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A NEW CHARACTERIZATION OF THE FIBONACCI NUMBERS

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1. INTRODUCTION

A theorem due to E. Zeckendorf (see [1] for proof and discussion) asserts that every positive integer n can be represented uniquely as a sum of distinct Fibonacci numbers such that no two consecutive Fibonacci numbers appear in the representation. With the definition $u_1 = 1$, $u_2 = 2$, and $u_{n+2} = u_{n+1} + u_n$ for $n \geq 1$, Zeckendorf's theorem yields an expansion for each positive integer n in the form

$$n = \sum_1^{\infty} \alpha_i u_i ,$$

where α_i is either 0 or 1 for each $i \geq 1$ and $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$. Thus each positive integer n can be associated with a binary sequence $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_i, \ldots$, where for given n, we see that $\alpha_i(n) = 1$ if u_i appears in the Zeckendorf expansion of n; otherwise $\alpha_i(n) = 0$. The constraint $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$ effectively states that two consecutive 1's can never appear in the binary sequence corresponding to n. For example, if $n = 17$, then $17 = 1 + 3 + 13 = u_1 + u_3 + u_6$, so that 17 is associated with the binary sequence 101001. (It is clear that each such expansion must have all zeros to the right of some $i = i_o$ depending on n and these noncontributing zeros are suppressed.)

The question arises as to what occurs if, instead of disallowing two consecutive non-zero coefficients in a Fibonacci expansion, we disallow two consecutive zero coefficients. In other words, we wish to consider representing an arbitrary positive integer n as a sum of distinct Fibonacci numbers,

$$n = \sum_1^{N} \beta_i u_i$$

with binary coefficients satisfying $\beta_N = 1$ and $\beta_i + \beta_{i+1} \geq 1$ for $i = 1, 2, \ldots, N-2$. In the following, Theorem 1 affords a result dual to the Zeckendorf theorem by showing that such an expansion always exists for every positive integer and moreover the expansion is unique under the imposed coefficient constraints. Theorem 2, which is our main result, shows that the expansion property of Theorem 1 together with the uniqueness requirement is sufficient to characterize the Fibonacci numbers. This converse theorem for the dual representation corresponds to Daykin's result [2] on the converse to Zeckendorf's theorem.

2. A DUAL-ZECKENDORF THEOREM

Definition 1: The Fibonacci sequence $\{u_i\}$ is defined by $u_1 = 1$, $u_2 = 2$, and $u_{n-2} = u_{n+1} + u_n$ for $n \geq 1$.

Lemma 1:

(a)
$$\begin{cases} u_{k+1} - 1 = u_k + u_{k-2} + u_{k-4} + \ldots + u_3 + u_1 & \text{for } k \text{ odd.} \\ u_{k+1} - 1 = u_k + u_{k-2} + u_{k-4} + \ldots + u_4 + u_2 & \text{for } k \text{ even.} \end{cases}$$

(b)
$$u_{k+1} = u_k + u_{k-2} + u_{k-4} + \ldots + u_{k-2p+2} + u_{k-2p+1} ,$$

where

$$p = 1, 2, \ldots, \frac{k}{2} \text{ for } k \text{ even and } p = 1, 2, \ldots, \frac{k-1}{2} \text{ for } k \text{ odd.}$$

(c)
$$u_{k+1} - 2 = \sum_{1}^{k-1} u_i \text{ for } k \geq 1, \text{ where } \sum_{m}^{\text{"}n\text{"}}$$

is defined to be zero for $n < m$.

Proof. The straightforward inductive proof is left to the reader.

Our first theorem, as noted in the introduction, is essentially a dual of the Zeckendorf theorem [1] :

Theorem 1: Every positive integer n has one and only one representation in the form

$$(1) \qquad n = \sum_{1}^{k} \beta_i u_i ,$$

where the β_i are binary digits satisfying

(2) $$\beta_i \beta_{i+1} \geq 1 \quad \text{for } i = 1, 2, \ldots, k-2$$

and

(3) $$\beta_k = 1 \quad .$$

For a given positive integer n, the value of k is determined as the unique integer for which the inequality,

(4) $$u_{k+1} - 2 < n \leq u_{k+2} - 2 \quad ,$$

is satisfied.

Convention: It will be assumed without explicit mention in the remainder of the paper that all expansion coefficients (subscripted variables α and β) are binary digits, that is, digits having either the value zero or the value unity.

Proof of Theorem: Let n be a positive integer satisfying inequality (4). From (c) of Lemma 1, we obtain the equivalent inequality,

(5) $$\sum_{1}^{k-1} u_i < n \leq \sum_{1}^{k} u_i \quad ,$$

so that by the Zeckendorf theorem, the non-negative integer

$$\sum_{1}^{k} u_i - n$$

possesses an expansion in the form,

(6) $$\sum_{1}^{k} u_i - n = \sum_{1}^{\infty} \alpha_i u_i \quad ,$$

with coefficients satisfying $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$.

Note from (5) that

$$\sum_{1}^{k} u_i - n < \sum_{1}^{k} u_i - \sum_{1}^{k-1} u_i = u_k \quad ,$$

which implies $\alpha_i = 0$ for $i \geq k$ in (6). In particular, $\alpha_k = 0$. Hence, (6) may be rewritten

$$\sum_1^k u_i - n = \sum_1^k \alpha_i u_i \quad \text{with } \alpha_k = 0 ,$$

or

$$n = \sum_1^k (1 - \alpha_i) u_i = \sum_1^k \beta_i u_i \quad \text{with } \beta_k = 1 , \tag{7}$$

where we have defined $\beta_i = 1 - \alpha_i$ for $i = 1, 2, \ldots, k$. It is clear from the relations $\alpha_i \alpha_{i+1} = 0$ and $\alpha_i \alpha_{i+1} = (1 - \beta_i)(1 - \beta_{i+1})$ that $\beta_i + \beta_{i+1} \geq 1$ for $i = 1, 2, \ldots, k - 2$, as required.

To show uniqueness, assume there exists a positive integer n with two representations:

$$n = \sum_1^m \beta_i u_i = \sum_1^p \beta_i' u_i , \tag{8}$$

where $\beta_m = \beta'_p = 1$, $\beta_i + \beta_{i+1} \geq 1$ for $i = 1, 2, \ldots, m - 2$ and $\beta_i' + \beta_{i+1}' \geq 1$ for $i = 1, 2, \ldots, p - 2$.

If $m \neq p$, then we assume $m > p$ without loss of generality, and from the coefficient constraints and Lemma 1, we have

$$\sum_1^m \beta_i u_i \geq u_m + u_{m-2} + u_{m-4} + \cdots + u_{1,2} = u_{m+1} - 1 ,$$

while

$$\sum_1^p \beta_i' u_i \leq \sum_1^p u_i \leq \sum_1^{m-1} u_i = u_{m+1} - 2 .$$

(Here and in what follows, the subscript notation $u_{1,2}$ serves to indicate the final term in a sum, the value of the final term being either u_1 or u_2 depending on the parity of the index associated with the initial term in the sum.) This is in evident contradiction to (8) and we conclude $m = p$.

Now, define $\alpha_i = 1 - \beta_i$ and $\alpha_i' = 1 - \beta_i'$ for $i = 1, 2, \ldots, p$. Then $\alpha_i \alpha_{i+1} = \alpha_i' \alpha_{i+1}' = 0$ for $i = 1, 2, \ldots, p - 1$, and (8) becomes

$$\sum_1^p (1 - \alpha_i)u_i = \sum_1^p (1 - \alpha_i')u_i$$

or

$$\sum_1^p \alpha_i u_i = \sum_1^p \alpha_i' u_i \quad . \tag{9}$$

Since both sides of (9) are admissible Zeckendorf representations, the uniqueness of such representations implies $\alpha_i = \alpha_i'$ for $i = 1, 2, \ldots, p$ or equivalently $\beta_i = \beta_i'$ for $i = 1, 2, \ldots, p$, which proves uniqueness of the dual representation and completes the proof of Theorem 1.

3. THE CONVERSE THEOREM

Next, we will show that the expansion property expressed in Theorem 1 actually provides a characterization of the Fibonacci numbers.

Definition 2: An arbitrary sequence of positive integers, $\{v_i\}$, $i = 1, 2, \ldots$ will be said to possess the dual unique representation property (Property D) if and only if every positive integer n has a unique representation in the form

$$n = \sum_1^p \beta_i v_i \quad \text{with} \quad \beta_p = 1, \quad \text{and} \tag{10}$$

$$\beta_i + \beta_{i+1} \geq 1 \quad \text{for } i = 1, 2, \ldots, p - 2 \quad . \tag{11}$$

Corollary 1: A sequence $\{v_i\}$ possessing Property D has distinct elements; that is, $v_m \neq v_n$ for $m \neq n$.

Proof: Assume $m \neq n$ and $v_m = v_n$. Take $m > n \geq 1$ without loss of generality; then,

$$\sum_{\substack{i=1 \\ i \neq n}}^{m} v_i = \sum_{1}^{m-1} v_i ,$$

contradicting the assumed uniqueness of expansions satisfying (10) and (11).

Lemma 2: If $\{v_i\}$ possesses Property D, then $v_1 = u_1$ and $v_2 = u_2$, where $\{u_i\}$ is the Fibonacci sequence of Definition 1.

Proof: In order to represent the integer 1 in the proper form [(10)-(11)], it is clear that either v_1 or v_2 must be equal to 1. If $v_1 = 1$, then $v_2 = 2$ necessarily and the Lemma is proved. In the remaining case, $v_1 = 2$, $v_2 = 1$ and it follows that $v_3 = 3$ and $v_4 = 6$. At this point, it is impossible to represent the integer 8 in proper form no matter how the remaining (distinct) v_i are chosen. Thus $v_1 = 1 = u_1$ and $v_2 = 2 = u_2$ as stated.

Theorem 2. If $\{v_i\}$, $i = 1, 2, \ldots$ is an arbitrary sequence of positive integers possessing Property D, then $v_i = u_i$ for all $i \geq 1$.

Proof: The assertion is true for $i = 1$ and $i = 2$ as proved in Lemma 2. Now, assume as an induction hypothesis that $v_1 = u_1$, $v_2 = u_2$, ..., $v_k = u_k$ for some $k \geq 2$. We wish to show that $v_{k+1} = u_{k+1}$ necessarily.

Recall from Theorem 1 (noting $v_i = u_i$ for $i = 1, 2, \ldots, k$ by the inductive assumption) that every positive integer n satisfying

$$0 \leq n \leq \sum_{1}^{k} v_i$$

has a representation

$$n = \sum_{1}^{m} \beta_i v_i ,$$

where $m \le k$, $\beta_m = 1$ and $\beta_i + \beta_{i+1} \ge 1$ for $i = 1, 2, \ldots, m-2$.

We show first that $v_{k+1} \ge u_{k+1}$. For, if not, then $v_{k+1} < u_{k+1}$ and

(12) $v_{k+1} + v_{k-1} + v_{k-3} + \ldots + v_{1,2} < u_{k+1} + u_{k-1} + u_{k-3} + \ldots + u_{1,2} = u_{k+2} - 1$,

which implies

$$v_{k+1} + v_{k-1} + \ldots + v_{1,2} \le u_{k+2} - 2 = \sum_1^k v_i .$$

From (12) and the remark in the preceding paragraph, we have

$$v_{k+1} + v_{k-1} + v_{k-3} + \ldots + v_{1,2} = \sum_1^m \beta_i v_i ,$$

with $m \le k$, $\beta_m = 1$ and $\beta_i + \beta_{i+1} \ge 1$ for $i = 1, 2, \ldots, m-2$. Since both sides are in the proper form and are not identical, uniqueness is contradicted. Therefore $v_{k+1} \ge u_{k+1}$ as asserted.

Now assume $v_{k+1} > u_{k+1}$. We shall show that this assumption also leads to a contradiction of uniqueness. If $v_{k+1} > u_{k+1}$, then the unique representation of the integer

$$\sum_1^k u_i + 1$$

has the form

(13) $$\sum_1^k u_i + 1 = \sum_1^{k+m} \beta_i v_i \quad \text{with } m \ge 2,\ \beta_{k+m} = 1 \text{ and } \beta_i + \beta_{i+1} \ge 1$$
$$\text{for } i = 1, 2, \ldots, k+m-2.$$

(For, if $m < 2$ in (13), then $m = 1$ since v_{k+1} must certainly appear with non-vanishing coefficient. But

$$\sum_1^{k+1} \beta_i v_i \ge v_{k+1} + u_{k-1} + u_{k-3} + \ldots + u_{1,2} > u_{k+1} + u_{k-1} + u_{k-3} + \ldots + u_{1,2} =$$
$$= \sum_1^k u_i + 1, \quad \text{so that (13) could not be valid.)}$$

The foregoing argument also shows $\beta_{k+1} = 0$ in (13); hence $\beta_k = \beta_{k+2} = 1$ from the coefficient constraints, and

$$\sum_{1}^{k} u_i + 1 = \sum_{1}^{k+m} \beta_i v_i \geq v_{k+2} + u_k + u_{k-2} + \cdots + u_{1,2} =$$

$$= v_{k+2} + u_{k+1} - 1 = v_{k+2} + \sum_{1}^{k-1} u_i + 1 ,$$

or $u_k \geq v_{k+2}$. From Corollary 1, we infer (note $v_k = u_k$) that

(14) $$v_{k+2} < u_k .$$

Now, consider the particular integer, $N = v_{k+2} + v_k + v_{k-2} + \cdots + v_{1,2}$, which is in this admissible form of (10) - (11). We have

(15) $$v_{k+2} + v_k + v_{k-2} + \cdots + v_{1,2} < u_k + (u_k + u_{k-2} + \cdots + u_{1,2}) =$$

$$= u_k + u_{k+1} - 1 = u_{k+2} - 1 ,$$

or

$$N = v_{k+2} + v_k + v_{k-2} + \cdots + v_{1,2} \leq u_{k+2} - 2 = \sum_{1}^{k} v_i .$$

Thus N also has a representation in admissible form involving at most the first k members of the sequence $\{v_i\}$, and uniqueness is contradicted.

The inequality $v_{k+1} > u_{k+1}$, is therefore untenable and we have shown $v_{k+1} = u_{k+1}$. The theorem then follows immediately by induction.

Thus, the dual unique representation property (Property D) is a property enjoyed only by the Fibonacci numbers and is therefore sufficient to characterize the sequence $\{u_i\}$.

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SOME RESULTS CONCERNING POLYOMINOES

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INTRODUCTORY REMARKS

An n-omino is a plane figure composed of n connected unit squares joined edge on edge. In the early nineteen hundreds, Henry Dudeney, the famous British puzzle expert, and the Fairy Chess Review popularized problems involving n-ominoes which they represented as figures cut from checkerboards. Solomon Golomb seems to have been the first mathematician to treat the subject seriously when as a graduate student at Harvard in 1954, he published "Checkerboards and Polyominoes" in the American Mathematical Monthly. Since 1954, several articles have appeared (see References); in particular, R. C. Read [9] and Murray Eden [2] have discussed the problem of finding or estimating the number p(n) of n-ominoes for a given n. From their results it is now known that for large n

$$c_1^n < p(n) < c_2^n$$

where c_1 and c_2 are certain positive constants greater than 1. In the first part of this paper we enumerate a subset of n-ominoes and provide an improved lower bound for p(n); later we discuss other problems of this sort and conclude with a brief exposition of problems dealing with configurations of n-ominoes.

MOSER'S BOARD PILE PROBLEM

In the following it will be convenient to have certain conventions. We say the region between $y = n-1$ and $y = n$ is the n^{th} row and call a rectangle of width one a strip. The first square on the left in a strip located in a row is called the initial square of the strip; an n-omino is located in the plane when some square in the n-omino exactly covers a square in the plane lattice. The set of all incongruent n-ominoes will be denoted by P(n) and for convenience we think of the elements of P(n) located in arbitrary regions of the plane. Ignoring changes in position due to translations, each element of P(n) has eight or less positions with respect to 90° rotations about the origin and reflections along the x or y axes; taking two n-ominoes to be distinct if one cannot be translated to cover the other, we find a new set $S(n)$ from $P(n)$ by including rotations and reflections of n-ominoes in $P(n)$ in $S(n)$.

The problem which is now to be discussed was probably first posed by Leo Moser in private correspondence with the present author; later he posed it in a different form at the 1963 Number Theory Conference held at the University of Colorado. Eden [2] also discusses the problem, but his results are not as complete as those given here. The problem is to enumerate a subset $B(n)$ of $S(n)$ which contains n-ominoes having the property that they can be translated in such a way that they are entirely in the first and second quadrants with exactly one strip in the first row with its initial square at the origin and each row after the first has no more than one strip in it. Such n-ominoes may be visualized as side elevations of board piles consisting of boards of various lengths which generally have not been stacked carefully, see Figure 1.

Moser noted that if $b(n)$ denotes the number of elements in $B(n)$, then

$$(1) \qquad b(n) = \sum (a_1 + a_2 - 1)(a_2 + a_3 - 1) \ldots (a_{i-1} + a_i - 1)$$

where the summation extends over all compositions $a_1 + a_2 \ldots + a_i = n$ of n. The relation in (1) can be established by the following combinatorial argument. For each composition $a_1 + a_2 + \ldots + a_i$ of n there is a subset of $B(n)$ consisting of n-ominoes which have a strip of a_t squares in the t^{th} row $(t = 1, 2, \ldots, i)$; the number of n-ominoes in each of these subsets is 1 if $i = 1$ which corresponds to the value of the empty product in the sum (in this there is a strip n units long in the first row) and $(a_1 + a_2 - 1)(a_2 + a_3 - 1) \ldots (a_{i-1} + a_i - 1)$ if $i \geq 2$. This follows since there are exactly $(a_{t-1} + a_t - 1)$ ways to join the strip of a_t squares in the t^{th} row to the strip of a_{t-1} squares in the row below and the total number of ways to connect up the strips to form an n-omino would be the product of all of these alternatives. The subsets corresponding to the compositions of n are exhaustive and disjoint in $B(n)$, so that $b(n)$ is the sum of the number of elements in each subset, which is (1).

The relation for $b(n)$ given by (1) does not furnish a very handy device for computing $b(n)$, but as Eden has shown it is helpful in estimating $b(n)$. Rather than attempt to sum (1) by purely algebraic manipulations, we retain the geometric interpretation of the problem so that combinatorial arguments can be more easily applied toward finding a recursion relation for $b(n)$.

