The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

TABLE OF CONTENTS

VOLUME 29 MAY 1991 NUMBER 2

PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

EDITORIAL POLICY

THE FIBONACCI QUARTERLY seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

SUBMITTING AN ARTICLE

Articles should be submitted in the format of the current issues of **THE FIBONACCI QUARTERLY.** They should be typewritten or reproduced typewritten copies, that are clearly readable, double spaced with wide margins and on only one side of the paper. The full name and address of the author must appear at the beginning of the paper directly under the title. Illustrations should be carefully drawn in India ink on separate sheets of bond paper or vellum, approximately twice the size they are to appear in print. Since the Fibonacci Association has adopted $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \geq 2$ and $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$, $n \geq 2$ as the standard definitions for The Fibonacci and Lucas sequences, these definitions *should not* be a part of future papers. However, the notations *must* be used.

Two copies of the manuscript should be submitted to: **GERALD E. BERGUM, EDITOR, THE FIBONACCI QUARTERLY, DEPARTMENT OF COMPUTER SCIENCE, SOUTH DAKOTA STATE UNIVERSITY, BOX 2201, BROOKINGS, SD 57007-0194.**

Authors are encouraged to keep a copy of their manuscripts for their own files as protection against loss. The editor will give immediate acknowledgment of all manuscripts received.

SUBSCRIPTIONS, ADDRESS CHANGE, AND REPRINT INFORMATION

Address all subscription correspondence, including notification of address change, to: **RICHARD VINE, SUBSCRIPTION MANAGER, THE FIBONACCI ASSOCIATION, SANTA CLARA UNIVERSITY, SANTA CLARA, CA 95053.**

Requests for reprint permission should be directed to the editor. However, general permission is granted to members of The Fibonacci Association for noncommercial reproduction of a limited quantity of individual articles (in whole or in part) provided complete reference is made to the source.

Annual domestic Fibonacci Association membership dues, which include a subscription to **THE FIBONACCI QUARTERLY,** are $35 for Regular Membership, $45 for Sustaining Membership, and $70 for Institutional Membership; foreign rates, which are based on international mailing rates, are somewhat higher than domestic rates; please write for details. **THE FIBONACCI QUARTERLY** is published each February, May, August and November.

All back issues of **THE FIBONACCI QUARTERLY** are available in microfilm or hard copy format from **UNIVERSITY MICROFILMS INTERNATIONAL, 300 NORTH ZEEB ROAD, DEPT. P.R., ANN ARBOR, MI 48106.** Reprints can also be purchased from **UMI CLEARING HOUSE** at the same address.

The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

A NOTE ON BERNOULLI POLYNOMIALS

Dario Castellanos
Universidad de Carabobo, Valencia, Venezuela
(Submitted January 1989)

1. Some General Remarks

Consider the function $x - [x] - \frac{1}{2}$ which is periodic with period 1. In the interval [0, 1] this function is simply $x - \frac{1}{2}$.

This function has the property that its integral in the interval [0, 1] is zero. Let us, then, with the same idea in mind define another function $\Phi_2(x)$, such that its derivative is $\Phi_1(x) = x - \frac{1}{2}$, and such that its integral in the interval [0, 1] is zero:

$$\int_0^1 \Phi_2(x)\,dx = 0.$$

Similarly, $\Phi_3'(x) = \Phi_2(x)$, and

$$\int_0^1 \Phi_3(x)\,dx = 0.$$

In general, we seek a sequence of functions $\Phi_n(x)$, $n = 1, 2, 3, \ldots$, such that

$$\Phi_1(x) = x - \tfrac{1}{2},\ \Phi_n'(x) = \Phi_{n-1}(x) \text{ for } n > 1,$$

and

$$\int_0^1 \Phi_n(x)\,dx = 0 \text{ for all } n \geq 1.$$

The constant multiples of these functions $n!\Phi_n(x) = B_n(x)$ are called Bernoulli polynomials after their discoverer [2]. They obey the relation

$$B_n'(x) = nB_{n-1}(x),\ n \geq 1,\ B_0(x) = 1. \tag{1.1}$$

The first few Bernoulli polynomials are

$$B_0(x) = 1,\ B_1(x) = x - 1/2,\ B_2(x) = x^2 - x + 1/6,$$
$$B_3(x) = x^3 - (3/2)x^2 + (1/2)x,\ B_4(x) = x^4 - 2x^3 + x^2 - 1/30, \text{ etc.}$$

It is clear from their construction that $B_n(x)$ is a polynomial of degree n. They are defined in the interval $0 \leq x \leq 1$. Their periodic continuation outside this interval are called Bernoulli functions.

The constant terms of the Bernoulli polynomials form a particularly interesting set of numbers. We set $B_n = B_n(0)$. It is obvious from the way the polynomials $B_n(x)$ are constructed that all the B_n are rational numbers. It can be shown that $B_{2n+1} = 0$ for $n \geq 1$, and is alternately positive and negative for even n. The B_n are called Bernoulli numbers, and the first few are

$$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42,$$
$$B_8 = -1/30,\ B_{10} = 5/66,\ B_{12} = -691/2730,\ B_{14} = 7/6, \text{ etc.}$$

Bernoulli polynomials and numbers are intimately related to the sum of the powers of the natural numbers.

Bernoulli polynomials possess the following generating function [5, 3],

$$te^{tx}(e^t - 1)^{-1} = \sum_{n=0}^{\infty} B_n(x)t^n/n!, \tag{1.2}$$

from which we find, on replacing x by $x + 1$ and then subtracting (1.2) from the resulting expression:

$$(1.3)\qquad \sum_{n=0}^{\infty}[B_n(x+1)-B_n(x)]t^n/n! = te^{tx}.$$

Using the Maclaurin expansion on the right-hand side and comparing powers of t, we find

$$(1.4)\qquad B_n(x+1)-B_n(x) = nx^{n-1},\ n = 2, 3, \ldots .$$

From (1.1) and (1.4) there follows

$$(1.5)\qquad \int_x^{x+1} B_n(s)\,ds = x^n,$$

from which we find [4]

$$(1.6)\qquad \sum_{k=0}^{r} k^n = \sum_{k=0}^{r}\int_k^{k+1} B_n(s)\,ds$$

$$= \int_0^{r+1} B_n(s)\,ds = \frac{B_{n+1}(r+1)-B_{n+1}}{n+1},\ n = 2, 3, 4, \ldots .$$

In the next section we will make use of the following property of Bernoulli polynomials [8]:

$$(1.7)\qquad \int_0^1 B_n(s)B_m(s)\,ds = (-1)^{n-1}\frac{m!n!}{(m+n)!}B_{n+m},$$

$$n = 1, 2, 3, \ldots;\ m = 1, 2, 3, \ldots .$$

Formula (1.7) is only apparently unsymmetrical in m and n. The reader can convince him- or herself of the symmetry of it by trying the different combinations of even and odd values of m and n.

2. An Expansion for Products of Bernoulli Polynomials

We wish to expand a product of two Bernoulli polynomials in series of Bernoulli polynomials [7]. It will simplify matters if we use the functions $\Phi_n(x)$ defined at the beginning of Section 1. We want, then, an expression of the form

$$(2.1)\qquad \Phi_n(x)\Phi_m(x) = \sum_{k=0}^{n+m} a_k\Phi_k(x),$$

where the Φ_n's are, we recall, Bernoulli polynomials divided by $n!$.

We will make use of the properties

$$(2.2)\qquad \int_0^1 \Phi_n(s)\,ds = 0 \text{ for } n \geq 1,$$

and (1.7), which now appears in the guise

$$(2.3)\qquad \int_0^1 \Phi_n(s)\Phi_m(s)\,ds = (-1)^{n-1}b_{n+m},\ n, m = 1, 2, \ldots,$$

where the b_n's are Bernoulli numbers divided by $n!$.

Also

$$(2.4)\qquad D\Phi_n(x) = \Phi_n' = \Phi_{n-1}.$$

Using Leibniz's theorem for the derivative of a product [1], we find from (2.1)

$$(2.5)\qquad D^s[\Phi_n(x)\Phi_m(x)] = \sum_{j=0}^{s}\binom{s}{j}D^j\Phi_n(x)D^{s-j}\Phi_m(x) = \sum_{k=0}^{n+m} a_k D^s\Phi_k(x).$$

That is,

$$(2.6)\qquad \sum_{k=s}^{n+m} a_k \Phi_{k-s}(x) = \sum_{k=0}^{n+m-s} a_{k+s}\Phi_k(x) = \sum \binom{s}{j}\Phi_{n-j}(x)\Phi_{m-s+j}(x),$$

with the restrictions that $n - j \geq 0$ and $m - s + j \geq 0$, i.e., $j \leq n$, $j \geq s - m$. Since the sum in (2.5) starts at $j = 0$ and ends at $j = s$, we must write (2.6) in the form

$$(2.7)\qquad \sum_{j=\max(0,\, s-m)}^{\min(s,\, n)} \binom{s}{j}\Phi_{n-j}(x)\Phi_{m-s+j}(x) = \sum_{k=0}^{n+m-s} a_{k+s}\Phi_k(x).$$

We now wish to integrate both sides of (2.7) from $x = 0$ to $x = 1$ and to apply properties (2.2) and (2.3). To do so, we must separate from the first sum in (2.7) the terms corresponding to $j = n$ and to $j = s - m$, since in both of these cases the corresponding index is zero and formula (2.3) does not apply.

This gives

$$(2.8)\qquad a_s = b_{n+m-s}(-1)^{n-1} \sum_{j=\max(0,\, s-m+1)}^{\min(s,\, n-1)} \binom{s}{j}(-1)^j, \quad s < m + n - 1.$$

If $s = m + n$, the first sum in (2.5) will contain only one term and we have

$$(2.9)\qquad a_{n+m} = \binom{n+m}{n}.$$

Similarly, if $s = m + n - 1$, then the sum will contain only two terms with non-zero index, both of which will integrate to zero and we have

$$(2.10)\qquad a_{n+m-1} = 0.$$

Expressing these results in terms of ordinary Bernoulli polynomials, we find, after dividing a_s by $s!$, the expressions

$$(2.11)\qquad B_n(x)B_m(x) = \sum_{k=0}^{n+m} \alpha_k B_k(x),$$

$$(2.12)\qquad \alpha_k = \frac{n!m!B_{n+m-k}}{(n+m-k)!}(-1)^{n-1} \sum_{j=\max(0,\, k-m+1)}^{\min(k,\, n-1)} \frac{(-1)^j}{(k-j)!j!}, \quad \begin{array}{l} k < n + m - 1, \\ m, n = 1, 2, \ldots, \end{array}$$

$$(2.13)\qquad \alpha_{n+m-1} = 0,$$

$$(2.14)\qquad \alpha_{n+m} = 1.$$

Equations (2.11)-(2.14) are the desired results. The reader may wish to look at reference [6] to see alternate ways of expressing these coefficients.

Since Bernoulli numbers of odd index greater than one are zero, we see that if n and m are of the same parity, then expansion (2.11) will only involve Bernoulli polynomials of even index. If n and m are of opposite parity, then expansion (2.11) will only involve Bernoulli polynomials of odd index.

If we define

$$(2.15)\qquad S_n(r) = \sum_{k=1}^{r} k^n,$$

and make use of (1.6), we can express (2.11) in terms of the S_n's:

$$(n + 1)(m + 1)S_n(r)S_m(r) = \sum_{k=1}^{n+m+2} k\alpha_k S_{k-1}(r) - (n + 1)B_{m+1}S_n(r)$$
$$- (m + 1)B_{n+1}S_m(r) - B_{m+1}B_{n+1} + \sum_{k=0}^{n+m+2} \alpha_k B_k.$$

Observe now that in the equation above $-B_{m+1}B_{n+1}$ cancels $\sum_{k=0}^{n+m+2}\alpha_k B_k$, since these expressions are the left- and right-hand sides of (2.11) with $x = 0$ and n and m replaced by $n + 1$ and $m + 1$, respectively.

The equation then takes the form

$$(2.16)\quad (n+1)(m+1)S_n(r)S_m(r) = \sum_{k=2}^{n+m+2} k\alpha_k S_{k-1}(r) - (n+1)B_{m+1}S_n(r) - (m+1)B_{n+1}S_m(r),$$

where the α_k's must now be written

$$(2.17)\quad \alpha_k = \frac{(n+1)!(m+1)!B_{n+m+2-k}}{(n+m+2-k)!}(-1)^n \sum_{j=\max(0,k-m)}^{\min(k,n)} \frac{(-1)^j}{(k-j)!j!},$$

$$k < n + m + 1,$$

$$(2.18)\quad \alpha_{n+m+1} = 0,$$

$$(2.19)\quad \alpha_{n+m+2} = 1,$$

and we have observed that $\alpha_1 = 0$.

Note now that the product of $S_n(r)$ and $S_m(r)$ will involve $S_k(r)$'s with odd index only if n and m are of the same parity, and $S_k(r)$'s with even index only if n and m are of opposite parity.

3. Some Examples

$$(3.1)\quad S_1(r)S_2(r) = \frac{5}{6}S_4(r) + \frac{1}{6}S_2(r),$$

$$(3.2)\quad S_1(r)S_3(r) = \frac{3}{4}S_5(r) + \frac{1}{4}S_3(r),$$

$$(3.3)\quad S_2(r)S_3(r) = \frac{7}{12}S_6(r) + \frac{5}{12}S_4(r),$$

$$(3.4)\quad S_2(r)S_4(r) = \frac{8}{15}S_7(r) + \frac{1}{2}S_5(r) - \frac{1}{30}S_3(r),$$

$$(3.5)\quad S_3(r)S_5(r) = \frac{5}{12}S_9(r) + \frac{2}{3}S_7(r) - \frac{1}{12}S_5(r),$$

$$(3.6)\quad S_3(r)S_7(r) = \frac{7}{24}S_{13}(r) + S_{11}(r) - \frac{3}{8}S_9(r) + \frac{1}{12}S_7(r),$$

$$(3.7)\quad S_1(r)S_3(r)S_5(r) = \frac{1}{4}S_{11} + \frac{35}{48}S_9(r) + \frac{1}{24}S_7(r) - \frac{1}{48}S_5(r).$$

Especially appealing are the formulas for powers of the $S_k(n)$'s. We obtain, for instance, the expressions

$$(3.8)\quad S_1(r)^2 = S_3(r),$$

$$(3.9)\quad S_2(r)^2 = \frac{2}{3}S_5(r) + \frac{1}{3}S_3(r),$$

$$(3.10)\quad S_3(r)^2 = \frac{1}{2}S_7(r) + \frac{1}{2}S_5(r),$$

$$(3.11)\quad S_4(r)^2 = \frac{2}{5}S_9(r) + \frac{2}{3}S_7(r) - \frac{1}{15}S_5(r),$$

$$(3.12)\quad S_5(r)^2 = \frac{1}{3}S_{11}(r) + \frac{5}{6}S_9(r) - \frac{1}{6}S_7(r),$$

(3.13) $S_1(r)^3 = \frac{3}{4}S_5(r) + \frac{1}{4}S_3(r),$

(3.14) $S_2(r)^3 = \frac{1}{3}S_8(r) + \frac{7}{12}S_6(r) + \frac{1}{12}S_4(r),$

(3.15) $S_3(r)^3 = \frac{3}{16}S_{11}(r) + \frac{5}{8}S_9(r) + \frac{3}{16}S_7(r),$

etc.

Formulas (3.8) through (3.11) have been known for a very long time. Formula (3.10) is attributed to Jacobi [9].

To the best of our knowledge, the only special case of (2.11) that is known is [10]

(3.16) $B_4(x) - B_4 = (B_2(x) - B_2)^2,$

and accounts for (3.8).

References

1. M. Abramowitz & I. Stegun, eds. *Handbook of Mathematical Functions*. Applied Mathematics Series 55. Washington, D.C.: National Bureau of Standards, 1964, p. 12.
2. J. Bernoulli. *Ars conjectandi* ... Basel, 1713, p. 97 (published posthumously).
3. D. Castellanos. "The Ubiquitous π, Part I." *Mathematics Magazine 61.2* (1988):70-71.
4. Erdélyi, Magnus, Oberhettinger, & Tricomi. *Higher Transcendental Functions*. Bateman Manuscript Project. New York: McGraw-Hill, 1953, Vol. 1, pp. 35-39.
5. L. Euler. "Methodus generalis summandi progressiones." *Coment. Academ. Aci. Petrop.* 6 (1738):68-97.
6. H. W. Gould. *Combinatorial Identities*. Morgantown, 1972. See esp. Eq. (1.4).
7. C. Jordan. *Calculus of Finite Differences*. 2nd ed. (1950), Section 84, p. 248.
8. Magnus, Oberhettinger, & Soni. *Formulas and Theorems for the Special Functions of Mathematical Physics*. 3rd ed. Springer-Verlag, 1966, p. 27.
9. A. Palacio Gros. "Lecciones de Análisis Matemático." Universidad Central de Venezuela, Caracas, 1951, p. 63.
10. H. Rademacher. *Topics in Analytic Number Theory*. Springer-Verlag, 1973, p. 9.

*****

FIBONACCI'S MATHEMATICAL LETTER TO MASTER THEODORUS

A. F. Horadam
University of New England, Armidale, Australia 2351
(Submitted March 1989)

1. Introduction

Sometime about 1225 A.D., Fibonacci—or Leonardo of Pisa, as he was known until relatively recent times—wrote an interesting, undated mathematical letter to Master Theodorus, philosopher at the court of the Holy Roman Emperor, Frederick II. The full title of this communication, written in medieval Tuscan Latin, is: *Epistola suprascripti Leonardi ad Magistrum Theodorum phylosophum domini Imperatoris*.

Our knowledge of this epistle comes from the nineteenth-century publication of Fibonacci's manuscripts by Boncompagni [1], which is the first printed record of his works. Boncompagni's printing of the *Epistola* [1, pp. 247-52] was prepared from a manuscript in the Biblioteca Ambrosiana di Milano. It has never been translated into English, though in 1919 McClenon [8] indicated his intention to do so.

Fibonacci's mathematical writings consist of five works (others having been lost). These are: (1) *Liber abbaci* (1202, revised 1228); (2) *Practica geometriae* (1220); (3) *Flos* (1225); (4) *Liber quadratorum* (1225), his greatest opus; and (5) the letter to Master Theodorus, the shortest of his extant writings. This useful letter has been somewhat neglected by historians of mathematics, a tendency I would like to see reversed.

To understand Fibonacci's outstanding contributions to knowledge, it is necessary to know something of the age in which he lived and of the mathematics that preceded him. Indeed, a study of his writings reminds one of the history of pre-medieval mathematics in microcosm. In an age of great commercial change and expansion, as well as political and religious struggle, he traveled widely throughout the Mediterranean area in pursuit of his business and mathematical interests. His writings reflect many sources of influence, principally the Greeks in geometry and number theory and the Arabs in algebraic techniques, while some of his problems reveal oriental influences emanating from China and India. Babylonian and Egyptian ideas are apparent in his calculations. For further information on Fibonacci's life and times one may consult, for example, Gies & Gies [3], Grimm [4], Herlihy [5], and Horadam [6].

In popular estimation, Fibonacci is best known for his introduction to Europe of the Hindu-Arabic numerals and, of course, for the set of integers associated (in the late nineteenth century) with his name. However, these popular images of Fibonacci obscure the consummate mastery he demonstrated in a wide range of mathematics.

2. The Letter to Master Theodorus

The mathematical contents of the *Epistola* are rather more speculative and recreational than is the material of his two major, earlier works which have an emphasis on practical arithmetic and geometry. After some general introductory remarks directed to Master Theodorus, Fibonacci proceeds to pose, and solve, a variety of problems.

(a) Problems of Buying Birds

In the first section of this document, Fibonacci's main subject is the "Problem of the 100 birds," a type of problem of oriental origin which he had previously discussed in *Liber abbaci*. Here, however, he develops a general method for solving indeterminate problems.

Fibonacci begins by discussing variations of the problem of buying a given number of birds (sparrows, turtledoves, and pigeons—let us label them x, y, and z—costing 1/3, 1/2, and 2 *denarii* each, respectively) with a given number of *denarii*, a *denarius* being a coin unit of currency. Details of the cases may be tabulated thus:

Denarii	Birds	Solution(s): x, y, z
30	30	9 10 11
29	29	3 16 10; 12 6 11
15	15	
16	15	3 6 6

Regarding the third case, which is insoluble in integers (mathematically, we obtain $4\frac{1}{2}$, 5, $5\frac{1}{2}$), Fibonacci remarks: "...*hoc esse non posse sine fractione avium demonstrabo*."

Next, Fibonacci varies the cost per bird when buying 30 birds of 3 kinds with 30 *denarii*. A bird of each variety now costs 1/3, 2, and 3 *denarii* respectively. He finds the unique solution to be 21, 4, 5 for the numbers of each kind of bird.

Finally, Fibonacci deals with the purchase with 24 *denarii* of birds of 4 kinds (sparrows, turtledoves, pigeons, and partridges) at a specified cost per bird, leading to the equations

$$x + y + z + t = 24,$$

$$\frac{1}{5}x + \frac{1}{3}y + 2z + 3t = 24,$$

for which the two solutions are 10, 6, 4, 4, and 5, 12, 2, 5, for x, y, z, t, respectively.

Admittedly, these problems become somewhat tedious because of their repetitive nature, but an insight into Fibonacci's mind is revealing. Remember that he had no algebraic symbolism to guide him. While his techniques, supplemented by tabulated information in the margin, are fairly standard for us in these problems, they might not have been easy for him.

(b) A Geometrical Problem

Following these algebraic problems, Fibonacci [1 (Vol. 2), p. 249] then proposes the geometrical construction of an equilateral pentagon in a given isosceles ("equicrural," i.e., equal legs) triangle. [Observe that an equilateral pentagon is only regular if it is also equiangular (108°).]

This problem in Euclidean geometry will be highlighted, for historical reasons, and for variety. Fibonacci states the problem in these words: "*De compositione pentagonj equilateri in triangulum equicrurium datum.*"

Our Figure 1 reproduces Fibonacci's diagram and notation. In it, Fibonacci takes $ab = ac = 10$, $bc = 12$, and draws di, ah, gl perpendicular to the base bc. The equilateral pentagon is $adefg$. Taking the side ad of the pentagon as the unknown *res* ("thing")—our x—to be determined, and using similar triangles, Fibonacci applies Pythagoras' Theorem to the triangle die to obtain

$$\left(8 - \frac{4}{5}x\right)^2 + \left(\frac{1}{10}x\right)^2 = x^2,$$

whence,

$$x^2 + 36\tfrac{4}{7}x = 182\tfrac{6}{7}$$

("*et sic reducta est questio ad unam ex regulis algebre*," he writes).

He obtains the approximate value

$$x = 4^0\, 27^{\mathrm{i}}\, 24^{\mathrm{ii}}\, 40^{\mathrm{iii}}\, 50^{\mathrm{iv}}$$

in Babylonian sexagesimal notation.

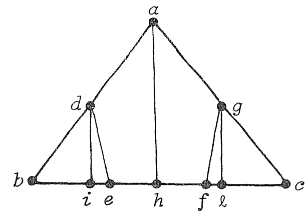

FIGURE 1

To achieve his solution, Fibonacci, with the visual aid of a geometrical diagram involving a square and rectangles, completes the square in the quadratic, i.e.,

$$\left(x + 18\tfrac{2}{7}\right)^2 = 517\tfrac{11}{49}\ \left[= 182\tfrac{6}{7} + \left(18\tfrac{2}{7}\right)^2\right],$$

then subtracts $18\frac{2}{7}$ from the square root of $517\frac{11}{49}$ (which he gives in sexagesimal notation as approximately "*22 et minuta 44 et secunda 23 et tertia 13 et quarta 7*," i.e., $22^0\, 44^{\mathrm{i}}\, 23^{\mathrm{ii}}\, 13^{\mathrm{iii}}\, 7^{\mathrm{iv}}$).

According to my computations using a calculator, Fibonacci's sexagesimal approximation agrees to six decimal places with my approximation (4.456855).

Fibonacci's problem is wrongly stated by some writers, for example, Van der Waerden [10, p. 40] and Vogel [11, p. 610], both of whom say: "A regular pentagon is (to be) inscribed in an equilateral triangle." (How can angles of 108° and 60° be equated?) It is all the more surprising to have Vogel immediately afterward praise Fibonacci's treatment as "a model for the early application of algebra in geometry" (a statement with which one cannot, of course, disagree). Perhaps the error is due to a mistranslation.

Loria [7, p. 231], who does give a proper account of the problem, states however that x is the length bd (which may be a misprint). But Fibonacci, after saying that he is taking each side of the pentagon to be *res*, continues "...*et auferam ad ex ab, scilicet rem de 10, remanebit db 10 minus re*" (i.e., $ab - ad = db = 10 - x$), i.e., $ad = x$.

Cantor [2] gives a correct interpretation and analysis of the problem. (Additionally, he extends Fibonacci's problem by finding the value of x in terms of equal sides of length a and base-length b for the general isosceles triangle.)

(c) Problems on the Distribution of Money

After this excursion into Euclidean geometry, Fibonacci reverts to money problems, in particular the distribution of money among five men—let us designate them by x, y, z, u, v—according to certain prescribed conditions.

In effect, we are required to solve the five equations

$$x + \frac{1}{2}y = 12,\ y + \frac{1}{3}z = 15,\ z + \frac{1}{4}u = 18,\ u + \frac{1}{5}v = 20,\ v + \frac{1}{6}x = 23.$$

To assist his explanation, Fibonacci arranges some of the information in tabular form. The answers are:

$$x = 6\tfrac{612}{721},\ y = 10\tfrac{218}{721},\ z = 14\tfrac{67}{721},\ u = 15\tfrac{453}{721},\ v = 21\tfrac{619}{721}.$$

However, Fibonacci presents his solutions in the Arabic form, i.e., the fractions precede the integer. For example, he gives v as $\frac{3}{7}\ \frac{88}{103}\ 21$, where the fractional part is to be interpreted as

$$\frac{3 + 88 \times 7}{7 \times 103}.$$

(I cannot reconcile my correct answer for x with the printed version of Fibonacci's answer which is not easy to decipher in my enlarged photocopy of the microfilmed text.) Fibonacci's argument in his solution indicates that he is thinking of the calculations for each man being performed in columns. Apart from this technique, his method of solution is the usual mechanical one of clearing the given equations of fractions and then adding or subtracting successive pairs of equations as appropriate.

The letter concludes with a variation of this problem. Fibonacci now requires the reader to solve the system of five equations:

$$x + \frac{1}{2}(y + z + u + v) = 12,$$

$$y + \frac{1}{3}(z + u + v + x) = 15,$$

$$z + \frac{1}{4}(u + v + x + y) = 18,$$

$$u + \frac{1}{5}(v + x + y + z) = 20,$$

$$v + \frac{1}{6}(x + y + z + u) = 23.$$

He does not tell us how he resolves the problem but finishes his correspondence with the comment: "*tunc questio esset insolubilis, nisi concederetur, primus habere debitum; quod debitum esset* $\frac{97}{197}$ 13."

His correct, unique solution is, in our notation,

$$x = -13\frac{97}{197},\ y = 3\frac{297}{394},\ z = 11\frac{99}{197},\ u = 15\frac{247}{394},\ v = 20\frac{20}{197}.$$

Much computational skill must have been required to achieve this solution. What is also important is the fact that Fibonacci was willing to acknowledge a negative number as a solution, this negative number being conceived in commercial terms as a debt. He did not, of course, use the minus sign which was introduced via mercantile arithmetic in Germany nearly three centuries later (also to represent a debt).

3. Concluding Remarks

While Fibonacci's letter to Master Theodorus does not reveal the true magnitude of his genius, it does exhibit some of his originality, versatility, and wide-ranging expertise, as well as some of his powerful methods.

He was, indeed, the *primum mobile* in pioneering the rejuvenation of mathematics in Christian Europe. He absorbed, and independently extended, the knowledge of his precursors, demonstrating a particular agility with computations and manipulations with indeterminate equations of the first and second degrees. In his geometrical expositions, he displayed a complete mastery of the content and rigor of Euclid's works and, moreover, he applied to problems of geometry the new techniques of algebra.

Unquestionably he was, as competent critics agree, the greatest creator and exponent of number theory for over a millennium between the time of Diophantus and that of Fermat.

To measure one's own mathematical ability against that of Fibonacci (born about 1175, died about 1240, while Pisa was still a prosperous maritime republic), the reader is invited to attempt some of the problems occurring in Fibonacci's writings, especially his *Liber quadratorum* (see Sigler [9]), e.g., the last problem in that book—proposed by Master Theodorus—namely, to solve the equations

$$x + y + z + x^2 = u^2,$$
$$x + y + z + x^2 + y^2 = v^2,$$
$$x + y + z + x^2 + y^2 + z^2 = w^2.$$

I conclude this short treatment of the *Epistola* with a chastening quote [4] from Fibonacci's best-known work, *Liber abbaci*, which expresses a sentiment reiterated in the Prologue of *Liber quadratorum* [9]:

> If I have perchance omitted anything more or less proper or necessary, I beg indulgence, since there is no one who is blameless and utterly provident in all things.

References

1. B. Boncompagni. *Scritti di Leonardo Pisano*. 2 vols. Rome: 1857 (Vol. I); 1862 (Vol. II).
2. M. Cantor. *Vorlesungen über Geschichte der Mathematik. Zweiter Band*. Leipzig: Teubner, 1913.
3. J. Gies & F. Gies. *Leonard of Pisa and the New Mathematics of the Middle Ages*. New York: Crowell, 1969.
4. R. E. Grimm. "The Autobiography of Leonardo Pisano." *Fibonacci Quarterly 11.1* (1973):99-104.
5. D. Herlihy. *Pisa in the Early Renaissance: A Study of Urban Growth*. New Haven: Yale University Press, 1958.
6. A. F. Horadam. "Eight Hundred Years Young." *The Australian Mathematics Teacher 31.4* (1975):123-34.
7. G. Loria. *Storia della Matematiche*. Milan: Editore Ulrico Hoepli, 1950.
8. R. B. McClenon. "Leonardo of Pisa and His *Liber Quadratorum*." *Amer. Math. Monthly 26.1* (1919):1-8.
9. L. E. Sigler. "The Book of Squares" (trans. of Fibonacci's *Liber quadratorum*). New York: Academic Press, 1987.
10. B. L. Van der Waerden. *A History of Algebra from al-Khowārismī to Emmy Noether*. New York: Springer-Verlag, 1985.
11. K. Vogel. "Fibonacci, Leonardo, or Leonardo of Pisa." *Dictionary of Scientific Biography*. Vol. 14. New York: Scribner, 1971, pp. 604-13.
12. A. P. Youschkevitch. *Geschichte der Mathematik im Mittelalter*. Leipzig: Teubner, 1964 (trans. from the Russian).

Addendum: A translation of Fibonacci's letter with commentary (both in Italian) appears in E. Picutti, "Il 'Flos' di Leonardo Pisano," *Physis 25* (1983): 293-387.

*****

ON MULTI-SETS

Supriya Mohanty
Bowling Green State University, Bowling Green, OH 43403
(Submitted April 1989)

The n^{th} Fibonacci number F_n and the n^{th} Lucas number L_n are defined by

$$F_1 = 1 = F_2 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3$$

and

$$L_1 = 1,\ L_2 = 3, \text{ and } L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 3,$$

respectively. Thus, the Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ..., and the Lucas sequence is 1, 3, 4, 7, 11, 18, 29, 47, 76, ... Here we have added two adjacent numbers of a sequence and put the result next in the line.

What happens if we put the result in the middle?

Given the initial sets $T_1 = \{1\}$ and $T_2 = \{1, 2\}$, we will get the following increasing sequences of T-sets. These sets are *multi-sets* and the elements are ordered.

$$T_3 = \{1, 3, 2\},\ T_4 = \{1, 4, 3, 5, 2\},\ T_5 = \{1, 5, 4, 7, 3, 8, 5, 7, 2\},$$
$$T_6 = \{1, 6, 5, 9, 4, 11, 7, 10, 3, 11, 8, 13, 5, 12, 7, 9, 2\},$$
$$T_7 = \{1, 7, 6, 11, 5, 14, 9, 13, 4, 15, 11, 18, 7, 17, 10, 13, 3, 14, 11, 19, 8, 21, 13, 18, 5, 17, 12, 19, 7, 16, 9, 11, 2\}, \ldots$$

We show in the following that these multi-sets have some nice and interesting properties.

Proposition 1: Let $|T_n|$ denote the cardinality of the multi-set T_n. Then $|T_n| = 2^{n-2} + 1$ for $n \geq 2$.

Proof: Since $|T_n| = 2^{n-2} + 1$ for $n = 2$, we consider the case $n > 2$ in the following. We obtain T_n from T_{n-1} by inserting a new number in between every pair of consecutive members of T_{n-1} which is their sum. If $|T_{n-1}| = m$, then there are $m - 1$ gaps. In each of these gaps a new number will be inserted to form T_n. Thus,

$$|T_n| = m + m - 1 = 2m - 1 = 2|T_{n-1}| - 1.$$

We have $|T_3| = 3$, $|T_4| = 5$, and $|T_5| = 9$. Looking at these numbers we conjecture that $|T_n| = 2^{n-2} + 1$ for $n > 2$. Our conjecture is true for $n = 3$, 4, and 5. Suppose it is true for $n = k$. Then $|T_k| = 2^{k-2} + 1$. Since $|T_{k+1}| = 2|T_k| - 1$,

$$|T_{k+1}| = 2(2^{k-2} + 1) - 1 = 2^{k-1} + 1 = 2^{(k+1)-2} + 1.$$

Thus, assuming the truth of the conjecture for $n = k$, we proved the truth of the conjecture for $n = k + 1$. Hence, by mathematical induction, our conjecture is true for all integers $n \geq 2$.

Proposition 2: The largest number present in the multi-set T_n is F_{n+1}. Furthermore, T_n contains all the Fibonacci numbers up to F_{n+1}.

Proof: Since we have only F_2 and F_3 in T_2, they will be separated by $F_2 + F_3 = F_4$ in T_3 and we shall have F_2, F_4, F_3 in T_3 with F_4 as the largest number and F_3 as the second largest number. Then, in T_4, F_4 and F_3 will be separated by $F_4 + F_3 = F_5$ and we shall have F_4, F_5, F_3 in T_4 with F_5 as the largest number and F_4, the second largest. By induction, we shall have F_n, F_{n+1} or F_{n+1}, F_n as consecutive members in T_n. Thus, the largest number present in T_n will be F_{n+1}.

Since $T_1 \subset T_2 \subset T_3 \subset \cdots \subset T_n$, T_n contains all of the Fibonacci numbers up to F_{n+1}.

Proposition 3: The multi-set T_n, $n \geq 3$ contains all of the Lucas numbers up to L_{n-1}.

Proof: The multi-set T_3 contains two consecutive members 1 and 3 which are L_1 and L_2. Then T_4 will contain L_1, $L_1 + L_2$, L_2, i.e., L_1, L_3, L_2 as consecutive members. T_5 will contain L_3, $L_3 + L_2$, L_2, i.e., L_3, L_4, L_2 as consecutive members. Thus, by induction, the highest Lucas number present in T_n will be L_{n-1}.

Since $T_1 \subset T_2 \subset \cdots \subset T_n$, T_n will contain all Lucas numbers up to L_{n-1}.

Proposition 4: Any two consecutive members in T_n, $n > 1$, are relatively prime.

Proof: The proposition is true for $n = 2$. Suppose it is true for T_{n-1}, i.e., $(a, b) = 1$ for every pair of consecutive members a and b in T_{n-1}. Let x and y be two consecutive members in T_n. Then, either $x - y$ and y (if $x > y$) or x and $y - x$ (if $y > x$) are consecutive members in T_{n-1}. By assumption, if $x - y$ and y are consecutive, then $(x - y, y) = 1$. Hence, $(x, y) = 1$. Similarly, if $(x, y - x) = 1$, then $(x, y) = 1$. By mathematical induction, the proposition holds for all n.

