ldots\}$, the positive root of $x^2 + (a-2)x - a = 0$.

In this paper, we present a new proof of Theorem 1 in [4] and Theorem 2 in [12] for the case $\alpha \doteq \{0, a, a, \ldots\}$, the positive root of $x^2 + ax - 1 = 0$ by exhibiting a connection between the characteristic of α and its string of gap types (see Theorems 5 and 8 below).

4.2 The Characteristic of $\alpha = \{0; a, a, \ldots\}$

From now on, unless otherwise stated, assume that $\alpha = \{0; a, a, \ldots\}$. For this case, we show how the string of gap types Φ_n is generated recursively and how it is related to the characteristic of α.

Theorem 4:

$$\Phi_n = \Phi_{n-1}^{*a}\Phi_{n-2} \ (n \geq 2), \ \Phi_0 = s, \ \Phi_1 = s^{a-1}l.$$

Proof: Theorem 1 implies the truth of the assertion for $n = 2, 3$. Using the induction hypothesis we show that the result holds in general by verifying it for $n = k + 1$, assuming that it holds for $n = k$ and $n = k - 1$.

$$\begin{aligned}\Phi_{k+1} &= P_{k+1}(\Phi_k)\\ &= P_{k+1}(\Phi_{k-1}^{*a}\Phi_{k-2})\\ &= P_{k+1}(\Phi_{k-1}^{*a})\ P_{k+1}(\Phi_{k-2})\\ &= P_{k-1}(\Phi_{k-1}^{*})^a P_{k-1}(\Phi_{k-2}).\end{aligned}$$

Thus,

$$\Phi_{k+1} = \Phi_k^{*a}\Phi_{k-1},$$

which follows from Theorem 1 and Proposition 2 for $\Theta = \Phi_{k-1}$ and the fact that $P_{k+1} = P_{k-1}$ for all $k \geq 2$. □

The following theorem shows how the string Φ_n is related to another string Ω_n which corresponds to the first q_n elements of the characteristic. One merely places the first element of Φ_n in the penultimate position of Φ_n to obtain Ω_n.

We let $A_n = \phi_{n,1}$, B_n be as in Proposition 3, and $C_n = \phi_{n,q_n}$. Now, let

$$\Omega_n = B_n A_n C_n.$$

Theorem 5:

$$\Omega_n = \Omega_{n-1}^a \Omega_{n-2} \ (n \geq 2), \ \Omega_0 = s, \ \Omega_1 = s^{a-1}l.$$

Proof: The result is readily shown to be true for $n = 2, 3$ from direct observation of the strings Ω_2 and Ω_3. These strings derive from Φ_2 and Φ_3, which may be written down using Theorem 4. In what follows, assume that $n > 3$.

Induction on n using Theorem 1 implies

$$A_n = C_{n-1} = \begin{cases} s, & n \text{ odd},\\ l, & n \text{ even}.\end{cases} \tag{17}$$

(This is actually true for $n \geq 2$.)

We are required to show that

$$B_n A_n C_n = (B_{n-1}A_{n-1}C_{n-1})^a B_{n-2}A_{n-2}C_{n-2},$$

or, using (17),

$$B_n = (B_{n-1}A_{n-1}C_{n-1})^a B_{n-2}. \tag{18}$$

Theorem 4 is equivalent to the statement

$$A_n B_n C_n = (C_{n-1}B_{n-1}^{*}A_{n-1})^a A_{n-2}B_{n-2}C_{n-2}.$$

Using (17) and rearranging terms leads to

$$B_n = B^*_{n-1}A_{n-1}(C_{n-1}B^*_{n-1}A_{n-1})^{a-1}A_{n-2}B_{n-2}.$$

Further manipulation gives

$$B_n = (B^*_{n-1}A_{n-1}C_{n-1})^{a}B_{n-2},$$

which is equivalent to (18). (Recall from Proposition 3 that $B^*_n = B_n$.) Thus, the theorem is proved. □

The following corollary gives the production rule for the string $\Omega_0\Omega_1\Omega_2\ldots$ The proof is by induction and is omitted. Note that the production rule is independent of n.

Corollary 6: Suppose that $Q(s) = s^{a-1}l$, $Q(l) = s^{a-1}ls$. Then

$$\Omega_n = Q(\Omega_{n-1}) = \Omega^{a}_{n-1}\Omega_{n-2}\ (n \geq 2),\ \Omega_0 = s,\ \Omega_1 = s^{a-1}l. \ \square$$

Example: For $\alpha = \tau$, we have $Q(s) = l$, $Q(l) = ls$, and $\Omega_n = \Omega_{n-1}\Omega_{n-2}$ for $n \geq 1$. Hence,

$$\begin{aligned}\Omega_0 &= s,\\ \Omega_1 &= l,\\ \Omega_2 &= ls,\\ \Omega_3 &= lsl,\\ \Omega_4 &= lslls,\\ \Omega_5 &= lsllslsl.\end{aligned}$$

The Golden Sequence is then $\lim_{n\to\infty}\Omega_n$. Comparing Theorem 5 with Fraenkel et al's result ([4], Theorem 1) [equivalent to our Equation (16)] identifies Ω_n as the first q_n elements of the characteristic. That is, $\Omega_n = \delta_n$, where δ_n is defined by (16). Thus, the string of gap types is generated in the same way as the characteristic, a result all the more surprising since it does not hold for all α. We proceed to verify the connection between Ω_n and the characteristic by exploiting the relationship between Φ_n and Ω_n. This, then (with Theorem 5), forms the new proof of Theorem 1 in [4] and Theorem 2 in [12] for the case $\alpha = \{0; a, a, \ldots\}$. The proof sheds light on the set of numbers for which the string of gap types corresponds to the characteristic. First, we need the following, which is proved in van Ravenstein ([17], Equation 5.12).

Lemma 7: $[k\alpha] = \left[k\dfrac{p_{n,i}}{q_{n,i}}\right]$, $k = 1, 2, \ldots, q_{n,i} - 1$ $(n \geq 2,\ 1 \leq i \leq a_n)$, where α is any irrational number. □

Let $\Omega_n = \omega_{n,1}\omega_{n,2}\ \cdots\ \omega_{n,q_n}$.

Theorem 8: For $n \geq 2$,

$$\omega_{n,j} = \begin{cases} s, & d_j = 0,\\ l, & d_j = 1,\end{cases}$$

where d_j is defined by (1) and $j = 1, 2, \ldots, q_n$.

Proof: Equation (14) is equivalent to

$$\begin{aligned}u_j &= q_n\left\{(-1)^{n-1}(j-1)\frac{q_{n-1}}{q_n}\right\},\\ &= (-1)^{n-1}(j-1)q_{n-1} - q_n\left[(-1)^{n-1}(j-1)\frac{q_{n-1}}{q_n}\right].\end{aligned}$$

Thus, for $j = 2, 3, \ldots, q_n - 1$,

$$u_{j+1} - u_j = (-1)^{n-1}q_{n-1} - (-1)^{n-1}q_n\left(\left[j\,\frac{q_{n-1}}{q_n}\right] - \left[(j-1)\frac{q_{n-1}}{q_n}\right]\right), \tag{19}$$

$$= (-1)^{n-1}q_{n-1} - (-1)^{n-1}q_n([j\alpha] - [(j-1)\alpha]). \tag{20}$$

The latter step follows from Lemma 7 and Equation (11). Hence, for $j = 2, 3, \ldots, q_n - 1$,

$$u_{j+1} - u_j = \begin{cases} (-1)^{n-1}q_{n-1}, & d_{j-1} = 0, \\ (-1)^{n-1}(q_{n-1} - q_n), & d_{j-1} = 1. \end{cases}$$

From (9) and (10), it may now be shown that

$$\phi_{n,j} = \begin{cases} s, & d_{j-1} = 0, \\ l, & d_{j-1} = 1, \end{cases} \tag{21}$$

where $j = 2, 3, \ldots, q_n - 1$.

To complete the proof, first note from (6) that

$$[q_n\alpha] = \begin{cases} p_n - 1, & n \text{ odd}, \\ p_n, & n \text{ even}. \end{cases}$$

From Lemma 7, $[(q_n - 1)\alpha] = [(q_n - 1)p_n/q_n] = p_n - 1$. Therefore,

$$d_{q_n-1} = [q_n\alpha] - [(q_n - 1)\alpha] = \begin{cases} 0, & n \text{ odd}, \\ 1, & n \text{ even}. \end{cases}$$

From (14) and (9) we have

$$\phi_{n,1} = \begin{cases} s, & d_{q_n-1} = 0, \\ l, & d_{q_n-1} = 0. \end{cases} \tag{22}$$

The result for ϕ_{n,q_n} follows similarly. From Lemma A1 (see Appendix), $[(q_n + 1)\alpha] = p_n$. Hence,

$$d_{q_n} = [(q_n + 1)\alpha] - [q_n\alpha] = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases}$$

and thus, from (14) and (9),

$$\phi_{n,q_n} = \begin{cases} s, & d_{q_n} = 0, \\ l, & d_{q_n} = 1. \end{cases} \tag{23}$$

Theorem 5 and Equations (21)-(23) establish the proof. □

Corollary 9: Suppose that $\alpha = \{0; a_1, a_2, \ldots\}$, where $a_j = a_{i-j+1}$ for $j = 1, 2, \ldots, i$. Then

$$\omega_{i,j} = \begin{cases} s, & d_j = 0, \\ l, & d_j = 1. \end{cases}$$

Proof: For this value of α,

$$\frac{p_i}{q_i} = \frac{q_{i-1}}{q_i} = \{0; a_i, a_{i-1}, \ldots, a_2, a_1\}.$$

The proof is then identical to the proof of Theorem 8; in particular, the step from (19) to (20) follows. □

The correspondence between Φ_n and the characteristic does not hold for all α, as the following (counter)example shows.

Let $\alpha = \{0; 1, 2, 3, 1 + \tau\} = \dfrac{2\tau + 9}{3\tau + 13}$. Then

$$\Phi_0 = s,\ \Phi_1 = l,\ \Phi_2 = lss,\ \Phi_3 = ssslsslssl,$$

and thus,

$$\Omega_0 = s,\ \Omega_1 = l,\ \Omega_2 = sls,\ \Omega_3 = sslsslsssl,$$

which does not correspond to the characteristic, since

$$\delta_0 = s,\ \delta_1 = l,\ \delta_2 = lls,\ \delta_3 = llsllsllsl. \ \square$$

Conjecture: The correspondence between Φ_n and the characteristic holds only for α equivalent to the number $\{0; a, a, a, \ldots\}$.

APPENDIX. The Evaluation of $[N\alpha]$, $N = 1, 2, \ldots$

We have shown how one may evaluate the integer parts of positive consecutive multiples of a number by forming its characteristic. Here, we present an alternative method by which we decompose the number into terms related to its continued fraction expansion. The method appears in Fraenkel et al. [3] and is central to their paper. We offer a new and shorter proof.

Lemma A1 (see Fraenkel et al. [4], Lemma 2): Suppose that $n > 0$ and $0 < q < q_n$. Then $[(q + q_{n-1})\alpha] = p_{n-1} + [q\alpha]$.

Lemma A2 (see, e.g., Fraenkel [5], Theorem 3): There is a unique decomposition of any natural number N in the form

$$N = \sum_{i=0}^{m} b_i q_i,$$

where the b_i's are integers; $0 \le b_0 < q_1$, $0 \le b_i \le a_{i+1}$, $i > 0$, and $b_i = a_{i+1}$, only if $b_{i-1} = 0$. Since this expansion is unique,

$$\text{(A)} \qquad \sum_{i=0}^{n} b_i q_i < q_{n+1}.$$

Theorem A3: If $N = \sum_{i=k}^{m} b_i q_i$, then

$$[N\alpha] = \begin{cases} \sum_{i=k}^{m} b_i p_i, & k \text{ even}, \\ -1 + \sum_{i=k}^{m} b_i p_i, & k \text{ odd} \end{cases}$$

where $b_k \ne 0$ (i.e., $k = \max\{j: b_j > 0\}$).

Proof: If $N = \sum_{i=k}^{m} b_i q_i$, $b_k \ne 0$, then

$$[N\alpha] = \left[\left(\sum_{i=k}^{m-1} b_i q_i + b_m q_m\right)\alpha\right] = \left[\left(\sum_{i=k}^{m-1} b_i q_i + (b_m - 1)q_m + q_m\right)\alpha\right].$$

From (A),

$$\sum_{i=k}^{m-1} b_i q_i + (b_m - 1)q_m < b_m q_m \le a_{m+1} q_m < q_{m+1}.$$

Hence, from Lemma A1,

$$[N\alpha] = p_m + \left[\left(\sum_{i=k}^{m-1} b_i q_i + (b_m - 1)q_m\right)\alpha\right].$$

Further application of (A) and the lemma leads to

$$[N\alpha] = b_m p_m + \left[\sum_{i=k}^{m-1} b_i q_i \alpha\right].$$

Clearly, we are led to

$$[N\alpha] = \sum_{i=k+1}^{m} b_m p_m + [b_k q_k \alpha].$$

From (6),

$$b_k q_k \alpha - b_k p_k = \frac{b_k(-1)^k}{t_{k+1}p_k + p_{k+1}}.$$

Thus, $-1 < b_k(q_k\alpha - p_k) < 1$, since $0 \le b_k \le a_{k+1}$. Hence,

$$[b_k q_k \alpha] = \begin{cases} b_k p_k, & k \text{ even,} \\ b_k p_k - 1, & k \text{ odd.} \end{cases}$$

This completes the proof. □

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Announcement

FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 30-August 3, 1990

Department of Mathematics and Computer Science

Wake Forest University

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The FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Wake Forest University, Winston-Salem, N.C., from July 30 to August 3, 1990. This Conference is sponsored jointly by the Fibonacci Association and Wake Forest University.

Forty-three papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations have been received. For a copy of the schedule of speakers or more information write to:

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A TWO-VARIABLE LAGRANGE-TYPE INVERSION FORMULA WITH APPLICATIONS TO EXPANSION AND CONVOLUTION IDENTITIES

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1. Introduction

Using Lagrange inversion, one obtains the formal power series (fps) expansions (cf. Riordan [12], Sec. 4.5)

$$\exp bz = \sum_{k=0}^{\infty} \frac{b(ak+b)^{k-1}}{k!} w^k, \tag{1.1}$$

$$\frac{\exp bz}{1-az} = \sum_{k=0}^{\infty} \frac{(ak+b)^k}{k!} w^k, \tag{1.2}$$

where $w = z \cdot \exp(-az)$, and

$$(1+z)^b = \sum_{k=0}^{\infty} \frac{b}{ak+b}\binom{ak+b}{k} v^k, \tag{1.3}$$

$$\frac{(1+z)^b}{1-\dfrac{az}{1+z}} = \sum_{k=0}^{\infty} \binom{ak+b}{k} v^k, \tag{1.4}$$

where $v = z/(1+z)^a$. With the help of these identities, Gould [6-8] obtained many convolution identities. Higher-dimensional extensions of (1.2) and (1.4) were studied and proved by Carlitz [1, 2] using MacMahon's Master Theorem. Finally, Carlitz's identities were embedded into a general theory by Joni [9]. The key for her results, again, is Lagrange inversion (this time the multivariable Lagrange-Good inversion formula, cf. Joni [10]).

In [5] Cohen & Hudson discovered two-variable generalizations of (1.1) and (1.2) that are different in nature from the corresponding results of Carlitz, and studied related convolution identities. Their proofs are based on a specific operator method also used in Cohen's papers [3] and [4]. Thus, the question remained open as to whether there might be a Lagrange-type inversion formula providing the background for Cohen & Hudson's results and yielding two-variable extensions of (1.3) and (1.4), in addition. This formula will be given in Section 3 (Theorem 1, Corollary 2) of this paper. Subsequently, we are able to derive all of Cohen & Hudson's results and, moreover, to give the "factorial" analogues that correspond to (1.3) and (1.4). This will be done in Section 4. For the purpose of illustration, we list some identities in the next section.

2. Some Expansion and Convolution Identities

To write our identities, it is convenient to adopt the usual multidimensional notations. Let $\underset{\sim}{k} = (k_1, k_2)$, $\underset{\sim}{n} = (n_1, n_2) \in \mathbb{Z}^2$ (pairs of integers) and $\underset{\sim}{z} = (z_1, z_2)$ be a pair of indeterminates, then we define $\underset{\sim}{k}! = k_1!k_2!$, $\underset{\sim}{n} \ge \underset{\sim}{k}$ if and only if $n_1 \ge k_1$ and $n_2 \ge k_2$, $\underset{\sim}{n} - \underset{\sim}{k} = (n_1 - k_1, n_2 - k_2)$, $\underset{\sim}{0} = (0, 0)$,

$$\underset{\sim}{z}^{\underset{\sim}{k}} = z_1^{k_1} z_2^{k_2} \quad \text{and} \quad \binom{\underset{\sim}{z}}{\underset{\sim}{k}} = \binom{z_1}{k_1}\binom{z_2}{k_2}.$$

Throughout this paper, for $\underset{\sim}{k} \in \mathbb{Z}^2$, λ_i, μ_i, α_i, $\beta_i \in \mathbb{C}$ (complex numbers), $i = 1, 2$, we shall write

$$R_1(\underset{\sim}{k}) = \frac{(\lambda_1 + \mu_1 k_1)(\alpha_2 + \beta_2 k_2)}{\lambda_2 + \mu_2 k_2}, \quad R_2(\underset{\sim}{k}) = \frac{(\lambda_2 + \mu_2 k_2)(\alpha_1 + \beta_1 k_1)}{\lambda_1 + \mu_1 k_1},$$

and

$$r_1(\underset{\sim}{k}) = \frac{(\lambda_1 + \mu_1 k_1)}{(\lambda_2 + \mu_2 k_2)}, \qquad r_2(\underset{\sim}{k}) = \frac{(\lambda_2 + \mu_2 k_2)}{(\lambda_1 + \mu_1 k_1)},$$

for short. Note that $r_i(\underset{\sim}{k})$ is equal to $R_i(\underset{\sim}{k})$ with $\beta_i = 0$ and $\alpha_i = 1$.

The first of Cohen & Hudson's identities, (1.3) in [5], is equivalent to

$$(2.1) \qquad \frac{1 - \dfrac{\mu_1\mu_2}{\lambda_1\lambda_2} z_1 z_2}{\left(1 - \dfrac{\mu_1}{\lambda_2} z_1\right)\left(1 - \dfrac{\mu_2}{\lambda_1} z_2\right)} = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{r_1(\underset{\sim}{k})^{k_1}}{k_1!} \frac{r_2(\underset{\sim}{k})^{k_2}}{k_2!} \underset{\sim}{z}^{\underset{\sim}{k}} \exp(-r_1(\underset{\sim}{k}) z_1 - r_2(\underset{\sim}{k}) z_2).$$

For $z_2 = 0$, (2.1) reduces to (1.2). The factorial analogue of (2.1) we prove is

$$(2.2) \qquad \frac{1 - \dfrac{\mu_1\mu_2}{\lambda_1\lambda_2} \dfrac{z_1 z_2}{(1 + z_1)(1 + z_2)}}{\left(1 - \dfrac{\mu_1}{\lambda_2} \dfrac{z_1}{1 + z_1}\right)\left(1 - \dfrac{\mu_2}{\lambda_1} \dfrac{z_2}{1 + z_2}\right)}$$

$$= \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \binom{r_1(\underset{\sim}{k})}{k_1}\binom{r_2(\underset{\sim}{k})}{k_2} \underset{\sim}{z}^{\underset{\sim}{k}} (1 + z_1)^{-r_1(\underset{\sim}{k})} (1 + z_2)^{-r_2(\underset{\sim}{k})}.$$

Equation (2.2) reduces to (1.4) for $z_2 = 0$. The "mixed" expansion,

$$(2.3) \qquad \frac{1 - \dfrac{\mu_1\mu_2}{\lambda_1\lambda_2} z_1 \dfrac{z_2}{1 + z_2}}{\left(1 - \dfrac{\mu_1}{\lambda_2} z_1\right)\left(1 - \dfrac{\mu_2}{\lambda_1} \dfrac{z_2}{1 + z_2}\right)} + \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{r_1(\underset{\sim}{k})^{k_1}}{k_1!} \binom{r_2(\underset{\sim}{k})}{k_2} \underset{\sim}{z}^{\underset{\sim}{k}} \exp(-r_1(\underset{\sim}{k}))(1 + z_2)^{-r_2(\underset{\sim}{k})},$$

is a two-variable generalization of (1.2) and (1.4) at the same time. This is seen by setting $z_2 = 0$ or $z_1 = 0$, respectively.

The second expansion of Cohen & Hudson, (1.5) in [5], is equivalent to

$$(2.4) \qquad \frac{1}{\lambda_1 - \mu_2\alpha_1 z_2} = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{1}{\lambda_1 + \mu_1 k_1} \frac{r_1(\underset{\sim}{k})^{k_1} R_2(\underset{\sim}{k})^{k_2}}{\underset{\sim}{k}!} \underset{\sim}{z}^{\underset{\sim}{k}} \exp(-r_1(\underset{\sim}{k}) z_1 - R_2(\underset{\sim}{k}) z_2).$$

Setting $z_2 = 0$ in (2.4) gives (1.1); setting $z_1 = 0$ gives (1.2). The factorial analogue of (2.4) is presented here:

$$(2.5) \qquad \frac{1}{\lambda_1 - \mu_2\alpha_1 \dfrac{z_2}{1 + z_2}} = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{1}{\lambda_1 + \mu_1 k_1} \binom{r_1(\underset{\sim}{k})}{k_1}\binom{R_2(\underset{\sim}{k})}{k_2} \underset{\sim}{z}^{\underset{\sim}{k}} (1 + z_1)^{-r_1(\underset{\sim}{k})} (1 + z_2)^{-R_2(\underset{\sim}{k})}.$$

This is a generalization of (1.3) and (1.4) at the same time. Similarly, a two-dimensional generalization of (1.1) and (1.4) is

$$(2.6) \qquad \frac{1}{\lambda_1 - \mu_2\alpha_1 \dfrac{z_2}{1 + z_2}}$$

$$= \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{1}{\lambda_1 + \mu_1 k_1} \frac{r_1(\underset{\sim}{k})^{k_1}}{k_1!} \binom{R_2(\underset{\sim}{k})}{k_2} \underset{\sim}{z}^{\underset{\sim}{k}} \exp(-r_1(\underset{\sim}{k}) z_1)(1 + z_2)^{-R_2(\underset{\sim}{k})},$$

and a generalization of (1.2) and (1.3) is

$$\frac{1}{\lambda_1 - \mu_2\alpha_1 z_2} = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{1}{\lambda_1 + \mu_1 k_1}\binom{r_1(\underset{\sim}{k})}{k_1}\frac{R_2(\underset{\sim}{k})^{k_2}}{k_2!}\underset{\sim}{z}^{\underset{\sim}{k}}(1 + z_1)^{-r_1(\underset{\sim}{k})}\exp(-R_2(\underset{\sim}{k})z_2). \tag{2.7}$$

We give another expansion, which Cohen & Hudson missed:

$$1 = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{\lambda_1\lambda_2 - k_1k_2\dfrac{(\alpha_1\mu_1 - \beta_1\lambda_1)(\alpha_2\mu_2 - \beta_2\lambda_2)}{(\alpha_1 + \beta_1k_1)(\alpha_2 + \beta_2k_2)}}{(\lambda_1 + \mu_1k_1)(\lambda_2 + \mu_2k_2)} \frac{R_1(\underset{\sim}{k})^{k_1}R_2(\underset{\sim}{k})^{k_2}}{\underset{\sim}{k}!} \cdot \underset{\sim}{z}^{\underset{\sim}{k}}\exp(-R_1(\underset{\sim}{k})z_1 - R_2(\underset{\sim}{k})z_2). \tag{2.8}$$

This is a two-variable generalization of (1.1). The corresponding generalization of (1.3) reads

$$1 = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{\lambda_1\lambda_2 - k_1k_2\dfrac{(\alpha_1\mu_1 - \beta_1\lambda_1)(\alpha_2\mu_2 - \beta_2\lambda_2)}{(\alpha_1 + \beta_1k_1)(\alpha_2 + \beta_2k_2)}}{(\lambda_1 + \mu_1k_1)(\lambda_2 + \mu_2k_2)} \binom{R_1(\underset{\sim}{k})}{k_1}\binom{R_2(\underset{\sim}{k})}{k_2} \cdot \underset{\sim}{z}^{\underset{\sim}{k}}(1 + z_1)^{-R_1(\underset{\sim}{k})}(1 + z_2)^{-R_2(\underset{\sim}{k})}, \tag{2.9}$$

that of (1.1) and (1.3),

$$1 = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{\lambda_1\lambda_2 - k_1k_2\dfrac{(\alpha_1\mu_1 - \beta_1\lambda_1)(\alpha_2\mu_2 - \beta_2\lambda_2)}{(\alpha_1 + \beta_1k_1)(\alpha_2 + \beta_2k_2)}}{(\lambda_1 + \mu_1k_1)(\lambda_2 + \mu_2k_2)} \frac{R_1(\underset{\sim}{k})^{k_1}}{k_1!}\binom{R_2(\underset{\sim}{k})}{k_2} \cdot \underset{\sim}{z}^{\underset{\sim}{k}}\exp(-R_1(\underset{\sim}{k})z_1)(1 + z_2)^{-R_2(\underset{\sim}{k})}. \tag{2.10}$$

Identities (2.1)-(2.10) will be proved in Section 4 by establishing three types of general expansions that underly (2.1)-(2.3), (2.4)-(2.7), and (2.8)-(2.10), respectively.

From each of the expansions (2.1)-(2.10), we may derive a convolution identity which generalizes Jensen's convolution or the Abel-type Jensen-Gould convolution identity, respectively (cf. [6], [7]). We shall give two examples; the remaining identities are obtained similarly.

Multiplying both sides of (2.1) with $\exp(s_1z_1 + s_2z_2)$ and comparing coefficients of $\underset{\sim}{z}^{\underset{\sim}{n}}$, we obtain (1.4) of [5].

$$\frac{s_2^{n_2}}{n_2!}\sum_{j_1=0}^{n_1}\left(\frac{\mu_1}{\lambda_2}\right)^{j_1}\frac{s_1^{n_1-j_1}}{(n_1 - j_1)!} + \frac{s_1^{n_1}}{n_1!}\sum_{j_2=0}^{n_2}\left(\frac{\mu_2}{\lambda_1}\right)^{j_2}\frac{s_2^{n_2-j_2}}{(n_2 - j_2)!} - \frac{\underset{\sim}{s}^{\underset{\sim}{n}}}{\underset{\sim}{n}!} = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{r_1(\underset{\sim}{k})^{k_1}r_2(\underset{\sim}{k})^{k_2}}{\underset{\sim}{k}!}\,\frac{(-r_1(\underset{\sim}{k}) + s_1)^{n_1-k_1}\cdot(-r_2(\underset{\sim}{k}) + s_2)^{n_2-k_2}}{(\underset{\sim}{n} - \underset{\sim}{k})!}. \tag{2.11}$$

The factorial analogue of (2.11), deduced by multiplying both sides of (2.2) with $(1 + z_1)^{s_1}(1 + z_2)^{s_2}$ and comparing coefficients of $\underset{\sim}{z}^{\underset{\sim}{n}}$ reads

$$\binom{s_2}{n_2}\sum_{j_1=0}^{n_1}\left(\frac{\mu_1}{\lambda_2}\right)^{j_1}\binom{s_1 - j_1}{n_1 - j_1} + \binom{s_1}{n_1}\sum_{j_2=0}^{n_2}\left(\frac{\mu_2}{\lambda_1}\right)^{j_2}\binom{s_2 - j_2}{n_2 - j_2} - \binom{\underset{\sim}{s}}{\underset{\sim}{n}} = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}}\binom{r_1(\underset{\sim}{k})}{k_1}\binom{r_2(\underset{\sim}{k})}{k_2}\binom{-r_1(\underset{\sim}{k}) + s_1}{n_1 - k_1}\binom{-r_2(\underset{\sim}{k}) + s_2}{n_2 - k_2}. \tag{2.12}$$

3. Lagrange Inversion

Let

$$\phi(\underset{\sim}{z}) = (\phi_1(z_1, z_2), \phi_2(z_1, z_2))$$

be some pair of fps in z_1 and z_2 with $\phi_i(0, 0) \neq 0$, $i = 1, 2$. Let

$$f(\underset{\sim}{z}) = (z_1/\phi_1(\underset{\sim}{z}),\ z_2/\phi_2(\underset{\sim}{z})).$$

The (two-variable) Lagrange-Good formula solves the problem of expanding a formal Laurent series $g(\underset{\sim}{z})$ of the form

$$(3.1)\qquad g(\underset{\sim}{z}) = \sum_{\underset{\sim}{j} \geq \underset{\sim}{m}} g_{\underset{\sim}{j}} \underset{\sim}{z}^{\underset{\sim}{j}},$$

for some $\underset{\sim}{m} \in \mathbf{Z}^2$, in terms of powers of $f(\underset{\sim}{z})$; namely, if

$$g(\underset{\sim}{z}) = \sum_{\underset{\sim}{k} \in \mathbf{Z}^2} c_{\underset{\sim}{k}} f^{\underset{\sim}{k}}(\underset{\sim}{z}),$$

then

$$c_{\underset{\sim}{k}} = \langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle g(\underset{\sim}{z}) \Delta(\underset{\sim}{z}) f^{-\underset{\sim}{k}}(\underset{\sim}{z}),$$

where $\langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle a(\underset{\sim}{z})$ denotes the coefficient of $\underset{\sim}{z}^{\underset{\sim}{0}}$ in $a(\underset{\sim}{z})$, and

$$\Delta(\underset{\sim}{z}) = \frac{\partial f}{\partial z}(\underset{\sim}{z})\phi_1(\underset{\sim}{z})\phi_2(\underset{\sim}{z}) \text{ with } \frac{\partial f}{\partial z}(\underset{\sim}{z}) \text{ the Jacobian of } f(\underset{\sim}{z}).$$

For formal Laurent series of the form (3.1), we shall use the abbreviation fLs.

