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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

LUCAS NUMBERS AND POLYNOMIALS OF ORDER K AND THE LENGTH OF THE LONGEST CIRCULAR SUCCESS RUN

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1. Introduction

The Lucas numbers L_n, $n = 0, 1, 2, \ldots$, may be defined by

$$L_n = L_{n-1} + L_{n-2},\ n = 2, 3, \ldots,\ L_0 = 2,\ L_1 = 1.$$

Among several combinatorial interpretations of the Lucas numbers in terms of permutations, combinations, compositions (ordered partitions), and distributions of objects into cells, the most commonly used as an alternative combinatorial definition of them is the following: The n^{th} Lucas number L_n, $n = 2, 3, \ldots$, is the number of combinations of n consecutive integers $\{1, 2, 3, \ldots, n\}$ placed on a circle (so that n and 1 are consecutive) with no two integers consecutive. Since

$$L(n, r, 2) = \frac{n}{n-r}\binom{n-r}{r},\ r = 0, 1, 2, \ldots, [n/2],\ n = 2, 3, \ldots,$$

where $[x]$ denotes the integral part of x, is the number of r-combinations of the n consecutive integers $\{1, 2, \ldots, n\}$, placed on a circle, with no two integers consecutive, it is clear that

$$L_n = \sum_{r=0}^{[n/2]} \frac{n}{n-r}\binom{n-r}{r},\ n = 1, 2, \ldots .$$

The polynomials

$$g_n(x) = \sum_{r=0}^{[n/2]} \frac{n}{n-r}\binom{n-r}{r} x^{n-2r},\ n = 1, 2, \ldots,$$

may be called Lucas polynomials. It is worth noting that these polynomials are related to the Chebyshev polynomials,

$$T_n(x) = \cos(n\theta),\ \cos\theta = x,$$

by $g_n(x) = 2i^{-n}T_n(ix/2)$, $i = \sqrt{-1}$. Riordan [8] considered the polynomials $h_n(x) = i^{-n}g_n(ix)$, $n = 1, 2, \ldots$, and the Lucas-type polynomials

$$L_n(x) = \sum_{r=0}^{[n/2]} \frac{n}{n-r}\binom{n-r}{r} x^{n-r} = x^{n/2} g(x^{1/2}),\ n = 1, 2, \ldots,$$

in a derivation of Chebyshev-type pairs of inverse relations.

The present paper is motivated by the problem of expressing the distribution function of the length of the longest run of successes in a circular sequence of n independent Bernoulli trials (Philippou & Marki [7]) and the reliability of a circular consecutive k-out-of-n failure system (Derman, Liebermann, & Ross [1]). An elegant solution to this problem is provided by the n^{th} Lucas-type polynomial of order k. This polynomial and the n^{th} Lucas number of order k, as a particular case of it, are examined in Section 2. As probabilistic applications, the above posed problems are discussed in Section 3.

2. Lucas Numbers and Polynomials of order K

Let $L(n, r, k)$ be the number of r-combinations of the n consecutive integers $\{1, 2, \ldots, n\}$ displaced on a circle, with no k integers consecutive. Moser & Abramson [3], essentially showed that

$$(2.1)\qquad L(n, r, k) = \begin{cases} \binom{n}{r}, & r = 0, 1, 2, \ldots, n,\ n = 0, 1, 2, \ldots, k-1,\ k = 2, 3, \ldots \\ \sum_{j=0}^{[n/k]} (-1)^j \binom{n-r}{j} \frac{n}{n-jk} \binom{n-jk}{n-r}, & r = 0, 1, 2, \ldots, [n - n/k],\\ & n = k, k+1, \ldots, k = 2, 3, \ldots \\ 0, & r > [n - n/k],\ n = k, k+1, \ldots, k = 2, 3, \ldots, \end{cases}$$

where $[x]$ denotes the integral part of x. As it can be easily shown, these numbers satisfy the recurrence relation

$$(2.2)\qquad L(n, r, k) = \begin{cases} \sum_{j=1}^{r+1} L(n-j, r-j+1, k), & r = 0, 1, 2, \ldots, n-1,\\ & n = 1, 2, \ldots, k,\ k = 2, 3, \ldots \\ \sum_{j=1}^{\min\{r+1, k\}} L(n-j, r-j+1, k), & r = 0, 1, 2, \ldots, [n - n/k],\\ & n = k+1, k+2, \ldots,\\ & k = 2, 3, \ldots . \end{cases}$$

The sum

$$(2.3)\qquad L_{n,k} = \sum_{r=0}^{[n-n/k]} L(n, r, k),\ n = 1, 2, \ldots, k = 2, 3, \ldots,$$

for $n = k, k+1, \ldots$, is the number of combinations of the n consecutive integers $\{1, 2, \ldots, n\}$ displaced on a circle, with no k integers consecutive. This number, which for $k = 2$ reduces to $L_{n,2} = L_n$, the n^{th} Lucas number, may be called the n^{th} Lucas number of order k.

The polynomial

$$(2.4)\qquad L_{n,k}(x) = \sum_{r=0}^{[n-n/k]} L(n, r, k)x^{n-r},\ n = 1, 2, \ldots, k = 2, 3, \ldots$$

may be called the n^{th} Lucas-type polynomial of order k. Clearly,

$$L_{n,k}(1) = L_{n,k}.$$

Recurrence relations, generating functions, and alternative algebraic expressions of these numbers and polynomials and also their connection with the corresponding Fibonacci numbers and polynomials are presented in the following theorems and corollaries.

Theorem 2.1: The sequence $L_{n,k}(x)$, $n = 1, 2, \ldots$, of Lucas-type polynomials of order k satisfies the recurrence relation

$$(2.5)\qquad L_{n,k}(x) = \begin{cases} x\left\{n + \sum_{j=1}^{n-1} L_{n-j,k}(x)\right\}, & n = 2, 3, \ldots, k,\ k = 2, 3, \ldots \\ x\sum_{j=1}^{k} L_{n-j,k}(x), & n = k+1, k+2, \ldots, k = 2, 3, \ldots, \end{cases}$$

with $L_{1,k}(x) = x$.

Proof: From (2.4), on using the recurrence relation (2.2), it follows that:

(a) for $n = 1, 2, \ldots, k$,

$$\begin{aligned}
L_{n,k}(x) &= \sum_{r=0}^{n-1} L(n, r, k)x^{n-r} = \sum_{r=0}^{n-1}\sum_{j=1}^{r+1} L(n-j, r-j+1, k)x^{n-r} \\
&= x\sum_{j=1}^{n}\sum_{r=j-1}^{n-1} L(n-j, r-j+1, k)x^{n-r-1} \\
&= x\left\{n + \sum_{j=1}^{n-1}\sum_{r=j}^{n-1} L(n-j, r-j+1, k)x^{n-r-1}\right\} \\
&= x\left\{n + \sum_{j=1}^{n-1} L_{n-j,k}(x)\right\};
\end{aligned}$$

(b) for $n = k+1, k+2, \ldots,$

$$\begin{aligned}
L_{n,k}(x) &= \sum_{r=0}^{[n-n/k]} L(n, r, k)x^{n-r} \\
&= \sum_{r=0}^{[n-n/k]}\ \sum_{j=1}^{\min\{r+1,k\}} L(n-j, r-j+1, k)x^{n-r} \\
&= x\sum_{j=1}^{k}\sum_{r=j-1}^{[n-n/k]} L(n-j, r-j+1, k)x^{n-k-1} \\
&= x\sum_{j=1}^{k} L_{n-j,k}(x);
\end{aligned}$$

and for $n = 1$,

$$L_{1,k}(x) = L(1, 0, k)x = x.$$

Remark 2.1: The n^{th} Lucas-type polynomial of order k, for $n = 2, 3, \ldots, k$, by virtue of (2.1) and (2.4) may be obtained as

$$L_{n,k}(x) = \sum_{r=0}^{n-1}\binom{n}{r}x^{n-r} = (1+x)^{n-1}. \tag{2.6}$$

Also, from (2.5), for $n = k+1, k+2, \ldots,$ it follows that

$$L_{n,k}(x) = (1+x)L_{n-1,k}(x) - xL_{n-k-1,k}(x). \tag{2.7}$$

Corollary 2.1: The sequence $L_{n,k}$, $n = 1, 2, \ldots,$ of the Lucas numbers of order k satisfies the recurrence relation

$$L_{n,k} = \begin{cases} n + \sum_{j=1}^{n-1} L_{n-j,k}, & n = 2, 3, \ldots, k,\ k = 2, 3, \ldots \\ \sum_{j=1}^{k} L_{n-j,k}, & n = k+1, k+2, \ldots,\ k+2, 3, \ldots, \end{cases} \tag{2.8}$$

with $L_{1,k} = 1$.

Theorem 2.2: The generating function of the sequence of Lucas-type polynomials of order k, $L_{n,k}(x)$, $n = 1, 2, \ldots,$ is given by

$$L_k(t; x) = \sum_{n=1}^{\infty} L_{n,k}(x)t^n = \left(x\sum_{j=1}^{k} jt^j\right)\left(1 - x\sum_{j=1}^{k} t^j\right)^{-1}. \tag{2.9}$$

Proof: Multiplying the recurrence relation (2.5) by t^n and summing for $n = 1, 2, \ldots,$ we find

$$\begin{aligned}
L_k(t;\ x) &= \sum_{n=1}^{\infty} L_{n,k}(x)t^n = xt + \sum_{n=2}^{k} L_{n,k}(x)t^n + \sum_{n=k+1}^{\infty} L_{n,k}(x)t^n \\
&= x\sum_{j=1}^{k} jt^j + x\sum_{n=2}^{k}\sum_{j=1}^{n-1} L_{n-j,k}(x)t^n + x\sum_{n=k+1}^{\infty}\sum_{j=1}^{k} L_{n-j,k}(x)t^n \\
&= x\sum_{j=1}^{k} jt^j + x\sum_{j=1}^{k-1}\sum_{n=j+1}^{k} L_{n-j,k}(x)t^n + x\sum_{j=1}^{k}\sum_{n=k+1}^{\infty} L_{n-j,k}(x)t^n \\
&= x\sum_{j=1}^{k} jt^j + x\sum_{j=1}^{k} t^j \sum_{n=j+1}^{\infty} L_{n-j,k}(x)t^{n-j} \\
&= x\sum_{j=1}^{k} jt^j + xL_k(t;\ x)\sum_{j=1}^{k} t^j,
\end{aligned}$$

from which (2.9) follows.

Corollary 2.2: The generating function of the sequence of Lucas numbers of order k, $L_{n,k}$, $n = 1, 2, \ldots$, is given by

$$(2.10)\qquad L_k(t) = \sum_{n=1}^{\infty} L_{n,k}t^n = \left(\sum_{j=1}^{k} jt^j\right)\left(1 - \sum_{j=1}^{k} t^j\right)^{-1}.$$

Theorem 2.3: The n^{th} Lucas-type polynomial of order k may be expressed as

$$(2.11)\qquad (a)\qquad L_{n,k}(x) = -1 + \sum_{j=0}^{[n/(k+1)]} (-1)^j \frac{n}{n-jk}\binom{n-jk}{j} x^j(1+x)^{n-j(k+1)}$$

$$(2.12)\qquad (b)\qquad L_{n,k}(x) = \sum \frac{r_1 + 2r_2 + \cdots + kr_k}{r_1 + r_2 + \cdots + r_k}\,\frac{(r_1 + r_2 + \cdots + r_k)!}{r_1!r_2!\ldots r_k!}\,x^{r_1+r_2+\cdots+r_k}$$

where the summation is extended over all partitions of n with no part greater than k, that is over all $r_i = 0, 1, 2, \ldots, n$, $i = 1, 2, \ldots, k$ such that

$$r_1 + 2r_2 + \cdots + kr_k = n.$$

Proof: The generating function (2.9) may be expanded into powers of t as

$$\begin{aligned}
L_k(t;\ x) &= -t\frac{d}{dt}\log\left(1 - x\sum_{j=1}^{k} t^j\right) \\
&= -t\frac{d}{dt}\log\{[1 - (1+x)t + xt^{k+1}](1-t)^{-1}\} \\
&= -t(1-t)^{-1} - t\frac{d}{dt}\log[1 - (1+x)t + xt^{k+1}] \\
&= -\sum_{n=1}^{\infty} t^n + t\frac{d}{dt}\sum_{r=1}^{\infty}[(1+x)t - xt^{k+1}]^r/r \\
&= -\sum_{n=1}^{\infty} t^n + t\frac{d}{dt}\sum_{r=1}^{\infty}\sum_{j=0}^{r}(-1)^j\frac{1}{r}\binom{r}{j}x^j(1+x)^{r-j}t^{r+jk} \\
&= -\sum_{n=1}^{\infty} t^n + \sum_{r=1}^{\infty}\sum_{j=0}^{r}(-1)^j\frac{r+jk}{r}\binom{r}{j}x^j(1+x)^{r-j}t^{r+jk} \\
&= -\sum_{n=1}^{\infty} t^n + \sum_{n=1}^{\infty}\sum_{j=1}^{[n/(k+1)]}(-1)^j\frac{n}{n-jk}\binom{n-jk}{j}x^j(1+x)^{n-j(k+1)}t^n
\end{aligned}$$

yielding (2.11).

A different expansion of (2.9) as

$$L_k(t;\, x) = -t\,\frac{d}{dt}\log\left(1 - x\sum_{j=1}^{k} t^j\right) = t\,\frac{d}{dt}\sum_{n=1}^{\infty}\left(x\sum_{j=1}^{k} t^j\right)^r\Big/ r$$

$$= t\,\frac{d}{dt}\sum_{r=1}^{\infty}\sum\frac{(r-1)!}{r_1!r_2!\,\ldots\,r_k!}\,x^{r_1+r_2+\cdots+r_k}t^{r_1+2r_2+\cdots+kr_k}$$

$$= \sum_{r=1}^{\infty}\sum\frac{(r_1+2r_2+\cdots+kr_k)(r-1)!}{r_1!r_2!\,\ldots\,r_k!}\,x^{r_1+r_2+\cdots+r_k}t^{r_1+2r_2+\cdots+kr_k}$$

where in the inner sums the summation is extended over all $r = 0, 1, 2, \ldots, r$, $i = 1, 2, \ldots, k$, such that $r_1 + r_2 + \cdots + r_k = r$, on putting

$$n = r - \sum_{j=1}^{k}(j-1)r_j$$

yields

$$L_k(t;\, x) = \sum_{n=1}^{\infty}\left\{\sum\frac{r_1+2r_2+\cdots+kr_k}{r_1+r_2+\cdots+r_k}\,\frac{(r_1+r_2+\cdots+r_k)!}{r_1!r_2!\,\ldots\,r_k!}\,x^{r_1+r_2+\cdots+r_k}\right\}t^n$$

where in the inner sum the summation is extended over all $r_i = 0, 1, 2, \ldots, n$, $i = 1, 2, \ldots, k$, such that $r_1 + 2r_2 + \cdots + kr_k = n$. The last expression implies (2.12).

Corollary 2.3: The n^{th} Lucas number of order k may be expressed as

$$(2.13)\quad (a)\quad L_{n,k} = -1 + \sum_{j=0}^{[n/(k+1)]}(-1)\,\frac{n}{n-jk}\binom{n-jk}{j}2^{n-j(k+1)},$$

$$(2.14)\quad (b)\quad L_{n,k} = \sum\frac{r_1+2r_2+\cdots+kr_k}{r_1+r_2+\cdots+r_k}\,\frac{(r_1+r_2+\cdots+r_k)!}{r_1!r_2!\,\ldots\,r_k!}$$

where the summation is extended over all $r_i = 0, 1, 2, \ldots, n$ such that

$$r_1 + 2r_2 + \cdots + kr_k = n.$$

Remark 2.2: A known expression for the n^{th} Lucas number L_n and two expressions for the n^{th} Lucas number of order 3, $H_n \equiv L_{n,3}$, may be deduced from the general expression (2.14). Setting $k = 2$ and introducing the variable $r = r_2$, it follows that

$$L_n = \sum_{r=0}^{[n/2]}\frac{n}{n-r}\binom{n-r}{r}.$$

Putting $k = 3$ and introducing the variables $r = r_2$, $j = r_3$, (2.14) reduces to

$$(2.15)\quad H_n = \sum_{r=0}^{[n/2]}\sum_{j=0}^{[(n-2r)/3]}\frac{n}{n-r-2j}\binom{n-r-2j}{r+j}\binom{r+j}{r}$$

while, introducing the variables $r = r_2 + 2r_3$, $j = r_3$, (2.20) becomes

$$(2.16)\quad H_n = \sum_{r=0}^{[2n/3]}\sum_{j=0}^{[r/3]}\frac{n}{n-r}\binom{n-r}{j}\binom{n-r-j}{r-2j}.$$

The Lucas numbers L_n are related to Fibonacci numbers F_n by

$$L_n = F_n + 2F_{n-1} = F_{n+1} + F_{n-1}.$$

An extension of this relation to the Lucas-type polynomials and the Fibonacci-type polynomials (see [6]) is obtained in the following theorem.

Theorem 2.4: The Lucas-type polynomials of order k, $L_{n,k}(x)$, $n = 1, 2, \ldots$, are expressed in terms of the Fibonacci-type polynomials of order k, $F_{n,k}(x)$, $n = 1, 2, \ldots$, by

$$(2.17) \quad L_{n,k}(x) = x \sum_{j=1}^{\min\{n,k\}} jF_{n-j+1,k}(x), \quad n = 1, 2, \ldots, \; k = 2, 3, \ldots .$$

Proof: Since (see [6])

$$\sum_{n=0}^{\infty} F_{n+1,k}(x)t^n = \left(1 - x\sum_{j=1}^{k} t^j\right)^{-1},$$

it follows from (2.9) that

$$\sum_{n=1}^{\infty} L_{n,k}(x)t^n = x\left(\sum_{j=1}^{k} jt^j\right)\left(\sum_{r=0}^{\infty} F_{r+1,k}(x)t^r\right)$$
$$= \sum_{n=1}^{\infty}\left\{x \sum_{j=1}^{\min\{n,k\}} jF_{n-j+1,k}(x)\right\}t^n,$$

which implies (2.17).

Corollary 2.4: The Lucas numbers of order k are expressed in terms of the Fibonacci numbers of order k by

$$(2.18) \quad L_{n,k} = \sum_{j=1}^{\min\{n,k\}} jF_{n-j+1,k}, \quad n = 1, 2, \ldots, \; k = 2, 3, \ldots .$$

Remark 2.3: The polynomial

$$(2.19) \quad g_{n,k}(x) = \sum_{r=0}^{[n-n/k]} L(n, r, k)x^{(n-r)k-n}, \quad n = 1, 2, \ldots, \; k = 2, 3, \ldots,$$

may be called the n^{th} Lucas polynomial of order k. It is related to the Lucas-type polynomial (2.4) by

$$(2.20) \quad g_{n,k}(x) = x^{-n}L_{n,k}(x^k), \quad n = 1, 2, \ldots, \; k = 2, 3, \ldots .$$

Expressions for these polynomials, analogous to (2.5), (2.9), (2.11) and (2.12), on using (2.20), may easily be deduced. Further,

$$(2.21) \quad g_{n,k}(x) = \sum_{j=1}^{\min\{n,k\}} jx^{k-j+1}f_{n-j+1,k}(x), \quad n = 1, 2, \ldots, \; k = 2, 3, \ldots,$$

where $f_{n,k}(x)$ is the n^{th} Fibonacci polynomial of order k (see [5] and [2] as k-bonacci polynomial). This relation may be deduced from (2.17) by virtue of (2.20) and [4],

$$f_{n,k}(x) = x^{-n+1}F_{n,k}(x^k).$$

3. Probabilistic Applications

Consider a circular sequence of n independent Bernoulli trials with constant success probability p and let $q = 1 - p$. Further, let C_n be the length of the longest circular run of successes and let S_n be the total number of successes. In Theorem 3.1, the conditional distribution function of C_n, given $S_n = r$, $P(C_n \le x/S_n = r)$, $-\infty < x < \infty$, is obtained in terms of the numbers $L(n, r, [x] + 1)$ and the distribution function of C_n, $P(C_n \le x)$, $-\infty < x < \infty$, is expressed in terms of the Lucas-type polynomials of order $[x] + 1$.

Theorem 3.1: Let C_n and S_n be the length of the longest run of successes and total number of successes, respectively, in a circular sequence of n independent Bernoulli trials with constant success probability p. Then,

$$(3.1)\qquad P(C_n \le x/S_n = r) = \begin{cases} 0 \\ L(n,\ r,\ k+1)\Big/\binom{n}{r}, & 0 \le x < r \le n,\ k = [x] \\ 1, & r \le x < \infty,\ r \le n. \end{cases}$$

$$(3.2)\qquad P(C_n \le x) = \begin{cases} 0, & -\infty < x < 0 \\ p^n L_{n,k+1}(q/p), & 0 \le x < n \\ 1, & n \le x < \infty. \end{cases}$$

Proof: The elements of the sample space are combinations $\{i_1,\ i_2,\ \ldots\}$ of the n consecutive integers $\{1,\ 2,\ \ldots,\ n\}$ displaced on a circle where i_m is the position of the m^{th} success, $m = 1,\ 2,\ \ldots$. The event $\{C_n \le x,\ S_n = r\}$ contains all the r-combinations of the n integers $\{1,\ 2,\ \ldots,\ n\}$ displaced on a circle, with no $k + 1 = [x] + 1$ integers consecutive. Clearly, the number of these r-combinations is given by $L(n,\ r,\ k + 1)$. Further, each of these r-combinations has probability $p^r q^{n-r}$. Hence,

$$(3.3)\qquad P(C_n \le x,\ S_n = r) = L(n,\ r,\ k+1)p^r q^{n-r},\quad k = [x],$$

and since

$$P(S_n = r) = \binom{n}{r}p^r q^{n-r},\quad r = 0,\ 1,\ 2,\ \ldots,\ n,$$

(3.1) follows.

Summing the probabilities (3.3) for $r = 0,\ 1,\ 2,\ \ldots,\ [n - n/(k+1)]$, on using (2.4), (3.2) is deduced.

Since $P(C_n = k) = P(C_n \le k) - P(C_n \le k - 1)$, $k = 0,\ 1,\ 2,\ \ldots$, on using (3.1), the next corollary is deduced.

Corollary 3.1: The probability function of the random variable C is given by

$$(3.4)\qquad P(C_n = k) = \begin{cases} q^n, & k = 0 \\ p^n, & k = n \\ p^n\{L_{n,k+1}(q/p) - L_{n,k}(q/p)\}, & k = 1,\ 2,\ \ldots,\ n-1. \end{cases}$$

Remark 3.1: A circular consecutive-k-out-of-n: F system is a system of n components displaced on a circle which fails when k consecutive components fail. Suppose that the probability for each component to function is p and to fail is $q = 1 - p$. Derman, Lieberman, and Ross (see [1]) expressed its reliability $R_c(p,\ n,\ k)$ as

$$R_c(p,\ k,\ n) = p^2\sum_{j=1}^{k} jq^{j-1}R_L(p,\ k,\ n-j-1),$$

where $R_L(p,\ k,\ n)$ denotes the reliability of a linear consecutive-k-out-of-n: F system.

Interpreting as a "success" the failure of a component, the reliability $R_c(p,\ k,\ n)$ is the probability that the length C_n of the longest circular run of successes in a circular sequence of n independent Bernoulli trials with constant success probability 1 is less than or equal to k. It is then clear from Theorem 3.1 that

$$(3.5)\qquad R_c(p,\ k,\ n) = q^n L_{n,k}(p/q) = \sum_{j=0}^{[n/(k+1)]}(-1)^j\frac{n}{n-jk}\binom{n-jk}{j}p^j q^{jk} - q^n$$

with the last equality by (2.11).

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CONTINUED FRACTIONS OF GIVEN SYMMETRIC PERIOD

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1. If $D > 1$ is a rational number, not a square, then $\sqrt{D}$ has a (simple) continued fraction expansion of the form

$$\sqrt{D} = [b_0, \overline{b_1, \ldots, b_{k-1}, 2b_0}]$$

with $k \geq 1$ and positive integers b_i such that the sequence $(b_1, \ldots, b_{k-1})$ is *symmetric*, i.e., $b_i = b_{k-i}$ for all $i \in \{1, \ldots, k-1\}$. Necessary and sufficient conditions on $b_0, \ldots, b_{k-1}$ which guarantee that D is an integer are stated in [3; §26]. Recently, C. Friesen [1] gave a fresh proof of these conditions. He deduced, moreover, that for a given symmetric sequence $(b_1, \ldots, b_{k-1})$ there is either no integral D such that the continued fraction expansion of $\sqrt{D}$ has the given sequence as its symmetric part or there are infinitely many squarefree such D.

In this paper, I shall prove a more precise statement. Starting with the conditions as in [3; §26] I will show that, given a symmetric sequence which meets these conditions, there are infinitely many D with prescribed p-adic exponent $v_p(D)$ for finitely many p and $p^2 \nmid D$ for all other p, such that $\sqrt{D}$ has the given sequence as the symmetric part of its continued fraction expansion. Moreover, I will show that about 2/3 (resp. 5/6) of all symmetric sequences of the given even (resp. odd) length are symmetric parts of the continued fraction expansion of $\sqrt{D}$ for some integral D. Finally, I consider the corresponding questions for the continued fraction expansion of $(1 + \sqrt{D})/2$ for an integral $D \equiv 1 \pmod 4$.

2. I begin by citing Satz [3; 3.17] in an appropriate form.

Theorem 1: Let $(b_1, \ldots, b_{k-1})$ $(k \geq 1)$ be a symmetric sequence in $\mathbb{N}_+$ and let $b_0 \in \mathbb{N}_+$. Then the following assertions are equivalent:

a) $[b_0, \overline{b_1, \ldots, b_{k-1}, 2b_0}] = \sqrt{D}$ with $D \in \mathbb{N}_+$;

b) $b_0 = \frac{1}{2} \cdot [me - (-1)^k fg]$ for some $m \in \mathbb{Z}$, where e, f, and g are defined by the matrix equation

$$(1) \qquad \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \prod_{i=1}^{k-1} \begin{pmatrix} b_i & 1 \\ 1 & 0 \end{pmatrix}.$$

If this condition is fulfilled, then

$$(2) \qquad D = b_0^2 + mf - (-1)^k g^2.$$

In order to state more precise results, I introduce the following notation.

Definition: For a symmetric sequence of positive integers $(b_1, \ldots, b_{k-1})$ $(k \geq 1)$ let

$$\mathscr{D}(b_1, \ldots, b_{k-1})$$

be the set of all $D \in \mathbb{N}_+$ with $\sqrt{D} = [b_0, \overline{b_1, \ldots, b_{k-1}, 2b_0}]$ for some $b_0 \in \mathbb{N}_+$.

Corollary 1: Let $(b_1, \ldots, b_{k-1})$ be a symmetric sequence in $\mathbf{N}_+$ and define e, f, g by (1). Then the following assertions are equivalent:

a) $\mathscr{D}(b_1, \ldots, b_{k-1}) \neq \emptyset$.

b) Either $e \equiv 1 \pmod 2$ or $e \equiv fg \equiv 0 \pmod 2$.

If b) is fulfilled, then $\mathscr{D}(b_1, \ldots, b_{k-1})$ consists of all $D \in \mathbf{N}_+$ which are of the form

$$D = \frac{e^2m^2}{4} + \left[f - (-1)^k \frac{efg}{2}\right] \cdot m + \left[\frac{f^2g^2}{4} - (-1)^k g^2\right] \tag{3}$$

with $m \in \mathbf{Z}$ satisfying $me - (-1)^k fg > 0$.

Proof: The conditions stated in b) are necessary and sufficient for the existence of $m \in \mathbf{Z}$ such that

$$b_0 = \frac{1}{2} \cdot [me - (-1)^k fg]$$

is a positive integer. Inserting this expression for b_0 in (2) yields (3). □

Applying Corollary 1 to the special sequence $(b_1, \ldots, b_{k-1}) = (1, \ldots, 1)$ gives

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} F_k & F_{k-1} \\ F_{k-1} & F_{k-2} \end{pmatrix},$$

where $(F_n)_{n \geq -1}$ is the ordinary Fibonacci sequence defined by

$$F_{-1} = 1,\ F_0 = 0,\ F_{n+1} = F_n + F_{n-1}.$$

Taking into account that $F_k \equiv 0 \pmod 2$ if and only if $k \equiv 0 \pmod 3$, I obtain

Corollary 2: $\mathscr{D}(\underbrace{1, \ldots, 1}_{(k-1)}) \neq \emptyset$ if and only if $k \not\equiv 0 \pmod 3$.

3. Now I investigate the possible prime powers dividing $D \in \mathscr{D}(b_1, \ldots, b_{k-1})$ for a given symmetric sequence $(b_1, \ldots, b_{k-1})$.

For $n \in \mathbf{Z}$, $n \neq 0$, and a prime p, set

$$v_p(n) = w \text{ if } p^w | n,\ p^{w+1} \nmid n \ (w \geq 0).$$

The following result is an immediate consequence of the arguments given in [2; §2].

Lemma: Let $F(X) = AX^2 + BX + C \in \mathbf{Z}[X]$ be a quadratic polynomial. For a prime p, set

$$E_p(F) = \{w \in \mathbf{N} \,|\, v_p(F(x)) = w \text{ for some } x \in \mathbf{Z}\}.$$

Let P be a finite set of primes, $w_p \in E_p(F)$ for $p \in P$, and suppose that, for every prime $p \notin P$, the congruence $F(x) \equiv 0 \pmod{p^2}$ has at most two solutions $x \pmod{p^2}$. Then there exist infinitely many $x \in \mathbf{N}$, such that

$$v_p(F(x)) = w_p \text{ for all } p \in P$$

and

$$v_p(F(x)) \leq 1 \text{ for all primes } p \notin P.$$

Now let $(b_1, \ldots, b_{k-1})$ $(k \geq 1)$ be a symmetric sequence of positive integers. Define e, f, and g by (1) and, depending on these numbers, for every prime p, a set $E_p = E_p(e, f, g, k) \subset \mathbf{N}$ of possible exponents as follows:

a) $p \neq 2$.

$$E_p = \begin{cases} \{0\}, & \text{if } e \equiv 1 \pmod 2,\ p \nmid e, \text{ and } \left(\frac{(-1)^k}{p}\right) = -1; \\ \mathbf{N}, & \text{otherwise.} \end{cases}$$

b) $p = 2$, $e \equiv 1 \pmod 2$:

$$E_2 = \begin{cases} \{0, 1\}, & \text{if } k \equiv 1 \pmod 2; \\ \mathbf{N}\backslash\{1, 2\}, & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

c) $p = 2$, $e \equiv fg \equiv 0 \pmod 2$:

$$E_2 = \begin{cases} \mathbf{N}_+, & \text{if } e \equiv 2,\ g \equiv 0 \pmod 4; \\ \mathbf{N}, & \text{otherwise.} \end{cases}$$

With these definitions, it is possible to state Theorem 2, which generalizes the results of [1]:

Theorem 2: Let $(b_1, \ldots, b_{k-1})$ $(k \geq 1)$ be a symmetric sequence of positive integers, define e, f, and g by (1), and suppose that either $e \equiv 1 \pmod 2$ or $e \equiv fg \equiv 0 \pmod 2$. For a prime p, let $E_p = E\,(e, f, g, k)$ be defined as above.

i) If $D \in \mathcal{D}(b_1, \ldots, b_{p-1})$, then $v_p(D) \in E_p$ for all primes p.

ii) Let P be a finite set of primes and $w_p \in E_p$ for $p \in P$. Then there are infinitely many $D \in \mathcal{D}(b_1, \ldots, b_{k-1})$ such that $v_p(D) = w_p$ for all $p \in P$ and $v_p(D) \leq 1$ for all primes $p \notin P$.

Proof:

Case 1. $e \equiv 1 \pmod 2$. By (1), $eg - f^2 = (-1)^{k+1}$ and thus $f + g \equiv 1 \pmod 2$. It follows from (3) that $D \in \mathbf{N}$ if and only if m is even. Set $m = 2n$; then, by (3),

$$D = D(n) = e^2n^2 + [2f - (-1)^k efg] \cdot n + \left[\frac{f^2g^2}{4} - (-1)^k g^2\right]. \tag{4}$$

By the above Lemma, it is enough to show that for every prime p the following two assertions are true:

1. $E_p = \{v_p(D(x)) \,|\, x \in \mathbf{Z}\}$.
2. The congruence $D(x) \equiv 0 \pmod{p^2}$ has at most two solutions $x \pmod{p^2}$.

From (4) I obtain, by an easy calculation,

$$e^2 \cdot D(n) = \left[e^2n + f - (-1)^k \frac{efg}{2}\right]^2 - (-1)^k,$$

$$D'(n) = 2e^2n + 2f - (-1)^k efg.$$

If $p|e$, $p \neq 2$, the congruence $D(x) \equiv 0 \pmod{p^w}$ has exactly one solution $x \pmod{p^w}$ for every $w \geq 1$ and thus there are $x \in \mathbf{Z}$ with $v_p(D(x)) = w$ for every $w \geq 0$. If $p \nmid e$, $p \neq 2$, and $[(-1)^k/p] = -1$, the congruence $D(x) \equiv 0 \pmod p$ has no solution. If $p \nmid e$, $p \neq 2$, and $[(-1)^k/p] = 1$, the congruence $D(x) \equiv 0 \pmod p$ has two different solutions; these satisfy $D'(x) \not\equiv 0 \pmod p$ and, therefore, for every $w \geq 0$, there are $x \in \mathbf{Z}$ with $v_p(D(x)) = w$, and the congruence $D(x) \equiv 0 \pmod{p^2}$ also has exactly two solutions modulo p^2.

If $k \equiv 1 \pmod 2$, the congruence $D(x) \equiv 0 \pmod 4$ is unsolvable, but since $D(0) \not\equiv D(1) \pmod 2$, there are $x \in \mathbf{Z}$ with $v_2(D(x)) = w$ for $w = 0$ and $w = 1$.

If $k \equiv 0 \pmod 2$, then

$$D(n) \equiv \left(n + f + \frac{efg}{2}\right)^2 - 1 \pmod 8;$$

thus $D(x) \equiv 0 \pmod 2$ already implies $D(x) \equiv 0 \pmod 8$, the congruence $D(x) \equiv 0 \pmod 4$ has exactly two solutions $x \pmod 4$, and for every $w \geq 3$ there are $x \in \mathbf{Z}$ with $v_2(D(x)) = w$.

