The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

SECOND-ORDER RECURRENCE AND ITERATES OF $[\alpha n + 1/2]$

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(Submitted June 1989)

The equation

$$[x[nx + 1/2] + 1/2] = [nx + 1/2] + n$$

determines a unique real number x, in the sense that there is only one value of x for which this equation holds for all positive integers n. This special value of x is the golden mean, $(1 + \sqrt{5})/2$.

The purpose of this note is to prove the above assertion in the more general form of Theorem 1 (of which it is the case when $a = b = 1$), and to give a necessary and sufficient condition that iterates of the function $[\alpha n + 1/2]$, in the sense of Theorem 2, form a second-order recurrence sequence.

Notation: Throughout, let $f(x) = x^2 - ax - b$, where a and b are nonzero integers satisfying $a^2 + 4b > 0$. Write the roots of $f(x)$ as

$$\alpha = (a + \sqrt{a^2 + 4b})/2 \quad \text{and} \quad \beta = a - \alpha.$$

Let $[a_0, a_1, a_2, \ldots]$ denote the continued fraction of the root α, with convergents p_k/q_k given in the usual way (e.g., Roberts [1], pp. 97-100) by

$$p_{-2} = 0,\ p_{-1} = 1,\ p_k = a_k p_{k-1} + p_{k-2} \text{ for } k \geq 0,$$
$$q_{-2} = 1,\ q_{-1} = 0,\ q_0 = 1,\ q_k = a_k q_{k-1} + q_{k-2} \text{ for } k \geq 1.$$

Lemma 1: $|\beta| < 1$ if and only if $|b - 1| < |a|$, and
$|\beta| = 1$ if and only if $|b - 1| = |a|$.

Proof: $|\beta| \leq 1$ if and only if

$$(1) \qquad a - 2 \leq \sqrt{a^2 + 4b} \leq a + 2,$$

with equality if and only if $|\beta| = 1$. This inequality shows that a cannot be less than or equal to -2, since $a^2 + 4b$ is positive. Moreover, if $a = -1$, then $b \geq 1$, so that $\sqrt{a^2 + 4b} > a + 2$, a contradiction. Therefore, $a \geq 1$. In case $a = 1$, we have

$$(2) \qquad 2 - a \leq \sqrt{a^2 + 4b} \leq a + 2,$$

and if $a \geq 2$, then the leftmost member of inequality (1) is nonnegative. So, if $a = 1$, square the members of inequality (2), and if $a \geq 2$, square those of inequality (1). In both cases, the resulting inequalities easily simplify to $-a \leq b - 1 \leq a$.

Lemma 2: There exists a positive integer K such that

$$[(p_k - aq_k - 1)(p_k - 1)/q_k + 1/2] < bq_k < [(p_k - aq_k + 1)(p_k + 1)/q_k + 1/2]$$

for all $k \geq K$.

Proof: It suffices to prove for all large enough k the inequalities

$$(p_k - aq_k - 1)(p_k - 1)/q_k + 1/2 < bq_k + 1$$

and

$$bq_k < (p_k - aq_k + 1)(p_k + 1)/q_k - 1/2,$$

which are equivalent to

(3) $\qquad [(p_k - 1)/q_k]^2 + a/q_k < ap_k/q_k + b + 1/2q_k < [(p_k + 1)/q_k]^2 - a/q_k.$

Substitute $\alpha + \varepsilon$ for p_k/q_k, where $|\varepsilon| < 1/q_k q_{k+1}$ (e.g., Roberts [1], p. 100), square where indicated, and use the fact that $\alpha^2 - a\alpha - b = 0$ to see that (3) is equivalent to

$$|\varepsilon q_k(a - 2\alpha - \varepsilon) + 1/2 - 1/q_k| < |a - 2\alpha - 2\varepsilon|,$$

which holds for all large enough k, since, as $k \to \infty$, the left member approaches $1/2$, while the right approaches $|a - 2\alpha| = \sqrt{a^2 + 4b} \geq 1$.

Lemma 3: If $|b - 1| < |a|$, then equation (4) below holds for $x = \alpha$ and for all $n \geq 1$.

Proof: By Lemma 1, $|\beta| < 1$, so that the fractional part $r = n\alpha + 1/2 - [n\alpha + 1/2]$ satisfies $|r - 1/2| < 1/2|\beta|$. Since $\beta = a - \alpha$, we have $-1 < (\alpha - a)(1 - 2r) < 1$, so that $0 < (\alpha - a + 1)/2 + (a - \alpha)r < 1$. Since $\alpha^2 = a\alpha + b$, we then have

$$0 < (\alpha - a)(n\alpha + 1/2 - r) + 1/2 - bn < 1,$$

or

$$bn < (\alpha - a)[n\alpha + 1/2] + 1/2 < 1 + bn,$$

so that equation (4) holds for $x = \alpha$.

Theorem 1: Suppose a and b are integers satisfying $|b - 1| < |a|$. Then there exists one and only one number x for which

(4) $\qquad [x[nx + 1/2] + 1/2] = a[nx + 1/2] + bn$

for all $n \geq 1$. Explicitly, $x = \alpha = (a + \sqrt{a^2 + 4b})/2$.

Proof: Let n be the denominator q_k of the k^{th} convergent p_k/q_k to the root α of $x^2 - ax - b$. We shall show that in order for (4) to hold for this choice of n, the number x must lie inside infinitely many intervals (L_k, R_k), where

$$L_k = (p_k - 1)/q_k \quad \text{and} \quad R_k = (p_k + 1)/q_k.$$

To see that $x \geq L_k$ for all large enough k, observe that, for $x < L_k$ and all large enough k, we have

$$[x[nx + 1/2] + 1/2 - a[nx + 1/2]] = [(x - a)[nx + 1/2] + 1/2]$$
$$\leq [((p_k - 1)/q_k - a)[p_k - 1 + 1/2] + 1/2]$$
$$\leq [(p_k - aq_k - 1)(p_k - 1)/q_k + 1/2] < bq_k,$$

by Lemma 2. This contradiction to (4) shows that $x \geq L_k$ for all large enough k. Similarly, Lemma 2 shows that $x \leq R_k$ for all large k. It follows that the only viable candidate for x is α, since only this number lies inside infinitely many of the intervals (L_k, R_k).

Lemma 3 shows that the root α does indeed satisfy (4) for all $n \geq 1$.

Theorem 2: For any positive integer n, the sequence $\{s_k\}$ given by

$$s_1 = n,\ s_2 = [\alpha n + 1/2],\ s_3 = [\alpha s_2 + 1/2],\ \ldots,\ s_k = [\alpha s_{k-1} + 1/2]$$

satisfies the recurrence relation $s_k = as_{k-1} + bs_{k-2}$ for all $n \geq 1$ and for all $k \geq 2$ if and only if $|b - 1| \leq |a|$.

Proof; If $|b - 1| < |a|$, then $s_3 = as_2 + bs_1$, according to (4). In fact, for any $k \geq 3$, substituting s_{k-2} for n into (4) yields $[\alpha s_k + 1/2] = as_{k-1} + bs_{k-2}$, as asserted.

Now, if $b - 1 = a$, then $\alpha = a + 1$, so that

$$s_2 = (a + 1)n \quad \text{and} \quad s_3 = (a + 1)s_2 = (a + 1)^2 n = as_2 + bs_1.$$

By induction,

$$s_k = (a + 1)^{k-1}n = as_{k-1} + bs_{k-2} \text{ for all } k \geq 3.$$

Similarly, if $b - 1 = -a$, then

$$s_k = (a - 1)^{k-1}n = as_{k-1} + bs_{k-2} \text{ for all } k \geq 3.$$

If $|b - 1| > |a|$, then $|\beta| > 1$ by Lemma 1. Then the well-known representation $a_1\alpha^m + b_1\beta^m$ for the m^{th} term of any recurrence sequence

$$t_m = at_{m-1} + bt_{m-2},$$

for which $a^2 + 4b > 0$, shows that the sequence

$$\alpha t_m - t_{m+1} = b_1\beta^m(\alpha - \beta)$$

diverges, so that the relation $t_{m+1} = [\alpha t_m + 1/2]$ cannot hold for all m.

In conclusion, we note that the well-known representation

$$F_n = [\alpha F_{n-1} + 1/2]$$

for the n^{th} Fibonacci number in terms of the golden mean, α, *and only one preceding term*, follows from Theorem 2 when $a = b = 1$. Theorem 2 reveals many other second-order recurrence sequences which lend themselves to this sort of first-order recurrence.

Reference

1. Joe Roberts. *Elementary Number Theory*. Cambridge, Mass.: MIT Press, 1977.

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ENTRY POINT RECIPROCITY OF CHARACTERISTIC CONJUGATE GENERALIZED FIBONACCI SEQUENCES

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(Submitted July 1989)

Introduction

Given a pair of integers, A, B, such that $(A, B) = 1$ and $0 < A < \frac{1}{2}B$, we define a generalized Fibonacci sequence as follows:

$$G_0 = B - A,\ G_1 = A,\ G_n = G_{n-1} + G_{n-2} \quad \text{for } n \geq 2.$$

Terms with negative indices can also be defined by:

$$G_{-n} = G_{2-n} - G_{1-n} \quad \text{for } n \geq 1.$$

We say that

$$|G_1^2 - G_0G_2| = |A^2 + AB - B^2|$$

is the *characteristic* of $\{G_n\}$. In addition, we define a *conjugate sequence* $\{H_n\}$ by:

$$H_0 = B - A,\ H_1 = B - 2A,\ H_n = H_{n-1} + H_{n-2} \quad \text{for } n \geq 2.$$

It is easily seen that:

1. $G_n > 0$ and $H_n > 0$ for all $n \geq 0$;
2. $H_n = (-1)^n G_{-n} = |G_{-n}|$;
3. $\{G_n\}$ and $\{H_n\}$ have the same characteristic;
4. $\{G_n\}$ and $\{H_n\}$ are distinct unless $A = 1$, $B = 3$, in which case $G_n = H_n = L_n$ (the n^{th} Lucas number; see [1]).

Let $\{T_n\} = \{G_n\}$ or $\{H_n\}$. If M is any positive integer, we say M enters $\{T_n\}$ if there exists $K > 0$ such that $M|T_K$. The least such K will be called the *entry point* of M in $\{T_n\}$, and denoted $T(M)$. The entry point of M in the original Fibonacci sequence $\{F_n\}$ (which is guaranteed to exist) is denoted $Z(M)$. The entry point of M (if it exists) in $\{L_n\}$, $\{G_n\}$, $\{H_n\}$ will be denoted $L(M)$, $G(M)$, $H(M)$, respectively.

In this paper we prove the following theorems.

Theorem 1: If $M|G_0$, then M enters $\{G_n\}$ and $\{H_n\}$, and $G(M) = H(M) = Z(M)$.

Theorem 2: If $M\nmid G_0$ but M enters $\{G_n\}$, then M also enters $\{H_n\}$, and $G(M) + H(M) = Z(M)$.

Theorem 2 may be considered an entry point reciprocity law. We will make use of the following identities.

(1) $\quad T_{m+n} = F_{m-1}T_n + F_mT_{n+1}$

(2) $\quad G_n = F_{n-2}A + F_{n-1}B$

(3) $\quad H_n = -F_{n+2}A + F_{n+1}B$

(4) $\quad (T_n, T_{n+1}) = (F_n, F_{n+1}) = 1$

(5) $\quad F_{-n} = (-1)^{n-1}F_n$

(6) $\quad F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$

The Main Results

Proof of Theorem 1: Since $G_0 = H_0 = B - A$, and $(G_0, G_1) = (H_0, H_1) = 1$, it suffices to show that, if $\{T_n\}$ is a sequence such that $M|T_0$ and $(T_0, T_1) = 1$, then M enters $\{T_n\}$ and $T(M) = Z(M)$. (1) implies $T_K = F_{K-1}T_0 + F_K T_1$; therefore, hypothesis implies $T_K \equiv F_K T_1 \pmod{M}$, so that

$$T_{Z(M)} \equiv F_{Z(M)}T_1 \equiv 0 \pmod{M}.$$

Thus, M enters $\{T_n\}$ and $T(M) \le Z(M)$. Also

$$F_{T(M)}T_1 \equiv T_{T(M)} \equiv 0 \pmod{M}.$$

But $(T_0, T_1) = 1$, so $(M, T_1) = 1$. Therefore, $F_{T(M)} \equiv 0 \pmod{M}$. This implies $Z(M) \le T(M)$, so $T(M) = Z(M)$.

Lemma 1: Let $\{T_n\} = \{G_n\}$ or $\{H_n\}$. If X is an integer such that $0 < X < Z(M)$ and $T_X \equiv 0 \pmod{M}$, then $X = T(M)$.

Proof: Hypothesis implies $T(M) \le X$. Suppose $T(M) = Y < X$. (1) implies

$$T_X = T_{(X-Y)+Y} = F_{X-Y-1}T_Y + F_{X-Y}T_{Y+1}.$$

Thus,

$$T_X \equiv F_{X-Y-1}T_Y + F_{X-Y}T_{Y+1} \pmod{M}.$$

But hypothesis implies $T_X \equiv T_Y \equiv 0 \pmod{M}$, so $F_{X-Y}T_{Y+1} \equiv 0 \pmod{M}$. Hypothesis and (4) imply $(T_Y, T_{Y+1}) = 1$, so that $(M, T_{Y+1}) = 1$. Therefore, $F_{X-Y} \equiv 0 \pmod{M}$. But $0 < X - Y < X < Z(M)$, which contradicts the definition of $Z(M)$. Hence, $T(M) = X$.

Proof of Theorem 2: Let $n = G(M)$. Hypothesis and (2) imply $F_{n-2}A + F_{n-1}B \equiv 0 \pmod{M}$. (3) implies

$$H_{Z(M)-n} = -F_{Z(M)+2-n}A + F_{Z(M)+1-n}B.$$

Now (6) implies

$$F_{Z(M)+2-n} = F_{1-n}F_{Z(M)} + F_{2-n}F_{Z(M)+1} \equiv F_{2-n}F_{Z(M)+1} \equiv (-1)^{n-1}F_{n-2}F_{Z(M)+1} \pmod{M};$$

$$F_{Z(M)+1-n} = F_{-n}F_{Z(M)} + F_{1-n}F_{Z(M)+1} \equiv F_{1-n}F_{Z(M)+1} \equiv (-1)^n F_{n-1}F_{Z(M)+1} \pmod{M}.$$

[The last steps involved use of (5).] Therefore,

$$\begin{aligned} H_{Z(M)-n} &\equiv (-1)^n F_{n-2}F_{Z(M)+1}A + (-1)^n F_{n-1}F_{Z(M)+1}B \\ &\equiv (-1)^n F_{Z(M)+1}(F_{n-2}A + F_{n-1}B) \equiv 0 \pmod{M}. \end{aligned}$$

Thus, by Lemma 1,

$$H(M) = Z(M) - n = Z(M) - G(M).$$

Corollary 1: For $\{T_n\}$, if $T(M)$ exists, then $T(M) \le Z(M)$; if $T(M) = Z(M)$, then $M|T_0$.

This follows from Theorems 1 and 2.

Corollary 2: If M enters $\{L_n\}$ and $M > 2$, then $L(M) = \frac{1}{2}Z(M)$; $L(2) = Z(2) = 3$. Moreover, if $M > 2$ and if $Z(M)$ is odd, then M does not enter $\{L_n\}$.

Proof: $2|L_0$, so Theorem 1 implies $L(2) = Z(2) = 3$. If $M > 2$ and M enters $\{L_n\}$, then $M \nmid L_0$. Since $\{L_n\}$ is self-conjugate, Theorem 2 implies $2L(M) = Z(M)$, so $L(M) = \frac{1}{2}Z(M)$. Hence, when $M > 2$, M enters $\{L_n\}$ only when $Z(M)$ is even.

Acknowledgment

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Reference

1. Charles H. King. "Conjugate Generalized Fibonacci Sequences." *Fibonacci Quarterly 6.1* (1968):46-49.

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Announcement

FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 20-24, 1992

Department of Mathematical and Computational Sciences
University of St. Andrews
St. Andrews KY16 9SS
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CALL FOR PAPERS

The FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of St. Andrews, St. Andrews, Scotland from July 20 to July 24, 1992. This Conference is sponsored jointly by the Fibonacci Association and The University of St. Andrews.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts are to be submitted by March 15, 1992. Manuscripts are due by May 30, 1992. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

Professor Gerald E. Bergum
The Fibonacci Quarterly
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SUMMATION OF CERTAIN RECIPROCAL SERIES RELATED TO FIBONACCI AND LUCAS NUMBERS

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(Submitted July 1989)

1. Introduction

Some years ago, R. Backstrom and B. Popov (see [1], [2], [3]) computed sums of the form

$$\sum \frac{1}{F_{an+b} + c} \quad \text{and} \quad \sum \frac{1}{L_{an+b} + c},$$

for certain values of a, b, and c. For instance, Backstrom obtained

$$(1) \qquad \sum_{n=0}^{\infty} \frac{1}{L_{2n} + 3} = \frac{2\sqrt{5} + 1}{10}, \quad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 1} = \frac{\sqrt{5}}{2},$$

and he also gave the estimate

$$(2) \qquad \sum_{n=0}^{\infty} \frac{1}{L_{2n} + 2} \approx \frac{1}{8} + \frac{1}{4 \log \alpha} = 0.64452\ldots,$$

where α is the golden ratio. Recently, G. Almkvist [4] has given an exact formula connecting the last sum with Jacobi's theta functions.

The aim of this note is to obtain new results of the same kind. For example, we show that

$$(3) \qquad \sum_{n=0}^{\infty} \frac{1}{L_{2n} + \sqrt{5}} = \frac{1}{\alpha} = 0.618\ldots,$$

which can be compared with (2), and the surprising result

$$(4) \qquad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 3/\sqrt{5}} = 1.$$

In the final section, following Almkvist's method, we express the series

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 2/\sqrt{5}},$$

in terms of the theta functions, with the estimate

$$(5) \qquad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 2/\sqrt{5}} \approx \frac{\sqrt{5}}{4 \log \alpha} - \frac{\sqrt{5} \cdot \Pi^2}{(\log \alpha)^2 (e^{\Pi^2/\log \alpha} + 2)}.$$

2. Main Result

Theorem: Let s be a positive integer. then

$$(6) \qquad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + L_s/\sqrt{5}} = \frac{s}{2F_s}, \quad s \text{ even}, \ s \neq 0,$$

and

(7) $$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + \sqrt{5}F_s} = \frac{1}{L_s}\left(\frac{s-1}{2} + \frac{1}{1+\alpha^{-s}}\right), \quad s \text{ odd}.$$

3. Preliminaries

As noticed by Almkvist, it is probably better for this study to use direct calculus rather than Fibonaccian identities.

Lemma 1: Let q, σ be real numbers, with $q > 1$, and let s be a positive integer. Then the following equality holds:

(8) $$\sum_{n=0}^{\infty} \frac{1}{q^{n+\sigma} + q^{-n-\sigma} + q^{s/2} + q^{-s/2}} = \frac{1}{q^{s/2} - q^{-s/2}} \sum_{n=0}^{s-1} \frac{1}{1 + q^{n+\sigma-s/2}}.$$

Proof: One can readily verify that

$$\frac{1}{q^{n+\sigma} + q^{-n-\sigma} + q^{s/2} + q^{-s/2}} = \frac{1}{q^{-s/2} - q^{s/2}}\left(\frac{1}{1 + q^{n+\sigma+s/2}} - \frac{1}{1 + q^{n+\sigma-s/2}}\right).$$

Hence, by the telescoping effect, for $N \geq s - 1$,

$$\sum_{n=0}^{N} \frac{1}{q^{n+\sigma} + q^{-n-\sigma} + q^{s/2} + q^{-s/2}} = \frac{1}{q^{-s/2} - q^{s/2}}\left(\sum_{n=N-s+1}^{N} \frac{1}{1 + q^{n+\sigma+s/2}} - \sum_{n=0}^{s-1} \frac{1}{1 + q^{n+\sigma-s/2}}\right).$$

Letting $N \to \infty$, we obtain (8) (since $q > 1$).

Lemma 2: Let a and s be positive integers, let b be any integer, and define $T_s(a, b)$ by

$$T_s(a, b) = \sum_{n=0}^{s-1} \frac{1}{1 + \alpha^{a(2n-s)+b}}.$$

Then

(9) $$T_s(1, 0) = \frac{s-1}{2} + \frac{1}{1 + \alpha^{-s}}$$

and

(10) $$T_s(1, 1) = \frac{s}{2}.$$

Remark: Here, α is the golden ratio or any positive real number.

Proof:

$$T_{2s}(1, 0) = \sum_{n=0}^{2s-1} \frac{1}{1 + \alpha^{2n-2s}} = \frac{1}{1 + \alpha^{-2s}} + \frac{1}{2} + \sum_{k=1}^{s-1}\left(\frac{1}{1 + \alpha^{2k}} + \frac{1}{1 + \alpha^{-2k}}\right)$$

$$= \frac{1}{1 + \alpha^{-2s}} + \frac{1}{2} + s - 1 = \frac{1}{1 + \alpha^{-2s}} + s - \frac{1}{2}.$$

On the other hand,

$$T_{2s+1}(1, 0) = \sum_{n=0}^{2s} \frac{1}{1 + \alpha^{2n-2s-1}}$$

$$= \frac{1}{1 + \alpha^{-2s-1}} + \sum_{k=1}^{s}\left(\frac{1}{1 + \alpha^{2k-1}} + \frac{1}{1 + \alpha^{-2k+1}}\right)$$

$$= \frac{1}{1 + \alpha^{-2s-1}} + s.$$

This concludes the proof of (9). The proof of (10) follows the same pattern.

4. Proof of The Theorem and of Other Identities

As usual, the Fibonacci and Lucas numbers are defined by

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - (-1)^n\alpha^{-n}), \quad L_n = \alpha^n + (-1)^n\alpha^{-n}.$$

Let a, b be integers, with $a \geq 1$. Put $\sigma = b/2a$ and $q = \alpha^{2a}$ in (8) to get

$$(11) \quad \sum_{n=0}^{\infty} \frac{1}{\alpha^{2an+b} + \alpha^{-2an-b} + \alpha^{as} + \alpha^{-as}} = \frac{1}{\alpha^{as} - \alpha^{-as}} T_s(a, b),$$

where $T_s(a, b)$ is defined above. Let us examine different cases according to the parity of a, b, and s.

First case: b even, s or a even. Since (11) can be written as

$$(12) \quad \sum_{n=0}^{\infty} \frac{1}{L_{2anb} + L_{as}} = \frac{1}{\sqrt{5}F_{as}} T_s(a, b),$$

letting $s = 1$ in (11), we obtain Backstrom's Theorem V. Namely,

$$\sum_{n=0}^{\infty} \frac{1}{L_{2an+b} + L_a} = \frac{1}{\sqrt{5}F_a}\,\frac{1}{1 + \alpha^{b-a}}, \quad a \text{ even}, \ b \text{ even}.$$

Letting $b = 0$, $a = 1$, and applying (9), we obtain Backstrom's Theorem IV, which is

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + L_s} = \frac{1}{\sqrt{5}F_s}\left(\frac{s-1}{2} + \frac{1}{1 + \alpha^{-s}}\right), \quad s \text{ even}.$$

Second case: b even, s and a odd. Formula (11) becomes

$$(13) \quad \sum_{n=0}^{\infty} \frac{1}{L_{2an+b} + \sqrt{5}F_{as}} = \frac{1}{L_{as}} T_s(a, b).$$

With $s = 1$ in (13), we have

$$\sum_{n=0}^{\infty} \frac{1}{L_{2an+b} + \sqrt{5}F_a} = \frac{1}{L_a}\,\frac{1}{1 + \alpha^{b-a}}, \quad b \text{ even}, \ a \text{ odd}.$$

Letting $a = 1$ and $b = 0$ in (13) and applying (9), we get (7) so that

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + \sqrt{5}F_s} = \frac{1}{L_s}\left(\frac{s-1}{2} + \frac{1}{1 + \alpha^{-s}}\right), \quad s \text{ odd}.$$

As special cases, we have

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + \sqrt{5}} = \frac{1}{\alpha} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{L_{2n} + 2\sqrt{5}} = \frac{1}{4} + \frac{\alpha}{8}.$$

Third case: b odd, s or a even. Formula (11) becomes

$$(14) \quad \sum_{n=0}^{\infty} \frac{1}{F_{2an+b} + L_{as}/\sqrt{5}} = \frac{1}{F_{as}} T_s(a, b).$$

With $s = 1$ in (14), we get

$$\sum_{n=0}^{\infty} \frac{1}{F_{2an+b} + L_a/\sqrt{5}} = \frac{1}{F_a}\,\frac{1}{1 + \alpha^{b-a}},\quad b \text{ odd},\ a \text{ even}.$$

When $a = 2$ and $b = 1$, we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{4n+1} + 3/\sqrt{5}} = \frac{1}{\alpha}.$$

Now, put $a = 1$, $b = 1$ in (14) and use (10) to get (6) so that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + L_s/\sqrt{5}} = \frac{s}{2F_s},\quad s \text{ even}.$$

As special cases, we mention (4) and

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 7/\sqrt{5}} = \frac{2}{3}.$$

Last case: b odd, s and a odd. Formula (11) becomes

$$(15)\qquad \sum_{n=0}^{\infty} \frac{1}{F_{2an+b} + F_{as}} = \frac{\sqrt{5}}{L_{as}}\,T_s(a, b).$$

Putting $s = 1$ in (15), we obtain Backstrom's Theorem II, which is

$$\sum_{n=0}^{\infty} \frac{1}{F_{2an+b} + F_a} = \frac{\sqrt{5}}{L_a}\,\frac{1}{1 + \alpha^{b-a}},\quad b \text{ odd},\ a \text{ odd}.$$

With $a = b = 1$ in (15), we obtain Backstrom's Theorem I, which is

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + F_s} = \frac{s\sqrt{5}}{2L_s},\quad s \text{ odd}.$$

Remark: Consider the recurrence relation

$$W_n = pW_{n-1} + W_{n-2},\quad n \geq 2,\ p > 0,$$

and the solutions

$$U_n = \frac{\alpha^n - (-1)^n\alpha^{-n}}{\sqrt{\Delta}},\qquad V_n = \alpha^n + (-1)^n\alpha^{-n},$$

where $\Delta = p^2 + 4$, $\alpha = \dfrac{p + \sqrt{\Delta}}{2} > 1$.

The results above could be generalized with U_n, V_n, $\sqrt{\Delta}$ in place of F_n, L_n, $\sqrt{5}$.

5. A New Tantalizing Problem

Let us return to (6). When putting $s = 0$ in the left-hand side, we obtain the convergent series

$$(16)\qquad \sum_{0}^{\infty} \frac{1}{F_{2n+1} + 2/\sqrt{5}} = 1.161685787\ldots,$$

where the number on the right is not one we recognize. Using the limit process introduced by Backstrom ([1], p. 20), we would have

$$\lim_{s \to 0} \frac{s}{2F_s} = \frac{\sqrt{5}}{4 \log \alpha} = 1.16168590\ldots,$$

so we see that $\sqrt{5}/(4 \log \alpha)$ is a good estimate of the sum (16).

Using the method introduced by Almkvist, we can now express (16) in terms of the theta functions. In fact, we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 2/\sqrt{5}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1 + q^{2n+1})^2},$$

where $q = \alpha^{-1}$. By a classical formula (see, e.g., [5], p. 471), we can write

$$\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1 + q^{2n+1})^2} = -\frac{1}{8\Pi^2} \frac{\nu_3''}{\nu_3},$$

where (with Almkvist's notations)

$$\nu_3 = \sqrt{-\frac{\Pi}{\log q}} \sum_n e^{\Pi^2 n^2/\log q}$$

and

$$\nu_3'' = \frac{2\Pi^2}{\log q}\sqrt{-\frac{\Pi}{\log q}} \sum_n \left(1 + \frac{2\Pi^2 n^2}{\log q}\right) e^{\Pi^2 n^2/\log q}.$$

(The summation is over all integers n.)

After some calculus, we obtain the final formula

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 2/\sqrt{5}} = \frac{\sqrt{5}}{4 \log \alpha} - \frac{\Pi^2\sqrt{5}}{(\log \alpha)^2} \frac{\sum_{n=1}^{\infty} n^2 e^{-\Pi^2 n^2/\log \alpha}}{1 + 2\sum_{n=1}^{\infty} e^{-\Pi^2 n^2/\log \alpha}},$$

which can be compared with Almkvist's formula for

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + 2}.$$

Limiting ourselves to the first term ($n = 1$), we get the estimate (5).

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SEQUENCES OF INTEGERS SATISFYING RECURRENCE RELATIONS

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Let us consider the recurrence relation

(1) $$n^3u_n - (34n^3 - 51n^2 + 27n - 5)u_{n-1} + (n - 1)^3u_{n-2} = 0.$$

Apery has proved that for $(u_0, u_1) = (1, 5)$ all of the u_n's are integers, and it is proved in [1] and [2] that, if all the numbers of a sequence satisfying (1) are integers, then $(u_0, u_1) = \lambda(1, 5)$, where λ is an integer. We give here a generalization of this result, with a simple proof, and applications to Apery's numbers as well as to the recurrence relation

(2) $$L_{n-1}F_nu_n - 5F_nF_{n-1}F_{2n-1}u_{n-1} - F_{n-1}L_nu_{n-2} = 0,$$

where F_n, L_n are the usual Fibonacci and Lucas numbers.

1. The Main Result

Let $\{a_n\}$, $\{b_n\}$ be two sequences of rational numbers with $\{u_n\}$ the sequence defined by (u_0, u_1) and the recurrence relation

(3) $$u_n = a_nu_{n-1} + b_nu_{n-2}, \ n \ge 2.$$

We then have two results.

Theorem 1: Suppose that

(4) a) For all integers $n \ge 2$, $b_n \ne 0$.

(5) b) There exists a real number P such that $\lim_{n\to\infty} \prod_{k=2}^{n} |b_k| = P$.

Then the recurrence relation (3) has two linearly independent integer solutions only if $|b_n| = 1$ for all large n.

THeorem 2: Suppose that

(6) a) For all $n \ge 2$, $b_n \ne 0$ and $|b_n| = 1$ for all large n.

(7) b) For all $n \ge 2$, $a_n \ne 0$ and $\lim_{n\to\infty}|a_n| = a$.

Then relation (3) has two linearly independent integer solutions only if $a_n = a$ for all large n, where a is an integer different from zero.

Remark: Recall that two sequences $\{p_n\}$ and $\{q_n\}$ are linearly dependent if two numbers (λ, μ) exist (not both zero) such that, for all n,

$$\lambda p_n + \mu q_n = 0.$$

In the other case, the sequences are linearly independent. It is easy to prove that $\{p_n\}$ and $\{q_n\}$, when satisfying (3), are linearly dependent if and only if

(8) $$p_0q_1 - p_1q_0 = 0.$$

2. Proof of Theorem 1

Let us suppose that $\{p_n\}$ and $\{q_n\}$ are two independent integer solutions of (3) and define the sequence Δ_n by

(9) $$\Delta_n = p_{n-1}q_n - p_nq_{n-1}, \ n \ge 1.$$

It is easily proved that

(10) $\Delta_n = -b_n\Delta_{n-1}$, $n \geq 2$.