The general two-dimensional Lagrange inversion problem can be formulated as follows: Let $F = (f_{\underset{\sim}{k}}(\underset{\sim}{z}))_{\underset{\sim}{k} \in \mathbf{Z}^2}$ be a "diagonal sequence," i.e., $f_{\underset{\sim}{k}}(\underset{\sim}{z})$ is of the form

$$f_{\underset{\sim}{k}}(\underset{\sim}{z}) = \sum_{\underset{\sim}{n} \geq \underset{\sim}{k}} f_{\underset{\sim}{n}\underset{\sim}{k}} \underset{\sim}{z}^{\underset{\sim}{k}}.$$

Then, for a given sequence F, one tries to find some sequence $\tilde{F} = (\tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}))_{\underset{\sim}{k} \in \mathbf{Z}^2}$ such that expanding an arbitrary fLs $g(\underset{\sim}{z})$ in terms of F,

$$(3.2\text{a})\qquad g(\underset{\sim}{z}) = \sum_{\underset{\sim}{k} \in \mathbf{Z}^2} c_{\underset{\sim}{k}} f_{\underset{\sim}{k}}(\underset{\sim}{z}),$$

the coefficients $c_{\underset{\sim}{k}}$ are given by

$$(3.2\text{b})\qquad c_{\underset{\sim}{k}} = \langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle g(\underset{\sim}{z}) \cdot \tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}).$$

Obviously, the sequence $\tilde{F}$ is uniquely determined by

$$(3.3)\qquad \langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle f_{\underset{\sim}{k}}(\underset{\sim}{z}) \cdot \tilde{f}_{\underset{\sim}{n}}(\underset{\sim}{z}) = \delta_{\underset{\sim}{n}\underset{\sim}{k}},$$

where $\delta_{\underset{\sim}{n}\underset{\sim}{k}}$ is the Kronecker delta. In this paper, we shall solve this Lagrange inversion problem for

$$(3.4)\qquad f_{\underset{\sim}{k}}(\underset{\sim}{z}) = \underset{\sim}{z}^{\underset{\sim}{k}} f_1(z_1)^{R_1(\underset{\sim}{k})} f_2(z_2)^{R_2(\underset{\sim}{k})},$$

where $f_1(t)$, $f_2(t)$ are fps in the single variable t with $f_i(0) \neq 0$, $i = 1, 2$. Evidently, $F = (f_{\underset{\sim}{k}}(\underset{\sim}{z}))_{\underset{\sim}{k} \in \mathbf{Z}^2}$ is a diagonal sequence.

Theorem 1: Let $F = (f_{\underset{\sim}{k}}(\underset{\sim}{z}))_{\underset{\sim}{k} \in \mathbf{Z}^2}$ be as defined in (3.4). The sequence

$$\tilde{F} = (\tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}))_{\underset{\sim}{k} \in \mathbf{Z}^2},$$

uniquely determined by (3.3), is given by

$$(3.5)\qquad \tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}) = \frac{\mu_1\mu_2}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} W f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1},$$

where W is the operator

$$(3.6)\qquad W = \det \begin{vmatrix} -z_1 D_1 + \dfrac{\lambda_1}{\mu_1} & \left(\alpha_2 - \dfrac{\beta_2}{\mu_2}\lambda_2\right) F_1(z_1) \\ \left(\alpha_1 - \dfrac{\beta_1}{\mu_1}\lambda_1\right) F_2(z_2) & -z_2 D_2 + \dfrac{\lambda_2}{\mu_2} \end{vmatrix}$$

with

$$F_i(z_i) = z_i \frac{\partial f_i}{\partial z_i}(z_i)/f_i(z_i), \quad i = 1, 2.$$

D_i stands for the differential operator with respect to z_i. Equivalently,

$$(3.7) \qquad \tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}) = \left[\left(1 + \mu_1 \frac{\alpha_2 + \beta_2 k_2}{\lambda_2 + \mu_2 k_2} F_1(z_1)\right)\left(1 + \mu_2 \frac{\alpha_1 + \beta_1 k_1}{\lambda_1 + \mu_1 k_1} F_2(z_2)\right) - \frac{(\alpha_1\mu_1 - \beta_1\lambda_1)(\alpha_2\mu_2 - \beta_2\lambda_2)}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} F_1(z_1)F_2(z_2)\right] f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}.$$

Proof: The proof is based on the method for treating Lagrange inversion problems introduced by the author [11]. For $i = 1, 2$, apply $(z_i D_i + (\lambda_i/\mu_i))$ on both sides of (3.4) to get

$$(3.8) \qquad \left(z_i D_i + \frac{\lambda_i}{\mu_i}\right) f_{\underset{\sim}{k}}(\underset{\sim}{z}) = \left[\frac{\lambda_i + \mu_i k_i}{\mu_i} + (\lambda_i + \mu_i k_i)\left(\frac{\beta_{3-i}}{\mu_{3-i}} + \frac{\alpha_{3-i} - \frac{\beta_{3-i}}{\mu_{3-i}}\lambda_{3-i}}{\lambda_{3-i} + \mu_{3-i}k_{3-i}}\right) F_i(z_i)\right] f_{\underset{\sim}{k}}(\underset{\sim}{z}), \quad i = 1, 2.$$

Writing $c_i(\underset{\sim}{k}) = (\lambda_i + \mu_i k_i)^{-1}$, $i = 1, 2$, simple manipulations show that the system (3.8) of two equations is equivalent to the system

$$(3.9) \qquad U_i f_{\underset{\sim}{k}}(\underset{\sim}{z}) = c_i(\underset{\sim}{k}) V f_{\underset{\sim}{k}}(\underset{\sim}{z}), \quad i = 1, 2,$$

where

$$U_i = \left(z_{3-i}D_{3-i} + \frac{\lambda_{3-i}}{\mu_{3-i}}\right)\left(\frac{1}{\mu_i} + \frac{\beta_{3-i}}{\mu_{3-i}} F_i(z_i)\right) + \left(\alpha_{3-i} - \frac{\beta_{3-i}}{\mu_{3-i}}\lambda_{3-i}\right) F_i(z_i)\left(\frac{1}{\mu_{3-i}} + \frac{\beta_i}{\mu_i} F_{3-i}(z_{3-i})\right)$$

and

$$(3.10) \qquad V = \det \begin{vmatrix} z_1 D_1 + \frac{\lambda_1}{\mu_1} & \left(\alpha_2 - \frac{\beta_2}{\mu_2}\lambda_2\right) F_1(z_1) \\ \left(\alpha_1 - \frac{\beta_1}{\mu_1}\lambda_1\right) F_2(z_2) & z_2 D_2 + \frac{\lambda_2}{\mu_2} \end{vmatrix}.$$

Now Theorem 1 of [11] with $A = \mathbb{C}$, $M_1 = M_2 =$ set of fLs, U_i, V, $c_i(\underset{\sim}{k})$ as above, may be applied. The bilinear form we need is defined by

$$(3.11) \qquad (a(\underset{\sim}{z}), b(\underset{\sim}{z})) = \langle \underset{\sim}{z}^0 \rangle a(\underset{\sim}{z}) b(\underset{\sim}{z}),$$

for fLs $a(\underset{\sim}{z})$ and $b(\underset{\sim}{z})$. Thus, by (4.4) of [11], the dual system

$$(3.12) \qquad U_i^* h_{\underset{\sim}{k}}(\underset{\sim}{z}) = c_i(\underset{\sim}{k}) W h_{\underset{\sim}{k}}(\underset{\sim}{z}), \quad i = 1, 2,$$

[note that $V^* = W$ as defined in (3.6), since $(z_i D_i)^* = -z_i D_i$] has to be solved first. It is a simple matter of fact that (3.12), the dual of (3.9), is equivalent to the dual of (3.8), which reads

$$\left(-z_i D_i + \frac{\lambda_i}{\mu_i}\right) h_{\underset{\sim}{k}}(\underset{\sim}{z}) = \left[\frac{\lambda_i + \mu_i k_i}{\mu_i} + (\lambda_i + \mu_i k_i)\left(\frac{\beta_{3-i}}{\mu_{3-i}} + \frac{\alpha_{3-i} - \frac{\beta_{3-i}}{\mu_{3-i}}\lambda_{3-i}}{\lambda_{3-i} + \mu_{3-i}k_{3-i}}\right) F_i(z_i)\right] h_{\underset{\sim}{k}}(\underset{\sim}{z}), \quad i = 1, 2.$$

A solution of this system of equations is seen to be $h_{\underset{\sim}{k}}(\underset{\sim}{z}) = f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$, hence, by (4.6) of [11], respecting $V^* = W$,

$$\tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}) = \frac{\mu_1\mu_2}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} Wh_{\underset{\sim}{k}}(\underset{\sim}{z}),$$

which establishes (3.5). A little bit of calculation from (3.5) leads to (3.7). □

Corollary 2 (Lagrange formula): Let $F = (f_{\underset{\sim}{k}}(\underset{\sim}{z}))_{\underset{\sim}{k} \in \mathbf{Z}^2}$ be as defined in (3.4). The coefficients in the expansion

(3.13) $\quad g(\underset{\sim}{z}) = \sum_{\underset{\sim}{k} \in \mathbf{Z}^2} c_{\underset{\sim}{k}} f_{\underset{\sim}{k}}(\underset{\sim}{z})$

are given by

(3.14) $\quad c_{\underset{\sim}{k}} = \langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle g(\underset{\sim}{z}) \tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}),$

with $\tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z})$ of (3.7), or

(3.15) $\quad c_{\underset{\sim}{k}} = \dfrac{\mu_1\mu_2}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} \langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1} Vg(\underset{\sim}{z}).$

Proof: Equation (3.14) is merely (3.2) for $f_{\underset{\sim}{k}}(\underset{\sim}{z})$ of (3.4), (3.15) is based on (3.14), (3.5), and $W^* = V$. □

As a first application, we shall prove (1.7) of Cohen & Hudson [5]. Take $f_1(t) = f_2(t) = \exp t$, which implies $F_1(t) = F_2(t) = t$. Let

$$g(\underset{\sim}{z}) = \frac{1}{\lambda_1\lambda_2} \sum_{j=0}^{\infty} \frac{\left[\left(\alpha_1 - \frac{\beta_1}{\mu_1}\lambda_1\right)\left(\alpha_2 - \frac{\beta_2}{\mu_2}\lambda_2\right) z_1 z_2\right]^j}{(\lambda_1/\mu_1 + 1)_j (\lambda_2/\mu_2 + 1)_j},$$

where $(a)_j = a(a+1) \ldots (a+j-1)$. For this choice of $f_i(t)$ [V depends on $F_i(t)$!], $g(\underset{\sim}{z})$ satisfies $Vg(\underset{\sim}{z}) = 1/\mu_1\mu_2$. Utilizing the Lagrange formula (3.15), from this fact we obtain

$$(3.16) \quad \frac{1}{\lambda_1\lambda_2} \sum_{j=0}^{\infty} \frac{\left[\left(\alpha_1 - \frac{\beta_1}{\mu_1}\lambda_1\right)\left(\alpha_2 - \frac{\beta_2}{\mu_2}\lambda_2\right) z_1 z_2\right]^j}{(\lambda_1/\mu_1 + 1)_j (\lambda_2/\mu_2 + 1)_j}$$

$$= \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} (\lambda_1 + \mu_1 k_1)^{-1} (\lambda_2 + \mu_2 k_2)^{-1} \frac{R_1(\underset{\sim}{k})^{k_1} R_2(\underset{\sim}{k})^{k_2}}{\underset{\sim}{k}!} (-\underset{\sim}{z})^{\underset{\sim}{k}} \exp(R_1(\underset{\sim}{k}) z_1 + R_2(\underset{\sim}{k}) z_2).$$

Equation (3.16) is another two-variable extension of (1.1) (set $z_2 = 0$).

4. Coefficient Formulas for Some Special Expansions

The following technical lemma turns out to be useful for further computations.

Lemma 3: Let $h(t)$ be an fLs in t. With the assumptions of Theorem 1,

(4.1) $\quad \langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle h(z_{3-i}) F_i(z_i) f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$

$$= -\frac{k_i(\lambda_{3-i} + \mu_{3-i} k_{3-i})}{(\lambda_i + \mu_i k_i)(\alpha_{3-i} + \beta_{3-i} k_{3-i})} \langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle h(z_{3-i}) f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}, \text{ for } i = 1, 2.$$

Proof: Without loss of generality we prove (4.1) for $i = 1$. We start with the identity

$$\langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle h(z_2)\left(1 + \mu_1 \frac{\alpha_2 + \beta_2 k_2}{\lambda_2 + \mu_2 k_2} F_1(z_1)\right) f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$$

$$= -\langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle h(z_2) z_2^{-k_2} f_2(z_2)^{-R_2(\underset{\sim}{k})} f_1(z_1)^{-\lambda_1(\alpha_2+\beta_2 k_2)/(\lambda_2+\mu_2 k_2)} \cdot$$

$$\cdot \frac{1}{k_1} z_1 D_1 \left[z_1^{-k_1} f_1(z_1)^{-\mu_1 k_1(\alpha_2+\beta_2 k_2)/(\lambda_2+\mu_2 k_2)}\right].$$

Because of $(z_1 D_1)^* = -z_1 D_1$ [with respect to the bilinear form of (3.11)] the right-hand side of this equation is equal to

$$\frac{1}{k_1}\langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle z_1 D_1\Big[h(z_2) z_2^{-k_2} f_2(z_2)^{-R_2(\underset{\sim}{k})} f_1(z_1)^{-\lambda_1(\alpha_2+\beta_2 k_2)/(\lambda_2+\mu_2 k_2)}\Big]\cdot$$
$$\cdot\, z_1^{-k_1} f_1(z_1)^{-\mu_1 k_1(\alpha_2+\beta_2 k_2)/(\lambda_2+\mu_2 k_2)}.$$

Together with a bit of manipulation, we finally arrive at (4.1). □

Corollary 4: Let $f_{\underset{\sim}{k}}(\underset{\sim}{z})$ be given by (3.4). Then

$$1 = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{\lambda_1\lambda_2 - k_1 k_2 \dfrac{(\alpha_1\mu_1 - \beta_1\lambda_1)(\alpha_2\mu_2 - \beta_2\lambda_2)}{(\alpha_1 + \beta_1 k_1)(\alpha_2 + \beta_2 k_2)}}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} d_{\underset{\sim}{k}} f_{\underset{\sim}{k}}(\underset{\sim}{z}), \tag{4.2}$$

where $d_{\underset{\sim}{k}} = \langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$.

Proof: By (3.14) we have to compute

$$\langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle 1 \cdot \tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}) = \langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle \tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}).$$

Using the form (3.7) of $\tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z})$, repeated application of (4.1) gives

$$\begin{aligned}\langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle \tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}) &= \langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle\Bigg[\left(1 - \mu_1\frac{k_1}{\lambda_1 + \mu_1 k_1}\right)\left(1 - \mu_2\frac{k_2}{\lambda_2 + \mu_2 k_2}\right) \\ &\qquad - \frac{(\alpha_1\mu_1 - \beta_1\lambda_1)(\alpha_2\mu_2 - \beta_2\lambda_2)}{(\alpha_1 + \beta_1 k_1)(\alpha_2 + \beta_2 k_2)}\cdot\frac{k_1 k_2}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)}\Bigg] f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1} \\ &= \frac{\lambda_1\lambda_2 - k_1 k_2\dfrac{(\alpha_1\mu_1 - \beta_1\lambda_1)(\alpha_2\mu_2 - \beta_2\lambda_2)}{(\alpha_1 + \beta_1 k_1)(\alpha_2 + \beta_2 k_2)}}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)}\langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1},\end{aligned}$$

which furnishes (4.2). □

For $f_i(z_i) = \exp(-z_i)$ and $f_i(z_i) = (1 + z_i)^{-1}$, respectively, the expansions (2.8) and (2.9) are obtained as special cases of (4.2). The mixed analogue (2.10) is (4.2) with $f_1(z_1) = \exp(-z_1)$ and $f_2(z_2) = (1 + z_2)^{-1}$.

Quite analogously, we prove

Corollary 5: If $f_{\underset{\sim}{k}}(\underset{\sim}{z}) = \underset{\sim}{z}^{\underset{\sim}{k}} f_1(z_1)^{r_1(\underset{\sim}{k})} f_2(z_2)^{R_2(\underset{\sim}{k})}$, then

$$\frac{1}{\lambda_1 + \mu_2\alpha_1 F_2(z_2)} = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} (\lambda_1 + \mu_1 k_1)^{-1} d_{\underset{\sim}{k}} f_{\underset{\sim}{k}}(\underset{\sim}{z}), \tag{4.3}$$

where $d_{\underset{\sim}{k}} = \langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$.

Sketch of Proof: Again using (3.14), (3.7), and (4.1) we proceed along the lines of the proof of the preceding corollary. □

The expansions (2.4)-(2.7) are special cases of (4.3). Finally, we have

Corollary 6: If $f_{\underset{\sim}{k}}(\underset{\sim}{z}) = \underset{\sim}{z}^{\underset{\sim}{k}} f_1(z_1)^{r_1(\underset{\sim}{k})} f_2(z_2)^{r_2(\underset{\sim}{k})}$, then

$$\frac{1 - \dfrac{\mu_1\mu_2}{\lambda_1\lambda_2} F_1(z_1) F_2(z_2)}{\left(1 + \dfrac{\mu_1}{\lambda_2} F_1(z_1)\right)\left(1 + \dfrac{\mu_2}{\lambda_1} F_2(z_2)\right)} = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} d_{\underset{\sim}{k}} f_{\underset{\sim}{k}}(\underset{\sim}{z}), \tag{4.4}$$

where $d_{\underset{\sim}{k}} = \langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$.

Proof: Observe that the left-hand side of (4.4) is equal to

$$1/\left(1+\frac{\mu_1}{\lambda_2}F_1(z_1)\right)+1/\left(1+\frac{\mu_2}{\lambda_1}F_2(z_2)\right)-1.$$

This in hand, the method used to prove Corollary 4 can be used again to settle (4.4). □

Equations (2.1)-(2.3) are special cases of (4.4).

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LAMBERT SERIES AND THE SUMMATION OF RECIPROCALS IN CERTAIN FIBONACCI-LUCAS-TYPE SEQUENCES

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1. Introduction

Consider the sequence of real numbers defined by the recurrence relation

(1.1) $W_n = pW_{n-1} + W_{n-2}$,

where p is a strictly positive real number. Special cases of (W_n) which interest us here are:

(1.2) $U_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ (Fibonacci-type sequence),

and

(1.3) $V_n = \alpha^n + \beta^n$ (Lucas-type sequence),

where

$$\alpha = \frac{p + \sqrt{p^2 + 4}}{2},$$

(1.4)
$$\beta = \frac{p - \sqrt{p^2 + 4}}{2}.$$

It is clear that

(1.5) $\alpha\beta = -1,\ \alpha > 1,\ -1 < \beta < 0.$

On the other hand, the Lambert series is defined by

(1.6) $L(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n},\quad |x| < 1.$

It has been known for a long time (see Horadam [1] for complete references) that

$$\sum_{n=1}^{\infty} \frac{1}{U_{2n}} = (\alpha - \beta)[L(\beta^2) - L(\beta^4)],$$

$$\sum_{n=1}^{\infty} \frac{1}{V_{2n-1}} = -L(\beta) + 2L(\beta^2) - L(\beta^4).$$

The purpose of this paper is to establish the following result.

Theorem 1:

(1.7) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+1}} = 2(\alpha - \beta)[L(\beta^2) - 2L(\beta^4) + 2L(\beta^8)] + \beta;$

(1.8) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{V_n V_{n+1}} = \frac{2}{\alpha - \beta}[L(\beta^2) - 2L(\beta^8)] + \frac{\beta}{(\alpha - \beta)p}.$

2. Preliminary Lemma

Lemma 1:

(2.1) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 - x^{2n+1}} = L(x) - L(x^2);$

$$(2.2)\quad \sum_{n=1}^{\infty} \frac{x^n}{1+x^n} = L(x) - 2L(x^2);$$

$$(2.3)\quad \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1+x^{2n+1}} = L(x) - 3L(x^2) + 2L(x^4).$$

(2.1) is obviously true, whereas (2.2) follows from the identity

$$\frac{x^n}{1+x^n} = \frac{x^n}{1-x^n} - \frac{2x^{2n}}{1-x^{2n}},$$

and (2.3) follows from

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{1+x^{2n+1}} = \sum_{n=1}^{\infty} \frac{x^n}{1+x^n} - \sum_{n=1}^{\infty} \frac{x^{2n}}{1+x^{2n}}.$$

3. Proof of Theorem 1

Lemma 2:

$$(3.1)\quad 2\sum_{n=1}^{\infty} \frac{1}{\alpha^n U_n} = \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+1}};$$

$$(3.2)\quad 2\sum_{n=1}^{\infty} \frac{1}{\alpha^n V_n} = \frac{1}{\alpha p} + \sum_{n=1}^{\infty} \frac{\alpha-\beta}{V_n V_{n+1}}$$

Proof: First, we have

$$\alpha U_{n+1} + U_n = \frac{1}{\alpha-\beta}\left[\alpha\left(\alpha^{n+1} - (-1)^{n+1}\frac{1}{\alpha^{n+1}}\right) + \alpha^n - (-1)^n \frac{1}{\alpha^n}\right]$$
$$= \frac{1}{\alpha-\beta}(\alpha^{n+2} + \alpha^n) = \frac{\alpha^{n+1}}{\alpha-\beta}\left(\alpha + \frac{1}{\alpha}\right) = \alpha^{n+1}.$$

Thus,

$$\frac{1}{\alpha^n U_n} + \frac{1}{\alpha^{n+1}U_{n+1}} = \frac{1}{U_n U_{n+1}}.$$

By adding this term by term, we find (3.1) since $U_1 = 1$. The proof of (3.2) follows the same pattern if we observe that

$$\alpha V_{n+1} + V_n = (\alpha - \beta)\alpha^{n+1}.$$

Thus,

$$\frac{1}{\alpha^n V_n} + \frac{1}{\alpha^{n+1}V_{n+1}} = \frac{\alpha-\beta}{V_n V_{n+1}}.$$

Now, adding this term by term, we find (3.2) since $V_1 = p$.

Lemma 3:

$$(3.3)\quad \sum_{n=1}^{\infty} \frac{1}{\alpha^n U_n} = (\alpha-\beta)[L(\beta^2) - 2L(\beta^4) + 2L(\beta^8)];$$

$$(3.4)\quad \sum_{n=1}^{\infty} \frac{1}{\alpha^n V_n} = L(\beta^2) - 2L(\beta^8).$$

Proof:

$$\frac{1}{\alpha-\beta}\sum_{n=1}^{\infty} \frac{1}{\alpha^n U_n} = \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n} - (-1)^n} = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{1-(-1)^n\beta^{2n}}$$
$$= \sum_{n=1}^{\infty} \frac{\beta^{4n}}{1-\beta^{4n}} + \sum_{n=0}^{\infty} \frac{\beta^{4n+2}}{1+\beta^{4n+2}}$$

Using (1.6) with $x = \beta^4$ and (2.3) with $x = \beta^2$, we find (3.3). On the other hand, we have:

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^n V_n} = \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n} + (-1)^n} = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{1 + (-1)^n \beta^{2n}} = \sum_{n=1}^{\infty} \frac{\beta^{4n}}{1 + \beta^{4n}} + \sum_{n=0}^{\infty} \frac{\beta^{4n+2}}{1 - \beta^{4n+2}}.$$

Using (2.2) with $x = \beta^4$ and (2.1) with $x = \beta^2$, we find (3.4). This concludes the proof of Lemma 3. Now the proof of the theorem follows immediately from Lemmas 2 and 3.

4. Special Cases

4.1 Fibonacci-Lucas Sequences

Let $p = 1$ in (1.1) to obtain

$$W_n = W_{n-1} + W_{n-2}, \quad \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

$U_n = F_n$ is the Fibonacci sequence and $V_n = L_n$ is the Lucas sequence. Equations (1.7) and (1.8) take the following form:

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} = 2\sqrt{5}\left[L\left(\frac{3 - \sqrt{5}}{2}\right) - 2L\left(\frac{7 - 3\sqrt{5}}{2}\right) + 2L\left(\frac{47 - 21\sqrt{5}}{2}\right)\right] + \frac{1 - \sqrt{5}}{2};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1}} = \frac{2}{\sqrt{5}}\left[L\left(\frac{3 - \sqrt{5}}{2}\right) - 2L\left(\frac{47 - 21\sqrt{5}}{2}\right)\right] + \frac{1 - \sqrt{5}}{2\sqrt{5}}.$$

4.2 Pell and Pell-Lucas Sequences

Let $p = 2$ in (1.1) to obtain

$$W_n = 2W_{n-1} + W_{n-2}, \quad \alpha = 1 + \sqrt{2}, \quad \beta = 1 - \sqrt{2}.$$

$U_n = P_n$ is the Pell sequence, $V_n = Q_n$ is the Pell-Lucas sequence. Equations (1.7) and (1.8) take the form:

$$\sum_{n=1}^{\infty} \frac{1}{P_n P_{n+1}} = 4\sqrt{2}[L(3 - 2\sqrt{2}) - 2L(17 - 12\sqrt{2}) + 2L(577 - 408\sqrt{2})] + 1 - \sqrt{2};$$

$$\sum_{n=1}^{\infty} \frac{1}{Q_n Q_{n+1}} = \frac{1}{\sqrt{2}}[L(3 - 2\sqrt{2}) - 2L(577 - 408\sqrt{2})] + \frac{1 - \sqrt{2}}{4\sqrt{2}}.$$

5. Generalization

The following theorem generalizes the above result. It is given without proof, since the methods required exactly parallel those of Section 3. We assume that K is an odd integer.

Theorem 2:

$$\sum_{n=1}^{\infty} \frac{1}{U_{kn} U_{k(n+1)}} = \frac{2(\alpha - \beta)}{U_k}[L(\beta^{2k}) - 2L(\beta^{4k}) + 2L(\beta^{8k})] + \frac{\beta^k}{U_k^2};$$

$$\sum_{n=1}^{\infty} \frac{1}{V_{kn} V_{k(n+1)}} = \frac{2}{(\alpha - \beta)U_k}[L(\beta^{2k}) - 2L(\beta^{8k})] + \frac{\beta^k}{(\alpha - \beta)U_k V_k}.$$

For the proof, the reader will need the following lemmas.

Lemma 2':

$$2\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} U_{kn}} = \frac{1}{\alpha^k U_k} + U_k \sum_{n=1}^{\infty} \frac{1}{U_{kn} U_{k(n+1)}};$$

$$2\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} V_{kn}} = \frac{1}{\alpha^k V_k} + (\alpha - \beta)U_k \sum_{n=1}^{\infty} \frac{1}{V_{kn} V_{k(n+1)}}.$$

Lemma 3':

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} U_{kn}} = (\alpha - \beta)[L(\beta^{2k}) - 2L(\beta^{4k}) + 2L(\beta^{8k})];$$

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} V_{kn}} = L(\beta^{2k}) - 2L(\beta^{8k}).$$

Reference

1. A. F. Horadam. "Elliptic Functions and Lambert Series in the Summation of Reciprocals in Certain Recurrence-Generated Sequences." *Fibonacci Quarterly 26.2* (1988):98-114.

