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The primary function of THHE FIBONACCI QUARTERLY is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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# The Fibonacci Quarterly 

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# A RADIX PRODUCT REPRESENTATION FOR REAL NUMBERS 

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(Submitted September 1988)

## Introduction

Two classical representations for real numbers in terms of integer "digits" are the series representation of Sylvester (1880) and the product representation of Cantor (1869): If $A$ denotes any real number ( $A>1$ in the product case), then these representations, respectively, take the forms:

$$
A=a_{0}+\frac{1}{a_{1}}+\frac{1}{a_{2}}+\frac{1}{a_{3}}+\cdots, \quad a_{i} \in \mathbb{N}
$$

where $\quad \alpha_{1} \geq 2, \alpha_{i+1} \geq \alpha_{i}\left(\alpha_{i}-1\right)+1$ for $i \geq 1$,
and $\quad A=2^{k} \prod_{i=1}^{\infty}\left(1+\frac{1}{a_{i}}\right), \quad a_{i} \in \mathbb{N}$,
where $k \in \mathbb{N}, \alpha_{1} \geq 2, \alpha_{i+1} \geq a^{2} \quad$ for $i \geq 1$.
For further details, see, for example, Perron [3].
Far more familiar to us than the above is of course the radix or decimaltype representation for $A$ to the base $q$, where here and throughout, $q$ denotes an integer greater than or equal to two. One of the advantages of this latter representation over the first two, is that the digits " $\alpha_{i}$ " all lie in the finite set $\{0,1, \ldots, q-1\}$ which allows us to conveniently express our decimal expansion base $q$ in the positional notation

$$
A=a_{n} a_{n-1} \cdots a_{1} a_{0} \cdot a_{-1} a_{-2} a_{-3} \cdots
$$

It seems therefore a natural question to ask whether we can derive a further product representation for a real number $A>1$ in the radix form

$$
A=\prod_{i=-m}^{\infty}\left(1+\frac{\alpha_{i}}{q^{i}}\right), \text { where } m \in \mathbb{N}, \alpha_{i} \in\{0,1, \ldots, q-1\} .
$$

The paper is set out as follows. In Section 2, we derive a more general type of radix product representation for real numbers $1<A<2$. The main interest of the radix product representation is that, like ordinary decimals (base q), it depends only on digits belonging to the set $\{0,1, \ldots, q-1\}$, thus allowing us to express the radix product

$$
A=\prod_{i=1}^{\infty}\left(1+\frac{a_{i}}{q^{i}}\right)
$$

as $0 * a_{1} a_{2} a_{3} \ldots$ say, just as in the decimal case. Furthermore, as shown in the paper, the rate of convergence of the radix product is basically the same as that of the ordinary decimal expansion. It is true that the Cantor product

$$
A=\Pi\left(1+\frac{1}{a_{i}}\right)
$$

converges more rapidly. However, by the same token, the Sylvester series

$$
A=\sum \frac{1}{\alpha_{i}}
$$

converges far more rapidly than the ordinary decimal expansion. Furthermore,
due to the exponential growth of the integers $\alpha_{i}$ in Sylvester's and Cantor's representation they are unwieldy to use in practice and each "digit" $\alpha_{i}$ must, in turn, be represented in the decimal system, a drawback which is absent in the case of the radix product. In Section 3, we introduce an alternative, computationally simpler algorithm which allows the computation of the radix product digits from the leading digits of a certain sequence of ordinary $q$ decimals. Finally, in Section 4, we investigate the possibility of an analogous radix product representation for real numbers $0<A<1$.

Throughout the paper, unless otherwise stated, lower case letters denote nonnegative integers.

## 2. Radix Products in a Varying Scale

Let $q_{1}, q_{2}, \ldots$ be an infinite sequence of natural numbers greater than one. Then, it is well known (see, e.g., Perron [3]) that every real number $A$ has a generalized decimal expansion

$$
A=\alpha_{0}+\frac{a_{1}}{q_{1}}+\frac{a_{2}}{q_{1} q_{2}}+\frac{a_{3}}{q_{1} q_{2} q_{3}}+\ldots
$$

where $\alpha_{0}=[A], 0 \leq \alpha_{i} \leq q_{i}-1$ for $i \geq 1$.
Using the product algorithm below, we derive an analogous generalized product representation: Let $1<A \equiv A_{1}<2$. Then, recursively define, for $n \geq 1$,

$$
\alpha_{n}=\left[\left(A_{n}-1\right) q_{1} q_{2} \cdots q_{n}\right], A_{n} \neq 1
$$

where

$$
A_{n+1}=\left(1+\frac{a_{n}}{q_{1} q_{2} \cdots q_{n}}\right)^{-1} A_{n}
$$

If $A_{n}=1$, then stop the algorithm. This leads to
Proposition 2.1: Let $1<A<2$. Then $A$ has a finite or infinite product representation

$$
A=\prod_{i=1}^{\infty}\left(1+\frac{\alpha_{i}}{q_{1} q_{2} \cdots q_{i}}\right)
$$

where the "digits" $\alpha_{i}$ satisfy $0 \leq \alpha_{i} \leq q_{i}-1$.
Proof: First, a repeated application of the recurrence yields

$$
\begin{aligned}
A \equiv A_{1} & =\left(1+\frac{a_{1}}{q_{1}}\right) A_{2}=\left(1+\frac{a_{1}}{q_{1}}\right)\left(1+\frac{\alpha_{2}}{q_{1} q_{2}}\right) A_{3}=\ldots \\
& =\left(1+\frac{\alpha_{1}}{q_{1}}\right)\left(1+\frac{\alpha_{2}}{q_{1} q_{2}}\right) \cdots\left(1+\frac{a_{n}}{q_{1} q_{2} \cdots q_{n}}\right) A_{n+1}
\end{aligned}
$$

if $A_{n} \neq 1$. Since $1<A_{1}<2,0 \leq \alpha_{1}=\left[\left(A_{1}-1\right) q_{1}\right]<q_{1}$. Suppose now, inductively, that $A_{i}>1$ and $0 \leq \alpha_{i} \leq q_{i}-1$ for $i \leq n$. From the definition

$$
a_{n}=\left[\left(A_{n}-1\right) q_{1} q_{2} \cdots q_{n}\right]
$$

we deduce that

$$
1+\frac{a_{n}}{q_{1} \cdots q_{n}} \leq A_{n}<1+\frac{a_{n}+1}{q_{1} \cdots q_{n}}
$$

and it follows that

Thus,

$$
\begin{aligned}
1 \leq A_{n+1}<\left(1+\frac{a_{n}+1}{q_{1} \cdots q_{n}}\right) /\left(1+\frac{a_{n}}{q_{1} \cdots q_{n}}\right) & =1+\frac{1}{q_{1} \cdots q_{n}+a_{n}} \\
& \leq 1+\frac{1}{q_{1} \cdots q_{n}} .
\end{aligned}
$$

as required. Now, either $A_{n}=1$ for some $n$, or

$$
1<A_{n}<1+\frac{1}{q_{1} q_{2} \cdots q_{n-1}} \leq 1+\frac{1}{2^{n-1}} \rightarrow 1 \text { as } n \rightarrow \infty, \text { for } n \geq 1
$$

The result follows.
Of particular interest to us is the decimal-type product representation obtained by setting $q_{1}=q_{2}=q_{3}=\cdots=q_{n}$ in the above. Before discussing this case in some detail, we briefly mention one further special product representation of interest, which arises from Proposition 2.1 by setting $q_{n}=n+1$ for $n \geq 1$.

Corollary 2.2: Every real number $1<A<2$ has a "factorial" product representation

$$
A=\prod_{i=1}^{\infty}\left(1+\frac{a_{i}}{(i+1)!}\right)
$$

where $0 \leq \alpha_{i} \leq i$ for $i \geq 1$.
In the sequel, however, we shall confine our attention to the most interesting case of Proposition 2.1 , obtained by setting $q_{n}=q$ for all $n \geq 1$.

Theorem 2.3: Every $A>1$ has a finite or infinite radix product representation (base $q$ ) of the form

$$
A=\prod_{n=-m}^{\infty}\left(1+\frac{a_{i}}{q^{i}}\right):=a_{n} a_{m-1} \ldots a_{1} a_{2} * a_{-1} a_{-2} \ldots
$$

where $m \in \mathbb{N}, a_{i} \in\{0,1, \ldots, q-1\}$.
Proof: It follows from Proposition 2.1 that we can represent every $1<A<2$ as

$$
A=\prod_{i=1}^{\infty}\left(1+\frac{a_{i}}{q^{i}}\right)
$$

A simple (nonunique) method of extending this product for $1<A<2$ to every $A>1$ is as follows: First, if $A^{\prime}<2 q$, then, for a suitable $0 \leq \alpha_{0} \leq q-1$, we can write

$$
A^{\prime}=\left(1+\frac{a_{0}}{q^{0}}\right) A
$$

where $1<A<2$. Now apply the algorithm to $A$. Next, if $A^{\prime \prime}>2 q$, then there exists $m \in \mathbb{N}$ such that $1+q^{m}<A^{\prime \prime} \leq 1+q^{m+1}$. Thus, we can write

$$
A^{\prime \prime}=\left(1+\frac{1}{q^{-m}}\right) A^{\prime}
$$

where $1<A^{\prime} \leq\left(1+q^{m+1}\right) /\left(1+q^{m}\right)<q$, and the product expansion for $A^{\prime \prime}$ now follows from that of $1<A^{\prime}<2 q$.
Remarks 2.4: Even in the case $1<A<2$ the radix product representation base $q$ is not necessarily unique. For example, to base 2,

$$
1+\frac{1}{2}=\left(1+\frac{1}{2^{2}}\right)\left(1+\frac{1}{2^{3}}\right)\left(1+\frac{1}{2^{4}}\right)\left(1+\frac{1}{2^{8}}\right) \ldots,
$$

where the one-term expansion on the left follows from applying the algorithm directly to $A=1.5$, while the algorithm applied to $A=1.2=1.5 / 1.25$ yields the expansion on the right.

Unfortunately, as these and other examples show, real numbers can have more than one expansion as a radix product subject only to the condition that the digits lie in $\{0,1, \ldots, q-1\}$. However, the constructive algorithm at the start of Section 3 produces a unique choice for the digits $\alpha_{i}$ at each step. For the digits produced by this algorithm, it follows from the proof of Proposition 2.1 that the following inequality holds for each $n \geq 1$ :
(*)

$$
\left(1+\frac{a_{n}+1}{q^{n}}\right)>\prod_{i=n}^{\infty}\left(1+\frac{a_{i}}{q^{i}}\right)
$$

Conversely, it can be shown that there is only one radix product expansion for a given $1<A<2$ for which ( $*$ ) holds for each $n \geq 1$. Thus, every $1<A<2$ has a unique radix product expansion

$$
A=\prod_{i=1}^{\infty}\left(1+\frac{a_{i}}{q^{i}}\right)
$$

s.t. for each $i \geq 1$,

$$
\left(1+\frac{a_{i}+1}{q^{i}}\right)>\prod_{n=i}^{\infty}\left(1+\frac{a_{n}}{q^{n}}\right)
$$

Furthermore, since the algorithm chooses the largest possible digit " $\alpha_{i}$ " at each stage, in general, this radix product expansion will converge faster than any other not satisfying ( $*$ ), and is thus the canonical expansion for $A$.

In addition, rational numbers need not have finite representations as $q$-radix products. As a particular case of Euler's product identity

$$
1+\frac{1}{y-1}=\prod_{n=1}^{\infty}\left(1+\frac{1}{y^{2^{n-1}}}\right), y \in \mathbb{R},|y|>1
$$

we have, for any $r \in \mathbb{N}$,

$$
A=1+\frac{1}{q^{r}-1}=\prod_{n=1}^{\infty}\left(1+\frac{1}{q^{2^{n-1} r}}\right)
$$

Note also that such $A$ have recurring ordinary $q$-radix expansions of the form $4=1 . \overline{00 . .01}$, where the period consists of $r-1$ zeros followed by a one. In general, however, other recurring decimals base $q$ need not have "nice" radix product representations, unlike the case above.

## 3. An Alternative Radix Product Algorithm

We can reformulate the general product algorithm of Section 2 in the case of a fixed base $q$, into the following computationally simpler form. It is easy to show that the new algorithm is equivalent to that of Section 2 in the case $q_{1}=q_{2}=\cdots=q$, provided we replace any real number with recurring decimal expansion

$$
A=1+\frac{a}{q^{s}}+\frac{q-1}{q^{s+1}}+\frac{q-1}{q^{s+2}}+\cdots, 0<a \leq q-1, s \in \mathbb{N}
$$

by the finite expression

$$
A=1+\frac{a+1}{q^{s}}
$$

In this form the algorithm determines only the nontrivial digits ( $\alpha_{i}>0$ ) in the radix product representation.

If $1<A<2$, let $\hat{A}_{1}=A$. Then, if the unique decimal expansion of $A$ (base q) is of the form

$$
A=1+\frac{b_{1}}{q^{r_{1}}}+\ldots, 1 \leq b_{1} \leq q-1, r_{1} \in \mathrm{~N}
$$

then we can write

If

$$
\hat{A}=1+\frac{b_{1}}{q^{r_{1}}} A_{1}^{\prime}, \text { where } 1 \leq A_{1}^{\prime}<1+\frac{1}{b_{1}} \leq 2
$$

$$
\hat{A}_{n}=1+\frac{b_{n}}{q^{r_{n}}} A_{n}^{\prime}
$$

has already been defined with

$$
1<A_{n}^{\prime}<1+\frac{1}{b_{n}} \leq 2
$$

then define

$$
\begin{aligned}
\hat{A}_{n+1} & =\left(1+\frac{b_{n}}{q^{r_{n}}}\right)^{-1} \hat{A}_{n}<\left(1+\frac{b_{n}}{q^{r_{n}}}\right)^{-1}\left(1+\frac{b_{n}+1}{q^{r_{n}}}\right) \\
& =1+\frac{1}{q^{r_{n}}+b_{n}}<1+\frac{1}{q^{r_{n}}} .
\end{aligned}
$$

It follows that we can write

$$
\hat{A}_{n+1}=1+\frac{b_{n+1}}{q^{r_{n+1}}} A_{n+1}^{\prime},
$$

where $r_{n+1}>r_{n}, 1 \leq b_{n+1} \leq q-1$ and
$1 \leq A_{n+1}^{\prime}<1+\frac{1}{b_{n+1}} \leq 2$.
If $A_{n}^{\prime}=1$, let $\hat{A}_{n+1}=1$ and stop the algorithm. Then

$$
A=\hat{A}_{1}=\left(1+\frac{b_{1}}{q^{r_{1}}}\right) \hat{A}_{2}=\cdots=\hat{A}_{n+1} \prod_{i=1}^{n}\left(1+\frac{b_{i}}{q^{r_{i}}}\right) .
$$

If the procedure does not terminate with some $\hat{A}_{n+1}=1$, then

$$
0<\hat{A}_{n+1}-1<\frac{1}{q^{r_{n+1}-1}} \leq \frac{1}{q^{r_{n}}} \leq \cdots \leq \frac{1}{q^{n}} \rightarrow 0 \text { as } n \rightarrow \infty .
$$

Thus, $\lim _{n \rightarrow \infty} \hat{A}_{n+1}=1$, and hence,

$$
A=\prod_{i=1}^{\infty}\left(1+\frac{b_{i}}{q^{r_{i}}}\right)
$$

$$
P_{n}=\left(1+\frac{b_{1}}{q^{r_{1}}}\right) \cdots\left(1+\frac{b_{n}}{q^{r_{n}}}\right)
$$

we also have

$$
A=\left(1+\frac{b_{n+1}}{q^{r_{n+1}}} A_{n+1}^{\prime}\right) P_{n}
$$

and so

$$
0<A-P_{n}=\frac{b_{n+1}}{q^{P_{n+1}}} A_{n+1}^{\prime} P_{n}<\frac{A}{q^{P_{n+1}-1}}<\frac{2}{q^{n}} .
$$

The above argument can therefore be used to give an alternative proof of Theorem 2.3 and, in addition, if for $A>1$,

$$
P_{n}=I_{n=-m}^{n}\left(1+\frac{b_{i}}{q^{r_{i}}}\right)
$$

then the rate of approximation to $A$ by the finite "decimal" $P_{n}$ is given by

$$
0<A-P_{n}<\frac{A}{q^{r_{n+1}-1}}<\frac{A}{q^{n}}, n \geq 1 .
$$

In order to appreciate how easily this algorithm can be applied in practice, we illustrate it with a numerical example. For convenience, we choose the base $q$ equal to ten: Let $A=\hat{A}_{1}=1.035124$. Then

$$
\begin{aligned}
& P_{1}=0 * 03, \hat{A}_{2}=(1.03)^{-1}(1.035124)=1.004974 \ldots \\
& P_{2}=0 * 034, \hat{A}_{3}=(1.004)^{-1}(1.004974 \ldots)=1.000970 \ldots, \\
& P_{3}=0 * 0349, \hat{A}_{4}=\ldots .
\end{aligned}
$$

To conclude this section, we make a few comments about radix product "fractions" base $q$, that is, radix products of the form $0 * \alpha_{1} \alpha_{2} \alpha_{3} \ldots, 0 \leq \alpha_{i} \leq$ q-1. It is clear from the above algorithms that any $1<A \leq 2$ has a representation as a fractional radix product. (To obtain a product expansion for $A=2$, we can apply the algorithm of Section 3 to $A=1.999 .$. ). However, fractional product representation also exists for certain real numbers greater than two. If we denote the largest such fraction (base $q$ ) by

$$
\epsilon_{q}=0 *(q-1)(q-1) \ldots
$$

then it follows from standard inequalities relating infinite series and products that

$$
1+\sum_{n=1}^{\infty} \frac{q-1}{q^{n}}<\epsilon_{q}<\exp \left(\sum_{n=1}^{\infty} \frac{q-1}{q^{n}}\right)
$$

which gives $2<\epsilon_{q}<e$ for every $q$. However, the actual value of $\epsilon_{q}$ varies with $q$. In the table below, we list approximations for $\epsilon_{q}$ for some small values of the base $q$.

TABLE 1. The Largest Radix Product Fraction Corresponding to Given Bases $q$

| $q$ | $\epsilon_{q}$ |
| ---: | :---: |
| 2 | 2.38423 |
| 3 | 2.26971 |
| 4 | 2.20963 |
| 5 | 2.17207 |
| 6 | 2.14619 |
| 7 | 2.12719 |
| 8 | 2.11263 |
| 9 | 2.10110 |
| 10 | 2.09172 |

Note that the values of $\epsilon_{q}$ listed correspond to those for the finite products

$$
\prod_{n=1}^{k}\left(1+\frac{q-1}{q^{n}}\right)
$$

for suitable values of $k$. If we denote such finite products by $\epsilon_{q}(k)$, then

$$
\begin{aligned}
\epsilon_{q}-\epsilon_{q}(k) & =\epsilon_{q}(k)\left(\prod_{i=k+1}^{\infty}\left(1+\frac{q-1}{q^{i}}\right)-1\right) \\
& <\epsilon_{q}\left(\exp \left((q-1) \prod_{i=k+1}^{\infty} \frac{1}{q^{i}}\right)-1\right)<e\left(e^{q^{-k}}-1\right) .
\end{aligned}
$$

With this as an upper bound for the error, large enough values of $k$ were chosen for each of the entries $q=2,3, \ldots, 10$ to give $\epsilon_{q}-\epsilon_{q}(k)<10^{-5}$. Examination of Table 1 suggests that $\epsilon_{q}$ is a decreasing function of $q$ for $q \geq 2$, a fact that can be verified by considering the derivative with respect to $q$, of $\log \epsilon_{q}$. Furthermore, using Theorem 5.7, in Hyslop [1] we see that the uniform convergence of the infinite series

$$
\sum_{i=1}^{\infty} \frac{q-1}{q^{i}}=1
$$

for $q \geq 2$, implies the uniform convergence of the product $\epsilon_{q}$, for $q \geq 2$, and it follows that $\lim _{q \rightarrow \infty} \in_{q}=2$.

## 4. Radix Product Expansions for Real Numbers Less than One

One immediate product representation for $0<A<1$ follows from the radix product expression for $A^{-1}>1$. Thus, if we have

$$
A^{-1}=\prod_{i=-m}^{\infty}\left(1+\frac{a_{i}}{q_{i}}\right), 0 \leq \alpha_{i} \leq q-1
$$

then

$$
A=\prod_{i=-m}^{\infty}\left(1+\frac{\alpha_{i}}{q^{i}}\right)^{-1}=\prod_{i=-m}^{\infty}\left(1-\frac{\alpha_{i}}{q^{i}+\alpha_{i}}\right)
$$

In particular, for $A>1 / 2$,

$$
\begin{equation*}
A=\prod_{i=1}^{\infty}\left(1-\frac{a_{i}}{q^{i}+a_{i}}\right) \tag{1}
\end{equation*}
$$

In this form, however, the product no longer has a denominator depending only on the base. This product does, however, suggest the possibility of representing every $0<A<1$ in the form

$$
A=\prod_{i=1}^{\infty}\left(1-\frac{b_{i}}{q^{i}}\right), 0 \leq b_{i} \leq q_{i}-1
$$

Unfortunately, it turns out that it is not possible to represent every 0 < $A<1$ or even $1 / 2<A<1$ in this manner.

To see this, let $\left\{\alpha_{k}\right\}$ be a sequence of real numbers with $\alpha_{k} \in(0,1)$ for every $k$. Then we deduce from Weierstrass's inequality (see Mitrinović [2], p. 210):

$$
\prod_{n=r}^{k}\left(1-a_{n}\right)>1-\sum_{n=r}^{k} a_{n}
$$

by taking limits that

$$
\prod_{n=2}^{\infty}\left(1-a_{n}\right) \geq 1-\sum_{n=2}^{\infty} a_{n}
$$

Hence,

$$
\prod_{n=1}^{\infty}\left(1-a_{n}\right) \geq\left(1-\sum_{n=2}^{\infty} a_{n}\right)\left(1-a_{1}\right)>1-\sum_{n=1}^{\infty} \alpha_{n}
$$

Applying this last inequality to $p_{1}=\prod_{i=1}^{\infty}\left(1-\frac{q-1}{q^{i}}\right)$, we obtain

$$
p_{1}>1-\sum_{i=1}^{\infty} \frac{q-1}{q^{i}}=0
$$

Since $P_{1}$ is the smallest number that can be represented in the form

$$
\prod_{i=1}^{\infty}\left(1-\frac{\alpha_{i}}{q^{i}}\right), 0 \leq \alpha_{i} \leq q-1
$$

it follows that there can be no such product representation for any $0<A<P_{1}$. Similarly, the largest real number that can be represented in the form

$$
\prod_{i=1}^{\infty}\left(1-\frac{\alpha_{i}}{q^{i}}\right), 0 \leq \alpha_{i} \leq q-1, \alpha_{1} \neq 0
$$

is $p_{2}=1-(1 / q)$, and the smallest real number that can be represented in the form

$$
\prod_{i=1}^{\infty}\left(1-\frac{a_{i}}{q^{i}}\right), \quad 0 \leq a_{i} \leq q-1, a_{1}=0
$$

is

$$
p_{3}=\prod_{i=2}^{\infty}\left(1-\frac{q-1}{q^{i}}\right)
$$

Since the inequality relating infinite products and series yields $p_{3}>p_{2}$ there can again be no such product representation for any real number $p_{3}>A>p_{2}$. In general, since

$$
\prod_{i=m+1}^{\infty}\left(1-\frac{q-1}{q^{i}}\right)>1-\frac{1}{q^{m}}
$$

there will be an infinite sequence of gaps in any representation system based upon products of this type.

A consideration of Equation (1) suggests that, for $1 / 2<A<1$, we can obtain a product expansion with digits in $\{0,1, \ldots, q-1\}$ and denominators independent of " $\alpha_{i}$ " consisting of terms

$$
\left(1-\frac{a_{i}}{q^{i}+q}\right), \quad i \geq 1
$$

To obtain such expansions, we introduce the following algorithm: Let

$$
\frac{1}{2}<A=A_{1}<1
$$

Then recursively define, for $n \geq 1$,

$$
\alpha_{n}=\left[\left(1-A_{n}\right)\left(q^{n}+q\right)\right], A_{n} \neq 1,
$$

where

$$
A_{n+1}=\left(1-\frac{a_{n}}{q^{n}+q}\right)^{-1} A_{n}
$$

If $A_{n}=1$, then stop the algorithm.
Using this we can show, in a similar manner to Proposition 2.1, that
Proposition 4.1: Every $1 / 2<A<1$ has a "near radix" product representation
$A=\prod_{n=1}^{\infty}\left(1-\frac{a_{n}}{q^{n}+q}\right)$
with "digits" $\alpha_{n}$ in the set $\{0,1, \ldots, q-1\}$.

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# PARTITIONS WITH "M(a) COPIES OF $a$ " 

## E. E. Guerin

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In [1], Agarwal \& Andrews studied partitions with " $\alpha$ copies of $\alpha$," and in [2], Agarwal \& Mullen studied partitions with " $d(\alpha)$ copies of $a$ " (where $d$ is the divisor function). In this note, partitions with " $M(\alpha)$ copies of $a$ " are considered; the maximum exponent function, $M$, is defined by

$$
M(\alpha)=\max \left(e_{1}, \ldots, e_{r}\right)
$$

if the integer $\alpha>1$ has canonical prime-power form $a=p_{1}^{e_{1}} \ldots p_{r}^{e_{r}}$, and $M(1)=$ 1.

Define $L$ to be the set of ordered pairs ( $a, b$ ) of positive integers with $1 \leq b \leq M(\alpha)$. We say $\pi$ is a partition of $n$ with $M(\alpha)$ copies of $\alpha$ if $\pi$ is a finite ordered collection $\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right), \ldots,\left(a_{k}, b_{k}\right)$ of elements of $L$ such that $\alpha_{1}+\alpha_{2}+\cdots+\alpha_{k}=n$ and, for $1 \leq i \leq j \leq k, \alpha_{i} \geq a_{j}$ with $b_{i} \leq b_{j}$ if $a_{i}=a_{j}$. If we replace $(\alpha, b)$ in $L$ by $a_{b}$, the partitions of $n$ with "M( $\alpha$ ) copies of $a^{\prime \prime}$ for $n=1,2,3,4$, can be represented, respectively, by

$$
\begin{aligned}
& 1_{1} ; 2_{1}, 1_{1}+1_{1} ; 3_{1}, 2_{1}+1_{1}, 1_{1}+1_{1}+1_{1} ; \\
& 4_{1}, 4_{2}, 3_{1}+1_{1}, 2_{1}+2_{1}, 2_{1}+1_{1}+1_{1}, 1_{1}+1_{1}+1_{1}+1_{1} .
\end{aligned}
$$

For the positive integer $n$, let $m(n)$ denote the number of partitions of $n$ with "M( $\alpha$ ) copies of $\alpha . "$ As in [3, Ch. 1] and [2], a generating function for such partitions is

$$
1+\sum_{n=1}^{\infty} m(n) q^{n}=\prod_{n=1}^{\infty}\left(1-q^{n}\right)^{-M(n)} .
$$

This is an immediate consequence of the following theorem [3, Th. 1.1]:
If $H$ is a set of positive integers, if "H" is the set of partitions with parts in $H$, and if $p(" H$ ", $n$ ) is the number of partitions of $n$ with parts in $H$, then for $|q|<1$,

$$
\sum_{n \geq 0} p\left({ }^{\prime \prime} H^{\prime \prime}, n\right) q^{n}=\prod_{n \in H}^{\infty}\left(1-q^{n}\right)^{-1}
$$

The factor $\left(1-q^{n}\right)^{-1}=1+q^{n}+q^{n+n}+\cdots$ is replaced by

$$
\begin{aligned}
\left(1-q^{n}\right)^{-M(n)}= & \left(1+q^{n}+q^{n+n}+\cdots\right)^{M(n)} \\
= & \left(1+q^{n_{1}}+q^{n_{1}+n_{1}}+\cdots\right)\left(1+q^{n_{2}}+q^{n_{2}+n_{2}}+\cdots\right) \\
& \cdots\left(1+q^{n_{M(n)}}+q^{n_{M(n)}+n_{M(n)}}+\cdots\right)
\end{aligned}
$$

for $n_{i}=n(1 \leq i \leq M(n))$; thus, the number of partitions of $n$ with ' $M(\alpha)$ copies of $a^{\prime \prime}$ is counted. For example, $m(4)$ is the coefficient of $q^{4}$ in

$$
\begin{aligned}
& (1-q)^{-1}\left(1-q^{2}\right)^{-1}\left(1-q^{3}\right)^{-1}\left(1-q^{4}\right)^{-2} \\
& =\left(1+q^{1_{1}}+q^{1_{1}+1_{1}}+q^{1_{1}+1_{1}+1_{1}}+q^{1_{1}+1_{1}+1_{1}+1_{1}}+\cdots\right)
\end{aligned}
$$

$\cdot\left(1+q^{2_{1}}+q^{2_{1}+2_{1}}+\cdots\right)\left(1+q^{3_{1}}+\cdots\right)\left(1+q^{4_{1}}+\cdots\right)\left(1+q^{4_{2}}+\cdots\right)$
for $1_{1}=1,2_{1}=2,3_{1}=3,4_{1}=42=4$; since

$$
q^{4}=q^{4_{1}}=q^{4_{2}}=q^{3_{1}+1_{1}}=q^{2_{1}+2_{1}}=q^{2_{1}+1_{1}+1_{1}}=q^{1_{1}+1_{1}+1_{1}+1_{1}}
$$

then $m(4)=6$, and the exponents

$$
4_{1}, 4_{2}, 3_{1}+1_{1}, 2_{1}+2_{1}, 2_{1}+1_{1}+1_{1}, 1_{1}+1_{1}+1_{1}+1_{1}
$$

are the six partitions of 4 with "M( $\alpha$ ) copies of $\alpha$."
If $p(n)$ is the number of unrestricted partitions of $n$, then

$$
\begin{aligned}
1+\sum_{n=1}^{\infty} m(n) q^{n} & =\prod_{n=1}^{\infty}\left(1-q^{n}\right)^{-1} \prod_{\substack{n>1 \\
M(n)>1}}(1-q)^{-(M(n)-1)} \\
& =\left(\sum_{n=0}^{\infty} p(n) q^{n} \prod_{\substack{n>1 \\
M(n)>1}}\left(\sum_{i=0}^{\infty} q^{i n}\right)^{M(n)-1}\right.
\end{aligned}
$$

Note that $M(n)=p(n)$ if $n=1,2,3$. Some values of $m(n)$ are shown below.

$$
\begin{array}{rrrrrrrrrrrrrrrrr}
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
m(n) & 1 & 2 & 3 & 6 & 8 & 13 & 18 & 30 & 41 & 60 & 82 & 121 & 162 & 226 & 302 & 422
\end{array}
$$

A recurrence formula for $m(n)$ is now given. Let $[r]$ denote the greatest integer less than or equal to the real number $r$; let

$$
\left(\sum_{i=0}^{\infty} q^{i}\right)^{k}=\sum_{v=0}^{\infty}(k)_{v} q^{v},
$$

where $k$ is a positive integer [so that $(k) v$ is the coefficient of $q v$ in the expanded form of $\left.\left(1+q+q^{2}+q^{3}+\ldots\right)^{k}\right)$; and let $s_{i}$ equal the $i^{\text {th }}$ nonsquarefree positive integer (with $s_{1}=4, s_{2}=8, s_{3}=9, s_{4}=12, s_{5}=16$, and so forth). Then, if $n \geq 4$,

$$
m(n)=\sum_{i=1}^{j} m_{n, s_{i}}
$$

where $j$ is the unique positive integer such that $s_{j} \leq n<s_{j+1}, m(0)$ is defined equal to 1 ,

$$
m_{n, 4}=p(n)+\sum_{i=1}^{[n / 4]} p(4-n i)
$$

for $n \geq 4$, and

$$
\begin{aligned}
m_{n, s_{j}}= & \left(M\left(s_{j}\right)-1\right)_{\left[n / s_{j}\right]} m\left(n-s_{j}\left[n / s_{j}\right]\right) \\
& +\sum_{i=1}^{\left[n / s_{j}\right]-1}\left(M\left(s_{j}\right)-1\right)_{i}\left(\sum_{v=1}^{j-1} m_{n-s_{j} i, s_{v}}\right)
\end{aligned}
$$

for $n \geq s_{j}>s_{1}$. For example,

$$
\begin{aligned}
m(16)= & m_{16,16}+m_{16,12}+m_{16,9}+m_{16,8}+m_{16,4} \\
= & (M(16)-1)_{1} m(0)+(M(12)-1)_{1} m_{1}(4)+(M(9)-1)_{1} m(7) \\
& +(M(8)-1)_{2} m(0)+(M(8)-1)_{1} m_{8,4} \\
& +(p(16)+p(12)+p(8)+p(4)+p(0)) \\
= & (3)_{1} \cdot 1+(1)_{1} \cdot 6+(1)_{1} \cdot 18+(2)_{2} \cdot 1 \\
& +(2)_{1}(p(8)+p(4)+p(0))+(231+77+22+5+1) \\
= & 3+6+18+3+56+336 \\
= & 422 .
\end{aligned}
$$

Combinatorial interpretations of partitions with "M( $\alpha$ ) copies of $\alpha$ " can be stated in terms of plane partitions [3, Ch. 11] and factorization patterns [2]. In [4], Mitchell considers plane partitions in which the number of parts equal to $j \geq 1$ in any row is not less than the number of parts equal to $j$ in the next row; we designate these plane partitions as Mitchell plane partitions (MPP's). Each MPP of a positive integer $n$ can be written uniquely in an "identical-element-column format" (ICF) of the type

$$
\begin{array}{llll}
\alpha_{11} & a_{21} & \cdots & a_{r 1} \\
\vdots & \vdots & & \vdots \\
a_{1 t_{1}} & \alpha_{2 t_{2}} & \cdots & a_{r t_{r}}
\end{array}
$$

with

$$
\sum_{i=1}^{r} \sum_{j=1}^{t_{i}} a_{i j}=n
$$

and

$$
\alpha_{i 1} \geq \alpha_{i+1,1}(i=1, \ldots, r-1)
$$

with

$$
\alpha_{i 1}=\cdots=\alpha_{i t_{i}}
$$

for each $i=1, \ldots, r$, and such that, if $a_{i 1}=\alpha_{k 1}$ for $i<k$, then $t_{i} \geq t_{k}$. If $n>1$, then $m(n)$ is the number of ICF's of the following types:
I. $a_{11} a_{21} \ldots a_{r l}\left(w i t h \sum_{i=1}^{r} a_{i 1}=n\right.$ and $a_{i 1} \geq a_{i+1,1}(i=1, \ldots, r-1)$; $t_{1}=\cdots=t_{r}=1$. LCF's of this type are unrestricted partitions of $n$. )
II. ICF's formed by first replacing one or more of any nonsquarefree $\alpha_{i l}$ ( $i=$ $1, \ldots, r$ ) in $I$, as indicated in (i) and (ii) below, and then rearranging these columns if necessary. (If $\alpha_{i l}$ is squarefree, then $\alpha_{i l}$ is the only acceptable form.)
(i) If $a_{i l} \neq a_{k l}$ for $k \neq i$, and $p$ is the smallest prime such that $p^{M\left(a_{i 1}\right)}$ divides $\alpha_{i l}$, then acceptable replacement forms for $\alpha_{i l}$ are those with $\alpha_{i 1} / p^{v}$ identical column entries, each entry $p^{v}\left(v=1, \ldots, M\left(\alpha_{i}\right)-1\right)$. If $\alpha_{i 1}=a_{i+1,1}=\ldots=a_{i+w, 1}, a_{i 1} \neq a_{k l}$ if $k \neq i, i+1, \ldots, i+w$ ( $1 \leq i<i+r \leq r$ ), then acceptable replacements are those with one or more of $a_{i l}, \ldots, a_{i+w, l}$ replaced by replacement forms specified in (i) under the condition that entries in the column replacing $\alpha_{b l}$ are greater than or equal to entries in the column replacing $\alpha_{c l}$ if $c>b(i \leq b<c \leq i+w)$.
Denote the set of ICF's of $n$ of these types by $\operatorname{MICF}(n)$, and $m(n)$ is the order of the set $\operatorname{MICF}(n)$.