Proposition 5: The second element of T_n is n and the last but one element of T_n is $2n - 3$.

Proof: The result follows by mathematical induction.

Proposition 6: The numbers 1, 2, 3, 4, and 6 appear once and only once in every T_n, $n \geq 6$ as follows:

(i) The number 1 appears in the first place and 1, n, $n - 1$ are consecutive members in T_n.

(ii) The number 2 appears in the $(2^{n-2} + 1)^{\text{th}}$ place and $2n - 5$, $2n - 3$, 2 are consecutive members in T_n.

(iii) The number 3 appears in the $(2^{n-3} + 1)^{\text{th}}$ place and $3n - 8$, 3, $3n - 7$ are consecutive members in T_n.

(iv) The number 4 appears in the $(2^{n-4} + 1)^{\text{th}}$ place and $4n - 15$, 5, $4n - 13$ are consecutive members in T_n.

Proof: Follows by induction.

Theorem 1: For $3 \leq m \leq n$, the multiplicity of m in multi-set T_n is $\frac{1}{2}\phi(m)$, where ϕ is Euler's function.

[$\phi(n)$ is the number of numbers less than n and relatively prime to n. We clearly have $\phi(P) = P - 1$ for a prime P. When n is composite with prime factorization $n = \prod_{i=1}^{r} P_i^{a_i}$, then

$$\phi(n) = n\prod_{i=1}^{r}\left(1 - \frac{1}{P_i}\right).]$$

Proof: To get an m in T_n, a pair (a, b) totalling m should appear in T_{n-1} as consecutive members. Since any two consecutive members in T_{n-1} are relatively prime (Proposition 4), the pair (a, b) must be relatively prime. So we need to know the number of pairs (a, b) with $(a, b) = 1$ and $a + b = m$. Consider $m = a + b$ with $(a, b) = 1$. Then, clearly, $(a, m) = 1 = (b, m)$. Since there are $\phi(m)$ numbers less than m and relatively prime to m, we can chose "a" in $\phi(m)$ ways. Once "a" is chosen, $b = m - a$ is fixed. Since the pairs (a, b) and (b, a) give the same total, we have $\frac{1}{2}\phi(m)$ pairs (a, b) satisfying $(a, b) = 1$ and $(a + b) = m$. Clearly $(1, m - 1)$ is one of the $\frac{1}{2}\phi(m)$ pairs, and this pair appears for the first time (and for the last as well) as consecutive members in T_{m-1}. This pair will yield an m in T_m. Thus, we are guaranteed an appearance of m in T_m.

A natural question is: How many times does m occur in T_m? Since m has $\frac{1}{2}\phi(m)$ pairs (a, b), m can appear at most $\frac{1}{2}\phi(m)$ times in T_m. We prove below that m occurs exactly $\frac{1}{2}\phi(m)$ times in T_m.

Consider a relatively prime pair $(a, m - a)$ with $a < m - a$, $a \neq 1$. Does it belong to T_n for some n? Since $(a, m - a) = 1$, the g.c.d. of "a" and "$m - a$" is 1. Then, by Euclid's g.c.d. algorithm, we have:

$$\begin{array}{l}
a \,\big)\; m - a \;\big(\; q_1 \\
\qquad\;\; aq_1 \\
\qquad \overline{\gamma_1 \,\big)\; a \;\big(\; q_2} \\
\qquad\qquad\;\; \gamma_1 q_2 \\
\qquad\qquad \overline{\gamma_2 \,\big)\; \gamma_1 \;\big(\; q_3} \\
\qquad\qquad\qquad\;\; \gamma_2 q_3 \\
\qquad\qquad\qquad \overline{\gamma_3 \,\big)\; \gamma_2 \;\big(\; q_4} \\
\qquad\qquad\qquad\qquad\;\; \gamma_3 q_4 \\
\qquad\qquad\qquad\qquad \overline{\gamma_4 \;\ddots} \qquad \ddots \\
\qquad\qquad\qquad\qquad\qquad \gamma_{t-1} \,\big)\; \gamma_{t-2} \;\big(\; q_t \\
\qquad\qquad\qquad\qquad\qquad\qquad\;\; \gamma_{t-1} q_t \\
\qquad\qquad\qquad\qquad\qquad \overline{1 = \gamma_t \,\big)\; \gamma_{t-1} \;\big(\; \gamma_{t-1}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \gamma_{t-1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \overline{\quad 0 \quad}
\end{array}$$

Thus, whenever $(a, m - a) = 1$, we have the last nonzero remainder $\gamma_t = 1$, with the last quotient γ_{t-1}. It is clear that γ_i $(i \neq t) > 1$.

From the algorithm, we obtain:

$$\begin{aligned}
m - a - \gamma_1 &= aq_1 \\
a - \gamma_2 &= \gamma_1 q_2 \\
\gamma_1 - \gamma_3 &= \gamma_2 q_3 \\
&\vdots \\
\gamma_{t-2} - \gamma_t &= \gamma_{t-1} q_t \\
\gamma_{t-1} &= \gamma_t \gamma_{t-1}, \text{ where } \gamma_t = 1 \text{ and } \gamma_i > 1 \text{ for } 1 < i < t.
\end{aligned}$$

Adding, we obtain:

$$m - \gamma_t = aq_1 + \gamma_1 q_2 + \gamma_2 q_3 + \cdots + \gamma_{t-1} q_t + \gamma_{t-1}$$

or $$m - 1 > q_1 + q_2 + q_3 + \cdots + q_t + \gamma_{t-1}.$$

If we start with two consecutive members, a, $m - a$ or $m - a$, a, and proceed backward, we reach the consecutive pair $(1, \gamma_{t-1})$ after $q_1 + q_2 + \cdots + q_t$ steps.

Conversely, if we start with two consecutive members, $1, \gamma_{t-1}$, we reach a consecutive member, a, $m - a$ or $m - a$, a, after $q_t + \cdots + q_3 + q_2 + q_1$ steps.

Since $1, \gamma_{t-1}$ are consecutive in the $T_{\gamma_{t-1}}$-set, and nowhere else, the pair $(a, m - a)$ appears as consecutive members in $T_{q_1 + q_2 + \cdots + q_t + \gamma_{t-1}}$.

Since $q_1 + q_2 + \cdots + \gamma_{t-1} < m - 1$, the pair $(a, m - a)$ or $(m - a, a)$ appears as consecutive members in T_i, $i < m - 1$. Thus, every pair $(a, m - a)$ with $(a, m) = 1$, excepting $(1, m - 1)$, appears as consecutive members in some T_i, $i < m - 1$ and the pair $(1, m - 1)$ appears as consecutive in T_m. Hence, for $3 \leq m \leq n$, the multiplicity of m in multi-set T_n is $\frac{1}{2}\phi(m)$. We shall see that, excepting the pair $(1, m - 1)$, other pairs appear in T_i, where $i < [(m + 3)/2]$.

Theorem 2: Every relatively prime pair $(a, m - a)$, $a \neq 1$, $a < m - a$ appears in T_i where $i < [(m + 1)/2]$, we have $i = (m + 1)/2$ in case m is odd.

Proof: We have $m - 1 = aq_1 + \gamma_1 q_2 + \gamma_2 q_3 + \cdots + \gamma_{t-1} q_t + \gamma_{t-1}$, where

$$a > \gamma_1 > \gamma_2 > \gamma_3 > \cdots > \gamma_{t-1} > \gamma_t = 1,$$

and each $q_i \geq 1$. If $\gamma_{t-1} = s$, then $m - 1 > s(q_1 + q_2 + q_3 + \cdots + q_t + 1)$, so

$$\frac{m-1}{s} > q_1 + q_2 + q_3 + \cdots + q_t + 1$$

or

$$\frac{m-1}{s} + s - 1 > q_1 + q_2 + q_3 + \cdots + q_t + s$$

or

$$q_1 + q_2 + q_3 + \cdots + q_t + s \leq \left[\frac{m-1}{s} + s - 1\right],$$

where $[x]$ stands for the greatest integer $\leq x$. The pair $(a, m - a)$ appears in the $(q_1 + q_2 + \cdots + q_t + s)^{\text{th}}$ multi-set. Hence, every pair $(a, m - a)$ of the required type terminating in 1 and s in the g.c.d. algorithm is present as consecutive members in the multi-set T_i, where $i \leq [(m - 1)/s + (s - 1)]$. For $s = 2$,

$$\left[\frac{m-1}{s} + s - 1\right] = \left[\frac{m-1}{2} + 2 - 1\right] = \left[\frac{m+1}{2}\right].$$

If m is odd and $s = 2$, then

$$\left[\frac{m-1}{s} + s - 1\right] = \frac{m+1}{2}.$$

For $s \neq 2$, the inequality

$$\frac{m-1}{s} + s - 1 \leq \frac{m+1}{2}$$

holds

$$\Leftrightarrow 2(m - 1 + s^2 - s) \leq sm + s$$

$$\Leftrightarrow 2s^2 - 3s - 2 \leq m(s - 2) \Leftrightarrow m \geq \frac{2s^2 - 3s - 2}{s - 2},\ s \neq 2,$$

$$\Leftrightarrow m \geq 2s + 1,$$

which is true because $m - a > a > s \Rightarrow m > 2s$, i.e., $m \geq 2s + 1$. Now, the above inequality yields

$$\left[\frac{m-1}{s} + s - 1\right] \leq \left[\frac{m+1}{2}\right].$$

Again, when m is odd, $s = (m - 1)/2$ is an integer and

$$\left[\frac{m-1}{s} + s - 1\right] = \left[2 + \frac{m-1}{2}\right] = \left[\frac{m+1}{2}\right] = \frac{m+1}{2}.$$

Thus, the bound $(m + 1)/2$ is attainable when m is odd and $s = (m - 1)/2$. For example, for $m = 43$, consider the pairs (2, 41) and (21, 22). Both appear in T_{22}. In the first case, $s = 2$; in the second case, $s = 21 = (43 - 1)/2$. Hence every relatively prime pair $(a, m - a)$, $a \neq 1$, $a < m - a$ appears in T_i, where $i \leq [(m + 1)/2]$.

From the above discussion, it is clear that i is much less than $[(m + 1)/2]$ when m is even. For $m = 90$, we have:

> (1, 89) in T_{89}; (7, 83) and (13, 77) in T_{18}; (23, 67) and (43, 47) in T_{15}; (11, 79), (29, 61), (31, 59), and (41, 49) in T_{14}; (17, 73), (19, 71), and (37, 53) in T_{11}. Thus, excepting (1, 89), all other pairs appear as consecutive members in T_i, $i \leq 18$. This is much less than $[(m + 1)/2] = 45$.

We discuss below the appearance of certain special pairs as consecutive members in the multi-sets.

(a) The pair $(1, a)$ is always relatively prime. This pair appears as consecutive members in T_a.

(b) The pair $(a + 1, a)$ is always relatively prime whether a is odd or even. This pair appears as consecutive member in T_{a+1}. For example, 4 and 5 appear as consecutive members in T_5, 9 and 10 in T_{10}.

(c) The pair $(2m - 1, 2)$ is always relatively prime. This pair appears as consecutive members in T_{m+1}, $[m + 1 = (2 + 2m - 1 + 1)/2]$. For example, 5 and 2 in T_4, 13 and 2 in $T_{(2+13+1)/2} = T_8$.

(d) The pair $(a, a + 2)$ is relatively prime if a is odd. We need $1 + (a - 1)/2$ steps to reach this pair if we start from the consecutive members 1, 2. Therefore, the pair $(a, a + 2)$ appears as consecutive members in $T_{[1+(a-1)/2]+2} = T_{(a+5)/2}$. For example, the pair 9 and 11 appear as consecutive members in $T_{(9+5)/2} = T_7$.

We use the above facts in the examples given in Table 1.

TABLE 1

m	Relatively Prime Pairs for a Total m	The Number of the T-Set Where the Pair Appears	The Number of the T-Set Where m Appears Separating This Pair
20	1, 19	19 by (a)	20
	3, 17	5 + 3 by (b) = 8	9
	7, 13	1 + 7 by (b) = 8	9
	9, 11	1 + 6 by (c) = 7	8
33	1, 19	32 by (a)	33
	2, 31	17 by (c)	18
	4, 29	7 + 4 by (a) = 11	12
	5, 28	5 + 1 + 3 by (b) or 5 + 4 by (d) = 9	10
	7, 26	3 + 5 by (d) = 8 or 3 + 1 + 4 by (c) = 8	9
	8, 25	3 + 8 by (a) = 1	12
	10, 23	2 + 3 + 3 by (a) = 8	9
	13, 20	1 + 1 + 7 by (b) or 1 + 1 + 1 + 6 by (a) = 9	10
	14, 19	1 + 2 + 5 by (b) or 1 + 2 + 1 + 4 by (a) = 8	9
	16. 17	1 + 16 by (a) = 17	18
40	1, 39	39 by (a)	40
	3, 37	12 + 3 by (a) = 15	16
	7, 33	4 + 5 by (d) = 9	10
	9, 31	3 + 2 + 4 by (a) = 9	10
	11, 29	2 + 1 + 1 + 4 by (b) = 8	9
	13, 27	2 + 13 by (a) = 15	16
	17. 23	1 + 2 + 6 by (b) or 1 + 2 + 1 + 5 by (a) = 9	10
	19, 21	1 + 9 + 2 by (a) or 1 + 11 by (c0 = 12	13
42	1, 41	41 by (a)	42
	5, 37	7 + 4 by (c) = 11	12
	11, 31	2 + 7 by (a) = 9	10
	13, 29	2 + 4 + 3 by (a) = 9	10
	17, 25	1 + 2 + 8 by (a) = 11	12
	19, 23	1 + 4 + 4 by (b) = 9	10

By Propositions 1 and 2, T_n $(n \geq 2)$ has $2^{n-2} + 1$ members with the highest number F_{n+1}. We have

$$2^{n-2} + 1 = F_{n+1} \text{ for } n = 2, 3, 4$$

and

$$2^{n-2} + 1 > F_{n+1} \text{ for } n > 4.$$

So, for $n > 4$, the multi-set T_n has more elements than the highest number present. Does it contain all numbers 1, 2, 3, 4, ... up to F_{n+1}? We see that T_5 omits 6, T_7 omits 20, and T_8 omits 28, 32, and 33. For 6 we have only one relatively prime pair (1, 5). This pair appears as consecutive members in T_5. So 6 will appear for the first time in T_6. From Table 1, we see that the relatively prime pair (9, 11) for 20 appears as consecutive members in T_7 and other pairs appear later. Therefore, 20 will appear for the first time in T_8. Again, the relatively prime pairs (7, 26), (10, 23), and (14, 19) for 33 appear as consecutive members in T_8 (see Table 1). Therefore, 33 will appear for the first time in T_9 and will appear thrice. Thus, given an integer m, we can always find the T_i where m appears for the first time, and given two integers m and i, we can always say whether m appears in T_i. But, for given i, we do not see how we can tell all the numbers which the multi-set T_i omits unless we construct T_i recursively, and this is a horrible task for large "i."

We conclude this paper with the following problem.

Problem 1: Given a positive integer i, find all numbers m that T_i omits without constructing T_i.

Reference

1. John Turner. Problem H-429. *Fibonacci Quarterly* *27.1* (1989):92.

*****

p-ADIC CONGRUENCES BETWEEN BINOMIAL COEFFICIENTS

V. Giambalvo
University of Connecticut, Storrs, CT 06268

Ray Mines
New Mexico State University, Las Cruces, NM 88003

David J. Pengelley
New Mexico State University, Las Cruces, NM 88003
(Submitted April 1989)

The study of identities and congruences involving binomial coefficients has a long history, not only because of the intrinsic beauty and apparent simplicity of many of the results, but also because applications for these abound in many fields, both inside and outside mathematics. The impetus for the present study came from work on classifying spaces in algebraic topology [3], where one needed to know how the 2-divisibility of $\binom{a+2^n}{b} - \binom{a}{b}$ depends on n, a, and b.

The basic question we would like to address is this: For a given prime p, and natural numbers a, b, $a \geq b \geq 1$, what is the p-divisibility of the difference

$$\binom{a+x}{b} - \binom{a}{b}$$

and how does it depend on the p-divisibility of x? For any integer k, let $v_p(k)$ denote the exponent of the highest power of p dividing k, and $v_p(k/n) = v_p(k) - v_p(n)$. We wish to consider

$$f(x) = v_p\left(\binom{a+x}{b} - \binom{a}{b}\right),$$

where x is any natural number. Now

$$F(x) = \binom{a+x}{b}$$

is a polynomial in x with $F'(0) \neq 0$, so it is elementary that, for $v_p(x)$ large (i.e., x near 0 in the p-adic metric),

$$f(x) = v_p(F(x) - F(0)) = v_p(x) + v_p(F'(0)).$$

In other words, f "stabilizes" for $v_p(x)$ sufficiently large. The aims of this note are threefold. First to determine exactly how large is sufficiently large, second to examine the behavior of f both in and near this range, and third to understand how the behavior of f is related to the divisibilities of $\binom{a+x}{b}$ and $\binom{a}{b}$. These three divisibilities are intimately connected by the fact that

$$v_p(y \pm z) \geq \min\{v_p(y), v_p(z)\},$$

with equality holding for $p = 2$ precisely when $v_2(y) \neq v_2(z)$. This creates some surprising phenomena when $p = 2$. The most striking is that while constancy of $v_2\binom{a+x}{b}$ for $v_2(x)$ large is necessary in order for f to exhibit stability, the latter always occurs before, not after, the former. One of our main aims is to understand the phenomena underlying this curious fact. Our Conclusion summarizes why this occurs.

Complete results will be given for $p = 2$, and some partial results will be obtained for odd primes, where the situation is much more complicated.

1. Preliminaries

First we look at $\binom{a+x}{b}$ and its p-divisibility. The basic result on divisibility is due to Kummer [4, pp. 115-16; 1, p. 270]: If $a = \sum a_i p^i$ and $b = \sum b_i p^i$ are the base p expansions of a and b (here, of course, a_i, $b_i \in [0, p)$), then $v_p\binom{a}{b}$, the p-divisibility of $\binom{a}{b}$, is the number of borrows in the base p subtraction $a - b$. A good general reference is [5]. Some related results can be found in [2]. Therefore,

$$v_p\binom{a+x}{b} = v_p\binom{a}{b} \text{ for } v_p(x) \text{ large,}$$

and we wish to quantify "large."

Definition 1: $M(a, b, p)$ is the smallest integer M such that

$$v_p\binom{a+x}{b} = v_p\binom{a}{b} \text{ whenever } v_p(x) \geq M.$$

For any integer n, let $\overline{n}_\ell$ be the residue of n modulo p^ℓ. From Kummer's theorem, it is clear that M is nothing other than $\min\{\ell \mid \overline{a}_\ell \geq b\}$. Let

$$S = \{a, a-1, \ldots, a-b+1\}$$

be the set of integers in the "numerator" of $\binom{a}{b}$. Let $s_1, s_2, \ldots, s_b$ be the elements of S arranged in order of decreasing p-divisibility, and let $d_i = v_p(s_i)$. So d_1 is the highest divisibility occurring in S, etc. Note that the d_i are not necessarily distinct. Our first lemma relates M to d_1.

Lemma 2: $v_p\binom{a+x}{b} = v_p\binom{a}{b}$ whenever $v_p(x) \geq M$, where $M = \min\{\ell \mid \overline{a}_\ell \geq b\} = d_1 + 1$.

Proof: Everything was done above, except the equality $M = d_1 + 1$. We show this by manipulating the base p expansion of a. Since $\overline{a}_M \geq b$, $\overline{a}_M$ can never be reduced to zero by subtracting something in the interval $[0, b)$, so no element of S is congruent to 0 mod p^M. Hence, $d_1 \leq M - 1$. To see that $d_1 \geq M - 1$, note that $\overline{a}_{M-1} < b$, and so there is an element of S which is zero mod p^{M-1}. Thus, $d_1 \geq M - 1$. $\square$

We now turn our attention to $f(x)$.

Definition 3: $N(a, b, p)$ is the smallest integer N such that

$$f(x) = v_p(x) + v_p(F'(0)) \text{ whenever } v_p(x) \geq N.$$

Since the equality

$$v_p\binom{a+x}{b} = v_p\binom{a}{b}$$

of Lemma 1 is clearly necessary for this stabilizing of f, one might expect that $N \geq M$. It is therefore surprising that, on the contrary, we will show that exactly the opposite occurs for $p = 2$, and that, for odd primes, M and N are more or less independent. The first step in computing N is to bound it from above. That is one purpose of the next section.

2. A Formula for f and a bound on $N(a, b, p)$

We start with the degree b polynomial

$$F(x) - F(0) = \binom{a+x}{b} - \binom{a}{b}.$$

Note that $f(x) = v_p(F(x) - F(0))$. Let S be as before and $S^{-1} = \{1/s \mid s \in S\}$. For any set of integers A let $\sigma_i(A)$ denote the i^{th}-elementary symmetric function on the elements of A and abbreviate $\sigma_k(S^{-1})$ by σ_k. Then expanding $F(x) - F(0)$, we obtain

$$F(x) - F(0) = \frac{1}{b!}\sum_{k=1}^{b}\sigma_{b-k}(S)x^k = \binom{a}{b}\sum_{k=1}^{b}\sigma_k(S^{-1})x^k = x\binom{a}{b}\sum_{k=1}^{b}\sigma_k x^{k-1}.$$

Clearly, for $v_p(x)$ large, v_p applied to the final sum leaves only $v_p(\sigma_1)$. This shows that f stabilizes as claimed, and gives our first formula for it.

Theorem 4: $f(x) = v_p(x) + v_p\binom{a}{b} + v_p(\sigma_1)$ for $v_p(x) \geq N$.

Our main interest is in what determines N, and in the curious way that this is related to $v_p\binom{a+x}{b}$ in and near the stable range when $p = 2$. Now to obtain a bound for N from the above, we need only determine how large $v_p(x)$ need be to ensure that

$$v_p\left(\sum_{k=1}^{b}\sigma_k x^{k-1}\right) = v_p(\sigma_1).$$

Theorem 5: $N(a, b, p) \leq v_p(\sigma_1) + d_1 + d_2 + 1.$

Proof: We will show that $v_p(\sigma_k) + v_p(x)(k - 1) > v_p(\sigma_1)$ for $k \geq 2$, as long as $v_p(x) > v_p(\sigma_1) + d_1 + d_2$.

Note that

$$v_p(\sigma_k) \geq -\sum_{i=1}^{k} d_i.$$

We then have

$$\begin{aligned} v_p(\sigma_k) + v_p(x)(k-1) &> -\sum_{i=1}^{k} d_i + (k-1)(v_p(\sigma_1) + d_1 + d_2) \\ &= v_p(\sigma_1) + (k-2)v_p(\sigma_1) + (k-2)(d_1 + d_2) - \sum_{i=3}^{k} d_i \\ &= v_p(\sigma_1) + (k-2)(v_p(\sigma_1) + d_1) + \sum_{i=3}^{k}(d_2 - d_i) \\ &\geq v_p(\sigma_1). \quad \square \end{aligned}$$

3. At the Prime 2

Henceforth, let $p = 2$ and let v stand for v_2. In this section we will simplify our formula for f in the stable range, show that $N = d_2 + 1$, and give a formula for N that is easily computed from a and b. This formula shows that N is almost determined by b.

We begin by obtaining more information about the behavior of $v\binom{a+x}{b}$.

Lemma 6: The following facts express how the relationship between $v\binom{a+x}{b}$ and $v\binom{a}{b}$ changes as $v(x)$ varies in relation to d_2, d_1, and M:

a. $d_1 > d_2$,

b. $v\binom{a+x}{b} = v(x) + v\binom{a}{b} - d_1$ when $d_2 < v(x) < M - 1 = d_1$,

c. $v\binom{a+x}{b} > v\binom{a}{b}$ when $v(x) = M - 1 = d_1$,

d. $v\binom{a+x}{b} = v\binom{a}{b}$ when $v(x) \geq M = d_1 + 1$.

Notice that Lemma 6 shows that $v\binom{a+x}{b}$ increases predictably for $d_2 < v(x) < d_1$, jumps sharply up when $v(x) = d_1$, and then drops to constancy for $v(x) > d_1$. Later, we will compare this behavior with that of $f(x)$.

Proof: We note first that $d_1 > d_2$, since between any two integers exactly divisible by 2^j lies one divisible by 2^{j+1}. For parts (b) and (c), we note that

$$v\binom{a+x}{b} - v\binom{a}{b} = v(s_1 + x) - v(s_1) + \sum_{i>1} v(s_i + x) - v(s_i).$$

Since $v(x) > d_2$, we have $v(s_i + x) = v(s_i)$ for all $i > 1$, so the sum evaporates. Then, if $v(x) = d_1$, we have $v(s_1 + s) > v(s_1)$, so the result is positive, while if $v(x) < d_1$, then $v(s_1 + x) = v(x)$, producing the result $v(x) - d_1$, as claimed. Part (d), which completes our description of the behavior of $v\binom{a+x}{b}$, is merely a restatement of Lemma 2. □

Now we can also strengthen our theorems about f and N, since we can actually compute $v(\sigma_1)$.

Corollary 7: $f(x) = v(x) + v\binom{a}{b} - d_1$ for $v(x) \geq N$, and $N \leq d_2 + 1 < d_1 + 1 = M$.

Proof: From Lemma 6, we know that $d_1 > d_2$. Hence, $v(\sigma_1) = -d_1$ and the result follows. □

This verifies that $N < M$, i.e., $f(x)$ stabilized before $v\binom{a+x}{b}$ becomes constant.

Next, we complete our determination of N with

Theorem 8: $N = d_2 + 1$. Moreover $f(x) > v(x) + v\binom{a}{b} - d_1$ whenever $v(x) = N - 1 = d_2$.

Proof: In view of Corollary 7, we need only show that

$$f(x) > d_2 + v\binom{a}{b} - d_1 \text{ if } v(x) = d_2.$$

Since

$$v\binom{a}{b} > d_2 + v\binom{a}{b} - d_1$$

from Lemma 6, this will follow if we also show that

$$v\binom{a+x}{b} > d_2 + v\binom{a}{b} - d_1.$$

Recalling that $v(x) = d_2 < d_1$, we have

$$\begin{aligned} v\binom{a+x}{b} - v\binom{a}{b} &= v(s_1 + x) - v(s_1) + \sum_{v(s_i) \leq d_2} (v(s_i + x) - v(s_i)) \\ &= d_2 - d_1 + \sum_{v(s_i) \leq d_2} (v(s_i + x) - v(s_i)) \\ &> d_2 - d_1, \end{aligned}$$

the last inequality holding, since each term in the sum if nonnegative, and at least one [with $v(s_2) = d_2$] is positive. □

We will now provide a formula for N more convenient for calculation. Let

$$k = k(b) = [\log_2(b)],$$

the greatest integer in $\log_2(b)$. Recall that, for any integer n, $\overline{n}_\ell$ denotes the residue of n modulo 2^ℓ. Let

$$g(a, b) = \begin{cases} k & \text{if } \overline{a}_k \geq \overline{b}_k, \\ k + 1 & \text{if } \overline{a}_k < \overline{b}_k. \end{cases}$$

Clearly, g is easy to compute from a and b, and it is almost determined by b.

Lemma 9: $N = d_2 + 1 = g$.

Proof: We need only show that $g = d_2 + 1$. First we show that $g \geq d_2 + 1$. Since $[\log_2(b)] = k$, we have $b \in [2^k, 2^{k+1})$. Since S is a sequence of b consecutive integers, S must contain exactly one or two multiples of 2^k. If only one, it is the element of highest two-divisibility d_1 in S, so $d_2 < k$; hence $d_2 + 1 \le k \le g$. If there are two, one is an even multiple of 2^k, of highest divisibility, the other is an odd multiple of 2^k. Hence, $d_2 = k$. Thus, we need to show that whenever $g = k$ (rather than $k + 1$), S has only one multiple of 2^k. But $g = k$ only when $\overline{b}_k \le \overline{a}_k$. We write $b = 2^k + \overline{b}_k$, $a = \beta 2^k + \overline{a}_k$, with $0 \le \overline{b}_k \le \overline{a}_k < 2^k$. Then

$$(\beta - 1)2^k \le a - b < \beta 2^k \le a < (\beta + 1)2^k,$$

so S has only one multiple of 2^k.

To show that $g \le d_2 + 1$, note that, since $b \in [2^k, 2^{k+1})$, there must be at least two multiples of 2^{k-1} in S. Thus, $d_2 \ge k - 1$, or $d_2 + 1 \ge k$. So we are done if $g = k$. If $g = k + 1$, then we need d_2 to be at least k. So we need two multiples of 2^k in S. We write a and b as before, but now $\overline{a}_k < \overline{b}_k < 2^k$, so we have

$$a - b = (\beta - 1)2^k + (\overline{a}_k - \overline{b}_k) < (\beta - 1)2^k < \beta 2^k \le a,$$

and we have exhibited two multiples of 2^k in S. $\square$

4. Conclusions

Our results for $p = 2$ provide a complete picture of the relationship among

$$v\binom{a}{b},\ v\binom{a+x}{b},\ \text{and } f(x) = v\left(\binom{a+x}{b} - \binom{a}{b}\right)$$

in the stable range. There are three possibilities:

$f(x)$ will equal $v\binom{a+x}{b}$ if $v\binom{a+x}{b} < v\binom{a}{b}$;

$f(x)$ will equal $v\binom{a}{b}$ if $v\binom{a}{b} < v\binom{a+x}{b}$;

$f(x)$ will exceed both of the above if they are equal.

We see that all three possibilities actually occur, in the order stated, as $v(x)$ increases through the stable range. This trio and order of behaviors is, in fact, the only way $f(x)$ can possibly achieve the formula

$$v(x) + v\binom{a}{b} - d_1$$

in a range that srarts earlier (at $N = d_2 + 1 = g$) than the constancy of $v\binom{a+x}{b}$ (at $M = d_1 + 1$).

For odd primes, the situation can be quite different. We illustrate the situation in the case of $b = 2$. Then

$$F(x) - F(0) = \binom{a+x}{2} - \binom{a}{2} = x(x + 2a - 1)/2.$$

Let j be a positive integer.

First, choose $a = p^j$. From Lemma 2, we have $M = d_1 + 1 = j + 1$, and since $v_p(\sigma_1) = -j$ and $d_2 = 0$, Theorem 5 says that $N \le 1$. Since

$$F(x) - F(0) = x(x + 2p^j - 1)/2,$$

we have that N is indeed 1. So, as above, $N < M$ and $N = [\log_p(b)] + 1$.

Next, choose $a = (p^j + 1)/2$. Here the situation is radically different. Since p is odd, $d_1 = d_2 = 0$, but $2a - 1 = p^j$, so $v_p(\sigma_1) = j$, and Theorem 5 says that $N \le j + 1$. From

$$F(x) - F(0) = x(x + p^j)/2,$$

we see that $N = j + 1$. But $M = d_1 + 1 = 1$.

There are patterns however, and the reader is invited to discover them.

References

1. L. E. Dickson. *History of the Theory of Numbers*. Vol. I. Washington, D.C.: Carnegie Institute of Washington, 1919.
2. Robert D. Fray. "Congruence Properties of Ordinary and q-Binomial Coefficients." *Duke J. Math.* *34* (1967):467-80.
3. V. Giambalvo, D. J. Pengelley, D. C. Ravenel. "A Fractal-Like Algebraic Splitting of the Classifying Space for Vector Bundles." *Trans. Amer. Math. Soc.* *307* (1988):433-55.
4. D. Kummer. "Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen." *Journal für die Reine und Angewandte Mathematik* *44* (1852):93-146.
5. D. Singmaster. "Divisibility of Binomial and Multinomial Coefficients by Primes and Prime Powers." *A Collection of Manuscripts Related to the Fibonacci Sequence: 18th Anniversary Volume*. Ed. V. E. Hoggatt, Jr., & M. Bicknell-Johnson. Santa Clara, Calif.: The Fibonacci Association, 1980, pp. 98-113.

*****

ZECKENDORF NUMBER SYSTEMS AND ASSOCIATED PARTITIONS

Clark Kimberling
University of Evansville, Evansville, IN 47702
(Submitted April 1989)

The binary number system lends itself to unrestricted ordered partitions, as indicated in Table 1.

TABLE 1. The Binary Case

n	Binary Representation	k	Associated Partition of k
1	1	1	1
2	10	2	2
3	11	2	11
4	100	3	3
5	101	3	21
6	110	3	12
7	111	3	111
8	1000	4	4
9	1001	4	31
10	1010	4	22
11	1011	4	211
12	1100	4	13
13	1101	4	121
14	1110	4	112
15	1111	4	1111
16	10000	5	8

Note that the partitions of $k = 4$, ranging from 4 to 1111, are in one-to-one correspondence with the integers from 8 to 15, for a total of 8 partitions. Similarly, there are 16 partitions of 5, 32 of 6, and generally, 2^{k-1} partitions of k. These are in one-to one correspondence with the binary representations of length k.

It is well known (Zeckendorf [1]) that the Fibonacci numbers

$$F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13,\ \ldots$$

serve as a basis for another zero-one number system, depending on unique sums of nonconsecutive Fibonacci numbers. These sums are often called Zeckendorf representations (see Table 2). The partitions of k that appear in this scheme are those in which only the last term can equal 1; that is,

$$k = r_1 + r_2 + \cdots + r_j, \text{ where } r_i \geq 2 \text{ for } i < j \text{ and } r_j \geq 1.$$

Table 2 suggests that, for any k, the number of partitions in which 1 is allowed only in the last place is the Fibonacci number F_k (e.g., 34 - 21 = 13 partitions of 7, ranging from 7 to 2221). This is nothing new, since the number of zero-one sequences of length k beginning with 1 and having no two consecutive 1's is well known to be F_k. It is less well known that these zero-one sequences correspond to partitions.

TABLE 2. The Zeckendorf Case

n	Zeckendorf Representation	Zero-One Representation	k	Associated Partition of k
1	1	1	1	1
2	2	10	2	2
3	3	100	3	3
4	3 + 1	101	3	21
5	5	1000	4	4
6	5 + 1	1001	4	31
7	5 + 2	1010	4	22
8	8	10000	5	5
⋮				
21	21	1000000	7	7
22	21 + 1	1000001	7	61
23	21 + 2	1000010	7	52
24	21 + 3	1000100	7	43
25	21 + 3 + 1	1000101	7	421
⋮				
32	21 + 8 + 3	1010100	7	223
33	21 + 8 + 3 + 1	1010101	7	2221
34	34	10000000	8	8

Here is a summary of the observations from Tables 1 and 2. The first-order recurrence sequence 1, 2, 4, 8, ... serves as a basis for unrestricted partitions, and the second-order recurrence sequence 1, 2, 3, 5, 8, ... serves as a basis for somewhat restricted partitions.

The purpose of this article is to extend these results to higher-order sequences, their zero-one number systems, and associated partitions. To this end, and for the remainder of the article, let m be an arbitrary fixed integer greater than 2.

Define a sequence $\{s_i\}$ inductively as follows:

$$s_i = 1 \quad \text{for } i = 1, 2, \ldots, m,$$

$$s_i = s_{i-1} + s_{i-m} \quad \text{for } i = m + 1, m + 2, \ldots .$$

Theorem 1: Every positive integer n is uniquely a sum

$$s_{i_1} + s_{i_2} + \cdots + s_{i_v}, \text{ where } i_t - i_u \ge m \text{ whenever } t > u.$$

Proof: The first m positive integers are one-term sums. Suppose, for $h \ge m + 1$, that the statement of the theorem holds for all $n \le h - 1$. Let i_1 be the greatest i for which $s_i \le h$. If $h - s_{i_1} = 0$, then the required sum is s_{i_1} itself.