Hence,

(11) $\Delta_n = (-1)^{n-1}b_2 \ldots b_n\Delta_1$, $n \geq 2$.

By the Remark above, $\Delta_1 = p_0q_1 - p_1q_0 \neq 0$, and by (5) we have

(12) $\lim_{n\to\infty}|\Delta_n| = |\Delta_1|P$;

thus, the sequence of integers $|\Delta_n|$ converges and we deduce from (12) that

(13) $|\Delta_n| = |\Delta_1|P$, for all large n.

By (11) we have $\Delta_n \neq 0$ for all n (since $b_n \neq 0$ and $\Delta_1 \neq 0$). Hence, (13) shows that $P \neq 0$. By (10) we have

$$1 = \frac{|\Delta_n|}{|\Delta_{n-1}|} = |b_n|, \text{ for all large } n.$$

This concludes the proof of Theorem 1.

3. Proof of Theorem 2

Suppose that $\{p_n\}$ and $\{q_n\}$ are two independent integer solutions of (3) and define the sequence D_n of integers by

$$D_n = p_{n-2}q_n - p_nq_{n-2}, \quad n \geq 2.$$

It is obvious that

(14) $D_n = a_n\Delta_{n-1}$, $n \geq 2$.

However, by (6) we have, for n large, since $|b_n| = 1$,

$$|\Delta_n| = |\Delta_1|P \neq 0.$$

Hence,

(15) $|D_n| = |a_n||\Delta_1|P \neq 0$, for all large n,

and by (7),

$$\lim_{n\to\infty}|D_n| = a|\Delta_1|P.$$

Thus, for all large n,

(16) $|D_n| = a|\Delta_1|P$.

Note that $a \neq 0$, since $D_n \neq 0$, and that a is a rational number by (16). Comparison of (15) and (16) shows that

$$|a_n| = a, \text{ for all large } n.$$

Let us now write $a = p/q$, where p and q are relatively prime integers. Without loss of generality, we can assume that

(17) $u_n = \pm\frac{p}{q}u_{n-1} \pm u_{n-2}$, for $n \geq 2$.

Consider the solution $\{v_n\}$ of (17) defined by the initial values (0, 1). Note that $\Delta_1 v_n$ is an integer, namely,

$$\Delta_1 v_n = -q_0p_n + p_0q_n.$$

The relation

$$\Delta_1 v_n = \pm\frac{p}{q}\Delta_1 v_{n-1} \pm \Delta_1 v_{n-2}$$

shows that

$$q \mid \Delta_1 v_{n-1}, \text{ for } n \geq 2.$$

By mathematical induction, it is easy to prove that for all integers $m \geq 1$ and $n \geq 1$, $q^m \mid \Delta_1 v_n$. Therefore, $q = 1$, and a is an integer.

4. Application

Suppose that $|b_n| = C_{n-1}/C_n$, with $C_n \neq 0$ for all n, $C_n \neq C_{n-1}$, and

$$\lim_{n\to\infty} C_n = C.$$

We can then write

$$\prod_{k=2}^{n} |b_k| = \frac{C_1}{C_n}, \text{ so that } P = \frac{C_1}{C}.$$

By Theorem 1, the sequence (3) cannot have two linearly independent solutions, since $|b_n| \neq 1$.

This result can be applied to (1) with $C_n = n^3$, and also to the recurrences

$$nu_n - (2m + 1)(2n - 1)u_{n-1} + (n - 1)u_{n-2} = 0, \tag{18}$$

and

$$n^2u_n - (11n^2 - 11n + 3)u_{n-1} - (n - 1)^2u_{n-2} = 0, \tag{19}$$

with $C_n = n$ in (18), $C_n = n^2$ in (19). Note that (18) and (19) admit integer solutions defined by the initial values $(1, 2m + 1)$ [resp. $(1, 3)$]. The integer solution of (18) is simply $u_n = P_n(-m)$, where

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}[x^n(1 - x)^n] = \prod_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k x^k$$

is the n^{th} Legendre polynomial over $[0, 1]$ (see [3] for another proof). Equations (1) and (19) appear in Apery's proof of the irrationality of $\zeta(3)$ and $\zeta(2)$.

Now, let us consider recurrence (2), in which we have

$$b_n = \frac{F_{n-1}L_n}{L_{n-1}F_n}.$$

Then

$$\prod_{k=2}^{n} b_k = \frac{L_n}{F_n} \quad \text{and} \quad P = \lim_{n\to\infty}\frac{L_n}{F_n} = \sqrt{5}.$$

By Theorem 1, the sequence (2) cannot have two linearly independent integer solutions. it will be shown below (and in [4]) that the solution $\{q_n\}$ defined by the initial values $(1, 0)$ is an integer sequence. On the other hand, the solution $\{p_n\}$ defined by the initial values $(0, 1)$ cannot be an integer sequence. Let us write the first few values of these two sequences in order to see this. They are:

n	0	1	2	3	4	5 ...
p_n	0	1	10	84	$\frac{8225}{3}$	$\frac{999146}{5}$ ...
q_n	1	0	3	25	816	59475 ...

It can also be shown that

$$\lim_{n\to\infty}\frac{p_n}{q_n} = \sum_{k=1}^{\infty}\frac{1}{F_k} = 3.35988566624\ldots .$$

Notice how quickly p_n/q_n converges. We have

$$\frac{p_4}{q_4} = 3.3598856\ldots \quad \text{and} \quad \frac{p_5}{q_5} = 3.35988566624\ldots\ .$$

One can deduce from this that $\sum_{k=1}^{\infty}(1/F_k)$ is irrational (see [4]).

5. Generalization

Consider the recurring sequence defined by $u_0, \ldots, u_{r-1}$ and

(20) $\quad u_n = a_n^1 u_{n-1} + a_n^2 u_{n-2} + \cdots + a_n^r u_{n-r},\ n \geq r,$

where r is a strictly positive integer, and where $\{a_n^1\}, \ldots, \{a_n^r\}$ are sequences of rational numbers. By analogy with Theorem 1, we have the following result.

Theorem 1': Suppose that

(a) For all $n \geq r$, $a_n^r \neq 0$.

(b) There exists a number P such that $\lim_{n\to\infty} \prod_{k=r}^{n} |a_k^r| = P$.

Then (20) has r linearly independent integer solutions only if $|a_n^r| = 1$ for all large n.

Proof: Suppose that $\{p_n^1\}, \ldots, \{p_n^r\}$ are r linearly independent integer sequence solutions of (20) and define the sequence Δ_n of integers by $r \times r$ determinant

$$\Delta_n = \left|p_{n-r+j}\right|_{\substack{1 \leq i \leq r\\ 1 \leq j \leq r}},\ n \geq r - 1.$$

It is easily proved that $\Delta_n = (-1)^{r-1} a_n \Delta_{n-1}$. Hence,

$$|\Delta_n| = |\Delta_{r-1}| \prod_{k=r}^{n} |a_k^r|,\ n \geq r.$$

We have $\Delta_{r-1} \neq 0$, since the $\{p_n^i\}$'s are independent, and the end of the proof is as in Theorem 1.

The reader can also find a theorem analogous to Theorem 2.

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CONJECTURES ABOUT s-ADDITIVE SEQUENCES

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A strictly increasing sequence of positive integers $a_1, a_2, \ldots$ is defined to be *s-additive* [1] if, for $n > 2s$, a_n is the least integer greater than a_{n-1} having precisely s representations $a_i + a_j = a_n$, $i < j$. The first $2s$ terms of an s-additive sequence are called the *base* of the sequence. An s-additive sequence, for a given base, may be either finite or infinite; the sequence is assumed to be maximal in the sense that the total number of terms is as large as possible. Consider, for example, the case in which $s = 1$, $a_1 = 1$, and $a_2 = 2$. The next fifteen terms of the sequence are 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53. The sequence is infinite (as is any 1-additive sequence) since $a_{n-3} + a_{n-1}$ is an integer greater than a_{n-1} with no other representation $a_i + a_j$ and, hence, there exists a least such integer. It is the archetypal s-additive sequence, and was first studied by Stanislaw Ulam [2]. An example of a 2-additive sequence is 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 17, 19, 29, 31, 33, 43, 44, 47, 51, ..., which also appears to be infinite (though a proof of this fact is not known). Not all 2-additive sequences are infinite, as illustrated by the sequence 1, 3, 5, 7, 8.

For $s \geq 1$, Raymond Queneau [1] showed that an s-additive sequence has at least $2s + 2$ terms if and only if there exist positive integers u and v such that the $2s$ numbers in the base (up to ordering) are $u, 2u, \ldots, su, v, u + v, 2u + v, \ldots, (s - 1)u + v$. This is called *Condition* u, v. We denote an s-additive sequence satisfying Condition u, v by the ordered triple (s, u, v). Note that the correspondence between such sequences and ordered triples is not one-to-one, since $(s, 1, s + 1) = (s, 2, 1)$. Queneau undertook a detailed examination of various properties of s-additive sequences, including conditions for sequences to be infinite and conditions for sequences to be regular (in a sense to be defined shortly). Some of the conjectures in [1] are consistent with conjectures presented here; some others are false due to several unfortunate errors in Queneau's computations.

We examine first conditions for s-additive sequences to be infinite.

Conjecture 1: A 2-additive sequence is infinite if and only if Condition u, v is met.

For $s \geq 3$, Condition u, v is necessary but not sufficient for infinitude, as evidenced by the finite 4-additive sequence $(4, 1, 5)$ = 1 (1) 10, 12 (2) 20, 23 (2) 31, 36, 38, 47, 48, 49, 51, 53, 60, 80, 85. In order to state Conjectures 2 through 4, we assume that Condition u, v is satisfied and that, without loss of generality, u and v are relatively prime. These two assumptions hold throughout the remainder of this paper.

Conjecture 2: An s-additive sequence, when $3 \leq s \leq 6$, is infinite if and only if

(a) $u = 1$ and v is as in Table 1,
(b) $u = 2$ and v is as in Table 2, or
(c) $u \geq 3$.

Conjecture 3: An s-additive sequence, when s is even and $8 \leq s \leq 20$, is infinite if and only if

(a) $u = 2$ and v is as in Table 3, or
(b) $u \geq 3$.

Table 1: Conditions associated with Conjecture 2(a)

s	Conditions on v
3	$v > 3$
4	$5 < v \neq 8, 12, 13, 17, 22$
5	$v = 6, 9, 13, 15$
6	$v = 8$

Table 2: Conditions associated with Conjecture 2(b)

s	Conditions on v
3	$v = 1, 5$ or $\equiv 3 \bmod 4$
4	$v \equiv 3, 7, 9 \bmod 10$
5	$v = 1$ or $\equiv 9 \bmod 12$
6	$v \equiv 3, 5, 7, 9, 11 \bmod 14$

Table 3: Conditions associated with Conjecture 3(a)

s	Conditions on v
8	$v = 3, 11, 21, 25, 29, 39, 57, 61, 65, 75, 83, 93, 97, 101, 111, 119, 129, 133, 137, 147$ or $151 < v \equiv 3, 7, 11 \bmod 18$
10	$v = 19, 23, 45, 51, 67, 89, 95, 107$ or $111 < v \equiv 1, 7, 19 \bmod 22$
12	$v = 47, 143, 169, 177, 183, 235, 261, 307, 313, 333, 339, 365, 391$ or $411 < v \equiv 1, 21 \bmod 26$
14	$v = 189, 249, 279, 309, 339, 369, 375, 399, 429, 459, 489, 519, 525, \ldots, 939$ or $945 < v \equiv 9, 15 \bmod 30$
16	$v = 187, 323, 663, 731, 833, 893, 935, 969, 995, 1003, 1029, 1037, 1063, \ldots, 1649$ or $1675 < v \equiv 9, 17 \bmod 34$
18	$v = 417, 645, 759, 873, 979, 987, 1101, 1215, 1329, 1443, 1519, 1557, \ldots, 3305$ or $3343 < v \equiv 37 \bmod 38$
20	$v = 439, 1333, 1343, 1543, 1573, 1615, 1627, 1637, 1657, 1699, 1741, 1783, 1867, 1889, \ldots, 4429$ or $4451 < v \equiv 19, 41 \bmod 42$

Conjecture 4: An s-additive sequence, when s is odd and $s \geq 7$, is infinite if and only if $u \geq 3$.

The sequence (24, 2, 1523) appears to be infinite, whereas (22, 2, v) is never infinite. Proof that certain sequences are finite is not difficult; for example, (3, 2, v) has $(7v + 53)/4$ terms ($a_{(7v+53)/4} = 10v + 10$) when $5 < v \equiv 1 \bmod 4$. However, no s-additive sequence, $s > 1$ and $u \leq 2$, has been proven to be infinite. Note that the example involving (3, 2, v) shows that arbitrarily long finite sequences exist. Long sequences are computationally unwieldy since all terms $a_1, \ldots, a_{n-1}$ must be considered when determining a_n. Thus, the computer evidence leading to Conjectures 1 through 4 is necessarily limited.

We turn now to regularity issues. An infinite s-additive sequence is *regular* if successive differences $a_{n+1} - a_n$ are eventually periodic; i.e., there is a positive integer N such that $a_{N+n+1} - a_{N+n} = a_{n+1} - a_n$ for all sufficiently large n. (The smallest such N is called the *period*.) An equivalent condition involves *arithmetic multiprogressions* [1] which are infinite sequences of the form

$$c_1, c_2, \ldots, c_k, b + c_1, b + c_2, \ldots, b + c_k,$$
$$2b + c_1, 2b + c_2, \ldots, 2b + c_k, \ldots,$$

where $0 < c_1 < c_2 < \cdots < c_k < b + c_1$. If some arithmetic multiprogression, after at most finitely many deletions of certain terms or insertions of additional terms, is equal to the s-additive sequence (s, u, v), then (s, u, v) is regular. We write this condition more compactly as

$$(s,\ u,\ v) \sim bn + [c_1,\ c_2,\ \ldots,\ c_k] \quad n = 0,\ 1,\ 2,\ \ldots,$$

in which the symbol ~ is to be interpreted as eventual equality. If greater precision is required, we write

$$(s,\ u,\ v) = bn + [c_1,\ c_2,\ \ldots,\ c_k] \oplus d_1,\ \ldots,\ d_p \ominus e_1,\ \ldots,\ e_q,$$

where $d_1, \ldots, d_p$ and $e_1, \ldots, e_q$ are, respectively, the inserted and deleted terms on the right-hand side that make equality hold.

The nature of Conjectures 2 through 4 might lead one to suspect that something is special about the case $u \geq 3$. This is true, in fact, as proved by Queneau in [1].

Theorem 1: If $s > 1$ and $u \geq 3$, then (s, u, v) is regular and

$$(s,\ u,\ v) = nu + [v] \oplus u,\ 2u,\ \ldots,\ su,\ (2s - 1)u + 2v.$$

A consequence of this result and Conjecture 4 is that there do not exist infinite irregular sequences when s is odd and $s \geq 7$. No analogous general formulas appear to hold for the remaining cases $s = 1$ or $u \leq 2$. A limited computer search for regular 1-additive sequences has uncovered many examples, some of which are exhibited in Table 4. (The first three of these were found by Queneau [1].) We conjecture that $(1, u, v)$ is regular for a wide variety of u and v. Though a proof is not known, a sensible argument might be based on Theorem 2 and (deceptively simple) Conjecture 5. Periods for $(1, 2, v)$ and for $(1, 4, v)$, as fascinatingly intricate functions of odd $v > 3$, are listed in Table 5. [Some cases have either incalculably long periods or long initial stretches before periodicity begins. For example, the period for $(1, 2, v)$, where $35 \leq v \leq 41$ is odd, probably exceeds 10^9.]

Table 4: Regular 1-additive sequences

$(1, 2, 5) = 126n + [5\ (2)\ 15,\ a_9 = 19,\ a_{10} = 23,\ \ldots,\ a_{34} = 119] \oplus 2, 12$

$(1, 2, 7) = 126n + [7\ (2)\ 21,\ a_{11} = 25,\ a_{12} = 29,\ \ldots,\ a_{28} = 117] \oplus 2, 16$

$(1, 2, 9) = 1778n + [9\ (2)\ 27,\ a_{13} = 31,\ a_{14} = 35,\ \ldots,\ a_{446} = 1767] \oplus 2, 20$

$(1, 2, 11) = 6510n + [11\ (2)\ 33,\ a_{15} = 37,\ a_{16} = 41,\ \ldots,\ a_{1630} = 6497] \oplus 2, 24$

$(1, 2, 13) = 23622n + [13\ (2)\ 39,\ a_{17} = 43,\ a_{18} = 47,\ \ldots,\ a_{5908} = 23607] \oplus 2, 28$

$(1, 2, 15) = 510n + [15\ (2)\ 45,\ a_{19} = 49,\ a_{20} = 53,\ \ldots,\ a_{82} = 493] \oplus 2, 32$

$(1, 2, 17) = 507842n + [17\ (2)\ 51,\ a_{21} = 55,\ a_{22} = 59,\ \ldots,$
$a_{126962} = 507823] \oplus 2, 36$

$(1, 4, 5) = 192n + [5\ (4)\ 17,\ 19,\ 21,\ a_{10} = 25,\ a_{11} = 27,\ \ldots,\ a_{35} = 173]$
$\oplus 4, 14, 24$

$(1, 4, 9) = 640n + [9\ (4)\ 29,\ 31,\ 33,\ 37\ (2)\ 41,\ a_{15} = 45,\ a_{16} = 47,\ \ldots,$
$a_{91} = 609] \oplus 4, 22, 40$

$(1, 4, 11) = 1318n + [11\ (4)\ 27,\ 37,\ 39,\ 43\ (2)\ 47,\ 51\ (2)\ 57,\ 61,\ 67,\ 69,\ 75,\ 77$
$83,\ 85,\ 89,\ 91,\ 99,\ 105,\ a_{29} = 111,\ a_{30} = 123,\ \ldots,\ a_{249} = 1309]$
$\oplus 4,\ 26,\ 31,\ 35,\ 48 \ominus 57,\ 105$

$(1, 4, 13) = 896n + [13\ (4)\ 41,\ 43,\ 45,\ 49\ (2)\ 53,\ a_{17} = 57,\ a_{18} = 59,\ \ldots,$
$a_{107} = 853] \oplus 4,\ 30,\ 56$

$(1, 4, 17) = 2304n + [17\ (4)\ 53,\ 55,\ 57,\ 61\ (2)\ 65,\ 69\ (2)\ 73,\ a_{22} = 77,$
$a_{23} = 79,\ \ldots,\ a_{251} = 2249] \oplus 4,\ 38,\ 72$

$(1, 4, 19) \sim 2560n + [a_{2552} = 14753,\ a_{2553} = 14761,\ \ldots,\ a_{2903} = 17275]$

$(1, 4, 21) = 2816n + [21\ (4)\ 65,\ 67,\ 69,\ 73\ (2)\ 77,\ 81\ (2)\ 85,\ a_{24} = 89,$
$a_{25} = 91,\ \ldots,\ a_{283} = 2749] \oplus 4,\ 46,\ 88$

Table 5: Periods for $(1, u, v)$, $u = 2$ and 4

v	$u = 2$	$u = 4$
5	32	32
7	26	-
9	444	88
11	1628	246
13	5906	104
15	80	-
17	126960	248
19	380882	352
21	2097152	280
23	1047588	5173
25	148814	304
27	8951040	10270
29	5406720	320
31	242	-
33	127842440	712
35	-	826
37	-	776
39	-	108966
41	-	824

Theorem 2: If a 1-additive sequence has only finitely many even terms, then the sequence is regular.

Proof; Let e denote the number of even terms in the 1-additive sequence a_1, a_2, a_3, ... Let $x_1 < x_2 < \cdots < x_e$ be the even terms and let $y_k = x_k/2$ for each k, where $1 \leq k \leq e$. Given an integer $n \geq y_e$, define

$$b_n = \text{the number of representations } a_i + a_j = 2n + 1,\ i < j.$$

Observe that $a_i + a_j = 2n + 1$ only if either a_i or a_j is equal to some x_k (since a sum of two integers is odd if and only if one of the integers is odd and the other is even). This observation gives rise to the following recursive formula:

$$b_n = \sum_{k=1}^{e} \delta(b_{n-y_k} - 1)$$

where $\delta(0) = 1$ and $\delta(r) = 0$ for $r \neq 0$. The summation simply counts the number of times (out of e) that $2n - x_k + 1$ is a term in a_1, a_2, ... Define now, for each $n \geq x_e$, a vector of y_e components

$$\beta_n = (b_{n-y_e}\ b_{n-y_e+1}\ b_{n-y_e+2}\ \cdots\ b_{n-1})^{\mathsf{T}}.$$

Regularity of the 1-additive sequence a_1, a_2, ... is clearly equivalent to eventual periodicity of the vector sequence β_{x_e}, β_{x_e+1}, ... The components of β_n obviously do not exceed e. Since the number of vectors of length y_e containing 0, 1, ..., $e - 1$ or e is $(e + 1)^{y_e} < \infty$, some β_n must recur, which, in turn, brings about periodicity by the recursive formula. This completes the proof.

Recall that u and v are assumed to be relatively prime. Assume, moreover, that $u < v$.

Conjecture 5:

$(1, 1, v)$ has infinitely many even terms.
$(1, 2, v)$ has two even terms (specifically $a_1 = 2$ and $a_{(v+7)/2} = 2v + 2$) when $v > 3$; it has infinitely many even terms when $v = 3$.
$(1, 3, v)$ has infinitely many even terms.

(1, 4, v) has four even terms when $v = 2^k - 1$ for some $k = 3, 4, 5, \ldots$;
otherwise, it has three even terms.
(1, 5, v) has thirteen even terms when $v = 6$;
otherwise, it has infinitely many even terms.
(1, u, v), for even $u \geq 6$, has $2 + u/2$ even terms.
(1, u, v), for odd $u \geq 7$, has $2 + v/2$ even terms when v is even;
otherwise, it has infinitely many even terms.

There is no reason for even terms to be small; for example, (1, 4, 255) has $a_{8750} = 260606$.

Other interesting trends exist in the distribution of successive differences $a_{n+1} - a_n$ for these sequences. Let us focus on (1, 2, v), $v > 3$, for definiteness. The successive differences are always even beyond a certain point. For most of a period, the successive differences remain relatively small. As the end of the period draws near, the successive differences seem to explode to a maximum value (= $2v + 2$), which concludes the period and a new period begins. In contrast, the sequence (1, 2, 3) appears to possess unbounded successive differences. This seems to occur as well for the sequence (s, 1, $s + 1$), for each $s = 1, 2, 3$, and 5; e.g., when $s = 2$, $a_{9384} - a_{9383} = 174886 - 174579 = 307$. Many questions arise. Is the converse of Theorem 2 true? Do there exist regular s-additive sequences for $s > 1$ and $u \leq 2$? Is it possible for successive differences of an infinite *irregular* s-additive sequence to be *bounded*?

Queneau also introduces several generalizations of s-additivity, of which we discuss one. (Replacing addition by multiplication in the definition of s-additivity defines s-multiplicativity. This has not been studied. Nor has substituting the condition $i < j$ by $i \leq j$.) A strictly increasing sequence of positive integers $a_1, a_2, \ldots$ is defined to be (s, t)-*additive with base* B if B consists of the first m terms $a_1, a_2, \ldots, a_m$ for some positive integer m and if, for $n > m$, a_n is the least integer greater than a_{n-1} having precisely s representations of the form

$$a_{i_1} + a_{i_2} + \cdots + a_{i_t} = a_n, \quad i_1 < i_2 < \cdots < i_t.$$

Note that an s-additive sequence is the same as an $(s, 2)$-additive sequence with $m = 2s$. Note also that, while $m \geq 2s$ is necessary for $(s, 2)$-additivity and $m \geq t$ is necessary for $(1, t)$-additivity, $m = 5$ is possible in conjunction with (2, 3)-additivity. Lacking a suitable analogue of Condition u, v for s-additivity, we write an (s, t)-additive sequence as $(s, t; a_1, \ldots, a_m)$. For example,

(2,3;1,2,3,4,5) = 1(1)5, 8(1)11, 25, 28, 29, 49, 66, 67, 69, 89, 92, 110, 111, ...

which appears to be infinite. As previously, any $(1, t)$-additive sequence, for $t \geq 2$, is infinite, while extension of the proof to (s, t)-additive sequences, for $s > 1$, does not seem possible. We conclude with several more arithmetic multiprogression formulas obtained by limited computer search for regular (1, 3)-additive sequences (see Table 6). The first of these was found by Peter N. Muller and also appears in [3].

Table 6: Regular (1, 3)-additive sequences

(1, 3; 1, 2, 3) ~	$25n$ + [80, 82, 104]
(1, 3; 1, 2, 9) ~	$572n$ + [581 (1) 590, 645 (1) 653, 708 (1) 717, 772 (1) 781, 836 (1) 844, 899 (1) 908, 963 (1) 972, 1027 (1) 1035, 1090 (1) 1098]
(1, 3; 1, 3, 4) ~	$219n$ + [411, 412, 444, 446, 481, 482, 517, 521, 554, 555, 591, 626]
(1, 3; 1, 3, 5) ~	$82n$ + [87, 89, 115, 117, 141, 143]
(1, 3; 1, 3, 6) ~	$51n$ + [164 (1) 167, 211 (1) 213]
(1, 3; 1, 3, 7) ~	$20n$ + [23]
(1, 3; 2, 3, 4) ~	$148n$ + [157, 159, 160, 203, 204, 206 (1) 208, 253 (1) 255, 258, 302]

Acknowledgment

I wish to thank Jane Hale [4] for introducing me to Queneau's literary work and for bringing my attention to the sequences discussed in this paper.

Postscript

Recent computations show that

$$(1, 4, 7) \sim 11301098n + [a_{13671499} = 80188457, \ldots, a_{15599457} = 91489549]$$

and

$$(1, 5, 6) \sim 1720n + [a_{156303} = 1579049, \ldots, a_{156510} = 1580767];$$

thus, (1, 4, 7) and (1, 5, 6) have periods 1927959 and 208, respectively. Further results on the regularity of certain 1-additive sequences will appear in a forthcoming paper.

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A NOTE ON EULER'S NUMBERS

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(Submitted July 1989)

Recently, Y. Imai, Y. Seto, S. Tanaka, and H. Yutani [1] defined the coefficients $Z(m, r)$ by

$$Z(m, r) = \sum_{k=1}^{r} (-1)^{r+k} \binom{m+1}{r-k} k^m \quad (m \geq 1,\ r = 1,\ \ldots,\ m), \tag{1}$$

$$Z(m, r) = 0 \quad (m \leq 0 \text{ or } r \leq 0 \text{ or } m < r),$$

and proved that

$$x^m = \sum_{r=1}^{m} \left(\frac{Z(m, r)}{m!} \prod_{i=1}^{m} (x + i - r) \right) \quad (x,\ m \in N),$$

$$Z(m, r) = Z(m,\ m + 1 - r),$$

$$\sum_{r=1}^{m} Z(m, r) = m! \quad (m \geq 1,\ r = 1,\ \ldots,\ m), \tag{2}$$

$$Z(m + 1,\ r) = (m - r + 2) Z(m,\ r - 1) + rZ(m, r).$$

In this short note we will show that the coefficients $Z(m, r)$ are just Euler's numbers $A_{m,r}$ introduced in 1755 by

$$A_{m,r} = \sum_{k=0}^{r-1} (-1)^k \binom{m+1}{k} (r - k)^m.$$

Indeed, using the substitution $j = r - k$, from (1) follows

$$\sum_{k=1}^{r} (-1)^{r+k} \binom{m+1}{r-k} k^m = \sum_{j=0}^{r-1} (-1)^j \binom{m+1}{j} (r - j)^m,$$

i.e., that $Z(m, r) = A_{m,r}$.

In [1] the authors mentioned that it would be interesting to find a connection between the coefficients $Z(m, r)$ and Stirling's numbers of the second kind $S(n, k)$. Since $Z(m, r) = A_{m,r}$, we have the following relations (see, e.g., [2], [3])

$$Z(m, r) = \sum_{k=0}^{m-r} (-1)^k \binom{r+k-1}{k-1} (m - k - r + 1)!\,S(m,\ m - r - k + 1),$$

$$Z(m, r) = \sum_{k=1}^{m} (-1)^{m-r+k-1} \binom{m-k}{k-1} k!\,S(m, k),$$

$$k!\,S(m, k) = \sum_{r=1}^{m} Z(m, r) \binom{m-r}{m-k}.$$

If we take $m = k$ in the last equality, we obtain

$$m!\,S(m, m) = \sum_{r=1}^{m} Z(m, r),$$

which is equivalent to (2), because $S(m, m) = 1$. This is Lemma 2 from [1].

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THE ZECKENDORF REPRESENTATION AND THE GOLDEN SEQUENCE

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Preamble

In what follows, we have

The Golden section: $\tau = \dfrac{\sqrt{5} - 1}{2} = 0.618\ldots$.

Fibonacci numbers: $F_0 = 0,\ F_1 = 1,\ F_i = F_{i-1} + F_{i-2},\ i \geq 2.$

The Zeckendorf representation of a number is simply the representation of that number as the sum of distinct Fibonacci numbers. If the number of terms of this sum is minimized, that representation is unique, as also is the representation when the number of terms is maximized. (See Brown [1] and [2].)

A *general* Zeckendorf representation will be written as

$$\sum_{j=1}^{h} F_{k_j}, \quad \text{where } k_1 > k_2 > \cdots > k_h \geq 2.$$

Thus, 16 can be represented as

$$F_7 + F_4,\ F_6 + F_5 + F_4,\ F_7 + F_3 + F_2, \text{ and } F_6 + F_5 + F_3 + F_2.$$

The first is the unique minimal representation; the last is the unique maximal representation. The others show that representations of any intermediate length need not be unique.

It is easy to show that only numbers of the form $F_n - 1$ have a unique Zeckendorf representation (i.e., one that is maximal and minimal).

From here on, we will refer to the minimal Zeckendorf representation and the maximal Zeckendorf representation as the *minimal* and *maximal*.

We define

Beta-sequence: $\{\beta_j\},\ j = 1, 2, 3, \ldots,\ \beta_j = [(j + 1)\tau] - [j\tau].$
This takes on only the values zero or unity.

Golden sequence: Any sequence such as *abaababa*... which is obtained from the Beta-sequence $\beta_1, \beta_2, \beta_3, \ldots,$ where "b" corresponds to a zero and "a" corresponds to a unit.

We will prove that the final term of each maximal representation is either F_2 or F_3 and show the pattern associated with the final terms in the representations of 1, 2, 3, 4, 5, 6, ..., namely: $F_2, F_3, F_2, F_2, F_3, F_2, \ldots$ is a Golden sequence with the term F_2 corresponding to a unit and the term F_3 corresponding to a zero.

More specifically, we will show that the last term in the maximal representation of the number n is $F_{3-\beta_n} = 2 - \beta_n$.

We note a similar result for the "modified" Zeckendorf representation which may include F_1 as well as F_2.

Main Results

Theorem 1: The maximal ends with F_2 or F_3.