Also, $m(n)$ is the number of restricted "maximum-exponent" factorization patterns (MFP's) of the type $b_{1}^{a_{1}} \ldots b_{r}^{a_{r}}$ with

$$
n=b_{1} a_{1}+\cdots+b_{r} a_{r}
$$

and

$$
b_{1}=\cdots=b_{k_{1}}>b_{k_{1}+1}=\cdots=b_{k_{2}}>\cdots>b_{k_{c-1}+1}=\cdots=b_{k_{e}}
$$

with

$$
k_{c}=r
$$

and

$$
\alpha_{1} \geq \ldots \geq a_{k_{1}}, a_{k_{1}+1} \geq \ldots \geq \alpha_{k_{2}}, \ldots, \alpha_{k_{c-1}+1} \geq \cdots \geq \alpha_{k_{c}},
$$

and in which, for $b_{v} a_{v}=w(1 \leq w \leq n)$ and for $v=1,2, \ldots, r, b_{v}^{a_{v}}$ has the following specified form:
(1) If $w$ is a squarefree positive integer, then $b_{v}^{a_{v}}=w^{1}$;
(2) If $w$ is not squarefree, and $p$ is the smallest prime such that $p^{M(w)}$ divides $w$, then $b_{v}^{a_{v}}=w^{l}$ or $b_{v}^{a_{v}}=\left(p^{t}\right)^{\left(w / p^{t}\right)}(t=1, \ldots, M(n)-1)$.
To illustrate, $m(8)=30$ and the elements of MICF (8) are

| 8 | 2 | 4 | 71 | 62 | 611 | 53 | 521 | 5111 | 44 | 42 | 22 | 431 | 321 | 422 | 222 |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 2 | 4 |  |  |  |  |  |  |  | 2 | 22 |  | 2 |  | 2 |  |
| 2 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 2 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |

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PARTITIONS WITH "M(a) COPIES OF \(a\) "
```

