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The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

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THE DETERMINATION OF A CLASS OF PRIMITIVE INTEGRAL TRIANGLES

Joseph E. Carroll and Ken Yanosko
Humboldt State University, Arcata, CA 95521
(Submitted December 1988)

One of the problems of classical number theory is the determination of all primitive integral right triangles. The well-known answer is that if $r > s$ are relatively prime positive integers, not both odd, then the triangle with sides $r^2 - s^2$, $2rs$, and $r^2 + s^2$ is such a triangle (easy to check) and any such triangle is of this form for some r and s. A simple proof of the latter half is given in [1]. This paper deals with a similar question that has a similar answer but a somewhat longer solution. The main tool in that solution is a thinly disguised version of the Chebyshev polynomials of the second kind.

Definition 1: Let $j \geq k$ be positive, relatively prime integers. A triangle will be called an $\langle j, k\rangle$ triangle if one of its angles is j/k times another.

It is easy to write down the primitive integral $\langle 1, 1\rangle$ (i.e., isosceles) triangles. These triangles have sides s, s, and r, where r and s are positive integers, $(r, s) = 1$, and $r < 2s$. The primitive integral $\langle 2, 1\rangle$ triangles have been determined by Luthar in [2]. If r and s are positive integers where $(r, s) = 1$ and $s < r < 2s$, then the triangle with sides rs, s^2, and $r^2 - s^2$ is a primitive integral $\langle 2, 1\rangle$ triangle, and all such triangles are of this form for suitable r and s. In this paper we shall determine all primitive integral $\langle j, k\rangle$ triangles for all j and k satisfying the criterion of Definition 1. Although this is hardly one of the burning mathematical questions of our time, it is hoped that the solution presented here will be of some interest, since it both draws ideas from several areas of mathematics and requires little background to understand.

First, let us fix j and k. It is clear that the $\langle j, k\rangle$ triangles are characterized by having angles $j\alpha$, $k\alpha$, and $\pi - (j + k)\alpha$ for some positive real number α such that $(j + k)\alpha < \pi$. Also, for any such α, there may or may not be a rational sided (hence, a primitive integral) triangle in the similarity class of $\langle j, k\rangle$ triangles associated with α in this way. The law of sines immediately gives us a triangle in that similarity class. If the triangle with sides a, b, c is denoted by the triple $\langle a, b, c\rangle$, then $\langle \sin j\alpha, \sin k\alpha, \sin(j + k)\alpha\rangle$ is in it. The following lemma leads us to a condition on α sufficient to ensure that there is a rational sided triangle similar to $\langle \sin j\alpha, \sin k\alpha, \sin(j + k)\alpha\rangle$.

Lemma 1: Define a sequence $\{p_n(x)\}_{n \geq 0}$ of polynomials with integer coefficients as follows: $p_0(x) \equiv 0$, $p_1(x) \equiv 1$, and, for $n \geq 2$,

$$p_n(x) = xp_{n-1}(x) - p_{n-2}(x).$$

Then, for any real number α which is not an integral multiple of π, we have

$$p_n(2\cos\alpha) = (\sin n\alpha)/(\sin\alpha).$$

Proof: The formula for the sine of a sum yields the following identities for $n \geq 2$:

$$\sin n\alpha = \sin(n - 1)\alpha\cos\alpha + \cos(n - 1)\alpha\sin\alpha$$

$$\sin(n - 2)\alpha = \sin(n - 1)\alpha\cos\alpha - \cos(n - 1)\alpha\sin\alpha$$

Adding these identities and dividing by $\sin\alpha$, we get:

$$(\sin n\alpha)/(\sin\alpha) = (2\cos\alpha)\cdot(\sin(n - 1)\alpha)/(\sin\alpha) - (\sin(n - 2)\alpha)/(\sin\alpha)$$

Thus, for any α which is not an integral multiple of π, the sequences

$$\{(\sin n\alpha)/(\sin \alpha)\}_{n \geq 0} \quad \text{and} \quad \{p_n(2 \cos \alpha)\}_{n \geq 0}$$

satisfy the same second-order linear recurrence relation. Furthermore, these sequences coincide on their first two terms. It follows that they are identical for all n.

Proposition 1: If $0 < \alpha < \pi/(j + k)$ and $\cos \alpha$ is a rational number, then there is a rational sided triangle with angles $j\alpha$, $k\alpha$, and $\pi - (j + k)\alpha$.

Proof: By Lemma 1, $\langle p_j(2 \cos \alpha), p_k(2 \cos \alpha), p_{j+k}(2 \cos \alpha)\rangle$ has the correct angles. Its sides are rational because $\cos \alpha$ is.

Remark 1: It is clear from the definition of $\{p_n\}$ that, for all $n \geq 1$, $p_n(x)$ is monic of degree $n - 1$. These polynomials, after a shift of subscripts and a change of variables, are none other than the Chebyshev polynomials of the second kind, $\{U_n(x)\}_{n \geq 0}$. For $n \geq 0$, $U_n(x) = p_{n+1}(2x)$. In fact, Lemma 1 is equivalent to a well-known property of U_n. It is proved again here to keep the discussion self-contained. The Chebyshev polynomials of the first kind, $\{T_n(x)\}_{n \geq 0}$, also deserve mention because they are used in the proof of the following lemma, which will lead us to the converse of Proposition 1. They can be defined by

$$T_0(x) \equiv 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \text{ for } n \geq 2.$$

Reasoning as in the proof of Lemma 1, one can show that, for any real number α, $T_n(\cos \alpha) = \cos n\alpha$.

Lemma 2: Let σ, τ be real numbers; then, for any integers m and n, $\cos(m\sigma + n\tau)$ is in the $\mathbf{Z}[\cos \sigma, \cos \tau]$ module generated by 1 and $\cos(\sigma + \tau)$.

Proof: Suppose that $m, n \geq 0$. Then

$$\begin{aligned}\cos(\pm(m\sigma \pm n\tau)) &= \cos m\sigma \cos n\tau \mp \sin m\sigma \sin n\tau \\ &= T_m(\cos \sigma)T_n(\cos \tau) \\ &\qquad \mp \sin \sigma\, p_m(2 \cos \sigma) \sin \tau\, p_n(2 \cos \tau).\end{aligned}$$

This follows from Lemma 1 and Remark 1 and is also true if σ or τ is an integral multiple of π. Using the formula for the cosine of a sum again, we deduce

$$\begin{aligned}\cos(\pm(m\sigma \pm n\tau)) &= T_m(\cos \sigma)T_n(\cos \tau) \\ &\qquad \pm p_m(2 \cos \sigma)p_n(2 \cos \tau)(\cos(\sigma + \tau) - \cos \sigma \cos \tau).\end{aligned}$$

Proposition 2: Suppose that for positive relatively prime integers $j \geq k$ with $0 < \alpha < \pi/(j + k)$ there is a rational sided triangle with angles $j\alpha$, $k\alpha$, and $\pi - (j + k)\alpha$. Then $\cos \alpha$ is a rational number.

Proof: If such a rational $\langle j, k\rangle$ triangle exists, then the law of cosines tells us that $\cos j\alpha$, $\cos k\alpha$, and $\cos(j + k)\alpha = -\cos(\pi - (j + k)\alpha)$ are all rational. Since j and k are relatively prime, there are integers m and n such that $mj + nk = 1$. Applying Lemma 2 for $\sigma = j\alpha$ and $\tau = k\alpha$, and using this m and n, we deduce that $\cos \alpha$ is rational, as claimed.

We now have necessary and sufficient conditions on α that there be a rational sided triangle with angles $j\alpha$, $k\alpha$, and $\pi - (j + k)\alpha$. When there is such a triangle, we need to find the primitive integral triangle in its similarity class. Properties of the sequence $\{p_n(x)\}$ and of a related sequence of homogeneous polynomials are the tools that will allow us to make that determination.

Proposition 3: The following are true for the sequence $\{p_n(x)\}$ defined in the statement of Lemma 1:

(a) $p_n(x) = \sum_{i=0}^{[(n-1)/2]} (-1)^i \binom{n-1-i}{i} x^{n-1-2i}$, for $n \geq 0$;

(b) $p_n(x) = \prod_{t=1}^{n-1} (x - 2\cos(t\pi/n))$, for $n \geq 1$;

(c) If $d|n$, then $p_d(x)|p_n(x)$ as polynomials in $\mathbf{Z}[x]$.

Proof: (a) A straightforward (if somewhat tedious) computation using a standard addition formula for binomial coefficients demonstrates that the sequence of candidate polynomials shown above satisfies the defining recurrence relation for the p_n. It is immediate that the two sequences coincide for $n = 0, 1$, so they must be the same for all n. Like Proposition 1, this is equivalent to a well-known statement about the U_n.

(b) Lemma 1 implies that $2\cos(t\pi/n)$ is a root of p_n for $t = 1, 2, \ldots, n - 1$ and, since the cosine is strictly decreasing on $[0, \pi]$, these roots are distinct. Since p_n has degree $n - 1$, the proposed equation is true up to multiplication by a constant. But, both p_n and the product above are monic, so the constant is 1.

(c) Part (b) implies this divisibility property as polynomials over the real numbers. If $p_n(x) = p_d(x)q(x)$, where $q(x)$ has real coefficients, the fact that p_d is integer monic and p_n is integral implies that q is integral. In fact, extending this reasoning, one can prove a stronger statement: if m and n are nonnegative integers, then $p_{(m,n)}(x)$ is the greatest common divisor of $p_m(x)$ and $p_n(x)$ in $\mathbf{Z}[x]$.

Remark 2: The field extension $\mathbf{Q}(e^{2\pi i/q})/\mathbf{Q}$ for q an odd prime is often used as an example in the teaching of Galois theory and algebraic number theory. It is shown that this extension is Galois of degree $q - 1$ with cyclic Galois group and that the irreducible polynomial of $e^{2\pi i/q}$ over $\mathbf{Q}$ is

$$\Phi_q(x) = x^{q-1} + \cdots + 1.$$

It is also shown that the unique subextension of index 2, which is the subfield fixed by complex conjugation, is generated by

$$2\cos(2\pi/q) = e^{2\pi i/q} + e^{-2\pi i/q},$$

an algebraic integer. Using Proposition 3(b), an identity satisfied by the $\{p_n\}$ that is easily proved, and some basic Galois theory, it can be shown that the irreducible polynomial of $2\cos(2\pi/q)$ over $\mathbf{Q}$ is

$$p_{(q+1)/2}(x) + p_{(q-1)/2}(x).$$

Proposition 3(a) then yields an explicit expression.

It is convenient to introduce a new sequence $\{P_n(x, y)\}_{n\geq 1}$ of homogeneous polynomials associated to $\{p_n(x)\}$. For $n \geq 1$, let

$$P_n(x, y) = y^{n-1}p_n(x/y) = \sum_{i=0}^{[(n-1)/2]} (-1)^i \binom{n-1-i}{i} x^{n-1-2i}y^{2i},$$

where the latter equation above follows from Proposition 3(a). Using Proposition 3(c), we immediately see that $d|n$ implies $P_d|P_n$ as polynomials in $\mathbf{Z}[x, y]$. We require a final lemma before stating and proving the main result of this paper.

Lemma 3: Let r and s be positive integers with $(r, s) = 1$ and let $n \geq 1$. Then

(a) $(s, P_n(r, s)) = 1$;

(b) $(P_n(r, s), P_{n+1}(r, s)) = 1$.

Proof: (a) First, we observe that $P_n(r, s) \equiv r^{n-1} \pmod{s}$. This follows either from the explicit expression for P_n given above or directly from the definition of P_n and the fact, noted in Remark 1, that p_n is integral monic of degree $n - 1$. Since $(r, s) = 1$, it follows that $(s, P_n(r, s)) = 1$ for $n \geq 1$.

(b) We prove this part by induction. Since $P_1(r, s) = 1$, the statement is true for $n = 1$. Let $n \geq 2$ and assume that the statement is true for $n - 1$. By the definition of the sequence $\{P_n(x, y)\}$, the defining recursion formula for $\{p_n(x)\}$ translates to

$$P_{n+1}(r, s) = rP_n(r, s) - s^2P_{n-1}(r, s).$$

Assume d is a positive integer such that $d|P_n(r, s)$ and $d|P_{n+1}(r, s)$. Then, by part (a), $(d, s) = 1$; by the equation above, $d|s^2P_{n-1}(r, s)$; thus $d|P_{n-1}(r, s)$. Therefore, by the induction assumption, $d = 1$.

Theorem 1: Let $j \geq k$ be positive integers with $(j, k) = 1$, and let r and s be positive integers with $(r, s) = 1$ and $\cos(\pi/(j + k)) < r/2s < 1$. Then

$$\langle s^kP_j(r, s),\ s^jP_k(r, s),\ P_{j+k}(r, s)\rangle$$

is a primitive integral $\langle j, k\rangle$ triangle with angles $j\alpha$, $k\alpha$, and $\pi - (j + k)$, for $\alpha = \arccos(r/2s)$, and all primitive integral $\langle j, k\rangle$ triangles are of this form for some such r and s.

Proof: By the proof of Proposition 1, for each r and s satisfying the conditions of the theorem, $\langle p_j(r/s),\ p_k(r/s),\ p_{j+k}(r/s)\rangle$ is a rational sided $\langle j, k\rangle$ triangle with the required angles. By Proposition 2, any similarity class of $\langle j, k\rangle$ triangles that includes a triangle with rational sides includes a triangle of this form for some r and s satisfying the hypotheses of the theorem. Our proposed triangle is clearly integer sided, and the definition of the P_n implies that it is similar to this one by a scale factor of s^{j+k-1}. Therefore, we need only prove that it is primitive. By Lemma 3(a), it suffices to show that, if u and v are positive integers with $(u, v) = 1$, then $(P_u(r, s), P_v(r, s)) = 1$. If $(u, v) = 1$, there are positive integers m and n such that mu and nv are consecutive integers. Then $(P_{mu}(r, s), P_{nv}(r, s)) = 1$ by Lemma 3(b). But, as noted above, $P_u|P_{mu}$ and $P_v|P_{nv}$. Thus, $(P_u(r, s), P_v(r, s)) = 1$, as required.

Example 1: To illustrate Theorem 1, we shall determine all primitive integral $\langle 3, 1\rangle$ triangles with no side longer than 100. Using Theorem 1, we know that they are of the form $\langle s(r^2 - s^2),\ s^3,\ r^3 - 2rs^2\rangle$ for r and s relatively prime positive integers with $\sqrt{2}/2 < r/2s < 1$. Since one side is s^3 and we are looking for those with sides no greater than 100, we must have $s = 1, 2, 3$, or 4. For $s = 1$, we would need $\sqrt{2} < r < 2$, which is not possible. For $s = 2$, we need $2\sqrt{2} < r < 4$, which is only possible for $r = 3$ and which gives us the triangle $\langle 10, 8, 3\rangle$. For $s = 3$, we need $3\sqrt{2} < r < 6$, which is only possible for $r = 5$ and which gives us the triangle $\langle 48, 27, 35\rangle$. For $s = 4$, we need $4\sqrt{2} < r < 8$, which is only possible for $r = 6, 7$. But 6 is not relatively prime to 4 and $r = 7$ gives us the triangle $\langle 132, 64, 119\rangle$, two sides of which are too large.

References

1. G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. 4th ed. Oxford: Oxford University Press, 1960, pp. 190-91.
2. R. S. Luthar. "Integer-Sided Triangles with One Angle Twice Another." *The College Mathematics Journal 15.3* (1984):55-56.

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AN ALGEBRAIC EXPRESSION FOR THE NUMBER OF KEKULÉ STRUCTURES OF BENZENOID CHAINS

Ratko Tǒsić and Olga Bodroža
University of Novi Sad, Dr. Ilije Djuričića 4, 21000 Novi Sad, Yugoslavia
(Submitted December 1988)

1. Introduction

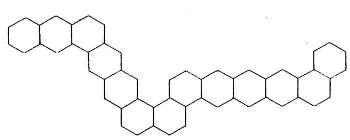
The enumeration of Kekulé structures for benzenoid polycyclic hydrocarbons is important because the stability and many other properties of these hydrocarbons have been found to correlate with the number of Kekulé structures. Starting with the algorithm proposed by Gordon & Davison [8], many papers have appeared on the problem of finding the "Kekulé structure count" K for such hydrocarbons. We can mention here only a few authors who contributed to this topic: Balaban & Tomescu [1, 2, 3, 4], Gutman [10, 11, 12], Herndon [13], Hosoya [12, 14], Sachs [16], Trinajstić [17], Farrell & Wahid [6], Fu-ji & Rong-si [8], Artemi [1], Yamaguchi [14]. A whole recent book [5] is devoted to Kekulé structures in benzenoid hydrocarbons.

In this paper we consider only undirected graphs comprised of 6-cycles. Let there be a total of m such cycles, which we shall denote as $C_1, C_2, \ldots, C_m$ in each graph of interest. Because the problem we treat arises from chemical studies of certain hydrocarbon molecules, we impose upon $C_1, C_2, \ldots, C_m$ the following conditions to reflect the underlying chemistry:

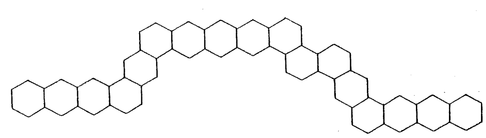
(i) Every C_i and C_{i+1} shall have a common edge denoted by e_i, for all $1 \le i \le m - 1$.

(ii) The edges e_i and e_j shall have no common vertex for any $1 \le i < j \le m - 1$.

Representing the 6-cycles as regular hexagons in the plane results in a graph such as that illustrated in Figures 1(a) and 1(b). In organic chemistry, such graphs correspond to benzenoid chains (each vertex represents a carbon atom or CH group, and no carbon atom is common to more than two 6-cycles).



(a)



(b)

FIGURE 1

2. Definitions and Notation

By $L(x_1, x_2, \ldots, x_n)$, we denote a benzenoid chain (i.e., a corresponding graph) composed from n linearly condensed portions (segments) consisting of $x_1, x_2, \ldots, x_n$ hexagons, respectively. Figures 1(a) and 1(b) show $L(3, 4, 2, 2, 5, 2)$ and $L(4, 3, 5, 2, 2, 3, 4)$, respectively.

Any two adjacent linear segments are considered as having a common hexagon. The common hexagon of two adjacent linear segments is called a "kink." The chain $L(x_1, x_2, \ldots, x_n)$ has exactly $n - 1$ kinks. So the total number of hexagons in $L(x_1, x_2, \ldots, x_n)$ is $m = x_1 + x_2 + \cdots + x_n - n + 1$. Observe that such notation implies $x_i \ge 2$, for $i = 1, 2, \ldots, n$.

We adopt the following notation:

$K_n(x_1, x_2, \ldots, x_n)$ is the number of Kekulé structures (perfect mathcings) of $L(x_1, x_2, \ldots, x_n)$.

F_i is the i^{th} Fibonacci number, defined as follows:

$F_{-2} = 1$, $F_{-1} = 0$; $F_k = F_{k-1} + F_{k-2}$, for $k \geq 0$.

For all other definitions, see [5].

3. Recurrence Relation and Algebraic Expression for $K_n(x_1, x_2, \ldots, x_n)$

It is easy to deduce the K formula for a single linear chain (polyacene) of x_1 hexagons, say $L(x_1)$ (see [5]):

(1) $K_1(x_1) = 1 + x_1$.

We define

(2) $K_0 = 1$.

It may be interpreted as the number of Kekulé structures for "no hexagons."

Theorem 1: If $n \geq 2$, then, for arbitrary $x_1 > 1$, $x_2 > 1$, $\ldots$, $x_n > 1$, the following recurrence relation holds:

(3) $$K_n(x_1, \ldots, x_{n-1}, x_n) = x_n K_{n-1}(x_1, \ldots, x_{n-1} - 1) + K_{n-2}(x_1, \ldots, x_{n-2} - 1).$$

Proof: Let H be the last kink of $L(x_1, x_2, \ldots, x_n)$. We apply the fundamental theorem for matching polynomials [7].

Let u and v be the vertices belonging only to hexagon (kink) H (Figure 2). Consider any perfect matching which contains the bond uv. The rest of such a perfect matching will be a perfect matching of the graph consisting of two components $L(x_n - 1)$ and $L(x_1, x_2, \ldots, x_{n-1} - 1)$. The number of such perfect matchings is

$$K_1(x_n - 1) \cdot K_{n-1}(x_1, x_2, \ldots, x_{n-1} - 1),$$

i.e., according to (1),

(4) $x_n K_{n-1}(x_1, x_2, \ldots, x_{n-1} - 1)$.

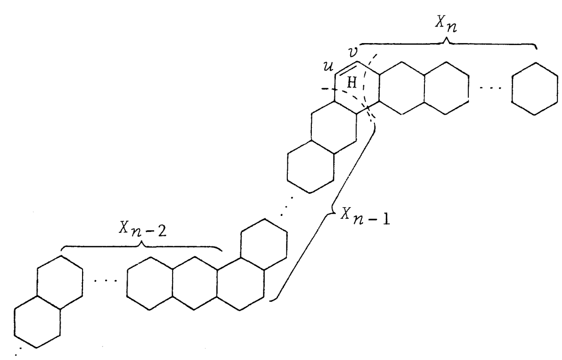


FIGURE 2

On the other hand, each perfect matching without the bond uv must contain all edges indicated in Figure 3. The rest of such a perfect matching will be a perfect matching of $L(x_1, x_2, \ldots, x_{n-2} - 1)$, the number of such perfect matching being

(5) $$K_{n-2}(x_1, x_2, \ldots, x_{n-2} - 1).$$

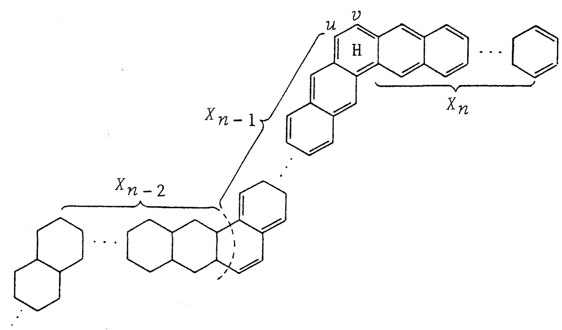


FIGURE 3

From (4) and (5), we obtain recurrence relation (3). □

Obviously, $K_n(x_1, x_2, \ldots, x_n)$ is a polynomial of the form

(6) $$K_n(x_1, \ldots, x_n) = g_n + \sum_{\substack{1 \le \ell_1 < \ell_2 < \cdots < \ell_p \le n \\ 1 \le p \le n}} g_n(\ell_1, \ldots, \ell_p) x_{\ell_1} \cdots x_{\ell_p}.$$

Clearly, $g_0 = 1$.

Now, we are going to determine the coefficients g_n and $g_n(\ell_1, \ldots, \ell_p)$.

First, we define an auxiliary polynomial

(7) $$Q_n(x_1, \ldots, x_{n-1}, x_n) = K_n(x_1, \ldots, x_{n-1}, x_n - 1).$$

For example, we have:

(8) $$Q_0 = 1,\ Q_1(x_1) = x_1,\ Q_2(x_1, x_2) = 1 - x_1 + x_1x_2.$$

From (3) and (7), we obtain the recurrence relation

$$Q_n(x_1, \ldots, x_{n-1}, x_n + 1) = x_nQ_{n-1}(x_1, \ldots, x_{n-1}) + Q_{n-2}(x_1, \ldots, x_{n-2}),$$

i.e.,

(9) $$Q_n(x_1, \ldots, x_{n-1}, x_n) = (x_n - 1)Q_{n-1}(x_1, \ldots, x_{n-1}) + Q_{n-2}(x_1, \ldots, x_{n-2}).$$

Let

(10) $$Q_n(x_1, \ldots, x_n) = S_n + \sum_{\substack{1 \le \ell_1 < \ell_2 < \cdots < \ell_p \le n \\ 1 \le p \le n}} S_n(\ell_1, \ldots, \ell_p) x_{\ell_1} \cdots x_{\ell_p}.$$

Clearly, $S_0 = 1$.

Now, we are going to determine the coefficients $S_n(\ell_1, \ldots, \ell_p)$ and S_n, for $n \ge 1$.

First, we prove the following lemmas.

Lemma 1: $S_n = (-1)^n F_{n-2}$.

Proof: The proof will be by induction on n. According to (8),

$$S_0 = 1 = (-1)^0 F_{-2}, \qquad S_1 = 0 = (-1)^1 F_{-1}.$$

Suppose that $S_i = (-1)^i F_{i-2}$, for $i \le k$. Then, according to (9),

$$S_k = -S_{k-1} + S_{k-2},$$

and by the induction hypothesis,

$$S_k = -(-1)^{k-1}F_{k-3} + (-1)^{k-2}F_{k-4}$$
$$= (-1)^{k-2}(F_{k-3} + F_{k-4}) = (-1)^k F_{k-2}. \quad \square$$

Lemma 2(a):

(11) $$S_n(\ell_1, \ldots, \ell_{p-1}, \ell_p) = (-1)^{n-\ell_p}F_{n-\ell_p}S_{\ell_p-1}(\ell_1, \ldots, \ell_{p-1}), \text{ for } p > 1.$$

(b):

(12) $$S_n(\ell_1) = (-1)^{n-\ell_1}F_{n-\ell_1}S_{\ell_1-1}.$$

Proof: It suffices to prove (a), since (b) is a particular case of (a). The proof will be by induction on $n - \ell_p$.

If $n - \ell_p = 0$ ($\ell_p = n$), then, according to (9),

(13) $$\begin{aligned} S_n(\ell_1, \ldots, \ell_{p-1}, \ell_p) &= S_{n-1}(\ell_1, \ldots, \ell_{p-1}) \\ &= (-1)^0F_0S_{n-1}(\ell_1, \ldots, \ell_{p-1}) \\ &= (-1)^{n-n}F_{n-n}S_{n-1}(\ell_1, \ldots, \ell_{p-1}). \end{aligned}$$

If $n - \ell_p = 1$ ($\ell_p = n - 1$), then, using (9) and (13), we have:

$$\begin{aligned} S_n(\ell_1, \ldots, \ell_p) &= -S_{n-1}(\ell_1, \ldots, \ell_p) \\ &= -S_{n-2}(\ell_1, \ldots, \ell_{p-1}) = (-1)^1F_1S_{n-2}(\ell_1, \ldots, \ell_{p-1}). \end{aligned}$$

Suppose that (11) is true for $n - \ell_p < k$ ($\ell_p > n - k$), $n - 1 \ge k \ge 2$. Then, for $n - \ell_p = k(\ell_p = n - k)$, according to (9),

$$S_n(\ell_1, \ldots, \ell_p) = -S_{n-1}(\ell_1, \ldots, \ell_p) + S_{n-2}(\ell_1, \ldots, \ell_p),$$

and, by the induction hypothesis,

$$\begin{aligned} S_n(\ell_1, \ldots, \ell_p) &= -(-1)^{n-1-\ell_p}F_{n-1-\ell_p}S_{\ell_p-1}(\ell_1, \ldots, \ell_{p-1}) \\ &\quad + (-1)^{n-2-\ell_p}F_{n-2-\ell_p}S_{\ell_p-1}(\ell_1, \ldots, \ell_{p-1}) \\ &= (-1)^{n-\ell_p}(F_{n-1-\ell_p} + F_{n-2-\ell_p})S_{\ell_p-1}(\ell_1, \ldots, \ell_{p-1}) \\ &= (-1)^{n-\ell_p}F_{n-\ell_p}S_{\ell_p-1}(\ell_1, \ldots, \ell_{p-1}). \quad \square \end{aligned}$$

Lemma 3: $S_n(\ell_1, \ldots, \ell_p) = (-1)^{n-p}F_{n-\ell_p}F_{\ell_p-\ell_{p-1}-1} \cdots F_{\ell_2-\ell_1-1}F_{\ell_1-3}$, for $p \ge 1$.

Proof: For $p = 1$, it follows, from (12) and Lemma 1, that

$$\begin{aligned} S_n(\ell_1) &= (-1)^{n-\ell_1}F_{n-\ell_1}S_{\ell_1-1} = (-1)^{n-\ell_1}F_{n-\ell_1}(-1)^{\ell_1-1}F_{\ell_1-3} \\ &= (-1)^{n-1}F_{n-\ell_1}F_{\ell_1-3}. \end{aligned}$$

For $1 < p \le n$, according to Lemmas 1 and 2,

$$S_n(\ell_1, \ldots, \ell_{p-1}, \ell_p) = (-1)^{n-\ell_p}F_{n-\ell_p}S_{\ell_p-1}(\ell_1, \ldots, \ell_{p-1}),$$

and now, by induction,

$$\begin{aligned} S_n(\ell_1, \ldots, \ell_{p-1}, \ell_p) &= (-1)^{n-\ell_p}F_{n-\ell_p}(-1)^{\ell_p-1-\ell_{p-1}}F_{\ell_p-\ell_{p-1}-1} \cdots \\ &\quad (-1)^{\ell_2-1-\ell_1}F_{\ell_2-\ell_1-1}(-1)^{\ell_1-1}F_{\ell_1-3} \\ &= (-1)^{n-p}F_{n-\ell_p}F_{\ell_p-\ell_{p-1}-1} \cdots F_{\ell_2-\ell_1-1}F_{\ell_1-3}. \quad \square \end{aligned}$$

Lemma 4(a): $g_n = (-1)^nF_{n-4}$,

(b): $g_n(\ell_1, \ldots, \ell_p) = (-1)^{n-p}F_{n-\ell_p-2}F_{\ell_p-\ell_{p-1}-1} \cdots F_{\ell_2-\ell_1-1}F_{\ell_1-3}$.

Proof: According to (7),

$$Q_n(x_1, \ldots, x_{n-1}, x_n + 1) = K_n(x_1, \ldots, x_{n-1}, x_n).$$

Hence,

(14) $$g_n(\ell_1, \ldots, \ell_p) = \begin{cases} S_n(\ell_1, \ldots, \ell_p), & \text{if } \ell_p = n, \\ S_n(\ell_1, \ldots, \ell_p) + S_n(\ell_1, \ldots, \ell_p, n), & \text{if } \ell_p < n. \end{cases}$$

Particularly, we have

(15) $g_n = S_n + S_n(n)$, for $n \geq 1$.

Now, from (15), Lemma 1, and Lemma 3, we have

$$g_n = (-1)\; F_{n-2} + (-1)^{n-1}F_{n-3} = (-1)^n(F_{n-2} - F_{n-3}) = (-1)^n F_{n-4},$$

and (a) is proved.

To prove (b), observe that, for $\ell_p = n$,

(16) $$g_n(\ell_1, \ldots, \ell_p) = S_n(\ell_1, \ldots, \ell_p) = (-1)^{n-p}F_{\ell_p - \ell_{p-1} - 1} \cdots F_{\ell_2 - \ell_1 - 1}F_{\ell_1 - 3},$$

and, for $\ell_p < n$,

$$\begin{aligned} g_n(\ell_1, \ldots, \ell_p) &= S_n(\ell_1, \ldots, \ell_p) + S_n(\ell_1, \ldots, \ell_p, n) \\ &= (-1)^{n-p}F_{n-\ell_p}F_{\ell_p - \ell_{p-1} - 1} \cdots F_{\ell_2 - \ell_1 - 1}F_{\ell_1 - 3} \\ &\quad + (-1)^{n-p-1}F_{n-\ell_p - 1}F_{\ell_p - \ell_{p-1} - 1} \cdots F_{\ell_2 - \ell_1 - 1}F_{\ell_1 - 3} \\ &= (-1)^{n-p}(F_{n-\ell_p} - F_{n-\ell_p - 1})F_{\ell_p - \ell_{p-1} - 1} \cdots F_{\ell_2 - \ell_1 - 1}F_{\ell_1 - 3}, \end{aligned}$$

i.e.,

(17) $$g_n(\ell_1, \ldots, \ell_p) = (-1)^{n-p}F_{n-\ell_p - 2}F_{\ell_p - \ell_{p-1} - 1} \cdots F_{\ell_2 - \ell_1 - 1}F_{\ell_1 - 3}.$$

Taking into account that, for $\ell_p = n$, $F_{n-\ell_p-2} = F_{-2} = 1$, (16) and (17) can be written together in the form

(18) $$g_n(\ell_1, \ldots, \ell_p) = (-1)^{n-p}F_{n-\ell_p - 2}F_{\ell_p - \ell_{p-1} - 1} \cdots F_{\ell_2 - \ell_1 - 1}F_{\ell_1 - 3}.$$

Theorem 2: $K_n(x_1, \ldots, x_n)$

$$= (-1)^n F_{n-4} + \sum_{\substack{1 \leq \ell_1 < \cdots < \ell_p \leq n \\ 1 \leq p \leq n}} g_n(\ell_1, \ldots, \ell_p)x_{\ell_1} \cdots x_{\ell_p},$$

where $g_n(\ell_1, \ldots, \ell_p)$ is given by (18).

Proof: Follows from Lemma 4. □

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References

1. A. T. Balaban, C. Artemi & I. Tomescu. "Algebraic Expressions for Kekulé Structure Counts of Non-Branched Regularly Cata-Condensed Benzenoid Hydrocarbons." *Mathematical Chemistry* *22* (1987):77-100.
2. A. T. Balaban & I. Tomescu. "Algebraic Expressions for the Number of Kekulé Structures of Isoarithmic Cata-Condensed Benzenoid Polycyclic Hydrocarbons." *Mathematical Chemistry* *14* (1983):155-82.
3. A. T. Balaban & I Tomescu. "Chemical Graphs, XL: Three Relations between the Fibonacci Sequence and the Numbers of Kekulé Structures for Non-Branched Cata-Condensed Polycyclic Aromatic Hydrocarbons." *Croatica Chemica Acta* *57.3* (1984):391-404.
4. A. T. Balaban & I Tomescu. "Chemical Graphs, XLI: Numbers of Conjugated Circuits and Kekulé Structures for Zig-Zag Catafusenes and (j, k)-hexes; Generalized Fibonacci Numbers." *Mathematical Chemistry* *17* (1985):91-120.