Proof: We note that F_3 cannot be replaced by $F_2 + F_1$ in a Zeckendorf expansion as $F_2 = F_1$. If F_k with $k > 3$ is the smallest term in an expansion of a number

n, then F_k can be replaced by $F_{k-1} + F_{k-2}$ and so the expansion is not maximal. Thus, if an expansion is maximal, it must end in F_2 or F_3.

Lemma 1: $[(j + F_i)\tau] = F_{i-1} + [j\tau]$ if $i \geq 2$ and $0 < j < F_{i+1}$.

Proof: Fraenkel, Muchkin, and Tassa proved in [3] that if θ is irrational, $0 < j < q_i$ and p_i/q_i is the i^{th} convergent to θ in the elementary theory of continued fractions, then

$$[(j + q_{i-1})\theta] = p_{i-1} + [j\theta], \quad i \geq 1.$$

As F_{i-1}/F_i is a convergent to τ, our result follows.

Lemma 2: If $\sum_{j=1}^{h} F_{k_j}$ is a Zeckendorf expansion, then $\sum_{j=2}^{h} F_{k_j} < F_{k_1+1} - 1$.

Proof: $\sum_{j=2}^{h} F_{k_j} \leq F_{k_1-1} + F_{k_1-2} + \cdots + F_2 = F_{k_1+1} - 2$, since $\sum_{i=1}^{n} F_i = F_{n+2} - 1$.

The result is now obvious.

Lemma 3: If j has a Zeckendorf expansion $\sum_{i=1}^{h} F_{k_i}$, then

(a) $[j\tau] = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_{h-1}-1} + [\tau F_{k_h}]$

(b) $[(j + 1)\tau] = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_{h-1}-1} + F_{k_h-1}$.

Proof:

(a) Let $m = \sum_{i=2}^{h} F_{k_i}$, then by Lemma 2, $m < F_{k_1+1}$ and so by Lemma 1,

$$[j\tau] = [(F_{k_1} + m)\tau] = F_{k_1-1} + [m\tau].$$

Similarly, if $n = \sum_{i=3}^{h} F_{k_i}$, $[m\tau] = F_{k_2-1} + [n\tau]$, so eventually

$$[j\tau] = F_{k_1-1} + \cdots + F_{k_{h-1}-1} + [\tau F_{k_h}].$$

(b) As in (a) (this time with $m + 1 < F_{k_1+1}$),

$$\begin{aligned}[(j + 1)\tau] &= [(F_{k_1} + \cdots + (F_{k_h} + 1))\tau] \\ &= F_{k_1-1} + \cdots + F_{k_{h-1}-1} + [(F_{k_h} + 1)\tau] \\ &= F_{k_1-1} + \cdots + F_{k_{h-1}-1} + F_{k_h-1} \text{ by Lemma 1.}\end{aligned}$$

Lemma 4: If j has a maximal $\sum_{i=1}^{h} F_{k_i}$, then

(a) $[j\tau] = F_{k_1-1} + \cdots + F_{k_{h-1}-1} + F_{k_h} - 1$.

(b) $\beta_j = 2 - F_{k_h}$.

Proof:

(a) If $k_h = 2$, then $[\tau F_{k_h}] = 0 = F_{k_h} - 1$.
If $k_h = 3$, then $[\tau F_{k_h}] = 1 = F_{k_h} - 1$,
so the result follows by Lemma 3(a).

(b) By Lemmas 4(a) and 3(b),

$$\beta_j = [(j + 1)\tau] - [j\tau] = F_{k_h-1} - F_{k_h} + 1 = 2 - F_{k_h},$$

as $k_h = 2$ or 3 and so $F_{k_h-1} = 1$.

Theorem 2: The last term in the maximal for j is $F_{3-\beta_j} = 2 - \beta_j$.

Proof: By Lemma 4(b), if F_{k_h} is the last term in a maximal for j, then

$$\beta_j = 2 - F_{k_h}.$$

If $k_h = 3$, then $\beta_j = 0$ and $F_{3-\beta_j} = F_3 = 2 - \beta_j$.

If $k_h = 2$, then $\beta_j = 1$ and $F_{3-\beta_j} = F_2 = 2 - \beta_j$.

We now see that the last term of the maximal for any integer j is either 1 or 2. It also follows immediately that the sequence of the last terms for the maximals for 1, 2, 3, 4, ... form a Golden sequence 1211212112..., where a unit is unchanged but a zero is replaced by 2.

Suppose we form the modified maximal from the maximal by forcing the last term to be unity; that is, the last two terms are $F_3 + F_2$, $F_3 + F_1$, or $F_2 + F_1$. Then it follows easily from the above that the second last terms of the modified maximals for 2, 3, 4, ... correspond to the same golden pattern as the last terms in the maximals for 1, 2, 3, ...

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THE PERIOD OF CONVERGENTS MODULO M OF REDUCED QUADRATIC IRRATIONALS*

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Introduction

The properties of the period lengths of the continued fraction convergents modulo m of reduced quadratic irrationals are studied in this paper. These period lengths vary wildly, yet will be shown to satisfy strong divisibility properties. Wall [6] studied these period lengths for the Fibonacci numbers that arise as convergents of the simple continued fraction with all partial quotients equal to 1. Many other papers, including [1], [3], [4], and [5], extend and complement those results. Some of the theorems in Wall extend in a direct manner to the continued fraction investigation given here; however, a key theorem of Wall about occurrences of zeros does not generalize so that new approaches are required. In some cases, known properties of continued fractions, for a reference see Rosen [2], yield simpler proofs for the analogs of theorems from Wall. Two theorems presented here give properties of the periods for reversals and rotations of the continued fractions which have no analogs from the Fibonacci numbers. Matrix computation of the convergents is developed and analyzed to produce further results including remarkably good bounds on the period lengths.

Definition of the Period

Reduced quadratic irrationals, denoted α in this paper, are those real numbers that have purely periodic simple continued fraction expansions. Consider such an α:

$$\alpha = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \; a_t + \cfrac{1}{\alpha}}}$$

where $a_i \in \mathbb{Z}^+$ and t is chosen as small as possible. This is abbreviated by $\alpha = [\overline{a_1, a_2, \ldots, a_t}]$, where t is said to be the *period* of α. Associated with each continued fraction are the p,q sequences defined in the following manner:

$$p_{-1} = 0,\; p_0 = 1,\; p_n = a_n p_{n-1} + p_{n-2},$$
$$q_{-1} = 1,\; q_0 = 0,\; q_n = a_n q_{n-1} + q_{n-2}.$$

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We illustrate the calculation of these sequences with $\alpha = (1 + \sqrt{3})/2 = [\overline{1, 2}]$.

a_n:			1	2	1	2	1	2	1	2	1	2
p_n:	0	1	1	3	4	11	15	41	56	153	209	571
q_n:	1	0	1	2	3	8	11	30	41	112	153	418

Below are the values in this table modulo 2. One can see that the sequence p_n, the sequence q_n, and both sequences taken together are all periodic.

a_n:			1	0	1	0	1	0	1	0	1	0
p_n:	0	1	1	1	0	1	1	1	0	1	1	1
q_n:	1	0	1	0	1	0	1	0	1	0	1	0

Theorem 1: The p,q sequence modulo m is purely periodic.

Proof: Consider the 2×2 block of p's and q's (mod m) at $st - 1$ and st, where $s = 0, 1, 2, \ldots$ and t is the period of α. Since there are only m^4 possibilities for this block, it eventually repeats so that, for some i, j, say with $i < j$,

$$p_{it-1} \equiv p_{jt-1}, \; p_{it} \equiv p_{jt},$$
$$q_{it-1} \equiv q_{jt-1}, \; q_{it} \equiv q_{jt} \pmod{m}.$$

Since $a_{it+n} = a_{jt+n}$, the defining relations give that the p,q sequence repeats. Also, from the defining relations, we see that

$$p_{it-2} = p_{it} - a_{it}p_{it-1} \equiv p_{jt} - a_{jt}p_{jt-1} = p_{jt-2}$$
$$p_{it-3} = p_{it-1} - a_{it-1}p_{it-2} \equiv p_{jt-1} - a_{jt-1}p_{jt-2} = p_{jt-3}$$
$$\vdots$$
$$p_{it-(it-1)} = p_1 \equiv p_{jt-(it-1)} = p_{(j-i)t+1}.$$

The same argument holds for the q sequence. Therefore, the p,q sequence is purely periodic. □

The *period of the p,q sequence modulo m* is denoted $k(\alpha, m)$, or $k(m)$, or k if no ambiguity occurs. It is evident from the proof of this theorem that $k(\alpha, m) \le m^4 t$. The remainder of this paper will explore the properties of $k(\alpha, m)$.

Elementary Properties

In light of the initial conditions for the p,q sequences and the definition of k, we get an immediate corollary.

Corollary 2: When $k = k(m)$, then $p_{k-1} \equiv 0$, $p_k \equiv 1$, $q_{k-1} \equiv 1$, and $q_k \equiv 0$ modulo m.

Next is a theorem which establishes that k is even for all moduli greater than 2.

Theorem 3: If $m > 2$, then $k(m)$ is even.

Proof: Suppose that $k = k(m)$ is odd. Then, by using the continued fraction identity $p_k q_{k-1} - p_{k-1} q_k = (-1)^k$ and substituting the values of the p,q sequence from the corollary into this equation, we have $(1)(1) - (0)(0) = -1$ mod m. Therefore, $2 \equiv 0$ mod m, which implies a modulus of 2. □

Theorem 4: If $m_1 | m_2$, then $k(m_1) | k(m_2)$.

Proof: Let $k = k(m_2)$ and $m_1 | m_2$, then $m_2 | p_{k-1}$ implies $m_1 | p_{k-1}$, and $m_2 | q_{k-1} - 1$ implies $m_1 | q_{k-1} - 1$. Likewise, for $p_k - 1$ and q_k. Hence, $k(m_1) | k(m_2)$. □

The following theorem shows that, if the periods of the prime power factors of a modulus are known, then the period of the modulus can readily be calculated.

Theorem 5: If m has the prime factorization $m = \Pi p_i^{e_i}$ and if k_i denotes the length of the period of the p,q sequence mod $p_i^{e_i}$, then $k(m) = \operatorname{lcm}[k_i]$.

Proof: Since $k_i | k$ for all i, $\operatorname{lcm}[k_i] | k$. On the other hand, since $p_k \equiv 1 \bmod p_i^{e_i}$ for all i, $p_k \equiv 1 \bmod \operatorname{lcm}[p_i^{e_i}]$. Similarly, $p_{k-1} \equiv 0$, $q_{k-1} \equiv 1$, $q_k \equiv 0$. Therefore, $k | \operatorname{lcm}[k_i]$. □

For the sequence of Fibonacci numbers modulo m, the zeros are known to be in arithmetic progression. The placement of zeros is not simple for continued fractions in general. Consider an example with $m = 3$:

a_n:			3 5 2	3 5 2	3 5 2	3 5 2	3 5 2	3 5 2	3 5 2	3 5 2
p_n:	0	1	0 1 2	1 1 0	1 2 2	2 0 2	0 2 1	2 2 0	2 1 1	1 0 1
q_n:	1	0	1 2 1	2 0 2	0 2 1	2 2 0	2 1 1	1 0 1	0 1 2	1 1 0

The theorem below begins giving insight into the structure of the convergents without controlling the zeros.

Notice that, for some α's and moduli m, the period of α reduces mod m. For example, $\alpha = [\overline{1, 2, 3, 4}]$ mod 2 is "the same as" $[\overline{1, 2}]$ mod 2. We say the period of α is *preserved* modulo m when this does not occur. It is frequently convenient to restrict consideration of $k(\alpha, m)$ to the case where the period of α is preserved modulo m. Of course, one can get information about $k(\alpha, m)$ when the period of α is not preserved. For example, one can consider $[\overline{1, 2}]$ instead of $[\overline{1, 2, 3, 4}]$ when the modulus is 2.

The next theorem states that $k(\alpha, m)$ is always a multiple of the period of α. This is useful information about the structure of the periods and also gives a trivial lower bound.

Theorem 6: If $\alpha = [\overline{a_1, a_2, \ldots, a_t}]$ and the period of α is preserved mod m, then $t | k(m)$.

Proof: Suppose that $k = k(m)$, then $p_n \equiv p_{n+k}$ for $n = 1, 2, \ldots$. So,

$$a_n p_{n-1} + p_{n-2} \equiv a_{n+k} p_{n+k-1} + p_{n+k-2} \bmod m.$$

Thus,

$$a_n p_{n-1} \equiv a_{n+k} p_{n+k-1} \equiv a_{n+k} p_{n-1}.$$

Similarly,

$$a_n q_{n-1} \equiv a_{n+k} q_{n-1} \bmod m.$$

Multiplying the congruences by q_n and p_n, respectively, and subtracting gives

$$a_n(-1)^{n-1} = a_n(q_n p_{n-1} - p_n q_{n-1}) \equiv a_{n+k}(q_n p_{n-1} - p_n q_{n-1}) = a_{n+k}(-1)^{n-1}.$$

It follows that $a_n \equiv a_{n+k} \bmod m$ and, therefore, $t | k(m)$. □

The hypothesis that the period of α is preserved mod m is indeed necessary, since for $\alpha = [\overline{1, 2, 3, 4, 5, 6}]$, $t = 6 \nmid 4 = k(\alpha, 2)$ and α reduced mod 2 is "the same as" $[\overline{1, 2}]$.

It is now known that in order to determine $k(m)$ one need only look at the $nt - 1$ and nt places in the p,q sequence, where $n = 1, 2, \ldots$.

Corollary 7: If the period of α is preserved mod m, the period length k is of the form $k = ct$, where c is the smallest positive integer with

$$p_{ct-1} \equiv 0,\ p_{ct} \equiv 1,\ q_{ct-1} \equiv 1,\ q_{ct} \equiv 0.$$

Matrix Formulation

The following theorems allow us to look at only these blocks of integers without going through the intermediate calculations. First, we establish the following lemma.

Lemma 8: Define $r_n = a_n r_{n-1} + r_{n-2}$ with initial conditions $r_{-1} = a$, $r_0 = b$, where $a, b \in \mathbb{Z}^+$. Then $r_n = bp_n + aq_n$.

Proof: For $n = -1$ and $n = 0$, the relation holds trivially. Now suppose that $r_n = bp_n + aq_n$ and $r_{n+1} = bp_{n+1} + aq_{n+1}$. Then,

$$r_{n+2} = a_{n+2}(bp_{n+1} + aq_{n+1}) + bp_n + aq_n = bp_{n+2} + aq_{n+2}. \ \square$$

We now define a matrix W called the *fundamental matrix* which depends only on α and that can be used to compute the blocks of convergents at the end of blocks of length t.

Theorem 9: Let

$$W = \begin{pmatrix} q_{t-1} & q_t \\ p_{t-1} & p_t \end{pmatrix}; \quad \text{then} \quad W^n = \begin{pmatrix} q_{nt-1} & q_{nt} \\ p_{nt-1} & p_{nt} \end{pmatrix}.$$

Proof: Consider the function $F_\alpha: \mathbb{Z}^2 \to \mathbb{Z}^2$ which takes an initial condition pair (a, b) to the pair (r_{t-1}, r_t) giving the last two terms resulting from applying one period of recursions $r_j = a_j r_{j-1} + r_{j-2}$, $j = 1, \ldots, t$, to initial conditions $r_{-1} = a$, $r_0 = b$. In light of the lemma, F_α can be written in matrix form:

$$F_\alpha(a, b) = (a, b)W.$$

On the other hand, applying n periods of the recursions is just n iterations of F_α and (p_{nt-1}, p_{nt}) is the result of applying n periods of the recursion to $(0, 1)$. Hence,

$$(p_{nt-1}, p_{nt}) = F_\alpha^n(0, 1) = (0, 1)W^n.$$

Likewise,

$$(q_{nt-1}, q_{nt}) = (1, 0)W^n$$

and the conclusion follows. $\square$

Notice in the example below that W, W^2, and W^3 appear upside down in the list of convergents of $\alpha = [\overline{3, 5, 2}]$.

$$W = \begin{pmatrix} 5 & 11 \\ 16 & 35 \end{pmatrix}, \quad W^2 = \begin{pmatrix} 201 & 440 \\ 640 & 1401 \end{pmatrix}, \quad W^3 = \begin{pmatrix} 8045 & 17611 \\ 25616 & 56075 \end{pmatrix},$$

a_k:			3	5	2	3	5	2	3	5	2
p_k:	0	1	3	16	35	121	640	1401	4843	25616	56075
q_k:	1	0	1	5	11	38	201	440	1521	8045	17611

The following corollary is a direct consequence of Theorem 9 and Corollary 7.

Corollary 10:

(i) If $W^n \equiv I \bmod m$, then $k(m) \mid nt$.

(ii) If the period of α is preserved mod m, then c is the smallest integer such that $W^c \equiv I \bmod m$ if and only if $k(m) = ct$.

Remark: If p is an odd prime, the order of the multiplicative group of matrices $\{A \in M_2(\mathbb{Z}_p) \mid \det(A) = \pm 1\}$ is $2(p + 1)p(p - 1)$ and it follows that

$$k(p) \mid 2(p + 1)p(p - 1)t.$$

This establishes a slightly better upper bound for $k(p)$ than the p^4t observed earlier. Furthermore, this remark limits the factors appearing in $k(p)$.

Reversals and Rotations

Given an $\alpha = [\overline{a_1, a_2, \ldots, a_t}]$, we construct other quadratic irrationals related to α: the *reversal* of α, $\alpha^\phi = [\overline{a_t, a_{t-1}, \ldots, a_1}]$ and the *rotation* of by one position, $\alpha^* = [\overline{a_t, a_1, a_2, \ldots, a_{t-1}}]$. The rotation of α by j positions to the right is indicated by α^{*j}. The following theorems show that $k(\alpha, m)$ is not changed when α is reversed or rotated. Thus, if we know $k(\alpha, m)$, then we really know $k(m)$ for up to $2t$ different quadratic irrationals.

Theorem 11: $k(\alpha, m) = k(\alpha^*, m) = k(\alpha^{*2}, m) = \cdots = k(\alpha^{*t-1}, m)$.

Proof: First, notice that if the period of α is not preserved mod m, then the period of α^{*j} is not preserved mod m for all j. If $\alpha = [\overline{a_1, a_2, \ldots, a_t}]$ degenerates into $\alpha' = [\overline{a_1, a_2, \ldots, a_{t'}}]$ mod m. That is, t' is the smallest positive integer so that for all j, $a_j \equiv a_{j+t'}$ mod m. Then for all j, $k(\alpha^{*j}, m) = k(\alpha'^{*j}, m)$, but the period of α' is preserved mod m. Thus, without loss of generality, we will assume the period of α is preserved mod m.

Let W be the fundamental matrix for α, let c be the smallest positive integer with $W^c \equiv I$ mod m, let F_α be the function as in the proof of Theorem 8 which gives the last two terms resulting from applying one α period of recursions to given initial conditions, and let p_n^*, q_n^* denote the p,q sequence for α^*.

Note that $\alpha^* = [a_t, \overline{a_1, a_2, \ldots, a_t}]$. Thus, (p_t^*, p_{t+1}^*) arise from applying one period of the α recursion relations to initial condition (p_0^*, p_1^*). That is,

$$(p_t^*, p_{t+1}^*) = F_\alpha(p_0^*, p_1^*) = (p_0^*, p_1^*)W$$

and applying "c" periods of the α recursions gives

$$(p_{ct}^*, p_{ct+1}^*) = F_\alpha^c(p_0^*, p_1^*) = (p_0^*, p_1^*)W^c.$$

Likewise for the q sequence. Thus, $k(\alpha^*, m) \mid k(\alpha, m)$. Applying this fact to further rotations gives

$$k(\alpha, m) = k(\alpha^{*t}, m) \mid k(\alpha^{*t-1}, m) \mid \ \ldots\ \mid k(\alpha^*, m) \mid k(\alpha, m)$$

and, hence, the required equalities must hold. □

Theorem 12: $k(\alpha, m) = k(\alpha^\phi, m)$.

Proof: If $k = k(\alpha, m)$, then, from well-known identities (see Rosen [2, p. 363]) of continued fractions $p_k^\phi/q_k^\phi = p_k/p_{k-1}$ and $p_{k-1}^\phi/q_{k-1}^\phi = q_k/q_{k-1}$. Therefore,

$$p_{k-1}^\phi = q_k \equiv 0,\ p_k^\phi = p_k \equiv 1,$$
$$q_{k-1}^\phi = q_{k-1} \equiv 1,\ q_k^\phi = p_{k-1} \equiv 0,$$

which implies $k(\alpha^\phi, m) \mid k(\alpha, m)$. It is evident that $k(\alpha^\phi, m) = k(\alpha, m)$ since, by applying the process on α^ϕ, we obtain $k(\alpha, m) \mid k(\alpha^\phi, m)$. □

Periods of Powers of Primes

The relation between $k(\alpha, p)$ and $k(\alpha, p^e)$ is explored next. Consider the periods of $\alpha = [\overline{1, 1, 1, 1, 1, 2}]$ for several prime power moduli.

p	$k(\alpha, p)$	$k(\alpha, p^2)$	$k(\alpha, p^3)$	$k(\alpha, p^4)$
2	12	24	48	96
3	18	18	18	54
5	36	36	180	900
7	84	588	4116	28812

Notice when the exponent of p in the modulus is increased by one the period seems to "increase" by a factor of p or 1. Indeed, the following theorems show that as the exponent of p increases the period $k(p^e)$ will increase by a factor of p after some initial constant sequence. An exception is $p = 2$, which is slightly more complicated.

It is interesting to note that for the analogous theorem of Wall [6] about the Fibonacci numbers there are no known examples with $k(p) = k(p^2)$. For the α given above, $k(p) = k(p^e)$ for some $e > 1$ does occur. Identifying when this occurs remains an open problem.

We now turn to proving the above properties. Let A be a matrix with integer entries. If p^e divides each element of A but p^{e+1} does not divide some element of A, we say p^e *exactly divides* A, and write $p^e \| A$. This means that A can be written $A = p^e S$ for some matrix S with integer entries where S contains an element which is not divisible by p.

Lemma 13: Let U be a matix with integer entries, I be the identity matrix, and p be an odd prime number. If $p^e \| U - I$ for some $e \geq 1$, then $p^{e+1} \| U^p - I$. Moreover, for $p = 2$, if $e \geq 2$ and $2^e \| U - I$, then $2^{e+1} \| U^2 - I$.

Proof: Suppose first that p is an odd prime with $p^e \| U - I$, so $U = I + p^e S$ where S is a matrix with integer entries and p does not divide some entry in S. The binomial theorem is not true for matrices in general, but it is true when one of the matrices is the identity. The third and higher terms of the binomial expansion below have at least two factors of p^e plus another factor of p coming from the binomial coefficient or from an additional factor of p^e. Thus, for some matrix T, we have

$$U^p = (I + p^e S)^p = \sum_{j=0}^{p} \binom{p}{j} p^{je} S^j = \binom{p}{0} I + \binom{p}{1} p^e S + p^{2e+1} T.$$

Thus, $U^p - I = p^{e+1} S + p^{2e+1} T$. Notice that $p^{e+1} | U^p - I$ and that if p^{e+2} did, then p would divide all the elements of S, which contradicts the hypothesis. Therefore, $p^{e+1} \| U^p - I$ as required.

Similarly, if $p = 2$ and $2^e \| U - I$, U has the same form as above and

$$U^2 = I + 2^{e+1} S + 2^{2e} S^2.$$

Thus, $2^{e+1} | U^2 - I$. Now, for $e \geq 2$, $2e \geq e + 2$ so that if $2^{e+2} | U^2 - I$ then $2 | S$, which is not so. Thus, $2^{e+1} \| U^2 - I$. □

Theorem 14: Let p be an odd prime which preserves the period of α. There is a positive integer e so that

$$k(p) = k(p^2) = \cdots = k(p^e) \quad \text{and} \quad k(p^{e+j}) = p^j k(p) \text{ for all } j \geq 1.$$

Moreover, for $p = 2$ there is an integer $e \geq 2$ such that

$$k(2^2) = k(2^3) = \cdots = k(2^e) \quad \text{and} \quad k(2^{e+j}) = 2^j k(2) \text{ for all } j \geq 1.$$

Also, $k(2) = k(4)$ or $k(2) = \frac{1}{2} k(4)$.

Proof: Let p be an odd prime and W be the fundamental matrix for α. Notice that p^n preserves the period of α for all n. So, by Corollary 10, $k(p^e) = nt$ if and only if n is the smallest positive integer with $W^n \equiv I \bmod p^e$. Select c to be the smallest exponent for which $W^c \equiv I \bmod p$. Then let e be the largest exponent (possibly 1) for which $k(p) = k(p^2) = \cdots = k(p^e)$. Notice that e must be finite, since for large enough e, p^e will be larger than the entries in W^c and, hence, $W^c \not\equiv I \bmod p^e$. Now $p^e \| W^c - I$ so that, by the lemma, $p^{e+1} \| W^{pc} - I$. Thus, $ct = k(p^e) | k(p^{e+1}) | pct$. So, $k(p^{e+1}) = ct$ or pct. If $k(p^{e+1}) = ct$, then $p^{e+1} | W^c - I$, which is impossible since $p^e \| W^c - I$. Therefore, $k(p^{e+1}) = pk(p^e)$. Continuing inductively gives $k(p^{e+j}) = p^j k(p^e)$.

Moreover, for $p = 2$, the same argument works beginning with $k(2^2)$, since the lemma used requires $e \geq 2$ in this case. Also, if $W^c \equiv I \bmod 2$, then for some matrix T, $W^c = I + 2T$; hence, $W^{2c} \equiv I \bmod 4$ and the ratio $k(4)/k(2)$ is 1 or 2. □

The special possibilities mentioned in the theorem for $p = 2$ do occur as indicated by the examples:

α	$k(2)$	$k(4)$	$k(8)$	$k(16)$	$k(32)$
$[\overline{1, 2}]$	4	8	8	16	32
$[\overline{1, 1, 2}]$	6	12	24	48	96
$[\overline{1, 2, 3}]$	6	6	12	24	48

Bounds for Prime Periods

It was shown in Corollary 10 that c is the smallest positive integer such that $W^c \equiv I \bmod m$ if $k(m) = ct$. To facilitate the analysis of W^c, we diagonalize the fundamental matrix. The eigenvalues of this matrix are

$$\lambda_1 = \frac{1}{2}[(p_t + q_{t-1}) + \sqrt{d}] \quad \text{and} \quad \lambda_2 = \frac{1}{2}[(p_t + q_{t-1}) - \sqrt{d}],$$

where

$$d = (p_t + q_{t-1})^2 + 4(-1)^{t-1}.$$

It is evident from the definitions of λ_1 and λ_2 that

$$\lambda_1\lambda_2 = (-1)^t \quad \text{and} \quad \lambda_1 + \lambda_2 = (p_t + q_{t-1}).$$

These identities are used in the following lemmas and theorems. Computing the eigenvectors and completing the diagonalization, we find the following form for W^n.

Theorem 15: Let W be the fundamental matrix for α and let $\mathscr{L}_n = (\lambda_1^n - \lambda_2^n)/\sqrt{d}$. Then,

$$W^n = \begin{bmatrix} (-1)^{t-1}\mathscr{L}_{n-1} + q_{t-1}\mathscr{L}_n & q_t\mathscr{L}_n \\ p_{t-1}\mathscr{L}_n & \mathscr{L}_{n+1} - q_{t-1}\mathscr{L}_n \end{bmatrix} \text{ for } n = 1, 2, \ldots .$$

Proof: The fundamental matrix can be diagonalized by the matrix P, where

$$P = \begin{bmatrix} q_t & q_t \\ \lambda_1 - q_{t-1} & \lambda_2 - q_{t-1} \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{-1}{q_t\sqrt{d}}\begin{bmatrix} \lambda_2 - q_{t-1} & -q_t \\ -(\lambda_1 - q_{t-1}) & q_t \end{bmatrix}.$$

Computing $W^n = PD^nP^{-1}$, where D is the diagonal matrix with λ_1 and λ_2 on the diagonal, we get

$$W^n = \frac{1}{q_t\sqrt{d}}\begin{bmatrix} q_t((\lambda_1 - q_{t-1})\lambda_2^n - (\lambda_2 - q_{t-1})\lambda_1^n) & q_t^2(\lambda_1^n - \lambda_2^n) \\ -(\lambda_1 - q_{t-1})(\lambda_2 - q_{t-1})(\lambda_1^n - \lambda_2^n) & q_t(\lambda_1^n(\lambda_1 - q_{t-1}) - \lambda_2^n(\lambda_2 - q_{t-1})) \end{bmatrix}.$$

This simplifies into the required matrix using the properties of the eigenvalues. □

Remark: An interesting consequence of this diagonalization is that

$$p_{nt-1}q_t = q_{nt}p_{t-1} \text{ for all } n = 1, 2, \ldots .$$

Lemma 16:

$$\mathscr{L}_{n-1} = (-1)^t[(p_t + q_{t-1})\mathscr{L}_n - \mathscr{L}_{n+1}] \quad \text{for } n = 1, 2, \ldots .$$

Proof: The eigenvalues λ_1 and λ_2 satisfy the characteristic equation of W. Thus, $\lambda_1^2 - (p_t + q_{t-1})\lambda_1 + (-1)^t = 0$ and, likewise, for λ_2. Multiplying these equations by λ_1^{n-1} and λ_2^{n-1}, respectively, and subtracting yields

$$\mathscr{L}_{n+1} - (p_t + q_{t-1})\mathscr{L}_n + (-1)^t \mathscr{L}_{n-1} = 0.$$

Solving for $\mathscr{L}_{n-1}$ gives the conclusion. □

Notice that $\mathscr{L}_0$ and $\mathscr{L}_1$ are integers and that $\mathscr{L}_{n+1}$ is an integer combination of $\mathscr{L}_n$ and $\mathscr{L}_{n-1}$. Therefore, $\mathscr{L}_n$ is an integer for $n = 1, 2, \ldots$.

Lemma 17: If p is an odd prime and $\left(\frac{d}{p}\right)$ is the Legendre symbol, then

(i) $\mathscr{L}_p \equiv \left(\frac{d}{p}\right) \bmod p$, and

(ii) $\mathscr{L}_{p+1} \equiv 2^{-1}(p_t + q_{t-1})\left[\left(\frac{d}{p}\right) + 1\right] \bmod p$.

Proof: By writing out λ_1^p and λ_2^p in their respective binomial expansions, cancelling the even terms, reducing modulo p, and applying Euler's criterion, we get that

$$\text{(i)}\quad \mathscr{L}_p = \frac{1}{\sqrt{d}}(\lambda_1^p - \lambda_2^p) = 2^{1-p}\sum_{\substack{1 \le j \le p \\ j \text{ odd}}}\binom{p}{j}(p_t + q_{t-1})^{p-j}d^{(j-1)/2}$$

$$\equiv \binom{p}{p}d^{(p-1)/2} \equiv \left(\frac{d}{p}\right) \bmod p, \text{ and}$$

$$\text{(ii)}\quad \mathscr{L}_{p+1} = \frac{1}{\sqrt{d}}(\lambda_1^{p+1} - \lambda_2^{p+1}) = 2^{-p}\sum_{\substack{1 \le j \le p \\ j \text{ odd}}}\binom{p+1}{j}(p_t + q_{t-1})^{p+1-j}d^{(j-1)/2}$$

$$\equiv 2^{-1}\left[\binom{p+1}{1}(p_t + q_{t-1}) + \binom{p+1}{p}(p_t + q_{t-1})d^{(p-1)/2}\right] \bmod p$$

$$\equiv 2^{-1}(p_t + q_{t-1})[(p_t + q_{t-1})^{p-1} + d^{(p-1)/2}] \bmod p$$

$$\equiv 2^{-1}(p_t + q_{t-1})\left[\left(\frac{d}{p}\right) + 1\right] \bmod p. \ \square$$

The following three corollaries are direct consequences of the previous two lemmas. They provide information about the entries in W^n when $n = p - \left(\frac{d}{p}\right)$.