```
4211
```

4211
221111 2111111 11111111

```
221111 2111111 11111111
```

The element 321 (obtained from 231 by a column rearrangement) has plane partition form 321 and corresponds to the MFP $3^{1} 2^{2} 1^{1}$.

For any positive integer $n$ there is a bijection between the set MICF( $n$ ) and the set of MFP's of $n$. Also, a bijection between the members of the multiset $\left\{1_{1}, 2_{1}, 3_{1}, 4_{1}, 4_{2}, \ldots, n_{1}, \ldots, n_{M(n)}\right\}$ and the set of MFP's of $n$ is indicated by $1_{1}$ corresponding to $1^{1}, 2_{1}$ to $2^{1}, 3_{1}$ to $3^{1}, 4_{1}$ to $2^{2}, 4_{2}$ to $4^{1}, \ldots$, and $n_{1}$ to $n^{1}$ if $M(n)=1$, or $n_{1}$ to $p^{n / p}$, ..., $n_{v(n)-1}$ to $\left(p^{M(n)-1)(n / p::(n)-1), ~} n_{M(n)}\right.$ to $n^{1}$ if $M(n)>1$ and $p$ is the smallest prime such that $p^{M(n)}$ divides $n$.

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# PERIODS IN DUCCI'S $n$-NUMBER GAME OF DIFFERENCES 

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## 1. Introduction

Let $A=\left(a_{1}, \alpha_{2}, \ldots, \alpha_{n}\right)$ be an $n$-tuple of nonnegative integers, and define

$$
D(A)=\left(\left|a_{1}-a_{2}\right|,\left|a_{2}-a_{3}\right|, \ldots,\left|a_{n-1}-a_{n}\right|,\left|a_{n}-a_{1}\right|\right)
$$

Note that in the definition of $D$ the $n$-tuple $A$ is regarded as written in a circle, so Ludington's title "Cycles of Differences" [3] is more suggestive than my "Columns of Differences" [1].

Sequences of the form $A, D(A), D^{2}(A), D^{3}(A), \ldots$ are called Ducci sequences here (see [5] and [7]). Some authors call them n-number games.

Since applying $D$ does not increase the maximum of the components of a tuple it follows that in a Ducci sequence there are just a finite number of different tuples. Let $D^{S}(A)$ be the first tuple which is equal to a previous tuple $D^{r}(A)$, then the tuples $D^{r}(A), D^{r+1}(A), \ldots, D^{s-1}(A)$ form a repeating cycle. The length of this cycle, $s-r$, is called the period of the sequence. If $R$ and $S$ are any two natural numbers such that $D^{R}(A)=D^{S}(A)$, then $s-r \mid S-R$.

This article deals mainly with the periods of Ducci sequences. (Authors who deal with the length of the part that precedes the cycle refer to that length as the length of the game.)

## 2. Maximal Periods

The components of every tuple in the periodic part of a Ducci sequence are all equal to either 0 or a constant $C$ which depends on the first tuple of the sequence (see [1], Th. 1 in [2], Lem. 3 in [3], item I in [7]). Since for every positive $\lambda, D(\lambda A)=\lambda D(A)$, one may assume without loss of generality that $C=1$. In other words, let us restrict our attention to $n$-tuples with components from $\{0,1\}$. In particular, the Ducci sequence that starts with the n-tuple ( $0, \ldots ., 0,1$ ) will be called a basic Ducci sequence, and the length of its period is denoted $P(n)$.

Let $H\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right)=\left(\alpha_{2}, \ldots, \alpha_{n}, \alpha_{1}\right)$, then $H$ is a linear transformation over $Z_{2}$. Since $|x-y| \equiv x+y(\bmod 2)$, it follows thar $D=I+H$, and $D$ is also a linear transformation over $Z_{2}^{n}$.
Theorem 1: For any $n$, the maximal period of Ducci's sequence of $n$-tuples is $P(n)$. Periods of other sequences divide this maximum.

If $D^{R}(A)=D^{S}(A)$ holds for $A=(0,0,0, \ldots, 0,1)$, it holds also for $A=(0,0, \ldots, 0,1,0)$, for $A=(0, \ldots, 0,1,0,0)$, etc. This follows from the cyclic character of $D$ (or, alternatively, from the commutativity of $D$ with $H$ ). Since $D$ is linear, it holds also for sums of these $A^{\prime}$ s. $\quad \square$

## 3. Upper Bounds for $P(n)$

Lemma 1: If $2^{m} \equiv t(\bmod n)$, then $D^{\left(2^{m}\right)}=I+H^{t}$.
Proof: By Induction on $m,(I+H)^{\left(2^{m}\right)}=I+H^{\left(2^{m}\right)}$. $H^{\left(2^{m}\right)}=H^{t}$ since, by the definition of $H, H^{n}=I$. $\square$
Note that Lemma 1 suggests an effective way to compute $D^{r}(A)$ for big $r^{\prime} s$ : Write $r$ as $\sum 2^{m_{1}}$, then compute $\left(\Pi\left(I+H^{t_{1}}\right)\right)(A)$.

Corollalry 1: If $n$ is a power of 2, then the cycle of Ducci's sequences consists of a single $n$-tuple ( $0,0, \ldots, 0$ ).
Proof: In this case, $D^{n}=I+H^{0}=I+I=0 . \square$
Corollary 2: If $n$ is not a power of 2, then the cycle of the basic Ducci sequence contains an $n$-tuple with exactly two l's.
Proof: Take any $m$ which is big enough to assure that $D^{\left(2^{m}\right)}(0,0, \ldots, 0,1)$ is in the periodic part of the sequence. Reducing $2^{m}$ modulo $n$ gives a $t \neq 0$. $H^{t}(0,0, \ldots, 0,1)$ has exactly one 1 , but it is not ( $0,0, \ldots, 0,1$ ). Thus, the result follows from Lemma 1.
Corollary 3: If $2^{m} \equiv 1(\bmod n)$ then $P(n)$ divides $2^{m}-1$.
Proof: In this case $D^{\left(2^{m}\right)}=I+H^{1}=D^{1}$.
Remark: Both Corollaries 1 and 3 are not new. Corollary 1 is item $D_{1}$ in [7] and appears in at least 19 of the 22 articles referred to there, sometimes only for $n=4$. The present proof is considerably shorter than the ones in [7] and in [6]. Corollary 3 is written without proof in [1] and is the "further" part of Theorem 3 in [3], restricted to odd n's.
Theorem 2: If $2^{M} \equiv-1(\bmod n)$, then $P(n)$ divides $n\left(2^{M}-1\right)$.
Proof: $D^{\left(2^{M}\right)}=I+H^{-1}=H^{-1}(H+I)=H^{-1} D$; hence, $D^{\left(n 2^{M}\right)}=H^{-n} D^{n}=D^{n} . \square$
Let us use the following abbreviations:
a. For an odd $n>1$, let $m(n)$ be the smallest $m>0$ such that $2^{m} \equiv 1$ (mod $n)$. [By Euler's theorem, such an $m$ does exist and $m(n) \mid \phi(n)$.
b. If for an odd $n>1$ there is an $M$ such that $2^{M} \equiv-1(\bmod n)$, then $n$ will be said to be "with a -1." When this occurs, the smallest such $M$ is $m(n) / 2$. If this does not occur, then we say that $n$ is "without a -1."
Facts:
For every odd $n$ with a -1 , from 3 to 163 except for 37 and 101 ,

$$
P(n)=n\left(2^{m(n) / 2}-1\right) .
$$

For every odd $n$ without a -1 , from 7 to 165 except for 95 and 111 , $P(n)=2^{m(n)}-1$.
For all of the four exceptions, $P(n)$ is $1 / 3$ of the "expected" value. I do not know whether any deeper thing is hidden behind this divisor 3.

These data were computed in the following way. Since for every odd $n$ there is an $m$ such that $2^{m} \equiv 1(\bmod n)$, and since $(0, \ldots, 0,1,1)=D(0, \ldots, 0$, $1)$, it follows from the proof of Corollary 3 that, for such an $n,(0, \ldots, 0$, $1,1)$ is in the periodic part of the basic Ducci sequence. The note just after Lemma 1 gives a fast way for checking whether $D^{r}(0, \ldots, 0,1,1)=(0, \ldots, 0$, 1, 1). By Corollary 3 and Theorem 2, one has to check only r's which divide $2-1$, and in many cases only the divisors of $n\left(2^{m(n) / 2}-1\right)$.

As an example, let us see how $P(37)$ is found. 18 is the smallest $M$ such that $2^{M} \equiv-1(\bmod 37)$. [In other words, 37 is with $\mathrm{a}-1$ and $\left.m(37)=36.\right]$ By Theorem 2,

$$
P(37) \mid 37 \cdot\left(2^{18}-1\right)=9699291.9699291=3 \cdot 3 \cdot 3 \cdot 7 \cdot 19 \cdot 37 \cdot 73 .
$$

A subroutine based on Lemma 1 is now called, and outputs $D^{r}(0, \ldots, 0,1,1)$ for $r=9699291 / 3,9699291 / 7,9699291 / 19,9699291 / 37$, and $9699291 / 73$. The first of these $r^{\prime}$ s returns ( $0, \ldots, 0,1,1$ ), while the other ones do not. Running this subroutine for $r=9699291 / 9$ does not return ( $0, \ldots, 0,1,1$ )
either; hence,

$$
P(37)=9699291 / 3=3233097 .
$$

Remark: The $D^{r}$ subroutine is quite fast. The reason for stopping the calculations at $P(165)$ was the time needed for the factoring. I thank Yehuda Kats of Levinsky College for Teachers, Tel-Aviv, for factoring the numbers that were needed in calculating $P(131), P(139)$, and $P(149)$.

## 4. More Properties of $P(n)$

Having seen that $P(n)$ may be a proper divisor of $2^{m(n)}-1$ or of $n\left(2^{m(n) / 2}-1\right)$ there is an interest in the following theorem.
Theorem 3: If $n$ is not a power of 2 , then $n \mid P(n)$.
Proof: Write the components of an $A \in Z_{2}^{n}$ on the vertices of a regular $n$-gon in a counterclockwise order, starting, say, at the highest vertex. For example, write ( $0,0,0,1,1$ ) as follows:


If $A$ has an axis of symmetry, then $D(A)$ also does, and its axis is obtained from that of $A$ by a rotation of $-180 / n$ degrees. [It is the bisector of the axis of $A$ and the axis of $H(A)$.$] If A$ has more than one axis, then each of the axes is transferred to the followers of $A$ in the same way.

By Corollary 2, there is an n-tuple with exactly two l's in the cycle of the basic Ducci sequence. Since this $n$-tuple has just one axis of symmetry, so do all of the tuples in the repeating cycle. During one cycle, the axis rotates a whole multiple of 180 degrees, so the period is a multiple of $n$.

In the proofs of the following theorems, I am going to cut a tuple into equal parts and write these parts one below the other in the form of a matrix. These matrices are not intended to represent linear transformations. They are just another way to write the tuples, and you may read them the same way you read an English text of more than one line. For example, for

$$
\begin{aligned}
& A=(a, b, c, d, e, f, g, h, i, j, k, l), \\
& H(A)=H\left[\begin{array}{llll}
a & b & c & d \\
e & f & g & h \\
i & j & k & l
\end{array}\right]=\left[\begin{array}{llll}
b & c & d & e \\
f & g & h & i \\
j & k & l & a
\end{array}\right]
\end{aligned}
$$

On the other hand, if the dimensions of the matrix form of $A$ are given, then each row and each column of $A$ are well defined tuples on their own, and our transformations $H$ and $D$ may apply to each of them. Let us define $\|_{H}(A)$ to be the matrix obtained from. A by replacing each row by $!l$ (that row), and define $D_{L}(A), H_{C}(A)$, and $E_{C}(A)$ in similar ways, with "D" instead of " $/ l^{\prime \prime}$ or with "column" instead of "row." Following our last example,

$$
H_{L}(A)=\left[\begin{array}{llll}
b & c & d & a \\
f & g & h & e \\
j & k & l & i
\end{array}\right] \quad H_{C}(A)=\left[\begin{array}{llll}
e & f & g & h \\
i & j & k & z \\
a & b & c & d
\end{array}\right] .
$$

Theorem 4: If $n=2^{m} k$, where $k$ is an odd number, then $P(n)=2^{m} P(k)$.
Let us write each $n$-tuple $A$ as a $k \times 2^{m}$ matrix. Since each row of $A$ is, now, of $2^{m}$ components, $H^{\left(2^{m}\right)}(A)=H_{C}(A)$. By Lemma 1 , this holds for $D^{\prime} s$ too.

To find $P(n)$, it is sufficient to inspect just every $2^{m}$ th element of the basic Ducci sequence since, by Theorem 3, $2^{m} \mid P(n)$. If we start our inspections with the first element of the entire sequence, i.e., with the $n$-tuple whose matrix is

$$
\left[\begin{array}{llll}
0, & \ldots & 0, & 0 \\
\vdots & \ldots & & \\
0, & \ldots, & 0, & 0 \\
0, & \ldots & 0, & 1
\end{array}\right]
$$

then the right column of the inspected elements forms a basic Ducci sequence of $k$-tuples, and the other columns are $0^{\prime}$ s (since $D^{\left(2^{m}\right)}=D_{C}$ ). The period of the inspected subsequence is, thus, $P(k)$, and the period of the entire sequence is $2^{m} P(k)$.

Theorem 5: If $k \mid n$, then $P(k) \mid P(n)$.
Proof: By Theorem 1, it is sufficient to find an $n$-tuple whose Ducci sequence has $P(k)$ for its period. We are going to see that the $n / k \times k$ matrix

$$
\left[\begin{array}{llll}
0, & \ldots, & 0, & 1 \\
\vdots & & \\
0, & \ldots, & 0, & 1
\end{array}\right]
$$

will do.
Indeed, if all of the rows of a matrix $A$ are equal to each other, then $H(A)$ $=H_{L}(A)$; hence, $D(A)=D_{L}(A)$. It follows that the Ducci sequence of $n$-tuples which starts with the above mentioned matrix, behaves like the basic Ducci sequence of $k$-tuples.

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# FIBONACCI NUMBERS OF THE FORM $C X^{2}$, WHERE $1 \leq C \leq 1000$ 

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## Introduction

Let $c$ and $n$ be natural numbers. Let $F_{n}$ denote the $n$th Fibonacci number, that is $F_{1}=F_{2}=1, F_{n}=F_{n-1}+F_{n-2}$ for $n \geq 3$. Consider the equation
(*) $\quad F_{n}=c x^{2}$.
In [1], Cohn solved (*) for $c=1,2$. In [9], we found all solutions of (*) such that $c$ is prime and either $c \equiv 3(\bmod 4)$ or $c<10,000$. Harborth \& Kemnitz [4] have asked for solutions of ( $*$ ) for composite values of $c$. Clearly, it suffices to consider only squarefree values of $c$.

If $c \leq 1000$, then $c$ has at most three distinct odd prime factors. Therefore $c=k p$ where $p$ is prime and $k=2,3,5,6,7,10,11,13,14,15,17,19$, $21,22,23,26,29,30,31,33,34,35,38,39,42,51,55,65,66$, or 70 . In this paper, we solve (*) for each of the above values of $c$. In the cases $k=$ $2,13,26,34$, our results are valid only for $p<10,000$; in the other cases, there are no restrictions on $p$. These results are listed in Table 1. Combining these new results with those from [1] and [9], we obtain all solutions of (*) such that $1 \leq c \leq 1000$. We list these solutions in Table 2.

## Preliminaries

Let $p$ denote a prime, $m$ a natural number. Let $L_{n}$ denote the $n^{\text {th }}$ Lucas number, that is $L_{1}=1, L_{2}=3, L_{n}=L_{n-1}+L_{n-2}$ for $n \geq 3$. Let $o_{p}(n)=k$ if $p^{k} \| n$, where $k \geq 0$. Let $(a / p)$ denote the Legendre symbol. Let $z(n)=\min \left\{m: n \mid F_{m}\right\}$. If $p$ is odd and $2 \mid z(p)$, let $y(p)=\frac{1}{2} z(p)$.

$$
\begin{equation*}
5 \nmid L_{n}, 13 \nmid L_{n}, 17 \nmid L_{n} \text { for all } n \text {. } \tag{15}
\end{equation*}
$$

If $m \geq 2$, then $m \mid F_{n}$ iff $z(m) \mid n$.

$$
\begin{equation*}
F_{2 n}=F_{n} L_{n} \tag{16}
\end{equation*}
$$

$$
\begin{equation*}
\text { If } m \geq 3 \text {, then } F_{m} \mid F_{n} \text { iff } m \mid n \tag{17}
\end{equation*}
$$

$$
\left(F_{n}, L_{n}\right)= \begin{cases}2 & \text { if } 3 \mid n  \tag{18}\\ 1 & \text { if } 3 \nmid n\end{cases}
$$

$$
\begin{align*}
& \text { (3) If } p \equiv 3(\bmod 4) \text {, then } F_{n}=p x^{2} \text { iff }\left(n, p, x^{2}\right)=(4,3,1) \text {. } \\
& \text { (4) If } p \equiv 1(\bmod 4) \text { and } p<10,000 \text {, then } F_{n}=p x^{2} \text { iff } \\
& (n, p)=(5,5),(7,13),(11,89),(13,233),(17,1597) \text {, or }(25,3001) \text {. } \\
& \text { (5) } \quad F_{n} \neq 6 x^{2} \text {. (6) } \quad L_{n}=x^{2} \text { iff } n=1 \text { or } 3 . \\
& \text { (7) } \quad L_{n}=2 x^{2} \text { iff } n=6 \text {. } \\
& \text { (8) } L_{n}=3 x^{2} \text { iff } n=2 \text {. } \\
& L_{n} \neq 6 x^{2} \text {. }  \tag{10}\\
& L_{n}=7 x^{2} \text { iff } n=4 \text {. } \\
& L_{n}=11 x^{2} \text { iff } n=5 \text {. } \\
& \text { (12) } L_{n}=19 x^{2} \text { iff } n=9 \\
& L_{n}=29 x^{2} \text { iff } n=7 \text {. }  \tag{14}\\
& L_{4 n} \equiv 7(\bmod 8) \text { if } 3 \nmid n \text { 。 }  \tag{13}\\
& F_{n}=x^{2} \text { iff } n=1,2 \text {, or } 12 .  \tag{1}\\
& F_{n}=2 x^{2} \text { iff } n=3 \text { or } 6 \text {. }  \tag{2}\\
& \text { If } p \equiv 3(\bmod 4) \text {, then } F_{n}=p x^{2} \text { iff }\left(n, p, x^{2}\right)=(4,3,1) \text {. } \\
& F_{n} \neq 6 x^{2} \text {. } \\
& L_{n}=3 x^{2} \text { iff } n=2 \text {. } \\
& L_{n}=7 x^{2} \text { iff } n=4 \text {. }
\end{align*}
$$

If $m \geq 2$ ，then $L_{m} \mid L_{n}$ iff $n / m$ is odd．

$$
\text { If } k \text { is odd, then }\left(L_{n}, L_{k n} / L_{n}\right) \mid k
$$

$$
o_{2}\left(L_{n}\right)= \begin{cases}2 & \text { if } n \equiv 3(\bmod 6), \\ 1 & \text { if } n \equiv 0(\bmod 6), \\ 0 & \text { otherwise }\end{cases}
$$

$$
\text { If } p \text { is odd, then } p \mid L_{n} \text { iff } n=k y(p), k \text { odd. }
$$

$$
F_{7 m} / F_{m}=125 F_{m}^{6}+175(-1)^{m} F_{m}^{4}+70 F_{m}^{2}+7(-1)^{m}
$$

$$
3 \mid F_{n} \text { iff } 4 \mid n
$$

Remarks：（6），（7），（1），and（2）are Theorems 1 through 4 in［1］．（3）and （4）are Corollary 1 and Theorem 3 in［9］，respectively．（5）and（9）follow from Lemmas 1 and 2 in［20］，respectively．（8）and（10）are established in ［2］，（11）through（13）in［11］．（32）is Theorem 1 in［7］．（28）is Lemma 16 in ［3］，while（34）follows from Theorem 2 in［3］．（41）follows from Theorem 4 in ［8］．（17），（18），（20），and（27）are $I_{7}$ ，Theorem III，Theorem V，and Theorem II in［5］，respectively．（40）follows from $I_{15}$ and $I_{18}$ in［5］．The other identi－ ties are elementary or well known．

## The Main Results

Lemma 1：$\quad L_{3 m} / L_{m}=x^{2}$ iff $m=1$.
Proof：If $L_{3 m} / L_{m}=x^{2}$ ，then（22）implies $L_{m}^{2}-3(-1)^{m}=x^{2}$ ．If $m$ is odd，then $L_{m}^{2}=1$ ，so $m=1$ ．If $m$ is even，then $L_{m}^{2}=4$ ，which is impossible，since $m$ is a natural number．Conversely，$L_{3} / L_{1}=4=2^{2}$ ．
Lemma 2：$L_{3 m} / L_{m} \neq 2 x^{2}$ ．
Proof：Assume the contrary．Then（22）implies $L_{m}^{2}-3(-1)^{m}=2 x^{2}$ ．If $3 \mid x$ ，then $3 \mid L_{m}$ ，so we get $\pm 3 \equiv 0$（mod 9），an impossibility．If $3 \nmid x$ ，then $L_{m}^{2} \equiv 2 x^{2} \equiv 2$ $(\bmod 3)$ ，an impossibility，since $(2 / 3)=-1$ ．
Lemma 3：If $p$ is odd，then $F_{m p} \equiv(5 / p) F_{m}(\bmod p)$ ．
Proof：This follows from（91）in［6］and Fermat＇s theorem，noting that $\Delta=5$ for the Fibonacci sequence．

$$
\begin{align*}
& F_{3 n} / F_{n}=L_{n}^{2}-(-1)^{n} \text {. } \\
& F_{5 n} / 5 F_{n}=5 F_{n}^{4}+5(-1)^{n} F_{n}^{2}+1 . \\
& L_{5 n} / L_{n}=L_{n}^{4}-5(-1)^{n} L_{n}^{2}+5 . \\
& z(p) \mid(p-e) \text { where } e=\left\{\begin{aligned}
1 & \text { if } p \equiv \pm 1(\bmod 5), \\
-1 & \text { if } p \equiv \pm 2(\bmod 5), \\
0 & \text { if } p=5 .
\end{aligned}\right. \\
& \left(F_{m}, F_{n}\right)=F_{(m, n)} .  \tag{28}\\
& \left(F_{n}, F_{5 n} / 5 F_{n}\right)=1 \text {. }  \tag{30}\\
& F_{4 n-1}+2=F_{2 n+1} L_{2 n-2} \text {. }  \tag{32}\\
& x^{2}-5 y^{2}=-4 \text { iff } x=L_{n}, y=F_{n} \text { for some odd } n \text {. } \\
& \text { If } p \text { is odd, } p \mid F_{m} \text {, and } p \nmid a \text {, then } o_{p}\left(F_{p^{k} a m} / F_{n}\right)=k \text {. } \\
& 2 \mid F_{3 m} / F_{m} \text { iff } 3 \text { 价. } \\
& 3 \mid L_{n} \text { iff } n \equiv 2(\bmod 4) \\
& \left(F_{n}, F_{5 n} / F_{n}\right)= \begin{cases}5 & \text { if } 5 \mid n, \\
1 & \text { if } 5 \nmid n .\end{cases} \\
& L_{3 n} / L_{n}=L_{n}^{2}-3(-1)^{n} . \\
& \text { 5 } X F_{3 n} / F_{n} \text { 。 }  \tag{22}\\
& \text { ( } \left.F_{n}, F_{k n} / F_{n}\right) \mid k . \\
& F_{4 n+1}+2=F_{2 n-1} L_{2 n+2} . \\
& \left(F_{m}, L_{m \pm n}\right) \mid L_{n} \text {. } \\
& \text { (36) } 2 \mid L_{n} \text { iff } 3 \mid n \text {. } \\
& \text { (38) } 4 \mid L_{n} \text { iff } n \equiv 3(\bmod 6) \text {. }
\end{align*}
$$

Lemma 4: If $p \equiv 3$ or $7(\bmod 20)$, then $F_{m p} / F_{m} \neq x^{2}$.
Proof: Let $F_{m p} / F_{m}=x^{2}$. If $p \mid F_{m}$, then (34) implies $o_{p}\left(F_{m p} / F_{m}\right)=1$, an impossibility. If $p \nmid F_{m}$, then Lemma 3 implies $F_{m p} / F_{m} \equiv(5 / p)(\bmod p)$, so $x^{2} \equiv(5 / p)$ $(\bmod p)$. If $p \equiv 3$ or $7(\bmod 20)$, then $(5 / p)=-1$, so $x^{2} \equiv-1(\bmod p)$ and $p \equiv 3$ (mod 4), an impossibility.
Lemma 5: If $F_{3 m} / F_{m}=2 x^{2}$, then $m$ is odd or $m=2$.
Proof: We must show that $F_{6 j} / F_{2 j}=2 x^{2}$ iff $j=1$. Now, $F_{6} / F_{2}=8=2(2)^{2}$. If $F_{6 j} / F_{2 j}=2 x^{2}$, then (17), (18), and (20) imply ( $F_{3 j} / F_{j}$ ) $\left(L_{3 j} / L_{j}\right)=2 x^{2}$. If $3 \nmid j$, then (35) implies $2 \mid F_{3 j} / F_{j}$, so

$$
\left(F_{3 j} / 2 F_{j}\right)\left(L_{3 j} / L_{j}\right)=x^{2}
$$

Let $d=\left(F_{3 j} / 2 F_{j}, L_{3 j} / L_{j}\right)$. Now, $d \mid\left(F_{3 j}, L_{3 j}\right)$, so (19) implies $d \mid 2$. We have

$$
F_{3 j} / 2 F_{j}=d y^{2}, \quad L_{3 j} / L_{j}=d z^{2}
$$

Lemma 2 implies $d \neq 2$. Therefore, $d=1$, so Lemma 1 implies $j=1$. If $j=3 k$, then (35) implies $2 \nmid F_{9 k} / F_{3 k}$. Let $g=\left(F_{9 k} / F_{3 k}, L_{9 k} / L_{3 k}\right)$. Then, $g \mid\left(F_{9 k}, L_{9 k}\right)$, so (19) implies $g \mid 2$. But $2 \nmid F_{9 k} / F_{3 k}$, so $g=1$. Therefore,

$$
F_{9 k} / F_{3 k}=y^{2}, \quad L_{9 k} / L_{3 k}=z^{2},
$$

which contradicts Lemma 1.
Lemma 6: $F_{3 m} / F_{m}=3 x^{2}$ iff $\left(m, x^{2}\right)=(4,16)$.
Proof: If $F_{3 m} / F_{m}=3 x^{2}$, then (16) implies $z(3) \mid 3 m$, so $m=4 k$. Now (21) implies $L_{4 k}^{2}-1=3 x^{2}$. If $3 \mid k$, then (36) implies $2 \mid L_{4 k}$, so $\left(L_{4 k}+1, L_{4 k}-1\right)=1$, so $L_{4 k} \pm 1=u^{2}$. Now (40) implies $L_{2 k}^{2}-1=u^{2}$ or $L_{2 k}^{2}-3=u^{2}$, so $L_{2 k}^{2}=1$ or 4, an impossibility. If $3 \nmid k$, then (36) implies $2 \nmid L_{4 k}$, so $\left(L_{4 k}+1, L_{4 k}-1\right)=2$. In fact, (14) implies

$$
\frac{L_{4 k}+1}{8} * \frac{L_{4 k}-1}{2}=3 y^{2}
$$

Since the factors on the left are coprime, one of them must be a square. If $\frac{1}{2}\left(L_{4 k}-1\right)=v^{2}$, then (40) implies $L_{2 k}^{2}-3=2 v^{2}$, an impossibility, since (2/3) $=-1$. Therefore,

$$
\left(L_{4 k}+1\right) / 8=u^{2} \quad \text { and } \quad \frac{1}{2}\left(L_{4 k}-1\right)=3 v^{2}
$$

Now $L_{4 k} \equiv 1(\bmod 6)$ implies $(6, k)=1$, so $L_{2 k} \equiv 3(\bmod 4)$. (40) implies $L_{2 k}^{2}-1$ $=8 u^{2}$, so $\left(L_{2 k}+1\right)\left(L_{2 k}-1\right)=8 u^{2}$. Since $2 \nmid L_{4 k}$, (40) also implies $2 \nmid L_{2 k}$, so $\left(L_{2 k}+1, L_{2 k}-1\right)=2$. Thus, we have

$$
L_{2 k}+1=4 a^{2}, \quad L_{2 k}-1=2 b^{2}
$$

Again (40) implies $L_{k}^{2}+3=4 a^{2}$, so that $L_{k}^{2}=1, k=1, m=4, x^{2}=16$. Conversely, $F_{12} / F_{4}=144 / 3=48=3(4)^{2}$.
Lemma 7: $F_{3 m} / F_{m} \neq 6 x^{2}$.
Proof: Assuming the contrary and reasoning as in the proof of Lemma 6 , we have $m:=4 k$ and $L_{4 k}^{2}-1=6 x^{2}$. Since $L_{4 k}$ is odd, (36) and (14) imply

$$
\left.\left(\left(L_{4 k}+1\right) / 8\right)\left(L_{4 k}-1\right) / 2\right)=6 w^{2}
$$

Since the factors on the left are coprime, we have

$$
\left(L_{4 k}+1\right) / 8=2 a y^{2}, \quad \frac{1}{2}\left(L_{4 k}-1\right)=b z^{2}, \quad a b=3
$$

If $a=1$, then $L_{4 k}=(4 y)^{2}-1$, which contradicts Theorem 5 in [6]. If $b=1$, then (14) implies $z^{2} \equiv 3(\bmod 4)$, an impossibility.

Lemma 8: If $p \mid F_{5 m} / F_{m}$, then $p=5$ or $p \equiv 1(\bmod 10)$.
Proof: If $p \mid F_{5 m} / F_{m}$ and $p \neq 5$, then $p \mid F_{5 m} / 5 F_{m}$, so (23) implies

$$
5 F_{m}^{4}+5(-1)^{m} F_{m}^{2}+1 \equiv 0(\bmod p)
$$

Since the discriminant is 5 , we must have $(5 / p)=1$; therefore, (26) implies $z(p) \mid(p-1)$. Now (16) implies $p \mid F_{p-1}$. The hypothesis implies $p \mid F_{5 m}$; hence, $p \mid\left(F_{5 m}, F_{p-1}\right)$. (27) implies $p \mid F_{(5 m, p-1)}$. (29) implies $p \nmid F_{m}$, so $p \nmid\left(F_{m}, F_{p-1}\right)$. (27) also implies $p \nmid F_{(m, p-1)}$; therefore, $(5 m, p-1) \neq(m, p-1)$, so $5 \mid(p-1)$. Thus, $p \neq 2$, so $p \equiv 1(\bmod 10)$.
Lemma 9: $L_{5 m} / L_{m} \neq x^{2}$.
Proof: Assume the contrary. Then (25) implies

$$
L_{m}^{4}-5(-1)^{m} L_{m}^{2}+5=x^{2}
$$

The discriminant is $25-4\left(5-x^{2}\right)=4 x^{2}+5$. Since our equation has integer roots, we must have $4 x^{2}+5=t^{2}$, so $x^{2}=1$, and $L_{m}^{2}=(-1)^{m}$ or $4(-1)^{m}$. But then $2 \mid m$ and $L_{m}^{2}=1$ or 4 , an impossibility.
Lemma 10: If $F_{n}=x^{2}-2$ and $n \neq 2(\bmod 4)$, then $\left(n, x^{2}\right)=(3,4)$ or $(9,36)$.
Proof:
Case 1. Let $n=4 m+1$. The hypothesis and (30) imply
$F_{2 m-1} L_{2 m+2}=x^{2}$.
Let $d=\left(F_{2 m-1}, L_{2 m+2}\right)$. (32) implies $d \mid L_{3}$, that is, $d \mid 4$. If $d=1$ or 4 , then $F_{2 m-1}$ and $L_{2 m+2}$ are squares, which contradicts (6). If $d=2$, then $F_{2 m-1}=2 y^{2}$ and $L_{2 m+2}=2 z^{2}$. (2) implies $2 m-1=3$, so $n=9$ and $x^{2}=36$.

Case 2. Let $n=4 m-1$. The hypothesis and (31) imply

$$
F_{2 m+1} L_{2 m-2}=x^{2}
$$

As in Case 1 , we must have $\left(F_{2 m+1}, L_{2 m-2}\right)=2$, so $F_{2 m+1}=2 y^{2}, L_{2 m-2}=2 z^{2}$. (2) implies $2 m+1=3$, so $n=3$ and $x^{2}=4$.

Case 3. Let $n=4 \mathrm{~m}$. Then $F_{n} \equiv 0,3$, or $5(\bmod 8)$. But $x^{2}-2 \equiv 6$, 7 , or 2 (mod 8). Therefore, $F_{n} \neq x^{2}-2$.
Lemma 11: $F_{5 m} / 5 F_{m}=x^{2}$ iff $m=x^{2}=1$.
Proof: Let $F_{5 m} / 5 F_{m}=x^{2}$. If $m=2 k$, then (17), (18), and (20) imp1y

$$
\left(F_{5 k} / 5 F_{k}\right)\left(L_{5 k} / L_{k}\right)=x^{2}
$$

Let $d=\left(F_{5 k} / 5 F_{k}, L_{5 k} / L_{k}\right)$. Then $d \mid\left(F_{5 k}, L_{5 k}\right)$, so (19) implies $d / 2$. But Lemma 8 implies $2 \nmid F_{5 m} / 5 F_{m}$, so $d=1$. Therefore, both $F_{5 k} / 5 F_{k}$ and $L_{5 k} / L_{k}$ are squares, which contradicts Lemma 9. If $2 \nmid m$, then (23) implies

$$
5 F_{m}^{4}-5 F_{m}^{2}+1=x^{2}
$$

The discriminant is $25-20\left(1-x^{2}\right)=20 x^{2}+5$. Since the preceding equation has integer roots, we must have $20 x^{2}+5=t^{2}$, but then $5 \mid t$, so $t^{2}=25 \omega^{2}$, and $4 x^{2}+1=5 w^{2}$. Therefore $(4 x)^{2}-5(2 w)^{2}=-4$. Now (33) implies that there exists odd $n$ such that $F_{n}=2 \omega, L_{n}=4 x$. Also

$$
F_{m}^{2}=(5 \pm 5 w) / 10=(1 \pm w) / 2
$$

Since $F_{m}^{2}>0$, we have $F_{m}^{2}=\frac{1}{2}(1+w)$. Therefore, $F_{n}=4 F_{m}^{2}-2$. Since $n$ is odd, Lemma 10 implies $F_{m}=1$ or 3. Now $m$ is odd, so $F_{m} \neq 3$. Therefore, $F_{m}=1$, so $m=x^{2}=1$. Conversely, $F_{5} / 5 F_{1}=1^{2}$.
Remark: Let $F_{m}=F_{m}^{*} F_{m}$, where $\left(F_{m^{\prime}}^{*}, F_{d}\right)=1$ for all $d<m$. $F_{m}^{*}$ is called the primitive part of $E_{m}$. In particular,

$$
F_{5 p}^{*}=F_{5 p} / F_{5} F_{p}=F_{5 p} / 5 F_{p} \quad(\text { if } p \neq 5)
$$

Lemma 11 implies $F_{5 p}^{*} \neq x^{2}$.
Lemma 12: $F_{9 m} / F_{m} \neq p x^{2}$ 。
Proof: If $F_{9 m} / F_{m}=p x^{2}$, let $d=\left(F_{3 m} / F_{m}, F_{9 m} / F_{3 m}\right)$. Now $d \mid\left(F_{3 m}, F_{9 m} / F_{3 m}\right)$; thus, (28) implies $d \mid 3$. If $d=1$, then $F_{3 m} / F_{m}$ or $F_{9 m} / F_{3 m}$ is a square, which contradicts Lemma 4. If $d=3$, then $F_{3 k} / F_{k}=3 y^{2}$, where $k=m$ or $3 m$. Lemma 6 implies $k=4=m$. But $F_{36} / F_{4} \neq p x^{2}$. The case $F_{9 m} / F_{m}=x^{2}$ is similar.
Lemma 13: $F_{7 m} / F_{m} \neq 7 x^{2}$.
Proof: Let $m$ be the least integer such that there exists $x$ such that $F_{7 m} / F_{m}=$ $7 x^{2}$. Now $7 \mid F_{7 m}$, so (16) implies $z(7) \mid 7 m$, so $8 \mid m$. Let $m=2 k$. (17), (18), and (20) imply

$$
\left(F_{7 k} / F_{k}\right)\left(L_{7 k} / L_{k}\right)=7 x^{2}
$$

Let $d=\left(F_{7 k} / F_{k}, L_{7 k} / L_{k}\right)$. Therefore, $d \mid\left(F_{7 k}, L_{7 k}\right)$, so (19) implies $d \mid 2$. But (44) implies $F_{7 k} / F_{k}$ is odd, so $d=1$. Therefore, $F_{7 k} / F_{k}=y^{2}$ or $7 y^{2}$. But the first possibility contradicts Lemma 4, while the second possibility contradicts the minimality of $m$.
Lemma 14: If $p$ and $y(p)$ are odd, then $L_{n} \neq 2 p x^{2}$.
Proof: If $L_{n}=2 p x^{2}$, then the hypothesis implies $o_{2}\left(L_{n}\right)$ is odd, so (42) implies $6 / n$. But the hypothesis and (43) imply $n$ is odd, a contradiction.
Lemma 15: If $p \equiv 5$ or $7(\bmod 8)$, then $L_{n} \neq 2 p x^{2}$.
Proof: Let $L_{n}=2 p x^{2}$. Then (36) implies $n=3 m$, so that $L_{m}\left(L_{3 m} / L_{m}\right)=2 p x^{2}$. Let $d=\left(L_{m}, L_{3 m} / L_{m}\right)$. (41) implies $d \mid 3$.

Case 1. $d=1$. (22) implies $3 \nmid L_{m}$, so (37) implies $m \not \equiv 2(\bmod 4)$. We have $L_{m}=a y^{2}, L_{3 m} / L_{m}=b z^{2}$, with $a b=2 p$, so $a \mid 2$ or $b \mid 2$. If $a=1$, then $b=2 p$ and (6) implies $m=1$ or 3. But $L_{3} / L_{1}=4 \neq 2 p z^{2} ; L_{g} / L_{3}=19 \neq 2 p z^{2}$. If $a=2$, then (7) implies $m=6$, an impossibility. If $b=1$, then $a=2 p$ and Lemma 1 implies $m=1$, so $L_{1}=1 \neq 2 p z^{2}$. Lemma 2 implies $b \neq 2$.

Case 2. $d=3$. Then $L_{m}=3 a y^{2}, L_{3 m} / L_{m}=3 b z^{2}$, with $a \mid 2$ or $b \mid 2$. If $a=1$, then $b=2 p$, and (8) implies $m=2$, but $L_{6} / L_{2}=6 \neq 6 p z^{2}$. (9) implies $a \neq 2$. (37) implies $m \equiv 2(\bmod 4)$, so (22) implies $L_{m}^{2}-3=3 b z^{2}$. Therefore, $3 b z^{2} \equiv-3$ (mod 9), so $b z^{2} \equiv-1(\bmod 3)$; thus, $b \neq 1$. If $b=2$, then $L_{m}^{2} \equiv 3$ (mod 6), which implies $m=12 k \pm 2$. Since $\alpha=p$, we have $L_{l 2 k \pm 1}^{2}=3 p y^{2}$. (40) implies $L_{6 k \pm 1}^{2}+2$ $=3 p y^{2}$. Therefore, $(-2 / p)=1$, which is impossible if $p \equiv 5$ or 7 (mod 8).
Lemma 16: Let $F_{n}=k p x^{2}$, where $2 \mid z(k)$. Then $2 \mid n$ and $F_{\frac{1}{2} n}=d a y^{2}$, $L_{\frac{1}{2} n}=d b z^{2}$, where

$$
d=\left(F_{\frac{1}{2} n}, L_{\frac{1}{2} n}\right)=\left\{\begin{array}{ll}
2 & \text { if } 3 \mid n \\
1 & \text { if } 3 \nmid n
\end{array}, a b=k p, \quad(a, b)=1, \text { and } d y z=x\right.
$$

Proof: The hypothesis, (16), and (17) imply $2 \mid n, F_{\frac{1}{2} n} L_{\frac{1}{2} n}=k p x^{2}$. The conclusion now follows from (19).
Theorem 1: $F_{n} \neq 6 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(6) \mid n$, so $n=12 \mathrm{~m}$. (38) and Lemma 16 imply $F_{6 m}=4 a y^{2}, L_{6 m}=2 b z^{2}, a b=3 p$. If $a=1, b=3 p$; hence, (37) implies $m$ is odd. But (1) implies $m=2$, an impossibility. (3) implies $a \neq 3$. If $b=1$, then $\alpha=3 p$, so (45) implies $2 \mid m$, but (7) implies $m=1$, an impossibility. (9) implies $b \neq 3$.
Theorem 2: $F_{n}=3 p x^{2}$ iff $\left(n, p, x^{2}\right)=(8,7,1)$ or $(12,3,16)$.

Proof: Assume $F_{n}=3 p x^{2}$. (16) implies $z(3) \mid n$, so $n=4 m$. Lemma 16 implies $F_{2 m}$ $=d a y^{2}, L_{2 m}=2 b z^{2}, d=\left(F_{2 m}, L_{2 m}\right), \alpha b=3 p$. If $3 \nmid m$, then (19) implies $d=1$, so either $F_{2 m}=y^{2}$ or $3 y^{2}$, or $L_{2 m}=z^{2}$ or $3 z^{2}$. (1), (3), (6), and (8) imply $2 m$ $=2$ or 4 , so $n=4$ or 8 . Now $F_{4}=3 \neq 3 p x^{2} . \quad F_{8}=21=3 p x^{2}$ implies $p=7, n=$ $8, x^{2}=1$. If $m=3 k$, then (19) implies $d=2$, so either $F_{6 k}=2 y^{2}$ or $6 y^{2}$, or $L_{2 k}=2 z^{2}$ or $6 z^{2}$. (2), (5), (7), and (9) imply $6 k=6$, so $n=12$. Now $F_{12}=$ 144, so $p=3, n=12, x^{2}=16$. Conversely, $F_{8}=21$ and $F_{12}=144$.
Theorem 3: Let $2<p<10^{4}$. Then $F_{n}=2 p x^{2}$ iff $\left(n, p, x^{2}\right)=(9,17,1)$.
Proof: If $F_{n}=2 p x^{2}$, then (16) implies $z(2) \mid n$, so $n=3 m$ and $F_{m}\left(F_{3 m} / F_{m}\right)=2 p x^{2}$. Let $d=\left(F_{m}, F_{3 m} / F_{m}\right)$. (28) implies $d \mid 3$. If $d=1$, then $F_{m}=\alpha y^{2}, F_{3 m} / F_{m}=b z^{2}$, $a b=2 p$. If $a=1$, then $2 F_{3 m} / F_{m}$. Therefore, (1) and (35) imply $m=1$ or 2 , so $n=3$ or 6. But $F_{3}=2 \neq 2 p x^{2} ; F_{6}=8 \neq 2 p x^{2}$. If $a=2$, then $b=p$ and (2) implies $m=3$ or 6 , so $n=9$ or 18 . Now $F_{18} / F_{6} \neq p x^{2} . F_{9} / F_{3}=17$, so, if $n=9$, then $p=17, x^{2}=1$. Lemma 4 implies $b=1$. If $b=2$, then $F_{m}=p y^{2}$. Since $F_{2}=1=p y^{2}$, Lemma 5 implies $m$ is odd. Therefore, (3), (4), and the hypothesis imply $m=5,7,11,13,17$, or 25 . But none of the corresponding values of $F_{3 m} / 2 F_{m}$ is a square. If $d=3$, then $F_{m}=3 a y^{2}, F_{3 m} / F_{m}=3 b z^{2}, a b=$ $2 p$. If $a=1$, then $b=2 p$. (3) implies $m=4$, but $F_{12} / F_{4}=48=6 p z^{2}$, so $p=$ 2, contrary to the hypothesis. (5) implies $\alpha \neq 2$. If $b=1$, then $\alpha=2 p$, which contradicts Theorem 1. If $b=2$, then $F_{3 m} / F_{m}=6 z^{2}$, which contradicts Lemma 7. Conversely, $F_{9}=34$.
Theorem 4: $F_{n}=5 p x^{2}$ iff $\left(n, p, x^{2}\right)=(10,11,1)$.
Proof: If $F_{n}=5 p x^{2}$, then (16) implies $z(5) \mid n$, so $n=5 m$, and $F_{m}\left(F_{5 m} / F_{m}\right)=5 p x^{2}$, so $F_{m}\left(F_{5 m} / 5 F_{m}\right)=p x^{2}$. Now (29) implies either (i) $F_{m}=y^{2}, F_{5 m} / 5 F_{m}=p z^{2}$, or (ii) $F_{m}=p y^{2}, F_{5 m} / 5 F_{m}=z^{2}$. If (i) holds, then (1) implies $m=1$, 2, or 12 . We get a contradiction unless $m=2, n=10, p=11, x^{2}=1$. If (ii) holds, then Lemma 11 implies $m=1$, so $F_{1}=1=p y^{2}$, an impossibility. Conversely, $F_{10}=55$.
Theorem 5: $F_{n}=7 p x^{2}$ iff $\left(n, p, x^{2}\right)=(8,3,1)$.
Proof: If $F_{n}=7 p x^{2}$, then (16) implies $z(7) \mid n$, so $n=8 m$. If $3 \nmid m$, then Lemma 16 implies $F_{4 m}=a y^{2}, L_{4 m}=b z^{2}, \alpha b=7 p$. If $a=1$, then (1) implies $m=3$, a contradiction. (3) implies $a \neq 7$. (6) implies $b \neq 1$. If $b=7$, then (10) implies $4 m=4$, so $n=8, p=3, x^{2}=1$. If $m=3 k$, then Lemma 16 implies $F_{12 k}$ $=2 a y^{2}, L_{12 k}=2 b z^{2}, a b=7 p$. (2) implies $a \neq 1$. Theorem 3 implies $a \neq 7$. (7) implies $b \neq 1$. Lemma 15 implies $b \neq 7$. Conversely, $F_{8}=21$.

Theorem 6: $F_{n} \neq 15 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(15) \mid n$, so $n=20 \mathrm{~m}$. If $3 \nmid m$, then (15) and Lemma 16 imply $F_{10 m}=5 a y^{2}, L_{10 m}=b z^{2}, \alpha b=3 p$. Now (4) implies $\alpha \neq 1$. Theorem 2 implies $a \neq 3$. (6) and (8) imply $b \neq 1$ and 3, respectively. If $m=3 k$, then (15) and Lemma 16 imply $F_{30 k}=10 \alpha y^{2}, L_{30 k}=2 b z^{2}, a b=3 p$ Theorems 3 and 1 imply $a \neq 1$ and 3, respectively. (7) and (9) imply $b \neq 1$ and 3, respectively.
Theorem 7: $F_{n}=10 p x^{2}$ iff $\left(n, p, x^{2}\right)=(15,61,1)$.
Proof: If $F_{n}=10 p x^{2}$, then (16) implies $z(10) \mid n$, so $n=15 m$, and $F_{5 m}\left(F_{15 m} / F_{5 m}\right)$ $=p x^{2}$. Let $d=\left(F_{5 m}, F_{15 m} / F_{5 m}\right)$. (28) implies $d \mid 3$. (24) implies $F_{5 m}=d a y^{2}$, $F_{15 m} / F_{5 m}=d b z^{2}, a b=2 p$. Suppose $d=1$. If $a=1$, then $b=2 p$ and (4) implies $5 m=5$, so $F_{15} / F_{5}=122=2 p z^{2}$. Therefore, $p=61, n=15, x^{2}=1$. Theorem 3 implies $a \neq 2$. Lemma 4 implies $b \neq 1$. If $b=2$, then $a=p$, so Theorem 4 implies $5 m=10$. But $F_{30} / F_{10} \neq 2 z^{2}$. Now suppose that $d=3$. Then $F_{5}=15 a y^{2}$, $F_{15} / F_{5}=3 b z^{2}, a b=2 p$. Theorems 2 and 1 imply, respectively, $a \neq 1$ and 2 . Lemmas 6 and 7 imply, respectively, $b \neq 1$ and 2 . Conversely, $F_{15}=610$.

Theorem 8: $F_{n}=11 p x^{2}$ iff $\left(n, p, x^{2}\right)=(10,5,1)$.
Proof: If $F_{n}=11 p x^{2}$, then (16) implies $z(11) \mid n$, so $n=10 m$. If $3 \nmid m$, then Lemma 16 implies $F_{5 m}=\alpha y^{2}, L_{5 m}=b z^{2}$, where $a b=11 p$, so $\alpha$ or $b=1$ or 11 . (1) and (3) imply, respectively, $a \neq 1$ and 11. (6) implies $b \neq 1$. If $b=11$, then $a=p$. Now (11) implies $5 m=5$, so $p=5, n=10$, and $x^{2}=1$. If $m=3 k$, then Lemma 16 implies $F_{15 k}=2 \alpha y^{2}, L_{15 k}=2 b z^{2}$, with $a$ and $b$ as above. (2) implies $a \neq 1$. Theorem 3 implies $\alpha \neq 11$. (6) implies $b \neq 1$. If $b=11$, then $L_{15 k}=22 z^{2}$. But since $y(11)=5$, this contradicts Lemma 14. Conversely, $F_{10}=55$.
Theorem 9: Let $p<10^{4}$. Then $E_{n}=13 p x^{2}$ iff $\left(n, p, x^{2}\right)=(14,29,1)$.
Proof: If $F_{n}=13 p x^{2}$, then (16) implies $z(13) \mid n$, so $n=7 m$, and $F_{m}\left(F_{7 m} / F_{m}\right)=$ 13px ${ }^{2}$. Let $d=\left(F_{m}, F_{7 m} / F_{m}\right)$. (28) implies $d 7$. If $d=1$, then $F_{m}=a y^{2}$, $F_{7 m} / F_{m}=b z^{2}, a b=13 p$, so $a$ or $b=1$ or 13. If $a=1$, then (1) implies $m=$ 1,2 , or 12. We get a contradiction unless $m=2$, in which case $n=14, p=$ 29, $x^{2}=1$. If $a=13$, then $b=p$ and (4) implies $m=7$. But $F_{49} / F_{7} \neq p z^{2}$. Lemma 4 implies $b \neq 1$. If $b=13$, then $a=p$. Now, the hypothesis and (4) imply $m=4,5,7,11,13,17$, or 25 . In each case, $F_{7 m} / F_{m} \neq p z^{2}$. If $d=7$, then (16) implies $z(7) \mid m$, so $m=8 k$, and we have $F_{8 k}=7 a y^{2}, F_{56 k} / F_{8 k}=7 b z^{2}$, $a b=13 p$. (3) implies $a \neq 1$. Theorem 5 implies $a \neq 13$. Lemma 13 implies $b \neq$ 1. If $b=13$, then $a=p$, so Theorem 5 implies $8 k=8$. But then $F_{56} / 91 F_{8}=$ $z^{2}$, an impossibility. Conversely, $F_{14}=377$.
Theorem 10: $F_{n}=14 p x^{2}$ iff $\left(n, p, x^{2}\right)=(24,23,144)$.
Proof: If $F_{n}=14 p x^{2}$, then (16) implies $z(14) \mid n$, so $n=24 m$. (38) and Lemma 16 imply $F_{12 m}=4 a y^{2}, L_{12 m}=2 b z^{2}, \alpha b=7 p$. If $a=1$, then (1) implies $12 m=12$, from which it follows that $n=24, p=23, x^{2}=144$. (3) implies $\alpha \neq 7$. (7) implies $b \neq 1$. Lemma 15 implies $b \neq 7$. Conversely, $F_{24}=46368$.
Theorem 11: $F_{n}=17 p x^{2}$ iff $\left(n, p, x^{2}\right)=(9,2,1)$.
