recursion relation (6) yields the other set of alternate Fibonacci numbers as the sequence of cumulative sums, the total particle count.

5. CONCLUDING REMARKS

One is directed to advanced problem H-50 December 1964, Fibonacci Quarterly, for the partitioning interpretation of the integer n of the model for $\phi(t) = kt$.

Suppose one defines two sets of Morgan-Voyce polynomials

$$b_0(x) = 1,\ b_1(x) = 1 + x;\ B_0(x) = 1,\ B_1(x) = 2 + x \quad ,$$

both sets satisfying

$$P_{n+2}(x) = (x + 2)\,P_{n+1}(x) - P_n(x), \quad n \geq 0 \quad . \tag{7}$$

It is easy to establish that

$$P_n(k) = \Delta_n = k\,B_{n-1}(k)$$

$$T_n(k) = \Delta_0 + \Delta_1 + \ldots + \Delta_n = b_n(k) \quad .$$

Thus for $k = 1$, we again find $B_{n-1}(1) = F_{2n}$ and $b_n(1) = F_{2n+1}$. See corrected problem B-26 with solution by Douglas Lind in the Elementary Problem Section of this issue, where the binomial coefficient relation mentioned in the note of Section 3 is shown. A future paper by Prof. M. N. S. Swamy dealing extensively with Morgan-Voyce polynomials will appear in an early issue of the Fibonacci Quarterly.

Acknowledgment: The author is completely indebted to Dr. V. E. Hoggatt, Jr., for bringing to his attention the theorem and its proof.

Additional references to work along the lines of generated compositions — some of which yield numbers with Fibonacci properties — will be found in the references at the end of this paper. (See note, page 94)

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H. Winthrop "A Theory of Behavioral Diffusion" A contribution to the Mathematical Biology of Social Phenomena. Unpublished thesis submitted to the Faculty of The New School for Social Research, 1953.

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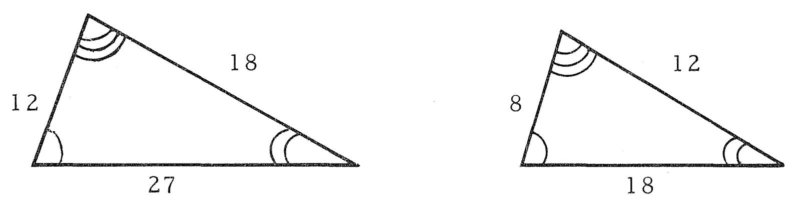
MYSTERY PUZZLER AND PHI

MARVIN H. HOLT
Wayzata, Minnesota

A problem proposed by Professor Hoggatt is as follows: Does there exist a pair of triangles which have five of their six parts equal but which are not congruent? (Here the six parts are the three sides and the three angles.) The initial impulsive answer is no! The problem also appears in [1] as well as in the MATH LOG.

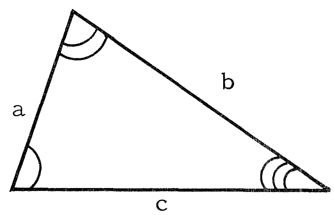
I have taken some time to work on the problem you suggested. I think you will agree that the solution I have is interesting. One problem, as you have stated it, is posed in a high school geometry text entitled, "Geometry" by Moise and Downs, published by Addison Wesley Company, (page 369).

In their solution key, they gave one possible pair of triangles that work:



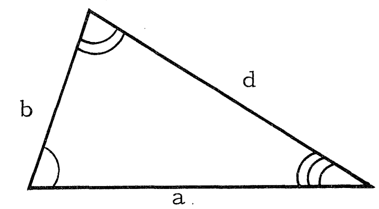
I discovered this after I solved the problem myself. But the above solution does not do justice to the problem at all, since my old friend τ is really the key to the solution. Note: Golden Mean = ϕ = τ in what follows.

I attacked the problem as follows: First, the five congruent parts cannot contain all three sides, since the triangles would then be congruent. Therefore, the five parts must be three angles and two sides which means that the two triangles are similar. But, the two sides cannot be in corresponding order, or the triangles would be congruent either by ASA or SAS. So, the situation must be one of two possibilities as I have sketched below: (My sketches are not to scale.)

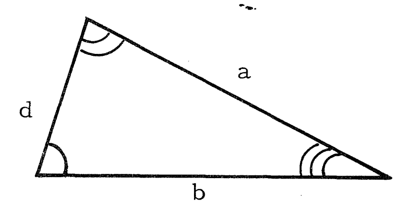


triangle 1

triangle 2



possibility 1



possibility 2

In both cases, by using relationships from similar triangles, it follows that $\frac{a}{b} = \frac{b}{c}$ or $b = ka$ and $c = kb = k^2a$ from possibility 2 and $\frac{a}{b} = \frac{b}{d}$ or $b = ka$ and $d = kb = k^2a$ from possibility 1.

So, the three sides of the triangle must be three consecutive members of a geometric series: a, ak, ak^2, where k is a proportionality constant and $k > 0$ and $k \neq 1$. If $k = 1$, the triangles would both be equilateral and thus congruent. Therefore, $k \neq 1$.

From my previous article on the Golden Section (Pentagon, Spring 1964) I worked out two problems on right triangles where the sides formed a geometric progression and the constants turned out to be $\sqrt{\tau}$ and $\sqrt{\frac{1}{\tau}}$. So, I knew of two more situations where the original problem could be solved. Then I began to consider various other values of k and I began to wonder what values of "k" will work. In other words, for what values of k will the numbers a, ak, and ak^2 be sides of a triangle. Once we know this, then another triangle with sides $\frac{a}{k}$, a, ak or ak, ak^2, ak^3 will have five parts congruent but the triangles would not be congruent.

In order for a, ak and ak^2 to be sides of a triangle, three statements must be true:

These are instances of the strict triangle inequality.

1. $a + ak > ak^2$ $(a + b > c)$

2. $a + ak^2 > ak$ $(a + c > b)$

3. $ak + ak^2 > a$ $(b + c > a)$

$[a > 0, \quad k > 0, \quad k \neq 1]$

For Case 1, consider $k > 1$

(a) $k > 1 \to k^2 > k \to 1 + k^2 > k$

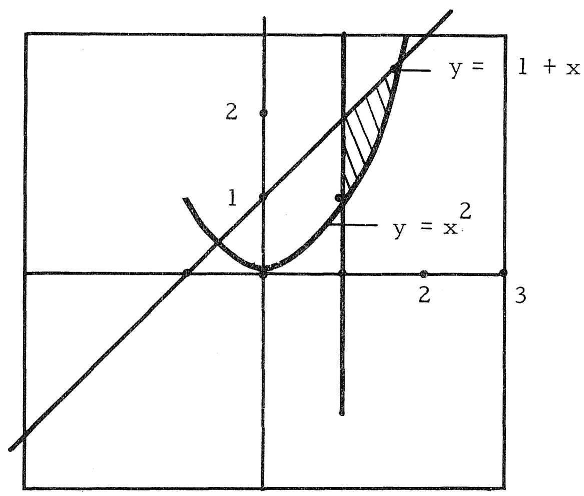
therefore, $a + ak^2 > ak$ (condition 2 above)

(b) $k > 1 \to k + 1 > 1 \to k^2 + k > 1$

therefore, $ak^2 + ak > a$ (condition 3 above)

(c) if $k > 1$ show $a + ak > ak^2$ (condition 1 above).

This part revolves around the problem of finding out when $1 + k > k^2$, or, graphically: For what $x > 1$ will $\underline{1 + x = y}$ be above $\underline{y = x^2}$?



Solving this problem produces the result that

$$k < \frac{1 + \sqrt{5}}{2} \quad \underline{\text{or}} \quad k < r:$$

So, if $1 < k < r$ then the numbers a, ak, ak^2 are the sides of the triangle that can be matched with $\frac{a}{k}$, a, ak or ak, ak^2, ak^3 to solve the original problem. (Incidentally: $1 < \sqrt{r} < r$. So this fits in here.)

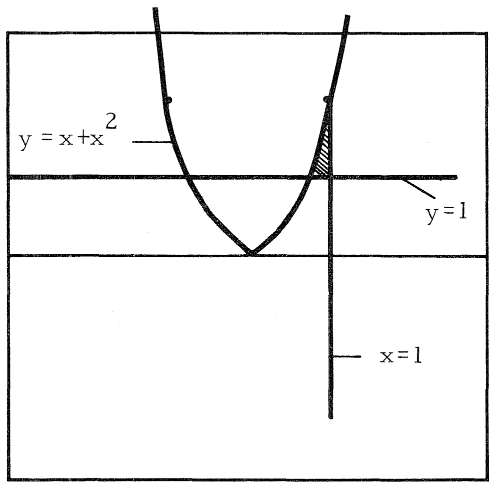
For Case 2, consider $k < 1$

(a) if $k < 1 \rightarrow k^2 < k \rightarrow k^2 < k + 1$ Therefore $ak^2 < ak + a$ (condition 1)

(b) if $k < 1 \rightarrow 1 + k > 1 \rightarrow a + ak^2 > ak$ (condition 2)

(c) Now, if $k < 1$ show $ak + ak^2 > a$. This is, essentially, finding what values of k make $k + k^2 > 1$.

Again, graphically, for what $x < 1$ will the parabola $y = x + x^2$ be above the line $y = 1$?



Solving this problem produces the result that $k > \frac{-1 + \sqrt{5}}{2}$. If you will follow this closely, $\frac{-1 + \sqrt{5}}{2}$ is the additive inverse of the conjugate of r. (i.e., $r = \frac{1 + \sqrt{5}}{2}$. Therefore, the conjugate of r is $\frac{1 - \sqrt{5}}{2}$ and its additive inverse is $\frac{-1 + \sqrt{5}}{2}$.) So, if $\frac{-1 + \sqrt{5}}{2} < k < 1$ the problem is again solved. (Again, $\frac{-1 + \sqrt{5}}{2} < \sqrt{\frac{1}{r}} < 1$, so my second problem fits here.)

Therefore, the complete solution can be summed up as follows, if k is a number such that $1 < k < \frac{1 + \sqrt{5}}{2} = r$ or $\frac{-1 + \sqrt{5}}{2} < k < 1$. Then the three sets of triangles with sides $\frac{a}{b}$, a, ak or a, ak, ak^2 or ak, ak^2, or ak^3 can be used to produce two triangles with five parts equal and the triangles themselves not congruent.

So, there are an infinite number of pairs of triangles that solve this problem and once again, r proves to be an interesting number and a key to the solution of interesting problems.

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1. Moise and Downs, Geometry, Addison-Wesley, p. 369.

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LADDER NETWORK ANALYSIS USING POLYNOMIALS

JOSEPH ARKIN
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In this paper we develop some ideas with the recurring series

(1) $$B_n = k_1 B_{n-1} + k_2 B_{n-2}, \quad B_0 = 1, \ (k_1 \text{ and } k_2 \neq 0) \quad ,$$

and show a relationship between this sequence and the simple network of resistors known as a ladder-network.

The ladder-network in Figure 1 is an important network in communication systems. The m-L sections in cascade that make up this network can be characterized by defining:

(2) a) the attenuation (input voltage/output voltage) = A,

b) the output impedance = z_0,

c) the input impedance = z_1.

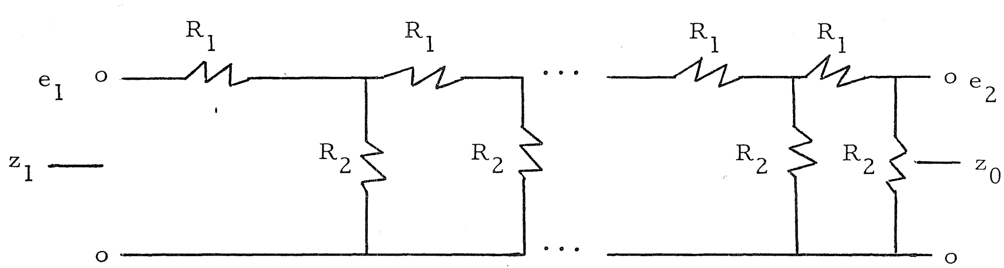

Figure 1

A result obtained by applying Kirchhoff's and Ohm's Laws to ladder-networks with $m = 1, 2, 3, \ldots, R_1 = R_2 k_1$, was tabulated with the results in Table 1, where setting $k_1 = 1$, $R_2 = 1$ ohm, the network in Figure 1 was analyzed by inspection [1].

m	z_0	A	z_1
1	R_2	(k_1+1)	$(k_1+1)R_2$
2	$\left(\frac{k_1+1}{k+2}\right)R_2$	$(k_1^2+3k_1+1)$	$\left(\frac{k_1^2+3k_1+1}{k_1+2}\right)R_2$
3	$\left(\frac{k_1^2+3k_1+1}{k_1^2+4k_1+3}\right)R_2$	$(k_1^3+5k_1^2+6k_1+1)$	$\left(\frac{k_1^3+5k_1^2+6k_1+1}{k_1^2+4k_1+3}\right)R_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1

We observe that the nth row in Table 1, may be written

m	z_0	A	z_1
n	$(C_{2n-2}/y_{2n-1})R_2$	C_{2n}	$(C_{2n}/y_{2n-1})R_2$

where,

(3) a) $C_n = k_1^{1/2}C_{n-1} + C_{n-2},\ C_0 = 1$,

b) $y_n = k_1^{1/2}y_{n-1} + y_{n-2},\ y_0 = 1/k_1^{1/2}$.

It then remains to solve for y_n and C_n in (3), to be able to analyze (Figure 1) by inspection for any value of k_1 ($k_1 \neq 0$), where $R_2 = 1$ ohm. So that, in (1), we let

(4) a) $w = (k_1 + (k_1^2 + 4k_2)^{1/2})/2$,

b) $v = (k_1 - (k^2 + 4k_2)^{1/2})/2$,

where it is evident,

c) $k_1 = w + v$,

and

d) $k_2 = -wv$.

Then, combining (c) and (d) with (1), leads to

$$B_n = ((w^2 - v^2)B_{n-1} - wv(w-v)B_{n-2})/(w-v) \quad , \tag{5}$$

$$B_n = ((w^3-v^3)B_{n-2} - wv(w^2-v^2)B_{n-3})/(w-v) \quad ,$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \quad ,$$

$$B_n = ((w^n - v^n)(w+v) - wv(w^{n-1} - v^{n-1})B_0)/(w-v) \quad ,$$

and we have

(6) $$B_n = \frac{w^{n+1} - v^{n+1}}{w - v} .$$

Where, in (1) we replace k_1 with $k_1^{1/2}$ and k_2 with 1, and combining this result with (3) and (6), leads to

(7) a) $$C_n = \frac{(k_1^{1/2} + (k_1+4)^{1/2})^{n+1} - (k_1^{1/2} - (k_1+4)^{1/2})^{n+1}}{((k_1+4)^{1/2})2^{n+1}} = \phi(k_1),$$

and

b) $$y_n = \phi(k_1)/k_1^{1/2} .$$

(8) Theorem.

The attenuation (input voltage/output voltage = A) of m-L sections in cascade in a ladder-network is given by

$$A^2 = \sum_{r=0}^{2m-2} C_r((-C_{2m-1})/C_{2m-2})^r) .$$

The proof of the theorem rests on the following

(9) Lemma.

The power series

$$(-1)^n \sum_{r=0}^{n} B_r x^r ,$$

is always a square, where B_r is defined in (1).

Proof of lemma.

Let

(10) $$1 = (1 - k_1 x - k_2 x^2)(\sum_{r=0}^{n} B_r x^r) ,$$

then, by comparing coefficients and by (1), we have

(11) $$x = \frac{-(B_n k_1 + B_{n-1} k_2)}{B_n k_2} = \frac{-B_{n+1}}{B_n k_2} ,$$

and replacing x with $(-B_{n+1})/(B_n k_2)$ in $(1-k_1x-k_2x^2)$, leads to

(12) $$1-k_1x-k_2x^2 = (B_n^2 k_2 + B_n B_{n+1} k_1 - B_{n+1}^2)/(B_n^2 k_2) .$$

By (4, d) and (6) it is easily verified

(13) $$B_n^2 - B_{n+1}B_{n-1} = (-k_2)^n ,$$

so that

(14) $$B_n^2 k_2 + B_n B_{n+1} k_1 - B_{n+1}^2 = (-1)^n k_2^{n+1} .$$

Then, replacing the numerator in (12) by the result in (14) leads to

(15) $$1 - k_1 x - k_2 x^2 = ((-1)^n k_2^n)/B_n^2 ,$$

so that (10) may be written as

(16) $$(-1)^n B_n^2 = \sum_{r=0}^{n} B_r x^r ,$$

which completes the proof of the lemma.

(17) The proof of the theorem is immediate, when in (11) and (16), we replace n with 2m-2, k_1 with $k_1^{1/2}$, k_2 with 1, and combine the result with (7, a) and the values of the attenuation in Table 1.

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1. a) S. L. Basin, "The Appearance of Fibonacci Numbers and the Q Matrix in Electrical Network Theory," Math Mag., 36(1963) pp. 84-97.

b) S. L. Basin, "The Fibonacci Sequence as it Appears in Nature," Fibonacci Quarterly, 1(1963) pp. 54-55.

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CONCERNING LATTICE PATHS AND FIBONACCI NUMBERS

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R. E. Greenwood [1] has investigated plane lattice paths from (0, 0) to (n, n) and has found a relationship between the number of paths in a certain restricted subclass of such paths and the Fibonacci sequence. Considering such paths and using a method of enumeration different from that used by Greenwood, an unusual representation of Fibonacci's sequence is suggested.