Corollary 18: If $\left(\frac{d}{p}\right) = 1$, then

(i) $\mathscr{L}_{p-2} \equiv (-1)^{t-1} \bmod p$,
(ii) $\mathscr{L}_{p-1} \equiv 0 \bmod p$, and
(iii) $\mathscr{L}_p \equiv 1 \bmod p$.

Corollary 19: If $\left(\frac{d}{p}\right) = 0$, then

(i) $\mathscr{L}_{p-1} \equiv 2^{-1}(-1)^{t-1}(p_t + q_{t-1}) \bmod p$,
(ii) $\mathscr{L}_p \equiv 0 \bmod p$, and
(iii) $\mathscr{L}_{p+1} \equiv 2^{-1}(p_t + q_{t-1}) \bmod p$.

Corollary 20: If $\left(\frac{d}{p}\right) = -1$, then

(i) $\mathscr{L}_p \equiv -1 \bmod p$,
(ii) $\mathscr{L}_{p+1} \equiv 0 \bmod p$, and
(iii) $\mathscr{L}_{p+2} \equiv (-1)^t \bmod p$.

Corollary 10 describes the relation of $k(p)$ to c such that $W^c \equiv I$ and Theorem 15 gives a form for W^n. These are combined to obtain divisibility properties for $k(p)$. These multiples of $k(p)$ also give upper bounds on $k(p)$.

Theorem 21: If p is an odd prime, then $k(p)$ divides $\begin{cases}(p-1)t & \text{if } \left(\frac{d}{p}\right) = 1,\\ 4pt & \text{if } \left(\frac{d}{p}\right) = 0,\\ 2(p+1)t & \text{if } \left(\frac{d}{p}\right) = -1.\end{cases}$

Proof:

Case 1. Suppose that $\left(\frac{d}{p}\right) = 1$, and then apply Corollary 18 to W^{p-1}:

$$W^{p-1} = \begin{bmatrix} (-1)^{t-1}\mathscr{L}_{p-2} + q_{t-1}\mathscr{L}_{p-1} & q_t\mathscr{L}_{p-1} \\ p_{t-1}\mathscr{L}_{p-1} & \mathscr{L}_p - q_{t-1}\mathscr{L}_{p-1} \end{bmatrix}$$

$$\equiv \begin{bmatrix} (-1)^{t-1}(-1)^{t-1} & 0 \\ 0 & 1 \end{bmatrix} = I \bmod p.$$

Therefore, $k(p)\,|\,(p-1)t$ for $\left(\frac{d}{p}\right) = 1$.

Case 2. Suppose that $\left(\frac{d}{p}\right) = 0$, and then apply Corollary 19 to W^p:

$$W^{p} = \begin{bmatrix} (-1)^{t-1}\mathscr{L}_{p-1} + q_{t-1}\mathscr{L}_{p} & q_t\mathscr{L}_{p} \\ p_{t-1}\mathscr{L}_{p} & \mathscr{L}_{p+1} - q_{t-1}\mathscr{L}_{p} \end{bmatrix}$$

$$\equiv \begin{bmatrix} 2^{-1}(p_t + q_{t-1}) & 0 \\ 0 & 2^{-1}(p_t + q_{t-1}) \end{bmatrix} \bmod p.$$

Thus, $W^{2p} \equiv 4^{-1}(p_t + q_{t-1})^2 I$, but since $(p_t + q_{t-1})^2 = d + 4(-1)^t \equiv 4(-1)^t \bmod p$ we have $W^{2p} \equiv (-1)^t I$. Therefore, $W^{4p} \equiv I$ and $k(p)\,|\,4pt$ in this case.

Case 3. Suppose that $\left(\frac{d}{p}\right) = -1$, and then apply Corollary 20 to W^{p+1}:

$$W^{p+1} = \begin{bmatrix} (-1)^{t-1}\mathscr{L}_{p} + q_{t-1}\mathscr{L}_{p+1} & q_t\mathscr{L}_{p+1} \\ p_{t-1}\mathscr{L}_{p+1} & \mathscr{L}_{p+2} - q_{t-1}\mathscr{L}_{p+1} \end{bmatrix} \equiv \begin{bmatrix} (-1)^t & 0 \\ 0 & (-1)^t \end{bmatrix} \bmod p.$$

Thus, $W^{2(p+1)} \equiv I$ and $k(p)\,|\,2(p+1)t$ in this case. □

The proof of the previous theorem allows tightening of the bound when the period of α is even.

Theorem 22: If t is even, then $k(p)$ divides $\begin{cases} (p-1)t & \text{if } \left(\frac{d}{p}\right) = 1, \\ 2pt & \text{if } \left(\frac{d}{p}\right) = 0, \\ (p+1)t & \text{if } \left(\frac{d}{p}\right) = -1. \end{cases}$

The bounds given by Theorems 21 and 22 are met with some frequency. For example, considering the primes less than 1000 for the modulus, the bounds are met about 66 percent of the time for $\alpha = [\overline{2, 1, 4, 3, 5}]$ and 35 percent of the time for $\alpha = [\overline{4, 5, 1, 3, 2, 5}]$.

Questions

We leave the reader with some questions. First, when does $k(p) = k(p^e)$? Wall stated that, for $\alpha = [\overline{1}]$, no examples for $k(p) = k(p^2)$ occur for $p < 10{,}000$ and we have checked this for $p < 100{,}000$. Does $k(p) = k(p^2)$ ever happen in that case? Given $\alpha = [\overline{a_1, a_2, \ldots, a_t}]$, can bounds be given on the p's for which $k(p) = k(p^2)$? Does t play a role in such bounds? Can anything be said for $k(p) = k(p^e)$ for $e = 3, 4, \ldots$?

Wall gives considerable discussion of the period length of the sequence of r_n's defined in Lemma 8 for the case in which $a_n = 1$ for all n. There, the period is often independent of the initial conditions a and b. To what extent does that theory work for periodic sequences of a_n's?

The next question concerns the upper bounds for $k(p)$ given by Theorems 21 and 22. We would like to know when $k(p)$ equals its upper bound. We conjecture that $k(p)$ is the upper bound with some frequency; perhaps two-thirds of the $k(p)$ equal their upper bound when t is a prime. Can the bounds be improved when t is composite?

Addendum on Lower Bounds

Theorem 6 gives a trivial lower bound on $k(p)$. It seems reasonable to expect $k(m) > c \log(m)$ for some constant c depending on α. Are such bounds possible? The referee offered the following solution. Let $a_1, a_2, \ldots, a_n$ be the complete list of the partial quotients for a given quadratic irrational α. Set $A = \max\{a_1, \ldots, a_n\} + 1$. Then

$$p_t \le (A - 1)p_{t-1} + p_{t-2} \le Ap_{t-1} \text{ for all } t \ge 2$$

and $p_1 = a_1 < A$ so that

$$p_t < A^t \text{ for all } t \ge 1.$$

For $A^t \le m < A^{t+1}$, this means that $k(m) \ge t$. It follows that

$$k(m) > \frac{\log m}{\log A} - 1 \text{ for all } m \ge 1.$$

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ON THE NOTION OF UNIFORM DISTRIBUTION MOD 1

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0. Introduction

The notion of a uniformly distributed sequence mod 1 is a classical tool of number theory (see, e.g., [1], [2]), but it is well known that there exist sequences which are not uniformly distributed; it turns out that this kind of sequence is more conveniently treated by notions other than the classical ones.

In this paper one such notion is used, which enables us to study the sequence formed by the fractional parts of decimal logarithms of the integers (it is well known that this sequence is not uniformly distributed in the classical sense; see, e.g., [1]).

With our result, we obtain a simple solution of the so-called first digit problem.

1. Preliminary Results

In this section we list some definitions and results used in the sequel. We begin with the definition of uniform distribution with respect to a measure on $\mathbb{N}^*$.

Definition 1.1: Let μ be a measure on $\mathbb{N}^*$, which we assume to be positive and unbounded; for each integer n, write

$$S_n = \mu([1, n]).$$

Now let $(x_n)_{n \geq 1}$ be a sequence of real numbers in $[0, 1]$. We say that $(x_n)_{n \geq 1}$ is μ-uniformly distributed in $[0, 1]$ if, for each function f in $C([0, 1])$, we have

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \mu\{k\} f(x_k)}{S_n} = \int_0^1 f(x)\, dx.$$

Remark 1.2: It is easily seen that we may replace (S_n) by any equivalent sequence.

Remark 1.3: The notion of uniform distribution in the sense of Definition (1.1) has been introduced by other authors, although they used different names and symbols.

It is also clear that it can be expressed by saying that the sequence of measures $(\nu_n)_{n \geq 1}$ on $[0, 1]$ defined by

$$\nu_n = \frac{\mu\{k\}}{S_n} \varepsilon_{x_k}$$

weakly converges to the Lebesgue measure on $[0, 1]$ (see, e.g., [8]).

In what follows, we shall use the following proposition, a direct consequence of well-known results concerning weak convergence; note that it is a straightforward generalization of a classical theorem in number theory (see [1], [2]).

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Proposition 1.4: The following conditions are equivalent:

(a) $(x_n)_{n\geq 1}$ is μ-uniformly distributed in $[0, 1]$;
(b) for every interval $[a, b[$ in $[0, 1]$, we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mu\{k\} 1_{[a,b[}(x_k)}{S_n} = b - a,$$

where $1_{[a,b[}$ stands for the indicator function of $[a, b[$.

For each integer n, write

(1.5) $H_n = \exp S_n$.

We shall assume that the sequence $(H_n)_{n\geq 1}$ is obtained by restriction to $\mathbb{N}^*$ of a function H defined on $\mathbb{R}^+$ and having the following property:

(1.6) There exists a positive constant ℓ and an increasing, slowly varying function L such that

$$H(y) = y^{\ell} L(y).$$

(We recall that L varies slowly at infinity if, for every $x > 0$, we have

$$\lim_{y\to+\infty} \frac{L(xy)}{L(y)} = 1.$$

For further properties, see [5].)

To handle the case $\ell = 0$, we make an additional assumption:

(1.7) For each (a_1, a_2, a_3, a_4) in $\mathbb{R}^4$, where a_1, a_2, a_3, a_4 are strictly positive numbers such that $a_1 \neq a_2$, $a_3 \neq a_4$, we have

$$\frac{L(a_1 y) - L(a_2 y)}{\log(a_1 a_2^{-1})} \sim \frac{L(a_3 y) - L(a_4 y)}{\log(a_3 a_4^{-1})}$$

as y converges to infinity.

We prove the following proposition.

Proposition 1.8:

(a) For every $x > 0$, we have $\lim_{y\to+\infty} \dfrac{L(x + y)}{L(y)} = 1$.

(b) In the case $\ell = 0$, for each (b_1, b_2, b_3, b_4) in $\mathbb{R}^4$, where b_1, b_2, b_3, b_4 are positive numbers, we have

$$\frac{L(a_1 y + b_1) - L(a_2 y + b_2)}{\log(a_1 a_2^{-1})} \sim \frac{L(a_3 y + b_3) - L(a_4 y + b_4)}{\log(a_3 a_4^{-1})},$$

as y converges to infinity.

Proof: Part (a) follows from the inequalities

$$1 \leq \frac{L(x + y)}{L(y)} \leq \frac{L(2y)}{L(y)},$$

the second of which holds for y sufficiently large.

The assertion of part (b) is proved by noting that, for every $\varepsilon > 0$, we have, for y sufficiently large

$$\frac{L(a_1 y) - L((a_2 + \varepsilon)y)}{L((a_3 + \varepsilon)y) - L(a_4 y)} \leq \frac{L(a_1 y + b_1) - L(a_2 y + b_2)}{L(a_3 y + b_3) - L(a_4 y + b_4)} \leq \frac{L((a_1 + \varepsilon)y) - L(a_2 y)}{L(a_3 y) - L((a_4 + \varepsilon)y)}.$$

Definition 1.9: Let μ be a measure on $\mathbb{N}^*$; we say that μ has property P if (1.6) holds [in the case $\ell = 0$, if (1.6) and (1.7) hold].

We shall also use some results concerned with the notion of density on $\mathbb{N}^*$, which is studied, for example, in [6].

Definition 1.10: Let μ be a measure on $\mathbb{N}^*$, and (S_n) its distribution function as defined in Definition (1.1).

Consider the density on $\mathbb{N}^*$ generated by the sequence of measures, $(\mu)_{n \geq 1}$, defined as follows:

$$\mu_n = \frac{1}{S_n} 1_{[1,n]} \cdot \mu;$$

this density will be called the μ-density.

Definition 1.11: For each $t > 0$, let $\tilde{\mu}_t$ be the measure on $\mathbb{N}^*$ defined by

$$\tilde{\mu}_t = \sum_{k \geq 1} [\exp(-tS_k) - \exp(-tS_{k+1})]\varepsilon_k.$$

The density generated by $(\tilde{\mu}_t)$ will be called the exponential density with respect to μ (or, more briefly, the μ-exponential density).

We state the following result, the proof of which is given in [6].

Proposition 1.12: Assume that the sequence $(\mu\{n\})_{n \geq 1}$ is bounded. Then the μ-density and the μ-exponential density agree everywhere.

The following theorem, proved in [7], gives a practical method for calculating an exponential density.

Theorem 1.13: Let $(\ell_n)_{n \geq 1}$, $(m_n)_{n \geq 1}$ be two sequences of positive real numbers, such that

(i) $\lim\limits_{n \to \infty} \ell_n = \lim\limits_{n \to \infty} m_n = +\infty$ and $\ell_n \leq m_n \leq \ell_{n+1}$ for every integer n;

(ii) the sequence $(m_n - \ell_n)_{n \geq 1}$ is bounded;

(iii) we have $m_n \sim m_{n+1}$, $\ell_n \sim \ell_{n+1}$ as n converges to infinity.

Last, let A be a real number, with $0 \leq A \leq 1$; then the following conditions are equivalent:

$$\text{(a)} \lim_{n \to \infty} \frac{\sum_{k=1}^{n} (\mu_k - \ell_k)}{m_n} = A; \qquad \text{(b)} \lim_{n \to \infty} \frac{\sum_{k=1}^{n} (\mu_k - \ell_k)}{\ell_n} = A;$$

$$\text{(c)} \lim_{n \to 0^+} \sum_{k \geq 1} [\exp(-t\ell_k) - \exp(-tm_k)] = A.$$

2. The Theorem of Uniform Distribution

We shall prove the following result.

Theorem 2.1: Let μ be a measure on $\mathbb{N}^*$, with property P. Then the sequence $(\{\log_{10} n\})_{n \geq 1}$ is μ-uniformly distributed in [0, 1].

Proof: Proposition (1.4) applies, so we can show that, for every interval $[a, b[$ in [0, 1], we have

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \mu\{k\} 1_{[a,b[}(\{\log_{10} k\})}{\log H_n} = b - a.$$

We can write

$$1_{[a,b[}(\{\log_{10}\}k) = 1_E(k),$$

where E is the subset of $\mathbb{N}^*$ the elements of which are the integers k satisfying the relation

$$10^{n+a} \le k < 10^{n+b}$$

for some integer n; hence, calculating the limit above amounts to finding the μ-density of E (in the sense of Definition 1.10).

First we note that, because of the relations

$$10^{n+b} - 10^{n+a} \ge 1 \quad \text{and} \quad 10^{n+a+1} - 10^{n+b} \ge 1,$$

which hold for n sufficiently large, E is neither finite nor cofinite. Denote by $(p_n)_{n\ge 1}$, $(q_n)_{n\ge 1}$ the two sequences of integers such that

$$E = \bigcup_{n\ge 1} [p_n,\ q_n[.$$

Moreover, for every $x > 0$, write

$$\vartheta(x) = \begin{cases} x & \text{if } x \text{ is an integer} \\ [x] + 1 & \text{otherwise,} \end{cases}$$

so that we have the obvious relations

$$p_n = \vartheta(10^{n+a}); \quad q_n = \vartheta(10^{n+b}).$$

Because of our hypotheses on μ, the sequence $(\mu\{n\})_{n\ge 1}$ is bounded; hence, Proposition 1.12 applies, and our goal is equivalent to finding the μ-exponential density of E, that is, we calculate the limit

$$\lim_{t\to 0^+} \sum_{n\ge 1} [\exp(-t \log H_{p_n}) - \exp(-t \log H_{q_n})];$$

we do this by means of Theorem 1.13, where we put

$$\ell_n = \log H_{p_n}; \quad m_n = \log H_{q_n}.$$

The inequalities $x \le \vartheta(x) \le x + 1$, together with Proposition 1.8, give

$$\lim_{n\to\infty} \frac{m_n - \ell_n}{m_n - m_{n-1}} = b - a;$$

now, a well-known theorem of Cesaro gives the same value for the limit we considered in Theorem 1.13(a).

Remark 2.2: Paper [3] treats, using different techniques, the particular case of the preceding theorem where $\mu\{n\} = 1/n$ (so that $S_n \sim \log n$). Paper [3] also contains a reference to another paper [4] in which the same particular case is studied. The same result is extended in a different direction in Theorem 7.16 on page 64 of [1].

Now, let r be an integer, with $1 \le r \le 9$ and, in the proof of Theorem 2.1, take $a = \log_{10} r$, $b = \log_{10}(r + 1)$; then E turns out to be the set of integers the decimal expansion of which has r as the first digit and the preceding proof gives

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mu\{k\} 1_E(k)}{\mu([1,\ n])} = \log_{10} \frac{r + 1}{r}.$$

This simple remark may be rephrased as follows:

Corollary 2.3: Let E be the set of integers the decimal expansion of which has r as the first digit; if μ is a measure on $\mathbb{N}^*$ satisfying the property P, then the μ-density of E is $\log_{10}(r + 1)/r$.

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SOME RECURSIVE ASYMPTOTES

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1. Introduction

We consider a generalized Fibonacci sequence $\{H_n\}$ defined by the linear homogeneous recurrence relation

$$(1.1) \qquad H_n = H_{n-1} + H_{n-2}, \; n > 2,$$

with initial conditions $H_1 = 1$, $H_2 = X$, where X can be real or complex. In [6] Horadam has studied the properties of these sequences. Among these properties, it is well known that

$$(1.2) \qquad \lim_{n\to\infty} H_{n+1}/H_n = \alpha,$$

where $\alpha = (1 + \sqrt{5})/2$ is the positive root of the associated auxiliary polynomial. The other root is $\beta = (1 - \sqrt{5})/2$.

The purpose of this paper is to look at two variations of (1.2) and at two curves that result. Before that, we recall that the general term for $\{H_n\}$ is given by

$$(1.3) \qquad H_n = A\alpha^n - B\beta^n,$$

where $A = (X - \beta)/\alpha\sqrt{5}$ and $B = (X - \alpha)/\beta\sqrt{5}$.

2. Curves

We next construct the function

$$(2.1) \qquad I(X) = \lim_{n\to\infty} \alpha^{2n} \left| \alpha - \frac{H_{n+1}}{H_n} \right|.$$

At first sight, this would appear to be indeterminate. However, with the use of (1.3), we can establish that

$$(2.2) \qquad I(X) = \pm \frac{\alpha - \beta}{A/B}$$

accordingly as n is even or odd. With the repeated use of $\alpha^2 = \alpha + 1$, (2.2) can be reduced to

$$(2.3) \qquad I(x) = \pm(3\alpha + 1)(X - \alpha)/(X - \beta).$$

Figure 1 is a sketch of $I(X)$ plotted on the Cartesian plane. We have a pair of intersecting hyperbolae with asymptotes given by $I(X) = \pm(3\alpha + 1)$ and $X = \beta$, and X-intercept of $X = \alpha$.

Now put $X = x + iy$, so that we have $I(X) \equiv I(x, y)$ and

$$(2.4) \qquad I(x, y) = \pm \frac{(3\alpha + 1)}{(\alpha + (x - 1))^2 + y^2}\{(x^2 + y^2 - x - 1) + iy(2\alpha - 1)\}.$$

Figure 2 is a sketch of $I(x, y)$ plotted on the Argand plane, holding y constant and varying x. We have a pair of parabolic pencils of coaxal circles. The radius of each circle is $(5 + 5\alpha)/2y$, and each is tangential to the real axis at the points $(\pm(3\alpha + 1), 0)$.

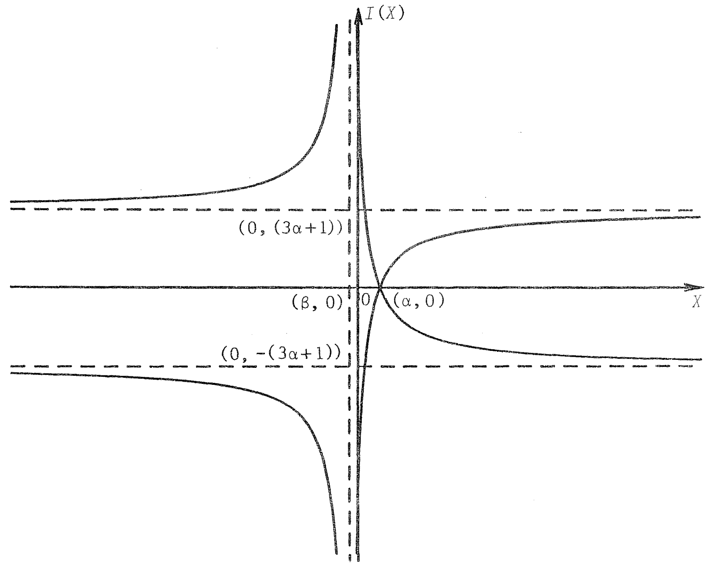

FIGURE 1. $I(X)$ vs. X

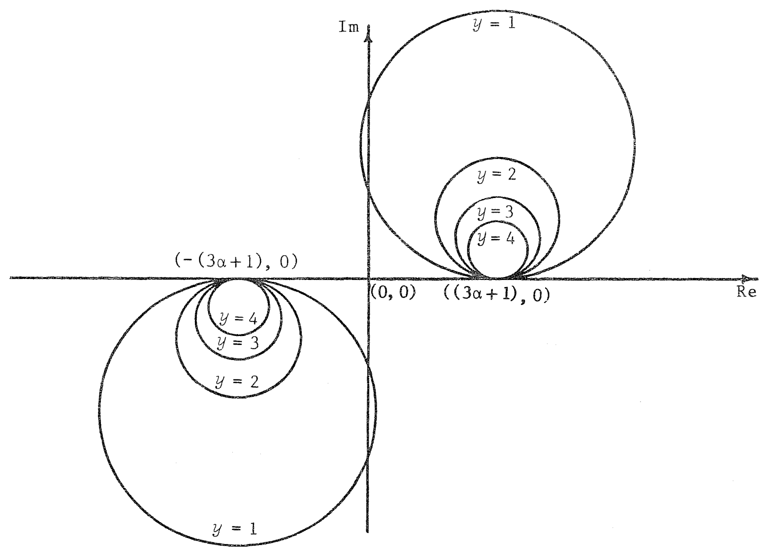

FIGURE 2. $I(x, y)$

In the terminology of Deakin [3], consider the following numbers:

complex: $x + iy$, $i^2 = -1$,
dual: $x + \varepsilon y$, $\varepsilon^2 = 0$,
duo: $x + \omega y$, $\omega^2 = 1$.

$I(x, y)$ generates circles in the complex plane, parabolas in the dual plane, and hyperbolas in the duo plane.

3. Other Generators of α

We define the iterative root sequence $\{\nu_n\} \equiv \{\nu_n(k, x; a, b)\}$ by means of the recurrence relation

(3.1) $\quad \nu_n(k, x; a, b) = (b\nu_{n-1}(k, x; a, b) + a)^{1/k}$

with initial term $\nu_1(k, x; a, b) = x^{1/k}$. For example (see [13]),

$$\lim_{n\to\infty} \nu_n(k, a; a, b) = \alpha.$$

It is known (see [2]) that if $\lim_{n\to\infty} \nu_n(k, a; a, b) = A$ then

(3.2) $\quad a + bA = A^k$

and

(3.3) $\quad \lim_{n\to\infty} \nu_n(k, F_{k-1}; F_{k-1}, F_k) = \alpha$

or

(3.4) $\quad F_{k-1} + F_k\alpha = \alpha^k,$

where F_k is the k^{th} Fibonacci number. An early example of (3.4) occurs in Basin & Hoggatt [1] and a later geometric illustration in Schoen [11]. For a background to this in the more usual context of continued fractions, see Hoggatt & Bruckman [5] and Kiss [9]. We wish to consider here the rate of convergence of (3.3).

Whitaker [12] recently showed that for sequences $\{\nu_n(k, x; a, b)\}$, a finite nonzero function $I(X)$ can be constructed in the form

$$I(X) = \lim_{n\to\infty}\left(\frac{kA^{k-1}}{b}\right)^n (A - \nu_n(k, X; a, b)).$$

The equivalent form for the Fibonacci case is

(3.5) $$I(X) = \lim_{n\to\infty}\left(\frac{k\alpha^{k-1}}{F_k}\right)^n (\alpha - \nu_n(k, X; F_{k-1}, F_k)).$$

As before, this can be considered on the real or complex planes, although there is no closed form for $I(X)$. Comparing (2.1) and (3.5), we can comment on the rates of convergence of the methods of generating α from the ratio of successive terms of the generalized Fibonacci sequence and from the iterative root sequence. The rate of convergence of the former method is proportional to α^2, whereas the other rate is proportional to $k\alpha^{k-1}/F_k$. If $k \ge 2$, $k\alpha^{k-1}/F_k > \alpha^2$, because

$$\alpha^2 F_k = (\alpha^3\alpha^{k-1} - \beta^{k-2})/\sqrt{5} < (\alpha^3/\sqrt{5})\alpha^{k-1} = (1.89)\alpha^{k-1}.$$

Thus, the iterative root sequences produce the faster convergence rate. If we consider noninteger values of k, we can find an iterative root sequence that converges to α at the same rate as the ratio of the generalized Fibonacci number; that is, we can find k, such that (i) $k\alpha^{k-1} = \alpha^2$ and (ii) $a + \alpha = \alpha^k$. This occurs when $k = 1.790048745$ and $a = 0.74841991$. Calculation shows that both H_{n+1}/H_n and $\nu_n(k, a; a, 1)$ with these values of k and a require 22 iterations to provide eight-figure accuracy for α.

4. Concluding Comments

The ideas presented here can be extended by altering the recurrence relation. One way is to include real coefficients, another is to increase the order. Kiss & Ticky [10] have determined the asymptotic distribution function for the ratios of the terms in the former case, and Goldstern, Tichy & Turnwald [4] in the latter. They have also established several estimates for the discrepancy or error term. Another generalization would be to consider

$$(4.1) \qquad I(X) = \lim_{n \to \infty} \alpha^{2n} \left| \alpha^k - \frac{H_{n+k}}{H_n} \right|$$

by analogy with (3.1) of Horadam [7]. In the Fibonacci sequence (4.1) can be rearranged as

$$(4.2) \qquad I(X)/(3\alpha + 1) = \pm F_k(X - \alpha)/(X - \beta).$$

Graphs of these are directly related to Fibonacci sequences as in Horadam & Shannon [8].

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SECOND-ORDER RECURRENCES AND THE SCHRÖDER-BERNSTEIN THEOREM

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1. Introduction

The Schröder-Bernstein theorem states that if f is a one-to-one mapping of X into Y and g is a one-to-one mapping of Y into X, then there exists a one-to-one mapping h of X onto Y; see, for example, [1].

The proof of the theorem involves the construction of three disjoint subsets of X satisfying certain criteria. Applied to a specific example, the subsets produced are unions of intervals bounded by ratios of successive Fibonacci and Lucas numbers and the singleton $\{2/(1+\sqrt{5})\}$ where $(1+\sqrt{5})/2$ is the golden ratio. More generally, the subsets produced are the unions of intervals bounded by ratios of successive elements of two general second-order recurrence sequences with the same characteristic equation and the singleton $\{1/\alpha\}$ where α denotes the positive root of the characteristic equation of the given recurrence.

As usual, we define the Fibonacci and Lucas sequences for all n by

(1) $\quad F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n$

and

(2) $\quad L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n.$

We further define the sequences $\{H_n\}$ and $\{K_n\}$ for all n by

(3) $\quad H_0 = c,\ H_1 = ac,\ H_{n+2} = aH_{n+1} + bH_n$

and

(4) $\quad K_0 = d,\ K_1 = e,\ K_{n+2} = aK_{n+1} + bK_n$

where a, b, c, d, and e are positive. Since we will need it later, we further require that

(5) $\quad a > \dfrac{e}{d}.$

2. Proof of the Schröder-Bernstein Theorem

Before showing how it is related to second-order recurrences, we outline the proof of the Schröder-Bernstein theorem.

With f and g as defined, let $g(Y)$ be the subset of X that is the image of Y under g. Let $A_0 = X - g(Y)$ and let $A_n = g(f(A_{n-1}))$ for each integer $n \geq 1$. Let $f(X)$ be the subset of Y that is the image of X under f, let $B_0 = g(Y - f(X))$, and let $B_n = g(f(B_{n-1}))$ for all $n \geq 1$. Finally, set

$$A = \bigcup_{i=0}^{\infty} A_i, \quad B = \bigcup_{i=0}^{\infty} B_i, \quad \text{and} \quad X_\infty = X - (A \cup B).$$

Then it is not difficult to show that A, B, and X_∞ are disjoint, that

$$X = A \cup B \cup X_\infty,$$

and that the function h, defined by

$$h(x) = \begin{cases} f(x) & \text{for } x \in A \cup X_\infty, \\ g^{-1}(x) & \text{for } x \in B, \end{cases}$$

is a one-to-one mapping from X onto Y.

3. An Example Involving Second-Order Recurrences

Theorem 1: Let $X = (0, 2)$, $Y = (1, \infty)$, $f(x) = x + 1$, and $g(y) = 1/y$. Then the sets A_n, B_n, and X_∞ of the proof of the Schröder-Bernstein theorem are given by the following:

$$(6)\qquad A_n = \begin{cases} \left[\dfrac{F_{n+1}}{F_{n+2}}, \dfrac{L_n}{L_{n+1}}\right) & n \geq 0,\ n \text{ even}, \\ \left(\dfrac{L_n}{L_{n+1}}, \dfrac{F_{n+1}}{F_{n+2}}\right] & n \geq 0,\ n \text{ odd}, \end{cases}$$

$$(7)\qquad B_n = \begin{cases} \left(\dfrac{F_n}{F_{n+1}}, \dfrac{L_{n+1}}{L_{n+2}}\right] & n \geq 0,\ n \text{ even}, \\ \left[\dfrac{L_{n+1}}{L_{n+2}}, \dfrac{F_n}{F_{n+1}}\right) & n \geq 0,\ n \text{ odd}, \end{cases}$$

and

$$(8)\qquad X_\infty = \left\{\frac{2}{1 + \sqrt{5}}\right\}.$$

More generally, the following theorem holds.