Otherwise, $h - s_{i_1}$ is, by the induction hypothesis, uniquely a sum $s_{i_2} + \cdots + s_{i_v}$ of the required sort, so that

$$h = s_{i_1} + s_{i_2} + \cdots + s_{i_v}. \tag{1}$$

Suppose $i_1 - i_2 \le m - 1$. Then

$$h \ge s_{i_1} + s_{i_2} \ge s_{i_1} + s_{i_1 - m + 1} = s_{i_1 + 1},$$

contrary to our choice of i_1 as the greatest i for which $h \ge s_i$.

Therefore, the sum in (1) has $i_t - i_u \ge m$ whenever $t > u$, and this sum is clearly unique with respect to this property. By the principle of mathematical induction, the proof of the theorem is finished.

Theorem 1 shows that the sequence $\{s_i\}$ serves as a basis for a "skip $m - i$ number system" analogous to the Zeckendorf, or Fibonacci, number system. The latter could be called the "skip 1 number system."

Examples: In the skip 1 system:

$$\begin{aligned} 31 &= 21 + 8 + 2 &&= 1010010\\ 32 &= 21 + 8 + 3 &&= 1010100\\ 33 &= 21 + 8 + 3 + 1 &&= 1010101\\ 34 &= 34 &&= 1000000 \end{aligned}$$

In the skip 2 system:

$$\begin{aligned} 57 &= 41 + 13 + 3 &&= 1001000100\\ 58 &= 41 + 13 + 4 &&= 1001001000\\ 59 &= 41 + 13 + 4 + 1 &&= 1001001001\\ 60 &= 60 &&= 10000000000 \end{aligned}$$

We turn now to partitions. For a quick glimpse of what is coming, notice that the zero-one representations for 57, 58, and 59, just above, lend themselves naturally to the partitions 343, 334, and 3331 of the integer 10.

In general, in the $m - 1$ system, for a given positive integer k, the digit one occurs at and only at places $i_1, i_2, \ldots, i_v$, where $k = s_{i_1} + s_{i_2} + \cdots + s_{i_v}$, and each pair of ones are separated by at least $m - 1$ zeros; therefore, to each k there is a unique ordered v-tuple of integers r_i defined by

$$(2)\qquad \begin{cases} r_1 = i_1, \text{ if } v = 1,\\ r_u = i_u - i_{u+1} \text{ for } u = 1, 2, \ldots, v - 1, \text{ if } v > 1 \text{ and } s_{i_v} \geq m,\\ r_u = i_u - i_{u+1} \text{ for } u = 1, 2, \ldots, v - 1 \text{ and } r_v = i_v,\\ \qquad\qquad\qquad \text{if } v > 1 \text{ and } s_{i_v} \leq m - 1. \end{cases}$$

We summarize these observations in Theorem 2.

Theorem 2: Let k be a positive integer, let $S_k = \{s_k, s_k + 1, \ldots, s_{k+1} - 1\}$, and let P_k be the set of partitions $r_1, r_2, \ldots, r_v$ of k that satisfy $r_v \geq 1$ and $r_i \geq m$ for $i = 1, 2, \ldots, m - 1$. Then equations (2) define a one-to-one correspondence between S_k and P_k, so that the number $p(k)$ of partitions in P_k is s_{k-m-1}.

Now for any positive integer k, and for $j = 1, 2, \ldots, m$, let $p(k, j)$ be the number of partitions $r_1, r_2, \ldots, r_v$ of k for which $r_v = j$ and $r_i \geq m$ for $1 = 1, 2, \ldots, v - 1$. As in Theorem 2, let $p(k)$ be the number of partitions of k for which $r_v \geq 1$ and $r_i \geq m$ for $i = 1, 2, \ldots, v - 1$. Let $q(k)$ be the number of partitions of k for which $r_i \geq m$ for *all* indices $i = 1, 2, \ldots, v - 1, v$.

Lemma 1:

$$p(k, j) = \begin{cases} 1 & \text{if } k = j \leq m,\\ 0 & \text{if } k \leq m,\ j \leq m, \text{ and } k \neq j. \end{cases}$$

Proof: For any given $k \leq m$, the partition of k is the number k by itself, so that $p(k, k) = 1$. Clearly, $p(k, j) = 0$ for $k \neq j$ since, in this case, no partition of the form described is possible.

Lemma 2: Suppose $i \leq j \leq m$. Then $p(k, j) = p(k - 1, j) + p(k - m, j)$ for $k = m + 1, m + 2, \ldots$.

Proof: Assume $k \geq m + 1$. Each of the $p(k - 1, j)$ partitions $r_1, r_2, \ldots, r_{v-1}, j$ of $k - 1$ yields a partition $r_1 + 1, r_2, \ldots, r_{v-1}, j$ of k. Moreover, $r_1 + 1 \geq m + 1$, so that every partition of k having first term $\geq m + 1$ corresponds in this manner to a partition of $k - 1$.

Each of the $p(k - m, j)$ partitions $r_2, r_3, \ldots, j$ of $k - m$ yields a partition $m, r_2, r_3, \ldots, j$ of k. Moreover, every partition of k having first term

m corresponds in this manner to a partition of $k - m$.

Since $p(k, j)$ counts partitions having first term $\geq m$, a proof that

$$p(k, j) = p(k - 1, j) + p(k - m, j)$$

is finished.

Theorem 3: Suppose k is a positive integer. The number $q(k)$ of partitions $r_1, r_2, \ldots, r_v$ of k having $r_i \geq m$ for $i = 1, 2, \ldots, v$ is given by the m^{th}-order linear recurrence $q(k) = q(k - 1) + q(k - m)$ for $k = m + 1, m + 2, \ldots$, where $q(j) = 0$ for $j = 1, 2, \ldots, m - 1$, and $q(m) = 1$.

Proof: The assertion follows directly from Lemma 2, since

$$q(k) = p(k) - \sum_{j=1}^{m-1} p(k, j).$$

Reference

1. E. Zeckendorf. "Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas." *Bull. Soc. Royale Sci. Liége 41* (1972):179-82.

*****

COMBINATORIAL REPRESENTATION OF GENERALIZED FIBONACCI NUMBERS*

Shmuel T. Klein
Bar-Ilan University, Ramat-Gan 52 900, Israel
(Submitted April 1989)

1. Introduction

A well-known combinatorial formula for the Fibonacci numbers F_n, defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, is

$$(1) \qquad \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} = F_{n+1} \text{ for } n \geq 0,$$

which can be shown by induction (see, for example, Knuth [11, Ex. 1.2.8-16]). The following proof, however, is easily generalizable to various other recursively defined sequences of integers.

The Fibonacci numbers $\{F_2, F_3, \ldots\}$ are the basis elements of the *binary Fibonacci numeration system* (see [11, Ex. 1.2.8-34] or Fraenkel [7]). Every integer K in the range $0 \leq K < F_{n+1}$ has a unique binary representation of $n - 1$ bits, $k_{n-1}k_{n-2} \ldots k_1$, such that

$$K = \sum_{i=1}^{n-1} k_i F_{i+1}$$

and such that there are no adjacent 1's in this representation of K (see Zeckendorf [17]). It follows that, for $n \geq 1$, F_{n+1} is the number of binary strings of length $n - 1$ having no adjacent 1's. The number of these strings with precisely i 1's, $0 \leq i \leq \lfloor n/2 \rfloor$, is evaluated using the fact that the number of possibilities to distribute a indistinguishable objects into $b + 1$ disjoint sets, of which $b - 1$ should contain at least one element, is $\binom{a+1}{b}$ (see Feller [6, Sec. II.5]). In our case, there are $n - 1 - i$ zeros to be partitioned into $i + 1$ runs, of which the $i - 1$ runs delimited on both sides by 1's should be nonempty; the number of these strings is therefore $\binom{n-i}{i}$.

In a similar way, counting strings of certain types, Philippou and Muwafi [15] derived a representation of Fibonacci numbers of order m, with $m \geq 2$, as a sum of multinomial coefficients; their formula coincides with that presented earlier by Miles [13].

The properties of the representation of integers in Fibonacci-type numeration systems were used by Kautz [10] for synchronization control. More recently, they were investigated in Pihko [16] and exploited in various applications, such as the compression of large sparse bit-strings (see Fraenkel and Klein [8]), the robust transmission of binary strings in which the length is in an unknown range (see Apostolico and Fraenkel [3]), and the evaluation of the potential number of phenotypes in a model of biological processing of genetic information based on the majority rule (see Agur, Fraenkel, and Klein [1]). In the present work, the properties of numeration systems are used to generate new combinatorial formulas. In the next section, this is done for the sequence based on the recurrence $a_i = a_{i-1} + a_{i-m}$, for some $m \geq 2$, which appears in certain applications to encoding algorithms for CD-ROM. Section 3

*This work was partially supported by a fellowship from the Ameritech Foundation.

deals with other generalizations of Fibonacci numbers, namely, sequences based on the recurrences $u_i = mu_{i-1} + u_{i-2}$ for $m \geq 1$, or $v_i = mv_{i-1} - v_{i-2}$ for $m \geq 3$, which are special cases of the sequences investigated by Horadam [9]. For certain values of m and with appropriate initial values, these two recurrence relations generate the subsequences of every k^{th} Fibonacci number, for all $k \geq 1$. For further details on the properties of numeration systems, the reader is referred to [7].

2. A Generalization of Fibonacci Numbers

Given a constant integer $m \geq 2$, consider the sequence defined by

$$A_n^{(m)} = n - 1 \qquad \text{for } 1 < n \leq m + 1,$$

$$A_n^{(m)} = A_{n-1}^{(m)} + A_{n-m}^{(m)} \text{ for } n > m + 1.$$

In particular, $F_n \equiv A_n^{(2)}$ are the standard Fibonacci numbers. It follows from [7, Th. 1] that, for fixed m, the numbers $\{A_2^{(m)}, A_3^{(m)}, \ldots\}$ are the basis elements of a binary numeration system with the following property: every integer K in the range $0 \leq K < A_{n+1}^{(m)}$ has a unique binary representation of $m - 1$ bits, $k_{n-1}k_{n-2}\ldots k_1$, such that

$$K = \sum_{i=1}^{n-1} k_i A_{i+1}^{(m)}$$

and such that there are at least $m - 1$ zeros between any two 1's in this representation of K. Hence, for $n \geq 1$, $A_{n+1}^{(m)}$ is the number of binary strings of length $n - 1$ having this property.

For $n = 2$, we again get the property that there are no adjacent ones in the binary representation.

An interesting application of the sequence $A_n^{(m)}$ is to analyze encoding methods for certain optical discs. A CD-ROM (compact disc-read only memory) is an optical storage medium able to store large amounts of digital data (about 550 MB or more). The information, represented by a spiral of almost two billion tiny pits separated by spaces, is molded onto the surface of the disc. A digit 1 is represented by a transition from a pit to a space or from a space to a pit, and the length of a pit or space indicates the number of zeros. Due to the physical limitations of the optical devices, the lengths of pits and spaces are restricted, implying that there are at least two 0's between any two 1's (for details, see, for example, Davies [4]): this is the case $m = 3$ of our sequence above. It follows that if we want to encode a standard ASCII byte (256 possibilities), we need at least 14 bits, which corresponds to $A_{16}^{(3)} = 277$. In fact, there is an additional restriction that no more than 11 consecutive zeros are allowed, which disqualifies 6 of the 277 strings, but 14 bits are still enough; indeed, the code used for CD-ROM is called EFM (eight to fourteen modulation).

We now derive a combinatorial formula for $A_{n+1}^{(m)}$. First, note that $A_{n+1}^{(m)}$ is also the number of binary strings of length $n + m - 2$, with zeros in its $m - 1$ rightmost bits, such that every 1 is immediately followed by $m - 1$ zeros. Let k be the number of 1's in such a string, so that k can take values from 0 to $\lfloor (n + m - 2)/m \rfloor$. We now consider the string consisting of elements of two types: blocks of the form 10...0 ($m - 1$ zeros) and single zeros; there are k elements of the first type and $(n + m - 2) - km$ of the second, which can be arranged in

$$\binom{n + m - 2 - (m - 1)k}{k}$$

ways. Thus, we have the following formula, holding for $m \geq 2$ and $n \geq 1$:

$$(2) \qquad \sum_{k=0}^{\lfloor (n+m-2)/m \rfloor} \binom{n+m-2-(m-1)k}{k} = A_{n+1}^{(m)}.$$

For $m = 2$, (2) reduces to formula (1). Using the example mentioned above for EFM codes, setting $m = 3$ and $n = 15$, we get:

$$\binom{16}{0} + \binom{14}{1} + \binom{12}{2} + \binom{10}{3} + \binom{8}{4} + \binom{6}{5}$$
$$= 1 + 14 + 66 + 120 + 70 + 6 = 277 = A_{16}^{(3)}.$$

3. Regular Fibonacci Subsequences

Let L_n be the n^{th} Lucas number, defined by $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. The standard extension to negative indices sets

$$L_{-n} = (-1)^n L_n \quad \text{and} \quad F_{-n} = (-1)^{n+1} F_n \text{ for } n \geq 1.$$

We are interested in the *regular subsequences* of the Fibonacci sequence obtained by scanning the latter in intervals of size k, i.e., the sequences $\{F_{kn+j}\}_{n=-\infty}^{\infty}$ for all constant integers $k \geq 2$ and $0 \leq j < k$. The following identity, which is easily checked and apparently due to Lucas (see Dickson [5, p. 395]), shows that all the subsequences with the same interval size k satisfy a simple recurrence relation: for all (positive, null, or negative) integers k and n,

$$(3) \qquad F_n = L_k F_{n-k} + (-1)^{k+1} F_{n-2k}.$$

It follows that all regular subsequences of the Fibonacci numbers can be generated by a recurrence relation of the type $u_i = mu_{i-1} \pm u_{i-2}$, for certain values of m, and with appropriate initial conditions. We now apply the above techniques to obtain combinatorial representations of these number sequences.

For fixed $m \geq 3$, define a sequence of integers by

$$U_0^{(m)} = 0, \; U_1^{(m)} = 1, \text{ and } U_n^{(m)} = mU_{n-1}^{(m)} - U_{n-2}^{(m)} \text{ for } n \geq 2.$$

The numbers $\{U_1^{(m)}, U_2^{(m)}, \ldots\}$ are the basis elements of an m-ary numeration system: every integer K in the range $0 \leq K < U_n^{(m)}$ has a representation of $n - 1$ "m-ary digits," $k_{n-1}k_{n-2}\ldots k_1$, with $0 \leq k_i \leq m - 1$, such that

$$K = \sum_{i=1}^{n-1} k_i U_i^{(m)};$$

this representation is unique if the following property holds: if, for some $1 \leq i < j \leq n - 1$, k_i and k_j both assume their maximal value $m - 1$, then there exists an index s satisfying $i < s < j$, for which $k_s \leq m - 3$ (see [7, Th. 4]). In particular, for $m = 3$, we get a ternary system based on the even-indexed Fibonacci numbers $\{1, 3, 8, 21, \ldots\}$, and in the representation of any integer using this sequence as basis elements, there is at least one zero between any two 2's.

For general m, we have that $U_n^{(m)}$ is the number of m-ary strings of length $n - 1$, such that, between any two $(m - 1)$'s, there is at least one of the digits $0, \ldots, (m - 3)$. For a given m-ary string A of length $n - 1$, let j_i be the number of i's in A, $0 \leq i \leq m - 1$, thus, $0 \leq j_i < n$ and

$$\sum_{i=0}^{m-1} j_i = n - 1.$$

To construct an m-ary string satisfying the condition, we first arrange the digits $0, \ldots, (m-3)$ in any order, which can be done in

$$\binom{\sum_{i=0}^{m-3} j_i}{j_0, j_1, \ldots, j_{m-3}}$$

ways. Then the j_{m-1} $(m-1)$'s have to be interspersed, with no two of them adjacent. In other words, the $\sum_{i=0}^{m-3} j_i$ smaller digits, which are now considered indistinguishable, are partitioned into $j_{m-1}+1$ sets, of which at least $j_{m-1}-1$ should be nonempty; there are

$$\binom{n - j_{m-2} - j_{m-1}}{j_{m-1}}$$

possibilities for this partition. Finally, the $(m-2)$'s can be added anywhere, in

$$\binom{n-1}{j_{m-2}}$$

ways. This yields the followinig formula, holding for $m \geq 3$ and $n \geq 1$:

$$\sum_{\substack{j_0, \ldots, j_{m-1} \geq 0 \\ j_0 + \cdots + j_{m-1} = n-1}} \binom{n - 1 - j_{m-2} - j_{m-1}}{j_0, j_1, \ldots, j_{m-3}} \binom{n-1}{j_{m-2}} \binom{n - j_{m-2} - j_{m-1}}{j_{m-1}} = U_n^{(m)}. \tag{4}$$

Using the fact that for integers a and b, $\binom{a}{b} = 0$ if $0 \leq a < b$, there is no need to impose further restrictions on the indices, but the rightmost binomial coefficient in (4) implies that j_{m-1} varies in fact in the range $0 \leq j_{m-1} \leq \lceil (n-1)/2 \rceil$. The sequence $(U_n^{(m)})$ corresponds to the sequence $(w_n(0, 1; m, 1))$ studied by Horadam [9], but formula (4) is different from Horadam's identity (3.20).

Remark: Noting that the definition and the multinomial expansion of the multivariate Fibonacci polynomials of order k $\{H_n^{(k)}(x_1, \ldots, x_k)\}$ of Philippou and Antzoulakos [14] may be trivially extended to $x_j \in R$ $(j = 1, \ldots, k)$, we readily get the following alternative to (4), namely,

$$U_n^{(m)} = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-j}{j} (-1)^j m^{n-1-2j}, \quad m \geq 3, \; n \geq 1,$$

since $\{U_n^{(m)}\} = \{H_n^{(2)}(m, -1)\}$ $(m \geq 3, n \geq 1)$.

From (3), we know that the regular subsequence $\{F_{k(n-1)+j}\}_{n=0}^{\infty}$ of the Fibonacci numbers, for constant even $k \geq 2$ and $0 \leq j < k$, is obtained by the same recurrence relation as the sequence $\{U_n^{(L_k)}\}_{n=0}^{\infty}$, with the difference that the first two elements (indexed 0 and 1) must be defined as F_{-k+j} and F_j instead of 0 and 1. Thus, we can express the Fibonacci subsequences with even interval size in terms of $U^{(m)}$:

Theorem 1: For any even constant $k \geq 2$ and any constant $0 \leq j < k$, the following identity holds for all $n \geq 1$:

$$F_{k(n-1)+j} = F_j U_n^{(L_k)} - F_{-k+j} U_{n-1}^{(L_k)}. \tag{5}$$

Proof: By induction on n. For $n = 1$,

$$F_j = F_j \times 1 - F_{-k+j} \times 0.$$

For $n = 2$,

$$F_{k+j} = L_k F_j + (-1)^{k+1} F_{-k+j} \quad \text{by (3)},$$

but since k is even, the right-hand side can be rewritten as

$$F_j U_2^{(L_k)} - F_{-k+j} U_1^{(L_k)}.$$

Suppose the identity holds for all integers $\le n$. Then

$$\begin{aligned} F_{kn+j} &= L_k F_{k(n-1)+j} - F_{k(n-2)+j} \\ &= L_k [F_j U_n^{(L_k)} - F_{-k+j} U_{n-1}^{(L_k)}] - F_j U_{n-1}^{(L_k)} + F_{-k+j} U_{n-2}^{(L_k)} \\ &= F_j U_{n+1}^{(L_k)} - F_{-k+j} U_n^{(L_k)}, \end{aligned}$$

so the identity holds also for $n + 1$, and therefore for all $n \ge 1$. □

In particular, for $j = 0$ and $k = 2$, we get the numbers $F_{2(n-1)}$, $n = 1, 2, \ldots$, which are the even-indexed Fibonacci numbers, and correspond by (5) to

$$U_{n-1}^{(L_2)} = U_{n-1}^{(3)}.$$

For $m = L_2 = 3$, the multinomial coefficient in (4) reduces to $\binom{j_0}{j_0} = 1$, and the equivalent of (4) can therefore be rewritten as:

$$\sum_{j_2=0}^{\lceil (n-1)/2 \rceil} \sum_{j_0=\max(0,\, j_2-1)}^{n-1-j_2} \binom{n-1}{j_0+j_2}\binom{j_0+1}{j_2} = \underline{U_n^{(3)} = F_{2n}}.$$

For example, for $n = 4$, we get:

$$\binom{3}{0}\binom{1}{0} + \binom{3}{1}\binom{2}{0} + \binom{3}{2}\binom{3}{0} + \binom{3}{3}\binom{4}{0} + \binom{3}{1}\binom{1}{1} + \binom{3}{2}\binom{2}{1} + \binom{3}{3}\binom{3}{1} + \binom{3}{3}\binom{2}{2}$$

$$= 1 + 3 + 3 + 1 + 3 + 6 + 3 + 1 = 21 = U_4^{(3)} = F_8.$$

For fixed $m \ge 1$, define a sequence of integers by

$$\underline{V_0^{(m)} = 1,\ V_1^{(m)} = 1, \text{ and } V_n^{(m)} = mV_{n-1}^{(m)} + V_{n-2}^{(m)} \text{ for } n \ge 2.}$$

The numbers $\{V_1^{(m)}, V_2^{(m)}, \ldots\}$ are the basis elements of an $(m+1)$-ary numeration system with the following property: every integer K in the range $0 \le K < V_n^{(m)}$ has a unique representation of $n - 1$ "$(m + 1)$-ary digits," $k_{n-1}k_{n-2}\ldots k_1$, with $0 \le k_i \le m$, such that

$$K = \sum_{i=1}^{n-1} k_i V_i^{(m)}$$

and such that, for $i \ge 1$, if k_{i+1} assumes its maximal value m, then $k \;= 0$ (see [7, Th. 3]). In particular, for $m = 1$, we get the binary numeration system based on the Fibonacci sequence and the condition that there are no adjacent 1's.

For general m, we have that $V_n^{(m)}$ is the number of $(m + 1)$-ary strings of length $n - 1$, such that when scanning the string from left to right, every appearance of the digit m, unless it is in the last position, is immediately followed by a digit 0. Special treatment of the rightmost digit is avoided by noting that $V_n^{(m)}$ is also the number of $(m + 1)$-ary strings of length n, with 0 in its rightmost position, and where each digit m is followed by a digit 0. For a given $(m + 1)$-ary string A of length n, let j_i be the number of i's in A, $0 \le i \le m$, thus $0 \le j_i \le n$ and

$$\sum_{i=0}^{m} j_i = n.$$

To construct an $(m + 1)$-ary string satisfying the condition, distribute the 0's in the spaces between the m's, such that every m is followed by at least one 0. In other words, the j_0 zeros have to be partitioned into $j_m + 1$ sets, of which at least j_m should be nonempty; there are $\binom{j_0}{j_m}$ possibilities for this partition.

We now consider the string obtained so far as consisting of j_0 units, where each unit is either one of the j_m pairs "$m0$" or one of the remaining $j_0 - j_m$ single zeros. The digits 1, ..., $(m - 1)$ are then to be distributed in the spaces between these units, including the space preceding the first unit, but not after the last unit, because the rightmost position must be 0. First the digits 1, ..., $(m - 1)$ are arranged in any order, which can be done in

$$\binom{\sum_{i=1}^{m-1} j_i}{j_1, \ldots, j_{m-1}}$$

ways; finally, these $\sum_{i=1}^{m-1} j_i$ digits, which are considered indistinguishable, are partitioned into j_0 sets, which can be done in

$$\binom{\sum_{i=0}^{m-1} j_i - 1}{j_0 - 1} = \binom{n - 1 - j_m}{n - j_0 - j_m}$$

ways. Summarizing, we get, for $m \geq 1$ and $n \geq 1$:

$$(6) \qquad \sum_{\substack{j_0 > 0,\ j_1, \ldots, j_m \geq 0 \\ j_0 + \cdots + j_m = n}} \binom{j_0}{j_m}\binom{n - 1 - j_m}{j_0 - 1,\ j_1,\ \ldots,\ j_{m-1}} = V_n^{(m)}.$$

For $m = 1$, the multinomial coefficient is $\binom{j_0 - 1}{j_0 - 1} = 1$, and we again get (1). For $m = 2$, the sequence $\{V_n^{(2)}\}$ is $\{1, 3, 7, 17, \ldots\}$, and the ternary numeration system based on this sequence is the system which yielded the best compression results in [8]. The sequence $\{V_n^{(m)}\}$ corresponds to $\{w_n(1, 1; m, -1)\}$ in [9], but again the combinatorial representation (6) is different from Horadam's formula (3.20). For $m = 2$, (6) reduces to:

$$\sum_{j_2=0}^{\lceil (n-1)/2 \rceil} \sum_{j_0=\max(1,\, j_2)}^{n-j_2} \binom{j_0}{j_2}\binom{n - 1 - j_2}{j_0 - 1} = V_n^{(2)}.$$

For example, for $n = 3$, we get:

$$\binom{1}{0}\binom{2}{0} + \binom{2}{0}\binom{2}{1} + \binom{3}{0}\binom{2}{2} + \binom{1}{1}\binom{1}{0} + \binom{2}{1}\binom{1}{1}$$
$$= 1 + 2 + 1 + 1 + 2 = 7 = V_3^{(2)}.$$

Returning to the regular subsequences of the Fibonacci numbers, we still need a combinatorial representation of the subsequences with odd interval size k, which by (3) satisfy the same recurrence relation as $V_n^{(L_k)}$, but possibly with other initial values. The counterpart of Theorem 1 for the odd intervals is:

Theorem 2: For any odd constant $k \geq 1$ and any constant $0 \leq j < k$, the following identity holds for all $n \geq 1$:

$$(7) \qquad F_{k(n-1)+j} = F_j V_n^{(L_k)} + (F_{-k+j} - F_j) \sum_{i=1}^{n-1} (-1)^{i+1} V_{n-i}^{(L_k)}.$$

Proof: By induction on n. For $n = 1$,

$$F_j = F_j \times 1 + (F_{-k+j} - F_j) \times 0.$$

For $n = 2$,

$$F_{k+j} = L_k F_j + F_{-k+j} = F_j(L_k + 1) + (F_{-k+j} - F_j)$$
$$= F_j V_2^{(L_k)} + (F_{-k+j} - F_j) V_1^{(L_k)}.$$

Suppose the identity holds for all integers $\leq n$. Then, denoting the constant $(F_{-k+j} - F_j)$ by α,

$$\begin{aligned}F_{kn+j} &= L_k F_{k(n-1)+j} + F_{k(n-2)+j}\\ &= L_k\left[F_j V_n^{(L_k)} + \alpha\sum_{i=1}^{n-1}(-1)^{i+1}V_{n-i}^{(L_k)}\right] + F_j V_{n-1}^{(L_k)} + \alpha\sum_{i=1}^{n-2}(-1)^{i+1}V_{n-1-i}^{(L_k)}\\ &= F_j V_{n+1}^{(L_k)} + \alpha\sum_{i=1}^{n-2}(-1)^{i+1}[L_k V_{n-i}^{(L_k)} + V_{n-1-i}^{(L_k)}] + \alpha L_k(-1)^n V_1^{(L_k)}.\end{aligned}$$

But the last term is

$$\alpha(-1)^n L_k = \alpha(-1)^n(V_2^{(L_k)} - V_1^{(L_k)}) = \alpha[(-1)^n V_2^{(L_k)} + (-1)^{n+1}V_1^{(L_k)}];$$

thus,

$$F_{kn+j} = F_j V_{n+1}^{(L_k)} + \alpha\sum_{i=1}^{n}(-1)^{i+1}V_{n+1-i}^{(L_k)},$$

and the identity holds also for $n + 1$, and therefore for all n. □

In particular, for $j = 2$ and $k = 3$, we get the numbers

$$\{F_{3(n-1)+2}\}_{n=1}^{\infty} = \{1, 5, 21, 89, \ldots\},$$

i.e., every third Fibonacci number, which correspond, by (7), to $V_n^{(L_3)} = V_n^{(4)}$. For example, using formula (6) with $m = L_3 = 4$, we get for $n = 3$ (writing in the multinomial coefficients the values $j_0, \ldots, j_4$ from left to right and collecting terms which differ only in the order of the values of j_1, j_2, j_3):

$$\begin{aligned}&\binom{3}{0}\binom{1}{1,0,0,0,1} + \binom{2}{1}\binom{2}{2,0,0,0,0} + 3\binom{1}{1}\binom{1}{0,1,0,0,1} + 3\binom{2}{0}\binom{2}{1,1,0,0,0}\\ &\qquad + 3\binom{1}{0}\binom{2}{0,2,0,0,0} + 3\binom{1}{0}\binom{2}{0,1,1,0,0}\\ &= 1 + 2 + 3 + 6 + 3 + 6 = 21 = V_3^{(4)} = F_8.\end{aligned}$$

4. Concluding Remarks

Combinatorial representations of several recursively defined sequences of integers were generated, using the special properties of the corresponding numeration systems. On the other hand, it may sometimes be desirable to evaluate directly the number of strings satisfying some constraints. The above techniques then suggest to try to define a numeration system accordingly. For example, in Agur and Kerszberg [2] a model of biological processing of genetic information is proposed, in which a binary string symbolizing a DNA sequence is transformed by repeatedly applying some transition function $\mathcal{M}$. For $\mathcal{M}$ being the majority rule, the number of possible final strings, or phenotypes, is evaluated in [1] using the binary numeration system based on the standard Fibonacci numbers. Other transition functions could be studied, and if the resulting phenotypes can be characterized as satisfying some constraints, the corresponding numeration system gives an easy way to evaluate the number of these strings.

Acknowledgment

I wish to thank Professors Abe Bookstein, Richard Guy, and Aviezri Fraenkel as well as an anonymous referee for their helpful comments.

References

1. Z. Agur, A. S. Fraenkel, & S. T. Klein. "The Number of Fixed Points of the Majority Rule." *Discrete Mathematics* *70* (1988):295-302.
2. Z. Agur & M. Kerszberg. "The Emergence of Phenotypic Novelties through Progressive Genetic Change." *Amer. Natur.* *129* (1987):862-75.
3. A. Apostolico & A. S. Fraenkel. "Robust Transmission of Unbounded Strings using Fibonacci Representations." *IEEE Trans. on Inf. Th.*, *IT-33* (1987): 238-45.
4. D. H. Davies. "The CD-ROM Medium." *J. Amer. Soc. Inf. Sc.* *39* (1988):34-42.
5. L. E. Dickson. *History of the Theory of Numbers*, Vol. 1. Washington, D.C.: Carnegie Institution of Washington, 1919.
6. W. Feller. *An Introduction to Probability Theory and Its Applications*, Vol. I. New York: John Wiley & Sons, 1950.
7. A. S. Fraenkel. "Systems of Numeration." *Amer. Math. Monthly* *92* (1985): 105-14.
8. A. S. Fraenkel & S. T. Klein. "Novel Compression of Sparse Bit-Strings—Preliminary Report." *Combinatorial Algorithms on Words*, NATO ASI Series Vol. F12. Berlin: Springer Verlag, 1985, pp. 169-83.
9. A. F. Horadam. "Basic Properties of a Certain Sequence of Numbers." *The Fibonacci Quarterly* *3.2* (1985):169-83.
10. W. H. Kautz. "Fibonacci Codes for Synchronization Control." *IEEE Trans. on Inf. Th.*, *IT-11* (1965):284-92.
11. D. E. Knuth. *The Art of Computer Programming*. Vol. I: *Fundamental Algorithms*. Reading, Mass.: Addison-Wesley, 1973.
12. D. E. Knuth. *The Art of Computer Programming*. Vol. III: *Sorting and Searching*. Reading, Mass.: Addison-Wesley, 1973.
13. E. P. Miles, Jr. "Generalized Fibonacci Numbers and Associated Matrices." *Amer. Math. Monthly* *67* (1960):745-52.
14. A. N. Philippou & D. L. Antzoulakos. "Multivariate Fibonacci Polynomials of Order k and the Multiparameter Negative Binomial Distribution of the Same Order." *Proc. Third International Conference on Fibonacci Numbers and Their Applications* (Pisa 1988). Ed. by G. E. Bergum, A. N. Philippou, and A. F. Horadam. Dordrecht: Kluwer Academic Publishers, 1989.
15. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order k." *The Fibonacci Quarterly* *20.1* (1982):28-32.
16. J. Pihko. "On Fibonacci and Lucas Representations and a Theorem of Lekkerkerker." *The Fibonacci Quarterly* *26.3* (1988):256-61.
17. E. Zeckendorf. "Representation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas." *Bull. Soc. Royale Sci. Liege* *41* (1972):179-82.

*****

A NOTE ON THE IRRATIONALITY OF CERTAIN LUCAS INFINITE SERIES

Richard André-Jeannin
Ecole Nationale D'Ingenieurs, Sfax, Tunisia
(Submitted May 1989)

1. Introduction

Recently, C. Badea [1] showed that

$$\sum_{n=0}^{\infty} \frac{1}{L_{2^n}}$$

is irrational, where L_n is the usual Lucas number. We shall extend here his result to other series, with a direct proof, and we shall also give a deeper result, namely,

$$\sum_{n=0}^{\infty} \frac{\varepsilon^n}{L_{2^n}} \notin \mathbb{Q}(\sqrt{5}), \text{ with } \varepsilon = \pm 1.$$

Consider the sequence of integers $\{w_n\}$ defined by the recurrence relation

(1.1) $\quad w_n = pw_{n-1} - qw_{n-2},$

where $p \geq 1$, $q \neq 0$ are integers with $d = p^2 - 4q > 0$. Roots of the characteristic polynomial of (1.1) are

$$\alpha = \frac{p + \sqrt{d}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{d}}{2},$$

where $\alpha + \beta = p$, $\alpha\beta = q$, and $\alpha - \beta = \sqrt{d} > 0$. Note that $\alpha > |\beta|$ and $\alpha > 1$ since $\alpha^2 > \alpha|\beta| = |q| \geq 1$.

Special cases of $\{w_n\}$ which interest us here are the generalized Fibonacci $\{U_n\}$ and Lucas $\{V_n\}$ sequences defined by

(1.2) $\quad U_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n.$

It is easily proved that $\{U_n\}$ and $\{V_n\}$ are increasing sequences of natural numbers (for $n \geq 1$) and that

$$U_n \sim \frac{\alpha^n}{\alpha - \beta}, \; V_n \sim \alpha^n, \; U_n \leq V_n$$

for all positive integers n.

We also have

(1.3) $\quad U_{2n} = U_n V_n,$

(1.4) $\quad \alpha U_n - U_{n+1} = -\beta^n.$

The purpose of this paper is to establish the following result.

Theorem: We assume that the above conditions are realized and that ε is fixed ($\varepsilon = \pm 1$). We then have:

1) $\theta = \displaystyle\sum_{n=0}^{\infty} \frac{\varepsilon^n}{V_{2^n}}$ is an irrational number;

2) If $\sqrt{d}$ is irrational and $|\beta| < 1$, then $1, \alpha, \theta$ are linearly independent over $\mathbb{Q}$ [or, in other words: $\theta \notin \mathbb{Q}(\sqrt{d})$].

Remark: When $q = \pm 1$, it is quite simple to prove that $|\beta| < 1$ and $\sqrt{d}$ is irrational. More generally, $|\beta| < 1$ if and only if $p + q > -1$ and $p - q > 1$ [since in that case $P(1) < 0$, $P(-1) > 0$, where P is the characteristic polynomial].