The paths considered here are comprised of steps of three types: (i) horizontal from (x, y) to (x + 1, y); (ii) vertical from (x, y) to (x, y + 1); and (iii) diagonal from (x, y) to (x + 1, y + 1).

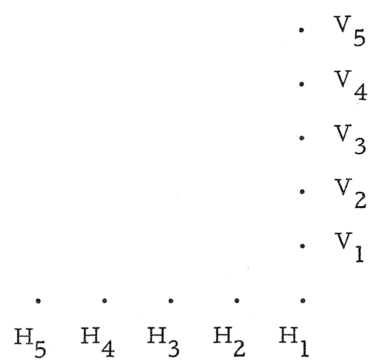

Figure 1

In the interest of simplicity of representation, we will here consider the paths from H_i to V_i, for each positive integer i. Note that the number of paths from H_i to V_i is the number of paths from (0, 0) to (i, i). However, instead of considering the total number of paths from H_i to V_i as was done by Greenwood, we will count only the number of paths from H_i to V_i which do not contain as subpaths any of the paths from H_j to V_j, for $j < i$. This number plus the number of paths from H_{i-1} to V_{i-1} is the total number of paths from H_i to V_i. The use of this counting device suggest the

Theorem:

Let

$$1_D = 1$$

$$2_D = \left[\frac{D-1}{2}\right], \text{ where } [\ ] \text{ denotes the greatest integer function}$$

$$3_D = 3_{D-1} + 2_{D-1}$$

$$4_D = 4_{D-2} + 3_{D-2}$$

$$\cdots$$

$$(2n)_D = (2n)_{D-2} + (2n-1)_{D-2}$$

$$(2n+1)_D = (2n+1)_{D-1} + (2n)_{D-1}$$

$$\cdots$$

with the restriction that $k_D = 0$ if $k > D$. For each positive integer D, let

$$f(D) = \sum_{k=1}^{D} k_D \quad .$$

The sequence $\{f(D) \mid D = 1, 2, 3, \ldots\}$ is the Fibonacci sequence.

The proof is direct and is therefore omitted.

The geometric interpretation of the numbers k_D and f(D) mentioned in the theorem is interesting. However, before considering this interpretation it is necessary to define a section of a path. For this purpose we will now consider a path as the point set to which p belongs if and only if for some step ((x, y), (u, v)) of the path, p belongs to the line interval whose end points are (x, y) and (u, v). A section of a path is a line interval which is a subset of the path and which is not a subset of any other line interval each of whose points is a point of the path.

The above mentioned geometric interpretation follows: By definition f(1) = 1. For each positive integer $D \geq 2$, let L_D denote the set of paths from H_D to V_D which do not contain as subpaths any of the paths from H_j to V_j, for $j < D$. f(D) is the number of paths belonging to the set L_D. k_D is the number of paths in the subset X of L_D such that x belongs to X if and only if x contains as subsets exactly k diagonal sections.

Figure 2 portrays the five paths which belong to L_5. In Figure 2a appears the one path of L_5 which contains only one diagonal section ($1_5 = 1$). The two paths of L_5 which contain exactly two diagonal sections appear in Figure 2b ($2_5 = 2$). In Figure 2c the two paths of L_5 which contain exactly three diagonal sections are shown ($3_5 = 2$). It is noted that $4_5 = 5_5 = 0$.

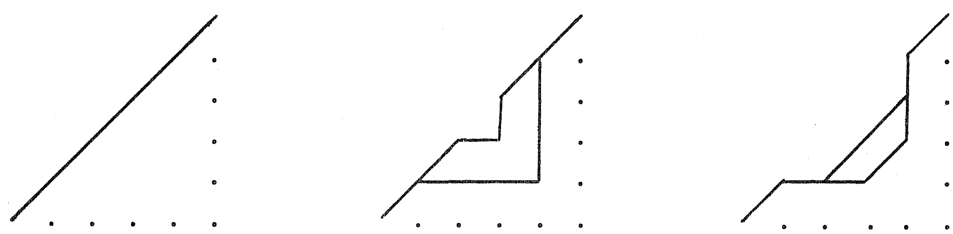

Fig. 2a $1_5 = 1$ | Fig. 2b $2_5 = 2$ | Fig. 2c $3_5 = 2$

$$f(5) = 1 + 2 + 2 + 0 + 0 = 5$$

Figure 2

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1. R. E. Greenwood, "Lattice Paths and Fibonacci Numbers," The Fibonacci Quarterly, Vol. 2, No. 1, pp. 13-14.

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REPLY TO EXPLORING FIBONACCI MAGIC SQUARES*

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Problem. For $n \geq 2$, show that there do not exist any nxn magic squares with distinct entries chosen from the set of Fibonacci numbers, $u_1 = 1$, $u_2 = 2$, $u_{n+2} = u_{n+1} + u_n$ for $n \geq 1$.

Proof. Trivial for $n = 2$.

If an nxn magic square existed for some $n \geq 3$ with distinct Fibonacci entries, then the requirement that the first three columns add to the same number would yield the equalities:

$$(*)\quad F_{i_1} + F_{i_2} + \ldots + F_{i_n} = F_{j_1} + F_{j_2} + \ldots + F_{j_n} = F_{k_1} + F_{k_2} + \ldots + F_{k_n} .$$

Since the entries are distinct, we may assume without loss of generality that $F_{i_1} > F_{i_2} > \ldots > F_{i_n}$, $F_{j_1} > F_{j_2} > \ldots > F_{j_n}$ and $F_{k_1} > F_{k_2} > \ldots > F_{k_n}$.

Noting that the columns contain no common elements, and by rearrangement if necessary, we assume $F_{i_1} > F_{j_1} > F_{k_1}$, again without losing generality; thus, $F_{i_1} \geq F_{k_1} + 2$.

Now

$$F_{i_1} + F_{i_2} + \ldots + F_{i_n} > F_{i_1} \geq F_{k_1+2} ,$$

while

$$F_{k_1} + F_{k_2} + \ldots + F_{k_n} \leq \sum_{1}^{k_1} F_i = F_{k_1+2} - 1 .$$

This contradicts the equality postulated in (*), and we conclude no magic squares in distinct Fibonacci numbers are possible.

*The Fibonacci Quarterly, October 1964, Page 216.

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THE FIBONACCI NUMBER F_u WHERE u IS NOT AN INTEGER

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INTRODUCTION

Fibonacci numbers, like factorials, are not naturally defined for any values except integer values. However the gamma function extends the concept of factorial to numbers that are not integers. Thus we find that $(1/2)! = \sqrt{\pi}/2$. This article develops a function which will give F_n for any integer n but which will furthermore give F_u for any rational number u. The article also defines a quantity $n\not\Delta^m$ and develops a function $f(x, y) = x\not\Delta^y$ where x and y need not be integers.

(1) DEFINITIONS

Let $n\not\Delta^0 = 1$ (Definitions (1) hold for all $n \in N$)

Let

$$n\not\Delta^1 \text{ (read "n cardinal") } = \sum_{k=1}^{n} k\not\Delta^0 = \sum_{k=1}^{n} 1 = n \quad .$$

This gives the cardinal numbers 1, 2, 3, ...

Let

$$n\not\Delta^2 \text{ (read "n triangular") } = \sum_{k=1}^{n} k\not\Delta^1 = \sum_{k=1}^{n} k \quad .$$

This gives the triangular numbers 1, 3, 6, 10, ...

Let

$$n\not\Delta^3 \text{ (read "n tetrahedral") } = \sum_{k=1}^{n} k\not\Delta^2 \quad .$$

This gives the tetrahedral numbers 1, 4, 10, 20, ...

In general, let

$$n\not\Delta^m \text{ (read "n delta-slash m") } = \sum_{k=1}^{n} k\not\Delta^{m-1} \quad .$$

This gives a figurate number series which can be assigned to the m-dimensional analog of the tetrahedron (which is the 3-dimensional analog of the triangle, etc.).

Let us construct an array $(a_{i,j})$, where we assign to each $a_{i,j}$ an appropriate coefficient of Pascal's triangle.

$$(a_{i,j}) = \begin{matrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \vdots & & & & & \end{matrix}$$

It is clear that in this arrangement the usual rule for forming Pascal's triangle is just

(2) $$a_{i,j} = a_{i,j-1} + a_{i-1,j} \quad .$$

But a comparison of this rule with the definitions (1) shows that Pascal's triangle can be written:

$$\begin{matrix} 1\Delta^0 & 1\Delta^1 & 1\Delta^2 & \cdots & 1\Delta^m & \cdots \\ 2\Delta^0 & 2\Delta^1 & 2\Delta^2 & \cdots & 2\Delta^m & \cdots \\ 3\Delta^0 & 3\Delta^1 & 3\Delta^2 & \cdots & 3\Delta^m & \cdots \\ \vdots & & & \cdots & & \cdots \\ n\Delta^0 & n\Delta^1 & n\Delta^2 & \cdots & n\Delta^m & \cdots \\ \vdots & & & & & \end{matrix}$$

where $a_{i,j} = i\Delta^{j-1}$. From the symmetry of Pascal's triangle, $a_{i,j} = a_{j,i}$. Therefore

(3) $$i\Delta^{j-i} = j\Delta^{i-1}; \quad n\Delta^m = (m+1)\Delta^{n-1} \quad .$$

Pascal's triangle is a well-known generator of Fibonacci numbers in the way shown in the following diagram.

$$
\begin{array}{cccccccl}
1 & 1 & 1 & 1 & 1 & \cdots & & 1 = 1 = F_1 \\
1 & 2 & 3 & 4 & 5 & \cdots & & 1 = 1 = F_2 \\
1 & 3 & 6 & 10 & 15 & \cdots & & 1+1 = 2 = F_3 \\
1 & 4 & 10 & 20 & 35 & \cdots & & 1+2 = 3 = F_4 \\
1 & 5 & 15 & 35 & 70 & \cdots & & 1+3+1 = 5 = F_5 \\
\vdots & & & & & & & \vdots
\end{array}
$$

We can apply the same course to our abstracted Pascal's triangle.

$$
\begin{array}{ccccccl}
1\not\Delta^0 & 1\not\Delta^1 & 1\not\Delta^2 & \cdots & & & F_1 = 1\not\Delta^0 \\
2\not\Delta^0 & 2\not\Delta^1 & 2\not\Delta^2 & \cdots & & & F_2 = 2\not\Delta^0 \\
3\not\Delta^0 & 3\not\Delta^1 & 3\not\Delta^2 & \cdots & & & F_3 = 3\not\Delta^0 + 1\not\Delta^1 \\
\vdots & & & & & & F_4 = 4\not\Delta^0 + 2\not\Delta^1 \\
 & & & & & & \vdots
\end{array}
$$

It is clear that, if we keep forming Fibonacci numbers from Pascal's triangle in this way, $F_n = n\not\Delta^0 + (n-2)\not\Delta^1 + (n-4)\not\Delta^2 + \ldots + (n-2m)\not\Delta^m$, or

$$
(4) \qquad F_n = \sum_{k=0}^{m} (n-2k)\not\Delta^k ,
$$

where we require that m be an integer and that $0 < n-2m \leq 2$, or in other words that $n/2 - 1 \leq m < n/2$. Now let us prove

(5) Theorem 1 $$n\not\Delta^m = \binom{n+m-1}{m}$$

Proof: It is sufficient to perform induction on n. Let the theorem be $E(n)$. Then if $n = 1$, $E(1)$ states

$$
\binom{n+m-1}{m} = \binom{1+m-1}{m} = \frac{m!}{m!} = 1 \quad .
$$

But by definition (1), $(m+1)\Delta^0 = 1$ for any $(m+1) \in N$. Then by equation (3) $1\Delta^m = 1$ for $m = 0, 1, 2, 3, \ldots$ and E(1) is true. Now let us assume that, for arbitrary $m \in N$, E(n) is true. Then

$$n\Delta^m = \binom{n+m-1}{m} .$$

From the definitions (1) it can be seen that

$$1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + n\Delta^{m-1} = n\Delta^m .$$

Therefore the induction hypothesis can be restated

$$1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} = \binom{n+m-1}{m} . \tag{6}$$

Add $\binom{n+m-1}{m-1}$ to both sides of equation (6) to obtain

$$\begin{aligned} 1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} + \binom{n+m-1}{m-1} \\ = \binom{n+m-1}{m} + \binom{n+m-1}{m-1} \end{aligned} \tag{7}$$

The right-hand side of equation (7) is $\binom{n+m}{m}$ by the standard identity for combinations, so we have

$$1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} + \binom{n+m-1}{m-1} = \binom{n+m}{m} ,$$

or

$$\begin{aligned} 1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} + \binom{(n+1)+m-2}{m-1} \\ = \binom{(n+1)+m-1}{m} , \end{aligned}$$

which is E(n+1). Therefore E(n) implies E(n+1) and Theorem 1 is true by mathematical induction.

Now let us prove

(8) Theorem 2 $$n\Delta^m = \left[(n+m)\int_0^1 x^{n-1}(1-x)^m dx\right]^{-1}$$

Proof: $\Gamma(n) = (n-1)!$ (gamma function)

$$B(m, n) = B(n, m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{(beta function)}$$

Therefore

$$\frac{1}{B(m,n)} = \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)} \quad ,$$

and

$$\frac{1}{B(m+1,\, n-m+1)} = \frac{\Gamma(n+2)}{\Gamma(m+1)\,\Gamma(n-m+1)} = \frac{(n+1)!}{m!\,(n-m)!}$$

$$= \frac{(n+1)n!}{m!\,(n-m)!} = (n+1)\binom{n}{m} \quad .$$

Then

$$(9) \qquad \binom{n}{m} = \frac{1}{(n+1)B(m+1,\, n-m+1)} = \left[(n+1)B(m+1,\, n-m+1)\right]^{-1} \quad .$$

We can now substitute the right-hand side of equation (5) into equation (9) to obtain

$$n\not\Delta^m = \binom{n+m-1}{m} = \left[(n+m)B(m+1,\, n)\right]^{-1} \quad ,$$

where

$$B(m+1,\, n) = B(n,\, m+1) = \int_0^1 x^{n-1}(1-x)^m dx \quad .$$

Therefore

$$n\not\Delta^m = \left[(n+m)\int_0^1 x^{n-1}(1-x)^m dx\right]^{-1} \quad .$$

Both equations (5) and (8) assert that $n\not\Delta^m = (m+1)\not\Delta^{n-1}$. Some interesting special cases of equation (5) are

$$n\not\Delta^0 = \binom{n-1}{0} = \frac{(n-1)!}{(n-1)!} = 1 \quad ,$$

$$n\not\Delta^1 = \binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n \quad ,$$

and

$$\sum_{k=1}^{n} k = n\not\Delta^2 = \binom{n+1}{2} = \frac{(n+1)!}{(n-1)!\,2!} = \frac{(n)(n+1)}{2} \quad .$$

Now we can put equation (8) into equation (4) to obtain

$$F_n = \sum_{k=0}^{m} \left[(n-k) \int_0^1 x^{n-2k-1}(1-x)^k dx\right]^{-1} , \tag{10}$$

where m is an integer, $n/2 - 1 \leq m < n/2$. But whereas equations (4) and (5) have meaning only for integer arguments, equations (8) and (10) can be used to find $x\Delta^y$ and F_u, where x, y, and u are any rational numbers.

In particular

$$F_u = \sum_{k=0}^{m} \left[(u-k) \int_0^1 x^{u-2k-1}(1-x)^k dx\right]^{-1} , \tag{11}$$

where m is an integer, $u/2 - 1 \leq m < u/2$. The equation (11), and the definite integral in it, are easily programmed for solution on a digital computer. A few values of F_u follow.

u	F_u
4.1000000	3.1550000
4.2000000	3.3200000
4.3000000	3.4950000
4.4000000	3.6800000
4.5000000	3.8750000
4.6000000	4.0800000
4.7000000	4.2950000
4.8000000	4.5200000
4.9000000	4.7550000
5.0000000	5.0000000
5.1000000	5.2550000
5.2000000	5.5200000
5.3000000	5.7950000
5.4000000	6.0800000
5.5000000	6.3750000
5.6000000	6.6800000
5.7000000	6.9950000
5.8000000	7.3200000
5.9000000	7.6550000
6.0000000	8.0000000

u	F_u
0.1	1.0
0.2	1.0
⋮	⋮
2.0	1.0
2.1	1.1
2.2	1.2
⋮	⋮
3.0	2.0
3.1	2.1
⋮	⋮
4.0	3.0

XXXXXXXXXXXXXXX

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. HILLMAN
University of Santa Clara, Santa Clara, California

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. Any problem believed to be new in the area of recurrent sequences and any new approaches to existing problems will be welcomed. The proposer should submit each problem with solution in legible form, preferably typed in double spacing with name and address of the proposer as a heading.

Solutions to problems should be submitted on separate sheets in the format used below within two months of publication.

B-64 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $L_n L_{n+1} = L_{2n+1} + (-1)^n$, where L_n is the n-th Lucas number defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$.

B-65 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let u_n and v_n be sequences satisfying $u_{n+2}+au_{n+1}+bu_n=0$ and $v_{n+2}+cv_{n+1}+dv_n=0$ where a, b, c, and d are constants and let $(E^2+aE+b)(E^2+cE+d) = E^4+pE^3+qE^2+rE+s$. Show that $y_n=u_n+v_n$ satisfies

$$y_{n+4}+py_{n+3}+qy_{n+2}+ry_{n+1}+sy_n = 0 \quad .$$

B-66 *Proposed by D.G. Mead, University of Santa Clara, Santa Clara, California*

Find constants p, q, r, and s such that

$$y_{n+4}+py_{n+3}+qy_{n+2}+ry_{n+1}+sy_n = 0$$

is a 4th order recursion relation for the term-by-term products $y_n=u_nv_n$ of solutions of $u_{n+2}-u_{n+1}-u_n = 0$ and $v_{n+2}-2v_{n+1}-v_n = 0$.