Case 2: $e \equiv fg \equiv 0 \pmod 2$. By (1), $eg - f^2 = (-1)^{k+1}$; thus, $k \equiv 0 \pmod 2$, $f \equiv 1 \pmod 2$, and $eg \equiv 0 \pmod 8$. It follows from (3) that $D \in \mathbf{Z}$ for all $m \in \mathbf{Z}$; therefore, I have to consider the polynomial

$$D = D(m) = \frac{e^2}{4} \cdot m^2 + \left(f - \frac{efg}{2}\right) \cdot m + \left(\frac{f^2g^2}{4} - g^2\right).$$

Again it is enough to show that for every prime p the following two assertions are true:

1. $E_p = \{v_p(D(x)) \mid x \in \mathbf{Z}\}$.
2. The congruence $D(x) \equiv 0 \pmod{p^2}$ has at most two solutions $x \pmod{p^2}$.

First, observe that

$$e^2D(m) = \left(\frac{e^2}{2} \cdot m + f - \frac{efg}{2}\right)^2 - 1.$$

If $p \neq 2$, the congruence $D(x) \equiv 0 \pmod{p}$ has at least one and at most two solutions $x \pmod{p}$, and these satisfy $D'(x) \not\equiv 0 \pmod{p}$. Therefore, for every $w \in \mathbf{N}$, there are $x \in \mathbf{Z}$ with $v_p(D(x)) = w$, and the congruence $D(x) \equiv 0 \pmod{p^2}$ has at most two solutions $x \pmod{p^2}$.

Suppose now that $e \equiv 2 \pmod{4}$ and $g \equiv 0 \pmod{4}$. Then $D(m) \equiv m^2 + fm \pmod{4}$, and it follows that $D(m) \equiv 0 \pmod{2}$ for all m, $D'(m) \equiv 1 \pmod{2}$ for all m, the congruence $D(x) \equiv 0 \pmod{4}$ has exactly two solutions $x \pmod{4}$, and for every $w \in \mathbf{N}$ there are $x \in \mathbf{Z}$ with $v_p(D(x)) = w$.

If $e \equiv 0 \pmod{4}$ or $g \equiv 2 \pmod{4}$, then the congruence $D(x) \equiv 0 \pmod{2}$ is soluble, and from $D'(x) \equiv 1 \pmod{2}$ for all x, it follows that the congruence $D(x) \equiv 0 \pmod{4}$ has at most two solutions $x \pmod{4}$ and that, for every $w \in \mathbf{N}$, there are $x \in \mathbf{Z}$ with $v_p(D(x)) = w$. □

4. In this section it will be shown that about 2/3 (resp. 5/6) of all symmetric integer sequences $(b_1, \ldots, b_{k-1})$ satisfy $\mathscr{D}(b_1, \ldots, b_{k-1}) \neq \emptyset$. To do this, define $\theta : \mathbf{Z} \to GL_2(\mathbf{F}_2)$ by

$$\theta(a) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \pmod{2};$$

for a finite sequence $(b_1, \ldots, b_m)$ define

$$\theta(b_1, \ldots, b_m) = \prod_{j=1}^{m} \theta(b_j) \in GL_2(\mathbf{F}_2).$$

Obviously, $\theta(b_1, \ldots, b_m)$ depends only on $b_1, \ldots, b_m \pmod{2}$. Put

$$\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbf{F}_2)$$

and find $\sigma^3 = \tau^2 = 1$, $\sigma\tau = \tau\sigma^2$ [as $GL_2(\mathbf{F}_2) \simeq \mathscr{S}_3$]. With these definitions, the following holds.

Theorem 3: Let $(b_1, \ldots, b_{k-1})$ $(k \geq 1)$ be a symmetric sequence of positive integers.

i) $(b_1, \ldots, b_{k-1}) \neq \emptyset$ if and only if $\theta(b_1, \ldots, b_{k-1}) \neq \sigma^2$.

ii) If k is even, $k = 2\ell$, then $\theta(b_1, \ldots, b_{k-1}) = \sigma^2$ if and only if

$$\theta(b_1, \ldots, b_{\ell-1}) \in \{\tau, \sigma^2\} \quad \text{and} \quad b_\ell \equiv 1 \pmod{2}.$$

Furthermore, if N_ℓ denotes the number of all

$$(b_1, \ldots, b_{\ell-1}) \in \{0, 1\}^{\ell-1} \text{ with } \theta(b_1, \ldots, b_{\ell-1}) \in \{\tau, \sigma^2\},$$

then

$$N_\ell = \frac{2^{\ell-1} + (-1)^\ell}{3}.$$

iii) If k is odd, $k = 2\ell + 1$, then $\theta(b_1, \ldots, b_{k-1}) = \sigma^2$ if and only if

$$\theta(b_1, \ldots, b_\ell) \in \{\sigma, \sigma\tau\}.$$

Furthermore, if N'_ℓ denotes the number of all

$\theta(b_1, \ldots, b_\ell) \in \{0, 1\}^\ell$ with $\theta(b_1, \ldots, b_\ell) \in \{\sigma, \sigma\tau\}$,

then

$$N'_\ell = N_{\ell+1}.$$

Proof: i) is an immediate consequence of Corollary 1. If $k = 2\ell$ and

$$\theta(b_1, \ldots, b_{\ell-1}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{F}_2),$$

then

$$\theta(b_1, \ldots, b_{k-1}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} b_\ell & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} ab_\ell & ab_\ell c + 1 \\ ab_\ell c + 1 & cb_\ell \end{pmatrix}$$

and thus

$$\theta(b_1, \ldots, b_{k-1}) = \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

if and only if $a = 0$, $c = b_\ell = 1$. Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{F}_2),$$

this implies also $b = 1$. Therefore, $\theta(b_1, \ldots, b_{k-1}) = \sigma^2$ if and only if

$$\theta(b_1, \ldots, b_{\ell-1}) = \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} \in \{\tau, \sigma^2\}.$$

If $k = 2\ell + 1$ and

$$\theta(b_1, \ldots, b_\ell) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{F}_2),$$

then

$$\theta(b_1, \ldots, b_{k-1}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a + b & ac + bd \\ ac + bd & c + d \end{pmatrix} = \sigma^2$$

if and only if $a = b = 1$ and $d = c + 1$, i.e.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \{\sigma, \sigma\tau\}.$$

To obtain the formulas for N_ℓ and N'_ℓ, consider the number

$$A_n(\xi) = \#\,(b_1, \ldots, b_n) \in \{0, 1\}^n \,|\, \theta(b_1, \ldots, b_n) = \xi\}$$

for any $n \in \mathbf{N}_+$ and $\xi \in GL_2(\mathbf{F}_2)$. These quantities satisfy the recursion formulas

$$A_1(\sigma) = A_1(\tau) = 1,$$
$$A_1(\xi) = 0 \text{ for all } \xi \in GL_2(\mathbf{F}_2)\backslash\{\sigma, \tau\},$$
$$A_{n+1}(\xi) = A_n(\xi\sigma^2) + A_n(\xi\tau) \text{ for all } \xi \in GL_2(\mathbf{F}_2),$$

which have the solution

$$A_n(\sigma) = A_n(\tau) = \frac{2^{n-1} + 2(-1)^{n-1}}{3},$$

$$A_n(\xi) = \frac{2^{n-1} + (-1)^n}{3} \text{ for } \xi \in GL_2(\mathbf{F}_2)\backslash\{\sigma, \tau\}.$$

Therefore, for $\ell \geq 2$,

$$N_\ell = A_{\ell-1}(\tau) + A_{\ell-1}(\sigma^2) = \frac{2^{\ell-1} + (-1)^\ell}{3},$$

$$N'_\ell = A_\ell(\sigma) + A_\ell(\sigma\tau) = \frac{2^\ell + (-1)^{\ell+1}}{3} = N_{\ell+1},$$

and these formulas remain true for $\ell = 1$. □

5. In this final section I formulate the corresponding results for the continued fraction expansion of $(1 + \sqrt{D})/2$ for $D \equiv 1 \pmod 4$; as the proofs are very similar to those for $\sqrt{D}$, I leave them to the reader. (For Theorem IA, see Satz [3; 3.34].)

Theorem 1A: Let $(b_1, \ldots, b_{k-1})$ $(k \geq 1)$ be a symmetric sequence in $\mathbf{N}_+$ and let $b_0 \in \mathbf{N}_+$. Then the following assertions are equivalent:

a) $[b_0, \overline{b_1, \ldots, b_{k-1}, 2b_0 - 1}] = \dfrac{1 + \sqrt{D}}{2}$ with $D \in \mathbf{N}_+$, $D \equiv 1 \pmod 4$.

b) $b_0 = \dfrac{1}{2} \cdot [1 + me - (-1)^k fg]$ for some $m \in \mathbf{Z}$, where e, f, and g are defined by (1).

If this condition is fulfilled, then

$$D = (2b_0 - 1)^2 + 4mf - 4 \cdot (-1)^k g^2.$$

Definition: For a symmetric sequence of positive integers $(b_1, \ldots, b_{k-1})$ $(k \geq 1)$ let $\mathscr{D}'(b_1, \ldots, b_{k-1})$ be the set of all $D \in \mathbf{N}_+$ with $D \equiv 1 \pmod 4$ and

$$\frac{1 + \sqrt{D}}{2} = [b_0, \overline{b_1, \ldots, b_{k-1}, 2b_0 - 1}] \quad \text{for some } b_0 \in \mathbf{N}_+.$$

Corollary 1A: Let $(b_1, \ldots, b_{k-1})$ be a symmetric sequence in $\mathbf{N}_+$ and define e, f, g by (1). Then the following assertions are equivalent:

a) $\mathscr{D}'(b_1, \ldots, b_{k-1}) \neq \emptyset$.

b) Either $e \equiv 1 \pmod 2$ or $e \equiv fg + 1 \equiv 0 \pmod 2$.

If b) is fulfilled, then $\mathscr{D}'(b_1, \ldots, b_{k-1})$ consists of all $D \in \mathbf{N}_+$, $D \equiv 1 \pmod 4$, which are of the form

$$D = e^2m^2 + [4f - 2 \cdot (-1)^k efg] \cdot m + [f^2g^2 - 4 \cdot (-1)^k g^2]$$

with $m \in \mathbf{Z}$ satisfying $1 + me - (-1)^k fg > 0$.

Corollary 2A: $\mathscr{D}'(1, \ldots, 1) \neq \emptyset$ (always).

Theorem 2A: Let $(b_1, \ldots, b_{k-1})$ $(k \geq 1)$ be a symmetric sequence of positive integers, define e, f, g by (1), and suppose that either $e \equiv 1 \pmod 2$ or $e \equiv fg + 1 \equiv 0 \pmod 2$. Let P' be the set of all odd primes p with $p \nmid e$ and

$$\left(\frac{(-1)^k}{p}\right) = -1.$$

i) If $D \in \mathscr{D}'(b_1, \ldots, b_{k-1})$ and $p \in P'$, then $p \nmid D$.

ii) Let P be a finite set of odd primes, $P \cap P' = \emptyset$ and $(w_p)_{p \in P}$ a sequence in $\mathbf{N}$. Then there are infinitely many $D \in \mathscr{D}'(b_1, \ldots, b_{k-1})$ such that $v_p(D) = w_p$ for all $p \in P$ and $v_p(D) \leq 1$ for all primes $p \notin P$.

Theorem 3A: Let $(b_1, \ldots, b_{k-1})$ $(k \geq 1)$ be a symmetric sequence of positive integers. Then $\mathscr{D}'(b_1, \ldots, b_{k-1}) = \emptyset$ if and only if k is even, $k = 2\ell$, and $b_\ell \equiv 0 \pmod 2$.

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ON DETERMINANTS WHOSE ELEMENTS ARE RECURRING SEQUENCES OF ARBITRARY ORDER

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Some years ago, Carlitz [1] and Zeitlin [2] calculated determinants of the form $|w^r_{a+k(i+j)}|$ $(i,\ j = 0, 1, \ldots, r)$, where $\{w_n\}$ is a second-order recurring sequence. More generally, the aim of this paper is to obtain a closed form for the $s \times s$ determinant

$$(1)\qquad \Delta_w\left[\begin{matrix} i_1, & \ldots, & i_r \\ j_1, & \ldots, & j_r \end{matrix}\middle| a\right] = \begin{vmatrix} w_a, & w_{a+j_1}, & \ldots, & w_{a+j_r} \\ w_{a+i_1}, & w_{a+i_1+j_1}, & \ldots, & w_{a+i_1+j_r} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ w_{a+i_r}, & w_{a+i_r+j_1}, & \ldots, & w_{a+i_r+j_r} \end{vmatrix},$$

where $s = r + 1$ and $a, i_1, \ldots, i_r, j_1, \ldots, j_r$ are integers, when $\{w_n\}$ satisfies the recurrence of order s,

$$(2)\qquad w_n = \sum_{k=1}^{s} (-1)^{k-1}\sigma_k w_{n-k}, \quad n \in \mathbf{Z},$$

where $\sigma_1, \sigma_2, \ldots, \sigma_s$ are complex numbers, with $\sigma_s \neq 0$.

We shall often write $\Delta^{i_1, i_2, \ldots, i_r}_{j_1, j_2, \ldots, j_r}$ instead of $\Delta_w\left[\begin{matrix} i_1, & \ldots, & i_r \\ j_1, & \ldots, & j_r \end{matrix}\middle| a\right]$.

We want to obtain an expression of Δ_w in terms of the Fibonacci solution $\{u^{(s)}_n\}$ of (2), whose initial conditions are:

$$(3)\qquad u^{(s)}_0 = u^{(s)}_1 = \cdots = u^{(s)}_{r-1} = 0; \quad u^{(s)}_r = 1.$$

We define the characteristic number e_w of the sequence $\{w_n\}$ by

$$(4)\qquad e_w = \Delta_w\left[\begin{matrix} 1, & 2, & \ldots, & r \\ 1, & 2, & \ldots, & r \end{matrix}\middle| 0\right] = |w_{i+j}| \quad (i,\ j = 0, 1, \ldots, r).$$

Note that, for the Fibonacci sequence $\{u^{(s)}_n\}$, we have, by (3) and (4),

$$e_{u^{(s)}} = (-1)^{\frac{r(r+1)}{2}} = (-1)^{\frac{rs}{2}}.$$

1. A Particular Case

In this section we assume that the characteristic polynomial of (2) admits distinct roots $\alpha_1, \ldots, \alpha_s$, and that α_i/α_j is not a root of unity, for distinct i and j. In that case, there exist complex numbers $C_1, \ldots, C_s$, such that

$$w_n = \sum_{i=1}^{s} C_i \alpha_i^n, \quad n \in \mathbf{Z}.$$

Notice also that

$$\sigma_s = \prod_{i=1}^{s} \alpha_i.$$

The statement of the main result of this section is

Theorem I: $$\Delta_w\begin{bmatrix} k, & 2k, & \ldots, & rk \\ k, & 2k, & \ldots, & rk \end{bmatrix}\!\!\left.\vphantom{\begin{matrix}k\\k\end{matrix}}\right| a \Big] = C_1 \ldots C_s\, \sigma_s^a\, V(\alpha_1^k, \ldots, \alpha_s^k)^2 = e_w\, \sigma_s^a \frac{V(\alpha_1^k, \ldots, \alpha_s^k)^2}{V(\alpha_1, \ldots, \alpha_s)^2},$$

where $V(\alpha_1, \ldots, \alpha_s) = \prod_{i>j}(\alpha_i - \alpha_j)$ is the Vandermonde determinant.

The proof will require the following result.

Lemma I: $e_w = C_1 \ldots C_s V(\alpha_1, \ldots, \alpha_s)^2$.

Proof: From the equality between matrices

$$[w_{i+j}] = [C_{j+1}\alpha_{j+1}^i][\alpha_{i+1}^j] \quad (i, j = 0, 1, \ldots, r),$$

and passing to determinants, we obtain

$$\begin{aligned} e_w &= |C_{j+1}\alpha_{j+1}^i|\,|\alpha_{i+1}^j| \quad (i, j = 0, 1, \ldots, r) \\ &= C_1 \ldots C_s |\alpha_{j+1}^i|^2 \\ &= C_1 \ldots C_s V(\alpha_1, \ldots, \alpha_s)^2. \quad \text{Q.E.D.} \end{aligned}$$

Proof of Theorem I: Let us consider the sequence $\{w_n'\}$, with $w_n' = w_{a+kn}$. Then we have

$$w_n' = \sum_{i=1}^{s} C_i \alpha_i^a (\alpha_i^k)^n, \tag{5}$$

and, since the α_i^k are distinct, $\{w_n'\}$ satisfies a recurrence

$$w_n' = \sum_{m=1}^{s} (-1)^{m-1} \sigma_m' w_{n-m}',$$

with

$$\sigma_m' = \sum_{1 \le i_1 < \cdots < i_m \le s} \alpha_{i_1}^k \ldots \alpha_{i_m}^k.$$

Clearly we have, with the above notations,

$$\Delta_w\left[\begin{matrix} k, & 2k, & \ldots, & rk \\ k, & 2k, & \ldots, & rk \end{matrix}\,\middle|\, a\right] = \Delta_{w'}\left[\begin{matrix} 1, & 2, & \ldots, & r \\ 1, & 2, & \ldots, & r \end{matrix}\,\middle|\, 0\right] = e_{w'}.$$

However, by Lemma I and (5), we have

$$\begin{aligned} e_{w'} &= \left[\prod_{i=1}^{s} C_i \alpha_i^a\right] V(\alpha_1^k, \ldots, \alpha_s^k)^2 \\ &= C_1 \ldots C_s \sigma_s^a V(\alpha_1^k, \ldots, \alpha_s^k)^2 = e_w \sigma_s^a \frac{V(\alpha_1^k, \ldots, \alpha_s^k)^2}{V(\alpha_1, \ldots, \alpha_s)^2}. \end{aligned}$$

Applications:

(i) Put $a = n - rk$ in the formula of Theorem I to get

$$\Delta_w\left[\begin{matrix} k, & 2k, & \ldots, & rk \\ k, & 2k, & \ldots, & rk \end{matrix}\,\middle|\, n - rk\right] = C_1 \ldots C_s \sigma_s^{n-rk} V(\alpha_1^k, \ldots, \alpha_s^k)^2 = e_w \sigma_s^{n-rk} \frac{V(\alpha_1^k, \ldots, \alpha_s^k)^2}{V(\alpha_1, \ldots, \alpha_s)^2}. \tag{6}$$

In the case $s = 2$, we obtain

$$\begin{aligned} w_{n-k}w_{n+k} - w_n^2 &= C_1C_2\sigma_2^{n-k}(\alpha_1^k - \alpha_2^k)^2 = e_w\sigma_2^{n-k}\frac{(\alpha_2^k - \alpha_1^k)^2}{(\alpha_2 - \alpha_1)^2} \\ &= e_w\sigma_2^{n-k}(u_k^{(2)})^2, \end{aligned}$$

which is the well-known Catalan relation; thus, (6) is a generalization of this result.

(ii) We can also study the sequence $\{w_n^r\}$, where $\{w_n\}$ satisfies the second-order recurrence

$$w_n = pw_{n-1} - qw_{n-2},$$

whence

$$w_n = C_1\alpha_1^n + C_2\alpha_2^n.$$

Assuming that α_1/α_2 is not a root of unity, we get

$$w_n^r = \sum_{i=0}^{r}\binom{r}{i}C_1^iC_2^{r-i}(\alpha_1^i\alpha_2^{r-i})^n, \tag{7}$$

where the $\alpha_1^i\alpha_2^{r-i}$ are distinct. Hence, $\{w_n^r\}$ satisfies a recurrence of type (2), with

$$\sigma_s = \prod_{i=0}^{r}\alpha_1^i\alpha_2^{r-i} = (\alpha_1\alpha_2)^{\frac{rs}{2}} = q^{\frac{rs}{2}}. \tag{8}$$

By application of Theorem I, we obtain a new proof of a known result (see [1], [2]).

Corollary I: $|w^r_{a+k(i+j)}| \qquad (i,\ j = 0,\ \ldots,\ r)$

$$= e_w^{\frac{rs}{2}}\, q^{\frac{ars}{2}+\frac{kr(r^2-1)}{3}}\sum_{i=0}^{r}\binom{r}{i}\sum_{i=1}^{r}(u_{ki}^{(2)})^2.$$

Proof: By Theorem I, (7), and (8), we get

$$\begin{aligned}
|w^r_{a+k(i+j)}| &= \Delta_{w^r}\left[\begin{matrix}k, & 2k, & \ldots, & rk\\ k, & 2k, & \ldots, & rk\end{matrix}\middle| a\right] \\
&= \prod_{i=0}^{r}\binom{r}{i}C_1^iC_2^{r-i}\cdot q^{\frac{ars}{2}}\cdot V(\alpha_2^r,\ \alpha_1\alpha_2^{r-1},\ \ldots,\ \alpha_1^r)^2 \\
&= \prod_{i=0}^{r}\binom{r}{i}\cdot (C_1C_2)^{\frac{rs}{2}}\cdot V(\alpha_2^r,\ \alpha_1\alpha_2^{r-1},\ \ldots,\ \alpha_1^r)^2,
\end{aligned}\tag{9}$$

and it can be shown (see [1], p. 130) that the value of the Vandermonde determinant is

$$(\alpha_1 - \alpha_2)^{\frac{rs}{2}}\, q^{\frac{kr(r^2-1)}{6}}\prod_{i=1}^{r}(u_{ki}^{(2)})^{r-i+1}. \tag{10}$$

The result follows now from (9) and (10) since, by Lemma I,

$$e_w = C_1C_2(\alpha_1 - \alpha_2)^2.$$

2. The General Results

In what follows, we do not make any assumption about the roots of the characteristic equation, and we put again $s = r + 1$. In this section we shall prove the following theorem.

Theorem II: Let $\{w_n\}$ be any solution of the recurrence (2). For all integers $a,\ i_1,\ i_2,\ \ldots,\ i_r,\ j_1,\ j_2,\ \ldots,\ j_r$, we have

$$\Delta_w\left[\begin{matrix}i_1, & \ldots, & i_r\\ j_1, & \ldots, & j_r\end{matrix}\middle| a\right] = \sigma_s^a\, e_w\delta_{i_1,\ldots,i_r}\delta_{j_1,\ldots,j_r}, \tag{11}$$

where $\delta_{i_1, \ldots, i_r}$ is the $r \times r$ determinant

$$\delta_{i_1, \ldots, i_r} = |u^{(s)}_{i_p+q-1}|, \quad (p,\ q = 1,\ 2,\ \ldots,\ r).$$

From Theorem II, we get a corollary which can be compared with (6).

Corollary II (Catalan's relation): For all integers n and k, we have

$$\Delta_w\begin{bmatrix} k, & 2k, & \ldots, & rk \\ k, & 2k, & \ldots, & rk \end{bmatrix}\!\!\!\!\begin{matrix} \\ n - rk \end{matrix}\Big] = \sigma_s^{n-rk} e_w \delta^2_{k, 2k, \ldots, rk}. \tag{12}$$

Proof: Put $a = n - rk$, $j_m = i_m = mk$, $1 \le m \le r$, in the general formula (11).

For example, in the case $s = 2$, (12) becomes

$$w_{n-k}w_{n+k} - w_n^2 = \sigma_2^{n-k}(u_k^{(2)})^2,$$

and, in the case $s = 3$,

$$\begin{vmatrix} w_{n-2k} & w_{n-k} & w_{n+k} \\ w_{n-k} & w_n & w_{n+k} \\ w_n & w_{n+k} & w_{n+2k} \end{vmatrix} = \sigma_3^{n-2k} e_w \begin{vmatrix} u_k^{(3)} & u_{2k}^{(3)} \\ u_{k+1}^{(3)} & u_{2k+1}^{(3)} \end{vmatrix}^2$$

3. Proof of Theorem II

We shall need the following results.

Lemma II:

(i) For all integers $i_1, \ldots, i_r, j_1, \ldots, j_r$,

$$\Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_r} = \Delta^{j_1, \ldots, j_r}_{i_1, \ldots, i_r}.$$

(ii) For all integers $i_1, \ldots, i_r, j_1, \ldots, j_r$, and all $1 \le p \le r$, we have

$$\Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_p, \ldots, j_r} = \sum_{k=1}^{s} (-1)^{k-1}\sigma_k \Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_p - k, \ldots, j_r},$$

and

$$\delta_{i_1, \ldots, i_p, \ldots, i_r} = \sum_{k=1}^{s} (-1)^{k-1}\sigma_k \delta_{i_1, \ldots, i_p - k, \ldots, i_r}.$$

(iii) If τ is a permutation of $\{1, 2, \ldots, r\}$ of sign $\varepsilon(\tau)$, then for all integers $i_1, \ldots, i_r, j_1, \ldots, j_r$,

$$\Delta^{i_1, \ldots, i_r}_{j_{\tau(1)}, \ldots, j_{\tau(r)}} = \varepsilon(\tau)\Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_r},$$

and

$$\delta_{j_{\tau(1)}, \ldots, j_{\tau(r)}} = \varepsilon(\tau)\delta_{j_1, \ldots, j_r}.$$

(iv) If $j_k = j_\ell$ for distinct k and ℓ or if there exists k such that $j_k = 0$, then

$$\Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_r} = \delta_{j_1, \ldots, j_r} = 0.$$

Proof: This is an immediate consequence of the properties of determinants.

Lemma III: Let us consider two sequences $\{X_n\}$ and $\{Y_n\}$, with $n = (n_1, \ldots, n_t) \in \mathbb{Z}^t$, such that, for all $n \in \mathbb{Z}^t$, and all $1 \le p \le t$,

$$X_{n_1, \ldots, n_p, \ldots, n_t} = \sum_{k=1}^{s} (-1)^{k-1}\sigma_k X_{n_1, \ldots, n_p - k, \ldots, n_t}, \tag{13}$$

and

$$Y_{n_1, \ldots, n_p, \ldots, n_t} = \sum_{k=1}^{s} (-1)^{k-1}\sigma_k Y_{n_1, \ldots, n_p - k, \ldots, n_t}. \tag{14}$$

If $X_n = Y_n$ holds for all n belonging to

(15) $$C_t = \{n \in \mathbf{Z}^t,\ 0 \le n_p \le r,\ 1 \le p \le t\},$$

then

(16) $$X_n = Y_n \text{ holds for all } n \in \mathbf{Z}^t.$$

Proof: By induction on t. The statement is well known for $t = 1$. Let us suppose that (16) holds up to a certain $t \ge 1$. For the inductive step $t \to t + 1$, fix an integer m and consider the sequences $\{x_n^{(m)}\}$ and $\{y_n^{(m)}\}$, with $n = (n_1, \ldots, n_t)$ defined by

$$x_n^{(m)} = X_{n_1, \ldots, n_t, m} \quad \text{and} \quad y_n^{(m)} = Y_{n_1, \ldots, n_t, m}.$$

By definition, $x_n^{(m)} = y_n^{(m)}$ holds for all $n \in C_t$ and all $0 \le m \le r$, and by the induction hypothesis,

$$x_n^{(m)} = y_n^{(m)} \text{ for } n \in \mathbf{Z}^t \text{ and } 0 \le m \le r.$$

Now, fix $n \in \mathbf{Z}^t$ and consider the sequences x'_m and y'_m, defined by

$$x'_m = X_{n_1, \ldots, n_t, m} \quad \text{and} \quad y'_m = Y_{n_1, \ldots, n_t, m}.$$

We have $x'_m = y'_m$ for $0 \le m \le r$, and the same equality holds for all integers m, since by (13) $\{x'_m\}$ and $\{y'_m\}$ satisfy a recurrence relation of order s. This concludes the proof of Lemma 3.

Proof of Theorem 2:

Step 1: We prove that, for all integers $i_1, \ldots, i_r, j_1, \ldots, j_r$,

(17) $$\Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_r} = \Delta^{i_1, \ldots, i_r}_{1, 2, \ldots, r} (-1)^{\frac{r(r-1)}{2}} \delta_{j_1, \ldots, j_r}.$$

Let us fix $i_1, \ldots, i_r$. By Lemma 2(ii) and Lemma 3, it suffices to show that (17) holds for $j_1, \ldots, j_r$ belonging to the set

$$C_r = \{(j_1, \ldots, j_r) \in \mathbf{Z}^r,\ 0 \le j \le r,\ 1 \le p \le r\}.$$

If one of the conditions of Lemma 2(iv) is satisfied, then (17) clearly holds. Therefore, we have only to consider the case where $(j_1, \ldots, j_r)$ is a permutation of $(1, 2, \ldots, r)$. By a direct calculation,

$$\delta_{1, \ldots, r} = (-1)^{\frac{r(r-1)}{2}},$$

whence (17) holds for $(j_1, \ldots, j_r) = (1, 2, \ldots, r)$, and by Lemma 2(iii), the equality holds for every permutation of $(1, 2, \ldots, r)$.

Step 2: By Lemma 2(i) and Step 1, the following statement holds:

$$\Delta^{i_1, \ldots, i_r}_{1, 2, \ldots, r} = \Delta^{1, 2, \ldots, r}_{i_1, \ldots, i_r} = \Delta^{1, 2, \ldots, r}_{1, 2, \ldots, r} (-1)^{\frac{r(r-1)}{2}} \delta_{i_1, \ldots, i_r}.$$

Hence, (17) becomes

(18) $$\Delta^{i_1, \ldots, i_r}_{j_1, \ldots, j_r} = \Delta^{1, 2, \ldots, r}_{1, 2, \ldots, r} \delta_{i_1, \ldots, i_r} \delta_{j_1, \ldots, j_r}.$$

Now, it is known (see [3], p. 99) that

$$\Delta^{1, 2, \ldots, r}_{1, 2, \ldots, r} = \delta_s^a e_w.$$

By this and (18), the proof is complete.

For a second-order recurring sequence, (11) becomes

$$w_a w_{a+i+j} - w_{a+i} w_{a+j} = \sigma_2^a e_w u_i^{(2)} u_j^{(2)}.$$

When giving particular values to a, i, and j, one can deduce from this some well-known identities.

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1. L. Carlitz. "Some Determinants Containing Powers of Fibonacci Numbers." *Fibonacci Quarterly* 4.2 (1966):129-34.
2. D. Zeitlin. "On Determinants Whose Elements Are Products of Recursive Sequences." *Fibonacci Quarterly* 8.4 (1970):350-59.
3. D. Jarden. *Recurring Sequences*. Jerusalem: Riveon Lematematika, 1973.

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Announcement

FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 20-24, 1992

Department of Mathematical and Computational Sciences
University of St. Andrews
St. Andrews KY16 9SS
Fife, Scotland

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CALL FOR PAPERS

The FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of St. Andrews, St. Andrews, Scotland from July 20 to July 24, 1992. This Conference is sponsored jointly by the Fibonacci Association and The University of St. Andrews.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts are to be submitted by March 15, 1992. Manuscripts are due by May 30, 1992. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

Professor Gerald E. Bergum
The Fibonacci Quarterly
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PERIODIC FIBONACCI AND LUCAS SEQUENCES

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(Submitted October 1989)

1. Introduction

In the early thirteenth century there appeared the book *Liber Abaci* by the mathematician Leonardo of Pisa [7], who also became known as Fibonacci (see also [2]). In it a problem concerning an ideal case of the reproduction of rabbits is treated, and the sequence

(1) $F = 1, 2, 3, 5, 8, \ldots$

is introduced. This sequence has since become known as the *Fibonacci Sequence*. One of its features is the recurrence relation

(2) $a_n = a_{n-1} + a_{n-2}$, for $n \geq 3$.

In the second half of the nineteenth century E. Lucas [8], who had actually coined the term *Fibonacci Numbers*, introduced a similar sequence connected closely to that of Fibonacci,

(3) $L = 1, 3, 4, 7, 11, \ldots,$

obeying the same recurrence relation as F. The sequence L has since become known as the *Lucas Sequence* [3] (see also [4]).

Since then the *generalized* sequences of both kinds have been introduced. For both, the recurrence relation is

$$a_n = aa_{n-1} + \sigma a_{n-2},$$

where a and σ are prescribed numbers.

We shall also stipulate $a_0 = 1$ or 2 according to whether the sequence is a generalized F or a generalized L, respectively. The recurrence relation holds already for $n = 2$ (see also [3]). In [10] Wall treated generalized Fibonacci sequences modulo an integer m and showed that some are periodic mod (m) (see also [6], [11], and [12]).

Now let a and σ be two arbitrary complex numbers and let the terms of the generalized Fibonacci (Lucas) sequence be $f_0 = 1$, $f_1 = a$ ($g_0 = 2$, $g_1 = a$). It turns out that in some cases such sequences are periodic. Put, for example, $a = 1$, $\sigma = -1$. Then both sequences are periodic of period 6.

In this paper we wish to characterize those sequences which are periodic; in other words, to specify precisely for which ordered pair (a, σ) the corresponding Fibonacci (Lucas) sequence is periodic. We shall also specify in each relevant case the period T, T being the least positive integer for which $a_{n+T} = a_n$ for every n.

Let us first look at degenerate cases. The case $a = \sigma = 0$ is trivial with $T = 0$. If just one of the two vanishes, the remaining parameter is necessarily a root of unity, a trivial case being $a = 1$, $\sigma = 0$, $T = 1$.

We may, therefore, assume both parameters to be nonzero.

2. Periodic Row-Column Matrices

Let $n > 1$ be a positive integer. Consider an $n \times n$-matrix $A = (a_{ij})$ over the complex field with $a_{ij} = 0$ if both i and j are greater than one. Put

$$a_{11} = a, \quad \sum_{j=2}^{n} a_{1j}a_{j1} = \sigma.$$

We shall name such a matrix a (one-row)-(one-column) matrix or, in short, an RCM.

The characteristic polynomial of A is $\lambda^n - a\lambda^{n-1} + \sigma\lambda^{n-2}$ so that the two nonzero eigenvalues of A satisfy the quadratic equation

(4) $\quad \lambda^2 - a\lambda - \sigma = 0$

whose roots are

$$\lambda_{1,2} = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 + \sigma}.$$

It follows that for $n \geq 2$ the spectrum of A depends solely on a and σ and is independent of n.

For $\sigma = a^2/4$, the matrix A is neither diagonalizable nor periodic for any nonzero value of a.

The polynomial $f(z) = z^2 - az - \sigma$ appears in a paper by M. Ward [11], among others. Ward also considers what he calls *degenerate* sequences in which zeros appear periodically, with periods 2, 3, 4, and 6, although the sequences as such are not periodic (see, e.g., [11, Th. 3]).

Except for the case $\sigma = -a^2/4$, the two nonvanishing eigenvalues of A are distinct. In addition, we have rank $A = 2$, and hence, A is diagonalizable. For $i = 1, 2$, we have

(5) $\quad \lambda_i^2 = a\lambda_i + \sigma,$

(6) $\quad \lambda_1 + \lambda_2 = a.$

Let j be a positive integer. Define

$$\gamma_j = \operatorname{Tr} A^j.$$

We have

$$\gamma_1 = a.$$

$$\gamma_2 = \lambda_1^2 + \lambda_2^2 = a\lambda_1 + \sigma + a\lambda_2 + \sigma = a^2 + 2\sigma.$$

Also, for $j \geq 3$, equalities (1) and (2) imply

(7) $$\gamma_j = \lambda_1^j + \lambda_2^j = \lambda_1^{j-2}\lambda_1^2 + \lambda_2^{j-2}\lambda_2^2 = a\lambda_1^{j-1} + \sigma\lambda_1^{j-2} + a\lambda_2^{j-1} + \sigma\lambda_2^{j-2}$$
$$= a\gamma_{j-1} + \sigma\gamma_{j-2}.$$

We thus have a recurrence formula for γ_j, $j \geq 3$, displaying a generalized Fibonacci sequence. We now turn to the possible periodicity of an RCM. A necessary condition for A to be periodic is $|\lambda_1| = |\lambda_2| = 1$. It also follows that A is periodic if and only if γ_k is periodic.