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THE DISTRIBUTION OF RESIDUES OF TWO-TERM RECURRENCE SEQUENCES

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Let U_0, U_1, A, B be integers and define, for $n \geq 2$,

$$U_n = AU_{n-1} + BU_{n-2}.$$

For an integer $m > 1$, the sequence (U_n) considered modulo m is eventually periodic. We say (U_n) is *uniformly distributed* modulo m [notation: u.d.(mod m)] if every residue modulo m occurs with the same frequency in any period. In this case, it is clear that the length of any period will be a multiple of m. Conditions that (U_n) be u.d.(mod m) can be found in [2, Theorem A]. Suppose (U_n) is u.d.(mod p^k) where p is a prime and $k > 0$. Let $M \geq 2$ be any integer. We study the relationship between the distribution of U_n (mod M) and U_n (mod $M \cdot p^k$). For integers $N \geq 2$ and $0 \leq c < N$, denote by $\nu(N, c)$ the number of times that c occurs as a residue in one shortest period of U_n (mod N). Our main result can now be stated.

Theorem: Let p be a prime and $k > 0$ be an integer such that U_n is u.d.(mod p^k). Say U_n has shortest period of length $p^k f$ modulo p^k. Let $M \geq 2$, and assume that U_n is purely periodic modulo M, with shortest period of length Q. Assume $p \nmid Q$. Then, for any $0 \leq a < M$, and $0 \leq b < M \cdot p^k$ with $b \equiv a \pmod{M}$,

$$\nu(M \cdot p^k, b) = \frac{f}{(Q, f)} \cdot \nu(M, a).$$

We remark that (,) denotes the GCD. Also, observe that the hypothesis $p \nmid Q$ yields $p \nmid M$. To prove the Theorem, we make use of a recent result of Vélez [2], which we state here for the reader's convenience.

Lemma: Suppose that U_n is u.d.(mod p^k) with shortest period of length $p^k f$. Then, for any integer $s \geq 0$, the sequence U_{s+qf}, $q = 0, 1, \ldots, p^k - 1$, consists of a complete residue system modulo p^k.

Proof of Theorem: Let $0 \leq a < M$ and let $\nu(M, a) = d$. As the Theorem is vacuous if $d = 0$, assume $d \geq 1$. Let $w_1, w_2, \ldots, w_d$ be all of the integers $0 \leq w_i < Q$ such that $U_{w_i} \equiv a \pmod{M}$. Let $0 \leq b < M \cdot p^k$, say $b \equiv r \pmod{p^k}$ with $0 \leq r < p^k$. Assume $b \equiv a \pmod{M}$. Note that U_n has period length

$$\text{LCM}(Q, fp^k) = \frac{f}{(Q, f)} \cdot Q \cdot p^k \text{ modulo } M \cdot p^k.$$

For ease of notation, we set $z = f/(Q, f)$. As $(M, p^k) = 1$, it suffices, by the Chinese Remainder Theorem, to show that the system

$$(1) \qquad U_n \equiv \begin{cases} a \pmod{M} \\ r \pmod{p^k} \end{cases}$$

has exactly $z \cdot d$ solutions, $0 \leq n < z \cdot Q \cdot p^k$.

We begin by producing, for each w_i, solutions $v_{i1}, v_{i2}, \ldots, v_{iz}$ of the system. Fix i. Then

$$U_{w_i + eQ} \equiv a \pmod{M} \text{ for all } 0 \leq e < z \cdot p^k - 1.$$

Let $0 \leq s_{i1} < s_{i2} < \cdots < s_{iz} < f$ be all of the distinct integers such that

$$w_i \equiv s_{i1} \equiv s_{i2} \equiv \cdots \equiv s_{iz} \pmod{(Q, f)}.$$

By Vélez's lemma, there exist integers $0 \le q_{i1}, q_{i2}, \ldots, q_{iz} \le p^k - 1$ such that

$$U_{s_{ij}+q_{ij}f} \equiv r \pmod{p^k}, \text{ for all } j.$$

Then, also, for any $0 \le t \le Q/(Q, f) - 1$, we have

$$U_{s_{ij}+(q_{ij}+tp^k)f} \equiv r \pmod{p^k}.$$

The bounds on e, t guarantee that these subscripts are less than $z \cdot Q \cdot p^k$. For each i, j, we seek $e = e_{ij}$, $t = t_{ij}$ in these bounds such that

$$w_i + e_{ij}Q = s_{ij} + (q_{ij} + t_{ij}p^k)f.$$

Write $s_{ij} - w_i = (Q, f)m_{ij}$. Note that since $\left(z \cdot p^k, \dfrac{Q}{(Q, f)}\right) = 1$, the linear congruence

$$t \cdot z \cdot p^k \equiv -(m_{ij} + q_{ij}z) \left(\text{mod } \frac{Q}{(Q, f)}\right)$$

has a unique solution $t = t_{ij}$ with $0 \le t_{ij} < \dfrac{Q}{(Q, f)} - 1$. But then

$$Q \mid (Q, f)(m_{ij} + q_{ij}z + t_{ij} \cdot z \cdot p^k);$$

thus, since $(Q, z \cdot Q \cdot p^k) = Q$, the linear congruence

$$eQ \equiv (Q, f)(m_{ij} + q_{ij}z + t_{ij} \cdot z \cdot p^k) \pmod{z \cdot Q \cdot p^k}$$

has Q solutions $0 \le e < z \cdot q \cdot p^k$. Hence, this congruence has a unique solution $e = e_{ij}$ satisfying $0 \le e_{ij} \le z \cdot p^k - 1$. With these values of e_{ij}, t_{ij}, we have

$$w_i + e_{ij}Q \equiv s_{ij} + (q_{ij} + t_{ij}p^k)f \pmod{z \cdot Q \cdot p^k},$$

so equality holds, since both sides are less than $z \cdot Q \cdot p^k$. Set $v_{ij} = w_i + e_{ij}Q$ for all i, j. Then $0 \le v_{ij} < z \cdot Q \cdot p^k$, and each v_{ij} is a subscript that satisfies the system (1), that is, $U_{v_{ij}} \equiv b \pmod{M \cdot p^k}$ for all i, j. We claim that the v_{ij} are distinct.

Suppose that $v_{ij} = v_{gh}$. Then $w_i + e_{ij}Q = w_g + e_{gh}Q$ implies $Q \mid (w_i - w_g)$. As $0 \le w_i, w_g < Q$, this gives $w_i = w_g$, so that $i = g$. Then

$$s_{ij} + (q_{ij} + t_{ij}p^k)f = s_{ih} + (q_{ih} + t_{ih}p^k)f,$$

so that $f \mid (s_{ij} - s_{ih})$. As $0 \le s_{ij}, s_{ih} < f$, we have that $s_{ij} = s_{ih}$; therefore, $j = h$. Thus, the v_{ij} are distinct. This shows that, for any $0 \le a < M$ and any $0 \le b < M \cdot p^k$, $\nu(M \cdot p^k, b) \ge z \cdot \nu(M, a)$. The proof is concluded by observing that

$$z \cdot Q \cdot p^k = \sum_{b=0}^{M \cdot p^k - 1} \nu(M \cdot p^k, b) = \sum_{a=0}^{M-1} \sum_{r=0}^{p^k-1} \nu(M \cdot p^k, b), \text{ where } b \equiv \begin{cases} a \pmod{M} \\ r \pmod{p^k} \end{cases}$$

$$\ge \sum_{a=0}^{M-1} \sum_{r=0}^{p^k-1} z \cdot \nu(M, a) = z \cdot p^k \sum_{a=0}^{M-1} \nu(M, a) = z \cdot p^k \cdot Q.$$

Hence, equality holds throughout, and the Theorem follows. □

Example: Let $A = B = 1$, $U_0 = 0$, $U_1 = 1$ so that U_n is the Fibonacci sequence. Then U_n is u.d.(mod 5). Take $M = 33$. Then U_n has period of length $Q = 40$ modulo 33, and one computes that $\nu(33, 1) = 5$, whereas $\nu(165, 1) = 3$. This justifies the hypothesis that $p \nmid Q$. Moreover, in this case, $\nu(33, a)$ assumes 5 values for $0 \le a < 33$, but $\nu(165, b)$ assumes only 4 values for $0 \le b < 165$.

In fact, our Theorem asserts that U_n has the same number of distinct distribution frequencies modulo M and $M \cdot p^k$, whenever M, p satisfy the hypotheses of the Theorem [that is, $\nu(M, *)$ and $\nu(M \cdot p^k, *)$ take on the same number of distinct values]. This provides an alternate method of obtaining the results in [1].

Note that the "purely periodic" hypothesis of the Theorem can be omitted if one substitutes asymptotic density for frequency, as the finite number of terms before U_n becomes periodic modulo M do not affect density. Our final result is well known but illustrates the Theorem's power.

Corollary: Suppose that U_n is u.d.(mod p^k) and is u.d.(mod M), where p is a prime that does not divide the length of the period of U_n (mod M). Then U_n is u.d.(mod $M \cdot p^k$).

References

1. E. Jacobson. "Almost Uniform Distribution of the Fibonacci Sequence." *Fibonacci Quarterly 27.4* (1989):335-37.
2. W. Y. Vélez. "Uniform Distribution of Two-Term Recurrence Sequences." *Trans. of the A.M.S. 301* (1987):37-45.

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GENERALIZATIONS OF THE DUAL ZECKENDORF INTEGER REPRESENTATION THEOREMS—DISCOVERY BY FIBONACCI TREES AND WORD PATTERNS

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"What is the use," thought Alice, *"of a book without pictures and conversations?..."*

Alice's Adventures in Wonderland,
—Lewis Carroll

1. Introduction

In this paper we show how the two well-known integer representation theorems which are associated with the name of Zeckendorf may be generalized as dual systems by constructing colored tree sequences whose shade sets partition $Z^+ = \{1, 2, \ldots\}$. Many interesting properties of the representations can be observed directly from the tree diagrams, and the proofs of the properties can truly be said to be "evident" or "obvious"; we shall not translate such proofs into other symbolic forms.

The Zeckendorf theorems are about representations of positive integers as sums of distinct elements of given number sequences. The first theorem is in Lekkerkerker [6], and a dual of it is given by Brown [2]. Early papers on properties of the Zeckendorf integer representations are Zeckendorf [12] and Brown [1]. Klarner [5] gives an excellent review of the literature to 1966, and extends many of the theories to that date. In [3] Carlitz et al. (1972) define Fibonacci representations of integers, and study their properties.

In Turner [7] we showed how to construct certain tree sequences and defined their *shade sets*, which together demonstrated the Zeckendorf representation theorems. In Zulauf & Turner [13], we showed how the shade sets could be defined in a set-theoretic notation, and proved the Zeckendorf theorems in a concise manner. In Turner [8] and Turner & Shannon [9] we defined Fibonacci word patterns and used them to study tree shade sets.

Notation and definitions for integer representations

(i) Let $\mathbf{c} = \{c_1, c_2, c_3, \ldots\}$ be any sequence of distinct real numbers, and let $N \in Z^+$ (i.e., N is a positive integer). We shall be concerned with representations of N of the form

$$N = \sum_{i=1}^{n} e_i c_i, \tag{1.1}$$

where $n \geq 1$ and $e_i \in \{0, 1\}$ for each i.

In this paper $\mathbf{c}$ will be a strictly increasing sequence of nonnegative integers. Once $\mathbf{c}$ is given, the vector $\mathbf{e} = \{e_1, \ldots, e_n\}$ determines the representation.

(ii) (as in [4]): The sequence $\mathbf{c}$ is *complete* with respect to the positive integers if and only if every integer $N \in Z^+$ has a representation of the form (1.1).

(iii) (see [4]): If the number of elements of $\mathbf{c}$ used in a representation is as small as possible, the representation is said to be *minimal*; if it is the largest possible, then the representation is *maximal*.

The dual Zeckendorf theorems

We shall use the notation Z and Z^* when referring to these theorems and related properties.

Theorem 1.1 (Z; [6], [12]): Every $N \in Z^+$ has one and only one representation in the form (1.1) with $c = \{u_n\} = \{F_{n+1}\} = \{1, 2, 3, 5, \ldots\}$, the Fibonacci sequence, and with the coefficients e_i satisfying $e_n = 1$, and $e_i e_{i+1} = 0$ if $1 \le i < n$.

Moreover, these representations are minimal; and for a given value of n there are F_n integers having Zeckendorf representations.

Theorem 1.2 (the dual, Z^*; [2]): Every $N \in Z^+$ has one and only one representation in the form (1.1) with $c = \{u_n\}$, $e_i \in \{0, 1\}$, $e_n = 1$, and $e_i + e_{i+1} \ne 0$ if $1 \le i < n$.

Moreover, these representations are maximal; and for a given value of n there are F_{n+1} integers having Zeckendorf representations.

The generalization in this paper:

In Section 2 we show how to construct classes of colored tree sequences whose shade sets exactly cover Z^+, and hence derive classes of complete sequences of integers (those used to color the trees). From these classes we select two which are dual in a sense that generalizes the dual conditions $e_i e_{i+1} = 0$ and $e_i + e_{i+1} \ne 0$ used in the Zeckendorf theorems 1.1 and 1.2, respectively. Thus, we obtain a class of dual integer representation theorems, of which the pair (Z, Z^*) is the simplest case.

2. Colored Trees and Their Shades

Definitions

(i) A *tree* is a set of n nodes (or points), and a set of $n - 1$ *edges* (lines joining pairs of the nodes), having no *cycles* (paths from a node which return to that node).

(ii) If one node in a tree is distinguished, and labelled as a *root*, we have a *rooted tree*.

(iii) If real numbers (in this paper integers) are assigned to the nodes of a rooted tree, we have a *number tree*. We call the numbers *colors* of the nodes.

(iv) A node, other than the root, which has only one edge attached to it is called a *leaf*. There is a unique path from the root to any given leaf. The sum of the colors on a root-to-leaf path is called the *shade* of the path.

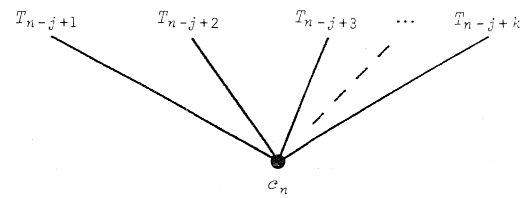
(v) The set of shades of all root-to-leaf paths in a rooted tree is the *shade set* (or *shade*) of the tree.

Generation of a 3-parameter class of colored tree sequences

Suppose we are given a coloring sequence of integers, denoted by $c \equiv c_0, c_1, c_2, \ldots, c_n, \ldots$; and also an initial sequence of i rooted trees denoted by $T_0, T_1, \ldots, T_{i-1}$, each of whose nodes is colored by a member of c.

Then we can continue the tree sequence in the following way.

For the n^{th} tree, take a k-fork (with $k \le n$) and color its root node c_n. Select an ordered subsequence of the $T_0, T_1, \ldots, T_{n-1}$, of length k and using consecutive members, and mount them one by one from left to right on the k prongs on the fork. The following diagram makes this construction clear:

(2.1) $T_n \equiv$ 

with $k < j \le n + 1$.

Any selection of values for the triple (i, j, k) will determine a sequence of colored trees, so the construction just defined determines a 3-parameter family of such trees. We shall also allow j, k to be functions of n.

Tree sequences with shade sets exactly equal to Z_0^+:

We investigate now the choices of **c**, the triple (i, j, k), and the initial trees such that they will lead to a sequence of trees having shade set $Z_0^+ = \{0, 1, 2, \ldots\}$. We shall require this to happen *exactly*; which is to say that if Z_m denotes the shade set of the m^{th} tree T_m of a sequence we shall require

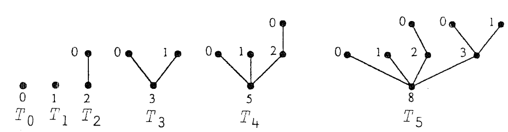
$$\bigcup_{m=0}^{\infty} Z_m = Z_0^+ \quad \text{and} \quad Z_u \cap Z_v = \emptyset \text{ when } u \ne v.$$

Examples

Before giving a general result, we shall give three examples to illustrate the various concepts introduced above. The first two provide graphical proofs of the dual Zeckendorf theorems; we treated these in [7] and [13]. The third gives an indication of the generalization we are aiming at, and we give the first seven trees of its sequence.

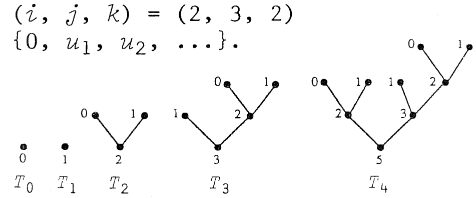
Example 1 (Zeckendorf, Z)

Parameter values: $(i, j, k) = (2, n + 1, n - 1)$ (for T_n)
Color sequence: $\{0, u_1, u_2, \ldots\}$ where $u_n = F_{n+1}$.



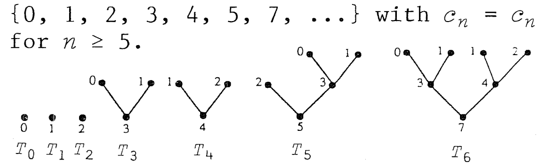
Example 2 (Zeckendorf dual, Z^*)

Parameter values: $(i, j, k) = (2, 3, 2)$
Color sequence: $\{0, u_1, u_2, \ldots\}$.



Example 3 (gap range 1, 2)

Parameter values: $(i, j, k) = (3, 4, 1)$
Color sequence: $\{0, 1, 2, 3, 4, 5, 7, \ldots\}$ with $c_n = c_{n-2} + c_{n-3}$, for $n \ge 5$.



For this last example, it may be observed that the sequence of tree shade sets $\{Z_n\}$, $n = 0, 1, 2, \ldots$ is

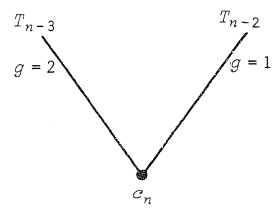
$$\{\{0\}, \{1\}, \{2\}, \{3, 4\}, \{5, 6\}, \{7, 8, 9\}, \{10, 11, 12, 13\}, \ldots\}.$$

It is also seen that the number of leaves in T_n is c_{n-2} after $n = 5$. It should be evident that with this color sequence and method of tree construction the shade set sequence will continue through the positive integers. Thus, the shade set covers Z_0^+ exactly. So, given any positive integer N, it will correspond to the shade of just one root-to-leaf path in one tree of this sequence of colored rooted trees. It is easy to derive a formula to tell which. We shall not do this here, but rather remark upon the fact that the numbers (colors) on that root-to-leaf path constitute a representation for N as a sum of distinct members of **c**. Thus, **c** is complete for Z^+. Moreover, because of the parameter values (i, j, k) and the construction process, we can state, from the following illustration, that the representation for N has gaps in its constituent colors of either 1 or 2 (i.e., at least a gap of 1 and at most a gap of 2).

Take $N = 12$, for example. This occurs in the shade of T_7. The third root-to-leaf path from the left gives the representation

$$12 = 1 + 4 + 7 = c_1 + c_4 + c_6.$$

The binary representation (i.e., the vector of e-values) is (0, 1, 0, 0, 1, 0, 1). The "gaps" referred to above can now be seen as runs of 0's occurring between the 1's. All representations from this tree sequence will have a gap of 1 zero or 2 zeros between every pair of adjacent 1's. A graphical "proof" of this is to write the construction rule thus, where the color gap sizes occurring are indicated on the fork edges:

$T_n =$ 

, for $n \geq 3$

It is evident that in every tree beyond the third a 2 or a 1 must occur on every edge, and hence only gaps of 2 and 1 occur in all root-to-leaf path sums.

The reader may care to check that similar reasoning applied to the trees of Examples 1 and 2 will verify the dual Zeckendorf theorems, with their gap properties that $e_i e_{i+1} = 0$ and $e_i + e_{i+1} \neq 0$, respectively.

Before going on to define the dual classes which generalize the Zeckendorf theorems, we give an indication of how studies of Fibonacci word patterns [8] occurring on the tree sequences can provide theorems about properties of integer representations. Referring to Example 2, for instance, suppose we wish to investigate the occurrence of integer representations with the Zeckendorf dual properties and which contain $u_1 = 1$ (i.e., also $e_1 = 1$). Examining the trees, we see that $u_1 = 1$ occurs only on leaf nodes. It is easy to derive formulas for the number of u_1's occurring in tree T_n (it is obviously F_n), and for the pattern of the occurrences. Details of the pattern are given in [8]; briefly, the pattern starting with T_1 is given by the Fibonacci word juxtaposition recurrence formula

$$W_{n+2} = W_n W_{n+1}, \text{ with } W_1 = 1 \text{ and } W_2 = 01.$$

This gives the pattern (which is the leaf-node color pattern) 1, 01, 101, 01101, ... The positions of the 1's are the places 1, 3, 4, 6, 8, 9, ... and of the 0's are 2, 5, 7, 10, ... These two sequences are the well-known Wythoff number sequences, given by $\{[\alpha n]\}$ and $\{[\alpha^2 n]\}$, respectively, where $\alpha = \frac{1}{2}(1 + \sqrt{5})$ and $[x]$ is the greatest integer function ([8] and [11]). Similar analyses lead to similar conclusions for the placings of the other colors in the tree sequence.

To study relative positions and frequencies of occurrences of integer representations from Example 3, it is necessary to solve the third-order recurrence given for c_n; and to study the corresponding word pattern recurrence $W_{n+3} = W_n W_{n+1}$ with initial words $W_1 = 0$, $W_2 = 1$, $W_3 = 2$.

3. Generalized Dual Zeckendorf Theorems

We choose parameter values for (i, j, k) in (2.1) so that two dual tree-sequence classes are defined. With suitable choices of initial trees, and of coloring sequences, we shall ensure that the first one (designated the GZ-class) generates integer representations such that all gaps in the e-vectors have *at least* g^* *zeroes*; and that the second one (designated the GZ^*-class) generates integer representations with all gaps having *at most* g^* *zeroes*. To be precise, we define a *gap* g to be a run of g zeroes occurring between two successive 1's in an e-vector. The conditions "at least g^*" and "at most g^*" on the gap sizes in the e-vector representations are the dual conditions. We note immediately that the GZ-class will contain the sequence of Example 1, since the conditions $e_j e_{j+1} = 0$ and $g \geq 1$ are equivalent. Likewise, the GZ^*-class contains the sequence of Example 2, since the conditions $e_j + e_{j+1} \neq 0$ and $g \leq 1$ are equivalent.

The following tables give definitions, and the first few color sequences and corresponding tree sequences as examples.

TABLE 1. Definitions

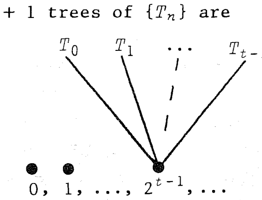
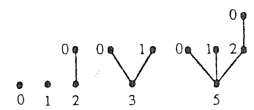
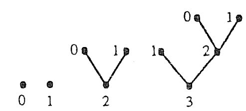
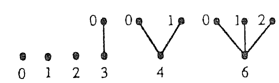
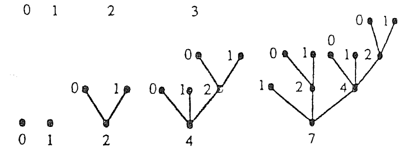
GZ-class	GZ^*-class
Gap sizes: $g \geq g^*$	$g \leq g^*$
Parameter: $(i, j, k) = (g^* + 1, n + 1, n - g^*)$ (for T_n)	$(i, j, k) = (g^* + 1, g^* + 2, g^* + 1)$
Color sequence: $c_{n+i} = c_n + c_{n+i-1}$	$c_{n+i} = c_n + c_{n+1} + \cdots + c_{n+i-1}$
Initial colors: 0, 1, 2, ..., i	0, 2^0, 2^1, 2^2, ..., 2^{i-1}
Initial trees: $\{\underset{r}{\bullet} \mid r = 0, \ldots, i - 1\}$	The first $i + 1$ trees of $\{T_n\}$ are given by:  with $2 \leq t \leq i$.
General solution for **c** is given in §4	General solution for **c** is given in §4

TABLE 2. Example Sequences

g^*	GZ-Class	GZ^*-Class
1	0, 1, 2; 3, 5, 8, 13, ...	0, 1, 2; 3, 5, 8, 13, ...
2	0, 1, 2, 3; 4, 6, 9, 13, ...	0, 1, 2, 4; 7, 13, 24, 44, ...
3	0, 1, 2, 3, 4; 5, 7, 10, 14, ...	0, 1, 2, 4, 8; 15, 29, 56, ...
	Tree Sequences ($g \geq g^*$)	Tree Sequences ($g \leq g^*$)
1		
2		

It should be noted that, since the shades of all the tree sequences in both the GZ- and GZ^*-classes exactly cover Z_0^+, all the color sequences used (with $c_0 = 0$ deleted) are complete for Z^+.

Within each pair of tree sequences, for each value of g^*, the root-to-leaf paths give integer representations using distinct colors and with gaps satisfying $g \geq g^*$ and $g \leq g^*$, respectively. Those on the tree sequences with $g^* = 1$ are the dual Zeckendorf representations.

As we said in the Introduction, there is hardly a need for formal proofs of the above statements about the integer representation properties. They all follow by induction, using the definitions of the procedures for constructing the colored trees. Study of the general tree diagram tells all! As Alice thought, in Wonderland: "What is the use of a book without pictures... ." However, to demonstrate the reason for the choices of (i, j, k) in the two classes, we shall give some details of the proofs. The key property to establish is that the shade sets of the trees in any sequence partition Z_0^+.

Theorem: Each tree-sequence in the two classes defined in Table 1 has shade set exactly equal to $Z_0^+ = \{0, 1, 2, \ldots\}$.

Proof: We shall use induction, for sequences in each class.

<u>Case (i)</u> Let $T = \{T_n\}$ be a tree-sequence in the GZ-class.

The first i trees in T have shades $0, 1, \ldots, (i - 1)$, respectively, by the definitions of initial colors and initial trees given in Table 1. The $(i + 1)^{\text{th}}$ tree is

$$T_i = \overset{T_0}{\bullet}\, i,$$

since $k = n - g = i - (i - 1) = 1$ (meaning there is a 1-fork), and $n - j + 1 = n - (n + 1) + 1 = 0$ (meaning that T_0 is mounted on it). Here we have used the formulas given in Table 1 for the parameters (i, j, k) in the GZ-class.

Thus, T_i has shade $0 + i = i$, which continues the shade sequence required by the theorem.

We now make the inductive hypothesis that the shade sets continue as for the theorem, up to the last (rightmost) branch of tree T_n, with $n > i$.

Referring to the construction diagram (2.1), inserting parameters $j = n + 1$ and $k = n - g^*$, we find that T_0 is mounted on the first (leftmost) branch of the k-fork used to construct T_n. This is also true for T_{n+1}, etc. Hence, the leftmost branch shade of T_{n+1} is $c_{n+1} + 0 = c_{n+1}$.

Now, the rightmost branch shade of tree T_n is c_n + (rightmost branch shade of T_{n-g^*-1}), which, by the inductive hypothesis, equals $c_n + (c_{n-g^*} - 1)$. Then, since $c_n + c_{n-g^*} = c_n + c_{n-i+1} = c_{n+1}$ (using parameter and color sequence definitions), we have shown that

(leftmost branch shade of T_{n+1}) = (rightmost branch shade of T_n) + 1.

Hence, the shade of T_{n+1} follows on in natural sequence from that of T_n. This completes the inductive proof.

Case (ii) Let $T = \{T_n\}$ belong to the GZ^*-class.

We proceed as for Case (i); we shall omit the details showing that the shades of $T_0, T_1, \ldots, T_{i+1}$ conform to the theorem.

Assume that the shade of the tree sequence $T_0, T_1, \ldots, T_n$, with $n > i + 1$, is a sequence $0, 1, 2, \ldots, r$. We shall show that the first element of the shade of T_{n+1} is $r + 1$.

Let us use the notation L_n, R_n to mean, respectively, the "leftmost branch shade of tree T_n" and the "rightmost branch shade of tree T_n." We have to show that $R_n + 1 = L_{n+1}$. From the construction diagram (2.1), and inserting the parameters for j, k from Table 1 for the GZ^*-class, we see that

$$R_n = c_n + R_n \quad (= c_n + L_n - 1);$$

and

$$L_{n+1} = c_{n+1} + L_{n-g^*} \quad (= c_n + c_{n-1} + \cdots + c_{n-g^*} + L_{n-g^*}).$$

Now

$$L_n = c_n + L_{n-g^*-1} = c_n + (L_{n-g^*} - c_{n-g^*-1}),$$

using the fact that, for $n \geq 1$, the cardinal number of the shade set of T_n is equal to c_n: this is easily established by induction, for the trees in GZ^*. So we have

$$\begin{aligned} R_n + 1 = c_n + L_n &= 2c_n - c_{n-g^*-1} + L_{n-g^*} \\ &= c_n + c_{n-1} + \cdots + c_{n-g^*} + L_{n-g^*} \\ &= c_{n+1} + L_{n-g^*} \\ &= L_{n+1}. \end{aligned}$$

The existence of the generalized dual Zeckendorf integer representations now follows immediately. The proof that the gap sizes satisfy conditions $g \geq g^*$ or $g \leq g^*$ for tree sequences in the GZ-class or GZ^*-class, respectively, rests on simple observations of the gaps that can occur [see diagram (2.1)] between c_n and the root colors of the k trees $T_{n-j+1}, \ldots, T_{n-j+k}$ used in the construction.