5. S. J. Cyvin & I. Gutman. *Kekulé Structures in Benzenoid Hydrocarbons*. Berlin: Springer-Verlag, 1988.
6. E. J. Farrell & S. A. Wahid. "Matchings in Benzene Chains." *Discrete Appl. Math.* *7* (1984):31-40.
7. E. J. Farrell. "An Introduction to Matching Polynomials." *J. Comb. Theory* Ser. B *27* (1979):75-86.
8. Z. Fu-ji & C. Rong-si. "A Theorem Concerning Polyhex Graphs." *Mathematical Chemistry* *19* (1986):179-88.
9. M. Gordon & W. H. T. Davison. "Resonance Topology of Fully Aromatic Hydrocarbons." *J. Chem. Phys.* *20* (1952):428-35.
10. I. Gutman. "Topological Properties of Benzenoid Hydrocarbons." *Bull. Soc. Chim. Beograd* *47.9* (1982):453-71.
11. I. Gutman. "Covering Hexagonal Systems with Hexagons." *Proceedings of the Fourth Yugoslav Seminar on Graph Theory*, Novi Sad, 1983, pp. 151-60.
12. I. Gutman & H. Hosoya. "On the Calculation of the Acyclic Polynomial." *Theor. Chim. Acta* *48* (1978):279-86.
13. W. C. Herndon. "Resonance Theory and the Enumeration of Kekulé Structures." *J. Chem. Educ.* *15* (1974):10-15.
14. H. Hosoya & T. Yamaguchi. "Sextet Polynomial: A New Enumeration and Proof Technique for Resonance Theory Applied to the Aromatic Hydrocarbons." *Tetrahedron Letters* (1975):4659-62.
15. L. Lovász & M. D. Plummer. *Matching Theory*. Budapest: Akademiai Kiado, 1986.
16. H. Sachs. "Perfect Matchings in Hexagonal Systems." *Combinatorica* *4.1* (1984):89-99.
17. N. Trinajstić. *Chemical Graph Theory*. Vol. 2. Boca Raton, Florida: CRC Press, 1983.

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GENERALIZED COMPLEX FIBONACCI AND LUCAS FUNCTIONS

Richard André-Jeannin
Ecole Nationale d'Ingenieurs de Sfax, Tunisia
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1. Introduction

Eric Halsey [3] has invented a method for defining the Fibonacci numbers $F(x)$, where x is a real number. Unfortunately, the Fibonacci identity

(1) $$F(x) = F(x-1) + F(x-2)$$

is destroyed. We shall return later to his method.

Francis Parker [6] defines the Fibonacci function by

$$F(x) = \frac{\alpha^x - \cos \pi x \alpha^{-x}}{\sqrt{5}},$$

where α is the golden ratio. In the same way, we can define a Lucas function

$$L(x) = \alpha^x + \cos \pi x \alpha^{-x}.$$

$F(x)$ and $L(x)$ coincide with the usual Fibonacci and Lucas numbers when x is an integer, and the relation (1) is verified. But the classical Fibonacci relations do not generalize. For instance, we do not have

$$F(2x) = F(x)L(x).$$

Horadam and Shannon [4] define Fibonacci and Lucas curves. They can be written, with complex notation

(2) $$F(x) = \frac{\alpha^x - e^{i\pi x}\alpha^{-x}}{\sqrt{5}},$$

(3) $$L(x) = \alpha^x + e^{i\pi x}\alpha^{-x}.$$

Again, we have $F(n) = F_n$, $L(n) = L_n$, for all integers n.

We shall prove in the sequel that the well-known identities for F_n and L_n are again true for all real numbers x, if $F(x)$ and $L(x)$ are defined by (2) and (3). For example, we have immediately

$$F(2x) = F(x)L(x).$$

We shall also relate these $F(x)$ and $L(x)$ to other Fibonacci properties as well as to Halsey's extension of the Fibonacci numbers.

2. Preliminary Lemma

Let us consider the set E of functions $w: \mathbb{R} \to \mathbb{C}$ such that

(4) $$\forall x \in \mathbb{R},\ w(x) = w(x-1) + w(x-2).$$

E is a complex vector space, and the following lemma is immediate.

Lemma 1: Let α be the positive root of $r^2 = r + 1$. Then the functions f and g, defined by

$$f(x) = \alpha^x, \quad g(x) = e^{i\pi x}\alpha^{-x}$$

are members of E.

Let us define now a subspace V of E by

$$V = \{w : \mathbb{R} \to \mathbb{C},\ w = \lambda f + \mu g,\ \lambda,\ \mu \in \mathbb{C}\}.$$

The functions F and L, defined by (2) and (3), are members of V.

Lemma 2: For all complex numbers a and b, there is a unique function w in V such that

$$w(0) = a, \quad w(1) = b.$$

Proof: We have

$$w(0) = \lambda + \mu = a, \quad w(1) = \lambda\alpha - \mu\alpha^{-1} = b.$$

By Cramer's rule, λ and μ exist and are unique.

Lemma 3: Let w be a member of V, and h a real number. Then the functions w_h and w_h', defined by

$$w_h(x) = w(x - h), \quad w_h'(x) = e^{i\pi x} w(h - x),$$

are members of V.

Proof: The proof is simple and therefore is omitted here.

Lemma 4: Let u and v be two elements of V and $\delta : \mathbb{R}^2 \to \mathbb{C}$, the function defined by

$$\delta(x, y) = \begin{vmatrix} u(x), & u(x+1) \\ v(y), & v(y+1) \end{vmatrix} = u(x)v(y+1) - u(x+1)v(y).$$

Then we have

$$\delta(x, y) = e^{i\pi y}\delta(x - y, 0). \tag{5}$$

Proof: First, we have

$$\delta(x, y) = \begin{vmatrix} u(x), & u(x) + u(x-1) \\ v(y), & v(y) + v(y-1) \end{vmatrix} = \begin{vmatrix} u(x), & u(x-1) \\ v(y), & v(y-1) \end{vmatrix} \tag{6}$$

$$= -\delta(x - 1, y - 1).$$

Now, let us define

$$\eta(x, y) = e^{i\pi y}\delta(x - y, 0) = e^{i\pi y}(u(x - y)v(1) - u(x - y + 1)v(0)).$$

Let x be a fixed real number. By Lemma 3, the functions

$$y \to \delta(x, y), \quad y \to \eta(x, y)$$

are members of V. We have

$$\delta(x, 0) = \eta(x, 0),$$

and, by (6),

$$\delta(x, 1) = -\delta(x - 1, 0) = \eta(x, 1).$$

By Lemma 2 we have, for all real numbers y,

$$\delta(x, y) = \eta(x, y).$$

This concludes the proof.

Lemma 5: Let F and L be the Fibonacci and Lucas functions defined by (2) and (3). Then, for all real numbers, we have:

$$L(x) = F(x + 1) + F(x - 1); \tag{7}$$

(8) $5F(x) = 2L(x+1) - L(x)$;

(9) $L(x) = 2F(x+1) - F(x)$.

The proofs readily follow from the lemmas and the definitions of the functions.

3. The Main Result

Theorem 1: Let u and v be two functions of V. Then, for all values of x, y, and z, we have

(10) $u(x)v(y+z) - u(x+z)v(y) = e^{i\pi y}F(z)[u(x-y)v(1) - u(x-y+1)v(0)]$,

where F is defined by (2).

Proof: For x and y fixed, consider the function Δ:

$$\Delta(z) = u(x)v(y+z) - u(x+z)v(y).$$

By Lemma 3, Δ is a member of V, and we have, with the notation of Lemma 4,

$$\Delta(0) = 0, \quad \Delta(1) = \delta(x, y).$$

Thus, we have, since the two members take the same values at $z = 0$, $z = 1$:

$$\Delta(z) = \delta(x, y)F(z).$$

The proof follows by Lemma 4.

4. Special Cases

Let us examine some particular cases of (10):

Case 1. $u = v = F$

Since $F(0) = 0$, $F(1) = 1$, we have

(11) $F(x)F(y+z) - F(x+z)F(y) = e^{i\pi y}F(z)F(x-y)$.

Case 2. $u = v = L$

Since $L(0) = 2$, $L(1) = 1$, we have, by (8),

(12) $L(x)L(y+z) - L(x+z)L(y) = -5e^{i\pi y}F(z)F(x-y)$.

Case 3. $u = F$, $v = L$

We have, by (9),

(13) $F(x)L(y+z) - F(x+z)L(y) = -e^{i\pi y}F(z)L(x-y)$.

Case 4. $u = L$, $v = F$

(14) $L(x)F(y+z) - L(x+z)F(y) = e^{i\pi y}F(z)L(x-y)$.

Case 5. Let $y = 0$ in (12) and (13) to get

(15) $2L(x+z) = L(x)L(z) + 5F(x)F(z)$,

(16) $2F(x+z) = F(x)L(z) + F(z)L(x)$.

Case 6. Let $y = 1$ in (11)-(14) to get

(17) $F(x+z) = F(x)F(z+1) + F(z)F(x-1)$,

(18) $L(x+z) = L(x)L(z+1) - 5F(z)F(x-1)$,

(19) $F(x+z) = F(x)L(z+1) - F(z)L(x-1)$,

(20) $L(x+z) = L(x)F(z+1) + F(z)L(x-1)$.

Case 7. Let $y = x - z$ in (11)-(14) to get

(21) $(F(x))^2 - F(x+z)F(x-z) = e^{i\pi(x-z)}(F(z))^2$,

(22) $(L(x))^2 - L(x+z)L(x-z) = -5e^{i\pi(x-z)}(F(z))^2$,

(23) $F(x)L(x) - F(x+z)L(x-z) = -e^{i\pi(x-z)}F(z)L(z)$,

(24) $F(x)L(x) - F(x-z)L(x+z) = e^{i\pi(x-z)}F(z)L(z)$.

Remark: (21) and (22) are Catalan's relations for $F(x)$, $L(x)$.

5. Application: A Reciprocal Series of Fibonacci Numbers

Theorem 2: Let x be a strictly positive real number and F the Fibonacci function. Then we have

$$\sum_{k=1}^{\infty} \frac{e^{i\pi 2^{k-1}x}}{F(x \cdot 2^k)} = \frac{e^{i\pi x}}{F(x)\alpha^x}.$$

Proof: We recall the relation attributed to De Morgan by Bromwich and to Catalan by Lucas,

$$(25)\quad \sum_{k=1}^{n} \frac{z^{2^{k-1}}}{1 - z^{2^k}} = \frac{1}{1-z}\,\frac{z - z^{2^n}}{1 - z^{2^n}},$$

where z is a complex number ($|z| \neq 1$). Now put $z = e^{i\pi x}\alpha^{-2x}$ in (25) to obtain:

$$(26)\quad \sum_{k=1}^{n} \frac{e^{i\pi 2^{k-1}x}\alpha^{-2^k x}}{1 - e^{i\pi 2^k x}\alpha^{-2^{k+1}x}} = \sum_{k=1}^{n} \frac{e^{i\pi 2^{k-1}x}}{\alpha^{2^k x} - e^{i\pi 2^k x}\alpha^{-2^k x}} = \frac{1}{\sqrt{5}} \sum_{k=1}^{n} \frac{e^{i\pi 2^{k-1}x}}{F(2^k x)}$$

On the other hand, the right member of (25) becomes

$$(27)\quad \frac{1}{1 - e^{i\pi x}\alpha^{-2x}} \cdot \frac{e^{i\pi x}\alpha^{-2x} - e^{i\pi 2^n x}\alpha^{-2^{n+1}x}}{1 - e^{i\pi 2^n x}\alpha^{-2^{n+1}x}} = \frac{1}{\sqrt{5}F(x)} \cdot \frac{e^{i\pi x}F((2^n - 1)x)}{F(x \cdot 2^n)}.$$

(26) and (27) give us

$$(28)\quad \sum_{k=1}^{n} \frac{e^{i\pi 2^{k-1}x}}{F(2^k x)} = \frac{e^{i\pi x}F((2^n - 1)x)}{F(2^n \cdot x)F(x)},$$

and so

$$(29)\quad \sum_{k=1}^{\infty} \frac{e^{i\pi 2^{k-1}x}}{F(2^k x)} = \frac{e^{i\pi x}}{F(x)\alpha^x}.$$

Remark: Put $x = m$ in (29), where m is a natural integer. After some calculations in the case m odd, we obtain the well-known formula:

$$(30)\quad \sum_{k=1}^{\infty} \frac{1}{F(2^k m)} = \frac{\sqrt{5}}{\alpha^{2m} - 1}.$$

Formula (30) was found by Lucas (see [5], p. 225) and was rediscovered by Brady [1]. See also Gould [2] for complete references.

6. Halsey's Fibonacci Function

First, we recall a well-known formula,

$$F_n = \sum_{k=0}^{m(n)} \binom{n-k-1}{k}, \quad n \geq 1,$$

where $m(n)$ is an integer such that $(n/2) - 1 \leq m(n) < (n/2)$.

We have used the binomial coefficients $\binom{n}{k}$ only when n is a positive integer but it is very convenient to extend their definitions. Then

$$\binom{x}{0} = 1, \quad \binom{x}{k} = \frac{x(x-1)\ \dots\ (x-k+1)}{k!}, \ k \geq 1,$$

defines the binomial coefficients for all values of x.

From this, we can introduce the function G,

$$(31) \qquad G(x) = \sum_{k=0}^{m(x)} \binom{x-k-1}{k}, \ x > 0,$$

where $m(x)$ is the integer defined by $(x/2) - 1 \leq m(x) < (x/2)$. Then, clearly, we have

$$G(n) = F_n, \ n \geq 1.$$

Theorem 3: G coincides with Halsey's extension of Fibonacci numbers, namely,

$$G(x) = \sum_{k=0}^{m(x)} [(x-k)B(x-2k, k+1)]^{-1}, \ x > 0,$$

where $B(x, y)$ is the beta-function:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x > 0, \ y > 0.$$

Proof: It is sufficient to show that

$$(32) \qquad \frac{1}{(x-k)B(x-2k, k+1)} = \binom{x-k-1}{k}.$$

In fact, the left member of (32) is

$$\frac{\Gamma(x-k+1)}{(x-k)\Gamma(x-2k)\Gamma(k+1)} = \frac{(x-k)(x-k-1)\ \dots\ (x-2k)\Gamma(x-2k)}{(x-k)\Gamma(x-2k)k!}$$
$$= \frac{(x-k-1)\ \dots\ (x-2k)}{k!} = \binom{x-k-1}{k},$$

in which we have used the well-known properties of the gamma-function:

$$\Gamma(x) = (x-1)\Gamma(x-1), \quad \Gamma(k) = (k-1)!$$

This concludes the proof.

Let p be a positive integer, and let G_p be the polynomial defined by

$$G_p(x) = \sum_{k=0}^{p} \binom{x-k-1}{k}.$$

We see, from (31), that

$$(33) \qquad G(x) = G_p(x), \quad 2p < x \leq 2p+2;$$

thus,

$$G_p(2p+1) = G(2p+1) = F_{2p+1},$$
$$G_p(2p+2) = G(2p+2) = F_{2p+2}.$$

In fact, we have a deeper result, which we state as the following theorem.

Theorem 4: $G_p(n) = F_n$ for $n = p+1, p+2, \dots, 2p+2$.

Proof: We shall prove this by mathematical induction. If $p = 0$, we have

$$G_0(1) = G_0(2) = 1.$$

Now we suppose that $G_{p-1}(n) = F_n$ $(n = p, \dots, 2p)$. Then we have

$$G_p(x) = G_{p-1}(x) + \binom{x-p-1}{p} = G_{p-1}(x) + \frac{(x-p-1)\ \ldots\ (x-2p)}{p!},$$

and thus,

$$G_p(n) = G_{p-1}(n) = F_n, \text{ for } n = p+1, \ldots, 2p;$$

but we have seen above that

$$G_p(2p+1) = F_{2p+1}, \quad G_p(2p+2) = F_{2p+2}.$$

This concludes the proof.

Corollary: G is continuous for all values of $x > 0$.

Proof: By (33), it is sufficient to show the continuity from the right at $x = 2p$. But

$$\lim_{\substack{x \to 2p \\ x > 2p}} G(x) = G_p(2p) = F_{2p} \quad \text{(by Theorem 4)}$$
$$= G(2p).$$

Finally, we see that Halsey's function is a continuous piecewise polynomial. For instance,

$$G(x) = 1, \qquad 0 < x \le 2,$$
$$G(x) = x - 1, \qquad 2 < x \le 4,$$
$$G(x) = \frac{x^2 - 5x + 10}{2}, \quad 4 < x \le 6.$$

References

1. W. G. Brady. "Addition to the Summation of Reciprocal Fibonacci and Lucas Series." *Fibonacci Quarterly* *9.4* (1971):402-04.
2. H. W. Gould. "A Rearrangement of Series Based on a Partition of the Natural numbers." *Fibonacci Quarterly* *15.1* (1977):66-77.
3. E. Halsey. "The Fibonacci Number F_u Where u Is Not an Integer." *Fibonacci Quarterly* *3.2* (1965):147-52.
4. A. F. Horadam & A. G. Shannon. "Fibonacci and Lucas Curves." *Fibonacci Quarterly* *26.1* (1988):3-13.
5. E. Lucas. "Theorie des fonctions numériques simplement périodiques." *Amer. J. Math* *1* (1878):184-240.
6. F. D. Parker. "A Fibonacci Function." *Fibonacci Quarterly* *6.1* (1968):1-2.

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MEASURES OF SETS PARTITIONING BOREL'S SIMPLY NORMAL NUMBERS TO BASE 2 IN [0, 1]

John Slivka
State University College at Buffalo, Buffalo, NY 14222

Norman C. Severo
State University of New York at Buffalo, Buffalo, NY 14214

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1. Introduction and Theorem

Let

$$\sum_{i=1}^{\infty} d_i(\omega)2^{-i}, \text{ where } d_i(\omega) = 0 \text{ or } 1 \text{ for } i = 1, 2, \ldots,$$

denote the dyadic expansion of any element ω in the closed unit interval [0, 1]. This expansion is unique except when ω is a dyadic rational

$$(2m - 1)2^{-n},\ m = 1, 2, \ldots, 2^{n-1},\ n = 1, 2, \ldots,$$

in which case there are two such expansions, the terminating one concluding with an unending succession of zeros and the nonterminating one concluding with an unending succession of ones. To insure uniqueness, we quite arbitrarily choose the terminating expansion in such a case.

Of particular interest is the asymptotic behavior of

$$p_m(\omega) \equiv m^{-1} \sum_{i=1}^{m} d_i(\omega),$$

the proportion of ones appearing among the first m dyadic places in the expansion of ω, for $m = 1, 2, \ldots$. Borel [2] asserted that "almost all" ω in [0, 1] have the property that the limiting value of this proportion is 1/2. More precisely, if ν is the Lebesgue measure on the class of Borel measurable subsets of [0, 1] and if

$$S \equiv \{\omega : 0 \le \omega \le 1, \lim_{m\to\infty} p_m(\omega) = 1/2\},$$

then $\nu(S) = 1$. Borel's arguments in support of this impressive fact were flawed, but valid proofs were supplied by later workers (see [1]). The set S defines those numbers in [0, 1] which are said to be simply normal to base 2.

The very definition of simply normal numbers induces rather natural families of partitions of [0, 1]. Motivated by the definition of S and the fact that, for each fixed positive real number ε less than 1/2 (to avoid triviality), the inequality

$$|p_m(\omega) - 1/2| > \varepsilon$$

holds for only finitely many values of m for every ω in S, we can sharpen Borel's landmark result by considering the following measurable functions which, moreover, can be defined for all ω in [0, 1]:

$$\ell(\omega, \varepsilon) \equiv \sup\{m : m = 1, 2, \ldots, \text{ and } p_m(\omega) > 1/2 + \varepsilon\}$$

and

$$n(\omega, \varepsilon) \equiv \sum_{m=1}^{\infty} I(\{\omega : 0 \le \omega \le 1,\ p_m(\omega) > 1/2 + \varepsilon\}),$$

where the supremum of the empty set is 0 and $I(A)$ is the indicator function of the set A. Thus, in the expansion of ω, $\ell(\omega, \varepsilon)$ is the "largest" dyadic place, and $n(\omega, \varepsilon)$ is the total "number" of dyadic places, at which the proportion of ones up to that place exceeds $1/2 + \varepsilon$. Note that these functions assume the value $+\infty$ for infinitely many ω in $[0, 1]$, but Borel's result implies that the sets on which they assume an infinite value have Lebesgue measure zero.

For every ω in S, the values of these functions are nonnegative integers. It is illuminating, therefore, to decompose S according to the values of each of these functions, creating the families of countable partitions $\mathfrak{L}(\varepsilon)$ and $\mathfrak{N}(\varepsilon)$ having respective members

$$L_j \equiv \{\omega : \omega \in S,\ \ell(\omega, \varepsilon) = j\}, \quad j = 0, 1, 2, \ldots,$$

and

$$N_j \equiv \{\omega : \omega \in S,\ n(\omega, \varepsilon) = j\}, \quad j = 0, 1, 2, \ldots .$$

The following theorem gives the Lebesgue measures of the members of each of these partitions when $\varepsilon = k/(2k + 4)$ for any positive integer k.

Theorem: Suppose $\varepsilon = k/(2k + 4)$ for some positive integer k. Then

$$\nu(L_0) = \nu(N_0) = 1 - \gamma_k,$$

and for $j = 1, 2, \ldots,$

$$\nu(L_j) = \left[1 - \gamma_k^{(k+2)([\![j/(k+2)]\!]+1)-j}\right]\binom{j}{[\![j/(k + 2)]\!]}2^{-(j+1)}$$

if $j \neq 0 \bmod (k + 2)$; whereas $\nu(L_j) = 0$ if $j = 0 \bmod (k + 2)$, and

$$\nu(N_j) = (1 - \gamma_k)2^{-j}\sum_{i=0}^{[\![j/(k+2)]\!]}[1 - (k + 2)i/j]\binom{j}{i}.$$

Here, γ_k is the unique solution of $x^{k+2} - 2x + 1 = 0$ in the open interval $(0, 1)$ and $[\![t]\!]$ is the greatest integer not exceeding t.

Remark 1: If $j = r \bmod (k + 2)$, where $r = 0, 1, \ldots, k + 1$, then we have that

$$(k + 2)([\![j/(k + 2)]\!] + 1) - j = k + 2 - r.$$

Remark 2: For $k = 1, 2, 3, 4$, and 5 and $k \to \infty$, the values of $\nu(L_j)$ are tabled in [3] for

$$j = 0, 1, \ldots, \inf\left\{h : \sum_{j=0}^{h}\nu(L_j) \geq 0.9999\right\},$$

and the values of $\nu(N_j)$ are tabled in [7] for

$$j = 0, 1, \ldots, \inf\left\{h : \sum_{j=0}^{h}\nu(N_j) \geq 0.9999\right\}.$$

Remark 3: Our theorem remains true if $p_m(\omega)$ is interpreted as the proportion of zeros appearing among the first m dyadic places in the expansion of ω for $m = 1, 2, \ldots$. Furthermore, since the proportion of zeros exceeds $1/2 + \varepsilon$ if and only if the proportion of ones is less than $1/2 - \varepsilon$, our theorem remains valid when the strict inequalities are reversed and ε is replaced by $-\varepsilon$ in the definitions of $\ell(\omega, \varepsilon)$ and $n(\omega, \varepsilon)$.

Note: Because

$$x^{k+2} - 2x + 1 = (x - 1)\left(\sum_{i=1}^{k+1}x^i - 1\right)$$

and, for $0 < x \leq 1/2$,

$$\sum_{i=1}^{k+1}x^i < 1,$$

γ_k is the unique solution of

$$\sum_{i=1}^{k+1} x^i = 1 \text{ in } (1/2,\ 1) \text{ for every positive integer } k.$$

We now show that $\gamma_k = r_{k+1}^{-1}$, the reciprocal of the $(k+1)^{\text{st}}$ Fibonacci root tabled in [5] for $k = 1, 2, \ldots, 18$. For any positive integer $K \geq 2$, consider the K-generalized Fibonacci numbers defined by $f_K(j) = 0$, for $j = 0, \ldots, K - 2$, $f_K(K - 1) = 1$, and

$$f_K(j) = \sum_{i=1}^{K} f_K(j - i) \text{ for } j = K, K + 1, \ldots,$$

and tabled in [5] for $K = 2, \ldots, 7$ and $j = 0, \ldots, 15$. Miles [6] proved that

$$\lim_{j \to \infty} f_K(j + 1)/f_K(j) = r_K,$$

where r_K is the unique solution of

$$\sum_{i=0}^{K-1} x^i = x^K \text{ in } (1,\ 2).$$

It follows that r_K^{-1} is the unique solution of

$$\sum_{i=1}^{K} x^i = 1 \text{ in } (1/2,\ 1);$$

hence, $\gamma_k = r_{k+1}^{-1}$ for $k = 1, 2, \ldots$.

2. Proof of the Theorem

If S^c denotes the complement of S with respect to [0, 1], then $\nu(S^c) = 0$, and since, for $j = 0, 1, 2, \ldots$,

$$\{\omega : 0 \leq \omega \leq 1,\ \ell(\omega, \varepsilon) = j\} = L_j \cup \{\omega : \omega \in S^c,\ \ell(\omega, \varepsilon) = j\},$$

it follows that

$$\nu(L_j) = \nu(\{\omega : 0 \leq \omega \leq 1,\ \ell(\omega, \varepsilon) = j\}).$$

Similarly, for every nonnegative integer j,

$$\nu(N_j) = \nu(\{\omega : 0 \leq \omega \leq 1,\ n(\omega, \varepsilon) = j\}).$$

Now it is well known (see, e.g., [4], Ex. 4, p. 56) that $\langle d_i(\omega) \rangle$ is a sequence of independent random variables (functions) on [0, 1] for which

$$p \equiv \nu(\{\omega : 0 \leq \omega \leq 1,\ d_i(\omega) = 1\}) = 1/2$$

and

$$q \equiv \nu(\{\omega : 0 \leq \omega \leq 1,\ d_i(\omega) = 0\}) = 1/2$$

for every positive integer i, since $d_i(\omega) = 1$ on 2^{i-1} disjoint intervals each of length 2^{-i}, and similarly for $d_i(\omega) = 0$. Note that

$$\{\langle d_i(\omega) \rangle : 0 \leq \omega \leq 1\}$$

differs from the set of all sequences of zeros and ones only by the set of sequences corresponding to the nonterminating expansions of the set of dyadic rationals mentioned above. As this latter set is countable and, hence, of measure zero, its inclusion or exclusion has no effect in our work.

If we define the Rademacher functions

$$x_i(\omega) = 2d_i(\omega) - 1,\ i = 1, 2, \ldots,$$

so that $\langle x_i(\omega) \rangle$ is a sequence of independent and identically distributed random variables such that $x_i(\omega) = +1$ or -1 with respective probabilities $p = 1/2$ and $q = 1/2$, then $p_m(\omega) > 1/2 + \varepsilon$ if and only if $s_m(\omega) > 2\varepsilon m$, where

$$s_m(\omega) \equiv \sum_{i=1}^{m} x_i(\omega) \text{ for every positive integer } m.$$

Our theorem then follows immediately from the theorems in [3] and [7], where

$$\mu = p - q = 0 \quad \text{and} \quad \lambda = 2\varepsilon = k/(k + 2),\ k = 1, 2, \ldots .$$

3. The Special Case $\varepsilon = 1/6$

The case in which $\varepsilon = 1/6$ ($k = 1$) is particularly attractive since it is the smallest ε dealt with by our theorem and since γ_1, the unique solution of $x^3 - 2x + 1 = 0$ in $(0, 1)$, is $\phi \equiv (\sqrt{5} - 1)/2$, the reciprocal of the ubiquitous golden ratio. In this case, our theorem yields $\nu(L_0) = 1 - \phi = \phi^2$ and, for $j = 0, 1, \ldots,$

$$\nu(L_{3j+1}) = \phi\binom{3j+1}{j}2^{-3j-2},$$

and

$$\nu(L_{3j+2}) = \phi^2\binom{3j+2}{j}2^{-3j-3} = [\phi(3j + 2)/(4j + 4)]\nu(L_{3j+1}),$$

with $\nu(L_{3j+3}) = 0$. Here, the successive values of $\nu(L_{3j+1})$ are most easily computed recursively using $\nu(L_1) = \phi/4$ and the relation

$$\nu(L_{3j+4}) = \frac{3(3j + 4)(3j + 2)}{16(j + 1)(2j + 3)}\nu(L_{3j+1}), \quad j = 0, 1, 2, \ldots .$$

It follows that, for $j = 0, 1, 2, \ldots,$

$$\nu(L_{3j+1}) > \nu(L_{3j+2}) > \nu(L_{3j+3}) = 0$$

and

$$\nu(L_{3j+1}) > \nu(L_{3j+4})$$

so that, for increasing values of the subscript, these measures exhibit an interesting "damped saw-tooth" pattern, each value of j corresponding to a single tooth.

It is noteworthy to observe that

$$\begin{aligned}\phi = 1 - \nu(L_0) &= 1 - \nu(\{\omega : \omega \in S,\ p_m(\omega) \le 2/3 \ \ \forall m = 1, 2, \ldots\}) \\ &= \nu(\{\omega : \omega \in S,\ p_m(\omega) > 2/3 \text{ for some } m = 1, 2, \ldots\}),\end{aligned}$$

that is, the set E of simply normal numbers to base 2 in [0, 1] having the property that the proportion of ones to some dyadic place in their expansion exceeds 2/3 has measure ϕ. Clearly, $S \cap [1/2, 1]$, with measure 1/2, is a subset of E. Yet, E is dense in [0, 1]. For if η is an arbitrarily small but fixed positive real number, then for any

$$\omega = \sum_{i=1}^{\infty} d_i(\omega)2^{-i} \text{ in } [0, 1],$$

consider

$$\omega' = \sum_{i=1}^{N} d_i(\omega)2^{-i} + \sum_{j=1}^{2N+1} 2^{-(N+j)} + \sum_{k=1}^{\infty} 2^{-(3N+2k)},$$

where N is the smallest positive integer such that $2^{-N} < \eta$. Here,

$$p_m(\omega') = m^{-1}\left[\sum_{i=1}^{N} d_i(\omega) + (2N + 1) + [\![(m - 3N)/2]\!]\right], \text{ for } m > 3N + 1,$$

so that $\lim_{m\to\infty} p_m(\omega') = 1/2$; hence, $\omega' \in S$. Moreover,

$$p_{3N+1}(\omega') = (3N + 1)^{-1}\left[\sum_{i=1}^{N} d_i(\omega) + (2N + 1)\right] \ge (2N + 1)/(3N + 1) > 2/3;$$

therefore, $\omega' \in E$. Finally, since ω and ω' agree in the first N dyadic places of their expansions, we have $|\omega' - \omega| \le 2^{-N} < \eta$.

It is also worth noting that the measures of the members of $\mathcal{L}(1/6)$ given above yield a simple formula expressing ϕ in terms of the series

$$y \equiv \sum_{j=0}^{\infty} \binom{3j+1}{j} 2^{-3j} \quad \text{and} \quad z \equiv \sum_{j=0}^{\infty} \binom{3j+2}{j} 2^{-3j}.$$

For,

$$\sum_{j=0}^{\infty} \nu(L_j) = \nu(S) = 1 = \phi^2 + \phi$$

implies $\phi y/4 + \phi^2 z/8 = \phi$; hence, $\phi = 2(4 - y)/z$. Note that

$$y/4 = 1/(\phi\sqrt{5}) \quad \text{and} \quad z/8 = 1/\sqrt{5}.$$

References

1. J. Barone & A. Novikoff. "A History of the Axiomatic Formulation of Probability from Borel to Kolmogorov: Part I." *Arch. Hist. Exact Sci.* *18* (1978): 123-90.
2. É. Borel. "Les probabilités dénombrables et leurs applications arithmétiques." *Rend. Circ. Mat. Palermo*, Ser. 1, *27* (1909):247-71.
3. C.-C. Chao & J. Slivka. "Some Exact Distributions of a Last One-Sided Exit Time in the Simple Random Walk." *J. Appl. Probab.* *23* (1986):332-40.
4. K. L. Chung. *A Course in Probability Theory.* 2nd ed. Orlando, Florida: Academic Press, 1974.
5. I. Flores. "Direct Calculation of k-Generalized Fibonacci Numbers." *Fibonacci Quarterly* *5.3* (1967):259-66.
6. E. P. Miles, Jr. "Generalized Fibonacci Numbers and Associated Matrices." *Amer. Math. Monthly* *67* (1960):745-52.
7. J. Slivka. "Some Density Functions of a Counting Variable in the Simple Random Walk." *Skand. Aktuarietidskr.* *53* (1970):51-57.

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Announcement

FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 20-24, 1992

Department of Mathematical and Computational Sciences
University of St. Andrews
St. Andrews KY169SS
Fife, Scotland

Local Committee

Dr. Colin M. Campbell, Co-Chairman
Dr. George M. Phillips, Co-Chairman

This conference will be sponsored jointly by the Fibonacci Association and the University of St. Andrews. Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations will be welcome. A call for papers will appear in the August 1991 issue of *The Fibonacci Quarterly* as will additional information on the Local and International Committees.

THE G.C.D. IN LUCAS SEQUENCES AND LEHMER NUMBER SEQUENCES

Wayne L. McDaniel
University of Missouri-St. Louis, St. Louis, MO 63121
(Submitted December 1988)

1. Introduction

Let P and Q be relatively prime integers, α and β $(\alpha > \beta)$ be the zeros of $x^2 - Px + Q$, and, for $k = 0, 1, 2, 3, \ldots$, let

$$(1) \qquad U_k = U_k(P, Q) = \frac{\alpha^k - \beta^k}{\alpha - \beta} \quad \text{and} \quad V_k = V_k(P, Q) = \alpha^k + \beta^k.$$

The following result is well known.

Theorem 0: Let m and n be positive integers, and $d = \gcd(m, n)$.

(i) $\gcd(U_m, U_n) = U_d$;

(ii) if $\frac{m}{d}$ and $\frac{n}{d}$ are odd, $\gcd(V_m, V_n) = V_d$;

(iii) if $m = n$, $\gcd(U_m, V_n) = 1$ or 2.

Using basic identities, Lucas proved Theorem 0 in the first of his two 1878 articles in which he developed the general theory of second-order linear recurrences [5]; Lucas had previously proven parts (i) and (iii) in his 1875 article [4]. Nearly four decades later, Carmichael [1] used the theory of cyclotomic polynomials to obtain both new results and results confirming and generalizing many of Lucas' theorems; Theorem 0 was among the results obtained using cyclotomic polynomials.

Curiously, the value of $\gcd(V_m, V_n)$ when m and n are not divisible by the same power of 2, and of $\gcd(U_m, V_n)$ for $m \neq n$, do not appear in the literature, and have, apparently, never been established. It is interesting that the values of all three of these gcd's can be rather easily found, for *all* pairs of positive integers m and n, by the application of an approach similar to that used in establishing the Euclidean algorithm to a single sequence of equations. We shall prove the following result.

Main Theorem: Let $m = 2^a m'$, $n = 2^b n'$, m' and n' odd, a and $b \geq 0$, and let $d = \gcd(m, n)$. Then

(i) $\gcd(U_m, U_n) = U_d$,

(ii) $\gcd(V_m, V_n) = \begin{cases} V_d & \text{if } a = b, \\ 1 \text{ or } 2 & \text{if } a \neq b; \end{cases}$

(iii) $\gcd(U_m, V_n) = \begin{cases} V_d & \text{if } a > b, \\ 1 \text{ or } 2 & \text{if } a \leq b. \end{cases}$

The value of $\gcd(V_m, V_n)$ is even if any only if Q is odd and either P is even or $3|d$; $\gcd(U_m, V_n)$ is even if and only if Q is odd and (1) P and d are even, or (2) P is odd and $3|d$.