Putting

$$\sqrt{\frac{a^2 + \sigma}{4}} = w,$$

we have

$$\lambda_1 = \frac{a}{2} + w, \quad \lambda_2 = \frac{a}{2} - w.$$

For both λ_1 and λ_2 to be on the unit circle, it is necessary that

$$|w| = \sqrt{1 - \frac{|a|^2}{4}} \quad \text{and} \quad \arg w = \arg a \pm \frac{\pi}{2}.$$

Set $\arg a = \phi$ and $\arg \lambda_1 - \phi = \psi$. Then $\arg \lambda_2 = \arg \lambda_1 - 2\psi$, so that

$$\arg \lambda_1 = a + \psi \quad \text{and} \quad \arg \lambda_2 = a = \psi \quad \text{(see Fig. 1).}$$

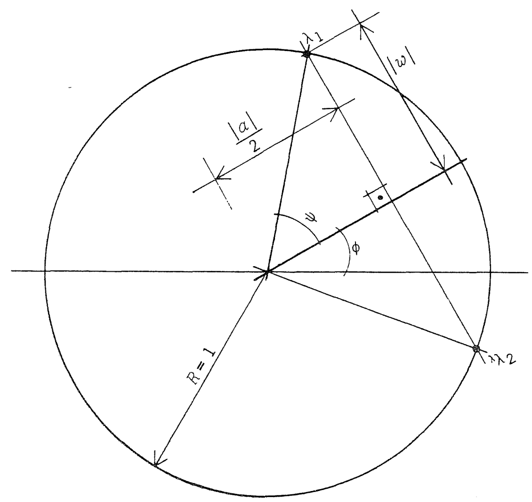

FIGURE 1

Then

$$\tan\psi = \frac{\sqrt{1 - \frac{|a|^2}{4}}}{\frac{|a|}{2}} = \sqrt{\frac{4}{|a|^2} - 1}.$$

Now set

$$\pm\psi + \phi = \arctan\left(\pm\sqrt{\frac{4}{|a|^2} - 1}\right) + \arg a = \frac{2\pi}{\rho_i}, \tag{8}$$

where $i = 1$ for the plus sign and $i = 2$ for the minus sign. A necessary and sufficient condition for A to be periodic is that both λ_1 and λ_2 be roots of unity. We also find that equation (4) implies

$$\sigma = \lambda^2 - a\lambda = \lambda(\lambda - a) = \frac{a}{2} \pm \left(i\sqrt{1 - \frac{|a|^2}{4}}e^{i\phi}\right)\left(-\frac{a}{2} \pm i\sqrt{1 - \frac{|a|^2}{4}}e^{i\phi}\right)$$

$$= \frac{a^2}{4} - \left(1 - \frac{|a|^2}{4}\right)e^{2i\phi} = \frac{|a|^2}{4}e^{2i\phi} - \frac{a^2}{4} - e^{2i\phi} = -e^{2i\phi}.$$

We thus have

Theorem 1: Let A be an RCM. Then A is periodic if and only if

(i) for both choices ($\pm$) we have $\pi^{-1}\left(\arg a \pm \arctan\sqrt{\frac{4}{|a|^2} - 1}\right)$ are rational;

(ii) $\sigma = -e^{2i \arg a}$.

Corollary 1: Let A be an RCM. Then A is periodic if and only if the following three conditions hold.

(i) $\pi^{-1} \arg a$ is rational;

(ii) $\pi^{-1} \arctan\sqrt{\frac{4}{|a|^2} - 1}$ is rational;

(iii) $\sigma = -e^{2i \arg a}$.

Corollary 2: Let A be a real RCM. Then A is periodic if and only if

$$\pi^{-1} \arctan\sqrt{\frac{4}{a^2} - 1} \text{ is rational and } \sigma = -1.$$

Corollary 3: A real RCM is periodic if and only if

$$\pi^{-1} \text{ arc tan} \sqrt{\frac{4}{a^2} - 1} \quad \text{and} \quad \sigma = -1.$$

Corollary 4: Let A be a purely imaginary RCM. Then A is periodic if and only if

$$\pi^{-1} \text{ arc tan} \sqrt{-\frac{4}{a^2} - 1} \text{ is rational} \quad \text{and} \quad \sigma = 1.$$

Corollary 5: A necessary condition for an RCM to be periodic is that a satisfy the inequality $0 < |a| < 2$.

Corollary 6: A necessary condition for an RCM to be periodic is $|\sigma| = 1$.

Let us now seek the period $T = T(A)$. It will clearly be the least integral for which both $T(\phi + \psi)$ and $T(\phi - \psi)$ are integral multiples of 2π. Put

$$\phi + \psi = \frac{2\pi}{\rho_1}, \quad \phi - \psi = \frac{2\pi}{\rho_2}.$$

For $i = 1, 2$, the ρ_i are necessarily rational, so that we may put

$$\rho_i = \frac{m_i}{n_i}, \text{ with } (m_i, n_i) = 1.$$

We then have

Theorem 2: Let A be a given periodic RCM. Then the period $T(A)$ is given by the formulas $T(A) = \text{L.C.M.}(m_1, m_2)$ where the m_i are defined as above.

We also have, for a periodic RCM, $(|a|/2) = \cos \psi$, so that we may write

$$a = 2 \cos \psi e^{i\phi}. \tag{9}$$

We may also write $\lambda_1 = e^{i(\phi+\psi)}$, $\lambda_2 = e^{i(\phi-\psi)}$, so that

$$\lambda_1 + \lambda_2 = e^{i\phi}(e^{i\psi} + e^{-i\psi}) = 2 \cos \psi e^{i\phi}.$$

Then it is easy to see that $\lambda_1^k = e^{ki(\phi+\psi)}$, $\lambda_2^k = e^{ki(\phi-\psi)}$ so that, likewise,

$$\gamma_k = \lambda_1^k + \lambda_2^k = 2 \cos(k\psi)e^{ki\phi},$$

thus proving that A is periodic if and only if the traces of the powers of A are periodic. We then have

Corollary 7: Let A be a periodic RCM with $a = 1$. Then A has period 6.

Proof: We have $\phi = 0$ and $\cos \psi = 1/2$, so that $\psi = \pi/3$. The result follows.

Let us consider two examples.

Example 1: Let $\phi = \frac{\pi}{20}$, $\psi = \frac{13}{60}\pi$. Then

$$a = 2 \cos \frac{13}{60} \pi e^{\frac{\pi i}{20}}, \quad \sigma = -e^{\frac{\pi i}{10}}.$$

We also have $\phi + \psi = \frac{4}{15}\pi$, $\phi - \psi = -\pi/6$, so that $m_1 = 15$, $m_2 = 12$, and hence,

$$T = \text{L.C.M.}(15, 12) = 60.$$

Example 2: Let $a = e^{\pi i/3}$. Then $= -e^{2\pi i/3}$. Also $\cos \psi = 1/2$ so that $\phi = \psi = \pi/3$; hence, $\phi + \psi = 2\pi/3$, $\phi - \psi = 2\pi$, $m_1 = 3$, $m_2 = 1$, and so $T = 3$.

3. The Leading Element of a Power of an RCM

Let A be an RCM. Put $A = (a_{ij})$. Let $a_{ij}^{(k)}$ denote the (i, j)-element of A^k. We consider $a_{11}^{(k)}$ for $k > 1$. Put $a_{ij} = \alpha_j$, $a_{i1} = \beta_i$. We then have $a_{11}^{(2)} = a^2 + \sigma$.

For $i \neq 1 \neq j$, we have

$$a_{1j}^{(2)} = a\alpha_j,\ a_{i1}^{(2)} = a\beta_i,\ a_{ij}^{(2)} = \beta_i\alpha_j,$$
$$a_{11}^{(3)} = a^3 + 2a\sigma,\ a_{1j}^{(3)} = (a^2 + \sigma)\alpha_j$$
$$a_{i1}^{(3)} = (a^2 + \sigma)\beta_i,\ a_{ij}^{(3)} = a\beta_i\alpha_j.$$

Put $f_0 = 1$, $f_1 = a$, $f_2 = a^2 + \sigma$.

Suppose that for some k we have

$$a_{11}^{(k)} = f_k,\ a_{1j}^{(k)} = \alpha_j f_{k-1}, \tag{10}$$
$$a_{i1}^{(k)} = \beta_i f_{k-1},\ a_{ij}^{(k)} = \beta_i\alpha_j f_{k-2} \quad \text{for } i \neq 1 \neq j.$$

Then

$$a_{11}^{(k+1)} = af_k + \sigma f_{k-1} = f_{k+1},$$
$$a_{1}^{(k+1)} = \alpha_j(af_{k-1} + \sigma f_{k-2}) = \alpha_j f_k,$$
$$a_{i1}^{(k+1)} = \beta_i f_k,$$
$$a_{ij}^{(k+1)} = \beta_i\alpha_j f_{k-1}.$$

We may use induction since 10 holds for $k = 2$. We thus have

Lemma 1: Let A be an RCM. Then equalities (10) hold for every $i, j > 1$ and for $k \geq 2$.

We thus obtain

Theorem 3: Let A be an RCM. Then the leading elements and the traces of the successive powers of A form a generalized Fibonacci sequence and a generalized Lucas sequence.

For $a = \sigma = 1$ we obtain the original Fibonacci and Lucas sequences appearing in (1) and (2). We may therefore look at RCM's as generating Fibonacci and Lucas sequences. A particular such case has already been treated in [5] and also in [1].

We may now combine the two aspects of RCM's, namely, periodicity on the one hand, and Fibonacci sequences on the other in order to draw the following conclusion.

Theorem 4: A generalized Fibonacci (Lucas) sequence with complex parameters a and σ is periodic if and only if both

$$\pi^{-1} \text{ arc tan}\sqrt{\frac{4}{|a|^2} - 1} \quad \text{and} \quad \pi^{-1} \arg a$$

are rational and $\sigma = -e^{2i \arg a}$.

Corollary 8: A generalized Fibonacci (Lucas) sequence with real parameter a is periodic if and only if

$$\pi^{-1} \text{ arc tan}\sqrt{\frac{4}{a^2} - 1}$$

is rational and $\sigma = -1$. The period T is determined as prescribed by Theorem 2.

Let $n \geq 2$ be an integer. Consider a generalized Fibonacci or Lucas sequence for which the parameters ϕ and ψ are $\phi = \psi = \pi/n$. Then

$$\phi + \psi = \frac{2\pi}{n},\ \phi - \psi = 2\pi$$

so that

$$a = 2\cos\frac{\pi}{n}e^{\frac{\pi i}{n}},\ \sigma = -e^{\frac{-2\pi i}{n}};$$

so we get a periodic sequence of period n. We may thus state

Corollary 9: Every positive integer ≥ 2 is a period for some generalized Fibonacci (Lucas) sequence.

For $n = 2$, we have to stipulate $a = 0$, $\sigma = 1$, since $\phi = \psi = \pi/2$. We may also state

Corollary 10: Every positive integer is a period for some RCM.

For $n = 1$ choose $a = 1$, $\sigma = 0$. The generalized Fibonacci sequence with parameters a and σ suggest that the traces γ_k be polynomials in a, σ of degree k, so that

$$\gamma_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \phi_{kj} a^{k-2j} \sigma^j.$$

The coefficients ϕ_{kj} may be established by graph-theoretical counting techniques. Induction may also be used to show that

$$\phi_{kj} = \binom{k-j}{j} + \binom{k-j-1}{j-1} = k\frac{(k-j-1)!}{j!(k-2j)!}.$$

The verification is left to the reader.

A similar formula may be found in [9].

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A NEW NUMERICAL TRIANGLE SHOWING LINKS WITH FIBONACCI NUMBERS

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0. Introduction

In the study of electrical networks, it is well known that the coefficients of the polynomial which characterizes the transfer function (ratio of output to input voltage) of a ladder network formed by a cascade of N identical uncoupled elementary cells belong to the $(N+1)^{\text{th}}$ row of Pascal's triangle. This circumstance allows us a direct and fast determination of the transfer function of the entire ladder network.

On the other hand, in the case of direct coupling among interacting elementary cells forming a ladder network, the polynomial coefficients are not those belonging to Pascal's triangle, but rather to another triangle named the "DFF triangle" from the initials of the authors who first dealt with it (see [3], [4]).

The DFF triangle also provides a noteworthy interest from the mathematical point of view, because some of its properties are connected with Fibonacci numbers.

1. The Generating Polynomials

The DFF triangle can be formed in the following manner ($a_{n,k}$ being the general coefficient).

We define (see [3], [4]):

(1.1) $\quad a_{n,k} = 0 \quad$ if $n < k$,

(1.2) $\quad a_{n,k} = 1 \quad$ if $n = k$, $k = 0$,

while the other elements of the triangle can be derived from the recursive formula

(1.3) $$a_{n,k} = a_{n-1,k} + \sum_{\alpha=0}^{n-1} a_{\alpha,k-1} \quad \text{if } n > k.$$

In this manner we have the DFF triangle for values of $a_{n,k}$:

$n \backslash k$	0	1	2	3	4	5	6	7 ...
0	1							
1	1	1						
2	1	3	1					
3	1	6	5	1				
4	1	10	15	7	1			
5	1	15	35	28	9	1		
6	1	21	70	84	45	11	1	
7	1	28	126	210	165	66	13	1
...	...	...	...	...	...	...	...	...

Thus, for example, $a_{3,2} = 5$ and $a_{7,5} = 66$.

The generating polynomial $P_n(x)$ is defined in [1] as

$$(1.4) \qquad P_n(x) = \sum_{k=0}^{n} \mathbf{a}_{n,k} x^k$$

where

$$(1.5) \qquad \mathbf{a}_{n,k} = \left. \frac{D^k P_n(x)}{k!} \right|_{x=0}.$$

From the DFF triangle it is possible to obtain the expression of the polynomial for small values of n:

$$(1.6) \qquad \begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 1 + x \\ P_2(x) &= 1 + 3x + x^2 \\ P_3(x) &= 1 + 6x + 5x^2 + x^3 \end{aligned}$$

and so on.

From (1.1), (1.2), (1.3), and (1.4) we have

$$(1.7) \qquad \sum_{k=0}^{n} \mathbf{a}_{n,k} x^k = \sum_{k=0}^{n} \mathbf{a}_{n-1,k} x^k + \sum_{k=0}^{n} \sum_{\alpha=0}^{n-1} \mathbf{a}_{\alpha,k-1} x^k$$

and

$$(1.8) \qquad \begin{aligned} P_n(x) &= \mathbf{a}_{n-1,n} x^n + \sum_{k=0}^{n-1} \mathbf{a}_{n-1,k} x^k + x \sum_{k=0}^{n} \sum_{\alpha=0}^{n-1} \mathbf{a}_{\alpha,k-1} x^{k-1} \\ &= P_{n-1}(x) + x \sum_{\alpha=0}^{n-1} \sum_{k=0}^{\alpha+1} \mathbf{a}_{\alpha,k-1} x^{k-1}, \end{aligned}$$

$$(1.9) \qquad P_n(x) = P_{n-1}(x) + x \sum_{\alpha=0}^{n-1} P_\alpha(x),$$

which is the recursive formula for the polynomials.

With the initial condition $P_0(x) = 1$, it is easy to obtain the polynomials (1.6). Furthermore, we can also use (1.5) to find the triangle coefficients.

In order to find the polynomials, we must apply the previous method. Let

$$(1.10) \qquad f(x, t) = \sum_{n=1}^{\infty} P_n(x) t^n.$$

Then

$$(1.11) \qquad P_n(x) = \left. \frac{D^n[f(x, t)]}{n!} \right|_{t=0}.$$

From (1.9) and (1.10) we have

$$(1.12) \qquad \begin{aligned} f(x, t) &= \sum_{n=1}^{\infty} P_{n-1}(x) t^n + x \sum_{n=1}^{\infty} \sum_{\alpha=0}^{n-1} P_\alpha(x) t^n \\ &= t \sum_{n=1}^{\infty} P_{n-1}(x) t^{n-1} + x \sum_{n=1}^{\infty} t^n [P_0 + P_1 + \cdots + P_{n-1}] \\ &= t[1 + f(x, t)] + x[1 + f(x, t)] \sum_{k=1}^{\infty} t^k = \frac{-t^2 + t(1 + x)}{t^2 - t(2 + x) + 1}. \end{aligned}$$

If we develop the denominator in (1.12) in partial fractions, we obtain

$$(1.13) \qquad f(x, t) = \frac{a(x) - 1/2}{t - b(x)/2} + \frac{-a(x) - 1/2}{t - c(x)/2} - 1,$$

where

$$y \equiv y(x) = (x^2 + 4x)^{1/2}, \quad a(x) = \frac{-y}{2(x + 4)},$$
$$b(x) = 2 + x + y, \quad \text{and} \quad c(x) = 2 + x - y.$$

From the binomial expansion in (1.13) and after simplification, we also have

$$(1.14)\quad f(x,\ t) = \frac{x+y+4}{(x+4)(x+y+2)}\sum_{n\ge1}\left[\frac{t}{b(x)/2}\right]^n + \frac{x-y+4}{(x+4)(x-y+2)}\sum_{n\ge1}\left[\frac{t}{c(x)/2}\right]^n = \sum_{n\ge1}\left[\frac{1+y/(x+4)}{(x+y+2)^{n+1}/2^n} + \frac{1-y/(x+4)}{(x-y+2)^{n+1}/2^n}\right]t^n$$

from which we have, using (1.10),

$$P_n(x) = \frac{1+y/(x+4)}{(x+y+2)^{n+1}/2^n} + \frac{1-y/(x+4)}{(x-y+2)^{n+1}/2^n} = \frac{(x-y+4)(x-y+2)^n + (x+y+4)(x+y+2)^n}{(x+4)2^{n+1}},$$

$$(1.15)\quad P_n(x) = \frac{1}{2^{n+1}}\left[\frac{x-y+4}{x+4}\sum_{h=0}^{n}(-1)^h\binom{n}{h}(x+2)^{n-h}y^h + \frac{x+y+4}{x+4}\sum_{h=0}^{n}\binom{n}{h}(x+2)^{n-h}y^h\right].$$

From this equation, on distinguishing the case of odd h from that of even h, and since $y = (x^2+4x)^{1/2}$, we can write

$$(1.16)\quad P_n(x) = \frac{1}{2^n}\left[\sum_{h\equiv0\ (\mathrm{mod}\ 2)}^{n}\binom{n}{h}(x+2)^{n-h}x^{h/2}(x+4)^{h/2} + \sum_{h\equiv1\ (\mathrm{mod}\ 2)}^{n}\binom{n}{h}(x+2)^{n-h}x^{(h+1)/2}(x+4)^{(h-1)/2}\right].$$

2. Determination of $\mathbf{a}_{n,k}$

From equations (1.5) and (1.16), and considering also Leibniz's formula

$$(2.1)\quad D^k[f(x)g(x)] = \sum_{j=0}^{k}\binom{k}{j}D^jf(x)D^{k-j}g(x),$$

we have

$$(2.2)\quad \mathbf{a}_{n,k} = \frac{1}{k!2^n}\left[\sum_{h\equiv0\ (\mathrm{mod}\ 2)}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{k}{j}D^j[x^{h/2}(x+4)^{h/2}]\,D^{k-j}[x+2]^{n-h} + \sum_{h\equiv1\ (\mathrm{mod}\ 2)}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{k}{j}D^j[x^{(h+1)/2}(x+4)^{(h-1)/2}]\,D^{k-j}[x+2]^{n-h}\right]_{x=0}$$

Then, from (2.1), it is possible to write

$$D^j[x^{h/2}(x+4)^{h/2}] = \sum_{m=0}^{j}\binom{j}{m}\binom{h/2}{m}m!x^{(h/2)-m} * \binom{h/2}{j-m}(j-m)!(x+4)^{(h/2)-j+m},$$

$$D^{k-j}[(x+2)^{n-h}] = \binom{n-h}{k-j}(k-j)!(x+2)^{n-h-k+j},$$

and

$$(2.3)\qquad D^j[x^{(h+1)/2}(x+4)^{(h-1)/2}] = \sum_{m=0}^{j}\binom{j}{m}\binom{(h+1)/2}{m}m!\,x^{((h+1)/2)-m} * \\ * \binom{(h-1)/2}{j-m}(j-m)!\,(x+4)^{((h-1)/2)-j+m}.$$

where here and in the following equations the * represents multiplication.

From (2.3) and from the properties of binomial coefficients, (2.2) becomes

$$(2.4)\qquad a_{n,k} = \frac{1}{2^n}\left\{\sum_{h\equiv 0 \pmod 2}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{n-h}{k-j}(x+2)^{n-h-k+j} * \right. \\ * \sum_{m=0}^{j}\binom{h/2}{m}\binom{h/2}{j-m}x^{(h/2)-m}(x+4)^{(h/2)-j+m} \\ + \sum_{h\equiv 1 \pmod 2}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{n-h}{k-j}(x+2)^{n-h-k+j}\sum_{m=0}^{j}\binom{(h+1)/2}{m}\binom{(h-1)/2}{j-m} * \\ \left. * \; x^{((h+1)/2)-m}(x+4)^{((h-1)/2)-j+m}\right\}_{x=0}.$$

When $x = 0$, the m-sum exists only if $m = h/2$ and $m = (h+1)/2$, respectively. So we can write

$$(2.5)\qquad a_{n,k} = \sum_{h=0}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{n-h}{k-j}2^{h-k-j}\left[\binom{h/2}{j-h/2} + \binom{(h-1)/2}{j-(h+1)/2}\right].$$

It is worth pointing out that $\binom{a}{b} = 0$ if $b \notin \mathbb{N}_0$, so

$$\binom{h/2}{j-h/2} \neq 0 \quad \text{only if } h \text{ is even}$$

and

$$\binom{(h-1)/2}{j-(h+1)/2} \neq 0 \quad \text{only if } h \text{ is odd.}$$

3. The Properties of $a_{n,k}$

3.1 The Asymptotic Expression of $a_{n,k}$

From [2], the asymptotic expression of the binomial coefficient is

$$(3.1)\qquad \binom{n}{k} \simeq \left(\frac{2}{\pi n}\right)^{1/2} 2^n \exp\left(-\frac{2((n/2)-k)^2}{n}\right)$$

and, from equation (2.5), we find that the asymptotic expression of $a_{n,k}$ can be expressed as

$$a_{n,k} \simeq \frac{2^{2n-k+2}}{\pi^{3/2}}\sum_{h=0}^{n}\frac{1}{(n(n-h))^{1/2}}\sum_{j=0}^{k}2^{-j} * \\ * \exp\left(\frac{-2(n-h)((n/2)-h)^2 - 2n[((n-h)/2)-(k-j)]^2}{n(n-h)}\right) * \\ * \left\{\frac{2^{h/2}}{h^{1/2}}\exp\left[-\frac{4}{h}\left(\frac{-3}{4}h - j\right)^2\right]_{h\text{ even}} \right. \\ \left. + \frac{2^{(h-1)/2}}{(h-1)^{1/2}}\exp\left[-\frac{4}{h-1}\left(\frac{3h+1}{4} - j\right)^2\right]_{h\text{ odd}}\right\}.$$

3.2 The Row Sums of the Triangle Are Equal to Fibonacci Numbers with Odd Subscripts

From the expression (1.16) for $P_n(x)$, when $x = 1$, we have

$$(3.3)\qquad P_n(1) = \frac{(5 + 5^{1/2})/5}{(3 + 5^{1/2})^{n+1}/2^n} + \frac{(5 - 5^{1/2})/5}{(3 - 5^{1/2})^{n+1}/2^n}$$

$$= \frac{1}{5^{1/2}}\left[\frac{1 + 5^{1/2}}{2}\left(\frac{3 - 5^{1/2}}{2}\right)^{n+1} - \frac{1 - 5^{1/2}}{2}\left(\frac{3 + 5^{1/2}}{2}\right)^{n+1}\right].$$

From Binet's formula, we have

$$(3.4)\qquad F_{2n+1} = \frac{1}{5^{1/2}}\left[\left(\frac{1 + 5^{1/2}}{2}\right)^{2n+1} - \left(\frac{1 - 5^{1/2}}{2}\right)^{2n+1}\right].$$

It is easy to show that $P_n(1) = F_{2n+1}$ (where $F_1 = 1$, $F_3 = 2$, $F_5 = 5$, ...).

This is the main result we were interested in showing in this paper. (It may also be verified in the table of the DFF triangle.)

3.3 The Sums of the Triangle Diagonals Give the Powers of 2

From a direct inspection of the DFF triangle and (1.3), we have that the sum of the elements of an upward-slanting diagonal is equal to the sum of all elements that are above this diagonal and, consequently, to the sum of all superior upward-slanting diagonals. This sum value is a power of 2.

In fact, if we define

$$\sum^n = \sum_{r=0}^{n} a_{n-r,\,r},$$

it is possible to write

$$\sum^n = \sum^{n-1} + \sum^{n-2} + \cdots + \sum^1 + 1$$

$$= 2\left(\sum^{n-2} + \sum^{n-3} + \cdots + \sum^1 + 1\right) = \cdots = 2^{n-2}\left(\sum^1 + 1\right) = 2^{n-1},$$

since $\sum^1 = 1$.

4. Conclusions

The principal aim of this paper has been the determination of a closed expression of the general coefficient $a_{n,k}$ of a new numerical triangle, named the DFF, which characterizes the transfer function of a ladder network whose elementary cells are directly coupled. Moreover, the authors present some of the triangle's interesting mathematical properties, one of which is connected to Fibonacci numbers.

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GENERALIZED MULTIVARIATE FIBONACCI POLYNOMIALS OF ORDER K AND THE MULTIVARIATE NEGATIVE BINOMIAL DISTRIBUTIONS OF THE SAME ORDER

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1. Introduction and Summary

In a recent paper, Philippou and Antzoulakos [4] introduced and studied the sequence of multivariate Fibonacci polynomials of order k and related them to the multiparameter negative binomial distribution of the same order of Philippou [3], in order to derive a recurrence relation for calculating its probabilities. This sequence of polynomials includes, as a special case, both the sequence of Fibonacci polynomials of order k and the sequence of Fibonacci-type polynomials of the same order of Philippou, Georghiou, and Philippou [9] and [10], respectively.

In this paper, we introduce a generalization of the sequence of multivariate Fibonacci polynomials of order k (see Definition 2.1), and we derive an expansion in terms of the multinomial coefficients and a recurrence for the general term of the $(r - 1)$-fold convolution of this sequence with itself (see Theorems 2.1 and 2.2). Next, we relate these polynomials to the multivariate negative binomial distribution of order k of Philippou, Antzoulakos, and Tripsiannis [8], and we derive a useful recurrence relation for calculating its probabilities (see Proposition 3.1 and Theorem 3.1). Analogous recurrences follow directly for the type I, type II, and extended multivariate negative binomial distributions of order k of [8] (see Corollaries 3.1-3.3).

The present paper generalizes results on multivariate Fibonacci polynomials of order k (see Remark 2.1) and Fibonacci-type polynomials of the same order (see Remark 2.2). At the same time, several results of Aki [1], Philippou and Georghiou [6], and Philippou and Antzoulakos [4] on recurrences for the probabilities of univariate geometric and negative binomial distributions of order k are generalized to the multivariate case.

Unless otherwise stated, in this paper k, m, and r are fixed positive integers, n_i $(1 \le i \le m)$ are integers, n_{ij} $(1 \le i \le m$ and $1 \le j \le k)$ are nonnegative integers as specified, x_{ij} $(1 \le i \le m$ and $1 \le j \le k)$ are real numbers in the interval $(0, \infty)$, $\underline{1}$ denotes the m-dimensional vector with a one in every position, and $\underline{j}_i$ $(1 \le i \le m$ and $1 \le j \le k)$ denotes the m-dimensional vector with a j in the i^{th} position and zeros elsewhere. Also, whenever sums and products are taken over i and j, ranging, respectively, from 1 to m and from 1 to k, we shall omit these limits for notational simplicity.

2. Generalized Multivariate Fibonacci Polynomials of Order k and Convolutions

In this section, we introduce the sequence of generalized multivariate Fibonacci polynomials of order k, to be denoted by

$$H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m),$$

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along with the $(r-1)$-fold convolution of $H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ with itself, to be denoted by

$$H_{\underline{n},r}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m),$$

and we derive a multinomial expansion and a recurrence for the $\underline{n}^{\text{th}}$ term of $H_{\underline{n},r}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$. In some instances, we shall use the notation $H_{\underline{n}}^{(k)}$ and $H_{\underline{n},r}^{(k)}$ instead of $H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ and $H_{\underline{n},r}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$, respectively.

Definition 2.1: The sequence of polynomials $H_{\underline{n}}^{(k)}(\underline{x}, \ldots, \underline{x}_m)$ is said to be the sequence of generalized multivariate Fibonacci polynomials of order k, if

$$H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m) = \begin{cases} 0, & \text{if some } n_i \le 0 \ (1 \le i \le m), \\ 1, & \text{if } \underline{n} = \underline{1}, \\ \sum_i \sum_j x_{ij} H_{\underline{n}-\underline{j}_i}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m), & \text{elsewhere,} \end{cases}$$

where $\underline{n} = (n_1, \ldots, n_m)$ and $\underline{x}_i = (x_{i1}, \ldots, x_{ik})$, $i = 1, \ldots, m$.

For $m = 1$, $n_1 = n$ (≥ 0) and $\underline{x}_1 = \underline{x}$, $H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ reduces to $H_n^{(k)}(\underline{x})$, the sequence of multivariate Fibonacci polynomials of order k of Philippou and Antzoulakos [4].

Lemma 2.1: Let $H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ be the sequence of generalized multivariate Fibonacci polynomials of order k, and denote its generating function by

$$g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m).$$

Then, for $0 < x_{ij} < 1$ $(1 \le i \le m$ and $1 \le j \le k)$ and $\sum_i \sum_j x_{ij} < 1$, we have

$$g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m) = \frac{t_1 \ldots t_m}{1 - \sum_i \sum_j x_{ij} t_i^j}, \quad |t_i| < 1, \quad i = 1, \ldots, m.$$

Proof: It can be shown by induction on $n_1, \ldots, n_m$ that $0 < x_{ij} < 1$ $(1 \le i \le m$ and $1 \le j \le k)$ and $\sum_i \sum_j x_{ij} < 1$ imply $0 \le H_{\underline{n}}^{(k)} \le 1$, which shows the convergence of $g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m)$ for at least $|t_i| < 1$, since for these t_i

$$\begin{aligned} g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m) &\le \sum_{n_1=1}^{\infty} \cdots \sum_{n_m=1}^{\infty} t_1^{n_1} \ldots t_m^{n_m} \\ &= \prod_i t_i (1 - t_i)^{-1}. \end{aligned}$$

Next, using Definition 2.1, we have

$$\begin{aligned} &g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m) \\ &= t_1 \ldots t_m + \sum_{\substack{n_1=1 \\ n_1 + \cdots + n_m \ge m+1}}^{\infty} \cdots \sum_{n_m=1}^{\infty} t_1^{n_1} \ldots t_m^{n_m} H_{\underline{n}}^{(k)} \\ &= t_1 \ldots t_m + \sum_{\substack{n_1=1 \\ n_1 + \cdots + n_m \ge m+1}}^{\infty} \cdots \sum_{n_m=1}^{\infty} t_1^{n_1} \ldots t_m^{n_m} \sum_i \sum_j x_{ij} H_{\underline{n}-\underline{j}_i}^{(k)} \\ &= t_1 \ldots t_m + \sum_i \sum_j x_{ij} \sum_{n_1=1}^{\infty} \cdots \sum_{n_m=1}^{\infty} t_1^{n_1} \ldots t_i^{n_i+j} \ldots t_m^{n_m} H_{\underline{n}}^{(k)} \\ &= t_1 \ldots t_m + \sum_i \sum_j x_{ij} t_i^j g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m), \end{aligned}$$

from which the lemma follows.

Now let $H_{\underline{n},r}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ be the $(r-1)$-fold convolution of the sequence $H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ with itself, i.e., $H_{\underline{n},r}^{(k)} = 0$ if some $n_i \le 0$ $(1 \le i \le m)$, and for $n_i \ge 1$ $(1 \le i \le m)$

$$(2.1) \qquad H_{\underline{n},r}^{(k)} = \begin{cases} H_{\underline{n}}^{(k)}, & \text{if } r = 1, \\ \displaystyle\sum_{c_1=1}^{n_1} \cdots \sum_{c_m=1}^{n_m} H_{\underline{c},r-1}^{(k)} H_{\underline{n}+\underline{1}-\underline{c}}^{(k)}, & \text{if } r \ge 2, \end{cases}$$

where $\underline{c} = (c_1, \ldots, c_m)$.

As a consequence of (2.1) and in view of Lemma 2.1, we have

$$(2.2) \qquad \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} t_1^{n_1} \cdots t_m^{n_m} H_{\underline{n}+\underline{1},r}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m) = \left(1 - \textstyle\sum_i \sum_j x_{ij} t_i^j\right)^{-r}.$$

Expanding (2.2) about $t_1 = \cdots = t_m = 0$ and using procedures similar to those of [5] and [8], we readily find the following closed formula for $H_{\underline{n},r}^{(k)}$, in terms of the multinomial coefficients.