Proof: If $F_{n}=17 p x^{2}$, then (16) implies $z(17) \mid n$, so $n=9 m$ and $F_{m}\left(F_{9 m} / F_{m}\right)=$ $17 p x^{2}$. Let $d=\left(F_{m}, F_{9 m} / F_{m}\right)$. (28) implies $d \mid 9$. Now $F_{m}=d a y^{2}, F_{9 m} / F_{m}=d b z^{2}$, $a b=17 p$. If $d=1$ or 9 , then Lemma 12 implies $b \neq 1,17, p$. Therefore, $b=$ $17 p$ and $a=1$, so (1) implies $m=1,2$, or 12 . We have a contradiction unless $m=1$, in which case $F_{9} / 17 F_{1}=2=p z^{2}$, so $p=2$, $n=9, x^{2}=1$. If $d=3$, then $O_{3}\left(F_{9 m} / F\right)$ is odd, but (34) implies $O_{3}\left(F_{9_{m}} / F_{m}\right)=2$. Conversely, $F_{9}=34$ 。
Theorem 12: $F_{n} \neq 19 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(19) \mid n$, so $n=18 \mathrm{~m}$. Lemma 16 implies $F_{9 m}=2 a y^{2}, L_{9 m}=2 b z^{2}, a b=19 p$. (2) implies $a \neq 1$. Theorem 3 implies $a \neq 19$. (7) implies $b \neq 1$. Since $y(19)=9$, Lemma 14 implies $b \neq 19$.
Theorem 13: $F_{n}=21 p x^{2}$ iff $\left(n, p, x^{2}\right)=(16,47,1)$.
Proof: If $F_{n}=2 l p x^{2}$, then (16) implies $z(21) \mid n$, so $n=8 m$. (37) implies $3 \nmid L_{4 m}$. If $3 \nmid m$, then Lemma 16 implies $F_{4 m}=3 a y^{2}$, $L_{4 m}=b z^{2}$, with $a b=7 p$. If $a=1$, then (3) implies $4 m=4$, so $L_{4}=7=7 p z^{2}$, an impossibility. If $a=7$, then Theorem 2 implies $4 m=8$ and $L_{8}=47=p z^{2}$, so $p=47, n=16$, and $x^{2}=1$. (6) implies $b \neq 1$. If $b=7$, then (10) implies $4 m=4$, so $F_{4}=3=3 p z^{2}$, an impossibility. If $m=3 k$, then Lemma 16 implies $F_{12 k}=6 \alpha y^{2}, L_{12 k}=2 b z^{2}, \alpha b=7 p$. (5) implies $\alpha \neq 1$. Theorem 1 implies $a \neq 7, a \neq p$. (7) implies $b \neq 1$. Conversely, $F_{16}=987$.
Theorem 14: $F_{n} \neq 22 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(22) \mid n$, so $n=30 \mathrm{~m}$. Lemma 16 implies $F_{15 m}=2 a y^{2}, L_{15 m}=2 b z^{2}, a b=22 p$, so $a \mid 22$ or $b \mid 22$. Now (2) and (1) imply $\alpha \neq 1$ and 2, respectively. Theorem 3 implies $\alpha \neq 11$. (3) implies
$a \neq 22$. (7) and (6) imply $b \neq 1$ and 2, respectively. Lemma 14 implies $b \neq 11$. (11) implies $b \neq 22$.

Theorem 15: $F_{n} \neq 23 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(23) \mid n$, so $n=24 m$. Lemma 16 implies $F_{12 m}=2 a y^{2}, L_{12 m}=2 b z^{2}, a b=23 p$. (2) implies $\alpha \neq 1$. Theorem 3 implies $a \neq 23$. (7) implies $b \neq 1$. Lemma 15 implies $b \neq 23$.
Theorem 16: Let $p<10^{4}$. Then $F_{n}=26 p x^{2}$ iff $\left(n, p, x^{2}\right)=(21,421,1)$.
Proof: If $F_{n}=26 p x^{2}$, then (16) implies $z(26) n$, so $n=21 m$ and $F_{7 m}\left(F_{21 m} / F_{7 m}\right)=$ $26 p x^{2}$. Let $d=\left(F_{7 m}, F_{21 m} / F_{7 m}\right)$. (28) implies $d \mid 3$. (34) implies $13 \chi_{F_{21 m}} / F_{7 m}$. Therefore, if $d=1$, we have $F_{7 m}=13 a y^{2},{ }_{21 m} / F_{7 m}=b z^{2}, a b=2 p$. If $a=1$, then (4) implies $7 m=7$, so $F_{2 l} / 2 F_{7}=421=p z^{2}$. Therefore, $p=421, n=21$, and $x^{2}=1$. Theorem 3 implies $\alpha \neq 2$. Lemma 4 implies $b \neq 1$. If $b=2$, then $F_{7}=13 p y^{2}$. The hypothesis and Theorem 9 imply $7 m=14$. But $F_{42} / F_{14} \neq 2 z^{2}$. If $d=3$, then (16) implies $z(3) \mid 7 m$, that is, $4 \mid 7 m$, so $7 m=28 k$. We now have $F_{28 k}=39 \alpha y^{2}, F_{84 k} / F_{28 k}=3 b z^{2}$, with $\alpha b=2 p$. Theorems 2 and 1 imply $a \neq 1$ and 2, respectively. Lemmas 6 and 7 imply $b \neq 1$ and 2, respectively. Conversely, $F_{21}=10346$.
Theorem 17: $F_{n}=29 p x^{2}$ iff $\left(n, p, x^{2}\right)=(14,13,1)$.
Proof: If $F_{n}=29 p x^{2}$, then (16) implies $z(29) \mid n$, so $n=14 m$. If $3 \nmid m$, then Lemma 16 implies $F_{7 m}=a y^{2}, L_{7 m}=b z^{2}, \alpha b=29 p$. (1) implies $a \neq 1$. (4) implies $\alpha \neq 29$. (6) implies $b \neq 1$. If $b=29$, then $F_{7 m}=p y^{2}$. (13) implies $7 m=7$, so $F_{7}=13=p y^{2}$. Therefore, $p=13, n=14, x^{2} \stackrel{m}{=}$. If $m=3 k$, then Lemma 16 implies $F_{2 l k}=2 a y^{2}, L_{2 l k}=2 b z^{2}, a b=29 p$. (2) implies $\alpha \neq 1$. Theorem 3 implies $a \neq 29$. (7) implies $b \neq 1$. Since $y(29)=7$, Lemma 14 implies $b \neq 29$. Conversely, $F_{14}=377$.
Theorem 18: $F_{n} \neq 30 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(30) \mid n$, so $n=60 \mathrm{~m}$. Lemma 16 implies $F_{30 m}=2 t y^{2}, L_{30 m}=2 b z^{2}$, tb $=30 p$. But (15) and (42) imply (b, 10) $=$ 1 , so $F_{30 \mathrm{~m}}=20 a y^{2}, L_{30 \mathrm{~m}}=2 b z^{2}, a b=3 p$. If $a=1$, then $F_{30 \mathrm{~m}}=5(2 y)^{2}$, which contradicts (4). Theorem 2 implies $a \neq 3$. (7) implies $b \neq 1$. (9) implies $b \neq$ 3.

Theorem 19: $F_{n} \neq 31 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(31) \mid n$, so $n=30 \mathrm{~m}$. Lemma 16 implies $F_{15 m}=2 a y^{2}, L_{15 m}=2 b z^{2}, \alpha b=31 p$. (2) implies $a \neq 1$. Theorem 3 implies $a \neq 31$. (7) implies $b \neq 1$. Lemma 15 implies $b \neq 31$.
Theorem 20: $F_{n} \neq 33 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(33) \mid n$, so $n=20 \mathrm{~m}$. (43) implies $11 \chi_{10 m}$. If $3 \nmid m$, then Lemma 16 implies $F_{10 m}=11 a y^{2}, L_{10 m}=b z^{2}$, $a b=3 p$. (3) implies $a \neq 1$. Theorem 2 implies $a \neq 3$. (6) and (8) imply $b \neq 1$ and 3, respectively. If $m=3 k$, then Lemma 16 implies $F_{30 k}=22 a y^{2}, L_{30 k}=2 b z^{2}, a b=3 p$. Theorems 3 and 1 imply $a \neq 1$ and 3, respectively. (7) and (9) imply $b \neq 1$ and 3 , respectively.
Theorem 21: If $p<10^{4}$, then $E_{n}=34 p x^{2}$ iff $\left(n, p, x^{2}\right)=(18,19,4)$.
Proof: If $F_{n}=34 p x^{2}$, then (16) implies $z(34) \mid n$, so $n=9 m$ and $F_{3 m}\left(F_{9 m} / F_{3 m}\right)=$ $34 p x^{2}$. Let $d=\left(F_{3 m}, F_{9 m} / F_{3 m}\right.$ ). (28) implies $d \mid 3$. (35) implies $2 \not{ }_{1} F_{9 m} / F_{3 m}$. If $\alpha=1$, then $F_{3 m}=2 \alpha y^{2}, F_{9 m} / F_{3 m}=b z^{2}, \alpha b=17 p$. If $\alpha=1$, then $b=17 p$ and (2) implies $3 m=3$ or 6 . If $3 m=3$, then $F_{g} / 17 F_{3}=1 \neq p z^{2}$. If $3 m=6$, then $F_{18} / 17 F_{6}=19=p z^{2}$, so $p=19$; hence, $n=18$ and $x^{2}=4$. If $\alpha=17$, then $b=p$, and Theorem 3 implies $3 m=9$. But $F_{27} / F_{9} \neq p z^{2}$. Lemma 4 implies $b \neq 1$.

If $b=17$, then $F_{3 m}=2 p y^{2}$. But the hypothesis and Theorem 3 imply $p=17$, so $17 \mid d$, an impossibility. If $d=3$, then (45) imlies $m=4 k$, so $F_{12 k}=6 a y^{2}$, $F_{36 k} / F_{12 k}=3 b z^{2}, \alpha b=17 p$. (5) implies $\alpha \neq 1$. Theorem 1 implies $\alpha \neq 17, p$. Lemma 6 implies $b \neq 1$. Conversely, $F_{18}=2584$.
Theorem 22: $F_{n} \neq 35 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(35) \mid n$, so $n=40 \mathrm{~m}$. If $3 \nmid \mathrm{~m}$, then (15) and lemma 16 imply $F_{20 m}=5 a y^{2}, L_{20 m}=b z^{2}, a b=7 p$. (4) implies $\alpha \neq$ 1. Theorem 4 implies $a \neq 7, \alpha \neq p$, so $b \neq 7$. (6) implies $b \neq 1$. If $m=3 k$, then (15) and Lemma 16 imply $F_{60 k}=10 \alpha y^{2}, L_{60 k}=2 b z^{2}, \alpha b=7 p$. Theorem 3 implies $\alpha \neq 1$. Theorem 7 implies $a \neq 7, a \neq p$, so $b \neq 7$. (7) implies $b \neq 1$.

We omit the proofs of the two following theorems (23 and 24) because they are similar to proofs of prior theorems.
Theorem 23: $F_{n}=38 p x^{2}$ iff $\left(n, p, x^{2}\right)=(18,17,4)$.
Theorem 24: $F_{n} \neq 39 p x^{2}$.
Theorem 25: $F_{n} \neq 42 p x^{2}$.
Proof: Assume the contrary. Then (16) implies z(42) $\mid n$, so $n=24 \mathrm{~m}$. (37) implies $3 \chi_{L_{12 m}}$; (38) implies $4 \chi_{12 m}$. Therefore, Lemma 16 implies $F_{12 m}=12 \alpha y^{2}$, $L_{12 m}=2 b z^{2}, a b=7 p$. (3) implies $a \neq 1$. Theorem 2 implies $a \neq 7$. (7) implies $b \neq 1$. Lemma 15 implies $b \neq 7$.
Theorem 26: $F_{n} \neq 51 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(51) n$, so $n=36 \mathrm{~m}$. (15), (37), and Lemma 16 imply $F_{18 m}=102 \alpha y^{2}, L_{18 m}=2 b z^{2}, \alpha b=p$. Theorem 1 implies $\alpha \neq 1$. (7) implies $b \neq 1$.

Theorem 27: $F_{n} \neq 55 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(55) \mid n$, so $n=10 \mathrm{~m}$. If $3 \nmid m$, then (15) and Lemma 16 imply $F_{5 m}=5 \alpha y^{2}, L_{5 m}=b z^{2}, \alpha b=11 p$. If $\alpha=1$, then Theorem 4 implies $5 m=5$, so $L_{5} / 11=1=p z^{2}$, an impossibility. If $a=11$, then Theorem 4 implies $5 m=10$, so $L_{10}=123=p z^{2}$, an impossibility. (6) implies $b \neq 1$. If $b=11$, then (11) implies $5 m=5$, so $F_{5 m} / 5=1=p y^{2}$, an impossibility. If $m=3 k$, then (15) and Lemma 16 imply $F_{15 k}=10 a y^{2}, L_{15 k}=2 b z^{2}$, $a b=11 p$. Theorem 3 implies $a \neq 1$. Theorem 7 implies $\alpha \neq 11$. (7) implies $b \neq$ 1. Lemma 14 implies $b \neq 11$.

Theorem 28: $F_{n}=65 p x^{2}$ iff $\left(n, p, x^{2}\right)=(35,141961,1)$.
Proof: $F_{35}=65 * 141961 * 1^{2}$. If $F_{n}=65 p x^{2}$, then (16) implies $z(65) \mid n$, so $n=$ $35 m$, and $F_{7 m}\left(F_{35 m} / F_{7 m}\right)=65 p x^{2}$. Let $d=\left(F_{7 m}, F_{35 m} / F_{7 m}\right)$. Now Lemma 8 implies $13 \nmid F_{35 m} / F_{7}$. If $5 \nmid m$, then (39) implies $d=1$, so $F_{7 m}=13 a y^{2}, F_{35 m} / F_{7 m}=5 b z^{2}$, $\alpha b=p$. If $\alpha=1$, then (4) implies $7 m=7$, so $F_{35} / 5 F_{7}=141961=p z^{2}$. Therefore $p=141961, n=35, x^{2}=1$. Lemma 11 implies $b \neq 1$. If $m=5 k$, then (39) implies $d=5$. (34) implies $5^{2} \nmid F_{175 k} / F_{35 k}$. Thus, $F_{35 k}=325 \alpha y^{2}, F_{175 k} / F_{35 k}=$ $5 b z^{2}, a b=p$. But (4) implies $a \neq 1$. Lemma 11 implies $b \neq 1$.
Theorem 29: $F_{n} \neq 66 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(66) \mid n$, so $n=60 \mathrm{~m}$. Now (43), (38), and Lemma 16 imply $F_{30 m}=44 a y^{2}, L_{30 m}=2 b z^{2}, a b=3 p$. (3) implies $a \neq$ 1. Theorem 2 implies $a \neq 3$. (7) and (9) imply $b \neq 1$ and 3, respectively.

Theorem 30: $F_{n} \neq 70 p x^{2}$.
Proof: Assume the contrary. Then (16) implies $z(70) \mid n$, so $n=120 \mathrm{~m}$. (15), (38), and Lemma 16 imply $F_{60 m}=20 a y^{2}, L_{60 m}=2 b z^{2}, a b=7 p$. (4) implies $a \neq 1$. Theorem 22 implies $a \neq 7$. (7) implies $b \neq 1$. Lemma 15 implies $b \neq 7$.

We summarize the results of Theorems 1 through 30 in Table 1 . For each listed value of $k$, we list all solutions of ( $\%$ ) with $c=k p$, if any. The cases $k=2,23,26,34$ are subject to the restriction that $p<10,000$.

TABLE 1

| $k\left(n, p, x^{2}\right)$ |  | ( $n, p, x^{2}$ ) |  | $\left(n, p, x^{2}\right)$ |  | ( $n, p, x^{2}$ ) | k | $\left(n, p, x^{2}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 (9, 17, 1) | 10 | $(15,61,1)$ | 21 | (16, 47, 1) | 31 | *********** | 42 | ************** |
| $3(8,7,1)$ | 11 | $(10,5,1)$ | 22 | ************ | 33 | *********** | 51 | ************** |
| $3(12,3,16)$ | 13 | (14, 29, 1) | 23 | ************ | 34 | (18, 19, 4) | 55 | *************** |
| $5(10,11,1)$ | 14 | (24, 23, 144) | 26 | $(21,421,1)$ | 35 | *********** | 65 | (35, 141961, 1) |
| 6 *********** | 15 | ************* | 29 | (14, 13, 1) | 38 | (18, 17, 4) | 66 | *************** |
| 7 (8, 3, 1) | $\begin{aligned} & 17 \\ & 19 \end{aligned}$ | $(9,2,1)$ <br> ************* | 30 | ************ | 39 | *********** | 70 | *************** |

Combining these new results with those of [1] and [9], we obtain Table 2, which lists all solutions of (*) such that $1 \leq C \leq 1000$.

TABLE 2

| $c$ | $\left(n, x^{2}\right)$ | $c$ | $\left(n, x^{2}\right)$ | $c$ | $\left(n, x^{2}\right)$ | $c$ | $\left(n, x^{2}\right)$ |
| :--- | :--- | ---: | :--- | :--- | :--- | :--- | :--- |
| 1 | $(1,1)$ | 3 | $(4,1)$ | 34 | $(9,1)$ | 322 | $(24,144)$ |
| 1 | $(2,1)$ | 5 | $(5,1)$ | 55 | $(10,1)$ | 377 | $(14,1)$ |
| 1 | $(12,144)$ | 8 | $(6,1)$ | 89 | $(11,1)$ | 610 | $(15,1)$ |
| 2 | $(3,1)$ | 13 | $(7,1)$ | 144 | $(12,1)$ | 646 | $(18,4)$ |
| 2 | $(6,4)$ | 21 | $(8,1)$ | 233 | $(13,1)$ | 987 | $(16,1)$ |

## Concluding Remarks

Notice that in every solution we have $x^{2}=1,4$, or 144 . This leads us to conjecture that in any solution of $(*)$ one must have $x^{2}=1,4$, or 144 .

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## *****

# VINOGRADOV'S INVERSION THEOREM FOR GENERALIZED ARITHMETICAL FUNCTIONS 

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1. Introduction

In this paper we introduce the Vinogradov [8] inversion theorem for functions defined on a finite partially ordered set. Our inversion theorem reduces to that by Vinogradov in the case of positive integers. For material relating to Vinogradov's inversion theorem, we refer to [2], [3], and [4].

As an example of our generalized Vinogradov inversion theorem we consider an inversion theorem relating to arithmetical functions and regular convolutions. As applications, we give expressions for certain restricted sums of Fibonacci and Lucas numbers. Special cases of the applications can be found in [4].

## 2. A Generalized Vinogradov Inversion Theorem

Let $(P, \subseteq)$ be a locally finite partially ordered set. A complex-valued function $f$ on $P \times P$ is said to be an incidence function of ( $P, \subseteq$ ) if $f(x, y)=$ 0 unless $x \subseteq y$. We denote by $I(\subseteq, P)$ the set of all incidence functions of ( $P$, $\subseteq)$. The convolution of $f, g \in I(\subseteq, P)$ is defined by

$$
(f \circ g)(x, y)=\sum_{x \subseteq z \subseteq y} f(x, z) g(z, y) .
$$

The inverse of $f \in I(\subseteq, P)$ is defined by

$$
f \circ f^{-1}=f^{-1} \circ f=\delta,
$$

where $\delta(x, x)=1$ and $\delta(x, y)=0$ if $x \neq y$. The inverse of $\zeta$, defined by $\zeta(x$, $y)=1$ whenever $x \subseteq y$, is denoted by $\mu$ and is called the Möbius function of $(P, \subseteq)$.

Now we are able to give our generalized Vinogradov inversion theorem. The original Vinogradov inversion theorem is reproduced in the remark of Theorem 2 in Section 3.

Theorem 1: Suppose ( $P, \subseteq$ ) and ( $P, \leq$ ) are locally finite partially ordered sets. Let $f_{x}$ be a complex-valued function of $x \in P$ and let $d_{x}$ be a function of $x \in P$ into $P$. Then, for all $a, b \in P$,

$$
\sum_{\substack{a \leq x \leq b \\ d_{x}=a}} f_{x}=\sum_{a \subseteq z} \mu(\alpha, z) \sum_{\substack{a \leq x \leq b \\ z \subseteq d_{x}}} f_{x},
$$

where $\mu$ is the Möbius function of ( $P, \subseteq$ ).
Proof: We have

$$
\sum_{\substack{a \leq x \leq b \\ d_{x}=a}} f_{x}=\sum_{a \leq x \leq b} f_{x} \delta\left(\alpha, d_{x}\right)=\sum_{a \leq x \leq j} f_{x} \sum_{a \subseteq z \subseteq d_{x}} \mu(\alpha, z)=\sum_{a \subseteq z} \mu(\alpha, z) \sum_{\substack{a \leq x \leq b \\ z \subseteq d_{x}}} f_{x},
$$

which was required.
Remark: We note that Theorem 1 implies the classical inversion theorem for incidence functions of ( $P, \subseteq$ ) stating that if, for all $\alpha, b \in P$,
then

$$
g(a, b)=\sum_{a \subseteq z \subseteq b} f(z, b),
$$

that is

$$
f(a, b)=\sum_{a \subseteq z \subseteq b} \mu(a, z) g(z, b) ;
$$

$$
\begin{equation*}
f(a, b)=\sum_{a \subseteq z \subseteq b} \mu(a, z) \sum_{z \subseteq y \subseteq b} f(y, b) \tag{1}
\end{equation*}
$$

In fact, let $a, b \in P$ with $a \subseteq b$. We assume $x \subseteq y \Rightarrow x \leq y$ for all $x, y \in P$ and denote

$$
\begin{aligned}
& \{x \in P: a \subseteq x \subseteq b\}=\left\{x_{1}(=a), x_{2}, \ldots, x_{m}(=b)\right\}, \\
& \{x \in P: a \leq x \leq b\}=\left\{y_{1}(=\alpha), y_{2}, \ldots, y_{n}(=b)\right\}, m \leq n .
\end{aligned}
$$

Then we take

$$
d_{y_{1}}=a, d_{y_{2}}=x_{2}, \ldots, d_{y_{m}}=b, d_{y_{m+1}}=\ldots=d_{y_{n}}=c
$$

where $c \notin b$, and $f_{x}=f\left(d_{x}, b\right)$. (If there does not exist an element $c \in P$ such that $c \notin b$, then we consider the set $P \cup\{c\}$.$) In this case,$
and

$$
\begin{aligned}
& \sum_{\substack{a \leq x \leq b \\
d_{x}=a}} f_{x}=\sum_{\substack{a \leq x \leq b \\
d_{x}=a}} f\left(d_{x}, b\right)=f(a, b) \\
& \sum_{a \subseteq z} \mu(a, z) \sum_{\substack{a \leq x \leq b \\
z \subseteq d_{x}}} f_{x}=\sum_{a \subseteq z} \mu(a, z) \sum_{\substack{a \leq x \leq b \\
z \subseteq d_{x}}} f\left(d_{x}, b\right)=\sum_{a \subseteq z} \mu(\alpha, z) \sum_{z \subseteq y \subseteq b} f(y, b)
\end{aligned}
$$

Thus, by Theorem 1 , we arrive at (1).

## 3. Regular Arithmetical Convolutions

Let $A$ be a mapping from the set $\mathbb{N}$ of positive integers to the set of subsets of $\mathbb{N}$ such that, for each $n \in \mathbb{N}, A(n)$ is a subset of the set of positive divisors of $n$. Then the $A$-convolution of two arithmetical functions $f$ and $g$ is defined by

$$
\left(f \circ_{A} g\right)(n)=\sum_{d \in A(n)} f(d) g(n / d)
$$

Narkiewicz [6] defined an $A$-convolution to be regular if:
(a) the set of arithmetical functions forms a commutative ring with unity with respect to the ordinary addition and the $A$-convolution;
(b) the $A$-convolution of multiplicative functions is multiplicative;
(c) the function $E$, defined by $E(n)=1$ for all $n \in \mathbb{N}$, has an inverse $\mu_{A}$ with respect to the $A$-convolution, and $\mu_{A}(n)=0$ or -1 whenever $n$ is a prime power.

The inverse of an arithmetical function $f$ such that $f(1) \neq 0$ with respect to the $A$-convolution is defined by

$$
f \circ_{A} f^{-1}=f^{-1} o_{A} f=E_{0}
$$

where $E_{0}(1)=1$ and $E_{0}(n)=0$ for $n>1$.
It can be proved (see [6]) that an $A$-convolution is regular if and only if
(i) $A(m n)=\{d e: d \in A(m), e \in A(n)\}$ whenever $(m, n)=1$,
(ii) for each prime power $p^{a}>1$ there exists a divisor $t=t_{A}\left(p^{a}\right)$ of $a$ such that

$$
A\left(p^{a}\right)=\left\{1, p^{t}, p^{2 t}, \ldots, p^{r t}\right\}
$$

where $r t=a$, and

$$
A\left(p^{i t}\right)=\left\{1, p^{t}, p^{2 t}, \ldots, p^{i t}\right\}, 0 \leq i<r .
$$

For example, the Dirichlet convolution $D$, where $D(n)$ is the set of all positive divisors of $n$, and the unitary convolution $U$, where

$$
U(n)=\{d>0: d \mid n,(d, n / d)=1\}
$$

are regular (see [1]). In this paper we confine ourselves to regular convolutions.

A positive integer $n$ is said to be $A$-primitive if $A(n)=\{1, n\}$. The generalized Möbius function $\mu_{A}$ is the multiplicative function given by (see [6])

$$
\mu_{A}\left(p^{a}\right)=\left\{\begin{aligned}
-1 & \text { if } p^{a}(>1) \text { is } A \text {-primitive } \\
0 & \text { if } p^{a} \text { is non- } A \text {-primitive }
\end{aligned}\right.
$$

For a positive integer $k$, we define

$$
A_{k}(n)=\left\{d>0: d^{k} \in A\left(n^{k}\right)\right\}
$$

It is known [7] that the $A_{k}$-convolution is regular whenever the $A$-convolution is regular. The symbol $(a, b)_{A, k}$ denotes the greatest $k^{\text {th }}$ power divisor of $a$ which belongs to $A(b)$. In particular, denote $(a, b)_{A, 1}=(a, b)_{A}$. Then

$$
(a, b)_{D}=(a, b)
$$

the greatest common divisor of $a$ and $b$.
Let $A$ be a regular arithmetical convolution. Then we define the relation $\subseteq$ on the set $\mathbb{N}$ of positive integers by

$$
m \subseteq n \Leftrightarrow m \in A(n)
$$

and denote by $\mathbb{N}_{A}$ the resulting locally finite partially ordered set.
Let $f$ be an arithmetical function, that is, a complex-valued function on $\mathbb{N}$. Then we can associate with $f$ an incidence function $f^{\prime}$ of $\mathbb{N}_{A}$ defined by

$$
f^{\prime}(m, n)= \begin{cases}f(n / m) & \text { if } m \in A(n), \\ 0 & \text { if } m \notin A(n) .\end{cases}
$$

The mapping $f \rightarrow f^{\prime}$ is one-one and

$$
\begin{equation*}
\left(f^{\prime} \circ g^{\prime}\right)(m, n)=\left(f \circ_{A} g\right)^{\prime}(m, n) \tag{2}
\end{equation*}
$$

(see [5], Ch. 7). Plainly

$$
\left(E_{0}\right)^{\prime}(m, n)=\delta(m, n), E^{\prime}(m, n)=\zeta(m, n)
$$

Therefore, by (2),

$$
\left(\mu_{A}\right)^{\prime}(m, n)=\mu(m, n)
$$

Now we are in a position to state Theorem 1 for regular convolutions. Letting $\leq$ be the natural ordering on $\mathbb{N}$, we can write
Theorem 2: Let $f_{i}$ be a complex-valued function of $i \in \mathbb{N}$ and 1 et $d_{i}$ be a function of $i \in \mathbb{N}$ into $\mathbb{N}$. Then, for all $n \in \mathbb{N}$,

$$
\sum_{\substack{i=1 \\ d_{i}=1}}^{n} f_{i}=\sum_{d \geq 1} \mu_{A}(d) \sum_{\substack{i=1 \\ d \in A\left(d_{i}\right)}}^{n} f_{i}
$$

Remark: If $A=D$ in Theorem 2, we obtain the original Vinogradov inversion theorem.
Corollary: Let $f_{i}$ be a complex-valued function of $i \in \mathbb{N}$. Then

$$
\sum_{\left(i, n^{k}\right)_{A, k}=1}^{n} f_{i}=\sum_{d \in A_{k,}(n)} \mu_{A_{k}}(d) \sum_{\substack{i=1 \\ d^{k} \mid i}}^{n} f_{i} .
$$

Proof: Replace $A$ by $A_{k}$ and take $d_{i}=\left(\left(i, n^{k}\right) A, k\right)^{1 / k}$ in Theorem 2. Since $d \in$ $A_{k}\left(\left(\left(i, n^{k}\right)_{A}, k\right)^{1 / k}\right)$ if and only if $d \in A_{k}(n), d^{k} \mid i$, we obtain the Corollary.
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## 4. Applications to Fibonacci and Lucas Numbers

Let $F_{i}$ be the $i$ th Fibonacci number, that is, $F_{1}=1, F_{2}=1, F_{n}=F_{n-1}+F_{n-2}$ $(n \geq 3)$, and let $L_{i}$ be the $i$ th Lucas number, that is, $L_{1}=1, L_{2}=3, L_{n}=L_{n-1}+$ $L_{n-2} \quad(n \geq 3)$.
Theorem 3: Let $A$ be a regular convolution and $k \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$,

$$
\begin{align*}
\sum_{\substack{i=1 \\
\left(i, n^{k}\right)_{, k}=1}}^{n} F_{i} & =\sum_{d \in A_{k}(n)} \mu_{A_{k}}(d) \frac{F_{m d^{k}+d^{k}}-(-1)^{d^{k}} F_{m d^{k}}-F_{d^{k}}}{L_{d^{k}}-(-1)^{d^{k}}-1},  \tag{3}\\
\sum_{\substack{i=1}}^{n} L_{i} & =\sum_{d \in A_{k}(n)} \mu_{A_{k}}(d) \frac{L_{m d^{k}+d^{k}}-(-1)^{d^{k}} L_{m d^{k}}-L_{d^{k}}+2(-1)^{d^{k}}}{L_{d^{k}}-(-1)^{d^{k}}-1},
\end{align*}
$$

where $m=\left[n / d^{k}\right]$, the greatest integer in $n / d^{k}$.
Proof: Plainly,

$$
\sum_{\substack{i=1 \\ d^{k} \mid i}}^{n} F_{i}=\sum_{1 \leq i \leq n / d^{k}} F_{i d^{k}} .
$$

Then, using the formulas

$$
F_{n}=\frac{1}{\sqrt{5}}\left(\alpha^{n}-\beta^{n}\right), \quad L_{n}=\alpha^{n}+\beta^{n},
$$

where

$$
\alpha=\frac{1}{2}(1+\sqrt{5}), \quad \beta=\frac{1}{2}(1-\sqrt{5}),
$$

we obtain, after some computations,

$$
\sum_{\substack{i=1 \\ d^{k} \mid i}}^{n} F_{i}=\frac{F_{m d^{k}}+d^{k}-(-1)^{d^{k}} F_{m d^{k}}-F_{d^{k}}}{L_{d^{k}}-(-1)^{d^{k}}-1}
$$

Thus, applying the Corollary of Theorem 2, we get (3). The proof of (4) goes through in a manner similar to that of (3).
Corollary: Let $A$ be a regular convolution. Then, for each $n \in \mathbb{N}$,

$$
\begin{aligned}
& \sum_{i=1}^{n} F_{i}=\sum_{d \in A(n)} \mu_{A}(d) \frac{F_{n+d}-(-1)^{d} F_{n}-F_{d}}{L_{d}-(-1)^{d}-1}, \\
&(i, n)_{A}=1 \\
&(i, n)_{d i}=1
\end{aligned} \sum_{i=1}^{n} L_{i}=\sum_{d(n)} \mu_{A}(d) \frac{L_{n+d}-(-1)^{d} L_{n}-L_{d}+2(-1)^{d}}{L_{d}-(-1)^{d}-1} .
$$

Theorem 4: Let $A$ be a regular convolution and $k \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$,

$$
\begin{align*}
& \sum_{\substack{i=1 \\
\left(i, n^{k}\right)_{h, k}>1}}^{n} F_{i}=F_{n+2}-\sum_{d \in A_{k}(n)} \mu_{A_{\dot{k}}}(d) \frac{F_{m d^{k}}+d^{k}-(-1)^{d^{k}} F_{m d^{k}}-F_{d^{k}}}{L_{d^{k}}-(-1)^{d^{k}}-1}-1 \text {, }  \tag{5}\\
& \sum_{\substack{\left.i=1 \\
n \\
n^{n}\right)}} L_{i}  \tag{6}\\
& =L_{n+2}^{\left(i, n^{k}\right)_{, k}>1} \sum_{d \in A_{k}(n)} \mu_{A_{k}}(d) \frac{L_{m d^{k}}+d^{k}-(-1)^{d^{k}} L_{m d^{k}}-L_{d^{k}}+2(-1)^{d^{k}}}{L_{d^{k}}-(-1)^{d^{k}}-1}-3 \text {, }
\end{align*}
$$

where $m=\left[n / \alpha^{k}\right]$.

Proof: We have

$$
\sum_{\substack{\left.i=1 \\ n^{n}\right)_{A, k}>1}}^{n} F_{i}=\sum_{i=1}^{n} F_{i}-\sum_{\substack{i=1 \\\left(i, n^{2}\right)_{A, k}=1}}^{n} F_{i} .
$$

Therefore, applying (3) and the identity

$$
\sum_{i=1}^{n} F_{i}=F_{n+2}-1,
$$

we obtain (5). Similarly, applying (4) and the identity

$$
\sum_{i=1}^{n} L_{i}=L_{n+2}-3
$$

we get (6).

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# TWO CLASSES OF NUMBERS APPEARING IN THE CONVOLUTION OF BINOMIAL-TRUNCATED POISSON AND POISSON-TRUNCATED BINOMIAL RANDOM VARIABLES 

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## 1. Introduction

The Stirling numbers of the second kind, known to mathematicians as the coefficients in the factorial expansion of powers, are of great importance in the calculus of finite differences, and have been extensively studied, especially with respect to their mathematical properties (see Jordan [14], Riordan [17] and references therein). Recently, several extensions and modifications were considered, which have proved useful in various combinatorial, probabilistic and statistical applications. Of the most interesting variations are the Lah numbers, Lah [16], and their generalization C-numbers, Charalambides [8], [9], appearing in the expansion of a factorial of $t$, scaled by a parameter $s$, in a sum of factorials of $t$.

The present paper was motivated by the problem of providing explicit expressions for the distribution of two-sample sums from Poisson and binomial distributions, one of which is left-truncated. Specifically, the distribution of the statistic $Z=X_{1}+\cdots+X_{v}+X_{v+1}+\cdots+X_{v+n}$, where
(a) $X_{1}, \ldots, X_{v}$ is a random sample from a Poisson and $X_{v+1}, \ldots, X_{v+n}$ an independent random sample from a left-truncated binomial distribution and
(b) $X_{1}, \ldots, X_{v}$ is a random sample from a binomial and $X_{v+1}, \ldots, X_{v+n}$ an independent random sample from a left-truncated Poisson distribution,
led to the introduction of two double sequences of Stirling and C-related numbers, obtained from the expansion of certain classes of polynomials in a series of factorials.

In Section 2, we discuss some general results relating the expansion of polynomials in factorials and the corresponding exponential generating functions (egf's). In Section 3, we consider two specific families of polynomials ( $r-q$ polynomials) and introduce two double sequences of numbers ( $R-Q$ numbers). Notice that in Tauber's [19] terminology these numbers might be called generalized Lah numbers. Next, the egf's of the $R-Q$ numbers are used to derive recurrence relations and initial conditions and the connection to well-known numbers is examined in Section 4. In Section 5, it is shown how $R-Q$ numbers can be used for the introduction of two new families of truncated discrete probability functions including binomial and hypergeometric distributions as special cases; also for the solution of the above-mentioned problems (a) and (b). An application to occupancy problems is also provided. Finally, in Section 6, a further generalization of the $R-Q$ numbers, through egf's, is also discussed, along with its properties and applications.

## 2. Preliminary General Results

Let $\left\{p_{m}(x), m=0,1, \ldots\right\}$ be a class of polynomials, and consider the double sequence $\{P(m, n), m=0,1, \ldots, n=0,1, \ldots, m\}$ obtained by expanding the polynomial $p_{m}(x)$ in a series of factorials, namely

$$
\begin{equation*}
p_{m}(x)=\sum_{n=0}^{m} P(m, n)(x)_{n} . \tag{2.1}
\end{equation*}
$$

Denote the egf of the numbers $P(m, n)$ with respect to the index $m$ by $f_{n}(t)$, and the egf of the polynomials $p_{m}(x)$ by $p(x ; t)$, that is

$$
\begin{equation*}
f_{n}(t)=\sum_{m=n}^{\infty} P(m, n) \frac{t^{m}}{m!}, \quad p(x ; t)=\sum_{m=0}^{\infty} p_{m}(x) \frac{t^{m}}{m!} . \tag{2.2}
\end{equation*}
$$

On using (2.1), we may easily verify that

$$
p(x, t)=\sum_{n=0}^{\infty} f_{n}(t)(x)_{n},
$$

and the next theorem is an immediate consequence of Newton's formula (see Jordan [14]).
Theorem 2.1: Let $p(x ; t)$ denote the egf of a class of polynomials $\left\{p_{m}(x)\right.$, $m=0,1, \ldots\}$ and $f_{n}(t)$ the egf of the corresponding numbers $P(m, n)$ as defined in (2.1). Then

$$
\begin{equation*}
f_{n}(t)=\frac{1}{n!}\left[\Delta_{x}^{n} p(x, t)\right]_{x=0} . \tag{2.3}
\end{equation*}
$$

We now state some general results referring to recurrence relations satisfied by the polynomials $p_{m}(x)$ and the numbers $P(m, n)$, when a certain partial differential equation holds true for the egf $p(x, t)$.

Theorem 2.2: If the egf $p(x, t)$ of the polynomials $p_{m}(x)$ satisfies the partial differential equation

$$
\begin{equation*}
\left(1+B t+C t^{2}\right) \frac{\partial p(x, t)}{\partial t}=(D+E t) p(x, t) \tag{2.4}
\end{equation*}
$$

where $B, C, D$, and $E$ may be functions of $x$, then there is a recurrence relation connecting three polynomials $p_{m}(x)$ with consecutive indices (degrees), namely,
(2.5) $\quad p_{m+1}(x)=(D-B m) p_{m}(x)+\left((E+C) m-C m^{2}\right) p_{m-1}(x)$.

Proof: Differentiate $p(x, t)$ of (2.2) term by term, substitute in (2.4) and equate the coefficients of $t^{m} / m$ ! in the right and left sides of the resulting identity.

Note that (2.5) is true for $m \geq 1$, while, for $m=0$, it reduces to
(2.6) $\quad p_{1}(x)=D p_{0}(x)$,
which suggests that $D=D(x)$ must be at least of order 1 with respect to $x$.
Theorem 2.3: If $p(x, t)$ satisfies the partial differential equation

$$
\begin{equation*}
(1+b t) \frac{\partial p(x, t)}{\partial t}=\left(c_{0}+c_{1} x+c_{2} t+c_{12} x t\right) p(x, t) \tag{2.7}
\end{equation*}
$$

with $b, c_{0}, c_{1}, c_{2}$, and $c_{12}$ being constants, then the polynomials $p_{m}(x)$ and the numbers $P(m, n)$ satisfy the recurrences
(2.8a) $p_{m+1}(x)=\left(c_{0}+c_{1} x-b m\right) p_{m}(x)+\left(c_{2}+c_{12} x\right) m p_{m-1}(x), m \geq 0$,
(2.8b) $\quad p_{1}(x)=\left(c_{0}+c_{1} x\right) p_{0}(x)$,
(2.9a) $P(m+1, n)=\left(c_{0}-b m+c_{1} n\right) P(m, n)+c_{1} P(m, n-1)$
$+m\left(c_{2}+n c_{12}\right) P(m-1, n)+c_{12} m P(m-1, n-1)$,
$1 \leq n \leq m-1$,
(2.9b) $P(m+1, m+1)=c_{1} P(m, m)$,
(2.9c) $P(m+1, m)=\left(c_{0}+\left(c_{1}-b\right) m\right) P(m, m)+c_{1} P(m, m-1)$
$+c_{12} m P(m-1, m-1)$.

Proof: For (2.8), apply Theorem 2.2 in the special case

$$
B(x)=b, c(x)=0, D(x)=c_{0}+c_{1} x, E(x)=c_{2}+c_{12} x
$$

For (2.9) observe that, after expanding $p_{m+1}(x), p_{m}(x)$, and $p_{m-1}(x)$ by (2.1), one obtains

$$
\begin{aligned}
\sum_{n=0}^{m+1} P(m+1, n)(x)_{n}= & \left(c_{0}-b m\right) \sum_{n=0}^{m} P(m, n)(x)_{n}+c_{1} \sum_{n=0}^{m} n P(m, n)(x)_{n} \\
& +c_{1} \sum_{n=1}^{m+1} P(m, n-1)(x)_{n}+m\left(c_{2}+n c_{12}\right) \sum_{n=0}^{m-1} P(m-1, n)(x)_{n} \\
& +m c_{12} \sum_{n=1}^{m} P(m-1, n-1)(x)_{n}
\end{aligned}
$$

which establishes the proof.
It is worth noticing that many classes of well-known numbers with special interest in statistical and combinatorial applications, have an egf $p(x, t)$ obeying the partial differential equation (2.7). For example,
$\alpha$. If $p_{m}(x)=x^{m}$, we obtain the Stirling numbers of the second kind (see Jordan [14]) and

$$
p(x, t)=\sum_{m=0}^{\infty} x^{m} \frac{t^{m}}{m!}=e^{t x}, \quad \frac{\partial p(x, t)}{\partial t}=x p(x, t)
$$

B. If $p_{m}(x)=(x-a)^{m}$, we are led to the noncentral Stirling numbers of the second kind, Koutras [15], or weighted Stirling numbers, Carlitz [4], [5], or $x$-Stirling numbers, Broder [1], with

$$
p(x, t)=\sum_{m=0}^{\infty}(x-\alpha)^{m} \frac{t^{m}}{m!}=e^{t(x-a)}, \frac{\partial p(x, t)}{\partial t}=(x-a) p(x, t)
$$

$\gamma$. If $p_{m}(x)=(-x)_{m}$, or more generally, $p_{m}(x)=(s x)_{m}$, we obtain the Lah or C-numbers, respectively, Lah [16], Charalambides [8], [9], with

$$
p(x, t)=\sum_{m=0}^{\infty}(s x)_{m} \frac{t^{m}}{m!}=(1+t)^{s x},(1+t) \frac{\partial p(x, t)}{\partial t}=\operatorname{sxp}(x, t)
$$

反. If $p_{m}(x)=(s x+r)_{m}$, the resulting numbers are the Gould and Hopper numbers studied by Charalambides \& Koutras [10]. In this case, we have

$$
p(x, t)=\sum_{m=0}^{\infty}(s x+r)_{m} \frac{t^{m}}{m!}=(1+t)^{s x+r},(1+t) \frac{\partial p(x, t)}{\partial t}=(s x+r) p(x, t)
$$

Notice how simple it is to compute the egf for any of the above-mentioned special cases. The egf $f_{n}(t)$ and the recurrences for the corresponding numbers are then easily obtained as a direct application of Theorems 2.1 and 2.3.

## 3. The $r-q$ Polynomials and Numbers-Generating Functions <br> and Recurrence Relations

Let us define two classes of polynomials by the formulas

$$
\begin{align*}
& r_{m}(x)=r_{m}(x ; s, \alpha)=e^{-a} \frac{d^{m}}{d t^{m}}\left[t^{s x} e^{a t}\right]_{t=1}  \tag{3.1}\\
& q_{m}(x)=q_{m}(x ; \alpha)=e^{-x} \frac{d^{m}}{d t^{m}}\left[t^{a} e^{x t}\right]_{t=1} \tag{3.2}
\end{align*}
$$

Thus, the first few $r-q$ polynomials are

$$
\begin{array}{ll}
r_{0}(x)=1, & r_{1}(x)=s x+\alpha, \quad r_{2}(x)=s^{2} x^{2}+(2 \alpha-1) s x+a^{2} \\
q_{0}(x)=1, & q_{1}(x)=x+\alpha, \quad q_{2}(x)=x^{2}+2 a x+\alpha(\alpha-1)
\end{array}
$$

Considering the Newton expansion of $r-q$ polynomials in a series of factorials, we may define the $R-Q$ numbers by

$$
\begin{align*}
& x_{m}(x ; \alpha, s)=\sum_{n=0}^{m} R(m, n ; s, \alpha)(x)_{n}=\sum_{n=0}^{m} R(m, n)(x)_{n},  \tag{3.3}\\
& q_{m}(x ; \alpha)=\sum_{n=0}^{m} Q(m, n ; \alpha)(x)_{n}=\sum_{n=0}^{m} Q(m, n)(x)_{n}
\end{align*}
$$

Since for $a=0$ the $r-q$ polynomials reduce to

$$
r_{m}(x)=(s x)_{m}, \quad q_{m}(x)=x^{m}
$$

it follows that

$$
R(m, n ; s, 0)=C(m, n, s)
$$

the $C$-numbers,

$$
R(m, n ;-1,0)=L(m, n)
$$

the Lah numbers, and

$$
Q(m, n ; 0)=S(m, n)
$$

the Stirling numbers of the second kind.
As a starting point, let us derive the egf of the $r-q$ polynomials and numbers, namely

$$
\begin{align*}
& r(x, t ; s, a)=\sum_{m=0}^{\infty} r_{m}(x ; s, \alpha) \frac{t^{m}}{m!}  \tag{3.5}\\
& f_{n}(t ; s, a)=\sum_{m=n}^{\infty} R(m, n ; s, \alpha) \frac{t^{m}}{m!} \\
& q(x, t ; \alpha)=\sum_{m=0}^{\infty} q_{m}(x ; \alpha) \frac{t^{m}}{m!}, \quad g_{n}(t ; \alpha)=\sum_{m=n}^{\infty} Q(m, n ; \alpha) \frac{t^{m}}{m!}
\end{align*}
$$

Regarding $t^{3 x} e^{a t}$ and $t^{a} e^{x t}$ as functions of $t$ and expanding in a Taylor series around $t=1$, we obtain

$$
\begin{aligned}
& t^{s x} e^{a t}=\sum_{m=0}^{\infty} \frac{d^{m}}{d t^{m}}\left[t^{s x} e^{a t}\right]_{t=1} \frac{(t-1)^{m}}{m!} \\
& t^{a} e^{x t}=\sum_{m=0}^{\infty} \frac{d^{m}}{d t^{m}}\left[t^{a} e^{x t}\right]_{t=1} \frac{(t-1)^{m}}{m!}
\end{aligned}
$$

and, using definitions (3.1) and (3.2), we get

$$
\begin{align*}
& r(x, t)=r(x, t ; s, \alpha)=(1+t)^{s x} e^{a t}  \tag{3.6}\\
& q(x, t)=q(x, t ; a)=(1+t)^{a} e^{x t}
\end{align*}
$$

As regards the egf's of $R-Q$ numbers, they may be obtained easily from Theorem 2.1 , which, in view of (3.6) and (3.7), gives

$$
\begin{aligned}
& f_{n}(t ; s, a)=\frac{1}{n!} e^{a t}\left[\Delta_{x}^{n}(1+t)^{s x}\right]_{x=0} \\
&=\frac{1}{n!} e^{a t}\left[(1+t)^{s x}\left\{(1+t)^{s}-1\right\}^{n}\right]_{x=0} \\
& g_{n}(t ; a)=\frac{1}{n!}(1+t)^{a}\left[\Delta_{x}^{n} e^{t x}\right]_{x=0}=\frac{1}{n!}(1+t)^{a}\left[e^{t x}\left\{e^{t}-1\right\}^{n}\right]_{x=0}
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
f_{n}(t)=f_{n}(t ; s, \alpha)=\frac{1}{n!} e^{a t}\left\{(1+t)^{s}-1\right\}^{n} \tag{3.8}
\end{equation*}
$$

$$
\begin{equation*}
g_{n}(t)=g_{n}(t ; \alpha)=\frac{1}{n!}(1+t)^{a}\left\{e^{t}-1\right\}^{n} . \tag{3.9}
\end{equation*}
$$

Differentiating (3.6) and (3.7) with respect to $t$, we obtain the partial differential equations

$$
(1+t) \frac{\partial r(x, t)}{\partial t}=(s x+\alpha t+\alpha) r(x, t)
$$

and

$$
(1+t) \frac{\partial q(x, t)}{\partial t}=(\alpha+x t+x) r(x, t),
$$

and using Theorem 2.3, we may establish the following recurrence relations for the $r-q$ polynomials and $R-Q$ numbers

$$
\begin{align*}
& r_{m+1}(x)=(a+s x-m) r_{m}(x)+a m r_{m-1}(x), m \geq 1, \\
& r_{1}(x)=(a+s x) r_{0}(x) ;  \tag{3.10}\\
& q_{m+1}(x)=(a+x-m) q_{m}(x)+m x q_{m-1}(x), m \geq 1, \\
& q_{1}(x)=(a+x) q_{0}(x) ;
\end{align*}
$$

(3.12) $R(m+1, n)=(\alpha+s n-m) R(m, n)+\alpha m R(m-1, n)$ $+s R(m, m-1), m \geq n+1 ;$
(3.13) $R(m+1, m)=s R(m, m-1)+(\alpha+s m-m) R(m, m)$;
(3.14) $R(m, m)=s R(m-1, m-1)$;
(3.15) $Q(m+1, n)=(\alpha+n-m) Q(m, n)+n m Q(m-1, n)$
$+Q(m, n-1)+m Q(m-1, n-1), m \geq n+1$;
(3.16) $Q(m+1, m)=\alpha Q(m, m)+Q(m, m-1)+m Q(m-1, m-1)$;
(3.17) $Q(m, m)=Q(m-1, m-1)$.

Notice that both relations (3.12) and (3.15) are not "triangular array recurrences" since, for the computation of the ( $m+1, n$ ) term, they require the value of the $(m-1, n)$ term. It is also obvious that, in order to compute all the terms of the double sequences $R(m, n)$ and $Q(m, n), m \geq n$ via recurrences (3.12) and (3.15), respectively, one should at least know the following "initial" (boundary) conditions