The final table gives the dual representations of $N = 1, 2, \ldots, 10$ for the cases $g^* = 1$ and $g^* = 2$. Note that they are, respectively, minimal and maximal representations. (See Table 3 below.)

4. Formulas for the Color Sequences in the Two Classes

In Table 1 we gave the initial values and general recurrence equations for the color sequences in the GZ- and GZ^*-classes. We end the paper by giving general solutions for the equations, which provide formulas for the terms of the sequences in terms of weighted sums of binomial coefficients. We also give geometrical interpretations for these weighted sums: they are related to the elements on certain diagonals of Pascal's triangle. Thus, in a very nice pictorial way, we have linked the generalizations of the dual Zeckendorf integer representations to generalizations of the Pascal-Lucas theorem which states that sums of the terms on the 45° upward diagonals of Pascal's triangle are Fibonacci numbers.

TABLE 3. Dual Integer Representations

GZ-Class			*GZ**-Class		
N	$g^* = 1$	$g^* = 2$	N	$g^* = 1$	$g^* = 2$
1	1 (1)	1 (1)	1	1 (1)	1 (1)
2	2 (01)	2 (01)	2	2 (01)	2 (01)
3	3 (001)	3 (001)	3	1 + 2 (11)	1 + 2 (11)
4	1 + 3 (101)	4 (0001)	4	1 + 3 (101)	4 (001)
5	5 (0001)	1 + 4 (1001)	5	2 + 3 (011)	1 + 4 (101)
6	1 + 5 (1001)	6 (00001)	6	1 + 2 + 3 (111)	2 + 4 (011)
7	2 + 5 (0101)	1 + 6 (10001)	7	2 + 5 (0101)	1 + 2 + 4 (111)
8	8 (00001)	2 + 6 (01001)	8	1 + 2 + 5 (1101)	1 + 7 (1001)
9	1 + 8 (10001)	9 (000001)	9	1 + 3 + 5 (1011)	2 + 7 (0101)
10	2 + 8 (01001)	1 + 9 (100001)	10	2 + 3 + 5 (0111)	1 + 2 + 7 (1101)

The recurrence for the *GZ*-class

We wish to index and refer to the sequences in the *GZ*-class, as g^* ranges over 1, 2, 3, etc. To this end we add a superscript in brackets, to the expression for the n^{th} term in the $g^{*\text{th}}$ sequence. Thus, $c_n^{(g^*)}$ denotes this expression. Since it is typographically clumsy to use g^* as the indexing letter, we shall replace g^* by i (note that the values for i as used here and subsequently are 1 less than the ones used for the parameter in Table 1).

The recurrence equation for $c_n^{(i)}$ (omitting $c_0^{(i)} = 0$ from each sequence) is

$$(4.1) \qquad c_{n+i+1}^{(i)} = c_{n+i}^{(i)} + c_n^{(i)}, \text{ for } n \geq 1,$$

with initial values

$$(4.2) \qquad c_s^{(i)} = s \text{ for } s = 1, 2, \ldots, i + 1.$$

[Note that for $i = 1$ this gives the Fibonacci sequence = $\{F_{n+1}\}$.]

The *general solution* to (4.1) and (4.2), given i, is

$$(4.3) \qquad c_{n-i+1}^{(i)} = \sum_{r=0}^{[n/i]} \binom{n - ri}{r}, \text{ for } n - i + 1 = 1, 2, 3, \ldots,$$

which can be demonstrated by direct substitution, and making use of the identity

$$\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$$

for the binomial coefficients.

$\left[\text{We use the normal convention that } \binom{a}{b} \equiv 0 \text{ when } a < b.\right]$

If i is small, say $i = 1, 2, 3$, or 4, then we can use Binet-type formulas to calculate the $c_n^{(i)}$ efficiently for any n. If i is large, then formula (4.3) above is probably the most efficient way to calculate $c_n^{(i)}$ exactly.

For example, if $i = 1000$, then $c_{2200}^{(1000)}$ is

$$\binom{3199}{0} + \binom{2199}{1} + \binom{1199}{2} + \binom{199}{3},$$

which equals

$$1 + 2199 + 1199 \times 1198/2 + 199 \times 198 \times 197/6 = 2014100.$$

Thus, to calculate $c_{2200}^{(1000)} = 2014100$ from the above formula requires only a few additions and multiplications; whereas to calculate it directly from the recurrence relation (4.1) could require that all the $c_n^{(1000)}$ be precomputed for $n = 1, 2, \ldots, 2199$. Clearly, formula (4.3) is much quicker.

The solution (4.3) also has a nice geometric interpretation, which we show in the final subsection.

The recurrence for the GZ^*-class

The recurrence equation for $c_n^{*(i)}$ (omitting $c_0^{*(i)} = 0$ from each sequence) is

$$(4.4) \qquad c_{n+i+1}^{*(i)} = c_n^{*(i)} + c_{n+1}^{*(i)} + \cdots + c_{n+1}^{*(i)}, \text{ for } n \geq 1,$$

with initial values

$$(4.5) \qquad c_s^{*(i)} = 2^{s-1} \text{ for } s = 1, 2, \ldots, i+1.$$

[Note that $i = 1$ again gives the Fibonacci sequence $\{F_{n+1}\}$.]

By considering $c_{n+i+1}^{*(i)} - c_{n+i}^{*(i)}$, and using (4.4), we find that

$$(4.6) \qquad c_{n+i+1}^{*(i)} = 2c_{n+i}^{*(i)} - c_{i-1}^{*(i)}, \; n = 1, 2, \ldots .$$

We used this equivalent form of the recurrence equation as a first step in obtaining a general solution. The details of our solution method are lengthy, and will be reported elsewhere. Our solution is given next, as (4.7): it may be checked by insertion into (4.4) or (4.6) and use of elementary algebra and manipulations with binomial coefficients.

The *general solution* to (4.4) and (4.5), given i, is

$$c_n^{*(i)} = u_n^{(i+1)} - u_{n-1}^{(i+1)} \text{ for } n = 1, 2, \ldots,$$

where

$$(4.7) \qquad u_n^{(i)} = \sum_{r=0}^{[n/i]} \frac{2^{n-ri}}{(-2)^r}\binom{n-ri}{r} \text{ for } n = 0, 1, 2, \ldots .$$

We show in the next subsection how the solutions for the dual pairs of recurrences in the GZ- and GZ^*-classes are neatly related to elements on the diagonals of increasing slope within Pascal's triangle.

Geometric interpretation

Consider Pascal's triangle, for the binomial coefficients, drawn as a 45° triangle rather than the usual equilateral triangle, thus:

$c_7^{(3)}$

$13 = c_6^{(1)}$, $21 = c_7^{(1)}$ } Lucas diagonal sums

n										
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	2	7			
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1

Then $c_{n+1-i}^{(i)}$ is just the sum of all the binomial coefficients on the line of *slope* i that starts on the left end of the n^{th} row. In particular, the n^{th} Fibonacci number is the sum of the numbers on a 45° line (slope $i = 1$) starting at the n^{th} row (these lines are the well-known Lucas diagonals). As an example, for the case $i = 3$,

$$c_7^{(3)} = c_{9+1-3}^{(3)} = \binom{9}{0} + \binom{6}{1} + \binom{3}{2} = 1 + 6 + 3 = 10.$$

The geometric interpretation also suggests the results $c_n^{(0)} = 2$ and $c_n^{(\infty)} = n$ corresponding to lines of slope 0 (horizontal) and ∞ (vertical), respectively.

As an example in the GZ^*-class, again taking $i = 3$, and with $n = 9$, we get

$$c_9^{*(3)} = u_9^{(4)} - u_8^{(4)} = \left[2^9\binom{9}{0} - \frac{2^5}{2}\binom{5}{1} \quad - \quad 2^8\binom{8}{0} - \frac{2^4}{2}\binom{4}{1}\right]$$

$$= 432 - 224 = 208.$$

Inspection of Pascal's triangle shows that $u_n^{(i)}$ is a weighted sum of the elements on the upward diagonal of slope i which begins at the first element of the n^{th} row: the weights are powers of 2 as given in (4.7).

Hence, $c_n^{*(i)}$ is the difference of weighted sums from the adjacent diagonals beginning on the n^{th} and $(n - 1)^{\text{th}}$ rows.

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THE FIBONACCI SEQUENCE AND THE TIME COMPLEXITY OF GENERATING THE CONWAY POLYNOMIAL AND RELATED TOPOLOGICAL INVARIANTS

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1. Introduction

In this discussion, only tame embeddings of S^1 in S^3 that are oriented will be considered. All knots will be represented as regular projections and any projection is assumed to be regular. The reader is expected to know "big O" notation; see [2] for example. Some knowledge of NP-Completeness is also useful.

The algorithm analysis of the complexities is relative to the number of crossings of a given knot projection. Also, the analytical creation of the Conway polynomial is in the Class P. This is shown by the presentation of a well-known algorithm used for computing the Alexander polynomial which can be easily suited to generate the Conway polynomial in better than $O(n^3)$ time.

The proof of the Conway algorithm having exponential worst case time complexity is based on showing the existence of n crossing knot projections. Given these particular projections, the Conway algorithm may perform $O(((1 + \sqrt{5})/2)^n)$ operations on the various knot projections which the algorithm derives in order to calculate the Conway polynomial.

Definition 1.1: The *crossing number* of a knot K is the minimum number of crossings for any regular projection of the knot K.

Definition 1.2: A *split link* (see [10], [11]) $L \subseteq S^3$ is a link $L = L_1 \cup L_2$ where L_1 and L_2 are nonempty sublinks and there exist two open balls, B_1 and B_2 in S^3 such that $B_1 \cap B_2 = \emptyset$; and $L_1 \subseteq B_1$ and $L_2 \subseteq B_2$.

Definition 1.3: A *tangle* (see [4]) is a portion of a knot diagram from which there emerge only four arcs from the four "directions" NE, NW, SE, and SW, and possibly some crossings inside.

Examples:

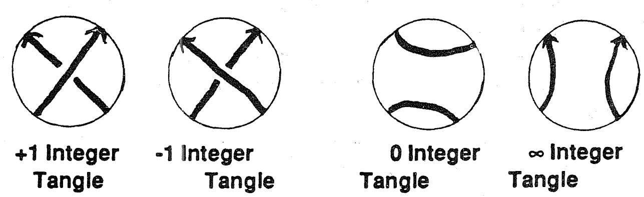

FIGURE 1

A tangle cannot have any knot arcs passing under or over it. The integer tangles pictured above, denoted by the integers +1 and -1 describe all crossings of any knot or link. The following three operations are the elementary knot crossing operations.

Definitions 1.4:

1. *Smoothing*—This operation takes an oriented +1 or -1 integer tangle and replaces it with an oriented ∞ integer tangle.

2. *Changing*—This operation takes an oriented +1 or −1 integer tangle and replaces it with an oriented −1 or +1 integer tangle, respectively.
3. *Deleting*—This operation takes a +1 or −1 integer tangle and replaces it with a 0 tangle.

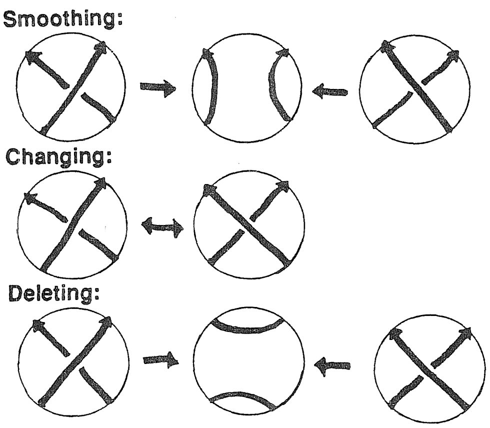

FIGURE 2

Fundamentally, the Conway algorithm is based on the first two knot operations. The operations of changing and smoothing crossings will be focused on, although smoothing and deleting [14] and changing and deleting will also be discussed.

Conceptually, there can be a *tree of projections* built during the application of elementary crossing operations. This tree of projections will become the basis of the complexity analysis.

A knot's tree of projections relative to changing and smoothing is built as follows:

If the knot K is the unknot, then the tree of projections is the trivial tree with an unknot projection as the root with no children.

Given a knot K that is not the unknot, the tree of projections is the binary tree with a projection of K as the root. Choose a crossing, call it X, change the knot projection K at the crossing X to produce the knot projection L. Take the knot projection K, and the crossing X. Smooth it to produce the link projection R. The right child of the root projection K is the smoothed knot projection R and the left child of the projection is the changed projection L. This process of smoothing and changing of knot projections and nonsplit link projections is recursively continued always placing changed knot or link projections as left subtrees and smoothed knot or link projections become right subtrees. When a subtree becomes the unknot or a split link it is a leaf having no more children. It may be observed that by changing a crossing one can reach either the unknot or a split link. A link's tree of projections is constructed similarly.

Examples:

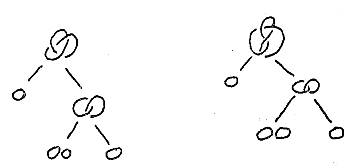

FIGURE 3

Notice that the trefoil and the figure eight knots have identically structured trees of projections. Any combinations of the three elementary operations consistently applied forms a tree of projections in a similar fashion. A tree of projections is assumed to be relative to changing and smoothing unless otherwise noted.

To quote Louis Kauffmann [10], a convenient way to get an unknot from a knot is to perform the following operations on the knot diagram [10, p. 79]:

> Choose a point p on the diagram and draw knot so that you first draw an over-crossing line at the first encounter with a crossing, under cross at the second encounter, and continue until you return to p.

Performing this to a knot projection and then applying Reidemeister moves can produce the unknot in familiar form.

Definition 1.5: An unknot projection developed in this fashion is designated a *descending* knot projection [6], [7]. A descending knot projection's mirror image is, of course, an *ascending* knot projection.

Just as there is a rather straightforward algorithm for creating a descending knot projection there is also a straightforward algorithm for detecting an ascending knot projection.

Definitions 1.6:

1. The *unknotting number* of a knot is the minimum number of changes that must be performed to produce the unknot starting from any knot projection. The unknotting number of a knot K is denoted by $u(K)$.
2. The *unsmoothing number* of a knot is the minimum number of smooths that must be performed to produce the unknot or split link from any knot projection. The unsmoothing number of a knot K is denoted by $us(K)$.
3. The *deleting number* of a knot is the minimum number of deletes that must be performed to the crossings to produce a split link or an unknot from any knot projection. The deleting number of a knot K is denoted $del(K)$.

In 1970, J. H. Conway defined a polynomial, $\nabla_K(z)$, with integer coefficients for oriented knots and links. The polynomial can be recursively calculated from a regular projection of a knot or link K by consistently applying the knot crossing operations of changing and smoothing.

Theorem 1.7: Every knot K has a well-defined Conway polynomial.

This is well known and a proof can be found in [11].

Surprisingly, the Conway polynomial is well defined independent of the particular sequence of smoothings and changings used to calculate it. One of the most important facts about Conway's algorithm is that it can be applied to any regular projection of a knot K and it will produce the same polynomial.

2. The Algorithm

Algorithm 2.1: (Conway's algorithm, see [4], [10], and [11])

Given: a projection of a knot K
Returning: the Conway polynomial of the knot K

1. Choose an orientation of the knot projection K.
2. If (K is the unknot), then:

   Conway polynomial of the unknot is 1: $\nabla_0 = 1$
3. If (K is a split link), then:

   Conway polynomial of the split link is 0: $\nabla_K = 0$

4. Otherwise, when K is not the unknot and not a split link, the tree of projections is built according to the following formula:

$$\nabla_K - \nabla_L = z\nabla_M$$

for K, L, and M knot and/or link projections identical in all respects except that they differ at one crossing in the following manner:

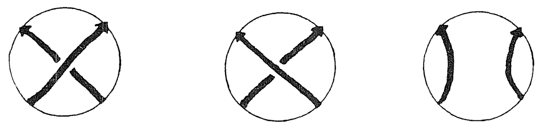

FIGURE 4

One must be sure to preserve the orientation and keep track of the multiple of the indeterminant on the right smoothing edge and the constant multiples on the left changing edges.
5. Recursively repeat steps 2 through 5 with the appropriate smoothed and changed knot and/or link projections until the entire tree of projections is built.
6. Return polynomial ∇_K.
7. End. □

Applying Algorithm 2.1 to a knot projection methodically generates its tree of projections by performing changes and smoothings to the knot projection until there are only projections of unknots or split links left. These changes and smoothings must be performed by adhering to the following formula,

$$\nabla_K - \nabla_L = z\nabla_M$$

where K, L, and M are identical knot projections in every respect except for the following significant differences at only one crossing:

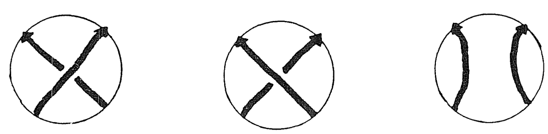

FIGURE 5

Using this relation recursively starting with any knot projection K, the projection eventually becomes resolved into either unknots or split links and, therefore, terminates. Notice that during step 4 of Algorithm 2.1 two new knots are created by changing and smoothing. These new knots may have many possible different projections, but any regular projection will suffice. The Conway polynomial of the unknot is defined to be one ($\nabla_0 = 1$) and any split link is defined to be zero ($\nabla_{\text{split link}} = 0$). The Conway polynomial of both the Trefoil and Figure Eight knots is $z + 1$.

The Conway algorithm terminates when all subtrees have become either unknots or split links.

The unknotting number is a lower bound of the number of crossing changes we must perform in order to construct the unknot from a given knot projection.

Similarly, the unsmoothing number is a lower bound. The next logical question might be: "Given any particular projection of a knot K, how fast can Conway's algorithm resolve the projection of the knot K into unknots and split links?"

Lemma 2.2: At most, $\lfloor (n/2) \rfloor$ is the unknotting number $\mathrm{u}(K)$ of any n crossing knot or link K.

Proof: First, taking the case of an alternating and oriented knot or link projection K of n crossings, orient the knot or link and then prepare to traverse it. If the traversal begins at an overcrossing, then set out to build a descending knot or link projection; otherwise, construct an ascending knot or link projection.

Without loss of generality, assume that the descending variety will be constructed. Starting traversal at an overcrossing of the knot or link projection every time a crossing is encountered, if it is the first encounter with the crossing, ensure that it is an overcrossing. If it is the first encounter with a crossing, being passed under, change it to an overcrossing. Otherwise, if it is the second encounter with this crossing, then proceed under the crossing. This will construct the descending knot or link in at most

$$\lceil (n/2) \rceil - 1 \le \lfloor (n/2) \rfloor$$

changes because it was assumed that the traversal started on an overcrossing.

Now we must contend with the nonalternating knot or link projection. Given a knot or link K with n crossings, if K has more overcrossings than undercrossings, then construct the descending knot projection in less than or equal to $\lfloor (n/2) \rfloor$ crossing changes; otherwise, construct the ascending knot projection similarly. □

Lemma 2.3: At most, $\lfloor (n/2) \rfloor$ is the deleting number $\mathrm{del}(K)$ of any n crossing knot or link K.

The proof is similar to the proof of Lemma 2.2, and is therefore omitted.

Lemma 2.4: Given an n crossing knot or link L, at most $n - 1$ is the unsmoothing number $\mathrm{us}(K)$.

Proof: Given a projection of the knot K with n crossings, if all n crossings are smoothed, then there will be only unknots and unlinks remaining. After $n - 1$ smooths, the knot K will have, at most, only one crossing left, however many associated unlinks are left. The knot with the one remaining crossing must be the unknot. □

Theorem 2.5: At most, given an n crossing knot, the tree of projections relative to changing and smoothing, and smoothing and deleting can have

$$O\{((1 + \sqrt{5})/2)^n\} \text{ projection nodes.}$$

[Note that $(1 + \sqrt{5})/2 = \phi$, the "golden" ratio.)

Proof: The tree of projections relative to changing and smoothing follows directly from Lemma 2.2 and Lemma 2.4. Given a knot projection K, build a tree of projections that can be described by the recurrence relation

$$f(n) = f(n - 1) + f(n - 2) + 1$$

where $f(p)$ is the number of nodes in any tree of projections of a given knot projection with n crossings and, of course, $f(0) = 1$, $f(1) = 1$, and $f(2) = 3$.

The tree of projections relative to smoothing and deleting follows identically from Lemmas 2.2 and 2.3. □

Theorem 2.5 supplies an upper bound on Conway's algorithm (Algorithm 2.1). This is due to the relation

(1) $\nabla_K = \nabla_L + z\nabla_M$

upon which Conway based his algorithm. Notice the similarity between (1) and the recurrence relation that defines the Fibonacci sequence.

An implementation of Conway's algorithm may not be bounded strictly by the upper bound established by Theorem 2.5, but any implementation of Conway's algorithm can be made to adhere to this upper bound.

A tree of projections relative to changing and deleting has an upper bound of $O(2^{(n/2)})$ given an n crossing knot. Additionally, a tree of projections relative to smoothing and smoothing, or changing and changing, or deleting and deleting all have exponential upper bounds. In summary, we have the following theorem.

Theorem 2.6: At most, the tree of projections relative to any consistently applied elementary knot operation has as an upper bound an exponential number of nodes.

In Theorems 2.5 and 2.6 we have not yet established the existence of any classes of knots which actually adhere to these bounds, i.e., How tight are these bounds?

3. The Complexity

A particular class of knots with specific projections illustrates that there exist knot projections whose trees of projection relative to smoothing and changing, and smoothing and deleting actually satisfy the upper bound of

$$O\{((1 + \sqrt{5})/2)^n\}.$$

The class of knots is the (2, n) torus knots and links and the Fibonacci knots [14] F_n = 1111...1 (n 1's in Conway's notation, see [4]). The (2, n) knots and links will all be assumed to have the projections and orientations described below.

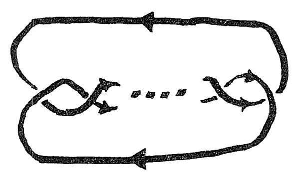

FIGURE 6

Standard projection of the (2, n) knots and links

Throughout the rest of this discussion, the (2, n) torus knots or links will denote the standard projections of the (2, n) knot or link. Also, given any standard projection of a (2, n) knot or link, smoothing and changing will produce standard projection knots or links. This is being done because standard projections can be maintained throughout the execution of Conway's algorithm. The Fibonacci Knot Class is defined by Turner in [14], which we follow. Just as in [14] a knot from the Fibonacci Knot Class will be denoted as F_n. In Conway's notation [4], F_n = 111...1 (n ones). This turns out to be useful in the worst case analysis. Several lemmas are now presented without proof, since they can easily be deduced via examination.

Lemma 3.1: Smoothing the specified projection of a torus knot (2, n), for $m \geq 1$ and $n = 2m + 1$, can produce the link projection (2, $n - 1$). Additionally, smoothing the link projection (2, n), for $n = 2m$ and $m \geq 1$, can produce a torus knot projection (2, $n - 1$). We also point out that changing a torus knot projection $(2, n)$, where $n = 2m + 1$ and $m \geq 1$, can produce the torus knot projection $(2, n - 2)$. And changing the link projection $(2, n)$, for $n = 2$ and $m \geq 1$, can produce the link projection $(2, n - 2)$.

Lemma 3.1 can be verified by examination of parts of the actual torus knot projections.

Lemma 3.2: (Turner [14]) Given a Fibonacci knot F_n deleting and smoothing a crossing can create the following projections F_{n-1} and F_{n-2}.

An n-trefoil is n trefoils connected in the following fashion:

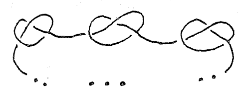

FIGURE 7

Lemma 3.3: Given an n-trefoil, any sequence of elementary knot operations forms an exponential tree of projections.

Lemma 3.3 can be proved by using induction.

Definition 3.4: An *AVL tree* [1], [2] is a binary tree which has the property that from any given node in the tree the depths of the right and left subtrees differ by at most one.

Examples:

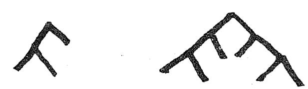

FIGURE 8

For brevity, from here the discussion is focused on the elementary operations of changing and smoothing.

The following induction establishes that applying Algorithm 2.1 to a $(2, n)$ knot projection can produce an AVL tree. The Conway algorithm, given a standard projection of a $(2, n)$ knot or link may maintain standard projections throughout smoothing and changing. It is easily proven that any AVL tree has exponential number of nodes, but an exact result will be found.

Theorem 3.5: A torus knot standard projection $(2, n)$, for $n = 2m + 1$ and $m \geq 1$, will produce a tree of projection with $\Theta\{((1 + \sqrt{5})/2)^n\}$ projection nodes provided standard projections are maintained throughout the computation of the Conway algorithm. The same is true for a $(2, n)$ link projection.

Proof: By Lemma 3.1, standard projections can be maintained throughout the construction of a tree of projections of a $(2, n)$ knot or link projection. Standard projections will be maintained throughout this proof.

Claim: When Conway's algorithm is applied to a $(2, n)$ knot or link in standard projection, it produces an AVL tree with the number of nodes described by the recurrence relation

$$f(n) = f(n - 1) + f(n - 2) + 1.$$

Proof by induction:

Basis. The unknot $(2, 1)$ produces a trivial AVL tree. The link $(2, 2)$ produces an AVL tree with 3 knot projections or nodes. The trefoil $(2, 3)$ produces a tree with 5 nodes.

Inductive hypothesis. Assume that for some number q and some function f the standard knot projection $(2, m)$ has an AVL tree of projections with $f(m)$ nodes as long as $m < q$. Additionally, the standard link projection $(2, m - 1)$ has an AVL tree of projections with $f(m - 1)$ nodes.

Inductive step. By the inductive hypothesis, the link or knot projection $(2, q - 1)$ has an AVL tree of projections of size $f(q - 1)$. The knot or link $(2, q - 2)$ has a tree of projections with $f(q - 2)$ nodes in its AVL tree. Now, by adding the +1 integer tangle appropriately to the link or knot projection $(2, q - 1)$, the knot or link projection $(2, q)$ may be produced. Performing a change to the standard projection of the knot or link $(2, q)$ can produce the knot or link projection $(2, q - 2)$. Smoothing a crossing of the knot or link $(2, q)$ may produce the link or knot $(2, q - 1)$ projection. Making the projection of the knot or link $(2, q)$ the root of the tree of projections with the subtrees created by the knot projections $(2, q - 1)$ and $(2, q - 2)$ can make a tree with a total of $f(q - 1) + f(q - 2) + 1$ projection nodes. This is true because the tree of projections of the link or knot $(2, q - 1)$ is an AVL tree, and the tree of projections of the knot or link $(2, q - 2)$ is also an AVL tree by the inductive hypothesis. The link or knot projection $(2, q - 1)$ has smoothing and changing subtrees consisting of the trees of projection $(2, q - 2)$ and $(2, q - 3)$, respectively. Since $\text{depth}((2, q - 2)) - \text{depth}((2, q - 3)) \le 1$ where $\text{depth}(K)$ is the depth of K's tree of projections, then $\text{depth}((2, q - 1)) - \text{depth}((2, q - 2)) \le 1$. Therefore, the knot or link projection $(2, q)$ can have an AVL tree of projections with number of nodes described by the recurrence relation

$$f(q) = f(q - 1) + f(q - 2) + 1.$$

This means that applying Conway's algorithm to the standard projection of $(2, n)$ and preserving standard projections throughout the calculation of the Conway polynomial will produce an AVL tree of projections with the number of nodes described by the recurrence relation

$$f(n) = f(n - 1) + f(n - 2) + 1.$$

End of induction.

Next, the exact number of nodes, in closed form, in this tree of projections can easily be derived by solving the nonhomogeneous, constant coefficient difference relation of second order:

$$f(n) - f(n - 1) - f(n - 2) = 1.$$

Given $f(n) - f(n - 1) - f(n - 2) = 1$, where $n \ge 2$, with the boundary conditions $f(0) = 1$, $f(1) = 1$, and $f(2) = 3$.