Theorem 2: Let a, b, c, d, e, $\{H_n\}_{n \geq 0}$, and $\{K_n\}_{n \geq 0}$ be as in the introduction. Let $X = (0, d/e)$, $Y = (a, \infty)$, $f(x) = bx + a$, $g(y) = 1/y$, and $H_{-1} = 0$. Then the sets A_n, B_n, and X_∞ of the proof of the Schroder-Bernstein theorem are given by

$$(9)\qquad A_n = \begin{cases} \left[\dfrac{H_n}{H_{n+1}}, \dfrac{K_n}{K_{n+1}}\right) & n \geq 0,\ n \text{ even}, \\ \left(\dfrac{K_n}{K_{n+1}}, \dfrac{H_n}{H_{n+1}}\right] & n \geq 0,\ n \text{ odd}, \end{cases}$$

$$(10)\qquad B_n = \begin{cases} \left(\dfrac{H_{n-1}}{H_n}, \dfrac{K_{n+1}}{K_{n+2}}\right] & n \geq 0,\ n \text{ even}, \\ \left[\dfrac{K_{n+1}}{K_{n+2}}, \dfrac{H_{n-1}}{H_n}\right) & n \geq 0,\ n \text{ odd}, \end{cases}$$

and

$$(11)\qquad X_\infty = \left\{\frac{2}{a + \sqrt{a^2 + 4b}}\right\}.$$

Before proving these theorems, it will be convenient to prove the following lemmas.

Lemma 1: If f, g, $\{H_n\}$, and $\{K_n\}$ are as defined in Theorem 2, then

$$g\left(f\left(\frac{H_{n-1}}{H_n}\right)\right) = \frac{H_n}{H_{n+1}} \quad \text{and} \quad g\left(f\left(\frac{K_n}{K_{n+1}}\right)\right) = \frac{K_{n+1}}{K_{n+2}}$$

for all $n \geq 0$.

Proof: Since $g(f(x)) = 1/(bx + a)$,

$$g\left(f\left(\frac{H_{n-1}}{H_n}\right)\right) = \frac{1}{b \cdot \dfrac{H_{n-1}}{H_n} + a} = \frac{H_n}{bH_{n-1} + aH_n} = \frac{H_n}{H_{n+1}}$$

by (3). Note that this is even true for $n = 0$, since $H_{-1} = 0$ is consistent with (3). Similarly, we show that

$$g\left(f\left(\frac{K_n}{K_{n+1}}\right)\right) = \frac{K_{n+1}}{K_{n+2}}.$$

Lemma 2: For the sequences $\{H_n\}$ and $\{K_n\}$ as defined in Theorem 2, the following inequalities hold.

$$(12) \qquad \frac{H_n}{H_{n+1}} > \frac{K_{n+1}}{K_{n+2}} \qquad n \geq 0,\ n \text{ even},$$

$$(13) \qquad \frac{H_n}{H_{n+1}} < \frac{K_{n+1}}{K_{n+2}} \qquad n \geq 0,\ n \text{ odd}.$$

Proof: Since a, b, c, d, and e are positive, it follows from (3) and (4) that

$$\frac{H_0}{H_1} = \frac{c}{ac} > \frac{e}{ae + bd} = \frac{K_1}{K_2}.$$

Since $g(f(x)) = 1/(bx + a)$ is a decreasing function and

$$f\left(\frac{H_0}{H_1}\right) = \frac{H_2}{H_1} \quad \text{and} \quad f\left(\frac{K_1}{K_2}\right) = \frac{K_3}{K_2},$$

it follows that

$$\frac{H_1}{H_2} < \frac{K_2}{K_3}$$

and the argument for all $n \geq 0$ is easily completed by induction.

Lemma 3: If X, A_n, and B_n are defined as in Theorem 2, then

$$X - \left[\left(\bigcup_{i=0}^{n} A_i\right) \cup \left(\bigcup_{i=0}^{n} B_i\right)\right] = \begin{cases} \left(\dfrac{K_{n+1}}{K_{n+2}}, \dfrac{H_n}{H_{n+1}}\right) & n \geq 0,\ n \text{ even}, \\[2ex] \left(\dfrac{H_n}{H_{n+1}}, \dfrac{K_{n+1}}{K_{n+2}}\right) & n \geq 0,\ n \text{ odd}. \end{cases}$$

Proof: For $n = 0$, it follows from (3) and (4) that

$$A_0 \cup B_0 = \left[\frac{H_0}{H_1}, \frac{K_0}{K_1}\right) \cup \left(\frac{H_{-1}}{H_0}, \frac{K_1}{K_2}\right] = \left[\frac{c}{ac}, \frac{d}{e}\right) \cup \left(\frac{0}{c}, \frac{e}{ae + bd}\right].$$

Thus, since $X = (0, d/e)$ and $e/(ae + bd) < c/ac$ as above,

$$X - (A_0 \cup B_0) = \left(\frac{e}{ae + bd}, \frac{c}{ac}\right) = \left(\frac{K_1}{K_2}, \frac{H_0}{H_1}\right)$$

as claimed. Assume that, for k even,

$$X - \left[\left(\bigcup_{i=0}^{k} A_i\right) \cup \left(\bigcup_{i=0}^{k} B_i\right)\right] = \left(\frac{K_{k+1}}{K_{k+2}}, \frac{H_k}{H_{k+1}}\right).$$

Then, since $H_{k+1}/H_{k+2} < K_{k+2}/K_{k+3}$ by Lemma 2, it follows that

$$X - \left[\left(\bigcup_{i=0}^{k+1} A_i\right) \cup \left(\bigcup_{i=0}^{k+1} B_i\right)\right] = \left(\frac{K_{k+1}}{K_{k+2}}, \frac{H_k}{H_{k+1}}\right) - \left(\frac{K_{k+1}}{K_{k+2}}, \frac{H_{k+1}}{H_{k+2}}\right] - \left[\frac{K_{k+2}}{K_{k+3}}, \frac{H_k}{H_{k+1}}\right) =$$

$$= \left(\frac{H_{k+1}}{H_{k+2}}, \frac{K_{k+2}}{K_{k+3}}\right).$$

This proves the result for $k + 1$. The proof for $k + 2$, which completes the induction, is the same as for $k + 1$ except that it requires the inequality

$$\frac{K_{k+3}}{K_{k+4}} < \frac{H_{k+2}}{H_{k+3}}$$

which also follows from Lemma 2, since k is even.

Proof of Theorem 2: As in the sketch of the proof of the Schröder-Bernstein theorem, we consider

$$g(Y) = g((a, \infty)) = \left(0, \frac{1}{a}\right)$$

and

$$A_0 = X - g(Y) = \left(0, \frac{d}{e}\right) - \left(0, \frac{1}{a}\right) = \left[\frac{1}{a}, \frac{d}{e}\right) = \left[\frac{H_0}{H_1}, \frac{K_0}{K_1}\right)$$

by (3) and (4), since $1/a < d/e$ by (5). Now, assume that

$$A_k = \left[\frac{H_k}{H_{k+1}}, \frac{K_k}{K_{k+1}}\right),$$

where $k \geq 0$ is even. Then

$$A_{k+1} = g(f(A_k)) = g\left(f\left(\left[\frac{H_k}{H_{k+1}}, \frac{K_k}{K_{k+1}}\right)\right)\right) = \left(\frac{K_{k+1}}{K_{k+2}}, \frac{H_{k+1}}{H_{k+2}}\right]$$

by Lemma 1 since, as noted above, $g(f(x))$ is a decreasing function. Repeating this argument with $k + 1$ replacing k, we have that

$$A_{k+2} = \left[\frac{H_{k+2}}{H_{k+3}}, \frac{K_{k+2}}{K_{k+3}}\right).$$

Thus, by mathematical induction, the A_n are as described in (9). Moreover, we note that we have also shown that

$$\frac{H_n}{H_{n+1}} < \frac{K_n}{K_{n+1}} \text{ for } n \text{ even} \quad \text{and} \quad \frac{H_n}{H_{n+1}} > \frac{K_n}{K_{n+1}} \text{ for } n \text{ odd.}$$

To prove (10), we recall from the sketch of the Schröder-Bernstein theorem that

$$B_0 = g(Y - f(X)) = g\left((a, \infty) - f\left(\left(0, \frac{d}{e}\right)\right)\right) = g\left((a, \infty) - \left(a, \frac{bd + ae}{e}\right)\right)$$
$$= g\left(\left[\frac{bd + ae}{e}, \infty\right)\right) = \left(0, \frac{e}{bd + ae}\right] = \left(\frac{H_{-1}}{H_0}, \frac{K_1}{K_2}\right],$$

since we take $H_{-1} = 0$ as noted in the proof of Lemma 1. The proof of (10) is now completed by induction exactly like the proof of (9). Finally, to prove (11), we use Lemma 3. As in the sketch of the Schröder-Bernstein theorem

$$X_\infty = X - (A \cup B)$$
$$= X - \left[\left(\bigcup_{i=0}^{\infty} A_i\right) \cup \left(\bigcup_{i=0}^{\infty} B_i\right)\right] = \lim_{n\to\infty}\left\{X - \left[\left(\bigcup_{i=0}^{n} A_i\right) \cup \left(\bigcup_{i=0}^{n} B_i\right)\right]\right\}$$
$$= \lim_{n\to\infty}\begin{cases}\left(\dfrac{K_{n+1}}{K_{n+2}}, \dfrac{H_n}{H_{n+1}}\right) & n \text{ even,}\\[2ex] \left(\dfrac{H_n}{H_{n+1}}, \dfrac{K_{n+1}}{K_{n+2}}\right) & n \text{ odd,}\end{cases} = \{1/\alpha\},$$

where $\alpha = (a + \sqrt{a^2 + 4b})/2$, since it is well known that

$$\lim_{n\to\infty} \frac{K_{n+1}}{K_n} = \lim_{n\to\infty} \frac{H_{n+1}}{H_n} = \alpha,$$

the positive root of the characteristic equation of the recurrences in (3) and (4). This completes the proof of Theorem 2.

It is interesting to note that, since α is a root of $x^2 = ax + b$,

$$f(g(\alpha)) = a + \frac{b}{\alpha} = \frac{a\alpha + b}{\alpha} = \frac{\alpha^2}{\alpha} = \alpha$$

and

$$g\left(f\left(\frac{1}{\alpha}\right)\right) = \frac{1}{(b/\alpha) + a} = \frac{\alpha}{b + a\alpha} = \frac{\alpha}{\alpha^2} = \frac{1}{\alpha}$$

so that α is a fixed point of $f(g(x))$ and $1/\alpha$ is a fixed point of $g(f(x))$.

Proof of Theorem 1: This follows immediately from Theorem 2 by taking $a = b = c = e = 1$ and $d = 2$.

Of course, similar results obtain for the Pell and other well-known sequences by other appropriate choices of a, b, c, d, and e.

Reference

1. George F. Simmons. *Introduction to Topology and Modern Analysis*. New York: McGraw-Hill, 1963, pp. 29-30.

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SETS OF TERMS THAT DETERMINE ALL THE TERMS OF A LINEAR RECURRENCE SEQUENCE

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A second-order linear homogeneous recurrence sequence u_0, u_1, u_2, ... is defined by a recurrence relation $u_n = au_{n-1} + bu_{n-2}$, where a and b are complex numbers with $b \neq 0$, and two initial terms u_0 and u_1. We raise the following question: for given a and b, what sets of terms, other than u_0 and u_1, are sufficient to determine the entire sequence? We shall see that *any* two terms are often sufficient, but not always. A comparable result will then be presented for recurrences of higher order.

Suppose a and b are given and v_p and v_q, where $p < q$, are known terms of a sequence satisfying $v_m = av_{m-1} + bv_{m-2}$. Then the terms u_0 and u_n of the sequence defined by $u_m = v_{m+p}$, where $n = q - p$, are known. Accordingly, without loss of generality, we recast the original question as follows: *under what conditions on a, b, and n do the values of u_0 and u_n determine the values of u_m for all $m \geq 0$?*

The answer depends on a sequence of bivariate polynomials defined recursively by $F_m(x, y) = xF_{m-1}(x, y) + yF_{m-2}(x, y)$, beginning with $F_1(x, y) = 1$ and $F_2(x, y) = x$. These are often called *Fibonacci polynomials*; indeed, $F_m(1, 1)$ is the m^{th} Fibonacci number.

Theorem 1: Suppose a and b are complex numbers satisfying $F_n(a, b) \neq 0$, where $b \neq 0$ and $F_n(x, y)$ denotes the Fibonacci polynomial of degree $n - 1$ in x. Then u_0 and u_n determine u_m for all $m \geq 0$.

Proof: If $n = 1$, then the recurrence $u_m = au_{m-1} + bu_{m-2}$ determines u_m inductively for all $m \geq 0$.

If $n = 2$, then the equation $u_2 = au_1 + bu_0$ yields $u_1 = (u_2 - bu_0)/a$, so that u_1 and hence all u_m are determined. [Note that $a \neq 0$, since $a = F_2(a, b)$.]

For $n \geq 3$, we have a system $u_s = au_{s-1} + bu_{s-2}$ of $n - 1$ equations, for $s = 2, 3, \ldots, n$. Write the first of these as $au_1 - u_2 = -bu_0$, the last as $bu_{n-2} + au_{n-1} = u_n$, and all the others as $bu_{s-2} + au_{s-1} - u_s = 0$. As an example, for $n = 5$, we have

$$\begin{aligned} au_1 - u_2 &= -bu_0 \\ bu_1 + au_2 - u_3 &= 0 \\ bu_2 + au_3 - u_4 &= 0 \\ bu_3 + au_4 &= u_5. \end{aligned}$$

The coefficient matrix of this system,

$$\begin{bmatrix} a & -1 & 0 & 0 \\ b & a & -1 & 0 \\ 0 & b & a & -1 \\ 0 & 0 & b & a \end{bmatrix}$$

clearly has determinant $F_5(a, b)$ given by Laplace expansion about the first column: $aF_4(a, b) + bF_3(a, b)$.

For the general case, $n \geq 3$, it easy to see, inductively, that the determinant of the coefficient matrix is $F_n(a, b)$. Accordingly, if $F_n(a, b) \neq 0$, then the system has a unique solution. In particular, u_1 is determined, so that u_m is determined for all $m \geq 0$.

Theorem 2: Suppose u_0 and u_n are known for some $n \geq 1$. Suppose, further, that a^2/b is a nonzero integer and one of the following holds:

(i) a^2/b does not equal -1, -2, or -3;
(ii) if $n \equiv 0 \bmod 3$, then $a^2 + b \neq 0$;
(iii) if $n \equiv 0 \bmod 4$, then $a^2 + 2b \neq 0$;
(iv) if $n \equiv 0 \bmod 6$, then $a^2 + 3b \neq 0$.

Then u_m is determined for all $m \geq 0$.

Proof: The polynomial $F_n(x, y)$ is an even function in x if n is odd, and odd in x if n is even. Accordingly, by the Fundamental Theorem of Algebra, this polynomial factors in the form

$$f_n(x^2, y) = (x^2 - c_1 y)(x^2 - c_2 y) \cdots (x^2 - c_{\left[\frac{n-1}{2}\right]} y)$$

if n is odd, and $xf_{n-1}(x^2, y)$ if n is even, where c_i is a complex number for $i = 1, 2, \ldots, n - 1$.

If a^2/b is a nonzero integer k, then $a^2 - kb = 0$, so that $c_i = a^2/b$ for some i. Thus, $x^2 - (a^2/b)y$ divides $F_n(x, y)$.

It is known ([1], Theorem 6) that the only divisors of $F_n(x, y)$ over the ring $I[x, y]$ that have degree 2 in x are the three second-degree Fibonacci-cyclotomic polynomials: $x^2 + y$, $x^2 + 2y$, $x^2 + 3y$, and that these are divisors if and only if n is divisible by 3, 4, or 6, respectively. Therefore, except for the four recognized cases, we have $F_n(a, b) \neq 0$, so that, by Theorem 1, u_m is determined for all $m \geq 0$.

Theorem 3: Suppose $a^2 + b = 0$ and u_0 is known. Then $u_m = (-a)^m u_0$ for every $m \equiv 0 \bmod 3$. Also, if u_k is known for some k not congruent to 0 modulo 3, then u_m is determined for all $m \geq 0$. In fact,

(1) $$u_m = (-a^3)^i u_j,$$

for $m = 3i + j$, $j = 0, 1, 2$, where $u_2 = au_1 - a^2 u_0$.

Proof: First, we shall establish equation (1). The statements

$$u_{3i} = (-1)^i a^{3i} u_0, \quad u_{3i+1} = (-1)^i a^{3i} u_1, \quad \text{and} \quad u_{3i+2} = (-1)^i a^{3i} u_2$$

are clearly true for $i = 0$. Assume them true for arbitrary $i \geq 0$. Then

$$\begin{aligned} u_{3i+3} &= au_{3i+2} + bu_{3i+1} \\ &= a(-1)^i a^{3i} u_2 - a^2(-1)^i a^{3i} u_1 \\ &= (-1)^i a^{3i+1}(u_2 - au_1) \\ &= (-1)^i a^{3i+1}(-a^2 u_0) \\ &= (-1)^{i+1} a^{3i+3} u_0, \end{aligned}$$

and, similarly,

$$u_{3i+4} = (-1)^{i+1} a^{3i+3} u_1 \quad \text{and} \quad u_{3i+5} = (-1)^{i+1} a^{3i+3} u_2.$$

By induction, therefore,

$$u_m = (-a^3)^i u_j \text{ for } m = 3i + j, \ j = 0, 1, 2.$$

Now equation (1) shows that u_0 determines those u_m for which m is a multiple of 3, and no others. However, if u_{3i+1} is also known for some i, then

$$u_{3i+1} = (-a^3)^i u_1,$$

so that u_1 is determined, and hence u_m is determined for all $m \geq 0$. A similar argument obviously applies if u_{3i+2} is known for some i.

Theorem 4: Suppose $a^2 + 2b = 0$ and u_0 is known. Then

$$u_m = (-1/4)^{m/4} a^m u_0 \text{ for every } m \equiv 0 \bmod 4.$$

If u_k is also known for some k not congruent to 0 modulo 4, then u_m is determined for all $m \geq 0$. In fact,

$$u_m = (-a^4/4)^i u_j \text{ for } m = 4i + j,\ j = 0,\ 1,\ 2,\ 3,$$

where $u_2 = au_1 - (a^2/2)u_0$ and $u_3 = (a^2/2)u_1 - (a^3/2)u_0$.

Proof: (The proof is similar to that of Theorem 3 and is omitted here.)

Theorem 5: Suppose $a^2 + 3b = 0$ and u_0 is known. Then

$$u_m = (-1/27)^{m/6} a^m u_0 \text{ for every } m \equiv 0 \bmod 6.$$

If u_k is also known for some k not congruent to 0 modulo 6, then u_m is determined for all $m \geq 0$. Explicitly,

$$u_2 = au_1 - (a^2/3)u_0,$$
$$u_3 = (2a^2/3)u_1 - (a^3/3)u_0,$$
$$u_4 = (a^3/3)u_1 - (2a^4/9)u_0,$$
$$u_5 = (a^4/9)u_1 - (a^5/9)u_0,$$

and $$u_m = (-a^6/27)^i u_j,$$

for $m = 6i + j = 0,\ 1,\ 2,\ 3,\ 4,\ 5$.

Proof: (The proof is similar to that of Theorem 3 and is omitted here.)

Second-order sequences for which $u_1 \neq 0$ and $u_0 = u_n = 0$ for some $n \geq 2$ are of special interest, since in this case $F_n(a, b) = 0$, so that Theorem 1 does not apply. Theorem 6 describes such sequences. [To see that $F_n(a, b) = 0$, note that the recurrence $u_m = au_{m-1} + bu_{m-2}$ gives

$$u_2 = au_1, \quad u_3 = au_2 + bu_1 = (a^2 + b)u_1 = u_1F_3(a, b),$$

and by induction, $u_n = u_1F_n(a, b)$.]

Theorem 6: Let $F_n(x, y)$ denote the n^{th} Fibonacci polynomial, where $n \geq 2$. If $u_1 = 0$ and $u_0 = u_n = 0$, then $F_n(a, b) = 0$, and there exist nonzero real numbers c, r and positive integers p, q such that

$$u_m = cr^m \sin mp\pi/q,$$

where n is an integer multiple of q, for $m = 0, 1, \ldots$.

Proof: From the Binet representation for the general term of a second-order homogeneous recurrence sequence,

$$u_m = w\alpha^m + z\overline{\alpha}^m.$$

It is easy to check that z must be a complex conjugate of w, so, after writing $w = a + bi$ and $\alpha = r(\cos\theta + i\sin\theta)$, we have

$$\begin{aligned} u_m &= (a + bi)r^m(\cos m\theta + i \sin m\theta) + (a - bi)r^m(\cos m\theta - i \sin m\theta) \\ &= 2r^m(a \cos m\theta - b \sin m\theta). \end{aligned}$$

Now a must equal 0, since $u_0 = 0$, and $\sin n\theta$ must equal 0, since $u_n = 0$. It follows that θ must be of the form $p\pi/q$, where n is a multiple of q. Thus, the asserted form for u_m has been demonstrated. Since u_m is not uniquely determined, Theorem 1 shows that $F\ (a, b) = 0$ (as was already proved differently just before the statement of Theorem 6).

Sequences of Higher Order

The method of proof of Theorem 1 extends readily to recurrence sequences of arbitrary order $k \geq 2$, as indicated by Theorem 7.

Theorem 7: Suppose $k \geq 2$, and suppose $c_0, c_1, \ldots, c_{k-1}$ are complex numbers satisfying $c_{k-1} \neq 0$. A set of k terms,

$$u_0, u_{m_1}, u_{m_2}, \ldots, u_{m_{k-1}},$$

where $0 < m_1 < m_2 < \cdots < m_{k-1}$, uniquely determine all the terms of a recurrence sequence given by

$$(2) \qquad u_n = c_{k-1}u_{n-1} + c_{k-2}u_{n-2} + \cdots + c_0 u_{n-k}$$

if and only if the matrix M defined below is nonsingular: let N denote the $(m_{k-1} - k + 1) \times (m_{k-1} + 1)$ matrix (a_{ij}) given by

$$a_{ij} = \begin{cases} c_{j-i+1} & \text{for } j = i-1, i, \ldots, i+k-2 \\ -1 & \text{for } j = i+k-1 \\ 0 & \text{for all remaining } j,\ 0 \leq j \leq m_{k-1}, \end{cases}$$

$$\text{for } i = 1, 2, \ldots, m_{k-1} - k + 1,$$

and define M to be the $(m_{k-1} - k + 1) \times (m_{k-1} - k + 1)$ matrix obtained by deleting from N the columns numbered $0, m_1, m_2, \ldots, m_{k-1}$.

Proof: Equation (2) generates, for $n = k, k+1, \ldots, m_{k-1}$, a system of $m_{k-1} - k + 1$ equations of the form

$$(3) \qquad c_{k-1}u_{n-1} + c_{k-2}u_{n-2} + \cdots + c_0 u_{n-k} - u_n = 0.$$

If all the terms $u_0, u_1, u_2, \ldots, u_{m_{k-1}}$ are regarded as unknowns, then the coefficient matrix of the system is N. If $u_0, u_{m_1}, u_{m_2}, \ldots, u_{m_{k-1}}$ are now regarded as known, and accordingly transposed to the right-hand side of each of the equations (3), then the coefficient matrix of the resulting system is M. By Cramer's Rule, this system has a unique solution if and only if $|M| \neq 0$.

As an example, consider a third-order recurrence

$$u_n = au_{n-1} + bu_{n-2} + cu_{n-3},$$

and suppose u_0, u_1, and u_m are known. (In the notation of Theorem 6, $k = 3$, $m_1 = 1$, and $m_2 = m$.) Define $T_1 = 1$, $T_2 = a$, and find for $m = 4$ that

$$N_4 = \begin{bmatrix} c & b & a & -1 & 0 \\ 0 & c & b & a & -1 \end{bmatrix},$$

which on deletion of columns 0, 1, and 4 leaves

$$M_4 = \begin{bmatrix} a & -1 \\ b & a \end{bmatrix}$$

with determinant $a^2 + b$. Define $T_3 = a^2 + b$. For $m = 5$,

$$N_5 = \begin{bmatrix} c & b & a & -1 & 0 & 0 \\ 0 & c & b & a & -1 & 0 \\ 0 & 0 & c & b & a & -1 \end{bmatrix} \text{ yields } M_5 = \begin{bmatrix} a & -1 & 0 \\ b & a & -1 \\ c & b & a \end{bmatrix},$$

with determinant $T_4 \equiv aT_3 + bT_2 + cT_1$. Continuing with $m = 6, 7, 8, \ldots$, we obtain recursively a sequence of trivariate polynomials:

$$T_m = aT_{m-1} + bT_{m-2} + cT_{m-1}.$$

Since, for example, $T_4(1, -1, 1) = 0$, Theorem 6 tells us that u_0, u_1, and u_5 are not sufficient to determine all the terms of a sequence obeying the recurrence $u_n = u_{n-1} - u_{n-2} + u_{n-3}$. On the other hand, as $T_5(1, -1, 1) \neq 0$, the terms u_0, u_1, and u_6 do determine the entire sequence.

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SOME CONVOLUTION-TYPE AND COMBINATORIAL IDENTITIES PERTAINING TO BINARY LINEAR RECURRENCES

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Introduction

Let sequences $\{r_n\}$ and $\{s_n\}$ be defined for $n \geq 0$. Letting

$$t_n = \sum_{k=0}^{n} r_k s_{n-k},$$

we obtain a sequence $\{t_n\}$ which is called the convolution of $\{r_n\}$ and $\{s_n\}$. In keeping with the ideas of V. E. Hoggatt, Jr. [7], one may define iterated convolution sequences as follows:

$$r_n^{(0)} = r_n; \quad r_n^{(j)} = \sum_{k=0}^{n} r_k r_{n-k}^{(j-1)} \quad \text{for } j \geq 1.$$

In particular, if $\{F_n\}$ denotes the Fibonacci sequence, then

$$F_n^{(1)} = \sum_{k=0}^{n} F_k F_{n-k}$$

is the convolution of the Fibonacci sequence with itself. Hoggatt [7] obtained the generating function:

$$x/(1 - x - x^2)^{j+1} = \sum_{n=0}^{\infty} F_n^{(j)} x^n.$$

The convolved sequence $F_n^{(1)}$ was also considered by Bicknell [2] and by Hoggatt & Bicknell-Johnson [8]. For related results, see also Bergum & Hoggatt [1] and Horadam and Mahon [9].

Let primary and secondary binary linear recurrences be defined, respectively, by

$$u_0 = 0,\ u_1 = 1,\ u_n = Pu_{n-1} - Qu_{n-2} \text{ for } n \geq 2;$$
$$v_0 = 2,\ v_1 = P,\ v_n = Pv_{n-1} - Qv_{n-2} \text{ for } n \geq 2,$$

where P and Q are nonzero, relatively prime integers such that $D = P^2 - 4Q \neq 0$. In this paper, we generalize prior results of Hoggatt and others by developing formulas for weighted convolutions of the type

$$\sum_{k=0}^{n} f(n, k) r_k s_{n-k},$$

where each of r_n and s_n is u_n or v_n and the weight function $f(n, k)$ is defined for $n \geq 0$ and $0 \leq k \leq n$ and satisfies the symmetry condition

$$f(n, n - k) = f(n, k) \text{ for all } k.$$

In addition, we prove some results about the sums

$$\sum_{k=0}^{n} \binom{n}{k} u_k \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k} v_k.$$

Preliminaries

Let the roots of the equation: $t^2 - Pt + Q = 0$ be

$$a = \tfrac{1}{2}(P + D^{\frac{1}{2}}), \quad b = \tfrac{1}{2}(P - D^{\frac{1}{2}}),$$

so that

(1) $a + b = P$

(2) $ab = Q$

(3) $a - b = D^{\frac{1}{2}}$

(4) $u_n = (a^n - b^n)/(a - b)$

(5) $v_n = a^n + b^n$

(6) $v_n = 2u_{n+1} - Pu_n$

(7) $v_n = Pu_n - 2Qu_{n-1}$

(8) $v_n = u_{n+1} - Qu_{n-1}$

(9) $t/(1 - Pt + Qt^2) = \sum_{n=0}^{\infty} u_n t^n$

(10) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

(11) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

(12) $\sum_{k=0}^{n} k(n - k) = \dfrac{n^3 - n}{6}$

(13) $\sum_{k=0}^{n} k^2(n - k)^2 = \dfrac{n^5 - n}{30}$

The Main Results

Theorem 1:

(a) $$\sum_{k=0}^{n} u_k u_{n-k} = \frac{(n + 1)v_n - 2u_{n+1}}{D} = \frac{nv_n - Pu_n}{D} = \frac{(n - 1)v_n - 2Qu_{n-1}}{D}$$

(b) $$\sum_{k=0}^{n} \binom{n}{k} u_k u_{n-k} = \frac{2^n v_n - 2P^n}{D}$$

Proof: Without restriction on $f(n, k)$, (4) implies

$$\begin{aligned}\sum_{k=0}^{n} f(n, k) u_k u_{n-k} &= \sum_{k=0}^{n} f(n, k)\left(\frac{a^k - b^k}{a - b}\right)\left(\frac{a^{n-k} - b^{n-k}}{a - b}\right)\\ &= (a - b)^{-2} \sum_{k=0}^{n} f(n, k)(a^n + b^n - a^k b^{n-k} - a^{n-k} b^k)\\ &= D^{-1}\left(v_n \sum_{k=0}^{n} f(n, k) - \sum_{k=0}^{n} f(n, k)(a^k b^{n-k} + a^{n-k} b^k)\right),\end{aligned}$$

using (3) and (5).

(a) If $f(n, k) = 1$, we get

$$\sum_{k=0}^{n} u_k u_{n-k} = D^{-1}\left((n+1)v_n - \sum_{k=0}^{n}(a^k b^{n-k} + a^{n-k}b^k)\right).$$

Now

$$\sum_{k=0}^{n} a^k b^{n-k} = \sum_{k=0}^{n} a^{n-k}b^k = b^n\left(\frac{(a/b)^{n+1} - 1}{(a/b) - 1}\right) = \frac{a^{n+1} - b^{n+1}}{a - b} = u_{n+1},$$

so

$$\sum_{k=0}^{n} u_k u_{n-k} = \frac{(n+1)v_n - 2u_{n+1}}{D}.$$

The other parts of (a) follow from (6) and (7), since

$$(n+1)v_n - 2u_{n+1} = nv_n + v_n - 2u_{n+1} = nv_n - Pu_n$$
$$= (n-1)v_n + v_n - Pu_n = (n-1)v_n - 2Qu_{n-1}.$$

(b) If $f(n, k) = \binom{n}{k}$, we get

$$\sum_{k=0}^{n}\binom{n}{k}u_k u_{n-k} = D^{-1}\left(v_n\sum_{k=0}^{n}\binom{n}{k} - \sum_{k=0}^{n}\binom{n}{k}(a^{n-k}b^k + a^k b^{n-k})\right)$$
$$= D^{-1}(2^n v_n - 2(a+b)^n) = \frac{2^n v_n - 2P^n}{D}$$

using (1) and (10).