To find a recursion relation for $b(n)$ we define subsets $B_r(n)$ $(r = 1, 2, \ldots, n)$ of $B(n)$ which contain n-ominoes with a strip of exactly r squares in the first row and let $b_r(n)$ denote the number of elements in $B_r(n)$. It is obvious that the subsets $B_r(n)$ $(r = 1, 2, \ldots, n)$ are exhaustive and disjoint in $B(n)$ so that we have immediately

$$b(n) = \sum_{r=1}^{n} b_r(n) \quad . \tag{2}$$

By definition of $B_n(n)$, $b_n(n) = 1$. Consider the elements of $B_r(n)$ with $r < n$; each element of $B_r(n)$ consists of a strip of r squares in the first row with some element of $B(n-r)$ located in the rows above the first. The situation can be appraised more concisely when one considers the number of ways an element of the subset $B_i(n-r)$ of $B(n-r)$ can be attached to the strip of r squares in the first row so that the n-ominoes formed will be an element of $B_r(n)$. Clearly this can be done in $r + i - 1$ ways, so that exactly $(r + i - 1)\, b_i(n-r)$ of the elements of $B_r(n)$ have an element of $B_i(n-r)$ connected to the strip of r squares in the first row. Since the subsets $B_i(n-r)$ $(i = 1, 2, \ldots, n-r)$ of $B(n-r)$ are exhaustive, disjoint subsets, it follows that

$$b_r(n) = \sum_{i=1}^{n-r} (r + i - 1)\, b_i(n-r) \quad \text{for } r < n \quad . \tag{3}$$

It will be seen presently that the relations in (2) and (3) are enough to find the desired recursion relation for $b(n)$. Before this result can be given, we have to prove a few lemmas.

Lemma 1: If $n > 1$, $b_r(n) - b_{r-1}(n-1) = b(n-r)$.

Proof: Using (3) it is seen that

$$b_r(n) - b_{r-1}(n-1) = \sum_{i=1}^{n-r} (r+i-1)b_i(n-r) - \sum_{i=1}^{n-r} (r+i-2)b_i(n-r)$$

$$= \sum_{i=1}^{n-r} b_i(n-r) \quad ,$$

but according to (2), the last expression is precisely $b(n-r)$, so the proof is finished.

Lemma 2: If $n > 1$, $b(n) = 2\,b(n-1) + b_1(n) - b_1(n-1)$.

Proof: Using relations for $b(n)$ and $b(n-1)$ given by (2), it is seen that

$$(5) \qquad b(n) - b(n-1) = \sum_{i=1}^{n} b_i(n) - \sum_{i=1}^{n-1} b_i(n-1)$$

$$= b_1(n) + \sum_{i=2}^{n-1} \left\{ b_i(n) - b_{i-1}(n-1) \right\} \quad ;$$

but according to Lemma 1, $b(n-i)$ can be substituted for $b_i(n) - b_{i-1}(n-1)$ in the last member of (5) so that making this substitution and transposing $-b(n-1)$ from the first to the last member gives

$$(6) \qquad b(n) = b_1(n) + \sum_{i=1}^{n-1} b(n-i) \quad .$$

Now using relations given by (6) for $b(n)$ and $b(n-1)$ we have

$$(7) \quad b(n) - b(n-1) = b_1(n) + \sum_{i=1}^{n-1} b(n-i) - b_1(n-1) - \sum_{i=1}^{n-2} b(n-1-i)$$

$$= b_1(n) - b_1(n-1) + b(n-1);$$

the desired result is obtained by adding $b(n-1)$ to the first and last members of (7).

Lemma 3: $b_1(n) = 4\, b_1(n-1) - 4\, b_1(n-2) + b_1(n-3) + 2\, b(n-3)$.

Proof: Taking $r = 1$ in (3) gives an expression for $b_1(n)$; namely,

$$b_1(n) = \sum_{i=1}^{n-1} i\, b_i(n-1) \quad . \tag{8}$$

Using relations for $b_1(n)$ and $b_1(n-1)$ given by (8) and substituting $b(n-2-i)$ for $b_{i+1}(n-1) - b_i(n-2)$ and $b(n-1)$ for

$$\sum_{i=1}^{n-1} b_i(n-1)$$

when they occur, it is seen that

$$\begin{aligned} b_1(n) - b_1(n-1) &= \sum_{i=1}^{n-1} i\, b_i(n-1) - \sum_{i=1}^{n-2} i\, b_i(n-2) \\ &= \sum_{i=1}^{n-1} b_i(n-1) + \sum_{i=1}^{n-2} i\, b_{i+1}(n-1) - \sum_{i=1}^{n-2} i\, b_i(n-2) \\ &= b(n-1) + \sum_{i=1}^{n-2} i \left\{ b_{i+1}(n-1) - b_i(n-2) \right\} \\ &= b(n-1) + \sum_{i=1}^{n-2} i\, b(n-2-i) \quad . \end{aligned} \tag{9}$$

Adding $b_1(n-1)$ to each member of the equality and dropping the last term in the sum in the right member of (9) (since $b(0) = 0$) a new relation for $b_1(n)$ is obtained:

$$b_1(n) = b_1(n-1) + b(n-1) + \sum_{i=1}^{n-3} (n-2-i)\, b(i) \quad . \tag{10}$$

This time using expressions for $b_1(n)$ and $b_1(n-1)$ given by (10) and again writing a relation for $b_1(n) - b_1(n-1)$, one obtains after a few algebraic manipulations

$$(11) \qquad b_1(n) = 2b_1(n-1) - b_1(n-2) - 2b(n-2) + \sum_{i=1}^{n-1} b(i) .$$

Repeating the same procedure as before only this time using expressions for $b_1(n)$ and $b_1(n-1)$ given by (11) yields

$$(12) \quad b_1(n) = 3b_1(n-1) - 3b_1(n-2) + b_1(n-3) + b(n-1) - 2b(n-2) + 2b(n-3);$$

but by Lemma 2, $b(n-1) - 2b(n-2) = b_1(n-1) - b_1(n-2)$ so that substituting the latter quantity for the former in (12) gives the desired result.

Theorem 1: $b(1) = 1$, $b(2) = 2$, $b(3) = 6$, $b(4) = 19$, and
$b(n) = 5b(n-1) - 7b(n-2) + 4b(n-3)$ for $n > 4$.

Proof: The values of $b(i)$ $(i = 1, 2, 3, 4)$ can be computed directly from (1) or by taking $b(1) = b_1(1) = 1$ the relations in (2) and (3) can be used together for the same purpose. Lemmas 2 and 3 provide the linear difference equations involving $b_1(n)$ and $b(n)$ which can be used to find

$$(13) \qquad b(n) = 5b(n-1) - 7b(n-2) + 4b(n-3) ,$$

$$(14) \qquad b_1(n) = 6b_1(n-1) - 12b_1(n-2) + 11b(n-3) - 4b(n-4),$$

which completes the proof.

The auxiliary equation for (13) has one real root greater than 3.2 so that for n sufficiently large

$$(15) \qquad b(n) > (3.2)^n .$$

We conclude from earlier remarks that $B(n)$ contains at least $b(n)/8$ incongruent n-ominoes, so that we can also replace $b(n)$ in (15) with $p(n)$.

Having disposed of the more difficult problem first, we now turn attention to solving an easier and related problem which was posed and solved by Moser.

Let $C(n)$ be the subset of $B(n)$ which contains all n-ominoes having the property that the initial square of the strip in the k^{th} row is no further to the left than the initial square of the strip in the $(k-1)^{st}$ row. Recall from the definition of $B(n)$ that the initial square of the strip in the first row is always located at the origin. Using a combinatorial argument similar to the one provided for the proof of (1), it is easy to prove

$$c(n) = \sum_{a_1+a_2+\ldots+a_i=n} a_1 a_2 \ldots a_{i-1} , \tag{16}$$

where $c(n)$ denotes the number of elements in $C(n)$. Applying the methods he gave in [8] , Moser was able to show from (16):

Theorem 2: $c(n)$ is equal to the $(2n-1)^{st}$ Fibonacci number.

We will give an alternate proof using the same idea used in the proof of Theorem 1. Let $C_i(n)$ be the subset of $C(n)$ which contains all n-ominoes having strips of exactly i squares in the first row. Clearly the subsets $C_i(n)$ $(i = 1, 2, \ldots, n)$ are exhaustive and disjoint in $C(n)$ so that letting $c_i(n)$ denote the number of elements in $C_i(n)$ we have

$$c(n) = \sum_{i=1}^{n} c_i(n) . \tag{17}$$

Next, it is easy to see that $c_n(n) = 1$, and for $i < n$, $c_i(n) = i\, c(n-i)$ since each element of $C(n-i)$ can be joined exactly i ways to the strip of i squares in the first row so as to form an element of $C_i(n)$; the n-ominoes thus formed obviously comprise all the elements of $C_i(n)$. Substituting the expressions just found for $c_i(n)$ into (17) we obtain

$$c(n) = 1 + \sum_{i=1}^{n-1} i\, c(n-i) . \tag{18}$$

Using expressions for $c(n)$ and $c(n-1)$ given by (18) we can combine the sums in $c(n) - c(n-1)$ to find

$$c(n) - c(n-1) = \sum_{i=1}^{n-1} c(i) ,$$

or

$$c(n) = c(n-1) + \sum_{i=1}^{n-1} c(i) .$$

Now using expressions for $c(n)$ and $c(n-1)$ given by (19) we can combine the sums in $c(n) - c(n-1)$ and deduce

$$c(n) = 3\,c(n-1) - c(n-2). \tag{20}$$

It is easy to prove that the Fibonacci numbers with odd indices satisfy the recurrence relation in (20). Also, using (16) we find $c(1) = f_1$ and $c(2) = f_3$ (f_i denotes the i^{th} Fibonacci number as usual) so that the sequences $\{c_i\}$ and $\{f_{2i-1}\}$ must be identical. Editorial Note: See H-50 Dec. 1964 and note notational differences.

N-OMINOES ENCLOSED IN RECTANGLES

R. C. Read [9] has treated the problem of enumerating the n-ominoes which "fit" into a $p \times q$ rectangle. An n-omino is said to <u>fit</u> in a $p \times q$ rectangle if it is the smallest rectangle in which the n-omino can be drawn with the sides of its squares parallel to the sides of the rectangle. Read's methods give exact counts of the n-ominoes in the sets considered; however, it is possible to obtain lower bounds for these numbers with less effort using similar ideas. To illustrate we will consider the problem of estimating from below the number $s_2(n)$ of n-ominoes which fit in a $2 \times k$ rectangle; we call this set of n-ominoes $S_2(n)$. Two elements are distinct if they are incongruent, so $S_2(n)$ is a subset of $P(n)$.

First, we observe that each element of $S_2(n)$ can be located entirely in the first quadrant in rows 1 and 2 with a square located at the origin. If each element of $S_2(n)$ is situated in the way just described in every way possible, a new set $U(n)$ is obtained where two elements are distinct if one does not exactly cover the other. Clearly, $u(n)$, the number of elements in $U(n)$, is less than or equal to $4s_2(n)$. Now $U(n)$ can be divided into two sets $U''(n)$ and $U'(n)$ consisting respectively of n-ominoes having and not having a square in the second row attached to the square at the origin. Let the number of elements in $U'(n)$ and $U''(n)$ be $u'(n)$ and $u''(n)$ respectively. Now it is easy to see that

$$u'(n) = u'(n-1) + u''(n-1) \tag{21}$$

since every element of $U''(n-1)$ and $U'(n-1)$ can be translated a unit to the right of the origin and a square located at the origin to give an element of $U'(n)$ and every element is obviously obtained in this fashion. It is also easy to prove

$$u''(n) = 2u'(n-2) + u''(n-2) \tag{22}$$

since every element of $U''(n-2)$ and every element of $U'(n-2)$ and its horizontal reflection can be translated a unit to the right of the origin and two squares added (one at the origin, the other attached above it) to form every element of $U''(n)$.

Using (21) and (22) we can find

$$u'(n) = u'(n-1) + u'(n-2) + u'(n-3) \tag{23}$$

and

$$u''(n) = u''(n-1) + u''(n-2) + u''(n-3), \tag{24}$$

so that it becomes evident from $u(n) = u'(n) + u''(n)$ that

$$u(n) = u(n-1) + u(n-2) + u(n-3) \quad . \tag{25}$$

Since $u(n)/4 \leq s_2(n)$, (25) provides a relation for estimating $s_2(n)$. The same procedure can be used for estimating the number of elements in $S_k(n)$ consisting of n-ominoes which fit in $k \times q$ rectangles.

N-OMINO CONFIGURATIONS

Problems involving n-omino configurations have enjoyed a great popularity among mathematical recreationists [4], [6]. We plan to devote a small amount of space to giving an exposition of problems which may be of interest to the mathematician. Generally these problems have the following form: given a region of area A and a set of n-ominoes having a combined area also A; can one cover the region with the set?

We say a set exactly covers a region when there is no overlap and no part of the region is left uncovered. It would be interesting to know necessary conditions that an n-omino be such that an unlimited number of copies could be used to exactly cover the plane. A related problem is to determine necessary conditions that some number of copies of a given n-omino could be used to exactly cover a rectangle. Thus, some easily proved necessary conditions are given by:

(i) if an n-omino has two lines of symmetry and a set of these n-ominoes exactly covers a rectangle, then the n-omino is itself a rectangle.

(ii) if an n-omino fits in a $p \times q$ rectangle and covers diagonally opposite corners of the rectangle, and a set of these n-ominoes can be used to exactly cover a rectangle, then the n-omino is itself a rectangle.

A rectangle exactly covered with a set of congruent n-ominoes is minimal when no rectangle of smaller area can be exactly covered with a set of the same n-ominoes containing fewer elements. It is easy to prove that there is an unlimited number of minimal rectangles involving either two or four n-ominoes. Figures 2, 3, 4 and 5 show instances of minimal rectangles involving more than four n-ominoes. Are there infinitely many cases of minimal rectangles which involve more than four n-ominoes (no two cases involving similar n-ominoes)? Are there minimal rectangles involving an odd number of n-ominoes which are not themselves rectangles?

Note that the configurations depicted in Figures 1, 2, 3 and 4 are symmetric with respect to the centers of the rectangles. Can this always be done in minimal rectangles?

GENERALIZATIONS OF N-OMINOES

In [5], Golomb suggests that one could try to determine or estimate the number of distinct ways n equilateral triangles or n regular hexagons could be simply connected edge on edge. Using 1, 2, 3, 4, 5 or 6 hexagons 1, 1, 3, 7, 22 or 83 combinations respectively result; so far no upper or lower bounds for the terms of this sequence have been given.

There is no reason why regular k-gons could not be used for cells in such combinatorial problems; overlapping of cells could be permitted so long as no cell exactly covered another. Thus, where at most four squares or three hexagons might have a vertex in common, at most ten pentagons might have a vertex in common. The number of distinct ways to join two regular k-gons is one; the number of ways to join three regular k-gons is the greatest integer in $k/2$. Perhaps it would not be difficult to determine in how many ways four or five regular k-gons could be joined together edge on edge so that distinct simply connected figures are formed.

Still another generalization of n-ominoes which seems not to have been considered is joining squares together edge on edge in three or more dimensions. The number of ways of joining k cubes face on face in three dimensions (including mirror images of some pieces) is 1, 1, 2, 8, 29, and 166 for $k = 1, 2, 3, 4, 5$, and 6 respectively; no bounds have been given for the terms of this sequence nor has much been done in a serious vein connected with the packing of space with these three dimensional analogues of polyominoes.

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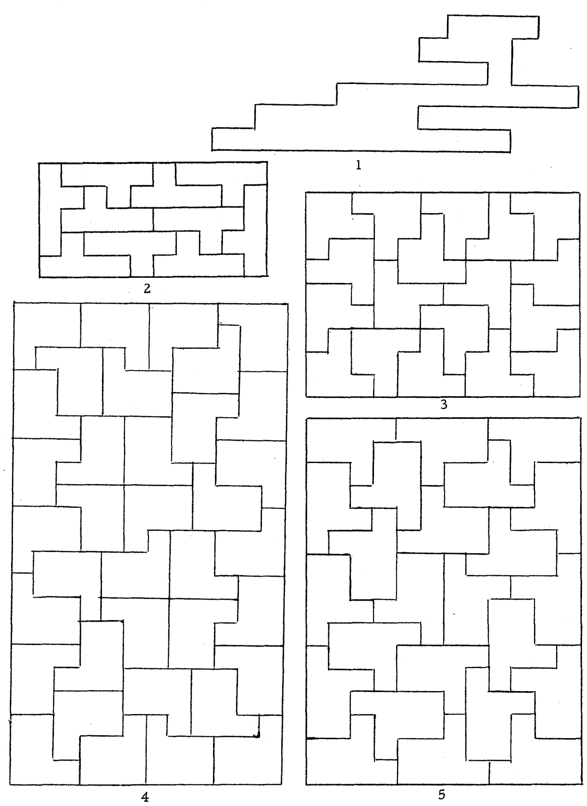

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ON THE REPRESENTATION OF INTEGERS AS SUMS OF DISTINCT FIBONACCI NUMBERS

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This paper gives an elementary discussion of the problem of expressing an arbitrary positive integer as a sum of distinct Fibonacci numbers. The recursive relation

$$F_{n+2} = F_{n+1} + F_n \tag{1}$$

together with $F_1 = F_2 = 1$ is used as the definition of Fibonacci numbers. No use will be made of F_1 in any representation.

As an example consider the integer 29. It may be expressed as a sum of Fibonacci numbers in the following ways:

$$\begin{aligned} 29 &= F_8+F_6 = F_8+F_5+F_4 = F_8+F_5+F_3+F_2 \\ &= F_7+F_6+F_5+F_4 = F_7+F_6+F_5+F_3+F_2 \end{aligned}$$

From this example it immediately becomes apparent that we shall need to impose some "ground rules" if we are to differentiate between the various types of representations. This leads us to the following definitions.

A representation will be called minimal if it contains no two consecutive Fibonacci numbers.

A representation is said to be maximal if no two consecutive Fibonacci numbers F_i and F_{i+1} are omitted from the representation, where $F_2 \leq F_i < F_{i+1} \leq F_n$ and F_n is the largest Fibonacci number involved in the representation.

Thus $F_8 + F_6$ is a minimal representation of the integer 29 while $F_7+F_6+F_5+F_3+F_2$ is a maximal representation.