```
a. m-axis values R(m, 0), Q(m, 0), m = 0, 1, ...,
b. first-diagonal values R(m,m),Q(m,m),m=1, 2, ...,
c. second-diagonal values R(m, m-1), Q(m, m-1), m = 1, 2, ... .
```

For (a), consider the egf's (3.8) and (3.9) which, in the special case $n=0$, give

$$
\begin{aligned}
& f_{0}(t)=\sum_{m=0}^{\infty} R(m, 0) \frac{t^{m}}{m!}=e^{a t}=\sum_{m=0}^{\infty} a^{m} \frac{t^{m}}{m!} \\
& g_{0}(t)=\sum_{m=0}^{\infty} Q(m, 0) \frac{t^{m}}{m!}=(1+t)^{a}=\sum_{m=0}^{\infty}\binom{a}{m} t^{m}
\end{aligned}
$$

Hence,
(3.18) $R(m, 0)=a^{m}, Q(m, 0)=(a)_{m}$.

The initial condition (b) is readily obtained through (3.14), (3.17), (3.18), as
(3.19) $R(m, m)=s^{m}, \quad Q(m, m)=1$.

As regards condition (c), we proceed as follows: relations (3.13) and (3.16), in view of (3.18), may be written in the form

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$$
\Delta_{m} s^{-m+1} R(m, m-1)=\alpha+(s-1) m, \quad \underset{m}{\Delta} Q(m, m-1)=\alpha+m,
$$

and inverting the difference operator $\Delta_{m}$, we obtain

$$
s^{-m+1} R(m, m-1)=a m+(s-1)\binom{m}{2}+k_{1}, \quad Q(m, m-1)=a m+\binom{m}{2}+k_{2}
$$

Since
and

$$
\begin{aligned}
& R(2,1)=2 R(1,0)+(\alpha+s-1) R(1,1)=(2 \alpha+s-1) s \\
& Q(2,1)=Q(1,0)+\alpha Q(1,1)+Q(0,0)=2 \alpha+1,
\end{aligned}
$$

both constants $k_{1}$ and $k_{2}$ should vanish, and we finally deduce that

$$
\begin{align*}
& R(m, m-1)=a m s^{m-1}+\binom{m}{2}(s)_{2} s^{m-2} \quad m=2,3, \ldots .  \tag{3.20}\\
& Q(m, m-1)=a m+\binom{m}{2}
\end{align*}
$$

It is obvious that the recurrences (3.12) and (3.15), along with initial conditions (3.18), (3.19) and (3.20) determine the double sequences $R(m, n), Q(m, n)$, $m \geq n$.

## 4. Connection with Other Numbers

Let us denote by

$$
s(m, n ; a)=\frac{1}{n!}\left[\frac{d}{d x^{n}}(x)_{m}\right]_{x=a}, \quad S(m, n ; a)=\frac{1}{n!}\left[\Delta^{n} x^{m}\right]_{x=a}
$$

the noncentral Stirling numbers of the first and second kind, respectively, and

$$
C(m, n ; s, \alpha)=\frac{1}{n!}\left[\Delta^{n}(s x+\alpha)_{m}\right]_{x=0}
$$

the noncentral $C$ or Gould and Hopper numbers.
The first class of numbers has been recently studied by Carlitz [4], [5] as weighted Sterling numbers, by Koutras [15], as noncentral Stirling numbers, by Broder [1] as $r$-Stirling numbers, and by Shanmugan [18]. The second class, which was introduced by Chak [6] and Gould \& Hopper [12], and subsequently investigated by Charalambides \& Koutras [10], is closely related to Howard's [13] degenerate weighted Stirling numbers $S_{1}(m, n, \lambda \mid \theta)$ and $S(m, n, \lambda \mid \theta)$ by

$$
\begin{aligned}
& S_{1}(m, n, \lambda \mid \theta)=(-1)^{m-n} C(m, n ; \theta-\lambda, \theta) / \theta^{n}, \\
& S(m, n, \lambda \mid \theta)=\theta^{m} C\left(m, n ; \lambda \theta^{-1}, \theta^{-1}\right) .
\end{aligned}
$$

In order to establish the connection between the $R-Q$ numbers and the abovementioned classes, let us denote by
and

$$
\begin{aligned}
& H_{n}(t)=H_{n}(t ; \alpha)=\sum_{m=n}^{\infty} S(m, n ; \alpha) \frac{t^{m}}{m!}=\frac{1}{n!} e^{a t}\left[e^{t}-1\right]^{n} \\
& C_{n}(t)=C_{n}(t ; s, \alpha)=\sum_{m=n}^{\infty} C(m, n ; s, \alpha) \frac{t^{m}}{m!}=\frac{1}{n!}(1+t)^{a}\left[(1+t)^{s}-1\right]^{n}
\end{aligned}
$$

the egf's of noncentral Stirling and $C$-numbers, respectively. Comparing with formulas (3.8) and (3.9), we obtain

$$
\begin{aligned}
& f_{n}(t ; a, s)=e^{a t} f_{n}(t ; 0, s)=e^{a t} C_{n}(t ; s, 0) \\
& g_{n}(t ; a)=(1+t)^{a} g_{n}(t ; 0)=(1+t)^{\alpha} H_{n}(t ; 0), \\
& (1+t)^{a} f_{n}(t ; s, a)=e^{a t} C_{n}(t ; s, a), \\
& e^{a t} g_{n}(t ; a)=(1+t)^{a} H_{n}(t ; a),
\end{aligned}
$$

which imply the corresponding relations

$$
\begin{equation*}
R(m, n ; s, \alpha)=\sum_{k=n}^{m}\binom{m}{k} a^{m-k} R(k, n ; s, 0)=\sum_{k=n}^{m}\binom{m}{k} a^{m-k} C(k, n, s) \tag{4.1}
\end{equation*}
$$

and

$$
\begin{align*}
& Q(m, n ; \alpha)=\sum_{k=n}^{m}\binom{m}{k}(\alpha)_{m-k} Q(k, n ; 0)=\sum_{k=n}^{m}\binom{m}{k}(\alpha)_{m-k} S(k, n) ; \\
& \sum_{k=n}^{m}\binom{m}{k}(\alpha)_{m-k} R(k, n ; s, \alpha)=\sum_{k=n}^{m}\binom{m}{k} \alpha^{m-k} C(k, n ; s, \alpha)  \tag{4.2}\\
& \sum_{k=n}^{m}\binom{m}{k} \alpha^{m-k} Q(k, n ; \alpha)=\sum_{k=n}^{m}\binom{m}{k}(\alpha)_{m-k} S(k, n ; \alpha) .
\end{align*}
$$

and

Note also that (4.1) leads to the inverse relations

$$
\begin{equation*}
C(m, n ; s)=\sum_{k=n}^{m}\binom{m}{k}(-\alpha)^{m-k} R(k, n ; s, a) \tag{4.3}
\end{equation*}
$$

and

$$
S(m, n)=\sum_{k=n}^{m}\binom{m}{k}(-\alpha)_{m-k} Q(k, n ; \alpha)
$$

which imply that the RHS sums are independent of the parameter $\alpha$.
Finally, we mention that, in view of (4.1), formulas (3.3) and (3.4) lead to the following explicit expressions for the $r-q$ polynomials

$$
\begin{equation*}
r_{m}(x ; s, a)=\sum_{k=0}^{m}\binom{m}{k} a^{m-k}(s x)_{k} \tag{4.4}
\end{equation*}
$$

and

$$
q_{m}(x ; \alpha)=\sum_{k=0}^{m}\binom{m}{k}(\alpha)_{m-k} x^{k}
$$

Remark 1: The proof of (4.4) could also be obtained through the egf's $p(x, t)$, $q(x, t)$, by expanding the RHS of (3.6) and (3.7) in a power series with respect to $t$.
Remark 2: Comparing (4.3) with the binomial and Vandermonde formulas,

$$
(a+x)^{m}=\sum_{k=0}^{m}\binom{m}{k} a^{m-k} x^{k}, \quad(a+s x)_{m}=\sum_{k=0}^{m}\binom{m}{k}(a)_{m-k}(s x)_{k}
$$

one might say that the $r-q$ polynomials are the "intermediate connectors" in the transition from powers to factorials and vice versa.

Another important formula for the $R-Q$ numbers may be obtained as follows: Multiplying (4.1) by $C\left(n, \lambda, s^{-1}\right)$ and summing up for $n=\lambda, \lambda+1, \ldots, m$, we have

$$
\begin{aligned}
\sum_{n=\lambda}^{m} R(m, n ; s, \alpha) C\left(n, \lambda, s^{-1}\right) & =\sum_{n=\lambda}^{m} \sum_{k=n}^{m}\binom{m}{k} a^{m-k} C(k, n, s) C\left(n, \lambda, s^{-1}\right) \\
& =\sum_{k=\lambda}^{m}\binom{m}{k} a^{m-k} \sum_{n=\lambda}^{k} C(k, n, s) C\left(n, \lambda, s^{-1}\right)
\end{aligned}
$$

and on using the orthogonality property of $C$-numbers, we deduce that

$$
\begin{equation*}
\sum_{n=\lambda}^{m} R(m, n ; s, \alpha) C\left(n, \lambda, s^{-1}\right)=\binom{m}{\lambda} \alpha^{m-\lambda} \tag{4.5}
\end{equation*}
$$

Similarly, the orthogonality property of Stirling numbers implies that

$$
\begin{equation*}
\sum_{n=\lambda}^{m} Q(m, n ; \alpha) s(n, \lambda)=\binom{m}{\lambda}(\alpha)_{m-\lambda} \tag{4.6}
\end{equation*}
$$

In matrix notation, formulas (4.5) and (4.6) could be stated as follows: If $R=\left(R_{m n}\right), Q=\left(Q_{m n}\right), C=\left(C_{m n}\right)$, and $s=\left(s_{m n}\right)$ are the infinite matrices with

$$
\begin{array}{lll}
R_{m n}=R(m, n ; s, \alpha), & Q_{m n}=Q(m, n ; \alpha), & m, n=0,1, \ldots, \\
C_{m n}=C\left(m, n, s^{-1}\right), & s_{m n}=s(m, n), & n, \lambda=0,1, \ldots,
\end{array}
$$

respectively, then

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$$
R C=\left(\binom{m}{\lambda} a^{m-\lambda}\right), \quad Q S=\left(\binom{m}{\lambda}(\alpha)_{m-\lambda}\right)
$$

## 5. Applications

## a. Two new families of discrete truncated distributions

It is obvious that the egf's (3.8) and (3.9) satisfy the relations

$$
f_{n+\lambda}(t ; s, a+b)=\frac{n!\lambda!}{(n+\lambda)!} f_{n}(t ; a, s) f_{\lambda}(t ; b, s)
$$

and

$$
g_{n+\lambda}(t ; \alpha+b)=\frac{n!\lambda!}{(n+\lambda)!} g_{n}(t ; \alpha) g_{\lambda}(t ; b),
$$

which imply the following addition theorems:

$$
\begin{aligned}
& R(m, n+\lambda ; s, a+b)=\binom{n+\lambda}{n}^{-1} \sum_{k=0}^{m}\binom{m}{k} R(k, n ; s, a) R(m-k, \lambda ; s, b) ; \\
& Q(m, n+\lambda ; a)=\binom{n+\lambda}{n}^{-1} \sum_{k=0}^{m}\binom{m}{k} Q(k, n ; a) Q(m-k, \lambda ; b) .
\end{aligned}
$$

For $\lambda=0$ one obtains, by virtue of (3.18),

$$
\begin{aligned}
& R(m, n ; s, a+b)=\sum_{k=0}^{m}\binom{m}{k} b^{m-k} R(k, n ; s, a), \\
& Q(m, n ; a+b)=\sum_{k=0}^{m}\binom{m}{k}(b)_{m-k} Q(k, n ; a),
\end{aligned}
$$

and, therefore, we are led to the conclusion that
(5.1) $f(x ; m, n ; a, b)=\binom{m}{x} \frac{b^{m-x} R(x, n ; s, a)}{R(m, n ; s, a+b)}, x=n, n+1, \ldots, m$,
and

$$
g(x ; m, n ; a, b)=\binom{m}{x} \frac{(b)_{m-x} Q(x, n ; a)}{Q(m, n ; a+b)}, \quad x=n, n+1, \ldots, m
$$

define families of multiparameter discrete distributions with range $R_{x}=\{n, n+1, \ldots, m\}$.

Note that probability functions (5.1) could be regarded as generalizations of binomial and hypergeometric laws, respectively, since

$$
\begin{aligned}
& f(x ; m, 0 ; a, b)=\binom{m}{x}\left(\frac{a}{a+b}\right)^{x}\left(\frac{b}{a+b}\right)^{m-x} \\
& g(x ; m, 0 ; a, b)=\binom{a}{x}\binom{b}{m-x} /\binom{a+b}{m}
\end{aligned}
$$

b. Convolution of binomial and Poisson distributions with truncation away from zero

Let $X_{1}, X_{2}, \ldots, X_{v}$ be a random sample from the binomial distribution (5.2) $P[X=x]=(1+\theta)^{-N}\binom{N}{x} \theta^{x}, \quad x=0,1,2, \ldots, N$, where $\theta=p /(1-p)>0$ and $N$ is a positive integer. It is well known that the sum $Z_{1}=X_{1}+\cdots+X_{v}$ is again a binomial variable $b(v N, P)$ with probability function

$$
\begin{equation*}
P\left[Z_{1}=z\right]=(1-\theta)^{-a}\binom{a}{z} \theta^{z}, \quad z=0,1, \ldots, a, \quad a=v N \tag{5.3}
\end{equation*}
$$

Assume further that another independent sample $X_{v+1}, \ldots, X_{v+n}$ coming from the zero-truncated Poisson distribution with parameter $\theta$ is available. For statistical inference purposes, it would be interesting to establish explicit formulas for the distribution of the two-sample sum $Z=X_{1}+\ldots+X_{v+n}$. To this end, we proceed as follows: The probability function of $Z_{2}=X_{v+1}+\ldots+X_{v}+n$ was obtained by Cacoullos [2] in the form

$$
P\left[Z_{2}=z\right]=\frac{n!S(z, n) \theta^{z}}{\left[e^{\theta}-1\right]^{n} z!}, \quad z=n, n+1, \ldots .
$$

Therefore,

$$
\begin{aligned}
P[Z=z] & =\sum_{x=n}^{z} P\left[Z_{1}=z-x\right] P\left[Z_{2}=x\right] \\
& =\frac{n!}{(1+\theta)^{a}\left(e^{\theta}-1\right)^{n}} \frac{\theta^{z}}{z!} \sum_{x=n}^{z}\binom{z}{x}(\alpha)_{z-x^{2}} S(x, n)
\end{aligned}
$$

which, on using (4.1), gives

$$
\begin{equation*}
P[Z=z]=\frac{n!Q(z, n ; a)}{(1+\theta)^{a}\left(e^{\theta}-1\right)^{n}} \frac{\theta^{z}}{z!}, \quad z=n, n+1, \ldots \tag{5.4}
\end{equation*}
$$

Expression (5.4) may be used to obtain an explicit formula for the (unique) unbiased estimator of the parametric function $\theta^{k}$ ( $k$ a positive integer) that is based on the two-sample sum Z. Thus, from the condition of unbiasedness

$$
E\left[h_{k}(Z)\right]=\theta^{k} \text { for every } \theta>0
$$

we obtain, by virtue of (5.4), (3.5), and (3.9),

$$
\sum_{z=n}^{\infty} h_{k}(z) Q(z, n ; \alpha) \frac{\theta^{z}}{z!}=\sum_{z=n+k}^{\infty}(z)_{k} Q(z-k, n ; \alpha) \frac{\theta^{z}}{z!}
$$

which implies that

$$
h_{k}(z)= \begin{cases}\frac{(z)_{k} Q(z-k, n ; \alpha)}{Q(z, n ; a)} & \text { if } z \geq n+k \\ 0 & \text { if } z<n+k\end{cases}
$$

Hence,

$$
h_{1}(Z)=Z Q(Z-1, n ; \alpha) / Q(Z, n ; \alpha), \quad Z \geq n+1
$$

is an unbiased estimator of $\theta$, and since

$$
\operatorname{Var}\left[h_{1}(Z)\right]=E\left[\left(h_{1}(Z)\right)^{2}\right]-\theta^{2}=E\left[\left(h_{1}(Z)\right)^{2}\right]-E\left[h_{2}(Z)\right]
$$

the statistic

$$
h^{*}(Z)=\left[h_{1}(Z)\right]^{2}-h_{2}(Z)
$$

will be an unbiased estimator of the variance of the unbiased estimator of $\theta$.
Consider the case where $X_{1}, X_{2}, \ldots, X_{v}$ is a random sample from the Poisson distribution with parameter 0 and $X_{v+1}, \ldots, X_{v+n}$ is an independent sample from the zero-truncated binomial law with probability function

$$
P[X=x]=\left[(1+\theta)^{N}-1\right]^{-1}\binom{N}{x} \theta^{x}, \quad x=1,2, \ldots .
$$

The distribution of $Z_{1}=X_{1}+\cdots+X_{v}$ is, of course, Poisson with parameter $v \theta$, while the probability function of $Z_{2}=X_{v+1}+\cdots+X_{v+n}$ is given by (see Cacoullos \& Charalambides [3])

$$
P\left[Z_{2}=x\right]=\frac{n!C(x, n, N)}{\left[(1+\theta)^{N}-1\right]^{n}} \frac{\theta^{x}}{x!}, \quad x=n, n+1, \ldots
$$

Therefore, the probability function of the two-sample sum $Z=X_{1}+\ldots+X_{v+n}$ is

$$
\begin{aligned}
P[Z=z] & =\sum_{x=n}^{m} P\left[Z_{1}=z-x\right] P\left[Z_{2}=x\right] \\
& =\frac{n!}{e^{v \theta}\left[(1+\theta)^{N}-1\right]^{n}} \frac{\theta^{z}}{z!} \sum_{x=n}^{z}\binom{z}{x} v^{z-x} C(x, n, N)
\end{aligned}
$$

which, on using (4.1), reduces to
(5.5) $P[Z=z]=\frac{n!R(z, n ; N, v)}{e^{v \theta}\left[(1+\theta)^{N}-1\right]^{n}} \frac{\theta^{z}}{z!}, z=n, n+1, \ldots$.

Following similar arguments with the binomial-zero truncated Poisson problem, one could easily verify that $h_{k}(Z)$ with

$$
h_{k}(Z)= \begin{cases}\frac{(z)_{k} R(z-k, n ; N, v)}{R(z, n ; N, v)} & \text { if } z \geq n+k, \\ 0 & \text { if } z<n+k,\end{cases}
$$

is an unbiased estimator of the parametric function $\theta^{k}$, while

$$
h^{*}(z)=\left[h_{1}(Z)\right]^{2}-h_{2}(Z)
$$

is an unbiased estimator of the variance of the unbiased estimator of $\theta$.

## c. Occupancy problems

Formula (4.1) implies the following combinatorial interpretation of the numbers $Q(m, n ; \alpha)$ : Consider $n$ identical cells with no capacity restrictions and a control cell of $a \in Z^{+}$different (distinguishable) compartments, each of capacity 1 . If $\alpha+m \geq n$, then $Q(m, n ; \alpha)$ is equal to the number of ways of distributing $m$ distinct balls into the cells so that none of the $n$ identical cells is empty.

## 6. The Generalized $R-Q$ Numbers

Following the technique used by Charalambides [8] and Charalambides \& Koutras [10], we may define the generalized $R-Q$ numbers

$$
R_{r}(m, n ; \alpha, s)=R_{r}(m, n) \text { and } Q_{r}(m, n ; \alpha)=Q_{r}(m, n)
$$

by their egf's as follows [cf. (3.8) and (3.9)],

$$
\begin{align*}
f_{n, r}(t) & =f_{n, r}(t ; s, \alpha)=\sum_{m=r n}^{\infty} R_{r}(m, n) \frac{t^{m}}{m!}  \tag{6.1}\\
& =\frac{1}{n!} e^{a t}\left\{(1+t)^{s}-\sum_{k=0}^{r-1}\binom{s}{k} t^{k}\right\}^{n}, \\
g_{n, r}(t) & =g_{n, r}(t ; \alpha)=\sum_{m=m}^{\infty} Q_{r}(m, n) \frac{t^{m}}{m!}  \tag{6.2}\\
& =\frac{1}{n!}(1+t)^{a}\left\{e^{t}-\sum_{k=0}^{r-1} \frac{t^{k}}{k!}\right\}^{n} .
\end{align*}
$$

The generalized $R-Q$ numbers retain many of the properties of the $R-Q$ numbers and may be studied in a similar way.

Thus, differentiating (6.1) and (6.2) with respect to $t$, we obtain the difference-differential equations
and

$$
(1+t) \frac{d}{d t} f_{n, r}(t)=(\alpha+s n+a t) f_{n, r}(t)+(s)_{r} \frac{t^{r-1}}{(r-1)!} f_{n, r-1}(t),
$$

$$
(1+t) \frac{d}{d t} g_{n, r}(t)=(a+n+n t) g_{n, r}(t)+(1+t) \frac{t^{r-1}}{(r-1)!} g_{n, r-1}(t)
$$

which imply the following recurrence relations:

$$
\begin{aligned}
& R_{r}(m+1, n)=(\alpha+s n-m) R_{r}(m, n)+\alpha m R_{r}(m-1, n) \\
& +\binom{m}{r-1}(s)_{r} R_{r}(m-r+1, n-1), m \geq r n+1 ; \\
& R_{r}(m+1, n)=(a+s n-m n) R_{r}(m, n)+\binom{m n}{r-1}(s)_{r} R_{r}(m-r+1, n-1) ; \\
& R_{r}(m, n)=\binom{m-1}{r-1}(s)_{r} R_{r}(m-r, n-1) ; \\
& Q_{r}(m+1, n)=(\alpha+n-m) Q_{r}(m, n)+n m Q_{r}(m-1, n) \\
& +\binom{m}{r-1} Q_{r}(m-r+1, n-1)+r\binom{m}{r} Q(m-r, n-1) \text {, } \\
& m \geq m+1 ; \\
& Q_{r}(m+1, n)=(\alpha+n-m) Q_{r}(m, n)+\binom{m n}{r-1} Q_{r}(r n-r+1, n-1) \\
& +r\binom{r n}{r} Q_{r}(m-r, n-1) \text {; } \\
& Q_{r}(m, n)=\binom{m}{r} r Q_{r}(m-r, n-1) .
\end{aligned}
$$

Notice also that the $m$-axis values for $R_{r}(m, n), Q_{r}(m, n)$ are

$$
R_{r}(m, 0)=\alpha^{m}, \quad Q_{r}(m, 0)=(\alpha)_{m},
$$

as may be readily verified from (6.1) and (6.2).
Another set of recurrences (with respect to $r$ ) useful for tabulation purposes is the following:

$$
\begin{aligned}
& R_{r+1}(m, n)=\sum_{k=0}^{n}(-1)^{k} \frac{(m)_{r k}}{k!}\binom{s}{r}^{k} R_{r}(m-r k, n-k) \\
& R_{r}(m, n)=\sum_{k=0}^{n} \frac{(m)_{r k}}{k!}\binom{s}{r}^{k} R_{r+1}(m-r k, n-k) \\
& Q_{r+1}(m, n)=\sum_{k=0}^{n}(-1)^{k} \frac{(m)_{r k}}{k!(r!)^{k}} Q_{r}(m-r k, n-k) \\
& Q_{r}(m, n)=\sum_{k=0}^{n} \frac{(m)_{r k}}{k!(r!)} Q_{r+1}(m-r k, n-k)
\end{aligned}
$$

This set of recurrences results from the formulas:

$$
\begin{aligned}
& f_{n, r+1}(t)=\sum_{k=0}^{n}(-1)^{k}\binom{s}{r}^{k} \frac{t^{r k}}{k!} f_{n-k, r}(t) ; \\
& f_{n, r}(t)=\sum_{k=0}^{n}\binom{s}{r}^{k} \frac{t^{r k}}{k!} f_{n-k, r+1}(t) ; \\
& g_{n, r+1}(t)=\sum_{k=0}^{n}(-1)^{k} \frac{t^{r k}}{k!(r!)^{k}} f_{n-k, r}(t) ; \\
& g_{n, r}(t)=\sum_{k=0}^{n} \frac{t^{r k}}{k!(r!)^{k}} f_{n-k, r+1}(t) .
\end{aligned}
$$

It is also worth noticing that:
a. The generalized $R-Q$ numbers are connected to the generalized $C$ and Stirling numbers (see [8]) by relations analogous to those of Section 4 for the "non-generalized" quantities.
b. The form of the egf's (6.1) and (6.2) imply "proper" addition theorems for the generalized $R-Q$ numbers, which lead to the definition of two multiparameter discrete distributions with probability functions

$$
f(x ; m, n ; a, b, r)=\binom{m}{x} \frac{b^{m-x} R_{r}(x, n ; s, \alpha)}{R_{r}(m, n ; s, a+b)}, x=m, r n+1, \ldots, m
$$

and

$$
g(x ; m, n ; a, b)=\binom{m}{x} \frac{(b)_{m-x} Q_{r}(x, n ; a)}{Q_{r}(m, n ; a+b)}, x=r n, r n+1, \ldots, m .
$$

c. The generalized $R-Q$ numbers appear in the convolution of two samples coming from a binomial and a Poisson law, when one of the distribution laws is truncated on the left away from a given nonnegative integer $r$. More precisely, we have:
(i) If $X_{1}, X_{2}, \ldots, X_{v}$ is a random sample from the binomial distribution $b(N, p)$ and $X_{v+1}, \ldots, X_{v+n}$ another independent sample from the Poisson distribution with parameter $\theta=p /(1-p)$, truncated away from $r$, i.e.,

$$
P\left[X_{i}=x\right]=\left[e^{\theta}-\sum_{k=0}^{r-1} \frac{\theta^{k}}{k!}\right]^{-1} \frac{\theta^{x}}{x!}, \quad \begin{aligned}
& x=r, r+1, \ldots, \\
& i=v+1, \ldots, v+n,
\end{aligned}
$$

then the distribution of the statistic $Z=X_{1}+\cdots+X_{v+n}$ is given by

$$
P[z=z]=\frac{Q_{r}(z, n ; a)}{g_{n, r}(\theta ; a)} \frac{\theta^{z}}{z!}, \quad z=m, m+1, \ldots .
$$

(ii) If $X_{1}, X_{2}, \ldots, X_{v}$ is a random sample from the Poisson distribution with parameter $\theta$, and $X_{v+1}, \ldots, X_{v+n}$ another independent sample from the binomial law with probability function

$$
P\left[X_{i}=x\right]=\left[(1+\theta)^{N}-\sum_{k=0}^{r-1}\binom{N}{x} \theta^{k}\right]^{-1}\binom{N}{x} \theta^{x}, \quad \begin{aligned}
& x=r, r+1, \ldots, \\
& i=v+1, \ldots, v+n,
\end{aligned}
$$

then the distribution of the statistic $Z=X_{1}+\cdots+X_{v+n}$ is given by

$$
P[z=z]=\frac{R_{r}(z, n ; N, n)}{f_{n, r}(\theta ; N, v)} \frac{\theta^{z}}{z!}, z=r n, m+1, \ldots .
$$

d. The numbers $Q_{r}(m, n ; a)$ admit a combinatorial interpretation similar to the one given for $Q(m, n ; a)$ in Section $5 c$. In the expression "none of the $n$ identical cells is empty," simply replace "is empty" by "contains less rhan $r$ balls."

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# ON THE NUMBER OF PROPAGATION PATHS IN MULTILAYER MEDIA 

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## 1. Introduction


#### Abstract

We consider the problem of enumerating paths of the type shown in Figure 1. A wave leaves $A$ and arrives at $B$ along a path that is straight line except perhaps at the intersection with horizontal lines where the wave may be reflected. The layers between lines represent homogeneous media through which the wave travels in a straight line only. At the boundary between layers, a wave is either reflected or transmitted. This models, for example, sound in sea water and electromagnetic waves in soil associated with power transmission, LaGracé et al. [6]. Figure 1 shows that points $A$ and $B$ are embedded between two layers, in which case a path may cross the $A B$ line some number of times. A similar problem exists in the reflection of light by adjacent panes of glass.




FIGURE 1. The path problem
It is known that the number of paths $P_{n}$ with $n$ reflections in two panes of glass forms a Fibonacci sequence [9, pp. 162-63, 3]. Extensions of this, including more panes and the addition of a mirror, have also been considered [2, 4, 5, 10].

Let there be $m_{1} \geq 0$ layers above the $A B$ line and $m_{2} \geq 0$ below. Any path from A to $B$ consists of an even number $2 n$ of traverses across layers. We seek $N_{m_{1}}, m_{2}(n)$, the number of paths from $A$ to $B$ consisting of $2 n$ traverses.

## 2. Special Cases

There are interesting special cases. The two layer model, $m_{1}=0$ and $m_{2}=$ 2 has been considered in electromagnetic wave propagation in soil, LaGracé et al. [7]. When $m_{1}, m_{2} \geq n$, the path problem is equivalent to the following.

Consider a city neighborhood that consists of $n$ by $n$ square blocks. How many different paths of minimum length are there from the northwest corner to the southeast corner?

View a path as an ordered sequence of $2 n$ letters, $n$ E's and $n$ S's. The path is determined as follows. Starting from the leftmost letter, consider each letter as a specification of whether to go east or south at the current intersection. At the end of the sequence, a traveler will have gone $n$ blocks east and $n$ blocks south. Since there is a one-to-one correspondence between paths and sequences, the number of paths is the number of ways to choose where in the sequence the S's should go, or

$$
N_{\geq n, \geq n}(n)=\binom{2 n}{n} .
$$

When $m_{1}=0$ and $m_{2} \geq n$, the problem is equivalent to Problem 33(a) of Lovasz
(see [8]):
How many monotonic mappings of $\{1, \ldots, n\}$ into itself satisfy the condition $f(x) \leq x$ for every $1 \leq x \leq n$ ?
A monotonic mapping can be represented as dots on a grid, as shown in Figure 2a. A path lying entirely on grid lines is drawn through the dots, beginning at (1, 1), point A, and ending at $(n+1, n+1)$, point B. If $f(x)=f(x+1)$, the segment $(x, f(x)) \rightarrow(x+1, f(x))$ is part of the path. If $f(x)<f(x+1)$, the subpaths $(x, f(x)) \rightarrow(x+1, f(x))$ and $(x+1, f(x)) \rightarrow(x+1, f(x+1))$ are part of the path. Also, subpaths $(n, f(n)) \rightarrow(n+1, f(n))$ and $(n+1, f(n))$ $\rightarrow((n+1),(n+1))$ are part of the path.

(a)

(b)

FIGURE 2. Monotonic mapping equivalence to the path problem
The restriction $f(x) \leq x$ for every $1 \leq x \leq n$ precludes a path from crossing the $A B$ line. It follows that the number of mappings is $N_{0,2 n}(n)$. An interesting argument [8, p. 163], yields a simple expression for this number. Figure 2b shows two additional points ( 0,1 ), point $A^{\prime}$, and ( $n+1, n+2$ ), point $B^{\prime}$. All paths from $A$ to $B$ below the $A B$ line that never cross it are precisely those paths from A to $B$ which never meet the A'B' line. The total number of paths between $A$ and $B$ is $\binom{2 n}{n}$, and if we subtract the number of paths which meet the $A^{\prime} B^{\prime}$ line, we have our result. Figure $2 b$ shows a path which meets the $A^{\prime} B^{\prime}$ line (and also crosses it). Let $C$ be the first point at which a path from $A$ to $B$ meets the $A^{\prime} B^{\prime}$ line. If we reflect the segment $A C$ about the $A^{\prime} B^{\prime}$ line, we obtain the segment $A^{\prime \prime} C$, where $A^{\prime \prime}$ is point $(0,2)$. Thus, any path A to B which meets the A'B' line can be converted to a path A"B. Further, the converse is true. Thus, the number of paths from A to $B$ meeting the A'B' line is equal to the number of unrestricted paths from $A^{\prime \prime}$ to $B$. This is

$$
\binom{n-1+n+1}{n-1}=\binom{2 n}{n-1} .
$$

It follows that,

$$
N_{0, \geq n}(n)=N_{\geq n, 0}(n)=\binom{2 n}{n}-\binom{2 n}{n-1}=\frac{1}{n+1}\binom{2 n}{n},
$$

which is a Catalan number. From this, it follows that the total number of paths from A to $B$ of $2 n$ traverses is reduced by $1 /(1+n)$ when no path is allowed to cross the $A B$ line. Consider now a more general case.

## 3. The Number of Paths Where the Media Is on One Side Only

Let

$$
F_{m}(x)=N_{m}(1) x+N_{m}(2) x^{2}+\cdots+N_{m}(i) x^{i}+\cdots
$$

be the ordinary generating function for $N_{0, m}(i)\left[=N_{m, 0}(i)\right]$. Then, $F^{2}(x)$ enumerates paths having one identified intersection with the $A B$ line; that is,
having one intersection on the $A B$ line distinct from all other such intersections. An end point is not considered an intersection. In a path with $p$ intersections with the $A B$ line, there are $\binom{p}{1}$ ways a single identified intersection can be chosen. Thus, such a path is counted $\binom{p}{1}$ times in $F_{m}^{2}(x)$. Specifically, $F_{m}(x)$ enumerates the ways the path to the left of the identified point can be chosen, $F_{m}(x)$ enumerates the ways to the right, and $F_{m}^{2}(x)$ enumerates the ways both can be chosen. In a similar manner, $F_{m}^{3}(x)$ enumerates paths with two identified intersections on the $A B$ line, etc. Consider

$$
G_{m}(x)=F_{m}^{2}(x)-F_{m}^{3}(x)+F_{m}^{4}(x)-\cdots=\frac{F_{m}^{2}(x)}{1+F_{m}(x)}
$$

$G_{m}(x)$ enumerates paths with at least one intersection with the $A B$ line. Specifically, a path with exactly $p$ intersections with the $A B$ line is counted ( $p_{1}$ ) times in $F_{m}^{2}(x),\binom{p}{2}$ times in $F^{3}(x), \ldots$, and $\binom{p}{p}$ times in $F_{m}^{p+1}(x)$. Thus, a path with exactly $p$ intersections is counted in $G_{m}(x)$ once:

$$
\binom{p}{1}-\binom{p}{2}+\cdots+(-1)^{p+1}\binom{p}{p}=1 .
$$

The number of paths having no intersection with the AB line is $N_{m-1}(n-1)$. This is enumerated in the ordinary generating function $x F_{m-1}(x)$. Thus,

$$
F_{m}(x)=\frac{F_{m}^{2}(x)}{1+F_{m}(x)}+x F_{m-1}(x)+x
$$

where the $+x$ term is the initial condition $N_{m}(1)=1$. Solving for $F_{m}(x)$ yields

$$
\begin{equation*}
F_{m}(x)=\frac{x}{\frac{1}{F_{m-1}(x)+1}-x} \tag{1}
\end{equation*}
$$

We can solve for $F_{m}(x)$ iteratively. When $m=1$, there is only one path and

$$
F_{1}(x)=x+x^{2}+x^{3}+\cdots=\frac{x}{1-x}
$$

$F_{2}(x)$ is obtained by substituting $x /(1-x)$ for $F_{m-1}(x)\left[=F_{1}(x)\right]$ in (1). $F_{3}(x)$ and other generating functions are obtained in a similar manner. Table 1 shows the generating functions $F_{m}(x)$ for $1 \leq m \leq 5$. Also shown is the corresponding power series expansion.

Let $F_{\infty}(x)$ be the generating function for the number of paths when there are arbitrarily many layers below the AB line. An expression for $F_{\infty}(x)$ can be obtained by substituting $F_{\infty}(x)$ for $F_{m}(x)$ and $F_{m-1}(x)$ in (1). This yields an expression that is quadratic in $F_{\infty}(x)$, which can be solved to produce the expression shown in Table 1.

We can find closed form expressions for the approximate number of paths by a manipulation of the generating function. We illustrate using $F_{3}(x)$.

$$
F_{3}(x)=x \frac{1-x}{1-3 x+x^{2}}=x \frac{\frac{\left(1-5^{1 / 2}\right)}{\left(5-2 \cdot 5^{1 / 2}\right)}}{1-x \frac{2}{3-5^{1 / 2}}}+x \frac{\frac{\left(1+5^{1 / 2}\right)}{\left(5+3 \cdot 5^{1 / 2}\right)}}{1-x \frac{2}{3+5^{1 / 2}}}
$$

Let $a_{n} \sim b_{n}$ mean $\lim _{n \rightarrow \infty} \frac{a_{n}}{b_{n}}=1$. Then, we can write,

$$
N_{3}(n) \sim \frac{\left(1-5^{1 / 2}\right)}{\left(5-3 \cdot 5^{1 / 2}\right)}\left(\frac{2}{3-5^{1 / 2}}\right)^{n-1}=0.724(2.618)^{n-1}
$$

TABLE 1. Generating functions, power series expansion, and closed form expressions for the number of paths with $2 n$ traverses when there are $m$ layers below the $A B$ line and none above the $A B$ line

| Generating Function $F_{m}(x)$ | Power Series | Closed Form Expression |
| :--- | :--- | :--- |
| $F_{1}(x)=\frac{x}{1-x}$ | $x+x^{2}+x^{3}+x^{4}+x^{5}+\cdots$ | $N_{1}(n)=1$ |
| $F_{2}(x)=\frac{x}{1-2 x}$ | $x+2 x^{2}+4 x^{3}+8 x^{4}+16 x^{5}+\cdots$ | $N_{2}(n)=2^{n-1}$ |
| $F_{3}(x)=\frac{x-x^{2}}{1-3 x+x^{2}}$ | $x+2 x^{2}+5 x^{3}+13 x^{4}+34 x^{5}+\cdots$ | $N_{3}(n) \sim 0.724(2.618)^{n-1}$ |
| $F_{4}(x)=\frac{x-2 x^{2}}{1-4 x+3 x^{2}}$ | $x+2 x^{2}+5 x^{3}+14 x^{4}+41 x^{5}+\cdots$ | $N_{4}(n) \sim 0.5(3)^{n-1}$ |
| $F_{5}(x)=\frac{x-3 x^{2}+x^{3}}{1-5 x+6 x^{2}-x^{3}}$ | $x+2 x^{2}+5 x^{3}+14 x^{4}+42 x^{5}+\cdots$ | $N_{4}(n) \sim 0.349(3.247)^{n-1}$ |
| $F_{\infty}(x)=x\left(F_{\infty}(x)+1\right)^{2}=\frac{1-2 x-(1-4 x)^{1 / 2}}{2 x}$ | $x+2 x^{2}+5 x^{3}+14 x^{4}+42 x^{5}+\ldots+\frac{1}{n+1}\left[\begin{array}{c}2 n \\ n\end{array}\right) x^{n}+\ldots$ | $N_{\infty}(n) \sim\left(\pi n^{3}\right)^{-1 / 2} 4^{n}=0.564 n^{-3 / 2} 4^{n}$ |

To find an approximation of a form similar to those given earlier for

$$
N_{\infty}(n)=\frac{1}{n+1}\binom{2 n}{n}
$$

we can represent $\binom{2 n}{n}$ in factorials and use Stirling's approximation. Alternatively, we can apply Theorem 5 of Bender [1] to the generating function for $N_{\infty}(n)$. In either case, we obtain

$$
N_{\infty}(n) \sim\left(\pi n^{3}\right)^{-1 / 2} 4^{n}
$$

Table 2 shows the values of the number of paths of $2 n$ traverses, where there are $m$ layers. These entries were obtained by a program to solve for the coefficients of the various generating functions $F_{m}(x)$ using a symbolic mathematical manipulation package.

TABLE 2. Number of paths with $2 n$ traverses when there are layers below the $A B$ line and none above the $A B$ line

| $m / n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| 3 | 1 | 2 | 5 | 13 | 34 | 89 | 233 | 610 | 1597 | 4181 |
| 4 | 1 | 2 | 5 | 14 | 41 | 122 | 365 | 1094 | 3281 | 9842 |
| 5 | 1 | 2 | 5 | 14 | 42 | 131 | 417 | 1341 | 4334 | 14041 |
| $\infty$ | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 | 4862 | 16796 |

## 4. The Number of Paths Where the Media Is on Both Sides

The calculation for $F_{m_{1}, m_{2}}(x)=N m_{1}, m_{2}(1) x+N m_{1}, m_{2}(2) x^{2}+\ldots$ can be made in terms of the case just considered. Specifically,

$$
\begin{equation*}
F_{m_{1}, m_{2}}(x)=\frac{F_{m_{1}}^{2}, m_{2}(x)}{1+F_{m_{1}, m_{2}}(x)}+x F_{m_{1}-1,0}(x)+x F_{0, m_{2}-1}(x)+2 x, \tag{2}
\end{equation*}
$$

for $m_{1}, m_{2} \geq 1$. $F_{m_{1}}^{2}, m_{2} /\left(1+F_{m_{1}}, m_{2}(x)\right)$ counts paths from A to $B$ with at least one intersection with the AB line. $x F_{m_{1}-1,0}(x)$ and $x F_{0, m_{2}-1}(x)$ count paths that are entirely above and below the $A B$ line, respectively. $+2 x$ represents the initial condition $N_{m_{1}}, m_{2}(1)=2$ when $m_{1}, m_{2} \geq 1$.

Solving (2) for $F_{m_{1}}, m_{2}(x)$ yields

$$
\begin{equation*}
F_{m_{1}, m_{2}}(x)=\frac{x}{\frac{1}{F_{m_{1}-1,0}(x)+F_{0, m_{2}-1}(x)+2}-x} . \tag{3}
\end{equation*}
$$

For the special case of $m_{1}=m_{2}=m$ we have, from (3),

$$
F_{m, m}(x)=\frac{x}{\frac{1}{2\left(F_{m-1}(x)+1\right)}-x} .
$$

Table 3 shows the generating functions for $F_{m, m}(x)$ for $1 \leq m \leq 5$ and $\infty$, and Table 4 shows the number of paths $N \geq n, \geq n(n)$ when there are layers above and below the $A B$ line. These show clearly the significantly larger number of paths which exist when they are allowed to cross the AB line.

TABLE 3. Generating functions, power series expansion, and closed form expressions for the number of paths with $2 n$ traverses when there are $m$ layers above and below the $A B$ line

| Generating Function $F_{m, m}(x)$ | Power Series | Closed Form Expression |
| :---: | :---: | :---: |
| $F_{1,1}(x)=\frac{2 x}{1-2 x}$ | $2 x+4 x^{2}+8 x^{3}+16 x^{4}+32 x^{5}+\ldots$ | $N_{1,1}(n)=2(2)^{n-1}$ |
| $F_{2,2}(x)=\frac{2 x}{1-3 x}$ | $2 x+6 x^{2}+18 x^{3}+54 x^{4}+162 x^{5}+\ldots$ | $N_{2,2}(n)=2(3)^{n-1}$ |
| $F_{3,3}(x)=\frac{2 x-2 x^{2}}{1-4 x+2 x^{2}}$ | $2 x+6 x^{2}+20 x^{3}+68 x^{4}+232 x^{5}+\ldots$ | $N_{3,3}(n) \sim 1.707(3.414)^{n-1}$ |
| $F_{4.4}(x)=\frac{2 x-4 x^{2}}{1-5 x+5 x^{2}}$ | $2 x+6 x^{2}+20 x^{3}+70 x^{4}+250 x^{5}+\ldots$ | $N_{4,4}(n) \sim 1.447(3.618)^{n-1}$ |
| $F_{5,5}(x)=\frac{2 x-6 x^{2}+2 x^{3}}{1-6 x+9 x^{2}-2 x^{3}}$ | $2 x+6 x^{2}+20 x^{3}+70 x^{4}+252 x^{5}+\ldots$ | $N_{4,4}(\boldsymbol{n})-1.244(3.732)^{n-1}$ |
| $F_{\ldots, \ldots}(x)=\frac{2 x\left(F_{-}(x)+1\right)}{1-2 x\left(F_{-( }(x)+1\right)}=(1-4 x)^{-1 / 2}-1$ | $2 x+6 x^{2}+20 x^{3}+70 x^{4}+252 x^{5}+\ldots+\binom{2 n}{n} x^{n}+\ldots$ | $N_{m, \infty}(n) \sim(\pi n)^{-1 / 2} 4^{n}=0.564 n^{-1 / 2} 4^{n}$ |

TABLE 4. Number of paths with $2 n$ traverses when there are layers above and below the $A B$ line

| $\mathrm{m} / \mathrm{n}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
| 2 | 2 | 6 | 18 | 54 | 162 | 486 | 1458 | 4374 | 13122 | 39366 |
| 3 | 2 | 6 | 20 | 68 | 232 | 792 | 2704 | 9232 | 31520 | 107616 |
| 4 | 2 | 6 | 20 | 70 | 250 | 900 | 3250 | 11750 | 42500 | 153750 |
| 5 | 2 | 6 | 20 | 70 | 252 | 922 | 3404 | 12630 | 46988 | 175066 |
| $\infty$ | 2 | 6 | 20 | 70 | 252 | 924 | 3432 | 12870 | 48620 | 184756 |

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# Applications of Fibonacci Numbers 

Volume 3<br>New Publication

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# ASYMPTOTIC POSITIVENESS OF LINEAR RECURRENCE SEQUENCES 

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## Dedicated to Professor L. Kuipers <br> on the occasion of his 80th birthday

Suppose that the first several terms of a sequence are given, then it is not so easy to predict the asymptotic behavior of this sequence. But once we know that this given sequence is a linear recurrence sequence, we can determine the asymptotic behavior through its recurrence formula.

Indeed, John R. Burke and William A. Webb [1] considered real linear recurrence sequences $\left\{u_{n}\right\}_{n=0}^{\infty}$ of order $d$ defined by

$$
\begin{equation*}
u_{n+d}=a_{d-1} u_{n+d-1}+a_{d-2} u_{n+d-2}+\cdots+a_{0} u_{n} \text { for } n \geq 0 \tag{1}
\end{equation*}
$$

where $\alpha_{0}, \alpha_{1}, \ldots, \alpha_{d-1}$ are real numbers, with its corresponding characteristic equation:

$$
\begin{equation*}
p(x)=x^{d}-a_{d-1} x^{d-1}-\cdots-a_{1} x-a_{0}=0 \tag{2}
\end{equation*}
$$

They obtained a criterion for the asymptotic positiveness of linear recurrence sequences (1) if the corresponding characteristic equation has distinct roots. Here we call a sequence $\left\{u_{n}\right\}_{n=0}^{\infty}$ asymptotically positive if there exists a natural number $n_{0}$ such that

$$
u_{n}>0 \text { for all } n \geq n_{0}
$$

In particular, if the above $n_{0}$ is equal to zero, we call this sequence $\left\{u_{n}\right\}_{n=0}^{\infty}$ totally positive.

In this note, we shall give a criterion of asymptotic positiveness of real linear recurrence sequences $\left\{u_{n}\right\}_{n=0}^{\infty}$ (1) of order $d$, when their characteristic equations have multiple roots.

Let us recall a general representation formula for $u_{n}$. We assume that the corresponding characteristic equation (2) of $\left\{u_{n}\right\}_{n=0}^{\infty}$ has roots $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{p}$ with corresponding multiplicities $m_{1}, m_{2}, \ldots, m_{p}$. Then there exist polynomials $b_{1}, b_{2}, \ldots, b_{p}$ with degree $b_{i} \leq m_{i}-1$ for $i=1,2, \ldots, p$, where the coefficients of polynomials $b_{1}, b_{2}, \ldots, b_{p}$ depend only on the roots of the characteristic equation (2) and the initial values of this recurrence sequence. Then, we have, for all $n \geq 0$,

$$
\begin{equation*}
u_{n}=b_{1}(n) \lambda_{1}^{n}+b_{2}(n) \lambda_{2}^{n}+\cdots+b_{p}(n) \lambda_{p}^{n} \tag{3}
\end{equation*}
$$

The detailed discussion of this representation (3) can be found, for example, in Władysław Narkiewicz [4] or Alecksei I. Markuševič [2].

Without loss of generality, we arrange the roots $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{p}$ according to their moduli as

$$
\left|\lambda_{1}\right| \geq\left|\lambda_{2}\right| \geq \cdots \geq\left|\lambda_{p}\right|
$$

Suppose first that $\lambda_{2}$ is the complex conjugate of $\lambda_{1}, \lambda_{1}$ is not real, and

$$
\begin{equation*}
\left|\lambda_{1}\right|=\left|\lambda_{2}\right|>\left|\lambda_{3}\right| \geq \cdots \geq\left|\lambda_{p}\right| \tag{4}
\end{equation*}
$$

We assume further that the sum of the first two terms of (3), denoted by

$$
\begin{equation*}
v_{n}=b_{1}(n) \lambda_{1}^{n}+b_{2}(n) \lambda_{2}^{n} \tag{5}
\end{equation*}
$$

does not vanish for infinitely many $n$. Then

$$
\text { (6) } \quad u_{n}=v_{n}+o\left(v_{n}\right)
$$

holds for all sufficiently large $n$ (see Nagasaka, Kanemitsu, \& Shiue [3]). Since $\left\{u_{n}\right\}_{n=0}^{\infty}$ is a real sequence, we get

$$
b_{2}(n)=\overline{b_{1}(n)}
$$

and

$$
\begin{aligned}
v_{n}=b_{1}(n) \lambda_{1}^{n}+b_{2}(n) \lambda_{2}^{n} & =b_{1}(n)\left(r e^{2 \pi i \theta}\right)^{n}+\overline{b_{1}(n)}\left(r e^{-2 \pi i \theta}\right)^{n} \\
& \left.=b_{1}(n) r^{n} e^{2 \pi i n \theta}+\overline{\left(b_{1}(n) r^{n} e^{2 \pi i n \theta}\right.}\right) \\
& =2 \operatorname{Re}\left\{b_{1}(n) r^{n} e^{2 \pi i n \theta}\right\},
\end{aligned}
$$

where $\lambda_{l}=r e^{2 \pi i \theta}$ and $\theta$ is not a multiple of $\pi$ (since $\lambda_{l}$ is not real). Now, if we write

$$
b_{1}(n)=c_{k} n^{k}+c_{k-1} n^{k-1}+\cdots+c_{0}
$$

where $c_{0}, c_{1}, \ldots, c_{k}$ are complex numbers determined by the roots $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{p}$, and initial values $u_{0}, u_{1}, \ldots, u_{d-1}$ with nonzero $c_{k}, k \leq m_{1}-1$. Then

$$
\begin{aligned}
v_{n} & =2 \operatorname{Re}\left(c_{k} n^{k} r^{n} e^{2 \pi i n \theta}\right)+o\left(n^{k_{r} n}\right) \\
& =2 n^{k} r^{n} \operatorname{Re}\left(c_{k}\right) \cos (2 \pi n \theta)+o\left(n^{k_{r}}\right) \text { for large } n
\end{aligned}
$$

Since $\theta$ is not a multiple of $\pi, v_{n}$ takes negative values for infinitely many $n$, by applying the same argument as in the proof of Theorem 1 in Burke $\&$ Webb [1]. Hence, by (6), the original linear recurrence sequence $\left\{u_{n}\right\}$ is not asymptotically positive for this case. Summarizing the above discussion, we have
Theorem 1: Suppose that the roots $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{p}$ of the characteristic equation of $\left\{u_{k^{\prime}}\right\}_{n=0}^{\infty}$ satisfy (4) and that $\lambda_{1}$ and $\lambda_{2}$ are complex conjugates of each other and are not real. Assume that $v_{n}$ does not vanish for infinitely many $n$, then the linear recurrence sequence $\left\{u_{n}\right\}_{n=0}^{\infty}$ is not asymptotically positive.

Secondly, we assume again the relation (4) with real $\lambda_{1}$ and $\lambda_{2}$, that is, $-\lambda_{2}=\lambda_{1}$. We denote the leading coefficients of the polynomials $b_{1}(n)+b_{2}(n)$ and $b_{1}(n)-b_{2}(n)$ by $A$ and $B$, respectively, and assume further that $A B \neq 0$ for all sufficiently large $n$. Say that $b_{1}(n)+b_{2}(n)$ has degree $k, b_{1}(n)-b_{2}(n)$ has degree $\ell$. Then (8) holds for all sufficiently large $n$. Hence, we have that, for all sufficiently large even $n$,
(7) $\quad u_{n}=A n^{k} \lambda_{l}^{n}+o\left(n^{k} \lambda_{l}^{n}\right)$
and, for all sufficiently large odd $n$, we get

$$
\begin{equation*}
u_{n}=B n^{\ell} \lambda_{1}^{n}+o\left(n^{l} \lambda_{1}^{n}\right) \tag{8}
\end{equation*}
$$

Thus, we obtain
Theorem 2: Suppose that the roots $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{p}$ of the characteristic equation of $\left\{u_{n}\right\}_{n=0}^{\infty}$ satisfy (4) and $0<\lambda_{1}=-\lambda_{2}$ that are real. Assume further that the leading coefficients $A$ and $B$ of the polynomials $b_{1}(n)+b_{2}(n)$ and $b_{1}(n)-b_{2}(n)$ are positive. Then $\left\{u_{n}\right\}_{n=0}^{\infty}$ is asymptotically positive.

We now leave assumption (4). Then, we have either

$$
\begin{equation*}
\left|\lambda_{1}\right|=\left|\lambda_{2}\right|=\left|\lambda_{3}\right|=\cdots=\left|\lambda_{j}\right|>\left|\lambda_{j+1}\right| \geq \cdots \geq\left|\lambda_{p}\right| \tag{9}
\end{equation*}
$$

for some $j>2$, or
(10) $\left|\lambda_{1}\right|>\left|\lambda_{2}\right| \geq \cdots \geq\left|\lambda_{p}\right|$.

First, let us consider the case (10). From the fact that the coefficients of the characteristic equation $\alpha_{0}, \alpha_{1}, \ldots, \alpha_{d-1}$ are all real, $\lambda_{1}$ must be real.

Also, if $b_{1}(n)$ is not identically zero, we get

$$
\begin{equation*}
u_{n}=C n^{m} \lambda_{1}^{n}+o\left(n^{m} \lambda_{1}^{n}\right), \tag{11}
\end{equation*}
$$

where $C$ is the leading coefficient of the polynomial $b_{1}(n)$ of degree $m \leq m_{1}-1$. Thus, we obtain
Theorem 3: Suppose that the roots $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{p}$ of the characteristic equation of $\left\{u_{n}\right\}_{n=0}^{\infty}$ satisfy (10). Assume further that the polynomial $b_{1}(n)$ is not identically zero, that $\lambda_{1}$ is positive, and that the leading coefficient $C$ of $b_{1}(n)$ is also positive. Then the linear recurrence sequence $\left\{u_{n}\right\}_{n=0}^{\infty}$ is asymptotically positive.

For the remaining case (9), we need to divide into the following three subcases:
(i) $j$ is even, all $\lambda_{\ell}$ are not real for $\ell=1,2, \ldots, j$ and $\lambda_{2 i}$ is the complex conjugate of $\lambda_{2 i-1}$ for $i=1,2, \ldots, j / 2$. We assume further that $b_{1}(n)$, $b_{2}(n), \ldots, b_{p}(n)$ do not vanish for all $n \geq n_{0}$.

Then, applying Theorem $1,\left\{u_{n}\right\}_{n=0}^{\infty}$ is not asymptotically positive.
(ii) $j$ is even, $0<\lambda_{1}=-\lambda_{2}$ are real, all other $\lambda_{\ell}$ for $\ell=3,4, \ldots, j$ are not real, and $\lambda_{2 i}$ is the complex conjugate of $\lambda_{2 i-1}$ for $i=2,3, \ldots, j / 2$. We suppose again that $b_{1}(n), b_{2}(n), \ldots, b_{j}(n)$ do not vanish for all $n \geq n_{0}$.

Then $\left\{u_{n}\right\}_{n=0}^{\infty}$ is asymptotically positive if the leading coefficients $A, B$ of $b_{1}(n)+b_{2}(n)$ and $b_{1}(n)-b_{2}(n)$, respectively, are both positive for all sufficiently large $n$ and either

$$
\begin{aligned}
& \min \left\{\operatorname{deg}\left(b_{1}(n)+b_{2}(n)\right), \operatorname{deg}\left(b_{1}(n)-b_{2}(n)\right)\right\} \text { is greater than } \\
& i=2, \max _{3, \ldots, j / 2}\left\{\operatorname{deg}\left(2 \operatorname{Re}\left(b_{2 i-1}(n)\right)\right)\right\}
\end{aligned}
$$

or
$\min (A, B)-1$ is greater than all the leading coefficients of $2 \operatorname{Re}\left(b_{2 i-1}(n)\right)$ for which

$$
\begin{aligned}
& \min \left\{\operatorname{deg}\left(b_{1}(n)+b_{2}(n)\right), \operatorname{deg}\left(b_{1}(n)-b_{2}(n)\right)\right\} \\
& =\operatorname{deg}\left\{2 \operatorname{Re}\left(b_{2 i-1}(n)\right)\right\} \text { for } i=2,3, \ldots, j / 2 .
\end{aligned}
$$

(iii) $j$ is odd, $0<\lambda_{1}$ is real, all other $\lambda_{\ell}$ are not real for $\ell=2,3, \ldots, j$ and $\lambda_{2 i+1}$ is the complex conjugate of $\lambda_{2 i}$ for $i=1,2, \ldots,[j / 2]$.

Then $\left\{u_{n}\right\}_{n=0}^{\infty}$ is asymptotically positive if the leading coefficient $C$ of $b_{1}(n)$ is positive and either $\operatorname{deg}\left(b_{1}(n)\right)$ is greater than

$$
i=1,2, \ldots,[j / 2]\left\{\operatorname{deg}\left(2 \operatorname{Re}\left(b_{2 i}(n)\right)\right)\right\}
$$

or $C-1$ is greater than all the leading coefficients of $2 \operatorname{Re}\left(b_{2 i}(n)\right)$ for which

$$
\operatorname{deg}\left(b_{1}(n)\right)=\operatorname{deg}\left(2 \operatorname{Re}\left(b_{2 i}(n)\right)\right) \text { for } i=1,2, \ldots,[j / 2] .
$$

We assume always the nonvanishing property of all $b_{\ell}(n)$ for $\ell=1,2, \ldots, j$, for the case (9). If some of the $b_{l}(n)$ are identically zero, say $b_{k}(n)$, then we simply ignore these terms $b_{k}(n) \lambda_{k}^{n}$, and it is sufficient to trace the above discussion.

Finally, we give explicit conditions for a real linear recurrence sequence of order 2 or of order 3 to be asymptotically positive.

We denote $\left\{s_{n}\right\}_{n=0}^{\infty}$ a linear recurrence sequence of order 2 with recurrence formula $s_{n+2}=a_{1} s_{n+1}+a_{0} s_{n}$. First, we assume that its corresponding characteristic equation of degree 2 has only one real double root $\alpha \neq 0$. Then, $\alpha_{1}=$ $2 \alpha$ and $\alpha_{0}=-\alpha^{2}$ and the $n^{\text {th }}$ term $s_{n}$ can be represented by

$$
s_{n}=\left(p_{1} n+p_{2}\right) \alpha^{n} \text { for } n \geq 0
$$

By solving the system of equations

$$
\left\{\begin{array}{l}
s_{0}=p_{2} \\
s_{1}=\left(p_{1}+p_{2}\right) \alpha
\end{array}\right.
$$

we obtain

$$
p_{1}=\left(s_{1}-s_{0} \alpha\right) / \alpha
$$

Applying the discussion of Theorem 3 above, we have
Theorem 4: Suppose the characteristic equation of a linear recurrence sequence $\left\{s_{n}\right\}_{n=0}^{\infty}$ has only one real double nonzero root $\alpha$. Sequence $\left\{s_{n}\right\}_{n=0}^{\infty}$ is asymptotically positive if and only if $\alpha>0$ and either $s_{1}>s_{0} \alpha$ or $s_{0}>0$ and $s_{1}=s_{0} \alpha$.
Corollary 4.1: Under the same assumption as in Theorem 4, the sequence $\left\{s_{n}\right\}_{n=0}^{\infty}$ is asymptotically positive if and only if $\alpha_{1}>0$ and either $2 s_{1}>\alpha_{1} s_{0}$ or $s_{0}>0$ and $2 s_{1}=a_{1} s_{0}$.

By using the relation between $\alpha$ and the $\alpha_{i}^{\prime}$ 's, this Corollary follows immediately from Theorem 4.

Let us recall the case where the characteristic equation of a linear recurrence sequence $\left\{s_{n}\right\}_{n=0}^{\infty}$, that is,

$$
\begin{equation*}
\lambda^{2}-a_{1} \lambda-a_{0}=0 \tag{12}
\end{equation*}
$$

has two distinct roots.
Theorem 5: Let $D=a_{1}^{2}+4 \alpha_{0}$ be the discriminant of equation (12) of degree 2 . Suppose the characteristic equation of $\left\{s_{n}\right\}_{n=0}^{\infty}$ has two distinct roots $\alpha_{l}$ and $\alpha_{2}$. This sequence $\left\{s_{n}\right\}_{n=0}^{\infty}$ is asymptotically positive if and only if $\sqrt{D}$ is real and one of the following four conditions is satisfied:
(i) $a_{1}=0, s_{0}>0, s_{1}>0$.
(ii) $\alpha_{1}>0,2 s_{1}>\left(a_{1}-\sqrt{D}\right) s_{0}$.

$$
\begin{align*}
& \alpha_{1}>0,2 s_{1}=\left(\alpha_{1}-\sqrt{D}\right) s_{0}, s_{0}>0, \alpha_{0}<0  \tag{iii}\\
& \alpha_{1}<0,2 s_{1}=\left(a_{1}+\sqrt{D}\right) s_{0}, s_{0}>0, \alpha_{0}>0
\end{align*}
$$

Proof: Suppose first that $\sqrt{D}$ is purely imaginary. Then $\alpha_{2}$ is the complex conjugate of $\alpha_{1}$ and the $n^{\text {th }}$ term $s_{n}$ can be represented by

$$
s_{n}=c_{1} \alpha_{1}^{n}+\bar{c}_{1} \bar{\alpha}_{1}^{n}
$$

since $\left\{s_{n}\right\}_{n=0}^{\infty}$ is a sequence of real numbers. We now apply Theorem 1 .
For $\left\{s_{n}\right\}_{n=0}^{\infty}, v_{n}$, as defined by (5), is identical to $s_{n}$. The nonvanishing assumption of $s_{n}=v_{n}$ is naturally satisfied, since otherwise $\left\{s_{n}\right\}_{n=0}^{\infty}$ becomes the sequence of $0^{\prime} s$ which is not asymptotically positive. Hence, all assumptions of Theorem 1 are fulfilled. Thus, for purely imaginary $\sqrt{D},\left\{s_{n}\right\}_{n=0}^{\infty}$ is not asymptotically positive by Theorem 1.

Now we get necessarily that if $\sqrt{D}$ is positive real then $\alpha_{1}>\alpha_{2}$. Condition (i) is already treated in the proof of Theorem 3 [1]. For the remaining cases, (ii), (iii), and (iv), we use a representation formula of $s_{n}$,

$$
s_{n}=c_{1} \alpha_{1}^{n}+c_{2} \alpha_{2}^{n}
$$

with

$$
c_{1}=\frac{s_{1}-s_{0} \alpha_{2}}{\alpha_{1}-\alpha_{2}}, \quad c_{2}=\frac{s_{0} \alpha_{1}-s_{1}}{\alpha_{1}-\alpha_{2}}
$$

In addition to case (ii) treated already in Theorem 3 [1], we are forced to add condition (iii), since $c_{1}$ may be zero. If $c_{1}=0$ with positive $\alpha_{1}$, then

$$
s_{n}=\frac{s_{0} \alpha_{1}-s_{1}}{\alpha_{1}-\alpha_{2}} \alpha_{2}^{n} .
$$

Thus, we require that $s_{0} \alpha_{1}-s_{1}>0$ and $\alpha_{2}>0$, from which we deduce $u_{0}>0$ and $a_{0}<0$.

If $\alpha_{1}<0$ with real positive $\sqrt{D}$, then $\alpha_{2}<0$ and $\left|\alpha_{1}\right|<\left|\alpha_{2}\right|$. For asymptotic positiveness of $\left\{s_{n}\right\}_{n=0}^{\infty}$, we require that $c_{2}=0, c_{1}>0$, and $\alpha_{1}>0$. Rewriting these three conditions, we obtain (iv).

The sufficiency part of Theorem 5 is almost immediate from the representation formula of $s_{n}$. Q.E.D.

Remark: Combining Theorems 4 and 5, we obtain a complete characterization for asymptotic positiveness of linear recurrence sequences $\left\{s_{n}\right\}_{n=0}^{\infty}$ of order 2 in terms only of the coefficients of the recurrence formula and of the initial values.

Now we consider a linear recurrence sequence $\left\{t_{n}\right\}_{n=0}^{\infty}$ of order 3 with recurrence relation

$$
t_{n+3}=a_{2} t_{n+2}+a_{1} t_{n+1}+a_{0} t_{n}
$$

Burke \& Webb [1] give a sufficient condition for $\left\{t_{n}\right\}_{n=0}^{\infty}$ to be asymptotically positive.
Theorem 6: Suppose the characteristic equation of $\left\{t_{n}\right\}_{n=0}^{\infty}$ has distinct roots $\alpha_{1}, \alpha_{2}, \alpha_{3}$ and that they satisfy either
(13) $\left|\alpha_{1}\right|>\left|\alpha_{2}\right| \geq\left|\alpha_{3}\right|$
or

$$
\left|\alpha_{1}\right|=\left|\alpha_{2}\right|>\left|\alpha_{3}\right| \text { and } \alpha_{2} \text { is the complex conjugate of } \alpha_{1} \text {. }
$$

If $\alpha_{1}>0$ and $c_{1}>0$, then $\left\{t_{n}\right\}_{n=0}^{\infty}$ is asymptotically positive where $t_{n}$ is written as

$$
\begin{equation*}
t_{n}=c_{1} \alpha_{1}^{n}+c_{2} \alpha_{2}^{n}+c_{3} \alpha_{3}^{n} . \tag{14}
\end{equation*}
$$

Keeping the assumption of distinct roots, Theorem 6 does not cover the following cases:
(i) $\alpha_{1}=-\alpha_{2}$ with real $\alpha_{1}$.
(ii) $\alpha_{2}$ is the complex conjugate of $\alpha_{1}$ and the roots satisfy

$$
\left|\alpha_{1}\right|=\left|\alpha_{2}\right|=\left|\alpha_{3}\right|
$$

Case (i) can be treated using Theorem 2; however, (ii) is a special case of (9) which brings certain difficulty to determine $\left\{t_{n}\right\}_{n=0}^{\infty}$ to be asymptotically positive.

Burke \& Webb give another elegant sufficient condition for $\left\{t_{n}\right\}_{n=0}^{\infty}$ to be asymptotically positive as Theorem 2 in [1], but they implicitly assume (13) and also that $c_{1} \neq 0$ in (14). In order to obtain the necessary and sufficient conditions for $\left\{t_{n}\right\}_{n=0}^{\infty}$ to be asymptotically positive as in Theorem 5 with the assumption of distinct roots, there are too many cases split according to the vanishingness of the coefficients in (14). We can treat all of these cases; however, we shall give necessary and sufficient conditions for $\left\{t_{n}\right\}_{n=0}^{\infty}$ to be asymptotically positive only when the characteristic equation has multiple roots, since originally we planned to generalize the results of Burke \& Webb [1] for multiple roots.

Thus, we assume that the characteristic equation of $\left\{t_{n}\right\}_{n=0}^{\infty}$ of order 3 has multiple roots. In order to determine conditions for $\left\{t_{n}\right\}_{n=0}^{\infty}$ to be asymptotically positive, Theorem 3 assumes that it is sufficient to consider only the following two cases:
(I) The corresponding characteristic equation of degree 3 has only one triple real root $\beta$.
(II) The corresponding characteristic equation of degree 3 has one double real root $\beta \neq 0$ and another real root $\gamma$ with $|\beta| \geq|\gamma|$.
Let us treat case (I). The $n^{\text {th }}$ term $t_{n}$ is represented by

$$
t_{n}=\left(q_{1} n^{2}+q_{2} n+q_{3}\right) \beta^{n} \text { for } n \geq 0
$$

Solving the system of equations

$$
\left\{\begin{array}{l}
t_{0}=q_{3} \\
t_{1}=\left(q_{1}+q_{2}+q_{3}\right) \beta \\
t_{2}=\left(4 q_{1}+2 q_{2}+q_{3}\right) \beta^{2}
\end{array}\right.
$$

we get

$$
q_{1}=\frac{t_{2}-2 t_{1}+t_{0} \beta^{2}}{2 \beta^{2}}, \quad q_{2}=\frac{-t_{2}+4 t_{1} \beta-3 t_{0} \beta^{2}}{2 \beta^{2}}, \quad q_{3}=t_{0}
$$

Thus, in case (I), the sequence $\left\{t_{n}\right\}_{n=0}^{\infty}$ is asymptotically positive if and only if $\beta>0$ and either

$$
\begin{equation*}
t_{2}-2 t_{1} \beta+t_{0} \beta^{2}>0 \tag{15}
\end{equation*}
$$

or

$$
\begin{equation*}
t_{2}-2 t_{1} \beta+t_{0} \beta^{2}=0 \text { and }-t_{2}+4 t_{1} \beta-3 t_{0} \beta^{2}>0 \tag{16}
\end{equation*}
$$

or
(17) $\quad t_{2}-2 t_{1} \beta+t_{0} \beta^{2}=-t_{2}+4 t_{1} \beta-3 t_{0} \beta^{2}=0$ and $t_{0}>0$.

Condition (16) can be reduced to

$$
\begin{equation*}
t_{1}>t_{0} \beta \text { and } t_{2}=2 t_{1} \beta-t_{0} \beta^{2} \tag{18}
\end{equation*}
$$

Condition (17) can also be reduced to

$$
\begin{equation*}
t_{2}=t_{0} \beta^{2}, \quad t_{1}=t_{0} \beta, \quad \text { and } \quad t_{0}>0 \tag{19}
\end{equation*}
$$

Summarizing the above argument, we have
Theorem 7: Let $\left\{t_{n}\right\}_{n=0}^{\omega}$ be a linear recurrence sequence of order 3. Suppose the characteristic equation of $\left\{t_{n}\right\}_{n=0}^{\prime \prime}$ has only one triple real root $\beta$. The sequence $\left\{t_{n}\right\}_{n=0}^{\prime \prime}$ is asymptotically positive if and only if $a_{0}>0, a_{2}>0$, and one of the following three conditions holds:
(i) $3 t_{2}-2 \alpha_{2} t_{1}-\alpha_{1} t_{0}>0$.
(ii) $3 t_{2}-2 a_{2} t_{1}-\alpha_{1} t_{0}=0$ and $3 t_{2}-4 a_{2} t_{1}-3 a_{1} t_{0}<0$.
(iii) $3 t_{2}-2 a_{2} t_{1}-\alpha_{1} t_{0}=3 t_{2}-4 a_{2} t_{1}-3 a_{1} t_{0}=0$ and $t_{0}>0$.

These three conditions are mentioned in (15), (18), and (19) above. We need only rewrite them as the relations

$$
\alpha_{2}=3 \beta, \quad \alpha_{1}=-3 \beta^{2}, \quad \alpha_{0}=\beta^{3}
$$

since $\beta$ is the triple multiple root of the characteristic equation

$$
\lambda^{3}-a_{2} \lambda^{2}-a_{1} \lambda-a_{0}=0 . \quad \text { Q.E.D. }
$$

$$
\begin{aligned}
& \text { For case (II), the } n^{\text {th }} \text { term } t_{n} \text { is represented by } \\
& t_{n}=\left(q_{1} n+q_{2}\right) \beta^{n}+h \gamma^{n} \text {. }
\end{aligned}
$$

Thus, we have

$$
h=\frac{\left(\beta^{2}+2 \gamma^{2}\right) t_{0}-2 \beta t_{1}+t_{2}}{(\beta-\gamma)^{2}}, \quad q_{1}=\frac{\gamma(\beta+2 \gamma) t_{0}-(\beta+\gamma) t_{1}+t_{2}}{\beta(\beta-\gamma)},
$$

and

$$
q_{2}=\frac{-\gamma(2 \beta+\gamma) t_{0}+2 \beta t_{1}-t_{2}}{(\beta-\gamma)^{2}} .
$$

We now divide into two subcases:
(IIa) $|\beta|>|\gamma|$.
In this case, the sequence $\left\{t_{n}\right\}_{n=0}^{\infty}$ is asymptotically positive if and only if $\beta>0$ and either $q_{1}>0$ or $q_{1}=0$ and $q_{2}>0$ or $q_{1}=q_{2}=0, h>0$, and $\gamma>0$.
(IIb) $|\beta|=|\gamma|$.
In this case, the sequence $\left\{t_{n}\right\}_{n=0}^{\infty}$ is asymptotically positive if and only if either $\beta>0$ and $q_{1}>0$ or $\beta>0, \gamma>0, q_{1}=0, q_{2}+h>0$, and $q_{2}>h$ or $\beta<0, q_{1}=0, q_{2}+h>0$, and $q_{2}<h$ or $q_{1}=q_{2}=0, h>0$, and $r>0$.
Remark: For an arbitrary given linear recurrence sequence $\left\{t_{n}\right\}_{n=0}^{\infty}$, we can give explicit conditions for $\left\{t_{n}\right\}_{n=0}^{\infty}$ to be asymptotically positive when the characteristic equation has one real double root $\beta$ and another real root $\gamma$ with $\gamma$ in terms of only the coefficients of the recurrence formula and of the initial values as in Theorem 6, since we have $\alpha_{2}=2 \beta+\gamma, a_{1}=-2 \beta \gamma-\beta^{2}$, and $\alpha_{0}=\beta^{2} \gamma$.

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# ON THE GENERALIZED FIBONACCI PSEUDOPRIMES 

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## 1. Introduction and Preliminaries

In this paper the results established by the first two authors in [3], [4], and [5] are extended and generalized.

After defining (in this section) classes of generalized Lucas numbers, $\left\{V_{n}(m)\right\}$, governed by the positive integral parameter $m$, the Fibonacci pseudoprimes of the $m^{\text {th }}$ Kind (m-F.Psps.) are characterized in Section 2. A method for constructing them is discussed in Section 3, while some numerical results concerning these pseudoprimes are presented in Section 4. Finally, in Section 5, some possible further investigations in this field are outlined.

Let $m$ be an arbitrary natural number. The generalized Lucas numbers $V_{n}(m)$ (or simply $V_{n}$, if there is no fear of confusion) are defined (e.g., see [1] and [7]) by the second-order linear recurrence relation
(1.1) $\quad V_{n+2}=m V_{n+1}+V_{n} ; V_{0}=2, V_{1}=m$.

These numbers can also be expressed by means of the closed form expression (Binet's form)
(1.2) $V_{n}=\alpha_{m}^{n}+\beta_{m}^{n}$,
where
(1.3) $\left\{\begin{array}{l}\Delta_{m}=\sqrt{m^{2}+4} \\ \alpha_{m}=\left(m+\Delta_{m}\right) / 2 \\ \beta_{m}=-1 / \alpha_{m}=\left(m-\Delta_{m}\right) / 2 .\end{array}\right.$

It can be noted that, letting $m=1$ in (1.1) and (1.2), the usual Lucas numbers $L_{n}$ are obtained.

The following fundamental property of the numbers $V_{n}$ has been established ([10], Eq. 108, p. 295): If $n$ is prime, then for all $m$,
$(1.4) \quad V_{n}(m) \equiv m(\bmod n)$.
2. The Fibonacci Pseudoprimes of the $\mathrm{m}^{\text {th }}$ kind:

Definition and Some Numerical Aspects
Rotkiewicz proved [15] that for each $m$, infinitely many odd composite numbers $n$ satisfy (1.4). Odd composite $n$ satisfying (1.4) are called Fibonacci pseudoprimes of the $m^{\text {th }}$ kind ( $m$-F.Psps.). Write $s_{k}(m)$ for the $k^{\text {th }}$ one. Note that $s_{1}(1)=705, s_{1}(2)=169$, and $s_{1}(3)=33$.

Some numerical aspects of the Fibonacci pseudoprimes of the $1^{\text {st }}$ kind $\left[s_{k}(1)\right.$ or 1 -F.Psps.] have been investigated by the authors in previous papers [3], [4], and [5]. In particular, we found that all l-F.Psps. below $10^{8}$ are squarefree and, as expected, most of them are congruent to 1 both modulo 4 (81.3\%) and modulo 10 (63.2\%). A heuristic argument to explain the popularity of the classes 1 modulo 4 and 1 modulo 10 can be constructed (cf. [12], p. 1018).

This work was carried out in the framework of an agreement between the Italian PT Administration and the Fondazione Ugo Bordoni.

Now, a question arises: "Do odd composites exist which are m-F.Psps. for distinct values of $m$ ?" The answer is the affirmative.

We define as strong Fibonacci pseudoprimes of the Mth Kind (M-sF.Psps.) all odd composites which are $m-$ F.Psps. for $m=1,2, \ldots, M$. Obviously, from this definition, it follows that $1-F . P s p s$. and $1-s F . P s p s$. coincide and an $M-s F . P s p$. is an $m$-sF.Psp. $(1 \leq m<M)$ as well. For information, the smallest $2-s F$.Psp. is $s_{14}(1)=34,561$, while the smallest 3 -sF.Psp. is $s_{89}(1)=1,034,881$. Note that Theorem 6 of [4] states that a l-F.Psp. is also a 4-F.Psp. so that all 3-sF.Psps. are also 4-sF.Psps.

A computer experiment was carried out [8] essentially to compile a table of 1-F.Psps. up to $10^{8}$ and to find $M$-sF.Psps. ( $M>1$ ) below this bound. The results can be summarized as follows. There are 852 l-F.Psps. below $10^{8}$ of which 48 are 2 -sF.Psps. Four among these numbers are 4 -sF.Psps. Among them, the rather exceptional number