Theorem 2:

(a) $$\sum_{k=0}^{n} v_k v_{n-k} = (n+1)v_n + 2u_{n+1}$$

(b) $$\sum_{k=0}^{n}\binom{n}{k}v_k v_{n-k} = 2^n v_n + 2P^n.$$

Proof: The proof is similar to that of Theorem 1, except that we use (5) instead of (4).

Theorem 3:

$$w_{n-1} = \sum_{k=0}^{n} u_k u_{n-k} \text{ for } n \geq 1$$

if and only if $w_0 = 0$, $w_1 = 1$, $w_n = Pw_{n-1} - Qw_{n-2} + u_n$ for $n \geq 2$.

Proof: (Sufficiency) Following Carlitz [4], let

$$W(t) = \sum_{n=0}^{\infty} w_n t^n.$$

Then

$$(1 - Pt + Qt^2)W(t) = w_0 + (w_1 - Pw_0)t + \sum_{n=2}^{\infty}(w_n - Pw_{n-1} + Qw_{n-2})t^n$$
$$= t + \sum_{n=2}^{\infty} u_n t^n = \sum_{n=0}^{\infty} u_n t^n = t/(1 - Pt + Qt^2),$$

so $W(t) = t/(1 - Pt + Qt^2)^2$, from which it follows by (9) that

$$w_{n-1} = \sum_{k=0}^{n} u_k u_{n-k}.$$

(Necessity) (Induction on n). Let

$$w_{n-1} = \sum_{k=0}^{n} u_k u_{n-k} \text{ for } n \geq 1.$$

By direct evaluation, we have

$$w_0 = 0,\ w_1 = 1,\ w_2 = 2P,\ w_3 = 3P^2 - 2Q.$$

Theorem 1(a) implies $w_{n-1} = D^{-1}(nv_n - Pu_n)$. Now

$$Pw_1 - Qw_0 + u_2 = P(1) - Q(0) + P = 2P = w_2;$$

$$Pw_2 - Qw_1 + u_3 = P(2P) - Q(1) + (P^2 - Q) = 3P^2 - 2Q = w_3.$$

$$\begin{aligned} Pw_{n-1} - Qw_{n-2} &= \frac{P}{D}(nv_n - Pu_n) - \frac{Q}{D}((n-1)v_{n-1} - Pu_{n-1}) \\ &= \frac{1}{D}(Pv_n + (n-1)(Pv_n - Qv_{n-1}) - P(Pu_n - Qu_{n-1})) \\ &= \frac{1}{D}(Pv_n + (n-1)v_{n+1} - Pu_{n+1}) \\ &= \frac{1}{D}(Pv_n - 2v_{n+1} + (n+1)v_{n+1} - Pu_{n+1}) \\ &= w_n - \frac{1}{D}(2v_{n+1} - Pv_n). \end{aligned}$$

But $2v_{n+1} - Pv_n = 2(a^{n+1} + b^{n+1}) - (a + b)(a^n + b^n) = a^{n+1} + b^{n+1} - ab^n - a^n b = (a^n - b^n)(a - b) = Du_n$. Therefore,

$$Pw_{n-1} - Qw_{n-2} + u_n = w_n - \frac{1}{D}(Du_n) + u_n = w_n.$$

Theorem 4: If

$$x_n = \sum_{k=0}^{n} v_k v_{n-k} \text{ for } n \geq 0,$$

then

$$x_0 = 4,\ x_1 = 4P,\ x_n = Px_{n-1} - Qx_{n-2} + Du_n \text{ for } n \geq 2.$$

Proof: This is similar to the proof of Necessity in Theorem 3, and therefore is omitted here.

Lemma 1: Let $f(n, k)$ be a function such that $f(n, n - k) = f(n, k)$ for all k such that $0 \leq k \leq n$, where n and k are nonnegative integers. Then

$$\sum_{k=0}^{n} Q^k f(n, k) u_{n-2k} = 0.$$

Proof: Let

$$S_n = \sum_{k=0}^{n} Q^k f(n, k) u_{n-2k},\ n^* = [\tfrac{1}{2}(n - 1)],\ S_1 = \sum_{k=0}^{n^*} Q^k f(n, k) u_{n-2k}.$$

Then

$$S_n - S_1 = \sum_{j=n-n^*}^{n^*} Q^j f(n, j) u_{n-2j}.$$

Letting $k = n - j$, we obtain

$$S_n - S_1 = \sum_{k=0}^{n^*} Q^{n-k} f(n, n - k) u_{2k-n} = \sum_{k=0}^{n^*} f(n, k) Q^{n-k}(-u_{n-2k}/Q^{n-2k}),$$

by (14), that is,

$$S_n - S_1 = -\sum_{k=0}^{n^*} Q^k f(n, k) u_{n-2k} = -S_1, \text{ so } S_n = 0.$$

Theorem 5: If $f(n, k)$ satisfies the hypothesis of Lemma 1 above, then

$$\sum_{k=0}^{n} f(n, k) u_k v_{n-k} = u_n\left(\sum_{k=0}^{n} f(n, k)\right).$$

Proof:
$$\begin{aligned}\sum_{k=0}^{n} f(n, k) u_k v_{n-k} &= \sum_{k=0}^{n} f(n, k)\left(\frac{a^k - b^k}{a - b}\right)(a^{n-k} + b^{n-k}) \\ &= \sum_{k=0}^{n} f(n, k)\left(\frac{a^n - b^n - a^{n-k}b^k + a^k b^{n-k}}{a - b}\right) \\ &= \sum_{k=0}^{n} f(n, k)\left(u_n - (ab)^k\left(\frac{a^{n-2k} - b^{n-2k}}{a - b}\right)\right) \\ &= u_n\left(\sum_{k=0}^{n} f(n, k)\right) - \sum_{k=0}^{n} Q^k f(n, k) u_{n-2k} = u_n\left(\sum_{k=0}^{n} f(n, k)\right),\end{aligned}$$

by Lemma 1.

Corollary 1:

(a) $\sum_{k=0}^{n} u_k v_{n-k} = (n + 1)u_n$ (b) $\sum_{k=0}^{n} \binom{n}{k} u_k v_{n-k} = 2^n u_n$

(c) $\sum_{k=0}^{n} \binom{n}{k}^2 u_k v_{n-k} = \binom{2n}{n} u_n$ (d) $\sum_{k=0}^{n} k(n - k) u_k v_{n-k} = \left(\frac{n^3 - n}{6}\right) u_n$

(e) $\sum_{k=0}^{n} k^2 (n - k)^2 u_k v_{n-k} = \left(\frac{n^5 - n}{30}\right) u_n$

Proof: This follows from Theorem 5 and (10) through (13).

Theorem 6: Let u_n and v_n be the primary and secondary binary linear recurrences, respectively, with parameters P and Q, as defined in the introduction, and with discriminant $D = P^2 - 4Q$. Define

$$U_n = \sum_{k=0}^{n} \binom{n}{k} u_k, \quad V_n = \sum_{k=0}^{n} \binom{n}{k} v_k.$$

Then, U_n and V_n are also primary and secondary binary linear recurrences, respectively, with parameters $P^* = P + 2$, $Q^* = P + Q + 1$, and discriminant $D^* = D$.

Proof:
$$\begin{aligned}U_n &= \sum_{k=0}^{n} \binom{n}{k} u_k = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{a^k - b^k}{a - b}\right) = D^{-\frac{1}{2}}\left(\sum_{k=0}^{n} \binom{n}{k} a^k - \sum_{k=0}^{n} \binom{n}{k} b^k\right) \\ &= \frac{(a + 1)^n - (b + 1)^n}{a - b} = \frac{(a + 1)^n - (b + 1)^n}{(a + 1) - (b + 1)}.\end{aligned}$$

If we let $A = a + 1$, $B = b + 1$, then

$$U_n = \frac{A^n - B^n}{A - B},$$

a primary binary linear recurrence with parameters

$$P^* = A + B = (a + 1) + (b + 1) = (a + b) + 2 = P + 2,$$

and

$$Q^* = AB = (a + 1)(b + 1) = ab + (a + b) + 1 = P + Q + 1.$$

Similarly, if

$$V_n = \sum_{k=0}^{n} \binom{n}{k} v_k,$$

then $V_n = A^n + B^n$, a secondary binary linear recurrence with A and B as above. Furthermore,

$$D^* = (P^*)^2 - 4Q^* = (P + 2)^2 - 4(P + Q + 1)$$
$$= P^2 + 4P + 4 - 4P - 4Q - 4 = P^2 - 4Q = D.$$

Theorem 7: Let $\{u_n\}$ and $\{v_n\}$ be primary and secondary binary linear recurrences with discriminant $D > 0$. Then there exists a positive integer, m, such that

$$\sum_{k=0}^{n} \binom{n}{k} u_k = u_{mn}, \quad \sum_{k=0}^{n} \binom{n}{k} v_k = v_{mn}$$

if and only if $m = 2$, $u_n = F_n$, $v_n = L_n$.

Proof: To prove sufficiency, we note that, if $P = -Q = 1$, so that $u_n = F_n$, $v_n = L_n$, then $a^2 = a + 1 = A$, $b^2 = b + 1 = B$, so that Theorem 6 implies

$$\sum_{k=0}^{n} \binom{n}{k} u_k = \frac{A^n - B^n}{A - B} = \frac{a^{2n} - b^{2n}}{a - b} = u_{2n},$$

$$\sum_{k=0}^{n} \binom{n}{k} v_k = A^n + B^n = a^{2n} + b^{2n} = v_{2n}.$$

To prove necessity, using the notation of Theorem 6, we note that hypothesis, (4), and (5) imply

$$\frac{A^n - B^n}{a - b} = \frac{a^{mn} - b^{mn}}{a - b}, \quad A^n + B^n = a^{mn} + b^{mn}.$$

Therefore, $A = a^m$, $B = b^m$, so that $a^m = a + 1$, $b^m = b + 1$. Let

$$f_m(x) = x^m - x - 1.$$

Then $f_m(a) = f_m(b) = 0$. If m is odd, then $f_m(x)$ has critical values at

$$x = \pm m^{[-1/(m-1)]}.$$

It is easily verified that $f_m(\pm m^{[-1/(m-1)]}) < 0$. Therefore, $f_m(x)$ has a unique real root, so $a = b$, which implies $D = 0$, contrary to hypothesis. If m is even, then $f_m(x)$ has a minimum at $x = m^{[-1/(m-1)]}$, and $f_m(m^{[-1/(m-1)]}) < 0$, so $f_m(x)$ has two real roots a and b with $a > b$. Now,

$$f_m(-1) = 1, \ f_m(0) = f_m(1) = -1, \ f_m(2) = 2^m - 3 > 0, \text{ for } m \geq 2,$$

so we must have $-1 < b < 0$ and $1 < a < 2$. Therefore, $0 < a + b < 2$ and $-2 < ab < 0$. Since $a + b$ and ab must be integers, we have $P = a + b = 1$, $Q = ab = -1$. It now follows that $u_n = F_n$, $v_n = L_n$, $a^m = a + 1 = a^2$, so $m = 2$.

Concluding Remarks

If $P = -Q = 1$, then $D = 5$, $u_n = F_n$, and $v_n = L_n$ (the n^{th} Lucas number). In this case, Theorems 1(a), 1(b), 2(a), 2(b), say, respectively:

$$\text{(I)} \quad \sum_{k=0}^{n} F_k F_{n-k} = \frac{nL_n - F_n}{5}$$

$$\text{(II)} \quad \sum_{k=0}^{n}\binom{n}{k}F_k F_{n-k} = \frac{2^n L_n - 2}{5}$$

$$\text{(III)} \quad \sum_{k=0}^{n} L_k L_{n-k} = (n + 1)L_n + 2F_{n+1}$$

$$\text{(IV)} \quad \sum_{k=0}^{n}\binom{n}{k}L_k L_{n-k} = 2^n L_n + 2$$

(I) was obtained by Hoggatt & Bicknell-Johnson [8]; an alternate form of (I) was given by Knuth [10]; (I) and (II) appeared without proof in Wall [11]; (II) and (IV) were given by Buschman [3].

Theorem 7 also yields the identities

$$\text{(V)} \quad \sum_{k=0}^{n}\binom{n}{k}F_k = F_{2n}; \quad \sum_{k=0}^{n}\binom{n}{k}L_k = L_{2n}.$$

(V) appeared in papers by Gould [6] and by Carlitz & Ferns [5].

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A NOTE ON A CLASS OF LUCAS SEQUENCES*

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1. Introduction

In a short communication that appeared in this jounal [12], Whitford considered the generalized Fibonacci sequence $\{G_n\}$ defined as

(1.1) $$G_n = (\alpha_d^n - \beta_d^n)/\sqrt{d},$$

where d is a positive odd integer of the form $4k + 1$ and

(1.2) $$\begin{cases} \alpha_d = (1 + \sqrt{d})/2 \\ \beta_d = (1 - \sqrt{d})/2. \end{cases}$$

The sequence $\{G_n\}$ can also be defined by the second-order linear recurrence relation

(1.3) $$G_{n+2} = G_{n+1} + ((d - 1)/4)G_n; \quad G_0 = 0, \quad G_1 = 1.$$

Monzingo observed [7] that, on the basis of the previous definitions, the analogous Lucas sequence $\{H_n\}$ can be defined either as

(1.4) $$H_{n+2} = H_{n+1} + ((d - 1)/4)H_n; \quad H_0 = 2, \quad H_1 = 1$$

or, by means of the *Binet form*

(1.5) $$H_n = \alpha_d^n + \beta_d^n.$$

Our principal aim is to extend the results established in [7] by finding further properties of the numbers H_n which, throughout this note, will be referred to as *Monzingo numbers*.

2. On the Monzingo Numbers $H_n(m)$

Letting

(2.1) $$(d - 1)/4 = m \in \mathbb{N}$$

in (1.3) and (1.4), we have

(2.2) $$G_{n+2}(m) = G_{n+1}(m) + mG_n(m); \quad G_0(m) = 0, \quad G_1(m) = 1$$

and the Monzingo numbers

(2.3) $$H_{n+2}(m) = H_{n+1}(m) + mH_n(m); \quad H_0(m) = 2, \quad H_1(m) = 1,$$

respectively. Note that both $\{G_n(m)\}$ and $\{H_n(m)\}$ are particular cases of the more general sequence $\{W_n(a, b; p, q)\}$ which has been intensively studied over the past years (e.g., see [3], [4], [5], and [6]). More precisely, we have

(2.4) $$\{H_n(m)\} = \{W_n(2, 1; 1, -m)\}.$$

The first few values of $H_n(m)$ are given in (2.5).

*Work carried out in the framework of the agreement between the Italian PT Administration and the Fondazione Ugo Bordoni.

$$(2.5)\quad \begin{aligned} H_0(m) &= 2\\ H_1(m) &= 1\\ H_2(m) &= (d+1)/2 = 2m+1\\ H_3(m) &= (3d+1)/4 = 3m+1\\ H_4(m) &= (d^2+6d+1)/8 = 2m^2+4m+1\\ H_5(m) &= (5d^2+10d+1)/16 = 5m^2+5m+1. \end{aligned}$$

Using Binet's form (1.5), (1.2), and the binomial theorem, the following general expression for $H_n(m)$ in terms of powers of d can readily be found to be

$$(2.6)\quad H_n(m) = H_n\left(\frac{d-1}{4}\right) = \frac{1}{2^{n-1}} \sum_{j=0}^{[n/2]} \binom{n}{2j} d^j,$$

where $[\cdot]$ denotes the greatest integer function.

From (2.3), it must be noted that $H_n(1)$ and the n^{th} Lucas numbers L_n coincide. As a special case, letting $m = 1$ (i.e., $d = 5$) in (2.6), we obtain

$$(2.7)\quad L_n = \frac{1}{2^{n-1}} \sum_{j=0}^{[n/2]} \binom{n}{2j} 5^j.$$

Countless identities involving the numbers $H_n(m)$ and $G_n(m)$ can be found with the aid of (1.1) and (1.5). A few examples of the various types are listed below.

$$(2.8)\quad H_n(m)H_{n+k}(m) = H_{2n+k}(m) + (-m)^n H_k(m) \quad \text{(cf. [7, (3)])},$$

whence *Simson's formula* for $\{H_n(m)\}$ turns out to be

$$(2.9)\quad H_{n-1}(m)H_{n+1}(m) - H_n^2(m) = (-m)^{n-1}(4m+1).$$

$$(2.10)\quad H_{2n}(m) = H_n^2(m) - 2(-m)^n,$$

$$(2.11)\quad G_{2n}(m) = G_n(m)H_n(m),$$

$$(2.12)\quad H_{2n+1}(m) = \frac{(4m+1)G_{2n}(m) + H_{2n}(m)}{2},$$

$$(2.13)\quad G_{2n+1}(m) = \frac{G_{2n}(m) + H_{2n}(m)}{2},$$

$$(2.14)\quad \sum_{j=0}^{n} H_{aj+b}(m) = \frac{(-m)^a (H_{an+b}(m) - H_{b-a}(m)) - H_{a(n+1)+b}(m) + H_b(m)}{(-m)^a + 1 - H_a(m)}.$$

Observe that (2.14) may involve the use of the negative-subscripted Monzingo numbers

$$(2.15)\quad H_{-n}(m) = (-m)^{-n} H_n(m).$$

$$(2.16)\quad C^{(1)} = \sum_{j=1}^{n} H_j(m)H_{n-j+1}(m) = \frac{[2m(2n-1)+n]H_{n+1}(m) - 2m^2 H_{n-1}(m)}{4m+1},$$

($\{C_n^{(1)}\}$, the *Monzingo 1^{st} Convolution Sequence*)

$$(2.17)\quad \sum_{j=1}^{n} jH_j(m) = \frac{nH_{n+4}(m) - (n+1)H_{n+3}(m) + 3m + 1}{m^2},$$

$$(2.18)\quad \sum_{j=0}^{\infty} \frac{H_j(m)}{j!} = \exp\left(\frac{1+\sqrt{4m+1}}{2}\right) + \exp\left(\frac{1-\sqrt{4m+1}}{2}\right).$$

The usefulness of (2.10)-(2.13) will be explained later.

Some properties of the Monzingo numbers can also be found by using appropriate matrices. As a minor example, we invite the reader to prove that

(2.19) $H_n(m) = \operatorname{tr} M^n$,

where tr A denotes the trace (sum of diagonal entries) of a generic square matrix A and

(2.20) $M = \begin{bmatrix} 1 & m \\ 1 & 0 \end{bmatrix}$.

Letting

(2.21) $m = k(k + 1) \quad (k \in \mathbb{N})$

in (2.3) leads to a simple but rather interesting case. In fact, we have [cf. (2.1)]

(2.22) $d = 4k^2 + 4k + 1 = (2k + 1)^2$,

so that [cf. (1.2)]

(2.23) $\alpha_d = k + 1 \quad$ and $\quad \beta_d = -k$

are integral and

(2.24) $H_n(k^2 + k) = (k + 1)^n + (-k)^n$.

On the basis of (2.24), it can readily be seen that the numbers $H_n(k^2 + k)$ can be expressed by means of the following first-order linear recurrence relation

(2.25) $H_n(k^2 + k) = (k + 1)H_{n-1}(k^2 + k) + (2k + 1)k^{n-1}(-1)^n;$
$H_0(k^2 + k) = 2.$

This suggests an analogous expression for $H_n(m)$ (m arbitrary). In fact, using (1.2), (1.5), and (2.1), it can be proved that

(2.26) $H_n(m) = \alpha_d H_{n-1}(m) - \sqrt{4m + 1}\,\beta_d^{n-1}; \quad H_0(m) = 2,$

whence, as a special case, we have

(2.27) $L_n = \alpha L_{n-1} - 5\beta^{n-1}; \quad L_0 = 2,$

where $\alpha = \alpha_5$ and $\beta = \beta_5$.

Now, let us consider a well-known (e.g., see [6], Cor. 7) divisibility property of the numbers $W_n(2, b; b, q)$ which, obviously, applies to the Monzingo numbers. Namely, we can write

(2.28) $H_r(m) \,|\, H_{r(2s+1)}(m)$

whence it follows that

Proposition 1: If $H_n(m)$ is a prime, then n is either a prime or a power of 2.

Proposition 1 and (2.24) give an alternative proof of a particular case (a and b, consecutive integers) of well-known number-theoretic statements concerning the divisors of $a^n \pm b^n$ (e.g., see [10], pp. 184ff.). More precisely, we can state

Proposition 2 (n odd): If $(k + 1)^n - k^n$ is a prime, then n is a prime.

Proposition 3 (n even): If $(k + 1)^n + k^n$ is a prime, then $n = 2^h$ $(h \in \mathbb{N})$.

It must be noted that, for $k = 1$, Proposition 2 is the well-known Mersenne's theorem, while Proposition 3 is related to a property concerning Fermat's numbers (e.g., see [10], p. 107). We point out that, from the said statements concerning the factors of $a^n \pm b^n$, it follows that, if p is an odd prime and

$H_p(k^2 + k)$ is composite, then its prime factors are of the form $2lp + 1$. For example, we can readily check that, for $k = 2$ and $p = 11$, we have

$$H_{11}(6) = 175099 = (2 \cdot 1 \cdot 11 + 1)^2(2 \cdot 15 \cdot 11 + 1).$$

Finally, let us consider the sum

$$(2.29) \quad S_{n,h} = \sum_{m=0}^{h} H_n(m)$$

and ask ourselves whether it is possible to find a closed form expression for (2.29) in terms of powers of h. A modest attempt in this direction is shown below. Taking into account that $H_n(0) = 1 \ \forall \ n > 0$, expressions valid for the first few values of n can easily be derived from (2.5) and from the calculation of $H_6(m) = 2m^3 + 9m^2 + 6m + 1$:

$$(2.30) \quad \begin{array}{ll} S_{1,h} = h + 1 & S_{4,h} = (2h^3 + 9h^2 + 10h + 3)/3 \\ S_{2,h} = h^2 + 2h + 1 & S_{5,h} = (5h^3 + 15h^2 + 13h + 3)/3 \\ S_{3,h} = (3h^2 + 5h + 2)/2 & S_{6,h} = (h^4 + 8h^3 + 16h^2 + 11h + 2)/2. \end{array}$$

3. Some Congruence and Divisibility Properties of the Monzingo Numbers

If we rewrite (2.6) as

$$(3.1) \quad 2^{n-1}H_n(m) = 1 + \sum_{j=1}^{[n/2]} \binom{n}{2j} d^j,$$

it is easily seen that

$$(3.2) \quad 2^{n-1}H_n(m) \equiv 1 \pmod{d}.$$

From (2.24), Proposition 1, and the definition of *perfect numbers* (e.g., see [9], p. 81), it follows that all even perfect numbers are given by $2^{p-1}H_p(2)$, where $H_p(2)$ is prime ($p \geq 3$, a prime). Since $m = 2$ implies $d = 9$, from (3.2) we can state

Proposition 4: Any even perfect number greater than 6 is congruent to 1 modulo 9.

By using either [1, (2)] or [2, (1.2)] and taking into account that [cf. (1.2)]

$$(3.3) \quad \begin{cases} \alpha_d + \beta_d = 1 \\ \alpha_d \beta_d = (1 - d)/4 = -m, \end{cases}$$

we obtain the following expression for $H_n(m)$ in terms of powers of m [cf. (2.5)]

$$(3.4) \quad H_n(m) = \sum_{j=0}^{[n/2]} nC_{n,j} m^j \quad (n \geq 1),$$

where

$$(3.5) \quad C_{n,j} = \frac{1}{n-j}\binom{n-j}{j}.$$

Rewrite (3.4) as

$$(3.6) \quad H_n(m) = 1 + n \sum_{j=1}^{[n/2]} C_{n,j} m^j \quad (n \geq 1)$$

and observe that, if n is a prime, then $C_{n,j}$ is integral. It follows that

$$(3.7) \quad H_n(m) \equiv 1 \pmod{n} \text{ if } n \text{ is prime.}$$

Note that (3.6) allows us to state that

(3.8) (i) $H_n(m) \equiv 1 \pmod{m} \quad (n \geq 1)$

(3.9) (ii) $H_n(2k)$ is odd $(n \geq 1)$,

(3.10) (iii) $H_n(2k+1) \equiv 1 + \sum_{j=1}^{[n/2]} nC_{n,j} = L_n \pmod{2}$,

(3.10′) that is to say, $H\,(2k+1)$ is even iff $n \equiv 0 \pmod{3}$.

Curiosity led us to investigate the divisibility of $H_n(m)$ by some primes $p > 2$. A computer experiment was carried out to determine the necessary and sufficient conditions on n for an odd prime $p \leq 47$ to be a divisor of $H_n(m)$ $(2 \leq m \leq 10)$. The case $m = 1$ has been disregarded, since the conditions on n for the congruence $L_n \equiv 0 \pmod{p}$ $(p \leq 47)$ to hold are well known. For p and m varying within the above said intervals, the results can be summarized as follows

(3.11) $H_n(m) \equiv 0 \pmod{p}$ iff $n \equiv r \pmod{2r}$.

The values of r are displayed in Table 1, where a blank value denotes that p is not a divisor of the Monzingo sequence $\{H_n(m)\}$.

TABLE 1. Values of r for $3 \leq p \leq 47$ and $2 \leq m \leq 10$

p \ m	2	3	4	5	6	7	8	9	10
3	-	-	2	-	-	2	-	-	2
5	2	3	-	-	-	2	3	-	-
7	3	2	4	-	-	-	4	3	2
11	-	6	6	2	-	3	-	5	-
13	6	-	3	7	2	6	7	-	-
17	4	8	-	8	8	9	2	-	9
19	-	-	9	10	3	10	5	2	5
23	11	11	12	6	11	-	4	12	-
29	14	14	-	-	7	-	14	-	5
31	5	4	16	16	-	16	15	16	3
37	18	-	-	18	18	-	18	-	-
41	10	21	-	-	20	21	10	5	-
43	-	-	21	-	21	-	-	22	7
47	23	8	23	-	23	8	24	23	6

Let us give an example of use of Table 1 by considering the case $m = 6$ and $p = 29$. For these two values, the table gives $r = 7$. It means that $H_n(6) \equiv 0 \pmod{29}$ iff $n \equiv 7 \pmod{14}$.

Of course, the above-mentioned experiment led us to discover also the repetition period $P_{m,p}$ of the Monzingo sequences reduced modulo p. Some values of $P_{m,p}$ are shown in Table 2.

TABLE 2. Values of $P_{m,p}$ for $3 \leq p \leq 47$ and $2 \leq m \leq 10$

p \ m	2	3	4	5	6	7	8	9	10
3	2	1	8	2	1	8	2	1	8
5	4	24	6	1	4	4	24	6	1
7	6	24	48	3	6	1	16	6	24
11	10	120	120	40	5	60	10	10	6
13	12	12	12	56	12	12	56	84	42
17	8	16	8	16	16	288	16	144	288
19	18	90	18	360	18	120	60	72	180
23	22	22	528	264	22	11	176	528	11
29	28	28	35	105	28	28	28	210	280
31	10	240	320	192	30	960	30	960	30
37	36	171	171	36	36	684	36	36	36
41	20	336	105	40	40	1680	20	40	20
43	14	42	42	42	42	33	77	1848	42
47	46	736	46	23	46	736	2208	46	552

3.1 The Numbers $H_n^{(1)}(m)$: A Divisibility Property

Both the definitions and most of the properties of the numbers $H_n(m)$ and $G_n(m)$ remain valid if m is an arbitrary (not necessarily integral) quantity. Let us define the numbers $H_n^{(1)}(m)$ as the first derivative of $H_n(m)$ with respect to m

$$(3.12)\quad H_n^{(1)}(m) = \frac{d}{dm} H_n(m).$$

From (3.12) and (3.4), we have

$$(3.13)\quad H^{(1)}(m) = \sum_{j=0}^{[n/2]} j \frac{n}{n-j}\binom{n-j}{j} m^{j-1} = \sum_{j=1}^{[n/2]} n \frac{(n-j-1)!}{(j-1)!(n-2j)!} m^{j-1}$$
$$= n \sum_{j=1}^{[n/2]} \binom{n-j-1}{j-1} m^{j-1} \quad (n \geq 1).$$

Now it is plain that $H_n^{(1)}(m) \equiv 0 \pmod{n}$. Moreover (cf. [6], p. 278), (3.13) leads to the following cute result

$$(3.14)\quad \frac{H_n^{(1)}(m)}{n} = G_{n-1}(m) \quad (n \geq 1).$$

4. The Monzingo Pseudoprimes

Of course, the converse of (3.7) is not always true. Let us define the odd composites satisfying (3.7) as *Monzingo Pseudoprimes of the* m^{th} *kind* and abbreviate them *m-M.Psps*. Incidentally, we note that the 1-M.Psps. and the Fibonacci pseudoprimes defined in [8] and investigated in [2] coincide.

For $m > 1$, the m-M.Psps. are not as rare as the Fibonacci pseudoprimes. Let $\mu_m(x)$ be the m-M.Psp.-counting function (i.e., the number of m-M.Psps. not exceeding x) and let $M_1(m)$ be the smallest among them. A computer experiment has been carried out to obtain $\mu_m(1000)$ and $M_1(m)$ for $1 \leq m \leq 25$. These quantities are shown against m in Tables 3 and 4, respectively.

TABLE 3. Values of $\mu_m(1000)$ for $1 \leq m \leq 25$

m	$\mu_m(1000)$	m	$\mu_m(1000)$
1	1	14	11
2	3	15	22
3	6	16	2
4	5	17	5
5	8	18	8
6	15	19	13
7	9	20	17
8	3	21	29
9	15	22	9
10	14	23	4
11	7	24	10
12	15	25	9
13	12		

TABLE 4. Values of $M_1(m)$ for $1 \leq m \leq 25$

m	$M_1(m)$	m	$M_1(m)$
1	705	14	21
2	341	15	9
3	9	16	85
4	25	17	51
5	15	18	9
6	9	19	25
7	49	20	15
8	231	21	9
9	9	22	33
10	25	23	69
11	33	24	9
12	9	25	25
13	49		

The reader who would enjoy discovering many more m-M.Psps. can use the simple computer algorithm described on pages 239-40 of [2], after replacing the identities (3.5)-(3-8) in [2] by the identities (2.10)-(2.13) shown in Section 2 above.

It can be proved that certain odd composites are m-M.Psps. In this note, we restrict ourselves to demonstrating that, for p an odd prime and s an integer greater than 1, p^s is a p-M.Psp.

Theorem 1: $H_{p^s}(p) \equiv 1 \pmod{p^s}$.