Our definition of U_k and V_k assures that the above result holds for all second-order linear recurring sequences $\{U_k\}$ and $\{V_k\}$ satisfying

$$U_0 = 0,\ U_1 = 1,\ U_{n+2} = PU_{n+1} - QU_n,$$

and

$$V_0 = 2,\ V_1 = P,\ V_{n+2} = PV_{n+1} - QV_n.$$

If $P = 1$ and $Q = -1$, the sequences are the Fibonacci and Lucas number sequences, respectively; for this case, a nice alternate proof of (ii) has been communicated to the author by Paulo Ribenboim, and appears now in [6]. If one defines the sequence $\{U_n\}$ more generally, by

$$U_1 = a,\ U_2 = b,\ U_{n+2} = cU_{n+1} + dU_n,$$

then Lucas' result [(i) above] will hold under certain circumstances: P. Horak & L. Skula [2] have characterized those sequences for which (i) holds.

In our last section, we shall observe that a result analogous to Theorem 1 holds for Lehmer numbers and the "associated" Lehmer numbers.

2. Preliminary Results

We base our proof on the following formulas, all of which are well-known, and are easily verified directly from the definition (1) of U_k and V_k.

Property L: Let $r > s \geq 0$, $e = \min\{r - s, s\}$, and $D = P^2 - 4Q$.

L(i) $\quad U_r = V_{r-s}U_s \pm Q^e U_{|r-2s|}$, where the + sign is used iff $r - 2s \geq 0$,

L(ii) $\quad V_r = V_{r-s}V_s - Q^e V_{|r-2s|}$,

L(iii) $\quad U_r = U_{r-s}V_s \pm Q^e U_{|r-2s|}$, where the + sign is used iff $r - 2s < 0$,

L(iv) $\quad V_r = DU_{r-s}U_s + Q^e V_{|r-2s|}$,

L(v) $\quad V_r^2 = DU_r^2 + 4Q^r$.

We will use the fact that, for $k > 0$,

(2) $\quad \gcd(U_k, Q) = \gcd(V_k, Q) = 1$,

which is also readily shown from (1) [or see [1], Th. I].

Finally, we require this result concerning the parity of U_k and V_k, which is easily deduced from (1), using $P = \alpha + \beta$ and $Q = \alpha\beta$ (or see [1], Th. III):

Parity Conditions: If $k = 0$, $U_k = 1$ and $V_k = 2$. Let $k > 0$.

(i) If Q is even, both U_k and V_k are odd;

(ii) If Q is odd and P is even, then V_k is even, and U_k is even iff k is;

(iii) If Q is odd and P is odd, then U_k and V_k are both even iff $3|k$.

3. The Basic Result

Let $\{\gamma_i\}$ and $\{\delta_i\}$ $(i \geq 0)$ be sequences of integers. Let $m_0 = 2^A M$ and $n_0 = 2^B N$ be positive integers with A and $B \geq 0$, M and N odd, and $m_0 > n_0$, and let

$$d_0 = |m_0 - 2n_0| \quad \text{and} \quad d = \gcd(m_0, n_0);$$

let G_{m_0} and H_{n_0} be integers, and K_{d_0} be defined by

$$G_{m_0} = \gamma_0 H_{n_0} + \delta_0 K_{d_0}.$$

Theorem 1: For $j = 1, 2, 3, \ldots$, let

$$m_j = n_{j-1},\ n_j = d_{j-1},\ G_{m_j} = H_{n_{j-1}} \text{ and } H_{n_j} = K_{d_{j-1}}, \quad \text{if } n_{j-1} \geq d_{j-1},$$

or

$$m_j = d_{j-1},\ n_j = n_{j-1},\ G_{m_j} = K_{d_{j-1}} \text{ and } H_{n_j} = H_{n_{j-1}}, \quad \text{if } n_{j-1} < d_{j-1},$$

let $d_j = |m_j - 2n_j|$, and let K_{d_j} be defined by

$$G_{m_j} = \gamma_j H_{n_j} + \delta_j K_{d_j}.$$

If, for $j \geq 0$, $\gcd(G_{m_j}, \delta_j) = 1$, then

$$\gcd(G_{m_0}, H_{n_0}) = \begin{cases}\gcd(H_d, K_d) & \text{if } A = B,\\ \gcd(H_d, K_0) & \text{if } A \neq B.\end{cases}$$

Proof: For each pair of integers r and s, we let (r, s) denote $\gcd(r, s)$. The definitions of m_j, n_j, and d_j imply that $\{m_j\}$ is a nonincreasing sequence of positive integers; let k be the least integer such that $m_{k-1} = m_k$. Now, it is clear, from our definitions above, that

$$(m_0, n_0) = (n_0, d_0) = (m_1, n_1) = (n_1, d_1) = \cdots$$
$$= (m_{k-1}, n_{k-1}) = (n_{k-1}, d_{k-1}).$$

Furthermore, by our assumptions that $G_{m_j} = \gamma_j H_{n_j} + \delta_j K_{d_j}$ and $(G_{m_j}, \delta_j) = 1$, we have, similarly

$$(G_{m_0}, H_{n_0}) = (H_{n_0}, K_{d_0}) = \cdots = (H_{n_{k-1}}, K_{d_{k-1}}).$$

Since, by definition, $m_k = \max\{n_{k-1}, d_{k-1}\}$, $m_{k-1} = n_{k-1}$ or d_{k-1}.

Case 1. If $m_{k-1} = n_{k-1}$, then $d_{k-1} = |m_{k-1} - 2n_{k-1}| = m_{k-1}$ also, so

$$(m_0, n_0) = (n_{k-1}, d_{k-1}) = m_{k-1};$$

that is, $d = m_{k-1} = n_{k-1} = d_{k-1}$. Hence, in Case 1,

$$(G_{m_0}, H_{n_0}) = (H_d, K_d).$$

Case 2. If $m_{k-1} = d_{k-1} \neq n_{k-1}$, then $d_{k-1} = |m_{k-1} - 2n_{k-1}|$ implies $n_{k-1} = 0$. But, then, since $n_{k-1} = \min\{n_{k-2}, d_{k-2}\}$, $d_{k-2} = 0$; this implies

$$d = (m_0, n_0) = (n_{k-2}, 0) = n_{k-2}.$$

Hence, in Case 2,

$$(G_{m_0}, H_{n_0}) = (H_{n_{k-2}}, K_{d_{k-2}}) = (H_d, K_0).$$

For $j \geq 0$, let $M_j = m_j/d$, $N_j = n_j/d$, and $D_j = d_j/d$. If $A = B$, M_0, N_0, and D_0 are each odd; consequently, M_j, N_j, and D_j are odd for $j = 0, 1, 2, 3, \ldots$. This is possible only in Case 1, since, in Case 2, $d_{k-2} = 0$, implying that D_{k-2} is even. If $A \neq B$, it is easy to see that, for each j, exactly one or exactly two of the three integers M_j, N_j, and D_j is (are) even, and this is possible only in Case 2, since, in Case 1, $M_{k-1} = N_{k-1} = D_{k-1}$. This proves the theorem.

4. Proof of the Main Theorem

For $j \geq 0$, we assume that m_j, n_j, d_j, G_{m_j}, H_{n_j}, and K_{d_j} are as defined in Section 3, and M_j, N_j, and D_j are as defined in the proof of Theorem 1. Let $S(r)$ denote the number of integers j, $0 < j \leq k$, such that $n_{j-1} \geq d_{j-1}$, and for each positive integer i, let $p(i)$ denote the parity of i.

Lemma 1: If $A \neq B$, and if there exists an integer k such that $d_k = 0$, then $S(k)$ is even if and only if $A > B$.

Proof: Assume $A \neq B$ and that there exists an integer k such that d_k (and hence, D_k) equals 0. It is clear that the number of integers j, $0 < j \leq k$ such that $N_{j-1} \geq D_{j-1}$ is $S(k)$. Now, $A \neq B$ implies that, for each j,

$$(p(M_j), p(N_j), p(D_j)) = (\text{even, odd, even}) \text{ or } (\text{odd, even, odd}),$$

and it is clear from the definitions of m_j and n_j that $S(k)$ is precisely the number of changes from one of these two forms to the other, as j assumes the values $0, 1, 2, \ldots, k$. Since $d_k = 0$,

$$(p(M_k), p(N_k), p(D_k)) = (\text{even, odd, even});$$

it follows that $S(k)$ is even if and only if M_0 is even; that is, if and only if $A > B$.

Proof of the Main Theorem: Let $e_j = \min\{m_j - n_j,\ n_j\}$.

(i) We assume without loss of generality that $m \geq n$, let $m = m_0$, $n = n_0$, and apply Theorem 1 with $G_{m_0} = U_{m_0}$, $H_{n_0} = U_{n_0}$, $\gamma_j = V_{m_j - n_j}$, and $\delta_j = \pm Q^{e_j}$, where the + sign is chosen if and only if $m_j - 2n_j \geq 0$, for $j \geq 0$. For each $j \geq 0$, $G_{m_j} = \gamma_j H_{n_j} + \delta_j K_{d_j}$ implies that $K_{d_j} = U_{d_j}$, by property L(i); since $(G_{m_j}, \delta_j) = 1$, as observed in Section 2,

$$\gcd(U_m, U_n) = \gcd(U_d, U_d) = U_d, \text{ if } a = b,$$

and

$$\gcd(U_m, U_n) = \gcd(U_d, U_0) = \gcd(U_d, 0) = U_d, \text{ if } a \neq b.$$

(ii) Assume, again without loss of generality, that $m \geq n$, and let $m = m_0$ and $n = n_0$. Defining G_{m_0}, H_{n_0}, K_{d_j}, γ_j, and δ_j as V_{m_0}, V_{n_0}, V_{d_j}, $V_{m_j - n_j}$, and $-Q^{e_j}$, for $j \geq 0$, respectively, we have, by Theorem 1 and L(ii),

$$\gcd(V_m, V_n) = \gcd(V_d, V_d) = V_d \text{ if } a = b,$$

and

$$\gcd(V_m, V_n) = \gcd(V_d, 2) = 1 \text{ or } 2 \text{ if } a \neq b,$$

proving (ii).

(iii) Case 1. Assume $m \geq n$, let $m = m_0$ and $n = n_0$, and define G_{m_0}, H_{n_0}, K_{d_0}, γ_0, and δ_0 as U_{m_0}, V_{n_0}, U_{d_0}, $U_{m_0 - n_0}$ and $\pm Q^{e_0}$, where the + sign is used if and only if $m_0 - 2n_0 < 0$. For $j = 1, 2, 3, \ldots$, let $\gamma_j = DU_{m_j - n_j}$, $\delta_j = Q^{e_j}$, and $K_{d_j} = V_{d_j}$ *if* $G_{m_j} = V_{n_{j-1}}$; and $\gamma_j = U_{m_j - n_j}$, $\delta_j = \pm Q^{e_j}$, and $K_{d_j} = U_{d_j}$ *if* $G_{m_j} = U_{n_{j-1}}$, where the + sign is used if and only if $m_j - 2n_j < 0$. Corresponding to each j ($j \geq 0$), then, $G_{m_j} = \gamma_j H_{n_j} + \delta_j K_{d_j}$ is either L(iii) or L(iv).

If $a = b$, Theorem 1 implies

$$\gcd(U_m, V_n) = \gcd(V_d, U_d) \text{ [or, } \gcd(U_d, V_d)],$$

and it is immediate from (2) and L(v) that this integer is either 1 or 2.

If $a \neq b$, Theorem 1 implies

$$\gcd(U_m, V_n) = \gcd(V_d, U_0) = \gcd(V_d, 0) = V_d,$$

or

$$\gcd(U_m, V_n) = \gcd(U_d, V_0) = \gcd(U_d, 2) = 1 \text{ or } 2.$$

Now, $G_{m_r} = \gamma_r H_{n_r} + \delta_r K_{d_r}$ changes from one of the forms L(iii) or L(iv) to the other as r changes from $j - 1$ to j if and only if $n_{j-1} \geq d_{j-1}$; hence, the number of such changes as j assumes the values $0, 1, 2, \ldots, k$, is $S(k)$. Since $K_{d_0} = U_{d_0}$, the integer k such that $K_{d_k} = U_0$ exists if and only if $S(k)$ is even, and, by Lemma 1, this happens if and only if $a > b$; that is, if $a \neq b$, $\gcd(U_m, V_n) = V_d$ if and only if $a > b$.

Case 2. Assume $n > m$, let $n = m_0$ and $m = n_0$, and define G_{m_0}, H_{n_0}, K_{d_0}, γ_0, and δ_0 to be V_{m_0}, U_{n_0}, V_{d_0}, $DU_{m_0 - n_0}$, and Q^{e_0}, respectively. All the remaining definitions parallel those in Case 1 in the obvious way, and the proof is similar.

The conditions determining whether $\gcd(V_m, V_n)$ or $\gcd(U_m, V_n)$ is 1 or 2 follow immediately from the parity conditions in Section 2.

Letting $F_k = U_k(1, -1)$ and $L_k = V_k(1, -1)$ represent the k^{th} Fibonacci and Lucas numbers, respectively, we have the following corollary.

Corollary: If $m = 2^a m'$, $n = 2^b n'$, m' and n' odd, a and $b \geq 0$, and $d = \gcd(m, n)$, then

(i) $\gcd(F_m, F_n) = F_d$;

(ii) $\gcd(L_m, L_n) = L_d$ if $a = b$, 2 if $a \neq b$ and $3 \mid d$, and 1 if $a \neq b$ and $3 \nmid d$;

(iii) $\gcd(F_m, L_n) = L_d$ if $a > b$, 2 if $a \le b$ and $3|d$, and 1 if $a \le b$ and $3\nmid d$.

5. Lehmer Numbers

Let R be an integer relatively prime to Q. We let α and β denote the zeros of $x^2 - \sqrt{R}x + Q$, and redefine

$$U_k = U_k(\sqrt{R}, Q) = \begin{cases} (\alpha^k - \beta^k)/(\alpha - \beta), & \text{if } k \text{ is odd,} \\ (\alpha^k - \beta^k)/(\alpha^2 - \beta^2), & \text{if } k \text{ is even,} \end{cases}$$

and

$$V_k = V_k(\sqrt{R}, Q) = \begin{cases} (\alpha^k + \beta^k)/(\alpha + \beta), & \text{if } k \text{ is odd,} \\ (\alpha^k + \beta^k), & \text{if } k \text{ is even.} \end{cases}$$

The numbers U_k and V_k were defined by Lehmer, who developed many of the properties of this generalization of Lucas sequences in his 1930 paper [3]. The numbers are known, respectively, as Lehmer numbers and the "associated" Lehmer numbers.

The Main Theorem is true for Lehmer numbers and the associated Lehmer numbers, except that appropriate changes must be made in the statement concerning the parity of the greatest common divisors. We shall not restate the theorem, and refer the reader to [3], Theorem 1.3, for the parity conditions for U_k and V_k.

Both U_k and V_k are prime to Q ([3], Th. 1.1), and it is not difficult to show, directly from the definitions above, the following counterpart of Property L:

Property L': Let $r > s \ge 0$, $e = \min\{r - s, s\}$, and $\Delta = R - 4Q$.

L'(i) $U_r = RV_{r-s}U_s \pm Q^eU_{|r-2s|}$, if r is odd and s is even,
$U_r = V_{r-s}U_s \pm Q^eU_{|r-2s|}$, otherwise;

L'(ii) $V_r = RV_{r-s}V_s - Q^eV_{|r-2s|}$, if r is even and s is odd,
$V_r = V_{r-s}V_s - Q^eV_{|r-2s|}$, otherwise;

L'(iii) $U_r = RU_{r-s}V_s \pm Q^eU_{|r-2s|}$, if r and s are odd,
$U_r = U_{r-s}V_s \pm Q^eU_{|r-2s|}$, otherwise;

L'(iv) $V_r = R\Delta U_{r-s}U_s + Q^eV_{|r-2s|}$, if r and s are even,
$V_r = \Delta U_{r-s}U_s + Q^eV_{|r-2s|}$, otherwise;

L'(v) $RV_r^2 = \Delta U_r^2 + 4Q^r$, if r is odd,
$V_r^2 = R\Delta U_r^2 + 4Q^r$, if r is even.

The + sign is used in L'(i) if and only if $r - 2s \ge 0$, and in L'(iii) if and only if $r - 2s < 0$.

Each of the identities L'(i) through L'(iv) is of the form

$$G_{m_j} = \gamma_j H_{n_j} + \delta_j K_{d_j}.$$

The proof that $\gcd(U_m, U_n)$, $\gcd(V_m, V_n)$, and $\gcd(U_m, V_n)$ are set forth in the Main Theorem is, then, precisely the same as that given in Section 4, with the slight changes required as the above identities replace the identities of Property L.

References

1. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$." *Annals of Math.* 15 (1913):30-70.
2. P. Horak & L. Skula. "A Characterization of the Second-Order Strong Divisibility Sequences." *Fibonacci Quarterly* 23 (1985):126-32.
3. D. H. Lehmer. "An Extended Theory of Lucas' Functions." *Annals of Math.* 31 (1930):419-48.
4. E. Lucas. "Sur la theorie des nombres premiers." *Atti R. Accad. Sc. Torino* (Math). 11 (1875-1876):928-37.
5. E. Lucas. "Theorie des fonctions numeriques simplement periodiques." *Amer. J. Math.* 1 (1878):184-240, 289-321.
6. P. Ribenboim. "Square Classes of Fibonacci and Lucas Numbers." *Port. Math.* 46 (1989):159-75.

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RECURRENT SEQUENCES INCLUDING N

J. H. E. Cohn
Royal Holloway & Bedford New College, Egham, Surrey TW20 0EX, England
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Introduction

Suppose a (large) integer N is given and we wish to choose positive integers A, B such that

(a) the sequence $\{w_n\}$ defined by $w_1 = A$, $w_2 = B$, and $w_{n+2} = w_{n+1} + w_n$, $n \geq 1$, contains the integer N,

(b) $s = A + B$ is minimal.

What can be said about s in relation to N, and how are A and B to be found? We also consider some generalizations.

The case $N = 1{,}000{,}000$ was recently the subject of a problem in a popular computing magazine [1]. Obviously, for $N \geq 2$, $A = 1$, $B = N - 1$ is one pair satisfying (a) and so the problem does have a solution for each N. Also $s \geq 2$, and equality here holds whenever $N = F_k$, a Fibonacci number. Hence,

$$\liminf s = 2 \text{ as } N \to \infty.$$

In the opposite direction, we shall show that $s > \gamma\sqrt{N}$ for infinitely many N, but that for all sufficiently large N, $s < \gamma\sqrt{N} + 0(N^{-1/2})$, where $\gamma = 2/\sqrt{\alpha}$ and $\alpha = (1 + \sqrt{5})/2$. We shall also show how to select A and B for each N.

The Original Problem

Clearly, for a solution to the problem $A \geq B > 0$, for if $B > A$, then the pair $A_1 = B - A$, $B_1 = A$ would yield a smaller s. Starting from A, B, we then obtain, successively, A, B, $A + B$, ..., t, N and we now define, for each $t < N$, the sequence

$$t_0 = N,\ t_1 = t,\ t_{n+2} = t_n - t_{n+1},\ n \geq 0,$$

i.e., work backwards, so to speak, until we arrive at

$$t_k = A + B,\ t_{k+1} = B,\ t_{k+2} = A,\ t_{k+3} \leq 0.$$

Thus, the only choice at our disposal is t; k is then characterized by being the smallest integer for which $t_{k+3} \leq 0$, and our object is to choose t so as to minimize $s = t_k$.

Let α and β be the roots of $\theta^2 = \theta + 1$. Then $\alpha\beta = -1$, $\alpha + \beta = 1$, and

$$F_n = (\alpha^n - \beta^n)/(\alpha - \beta).$$

Then the roots of $\theta^2 = 1 - \theta$ are $-\alpha$ and $-\beta$, so that, for suitable constants c and d,

$$t_n = (-1)^n\{c\alpha^n + d\beta^n\}.$$

Using the initial conditions $t_0 = N$, $t_1 = t$, we then find that

$$(1)\qquad t_n = (-1)^n\{NF_{n-1} - tF_n\}.$$

Also, for $n > 0$,

$$(2)\qquad \alpha F_{n-1} - F_n = -\beta^{n-1} = (-1)^n\alpha^{-n+1},$$

and so

(3) $\quad (-1)^n\{\alpha F_{n-1} - F_n\} > 0.$

We now prove the following.

Theorem: Let

$$t_n = (-1)^n\{NF_{n-1} - tF_n\},$$

where $t_k = A + B$, $t_{k+1} = B$, $t_{k+2} = A$. Then $t = [n/\alpha]$ gives the smallest value for $t_k = A + B = s$ and

$$s < 2\sqrt{(N/\alpha)} \simeq 1.5723\sqrt{N}.$$

There are two cases. Suppose first that $N > \alpha t$. Then

$$t_n = (-1)^n t\{\alpha F_{n-1} - F_n\} + (-1)^n\{N - \alpha t\}F_{n-1} > (-1)^n\{N - \alpha t\}F_{n-1},$$

so t_n can be negative or zero only if n is odd. Thus, k must be even, and if $k = 2K$, then $t_{2K+1} > 0$, $t_{2K+3} \le 0$. Thus, from (1)

$$\frac{F_{2K}}{F_{2K+1}} < \frac{t}{n} \le \frac{F_{2K+2}}{F_{2K+3}}$$

and defining $\rho = N/\alpha - t > 0$, we have

$$\frac{F_{2K+3} - \alpha F_{2K+2}}{\alpha F_{2K+3}} \le \frac{\rho}{N} < \frac{F_{2K+1} - \alpha F_{2K}}{\alpha F_{2K+1}}$$

i.e., in view of (2),

(4) $\quad \alpha^{2K+1}F_{2K+1} < N/\rho \le \alpha^{2K+3}F_{2K+3},$

whence,

$$\alpha^{4K+2} + 1 = \alpha^{2K+1}(\alpha^{2K+1} - \beta^{2K+1}) < N\sqrt{5}/\rho$$
$$\le \alpha^{2K+3}(\alpha^{2K+3} - \beta^{2K+3}) = \alpha^{4K+6} + 1;$$

so

(5) $\quad \alpha^{4K+2} < N\sqrt{5}/\rho - 1 \le \alpha^{4K+6}.$

Also, in this case,

(6) $$\begin{aligned} s = t_{2K} &= NF_{2K-1} - tF_{2K} \\ &= N(F_{2K-1} - F_{2K}/\alpha) + \rho F_{2K} \\ &= N/\alpha^{2K} + \rho F_{2K} = \xi + \eta, \text{ say.}\end{aligned}$$

Of these two terms, ξ is always the larger; in fact, from (4), we have

(7) $$\frac{\alpha F_{2K+1}}{F_{2K}} < \frac{\xi}{\eta} = \frac{N}{\rho\alpha^{2K}F_{2K}} \le \frac{\alpha^3 F_{2K+3}}{F_{2K}},$$

whence

(8) $\quad \alpha^2 < \xi/\eta \le \alpha^6 + 2|\beta|^{2K-3}/F_{2K}.$

We now show that, for all $t < N/\alpha$, $t = [N/\alpha]$ gives the smallest value for s. For, let $t = [N/\alpha]$ and $t' < t$ be any other integer, yielding, respectively, ρ, K, ξ, η, s and ρ', K', ξ', η', and s'. Then $t' \le t - 1$, whence $\rho' \ge \rho + 1$ and, in view of (5), $K' \le K$. If $K' = K$, then $\xi' = \xi$ and $\eta' > \eta$, which gives $s' > s$, whereas, if $K' < K$, then

$$s' = \xi' + \eta' > \xi' \ge \alpha^2\xi = (\alpha + 1)\xi > \xi + \eta = s,$$

in view of (8). Moreover, using (7), we see that

$$\begin{aligned}\frac{s^2}{N} &= \frac{(\xi + \eta)^2}{N} = \frac{\xi\eta}{N}\left\{\frac{\xi}{\eta} + 2 + \frac{\eta}{\xi}\right\} \le \frac{\rho F_{2K}}{\alpha^2}\left\{\frac{\alpha^3 F_{2K+3}}{F_2} + 2 + \frac{F_{2K}}{\alpha^3 F_{2K+3}}\right\}\\ &= \frac{\rho}{\alpha^{2K+3}F_{2K+3}}(\alpha^3 F_{2K+3} + F_{2K})^2\\ &= \frac{\rho(\alpha - \beta)}{\alpha^{4K+6} + 1}\left\{\frac{\alpha^3(\alpha^{2K+3} - \beta^{2K+3}) + (\alpha^{2K} - \beta^{2K})}{(\alpha - \beta)}\right\}^2\\ &= \frac{\rho\alpha^{4K+6}}{\alpha^{4K+6} + 1} \cdot \frac{(\alpha^3 - \beta^3)^2}{(\alpha - \beta)} < 4\rho\sqrt{5}.\end{aligned}$$

Thus,

(9) $\quad s < 2N^{1/2}\rho^{1/2}5^{1/4}$.

The case in which $N < \alpha t$ is entirely similar. Suppressing the details we find that k must be odd, and if $k = 2M - 1$, then with $\sigma = t - N/\alpha$, we obtain

(4′) $\quad \alpha^{2M}F_{2M} < N/\sigma \le \alpha^{2M+2}F_{2M+2}$,

(5′) $\quad \alpha^{4M} < N\sqrt{5}/\sigma + 1 \le \alpha^{4M+4}$,

(6′) $\quad s = N/\alpha^{2M-1} + \sigma F_{2M-1} = \xi + \eta$, say.

(7′) $\quad \dfrac{F_{2M}}{F_{2M-1}} < \dfrac{\xi}{\eta} = \dfrac{N}{\sigma\alpha^{2M-1}F_{2M-1}} \le \dfrac{\alpha^3 F_{2M+2}}{F_{2M-1}}$,

(8′) $\quad \alpha^2 - \beta^{4M-3}\sqrt{5} < \xi/\eta < \alpha^6$.

For all sufficiently large N,

(9′) $\quad s < 2N^{1/2}\sigma^{1/2}5^{1/4} + O(N^{-1/2})$.

At this stage we may immediately make the observation that, for any N, one of σ and ρ lies below 1/2, and so (9) and (9′) immediately give an upper bound of $(2N\sqrt{5})^{1/2} + O(N^{-1/2})$ or approximately $2 \cdot 115N^{1/2}$. It is, however, possible to improve this.

Let us suppose that $\rho/\sigma = \alpha^{-2\theta}$, so that

(10) $\quad \rho = 1/(1 + \alpha^{2\theta})$ and $\sigma = \alpha^{2\theta}/(1 + \alpha^{2\theta})$,

since $\sigma + \rho = 1$. Then, if $\theta \ge 1 - 1/N$, i.e., ρ is small, we use the inequality (9), and if $\theta \le -1 + 1/N$, i.e., σ is small, we use the inequality (9′) and, in either case, obtain

(11) $\quad s < 2N^{1/2}5^{1/4}/(1 + \alpha^2)^{1/2} + O(N^{-1/2}) = \gamma N^{1/2} + O(N^{-1/2})$,

as required. The remaining case is $|\theta| < 1 - 1/N$. Let $N\sqrt{5}/\rho - 1 = \alpha^\lambda$, and let $N\sqrt{5}/\sigma + 1 = \alpha^\mu$. Then a little manipulation yields

$$2\theta > \lambda - \mu > 2\theta - 1/N,$$

and so, certainly, $|\lambda - \mu| < 2$. Then we have, from (5), that $\alpha^\lambda > \alpha^{4K+2}$, i.e., $\lambda > 4K + 2$ and, from (5′), that $4M + 4 > \mu$. Since $\mu + 2 > \lambda$, $4M + 6 > 4K + 2$ and so $M \ge K$. Similarly, we find that $M \le K - 1$, and so all in all $M = K$ or $K - 1$; in other words, the values of k obtained from ρ or σ differ by exactly one. It is easy to see that whichever is the larger value would give the sharper bound for s, but there is no a priori way to determine which does indeed give the larger k. If it is $2K$, then we can improve the bound given by (9), by observing that

$$\lambda < \mu + 2\theta < 4M + 4 + 2\theta,$$

and so the upper bound for ξ/η given by (8) can be improved to $\alpha^{4+2\theta} + O(1/N)$ and then the same argument which led to (9) now leads to

$$\frac{s^2}{N} < \frac{\rho(\alpha^{2+\theta} + \alpha^{-2-\theta})^2}{(\alpha - \beta)} + O(1/N) = \frac{(\alpha^{2\theta} + \alpha^{-2-\theta})^{2\theta}}{(\alpha^2 + 1)\sqrt{5}} + O(1/N)$$
$$= f(\theta) + O(1/N), \text{ say.}$$

In the same way, it is possible to improve the bound if the larger value is given by $2M - 1$, and the corresponding bound for s^2/N is just $f(-\theta) + O(1/N)$. Since we do not know which of these will apply, we must take the larger one, i.e., $g(\theta) = \max\{f(\theta), f(-\theta)\}$. It is quite simple to see that $f(\theta)$ is an increasing function of θ and so the worst case arises from $(1 - 1/N)$, the upper bound for $|\theta|$, giving

$$s^2/N < 4/\alpha + O(1/N),$$

yielding (11) again. This concludes the proof of the theorem.

Now, we show that this bound cannot be reduced. Choosing $N = F_{2n+1}F_{2n+2}$, we find that

$$[N/\alpha] = (\alpha^{4n+2} + \beta^{4n+2} - 3)/5, \quad \rho = (\alpha + \beta^{4n+3})/\sqrt{5},$$
$$\sigma = -(\beta + \beta^{4n+3})/\sqrt{5}, \quad \lambda = 4n + 2, \quad \mu = 4n + 4,$$

and so $K = n - 1$ and $M = n$. Therefore, it follows that the latter gives the larger value for k, and that, in view of (9′),

$$\begin{aligned} s &= N\alpha^{1-2n} + \rho F_{2n-1} \\ &= \frac{(\alpha^{2n+1} - \beta^{2n+1})(\alpha^{2n+2} - \beta^{2n+2})}{5\alpha^{2n-1}} - \frac{(\beta + \beta^{4n+3})(\alpha^{2n-1} - \beta^{2n-1})}{5} \\ &= \frac{1}{5}\{\alpha^{2n+4} + \beta^{2n-2} - \beta^{2n} - \beta^{6n+4} + \alpha^{2n-2} + \beta^{2n} + \beta^{2n+4} + \beta^{6n+4}\} \\ &= \frac{1}{5}(\alpha^{2n+1} - \beta^{2n+1})(\alpha^3 - \beta^3) = F_3F_{2n+1} = 2F_{2n+1}, \end{aligned}$$

and now

$$\frac{s^2}{N} = \frac{4F_{2n+1}}{F_{2n+2}} = \frac{4(\alpha^{2n+1} - \beta^{2n+1})}{(\alpha^{2n+2} - \beta^{2n+2})} > \frac{4}{\alpha}.$$

This concludes the discussion of the original problem.

Generalizations

Several generalizations are now possible. the simplest of these consists of choosing a given integer $a \geq 1$ and replacing the original relations by

(a1) the sequence $\{w_n\}$ defined by $w_1 = A$, $w_2 = B$, and $w_{n+2} = aw_{n+1} + w_n$, $n \geq 1$, contains the integer N,

(b1) $s = aB + A$ is minimal.

This creates but minor changes in the working above. We now let $\alpha > 0$ and $\beta < 0$ be the roots of $\theta^2 = a\theta + 1$ and then $\alpha\beta = -1$, $\alpha + \beta = a$, $\alpha - \beta = (a^2 + 4)^{1/2}$. We define F_n as before *in terms of* α *and* β, although, of course, F_n will no longer be the Fibonacci number. The effects of this are to replace $\sqrt{5}$ wherever it occurs by the new value of $\alpha - \beta$, and to replace the number $2 = F_3$ in formulas (9), (9′), and (11) and in the value of γ, by $a^2 + 1$. The form of the result remains identical, with

$$\gamma = (a^2 + 1)/\sqrt{\alpha} \quad \text{and} \quad \alpha = (a + (a^2 + 4)^{1/2})/2.$$

The details are omitted.

The next generalization we consider consists of replacing the original relations by (a2) the sequence $\{w_n\}$ defined by $w_1 = A$, $w_2 = B$, and $w_{n+2} = aw_{n+1} - w_n$, $n \geq 1$, contains the integer N,

(b2) $s = aB - A > 0$ is minimal.

Here the integer a cannot be 1, otherwise any such sequence would contain only six distinct numbers, or 2, otherwise the problem becomes trivial since we could always take $w_1 = 1$, $w_2 = 2$, and then $w_N = N$ with $s = 3$. So we assume that $a \geq 3$. We now let the roots of $\theta^2 = a\theta - 1$ be

$$\alpha = (a + (a^2 - 4)^{1/2})/2 \quad \text{and} \quad \beta = (a - (a^2 - 4)^{1/2})/2,$$

and then $\alpha\beta = 1$, $\alpha + \beta = a$, with

$$0 < \beta < 1 < \alpha \quad \text{and} \quad \alpha - \beta = \alpha = (a^2 - 4)^{1/2}.$$

Again we let $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, and proceeding as before we let the integer in the sequence before N be t, and obtain A, B, $aB - A$, ..., t, N, and so, if $t_0 = N$, $t_1 = t$, $t_{n+2} = at_{n+1} - t_n$, we get a reverse sequence where

(12) $\quad t_n = tF_n - NF_{n-1}$,

(13) $\quad F_n - \alpha F_{n-1} = \beta^{n-1} > 0$,

(14) $\quad t_n = -(N - t\alpha)F_{n-1} + t\beta^{n-1}$.

What happens now depends on the sign of $(N - t\alpha)$.

Case I. $N > t\alpha$. Then, eventually, t_n becomes negative, and we find that

$$s = F_k,\ F_{k+1} = B,\ F_{k+2} = A, \text{ and } F_{k+3} \leq 0.$$

All this parallels the previous work with only minor differences, and if $\rho = N/\alpha - t$, then we find that

(15) $\quad \alpha^{2k+6} \geq 1 + N(\alpha - \beta)/\rho > \alpha^{2k+4}$,

(16) $$\begin{aligned} s = t_k &= tF_k - NF_{k-1} \\ &= N(F_k/\alpha - F_{k-1}) - \rho F_k \\ &= N/\alpha^k - \rho F_k = \xi - \eta, \text{ say.} \end{aligned}$$

(17) $\quad \alpha^4 < \xi/\eta < \alpha^6 + O(1/N)$.

Unfortunately, it is no longer necessarily the case that $s' > s$ whenever $t' < t = [N/\alpha]$. For we have $t' \leq t - 1$, whence $\rho' \geq \rho + 1$, and so, in view of (15), $k' \leq k$. Now, if indeed $k' < k$, then $s' > s$, for

$$s' = \xi' - \eta' > \xi'(1 - 1/\alpha^4) \geq \alpha\xi(1 - 1/\alpha^4) > \xi > \xi - \eta = s.$$

However, if $k' = k$, then $s' < s$, since now $\rho' > \rho$. Although this is true, we shall see presently that it causes no problems, for then $\rho' > 1$, and in such a case a choice with $t > N/\alpha$ would always yield a smaller s. In any event, we obtain a result analogous to (9),

(18) $\quad s < (a^2 - 1)N^{1/2}\rho^{1/2}(a^2 - 4)^{1/4} + O(N^{-1/2})$.