The closed form solution is easily derived using the method of variation of constants:

$$f(n) = \frac{1 + \sqrt{5}}{2\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^n + \frac{-1 + \sqrt{5}}{2\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^n \sum_{i=0}^{n-1}\left(\frac{2}{1 - \sqrt{5}}\right)^{i+1}$$
$$+ \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^n \sum_{i=0}^{n-1}\left(\frac{2}{1 + \sqrt{5}}\right)^{i+1}$$

Taking the identity

$$(*) \qquad \sum_{i=1}^{n-2}\left(\frac{2}{1 + \sqrt{5}}\right)^{i+1} = \frac{1 - \sqrt{5}}{1 + \sqrt{5}}\left[\left(\frac{2}{1 + \sqrt{5}}\right)^{n-1} - \frac{2}{1 + \sqrt{5}}\right]$$

which converges to 0 as n approaches ∞, noting $(1 - \sqrt{5})/2 < 1$, $2/(1 + \sqrt{5}) < 1$, and, for some integer k and some constant A when $n \ge k$,

$$f(n) \leq A\left(\frac{1 + \sqrt{5}}{2}\right)^n,$$

we obtain

$$f(n) = O\left(\left(\frac{1 + \sqrt{5}}{2}\right)^n\right).$$

Now, to show $((1 + \sqrt{5})/2) = O(f(n))$: Given equation (*) and the fact that $(1 - \sqrt{5})/2 < 1$, then choosing a large enough constant A and some number k when $n \geq k$,

$$((1 + \sqrt{5})/2)^n \leq A((1 + \sqrt{5})/2)^{n+1} \text{ as } n \to \infty.$$

Hence,

$$f(n) = \Theta\{((1 + \sqrt{5})/2)^n\}.$$

This completes the proof. □

Since the tree of projections for a $(2, n)$ knot projection whose children remain as standard projections has depth of at least $\lfloor(n/2)\rfloor$, it is quite easy to show $f(n) = O(2^{\lfloor(n/2)\rfloor})$, alternatively.

Noting Theorem 2.5 and Theorem 3.5 and given any n crossing knot K, call its tree of projections relative to smoothing and changing PK. Then the standard projection of the $(2, n)$ knot can have a tree of projections $P(2, n)$ that is larger than or equal to PK.

A similar argument, along with Theorem 2.6, illustrates that the Fibonacci Knot Class forms an upper bound on the size of a tree of projections relative to smoothing and deleting. So we have proved

Theorem 3.6: The $(2, n)$ knots and the Fibonacci knots F_n have trees of projections relative to changing and smoothing, and smoothing and deleting, respectively, which can be the largest possible given a knot with n crossings.

It is left to the reader to show that the n-trefoil will produce a knot which can have the largest possible tree of projections relative to changing and deleting within a constant.

It has been established that there are knot projections (and link projections, for that matter) whose trees of projections can have an exponential number of nodes relative to the number of crossings in the projections. The tree of projections may be built as needed and taken apart immediately afterward.

Any operation on the nodes of the tree of projections of a knot can be considered the fundamental operation. The operation of checking for an unknot or split link seems to fit the bill best (see Algorithm 2.1). This is because the appearance of the unknot or split link indicates a leaf node in the tree of projections with no children; hence, the algorithm may stop smoothing and changing down that particular branch of the tree.

And now a nice application: Denoting the Conway algorithm by C.

Theorem 3.7: $C_{\text{worst}}(n) = O((1 + \sqrt{5})/2)^n)$; or the Conway algorithm has exponential worst cast time complexity.

The Conway algorithm, given any knot projection K is invariant since the Conway polynomial of the knot K is well defined given any knot projection of K (Theorem 1.7). Let n denote the number of crossings of a particular knot. By Theorem 3.5, using C in constructing the Conway polynomial of the standard projection of the torus knot $(2, n)$, it is possible to produce an exponential tree of projections. So, $C_{\text{worst}}(n)$ has exponential time complexity dominated by $O(((1 + \sqrt{5})/2)\ )$ unknotting checks. □

Corollary 3.8: Any polynomial invariant constructed by any consistent combination of knot operations (changing, smoothing, and deleting) on a knot projection is of exponential worst case cost. Provided the operations are performed to the given knot projection until all of the derived knot or link projections are resolved into unknots and split links at least.

This is true, since these operations are irrespective of the unknown's and their coefficients.

For example, take the Homfly polynomial [6] of a knot K, $\text{Homfly}_K(x, y, z)$, which can be calculated in a similar fashion to Algorithm 2.1. The Conway polynomial of the same knot $\nabla_K(z)$ can then be created by setting $x = 1$ and $y = 1$, resulting in Homfly $(1, 1, z) = \nabla_K(z)$.

It might be noted that Conway's algorithm has a constant best case time complexity. It seems to be a very hard problem to decide the average case time complexity of Conway's algorithm.

Theorem 3.9: The act of creating the Conway polynomial is in the class P.

This will be proved by the presentation of an algorithm which can determine the Conway polynomial in better than $O(n^3)$ time.

Algorithm 3.10: (A version of Alexander's Algorithm, [3]; see also [13])

Given: a projection of a knot K.
Returning: the Conway polynomial of the knot K.

1. Choose an orientation of the knot projection K. Label the crossings from X_1 to X_n for a knot with n crossings.
2. Create an n by n matrix, calling it mat, filling all entries of mat with zeros.
3. Fill the entries of the matrix as follows:
   Each crossing is associated with a column of the matrix.
   for col : = 1 to n do
   Say crossing col is:

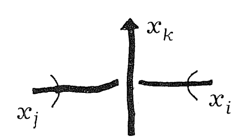

FIGURE 9

Then let

mat[k, col] : = $1 - z$
mat[i, col] : = z
mat[j, col] : = -1

endfor

4. Disregard any row and column of the matrix mat producing a $(n-1) \times (n-1)$ matrix.
5. Calculate the determinant of the matrix produced in step 4.
6. The polynomial created by computing the determinant is the Alexander polynomial.
7. The Alexander polynomial is converted to the Conway polynomial by noting $\Delta_K(z^2) = \nabla_K(z - z^{-1})$ where equality is up to multiples of $\pm z^n$.
8. Normalize $\Delta_K(z^2)$ into $\nabla_K(z)$. Return $\nabla_K(z)$.
9. End.

This algorithm clearly terminates, and the computationally time consuming part of Algorithm 3.10 is step 5, calculating the determinant. Step 5 can be done by a straightforward algorithm in $O(n^3)$ time. For example, performing

Gaussian elimination followed by multiplication along the diagonal computes a determinant in $O(n^3)$ time. There are other algorithms which can compute this in slightly better time. The fact that the matrix is sparse, only having three nonzero elements in each column, can also be taken into account.

This $O(n^3)$ complexity is the same given any n crossing knot projection. Therefore, denoting Alexander's algorithm as Alex, it must be that

$$\text{Alex}_{\text{best}}(n) = \text{Alex}_{\text{worst}}(n) = \text{Alex}_{\text{average}}(n) = O(n^3),$$

assuming Gaussian elimination followed by multiplication along the diagonal is used to calculate the determinant. Hence, the act of creating the Conway polynomial is in the class P.

4. Conclusion

In this paper it is established that the consistent application of elementary knot operations may lead to an exponential number of derived knot projections. This illustrates that Conway's algorithm has exponential worst case time complexity. Moreover, the nonvacuous upper bound on the worst case complexity is based on the golden ratio. It was then illustrated that a determinant based algorithm given by Alexander is of polynomial time complexity; hence, calculating the Conway polynomial is in the class P. It is interesting to note that Jaeger [8] has shown the calculation of the Homfly polynomial to be in the class NP-Hard.

Corollary 3.8 shows that this complexity analysis can be applied to the calculations of the Jones polynomial [9], the Homfly polynomial [6], the Kauffman polynomial [12, Appendix], and many other polynomial invariants of knots and links.

Hopefully, algorithms for the calculation of knot polynomials will receive more attention in the future. Many interesting questions remain open. It is presently unknown whether any knot polynomial can detect knottedness. Yet Fellows & Langston [5] have recently nonconstructively shown that detecting knotlessness is in P. So the quest is on to find some invariant, knot polynomial or otherwise, that can recognize the unknot in polynomial time.

Acknowledgments

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AN ALGEBRAIC IDENTITY AND SOME PARTIAL CONVOLUTIONS

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Let $\{a_i\}_{i \geq 0}$ and $\{b_i\}_{i \geq 0}$ be any two real or complex number sequences satisfying $a_i \neq 0$ and $b_i \neq 0$ for $i > 0$. Assume that x, y, and z are three formal variables. For any natural number n, define a formal binomial coefficient as follows:

$$(1) \qquad \binom{x}{n}_{(a)} = \prod_{k=1}^{n} \frac{x - a_{k-1}}{a_k}, \text{ where } \binom{x}{0}_{(a)} = 1.$$

It is obvious that when $a_k = k$ $(k = 0, 1, \ldots)$, $\binom{x}{n}_{(a)}$ reduces to the ordinary binomial coefficient. If we replace x by $1 - q^x$ and a_k by $1 - q^k$ $(k = 0, 1, \ldots)$ instead, then $\binom{x}{n}_{(a)}$ becomes the Gaussian binomial coefficient

$$q^{\binom{n}{2}}\begin{bmatrix} x \\ n \end{bmatrix}.$$

Based on these preliminaries, we are ready to state our main result.

Theorem: Let $0 \leq m \leq n \leq r$ be three natural numbers. Then the following algebraic identity holds:

$$(2) \qquad \sum_{k=m}^{n} \{b_{r-k}(x - a_k)z - a_k(y - b_{r-k})\}\binom{x}{k}_{(a)}\binom{y}{r-k}_{(b)} z^k$$
$$= a_{n+1}b_{r-n}\binom{x}{n+1}_{(a)}\binom{y}{r-n}_{(b)} z^{n+1} - a_m b_{r-m+1}\binom{x}{m}_{(a)}\binom{y}{r-m+1}_{(b)} z^m.$$

This identity follows from splitting the summand

$$\{b_{r-k}(x - a_k)z - a_k(y - b_{r-k})\}\binom{x}{k}_{(a)}\binom{y}{r-k}_{(b)} z^k$$
$$= a_{k+1}b_{r-k}\binom{x}{k+1}_{(a)}\binom{y}{r-k}_{(b)} z^{k+1} - a_k b_{r-k+1}\binom{x}{k}_{(a)}\binom{y}{r-k+1}_{(b)} z^k$$

and diagonal cancellation.

Taking $z = 1$, (2) reduces to the following.

Corollary: Let $0 \leq m \leq n \leq r$ be three natural numbers, then

$$(3) \qquad \sum_{k=m}^{n} (b_{r-k}x - a_k y)\binom{x}{k}_{(a)}\binom{y}{r-k}_{(b)}$$
$$= a_{n+1}b_{r-n}\binom{x}{n+1}_{(a)}\binom{y}{r-n}_{(b)} - a_m b_{r-m+1}\binom{x}{m}_{(a)}\binom{y}{r-m+1}_{(b)}.$$

For the remainder of the paper, we shall discuss the applications of (2) and (3) to combinatorial identities.

First, letting $m = 0$, $y = m - x$, and $a_k = b_k = k$ in (3) gives

$$(4) \qquad \sum_{k=0}^{n} \frac{rx - mk}{rx}\binom{x}{k}\binom{m - x}{r - k} = \frac{r - n}{r}\binom{x - 1}{n}\binom{m - x}{r - n}.$$

If we define a partial convolution by

$$(5) \qquad S_m(x, r, n) = \sum_{k=0}^{n} \binom{x}{k}\binom{m - x}{r - k},$$

then (4) generates the following recurrence:

$$S_m(x,\ r,\ n) = \frac{r-n}{r}\binom{x-1}{n}\binom{m-x}{r-n} + \frac{m}{r}S_{m-1}(x-1,\ r-1,\ n-1).$$

Performing iteration on this recurrence and noting that the closed form of $S_0(x,\ r,\ n)$ from (4) and (5) is

$$\sum_{k=0}^{n}\binom{x}{k}\binom{-x}{r-k} = \frac{r-n}{r}\binom{x-1}{n}\binom{-x}{r-n}, \tag{6}$$

we have

$$S_m(x,\ r,\ n) = \sum_{k=0}^{m}\frac{r-n}{r-k}\binom{m}{k}\binom{r}{k}^{-1}\binom{x-k-1}{n-k}\binom{m-x}{r-n}, \tag{7}$$

which contains the following interesting example (cf. Anderson [1]):

$$\sum_{k=0}^{n}\binom{x}{k}\binom{1-x}{r-k} = \frac{(1-x)(r-1)-n}{r(r-1)}\binom{x-1}{n}\binom{-x}{r-n-1}. \tag{8}$$

This identity and (6) are the main results of [1] established by the induction principle.

Rewriting (7) in the form

$$S_m(x,\ r,\ n) = (-1)^{m+n}\,\frac{r-n}{r}\binom{m-x}{r-n}\binom{r-1}{m}^{-1}\sum_{k=0}^{m}\binom{m-r}{m-k}\binom{n-x}{n-k},$$

and making some trivial modifications, it may be reformulated as

$$\sum_{i=0}^{n}\binom{n}{i}^{-1}\binom{x}{i}\binom{m-x-r+n}{n-i}\binom{r}{i} = \frac{\binom{n-r}{n}}{\binom{m-r}{m}}\sum_{k=0}^{n}\binom{m-r}{m-k}\binom{n-x}{n-k}. \tag{9}$$

Since (9) is a polynomial identity in r, it is also true if we replace r by a continuous variable y which provides an algebraic identity. The particular case of $m = 0$ in (9) yields the following combinatorial identity:

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{k}\binom{n}{k}^{-1}\binom{n-x-y}{n-k} = \binom{n-x}{n}\binom{n-y}{n}. \tag{10}$$

Next, taking $a_i = b_i = i$ and replacing r and n by $m + n$ in (3), we have

$$\sum_{k=0}^{n}\{(m+k)y - (n-k)x\}\binom{x}{m+k}\binom{x-1}{n-k} = my\binom{x}{m}\binom{y-1}{n}. \tag{11}$$

Putting $x = y$ and $m = n + 1$ in (11), we obtain the following identity,

$$\sum_{k=0}^{n}\frac{2k+1}{n+k+1}\binom{x}{n-k}\binom{x-1}{n+k} = \binom{x-1}{n}^2, \tag{12}$$

which reduces to an identity of Riordan ([3], p. 18) for $x = n - m$.

If we let $m \to m + s$, $x \to 2m + s$, and $y \to 2n + s$, alternatively, then (11) degenerates to Prodinger's generalization for Riordan's identity (cf. [2], and [3], p. 89):

$$\sum_{k\ge 0}(2k+s)\binom{2m+s}{m-k}\binom{2n+s}{n-k} = \frac{(m+s)(n+s)}{m+n+s}\binom{2m+s}{m}\binom{2n+s}{n}. \tag{13}$$

Finally, letting $x = y = 1 - q^t$, $a_i = b_i = 1 - q^i$, and $m = 0$ in (3), we obtain the following q-binomial convolution formula by simple computation:

$$\sum_{k=0}^{n}\frac{1-q^{r-2k}}{1-q^{r-n}}\begin{bmatrix}t\\k\end{bmatrix}\begin{bmatrix}t\\r-k\end{bmatrix}q^{(n-k)(r-n-k-1)} = \begin{bmatrix}t-1\\n\end{bmatrix}\begin{bmatrix}t\\r-n\end{bmatrix}. \tag{14}$$

When $q \to 1$, (14) reduces to the ordinary binomial identity:

$$\sum_{k=0}^{n}\frac{r-2k}{r-n}\binom{x}{k}\binom{x}{r-k} = \binom{x-1}{n}\binom{x}{r-n}. \tag{15}$$

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ON CERTAIN NUMBER-THEORETIC INEQUALITIES

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1. Introduction

This note deals with certain inequalities involving the elementary arithmetic functions $d(r)$, $\phi(r)$, and $\sigma_k(r)$ and their unitary analogues. We recall that a divisor d of r is called unitary [2] if $(d, r/d) = 1$. Let $e \equiv 1$, $I_k(r) = r^k$ $(k \geq 0)$, and μ denote the Möbius function. In terms of Dirichlet convolution, denoted by $(\cdot)$, we have [1]:

$$\left.\begin{aligned} d(r) &= (e \cdot e)(r) \\ \phi(r) &= (I \cdot \mu)(r) \\ \sigma_k(r) &= (I_k \cdot e)(r) \end{aligned}\right\} \text{ where } I(r) = r.$$

The unitary convolution of two arithmetic functions f and g is defined by

$$(1.1) \qquad (f \oplus g)(r) = \sum_{d \| r} f(d) g\left(\frac{r}{d}\right),$$

where $d \| r$ means that d runs through the unitary divisors of r. The unitary analogue μ^* of μ is given by [2]

$$(1.2) \qquad \mu^*(r) = (-1)^{\omega(r)},$$

where $\omega(r)$ denotes the number of distinct prime factors of r with $\omega(1) = 0$. The unitary analogue ϕ^* [2] of the Euler totient is given by

$$(1.3) \qquad \phi^*(r) = (I \oplus \mu^*)(r).$$

The unitary analogues of d and σ_k are d^* and σ_k^* and

$$(1.4) \qquad d^*(r) = 2^{\omega(r)},$$

$\omega(r)$ being as defined in (1.2);

$$\sigma_k^*(r) = (I_k \oplus e)(r).$$

For properties of σ_k^*, see [5]. It is known that d^*, ϕ^*, and σ_k^* are multiplicative functions. Further, given a prime p, $m \geq 1$,

$$(1.5) \qquad \begin{cases} d^*(p^m) = 2 \\ \phi^*(p^m) = p^m - 1 \\ \sigma_k^*(p^m) = p^{mk} + 1 \end{cases}$$

Let $\phi_k = (I_k \cdot \mu)$. $\phi_k(r)$ is multiplicative in r.

From the structure of ϕ_k and σ_k, we note that

$$\begin{aligned} (\phi_k \cdot \sigma_k) &= (I_k \cdot \mu) \cdot (I_k \cdot e) \\ &= (I_k \cdot I_k) \cdot (\mu \cdot e) \\ &= (I_k \cdot I_k) \text{ as } \mu \text{ is the Dirichlet inverse of } e. \end{aligned}$$

or

$$(1.6) \qquad \sum_{d | r} \phi_k(d) \sigma_k\left(\frac{r}{d}\right) = r^k d(r) \quad (k \geq 1).$$

It follows that

$$\phi_k(r) + \sigma_k(r) = \sum_{\substack{d|r \\ d \neq 1,\ d \neq r}} \phi_k(d)\sigma_k\left(\frac{r}{d}\right) = r^k d(r).$$

Therefore,

$$(1.7) \quad \phi_k(r) + \sigma_k(r) \leq r^k d(r)$$

with equality if and only if r is a prime.

In arriving at (1.7), we have used the fact that ϕ_k and σ_k assume only positive values.

Defining $\phi_k^* = I_k \oplus \mu^*$, and noting that

$$(1.8) \quad \phi_k^* \oplus \sigma_k^* = r^k d^*(r),$$

we have

Theorem 1: $\phi_k^*(r) + \sigma_k^*(r) \leq r^k d^*(r)$ with equality if and only if r is a prime power.

Further, using the fact that

$$\phi_k^* \oplus d^* = \sigma_k^*,$$

we also obtain

Theorem 2: $\phi_k^*(r) + d^*(r) \leq \sigma_k^*(r)$ with equality if and only if r is a prime power.

We remark that Theorem 2 is analogous to the inequality involving ϕ, d, and σ, see [4], [6].

Using the multiplicativity of ϕ_k^* and σ_k^*, one could also prove

Theorem 3: For $k \geq 1$,

$$\frac{1}{\zeta(2k)} < \frac{\sigma_k^*(r)\phi_k^*(r)}{r^2} < 1,$$

where $\zeta(s)$ is the Riemann ζ-function.

Now, the AM-GM inequality yields

$$(1.9) \quad \frac{\sigma_k(r)}{d(r)} \geq r^{k/2} \quad \text{(see [9])}$$

and

$$(1.10) \quad \frac{\sigma_k^*(r)}{d^*(r)} \geq r^{k/2}.$$

The aim of this note is to establish a few more inequalities which come out as special cases of certain general inequalities found in [3] and [7].

Let

$$\begin{matrix} 0 < a \leq a_i \leq A \\ 0 < b \leq b_i \leq B \end{matrix} \qquad (i = 1, 2, \ldots, s)$$

where a_i, b_i $(i = 1, 2, \ldots, s)$, a, A, b, B are real numbers. Then, from [7],

$$(1.11) \quad \frac{\left(\sum_{i=1}^{s} a_i^2\right)\left(\sum_{i=1}^{s} b_i^2\right)}{\left(\sum_{i=1}^{s} a_i b_i\right)^2} \leq \frac{(AB + ab)^2}{4ABab}$$

Next, let

$$0 \leq a_1^{(k)} \leq a_2^{(k)} \leq \cdots \leq a_s^{(k)} \quad (k = 1, 2, \ldots, m).$$

Then, an inequality due to Tchebychef [3] states that:

$$(1.12)\qquad \left(\frac{\sum_{i=1}^{s} a_i^{(1)}}{s}\right)\cdots\left(\frac{\sum_{i=1}^{s} a_i^{(m)}}{s}\right) \le \frac{\sum_{i=1}^{s} a_i^{(1)}\ \cdots\ a_i^{(m)}}{s}$$

The inequalities derived in Section 2 are essentially illustrations of (1.11) and (1.12).

2. Inequalities

Theorem 4: For $k \ge 0$,

$$(2.1)\qquad \frac{\sigma_k(r)}{d(r)} \le \frac{r^k + 1}{2}$$

and

$$(2.2)\qquad \frac{\sigma_k^*(r)}{d^*(r)} \le \frac{r^k + 1}{2}.$$

Proof of (2.1): Let $d_1, \ldots, d_s$ be the divisors of r. We appeal to (1.11) by taking $a_i = d_i^{k/2}$, $b_i = d_i^{-m/2}$, $A = r^{k/2}$, $a = 1$, $b = r^{-m/2}$, $B = 1$. Then

$$\frac{\sigma_k(r)\sigma_m(r)}{r^m\left(\sigma^{(r)}_{(k-m)/2}\right)^2} \le \frac{(r^{k/2} + r^{-m/2})^2}{4r^{k/2 - m/2}}$$

or

$$(2.3)\qquad \frac{(\sigma_k(r)\sigma_m(r))^{1/2}}{\sigma^{(r)}_{(k-m)/2}} \le \frac{1}{2r^{(k-m)/2}}(r^{(k+m)/2} + 1)$$

Setting $m = k$ in (2.3), we obtain (2.1).

Similarly, by considering the unitary divisors of r, we arrive at (2.2).

In view of (1.9) and (1.10), we also have

Corollary:

$$(2.4)\qquad r^{k/2} \le \frac{\sigma_k(r)}{d(r)} \le \frac{r^k + 1}{2}$$

and

$$(2.5)\qquad r^{k/2} \le \frac{\sigma_k^*(r)}{d^*(r)} \le \frac{r^k + 1}{2}$$

Theorem 5: For $k, m \ge 0$,

$$(2.6)\qquad \frac{\sigma_{k+m}(r)}{\sigma_m(r)} \ge r^{k/2}$$

and

$$(2.7)\qquad \frac{\sigma_{k+m}^*(r)}{\sigma_m^*(r)} \ge r^{k/2}.$$

Proof of (2.6): Let $d_1, \ldots, d_s$ be the divisors of r. We appeal to (1.12) with

$$a_i^{(1)} = d_i^{k_1}, \ \ldots, \ a_i^{(m)} = d_i^{k_m} \ (i = 1, 2, \ldots, s)$$

where $k_1, \ldots, k_m$ are positive numbers.

Then,

$$\frac{\sigma_{k_1 + \cdots + k_m}(r)}{s} \ge \frac{\sigma_{k_1}(r)}{s} \cdots \frac{\sigma_{k_m}(r)}{s}$$

with $s = d(r)$. From (1.9), we obtain

$$(2.8)\qquad \frac{\sigma_{k_1 + \cdots + k_m}(r)}{\sigma_{k_i}(r)} \ge r^{\frac{1}{2}\sum_{j \ne i} k_j}.$$

Writing $m = 2$, we get

$$\frac{\sigma_{k_1+k_2}(r)}{\sigma_{k_2}(r)} \geq r^{k_1/2},$$

which proves (2.6).

The proof of (2.7) is similar and is omitted here.

Remark: Inequalities (2.6) and (2.7) generalize (1.9) and (1.10), respectively.

In this connection, we point out that analogous to the inequality $\phi(r)d(r) \geq r$ [8], one could prove using multiplicativity of ϕ_k^* and d^* that

Theorem 6: For $k \geq 1$,

(2.9) $\quad d^*(r)r^k \leq \phi_k^*(r)(d^*(r))^2 \leq r^{2k}.$

The proof of (2.9) is omitted.

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LENGTH OF THE n-NUMBER GAME

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The n-number game is defined as follows. Let $S = (s_1, s_2, \ldots, s_n)$ be an n-tuple of nonnegative integers. A new n-tuple $D(S) = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n)$ is obtained by taking numerical differences; that is, $\hat{s}_i = |s_i - s_{i+1}|$. Subscripts are reduced modulo n so that $\hat{s}_n = |s_n - s_1|$. The sequence S, $D(S)$, $D^2(S)$, ... is called *the n-number game generated by S*. To see that a game contains only a finite number of distinct tuples let $|S| = \max\{s_i\}$ and observe that $|S| \geq |D(S)|$. Since there are only a finite number of n-tuples with entries less than or equal to $|S|$, eventually repetition must take place. When $n = 2^w$, it is well known that every game terminates with $(0, 0, \ldots, 0)$. That this is not the case for other values of n is easily seen by considering the following 3-tuple:

$$\begin{aligned} R &= (1, 0, 0) \\ D(R) &= (1, 0, 1) \\ D^2(R) &= (1, 1, 0) \\ D^3(R) &= (0, 1, 1) \\ D^4(R) &= (1, 0, 1) = D(R) \end{aligned}$$

The tuples $D(R)$, $D^2(R)$, and $D^3(R)$ form what is called a *cycle*.

For any n-tuple S, we say the game generated by S has *length* λ, denoted by $L(S)$, if $D^{\lambda}(S)$ is in a cycle, but $D^{\lambda-1}(S)$ is not. Thus, in the example above, $L(R) = 1$, while $L(D(R)) = 0$. For each n, the length of games is unbounded. That is, for any λ, there exists an n-tuple S such that $L(S) > \lambda$. On the other hand, for tuples S with $|S| \leq m$, there is a game of maximum length, since there are only a finite number of such tuples. We introduce the following notation:

$$\begin{aligned} \mathcal{S}_n(m) &= \{S \mid S \text{ is an } n\text{-tuple with } |S| = m\}, \\ \mathcal{L}_n(m) &= \max\{L(S) \mid S \in \mathcal{S}_n(m)\}. \end{aligned}$$

On occasion, when the context is clear, we will drop the subscript. The values of $\mathcal{L}_4(m)$ and $\mathcal{L}_7(m)$, along with tuples giving games of maximum lengths, have been determined in [10] and [6]. We consider this question when n is not a power of 2. We first find an upper bound on $\mathcal{L}_n(m)$. Then we show that this bound is actually realized when $n = 2^w + 1$.

Before proceeding, a few additional comments are in order. Observe that, for any tuple S, if we multiply all the entries by a constant c and denote the resulting tuple by cS, then

$$(1) \qquad D(cS) = cD(S).$$

Additionally, if all the nonzero entries of S are equal with $S \in \mathcal{S}(m)$, then $S = mE$ for some $E \in \mathcal{S}(1)$. In particular, an entry e_i in E equals 1 if and only if the corresponding entry in S, s_i, equals m.

Since a game concludes when a cycle is reached, it is important to be able to identify those tuples which occur in a cycle. This author did that in [5]. The following theorem gives the salient facts from that work. We say that an n-tuple S has *a predecessor* if $S = D(R)$ for some n-tuple R.

Theorem 1: Let $n = kr$ where $k = 2^k$ and r is odd with $r > 1$. Suppose S is an n-tuple. Then

(i) S has a predecessor if and only if there exist values $\varepsilon_\ell \in \{-1, 1\}$, $\ell = 1, 2, n$, such that

$$\sum_{\ell=1}^{n} \varepsilon_\ell s_\ell = 0.$$

(ii) S is in a cycle if and only if all its entries are 0 or $|S|$ and

$$\sum_{j=0}^{r-1} e_{i+jk} \equiv 0 \pmod 2, \text{ for } i = 1, \ldots, k,$$

where $S = |S| \cdot E$ with $E = (e_1, e_2, \ldots, e_n) \in \mathscr{S}(1)$.

Part (ii) guarantees that when n is not a power of 2, there are nontrivial cycles; indeed, for n odd, the tuple $\mathbb{E} = (0, \ldots, 0, 1, 1)$ is in a cycle. Moreover, (ii) along with (1) gives

(2) $\quad L(cS) = L(S)$.