2. Preliminary Lemmas

Let $\{p_n\}$ and $\{q_n\}$ be two sequences of integers defined by

$$S_n = \sum_{k=0}^{n} \frac{\varepsilon^k}{V_{2^k}} = \frac{p_n}{q_n}, \text{ with } q_n = \prod_{k=0}^{n} V_{2^k}.$$

By (1.3), we have

(2.1) $\quad q_n = U_{2^{n+1}}.$

We need the following lemmas.

Lemma 1: $\left|\theta - \dfrac{p_n}{q_n}\right| = \varepsilon^{n+1}\left(\theta - \dfrac{p_n}{q_n}\right).$

Proof: The result is obvious when $\varepsilon = 1$. In the other case, since V_n is increasing, we have:

$$\frac{p_{2n}}{q_{2n}} > \theta, \quad \frac{p_{2n+1}}{q_{2n+1}} < \theta.$$

Lemma 2: $p_n q_{n-1} - p_{n-1} q_n = \varepsilon^n U_{2^n}^2.$

Proof: $\dfrac{\varepsilon^n}{V_{2^n}} = S_n - S_{n-1} = \dfrac{p_n q_{n-1} - p_{n-1} q_n}{q_n q_{n-1}}.$ Hence, by (2.1) and (1.3),

$$p_n q_{n-1} - p_{n-1} q_n = \frac{\varepsilon^n}{V_{2^n}} q_n q_{n-1} = \frac{\varepsilon^n}{V_{2^n}} U_{2^{n+1}} U_{2^n} = \varepsilon^n U_{2^n}^2.$$

Lemma 3: For all positive integers n and k, we have

$$\frac{U_{2^{n+1}}}{V_{2^{n+k+1}}} \le \left(\frac{1}{V_{2^{n+1}}}\right)^k.$$

Proof: Using (1.3), we can show that

$$U_{2^{n+1}} \prod_{i=1}^{k} V_{2^{n+i}} = U_{2^{n+k+1}} \le V_{2^{n+k+1}}$$

and so

$$\frac{U_{2^{n+1}}}{V_{2^{n+k+1}}} \le \frac{1}{\prod_{i=1}^{k} V_{2^{n+i}}} \le \left(\frac{1}{V_{2^{n+1}}}\right)^k,$$

since V_n is increasing.

Lemma 4: $\lim_{n\to\infty} |q_n\theta - p_n| = \dfrac{1}{\alpha - \beta}$, where $\{p_n\}$ and $\{q_n\}$ are defined as above.

Proof: $\left|\theta - \dfrac{p_n}{q_n}\right| = \varepsilon^{n+1}\left(\theta - \dfrac{p_n}{q_n}\right) = \varepsilon^{n+1}(\theta - S_n)$

$$= \varepsilon^{n+1} \sum_{k=0}^{\infty} \frac{\varepsilon^{n+k+1}}{V_{2^{n+k+1}}} = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{V_{2^{n+k+1}}}.$$

Hence,
$$|q_n\theta - p_n| = \sum_{k=0}^{\infty} \frac{\varepsilon^k q_n}{V_{2^{n+k+1}}} = \sum_{k=0}^{\infty} \frac{\varepsilon^k U_{2^{n+1}}}{V_{2^{n+k+1}}} = \frac{U_{2^{n+1}}}{V_{2^{n+1}}} + R_n,$$

with $R_n = \sum_{k=1}^{\infty} \frac{\varepsilon^k U_{2^{n+1}}}{V_{2^{n+k+1}}}$.

However, by Lemma 3, we have

$$|R_n| \le \sum_{k=1}^{\infty} \frac{U_{2^{n+1}}}{V_{2^{n+k+1}}} \le \sum_{k=1}^{\infty} \left(\frac{1}{V_{2^{n+1}}}\right)^k = \frac{1}{V_{2^{n+1}} - 1},$$

so that $\lim_{n\to\infty} R_n = 0$ and

$$\lim_{n\to\infty}|q_n\theta - p_n| = \lim_{n\to\infty} \frac{U_{2^{n+1}}}{V_{2^{n+1}}} = \frac{1}{\alpha - \beta}.$$

3. Proof of the First Part of the Theorem

Recall that a convergent sequence of integers is stationary, and suppose that $\theta = a/b$ (a and b integers, $b > 0$). By Lemma 4, the sequence of positive integers $|q_n a - p_n b|$ tends to the limit $c = b/(\alpha - \beta)$. When $(\alpha - \beta)$ is irrational, this is clearly impossible. In the other case we have, for all large n, since the sequence is stationary,

$$\left|q_n \frac{a}{b} - p_n\right| = \varepsilon^{n+1}\left(q_n \frac{a}{b} - p_n\right) = \frac{1}{\alpha - \beta},$$

and so, for all large n,

$$q_n \frac{a}{b} - p_n = \frac{\varepsilon^{n+1}}{\alpha - \beta}. \tag{3.1}$$

Using (3.1) for n and $n - 1$, we have

$$p_n q_{n-1} - p_{n-1} q_n = \frac{\varepsilon^n}{\alpha - \beta}(q_n - \varepsilon q_{n-1}).$$

By (2.1), (1.3), and Lemma 2, we obtain

$$U_{2^n}^2 = \frac{1}{\alpha - \beta}(U_{2^{n+1}} - \varepsilon U_{2^n}) = \frac{U_{2^n}}{\alpha - \beta}(V_{2^n} - \varepsilon),$$

and so

$$U_{2^n} = \frac{1}{\alpha - \beta}(V_{2^n} - \varepsilon).$$

It follows from this and (1.2) that

$$\alpha^{2^n} - \beta^{2^n} = \alpha^{2^n} + \beta^{2^n} - \varepsilon \quad \text{or} \quad \beta^{2^n} = \varepsilon/2,$$

for all large n. This is clearly impossible, since

$$\lim_{n\to+\infty} |\beta|^{2^n} \in \{0, 1, +\infty\}.$$

This concludes the proof.

Examples:

a) $\sum_{n=0}^{\infty} \frac{\varepsilon^n}{L_{2^n}}$ is irrational (the case $\varepsilon = 1$ is Badea's).

b) $\sum_{n=0}^{\infty} \frac{\varepsilon^n}{2^{2^n} + 1}$ is irrational (the case $\varepsilon = 1$ was discovered by Golomb [2]).

4. Proof of the Second Part of the Theorem

Suppose that we can find a relation

(4.1) $k_0 + k_1\alpha + k_2\theta = 0,\ k_i \in \mathbb{Q}.$

We can limit ourselves to the case of $k_i \in Z$. Replacing n by 2^{n+1} in (1.4) and putting $x_n = U_{2^{n+1}+1}$, we have

(4.2) $\lim_{n\to\infty}(\alpha q_n - x_n) = 0,$

since $|\beta| < 1$.

By (4.1), it follows that

$$k_0 q_n + k_1(q_n\alpha - x_n) + k_2(q_n\theta - p_n) + k_1 x_n + k_2 p_n = 0$$

or, for all positive integers n,

$$k_1(q_n\alpha - x_n) + k_2(q_n\theta - p_n) \in Z.$$

Hence, by Lemma 1,

$$k_1\varepsilon^{n+1}(q_n\alpha - x_n) + k_2|q_n\theta - p_n| \in Z.$$

Using Lemma 4 and (4.2), it follows that

$$\lim_{n\to\infty}(k_1\varepsilon^{n+1}(q_n\alpha - x_n) + k_2|q_n\theta - p_n|) = \frac{k_2}{\alpha - \beta} \in Z.$$

Thus, we have $k_2 = 0$ (since $\alpha - \beta$ is irrational) and, by (4.1),

$$k_1 = k_0 = 0,$$

since $\alpha = (p + \sqrt{d})/2$ is irrational. This concludes the proof.

Example: $\sum_{n=0}^{\infty} \frac{\varepsilon^n}{L_{2^n}} \notin \mathbb{Q}(\sqrt{5})$.

Corollary: Let r be a positive integer. With the hypotheses of the theorem, we have:

1) $\theta_r = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{V_{r\cdot 2^n}}$ is an irrational number;

2) If $\sqrt{d}$ is irrational and $|\beta| < 1$, then $1, \alpha, \theta_r$ are linearly independent over $\mathbb{Q}$.

Define the sequence $\{V_n'\}$ by

$$V_n' = V_{rn} = (\alpha^r)^n + (\beta^r)^n.$$

$\{V_n'\}$ is the Lucas generalized sequence, with real roots α^r and β^r, which is associated with the recurrence

$$W_n' = (\alpha^r + \beta^r)W_{n-1}' - \alpha^r\beta^r W_{n-2}' = V_r W_{n-1}' - q^r W_{n-2}'.$$

We can apply the result of the Theorem to the sequence $\{V_{2^n}'\}$. In fact, we have

$$V_r \ge V_1 = p \ge 1,\quad |\beta|^r < 1\quad (\text{since } |\beta| < 1)$$

and the discriminant d' of the recurrence is

$$d' = V_r^2 - 4q^r = (\alpha^r - \beta^r)^2 = (\alpha - \beta)^2 U_r^2.$$

From this, we have

$$\sqrt{d'} = (\alpha - \beta)U_r = \sqrt{d}U_r.$$

Thus, $\sqrt{d'}$ is an irrational number because $\sqrt{d}$ is.

References

1. C. Badea. "The Irrationality of Certain Infinite Series." *Glasgow Math. J.* *29* (1987):221-28.
2. S. W. Golomb. "On the Sum of the Reciprocals of the Fermat Numbers and Related irrationalities." *Can. J. Math.* *15* (1963):475-78.

*****

Announcement

FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 20-24, 1992

Department of Mathematical and Computational Sciences
University of St. Andrews
St. Andrews KY16 9SS
Fife, Scotland

LOCAL COMMITTEE
Colin M. Campbell, Co-chairman
George M. Phillips, Co-chairman
Dorothy M.E. Foster
John H. McCabe
John J. O'Connor
Edmund F. Robertson

INTERNATIONAL COMMITTEE
A.F. Horadam (Australia), Co-chair
A.N. Philippou (Cyprus), Co-chair
S. Ando (Japan)
G.E. Bergum (U.S.A.)
P. Filipponi (Italy)
H. Harborth (Germany)
M. Johnson (U.S.A.)
P. Kiss (Hungary)
C.T. Long (U.S.A.)
B.S. Popov (Yugoslavia)
J. Turner (New Zealand)
M.E. Waddill (U.S.A.)

LOCAL INFORMATION

For information on local housing, food, local tours, etc. please contact:

Dr. G.M. Phillips
Mathematical Institute
University of St. Andrews
North Haugh
ST. ANDREWS, Fife KY16 9SS
Scotland

CALL FOR PAPERS

The FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of St. Andrews, St. Andrews, Scotland from July 20 to July 24, 1992. This Conference is sponsored jointly by the Fibonacci Association and The University of St. Andrews.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts are to be submitted by March 15, 1992. Manuscripts are due by May 30, 1992. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

Professor Gerald E. Bergum
The Fibonacci Quarterly
Department of Computer Science, South Dakota State University
P.O. Box 2201, Brookings, South Dakota 57007-0194

COMBINATORIAL INTERPRETATIONS OF THE q-ANALOGUES OF L_{2n+1}

A. K. Agarwal*
Indian Institute of Technology, Delhi, New Delhi-110016, India
(Submitted May 1989)

1. Introduction

Recently in [1], two different q-analogues of L_{2n+1} were found. Our object here is to interpret these q-analogues as generating functions. As usual, $\begin{bmatrix} n \\ m \end{bmatrix}$ will denote the Gaussian polynomial, which is defined by

$$(1.1)\qquad \begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} (q;q)_n/(q;q)_m(q;q)_{n-m}, & \text{if } 0 \le m \le n, \\ 0, & \text{otherwise} \end{cases}$$

where

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1 - aq^i)}{(1 - aq^{n+i})}.$$

We shall also need the following well-known properties of $\begin{bmatrix} n \\ m \end{bmatrix}$:

$$(1.2)\qquad \begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n - m \end{bmatrix};$$

$$(1.3)\qquad \begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n - 1 \\ m \end{bmatrix} + q^{n-m}\begin{bmatrix} n - 1 \\ m - 1 \end{bmatrix}.$$

In [1], we studied two different q-analogues of L_{2n+1} denoted by $C_n(q)$ and $\overline{C}_n(q)$, respectively. These were defined by

$$(1.4)\qquad C_n(q) = \sum_{j=0}^{n} A_{n,j}(q),$$

where

$$(1.5)\qquad A_{n,j}(q) = \begin{bmatrix} 2n - j \\ j \end{bmatrix} q^{\binom{j}{2}} + (1 + q^j)\begin{bmatrix} 2n - j \\ j - 1 \end{bmatrix} q^{2n-2j+1+\binom{j}{2}}$$

and

$$(1.6)\qquad \overline{C}_n(q) = D_n(q) + D_{n-1}(q).$$

where

$$(1.7)\qquad D_n(q) = \sum_{m=0}^{n} B_{n,m}(q)$$

in which $B_{n,m}(q)$ are defined by

$$(1.8)\qquad B_{n,m}(q) = q^{m^2}\begin{bmatrix} n + m + 1 \\ 2m + 1 \end{bmatrix}.$$

Remark 1: $A_{n,j}(q)$ defined by (1.5) above are $D_{n,j}(q)$ in [1, p. 171] with j replaced by $n - j$. This only reverses the order of summation in (1.4).

Remark 2: Equation (1.8) is (3.6) in [1, p. 172] with m replaced by $n - m$ and (1.2) applied.

---

*This paper was presented at the 853rd Meeting of the American Mathematical Society, University of California at Los Angeles, November 18-19, 1989.

Several combinatorial interpretations of the polynomials $C_n(q)$, $A_{n,m}(q)$, $\overline{C}_n(q)$, $D_n(q)$, and $B_{n,m}(q)$, for $q = 1$, were given in [1]. In this paper, we refine our results for the general value of q, or, in other words, we interpret these polynomials as generating functions. In Section 2, we shall state and prove our main results.

2. The Main Results

In this section, we first state two theorems and three corollaries. The proofs then follow.

Theorem 1: Let $P(m, n, N)$ denote the number of partitions of N into $m - 1$ distinct parts, where the value of each part is less than or equal to $2n - m$, or the number of partitions of N into m distinct parts where each part has a value which is less than or equal to $2n - m + 1$. Then

$$(2.1)\qquad A_{n,m}(q) = \sum_{N=0}^{r} P(m, n, N)q^N,$$

where

$$r = 2nm - 3\binom{m}{2}.$$

Example: The coefficient of q^7 in $A_{5\ 2}(q)$ is 4 (see below); also, $p(2, 5, 7) = 4$, since the relevant partitions are 7, 6 + 1, 5 + 2, and 4 + 3.

$$A_{5,2}(q) = q^{17} + q^{16} + 2q^{15} + 2q^{14} + 3q^{13} + 3q^{12} + 4q^{11} + 4q^{10} + 4q^9$$
$$+ 4q^8 + 4q^7 + 3q^6 + 3q^5 + 2q^4 + 2q^3 + q^2 + 2$$

Corollary 1:

$$(2.2)\qquad C_n(q) = \sum_{N=0}^{s} P(n, N)q^N,$$

where

$$(2.3)\qquad P(n, N) = \sum_{m=0}^{n} P(m, n, N)$$

and

$$s = \max\left\{2nm - 3\binom{m}{2}\right\},\ 1 \le m \le n.$$

Theorem 2: Let $Q(m, n, N)$ denote the number of partitions of N of the form $\pi = b_1 + b_2 + \cdots + b_t$, such that $m \le t \le 2m + 1$:

$$b_{i-1} - b_i \ge 2 \qquad \text{if } 2 \le i \le m$$
$$b_m - b_{m+1} \ge 1$$
$$b_{i-1} \ge b_i \qquad \text{if } i > m + 1$$
$$b_1 \le n + m - 1.$$

Then,

$$(2.4)\qquad B_{n,m}(q) = \sum_{N=0}^{u} Q(m, n, N)q^N,$$

where

$$u = n^2 + (n - m) - (n - m)^2.$$

Corollary 2:

$$(2.5)\qquad D_n(q) = \sum_{N=0}^{n^2} Q(n, N)q^N,$$

where

$$(2.6)\qquad Q(n, N) = \sum_{m=0}^{n} Q(m, n, N).$$

Corollary 3:

$$(2.7)\qquad \bar{C}_n(q) = \sum_{N=0}^{n^2} R(n, N)q^N,$$

where

$$(2.8)\qquad R(n, N) = Q(n, N) + Q(n-1, N).$$

Proof of Theorem 1: Letting $j = m$ in (1.5), we have

$$\begin{aligned} A_{n,m}(q) &= \left(\begin{bmatrix} 2n-m \\ m \end{bmatrix} + \begin{bmatrix} 2n-m \\ m-1 \end{bmatrix} q^{2n-2m+1}\right) q^{\binom{m}{2}} + \begin{bmatrix} 2n-m \\ m-1 \end{bmatrix} q^{2n-2m+1+m+\binom{m}{2}} \\ &= \begin{bmatrix} 2n-m+1 \\ m \end{bmatrix} q^{\binom{m}{2}} + \left(\begin{bmatrix} 2n-m+1 \\ m \end{bmatrix} - \begin{bmatrix} 2n-m \\ m \end{bmatrix}\right) q^{\binom{m+1}{2}}, \end{aligned}$$

where the last step comes by using (1.3) with n replaced by $2n - m + 1$ and noting that

$$m + \binom{m}{2} = \binom{m+1}{2}.$$

Since $A_{n,m}(q)$ is a polynomial, the degree of $A_{n,m}(q)$ is the degree of

$$\begin{bmatrix} 2n-m+1 \\ 2 \end{bmatrix} q^{\binom{m+1}{2}}, \text{ which is } 2nm - 3\binom{m}{2}.$$

It is easily seen that

$$\begin{bmatrix} 2n-m+1 \\ m \end{bmatrix} q^{\binom{m}{n}}$$

generates partitions into $m - 1$ or m distinct parts, where each part has a value less than or equal to $2n - m$, and

$$\left(\begin{bmatrix} 2n-m+1 \\ m \end{bmatrix} - \begin{bmatrix} 2n-m \\ m \end{bmatrix}\right) q^{\binom{m+1}{2}}$$

generates partitions into m distinct parts with the largest part equal to $2n - m + 1$. Combining these results, we see that $A_{n,m}(q)$ generates $P(m, n, N)$. The proof of Corollary 1 is now obvious.

Proof of Theorem 2: By the definition of the Gaussian polynomial, it is clear that

$$\begin{bmatrix} n+m+1 \\ 2m+1 \end{bmatrix}$$

generates partitions into at most $2m + 1$ parts where each part has a value less than or equal to $n - m$. Multiplication of $\left[\begin{smallmatrix} n+m+1 \\ 2m+1 \end{smallmatrix}\right]$ by $q^{m^2} = q^{1+3+\cdots+2m-1}$ means that we are adding $2m - 1$ to the largest part, $2m - 3$ to the next largest part, $2m - 5$ to the next largest part, etc. Since the largest part is less than or equal to $n - m + (2m - 1) = n + m - 1$, there are at least m parts where the minimal difference of the first m parts (with the parts arranged in nonincreasing order) is 2. The m^{th} and the $(m + 1)^{\text{th}}$ parts are distinct. Obviously, the degree of $B_{n,m}(q)$ is

$$m^2 + (2m+1)(n+m+1-2m-1) = n^2 + (n-m) - (n-m)^2.$$

This completes the proof of Theorem 2.

Corollaries 2 and 3 are now direct results of Theorem 2.

3. Conclusions

In the literature, we find several combinatorial interpretations of the q-analogues of the Fibonacci numbers. The Catalan numbers and Stirling numbers are other good examples. The most obvious question that arises here is: Do the polynomials $A_{n,m}(q)$, $C_n(q)$, $B_{n,m}(q)$, $D_n(q)$, and $\overline{C}_n(q)$ have combinatorial interpretations other than those presented in this paper? So far, we know one more combinatorial interpretation of the polynomials $D_n(q)$. Before we state it in the form of a theorem, we recall the following definitions from [2].

Definition 1: Let π be a partition. Let γ_{ij} be the node of π in the i^{th} row and j^{th} column of Ferrers' graph of π. We say that γ_{ij} lies on the diagonal δ if $i - j = \delta$.

Definition 2: Let π be a partition whose Ferrers graph is embedded in the fourth quadrant. Each node (i, j) of the fourth quadrant which is not in the Ferrers graph of π is said to possess an anti-hook difference $\rho_i - k_j$ relative to π, where ρ_i is the number of nodes on the i^{th} row of the fourth quadrant to the left of the node (i, j) that are not in the Ferrers graph of π and k_j is the number of nodes in the j^{th} column of the fourth quadrant that lie above node (i, j) and are not in the Ferrers graph of π.

Remark: By the Ferrers graph of a partition, in the above definitions, we mean its graphical representation. If $\pi = a_1 + a_2 + \cdots + a_n$ (with $a_i > a_{i+1}$, $1 < i < n - 1$ is a partition, then the i^{th} row of the graphical representation of this partition contains a_i points (or dots, or nodes). The graphical representation of the partition 5 + 3 + 1 of 9, thus, is:

```
. . . . .
. . .
.
```

We now present the other combinatorial interpretation of the polynomials $D_n(q)$ in the following form.

Theorem 3: Let $f(n, k)$ denote the number of partitions of k with the largest part $\le n$ and the number of parts $\le n$, which have all anti-hook differences on the 0 diagonal equal to 0 or 1. Let $g(n, k)$ denote the number of partitions of k with the largest part $\le n + 1$ and the number of parts $\le n - 1$, which have all anti-hook differences on the -2 diagonal equal to 1 or 2. For $k \ge 1$, let $h(n, k) = f(n, k) + g(n, k - 1)$. Then

$$D_n(q) = 1 + \sum_{k=1}^{n^2} h(n, k)q^k.$$

Note: For the proof of Theorem 3, see [2, Th. 2, pts. (1) and (4), p. 11]. We remark here that part (3) of Theorem 2 in [2] was incorrectly stated:

$$q^{n^2+n}d_{2n-1}(q^{-1}) \text{ should be replaced by } q^{n^2+n}d_{2n}(q^{-1}).$$

References

1. A. K. Agarwal. "Properties of a Recurring Sequence." *The Fibonacci Quarterly 27.2* (1989):169-75.
2. A. K. Agarwal & G. E. Andrews. "Hook Differences and Lattice Paths." *J. Statist. Plan. Inference 14* (1986):5-14.

*****

SUBSETS WITHOUT UNIT SEPARATION AND PRODUCTS OF FIBONACCI NUMBERS

John Konvalina and Yi-Hsin Liu
University of Nebraska at Omaha, Omaha, NE 68182
(Submitted June 1989)

1. Introduction

It is well known that the Fibonacci numbers are intimately related to subsets of $\{1, 2, 3, \ldots, n\}$ not containing a pair of consecutive integers. More precisely, let F_n denote the n^{th} Fibonacci number determined by the recurrence relation

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n \quad (n \geq 1).$$

Then the total number of subsets of $\{1, 2, 3, \ldots, n\}$ not containing a pair of consecutive integers is F_{n+2}. This result can also be expressed in terms of a well-known combinatorial identity. Kaplansky [2] showed that the number of k-subsets of $\{1, 2, 3, \ldots, n\}$ not containing a pair of consecutive integers is

$$\binom{n+1-k}{k}.$$

Consequently, summing over k we obtain the identity

$$(1) \qquad \sum_{k \geq 0} \binom{n+1-k}{k} = F_{n+2}.$$

In this paper we will derive a combinatorial identity expressing the square of a Fibonacci number and the product of two consecutive Fibonacci numbers in terms of the number of subsets of $\{1, 2, 3, \ldots, n\}$ without unit separation. Two objects are called *uniseparate* if they contain exactly one object between them. For example, the following pairs of integers are uniseparate: (1, 3), (2, 4), (3, 5), etc. Konvalina [3] showed that the number of k-subsets of $\{1, 2, 3, \ldots n\}$ not containing a pair of uniseparate integers is

$$(2) \qquad \begin{cases} \displaystyle\sum_{i=0}^{[k/2]} \binom{n+1-k-2i}{k-2i} & \text{if } n \geq 2(k-1), \\ 0 & \text{if } n < 2(k-1). \end{cases}$$

Let T_n denote the total number of subsets of $\{1, 2, 3, \ldots, n\}$ without unit separation. Then, summing over k, we have

$$(3) \qquad T_n = \sum_{k \geq 0} \sum_{i=0}^{[k/2]} \binom{n+1-k-2i}{k-2i}.$$

We will prove that if n is even then T_n is the square of a Fibonacci number; while, if n is odd T_n is the product of two consecutive Fibonacci numbers.

2. Main Result

Theorem: If $n \geq 1$, then
$$T_{2n} = F_{n+2}^2$$
$$T_{2n+1} = F_{n+2}F_{n+3}.$$

Proof: The following identities on summing every fourth Fibonacci number are needed in obtaining the result:

$$\sum_{j=1}^{n} F_{4j} = F_{2n+1}^2 - 1, \tag{4}$$

$$\sum_{j=1}^{n} F_{4j-2} = F_{2n}^2; \tag{5}$$

$$\sum_{j=1}^{n} F_{4j-3} = F_{2n-1}F_{2n}; \tag{6}$$

$$\sum_{j=1}^{n} F_{4j-1} = F_{2n}F_{2n+1}. \tag{7}$$

These identities are easily proved by induction and the following well-known Fibonacci identities (see Hoggatt [1]):

$$F_{n+1}^2 - F_{n-1}^2 = F_{2n},$$
$$F_nF_{n+1} - F_{n-1}F_{n-2} = F_{2n-1}.$$

Now, evaluating T_n in (3), we obtain

$$T_n = \sum_{k \ge 0} \sum_{i=0}^{[k/2]} \binom{n+1-k-2i}{k-2i} = \sum_{k=0}^{[(n+2)/2]} \sum_{i \ge 0} \binom{n+1-k-2i}{k-2i}.$$

Now, replacing k by $k + 2i$, since $k - 2i$ contributes zero to the sum, we obtain the key identity

$$T_n = \sum_{i \ge 0} \sum_{k=0}^{m} \binom{n+1-k-4i}{k}, \tag{8}$$

where $m = [(n+2)/2] - 2i$.

Next, we will apply (1) and the Fibonacci identities (4), (5), (6), and (7) to evaluate (8). First, identity (1) can be expressed as follows:

$$\sum_{k \ge 0} \binom{n+1-k}{k} = \sum_{k=0}^{[(n+1)/2]} \binom{n+1-k}{k} = F_{n+2}. \tag{9}$$

Replacing n by $n - 4i$, identity (9) becomes

$$\sum_{k=0}^{p} \binom{n+1-k-4i}{k} = F_{n+2-4i}, \tag{10}$$

where $p = [(n+1)/2] - 2i$.

To complete the proof, we will evaluate (8) based on whether $n \equiv 0, 1, 2,$ or 3 (mod 4).

Odd Case: If n is odd, then $[(n+2)/2] = [(n+1)/2]$, so $m = p$ and, applying (10) to (8), we have a sum involving every fourth Fibonacci number.

$$T_n = \sum_{i \ge 0} F_{n+2-4i}. \tag{11}$$

Case 1. $n \equiv 1 \pmod 4$

In this case we have $n + 2 = 4t - 1$ for some integer t. Substitute $t = (n+3)/4$ for n in (7) and apply to (11) to obtain

$$T_n = F_{(n+3)/2}F_{(n+3)/2+1}.$$

Since n is odd, replace n by $2n + 1$, and the desired result

$$T_{2n+1} = F_{n+2}F_{n+3}$$

is obtained.

Case 2. $n \equiv 3 \pmod 4$

In this case we have $n + 2 = 4t - 3$ for some t. Substitute $t = (n + 5)/4$ for n in (6) and apply to (11) to obtain

$$T = F_{(n+5)/2-1}F_{(n+5)/2}.$$

Replace n by $2n + 1$ and the result is the same as in the previous case:

$$T_{2n+1} = F_{n+2}F_{n+3}.$$

Even Case: If n is even, $m = p + 1$, and applying (10) to (8) we have

$$(12)\qquad T_n = \sum_{i\ge 0}\sum_{k=0}^{p+1}\binom{n+1-k-4i}{k}$$

$$= \sum_{i\ge 0}\sum_{k=0}^{p}\binom{n+1-k-4i}{k} + \sum_{i\ge 0}\binom{n+1-(p+1)-4i}{p+1}$$

$$= \sum_{i\ge 0}F_{n+2-4i} + \sum_{i\ge 0}\binom{n/2-2i}{n/2-2i+1}.$$

Observe that the last summation is zero except when $n/2 - 2i + 1 = 0$. That is, when $i = (n + 2)/4$ or $n + 2 \equiv 0 \pmod 4$. In this case, the last sum is 1.

Case 3. $n \equiv 2 \pmod 4$

Here $n + 2 = 4t$ for some t. Substitute $t = (n + 2)/4$ for n in (4) and apply to (12) to obtain

$$T_n = (F^2_{(n+4)/2} - 1) + 1 = F^2_{(n+4)/2}.$$

Since n is even, replace n by $2n$ and the desired result is obtained:

$$T_{2n} = F^2_{n+2}.$$

Case 4. $n \equiv 0 \pmod 4$

Here $n + 2 = 4t - 2$ for some t. Substitute $t = (n + 4)/4$ for n in (5) and apply to (12) to obtain

$$T_n = F^2_{(n+4)/2}.$$

Replace n by $2n$ and the result is the same as in the previous case.

Table 1

n	F_n	F_n^2	F_nF_{n+1}	T_n	n	F_n	F_n^2	F_nF_{n+1}	T_n
1	1	1	1	1	7	13	169	273	40
2	1	1	2	4	8	21	441	714	64
3	2	4	6	6	9	34	1156	1870	104
4	3	9	15	9	10	55	3025	4895	169
5	5	25	40	15	11	89	7921	12816	273
6	8	64	104	25	12	144	20736	33552	441

References

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.
2. I. Kaplansky. "Solution of the "Probleme des menages." *Bull. Amer. Math. Soc.* *49* (1943):784-85.
3. J. Konvalina. "On the Number of Combinations without Unit Separation." *J. Combin. Theory*, Ser. *A31* (1981):101-07.

*****

AN EXTENSION OF A THEOREM BY CHEO AND YIEN CONCERNING DIGITAL SUMS

Robert E. Kennedy and Curtis N. Cooper
Central Missouri State University, Warrensburg, MO 64093
(Submitted June 1989)

1. Introduction

For a nonnegative integer k, let $s(k)$ denote the digital sum of k. In [1], Cheo and Yien prove that, for a nonnegative integer x,

$$\sum_{k=0}^{x-1} s(k) = (4.5)x \log x + O(x). \tag{1.1}$$

Here $O(f(x))$ is the useful "big-oh" notation and denotes some unspecified function $g(x)$ such that $g(x)/f(x)$ is eventually bounded. We usually write

$$g(x) = O(f(x))$$

and read, "$g(x)$ is big-oh of $f(x)$." For an introduction to this important notation, see [3]. In this paper we determine a formula for

$$\sum_{k=0}^{x-1} (s(k))^2. \tag{1.2}$$

The resulting formula will be used to calculate the mean and variance of the sequence of digital sums. First, in order to facilitate the discussion, we introduce some notation.

2. Notation

For each positive integer x, let $[0, x)$ denote the set of nonnegative integers strictly less than x. In addition, we will let $s([0, x))$ be the sequence

$$s(0),\ s(1),\ s(2),\ \ldots,\ s(x-1). \tag{2.1}$$

That is, we have not only taken into account $s(k)$, but also the frequency of $s(k)$. Then, letting μ and σ^2 be the mean and variance of $s([0, x))$, respectively, we have

$$\mu = \frac{1}{x}\sum_{k=0}^{x-1} s(k)$$

and

$$\sigma^2 = \frac{1}{x}\sum_{k=0}^{x-1} (s(k))^2 - \mu^2. \tag{2.2}$$

If x is a power of 10, then the following lemma gives the exact value of μ and σ^2. Its proof is given in [2] and will not be reproduced here.

Lemma 2.1: Let $x = 10^n$ for a positive integer n. Then

$$\mu = \text{the mean of } s([0, x)) = 4.5n$$

and

$$\sigma^2 = \text{the variance of } s([0, x)) = 8.25n.$$

3. A Formula for (1.2)

The following theorem gives a formula for the expression (1.2).

Theorem 3.1: Let s be the digital sum function and let x be a positive integer. Then

$$\sum_{k=0}^{x-1}(s(k))^2 = 20.25x\log^2 x + O(x\log x). \tag{3.1}$$

Proof: For each positive integer x, define

$$A(x) = \sum_{k=0}^{x-1}s(k) \quad\text{and}\quad B(x) = \sum_{k=0}^{x-1}(s(k))^2.$$

In [1], Cheo and Yien showed that for any positive integer n and for any decimal digit $c(0, 1, 2, 3, \ldots, \text{or } 9)$,

$$A(c\cdot 10^n) = \left(4.5cn + \frac{c(c-1)}{2}\right)10^n. \tag{3.2}$$

Using Lemma 2.1 and formula (2.2) for the variance of $s([0, x))$, we have

$$B(10^n) = 20.25n^2 10^n + 8.25n10^n. \tag{3.3}$$

Therefore, for a positive integer n and a decimal digit c, we can calculate $B(c\cdot 10^n)$. That is,

$$B(c\cdot 10^n) = \sum_{k=0}^{c\cdot 10^n-1}(s(k))^2.$$

Since for $0 \le k < 10^n$ and $0 \le j \le c-1$ we have that $s(j\cdot 10^n + k) = j + s(k)$. Thus, the above single sum can be rewritten as a double sum

$$B(c\cdot 10^n) = \sum_{j=0}^{c-1}\sum_{k=0}^{10^n-1}(j+s(k))^2 = \sum_{j=0}^{c-1}\sum_{k=0}^{10^n-1}(j^2 + 2js(k) + (s(k))^2)$$

$$= \sum_{j=0}^{c-1}j^2 10^n + 2\sum_{j=0}^{c-1}j\left(\sum_{k=0}^{10^n-1}s(k)\right) + \sum_{k=0}^{10^n-1}c(s(k))^2.$$

We may now apply (3.2) and (3.3) to obtain

$$B(c\cdot 10^n) = \frac{(c-1)c(2c-1)}{6}10^n + 2\sum_{j=0}^{c-1}j(4.5)n10^n + c(8.25n + 20.25n^2)10^n.$$

Continuing, we have

$$B(c\cdot 10^n) = \frac{(c-1)c(2c-1)}{6}10^n + (c-1)c(4.5n)10^n + c(20.25n^2 + 8.25n)10^n. \tag{3.4}$$

Since

$$\sum_{i=0}^{n-1}i10^i = \frac{1}{9^2}(10^n(9n-10)+10) \tag{3.5}$$

and

$$\sum_{i=0}^{n-1}i^2 10^i = \frac{1}{9^3}(10^n(81n^2 - 180n + 110) - 110), \tag{3.6}$$

we can now prove (3.1). Let

$$x = a_n 10^{n-1} + a_{n-1}10^{n-2} + \cdots + a_1 10^0$$

be the decimal representation of the nonnegative integer x. Then

$$B(x) = \sum_{k=0}^{x-1} (s(k))^2.$$

Similarly, as in the determination of $B(c \cdot 10^n)$, this single sum can be written as the following sum of single sums

$$\begin{aligned} B(x) &= \sum_{k=0}^{a_n 10^{n-1}-1} (s(k))^2 + \sum_{k=0}^{a_{n-1}10^{n-2}-1} (a_n + s(k))^2 \\ &\quad + \cdots + \sum_{k=0}^{a_1 10^0 - 1} (a_n + a_{n-1} + \cdots + a_2 + s(k))^2 \\ &= \sum_{i=1}^{n} B(a_i 10^{i-1}) + 2\sum_{k=1}^{n-1}\left(\sum_{i=k+1}^{n} a_i\right) A(a_k 10^{k-1}) + \sum_{k=1}^{n-1}\left(\sum_{i=k+1}^{n} a_i\right)^2 a_k 10^{k-1}. \end{aligned}$$

Using (3.4), we have that

$$B(x) = t_1 - t_2 + t_3 + t_4 + t_5 + t_6,$$

where

$$t_1 = 20.25(n-1)^2 x,$$

$$t_2 = 20.25\sum_{i=1}^{n}((n-1)^2 - (i-1)^2)a\ 10^{i-1},$$

$$t_3 = \sum_{i=1}^{n}(4.5a_i^2 + 3.75a_i)(i-1)10^{i-1},$$

$$t_4 = \sum_{i=1}^{n} \frac{(a_i - 1)(a_i)(2a_i - 1)}{6} 10^{i-1},$$

$$t_5 = \sum_{k=1}^{n-1}\left(\sum_{i=k+1}^{n} a_i\right)^2 a_k 10^{k-1},$$

and

$$t_6 = 2\sum_{k=1}^{n-1}\left(\sum_{i=k+1}^{n} a_i\right) A(a_k 10^{k-1}).$$

It can be shown without difficulty that

$$t_3 = O(x \log x), \quad t_4 = O(x),$$

and since the calculation of t_5 and t_6 are similar, only t_6 will be calculated here. Thus,

$$\begin{aligned} t_6 &= 2\sum_{k=1}^{n-1}\left(\sum_{i=k+1}^{n} a_i\right) A(a_k 10^{k-1}) \le 18\sum_{k=1}^{n-1}(n-k)A(a_k 10^{k-1}) \\ &= 18\sum_{k=1}^{n-1}(n-k)\left(4.5a\ (k-1) + \frac{a_k(a_k - 1)}{2}\right)10^{k-1} \end{aligned}$$

by (3.2). Hence,

$$\begin{aligned} (3.7) \qquad t_6 &\le 729\sum_{k=1}^{n-1}(-k^2 + (n+1)k - n)10^{k-1} + 729\sum_{k=1}^{n-1}(n-k)10^{k-1} \\ &= 729\left(\frac{-1}{10}\sum_{k=1}^{n-1} k^2 10^k + \frac{n}{10}\sum_{k=1}^{n-1} k 10^k\right), \end{aligned}$$

and using (3.5) and (3.6) we have, after simplification,

$$t_6 \le 9n10^n - 11 \cdot 10^n + 9n + 11,$$

and so

$$t_6 = O(x \log x),$$

since $n = O(\log x)$ and $10^n = O(x)$. Similarly,

$$t_5 = O(x).$$

Thus $t_2 = O(x \log x)$ follows, since

$$\begin{aligned} t_2 &= 20.25 \sum_{i=1}^{n} ((n-1)^2 - (i-1)^2) a_i 10i - 1 \\ &\le (20.25)(n-1)^2(10^n - 1) - (20.25)(9) \sum_{i=1}^{n-1} i^2 10^i, \end{aligned}$$

and so, by (3.6),

$$t_2 \le (20.25)(n-1)^2(10^n - 1) - \frac{1}{4}(10^n(81n^2 - 180n + 110) - 110)$$

and, after simplification, we obtain

$$t_2 = O(x \log x).$$

Therefore,

$$\begin{aligned} B(x) &= 20.25(n-1)\ x - O(x \log x) + O(x \log x) + O(x) \\ &\qquad + O(x) + O(x \log x) \\ &= 20.25(n-1)^2 x + O(x \log x) \\ &= 20.25 x \log^2 x + O(x \log x), \end{aligned}$$

since $n = \log x + O(1)$, and (3.1) has been proven.