```
s802(1) = 87,318,001 = 17•71• 73.991
```

is a 7-sF.Psp. and is, at the same time, a Carmichael number. Carmichael numbers are composite numbers $n$ which satisfy the Fermat congruence $b^{n-1} \equiv 1$ (mod $n$ ) for each $b$ relatively prime to $n$. Denoting the $k^{\text {th }}$ Carmichael number by $C_{k}$, we found that

$$
s_{802}(1)=C_{244}
$$

### 2.1 Tables of 1-F.Psps: A Brief Historical Note

Earlier authors investigated the 1-F.Psps. and compiled tables of them up to certain bounds. To the best of our knowledge, apart from the sporadical discoveries of the first few l-F.Psps. (e.g., see [11]; [5], Sec. 2), the oldest table (up to 555,200 ) containing, among other numbers, such pseudoprimes was compiled by Duparc [6] in 1955. In 1976 Yorinaga [17] compiled an analogous table to 707,000, and in 1983 Singmaster [16] published a table of 1F.Psps. to 100,000 (these numbers were defined as Lucas pseudoprimes by the author). A table of $1-\mathrm{F}$. Psps. up to $10^{6}$ was given by the first two authors [5] in 1987.

The second author extended this table up to $10^{8}$ [8]. Copies of it will be sent, free of charge, upon request.

## 3. A Method To Obtain m-F.Psps.

In this section we offer a method to obtain generating formulas for the $m$-F.Psps. and, as a particular instance, we work out formulas for generating $M$-sF.Psps. ( $M=1,2,3,4,5$ ). The case $M=1$ concerns, of course, numbers that are simply l-F.Psps. Some numerical examples are also given.

First, let us state the following propositions.
Proposition 1: Let $p_{i}=5 k_{i} \pm 1$ and $q_{j}=5 h_{j} \pm 2$ be odd rational primes. Let

$$
n=\prod_{i, j} p_{i}^{a} q_{j}^{a} \quad(a \in\{0,1\})
$$

be an odd composite and $\Lambda(n)=1 \mathrm{~cm}\left(p_{i}-1,2 q_{j}+2\right)_{i, j}$.
If $n-1 \equiv 0[\bmod \Lambda(n)]$, then $L_{n} \equiv 1(\bmod n)$, that is, $n$ is a $1-F . P s p$.
Proposition 2: If $p_{i}=8 k_{i} \pm 1, q_{j}=8 h_{j} \pm 3$, and $n-1 \equiv 0[\bmod \Lambda(n)]$, then $n$ is a 2-F.Psp.
Proposition 3: If $p_{i}=13 k_{i} \pm u(u=1,3,4), q_{j}=13 h_{j} \pm v(v=2,5,6)$, and $n-1 \equiv 0[\bmod \Lambda(n)]$, then $n$ is a $3-$ F.Psp.

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Proposition 4: If $p_{i}=29 k_{i} \pm u(u=1,4,5,6,9,13), q_{j}=29 h_{j} \pm v(v=2,3$, $8,10,11,12,14)$, and $n-1 \equiv 0[\bmod \Lambda(n)]$, then $n$ is a $5-\mathrm{F} . \mathrm{Psp}$.
Proof of Proposition 1: Since $\alpha_{1}$ and $\beta_{1}$, see (1.3), are integers (more precisely, unities) over the quadratic field $k(\sqrt{5})$, we have (see [9], p. 222)

$$
\begin{equation*}
\alpha_{1}^{p_{i}-1} \equiv \beta_{1}^{p_{i}-1} \equiv 1\left(\bmod p_{i}\right) \tag{3.1}
\end{equation*}
$$

and (from [9], p. 223)
(3.2) $\left\{\begin{array}{l}\alpha_{1}^{q_{j}+1} \equiv N \alpha_{1}=\alpha_{1} \beta_{1}=-1\left(\bmod q_{j}\right), \\ \beta_{1}^{q_{i}+1} \equiv N \beta_{1}=\beta_{1} \alpha_{1}=-1\left(\bmod q_{j}\right),\end{array}\right.$
$N \xi$ being the norm of the element $\xi$ of a generic quadratic field.
If $n-1 \equiv 0[\bmod \Lambda(n)]$, then by (3.1) we can write
(3.3) $\left.\alpha_{1}^{n-1}=\alpha_{1}^{t_{i}\left(p_{i}-1\right)}=\left(\alpha_{1}^{p}\right)^{-1}\right)^{t_{i}} \equiv 1\left(\bmod p_{i}\right) \quad\left(t_{i} \in \mathbb{N}=\{0,1,2, \ldots\}\right)$
and, analogously,
(3.4) $\beta_{1}^{n-1} \equiv 1\left(\bmod p_{i}\right)$.

Under the same condition, by (3.2) we have
(3.5) $\alpha_{1}^{n-1}=\alpha_{1}^{r_{j}\left(2 q_{i}+2\right)}=\left(\alpha_{i}^{q_{j}+1}\right)^{2 r_{j}} \equiv 1\left(\bmod q_{j}\right) \quad\left(r_{j} \in \mathbb{N}\right)$
and
(3.6) $\beta_{1}^{n-1} \equiv 1\left(\bmod q_{j}\right)$.

Then, by (3.3)-(3.6) we obtain the congruences

$$
\begin{equation*}
\left.\alpha_{1}^{n} \equiv \alpha_{1}\left(\bmod \prod_{i, j} p_{i}^{a} q_{j}^{a}\right) \quad \text { (i.e., } \bmod n\right) \tag{3.7}
\end{equation*}
$$

and
(3.8) $\quad \beta_{1}^{n} \equiv \beta_{1}(\bmod n)$.

Finally, by (3.7) and (3.8) we have

$$
L_{n}=\alpha_{1}^{n}+\beta_{1}^{n} \equiv \alpha_{1}+\beta_{1}=1(\bmod n) \cdot \text { Q.E.D. }
$$

The proofs of Propositions 2, 3, and 4 are similar to that of Proposition 1 and are omitted for brevity.

### 3.1. Generating 1-F.Psps.

The first two examples offered in this subsection follow directly from Theorem 4 of [4] and give formulas for generating l-F.Psps. which are, in addition, Carmichael numbers. The above mentioned theorem states that, if $n=$ $p_{1} p_{2} \ldots p_{s}$, with $p_{i}$ a prime of the form $5 k_{i} \pm 1(1 \leq i \leq s)$, is a Carmichael number, then $n$ is also a l-F.Psp. Note that Proposition 1 generalizes this theorem.
Example 1: $n=p_{1} p_{2} p_{3}$
In 1939 Chernick invented universal forms for generating Carmichael numbers [2]. In this paper we refer to Ore's book [10] where these formulas are reported.

For constructing numbers $n$ of the above form (see [10], pp. 334-336), a suitable choice of the integral parameters $P_{1}, \mu_{2}$, and $P_{3}$ [ibid.] is necessary. For instance, for $P_{1}=5, P_{2}=1$, and $P_{3}=6$, we obtain (3.9) $n(t)=(30 t+19)(150 t+91)(180 t+109)(t \in \mathbb{N})$.

For all values of $t$ such that all three factors on the right-hand side of (3.9) are prime (necessarily of the form $5 k_{i} \pm 1$ ), $n(t)$ is both a $1-\mathrm{F} . \mathrm{Psp}$. and a Carmichael number. The smallest among such numbers is

$$
n(4)=79,624,621=s_{766}(1)=C_{233}
$$

Example 2: $n=p_{1} p_{2} p_{3} p_{4}$
A formula yielding Carmichael numbers with four factors can be readily obtained from ([13], p. 99):
(3.10) $n(t)=(30 t+1)(60 t+1)(90 t+1)(180 t+1) \quad(t \in \mathbb{N})$.

For all values of $t$ such that all four factors on the right-hand side of (3.10) are prime (necessarily of the form $5 k_{i} \pm 1$ ), $n(t)$ is both a $1-F . P s p$. and a Carmichael number. The smallest among such numbers is

$$
n(9)=192,739,365,541=C_{4568}
$$

Example 3: $n=p q_{1} q_{2}$
The following example is based on Proposition 1 . Let $p=5 k \pm 1$ and $q_{j}=$ $5 h_{j} \pm 2$. It can be readily proved that, if $n-1 \equiv 0[\bmod \Lambda(n)]$, then any two of the three numbers $p-1, q_{1}+1$, and $q_{2}+1$ have the same greatest common divisor $d$. Therefore, we can write

$$
\begin{equation*}
p-1=d P, q_{1}+1=d Q_{1}, q_{2}+1=d Q_{2} \tag{3.11}
\end{equation*}
$$

or

$$
p-1=d P, 2 q_{1}+2=2 d Q_{1}, 2 q_{2}+2=2 d Q_{2}
$$

where the numbers $P, Q_{1}$, and $Q_{2}$ are relatively prime in pairs. Consequently, we have

$$
\Lambda(n)=1 \mathrm{~cm}\left(p-1,2 q_{1}+2,2 q_{2}+2\right)=2 d P Q_{1} Q_{2}
$$

and the sufficient condition for $n$ to be a l-F.Psp. (see Proposition 1 ) takes the form
(3.12) $n=p q_{1} q_{2} \equiv 1 \quad\left(\bmod 2 d P Q_{1} Q_{2}\right)$.

Following Ore (see [10], pp. 335-336), let us replace the values of $p$, $q_{l}$, and $q_{2}$ on the left-hand side of (3.12) by the corresponding values obtainable by (3.11). After some manipulations, omitted for brevity, we obtain the congruence

$$
\begin{equation*}
d\left(Q_{1} Q_{2}-P Q_{1}-P Q_{2}\right)+P-Q_{1}-Q_{2} \equiv 0 \quad\left(\bmod 2 P Q_{1} Q_{2}\right) \tag{3.13}
\end{equation*}
$$

After choosing suitable values for $P, Q_{1}$, and $Q_{2}$, we find the smallest positive solution $d_{0}$ to the congruence (3.13) so that, by (3.11), we can write
(3.14) $\left\{\begin{aligned} p & =\left(d_{0}+2 t P Q_{1} Q_{2}\right) P+1, \\ q_{1} & =\left(d_{0}+2 t P Q_{1} Q_{2}\right) Q_{1}-1, \\ q_{2} & =\left(d_{0}+2 t P Q_{1} Q_{2}\right) Q_{2}-1 .\end{aligned} \quad(t \in \mathbb{N})\right.$

The choice of $P, Q_{1}$, and $Q_{2}$ must yield a value of $d_{0}$ such that $p=5 k \pm 1$ and $q_{j}=5 h_{j} \pm 2(j=1,2)$. For all values of $t$ such that all three numbers $p$, $q_{1}$, and $q_{2}$ are prime, $n$ is a l-F.Psp. (but, in general, it is not a Carmichael number).

For instance, putting $P=5, Q_{1}=1$, and $Q_{2}=2$ in (3.13), we obtain $d_{0}=14$ and, by (3.14),
(3.15) $n(t)=q_{1} q_{2} p=(20 t+13)(40 t+27)(100 t+71) \quad(t \in \mathbb{N})$.

For $t \leq 100,000$ there exist 641 1-F.Psps. of the above form. The smallest among them is

$$
n(2)=s_{114}(1)=1,536,841
$$

while the largest is

$$
n(99,992)=79,982,429,286,524,601,241
$$

Many more formulas for generating l-F.Psps. can be obtained by means of other suitable choices of $P, Q_{1}$, and $Q_{2}$ in (3.13). As a further example, letting $P=5, Q_{1}=2$, and $Q_{2}=9$, we get
(3.16) $n(t)=(360 t+203)(900 t+511)(1620 t+917) \quad(t \in \mathbb{N})$.

For $t \leq 100,000$ there exist $12551-F$.Psps. of this form. The smallest among them is

$$
n(10)=619,127,589,961
$$

while the largest is

$$
n(99,994)=524,794,437,221,730,602,894,281
$$

It must be noted that the sets containing the $1-\mathrm{F} . \mathrm{Psps}$. of the forms (3.15) and (3.16) are disjoint.

### 3.2 Generating $m$-F.Psps. $(m>1)$

Using the results established in Section 3.1 and Propositions 2-4, we can derive formulas for generating $M$-sF.Psps. $(M=2,3,4,5)$.

For example, let us consider expression (3.15) which generates $1-\mathrm{F} . \mathrm{Psps}$. and impose that $q_{1}$ (and $q_{2}$ ) and $p$ are of the forms $8 h \pm 3$ and $8 k \pm 1$, respectively (see Proposition 2). As a particular instance, if we impose that $p \equiv-1$ (mod 8), then the congruence $t \equiv 0(\bmod 2)$ must necessarily hold. For such values of $t$, the relations $q_{1}=8 h_{1}-3$ and $q_{2}=8 h_{2}+3$ turn out, so that the conditions of Proposition 2 are fulfilled (the congruence $n-1 \equiv 0[\bmod \Lambda(n)]$ holds in (3.15), by construction).

Consequently, the numbers

$$
\begin{align*}
n(t) & =q_{1} q_{2} p=(20 \cdot 2 t+13)(40 \cdot 2 t+27)(100 \cdot 2 t+71)  \tag{3.17}\\
& =(40 t+13)(80 t+27)(200 t+71) \quad(t \in \mathbb{N})
\end{align*}
$$

are 2 -sF.Psps. for all values of $t$ such that all three factors on the righthand side of (3.17) are prime. For $t \leq 50,000$, there exist 3292-sF.Psps. of this form. The smallest (largest) among them and the smallest (largest) 1F.Psp. obtainable by (3.15) (for $t \leq 100,000$ ) obviously coincide.

Analogously, by imposing the condition $p \equiv 3$ (mod 13) (see Proposition 3) to (3.17), we obtain the numbers
(3.18) $n(t)=(520 t+93)(1040 t+187)(2600 t+471) \quad(t \in \mathbb{N})$,
which, for all values of $t$ such that all three factors are prime, are 3sF.Psps. and, consequently (cf. the end of the fourth paragraph in Section 2), are also 4-sF.Psps. For $t \leq 50,000$ there exist 256 such numbers. The smallest among them is

$$
n(59)=291,424,493,801,801
$$

while the largest is

$$
n(49,976)=175,508,922,783,506,139,921,721
$$

Finally, by imposing the condition $p \equiv-4$ (mod 29) (see Proposition 4) on (3.18), we obtain the numbers

1990]
(3.19) $n(t)=(15,080 t+2173)(30,160 t+4347)(75,400 t+10,871) \quad(t \in \mathbb{N})$
which, for all values of $t$ such that all three factors are prime, are 5-sF.Psps. For $t \leq 25,000$ there exist 73 such numbers. The smallest among them is

$$
n(47)=3,593,246,900,779,046,281,
$$

while the largest is
$n(24,791)=522,508,952,184,890,040,253,388,041$.
It can be proved that numbers of the form (3.19) cannot be 6-F.Psps.

## 4. Carmichael Numbers and Generalized Fibonacci Pseudoprimes:

## A Computer Experiment

By means of this experiment, we sought numbers which are M-sF.Psps. for comparatively large $M$. Since the largest value of $M$ which we were aware of (namely, $M=7$ ) pertains to a Carmichael number (namely, $C_{244}=87,318,001$ ), we submitted all numbers $C_{k}<25 \cdot 10^{9}$ to the test
(4.1) $\quad V_{0}:(m) \equiv m\left(\bmod C_{k}\right)$
for $m=1,2,3, \ldots$, with the aid of an efficient computer algorithm which finds $V_{n}$ reduced modulo $n$ after $\left[\log _{2} n\right]$ recursive calculations (cf. [14], pp. $114 \mathrm{ff}$. .). We could carry out this experiment by virtue of the courtesy of the editor of this journal who placed the table of Carmichael numbers compiled by S. Wagstaff (Purdue University) (cf. [12]) at our disposal.

While this paper was being refereed, Professor Wilfrid Keller (Rechenzentrum der Universitaet Hamburg, FRG) kindly provided us with a table of all $C_{k} \leq 10^{13}$ compiled by him. Submitting these numbers to the test (4.1) yielded the following update to the results obtained from Wagstaff's table. There exist 19,278 Carmichael numbers below $10^{13}$ :

| 3518 among them are 1-F.Psps. | 3518 among them are 1-sF.Psps. |
| :---: | :---: |
| 2767 are 2-F.Psps. | 599 are 2-sF.Psps. |
| 1735 are 3-F.Psps. | 63 are 3-sF.Psps. |
| 3679 are 4-F.Psps. | 63 are 4-sF.Psps. |
| 1104 are 5-F.Psps. | 9 are 5-sF.Psps. |
| 1643 are 6-F.Psps. | 8 are 6-sF.Psps. |
| 1258 are 7-F.Psps. | 4 are 7-sF.Psps. |
| 1307 are 8-F.Psps. | None of them is an 8-sF.Psp. |
| 1443 are 9-F.Psps. |  |
| 1324 are 10-F.Psps. |  |

The three additional 7-sF.Psps. we found are