Theorem 2.1: Let $H_{\underline{n},r}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ be the $(r-1)$-fold convolution of the sequence $H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ with itself. Then

$$H_{\underline{n}+\underline{1},r}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m) = \sum_{\sum_j j n_{ij} = n_i} \binom{n_{11} + \cdots + n_{mk} + r - 1}{n_{11}, \ldots, n_{mk}, r-1} \Pi_i \Pi_j x_{ij}^{n_{ij}},$$
$$n_i = 0, 1, \ldots \ (1 \le i \le m).$$

Proof: Let $|t_i| < 1$ $(1 \le i \le m)$, $0 < x_{ij} < 1$ $(1 \le i \le m$ and $1 \le j \le k)$, and let $\sum_i \sum_j x_{ij} < 1$. Then

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} t_1^{n_1} \cdots t_m^{n_m} H_{\underline{n}+\underline{1},r}(\underline{x}_1, \ldots, \underline{x}_m)$$
$$= \left(1 - \textstyle\sum_i \sum_j x_{ij} t_i^j\right)^{-r}, \quad \text{by (2.2)},$$
$$= \sum_{n=0}^{\infty} \binom{n+r-1}{n} \left(\textstyle\sum_i \sum_j x_{ij} t_i^j\right)^n, \quad \text{since } \left|\textstyle\sum_i \sum_j x_{ij} t_i^j\right| < 1,$$
$$= \sum_{n=0}^{\infty} \binom{n+r-1}{n} \sum_{\sum_i \sum_j n_{ij} = n} \binom{n}{n_{11}, \ldots, n_{mk}} \Pi_i \Pi_j (x_{ij} t_i^j)^{n_{ij}},$$
by the multinomial theorem,
$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \sum_{\substack{\sum_j n_{ij} = n_i \\ i=1,\ldots,m}} \binom{n_{11} + \cdots + n_{mk} + r - 1}{n_{11}, \ldots, n_{mk}, r-1} \Pi_i \Pi_j (x_{ij} t_i^j)^{n_{ij}}$$
$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} t_1^{n_1} \cdots t_m^{n_m} \sum_{\substack{\sum_j j n_{ij} = n_i \\ i=1,\ldots,m}} \binom{n_{11} + \cdots + n_{mk} + r - 1}{n_{11}, \ldots, n_{mk}, r-1} \Pi_i \Pi_j x_{ij}^{n_{ij}},$$

by replacing n_i by $n_i - \sum_j (j-1) n_{ij}$ $(1 \le i \le m)$. The theorem follows.

We proceed next to show that $H_{\underline{n},r}^{(k)}$ satisfies the following linear recurrence with variable coefficients, using procedures similar to those of [4] and [6].

Theorem 2.2: Let $H_{\underline{n},r}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ be the $(r-1)$-fold convolution of the sequence $H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ with itself. Then

$$H_{\underline{n},r}^{(k)} = 0, \quad \text{if some } n_i \le 0 \ (1 \le i \le m), \quad H_{\underline{1},r}^{(k)} = 1,$$

and

$$H_{\underline{n}+\underline{1},r}^{(k)} = \textstyle\sum_i \sum_j x_{ij} H_{\underline{n}+\underline{1}-\underline{j}_i,r}^{(k)} + \dfrac{r-1}{n_s} \textstyle\sum_j j x_{sj} H_{\underline{n}+\underline{1}-\underline{j}_s,r}^{(k)},$$

if $n_i \ge 0$ and some $n_s \ge 1$ $(1 \le i, s \le m)$.

Proof: From the definition of $H_{\underline{n}, r}^{(k)}$, we have

(2.3) $\quad H_{\underline{n}, r}^{(k)} = 0, \quad$ if some $n_i \le 0$ $(1 \le i \le m)$ and $H_{\underline{1}, r}^{(k)} = 1$.

Now, using (2.2) twice, we have

(2.4) $\quad H_{\underline{n}+\underline{1}, r}^{(k)} = H_{\underline{n}+\underline{1}, r+1}^{(k)} - \sum_i \sum_j x_{ij} H_{\underline{n}+\underline{1}-\underline{j}_i, r+1}^{(k)}, \; n_i \ge 0 \; (1 \le i \le m),$

since the generating function of the right-hand side reduces to that of $H_{\underline{n}+\underline{1}, r}^{(k)}$. Next, differentiating both sides of (2.2) with respect to t_s $(1 \le s \le m)$, we get

(2.5) $\quad n_s H_{\underline{n}+\underline{1}, r}^{(k)} = r \sum_j j x_{sj} H_{\underline{n}+\underline{1}-\underline{j}_s, r+1}^{(k)}, \; n_i \ge 0$ and $n_s \ge 1$ $(1 \le i \ne s \le m)$.

Combining (2.4) and (2.5), we obtain

$$H_{\underline{n}+\underline{1}, r}^{(k)} = \sum_i \sum_j x_{ij} H_{\underline{n}+\underline{1}-\underline{j}_i, r}^{(k)} + \frac{r-1}{n_s} \sum_j j x_{sj} H_{\underline{n}+\underline{1}-\underline{j}_s, r}^{(k)},$$

$$\text{if } n_i \ge 0 \text{ and some } n_s \ge 1 \; (1 \le i, s \le m),$$

by means of (2.1), which along with (2.3) establishes the theorem.

Remark 2.1: For $m = 1$, $n_1 = n$, and $\underline{x}_1 = \underline{x} = (x_1, \ldots, x_k)$, Theorems 2.1 and 2.2 reduce to the main results of Philippou and Antzoulakos [4] on multivariate Fibonacci polynomials of order k (see Theorems 2.2 and 2.3), namely,

(2.6) $\quad H_{n+1, r}^{(k)}(\underline{x}) = \sum_{\sum_j j n_j = n} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ldots, n_k, r-1} \prod_j x_j^{n_j}, \; n \ge 0,$

and

(2.7) $\quad H_{n+1, r}^{(k)}(\underline{x}) = \sum_j \frac{x_j}{n}[n + j(r-1)] H_{n+1-j, r}^{(k)}(\underline{x}), \; n \ge 1.$

Remark 2.2: For $m = 1$, $n_1 = n$, and $\underline{x}_1 = (x, \ldots, x)$, Theorems 2.1 and 2.2 reduce to Theorems 2.1(a) and 2.2 of Philippou and Georghiou [6], respectively, since for these values

$$H_{n_1, r}^{(k)}(\underline{x}_1) = F_{n, r}^{(k)}(x),$$

where $F_{n, r}^{(k)}(x)$ denotes the $(r-1)$-fold convolution of the sequence of Fibonacci-type polynomials of order k with itself.

We note in ending this section that the sequence $F_{\underline{n}}^{(k)}$ defined by

$$F_{\underline{n}}^{(k)} = \begin{cases} 0, & \text{if some } n_i \le 0 \; (1 \le i \le m), \\ 1, & \text{if } \underline{n} = \underline{1}, \\ \sum_i \sum_j F_{\underline{n}-\underline{j}_i}^{(k)}, & \text{elsewhere}, \end{cases}$$

may be called the multiple Fibonacci sequence of order k, since for $m = 1$ and $n_1 = n$ (≥ 0) it reduces to $F_n^{(k)}$, the Fibonacci sequence of order k (see, e.g., Philippou and Muwafi [7]). It may be noted that

(2.8) $\quad F_{\underline{n}+\underline{1}}^{(k)} = \sum_{\sum_j j n_{ij} = n_i} \binom{n_{11} + \cdots + n_{mk}}{n_{11}, \ldots, n_{mk}}, \; n_i = 0, 1, \ldots \; (1 \le i \le m).$

which follows from Theorem 2.1 for $r = 1$ and $x_{ij} = 1$ $(1 \le i \le m$ and $1 \le j \le k)$.

3. Recurrence Relations for the Multivariate Negative Binomial Distributions of Order k

In this section, we employ Theorems 2.1 and 2.2 to derive a recurrence relation for calculating the probabilities of the following multivariate negative binomial distribution of order k of Philippou, Antzoulakos, and Tripsiannis [8].

Definition 3.1: A random vector $\underline{N} = (N_1, \ldots, N_m)$ is said to have the multivariate negative binomial distribution of order k with parameters r, $q_{11}, \ldots, q_{mk}$ ($r > 0$, $0 < q \quad < 1$ for $1 \le i \le m$ and $1 \le j \le k$, and $q_{11} + \cdots + q_{ij} < 1$), to be denoted by $\mathrm{MNB}_k(r;\, q_{11}, \ldots, q_{mk})$, if

$$P(N_1 = n_1, \ldots, N_m = n_m)$$
$$= p^r \sum_{\sum_j j n_{ij} = n_i} \binom{n_{11} + \cdots + n_{mk} + r - 1}{n_{11}, \ldots, n_{mk}, r - 1} \prod_i \prod_j q_{ij}^{n_{ij}},$$
$$n_i = 0, 1, \ldots \; (1 \le i \le m),$$

where $p = 1 - q_{11} - \cdots - q_{mk}$.

Analogous recurrences are also given for the type I, type II, and extended multivariate negative binomial distributions of order k of [8], denoted by

$$\overline{\mathrm{MNB}}_{k,\,\mathrm{I}}(r;\, Q_1, \ldots, Q_m),\ \mathrm{MNB}_{k,\,\mathrm{II}}(r;\, Q_1, \ldots, Q_m),\text{ and}$$
$$\overline{\mathrm{MENB}}_k(r;\, q_{11}, \ldots, q_{mk}).$$

These distributions result by applying to the parameters of $\mathrm{MNB}_k(r;\, q_{11}, \ldots, q_{mk})$ the following transformations, respectively:

(a) $q_{ij} = P^{j-1}Q_i$ ($0 < Q_i < 1$ for $1 \le i \le m$, $\sum_i Q_i < 1$ and $P = 1 - \sum_i Q_i$);

(b) $q_{ij} = Q_i/k$ ($0 < Q_i < 1$ for $1 \le i \le m$, $\sum_i Q_i < 1$ and $P = 1 - \sum_i Q_i$);

(c) $q_{ij} = P_1P_2 \ldots P_{j-1}Q_{ij}$ ($P_0 = 1$, $0 < Q_{ij} < 1$ for $1 \le i \le m$ and $1 \le j \le k$, $\sum_i Q_{ij} < 1$ and $P_j = 1 - \sum_i Q_{ij}$ for $1 \le j \le k$).

We note first the following proposition that relates the multivariate negative binomial distribution of order k to the generalized multivariate Fibonacci polynomials of the same order.

Proposition 3.1: Let $\underline{N} = (N_1, \ldots, N_m)$ be a random vector distributed as

$$\mathrm{MNB}_k(r;\, q_{11}, \ldots, q_{mk}),$$

and let $H^{(k)}_{\underline{n},\, r}$ be the $(r - 1)$-fold convolution of the sequence $H^{(k)}_{\underline{n}}$ with itself. Then

$$P(N_1 = n_1, \ldots, N_m = n_m) = p^r H^{(k)}_{\underline{n}+\underline{1},\, r}(\underline{q}_1, \ldots, \underline{q}_m),$$
$$n_i = 0, 1, \ldots, 1 \le i \le m,$$

where $\underline{q}_i = (q_{i1}, \ldots, q_{ik})$, $i = 1, \ldots, m$.

Proof: The proof is a direct consequence of Theorem 2.1 and Definition 3.1.

We proceed now to derive a recurrence relation for calculating the probabilities of $\mathrm{MNB}_k(r;\, q_{11}, \ldots, q_{mk})$.

Theorem 3.1: Let $\underline{N} = (N_1, \ldots, N_m)$ be a random vector distributed as

$$\mathrm{MNB}_k(r;\, q_{11}, \ldots, q_{mk}),$$

and set

$$P_{\underline{n},\, r} = P(N_1 = n_1, \ldots, N_m = n_m).$$

Then

$$P_{\underline{n},\, r} = \begin{cases} 0, & \text{if some } n_i \le -1 \ (1 \le i \le m), \\ p^r, & \text{if } n_1 = \cdots = n_m = 0, \\ \sum_i \sum_j q_{ij} P_{\underline{n}-\underline{j}_i,\, r} + \dfrac{r - 1}{n_s} \sum_j j q_{sj} P_{\underline{n}-\underline{j}_s,\, r}, & \text{if } n_i \ge 0 \text{ and some } n_s \ge 1 \ (1 \le i,\, s \le m). \end{cases}$$

Proof: If some $n_i \le -1$ $(1 \le i \le m)$, $(N_1 = n_1, \ldots, N_m = n_m) = \emptyset$, which implies $P_{\underline{n},r} = P(\emptyset) = 0$. If $n_1 = \cdots = n_m = 0$, Definition 3.1 gives $P_{\underline{n},r} = p^r$. If $n_i \ge 0$ and some $n_s \ge 1$ $(1 \le i, s \le m)$, we have

$$\begin{aligned} P_{\underline{n},r} &= p^r H^{(k)}_{\underline{n}+\underline{1},r}(\underline{q}_1, \ldots, \underline{q}_m), \text{ by Proposition 3.1,} \\ &= p^r \Big\{ \textstyle\sum_i \sum_j q_{ij} H^{(k)}_{\underline{n}+\underline{1}-\underline{j}_i,r}(\underline{q}_1, \ldots, \underline{q}_m) \\ &\quad + \frac{r-1}{n_s} \textstyle\sum_j j q_{sj} H^{(k)}_{\underline{n}+\underline{1}-\underline{j}_s,r}(\underline{q}_1, \ldots, \underline{q}_m) \Big\}, \text{ by Theorem 2.2,} \\ &= \textstyle\sum_i \sum_j q_{ij} P_{\underline{n}-\underline{j}_i,r} + \frac{r-1}{n_s} \textstyle\sum_j j q_{sj} P_{\underline{n}-\underline{j}_s,r}, \text{ by Proposition 3.1.} \end{aligned}$$

Using the transformations (a), (b), and (c), respectively, Theorem 3.1 now reduces to the following corollaries.

Corollary 3.1: Let $\underline{N} = (N_1, \ldots, N_m)$ be a random vector distributed as

$$\overline{\text{MNB}}_{k,\text{I}}(r; q_1, \ldots, q_m),$$

and set

$$P_{\underline{n},r} = P(N_1 = n_1, \ldots, N_m = n_m).$$

Then

$$P_{\underline{n},r} = \begin{cases} 0, & \text{if some } n_i \le -1 \ (1 \le i \le m), \\ p^{kr}, & \text{if } n_1 = \cdots = n_m = 0, \\ \sum_i \sum_j p^{j-1} q_i P_{\underline{n}-\underline{j}_i,r} + \frac{r-1}{n_s} \sum_j j p^{j-1} q_s P_{\underline{n}-\underline{j}_s,r}, & \text{if } n_i \ge 0 \text{ and some } n_s \ge 1 \ (1 \le i, s \le m). \end{cases}$$

Corollary 3.2: Let $\underline{N} = (N_1, \ldots, N_m)$ be a random vector distributed as

$$\text{MNB}_{k,\text{II}}(r; q_1, \ldots, q_m),$$

and set

$$P_{\underline{n},r} = P(N_1 = n_1, \ldots, N_m = n_m).$$

Then

$$P_{\underline{n},r} = \begin{cases} 0, & \text{if some } n_i \le -1 \ (1 \le i \le m), \\ p^{r}, & \text{if } n_1 = \cdots = n_m = 0, \\ \sum_i \sum_j \frac{q_i}{k} P_{\underline{n}-\underline{j}_i,r} + \frac{r-1}{n_s} \sum_j j \frac{q_s}{k} P_{\underline{n}-\underline{j}_s,r}, & \text{if } n_i \ge 0 \text{ and some } n_s \ge 1 \ (1 \le i, s \le m). \end{cases}$$

Corollary 3.3: Let $\underline{N} = (N_1, \ldots, N_m)$ be a random vector distributed as

$$\overline{\text{MENB}}_k(r; q_{11}, \ldots, q_{mk}),$$

and set

$$P_{\underline{n},r} = P(N_1 = n_1, \ldots, N_m = n_m).$$

Then

$$P_{\underline{n},r} = \begin{cases} 0, & \text{if some } n_i \le -1 \ (1 \le i \le m), \\ (p_1 \cdots p_k)^{r}, & \text{if } n_1 = \cdots = n_m = 0, \\ \sum_i \sum_j p_1 \cdots p_{j-1} q_{ij} P_{\underline{n}-\underline{j}_i,r} + \frac{r-1}{n_s} \sum_j p_1 \cdots p_{j-1} q_{sj} P_{\underline{n}-\underline{j}_s,r}, & \text{if } n_i \ge 0 \text{ and some } n_s \ge 1 \ (1 \le i, s \le m). \end{cases}$$

For $m = 1$, Theorem 3.1 and Corollaries 3.1-3.3 reduce to known recurrences concerning respective univariate negative binomial distributions of order k (see [4] and [6]). For $k = 1$, Theorem 3.1 (or any one of Corollaries 3.1-3.3) provides the following recurrence for the probabilities of $\text{MNB}(r; q_1, \ldots, q_m)$, the usual multivariate negative binomial distribution,

$$P_{\underline{n},\,r} = \begin{cases} 0, & \text{if some } n_i \le -1 \ (1 \le i \le m), \\ p^r, & \text{if } n_1 = \cdots = n_m = 0, \\ \sum_i q_i P_{\underline{n}-\underline{1}_i,\,r} + \dfrac{r-1}{n_s} q_s P_{\underline{n}-\underline{1}_s,\,r}, & \\ \qquad \text{if } n_i \ge 0 \text{ and some } n_s \ge 1 \ (1 \le i,\ s \le m), \end{cases}$$

which does not seem to have been noticed before.

Remark 3.1: For $r = 1$, Theorem 3.1 and Corollaries 3.1-3.3 provide recurrences for the probabilities of respective multivariate geometric distributions of order k of [8], defined by

$$MG_k(q_{11}, \ldots, q_{mk}) = MNB_k(1; q_{11}, \ldots, q_{mk}),$$

$$\overline{MG}_{k,\,\mathrm{I}}(q_1, \ldots, q_m) = \overline{MNB}_{k,\,\mathrm{I}}(1; q_1, \ldots, q_m),$$

$$MG_{k,\,\mathrm{II}}(q_1, \ldots, q_m) = MNB_{k,\,\mathrm{II}}(1; q_1, \ldots, q_m),$$

and $$\overline{MEG}_k(q_{11}, \ldots, q_{mk}) = \overline{MENB}\,(1; q_{11}, \ldots, q_{mk}).$$

The resulting recurrence for $\overline{MEG}_k(q_{11}, \ldots, q_{mk})$ has also been obtained in [5], via a different method.

We note in ending this paper that another derivation of Theorem 3.1, without employing the sequence of generalized multivariate Fibonacci polynomials of order k, has been obtained by Antzoulakos and Philippou (see [2]).

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ON GENERATING FUNCTIONS FOR POWERS OF RECURRENCE SEQUENCES

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1. Introduction

Let $\{w_q\}$ be a recurrence sequence of order n ($n \in \mathbf{N}$) and let its generating function be given by

$$(1) \qquad w(z) \equiv \sum_{q=0}^{\infty} w_q z^q = \frac{W_1(z)}{\prod_{j=1}^{n}(1 - b_j z)},$$

where $W_1(z)$ is a polynomial in z with deg $W_1(z) = m$. For a positive integer k, let $w_k(z)$ denote the generating function of the sequence $\{w_q^k\}$ of the k^{th} powers of w_q. It is known that $w_k(z)$ is a rational function in z (see [6] or [8]). The aim of this paper is to study the degrees of polynomials in the numerator and denumerator of $w_k(z)$. This paper is similar in character to [4].

The function $w_k(z)$ has been studied with $m = n - 1$ in [8] and [11]. Generating functions for powers of third-order recurrence sequences have been studied in [13], and those of second-order recurrence sequences in [1], [3], [5], [7], [9], [10], and [12].

The proof of our result is based on the following theorem by Hadamard:

If $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$, and $C(z) = \sum_{n=0}^{\infty} a_n b_n z^n$,
then

$$C(z) = \frac{1}{2\pi i}\int_{\gamma} A(s)B(z/s)\frac{ds}{s},$$

where γ is a contour in the s plane, which includes the singularities of $B(z/s)/s$ and excludes the singularities of $A(s)$. If the radius of convergence of $A(z)$ [resp. $B(z)$] is R (resp. R'), then the radius of convergence of $C(z)$ is at least RR', and γ may, for example, be any circle between $|s| = R$ and $|s| = |z|/R'$ (see [6], p. 813, [14], pp. 157-59).

2. The Generating Function $w_k(z)$

Theorem: Let $\{w_q\}$ be a recurrence sequence of order n and let its generating function be given by (1). Then

$$(2) \qquad w_k(z) = \frac{W_k(z)}{D_k(z)},$$

where

$$D_k(z) = \prod_{\substack{(r_1, \ldots, r_n) \in \mathbf{N}_0^n \\ r_1 + \cdots + r_n = k}} (1 - b_1^{r_1} \cdots b_n^{r_n} z), \quad \mathbf{N}_0 = \mathbf{N} \cup \{0\},$$

and $W_k(z)$ is a polynomial in z with

$$\deg W_k(z) \le \binom{n + k - 1}{k} - n + m.$$

Proof: Clearly $W_1(z)$ can be written in the form

$$W_1(z) = w_p z^p \prod_{i=1}^{m-p}(1 - a_i z), \quad 0 \le p \le m,$$

where p is the least integer such that $w_p \ne 0$. Assume first that $b_{j_1} \ne b_{j_2}$ for $j_1 \ne j_2$ and $b_j \ne 0$ for $j = 1, 2, \ldots, n$. Then we distinguish two cases: $m < n$, $m \ge n$.

Case 1. Let $m < n$. We proceed by induction on k. If $k = 1$, the theorem holds. Assume it holds for $k = K$ $(K \ge 1)$. We shall prove that it holds for $k = K + 1$. Applying Hadamard's theorem and the Cauchy residue theorem and noting that the appropriate winding numbers are = 1, we obtain

$$\begin{aligned} w_{K+1}(z) &= \frac{1}{2\pi i}\int_\gamma w_K(s)w(z/s)\frac{ds}{s} \\ &= \frac{1}{2\pi i}\int_\gamma \frac{W_K(s)w_p z^p \prod_{i=1}^{m-p}(s - a_i z)}{\prod_{r_1+\cdots+r_n=K}(1 - b_1^{r_1}\cdots b_n^{r_n}s)\prod_{j=1}^{n}(s - b_j z)} s^{n-m-1}ds \\ &= \sum_{h=1}^{n} \frac{W_K(b_h z)w_p \prod_{i=1}^{m-p}(b_h - a_i)}{\prod_{r_1+\cdots+r_n=K}(1 - b_1^{r_1}\cdots b_n^{r_n}b_h z)\prod_{\substack{j=1\\ j\ne h}}^{n}(b_h - b_j)} b_h^{n-m-1}. \end{aligned}$$

Denote briefly

$$C_h = w_p\prod_{i=1}^{m-p}(b_h - a_i)\prod_{\substack{j=1\\ j\ne h}}^{n}(b_h - b_j)^{-1}b_h^{n-m-1},$$

$$E_{K+1}^{(h)}(z) = \prod_{r_1+\cdots+r_{h-1}+r_{h+1}+\cdots+r_n=K+1}\left(1 - b_1^{r_1}\cdots b_{h-1}^{r_{h-1}}b_{h+1}^{r_{h+1}}\cdots b_n^{r_n}z\right).$$

Converting the fraction in the sum over h by $E_{K+1}^{(h)}(z)$, we obtain

$$w_{K+1}(z) = \frac{\sum_{h=1}^{n} C_h W_K(b_h z)E_{K+1}^{(h)}(z)}{D_{K+1}(z)}. \tag{3}$$

The number of solutions of the equation $r_1 + \cdots + r_n = K$ in $(r_1, \ldots, r_n) \in \mathbf{N}_0^n$ is equal to

$$\binom{n + K - 1}{K}.$$

Thus, the number of solutions of the equation $r_1 + \cdots + r_{h-1} + r_{h+1} + \cdots + r_n = K + 1$ in $(r_1, \ldots, r_{h-1}, r_{h+1}, \ldots, r_n) \in \mathbf{N}_0^{n-1}$ is equal to

$$\binom{n + K - 1}{K + 1}.$$

This is plainly the degree of the polynomial $E_{K+1}^{(h)}(z)$. Thus, the degree of the polynomial in the numerator of the fraction of (3) is less than or equal to

$$\binom{n + K - 1}{K} - n + m + \binom{n + K - 1}{K + 1},$$

that is, less than or equal to

$$\binom{n + (K + 1) - 1}{K + 1} - n + m.$$

This proves the theorem in Case 1.

Case 2. Let $m \geq n$. We proceed by induction on k in this case, too. The theorem holds for $k = 1$. Assume it holds for $k = K$. Then the series $w_K(z)$ can be written in the form

$$w_K(z) = \sum_{i=0}^{a-b} u_i z^i + \frac{U_K(z)}{D_K(z)},$$

where

$$a = \deg W_K(z) \leq \binom{n + K - 1}{K} - n + m, \quad b = \binom{n + K - 1}{K}$$

and $U_K(z)$ is a polynomial in z of degree $< b$. Note that $a - b \leq m - n$. The series $w(z)$ can be written in the form

$$w(z) = \sum_{j=0}^{m-n} v_j z^j + \sum_{\ell=0}^{n} \frac{A_\ell}{1 - b_\ell z}.$$

Applying Hadamard's theorem and the Cauchy residue theorem and noting that the appropriate winding numbers are =1, we obtain

$$\begin{aligned} w_{K+1}(z) &= \frac{1}{2\pi i}\int_\gamma w_K(s) w(z/s)\frac{ds}{s} \\ &= \frac{1}{2\pi i}\int_\gamma \sum_{i=0}^{a-b}\sum_{j=0}^{m-n} u_i v_j s^i \frac{z^j}{s^{j+1}}\,ds + \frac{1}{2\pi i}\int_\gamma \sum_{i=0}^{a-b}\sum_{\ell=0}^{n} u_i A_\ell \frac{s^i}{s - b_\ell z}\,ds \\ &\quad + \frac{1}{2\pi i}\int_\gamma \sum_{j=0}^{m-n} \frac{U_K(s)}{D_K(s)} v_j \frac{z^j}{s^{j+1}}\,ds + \frac{1}{2\pi i}\int_\gamma \sum_{\ell=0}^{n} \frac{U_K(s)}{D_K(s)} \frac{A_\ell}{s - b_\ell z}\,ds \\ &= \sum_{i=0}^{a-b} u_i v_i z^i + \sum_{i=0}^{a-b}\sum_{\ell=0}^{n} u_i A_\ell b_\ell^j z^i + \sum_{j=0}^{m-n} B_j v_j z^j + \sum_{\ell=0}^{n} \frac{U_K(b_\ell z)}{D_K(b_\ell z)} A_\ell, \end{aligned}$$

where B_j ($j = 0, 1, \ldots, m - n$) is a complex constant. Now we can see, after some calculations, that $w_{K+1}(z)$ can be written in the form

$$w_{K+1}(z) = \frac{W_{K+1}(z)}{D_{K+1}(z)},$$

where

$$\deg W_{K+1}(z) \leq \binom{n + (K + 1) - 1}{K + 1} - n + m.$$

This proves the theorem in Case 2.

Now the theorem is proved when $b_{j_1} \neq b_{j_2}$ for $j_1 \neq j_2$ and $b_j \neq 0$ for $j = 1, 2, \ldots, n$. But the coefficients of z^q ($q = 0, 1, \ldots$) in the series $w_k(z)$ and in the polynomials $W_k(z)$ and $D_k(z)$ are polynomials in the variables w_p, a_i, and b_j. Thus, taking limits $b_{j_1} \to b_{j_2}$, $b_j \to 0$ proves that the theorem holds for all $b_1, \ldots, b_n$. This completes the proof.

Remark: It should be noted that, in the case in which two or more of the b_j are equal, the treatment used at the end of the proof does not have to give the best possible result (cf. [8], Sec. 7). However, application of Hadamard's theorem and Cauchy's residue theorem would be too laborious in that case.

Example: Let $\{w_q\} \equiv \{F_q\}$, the Fibonacci sequence, and let $\alpha = (1 + \sqrt{5})/2$, and $\beta = (1 - \sqrt{5})/2$. Then, for $K = 1$, formula (3) is

$$F_2(z) = \frac{\alpha(\alpha - \beta)^{-1}(1 - \beta^2 z) + \beta(\beta - \alpha)^{-1}(1 - \alpha^2 z)}{(1 - \alpha^2 z)(1 - \alpha\beta z)(1 - \beta^2 z)},$$

which gives the well-known formula

$$F_2(z) = \frac{1 - z}{1 - 2z - 2z^2 + z^3}$$

(see, e.g., [2]; [13], p. 794).

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A NOTE ON A THEOREM OF SCHINZEL

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1. Introduction

Consider a sequence defined by the condition

(1.1) $u_0 = 0,\ u_1 = 1,\ u_{n+2} = au_{n+1} + u_n,\ n = 0, 1, 2, \ldots \quad (a \in \mathbb{Z}).$

If $a = 1$, then $u_n = F_n$, that is, we get the sequence of Fibonacci numbers. If p is a fixed prime, we also consider the sequence $\overline{u}_0, \overline{u}_1, \overline{u}_2, \ldots$ defined by the same condition in $\mathbb{F}_p$, the finite field of p elements. Let $k = k(p)$ be the length of the shortest period of the sequence $\overline{u}_0, \overline{u}_1, \overline{u}_2, \ldots$. A Schinzel [1] has proved the following result.

Theorem 1.1 (Schinzel): Let $S = S(p)$ be the set of frequencies with which different residues occur in the sequence $\overline{u}_n$ $[0 \le n < k(p)]$. For $p > 7$, $p \nmid a(a^2 + 4)$ we have

$$S = \{0, 1, 2\} \text{ or } \{0, 1, 2, 3\} \text{ if } k(p) \not\equiv 0 \pmod 4,$$
$$S = \{0, 2, 4\} \text{ if } k(p) \equiv 4 \pmod 8,$$
$$S = \{0, 1, 2\} \text{ or } \{0, 2, 3\} \text{ or } \{0, 1, 2, 4\} \text{ or } \{0, 2, 3, 4\} \text{ if } k(p) \equiv 0 \pmod 8.$$

The purpose of this note is to show how this result can be extended, using the same method, with some minor modifications. Consider the sequence defined by the condition

(1.2) $v_0 = 2,\ v_1 = a,\ v_{n+2} = av_{n+1} + v_n,\ n = 0, 1, 2, \ldots$.

If $a = 1$, then $v_n = L_n$, that is, we get the sequence of Lucas numbers. Consider also the sequence $\overline{v}_0, \overline{v}_1, \overline{v}_2, \ldots$ defined by the same condition in $\mathbb{F}_p$. Let $k' = k'(p)$ be the length of the shortest period of the sequence $\overline{v}_0, \overline{v}_1, \overline{v}_2, \ldots$. We prove that $k' = k$ (Lemma 2.1 below) and get the following result.

Theorem 1.2: Let $S' = S'(p)$ be the set of frequencies with which different residues occur in the sequence v_n $[0 \le n < k(p)]$. For $p > 7$, $p \nmid a(a^2 + 4)$ we have

$$S' = \{0, 1, 2\} \text{ or } \{0, 1, 2, 3\} \text{ if } k(p) \not\equiv 0 \pmod 4,$$
$$S' = \{0, 1, 2\} \text{ or } \{0, 2, 3\} \text{ or } \{0, 1, 2, 4\} \text{ or } \{0, 2, 3, 4\} \text{ if } k(p) \equiv 0 \pmod 4.$$

Moreover,

(1.3) $S' = S$ if $k(p) \not\equiv 4 \pmod 8$.

Corresponding to Schinzel's three corollaries, we deduce from Theorem 1.2 the following corollaries.

Corollary 1.3: If $p > 7$, $p \nmid a^2 + 4$, then at least one residue mod p does not occur in the sequence $\overline{v}_n$.

Corollary 1.4: If $p \ne 5$, $p \nmid a(a^2 + 4)$, then at least one residue mod p occurs exactly twice in the shortest period of the sequence $\overline{v}_n$.

Corollary 1.5: For $a = 1$, $p > 7$,

$$S' = \{0, 1, 2, 3\} \text{ if } k(p) \not\equiv 0 \pmod 4,$$
$$S' = \{0, 1, 2\} \text{ or } \{0, 2, 3\} \text{ or } \{0, 1, 2, 4\} \text{ or } \{0, 2, 3, 4\} \quad \text{if } k(p) \equiv 4 \pmod 8,$$
$$S' = \{0, 1, 2, 4\} \text{ or } \{0, 2, 3, 4\} \text{ if } k(p) \equiv 0 \pmod 8.$$

L. Somer [2] has proved Corollary 1.3 except for the case where $p \equiv 1$ or $9 \pmod{20}$.

2. Some Lemmas

Let $D = a^2 + 4$ and let ξ be a zero of $x^2 - ax - 1$ in the finite field $\mathbb{F}_q$, where $q = p$ if $\left(\frac{D}{p}\right) = 1$ and $q = p^2$ if $\left(\frac{D}{p}\right) = -1$ (we exclude the case $p \mid D$).

For $\overline{u}_n$ and $\overline{v}_n$ we have the formulas

$$\text{(2.1)} \qquad \overline{u}_n = \frac{\xi^n - (-\xi^{-1})^n}{\xi + \xi^{-1}}, \ \overline{v}_n = \xi^n + (-\xi^{-1})^n.$$

Let δ be the least positive exponent such that $\xi^\delta = 1$.

The following seven lemmas correspond to the lemmas in [1].

Lemma 2.1: For $p \nmid 2D$, we have $k'(p) = [\delta, 2] = k(p)$. (Here, the symbol $[\delta, 2]$ means the least common multiple of δ and 2.)

Proof: The second equation above is the content of Lemma 1 in [1]. The first equation follows by exactly analogous considerations using (2.1). □

Lemma 2.2: Let $p \nmid 2D$. The conditions

$$n \equiv m \pmod 2 \text{ and } \overline{v}_n = \overline{v}_m$$

hold if and only if either $n \equiv m \pmod k$ or $n \equiv m \equiv 0 \pmod 2$ and $n + m \equiv 0 \pmod k$ or $k \equiv 0 \pmod 4$, $n \equiv m \equiv 1 \pmod 2$ and $n + m \equiv k/2 \pmod k$.

Proof: We use (2.1) and combine arguments in the proofs of Lemma 2 and Lemma 3 in [1]. □

Lemma 2.3: Let $p \nmid 2D$. The conditions

$$n \equiv m \pmod 2 \text{ and } \overline{v}_n = -\overline{v}_m$$

are equivalent to

$$n \equiv m \equiv 1 \pmod 2 \text{ and } n + m \equiv 0 \pmod k \text{ if } k \equiv 2 \pmod 4,$$

$$n \equiv m + k/2 \pmod 2 \text{ and } \overline{v}_n = \overline{v}_{m+k/2} \text{ if } k \equiv 0 \pmod 4.$$

Proof: We use (2.1) and combine arguments in the proofs of Lemma 2 and Lemma 3 in [1]. □

Lemma 2.4: Let $p \nmid 2D$ and let $0 \le n < k$. We have $\overline{v}_n = 0$ if and only if

$$k \equiv 2 \pmod 4 \text{ and } n = k/2,$$
$$k \equiv 0 \pmod 8 \text{ and } n = k/4 \text{ or } n = 3k/4.$$

Proof: Analogous to the proof of Lemma 4 in [1]. □

Lemma 2.5: Let $p \nmid 2D$. We have

$$k \mid p - 1 \text{ if } \left(\frac{D}{p}\right) = 1, \quad k \mid 2(p + 1) \text{ if } \left(\frac{D}{p}\right) = -1.$$

Proof: In view of Lemma 2.1, this is exactly the same as Lemma 5 in [1]. □

Lemma 2.6: If $k = 2(p+1) \equiv 0 \pmod 8$, then for every nonnegative integer e there is an n such that

$$\bar{v}_{n+e} = \bar{v}_n. \tag{2.2}$$

Proof: If $\bar{u}_e \neq 0$, we use the identity

$$v_n v_{m+e} - v_m v_{n+e} = (-1)^{m+1} D u_e u_{n-m}$$

and find by virtue of Lemma 4 in [1] that the quotients

$$\frac{\bar{v}_{n+e}}{\bar{v}_n} \text{ for } 0 \le n < \frac{k}{2},\ n \neq \frac{k}{4}$$

are all distinct. Since $k/2 = p+1$, we have p distinct elements of $\mathbb{F}_p$. One of them must be 1, which gives (2.2).