It follows that a maximal (minimal) representation may be transformed into a minimal (maximal) one by the application or repeated application of (1).

We shall first restrict our attention to minimal representations.

As an illustrative example of minimal representations we consider the representations of all integers N such that

$$F_7 \leq N < F_8$$

Thus N may be any one of the eight integers 13, 14, 15, 16, 17,18, 19, or 20. Now the smallest integer in this set, namely 13, cannot be represented by the Fibonacci numbers F_2, F_3, F_4, F_5 and F_6, since the largest integer under the minimal representation rule which they can represent is

$$F_6+F_4+F_2 = 12$$

Hence to represent all integers of this set requires F_2, F_3, F_4, F_5, F_6 and F_7.

By trial we obtain the following representations

$$13 = F_7;\ 14 = F_7+F_2;\ 15 = F_7+F_3;\ 16 = F_7+F_4;$$

$$17 = F_7+F_4+F_2;\ 18 = F_7+F_5;\ 19 = F_7+F_5+F_2;$$

$$20 = F_7+F_5+F_3 \quad .$$

One of these integers, namely 13, requires only one Fibonacci number to represent it. Four of them, namely, 14, 15, 16 and 18 require two and three of them 17, 19, and 20 require three.

Let $U_{n,m}$ denote the number of integers N in the range $F_n \leq N < F_{n+1}$ which require m Fibonacci numbers to represent them.

Then

$$U_{7,1} = 1;\ U_{7,2} = 4;\ U_{7,3} = 3 \quad .$$

It is also evident that

$$U_{7,1} + U_{7,2} + U_{7,3} = F_8 - F_7 = F_6 = 8 \quad .$$

Now it is known (1) that

$$U_{n,m} = 0, \text{ if } m > \left[\tfrac{n}{2}\right]$$

Thus we may write

$$\sum_{i=1}^{n} U_{n,i} = F_{n+1} - F_n = F_{n-1} \quad .$$

Table I gives values of $U_{n,m}$ for $n = 1, 2, 3, \ldots, 8$; $m = 1, 2, 3, \ldots, 4$.

We now discuss some properties of the function $U_{n,m}$.

Consider integers P, Q and R, defined by the following relations

$$F_n \leq P < F_{n+1};\ F_{n-1} \leq Q < F_n;\ F_{n-2} \leq R < F_{n-1}$$

Thus

$$P = F_n + p, \quad p = 0, 1, 2, \ldots, F_{n-1} - 1 \tag{2}$$

$$Q = F_{n-1} + q, \quad q = 0, 1, 2, \ldots, F_{n-2} - 1 \tag{3}$$

$$R = F_{n-2} + r, \quad r = 0, 1, 2, \ldots, F_{n-3} - 1 \tag{4}$$

We arrange the integers P in two sets (A) and (B) as follows.

$$P = F_n + p_1, \quad p_1 = 0, 1, 2, \ldots, F_{n-2} - 1 \tag{A}$$

$$\text{(B)}\quad P = F_n + p_2,\ p_2 = F_{n-2}, F_{n-2}+1, F_{n-2}+2, \ldots, F_{n-2} + (F_{n-1} - F_{n-2} - 1)$$

$$= F_{n-2} + r,\ r = 0, 1, 2, \ldots, F_{n-3} - 1$$

If k is a positive integer,(1) implies that

$$F_n + k = F_{n-1} + k + F_{n-2}$$

Hence for the set (A)

$$F_n + p_1 = F_{n-1} + p_1 + F_{n-2}$$

$$P = F_{n-1} + q + F_{n-2}$$

$$P = F_n + q \ .$$

Comparing the last equation with (2) and (3) we see that the representation of an integer Q may be converted into a representation of an integer P of the set (A) by replacing F_{n-1} in the former by F_n.

By this operation we may derive the representations of F_{n-2} of the integers P from the representations of the integers Q. Derivation of the representations of P in this manner leaves the number of Fibonacci numbers unchanged.

We have now to consider the integers P in the set (B).

We have

$$P = F_n + p_2, \; p_2 = F_{n-2}, \; F_{n-2}+1, \; F_{n-2}+2, \ldots, \; F_{n-2}+(F_{n-1}-F_{n-2}-1)$$

$$= F_n + F_{n-2} + r, \; r = 0, 1, 2, \ldots, \; F_{n-3}-1$$

$$= F_n + R \text{ by (4)}$$

The last result implies that the representations of the integers P in the set (B) may be derived from the representations of the integers R by adding F_n to each of the latter. This operation increases by one the number of the F_i in the representation of P over that of R from which it is derived.

By these two operations the representations of the F_{n-1} integers in $F_n \leq P < F_{n+1}$ can be derived from the representations of the F_{n-2} integers in $F_{n-1} \leq Q < F_n$ and the F_{n-3} integers in $F_{n-2} \leq Q < F_{n-1}$.

The following equations follow from the above discussion:

$$u_{n,m} = u_{n-1,m} + u_{n-2,m-1} \qquad (n > 2, m > 1)$$

$$u_{n,1} = 1 \tag{5}$$

$$u_{n,m} = 0 \text{ for } 2m > n.$$

These equations indicate that the $u_{n,m}$ may be related to the binomial coefficients $\binom{r}{k}$, which have the following properties:

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$$

$$\binom{r}{0} = 1$$

$$\binom{r}{k} = 0 \text{ for } k > r.$$

Letting $U_{n,m} = \binom{n-m-1}{m-1}$, these relations for the $\binom{r}{k}$ become the relations (5) with the $U_{n,m}$ substituted for the $u_{n,m}$. Since (5) makes it possible to calculate any $u_{n,m}$ with $n > 2$ and $m > 1$, these relations characterize the $u_{n,m}$ and so $u_{n,m} = U_{n,m} = \binom{n-m-1}{m-1}$ for $n > 2$ and $m > 1$.

The reader is referred to the paper "A Combinational Problem" by Lafer and Long in the November 1962 issue of the American Mathematical Monthly for an expository account of the inductive and deductive aspects of a similar problem. [3]

The proof of this is left to the reader.

We turn now to a discussion of maximal representation of integers as sums of Fibonacci numbers. In this discussion we shall use a different technique, one that could have been used equally well in the discussion of minimal representations.

As an example we consider the integers N such that $F_7 - 1 \leq N < F_8 - 1$. These are, 13, 14, 15, 16, 17, 18, 19 and 20. The reason for using the range $F_7 - 1 \leq N < F_{8-1}$ instead of $F_7 \leq N < F_8$ will become evident later.

Bearing in mind the definition of maximal representation we derive the following representations

$$12 = F_6+F_4+F_2;\ 13 = F_6+F_4+F_3;\ 14 = F_6+F_4+F_3+F_2;$$

$$15 = F_6+F_5+F_3;\ 16 = F_6+F_5+F_3+F_2;\ 17 = F_6+F_5+F_4+F_2;$$

$$18 = F_6+F_5+F_4+F_3;\ 19 = F_6+F_5+F_4+F_3+F_2 \quad .$$

These eight representations may be written compactly in the following form.

$$\begin{array}{ccccccc} & & F_6 & F_5 & F_4 & F_3 & F_2 \\ 12 & = & (\ 1 & 0 & 1 & 0 & 1\ ) \\ 13 & = & (\ 1 & 0 & 1 & 1 & 0\ ) \\ 14 & = & (\ 1 & 0 & 1 & 1 & 1\ ) \\ 15 & = & (\ 1 & 1 & 0 & 1 & 0\ ) \\ 16 & = & (\ 1 & 1 & 0 & 1 & 1\ ) \\ 17 & = & (\ 1 & 1 & 1 & 0 & 1\ ) \\ 18 & = & (\ 1 & 1 & 1 & 1 & 0\ ) \\ 19 & = & (\ 1 & 1 & 1 & 1 & 1\ ) \end{array}$$

In this display we have used the binary digits in conjunction with Fibonacci numbers denoting place values. It should be noticed that with this scheme two zeros cannot be in adjacent places in maximal representation. For example (...100...) must be replaced by (...011...) since $F_i = F_{i-1} + F_{i-2}$. Also the Fibonacci numbers denoting the positional values are arranged in ascending order from the right to left beginning with F_2.

We now consider the general case. Let N be an integer defined by

$$F_n - 1 \leq N < F_{n+1} - 1$$

Let $V_{n,m}$ denote the number of integers N in this interval which require m Fibonacci numbers to represent them in maximal representation.

Thus for the illustrative example given above

$$V_{7,3} = 3; \; V_{7,4} = 4; \; V_{7,5} = 1$$

Also

$$V_{7,3} + F_{7,4} + V_{7,5} = F_8 - F_7 = F_6 = 8$$

The largest integer in the interval $F_n - 1 \leq N < F_{n+1} - 1$ is $F_{n+1} - 2$ and since (2)

$$\sum_{i=2}^{n-1} F_i = F_{n+2} - 2$$

it follows that

$$F_{n+1} - 2 = (111\ldots11) \quad \text{(n-2 digits)}$$

in which no zeros appear and in which the left hand positional value is F_{n-1}. This explains the reason for taking the upper bound of N to be $F_{n+1} - 1$ instead of F_{n+1}.

The smallest integer in the range in question is $F_n - 1$ and since

$$\sum_{i=2}^{n-2} F_i = F_n - 2 < F_n - 1$$

it follows that there must be a "one" in the first (left hand position) defined by F_{n-1}. Further since (2)

$$F_2 + F_4 + F_6 + \ldots + F_n = F_{n+1} \quad \text{(n even)}$$

$$F_3 + F_5 + F_7 + \ldots + F_n = F_{n+1} \quad \text{(n odd)}$$

it follows that the smallest integer in the range in question is indicated by

$$(1010\ldots10) \quad \text{or} \quad (1010\ldots101)$$

according as n is odd or even.

From these observations we conclude that

$$V_{n,m} = 0 \quad \text{if} \begin{cases} m > n-2 \quad \text{or} \quad n < m+2 \\ m < \left[\frac{n-1}{2}\right] \quad \text{or} \quad n > 2(m+1) \end{cases}$$

$$\sum_{i=\left[\frac{n-1}{2}\right]}^{n-2} V_{n,i} = F_{n+1} - F_n = F_{n-1}$$

Table II gives values of $V_{n,m}$ for $n = 2, 3, \ldots 12; m = 1, 2, \ldots, 10$.

We now establish the recursive relation

(6) $$V_{n,m} = V_{n-1,m-1} + V_{n-2,m-1}$$

Consider integers P, Q and R defined by

$$F_n - 1 \le P < F_{n+1} - 1$$

$$F_{n-1} - 1 \le Q < F_n - 1$$

$$F_{n-2} - 1 \le R < F_{n-1} - 1$$

The Fibonacci positional representation of the integers Q are of the type

$$\begin{array}{cccccc} & F_{n-2} & F_{n-3} & F_{n-4} & \cdots & F_2 \\ Q = (& 1 & a & b & \ldots & c\ ) \end{array}$$

Adding F_{n-1} to each integer Q will produce F_{n-2} integers all of which will be within the interval

$$F_{n-1}+F_{n-1}-1 \leq Q + F_{n-1} < F_{n-1}+F_n-1 \quad .$$

This is equivalent to

$$F_{n+1}-F_n+F_{n-1}-1 \leq Q + F_{n-1} < F_{n+1}-1$$

$$F_{n+1}-F_{n-1}-F_{n-2}+F_{n-1}-1 \leq Q + F_{n-1} < F_{n+1}-1$$

$$F_{n+1}-F_{n-2}-1 \leq Q + F_{n-1} < F_{n+1}-1$$

$$F_n+F_{n-3}-1 \leq Q + F_{n-1} < F_{n+1}-1$$

These F_{n-2} integers $Q + F_{n-1}$ are all in the interval $F_n-1 \leq P < F_{n+1}-1$. Their positional representation takes the form

$$\begin{array}{ccccccc} & F_{n-1} & F_{n-2} & F_{n-3} & F_{n-4} & \cdots & F_2 \\ Q + F_{n-1} = (& 1 & 1 & a & b & \ldots & c\ ) \end{array}$$

Hence the representations of F_{n-2} of the integers P may be derived from the integers Q by creating an additional position defined as F_{n-1}.

The F_{n-3} integers R have positional representations of the form

$$\begin{array}{cccccc} & F_{n-3} & F_{n-4} & F_{n-5} & \cdots & F_2 \\ R = (& 1 & d & e & \ldots & f\ ) \end{array}$$

Adding F_{n-1} to each of these F_{n-3} integers will result in integers all in the interval

$$F_{n-1}+F_{n-2}-1 \leq R + F_{n-1} < F_{n-1}+F_{n-1}-1$$

$$F_n-1 \leq R + F_{n-1} < F_{n+1}-F_{n-2}-1 \quad .$$

That is, these F_{n-3} integers are all within the interval $F_n - 1 \leq P < F_{n+1} - 1$. Each of them will have representations of the form

$$\begin{array}{lcccccccc} & F_{n-1} & F_{n-2} & F_{n-3} & F_{n-4} & F_{n-5} & \cdots & F_2 & \\ R + F_{n-1} = (& 1 & 0 & 1 & d & e & \ldots & f &) \end{array}$$

Hence the representations of F_{n-3} of the integers P may be obtained from the representations of R by adding on the left two positional values namely F_{n-1} and F_{n-2}.

Since the first operation results in a representation which has a "one" in the second (from the left) place while the second operation gives a representation with a zero in that place the two representations are disjoint. Thus there is no overlapping and all integers P are accounted for by these two operations.

This completes the proof of (6).

It is readily verified that

$$V_{n,m} = \binom{m}{n-m-2} \tag{7}$$

satisfies the recursive relation (6).

From (7) we find that

$$\sum_{i=m+2}^{2(m+1)} V_{i,m} = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \ldots + \binom{m}{m}$$

Also from the paragraph following (5) and (7) we see that

$$V_{n,m} = U_{n,n-m-1} \quad .$$

REFERENCES

1. P. Lafer, "Exploring the Fibonacci Representation of Integers," The Fibonacci Quarterly, April 1964, p. 114.

2. Cf. N. N. Vorob'ev, Fibonacci Numbers, New York, 1961, pp. 6, 7.

3. P. Lafer and C. T. Long, "A Combinatorial Problem," The American Mathematical Monthly, Nov. 1962, pp. 876-883.

Table I

Values of $U_{n,m}$ $n = 1, 2, \ldots, 8$ $m = 1, 2, \ldots, 4$

	$n \backslash m$	1	2	3	4
$F_1 \le N < F_2$	1	0	0	0	0
$F_2 \le N < F_3$	2	1	0	0	0
$F_3 \le N < F_4$	3	1	0	0	0
$F_4 \le N < F_5$	4	1	1	0	0
$F_5 \le N < F_6$	5	1	2	0	0
$F_6 \le N < F_7$	6	1	3	1	0
$F_7 \le N < F_8$	7	1	4	3	0
$F_8 \le N < F_9$	8	1	5	6	1

Table II

Values of $V_{n,m}$ $n = 2, 3, 4, \ldots, 12;$ $m = 1, 2, 3, \ldots, 10$

	$n \backslash m$	1	2	3	4	5	6	7	8	9	10
$F_2 - 1 \le N < F_3 - 1$	2	0	0	0	0	0	0	0	0	0	0
$F_3 - 1 \le N < F_4 - 1$	3	1	0	0	0	0	0	0	0	0	0
$F_4 - 1 \le N < F_5 - 1$	4	1	1	0	0	0	0	0	0	0	0
$F_5 - 1 \le N < F_6 - 1$	5	0	2	1	0	0	0	0	0	0	0
$F_6 - 1 \le N < F_7 - 1$	6	0	1	3	1	0	0	0	0	0	0
$F_7 - 1 \le N < F_8 - 1$	7	0	0	3	4	1	0	0	0	0	0
$F_8 - 1 \le N < F_9 - 1$	8	0	0	1	6	5	1	0	0	0	0
$F_9 - 1 \le N < F_{10} - 1$	9	0	0	0	4	10	6	1	0	0	0
$F_{10} - 1 \le N < F_{11} - 1$	10	0	0	0	1	10	15	7	1	0	0
$F_{11} - 1 \le N < F_{12} - 1$	11	0	0	0	0	5	20	21	8	1	0
$F_{12} - 1 \le N < F_{13} - 1$	12	0	0	0	0	1	15	35	28	9	1

N. B. The entries in the vertical columns are rows of PASCAL's arithmetic triangle so that the table may be easily extended.

XXXXXXXXXXXXXX

ON A GENERAL FIBONACCI IDENTITY

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1. The Fibonacci sequence is defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n \quad , \tag{1}$$

together with the particular values

$$F_0 = 0, \; F_1 = 1 \quad . \tag{2}$$

It is easily verified that the unique solution* of (1) and (2) is given by

$$F_n = (\alpha^n - \beta^n)/(\alpha - \beta) \quad , \tag{3}$$

where α and β are the roots of the equation

$$x^2 = x + 1 \quad , \tag{4}$$

namely

$$\alpha = \frac{1}{2}(1 + \sqrt{5}), \; \beta = \frac{1}{2}(1 - \sqrt{5}) = -\alpha^{-1} \quad . \tag{5}$$

The sequence is thus defined for all integers n, positive or negative or zero. From (1) and (2), we infer that (3) takes integer values for all n, and we observe, by (3) and (5), that

$$F_{-n} = (-1)^{n+1} F_n \quad . \tag{6}$$

This sequence and its generalizations have been the subject of a vast literature, and a very large number of identities of different kinds, involving the Fibonacci numbers, can be demonstrated. It is the purpose of this paper to show how a considerable body of these may be obtained as particular cases of a single identity.

*Direct substitution shows that (3) is a solution of (1) and (2). If F'_n were another solution, $f_n = F_n - F'_n$ would satisfy a relation (1), with $f_0 = f_1 = 0$. Induction on n now shows that $f_n = 0$ for all n, so that (3) is the unique solution, as stated.