2. A Bound on $\mathscr{L}_n(m)$

For $S \in \mathscr{S}_n(m)$, we say that S *has* μ *0's and m's in a row*, denoted by $\mu(S)$, if the following conditions are met: there exists an integer η such that $s_i \in \{0, m\}$ for $i = \eta, \eta + 1, \ldots, \eta + \mu - 1$, at least one of these s_i equals m, and μ is as large as possible. As usual, we reduce subscripts modulo n. Thus, for example,

$$\mu(S) = 6 \text{ when } S = (3, 2, 3, 0, 1, 3, 0, 3, 0, 0).$$

Loosely speaking, a tuple S will produce a long game if, at each step, $\mu(D^k(S))$ is as large as possible. In determining an upper bound on $\mathscr{L}_n(m)$, the following lemmas will be useful.

Lemma 1: Let $S \in \mathscr{S}_n(m)$, $\mu(S) = t$, and $t < n$. If $D(S) \in \mathscr{S}_n(m)$, then $\mu(D(S)) \le t - 1$.

Proof: By hypothesis, for some η, we have

$$\begin{aligned} &s_i \in \{0, m\} && \text{for } i = \eta, \eta + 1, \ldots, \eta + t - 1, \\ &s_i = m && \text{for some } i,\ \eta \le i \le \eta + t - 1, \\ &1 \le s_{\eta-1},\ s_{\eta+t} \le m - 1. \end{aligned}$$

As before, let $D(S) = (\hat{s}_1, \ldots, \hat{s}_n)$. Then

$$\begin{aligned} &\hat{s}_i \in \{0, m\} && \text{for } i = \eta, \eta + 1, \ldots, \eta + t - 2, \\ &1 \le \hat{s}_{\eta-1},\ \hat{s}_{\eta+t-1} \le m - 1. \end{aligned}$$

Hence, if $|D(S)| = m$, then $\mu(D(S)) \le t - 1$. $\square$

At first glance, it might seem in Lemma 1 that, if $|D(S)| = m$, then $\mu(D(S))$ must equal $t - 1$. It is possible, however, to have strict inequality. This would occur if $\hat{s}_i = 0$ for $\eta \le i \le \eta + t - 2$, while $\hat{s}_j = m$ for some other j.

Lemma 2: Suppose that $S \in \mathscr{S}_n(m)$ and not all the nonzero entries equal m. Then $|D^{n-1}(S)| \le m - 1$. Further, if S has a predecessor, then $|D^{n-2}(S)| \le m - 1$.

Proof: Let $\mu(S) = t$. By hypothesis, $t \le n - 1$, and if S has a predecessor, then by Theorem 1(i), $t \le n - 2$. In either case, Lemma 1 applies. So, if $|D^i(S)| = m$, for $i = 1, \ldots, t - 1$, then $\mu(D^i(S)) \le t - i$. Of course, if $\mu(D^j(S)) = 1$, then $|D^{j+1}(S)| \le m - 1$. Thus, $|D^t(S)| \le m - 1$. $\square$

In a moment we will consider those tuples in which all nonzero entries equal m. In that case, $S = mE$ for some $E \in \mathscr{S}(1)$. For tuples in $\mathscr{S}_n(1)$, the following is useful. Let $\mathbb{A} = \mathbb{Z}_2[t]/\mathscr{I}$ where $\mathbb{Z}_2[t]$ is the polynomial ring over

$\mathbb{Z}_2$ and $\mathcal{I}$ is the principal ideal generated by $t^n + 1$. We associate with $E = (e_1, \ldots, e_n) \in \mathcal{S}_n(1)$, the polynomial

$$\mathcal{P}_E(t) = e_n + e_{n-1}t + \cdots + e_2 t^{n-2} + e_1 t^{n-1} \text{ in } \mathbb{A}.$$

Since $\hat{e}_i = |e_i - e_{i+1}| = e_i + e_{i+1}$ in $\mathbb{Z}_2$ and $t^n = 1$ in $\mathbb{A}$,

$$\begin{aligned}\mathcal{P}_{D(E)}(t) &= (e_n + e_1) + (e_{n-1} + e_n)t + \cdots + (e_2 + e_3)t^{n-2} + (e_1 + e_2)t^{n-1} \\ &= (1 + t)\mathcal{P}_E(t).\end{aligned} \tag{3}$$

Lemma 3: Let $n = kr$, where $k = 2^k$ and r is odd with $r > 1$. Suppose $S \in \mathcal{S}_n(m)$ and all the nonzero entries equal m. Then $L(S) \le k$. Further, if S has a predecessor, then $L(S) \le k - 1$.

Proof: As usual, we let $S = mE$, where $E = (e_1, \ldots, e_n) \in \mathcal{S}(1)$. For the first part, by (2), we need only show that $D^k(E)$ is in a cycle. Using (3), we find

$$\begin{aligned}\mathcal{P}_{D^k(E)}(t) &= (1 + t)^k \mathcal{P}_E(t) \\ &= (1 + t^k)\mathcal{P}_E(t) \\ &= (1 + t^k)(e_n + e_{n-1}t + \cdots + e_2t^{n-2} + e_1t^{n-1}) \\ &= \sum_{\ell=0}^{k-1}(e_{n-\ell} + e_{k-\ell})t^\ell + \sum_{\ell=k}^{n-1}(e_{n-\ell} + e_{n+k-\ell})t^\ell \text{ in } \mathbb{A}.\end{aligned}$$

The second equality holds since k is a power of 2 and so all the binomial coefficients in $(1 + t)^k$ except for the first and last are even. From the above, we see that

$$D^k(E) = (e_1 + e_{k+1}, e_2 + e_{k+2}, \ldots, e_{n-k} + e_n, e_{n-k+1} + e_1, \ldots, e_n + e_k).$$

We now check condition (ii) of Theorem 1. In doing so, we use the fact that $n - k = (r - 1)k$. For $i = 1$, we have

$$(e_1 + e_{k+1}) + (e_{k+1} + e_{2k+1}) + \cdots + (e_{n-k+1} + e_1) \equiv 0 \pmod 2.$$

Similarly, (ii) holds for all other values of i. Thus, $D^k(E)$ is in a cycle and $L(E) \le k$.

For the second part, it is also sufficient to show that $L(E) \le k - 1$. Consider the tuple $F = (f_1, f_2, \ldots, f_n) \in \mathcal{S}(1)$ defined by

$$f_1 = 0,\ f_i = e_1 + e_2 + \cdots + e_{i-1} \pmod 2,\ i = 2, \ldots, n.$$

Since S has a predecessor, E does as well; because the entries of E are either 0 or 1, Theorem 1(i) gives

$$e_1 + e_2 + \cdots + e_n \equiv 0 \pmod 2.$$

This means that $f_n = e_n$ and so $D(F) = E$. Thus, $L(E) = L(D(F)) \le k - 1$. □

Theorem 2: Let $n = kr$, where $k = 2^k$ and r is odd with $r > 1$. Then $\mathcal{L}_n(m) \le (m - 1)(n - 2) + k$.

Proof: Let $S \in \mathcal{S}_n(m)$. If all the nonzero entries of S are equal, then by Lemma 3, $L(S) \le k$ and so the theorem holds. Otherwise, by Lemma 2, $|D^{n-1}(S)| \le m - 1$. Continuing, suppose that, for some $\ell = 1, \ldots, m - 2$, all the nonzero entries of $D^{\ell(n-2)+1}(S)$ are equal. Then, again by Lemma 3, $L(D^{\ell(n-2)+1}(S)) \le k - 1$, which means $L(S) \le \ell(n - 2) + k$. On the other hand, if the latter condition does not hold, then, by Lemma 2, $|D^{(m-1)(n-2)+1}(S)| \le 1$. Another application of Lemma 3 gives the desires result. □

If there is a tuple $S \in \mathcal{S}_n(m)$ with $L(S) = (m - 1)(n - 2) + k$, then the proof of Theorem 2 tells us what the tuples in the game must look like.

Corollary 1: Let $n = kr$, where $k = 2^\kappa$ and r is odd with $r > 1$. If

$$\mathscr{L}_n(m) = (m-1)(n-2) + k,$$

then there exists $S \in \mathscr{S}_n(m)$ such that

(i) $|D^{\ell(n-2)+1}(S)| = m - \ell$ and $\mu(D^{\ell(n-2)+1}(S)) = n - 2$ for $\ell = 0, \ldots, m-1$,

(ii) $L(D^{(m-1)(n-2)+1}(S)) = k - 1$.

Proof: This follows immediately from the proof of Theorem 2. □

In a moment we will state a condition for the existence of a game of maximum length in terms of the n-tuple $(0, \ldots, 0, 1, 1)$. Before proceeding, two comments are in order. First, if the entries of an n-tuple are rearranged so that adjacent elements remain adjacent, then similar games result. Or, more precisely, if $S = (s_1, s_2, \ldots, s_n)$ and σ_1 is a permutation contained in the dihedral group $\mathscr{D}_n$, then

$$D(\sigma_1(S)) = \sigma_2(D(S)) \text{ for some } \sigma_2 \in \mathscr{D}_n. \tag{4}$$

Second, it is convenient to associate with $S = (s_1, s_2, \ldots, s_n)$ an n-tuple $\mathscr{M}(S) \in \mathscr{S}(1)$ which is related to the parity of the entries of S. We define $\mathscr{M}(S) = (m_1, m_2, \ldots, m_n)$ in the obvious way with $m_i \equiv s_i \pmod 2$. Observe that

$$\mathscr{M}(D(S)) = D(\mathscr{M}(S)). \tag{5}$$

Theorem 3: Let $n = kr$, where $k = 2^\kappa$ and r is odd with $r > 1$. Suppose for $m \geq 4$, $\mathscr{L}_n(m) = (m-1)(n-2) + k$. Then, for some $\sigma \in \mathscr{D}_n$,

$$D^{2(n-2)}(\mathbb{E}) = \sigma(\mathbb{E}), \text{ where } \mathbb{E} = (0, \ldots, 0, 1, 1).$$

Proof: By hypothesis, there exists an n-tuple S with $|S| = m$ and

$$L(S) = (m-1)(n-2) + k.$$

Let $T = D^{(m-4)(n-2)+1}(S)$. Corollary 1 implies that

$$|T| = |D^{(m-4)(n-2)+1}(S)| = 4,\ \mu(T) = n - 2, \text{ and } |D^{(n-2)}(T)| = 3.$$

Since $\mu(T) = n - 2$, T has exactly two adjacent entries with values in $\{1,2,3\}$. One of these must equal either 1 or 3; for, if not, then $|D^{(n-2)}(T)| \leq 2$. Moreover, since T has a predecessor, Theorem 1(i) guarantees that both are in $\{1,3\}$. This shows that

$$\mathscr{M}(T) = \sigma_1(\mathbb{E}) \text{ for some } \sigma_1 \in \mathscr{D}_n.$$

Similarly,

$$\mathscr{M}(D^{2(n-2)}(T)) = \sigma_2(\mathbb{E}) \text{ for } \sigma_2 \in \mathscr{D}_n.$$

Hence,

$$\begin{aligned}\sigma_2(\mathbb{E}) &= \mathscr{M}(D^{2(n-2)}(T))\\ &= D^{2(n-2)}(\mathscr{M}(T))\\ &= D^{2(n-2)}(\sigma_1(\mathbb{E}))\\ &= \sigma_3(D^{2(n-2)}(\mathbb{E}))\end{aligned}$$

The second equality follows from (5); the last, from (4). Thus, for $\sigma = \sigma_3^{-1}\sigma_2 \in \mathscr{D}_n$, $D^{2(n-2)}(\mathbb{E}) = \sigma(\mathbb{E})$. □

Theorem 3 is the heart of the matter. Whether or not there exists an n-tuple which has the maximum possible length depends in large part on $\mathbb{E}$. Since $\mathbb{E} \in \mathscr{S}(1)$, Theorem 3 can be recast in terms of polynomials in $\mathbb{A}$. Using (3), we see that, in order to have an n-tuple of maximum length,

$$(1+t)^{2(n-2)}\mathscr{P}_{\mathbb{E}}(t) = \mathscr{P}_{\sigma(\mathbb{E})}(t).$$

Since $\mathscr{P}_{\mathbb{E}}(t) = 1 + t$, $\mathscr{P}_{\sigma(\mathbb{E})}(t) = t^j + t^{j+1}$ for some j, where, if necessary, the exponent $j + 1$ is reduced modulo n. Thus, we have

Corollary 2: Let $n = kr$, where $k = 2^{\kappa}$ and r is odd with $r > 1$. Suppose that, for $m \geq 4$, $\mathscr{L}_n(m) = (m - 1)(n - 2) + k$. Then, for some j,

$$(1+t)^{2n-3} = t^j(1+t) \tag{6}$$

in $\mathbb{A}$. □

Theorem 4: Let n be an integer such that $n \neq 2^w$ and $n \neq 2^w + 1$ for any w. Then, for $m \geq 4$, $\mathscr{L}_n(m) < (m - 1)(n - 2) + k$.

Proof: First, suppose n is even. By Theorem 1(ii), $\mathbb{E} = (0, \ldots, 0, 1, 1)$ is not in a cycle. Thus, $D^i(\mathbb{E}) \neq \mathbb{E}$ for any i. Now, if $D^{2(n-2)}(\mathbb{E}) = \sigma(\mathbb{E})$ for some $\sigma \in \mathscr{D}_n$, then $D^{2(n-2)p}(\mathbb{E}) = \mathbb{E}$ where p is the order of σ in $\mathscr{D}_n$. Consequently, the conclusion of Theorem 3 cannot hold.

For n odd, we will expand $(1 + t)^{2n-3}$ denoting the ℓ^{th} binomial coefficient by c_ℓ.

$$\begin{aligned}(1+t)^{2n-3} &= \sum_{\ell=0}^{2n-3} c_\ell t^\ell = \sum_{\ell=0}^{n-3}(c_\ell + c_{\ell+n})t^\ell + (c_{n-2}t^{n-2} + c_{n-1}t^{n-1})\\ &= \sum_{\ell=0}^{n-3}(c_\ell + c_{n-3-\ell})t^\ell + (c_{n-2}t^{n-2} + c_{n-2}t^{n-1})\\ &= \sum_{\ell=0}^{\frac{n-5}{2}}(c_\ell + c_{n-3-\ell})(t^\ell + t^{n-3-\ell})\\ &\qquad + 2c_{\frac{n-3}{2}}t^{\frac{n-3}{2}} + c_{n-2}(t^{n-2} + t^{n-1}).\end{aligned}$$

The second equality follows by using $t^n = 1$; the third, from $c_\ell = c_{2n-3-\ell}$. Now when $2n - 3 = 2^v - 1$ for some v, all the binomial coefficients are odd, so that we have

$$(1+t)^{2n-3} = t^{n-2}(1+t).$$

Thus, (6) holds for $n = 2^w + 1$, where $w = v - 1$. On the other hand, when $2n - 3 \neq 2^v - 1$ for any v, then c_{n-2} is even. So, if t^ℓ is present in the expansion of $(1 + t)^{2n-3}$, then so also is $t^{n-3-\ell}$. Hence, (6) cannot hold. □

3. The Case $n = 2^w + 1$

We now consider the case in which $n = 2^w + 1$. Corollary 2 and Theorem 4 imply that a game of maximum length is possible. We show that this actually occurs. Before examining the general case, we consider the special case $n = 3$.

Lemma 4: Let $n = 3$ and define $T_m = (m - 1, 1, m)$. Then, for $m \geq 2$, $D(T_m) = \sigma(T_{m-1})$ for some $\sigma \in \mathscr{D}_3$.

Proof: The result is immediate since $D(T_m) = (m - 2, m - 1, 1)$. □

Lemma 5: Suppose $n = 2^w + 1$, $w \geq 2$. Let $T_m = (0, 0, \ldots, 0, m - 1, 1, m)$. Then, for $m \geq 2$,

$$\begin{aligned}D^{n-4}(T_m) &= (0, m-1, t_3, t_4, \ldots, t_{n-1}, m)\\ D^{n-3}(T_m) &= (m-1, 1, \ldots, 1, m)\\ D^{n-2}(T_m) &= (m-2, 0, \ldots, 0, m-1, 1)\end{aligned}$$

where the entries in $D^{n-4}(T_m)$ have the property that $|t_i - t_{i+1}| = 1$ for $i = 2, \ldots, n - 1$.

Proof: The proof proceeds by induction on w. Suppose that $w = 2$ so that $n = 5$. Then $T_m = (0, 0, m - 1, 1, m)$ and it is easily seen that $D(T_m) = (0, m - 1, m - 2, m - 1, m)$.

Suppose that the Lemma 5 holds for $w - 1$; more specifically, suppose that

$$D^{\ell-4}(\overline{T}_m) = (0, m - 1, r_3, r_4, r_{\ell-1}, m), \text{ and}$$

$$D^{\ell-4}(\overline{T}_{m-1}) = (0, m - 2, s_3, s_4, \ldots, s_{\ell-1}, m - 1),$$

where $\ell = 2^{w-1} + 1$, $|r_i - r_{i+1}| = 1$, and $|s_i - s_{i+1}| = 1$ for $i = 2, \ldots, \ell - 1$. Consider the $(2^w + 1)$-tuple T_m. We can view T_m as a 2^{w-1} zero-tuple concatenated with a $2^{w-1} + 1$ "T_m-type-tuple." Thus, when we compute $D^k(T_m)$ for k less than $2^{w-1} - 2$, we have the same pattern we have for the $2^{w-1} + 1$ case. Thus, we have

k	$D\ (T\ )$
$2^{w-1} - 3$	$(0, 0, \ldots, 0, 0, 0, m - 1, r_3, r_4, \ldots, r_{\ell-1}, m)$
$2^{w-1} - 2$	$(0, 0, \ldots, 0, 0, m - 1, 1, 1, 1, \ldots, 1, m)$
$2^{w-1} - 1$	$(0, 0, \ldots, 0, m - 1, m - 2, 0, 0, \ldots, 0, m - 1, m)$
2^{w-1}	$(0, 0, \ldots, m - 1, 1, m - 2, 0, \ldots, 0, m - 1, 1, m)$
$\vdots$	$\vdots$
$2^w - 3$	$(0, m - 1, s_{\ell-1}, \ldots, s_3, m - 2, m - 1, r_3, r_4, \ldots, r_{\ell-1}, m)$.

Note that for $k = 2^{w-1}$, $D^k(T_m)$ may be viewed as the $2^{w-1} + 1$ "T_{m-1}-type" tuple, $(0, \ldots, 0, m - 1, 1, m - 2)$, concatenated with the 2^{w-1} tuple, $(0, \ldots, 0, m - 1, 1, m)$. The latter is like the $2^{w-1} + 1$ "T_m-type" tuple, except that it is missing the leading zero. By induction, the second through $(n - 1)^{\text{st}}$ entries in $D^k(T_m)$, $k = 2^w - 3 = n - 4$, differ from the next one by 1. Thus, $D^{n-4}(T_m)$ has the proper form. The conclusion for $D^{n-3}(T_m)$ and $D^{n-2}(T_m)$ follows immediately. □

Theorem 5: Suppose $n = 2^w + 1$ for $w \geq 1$. Define $R_m = (0, 0, \ldots, 0, m - 1, m)$ for $m \geq 1$. Then $L(R_m) = (m - 1)(n - 2) + 1$.

Proof: Note that $D(R_m) = T_m = (0, \ldots, 0, m - 1, 1, m)$. Now, by Lemmas 4 and 5, $D^{n-2}(T_m) = \sigma(T_{m-1})$ for some $\sigma \in \mathscr{D}_n$ and $m \geq 2$. Further, T_1 is contained in a cycle, but no other T_m is. Thus, we have $L(R_m) = (m - 1)(n - 2) + 1$.

4. Remaining Questions

For n not a power of 2 and $n \neq 2^w + 1$, how large is $\mathscr{L}_n(m)$? What tuple produces the longest game? Only for $n = 7$ are the answers to these questions known [6].

Because Theorem 3 cannot hold for even n, it is tempting to try to prove a related version using $\mathbb{E} = (0, \ldots, 0, 1, 0, \ldots, 0, 1)$, where the 1's occur in the $(n - k)^{\text{th}}$ and n^{th} places. All efforts to date have failed. What relation, if any, does $\mathscr{L}_{2n}(m)$ have to $\mathscr{L}_n(m)$? The following is a limited answer to that question.

Theorem 6: $2\mathscr{L}_n(m) \leq \mathscr{L}_{2n}(m)$.

Proof: Let $S \in \mathscr{S}_n(m)$ with $L(S) = \mathscr{L}_n(m)$. Then the tuple $S \wedge 0$, where

$$S \wedge 0 = (0, s_1, 0, s_2, 0, s_3, \ldots, 0, s_n)$$

is in $\mathscr{S}_{2n}(m)$. By Theorem 1(ii), $D(S \wedge 0)$ is in a cycle if and only if S is. Further, $D^2(S \wedge 0) = D(S) \wedge 0$. Thus, $L(S \wedge 0) = 2L(S)$. □

Unfortunately, from the few cases studied, it appears that the above inequality is a strict one.

The n-number game has been studied extensively; indeed, many key results keep reappearing in the literature and being reproved. An extensive bibliography appears in [7]. In the interest of completeness, additional references which either do not appear in that article or were published after 1982 are listed below.

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TILING THE k^{th} POWER OF A POWER SERIES

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In this paper we consider the problem of dividing a rectangle into non-overlapping squares and rectangles using recurring-sequence tiling. The results obtained herein are illustrated with appropriate figures. These results, with the exception of basic introductory material, are believed to be new. There seem to be no such results in the literature.

Among the many generating functions possible, we use the following:

(1) $$G(x)^{-k} = 1/(1 - x - x^2 - \cdots - x)$$
(where $m = 2, 3, 4, \ldots$, and $k = 1, 2, 3, \ldots$).

Note that we can write $G(x)^{-k}$ as a power series in x in the form

(2) $$G(x)^{-k} = F_{m,k}(0) + F_{m,k}(1)x + F_{m,k}(2)x^2 + \cdots + F_{m,k}(n)x^n + \cdots$$
(where $F_{m,k}(0) = 1$, for all m and k).

We develop a general construction method for performing the tiling using $k \geq 2$. Our work is an extension of the tiling done in [7] for $k = 1$.

Using (2),

(3) $$G(x)^{-(k-1)} = (1 - x - x^2 - \cdots - x^m)(F_{m,k}(0) + F_{m,k}(1)x + F_{m,k}(2)x^2 + \cdots + F_{m,k}(n)x^n + \cdots).$$

Now, combining coefficients in Equation (3) leads to

(4) $$F_{m,k}(n) = F_{m,k}(n-1) + F_{m,k}(n-2) + \cdots + F_{m,k}(n-m) + F_{m,k-1}(n).$$

The last term of Equation (4) is important. To preserve the geometry of the method of tiling we have used in this paper, it is necessary that

(5) $$F_{m,k-1}(n) < F_{m,k}(n-m),$$

where $F_{m,k-1}(n)$ is the value of the initial tile placed in the construction.

First let $m = 2$; we shall find the sizes of tiles corresponding to values of $F_{2,k}(n)$ for various values of k. In Table 1, we have outlined the value of the initial tiles generated by the necessary condition that

$$F_{m,k-1}(n) < F_{m,k}(n-m)$$

as discussed above. For example, note that 420 < 474, 23109 < 25088, etc.

The values in Table 1 can be used in an example tiling construction for $m = 2$ and $k = 2$, as shown in Figure 2. Note that each shape is a square.

For higher-order constructions, the difficulty lies in choosing the initial tile. We now concern ourselves with finding the value of that required initial tile. A repeated use of Equation (4) above, in example cases of $m = 2$ and $m = 3$, is summarized in Tables 3 and 4.

TABLE 1. Coefficients $F_k(n)$ when $m = 2$

n	k = 1	k = 2	k = 3	k = 4	k = 5
0	1	1	1	1	1
1	1	2	3	4	5
2	2	5	9	14	20
3	3	10	22	40	65
4	5	20	51	105	190
5	8	38	111	256	511
6	13	71	233	594	1295
7	21	130	474	1324	3130
8	34	235	942	2860	7285
9	55	$\boxed{420}$	1836	6020	16435
10	89	744	3522	12402	36122
11	144	1308	6666	25088	77645
12	233	2285	12473	49963	163730
13	377	3970	$\boxed{23109}$	98160	339535
14	610	6865	42447	190570	693835
15	987	11822	77378	366108	1399478
16	1597	20284	140109	696787	2790100
17	2584	34690	252177	$\boxed{1315072}$	5504650
18	4181	59155	451441	2463300	10758050
19	6765	100610	804228	4582600	20845300
20	10946	170711	1426380	8472280	40075630

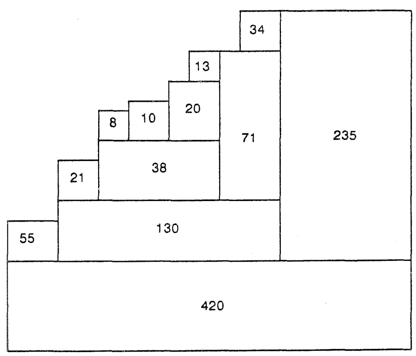

FIGURE 2

TABLE 3. Calculating the Initial Tile for $m = 2$

k = 2	$F_{2,1}(5) < F_{2,2}(3)$
k = 3	$F_{2,2}(9) < F_{2,3}(7)$
k = 4	$F_{2,3}(13) < F_{2,4}(11)$
k = 5	$F_{2,4}(17) < F_{2,5}(15)$
k = 6	$F_{2,5}(21) < F_{2,6}(19)$

TABLE 4. Calculating the Initial Tile for $m = 3$

k = 2	$F_{3,1}(12) < F_{3,2}(9)$
k = 3	$F_{3,2}(23) < F_{3,3}(20)$
k = 4	$F_{3,3}(34) < F_{3,4}(31)$
k = 5	$F_{3,4}(45) < F_{3,5}(42)$
k = 6	$F_{3,5}(56) < F_{3,6}(53)$

Surveying the values above suggests the following relationships. Define $Q_{m,k}$ to be the smallest number such that

$$F_{m,k-1}(Q_{m,k} + m) < F_{m,k}(Q_{m,k}),$$

and let $Q_m = Q_{m,1}$. Let $Q_0 = Q_1 = 1$. Then from Table 3, where $m = 2$, we observe that

(6) $\quad Q_2 = 3Q_1 - 2Q_0 + 2.$

Further, we observe from Table 4, where $m = 3$, that

(7) $\quad Q_3 = 3Q_2 - 2Q_1 + 2.$

We then generalize that

(8) $\quad Q_m = 3Q_{m-1} - 2Q_{m-2} + 2.$

By elementary means we find that Equation (8) can be stated as

(9) $\quad Q_m = 2^{m+1} - 2m - 1.$

Then $F_{m,k}(Z_m)$ is the initial tile I, where $I = Z_m = 2^{m+1} - m - 1$, for $m \geq 2$.

An examination of Tables 3 and 4 will show that a pattern emerges as k changes and one looks for a value of n which will result in the next initial tile value. As k changes by one, the value of n changes by a constant amount. That constant is equal to

(10) $\quad P_m = 2^{m+1} - m - 2.$

It can be shown inductively, step by step, that the values of the initial tiles are

(11) $\quad F_{m,k-1}[Z_m + (k-2)P_m] < F_{m,k}[Q_m + (k-2)P_m]$, where $m \geq 2$ and $k \geq 1$.

Then, for example, the next tile values in Table 3, using (11), are seen to be

$$F_{2,6}(25) < F_{2,7}(23).$$

Finally, we now show the general case of placing tiles in the construction. We place the sets of squares in the order Set 1, Set 2, etc., where the Sets are defined as

(12) Set 1: $F_{m,k}(n) = F_{m,k}(n-1) + F_{m,k}(n-2) + \cdots + F_{m,k}(n-m) + F_{m,k-1}(n)$

(13) Set 2: $F_{m,k}(n+m-1) = F_{m,k}(n+m-2) + F_{m,k}(n+m-3) + \cdots + F_{m,k}(n-1) + F_{m,k-1}(n+m-1)$

(14) Set 3: $F_{m,k}(n+2m-2) = F_{m,k}(n+2m-3) + F_{m,k}(n+2m-4) + \cdots + F_{m,k}(n+m-2) + F_{m,k-1}(n+2m-2)$

...

(15) Set $j+1$: $F_{m,k}(n + j(m-1)) = F_{m,k}(n-1+j(m-1)) + F_{m,k}(n-2+j(m-1)) + \cdots + F_{m,k}(n+m+j(m-1)) + F_{m,k-1}(n+j(m-1))$,

where $j = 0, 1, 2, \ldots$.