Suppose now that $\bar{u}_e = 0$. By Lemma 4 in [1], $e \equiv 0 \pmod{k/2}$. It follows from Lemma 2.4 that we can take $n = k/4$. □

Lemma 2.7: Let $p \nmid 2D$. We have

$$\sum_{j=0}^{k/2-1} \bar{v}_{2j}^2 = k, \quad \sum_{j=0}^{k/2-1} \bar{v}_{2j+1}^2 = -k, \quad \sum_{j=0}^{k-1} \bar{v}_j^4 = 6k.$$

Proof: Analogous to the proof of Lemma 7 in [1]. □

We remark that Lemma 2.6 and the last equation in Lemma 2.7 will not be used in this paper.

3. Proof of Theorem 1.2

To prove Theorem 1.2 we shall consider successively the cases $k \not\equiv 4 \pmod 8$ and $k \equiv 4 \pmod 8$. In the first case we prove (1.3).

1. Let $k \not\equiv 4 \pmod 8$. It follows from Lemma 2.4 that 0 occurs in the sequence $\bar{v}_n$ $(0 \le n < k)$. Thus, the sequence $\bar{v}_n$ $(0 \le n < k)$ is a non-zero multiple of a translation of the sequence $\bar{u}_n$ $(0 \le n < k)$. In fact, if t is the least positive integer such that $\bar{v}_n = 0$, then $-t$ is the amount by which the sequence $\bar{u}_n$ $(0 \le n < k)$ is translated and $\bar{v}_{t+1}$ is the constant multiplier. It then follows immediately that the sequences $\bar{v}_n$ $(0 \le n < k)$ and $\bar{u}_n$ $(0 \le n < k)$ have the same frequency pattern of residues appearing in these sequences. (1.3) now follows immediately.

2. Let $k \equiv 4 \pmod 8$. According to Lemma 2.4, 0 does not occur in the sequence $\bar{v}_n$ $(0 \le n < k)$ so that $0 \in S'$.

According to Lemma 2.2, every element in the sequence $\bar{v}_{2j}$ $(0 \le 2j < k)$ occurs there exactly twice, except for the elements $\bar{v}_0$ and $\bar{v}_{k/2}$, which occur once. Moreover, $\bar{v}_{k/2} = -\bar{v}_0$ by Lemma 2.3. Similarly, every element in the sequence $\bar{v}_{2j+1}(0 \le j < k/2)$ occurs there exactly twice, except for the elements $\bar{v}_{k/4}$ and $\bar{v}_{3k/4} = -\bar{v}_{k/4}$, which occur once.

Since $k \equiv 0 \pmod 4$, it follows from Lemma 2.1 that $\delta = k$ and, therefore, $\xi^{k/2} = -1$. Using (2.1), we see that

$$\bar{v}_{k/4}^2 = \bar{v}_{3k/4}^2 = -4. \tag{3.1}$$

We assume now that $2 \notin S'$. Consider the elements $\bar{v}_{2j}(0 < 2j < k/2)$. These must occur in the sequence $\bar{v}_{2j+1}$ $(0 \le 2j+1 < k)$. Since by Lemma 2.3

$$\bar{v}_{2j} = -\bar{v}_{k/2-2j}$$

there are two cases:

I. $\overline{v}_{2j} \neq \pm\overline{v}_{k/4}$ $(0 < 2j < k/2)$,

and

II. $\overline{v}_{2j'} = \overline{v}_{k/4}$ and $\overline{v}_{k/2-2j'} = \overline{v}_{3k/4}$ for some j' $(0 < 2j' < k/2)$.

We shall consider these two cases separately.

Case I: In this case of the two sequences

$\overline{v}_{2j}$ $(0 \le 2j < k,\ j \neq 0,\ j \neq k/4)$

and

$\overline{v}_{2j+1}$ $(0 \le 2j+1 < k,\ 2j+1 \neq k/4,\ 2j+1 \neq 3k/4)$

one is a permutation of the other. Using (3.1), it follows that

$$\sum_{j=0}^{k/2-1} \overline{v}_{2j}^2 - 2(4) = \sum_{j=0}^{k/2-1} \overline{v}_{2j+1}^2 - 2(-4),$$

from which we infer, using Lemma 2.7, that $2k \equiv 16 \pmod{p}$, $k \equiv 8 \pmod{p}$.
It follows from Lemma 2.5 that either

$k = 2(p+1)$ or $k \le p + 1$.

If $k = 2(p+1)$, then $k \equiv 8 \pmod{p}$ implies $3 \equiv 0 \pmod{p}$, which contradicts the assumption $p > 7$. If $k \le p + 1$, then we must have $k = 8$, which contradicts the assumption $k \equiv 4 \pmod{8}$.

Case II: In this case, there are two different elements in the sequence $\overline{v}_{2j+1}$ $(0 \le 2j+1 < k)$ which occur twice in this sequence and which are not equal to any element $\overline{v}_{2j}$ $(0 < 2j < k/2)$. Since we are assuming that $2 \notin S'$, these elements must appear in the sequence $\overline{v}_{2j}$ $(0 \le 2j < k)$ and, therefore, they must be $\overline{v}_0$ and $\overline{v}_{k/2} = -\overline{v}_0$. It follows that the sequences $\overline{v}_{2j}$ $(0 \le 2j < k)$ and $\overline{v}_{2j+1}$ $(0 \le 2j+1 < k)$ consist of the same elements. Moreover, $\overline{v}_0$ and $\overline{v}_{k/2}$, which occur in the former sequence once, occur in the latter sequence twice and the elements $\overline{v}_{2j'} = \overline{v}_{k/4}$ and $\overline{v}_{k/2-2j'} = \overline{v}_{3k/4}$, occurring in the former sequence twice, occur in the latter sequence once. It follows that

$$\sum_{j=0}^{k/2-1} \overline{v}_{2j}^2 - 2(4) - 4(-4) = \sum_{j=0}^{k/2-1} \overline{v}_{2j+1}^2 - 4(2) - 2(-4),$$

from which we obtain, using Lemma 2.7, that $2k \equiv -16 \pmod{p}$, $k \equiv -8 \pmod{p}$. In a similar manner to that in Case I, we conclude that either $5 \equiv 0 \pmod{p}$, a contradiction, or $k = p - 8 \equiv 1 \pmod{2}$, which contradicts Lemma 2.1.

The assumption $2 \notin S'$ thus leads to a contradiction in every case, so that we have proved that $2 \in S'$.

Now we prove that either $1 \in S'$ or $3 \in S'$ but not both. We must again look at the four elements $\overline{v}_0$, $\overline{v}_{k/2}$, $\overline{v}_{k/4}$, and $\overline{v}_{3k/4}$. It is clear that our assertion is true if we prove that the following four conditions are equivalent:

(3.2) $\exists n \equiv 1 \pmod{2}$ such that $\overline{v}_n = \overline{v}_0$,

(3.3) $\exists n \equiv 1 \pmod{2}$ such that $\overline{v}_n = \overline{v}_{k/2}$,

(3.4) $\exists n \equiv 0 \pmod{2}$ such that $\overline{v}_n = \overline{v}_{k/4}$,

(3.5) $\exists n \equiv 0 \pmod{2}$ such that $\overline{v}_n = \overline{v}_{3k/4}$.

Since $\overline{v}_{k/2} = -\overline{v}_0$ and $\overline{v}_{3k/4} = -\overline{v}_{k/4}$ it follows from Lemma 2.3 that

$(3.2) \Leftrightarrow (3.3)$ and $(3.4) \Leftrightarrow (3.5)$.

It remains to be proved that

$(3.2) \Leftrightarrow (3.4)$.

<u>(3.2) ⇒ (3.4)</u> Suppose that $n \equiv 1 \pmod 2$, $\overline{v}_n = \overline{v}_0$. We prove that

(3.6) $\quad \overline{v}_{n+k/4} = \overline{v}_{k/4}.$

Since $k/4 \equiv 1 \pmod 2$, this will prove (3.4). It follows from (2.1) that

(3.7) $\quad \xi^n - 1 = \xi^{-n} + 1$

and that (3.6) is equivalent to the equation

$$\xi^{n+k/4} + \xi^{-n-k/4} = \xi^{k/4} - \xi^{-k/4},$$

which, using (3.7), can be written as

(3.8) $\quad (\xi^n - 1)(\xi^{k/4} + \xi^{-k/4}) = 0.$

It follows from Lemma 4 in [1] that $\overline{u}_{k/4} = 0$. This, by (2.1), implies that (3.8) holds. Therefore, also (3.6) holds and we have proved the implication (3.2) ⇒ (3.4).

<u>(3.4) ⇒ (3.2)</u> Suppose that $n \equiv 0 \pmod 2$ and $\overline{v}_n = \overline{v}_{k/4}$. We prove that

(3.9) $\quad \overline{v}_{n+3k/4} = \overline{v}_0.$

Using (2.1), the equation (3.9) can be written as

(3.10) $\quad \xi^{n+3k/4} - \xi^{-n-3k/4} = 2.$

We find

$$\begin{aligned}\xi^{n+3k/4} &= (-\xi^{-n} + \xi^{k/4} - \xi^{-k/4})\xi^{3k/4} = -\xi^{-n+3k/4} + \xi^k - \xi^{k/2}\\ &= -\xi^{-n+3k/4} + 1 - (-1),\end{aligned}$$

so that (3.10) will follow if we show that

(3.11) $\quad \xi^{-n+3k/4} + \xi^{-n-3k/4} = \xi^{-n}(\xi^{3k/4} + \xi^{-3k/4}) = 0.$

But

$$(\xi^{3k/4} + \xi^{-3k/4})^2 = (\xi^{k/2})^3 + 2 + (\xi^{-k/2})^3 = (-1)^3 + 2 + (-1)^3 = 0,$$

so that (3.11) follows and the implication (3.4) ⇒ (3.2) is proved.

It has now been proved that the conditions (3.2)-(3.5) are all equivalent.

Since every residue occurs at most twice among $\overline{v}_{2j}$ $(0 \le 2j < k)$ and at most twice among $\overline{v}_{2j+1}$ $(0 < 2j+1 < k)$ it occurs at most four times among $\overline{v}_n$ $(0 \le n < k)$. It follows from what has been proved that, in the case $k \equiv 4 \pmod 8$, we have

$$S' = \{0, 1, 2\} \text{ or } \{0, 2, 3\} \text{ or } \{0, 1, 2, 4\} \text{ or } \{0, 2, 3, 4\}.$$

This completes the proof of Theorem 1.2. □

Proof of Corollary 1.3: For $p \nmid a$, this corollary follows directly from Theorem 1.2. For $p \mid a$, we have $\overline{v}_n = 0$ or 2; hence, $0 \in S'$. □

Proof of Corollary 1.4: If $k \not\equiv 4 \pmod 8$, then $S' = S$ by (1.3) and $2 \in S'$ follows from Schinzel's Corollary 2. Corollary 1.4 clearly holds for $p = 2$ by inspection. If $k \equiv 4 \pmod 8$, then the proof that $2 \in S'$ in the proof of Theorem 1.2 holds if $p > 7$. However, by (3.1), if $k \equiv 4 \pmod 8$, then

$$\overline{v}^2_{k/4} = \overline{v}^2_{3k/4} = -4,$$

which implies $p = 2$ or $p \equiv 1 \pmod 4$. Thus, $2 \notin S'$ can hold only if $p = 5$. □

Remark 3.1: Corollary 1.4 is not formulated as generally as the corresponding Corollary 2 in [1]. Example 3.2 shows that $2 \notin S'$ can occur if $p = 5$.

Example 3.2: Take $a = 2$ and $p = 5$, $p \nmid a(a^2 + 4) = 16$. Then $S' = \{0, 3\}$. In fact, the shortest period consists of the residues 2, 2, 1, 4, 4, 2, 3, 3, 4, 1, 1, 3. Note that in this case $k = 2p + 2 = 12 \equiv -8 \pmod{p}$ which was a possibility in Case II.

Proof of Corollary 1.5: This corollary follows from Corollary 3 in [1] and Theorem 1.2. $\square$

We conclude this note by making the following observation. We can look at Corollary 2 in [1] and the corresponding Corollary 1.4 at the same time and calculate the *smallest residue which appears exactly twice in the shortest period*. Keeping the integer a fixed and considering primes $p > 5$, $p \nmid a(a^2 + 4)$ let us denote these residues by $sr_2\overline{u}(p)$ and $sr_2\overline{v}(p)$. It therefore follows from Lemma 4 in [1] and Lemma 2.4 above that we have the following result:

$$sr_2\overline{u}(p) = 0 \Leftrightarrow sr_2\overline{v}(p) = 0 \Leftrightarrow k(p) \equiv 0 \pmod{8}.$$

Acknowledgment

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SECOND-ORDER STOLARSKY ARRAYS

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In 1977, Kenneth B. Stolarsky [6] introduced an array $s(i, j)$ of positive integers such that every positive integer occurs exactly once in the array, and every row satisfies the familiar Fibonacci recurrence:

$$s(i, j) = s(i, j - 1) + s(i, j - 2) \text{ for all } j \geq 3 \text{ for all } i \geq 1.$$

The first seven rows of Stolarsky's array begin as shown here:

1	2	3	5	8	13	21	...
4	6	10	16	26	42	68	...
7	11	18	29	47	76	123	...
9	15	24	39	63	102	165	...
12	19	31	50	81	131	212	...
14	23	37	60	97	157	254	...
17	28	45	73	118	191	309	...

Hendy [4], Butcher [2], and Gbur [3] considered Stolarsky's array, and Morrison [5] and Burke and Bergum [1, p. 146] considered closely related arrays. In particular, Gbur discussed arrays whose row recurrence is given by

$$s(i, j) = as(i, j - 1) + s(i, j - 2),$$

which, for $a = 1$, is the row recurrence for Stolarsky's original array. In this note, we show that any one of a larger class of second-order recurrences can be used to construct infinitely many Stolarsky arrays.

Define a *Stolarsky pre-array (of q rows)* as an array $s(i, j)$ of distinct positive integers satisfying

$$s(i, j) = as(i, j - 1) + bs(i, j - 2) \text{ for all } j \geq 3 \text{ for } 1 \leq i \leq q,$$

where a and b are integers satisfying $1 \leq b \leq a$, and the numbers 1, 2, 3, ..., q are all present in the array. By a *Stolarsky array* we shall mean an array $s(i, j)$ whose first q rows comprise a Stolarsky pre-array for every positive integer q. For the following Stolarsky pre-array, $q = 2$, $a = 1$, and $b = 1$:

1	4	5	9	12	23	37	60	...
2	8	10	18	28	46	74	120	...

In order to construct Row 3 beginning with $s(3, 1) = 3$, note that $s(3, 2)$ cannot be 4 or 5, as these appear in Row 1; nor 6, as then $s(3, 3) = 9$, already in Row 1; nor 7 nor 8 nor 9 nor 10 nor 11. These observations illustrate the problem: *once q rows of a (prospective) Stolarsky array have been constructed, can Row $q + 1$ always be constructed*? We shall show that the answer is *yes*, and that, actually, Row $q + 1$ can be constructed in infinitely many ways.

The symbols s_1, s_2, ... will always represent a sequence of the following kind:

(i) $s_1 > 0$, $s_2 > 0$, and $s_n = as_{n-1} + bs_{n-2}$ for $n \geq 3$,

where a and b are integers satisfying $1 \leq b \leq a$. Let

$$\alpha = \frac{a + \sqrt{a^2 + 4b}}{2} \quad \text{and} \quad \beta = a - \alpha,$$

so that $\alpha > 1$, $-1 < \beta < 0$, and the identities $\alpha^2 = a\alpha + b$ and $\beta^2 = a\beta + b$ yield

(ii) $s_n = a_1\alpha^n + b_1\beta^n$ for all $n \geq 1$, where

$$a_1 = \frac{s_1\beta - s_2}{\alpha(\beta - \alpha)} \quad \text{and} \quad b_1 = \frac{s_2 - s_1\alpha}{\beta(\beta - \alpha)}.$$

Similarly, the symbols $t_1, t_2, \ldots$ will always mean a sequence given by

$$t_n = at_{n-1} + bt_{n-2} = a_2\alpha^n + b_2\beta^n,$$

where

$$a_2 = \frac{t_1\beta - t_2}{\alpha(\beta - \alpha)} \quad \text{and} \quad b_2 = \frac{t_2 - t_1\alpha}{\beta(\beta - \alpha)}, \quad \text{and } t_1 > 0,\ t_2 > 0.$$

Lemma 1.1: There exists a positive integer N such that $s_{n+1} = [\alpha s_n + \frac{1}{2}]$ for every $n \geq N$. The least such N is $2 + [\log_{\alpha/b} 2|\alpha s_1 - s_2|]$.

Proof: $\alpha s_n = \alpha(a_1\alpha^n + b_1\beta^n) = a_1\alpha^{n+1} + b_1\beta^{n+1} + \alpha b_1\beta^n - b_1\beta^{n+1}$

$$= s_{n+1} + b_1\beta^n(\alpha - \beta),$$

so that $s_{n+1} = [\alpha s_n + \frac{1}{2}]$ if and only if $0 < b_1\beta^n(\alpha - \beta) + \frac{1}{2} < 1$. This is equivalent to $-1 < 2(\alpha s_1 - s_2)\beta^{n-1} < 1$, hence to

$$\left(\frac{b}{\alpha}\right)^{n-1} = |\beta^{n-1}| < \frac{1}{2|\alpha s_1 - s_2|},$$

and hence equivalent to $n - 1 \geq \log_{\alpha/b} 2|\alpha s_1 - s_2|$, as required.

Lemma 1.2: Suppose s_1 is not among $t_1, t_2, \ldots$, and t_1 is not among $s_1, s_2, \ldots$. Let

$$M = 2 + [\log_{\alpha/b} 2|\alpha s_1 - s_2|] \quad \text{and} \quad N = 2 + [\log_{\alpha/b} 2|\alpha t_1 - t_2|].$$

If $m \geq M$, $n \geq N$, and $s_m < t_n \leq s_{m+1}$, then $s_m < t_n < s_{m+1} < t_{n+1} < s_{m+2} < \cdots$.

Proof: Suppose $m \geq M$ and $n \geq N$. By Lemma 1.1, $s_{i+1} = [\alpha s_i + \frac{1}{2}]$ for every $i \geq m$ and $t_{i+1} = [\alpha t_i + \frac{1}{2}]$ for every $i \geq n$. So, if $t_n = s_{m+1}$, then

$$[\alpha t_n + \tfrac{1}{2}] = [\alpha s_{m+1} + \tfrac{1}{2}],$$

so that $t_{n+1} = s_{m+2}$. But then $at_n + bt_{n-1} = as_{m+1} + bs_m$, so that $t_{n-1} = s_m$. But then $at_{n-1} + bt_{n-2} = as_m + bs_{m-1}$, so that $t_{n-2} = s_{m-1}$. Continuing, we eventually reach $t_1 = s_p$ for some $p \geq 1$ or else $t_q = s_1$ for some $q \geq 1$, contrary to the hypothesis.

Now that we have $s_m < t_n$ and $t_n < s_{m+1}$, the remaining inequalities in the asserted chain follow by induction: $s_p < t_q$ implies

$$[\alpha s_p + \tfrac{1}{2}] < [\alpha t_q + \tfrac{1}{2}],$$

so that $s_{p+1} < t_{q+1}$, and $t_q < s_r$ similarly implies $t_{q+1} < s_{r+1}$.

Lemma 1.3: Suppose s_1, s_2, and t_1 are given and $t_1 > s_1$. For $k \geq 1$, let $t_j^{(k)}$ denote the sequence t_1, $t_2 = t_1 + k$, $t_3 = at_2 + bt_1$, $\ldots$. Then there exist positive integers C and K, both independent of k, such that if $k > K$ and $m > C[\log_\alpha k]$ and n is the index satisfying $s_m < t_n^{(k)} \leq s_{m+1}$, then

$$s_m < t_n^{(k)} < s_{m+1} < t_{n+1}^{(k)} < s_{m+1} < \cdots .$$

Proof: Let

$$M = 2 + [\log_{\alpha/b} 2|\alpha s_1 - s_2|] \quad \text{and} \quad N(k) = 2 + [\log_{\alpha/b} 2|\alpha t_1 - t_1 - k|].$$

Let $p(k)$ be the index satisfying

$$s_{p(k)} < t_{N(k)}^{(k)} \leq s_{p(k)+1}.$$

Clearly, there is a positive integer K_1 so large that $p(k) \geq M$ for all $k \geq K_1$. For such k, Lemma 1.2 gives

(1) $\quad s_{p(k)+h} < t^{(k)}_{N(k)+h} < s_{p(k)+1+h}$ for all $h \geq 0$.

Also, for all $k \geq K_1$,

$$a_1\alpha^{p(k)} + b_1\beta^{p(k)} = s_{p(k)} < t^{(k)}_{N(k)} = a_2\alpha^{N(k)} + b_2\beta^{N(k)} < (a_2 + |b_2|)\alpha^{N(k)}.$$

Let A, B, K_2 be positive integers, with $K_2 > K_1$, all independent of k, satisfying $a_2 + |b_2| < A + Bk$ for all $k > K_2$; to see that such A and B exist, observe

$$a_2 = \frac{t_1\beta - (t_1 + k)}{\alpha(\beta - \alpha)} \quad \text{and} \quad b_2 = \frac{t_1 + k - t_1\alpha}{\beta(\beta - \alpha)}.$$

For all such k,

$$a_1\alpha^{p(k)} < (A + Bk)\alpha^{N(k)} + Q(k), \text{ where } Q(k) = 1 + |b_1\beta^{p(k)}|.$$

Then

$$a_1\alpha^{p(k)} < Q(k) + (A + Bk)\alpha^{2+\log_{\alpha/b} 2|\alpha t_1 - t_1 - k|},$$

so that

$$a_1\alpha^{p(k)} < Q(k) + \alpha^2(A + Bk)(2|\alpha t_1 - t_1 - k|)^{\frac{1}{1-\log_\alpha b}}.$$

Applying $\log_\alpha$ to both sides and the inequality $\log_\alpha(x + y) < \log_\alpha x + \log_\alpha y$ to the resulting right-hand side yields

$$p(k) + \log_\alpha a_1 < \log_\alpha Q(k) + 2 + \log_\alpha(A + Bk) + \frac{1}{1 - \log_\alpha b}\log_\alpha(2|\alpha t_1 - t_1 - k|).$$

Now $\lim_{k\to\infty} Q(k) = 1$, so that there must exist positive integers C and K_3, independent of k, with $K_3 > K_2$, such that

$$p(k) + 1 < C[\log_\alpha k] \text{ for all } k > K_3.$$

For such k, if m is any integer that exceeds $C[\log k]$, then $m = p(k) + h$ for some $h \geq 1$. For $n = N(k) + m - p(k)$, the stated chain of inequalities follows from (1).

Theorem: Let $S = \{s(x, y): 1 \leq x \leq q, y \geq 1\}$ be a Stolarsky pre-array. Suppose $t_1 \notin S$ and $t_1 > \max\{s(x, 1): 1 \leq x \leq q\}$. Then there exist infinitely many numbers t_2 such that no term of the sequence $t_1, t_2, t_3 = at_2 + bt_1, \ldots$ lies in S.

Proof: Suppose, to the contrary, that there are at most finitely many numbers $k \geq 1$ for which the sequence t_1, $t_2 = t_1 + k$, $t_3 = at_2 + bt_1$, ... contains no element of S. Let k_1 be the greatest of these k. Let $t^{(k)}_1, t^{(k)}_2, \ldots$ denote the (a, b)-recurrence sequence whose first two terms are t_1 and $t_2 = t_1 + k_1 + k$. Then, for every positive integer k, the sequence $t^{(k)}_1, t^{(k)}_2, \ldots$ contains a term of S. That is, there exist indices $j(k)$, $x(k)$, and $y(k)$ for which

(2) $\quad t^{(k)}_{j(k)} = s(x(k), y(k))$, where

(3) $\quad 1 \leq x(k) \leq q$.

On the other hand, by Lemma 1.3, there exist constants $C_1, C_2, \ldots, C_q$ and $K_1, K_2, \ldots, K_q$, all independent of k, such that for $x = 1, 2, \ldots, q$, if

$$y_x > C_x[\log_\alpha k]$$

where $k > K_x$ and j_x is the index for which

$$s(x, y_x) < t^{(k)}_{j_x} \leq s(x, y_x + 1),$$

then equation (2) cannot hold for any $j(k) \leq j_x$. Accordingly, (2) implies

(4) $\quad 1 \le y(k) \le C_{x(k)}[\log k]$ for all $k > K = \max\{K_1, K_2, \ldots, K_q\}$.

Now, since the index $x(k)$ in (2) is $\le q$, we have $s(x(k), 1) < t_1^{(k)}$ for all k, by hypothesis, and also $s(x(k), 2) < t_2^{(k)}$ for all k larger than some K^*. Therefore, in equation (2), $j(k) \le y(k)$, so that

(5) $\quad 1 \le j(k) \le C_{x(k)}[\log_\alpha k]$ for all $k > K^*$.

Let $m(k) = [\log_\alpha k] \max\{C_1, C_2, \ldots, C_q\}$. Then, for all $k > \mathbb{K} = \max\{K, K^*\}$, we have

$$1 \le x(k) \le q,\ 1 \le y(k) \le m(k),\ 1 \le j(k) \le m(k).$$

Let k' be any integer large enough that $k' > q[m(\mathbb{K} + k')]^2$. Then, for $k = 1, 2, 3, \ldots, k'$, we have

$$1 \le x(\mathbb{K} + k) \le q,\ 1 \le y(\mathbb{K} + k) \le m(\mathbb{K} + k'),\ 1 \le j(\mathbb{K} + k) \le m(\mathbb{K} + k').$$

Now, the total number of *distinct* triples (x, y, j) that can satisfy three such inequalities is the product $q[m(\mathbb{K} + k')]^2$, but we have more than this number. Therefore, there exist distinct k_u and k_v for which

$$x(k_u) = x(k_v),\ y(k_u) = y(k_v),\ j(k_u) = j(k_v).$$

This means that the sequences

$$t_1,\ t_2^{(k_u)},\ \ldots,\ t_{j(k_u)}^{(k_u)},\ \ldots \quad \text{and } t_1,\ t_2^{(k_v)},\ \ldots,\ t_{j(k_v)}^{(k_v)},\ \ldots$$

have identical first terms and identical $j(k_u)^{\text{th}}$ terms. But this implies

$$t_2^{(k_u)} = t_2^{(k_v)},$$

contrary to $k_u \ne k_v$. This contradiction finishes the proof.

Conclusion

An obvious consequence of the theorem is that any Stolarsky pre-array can be extended to a Stolarsky array. For each new row, one need only choose t_1 to be the *least* positive integer satisfying the hypothesis of the theorem; that is, the least not yet present in the array being constructed. This choice ensures that every positive integer must occur in the constructed Stolarsky array.

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ARITHMETIC SEQUENCES AND FIBONACCI QUADRATICS

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1. Introduction

It is known [1] that the equation $F_n x^2 + F_{n+1}x - F_{n+2} = 0$ has solutions -1 and F_{n+2}/F_n, where $\{F_n\}_{n \geq 1}$ denotes the Fibonacci sequence. One wonders if other interesting results might be obtained if the coefficients of the quadratic equation were some other functions of the Fibonacci numbers. The answer, as might be expected, is in the affirmative. Surprisingly, however, the results in this paper arise in response to the following quite different question. Under what conditions does the quadratic equation $ax^2 + bx - c = 0$ have rational roots given that a, b, and c are represented by the arithmetic sequence n, $n + r$, $n + 2r$ in some order, where n and r are positive integers? In this paper, we treat only the case $r = 1$.

As usual, $\{L_n\}_{n \geq 1}$ will denote the Lucas sequence and α the golden ratio. Moreover, we will have occasion to use such well-known results as

$$L_n = F_{n+1} + F_{n-1},\ L_n + F_n = 2F_{n+1},\ L_n - F_n = 2F_{n-1},\ \alpha^n = (L_n + F_n\sqrt{5})/2$$

(see [2]). Note that $L_n = F_{n+1} + F_{n-1}$ can be written as

(1) $\quad L_n = 2F_{n-1} + F_n.$

Also, we will need the following identities from [2]:

(2a) $\quad F_{n+1}^2 = F_n F_{n+2} + (-1)^n;$

(2b) $\quad F_{n+1}F_{n-2} = F_n F_{n-1} + (-1)^{n+1}.$

2. Fibonacci Quadratics

The equations

$$ax^2 + bx - c = 0,\quad ax^2 - bx - c = 0,$$
$$cx^2 + bx - a = 0,\quad \text{and } cx^2 - bx - a = 0$$

have the same discriminant. Therefore, we shall study only the first one. Let us consider the case $r = 1$.

Theorem 1: Rational solutions to

(3) $\quad nx^2 + (n + 1)x - (n + 2) = 0$

exist *if and only if*

(4a) $\quad n = F_{2m+1} - 1 \quad (m \geq 1)$

and they are

(4b) $\quad F_{2m}/(F_{2m+1} - 1),\quad -F_{2m+2}/(F_{2m+1} - 1).$

Proof: The discriminant of (3) is

$$D_1 = (n + 1)^2 + 4n(n + 2)$$
$$= 5(n + 1)^2 - 4.$$

Rational solutions of (3) exist if and only if D_1 is a perfect square, say, for example, $D_1 = t^2$. Then we have

(4c) $\quad t^2 - 5(n + 1)^2 = -4,$

which has positive solutions $t = L_{2m+1}$ and $n = F_{2m+1} - 1$ with $m \geq 1$ for $n \neq 0$, as shown by Long and Jordan [4, Lemma 1], although their proof can be considerably simplified by the use of the identity $\alpha_n = (L_n + F_n\sqrt{5})/2$. But, by (1), $t = 2F_{2m} + F_{2m+1}$ and $b = n + 1 = F_{2m+1}$. Using these values in

$$x = (-b \pm t)/2n,$$

we get (4b). It is interesting to note that the solutions are proportional to F_{2m} and F_{2m+2}, which precede and follow F_{2m+1}, respectively.

Theorem 2: Rational solutions to

(5) $\quad nx^2 + (n + 2)x - (n + 1) = 0$

exist *if and only if*

(6a) $\quad n = F_{2m+3}F_{2m} \quad (m \geq 1)$

and they are

(6b) $\quad F_{2m+2}/F_{2m+3}, \quad -F_{2m+1}/F_{2m}.$

Proof: The discriminant of (5) is

$$\begin{aligned} D_2 &= (n + 2)^2 + 4n(n + 1) \\ &= n^2 + 4(n + 1)^2. \end{aligned}$$

Rational solutions of (5) exist if and only if D_2 is a perfect square, $D_2 = t^2$. Thus, $[n, 2(n + 1), t]$ form a Pythagorean triplet, not necessarily primitive. We represent the triplet as $(g^2 - h^2, 2gh, g^2 + h^2)$ to get

(6c) $\quad g^2 - gh - (h^2 - 1) = 0.$

[Note that if it were represented as $(2gh, g^2 - h^2, g^2 + h^2)$ then $g^2 - h^2 = 4gh + 2$ and this implies $g^2 - h^2 \equiv 2 \pmod 4$, an impossibility.] But, again, g is an integer if and only if the discriminant of (6c) is a perfect square:

$$h^2 + 4(h^2 - 1) = 5h^2 - 4 = s^2$$

or

(6d) $\quad s^2 - 5h^2 = -4.$

This is the same Pell equation as before and so has solutions $s = L_{2m+1}$ and $h = F_{2m+1}$. Now

$$g = (h \pm s)/2 = [F_{2m+1} \pm L_{2m+1}]/2 = (F_{2m+1} + F_{2m}), -F_{2m} = F_{2m+2}, -F_{2m}.$$

Since only the first solution gives positive n,

$$n = g^2 - h^2 = F^2_{2m+2} - F^2_{2m+1} = F_{2m+3}F_{2m},$$

with $m \geq 1$, for $n \neq 0$. In this case, using (2b) and (2a), we obtain

$$\begin{aligned} b = F_{2m+3}F_{2m} + 2 &= F_{2m+2}F_{2m+1} + 1 = F_{2m+2}(F_{2m+2} - F_{2m}) + 1 \\ &= F^2_{2m+2} - F_{2m+2}F_{2m} + 1 = F_{2m+3}F_{2m+1} - F_{2m+2}F_{2m} \end{aligned}$$

and

$$t = g^2 + h^2 = F^2_{2m+2} + F^2_{2m+1} = F_{2m+3}F_{2m+1} + F_{2m+2}F_{2m}.$$

Using these in $x = (-b \pm t)/2n$, we obtain the solutions (6b) as claimed.

The last equation to be considered is

$$(n+1)x^2 + nx - (n+2) = 0.$$

Instead, we investigate the equivalent equation

$$nx^2 + (n-1)x - (n+1) = 0.$$

Theorem 3: Rational solutions to

(7) $$nx^2 + (n-1)x - (n+1) = 0$$

exist *if and only if*

(8a) $$n = F_{2m+1}F_{2m} \quad (m \geq 1)$$

and they are

(8b) $$F_{2m-1}/F_{2m}, \quad -F_{2m+2}/F_{2m+1}.$$

Proof: The discriminant of (7) is

$$D_3 = (n-1)^2 + 4n(n+1) = 4n^2 + (n+1)^2.$$

Rational solutions of (7) exist if and only if D_3 is a perfect square, $D_3 = t^2$. Thus, $(2n, n+1, t)$ form a Pythagorean triplet. We represent the triplet as $(2gh, g^2 - h^2, g^2 + h^2)$ to get

(8c) $$g^2 - gh - (h^2 + 1) = 0.$$

[Note that if it were represented as $(g^2 - h^2, 2gh, g^2 + h^2)$ then we would have $4gh - 2 = g^2 - h^2$ and this implies $g^2 - h^2 \equiv 2 \pmod 4$, an impossibility.] As before, g is an integer if and only if the discriminant of (8c) is a perfect square:

$$h^2 + 4(h^2 + 1) = 5h^2 + 4 = s^2$$

or

(8d) $$s^2 - 5h^2 = 4$$

which has positive solutions $s = L_{2m}$ and $h = F_{2m}$ for $m \geq 1$ by [4, Lemma 2]. Since

$$g = (h \pm s)/2 = (F_{2m} \pm L_{2m})/2 = (F_{2m} + F_{2m-1}), -F_{2m-1} = F_{2m+1}, -F_{2m-1}.$$

Only the first solution gives positive n:

$$n = gh = F_{2m+1}F_{2m}$$

with $m \geq 1$, for $n \neq 0$. In this case, using (2a) and (2b), we have that

$$b = F_{2m+1}F_{2m} - 1 = F_{2m}(F_{2m+2} - F_{2m}) - 1 = F_{2m+2}F_{2m} - (F_{2m}^2 + 1)$$
$$= F_{2m+2}F_{2m} - F_{2m+1}F_{2m-1}$$

and

$$t = g^2 + h^2 = F_{2m+1}^2 + F_{2m}^2 = F_{2m+2}F_{2m} + F_{2m+1}F_{2m-1}.$$

Using these in $x = (-b \pm t)/2n$, we obtain the solutions (8b) as claimed.