Using (1.1), we have an immediate corollary to Theorem 3.1.

Corollary 3.2: Let s be the digital sum function and x be a positive integer. Then

$$\mu = 4.5 \log x + O(1) \tag{3.8}$$

and

$$\sigma^2 = O(\log x). \tag{3.9}$$

Proof: The proof of (3.8) follows immediately from (1.1) by dividing through by x. To prove (3.9), we use (3.8), (3.7), and (2.2) to obtain

$$\sigma^2 = 20.25 \log^2 x + O(\log x) - (4.5 \log x + O(1))^2.$$

This implies that

$$\sigma^2 = O(\log x).$$

4. Conclusion

In [1], Cheo and Yien proved that $O(x)$ is the best possible residual for the relation (1.1). Here, the second term of (3.3) implies that $O(x \log x)$ is the best residual given by (3.1). It should also be mentioned here that we have restricted our discussion to base ten numbers. Cheo and Yien, however, give their results for any positive integer base greater than one. By substituting base 10 by base b, the base 10 digit 9 by $b - 1$, and base 10 logarithms by base b logarithms, the results of this paper can, in like manner, be given without restricting base. Finally, we note that the determination of a formula for

$$\sum_{k=0}^{x-1}(s(k))^n, \text{ for } n \geq 3,$$

appears to be complicated and is left as an open problem.

References

1. P. Cheo & S. Yien. "A Problem on the K-adic Representation of Positive Integers." *Acta Math. Sinica* 5 (1955):433-38.
2. R. Kennedy & C. Cooper. "On the Natural Density of the Niven Numbers." *College Math. Journal* 15 (1984):309-12.
3. D. E. Knuth. *The Art of Computer Programming*, Vol. I. New York: Addison-Wesley Publishing Company, 1969.

*****

ZEROS OF CERTAIN CYCLOTOMY-GENERATED POLYNOMIALS

Karl Dilcher
Dalhousie University, Halifax, Nova Scotia, Canada, B3H 3J5
(Submitted June 1989)

1. Introduction

The characteristic equation of the sequence of Fibonacci numbers is

$$(1.1)\qquad x^2 - x - 1 = 0;$$

its roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ play an important role in the theory of Fibonacci numbers and other related matters. The Fibonacci numbers have been generalized in various ways. One such generalization and the corresponding characteristic equations were recently studied by Horadam and Shannon [3]:

Define the polynomials $\phi_n(x)$ by $\phi_0 \equiv 0$, and

$$(1.2)\qquad \phi_n(x) = x^{n-1} + x^{n-2} + \cdots + x + 1 \quad (n \geq 1).$$

The "cyclotomy-generated polynomial of Fibonacci type" of degree $n^2 + n$ is then defined by

$$(1.3)\qquad p_n(x) = x^{n^2+n} - \phi_{n^2+n} - x^{2n+1}\,\frac{\phi_{n^2-1}}{\phi_{n+1}} + x^{2n}\,\frac{\phi_{n^2-n}}{\phi_n}.$$

It is easy to see that $p_1(x)$ is the left-hand side of (1.1).

In [3], both real and complex zeros of $p_n(x)$ were studied. However, some of the more interesting properties were given only in the form of conjectures. It is the purpose of this paper to provide proofs of these conjectures, based on some classical results from the geometry of polynomials. Furthermore, it will be shown that the main factor of $p_n(x)$ is irreducible over the rationals for all n, and that the unique positive zeros of $p_n(x)$ are Pisot numbers.

2. Roots of Unity

Horadam and Shannon [3] observed that $n^2 - n$ complex zeros of $p_n(z)$ lie on the unit circle for small n; they conjectured that this is true for all n. The following proves this conjecture.

Proposition 1: $p_n(z)$ has the $n^2 - n$ zeros $z_k = \exp(2\pi ik/n(n + 1))$, where $k = 1, 2, \ldots, n^2 + n - 1$, excluding multiples of n and of $n + 1$.

Proof: Note that we may write $\phi_n(x) = (x^n - 1)/(x - 1)$ for $x \neq 1$. With (1.3) we get

$$\begin{aligned}
&(x^{n+1} - 1)(x^n - 1)(x - 1)p_n(x)\\
&= x^{n^2+n}(x^{n+1} - 1)(x^n - 1)(x - 1) - (x^{n^2+n} - 1)(x^{n+1} - 1)(x^n - 1)\\
&\quad - x^{2n+1}(x^{n^2-1} - 1)(x^n - 1)(x - 1) + x^{2n}(x^{n^2-} - 1)(x^{n+1} - 1)(x - 1)\\
&= x^{n^2+3+2} - 3x^{n^2+3+1} + x^{n^2+3} + x^{n^2+2+1} + x^{n^2+2} - x^{n^2+n}\\
&\quad - x^{2n+2} + 3x^{2n+1} - x^{2n} - x^{n+1} - x^n + 1\\
&= (x^{n^2+n} - 1)(x^{2n+2} - 3x^{2n+1} + x^{2n} + x^{n+1} + x^n - 1),
\end{aligned}$$

and, therefore,

$$(2.1) \qquad p_n(x) = \frac{x^{n^2+n} - 1}{(x^{n+1} - 1)(x^n - 1)(x - 1)}(x^{2n+2} - 3x^{2n+1} + x^{2n} + x^{n+1} + x^n - 1).$$

This proves the proposition, since $x^{n^2+n} - 1$ has zeros $z_k = \exp(2\pi ik/n(n + 1))$, where z_n, z_{2n}, ... are cancelled by the zeros of $x^{n+1} - 1$, and z_{n+1}, $z_{2(n+1)}$, ... are cancelled by the zeros of $x^n - 1$.

3. Roots within the Unit Circle

It is clear from (2.1) that the remaining zeros of $p_n(z)$ are those of

$$(3.1) \qquad f_n(z) := z^{2n+2} - 3z^{2n+1} + z^{2n} + z^{n+1} + z^n - 1.$$

First, we note that $f_n(z)$ has a double zero at $z = 1$, since $z^{n+1} - 1$ and $z^n - 1$ have simple zeros at $z = 1$, while $p_n(1) = 1 - n^2 - n \neq 0$, by (1.3). Hence, we may consider

$$(3.2) \qquad \begin{aligned} r_n(z) &:= f_n(z)/(z - 1)^2 \\ &= z^{2n} - z^{2n-1} - 2z^{2n-2} - \cdots - nz^n - nz^{n-1} - (n - 1)z^{n-2} \\ &\qquad - \cdots - 2z - 1 \end{aligned}$$

(see also [3, p. 91]). We note that we can write

$$(3.3) \qquad r_n(z) = z^{2n} - \frac{(1 - z^n)(1 - z^{n+1})}{(1 - z)^2}.$$

The following three propositions show that all but one of the zeros of $r_n(z)$ lie in a narrow annular region just inside the unit circle, and that the arguments of all $2n$ zeros are quite evenly distributed.

Proposition 2: For all $n \geq 1$, the zeros of $r_n(z)$ lie outside the circle

$$|z| = (1/3)^{1/n}.$$

Proof: We apply Rouché's Theorem (see, e.g., [4, p. 2]). Departing from (3.3), we let

$$P(z) := z^{2n} \quad \text{and} \quad Q(z) := -(1 - z^n)(1 - z^{n+1})/(1 - z)^2.$$

Set $t := |z|$. Now, for $t < 1$,

$$|Q(z)| \geq \frac{(1 - t^n)(1 - t^{n+1})}{(1 + t)^2} = \frac{1 - t^n - t^{n+1} + t^{2n+1}}{(1 + t)^2},$$

while

$$|P(z)| = t^{2n}.$$

Hence, we have $|Q(z)| > |P(z)|$ when

$$\frac{1 - t^n - t^{n+1} + t^{2n+1}}{(1 + t)^2} > t^{2n},$$

which is equivalent to

$$t^n(1 + t + t^n + t^{n+1} + t^{n+2}) < 1;$$

this holds when

$$t^n(2 + 3t^n) \leq 1$$

(since $t < 1$). But this last inequality is satisfied for $t^n = 1/3$. Hence, by Rouché's Theorem, $r_n(z) = P(z) + Q(z)$ has the same number of zeros within the circle $|z| = (1/3)^{1/n}$ as does $Q(z)$, namely, none at all, since all the zeros of $Q(z)$ have modulus 1. Also, the above inequalities show that there can be no zero on this circle. The proof is now complete.

Proposition 3: For $n \geq 1$, $r_n(z)$ has $2n - 1$ zeros within the unit circle.

Proof: It is easy to verify the following factorization. For any a,

$$(3.4)\qquad (a + 1)az^{2n} - z^{2n-1} - 2z^{2n-2} - \cdots - nz^n - nz^{n-1} - (n - 1)z^{n-2} - \cdots - 2z - 1$$
$$= [(a + 1)z^n + z^{n-1} + \cdots + z + 1][az^n - z^{n-1} - \cdots - z - 1].$$

In particular, if we set $a = (\sqrt{5} - 1)/2$, then $(a + 1)a = 1$, and with (3.1) we get

$$(3.5)\qquad r_n(z) = g_n(z)h_n(z),$$

where

$$g_n(z) = \frac{\sqrt{5} - 1}{2}z^n - z^{n-1} - \cdots - z - 1,$$

and

$$h_n(z) = \frac{\sqrt{5} + 1}{2}z^n + z^{n-1} + \cdots + z + 1.$$

The Kakeya-Eneström Theorem (see, e.g., [4, p. 136] or [7, Prob. III.22]) now shows immediately that all n zeros of $h_n(z)$ lie within the unit circle. To deal with $g_n(z)$, we consider

$$(3.6)\qquad (z - 1)g_n(z) = \frac{\sqrt{5} - 1}{2}z^{n+1} - \frac{\sqrt{5} + 1}{2}z^n + 1.$$

By Pellet's Theorem (see, e.g., [4, p. 128]), n zeros of $(z - 1)g_n(z)$ lie on or within the unit circle. But $z = 1$ is the only zero on the unit cricle, since the difference of the complex vectors $((\sqrt{5} - 1)/2)z^{n+1}$ and $((\sqrt{5} - 1)/2z^n$ has length one only if they are collinear; (3.6) then implies $z = 1$. Hence, $g_n(z)$ has $n - 1$ zeros within the unit circle. (We remark that this fact also follows directly from Theorem 2.1 in [2].) The proof is now complete, with (3.5).

It was remarked in [3] that the complex zeros of $p_n(z)$ not located on the unit circle appear to lie close to the "missing" roots of unity (see Proposition 1 above). With regard to this, we have the following result.

Proposition 4: For $n \geq 1$, $r_n(z)$ has at least one zero in each sector

$$\left|\arg z - \frac{k}{n}\pi\right| \leq \frac{\pi}{n + 1},\ k = 0, 1, \ldots, 2n - 1.$$

Proof: We use the factorization (3.5). In analogy to (3.6), we have

$$(3.7)\qquad (z - 1)h_n(z) = \frac{\sqrt{5} + 1}{2}z^{n+1} - \frac{\sqrt{5} - 1}{2}z^n - 1.$$

Equation (3.7) can be brought into the form $az^{n+1} + z^n + 1$ by replacing z by $((\sqrt{5} + 1)/2)^{1/n}z$. The result of [4, p. 165, Ex. 3] implies that $(z - 1)h_n(z)$ has at least one zero in each of the sectors

$$(3.8)\qquad \left|\arg z - \frac{2k + 1}{n}\pi\right| \leq \frac{\pi}{n + 1},\ k = 0, 1, \ldots, n - 1.$$

The trivial zero $z = 1$ (i.e., $\arg z = 0$) is not contained in any of these sectors; hence, exactly one zero of $h_n(z)$ lies in each of the sectors (3.8).

To deal with the factor $g_n(z)$, we consider (3.6) and replace z by

$$(e^{i\pi}(\sqrt{5} - 1)/2)^{1/n}z.$$

This brings the right-hand side of (3.6) into the form $a'z^{n+1} + z^n + 1$ for some complex a'. We now apply a well-known result on the angular distribution of the zeros of certain trinomials (see [4, p. 165, Ex. 3]) and "rotate" the resulting sectors by an angle of π/n. This shows that $(z - 1)g_n(z)$ has at least one zero in each of the sectors

$$(3.9) \qquad \left|\arg z - \frac{2k}{n}\pi\right| \le \frac{\pi}{n+1}, \quad k = 0, 1, \ldots, n - 1.$$

But the sector belonging to $k = 0$ contains two zeros, namely, $z = 1$ and the unique positive zero of $g_n(z)$ (by Descartes's Rule of Signs; see, e.g., [6, p. 45]). Hence, each sector (3.9) contains exactly one zero of $g_n(z)$. This proves Proposition 4. [We have actually proved a slightly stronger statement; but the sectors (3.8) and (3.9) are overlapping.]

Remark: As we just saw, the trinomials on the right-hand sides of (3.6) and (3.7) can be brought into the form $f(z) = az^{n+1} + z^n + 1$. One could also consider the inverted polynomial

$$f^*(z) = z^{n+1}\overline{f}(1/z) = z^{n+1} + z + \overline{a},$$

the zeros z_j^* of which are the inverses of the zeros z_j of $f(z)$ relative to the unit circle (i.e., $z_j^* = 1/\overline{z}_j$; see [4, p. 194]). In this regard, we mention that the trinomials $z^{n+1} - (n + 1)z + n = 0$ were studied in [5]; very exact bounds on the arguments and the moduli of the zeros of these trinomials were obtained. Probably the methods in [5] could be used to obtain similar results for the trinomials in (3.6) and (3.7).

4. Real Roots

Horadam and Shannon [3] showed that $p_n(z)$ has exactly one positive zero x_{1n} with

$$(4.1) \qquad \lim_{n\to\infty} x_{1n} = \frac{\sqrt{5}+3}{2} \left[= \left(\frac{\sqrt{5}+1}{2}\right)^2\right];$$

this is the one zero not covered by Proposition 3. They also conjectured that $p_n(z)$ has exactly one negative zero x_{2n} with $-1 < x_{2n} < 0$. They proved this conjecture under the condition that the factorization (2.1) above is true; hence, the existence of this negative zero is established. It was also conjectured in [3] that

$$(4.2) \qquad \lim_{n\to\infty} x_{2n} = -1;$$

this follows immediately from Propositions 2 and 3. Our aim in this section is to give quantitative versions of (4.1) and (4.2).

Let $G_n(z)$ and $H_n(z)$ denote the trinomials on the right-hand sides of (3.6) and (3.7), respectively. By Descartes's Rule of Signs, $G_n(z)$ has two positive zeros ($z = 1$ and $z = x_{1n}$), while $H_n(z)$ has only one positive zero ($z = 1$). As to the negative zeros, we consider $G_n(-z)$ and $H_n(-z)$. The signs of the coefficients of $G_n(-z)$ are $(-1)^{n+1}$, $(-1)^{n+1}$, 1; that is, there is one sign change when n is even and none when n is odd. Hence, $G_n(z)$ has a negative zero (namely, x_{2n}) only when n is even. The signs of the coefficients of $H_n(-z)$ are $(-1)^{n+1}$, $(-1)^{n+1}$, -1; this implies that $H_n(z)$ has a negative zero (namely, x_{2n}) only when n is odd.

The following results give estimates on the location of these zeros.

Proposition 5: Let $\alpha := (\sqrt{5} + 1)/2$. Then, for all $n \ge 1$,

$$\alpha^2(1 - \alpha^{-2n}) \le x_{1n} < \alpha^2(1 - \alpha^{-2n-1}),$$

with equality only for $n = 1$. Furthermore, we have, asymptotically,

$$x_{1n} \sim \alpha^2(1 - \alpha^{-2n-1}) \text{ as } n \to \infty.$$

Proof: It suffices to find two points at which $G_n(z)$ has opposite signs. It is easy to see that, for any ε, we have

$$G_n(\alpha^2 - \varepsilon) = -\varepsilon \frac{\sqrt{5} - 1}{2}(\alpha^2 - \varepsilon)^n + 1,$$

and therefore, for arbitrary numbers a,

$$G_n(\alpha^2 - a\alpha^{-2n}) = -a \frac{\sqrt{5} - 1}{2}(1 - a\alpha^{-2n-2})^n + 1. \tag{4.3}$$

First, we let $a = \alpha$. Since $(1 - \alpha^{-2n-1})^n < 1$ for all n, we get

$$G_n(\alpha^2 - \alpha^{1-2n}) > 0 \text{ for } n \geq 1. \tag{4.4}$$

In the other direction, we set $a = \alpha^2$. It is easy to see (using calculus) that $(1 - \alpha^{-2n})^n$ is an increasing sequence for $n \geq 1$. Thus, we get, with (4.3),

$$G_n(\alpha^2 - \alpha^{2-2n}) \leq G_1(\alpha^2 - 1) = 0,$$

with equality only for $n = 1$. This, together with (4.4), proves the first statement of the proposition. The asymptotic expression follows from the fact that, for any real a, we have $(1 - a\alpha^{-2n-2})^n \to 1$ as $n \to \infty$.

Proposition 6: For all $n \geq 2$, we have

$$-1 + \frac{1}{2n} < x_{2n} < -1 + \frac{\log 5}{2n}, \tag{4.5}$$

and we have, asymptotically,

$$x_{2n} \sim -1 + \frac{\log 5}{2n} \text{ as } n \to \infty.$$

Proof: First, let n be even. Then, for any a, we have

$$G_n\left(-1 + \frac{a}{n}\right) = -\left(1 - \frac{a}{n}\right)^n\left[\sqrt{5} - \frac{a}{n}\frac{\sqrt{5} - 1}{2}\right] + 1. \tag{4.6}$$

We note that $(1 - a/n)^n$ is an increasing sequence for $n \geq 2$, at least when $a = 1/2$. Hence, for all $n \geq 2$,

$$G_n\left(-1 + \frac{1/2}{n}\right) \leq G_2\left(-1 + \frac{1}{4}\right) = -\left(1 - \frac{1}{4}\right)^2\left[\sqrt{5} - \frac{\sqrt{5} - 1}{8}\right] + 1 < 0.$$

In the other direction, we use the fact that $(1 - (\log 5)/2n)^n < 1/\sqrt{5}$ for all n. Hence, with (4.6),

$$G_n\left(-1 + \frac{\log 5}{2n}\right) > -\frac{1}{\sqrt{5}}\sqrt{5} + 1 = 0.$$

This proves (4.5) for even n. If n is odd, we have, for arbitrary a,

$$H_n\left(-1 + \frac{a}{n}\right) = \left(1 - \frac{a}{n}\right)^n\left[\sqrt{5} - \frac{a}{n}\frac{\sqrt{5} + 1}{2}\right] - 1. \tag{4.7}$$

We find that, for $n \geq 3$,

$$H_n\left(-1 + \frac{1/2}{n}\right) \geq H_3\left(-1 + \frac{1}{6}\right) = \left(1 - \frac{1}{6}\right)^3\left[\sqrt{5} - \frac{\sqrt{5} + 1}{12}\right] - 1 > 0,$$

while, again with (4.7),

$$H_n\left(-1 + \frac{\log 5}{2n}\right) < \frac{1}{\sqrt{5}}\sqrt{5} - 1 = 0.$$

This completes the proof of (4.5). The asymptotic expression follows from (4.6) and (4.7), and from the fact that $(1 - a/n)^n \to e^{-a}$ as $n \to \infty$.

Remarks: *(1)* The zero $x_{21} = -1/\alpha \simeq -0.61803$; it does not satisfy (4.5).

(2) As an illustration for Propositions 5 and 6, see Table 3 in [3]. The two results also explain the observation in [3] that "the negative root approaches its lower bound more slowly than the positive root approaches its upper bound."

5. Some Algebra

In the theory of uniform distribution modulo 1, sequences of the type ω^n supply important special cases (see, e.g., [8, p. 2]). For instance, it is known that ω^n is uniformly distributed modulo 1 for almost all (in the Lebesgue sense) numbers $\omega > 1$, but very little is known for particular values of ω. On the other hand, it is of interest to study "bad" examples of ω, namely, those for which the sequence ω^n is very "unevenly" distributed modulo 1.

One such example is $\omega = \alpha = (1 + \sqrt{5})/2$; its conjugate is $\beta = (1 - \sqrt{5})/2$. Now $\alpha^n + \beta^n$ are the Lucas numbers 2, 1, 3, 4, 7, ... and thus are rational integers, so that

$$\alpha^n + \beta^n \equiv 0 \pmod 1.$$

But $|\beta| < 1$, and so $\beta^n \to 0$ as $n \to \infty$, which implies that $\alpha^n \to 0 \pmod 1$.

Hence, α^n (modulo 1) has a single accumulation point. α shares this property with a wider class of algebraic numbers (see [8] or [1]).

Definition: A *Pisot number* is an algebraic integer $\theta > 1$ such that all of its conjugates have moduli strictly less than 1.

Theorem (Salem [8]): If θ is a Pisot number, then $\theta \to 0 \pmod 1$ as $n \to \infty$.

The proof of this theorem is similar to the above discussion on the properties of α^n.

It is our aim now to show that the unique positive zeros x_{1n} of the polynomials $p_n(x)$ are Pisot numbers. First, we need the following result,

Proposition 7: The polynomials $r_n(x)$ are irreducible over the rationals.

We have seen in the previous sections that $r_n(z)$ has $2n - 1$ zeros satisfying $|z| < 1$ and one zero satisfying $|z| > 1$. Also, $r_n(z)$ is a monic polynomial with rational integer coefficients. If $r_n(z)$ were reducible over the rationals, then, by Gauss's Lemma, $r_n(z) = G(z)H(z)$ for suitable monic polynomials $G(z)$, $H(z)$ of positive degrees with rational integer coefficients. One of these polynomials, say $G(z)$, must have all its zeros of modulus strictly less than one. Hence, the constant term of $G(z)$ (the product of all its zeros) has modulus $|G(0)| < 1$. But this contradicts the fact that the constant term of $G(z)$ is a nonzero integer.

Remark: The proof of Proposition 7 is taken from [9] where, by the way, a trinomial similar to (3.6) and (3.7) is considered. See also the remark on page 12 in [1].

Proposition 8: The unique positive zeros x_{1n} of $r_n(z)$ are Pisot numbers for all $n \geq 1$.

Proof: This follows from Proposition 7 and the results on the zeros of $r_n(z)$ in the previous sections.

We close with a factorization involving Fibonacci numbers. Equation (3.4) shows that the left-hand side of (3.4) splits into two factors of equal degree if a is rational. On the other hand, Proposition 7 shows that this polynomial is irreducible over $\mathbb{Q}$ for $a = (\sqrt{5} - 1)/2$. These remarks suggest that we set

$a = F_k/F_{k+1}$ (where F_k is the k^{th} Fibonacci number), as it is well known that

$$F_k/F_{k+1} \to (\sqrt{5} - 1)/2 \text{ for } k \to \infty;$$

these fractions are actually the best rational approximations to $(\sqrt{5} - 1)/2$. If we take into account

$$a + 1 = F_k/F_{k+1} + 1 = F_{k+2}/F_{k+1}$$

and

$$(a + 1)a = F_{k+2}F_k/F_{k+1}^2 = (F_{k+1}^2 - 1)/F_{k+1}^2,$$

we obtain the factorization

$$\begin{aligned}(1 - F_{k+1}^{-2})z^{2n} - z^{2n-1} - 2z^{2n-2} - \cdots - nz^n - nz^{n-1} - (n-1)z^{n-2} \\ - \cdots - 2z - 1 \\ = [(F_{k+2}/F_{k+1})z^n + z^{n-1} + \cdots + z + 1][(F_k/F_{k+1})z^n - z^{n-1} \\ - \cdots - z - 1].\end{aligned}$$

We note that the left-hand side of this factorization converges quite rapidly to $r_n(z)$ as $k \to \infty$, uniformly on compact subsets of $\mathbb{C}$.

References

1. M. J. Bertin & M. Pathiaux-Delefosse. *Conjecture de Lehmer et petits nombres de Salem*. Queen's Papers in Pure and Applied Mathemetics, No. 81. Kingston, Ontario, 1989.
2. K. Dilcher, J. D. Nulton, & K. B. Stolarsky. "The Zeros of a Certain Family of Trinomials." *Glasgow Math. J.* (to appear).
3. A. F. Horadam & A. G. Shannon. "Cyclotomy-Generated Polynomials of Fibonacci Type." In *Fibonacci Numbers and Their Applications*, pp. 81-97, edited by A. N. Philippou et al. Dordrecht: D. Reidel, 1986.
4. M. Marden. *Geometry of Polynomials*. American Mathematical Society, Mathematical Surveys No. 3, 1966.
5. J. L. Nicolas & A. Schinzel. "Localisation des zéros de polynômes intervenant en théorie du signal." In *Cinquante Aus de Polynômes: Fifty Years of Polynomials*, pp. 167-79, ed. M. Langevin and M. Waldschmidt. Lecture Notes in Mathematics, No. 1415. Berlin and Heidelberg: Springer-Verlag, 1990.
6. N. Obreschkoff. *Verteilung und Berechnung der Nullstellen reeller Polynome*. Berlin: VEB Deutscher Verlag der Wissenschaften, 1963.
7. G. Pólya & G. Szegö. *Problems and Theorems in Analysis*. New York: Springer-Verlag, 1972.
8. R. Salem. *Algebraic Numbers and Fourier Analysis*. Boston: D. C. Heath, 1963.
9. Solution to Problem E3008: An Irreducible Polynomial. *Amer. Math. Monthly 96.2* (1989): 155-56.

*****

SOME SEQUENCES ASSOCIATED WITH THE GOLDEN RATIO*

Tom C. Brown and Allen R. Freedman
Simon Fraser University, Burnaby, B.C., Canada V5A IS6
(Submitted June 1989)

A number of people have considered the arithmetical, combinatorial, geometrical and other properties of sequences of the form $([n\alpha]: n \geq 1)$, where α is a positive irrational number and $[\ ]$ denotes the greatest integer function. (See, e.g., [1]-[16]) and the references contained in those papers, especially [8] and [16].)

There are several other sequences which may be naturally associated with the sequence $([n\alpha]: n \geq 1)$. They are the *difference sequence*

$$f_\alpha(n) = [(n+1)\alpha] - [n\alpha] - [\alpha]$$

(the difference sequence is "normalized" by subtracting $[\alpha]$ so that its values are 0 and 1), the *characteristic function*

$$g_\alpha(n) \quad (g_\alpha(n) = 1 \text{ if } n = [k\alpha] \text{ for some } k, \text{ and } g_\alpha(n) = 0 \text{ otherwise}),$$

and the *hit sequence*

$$h_\alpha(n),$$

where $h_\alpha(n)$ is the number of different values of k such that $[k\alpha] = n$.

We use the notation

$$f_\alpha = (f_\alpha(n): n \geq 1),\ g_\alpha = (g_\alpha(n): n \geq 1),\ h_\alpha = (h_\alpha(n): n \geq 1).$$

Note that $f_\alpha = f_{\alpha+k}$ for any integer $k \geq 1$. In particular, $f_\alpha = f_{\alpha-1}$ if $\alpha > 1$.

Special properties of these sequences in the case where α equals τ, the golden mean, $\tau = (1+\sqrt{5})/2$, are considered in [5], [12], [14], and [16]. For example, the following is observed in [12]. Let $u_n = [n\tau]$, $n \geq 1$, and let F_k denote the k^{th} Fibonacci number. Given k, let $r = F_{2k}$, $s = F_{2k+1}$, $t = F_{2k+2}$. Then

$$u_r = s,\ u_{2r} = 2s,\ u_{3r} = 3s,\ \ldots,\ u_{(t-2)r} = (t-2)s;$$

thus, the sequence $([n\tau])$ contains the $(t-2)$-term arithmetic progression $(s, 2s, 3s, \ldots, (t-2)s)$.

It was shown in [16], using a theorem of A. A. Markov [11] (which describes the sequence f_α (for any α) explicitly in terms of the simple continued fraction expansion of α), that the difference sequence f_τ has a certain "substitution property." We give a simple proof of this below (Theorem 2) without using Markov's theorem. We also make several observations concerning the three sequences f_τ, g_τ, and h_τ.

Theorem 1: The golden mean τ is the smallest positive irrational real number α such that $f_\alpha = g_\alpha = h_\alpha$. In fact, $f_\alpha = g_\alpha = h_\alpha$ exactly when $\alpha^2 = k\alpha + 1$, where $k = [\alpha] \geq 1$.

Proof: It follows directly from the definitions (we omit the details) that if α is irrational and $\alpha > 1$, then $h_\alpha = g_\alpha = f_{1/\alpha}$. (The fact that $g_\alpha = f_{1/\alpha}$ is mentioned in [8]. It is straightforward to show that

$$g_\alpha(n) = 1 \Rightarrow f_{1/\alpha}(n) = 1 \quad\text{and}\quad g_\alpha(n) = 0 \Rightarrow f_{1/\alpha}(n) = 0.)$$

*Research partially supported by NSERC.

Also, if α is irrational and $\alpha > 0$, then

$$h_\alpha(n) = f_{1/\alpha}(n) + [1/\alpha] \text{ for all } n \geq 1.$$

Thus, if α is irrational and $f_\alpha = g_\alpha = h_\alpha$, then $\alpha > 1$ (otherwise, g_α is identically equal to 1, and f_α is not) and

$$f_{\alpha - [\alpha]}(n) = f_\alpha(n) = g_\alpha(n) = f_{1/\alpha}(n) \text{ for all } n \geq 1.$$

Since the sequence f_β determines β if $\beta < 1$, this gives $\alpha - [\alpha] = 1/\alpha$, and the result follows.

Definition: For any finite or infinite sequence w consisting of 0's and 1's, let $\overline{w}$ be the sequence obtained from w by replacing each 0 in w by 1, and each 1 by 10. For example, $\overline{10110} = 10110101$. (Compare "Fibonacci strings" [10, p. 85].)

Note that $\overline{uv} = \overline{u} \cdot \overline{v}$, and that $\overline{u} = \overline{v} \Rightarrow u = v$ by induction on the length of v.

Theorem 2: The sequences f_τ and $\overline{f_\tau}$ are identical.

Proof: First, we show that if $0 < \alpha < 1$, then $\overline{f_\alpha} = g_{1+\alpha}$. Let $L(w)$ denote the *length* of the finite sequence w, so that if $w = f_\alpha(1) f_\alpha(2) \ldots f_\alpha(k)$, then

$$L(\overline{w}) = k + f_\alpha(1) + \cdots + f_\alpha(k) = k + [(k+1)\alpha].$$

Thus,

$$\begin{aligned}[\overline{f_\alpha}(n) = 1] &\Leftrightarrow [n = L(\overline{w}) + 1 \text{ for some initial segment } w \text{ of } f_\alpha] \\ &\Leftrightarrow [n = [(k+1)(1+\alpha)] \text{ for some } k \geq 0] \Leftrightarrow [g_{1+\alpha}(n) = 1].\end{aligned}$$

Therefore, $\overline{f_\tau} = \overline{f_{\tau - 1}} = g_\tau = f_{1/\tau} = f_{\tau - 1} = f_\tau$.

Corollary 1: The sequence f_τ can be generated by starting with $w = 1$ and repeatedly replacing w by $\overline{w}$.

Proof: If we define $E_1 = 1$ and $E_{k+1} = \overline{E_k}$, then, since $\overline{1} = 10$ begins with a 1, it follows that, for each k, E_k is an initial segment of E_{k+1}. By Theorem 2 and induction, each E_k is an initial segment of f_τ. Thus,

$$E_1 = 1,\ E_2 = \overline{E_1} = 10,\ E_3 = \overline{E_2} = 101,\ E_4 = \overline{E_3} = 10110,$$
$$E_5 = \overline{E_4} = 10110101, \text{ etc.},$$

are all initial segments of f_τ. (These blocks naturally have lengths 1, 2, 3, 5, 8, ...)

Corollary 2: For each $i \geq 1$, let x_i denote the number of 1's in the sequence f_τ which lie between the i^{th} and $(i+1)^{\text{st}}$ 0's. Thus,

$$f_\tau = 1011010110110101101011011010110 11\ldots,$$
$$(x_n) = \quad 2\ 1\ \ 2\ \ 2\ 1\ \ 2\ 1\ \ 2\ \ 2\ 1\ \ 2\ \ 2\ldots\ .$$

Then the sequences $(x_n - 1)$ and f_τ are identical.