Proof: By observing (3.6), it is plain that it suffices to prove that $C_{p^s,j}\,p^j$ is integral for $1 \le j \le (p^s - 1)/2$. More precisely [cf. (3.5)], if $(p, j) = 1$, then $C_{p^s,j}$ is an integer; thus, it suffices to prove that the power a with which p enters into $j!$ is less than j. This is true for any j and p (odd). In fact, it is known (e.g., see [11], p. 21) that

$$a = \sum_{i=1}^{\infty} [j/p^i], \tag{4.1}$$

whence we can write

$$a < \sum_{i=1}^{\infty} j/p^i = j/(p - 1) < j. \quad \text{Q.E.D.}$$

Let us conclude this note by pointing out that the numerical evidence turning out from the above said computer experiment suggests the following

Conjecture 1: If $p \ge 5$ is a prime and s is an integer greater than 1, then p^s is a $(p - 1)$-M.Psp., that is

$$H_{p^s}(p - 1) \equiv 1 \pmod{p^s}. \tag{4.2}$$

For some values of p, we checked Conjecture 1 by ascertaining that, while the addends $C_{p^s,j}\,(p - 1)^j$ are in general *not* integral, the sum

$$\sum_{j=1}^{(p^s-1)/2} C_{p^s,j}(p - 1)^j \tag{4.3}$$

is. For example, let us consider the case $p = 7$, $s = 2$ and show that (4.3) is integral. The nonintegral addends in (4.3) are those for which g.c.d.$(p^s - j, j) \ne 1$, that is

$$A_1 = \frac{1}{42}\binom{42}{7}6^7, \quad A_2 = \frac{1}{35}\binom{35}{14}6^{14}, \quad A_3 = \frac{1}{28}\binom{28}{21}6^{21}. \tag{4.4}$$

Let us write

$$A_1 + A_2 + A_3 = \frac{41 \cdot 39 \cdot 38 \cdot 37 \cdot 2}{7}6^7 + \frac{34 \cdot 31 \cdot 29 \cdot 23 \cdot 11 \cdot 5 \cdot 4 \cdot 3}{7}6^{14} + \frac{26 \cdot 23 \cdot 11 \cdot 9 \cdot 5}{7}6^{21} \tag{4.5}$$

and reduce the sum of the numerators on the right-hand side of (4.5) modulo 7

$$6 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 6 + 6 \cdot 3 \cdot 1 \cdot 2 \cdot 4 \cdot 5 \cdot 4 \cdot 3 \cdot 1 + 5 \cdot 2 \cdot 4 \cdot 2 \cdot 5 \cdot 6$$
$$\equiv 6 + 2 + 6 \equiv 0 \pmod 7.$$

It follows that $A_1 + A_2 + A_3$ is integral, so that 49 is a 6-M.Psp.

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MULTIPLICATIVE PARTITIONS OF BIPARTITE NUMBERS

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1. Introduction

For a positive integer n, let $f(n)$ be the number of multiplicative partitions of n. That is, $f(n)$ represents the number of different factorizations of n, where two factorizations are considered the same if they differ only in the order of the factors. For example, $f(12) = 4$, since $12 = 6 \cdot 2 = 4 \cdot 3 = 3 \cdot 2 \cdot 2$ are the four multiplicative partitions of 12. Hughes & Shallit [2] showed that $f(n) \le 2n^{\sqrt{2}}$ for all n. Mattics & Dodd [3] improved this to $f(n) \le n$, and in [4] they further improved this to $f(n) \le n/\log(n)$ for $n \ne 144$. In this paper, we generalize the notion of multiplicative partitions to bipartite numbers and obtain a corresponding bound.

By a j-partite number, we mean an ordered j-tuple $(n_1, \ldots, n_j)$, where all n_i are positive integers. Bipartite refers to the case $j = 2$. We can extend the idea of multiplicative partitions to bipartite numbers as follows. For positive integers m and n, define $f_2(m, n)$ to be the number of different ways to write (m, n) as a product $(a_1, b_1)(a_2, b_2)\ldots(a_k, b_k)$, where the multiplication is done coordinate-wise, all a_i and b_i are positive integers, $(1, 1)$ is not used as a factor of $(m, n) \ne (1, 1)$, and two such factorizations are considered the same if they differ only in the order of the factors. Hence, $(2, 1)(2, 1)(1, 4)$ and $(1, 4)(2, 1)(2, 1)$ are considered the same factorizations of $(4, 4)$, while $(2, 1)(2, 1)(1, 4)$ and $(1, 2)(1, 2)(4, 1)$ are considered different. Thus, for example, $f_2(6, 2) = 5$, since the five multiplicative partitions $(6, 2)$ are

$$(6, 2) = (6, 1)(1, 2) = (3, 2)(2, 1) = (3, 1)(2, 2)$$
$$= (3, 1)(2, 1)(1, 2).$$

It is clear that $f(n) = f_2(n, 1)$ for all n. In Section 2, we give an upper bound for $f_2(m, n)$. The definition of $f_2(m, n)$ may be extended to $f_j(n_1, \ldots, n_j)$ in an obvious way.

Throughout this paper, unless otherwise stated, $p_1 = 2$, $p_2 = 3$, ... will represent the sequence of primes.

2. An Upper Bound for $f_2(m, n)$

When first considering the function $f_2(m, n)$, some conjectures immediately came to mind:

(1) $f_2(m, n) = f(m)f(n)$ (2) $f_2(m, n) \le f(m)f(n)$
(3) $f_2(m, n) = f(mn)$ (4) $f_2(m, n) \le f(mn)$
(5) $f_2(m, n) \le mn/\log(mn)$ (6) $f_2(m, n) \le mn$.

Surprisingly, none of these is true. The values $f(2) = 1$, $f(6) = 2$, $f(12) = 4$, and $f_2(6, 2) = 5$ provide counterexamples to (1)-(5). As it turns out, (6) is also false (see Section 3).

In the next theorem, we establish an upper bound on $f_2(m, n)$. We will first need the following three lemmas.

Lemma 1: Let $\{p_1, \ldots, p_j\}$, $\{q_1, \ldots, q_k\}$, and $\{r_1, \ldots, r_{j+k}\}$ each be a set of distinct primes, and let

$$x = p_1^{a_1} \ldots p_j^{a_j}, \quad y = q_1^{b_1} \ldots q_k^{b_k}, \quad z = r_1^{a_1} \ldots r_j^{a_j} r_{j+1}^{b_1} \ldots r_{j+k}^{b_k},$$

where all a_i and b_i are positive integers. Then, $f(z) = f_2(x, y)$.

Proof: With each factorization

$$z = [r_1^{c_{11}} r_2^{c_{12}} \ldots r_{j+k}^{c_{1,j+k}}][r_1^{c_{21}} r_2^{c_{22}} \ldots r_{j+k}^{c_{2,j+k}}] \ldots [r_1^{c_{t1}} \ldots r_{j+k}^{c_{t,j+k}}]$$

we associate the following factorization of (x, y):

$$[p_1^{c_{11}} \ldots p_j^{c_{1j}},\ q_1^{c_{1,j+1}} \ldots q_k^{c_{1,j+k}}] \ldots [p_1^{c_{t1}} \ldots p_j^{c_{tj}},\ q_1^{c_{t,j+1}} \ldots q_k^{c_{t,j+k}}].$$

This association is obviously a one-to-one correspondence.

Lemma 1 can easily be extended to j-partite numbers. Thus, for example, $f_2(12, 4) = f(180) = f_3(4, 4, 2) = f_2(36, 2)$.

It is well known that

(a) $p_n > n \log n$ for $n \geq 1$, and
(b) $p_n < n(\log n + \log\log n)$ for $n \geq 6$ (see [5]).

As a consequence, we have the following lemma.

Lemma 2: For $n \geq 4$, $p_{2n-1}p_{2n} < p^{2.97}$.

Proof: Direct computation shows the inequality holds for $n = 4$, 5, and 6. Note that, for $n \geq 7$,

$$(2n-1)(\log(2n-1) + \log\log(2n-1))2n(\log 2n + \log\log 2n) < (n \log n)^{2.97}.$$

Thus, from (a) and (b) above, $p_{2n-1}p_{2n} < (n \log n)^{2.97} < p^{2.97}$.

Lemma 3: Let $c_1 \geq c_2 \geq \cdots \geq c_k > 0$. Then

$$\prod_{i=1}^{k} [p_{2i-1}p_{2i}]^{c_i} < \prod_{i=1}^{k} p_i^{3.032c_i}.$$

Proof: If $k = 1$, the inequality holds, since $p_1p_2 < p_1^{2.585}$. For $k = 2$, since $p_3p_4 < p_2^{3.237}$, we have

$$[p_1p_2]^{c_1}\,[p_3p_4]^{c_2} < p_1^{2.585c_1} p_2^{3.237c_2}\,[p_1^{.4c_1}/p_2^{.252c_2}] = [p_1^{c_1}p_2^{c_2}]^{2.985}.$$

If $k = 3$,

$$(1) \qquad \prod_{i=1}^{3} [p_{2i-1}p_{2i}]^{c_i} < [p_1^{c_1}p_2^{c_2}]^{2.985} p_3^{3.084c_3}\,[p_1^{c_1}p_2^{c_2}]^{.047}/p_3^{.052c_3}$$

$$= \prod_{i=1}^{3} p_i^{3.032c_i}.$$

If $k \geq 4$, the inequality follows easily from (1) and Lemma 2.

Theorem 1: Let m and n be positive integers with $(m, n) \neq (1, 1)$. Then

$$f_2(m, n) < (mn)^{1.516}/\log(mn).$$

Proof: We can assume that $m = p_1^{a_1} \ldots p_k^{a_k}$ and $n = p_1^{b_1} \ldots p_r^{b_r}$, where $k \geq r$ and $a_i \geq a_{i+1}$, $b_i \geq b_{i+1}$ for each i. Then, by Lemma 1,

$$f_2(m, n) = f(p_1^{a_1}p_2^{b_1}p_3^{a_2}p_4^{b_2} \ldots p_{2k-1}^{a_k}p_{2k}^{b_k}),$$

where $b_i = 0$ if $i > r$. For $i = 1, \ldots, k$, let

$$\alpha_i = \max\{a_i, b_i\},\ \beta_i = \min\{a_i, b_i\},\ c_i = (a_i + b_i)/2.$$

We first consider the case in which

$$\prod_{i=1}^{k} p_{2i-1}^{\alpha_i} p_{2i}^{\beta_i} \neq 144.$$

Then, by Lemmas 1 and 3 and the known bound for $f(n)$,

$$f_2(m, n) = f[p_1^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_4^{\beta_2} \ldots p_{2k-1}^{\alpha_k} p_{2k}^{\beta_k}] \leq \frac{\prod_{i=1}^{k} [p_{2i-1}^{\alpha_i} p_{2i}^{\beta_i}]}{\log\left[\prod_{i=1}^{k} [p_{2i-1}^{\alpha_i} p_{2i}^{\beta_i}]\right]}$$

$$\leq \frac{\prod_{i=1}^{k} [p_{2i-1}^{\alpha_i} p_{2i}^{\beta_i}]}{\log(mn)} \leq \frac{\prod_{i=1}^{k} (p_{2i-1} p_{2i})^{c_i}}{\log(mn)} \leq \frac{\prod_{i=1}^{k} p_i^{3.032c_i}}{\log(mn)}$$

$$= \frac{\prod_{i=1}^{k} (p_i^{a_i + b_i})^{1.516}}{\log(mn)} = \frac{(mn)^{1.516}}{\log(mn)}.$$

In case $\prod_{i=1}^{k} p_{2i-1}^{\alpha_i} p_{2i}^{\beta_i} = 144$, it then follows by Lemma 1 that $mn \geq 2^6$. Noting that $f(144) = 29$, we see that the theorem is true in this case as well.

3. Remarks and Computations

3.1. Using the algorithm from [1], the values of $f_2(m, n)$ were found for all m and n such that $mn \leq 2{,}000{,}000$ and for other selected values of m and n with mn as large as 167,961,600 by calculating the corresponding values of f as described in Lemma 1. Since large values of m and n tended to give the greatest values for the ratio $f_2(m, n)/mn$, and since these are the values that require the greatest computing time, we used the observations made in Remark 3.2 below to determine which pairs (m, n) to study.

3.2. Using the notation in the proof of Theorem 1, the pairs (m, n) can be described by the ordered $2k$-tuple $a_1b_1 \ldots a_kb_k$. In Table I below, we use this notation to list those $2k$-tuples we have found for which there exist ordered pairs (m, n) having ratios $r(a_1b_1 \ldots a_kb_k) = f_2(m, n)/mn > 1.5$ [given the $2k$-tuple, m and n are chosen so as to maximize $f_2(m, n)$].

TABLE I. Forms Yielding Large Ratios $f_2(m, n)/mn$

$a_1b_1 \ldots a_kb_k$	$f_2(m, n)$	$f_2(m, n)/(m, n)$
663311	162,075,802	2.17115
772211	61,926,494	1.86652
762211	30,449,294	1.83553
662211	15,173,348	1.82935
872211	119,957,268	1.80781
553311	33,439,034	1.79179
862211	58,256,195	1.75589
652211	7,126,811	1.71846
752211	14,096,512	1.69952
553211	10,511,373	1.68971
552211	3,400,292	1.63980
643311	30,428,542	1.63047
962211	107,097,889	1.61401
852211	26,610,876	1.60415
643211	9,584,844	1.54077
554411	255,339,989	1.52023
543311	14,162,812	1.51779

The prevalence of the forms $aabb11$ in the table is noteworthy. Although the forms $(a + 1)(a - 1)bb11$ also appear, the ratio is higher for $aabb11$. Similarly, the forms $(a + 1)abb11$ have higher ratios than $(a + 2)(a - 1)bb11$. We suspect that sequences of the form $aabbcc11$ also have large ratios, but the lengthy computation time made this infeasible to verify. A result which helps explain the prevalence of trailing 1's in the sequence $a_1b_1 \ldots a_kb_k$ is as follows: Let

$$j = \begin{cases} 2k & \text{if } b_k \neq 0 \\ 2k - 1 & \text{if } b_k = 0 \end{cases}$$

and let $c_1 \ldots c_j$ denote $a_1b_1 \ldots a_kb_k$. Then, if $1 \le i \le 2k$,

$$[6p_{[(i+2)/2]}/5p_{[(j+1)/2]}] \quad r(c_1 \ldots c_i \ldots c_j)$$
$$\le \quad r(c_1 \ldots c_{i-1}c_{i+1} \ldots c_j1) \text{ when } c_i \ge 2,$$

where [ ] denotes the greatest integer function. This result follows easily from the lemma on page 22 of [1].

<u>3.3</u>. For the more general function $f_j(n_1, \ldots, n_j)$, note that

$$f_j(q_1, \ldots, q_j) = f(p_1, \ldots, p_j) = B(j),$$

where $B(j)$ is the j^{th} Bell number and the q_i are any primes. ($B(j)$ grows very fast. See, e.g., [6].)

<u>3.4</u>. If we set

$$f_2(m, n) = (mn)^{\alpha}/\log(mn),$$

then, for all m and n for which $f_2(m, n)$ was calculated, $\alpha < 1.251$. The largest value of α occurred when $m = n = 24$ with $f_2(24, 24) = 444$. (This was the only case in which $\alpha > 5/4$.) Based on these data, we propose the following

Conjecture: $f_2(m, n) < (mn)^{1.251}/\log(mn)$ for all m and n.

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A COMBINATORIAL INTERPRETATION OF THE SQUARE OF A LUCAS NUMBER

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1. Introduction

The Fibonacci numbers have a well-known combinatorial interpretation in terms of the total number of subsets of $\{1, 2, 3, \ldots, n\}$ not containing a pair of consecutive integers. Recently, Konvalina & Liu [4] showed that the squares of the Fibonacci numbers have a combinatorial interpretation in terms of the total number of subsets of $\{1, 2, 3, \ldots, 2n\}$ without unit separation. Two integers are called *uniseparate* if they contain exactly one integer between them. For example, the following pairs of integers are uniseparate: (1, 3), (2, 4), (3, 5), (4, 6), etc.

In this paper, we will show that the squares of the Lucas numbers also have a combinatorial interpretation in terms of subsets of $\{1, 2, \ldots, 2n\}$ without unit separation if the integers $\{1, 2, 3, \ldots, 2n\}$ are arranged in a circle instead of a line.

Let F_n denote the n^{th} Fibonacci number determined by the recurrence relation:

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n \quad (n \geq 1).$$

Kaplansky [2] showed that the numbers of k-subsets of $\{1, 2, 3, \ldots, n\}$ not containing a pair of consecutive integers is

$$\binom{n+1-k}{k}.$$

Summing over all k-subsets, we obtain the well-known identity

$$(1) \qquad \sum_{k \geq 0} \binom{n+1-k}{k} = F_{n+2}.$$

Let $f(n, k)$ denote the number of k-subsets of $\{1, 2, 3, \ldots, n\}$ without unit separation. Konvalina [3] proved

$$(2) \qquad f(n, k) = \begin{cases} \displaystyle\sum_{i=0}^{[k/2]} \binom{n+1-k-2i}{k-2i} & \text{if } n \geq 2(k-1), \\ 0 & \text{if } n < 2(k-1). \end{cases}$$

Summing over all k-subsets, Konvalina & Liu [4] showed

$$(3) \qquad \sum_{k \geq 0} f(n, k) = \begin{cases} F_{m+2}^2 & \text{if } n = 2m, \\ F_{m+2}F_{m+3} & \text{if } n = 2m+1 \end{cases}$$

Next, let L_n denote the n^{th} Lucas number determined by the recurrence relation:

$$L_1 = 1,\ L_2 = 3,\ L_{n+2} = L_{n+1} + L_n \quad (n \geq 1).$$

The following identity expressing a Lucas number in terms of the sum of two Fibonacci numbers is well known (see Hoggatt [1]):

$$(4) \qquad L_n = F_{n+1} + F_{n-1}.$$

The Lucas numbers have a combinatorial interpretation in terms of the total number of subsets of $\{1, 2, 3, \ldots, n\}$ arranged in a circle and not containing a pair of consecutive integers (n and 1 are consecutive). One way to prove this is as follows: Kaplansky [2] showed that the number of k-subsets of $\{1, 2, 3, \ldots, n\}$ arranged in a circle and not containing a pair of consecutive integers is

$$(5) \qquad \frac{n}{k}\binom{n-k-1}{k-1} = \binom{n-k}{k} + \binom{n-k-1}{k-1}.$$

Summing over all k-subsets and applying (1) and (4), we obtain:

$$\sum_{k\ge 0}\frac{n}{k}\binom{n-k-1}{k-1} = \sum_{k\ge 0}\binom{n-k}{k} + \sum_{k\ge 0}\binom{n-k-1}{k-1} = F_{n+1} + F_{n-1} = L_n.$$

2. The Main Result

Let $g(n, k)$ denote the number of k-subsets of $\{1, 2, 3, \ldots, n\}$ arranged in a circle and without unit separation. Konvalina [3] proved the following identity:

$$(6) \qquad g(n, k) = f(n-2, k) + 2f(n-5, k-1) + 3f(n-6, k-2).$$

Let C_n denote the total number of subsets of $\{1, 2, 3, \ldots, n\}$ arranged in a circle and without unit separation. The following result relates the square of a Lucas number and C_{2m}.

Theorem: If $n > 2$, then

$$C_n = \begin{cases} L_m^2 & \text{if } n = 2m, \\ L_n & \text{if } n = 2m+1. \end{cases}$$

Proof: Summing over all k-subsets and applying (6), we have

$$(7) \qquad C_n = \sum_{k\ge 0} g(n, k) = \sum_{k\ge 0} f(n-2, k) + 2f(n-5, k-1) + 3f(n-6, k-2).$$

Even Case: $n = 2m$

Applying identity (3) to (7), we obtain

$$\begin{aligned} C_n &= \sum_{k\ge 0} f(n-2, k) + 2\sum_{k\ge 0} f(n-5, k-1) + 3\sum_{k\ge 0} f(n-6, k-2) \\ &= F_{m+1}^2 + 2F_{m-1}F_m + 3F_{m-1}^2 \\ &= F_{m+1}^2 + 2F_{m-1}(F_m + F_{m-1}) + F_{m-1}^2 \\ &= (F_{m+1} + F_{m-1})^2 = L_m^2. \end{aligned}$$

Odd Case: $n = 2m + 1$

Applying identity (3) to (7), we have

$$\begin{aligned} C_n &= F_{m+1}F_{m+2} + 2F_m^2 + 3F_{m-1}F_m \\ &= F_{m+1}F_{m+2} + 2F_m(F_m + F_{m-1}) + F_{m-1}F_m \\ &= F_{m+1}F_{m+2} + 2F_mF_{m+1} + F_{m-1}F_m \\ &= (F_{m+1}F_{m+2} + F_mF_{m+1}) + (F_mF_{m+1} + F_{m-1}F_m) \\ &= F_{2m+2} + F_{2m} = L_{2m+1} = L_n. \end{aligned}$$

Note: We have applied the following known identity (see [1], p. 59, identity I_{26} with $m = n - 1$): $F_{2n} = F_nF_{n+1} + F_{n-1}F_n$.

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RECIPROCAL GCD MATRICES AND LCM MATRICES

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1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be an ordered set of distinct positive integers. The $n \times n$ matrix $[S] = (s_{ij})$, where $s_{ij} = (x_i, x_j)$, the greatest common divisor of x_i and x_j, is called the greatest common divisor (GCD) matrix on S. The study of GCD matrices was initiated in [1]. In that paper, the authors obtained a structure theorem for GCD matrices and showed that each is positive definite, and hence nonsingular. A corollary of these results yielded a proof that, if S is factor-closed, then the determinant of S, $\det[S]$, is equal to $\phi(x_1)\phi(x_2) \ldots \phi(x_n)$, where $\phi(x)$ is Euler's totient. The set S is said to be factor-closed (FC) if all positive factors of any member of S belong to S.

In [4], Z. Li used the structure in [1] to compute a formula for the determinant of an arbitrary GCD matrix.

In this paper, we define a natural analog of the GCD matrix on S. Let $[[S]] = (t_{ij})$ be the $n \times n$ matrix with $t_{ij} = [x_i, x_j]$, the least common multiple of x_i and x_j. We shall obtain a structure theorem for $[[S]]$ and show that it is nonsingular, but never positive definite. As it turns out, the matrix factorization of $[[S]]$ emerges from the structure of the related *reciprocal GCD matrix* $1/[S]$, the i, j-entry of which is $1/(x_i, x_j)$. Reciprocal GCD matrices are addressed in the next section.

2. Reciprocal GCD Matrices

Definition 1: Let $S = \{x_1, x_2, \ldots, x_n\}$ be an ordered set of distinct positive integers. The matrix $1/[S]$ is the $n \times n$ matrix whose i, j-entry is $1/(x_i, x_j)$. We call $1/[S]$ the reciprocal GCD matrix on S.

Clearly reciprocal GCD matrices are symmetric. Furthermore, rearrangements of the elements of S yield similar matrices. Hence, as in [1] and [2], we may always assume $x_1 < x_2 < \ldots < x_n$.

We shall show that each reciprocal GCD matrix can be written as a product of A and A^T, the transpose of A, for some matrix A with complex number entries.

In what follows, we let $\mu(n)$ denote the Moebius function

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1 p_2 \ldots p_r, \text{ distinct prime factors} \\ 0 & \text{otherwise.} \end{cases}$$

The lower-case letter "p" will always denote a positive prime.

Definition 2: If n is a positive integer, we denote by $g(n)$ the sum

$$g(n) = \frac{1}{n} \cdot \sum_{e|n} e \cdot \mu(e).$$

We observe that $g(n) = f(n)h(n)$, where $f(n) = 1/n$ and $h(n) = \sum_{e|n} e \cdot \mu(e)$. Since f and h are multiplicative functions, g is multiplicative. Furthermore, if p is a prime, $h(p^m) = 1 - p$. Hence, $g(p^m) = (1 - p)/p^m$. It follows that

$$g(n) = \frac{1}{n} \prod_{p|n} (1 - p) = \frac{\phi(n)}{n^2} \prod_{p|n} (-p).$$

Moreover, by the Moebius Inversion Formula (see, e.g., [5]), it is true that

$$f(n) = 1/n = \sum_{e|n} g(e).$$

These results are summarized in the following lemma.

Lemma 1: Let n be a positive integer. Then $g(n) = 1$ if $n = 1$, and

$$g(n) = \frac{1}{n} \prod_{p|n} (1 - p) \text{ if } n > 1.$$

Moreover,

$$1/n = \sum_{e|n} g(e). \ \square$$

It is clear that any set of positive integers is contained in an (minimal) FC set. We obtain the following structure theorem for reciprocal GCD matrices.

Theorem 1: Let $S = \{x_1, x_2, \ldots, x_n\}$ be ordered by $x_1 < x_2 < \ldots < x_n$. Then the reciprocal GCD matrix $1/[S]$ is the product of an $n \times m$ complex matrix A and the $m \times n$ matrix A^T, where the nonzero entries of A are of the form $\sqrt{g(d)}$ for some d in an FC set that contains S.

Proof: Suppose $F = \{d_1, d_2, \ldots, d_m\}$ is an FC set containing S. Let the complex matrix $A = (a_{ij})$ be defined as follows:

$$a_{ij} = \begin{cases} \sqrt{g(d_j)} & \text{if } d_j \text{ divides } x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(AA^T)_{ij} = \sum_{k=1}^{m} a_{ik} a_{jk} = \sum_{\substack{d_k | x_i \\ d_k | x_j}} \sqrt{g(d_k)} \cdot \sqrt{g(d_k)} = \sum_{d_k | (x_i, x_j)} g(d_k) = \frac{1}{(x_i, x_j)},$$

since F is factor-closed. Thus, $1/[S] = AA^T$. $\square$

Remark 1: Some of the entries $\sqrt{g(d_j)}$ of A in Theorem 1 may be imaginary complex numbers. A real matrix factorization for $1/[S]$ could be obtained by defining $B = (b_{ij})$ via

$$b_{ij} = \begin{cases} g(d_j) & \text{if } d_j \text{ divides } x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, if C is the incidence matrix corresponding to B, it is true that $1/[S] = B \cdot C^T$.

Corollary 1: Let S be an FC set. Then

$$\det(1/[S]) = g(x_1)g(x_2) \ldots g(x_n).$$

Proof: In Theorem 1, take $F = S$; then A and A^T are lower triangular and upper triangular, respectively. So

$$\det(1/[S]) = \det(A) \cdot \det(A^T) = (\det(A))^2 = g(x_1)g(x_2) \ldots g(x_n). \ \square$$

Remark 2: The set F in Theorem 1 may be chosen so that $d_1 = x_1$, $d_2 = x_2$, $\ldots$, $d_n = x_n$. Hence $A = [A_1, A_2]$, where A_1 is an $n \times n$ lower triangular matrix of the form

$$\begin{bmatrix} \sqrt{g(x_1)} & & & 0 \\ & \sqrt{g(x_2)} & & \\ & & \ddots & \\ * & & & \sqrt{g(x_n)} \end{bmatrix}.$$

Therefore, rank $(A) = n$. However, since A has nonreal entries, we cannot conclude that AA^T is nonsingular.

Remark 3: Unlike GCD matrices, reciprocal GCD matrices are *never* positive definite. Recall that the AA^T factorization in Theorem 1 is a complex matrix product, whereas, in [1], A is real. The fact that a reciprocal GCD matrix is not positive definite follows readily from the observation that its leading principal 2×2 minor

$$\frac{1}{x_1 x_2} - \frac{1}{(x_1,\ x_2)^2}$$

is negative.

Remark 4: As in [4], a sum formula for the determinant of an arbitrary reciprocal GCD matrix may be obtained from the Cauchy-Binet Formula (see, e.g., [3]) and the factorization AA^T. We omit this formula due to its length.

3. LCM Matrices

Definition 3: Let $S = \{x_1, x_2, \ldots, x_n\}$ be an ordered set of distinct positive integers. The $n \times n$ matrix $[[S]] = (t_{ij})$, where $t_{ij} = [x_i, x_j]$, the least common multiple of x_i and x_j, is called the least common multiple [LCM] matrix on S.

The structure and determinants of LCM matrices come directly from results on reciprocal GCD matrices, since

$$[x_i,\ x_j] = \frac{x_i x_j}{(x_i,\ x_j)}.$$

If $[[S]]$ is an LCM matrix, we may factor out x_i from Row i and x_j from Column j to obtain $1/[S]$. Hence, every LCM matrix results from performing elementary row and column operations on the corresponding reciprocal GCD matrix.

The following theorem is a direct consequence of the preceding remarks.

Theorem 2: Let $S = \{x_1, x_2, \ldots, x_n\}$ be ordered by $x_1 < x_2 < \cdots < x_n$, and let A be the $n \times n$ matrix in Theorem 1. Then

$$[[S]] = D \cdot AA^T \cdot D = D \cdot (1/[S]) \cdot D,$$

where D is the $n \times n$ diagonal matrix $\operatorname{diag}(x_1, x_2, \ldots, x_n)$. □

Corollary 2: An LCM matrix is not positive definite. □

Corollary 3: If S is an FC set, then

$$\det[[S]] = x_1^2 \ldots x_n^2 \cdot g(x_1) \ldots g(x_n) = \prod_{i=1}^{n} \left[\phi(x_i) \cdot \prod_{p \mid x_i} (-p) \right]. \quad \square$$

As before, the Cauchy-Binet formula may be used to obtain a sum formula for $\det[[S]]$, S arbitrary.

Remark 5: We know from Corollary 3 that $\det[[S]] \neq 0$ when S is FC. A natural question arises: When is $\det[[S]]$ zero? For instance, when $S = \{1, 2, 15, 42\}$, $\det[[S]] = 0$. Furthermore, when is $\det[[S]]$ positive? This does not depend entirely upon the parity of n, even in the factor-closed case. For example, when $S = \{1, 2, 4, 8\}$, $\det[[S]] < 0$, but when $S = \{1, 2, 3, 6\}$, $\det[[S]] > 0$. In view of these comments, we leave the following as a problem.

Problem: For which sets S is $\det[[S]]$ positive? For which FC sets S is $\det[[S]]$ positive? For which sets S is $\det[[S]] = 0$?

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SUMMING INFINITE SERIES WITH SEX

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The usual way of computing the sum of the series $\sum_{n=1}^{\infty} nx^n$ for particular choices of x, $|x| < 1$, is to start with the geometric series

$$(1) \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

and then appeal to uniform convergence and interval of convergence properties to obtain

$$x\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} nx^n = x\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}.$$

What follows is a more insightful proof for $0 < x < 1$ that is accessible to students in finite mathematics classes who are familiar with neither infinite series nor calculus.