The case $r > 1$ is under consideration.

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ϕ-PARTITIONS

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The purpose of this paper is to study partitions of positive integers for which Euler's totient function is endomorphic. That is, $n = a_1 + \cdots + a_i$ is a ϕ-partition if $i \geq 2$, and $\phi(n) = \phi(a_1) + \cdots + \phi(a_i)$.

Questions related to two-summand ϕ-partitions have been considered by the present author [2] and by Makowski [3]; here, we generalize to ϕ-partitions with an arbitrary number of summands. Results include: characterizations of positive integers which have at least one ϕ-partition and of those which have only one ϕ-partition; constructive proof that any prime p has exactly $\pi(p)$ ϕ-partitions; and techniques for constructing ϕ-partitions and reduced ϕ-partitions for various types of positive integers.

Throughout the paper, p and q will denote distinct primes and n will denote a positive integer.

Definition 1: A square-free n is *simple* if $n = 1$ or n has maximal prime divisor p and $q|n$ for every prime $q < p$.

Lemma 2: If s is simple, $n < 2s$, and $n \neq s$, then $\dfrac{s}{\phi(s)} > \dfrac{n}{\phi(n)}$.

Proof: Let $s = 2 \cdot 3 \cdot \cdots \cdot p_i$, and let $2s > n = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$ for $q_1 < \cdots < q_k$. Since $n < 2s$, we have $k \leq i$, and since s is simple, we have $q_j \geq p_j$ for each $1 \leq j \leq k$. If $k = i$ and $q_j = p_j$ for every $1 \leq j \leq k$, then $n = s$. Thus, $k < i$ or $q_j > p_j$ for some $1 \leq j \leq k$. In either case,

$$\frac{n}{\phi(n)} = \frac{q_1 \cdots q_k}{(q_1 - 1) \cdots (q_k - 1)} < \frac{1 \cdot 2 \cdot \cdots \cdot p_i}{1 \cdot 2 \cdot \cdots \cdot (p_i - 1)} = \frac{s}{\phi(s)}.$$

Theorem 3: n has at least one ϕ-partition iff n is not simple.

Proof: (i) Let n be nonsimple. Then there exists a prime p such that $p^\alpha | n$ for $\alpha > 1$, or n is square-free with maximal prime divisor p and there exists $q < p$ such that $q \nmid n$.

Suppose $p^\alpha \| n$ for $\alpha > 1$, and let $n = p^\alpha t$. Then $\phi(n) = \phi(p^\alpha t) = p\phi(p^{\alpha-1}t)$. Hence, $n = \underbrace{p^{\alpha-1}t + \cdots + p^{\alpha-1}t}_{p \text{ summands}}$ is a ϕ-partition.

Now suppose n is square-free with maximal prime divisor p and there exists $q < p$ such that $q \nmid n$. Let $n = pj$ and $p - q = a$. Then

$$\begin{aligned}\phi(pj) &= \phi(p)\phi(j) = (p - 1)\phi(j) = (a + q - 1)\phi(j)\\ &= a\phi(j) + (q - 1)\phi(j) = a\phi(j) + \phi(qj).\end{aligned}$$

Hence, $n = \underbrace{j + \cdots + j}_{a \text{ summands}} + qj$ is a ϕ-partition.

(ii) Suppose $n = 2 \cdot 3 \cdot \cdots \cdot p_k$ is simple and $n = a_1 + \cdots + a_i$ is a ϕ-partition. Let a_j be a summand of the partition. Since $a_j < n$, it follows from Lemma 2 that

$$\frac{a_j}{\phi(a_j)} < \frac{n}{\phi(n)}.$$

Hence,

$$n = \frac{n}{\phi(n)}\phi(n) = \frac{n}{\phi(n)}\phi(a_1) + \cdots + \frac{n}{\phi(n)}\phi(a_i)$$

$$> \frac{a_1}{\phi(a_1)}\phi(a_1) + \cdots + \frac{a_i}{\phi(a_i)}\phi(a_i) = a_1 + \cdots + a_i.$$

This contradiction completes the proof.

Lemma 4: If $n = a_1 + \cdots + a_i$ is a unique ϕ-partition of n, then each summand is simple.

Proof: Suppose $n = a_1 + \cdots + a_i$ is a unique ϕ-partition and some summand a_j is not simple. Then, by Theorem 3, a_j has a ϕ-partition $a_j = b_1 + \cdots + b_k$; thus, $n = a_1 + \cdots + a_{j-1} + b_1 + \cdots + b_k + a_{j+1} + \cdots + a_i$ is a ϕ-partition of n which is different from $n = a_1 + \cdots + a_i$.

Lemma 5: If a unique ϕ-partition of n has two equal summands, then $n = 2s$ for s simple.

Proof: Suppose $n = s + s + a_1 + \cdots + a_i$ is a unique ϕ-partition of n. If some summand $a_j \neq 0$, then $n = 2s + a_1 + \cdots + a_i$ is a different ϕ-partition of n. Therefore, each $a_j = 0$ and $n = 2s$. By Lemma 4, s is simple.

Theorem 6: n has a unique ϕ-partition iff $n = 2s$ for s simple or $n = 3$.

Proof: (i) Suppose n has a unique ϕ-partition. Then, by Theorem 3, n is not simple.

If n is square-free with maximum prime divisor p and $q < p$ such that $q \nmid n$, let $n = pj$ and $p - q = a$. Then, from the proof of Theorem 3(i), we have

$$n = \underbrace{j + \cdots + j}_{a \text{ summands}} + qj \text{ is a } \phi\text{-partition.}$$

And since it is unique, Lemma 4 implies that j is simple and Lemma 5 implies that $a = 1$. Thus, $p - q = 1$. Hence, we have $p = 3$, $q = 2$, and $n = 3$.

Now suppose $p^\alpha \| n$ for $\alpha > 1$ and $n = p^\alpha t$. Then

$$n = \underbrace{p^{\alpha-1}t + \cdots + p^{\alpha-1}t}_{p \text{ summands}} \text{ is a } \phi\text{-partition,}$$

and since it is unique, we have that $p^{\alpha-1}t$ is simple (Lemma 4). Therefore, by Lemma 5, $n = 2s$ for s simple.

(ii) It is obvious that $3 = 1 + 2$ is a unique ϕ-partition of 3.

Let $n = 2s$ for s simple. Clearly, $2s = s + s$ is a ϕ-partition. Suppose $2s = a_1 + \cdots + a_i$ is a different ϕ-partition. Then there exists a summand $a_j \neq s$. Since $a_j < 2s$, we have, by Lemma 2, that

$$\frac{a_j}{\phi(a_j)} < \frac{s}{\phi(s)}.$$

This gives the contradiction,

$$2s = \frac{2s\phi(s)}{\phi(s)} = \frac{s\phi(2s)}{\phi(s)} = \frac{s}{\phi(s)}(\phi(a_1) + \cdots + (a_i))$$

$$= \frac{s}{\phi(s)}\phi(a_1) + \cdots + \frac{s}{\phi(s)}\phi(a_i) > \frac{a_1}{\phi(a_1)}\phi(a_1) + \cdots + \frac{a_i}{\phi(a_i)}\phi(a_i)$$

$$= a_1 + \cdots + a_i.$$

Hence, $2s = s + s$ is a unique ϕ-partition of n.

Theorem 7: $p = a_1 + \cdots + a_i$ is a ϕ-partition iff one summand is prime and every other summand is 1.

Proof: (i) $p = \underbrace{1 + \cdots + 1}_{p-q \text{ summands}} + q$ is clearly a ϕ-partition for every prime $q < p$.

(ii) Let $p = a_1 + \cdots + a_i$ be a ϕ-partition. It is obvious that at least one summand is greater than 1. Suppose the two summands, a_1 and a_2, are each greater than 1. Then $\phi(a_1) \le a_1 - 1$ and $\phi(a_2) \le a_2 - 1$. Therefore, we have the contradiction

$$a_1 + \cdots + a_i - 1 = p - 1 = \phi(p)$$
$$= \phi(a_1) + \cdots + \phi(a_i) \le a_1 + \cdots + a_i - 2.$$

Assume $a_1 > 1$. Then $a_1 = p - i + 1$, and

$$p - 1 = \phi(p) = \underbrace{\phi(1) + \cdots + \phi(1)}_{i-1 \text{ summands}} + \phi(a_1) = i - 1 + \phi(a_1).$$

Hence, $\phi(a_1) = p - i = a_1 - 1$. Therefore, a_1 is prime.

As an immediate consequence of this theorem, we get

Corollary 8: A prime p has exactly $\pi(p)$ ϕ-partitions.

We now provide two very general techniques for constructing ϕ-partitions for a particular n.

1. If n is even, $p \| n$, $p = 2^{a_1} + \cdots + 2^{a_i} + q$, $q \nmid n$, and $n = 2^\alpha pm$, then $n = 2^{a_1 + \alpha}m + \cdots + 2^{a_i + \alpha}m + 2^\alpha mq$ is a ϕ-partition.

Some results regarding how many ways a particular prime p can be written as the sum of a prime and powers of 2 are given in [1].

Definition 9: A positive integer m is *prime dependent* on n if every prime divisor of m is a divisor of n.

Notice that if m is prime dependent on n then $\phi(mn) = m\phi(n)$.

2. If $n = p^\alpha t$ where $\alpha > 1$ and $p \nmid t$, and $p = a_1 + \cdots + a_i$ such that each summand is prime dependent on n, then

$$n = a_1 p^{\alpha-1}t + \cdots + a_i p^{\alpha-1}t \text{ is a } \phi\text{-partition.}$$

Notice that for every p such that $p^\alpha | n$ for $\alpha > 1$ we get a ϕ-partition of n with p summands by letting

$$p = \underbrace{1 + \cdots + 1}_{p \text{ summands}}$$

in construction 2. If n is even, for each such p we can get ϕ-partitions with x summands for every x satisfying $a \le x \le p$, where a is the number of nonzero digits in the binary representation of p.

Definition 10: If $n = a_1 + \cdots + a_i$ and $a_1 = b_1 + \cdots + b_j$ are ϕ-partitions, then $n = b_1 + \cdots + b_j + a_2 + \cdots + a_i$ is an expansion of $n = a_1 + \cdots + a_i$.

Expansions are clearly ϕ-partitions.

Definition 11: A ϕ-partition is *reduced* if each of its summands is simple.

It is obvious that a ϕ-partition can be expanded iff it is not reduced. So every nonsimple number has at least one reduced ϕ-partition. The following are examples of reduced ϕ-partitions for various types of n:

(i) $2^a = \underbrace{2 + \cdots + 2}_{2^{a-1} \text{ summands}}$

(ii) $p^\alpha = \underbrace{1 + \cdots + 1}_{\substack{p^{\alpha-1}(p-2) \\ \text{summands}}} + \underbrace{2 + \cdots + 2}_{\substack{p^{\alpha-1} \\ \text{summands}}}$

$$\text{(iii)}\quad 2^a p^\alpha = \underbrace{2 + \cdots + 2}_{\substack{2^{a-1}p^{\alpha-1}(p-3)\\ \text{summands}}} + \underbrace{6 + \cdots + 6}_{\substack{2^{a-1}p^{\alpha-1}\\ \text{summands}}}$$

$$\text{(iv)}\quad pq = \underbrace{1 + \cdots + 1}_{\substack{(p-2)(q-2)\\ \text{summands}}} + \underbrace{2 + \cdots + 2}_{\substack{p+q-5\\ \text{summands}}} + 6$$

Several open questions about two-summand ϕ-partitions could be resolved if it can be shown that reduction is unique. Evidence and intuition strongly suggest that it is; but it seems that a proof may be quite difficult. We close with the conjecture: Every nonsimple number has exactly one reduced ϕ-partition.

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ITERATIONS OF A KIND OF EXPONENTIALS

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1. Introduction

We shall study a sequence of numbers defined recursively. Let ln denote the principal branch of the natural logarithm, i.e., $\ln(re^{i\theta}) = \ln r + i\phi$, $r > 0$, with $\phi \equiv \theta \pmod{2\pi}$, $-\pi < \phi \le \pi$. We put $0 \square z := z$, $1 \square z := z^z (:= e^{z \ln z})$ and

$$(1) \qquad (n+1) \square z = (n \square z)^{(n \square z)}, \quad n = 0, 1, 2, \ldots$$

$$(n \square 1 = 1, \; n \square (-1) = -1, \; 1 \square i = e^{-\pi/2}).$$

We consider, in fact, a more general operation defined by

$$\alpha_0(a, b) := b, \; \alpha_1(a, b) := b^{b^a}$$

and

$$(2) \qquad \alpha_{n+1}(a, b) := \alpha_n(a, b)^{\alpha_n^a(a, b)}, \quad n = 0, 1, 2, \ldots$$

$$\left(\alpha_n(1, z) = n \square z, \; n \square i = \alpha_{n-1}^{-\pi/2}\left(-\frac{\pi}{2}, e\right)\right).$$

By mathematical induction, we obtain the

Proposition: The following algebraic relations hold for all $n, m \in \mathbb{N}$ and $a, b, c, z \in \mathbb{C}$:

a) $\alpha_{n+m}(a, b) = \alpha_n(a, \alpha_m(a, b))$ [in particular $(n + m) \square z = n \square (m \square z)$].

b) $\alpha_n(a, b^c) = \alpha_n^c(ac, b)$ [in particular $n \square z^c = \alpha_n^c(c, z)$ and $\alpha_n(a, b^a) = \alpha_n^a(a^2, b)$].

c) $\alpha_n(a, b) = b^{\prod_{k=0}^{n-1} \alpha_k^a(a, b)}$ $\left(\text{in particular } n \square z = z^{\prod_{k=0}^{n-1} k \square z}\right)$.

It will be proved in the paper that

$$(3) \qquad \lim_{n \to \infty} n \square e^{z/n} = 1, \quad |z| < \frac{1}{e}, \; z \in \mathbb{C}.$$

Moreover, the inverse function of ψ, $\psi(z) := n \square z$, is explicitly calculated for $|z| \le 1/e$, and we examine the possibility to extend the definition of $\zeta \square z$ to complex values of ζ.

2. The Evaluation of a Limit

The evaluation (3) is an immediate consequence of

Theorem 1: For all positive integers n and complex numbers z such that $|z| < 1/e$, we have

$$(4) \qquad |\ln(n \square e^{z/n})| \le \frac{1}{n} \sum_{\nu=1}^{\infty} \frac{\nu^\nu}{\nu!} |z|^\nu.$$

The following lemma is useful to prove (4) (in [2], see formula (15) and section 4.1).

Lemma 1: Let $f_0^{(4)} := f$, $f_1^{(4)}(z) := \exp\left(\frac{zf'(z)}{f(z)}\right)$ and $f_{m+1}^{(4)} := (f_m^{(4)})_1^{(4)}$, $m = 1, 2, 3, \ldots$. We have

(5) $$(f(z^z))_m^{(4)} = \prod_{k=1}^{m} \prod_{j=0}^{k} f_k^{(4)}(z^z)^{\omega(m,k,j)z^k(\ln z)^j}$$

where

$$j!\,\omega(m, k, j) = \binom{m}{k-j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-s)^{m-k+j}, \quad 0 \le j \le k, \quad 1 \le k \le m.$$

In particular,

(6) $$(f(z^z))_m^{(4)}(z = 1) = \prod_{k=1}^{m} f_{k(1)}^{(4)\binom{m}{k}k^{m-k}}.$$

Proof of Theorem 1: We apply (6) recursively to

$$f(\zeta) = (n-1) \,\square\, \zeta,\ (n-2) \,\square\, \zeta,\ \ldots,\ 1 \,\square\, \zeta.$$

Using $n \,\square\, \zeta = (n-1) \,\square\, \zeta^{\zeta}$, we get

(7) $$(n \,\square\, \zeta)^{(4)}(\zeta = 1) = \prod_{k=1}^{m} ((n-1) \,\square\, \zeta)_k^{(4)^{k^{m-k}\binom{m}{k}}}(\zeta = 1).$$

At the r^{th} step, we obtain ($k_0 := m$):

(8) $$(n \,\square\, \zeta)_m^{(4)}(\zeta = 1)$$

$$= \prod_{k_1=1}^{m} \cdots \prod_{k_r=1}^{k_{r-1}} \Big((n-r) \,\square\, \zeta\Big)_{k_r}^{(4)}(\zeta = 1)^{k^{m-k_1}\binom{m}{k_1}\cdots k_r^{k_{r-1}-k_r}\binom{k_{r-1}}{k_r}},$$

whence, since $(1 \,\square\, \zeta)_\nu^{(4)}(\zeta = 1) = e^\nu$, $\nu = 0, 1, 2, \ldots,$

(9) $$(n \,\square\, \zeta)_m^{(4)}(\zeta = 1)$$

$$= \prod_{k_1=1}^{m} \cdots \prod_{k_{n-1}=1}^{k_{n-2}} \exp\left(k_{n-1} \cdot k_{n-1}^{k_{n-2}-k_{n-1}}\binom{k_{n-2}}{k_{n-1}} \cdots k_1^{m-k_1} \cdot \binom{m}{k_1}\right)$$

It follows from (9) that

(10) $$\exp(m \cdot n^{m-1}) \le (n \,\square\, \zeta)_m^{(4)}(\zeta = 1)$$

$$\le \exp\left(m^{m-1} \cdot \sum_{k_1=1}^{m} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} k_{n-1} \cdot \binom{k_{n-2}}{k_{n-1}} \cdots \binom{m}{k_1}\right)$$

$$= \exp(m^m \cdot n^{m-1}) \quad \left(\text{we use } \sum_{j=1}^{N} j\binom{N}{j}x^{j-1} = N(1+x)^{N-1}\right).$$

Thus, the series

$$\sum_{m=1}^{\infty} \frac{\ln((n \,\square\, \zeta)_m^{(4)}(\zeta = 1))Z^m}{m!} \text{ converges for } |Z| < \frac{1}{ne} \text{ and}$$

(11) $$\left|\sum_{m=1}^{\infty} \frac{\ln((n \,\square\, \zeta)_m^{(4)}(\zeta = 1))}{m!\,n^m} z^m\right| \le \frac{1}{n}\sum_{m=1}^{\infty}\frac{m^m}{m!}|z|^m, \quad |z| < \frac{1}{e}.$$

Let us observe that, in general,

(12) $$F_m^{(4)}(z_0) = \exp\left(\frac{\partial^m}{\partial\omega^m}\ln F(z_0 e^w)\bigg|_{w=0}\right).$$

In our case

$$\ln(n \,\square\, \zeta)_m^{(4)}(\zeta = 1) = \frac{\partial^m}{\partial w^m}\ln(n \,\square\, e^w)\bigg|_{w=0}$$

so that the MacLaurin expansion of $\ln(n \,\square\, e^{z/n})$, namely,

(13) $$\ln(n \,\square\, e^{z/n}) = \sum_{m=1}^{\infty} \frac{\ln(n \,\square\, \zeta)_m^{(4)}(\zeta = 1)}{m!\,n^m} z^m,$$

is valid for $|z| < 1/e$ in view of (11). This completes the proof of Theorem 1, since (4) follows from (11) and (13). □

3. The Inverse Function

If $\zeta = n \,\square\, z$, $n = 1, 2, 3, \ldots$, then we write $z = (-n) \,\square\, \zeta$ in a domain where the inverse function is defined (this is essentially what is called "partial inverse" in [3]). The inverse function is defined in such a way that

$$(14) \qquad (n + m) \,\square\, z = n \,\square\, (m \,\square\, z), \quad n, m \in \mathbb{Z}.$$

To prove the next theorem, we need the following lemma.

Lemma 2: For all complex numbers $A_1, A_2, \ldots, A_m$, we have

$$(15) \qquad \sum_{\pi(m,r)} \frac{r!}{k_1! \ldots k_m!} \prod_{j=1}^{m} A_j^{k_j} = \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \geq 1}} \prod_{\ell=1}^{r} A_{\nu_\ell}, \quad 1 \leq r \leq m.$$

Here and in what follows, $\pi(m, r)$ means that the summation is extended over the numbers $k_1, \ldots, k_m$ such that

$$k_1 + 2k_2 + \cdots + mk_m = m, \; k_1 + k_2 + \cdots + k_m = r,$$

with $k_j \geq 0$, $1 \leq j \leq m$.

Proof: Let

$$f(z) := \sum_{m=1}^{\infty} B_m z^m, \quad g(z) := \sum_{m=1}^{\infty} A_m z^m$$

be two analytic functions in a neighborhood of $z = 0$ such that $f(0) = g(0) = 0$. We have

$$f(g(z)) = \sum_{m=1}^{\infty} B_m (g(z))^m = \sum_{m=1}^{\infty} \sum_{\nu_1=1}^{\infty} \cdots \sum_{\nu_m=1}^{\infty} B_m A_{\nu_1} \ldots A_{\nu_m} z^{\nu_1 + \cdots + \nu_m}$$

$$= \sum_{m=1}^{\infty} \sum_{r=m}^{\infty} \sum_{\substack{\nu_1 + \cdots + \nu_m = r \\ \nu_\ell \geq 1}} B_m A_{\nu_1} \ldots A_{\nu_m} z^{\nu_1 + \cdots + \nu_m},$$

whence

$$(16) \qquad f(g(z)) = \sum_{m=1}^{\infty} \sum_{r=1}^{m} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \geq 1}} B_r \prod_{\ell=1}^{r} A_{\nu_\ell} \cdot z^m$$

i.e.,

$$(17) \qquad \frac{(f(g(z)))^{(m)}(z = 0)}{m!} = \sum_{r=1}^{m} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \geq 1}} B_r \prod_{\ell=1}^{r} A_{\nu_\ell}.$$

On the other hand, we compute $(f(g(z)))^{(m)}$ using the Faa di Bruno formula [5, p. 177], namely,

$$(18) \qquad (f(g(z)))^{(m)} = \sum_{r=1}^{m} \sum_{\pi(m,r)} \frac{m!}{k_1! \ldots k_m!} \prod_{j=1}^{m} \left(\frac{g^{(j)}(z)}{j!}\right)^{k_j} \cdot f^{(r)}(g(z)).$$

It gives us

$$(19) \qquad \frac{(f(g(z)))^{(m)}(z = 0)}{m!} = \sum_{r=1}^{m} \sum_{\pi(m,r)} \frac{r!}{k_1! \ldots k_m!} B_r \prod_{j=1}^{m} A_j^{k_j},$$

and the result follows by comparison of (17) and (19).

Remark: Formula (15) gives a variant of (18):

$$(20) \qquad \frac{(f(g(z)))^{(m)}}{m!} = \sum_{r=1}^{m} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \geq 1}} \prod_{\ell=1}^{r} \left(\frac{g^{(\nu_\ell)}(z)}{\nu_\ell!}\right) \cdot \frac{f^{(r)}(g(z))}{r!}.$$

We shall also need

Lemma 3 [2, p. 238]: For all analytic functions $\phi(z)$, we have

$$(21)\qquad \sum_{\pi(m,r)} \frac{m!}{k_1! \ldots k_m!} \prod_{j=1}^{m} \left(\frac{(\phi^j(z))^{(j-1)}}{j!}\right)^{k_j} = \binom{m-1}{r-1} (\phi^m(z))^{(m-r)}, \quad 1 \le r \le m.$$

A representation of $(-1) \square y$ is obtainable from the results of [3] (an interesting list of references is given in that paper). It is proved that the function

$$x = h(z) = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}$$

converges when $e^{-e} \le z \le e^{1/e}$; moreover,

$$g(h(z)) = z \quad \text{and} \quad h(g(x)) = x, \quad e^{-1} \le x \le e,$$

where

$$g(x) = x^{1/x}.$$

But

$$\frac{1}{g\left(\frac{1}{x}\right)} = 1 \square x =: y,$$

whence

$$g\left(\frac{1}{x}\right) = \frac{1}{y}, \quad \frac{1}{x} = h\left(\frac{1}{y}\right) \quad \text{for } e^{-1/e} \le y \le e^e,$$

i.e.,

$$x = \frac{1}{h\left(\frac{1}{y}\right)} = (-1) \square y,$$

whence

$$(22)\qquad (-1) \square y = y^{y^{-y^{-y^{-y^{\cdot^{\cdot^{\cdot}}}}}}}, \quad e^{-1/e} \le y \le e^e.$$

Replacing y by $(-1) \square y$ gives a similar representation for $(-2) \square y$, and so on. We give here another kind of representation for $(-m) \square z$, $m = 1, 2, 3, \ldots$.

Theorem 2: For all positive integers m and complex numbers z such that

$$|\ln z| \le \frac{1}{me},$$

we have

$$(23)\qquad (-m) \square z = \prod_{\nu=1}^{\infty} \prod_{\nu_1=1}^{\nu} \cdots \prod_{\nu_{m-1}=1}^{\nu_{m-2}} \exp\left(\frac{(-1)^{\nu-1}}{\nu!} \cdot \binom{\nu-1}{\nu_1-1} \cdots \binom{\nu_{m-2}-1}{\nu_{m-1}-1} \cdot \nu^{\nu-\nu_1} \cdots \nu_{m-2}^{\nu_{m-2}-\nu_{m-1}} \cdot \nu_{m-1}^{\nu_{m-1}-1} \cdot (\ln z)^{\nu}\right).$$

Proof: According to the Lagrange expansion theorem, the root z of the equation $z \ln z = \ln \zeta$ which tends to 1 with ζ is given by

$$\ln z = \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{\nu^{\nu-1}}{\nu!} (\ln \zeta)^{\nu}, \quad |\ln \zeta| \le \frac{1}{e}.$$

Since $z \ln z = \ln \zeta$ implies $\zeta = z^z = 1 \square z$, we obtain

$$(24)\qquad \ln((-1) \square \zeta) = \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{\nu^{\nu-1}}{\nu!} (\ln \zeta)^{\nu}, \quad |\ln \zeta| \le \frac{1}{e},$$

which corresponds to (23) for $m = 1$.

Now we replace ζ by $(-1) \square \zeta$ in (24) to obtain

$$\ln((-2) \square \zeta) = \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{\nu^{\nu-1}}{\nu!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_\nu=1}^{\infty} (-1)^{k_1+\cdots+k_\nu-\nu} \cdot \frac{k_1^{k_1-1} \cdots k_\nu^{k_\nu-1}}{k_1! \cdots k_\nu!} \cdot (\ln \zeta)^{k_1+\cdots+k_\nu}$$

$$= \sum_{\nu=1}^{\infty} \sum_{\mu=\nu}^{\infty} \sum_{\substack{k_1+\cdots+k_\nu=\mu \\ k_\ell \ge 1}} (-1)^{\mu-1} \frac{\nu^{\nu-1}}{\nu!} \prod_{\ell=1}^{\nu} \left(\frac{k_\ell^{k_\ell-1}}{k_\ell!}\right) \cdot (\ln \zeta)^{\mu},$$

i.e.,

$$\ln((-2) \square \zeta) = \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\nu} \sum_{\substack{k_1+\cdots+k_\mu=\nu \\ k_\ell \ge 1}} (-1)^{\nu-1} \frac{\mu^{\mu-1}}{\mu!} \prod_{\ell=1}^{\mu} \left(\frac{k_\ell^{k_\ell-1}}{k_\ell!}\right) \cdot (\ln \zeta)^{\nu}. \tag{25}$$

The identity (15) with $A_j = \dfrac{j^{j-1}}{j!}$ gives

$$\sum_{\substack{k_1+\cdots+k_\mu=\nu \\ k_\ell \ge 1}} \prod_{\ell=1}^{\mu} \left(\frac{k_\ell^{k_\ell-1}}{k_\ell!}\right) = \sum_{\pi(\nu,\mu)} \frac{\mu!}{k_1! \cdots k_\nu!} \prod_{j=1}^{\nu} \left(\frac{j^{j-1}}{j!}\right)^{k_j}, \tag{26}$$

while (21) [with $\phi(z) = e^z$] gives

$$\sum_{\pi(\nu,\mu)} \frac{\nu!}{k_1! \cdots k_\nu!} \prod_{j=1}^{\nu} \left(\frac{j^{j-1}}{j!}\right)^{k_j} = \binom{\nu-1}{\mu-1} \nu^{\nu-\mu}, \quad 1 \le \mu \le \nu. \tag{27}$$

We obtain

$$\sum_{\substack{k_1+\cdots+k_\mu=\nu \\ k_\ell \ge 1}} \prod_{\ell=1}^{\mu} \left(\frac{k_\ell^{k_\ell-1}}{k_\ell!}\right) = \frac{\mu!}{\nu!} \binom{\nu-1}{\mu-1} \nu^{\nu-\mu}, \quad 1 \le \mu \le \nu, \tag{28}$$

and it follows from (25) that

$$\ln((-2) \square \zeta) = \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\nu} \frac{(-1)^{\nu-1}}{\nu!} \nu^{\nu-\mu} \mu^{\mu-1} \binom{\nu-1}{\mu-1} (\ln \zeta)^{\nu}. \tag{29}$$

It is readily seen that the coefficients in the summation over ν of (29) are bounded by

$$\frac{\nu^{\nu-1}}{2\nu!}(2|\ln \zeta|)^{\nu},$$

so that (29) is valid for $|\ln \zeta| \le 1/2e$.

The proof is easily completed by mathematical induction. We write

$$\ln((-(m+1)) \square \zeta) = \ln((-m) \square ((-1) \square \zeta)),$$

substitute z to $(-1) \square \zeta$ in (23), and use (28) to simplify the coefficients. The estimation

$$|(-m) \square \zeta| \le \exp\left(\frac{1}{m} \sum_{\nu=1}^{\infty} \frac{\nu^{\nu-1}}{\nu!} |m \ln \zeta|^{\nu}\right) \le e^{1/m} \tag{30}$$

holds for $|\ln \zeta| \le 1/me$. $\square$

Remark: It follows from the proof of Theorem 2 that

$$\lim_{m\to\infty} (-m) \square \zeta^{1/m} = 1, \quad |\ln \zeta| \le \frac{1}{e}. \tag{31}$$

4. Extension of the Definition

In this section, we consider the possibility to define $\zeta \square z$ for complex values of ζ. We give only partial results, but it is interesting to observe that it seems quite possible to extend $\zeta \square z$ to a bianalytic function of z, ζ. All along the process, the relation

(32) $\quad (\zeta_1 + \zeta_2) \square z = \zeta_1 \square (\zeta_2 \square z)$

should remain valid in some domains of the complex plane.

4.1 Extension to Rational Numbers

First, we try to see how $\frac{1}{2} \square z$ can be defined. Let us consider a more general question. Given $z_0 \in \mathbb{C}$ and

$$g(z) := \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad a_0 := z_0,$$

analytic in a neighborhood of z_0 (this fact will be abbreviated $z \circlearrowleft z_0$ in what follows), does there exist an analytic function

$$f(z) := \sum_{k=0}^{\infty} b_k (z - z_0)^k, \quad b_0 := z_0,$$

such that the functional equation

(33) $\quad f(f(z)) = g(z)$

is valid for $z \circlearrowleft z_0$?

A solution is not always possible, as shown by the example

$$g(z) = z^2, \quad z_0 = 0.$$

An affirmative answer for $g(z) = z^z$, $z_0 = 1$, would imply that the solution $f(z) =: \frac{1}{2} \square z$ satisfies the relation

$$\frac{1}{2} \square \left(\frac{1}{2} \square z\right) = f(f(z)) = 1 \square z.$$

To solve the functional equation

(34) $\quad f(f(z)) = z^z, \quad f(1) = f'(1) = 1,$

we seek a solution of the form

$$f(z) = 1 + \sum_{k=1}^{\infty} b_k (z - 1)^k.$$

Substituting z to $f(z)$, we obtain

$$z^z =: 1 + \sum_{k=1}^{\infty} a_k (z - 1)^k = 1 + \sum_{k=1}^{\infty} b_k (f(z) - 1)^k$$

$$= 1 + \sum_{k=1}^{\infty} \sum_{\ell=1}^{k} \sum_{\substack{\nu_1 + \cdots + \nu_\ell = k \\ \nu_j \geq 1}} b_\ell \prod_{j=1}^{\ell} b_{\nu_j} \cdot (z - 1)^k$$

(in the context of [2], it is not difficult to verify that $|a_k| \leq 1$ for all $k \in \mathbb{N}$). It is then readily seen that the aforesaid question can be answered in the affirmative if we find a practical way to solve the following two problems:

1. Express $b_1, b_2, \ldots, b_k$ in terms of $a_1, a_2, \ldots, a_k$ in the relations $a_1 = b_1 = 1$,

$$a_k = \sum_{r=1}^{k} \sum_{\substack{\nu_1 + \cdots + \nu_r = k \\ \nu_\ell \geq 1}} b_r \prod_{\ell=1}^{r} b_{\nu_\ell}, \quad k = 1, 2, 3, \ldots .$$

2. Show that the radius of convergence of $\sum_{k=1}^{\infty} b_k(z-1)^k$ is positive.

We assume in the remainder of the paper that the radius of convergence is positive in the case $g(z) = z^z$, $z_0 = 1$. Unfortunately, this fact is not proved but it seems very likely that it is ≥ 1.