B-67 *Proposed by D.G. Mead, University of Santa Clara, Santa Clara, California*

Find the sum $1\cdot 1+1\cdot 2+2\cdot 5+3\cdot 12+\ldots+F_nG_n$, where $F_{n+2}=F_{n+1}+F_n$ and $G_{n+2}=2G_{n+1}+G_n$.

B-68 *Proposed by Walter W. Horner, Pittsburgh, Pennsylvania*

Find expressions in terms of Fibonacci numbers which will generate integers for the dimensions and diagonal of a rectangular parallelopiped, i.e., solutions of

$$a^2+b^2+c^2 = d^2 \quad .$$

B-69 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Solve the system of simultaneous equations:

$$xF_{n+1}+yF_n = x^2+y^2$$

$$xF_{n+2}+yF_{n+1} = x^2+2xy$$

where F_n is the n-th Fibonacci number.

SOLUTIONS

CHEBYSHEV POLYNOMIALS

B-27 *Proposed by D.C. Cross, Exeter, England*

Corrected and restated from Vol. 1, No. 4: The Chebyshev Polynomials $P_n(x)$ are defined by $P_n(x) = \cos(n\text{Arccos } x)$. Letting $\phi = \text{Arccos } x$, we have

$$\cos \phi = x = P_1(x),$$

$$\cos (2\phi) = 2\cos^2\phi - 1 = 2x^2 - 1 = P_2(x),$$

$$\cos (3\phi) = 4\cos^3\phi - 3\cos\phi = 4x^3 - 3x = P_3(x),$$

$$\cos (4\phi) = 8\cos^4\phi - 8\cos^2\phi + 1 = 8x^4 - 8x^2 + 1 = P_4(x), \quad \text{etc.}$$

It is well known that

$$P_{n+2}(x) = 2xP_{n+1}(x) - P_n(x) \quad .$$

Show that

$$P_n(x) = \sum_{j=0}^{m} B_{jn}x^{n-2j}$$

where

$$m = \left[n/2 \right] ,$$

the greatest integer not exceeding $n/2$, and

(1) $B_{on} = 2^{n-1}$

(2) $B_{j+1,\, n+1} = 2B_{j+1,\, n} - B_{j,\, n-1}$

(3) If $S_n = |B_{on}| + |B_{1n}| + \ldots + |B_{mn}|$, then $S_{n+2} = 2S_{n+1} + S_n$.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

By De Moivre's Theorem,

$$(\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi \quad .$$

Letting $x = \cos \phi$, and expanding the left side,

$$\cos n\phi + i \sin n\phi = (x + i \sqrt{1 - x^2})^n$$

$$= \sum_{j=0}^{n} (-1)^{j/2} \binom{n}{j} x^{n-j} (1 - x^2)^{j/2} \quad .$$

We equate real parts, noting that only the even terms of the sum are real,

$$\cos n\phi = P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} x^{n-2k}(1 - x^2)^k \quad .$$

We may prove from this (cf. Formula (22), p. 185, Higher Transcendtal Functions, Vol. 2 by Erdelyi et al; R. G. Buschman, "Fibonacci Numbers, Chebyshev Polynomials, Generalizations and Difference Equations," Fibonacci Quarterly, Vol. 1, No. 4, p. 2) that

(*) $$B_{j,\, n} = \frac{n\, (-1)^j\, 2^{n-2j-1}\, (n-j-1)!}{j!\, (n-2j)!} \quad .$$

From this, we have

(1) $$B_{o,\, n} = 2^{n-1} \quad .$$

It is also easy to show from (*) that

$$(2) \qquad B_{j+1,\,n+1} = 2\,B_{j+1,\,n} - B_{j,\,n-1} \quad .$$

Now (*) implies

$$B_{j,\,n} = (-1)^j \,|B_{j,\,n}| \quad ,$$

so that (2) becomes

$$(-1)^{j+1}\,|B_{j+1,\,n+1}| = 2\,(-1)^{j+1}\,|B_{j+1,\,n}| + (-1)^{j+1}\,|B_{j,\,n-1}| \quad ,$$

or

$$|B_{j+1,\,n+1}| = 2\,|B_{j+1,\,n}| + |B_{j,\,n-1}| \quad .$$

Summing both sides for j to $\left[\frac{n+1}{2}\right]$, we have

$$(3) \qquad S_{n+1} = 2\,S_n + S_{n-1} \quad .$$

Also solved by the proposer.

A SPECIAL CASE

B-52 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $F_{n-2}F_{n+2} - F_n^2 = (-1)^{n+1}$, where F_n is the n-th Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

Solution by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

Identity XXII (Fibonacci Quarterly, Vol. 1, No. 2, April 1963, p. 68) states:

$$F_nF_m - F_{n-k}F_{m+k} = (-1)^{n-k}\,F_kF_{m+k-n} \quad .$$

The proposed identity is immediate on taking $m = n$ and $k = 2$.

More generally, we have

$$F_n^2 - F_{n-k}F_{n+k} = (-1)^{n-k}F_k^2 \quad \text{for} \quad 0 \le k \le n \quad .$$

Also solved by Marjorie Bicknell, Herta T. Freitag, John E. Homer, Jr., J.A.H. Hunter, Douglas Lind, Gary C. MacDonald, Robert McGee, C.B.A. Peck, Howard Walton, John Wessner, Charles Ziegenfus, and the proposer.

SUMMING MULTIPLES OF SQUARES

B-53 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that

$$(2n - 1)F_1^2 + (2n - 2)F_2^2 + \dots + F_{2n-1}^2 = F_{2n}^2 \quad .$$

Solution by James D. Mooney, University of Notre Dame, Notre Dame, Indiana

Remembering that

$$\sum_{k=0}^{n} F_k^2 = F_n F_{n+1} \quad ,$$

we may proceed by induction. Clearly for $n = 1$, $F_1^2 = 1 = F_2^2$. Assume

$$\left[2(n-1) - 1\right] F_1^2 + \left[2(n-1) - 2\right] F_2^2 + \dots + F_{2(n-1)-1} =$$

$$= (2n-3)F_1^2 + (2n-4)F_2^2 + \dots + F_{2n-3} = F_{2n-2}^2 \quad .$$

Then

$$(2n-1)F_1^2 + \dots + F_{2n-1} = \left[(2n-3)F_1^2 + \dots + F_{2n-3}\right] +$$

$$2(F_1^2 + \dots + F_{2n-2}^2) + F_{2n-1}^2 = F_{2n-2}^2 + \sum_{k=0}^{2n-2} F_k^2 + \sum_{k=0}^{2n-1} F_k^2 =$$

$$F_{2n-2}^2 + F_{2n-2}F_{2n-1} + F_{2n-1}F_{2n} = F_{2n-2}^2 + F_{2n-2}F_{2n-1} +$$

$$+ F_{2n-1}(F_{2n-2} + F_{2n-1}) = F_{2n-2}^2 + 2F_{2n-2}F_{2n-1} + F_{2n-1}^2 =$$

$$(F_{2n-2} + F_{2n-1})^2 = F_{2n}^2 \quad . \quad \text{Q.E.D.}$$

Also solved by Marjorie Bicknell, J.L. Brown, Jr., Douglas Lind, John E. Homer, Jr., Robert McGee, C.B.A. Peck, Howard Walton, David Zeitlin, Charles Ziegenfus, and the proposer.

RECURRENCE RELATION FOR DETERMINANTS

B-54 *Proposed by C.A. Church, Jr., Duke University, Durham, N. Carolina*

Show that the n-th order determinant

$$f(n) = \begin{vmatrix} a_1 & 1 & 0 & 0 & & 0 & 0 \\ -1 & a_2 & 1 & 0 & & 0 & 0 \\ 0 & -1 & a_3 & 1 & & 0 & 0 \\ 0 & 0 & -1 & a_4 & \cdots & 0 & 0 \\ \cdots & & & & & & \\ \cdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & a_n \end{vmatrix}$$

satisfies the recurrence $f(n) = a_n f(n-1) + f(n-2)$ for $n > 2$.

Solution by John E. Homer, Jr., La Crosse, Wisconsin

Expanding by elements of the n-th column yields the desired relation immediately.

Also solved by Marjorie Bicknell, Douglas Lind, Robert McGee, C.B.A. Peck, Charles Ziegenfus, and the proposer.

AN EQUATION FOR THE GOLDEN MEAN

B-55 *From a proposal by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Show that $x^n - xF_n - F_{n-1} = 0$ has no solution greater than a, where $a = (1 + \sqrt{5})/2$, F_n is the n-th Fibonacci number, and $n > 1$.

Solution by G.L. Alexanderson, University of Santa Clara, California

For $n > 1$ let $p(x, n) = x^n - xF_n - F_{n-1}$, $g(x) = x^2 - x - 1$, and $h(x, n) = x^{n-2} + x^{n-3} + 2x^{n-4} + \ldots + F_k x^{n-k-1} + \ldots + F_{n-2}x + F_{n-1}$. It is easily seen that $p(x, n) = g(x)h(x, n)$, $g(x) < 0$ for $-1/a < x < a$, $g(a) = 0$, $g(x) > 0$ for $x > a$, and $h(x, n) > 0$ for $x \geq 0$. Hence $x = a$ is the unique positive root of $p(x, n) = 0$.

Also solved by J.L. Brown, Jr., Douglas Lind, C.B.A. Peck, and the proposer.

GOLDEN MEAN AS A LIMIT

B-56 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Let F_n be the n-th Fibonacci number. Let $x_0 \geq 0$ and define $x_1, x_2, \ldots$ by $x_{k+1} = f(x_k)$ where

$$f(x) = \sqrt[n]{F_{n-1} + xF_n} \ .$$

For $n > 1$, prove that the limit of x_k as k goes to infinity exists and find the limit. (See B-43 and B-55.)

Solution by G.L. Alexanderson, University of Santa Clara, Santa Clara, California

For $n > 1$ let $p(x) = x^n - xF_n - F_{n-1}$. Let $a = (1 + \sqrt{5})/2$. As in the proof of B-55, one sees that $p(x) > 0$ for $x > a$ and that $p(x) < 0$ for $0 \leq x < a$. If $x_k > a$, we then have

$$(x_k)^n > x_kF_n + F_{n-1} = (x_{k+1})^n$$

and so $x_k > x_{k+1}$. It is also clear that $x_k > a$ implies

$$(x_{k+1})^n = x_kF_n + F_{n-1} > aF_n + F_{n-1} = a^n$$

and hence $x_{k+1} > a$. Thus $x_o > a$ implies $x_o > x_1 > x_2 > \ldots > a$. Similarly, $0 \leq x_o < a$ implies $0 \leq x_o < x_1 < x_2 < \ldots < a$. In both cases the sequence $x_o, x_1, \ldots$ is monotonic and bounded. Hence x_k has a limit $L > 0$ as k goes to infinity. Since L satisfies

$$L = \sqrt[n]{F_{n-1} + LF_n} \ ,$$

L must be the unique positive solution of $p(x) = 0$.

Also solved by Douglas Lind and the proposer.

A FIBONACCI-LUCAS INEQUALITY

B-57 *Proposed by G.L. Alexanderson, University of Santa Clara, Santa Clara, California*

Let F_n and L_n be the n-th Fibonacci and n-th Lucas number respectively. Prove that

$$(F_{4n}/n)^n > L_2L_6L_{10} \cdots L_{4n-2}$$

for all integers $n > 2$.

Solution by David Zeitlin, Minneapolis, Minnesota

Using mathematical induction, one may show that

$$F_{4n} = \sum_{k=1}^{n} L_{4k-2}, \quad n = 1, 2, \ldots \quad .$$

If we apply the well-known arithmetic-geometric inequality to the unequal positive numbers $L_2, L_6, L_{10}, \ldots, L_{4n-2}$, we obtain for $n = 2, 3, \ldots,$

$$\frac{F_{4n}}{n} = \frac{\sum_{k=1}^{n} L_{4k-2}}{n} = \sqrt[n]{L_2 L_6 L_{10} \cdots L_{4n-2}} \quad ,$$

which is the desired inequality.

Also solved by Douglas Lind and the proposer.

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ACKNOWLEDGMENT

It is a pleasure to acknowledge the assistance furnished by Prof. Verner E. Hoggatt, Jr. concerning the essential idea of "Maximal Sets" and the line of proof suggested in the latter part of my article "On the Representations of Integers as Distinct Sums of Fibonacci Numbers." The article appeared in Feb.,1965. H. H. Ferns

CORRECTION Volume 3, Number 1

Page 26, line 10 from bottom of page

$$V_{7,3} + V_{7,4} + V_{7,5} = F_8 - F_7 = F_6 = 8$$

Page 27, lines 4 and 5

$$F_2 + F_4 + F_6 + \ldots + F_n = F_{n+1} - 1 \quad \text{(n even)}$$

$$F_3 + F_5 + F_7 + \ldots + F_n = F_{n+1} - 1 \quad \text{(n odd)}$$

ACKNOWLEDGMENT

Both the papers "Fibonacci Residues" and "On a General Fibonacci Identity," by John H. Halton, were supported in part by NSF grant GP2163.

CORRECTION Volume 3, Number 1

Page 40, Equation (81), the R. H. S. should have an additional term

$$-v^2 F_{v+2}$$

BASIC PROPERTIES OF A CERTAIN GENERALIZED SEQUENCE OF NUMBERS

A. F. HORADAM
The University of North Carolina, Chapel Hill, N. C.

1. INTRODUCTION

Let α, β be the roots of

$$(1.1) \qquad x^2 - px + q = 0$$

where p, q are arbitrary integers. Usually, we think of α, β as being real, though this need not be so.

Write

$$(1.2) \qquad d = (p^2 - 4q)^{1/2}.$$

Then

$$(1.3) \qquad \alpha = (p + d)/2, \ \beta = (p - d)/2$$

so that

$$(1.4) \qquad \alpha + \beta = p, \ \alpha\beta = q, \ \alpha - \beta = d.$$

Recently [6], a certain generalized sequence $\{w_n\}$ was defined:

$$(1.5) \qquad \{w_n\} \equiv \{w_n (a, b;p, q)\} : \ w_0 = a, \ w_1 = b, \ w_n = pw_{n-1} - qw_{n-2} \ (n \geqslant 2)$$

in which

$$(1.6) \qquad w_n = A\alpha^n + B\beta^n ,$$

where

$$(1.7) \qquad A = \frac{b - a\beta}{\alpha - \beta}, \ B = \frac{a\alpha - b}{\alpha - \beta}$$

whence

$$(1.8) \qquad A + B = a, \ A - B = (2b - pa)d^{-1} , \ A B = e \, d^{-2}$$

in which we have written

$$(1.9) \qquad e = pab - qa^2 - b^2.$$

Sequences like $\{w_n\}$ have been previously introduced by, for example, Bessel-Hagen [1] and Tagiuri [11], though in the available literature I cannot find evidence of much progress from the definition [11] to have discovered a few of the results listed hereunder.

The purpose of [6] was to determine a recurrence relation for the k^{th} powers of w_n (k an integer), that is, to obtain an explicit form for

$$w_k(x) \quad \sum_{n=0}^{\infty} w_n^k x^n .$$

Here, we propose to examine some of the fundamental arithmetical properties of $\{w_n\}$. No attempt at all is made to analyze congruence or prime number features of $\{w_n\}$. In selecting properties to generalize we have been guided by those properties of the related sequences (see 2. below) which in the literature and from experience seem most basic. Naturally, the list could be extended as far as the reader's enthusiasm persists.

It is intended that this paper should be the first of a series investigating aspects of $\{w_n\}$. Organization of the material is as follows: in 2., various special (known) sequences related to $\{w_n\}$ are introduced, while in 3. some linear formulas involving $\{w_n\}$ are established, and in 4. some non-linear expressions are obtained. Finally, in 5., some comments on the degenerate case $p^2 = 4q$ are offered.

2. RELATED SEQUENCES

Particular cases of $\{w_n\}$ are the sequences $\{u_n\}$, $\{v_n\}$, $\{h_n\}$, $\{f_n\}$, $\{l_n\}$ given by:

(2.1) $$w_n(1, \quad p; p, \quad q) = u_n(p, q)$$

(2.2) $$w_n(2, \quad p; p, \quad q) = v_n(p, q)$$

(2.3) $$w_n(r, r+s; 1, -1) = h_n(r, s)$$

(2.4) $$w_n(1, \quad 1;\ 1,\ -1) = f_n \ (= u_n(1,-1) = h_n(1,0))$$

(2.5) $$w_n(2, \quad 1;\ 1,\ -1) = l_n \ (= v_n(1,-1) = h_n(2,-1)).$$

Historical information about these second order recurrence sequences may be found in Dickson [3]. Of course, $\{f_n\}$ is the famous Fibonacci sequence, $\{l_n\}$ is the Lucas sequence, and $\{u_n\}$ and $\{v_n\}$ are generalizations of these, while $\{h_n\}$ discussed in [4] is a different generalization of them. Chief properties of $\{u_n\}$, $\{v_n\}$, $\{f_n\}$ and $\{l_n\}$ may be found in, for instance, Jarden [7], Lucas [8] and Tagiuri [10] and [11], those of $\{f_n\}$ especially being featured in Subba Rao [9] and Vorob'ev [12].