2. We begin by defining the function

$$S_0(n) = F_n + F_{n+1} - F_{n+2} \quad . \tag{7}$$

Then, immediately, by (1), we see that, for all integers n,

$$S_0(n) = 0 \quad . \tag{8}$$

Now consider the function

$$S_1(m, n) = F_m F_n + F_{m+1} F_{n+1} - F_{m+n+1} \quad . \tag{9}$$

Then, by (1), for any m and n,

$$S_1(m+1, n) = S_1(m, n) + S_1(m-1, n) \quad . \tag{10}$$

Also, by (1), (2), (7), and (8), we have that

(11) and
$$\begin{cases} S_1(0, n) = F_{n+1} - F_{n+1} = 0 \\ S_1(1, n) = S_0(n) = 0; \end{cases}$$

whence (10) yields, by upward and downward induction on m, that, for all integers m and n,*

$$S_1(m, n) = 0 \quad . \tag{12}$$

Next consider the function

$$S_2(r, m, n) = F_m F_n - (-1)^r (F_{m+r} F_{n+r} - F_r F_{m+n+r}) \quad . \tag{13}$$

Again applying (1), we see that

$$S_2(r+1, m, n) = S_2(r-1, m, n+2) - S_2(r, m, n+1) \quad . \tag{14}$$

Now, for any m or n, by (2), (9), and (12),

(15) and
$$\begin{cases} S_2(0, m, n) = F_m F_n - F_m F_n = 0 \\ S_2(1, m, n) = S_1(m, n) = 0 \quad . \end{cases}$$

*We may also note that, for any fixed n, (10) is a relation of the form (1). Thus, as in the previous footnote, we get (12), for all m and n.

Thus, by upward and downward induction on r in (14),* we find that, for all integers r, m, and n,

$$S_2(r,\ m,\ n) = 0 \quad . \tag{16}$$

Finally consider the function (with $k \geq 0$)

$$S_3(k, r, m, n) = F_m^k F_n - (-1)^{kr} \sum_{h=0}^{k} \binom{k}{n} (-1)^h F_r^h F_{r+m}^{k-h} F_{n+kr+hm} \quad . \tag{17}$$

It is well known that

$$\binom{k+1}{h} = \binom{k}{h} + \binom{k}{h-1} \tag{18}$$

and

$$\binom{k}{h} = 0 \text{ when } h < 0 \text{ or } 0 \leq k < h \quad . \tag{19}$$

Thus we can show, by (13), (16), (18), and (19), that

$$S_3(k+1,\ r,\ m,\ n) = F_m S_3(k,\ r,\ m,\ n) \quad . \tag{20}$$

Also, by (13), (16), (17), and (19),

$$(21) \quad \left\{ \begin{array}{l} S_3(0,\ r,\ m,\ n) = F_n - F_n = 0 \\ \\ S_3(1,\ r,\ m,\ n) = S_2(r,\ m,\ n) = 0 \quad . \end{array} \right.$$

and

Thus, by upward induction on k in (20), we get that, for all integers r, m, and n, and all integers $k \geq 0$,

$$S_3(k,\ r,\ m,\ n) = 0 \quad . \tag{22}$$

*We observe that, while the inductive argument leading to (12) assumes an arbitrarily chosen and fixed n; the corresponding argument yielding (16) assumes, at each step, that (16) holds for a consecutive pair of values of r, an arbitrary fixed value of m, and <u>all</u> values of n.

This is the general identity promised above:

$$\text{(23)} \qquad \boxed{F_m^k F_n = (-1)^{k r} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_r^h F_{r+m}^{k-h} F_{n+kr+hm}} .$$

3. We may now look at some of the identities which are obtained as particular cases of (23). On the left of each identity below, the vector (k, r , m, n) is shown. In some cases, the identity (6) is used to remove negative subscripts.

$$\text{(24)} \quad (k, r, -m, -n):\ F_m^k F_n = \sum_{h=0}^{k} \binom{k}{h} (-1)^{(m-1)(k-h)} F_r^h F_{r-m}^{k-h} F_{n-kr+hm}$$

$$\text{(25)} \quad (k, r, -m, -kr-n):\ F_m^k F_{n+kr} = \sum_{h=0}^{k} \binom{k}{h} (-1)^{(m-1)(k-h)} F_r^h F_{r-m}^{k-h} F_{n+hm}$$

$$\text{(26)} \quad (k, r, m, -kr):\ F_m^k F_{kr} = \sum_{h=1}^{k} \binom{k}{h} (-1)^{h-1} F_r^h F_{r+m}^{k-h} F_{hm}$$

$$\text{(27)} \quad (k, r, m, m):\ F_m^{k+1} = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_r^h F_{r+m}^{k-h} F_{kr+(h+1)m}$$

$$\text{(28)} \quad (k, r, m, nr):\ F_m^k F_{nr} = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_r^h F_{r+m}^{k-h} F_{(n+k)r+hm}$$

$$\text{(29)} \quad (k, r, mr, n):\ F_{mr}^k F_n = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_r^h F_{(m+1)r}^{k-h} F_{n+(k+hm)r}$$

$$\text{(30)} \quad (k, r, m, 0):\ \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_r^h F_{r+m}^{k-h} F_{kr+hm} = 0$$

$$\text{(31)} \quad (k, r, m, \pm 1):\ F_m^k = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_r^h F_{r+m}^{k-h} F_{kr+hm\pm 1}$$

(32) $(k, r, mr, 0)$: $\sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{(m+1)r}^{k-h} F_{(k+hm)r} = 0$

(33) $(k, r, \pm 1, n)$: $F_n = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{r\pm 1}^{k-h} F_{n+kr\pm h}$

(34) $(k, r, \pm 1, -kr-n)$: $F_{n+kr} = \sum_{h=0}^{k} \binom{k}{h} F_r^h F_{r\pm 1}^{k-h} F_{n\mp h}$

(35) $(k, r, \pm 1, -kr)$: $F_{kr} = \sum_{h=1}^{k} \binom{k}{h} F_r^h F_{r\pm 1}^{k-h} F_{\mp h}$

(36) $(k, r, \pm 1, nr)$: $F_{nr} = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{r\pm 1}^{k-h} F_{(n+k)r\pm h}$

(37) $(k, r, \pm 2, n)$: $F_n = (\pm 1)^k(-1)^{kr} \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{r\pm 2}^{k-h} F_{n+kr\pm 2h}$

(38) $(k, r, \pm 2, -kr)$: $F_{kr} = (\pm 1)^k \sum_{h=1}^{k} \binom{k}{h}(-1)^{h-1} F_r^h F_{r\pm 2}^{k-h} F_{\pm 2h}$

(39) $(k, \pm 1, m, n)$: $F_m^k F_n = (-1)^k \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_{m\pm 1}^{k-h} F_{n\pm k+hm}$

(40) $(k, \pm 1, m, \mp k)$: $F_m^k F_k = (\pm 1)^k \sum_{h=1}^{k} \binom{k}{h}(-1)^{h-1} F_{m\pm 1}^{k-h} F_{hm}$

(41) $(k, \pm 2, m, n)$: $F_m^k F_n = \sum_{h=0}^{k} \binom{k}{h}(\mp 1)^h F_{m\pm 2}^{k-h} F_{n\pm 2k+hm}$

(42) $(k, r, \pm 1, 0)$: $\sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{r\pm 1}^{k-h} F_{kr\pm h} = 0$

(43) $(k, r, \pm 2, 0)$: $\sum_{h=0}^{k} \binom{k}{h}(-1)^h F_r^h F_{r\pm 2}^{k-h} F_{kr\pm 2h} = 0$

(44) $(k, \pm 1, m, 0)$: $\sum_{h=0}^{k} \binom{k}{h}(-1)^h F_{m\pm 1}^{k-h} F_{hm\pm k} = 0$

(45) $(k, \pm 1, m, \pm' 1)$: $F_m^k = (-1)^k \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_{m\pm 1}^{k-h} F_{hm\pm k\pm' 1}$

(46) $(k, 1, 1, -n)$: $F_n = \sum_{h=0}^{k} \binom{k}{h} F_{n-k-h}$

(47) $(k, 1, 1, -nk)$: $F_{nk} = \sum_{h=0}^{k} \binom{k}{h} F_{(n-1)k-h}$

(48) $(k, 1, 1, -k)$: $F_k = \sum_{h=0}^{k} \binom{k}{h}(-1)^{h-1} F_h$

(49) $(k, 2, -1, -n)$: $F_n = \sum_{h=0}^{k} \binom{k}{h} F_{n-2k+h}$

(50) $(k, -1, 2, -n)$: $F_n = \sum_{h=0}^{k} \binom{k}{h}(-1)^h F_{n+k-2h}$

(51) $(k, 2, -1, -2k)$: $F_{2k} = \sum_{h=0}^{k} \binom{k}{h} F_h$

(52) $(k, -1, 2, k)$: $F_k = \sum_{h=0}^{k} \binom{k}{h}(-1)^{k-h}F_{2h}$

(53) $(k, 1, 1, 0)$: $\sum_{h=0}^{k} \binom{k}{h}(-1)^{h}F_{k+h} = 0$

(54) $(k, 2, -1, 0)$: $\sum_{h=0}^{k} \binom{k}{h}(-1)^{h}F_{2k-h} = 0$

(55) $(k, -1, 2, 0)$: $\sum_{h=0}^{k} \binom{k}{h}(-1)^{h}F_{k-2h} = 0$

(56) $(1, r, m, n)$: $F_mF_n = (-1)^r(F_{m+r}F_{n+r} - F_rF_{m+n+r})$

(57) $(1, r, m, -n)$: $F_mF_n - F_{m+r}F_{n-r} = (-1)^{n-r}F_rF_{m-n+r}$

(58) $(1, r, m, -m)$: $F_m^2 - F_{m+r}F_{m-r} = (-1)^{m-r}F_r^2$

(59) $(1, r, m, n-r)$: $F_rF_{m+n} = F_{r+m}F_n - (-1)^rF_mF_{n-r}$

(60) $(1, r-k, m+k, -m+k)$: $F_{m+k}F_{m-k} - F_{m+r}F_{m-r} = (-1)^{m-r}F_rF_{k+r}$

(61) $(1, \pm 1, m, n)$: $F_{m+n\pm 1} = F_mF_n + F_{m\pm 1}F_{n\pm 1}$

(62) $(1, 1, m, n-1)$: $F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$

(63) $(1, 2, m-1, n-1)$: $F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1}$

(64) $(1, \pm 1, m, m)$: $F_{2m\pm 1} = F_m^2 + F_{m\pm 1}^2$

(65) $(1, 1, m, m-1)$: $F_{2m} = F_m(F_{m+1} + F_{m-1})$

(66) $(1, 2, m-1, m-1)$: $F_{2m} = F_{m+1}^2 - F_{m-1}^2$

(67) $(1, 1, m, -m)$: $F_m^2 - F_{m+1}F_{m-1} = (-1)^{m-1}$

(68) $(1, 2, m, -m)$: $F_m^2 - F_{m+2}F_{m-2} = (-1)^m$

(69) $(1, 1, m+1, -m+1)$: $F_{m+1}F_{m-1} - F_{m+2}F_{m-2} = 2(-1)^m$

This rather long list includes most of the identities, derivable as special cases of (23), which I have found in the literature, and a number of others (including (23) itself, (24)-(32), (37)-(45), and (60)), which I believe to be new and useful.*

4. We may now ask what else can be done with the family of identities (23)-(69). Some of the further developments will be demonstrated below.

Putting $n = m$ in (59) and dividing by F_m, we obtain, by (65) and (6), that

$$(70) \qquad (F_{m+1} + F_{m-1})F_\tau = F_{\tau+m} + (-1)^m F_{\tau-m} \quad .$$

Thus

$$(71) \left[F_{m+1} + F_{m-1} - 1 - (-1)^m\right] F_\tau = (F_{\tau+m} - F_\tau) - (-1)^m(F_\tau - F_{\tau-m}) \quad .$$

The usefulness of this identity is seen when we put $\tau = rm + n$ and sum from $r = 1$ to $r = t$. The right-hand side telescopes to yield

$$(72) \qquad \sum_{r=1}^{t} F_{rm+n} = \frac{F_{(t+1)m+n} - F_{m+n} - (-1)^m(F_{tm+n} - F_n)}{F_{m+1} + F_{m-1} - 1 - (-1)^m}$$

(This result is known [1], but I believe that the line of proof is new.) Certain particular cases have been known for a long time; for instance,

*EDITORIAL NOTE: A different form of the identity (23) appears in an unpublished Master's Thesis entitled "Moduls m properties of the Fibonacci numbers," written by John Vinson at Oregon State University in 1961. (Other parts of that thesis appear as a paper by John Vinson in the Fibonacci Quarterly, 1(1963) No. 2, pp. 37-45.)

$$\sum_{r=1}^{t} F_{r+s} = F_{t+1+2} - F_{1+s} + F_{t+s} - F_s = F_{t+s+2} - F_{s+2} \quad , \tag{73}$$

$$\sum_{r=1}^{t} F_{2(r+s)} = F_{2(t+1+s)} - F_{2(1+s)} - F_{2(t+s)} + F_{2s} = F_{2(t+s)+1} - F_{2s+1} , \tag{74}$$

$$\sum_{r=1}^{t} F_{2(r+s)-1} = F_{2(t+s)} - F_{2s} , \tag{75}$$

and

$$\begin{aligned} \sum_{r=1}^{t} F_{3(r+s)} &= \frac{1}{4}(F_{3(t+1+s)} - F_{3(1+s)} + \\ &+ F_{3(t+s)} - F_{3s}) = \frac{1}{2}(F_{3(t+s)+2} - F_{3s+2}). \end{aligned} \tag{76}$$

If we sum (64) from $m = s + 1$ to $m = s + t$, put $r = m - s$, use (75), and slightly rearrange the result, we obtain that

$$\sum_{r=1}^{t} F_{r+s}^2 = \frac{1}{2}(F_{2(t+s)} + F_{t+s}^2 - F_{2s} - F_s^2) \quad . \tag{77}$$

Now rewrite (65), using (1), in the form

$$F_{2m} \pm F_m^2 = 2F_m F_{m\pm 1} \quad , \tag{78}$$

and sum (78) as before, using (74) and (77); then we get

$$\sum_{r=1}^{t} F_{r+s}F_{r+s+1} = \frac{1}{4}(F_{2(t+s)+3} + F_{t+s}^2 - F_{2s+3} - F_s^2) \quad . \tag{79}$$

If we sum (73) with $t = w - s$, from $s = v$ to $s = w - 1$, we get that

$$\sum_{s=v}^{w-1} \sum_{u=s+1}^{w} F_u = \sum_{u=v+1}^{w} \sum_{s=v}^{u-1} F_u = \sum_{u=v+1}^{w} (u-v)F_u = \sum_{u=v+1}^{w} uF_u + vF_{v+2} -$$

$$- vF_{w+2} = \sum_{s=v}^{w-1} (F_{w+2} - F_{s+2}) = (w - v)F_{w+2} - F_{w+3} + F_{v+3} ,$$

which yields

$$\sum_{u=v+1}^{w} uF_u = wF_{w+2} - F_{w+3} - vF_{v+2} + F_{v+3} . \tag{80}$$

The same process of summation applied to (80) yields

$$\sum_{u=v+1}^{w} u^2F_u = w^2F_{w+2} - (2w-1)F_{w+3} + 2F_{w+4} + (2v-1)F_{v+3} - 2F_{v+4} \tag{81}$$

and we can evidently iterate the procedure to obtain the sum of $u^m F_u$ for any positive integer m.

Again, replace m by $r + m$ in (63) and apply (61) to the result. This gives

$$F_{r+m+n} = (F_{r+1}F_{m+1} + F_r F_m)F_{n+1} - (F_r F_m + F_{r-1}F_{m-1})F_{n-1}$$

or, by (1),

$$F_{r+m+n} = F_{r+1}F_{m+1}F_{n+1} + F_r F_m F_n - F_{r-1}F_{m-1}F_{n-1} . \tag{82}$$

In particular,

$$F_{3m} = F_{m+1}^3 + F_m^3 - F_{m-1}^3 \tag{83}$$

and

$$F_{3m} = F_m F_{m+1} F_{m+2} + F_{m-1}F_m F_{m+1} - F_{m-2}F_{m-1}F_m . \tag{84}$$

We may note, at this point, that (83) can be put in yet another form, with the help of (67):

$$F_{3m} = F_m^3 + (F_{m+1} - F_{m-1})(F_{m+1}^2 + F_{m+1}F_{m-1} + F_{m-1}^2) =$$

$$= F_m^3 + F_m\left[(F_{m+1} - F_{m-1})^2 + 3F_{m+1}F_{m-1}\right] =$$

$$= F_m^3 + F_m\left\{F_m^2 + 3\left[F_m^2 + (-1)^m\right]\right\}$$

or

(85) $$F_{3m} = 5F_m^3 + 3(-1)^m F_m \quad .$$

By summing (83) and (84) from $m = s + 1$ to $m = s + t$, and using (76), we obtain respectively that

(86) $$\sum_{r=1}^{t} F_{r+s}^3 = \frac{1}{2}(F_{3(t+s)-1} - 2F_{t+s-1}^3 - F_{3s-1} + 2F_{s-1}^3)$$

and

(87) $$\sum_{r=1}^{t} F_{r+s-1}F_{r+s}F_{r+s+1} = \sum_{r=1}^{t} F_{r+s}^3 + (-1)^{t+s}F_{t+s-1} - (-1)^s F_{s-1} \quad .$$

If we multiply (67) by F_m and sum from $m = s + 1$ to $m = s + t$, we get that

$$\sum_{r=1}^{t} F_{r+s}^3 - \sum_{r=1}^{t} F_{r+s-1}F_{r+s}F_{r+s+1} = \sum_{r=1}^{t} (-1)^{r+s-1}F_{r+s} \quad .$$

A comparison of this last result with (87) yields

(88) $$\sum_{r=1}^{t} (-1)^{r+s}F_{r+s} = (-1)^{t+s}F_{t+s-1} - (-1)^s F_{s-1} \quad .$$

This last result may be verified by combining (73) and (74), or by summing the identity (derived from (1)),

(89) $$(-1)^{r+s}F_{r+s} = (-1)^{r+s-2}F_{r+s-2} - (-1)^{r+s-1}F_{r+s-1} \quad .$$

As a final illustration of the large family of identities springing from (23), we consider the generalizations of (66) and (83), analogous to that of (1) in (70). First we obtain a few results analogous to (85). Clearly

$$(F_{m+1} + F_{m-1})^2 = (F_{m+1} - F_{m-1})^2 + 4F_{m+1}F_{m-1} \quad ;$$

thus, by (1) and (67),

$$(90) \qquad (F_{m+1} + F_{m-1})^2 = 5F_m^2 + 4(-1)^m \quad ,$$

and therefore, by (65),

$$(91) \qquad F_{2m}^2 = 5F_m^4 + 4(-1)^m F_m^2 \quad .$$

Also, by (1),

$$(92) \qquad F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2} = \frac{1}{2}(F_{n+3} + F_{n-3}) \quad ;$$

whence, by (85) and (67),

$$(93) \qquad F_{3m+1} + F_{3m-1} = (F_{m+1} + F_{m-1})\left[5F_m^2 + (-1)^m\right] \quad .$$

Putting $2r$ for r and $2m$ for m in (70), we get, by (64), (66), and (67), that

$$(94) \left[5F_m^2 + 2(-1)^m\right]F_{2r} = F_{r+m+1}^2 - F_{r+m-1}^2 + F_{r-m+1}^2 - F_{r-m-1}^2 \quad .$$

Alternatively, on squaring (70), we obtain, by (58) and (90), that

$$\left[5F_m^2 + 4(-1)^m\right]F_r^2 = F_{r+m}^2 + F_{r-m}^2 + 2(-1)^m\left[F_r^2 - (-1)^{r+m}F_m^2\right] \quad ,$$

whence

$$(95) \qquad \left[5F_m^2 + 2(-1)^m\right]F_r^2 + 2(-1)^r F_m^2 = F_{r+m}^2 + F_{r-m}^2 \quad .$$

We see that (94) yields (66) on putting one for m and m for r. Finally, put $3r$ for r and 3m for m in (70). Then, by (85) and (93), we get

$$(F_{m+1} + F_{m-1})\left[5F_m^2 + (-1)^m\right]F_{3r} = 5F_{r+m}^3 + 3(-1)^{r+m}F_{r+m} +$$
$$+ 5(-1)^m F_{r-m}^3 + 3(-1)^r F_{r-m}$$
$$= 5F_{r+m}^3 + 5(-1)^m F_{r-m}^3 + 3(-1)^{r+m}(F_{m+1} + F_{m-1})F_r$$
$$= 5F_{r+m}^3 + 5(-1)^m F_{r-m}^3 + (-1)^m (F_{m+1} + F_{m-1})(F_{3r} - 5F_r^3) \quad .$$

Thus, by (65),

$$(96) \qquad F_m F_{2m} F_{3r} = F_{r+m}^3 - (-1)^m (F_{m+1} + F_{m-1})F_r^3 + (-1)^m F_{r-m}^3 \quad .$$

This identity is new, but we can find in the literature [2] the particular cases when $m = 1$ and $m = 2$, namely (83) (with r for m) and

$$(97) \qquad 3F_{3r} = F_{r+2}^3 - 3F_r^3 + F_{r-2}^3 \quad .$$

REFERENCES

1. K. Siler, "Fibonacci Summations," Fibonacci Quarterly, 1(1963) No. 3, pp. 67-69.
2. Fibonacci Quarterly, 1(1963) No. 2, p. 60.