We generalize one step further and ask for an analytic solution of

$$(35)\qquad f_q(z) = z^z, \quad f(1) = f'(1) = 1, \text{ where } f_q(z) = \underbrace{f(f(\cdots f(z)\cdots))}_{q \text{ times}}.$$

This leads us to define

$$(36)\qquad \frac{1}{q} \square z := f(z) := 1 + \sum_{k=1}^{\infty} b_k\left(\frac{1}{q}\right)(z-1)^k, \quad z \circlearrowright 1,$$

for $q = 1, 2, 3, \ldots$ (the domain of validity should contain $|z - 1| < q/2$). It is then possible to define $p/q \square z$ for $p/q \in \mathbb{Q}_+$. Simply:

$$(37)\qquad \frac{p}{q} \square z := \underbrace{\frac{1}{q} \square \left(\frac{1}{q} \square \cdots \square \left(\frac{1}{q} \square z\right)\cdots\right)}_{p \text{ times}} =: 1 + \sum_{k=1}^{\infty} b_k(p, q)(z-1)^k, \quad z \circlearrowright 1.$$

It appears that $b_k(p, q) = b_k(p/q)$. There is no problem defining $p/q \square z$ for $p/q \in \mathbb{Q}_-$. We construct $(-1)/q \square z$ by requiring

$$\frac{(-1)}{q} \square \left(\frac{1}{q} \square z\right) \equiv z$$

and we observe that (32) remains true for all rationals ζ_1, ζ_2. Here, we can write

$$(38)\qquad \frac{p}{q} \square z = z + \frac{p}{q}(z-1)^2 + \frac{p}{2q}\left(\frac{2p}{q} - 1\right)(z-1)^3 + \cdots, \quad z \circlearrowright 1.$$

4.2 Extension to Complex Numbers

It is reasonable to expect that a passage to the limit can be justified in (38). This would permit us to define $t \square z$ for $t \in \mathbb{R}$ by

$$(39)\qquad t \square z := \lim_{j\to\infty} \sum_{k=0}^{\infty} b_k\left(\frac{p_j}{q_j}\right) \cdot (z-1)^k = \sum_{k=0}^{\infty} b_k(t) \cdot (z-1)^k, \quad z \circlearrowright 1,$$

where p_j/q_j, $j = 1, 2, 3, \ldots$, is any sequence of rational numbers converging to t [note that the coefficients $b_k(t)$ are reals for real values of t).

Finally, (39) is extended to complex values of t by analytic continuation and (32) remains valid. We do not give details of our calculations, since the question concerning the radius of convergence is open. At the end of the process we obtain a representation of the form

$$(40)\qquad \zeta \square z = z + \zeta(z-1)^2 + \zeta\left(\zeta - \frac{1}{2}\right)(z-1)^3 + \cdots, \quad \zeta \circlearrowright 0, \; z \circlearrowright 1.$$

We can define $\alpha_\zeta(a, z)$ [see (2)] by requiring

$$\alpha_\zeta^a(a, z) = \zeta \square z^a.$$

5. Some Observations

5.1 Solution of a Functional Equation

We observe that the functional equation

$$(41)\qquad f_q(z) = z^N, \quad f(0) = 0, \quad N \in \mathbb{N}$$

can be solved.

Theorem 3: Let $N > 1$ be an integer. There exists an analytic solution, in a neighborhood of the origin, of the equation (41) if and only if $N = M^q$, $M \in \mathbb{N}$. The solution is unique up to a multiplicative constant which must be an $\left(\frac{N-1}{M-1}\right)^{\text{th}}$ root of unity.

Proof: If $N = M^q$, then a solution of (41) is

$$f(z) = cz^M, \quad c^{\frac{N-1}{M-1}} = 1.$$

We must prove that an analytic solution $f(z)$, $z \circlearrowleft 0$, exists only in that case.

Equation (41) implies

$$(42) \qquad f(z^N) = (f(z))^N, \quad f(0) = 0, \quad (N > 1).$$

Let us assume for a moment that the solutions of (42) are of the form

$$f(z) = cz^M, \quad c^N = c,$$

for some positive integer M. Substituting in (41), we find that

$$z^N = c^{1+M+\cdots+M^{q-1}} \cdot z^{M^q},$$

i.e., $N = M^q$ and $c^{\frac{N-1}{M-1}} = 1$. Hence, we need only to prove that all the analytic solutions of (42) are of the indicated form. Let

$$f(z) = \sum_{m=1}^{\infty} A_m z^m$$

be a solution of (42). We have

$$f(z^N) = \sum_{k=1}^{\infty} A_k z^{Nk} = (f(z))^N = \sum_{\nu_1=1}^{\infty} \cdots \sum_{\nu_N=1}^{\infty} A_{\nu_1} \cdots A_{\nu_N} \cdot z^{\nu_1+\cdots+\nu_N}$$

$$= \sum_{m=N}^{\infty} \sum_{\substack{\nu_1+\cdots+\nu_N=m \\ \nu_\ell \geq 1}} \prod_{\ell=1}^{N} A_{\nu_\ell} \cdot z^m,$$

whence

$$(43) \qquad \sum_{\substack{\nu_1+\cdots+\nu_N=m \\ \nu_\ell \geq 1}} \prod_{\ell=1}^{N} A_{\nu_\ell} = \begin{cases} A_k & \text{if } m = kN,\ k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

This relation, for $m = N$, gives $A_1^N = A_1$, i.e., $A_1 = 0$ or $A_1^{N-1} = 1$. The following reasoning is easily adapted to the case $A_1 \neq 0$ [we obtain the solution $f(z) = A_1 z$]. Let us suppose that $A_1 = 0$. Let $k_0 > 1$ be the first index for which $A_{k_0} \neq 0$. We prove by mathematical induction that $A_{k_0+\ell} = 0$, $\ell = 1, 2, 3, \ldots$ [this gives us the solution $f(z) = A_{k_0} z^{k_0}$, $A_{k_0}^N = A_{k_0}$).

First, we examine the relation (43) with $m = Nk_0 + 1$. If a ν_ℓ is less than k_0, then the corresponding term, in the left-hand member of (43), is equal to zero. Thus, we examine only the solutions of

$$(44) \qquad \nu_1 + \nu_2 + \cdots + \nu_N = Nk_0 + 1, \ \nu_\ell \geq k_0, \ 1 \leq \ell \leq N.$$

Let $\nu_{\ell_1} = \cdots = \nu_{\ell_s} = k_0$ $(s < N)$ and $\nu_j \geq k_0 + 1$, $j \neq \ell_1, \ldots, \ell_s$. In view of (44), we have

$$Nk_0 + 1 \geq sk_0 + (N - s)(k_0 + 1),$$

whence $s \geq N - 1$ and, in fact, $s = N - 1$. Since the right-hand member of (43) is zero, this relation is reduced to $A_{k_0}^{N-1} \cdot A_{k_0+1} = 0$, i.e., $A_{k_0+1} = 0$.

Now we suppose that $A_{k_0+1} = \cdots = A_{k_0+\ell-1} = 0$, $\ell > 1$, and examine the relation (43) with $m = Nk_0 + \ell$. Let us consider the equation

$$(45) \qquad \nu_1 + \nu_2 + \cdots + \nu_N = Nk_0 + \ell, \ \nu_\ell \geq k_0, \ 1 \leq \ell \leq N.$$

If $\nu_{\ell_1} = \cdots = \nu_{\ell_r} = k_0$ $(r < N)$, then $\nu_j \geq k_0 + \ell$ for $j \neq \ell_1, \ldots, \ell_r$ (in order to have $A_{\nu_1} \ldots A_{\nu_N} \neq 0$), so that $Nk_0 + \ell \geq rk_0 + (N - r)(k_0 + \ell)$, whence $r = N - 1$ and (43) is reduced to

$$NA_{k_0}^{N-1} \cdot A_{k_0+\ell} = \begin{cases} A_k & \text{if } Nk_0 + \ell = kN \\ 0 & \text{otherwise,} \end{cases}$$

for some integer k. The possibility $Nk_0 + \ell = kN$ implies $k = k_0 + \ell/N$; but

$$k_0 < k_0 + \frac{1}{N} < k_0 + \ell,$$

so that $A_k = 0$ by hypothesis. In both cases, we conclude that $A_{k_0+\ell} = 0$. $\square$

Remarks: The examples

$$f(z) = \frac{z}{(1-\omega)z + \omega}, \quad \omega^q = 1,$$

show that other solutions of (41) are possible for $N = 1$. We may compare (42) with Wedderburn's functional equation $g(x^2) = [g(x)]^2 + 2ax$ (see [1] for references).

5.2 Solution of a Recurrence Relation

There is a relation similar to 1 which may be solved without difficulty. Let A_m, B_m, $m = 1, 2, 3, \ldots$ be two sequences of complex numbers related by

$$(46) \qquad A_m = \sum_{r=1}^{m} \sum_{\substack{\nu_1+\cdots+\nu_r=m \\ \nu_\ell \ge 1}} \prod_{\ell=1}^{r} B_{\nu_\ell}, \quad m = 1, 2, 3, \ldots .$$

We have

$$(47) \qquad B_m = \sum_{r=1}^{m} \sum_{\substack{\nu_1+\cdots+\nu_r=m \\ \nu_\ell \ge 1}} (-1)^{r-1} \prod_{\ell=1}^{r} A_{\nu_\ell}, \quad m = 1, 2, 3, \ldots .$$

Proof: Let

$$f(z) := (1-z)^{-1}, \quad g(z) := \sum_{m=1}^{\infty} B_m z^m.$$

Using Faa di Bruno's formula in the form (20), we obtain

$$\frac{(f(g(z)))^{(m)}(z=0)}{m!} = \sum_{r=1}^{m} \sum_{\substack{\nu_1+\cdots+\nu_r=m \\ \nu_\ell \ge 1}} \prod_{\ell=1}^{r} B_{\nu_\ell} = A_m,$$

whence

$$f(g(z)) = 1 + \sum_{m=1}^{\infty} A_m z^m = \frac{1}{1-g(z)} = \frac{1}{1 - \sum_{m=1}^{\infty} B_m z^m}.$$

It follows that

$$\left(1 + \sum_{m=1}^{\infty} A_m z^m\right)\left(1 - \sum_{m=1}^{\infty} B_m z^m\right) \equiv 1,$$

and by comparison of the coefficients:

$$(48) \qquad B_m = A_m - \sum_{s=1}^{m-1} A_{m-s} B_s, \quad m \ge 2.$$

Thus,

$$\begin{aligned} B_m &= A_m - A_{m-1}A_1 - \sum_{s=2}^{m-1} A_{m-s}\left(A_s - \sum_{t=1}^{s-1} A_{s-t}B_t\right) \\ &= A_m - \sum_{s=1}^{m-1} A_{m-s}A_s + \sum_{s=2}^{m-1} \sum_{t=1}^{s-1} A_{m-s}A_{s-t}B_t \\ &= \sum_{\nu_1=m} A_{\nu_1} - \sum_{\nu_1+\nu_2=m} A_{\nu_1}A_{\nu_2} + \sum_{s=2}^{m-1} \sum_{t=1}^{s-1} A_{m-s}A_{s-t}B_t. \end{aligned}$$

At the n^{th} step, we obtain

$$B_m = \sum_{r=1}^{n} (-1)^{r-1} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \geq 1}} \prod_{\ell=1}^{r} A_{\nu_\ell} + (-1)^n \sum_{s_1=n}^{m-1} \sum_{s_2=n-1}^{s_1-1} \cdots \sum_{s_n=1}^{s_{n-1}-1} A_{m-s_1} \cdots A_{s_{n-1}-s_n} \cdot B_{s_n}, \text{ for } n = 1, 2, \ldots, (m-1).$$

This gives us

$$B_m = \sum_{r=1}^{m-1} (-1)^{r-1} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \geq 1}} \prod_{\ell=1}^{r} A_{\nu_\ell} + (-1)^{m-1} A_1^{m-1} B_1$$

$$= \sum_{r=1}^{m} (-1)^{r-1} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \geq 1}} \prod_{\ell=1}^{r} A_{\nu_\ell}. \quad \square$$

5.3 An Identity

Using (32), we can write

$$\frac{\partial}{\partial a}(a \,\square\, z) = \lim_{h \to 0} \frac{((a + h) \,\square\, z) - (a \,\square\, z)}{h} = \lim_{h \to 0} \frac{(h \,\square\, (a \,\square\, z)) - (a \,\square\, z)}{h},$$

and (40) [with $\zeta = h$ and z replaced by $(a \,\square\, z)$] gives

(49) $$\frac{\partial}{\partial a}(a \,\square\, z) = ((a \,\square\, z) - 1)^2 - \frac{1}{2}((a \,\square\, z) - 1)^3 + \cdots .$$

On the other hand, (40) gives directly

(50) $$\frac{\partial}{\partial a}(a \,\square\, z) = (z - 1)^2 + \left(2a - \frac{1}{2}\right)(z - 1)^3 + \cdots,$$

whence

(51) $$((a \,\square\, z) - 1)^2 - \frac{1}{2}((a \,\square\, z) - 1)^3 + \cdots$$
$$= (z - 1)^2 + \left(2a - \frac{1}{2}\right)(z - 1)^3 + \cdots, \quad z \circlearrowleft 1, \ a \circlearrowleft 0.$$

5.4 An Appearance of the Fibonacci Numbers

The recurrence relation 1 (section 4.1) may be written in the form

(52) $$b_k = \frac{1}{2}a_k - \frac{1}{2}\sum_{r=2}^{k-1} \sum_{\substack{\nu_1 + \cdots + \nu_r = k \\ \nu_\ell \geq 1}} b_r \cdot \prod_{\ell=1}^{r} b_{\nu_\ell}, \quad k \geq 3.$$

To find a bound for $|b_k|(|a_k| \leq 1)$, we may try to use (52) with $k = r$, $k = \nu_\ell$ and make the substitutions. To do that, we need to take into account that (52) holds only for $k \geq 3$. In particular, we must examine, separately, the solutions of $\nu_1 + \cdots + \nu_r = k$ with $1 \leq \nu_e \leq 2$, $1 \leq \ell \leq r$. This leads us to evaluate the summation

(53) $$\sum_{\frac{k}{2} \leq r \leq k} \sum_{\substack{\nu_1 + \cdots + \nu_r = k \\ 1 \leq \nu_\ell \leq 2}} 1 =: \sum_{\frac{k}{2} \leq r \leq k} p_r(k, 2),$$

where $p_r(k, 2)$ is the number of solutions of $\nu_1 + \cdots + \nu_r = k$, $1 \leq \nu_\ell \leq 2$. This number is $\binom{r}{k-r}$; indeed, if $\nu_{\ell_1} = \cdots = \nu_{\ell_s} = 1$ and $\nu_\ell = 2$, $\ell \neq \ell_1, \ldots, \ell_s$, then $s \cdot 1 + (r - s) \cdot 2 = k$, so that $s = 2r - k$ and the number of solutions is

$$\binom{r}{s} = \binom{r}{2r - k} = \binom{r}{k - r}$$

(see also the Remark below). Hence, we obtain (see [4], p. 14, Problem 1):

(54) $$\sum_{\frac{k}{2} \le r \le k} p_r(k,\ 2) = \sum_{\frac{k}{2} \le r \le k} \binom{r}{k - r} = f_k, \quad k = 0,\ 1,\ 2,\ \ldots,$$

the k^{th} Fibonacci number.

Remark: Using the generating function

$$\frac{z^r(z^M - 1)^r}{(z - 1)^r} = \left(\sum_{k=1}^{M} z^k\right)^r = \sum_{k=r}^{rM} p_r(k,\ M)z^k$$

and the Leibniz formula, we deduce that the number of solutions, $p_r(k,\ M)$, of the equation $\nu_1 + \cdots + \nu_r = k$, $1 \le \nu_\ell \le M$, is equal to

(55) $$p_r(k,\ M) = \sum_{j=0}^{\left[\frac{k-r}{M}\right]} (-1)^j \binom{r}{j} \binom{k - jM - 1}{r - 1}, \quad r \le k \le rM.$$

In particular,

$$p_r(k,\ 2) = \sum_{j=0}^{\left[\frac{k-r}{2}\right]} (-1)^j \binom{r}{j} \binom{k - 2j - 1}{r - 1} = \binom{r}{k - r}, \quad r \le k \le 2r.$$

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A NEW FORMULA FOR LUCAS NUMBERS

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Introduction

The Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ are well-known to the readers of this Journal. Several closed form formulas exist for Fibonacci and Lucas numbers, namely:

$$(1)\qquad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad\qquad (2)\qquad L_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1}{2}(1 + 5^{\frac{1}{2}})$, $\beta = \frac{1}{2}(1 - 5^{\frac{1}{2}})$.

$$(3)\qquad F_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} 5^k, \qquad\qquad (4)\qquad L_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} 5^k,$$

$$(5)\qquad F_{n+1} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} \qquad\qquad (6)\qquad L_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k}\binom{n-k}{k}.$$

George E. Andrews, [1] and [2], derived an additional explicit formula for the Fibonacci numbers, which can be written as

$$(7)\qquad F_n = \sum_{k=-\left[\frac{n+1}{5}\right]}^{\left[\frac{n}{5}\right]} (-1)^k \binom{n}{[\frac{1}{2}(n-5k)]}.$$

In [1], Andrews proved (7) by using a relation between the Fibonacci numbers and the primitive fifth roots of unity, namely:

$$\alpha = -2\cos(4\pi/5), \quad \beta = -2\cos(2\pi/5).$$

In [2], Andrews obtained (7) as a consequence of a polynomial identity. In this note, following Andrews, we derive a corresponding explicit formula for the Lucas numbers which is

$$(8)\qquad L_n = \sum_{k=-\left[\frac{n+1}{5}\right]}^{\left[\frac{n}{5}\right]} (-1)^k \frac{n + [\frac{1}{2}(n-5k)]}{n} \binom{n}{[\frac{1}{2}(n-5k)]}.$$

Preliminaries

$$(9)\qquad \sum_{j=0}^{\left[\frac{n}{2}\right]} x^{j^2+j} \prod_{k=1}^{j} \frac{x^{n+1-j-k} - 1}{x^k - 1} = \sum (-1)^t x^{\frac{1}{2}t(5t-3)} \prod_{k=1}^{\left[\frac{n+3-5t}{2}\right]} \frac{x^{n+2-k} - 1}{x^k - 1}.$$

$$(10)\qquad F_{n+1} = \sum (-1)^k \binom{n+1}{[\frac{1}{2}(n+1-5k)] + 1}.$$

$$(11)\qquad \binom{n}{k} = \binom{n}{n-k}.$$

(12) $m = \left[\frac{m + r}{2}\right] + \left[\frac{m + 1 - r}{2}\right]$ for all m, r.

(13) $\binom{m - 1}{r - 1} = \frac{r}{m}\binom{m}{r}$ if $1 \le r \le m$.

(14) $L_n = F_{n+1} + F_{n-1}$.

Remarks: Equation (9) is the Theorem from [2] with $\alpha = -1$. Equation (10) is obtained by taking the limit as x approaches 1 in (9) and then applying (5). Equations (11) through (14) are elementary.

Proof of (8): Equation (10) implies that

(15) $$F_{n-1} = \sum (-1)^k \binom{n - 1}{[\frac{1}{2}(n - 1 - 5k)] + 1}.$$

Replacing k by $-k$ in (15), we get

(16) $$F_{n-1} = \sum (-1)^k \binom{n - 1}{[\frac{1}{2}(n - 1 + 5k)] + 1}.$$

which implies, by using (11), that

(17) $$F_{n-1} = \sum (-1)^k \binom{n - 1}{n - 2 - [\frac{1}{2}(n - 1 + 5k)]}.$$

If we now use equation (12), we see that

(18) $$F_{n-1} = \sum (-1)^k \binom{n - 1}{[\frac{1}{2}(n - 5k)]}.$$

Applying (13) to equation (18), we obtain

(19) $$F_{n-1} = \sum (-1)^k \frac{[\frac{1}{2}(n - 5k)]}{n}\binom{n}{[\frac{1}{2}(n - 5k)]}.$$

Equation (19) together with equations (7) and (14) yields

(20) $$L_n = \sum (-1)^k \frac{n + [\frac{1}{2}(n - 5k)]}{n}\binom{n}{[\frac{1}{2}(n - 5k)]},$$

which is the same as (8) and the proof is complete. (The limits of summation in (8) are determined by the criterion that $0 \le [\frac{1}{2}(n - 5k)] \le n$.)

Concluding Remarks

The reader who consults [1] should take note that (i) Andrews' middle initial is erroneously given as H.; (ii) on pages 113 and 117, the name "Einstein" should be "Eisenstein." Both errors were made without consulting Andrews and were not in his original manuscript.

Acknowledgment

I wish to thank the referee for his suggestions, which led to a simpler proof of (8).

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DIVISIBILITY OF GENERALIZED FIBONACCI AND LUCAS NUMBERS BY THEIR SUBSCRIPTS

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1. Introduction

In this paper, we shall extend some previous results ([2], [3], [4]) concerning divisibility of terms of certain recurring sequences based on their subscripts. We shall use the generalized Fibonacci and Lucas numbers, defined for $n \geq 0$ by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where α and β are two complex numbers such that $P = \alpha + \beta$ and $Q = \alpha\beta$ are relatively prime nonzero integers. We shall exclude from consideration the case in which and are roots of unity. Then U_n and V_n are always different from zero [1]. We shall also give some applications to the equation

$$a^n \pm b^n \equiv 0 \pmod{n},$$

where $a > b \geq 1$ are relatively prime *integers*.

In what follows, $\omega(q)$ [resp. $\overline{\omega}(q)$] denotes the rank of apparition of the positive integer q in the sequence $\{U_m\}$ (resp. $\{V_m\}$), i.e., the least positive index ω (resp. $\overline{\omega}$) for which $q | U_\omega$ (resp. $q | V_{\overline{\omega}}$). Recall that the integer b is an odd multiple of the integer a if $a | b$ and $2 \nmid (b/a)$. The main result, which generalizes the one of Jarden [3], can be stated as follows.

Theorem 1: Let $n = p_1^{\lambda_1} p_2^{\lambda_2} \ldots p_k^{\lambda_k} \geq 2$ be a natural integer.

(i) If $n \geq 2$ divides some member of the sequence $\{U_m\}$, then $U_n \equiv 0 \pmod{n}$ if and only if the rank of apparition of any prime divisor of n also divides n.

(ii) If $n \geq 3$ divides some member of the sequence $\{V_m\}$, then $V_n \equiv 0 \pmod{n}$ if and only if n is an odd multiple of $\operatorname{lcm}(\overline{\omega}(p_1), \ldots, \overline{\omega}(p_k))$.

2. Preliminary Results

The following well-known properties will be necessary for our future proofs. Proofs of these results can be found in the papers of Lucas [5] or Carmichael [1].

(i) For each integer $n \geq 1$, $\gcd(U_n, Q) = \gcd(V_n, Q) = 1$.

(ii) If p is a prime number such that $p \nmid Q$, then $\omega(p) = p$ if and only if $p | (\alpha - \beta)^2$, and $\gcd(\omega(p), p) = 1$ otherwise.

(iii) If q is a prime divisor of $\omega(p)$, with $p \neq 2$ and $p \nmid (\alpha - \beta)^2$, then $q < p$. Moreover, we have

(a) $\omega(p^\lambda) = \omega(p)p^\mu$, $0 \leq \mu < \lambda$,

(b) $\omega(p_1^{\lambda_1} \ldots p_k^{\lambda_k}) = \operatorname{lcm}(\omega(p_1^{\lambda_1}), \ldots, \omega(p_k^{\lambda_k}))$, and

(c) $n | U_m$ if and only if $\omega(n) | m$.

(iv) If the prime number p divides some member of the sequence $\{V_m\}$, then

(a) $\overline{\omega}(p) < p$,

(b) $\gcd(\overline{\omega}(p), p) = 1$,

(c) $\overline{\omega}(p^\lambda) = \overline{\omega}(p)p^\mu$, $0 \le \mu < \lambda$, p odd,

(d) If $2^\lambda | V_m$, then $\overline{\omega}(2) = \overline{\omega}(2^\lambda)$, and

(e) If $n = p_1^{\lambda_1} \ldots p_k^{\lambda_k}$ divides some member of the sequence $\{V_m\}$, then $\overline{\omega}(n) = \mathrm{lcm}(\overline{\omega}(p_1^{\lambda_1}), \ldots, \overline{\omega}(p_k^{\lambda_k}))$, and, for $n \ge 3$, $n | V_m$ if and only if m is an odd multiple of $\overline{\omega}(n)$.

3. Proof of Theorem 1

(i) Let $n = p_1^{\lambda_1} \ldots p_k^{\lambda_k} \ge 2$ be an integer which divides some member of the sequence $\{U_m\}$. First, assume that $n | U_n$. Then, for each $1 \le i \le k$, $p_i | U_n$, and $\omega(p_i) | n$. Second, assume that, for each i, $\omega(p_i) | n$.

If $p_i | (\alpha - \beta)^2$, then

$$\omega(p_i^{\lambda_i}) = \omega(p_i)p_i^{\mu_i} = p_i^{\mu_i+1} | n,$$

since $\mu_i < \lambda_i$; otherwise,

$$\omega(p_i^{\lambda_i}) = \omega(p_i)p_i^{\mu_i} | n,$$

since $\gcd(\omega(p_i), p_i) = 1$, and $\mu_i < \lambda_i$. Thus,

$$\omega(n) = \mathrm{lcm}(\omega(p_1^{\lambda_1}), \ldots, \omega(p_k^{\lambda_k})) | n, \quad \text{and} \quad n | U_n.$$

(ii) Now, let $n = p_1^{\lambda_1} \ldots p_k^{\lambda_k} \ge 3$ be an integer which divides some member of the sequence $\{V_m\}$. First, assume that n is an odd multiple of $\mathrm{lcm}(\overline{\omega}(p_1), \ldots, \overline{\omega}(p_k))$. If $p = 2$, then $\overline{\omega}(p_i^{\lambda_i}) = \overline{\omega}(p_i) | n$, whereas if $p_i \ne 2$, then $\overline{\omega}(p_i^{\lambda_i}) = \overline{\omega}(p_i)p_i^{\mu_i} | n$, since $\gcd(\overline{\omega}(p_i), p_i) = 1$, and $\mu_i < \lambda_i$. Therefore, n is an odd multiple of $\overline{\omega}(n) = \mathrm{lcm}(\overline{\omega}(p_1^{\lambda_1}), \ldots, \overline{\omega}(p_k^{\lambda_k})$, since n is an odd multiple of $\mathrm{lcm}(\overline{\omega}(p_1), \ldots, \overline{\omega}(p_k))$. Second, assume that $n | V_n$, with $n \ge 3$. We know that n is an odd multiple of $\mathrm{lcm}(\overline{\omega}(p_1^{\lambda_1}), \ldots, \overline{\omega}(p_k^{\lambda_k})) = \overline{\omega}(n)$. Therefore, n is an odd multiple of $\mathrm{lcm}(\overline{\omega}(p_1), \ldots, \overline{\omega}(p_k))$, since $\overline{\omega}(p_i^{\lambda_i}) = \overline{\omega}(p_i)p_i^{\mu_i}$, p_i odd, or $\overline{\omega}(p_i^{\lambda_i}) = \overline{\omega}(p_i)$, if $p_i = 2$. This concludes the proof of Theorem 1.

Theorem 1 immediately yields the following Corollary, due to Jarden [3].

Corollary 1: (i) If $U_n \equiv 0 \pmod{n}$, and m is composed of only prime factors of n, then also $U_{mn} \equiv 0 \pmod{mn}$.

(ii) If $V_n \equiv 0 \pmod{n}$, and m is composed of only odd prime factors of n, then also $V_{mn} \equiv 0 \pmod{mn}$.

Remark 1: By application of Theorem 1 and Corollary 1, numerical examples can be obtained. For instance, let $n = p_1^{\lambda_1} \ldots p_k^{\lambda_k}$ be an odd number, such that $3 \le p_1 < \cdots < p_k$, and $n | U_n$. We have $\omega(p_1) \ne 1$, since $U_1 = 1$, and by §2(iii), $\omega(p_1) = p_1$, and $p_1 | (\alpha - \beta)^2$, since $\omega(p_1)$ is a factor of n. This case can occur only if $(\alpha - \beta)^2$ admits an odd prime divisor. Moreover, we have

$$\omega(p_i) = p_i,$$

or

$$\omega(p_i) = p_1^{\mu_1} \ldots p_{i-1}^{\mu_{i-1}}, \quad i = 2, \ldots, k; \quad \mu_j \le \lambda_j, \quad j = 1, \ldots, i - 1.$$

Theorem 1 also yields the following Corollary.

Corollary 2: If $n | U_n$, then $U_n | U_{U_n}$.

Proof: If $n | U_n$, and if p is a prime number such that $p | U_n$, then $\omega(p) | n | U_n$, and the result follows by Theorem 1.

4. The Congruence $a^n \pm b^n \equiv 0 \pmod{n}$

In what follows, we assume that $a > b \geq 1$ are relatively prime integers and that $e(n)$ denotes the rank of apparition of n in the sequence $\{a^m - b^m\}$. The next result generalizes the main theorem of [4].

Theorem 2: Let n and ab be relatively prime. Then the following statements are equivalent:

(i) $U_n \equiv 0 \pmod{n}$.
(ii) $a^n - b^n \equiv 0 \pmod{n}$.
(iii) $n \equiv 0$ [mod $e(n)$].
(iv) $n \equiv 0$ [mod $e(p)$], for each prime factor p of n.

Proof: It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Now, assume that $n \equiv 0$ [mod $e(p)$] for each prime factor of n. If $p \mid a - b$, then [§2(ii)] $\omega(p) = p \mid n$. On the other hand, if $p \nmid a - b$, then $p \mid U_n$ if and only if $p \mid a^n - b^n$. Thus, $\omega(p) = e(p) \mid n$. The conclusion follows by Theorem 1.

Corollary 3: The equation $a^n - b^n \equiv 0 \pmod{n}$ has

(i) no solution if $a = b + 1$ and $n \geq 2$,
(ii) infinitely many solutions otherwise.

Proof: If $a - b$ admits at least one prime divisor p, then $p^\lambda \mid U_{p^\lambda}$, for each positive integer λ, by Corollary 1. On the other hand, if $a - b = 1$, then $Q = ab$ is even and n must be odd. But this case cannot occur since, if p was the least prime factor of n, we would have, by Remark 1 above,

$$\omega(p) \mid (a - b)^2. \quad \text{Q.E.D.}$$

Corollary 4: The equation $a^n + b^n \equiv 0 \pmod{n}$ admits infinitely many solutions.

Proof: If $V_1 = a + b$ admits an odd prime divisor p, then $p^\lambda \mid V_{p^\lambda}$, for each $\lambda \geq 1$, by Theorem 1 and Corollary 1. On the other hand, suppose that

$$V_1 = a + b = 2^m, \quad m \geq 2.$$

Thus a and b are odd and

$$V_2 = (a + b)^2 - 2ab = 2(2^{2m-1} - Q),$$

where $2^{2m-1} - Q > 1$ is odd, since Q is also odd. Thus, V_2 admits an odd prime divisor p, and $2p$ is an odd multiple of $\operatorname{lcm}(\overline{\omega}(2), \overline{\omega}(p)) = 2$. By Theorem 1 and Corollary 1, we have

$$2p^\alpha \mid V_{2p^\alpha}, \quad \alpha \geq 1. \quad \text{Q.E.D.}$$

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THE STATISTICS OF THE SMALLEST SPACE ON A LOTTERY TICKET

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Introduction

A day hardly goes by without hearing that some lucky person has become an "instant millionaire" by winning a lottery. Recently, one of the authors was visiting relatives in Florida when a sequence of winning lottery numbers was announced. (In the Florida state lottery, one chooses six distinct integers from 1 to 49.) Someone suggested that a person might just as well choose 1, 2, 3, 4, 5, and 6 as any other sequence. In fact, why not choose any six consecutive integers . . . what difference does it make? The chances are the same as any other sequence of six distinct integers!

This led to the following analysis of the least interval between consecutive members of a sequence of six integers. Here, we are concerned with the set of possible lottery tickets for the Florida state lottery. That is, the set of all possible six distinct integers from 1 to 49. The calculation given below can be generalized to "r integers from 1 to n are chosen." The generalization will be given at the end of this article. For clarity, however, we will use Florida's lottery as an example of the technique involved.

In what follows, we let L be the set of all possible Florida lottery tickets. That is,

$$L = \{(t_1, t_2, t_3, t_4, t_5, t_6) : 1 \le t_1 < t_2 < t_3 < t_4 < t_5 < t_6 \le 49\}.$$

We also define the function f on L by:

$$f(t_1, t_2, t_3, t_4, t_5, t_6) = \min\{t_{i+1} - t_i : i = 1, 2, 3, 4, 5\}.$$

Thus, if $t \in L$, we can think of $f(t)$ as the "smallest space" on the ticket t. Our purpose is to determine the mean smallest space with respect to the members of L. That is,

$$\frac{\sum_{t \in L} f(t)}{\binom{49}{6}}$$

will be determined.

Determination of the Mean of the Smallest Spaces of L

Consider the set of 5-tuples,

$$D = \{(d_1, d_2, d_3, d_4, d_5) : 5 \le d_1 + d_2 + d_3 + d_4 + d_5 \le 48;\ d_i \ge 1\},$$

and the function $F : L \to D$ defined by

$$F((t_1, t_2, t_3, t_4, t_5, t_6)) = (t_2 - t_1, t_3 - t_2, t_4 - t_3, t_5 - t_4, t_6 - t_5).$$

It is clear that F is a function from L *onto* D. This will enable us to efficiently determine

$$\sum_{t \in L} f(t)$$

by use of a particular partition of D. If $d \in D$, we note that

$$\#\{t \in L : F(t) = d\} = 49 - s,$$

where $\#$ is used to denote the number of elements in a set and where

$$d = (d_1, d_2, d_3, d_4, d_5) \quad \text{and} \quad s = d_1 + d_2 + d_3 + d_4 + d_5.$$

For $d = (d_1, d_2, d_3, d_4, d_5) \in D$, we define

$$s(d) = d_1 + d_2 + d_3 + d_4 + d_5$$
$$a(d) = \min\{d_1, d_2, d_3, d_4, d_5\}$$
$$m(d) = \#\{i : d_i = a(d)\}.$$

When the context is clear, we will just write s, a, or m. We now see that

$$f(t) = a(F(t)),$$

and that

$$5 \le s \le 48,$$
$$1 \le a \le 9,$$

and $\quad 1 \le m \le 5.$

For each triple (i, j, k) with $5 \le i \le 48$, $1 \le j \le 9$, and $1 \le m \le 5$, we define

$$D_{ijk} = \{d \in D : s(d) = i;\ a(d) = j;\ m(d) = k\}$$

and note that

$$\mathscr{D} = \{D_{ijk} : 5 \le i \le 48;\ 1 \le j \le 9;\ 1 \le k \le 5\}$$

is a partition of D. Since

$$(*) \qquad \sum_{t \in D} f(t) = \sum_{D_{ijk} \in \mathscr{D}} (49 - i)j(\#D_{ijk}),$$

we proceed to determine the right side of (*) by first considering each $k = 1$, 2, 3, 4, and 5. For this, we use the following theorem. Its statement and proof are found in [1: Theorem 2.4.3; pp. 145-46].