Proof: If we start with the sequence (x_n) and replace each 1 by 10 and each 2 by 101, we obtain the sequence f_τ. Since $\overline{\overline{0}} = 10$ and $\overline{\overline{1}} = 101$, this shows that $(\overline{\overline{x_n - 1}}) = f_\tau = \overline{\overline{f_\tau}}$. Therefore, $(\overline{x_n - 1}) = \overline{f_\tau}$, and, finally, $(x_n - 1) = f_\tau$.

References

1. T. Bang. "On the Sequence $[n\alpha]$." *Math. Scand.* 5 (1957):69-76.
2. I. G. Connell. "A Generalization of Wythoff's Game." *Can. Math. Bull.* 2 (1959):181-90.
3. I. G. Connell. "Some Properties of Beatty Sequences I." *Can. Math. Bull.* 2 (1959):190-97.

4. I. G. Connell. "Some Properties of Beatty Sequences II." *Can. Math. Bull.* *3* (1960):17-22.
5. H. S. M. Coxeter. "The Golden Section, Phyllotzxis, and Wythoff's Game." *Scripta Math.* *19* (1953):135-43.
6. A. S. Fraenkel. "The Bracket Function and Complementary Sets of Integers." *Can. J. Math.* *21* (1969):6-27.
7. A. S. Fraenkel & J. Levitt. "Characterization of the Set of Values $f(n) = [n\alpha]$, $n = 1, 2, \ldots$." *Discrete Math.* *2* (1972):335-45.
8. A. S. Fraenkel, M. Mushkin, & U. Tassa. "Determination of $[n\theta]$ by Its Sequence of Differences." *Canad. Math. Bull.* *21* (1978):441-46.
9. R. L. Graham. "Covering the Positive Integers by Disjoint Sets of the Form $\{[n\alpha + \beta]: n = 1, 2, \ldots\}$." *J. Combinatorial Th. (A)* *15* (1973):354-58.
10. D. Knuth. *The Art of Computer Programming*. Vol. I. 2nd ed. New York: Addison-Wesley, 1973.
11. A. A. Markoff. "Sur une question de Jean Bernoulli." *Math. Ann.* *19* (1882): 27-36.
12. N. S. Mendelsohn. "The Golden Ratio and van der Waerden's Theorem." *Proc. 5th SE Conf. on Combinatorics, Graph Theory, and Computing, Congressus Numerantium* *10* (1974):93-109.
13. I. Niven. *Diophantine Approximations*. New York: Wiley, 1963.
14. J. Rosenblatt. "The Sequence of Greatest Integers of an Arithmetic Progression." *J. London Math. Soc. (2)* *17* (1978):213-18.
15. Th. Skolem. "On Certain Distributions of Integers in Pairs with Given Differences." *Math. Scand.* *5* (1957):57-68.
16. K. B. Stolarsky. "Beatty Sequences, Continued Fractions, and Certain Shift Operators." *Canad. Math. Bull.* *19* (1976):473-82.

*****

q-DETERMINANTS AND PERMUTATIONS

Kung-Wei Yang
Western Michigan University, Kalamazoo, MI 49008
(Submitted June 1989)

1. Permutations

We write a *permutation* p of $\{1, 2, \ldots, n\}$ in the form $p(1)p(2)\ldots p(n)$. An *inversion* of the permutation $p(1)p(2)\ldots p(n)$ is a pair $(p(i), p(j))$ such that $p(i) > p(j)$ and $i < j$. We let $i(p)$ denote the number of inversions of p. For example, there are four inversions in the permutation $p = 2431$: $(2, 1)$, $(3, 1)$, $(4, 1)$, $(4, 3)$; hence, $i(p) = 4$.

For applications to other areas (computer science, chemistry, physics), it is useful to note that the number of inversions of the permutation $p(1)p(2)\ldots p(n)$ is the same as the minimum number of interchanges of adjacent numbers required to restore $p(1)p(2)\ldots p(n)$ to its natural order $1\,2\ldots n$.

2. Definitions

Let K be a field of characteristic 0, $K[q]$ the polynomial ring, and R a commutative ring with identity containing $K[q]$. Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in R. The ordinary determinant of A is given by the familiar formula [3, p. 14]

$$\det(A) = \sum (-1)^{i(p)} a_{1p(1)} a_{2p(2)} \ldots a_{np(n)},$$

where the summation is extended over all permutations p, and $i(p)$ is the number of inversions of the permutation p. The *q-determinant of A* is defined by the same expression with (-1) replaced by the indeterminate q:

$$\det_q(A) = \sum q^{i(p)} a_{1p(1)} a_{2p(2)} \ldots a_{np(n)}.$$

This makes q a marker for the number of inversions of a permutation.

Now, just as one can approach the subject of determinants from the point of view of Grassmann algebras, we can approach the subject of q-determinants from the point of view of q-Grassmann algebras. A *q-Grassmann algebra* (cf. [6]) is the associative $K[q]$-algebra generated by $x_1, x_2, \ldots, x_n$, satisfying the relations $x_i^2 = 0$ and $x_j x_i = q x_i x_j$, if $i < j$. Clearly, in this algebra, every monomial can be written in the *normal form*

$$c x_{i_1} x_{i_2} \ldots x_{i_r}$$

where c is in $K[q]$ and $i_1 < i_2 < \cdots < i_r$. Hence, in normal form we have

$$(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n)$$
$$\cdots (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n) = \det_q(a_{ij}) x_1 x_2 x_3 \ldots x_n.$$

3. Properties

Theorem 1:

(1) The q-determinant is a multilinear function of the rows and columns.
(2) The q-determinant of a block triangular matrix is the product of the q-determinants of the diagonal blocks.
(3) $\det_q(A) = \det_q(A^T)$, where A^T is the transpose of A.
(4) (Expansion Theorem) Let A_{ij} denote the (i, j)-minor of A. Then,

$$\begin{aligned}\det_q(A) &= a_{11}\det_q(A_{11}) + qa_{21}\det_q(A_{21}) + q^2a_{31}\det_q(A_{31}) + \cdots \\ &\qquad + q^{n-1}a_{n1}\det_q(A_{n1}) \\ &= a_{nn}\det_q(A_{nn}) + qa_{(n-1)n}\det_q(A_{(n-1)n}) + \cdots + q^{n-1}a_{1n}\det_q(A_{1n}) \\ &= a_{11}\det_q(A_{11}) + qa_{12}\det_q(A_{12}) + q^2a_{13}\det_q(A_{13}) + \cdots \\ &\qquad + q^{n-1}a_{1n}\det_q(A_{1n}) \\ &= a_{nn}\det_q(A_{nn}) + qa_{n(n-1)}\det_q(A_{n(n-1)}) + \cdots + q^{n-1}a_{n1}\det_q(A_{n1}).\end{aligned}$$

Proof: Parts (1) and (2) are obvious; (3) follows from $i(p) = i(p^{-1})$. The four equalities in (4) represent four ways of sorting the terms of $\det_q(A)$. They follow from the q-Grassmann algebra formulation of the q-determinants. [The last two equalities also follow from the first two and part (3).] Q.E.D.

4. Fibonacci Polynomials

There are several related polynomial sequences all named Fibonacci polynomials. Here by Fibonacci polynomials we mean the polynomials Riordan called $L_n(x)$ in his book [4, pp. 182-83]. They were later reintroduced by Doman and Williams in [1]. It is interesting to note that Doman and Williams were led to the definition of these polynomials from a study of a one-dimensional Ising chain in physics.

Fibonacci polynomials $F_n(q)$ are defined by the recurrence relation

$$F_{n+1}(q) = F_n(q) + qF_{n-1}(q),$$

and the initial conditions $F_0(q) = 0$, $F_1(q) = 1$. They are, in fact, expressible as

$$F_{n+1}(q) = \sum_{i=0}^{h}\binom{n-i}{i}q^i,$$

where h is the integer part of $n/2$ (for $n > 0$). As we shall show in the following, there are also the generating functions of the number of inversions of permutations p satisfying $|i - p(i)| < 2$, for all i.

5. Generating Functions

In this section, we derive several generating functions of the number of inversions of permutations by applying q-determinants to (0, 1)-matrices. We let K be the rational field, and we use the abbreviations:

$$[n] = (1 + q + q^2 + \cdots + q^{n-1}),$$

$$[n]! = [1][2][3]\ldots[n].$$

Theorem 2: The generating functions of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ is $[n]!$ ([5, p. 21]).

Proof: Let J_n denote the $n \times n$ matrix whose every entry is equal to 1. By the Expansion Theorem,

$$\sum q^{i(p)} = \det_q(J_n) = (1 + q + q^2 + \cdots + q^{n-1})\det_q(J_{n-1}) = [n]!$$

Here the summation is taken over all permutations. Q.E.D.

Theorem 3: The generating functions of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ satisfying $(i - p(i)) < r$, for all i, where $r \le n$, is $[r]^{n-r}[r]!$.

Proof: Let $K_n(r) = (k_{ij})$ denote the $n \times n$ matrix defined by

$$k_{ij} = \begin{cases} 1, & \text{if } i - j < r, \\ 0, & \text{otherwise.} \end{cases}$$

Again, by the Expansion Theorem,

$$\begin{aligned}\sum_{i - p(i) < r} q^{i(p)} &= \det_q(K_n(r)) = (1 + q + q^2 + \cdots + q^{r-1})\det_q(K_{n-1}(r)) \\ &= [r]^{n-r}\det_q(K_r(r)) = [r]^{n-r}[r]! \quad \text{Q.E.D.}\end{aligned}$$

Theorem 4: The generating functions of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ satisfying $|i - p(i)| < 2$, for all i, is the Fibonacci polynomial $F_{n+1}(q)$.

Proof: Let $L_n = (f_{ij})$ denote the $n \times n$ matrix defined by

$$f_{ij} = \begin{cases} 1, & \text{if } |i - j| < 2, \\ 0, & \text{otherwise.} \end{cases}$$

The desired generating function is then

$$\sum_{|i - p(i)| < 2} q^{i(p)} = \det_q(L_n).$$

By the Expansion Theorem, $\det_q(L_n)$ satisfies the recurrence

$$\det_q(L_{n+1}) = \det_q(L_n) + q\det_q(L_{n-1}),$$

and the initial conditions $\det_q(L_1) = 1$, $\det_q(L_2) = 1 + q$. Hence, the generating function is $F_{n+1}(q)$. Q.E.D.

We note that, since $F_{n+1}(1) = F_{n+1}$ is the Fibonacci number, the number of permutations satisfying $|i - p(i)| \le 1$ is F_{n+1} (see Example 4.7.7 of [5] and the related references given there).

Now, call $A \le B$, if $A = (a_{ij})$, $B = (b_{ij})$ are matrices with rational entries and $a_{ij} \le b_{ij}$ for all i, j. Similarly, define $f(q) \le g(q)$, if $f(q)$, $g(q)$ are polynomials with rational coefficients and the coefficient of every term q^i in $f(q)$ is less than or equal to the coefficient of the corresponding term q^i in $g(q)$. It is easy to see that if A and B are (0, 1)-matrices and $A \le B$, then det $(A) \le \det_q(B)$ and, therefore, $0 \le \det_q(A) - \det_q(B)$.

Corollary 1: The generating function of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ such that $i - p(i) \ge r$ for some i is given by

$$[n]! - [r]^{n-r}[r]!$$

When $r = 2$, the generating function is

$$[n]! - [2]^{n-1} = [n]! - (1 + q)^{n-1},$$

and when $r = n - 1$, it is

$$[n]! - [n - 1][n - 1]! = q^{n-1}[n - 1]!$$

which is obvious from the given condition.

Corollary 2: The generating function of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ such that $|i - p(i)| \geq 2$ for some i is given by

$$[n]! - F_{n+1}(q).$$

Corollary 3: Let r be ≥ 2. The generating function of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ such that $(i - p(i)) < r$ for all i and $|i - p(i)| \geq 2$ for some i is given by

$$[r]^{n-r}[r]! - F_{n+1}(q).$$

The special case $r = 2$ of Corollary 3 is of particular interest. It says the generating function of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ such that $(i - p(i)) < 2$ for all r and $|i - p(i)| \geq 2$ for some i is given by

$$(1 + q)^{n-1} - F_{n+1}(q) = \sum_{i=0}^{n-1} \left\{\binom{n-1}{i} - \binom{n-i}{i}\right\} q^i,$$

where it is understood that $\binom{r}{i} = 0$ if $r < i$.

6. Remarks

From a preprint ("Quantum Deformation of Flag Schemes and Grassmann Schemes I: A q-Deformation of the Shape-Algebra for $GL(n)$" by Earl Taft & Jacob Towber) which we received from Professor Earl Taft recently, we learned that another q-analogue of determinant (essentially replacing q by $-q^{-1}$) has been developed by Yu I. Manin.

We should also point out that the evaluation of a q-determinant is in general difficult, for the evaluation of even one of its specializations ($q = 1$), the permanent, is difficult (see [2]).

References

1. B. G. S. Doman & J. K. Williams. "Fibonacci and Lucas Polynomials." *Math. Proc. Camb. Phil. Soc.* *90* (1981):385-87.
2. M. Marcus & H. Minc. "Permanents." *Amer. Math. Monthly* *72* (1965):577-591.
3. T. Muir. *A Treatise on the Theory of Determinants*. New York: Dover, 1960.
4. J. Riordan. *An Introduction to Combinatorial Analysis*. New York: Wiley & Sons, 1958.
5. R. P. Stanley. *Enumerative Combinatorics*. Monterey: Wadsworth & Brooks /Cole, 1986.
6. K.-W. Yang. "Solution of q-Difference Equations." *Bull. London Math. Soc.* *20* (1988):1-4.

*****

CHOLESKY ALGORITHM MATRICES OF FIBONACCI TYPE AND PROPERTIES OF GENERALIZED SEQUENCES*

Alwyn F. Horadam

University of New England, Armidale, Australia

Piero Filipponi

Fondazione Ugo Bordoni, Rome, Italy

(Submitted June 1989)

1. Introduction

Many properties of the generalized Fibonacci numbers U_n and the generalized Lucas numbers V_n (e.g., see [3]-[5], [8]-[10], [12], [14], [15]) have been obtained by altering their recurrence relation and/or the initial conditions. Here we offer a somewhat new matrix approach for developing properties of this nature.

The aim of this paper is to use the 2-by-2 matrix $\boldsymbol{M}_k$ determined by the Cholesky LR decomposition algorithm to establish a large number of identities involving U_n and V_n. Some of these identities, most of which we believe to be new, extend the results obtained in [6] and elsewhere.

Particular examples of the use of the matrix $\boldsymbol{M}_k$, including summation of some finite series involving U_n and V_n, are exhibited. A method for evaluating some infinite series is then presented which is based on the use of a closed form expression for certain functions of the matrix $x\boldsymbol{M}_k^n$.

2. Generalities

In this section some definitions are given and some results are established which will be used throughout the paper.

2.1. The Numbers U_n and the Matrix M

Letting m be a natural number, we define (see [4]) the integers $U_n(m)$ (or more simply U_n if there is no fear of confusion) by the second-order recurrence relation

(2.1) $\quad U_{n+2} = mU_{n+1} + U_n; \quad U_0 = 0,\ U_1 = 1 \quad \forall\, m.$

The first few numbers of the sequence $\{U_n\}$ are:

U_0	U_1	U_2	U_3	U_4	U_5	U_6	...
0	1	m	m^2+1	m^3+2m	m^4+3m^2+1	m^5+4m^3+3m	... .

We recall [4] that the numbers U_n can be expressed in the closed form (Binet form)

(2.2) $\quad U_n = (\alpha_m^n - \beta_m^n)/\Delta_m,$

where

*Work carried out in the framework of the agreement between the Italian PT Administration and the Fondazione Ugo Bordoni.

$$(2.3)\qquad \begin{cases} \Delta_m = \sqrt{m^2+4} \\ \alpha_m = (m+\Delta_m)/2 \\ \beta_m = (m-\Delta_m)/2. \end{cases}$$

From (2.3) it can be noted that

$$(2.4)\qquad \begin{cases} \alpha_m\beta_m = -1 \\ \alpha_m + \beta_m = m \\ \alpha_m - \beta_m = \Delta_m. \end{cases}$$

We also recall [4] that

$$(2.5)\qquad U_n = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} m^{n-2j-1},$$

where [•] denotes the greatest integer function. Moreover, as we sometimes require negative-valued subscripts, from (2.2) and (2.4) we deduce that

$$(2.6)\qquad U_{-n} = (-1)^{n+1}U_n.$$

From (2.1) it can be noted that the numbers $U_n(1)$ are the Fibonacci numbers F_n and the numbers $U_n(2)$ are the Pell numbers P_n.

Analogously, the numbers $V_n(m)$ (or more simply V_n) can be defined [4] as

$$(2.7)\qquad V_n = \alpha_m^n - \beta_m^n = U_{n-1} + U_{n+1}.$$

The first few numbers of the sequence $\{V_n\}$ are:

V_0	V_1	V_2	V_3	V_4	V_5	V_6	...
2	m	m^2+2	m^3+3m	m^4+4m^2+2	m^5+5m^3+5m	$m^6+6m^4+9m^2+2$	... .

These numbers satisfy the recurrence relation

$$(2.8)\qquad V_{n+2} = mV_{n+1} + V_n; \quad V_0 = 2, \; V_1 = m \quad \forall m.$$

From (2.7) and (2.4) we have

$$(2.9)\qquad V_{-n} = (-1)^n V_n,$$

and it is apparent that the numbers $V_n(1)$ are the Lucas numbers L_n while the numbers $V_n(2)$ are the Pell-Lucas numbers Q_n [11].

Definitions (2.1) and (2.8) can be extended to an arbitrary generating parameter, leading in particular to the double-ended complex sequences $\{U_n(z)\}_{-\infty}^{\infty}$ and $\{V_n(z)\}_{-\infty}^{\infty}$. Later we shall make use of the numbers $U_n(z)$.

Let us now consider the 2-by-2 symmetric matrix

$$(2.10)\qquad \boldsymbol{M} = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}$$

which is governed by the parameter m and of which the eigenvalues are α_m and β_m. For n a nonnegative integer, it can be proved by induction [6] that

$$(2.11)\qquad \boldsymbol{M}^n = \begin{bmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{bmatrix}.$$

Also, the matrix $\boldsymbol{M}$ can obviously be extended to the case where the parameter m is arbitrary (e.g., complex).

2.2 A Cholesky Decomposition of the Matrix M: The Matrix M_k

Let us put

$$(2.12) \quad M_1 = M = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}$$

and decompose M_1 as

$$(2.13) \quad M_1 = T_1 T_1' = \begin{bmatrix} a_1 & 0 \\ c_1 & b_1 \end{bmatrix} \begin{bmatrix} a_1 & c_1 \\ 0 & b_1 \end{bmatrix},$$

where T_1 is a lower triangular matrix and the superscript (′) denotes transposition, so that T_1' is an upper triangular matrix. The values of the entries a_1, b_1, and c_1 of T_1 can be readily obtained by applying the usual matrix multiplication rule on the right-hand side of the matrix equation (2.13). In fact, the system

$$(2.14) \quad \begin{cases} a_1^2 = m \\ a_1 c_1 = 1 \\ b_1^2 + c_1^2 = 0 \end{cases}$$

can be written, whose solution is

$$(2.15) \quad \begin{cases} a_1 = \pm\sqrt{m} \\ c_1 = 1/a_1 \\ b_1 = \pm i c_1, \end{cases}$$

where $i = \sqrt{-1}$.

Any of these four solutions leads to a *Cholesky LR decomposition* [17] of the symmetric matrix M_1.

On the other hand, it is known [7] that a lower triangular matrix and an upper triangular matrix do not commute, so that the reverse product $T_1' T_1$ leads to the symmetric matrix M_2 which is similar to but different from M_1. If we take $b_1 = +ic_1$ [cf. (2.15)], we have

$$(2.16) \quad M_2 = \frac{1}{m} \begin{bmatrix} m^2 + 1 & i \\ i & -1 \end{bmatrix},$$

while, if we take $b_1 = -ic_1$, the off diagonal entries of M_2 become negative.

In turn, M_2 can be decomposed in a similar way, thus getting

$$M_2 = T_2 T_2' = \begin{bmatrix} a_2 & 0 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} a_2 & c_2 \\ 0 & b_2 \end{bmatrix},$$

where

$$\begin{cases} a_2 = \pm\sqrt{(m^2 + 1)/m} \\ c_2 = 1/a_2 \\ b_2 = \pm i c_2. \end{cases}$$

The reverse product $T_2' T_2$ leads to a matrix M_3 with sign of the off diagonal entries depending on whether $b_2 = +ic_2$ or $-ic_2$ has been considered.

If we repeat such a procedure *ad infinitum*, we obtain the set $\{M_k\}_1^\infty$ of 2-by-2 symmetric matrices

(2.17) $$\boldsymbol{M}_k = \frac{1}{U_k}\begin{bmatrix} U_{k+1} & i^{k-1} \\ i^{k-1} & -U_{k-1} \end{bmatrix} \quad (k \geq 1).$$

Because of the ambiguity of signs that arises in the Cholesky factorization, (2.17) is not the only possible result of k applications of the LR algorithm to $\boldsymbol{M}$. However, the only other possible result differs from that shown in (2.17) only in the sign of the off diagonal entries. From here on, we will consider only the sequence $\{\boldsymbol{M}_k\}$ given by equation (2.17).

Since the matrices $\boldsymbol{M}_k$ are similar, their eigenvalues coincide. $\boldsymbol{M}_k$ tends to a diagonal matrix containing these eigenvalues (namely, α_m and β_m) as k tends to infinity.

To establish the general validity of (2.17), consider the Cholesky decomposition

$$\boldsymbol{M}_k = \frac{1}{U_k U_{k+1}}\begin{bmatrix} U_{k+1} & 0 \\ i^{k-1} & iU_k \end{bmatrix}\begin{bmatrix} U_{k+1} & i^{k-1} \\ 0 & iU_k \end{bmatrix}$$

where *Simson's formula*

(2.18) $$U_{k+1}U_{k-1} - U_k^2 = (-1)^k$$

has been invoked. Simson's formula may be quite readily established by using the Binet form (2.2) for U_n and the properties (2.4) of α_m and β_m. On the other hand, from (2.11) and (2.10), it is seen that

$$U_{k+1}U_{k-1} - U_k^2 = \det(\boldsymbol{M}^k) = (\det \boldsymbol{M})^k = (-1)^k.$$

Reversing the order of multiplication, we get

$$\frac{1}{U_k U_{k+1}}\begin{bmatrix} U_{k+1} & i^{k-1} \\ 0 & iU_k \end{bmatrix}\begin{bmatrix} U_{k+1} & 0 \\ i^{k-1} & iU_k \end{bmatrix} \underset{[\text{by } (2.18)]}{=} \frac{1}{U_{k+1}}\begin{bmatrix} U_{k+2} & i^k \\ i^k & -U_k \end{bmatrix} \underset{[\text{by } (2.17)]}{=} \boldsymbol{M}_{k+1}.$$

Thus, if the matrix for $\boldsymbol{M}_k$ is valid, then so is the matrix for $\boldsymbol{M}_{k+1}$.

For convenience, $\boldsymbol{M}_k$ may be called the *Cholesky algorithm matrix of Fibonacci type* of order k.

Furthermore, if we apply the Cholesky algorithm to $\boldsymbol{M}^n$ [see (2.11)] rather than to $\boldsymbol{M}$, we obtain

(2.19) $$(\boldsymbol{M}^n)_k = \frac{1}{U_k}\begin{bmatrix} U_{k+n} & i^{k-1}U_n \\ i^{k-1}U_n & (-1)^n U_{k-n} \end{bmatrix} = \frac{1}{U_k}\begin{bmatrix} U_{n+k} & i^{k-1}U_n \\ i^{k-1}U_n & (-1)^{k-1}U_{n-k} \end{bmatrix}.$$

Observe that

(2.20) $$(\boldsymbol{M}^n)_k = (\boldsymbol{M}_k)^n = \boldsymbol{M}_k^n.$$

Validation of this statement may be achieved through an inductive argument. Assume (2.20) is true for some value of n, say N. Thus,

(2.21) $$(\boldsymbol{M}_k)^N = (\boldsymbol{M}^N)_k.$$

Then,

$$(\boldsymbol{M}_k)^{N+1} = \boldsymbol{M}_k(\boldsymbol{M}_k)^N \underset{[\text{by } (2.21)]}{=} \boldsymbol{M}_k(\boldsymbol{M}^N)_k = (\boldsymbol{M}^{N+1})_k$$

after a good deal of calculation, so that if (2.20) is true for N, it is also true for $N+1$. In the calculations, it is necessary to derive certain identities among the U_n by using (2.2) and (2.4).

2.3 Functions of the Matrix xM_k^n

From the theory of functions of matrices [7], it is known that if f is a function defined on the spectrum of a 2-by-2 matrix $A = [a_{ij}]$ with distinct eigenvalues λ_1 and λ_2, then

$$f(A) = X = [x_{ij}] = c_0 I + c_1 A, \tag{2.22}$$

where I is the 2-by-2 identity matrix and the coefficients c_0 and c_1 are given by the solution of the system

$$\begin{cases} c_0 + c_1\lambda_1 = f(\lambda_1) \\ c_0 + c_1\lambda_2 = f(\lambda_2). \end{cases} \tag{2.23}$$

Solving (2.23) and using (2.22), after some manipulations we obtain

$$x_{11} = [(a_{11} - \lambda_1)f(\lambda_2) - (a_{11} - \lambda_2)f(\lambda_1)]/(\lambda_2 - \lambda_1) \tag{2.24}$$

$$x_{12} = a_{12}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2 - \lambda_1) \tag{2.25}$$

$$x_{21} = a_{21}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2 - \lambda_1) \tag{2.26}$$

$$x_{22} = [(a_{22} - \lambda_1)f(\lambda_2) - (a_{22} - \lambda_2)f(\lambda_1)]/(\lambda_2 - \lambda_1). \tag{2.27}$$

For x an arbitrary quantity, let us consider the matrix xM_k^n having eigenvalues

$$\begin{cases} \lambda_1 = x\alpha_m^n \\ \lambda_2 = x\beta_m^n \end{cases} \tag{2.28}$$

and let us find closed form expressions for the entries y_{ij} of

$$Y = [y_{ij}] = f(xM_k^n).$$

by (2.24)-(2.27), after some tedious manipulations involving the use of certain identities easily derivable from (2.2) and (2.3), we get

$$y_{11} = [\alpha_m^k f(x\alpha_m^n) - \beta_m^k f(x\beta_m^n)]/(\Delta_m U_k) \tag{2.29}$$

$$y_{12} = y_{21} = i^{k-1}[f(x\alpha_m^n) - f(x\beta_m^n)]/(\Delta_m U_k) \tag{2.30}$$

$$y_{22} = [\alpha_m^k f(x\beta_m^n) - \beta_m^k f(x\alpha_m^n)]/(\Delta_m U_k). \tag{2.31}$$

As an illustrative example, let f be the inverse function. Then, from (2.29)-(2.31) we obtain

$$(xM_k^n)^{-1} = \frac{(-1)^n}{xU_k}\begin{bmatrix} (-1)^{k-1}U_{n-k} & -i^{k-1}U_n \\ -i^{k-1}U_n & U_{n+k} \end{bmatrix} \tag{2.32}$$

$$= \frac{1}{x}M_k^{-n} \quad \text{[using (2.19) and (2.6)].} \tag{2.33}$$

3. Some Applications of the Matrix M_k

In this and later sections some identities involving U_n and V_n are worked out as illustrations of the use of our Cholesky algorithm matrix of Fibonacci type M_k.

Example 1: From (2.19) we can write

$$M_k^k = \frac{1}{U_k}\begin{bmatrix} U_{2k} & i^{k-1}U_k \\ i^{k-1}U_k & 0 \end{bmatrix} = \begin{bmatrix} V_k & i^{k-1} \\ i^{k-1} & 0 \end{bmatrix} = R_k = [r_{ij}] \; (= [r_{ij}^{(1)}]), \tag{3.1}$$

whence

(3.2) $\quad M_k^{nk} = R_k^n = [r_{ij}^{(n)}].$

Thus, $r_{11} = V_k$, $r_{12} = r_{21} = i^{k-1}$, $r_{22} = 0$. Take $r_{11}^{(0)} = 1$. By induction on n, with the aid of Pascal's formula for binomial coefficients, it can be proved that

$$(3.3)\qquad \begin{cases} r_{11}^{(n)} = \displaystyle\sum_{j=0}^{[n/2]} (-1)^{j(k-1)}\binom{n-j}{j}V_k^{n-2j} \\ r_{12}^{(n)} = r_{21}^{(n)} = i^{k-1}r_{11}^{(n-1)} \\ r_{22}^{(n)} = (-1)^{k-1}r_{11}^{(n-2)} \qquad (n \ge 2). \end{cases}$$

On the other hand, the matrix M_k^{nk} can be expressed also [cf. (2.19)] as

$$(3.4)\qquad M_k^{nk} = \frac{1}{U_k}\begin{bmatrix} U_{k(n+1)} & i^{k-1}U_{nk} \\ i^{k-1}U_{nk} & (-1)^{k-1}U_{k(n-1)} \end{bmatrix}.$$

Equating the upper left entries on the right-hand sides of (3.2) and (3.4), by (3.3) we obtain the identity

$$(3.5)\qquad U_{k(n+1)}/U_k = \sum_{j=0}^{[n/2]} (-1)^{j(k-1)}\binom{n-j}{j}V_k^{n-2j},$$

i.e., $U_k \mid U_{k(n+1)}$, as we would expect.

Furthermore, from (3.1) we can write

$$(3.6)\qquad [(-i)^{k-1}M_k^k]^n = \begin{bmatrix} (-i)^{k-1}V_k & 1 \\ 1 & 0 \end{bmatrix}^n = Z_k^n = [z_{ij}^{(n)}],$$

where $Z_k = [z_{ij}]$ is an *extended* M *matrix* depending on the complex parameter

(3.7) $\quad z = (-i)^{k-1}V_k(m).$

From (2.11) we have

(3.8) $\quad z_{12}^{(n)} = U_n(z),$

and by equating $z_{12}^{(n)}$ and the upper right entry of $[(-i)^{k-1}M_k^k]^n$ obtained by (3.4) we can write

$$(3.9)\qquad \begin{aligned}(-i)^{n(k-1)}i^{k-1}U_{nk}(m)/U_k(m) &= (-1)^{n(k-1)}i^{(n+1)(k-1)}U_{nk}(m)/U_k(m) \\ &= U_n(z).\end{aligned}$$

From (2.5) and (3.7) it can be verified that

$$(3.10)\qquad U_n(z) = \begin{cases} U_n(V_k(m)) & (k \text{ odd, } n \text{ odd}) \\ (-1)^{(k-1)/2}U_n(V_k(m)) & (k \text{ odd, } n \text{ even}). \end{cases}$$

Therefore, from (3.9) and (3.10) we obtain the noteworthy identity

(3.11) $\quad U_{nk}(m)/U_k(m) = U_n(V_k(m)) \qquad (k \text{ odd})$

which connects numbers defined by (2.1) having different generating parameters.

For instance,

$$(m^3+3m)^2+1 = m^6+6m^4+9m^2+1 = (m^8+7m^6+15m^4+10m^2+1)/(m^2+1)$$
$$= U_3(V_3(m)) = U_9(m)/U_3(m)$$

which simultaneously verifies (3.5) and (3.11).

Example 2: Following [2], from (2.19) we can write either

$$(3.12)\quad M_k^r M_k^s = \frac{1}{U_k^2}\begin{bmatrix} U_{r+k}U_{s+k} - (-1)^k U_r U_s & i^{k-1}[U_{r+k}U_s - (-1)^k U_r U_{s-k}] \\ i^{k-1}[U_{s+k}U_r - (-1)^k U_s U_{r-k}] & U_{r-k}U_{s-k} - (-1)^k U_r U_s \end{bmatrix}$$

or

$$(3.13)\quad M_k^{r+s} = \frac{1}{U_k}\begin{bmatrix} U_{r+s+k} & i^{k-1}U_{r+s} \\ i^{k-1}U_{r+s} & (-1)^{k-1}U_{r+s-k} \end{bmatrix}.$$

By equating the upper right entries on the right-hand sides of (3.12) and (3.13) we obtain

$$(3.14)\quad U_k U_{r+s} = U_{r+k}U_s - (-1)^k U_r U_{s-k}$$
$$= U_{s+k}U_r - (-1)^k U_s U_{r-k} \text{ also.}$$

4. Evaluation of Some Finite Series

In this section the sums of certain finite series involving U_n and V_n are found on the basis of some properties of the Fibonacci-type Cholesky algorithm matrix M_k.

It is readily seen from (2.17) and (2.19), with the aid of (2.1), that

$$(4.1)\quad M_k^2 = mM_k + I,$$

whence

$$(4.2)\quad M_k^{-1} = M_k - mI.$$

Moreover, using the identity

$$(4.3)\quad V_n U_p - U_{n+p} = (-1)^{p-1}U_{n-p},$$

which can be easily proved using (2.2) and (2.7), we can verify that

$$(4.4)\quad (xM_k^n - I)^{-1} = \frac{xM_k^n - (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1}$$

where x is an arbitrary quantity subject by (2.28) to the restrictions

$$(4.5)\quad x \neq \begin{cases} 1/\alpha_m^n \\ 1/\beta_m^n. \end{cases}$$

A) From (4.1) we can write

$$(M_k^2 - I)^n = (mM_k)^n$$

and, therefore,

$$\sum_{j=0}^{n}(-1)^n\binom{n}{j}M_k^{2j} = m^n M_k^n,$$

whence, by (2.19), we obtain a set of identities which can be summarized by

(4.6) $$\sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} U_{2j+s} = m^n U_{n+s},$$

where n is a nonnegative integer and s an arbitrary integer. Replacing s by $s \pm 1$ in (4.6) and combining the results obtained, from (2.7) we have

(4.7) $$\sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} V_{2j+s} = m^n V_{n+s}.$$

Furthermore, following [13], from (4.1) we can write

(4.8) $$(mM_k + I)^n M_k^s = M_k^{2n+s}.$$

Equating appropriate entries on both sides of (4.8), with the aid of (2.19), we obtain

(4.9) $$\sum_{j=0}^{n} \binom{n}{j} m^j U_{j+s} = U_{2n+s},$$

whence, replacing s by $s \pm 1$ as earlier, we get

(4.10) $$\sum_{j=0}^{n} \binom{n}{j} m^j V_{j+s} = V_{2n+s}.$$

B) From (4.2) we can write

$$(M_k - mI)^n = (M_k^n)^{-1},$$

whence, by (2.19) and (2.32), after some manipulations, we obtain a set of identities which can be summarized by

(4.11) $$\sum_{j=0}^{n} (-1)^j m^{n-j}\binom{n}{j} U_{j+s} = (-1)^{s-1} U_{n-s}.$$

C) Finally, let us consider the identity

(4.12) $$(xA^n - I)\sum_{j=0}^{h} x^j A^{nj} = x^{h+1}A^{n(h+1)} - I,$$

which holds for any square matrix A. From (4.12) and (4.4) we can write, for the Cholesky algorithm matrix M_k of Fibonacci type,

(4.13) $$\begin{aligned}\sum_{j=0}^{h} x^j M_k^{nj} &= (xM_k^n - I)^{-1}(x^{h+1}M_k^{n(h+1)} - I)\\ &= \frac{xM_k^n - (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1}(x^{h+1}M_k^{n(h+1)} - I)\\ &= \frac{x^{h+2}M_k^{n(h+2)} - xM_k^n - x^{h+1}(V_n x - 1)M_k^{n(h+1)} + (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1}.\end{aligned}$$

After some manipulations involving the use of (4.3), from (4.13) and (2.19) we obtain a set of identities which can be summarized as

(4.14) $$\sum_{j=0}^{h} x^j U_{nj+s} = \frac{(-1)^{n-1}x^{h+2}U_{nh+s} + x^{h+1}U_{n(h+1)+s} - (-1)^s xU_{n-s} - U_s}{(-1)^{n-1}x^2 + V_n x - 1},$$

where n is a nonnegative integer and s is an arbitrary integer. Replacing s by $s \pm 1$ in (4.14), by (2.7) we can derive

(4.15) $$\sum_{j=0}^{h} x^j V_{nj+s} = \frac{(-1)^{n-1}x^{h+2}V_{nh+s} + x^{h+1}V_{n(h+1)+s} + (-1)^s xV_{n-s} - V_s}{(-1)^{n-1}x^2 + V_n x - 1}.$$

We point out that (4.14) and (4.15) obivously hold under the restrictions (4.5).