Case II. $N < t\alpha$, is entirely different. Let $t = N/\alpha + \sigma$. Then

$$t_n = t\beta^{n-1} + \sigma\alpha F_{n-1}$$

is positive for all $n > 0$, and we now need to choose $k = K$ to minimize $s = t_k$. Then $t_K \leq t_{K+1}$ gives, in view of (12),

$$N(F_K - F_{K-1}) \leq (F_{K+1} - F_K)t = (F_{K+1} - F_K)(N/\alpha + \sigma)$$

and so, using (13),

$$(F_{K+1} - F_K)\sigma \geq N(\beta^K - \beta^{K+1})$$

and so

$$(1 - \beta)(\alpha^{K+1} + \beta^{K})\sigma \geq N(\alpha - \beta)(1 - \beta)\beta^{K}$$

which, together with a similar inequality obtained from $t_K \leq t_{K-1}$ yields

(19) $\quad \alpha^{2K-1} \leq N(\alpha - \beta)/\sigma - 1 \leq \alpha^{2K+1},$

and then

$$\begin{aligned} s = t_K &= tF_K - NF_{K-1} \\ &= (N/\alpha + \sigma)F_K - NF_{K-1} \\ &= N/\alpha^K + \sigma F_K = \xi + \eta, \text{ say.} \end{aligned}$$

In this case, it is clear that the smallest s is provided by taking σ as small as possible, and we find, using (19), that the ratio η/ξ lies between α and $(\alpha^{2K} - 1)/(\alpha^{2K+1} + 1) < 1/\alpha$, and so we obtain, as before,

$$\begin{aligned} \frac{s^2}{N} &= \frac{(\xi + \eta)^2}{N} = \frac{\xi\eta}{N}\left\{\frac{\xi}{\eta} + 2 + \frac{\eta}{\xi}\right\} \\ &\leq \frac{\sigma F_K}{\alpha^K}\left\{\frac{\alpha^{2K} - 1}{\alpha^{2K+1} + 1} + 2 + \frac{\alpha^{2K+1} + 1}{\alpha^{2K} + 1}\right\} \\ &= \frac{\sigma\alpha^{2K+1}(\quad + 1)}{(\alpha^{2K+1} + 1)(\alpha - 1)} < \frac{\sigma(\alpha + 1)}{(\alpha - 1)}. \end{aligned}$$

Thus,

(20) $\quad s < N^{1/2}\sigma^{1/2}\left\{\dfrac{1 + \beta}{1 - \beta}\right\}^{1/2}$

and this bound is much better than that provided by (18) unless ρ is extremely small, certainly less than 1. This justifies our earlier remark that we need only consider the smallest value of ρ. Since, at any rate, we can always take $\sigma < 1$ in (20), we obtain immediately

$$s < N^{1/2}\left\{\frac{1 + \beta}{1 + \beta}\right\}^{1/2}.$$

This can be improved slightly, and we prove that $s < N^{1/2}\delta$, where

$$\delta^2 = \frac{1 + \beta}{1 - \beta}\,\frac{1}{1 + \beta^3}.$$

As before, we define θ by $\rho/\sigma = \alpha^{-2\theta}$ obtaining (10), and define λ and μ by

$$N(\alpha - \beta)/\rho - 1 = \alpha^{\lambda} \quad \text{and} \quad N(\alpha - \beta)/\sigma + 1 = \alpha^{\mu},$$

whence

$$2\theta < \lambda - \mu < 2\theta + 1/N.$$

If now $\theta \leq 3/2$, then $\sigma \leq (1 + \beta^3)^{-1/2}$ and then (20) gives the required result, whereas if $\theta > 5/2 - 1/N$, then we find that

$$\rho^2 < \beta^5/(1 + \beta^5) + O(1/N)$$

and then, using (18), we find that

$$\frac{s^2}{N} < \frac{(\alpha^3 - \beta^3)^2}{\alpha - \beta}\,\frac{\beta^5}{1 + \beta^5} + O(1/N),$$

and since $\beta < 1$, the result easily follows. The remaining case is where

$$3 < \lambda - \mu < 5$$

and then, in view of (15) and (19), we find that $2k < \lambda - 4$ and $2K \geq \mu - 1$, whence

$$2(K - k) > \mu - \lambda + 3 > -2,$$

and so, since both k and K are integers, $K \geq k$. Thus, from (16) and (19), we find that

$$s < N/\alpha^k \leq N/\alpha^K < N^{1/2}(1 - \beta^2)^{-1/2} + O(N^{-1/2})$$

and again the result follows.

The following example shows that the result is best possible. Let $N = (F_{n+1} - F_n)L$, where the integer L is to be chosen later. Then

$$N/\alpha = \frac{L}{\alpha - \beta}\{\alpha^n - \alpha^{n-1} - \beta^{n+2} + \beta^{n+1}\}$$
$$= (F_n - F_{n-1})L - L\beta^n(1 - \beta),$$

and so

$$[N/\alpha] = (F_n - F_{n-1})L - 1,$$

provided that $L\beta^n(1 - \beta) < 1$. It is easily seen that this latter condition is equivalent to $L \leq F_{n+1} + F_n$, so we let $L = F_{n+1} + F_n - x$, where $x \geq 0$ is to be chosen later. If we now take $t = (F_n - F_{n-1})L = [N/\alpha]$, then

$$t_r = (F_{n-r+1} - F_{n-r})L,$$

so the least $t_r = t_n = t_{n+1} = L$. On the other hand, if $t = [N/\alpha] - 1$, then

$$t_r = (F_{n-r+1} - F_{n-r})L - F_r,$$

so

$$t_n = L - F_n = F_{n+1} - x,$$
$$t_{n+1} = L - F_{n+1} = F_n - x,$$

and

$$t_{n+2} = F_{n-1} - x(\alpha - 1).$$

Now, if we choose x to be the least integer $\geq F_{n-1}/(\alpha - 1)$, then we find that $k = n - 1$, and the value of t_k exceeds L, the value given for s by the other choice. Hence, for such an N, we obtain

$$\frac{s^2}{N} = \frac{F_{n+1} + F_n - x}{F_{n+1} - F_n} = \frac{(\alpha - 1)(F_{n+1} + F_n) - F_{n-1}}{(\alpha - 1)(F_{n+1} - F_n)} + O(1)$$
$$= \frac{\alpha F_{n+1} - F_n}{(\alpha - 1)(F_{n+1} - F_n)} + O(1)$$
$$= \frac{(1 + \beta^2) - \beta^2}{(1 - \beta + \beta^2)(1 - \beta)} + O(1)$$
$$= \frac{1 + \beta}{1 - \beta}\,\frac{1}{1 + \beta^3} + O(1) = \delta^2 + O(1), \text{ say.}$$

Thus, letting $n \to \infty$, we find that $s < N^{1/2}\delta + O(N^{-1/2})$.

Reference

1. "Leisure Lines." *Personal Computer World* *11.3* (1988):211.

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CONTINUED POWERS AND ROOTS

Dixon J. Jones
University of Alaska Fairbanks, Fairbanks, AK 99775-1110
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1. Introduction

For select real values of p and for real x_i, the expression

$$(1)\qquad \lim_{k\to\infty} x_0 + (x_1 + (x_2 + (\cdots + (x_k)^p \cdots)^p)^p)^p$$

is practically ubiquitous in mathematics. For instance, (1) represents nothing more than the old familiar $\sum_{k=0}^{\infty} x_k$ when $p = 1$. When $p = -1$, it becomes a novel notation for the continued fraction

$$x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots}}}$$

When $p = 0$, the expression is identically 1 (provided that the terms are not all 0).

Not quite ubiquitous, but certainly not rare, is the case $p = 1/2$, in which (1) becomes

$$(2)\qquad \lim_{k\to\infty} x_0 + \sqrt{x_1 + \sqrt{x_2 + \sqrt{\cdots + \sqrt{x_k}}}},$$

a form variously known as an "iterated radical," "infinite radical," "nested root," or "continued root." The literature reveals an assortment of problems involving (2) but only a smattering of other direct references. Of the few treatments of nested square roots as a research topic, one of the sharpest and most thorough is a paper by A. Herschfeld from 1935 [4], wherein he refers to (2) as a "right infinite radical" and derives necessary and sufficient conditions for its convergence. Recently, some of Herschfeld's results have been independently rediscovered [10].

A mathematical construct which includes infinite series, continued fractions, and infinite nested radicals as special cases ought to merit serious investigation. On the other hand, cases of (1) for other powers, for instance $p = 2$, seem likely to produce little more than irritating thickets of nested parentheses, and integer x_k clearly cause rapid divergence. [Herschfeld mentions the form (1), calls it a "generalized right infinite radical," notes the cases $p = 1$ and $p = -1$, states without proof what amounts to a necessary and sufficient condition for the convergence of (1) for $0 < p < 1$, and drops the subject there.] Yet, surprisingly, it turns out that (1) may converge even for very large p; even more surprisingly, there is a sense in which the convergence gets "better" the larger p grows.

In this article we gather and derive some basic properties of expression (1), especially its necessary and sufficient conditions for convergence. (For logistical reasons, we will deal only with positive powers p and nonnegative terms x_k; negative powers, complex terms, and interconnections between the variations represent unmapped territories which appear to be inhabited by interesting results.) We note the peculiar fickleness of infinite series in this context, and we conclude with a few comments interpreting (1) as a special composition of functions.

2. Definitions, Notation, and Qualitative Aspects

Given a sequence $\{x_n | n = 0, 1, 2, \ldots\}$ of real numbers (called *terms*), and given a real number p, define a sequence $\{y_n\}$ by

$$y_k = \mathop{C}_{i=0}^{k}(p, x_i) = x_0 + (x_1 + (x_2 + (\cdots + (x_k)^p \cdots)^p)^p)^p. \tag{3}$$

The limit of y_k as $k \to \infty$ will be called a *continued* (p^{th}) *power*, denoted by $C_{i=0}^{\infty}(p, x_i)$. If the limit exists, the continued power will be said to converge to that limit. (We do not insist that the limit be real, although it will be in what follows, given the assumption of positive terms and powers.) Borrowing from the jargon of continued fractions, $C_{i=0}^{k}(p, x_i)$ will be called the k^{th} *approximant* of the continued power. With the intent of both emphasizing and streamlining their retrograde associativity, we will make a slight deviation from standard notation and write continued powers and their k^{th} approximants, respectively, as

$$\mathop{C}_{i=0}^{\infty}(p, x_i) = x_0 + {}^p(x_1 + {}^p(x_2 + \cdots))$$

and

$$\mathop{C}_{i=0}^{k}(p, x_i) = x_0 + {}^p(x_1 + {}^p(\cdots + {}^p(x_k) \cdots)).$$

Implicit in this notation is the convention ${}^p(x) = x^p$, and the raising of quantities to powers will be effected both ways. For $j \geq 1$, we will call

$$\mathop{C}_{i=j}^{\infty}(p, x_i) = x_j + {}^p(x_{j+1} + {}^p(x_{j+2} + \cdots))$$

and

$$\mathop{C}_{i=j}^{k}(p, x_i) = x_j + {}^p(x_{j+1} + {}^p(\cdots + {}^p(x_k) \cdots))$$

the *truncation at* x_j of a continued power and of its k^{th} approximant, respectively. If the arguments p and x_i are understood in a given discussion, then $C_{i=j}^{k}(p, x_i)$ will be shortened to C_j^k. Note that

$$\mathop{C}_{k}^{k} = x_k \quad (k \geq 0),$$

$$\mathop{C}_{j}^{k} = x_j + {}^p\left(\mathop{C}_{j+1}^{k}\right) \quad (0 \leq j < k).$$

In the event that $p = 1/m$, m a positive integer [or, more loosely, for $m \in (1, \infty)$], we may use the notation developed in [10]:

$$\mathop{C}_{i=0}^{\infty}(p, x_i) = x_0 + \sqrt[m]{(x_1 + \sqrt[m]{(x_2 + \sqrt[m]{(\cdots}}}$$

and will call such an expression a *continued root* (dropping the m, of course, when $m = 2$).

The contrary associativity of a continued power is at the outset perhaps its most prominent and daunting feature. Not only must the evaluation of a finite approximant be performed from right to left, but the kth approximant cannot in general be obtained as a simple function of the $(k-1)^{\text{st}}$; that is, there is in general no simple recursion formula relating C_0^{k-1} to C_0^k. To manipulators of infinite series and continued fractions, this annoyance is less severe than it is to us, because the essentially linear and fractional nature of series and continued fractions permits the elimination of nested parentheses. For most continued powers, however, nonlinearity will subvert or preclude such simplification.

Since computation of the k^{th} approximant "begins" at x_k and "ends" at x_0, one might say that continued powers "end, but never begin" as the number of terms increases without bound. This is in stark contrast to most other infinite constructs (borne for the most part by truly iterated processes) which "begin, but never end." To have an end, but no beginning, seems rather bizarre; perhaps this is because our intuition, abstracted from the natural world, prefers infinite processes with finite origins. After all, anyone who is born can wish never to die, but what sense can be made of the possibility of dying, having never been born? For now, we will accept the informal idea of expressions that "end, but never begin" without dwelling on its deeper implications, lest by sheer grammatical duality the familiar processes that "begin, but never end" come to look equally doubtful.

3. Continued Powers of Constant Terms

Continued powers turn up in the literature often as continued square roots having constant terms, as in the formula (mentioned in [8]) for the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2} = \sqrt{}(1 + \sqrt{}(1 + \sqrt{}(1 + \sqrt{}(\cdots$$

Such expressions invite consideration of continued powers of the form

$$\mathop{\mathrm{C}}_{i=0}^{\infty}(p,\, a) = a + {}^{p}(a + {}^{p}(a + \cdots\ )).$$

For a given $p > 0$, what values of $a \geq 0$ (if any) will make this continued power converge?

To answer this question, we conjure up an insight so useful that in one way or another it makes possible all of our later results: namely, *the order of operations can be reversed in a continued power of constant terms*. That is, the evaluation of a finite approximant may be performed by associating either to the right or to the left when all the terms are equal, as the following construction demonstrates:

$$\begin{aligned} a &= a \\ a + {}^{p}(a) &= (a)^{p} + a \\ &\ \vdots \\ a + {}^{p}(\cdots + {}^{p}(a + {}^{p}(a))\ \cdots) &= (\cdots\ ((a)^{p} + a)^{p} + \cdots)^{p} + a \end{aligned} \tag{4}$$

where each side of the last line has the same number of terms. *Note that this does not work if the terms are not equal*. If you index the terms as you add them, you will find that neither the left- nor right-hand expressions are approximants of a continued power.

As mentioned in Section 2, associativity in the "wrong" direction is the main impediment to the study of continued powers in general. The appeal of the present situation lies in the fact that a continued power of constant terms is equivalent to a form whose associativity proceeds in the "right" direction, and whose convergence can be studied using known techniques. The tool we will make most use of is the algorithm known in numerical analysis as "successive approximation" or "fixed-point iteration"; for those whom it may benefit, we briefly synopsize this algorithm and its properties. In fixed-point iteration, a generating function g is defined on an interval I, a starting point w_0 is chosen in I, and a sequence $\{w_k\}$ is generated by $w_k = g(w_{k-1})$ for $k = 1, 2, 3, \ldots$. The sequence $\{w_k\}$ converges to an (*attracting*) *fixed point* λ in I [with the property that $g(\lambda) = \lambda$], provided that g and I satisfy certain conditions. For our purposes the following conditions due to Tricomi (mentioned in [3]) will suffice, although others are known (cf. [5]):

(i) $g(x)$ must be continuous on the (closed, half-open, or open) interval I;

(ii) there must exist a number $\lambda \in I$ such that $g(\lambda) = \lambda$;

(iii) $|g(x) - \lambda| < |x - \lambda|$ for all $x \in I$, $x \neq \lambda$.

Despite notational vagaries, it is no secret ([7], [9]) that, for $p = 1/2$, the expression $(((a)^p + a)^p + a)^p + \cdots$ is simply an "unabbreviated" fixed-point algorithm generated by $g(x) = g_a(x) = x^p + a$ at the starting point $x = 0$. Extending this interpretation to the general case, we invoke the identity (4) to claim that the convergence of $C_{i=0}^{\infty}(p, a)$ depends only on $g_a(x)$ and a suitable interval I containing the starting point $x = 0$ and the fixed point λ. In fact, C_0^{∞} converges just when g and I conform to conditions (i), (ii), and (iii) above. With this strategy in hand we obtain

Theorem 1: The continued p^{th} power with nonnegative constant terms $x_n = a$ converges if and only if

$a \geq 0$ for $0 < p < 1$;
$a = 0$ for $p = 1$; and
$0 \leq a \leq R$ for $p > 1$

where
$$R = \sqrt[p-1]{\frac{(p-1)^{p-1}}{p^p}}$$

The set $[0, \infty)$ will be called the *interval of convergence* for a continued p^{th} power, $0 < p < 1$. Likewise $\{0\}$ and $[0, R]$ will be the intervals of convergence for $p = 1$ and $p > 1$, respectively.

Proof: The case $p = 1$ is trivial, since the only value of a for which $\sum_{i=0}^{\infty} a$ is finite is $a = 0$, and $R = 0$ when $p = 1$. Indeed, $C_{i=0}^{\infty}(p, a)$ converges whenever $a = 0$ for any $p > 0$.

For $g_a(x) = x^p + a$ and $p > 0$, continuity is not an issue for x and a in R^+. Condition (i) is satisfied by any positive interval.

Condition (iii) is fulfilled for $0 < p < 1$ and $p > 1$, since in both cases the function $g_a(x) = x^p + a$ is strictly increasing, and it is easily shown that either $\lambda > g_a(x) > x$ or $\lambda < g_a(x) < x$ for $x \neq \lambda$ in the interval(s) I which pertain.

The remainder of the proof, then, involves determining those intervals I and establishing the existence of $\lambda \in I$ for positive $p \neq 1$. Because the functions involved are very well-behaved, we offer remarks about their graphs rather than detailed derivations of their properties. Essentially, the problem is to determine how far a power function can be vertically translated so that it always possesses an attracting fixed point.

$0 < p < 1$. The curve $y = g_a(x) = x^p + a$ (typified by $y = \sqrt{x} + a$) is strictly increasing, concave downward, and vertically translated $+a$ units. For $a > 0$, take $I = [0, \infty)$. From a graph, it is clear that $y = g_a(x)$ intersects $y = x$ exactly once in I, at the point $x = \lambda = g_a(x)$. (For a treatment of this case when $p = 1/2$ and a is complex, see [11].)

$p > 1$. Here the curve $y = g_a(x)$ is exemplified by $y = x^2 + a$; it is concave upward, strictly increasing, and elevated a units. There is a point $a = R$ at which $y = g_a(x)$ is tangent to $y = x$; for $a > R$, the two curves do not intersect; hence, no $\lambda = g_a(\lambda)$ exist.

When $a = R$, λ is the point of tangency of $y = g_R(x)$ and $y = x$. The derivative of $g_R(x)$ is 1 at $x = \lambda$, whereby $\lambda = X = (1/p)^{1/(p-1)}$. Then, from $\lambda = g_R(\lambda) = \lambda^p + R$ and $\lambda = X$, we find
$$R = X - X^p = \left(\frac{1}{p}\right)^{\frac{1}{p-1}} - \left(\frac{1}{p}\right)^{\frac{p}{p-1}} = \left(\frac{1}{p}\right)^{\frac{p}{p-1}}(p-1) = \sqrt[p-1]{\frac{(p-1)^{p-1}}{p^p}}.$$

This form for R was chosen to foreshadow a recurrent theme in the field of continued powers, namely the persistent appearance of expressions of the form A^A/B^B. At any rate, for $a = R$, take $I = [0, X]$.

Finally, when $0 \le a < R$, $y = g_a(x)$ intersects $y = x$ at two points lying on either side of the point X. Take $I = [0, X]$, so that the single intersection point less than X is the point $\lambda \in I$ satisfying condition (ii). We have shown that $g_a(x)$ generates convergent fixed-point algorithms over $I = [0, X]$ for $0 \le a \le R$, which ends the proof.

Theorem 1 reveals that, for instance

$$\overset{\infty}{\underset{i=0}{\mathcal{C}}} (2, a) = a + {}^2(a + {}^2(a + \cdots))$$

converges for any $a \in [0, 1/4]$; the proof shows that

$$\overset{\infty}{\underset{i=0}{\mathcal{C}}} \left(2, \frac{1}{4}\right) = \frac{1}{2}.$$

One may show that as $p \to \infty$ the point $R \to 1$, hence the interval of convergence grows larger as p increases beyond 1. In this context, we can reasonably say that the convergence of a continued p^{th} power gets "better" as p grows large, and is "worst" for the famous case $p = 1$, namely infinite series.

4. Continued Powers of Arbitrary Terms; $0 < p < 1$

The first discussion of the convergence of the continued square root

$$\overset{\infty}{\underset{i=0}{\mathcal{C}}} \left(\frac{1}{2}, x_i\right) = x_0 + \surd(x_1 + \surd(x_2 + \surd(\cdots$$

appears to have been made in 1916 by Pólya & Szegö [8], who showed that it converges or diverges accordingly as

$$\limsup_{n \to \infty} \frac{\log \log x_n}{n}$$

is less than or greater than log 2. This result was encompassed by a theorem of Herschfeld, which gives a necessary and sufficient condition for the convergence of a continued square root and which easily generalizes to the main theorem of this section.

Theorem 2: For $0 < p < 1$, the continued p^{th} power with terms $x_n \ge 0$ converges if and only if $\{x_n^{p^n}\}$ is bounded.

This follows simply by substitution of $1/p^{\text{th}}$ roots for square roots in Herschfeld's proof of the case $p = 1/2$. In lieu of a proof by plagiarism, we merely convey the proof's salient features; and to that end, let us take a moment to establish three useful properties of approximants and their truncations. (Remember that $\{x_n\}$ is nonnegative and p is positive in what follows.) First, successive truncations of the approximant C_0^k conform to the inequality

$$\overset{k}{\underset{j}{\mathcal{C}}} \ge \left(\overset{k}{\underset{j+1}{\mathcal{C}}}\right)^p \quad (0 \le j \le k - 1)$$

which follows from $C_j^k = x_j + (C_{j+1}^k)^p$. Furthermore, the approximants form a nondecreasing sequence:

$$\overset{k+1}{\underset{0}{\mathcal{C}}} \ge \overset{k}{\underset{0}{\mathcal{C}}} \quad (k \ge 0).$$

To see this, start with $x_k + {}^p(x_{k+1}) \ge x_k$ and construct each approximant backwards to x_0. Finally, from the formula

$$\overset{k}{\underset{0}{C}} = x_0 + {}^p\!\left(x_1 + {}^p\!\left(\cdots + {}^p\!\left(x_{j-1} + {}^p\!\left(\overset{k}{\underset{j}{C}}\right)\right)\ \cdots\right)\right)$$

it is clear that a continued power converges if any truncation converges.

The necessity of Theorem 2 is easily proved by applying the inequality for successive truncations n times to C_0^n and letting $n \to \infty$:

$$\overset{n}{\underset{0}{C}} \geq \left(\overset{n}{\underset{n}{C}}\right)^{p^n} = x_n^{p^n}$$

$$\overset{\infty}{\underset{0}{C}} \geq \lim_{n\to\infty} x_n^{p^n}.$$

C_0^∞ converges; hence, $\{x_n^{p^n}\}$ is bounded.

On the other hand, suppose there is an $M > 0$ such that $x_n^{p^n} \leq M$ for all $n \geq 0$ or, equivalently, $x_n \leq M^{p^{-n}}$. With this, one can construct the inequality

$$x_0 + {}^p(x_1 + {}^p(\cdots + {}^p(x_n)\ \cdots)) \leq M + {}^p(M^{p^{-1}} + {}^p(\cdots + {}^p(M^{p^{-n}})\ \cdots)).$$

Multiplying the right side by M/M and distributing the denominator through the successive parentheses results in

$$\overset{n}{\underset{i=0}{C}}(p,\ x_i) \leq M[1 + {}^p(1 + {}^p(\cdots + {}^p(1)\ \cdots))]$$

or

$$\overset{n}{\underset{i=0}{C}}(p,\ x_i) \leq M\overset{n}{\underset{i=0}{C}}(p,\ 1).$$

The continued root on the right converges as $n \to \infty$, because 1 is in the set of constants for all continued roots. The nondecreasing approximants on the left are therefore bounded; hence, $C_{i=0}^\infty(p,\ x_i)$ converges, which finishes the proof.

The condition of Theorem 3 is met by most common sequences. An example of a divergent continued root is

$$\overset{\infty}{\underset{i=0}{C}}\left(\frac{1}{3},\ 2^{4^i}\right) = 2 + \sqrt[3]{(2^4 + \sqrt[3]{(2^{16} + \sqrt[3]{(2^{64} + \sqrt[3]{(\ \cdots}}}}$$

where the sequence of terms fails the "upper bound" test: $(2^{4^n})^{p^n} = 2^{(4/3)^n} \to \infty$.

5. Continued Powers of Arbitrary Terms; $p \geq 1$

As p exceeds the critical value 1, continued p^{th} powers converge with markedly lower enthusiasm. They behave stubbornly, although not pathologically—for, given the hypotheses of this discourse, we are favored at least with a nondecreasing sequence of approximants—and in one sense the most reticent examples are infinite series ($p = 1$). In this section we will show that, among other things, the better-known convergence tests for series are just limiting cases of conditions which hold for general continued p^{th} powers ($p > 1$).

For instance, it is common knowledge that, if an infinite series converges, then its n^{th} term must approach zero. The analogous property for continued powers is summarized in

Theorem 3: For $p > 1$, the continued p^{th} power with terms $x_n \geq 0$ and interval of convergence $[0,\ R]$ converges if $\limsup x_n < R$. For $p \geq 1$, it diverges if $\liminf x_n > R$.

Proof: We first prove the latter assertion. If $\liminf x_n = B > R$, then for each $\varepsilon > 0$ there is a natural number N such that $B - \varepsilon < x_n$ for all $n \geq N$. In particular, choose $\varepsilon = \varepsilon_0 > 0$ such that $R < B - \varepsilon_0 < x_n$, and for convenience, set $v = B - \varepsilon_0$. Then use $v < x_n$ for all $n \geq N$ to construct

$$v + {}^p(v + {}^p(\cdots + {}^p(v)\ \cdots)) < x_N + {}^p(x_{N+1} + {}^p(\cdots + {}^p(x_n)\ \cdots)).$$

More compactly we have, in the limiting case,

$$\underset{i=N}{\overset{\infty}{C}} (p, v) \le \underset{i=N}{\overset{\infty}{C}} (p, x_i).$$

But $C_{i=N}^{\infty}(p, v)$ diverges, because $v = B - \varepsilon_0$ is greater than R and not in the interval of convergence. Therefore, the truncation $C_{i=N}^{\infty}(p, x_i)$ diverges, and likewise the entire continued power.

A similar argument shows that, if $\limsup x_n = B < R$, the continued power converges. However, if $R = 0$, we would be assuming that $\limsup x_n = B < 0$, which for a nonnegative sequence is a malfeasance. By excluding the case $p = 1$ (for which $R = 0$), we salvage this argument and complete the proof.

We come now to a situation wherein continued powers show substantially greater resistance to examination. The deep questions of our present line of inquiry involve powers greater than one and terms x_n for which

$$\liminf x_n \le R \le \limsup x_n.$$

One of the simplest examples with these features is the continued square

$$\underset{i=0}{\overset{\infty}{C}} (2, t_i),$$

where we have nonnegative constants a and b such that $t_{2i+1} = a$, $t_{2i} = b$, and $a \le 1/4 \le b$ ($R = 1/4$ for a continued square). That is,

$$\underset{i=0}{\overset{\infty}{C}} (2, t\ ) = b + {}^2(a + {}^2(b + {}^2(a + \cdots))).$$

Our approach to this example parallels the development of Section 3. The problem of "backwards" associativity is overcome by the identities

$$\begin{aligned}(5)\qquad & b + {}^2(a + {}^2(\cdots + {}^2(a + {}^2(b))\cdots)) \\ &\qquad = ((\cdots((b)^2 + a)^2 + \cdots)^2 + a)^2 + b,\end{aligned}$$

where each side has the same odd number of terms, and

$$\begin{aligned}(6)\qquad & b + {}^2(a + {}^2(\cdots + {}^2(b + {}^2(a))\cdots)) \\ &\qquad = ((\cdots((a)^2 + b)^2 + \cdots)^2 + a)^2 + b,\end{aligned}$$

where each side has the same even number of terms. The right-hand sides of these equations can each be thought of as an unabbreviated fixed-point algorithm generated by the function $g_{a,b}(x) = (x^2 + a)^2 + b$; in equation (5) the starting point is $x = b$, while in (6) it is $x = 0$. We want this algorithm to converge to the same limit regardless of the point at which it starts. Under our hypotheses, $g_{a,b}$ is positive, strictly increasing, and "concave upwards" in $\mathbf{R}^+$; a and b are not both 0; thus, it follows that there is a unique point in $\mathbf{R}^+$ where the derivative of $g_{a,b}$ equals 1. This leads to the equation $4x^3 + 4ax - 1 = 0$, having a single positive real solution which we call γ (stated explicitly below).

The convergence of the fixed-point algorithm using $g_{a,b}$ can now be assured. For $b = \gamma - (\gamma^2 + a)^2$, the unique attracting fixed point in $\mathbf{R}^+$ of $g_{a,b}$ is the point of tangency of $y = g_{a,b}(x)$ and $y = x$. When $b < \gamma - (\gamma^2 + a)^2$, $y = g_{a,b}(x)$ intersects $y = x$ in two points lying on either side of $x = \gamma$, and the left one is the desired attracting fixed point. The interval $I = [0, \gamma]$ maps into itself, and since both 0 and b are contained in I, they may be used as starting points for a convergent fixed-point algorithm using $g_{a,b}$. Thus, we are led to the following

Proposition: For $0 \le a \le 1/4 \le b$, the continued square

$$b + {}^2(a + {}^2(b + {}^2(a + \cdots)))$$

converges if and only if $b \le \gamma - (\gamma^2 + a)^2$, where

$$\gamma = \sqrt[3]{\frac{1}{8} + \sqrt{\frac{1}{64} + \frac{a^3}{27}}} + \sqrt[3]{\frac{1}{8} - \sqrt{\frac{1}{64} + \frac{a^3}{27}}}.$$

(The reader may find it entertaining to show by this Proposition that $b = 1/4$ if $a = 1/4$, as Theorem 1 requires.) This is not a particularly graceful conclusion to an admittedly rough sketch. But not much more elegant, and considerably less specific, is the generalization to powers other than 2, via the same argument.

Theorem 4: Given $p > 1$, interval of convergence $[0, R]$, and $0 \le a \le R \le b$, the continued p^{th} power

$$b + {}^p(a + {}^p(b + {}^p(a + \cdots)))$$

converges if and only if $b \le \gamma - (\gamma^p + a)^p$, where γ is the unique root in $\mathbf{R}^+$ of $p^2(x^{p+1} + ax)^{p-1} - 1 = 0$.

And so the simplest continued power for which $\liminf x_n \le R \le \limsup x_n$ leads to a result whose application will in most cases require solution of an equation by numerical approximation. Worse yet, note that Theorem 4 has virtually no relevance to

$$b + {}^p(b + {}^p(a + {}^p(b + {}^p(b + {}^p(a + \cdots)))))$$

or to similar constructions in which various arrangements of two constants make up the sequence of terms. Such apparitions are manageable to the extent that we *can* find generating functions for equivalent fixed-point algorithms; these functions and their derivatives, however, are not likely to be pleasant to work with, especially for noninteger p.

On the other hand, one should not be left believing that the situation is near hopeless when $\liminf x_n \le R \le \limsup x_n$. For instance, satisfying results are attainable for a continued power whose terms monotonically decrease to R. Subsumed by this special case are (not necessarily convergent) infinite series whose terms decrease to 0. Just as the ratio of consecutive terms sometimes imparts useful information about the convergence of series, so too does a kind of "souped-up" ratio test apply to continued p^{th} powers. In fact, the continued powers test almost reduces to d'Alembert's ratio test for series as $p \to 1$, but the precarious nature of infinite sums considered as special continued powers causes an interesting and instructive discrepancy.

Theorem 5: For $p > 1$, the continued p^{th} power with terms $x_n > 0$ converges if

$$\frac{(x_{n+1})^p}{x_n} \le \frac{(p-1)^{p-1}}{p^p}$$

for all sufficiently large values of n.

Proof: Assume the validity of the ratio test (for $n \ge 0$, without loss of generality) in the form $(x_{n+1})^p \le cx_n$, where $c = (p-1)^{p-1}/p^p$. Using this inequality, a proof by induction on the index k ($k \le n$) shows that

$$\text{(7)} \qquad \underset{n-k}{\overset{n}{\mathbf{C}}} \le (x_{n-k})[1 + c\ {}^p(1 + c\ {}^p(\cdots + c\ {}^p(1 + c)\ \cdots))],$$

where the number of c's on the right is k. When $k = n$, (7) becomes

$$\text{(8)} \qquad \underset{0}{\overset{n}{\mathbf{C}}} \le x_0[1 + c\ {}^p(1 + c\ {}^p(\ \cdots + c\ {}^p(1 + c)\ \cdots))],$$

where the number of c's is now n. The right side of (8) contains a variation on a continued power of constants, equivalent to an unabbreviated fixed-point algorithm generated by the function $g(x) = 1 + cx^p$ at the starting point $x = 0$:

$$(9) \qquad 1 + c^{\,p}(1 + c^{\,p}(\cdots + c^{\,p}(1 + c) \cdots)) = ((\cdots (c + 1)^p c + \cdots)^p c + 1)^p c + 1,$$

where both sides are of equal length. By applying the conditions (i), (ii), and (iii) from Section 3, this algorithm can be shown to converge on the interval $I = [0, p/(p - 1)]$, which just manages to include both the starting point $x = 0$ and the fixed point $\lambda = p/(p - 1)$. Thus, the right side of (9) converges in the limiting case to $p/(p - 1)$, which when combined with (8) shows that

$$(10) \qquad \lim_{n\to\infty} \overset{n}{\underset{0}{C}} \le x_0\left(\frac{p}{p-1}\right).$$

We therefore infer the congruence of C_0^∞, which completes the proof.