Two rather interesting specializations of (2.1) and (2.2) are the Fermat sequences $\{u_n(3,\ 2)\} = \{2^{n+1} - 1\}$ and $\{v_n(3,2)\} = \{2^n + 1\}$, and the Pell sequences $\{u_n(2,\ -1)\}$ and $\{v_n(2,\ -1)\}$. (See [1] or [8]).

From (1.6), (1.7) and (2.1) - (2.5) it follows that

(2.6) $$u_n = \frac{\alpha^{n+1} - \beta^{n+1}}{d}$$

(2.7) $$v_n = \alpha^n + \beta^n$$

(2.8) $$h_n = \frac{(r + s - r\beta_1)\alpha_1^n - (r + s - r\alpha_1)\beta_1^n}{\sqrt{5}}$$

(2.9) $$f_n = \frac{\alpha_1^{n+1} - \beta_1^{n+1}}{\sqrt{5}}$$

(2.10) $$l_n = \alpha_1^n + \beta_1^n$$

wherein

(2.11) $$\alpha_1 = \frac{1 + \sqrt{5}}{2}, \quad \beta_1 = \frac{1 - \sqrt{5}}{2},$$

that is, α_1, β_1 are the roots of

(2.12) $$x^2 - x - 1 = 0.$$

Consequently, by (1.4)

$$\alpha_1 + \beta_1 = 1, \ \alpha_1\beta_1 = -1, \ \alpha_1 - \beta_1 = \ 5. \tag{2.13}$$

To assist the reader, and as a source of ready reference, the full set of results for the five specializations of $\{w_n\}$ will often be written down, as in (2.6) - (2.10).

Obviously from (1.9), e characterizes the various sequences. For $\{u_n\}$, $\{v_n\}$, $\{h_n\}$, $\{f_n\}$, $\{l_n\}$ we derive $e = -q$, $p^2 - 4q$, $r^2 - rs - s^2$, 1, 5 respectively.

By (1.6), (1.7) and (2.6) we have

$$w_n = au_n + (b - pa)\, u_{n-1} = bu_{n-1} - qa\, u_{n-2}\,, \tag{2.14}$$

with, in particular, the known [8] expressions

$$v_n = 2u_n - pu_{n-1} = pu_{n-1} - 2q\, u_{n-2}\,. \tag{2.15}$$

(Ultimately, of course, these yield $l_n = 2f_n - f_{n-1} + 2f_{n-2}$.)

Putting $n = 0$ in (2.14) requires the existence of values for negative subscripts, as yet not defined. Allowing unrestricted values of n therefore in (1.6) we obtain

$$\left\{ \begin{aligned} w_{-n} &= A\,\alpha^{-n} + B\,\beta^{-n} \\ &= q^{-n}\,(au_n - bu_{n-1}) \end{aligned} \right. \tag{2.16}$$

after simplification using

$$u_{-n} = q^{-n+1}\, u_{n-2}\,, \tag{2.17}$$

which follows from (2.6).

Combining (2.14) and (2.16) we have

$$w_{-n} = q^{-n}\, \frac{(au_n - bu_{n-1})}{au_n + (b - pa)u_{n-1}}\, w_n \tag{2.18}$$

whence it follows from (2.2) - (2.5) that

$$v_{-n} = q^n\, v_n \tag{2.19}$$

(2.20) $$h_{-n} = (-1)^n \frac{\{r(u_n - u_{n-1}) - su_{n-1}\}}{ru_n + su_{n-1}} h_n$$

(2.21) $$f_{-n} = (-1)^n f_{n-2}$$

In particular,

(2.23) $$w_{-1} = A\alpha^{-1} + \beta^{-1} = \frac{pa - b}{q}$$

so that

(2.24) $$u_{-1} = 0$$

(2.25) $$v^{-1} = \frac{p}{q}$$

(2.26) $$h_{-1} = s$$

(2.27) $$f_{-1} = 0$$

(2.28) $$l_{-1} = -1$$

Many of the simplest $\{w_n\}$ are expressible in terms of $\{f_n\}$. Besides (2.4) we have

(2.29) $$w_n(-1, 1; -1, -1) = (-1)^{n-1} f_n$$

(2.30) $$w_n(1, -1; 1, -1) = -f_{n-3}$$

(2.31) $$w_n(1, 1; -1, -1) = (-1)^{n-1} f_{n-3}.$$

More generally,

(2.32) $$w_n(a, b; 1, -1) = af_{n-2} + bf_{n-1}$$

(2.33) $$w_n(a, b; -1, -1) = (-1)^n \{af_{n-2} - bf_{n-1}\}$$

Notice that

(2.34) $$\begin{cases} w_n(a_1, b_1; p_1, q_1) = -w_n(a_2, b_2; p_2, q_2) \\ \text{provided} \\ a_2 = -a_1,\ b_2 = -b_1,\ p_2 = p_1,\ q_2 = q_1. \end{cases}$$

Some sequences are cyclic. Examples are

$$w_n\,(a,\ b;\ -1,\ 1) \tag{2.35}$$

for which α, β $(= \alpha^2)$ are the complex cube roots of 1 and

$$w_n\,(a,\ b;\ 1,\ 1) \tag{2.36}$$

for which α, β $(= \alpha^2)$ are the complex cube roots of -1. Sequence (2.35) is cyclic of order 3 (with terms a, b, $-a - b$) since $\alpha^{3n} = \beta^{3n} = 1$, while sequence (2.36) is cyclic of order 6 (with terms a, b, $-a + b$, $-a$, $-b$, $a - b$) since $\alpha^{3n} = \beta^{3n} = -1$, so $\alpha^{6n} = \beta^{6n} = 1$ (n odd in this case). (Refer (1.6)).

Geometric-type sequences arise when $p = 0$ (so that by (1.5) $w_{n+1} = -qw_{n-1}$) and $q = 0$ (so that $w_{n+1} = pw_n$).

3. LINEAR PROPERTIES

From (1.5) and (1.6) it follows that

$$\frac{w_n}{w_{n-1}} \to \begin{cases} \alpha \\ \beta \end{cases}, \quad \frac{w_n}{w_{n-k}} \to \begin{cases} \alpha^k & \text{if } -1 \le \beta \le 1, \\ \beta^k & \text{if } -1 \le \alpha \le 1, \end{cases} \tag{3.1}$$

$$w_{n+2} - (p^2 - q)\,w_n + pq\,w_{n-1} = 0\,, \tag{3.2}$$

and

$$pw_{n+2} - (p^2 - q)\,w_{n+1} + q^2\,w_{n-1} = 0\,. \tag{3.3}$$

Repeated use of $qw_{k-1} = -w_{k+1} + pw_,$ $(k = 1, \ldots, n)$ leads to the sum of the first n terms

$$q\sum_{j=0}^{n-1} w_j = (p - 1)\,(w_2 + w_3 + \ldots\ldots + w_n) - w_{n+1} + pw_1 \tag{3.4}$$

whence

$$(p - q - 1)\sum_{j=0}^{n-1} w_j = w_{n+1} - w_1 - (p - 1)\,(w_n - w_0) \tag{3.5}$$

while the corresponding results for differences are

$$(3.6) \quad q \sum_{j=0}^{n-1} (-1)^j w_j = (p+1)(-w_2 + w_3 - \dots + (-1)^{n-1} w_n) + (-1)^n w_{n+1} + pw_1$$

and

$$(3.7) \quad (p - q + 1) \sum_{j=0}^{n-1} (-1)^j w_j = (-1)^{n+1} w_{n+1} + w_1 - (p+1) \left\{ (-1)^{n+1} w_n + w_0 \right\} .$$

Replace n by 2n in (3.4), (3.5) (3.6) and (3.7). Write

$$(3.8) \quad \sigma = w_0 + w_2 + \dots + w_{2n-2} ,$$

and

$$(3.9) \quad \rho = w_1 + w_3 + \dots + w_{2n-1} .$$

Adding and subtracting (3.4), (3.6) give

$$(3.10) \quad (1 + q)\sigma = p\rho - (w_{2n} - w_0)$$

and

$$(3.11) \quad (1 + q)\rho = p\sigma + q(w_{2n-1} - w_{-1})$$

for the sum of the even - (odd -) indexed terms of $\{w_n\}$. Clearly by (1.5) addition of (3.10) and (3.11) yields the sum of the first 2n terms (3.4) as expected. Solve (3.10) and (3.11) so that

$$(3.12) \quad \left\{p^2 - (1+q)^2\right\} \sigma = (1+q)(w_{2n} - w_0) - pq(w_{2n-1} - w_{-1})$$

and

$$(3.13) \quad \left\{p^2 - (1+q)^2\right\} \rho = p(w_{2n} - w_0) - q(1+q)(w_{2n-1} - w_{-1}) .$$

Using the alternative expression $w_n = bu_{n-1} - qau_{n-2}$ (2.14), we have

$$\begin{cases} w_{n+1} = w_1 u_n - q w_0 u_{n-1} \\ w_{n+2} = w_2 u_n - q w_1 u_{n-1} \\ w_{n+3} = w_3 u_n - q w_2 u_{n-1} \end{cases}$$

whence

$$(3.14)\qquad \begin{cases} w_{n+r} = w_r u_n - q w_{r-1} u_{n-1} \\ \qquad\;\; = w_n u_r - q w_{n-1} u_{r-1} \end{cases}$$

on interchanging n and r. Equations (3.14) may also be obtained from (1.5), (2.1) and (2.14). Of course

$$(3.15)\qquad \begin{cases} w_{n+r} = w_{r-j} u_{n+j} - q w_{r-j-1} u_{n+j-1} \\ \qquad\;\; = w_{n+j} u_{r-j} - q w_{n+j-1} u_{r-j-1} \end{cases}$$

also.

Further, from (1.6) and (2.7) it follows that

$$(3.16)\qquad \frac{w_{n+r} + q^r w_{n-r}}{w_n} = v_r$$

that is, the expression on the left is independent of a, b, n. Interchange r and n in (3.16) and then set $r = 0$. Accordingly,

$$(3.17)\qquad w_n + q^n w_{-n} = a v_n .$$

Observe also from (1.6) and (2.6) that

$$(3.18)\qquad \frac{w_{n+r} - q^r w_{n-r}}{w_{n+s} - q^s w_{n-s}} = \frac{u_{r-1}}{u_{s-1}}$$

which [10] is an integer provided s divides r.

Two binomial results of interest may be noted. Firstly, from (1.6) it follows that

$$(3.19)\qquad w_{2n} = (-q)^n \sum_{j=0}^{n} \binom{n}{j} \left(-\frac{p}{q}\right)^{n-j} w_{n-j}$$

where we have used the fact $\alpha^2 - p\alpha + q = 0$, $\beta^2 - p\beta + q = 0$.

Starting from (1.3) and (1.6), we readily derive

$$2^n w_n = A(p + d)^n + B(p - d)^n .$$

$$(3.20)\quad 2^n w_n = a \sum_{j=0}^{[n/2]} p^{n-2j} d^{2j} \binom{n}{2j} + (2b - pa) \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2j+1} p^{n-2j-1} d^{2j}$$

whence follow the known [1] expressions

$$(3.21)\quad 2^n u_n = \sum_{j=0}^{[n/2]} \binom{n+1}{2j+1} p^{n-2j} d^{2j}$$

$$(3.22)\quad 2^{n-1} v_n = \sum_{j=0}^{[n/2]} \binom{n}{2j} p^{n-2j} d^{2j}$$

$$(3.23)\quad 2^n f_n = \sum_{j=0}^{[n/2]} \binom{n+1}{2j+1} 5^j$$

$$(3.24)\quad 2^{n-1} l_n = \sum_{j=0}^{[n/2]} \binom{n}{2j} 5^j .$$

Suitable substitutions in the above results lead to the special cases for $\{u_n\}$, $\{v_n\}$, $\{h_n\}$, $\{f_n\}$ and $\{l_n\}$; for example, for $\{f_n\}$, in (3.4)

$$\sigma + \rho = f_{2n+1} - 1,$$

and in (3.14) with $r = n$,

$$f_n^2 + f_{n-1}^2 = f_{2n} = \sum_{k=0}^{n} \binom{n}{k} f_{n-k},$$

using (3.19).

If we write

$$(3.25)\quad \frac{w_n}{w_{n+1}} = r_n$$

so that, by (1.5),

$$(3.26)\quad r_n = \frac{1}{p - q\, r_{n-1}}, \quad r_{n-1} = \frac{1}{p - q\, r_{n-2}}, \quad \ldots\ldots\ldots,$$

enabling us to express the limit of the ratio as a continued fraction.

Sometimes, when $q = -1$, it is notationally convenient to write

$$(3.27)\qquad \begin{cases} \alpha_o = e^{\eta_o} = \sinh \eta_o + \cosh \eta_o \\ \beta_o = -e^{-\eta_o} = \sinh \eta_o - \cosh \eta_o \end{cases}$$

where (1.2)

$$(3.28)\qquad \cosh \eta_o = \frac{d_o}{2}, \quad \sinh \eta_o = \frac{p}{2}, \quad \tanh \eta_o = p \quad d_o^{-1} .$$

Zero suffices signify that $q = -1$.

Combining this hyperbolic notation with the remarks immediately preceding (3.27), and proceeding to the limit (refer (3.1)), we see that for $p = 1$, $q = -1$, that is, for $\{h_n\}$ (and its specializations $\{f_n\}$, $\{l_n\}$),

$$\frac{h_n}{h_{n+1}} \longrightarrow \frac{1}{\alpha_1} = e^{-\eta_1}$$

$$= \cosh \eta_1 - \sinh \eta_1$$

$$= \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}$$

(observe that by (2.12) $\frac{1}{\alpha_1} = g$ is a root of $x^2 + x - 1 = 0$ so that $g = \frac{1}{1 + g}$, leading to the continued fraction.)

Furthermore, (3.27) and (3.28), with (1.5), imply

$$(3.30)\qquad w_{o,n} = (A_o + (-1)^n B_o) \sinh n\, \eta_o + (A_o - (-1)^n B_o) \cosh n\, \eta_o .$$

Hyperbolic expressions for the specialized sequences are then, from (2.6), (2.7), (2.9), (2.10),

$$(3.31)\qquad \begin{cases} u_n = \dfrac{\sinh (n+1)\, \eta_o}{\cosh \ _o} & (n \text{ odd}) \\[2ex] \quad\ = \dfrac{\cosh (n+1)\, \eta_o}{\cosh \ _o} & (n \text{ even}) \end{cases}$$

$$(3.32)\qquad \begin{cases} v_n = 2 \sinh n\ \eta_o & (n \text{ even}) \\ \quad = 2 \cosh n\ \eta_o & (n \text{ odd}) \end{cases}$$

with corresponding expressions for f_n, l_n respectively, in which η_o is replaced by η_1. A hyperbolic expression for h_n is given in [5].

4. NON-LINEAR PROPERTIES

Essentially, the problem in obtaining non-linear formulas (as in the linear case) is to detect the appropriate coefficients (functions of p, q) of w_n^k. Basic non-linear (quadratic) results have already been recorded in [6], namely:

$$(4.1)\qquad a w_{m+n} + (b-pa)\, w_{m+n-1} = w_m\, w_n - q w_{m-1}\, w_{n-1}\ ,$$

$$(4.2)\qquad a w_{2n} + (b-pa)\, w_{2n-1} = w_n^2 - q w_{n-1}^2 = w_{n+1}\, w_{n-1} - q w_n\, w_{n-2}\ ,$$

$$(4.3)\qquad w_{n+1}\, w_{n-1} - w_n^2 = q^{n-1}\, e\ .$$

Obviously, from (4.3) with $n = 0$,

$$(4.4)\qquad e = q\,(w_1\, w_{-1} - w_0^2)$$

which may be compared with (1.9), using (1.5) and (2.23).

An extension of (4.3) is, by (1.6) and (2.6),

$$(4.5)\qquad w_{n+r}\, w_{n-r} - w_n^2 = e\, q^{n-r}\, u_{r-1}^2\ .$$

Putting $r = n$ in (4.5), we have

$$(4.6)\qquad w_n^2 + e\, u_{n-1}^2 = a\, w_{2n}\ .$$

Interchange r and n in (4.5), then suppose $r = 0$. We deduce

$$(4.7)\qquad w_n\, w_{-n} = a^2 + e\, q^{-n}\, u_{n-1}^2\ .$$

($n = 1$ reduces (4.7) to (4.4).)

Specializations of (4.1) are, on multiplication by 2 and use of (1.2), (1.4), (2.6), (2.7) and (2.15), the known [8] results

(4.8) $$2\, u_{m+n-1} = u_{m-1}\, v_n + u_{n-1}\, v_m$$

and

(4.9) $$2\, v_{m+n} = v_m\, v_n + d^2\, u_{m-1}\, u_{n-1} \; .$$

Next, by (4.6), we derive, using (2.6), (2.7), (1.2) and (1.4),

(4.10) $$u_{2n-1} = u_{n-1}\, v_n$$

and

(4.11) $$2\, v_{2n} = v_n^2 + d^2\, u_{n-1}^2$$

with

(4.12) $$v_{2n} = v_n^2 - 2q^n = d^2\, u_{n-1}^2 + 2q^n \; .$$

Again, (4.1) with $m = 2n$ gives an expression for w_{3n} from which we deduce, by (4.10), (2.6), (2.7) and the recurrence relation for v_{3n},

(4.13) $$\frac{u_{3n-1}}{u_{n-1}} = v_n^2 - q^n$$

and

(4.14) $$\frac{v_{3n}}{v_n} = v_n^2 - 3q^n \; .$$

Results (4.10) - (4.14) occur in Lucas [8] in a slightly adjusted notation.

Coming now to the sum of the first n terms, we use the first half of (4.2).