Using this method of set placement, the final general construction will look like Figure 5 below. Note that all shapes are squares, and that as of yet we have not fully tiled the rectangle; there are still rectangular gaps in the construction.

We now proceed to find the filler rectangles used to fill in the gaps left after placing the square tiles. We note first that the general coefficients $F_{m,k}(n)$ may be listed as follows, where $m \geq 2$, $k \geq 1$, and $n \geq 0$.

(16)
$F_{m,k}(0)$
$F_{m,k}(1)$
$F_{m,k}(2)$
...
$F_{m,k}(I)$
...

where $F_{m,k}(I) = F_{m,k}(I-1) + F_{m,k}(I-2) + \cdots + F_{m,k}(I-m+1) + F_{m,k}(I-m) + F_{m,k-1}(I)$; $I = Z_m + (k-1)P_m$; $F_{m,0}(I) = 0$, $F_{m,k}(0) = 1$, and $F_{m,k}(1) = k$.

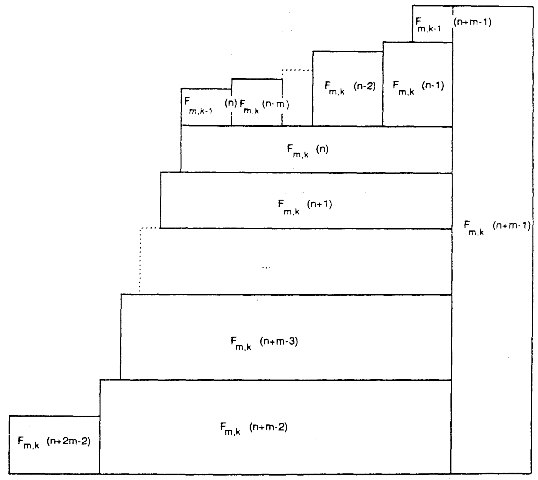

FIGURE 5

Note that I is the initial tile value, and it is evident that the value of I depends on the values of k and m.

First we examine the case of $m = 2$ and $k = 2$ [in other words, $1/(1 - x - x^2)^2$]. Figure 2 may now be redrawn using function notation rather than actual numbers, and showing the gap rectangles. We use the notation H and V to denote filler rectangles that appear to be oriented horizontally and vertically, respectively.

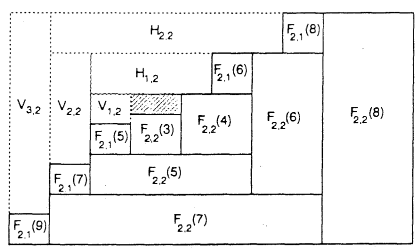

FIGURE 6

One can see the following rectangle sizes from Figure 6.

$$
\begin{aligned}
(17)\qquad & H_{1,k}[F_{2,2}(5) - F_{2,1}(6),\ F_{2,1}(6)] \\
& H_{2,k}[F_{2,2}(7) - F_{2,1}(8),\ F_{2,1}(8)] \\
& \ldots \\
& H_{n,k}[F_{2,2}(2n+5) - F_{2,1}(2n+6),\ F_{2,1}(2n+6)] \\
& V_{1,k}[F_{2,1}(5),\ F_{2,2}(4) - F_{2,1}(5)] \\
& V_{2,k}[F_{2,1}(7),\ F_{2,2}(6) - F_{2,1}(7)] \\
& \ldots \\
& V_{n,k}[F_{2,1}(2n+3),\ F_{2,2}(2n+2) - F_{2,1}(2n+5)], \\
& \text{where } n \geq 0,\ m = 2, \text{ and } k = 2.
\end{aligned}
$$

We can generalize this idea for any k. To use the I notation, Figure 6 may be redrawn as follows:

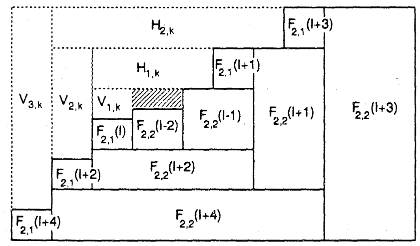

FIGURE 7

Furthermore, (17) can be rewritten as

(18) $H_{1,k}[F_{2,k}(I) - F_{2,k}(I+1), F_{2,k-1}(I+1)]$
$H_{2,k}[F_{2,k}(I+2) - F_{2,k-1}(I+3), F_{2,k-1}(I+3)]$
...
$H_{n,k}[F_{2,k}(2n+I) - F_{2,k-1}(2n+I+1), F_{2,k-1}(2n+I+1)$
$V_{1,k}[F_{2,k-1}(I), F_{2,k}(I-1) - F_{2,k-1}(I)]$
$V_{2,k}[F_{2,k-1}(I+2), F_{2,k}(I+1) - F_{2,k-1}(I+2)]$
...
$V_{n,k}[F_{2,k-1}(2n+I-2), F_{2,k}(2n+I-3) - F_{2,k-1}(2n+I-2)$,
where $n \geq 0$, $m = 2$, and $k \geq 2$.

We now show an example for $m > 2$, in particular for $m = 3$, $k = 2$. In this case, the tiling construction begins with the framework shown in Figure 8, using (4) for recursion and Table 4 to determine the initial tile.

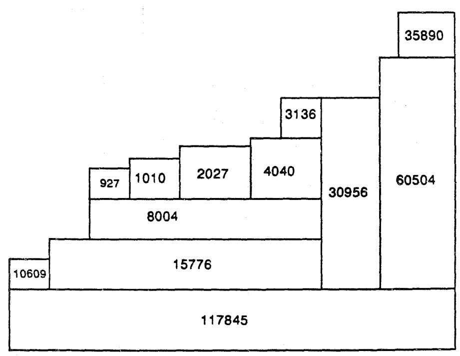

FIGURE 8

The filler rectangles are formed in a similar manner as the case of $m = 2$, except an alternating pattern containing two different constructions is formed. The complete construction for $k = 2$, $m = 3$ is shown in Figure 9. This same pattern is followed for any k, where only the value of the arguments of the function F is changed as influenced by the change in the initial tile.

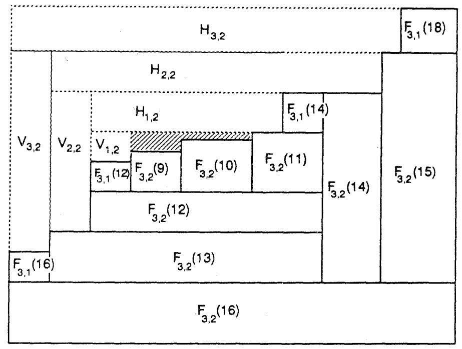

FIGURE 9

One can see the following rectangle sizes from Figure 9:

(19) $H_{1,3}[F_{3,2}(12) - F_{3,1}(14),\ F_{3,1}(14)]$
$H_{2,2}[F_{3,2}(13) + F_{3,2}(14),\ F_{3,2}(15) - F_{3,2}(14)]$
$\ldots$
$V_{1,2}[F_{3,1}(12),\ F_{3,2}(11) - F_{3,1}(12)]$
$V_{2,2}[F_{3,2}(13) - F_{3,2}(12),\ F_{3,2}(14) - F_{3,2}(13)]$
$\ldots$.

The pattern of the pair of equations in both the horizontal and vertical rectangles is repeated. These formulas would also be valid for any value of k, not just $k = 2$.

We can now generalize this idea for any m. The pattern is repeated every $m - 1$ rectangles. Therefore, the difference in the argument where this pattern repeats is $2m - 2$. Figure 10 shows the general construction.

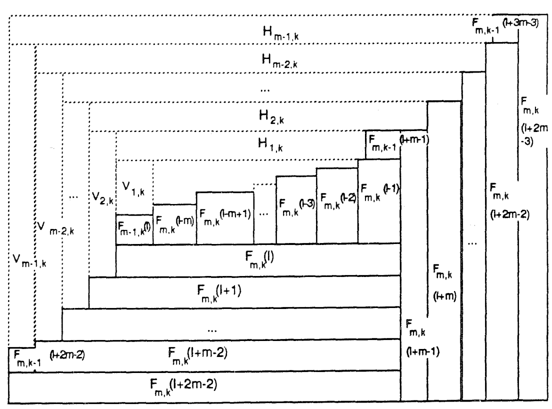

FIGURE 10

The general formulas for the horizontal and vertical rectangles are:

(20) $H_{1,k}[F_{m,k}(I) - F_{m,k-1}(I + m - 1),\ F_{m,k-1}(I + m - 1)]$
$H_{2,k}[F_{m,k}(I + 1) + F_{m,k}(I + m - 1),\ F_{m,k}(I + m) - F_{m,k}(I + m - 1)]$
$H_{3,k}[F_{m,k}(I + 2) + F_{m,k}(I + m - 1) + F_{m,k}(I + m),$
$\quad F_{m,k}(I + m + 1) - F_{m,k}(L + m)]$
$\ldots$
$H_{m-1,k}[F_{m,k}(I + m - 2) + F_{m,k}(I + m - 1) + F_{m,k}(I + m)$
$\quad + \cdots + F_{m,k}(I + 2m - 4),\ F_{m,k}(I + 2m - 3) - F_{m,k}(I + 2m - 4)]$

$$
\begin{aligned}
&\ldots\\
&V_{1,k}[F_{m,k-1}(I),\ F_{m,k}(I-1) - F_{m,k-1}(I)]\\
&V_{2,k}[F_{m,k}(I+1) - F_{m,k}(I),\ F_{m,k}(I+m-1) - F_{m,k}(I+m-2)\\
&\qquad - \cdots - F_{m,k}(m+2) - F_{m,k}(m+1)]\\
&V_{3,k}[F_{m,k}(I+2) - F_{m,k}(I+1),\ F_{m,k}(I+m) - F_{m,k}(I+m-2)\\
&\qquad - F_{m,k}(I+m-3) - \cdots - F_{m,k}(I+2)]\\
&\ldots\\
&V_{m-2,k}[F_{m,k}(I+m-3) - F_{m,k}(I+m-4),\\
&\qquad\quad F_{m,k}(I+2m-5) - F_{m,k}(I+m-2) - F_{m,k}(I+m-3)]\\
&V_{m-1,k}[F_{m,k}(I+m-2) - F_{m,k}(I+m-3),\\
&\qquad\quad F_{m,k}(I+2m-4) - F_{m,k}(I+m-2)].
\end{aligned}
$$

Then the pattern of $m - 1$ rectangles repeats with the argument (I) incremented by $2m - 2$.

This construction, shown in Figure 10, generalizes the recurring-sequence tiling for any k and m using the extended Fibonacci numbers generated with (4). The initial tile is selected by the criterion of (5) and the horizontal and vertical filler rectangles have the dimensions described in (20).

Acknowledgments

The authors wish to thank Colonel David H. Cameron, Professor Paul Erdös, and Professor M. V. Subbarao for their interest and encouragement in the preparation of this paper.

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*****

ON A HOGGATT-BERGUM PAPER WITH TOTIENT FUNCTION APPROACH FOR DIVISIBILITY AND CONGRUENCE RELATIONS

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(Submitted August 1988)

During their discussion of divisibility and congruence relations of the Fibonacci and Lucas numbers, Hoggatt & Bergum found values of n satisfying the congruences $F_n \equiv 0 \pmod{n}$ or $L_n \equiv 0 \pmod{n}$. In this connection, Hoggatt & Bergum's research appears in Theorems 1, 3, 5, 6, and 7 of [4]. The present paper originated on the same lines in search of values of n that satisfy $\phi(F_n) \equiv 0 \pmod{n}$ or $\phi(L_n) \equiv 0 \pmod{n}$, where ϕ is the totient function. Before going into the analysis of the problem, we state some results that will be quoted frequently.

(a) (i) For $n > 2$, $\phi(n)$ is even; (ii) if $m|n$, then $\phi(m)|\phi(n)$ ([3], pp. 140-41).
(b) All odd prime divisors of L_{2n+1} are of the form $10m \pm 1$ ([4], p. 193).
(c) Every F_n with $n > 12$ and L_n with $n > 6$ has at least one primitive prime divisor ([6], p. 15).
(d) Let p be a primitive prime divisor of F_n $(n > 5)$; if $n \equiv 5 \pmod{10}$, then $p \equiv 1 \pmod{4n}$ ([6], p. 10).
(e) A primitive prime divisor p of L_{5n} with $n \geq 1$ satisfies $p \equiv 1 \pmod{10n}$ ([6], p. 11).
(f) Let n be odd and p an odd primitive prime divisor of F_n; if $p \equiv \pm 1$ (mod 10), then $p \equiv 1 \pmod{4n}$ ([1], p. 254).
(g) Let p be an odd primitive prime divisor of L_n; if $p \equiv \pm 1 \pmod{10}$, then $p \equiv 1 \pmod{2n}$ ([1], p. 255).

We begin our discussion by proving the following theorem.

Theorem 1: If n is an odd integer greater than 3, then
(i) $\phi(L_n) \equiv 0 \pmod{n}$;
(ii) $\phi(F_{2n}) \equiv 0 \pmod{2n}$.

Proof: Both results are true when $n = 5$. Thus, we choose $n \geq 7$.

(i) Based on (b) and (c), we have the existence of at least one primitive prime divisor p of L_n of the form $10m \pm 1$. Consequently, by (g):

(1) $p \equiv 1 \pmod{2n}$.

Since $p|L_n$ and $\phi(p) \equiv 0 \pmod{2n}$ is true from (1), we have, using (a),

$$\phi(L_n) \equiv 0 \pmod{2n} \Rightarrow \phi(L_n) \equiv 0 \pmod{n}$$

(ii) Since $F_{2n} = F_n L_n$ and $\phi(L_n) \equiv 0 \pmod{2n}$, we have $2n|\phi(F_{2n})$.

Note: From the above, with odd $n > 3$, $\phi(L_n) \equiv 0 \pmod{2n}$ and $\phi(F_{2n}) \equiv 0 \pmod{4n}$ are both true. The second part follows from [5].

Corollary: If n is odd, $n > 3$, and $3|n$, then $4n|\phi(L_n)$.

Proof: By Lemma 1 of [4] (p. 193), $4|L_n$. From Theorem 1, $p|L_n$, where $p \equiv 1 \pmod{2n}$; consequently, by (a), $\phi(4p)|\phi(L_n)$. This proves our result.

In regard to Fibonacci numbers with even subscripts, we prove the following theorem.

Theorem 2: The congruence $\phi(F_{2N}) \equiv 0 \pmod{2N}$ is true for all positive integers N, except when $N = 1, 2, 3, 4, 8, 16$.

Proof: It is easy to verify that, for $N = 1, 2, 4, 8, 16$, the congruence $\phi(F_{2N}) \equiv N \pmod{2N}$ holds and, for $N = 3$, the result $\phi(F_6) \equiv 4 \pmod 6$ is true. Excluding these values, we complete the proof by considering the following cases:

<u>Case 1</u>. If N is odd and greater than 3, the result follows from Theorem 1.

<u>Case 2</u>. For even values of N, we discuss the proof in two parts:

<u>Part 1</u>. Let $N = 2^{n-1}$, $n \geq 6$. For $n = 6$, the result

$$\phi(F_{64}) \equiv 0 \pmod{64}$$

is true (see [1], App. Table). For $n > 6$, we apply induction on n.

$\phi(F_{2^n}) \equiv 0 \pmod{2^n}$ is true by inductive hypothesis.

$\phi(L_{2^n}) \equiv 0 \pmod 2$ is true by (a), and

$\phi(F_{2^{n+1}}) = \phi(F_{2^n})\phi(L_{2^n})$; therefore,

$\phi(F_{2^{n+1}}) \equiv 0 \pmod{2^{n+1}}$.

<u>Part 2</u>. Let $N = 2^{n-1}t$, where $n \geq 1$, t is odd, $t > 1$, and $2N \neq 6$. If $n = 1$, see Theorem 1. If $n > 1$, we use induction on n.

$F_{2^{n+1}\cdot t} = F_{2^n\cdot t}L_{2^n\cdot t}$; $L_{2^n} | L_{2^n\cdot t}$ and L_{2^n} & $F_{2^n\cdot t}$ are relatively prime; therefore,

$\phi(F_{2^n\cdot t}) \cdot \phi(L_{2^n})$ divides $\phi(F_{2^{n+1}\cdot t})$.

Repeating the argument of Part 1 above, we observe that

$\phi(F_{2^n\cdot t}) \equiv 0 \pmod{2^n\cdot t}$

is true by the inductive hypothesis,

$\phi(L_{2^n}) \equiv 0 \pmod 2$

follows from (a). Hence, the proof is complete.

For examination of Lucas numbers with even subscripts, it is important to study the values of $\phi(L_{2^n})$. By verification, it follows that $\phi(L_{2^n}) \equiv 0 \pmod{2^n}$ is true when $n = 1, 5, 6, 7, 8$ and false for $n = 2, 3, 4$. It remains an open question whether $\phi(L_{2^n}) \equiv 0 \pmod{2^n}$ would be true for all $n \geq 5$ or for infinitely many n or only for a finite number n.

Since $2^n \equiv 0 \pmod 4$ is true for $n \geq 5$, every odd prime divisor of L_{2^n} is one of the forms $40m + 1$, $40m + 7$, $40m + 9$, $40m + 23$ ([6], p. 11). In [7] it is proved that $L_{2^n} \equiv 3 \pmod 4$ and, hence, contains an odd number of primes of the form $4m + 3$.

In view of this, we conclude that, if L_{2^n} is the product of an even number of primes, then it must contain at least one prime p of the type $40m + 1$ or $40m + 9$. If this prime p is primitive, then $p \equiv 1 \pmod{2^{n+1}}$ by (g). In this case, $2^n | \phi(p)$ and, consequently, $\phi(L_{2^n}) \equiv 0 \pmod{2^n}$ is true.

Based on this discussion, we are led to make the following conjecture.

Conjecture: There may exist infinitely many n such that $\phi(L_{2^n}) \equiv 0 \pmod{2^n}$ is true.

It is interesting to note that the following allied result holds.

Theorem 3: For all positive integers n, $\phi(L_{2^n} + 1) \equiv 0 \pmod{2^n}$ is true.

Proof: Using the Binet form, it is easy to see that

$$L_{2^{n+1}} + 1 = (L_{2^n} + 1)(L_{2^n} - 1).$$

Since $(L_{2^n} - 1)$ is always even, it follows by induction that $L_{2^n} + 1 \equiv 0 \pmod{2^{n+1}}$ is always true. Hence, $2^n = \phi(2^{n+1})$ divides $\phi(L_{2^n} + 1)$.

Some special cases are discussed in the following theorems.

Theorem 4: If n is a positive integer, the following congruences hold:

(i) $\phi(L_{5n}) \equiv 0 \pmod{10n}$.

(ii) $\phi(F_{5n}) \equiv 0 \pmod{80n}$; n is odd, $n > 1$.

Proof:

(i) The proof follows from (c) and (e) and the fact that $\phi(L_5) \equiv 0 \pmod{10}$ when $n = 1$.

(ii) Since, for odd n, $5n \equiv 5 \pmod{10}$ is true when $n > 1$, by (c) and (d) there exists a primitive prime divisor p of F_{5n} satisfying $p \equiv 1 \pmod{20n}$. Since 5 and p are both relatively prime factors of F_{5n}, $80n$ divides $\phi(F_{5n})$ by (a).

Theorem 5:

(i) If $k \geq 2$, then $\phi(F_{5^k}) \equiv 0 \pmod{16 \cdot 5^{2k-1}}$.

(ii) If $n = 2^{r+1} \cdot 3^m \cdot 5^k$ with $r \geq 1$, $m \geq 1$, $k \geq 1$, then $\phi(F_n) \equiv 0 \pmod{4n}$.

Proof:

(i) From [4], p. 192, we have $5^k | F_{5^k}$. Since $5^k \equiv 5 \pmod{10}$ is true, by (d) there exists a primitive prime divisor p satisfying $p \equiv 1 \pmod{4.5^k}$. As 5^k and p are relatively prime, $\phi(5^k) \cdot \phi(p)$ divides $\phi(F_k)$.

(ii) By [4], p. 192, we have $n | F_n$. This, along with Theorem 2, completes the proof.

A Final Note: It is desirable to shed some light on the cases not discussed thus far and on the difficulties encountered in the generalization process. This is done by showing that the following two congruences are not valid in general for a positive integer n.

(i) $\phi(L_{2n}) \equiv 0 \pmod{2n}$

(ii) $\phi(F_{2n+1}) \equiv 0 \pmod{2n + 1}$

In regard to (i), we observe that if, for a composite m, L_m is prime, then m must be of the form 2^t, where $t \geq 2$ (see [2]). Consequently, with $t \geq 2$ when L_{2^t} is prime, which is primitive, we have $\phi(L_{2^t}) = L_{2^t} - 1$.

As proved in Theorem 3, $L_{2^t} \equiv -1 \pmod{2^t}$; therefore, we can conclude that $\phi(L_{2^t}) \equiv -2 \pmod{2^t}$. Thus, it follows that $\phi(L_{2^t}) \not\equiv 0 \pmod{2^t}$ when L_{2^t} is prime and $t \geq 2$. Besides this, there may exist other Lucas numbers connected with this which may not satisfy the congruence of (i). One such illustration will be the members of the type $L_{2^t \cdot p}$, where p is an odd prime and $t \geq 1$. We observe that, for $n < 50$, $\phi(L_{2n}) \not\equiv 0 \pmod{2n}$ when $n = 2, 4, 8, 11, 12, 17, 26, 29, 37, 46$. In view of this, we conclude that the congruence relation in (i) is not true in general.

For case (ii), we observe that for odd n, if F_n is prime p, where $p \equiv \pm 3 \pmod{10}$, then $p \equiv (2n - 1) \pmod{4n}$ (see [1], p. 254).

Thus, under this hypothesis of primality, $\phi(F_n) \equiv -2 \pmod{n}$. It is easy to see that F_n is a prime of this type when $n = 7, 13, 17, 23, 43, 47, 83$. It is interesting to observe that if, for a prime subscript n, F_n is the product of two primitive primes each $\equiv \pm 3 \pmod{10}$, then $\phi(F_n) \equiv 4 \pmod{n}$. This is true when $n = 59, 61, 71, 79, 101, 109$.

Based on this, there may exist Fibonacci numbers of odd subscripts n, where n is composite, which may also not satisfy relation (ii). One such example is F_{161}, where $\phi(F_{161}) \equiv 16 \pmod{161}$. Consequently, we are justified to say that the congruence relation of (ii) is also not true in general.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n, satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-670 *Proposed by Russell Euler, Northwest Missouri State U., Marysville, MO*

Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{nF_n}{2^n}$.

B-671 *Proposed by Herta T. Freitag, Roanoke, VA*

Show that all even perfect numbers are hexagonal and hence are all triangular. [A perfect number is a positive integer which is the sum of its proper positive integral divisors. The hexagonal numbers are {1, 6, 15, 28, 45, ...} and the triangular numbers are {1, 3, 6, 10, 15, ...}.]

B-672 *Proposed by Philip L. Mana, Albuquerque, NM*

Let S consist of all positive integers n such that $n = 10p$ and $n + 1 = 11q$, with p and q primes. What is the largest positive integer d such that every n in S is a term in an arithmetic progression a, $a + d$, $a + 2d$, ...?

B-673 *Proposed by Paul S. Bruckman, Edmonds, WA*

Evaluate the infinite product $\displaystyle\prod_{n=2}^{\infty} \frac{F_{2n} + 1}{F_{2n} - 1}$.

B-674 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Define the sequence $\{u_n\}$ by

$$u_0 = 0,\ u_1 = 1,\ u_n = gu_{n-1} - u_{n-2},\ \text{for } n \text{ in } \{2, 3, \ldots\},$$

where g is a root of $x^2 - x - 1 = 0$. Compute u_n for n in {2, 3, 4, 5} and then deduce that $(1 + \sqrt{5})/2 = 2\cos(\pi/5)$ and $(1 - \sqrt{5})/2 = 2\cos(3\pi/5)$.

<u>B-675</u> *Proposed by Richard André-Jeannin, Sfax, Tunisia*

In a manner analogous to that for the previous problem, show that

$$\sqrt{2+\sqrt{2}} = 2\cos\frac{\pi}{8} \quad \text{and} \quad \sqrt{2-\sqrt{2}} = 2\cos\frac{3\pi}{8}.$$

SOLUTIONS

Not True Asymptotically

<u>B-645</u> *Proposed by R. Tošić, U. of Novi Sad, Yugoslavia*

Let $G_{2m} = \binom{2m-1}{m} - 2\binom{2m-1}{m-3} + \binom{2m}{m-5}$ for $m = 1, 2, 3, \ldots,$

$$G_{2m+1} = \binom{2m}{m} - \binom{2m+1}{m-2} + 2\binom{2m}{m-5} \qquad \text{for } m = 0, 1, 2, \ldots,$$

where $\binom{n}{k} = 0$ for $k < 0$. Prove or disprove that $G_n = F_n$ for $n = 0, 1, 2, \ldots$.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Let us study the asymptotic growth of G_n. It is evident that

$$G_{2m} \sim \binom{2m-1}{m} \quad \text{and} \quad G_{2m+1} \sim \binom{2m}{m}.$$

Hence,

$$\frac{G_{2m+1}}{G_{2m}} \sim \frac{\binom{2m}{m}}{\binom{2m-1}{m}} = \frac{(2m)!}{m!m!}\cdot\frac{m!(m-1)!}{(2m-1)!} = 2$$

and

$$\frac{G_{2m+2}}{G_{2m+1}} \sim \frac{\binom{2m+1}{m+1}}{\binom{2m}{m}} = \frac{(2m+1)!}{(m+1)!m!}\cdot\frac{m!m!}{(2m)!} = \frac{2m+1}{m+1} \sim 2,$$

so that $G_n/G_{n-1} \sim 2$. However, it is well known that

$$F_n/F_{n-1} \sim (1+\sqrt{5})/2.$$

Thus, $G_n \neq F_n$ for sufficiently large n. In fact, from numerical computations, we have $G_n = F_n$ for $0 < n \le 14$, and $G_n > F_n$ for $n \ge 15$.

Also solved by Charles Ashbacher, Paul S. Bruckman, James E. Desmond, Piero Filipponi, L. Kuipers, and the proposer.

Triangular Number Analogue

<u>B-646</u> *Proposed by A. P. Hillman in memory of Gloria C. Padilla*

We know that $F_{2n} = F_nL_n = F_n(F_{n-1} + F_{n+1})$. Find m as a function of n so as to have the analogous formula $T_m = T_n(T_{n-1} + T_{n+1})$, where T_n is the triangular number $n(n+1)/2$.

Solution by H.-J. Seiffert, Berlin, Germany

We have: $T_n(T_{n-1} + T_{n+1}) = T_n(T_n - n + T_n + n + 1) = T_n(2T_n + 1)$

$$= n(n+1)(n(n+1)+1)/2 = T_{n(n+1)}.$$

Also solved by Richard André-Jeannin, Wray G. Brady, Paul S. Bruckman, Nicos D. Diamantis, Russell Euler, Piero Filipponi, Herta T. Freitag, Russell Jay Hendel, L. Kuipers, Jack Lee, Carl Libis, Bob Prielipp, Jesse Nemoyer & Joseph J. Kostal & Durbha Subramanyam, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

Much Ado about Zero

B-647 *Proposed by L. Kuipers, Serre, Switzerland*

Simplify

$$[L_{2n} + 7(-1)^n][L_{3n+3} - 2(-1)^n L_n] - 3(-1)^n L_{n-2}L_{n+2}^2 - L_{n-2}L_{n-1}L_{n+2}^3.$$

Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA

The given expression simplifies to zero. By using the Binet form of Lucas numbers, it follows that $L_{2n} + 7(-1)^n = L_{n-2}L_{n+2}$. In view of this, the given expression is

$$L_{n-2}L_{n+2}[L_{3n+3} - 2(-1)^n L_n - (L_{n+2}^2 L_{n-1} + 3(-1)^n L_{n+2})].$$

Again, applying the Binet form of Lucas numbers, we see that

$$L_{n+2}^2 L_{n-1} + 3(-1)^n L_{n+2} = L_{3n+3} - 2(-1)^n L_n.$$

Hence, the required conclusion follows.

Also solved by Paul S. Bruckman, Herta T. Freitag, Y. H. Harris Kwong, Carl Libis, Bob Prielipp, H.-J. Seiffert, M. Wachtel, Gregory Wulczyn, and the proposer.

Pell Primitive Pythagorean Triples

B-648 *Proposed by M. Wachtel, Zurich, Switzerland*

The Pell numbers P_n and Q_n are defined by

$$P_{n+2} = 2P_{n+1} + P_n,\ P_0 = 0,\ P_1 = 1;\ Q_{n+2} = 2Q_{n+1} + Q_n,\ Q_0 = 1 = Q_1.$$

Show that $(P_{4n}, P_{2n}^2 + 1, 3P_{2n}^2 + 1)$ is a primitive Pythagorean triple for n in $\{1, 2, \ldots\}$.