The continued square $C_{i=0}^\infty(2, 4^{(2^{-i}-1)})$ is an example of a continued power which converges by the test of Theorem 5. The sequence of terms

$$\{1, 4^{-1/2}, 4^{-3/4}, 4^{-7/8}, \ldots\}$$

satisfies the inequality $(x_{n+1})^2/x_n \le 1/4$; in fact, equality holds for all n. That the ratio test is not necessary for convergence, even when the terms decrease monotonically, is demonstrated by

$$\overset{\infty}{\underset{i=0}{C}} \left(2, \frac{1}{2} + 2^{-i}\right),$$

which converges by comparison with the other continued square mentioned above. (The proof depends on the inequality

$$\frac{1}{4} + 2^{-(n+2)} < 4^{(2^{-n}-1)},$$

whose verification is a mildly interesting exercise in its own right.) The terms $x_n = 1/4 + 2^{-n}$ satisfy the necessary condition lim inf $x_n = 1/4$, but fail the ratio test for all n because

$$(x_{n+1})^2/x_n = \frac{1}{4} + 1/(2^{2n} + 2^{n+2}).$$

Since $(p - 1)^{p-1}/p^p \to 1$ as $p \to 1$, Theorem 5 seems to tell us that an infinite series converges if $x_{n+1}/x_n \le 1$. The many erroneous aspects of this conclusion arise because the fixed point of $g(x) = 1 + cx^p$, namely $\lambda = p/(p - 1)$, ceases to be finite when $p = 1$. Thus, in the inequality (10), the series is not bounded, and the construction used to prove the ratio test becomes indeterminate.

6. Continued Powers as Function Compositions

The analytic theory of continued fractions has long recognized that continued fractions, infinite series, and even infinite products can be defined in the complex plane by means of the composition

$$(11) \qquad F_k(w_0) = f_0 \circ f_1 \circ \cdots \circ f_k(w_0)$$

of linear fractional transformations

$$f_k(w) = \frac{a_k + c_k w}{b_k + d_k w}, \quad k = 0, 1, 2, \ldots,$$

by suitable choices of a_k, b_k, c_k, and d_k [6]. Many other constructs can be defined similarly using different functions for the f_k. For instance $f_k(w) = a_k + tw$ and $w_0 = 0$ produces polynomials in t. For real x, $f_k(x) = (a_k)^x$, with $a_k > 0$, $k = 0, 1, 2, \ldots$, generates what is sometimes called a "tower" or a "continued exponential":

$$F_k(1) = a_0^{a_1^{a_2^{\cdot^{\cdot^{\cdot^{a_k}}}}}}$$

where evaluation is made from the top down ([1], [2]).

This paper has investigated the limiting behavior of (11) when

$$f_k(x) = x_k + x^p, \text{ with } x \geq 0,\ p > 0, \text{ and } x_k \geq 0 \text{ for } k = 0, 1, 2, \ldots .$$

The order of composition in (11) is synonymous with the problematical associativity of continued powers. In retrospect, our progress depended on establishing the convergence of (11) for the special case $f_0 = f_1 = \cdots = f_k = g$, where we variously used $g(x) = x^p + a$, $g(x) = (x^p + a)^p + b$, and $g(x) = 1 + cx^p$. In these cases the composition (11) reduces to

$$F_k(0) = g \circ g \circ \cdots \circ g(0)$$

whose handy recursion formula

$$F_k(0) = g \circ F_{k-1}(0)$$

paves the way for conquest by fixed-point algorithms. This method promises to be helpful in exploring continued negative powers and other function-compositional objects that distinguish themselves by uncooperatively nesting their operations.

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References

1. D. F. Barrow. "Infinite Exponentials." *Amer. Math. Monthly* *43* (1936):150-160.
2. M. Creutz & R. M. Sternheimer. "On the Convergence of Iterated Exponentiation—I." *Fibonacci Quarterly* *18* (1980):341-47.
3. J. B. Diaz & F. T. Metcalf. "On the Set of Subsequential Limit Points of Successive Approximations." *Trans. Amer. Math. Soc.* *135* (1969):459-85.
4. A. Herschfeld. "On Infinite Radicals." *Amer. Math. Monthly* *42* (1935):419-429.
5. E. Isaacson & H. B. Keller. *Analysis of Numerical Methods*. New York: John Wiley, 1966, pp. 85-91.
6. W. B. Jones & W. J. Thron. *Continued Fractions: Analytic Theory and Applications*. London: Addison-Wesley, 1980, p. 11.
7. C. S. Ogilvy. "To What Limits Do Complex Iterated Radicals Converge?" *Amer. Math. Monthly* *77* (1980):388-89.
8. G. Pólya & G. Szegö. *Problems and Theorems in Analysis*. Vol. I. New York: Springer-Verlag, 1972.
9. E. J. Purcell & D. Varberg. *Calculus with Analytic Geometry*. 4th ed. Englewood Cliffs, N.J.: Prentice Hall, 1984, p. 448.
10. W. S. Sizer. "Continued Roots." *Math. Magazine* *59* (1986):23-27.
11. P. L. Walker. "Iterated Complex Radicals." *Math. Gazette* *67* (1983):269-73.

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GENERALIZED STAGGERED SUMS

A. G. Shannon
University of Technology, Sydney, 2007, Australia

A. F. Horadam
The university of New England, Armidale, 2351, Australia
(Submitted January 1989)

1. Introduction

Wiliam [8] showed that, for the recurring sequence defined by $u_1 = 0$, $u_2 = 1$, and

(1.1) $$u_{n+2} = au_n + bu_{n+1},$$

(1.2) $$\sum_{n=1}^{\infty} u_n/10^n = 1/(100 - 10b - a),$$

where $(b+a)/20$ and $(b-a)/20$ are less than 1 and $b = \sqrt{b^2 + 4a}$ (cf. [8]). Thus, for the Fibonacci numbers defined by the same initial conditions and $a = b = 1$, we get the "staggered sum" of Wiliam:

(1.3) $$.0 + .01 + .001 + .0002 + .00003 + \cdots = 1/89.$$

It is the purpose of this note to generalize the result for arbitrary-order recurring sequences, and to relate it to an arithmetic function of Atanassov [1].

2. Arbitrary-Order Sequence

More generally, for the linear recursive sequence of order k, defined by the recurrence relation

(2.1) $$u_n = \sum_{j=1}^{k} (-1)^{j+1} P_j u_{n-j}, \quad n > 1,$$

where the P_j are integers, and with initial conditions $u_0 = 1$ and $u_n = 0$ for $n < 0$, we can establish that the formal generating function is given by

(2.2) $$\sum_{n=0}^{\infty} u_n x^n = (x^k f(1/x))^{-1},$$

where $f(x)$ denotes the auxiliary polynomial

(2.3) $$f(x) = x^k + \sum_{j=1}^{k} (-1)^j P_j x^{k-j}.$$

Proof: If $$u(x) = u_0 + u_1 x + u_2 x^2 + \cdots + u_k x^k + \cdots,$$
then $$-P_1 x u(x) = -P_1 u_0 x - P_1 u_1 x^2 - \cdots - P_1 u_{k-1} x^k - \cdots,$$
and $$(-1)^k x^k P_k u(x) = (-1)^k P_k u_0 x^k + \cdots,$$
so that
$$u(x)\left(1 + \sum_{j=1}^{k} (-1)^j P_j x^j\right) = u_0 \quad \text{or} \quad u(x) x^k \left(x^{-k} + \sum_{j=1}^{k} (-1) P_j x^{j-k}\right) = 1$$
or
$$u(x) x^k f(1/x) = 1.$$

We see then that, for $k = 2$ and $P_1 = -P_2 = 1$, we get Wiliam's case in which $x = 1/10$, namely

$$\sum_{n=0}^{\infty} u_n/10^n = 1/10^{-2}f(10) = 1/\frac{1}{100}(100 - 10b - a),$$

or

$$\sum_{n=0}^{\infty} u_n/10^{2+n} = 1/(100 - 10b - a)$$

(where his initial values are displaced by 2 from those here).

3. Atanassov's Arithmetic Functions

Atanassov [1] has defined arithmetic functions ϕ and Ψ as follows. For

$$\begin{aligned} n &= \sum_{i=1}^{j} a_i 10^{j-i}, \quad a_i \in \mathbb{N}, \\ &\equiv a_1 a_2 \ \ldots \ a_j, \quad 0 \le a_i \le 9, \end{aligned}$$

let $\phi\colon \mathbb{N} \to \mathbb{N}$ be defined by

$$\phi(n) = \begin{cases} 0 & \text{for } n = 0, \\ \sum_{i=1}^{j} a_i & \text{otherwise;} \end{cases}$$

and for the sequence of functions $\phi_0, \phi_1, \phi_2, \ldots,$

$$\phi_0(n) = n, \ \phi_{\ell+1}(n) = \phi(\phi_\ell(n)),$$

let $\Psi\colon \mathbb{N} \to \Delta = \{0, 1, 2, \ldots, 9\}$ be defined by $\Psi(n) = \phi_\ell(n)$, in which

$$\phi_\ell(n) = \phi_{\ell+1}(n).$$

For example, $\phi(889) = 25$, $\Psi(889) = 7$, since

$$\begin{aligned} \phi_0(889) &= 889, \\ \phi_1(889) &= 25, \\ \phi_2(889) &= 7 \\ &= \phi_3(889). \end{aligned}$$

It then follows that

(3.1) $\quad \Psi(\Psi(10^k/u(0.1)) + k) = 1,$

as Table 1 illustrates.

TABLE 1

k	2	3	4	5	6	7	8	10	11
$\Psi(\underbrace{8 \ldots 8}_{k-1 \text{ times}}9)$	8	7	6	5	4	3	2	9	8

The result follows from Theorem 1 and 5 of Atanassov, which are, respectively,

(3.2) $\quad \Psi(n + 1) = \Psi(\Psi(n) + 1);$

(3.3) $\quad \Psi(n + 9) = \Psi(n).$

Thus, $10^k/u(1/10) = \underbrace{8 \ldots 8}_{k-1 \text{ times}}9$, and so,

$$\Psi(10^k/u(1/10)) = 8(k - 1) + 8 + 1 = 8k + 1,$$

and $\quad \Psi(\Psi(10^k/u(0.1)) + k) = \Psi(9k + 1) = \Psi(9 + 1) = 1$, as required.

4. Other Values of X

The foregoing was for $x = 1/10$. In Table 2, we list the values of $\Psi(f(x))$ for integer values of k and $1/x = X$ from 2 to 10 when $P_j = -1$, $j = 1, 2, \ldots, k$, in the appropriate recurrence relation.

TABLE 2

X/k	2	3	4	5	6	7	8	9	10
2	1	1	1	1	1	1	1	1	1
3	5	5	5	5	5	5	5	5	5
4	2	7	9	8	4	6	5	1	3
5	1	4	1	4	1	4	1	4	1
6	2	2	2	2	2	2	2	2	2
7	5	7	3	2	4	9	8	1	6
8	1	7	1	7	1	7	1	7	1
9	8	8	8	8	8	8	8	8	8
10	8	7	6	5	4	3	2	1	9

To prove these results, we let $x = 1/X$ and so

(4.1) $\quad f(X) = X^k - X^{k-1} - X^{k-2} - \dots - X^2 - X - 1.$

The calculations which follow are mod 9. Thus, $3^t \equiv 0$, $6^t \equiv 0$, $9^t \equiv 0$ when $t \geq 2$. (Of course, $9^t \equiv 0$ when $t = 1$.)

Case A: $X = 3, 6, 9 = N$,

$f(N) \equiv -N - 1 \pmod 9$ for all k,
$f(3) \equiv -4 \equiv 5$,
$f(6) \equiv -7 \equiv 2$,
$f(9) \equiv -1 \equiv 8$ as in the appropriate rows of Table 2.

Case B: $\quad X = 4, 7, 10 = 3 + 1, 6 + 1, 9 + 1 = N + 1$,

(4.2) $\quad f(N + 1) = (N + 1)^k - (N + 1)^{k-1} - \dots - (N + 1)^2 - (N + 1) - 1.$

The only terms that interest us, mod 9, in the expansions are the second last and last in each expansion. Then (4.2) becomes

$$Nk - N(k - 1) - N(k - 2) - \dots - N \cdot 3 - N \cdot 2 - N \cdot 1$$
$$+ 1 - 1 \underbrace{- 1 - 1 - \dots - 1 - 1 - 1}_{k-2 \text{ times}} - 1$$
$$= Nk - N\sum_{n=1}^{k} n - (k - 2) - 1$$
$$= Nk - \frac{1}{2} N(k - 1)k - (k - 1)$$
$$= Nk\left\{1 - \frac{1}{2}(k - 1)\right\} - (k - 1)$$
$$\equiv Nk^2 - (k - 1) \text{ since } -N \equiv 2N \text{ for } N = 3, 6, 9.$$

Thus, $\quad f(4) = 3k^2 - k + 1$
$f(7) = 6k^2 - k + 1$
$f(10) = -k + 1$ since $9k^2 \equiv 0$.

Substitution of the values $k = 2, 3, \dots, 10$ gives the tabulated values.

Case C: $\quad X = 2, 5, 8 = 3 - 1, 6 - 1, 9 - 1, = N - 1$,

(4.3) $\quad f(N - 1) = (N - 1)^k - (N - 1)^{k-1} - \dots - (N - 1)^2 - (N - 1) - 1.$

As in Case B, this becomes

$$Nk(-1)^{k-1} - N(k - 1)(-1)^{k-2} - N(k - 2)(-1)^{k-3} - \dots - N \cdot 2(-1)^1$$
$$- N \cdot 1(-1)^0 + (-1)^k - (-1)^{k-1} - (-1)^{k-2} - \dots - 1 + 1 - 1.$$

When k is even, this becomes

$$\begin{aligned}
&\underbrace{-Nk - N(k-1)}\ = \underbrace{N(k-2) - N(k-3)} + \cdots + \underbrace{2N - N}\ \underbrace{+ 1 + 1} \\
&\quad \underbrace{- 1 + 1} + \cdots \underbrace{- 1 + 1} - 1 = -2NK + \underbrace{N + N + \cdots + N}_{k/2 \text{ terms}} + 1 \\
&\qquad = -2NK + \frac{1}{2}kN + 1 \\
&\qquad = -\frac{3}{2}Nk + 1 \\
&\qquad \equiv 3Nk + 1 \quad \text{since } -3 \equiv 6 \\
&\qquad \equiv 1 \qquad\quad\ \ \text{since } 3N \equiv 0,
\end{aligned}$$

which agrees with the appropriate entries of Table 2.

When k is odd, (4.3) becomes

$$\begin{aligned}
&Nk + N(k-1) - N(k-2) + N(k-3) - \cdots - N \cdot 2(-1)^1 - N \cdot 1(-1)^0 - 1 \\
&\quad \underbrace{-1+1}\,\underbrace{-1+1} - \cdots \underbrace{-1+1} - 1 = Nk + \underbrace{N + N + \cdots + N}_{(k-1)/2 \text{ terms}} - 2 \\
&\qquad = Nk + \frac{1}{2}(k-1)N - 2 \\
&\qquad = \frac{3}{2}Nk - \frac{1}{2}N - 2 \\
&\qquad \equiv -3Nk + 4N - 2 \quad \text{since } 3 \equiv -6,\ -1 \equiv 8 \\
&\qquad \equiv 4N - 2 \qquad\qquad \text{since } -3N \equiv 0.
\end{aligned}$$

Thus,

$f(2) \equiv 1$,
$f(5) \equiv 4$,
$f(8) \equiv 7$, as required.

5. Concluding Comments

Wiliam's staggered sum for Pell numbers [4] can be written as

(5.1) $$.0 + .01 + .002 + .0005 + .00012 + .000029 + \cdots = 1/79.$$

This is a particular case of Hulbert [5] who also noted a result like (1.3) which can be found in Reichmann [6]. Hulbert stated, without proof, that

(5.2) $$\sum_{n=1}^{\infty} 10^{-n}F_n = 1/(9.9 - k)$$

for

(5.3) $$F_{n+1} = kF_n + F_{n-1} \quad \text{with } F_1 - 1,\ F_2 = k\ (k = 1, 2, \ldots, 8).$$

When $k = 2$, we have the Pell case. We can generalize the Pell sequence by setting $P_1 = 2$, $P_j = -1$, $j = 2, \ldots, k, \ldots$. Then we may extend the work of Section 4 by the addition of a term $-X^{k-1}$ in $f(X)$, for $X = 2, 3, \ldots, 10$.

Hulbert also noted a staggered sum formed from

(5.4) $$\sum_{n=1}^{\infty} 10^{-n}\binom{r+n-1}{r} = 10^{-1}(0.9)^{-r+1} \quad (r = 0, 1, 2, \ldots).$$

This is a particular case of Equation (1.3) of Gould [2], namely

(5.5) $$\sum_{r=0}^{\infty} \binom{r+n}{r} x^r = (1-x)^{-n-1}.$$

Curiously, the same issue of the Bulletin where Hulbert's note appeared had in its Puzzle Corner the problem of finding

$$(5.6)\qquad \binom{n}{0} + \binom{n-2}{2} + \binom{n-4}{4} + \cdots$$

the series terminating when the binomial coefficients become improper. This, too, follows from Gould whose Equations (1.74) and (1.75) are, respectively

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} = (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta),$$

$$\sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} = \frac{1}{2}((-1)^{[n/3]} + (-1)^{[(n+1)/3]}),$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, and $[\cdot]$ represents the greatest integer function. It can be seen then that the series (5.6) equals

$$\frac{1}{2}\sum_{k=0}^{[n/2]} (1 + (-1)^k)\binom{n-k}{k}$$
$$= (\alpha^{n+1} - \beta^{n+1})/2(\alpha - \beta) + ((-1)^{[n/3]} + (-1)^{[2(n+1)/3]})/4.$$

It is also of interest to note that the generalized sequences of Section 2 are related to statistical studies of such gambling events as success runs [7] and expected numbers of consecutive heads [3].

References

1. K. T. Atanassov. "An Arithmetic Function and Some of Its Applications." *Bulletin of Number Theory* *9* (1985):18-27.
2. H. W. Gould. *Combinatorial Identities*. Morgantown: West Virginia University, 1972.
3. K. Hirst. "Are *m* Heads Better than Two?" *International Journal of Mathematical Education in Science and Technology* *19* (1988):687-90.
4. A. F. Horadam. "Pell Identities." *Fibonacci Quarterly* *9* (1971):245-63.
5. B. J. Hulbert. "Fibonacci Sequences." *Bulletin of the Institute of Mathematics and Its Applications* *14* (1978):187.
6. W. J. Reichmann. *The Spell of Mathematics* (quoted in Hulbert).
7. A. Tomkins & D. Pitt. "Runs and the Generalised Fibonacci Sequence." *Mathematical Gazette* *69* (1985):109-13.
8. D. Wiliam. "A Fibonacci Sum." *Mathematical Gazette* *69* (1985):29-31.

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SOLUTIONS OF FERMAT'S LAST EQUATION IN TERMS OF WRIGHT'S HYPERGEOMETRIC FUNCTION

Allen R. Miller
Naval Research Laboratory, Washington, D.C. 20375-5000
(Submitted January 1989)

Introduction

In this paper we study a problem related to Fermat's last theorem. Suppose that X, Y, and Z are positive numbers where

$$X^a + Y^a = Z^a. \tag{1}$$

We show that we can solve this equation for a; that is, we find a unique

$$a = a(X, Y, Z)$$

in closed form. The method of solution is rather elementary, and we employ Wright's generalized hypergeometric function in one variable [1], as defined below:

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{matrix}\; z\right] \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i n)}{\prod_{i=1}^{q}\Gamma(\beta_i + B_i n)} \frac{z^n}{n!}.$$

When $p = q = 1$, we see that

$${}_1\Psi_1\left[\begin{matrix}(\alpha, A); \\ (\beta, B);\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + An)}{\Gamma(\beta + Bn)} \frac{z^n}{n!}, \tag{2}$$

which is a generalization of the confluent hypergeometric function ${}_1F_1[\alpha; \beta; z]$.

An Equivalent Form of Equation (1)

In Equation (1), the case $X = Y$ is not interesting since, clearly,

$$a = \frac{\ln(1/2)}{\ln(X/Z)}.$$

Therefore, we shall assume, without loss of generality, that

$$Z > Y > X > 0,$$

and write Equation (1) as

$$e^{a\ln(X/Z)} + e^{a\ln(Y/Z)} - 1 = 0.$$

Now, making the transformation

$$e^{a\ln(Y/Z)} \equiv y, \tag{3}$$

we obtain

$$y^{\frac{\ln(X/Z)}{\ln(Y/Z)}} + y - 1 = 0,$$

and since

$$\frac{\ln(X/Z)}{\ln(Y/Z)} = \frac{\ln(Z/X)}{\ln(Z/Y)} > 1,$$

we arrive at

$$y^{\frac{\ln(Z/X)}{\ln(Z/Y)}} + y - 1 = 0. \tag{4}$$

Equation (4) is then equivalent to Equation (1), and our aim is to solve this equation for y, thereby obtaining a. We note that it is not difficult to verify that Equation (4) has a unique positive root y in the interval (1/2, 1).

Solution of Equation (4)

In 1915, Mellin [2, 3] investigated certain transform integrals named after him in connection with his study of the trinomial equation

$$y^N + xy^P - 1 = 0, \; N > P, \tag{5}$$

where x is a real number and N, P are positive integers. Mellin showed that, for appropriately bounded x, a positive root of Equation (5) is given by

$$y = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)x^{-z}\,dz, \; 0 < c < 1/P, \tag{6}$$

where

$$F(z) = \frac{\Gamma(z)\Gamma\left(\frac{1}{N} - \frac{P}{N}z\right)}{N\Gamma\left[1 + \frac{1}{N} + \left(1 - \frac{P}{N}\right)z\right]}$$

and

$$|x| < (P/N)^{-P/N}(1 - P/N)^{P/N-1} \le 2. \tag{7}$$

The inverse Mellin transform, Equation (6), is evaluated by choosing an appropriate closed contour and using residue integration to find that

$$y = \frac{1}{N}\sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{N} + \frac{P}{N}n\right)}{\Gamma\left[1 + \frac{1}{N} + \left(\frac{P}{N} - 1\right)n\right]}\,\frac{(-x)^n}{n!}. \tag{8}$$

Under the condition shown in Equation (7), Mellin, in fact, found all of the roots of Equation (5). However, suppose we relax the restriction that N and P are positive integers. Instead, let N and P be positive numbers. We then observe that Equation (8) gives *a fortiori* a positive root of Equation (5) for positive numbers N and P. Further, without loss of generality, we set $P = 1$, $N = \omega$. Then, using the Wright function defined by Equation (2), we arrive at the following. The unique positive root of the transcendental equation

$$y^{\omega} + xy - 1 = 0, \; \omega > 1, \tag{9}$$

where

$$|x| < \omega/(\omega - 1)^{1-1/\omega}$$

is given by

$$y = \frac{1}{\omega}\, {}_1\Psi_1\left[\begin{matrix}\left(\frac{1}{\omega}, \frac{1}{\omega}\right) & ; \\ \left(\frac{1}{\omega} + 1, \frac{1}{\omega} - 1\right); & -x\end{matrix}\right]. \tag{10}$$

We observe that for any $|x| < \infty$, Equation (9) has a unique positive root y. Equations (9) and (10) may also be obtained from Equation (30) on page 713 of [4].

Let us now apply the latter result to Equation (4). On setting

$$x = 1, \; \omega^{-1} = \frac{\ln(Z/Y)}{\ln(Z/X)} \equiv \lambda,$$

and noting that $1 < \omega/(\omega - 1)^{1-1/\omega}$, we find

$$(11)\qquad y = \lambda\,{}_1\Psi_1\begin{bmatrix}(\lambda,\ \lambda) & ; & \\ & & -1\\ (\lambda + 1,\ \lambda - 1); & & \end{bmatrix},\ 0 < \lambda < 1.$$

Solution of Equation (1)

We now solve Equation (1) for a. From the transformation Equation (3), we see that

$$(12)\qquad a\ \ln(Y/Z) = \ln\ y.$$

Then, using Equation (11), we arrive at the following. If $Z > Y > X > 0$ are such that

$$X^a + Y^a = Z^a,$$

then

$$(13)\qquad a = \frac{\ln\left\{\lambda\,{}_1\Psi_1\begin{bmatrix}(\lambda,\ \lambda) & ; & \\ & & -1\\ (\lambda + 1,\ \lambda - 1); & & \end{bmatrix}\right\}}{\ln(Y/Z)},$$

where

$$(14)\qquad \lambda \equiv \frac{\ln(Z/Y)}{\ln(Z/X)},\ 0 < \lambda < 1.$$

We now prove the following. Consider for $X < Y$, $M \geq 1$, the diophantine equation

$$X^M + Y^M = Z^M.$$

Then the positive integers X, Y, and Z must satisfy

$$(15)\qquad X^{\lambda}Y^{-1}Z^{1-\lambda} = 1,$$

where λ is an irrational number such that $0 < \lambda < 1$.

From Equation (12) we have

$$(16)\qquad (Y/Z)^M = y,$$

so that y is a rational number in the interval $1/2 < y < 1$ as we noted earlier. If λ is rational, there exist relatively prime integers s and t such that

$$\lambda = \omega^{-1} = s/t.$$

Hence, y is the unique positive root of

$$y^{t/s} + y - 1 = 0.$$

Now, since $\lambda < 1$, then $s < t$, and we obtain the polynomial equation of degree t with integer coefficients:

$$y^t + (-1)^s y^s + \cdots + 1 = 0.$$

The only positive rational root that this equation may have is $y = 1$ (see [5], p. 67). But $y < 1$, so the assumption that λ is rational leads to a contradiction. We have then that λ is irrational, and Equation (15) follows from Equation (14). This proves our result. W. P. Wardlaw has given another proof that λ is irrational in [6].

The Wright function ${}_1\Psi_1$ appearing in Equation (13) depends only on the parameter λ. Thus, for brevity, we define

$$\Psi(\lambda) \equiv {}_1\Psi_1\begin{bmatrix}(\lambda,\ \lambda) & ; & \\ & & -1\\ (\lambda + 1,\ \lambda - 1); & & \end{bmatrix},\ 0 < \lambda < 1.$$

From our previous result, we see that, if Fermat's theorem* is false, then there exist positive integers $X < Y < Z$ such that λ is irrational.

Therefore, Fermat's theorem is false if and only if there exist positive integers $Y < Z$, $M > 2$, and an irrational number λ $(0 < \lambda < 1)$ such that

$$(Y/Z)^M = \lambda\Psi(\lambda).$$

Thus, Fermat's conjecture may be posed as a problem involving the special function $\lambda\Psi(\lambda)$. We remark that recently, Fermat's conjecture has been given in combinatorial form [7].

Some Elementary Properties of $\lambda\Psi(\lambda)$

Although the series representation for $\lambda\Psi(\lambda)$, which follows below in Equation (17), does not converge for $\lambda = 0, 1$, it is natural to define

$$\lambda\Psi(\lambda)\Big|_{\lambda=1} = 1/2, \quad \lambda\Psi(\lambda)\Big|_{\lambda=0} = 1.$$

Using this definition, we give a brief table of values for $\lambda\Psi(\lambda)$, which is correct to five significant figures:

λ	$\lambda\Psi(\lambda)$	λ	$\lambda\Psi(\lambda)$
0.0	1.00000	0.6	0.58768
0.1	0.83508	0.7	0.56152
0.2	0.75488	0.8	0.53860
0.3	0.69814	0.9	0.51825
0.4	0.65404	1.0	0.50000
0.5	0.61803		

Observe that we may write the inverse relation

$$\lambda = \ln \lambda\Psi(\lambda)/\ln[1 - \lambda\Psi(\lambda)].$$

Note also that when $\lambda = 1/2$, $\omega = 2$ and Equation (9) becomes $y^2 + y - 1 = 0$, whose positive root is $(-1 + \sqrt{5})/2$.

The following series representations for $\lambda\Psi(\lambda)$, $0 < \lambda < 1$ may easily be derived from the first one below:

$$(17) \qquad \lambda\, {}_1\Psi_1\left[\begin{matrix}(\lambda,\ \lambda) & ; \\ & -1 \\ (\lambda + 1,\ \lambda - 1); & \end{matrix}\right] = \lambda \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(\lambda + \lambda n)}{\Gamma(\lambda + 1 + (\lambda - 1)n)}$$

$$(18) \qquad = \frac{\lambda}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(1 - \lambda)n - 1} \sin[\pi(1 - \lambda)n] B(\lambda n,\ n - \lambda n)$$

$$(19) \qquad = 1 - \lambda \sum_{n=0}^{\infty} (-1)^n {}_2F_1[-n,\ (1 - \lambda)(n + 2);\ 2;\ 1]$$

$$(20) \qquad = 1 + \lambda \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \binom{\lambda(1 + n) - 1}{n - 1}.$$

Equation (18) follows from Equation (17) by using

$$\Gamma(z)\Gamma(-z) = -\pi/z \sin \pi z;$$

$B(x, y)$ is the beta function. Equation (19) follows from Equation (17) by using Gauss's theorem for ${}_2F_1[a, b; c; 1]$. Equation (20) follows from Equation (17) by using

$$\binom{\alpha}{m} = \Gamma(1 + \alpha)/m!\Gamma(1 + \alpha - m).$$

*Fermat's theorem states that there are no integers $x, y, z > 0$, $n > 2$ such that $x^n + y^n = z^n$.

Equation (20), for $1/\lambda$ an integer greater than one, is due to Lagrange ([2], p. 56).

Conclusion

The equation $X^a + Y^a = Z^a$ has been solved for a as a function of X, Y, and Z in terms of a Wright function ${}_1\Psi_1$ with negative unit argument. An equivalent form of Fermat's last theorem has been given using this function. Further, some elementary properties of ${}_1\Psi_1$ have been stated.

References

1. H. M. Srivastava & H. L. Manocha. *A Treatise on Generating Functions.* Halsted Press, 1984.
2. M. G. Belardinelli. "Résolution analytique des equations algebraiques generales." *Mémorial des Sciences Mathematiques*, Fascicule 145. Gauthiers-Villars, 1960.
3. H. Hochstadt. *The Functions of Mathematical Physics.* Wiley, 1971.
4. A. P. Prudnikov, Yu. A. Brychkov, & O. I. Marichev. *Integrals and Series*, Vol. 1. Gordon and Breach, 1986.
5. S. Borofsky. *Elementary Theory of Equations.* Macmillan, 1950.
6. W. P. Wardlaw, personal communication.
7. W. V. Quine. "Fermat's Last Theorem in Combinatorial Form." *Amer. Math. Monthly* 95 (1988):636.

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A GENERALIZATION OF A RESULT OF SHANNON AND HORADAM

Dario Castellanos
Universidad de Carabobo, Valencia, Venezuela
(Submitted January 1989)

1. Introduction

In a recent note in this magazine [5] Professors A. G. Shannon and A. F. Horadam generalize a result proposed by Eisenstein [2] and solved by Lord [4] to the effect that

$$(1.1)\qquad L_n - \frac{(-1)^n}{L_n} - \frac{(-1)^n}{L_n} - \cdots = \alpha^n,$$

where L_n is the n^{th} Lucas number and α is the positive root of $x^2 - x - 1 = 0$.

They introduce the sequence $\{w_n\} \equiv \{w_n(a, b; p, q)\}$ defined by the initial conditions $w_0 = a$, $w_1 = b$, and the recurrence relation

$$(1.2)\qquad w_n = pw_{n-1} - qw_{n-2},\ n \geq 2,$$

where p and q are arbitrary integers.

They let $\alpha = (p + \sqrt{(p^2 - 4q)})/2$, $\beta = (p - \sqrt{(p^2 - 4q)})/2$, for $|\beta| < 1$, be the roots of

$$(1.3)\qquad x^2 - px + q = 0,$$

so that $\{w_n\}$ has the general term

$$(1.4)\qquad w_n = A\alpha^n + B\beta^n,$$

where

$$A = (b - a\beta)/d,\ B = (a\alpha - b)/d,\ AB = e/d^2;$$

$$e = pab - qa^2 - b^2,\ d = \alpha - \beta,\ p = \alpha + \beta,\ q = \alpha\beta.$$

They also let $Q_n = ABq^n$.

The Fibonacci sequence is

$$\{F_n\} \equiv \{w_n(0, 1; 1, -1)\},\ Q_n = (-1)^{n+1}/5;$$

the Lucas sequence is

$$\{L_n\} \equiv \{w_n(2, 1; 1, -1)\},\ Q_n = (-1)^n;$$

the Pell sequence is

$$\{P_n\} \equiv \{w_n(0, 1; 2, -1)\},\ Q_n = (-1)^n/8.$$

Shannon and Horadam's result is

$$(1.5)\qquad w_n - \frac{Q_n}{w_n} - \frac{Q_n}{w_n} - \cdots = A\alpha^n.$$

They establish this result by finding a general expression for the convergents of the continued fraction (1.5) and determining the limiting form with an appeal to some results of Khovanskii [3].

2. An Alternate Approach

Consider the identity

$$(2.1)\qquad \sqrt{s} - t = (s - t^2)/(2t + (\sqrt{s} - t)),$$

which gives at once the continued fraction (see [1])

$$\sqrt{s} = t + \frac{s - t^2}{2t\ +}\ \frac{s - t^2}{2t\ +}\ \frac{s - t^2}{2t\ +} \cdots . \tag{2.2}$$

In (2.2), replace s and t by $\frac{1}{4}t^2 - s$ and $\frac{1}{2}t$, respectively, to obtain

$$\sqrt{(\tfrac{1}{4}t^2 - s)} - \tfrac{1}{2}t = \frac{-s}{t\ +}\ \frac{-s}{t\ +}\ \frac{-s}{t\ +} \cdots,$$

or equivalently,

$$\sqrt{(\tfrac{1}{4}t^2 - s)} + \tfrac{1}{2}t = t - \frac{s}{t\ -}\ \frac{s}{t\ -}\ \frac{s}{t\ -} \cdots . \tag{2.3}$$

With the notation of Section 1, let $s = Q_n = AB(\alpha\beta)^n$, $t = w_n = A\alpha^n + B\beta^n$. Simple arithmetic shows that the left-hand side of (2.3) becomes $A\alpha^n$, and we find

$$A\alpha^n = w_n - \frac{Q_n}{w_n\ -}\ \frac{Q_n}{w_n\ -} \cdots, \tag{2.4}$$

which is the result of Shannon and Horadam.

Similarly, let $s = (-1)^{n+1}$, $t = 2F_n$, and recall that $F_n^2 + (-1)^n = F_{n-1}F_{n+1}$, and (2.3) gives

$$\sqrt{(F_{n-1}F_{n+1})} - F_n = \frac{(-1)^n}{2F_n\ +}\ \frac{(-1)^n}{2F_n\ +} \cdots . \tag{2.5}$$

As the reader no doubt knows, $\sqrt{(F_{n-1}F_{n+1})}$ is approximated by F_n, the approximation becoming better as n increases. The continued fraction in the right-hand side of (2.5) gives the error committed in the approximation.

Classes of expressions can be found by choosing suitable values of s and t. Especially interesting is the choice

$$t = a_1 w_{n_1}^{k_1} + a_2 w_{n_2}^{k_2} + \cdots + a_m w_{n_m}^{k_m},$$

where $k_1, k_2, \ldots, k_m, n_1, n_2, \ldots, n_m$ are arbitrary integers, $a_1, a_2, \ldots, a_m$ are arbitrary real numbers, and s is an arbitrary parameter.