Write

(4.15) $$\tau = \sum_{j=0}^{n-1} w_j^2 \; .$$

Then, it follows that

(4.16) $$(1-q)\, \tau = a\sigma + (b-pa)\rho - \left\{ q w_{n-1}^2 + (b-pa)\, w_{2n-1} \right\} \; ,$$

whence r may be found from (3.12) and (3.13).

Repeating the first half of (4.2) leads to

$$(4.17) \qquad w_{n+1}^2 - q^2 w_{n-1}^2 = b\, w_{2n+1} + (b - pa)\, q\, w_{2n-1} \quad .$$

From (1.6), (1.8) and (2.6),

$$(4.18) \qquad w_{n-r}\, w_{n+r+t} - w_n\, w_{n+t} = q^{n-r}\, e\, u_{r-1}\, u_{r+t-1}$$

whence $t = 0$ gives (4.5).

Replacing w_n by u_n in (3.14) and (3.15) (with $-j$ substituted for j) yields

$$(4.19) \quad u_{n+r} = u_n\, u_r - q\, u_{n-1}\, u_{r-1} = u_{n-j}\, u_{r+j} - q\, u_{n-j-1}\, u_{r+j-1}$$

whence

$$(4.20) \quad \left\{ \begin{aligned} u_n\, u_r - u_{n-j}\, u_{r+j} &= q\,(u_{n-1}\, u_{r-1} - u_{n-j-1}\, u_{r+j-1}) \\ &= q^{n-j}\,(u_j\, u_{r-n+j} - u_{r-n+2j}) \\ &= q^{n-j+1}\, u_{j-1}\, u_{r-n+j-1} \end{aligned} \right.$$

by repeated application of (4.19) and replacement in the first half of (4.19) of n by $r-n+j$ and r by j to obtain an expression for u_{r-n+2j} $(u_0 = 1)$. Note that (4.20) is the special case of (4.18) for which $w_n = u_n$ so that $e = -q$ (n, r, j in (4.20) replaced by $n - r$, $n + r + t$, respectively and (2.17) used).

In particular, it follows from (4.20) with $j = 1$ that

$$(4.21) \qquad u_{n-1}\, u_{r-2} - u_{n-2}\, u_{r-1} = q^{n-1}\, u_{r-n-1} \quad .$$

Moreover, (4.21) and $w_n = b\, u_{n-1} - q\, a\, u_{n-2}$ give for the sequences $\{w_n\}$ and $\{w'_n\}$

$$(4.22) \quad \begin{aligned} w'_n\, w_r - w_n\, w'_r &= q\,(a'\, b - a\, b')(u_{n-1}\, u_{r-2} - u_{n-2}\, u_{r-1}) \\ &= q^n\,(a'b - a\, b')\, u_{r-n-1} \end{aligned}$$

Cubic expressions in w_n are generally quite complicated, so we derive only the sum of the first n cubes. Cube both sides of (1.5) and then use (1.5) again. Thus

(4.23) $$w_{n+1}^3 = p^3 w_n^3 - q^3 w_{n-1}^3 - 3pq\, w_{n-1} w_n w_{n+1} \quad .$$

But, from (4.3),

(4.24) $$w_{n-1} w_n w_{n+1} = w_n^3 + q^{n-1} e\, w_n \quad ,$$

so that from (4.23) and (4.24) it follows that

(4.25) $$w_{n+1}^3 + (3pq - p^3) w_n^3 + q^3 w_{n-1}^3 = -3pq^n e\, w_n \quad .$$

Now a calculation involving (1.6) and the summation of geometric series leads to

(4.26) $$\sum_{j=1}^{n-1} q^j w_j = \frac{q}{1-pq+q^3} \left\{ w_1 - q^2 w_0 - q^{n-1} (w_n - q^2 w_{n-1}) \right\} \quad .$$

Write

(4.27) $$\omega = \sum_{j=0}^{n-1} w_j^3 \quad .$$

Combining (4.25), (4.26) and (4.27), we find

(4.28) $$(1+3pq-p^3+q^3)\,\omega = \frac{-3pqe}{1-pq+q^3} \left\{ w_1 - q^2 w_0 - q^{n-1}(w_n \; q^2 w_{n-1}) \right\}$$
$$+q^3 w_{n-1}^3 - w_n^3 + (1+3pq-p^3)\, w_0^3$$

Appropriate substitution in the above formulas of 4. lead to corresponding results for the special sequences (2.1) - (2.5). For instance, applying (4.16) and (4.28) to $\{f_n\}$, we have $r = \frac{1}{2}\{f_{2n-1} - f_{n-1}^2\}$,

$$\omega = \frac{1}{4} \left\{ f_{n-1}^3 + f_n^3 + 3(-1)^{n-1} f_{n-2} + 2 \right\}$$

respectively.

5. DEGENERATE CASE

Throughout the analysis of the nature of $\{w_n\}$, the hypothesis that $p^2 \neq 4q$ has been assumed. But suppose now that $p^2 = 4q$. The simplest degenerate case occurs when $p = 2$, $q = 1$ ($\alpha = \beta = 1$) for which exists the trivial sequence ($n \geq 0$)

$$(5.1) \qquad v_n(2,\ 1) : 2,\ 2,\ 2,\ 2,\ 2,\ \ldots\ .$$

and the sequence of natural numbers ($n \geqslant 0$)

$$(5.2) \qquad u_n(2,\ 1) : 1,\ 2,\ 3,\ 4,\ 5,\ \ldots\ ,$$

that is, $u_n = n+1$ and $v_n = 2$. For negative n, (2.19) implies $v_{-n} = v_n$, that is, every element of $\{u_n\ (2,1)\}$ is 2, while (2.17) implies $u_{-n} = -u_{n-2}$, that is, like elements of $\{u_n\ (2,1)\}$ are the positive and negative integers in order.

Generally, in the degenerate case,

$$(5.3) \qquad \alpha = \beta = \frac{p}{2}\ .$$

The main features of the degenerate case, as they apply to $\{u_n\}$ and $\{v_n\}$ are discussed in Carlitz [2], with acknowledgement to Riordan. Brief comments, as they relate to $\{w_n\}$, are made in [6]. In passing, we note that Carlitz [2] has established the interesting relationship between degenerate

$$\{u_n\ (p,\ \frac{p^2}{4})\}$$

and the Eulerian polynomial $A_k(x)$ which satisfies the differential equation

$$A_{n+1}(x) = (1 + nx)\ A_n(x) + x(1 - x)\ \frac{d}{dx}\ A_n(x)\ ,$$

where $A_0(x) = A_1(x) = 1$, $A_2(x) = 1+x$, $A_3(x) = 1 + 4x + x^2$.

Finally, it must be emphasized that $\{h_n\}$ and its specializations $\{f_n\}$ and $\{l_n\}$ can have no such degenerate cases, because $p^2 - 4q$ then equals 5 ($\neq 0$).

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NON-FIBONACCI NUMBERS

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In order to understand the properties of a set it is often worth while to study the complement of the set. When The Fibonacci Association and this Quarterly were being established, the writer began to think about non-Fibonacci numbers as well as about Fibonacci numbers, but what is known about non-Fibonacci numbers? With the hope of generating more interest in non-Fibonacci numbers, I posed as the first problem in this Quarterly, problem H-1, the question of finding a formula for the n-th non-Fibonacci number. The purpose of the present paper is to discuss the problem and give a solution to it.

We begin with the concept of complementary sequences. A sequence is an ordered set. Two sets of natural numbers, say A and B, are called complementary if they are disjoint and their union is the set of all natural numbers. Many examples are available: Even numbers and odd numbers; primes and non-primes; k-th powers and non k-th powers. But the reader may not realize that formulas can be written down for such sequences. Of course, even and odd numbers are generated easily by $2n$ and $2n-1$ where n is any natural number, but it is not as well known that a bonafide formula for the n-th non k-th power is given by the expression

$$n + \left[\sqrt[k]{n + \left[\sqrt[k]{n} \right]} \right], \qquad k \geq 2 \ ,$$

where square brackets indicate the integral part of a number. Such a formula is quite entertaining, and is a special case given by Lambek and Moser [11] in a general study of complementary sequences. They give seven examples, as well as a general result.

A remarkable pair of complementary sequences was discovered about forty years ago by Samuel Beatty at the University of Toronto. He posed his discovery as a problem in the American Mathematical Monthly [2]. We may state Beatty's theorem in the following equivalent form. If x and y are irrational numbers such that $1/x + 1/y = 1$, then the sequences $[nx]$ and $[ny]$, $n = 1, 2, 3, \ldots$, are complementary.

This theorem has been rediscovered a number of times since 1926. The short list of references at the end of this paper will give some idea of what is known about complementary sequences. Beatty's result has been fairly popular in Canada. Besides the work in Canada by Lambek and Moser, there was the work of Coxeter, and the master's thesis by Ian Connell (published in part in [3]). The interesting extension by Myer Angel [1] was written when he was a second year student at McGill University. Our main interest here is in the 1954 paper of Lambek and Moser.

Let $f(n)$, $n = 1, 2, 3, \ldots$, be a non-decreasing sequence of positive integers and define, as in [11] and [8, editor's remarks], the 'inverse' f^* by

$$f^*(n) = \text{number of } k \text{ such that } f(k) < n = \sum_{\substack{1 \leq k \\ f(k) < n}} 1 .$$

Thus f^* is the distribution function which one would expect to study in connection with any sequence. If f defines the sequence of prime numbers, then $f^*(n) = \pi(n-1)$ = number of primes $< n$. Note also that $f^{**} = f$. We shall also define $F(n) = f^*(n+1)$. Next, define recursively

$$F_0(n) = n; \quad F_k(n) = n + F(F_{k-1}(n)), \quad k > 0 .$$

Moser and Lambek showed that if $Cf(n)$ is the sequence complementary to $f(n)$, then

$$Cf(n) = \lim_{k \to \infty} F_k(n) .$$

What is more, they showed that the sequence $F_k(n)$ attains its limit $Cf(n)$ in a finite number of steps when this limit is finite. In fact one need not go beyond $k = Cf(n) - n$.

Thus the n-th non-prime number is the limit of the sequence n, $n + \pi(n)$, $n + \pi(n + \pi(n))$, $\ldots$. Often two steps are sufficient to attain the limit. Thus the n-th natural number which is not a perfect k-th power is given by the expression enunciated at the outset of this paper.

The n-th natural number not of the form $[e^m]$ with $m \geq 1$ is $n + [\log(n + 1 + [\log(n+1)]))]$.

As for the Fibonacci and non-Fibonacci numbers, let $f(n) = f_n$ be a Fibonacci number, defined recursively by $f_{n+1} = f_n + f_{n-1}$ with $f_1 = 1$, $f_2 = 2$. Let g_n designate the non-Fibonacci numbers. The following table will illustrate the calculations involved.

n	f_n	$f^*(n)$	F(n)	A	B	C	D	g_n = E	F
1	1	0	1	2	2	3	3	4	0.67
2	2	1	2	4	3	5	4	6	2.10
3	3	2	3	6	4	7	4	7	2.95
4	5	3	3	7	4	8	5	9	3.55
5	8	3	4	9	5	10	5	10	4.02
6	13	4	4	10	5	11	5	11	4.39
7	21	4	4	11	5	12	5	12	4.71
8	34	4	5	13	6	14	6	14	4.99
9	55	5	5	14	6	15	6	15	5.24
10	89	5	5	15	6	16	6	16	5.45
11	144	5	5	16	6	17	6	17	5.65
12	233	5	5	17	6	18	6	18	5.84
13	377	5	6	19	6	19	6	19	6.00
14	610	6	6	20	6	20	6	20	6.15
15	987	6	6	21	7	22	7	22	6.30
16		6	6	22	7	23	7	23	6.43
17		6	6	23	7	24	7	24	6.55
18		6	6	24	7	25	7	25	6.67
19		6	6	25	7	26	7	26	6.79
20		6	6	26	7	27	7	27	6.90
21		6	7	28	7	28	7	28	7.00
22		7	7	29	7	29	7	29	7.09
23		7	7	30	7	30	7	30	7.19
24		7	7	31	7	31	7	31	7.28
25		7	7	32	7	32	7	32	7.36
26		7	7	33	7	33	7	33	7.44
27		7	7	34	8	35	8	35	7.52

In the table, successive columns indicate the steps in evaluation of the limit $g_n = Cf(n)$ as follows:

$$A = n + F(n),$$
$$B = F(n + F(n)),$$
$$C = n + F(n + F(n)),$$
$$D = F(n + F(n + F(n))),$$
$$E = n + F(n + F(n + F(n))) .$$

Three iterations were found necessary to generate the non-Fibonacci numbers g_n, at least up to $n = 40$. It is left as a research problem for the reader to determine if more than three iterations are ever necessary.

It is evident that to obtain an elegant formula for g_n we have then two problems: (a) the number of steps required to find $Cf(n)$; (b) a neat formula for the distribution function $F(n)$ or equivalently the inverse $f^*(n)$.

The study of F or f^* corresponds to the study of the distribution of prime numbers, but because of the regular pattern of distribution we can supply a fairly neat formula for $F(n)$. It was noted by K. Subba Rao [13] that we have the asymptotic result:

$$F(n) \sim \frac{\log n}{\log a} , \quad \text{as} \quad n \to \infty ,$$

where

$$a = \frac{1 + \sqrt{5}}{2} .$$

As a matter of fact one can prove much more. We have the following THEOREM. Let $F(n)$ = number of Fibonacci numbers $f_k \le n$. Then

$$F(n) \sim \frac{\log n}{\log a} + \log_a \sqrt{5} - 1 \doteq 2.08 \log n + 0.67$$

and, for $n > n_o$, $F(n)$ is the greatest integer $\le$ this value. Column F in the table gives the value of the expression $2.08 \log n + 0.67$ as computed from a standard 10-inch slide rule. Even this crude calculation is good enough to show how closely the formula comes to $F(n)$.

Thus we have the following approximate formula for the n-th non-Fibonacci number:

$$g_n = n + F(n + F(n + F(n))) ,$$

with

$$F(n) = \left[\log_a n + \frac{1}{2} \log_a 5 - 1\right] \quad \text{for } n \geq 2 ,$$

$$\doteq \left[2.08 \log_e n + 0.67\right]$$

We shall conclude by noting some curious generating functions for the distribution function (or inverse) $f^*(n)$. For any non-decreasing sequence of positive integers $f(n)$, we have [8, editor's remarks]

$$x \sum_{n=1}^{\infty} x^{f^*(n)} = (1 - x) \sum_{n=1}^{\infty} f(n)\, x^n ,$$

and

$$\sum_{k=1}^{n} \sum_{j=1}^{f(k)} A_{j,k} = \sum_{j=1}^{f(n)} \sum_{k=1+f^*(j)}^{n} A_{j,k} ,$$

the latter identity holding for an arbitrary array of numbers $A_{j,k}$, being merely an example of summing in the one case by rows and in the other case by columns first. As an example with application to formulas involving the Fibonacci numbers we may note that

$$\sum_{k=1}^{n} \sum_{j=1}^{f_k} A_{j,k} = \sum_{j=1}^{f_n} \sum_{k=1+F(j-1)}^{n} A_{j,k} .$$

In this formula, take $A_{j,k} = 1$ identically. Then we find the formula

$$\sum_{k=1}^{f_n} F(k - 1) = nf_n - f_{n+2} + 2 , \qquad (F(0) = 0)$$

this being but one of many interesting relations connecting f_n and $F(n)$. From Theorem 2 of [11] we have that the sequences $n + f_n$ and $n + F(n-1)$ are complementary. The reader may find it of interest to develop the corresponding formulas for non-Lucas numbers or other recurrent sequences. In a forthcoming paper [10] Holladay has given a very general and closely reasoned account of some remarkable results for complementary sequences. If a personal remark be allowed, his paper is an outgrowth of discussions concerning problem H-1 and the application of complementary sequences to certain problems in game theory.

As a final remark, there is the question of the distribution of non-Fibonacci numbers and identities which they may satisfy. It is hoped to discuss other properties of non-Fibonacci numbers and other formulas for them in a later paper.

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Corrections to "Summation Formulae for Multinomial Coefficients" by Selmo Tauber, Vol. III, No. 2:

(5) line 3 (p. 97) $\binom{N+1}{k_1, k_2, \ldots, k_p, \ldots, k_n}$

(6) last line (p. 97) $2 \sum_{\alpha=1}^{k_p} (-1)^{\alpha} (k_1, k_2, \ldots$ etc.

(8) lines 3 and 4, upper index of mult. coeff. (p. 99)

N+h+1

N+q-1

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TWO FIBONACCI CONJECTURES

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Consider the problem of solving, in positive integers, the following Diophantine equation (suggested by the Editor):

$$F_n x + F_{n+1} y = x^2 + y^2. \tag{1}$$

First let us note that (1) always has the <u>trivial</u> solution (F_n, F_{n+1}), i.e., $x = F_n$, $y = F_{n+1}$. Does (1) ever have a non-trivial solution? If n is fixed, we know from analytic geometry that there are at most a finite number of solutions. However, we shall soon see that for infinitely many n (1) has at least two non-trivial solutions.

<u>Theorem 1.</u> If $n > 1$ and $n \equiv 1 \pmod 3$, then

$$\left(\frac{F_{n+2}}{2}, \frac{F_{n+2}}{2}\right)$$

and

$$\left(\frac{F_{n+2}}{2}, \frac{F_{n-1}}{2}\right)$$

are non-trivial solutions of (1).

Proof. Since $n \equiv 1 \pmod 3$, $F_{n-1} \equiv F_{n+2} \equiv 0 \pmod 2$ which guarantees that the quotients involved are indeed integers. One may immediately verify that they satisfy (1).