5. Evaluation of Some Infinite Series

In this section a method for finding the sums of certain infinite series involving U_n and V_n is shown which is based on the use of functions of the matrix xM_k^n (see Section 2.3).

Under certain restrictions, some sums can be worked out by using the results established in Section 4 above. For example, if

$$-1/\alpha_m^n < x < 1/\alpha_m^n, \tag{5.1}$$

we can take the limit of both sides of (4.14) and (4.15) as h tends to infinity thus getting

$$\sum_{j=0}^{\infty} x^j U_{nj+s} = \frac{(-1)^{s-1} x U_{n-s} - U_s}{(-1)^{n-1}x^2 + V_n x - 1}, \tag{5.2}$$

$$\sum_{j=0}^{\infty} x^j V_{nj+s} = \frac{(-1)^{s} x V_{n-s} - V_s}{(-1)^{n-1}x^2 + V_n x - 1}. \tag{5.3}$$

5.1 Use of Certain Functions of xM_k^n

Following [6], we consider the power series expansion of $\exp(xM_k^n)$ [7],

$$Y = \exp(xM_k^n) = \sum_{j=0}^{\infty} \frac{x^j M_k^{jn}}{j!} \tag{5.4}$$

and the closed form expressions of the entries y_{ij} of Y derivable from (2.29)-(2.31) by letting f be the exponential function. Equating y_{ij} and the corresponding entry of Y on the right-hand side of (5.4), from (2.19) we obtain the identities

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn+k}}{j!} = [\alpha_m^k \exp(x\alpha_m^n) - \beta_m^k \exp(x\beta_m^n)]/\Delta_m, \tag{5.5}$$

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn}}{j!} = [\exp(x\alpha_m^n) - \exp(x\beta_m^n)]/\Delta_m, \tag{5.6}$$

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn-k}}{j!} = (-1)^{k-1}[\alpha_m^k \exp(x\beta_m^n) - \beta_m^k \exp(x\alpha_m^n)]/\Delta_m, \tag{5.7}$$

which, by using the identity $(-1)^{k-1}\alpha_m^{-k} = -\beta_m^k$ [see (2.4)], can be summarized as

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn+s}}{j!} = [\alpha_m^s \exp(x\alpha_m^n) - \beta_m^s \exp(x\beta_m^n)]/\Delta_m, \tag{5.8}$$

where n is a nonnegative integer and s is an arbitrary integer.

From (5.8), (2.7), and (2.3) we can readily derive

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{x^j V_{jn+s}}{j!} &= [\alpha_m^{s-1}\exp(x\alpha_m^n)(1 + \alpha_m^2) - \beta_m^{s-1}\exp(x\beta_m^n)(1 + \beta_m^2)]/\Delta_m \\ &= \alpha_m^{s-1}\exp(x\alpha_m^n)(\Delta_m + m)/2 - \beta_m^{s-1}\exp(x\beta_m^n)(\Delta_m - m)/2 \\ &= \alpha_m^s \exp(x\alpha_m^n) + \beta_m^s \exp(x\beta_m^n).\end{aligned} \tag{5.9}$$

By considering power series expansions [1], [16], [7] of other functions of the matrix xM_k^n, the above presented technique allows us to evaluate a very large amount of infinite series involving numbers U_n and V_n. We confine ourselves to showing an example derived from the expansion of $\tan^{-1}y$ (see [1] and [7, p. 113].

Under the restriction

$$(5.10)\quad -1/\alpha_m^n \le x \le 1/\alpha_m^n,$$

we have

$$(5.11)\quad \sum_{j=1}^{\infty} \frac{(-1)^{j+1}x^{2j-1}U_{n(2j-1)+s}}{2j-1} = [\alpha_m^s \tan^{-1}(x\alpha_m^n) - \beta_m^s \tan^{-1}(x\beta_m^n)]/\Delta_m.$$

6. Conclusion

The identities established in this paper represent only a small sample of the possibilities available to us. We believe that the Cholesky decomposition matrix M_k is a useful tool for discovering many more identities. Further investigations into the properties of matrices of this type are warranted.

Acknowledgment

The authors wish to thank the referee for several suggestions that helped to improve the presentation of this paper.

References

1. M. Abramowitz & I. Stegun. *Handbook of Mathematical Functions*. New York: Dover, 1972.
2. S. L. Basin & V. E. Hoggatt, Jr. "A Primer on the Fibonacci Sequence: Part II." *Fibonacci Quarterly 1.2* (1963):47-52.
3. G. E. Bergum & V. E. Hoggatt, Jr. "An Application of the Characteristic of the Generalized Fibonacci Sequence." *Fibonacci Quarterly 15.3* (1977): 215-20.
4. M. Bicknell. "A Primer on the Pell Sequence and Related Sequences." *Fibonacci Quarterly 13.4* (1975):345-49.
5. J. H. Clarke & A. G. Shannon. "Some Generalized Lucas Sequences." *Fibonacci Quarterly 23.2* (1985):120-25.
6. P. Filipponi & A. F. Horadam. "A Matrix Approach to Certain Identities." *Fibonacci Quarterly 26.2* (1988):115-26.
7. F. R. Gantmacher. *The Theory of Matrices*. New York: Chelsea, 1960.
8. V. E. Hoggatt, Jr., & M. Bicknell-Johnson. "Generalized Lucas Sequences." *Fibonacci Quarterly 15.2* (1977):131-39.
9. A. F. Horadam. "A Generalized Fibonacci Sequence." *Amer. Math. Monthly 68.5* (1961):455-59.
10. A. F. Horadam. "Generating Functions for Powers of Certain Generalized Sequences of Numbers." *Duke Math. J. 32* (1965):437-59.
11. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *Fibonacci Quarterly 23.1* (1985):7-20.
12. E. Lucas. *Théorie des nombres*. Paris: Blanchard, 1961.
13. Bro. J. M. Mahon & A. F. Horadam. "Matrix and Other Summation Techniques for Pell Polynomials." *Fibonacci Quarterly 24.4* (1986):290-308.
14. S. Pethe & A. F. Horadam. "Generalized Gaussian Fibonacci Numbers." *Bull. of the Australian Math. Soc. 33.1* (1986):37-48.
15. A. G. Shannon. "Fibonacci Analogs of the Classical Polynomials." *Math. Magazine 48* (1975):123-30.
16. M. R. Siegel. *Manuale di Matematica*. Milan: ETAS Libri, S.p.A., 1974.
17. J. H. Wilkinson. *The Algebraic Eigenvalue Problem*. Oxford: Clarendon Press, 1965.

*****

A NEW CHARACTERIZATION OF THE FIBONACCI-FREE PARTITION

Timothy Chow
226 Shady Avenue, Apt. 308, Pittsburgh, PA 15206
(Submitted July 1989)

1. Introduction

There exists a unique partition of the positive integers into two disjoint sets A and B such that no two distinct integers from the same set sum to a Fibonacci number (see [1], [2], and [3]). For the purposes of this paper, we shall refer to this partition as the "Fibonacci-free partition." The first few numbers in the sets A and B are:

$$A = \{1, 3, 6, 8, 9, 11, \ldots\},$$

$$B = \{2, 4, 5, 7, 10, 12, \ldots\}.$$

In this paper, we shall prove that the sets A and B can be written in the form

$$A = \{[n\phi]\} - \{[m\phi] | \mathrm{fp}(m\phi) > \phi/2\},$$

$$B = \{[n\phi^2]\} \cup \{[m\phi] | \mathrm{fp}(m\phi) > \phi/2\},$$

where m is a positive integer, $\phi = (1 + \sqrt{5})/2$, n ranges over all the positive integers, and $\mathrm{fp}(x)$ denotes the fractional part of x. (We depart from the standard notation where (x) denotes the fractional part of x to avoid confusion in complicated expressions. See Lemma 4.4 below, for instance.) We shall also prove the following conjecture of Chris Long [4]: the set A satisfies the equality $A = \{[n\phi]\} - A'$, where $A' = \{[s\phi^3] | s \in A\}$.

We remark that in [3] it is shown that the Fibonacci-free partition cannot be expressed in the form $A = \{[na]\}$, $B = \{[nb]\}$ for any a and b, but that the above result shows that such a representation is "almost" possible.

A note on notation: in this paper, unless otherwise specified, F_n denotes the n^{th} Fibonacci number, $\lceil x \rceil$ denotes the least integer $\geq x$, and $\mathrm{dist}(x)$ is the distance of x from the nearest integer, i.e.,

$$\mathrm{dist}(x) = \min\{x - [x], \lceil x \rceil - x\}.$$

2. An Important Lemma

Definition: A positive integer a is said to have the *distance property* if

$$\mathrm{dist}(a\phi) > \mathrm{dist}(\phi F)$$

for all Fibonacci numbers $F > a$.

Lemma 2.1: All positive integers have the distance property.

This crucial lemma is the key to the proof of Theorem 3.4 below. It will be used in the proofs of all three lemmas in the next section.

Proof: We proceed by induction. Note first of all that 1 has the distance property. So now suppose that there exists $F_n \geq 2$ such that all integers $\leq F_n$ have the distance property. We have to show that all integers $\leq F_{n+1}$ also have the distance property. It is well known that Fibonacci numbers have the distance property. So we need only check that if k is any integer such that $F_n < k < F_{n+1}$, then $\mathrm{dist}(k\phi) > \mathrm{dist}(\phi F_{n+1})$.

The case $k = F_{n+1}$ is clear; therefore, we can safely assume $k < F_{n+1}$. Let $m = k - F_n$. Then m is a positive integer $< F_{n-1}$ so that $\operatorname{dist}(m\phi) > \operatorname{dist}(\phi F_{n-1})$, by the induction hypothesis. There are two cases to consider:

(1) $\phi F_{n+1} > F_{n+2}$. Then $\operatorname{dist}(\phi F_{n+1}) = \operatorname{fp}(\phi F_{n+1})$. So to show that $\operatorname{dist}(\phi F_{n+1}) < \operatorname{dist}(k\phi)$, we just need to show two things:

$$\operatorname{fp}(\phi F_{n+1}) < \operatorname{fp}(k\phi) \quad \text{and} \quad \operatorname{fp}(\phi F_{n+1}) < 1 - \operatorname{fp}(k\phi).$$

First of all, note that $\operatorname{fp}(m\phi) > \operatorname{fp}(\phi F_{n-1})$, since

$$\operatorname{fp}(m\phi) \geq \operatorname{dist}(m\phi) > \operatorname{dist}(\phi F_{n-1}) = \operatorname{fp}(\phi F_{n-1}).$$

Now, $\operatorname{dist}(\phi F_n) < \operatorname{dist}(\phi F_{n-1})$, and since $\operatorname{dist}(\phi F_n) = F_{n+1} - \phi F_n$, this means that $\operatorname{fp}(\phi F_{n-1}) - (F_{n+1} - \phi F_n) > 0$. Thus,

$$\begin{aligned} \operatorname{fp}(\phi F_n) + \operatorname{fp}(\phi F_{n-1}) &= \phi F_n - (F_{n+1} - 1) + \operatorname{fp}(\phi F_{n-1}) \\ &= 1 + \operatorname{fp}(\phi F_{n-1}) - (F_{n+1} - \phi F_n) \\ &> 1. \end{aligned}$$

But $\operatorname{fp}(m\phi) > \operatorname{fp}(\phi F_{n-1})$, so $\operatorname{fp}(\phi F_n) + \operatorname{fp}(m\phi) > 1$. By the definition of m above, $k\phi = \phi F_n + m\phi$. It follows that

$$\begin{aligned} \operatorname{fp}(k\phi) &= \operatorname{fp}(\phi F_n) + \operatorname{fp}(m\phi) - 1 \\ &> \operatorname{fp}(\phi F_n) + \operatorname{fp}(\phi F_{n-1}) - 1 \\ &= \operatorname{fp}(\phi F_{n+1}). \end{aligned}$$

It remains to be shown that $\operatorname{fp}(\phi F_{n+1}) < 1 - \operatorname{fp}(k\phi)$. We have

$$\begin{aligned} \operatorname{fp}(k\phi) &= \operatorname{fp}(\phi F_n) + \operatorname{fp}(m\phi) - 1 \\ &< \operatorname{fp}(\phi F_n) \\ &= 1 - \operatorname{dist}(\phi F_n) \\ &< 1 - \operatorname{dist}(\phi F_{n+1}) \\ &= 1 - \operatorname{fp}(\phi F_{n+1}), \end{aligned}$$

i.e., $1 - \operatorname{fp}(k\phi) > \operatorname{fp}(\phi F_{n+1})$, so we are done.

(2) $\phi F_{n+1} < F_{n+2}$. In this case, $\operatorname{dist}(\phi F_{n+1}) = 1 - \operatorname{fp}(\phi F_{n+1})$; thus, we need to show that $\operatorname{fp}(\phi F_{n+1}) > \operatorname{fp}(k\phi)$ and that $\operatorname{fp}(\phi F_{n+1}) > 1 - \operatorname{fp}(k\phi)$. The arguments are almost the same as in case (1), so we will not repeat them here. Again we find that $\operatorname{dist}(\phi F_{n+1}) < \operatorname{dist}(k\phi)$.

This completes the proof.

3. The New Characterization

Lemma 3.1: $[m\phi^2] + [n\phi^2]$ is never a Fibonacci number (m, n positive integers).

Proof: Suppose $[m\phi^2] + [n\phi^2] = F_i$ for some i. Since m and n are positive, $F_i \geq 5$ (just let $m = n = 1$). Now,

$$\begin{aligned} \left[\frac{F_i}{\phi^2}\right] = \left[\frac{F_i}{\phi}(\phi - 1)\right] = \left[F_i - \frac{F_i}{\phi}\right] &= F_i - [F_i/\phi] - 1 = F_i - 1 - [\phi F_i - F_i] \\ &= F_i - 1 - ([\phi F_i] - F_i), \end{aligned}$$

which equals either $F_i - 1 - F_{i-1} = F_{i-2} - 1$ or $F_i - 1 - (F_{i-1} - 1) = F_{i-2}$. But we also have

$$\left[\frac{F_i}{\phi^2}\right] = \left[\frac{[m(1+\phi)] + [n(1+\phi)]}{(m+n)(1+\phi)} \cdot \frac{(m+n)(1+\phi)}{\phi^2}\right] =$$

$$= \left[\frac{m + n + [m\phi] + [n\phi]}{m + n + m\phi + n\phi} \cdot (m + n)\right].$$

To evaluate this expression, note that the denominator of the big fraction here exceeds the numerator by no more than 2, and the denominator is more than $2(m + n)$, so the fraction is greater than

$$\frac{2(m + n) - 2}{2(m + n)} = \frac{m + n - 1}{m + n}.$$

Hence, multiplying the fraction by $m + n$ will give a number between $m + n - 1$ and $m + n$, so the entire expression (after flooring) evaluates to $m + n - 1$.

Equating the two expressions for $[F_i/\phi^2]$, we see that $m + n - 1$ equals either $F_{i-2} - 1$ or F_{i-2}. In other words, there are two cases to be considered:

$$m + n = F_{i-2} \quad \text{and} \quad m + n = F_{i-2} + 1.$$

Suppose first that $m + n = F_{i-2}$. There are two subcases:

(1) $(m + n)\phi < F_{i-1}$. Then $[m\phi] + [n\phi]$ must equal either $F_{i-1} - 1$ or $F_{i-1} - 2$. But if $[m\phi] + [n\phi]$ were equal to $F_{i-1} - 2$, then $\mathrm{fp}(m\phi) + \mathrm{fp}(n\phi)$ would have to equal $1 + \mathrm{fp}(\phi F_{i-2})$, so that either $\mathrm{fp}(m\phi)$ or $\mathrm{fp}(n\phi)$ would have to be greater than $\mathrm{fp}(\phi F_{i-2})$. But $\mathrm{fp}(\phi F_{i-2}) = 1 - \mathrm{dist}(\phi F_{i-2})$, so this would mean that either m or n would not have the distance property, contradicting Lemma 2.1. Hence, $[m\phi] + [n\phi] = F_{i-1} - 1$.

(2) $(m + n)\phi > F_{i-1}$. Then $[m\phi] + [n\phi]$ must equal either $F_{i-1} - 1$ or F_{i-1}. But if $[m\phi] + [n\phi]$ were equal to F_{i-1}, we would have $\mathrm{fp}(m\phi) + \mathrm{fp}(n\phi) = \mathrm{fp}(\phi F_{i-2})$, which would imply that either $\mathrm{fp}(m\phi)$ or $\mathrm{fp}(n\phi)$ was less than $\mathrm{fp}(\phi F_{i-2})$. But $\mathrm{fp}(\phi F_{i-2}) = \mathrm{dist}(\phi F_{i-2})$, so this would mean that either m or n would not have the distance property, contradicting Lemma 2.1. So again we have $[m\phi] + [n\phi] = F_{i-1} - 1$.

It follows that

$$[m\phi^2] + [n\phi^2] = m + n + [m\phi] + [n\phi] = F_{i-2} + F_{i-1} - 1 = F_i - 1,$$

contrary to the assumption that $[m\phi^2] + [n\phi^2] = F_i$.

Suppose now that $m + n = F_{i-2} + 1$. Then $[m\phi] + [n\phi]$ is either $F_{i-1} + 1$ or F_{i-1}, and

$$[m\phi^2] + [n\phi^2] = m + n + [m\phi] + [n\phi] = F_{i-2} + 1 + F_{i-1} + r = F_i + 1 + r,$$

where $r = 0$ or 1, again contrary to the assumption that $[m\phi^2] + [n\phi^2] = F_i$. This establishes Lemma 3.1.

Lemma 3.2: If $[m\phi] + [n\phi]$ is a Fibonacci number (where m and n are distinct positive integers), then either $\mathrm{fp}(m\phi) > \phi/2$ or $\mathrm{fp}(n\phi) > \phi/2$, but not both.

Proof: Suppose $[m\phi] + [n\phi] = F_k$ for some k. Now $[(m + n)\phi]$ exceeds $[m\phi] + [n\phi]$ by at most one, so $[(m + n)\phi]$ is either F_k or $F_k + 1$. Let us write $[(m + n)\phi]$ as $F_k + r$, where $r = 0$ or 1. Then

$$(m + n)\phi - \mathrm{fp}((m + n)\phi) = [(m + n)\phi] = F_k + r,$$

so

$$\begin{aligned} m + n &= \frac{\mathrm{fp}((m + n)\phi)}{\phi} + \frac{F_k + r}{\phi} \\ &= \frac{\mathrm{fp}((m + n)\phi)}{\phi} + \frac{r}{\phi} + \phi F_k - F_{k+1} + F_{k+1} - F_k \\ &= \frac{\mathrm{fp}((m + n)\phi)}{\phi} + \frac{r}{\phi} \pm \mathrm{dist}(\phi F_k) + F_{k-1}. \end{aligned}$$

Let x denote the sum of the first three terms in this expression. Since $\operatorname{dist}(\phi F_k) \le \frac{1}{2}$, we have $x > -1$. Moreover,

$$\frac{\operatorname{fp}((m+n)\phi)}{\phi} + \frac{r}{\phi} < \frac{2}{\phi},$$

so $x < (2/\phi) + \frac{1}{2} < 2$. It follows that $m + n$ equals either F_{k-1} or $F_{k-1} + 1$.

Suppose $m + n$ is F_{k-1}. Then $[(m + n)\phi] = [\phi F_{k-1}]$, which cannot equal $F_k + 1$ and, therefore, must equal F_k. Thus, $\operatorname{dist}(\phi F_{k-1}) = \operatorname{fp}(\phi F_{k-1})$. Then, by Lemma 2.1, $\operatorname{fp}(m\phi)$ and $\operatorname{fp}(n\phi)$ must both be greater than $\operatorname{fp}(\phi F_{k-1})$. But

$$\begin{aligned}[(m+n)\phi] - [m\phi] - [n\phi] &= \operatorname{fp}(m\phi) + \operatorname{fp}(n\phi) - \operatorname{fp}((m+n)\phi)\\ &= \operatorname{fp}(m\phi) + \operatorname{fp}(n\phi) - \operatorname{fp}(\phi F_{k-1})\end{aligned}$$

must be an integer, so it must equal 1. In other words,

$$[m\phi] + [n\phi] = [(m+n)\phi] - 1 = F_k - 1,$$

but this is a contradiction.

So $m + n = F_{k-1} + 1$. Then $m\phi + n\phi = \phi F_{k-1} + \phi$. We split into two cases:

(1) $\phi F_{k-1} > F_k$. We have

$$\begin{aligned}F_k = [m\phi] + [n\phi] &= m\phi + n\phi - \operatorname{fp}(m\phi) - \operatorname{fp}(n\phi)\\ &= \phi F_{k-1} + \phi - \operatorname{fp}(m\phi) - \operatorname{fp}(n\phi)\\ \Rightarrow \quad \operatorname{fp}(m\phi) + \operatorname{fp}(n\phi) &= \phi F_{k-1} - F_k + \phi\\ &= \operatorname{dist}(\phi F_{k-1}) + \phi.\end{aligned}$$

So it is clearly impossible for both $\operatorname{fp}(m\phi)$ and $\operatorname{fp}(n\phi)$ to be less than $\phi/2$; we need to show that they cannot both be greater than $\phi/2$. Suppose $\operatorname{fp}(m\phi) = \phi/2 + \varepsilon_1$ and $\operatorname{fp}(n\phi) = \phi/2 + \varepsilon_2$, where ε_1 and ε_2 are both positive and $\varepsilon_1 + \varepsilon_2 = \operatorname{dist}(\phi F_{k-1})$. Then $\operatorname{fp}(|m - n|\phi) = |\varepsilon_1 - \varepsilon_2| < \operatorname{dist}(\phi F_{k-1})$, but since m and n are distinct (and this is where distinctness is really crucial), $|m - n|$ is strictly positive, and this contradicts Lemma 2.1 (since $|m - n|$ is a positive integer less than F_{k-1}). Hence, $\operatorname{fp}(m\phi)$ and $\operatorname{fp}(n\phi)$ cannot both be greater than $\phi/2$.

(2) $\phi F_{k-1} < F_k$. The argument is similar, except that here

$$\operatorname{fp}(m\phi) + \operatorname{fp}(n\phi) = \phi - \operatorname{dist}(\phi F_{k-1}).$$

Then clearly we cannot have both $\operatorname{fp}(m\phi)$ and $\operatorname{fp}(n\phi)$ greater than $\phi/2$, and writing $\operatorname{fp}(m\phi) = \phi/2 - \varepsilon_1$ and $\operatorname{fp}(n\phi) = \phi/2 - \varepsilon_2$ leads to the same contradiction as before.

Lemma 3.3: If $\operatorname{fp}(m\phi) > \phi/2$, then $[m\phi] + [n\phi^2]$ is not a Fibonacci number. (m and n are positive integers but not necessarily distinct.)

Proof: We show that $F_k - [m\phi]$ can be written in the form $[n\phi]$ for some n if $F_k > [m\phi]$. There are two cases:

(1) $F_k = [\phi F_{k-1}] + 1$. If $F_{k-1} = m$, then $F_k - [m\phi] = 1$, which is of the form $[1 \cdot \phi]$. Otherwise, $F_{k-1} > m$. Moreover, by Lemma 2.1,

$$1 - \operatorname{fp}(m\phi) \ge \operatorname{dist}(m\phi) > \operatorname{dist}(\phi F_{k-1}) = 1 - \operatorname{fp}(\phi F_{k-1}),$$

i.e., $\operatorname{fp}(m\phi) < \operatorname{fp}(\phi F_{k-1})$. Let $d = (F_{k-1} - m)\phi$. Then

$$\operatorname{fp}(d) = \operatorname{fp}(\phi F_{k-1}) - \operatorname{fp}(m\phi) < 1 - \phi/2.$$

It follows that $[d + \phi] = [d] + 1$, and also that $[d] = [\phi F_{k-1}] - [m\phi]$. Thus,

$$F_k - [m\phi] = [\phi F_{k-1}] + 1 - [m\phi] = [d] + 1 = [d + \phi] = [(F_{k-1} - n + 1)\phi],$$

so we can just set $n = F_{k-1} - m + 1$.

(2) $F_k = [\phi F_{k-1}]$. Now the smallest value of m for which $\text{fp}(m\phi) > \phi/2$ is $m = 3$, so the smallest value of F_k for which this case can occur is $F_k = 8$. Since $\text{fp}(m\phi) = \text{dist}(m\phi)$, we have $\text{fp}(\phi F_{k-1}) < \text{fp}(5\phi) < 0.091$. So $\text{fp}((F_{k-1} - m)\phi) < 1 < \phi/2 + 0.091$; thus,

$$[(F_{k-1} - m + 1)\phi] = [(F_{k-1} - m)\phi] + 1.$$

Since $\text{fp}(\phi F_{k-1}) < \text{fp}(m\phi)$,

$$F_k - [m\phi] = [\phi F_{k-1}] - [m\phi] = [(F_{k-1} - m)\phi] + 1 = [(F_{k-1} - m + 1)\phi]$$

as we just proved, so we can just set $n = F_{k-1} - m + 1$, as before.

Now, since $1/\phi + 1/\phi^2 = 1$, we can apply Beatty's theorem, which states that $\{[na]\}$ and $\{[nb]\}$ form a partition of the positive integers if and only if a and b are irrational and $1/a + 1/b = 1$ (see [5], [6]). It follows that any number that can be written in the form $[n\phi]$ cannot be written in the form $[s\phi^2]$ for any s, so that Lemma 3.3 follows immediately.

Theorem 3.4: The two sets A and B of the Fibonacci-free partition can be written in the form

$$A = \{[n\phi]\} - \{[m\phi] \mid \text{fp}(m\phi) > \phi/2\},$$

$$B = \{[n\phi]\} \cup \{[m\phi] \mid \text{fp}(m\phi) > \phi/2\}.$$

Proof: First of all, we note that, by Beatty's theorem, A and B do indeed form a partition of the positive integers. From Lemmas 3.1-3.3, we see that this partition has the property that no two distinct integers from the same set sum to a Fibonacci number. The theorem then follows from the uniqueness of the Fibonacci-free partition.

4. Long's Conjecture

Lemma 4.1: If n is a positive integer such that $\text{fp}(n\phi) > \phi/2$, then there exists a positive integer k such that $n = [k\phi]$ and $\text{fp}(k\phi) < (\phi - 1)/2$.

Proof: First, note that

$$\text{fp}(n/\phi) = \text{fp}(n\phi - n) = \text{fp}(n\phi) > \phi/2.$$

Let $a = 1 - \text{fp}(n/\phi)$. Note that $a < 1 - \phi/2$. Then

$$\phi\lceil n/\phi\rceil = \phi(n/\phi) + a\phi = n + a\phi < n + (1 - \phi/2)\phi = n + (\phi - 1)/2.$$

Now set $k = \lceil n/\phi\rceil$. It is clear that k has the desired properties.

Lemma 4.2: If k is a positive integer such that $\text{fp}(k\phi) < (\phi - 1)/2$, then $\text{fp}([k\phi]\phi) > \phi/2$.

Proof:
$$\begin{aligned}\text{fp}([k\phi]\phi) &= \text{fp}(k\phi^2 - \phi\,\text{fp}(k\phi))\\ &= \text{fp}(k\phi + k - \phi\,\text{fp}(k\phi))\\ &= \text{fp}(k\phi - \phi\,\text{fp}(k\phi))\\ &= \text{fp}(k\phi - \text{fp}(k\phi) - (\phi - 1)\text{fp}(k\phi))\\ &= 1 - \text{fp}((\phi - 1)\text{fp}(k\phi))\\ &> 1 - (\phi - 1)(\phi - 1)/2\\ &= \phi/2.\end{aligned}$$

Lemma 4.3: If k is a positive integer such that $\text{fp}(k\phi) < (\phi - 1)/2$, then $[[k\phi]\phi] = [s\phi^3]$, where $s = [[k\phi](2 - \phi)]$.

Proof: By Lemma 4.2, $\text{fp}([k\phi]\phi) > \phi/2 = (2\phi + 1)/(2\phi + 2)$. Thus,

$$2(\phi + 1)\text{fp}([k\phi]\phi) > 2\phi + 1 \Rightarrow (1 + \phi^3)\text{fp}([k\phi]\phi) > \phi^3$$

$$\Rightarrow \text{fp}([k\phi]\phi) > \phi^3(1 - \text{fp}([k\phi]\phi)) = \phi^3\text{fp}([k\phi](2 - \phi))$$

$$\Rightarrow [[k\phi]\phi] = [[k\phi]\phi - \phi^3\text{fp}([k\phi](2 - \phi))].$$

Now

$$[k\phi]\phi = [k\phi](2 - \phi)\phi^3 = s\phi^3 + \phi^3\text{fp}([k\phi](2 - \phi)).$$

Subtracting $\phi^3\text{fp}([k\phi](2 - \phi))$ from both sides and then flooring both sides gives the required result.

Lemma 4.4: If n is a positive integer such that $\text{fp}(n\phi) > \phi/2$, there exists a positive integer m such that $[n\phi] = [[m\phi]\phi^3]$ and $\text{fp}(m\phi) < \phi/2$.

Proof: In view of Lemmas 4.1-4.3, we need to show that $[n(2 - \phi)]$ can be written in the form $[m\phi]$ with $\text{fp}(m\phi) < \phi/2$. Now $[n(2 - \phi)] = [n(2\phi - 3)\phi]$. Let $m = \lceil n(2\phi - 3)\rceil$. We claim that this is the desired m. For

$$\begin{aligned} m\phi &= n(2\phi - 3)\phi + \phi(1 - \text{fp}(n(2\phi - 3))) \\ &= n(2 - \phi) + \phi(1 - \text{fp}(n(2\phi - 3))). \end{aligned}$$

Now $\text{fp}(n(2\phi - 3)) = \text{fp}(2n\phi) > \text{fp}(\phi)$ so that $\phi(1 - \text{fp}(n(2\phi - 3))) < \phi - 1$. Furthermore, $\text{fp}(n(2 - \phi)) = 1 - \text{fp}(n\phi) < 1 - \phi/2$. Thus,

$$\begin{aligned} n(2 - \phi) + \phi(1 - \text{fp}(n(2\phi - 3))) &< [n(2 - \phi)] + 1 - \phi/2 + \phi - 1 \\ &= [n(2 - \phi)] + \phi/2. \end{aligned}$$

Hence, $[m\phi] = [n(2 - \phi)]$ and $\text{fp}(m\phi) < \phi/2$, as required.

Lemma 4.5: If n is a positive integer such that $\text{fp}(n\phi) < \phi/2$, there exists a positive integer m such that $[[n\phi]\phi^3] = [m\phi]$ and $\text{fp}(m\phi) > \phi/2$.

Proof: First, we note that

$$\text{fp}([n\phi]\phi) = 1 - \text{fp}((\phi - 1)\text{fp}(n\phi)) > 1 - (\phi - 1)\phi/2 = \frac{1}{2}.$$

(For a justification of the first equality in the above derivation, see the first five lines of the proof of Lemma 4.2 above.) Now

$$\text{fp}([n\phi]\phi^3) = \text{fp}([n\phi](2\phi + 1)) = \text{fp}(2[n\phi]\phi).$$

Since $\text{fp}([n\phi]\phi) > \frac{1}{2}$, it follows that $\text{fp}(2[n\phi]\phi) = 2\text{fp}([n\phi]\phi) - 1$. So we have $\text{fp}([n\phi]\phi^3) = 2\text{fp}([n\phi]\phi) - 1$. Thus,

$$(*) \qquad \begin{aligned} \text{fp}([n\phi]\phi^3) + \phi(1 - \text{fp}([n\phi]\phi)) &= 2\text{fp}([n\phi]\phi) - 1 + \phi - \phi\text{fp}([n\phi]\phi) \\ &= \phi - 1 + (2 - \phi)\text{fp}([n\phi]\phi) \\ &< \phi - 1 + 2 - \phi = 1. \end{aligned}$$

Now let $m = \lceil [n\phi]\phi^2\rceil$. We claim that this is the desired m. For

$$\begin{aligned} [m\phi] &= [\phi([n\phi]\phi^2 + 1 - \text{fp}([n\phi]\phi^2))] \\ &= [[n\phi]\phi^3 + \phi - \phi\text{fp}([n\phi]\phi)] = [[n\phi]\phi^3]. \end{aligned}$$

The last equality follows from equation (*) above.

It remains to show that $\text{fp}(m\phi) > \phi/2$. We have

$$[m\phi] = [[n\phi]\phi^3] = [[n\phi]\phi^2\phi] = [m\phi - \phi + \phi\text{fp}([n\phi]\phi^2)].$$

But

$$\begin{aligned} -\phi + \phi\text{fp}([n\phi]\phi^2) &= -\phi + \phi\text{fp}([n\phi]\phi) \\ &= \phi(\text{fp}([n\phi]\phi) - 1) > \phi\left(\frac{1}{2} - 1\right) = -\phi/2. \end{aligned}$$

It follows immediately that $\text{fp}(m\phi) > \phi/2$, as required.

Theorem 4.6: The set A of the Fibonacci-free partition (defined in Theorem 3.4 above) satisfies the equality $A = \{[n\phi]\} - A'$, where $A' = \{[s\phi^3] \mid s \in A\}$.

Proof: From Lemma 4.4, we see that $\{[n\phi]\} - A' \subset A$, and from Lemma 4.5, we see that $A \subset \{[n\phi]\} - A'$.

References

1. V. E. Hoggatt, Jr. "Additive Partitions of the Positive Integers." *Fibonacci Quarterly* *18.3* (1980):220-25.
2. V. E. Hoggatt, Jr., et al. "Additive Partitions of the Positive Integers and Generalized Fibonacci Representations." *Fibonacci Quarterly 22.1* (1984): 2-21.
3. K. Alladi, P. Erdös, & V. E. Hoggatt, Jr. "On Additive Partitions of Integers." *Discrete Math. 22.3* (1978):201-11.
4. C. Long. Rutgers University, private correspondence.
5. S. Beatty. "Problem 3173." *Amer. Math. Monthly 33.3* (1926):159; "Solutions." *Ibid. 34.3* (1927):159.
6. A. Fraenkel. "Complementary Systems of Integers." *Amer. Math. Monthly 84.2* (1977):114-15.

*****

Elementary Problems Editor Retires

After 27 years of dedicated service as the Editor of the Elementary Problems Section of *The Fibonacci Quarterly,* Dr. Abe Hillman has made the decision to retire. His replacement will be

Dr. Stanley Rabinowitz
12 Vine Brook Road
Westford, MA 01886

As of now, all problems for the Elementary Problems Section should be sent to Dr. Rabinowitz at the above address.

As the Editor of the journal, I would like to take this opportunity to thank Dr. Hillman for his cooperation and a job well done. Here's wishing you the best of luck in your retirement years!

Dr. Gerald E. Bergum

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-688 *Proposed by Russell Euler, Northwest Missouri State U., Maryville, MO*

Find the number of increasing sequences of integers such that 1 is the first term, n is the last term, and the difference between successive terms is 1 or 2. [For example, if $n = 8$, then one such sequence is 1, 2, 3, 5, 6, 8 and another is 1, 3, 4, 6, 8.]