Many other expressions can be found by giving appropriate values to s and t. It is left to the reader to discover them.

Acknowledgment

The author wishes to thank the referee for many helpful comments and stylistic improvements.

References

1. D. Castellanos. "A Generalization of Binet's Formula and Some of Its Consequences." *Fibonacci Quarterly* 27.5 (1989):424-38. Equation (2.3) was discovered by the author. Joseph Ehrenfried Hofmann's *Geschichte der Mathematik* seems to indicate that a formula essentially equivalent to it was originally discovered by Michel Rolle in his *Mémoires de mathêmatiques et de physiques*, vol. 3 (Paris, 1692). C. D. Olds makes the claim, in his *Continued Fractions*, that the formula may have been known to Rafael Bombelli, a native of Bologna and a disciple of Girolamo Cardano, as far back as 1572.
2. M. Eisenstein. Problems B-530 and B-531. *Fibonacci Quarterly* 22 (1984):274.
3. A. N. Khovanskii. *The Application of Continued Fractions*. Tr. from Russian by Peter Wynn. Gronigen: Noordhoff, 1963.
4. G. Lord. Solutions to B-530 and B-531. *Fibonacci Quarterly* 23 (1985):280-81.
5. A. G. Shannon & A. F. Horadam. "Generalized Fibonacci Continued Fractions." *Fibonacci Quarterly* 26 (1988):219-23.

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FIBONACCI NUMBERS ARE NOT CONTEXT-FREE

Richard J. Moll and Shankar M. Venkatesan
Department of Computer Science, University of Minnesota
(Submitted February 1989)

The Fibonacci numbers, given by the recurrence relation

$$F(n + 2) = F(n + 1) + F(n),\ F(1) = 1,\ F(2) = 1,$$

are considered to be written in base b, so "trailing zeros" correspond exactly to "factors of b." From [4], Theorem 5, page 527, it follows that, for any prime p, there exists n s.t. $F(k \times n) \equiv 0 \pmod{p^i}$ for positive i and k. The existence of j s.t. $F(j) \equiv 0 \pmod{b^i}$, for arbitrary positive b, follows by applying the above to the prime factoring of b and choosing j to be the least common multiple of the n. Thus, in any base, there exist Fibonacci numbers with arbitrarily many trailing zeros.

In the proof of this same theorem [4], it is established for any prime p that, if $F(n)$ is the first term $\equiv 0 \pmod{p^e}$ but $\not\equiv 0 \pmod{p^{e+1}}$, then $F(p \times n)$ is the first term $\equiv 0 \pmod{p^{e+1}}$, also $F(p \times n) \not\equiv 0 \pmod{p^{e+2}}$.

This establishes, for each prime base p, a lower bound on n which increases exponentially with the number of trailing zeros in $F(n)$ base p. This bound generalizes to composite bases because when $F(n)$ has e trailing zeros in base b it must also have e trailing zeros in all bases p, where p is a prime factor of b. Specifically, there is some constant k such that, for all sufficiently large n,

$$TZ(F(n)) < k \times \log(n),$$

where $TZ(x)$ is the number of trailing zeros in x.

Since the Fibonacci sequence is asymptotically exponential, there is some constant c s.t. $n < c \times |F(n)|$, where $|F(n)|$ denotes the *length* of $F(n)$ as a string, i.e., the number of digits in $F(n)$ in base b. Combining these, and adjusting k to also account for c, gives

$$TZ(F(n)) < k \times \log(|F(n)|). \tag{1}$$

These facts can be used to show that the Fibonacci numbers do not form a *context-free* set. A set of strings is context-free iff it is the set generated by some context-free grammar or, equivalently, a set of strings is context-free iff it is the set recognized by some pushdown automaton. Ogden's Lemma, stated below, gives a property true of all context-free sets, and is used in Lemma 1 to show a set of strings closely related to the Fibonacci numbers to be not context-free.

Ogden's Lemma [2]: Let Q be a context-free set. Then there is a constant j such that, if α is any string in Q and we mark any j or more positions of α "distinguished," then we can write $\alpha = uvwxy$, such that:

1) v and x together have at least one distinguished position,
2) vwx has at most n distinguished positions, and
3) for all $i \geq 0$, uv^iwx^iy is in Q.

Lemma 1: Let Q be the set of strings such that the members of Q are the Fibonacci numbers written in base b with a new symbol "#" inserted immediately following the last nonzero digit. The set Q is not context-free.

Proof: The proof is by contradiction. Assume that Q is context-free.

Let j be the number of "distinguished" positions required for Ogden's Lemma (see [2] for a description of Ogden's Lemma). Since we know there are

Fibonacci numbers with arbitrarily many trailing zeros, let α be a member of Q corresponding to a Fibonacci number with at least j trailing zeros. The trailing zeros, which follow the "#," are used as the distinguished positions for purposes of Ogden's Lemma.

Applying Ogden's Lemma, α may be partitioned as follows:

$$\alpha = uvwxy,$$

where x contains at least one of the trailing zeros. Further, for all $i \geq 0$, $\beta_i = uv^iwx^iy$ must also be in the set Q, and thus correspond to some Fibonacci number satisfying (1).

If x contained the "#," then clearly β_2 would contain two "#" symbols and, thus, could not be a member of Q. Therefore, x contains only "0"s, so β_i has at least $j + i - 1$ trailing zeros.

Since v and x together can be no longer than α, then β_i can be no more than i times as long as α: So $|\beta_i| \leq i \times |\alpha|$. Applying (1) to these bounds gives:

$$j + i - 1 < k \times \log(i \times |\alpha|).$$

Choosing $i = 2k^2|\alpha| + 1$ produces a contradiction. □

Theorem: For all integers $b \geq 2$, the set of Fibonacci numbers in base b, considered as strings over the alphabet 0, 1, ..., $b - 1$, is not context-free.

Proof: Assume M is a pushdown automaton (PDA) recognizing the set of Fibonacci numbers. We modify the finite control to give another PDA M', recognizing the set Q, thus contradicting Lemma 4. An informal description of M' follows.

M' contains a copy of the machine M, plus additional logic in the finite control to filter the input and pass it to this internal copy of M. M' accepts only when this internal M accepts the string passed to it.

behaves as follows:

- M' rejects if the input does not contain exactly one "#," if the "#" does not immediately follow a nonzero digit, or if there are any nonzero digits following the "#." Otherwise, M' accepts if and only if its inter-internal simulation of M accepts.
- When M' reads a digit (any symbol except "#") from the input, it passes that digit to M. The "#" symbol, having been checked as above, is otherwise ignored and is not passed to M.

By the above rules, if M' accepts, then the input must be a Fibonacci number with a "#" inserted following the last nonzero digit. Thus, the input is in the set Q.

Conversely, if the input is in the set Q, then M' will pass the Fibonacci number to M and thus accept.

Therefore, M' accepts the set Q, a contradiction by Lemma 4; hence, the set of Fibonacci numbers is not context-free. □

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References

1. A. Cobham. "Uniform Tag Sequences." *Mathematical Systems Theory* (1972):164-191.
2. J. Hopcroft & J. Ullman. *Introduction to Automata Theory, Languages and Computation*. New York: Addison-Wesley, 1979.

3. R. J. Moll. "Factors of Two in the Fibonacci Numbers." Master's Project Report, University of Minnesota, 1989.
4. D. D. Wall. "Fibonacci Numbers Modulo *m*." *Amer. Math. Monthly* (1960):525-532.

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ON FERMAT'S EQUATION

Krystyna Białek and Aleksander Grytczuk
Pedagogical University, Zielona Góra, Poland
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1. Introduction

In 1856 I. A. Grünert ([6], see also [9], p. 226) proved that if n is an integer, $n \geq 2$ and $0 < x < y < z$ are real numbers satisfying the equation

(1.1) $\quad x^n + y^n = z^n$

then

(1.2) $\quad z - y < \dfrac{x}{n}.$

This result was rediscovered by G. Towes [10], and then by D. Zeitlin [11].

In 1979 L. Meres [7] improved the result of Grünert, replacing (1.2) by

(1.3) $\quad z - y < \dfrac{x}{a}$, for $a = n + 1 - n^{2-n}$, $n \geq 2$.

In [1], we improved the result of Meres, replacing (1.3) by

(1.4) $\quad z - y < \dfrac{x}{n+1}$, for $n \geq 4$.

Next, in [2], it has been proved that if k is a positive integer and, for $n > [(2k+1)C_1]$, $C_1 = (\log 2)/[2(1 - \log 2)]$, Equation (1.1) has a solution in real numbers $0 < x < y < z$, then

(1.5) $\quad z - y < \dfrac{x}{n+k}.$

Fell, Graz, & Paasche [5] have proved that, if (1.1) has a solution in positive integers $x < y < z$, where $n \geq 2$, then

(1.6) $\quad x^2 > 2y + 1.$

In 1969, M. Perisastri ([8], cf. [9], p. 226) proved that

(1.7) $\quad x^2 > z.$

In [2], it has been proved that

(1.8) $\quad x^2 > 2z + 1.$

A. Choudhry, in [4], improved the inequality (1.8) to the form

(1.9) $\quad x^{1+\frac{1}{n-1}} > z.$

In fact, A. Choudhry proved that

(1.10) $\quad z < C(n) \cdot x^{1+\frac{1}{n-1}},$

where

(1.11) $\quad C(n) = \dfrac{2^{\frac{1}{n}}}{n^{\frac{1}{n-1}}}$, for $n \geq 2$.

First we remark that inequality (1.9) in the Theorem of Choudhry follows immediately from (1.1) and the assumption that $0 < x < y < z$. Really, we have

$$x^n = z^n - y^n = (z - y)(z^{n-1} + z^{n-2}y + \cdots + y^{n-1}) > z^{n-1},$$

and (1.9) follows.

In this paper we prove the following theorems.

Theorem 1: If the equation (1.1) has a solution in positive integers $x < y < z$ where $n \geq 2$, then

$$(1.12) \quad z < C_1(n) \cdot x^{1+\frac{1}{n-1}}$$

where

$$(1.13) \quad C_1(n) = \frac{2^{\frac{1}{2n}}}{n^{\frac{1}{n-1}}}.$$

We remark that $C_1(n) < C(n) < 1$.

Next, we have the following theorem.

Theorem 2: If $z - x \leq C$, then (1.1) has only a finite number of solutions in positive integers $x < y < z$ and

$$(1.14) \quad z < C\left(n \cdot 2^{\frac{n-1}{n}} + 1\right).$$

We remark that, from Theorem 1 (see [2]) and the inequality (1.5), we get the following corollary.

Corollary: If k is a positive integer (1.1) has a solution in positive integers $x < y < z$ for $n > [(2k + 1)C_1]$, $C_1 = (\log 2)/[2(1 - \log 2)]$, then

$$x > k + [(2k + 1)C_1].$$

Let $G_2(k)$ be the set of all matrices of the form

$$\begin{pmatrix} r & s \\ ks & r \end{pmatrix},$$

where $k \neq 0$ is a fixed integer and r, $s \neq 0$ are arbitrary integers.

Let R_K denote the ring of all integers of the field $K = Q(\sqrt{k})$. Then, in [3], we proved the following theorem.

Theorem 3: A necessary and sufficient condition for (1.1) to have a solution in elements $A, B, C \in G_2(k)$ is the existence of the numbers $\alpha, \beta, \gamma \in R_K$, where $K = Q(\sqrt{k})$ such that $\alpha^n + \beta^n = \gamma^n$. The proof of Theorem 3 in [3] is based on some properties of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ with } a, b, c, d \in Z.$$

In this paper we give a very simple proof of this theorem.

2. Proof of Theorems

2.1 Proof of Theorem 1

For the proof of Theorem 1, we note that

$$(2.1) \quad z^{n-1} + z^{n-2}y + \cdots + zy^{n-2} + y^{n-1} > n(zy)^{\frac{n-1}{2}}.$$

From (1.1) and $x < y < z$ we have $z^n < 2y^n$; hence,

$$(2.2) \quad y > \left(\frac{1}{2}\right)^{\frac{1}{n}} \cdot z.$$

Since

$$(2.3) \quad x^n = (z - y)(z^{n-1} + z^{n-2}y + \cdots + zy^{n-2} + y^{n-1}),$$

we see, by (2.1), (2.2), and (2.3), that it follows that

(2.4) $$x^n > n \cdot z^{n-1}\left(\frac{1}{2}\right)^{\frac{n-1}{2n}}.$$

From (2.4), we get

$$z < \frac{2^{\frac{1}{2n}}}{n^{\frac{1}{n-1}}} \cdot x^{1+\frac{1}{n-1}}$$

and the proof is complete.

2.2 Proof of Theorem 2

From (1.1), we have

(2.5) $$y^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}).$$

Since $x < y < z$, then by (2.5) it follows that

(2.6) $$y^n < (z - x)n \cdot z^{n-1}.$$

From (2.6) and (2.2), we get

(2.7) $$y^n < (z - x)n\left(2^{\frac{1}{n}}y\right)^{n-1} = n \cdot 2^{\frac{n-1}{n}}(z - x)y^{n-1}.$$

From (2.7), we get

(2.8) $$y < n \cdot 2^{\frac{n-1}{n}}(z - x).$$

From (2.8) and our assumption that $z - x \le C$, we have

(2.9) $$y < n \cdot 2^{\frac{n-1}{n}}C.$$

Since $x < y$, we see by (2.9) that $x < n \cdot 2^{\frac{n-1}{n}}C$. From our assumption, it now follows that

$$z \le x + C < n \cdot 2^{\frac{n-1}{n}}C + C = C\left(1 + n \cdot 2^{\frac{n-1}{n}}\right)$$

and the proof is finished.

2.3 Proof of Theorem 3

First we remark that it suffices to prove that the set $G_2(k)$ is isomorphic to R_K, where $K = Q(\sqrt{k})$. Let

$$\phi\colon G_2(k) \to R_K,\ K = Q(\sqrt{k}),$$

and

$$\phi\left(\begin{pmatrix} r & s \\ ks & r \end{pmatrix}\right) = r + s\sqrt{k}.$$

Then we prove that ϕ is an isomorphism. Indeed, we have, for $A, B \in G_2(k)$,

$$\phi(A \cdot B) = \phi(A) \cdot \phi(B) \quad \text{and} \quad \phi(A + B) = \phi(A) + \phi(B);$$

therefore, $G_2(k) \simeq R_K$, where $K = Q(\sqrt{k})$. The proof is complete.

References

1. K. Bialek. "Remark of Fermat's Equation." *Discuss. Math.* *7* (1985):119-22.
2. K. Bialek. "On Some Inequalities Connected with Fermat's Equation." *El. Math.* *43.3* (1988):78-83.
3. K. Bialek & A. Grytczuk. "The Equation of Fermat in $G_2(k)$ and $Q(\sqrt{k})$." *Acta Acad. Paed. Agriens* *18.11* (Matematika-Eger Hungary)(1987):81-90.
4. A. Choudhry. "On Fermat's Last Theorem." *Elem. Math.* *43.6* (1988):182-83.
5. A. Fell & I. Paasche. "P66 Fermatproblem." *Praxis der Math.* *3* (1961):80.
6. I. A. Grünert. "Wenn $n > 1$, so gibt es unter den ganzen Zahlen von 1 bis nicht zwei Werte von x und y für welche wenn z einen ganzen Wert bezeichnet, $x^n + y^n = z^n$ ist." *Archiv. Math. Phys.* *27* (1856):119-20.

7. L. Meres. "About Certain Inequalities Connected with the Great Theorem of Fermat." *Zeszyt. Nauk. Polit. Slaskiej 560 Mat.-Fiz.* *30* (1979):215-18.
8. M. Perisastri. "On Fermat's Last Theorem." *Amer. Math. Monthly* *76* (1969): 671-75.
9. P. Ribenboim. *13 Lectures on Fermat's Last Theorem.* New York-Heidelberg-Berlin: Springer Verlag, 1980.
10. G. Towes. "On Fermat's Last Theorem." *Amer. Math. Monthly* *41* (1934):419-424.
11. D. Zeitlin. "A Note on Fermat's Last Theorem." *Fibonacci Quarterly* *12* (1974):368-402.

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LUCAS PRIMITIVE ROOTS

Bui Minh Phong*

Computer Center of E. Lorand University
Budapest H-1117, Bogdanfy u.10/B, Hungary

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1. Introduction

Let $U = \{U_n\}_{n=0}^{\infty}$ be a Lucas sequence defined by integers $U_0 = 0$, $U_1 = 1$, P, Q, and by the recursion

$$U_{n+1} = PU_n - QU_{n-1}, \text{ for } n \geq 1.$$

The polynomial

$$f(x) = x^2 - Px + Q$$

with discriminant

$$D = P^2 - 4Q$$

is called the characteristic polynomial of the sequence U. In the case where $P = -Q = 1$, the sequence U is the Fibonacci sequence and we denote its terms by $F_0, F_1, F_2, \ldots$.

Let p be an odd prime with $p \nmid Q$ and let $e \geq 1$ be an integer. The positive integer $u = u(p^e)$ is called the rank of apparition of p^e in the sequence U if $p^e | U_u$ and $p^e \nmid U_m$ for $0 < m < u$; furthermore, $\bar{u} = \bar{u}(p^e)$ is called the period of the sequence U modulo p^e if it is the smallest positive integer for which $U_{\bar{u}} \equiv 0$ and $U_{\bar{u}+1} \equiv 1 \pmod{p^e}$. In the Fibonacci sequence, we denote the rank of apparition of p^e and period of F modulo p^e by $f(p^e)$ and $\bar{f}(p^e)$, respectively.

Let the number g be a primitive root (mod p^e). If $x = g$ satisfies the congruence

$$(1) \qquad f(x) = x^2 - Px + Q \equiv 0 \pmod{p^e},$$

then we say that g is a Lucas primitive root (mod p^e) with parameters P and Q. Throughout this paper, we shall write "Lucas primitive root mod p^e" without including the phrase, "with parameters P and Q," if the sequence U is given. This is the generalization of the definition of Fibonacci primitive roots (FPR) modulo p that was given by D. Shanks [6] for the case $P = -Q = 1$.

The conditions for the existence of FPR (mod p) and their properties were studied by several authors. For example, D. Shanks [6] proved that if there exists a FPR (mod p) then $p = 5$ or $p \equiv \pm 1 \pmod{10}$; furthermore, if $p \neq 5$ and there are FPR's (mod p), then the number of FPR's is two or one, according to whether $p \equiv 1 \pmod 4$ or $p \equiv -1 \pmod 4$. In [7], D. Shanks & L. Taylor have shown that if g is a FPR (mod p) then $g - 1$ is a primitive root (mod p). M. J. DeLeon [4] proved that there is a FPR (mod p) if and only if $\bar{f}(p) = p - 1$. In [2] we studied the connection between the rank of apparition of a prime p and the existence of FPR's (mod p). We proved that there is exactly one FPR (mod p) if and only if $f(p) = p - 1$ or $p = 5$; moreover, if $p \equiv 1 \pmod{10}$ and there exist two FPR's (mod p) or no FPR exists, then $f(p) < p - 1$. M. E. Mays [5] showed that if both $p = 60k - 1$ and $q = 30k - 1$ are primes then there is a FPR (mod p).

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The purpose of this paper is to give some connections among the rank of apparition of p^e in the Lucas sequence U, the period of U modulo p^e, and the Lucas primitive roots (mod p^e); furthermore, we show necessary and sufficient conditions for the existence of Lucas primitive roots (mod p^e). In the case in which $P = -Q = e = 1$, our results reproduce and improve upon some results for FPR's (mod p) mentioned above.

We shall prove the following two theorems.

Theorem 1: Let U be a Lucas sequence defined by integers $P \neq 0$ and $Q = -1$, let p be an odd prime with $p \nmid D = P^2 + 4$, and let $e \geq 1$ be an integer. Then there is a Lucas primitive root (mod p^e) if and only if

$$\overline{u}(p^e) = \phi(p^e),$$

where ϕ denotes the Euler function. There is exactly one Lucas primitive root (mod p^e) if $\overline{u}(p^e) = \phi(p^e)$ and $p \equiv -1 \pmod 4$, and there are exactly two Lucas primitive roots (mod p^e) if $\overline{u}(p^e) = \phi(p^e)$ and $p \equiv 1 \pmod 4$.

Theorem 2: Let U be a Lucas sequence defined by integers $P \neq 0$ and $Q = -1$, let p be an odd prime with $p \nmid D = P^2 + 4$, and let $e \geq 1$ be an integer. Then there is exactly one Lucas primitive root (mod p^e) if and only if $u(p^e) = \phi(p^e)$ and $p \equiv -1 \pmod 4$, and exactly two Lucas primitive roots (mod p^e) exist if and only if

$$u(p^e) = \phi(p^e)/2 \quad \text{and} \quad p \equiv 1 \pmod 8$$

or

$$u(p^e) = \phi(p^e)/4 \quad \text{and} \quad p \equiv 5 \pmod 8.$$

From these theorems, some other results follow.

Corollary 1: If U, p, and e satisfy the conditions of Theorem 2 and

$$u(p^e) = \phi(p^e),$$

then g is a Lucas primitive root (mod p^e) if and only if $x = g$ satisfies the congruence

$$U_n x + U_{n-1} \equiv -1 \pmod{p^e}, \tag{2}$$

where $n = \phi(p^e)/2$.

Corollary 2: If U, p, and e satisfy the conditions of Theorem 2 and g is a Lucas primitive root (mod p^e), then $g - P$ is a primitive root (mod p^e).

Corollary 3: If $P \neq 0$ is an integer and both q and $p = 2q + 1$ are primes with conditions $p \nmid P$ and $(D/p) = 1$, where $D = P^2 + 4$ and (D/p) is the Legendre symbol, then there is exactly one Lucas primitive root (mod p) with parameters P and $Q = -1$.

2. Known Results and Lemmas

Let U be a Lucas sequence defined by nonzero integers P and Q, and let $D = P^2 - 4Q$ be the discriminant of the characteristic polynomial of U. If p is an odd prime with $p \nmid Q$ and $e \geq 1$ is an integer, then, as is well known, we have:

(i) $U_n \equiv 0 \pmod{p^e}$ if and only if $u(p^e) \mid n$;

(ii) $U_n \equiv 0$ and $U_{n+1} \equiv 1 \pmod{p^e}$ if and only if $\overline{u}(p^e) \mid n$;

(iii) $u(p) = p$ if $p \mid D$,
$u(p) \mid p - (D/p)$ if $p \nmid D$, where (D/p) is the Legendre symbol;

(iv) $\overline{u}(p^e) = \overline{u}(p) \cdot p^{e-k}$ if $\overline{u}(p) = \cdots = \overline{u}(p^k) \neq \overline{u}(p^{k+1})$ and $e \geq k$;

(v) $u(p) \mid \overline{u}(p)$;

(vi) Let $u(p^e) = 2^a u'$ and $d(p^e) = 2^b d'$, where $d(p^e)$ denotes the least positive integer d for which $Q^d \equiv 1 \pmod{p^e}$ and u', d' are odd integers. We have

$$\bar{u}(p^e) = \begin{cases} [u(p^e),\ d(p^e)] & \text{if } a = b > 0, \\ 2[u(p^e),\ d(p^e)] & \text{if } a \neq b, \end{cases}$$

where [x, y] denotes the least common multiple of integers x and y. (For these properties of Lucas sequences, we refer to [1], [3], [8]).

First, we note that congruence (1) is solvable if and only if the congruence $y^2 \equiv D = P^2 - 4Q \pmod{p^e}$ has solutions. Thus, in case $p \nmid D$, congruence (1) is solvable if and only if $(D/p) = 1$; furthermore, if $(D/p) = 1$, then (1) has two distinct solutions $\pmod{p^e}$.

Let p be an odd prime for which $(D/p) = 1$ and let g_1 and g_2 be the two distinct solutions of (1). Then we have

(3) $\qquad g_1 - g_2 \not\equiv 0 \pmod{p}$,

(4) $\qquad g_1 + g_2 \equiv P,\ g_1 g_2 \equiv Q \pmod{p^e}$;

furthermore, it can easily be seen by induction that

(5) $\qquad g_i^n \equiv U_n g_i - QU_{n-1} \pmod{p^e} \quad (i = 1,\ 2)$

for every integer $n \geq 1$. Let $n_i = n_i(p^e)$ be the least positive integer for which

$$g_i^{n_i} \equiv 1 \pmod{p^e}.$$

We may assume that $n_1(p^e) \geq n_2(p^e)$.

Lemma 1: If p is an odd prime with conditions $p \nmid Q$, $(D/p) = 1$, and e is a positive integer, then

$$\bar{u}(p^e) = [n_1(p^e),\ n_2(p^e)].$$

Proof: Since $(D/p) = 1$, congruence (1) has two distinct solutions g_1 and g_2 which belong to the exponents $n_1 = n_1(p^e)$ and $n_2 = n_2(p^e)$ $\pmod{p^e}$. Let $\bar{u} = \bar{u}(p^e)$ and $q = [n_1,\ n_2]$. The definition of $\bar{u}$ implies that

$$1 \equiv U_{\bar{u}+1} = PU_{\bar{u}} - QU_{\bar{u}-1} \equiv -QU_{\bar{u}-1} \pmod{p^e};$$

therefore, by (5), for $i = 1$ and $i = 2$, we have

$$g_i^{\bar{u}} \equiv U_{\bar{u}} g_i - QU_{\bar{u}-1} \equiv -QU_{\bar{u}-1} \equiv 1 \pmod{p^e}$$

and so $q \mid \bar{u}$ follows.

On the other hand, by (5) and the definition of q, we have

$$U_q g_1 - U_q g_2 \equiv g_1^q - g_2^q \equiv 0 \pmod{p^e},$$

which with (3) implies $U_q \equiv 0 \pmod{p^e}$. Thus,

$$U_{q+1} = PU_q - QU_{q-1} \equiv -QU_{q-1} \equiv U_q g_1 - QU_{q-1} \equiv g_1^q \equiv 1 \pmod{p^e},$$

and so, by (ii), we have $\bar{u} = q$.

Lemma 2: Let $Q = -1$ and $D = P^2 + 4$. If p is an odd prime with $(D/p) = 1$ and e is a positive integer, then

$$\bar{u}(p^e) = \begin{cases} n_1(p^e) = n_2(p^e) = 4u(p^e) & \text{if } u(p^e) \not\equiv 0 \pmod{2} \\ n_1(p^e) = n_2(p^e) = 2u(p^e) & \text{if } u(p^e) \equiv 0 \pmod{4} \\ n_1(p^e) = 2n_2(p^e) = u(p^e) & \text{if } u(p^e) \equiv 2 \pmod{4}. \end{cases}$$

Proof: Since $Q = -1$ and p is an odd prime, we have $d(p^e) = 2$. Thus, by (vi), we have

$$(6) \qquad \bar{u} = \bar{u}(p^e) = \begin{cases} 4u & \text{if } u = u(p^e) \not\equiv 0 \pmod 2 \\ 2u & \text{if } u = u(p^e) \equiv 0 \pmod 4 \\ u & \text{if } u = u(p^e) \equiv 2 \pmod 4. \end{cases}$$

Since $(D/p) = 1$, congruence (1) has two distinct solutions, g_1 and g_2, which belong to exponents $n_1 = n_1(p^e)$ and $n_2 = n_2(p^e)$ modulo p^e.

If $n_1 = n_2 = n$, then, by (4), we have

$$1 \equiv (g_1 g_2)^n \equiv Q^n \equiv (-1)^n \pmod{p^e}$$

and so $n = 2m$, where m is a positive integer. Now it can easily be seen that $g_1^m \equiv g_2^m \equiv -1 \pmod{p^e}$; thus, by (5), it follows that

$$U_m g_1 - U_m g_2 \equiv g_1^m - g_2^m \equiv 0 \pmod{p^e}.$$

By (3) and (i), it follows that $u|m$. Hence, $2u|n$. On the other hand, by Lemma 1, $\bar{u} = n$ and so $2u|\bar{u}$; therefore, by (6), we have $\bar{u} = n = 4u$ if $u \not\equiv 0 \pmod 2$ or $\bar{u} = n = 2u$ if $u \equiv 0 \pmod 4$, since in the third case the relation $2u|\bar{u}$ cannot be satisfied.

Now let $n_1 > n_2$. In this case, we have $g_1^{2n_2} \equiv 1 \pmod{p^e}$ and

$$1 \not\equiv g_1^{n_2} \equiv (g_1 g_2)^{n_2} \equiv Q^{n_2} = (-1)^{n_2} \pmod{p^e}.$$

Thus, n_2 is an odd integer; furthermore, $n_1|2n_2$. By our assumption, it follows that $n_1 = 2n_2$. Thus, by Lemma 1, $\bar{u} = n_1 = 2n_2$ follows, and, by (6), we obtain $\bar{u} = n_1 = 2n_2 = u$, because $\bar{u} = 2n_2 \equiv 2 \pmod 4$. This completes the proof.

3. Proofs of Results

Proof of Theorem 1: If there exists a Lucas primitive root (mod p^e), that is, if congruence (1) is solvable and $n_1(p^e) = \phi(p^e)$ or $n_2(p^e) = \phi(p^e)$, then $(D/p) = 1$ and, by Lemma 1, using the relation $n_i|\phi(p^e)$, we get

$$\bar{u}(p^e) = \phi(p^e).$$

Now assume that $\bar{u}(p^e) = \phi(p^e) = p^{e-1}(p-1)$. Using (iv) we get $\bar{u}(p) = p - 1$ and using (iii) and (v) we have

$$u(p) \,|\, (p - 1, p - (D/p)).$$

If $(D/p) = -1$, then $u(p) = 2$ and so $p|P = U_2$. From this

$$(D/p) = ((P^2 + 4)/p) = (4/p) = 1,$$

a contradiction. Thus, $(D/p) = 1$ and (1) is solvable.

If $p \equiv -1 \pmod 4$, then $\bar{u}(p^e) \equiv 2 \pmod 4$. By Lemma 2, we have

$$\bar{u}(p^e) = n_1(p^e) = 2n_2(p^e) = \phi(p^e),$$

which proves that in this case there is exactly one Lucas primitive root (mod p^e).

If $p \equiv 1 \pmod 4$, then $\bar{u}(p^e) \equiv 0 \pmod 4$. In this case, by Lemma 2,

$$\bar{u}(p^e) = n_1(p^e) = n_2(p^e) = \phi(p^e),$$

which proves that there are exactly two Lucas primitive roots (mod p^e). This completes the proof.

Proof of Theorem 2: If there is exactly one Lucas primitive root mod p^e, that is, congruence (1) is solvable and $n_1(p^e) = \phi(p^e)$, $n_2(p^e) < \phi(p^e)$, then $(D/p) = 1$. By Lemma 2, we have

$$\bar{u}(p^e) = n_1(p^e) = 2n_2(p^e) = u(p^e) = \phi(p^e)$$

and $p \equiv -1 \pmod 4$.

If $u(p^e) = \phi(p^e)$ and $p \equiv -1 \pmod 4$, then $u(p^e) \equiv 2 \pmod 4$. Using (6), we have $\overline{u}(p^e) = u(p^e) = \phi(p^e)$; thus, by Theorem 1, it follows that there exists exactly one Lucas primitive root (mod p^e).

Now we assume that there are exactly two Lucas primitive roots (mod p^e). Then $(D/p) = 1$ and, by Lemma 2, we have

$$u(p^e) = \phi(p^e)/2 \quad \text{if } \phi(p^e)/2 \equiv 0 \pmod 4$$

or

$$u(p^e) = \phi(p^e)/4 \quad \text{if } \phi(p^e)/4 \not\equiv 0 \pmod 2.$$

It follows that $u(p^e) = \phi(p^e)/2$ and $p \equiv 1 \pmod 8$ or $u(p^e) = \phi(p^e)/4$ and $p \equiv 5 \pmod 8$.

If $u(p^e) = \phi(p^e)/2$ and $p \equiv 1 \pmod 8$ or $u(p^e) = \phi(p^e)/4$ and $p \equiv 5 \pmod 8$, then $u(p^e) \equiv 0 \pmod 4$ or $u(p^e) \not\equiv 0 \pmod 2$. By (6), we get $\overline{u}(p^e) = \phi(p^e)$. From this, using Theorem 1, it follows that in this case there are exactly two Lucas primitive roots (mod p^e).

Proof of Corollary 1: If g is a Lucas primitive root (mod p^e), then

$$g^{\phi(p^e)/2} \equiv -1 \pmod{p^e};$$

thus, by (5), $x = g$ satisfies congruence (2).

Let $n = \phi(p^e)/2$ and let g be an integer satisfying the congruence

$$U_n g + U_{n-1} \equiv -1 \pmod{p^e}. \tag{7}$$

From this it follows that

$$(U_n g + U_{n-1})^2 = U_n^2(g^2 - Pg - 1) + U_n g(PU_n + 2U_{n-1}) + (U_n^2 + U_{n-1}^2) \equiv 1 \pmod{p^e}. \tag{8}$$

It is well known that

$$U_n(PU_n - 2QU_{n-1}) = U_{2n} \quad \text{and} \quad U_n^2 - QU_{n-1}^2 = U_{2n-1} \tag{9}$$

for any integer $n \geq 1$. In our case, $Q = -1$ and $u(p^e) = \phi(p^e) = 2n$; therefore, by (8) and (9)

$$U_n^2(g^2 - Pg - 1) + U_{2n-1} \equiv 1 \pmod{p^e} \tag{10}$$

follows. But

$$U_{2n-1} = U_{2n+1} - PU_{2n} \equiv U_{2n+1} \equiv 1 \pmod{p^e}, \tag{11}$$

since, by the condition $u(p^e) = \phi(p^e) = 2n$, as we have seen above, we have $u(p^e) = \phi(p^e) = 2n = \overline{u}(p^e)$; furthermore, it can easily be seen that $p \nmid U_n$, so, by (10) and (11), we get

$$g^2 - Pg - 1 \equiv 0 \pmod{p^e}.$$

Thus, by (5) and (7), we have

$$g^n \equiv U_n g + U_{n-1} \equiv -1 \pmod{p^e}. \tag{12}$$

By Lemma 2, using the condition $u(p^e) = \phi(p^e)$ and (12), it follows that g belongs to the exponent $u(p^e) = \phi(p^e)$ modulo p^e, that is, g is a Lucas primitive root (mod p^e).

Proof of Corollary 2: If g is a primitive root (mod p^e) and $g^2 \equiv Pg + 1$ (mod p^e), then $g(g - P) \equiv 1 \pmod{p^e}$. This shows that $g - P$ is a primitive root (mod p^e).

Proof of Corollary 3: Using Lemma 2, by our assumptions we have

$$u(p) = 2q = p - 1.$$

Using Theorem 2, this proves that there exists exactly one Lucas primitive root (mod p).