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REQUEST

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San Jose State College
San Jose, California

ADVANCED PROBLEMS AND SOLUTIONS

Edited by VERNER E. HOGGATT, JR.
San Jose State College, San Jose, California

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-52 *Proposed by Brother U. Alfred, St. Mary's College, California*

Prove that the value of the determinant:

$$\begin{vmatrix} u_n^2 & u_{n+2}^2 & u_{n+4}^2 \\ u_{n+2}^2 & u_{n+4}^2 & u_{n+6}^2 \\ u_{n+4}^2 & u_{n+6}^2 & u_{n+8}^2 \end{vmatrix}$$

is $18\,(-1)^{n+1}$.

H-53 *Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, California and S.L. Basin, Sylvania Electronics Systems, Mt. View, California*

The Lucas sequence, $L_1 = 1$, $L_2 = 3$; $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$, is incomplete (see V. E. Hoggatt, Jr. and C. King, Problem E-1424 American Mathematical Monthly Vol. 67, No. 6, June-July 1960 p. 593) since every integer n, is not the sum of distinct Lucas numbers. OBSERVE THAT 2, 6, 9, 13, 17, ... cannot be so represented. Let M(n) be the number of positive integers less than n which cannot be so represented. Show

$$M(L_n) = F_{n-1} .$$

Find, if possible, a closed form solution for M(n).

H-54 *Proposed by Douglas Lind, Falls Church, Va.*

If F_n is the nth Fibonacci number, show that

$$\phi(F_n) \equiv 0 \pmod 4), \qquad n > 4$$

where $\phi(n)$ is Euler's function.

H-55 *Proposed by Raymond Whitney, Lock Haven State College, Lock Haven, Penn.*

Let $F(n)$ and $L(n)$ denote the nth Fibonacci and nth Lucas numbers, respectively.

Given $U(n) = F(F(n))$, $V(n) = F(L(n))$, $W(n) = L(L(n))$ and $X(n) = L(F_n)$, find recurrence relations for the sequences $U(n)$, $V(n)$, $W(n)$ and $X(n)$.

H-56 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Show

$$\sum_{n=1}^{\infty} \frac{F_k^n}{F_n F_{n+2} \cdots F_{n+k} F_{n+k+1}} = \frac{(F_k/F_{k+1})}{\prod_{i=1}^{k+1} F_i}, \quad k \geq 1 .$$

H-57 *Proposed by George Ledin, Jr., San Francisco, California*

If F_n is the nth Fibonacci number, define

$$G_n = \left(\sum_{k=1}^{n} k F_k\right) \Big/ \left(\sum_{k=1}^{n} F_k\right)$$

and show

$$\text{(i)} \quad \lim_{n \to \infty} (G_{n+1} - G_n) = 1$$

$$\text{(ii)} \quad \lim_{n \to \infty} (G_{n+1}/G_n) = 1 .$$

Generalize.

H-58 *Proposed by John L. Brown, Jr., Ordnance Research Laboratory, The Penn. State University, State College, Penn.*

Evaluate, as a function of n and k, the sum

$$\sum_{i_1+i_2+\cdots+i_{k+1}=n} F_{2i_1+2} F_{2i_2+2} \cdots F_{2i_k+2} F_{2i_{k+1}+2} ,$$

where $i_1, i_2, i_3, \ldots, i_{k+1}$ constitute an ordered set of indices which take on the values of all permutations of all sets of $k+1$ <u>nonnegative</u> integers whose sum is n.

REPROPOSED CHALLENGE

H-22 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

$$\text{If } P(x) = \prod_{i=1}^{\infty} (1 + x^{F_i}) = \sum_{n=0}^{\infty} R(n)\, x^n ,$$

then show

(i) $$R(F_{2n} - 1) = n$$

(ii) $$R(N) > n \text{ if } N > F_{2n} - 1 \quad .$$

INVERSION OF FIBONACCI POLYNOMIALS

P-3 *Proposed by Paul F. Byrd, San Jose State College, San Jose, California* in "Expansion of Analytic Functions in Polynomials Associated with Fibonacci Numbers," Vol. I, No. 1, Feb. 1963, pp. 16-29.

Verify the reciprocal relationship

$$x^n = \left(\frac{1}{2}\right)^n \sum_{r=0}^{[n/2]} (-1)^r \binom{n}{r} \frac{n-2r+1}{n-r+1} \gamma_{n+1-2r}(x), \qquad (n \geq 0)$$

where

$$\gamma_{k+1}(x) = \sum_{m=0}^{[k/2]} \binom{k-m}{m} (2x)^{k-2m} \qquad (k \geq 0)$$

Solution by Gary McDonald, St. Mary's College, Winona, Minnesota

Verification by induction: (Equation numbers refer to P. F. Byrd's article,

For $n = 0$, we have $1 = \gamma_1(x)$ which agrees with (2.2). Assuming (1) true for $n = k$, we can write

$$x^{k+1} = \frac{1}{2^{k+1}} \sum_{r=0}^{[k/2]} (-1)^r \binom{k}{r} \frac{k-2r+1}{k-r+1} (2x)\, \gamma_{k+1-2r}(x) \quad .$$

Recalling (2.1), we have

$$x^{k+1} = \frac{1}{2^{k+1}} \left[\sum_{r=0}^{[k/2]} (-1)^r \binom{k}{r} \frac{k-2r+1}{k-r+1} \gamma_{k+2-2r}(x) - \sum_{r=0}^{[k/2]} (-1)^r \binom{k}{r} \frac{k-2r+1}{k-r+1} \gamma_{k-2r}(x) \right]$$

$$= \frac{1}{2^{k+1}} \left[\gamma_{k+2}(x) + \sum_{r=1}^{[k/2]} (-1)^r \binom{k}{r} \frac{k-2r+1}{k-r+1} \gamma_{k+2-2r}(x) - \right.$$

$$\left. - \sum_{r=0}^{[k/2]-1} (-1)^r \binom{k}{r} \frac{k-2r+1}{k-r+1} \gamma_{k-2r}(x) + C \right]$$

where

$$C = (-1)^{[k/2]+1}\binom{k}{[k/2]}\frac{k-2[k/2]+1}{k-[k/2]+1}\,\gamma_{k-2[k/2]}(x) \;.$$

Letting $j = r + 1$ in the second Σ for x^{k+1},

$$x^{k+1} = \frac{1}{2^{k+1}}\left[\gamma_{k+2}(x) + \sum_{r=1}^{[k/2]}(-1)^r\binom{k}{r}\frac{k-2r+1}{k-r+1}\,\gamma_{k+2-2r}(x) + \right.$$

$$\left. + \sum_{j=1}^{[k/2]}(-1)^j\binom{k}{j-1}\frac{k+3-2j}{k+2-j}\,\gamma_{k+2-2j}(x) + C\right] \;,$$

or combining coefficients of $\gamma_i(x)$

$$(2)\qquad x^{k+1} = \frac{1}{2^{k+1}}\left\{\gamma_{k+2}(x) + \sum_{r=1}^{[k/2]}(-1)^r\left[\binom{k}{r}\frac{k-2r+1}{k-r+1} + \right.\right.$$

$$\left.\left. + \binom{k}{r-1}\frac{k+3-2r}{k+2-r}\right]\gamma_{k+2-2r}(x) + C\right\}$$

We can reduce the quantity in brackets as follows:

$$\binom{k}{r}\frac{k-2r+1}{k-r+1} + \binom{k}{r-1}\frac{k+3-2r}{k+2-r} = \left[\binom{k}{r}\frac{(k-2r+1)(k+2-r)}{k-r+1} + \binom{k}{r-1}(k+3-2r)\right]\left(\frac{1}{k+2-r}\right)$$

$$= \left[\frac{k!}{(k-r+1)!\,r!}\,(k-2r+1)(k+2-r) + \frac{k!\,(k+3-2r)}{(k-r+1)!\,(r-1)!}\right]\left(\frac{1}{k+2-r}\right)$$

$$= \left[\frac{(k-2r+1)(k+2-r) + r(k+3-2r)}{k+1}\right]\frac{(k+1)!}{(k-r+1)!\,r!\,(k+2-r)}$$

$$= \left[\frac{k^2+3k-2rk-2r+2}{(k+1)}\right]\frac{(k+1)!}{(k-r+1)!\,r!\,(k+2-r)}$$

$$= \frac{k+2-2r}{k+2-r}\binom{k+1}{r} \;.$$

Therefore from (2),

$$(3)\qquad x^{k+1} = \frac{1}{2^{k+1}}\left[\gamma_{k+2}(x) + \sum_{r=1}^{[k/2]}(-1)^r\binom{k+1}{r}\frac{k+2-2r}{k+2-r}\,\gamma_{k+2-2r}(x) + C\right].$$

Note that:

a) $$\gamma_{k+2}(x) = (-1)^0\binom{k+1}{0}\frac{k+2-0}{k+2-0}\,\gamma_{k+2-0}(x) \;.$$

b) When k is even $C = 0$, and $\left[\frac{k}{2}\right] = \left[\frac{k+1}{2}\right]$.

When k is odd, then $\left[\frac{k}{2}\right] = \frac{k-1}{2}$ and

$$C = (-1)^{\left[\frac{k+1}{2}\right]} \binom{k}{\frac{k-1}{2}} \frac{2}{\frac{3+k}{2}} \gamma_1(x)$$

$$= (-1)^{\left[\frac{k+1}{2}\right]} \frac{k!}{(\frac{k-1}{2})!\,(\frac{k+1}{2})!} \frac{4}{3+k} \gamma_1(x)$$

If we let $r = \left[\frac{k+1}{2}\right]$ in the Σ of equation (3), we have

$$(-1)^{\left[\frac{k+1}{2}\right]} \frac{(k+1)!}{(\frac{k+1}{2})!\,(\frac{k+1}{2})!} \frac{2}{k+3} \gamma_1(x) = (-1)^{\left[\frac{k+1}{2}\right]} \frac{(k+1)k!}{\frac{2(k+1)}{2}(\frac{k-1}{2})!\,(\frac{k+1}{2})!} \frac{4}{k+3} \gamma_1(x)$$

$$= (-1)^{\left[\frac{k+1}{2}\right]} \frac{k!}{(\frac{k-1}{2})!\,(\frac{k+1}{2})!} \frac{4}{k+3} \gamma_1(x)$$

$$= C, \quad k \text{ odd.}$$

Therefore, we may combine $\gamma_{k+2}(x)$ and C into the Σ in (3) and write

$$x^{k+1} = \frac{1}{2^{k+1}} \sum_{r=0}^{\left[\frac{k+1}{2}\right]} (-1)^r \binom{k+1}{r} \frac{k+2-2r}{k+2-r} \gamma_{k+2-2r}(x) \quad ,$$

from which we conclude

$$x^n = \frac{1}{2^n} \sum_{r=0}^{[n/2]} (-1)^r \binom{n}{r} \frac{n-2r+1}{n-r+1} \gamma_{n+1-2r}(x), \qquad n \geq 0 \quad .$$

DEFERRED ANSWER

H-34 *Proposed by P.F. Byrd, San Jose State College*

Derive the series expansions

$$J_{2k}(\alpha) = I_k^2(\alpha) + \sum_{m=1}^{\infty} (-1)^{m+k} I_{m+k}(\alpha) I_{m-k}(\alpha) L_{2m}$$

($k = 0, 1, 2, 3, \ldots$) for the Bessel functions J_{2k} of all even orders, where L_n are Lucas numbers and I_n are modified Bessel functions.

The solution will appear in a fine paper by the proposer to appear later in the Quarterly.

FIBONACCI AND MAGIC SQUARES

H-35 *Proposed by Walter W. Horner, Pittsburgh, Pa.*

Select any nine consecutive terms of the Fibonacci sequence and form the magic square

u_8	u_1	u_6
u_3	u_5	u_7
u_4	u_9	u_2

show

$$u_8u_1u_6 + u_3u_5u_7 + u_4u_9u_2 = u_8u_3u_4 + u_1u_5u_9 + u_6u_7u_2 \quad .$$

Generalize.

Solution by Maxey Brooke, Sweeny, Texas and F.D. Parker, SUNY, Buffalo, N.Y.

If U_n satisfies the general second order difference equation, then

$$\begin{vmatrix} U_1 & U_2 & U_3 \\ U_4 & U_5 & U_6 \\ U_7 & U_8 & U_9 \end{vmatrix} = 0$$

since $U_{n+2} = \alpha U_{n+1} + \beta U_n$ with U_1 and U_2 arbitrary. The expansion of this determinant yields products whose subscripts add up to the requisite 15 and yields the equality asked for in the problem.

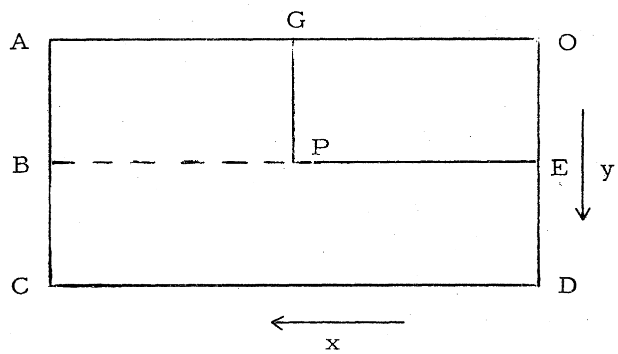
Also solved by the proposer.

GOLDEN SECTION IN CENTROIDS

H-36 *Proposed by J.D.E. Konhauser, State College, Pa.*

Consider a rectangle R. From the upper right corner of R remove a rectangle S (similar to R and with sides parallel to the sides of R. Determine the linear ratio $K = L_R/L_S$ if the centroid of the remaining L shaped region is where the lower left corner of the removed rectangle was.

Solution by John Wessner, Melbourne High School, Melbourne, Florida



$GO = L_R$

$CD = L_S$

$OD = \alpha L_S$

$OE = \alpha L_R$

The centroid of AGPB is at $\frac{1}{2}\{L_S + L_R\}$, $\frac{1}{2}\ \alpha L_R$ and has weight $\alpha L_R\{L_S - L_R\}$. Similarly the centroid of BEDC is at $\frac{1}{2} L_S$, $\frac{1}{2}\ \alpha\{L_S + L_R\}$ and has weight $\alpha L_S\{L_S - L_R\}$. The centroid of this remainder must be at P and have x-coordinate

$$L_R = \frac{\frac{1}{2}\ L_S^2\alpha\{L_S - L_R\} + \frac{1}{2}\ \alpha\{L_S + L_R\}\ \{L_S - L_R\}\ L_R}{\alpha\{L_S^2 - L_R^2\}}$$

or upon expanding

$$2L_R\{L_S^2 - L_R^2\} = L_S^3 - L_R^3 \quad .$$

Division by L_R^3 gives

$$2\{K^2 - 1\} = K^3 - 1 \quad .$$

After removing the obvious root +1 we have $K^2 - K - 1 = 0$ which has as its positive root $\gamma = (1 + \sqrt{5})/2$.

Editorial Comment: The above property is shared by many geometric figures including the ellipse. A short paper later will show this.

Also solved by David Sowers and the proposer.

A FASCINATING RECURRENCE

H-37 *Proposed by H.W. Gould, West Virginia University, Morgantown, West. Va.*

Find a triangle with sides n+1, n, n-1 having integral area. The first two examples appear to be 3, 4, 5 with area 6; and 13, 14, 15 with area 84.

The proposer's paper comprehensively discussing this problem will soon appear in the Quarterly.

NO SOLUTIONS RECEIVED

H-38 *Proposed by R.G. Buschman, SUNY, Buffalo, N.Y.*

(See Fibonacci Numbers, Chebyshev Polynomials, Generalizations and Difference Equations Vol. 1, No. 4, Dec. 1963, pp. 1-7.)