```
C1092 = 3,998,554,561 = 31 • 41 • 199 • 15,809,
C 3662 = 103,964,580,721 = 37. 41 . 43 | 199 • 8009,
C}\mp@subsup{C}{7122}{\prime}=669,923,876,161=17•43•97•197•199•241
```

Since none of these numbers is an 8-sF.Psp., the record was not beaten! We offered [4] a prize of 50,000 Italian lire to the first person who would communicate to us an 8-sF.Psp. (below $10^{100}$ ). Of course, at least one of its factors was also requested.

Currently, the smallest 8 -sF.Psp. which we were able to construct [see Sec. 5(iv)] is the 29-digit Carmichael number

$$
34,613,972,314,979,099,337,871,392,961
$$

(three factors). Actually, this number is an 11-sF.Psp. The first author won the prize.

Incidentally, we used the above mentioned algorithm also to submit all composite Lucas numbers $L_{p}(2 \leq p \leq 953, p$ either a prime or a power of 2 ) to the test
(4.2) $\quad V_{L_{p}}(m) \equiv m\left(\bmod L_{p}\right)$
for $m=2$. We recall (see Corollaries 1 and 3 of [4]) that (4.2) holds for any $p$ if $m=1$. The result of this experiment led us to formulate the following
Conjecture 1: No composite $L_{p}$ is a 2-F.Psp.
which implies the equivalent "Lp is prime iff (4.2) holds for $m=2 . "$ If Conjecture 1 were proved, then a powerful tool for finding very large Lucas primes would have been discovered.

## 5. Future Work

The authors intend to continue their study on the properties of $m$-F.Psps. The principal aim of this further work is:
(i) to find the smallest $M$-sF.Psps. for $8 \leq M \leq 15$;
(ii) to evaluate the order of magnitude of the smallest $M$-sF.Psps. for $M>15$;
(iii) to find the smallest $M$-sF.Psps. ( $M>2$ ) (if any) which are the product of exactly two distinct primes (the smallest 1-F.Psp. and 2sF.Psp. of this form are $s_{5}(1)=F_{19}=4181$ and $s_{202}(1)=4,403,027$, respectively).
(iv) to establish formulas for generating M-sF.Psps. ( $M \geq 2$ ) which are, at the same time, Carmichael numbers.

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## REPORT ON THE FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Herta T. Freitag

Sponsored jointly by the Fibonacci Association and Wake Forest University, The Fourth International Conference on Fibonacci Numbers and Their Applications was held from July 30 to August 3, 1990. As the Conference took place at Wake Forest University, our foreign visitors especially gained a most enjoyable insight into one of America's delightful set-ups: a small, highly esteemed, liberal arts University, nestled at the outskirts of a faithfully restored eighteenth-century town, Winston-Salem, North Carolina.

Immediately upon arrival it became clear to us how carefully and compe-tently-under the leadership of the co-chairmen of the International Committee, A. F. Horadam (Australia) and A. N. Philippou (Cyprus), as well as of the cochairmen of the Local Committee, F. T. Howard and M. E. Waddill-our Conference had been planned and prepared. Special thanks must also go to G. E. Bergum, editor of our Fibonacci Quarterly, for arranging an outstanding program.

There were about 50 participants, 40 of whom presented papers. Of these, two were women. From some 13 different lands they came; beside the U.S., the host country, Italy would have won the prize for maximum attendance, then Canada and Scotland, closely followed by Australia and Japan.

Papers related to the Fibonacci numbers and their ramifications, and to recursive sequences and their generalizations, as well as those that analyzed and explained number relationships, were presented. Once again, as had been the case in our previous conferences, the diversity of the papers gave testimony to the fertility of Fibonacci-related mathematics, as well as to the fructification of ideas, brought about through our mutual but, at the same time, disparate interests. The interplay between theoretically oriented manuscripts and those that highlighted practical aspects was, again, conspicuous and fascinating.

The Conference was held in the new Olin Physical Laboratory, which was accessible via overcoming several road hurdles that were necessitated by construction work across the campus. Although our hosts were most apologetic about this, we saw it as a sign of a vital, dynamic and, indeed, growing University.
(Please turn to page 382)

# EXTENSIONS OF CONGRUENCES OF GLAISHER AND NIELSEN CONCERNING STIRLING NUMBERS 

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## 1. Introduction

Let $s(n, k)$ and $S(n, k)$ be the (unsigned) Stirling numbers of the first and second kinds, respectively. These numbers are well known and have been extensively studied; see, for example, [5, Ch. 5]. The generating functions are
(1.1) $\quad(-\log (1-x))^{k}=k!\sum_{n=k}^{\infty} s(n, k) x^{n} / n!$,
(1.2) $\quad\left(e^{x}-1\right)^{k}=k!\sum_{n=k}^{\infty} S(n, k) x^{n} / n!$.

Congruences for the Stirling numbers are apparently not well known. A few congruences for prime moduli can be found in [5, pp. 218-19, 229] and other books, but surprisingly little work has been done on this problem. Carlitz [4] worked out a method for finding congruences for $S(n, k)(\bmod p)$, where $p$ is prime, and recent papers by the author [8], Kwong [10], Nijenhuis \& Wilf [12], and Peele [14] indicate an increased interest in Stirling number congruences.

The main purpose of this paper is to extend results of Glaisher [6] and Nielsen [11, p. 338] by proving the following congruences: Let $p$ be an odd prime, let $n$ be a positive integer, and suppose $p^{t} \| n$; that is, $p^{t}$ is the highest power of $p$ dividing $n$. Let $B_{2 r}$ be the $2 r^{\text {th }}$ Bernoulli number. For $0<$ $2 r<2 p-2$ and $1<2 r+1<2 p-2$, we have
(1.3) $s(n, n-2 r) \equiv \frac{-n}{2 r}\binom{n-1}{2 r} B_{2 r^{\prime}} \quad\left(\bmod p^{2 t}\right)$,
(1.4) $s(n, n-2 r-1) \equiv-\frac{n^{2}(2 r+1)}{4 r}\binom{n-1}{2 r+1} B_{2 r}\left(\bmod p^{3 t}\right)$,
(1.5) $\quad S(n+2 r, n) \equiv \frac{n}{2 r}\binom{n+2 r}{2 r} B_{2 r}\left(\bmod p^{2 t}\right)$,
$S(n+2 r+1, n) \equiv \frac{n^{2}(2 r+1)}{4 r}\binom{n+2 r+1}{2 r+1} B_{2 r}\left(\bmod p^{3 t}\right)$.
When $n=p$ and $0<2 r<p-1,1<2 r+1<p-1$, congruences (1.3) and (1.4) reduce to the previously mentioned theorem of Glaisher, while (1.5) and (1.6) reduce to the results of Nielsen. Since extensive tables of the Bernoulli numbers are available (the first sixty are listed in [9, p. 234], for example) and, since the properties of the Bernoulli numbers are well known, perhaps congruences like (1.3)-(1.6) can give us information about the structure of the Stirling numbers. We note that applications of Glaisher's congruence are given in [2] and [6].

We also prove in this paper that, for $0<m<2 p-2$ and $p^{t} \| n$,

$$
\begin{align*}
& s(n+m, n) \equiv \frac{n}{m}\binom{n+m}{m}(-1)^{m} B_{m}^{(m)} \quad\left(\bmod p^{2 t}\right),  \tag{1.7}\\
& S(n, n-m) \equiv-\frac{n}{m}\binom{n-1}{m} B_{m}^{(m)}\left(\bmod p^{2 t}\right),
\end{align*}
$$

where $B_{m}^{(m)}$ is a Bernoulli number of higher order. The numbers $B_{m}^{(m)}$ are discussed in [13, pp. 150-51, 461] and a table of the first thirteen values is given.

We shall actually prove (1.3)-(1.8) in a more general setting. Since it is just as easy to do so, we shall prove congruences for the degenerate Stirling numbers $s(n, k \mid \lambda)$ and $S(n, k \mid \lambda)$ of Carlitz [3]. By letting $\lambda=0$, we obtain (1.3)-(1.8). We shall also show how to extend the range of $r$ to $0<2 r \leq$ ( $p-1) p^{t}$ and $1<2 p+1<(p-1) p^{t}$, although the congruences become more complicated.

A summary by sections follows. Section 2 is a preliminary section in which we give the definitions and basic properties of the special numbers we need and state a theorem of Carlitz that is necessary for most of the results of this paper. In Section 3 we prove congruences (1.3)-(1.6) in terms of degenerate Stirling and Bernoulli numbers. In Section 4 we prove (1.7) and (1.8) in a more general setting. In Section 5 we extend (1.3)-(1.6) by increasing the range of $r$.

## 2. Preliminaries

The primary tool of this paper is the following theorem of Carlitz [2], who used it to prove the Glaisher and Nielsen congruences, as well as congruences for other special numbers.
Theorem 2.1 (Carlitz) : Take

$$
f=f(x)=\sum_{m=1}^{\infty} c_{m} x^{m} / m!\quad\left(c_{1}=1\right)
$$

where the $c_{m}$ are rational numbers and, for $k \geq 1$, define

$$
\left(\frac{x}{f}\right)^{k}=\sum_{m=0}^{\infty} \alpha_{m}^{(k)} x^{m} / m!,
$$

with $\alpha_{m}^{(1)}=\alpha_{m}$. Define $\delta_{m}$ by means of

$$
\frac{x f^{\prime}}{f}=\sum_{m=0}^{\infty} \delta_{m} x^{m} / m!
$$

Then

$$
\begin{equation*}
m a_{m}^{(k)}=-k \sum_{r=1}^{m}\binom{m}{r} \delta_{r} a_{m-r}^{(k)} \tag{2.1}
\end{equation*}
$$

Next we define and give properties of the degenerate Stirling numbers, the degenerate Bernoulli numbers, and other special numbers that we need.

Carlitz [3] defined the degenerate Stirling numbers of the first and second kinds, $s(n, k \mid \lambda)$ and $S(n, k \mid \lambda)$, by means of

$$
\begin{align*}
& \left(\frac{1-(1-x)^{\lambda}}{\lambda}\right)^{k}=k!\sum_{n=k}^{\infty} s(n, k \mid \lambda) x^{n} / n!  \tag{2.2}\\
& \left((1+\lambda x)^{\mu}-1\right)^{k}=k!\sum_{n=k}^{\infty} S(n, k \mid \lambda) x^{n} / n! \tag{2.3}
\end{align*}
$$

where $\lambda \mu=1$. Comparing (2.2) and (2.3) with (1.1) and (1.2), we see that the limiting case $\lambda=0$ gives the ordinary Stirling numbers. Carlitz [3] also defined $\beta_{m}^{(k)}(\lambda, z)$ by means of

$$
\begin{equation*}
\left(\frac{x}{(1+\lambda x)^{u}-1}\right)^{k}(1+\lambda x)^{\mu z}=\sum_{m=0}^{\infty} \beta_{m}^{(k)}(\lambda, z) x^{m} / m! \tag{2.4}
\end{equation*}
$$

with the notation

$$
\begin{equation*}
\beta_{m}^{(k)}(\lambda)=\beta_{m}^{(k)}(\lambda, 0) . \tag{2.5}
\end{equation*}
$$

We use the notation $\beta_{m}^{(1)}(\lambda, z)=\beta_{m}(\lambda, z)$. Thus,
(2.6) $\beta_{m}(\lambda, 0)=\beta_{m}(\lambda)$,
the degenerate Bernoulli number [1], and $\beta_{m}(0,0)=B_{m}$, the ordinary Bernoulli number [5, p. 48]. It is known [3], that
(2.7) $s(k, k-m \mid \lambda)=(-1)^{m}\binom{k-1}{m} \beta_{m}^{(k)}(\lambda)$,
(2.8) $\quad S(k+m, k \mid \lambda)=\binom{k+m}{m} \beta_{m}^{(-k)}(\lambda)$.

The author [7] defined $\alpha_{m}^{(k)}(\lambda)$ by means of

$$
\begin{equation*}
\left(\frac{\lambda x}{1-(1-x)^{\lambda}}\right)^{k}=\sum_{m=0}^{\infty} \alpha_{m}^{(k)}(\lambda) x^{m} / m! \tag{2.9}
\end{equation*}
$$

and showed that
(2.10) $s(k+m, k \mid \lambda)=\binom{k+m}{k} \alpha_{m}^{(-k)}(\lambda)$,
(2.11) $S(k, k-m \mid \lambda)=(-1)^{m}\binom{k-1}{m} \alpha_{m}^{(k)}(\lambda)$.

We shall make use of the numbers $\beta_{m}(\lambda, 1-\lambda)$. It follows from (2.4) that, when $\lambda=0$, we have

$$
\beta_{m}(\lambda, 1-\lambda)=\beta_{m}(0,1)=\left\{\begin{aligned}
B_{m} & \text { when } m>1 \\
-B_{1} & \text { when } m=1
\end{aligned}\right.
$$

Also, from (2.4), we see that $\beta_{0}(\lambda, 1-\lambda)=1, \beta_{1}(\lambda, 1-\lambda)=(1-\lambda) / 2$, and, for $m>1$,
(2.12) $\quad \beta_{m}(\lambda, 1-\lambda)=\beta_{m}(\lambda)-m \lambda \beta_{m-1}(\lambda)$.

It follows from (2.12), by induction on $m$, that $\beta_{m}(\lambda, 1-\lambda)$ satisfies a degenerate Staudt-Clausen theorem in exactly the same way that $\beta_{m}(\lambda)$ does [1]. Thus, we can say that, if $p$ is a prime number and if $\lambda$ is rational, $\lambda=a / b$ with $b$ not divisible by $p$, then, for $r>0$,
(2.13) $p \beta_{2 r}(\lambda, 1-\lambda) \equiv\left\{\begin{aligned}-1(\bmod p) & \text { if }(p-1) \mid 2 r \text { and } p \mid \alpha, \\ 0(\bmod p) & \text { otherwise. }\end{aligned}\right.$
(2.14) $2 p \beta_{2 r+1}(\lambda, 1-\lambda) \equiv 0(\bmod p)$.

Note that, if $\lambda$ is integral $(\bmod p)$ and $\lambda \not \equiv 0(\bmod p)$, then $\beta_{m}(\lambda, 1-\lambda)$ is integral $(\bmod p)$. It follows that, if $\lambda$ is integral (mod $p$ ), then

$$
m \lambda \beta_{m-1}(\lambda, 1-\lambda) \equiv 0(\bmod m) .
$$

Now suppose $p$ is an odd prime and $m \equiv 0\left(\bmod p^{w}\right)$. It follows from (2.12) and properties of $\beta_{m}(\lambda)$ [1] that, if $m \not \equiv 0(\bmod p-1)$ and/or $a \not \equiv 0(\bmod p)$, then (2.15) $\beta_{m}(\lambda, 1-\lambda) \equiv 0\left(\bmod p^{w}\right)$.

## 3. Extensions of the Glaisher-Nielsen Results

In this section, and in Sections 4 and 5, we always assume that $p$ is an odd prime, $n$ is a positive integer, and $p^{t} \| n$. We also assume $\lambda$ is rational and integral $(\bmod p)$; that is, $\lambda=a / b$ with $b$ not divisible by $p$.

If we apply Theorem 2.1 with $f(x)=(1+\lambda x)^{\mu}-1$, and $\lambda \mu=1$, we see that

$$
a_{m}^{(k)}=\beta_{m}^{(k)}(\lambda), \quad \delta_{m}=\beta_{m}(\lambda, 1-\lambda) \quad(m \geq 1),
$$

where $\beta_{m}^{(k)}(\lambda)$ is defined by (2.4) and (2.5) and $\beta_{m}(\lambda, 1-\lambda)$ is defined by (2.4) with $\mathcal{K}=1$. Thus, (2.1) becomes

$$
\begin{equation*}
\beta_{m}^{(k)}(\lambda)=-\frac{k}{m} \sum_{r=1}^{m}\binom{m}{r} \beta_{r}(\lambda, 1-\lambda) \beta_{m-r}^{(k)}(\lambda), \text { with } \beta_{0}^{(k)}(\lambda)=1 \tag{3.1}
\end{equation*}
$$

Note that $\beta_{m}^{(n)}(\lambda)$ and $\beta_{m}^{(-n)}(\lambda)$ are integral $(\bmod p)$ for $m<(p-1) p^{t}$.
Theorem 3.1: For $m=1, \ldots, 2 p-3$,

$$
-\beta_{m}^{(-n)}(\lambda) \equiv \beta_{m}^{(n)}(\lambda) \equiv-\frac{n}{m} \beta_{m}(\lambda, 1-\lambda) \quad\left(\bmod p^{2 t}\right)
$$

If $\lambda \not \equiv 0(\bmod p)$, the congruence is valid for $m=1, \ldots,(p-1) p^{t}$.
Proof: From (3.1) and the properties (2.13)-(2.15) of $\beta_{r}(\lambda, 1-\lambda)$, we see that (3.2) $\beta_{m}^{(-n)}(\lambda) \equiv \beta_{m}^{(n)}(\lambda) \equiv 0\left(\bmod p^{t}\right)$, for $m=1,2, \ldots, 2 p-3, m \neq p-1$, if $\lambda \equiv 0(\bmod p)$. If $m=p-1$, then $\beta_{p-1}^{(-n)}(\lambda) \equiv \beta_{p-1}^{(n)}(\lambda) \equiv 0\left(\bmod p^{t-1}\right)$.
If $\lambda \not \equiv 0(\bmod p)$, congruence (3.2) holds for $m=1, \ldots,(p-1) p^{t}$. We note that $(p-1) \equiv 0(\bmod p)$ for $m=1,2, \ldots, 2 p-3, m \neq p-1$. Thus, letting $k=n$ or $k=-n$, we see that every term on the right side of (3.1), with the exception of of the $r=m$ term, is divisible by $p^{2 t}$. This completes the proof.

The following corollary is immediate from (2.7) and (2.8).
Corollary 3.1: For $m=1, \ldots, 2 p-3$,

$$
\begin{aligned}
& s(n, n-m \mid \lambda) \equiv(-1)^{m-1} \frac{n}{m}\binom{n-1}{m} \beta_{m}(\lambda, 1-\lambda) \quad\left(\bmod p^{2 t}\right) \\
& S(n+m, n \mid \lambda) \equiv \frac{n}{m}\binom{n+m}{m} \beta_{m}(\lambda, 1-\lambda) \quad\left(\bmod p^{2 t}\right)
\end{aligned}
$$

If $\lambda \not \equiv 0(\bmod p)$, the congruences are valid for $m=1,2, \ldots,(p-1) p^{t}$.
If $m$ is even and we let $\lambda=0$ in Corollary 3.1 , we obtain congruences (1.3) and (1.5). If $m$ is odd and $m>1$, we see from Theorem 3.1 that

$$
B_{m}^{(-n)} \equiv B_{m}^{(n)} \equiv 0 \quad\left(\bmod p^{2 t}\right)
$$

where $B_{m}^{(k)}$ is the Bernoulli number of order $k$, defined by (2.5) with $\lambda=0$. This is true because the Bernoulli number $B_{m}$ is 0 when $m$ is odd, $m>1$. Thus, when $\lambda=0$, each term on the right side of (3.1), with the exception of the $r=$ 1 and $r=m-1$ terms, is divisible by $p^{3 t}$. Hence,

Similarly,

$$
B_{2 r+1}^{(-n)} \equiv \frac{n^{2}(2 r+1)}{4 r} B_{2 r}\left(\bmod p^{3 t}\right)
$$

Thus, we can state the following corollary.
Corollary 3.2: The ordinary Stirling numbers satisfy congruences (1.3)-(1.6).
It is not difficult to extend the range of $m$ in Theorem 3.1. We do this in Section 5.

## 4. Further Congruences for the Stirling Numbers

In this section we prove congruences (1.7) and (1.8). Throughout the section, we still have the assumptions concerning $n, p, t$, and $\lambda$ stated at the beginning of Section 3 .

We first apply Theorem 2.1 to

$$
f(x)=\frac{1-(1-x)^{\lambda}}{\lambda} .
$$

We define $\alpha_{m}^{(k)}(\lambda)$ by (2.9), and we define $A_{m}(\lambda)$ by means of

$$
\begin{equation*}
\frac{x}{f} \cdot f^{\prime}=\frac{x \lambda}{\left[1-(1-x)^{\lambda}\right]\left[(1-x)^{1-\lambda}\right]}=\sum_{m=0}^{\infty} A_{m}(\lambda) x^{m} / m! \tag{4.1}
\end{equation*}
$$

Then we have, by Theorem 2.1,

$$
\begin{equation*}
\alpha_{m}^{(k)}(\lambda)=-\frac{k}{m} \sum_{r=1}^{m}\binom{m}{r} A_{r}(\lambda) \alpha_{m-r}^{(k)}(\lambda) \tag{4.2}
\end{equation*}
$$

When $\lambda=0$, (4.1) reduces to

$$
\frac{x}{(1-x) \ln (1-x)}=\sum_{m=0}^{\infty}(-1)^{m} B_{m}^{(m)} x^{m} / m!
$$

where $B_{m}^{(m)}$ is the Bernoulli number of higher order [10, pp. 150-51].
Lemma 4.1: If $A_{m}(\lambda)$ is defined by (4.1) and $\beta_{m}^{(k)}(\lambda)$ by (2.4) and (2.5), then

$$
A_{m}(\lambda)=(-1)^{m} \beta_{m}^{(m)}(\lambda) \quad(m=0,1,2, \ldots)
$$

Proof: We first note that $\beta_{m}^{(m)}(0)=B_{m}^{(m)}$.
Using (2.5) and (2.6), we can prove by induction on $z$ that, for all positive integers $z$,
(4.3) $\quad \beta_{m}^{(m)}(\lambda, z)=\sum_{j=0}^{m}\binom{m}{j}\binom{z}{j} j!\beta_{m-j}^{(m-j)}(\lambda)$.

Equation (4.3) is valid for all real $z$ since $B_{m}^{(m)}(\lambda, z)$ is a polynomial in $z$. We note that a more general result could be proved for numbers generated by $(x / f)^{k}(f+1)^{z}$, with $f$ defined by Theorem 2.1. From (2.4), we also have
(4.4) $\beta_{m}^{(m)}(\lambda, \lambda)-\beta_{m}^{(m)}(\lambda)=m \lambda \beta_{m-1}^{(m)}(\lambda)=m \lambda(\lambda-1) \cdots(\lambda-m+1)$.

Simplifying (4.3), with the aid of (4.4), we have, for $\lambda \neq 0$,
(4.5) $\sum_{j=0}^{m} \frac{(\lambda-1)(\lambda-2) \cdots(\lambda-m+j)}{(m-j+1)!j!} \beta_{j}^{(j)}(\lambda)=\frac{(\lambda-1)(\lambda-2) \cdots(\lambda-m)}{m!}$,
with $\beta_{0}^{(0)}(\lambda)=1$. By means of (4.1) we can show that $(-1)^{j} A_{j}(\lambda)$ satisfies the same recurrence with $A_{0}(\lambda)=1$. This completes the proof.

Thus, we can write, for all $\lambda$,

$$
\begin{equation*}
\alpha_{m}^{(k)}(\lambda)=\frac{-k}{m} \sum_{r=1}^{m}\binom{m}{r}(-1)^{r} \beta_{r}^{(r)}(\lambda) \alpha_{m-r}^{(k)}(\lambda) \tag{4.6}
\end{equation*}
$$

Before proving the main result of this section, we need to examine the properties of $\beta_{r}^{(r)}(\lambda)$. The first few values are given in the following table.

| $r$ | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| $\beta_{r}^{(r)}(\lambda)$ | 1 | $(\lambda-1) / 2$ | $(\lambda-1)(\lambda-5) / 6$ | $-3(\lambda-1)(\lambda-3) / 4$ |

For $\lambda=0$, the first thirteen values are given in [11, p. 461]. By the recurrence (4.5), we see that if $p$ is an odd prime, 1990]

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$$
\begin{aligned}
& p \beta_{m}^{(m)}(\lambda) \equiv 0(\bmod p) \quad(m=1, \ldots, p-2), \\
& p \beta_{p-1}^{(p-1)}(\lambda) \equiv \begin{cases}0(\bmod p) & \text { if } \lambda \not \equiv 0(\bmod p), \\
1(\bmod p) & \text { if } \lambda \equiv 0(\bmod p),\end{cases} \\
& \beta_{p}^{(p)}(\lambda) \equiv 0(\bmod p), \\
& p \beta_{p+k}^{(p+k)}(\lambda) \equiv 0(\bmod p) \quad(k=1, \ldots, p-3) \\
& p \beta_{2 p-2}^{(2 p-2)}(\lambda) \equiv\left\{\begin{array}{lll}
0(\bmod p) & \text { if } \lambda \neq 0(\bmod p) \\
-1(\bmod p) & \text { if } \lambda \equiv 0(\bmod p),
\end{array}\right. \\
& p \beta_{2 p-1}^{(2 p-1)}(\lambda) \equiv \begin{cases}0(\bmod p) & \text { if } \lambda \not \equiv 0(\bmod p) \\
1 / 2(\bmod p) & \text { if } \lambda \equiv 0(\bmod p),\end{cases} \\
& \beta_{2 p}^{(2 p)}(\lambda) \equiv 0(\bmod p) \quad(p>3) .
\end{aligned}
$$

Theorem 4.1: For $p>2$ and $m=1, \ldots, 2 p-3$,

$$
\alpha_{m}^{(n)}(\lambda) \equiv-\alpha_{m}^{(-n)}(\lambda) \equiv(-1)^{m+1} \frac{n}{m} \beta_{m}^{(m)}(\lambda) \quad\left(\bmod p^{2 t}\right) .
$$

Proof: From (4.6) and properties of $\beta_{r}^{(r)}(\lambda)$, we see that, for $1 \leq m \leq 2 p-3$, $m \neq p-1$,

$$
\alpha_{m}^{(-n)}(\lambda) \equiv \alpha_{m}^{(n)}(\lambda) \equiv 0 \quad\left(\bmod p^{t}\right) .
$$

Also

$$
\alpha_{p-1}^{(n)}(\lambda) \equiv 0\left(\bmod p^{t-1}\right) .
$$

Since $\left(p^{m}-1\right) \equiv 0(\bmod p)$ for $1 \leq m \leq 2 p-3, m \neq p-1$, we see that every term on the right side of (4.6) (when $k=n$ or $k=-n$ ) is divisible by $p^{2 t}$, except the $r=m$ term. This completes the proof.

The next corollary follows immediately from (2.10) and (2.11).
Corollary 4.1: For $m=1, \ldots, 2 p-3$,

$$
\begin{aligned}
& s(n+m, n \mid \lambda) \equiv \frac{n}{m}\binom{n+m}{n}(-1)^{m} \beta_{m}^{(m)}(\lambda)\left(\bmod p^{2 t}\right), \\
& S(n, n-m \mid \lambda) \equiv-\frac{n}{m}\binom{n-1}{m} \beta_{m}^{(m)}(\lambda)\left(\bmod p^{2 t}\right) .
\end{aligned}
$$

Corollary 4.2: The ordinary Stirling numbers satisfy congruences (1.7) and (1.8).

## 5. Extensions of Congruences (1.3)-(1.6)

Let $n, p, t$, and $\lambda$ be defined as in Section 3 . Suppose $m$ and $h$ are such that $2 p-2 \leq m$ and

$$
(p-1) p^{h-1}<m \leq(p-1) p^{h}<(p-1) p^{t} .
$$

Then we define $f(t, h)$ by

$$
f(t, h)=\left\{\begin{array}{l}
2 t-h \text { if } m \not \equiv 0(\bmod p-1), \\
2 t-1 \text { if } m \equiv 0(\bmod p-1), m \neq 0(\bmod p), h=1, \\
2 t-h-1 \text { if } m 0(\bmod p-1), m \not \equiv 0(\bmod p), h>1, \\
2 t-h-u-1 \text { if } m \equiv 0(\bmod p(p-1)), p^{u} \| m .
\end{array}\right.
$$

We now extend Theorem 3.1.
Theorem 5.1: Suppose $\lambda \equiv 0(\bmod p)$. With $m, h$, and $f(t, h)$ defined as above, we have

$$
-\beta_{m}^{(-n)}(\lambda) \equiv \beta_{m}^{(n)}(\lambda) \equiv-\frac{n}{m} \beta_{m}(\lambda, 1-\lambda) \quad\left(\bmod p^{f(t, h)}\right) .
$$

[Nov.

Proof: We first note that Theorem 5.1 implies that, if $m$ is restricted as in the statement of the theorem, then

$$
\beta_{m}^{(-n)}(\lambda) \equiv \beta_{m}^{(n)}(\lambda) \equiv 0\left\{\begin{array}{l}
\left(\bmod p^{t}\right) \text { if } m \not \equiv 0(\bmod p-1),  \tag{5.1}\\
\left(\bmod p^{t-1}\right) \text { if } m \equiv 0(\bmod p-1), m \not \equiv 0(\bmod p), \\
\left(\bmod p^{t-u-1}\right) \text { if } m \equiv 0(\bmod p(p-1)), p^{u \| m}
\end{array}\right.
$$

We first look at the case $m=2 p-2$. In (3.1), with $k=n$ or $k=-n$, and $m=$ $2 p-2$, all terms on the right side with $r<m$ are divisible by $p^{2 t}$ except the term $r=p-1$. We have

$$
\frac{n}{2 p-2}\binom{2 p-2}{p-1} \beta_{p-1}(\lambda, 1-\lambda) \beta_{p-1}^{( \pm n)}(\lambda) \equiv 0\left(\bmod p^{2 t-1}\right)
$$

So

$$
-\beta_{2 p-2}^{(-n)}(\lambda) \equiv \beta_{2 p-2}^{(n)}(\lambda) \equiv-\frac{n}{2 p-2} \beta_{2 p-2}(\lambda, 1-\lambda)\left(\bmod p^{2 t-1}\right)
$$

We now use induction on $m$. Assume Theorem 5.1 is true for all positive integers $r$ such that $2 p-2 \leq r<m$. In particular, assume congruences (5.1) hold with $m$ replaced by $r$. The problem is to show in (3.1) that, for $r=1$, .., $m-1$,
(5.2) $\frac{n}{m}\binom{m}{r} \beta_{r}(\lambda, 1-\lambda) \beta_{m-r}^{( \pm n)}(\lambda) \equiv 0\left(\bmod p^{f(t, h)}\right)$.

By using the induction hypothesis and the properties of $\beta_{r}(\lambda, 1-\lambda)$ discussed in Section 2, we can routinely show that (5.2) holds for all cases of $m$ given in the definition of $f(t, h)$.
Corollary 5.1: With the hypotheses of Theorem 5.1,

$$
\begin{aligned}
& s(n, n-m \mid \lambda) \equiv \frac{n}{m}(-1)^{m-1}\binom{n-1}{m} \beta_{m}(\lambda, 1-\lambda)\left(\bmod p^{f(t, h)}\right) \\
& S(n+m, n \mid \lambda) \equiv \frac{n}{m}\binom{n+m}{m} \beta_{m}(\lambda, 1-\lambda)\left(\bmod p^{f(t, h)}\right)
\end{aligned}
$$

Now let

$$
g(t, h)=\left\{\begin{array}{l}
3 t-h-w-1 \text { if } 2 r \equiv 0(\bmod p(p-1)), p^{w} \| 2 r \\
3 t-h-u-1 \text { if } p^{u} \|(2 r+1), u \geq 1 \\
3 t-h-1 \text { in all other cases. }
\end{array}\right.
$$

By letting $\lambda=0$ in Corollary 5.1, we can now prove the following extensions of (1.3)-(1.6).

Corollary 5.2: Let $2 r$ and $2 r+1$ be restricted as $m$ is restricted in Theorem 5.1. Then

$$
\begin{aligned}
& s(n, n-2 r) \equiv \frac{-n}{2 r}\binom{n-1}{2 r} B_{2 r}\left(\bmod p^{f(t, h)}\right) \\
& S(n+2 r, n) \equiv \frac{n}{2 r}\binom{n+2 r}{2 r} B_{2 r}\left(\bmod p^{f(t, h)}\right) \\
& s(n, n-2 r-1) \equiv \frac{-n^{2}(2 r+1)}{4 r}\binom{n-1}{2 r+1} B_{2 r}\left(\bmod p^{g(t, h)}\right) \\
& S(n+2 r+1, h) \equiv \frac{n^{2}(2 r+1)}{4 r}\binom{n+2 r+1}{2 r+1} B_{2 r}\left(\bmod p^{g(t, h)}\right) .
\end{aligned}
$$

If $m>(p-1) p^{t-1}$, the congruences become more complicated. However, using the same kind of reasoning as before, we can state the following result. We let $f(t, t)$ be defined as in Theorem 5.1 and define $y_{1}$ and $y_{2}$ by

$$
p^{y_{1}}\left\|\binom{n-1}{m}, \quad p^{y_{2}}\right\|\binom{n+m}{m} .
$$

Theorem 5.2: Let $(p-1) p^{t-1}<m \leq(p-1) p^{t}$ and $m \not \equiv 0\left(\bmod (p-1) p^{t-1}\right)$. Then

$$
\begin{aligned}
& s(n, n-m \mid \lambda) \equiv \frac{n}{m}(-1)^{m-1}\binom{n-1}{m} \beta_{m}(\lambda, 1-\lambda)\left(\bmod p^{y_{1}+f(t, t)}\right) \\
& S(n+m, n \mid \lambda) \equiv \frac{n}{m}\binom{n+m}{m} \beta_{m}(\lambda, 1-\lambda)\left(\bmod p^{y_{2}+f(t, t)}\right)
\end{aligned}
$$

By letting $\lambda=0$ in Theorem 5.2, we get the corresponding congruences for the ordinary Stirling numbers.

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# GENERALIZED FIBONACCI SEQUENCES VIA ARITHMETICAL FUNCTIONS 

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Haukkanen has pointed out in [3] the connection that exists between the specially multiplicative arithmetical functions (to be defined in Section 2) and the Fibonacci sequence. In this paper we shall discuss the similar connection that exists between certain arithmetical functions and the generalized sequences $\left\{w_{n}\right\}$, where $w_{n}=w_{n}(a, b ; p, q)$, i.e.,
$w_{0}=a, w_{1}=b$, and $w_{n}=p w_{n-1}-q w_{n-2}$, for $n \geq 2$,
which have been studied by Horadam [5], [6], [7], and others (see, for example, [10]). Here, $a, b, p$, and $q$ are arbitrary complex numbers, except that $a \neq 0$.

Our aim is to characterize the family of generalized sequences in terms of a family of arithmetical functions, and to illustrate how certain properties of the sequences reflect properties of the arithmetical functions. This work was done while the second author was a visiting Stouffer professor at the University of Kansas during the 1987-1988 academic year.

General background material on arithmetical functions can be found in most texts on number theory, and more specialized material is in the books by Apostol [1] and McCarthy [14]. We shall review here and in the following section several concepts which are used in this paper.

A (complex-valued) arithmetical function, $f$, is called multiplicative if $f(1)=1$ and $f(r s)=f(r) f(s)$ whenever $(r, s)=1$ : it is called completely multiplicative if $f(1)=1$ and $f(r s)=f(r) f(s)$ for all positive integers $r$ and $s$. If $f$ is an arithmetical function and $t$ is a prime, then the formal power series

$$
f_{(t)}(x)=f(1)+f(t) x+f\left(t^{2}\right) x^{2}+\cdots
$$

is called the Bell series of $f$ at $t$. Bell series are discussed on pages $42-45$ of [1], and in several exercises (1.97-1.102) of [14]. If $f$ is multiplicative, then $f$ is determined completely by its Bell series (at all primes $t$ ). If $f$ is completely multiplicative, its Bell series at $t$ is

$$
f_{(t)}(x)=1+f(t) x+f(t)^{2} x^{2}+\cdots=\frac{1}{1-f(t) x}
$$

We shall abuse the language and refer to the closed form of the Bell series as the Bell series itself. It is the relation between arithmetical functions and their Bell series that allows us to make the connection between arithmetical functions and generalized sequences.

$$
\text { 2. The Sequences }\left\{w_{n}\right\}
$$

An arithmetical function, $f$, is called specially multiplicative if there exist completely multiplicative functions $g_{1}$ and $g_{2}$ such that $f=g_{1} * g_{2}$, the Dirichlet convolution of $g_{1}$ and $g_{2}$, i.e.,
for all positive integers $r$, where $d$ runs over all of the positive divisors of $r$. Specially multiplicative functions arise naturally in several contexts in number theory. However, we emphasize that examples can be constructed in a completely arbitrary manner, as follows. For each prime $t$, let $\alpha_{t}$ and $\beta_{t}$ be complex numbers. Let $g_{1}$ and $g_{2}$ be the completely multiplicative functions such that, for each prime $t, g_{1}(t)=\alpha_{t}$ and $g_{2}(t)=\beta_{t}$. Let $f=g_{1} * g_{2}$. Then $f$ is specially multiplicative and, for each prime $t$ and $n \geq 1$,

$$
f\left(t^{n}\right)=\sum_{j=0}^{n} \alpha_{t}^{j} \beta_{t}^{n-j}
$$

Specially multiplicative functions were studied first by Vaidyanathaswamy [20] under the name "quadratic functions," and the name "specially multiplicative functions" was given to them by Lehmer [11]. These functions are discussed on pages $18-27$ and $65-68$ of [14] and in papers by Kesava Menon [8], McCarthy [12], [13], Mercier [15], Ramanathan [16], Rankin [17], Redmond \& Sivaramakrishnan [18], and Sivaramakrishnan [19].

If $f$ is specially multiplicative, the Bell series of $f$ at a prime $t$ is given by

$$
f_{(t)}(x)=\frac{1}{1-f(t) x+B(t) x^{2}}
$$

where $B$ is the completely multiplicative function for which $B(t)=g_{1}(t) g_{2}(t)$ for each $t$ : we note that $f(t)=g_{1}(t)+g_{2}(t)$. Furthermore, if $f$ is a multiplicative function such that, for each prime $t$, its Bell series at $t$ is given by

$$
f_{(t)}(x)=\frac{1}{1-c_{t} x+d_{t} x^{2}}
$$

for some complex numbers $c_{t}$ and $d_{t}$, then $f$ is specially multiplicative, as it was described earlier in this section, with $\alpha_{t}$ and $\beta_{t}$ the (possibly equal) roots of $X^{2}-c_{t} X+d_{t}$.

In [9], Lahiri defined an arithmetical function $f$ to be quasimultiplicative if $f(1) \neq 0$ and if there is a complex number $k \neq 0$ such that $f(r) f(s)=k f(r s)$ whenever $(r, s)=1$. It follows immediately that $k=f(1)$ and $k^{-1} f$ is multiplicative. In fact, an arithmetical function $f$ with $f(1) \neq 0$ is quasimultiplicative if and only if $f(1)^{-1} f$ is multiplicative.

Now we can make precise the connection between the generalized sequences $\left\{\omega_{n}\right\}$ and certain arithmetical functions.
Theorem 1: For a sequence of complex numbers $\left\{c_{n}\right\}, n \geq 0$, there exist complex numbers $a, b, p$, and $q$ such that $c_{n}=w_{n}(\alpha, b ; p, q)$ for all $n \geq 0$ if and only if there is a quasimultiplicative function $f$ and a prime $t$ such that
(i) $f(1)^{-1} f=g_{1} * \mu g_{2}$, where $g_{1}$ is specially multiplicative, $g_{2}$ is completely multiplicative, and $\mu$ is the Möbius function, and
(ii) $c_{n}=f\left(t^{n}\right)$ for all $n \geq 0$.

Proof: The generating function of the generalized sequence $\left\{w_{n}\right\}$, where $w_{n}=$ $\omega_{n}(a, b ; p, q)$ is, from [6],

$$
\sum_{n=0}^{\infty} w_{n} x^{n}=\frac{a+(b-p a) x}{1-p x+q x^{2}}
$$

Let $t$ be an arbitrary prime, and let $g_{l}$ be a specially multiplicative function such that

$$
g_{l(t)}(x)=\frac{1}{1-p x+q x^{2}},
$$

and let $g_{1}$ be a completely multiplicative function such that $g_{2}(t)=(p a-b) / a$,
so that

$$
g_{2(t)}(x)=\frac{a}{a+(b-p a) x}
$$

The inverse $g_{2}^{-1}$ of $g_{2}$ with respect to Dirichlet convolution is $\mu g_{2}$ (see Prop. 1.8 in [14]), and $g_{2(t)}^{-1}(x)=\left(g_{2(t)}(x)\right)^{-1}$ (see Th. 2.25 in [1]). Therefore, if $f$ is a quasimultiplicative function given by $f(r)=\alpha\left(g_{1} * \mu g_{2}\right)(r)$ for all $r$, when $w_{n}=f\left(t^{n}\right)$ for all $n \geq 0$.

Conversely, let $f$ be a quasimultiplicative function for which (i) holds, and suppose that, for some prime $t$,

$$
c_{n}=f\left(t^{n}\right) \text { for all } n \geq 0
$$

Then $c_{n}=w_{n}(a, b ; p, q)$ for all $n \geq 0$, where

$$
a=f(1), p=g_{1}(t), b=a\left(g_{1}(t)-g_{2}(t)\right), \text { and } q=h_{1}(t) h_{2}(t)
$$

where $h_{1}$ and $h_{2}$ are completely multiplicative functions such that $g_{1}=h_{1} * h_{2}$.

## 3. Some Examples

Horadam pointed out in [7] that several sequences of general interest are of the kind considered in Section 2.
(A) $\quad w_{n}=w_{n}(1,2 ; 2,1) .\left\{w_{n}\right\}$ is the sequence of positive integers. The quasimultiplicative function is $\tau$, where $\tau(r)$ is the number of divisors of $r$.
(B) $\quad w_{n}=w_{n}(1,3 ; 2,1) .\left\{w_{n}\right\}$ is the sequence of odd positive integers. The function is $\tau * \mu \lambda$, where $\lambda$ is Liouville's function (see [14], p. 45) .
(C) $w_{n}=w_{n}(a, \alpha+d ; 2,1) .\left\{w_{n}\right\}$ is the arithmetical progression

$$
a, a+d, a+2 d, \ldots
$$

The function is $a(\tau * \mu g)$, where $g$ is the completely multiplicative function with $g(t)=1-d / a$. Here, and in other examples, $t$ is an arbitrary prime.
(D) $w_{n}=w_{n}(\alpha, a q ; q+1, q) .\left\{w_{n}\right\}$ is the geometric progression

$$
a, a q, a q^{2}, \ldots
$$

The function is $a h$, where $h$ is the completely multiplicative function with $h(t)=q$.
(E) The Fermat sequences $\left\{u_{n}\right\}$ and $\left\{v_{n}\right\}$, where

$$
u_{n}=w_{n}(1,3 ; 3,2)=2^{n+1}-1 \text { and } v_{n}=w_{n}(2,3 ; 3,2)=2^{n}+1
$$

The functions are, respectively, $h_{1} * h_{2}$ and $2\left(h_{1} * h_{2} * \mu g_{2}\right)$, where $h_{1}, h_{2}$, and $g_{2}$ are completely multiplicative functions with

$$
h_{1}(t)=1, h_{2}(t)=2, \text { and } g_{2}(t)=3 / 2
$$

(F) The Pell sequences $\left\{u_{n}\right\}$ and $\left\{v_{n}\right\}$, where

$$
u_{n}=w_{n}(1,2 ; 2,-1) \text { and } v_{n}=w_{n}(2,2 ; 2,-1)
$$

The functions are, respectively, $h_{1} * h_{2}$ and $2\left(h_{1} * h_{2} * \mu g_{2}\right)$, where $h_{1}, h_{2}$, and $g_{2}$ are completely multiplicative functions with
$h_{1}(t)=1+\sqrt{2}, h_{2}(t)=1-\sqrt{2}$, and $g_{2}(t)=1$.

One more example. In [4], Horadam considered the sequence $\left\{w_{n}\right\}$, where $w_{n}=$ $\omega_{n}(r, r+s ; 1,-1)$. The function is $r\left(h_{1} * h_{2} * \mu g_{2}\right)$, where $h_{1}$, $h_{2}$, and $g_{2}$ are completely multiplicative functions with

$$
h_{1}(t)=(1+\sqrt{5}) / 2, \quad h_{2}(t)=(1-\sqrt{5}) / 2, \text { and } g_{2}(t)=s / r
$$

With $r=1$ and $s=0$ this is, of course, the Fibonacci sequence.
In several of the examples, $\alpha=1$ and $b=p$. Sequences for which this is true are of special interest, and they will be discussed in the following section. Thus, we shall consider sequences $\left\{u_{n}\right\}$, where

$$
u_{n}=u_{n}(p, q)=w_{n}(1, p ; p, q)
$$

These are the sequences for which the corresponding arithmetical functions are specially multiplicative.

## 4. The Sequences $\left\{u_{n}\right\}$

There exist various characterizations of specially multiplicative functions, and each of them furnishes us with a characterization of the class of sequences $\left\{u_{n}\right\}$. Thus, we have the following theorem; no proof will be given, and the reader is referred instead to Theorem 1.12 and Exercises 1.101 and 1.102 in [14].

Theorem 2: For a sequence of complex numbers $\left\{c_{n}\right\}, n \geq 0$, the following statements are equivalent:
(i) $c_{n}=u_{n}(p, q)$ for complex numbers $p$ and $q$, and all $n \geq 0$.
(ii) There is a specially multiplicative function $f$ and a prime $t$ such that $c_{n}=f\left(t^{n}\right)$ for all $n \geq 0$.
(iii) $c_{0}=1$ and there is a complex number $a$ such that, for all $m, n \geq 1$, $c_{m+n}=c_{m} c_{n}-a c_{m-1} c_{n-1}$.
(iv) $c_{0}=1$ and there is a complex number $b$ such that, for all $m, n \geq 0$ with $m \leq n$,

$$
c_{m} c_{n}=\sum_{i=0}^{m} c_{m+n-2 i} b^{i}
$$

(v) There are complex numbers $d$ and $e$ such that

$$
\sum_{n=0}^{\infty} c_{n} x^{n}=\frac{1}{1-d x+e x^{2}}
$$

From the details of the proof of this theorem it emerges that, if (i)-(v) hold for a sequence $\left\{c_{n}\right\}$, then $d=p=f(t)$ and $a=b=e=q=f(t)^{2}-f\left(t^{2}\right)$.

Results about specially multiplicative functions now yield results about the sequences $\left\{u_{n}\right\}$, and vice versa, of course. For example, by Theorem 2 of [18], if $u_{n}=u_{n}(p, q)$ for $n \geq 0$, then, for all $n \geq 1$,

$$
u_{n}=\sum_{j=0}^{[n / 2]}(-1)^{j}\left(n-j^{j}\right) p^{n-2 j} q^{j}
$$

This is an old result about these sequences. The original reference is given on page 394 of [2].

The identities of (iii) and (iv) are special cases of the same general identity. The latter is obtained from an arithmetical identity involving specially multiplicative functions given first in [13] (see also Ex. 1.79 in [14]). Let $u_{n}=u_{n}(p, q)$ for $n \geq 0$. If $g$ is an arbitrary arithmetical function and if $G=g * \mu$, and if $t$ is any prime, then for all $m, n \geq 0$ with $m \leq n$,

$$
\begin{equation*}
\sum_{i=0}^{m} G\left(t^{i}\right) q^{i} u_{m-i} u_{n-i}=\sum_{i=0}^{m} g\left(t^{i}\right) q^{i} u_{m+n-2 i} \tag{1}
\end{equation*}
$$

If $g=\zeta$, where $\zeta(r)=1$ for all $r$, then $G=\delta$, where $\delta(1)=1$ and $\delta(r)=0$ for all $r>1$, and (1) is the identity of (iv). If $g=\delta$, then $G=\mu$, and (1) is the identity of (iii). If $g=\tau=\zeta * \zeta$, then $G=\zeta$, and we obtain from (1) the identity

$$
\sum_{i=0}^{m} q^{i} u_{m-i} u_{n-i}=\sum_{i=0}^{m}(i+1) q^{i} u_{m+n-2 i}
$$

in particular, with $m=n$,

$$
\sum_{i=0}^{n} q^{n-i} u_{i}^{2}=\sum_{i=0}^{n}(n-i+1) q^{n-i} u_{2 i}
$$

Kesava Menon [8] associated with a multiplicative function $f$ another multiplicative function $f^{*}$, which he called the norm of $f$. The definition of $f^{*}$ can be found on page 50 of [14] and in several of the papers in our list of references. For our purposes, it suffices to note that if $g$ and $h$ are completely multiplicative functions, and if $f=g * h$, then $f^{*}$ is also specially multiplicative and, in fact, $f^{*}=g^{2} * h^{2}$. Thus, if the sequence $\left\{u_{n}\right\}$, where $u_{n}=u_{n}(p, q)$, is given by $u_{n}=f\left(t^{n}\right)$ for a prime $t$, then we can associate with $\left\{u_{n}\right\}$ the sequence $\left\{u_{n}^{*}\right\}$, where $u_{n}^{*}=f^{* *}\left(t^{n}\right)$. We have $u_{n}^{*}=u_{n}\left(p^{*}, q^{*}\right)$, where

$$
p^{*}=f^{*}(t)=g(t)^{2}+h(t)^{2}=p^{2}-2 q \text { and } q^{*}=g(t)^{2} h(t)^{2}=q^{2}
$$

Thus,

$$
u_{n}^{*}=u_{n}\left(p^{2}-2 q, q^{2}\right)
$$

From Theorems 4.1 and 4.2 of Sivaramakrishnan [19], which relate the functions $f$ and $f^{*}$, we obtain two identities relating the sequences $\left\{u_{n}\right\}$ and $\left\{u_{n}^{*}\right\}$ :

$$
\begin{align*}
& u_{n}^{2}=u_{n}^{*}+2 \sum_{i=1}^{n} q^{i} u_{n-i}^{*}  \tag{2}\\
& \sum_{i=0}^{n}(-1)^{i} u_{i}^{2} u_{n-i}^{2}=\sum_{i=0}^{n}(-1)^{i} u_{i}^{*} u_{n-i}^{*} \tag{3}
\end{align*}
$$

and

## 5. Generating Functions

We can obtain some information about the generating functions of the sequences $\left\{u_{n}\right\}$, and related sequences, from the Dirichlet series generating functions of corresponding arithmetical functions. In this section we assume that the reader is familiar with at least some of the material in Chapter 5 of [14] on Dirichlet series. Theorems about Dirichlet series generating functions involve hypotheses concerning the convergence of the series: we shall assume that whatever convergence is required does hold.

It will suffice to give several examples. Mercier ([15], Th. 3) gave the generating function of the product of two specially multiplicative functions, and we shall use the form of his result given on page 104 of his paper. From Mercier's result we obtain the following: if $u_{n}=u_{n}(p, q)$ and $u_{n}^{\prime}=u_{n}\left(p^{\prime}, q^{\prime}\right)$ for all $n \geq 0$, then

$$
\sum_{n=0}^{\infty} u_{n} u_{n}^{\prime} x^{n}=\frac{1-q q^{\prime} x^{2}}{1-p p^{\prime} x+\left[\left(p^{2}-q\right) q^{\prime}+\left(p^{\prime 2}-q^{\prime}\right) q\right] x^{2}-p p^{\prime} q q^{\prime} x^{3}+q^{2} q^{\prime 2} x^{4}}
$$

In particular,

$$
\sum_{n=0}^{\infty} u_{n} u_{n}^{*} x^{n}=\frac{1-q^{3} x^{2}}{1-p\left(p^{2}-2 q\right) x+\left(p^{4}-3 p^{2} q+2 q^{2}\right) q x^{2}-p\left(p^{2}-2 q\right) q^{3} x^{3}+q^{6} x^{4}}
$$

and
1990]

$$
\begin{equation*}
\sum_{n=0}^{\infty} u_{n}^{2} x^{n}=\frac{1-q^{2} x^{2}}{1-p^{2} x+2\left(p^{2}-q\right) q x^{2}-p^{2} q^{2} x^{3}+q^{4} x^{4}} \tag{4}
\end{equation*}
$$

The denominator on the right-hand side of (4) factors into the product of two quadratics, one of which is $(1-q x)^{2}$. Hence,

$$
\begin{equation*}
\sum_{n=0}^{\infty} u_{n}^{2} x^{n}=\frac{1+q x}{\left(1-\left(p^{2}-2 q\right) x+q^{2} x^{2}\right)(1-q x)} \tag{5}
\end{equation*}
$$

and we note that

$$
\begin{equation*}
\sum_{n=0}^{\infty} u_{n}^{*} x^{n}=\frac{1}{1-\left(p^{2}-2 q\right) x+q^{2} x^{2}} \tag{6}
\end{equation*}
$$

The generating function (6) can be obtained also from the Corollary to Theorem 7 of Redmond \& Sivaramakrishnan [18].

From (5) and (6), we have

$$
\begin{equation*}
\sum_{n=0}^{\infty} u_{n}^{2} x^{n}=\frac{1+q x}{1-q x} \sum_{n=0}^{\infty} u_{n}^{*} x^{n} \tag{7}
\end{equation*}
$$

Now,

$$
\frac{1+q x}{1-q x}=1+\sum_{n=1}^{\infty} 2 q^{n} x^{n} .
$$

Thus, if we multiply out the right-hand side of (7) and compare coefficients of $x^{n}$, we obtain (2). If we replace $x$ by $-x$ in (7) and multiply the left- and right-hand sides of the resulting equation by the left- and right-hand sides, respectively, of (7), and then compare coefficients of $x^{n}$, we obtain (3).

From the Corollary to Theorem 8 of Redmond \& Sivaramakrishnan [18], we have

$$
\sum_{n=0}^{\infty} u_{2 n} x^{n}=(1+q x) \sum_{n=0}^{\infty} u_{n}^{\star} x^{n}
$$

Combining this with (7) gives

$$
\sum_{n=0}^{\infty} u_{n}^{2} x^{n}=\left(\sum_{n=0}^{\infty} q^{n} x^{n}\right)\left(\sum_{n=0}^{\infty} u_{2 n} x^{n}\right)
$$

and if we multiply this out and compare coefficients of $x^{n}$, we obtain

$$
u_{n}^{2}=\sum_{i=0}^{n} q^{n-i} u_{2 i} .
$$

From Theorem 9 of the same paper, we see that, for a fixed $m \geq 1$,

$$
\begin{equation*}
\sum_{n=0}^{\infty} u_{m+n} x^{n}=\left(u_{m}-q u_{m-1} x\right) \sum_{n=0}^{\infty} u_{n} x^{n} . \tag{8}
\end{equation*}
$$

If we multiply out the right-hand side and compare coefficients of $x^{n}$, we find that (8) is simply the expression in series form of the identity in (iii) of Theorem 2.

## 6. Linear Combinations of Sequences

In this section we shall obtain a result suggested by a theorem of Rankin on specially multiplicative functions ([17], Th. 5). We shall say that sequences $\left\{\alpha_{n}^{(j)}\right\}, j=1, \ldots, k$ are linearly independent if the only complex numbers $c_{1}, \ldots, c_{k}$ for which
$c_{1} \alpha_{n}^{(1)}+\cdots+c_{k} \alpha_{n}^{(k)}=0$, for all $n \geq 0$,
for all $n \geq 0$ are $c_{1}=\cdots=c_{k}=0$.

Theorem 3: If $p_{1}, \ldots, p_{k}$ are distinct complex numbers and if $u_{n}^{(j)}=u_{n}\left(p_{j}, q\right)$, for $j=1, \ldots, k$, then the sequences $\left\{u_{n}^{(j)}\right\}$ are linearly independent.
Proof: Suppose that $c_{1} u_{n}^{(1)}+\cdots+c_{k} u_{n}^{(k)}=0$, for $n \geq 0$. Then the first $k$ of these equations form a system of $k$ linear equations with $c_{1}, \ldots, c_{k}$ as the unknowns, and the matrix of coefficients is $\left[\mathcal{L}_{i}^{(j)}\right]$, where $i=0,1, \ldots, k-1$ and $j=1, \ldots, k$. Its first row is $1,1, \ldots, 1$ and its second row is $p_{1}, p_{2}$, $\ldots, p_{k}$. Furthermore, as we have noted in Section 4,

$$
u_{i}^{(j)}=\sum_{r=0}^{[i / 2]}(-1)^{r}\binom{i-r}{p} E_{j}^{i-2 r} q^{r}
$$

Thus, if $i \geq 2$, then by adding appropriate multiples of rows $i-2[i / 2], \ldots$, $i-2$ to row $i$, the matrix can be transformed into one having $p_{1}^{i}, \ldots, p_{k}^{i}$ for its $i$ th row. The determinant of the matrix of coefficients is unchanged by this transformation. Thus,

$$
\operatorname{det}\left[u_{i}^{(j)}\right]=\left|\begin{array}{llll}
1 & 1 & \cdots & 1 \\
p_{1} & p_{2} & \cdots & p_{k} \\
\vdots & \vdots & & \vdots \\
p_{1}^{k-1} & p_{2}^{k-1} & & p_{k}^{k-1}
\end{array}\right|=\prod_{1 \leq i<j \leq k}\left(p_{j}-p_{i}\right) \neq 0
$$

Therefore, $c_{1}=\cdots=c_{k}=0$.
Theorem 4: Let $p_{1}, \ldots, p_{k}$ be distinct complex numbers and let $u_{n}^{(j)}=u_{n}\left(p_{j}, q\right)$ for $j=1, \ldots, k$. If, for complex numbers $c_{1}, \ldots, c_{k}$, we have

$$
u_{n}=c_{1} u_{n}^{(1)}+\cdots+c_{k} u_{n}^{(k)}=u_{n}(p, q),
$$

for some $p$ and for all $n \geq 0$, then for some $h$ with $1 \leq h \leq k$, we have $c_{h}=1$, $c_{j}=0$, for $j \neq h$ and $p=p_{h}$.
Proof: We shall use identity (iv) of Theorem 2. We have, for all $m, n \geq 0$ with $m \leq n$,

$$
\begin{aligned}
u_{m} u_{n}=\sum_{i=0}^{m} u_{m+n-2 i} q^{i} & =\sum_{i=0}^{m} q^{i} \sum_{j=1}^{k} c_{j} u_{n i+n-2 i}^{(j)} \\
& =\sum_{j=1}^{k} c_{j} \sum_{i=0}^{m!} u_{n+n+2 i}^{(j)} q^{i}=\sum_{j=1}^{k} c_{j} u_{m,}^{(j)} u_{n}^{(j)} .
\end{aligned}
$$

Also,

$$
u_{m} u_{n}=\sum_{j=1}^{k} c_{j} u_{n}^{(j)} u_{n}=\sum_{j=1}^{k} \sigma_{j} u_{n} u_{n}^{(j)} .
$$

Thus, if $m \leq n$,
and alsó

$$
\begin{equation*}
\sum_{i=1}^{k} c_{i}^{k}\left(u_{n}-u_{n}^{(j)}\right) u_{m_{i}}^{(j)}=0 \tag{9}
\end{equation*}
$$

$$
\sum_{j=1}^{k} c_{j}\left(v_{m}-u_{m}^{(j)}\right) u_{n}^{(j)}=0 .
$$

Therefore, (9) holds for all $m, n \geq 0$, without regard to the relative sizes of $m$ and $n$. Hence, for each (fixed) $n$,

$$
c_{j}\left(u_{n}-u_{n}^{(j)}\right)=0, \text { for } j=1, \ldots, k
$$

Since $u_{0}=1$, we must have $c_{j} \neq 0$ for some $j$. Suppose $c_{n} \neq 0$; then $u_{n}=u_{n}^{(h)}$ for all $n \geq 0$, and

$$
\sum_{\substack{j=1 \\ j \neq h}} c_{j} u_{n}^{(j)}+\left(c_{h}-1\right) u_{n}^{(h)}=0, \text { for all } n \geq 0
$$

Thus, $c_{h}=1$ and $c_{j}=0$, for $j \neq h$. Further,

$$
p_{h}=u_{1}^{(h)}=u_{1}=p
$$

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# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by

## A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

## BASIC FORMULAS

The Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ satisfy
$F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1 ;$
$L_{n+2}=L_{n+1}+L_{n}, L_{0}=2, L_{1}=1$.
Also, $\alpha=(1+\sqrt{5}) / 2, \beta=(1-\sqrt{5}) / 2, F_{n}=\left(\alpha^{n}-\beta^{n}\right) / \sqrt{5}$, and $L_{n}=\alpha^{n}+\beta^{n}$.

PROBLEMS PROPOSED IN THIS ISSUE
B-676 Proposed by Herta T. Freitag, Roanoke, VA
Let $T_{n}$ be the $n^{\text {th }}$ triangular number $n(n+1) / 2$. Characterize the positive integers $n$ such that

$$
T_{n} \mid \sum_{i=1}^{n} T_{i}
$$

B-677 Proposed by Herta T. Freitag, Roanoke, VA
Let $T_{n}=n(n+1) / 2$. Characterize the positive integers $n$ with

$$
\sum_{i=1}^{n} T_{i} \mid \sum_{i=1}^{n} T_{i}^{2}
$$

B-678 Proposed by $R$. André-Jeannin, Sfax, Tunisia
Show that $L_{4 n}$ and $L_{4 n+3}$ are never triangular numbers.
B-679 Proposed by $R$. André-Jeannin, Sfax, Tunisia
Express $L_{n-2} L_{n-1} L_{n+1} L_{n+2}$ as a polynomial in $L_{n}$.

B-680 Proposed by Russell Jay Hendel \& Sandra A. Monteferrante, Dowling College, Oakdale, NY

For an integer $\alpha \geq 0$, define a sequence $x_{0}, x_{1}, \ldots$ by $x_{0}=0, x_{1}=1$, and $x_{n+2}=\alpha x_{n+1}+x_{n}$ for $n \geq 0$. Let $d=\left(\alpha^{2}+4\right)^{1 / 2}$. For $n \geq 2$, what is the nearest integer to $d x_{n}$ ?

B-681 Proposed by H.-J. Seiffert, Berlin, Germany
Let $n$ be a nonnegative integer, $k \geq 2$ an even integer, and $r \in\{0,1, \ldots, k-1\}$. Show that

$$
F_{k n+r^{\prime}} \equiv\left(F_{k+r}-F_{r}\right) n+F_{r} \quad\left(\bmod L_{k}-2\right)
$$

## SOLUTIONS

## Golden Geometric Progressions

B-652 Proposed by Herta T. Freitag, Roanoke, VA
Let $\alpha=(1+\sqrt{5}) / 2$,

$$
S_{1}(n)=\sum_{k=1}^{n} \alpha^{k} \quad \text { and } \quad S_{2}(n)=\sum_{k=1}^{n} \alpha^{-k} .
$$

Determine $m$ as a function of $n$ such that $\frac{S_{1}(n)}{S_{2}(n)}-\alpha F_{m}$ is a Fibonacci number.
Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY
Both $S_{1}(n)$ and $S_{2}(n)$ are geometric series, whose sums are

$$
S_{1}(n)=\frac{\alpha\left(\alpha^{n}-1\right)}{\alpha-1} \quad \text { and } \quad S_{2}(n)=\frac{1}{\alpha^{n}} \cdot \frac{\alpha^{n}-1}{\alpha-1} .
$$

respectively. Hence, if $\beta$ denotes $(1-\sqrt{5}) / 2$, then

$$
\frac{S_{1}(n)}{S_{2}(n)}-\alpha F_{m}=\alpha\left(\alpha^{n}-F_{m}\right)=\frac{\alpha^{2}\left(\alpha^{n}-\alpha^{m-1}\right)+\left(\alpha^{n}-\beta^{m-1}\right)}{\alpha-\beta}=F_{n}
$$

when $m=n+1$.
Also solved by R. André-Jeannin, Paul S. Bruckman, L. Cseh, Piero Filipponi, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

## Pythagorean Triples

B-653 Proposed by Herta T. Freitag, Roanoke, VA
The sides of a triangle are $a=F_{2 n+3}, b=F_{n+3} F_{n}$, and $c=F_{3} F_{n+2} F_{n+1}$, with $n$ a positive integer.
(i) Is the triangle acute, right, or obtuse?
(ii) Express the area as a product of Fibonacci numbers.

Solution by Paul S. Bruckman, Edmonds, WA
Note that
$b=\left(F_{n+2}+F_{n+1}\right)\left(F_{n+2}-F_{n+1}\right)=F_{n+2}^{2}-F_{n+1}^{2} ;$
$c=2 F_{n+2} F_{n+1}$;
and $\quad a=F_{n+2}^{2}+F_{n+1}^{2}$.
We readily see that the given triangle is a Pythagorean (right) triangle, and that it satisfies: $a^{2}=b^{2}+c^{2}$, i.e., $a$ is the hypotenuse.

If $A$ is its area, then

$$
A=\frac{1}{2} b c=F_{n} F_{n+1} F_{n+2} F_{n+3} .
$$

Also solved by L. Cseh, Piero Filipponi, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

Infinite Series
B-654 Proposed by Alejandro Necochea, Pan American U., Edinburgh, TX
Sum the infinite series

$$
\sum_{k=1}^{\infty} \frac{1+2^{k}}{2^{2 k}} F_{k}
$$

Solution by Wray Brady, Axixic Jalisco, Mexico
$f(x)=x /\left(1-x-x^{2}\right)$ is the generating function for the series

$$
\sum_{k=1}^{\infty} F_{k} x^{k}
$$

which series converges if $|x|<1 / a$. Thus, the sum of the series is

$$
f(1 / 2)+f(1 / 4)=26 / 11
$$

Also solved by R. André-Jeannin, Paul S. Bruckman, L. Cseh, Russell Euler, Piero Filipponi, Herta T. Freitag, Russell Jay Hendel, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, B. S. Popov, H.-J. Seiffert, Sahib Singh, and the proposer.

## Farey Fractions

B-655 Proposed by L. Kuipers, Sierre, Switzerland
Prove that the ratio of integers $x / y$ such that

$$
\frac{F_{2 n}}{F_{2 n+2}}<\frac{x}{y}<\frac{F_{2 n+1}}{F_{2 n+3}}
$$

and with smallest denominator $y$ is $\left(F_{2 n}+F_{2 n+1}\right) /\left(F_{2 n+2}+F_{2 n+3}\right)$.

Solution by Sahib Singh, Clarion University, Clarion, PA

Since

$$
\frac{F_{2 n+1}}{F_{2 n+3}}-\frac{F_{2 n}}{F_{2 n+2}}=\frac{1}{F_{2 n+2 F_{2 n+3}}}
$$

therefore

$$
\frac{F_{2 n}}{F_{2 n+2}} \quad \text { and } \quad \frac{F_{2 n+1}}{F_{2 n+3}}
$$

can be regarded as adjacent fractions of the Farey sequence of order $F_{2 n+3}$ (see Question 5 on page 173 of An Introduction to the Theory of Numbers by Ivan Niven and H. S. Zuckerman, $4^{\text {th }}$ ed. [New York: Wiley \& Sons, 1980]). Hence, by Theorem 6.4 (Ibid., page 171), the desired conclusion follows.

Also solved by R. André-Jeannin, Paul S. Bruckman, B. S. Popov, and the proposer.

## Closed Form

B-656 Proposed by Richard André-Jeannin, Sfax, Tunisia
Find a closed form for the sum

$$
S_{n}=\sum_{k=0}^{n} w_{k} p^{n-k}
$$

where $w_{n}$ satisfies $w_{n}=p w_{n-1}-q w_{n-2}$ for $n$ in $\{2,3, \ldots\}$, with $p$ and $q$ nonzero constants.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY
It is a routine exercise to show that

$$
w_{k}=\frac{\alpha \alpha^{k}+b \beta^{k}}{\alpha-\beta}
$$

where

$$
\begin{aligned}
& \alpha=\left(p+\sqrt{p^{2}-4 q}\right) / 2, \quad \beta=\left(p-\sqrt{p^{2}-4 q}\right) / 2, \\
& \alpha=w_{1}-\beta w_{0}, \text { and } b=\alpha w_{0}-w_{1} .
\end{aligned}
$$

The formula for $w_{k}$ leads to

$$
S_{n}=\sum_{k=0}^{n} \omega_{k} p^{n-k}=\frac{\alpha}{\alpha-\beta} \cdot \frac{p^{n+1}-\alpha^{n+1}}{p-\alpha}+\frac{b}{\alpha-\beta} \cdot \frac{p^{n+1}-\beta^{n+1}}{p-\beta} .
$$

Since $\alpha+\beta=p$ and $\alpha \beta=q$, we have

$$
\begin{aligned}
S_{n} & =\frac{\alpha \alpha\left(p^{n+1}-\alpha^{n+1}\right)+b \beta\left(p^{n+1}-\beta^{n+1}\right)}{(\alpha-\beta) \alpha \beta} \\
& =\frac{p^{n+1}(\alpha \alpha+b \beta)-\left(\alpha \alpha^{n+2}+b \beta^{n+2}\right)}{q(\alpha-\beta)}=\frac{p^{n+1} w_{1}-w_{n+2}}{q} .
\end{aligned}
$$

Also solved by Paul S. Bruckman, L. Cseh, Russell Euler, Piero Filipponi, L. Kuipers, B. S. Popov, H.-J. Seiffert, and the proposer.

## Disjoint Increasing Sequences

B-657 Proposed by Clark Kimberling, U. of Evansville, Evansville, IN
Let $m$ be an integer and $m \geq 3$. Prove that no two of the integers

$$
k\left(m F_{n}+F_{n-1}\right) \text { for } k=1,2, \ldots, m-1 \text { and } n=0,1,2, \ldots
$$

are equal. Here $F_{-1}=1$.
Composite of solutions by Paul S. Bruckman, Edmonds, WA, and Philip L. Mana, Albuquerque, $N M$