<u>Theorem 2.</u> If (x_o, y_o) is a solution of (1), than $u = 2x_o - F_n$, $v = 2y_o - F_{n+1}$ is a solution of

$$u^2 + v^2 = F_{2n+1}. \tag{2}$$

Proof. This is an immediate consequence of the identity (Lucas, 1876)

$$F_n^2 + F_{n+1}^2 = F_{2n+1}.$$

If $u = u_o$ and $v = v_o$ is a solution of (2) with $(u_o, v_o) \neq (F_n, F_{n+1})$ (or any of the other 7 solutions of (2) obtained by changing signs or interchanging F_n and F_{n+1}) we shall call (u_o, v_o) a <u>non-trivial solution</u> of (2).

Theorem 3. If $n \not\equiv 1 \pmod 3$, then (1) has a non-trivial solution if and only if (2) has a non-trivial solution.

Proof.

(a) If (x_o, y_o) is a non-trivial solution of (1), then by Theorem 2, $u = 2x_o - F_n$, $v = 2y_o - F_{n+1}$ is a solution of (2). If $2x_o - F_n = \pm F_n$, then $x_o = F_n$ (and hence $y_o = F_{n+1}$ or 0) or $x_o = 0$, a contradiction. If $2x_o - F_n = \pm F_{n+1}$, $x_o = F_{n+2}/2$ or $x_o < 0$ which is impossible since F_{n+2} is odd and we are considering only positive solutions of (1).

(b) Let us assume $u_o > 0$, $v_o > 0$ is a non-trivial solution of (2). F_w is even if and only if $w \equiv 0 \pmod 3$; thus by hypothesis F_{2n+1} is odd. But $F_{2n+1} = u_o^2 + v_o^2$, hence u_o and v_o must be of different parity. Moreover, for the same reason F_n and F_{n+1} must also be of different parity. Thus (interchanging names if necessary) we may be sure that

$$\left(\frac{u_o + F_n}{2}, \frac{v_o + F_{n+1}}{2}\right)$$

is an integral solution of (1). If

$$\frac{u_o + F_n}{2} = F_n$$

we would, as before, get a contradiction.

The reader is invited to show that the number of non-trivial solutions of (1) is always even.

Now the problem of representing a number as the sum of two squares has received considerable attention. The following result, known to Fermat and others was proved by Euler:

If $N = bc^2 > 0$, where b is square-free, then N is representable as the sum of two squares if and only if b has no prime factors of the form $4k + 3$.

Theorems on the number of such representations can be found in virtually every introductory text on number theory.

Thus by Theorem 3 if $n \not\equiv 1 \pmod 3$ and F_{2n+1} is a prime of form $4k+1$, the only solution of (1) in positive integers is (F_n, F_{n+1}) since every prime of the form $4k+1$ is the sum of two squares in essentially only one way.

It is interesting to note that the pertinent identity

$$(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2$$

was given by Fibonacci in his Liber Abaci of 1202. This can be used to expedite numerical investigations. However, one needs to beware (or at least be aware) of such accidents as the following:

Let $n > 0$, $n \equiv 2 \pmod 3$;

(i) If $n < 32$ (and $n \neq 17$), then $2n+1$ is a prime.

(ii) If $n < 17$, both $2n+1$ and F_{2n+1} are primes!

Another useful result is

Theorem 4. $F_{2n+1} \equiv 1,\ 2,$ or $5 \pmod 8$.

Proof. We shall use the identity $F_{2n+1} = F_n^2 + F_{n+1}^2$. If g is odd, then $g^2 \equiv 1 \pmod 8$. Thus if F_n and F_{n+1} are both odd, $F_{2n+1} \equiv 2 \pmod 8$. Since two consecutive Fibonacci numbers are relatively prime, the only remaining possibility is that F_n and F_{n+1} are of different parity; in this case we get $F_{2n+1} \equiv 1$ or $5 \pmod 8$.

The reader may prove that in general $F_n \not\equiv 4 \pmod 8$.

Finally, this problem suggests the following conjectures:

Conjecture 1. There are infinitely many values of n for which (1) has only a trivial solution.

Conjecture 2. F_{2n+1} is never divisible by a prime of the form $4k+3$.

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FIBONACCIOUS FACTORS

ROBERT B. ELY, III

1. INTRODUCTION

In earlier issues of the Quarterly there have been shown and proven answers to the following questions about the basic series (1, 1, 2, 3, 5 ---).

(1) By what primes are the various terms, U_n, divisible?

(2) At what points do various primes first appear as factors?

(3) At what periods do they reappear?

In this paper we deal with answers to the same questions as to the general series (a, b, a + b, a + 2b, 2a + 3b ---).

2. PERIODS OF REAPPEARANCE ARE THE SAME

Our task is simplified if we answer the last question first:

> If k is the period at which a prime repeats its zero residues in the basic series, k is also the period of zero residues in any general series.

Suppose that a prime first divides the nth term of a given series (a, b, a + b ---) and let the (n-1)th term be c. Then modulo p, (which we hereafter abbreviate to "[p") the series runs in this neighborhood as c, 0, c, c, 2c, 3c etc. The terms after the zero are those of the basic series each multiplied by c. Now if $x \not\equiv o$ [p, so also $cx \not\equiv o$ if $c \not\equiv o$ [p. Again, if $x \equiv o$ [p, so also $cx \equiv o$ [p. This means that in the two series (1, 1, 2, ---) and (c, c, 2c ---) the zeros appear at the same terms

3. SUMMARY OF PREVIOUS RESULTS AS TO FIRST APPEARANCES

(1) There are some terms of the basic series divisible by any prime one may choose.

(2) The term U_{abc} ... is divisible by U_a, U_b U_c ... E.G.

$U_{12} = 144$

is divisible by

U $U_2 = 1$
$U_3 = 2$
$U_4 = 3$
$U_6 = 8$

(3) Such a term U_n, for which n is composite, may also have other factors, called "primitive prime divisors;" and the general form of these primes is determined by the following rules (but their identity must be found by trial and error).

(A) If n is odd; p is of the form $2kn \pm 1$

(B) If $n = 2(2r + 1)$; p is of the form $nk \pm 1$

(C) If $n = 2^m(2r + 1)$; p is of the form $nk/2 - 1$

Examples are listed in the February 1963 Quarterly at pp. 44-45.

(4) The fact that n is prime does not imply that U_n is prime. E.g., U U 19 = 4181 = 37 x 113; even though 19 is prime. However, the converse is true: If U_n is prime, so also is n.

(5) The even prime, 2, is a factor of every third term of the series; and the odd prime 5 is a factor of every 5th term.

(6) All other odd primes are of the forms ± 1 and ± 3 [10. They appear and reappear as factors according to the following rules:

(a) If $p \equiv \pm 1$ [10, it will first appear when the n of $U_n = \frac{p-1}{d}$, d being some positive integer; and will reappear every nth term thereafter;

(b) If $p \equiv \pm 3$ [10, it will first appear when $n = \frac{p+1}{d}$, and every nth term thereafter, d again being some positive integer. E.g.,

3	divides	U_4	and	every	4th	term	thereafter
7	"	U_8	"	"	8th	"	"
11	"	U_{10}	"	"	10th	"	"
13	"	U_7	"	"	7th	"	"
17	"	U_9	"	"	9th	"	"
19	"	U_{18}	"	"	18th	"	"

(c) The rules for determining the divisor, d, of $p \pm 1$ in (6) have not yet been given. Examination of the primes less than 80 give d = 1, 2 or 4 in all cases except 47, where it is 3. However, in the range from p = 2,000 to 3,000, given in the February 1963 issue at pp. 36-40, d has values ranging from 1 to 78.

(7) Nothing has thus far been said about the appearances and periods of composite factors, ab $(a \neq b)$, nor factors which are powers, p^c.

4. NEW ANSWERS TO THE QUESTION OF FIRST APPEARANCES

(1) "By what factors are the terms of the general series $(a, b, a+b, a+2b, +3b \ldots)$ divisible?"

It can be shown that if A, B and C denote any three successive terms in this series, then $B^2 - AC = \pm$ a constant, no matter which three terms are chosen, and no matter what the values of A and B (the first two terms).

Specifically, work on the first few terms of the general series shows what this constant must be

$$b^2 - a(a+b) = b^2 - ab - a^2$$

$$\text{or} \quad (a+b)^2 - b(a+2b) = a^2 + 2ab + b^2 \quad - ab - ab^2$$

$$= -b^2 + ab + a^2$$

$$= -(b^2 - ab - a^2)$$

How can we make use of this constancy of $B^2 - AC$ to determine the possibility of a given prime, p, as a factor of some term in the general series? By changing the equation to a congruence [p. If any term, C, of the series is divisible by p; then C and its two immediate predecessors must satisfy the congruence

$$B^2 - AC \equiv \pm (b^2 - ab - a^2) \ [p$$

But we are assuming $C \equiv 0$ [p. This eliminates the term $-AC$. Hence we must have $B^2 \equiv \pm (b^2 - ab - a^2)$ [p.

In other words, once we know the first two terms, a and b of a general series; we know that the only possible factors for terms of the series are those for which $\pm (b^2 - ab - a^2)$ is a quadratic residue. Primes of which this is not true cannot be the modulus in the congruence

$$B^2 \equiv \pm (b^2 - ab - a^2) \ [p.$$

However, it does not follow from the necessity of this condition that it is also sufficient. E.g., 1, 4, 5 ... is never divisible by 89. Nevertheless, Brother Alfred has shown that there are some primes which are factors of all Fibonaccious series.

(2) We can no longer say that U_{abc} is divisible by U_a, U_b and U_c, as a single example will show. Consider 3, 7, 10, 17, 27, 44, 71, 115, 186, 301. $U_{10} = 301$ is divisible by $U_2 = 7$, but not by $U_5 = 27$.

(3) Neither can we say of a general series that if U_n is prime, so too is n. Vide 2, 5, 7, 12, 19, 31 ... for which U_6 is prime but 6 is not.

(4) (a) Nor do we have in the general series a set of primitive prime factors, in view of (2) above.

(b) Thus we are fairly limited, as to rules for the forms of certain, possible or impossible prime factors of the general series. We make here only two observations:

(i) For primes of the form $p = 4k + 3$, either a or -a is a residue for any value of a. Hence these primes are possible, but not necessarily certain factors of any general series.

(ii) On the other hand, for primes of the form $p = 4k + 1$, there can be values of a for which neither a nor -a is a residue. E.g., neither 2 or -2 is a residue [5; and neither ± 2 nor ± 5 nor ± 6 are residues [13. Hence these primes are impossible factors of general series for which the initial terms are correctly chosen.

E.g., no terms of the series 1, 63, 64, 127 are ever divisible by 5, 11, 13 or 17, since $\pm(63^2 - 64) = \pm(3969-64) = \pm 3905$ is a non-residue of each of these primes.

Hence let us put aside for the moment the more particular rules of forms of factors of the general series, and turn to the place of first appearance of possible factors. The intervals of reappearance are as in the basic series.

(5) First let us review 2 and 5. If any series is reduced 2, we have only four patterns, depending on choice of initial terms

1, 1, 0, 1, 1, 0, 1, 1, 0, ...
0, 0, 0, 0, 0, 0, 0, 0 ......
1, 0, 1, 1, 0, 1, 1, 0 ....
0, 1, 1, 0 ......

That is to say: one of the first three terms must be even; and thereafter either all or every 3rd term is even.

For 5, the situation is a little more complex. Actual computation of first appearances for the various combinations of remainders of the first two terms enables us to make the following table:

If the second term has a remainder of

and the first a remainder of	0	1	2	3	4	[5
0	1	1	1	1	1	
1	2	5	4	N	3	
2	2	N	5	3	4	
3	2	4	3	5	N	
4	2	3	N	4	5	

the entries show the number of the smallest term divisible by 5, where N signifies "none." Thus we see that 5 may first appear as a factor of any term from the 1st to the 5th, or be suppressed entirely; by proper choice of first terms. However, as the reader can easily verify, if 5 appears once as a factor, it reappears in every 5th term thereafter.

(6) Now, as before, let us turn from these two special cases of 2 (the only even prime) and 5 (the only one $\equiv 5$ [10) and consider the remaining ones of the forms ± 1 and ± 3 [10. We make the following conjectures:

(a) By proper choice of initial terms we can make any such prime, p, first appear as a factor of any term whose number (rank)$<$p; or, if p is of the form $4k + 1$, we can suppress it altogether.

(b) If such a prime appears at all, it will reappear at the same interval as in the basic series.

To test these conjectures, let us make tables, as for 5, for 7 and 11.

	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	2	8	7	4	5	6	3
2	2	5	8	6	7	3	4
3	2	6	4	8	3	5	7
4	2	7	5	3	8	4	6
5	2	4	3	7	6	8	5
6	2	3	6	5	4	7	8

Note the absence of N's; since 7 is always a factor of some terms of any general series.

For 11: b,

Second term

	0	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1	1
1	2	10	9	5	N	4	6	8	N	7	3
2	2	6	10	8	9	N	5	7	N	3	4
3	2	N	N	10	4	7	9	6	3	5	8
4	2	5	6	7	10	N	8	3	9	4	N
5	2	7	8	4	5	10	3	N	6	N	9
6	2	9	N	6	N	3	10	5	4	8	7
7	2	N	4	9	3	8	N	10	7	6	5
8	2	8	5	3	6	9	7	4	10	N	N
9	2	4	3	N	7	5	N	9	8	10	6
10	2	3	7	N	8	6	4	N	5	9	10

Observing these three tables, we see the following common features:

(i) The top line is always all 1's;

(ii) The left column is always all 2's, except for the top entry.

(iii) One diagonal is all 3's.

(iv) The other diagonal is all k's (where k will be seen to be the constant of reappearance, in this case 10), except for the upper left corner.

(v) The nth line (except the top) is line 1 "spaced out" at intervals of m from the 3.

(vi) Hence only line 1 need be computed.

Some of the features are obvious:

(i) The top line of 1's mean only that a (in the series a, b, a + b ...) $\equiv$ o [p. Hence the first zero is at the first term.

(ii) The left column of 2's is similarly explicable. The exception of 1 at the top left corner is because both a and b $\equiv$ o, and the earlier of the two is a, the 1st term.

(iii) The diagonal of 3's is due to their representing series in which the first two terms are a, p-a, p. The a's vary; but p in the 3rd term does not.

(iv) The identities in the other diagonal represent general series of which the first two terms are both a (2, 2, 4 ..., 3, 3, 6 ..., 4, 4, 8 ...). The terms of each of these series are those of the basic (1, 1, ...) each multiplied by a. Consequently if any term in the basic series gave a remainder [p it would also give a remainder (usually different) when multiplied by a constant. On the other hand, if the nth term, U_n, of the basic series $\equiv$ o [p; so also a U_n $\equiv$ o [p. That is to say, the earliest zero remainder in (a, a, 2a ...) occurs at the same term, regardless of the value of a.

(v) The "spacing out" of Line 1 to get the entries in Line n of the table is explicable similarly. If x $\equiv$ o [p so also kx $\equiv$ o [p while if x $\not\equiv$ o [p so also kx $\not\equiv$ o [p, in the first case for any value of k, and in the second so long as k $\not\equiv$ o [p.

This means that the occurrence of zeros in any series (a, b, a + b ...) is unchanged if each term in the series is multiplied by the same constant, k $\not\equiv$ o [p. In other words, while non-zero remainders may vary, p will occur as a factor of precisely the same terms in series (1, 2, 3, 5 ...), (2, 4, 6, 10 ...), (3, 6, 9, 15 ...) etc. Hence the entries in line 1 and col. 2, line 2 and col. 4, line 3 and col. 6 of the table must be the same; and similar reasoning shows how the rest of the spacing out follows the same pattern.

(vi) Finally we must consider line "1" of the table. To fill it out the hard and obvious way requires us to run out, reduced [p, the various series (1, 2, 3, 5 ...), (1, 3, 4, 7 ...), (1, 4, 5, 9 ...) until we reach a zero in each; and then make corresponding entries in line 1. This done, spacing out as per (v) will complete the table.

An alternative, or a cross-check can be made as follows: Suppose we run out the basic series for a prime we have not yet considered, 13. The series reduced [13 to the first zero is 1, 1, 2, 3, 5, 8, 0.

Attached is a table partially filled in, with the invariable 1st row of 1's, left column of 2's, diagonal of 3's, and diagonal of 7's (the zero period of the basic series). There are other entries, which we now explain.

For [13

Remainder of Second Term (b)

Remainder of First Term (a)	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	7	6										3
2	2		7	5								3	4
3	2			7		4					3	5	
4	2				7					3			
5	2					7			3		6		
6	2						7	3					
7	2						3	7					
8	2			6		3			7				
9	2				3					7			
10	2		5	3					4		7		
11	2	4	3								5	7	
12	2	3										6	7

1	1	2	3	5	8	0	marked	///
8	8	3	11	1	12	0	"	\\\
12	12	11	10	8	5	0	"	≡
5	5	10	2	12	1	0	"	||||

The entry in (1, 1) is 7; because we have just seen that 7 is the zero-period of the basic series. There is similarly a 6 in the square (1, 2) because after a look at the basic series, we see that if we start a new series with first terms 1, 2, instead of 1, 1; we arrive at 0 after 6 terms instead of 7. In fact, as the 7 and 11 tables have illustrated already, the entry in square (1, 2) of the table is always k-1, where k is the number of the first zero term in the basic series. Similarly the entry in the square (2, 3) is always k-2; and in the square (3, 5) it is k-3; etc.; because as we select later and later pairs of terms in the basic series to start new series, we reduce one by one the number of the first term in which zero appears. Hence we can, without further computation than the basic series reduced [p, fill in a number of entries on various lines of the zero appearance table (see the attached figure for 13).