Show $$(u_{n+r} + (-b)^r u_{n-r})/u_n = \lambda_r .$$

CORRECTED

H-40 *Proposed by Walter Blumberg, New Hyde Park, L.I., N.Y.*

Let U, V, A and B be integers, subject to the following conditions (i) $U > 1$, (ii) $(U, 3) = 1$; (iii) $(A, V) = 1$;

$$\text{(iv)} \quad V = \sqrt{(U^2 - 1)/5} \quad .$$

Show A^2U+BV is not a square.

CONVOLUTIONS AND OPTICAL 2-STACK

H-39 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Solve the difference equation in closed form

$$C_{n+2} = C_{n+1} + C_n + F_{n+2} \quad ,$$

where $C_1 = 1$, $C_2 = 2$, and F_n is the nth Fibonacci number. Give two separate characterizations of these numbers.

Solution by L. Carlitz, Duke University, Durham, N.C.

Since $C_2 = C_1 + C_0 + F_2$ we have also $C_0 = 0$. If we put

$$C(t) = \sum_0^\infty C_n t^n \quad ,$$

then it follows from

$$C_{n+2} = C_{n+1} + C_n + F_{n+2}$$

that

$$(1-t-t^2)C(t) = \sum_0^\infty F_n t^n = \frac{t}{1-t-t^2} \quad .$$

Thus

$$\text{(1)} \qquad C(t) = \frac{t}{(1-t-t^2)^2} \quad .$$

Expanding we get

$$C(t) = t \sum_{r=0}^{\infty} (r+1)(t+t^2)^r$$

$$= \sum_{r=0}^{\infty} (r+1)t^{r+1} \sum_{s=0}^{r} \binom{r}{s} t^s$$

$$= \sum_{n=0}^{\infty} t^{n+1} \sum_{r=0}^{n} (r+1)\binom{r}{n-r} ,$$

so that

$$C_{n+1} = \sum_{r=0}^{n} (r+1)\binom{r}{n-r} = \sum_{2r \le n} (n-r+1)\binom{n-r}{r} .$$

Another explicit expression that follows from (1) is

$$C_{n-1} = \sum_{r=1}^{n-1} F_r F_{n-r} .$$

Next is we differentiate

$$\frac{t}{1-t-t^2} = \sum_0^{\infty} F_n t^n$$

we get

$$\frac{1+t^2}{(1-t-t^2)^2} = \sum_0^{\infty} (n+1)F_{n+1} t^n ,$$

which yields

$$C_n + C_{n-2} = (n+1)F_{n+1} .$$

A consequence of this is

$$C_n = \sum_{2k \le n} (-1)^k (n-2k+1)F_{n-2k+1} .$$

Finally consider the number

$$C'_n = AnF_n + BnL_n .$$

We find that

$$C'_{n+2} - C'_{n+1} - C'_n = A(F_{n+2}+F_n) + B(L_{n+2}+L_n) .$$

Since

$$L_{n+2} + L_n = 5(F_{n+2}-F_n)$$

we get

$$C'_{n+2} - C'_{n+1} - C'_n = (A+5B)F_{n+2} + (A-5B)F_n .$$

Hence for $A = 6$, $B = 1/10$ it follows that

$$C'_{n+2} - C'_{n+1} - C'_n = F_{n+2} \quad .$$

Clearly

$$C_n = n/2\ F_n + n/10\ L_n + aF_n + bL_n \quad .$$

Taking $n = 0$ we get $b = 0$. For $n = 1$ we get $a = 2/3$. Therefore we have

$$C_n = n/2\ F_n + n/10\ L_n + 2/5\ F_n = \frac{n\ L_{n+1} + 2F_n}{5} \quad .$$

Also solved by Ronald Weimshenk, John L. Brown, Jr., Donald Knuth, H.H. Ferns and the proposer.

Editorial Note: Another characterization, besides the convolution

$$C_{n+1} = \sum_{r=1}^{n+1} F_r F_{n-r} = \frac{(n+1)L_{n+2} + 2F_{n+1}}{5} \quad ,$$

is the number of crossings of the interface, in the optical stack in problem B-6, Dec. 1963, p. 75, for all rays which are reflected n-times.

If $\quad f_0(x) = 0,\ \ f_1(x) = 1,\ \ \text{and}\ \ f_{n+2}(x) = xf_{n+1}(x) + f_n(x) \quad ,$

the Fibonacci polynomials, then

$$f_n(1) = F_n \quad \text{and} \quad f_n'(1) = C_{n-1} \quad .$$

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MATH MORALS

Brother U. Alfred

A tutor who tutored two rabbits,
Was intent on reforming their habits.
Said the two to the tutor,
"There are rabbits much cuter,
But non-Fibonacci, dagnabits."*

*The author has just taken out poetic license $\#F_{97}$ according to one clause of which it is permissible to corrupt corrupted words.

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PRODUCTS OF ODDS

SHERYL B. TADLOCK*
Madison College, Harrisonburg, Virginia

The following are used in the proofs of the identities:

$$F_n = \frac{\beta^n - \alpha^n}{\beta - \alpha}, \quad L_n = \beta^n + \alpha^n, \quad (\beta\alpha)^n = (-1)^n,$$

where

$$\beta = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \alpha = \frac{1-\sqrt{5}}{2}$$

1. $F_{2k+1}F_{2j+1} = F_{k+j+1}^2 + F_{k-j}^2$

$$F_{k+j+1}^2 + F_{k-j}^2 = \left[\frac{\beta^{k+j+1} - \alpha^{k+j+1}}{\beta - \alpha}\right]^2 + \left[\frac{\beta^{k-j} - \alpha^{k-j}}{\beta - \alpha}\right]^2$$

$$= \frac{\beta^{2k+2j+2} + \alpha^{2k+2j+2} - 2(\beta\alpha)^{k+j+1} - 2(\beta\alpha)^{k-j} + \beta^{2k-2j} + \alpha^{2k-2j}}{(\beta - \alpha)^2}$$

$$= \frac{\beta^{2k+1}(\beta^{2j+1} + \beta^{-2j-1}) + \alpha^{2k+1}(\alpha^{2j+1} + \alpha^{-2j-1})}{(\beta - \alpha)^2}$$

$$- \frac{2(\beta\alpha)^{k+j}(\beta\alpha + (\beta\alpha)^{-2j})}{(\beta - \alpha)^2}$$

Recalling that $\beta^{-2j-1} = (-1)^{-2j-1}\alpha^{2j+1} = -\alpha^{2j+1}$

and $\alpha^{-2j-1} = (-1)^{-2j-1}\beta^{2j+1} = -\beta^{2j+1}$

and that the last term has the value of 0, the above expression becomes

$$F_{k+j+1}^2 + F_{k-j}^2 = \frac{\beta^{2k+1}(\beta^{2j+1} - \alpha^{2j+1}) - \alpha^{2k+1}(\beta^{2j+1} - \alpha^{2j+1})}{(\beta - \alpha)^2}$$

$$= \frac{(\beta^{2k+1} - \alpha^{2k+1})(\beta^{2j+1} - \alpha^{2j+1})}{(\beta - \alpha)\qquad\qquad(\beta - \alpha)}$$

But the right-hand side is of the form $F_{2k+1}F_{2j+1}$. Therefore,

$$F_{k+j+1}^2 + F_{k-j}^2 = F_{2k+1}F_{2j+1} \quad .$$

*Student

2. $L_{2k+1}L_{2j+1} = L^2_{k+j+1} - L^2_{k-j} + 4(-1)^{k-j}$

$$
\begin{aligned}
L_{2k+1}L_{2j+1} &= (\beta^{2k+1} + \alpha^{2k+1})(\beta^{2j+1} + \alpha^{2j+1}) \\
&= \beta^{2k+2j+2} + \alpha^{2k+2j+2} + \beta^{2k+1}\alpha^{2j+1} + \alpha^{2k+1}\beta^{2j+1} \\
&= \beta^{2k+2j+2} + \alpha^{2k+2j+2} + \beta^{k+j+1}\alpha^{k+j+1}(\beta^{k-j}\alpha^{j+k} + \alpha^{k-j}\beta^{j-k})
\end{aligned}
$$

Observing that $\beta^{k-j}\alpha^{j-k} = (-1)^{j-k}\beta^{k-j}\beta^{k-j} = (-1)^{j-k}\beta^{2k-2j}$

and $\alpha^{k-j}\beta^{j-k} = (-1)^{j-k}\alpha^{k-j}\alpha^{k-j} = (-1)^{j-k}\alpha^{2k-2j}$

and adding and subtracting $2\beta^{k+j+1}\alpha^{k+j+1}$ the above expression becomes

$$
\begin{aligned}
L_{2k+1}L_{2j+1} &= \beta^{2k+2j+2} + \alpha^{2k+2j+2} + 2\beta^{k+j+1}\alpha^{k+j+1} - 2\beta^{k-j-1}\alpha^{k-j+1} \\
&\quad + (\beta\alpha)^{k+j+1}((-1)^{j-k}\beta^{2k-2j} + (-1)^{j-k}\alpha^{2k-2j}) \\
&= \beta^{2k+2j+2} + 2\beta^{k+j+1}\alpha^{k+j+1} + \alpha^{2k+2j+2} \\
&\quad + (\beta\alpha)^{k+j+1}(-2 + (-1)^{j-k}\beta^{2k-2j} + (-1)^{j-k}\alpha^{2k-2j}) \\
&= (\beta^{k+j-1} + \alpha^{k+j+1})^2 + (-1)^{k+j+1}(-1)^{j-k}(\beta^{2k-2j} + \alpha^{2k-2j} - 2(-1)^{k-j}) \\
&= L^2_{k+j+1} + (-1)^{2j+1}(\beta^{2k-2j} + 2\beta^{k-j}\alpha^{k-j} + \alpha^{2k-2j} - 2\beta^{k-j}\alpha^{k-j}) \\
&\quad - 2(-1)^{k-j}(-1)^{2j+1} .
\end{aligned}
$$

Noting that $-2(-1)^{k-j}(-1)^{2j+1} = -2(-1)^{k-j}(-1) = 2(-1)^{k-j}$

we have

$$
\begin{aligned}
L_{2k+1}L_{2j+1} &= L^2_{k+j+1} - \left[(\beta^{k-j} + \alpha^{k-j})^2 - 2(\beta\alpha)^{k-j}\right] + 2(-1)^{k-j} \\
&= L^2_{k+j+1} - (L^2_{k-j} - 2(-1)^{k-j}) + 2(-1)^{k-j} \\
&= L^2_{k+j+1} - L^2_{k-j} + 2(-1)^{k-j} + 2(-1)^{k-j} .
\end{aligned}
$$

Therefore,

$$L_{2k+1}L_{2j+1} = L^2_{k+j+1} - L^2_{k-j} + 4(-1)^{k-j} .$$

By using the identity $L_n^2 - 5F_n^2 = 4(-1)^n$ (Vol. 1, No. 1, p. 66, this Quarterly), it can easily be shown that

$$L_{k+j+1}^2 - L_{k-j}^2 + 4(-1)^{k-j} = L_{k+j+1}^2 - 5F_{k-j}^2 = 5F_{k+j+1}^2 - L_{k-j}^2 .$$

Thus, we have proofs of the following Fibonacci identity and the analogous Lucas identities for products of odds:

(1) $$F_{2k+1}F_{2j+1} = F_{k+j+1}^2 + F_{k-j}^2$$

(2) $$L_{2k+1}L_{2j+1} = L_{k+j+1}^2 - L_{k-j}^2 + 4(-1)^{k-j}$$

(3) $$L_{2k+1}L_{2j+1} = L_{k+j+1}^2 - 5F_{k-j}^2$$

(4) $$L_{2k+1}L_{2j+1} = 5F_{k+j+1}^2 - L_{k-j}^2 .$$

These four identities correspond closely to those given for products of evens in this Quarterly, Vol. 2, No. 1, p. 78.

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EXPLORING SCALENE FIBONACCI POLYGONS

Proposed by BROTHER U. ALFRED on page 60, October 1963
"The Fibonacci Quarterly"

C.B.A. Peck

The sequence of Fibonacci numbers may be defined by

(1) $$F_0 = 0, \; F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geqq 2.$$

We describe a way of deciding when a set S of m distinct numbers drawn from the sequence $F_2, F_3, F_4, \ldots$ corresponds to the sides of some plane polygon with m sides. If they do we call S a (scalene Fibonacci) polygon, for short.

To prove the result, we find it convenient to use the following identities, easily proved from (1) by induction on k and n, respectively:

(2) $$F_n = F_{n-2k} + \sum_{i=1}^{k} F_{n-2i+1} \quad \text{for } n \geqq 4 \text{ and } 0 < 2k < n,$$

(3) $$F_n > \sum_{i=2}^{n-2} F_i \quad \text{for } n \geqq 1 \text{ (the sum is zero if } n = 1, 2, 3).$$

Suppose once and for all that F_n is the largest number in S. If we denote by S(n, k) the set of numbers appearing in (2), then S is a polygon if and only if it properly contains some S(n, k). If it equals S(n, k) for some k we call it a degenerate polygon.

Proof: If $F_{n-1} \notin S$, then by (2) S contains no S(n, k). By (3) F_n exceeds the sum of the other numbers in S, which shows that S is not even a degenerate polygon. Now suppose that $F_{n-1} \in S$ (so that $n \geqq 3$) and proceed downward through the sequence in (1), starting with F_{n-1} and stopping short of F_1. The numbers alternate in and out of S until one of two things happens.

1. S is found to contain no S(n, k), either because the alternation stops at an adjacent pair not in S, say F_{n-2j}, F_{n-2j-1} with $n-2j-1 \geqq 2$, or continues to the bottom (here we set $n-2j-1 = 1$ or 0 according as n is even or odd). Then every number in S other than F_n occurs in either

$$\sum_{i=2}^{n-2j-2} F_i \quad \text{or} \quad \sum_{i=1}^{j} F_{n-2i+1} .$$

The first sum $< F_{n-2j}$ by (3), whence the sum of each $< F_n$ by (2). Thus S is again not even a degernate polygon.

2. The alternation stops with an adjacent pair in S, say F_{n-2k+1}, F_{n-2k} with $n-2k \geqq 2$, so that S(n, k) is in S. Then (2) shows that S is a (degenerate) polygon if there are (no) numbers in S besides those in S(n, k), on the grounds that F_n ($\leqq$) the sum of the other numbers in S.

Could two sets of numbers drawn from F_2, F_3, F_4, ... be proportional to the lengths of the sides of a single polygon? This is not possible, at any rate, when the numbers in each set are distinct, for suppose that we did have two scalene Fibonacci polygons with largest sides F_n and F_N ($N > n$) proportional to a third polygon, hence to each other, say in the ratio $P > 1$. We have just seen that if F_n is the largest number in such a set, then F_{n-1} must belong to it. Since the largest and second largest sides correspond, we have $PF_n = F_N$ and $PF_{n-1} = F_{N-1}$. By (1), we have, then, $PF_{n-2} = F_{N-2}$, and n-2 further applications of (1) yield finally $PF_0 = F_{N-n}$. By (1), the l.h.s. is zero and the r.h.s. positive, which is absurd.

An interesting exercise is to use this argument (with suitable amplification of the last sentence) on any two Fibonacci polygons such that in at least one of them there are numbers whose subscripts differ by only one or two. We need something stronger for such polygons as F_n, F_{n-3}, F_{n-3}, F_{n-3}, F_{n-3}, F_{n-3}.

The generalization of (2) which seems to be called for is some characterization of the coefficients in inequalities of the form

$$F_n \leqq \sum_{i=2}^{n} a_i F_i$$

where the a_i's are nonnegative integers.

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NOTE ON THIRD ORDER DETERMINANTS

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The recent exhaustive investigation of nine-digit determinants by Bicknell and Hoggatt that appeared in the Mathematics Magazine of May-June, 1963, raises an interesting question [1]. Given that

$$\begin{vmatrix} 9 & 4 & 2 \\ 3 & 8 & 6 \\ 5 & 1 & 7 \end{vmatrix}$$

or any equivalent arrangement producing the same set of products has a maximum value of 412, would we obtain a maximum for any other nine consecutive positive integers using the same relative arrangement? This note will offer a negative answer and indicate the maximum for all positive values.

First, a small amount of theory is in order. If a third order determinant has elements a_i and a fixed quantity b is added to each element the resulting determinant would be:

$$\begin{vmatrix} a_1 + b & a_2 + b & a_3 + b \\ a_4 + b & a_5 + b & a_6 + b \\ a_7 + b & a_8 + b & a_9 + b \end{vmatrix}$$

Subtract the second column from the third and the first from the second to obtain

$$\begin{vmatrix} a_1 + b & a_2 - a_1 & a_3 - a_2 \\ a_4 + b & a_5 - a_4 & a_6 - a_5 \\ a_7 + b & a_8 - a_7 & a_9 - a_8 \end{vmatrix}$$

from which it is evident that the value of the altered determinant is

$$D + \lambda b ,$$

where D is the value of the original determinant and λ is the sum of the three minors formed from the second and third columns. Fxpanded and grouped appropriately we obtain

$$\lambda = (a_1a_5+a_5a_9+a_9a_1) + (a_2a_6+a_6a_7+a_7a_2) + (a_3a_4+a_4a_8+a_8a_3)$$
$$- (a_1a_6+a_6a_8+a_8a_1) - (a_3a_5+a_5a_7+a_7a_3) - (a_2a_4+a_4a_9+a_9a_2)$$

This coefficient λ gives the change in the value of the determinant as we add 1 to each of its elements. See [2] for another use.

It should be noted that the groups in λ are the same as those for the positive and negative terms of the determinant expansion and hence any alteration of the arrangement of determinant elements which leaves the expansion unchanged will also be without effect on λ.

An independent investigation shows that the maximum value of λ is 81 when the elements of the determinant are the nine digits, while the value of λ for the determinant giving a maximum of 412 is only 80. Thus, the smaller valued determinant with $\lambda = 81$ will eventually overtake the larger as the elements of the determinants are increased uniformly.

By calculating λ for the largest values given in the table of Bicknell and Hoggatt (Ref. 1, p. 152) λ is found to be 81 for 405 = 630-225a and 630-225c. Adding n to each element of 630-225a, for example, will produce a determinant of value 405 + 81n; doing likewise for the original maximum determinant of value 412 produces a value of 412 + 80n. To find when these will be equal, set

$$405 + 81n = 412 + 80n$$

the solution being $n = 7$.