The expected value of a finite random variable $X = \{x_1, \ldots, x_n\}$ with associated probabilities $\{f(x_1), \ldots, f(x_n)\}$ is

$$\sum_{i=1}^{n} x_i f(x_i).$$

Consider the problem of determining the number of children a couple would expect to have if they continued to reproduce until a girl was born. The probability of having exactly n children would be

$$\left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^n,$$

which means that

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

children would be expected. But, since *exactly one* girl is expected and a boy is as likely as a girl, this sum must be "two." More generally, suppose the probability of a boy is x and of a girl is $1 - x$. Then, for every girl, we would expect $x/(1 - x)$ boys, so the expected number of children $x/(1 - x) + 1$ could be expressed as

$$\sum_{n=1}^{\infty} nx^{n-1}(1-x) = \frac{x}{1-x} + 1 = \frac{1}{1-x},$$

from which we conclude that

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

One could similarly find different sums by specifying other gender restrictions. For instance, the probability of the k^{th} girl being the n^{th} child is

$$\binom{n-1}{k-1}x^{n-k}(1-x)^k.$$

Therefore,

$$\sum_{n=k}^{\infty} n\binom{n-1}{k-1}x^{n-k}(1-x)^k = k\left(\frac{x}{1-x}+1\right) = \frac{k}{1-x}$$

or, equivalently,

$$\text{(2)} \qquad \sum_{n=k}^{\infty}\binom{n}{k}x^n = \frac{x^k}{(1-x)^{k+1}}.$$

Since the probability of a girl being born

$$\left(\sum_{n=1}^{\infty} x^{n-1}(1-x)\right)$$

must be "one," we have an alternate proof of (1). Note that (2) may also be established by differentiating k times the identity (1).

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Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n, satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-694 *Proposed by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA*

Prove that $L_{2^n} \equiv 7 \pmod{40}$ for $n \ge 2$.

B-695 *Proposed by Russell Euler, Northwest Missouri State U., Maryville, MO*

Define the sequences $\{P_n\}$ and $\{Q_n\}$ by

$$P_0 = 0, \; P_1 = 1, \; P_{n+2} = 2P_{n+1} + P_n \quad \text{for } n \ge 0$$

and

$$Q_0 = 1, \; Q_1 = 1, \; Q_{n+2} = 2Q_{n+1} + Q_n \quad \text{for } n \ge 0.$$

Find a simple formula expressing Q_n in terms of P_n.

B-696 *Proposed by Herta T. Freitag, Roanoke, VA*

Let (a, b, c) be a Pythagorean triple with the hypotenuse $c = 5F_{2n+3}$ and $a = L_{2n+3} + 4(-1)^{n+1}$.

(a) Determine b.
(b) For what values of n, if any, is the triple primitive? [The elements of a primitive triple have no common factor.]

B-697 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Find a closed form for the sum

$$S_n = \sum_{k=1}^{n} \frac{q^{k-1}}{w_k w_{k+1}}$$

where $w_n \neq 0$ for all n and $w_n = pw_{n-1} - qw_{n-2}$ for $n \geq 2$, with p and q nonzero constants.

B-698 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Consider the sequence of real numbers $a_1, a_2, \ldots$, where $a_1 > 2$ and

$$a_{n+1} = a_n^2 - 2 \quad \text{for } n \geq 1.$$

Find $\lim_{n\to\infty} b_n$, where

$$b_n = \frac{a_{n+1}}{a_1 a_2 \ldots a_n} \quad \text{for } n \geq 1.$$

B-699 *Proposed by Larry Blaine, Plymouth State College, Plymouth, NH*

Let a be an integer greater than 1. Define a function $p(n)$ by

$$p(1) = a - 1 \quad \text{and} \quad p(n) = a^n - 1 - \sum p(d) \quad \text{for } n \geq 2,$$

where $\sum$ denotes the sum over all d with $1 \leq d < n$ and $d|n$.

Prove or disprove that $n|p(n)$ for all positive integers n.

SOLUTIONS

edited by A. P. Hillman

Application of Generating Functions

B-670 *Proposed by Russell Euler, Northwest Missouri State U., Maryville, MO*

Evaluate $\sum_{n=1}^{\infty} \frac{nF_n}{2^n}$.

Solution by Russell Jay Hendel, Dowling College, Oakdale, NY

The generating function

$$F(x) \equiv \sum F_n x^n = -\frac{x}{x^2 + x - 1}$$

has radius of convergence α^{-1}. Differentiating both sides with respect to x and then multiplying by x gives:

$$\sum_{n=1}^{\infty} nF_n x^n = F(x) + (F(x))^2(2x + 1), \quad \text{for } |x| < \alpha^{-1}.$$

Therefore, letting $x = .5$ in the last equation, we find

$$\sum_{n=1}^{\infty} \frac{nF_n}{2^n} = 10.$$

Also solved by Richard André-Jeannin, Barry Booton, Paul S. Bruckman, Joe Howard, Hans Kappus, Joseph J. Kostal, Y. H. Harris Kwong, Alex Necochea, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Even Perfect Numbers Are Hexagonal and Triangular

B-671 *Proposed by Herta T. Freitag, Roanoke, VA*

Show that all even perfect numbers are hexagonal and hence are all triangular. [A perfect number is a positive integer which is the sum of its proper positive integral divisors. The hexagonal numbers are {1, 6, 15, 28, 45, ...} and the triangular numbers are {1, 3, 6, 10, 15, ...}.]

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

The formulas for the k^{th} triangular number T_k and the k^{th} hexagonal number H_k are

$$T_k = \frac{k(k+1)}{2} \quad \text{and} \quad H_k = k(2k-1) = T_{2k-1},$$

respectively. It is well known that every even perfect number n is of the form

$$n = 2^{p-1}(2^p - 1),$$

where $2^p - 1$ is prime; so n is the $(2^{p-1})^{\text{th}}$ hexagonal number, which is also triangular.

Also solved by Richard André-Jeannin, Charles Ashbacher, Paul S. Bruckman, Russell Euler, Russell Jay Hendel, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Proposal in 10·199, Solution in 11·181

B-672 *Proposed by Philip L. Mana, Albuquerque, NM*

Let S consist of all positive integers n such that $n = 10p$ and $n + 1 = 11q$, with p and q primes. What is the largest positive integer d such that every n in S is a term in an arithmetic progression a, $a + d$, $a + 2d$, ...?

Solution by Richard André-Jeannin, Sfax, Tunisia

Let n be a member of S. It is clear that $11(q - 1) = 10(p - 1)$; hence,

$$p = 11r + 1 \quad \text{and} \quad q = 10r + 1.$$

Since p, q are prime numbers, it is easily proved that r is even and $r \equiv 0 \pmod 3$. Hence, $r = 6s$, $p = 66s + 1$, $q = 60s + 1$, and the members of S are terms in the arithmetical progression $u_s = 660s + 10$.

Now we have

$$u_{10} = 10 \cdot 661, \quad u_{10} + 1 = 11 \cdot 601,$$

and

$$u_{11} = 10 \cdot 727, \quad u_{11} + 1 = 11 \cdot 661;$$

hence, u_{10} and u_{11} are members of S, and the largest d such that every n in S is in an arithmetical progression is $d = 660$.

Also solved by Charles Ashbacher, Paul S. Bruckman, Y. H. Harris Kwong, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Fibonacci Infinite Product

B-673 *Proposed by Paul S. Bruckman, Edmonds, WA*

Evaluate the infinite product $\displaystyle\prod_{n=2}^{\infty} \frac{F_{2n} + 1}{F_{2n} - 1}$.

Solution by Joseph J. Kostal, U. of Illinois at Chicago, IL

$$\prod_{n=2}^{\infty}\frac{F_{2n}+1}{F_{2n}-1}=\prod_{n=1}^{\infty}\left(\frac{F_{4n}+1}{F_{4n}-1}\cdot\frac{F_{4n+2}+1}{F_{4n+2}-1}\right)$$
$$=\prod_{n=1}^{\infty}\left(\frac{F_{2n-1}L_{2n+1}}{F_{2n+1}L_{2n-1}}\cdot\frac{F_{2n+2}L_{2n}}{F_{2n}L_{2n+2}}\right)$$
$$=\prod_{n=1}^{\infty}\left(\frac{F_{2n-1}F_{2n+2}}{F_{2n}F_{2n+1}}\cdot\frac{L_{2n}L_{2n+1}}{L_{2n-1}L_{2n+2}}\right)$$
$$=\prod_{n=1}^{\infty}\frac{F_{2n-1}F_{2n+2}}{F_{2n}F_{2n+1}}\cdot\prod_{n=1}^{\infty}\frac{L_{2n}L_{2n+1}}{L_{2n-1}L_{2n+2}}$$
$$=\frac{F_1}{F_2}\cdot\frac{L_2}{L_1}=\frac{1}{1}\cdot\frac{3}{1}=3.$$

Also solved by R. André-Jeannin, Bob Prielipp, H.-J. Seiffert, and the proposer.

Trigonometric Recursion

B-674 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Define the sequence $\{u_n\}$ by

$$u_0=0,\ u_1=1,\ u_n=gu_{n-1}-u_{n-2},\ \text{for } n \text{ in } \{2,3,\ldots\},$$

where g is a root of $x^2-x-1=0$. Compute u_n for n in $\{2, 3, 4, 5\}$ and then deduce that $(1+\sqrt{5})/2 = 2\cos(\pi/5)$ and $(1-\sqrt{5})/2 = 2\cos(3\pi/5)$.

Solution by Paul S. Bruckman, Edmonds, WA

Since g satisfies the equation

(1) $\quad g^2 = g + 1,$

we have

(2) $\quad g=\alpha=\frac{1}{2}(1+\sqrt{5})\quad\text{or}\quad g=\beta=\frac{1}{2}(1-\sqrt{5}).$

The characteristic equation of the given recurrence is

(3) $\quad z^2 - gz + 1 = 0,$

which has roots z_1 and z_2 given by

(4) $\quad z_1=\frac{1}{2}(g+(g^2-4)^{1/2}),\quad z_2=\frac{1}{2}(g-(g^2-4)^{1/2}).$

Making the substitution $g = 2\cos\theta$, we may express the roots in (4) as follows:

(5) $\quad z_1=\exp(i\theta),\quad z_2=\exp(-i\theta).$

From the initial conditions, we find that we may express u_n in the following Binet form:

(6) $\quad u_n=\dfrac{z_1^n-z_2^n}{z_1-z_2},\quad n=0,1,2,\ldots.$

Equivalently, using (5), we obtain

(7) $\quad u_n=\dfrac{\sin n\theta}{\sin\theta},\quad n=0,1,2,\ldots.$

Using (1) and the given recurrence, we find the following values:

$$u_2=g\cdot1-0=g;\quad u_3=g\cdot g-1=g;\quad u_4=g\cdot g-g=1;$$

$$u_5 = g \cdot 1 - g = 0; \quad u_6 = g \cdot 0 - 1 = -1; \quad u_7 = g(-1) - 0 = -g, \text{ etc.}$$

Clearly, from (2), $g \neq \pm 2$; hence, $\theta \neq m\pi$, and $\sin\theta \neq 0$. Since

$$u_5 = \sin 5\theta / \sin\theta = 0,$$

we see that $\theta = m\pi/5$ for some integer m, not a multiple of 5. We may restrict m to the residues (mod 10), since $10\theta = 2m\pi$. Also,

$$u_2 = 2\cos\theta, \quad u_7 = \sin(2\theta + m\pi)/\sin\theta = (-1)^m \sin 2\theta/\sin\theta = (-1)^m u_2.$$

However, as we have seen, $u_7 = -u_2$; therefore, m must be odd. Moreover, since $\cos(2\pi - \theta) = \cos\theta$, we may eliminate the values $m = 7$ and 9. Therefore, $m = 1$ or 3. Then, α and β must be equal to $2\cos\pi/5$ and $2\cos 3\pi/5$, in some order. Clearly, $\alpha > 0$ and $\beta < 0$; also, $2\cos\pi/5 > 0$ and $2\cos 3\pi/5 < 0$. Therefore,

(8) $\qquad \alpha = 2\cos\pi/5, \quad \beta = 2\cos 3\pi/5.$

Also solved by Herta T. Freitag, Hans Kappus, L. Kuipers, and the proposer.

Another Sine Recursion

B-675 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

In a manner analogous to that for the previous problem, show that

$$\sqrt{2+\sqrt{2}} = 2\cos\frac{\pi}{8} \quad\text{and}\quad \sqrt{2-\sqrt{2}} = 2\cos\frac{3\pi}{8}.$$

Solution by Paul S. Bruckman, Edmonds, WA

We have the same characteristic equation for z and the same substitutions as in B-674; however, in this case, g satisfies the equation

(1) $\qquad g^4 = 4g^2 - 2.$

In this case, we may obtain the following values:

$$u_2 = g,\ u_3 = g^2 - 1,\ u_4 = g^3 - 2g,\ u_5 = g^4 - 3g^2 + 1 = g^2 - 1,$$

$$u_6 = g,\ u_7 = 1,\ u_8 = 0,\ u_9 = -1,\ u_{10} = -g, \text{ etc.}$$

As before,

(2) $\qquad u_n = \sin n\theta/\sin\theta, \quad n = 0, 1, 2, \ldots, \text{ where } g = 2\cos\theta.$

Again, we note that $g \neq \pm 2$, so $\sin\theta \neq 0$. Since $u_8 = 0$, therefore $8\theta = m\pi$, or $\theta = m\pi/8$, for some integer m (not a multiple of 8). We see, from above, that $u_{10} = -u_2$. But

$$u_{10} = \sin(2\theta + m\pi)/\sin\theta = (-1)^m u_2;$$

hence, m must be odd. Again, we may restrict m to the residues of the period, in this case, mod 16; moreover, we may eliminate the values $m = 9$, 11, 13, and 15, since $\cos(2\pi - \theta) = \cos\theta$. Therefore, we may restrict m to the values $m = 1$, 3, 5, or 7. The roots of (1) are given by $\pm\sqrt{2+\sqrt{2}}$ and $\pm\sqrt{2-\sqrt{2}}$; thus, these must be equal to $2\cos m\pi/8$, $m = 1, 3, 5, 7$, in some order. Since

$$\frac{1}{2}\pi < m\pi/8 < \pi, \quad \text{for } m = 5 \text{ or } 7,$$

it is clear that the positive roots (which are the ones we are interested in) are generated by $m = 1$ or 3. Also, $\cos x$ decreases over the interval $[0, \frac{1}{2}\pi]$, from which it follows that

$$2\cos\pi/8 = \sqrt{2+\sqrt{2}}, \quad 2\cos 3\pi/8 = \sqrt{2-\sqrt{2}}.$$

Also solved by Herta T. Freitag, Hans Kappus, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-455 *Proposed by T. V. Padma Kumar, Trivandrum, South India*

Characterize, as completely as possible, all "Magic Squares" of the form

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

subject to the following constraints:

1. Rows, columns, and diagonals have the same sum
2. $a_1 + a_4 + d_1 + d_4 = b_2 + b_3 + c_2 + c_3 = a_1 + b_1 + a_4 + b_4 = K$
3. $c_1 + d_1 + c_4 + d_4 = a_2 + a_3 + b_2 + b_3 = c_2 + c_3 + d_2 + d_3 = K$
4. $a_1 + a_2 + b_1 + b_2 = c_1 + c_2 + d_1 + d_2 = a_3 + a_4 + b_3 + b_4 = K$
5. $c_3 + c_4 + d_3 + d_4 = c_1 + d_2 + a_3 + b_4 = a_1 + a_2 + d_1 + d_2 = K$
6. $a_3 + a_4 + d_3 + d_4 = b_1 + b_2 + c_1 + c_2 = b_3 + b_4 + c_3 + c_4 = K$
7. $a_2 + a_3 + d_2 + d_3 = b_1 + c_1 + b_4 + c_4 = K$
8. $a_1 + b_1 + c_1 + a_2 + b_2 + a_3 = b_4 + c_3 + c_4 + d_2 + d_3 + d_4 = 3K/2$
9. $b_1 + c_1 + d_1 + c_2 + d_2 + d_3 = a_2 + a_3 + a_4 + b_3 + b_4 + c_4 = 3K/2$
10. $a_2^2 + a_3^2 + d_2^2 + d_3^2 = b_1^2 + c_1^2 + b_4^2 + c_4^2$
11. $c_1^2 + c_2^2 + d_1^2 + d_2^2 = a_3^2 + b_3^2 + a_4^2 + b_4^2$
12. $c_3^2 + c_4^2 + d_3^2 + d_4^2 = a_1^2 + b_1^2 + a_2^2 + b_2^2$
13. $a_1^2 + a_2^2 + a_3^2 + a_4^2 + b_1^2 + b_2^2 + b_3^2 + b_4^2 = M$
14. $c_1^2 + c_2^2 + c_3^2 + c_4^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 = M$
15. $a_1^2 + b_1^2 + c_1^2 + d_1^2 + a_2^2 + b_2^2 + c_2^2 + d_2^2 = M$
16. $a_3^2 + b_3^2 + c_3^2 + d_3^2 + a_4^2 + b_4^2 + c_4^2 + d_4^2 = M$
17. $a_1 + b_2 + c_3 + d_4 + d_1 + c_2 + b_3 + a_4 = b_1 + c_1 + a_2 + d_2 + a_3 + d_3 + b_4 + c_4$
18. $a_1a_2 + a_3a_4 + b_1b_2 + b_3b_4 = c_1c_2 + c_3c_4 + d_1d_2 + d_3d_4$
19. $a_1b_1 + c_1d_1 + a_2b_2 + c_2d_2 = a_3b_3 + c_3d_3 + a_4b_4 + c_4d_4$

H-456 *Proposed by David Singmaster, Polytechnic of the South Bank, London, England*

Among the Fibonacci numbers, F_n, it is known that: 0, 1, 144 are the only squares; 0, 1, 8 are the only cubes; 0, 1, 3, 21, 55 are the only triangular numbers. (See Luo Ming's article in *The Fibonacci Quarterly* 27.2 [May 1989]: 98-108.)

A. Let $p(m)$ be a polynomial of degree at least 2 in m. Is it true that $p(m) = F_n$ has only finitely many solutions?

B. If we replace F_n by an arbitrary recurrent sequence f_n, we cannot expect a similar result, since f_n can easily be a polynomial in n. Even if we assume the auxiliary equation of our recurrence has no repeated roots, we still cannot expect such a result. For example, if

$$f_n = 6f_{n-1} - 8f_{n-2},\ f_0 = 2,\ f_1 = 6,$$

then

$$f_n = 2^n + 4^n,$$

so every f_n is of the form $p(m) = m^2 + m$. What restriction(s) on f_n is(are) needed to make $f_n = p(m)$ have only finitely many solutions?

Comments: The results quoted have been difficult to establish, so Part A is likely to be quite hard and, hence, Part B may well be extremely hard.

H-457 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $f(N)$ denote the number of addends in the Zeckendorf decomposition of N. The numerical evidence resulting from a computer experiment suggests the following two conjectures. Can they be proved?

Conjecture 1: For given positive integers k and n, there exists a positive integer n_k (depending on k) such that $f(kF_n)$ has a constant value for $n \geq n_k$.

For example,

$$24F_n = F_{n+6} + F_{n+3} + F_{n+1} + F_{n-4} + F_{n-6} \text{ for } n \geq 8.$$

By inspection, we see that $n_1 = 1$, $n_k = 2$ for $k = 2$ or 3, $n_4 = 4$ and $n_k = 5$ for $5 \leq k \leq 8$.

Conjecture 2: For $k \geq 6$, let us define:

(i) μ, the subscript of the smallest odd-subscripted Lucas number such that $k \leq L_\mu$,

(ii) ν, the subscript of the largest Fibonacci number such that $k > F_\nu + F_{\nu-6}$.

Then, $n_k = \max(\mu, \nu)$.

H-458 *Proposed by Paul Bruckman, Edmonds, WA*

Given an integer $m \geq 0$ and a sequence of natural numbers $a_0, a_1, \ldots, a_m$, form the periodic simple continued fraction (s.c.f.) given by:

(1) $$\theta = [a_0; \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}].$$

The period is symmetric, except for the final term $2a_0$, and may or may not contain a central term [that is, a_m occurs either once or twice in (1)]. Evaluate θ in terms of nonperiodic s.c.f.'s.

SOLUTIONS

No Doubt

H-437 *Proposed by L. Kuipers, Sierre, Switzerland*
(Vol. 28, no. 1, February 1990)

Let x, y, n be Natural numbers, where n is odd. If

(*) $L_n/L_{n+2} < x/y < L_{n+1}/L_{n+3}$, show that $y > (1/5)L_{n+4}$.

Are there fractions, x/y, satisfying (*) for which $y < L_{n+4}$?

Solution by Russell Jay Hendel & Sandra A. Monteferrante, Dowling College, Oakdale, NY

We prove that the rational number with smallest denominator satisfying (*) is F_{n+1}/F_{n+3}. An easy induction then shows that $5F_{n+3} > L_{n+4}$, from which the first assertion readily follows. For $n \geq 1$, $F_{n+3} < L_{n+2} < L_{n+4}$. This answers the second question in the affirmative.

Proof: If $n = 1$, an inspection shows that 1/3 is the rational number with the smallest denominator between 1/4 and 3/7. We therefore assume $n \geq 2$.

First

$$L_n/L_{n+1} = 1/(1 + L_{n-1}/L_n).$$

Hence, the continued fraction expansion of L_n is [0; 1, ..., 1, 3] with $n - 1$ ones. Similarly,

$$L_n/L_{n+2} = 1/(2 + L_{n-1}/L_n)$$

and, therefore, $L_n/L_{n+2} = [0; 2, 1, \ldots, 1, 3]$ with $n - 2$ ones.

Next, let z be a real variable and fix an odd n. Define

$$P_0 = 0,\ Q_0 = 1,\ P_1 = 1,\ Q_1 = 2,\ P_i = F_i,\ Q_i = F_{i+2}\ (\text{for } 2 \leq i \leq n-1),$$

$$P_n(z) = zF_{n-1} + F_{n-2}, \quad \text{and} \quad Q_n(z) = zF_{n+1} + F_n.$$

Define the function $f(z) = P_n(z)/Q_n(z) = [0; 2, 1, \ldots, 1, z]$ with $n - 2$ ones. Then $f(3) = L_n/L_{n+2}$, $f(4/3) = L_{n+1}/L_{n+3}$, and $f(\ )$ maps the open interval, $4/3 < z < 3$ onto the open interval $(L_n/L_{n+2}, L_{n+1}/L_{n+3})$.

It follows that, if $f(z)$ is a rational inside the interval $(f(3), f(4/3))$, then its continued fraction must begin [0; 2, 1, ..., 1, 2, ...]. Clearly, among all such continued fractions, $f(2)$ has the smallest denominator. Since

$$f(2) = P_n(2)/Q_n(2) = F_{n+1}/F_{n+3},$$

the proof is complete.

The above analysis can be generalized to describe other rationals with small denominators. For example: $F_m/F_{m+2} = [0; 2, 1, \ldots, 1, 2]$ with $m - 3$ ones where m is an integer bigger than 3. It follows that F_m/F_{m+2} is always in the open interval $(L_n/L_{n+2}, L_{n+1}/L_{n+3})$, if $m \geq n + 1$. In particular, F_m/F_{m+2} satisfies (*) with $F_{m+2} \leq L_{n+4}$, if $n + 1 \leq m \leq n + 3$.

Also solved by P. Bruckman, R. André-Jeannin, and the proposer.

A Fibonacc-ious Integral

H-438 *Proposed by H.-J. Seiffert, Berlin, Germany*
(Vol. 28, no. 1, February 1990)

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \text{ for } n \ge 2.$$

Show that, for all odd integers $n \ge 3$,

$$\int_{-\infty}^{+\infty} \frac{dx}{F_n(x)} = \frac{\pi}{n}\left(1 + 1/\cos\frac{\pi}{n}\right).$$

Solution by Paul S. Bruckman, Edmonds, WA

As is readily established,

(1) $$F_n(x) = \frac{u^n - v^n}{u - v}, \quad n = 0, 1, 2, \ldots,$$

where

(2) $$u = u(x) = \tfrac{1}{2}(x + \sqrt{x^2 + 4}),\ v = v(x) = \tfrac{1}{2}(x - \sqrt{x^2 + 4}).$$

Let

(3) $$I_n = \int_{-\infty}^{\infty} \frac{dx}{F_n(x)}, \quad \text{for odd } n \ge 3.$$

Note that $F_n(x)$ is an even polynomial (for odd n); hence,

(4) $$I_n = 2\int_0^{\infty} \frac{dx}{F_n(x)}.$$

We may make the substitution: $x = 2 \sinh \theta$ in (4); then $u(x) = e^{\theta}$, $v(x) = -e^{-\theta}$, $F_n(x) = \cosh\theta/\cosh n\theta$, and $dx = 2 \cosh\theta\, d\theta$. Therefore,

(5) $$I = 4\int_0^{\infty} \cosh^2\theta/\cosh n\theta\, d\theta.$$

Since $n \ge 3$, we see that (5) is well defined; indeed, the integrand may be expanded into a uniformly convergent series. We do so, as follows:

$$\begin{aligned} 4\cosh^2\theta/\cosh n\theta &= 2(e^{2\theta} + 2 + e^{-2\theta})/(e^{n\theta} + e^{-n\theta}) \\ &= 2e^{(2-n)\theta}\left\{\frac{1 + 2e^{-2\theta} + e^{-4\theta}}{1 + e^{-2n\theta}}\right\} \\ &= 2e^{(2-n)\theta}(1 + 2e^{-2\theta} + e^{-4\theta})\sum_{k=0}^{\infty}(-1)^k e^{-2nk\theta}. \end{aligned}$$

Hence, I_n is equal to:

$$2\sum_{k=0}^{\infty}(-1)^k\int_0^{\infty}[e^{-n\theta(2k+1)+2\theta} + 2e^{-n\theta(2k+1)} + e^{-n\theta(2k+1)-2\theta}]d\theta$$

$$= 2\sum_{k=0}^{\infty}(-1)^k[(n(2k+1) - 2)^{-1} + 2(n(2k+1))^{-1} + (n(2k+1) + 2)^{-1}],$$

or, after some simplification:

(6) $$I_n = \frac{4}{n}\sum_{k=0}^{\infty}(-1)^k/(2k+1) + 4n\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)}{(2k+1)^2n^2 - 4}.$$

The first series in (6) is the well-known Leibnitz series for $\frac{1}{4}\pi$.

The second series in (6) may be evaluated from the Mittag-Leffler formula (see [1]):

(7) $$\pi \sec \pi z = \sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)}{(k+\frac{1}{2})^2 - z^2}, \text{ provided } (z - \tfrac{1}{2}) \notin \mathbb{Z}.$$

Setting $z = 1/n$ in (7), we obtain:

$$\pi \sec \pi/n = 4n^2 \sum_{k=0}^{\infty} (-1)^k (2k + 1)[(2k + 1)^2 n^2 - 4]^{-1}.$$

Comparing this with the second series in (6) yields the desired result:

$$I_n = \pi/n(1 + \sec \pi/n).$$

NOTE: By similar methods, we may prove the following result:

$$\int_{-\infty}^{\infty} x\, dx/F_n(x) = \pi/n(\tan \pi/2n + \tan 3\pi/2n), \text{ if } n \geq 4 \text{ is even.}$$

Reference

1. Louis L. Pennisi. *Elements of Complex Variables*, 2nd ed. Urbana: University of Illinois, 1976, p. 336.

Also solved by P. Byrd, R. André-Jeannin, Y. H. Kwong, N. A. Volodin, and the proposer.

Another Lucas Congruence

H-439 *Proposed by Richard André-Jeannin, ENIS BP W, Tunisia (Vol. 28, no. 1, February 1990)*

Let p be a prime number ($p \neq 2$) and m a Natural number. Show that

$$L_{2p^m} + L_{4p^m} + \cdots + L_{(p-1)p^m} \equiv 0 \pmod{p^{m+1}}.$$

Solution by the proposer

From the formula:

$$a^p + b^p = (a + b)\left[(-1)^{\frac{p-1}{2}}(ab)^{\frac{p-1}{2}} + \sum_{k=1}^{\frac{p-1}{2}} (-1)^{k-1}(ab)^{k-1}(a^{p-2k+1} + b^{p-2k+1})\right],$$

we get, when taking $a = \alpha^{p^m}$, $b = \beta^{p^m}$,

$$L_{p^{m+1}} = L_{p^m}[1 + L_{(p-1)p^m} + L_{(p-3)p^m} + \cdots + L_{2p^m}],$$

hence:

(1) $\qquad L_{p^{m+1}} - L_{p^m} = L_{p^m}[L_{(p-1)p^m} + \cdots + L_{2p^m}].$

But it is known (see Jarden, *Recurring Sequences*, p. 111) that:

(2) $\qquad L_{p^{m+1}} \equiv L_{p^m} \pmod{p^{m+1}}$

and thus (1) becomes:

(3) $\qquad 0 \equiv L_{p^m}[L_{(p-1)p^m} + \cdots + L_{2p^m}] \pmod{p^{m+1}}.$

Now we have: $\gcd(p, L_{p^m}) = 1$ [since, by (2): $L_{p^m} \equiv 1 \pmod{p}$]. Thus, (3) shows that

$$L_{(p-1)p^m} + \cdots + L_{2p^m} \equiv 0 \pmod{p^{m+1}}.$$

Also solved by P. Bruckman and G. Wulczyn.

A Square Product

H-440 *Proposed by T. V. Padma Kumar, Trivandrum, South India (Vol. 28, no. 2, May 1990)*

NOTE: This is the same as H-448.

If $a_1, a_2, \ldots, a_m, n$ are positive integers such that $n > a_1, a_2, \ldots, a_m$ and $\emptyset(n) = m$ and a_i is relatively prime to n for $i = 1, 2, 3, \ldots, m$, prove

$$\left(\prod_{i=1}^{m} a_i\right)^2 \equiv 1 \pmod{n}.$$

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

Consider the ring $(Z_n, +_n, \cdot_n)$ with $Z_n = \{0, 1, 2, \ldots, (n-1)\}$, where the operations are addition modulo n and multiplication modulo n, respectively. Under this hypothesis, the given members $a_1, a_2, \ldots, a_m$ are precisely the members of the multiplicative group of units of this ring. These m units can be partitioned into two classes. The first class consists of those members a_i (as well as a_t) such that

$$a_i a_t \equiv 1 \pmod{n}, \text{ where } i \neq t;\ 1 \leq i,\ t \leq m.$$

The second class contains the remaining members a_j that satisfy $a_j^2 \equiv 1 \pmod{n}$.

Without loss of generality, we can name the members of the first class as $a_1, a_2, \ldots, a_k$ and the members of the second class as $a_{k+1}, a_{k+2}, \ldots, a_m$. (Note that it is possible that the first class is empty, so that $k = 0$: this can be verified when $n = 8$.)

Consequently,

$$\prod_{i=1}^{m} a_i = \left(\prod_{i=1}^{k} a_i\right)(a_{k+1} \cdot a_{k+2} \cdot \cdots \cdot a_m).$$

Since $\prod_{i=1}^{k} a_i \equiv 1 \pmod{n}$, we conclude that:

$$\left(\prod_{i=1}^{m} a_i\right)^2 = \left(\prod_{i=1}^{k} a_i\right)^2 (a_{k+1} \cdot a_{k+2} \cdot \cdots \cdot a_m)^2 \equiv 1 \pmod{n}.$$

Also solved by P. Bruckman, B. Prielipp, and L. Somer.

<u>Editorial Notes:</u>

1. Lawrence Somer's name was inadvertently omitted as a solver of H-424.

2. A number of readers pointed out that H-451 is the same as B-643.

3. Paul Bruckman's name was inadvertently omitted as a solver of H-434. He mentioned that line one of the solution should read "$[c_1 r_1^n + \frac{1}{2}]$" and that the value reported for the approximation of c_1 should be 1.22041 not 1.22144.

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A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

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