Moreover, we can use these entries, with a little more trial and error, to work back to values in line 1 of the table. For example, let us again look at the 13 table. The period of zero-appearances being 7 (as we have seen from the basic series) and 3, 5 being the 4th and 5th terms in the basic series, we know that 0 appears at the (7-3)th term in a new series (3, 5, 8, 0 ...). Suppose we multiply the new series, term by term, by such a factor (9) as makes a still newer series with the first term 1.

We have	3 x 9,	5 x 9,	8 x 9,	0x 9 ---	[13
or	27 ,	45,	72,	0 ---	[13
or	1 ,	6,	7,	0 ---	[13

Hence from the entry of 4 in square (3, 5) we can check the same entry in (1, 6); both must be and are 4.

Here we note an interesting point. Still working with modulus 13, we have the basic series

	1, 1, 2, 3, 5, 8, 8,	first zero 7
from which we get	1, 2, 3, 5, 8, 0,	zero 6
	2, 3, 5, 8, 0	zero 5
	3, 5, 8, 0	zero 4
	5, 8	zero 3

We have found the entry in the table (first zero) for 3, 5 was the same as for (1, 6). Similarly we have seen that (1, 2) is simply 1 less than (1, 1). Again (2, 3) $\equiv$ (14, 21) $\equiv$ (1, 8); and (5, 8) $\equiv$ (40, 64) $\equiv$ (1, 12). However, this gives us entries in line 1 only for claims 1, 2, 6, 8 and 12. We have no data for the remaining columns, i.e. for series beginning (1, 3) (1, 4) (1, 5) (1, 7) (1, 9) (1, 10) and (1, 11).

One might at first imagine that these deficiencies were due to the fact that we had only run our basic series out to the first zero, instead of continuing beyond this restricted period to the full period, when not only zero but all remainders [13 repeat: 1, 1, 2, 3, 5, 8, 0, 8, 8, 3, 11, 1, 12, 0, 12, 12, 11, 10, 8, 5, 0, 5, 5, 10, 2, 12, 1, 0. However, the reader will find that the new entries in squares (8, 8) (8, 3) (3, 11) etc. still "run back" to the same set of 5 entries on line 1.

There are no entries on line 1 in columns 3, 4, 5, 7, 9, 10 and 11; because series with first terms 1 and second terms 3, 4, 5, 7, 9, 10 and 11 have no terms divisible by 13! Recall our test, of whether p could be a factor of a series beginning (1, b, 1 + b), i.e., is $\pm b^2 - b - 1$ a residue of p? It will be found that

$$\pm(3^2 - 3 - 1) \equiv 5 \text{ or } 8$$

$$\pm(4^2 - 4 - 1) \equiv 11 \text{ or } 2$$

$$\pm(5^2 - 5 - 1) = \pm 19 \equiv 6 \text{ or } 7$$

$$\pm(7^2 - 7 - 1) = \pm 41 \equiv 2 \text{ or } 11$$

$$\pm(9^2 - 9 - 1) = \pm 71 \equiv 6 \text{ or } 7$$

$$\pm(10^2 - 10 - 1) = \pm 89 \equiv 11 \text{ or } 2$$

$$\pm(11^2 - 11 - 1) = \pm 109 \equiv 5 \text{ or } 8$$

are none of them residues [13.

Consequently there must be entries of N (for "never") in each of Columns 3, 4, 5, 7, 9, 10 and 11 of Line 1.

TO SUMMARIZE as to the appearances of p as a factor of terms in a general series (a, b, a + b).

If p is prime

(i) It will never appear unless $\pm(b^2 - ab - a^2)$ is a residue of p.

(ii) If it can appear per (i), and does so, it will reappear at the same interval as in the basic series.

(iii) To determine the place of first appearance there is no simpler method known to the writer than to reduce a and b [p and then run the series out to the first zero. However, this can be quite a bit simpler than running out the series, itself. E.g., what, if any, terms

are divisible by 19 in the series 119, 231, 350, 581? Note
$119 \equiv 5 \quad [19 \qquad 231 \equiv 3 \quad [19$
Hence the first 3 terms $\equiv$ 5, 3, 8 and $3^2 - 5.8 = -31 \equiv 112 \equiv 7$, a residue; so that 19 is a possible factor, then we have 5, 3, 8, 11, 0. I.e., the 5th term 931 is so divisible. Moreover, since the zero period of the basic series is 18; this is also the period in our given series; and the 23rd, 41st and every 18th term thereafter is divisible by 19.

If p is composite, the rules for zero appearances can be derived from the rules of its prime factors in a manner easily illustrated by two examples:

(1) What, if any, terms are divisible by 143 in the series

$$1, 6, 7, 13 \text{ --- } ?$$

Since 143 = 11 x 13 we first check possibility of both primes as factors

$$6^2 - 6 - 1 = 29 \equiv 7 \ [11 \text{ and } 3 \ [13$$

$$-7 \equiv 4 \text{ is a residue of } 11; \text{ and } 3 \text{ is a residue of } 13.$$

Hence both primes are possible factors

Moreover, it can easily be found that zero [11 appears at the 6th term with a period of 10; while zero [13 appears at the 4th term with a period of 7.

Hence the number n, of the first term divisible by 143 must satisfy the congruences.

$$n \equiv 6 \quad [10$$
$$\text{and } n \equiv 4 \quad [13$$

The minimum solution is 56. Hence the 56th term is the smallest divisible as required by 143.

(2) On the other hand, there are cases in which, while there may be terms of a series divisible by each of two (or more) primes, there may be none divisible by both (or all). Consider

$$1, 7, 8, 15, 23$$

As the reader can check, the 4th term and every 5th thereafter is divisible by 5; while the 8th term (99) and every 10th thereafter is divisible by 11. However, there is no term divisible by 55. This is due to the fact that there is no solution to the simultaneous congruences

$$n \equiv 4 \; [5 \quad \text{(a number ending in 4 or 9)}$$

$$n \equiv 8 \; [10 \quad \text{(a number ending in 8)}$$

No number satisfies both conditions.

Thus there is no fixed and simple test for divisibility of a general series by a composite number. One must determine for each prime factor of the composite modulus, (i) the term at which it first appears and (ii) the period at which it reappears thereafter. Then one must test the congruences expressing these two conditions for each prime in the composite modulus; and either solve them or find them to be insoluble.

To complete this analysis would require attack on the problem of zero appearances in both the basic and general series for moduli which are powers of primes, p^c. However, this discussion is postponed pending publication of a proof by J. H. E. Cohn that in the basic series no terms are exact squares, except U_1, U_2 and U_7.

Beyond this we offer only these Conjectures:

In the basic series

(i) If the k^{th} term is the first one divisible by p, then the choice of first two terms, and will not be greater than the (p^{c-1})th term.

(ii) There will be no first appearance, if the first terms are chosen so that $\pm (b^2 - ba - a^2)$ are nonresidues $[p^c$.

(iii) If there is a first appearance, there will be reappearances at the same period as in the basic series.

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A GENERATING FUNCTION FOR FIBONACCI NUMBERS

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Since interesting identities for certain number theoretic functions can be derived from their generating functions, in particular generating functions for Dirichlet series, the following problem seemed to be of interest.

Problem: Find a generating function G which yields the Fibonacci numbers in the coefficients of a Dirichlet series.

First we note that we must write the series in the form

$$(1) \qquad G(s) = \sum_{n=1}^{\infty} f_n c_n n^{-s} ,$$

since the series diverges for $c_n \equiv 1$, the f_n's increase too rapidly. Part of the goal is, as a result, to find a simple expression to use for c_n.

One attempt at the solution proceeds as follows. Consider the more general difference equation,

$$(2) \qquad u_0,\ u_1,\ u_{n+1} = au_n + bu_{n-1} \quad (n \geqq 1) ,$$

from which we can write

$$u_n = [z_2^n(u_1 - z_1u_0) - z_1^n(u_1 - z_2u_0)]/(z_2 - z_1)$$

with $z_1z_2 = -b$, $z_1+z_2 = a$, $z_1 \neq z_2$. Substituting into the Dirichlet series we have

$$(3) \qquad \sum_{n=1}^{\infty} u_n c_n n^{-s} = A(z_1) \sum_{n=1}^{\infty} c_n z_2^n n^{-s} + A(z_2) \sum_{n=1}^{\infty} c_n z_1^n n^{-s}$$

where the function A is defined by

$$A(z_1) = (u_0z_1^2 - u_1z_1)/(z_1^2+b) = (u_1 - z_1u_0)/(z_2 - z_1) .$$

Since c_n must be chosen to guarantee the convergence of the series in (3), it is convenient to select $c_n = c$ and then $|cz_2| < 1$, $|cz_1| < 1$. Hence equation (3) can be written

$$(4) \qquad \sum_{n=1}^{\infty} u_n c^n n^{-s} = A(z_1)\, F(az_2, s) + A(z_2)\, F(az_1, s) \quad ,$$

where $F(z, s)$ is a function discussed by Truesdell [2]. Further

$$F(z, s) = \sum_{n=1}^{\infty} z^n n^{-s} = z\, \Phi(z, s, 1),$$

where Φ denotes the Lerch Zeta-function - some of the properties of which are known [1:1.11]. This allows the result to be expressed in various forms.

The difference equation (2) can be rewritten for $c^n u_n = v_n$ in the form

$$v_0,\ v_1,\ v_{n+1} = acv_n + bc^2 v_{n-1} \quad (n \geqq 1) \ .$$

For the Fibonacci case it is convenient to set $c = 1/2$, so that the generating function for $2^{-n} f_n$, that is

$$G(s) = \sum_{n=1}^{\infty} (2^{-n} f_n)\, n^{-s} \quad ,$$

can be written in the form

$$(5) \qquad G(s) = (2/\sqrt{5}) \left\{ F\left[(1+\sqrt{5})/4, s\right] - F\left[(1-\sqrt{5})/4, s\right] \right\} \ .$$

To make efficient use of this generating function one needs to have available identities involving the function $F(z, s)$, especially such identities as involve products. Analogous to the ζ-function, an infinite product expansion for $F(z, s)$ in terms of s, with fixed z, might be helpful.

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2. C. A. Truesdell, "On a Function which occurs in the Theory of the Structure of Polymers," Ann. of Math. 46(1945), pp. 144-151.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by VERNER E. HOGGATT, JR.
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Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-61 *Proposed by P. F. Byrd, San Jose State College, San Jose, California (corrected)*

Let

$$f_{n,k} = 0 \quad \text{for} \quad 0 \le n \le k-2, \; f_{k-1,k} = 1 \quad \text{and}$$

$$f_{n,k} = \sum_{j=1}^{k} f_{n-j,k} \quad \text{for} \quad n \ge k \; .$$

Show that

$$\frac{1}{2} < \frac{f_{n,k}}{f_{n+1,k}} < \frac{1}{2} + \frac{1}{2k} \quad \text{as} \quad n \longrightarrow \infty \; .$$

Hence

$$\lim_{k \to \infty} \; \lim_{n \to \infty} \; \frac{f_{n,k}}{f_{n+1,k}} = \frac{1}{2} \; .$$

H-65 *Proposed by J. Wlodarski, Porz-Westhoven, Federal Republic of Germany*

The units digit of a positive integer, M, is 9. Take the 9 and put it on the left of the remaining digits of M forming a new integer, N, such that N = 9M. Find the smallest M for which this is possible.

H-66 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va., and Raymond Whitney, Pennsylvania State University, Hazelton Campus, Hazelton, Pa.*

Let

$$\sum_{j=0}^{k} a_j y_{n+j} = 0$$

be a linear homogeneous recurrence relation with constant coefficients a_j. Let the roots of the auxiliary polynomial

$$\sum_{j=0}^{k} a_j x^j = 0 \quad \text{be} \quad r_1,\ r_2,\ \ldots,\ r_m$$

and each root r_i be of multiplicity m_i $(i = 1, 2, \ldots, m)$. Jeske (Linear Recurrence Relations - Part 1, Fibonacci Quarterly, Vol. 1, No. 2, pp. 69-74) showed that

$$\sum_{n=0}^{\infty} y_n \frac{t^n}{n!} = \sum_{i=1}^{m} e^{r_i t} \sum_{j=0}^{m_i - 1} b_{ij} t^j \ .$$

He also stated that from this we may obtain

$$(*) \qquad y_n = \sum_{i=1}^{m} r_i^n \sum_{j=0}^{m_i - 1} b_{ij} n^j \ .$$

(i) Show that (*) is in general incorrect, (ii) state under what conditions it yields the correct result, and (iii) give the correct formulation.

H-67 *Proposed by J. W. Gootherts, Sunnyvale, California*

Let $B = (B_0, B_1, \ldots B_n)$ and $V = (F_m, F_{m+1}, \ldots F_{m+n})$ be two vectors in Euclidian $n + 1$ space. The B_i's are binomial coefficients of degree n and the F_{m+i}'s are consecutive Fibonacci numbers starting at any integer m.

Find the limit of the angle between these vectors as n approaches infinity.

H-68 *Proposed by H. W. Gould, West Virginia University, Morgantown, West Virginia*

Prove that

$$\sum_{k=1}^{n} \frac{1}{F_k} \geq \frac{n^2}{F_{n+2}-1}, \quad n \geq 1$$

with equality only for $n = 1, 2$.

H-62 *Proposed by H. W. Gould, West Virginia University, Morgantown, West Virginia (corrected)*

Find all polynomials $f(x)$ and $g(x)$, of the form

$$f(x+1) = \sum_{j=0}^{r} a_j x^j, \quad a_j \text{ an integer}$$

$$g(x) = \sum_{j=0}^{s} b_j x^j, \quad b_j \text{ an integer}$$

such that

$$2\left\{x^2 f^3(x+1) - (x+1)^2 g^3(x)\right\} + 3\left\{x^2 f^2(x+1) - (x+1)^2 g^2(x)\right\}$$
$$+(2x+1)\left\{xf(x+1) - (x+1)g(x)\right\} = 0 .$$

H-69 *Proposed by M. N. S. Swamy, University of Saskatchewan, Regina, Canada*

Given the polynomials $B_n(x)$ and $b_n(x)$ defined by,

$$b_n(x) = x B_{n-1}(x) + b_{n-1}(x) \quad (n > 0)$$
$$B_n(x) = (x + 1) B_{n-1}(x) + b_{n-1}(x) \quad (n > 0)$$
$$b_0(x) = B_0(x) = 1$$

it is possible to show that,

$$B_n(x) = \sum_{r=0}^{n} \binom{n+r+1}{n-r} x^r ,$$

and

$$b_n(x) = \sum_{r=0}^{n} \binom{n+r}{n-r} x^r .$$

It can also be shown that the zeros of $B_n(x)$ or $b_n(x)$ are all real, negative and distinct. The problem is whether it is possible to factorize $B_n(x)$ and $b_n(x)$. I have found that for the first few values of n, the result

$$B_n(x) = \prod_{r=1}^{n} \left[x + 4 \cos^2 \left(\frac{r}{n+1}\right) \cdot \frac{\pi}{2} \right]$$

holds. Does this result hold good for all n? Is it possible to find a similar result for $b_n(x)$?

SOLUTIONS

FROM BEST SET OF K TO BEST SET OF K+1 ?

H-42 *Proposed by J. D. E. Konhauser, State College, Pa.*

A set of nine integers having the property that no two pairs have the same sum is the set consisting of the nine consecutive Fibonacci numbers, 1, 2, 3, 5, 8, 13, 21, 34, 55 with total sum 142. Starting with 1, and annexing at each step the smallest positive integer which produces a set with the stated property yields the set 1, 2, 3, 5, 8, 13, 21, 30, 39 with sum 122. Is this the best result? Can a set with lower total sum be found?

Partial solution by the proposer.

Partial answer. The set 1, 2, 4, 5, 9, 14, 20, 26, 35 has total sum 116. For eight numbers the best set appears to be 1, 2, 3, 5, 9, 15, 20, 25 with sum 80. Annexing the lowest possible integer to extend the set to nine members requires annexing 38 which produces a set with sum 118. It is not clear (to me, at least) how to progress from a best set of k integers to a best set for k + 1 integers.

Comments by Murray Berg, Oakland, California

The set given above in the partial solution is invalid since $1+5 = 4+2 = 6$ and the problem asks for distinct sums for different pairs.

Comments by the Editor

An apparent solution summing to 118 was received but was discarded since the sum was larger than the partial solution given above. Please resubmit if you read this.