References

1. P. Bundschuh & J. S. Shiue. "A Generalization of a Paper by D. D. Wall." *Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat.* Ser II *56* (1974): 135-44.
2. P. Kiss & B. M. Phong. "On the Connection between the Rank of Apparition of a Prime p in the Fibonacci Sequence and the Fibonacci Primitive Roots." *Fibonacci Quarterly* *15* (1977):347-49.
3. D. H. Lehmer. "An Extended Theory of Lucas' Functions." *Ann. of Math.* *31* (1930):419-48.
4. M. J. DeLeon. "Fibonacci Primitive Roots and Period of the Fibonacci Numbers Modulo p." *Fibonacci Quarterly* *15* (1977):353-55.
5. M. E. Mays. "A Note on Fibonacci Primitive Roots." *Fibonacci Quarterly* *20* (1982):111.
6. D. Shanks. "Fibonacci Primitive Roots." *Fibonacci Quarterly* *10* (1972):163-168, 181.
7. D. Shanks & L. Taylor. "An Observation of Fibonacci Primitive Roots." *Fibonacci Quarterly* *11* (1973):159-60.
8. O. Wyler. "On Second Order Recurrences." *Amer. Math. Monthly* *72* (1965):500-506.

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DISTRIBUTION OF RESIDUES OF CERTAIN SECOND-ORDER LINEAR RECURRENCES MODULO p—II

Lawrence Somer
Catholic University of America, Washington, D.C. 20064
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1. Introduction

Let $(u) = u(a, b)$, called the Lucas sequence of the first kind (LSFK), be a second-order linear recurrence satisfying the relation

(1) $\qquad u_{n+2} = au_{n+1} + bu_n,$

where $u_0 = 0$, $u_1 = 1$, and the parameters a and b are integers. Let $D = a^2 + 4b$ be the discriminant of $u(a, b)$. Let $(v) = v(a, b)$, called the Lucas sequence of the second kind (LSSK), be a recurrence satisfying (1) with initial terms $v_0 = 2$, $v_1 = a$. Throughout this paper, p will denote an odd prime unless specified otherwise. Further, d will always denote a residue modulo p. The *period* of $u(a, b)$ modulo p will be denoted by $\mu(p)$. It is known (see [5]) that, if $p \nmid b$, then $u(a, b)$ is purely periodic modulo p. We will always assume that, in the LSFK $u(a, b)$, $p \nmid b$. The *restricted period* of $u(a, b)$ modulo p, denoted by $\alpha(p)$, is the least positive integer t such that $u_{n+t} \equiv su_n \pmod p$ for all nonnegative integers n and some nonzero residue s. Then s is called the *principal multiplier* of (u) modulo p. It is easy to see that $\alpha(p) \mid \mu(p)$ and that $\beta(p) = \mu(p)/\alpha(p)$ is the exponent of the principal multiplier s of (u) modulo p.

We will let $A(d)$ denote the number of times the residue d appears in a full period of $u(a, b)$ modulo p and $N(p)$ denote the number of distinct residues appearing in $u(a, b)$ modulo p. In a previous paper [13], the author considered the LSFK $u(a, 1)$ modulo p and gave constraints for the values which $A(d)$ can attain. In particular, it was shown that $A(d) \le 4$ for all d. Upper and lower bounds for $N(p)$ were given in terms of $\alpha(p)$. Schinzel [8] improved on the constraints given in [13] for the values $A(d)$ can have in the LSFK $u(a, 1)$ modulo p.

In this paper we will consider the LSFK $u(a, -1)$ modulo p and determine the possible values for $A(d)$. In particular, we will show that $A(d) \le 2$ for all d. We will also obtain upper bounds for $N(p)$. If $\alpha(p)$ is known, we will determine $N(p)$ exactly. Schinzel [8] also presented results concerning $A(d)$ for the LSFK $u(a, -1) \pmod p$, citing a preprint on which the present paper is based.

In [12], the author obtained the following partial results concerning $A(d)$ in the LSFK $u(a, -1) \pmod p$.

Theorem 1: Consider the LSFK $u(a, -1)$ modulo p with discriminant $D = a^2 - 4$.

(i) If $p \ge 5$ and $p \nmid D$, then there exists a residue d such that $A(d) = 0$.
(ii) If $p \mid D$, then $A(d) \ne 0$ for any d. In particular, we must have that $a \equiv \pm 2 \pmod p$. If $a \equiv 2 \pmod p$, then

$$u_n \equiv n \pmod p$$

and $A(d) = 1$ for all d. If $a \equiv -2 \pmod p$, then

$$u_n \equiv (-1)^{n+1} n \pmod p$$

and $A(d) = 2$ for all d.

2. Preliminaries

A *general multiplier* of $u(a, b)$ (mod p) is any nonzero residue s' such that

$$u_{n+t} \equiv s'u_n \pmod p$$

for some fixed positive integer t' and all nonzero integers n. It is known that, if s is the principal multiplier of $u(a, b)$ (mod p) and s' is a general multiplier of $u(a, b)$ (mod p), then

$$s' \equiv s^i \pmod p$$

for some i such that $0 \le i \le \beta(p) - 1$.

For the LSFK $u(a, b)$, let $k = \alpha(p)$. We will let $A_i(d)$ denote the number of times the residue d appears among the terms

$$u_{ki}, u_{ki+1}, \ldots, u_{ki+k-1} \text{ modulo } p,$$

where $0 \le i \le \beta(p) - 1$. Results concerning $A_i(d)$ will be obtained for the LSFK $u(a, -1)$ (mod p).

The following results concerning $u(a, b)$ and $v(a, b)$ are well known:

(2) $$v_n^2 - Du_n^2 = 4(-b)^n;$$

(3) $$u_{2n} = u_n v_n.$$

Proofs can be found in [4].

3. The Main Theorems

Our results concerning the distribution of residues in the LSFK $u(a, -1)$ modulo p will depend on knowledge of the values of $\alpha(p)$, $\beta(p)$, and (D/p), where (D/p) denotes the Legendre symbol. Theorems 2 and 3 will provide information on the values $\mu(p)$, $\alpha(p)$, and $\beta(p)$ can take for the LSFK $u(a, -1)$ depending on whether $(D/p) = 0$, 1, or -1.

Theorem 2: Let $u(a, b)$ be a LSFK. Then

(4) $$\alpha(p) \mid p - (D/p).$$

Further, if $p \nmid D$, then

(5) $$\alpha(p) \mid (p - (D/p))/2$$

if and only if $(-b/p) = 1$. Moreover, if $(D/p) = 1$, then

(6) $$\mu(p) \mid p - 1.$$

Proof: Proofs of (4) and (6) are given in [4, pp. 44-45] and [1, pp. 315-17]. Proofs of (5) are given in [6, p. 441] and [1, pp. 318-19].

Theorem 3: Consider the LSFK $u(a, -1)$ with discriminant D. Suppose that $p \nmid D$. Let D' be the square-free part of D. If $|a| \ge 3$, let ε be the funcamental unit of $Q(\sqrt{D'})$. Let s be the principal multiplier of $u(a, -1)$ modulo p.

(i) $\beta(p) = 1$ or 2; $s \equiv 1$ or $-1 \pmod p$.
(ii) If $\alpha(p) \equiv 0 \pmod 2$, then $\beta(p) = 2$.
(iii) If $\alpha(p) \equiv 1 \pmod 2$, then $\beta(p)$ may be 1 or 2.
(iv) If $(2 - a/p) = (2 + a/p) = -1$, then $\alpha(p) \equiv 0 \pmod 2$ and $\beta(p) = 2$.
(v) If $(2 - a/p) = 1$ and $(2 + a/p) = -1$, then $\alpha(p) \equiv 1 \pmod 2$ and $\beta(p) = 2$.
(vi) If $(2 - a/p) = -1$ and $(2 + a/p) = 1$, then $\alpha(p) \equiv 1 \pmod 2$ and $\beta(p) = 1$.
(vii) If $p \equiv 1 \pmod 4$, $(D/p) = 1$, and the norm of ε is -1, then $\alpha(p) \mid (p-1)/4$.

Proof: This is proved in [11, pp. 328-31].

We are now ready for the statement of our principal theorems. Following the notation introduced by Schinzel in [8], we will let $S = S(p)$ denote the set of all the values which $A(d)$ attains in the LSFK $u(a, -1)$ modulo p.

Theorem 4: Let $u(a, -1)$ be an LSFK. Suppose that $\beta(p) = 1$, and let $k = \alpha(p)$. Then $k \equiv 1 \pmod 2$. Let $A_0'(d)$ denote the number of times the residue d appears among the terms $u_0, u_1, \ldots, u_{(k-1)/2}$ modulo p. Let $A_1'(d)$ denote the number of times the residue d appears among the terms $u_{(k+1)/2}, u_{(k+3)/2}, \ldots, u_k$ modulo p.

(i) $A(d) = A(-d)$.
(ii) If $p \geq 5$, then $S = \{0, 1\}$.
(iii) $A'(d) = 0$ or 1 for $i = 0, 1$.
(iv) $A_0'(d) = A_1'(-d)$.

Theorem 5: Let $u(a, -1)$ be an LSFK. Suppose that $\alpha(p) \equiv 1 \pmod 2$ and $\beta(p) = 2$.

(i) $A(d) = A(-d)$.
(ii) If $p \geq 5$, then $S = \{0, 2\}$.
(iii) If $d \not\equiv 0 \pmod p$, then $A_i(d) = 0$ or 2 for $i = 0, 1$.
(iv) $A_0(0) = A_1(0) = 1$.
(v) $A_0(d) = A_1(-d)$.

Theorem 6: Let $u(a, -1)$ be an LSFK with discriminant D. Suppose $\alpha(p) \equiv 0 \pmod 2$. Then $\beta(p) = 2$ and $(-D/p) = 1$.

(i) $A(d) = A(-d)$.
(ii) $A(d) = 1$ if and only if $d \equiv \pm 2/\sqrt{-D} \pmod p$.
(iii) If $p \geq 5$, then $S = \{0, 1, 2\}$.
(iv) If $d \not\equiv 0$ or $\pm 2/\sqrt{-D} \pmod p$, then $A_i(d) = 0$ or 2 for $i = 0, 1$.
(v) If $d \equiv 0$ or $\pm 2/\sqrt{-D} \pmod p$, then $A_i(d) = 1$ for $i = 0, 1$.
(vi) $A_0(d) = A_1(-d)$.

Theorem 7: Let $u(a, -1)$ be an LSFK. Suppose that $p \nmid D$ and $a \not\equiv 0, 1$, or $-1 \pmod p$. Let D' be the square-free part of D. Let ε be the fundamental unit of $Q(\sqrt{D'})$. Let $c_1 = 0$ if $\alpha(p) \equiv 1 \pmod 2$ and $c_1 = 1$ if $\alpha(p) \equiv 0 \pmod 2$.

(i) $N(p) \equiv 1 \pmod 2$.
(ii) $N(p) \leq (p - (D/p))/2 + c_1$.
(iii) If $p \equiv 1 \pmod 4$, $(D/p) = 1$, and ε has norm -1, then

$$N(p) \leq (p - 1)/4 + c_1.$$

(iv) $N(p) = \alpha(p) + c_1$.

4. Necessary Lemmas

The following lemmas will be needed for the proofs of Theorems 4-7.

Lemma 1: Let $u(a, b)$ be an LSFK. Let s be the principal multiplier of (u) modulo p and let $k = \alpha(p)$. Then

$$u_{k-n} \equiv (-1)^{n+1} s u_n / b^n \pmod p, \tag{7}$$

for $0 \leq n \leq k$. In particular, if $b \equiv -1 \pmod p$, then

$$u_{k-n} \equiv -s u_n \pmod p, \tag{8}$$

for $0 \leq n \leq k$.

Proof: We proceed by induction. Clearly,

$$u_{k-0} \equiv 0 \equiv (-1)^{0+1} s u_0 / b^0 \equiv 0 \equiv u_0 \pmod p.$$

Also,

$$u_{k-1} \equiv b^{-1}(u_{k+1} - a u_k) \equiv b^{-1}(s u_1 - a \cdot 0) \equiv (-1)^{1+1} s u_1 / b^1 \pmod p.$$

Now assume that

$$u_{k-n} \equiv (-1)^{n+1} s u_n / b^n \pmod{p}$$

and

$$u_{k-(n+1)} \equiv (-1)^{n+2} s u_{n+1} / b^{n+1} \pmod{p}.$$

Then

$$\begin{aligned} u_{k-(n+2)} &\equiv b^{-1}(u_{k-n} - a u_{k-(n+1)}) \\ &\equiv b^{-1}(-1)^{n+1} s[(b u_n / b^{n+1}) + (a u_{n+1} / b^{n+1})] \\ &= b^{-1}(-1)^{n+1} s(u_{n+2}/b^{n+1}) \equiv (-1)^{n+3} s u_{n+2}/b^{n+2} \pmod{p}. \end{aligned}$$

The result for $b \equiv -1 \pmod{p}$ follows by inspection.

Lemma 2: Let $u(a, b)$ be an LSFK. Let n and c be positive integers such that $n + c \le \alpha(p) - 1$. Let $k = \alpha(p)$. Then

(9) $$(u_{n+c}/u_n)(u_{k-n}/u_{k-n-c}) \equiv (-b)^c \pmod{p}.$$

Proof: This follows from congruence (7) in Lemma 1. Another proof is given in [12, p. 123].

Lemma 3: Consider the LSFK $u(a, b)$. Let c be a fixed integer such that $1 \le c \le \alpha(p) - 1$. Then the ratios u_{n+c}/u_n are all distinct modulo p for $1 \le n \le \alpha(p) - 1$.

Proof: This is proved in [12, pp. 120-21].

Lemma 4: Let $u(a, -1)$ be an LSFK and let $k = \alpha(p)$. Then

$$u_n \not\equiv \pm u_{n+c} \pmod{p}$$

for any positive integers n and c such that either $n + c \le k/2$ or it is the case that $n \ge k/2$ and $n + c \le k - 1$.

Proof: Suppose there exist positive integers n and c such that $n + c \le k - 1$ and

$$u_n \equiv \pm u_{n+c} \pmod{p}.$$

Then

$$u_{n+c}/u_n \equiv \pm 1 \pmod{p}.$$

By Lemma 2,

$$(u_{n+c}/u_n)(u_{k-n}/u_{k-n-c}) \equiv 1^c \equiv 1 \pmod{p};$$

hence,

$$u_{k-n}/u_{k-n-c} \equiv u_{n+c}/u_n \equiv \pm 1 \pmod{p}.$$

Thus, by Lemma 3,

$$n + c = k - n$$

leading to

$$n = (k - c)/2.$$

Consequently,

$$n = (k - c)/2 \text{ and } n + c = (k + c)/2.$$

The result now follows.

Lemma 5: Let $u(a, -1)$ be an LSFK and let $k = \alpha(p)$. Let N_1 be the largest integer t such that there exist integers $n_1, n_2, \ldots, n_t$ for which $1 \le n_i \le [k/2]$ and $u_{n_i} \not\equiv \pm u_{n_j} \pmod{p}$ if $1 \le i < j \le [k/2]$, where $[x]$ is the greatest integer less than or equal to x. Then

(10) $$N(p) = 2N_1 + 1.$$

Proof: By Theorem 3, $\beta(p) = 1$ or 2. First, suppose that $\beta(p) = 2$. Then -1 is the principal multiplier of (u) modulo p and the residue $-d$ appears in (u)

modulo p if and only if d appears in (u) modulo p. Moreover, it follows from Lemma 1 and the fact that -1 is a principal multiplier of (u) modulo p that if $d \not\equiv 0 \pmod{p}$ and d appears in $(u) \pmod{p}$, then $d \equiv \pm u_{n_i} \pmod{p}$ for some i such that $1 \le i \le N_1$. Including the residue 0, we see that (10) holds.

Now suppose that $\beta(p) = 1$. By congruence (8) in Lemma 1, the residue $-d$ appears in (u) modulo p if and only if d appears in (u) modulo p. It also follows from Lemma 1 that, if $d \not\equiv 0 \pmod{p}$ and d appears in (u) modulo p, then $d \equiv \pm u_{n_i} \pmod{p}$ for some i such that $1 \le i \le N_1$. Counting the residue 0, we see that the result follows.

Lemma 6: Let $u(a, -1)$ be an LSFK. Let $k = \alpha(p)$. Let $A'(d)$ denote the number of times the residue d appears among the terms $n_1, n_2, \ldots, n_{[k/2]}$ modulo p. Let N_1 be defined as in Lemma 5.

(i) $A'(d) + A'(-d) = 0$ or 1.
(ii) $N_1 = [k/2]$.

Proof: (i) follows from Lemma 4; (ii) follows from (i).

Lemma 7: Let $u(a, b)$ be an LSFK. Suppose that $p \nmid b$. Let s be the principal multiplier of (u) modulo p and s^j be a general multiplier of $(u) \pmod{p}$, where $1 \le j \le \beta(p) - 1$. Then

$$A(d) = A(s^j d).$$

Proof: This is proved in [13].

Lemma 8: Let $u(a, -1)$ be an LSFK with discriminant D. Suppose that $\alpha(p) \equiv 0 \pmod{2}$. Let $k = \alpha(p)$. Then

$$u_{k/2} \equiv \pm 2/\sqrt{-D} \pmod{p}.$$

Proof: Since $\alpha(p) \equiv 0 \pmod{2}$, it follows from (4) that $p \nmid D$. By (2), it follows that

$$v_{k/2}^2 - Du_{k/2}^2 = 4(1)^{k/2} = 4. \tag{11}$$

Now, $u_{k/2} \not\equiv 0 \pmod{p}$. Thus, by (3), $v_{k/2} \equiv 0 \pmod{p}$. Hence, by (11),

$$-Du_{k/2}^2 \equiv 4 \pmod{p}$$

and the result follows.

5. Proofs of the Main Theorems

We are finally ready to prove Theorems 4-7.

Proof of Theorem 4: The fact that $\alpha(p) \equiv 1 \pmod{2}$ follows from Theorem 3.

(i) and (iv) follow from Lemma 1; (ii) follows from Theorem 1(i), Lemma 6(i), and Lemma 1; (iii) follows from Lemma 6(i) and the fact that $A(0) = 1$.

Proof of Theorem 5: (i) follows from Lemma 7; (ii) and (iii) follow from Theorem 1(i), Lemma 6(i), Lemma 1, and the fact that -1 is the principal multiplier of $u(a, -1)$ modulo p; (iv) follows by inspection; and (v) follows from the fact that -1 is the principal multiplier of (u) modulo p.

Proof of Theorem 6: The fact that $\beta(p) = 2$ follows from Theorem 3. The fact that $(-D/p) = 1$ follows from Lemma 8.

(i) follows from Lemma 7; (ii), (iv), and (v) follow from Lemmas 8, 6(i), and 1 and the fact that -1 is the principal multiplier of (u) modulo p; (iii) follows from Theorem 1(i), Lemma 6(i), Lemma 1 and the fact that -1 is the principal multiplier of $u(a, -1)$ modulo p; and (vi) follows from the fact that -1 is the principal multiplier of (u) modulo p.

Remark: Note that Theorem 3 gives conditions for the hypotheses of Theorems 4-6 to be satisfied.

Proof of Theorem 7: (i) follows from Lemma 5; (ii) follows from Lemma 5, Lemma 6(ii), and Theorem 2; (iii) This follows from Lemma 5, Lemma 6(ii), and Theorem 3(vii); and (iv) follows from Lemmas 5 and 6(ii).

6. Special Cases

For completeness, we present Theorems 8 and 9 which detail special cases we have not treated thus far. For these theorems, p will designate a prime, not necessarily odd.

Theorem 8: Let $u(a, -1)$ be an LSFK. Suppose $p \nmid D$.

(i) If $a \equiv 0 \pmod p$, then $\alpha(p) = 2$, $\beta(p) = 2$, $N(p) = 3$, $A(0) = 2$, $A(1) = A(-1) = 1$, and $A(d) = 0$ if $d \not\equiv 0, 1$, or $-1 \pmod p$.

(ii) If $a \equiv 1 \pmod p$ and $p > 2$, then $\alpha(p) = 3$, $\beta(p) = 2$, $N(p) = 3$, $A(0) = A(1) = A(-1) = 2$, and $A(d) = 0$ if $d \not\equiv 0, 1$, or $-1 \pmod p$.

(iii) If $a \equiv 1 \pmod p$ and $p = 2$, then $\alpha(p) = 3$, $\beta(p) = 1$, $N(p) = 2$, $A(0) = 1$, and $A(1) = 2$.

(iv) If $a \equiv -1 \pmod p$ and $p > 2$, then $\alpha(p) = 3$, $\beta(p) = 1$, $N(p) = 3$, $A(0) = A(1) = A(-1) = 1$, and $A(d) = 0$ if $d \not\equiv 0, 1$, or $-1 \pmod p$.

Proof: (i)-(iv) follow by inspection.

Theorem 9: Let $u(a, -1)$ be an LSFK. Suppose that $p \mid D$. Then $a \equiv \pm 2 \pmod p$. If $a \equiv 2 \pmod p$, then $\alpha(p) = p$, $\beta(p) = 1$, $N(p) = p$, and $A(d) = 1$ for all residues d modulo p. If $p > 2$ and $a \equiv -2 \pmod p$, then $\alpha(p) = p$, $\beta(p) = 2$, $N(p) = p$, and $A(d) = 2$ for all residues d modulo p.

Proof: This follows from Theorem 1(ii).

Remark: If $D \equiv 0 \pmod p$, we see from Theorem 9 that the residues of $u(a, -1)$ are equidistributed modulo p. See [7, p. 463] for a comprehensive list of references on equidistributed linear recurrences.

7. Concluding Remarks

In [8] and [13] it was shown that, for the LSFK $u(a, 1)$ modulo p, $A(d) \le 4$. In the present paper it was shown that, for the LSFK $u(a, -1)$ modulo p, $A(d) \le 2$. In [14] we extend these results considerably. Specifically, let $w(a, b)$ be a second-order linear recurrence with arbitrary initial terms w_0, w_1 over the finite field F_q satisfying the relation

$$w_{n+2} = aw_{n+1} + bw_n.$$

where $b \ne 0$. Then

$$A(d) \le 2 \cdot \operatorname{ord}(-b)$$

for all elements $d \in F_q$, where $\operatorname{ord}(x)$ denotes the order of x in F_q.

References

1. R. P. Backstrom. "On the Determination of the Zeros of the Fibonacci Sequence." *Fibonacci Quarterly* 4.4 (1966):313-22.
2. G. Bruckner. "Fibonacci Sequences Modulo a Prime $p \equiv 3 \pmod 4$." *Fibonacci Quarterly* 8.2 (1970):217-20.
3. S. A. Burr. "On Moduli for Which the Fibonacci Sequence Contains a Complete System of Residues." *Fibonacci Quarterly* 9.4 (1971):497-504.

4. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$." *Ann. Math. Second Series* *15* (1913):30-70.
5. R. D. Carmichael. "On Sequences of Integers Defined by Recurrence Relations." *Quart. J. Pure Appl. Math.* *48* (1920):343-72.
6. D. H. Lehmer. "An Extended Theory of Lucas' Functions." *Ann. Math. Second Series* *31* (1930):419-48.
7. R. Lidl & H. Niederreiter. *Finite Fields*. Reading, Mass.: Addison-Wesley, 1983.
8. A. Schinzel. "Special Lucas Sequences, Including the Fibonacci Sequence, Modulo a Prime." To appear.
9. A. P. Shah. "Fibonacci Sequences Modulo m." *Fibonacci Quarterly* *6.1* (1968): 139-41.
10. L. Somer. "The Fibonacci Ratios F_{k+1}/F_k Modulo p." *Fibonacci Quarterly* *13.4* (1975):322-24.
11. L. Somer. "The Divisibility Properties of Primary Lucas Recurrences with Respect to Primes." *Fibonacci Quarterly* *18.4* (1980):316-34.
12. L. Somer. "Primes Having an Incomplete System of Residues for a Class of Second-Order Linear Recurrences." *Applications of Fibonacci Numbers*. Ed. by A. N. Philippou, A. F. Horadam, & G. E. Bergum. Dordrecht, Holland: Kluwer Academic Publishers, 1988, pp. 113-41.
13. L. Somer. "Distribution of Residues of Certain Second-Order Linear Recurrences Modulo p." *Applications of Fibonacci Numbers*, Vol. 3. Ed. G. E. Bergum, A. N. Philippou, and A. F. Horadam. Dordrecht, Holland: Kluwer Academic Publishers, 1990, pp. 311-24.
14. L. Somer, H. Niederreiter, * A. Schinzel. "Maximal Frequencies of Elements in Second-Order Recurrences Over a Finite Field." To appear.

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PASCAL'S TRIANGLE MODULO 4

Kenneth S. Davis
Albion College, Albion, MI 49224
William A. Webb
Washington State University, Pullman, WA 99164-2930
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Introduction

Pascal's triangle has a seemingly endless list of fascinating properties. One such property which has been extensively studied is the fact that the number of odd entries in the n^{th} row is equal to 2^t where t is the number of ones in the base two representation of n (see [1], [2], and [3]).

Generalizations of this property seem surprisingly difficult. For a prime modulus, Hexel & Sachs [4] obtain a rather involved expression for the number of occurrences of each residue. Explicit formulas are obtained for $p = 3$ and 5. In particular, for a prime modulus p, the number of occurrences for a given residue in row n depends only on the number of times each digit appears in the base p representation of n. However, it is easily seen that composite moduli do not satisfy this property. In this article we consider Pascal's triangle modulo 4 and obtain explicit formulas for the number of occurrences of each residue modulo 4.

Notation and Conventions

The letters n, j, k, ℓ will denote nonnegative integers. The letter n will typically refer to an arbitrary row of Pascal's triangle. We will need detailed information on the base two representation of n. The following definitions will be useful.

Let

$$n = \sum_{i=0}^{k} a_i 2^i, \text{ where } a_i = 0 \text{ or } 1, \text{ and } B(n) = \sum_{i=0}^{k} a_i.$$

We also define

$$c_i = 1 \text{ if and only if } a_{i+1} = 1 \text{ and } a_i = 0, \text{ where } a_{k+1} = 0.$$

We then define

$$C(n) = \sum_{i=0}^{k} c_i.$$

Similarly, we define

$$d_i = (a_{i+1})(a_i) \text{ and } D(n) = \sum_{i=0}^{k} d_i.$$

Clearly, $B(n)$ is the number of "1"; $C(n)$ is the number of "10"; and $D(n)$ is the number of "11" blocks, not necessarily disjoint, in the base two representation of n.

For our purposes,

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

is defined for integer values of n and j; further,

$$\binom{n}{j} = 0 \text{ if } j < 0 \text{ or } j > n.$$

We define

$$\left\langle {n \atop j} \right\rangle = r \text{ if and only if } \binom{n}{j} \equiv r \pmod 4.$$

Let $N(n) = (a, b, c)$, where $N_1(n) = a$ is the number of ones, $N_2(n) = b$ is the number of twos, and $N_3(n) = c$ is the number of threes in the n^{th} row of Pascal's triangle.

We will make use of several well-known results found in Singmaster [5].

Lemma 1: $p^e \Big\| \binom{n}{j}$ if and only if the p-ary subtraction $n - j$ has e borrows.

Lemma 2: The number of odd binomial coefficients in the n^{th} level of Pascal's triangle is $2^{B(n)}$.

We begin our work with an easy result which we prove for completeness.

Lemma 3: $N(2^k) = (2, 1, 0)$ when $k \geq 1$.

Proof: Clearly

$$\left\langle {2^k \atop 0} \right\rangle = \left\langle {2^k \atop 2^k} \right\rangle = 1$$

so $N_1(2^k) \geq 2$. By Lemma 2,

$$N_1(2^k) + N_3(2^k) = 2.$$

So $N_1(2^k) = 2$ and $N_3(2^k) = 0$. Further, for $0 < j < 2^{k-1}$, $2^k - j$ will have at least two borrows when performed in base two. Thus,

$$4 \Big| \binom{2^k}{j}; \text{ hence, } \left\langle {2^k \atop j} \right\rangle = 0.$$

Similarly, for $2^{k-1} < j < 2^k$. Noticing

$$\left\langle {2^k \atop 2^{k-1}} \right\rangle = 2,$$

we conclude $N_2(2^k) = 1$. □

Lemma 4: Let $n = 2^k + \ell$, where $0 < \ell < 2^k$.

(i) If $\ell < j < 2^{k-1}$, then $\left\langle {n \atop j} \right\rangle = 0$.

(ii) If $\ell < j < 2^k$, then $\left\langle {n \atop j} \right\rangle = 0$ or 2.

Proof: In case (i), we must borrow at least twice in subtracting $n - j$, and in case (ii), at least one borrow must take place.

By Lemmas 3 and 4, it is clear that Pascal's triangle modulo 4 has the following form:

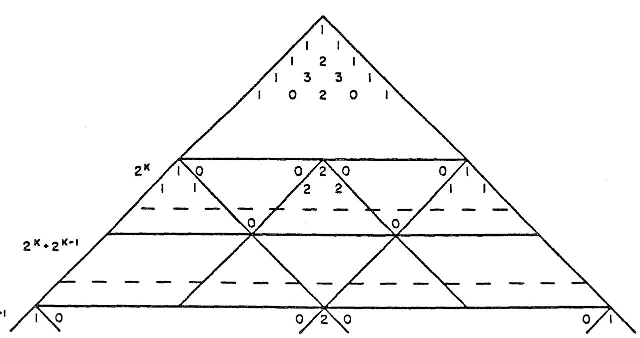

Figure 1

The standard identity

$$\left\langle {n \atop j - 1} \right\rangle + \left\langle {n \atop j} \right\rangle = \left\langle {n + 1 \atop j} \right\rangle$$

shows that any row in Figure 1 completely determines all subsequent rows. This identity and Lemma 3 yield the following recursive relations.

Part 1: If $n = 2^k + \ell$, where $0 \le \ell < 2^{k-1}$ (see upper dashed line in Fig. 1):

(i) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle$ for $0 \le j \le \ell$;

(ii) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle = 0$ for $\ell + 1 \le j < 2^{k-1}$;

(iii) $\left\langle {n \atop j} \right\rangle = 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle$ for $2^{k-1} \le j \le 2^{k-1} + \ell$;

(iv) $\left\langle {n \atop j} \right\rangle = 0$ for $2^{k-1} + \ell < j < 2^k$;

(v) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j - 2^k} \right\rangle$ for $2^k \le j \le n$.

Part 2: If $n = 2^k + \ell$, where $2^{k-1} \le \ell < 2^k$ (see lower dashed line in Fig. 1):

(vi) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle$ for $0 \le j < 2^{k-1}$;

(vii) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle + 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle$ for $2^{k-1} \le j \le \ell$;

(viii) $\left\langle {n \atop j} \right\rangle = 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle$ for $\ell < j < 2^k$;

(ix) $\left\langle {n \atop j} \right\rangle = 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle + \left\langle {\ell \atop j - 2^k} \right\rangle$ for $2^k \le j \le \ell + 2^k$

(x) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j - 2^k} \right\rangle$ for $2^{k-1} + \ell < j \le n$.

All of the expressions above are considered modulo 4.

We are now in a position to count the number of ones and threes modulo 4. Recall that $D(n) > 0$ if and only if the base two representation of n has a "11" block.

Theorem 5: If $D(n) = 0$, then $N_1(n) = 2^{B(n)}$ and $N_3(n) = 0$.

Proof: We use induction on n. The theorem is true for $n \le 3$. Since $D(n) = 0$, we know $n = 2^k + \ell$, where $\ell < 2^{k-1}$ and $D(\ell) = 0$. Using (iii) of the recursion, we have

$$\left\langle {n \atop j} \right\rangle \equiv 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle \pmod 4$$

for $2^{k-1} \le j < 2^k$. Thus, there are no threes in this section of the n^{th} row of Pascal's triangle. By (i) and (v), we see

$$\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle \text{ for } j < 2^{k-1} \quad \text{and} \quad \left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j - 2^{k-1}} \right\rangle \text{ for } j > 2^k.$$

Thus, $N_3(n) = 2N_3(\ell)$. But by induction, $N_3(\ell) = 0$. The theorem now follows from Lemma 2. □

Theorem 6: If $D(n) > 0$, then $N_1(n) = N_3(n) = 2^{B(n)-1}$.

Proof: The result is clear for $n \le 4$.

Case 1: $n = 2^k + \ell$, where $\ell < 2^{k-1}$. Clearly, $D(\ell) > 0$. When considering $\langle{n \atop j}\rangle$, by the recursion, we need only consider $j \le \ell$ or $2^k \le j$. For $0 \le j \le \ell$, there are as many ones and threes as in row ℓ. By symmetry, there are as many for $2^k \le j$. Thus, $N_1(n) = 2N_1(\ell)$ and $N_3(n) = 2N_3(\ell)$, so the result holds by induction.

Case 2: $n = 2^k + \ell$, where $2^{k-1} \le \ell < 2^k$. Let $\ell = 2^{k-1} + r$. Consider the five sections of row n:

A. $0 \le j < 2^{k-1}$;

B. $2^{k-1} \le j \le \ell$;

C. $\ell < j < 2^k$;

D. $2^k \le j \le \ell + 2^{k-1}$;

E. $\ell + 2^{k-1} < j \le \ell + 2^k = n$.

By symmetry, $A = E$ and $B = D$. In section C, by (viii),

$$\left\langle{n \atop j}\right\rangle = 2\left\langle{\ell \atop j - 2^{k-1}}\right\rangle,$$

and there are no ones or threes in C.

In section A,

$$\left\langle{n \atop j}\right\rangle = \left\langle{\ell \atop j}\right\rangle \quad \text{for } 0 \le j < 2^{k-1}.$$

Since we are trying to count the number of times $\langle{\ell \atop j}\rangle = 1$ or 3, by Lemma 4, we need only consider $j \le r$.

In section B,

$$\left\langle{n \atop j}\right\rangle = \left\langle{\ell \atop j}\right\rangle + 2\left\langle{\ell \atop j - 2^{k-1}}\right\rangle.$$

Now, by Lemma 1, $\langle{\ell \atop j}\rangle$ and $\langle{\ell \atop j - 2^{k-1}}\rangle$ are both odd or both even. We need only consider the case when they are both odd. Thus,

$$2\left\langle{\ell \atop j - 2^{k-1}}\right\rangle \equiv 2 \pmod{4}.$$

Observing $x + 2 \equiv 3x$ if $x \equiv 1$ or 3 (modulo 4), we have

$$\left\langle{n \atop j}\right\rangle \equiv 3\left\langle{\ell \atop j}\right\rangle \equiv 3\left\langle{\ell \atop \ell - j}\right\rangle \pmod{4}.$$

Since we are in section B, $2^{k-1} \le j \le \ell$, and recalling that $\ell = 2^{k-1} + r$, we see that $0 \le \ell - j \le r$, that is, $\langle{\ell \atop \ell - j}\rangle$ is in section A.

This implies the number of ones in section A equals the number of threes in section B and the number of threes in section A equals the number of ones in section B. Hence, there are an equal number of ones and threes in the combined sections of A and B; thus, $N_1(n) = N_3(n)$. The theorem now follows from Lemma 2. □

Theorem 7: $N_2(n) = C(n)2^{B(n)-1}$.