Theorem: For integers $r, r_1, r_2, \ldots, r_n$, the number of solutions to

$$x_1 + x_2 + \cdots + x_n = r$$
$$x_i \ge r_i \text{ for } i = 1, 2, \ldots, n$$

is

$$\binom{n - 1 + r - r_1 - r_2 - \cdots - r_n}{n - 1}$$

Thus, if we let s and a be given, we use the above theorem to find the number of solutions to

$$d_1 + d_2 + d_3 + d_4 + d_5 = s$$
$$d_i \ge r_i;\ i = 1, 2, 3, 4, 5,$$

for $m = 1$, 2, 3, 4, and 5. For example, if $m = 1$, $d_i = a$ for some i and $d_j \ge a + 1$ for $j \ne i$. Since there are $\binom{5}{1}$ ways to choose the d_i and, by the theorem,

$$x_1 + x_2 + x_3 + x_4 = s - a$$
$$x_i \ge a + 1;\ i = 1, 2, 3, 4$$

has

$$\binom{s - 5a - 1}{3}$$

solutions, it follows that

$$\#D_{s,a,1} = 5\binom{s - 5a - 1}{3}.$$

Similarly, we obtain

$$\#D_{s,a,2} = \binom{5}{2}\binom{s-5a-1}{2} = 10\binom{s-5a-1}{2},$$

$$\#D_{s,a,3} = \binom{5}{3}\binom{s-5a-1}{1} = 10\binom{s-5a-1}{1},$$

$$\#D_{s,a,4} = \binom{5}{4}\binom{s-5a-1}{0} = 5\binom{s-5a-1}{0},$$

and

$$\#D_{s,a,5} = \binom{5}{5}\binom{s-5a-1}{-1} = \binom{s-5a-1}{-1}.$$

It should be noted here that we will use the convention

$$\binom{n}{-1} = \begin{cases} 1, & \text{if } n = -1; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\binom{i}{k} = 0 \quad \text{if } i < k.$$

Thus, for fixed s and a,

$$\sum_{k=1}^{5} \#D_{sak} = \sum_{k=1}^{5} \binom{5}{k}\binom{s-5a-1}{4-k},$$

and by [2: Formula 21; p. 58], we have

$$\sum_{k=1}^{5} \#D_{sak} = \binom{s-5a+4}{4} - \binom{s-5a-1}{4}$$

Hence,

$$\sum_{t \in L} f(t) = \sum_{i=5}^{48} \sum_{j=1}^{9} \sum_{k=1}^{5} (49-i)j(\#D_{ijk}) = \sum_{i=5}^{48} \sum_{j=1}^{9} (49-i)j \sum_{k=1}^{5} \#D_{ijk}$$

which is, by the above,

$$= \sum_{i=5}^{48} (49-i) \sum_{j=1}^{9} j\left[\binom{i-5j+4}{4} - \binom{i-5j-1}{4}\right]$$

and by telescoping the inner sum,

$$= \sum_{i=5}^{48} \sum_{j=1}^{9} (49-i)\binom{i-5j+4}{4} = \sum_{j=1}^{9} \sum_{i=0}^{49} \binom{49-i}{1}\binom{i-5j+4}{4}$$

$$= \sum_{j=1}^{9} \binom{54-5j}{6}$$

by [2: Formula 25; p. 58]. We have, then, that the mean "smallest space" on a (Florida) lottery ticket is

$$\frac{\sum_{t \in L} f(t)}{\binom{49}{6}} = \frac{\sum_{j=1}^{9} \binom{54-5j}{6}}{\binom{49}{6}}$$

which is approximately 1.88.

Distribution

Of interest, also, would be a list of how lottery tickets are distributed with respect to the "smallest space" concept. For example, how many of the $\binom{49}{6}$ Florida state lottery tickets have a "smallest space" of 3?

This can be answered readily by noting that by omitting the j factor in the summand of (*) and summing with a fixed j, we have that the number of Florida lottery tickets with a "smallest space" of a is

$$\sum_{\substack{D_{ijk}\in\mathscr{D}\\ j=a}} (49 - i)\#D_{ijk},$$

which simplifies to

$$\binom{54 - 5a}{6} - \binom{49 - 5a}{6}$$

Hence, we can construct the following list of how the Florida lottery tickets are distributed with respect to the "smallest space" idea.

smallest space	*number of such tickets*
1	6924764
2	3796429
3	1917719
4	869884
5	340424
6	107464
7	24129
8	2919
9	84

Thus, it can be observed that close to 91% of all possible Florida state lottery tickets have a "smallest space" of 1, 2, or 3. It seems, then, that it might be wise to choose a lottery ticket that has a "smallest space" of 1, 2, or 3 and avoid those with a "smallest space" greater than 3.

Conclusion

As stated earlier, the above could be generalized to a lottery where r numbers from the sequence 1, 2, 3, ..., n are chosen. Using the same technique as before, it is easily shown that the mean of the "smallest space" of all possible lottery tickets where r numbers are chosen from 1, 2, 3, ..., n is

$$\frac{\displaystyle\sum_{j=1}^{\left[\frac{n-1}{r-1}\right]} \binom{n - (r - 1)(j - 1)}{r}}{\binom{n}{r}}$$

and that the number of such lottery tickets with a "least space" of a is

$$\binom{n - (r - 1)(a - 1)}{r} - \binom{n - (r - 1)a}{r}.$$

Of course, another approach in investigating lottery tickets might be to analyze the collection of lottery tickets with respect to the "largest space" on a ticket. This should also be of interest, and we encourage the reader to make such an analysis.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

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BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n, satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-700 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that for positive integers m and n,

$$\alpha^m(\alpha L_n + L_{n-1}) = \alpha^n(\alpha L_m + L_{m-1}).$$

B-701 *Proposed by Herta T. Freitag, Roanoke, VA*

In triangles ABC and DEF, $AC = DF = 5F_{2n}$, $BC = L_{n+2}L_{n-1}$, $EF = L_{n+1}L_{n-2}$, and $AB = DE = 5F_{2n+1} - L_{2n+1} + (-1)^{n-1}$. Prove that $\angle ACB = \angle DFE$.

B-702 *Proposed by L. Kuipers, Sierre, Switzerland*

For n a positive integer, let

$$x_n = F_n + \cfrac{1}{L_n + \cfrac{1}{F_n + \cfrac{1}{L_n + \cfrac{1}{\ddots}}}} \quad \text{and} \quad y_n = F_n + \cfrac{1}{F_{n+1} + \cfrac{1}{F_n + \cfrac{1}{F_{n+1} + \cfrac{1}{\ddots}}}}.$$

(a) Find closed form expressions for x_n and y_n.
(b) Prove that $x_n < y_n$ when $n > 1$.

B-703 *Proposed by H.-J. Seiffert, Berlin, Germany*

Prove that for all positive integers n,

$$\sum_{k=1}^{n} 4^{n-k} F_{2^k}^4 = \frac{F_{2^{n+1}}^2 - 4^n}{5}.$$

B-704 *Proposed by Paul S. Bruckman, Edmonds, WA*

Let a and b be fixed integers. Show that if three integers are of the form $ax^2 + by^2$ for some integers x and y, then their product is also of this form.

B-705 *Proposed by H.-J. Seiffert, Berlin, Germany*

(a) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{L_{2n}}{n^2\binom{2n}{n}} = \frac{\pi^2}{5}$.

(b) Find the value of $\displaystyle\sum_{n=1}^{\infty} \frac{F_{2n}}{n^2\binom{2n}{n}}$.

SOLUTIONS

edited by **A. P. Hillman**

Triangular Divisibility

B-676 *Proposed by Herta T. Freitag, Roanoke, VA*

Let T_n be the n^{th} triangular number $n(n+1)/2$. Characterize the positive integers n such that

$$T_n \,\Big|\, \sum_{i=1}^{n} T_i.$$

Solution by Hans Kappus, Rodersdorf, Switzerland

It is immediate that

$$\sum_{i=1}^{n} T_i = (n+2)T_n/3.$$

Therefore, T_n divides $\sum_{i=1}^{n} T_i$ if and only if $n \equiv 1 \pmod 3$.

Also solved by R. André-Jeannin, Charles Ashbacher, Wray Brady, Paul S. Bruckman, Russell Euler, Guo-Gang Gao, Russell Jay Hendel, Joseph J. Kostal, L. Kuipers, Carl Libis, Graham Lord, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Paul Smith, Lawrence Somer, W. R. Utz, and the proposer.

More Triangular Divisibility

B-677 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $T_n = n(n+1)/2$. Characterize the positive integers n with

$$\sum_{i=1}^{n} T_i \,\Big|\, \sum_{i=1}^{n} T_i^2.$$

Solution by Hans Kappus, Rodersdorf, Switzerland

A straightforward calculation shows that

$$\sum_{i=1}^{n} T_i^2 = \frac{3n^2 + 6n + 1}{10} \cdot \frac{n + 2}{3} T_n = \frac{3n^2 + 6n + 1}{10} \sum_{i=1}^{n} T_i,$$

by the result of B-676. Working mod 10, we see that $3n^2 + 6n + 1$ is a multiple of 10 if and only if

$$n \equiv 1 \pmod{10} \quad \text{or} \quad n \equiv 7 \pmod{10}.$$

Also solved by R. André-Jeannin, Charles Ashbacher, Paul S. Bruckman, Russell Euler, Joseph J. Kostal, L. Kuipers, Carl Libis, Graham Lord, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Smith, and the proposer.

Nontriangular Numbers

B-678 *Proposed by R. André-Jeannin, Sfax, Tunisia*

Show that L_{4n} and L_{4n+3} are never triangular numbers.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall use the following known results in our solution:

(1) $L_{4n} - 2 = 5F_{2n}^2$ for each positive integer n;

(2) $L_{4n+2} + 2 = 5F_{2n+1}^2$ for each nonnegative integer n.

Note: (1) is (I_{16}) and (2) is (I_{17}) on p. 59 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr. (Boston: Houghton Mifflin, 1969).

As immediate corollaries, we have:

(1′) $L_{4n} \equiv 2 \pmod 5$;

(2′) $L_{4n+2} \equiv 3 \pmod 5$.

Next, we establish the following results.

Lemma 1: The sequence of triangular numbers T_n is periodic modulo 5 with a period of 5.

Proof: It suffices to show that $T_{n+5} \equiv T_n \pmod 5$ where n is an arbitrary positive integer.

$$T_{n+5} - T_n = \frac{(n+5)(n+6)}{2} - \frac{n(n+1)}{2} = \frac{(n^2 + 11n + 30) - (n^2 + n)}{2}$$
$$= 5n + 15 \equiv 0 \pmod 5.$$

Lemma 2: Let n be a positive integer. Then T_n is congruent to 0, 1, or 3 modulo 5.

Proof: The claimed result follows from Lemma 1 and the table given below.

n	1	2	3	4	5
T_n	1	3	6	10	15
T_n (mod 5)	1	3	1	0	0

The fact that L_{4n} is never a triangular number follows from (1′) and Lemma 2.

Since, from (1′) and (2′),

$$L_{4n+3} = 2L_{4n+2} - L_{4n},\ L_{4n+3} \equiv 2(3) - 2 \pmod{5},$$

we have

$$L_{4n+3} \equiv 4 \pmod{5}.$$

Thus, L_{4n+3} is never a triangular number by Lemma 2.

Also solved by Paul S. Bruckman, H.-J. Seiffert, Sahib Singh, and the proposers.

Product of 4 Lucas Numbers

B-679 *Proposed by R. André-Jeannin, Sfax, Tunisia*

Express $L_{n-2}L_{n-1}L_{n+1}L_{n+2}$ as a polynomial in L_n.

Solution by Guo-Gang Gao, Université de Montréal, Montréal, Canada

It is easy to prove that $L_{2n} = L_n^2 - (-1)^n 2$. Then

$$\begin{aligned} L_{n-2}L_{n+2} &= (\alpha^{n-2} + \beta^{n-2})(\alpha^{n+2} + \beta^{n+2}) \\ &= L_{2n} + (-1)^{n-2}L_4 \\ &= L_n^2 + (-1)^n 5. \end{aligned}$$

Similarly,

$$L_{n-1}L_{n+1} = L_n^2 - (-1)^n 5.$$

Therefore,

$$L_{n-2}L_{n-1}L_{n+1}L_{n+2} = L_n^4 - 25.$$

Also solved by Paul S. Bruckman, Russell Euler, Herta T. Freitag, Russell Jay Hendel, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Smith, Lawrence Somer, and the proposer.

Congruence

B-681 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let n be a nonnegative integer, $k \geq 2$ an even integer, and $r \in \{0, 1, \ldots, k-1\}$. Show that

$$F_{kn+r} \equiv (F_{k+r} - F_r)n + F_r \pmod{L_k - 2}.$$

Solution by Guo-Gang Gao, Université de Montréal, Montréal, Canada

Let us first prove that

$$F_{k(n+1)+r} = F_{kn+r}L_k - F_{k(n-1)+r},$$

where $k \geq 2$ is an even integer and $r \geq 0$. Notice that

$$(\alpha \times \beta)^k = (-1)^k = 1.$$

$$F_{kn+r}L_k = \frac{1}{\sqrt{5}}(\alpha^{kn+r} - \beta^{kn+r})(\alpha^k + \beta^k)$$

$$= \frac{1}{\sqrt{5}}(\alpha^{k(n+1)+r} - \beta^{k(n+1)+r}) + \frac{1}{\sqrt{5}}(\alpha^{k(n-1)+r} - \beta^{k(n-1)+r})$$

$$= F_{k(n+1)+r} + F_{k(n-1)+r}.$$

Use mathematical induction for the proof:

(1) It is trivially true when $n = 0, 1$.

(2) Assume that the claim holds for up to n.

Then, by the inductive hypothesis, we have the following:

$$\begin{aligned} F_{k(n+1)+r} &= F_{kn+r}L_k - F_{k(n-1)+r} \\ &\equiv ((F_{k+r} - F_r)n + F_r)L_k \\ &\qquad - ((F_{k+r} - F_r)(n-1) + F_r) \pmod{L_k - 2} \\ &\equiv 2((F_{k+r} - F_r)n + F_r) \\ &\qquad - ((F_{k+r} - F_r)(n-1) + F_r) \pmod{L_k - 2} \\ &\equiv (F_{k+r} - F_r)(n+1) + F_r \pmod{L_k - 2}. \end{aligned}$$

This completes the proof.

Also solved by Paul S. Bruckman, Bob Prielipp, and the proposer.

Lucas Triangular Numbers

B-682 *Proposed by Joseph J. Kostal, University of Illinois, Chicago, IL*

Let $T(n)$ be the triangular number $n(n+1)/2$. Show that

$$T(L_{2n}) - 1 = \frac{1}{2}(L_{4n} + L_{2n}).$$

Solution by C. Georghiou, University of Patras, Patras, Greece

We have

$$T(L_{2n}) - 1 = (L_{2n}^2 + L_{2n} - 2)/2 = (L_{4n} + L_{2n})/2,$$

since it is well known that $L_{2n}^2 - 2 = L_{4n}$.

Also solved by Charles Ashbacher, Scott H. Brown, Paul S. Bruckman, David M. Burton, Russell Euler, Piero Filipponi, Herta T. Freitag, Guo-Gang Gao, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Bob Prielipp, Don Redmond, H.-J. Seiffert, Mohammad Parvez Shaikh, Sahib Singh, Lawrence Somer, and the proposer.

LT-Composite

B-683 *Proposed by Joseph J. Kostal, University of Illinois, Chicago, IL*

Let $L(n) = L_n$ and $T_n = n(n+1)/2$. Show that

$$L(T_{2n}) = L(2n^2)L(n) + (-1)^{n+1}L(2n^2 - n).$$

Solution by C. Georghiou, University of Patras, Patras, Greece

We have $L(T_{2n}) = L(2n^2 + n)$. But

$$\begin{aligned} L(2n^2+n) - L(2n^2)L(n) &= \alpha^{2n^2+n} + \beta^{2n^2+n} - \alpha^{2n^2+n} - \beta^{2n^2+n} \\ &\qquad - \alpha^{2n^2}\beta^n - \alpha^n\beta^{2n^2} \\ &= -(\alpha\beta)^n[\alpha^{2n^2-n} + \beta^{2n^2-n}] \\ &= (-1)^{n+1}L(2n^2-n), \end{aligned}$$

which proves the assertion.

Also solved by Charles Ashbacher, Paul S. Bruckman, David M. Burton, Russell Euler, Piero Filipponi, Herta T. Freitag, Guo-Gang Gao, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

B-680 ***Will be published in the next issue as an error was detected just before publication.***

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

<u>H-459</u> *Proposed by Stanley Rabinowitz, Westford, MA*

Prove that for all $n > 3$,

$$\frac{13\sqrt{5} - 19}{10}L_{2n+1} + 4.4(-1)^n$$

is very close to the square of an integer.

<u>H-460</u> *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_{n+2}(x) = xF_{n+1}(x) + F_n(x).$$

Show that, for all positive reals x,

(a) $$\sum_{k=1}^{n-1} 1/\left(x^2 + \sin^2\frac{k\pi}{2n}\right) = \frac{(2n-1)F_{2n+1}(2x) + (2n+1)F_{2n-1}(2x)}{4x(x^2+1)F_{2n}(2x)} - \frac{1}{2x^2},$$

(b) $$\sum_{k=1}^{n-1} 1/\left(x^2 + \sin^2\frac{k\pi}{2n}\right) \sim n/(x\sqrt{x^2+1}),\ \text{as } n \to \infty,$$

(c) $$\sum_{k=1}^{n-1} 1/\sin^2\frac{k\pi}{2n} = 2(n^2 - 1)/3.$$

<u>H-461</u> *Proposed by Lawrence Somer, Washington, D.C.*

Let $\{u_n\} = u(a, b)$ denote the Lucas sequence of the first kind satisfying the recursion relation

$$u_{n+2} = au_{n+1} + bu_n,$$

where a and b are nonzero integers and the initial terms are $u_0 = 0$ and $u_1 = 1$. The prime p is a primitive divisor of u_n if $p \mid u_n$ but $p \nmid u_m$ for $1 \le m \le n - 1$. It is known (see [1], p. 200) for the Fibonacci sequence $\{F_n\} = u(1, 1)$ that, if p is an odd prime divisor of F_{2n+1}, where $n \ge 1$, then $p \equiv 1 \pmod 4$.

(i) Find an infinite number of recurrences $u(a, b)$ such that every odd primitive prime divisor p of any term of the form u_{2n+1} or u_{4n} satisfies $p \equiv 1 \pmod 4$, where $n \geq 1$.

(ii) Find an infinite number of recurrences $u(a, b)$ such that every odd primitive prime divisor p of any term of the form u_{4n} or u_{4n+2} satisfies $p \equiv 1 \pmod 4$, where $n \geq 1$.

Reference

1. E. Lucas. "Théorie des Fonctions Numériques Simplement Périodiques." *Amer. J. Math.* *1* (1878):184-240, 289-321.

SOLUTION

Either Way

H-441 *Proposed by Albert A. Mullin, Huntsville, AL (Vol. 28, no. 2, May 1990)*

By analogy with palindrome, a *validrome* is a sentence, formula, relation, or verse that remains valid whether read forward or backward. For example, relative to *prime* factorization, 341 is a factorably validromic number since $341 = 11 \cdot 31$, and when backward gives $13 \cdot 11 = 143$, which is also correct. (1) What is the largest factorably validromic square you can find? (2) What is the largest factorably validromic square, *avoiding* palindromic numbers, you can find? Here are three examples of factorably validromic squares:

$$13 \cdot 13, \quad 101 \cdot 101, \quad 311 \cdot 311.$$

Solution by Paul S. Bruckman, Edmonds, WA

Suppose $n = \theta_1\theta_2\ldots\theta_r$ is in denary notation; we write

$$n' = \theta_r\theta_{r-1}\ldots\theta_1.$$

Given two natural numbers m and n, we say the product $m \times n$ is *validromic* if and only if $m \times n = m' \times n'$. A natural number n is said to be a *validromic square root* if and only if:

(1) $\quad (n^2)' = (n')^2.$

Let V denote the set of validromic square roots; we also write $n \in V$ if equation (1) holds. In this case, we also call n^2 a *validromic square*.

Some interesting properties of such numbers are derived by analyzing the familiar "long multiplication" process, somewhat modified. The multiplication for $n^2 = n \times n$ is indicated below:

$$(2) \qquad \begin{array}{cccccccccc}
 & & & & & \theta_1 & \theta_2 & \cdots & \theta_r & \\
 & & & & \times & \theta_1 & \theta_2 & \cdots & \theta_r & \\
\hline
 & & & & & \theta_1\theta_r & \theta_2\theta_r & \cdots & \theta_{r-1}\theta_r & \theta_r^2 \\
 & & & & \theta_1\theta_{r-1} & \theta_2\theta_{r-1} & \theta_3\theta_{r-1} & \cdots & \theta_r\theta_{r-1} & \\
 & & & \iddots & \vdots & \vdots & \vdots & \iddots & & \\
 & & \theta_1\theta_2 & \cdots & \theta_{r-2}\theta_2 & \theta_{r-1}\theta_2 & \theta_r\theta_2 & & & \\
 & \theta_1^2 & \theta_2\theta_1 & \cdots & \theta_{r-1}\theta_1 & \theta_r\theta_1 & & & & \\
\hline
 & s_1 & s_2 & \cdots & s_{r-1} & s_r & s_{r+1} & \cdots & s_{2r-2} & s_{2r-1} \\
a_0 & a_1 & a_2 & \cdots & a_{r-1} & a_r & a_{r+1} & \cdots & a_{2r-2} & a_{2r-1}
\end{array}$$

In this product, the terms $\theta_i\theta_j$ are *not* reduced (mod 10) as they would normally be, nor are the columnar sums s_k. Therefore,

$$s_k = \sum_{\substack{i+j=k+1\\ 1\le i,\ j\le r}} \theta_i\theta_j, \text{ or more precisely,}$$

$$(3) \qquad s_k = \sum_{i=\max(k-r+1,\,1)}^{\min(k,\,r)} \theta_i\theta_{k+1-i}, \; k = 1, 2, \ldots, 2r-1.$$

Thus, the terms $\theta_i\theta_j$ and the sums s_k are not necessarily denary digits. However, the a_k's (indicated below the s_k's) *are* denary digits, obtained by the process of "carrying forward and bringing down" familiar to any schoolchild. We do not preclude the possibility $a_0 = 0$.

Next, we carry out the similar multiplication for $(n')^2 = n' \times n'$:

$$(4) \qquad \begin{array}{cccccccccc}
 & & & & & \theta_r & \theta_{r-1} & \cdots & & \theta_1 \\
 & & & & & \times\ \theta_r & \theta_{r-1} & \cdots & & \theta_1 \\
\cline{6-10}
 & & & & & \theta_r\theta_1 & \theta_{r-1}\theta_1 & \cdots & \theta_2\theta_1 & \theta_1^2 \\
 & & & & \theta_r\theta_2 & \theta_{r-1}\theta_2 & \theta_{r-2}\theta_2 & \cdots & \theta_1\theta_2 & \\
 & & & \iddots & \vdots & \vdots & \vdots & \iddots & & \\
 & & \theta_r\theta_{r-1} & \cdots & \theta_3\theta_{r-1} & \theta_2\theta_{r-1} & \theta_1\theta_{r-1} & & & \\
 & \theta_r^2 & \theta_{r-1}\theta_r & \cdots & \theta_2\theta_r & \theta_1\theta_r & & & & \\
\hline
 & s_{2r-1} & s_{2r-2} & \cdots & s_{r+1} & s_r & s_{r-1} & \cdots & s_2 & s_1 \\
b_0 & b_1 & b_2 & \cdots & b_{r-1} & b_r & b_{r+1} & \cdots & b_{2r-2} & b_{2r-1}
\end{array}$$

As in the first product, we allow $b_0 = 0$. The observation that the columnar sums s_k in (4) are identical to those in (2) (except in reverse order) is a consequence of their consisting of the same components $\theta_i\theta_j$, albeit in permuted order. In fact, we see that if we reverse the order of the r "product-rows" in (4), then reverse the order of the digits in each such row, we obtain the corresponding product-rows of (2).

Using the notation introduced, we call the product $n \times n$ *proper* if and only if, for all $i, j \in \{1, 2, \ldots, r\}$, $k \in \{1, 2, \ldots, 2r-1\}$, the products $\theta_i\theta_j$ and the s_k's are all denary digits. Otherwise, we say that the product $n \times n$ is *improper*. We now prove a useful characterization of validromic square roots.

Theorem 1: $n \in V$ if and only if $n \times n$ is proper.

Proof: First, suppose $n \times n$ is proper. Looking at (2) and (4), it is clear that $a_0 = b_0 = 0$, and moreover that $a_k = s_k = b_{2r-k}$, $k = 1, 2, \ldots, 2r-1$. Equivalently, $(n^2)' = (n')^2$, or $n \in V$.

Conversely, suppose that $s_k \ne a_k$ for some k. Let $s_k = a_k$, for all $k > h$, but $s_h = a_h + 10d_h$ for some $d_h > 1$. Inspection of (4) yields:

$$b_{2r-h} = a_h, \text{ but } b_{2r-h-1} \equiv s_{h+1} + d_h \pmod{10};$$

if $h = 2r-1$, we define $s_{2r} = a_{2r} = 0$. If $d_h \not\equiv 0 \pmod{10}$, then

$$b_{2r-h-1} \equiv a_{h+1} + d_h \pmod{10}, \text{ so } b_{2r-h-1} \ne a_{h+1}.$$

If $d_h \equiv 0 \pmod{10}$, then

$$b_{2r-h-1} = a_{h+1}, \text{ but } b_{2r-h-2} \equiv s_{h+2} + d_{h+1} \equiv a_{h+2} + d_{h+1} \pmod{10},$$

where $d_{h+1} = d_h/10$. We apply the same argument until we find a nonzero remainder that is *not* a multiple of 10; eventually, there exists a value of k such that

$b_{2r-k} \neq a_k$. Thus, if $n \times n$ is improper, then $n \notin V$. This completes the proof of Theorem 1.

The theorem just proved greatly facilitates the search for validromic square roots (and validromic squares). A by-product of its proof is that if $n \in V$ and n has r digits, then n^2 has $2r - 1$ digits; to avoid trivial variants, we adopt the convention that, if $n = \theta_1\theta_2\ldots\theta_r \in V$, then $\theta_1 \neq 0$, $\theta_r \neq 0$. Thus, $n^2 < 10^{2r-1}$, which implies the following

Corollary: If $n \in V$ has r digits, then $n \le [10^{r-\frac{1}{2}}] = \underbrace{3162\ldots}_{(r\text{ digits})}$.

Let n_r denote the largest r-digit validromic square root. Then, by the Corollary, $n_1 \le 3$, $n_2 \le 31$, $n_3 \le 316$, etc. We readily find that $n_1 = 3$ (trivially), and $n_2 = 31$. There are other useful observations that may be made to facilitate extension of these initial results.

In what follows, we suppose that $n_r = \theta_1\theta_2\ldots\theta_r \in V$ (with the conventions as described previously). First, we surmise that $\theta_1 = 3$ for all r; this is easily proved. Clearly, this is true for $r = 1$ and $r = 2$. If $r > 2$, define

$$m_r = 3\underbrace{0\,0\,\ldots\,0}_{r-2}1;$$

then

$$m_r^2 = 9\underbrace{0\,0\,\ldots\,0}_{r-2}6\underbrace{0\,0\,\ldots\,0}_{r-2}1, \text{ and } (m_r')^2 = 1\underbrace{0\,0\,\ldots\,0}_{r-2}6\underbrace{0\,0\,\ldots\,0}_{r-2}9,$$

so $m_r \in V$. Since $n_r \ge m_r$, by definition of n_r, thus $\theta_1 \ge 3$. But the Corollary implies $\theta_1 \le 3$. Hence, $\theta_1 = 3$.

Clearly, if $n > m$ and $m \times m$ is improper, so is $n \times n$. This observation allows us to reject all candidates for n_r which exceed a previously excluded candidate and differ from it in only one or more digits. However, a much more powerful result may be inferred, which greatly reduces our search for n_r. Given that $\theta_1 = 3$ and $\theta_r = 1$, then the formula in (3) implies:

$$s_k \ge 2\theta_1\theta_k = 6\theta_k, \text{ for } k = 2, 3, \ldots, r.$$

However, s_k must be a digit; this implies $\theta_k = 0$ or 1. Therefore, n_r must be composed of "binary" digits, except for its leading digit, which is 3, and its last digit must equal 1.

Proceeding largely by trial and error, with the tools developed thus far, we find n_r, at least for the initial values of r. We begin from the left with $\theta_1 = 3$, then affix as many consecutive 1's as possible to the right. When one or more 0's must be used, we try to minimize the number of such 0's, and to push them as far to the right as possible, subject only to the condition that $\theta_r = 1$. As we proceed, we keep track of the rejected candidates, so as to reduce our search. Thus, if $\theta_1'\theta_2'\ldots\theta_r'$ is such a rejected value for n_r, then we know that $n_{r+1} < \theta_1'\theta_2'\ldots\theta_r'1$. Proceeding in this fashion, we find the following values of n_r, up to $r = 15$ (though we could have continued the table, by these same methods):

(5)

r	n_r	r	n_r	r	n_r
1	3	6	311101	11	31111 01000 1
2	31	7	31111 01	12	31111 01010 01
3	311	8	31111 001	13	31111 01010 001
4	3111	9	31111 0101	14	31111 01010 0011
5	31111	10	31111 01001	15	31111 01010 00001

Inspection of the foregoing table leads to the conjecture that θ_k is constant for all sufficiently large r; a rigorous proof of this premise seems possible but was not attempted. A related observation is that, for sufficiently large k, the values of θ_k do not affect the leading digits of n_r^2.

To stress dependence of r (as well as k), we use the expanded notation:

$$\theta_k^{(r)} \equiv \theta_k, \quad s_k^{(r)} \equiv s_k.$$

If r_k represents the minimum value of r such that $\theta_k^{(r)} = \theta_k$, a constant for all $r \ge r_k$, we can tabulate our *apparent* results as follows:

(6)

k	r_k	θ_k	k	r_k	θ_k	k	r_k	θ_k
1	1	3	7	9	1	13	15	0
2	2	1	8	9	0	14	21	1
3	3	1	9	12	1	15	16	0
4	4	1	10	11	0	16	17	0
5	7	1	11	12	0	17	26	1
6	7	0	12	16	1	18	19	0 etc.

Of course, in order to form this table, we first need to compute n_r for $r \gg 18$; even then, we cannot always be certain that the values in (6) are correct, at least for the higher values of k. However, if we *can* accept these values as gospel, we may then extend the table of n_r's.

The number of terms $\theta_i\theta_j$ in $s_k^{(r)}$ is maximized when $k = r$, and such number is r. A necessary (but not sufficient) test, therefore, is that $s_r^{(r)}$ be a digit. Other values of $s_r^{(r)}$ also need to be tested, of course; since the ones most likely to fail are the ones whose terms contain $\theta_1 = 3$, we test those first.

We illustrate by finding n_{27}, assuming that (6) is correct. We note that

$$s_{27}^{(27)} = \sum_{i=1}^{27} \theta_i^{(27)}\ \theta_{28-i}^{(27)} = 2\sum_{i=1}^{13} \theta_i \theta_{28-i}^{(27)} + \theta_{14}^2, \text{ with } \theta_{27}^{(27)} = 1;$$

thus,

$$s_{27}^{(27)} = 2(\theta_{27}^{(27)}\theta_1 + \theta_{26}^{(27)}\theta_2 + \theta_{25}^{(27)}\theta_3 + \theta_{24}^{(27)}\theta_4 + \theta_{23}^{(27)}\theta_5 + \theta_{21}^{(27)}\theta_7 + \theta_{19}^{(27)}\theta_9) + \theta_{14}^2$$

$$= 2(3 + \theta_{26}^{(27)} + \theta_{25}^{(27)} + \theta_{24}^{(27)} + \theta_{23}^{(27)} + \theta_{21}^{(27)} + \theta_{19}^{(27)}) + 1.$$

To maximize n_{27}, we may *attempt* $\theta_{19}^{(27)} = 1$; however, since $s_{27}^{(27)}$ is to be a digit, this forces $\theta_{21}^{(27)} = \theta_{23}^{(27)} = \theta_{24}^{(27)} = \theta_{25}^{(27)} = \theta_{26}^{(27)} = 0$. At this point, nothing can be inferred about $\theta_{20}^{(27)}$ or $\theta_{22}^{(27)}$; for this, we need to consider the following:

$$s_{20}^{(27)} = \sum_{i=1}^{20} \theta_i^{(27)}\theta_{21-i}^{(27)} = 2(\theta_{20}^{(27)}\theta_1 + \theta_{19}^{(27)}\theta_2 + \theta_{17}\theta_4 + \theta_{14}\theta_7 + \theta_{12}\theta_9)$$

$$= 2(\theta_{20}^{(27)} + 1 + 1 + 1 + 1),$$

assuming $\theta_{19}^{(27)} = 1$. In order for this last expression to be a digit, we must have $\theta_{20}^{(27)} = 0$. Likewise, we find that $\theta_{19}^{(27)} = 1$ implies $s_{22}^{(27)} = 2(\theta_{22}^{(27)} + 1 + 1 + 1 + 1)$, which can only be a digit if $\theta_{22}^{(27)} = 0$. Therefore, we surmise that n_{27} is given by using the values of θ_k shown by (6) for its first 18 digits, then, with $\theta_{19}^{(27)}\theta_{20}^{(27)}\ldots\theta_{27}^{(27)} = 100000001$. Testing this as a candidate for n_{27}, we find that it works; hence, we conclude that n_{27} is as just described.

Continuing in this fashion, we may extend (5) and (6) by alternating back and forth between tables. With considerable effort, the following additional values of n_r were derived (manually) by these methods:

r	n_r
16	31111 01010 01000 1
17	31111 01010 01010 01
18	31111 01010 01000 001
19	31111 01010 01010 0101
20	31111 01010 01000 00001
21	31111 01010 01010 01000 1
22	31111 01010 01010 01000 01
23	31111 01010 01010 01000 001
24	31111 01010 01010 01010 0001
25	31111 01010 01010 00000 00001
26	31111 01010 01010 01010 00010 1
27	31111 01010 01010 01010 00000 01
28	31111 01010 01010 01010 00000 001
29	31111 01010 01010 01010 00010 0001
30	31111 01010 01010 01000 10000 00001

In theory, one could extend these results indefinitely, however, without the aid of a computer, human endurance wanes. It seems quite plausible that a program might be devised, enabling extension of the foregoing tables to an arbitrary degree. The aim of such extension would be to discover any lurking pattern in the sequence of "binary" digits among the θ_k's, as k increases. It may be surmised that, having discovered such a pattern, one might be able to prove its validity rigorously. This exercise is left for the interested reader.

As for this particular solver, he gave up the effort at $r = 30$. Therefore, to "answer" both parts of the problem simultaneously (since neither n_r nor n_r^2, clearly, are palindromes), the largest validromic square found was n_{30}^2, where

$$n_{30} = 31111\ 01010\ 01010\ 01000\ 10000\ 00001.$$

<u>Note</u>: The proposer noted that $441 = 21 \cdot 21$, so that the restriction of factors to squares is unnecessary.

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