Solution by Paul S. Bruckman, Edmonds, WA

The Pell numbers satisfy the following identities:

(1) $\quad 2P_{2n}Q_{2n} = P_{4n}$;

(2) $\quad Q_{2n}^2 - 2P_{2n}^2 = 1$.

Hence,

(3) $\quad Q_{2n}^2 - P_{2n}^2 = P_{2n}^2 + 1$.

It is known that primitive Pythagoren triples are generated by

(4) $\quad (2ab, a^2 - b^2, a^2 + b^2)$, where g.c.d. $(a, b) = 1$.

We may let $a = Q_{2n}$, $b = P_{2n}$. We see from (2) that g.c.d. $(a, b) = 1$. Also

$2ab = P_{4n}$ [using (1)];

$a^2 - b^2 = P_{2n}^2 + 1$ [using (3)];

and

$$a^2 + b^2 = 3P_{2n}^2 + 1 \quad \text{[adding } 2b^2 \text{ to both sides of (3)]}.$$

This proves the assertion.

Also solved by Nicos D. Diamantis, Ernest J. Eckert, Russell Euler, Piero Filipponi, Herta T. Freitag, Russell Jay Hendel, L. Kuipers, Jesse Nemoyer & Joseph J. Kostal & Durbha Subramanyam, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

Sides Differing by 17

B-649 *Proposed by M. Wachtel, Zurich, Switzerland*

Give a rule for constructing a sequence of primitive Pythagorean triples (a_n, b_n, c_n) whose first few triples are in the table

n	1	2	3	4	5	6	7	8
a_n	24	28	88	224	572	1248	3276	7332
b_n	7	45	105	207	555	1265	3293	7315
c_n	25	53	137	305	797	1777	4645	10357

and which satisfy

$$|a_n - b_n| = 17,$$

$$a_{2n-1} + a_{2n} = 26P_{2n} = b_{2n-1} + b_{2n},$$

and $$c_{2n-1} + c_{2n} = 26Q_{2n}.$$

[P_n and Q_n are the Pell numbers of B-648.]

Rule by Paul S. Bruckman, Edmonds, WA

$$(a_{2n-1}, b_{2n-1}, c_{2n-1})$$
$$= (10P_n^2 + 26P_nQ_n - 12Q_n^2, -24P_n^2 + 26P_nQ_n + 5Q_n^2, 26P_n^2 - 14P_nQ_n + 13Q_n^2),$$

$$(a_{2n}, b_{2n}, c_{2n})$$
$$= (-10P_n^2 + 26P_nQ_n + 12Q_n^2, 24P_n^2 + 26P_nQ_n - 5Q_n^2, 26P_n^2 + 14P_nQ_n + 13Q_n^2).$$

Rule by Ernest J. Eckert, U. of South Carolina, Aiken, SC

Let $(a_1, b_1, c_1) = (24, 7, 25)$, $(a_2, b_2, c_2) = (28, 45, 53)$ and A denote the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

Then $[a_{2n-1} \quad b_{2n-1} \quad c_{2n-1}]$ is the matrix product $[a_1 \quad b_1 \quad c_1]A^{n-1}$ and

$$[a_{2n} \quad b_{2n} \quad c_{2n}] = [a_2 \quad b_2 \quad c_2]A^{n-1}.$$

Editor's note: The derivations and proofs given by Bruckman and Eckert are not included because of space limitations; however, since each term in the required equations satisfies the same 3rd order linear homogeneous recursion

$$w_{n+3} = 5(w_{n+2} + w_{n+1}) - w_n,$$

it suffices to verify the rules for $n = 1, 2,$ and 3.

Also solved by Gregory Wulczyn and the proposer.

Average Age of Generalized Rabbits

B-650 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome Italy & David Singmaster, Polytechnic of the South Bank, London, UK*

Let us introduce a pair of 1-month-old rabbits into an enclosure on the first day of a certain month. At the end of one month, rabbits are mature and each pair produces $k - 1$ pairs of offspring. Thus, at the beginning of the second month there is 1 pair of 2-month-old rabbits and $k - 1$ pairs of 0-month-olds. At the beginning of the third month, there is 1 pair of 3-month-olds, $k - 1$ pairs of 1-month-olds, and $k(k - 1)$ pairs of 0-month-olds. Assuming that the rabbits are immortal, what is their average age A_n at the end of the n^{th} month? Specialize to the first few values of k. What happens as $n \to \infty$?

Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA

If A_i denotes the average age at the end of the i^{th} month, then we have the following recurrence relation:

$$A_{i+1} = \frac{1}{k}(1 + A_i), \text{ where } A_1 = \frac{2}{k};\ k > 1.$$

Using this, we conclude that

$$A_n = \frac{1}{k^n}\left(1 + \sum_{i=0}^{n-1} k^i\right) = \frac{k^n + k - 2}{k^n(k - 1)}$$

Thus,

$$A_2 = \frac{k + 2}{k^2};\quad A_3 = \frac{k^2 + k + 2}{k^3}, \text{ etc.}$$

$$\operatorname*{Limit}_{n \to \infty} A_n = \frac{1}{k - 1}.$$

Also solved by Paul S. Bruckman and the proposers.

Multiples of a Prime p

B-651 *Proposed by L. Van Hamme, Vrije Universiteit, Brussels, Belgium*

Let $u_0, u_1, \ldots$ be defined by $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = u_{n+1} - u_n$. Also let p be a prime greater than 3, and for n in $X = \{1, 2, \ldots, p - 1\}$, let n^{-1} denote the v in X with $nv \equiv 1 \pmod p$. Prove that

$$\sum_{n=1}^{p-1} (n^{-1}u_{n+k}) \equiv 0 \pmod p$$

for all nonnegative integers k.

Solution by the proposer.

Let ρ be a zero of $1 + X + X^2$. Hence, $\rho^3 = 1$. Since

$$\begin{aligned}(1 + \rho)^p - 1 - \rho^p &= -\rho^{2p} - 1 - \rho^p \\ &= -(\rho^2 + 1 + \rho) = 0 \qquad \text{if } p \equiv 1 \pmod 3 \\ &= -(\rho^{-2} + 1 + \rho^{-1}) = 0 \quad \text{if } p \equiv -1 \pmod 3,\end{aligned}$$

ρ is also a zero of $(1 + X)^p - 1 - X$. Hence,

$$\sum_{n=1}^{p-1}\binom{p}{n}\rho^n = 0, \quad \sum_{n=1}^{p-1}\binom{p}{n}\rho^{-n} = 0.$$

Multiplying the first equation with ρ^k, the second with ρ^{-k}, and using the easily verified formula

$$u_n = \frac{(-1)^{n-1}}{\sqrt{-3}}(\rho^n - \rho^{-n}),$$

we get

$$\sum_{n=1}^{p-1}(-1)^{n-1}u_{n+k}\binom{p}{n} = 0.$$

Dividing by p and using

$$\frac{1}{p}\binom{p}{n} \equiv \frac{(-1)^{n-1}}{n} \pmod{p}, \quad 1 \le n \le p-1,$$

we get the assertion.

Also solved by Paul S. Bruckman.

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(continued from page 288)

$Z_i(t)$ represents the number of zeros of f_t which are ε-close to η_i. By invariance of the complex integral, the functions $Z_i(t)$ are constant since the functions f_t vary continuously and do not vanish on the path of integration. Hence, $Z_i(0) = Z_i(1)$ for each i. This says that in a small neighborhood of each zero of f_1, there is a one-to-one correspondence of zeros of f_1 with zeros of f_0, in the required manner. □

In the case of our given functions, we find that the zeros of the polynomial $f_n(z)$ are close to the zeros of $g_n(z)$, which lie on the circle $|z| = \alpha$, as required, and the zeros of f_n get closer to the circle as $n \to \infty$. ■

Also solved by P. Bruckman, O. Brugia & P. Filipponi, L. Kuipers, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-443 *Proposed by Richard Andre-Jeannin, Sfax, Tunisia*

Let us consider the recurrence

$$w_n = mw_{n-1} + w_{n-2},$$

where $m > 0$ is an integer and U_n, V_n the solutions defined by

$$U_0 = 0,\ U_1 = 1;\ V_0 = 2,\ V_1 = m.$$

Show that, if q is an odd divisor of $m^2 + 1$, then

$$V_q \equiv m \pmod q.$$

H-444 *Proposed by H.-J. Seiffert, Berlin, Germany*

Show that, for $n = 0, 1, 2, \ldots,$

$$F_n = \sum_{\substack{k=0 \\ (5,\, n-2k)=1}}^{[n/2]} (-1)^{[(n-2k+2)/5]} \binom{n}{k},$$

where (r, s) denotes the greatest common divisor of r and s and [ ] the greatest integer function.

H-445 *Proposed by Paul S. Bruckman, Edmonds, WA*

Establish the identity:

$$(1)\qquad \sum_{n=1}^{\infty} \mu(n)\left(\frac{z^n}{1 - z^{2n}}\right) = z - z^2,\ z \in \mathbb{C},\ |z| < 1, \text{ and } \mu \text{ is the Möbius function.}$$

As special cases of (1), obtain the following identities:

$$(2)\qquad \sum_{n=1}^{\infty} \mu(2n)/F_{2ns} = -\beta^{2s}\sqrt{5},\ s = 1, 3, 5, \ldots,\ \beta = \tfrac{1}{2}(1 - \sqrt{5});$$

$$(3)\qquad \sum_{n=1}^{\infty} \mu(2n-1)/L_{(2n-1)s} = -\beta^{s},\ s = 1, 3, 5, \ldots;$$

$$(4)\qquad \sum_{n=1}^{\infty} \mu(n)/F_{ns} = (\beta^{s} - \beta^{2s})\sqrt{5},\ s = 2, 4, 6, \ldots;$$

(5) $\sum_{n=1}^{\infty} (-1)^{n-1}\mu(n)/F_{ns} = (\beta^s + \beta^{2s})\sqrt{5}$, $s = 2, 4, 6, \ldots$;

(6) $\sum_{n=1}^{\infty} (-1)^{n-1}\mu(2n - 1)/F_{(2n-1)s} = -\beta^s\sqrt{5}$, $s = 1, 3, 5, \ldots$;

(7) $\sum_{n=1}^{\infty} (-1)^{n-1}\mu(2n - 1)/L_{(2n-1)s} = \beta^s$, $s = 2, 4, 6, \ldots$.

SOLUTIONS

Rather Compact

H-421 *Proposed by Piero Filipponi, Rome, Italy*
(Vol. 26, no. 2, May 1988)

Let the numbers $U_n(m)$ (or merely U_n) be defined by the recurrence relation [1]

$$U_{n+2} = mU_{n+1} + U_n;\ U_0 = 0,\ U_1 = 1,$$

where $m \in N = \{1, 2, \ldots\}$.

Find a compact form for

$$S(k, h, n) = \sum_{j=0}^{n-1} U_{k+jh}U_{k+(n-1-j)h}\quad (k, h, n \in N).$$

Note that, in the particular case $m = 1$, $S(1, 1, n) = F_n^{(1)}$ is the n^{th} term of the Fibonacci first convolution sequence [2].

References

1. M. Bicknell. "A Primer on the Pell Sequence and Related Sequences." *Fibonacci Quarterly 13.4* (1975):345-49.
2. V. E. Hoggatt, Jr. "Convolution Triangles for Generalized Fibonacci Numbers." *Fibonacci Quarterly 8.2* (1970):158-71.

Solution by the proposer

It is known [1] that

(1) $U_n = (\alpha^n - \beta^n)/\Delta$ (Binet form)

where

(2) $$\begin{cases} \Delta = (m^2 + 4)^{1/2}, \\ \alpha = (m + \Delta)/2, \\ \beta = (m - \Delta)/2. \end{cases}$$

Analogously, the numbers $V_n(m)$ (or merely V_n) can be defined as either

(3) $V_{n+2} = mV_{n+1} + V_n;\ V_0 = 2,\ V_1 = m,$

or

(4) $V_n = U_{n-1} + U_{n+1} = \alpha^n + \beta^n$ (Binet form).

The following identities will be used throughout the solution:

(5) $V_{j+k} - (-1)^k V_{j-k} = \Delta^2 U_j U_k$;

(6) $$\begin{cases} U_{-n} = (-1)^{n+1}U_n, \\ V_{-n} = (-1)^n V_n; \end{cases}$$

$$(7) \qquad \sum_{j=0}^{r} x^j V_{sj+t} = \frac{(-1)^{s-1}x^{r+2}V_{sr+t} + x^{r+1}V_{s(r+1)+t} + (-1)^t x V_{s-t} - V_t}{(-1)^{s-1}x^2 + V_s x - 1},$$

where s and t are arbitrary integers and x is an arbitrary quantity subject to the restriction $x \neq \alpha^{-s}, \beta^{-s}$.

Identities (5) and (6) can be readily proven with the aid of (1), (2), and (4). The proof of (7) is slightly more complicated but several approaches are possible. One of these proofs is given in "A Matrix Approach to Certain Identities" by P. Filipponi & A. F. Horadam (*Fibonacci Quarterly* 26.2 [1988]:115-26).

Now, from (5), we can write

$$U_{k+jh}U_{k+(n-1-j)h} = [V_{2k+(n-1)h} - (-1)^{k+(n-j-1)h}V_{(2j-n+1)h}]/\Delta^2,$$

whence

$$(8) \qquad S(k, h, n) = \frac{nV_{2k+(n-1)h}}{\Delta^2} - \frac{(-1)^{k+(n-1)h}}{\Delta^2}\sum_{j=0}^{n-1}(-1)^{jh}V_{2hj-(n-1)h}$$

$$= \frac{nV_{2k+(n-1)h}}{\Delta^2} - \frac{(-1)^{k+(n-1)h}}{\Delta^2}X_{h,n}.$$

Using (7), (6), and (5), let us calculate the quantity $X_{h,n}$:

<u>Case 1</u>: h is odd [$x = -1$ in (7)]

$$(9) \qquad X_{n,h} = \frac{2(-1)^{n-1}[V_{h(n+1)} + V_{h(n-1)}]}{V_{2h} + 2} = \frac{2(-1)^{n-1}\Delta^2 U_{hn}U_h}{V_{2h} + 2}.$$

Using (1) and (4), (9) becomes

$$(10) \qquad X_{h,n} = 2(-1)^{n-1}U_{hn}/U_h.$$

<u>Case 2</u>: h is even [$x = 1$ in (7)]

$$(11) \qquad X_{h,n} = \frac{2[V_{h(n+1)} - V_{h(n-1)}]}{V_{2h} - 2} = \frac{2\Delta^2 U_{hn}U_h}{V_{2h} - 2} = 2U_{hn}/U_h.$$

From (8), (9), and (10), we obtain

$$(12) \qquad S(k, h, n) = [nV_{2k+(n-1)h} - 2(-1)^k U_{hn}/U_h]/\Delta^2.$$

The relationship (4) allows us to express $S(k, h, n)$ merely in terms of numbers U_n.

As a particular case, we have

$$(13) \qquad S(1, 1, n) = [nV_{n+1} + 2U_n]/\Delta^2 = [nU_{n+2} + (n + 2)U_n]/\Delta^2.$$

Also solved by P. Bruckman, L. Kuipers, H.-J. Seiffert, and N. A. Volodin.

<u>Lotsa Sequences</u>

<u>H-422</u> *Proposed by Larry Taylor, Rego Park, NY (Vol. 26, no. 2, May 1988)*

(A1) Generalize the numbers (2, 2, 2, 2, 2, 2, 2) to form a seven-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference F_n.

(A2) Generalize the numbers (1, 1, 1, 1, 1, 1) to form a six-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference F_n.

(A3) Generalize the numbers (4, 4, 4, 4, 4) to form a five-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference $5F_n$.

(A4) Generalize the numbers (3, 3, 3, 3), (3, 3, 3, 3), (3, 3, 3, 3) to form three four-term arithmetic progressions of integral multiples of Fibonacci and/or Lucas numbers with common differences F_n, $5F_n$, F_n.

(B) Generalize the Fibonacci and Lucas numbers in such a way that, if the Fibonacci numbers are replaced by the generalized Fibonacci numbers and the Lucas numbers are replaced by the generalized Lucas numbers, the arithmetic progressions still hold.

Solution by Paul S. Bruckman, Edmonds, WA

We indicate below the solutions to parts (A1)-(A4) of the problem:

(A1) $(-2F_{n-2},\ F_{n-3},\ 2F_{n-1},\ L_n,\ 2F_{n+1},\ F_{n+3},\ 2F_{n+2})$;

(A2) $(-L_{n-1},\ -F_{n-2},\ F_{n-1},\ F_{n+1},\ F_{n+2},\ L_{n+1})$;

(A3) $(-4L_{n-1},\ -L_{n-3},\ 2L_n,\ L_{n+3},\ 4L_{n+1})$;

(A4) (i) $(3F_{n+1},\ L_{n+2},\ F_{n+4},\ 3F_{n+2})$;

(ii) $(-3L_{n-1},\ L_{n-2},\ L_{n+2},\ 3L_{n+1})$;

(iii) $(-3F_{n-2},\ -F_{n-4},\ L_{n-2},\ 3F_{n-1})$.

First, we verify that the above yield the desired solutions:

(A1)
$$\begin{aligned}
-2F_{n-2} + F_n &= -2F_{n-2} + 2F_{n-2} + F_{n-3} = F_{n-3};\\
F_{n-3} + F_n &= F_{n-1} - F_{n-2} + F_{n-1} + F_{n-2} = 2F_{n-1};\\
2F_{n-1} + F_n &= F_{n-1} + F_{n+1} = L_n;\\
L_n + F_n &= F_{n-1} + F_n + F_{n+1} = 2F_{n+1};\\
2F_{n+1} + F_n &= F_{n+1} + F_{n+2} = F_{n+3};\\
F_{n+3} + F_n &= F_{n+2} + F_{n+1} + F_{n+2} - F_{n+1} = 2F_{n+2}. \quad \text{Q.E.D.}
\end{aligned}$$

(A2)
$$\begin{aligned}
&-L_{n-1} + F_n = -F_{n-2} - F_n + F_n = -F_{n-2}; \quad -F_{n-2} + F_n = F_{n-1};\\
&F_{n-1} + F_n = F_{n+1}; \quad F_{n+1} + F_n = F_{n+2}; \quad F_{n+2} + F_n = L_{n+1}. \quad \text{Q.E.D.}
\end{aligned}$$

(A3)
$$\begin{aligned}
-4L_{n-1} + 5F_n &= -4L_{n-1} + L_{n+1} + L_{n-1} = L_n + L_{n-1} - 3L_{n-1}\\
&= L_{n-1} + L_{n-2} - 2L_{n-1} = L_{n-2} - L_{n-1} = -L_{n-3};\\
-L_{n-3} + 5F_n &= -L_{n-3} + L_{n+1} + L_{n-1} = -L_{n-1} + L_{n-2} + L_n + 2L_{n-1}\\
&= L_{n-2} + L_{n-1} + L_n = 2L_n;\\
2L_n + 5F_n &= 2L_n + L_{n-1} + L_{n+1} = L_n + L_{n-1} + L_n + L_{n+1}\\
&= L_{n+1} + L_{n+2} = L_{n+3};\\
L_{n+3} + 5F_n &= L_{n+3} + L_{n-1} + L_{n+1} = L_{n+2} + L_{n+1} + L_{n+1} - L_n + L_{n+1}\\
&= L_{n+1} + L_n + 3L_{n+1} - L_n = 4L_{n+1}. \quad \text{Q.E.D.}
\end{aligned}$$

(A4) (i)
$$\begin{aligned}
3F_{n+1} + F_n &= 2F_{n+1} + F_{n+2} = F_{n+1} + F_{n+3} = L_{n+2};\\
L_{n+2} + F_n &= F_{n+3} + F_{n+1} + F_n = F_{n+3} + F_{n+2} = F_{n+4};\\
F_{n+4} + F_n &= 2F_{n+2} + F_{n+1} + F_{n+2} - F_{n+1} = 3F_{n+2}. \quad \text{Q.E.D.}
\end{aligned}$$

(ii)
$$\begin{aligned}
-3L_{n-1} + 5F_n &= -3L_{n-1} + L_{n+1} + L_{n-1} = L_n + L_{n-1} - 2L_{n-1}\\
&= L_n - L_{n-1} = L_{n-2};\\
L_{n-2} + 5F_n &= L_{n-2} + L_{n-1} + L_{n+1} = L_n + L_{n+1} = L_{n+2};\\
L_{n+2} + 5F_n &= L_{n+1} + L_n + L_{n+1} + L_{n-1} = 3L_{n+1}. \quad \text{Q.E.D.}
\end{aligned}$$

(iii)
$$\begin{aligned}
-3F_{n-2} + F_n &= -3F_{n-3} - 3F_{n-4} + 2F_{n-2} + F_{n-3}\\
&= 2F_{n-2} - 2F_{n-3} - 3F_{n-4} = -F_{n-4};\\
-F_{n-4} + F_n &= -F_{n-2} + F_{n-3} + F_{n-1} + F_{n-2} = L_{n-2};
\end{aligned}$$

$$L_{n-2} + F_n = F_{n-3} + F_{n-1} + F_n = F_{n-1} - F_{n-2} + F_{n-1} + F_{n-1} + F_{n-2}$$
$$= 3F_{n-1}. \quad \text{Q.E.D.}$$

Although not required, it is informative to show how the preceding progressions were discovered. We illustrate the method for part (A1) of the problem. First, we note that the value 2 can be assumed only by the following seven admissible terms: $(F_{-3}, -2F_{-2}, 2F_{-1}, L_0, 2F_1, 2F_2, F_3)$. If we suppose that these are special cases of the desired terms, not necessarily in proper order, we surmise that the general terms of the desired solution may be formed by adding n to each suffix of the preceding list. If so, the asymptotic values of such terms are as follows, again, not necessarily in proper order:

$$\alpha^n 5^{-1/2} \cdot (\alpha^{-3}, -2\alpha^{-2}, 2\alpha^{-1}, 5^{1/2}, 2\alpha, 2\alpha^2, \alpha^3).$$

The terms in parentheses may be crudely approximated as follows: (.24, -.76, 1.24, 2.24, 3.24, 5.24, 4.24). We now rearrange these last terms in ascending order of magnitude: (-.76, .24, 1.24, 2.24, 3.24, 4.24, 5.24), and note that all the terms are indeed in A.P. We now write down the terms of the first list corresponding to these last terms, as follows: $(-2F_{-2}, F_{-3}, 2F_{-1}, L_0, 2F_1, F_3, 2F_2)$. Finally, we add n to each suffix in this last septet, thereby forming the candidate for the desired general solution; as we have verified, this indeed generates the correct solution.

A similar process yields the solutions of the other parts of the problem, though in parts (A3) and (A4) the process is complicated by the fact that the choice of terms forming an A.P. is not unique; moreover, in (A4), a pair of "red herrings" occur, which cannot be used to form an A.P., but these are readily identifiable as such and may quickly be eliminated from consideration.

(B) The appropriate generalization is readily obtained by using the generalized Fibonacci and Lucas numbers defined as follows, for arbitrary constants r and s:

$$U_n = rF_n + sF_{n-1}, \quad V_n = rL_n + sL_{n-1}, \text{ for all integers } n.$$

It is easy to see that the U_n's and V_n's satisfy the Fibonacci recurrence, but have different initial values, in general. From this, we see that the desired generalization is obtained by replacing F by U and L by V in (A1)-(A4); the differences in each A.P. will then be an appropriate multiple (either 1 or 5) of U_n, rather than of F_n. We illustrate only with case (A4)(i):

$$\begin{aligned}&(3U_{n+1}, V_{n+2}, U_{n+4}, 3U_{n+2})\\ &= (3(rF_{n+1} + sF_n), (rL_{n+2} + sL_{n+1}), (rF_{n+4} + sF_{n+3}), 3(rF_{n+2} + sF_{n+1}))\\ &= r(3F_{n+1}, L_{n+2}, F_{n+4}, 3F_{n+2}) + s(3F_n, L_{n+1}, F_{n+3}, 3F_{n+1});\end{aligned}$$

from (A4)(i), each quadruplet in parentheses is in A.P., with common difference F_n and F_{n-1}, respectively. Due to linearity, the general terms are also in A.P., with common difference $= rF_n + sF_{n-1} = U_n$. Q.E.D.

Also solved by L. Kuipers and the proposer.

A Golden Result

H-423 *Proposed by Stanley Rabinowitz, Littleton, MA*
(Vol. 26, no. 3, August 1988)

Prove that each root of the equation

$$F_n x^n + F_{n+1}x^{n-1} + F_{n+2}x^{n-2} + \cdots + F_{2n-1}x + F_{2n} = 0$$

has an absolute value near ϕ, the golden ratio.

Solution by Tad White, University of California, Los Angeles, CA

Problem: Show that the zeros of the polynomial $F_n z^n + \cdots + F_{2n}$ lie near the circle $|z| = \alpha$, where α is a positive root of $z^2 - z - 1 = 0$.

Solution: First divide through by F_n to obtain a monic polynomial; we will examine the roots of

$$f_n(z) = z^n + \frac{F_{n+1}}{F_n} z^{n-1} + \cdots + \frac{F_{2n}}{F_n}.$$

The following lemma gives us information about the coefficients of f_n.

Lemma 1: If β is the negative root of $z^2 - z - 1 = 0$, then

$$\left|\frac{F_{n+k}}{F_n} - \alpha^k\right| \le \frac{|\beta^n| F_k}{F_n}, \text{ for all } n, k.$$

Proof: Using Binet's formula for F_n, we can write

$$\sqrt{5}(F_{n+k} - F_n\alpha^k) = (\alpha^{n+k} - \beta^{n+k}) - \alpha^k(\alpha^n - \beta^n) = \beta^n(\beta^k - \alpha^k) = -\sqrt{5}\beta^n F_k;$$

dividing by $\sqrt{5}F_n$ and taking absolute values completes the proof. □

If we define $g_n(z) = z^n + \alpha z^{n-1} + \cdots + \alpha^n$, then Lemma 1 tells us that the coefficients of f_n and g_n are close. To make this precise, we can define a norm on the vector space P_n of complex polynomials of degree $\le n$ via

$$\|f\| = \sum_{k=0}^{n} |\alpha_k| \text{ if } f(z) = \sum_{k=0}^{n} \alpha_k z^k.$$

Then Lemma 1 says that

$$\|f_n - g_n\| \le \beta^n \sum_{k=0}^{n} \frac{F_k}{F_n} < \beta^n \frac{F_{n+2}}{F_n} < 3\beta^n.$$

Since $|\beta| < 1$, this says $\|f_n - g_n\| \to 0$ as $n \to \infty$. Note also that

$$g_n(z) = \frac{z^{n+1} - \alpha^{n+1}}{z - \alpha},$$

so the roots of g_n lie on the circle $|z| = \alpha$. Hence, we need only show that the locations of the zeros of a polynomial vary in some sense continuously with the coefficients. This can be made precise via the following lemma.

Lemma 2: Given a sufficiently small $\varepsilon > 0$ and $f_1 \in P_n$, there exists $\delta > 0$ such that if f_0 is an element of P_n with $\|f_0 - f_1\| < \delta$, there exists a one-to-one correspondence between the roots ζ_i of f_0 and the roots η_i of f_1 such that $|\zeta_i - \eta_i| < \varepsilon$ for each i.

Proof: Let $f_t = (1 - t)f_0 + tf_1$ for $0 \le t \le 1$; note that $f_t \in P_n$ for each t. Since the set of zeros of f_1 is discrete, and since ε is small, f_1 does not vanish in the closed punctured ball of radius ε around η_i. Observe that the evaluation maps $e_z : P_n \to \mathbb{C}$, given by $e_z(f) = f(z)$ are continuous with respect to the norm $\|\cdot\|$, and in fact uniformly continuous if we restrict z to the compact set $|z| < 2$ (which contains all of the roots η_i in its interior). Therefore, we can choose δ such that $\|f_0 - f_1\| < \delta$ implies that f_0 does not vanish on $\partial B(\eta_i, \varepsilon)$, where $B(\eta_i, \varepsilon)$ is the closed ball of radius ε at η_i. Since $\|f_0 - f_1\|$ is a monotonic function of t, we have that no f_t vanishes on $\partial B(\eta_i, \varepsilon)$.

Assume further that ε is small enough that the paths $\partial B(\eta_i, \varepsilon)$ are disjoint; then define the functions

$$Z_i(t) = \frac{1}{2\pi i}\int_{\partial B(\eta_i, \varepsilon)} \frac{f_t'(z)}{f_t(z)} dz;$$

(please turn to page 282)

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

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Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

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A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

Applications of Fibonacci Numbers. Edited by A.N. Philippou, A.F. Horadam and G.E. Bergum.

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