Proof: Recall that

$$\left\langle{n \atop j}\right\rangle = 2 \text{ if and only if } 2 \,\Big\|\, \binom{n}{j},$$

which occurs if and only if $n - j$ has exactly one borrow in base two. Thus, we wish to count the number of j's such that $n - j$ has exactly one borrow. Suppose the borrow occurs from position $i + 1$ to position i. If

$$n = \sum_{i=0}^{k} a_i 2^i \quad \text{and} \quad j = \sum_{i=0}^{k} b_i 2^i,$$

then $a_{i+1} = 1$ and $a_i = 0$, $b_{i+1} = 0$ and $b_i = 1$. Thus, if $C(n) = 0$, it follows that $N_2(n) = 0$.

So we assume $C(n) \geq 1$. To ensure no other borrow occurs, it must be the case that $b_\ell = 0$ when $a_\ell = 0$ for $\ell \neq i$. When $a_\ell = 1$, $\ell \neq i + 1$, b_ℓ may equal 0 or 1. So for each "10" in n's representation, there are $2^{B(n)-1}$ j's for which $\langle {n \atop j} \rangle = 2$. Thus, $N_2(n) = C(n)2^{B(n)-1}$. □

To summarize, we have

$$N(n) = \begin{cases} (2^{B(n)},\ C(n)2^{B(n)-1},\ 0) & \text{if } D(n) = 0, \\ (2^{B(n)-1},\ C(n)2^{B(n)-1},\ 2^{B(n)-1}) & \text{if } D(n) > 0. \end{cases}$$

Recurrences of the type used here are possible for other composite moduli, but they become increasingly complex. A complete characterization of the residues modulo 6 would be interesting, since 6 is not a prime power. Also, the question of general results for arbitrary composite moduli remains open.

References

1. L. Carlitz. "The Number of Binomial Coefficients Divisible by a Fixed Power of a Prime." *Rend. Circ. Mat. Palermo (II) 16* (1967):299-320.
2. N. J. Fine. "Binomial Coefficients Modulo a Prime." *Amer. Math. Monthly 54* (1947):589-92.
3. H. Harborth. "Number of Odd Binomial Coefficients." *Proc. Amer. Math. Soc. 62* (1977):19-24.
4. E. Hexel & H. Sachs. "Counting Residues Modulo a Prime in Pascal's Triangle." *Indian J. Math. 20* (1978):91-105.
5. D. Singmaster. "Divisibility of Binomial and Multinomial Coefficients by Primes and Prime Powers." *A Collection of Manuscripts Related to the Fibonacci Sequence.* Santa Clara, Calif: The Fibonacci Association, 1980, pp. 98-113.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-682 *Proposed by Joseph J. Kostal, U. of Illinois at Chicago*

Let $T(n)$ be the triangular number $n(n+1)/2$. Show that

$$T(L_{2n}) - 1 = \frac{1}{2}(L_{4n} + L_{2n}).$$

B-683 *Proposed by Joseph J. Kostal, U. of Illinois at Chicago*

Let $L(n) = L_n$ and $T_n = n(n+1)/2$. Show that

$$L(T_{2n}) = L(2n^2)L(n) + (-1)^{n+1}L(2n^2 - n).$$

B-684 *Proposed by L. Kuipers, Sierre, Switzerland*

(a) Find a straight line in the Cartesian plane such that (F_n, F_{n+1}) and (F_{n+1}, F_{n+2}) are on opposite sides of the line for all positive integers n.

(b) Is the line unique?

B-685 *Proposed by Stanley Rabinowitz, Westford, Massachusetts, and Gareth Griffith, U. of Saskatchewan, Saskatoon, Saskatchewan, Canada*

For integers $n \geq 2$, find k as a function of n such that

$$F_{k-1} \leq n < F_k.$$

B-686 *Proposed by Jeffrey Shallit, U. of Waterloo, Ontario, Canada*

Let a and b be integers with $0 < a \le b$. Set $c_0 = a$, $c_1 = b$, and for $n \ge 2$ define c_n to be the least integer with $c_n/c_{n-1} > c_{n-1}/c_{n-2}$. Find a closed form for c_n in the cases:

(a) $a = 1$, $b = 2$; (b) $a = 2$, $b = 3$.

B-687 *Proposed by Jeffrey Shallit, U. of Waterloo, Ontario, Canada*

Let c_n be as in Problem B-686. Find a closed form for c_n in the case with $a = 1$ and b an integer greater than 1.

SOLUTIONS

Pell Parity Problem

B-658 *Proposed by Joseph J. Kostal, U. of Illinois at Chicago*

Prove that $Q_1^2 + Q_2^2 + \cdots + Q_n^2 \equiv P_n^2 \pmod 2$, where the P_n and Q_n are the Pell numbers defined by

$$P_{n+2} = 2P_{n+1} + P_n,\ P_0 = 0,\ P_1 = 1;$$

$$Q_{n+2} = 2Q_{n+1} + Q_n,\ Q_0 = 1,\ Q_1 = 1.$$

Solution by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

More generally, it can be proved that

$$S = \sum_{i=1}^{n} Q_i^{k_i} \equiv P^h \pmod 2,$$

where $k_1, k_2, \ldots, k_n$ and h are arbitrary positive integers. Using the recurrence relation, it is readily seen that Q_i is odd for all i, so that $Q_i^{k_i}$ is. Therefore, S is odd (even) if n is odd (even). On the other hand, it is known that the Pell numbers P_n (and any power of them) are odd (even) if n is odd (even).

Also solved by Richard André-Jeannin, Charles Ashbacher, Wray Brady, Paul S. Bruckman, Russell Euler, Herta T. Freitag, C. Georghiou, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Nearest Integer

B-659 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

For $n \ge 3$, what is the nearest integer to $F_n\sqrt{5}$?

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, NY

For $n \ge 3$, L_n is the nearest integer to $F_n\sqrt{5}$, since

$$|F_n\sqrt{5} - L_n| = 2|\beta|^n \le 2|\beta|^3 < 1/2.$$

Also solved by Charles Ashbacher, Wray Brady, Paul S. Bruckman, Russell Euler, Piero Filipponi, Herta T. Freitag, C. Georghiou, Russell Jay Hendel & Sandra A. Monteferrante, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Binomial Expansions

B-660 *Proposed by Herta T. Freitag, Roanoke, VA*

Find closed forms for:

$$\text{(i)}\quad 2^{1-n}\sum_{i=0}^{[n/2]}\binom{n}{2i}5^i, \quad \text{(ii)}\quad 2^{1-n}\sum_{i=1}^{[(n+1)/2]}\binom{n}{2i-1}5^{i-1},$$

where $[t]$ is the greatest integer in t.

Solution by Lawrence Somer, Washington, D.C.

The answer to (i) is L_n; the answer to (ii) is F_n. These representations are obtained from the binomial expansions for

$$L_n = ((1+\sqrt{5})/2)^n + ((1-\sqrt{5})/2)^n$$

and

$$F_n = (1/\sqrt{5})[((1+\sqrt{5})/2)^n - ((1-\sqrt{5})/2)^n],$$

respectively. The representation for F_n in (ii) was given by E. Catalan in 1857 in *Manuel des Candidats a l'Ecole Polytechnique*. A proof for the representation of L in (i) can be found in [2, p. 69]. Proofs for the representation of F in (ii) can be found in [1, p. 150] and [2, p. 68].

References

1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*, 4th ed. London: Oxford University Press, 1960.
2. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. New York: Halsted Press, 1989.

Also solved by Richard André-Jeannin, Wray Brady, Paul S. Bruckman, Piero Filipponi, C. Georghiou, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, Dan Redmond, H.-J. Seiffert, Sahib Singh, and the proposer.

Integral Divisor

B-661 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $T(n) = n(n+1)/2$. In Problem B-646, it was seen that $T(n)$ is an integral divisor of $T(2T(n))$ for all n in $Z^+ = \{1, 2, \ldots\}$. Find the n in Z^+ such that $T(n)$ is an integral divisor of

$$\sum_{i=1}^{n} T(2T(i)).$$

Solution by C. Georghiou, University of Patras, Greece

We have $T(2T(i)) = (i + 2i^2 + 2i^3 + i^4)/2$ and, therefore,

$$\sum_{i=1}^{n} T(2T(i)) = T(n)\frac{(n^3 + 4n^2 + 6n + 4)}{5}.$$

But $n^3 + 4n^2 + 6n + 4 \equiv (n - 1)(n^2 + 1) \pmod 5$, from which it follows that $T(n)$ is a divisor of the given sum iff $n \equiv 1$, 2, or 3 (mod 5).

Also solved by Richard André-Jeannin, Paul S. Bruckman, David M. Burton, Russell Euler, Piero Filipponi, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Smith, Gregory Wulczyn, and the proposer.

Congruences Modulo 9

B-662 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let $H_n = L_nP_n$, where the L_n and P_n are the Lucas and Pell numbers, respectively. Prove the following congruences modulo 9:

(1) $H_{4n} \equiv 3n$; (2) $H_{4n+1} \equiv 3n + 1$;

(3) $H_{4n+2} \equiv 3n + 6$; (4) $H_{4n+3} \equiv 3n + 2$.

Solution by C. Georghiou, University of Patras, Greece

More generally, we show that for any integer m we have

$$H_{4n+m} \equiv L_mP_m - 3nL_{m+2}P_m - 6nL_mP_{m+2} \pmod 9.$$

Indeed, we have

$$\begin{aligned} L_{4n+m} &= \alpha^{4n+m} + \beta^{4n+m} = \alpha^m(3\alpha^2 - 1)^n + \beta^m(3\beta^2 - 1)^n \\ &= \sum_{i=0}^{n}\binom{n}{i}3^i(-1)^{n-i}[\alpha^{2i+m} + \beta^{2i+m}] \\ &= \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}3^iL_{2i+m} \\ &\equiv (-1)^n[L_m - 3nL_{m+2}] \pmod 9. \end{aligned}$$

Similarly, if $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$, we have

$$\begin{aligned} P_{4n+m} &= (\gamma^{4n+m} - \delta^{4n+m})/2\sqrt{2} = [\gamma^m(6\gamma^2 - 1)^n - \delta^m(6\delta^2 - 1)^n]/2\sqrt{2} \\ &= 2^{-3/2}\sum_{i=0}^{n}\binom{n}{i}6^i(-1)^{n-i}[\gamma^{2i+m} - \delta^{2i+m}] \\ &= \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}6^iP_{2i+m} \\ &\equiv (-1)^n[P_m - 6nP_{m+2}] \pmod 9, \end{aligned}$$

from which the assertion follows immediately.

Now, by setting $m = 0$, 1, 2, and 3, we find congruences (1)-(4), respectively.

Also solved by Paul S. Bruckman, Piero Filipponi, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Lawrence Somer, Gregory Wulczyn, and the proposer.

Dense in an Interval

B-663 *Proposed by Clark Kimberling, U. of Evansville, Indiana*

Let $t_1 = 1$, $t_2 = 2$, and $t_n = (3/2)t_{n-1} - t_{n-2}$ for $n = 3, 4, \ldots$. Determine lim sup t_n.

Solution by Hans Kappus, Rodersdorf, Switzerland

Solving the given difference equation by standard techniques, one easily obtains

$$t_n = (32/7)^{1/2}\sin(na - b),$$

where

$$a = \arctan(\sqrt{7}/3),\ b = \arctan(\sqrt{7}/11).$$

Now, since $\cos a = 3/4$, we conclude that a is not a rational multiple of π, and hence (t_n) is not periodic. Therefore, by a well-known theorem, the numbers t_n are everywhere dense in the interval $|t| \le (32/7)^{1/2}$. It follows that

$$\limsup t_n = (32/7)^{1/2}.$$

Also solved by Richard André-Jeannin, Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

<u>H-449</u> *Proposed by Ioan Sadoveanu, Ellensburg, WA*

Let $G(x) = x^k + a_1x^{k-1} + \cdots + a_k$ be a polynomial with c as a root of order p. If $G^{(p)}(x)$ denotes the p^{th} derivative of $G(x)$, show that

$$\left\{\frac{n^p c^{n-p}}{G^{(p)}(c)}\right\} \text{ is a solution to the recurrence}$$

$$u_n = c^{n-k} - a_1u_{n-1} - a_2u_{n-2} - \cdots - a_ku_{n-k}.$$

<u>H-450</u> *Proposed by R. André-Jeannin, Sfax, Tunisia*

Compare the numbers

$$\Theta = \sum_{n=1}^{\infty} \frac{1}{F_n}$$

and

$$\Theta' = 2 + \sum_{n=1}^{\infty} \frac{1}{F_n(2F_{n-1}^2 + (-1)^{n-1})(2F_n^2 + (-1)^n)}.$$

<u>H-451</u> *Proposed by T. V. Padmakumar, Trivandrum, South India*

If p is a prime and x and a are positive integers, show

$$\binom{x + ap}{p} - \binom{x}{p} \equiv a \pmod{p}.$$

SOLUTIONS

<u>Pell Mell</u>

<u>H-424</u> *Proposed by Piero Filipponi & Adina Di Porto, Rome, Italy (Vol. 26, no. 3, August 1988)*

Let F_n and P_n denote the Fibonacci and Pell numbers, respectively.

Prove that, if F_p is a prime ($p > 3$), then either $F_p | P_H$ or $F_p | P_{H+1}$, where $H = (F_p - 1)/2$.

Solution by Paul S. Bruckman, Edmonds, WA

Let $q = F_p > 3$, a prime. Since $p \equiv \pm 1 \pmod 6$, it is clear from a table of congruences (mod 4) that $q = F_p \equiv 1 \pmod 4$. Hence, $H = \frac{1}{2}(q - 1)$ is even. We will consider two separate cases, but first we indicate some results which involve Pell numbers (and their "Lucas-Pell" counterparts):

(1) $\quad a = 1 + \sqrt{2},\ b = 1 - \sqrt{2};$

(2) $\quad P_n = \dfrac{a^n - b^n}{a - b},\ Q_n = a^n + b^n,\ n = 0, 1, 2, \ldots;$

(3) $\quad P_{2n} = P_n Q_n;$

(4) $\quad Q_n^2 = Q_{2n} + 2(-1)^n.$

Also, if P is an odd prime, the following congruences may be shown to be valid (see "Some Divisibility Properties of Generalized Fibonacci Sequences" by Paul S. Bruckman, *The Fibonacci Quarterly 17.1* (1979), 42-49):

(5) $\quad a^P \equiv a,\ b^P \equiv b \pmod P$, iff $\left(\frac{2}{P}\right) = 1;$

(6) $\quad a^P \equiv b,\ b^P \equiv a \pmod P$, iff $\left(\frac{2}{P}\right) = -1.$

But $(2|P) = 1$ iff $P \equiv \pm 1 \pmod 8$; we may now complete the proof of the desired result.

<u>**Case I**</u>: $H \equiv 0 \pmod 4$. Then $q = 2H + 1 \equiv 1 \pmod 8$; using (5), we have

$$a^q \equiv a,\ b^q \equiv b \pmod q,$$

so

$$a^{q-1} = a^{2H} \equiv b^{q-1} = b^{2H} \equiv 1 \pmod q.$$

Hence,

(7) $\quad P_{2H} \equiv 0,\ Q_{2H} \equiv 2 \pmod q.$

Also, using (3), (4), and (7), we have

(8) $\quad P_{2H} = P_H Q_H \equiv 0 \pmod q;$

(9) $\quad Q_H^2 = Q_{2H} + 2 \equiv 4 \pmod q.$

Since $Q_H \not\equiv 0 \pmod q$ and $q | P_H Q_H$, it follows that $q | P_H$ in this case.

<u>**Case II**</u>: $H \equiv 2 \pmod 4$. Then $q = 2H + 1 \equiv 5 \pmod 8$. Hence, using (6),

$$a^q \equiv b,\ b^q \equiv a \pmod q;$$

thus

$$a^{q+1} = a^{2H+2} \equiv b^{q+1} = b^{2H+2} \equiv -1 \pmod q.$$

Therefore,

(10) $\quad P_{2H+2} \equiv 0,\ Q_{2H+2} \equiv -2 \pmod q.$

Using (3), (4), and (10), we have

(11) $\quad P_{2H+2} = P_{H+1}Q_{H+1} \equiv 0 \pmod q;$

(12) $\quad Q_{H+1}^2 = Q_{2H+2} - 2 \equiv -4 \pmod q.$

Since $Q_{H+1} \not\equiv 0 \pmod q$ and $q | P_{H+1}Q_{H+1}$, it follows that $q | P_{H+1}$. Q.E.D.

Also solved by P. Tzermias and the proposers.

Two and Two Make ϕ

H-429 *Proposed by John Turner, Hamilton, New Zealand*
(Vol. 27, no. 1, February 1989)

Fibonacci enthusiasts know what happens when they add two adjacent numbers of a sequence and put the result next in line.

Have they considered what happens if they put the results *in the middle*?

They will get the following increasing sequence of T-sets (multi-sets):

$$\left.\begin{array}{l} T_1 = \{1\} \\ T_2 = \{1, 2\} \end{array}\right\} \text{given initial sets}$$

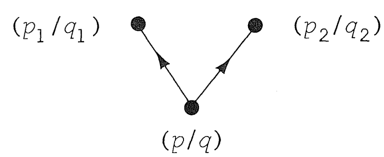
$T_3 = \{1, 3, 2\}$,
$T_4 = \{1, 4, 3, 5, 2\}$,
$T_5 = \{1, 5, 4, 7, 3, 8, 5, 7, 2\}$,
$T_6 = \{1, 6, 5, 9, 4, 11, 7, 10, 3, 11, 8, 13, 5, 12, 7, 9, 2\}$,
etc.

Prove that for $3 \le i \le n$ the multiplicity of i in multi-set T_n is $\frac{1}{2}\phi(i)$, where ϕ is Euler's function.

Solution by the proposer

A binary tree can be grown, and rational numbers assigned to its nodes, as follows:

Assign (1/1) to the root node; then from each node in the tree grow a left-branch and a right-branch and assign rational numbers to the new nodes as done below:



Assignment rule:

If $(p/q) = [a_0; a_1, a_2, \ldots, a_n, 1]$ (simple continued fraction);
$(1/1) = [0; 1]$; then assign

$(p_1/q_1) = [a_0; a_1, a_2, \ldots, a_n, 1, 1]$ (on left-branch node),
and $(p_2/q_2) = [a_0; a_1, a_2, \ldots, a_n + 1, 1]$ (on right-branch node).

It is easy to show [1] that all rational numbers are generated uniquely by this process (there is a one-to-one correspondence between the node values and the set of simple continued fractions whose last component is 1).

If the rational numbers (p/q) on the nodes in the left-hand subtree are considered, it will be seen that they will constitute the set of all rational numbers in the interval (0, 1) as the growth process continues ad infinitum. Hence, each q-value will occur $\phi(q)$ times, for $q > 2$.

The formation of the q-values in the tree, above the node (1/2), and in the left subtree from there corresponds to the formation of the integer values included in the T-sets at each stage.

The right subtree about (1/2) generates an identical sequence of sets of q-values (in different order at each tree level).

The result of the problem follows immediately.

(Drawing the tree up to the fourth level will make all the above statements clear.)

Reference

1. A. G. Shaake & J. C. Turner. "A New Theory of Braiding (RR1/1)." *Research Report No. 165* (1988), 1-42.

Also solved by P. Bruckman, S. Mohanty, and S. Shirali.

And More Identities

H-430 *Proposed by Larry Taylor, Rego Park, NY (Vol. 27, no. 2, May 1989)*

Find integers j, k ($\neq 0, \pm 1, \pm 2$), m_i and n_i such that:

(A) $5F_{m_i}F_{n_i} = L_k + L_{j+i}$, for $i = 1, 5, 9, 13, 17, 21$;

(B) $5F_{m_i}F_{n_i} = L_k - L_{j+i}$, for $i = 3, 7, 11, 15, 19, 23$;

(C) $F_{m_i}L_{n_i} = F_k + F_{j+i}$, for $i = 1, 2, \ldots, 22, 23$;

(D) $L_{m_i}F_{n_i} = F_k - F_{j+i}$, for $i = 1, 3, \ldots, 21, 23$;

(E) $L_{m_i}L_{n_i} = L_k - L_{j+i}$, for $i = 1, 5, 9, 13, 17, 21$;

(F) $L_{m_i}L_{n_i} = L_{-k} + L_{j+i}$, for $i = 2, 4, 6, 8$;

(G) $L_{m_i}L_{n_i} = L_k + L_{j+i}$, for $i = 3, 7, 11, 15, 16, 18, 19, 20, 22, 23$;

(H) $L_{m_i}L_{n_i} = L_k + F_{j+i}$, for $i = 10$;

(I) $L_{m_i}F_{n_i} = L_k + F_{j+i}$, for $i = 12$;

(J) $5F_{m_i}F_{n_i} = L_k + F_{j+i}$, for $i = 14$.

Solution by Paul S. Bruckman, Edmonds, WA

Although there is some method to the process whereby j and k are discovered, there is also a lot of trial and error involved. It is easier to simply indicate, without further ado, the results of our search:

(1) $j = -12$, $k = 7$.

With these values, we find that the problem has solutions m_i and n_i, which are indicated below; no claim is made that other values of j and k cannot work equally well, though this seems likely.

(A) $L_7 + L_{-11} = 29 - 199 = -170 = 5(34)(-1) = 5F_9F_{-2}$;

$L_7 + L_{-7} = 29 - 29 = 0 = 5F_7F_0$;

$L_7 + L_{-3} = 29 - 4 = 25 = 5(5)(1) = 5F_5F_2$;

$L_7 + L_1 = 29 + 1 = 30 = 5(2)(3) = 5F_3F_4$;

$L_7 + L_5 = 29 + 11 = 40 = 5(1)(8) = 5F_1F_6$;

$L_7 + L_9 = 29 + 76 = 105 = 5(1)(21) = 5F_{-1}F_8$.

Note that we may take $m_i = \frac{1}{2}(19 - i)$, $n_i = 7 - n_i = \frac{1}{2}(i - 5)$, for all given i.

(B) $L_7 - L_{-9} = 29 + 76 = 105 = 5(21)(1) = 5F_8F_{-1}$, etc.,

i.e., this yields the same results as part (A), in reverse order. With the same functions m_i and n_i as in part (A), we obtain the same identities.

(C) $F_7 + F_{-11} = F_7 + F_{11} = 13 + 89 = 102 = 34 \cdot 3 = F_9L_2$ (i = 1, 23);

$F_7 + F_{-10} = 13 - 55 = -42 = (-21)(2) = F_{-8}L_0$ (i = 2);

$F_7 + F_{-9} = F_7 + F_9 = 13 + 34 = 47 = 1 \cdot 47 = F_1L_8$ (i = 3, 21);

$F_7 + F_{-8} = 13 - 21 = -8 = (2)(-4) = (-8)(1) = F_3L_{-3} = F_{-6}L_1$ (i = 4);

$F_7 + F_{-7} = F_7 + F_7 = 13 + 13 = 26 = 13 \cdot 2 = F_7L_0$ (i = 5, 19);

$F_7 + F_{-6} = 13 - 8 = 5 = 5 \cdot 1 = F_5L_1$ (i = 6);

$F_7 + F_{-5} = F_7 + F_5 = 13 + 5 = 18 = 1 \cdot 18 = F_1L_6$ (i = 7, 17);

$F_7 + F_{-4} = 13 - 3 = 10 = 5 \cdot 2 = F_5L_0$ (i = 8);

$F_7 + F_{-3} = F_7 + F_3 = 13 + 2 = 15 = 5 \cdot 3 = F_5L_2$ (i = 9, 15);

$F_7 + F_{-2} = 13 - 1 = 12 = 3 \cdot 4 = F_4L_3$ (i = 10);

$F_7 + F_{-1} = F_7 + F_1 = 13 + 1 = 14 = 2 \cdot 7 = F_3L_4$ (i = 11, 13);

$F_7 + F_0 = 13 = 13 \cdot 1 = F_7L_1$ (i = 12);

$F_7 + F_2 = 13 + 1 = 14 = F_3L_4$ (i = 14);

$F_7 + F_4 = 13 + 3 = 16 = 8 \cdot 2 = F_6L_0$ (i = 16);

$F_7 + F_6 = 13 + 8 = 21 = 21 \cdot 1 = 3 \cdot 7 = F_8L_1 = F_4L_4$ (i = 18);

$F_7 + F_8 = 13 + 21 = 34 = 34 \cdot 1 = F_9L_1$ (i = 20);

$F_7 + F_{10} = 13 + 55 = 68 = 34 \cdot 2 = F_9L_0$ (i = 22).

(D) $F_7 - F_{-11} = F_7 - F_{11} = 13 - 89 = -76 = 76(-1) = L_9F_{-2}$ (i = 1, 23);

$F_7 - F_{-9} = F_7 - F_9 = 13 - 34 = -21 = (-1)(21) = L_{-1}F_8$ (i = 3, 21);

$F_7 - F_{-7} = F_7 - F_7 = 0 = L_7F_0$ (i = 5, 19);

$F_7 - F_{-5} = F_7 - F_5 = 13 - 5 = 8 = L_3F_3 = L_1F_6$ (i = 7, 17);

$F_7 - F_{-3} = F_7 - F_3 = 13 - 2 = 11 = L_5F_2$ (i = 9, 15);

$F_7 - F_{-1} = F_7 - F_1 = 13 - 1 = 12 = 4 \cdot 3 = L_3F_4$ (i = 11, 13).

In this case, $m_i = \frac{1}{2}(19 - i)$, $n_i = \frac{1}{2}(i - 5)$, i = 1, 5, 9, 13, 17, 21;

$m_i = \frac{1}{2}(i - 5)$, $n_i = \frac{1}{2}(19 - i)$, i = 3, 7, 11, 15, 19, 23.

(E) $L_7 - L_{-11} = 29 + 199 = 228 = 76 \cdot 3 = L_9L_{-2}$;

$L_7 - L_{-7} = 29 + 29 = 58 = 29 \cdot 2 = L_7L_0$;

$L_7 - L_{-3} = 29 + 4 = 33 = 11 \cdot 3 = L_5L_2$;

$L_7 - L_1 = 29 - 1 = 28 = 4 \cdot 7 = L_3L_4$;

$L_7 - L_5 = 29 - 11 = 18 = 1 \cdot 18 = L_1L_6$;

$L_7 - L_9 = 29 - 76 = -47 = (-1)(47) = L_{-1}L_8$.

In this case, $m_i = \frac{1}{2}(19 - i)$, $n_i = \frac{1}{2}(i - 5)$.

(F) $L_{-7} + L_{-10} = -29 + 123 = 94 = L_8L_0$;

$L_{-7} + L_{-8} = -29 + 47 = 18 = 18 \cdot 1 = L_6L_1$;

$L_{-7} + L_{-6} = -29 + 18 = -11 = 11(-1) = L_5L_{-1}$;

$L_{-7} + L_{-4} = -29 + 7 = -22 = (-11)(2) = L_{-5}L_0$.

(G) $L_7 + L_{-9} = 29 - 76 = -47 = (-1)(47) = L_{-1}L_8$;

$L_7 + L_{-5} = 29 - 11 = 18 = 1 \cdot 18 = L_1L_6$;

$L_7 + L_{-1} = 29 - 1 = 28 = 4 \cdot 7 = L_3L_4$;

$L_7 + L_3 = 29 + 4 = 33 = 11 \cdot 3 = L_5L_2$;

$L_7 + L_4 = 29 + 7 = 36 = 18 \cdot 2 = L_6L_0$;

$L_7 + L_6 = 29 + 18 = 47 = 47 \cdot 1 = L_8L_1$;

$L_7 + L_7 = 29 + 29 = 58 = 29 \cdot 2 = L_7L_0$;

$L_7 + L_8 = 29 + 47 = 76 + 76 \cdot 1 = L_9L_1$;

$L_7 + L_{10} = 29 + 123 = 152 = 76 \cdot 2 = L_9L_0$;

$L_7 + L_{11} = 29 + 199 = 228 = 76 \cdot 3 = L_9L_2$.

(H) $L_7 + F_{-2} = 29 - 1 = 28 = 4 \cdot 7 = L_3L_4$.

(I) $L_7 + F_0 = 29 + 0 = 29 \cdot 1 = L_7F_1$.

(J) $L_7 + F_2 = 29 + 1 = 30 = 5 \cdot 2 \cdot 3 = 5F_3F_4$.

Also solved by L. Kuipers and the proposer.

Count to Five

H-432 *Proposed by Piero Filipponi, Rome, Italy (Vol. 27, no. 2, May 1989)*

For k and n nonnegative integers and m a positive integer, let $M(k, n, m)$ denote the arithmetic mean taken over the k^{th} powers of m consecutive Lucas numbers of which the smallest is L_n.

$$M(k, n, m) = \frac{1}{m}\sum_{j=n}^{n+m-1} L_j^k.$$

For $k = 2^h$ ($h = 0, 1, 2, 3$), find the smallest nontrivial value m_h ($m_h > 1$) of m for which $M(k, n, m)$ is integral for every n.

Solution by the proposer

Let

$$L(k, n, m) = \sum_{j=n}^{n+m-1} L_j^k.$$

First, with the aid of Binet forms for F_s and L_s and use of the geometric series formula, we obtain the following general expression for $L(2t, 0, s + 1)$ ($t = 0, 1, \ldots$):

(1) $$L(2t,\ 0,\ s+1) = \sum_{j=0}^{s} L_j^{2t} = \binom{2t}{t} X_{s,t} + \sum_{i=0}^{t-1} \binom{2t}{i} [(-1)^{is} L_{2(s+1)(t-i)} - (-1)^{i(s+1)} L_{2s(t-i)} + L_{2(t-i)} - 2(-1)^i] / [L_{2(t-i)} - 2(-1)^i],$$

where

(1′) $$X_{s,t} = \begin{cases} s+1 & \text{if } t \text{ is even,} \\ [1 + (-1)^s]/2 & \text{if } t \text{ is odd.} \end{cases}$$

Then, specializing (1) and (1′) to $t = 1$, 2, and 4, after some simple but tedious manipulations involving the use of certain elementary Fibonacci identities (see V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*), we obtain

(2) $$L(1,\ 0,\ s+1) = L_{s+2} - 1;$$

(3) $$L(2,\ 0,\ s+1) = L_{2s+1} + 2 + (-1)^s;$$

(4) $$L(4,\ 0,\ s+1) = F_{4s+2} + 4(-1)^s F_{2s+1} + 6s + 11;$$

(5) $$L(8,\ 0,\ s+1) = [F_{8s+4} + 84F_{4s+2} + 12(-1)^s(F_{6s+3} + 14F_{2s+1}) + 3(70s + 163)]/3,$$

respectively. We point out that (2) has been obtained separately.

Case (i): $k = 1$ $(h = 0)$

From (2) we can write

(6) $$L(1,\ n,\ m) = L(1,\ 0,\ n+m) - L(1,\ 0,\ n) = L_{n+m+1} - L_{n+1}.$$

If $m = 24$, using Hoggatt's identities I_{24} and I_{32}, from (6) we can write

$$L(1,\ n,\ 24) = 5F_{12}F_{n+13}$$

whence

(7) $$M(1,\ n,\ 24) = L(1,\ n,\ 24)/24 = 30F_{n+13}$$

appears to be integral independently of n. Moreover, it can be readily verified that

$M(1,\ 0,\ m)$ is not integral for $m = 2$, 4-23;

$M(1,\ 1,\ 3)$ is not integral.

It follows that $m_0 = 24$.

Case (ii): $k = 2$ $(h = 1)$

From (3) we can write

(8) $$L(2,\ n,\ m) = L(2,\ 0,\ n+m) - L(2,\ 0,\ n) = L_{2n+2m-1} - L_{2n-1} + (-1)^{n-m-1} - (-1)^{n-1}.$$

If $m = 12$, using Hoggatt's identities I_{24} and I_{32}, from (8) we can write

$$L(2,\ n,\ 12) = 5F_{12}F_{2n+11},$$

whence

(9) $$M(2,\ n,\ 12) = L(2,\ n,\ 12)/12 = 60F_{2n+11}$$

appears to be integral independently of n. Moreover, it can be readily verified that

$M(2, 0, m)$ is not integral for $m = 2$-9, 11;

$M(2, 1, 10)$ is not integral.

It follows that $m_1 = 12$.

Case (iii): $k = 4$ $(h = 2)$

From (4) we can write

$$(10) \quad L(4, n, m) = L(4, 0, n + m) - L(4, 0, n)$$
$$= F_{4n+4m-2} + 4(-1)^{m+n-1}F_{2n+2m-1} - 4(-1)^{n-1}F_{2n-1} + 6m.$$

If $m = 5$, using Hoggatt's identities I_{24}, I_{22}, and I_7, (10) can be rewritten as

$$L(4, n, 5) = F_5[L_{4(n+2)}L_5 + 4(-1)^n L_{2(n+2)}] + 30,$$

whence

$$(11) \quad M(4, n, 5) = L(4, n, 5)/5 = L_{4(n+2)}L_5 + 4(-1)^n L_{2(n+2)} + 6$$

appears to be integral independently of n. Moreover, it can be readily verified that

$M(4, 0, m)$ is not integral for $m = 2, 3, 4$.

It follows that $m_2 = 5$.

Case (iv): $k = 8$ $(h = 3)$

Letting

$$(12) \quad r = 2n + m - 1$$

and omitting the intermediate steps for brevity, from (5) we can write

$$(13) \quad L(8, n, m) = L(8, 0, n + m) - L(8, 0, n)$$
$$= [L_{4r}F_{4m} + 84L_{2r}F_{2m} - 12(-1)^{n+m}(L_{3r}F_{3m} + 14L_rF_m) + 210m]/3$$
$$= F_m[L_{4r}L_{2m}L_m + 84L_{2r}L_m - 12(-1)^{n+m}(L_{3r}F_{3m}/F_m + 14L_r)]/3 + 70m.$$

Letting $m = 5$ in both (12) and (13), we have

$$L(8, n, 5) = F_5[1353L_{8(n+2)} + 924L_{4(n+2)} + 12(-1)^n(122L_{6(n+2)} + 14L_{2(n+2)})]/3 + 350$$
$$= 5[451L_{8(n+2)} + 308L_{4(n+2)} + 4(-1)^n(122L_{6(n+2)} + 14L_{2(n+2)}] + 350,$$

whence

$$(14) \quad M(8, n, 5) = L(8, n, 5)/5 = 451L_{8(n+2)} + 308L_{4(n+2)} + 4(-1)^n(122L_{6(n+2)} + 14L_{2(n+2)}) + 70$$

appears to be integral independently of n. Moreover, it can be readily verified that

$M(8, 0, m)$ is not integral for $m = 2, 3, 4$.

It follows that $m_3 = 5$.

Also solved by P. Bruckman.

Editorial Note: A number of readers have pointed out that H-440 and H-448 are essentially the same. Sorry about that.

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BOOKS AVAILABLE
THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

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