Thus, if we have nine consecutive positive integers beginning with m, the maximum value that can be achieved is $412 + 80m$ if $m \leq 8$; the maximum possible is $405 + 81m$ if $m \geq 8$.

REFERENCES

1. Marjorie Bicknell and Verner E. Hoggatt, Jr., "An Investigation of Nine-Digit Determinants," Mathematics Magazine, 36(1963), 147-152.

2. Marjorie Bicknell and Verner E. Hoggatt, Jr., "Fibonacci Matrices and Lambda Functions," The Fibonacci Quarterly, 1(1963) April, pp. 47-52.

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MORE ON FIBONACCI NIM

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Fibonacci Nim [1] was originally stated as follows:

> "Consider a game involving two players in which initially there is a group of 100 or less objects. The first player may reduce the pile by any Fibonacci number (member of the series 1, 1, 2, 3, 5, 8, 13, 21, ...). The second player does likewise. The player who makes the last move wins the game."

Let persons A and B be playing the game which A wins. If A is to win he must be able to reduce the pile to zero on his final move. Thus A must draw from $0+F_n$ $(n = 1, 2, 3, \ldots)$ on his final move.

Looking at the sequence of the number of objects from which A must draw to win on the final move, 1, 2, 3, 5, ..., we see that 4 is the first positive integer missing. If B is forced to play with 4 objects remaining, A can certainly win the game.

Now suppose A gets the opportunity to draw from $4+F_n$(5, 6, 7, 9, 12, ...). A will be able to reduce the pile to 4 objects and can continue to win.

The smallest positive integer that is not contained in the union of the sets $\{0+F_n\}$ and $\{4+F_n\}$ is 10. If B is forced to draw from a pile of 10 objects, B cannot reduce the pile to 4 or 0 but B will leave A in a position to reduce the pile to 4 or 0 and thus A can win.

Now we wish to generate the sequence of positions from which it is unsafe to draw (0, 4, 10, ...). Let $U_1=0$. Then U_2 is the smallest positive integer which is not equal to U_1+F_n $(n = 2, 3, \ldots)$. U_3 is the smallest positive integer which is not equal to U_1+F_n or U_2+F_n $(n = 2, 3, \ldots)$.

Therefore U_r $(r = 2, 3, \ldots)$ is the smallest positive integer which is not equal to U_t+F_n, where $t = 1, 2, \ldots, r-1$ and $n = 2, 3, \ldots$

*Student

U_r

F_n	0	1	2	3	5	8	13	21	34	55	89
	4	5	6	7	9	12	17	25	38	59	93
	10	11	12	13	15	18	23	31	44	65	99
	14	15	16	17	19	22	27	35	48	69	
	20	21	22	23	25	28	33	41	54	75	
	24	25	26	27	29	32	37	45	58	79	
	30	31	32	33	35	38	43	51	64	85	
	36	37	38	39	41	44	49	57	70	91	
	40	41	42	43	45	48	53	61	74	95	
	46	47	48	49	51	54	59	67	80		
	50	51	52	53	55	58	63	71	84		
	56	57	58	59	61	64	69	77	90		
	60	61	62	63	65	68	73	81	94		
	66	67	68	69	71	74	79	87	100		
	72	73	74	75	77	80	85	93			
	76	77	78	79	81	84	89	97			
	82	83	84	85	87	90	95				
	86	87	88	89	91	94	99				
	92	93	94	95	97	100					
	96	97	98	99							

The first player can always win if he starts on some position not equal to U_r ($r = 1, 2, \ldots$) and always reduces the pile to some U_r.

Here are all the values of U_r thus far computed:

0	4	10	14	20	24	30	36	40	46	50	56	60	66	72
76	82	86	92	96	102	108	112	118	122	128	132	138	150	160
169	176	186	192	196	202	206	212	218	222	228	232	238	242	248
254	260	264	270	274	280	284	290	296	300	306	310	316	322	326
332	338	342	348	352	358	364	368	374	378	384	388	394	400	406
410	416	420	426	430	436	442	446	452	456	462	468	472	478	484
488	494	498	504	510	514	520	524	530	534	540	552	556	562	566
572	576													

The following observations can be made:

1. $U_{r+1} = U_r +$ some non-Fibonacci number.
2. If $U_{r+1} - U_r = 4$, then $U_{r+2} - U_{r+1} \neq 4$ since $4+4=8=F_6$.
3. Thus the average difference of $U_{r+1} - U_r \geqq 5$, $r = 1, 2, 3, \ldots$
4. The density of $\{U_r\}$ in the positive integers must be $\leqq 1/5$.
5. The probability that the starting person can win is $\geqq 4/5$ if nothing is known about the starting position of the game.

The following questions are left unanswered:

1. Is there a closed form solution for $\{U_r\}$?

2. What is the limiting density of $\{U_r\}$ in the positive integers?

Similar results are found when one considers "Lucas Nim" analogous to Fibonacci Nim.

REFERENCES

1. Brother U. Alfred, "Research Project: Fibonacci Nim," Fibonacci Quarterly, 1(1963), No. 1, p. 63.
2. Michael J. Whinihan, "Fibonacci Nim," Fibonacci Quarterly, 1(1963), No. 4, pp. 9-13.

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ON SUMS $F_x^2 \pm F_y^2$

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Formulas for the sum of the squares of Fibonacci numbers are:

(1) $$F^2_{n+2k} + F^2_n = F_{n+2k-2}\, F_{n+2k+1} + F_{2k-1}\, F_{2n+2k-1}$$

(2) $$F^2_{n+2k+1} + F^2_n = F_{2k+1}\, F_{2n+2k+1}$$

(3) $$F^2_{n+2k} - F^2_n = F_{2k}\, F_{2n+2k}$$

(4) $$F^2_{n+2k+1} - F^2_n = F_{n-1}\, F_{n+2} + F_{2k}\, F_{2n+2k+2}$$

Validity of the above is established by using:

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n),\ L_n = \alpha^n + \beta^n,\ \alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2},\ \alpha\beta = -1\ .$$

For example:

$$5(F^2_{n+2k+1} - F^2_n) =$$

$$(\alpha^{2n+4k+2} + \beta^{2n+4k+2}) - (\alpha^{2n} + \beta^{2n}) - 2\alpha^n\beta^n(\alpha^{2k+1}\beta^{2k+1} - 1) =$$

$$L_{2n+4k+2} - L_{2n} - 2(-1)^n(-2) = L_{2n+4k+2} - L_{2n} - (-1)^{n-1}\, L_3$$

$$5(F_{n-1}\, F_{n+2} + F_{2k}\, F_{2n+2k+2}) =$$

$$(\alpha^{2n+1} + \beta^{2n+1}) + (\alpha^{2n+4k+2} + \beta^{2n+4k+2}) - \alpha^{n-1}\beta^{n-1}(\alpha^3 + \beta^3) -$$

$$- \alpha^{2k}\beta^{2k}(\alpha^{2n+2} + \beta^{2n+2}) =$$

$$L_{2n+4k+2} + (L_{2n+1} - L_{2n+2}) - (-1)^{n-1}\, L_3 = L_{2n+4k+2} - L_{2n} - (-1)^{n-1}\, L_3.$$

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PHYLLOTAXIS

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Leaves are commonly arranged on the plant stem according to a pattern. If the pattern is "whorled," several leaves arise from the same node, at intervals along the stem. If the pattern is "distichous," the arrangement is two-ranked. However, the most common pattern of arrangement is "spiral."

The most accurate method for studying plant phyllotaxis is by transecting the apical bud and making observations on the cross-section. When one examines such a cross-section, the most striking feature to meet the eye is the spiral appearance of the arrangement of leaf primordia. It has been found that there is a definite, heritable spiral appearance of the arrangement of leaf primordia. It has been found that there is a definite, heritable spiral arrangement which can be designated (in most cases) by two numbers: the number of spirals which turn in one direction, and the number which turn in the other (these curves are called "parastichies"). The intersections of these two spiral systems delineate "quasi-squares," within which are found the leaf primordia (2, 4, 40).

In an overwhelming number of species (434 species in the Angiospermae and 44 species in the Gymnospermae were found by T. Fujita in 1938) the parastichy numbers fall in the Fibonacci Sequence, the most common pairs of numbers being 2:3 and 3:5 (see Appendix) (40). When the parastichy numbers do not fall in the Fibonacci Sequence, they regularly fall into one of the other summation series (see Appendix, footnote).

It has also been found by investigators in the field (2, 14, 40) that the angle between adjacent leaf primordia is, in a convincing number of cases, approximately $137^{\circ}30'$. This is variously called the "ideal angle," the "divergence angle," and the "Limitdivergenz." This angle can be obtained mathematically by applying the limiting value of the Fibonacci Sequence u_n/u_{n+1}:

$$360^{\circ} - (0.6180)(360^{\circ}) = 137^{\circ}30'.$$

Phyllotaxis has been a field of interest for centuries. Since 1900 several theories have been offered as explanation of some of the phenomena of phyllotaxis. Some experimentation has been done to determine the effect of environment or mechanical damage on phyllotaxis (6, 11, 21, 25, 42); and some X-ray and chemical effects on the development of leaf arrangement have been noted (17, 20, 22, 23). However, after 1920 very little has been published on this subject; perhaps the feeling is that there are so many more fruitful and less "mysterious" areas of botanical interest, that this one is best left alone. Also, the subject seems to lie more properly in the realm of biophysics, which is a relatively new field.

The spiral arrangement discussed in (1) above is not peculiar to plants. It is also found in the shells of foraminifera (4), nautili, and other animals. It is the opinion of Church (4, p. 48) that the factor common to both plants and the foraminifera is "the building of new units one at a time, — and it thus appears that this is the essential factor behind all such presentation of Fibonacci relations, to all time."

Church also feels that the Fibonacci phyllotaxis is phylogenetically primitive (4, p. 13).

> "... very admirable spiral arrangements, in which Fibonacci symmetry may be distinctly traced, obtain in the case of many of the more massive Brown Seaweeds (Phaeophyceae-Fucoideae), in the orientation of the more or less frondose or leaf-like lateral ramuli; leaving little doubt that the phyllotaxis-mechanism is, in fact, a still older function of the axis of marine types of vegetation, and that the presentation of such phenomena, even in a more elaborated and special form, can be but the continuation and amplification of factors of marine phytobenthon; and that it is to the sea that one must look for the origin and primary intention of this remarkable relation." (4, pp. 37-38)

There seems to be little doubt that the primary mechanism responsible for Fibonacci phyllotaxy is genetic in nature, rather than being a function of growth conditions such as availability of, and need for, illumination. In the words of Church,

> "It can only be concluded that the plant is somehow biased from the first in favour of members arranged one by one in a Fibonacci sequence; and the suggestion immediately offers that this may be in some way the expression of the inheritance of the equipment of a preceding phase and the solution of a much older problem." (4, p. 53)

I suggest that, by a consideration of a type of order and symmetry so basic to living matter, one may perhaps gain some insight into the problem of the origin of that order.

Editorial Comment: A mimeographed 46 entry annotated bibliography is available on request from the Fibonacci Association. Send requests directly to Brother U. Alfred, St. Mary's College, California 94575.

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CORRECTIONS: Volume 2, Number 3

In the poem "A Digit Muses" by Brother U. Alfred, page 210, we wish to say "oh pshaw, no phi!" since PHI was omitted from the end of the sixth line.

Page 204: The symbol ϕ was omitted from the numerator of the last displayed equation. The numerator is, of course, $\phi(x)$.

CORRECTIONS: Volume 2, Number 4

Page 290: Title should have π in the blank after the second word in the title and the eighth displayed equation should have a + and a - respectively in the blanks between the second and third terms and the third and fourth terms.

Page 281: Missing symbol in the first displayed equation is, of course, a summation symbol.

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FIBONACCI FANTASY: THE SQUARE ROOT OF THE Q MATRIX

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The matrix

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

has many well known fascinating properties, one being that

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

where F_n is the nth Fibonacci number. The Q matrix also has a Fibonacci square root, which can be exhibited after making a simple definition.

We extend the relationships

$$L_k = \alpha^k + \beta^k$$

$$F_k = (\alpha^k - \beta^k)/(\alpha - \beta), \quad \alpha = (1 + \sqrt{5})/2, \quad \beta = (1 - \sqrt{5})/2,$$

to allow k to equal any integral multiple of one-half. Considering odd multiples of one-half for a moment, it is easy algebraically to obtain

$$F^2_{(2n+1)/2} = \left[L_{2n+1} + 2i(-1)^{n+1}\right]/5$$

and

$$L^2_{(2n+1)/2} = L_{2n+1} + 2i(-1)^n, \quad i = \sqrt{-1},$$

directly from the extended definition. So then, all Fibonacci or Lucas numbers whose subscripts are odd multiples of one-half are complex. Also, combining the two equations directly above yields

$$L^2_{(2n+1)/2} - 5F^2_{(2n+1)/2} = 4i(-1)^n.$$

Returning to the Q matrix, a square root of Q is given by [1]

$$Q^{1/2} = \begin{bmatrix} F_{3/2} & F_{1/2} \\ F_{1/2} & F_{-1/2} \end{bmatrix} ,$$

for, by applying the extended definition and simplifying,

$$F_{3/2}^2 + F_{1/2}^2 = 1, \quad F_{1/2}^2 + F_{-1/2}^2 = 0 ,$$

and

$$F_{3/2}F_{1/2} + F_{1/2}F_{-1/2} = 1 .$$

As suggested by the second of these equalities, we can write

$$F_{-1/2} = iF_{1/2} .$$

By taking the determinant of the square root of Q,

$$i = F_{3/2}F_{-1/2} - F_{1/2}^2 .$$

Also, that

$$Q^{n/2} = \begin{bmatrix} F_{(n+2)/2} & F_{n/2} \\ F_{n/2} & F_{(n-2)/2} \end{bmatrix}$$

can be established by induction, using the extended definition and algebraic manipulation. By equating corresponding elements of equal matrices, from $(Q^{n/2})^2 = Q^n$, we obtain

$$F_{(n+2)/2}^2 + F_{n/2}^2 = F_{n+1}$$

and

$$F_{(n+2)/2}F_{n/2} + F_{n/2}F_{(n-2)/2} = F_n .$$

Taking the determinant of $Q^{n/2}$ yields

$$F_{(n+2)/2}F_{(n-2)/2} - F_{n/2}^2 = (-1)^{n/2} = i^n .$$

The reader can easily establish that

$$F_{(2n+3)/2} = F_{3/2}F_{n+1} + F_nF_{1/2} ,$$

$$F_{(2n+3)/2} = F_{(n+3)/2}F_{(n+2)/2} + F_{(n+1)/2}F_{n/2} ,$$

$$F_{2n+1} = F_{(2n+1)/2}L_{(2n+1)/2} ,$$

$$F_{(2n+1)/2} = F_{(n+3)/2}F_{n/2} + F_{(n+1)/2}F_{(n-2)/2} .$$

Let us pursue a more general result. It can be established by induction that

$$Q^{p/r} = \begin{bmatrix} F_{(p+r)/r} & F_{p/r} \\ F_{p/r} & F_{(p-r)/r} \end{bmatrix} , \quad r \neq 0 ,$$

if we further extend the definition of Fibonacci numbers to include subscripts which are rational numbers. Taking the determinant yields

$$(-1)^{p/r} = F_{(p+r)/r}F_{(p-r)/r} - F_{p/r}^2 .$$

As an example, since

$$Q^{p/r}\, Q^{r/p} = Q^{(p^2+r^2)/rp},$$

consideration of the elements of these matrices leads to

$$F_{(p^2+r^2+rp)/rp} = F_{(p+r)/r}F_{(r+p)/p} + F_{p/r}F_{r/p} ,$$

which is a general case of the familiar identity

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n .$$

In general, it seems that identities which hold for integral subscripts also hold for our specialized rational subscripts. What if the Fibonacci subscript were a complex number? J. C. Amson [2] has answered this question while demonstrating an analogy to the familiar circular and hyperbolic functions.

Amson defined modified Lucas functions as

$$\text{luc } z = (w^z - \overline{w}^z)/2\Delta, \qquad \text{coluc } z = (w^z + \overline{w}^z)/2,$$

where z is a complex number and w and $\overline{w}$ are the roots of the quadratic equation $x^2 = Px - Q$, with discriminant $\Delta^2 = P^2 - 4Q$. (Notice that luc $z = (F_z)/2$, coluc $z = (L_z)/2$ when $P = 1$, $Q = -1$.) Algebraically, we see that, among other identities,

$$Q^z \text{luc}(-z) = -\text{ luc } z$$

$$Q^z \text{coluc}(-z) = \text{coluc } z$$

$$\text{luc } 0 = 0, \quad \text{coluc } 0 = 1$$

$$\text{luc } 2z = 2 \text{ luc } z \text{ coluc } z$$

$$\text{luc}(z_1 + z_2) = \text{luc } z_1 \text{ coluc } z_2 + \text{coluc } z_1 \text{ luc } z_2$$

$$Q^z \text{ luc}(z_1 - z_2) = \text{luc } z_1 \text{ coluc } z_2 - \text{coluc } z_1 \text{ luc } z_2$$

$$\text{coluc}(z_1 + z_2) = \text{coluc } z_1 \text{ coluc } z_2 + \Delta^2 \text{luc } z_1 \text{ luc } z_2$$

$$Q^z \text{ coluc}(z_1 - z_2) = \text{coluc } z_1 \text{ coluc } z_2 - \Delta^2 \text{luc } z_1 \text{ luc } z_2$$

$$\text{coluc}^2 z + \Delta^2 \text{luc}^2 z = \text{coluc } 2z$$

$$\text{coluc}^2 z - \Delta^2 \text{luc}^2 z = Q^z$$

$$(\text{coluc } z + \Delta \text{ luc } z)^n = \text{coluc } nz + \Delta \text{ luc } nz.$$

Comparison of these Lucas functions with those derived from the circular functions defined by

$$\sin z = (e^{iz} - e^{-iz})/2i, \qquad \cos z = (e^{iz} + e^{-iz})/2$$

or those from the hyperbolic functions defined similarly by

$$\sinh z = (e^z - e^{-z})/2, \qquad \cosh z = (e^z + e^{-z})/2$$

reveals a close analogy. Also, in the special case that the quadratic equation is $x^2 = x + 1$, we see a familiar list of Fibonacci identities emerging for complex subscripts. This fine reference [2] was brought to our attention by Prof. Tyre A. Newton.