```
Assume that m\geq 3;u,v\inN={0, 1, ...};
(1) j, k f {1, 2, ..., m-1};
    j(mF
```

We wish to show that $j=k$ and $u=v$. It is easily seen that $m F_{n}+F_{n-1}>0$, for $n \geq 0$.
Therefore, if $u=v$, then $j=k$ as desired. Now there is no loss of generality in assuming that $0 \leq u<v$.

If $u=0$, then $v>0$ and (2) gives
$j=k\left(m F_{v}+F_{v-1}\right) \geq m$,
which contradicts (1). If $u=1$, then $v>1$, and (2) gives

$$
m j=k\left(m F_{v}+F_{v-1}\right) .
$$

Thus, $m\left(j-k F_{v}\right)=k F_{v-1} \geq 1$. So
$j-k F_{v} \geq 1$ and $j>k F_{v} \geq k F_{v-1}=m\left(j-k F_{v}\right) \geq m$,
again contradicting (1).
Now we can assume that $2 \leq u<v$. Also, we assume that $\operatorname{gcd}(j, k)=1$ since this is the situation when $j$ and $k$ are divided by $\operatorname{gcd}(j, k)$ in (1). Then (2) shows that

$$
j \mid\left(m F_{v}+F_{v-1}\right)
$$

and we let $m F_{v}+F_{v-1}=c j$. This leads to $m F_{u}+F_{u-1}=c k$ and

$$
\begin{aligned}
c\left(k F_{v}-j F_{u}\right) & =\left(m F_{u}+F_{u-1}\right) F_{v}-\left(m F_{v}+F_{v-1}\right) F_{u} \\
& =F_{u-1} F_{v}-F_{v-1} F_{u}=(-1)^{u} F_{v-u} .
\end{aligned}
$$

Hence, $c \mid F_{v-u}$, and we let $d=F_{v-u} / c$. Now $k F_{v}-j F_{u}=(-1)^{u} d$; therefore,

$$
\begin{equation*}
j F_{u}=k F_{v}-(-1)^{u} d=\left[\left(m F_{u}+F_{u-1}\right) / c\right] F_{v}-(-1)^{u} d . \tag{3}
\end{equation*}
$$

Since $v \geq 3$ and $v-u<v$, we have $F_{v}>F_{v-u}=c d$. Hence, $u \geq 2$, and (3) gives $j F_{u}>\left(m F_{u}+F_{u-1}\right) d-d \geq\left(m F_{u}+1\right) d-d=m d F_{u}$.
Thus, $j>m d \geq m$. This contradiction and the previous work show that $u=v$ and $j=k$.

Also solved by Piero Filipponi and the proposer.

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by
Raymond E. Whitney
Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

## PROBLEMS PROPOSED IN THIS ISSUE

H-446 Proposed by J. A. Sjogren, University of Santa Clara, Santa Clara, CA
Establish the following result:
Let $n$ be a whole number and, for any rational number $q$, let $[q$ ] be the greatest integer contained in $q$. Then

$$
f_{n}=\left[\frac{n-1}{2}\right] \prod_{k=1}^{2}\left(3+2 \cos \frac{2 \pi k}{n}\right)
$$

Here, an empty product is to be interpreted as unity.
H-447 Proposed by Albert A. Mullin, Huntsville, AL
Determine the minimal number of one-ohm resistors necessary to realize a two-terminal circuit to within $10^{-6}$ ohms of $e$ ohms of resistance. The twoterminal circuit is permitted to be non-series-parallel; i.e., we allow bridgetype sub-circuits, among others. (2) How is this minimal number of unit resistors increased if only series-parallel sub-circuits are permitted? Of course, $e$ is the usual transcendental number.

H-448 Proposed by T. V. Padmakumar, Trivandrum, India
If $n$ is any number and $a_{1}, a_{2}, \ldots, a_{m}$ are prime to $n$ ( $n>a_{1}, \alpha_{2}, \ldots, a_{m}$ ), then $\left(\alpha_{1} a_{2} \ldots a_{m}\right)^{2} \equiv 1(\bmod n)$. [The number of positive integers less than $n$ and prime to it is denoted by $\phi(n)=m$.]

## SOLUTIONS

## Sum Integrals

H-425 Proposed by Stanley Rabinowitz, Littleton, MA (Vol. 26, no. 4, November 1988)

Let $F_{n}(x)$ be the $n^{\text {th }}$ Fibonacci polynomial $\left[F_{1}(x)=1, \quad F_{2}(x)=x, F_{n+2}(x)=\right.$ $\left.x F_{n+1}(x)+F_{n}(x).\right]$
Evaluate: (a) $\int_{0}^{1} F_{n}(x) d x$
(b) $\int_{0}^{1} F_{n}^{2}(x) d x$

Solution by Paul S. Bruckman, Edmonds, WA
It may readily be shown that

$$
\begin{equation*}
F_{n}(x)=\frac{a^{n}-b^{n}}{a-b}, n=1,2,3, \ldots, \tag{1}
\end{equation*}
$$

where

$$
\begin{equation*}
a \equiv \alpha(x) \equiv \frac{1}{2}\left(x+\sqrt{x^{2}+4}\right), \quad b \equiv b(x) \equiv \frac{1}{2}\left(x-\sqrt{x^{2}+4}\right) . \tag{2}
\end{equation*}
$$

Note
(3)

$$
a+b=x, \quad a-b=\left(x^{2}+4\right)^{\frac{1}{2}}, \quad a b=-1
$$

We may also define the Lucas polynomials as follows:

$$
\begin{equation*}
L_{1}(x)=x, \quad L_{2}(x)=x^{2}+2, \quad L_{n+2}(x)=x L_{n+1}(x)+L_{n}(x) \tag{4}
\end{equation*}
$$

Therefore, we find that

$$
\begin{equation*}
L_{n}(x)=a^{n}+b^{n}, n=1,2,3, \ldots \tag{5}
\end{equation*}
$$

It is easy to differentiate $a$ and $b$ (with respect to $x$ ), and we find:
(6) $\quad a^{\prime}(x)=a /(a-b), \quad b^{\prime}(x)=-b /(a-b)$.

From (6), it follows that

$$
\begin{equation*}
L_{n}^{\prime}(x)=n F_{n}(x) . \tag{7}
\end{equation*}
$$

This implies the indefinite integral:

$$
\begin{equation*}
\int F_{n}(x) d x=\frac{1}{n} L_{n}(x) \tag{8}
\end{equation*}
$$

Now $L_{n}(1)=L_{n}$, the standard Lucas numbers, while $L_{n}(0)=1+(-1)^{n}=2 e_{n}$, where $e_{n}$ is the characteristic function of the even integers. Therefore, we obtain the solution to part (a):

$$
\begin{equation*}
\int_{0}^{1} F_{n}(x) d x=\frac{1}{n}\left(L_{n}-2 e_{n}\right) . \tag{9}
\end{equation*}
$$

Solution to part (b):
Consider the expression $S_{n}(x)$ defined as follows:

Differentiating $S_{n}$ term by term, we obtain [using (7) and (1)]:

$$
\begin{aligned}
S_{n}^{\prime}(x) & =\sum_{k=0}^{n-1}(-1)^{n-1-k} F_{2 k+1}(x)=\sum_{k=0}^{n-1}(-1)^{n-1-k}\left[\frac{a^{2 k+1}-b^{2 k+1}}{a-b}\right] \\
& =\frac{a}{(a-b)}\left(a^{2}+1\right)^{-1}\left(a^{2 n}-(-1)^{n}\right)-\frac{b}{(a-b)}\left(b^{2}+1\right)^{-1}\left(b^{2 n}-(-1)^{n}\right) \\
& =(a-b)^{-2}\left(a^{2 n}-(-1)^{n}+b^{2 n}-(-1)^{n}\right) \\
& =\left(a^{n}-b^{n}\right)^{2}(a-b)^{-2},
\end{aligned}
$$

or
(11) $\quad S_{n}^{\prime}(x)=F_{n}^{2}(x)$.

It follows from (11) that

$$
\begin{equation*}
\int_{0}^{1} F_{n}^{2}(x) d x=S_{n}(1)-S_{n}(0) \tag{12}
\end{equation*}
$$

Since $L_{2 k+1}(0)=2 e_{2 k+1}=0$, we see that $S_{n}(0)=0$. Hence,

$$
\begin{equation*}
\int_{0}^{1} F_{n}^{2}(x) d x=\sum_{k=0}^{n-1}(-1)^{n-1-k} \frac{L_{2 k+1}}{2 k+1} \tag{13}
\end{equation*}
$$

It does not appear possible to simplify the foregoing expression further, into some kind of closed form.

Also solved by O. Brugia \& P. Filipponi, R. Euler, C. Georghiou, R. AndreJeannin, L. Kuipers, H.-J. Seiffert, J. Shallit, and D. Zeitlin. Georghiou mentioned that part (a) is identical to $\mathrm{H}-410$.

## Another Identity

H-426 Proposed by Larry Taylor, Rego Park, NY (Vol. 26, no. 4, November 1988)

Let $j, k, m$, and $n$ be integers. Prove that

$$
\left(F_{n} F_{m+k-j}-F_{m} F_{n+k-j}\right)(-1)^{m}=\left(F_{k} F_{j+n-m}-F_{j} F_{k+n-m}\right)(-1)^{j}
$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI
Our solution will use the following known results:

$$
\begin{equation*}
5 F_{t} F_{s}=5 F_{s} F_{t}=L_{s+t}-(-1)^{t} L_{s-t} \tag{1}
\end{equation*}
$$

and
(2) $(-1)^{t} L_{t}=L_{-t}$.
[For (1), see (10) and (12) on page 115 of the April 1975 issue of this journal; for (2), see exercises 3 and 9 on page 29 of Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. (Boston: Houghton Mifflin, 1969.]

We now use the above information to produce the following collection of equations, each of which is equivalent to the desired result.

$$
\begin{aligned}
& 5\left(F_{n k} F_{m+k-j}-F_{m} F_{n+k-j}\right)(-1)^{m} \\
& =5\left(F_{k} F_{i+n-m}-F_{i} F_{k+n-m}\right)(-1)^{\dot{j}}+\left[L_{n+m+k-i}-(-1)^{n} L_{m+k-i-n}-L_{m+n+k-i}^{i}\right. \\
& \left.+(-1)^{m} L_{n+k-j-m}\right](-1)^{m} \\
& =\left[L_{n+k+j-m}-(-1)^{k} L_{j+n-m-k}-L_{n+i}+k-m+(-1)^{\dot{i}} L_{k+n-m-j}\right](-1)^{j} \\
& -(-1)^{m_{i}+n_{i}} L_{m+k-j-n}+L_{n}+k-j-m_{i} \\
& =-(-1)^{\dot{j}+\dot{j} L_{j}+n-m_{i}-k+L_{k}+n-m-j+(-1)^{m+n-k-j} L_{m+k-j-n}, ~} \\
& =L_{-\left(m+k-j-r_{n}\right)}+(-1)^{2 k}(-1)^{-2 r}(-1)^{m+n-k-i} L_{m}+k-j-r \\
& =L_{-(m+k-j-n)}+(-1)^{m+j-i-n} L_{m+i-i-n} \\
& =L_{-\left(n_{i}+k-j-n\right)}
\end{aligned}
$$

Since the last equality holds by (2), our solution is complete.

Also solved by P. Bruckman, P. Filipponi, C. Georghiou, R. Hendel, R. Andre-Jeannin, L. Kuipers, H.-J. Seiffert, S. Singh, and the proposer.

## A Recurrent Composition

H-427 Proposed by Piero Filipponi, Rome, Italy (Vol. 26, no. 4, November 1988)

Let $C(n, k)=C_{1}(n, k)$ denote the binomial coefficient $\binom{n}{k}$.
Let $C_{2}(n, k)=C[C(n, k), k]$ and, in general, $C_{i}(n, k)=C(C\{\ldots[C(n, k), k]\})$.
For given $n$ and $i$, is it possible to determine the value $k_{0}$ of $k$ for which $C_{i}\left(n, k_{0}\right)>C_{i}(n, k) \quad\left(k=0,1, \ldots, n ; k \neq k_{0}\right)$ ?

Solution by Paul S. Bruckman, Edmonds, WA
We may make the problem a bit more precise by redefining $k_{0}$ uniquely as follows:
$C_{i}\left(n, k_{0}\right) \geq C_{i}(n, k), k=0,1,2, \ldots, n$,
with $k_{0}$ being the smallest integer with this property.
Of course, $k_{0}=k_{0}(n, i)$, dependent on the values of $n$ and $i$.
We may readily show that $k_{0}$ as thus defined is uniquely determined. We see that, for all $n>1$,

$$
\begin{equation*}
C_{i}(n, 0)=1 ; \quad C_{i}(n, 1)=C_{i}(n, n-1)=n ; \quad C_{i}(n, n)=0, \tag{2}
\end{equation*}
$$

from which the conclusion follows.
The construction of $k_{0}$ is much more difficult in the general case; however, for $i=1$, the solution is well known, namely:
(3) $\quad k_{0}(n, 1)=[n / 2]$.

In other words, given $n$, the maximum value of $\binom{n}{k}$ is assumed at $k=[n / 2]$ (and also at $k=\left[\frac{1}{2}(n+1)\right]$, which we ignore, due to our uniqueness definition).

Even the case $i=2$ readily becomes formidable; however, we may make some statistical inferences, by means of Stirling's formula, which may have some validity as $n \rightarrow \infty$. By means of a TI-60 Scientific Calculator, the following table was obtained:

| $\underline{n}$ | $\underline{k_{0}(n, 2)}$ | $\underline{n}$ | $\underline{k_{0}(n, 2)}$ | $n$ | $\underline{k_{0}(n, 2)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 9 | 6 | 18 | 12 |
| 1 | 0 | 10 | 6 | 19 | 13 |
| 2 | 1 | 11 | 7 | 20 | 13 |
| 3 | 1 | 12 | 8 | 21 | 14 |
| 4 | 2 | 13 | 8 | 22 | 15 |
| 5 | 3 | 14 | 9 | 23 | 15 |
| 6 | 4 | 15 | 10 | 24 | 16 |
| 7 | 4 | 16 | 10 | 25 | 17 |
| 8 | 5 | 17 | 11 | 26 | 17 |
|  |  |  |  | 27 | 18 |

It appears from the table that $k_{0}(n, 2) \sim 2 n / 3$, at least asymptotically. We conjecture that, given $i$, a constant $\theta(i)$ exists such that

$$
\begin{equation*}
k_{0}(n, i) \sim n \cdot \theta(i), \text { as } n \rightarrow \infty \text {. } \tag{4}
\end{equation*}
$$

To test this hypothesis, we note that, for most values of $k, C_{i}(n, k)$ is of a much higher order of magnitude than $k$. Using the approximate relation

$$
\begin{equation*}
\binom{n}{k} \sim \frac{n^{k}}{k!} \text {, valid when } n / k \text { is large, } \tag{5}
\end{equation*}
$$

as well as the obvious recursion

$$
\begin{equation*}
C_{i+1}(n, k)=\binom{C_{i}(n, k)}{k}, \tag{6}
\end{equation*}
$$

we obtain the asymptotic relation
(7) $\quad C_{i+1}(n, k) \sim \frac{\left(C_{i}(n, k)\right)^{k}}{k!}$, for all except the extreme values of $k$.

We may make another observation, namely, that the sequence

$$
K_{n}=\left(k_{0}(n, i)\right)_{i=1}^{\infty}
$$

is nondecreasing for a given $n$. To see this, let $u=k_{0}(n, i), v=k_{0}(n, i+1)$. Then, by definition, $C_{i}(n, v) \leq C_{i}(n, u)$ and $C_{i+1}(n, u) \leq C_{i+1}(n, v)$. Thus,

$$
\binom{C_{i}(n, v)}{u} \leq\binom{ C_{i}(n, u)}{u} \leq\binom{ C_{i}(n, v)}{v},
$$

which implies

$$
\begin{equation*}
u \leq v . \tag{8}
\end{equation*}
$$

We assume the relationship in (4); letting $\theta=\theta(2)$ and applying Stirling's formula, we obtain:

$$
\begin{equation*}
C_{1}(n, \theta n) \sim A_{1} n^{-\frac{1}{2}} B_{1}^{n}, \text { where } A_{1}=(2 \pi \theta(1-\theta))^{-\frac{1}{2}}, B_{1}=\left(\theta^{\theta}(1-\theta)^{1-\theta}\right)^{-1} \tag{9}
\end{equation*}
$$

Since $C_{1}(n, \theta n)$ is generally much larger than $\theta n$, we may apply (5) with $k$ set to equal $\theta$ n and obtain, after a second application of Stirling's formula:

$$
\begin{align*}
C_{2}(n, \theta n) \sim A_{2} n^{-\frac{1}{2}-\frac{3}{2} n \theta} B_{2}^{n} C_{2}^{n^{2}}, \text { where } A_{2} & =(2 \pi \theta)^{-\frac{1}{2}},  \tag{10}\\
B_{2} & =\left(e A_{1} / \theta\right)^{\theta}, \text { and } C_{2}=B_{1}^{\theta} .
\end{align*}
$$

Note that $C_{2}(n, \theta n)$ is dominated by the term $C_{2}^{n^{2}}=\left(B_{1}^{\theta}\right)^{n^{2}}$. We may perform a similar computation, only this time letting $\theta=\theta(3)$; we then find that

$$
C_{3}(n, \theta n) \sim A_{1} n^{-\frac{1}{2}\left(1+3 n \theta+n^{2} \theta^{2}\right)} B_{3}^{n} C_{3}^{n^{2}} D_{3}^{n^{3}}
$$

where $B_{3}$ and $C_{3}$ are constants and $D_{3}=B_{1}^{\Theta^{2}}$ [note that $A_{1}$ and $B_{1}$ are defined as in (9), but have different values, since a different value of $\theta$ is used to compute them]. We note that $C_{3}(n, \theta n)$ is dominated by the term

$$
D_{3}^{n^{3}}=\left(B_{1}^{\theta^{2}}\right)^{n^{3}}
$$

It appears that if we can determine $\theta$ such that $B_{1}^{\theta}$ and $B_{1}^{\theta^{2}}$ are maximized, we can determine $k_{0}(n, 2)$ and $k_{0}(n, 3)$, respectively. Generalizing further, we see that $\theta(i)=\theta$ may be determined, according to this line of reasoning, by maximizing the expression

$$
\begin{equation*}
B_{1}^{\theta^{i-1}}=\theta^{-\theta^{i}}(1-\theta)^{-(1-\theta) \theta^{i-1}} \tag{11}
\end{equation*}
$$

Since $\theta \in(0,1)$, we see that the above expression exceeds unity; hence, its logarithm is positive. Letting $f(\theta, i)$ equal the logarithm, we thus seek to maximize the expression:

$$
\begin{equation*}
f(\theta, i)=-\theta^{i} \log \theta-\theta^{i-1}(1-\theta) \log (1-\theta) . \tag{12}
\end{equation*}
$$

Note that $f>0$ if $\theta \in(0,1)$, while

$$
f(0, i)=\lim _{\theta \rightarrow 0^{+}} f(\theta, i)=0 \text { and } f(1, i)=\lim _{\theta \rightarrow 1^{-}} f(\theta, i)=0
$$

hence, $f$ does indeed assume a maximum value for some $\theta \in(0,1)$. To find this value of $\theta(i)$, we need to solve the equation $f^{\prime}(\theta, i)=0$, which yields the transcendental equation

$$
\begin{equation*}
\left[\theta-\left(\frac{i-1}{i}\right)\right] \log (1-\theta)-\theta \log \theta=0 \tag{13}
\end{equation*}
$$

Clearly, for $i=1$, we obtain the value $\theta=\theta(1)=\frac{1}{2}$, which is correct. For $i=2$, we obtain, after some computation on the TI-60, the value

$$
\begin{equation*}
\theta(2) \doteq .7035060764 \tag{14}
\end{equation*}
$$

For a few other values of $i$, the following values were obtained, as the solutions of (13):

$$
\begin{align*}
& \theta(3) \doteq .78783 \text { 98702, } \quad \theta(4) \doteq .8341745130  \tag{15}\\
& \theta(5) \doteq .8635823417, \quad \theta(6) \doteq .8839571002
\end{align*}
$$

It is conjectured that, asymptotically at least, the above algorithm yields the appropriate value of $\theta(i)$, such that $k_{0}(n, i)$ is validly obtained in (4). As for the smaller values of $n$, no inference is provided by this procedure.

## Same Difference

H-428 Proposed by Larry Taylor, Rego Park, NY
(Vol. 27, no. 1, February 1989)
Let $j, m$, and $n$ be integers. Let $a$ and $b$ be relatively prime even-odd integers with $b$ not divisible by 5. Let $A_{n}=a L_{n}+b F_{n}$. Then $A_{n}=A_{n+1}-A_{n-1}$ with initial values $A_{1}=b+a, A_{-1}=b-a$.

Prove that the following three numbers

$$
\left(2 F_{n-j} A_{m-j}, \quad F_{n+j} A_{m+j}, \quad 2 F_{2 j} A_{n+m}\right)
$$

are in arithmetic progression.
Solution by Russell Jay Hendel, Dowling College, Oakdale, NY
We solve the problem without the divisibility restrictions on $\alpha$ and $b$. First, observe that $[2 x, y, 2 z]$ are in arithmetic progression iff $x+z=y$. Next, observe that $A_{n}$ are linear combinations of the $F_{n}$ and $L_{n}$. Therefore, it suffices to prove the two equations,

$$
F_{n-j} L_{m-j}+F_{2 j} L_{n+m}=F_{n+j} L_{m+j} \text { and } F_{n-j} F_{m-j}+F_{2 j} F_{n+m}=F_{n+j} F_{m+j}
$$

We will prove only second equation, proof of the first equation being similar.
By Binet's formula (which clearly holds also for subscripts with negative values), we reduce proof of this equation to proof of the equality

$$
\begin{aligned}
& {\left[p^{n+m-2 j}+q^{n+m-2 j}\right]-\left(p^{n+m} q^{2 j}+p^{2 j} q^{n+m}\right)} \\
& =\left[q^{n-j} p^{m-j}+p^{n-j} q^{m-j}\right]-\left(p^{n+j} q^{m+j}+p^{m+j} q^{n+j}\right)
\end{aligned}
$$

with $p$ and $q$ the roots of $x^{2}-x-1=0$. To complete the proof, we multiply the bracketed expression on each side of this equality by $1=(p q)^{2 j}$. This shows that both sides of the equation are zero and completes the proof.

Also solved by P. Bruckman, P. Filipponi, R. Andre-Jeannin, L. Kuipers, Y. H. Harris Kwong, and the proposer.

## Editorial Note:

The editor welcomes solutions of any previously proposed problem. Also, in order to avoid misreading proposals or solutions, it would be appreciated if submitted material is typed or printed.

## *****

(Continued from page 354)
Once in our medium-sized auditorium, we were intrigued (and assisted) by "the wonders technology had wrought": there were two overhead projectors, and blackboards-ugh, whiteboards!-came from everywhere; up and down they went, above and below, over and across, sometimes interceded by a screen that appeared from nowhere..., and all of it happened by the touch of a button, skillfully activated by the cognoscenti.

Of course, there was not only food for the mind and the soul, but also for the stomach. Wake Forest University graciously treated us to daily morning and afternoon coffee breaks, and the president, Dr. Thomas K. Hearn, Jr., hosted a wine and cheese reception on campus.

Even though our daily meetings took place from 9:00 a.m. till noon, and from 2:00 p.m. to 5:00 p.m., we did not ALWAYS work. In midweek, the afternoon was freed, and we took off to Doughton Park in the beautiful Blue Ridge Mountains of North Carolina. There the group dispersed to enjoy the magnificent scenery with a choice of several hiking trails that offer spectacular vistas. Those of us who preferred less energetic activities, relaxed at a coffee shop where we did, what we seem to do best, or at least most often, and with pleasure: exchange mathematical ideas. All this was followed by a lavish, typically North Carolinian dinner at Shatley Springs.

The next day we celebrated our customary evening banquet. It was held on campus, and was at once elegant and friendly, somehow reflecting the spirit of our group. We speak with many different foreign accents, yet we all understand each other, professionally and personally. The magnetism of our beloved discipline has somehow promoted a very special bond of friendship. Many of us had been together at some of the past conferences. Quite a few papers exhibited the resulting kinding of common mathematical interests which culminated in joint authorships.

Maybe several of you are already gathering your thoughts for our next Conference. "Auf Wiedersehen," then, in 1992 at St. Andrews University, Scotland.

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Announcement

# FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS 

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.
A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.
Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.
Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence-18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

Applications of Fibonacci Numbers. Edited by A.N. Philippou, A.F. Horadam and G.E. Bergum.

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