B-689 *Proposed by Philip L. Mana, Albuquerque, NM*

Show that $F_n^2 - 1$ is a sum of Fibonacci numbers with distinct positive even subscripts for all integers $n \geq 3$.

B-690 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_k = \alpha^{10k+1} + \alpha^{10k+2} + \alpha^{10k+3} + \cdots + \alpha^{10k+10}$, where $\alpha = (1 + \sqrt{5})/2$. Find positive integers b and c such that $S_k/\alpha^{10k+b} = c$ for all nonnegative integers k.

B-691 *Proposed by Heiko Harborth, Technische Universität Braunschweig, West Germany*

Herta T. Freitag asked whether a golden rectangle can be inscribed into a larger golden rectangle (all four vertices of the smaller are points on the sides of the larger one). An answer follows from the solution of the generalized problem: Which rectangles can be inscribed into larger similar rectangles?

B-692 *Proposed by Gregory Wulczyn, Lewisburg, PA*

Let $G(a, b, c) = -4 + L_{2a}^2 + L_{2b}^2 + L_{2c}^2 + L_{2a}L_{2b}L_{2c}$. Prove or disprove that each of F_{a+b+c}, F_{b+c-a}, F_{c+a-b}, and F_{a+b-c} is an integral divisor of $G(a, b, c)$ for all odd positive integers a, b, and c.

B-693 *Proposed by Daniel C. Fielder & Cecil O. Alford, Georgia Tech, Atlanta, GA*

Let A consist of all pairs $\{x, y\}$ chosen from $\{1, 2, \ldots, 2n\}$, B consist of all pairs from $\{1, 2, \ldots, n\}$, and C of all pairs from $\{n + 1, n + 2, \ldots, 2n\}$. Let S consist of all sets $T = \{P_1, P_2, \ldots, P_k\}$ with the P_i (distinct) pairs in A. How many of the T in S satisfy at least one the the conditions:

(i) $P_i \cap P_j \neq \emptyset$ for some i and j, with $i \neq j$,

(ii) $P_i \in B$ for some i, or

(iii) $P_i \in C$ for some i?

SOLUTIONS

Limit of Nested Square Roots

B-664 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

Let $a_0 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for n in $\{0, 1, \ldots\}$. Show that

$$\lim_{n\to\infty} a_n = \sum_{i=0}^{\infty}\left[\sum_{j=0}^{i}\binom{i}{j}\right]^{-1}.$$

Solution by Russell Jay Hendel, Dowling College, Oakdale, NY

First

$$\sum_{0\le j\le i}\binom{i}{j} = 2^i \quad \text{and} \quad \sum_{0\le i\le\infty} 2^{-i} = 2.$$

Next,

$$a_{n+1}^2 > a_n^2 \text{ iff } a_n + 2 > a_{n-1} + 2,$$

and similarly,

$$a_{n+1} < 2 \text{ iff } a_{n+1}^2 = a_n + 2 < 4,$$

implying $a_n < 2$. Thus, an induction shows that the sequence $\{a_n\}$ is monotonely increasing and bounded; hence, the limit, L, exists. Squaring the defining recursion and taking limits we find $L^2 = L + 2$ or $L = 2$.

This solves the problem, since both sides of the problem equation have a value of 2.

Also solved by R. André-Jeannin, Paul S. Bruckman, Pat Costello, Piero Filipponi & Adina DiPorto, C. Georghiou, Norbert Jensen & Uwe Pettke, Hans Kappus, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, and the proposer.

Unique Real Solutions of Cubics

B-665 *Proposed by Christopher C. Street, Morris Plains, NJ*

Show that $AB = 9$, where

$$A = (19 + 3\sqrt{33})^{1/3} + (19 - 3\sqrt{33})^{1/3} + 1,$$
$$B = (17 + 3\sqrt{33})^{1/3} + (17 - 3\sqrt{33})^{1/3} - 1.$$

Solution by Hans Kappus, Rodersdorf, Switzerland

Put

$$(19 + 3\sqrt{33})^{1/3} = a, \quad (19 - 3\sqrt{33})^{1/3} = a'.$$

Then $a^3 + (a')^3 = 38$, $aa' = 4$, and we have

$$(A - 1)^3 = (a + a')^3 = 3aa'(a + a') + a^3 + (a')^3 = 12(A - 1) + 38.$$

Therefore, $f(A) = 0$, where

$$f(x) = x^3 - 3x^2 - 9x - 27.$$

In the same way, we find that $g(B) = 0$, where

$$g(x) = x^3 + 3x^2 + 9x - 27.$$

It is easily seen that the polynomials f and g have exactly one real zero each, which must therefore be A and B, respectively. On the other hand,

$$x^3 f(9/x) = -27g(x);$$

hence, $f(9/B) = 0$, and therefore $A = 9/B$.

Also solved by Paul S. Bruckman, C. Georghiou, Norbert Jensen & Uwe Pettke, L. Kuipers, Y. H. Harris Kwong, Carl Libis, and the proposer.

Diagonal p of Pascal Triangle Modulo p

B-666 *Taken from solutions to B-643 by Russell Jay Hendel, Dowling College, Oakdale, NY, and by Lawrence Somer, Washington, D.C.*

For primes p, prove that

$$\binom{n}{p} \equiv [n/p] \pmod{p},$$

where $[x]$ is the greatest integer in x.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

The result follows easily from the same formula of E. Lucas used in the solution to B-643 [vol. 28 (1990), p. 185]. Alternately, let t be the integer satisfying $0 \le t \le p - 1$ and $p \mid (n - t)$. Then $(n - t)/p = [n/p]$ and

$$n(n - 1) \cdots (n - t + 1)(n - t - 1) \cdots (n - p + 1) \equiv (p - 1)! \pmod{p}.$$

Therefore, in the field Z_p of the integers modulo p,

$$\binom{n}{p} = [n/p]\frac{n \cdots (n - t + 1)(n - t - 1) \cdots (n - p + 1)}{(p - 1)!} = \left[\frac{n}{p}\right]$$

Hence, $\binom{n}{p} \equiv [n/p] \pmod{p}$.

Also solved by R. André-Jeannin, Paul S. Bruckman, C. Georghiou, Norbert Jensen & Uwe Pettke, Bob Prielipp, Sahib Singh, Amitabha Tripathi, and the proposer.

Cyclic Permutation of Digits

B-667 *Proposed by Herta T. Freitag, Roanoke, VA*

Let p be a prime, $p \neq 2$, $p \neq 5$, and m be the least positive integer such that $10^m \equiv 1 \pmod{p}$. Prove that each m-digit (integral) multiple of p remains a multiple of p when its digits are permuted cyclically.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Suppose

$$n = a_{m-1}10^{m-1} + \cdots + a_1 10 + a_0 \equiv 0 \pmod{p}.$$

Let n_t be the integer obtained from n by permuting its digits cyclically by t positions. More specifically,

$$n_t = a_{m-t-1}10^{m-1} + \cdots + a_0 10^t + a_{m-1}10^{t-1} + \cdots + a_{m-t},$$

where $0 \leq t \leq m-1$. Since $10^m \equiv 1 \pmod{p}$, we have

$$\begin{aligned} n_t &\equiv a_{m-t-1}10^{m-1} + \cdots + a_0 10^t + a_{m-1}10^{m+t-1} + \cdots + a_{m-t}10^m \\ &= 10^t(a_{m-t-1}10^{m-t-1} + \cdots + a_0 + a_{m-1}10^{m-1} + \cdots + a_{m-t}10^{m-t}) \\ &= 10^t n \equiv 0 \pmod{p}. \end{aligned}$$

Also solved by R. André-Jeannin, Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, Norbert Jensen & Uwe Pettke, L. Kuipers, Bob Prielipp, Sahib Singh, Lawrence Somer, and the proposer.

Base 9 Modular Arithmetic Progression

B-668 *Proposed by A. P. Hillman in memory of Gloria C. Padilla*

Let h be the positive integer whose base 9 numeral

100101102...887888

is obtained by placing all the 3-digit base 9 numerals end-to-end as indicated.

(a) What is the remainder when h is divided by the base 9 integer 14?

(b) What is the remainder when h is divided by the base 9 integer 81?

Solution by C. Georghiou, University of Patras, Greece

It is easy to see that

$$\begin{aligned} (100101102\ldots887888)_9 &= (888)_9 9^0 + (887)_9 9^3 + \cdots + (100)_9 9^{3\cdot 647} \\ &= (729-1)9^0 + (729-2)9^3 + \cdots + (729-648)9^{3\cdot 647} \\ &= \sum_{n=1}^{648}(9^3 - n)9^{3n-3}. \end{aligned}$$

(a) Now, since $(14)_9 = 13$ and $9^3 \equiv 1 \pmod{13}$, we have

$$h = \sum_{n=1}^{648}(9^3 - n)9^{3n-3} \equiv \sum_{n=1}^{648}(1 - n) = \frac{647 \cdot 648}{2} \equiv -3 \pmod{13}.$$

(b) Now we have $(81)_9 = 73$ and $9^3 \equiv -1 \pmod{73}$. Therefore,

$$h = \sum_{n=1}^{648}(9^3 - n)9^{3n-3} \equiv -2 + \sum_{n=2}^{648}(-1 - n)(-1)^{n-1} = -1 - \sum_{n=1}^{649} n(-1)^n$$

$$= -1 - \frac{(-1)^{649}(2 \cdot 649 + 1) - 1}{4} = 324 \equiv 32 \pmod{73}.$$

Also solved by Charles Ashbacher, Paul S. Bruckman, and the proposer.

Fibonacci and Lucas Identities

B-669 *Proposed by Gregory Wulczyn, Lewisberg, PA*

Do the equations

$$25F_{a+b+c}F_{a+b-c}F_{b+c-a}F_{c+a-b} = 4 - L_{2a}^2 - L_{2b}^2 - L_{2c}^2 + L_{2a}L_{2b}L_{2c}$$

$$L_{a+b+c}L_{a+b-c}L_{b+c-a}L_{c+a-b} = -4 + L_{2a}^2 + L_{2b}^2 + L_{2c}^2 + L_{2a}L_{2b}L_{2c}$$

hold for all even integers a, b, and c?

Solution by C. Georghiou, University of Patras, Greece

The answer is "Yes"! From the identity

$$5F_{m+n}F_{m-n} = L_{2m} - (-1)^{m+n}L_{2n}$$

we get [setting $(-1)^{a+b+c} = e$)

$$\begin{aligned}25F_{a+b+c}F_{a+b-c}F_{a-b+c}F_{c+b-a} &= [L_{2a+2b} - eL_{2c}][L_{2c} - eL_{2a-2b}]\\ &= L_{2c}[L_{2a+2b} + L_{2a-2b}] - eL_{2c}^2 - eL_{2a+2b}L_{2a-2b}\\ &= L_{2c}[L_{2a}L_{2b}] - eL_{2c}^2 - e[L_{4a} + L_{4b}]\\ &= L_{2a}L_{2b}L_{2c} - e[L_{2a}^2 + L_{2b}^2 + L_{2c}^2 - 4],\end{aligned}$$

and for a, b, and c even (actually for $a + b + c$ even), the given identity is established.

In a similar way, using the identity

$$L_{m+n}L_{m-n} = L_{2m} + (-1)^{m+n}L_{2n}$$

we find

$$\begin{aligned}L_{a+b+c}L_{a+b-c}L_{a-b+c}L_{c+b-a} &= [L_{2a+2b} + eL_{2c}][L_{2c} + eL_{2a-2b}]\\ &= L_{2c}[L_{2a+2b} + L_{2a-2b}] + eL_{2c}^2 + eL_{2a+2b}L_{2a-2b}\\ &= L_{2a}L_{2b}L_{2c} + e[L_{2a}^2 + L_{2b}^2 + L_{2c}^2 - 4],\end{aligned}$$

which establishes the second identity.

Also solved by R. André-Jeannin, Paul S. Bruckman, Herta T. Freitag, Norbert Jensen & Uwe Pettke, L. Kuipers, Bob Prielipp, and the proposer.

*****

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-452 *Proposed by Don Redmond, Southern Illinois U., Carbondale, IL*

Let $p_r(m)$ denote the m^{th} r-gonal number $(m/2)\{2 + (r - 2)(m - 1)\}$. Characterize the values of r and m such that

$$p_r(m) \left| \sum_{k=1}^{m} p_r(k) \right. .$$

H-453 *Proposed by James E. Desmond, Pensacola Jr. College, Pensacola, FL*

Show that for positive integers m and n,

$$\frac{L_{(2m+1)n}}{L_n} = \sum_{j=1}^{m} (-1)^{(n+1)(m-j)} L_{2nj} + (-1)^{m(n+1)}$$

and

$$\frac{F_{2mn}}{L_n} = \sum_{j=1}^{m} (-1)^{(n+1)(m-j)} F_{n(2j-1)}.$$

H-454 *Proposed by Larry Taylor, Rego Park, NY*

Construct six distinct Fibonacci-Lucas identities such that

(a) Each identity consists of three terms;
(b) Each term is the product of two Fibonacci numbers;
(c) Each subscript is either a Fibonacci or a Lucas number.

SOLUTIONS

An Old-Timer

H-91 *Proposed by Douglas Lind, U. of Virginia, Charlottesville, VA (Vol. 4, no. 3, October 1966) [corrected]*

Let $m = \left[\frac{k}{2}\right]$, then show

$$\frac{F_{kn}}{F_n} = \sum_{j=0}^{m-1} (-1)^{jn} L_{n(k-1-2j)} + e_n,$$

where

$$e_n = \begin{cases} (-1)^{mn} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

and $[x]$ is the greatest integer not exceeding x.

Solution by James E. Desmond, Pensacola Jr. College, Pensacola, FL

Using the well-known algebraic identity,

$$\frac{x^k - y^k}{x - y} = \sum_{j=0}^{\left[\frac{k}{2}\right]-1} x^j y^j (x^{k-1-2j} + y^{k-1-2j}) + x^{\left[\frac{k}{2}\right]} y^{\left[\frac{k}{2}\right]} \frac{1 + (-1)^{k+1}}{2}$$

for all positive integers k and nonzero real numbers x, y with $x \neq y$; let $x = \alpha^n$ and $y = \beta^n$ where n is a positive integer. We obtain

$$\frac{\alpha^{nk} - \beta^{nk}}{\alpha^n - \beta^n} = \sum_{j=0}^{\left[\frac{k}{2}\right]-1} \alpha^{nj} \beta^{nj} (\alpha^{n(k-1-2j)} + \beta^{n(k-1-2j)}) + \alpha^{n\left[\frac{k}{2}\right]} \beta^{n\left[\frac{k}{2}\right]} \frac{1 + (-1)^{k+1}}{2}.$$

That is,

$$\frac{F_{nk}}{F_n} = \sum_{j=0}^{\left[\frac{k}{2}\right]-1} (-1)^{nj} L_{n(k-1-2j)} + (-1)^{n\left[\frac{k}{2}\right]} \frac{1 + (-1)^{k+1}}{2}$$

$$= \sum_{j=0}^{m-1} (-1)^{jn} L_{n(k-1-2j)} + e_n.$$

Pell-Mell

<u>H-433</u> *Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 27, no. 4, August 1989)*

Let P_0, P_1, ... be the Pell numbers defined by

$$P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2.$$

Show that, for $n = 1, 2, \ldots,$

$$6(n + 1)P_{n-1} + P_{n+1} \equiv (-1)^{n+1}(9n^2 - 7)F_{n+1} \pmod{27}.$$

Solution by Robert B. Israel, U. of British Columbia, Vancouver, B.C.

The congruence

(1) $\quad P_n \equiv (-1)^n((18n^2 + 21n + 2)F_n + 12nF_{n+1}) \pmod{27}$, for all $n \geq 0$,

can be established by checking that the right-hand side obeys the defining equations for P_n mod 27. Some tedious but straightforward manipulations then lead to the desired result.

Not content to let the matter rest there, we generalize it. Let p and k be natural numbers, and define U_n by

$$U_0 = 0,\ U_1 = 1,\ U_n = (p - 1)U_{n-1} + U_{n-2}$$

(so that the Pell numbers are the case $p = 3$).

Theorem: If no prime factors of p are equal to 5 or less than k, there is a congruence

$$(2) \qquad U_n \equiv (-1)^n \sum_{j=0}^{k-1} n^j (a_j F_n + b_j F_{n+1}) \pmod{p^k}, \text{ for all } n \ge 0,$$

where a_j and b_j are integers.

It is more convenient to work with $V_n = (-1)^{n+1}U_n$. The generating functions of F_n and V_n are, respectively,

$$F(z) = \frac{z}{1 - z - z^2} \quad \text{and} \quad V(z) = \frac{z}{1 + (p-1)z - z^2}.$$

Letting $x = z^{-1} - 1 - z$, we have $F(z) = x^{-1}$ and

$$V(z) = \frac{1}{x + p} \equiv \sum_{j=0}^{k-1} (-p)^j x^{-1-j} \pmod{p^k}$$

(this being interpreted as a statement about formal power series in the indeterminate z with coefficients in the integers mod p^k). The generating function of $n^j F_n$ is

$$G_j(z) = \left(z \frac{d}{dz}\right)^j F(z).$$

The generating function of $(n+1)^j F_{n+1}$ is $z^{-1}G_j(z)$. To prove the theorem, it is enough to prove that for $2 \le j \le k$ there are congruences

$$(3) \qquad x^{-j} \equiv \sum_{i=0}^{j-1} (c_i + d_i z^{-1}) G_i(z) \pmod{p^k}.$$

Let $w = z^{-1} + z$. I claim that

$$(4) \qquad G_{2j}(z) = \sum_{i=1}^{2j+1} \frac{c_{i,2j}}{x^i} \quad \text{with } c_{2j+1,2j} = (2j)!5^j,$$

$$(5) \qquad G_{2j+1}(z) = \sum_{i=2}^{2j+2} c_{i,2j+1} \frac{w}{x^i} \quad \text{with } c_{2j+2,2j} = (2j+1)!5^j,$$

where $c_{i,j}$ are integers. The proof is by induction, using the identities

$$z\frac{dx}{dz} = -w, \quad z\frac{dw}{dz} = -x - 1, \quad w^2 = (x+1)^2 + 4 = 5 + 2x + x^2.$$

Equation (4) allows us to express $1/x^{2j+1}$ mod p^k in terms of $G_{2j}(z)$ and lower powers of $1/x$, as long as $(2j)!5^j$ is invertible mod p^k. To treat $1/x^{2j+2}$ similarly, we first use the identity $(2z^{-1} - 1)w = (2z^{-1} + 1)x + 5$ and (5) to get

$$(6) \qquad \frac{(2j+1)!5^{j+1}}{x^{2j+2}} = (2z^{-1} - 1)G_{2j+1} - \sum_{i=1}^{2j+1} \frac{5c_{i,2j+1} + c_{i+1,2j+1}}{x^i} - 2\sum_{i=1}^{2j+1} c_{i+1,2j+1}\frac{z^{-1}}{x^i},$$

where $c_{1,2j+1} = 0$. The factors of z^{-1} that arise here are harmless. To avoid factors of z^{-2}, however, we can use the identity $(z^{-1} + 2)w = (z^{-1} - 2)x + 5z^{-1}$ together with (5) to get

$$\frac{(2j+1)!5^{j+1}z^{-1}}{x^{2j+2}} = (z^{-1} + 2)G_{2j+1} - \sum_{i=1}^{2j+1} (5c_{i,2j+1} + c_{i+1,2j+1})\frac{z^{-1}}{x^i} + 2\sum_{i=1}^{2j+1} \frac{c_{i+1,2j+1}}{x^i}.$$

Repeated use of these formulas results in the desired congruences (3).

In the case $k = 3$, for example, the result of all of this is

$$U_n \equiv (-1)^n\left(\left(-\frac{p^2n^2}{10} - \frac{3p^2n}{50} - \frac{pn}{5} + \frac{p^2}{25} - \frac{p}{5} - 1\right)F_n + \left(\frac{3p^2n}{25} + \frac{2pn}{5}\right)F_{n+1}\right) \pmod{p^3}$$

if $(p, 10) = 1$. With $p = 3$, this yields (1).

Also solved by P. Bruckman, R. J. Hendel, L. Kuipers, G. Wulczyn, and the proposer.

Strange Sex

H-434 *Proposed by Piero Filipponi & Odoardo Brugia, Rome, Italy (Vol. 27, no. 4, August 1989)*

Strange creatures live on a planet orbiting around a star in a remote galaxy. Such beings have three sexes (namely, sex A, sex B, and sex C) and are reproduced as follows:

(i) An individual of sex A (or simply A) generates individuals of sex C by parthenogenesis.

(ii) If A is fertilized by an individual of sex B, then A generates individuals of sex B.

(iii) In order to generate individuals of sex A, A must be fertilized by an individual of sex A, an individual of sex B, and an individual of sex C.

Find a closed form expression for the number T_n of ancestors of an individual of sex A in the n^{th} generation. Note that, according to (i), (ii), and (iii), A has three parents ($T_1 = 3$) and six grandparents ($T_2 = 6$).

Solution by Russell Jay Hendel, Dowling College, Oakdale, NY

We claim

$$T_n = c_1\left[r_1^n + \frac{1}{2}\right], \quad \text{for all } n \geq 0;$$

with $r_1 \geq r_2 \geq 0 \geq r_3$ the three roots of $p(z) = z^3 - 2z^2 - z + 1$; and

$$c_1 = \frac{r_1^2 + r_1 - 1}{(r_2 - r_1)(r_3 - r_1)} \approx 1.22144\ldots .$$

The proof will use complex variable methods to derive the value of the c_i and linear algebra methods to derive the value of T_n.

First, define a homomorphism, H, on the free monoid on the letters {A, B, C} by $H(\text{C}) = \text{A}$, $H(\text{B}) = \text{AB}$, and $H(\text{A}) = \text{ABC}$, so that T_n equals the length of the string $P^n(\text{A})$. Following Rauzy [2], a convenient way to study this length is by letting M be the 3×3, 1-0, upper triangular matrix, defined by $M(i, j) = 1$ if $i + j \geq 4$, and 0 otherwise.

Following Rorres & Anton [3], define vectors

$$\mathbf{v}_{-1} = (1, 0, 0)^* \quad \text{and} \quad M\mathbf{v}_{n-1} = \mathbf{v}_n = (x_n, y_n, z_n)^*,$$

with * denoting vector transpose. Thus, $x_n = T_n$, and

$$\mathbf{v}_n = M^{n+1}\mathbf{v}_{-1} = PD^{n+1}P^{-1}\mathbf{v}^{-1},$$

with $M = PDP^{-1}$, a diagonal decomposition of M. Since the characteristic polynomial of M is $p(z)$, some straightforward manipulation yields

$$x_n = \sum_{i=1}^{3} c_i r_i^n,$$

for some constants c_i, $1 \le i \le 3$.

To find closed formulas for the c_i, we study the generating function

$$T(z) = \sum_{i=0}^{\infty} T_i z^i = \frac{1 + z - z^2}{z^3 p(z^{-1})} = \frac{1 + z - z^2}{\prod_{i=1}^{3}(z - r^{-1})}.$$

Following Hagis [1], we employ the Residue Theorem to yield:

$$\frac{1}{2\pi i}\int_{C_s} \frac{T(z)}{z^{n+1}}\,dz = \frac{1}{2\pi i}\int_{C_0} \frac{T(z)}{z^{n+1}}\,dz + \frac{1}{2\pi i}\sum_{i=1}^{3}\int_{C_i} \frac{T(z)}{z^{n+1}}\,dz$$

where C_S is the circle of radius S about the origin, and C_0 and C_i are circles of radius .1 around the origin and the r_i^{-1}, respectively. By the triangle inequality for integrals, as S goes to infinity we have

$$\left|\frac{1}{2\pi i}\int_{C_s} \frac{T(z)}{z^{n+1}}\,dz\right| \le O(S^{-1}) \to 0.$$

By the Cauchy Integral Formula for derivatives, we have

$$\frac{1}{2\pi i}\int_{C_0} \frac{T(z)}{z^{n+1}}\,dz = \frac{T^{(n)}(0)}{n!} = T_n.$$

Finally, by the Cauchy Integral Formula and some manipulations, we have

$$\frac{1}{2\pi i}\int_{C_i} \frac{T(z)}{z^{n+1}}\,dz = \text{Residue at } r_i^{-1} = \frac{r_i^2 + r_i - 1}{\prod_{j \ne i}(r_j - r_i)} r_i^n.$$

Combining the above, we have an alternate derivation of the preliminary formula for the T_n with closed expression for the c_i.

To complete the proof, simply observe that, for large n,

$$T_n - c_1 r_1^n = c_2 r_2^n + c_3 r_3^n = O(|r_3|)^n \to 0.$$

For small n, a calculator can be used to verify that an upper bound for the absolute value of the preceding expression is bounded by 1/2. The details are left to the reader. (In passing, we note that it is straightforward to prove that $T_n - c_1 r_1^n$ is oscillating and monotone in opposite directions for even and odd n.)

References

1. P. Hagis. "An Analytic Proof of the Formula for F_n." *Fibonacci Quarterly 2* (1964):267-68.
2. Rauzy. "Nombres algebriques et substitutions." *Bull. Soc. Math. France 110* (1982):147-78.
3. C. Rorres & H. Anton. *Applications of Linear Algebra*. New York: Wiley, 1984.

Probably

H-436 *Proposed by Piero Filipponi, Rome, Italy*
(Vol. 27, no. 5, November 1989)

For p an arbitrary prime number, it is known that

$$(p - 1)! \equiv p - 1 \pmod{p}, \quad (p - 2)! \equiv 1 \pmod{p},$$

and

$$(p - 3)! \equiv (p - 1)/2 \pmod{p}.$$

Let k_0 be the smallest value of an integer k for which $k! > p$.

The numerical evidence turning out from computer experiments suggests that the probability that, for k varying within the interval $[k_0, p - 3]$, $k!$ reduced modulo p is either even or odd is 1/2. Can this conjecture be proved?

Solution by Paul S. Bruckman, Edmonds, WA

We will show that the proposer's conjecture is equivalent to the proposition that the primes are somehow equally distributed, a concept which we will define more precisely later. First, we form the following short table of $k_0 = k_0(p)$, for the first few primes p:

p	k_0
2	3
3	3
5	3
7	4
11	4

Clearly, $k_0 \le p - 3$ only if $p \ge 7$; suppose then that $p \ge 7$ henceforth. Now any such prime must be of one of the two forms: $4a + 1$ or $4a + 3$. Then

$$(p - 3)! \equiv \frac{1}{2}(p - 1) \equiv 2a \text{ or } 2a + 1.$$

Note that these are *proper* residues (mod p), that is, lie in the interval $[1, p - 1]$. We introduce the notation: $\underline{f(p)} \equiv x$ to mean that $f(p) \equiv x \pmod{p}$, and $x \in [1, p - 1]$. If we can expect that a prime is equally likely to be of either form, it would then follow that $\Pr[\underline{(p - 3)!}$ is even$] = 1/2$. This seems a plausible supposition, but is apparently an unproven proposition.

We now tackle the general case. Consider $(p - r - 1)!$, where r is chosen so that $r \in [2, p - 1 - k_0]$. Then

$$\begin{aligned}(p - r - 1)! &= (p - 2)!/(p - 2)(p - 3) \ldots (p - r)\\ &\equiv 1/(-1)^{r-1}2 \cdot 3 \cdot \cdots \cdot r,\end{aligned}$$

or

(1) $$(p - r - 1)! \equiv (-1)^{r-1}(r!)^{-1} \pmod{p}.$$

Since g.c.d.$(p, r!) = 1$, there exists some integer b such that

(2) $$p \equiv b \pmod{2(r!)}.$$

As b assumes all values in $[1, 2(r!) - 1]$ with g.c.d.$(b, r!) = 1$, it is clear that any prime p must be of one of those forms [there are $2\phi(r)$ such, where ϕ is the Euler (totient) function]. Again, we may reasonably conjecture that each choice of b is equally probable, as p is randomly chosen. For example, for $r = 3$, there are $2\phi(3) = 4$ choices: $p \equiv 1$, 5, 7, or 11 (mod 12), and we may plausibly suppose that each form of p is equally likely.

Now, there are infinitely many integers x such that congruence $px \equiv (-1)^r$ (mod $r!$) has solutions. However, if we restrict x to the interval $(0, r!)$, then $x = c$ is uniquely determined. Hence,

$$(cp - (-1)^r)/r! \equiv (-1)^{r-1}(r!)^{-1} \pmod{p};$$

therefore, from (1), we have:

(3) $$(p - r - 1)! \equiv (cp - (-1)^r)/r! \pmod{p}.$$

Moreover,

$$(cp - (-1)^r)/r! \geq (p - 1)/r! \geq \frac{2(r!) + 1 - 1}{r!} \geq 2,$$

and

$$(cp - (-1)^r)/r! \leq \frac{(r! - 1)p + 1}{r!} = p - (p - 1)/r!$$
$$\leq p - (2(r!) + 1 - 1)/r! = p - 2.$$

This shows that $(cp - (-1)^r)/r!$ is a proper residue (mod p). We have proven the following result.

Lemma 1:

(4) $\quad \underline{(p - r - 1)!} \equiv (cp - (-1)^r)/r!, \; r = 2, 3, \ldots, p - 1 - k_0,$

where c is uniquely determined by $c \equiv (-1)^r p^{-1}$ (mod $r!$), $0 < c < r!$.

Now, given r, suppose we choose b such that $0 < b < r!$, and that $p \equiv b$ (mod $2(r!)$), for some prime p. Also suppose that p' is prime, where $p' \equiv b'$ (mod $2(r!)$), and $b' = b + r!$ [hence, $r! < b' < 2(r!)$ and g.c.d.$(b', r!) = 1$). Let c and c' denote the values determined from Lemma 1, with p and p', respectively. Thus, $p = 2a(r!) + b$, $p' = 2a'(r!) + b'$ for some integers a and a'. From Lemma 1,

$$c \equiv (-1)^r p^{-1} \equiv (-1)^r/[2a(r!) + b] \equiv (-1)^r b^{-1} \pmod{r!};$$

also,

$$c' \equiv (-1)^r (p')^{-1} \equiv (-1)^r/[2a'(r!) + b + r!] \equiv (-1)^r b^{-1} \pmod{r!}.$$

Hence, $c' \equiv c$ (mod $r!$). However, since $0 < c < r!$ and $0 < c' < r!$, it follows that $c' = c$. Also, from Lemma 1,

$$\underline{(p' - r - 1)!} \equiv [cp' - (-1)^r]/r! = \frac{c[2a'(r!) + b + r!] - (-1)^r}{r!}$$
$$= \frac{c[2a(r!) + b] + (2a' - 2a)cr! - (-1)^r}{r!} + c$$
$$= [cp - (-1)^r]/r! + (2a' - 2a + 1)c.$$

Note that c must be an odd number, since $r!$ is even and $r!$ divides $(cp - (-1)^r)$. Hence, we have proven the following result.

Lemma 2: Given primes p and p',

$$2 \leq r \leq \min\{(p - 1 - k_0), (p' - 1 - k_0')\},$$

where $k_0' = k_0(p')$ and $p' \equiv p + r!$ (mod $2(r!)$), then $\underline{(p' - r - 1)!}$ and $\underline{(p - r - 1)!}$ are disparate.

If it is true that each prime p of the form $p \equiv b$ (mod $2(r!)$) is equally likely, as b varies over its $2\phi(r!)$ possible values, then it would follow from Lemma 2 that Prob$[\underline{(p - r - 1)!}$ is even$] = 1/2$. Letting r vary over its possible values $r = 2, 3, \ldots, p - 1 - k_0$, we could then conclude that

$$\text{Prob}(\underline{k!} \text{ is even}) = \frac{1}{2}, \text{ for } k = k_0, k_0 + 1, \ldots, p - 3,$$

where p is a random prime. Thus, given integers $r \geq 2$ and $b \in [1, 2(r!) - 1]$, with g.c.d.$(b, r!) = 1$, the following results are equivalent, for random primes p:

(a) Prob$(p \equiv b \pmod{2(r!)}) = 1/2\phi(r!)$;

(b) Prob$[\underline{(p - r - 1)!}$ is even$] = \frac{1}{2}$.

The result conjectured in (a) seems plausible enough; however, as far as is known, it remains unproven.

*****

SUSTAINING MEMBERS

M.H. Ahmadi
*A.L. Alder
G.L. Alexanderson
S. Ando
R. Andre-Jeannin
*J. Arkin
M.K. Azarian
L. Bankoff
M. Berg
J.G. Bergart
G. Bergum
G. Berzsenyi
*M. Bicknell-Johnson
P.S. Bruckman
M.F. Bryn
P.F. Byrd
G.D. Chakerian
C.K. Cook
J.W. Creely
P.A. DeCaux
M.J. DeLeon
J. Desmond
H. Diehl
T.H. Engel
D.R. Farmer
D.C. Fielder
Emerson Frost
Anthony Gioia
R.M. Giuli
P. Hagis, Jr.
H. Harborth
A.P. Hillman
*A.F. Horadam
F.T. Howard
R.J. Howell
C.H. Kimberling
R.P. Kovach
J. Lahr
J.C. Lagarias
L.H. Lange
C.T. Long
Br. J.M. Mahon
*J. Maxwell
F.U. Mendizabal
L. Miller
M.G. Monzingo
J.F. Morrison
K. Nagasaka
S.A. Obaid
A. Prince
S. Rabinowitz
J.A. Schumaker
A.G. Shannon
L.W. Shapiro
J.R. Siler
D. Singmaster
J. Sjoberg
L. Somer
M.N.S. Swamy
*D. Thoro
J.C. Turner
T.P. Vaughan
J.N. Vitale
R. Vogel
M. Waddill
J.E. Walton
G. Weekly
R.E. Whitney
B.E. Williams
A.C. Yanoviak

*Charter Members

INSTITUTIONAL MEMBERS

ACADIA UNIVERSITY LIBRARY
Wolfville, Nova Scotia

THE BAKER STORE EQUIPMENT COMPANY
Cleveland, Ohio

CALIFORNIA STATE UNIVERSITY SACRAMENTO
Sacramento, California

ETH-BIBLIOTHEK
Zurich, Switzerland

FERNUNIVERSITAET HAGEN
Hagen, West Germany

HOWELL ENGINEERING COMPANY
Bryn Mawr, California

MATHEMATICS SOFTWARE COMPANY
Evansville, Indiana

PRINCETON UNIVERSITY
Princeton, New Jersey

SAN JOSE STATE UNIVERSITY
San Jose, California

SANTA CLARA UNIVERSITY
Santa Clara, California

KLEPCO, INC.
Sparks, Nevada

TECHNOLOGICAL EDUCATION INSTITUTE
Larissa, Greece

UNIVERSITY OF CALIFORNIA, SANTA CRUZ
Santa Cruz, California

UNIVERSITY OF NEW ENGLAND
Armidale, N.S.W. Australia

UNIVERSITY OF NEW ORLEANS
New Orleans, Louisiana

UNIVERSITY OF ROMA "LA SAPIENZA"
Roma, Italy

WAKE FOREST UNIVERSITY
Winston-Salem, North Carolina

WASHINGTON STATE UNIVERSITY
Pullman, Washington

JOVE STATISTICAL TYPING SERVICE
2088 Orestes Way
Campbell, California 95008

BOOKS AVAILABLE
THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Fibonacci Numbers and Their Applications. Edited by A.N. Philippou, G.E. Bergum and A.F. Horadam.

Applications of Fibonacci Numbers. Edited by A.N. Philippou, A.F. Horadam and G.E. Bergum.

Please write to the Fibonacci Association, Santa Clara University, Santa Clara CA 95053, U.S.A., for current prices.