REFERENCES

1. The square root of Q was suggested by Maxey Brooke in a letter.
2. J. C. Amson, "Lucas Functions," Eureka: The Journal of the Archimedeans, (Cambridge University), No. 26, October, 1963, pp. 21-25.
3. S. L. Basin and Verner E. Hoggatt, Jr., "A Primer on the Fibonacci Sequence, Part II," Fibonacci Quarterly, 1:2, pp. 61-68.

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LETTER TO THE EDITOR

P. NAOR
The University of North Carolina
Chapel Hill, N.C.

I read with great interest your recent paper "On the Ordering of the Fibonacci Sequences." The general idea underlying your ordering procedure is excellent, but the representation can be improved and (possibly obscure) relationships may be brought to light.

Consider (for the time being) sequences for which $D \geq 11$. For reasons which will soon become clear I prefer to define the number f_o (in your notation) as the first term in the sequence, ϕ, say. You correctly pointed out that "a negative sequence may be obtained from a positive sequence by changing the signs of all terms"...; however, there is another (rather simple) operation which establishes an equivalence between two sequences. Consider a sequence

$$\ldots\ldots \phi_{-4}, \phi_{-3}, \phi_{-2}, \phi_{-1}, \phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \ldots\ldots$$

and assume, for convenience, that the monotonic portion is positive. It is easy to verify that ϕ_{-n} is positive (negative) if n is even (odd) where n is a non-negative integer. Next view an associated sequence $\{\phi'\}$ defined by

$$\phi'_{\pm n} = \phi_{\mp n} \quad \text{if } n \text{ is even}$$

$$\phi'_{\pm n} = \phi_{\mp n} \quad \text{if } n \text{ is odd.}$$

It is elementary to show that $\{\phi'\}$ is a Fibonacci sequence (with the monotonic part positive) - thus Fibonacci sequences typically appear in pairs - $\{\phi''\}$ being $\{\phi\}$ - although the possibility of a sequence being self-associated (and thus appearing to be single) cannot be ruled out a priori. Now for a series to be self-associated in the following must hold - $\phi_{-1} = +\phi_{+1}$ so that the central term ϕ_o becomes

$$\phi_o = \phi_{+1} - \phi_{-1} = 2\phi_{+1}$$

and, if we are interested only in sequences which are not integral multiples of other sequences, it becomes clear that one, and only one, such sequence exists, to wit

$$\ldots\ldots, -4,\ 3,\ -1,\ 2,\ 1,\ 3,\ 4,\ \ldots\ldots$$

whose D equals 5. Let this sequence be denoted as the ordinary self-associated sequence. However, there exists in addition an extraordinary self-associated sequence. If we admit (which we did not before) the possibility $\phi_o = 0$, we have

$$\phi_{-1} = \phi_{+1}$$

and the only "prime" solution is the Fibonacci sequence

$$\ldots\ldots, -3,\ 2,\ -1,\ 1,\ 0,\ 1,\ 1,\ 2,\ 3,\ \ldots\ldots$$

You note, of course, that in this single case f_o equals ϕ_o and not ϕ_{+1}. This sequence is indeed extraordinary in several respects: In contradistinction to all other sequences it has the property that ϕ_{-n} is positive (negative) if n is odd (even). Also $\phi_o < |\phi_{\pm 1}|$ is true in this case whereas in all other cases the inequality holds in the opposite direction. An exceptional behavior of D will be discussed in this letter.

It is then my proposal to characterize the Fibonacci sequences not by (f_o, f_1) but rather by (ϕ_{-1}, ϕ_{+1}). This representation has numerous advantages: The two mutually dual Fibonacci sequences may be represented by one pair of brackets, e.g., what you represent as (1, 4) and (2, 5) would become in my notation (-2, 1) and (-1, 2); both in one representation would be written as [2, 1] with the agreement that the larger (in absolute value) number precedes the smaller number. The ordinary self-associated (or self-dual) sequence would be [1, 1] whereas the extraordinary self-dual sequence deserves special notation, e.g. (1, 1).

Consider now the quantity D as defined in your paper

$$D = f_1^2 - f_1 f_o - f_o^2$$

In terms of ϕ_{-1} and ϕ_{+1} this becomes

$$D = \phi_{-1}^2 - 3\phi_{-1}\phi_{+1} + \phi_{+1}^2 = (\phi_{+1} - \phi_{-1})^2 - \phi_{-1}\phi_{+1} .$$

Again since for the original Fibonacci sequence ϕ_0 is differently defined (in terms of your f's) we get $D = -1$ in this case on my definition but this is not disconcerting. To my mind the original Fibonacci sequence is sufficiently extraordinary (on comparison with other such sequences) that it deserves a D with a sign different from that of the others. Inspection of the D's as presented in your paper leads me to the following conjectures (I am inclined to think that (a) it is not difficult to prove them, (b) it has been done so before - thus I have not taken the trouble).

Let prime numbers of the form $10n \pm 1$ be represented by g_k. We have then

(1) The set of feasible D's is made up of -1, 5, all g_k's, all products of g_k's which we denote by Q_m (i.e. $Q_m = g_i^a g_j^b \ldots g_k^c$), all numbers of the form $5g_k$, and all numbers of the form $5Q_m$. In other words a necessary condition for an integer to be a D is that it belongs to the set $\{-1, 5, g_k, Q_m, 5g_k, 5Q_m\}$.

(2) Each number in the above set may indeed be found in the list of D's. In other words, the above is also a sufficient condition.

(3) The number of sequences associated with a given value of D is simply related to its factorization properties. I reserve a final formulation of my conjecture on this part until I have seen more "experimental material," i.e., a table of D's (with associated sequences) between 1000 and 2000. It is already obvious that for -1 and 5 we get the self-dual sequences and for each g_k and $5g_k$ we have one pair of dual sequences. As for a Q_m it is obvious that if it equals g_k^a $(a > 1)$ we have again one associated pair, but for the case $Q_m = g_i^a g_j^b \ldots a_k^c$ the number of associated pairs is a function of the degree of "compositeness" and this should be looked into a little more carefully by means of an extended Table. Finally, the number of pairs of Fibonacci sequences associated with $5Q_m$ is identical with the number of pairs associated with Q_m.

If you are aware of literature relating to these conjectures, kindly let me know. Also if you have an extended table of the D's I should appreciate a copy.

I hope some of my remarks may have been of use for ordering and classification purposes.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. HILLMAN
University of Santa Clara, Santa Clara, California

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. Any problem believed to be new in the area of recurrent sequences and any new approaches to existing problems will be welcomed. The proposer should submit each problem with solution in legible form, preferably typed in double spacing with name and address of the proposer as a heading.

Solutions to problems listed below should be submitted on separate signed sheets within two months of publication.

B-58 *Proposed by Sidney Kravitz, Dover, New Jersey*

Show that no Fibonacci number other than 1, 2, or 3 is equal to a Lucas number.

B-59 *Proposed by Brother U. Alfred, St. Mary's College, California*

Show that the volume of a truncated right circular cone of slant height F_n with F_{n-1} and F_{n+1} the diameters of the bases is

$$\sqrt{3}\,\pi(F_{n+1}^3 - F_{n-1}^3)/24 .$$

B-60 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $L_{2n}L_{2n+2} - 5F_{2n+1}^2 = 1$, where F_n and L_n are the n-th Fibonacci number and Lucas number, respectively.

B-61 *Proposed by J.A.H. Hunter, Toronto, Ontario*

Define a sequence $U_1, U_2, \ldots$ by $U_1 = 3$ and

$$U_n = U_{n-1} + n^2 + n + 1 \quad \text{for} \quad n > 1 .$$

Prove that $U_n \equiv 0 \pmod{n}$ if $n \not\equiv 0 \pmod{3}$.

B-62 *Proposed by Brother U. Alfred, St. Mary's College, California*

Prove that a Fibonacci number with odd subscript cannot be represented as the sum of squares of two Fibonacci numbers in more than one way.

B-63 *An old problem whose source is unknown, suggested by Sidney Kravitz, Dover, New Jersey*

In ΔABC let sides AB and AC be equal. Let there be a point D on side AB such that AD = CD = BC. Show that

$$2\cos \measuredangle A = AB/BC = (1 + \sqrt{5})/2 \quad ,$$

the golden mean.

SOLUTIONS

A BOUND ON BOUNDED FIBONACCI NUMBERS

B-44 *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that for every positive integer k there are no more than n Fibonacci numbers between n^k and n^{k+1}.

Solution by the proposer.

Assume the maximum,

$$(1) \qquad n^k < F_{r+1},\ F_{r+2},\ \ldots,\ F_{r+n} < n^{k+1} \quad .$$

Now

$$\sum_{j=1}^{n-1} F_{r+j} = \sum_{j=1}^{r+n-1} F_j - \sum_{j=1}^{r} F_j$$

$$= F_{r+n+1} - F_{r+2} \quad .$$

But by (1),

$$\sum_{j=1}^{n-1} F_{r+j} + F_{r+2} > n \cdot n^k$$

and hence

$$F_{r+n+1} > n^{k+1} \quad ,$$

thus proving the proposition.

ANOTHER SUM

B-45 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Let H_n be the n-th generalized Fibonacci number, i.e., let H_1 and H_2 be arbitrary and $H_{n+2} = H_{n+1} + H_n$ for $n > 0$. Show that $nH_1 + (n-1)H_2 + (n-2)H_3 + \ldots + H_n = H_{n+4} - (n+2)H_2 - H_1$.

Solution by David Zeitlin, Minneapolis, Minnesota.

In B-20 (see Fibonacci Quarterly, 2(1964) p. 77), it was shown that

$$\sum_{j=1}^{n} H_j = H_{n+2} - H_2 \quad .$$

In B-40 (see Fibonacci Quarterly, 2(1964), p. 155), Wall proposed that

$$\sum_{j=1}^{n} jH_j = (n+1)H_{n+2} - H_{n+4} + H_1 + H_2 \quad .$$

Thus, the desired sum

$$\begin{aligned}\sum_{j=1}^{n} [(n+1) - j] H_j &= (n+1) \sum_{j=1}^{n} H_j - \sum_{j=1}^{n} jH_j \\ &= [(n+1)H_{n+2} - (n+1)H_2] - [(n+1)H_{n+2} - H_{n+4} + H_1 + H_2] \\ &= H_{n+4} - (n+2)H_2 - H_1 \quad .\end{aligned}$$

Also solved by Douglas Lind, Kenneth E. Newcomer, Farid K. Shuayto, Sheryl B. Tadlock, Howard L. Walton, Charles Zeigenfus, and the proposer.

A CONTINUANT DETERMINANT

B-46 *Proposed by C.A. Church, Jr., Duke University, Durham, North Carolina*

Evaluate the n-th order determinant

$$D_n = \begin{vmatrix} a+b & ab & 0 & 0 & \cdots \\ 1 & a+b & ab & 0 & \cdots \\ 0 & 1 & a+b & ab & \cdots \\ 0 & 0 & 1 & a+b & \cdots \\ \cdots & & & & \\ \cdots & & & & \\ \cdots & & & & \end{vmatrix} .$$

Solution by F.D. Parker, SUNY, Buffalo, N.Y.

We denote the value of the determinant of order n by D(n), and notice that $D(1) = a + b$ and $D(2) = a^2 + ab + b^2$. Expanding D(n) by the first row, we see that

$$D(n) = (a + b)\ D(n-1) - ab\ D(n-2) \quad .$$

This is a homogeneous linear second order difference equation; if $a \neq b$, the solution which fits the initial conditions is

$$D(n) = (a^{n+1} - b^{n+1})/(a-b) \quad .$$

If $a = b$, the solution which fits the initial conditions is $D(n) = (1 + n)\ a^n$.

Also solved by Joel L. Brenner, Douglas Lind, C.B.A. Peck, David Zeitlin, and the proposer.

Lind, Peck, and Zeitlin pointed out that B-46 is a special case of B-13. Peck also noted that B-46 is an example of a class of continuants mentioned by J. J. Sylvester in the Philosophical Magazine, Series 4, 5 (1853)446-457. (See T. Muir, History of the Theory of Determinants (Dover) Vol. I, p. 418.) Brenner noted that B-46 and similar problems occur as Nos. 217, 225, 234, etc. in Faddeev and Sominski, Problems in Higher Algebra, a translation of which will soon be published by W. H. Freeman.

CONSECUTIVE COMPOSITE FIBONACCI NUMBERS

B-47 *Proposed by Barry Litvack, University of Michigan, Ann Arbor, Michigan*

Prove that for every positive integer k there are k consecutive Fibonacci numbers each of which is composite.

Solution by Sidney Kravitz, Dover, New Jersey

Let F_n be the n-th Fibonacci number. We note that $F_n > 1$ for $n > 2$, that F_j divides F_{mj} and that j is a divisor of $(k+2)! + j$ for $3 \leq j \leq k+2$. Thus the k consecutive Fibonacci numbers

$$F_{(k+2)!+3},\ F_{(k+2)!+4},\ \ldots,\ F_{(k+2)!+k+2}$$

are divisible by F_3, F_4, ..., F_{k+2} respectively.

Also solved by R.W. Castown, Douglas Lind, F.D. Parker, and the proposer.

A BINOMIAL EXPANSION

B-48 *Proposed by H.H. Ferns, University of Victoria, Victoria, British Columbia, Canada*

Prove that

$$\sum_{k=1}^{r-1} (-2)^k \binom{r}{k} F_k = \begin{cases} -2^r F_r & \text{if } r \text{ is an even positive integer} \\ 2^r F_r - 2(5)^{(r-1)/2} & \text{if } r \text{ is an odd positive integer,} \end{cases}$$

where $F_{n+2} = F_{n+1} + F_n (F_1 = F_2 = 1)$ and find the corresponding sum in which the F_k are replaced by the Lucas numbers L_k.

Solution by D.G. Mead, University of Santa Clara, Santa Clara, California

Let S be the given sum. By the Binet formula,

$$F_n = (a^n - b^n)/(a-b)$$

where $a = (1 + \sqrt{5})/2$ and $b = 1 - a$. Then $a - b = \sqrt{5} = 1 - 2a = 2b - 1$, and

$$S + (-2)^r F_r = \sum_{k=0}^{r} \binom{r}{k} (-2)^k F_k$$

$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{r} \binom{r}{k} \left[(-2a)^k - (-2b)^k\right]$$

$$= \frac{(1 - 2a)^r - (1 - 2b)^r}{\sqrt{5}}$$

$$= \frac{(\sqrt{5})^r \left[1 - (-1)^r\right]}{\sqrt{5}} \quad .$$

The desired conclusion follows immediately.

Similarly one sees from $L_n = a^n + b^n$ that the corresponding sum for the Lucas numbers is $-2 - 2^r L_r + 2(\sqrt{5})^r$ for r even and $-2 + 2^r L_r$ for r odd.

Also solved by the proposer.

AN ALPHAMETIC

B-49 *Proposed by Anton Glaser, Pennsylvania State University, Abington, Pennsylvania*

Let ϕ represent the <u>letter</u> "oh".	TWϕ
Given that T, W, ϕ, L, V, P, and TWϕ are	IS
Fibonacci numbers, solve the cryptarithm	THE
in the base 14, introducing the digits	ϕNLY
α, β, γ, and δ in base 14 for 10, 11,	EVEN
12, and 13 in base 10.	$\overline{\text{PRIME}}$

Solution by Charles Ziegenfus, Madison College, Harrisonburg, Virginia

With a little calculation one observes that the Fibonacci number corresponding to TWϕ is 2584. Thus, $T = \delta$, $W = 2$, $\phi = 8$, $P = 1$, $L = 3$ or 5, and $V = 3$ or 5. Next we note that $8 + E + (2 \text{ or } 3) = 1R$, so that $E = 4 + R$ or $E = 3 + R$. Tabulating these results:

R	0	6	7	4	6	7	9
E	4	α	β	7	9	α	γ

Further, $8 + S + E + Y + N = kE$ or $S + Y + N = k0 - 8 = 6$, 16, or 26 in base 14. There are no possible choices for S, Y, N such that $S + Y + N = 6$ or 26. Thus S, Y, N can be chosen from $\{0, 9, \beta\}$; $\{4, 7, 9\}$; $\{4, 6, \alpha\}$. Tabulating this with the previous result we obtain:

R	6	7	4	7	9
E	α	β	7	α	γ
S, Y, N	4, 7, 9 or 0, 9, β	4, 6, α	0, 9, β	0, 9, β	4, 6, α

Further,

$$T + W + \phi + I + S + T + H + E + \phi + N + L + Y + E + V + E + N$$

$$- P - R - I - M - E \text{ is a multiple of } \delta.$$

We reduce the above to $6 + (2 \cdot E - R) + H + N - M = \delta \cdot k$.

On substituting the possible values for R and E we further reduce this problem to the following cases:

a. $R = 6$ and $E = \alpha$, $7 + N + H - M = \delta \cdot k$.

b. $R = 7$ and $E = \beta$, $8 + N + H - M = \delta \cdot k$.

c. $R = 4$ and $E = 7$, $3 + N + H - M = \delta \cdot k$.

d. $R = 7$ and $E = \alpha$, $6 + N + H - M = \delta \cdot k$.

e. $R = 9$ and $E = \gamma$, $8 + H + N - M = \delta \cdot k$.

From the previous table we observe that there are exactly three choices for N. Using these in the above cases reduces the problem to an equation involving only H and M and only three choices for these. Thus we obtain two distinct solutions (actually four since S and Y can be interchanged).

0	1	2	3	4	5	6	7	8	9	α	β	γ	δ
S or Y	P	W	V	H	L	R	M	ϕ	N	E	S or Y	I	T
M	P	W	V	S or Y	L	R	N	ϕ	S or Y	E	I	H	T

Also solved by the proposer and partially solved by J.A.H. Hunter.

AND ANOTHER SUM

B-50 *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that

$$\sum_{j=0}^{n} \left[2F_j^2 - \binom{n}{j}F_j\right] = F_n^2 .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Since

$$\sum_{j=0}^{n} F_j^2 = F_n F_{n+1},$$

$$\sum_{j=0}^{n} \binom{n}{j} F_j = F_{2n} = F_n L_n = F_n(F_{n+1} + F_{n-1}),$$

the desired sum is

$$2F_nF_{n+1} - F_nF_{n+1} - F_nF_{n-1} = F_n(F_{n+1} - F_{n-1}) = F_n^2 .$$

Also solved by H.H. Ferns, Farid K. Shuayto, Sheryl B. Tadlock, and the proposer.

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