AT LAST A SOLUTION

H-26 *Proposed by L. Carlitz, Duke University (corrected)*

Let $R_k = (b_{rs})$, where $b_{rs} = \binom{r-1}{k+1-2}$ $(r, s = 1, 2, \ldots, k+1)$. Then show

$$R_k^n = \left(\sum_{j=1}^{s} \binom{r-1}{j-1} \binom{k+1-r}{s-j} F_{n-1}^{k+1-r-s+j} F_n^{r+s-2j} F_{n+1}^{j-1} \right)$$

Letting $R_k^n = (a_{rs})$, we evaluate a_{rs} by extending the proposer's method of solving B-16 (Fibonacci Quarterly, Vol. 2, No. 2, pp. 155-157). Using Carlitz's notation, we may easily show by induction that the transformation

$$T_1 : \begin{cases} x' = y \\ y' = x + y \end{cases}$$

induces the transformation

$$T_k: \begin{cases} x'^k = \binom{0}{0} y^k \\ x'^{k-1} y' = \binom{1}{1} x y^{k-1} + \binom{1}{0} y^k \\ \vdots \\ x' y'^{k-1} = \binom{k-1}{k-1} x^{k-1} y + \ldots + \binom{k-1}{1} x y^{k-1} + \binom{k-1}{0} y^k \\ y'^k = \binom{k}{k} x^k + \binom{k}{k-1} x^{k-1} y + \ldots + \binom{k}{1} x y^{k-1} + \binom{k}{0} y^k \end{cases}$$

Carlitz also showed that the T_1^n is given by

$$(1) \qquad T_1^n : \begin{cases} x^{(n)} = F_{n-1} x + F_n y \\ y^{(n)} = F_n x + F_{n+1} y \end{cases}$$

so that T_k^n induces the transformation

$$(2) \quad T_k^n: \left\{ (x^{(n)})^{k-r+1} (y^{(n)})^{r-1} = \sum_{j=1}^{k+1} a_{rs} x^{k+1-s} y^{s-1} \quad (r = 1, 2, \ldots, k+1), \right.$$

We note here a misprint in the B-16 solution: the last transformation should begin with T_2^n instead of T_1^n. To evaluate a_{rs}, we substitute (1) into (2) to obtain

$$\sum_{s=1}^{k+1} a_{rs} x^{k+1-s} y^{s-1} = (F_{n-1} x + F_n y)^{k+1-r} (F_n x + F_{n+1} y)^{r-1}$$

$$= \sum_{i=0}^{k+1-r} \binom{k+1-r}{i} F_{n-1}^{k-1-r-i} F_n^i x^{k+1-r-i} y^i$$

$$\times \sum_{j=0}^{r-1} \binom{r-1}{j} F_n^{r-1-j} F_{n+1}^j x^{r-1-j} y^j$$

$$= \sum_{i=0}^{k+1-r} \sum_{j=0}^{r-1} \binom{k+1-r}{i} \binom{r-1}{j} F_{n-1}^{k+1-r-i} F_n^{r-1+i-j} F_{n+1}^j x^{k-i-j} y^{i+j} .$$

We equate coefficients of $x^{k+1-s}y^{s-1}$, summing all terms of the last sum with $i+j = s-1$, and since $j \le s-1$ we find

$$a_{rs} = \sum_{j=0}^{s-1} \binom{k+1-r}{s-1-j}\binom{r-1}{j-1} F_{n-1}^{k+2-r-s+j} F_n^{r+s-2-2j} F_{n+1}^{j}$$

$$= \sum_{j=1}^{s} \binom{k+1-r}{s-j}\binom{r-1}{j-1} F_{n-1}^{k+1-r-s+j} F_n^{r+s-2j} F_{n+1}^{j-1} .$$

ANOTHER LATE ONE

H-38 *Proposed by R. G. Buschman, SUNY, Buffalo, N. Y.*

(See Fibonacci Numbers, Chebyshev Polynomials, Generalizations and Difference Equations; Vol. 1, No. 4, Dec. 1963, pp. 1-7.)

Show

$$(u_{n+r} + (-b)^r u_{n-r})/u_n = \lambda_r .$$

Solution by Douglas Lind

Let $z_1 \neq z_2$ be the roots of $z^2 - az - b = 0$, and note $a = z_1 + z_2$, $-b = z_1 z_2$. We recall from the article that

$$u_n = \left\{ (u_1 - z_1 u_0) z_2^n - (u_1 - z_2 u_0) z_1^n \right\} / (z_2 - z_1)$$

and

$$\lambda_n = \left\{ (a - 2z_1) z_2^n - (a - 2z_2) z_1^n \right\} / (z_2 - z_1) .$$

Now

$$\lambda_n = z_2^n + z_1^n$$

since $a - 2z_1 = z_2 - z_1 = -(a - 2z_2)$, so that

$$u_n \lambda_r = \left\{ (u_1 - z_1 u_0) z_2^{n+r} - (u_1 - z_2 u_0) z_1^{n+r} + (-b)^r (u_1 - z_1 u_0) z_2^{n-r} - (-b)^r (u_1 - z_2 u_0) z_1^{n-r} \right\} / (z_2 - z_1)$$

$$= u_{n+r} + (-b)^r u_{n-r}$$

the desired result.

Also solved by Clyde Bridger and the proposer.

THE FIBONACCI NUMBERS AND THE "MAGIC" NUMBERS

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It was reported here (The Fibonacci Quarterly, issue 4, 1963) that one of the fundamental asymmetries in the world of atoms is asymmetrical distribution of fission fragments by mass numbers resulting from the bombardment of most heavy nuclei (by thermal neutrons).

The problem of this type of the asymmetry is one of most difficult problems in the branch of fission-physics.

It seems that by the here mentioned asymmetry there is a connection between the Fibonacci numbers (... 34, 55, 89, 144, ...) and the so-called "magic" numbers (2, 8, 20, 28, 50, 82 for protons and 2, 8, 20, 28, 50, 82, 126 for neutrons), which are well known in nuclear physics.

As a matter of fact the fission-nucleus ${}_{92}U^{236}$ possesses 144 neutrons and consequently a sufficient quantity of neutrons to form two neutron-shells: one with 50 neutrons and the other with 82 neutrons. If the rest of 12 neutrons [144 - (50 + 82)] divide in two equal parts, the whole number of neutrons in the heavy fragment is 82 + 6 = 88 (89) + and in the light fragments 50 + 6 = 56 (55).[1)]

The 92 protons of the nucleus ${}_{92}U^{236}$ can also form two shells with "magic" numbers of protons: 28 and 50 respectively. If the rest of protons [92 - (28 + 50)] = 14 divide in the same manner as the rest of the 12 neutrons, the whole number of protons in light fission-fragment should be: 28 + 7 = 35(34) and in the heavy fragment: 50 + 7 = 57(55).

These numbers of protons (35 and 57) and the neutrons (56 and 88) in both fission-fragments of the nucleus ${}_{92}U^{236}$ conform rather well the most experimental results.

\+ The number in parenthesis is the nearest Fibonacci number.

1) Mukhin, K. N., Introduction to Nuclear Physics. Moskow, USSR (1963), p. 350.

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AN ELEMENTARY METHOD OF SUMMATION

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The purpose of this note is to present an elementary method for summing the first n terms of a sequence which satisfies a given homogeneous linear recursion relation. The method is, in fact, a simple extension of that normally used for summing a geometric progression, which we first recall.

Let:

$$S = a + ar + ar^2 + \ldots + ar^n$$

Then:

$$-rS = \quad - ar - ar^2 - \ldots - ar^n - ar^{n+1}$$

Therefore:

$$S(1 - r) = a - ar^{n+1}$$

and if $r \neq 1$,

$$S = \frac{a - ar^{n+1}}{1 - r} .$$

We now turn to the general case. If for every positive integer j, G_j satisfies

$$G_{j+k} + \sum_{i=1}^{k} c_i G_{j+k-i} = 0, \tag{1}$$

where the c_i are fixed quantities, we write, as above

$$\begin{array}{llllll}
S = G_1 + G_2 + G_3 + \ldots & + \; G_{k+1} & + G_{k+2} & + \ldots + G_n & & \\
c_1 S = c_1 G_1 + c_1 G_2 + \ldots & + \; c_1 G_k & + c_1 G_{k+1} & + \ldots + c_1 G_{n-1} & + c_1 G_n & \\
c_2 S = \qquad c_2 G_1 + \ldots & + \; c_2 G_{k-1} & + c_2 G_k & + \ldots + c_2 G_{n-2} & + c_2 G_{n-1} & + c_2 G_n \\
\ldots & & & & & \\
c_k S = & \quad c_k G_1 & + c_k G_2 & + \ldots + c_k G_{n-k} & + \ldots + c_k G_n &
\end{array}$$

Since, adding vertically and using (1), the sum of the terms inside the dotted lines is zero, we see:

$$S(1+c_1+\ldots+c_k) = G_1(1+c_1+\ldots+c_{k-1}) + G_2(1+c_1+\ldots+c_{k-2}) + \ldots + G_k$$
$$+ G_n(c_1+c_2+\ldots+c_k) + G_{n-1}(c_2+\ldots+c_k) + \ldots + c_k G_{n-k+1}.$$

If $1+c_1+c_2+\ldots+c_{k-1} \neq 0$, we can solve for S.

The same method can be used to find

$$\sum_{i=1}^{n} i^t G_i \,, \text{ for a given } t,$$

if the G_i satisfy (1). To facilitate the presentation, we collect some terminology and facts.

Let E be the operator with the property that

$$EG_i = G_{i+1} \;.$$

To say that G_j satisfies (1) is equivalent to the statement that the operator

$$\phi(E) = E^k + \sum_{i=1}^{k} c_i E^{k-i}$$

when applied to any G_j, yields zero (E^0 being the identity operator). The associated polynomial

$$\phi(x) = x^k + \sum_{i=1}^{k} c_i x^{k-i}$$

is called the characteristic polynomial.* The special role of the number one in our generalization is now easily stated, for

$$1 + c_1 + \ldots + c_k \neq 0$$

if and only if unity is not a root of the characteristic polynomial.

*$\phi(x)$ is unique if we assume $\psi(E)\,G_j = 0$ for all positive j implies the degree of $\psi(x) \geq k$.

It is known ([2] , pp. 548-552) that if $\phi(E)G_j = 0$, then $B_j = j^{t-1} G_j$ satisfies

$$[\phi(E)]^t B_j = 0, \text{ for } t \geq 1.$$

If $\phi(1) \neq 0$ then $\psi(1) \neq 0$, where $\psi(x) = [\phi(x)]^t$, and the method just described can be used to find

$$T = \sum_{j=1}^{n} B_j = \sum_{j=1}^{n} j^{t-1} G_j .$$

Writing

$$\psi(x) = x^{kt} + \sum_{i=1}^{kt} d_i x^{kt-i},$$

we find:

$$p_0 T = \sum_{j=1}^{kt-j} p_j B_j + \sum_{j=0}^{kt-1} r_j B_{n-j}$$

where

$$p_j = 1 + \sum_{i=1}^{kt-j} d_i \text{ and } r_j = \sum_{i=j+1}^{kt} d_i .$$

Since $\phi(E)G_j = 0$ and $B_j = j^t G_j$, one can easily obtain T in terms of $G_1, \ldots, G_{k-1}; G_{n-k+2}, \ldots, G_n$.

The assumption that unity not be a root of the characteristic polynomial has been critical to our discussion so far. We now assume $\{G_j\}$ satisfies

$$\chi(E)\, G_j = 0$$

where $\chi(E)$ is a polynomial with $\chi(1) = 0$. Factoring out all the factors $x - 1$ in $\chi(x)$, we obtain

$$\chi(x) = (x - 1)^a\, \phi(x), \text{ where } \phi(1) \neq 0.$$

Letting $C_j = \phi(E)G_j$, we note:

$$\phi(1)S = \phi\frac{(1)}{1}\sum_{j=1}^{n} G_j = \sum_{j=1}^{k} G_j q_j + \sum_{j=1}^{n-k} C_j + \sum_{j=0}^{k-1} G_{n-j} s_j ,$$

where

$$\chi(x) = x^k + \sum_{i=1}^{k} c_i x^{k-i},$$

$$q_j = 1 + \sum_{i=1}^{k-j} c_i \text{ and } s_j = \sum_{i=j+1}^{k} c_i .$$

However, it is known ([2], pp. 548-552) that if $(E - 1)^a C_j = 0$, then C_j is a polynomial of degree $\leq a - 1$. If we assume the formulas for

$$\sum_{j=1}^{n} j^p$$

are known, for j a positive integer, the only problem remaining is that of determining the polynomial $C_j = d_0 + d_1 j + \ldots + d_{a-1} j^{a-1}$. It is easy to show that the difference operator $E-1$ when applied to a polynomial of degree r yields a polynomial of degree $r - 1$. Therefore $(E - 1)^j C_1$ involves only $d_{a-1}, d_{a-2} \ldots, d_j$ and the system of linear equations on the d_i obtained by computing $(E - 1)^j C_1$, $j = 0, 1, 2, \ldots, a - 1$ can clearly be solved for the d_i.

The above is a generalization of the technique used by Erbacher and Fuchs to solve problem H-17. [4]

Example: Assume that for each positive integer j, G_j satisfies $\chi(E)G_j = 0$, where $\chi(x) = (x - 1)^3 (x^3 - 3x^2 + 4x + 2) = (x-1)^3 \phi(x)$. If $G_1 = G_2 = G_3 = G_4 = G_5 = 0$, $G_6 = 1$, then $C_1 = \phi(E)G_1 = 0$, $C_2 = \phi(E) G_2 = 0$, $C_3 = \phi(E) G_3 = 1$. With $C_j = d_0 + d_1 j + d_2 j^2$, we find $(E - 1)^2 C_1 = 2d_2 = 1$, $(E - 1) C_1 = d_1 + 3d_2 = 0$ and $C_1 = d_0 + d_1 + d_2$. Hence $C_j = 1 - (3/2) j + j^2/2$ and

$$\phi(1)\, S = 4S = 2G_1 - 2G_2 + G_3 + \sum_{j=1}^{n-3} (1 - (3j)/2 + j^2/2) + 3G_n + 6G_{n-1} + 2G_{n-2}$$

$$= \sum_{j=1}^{n-3} (1 - (3j)/2 + j^2/2) + 3G_n + 6G_{n-1} + 2G_{n-2} \;.$$

In conclusion, we have seen how the elementary method used to sum a geometric progression can be generalized to find the sum of the first n terms of a sequence which satisfies a linear homogeneous recursion relation. It may be worth stating that this method is applicable to a sequence whose terms are products of corresponding terms of sequences each of which satisfy a linear homogeneous recursion relation (see [1] pp. 42-45 for a special case).

We propose as a problem for the reader: Find in closed form the sum of the first n terms of the sequence $\{w_n\}$:

$$1, 2, 10, 36, 145, \ldots$$

where $w_n = F_n G_n$ with $F_{n+2} = F_{n+1} + F_n$ $(F_1 = F_2 = 1)$ and $G_{n+2} = 2G_{n+1} + G_n$ $(G_1 = 1,\ G_2 = 2)$.

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2. C. Jordan, "Calculus of Finite Differences," Chelsea, New York, Ed. 1950.

3. James A. Jeske, "Linear Recurrence Relations, Part I," The Fibonacci Quarterly, Vol. 1, No. 2, pp. 69-74.

4. Problem H-17, The Fibonacci Quarterly, Proposed in Vol. 1, No. 2, 1963, p. 55 and solved by Joseph Erbacher and John Allen Fuchs in Vol. 2, No. 1, 1964, p. 51.

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ON IDENTITIES INVOLVING FIBONACCI NUMBERS

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Rather extensive lists of identities involving Fibonacci numbers have been given by K. Subba Rao [1] and by David Zeitlin [2]. Additional identities are presented here, with the feature that summation by parts has been used for effecting the proofs (except for identity 23).

Let $f_o = 0$ and $f_1 = 1$ and let $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Then

(1) $$\sum_{k=0}^{n} kf_k = nf_{n+2} - f_{n+3} + 2$$

(2) $$\sum_{k=0}^{n} (1)^k kf_k = (-1)^n (n+1) f_{n-1} + (-1)^{n-1} f_{n-2} - 2, \quad n \geqq 2$$

(3) $$\sum_{k=0}^{n} (-1)^k f_{2k} = \left[(-1)^n (f_{2n+2} + f_{2n}) - 1 \right] /5$$

(4) $$\sum_{k=0}^{n} (-1)^k f_{2k+1} = \left[(-1)^n (f_{2n+3} + f_{2n+1}) + 2 \right] /5$$

(5) $$\sum_{k=0}^{n} kf_{2k} = (n+1) f_{2n+1} - f_{2n+2}$$

(6) $$\sum_{k=0}^{n} kf_{2k+1} = (n+1) f_{2n+2} - f_{2n+3} + 1$$

(7) $$\sum_{k=0}^{n} (-1)^k kf_{2k} = (-1)^n (nf_{2n+2} + (n+1) f_{2n})/5$$

(8) $$\sum_{k=0}^{n} (-1)^k kf_{2k+1} = (-1)^n (nf_{2n+3} + (n+1) f_{2n+1})/5 - 1/5$$

(9) $$\sum_{k=0}^{n} k^2 f_{2k} = (n^2+2)f_{2n+1} - (2n+1)f_{2n} - 2$$

(10) $$\sum_{k=0}^{n} k^2 f_{2k+1} = (n^2+2)f_{2n+2} - (2n+1)f_{2n+1} - 1$$

(11) $$\sum_{k=0}^{2n} k f_k^2 = (2n+1)f_{2n}f_{2n+1} - f_{2n+1}^2 + 1$$

(12) $$2 \sum_{k=0}^{n} (-1)^k k f_{m+3k} = (-1)^n (n+1) f_{m+3n+1} - ((-1)^n f_{m+3n+2} + f_{m-1})/2, \quad m=2, 3, \ldots$$

(13) $$3 \sum_{k=0}^{n} (-1)^k k f_{m+4k} = (-1)^n (n+1) f_{m+4n+2} - ((-1)^n f_{m+4n+4} + f_m)/3, \quad m=2, 3, \ldots$$

(14) $$121 \sum_{k=0}^{n} (-1)^k k f_{m+5k} = (-1)^n \left[(55n+35)f_{m+5n+1} - 25 f_{m+5n+2} + (22n+18) f_{m+5n}\right] - \left[20 f_{m+1} - 17 f_m - 10 f_{m-1}\right], \quad m=1, 2, \ldots$$

(15) $$\sum_{k=0}^{n} \sum_{k_1=0}^{k} f_{k_1} = f_{n+4} - (n+3)$$

(16) $$\sum_{k=0}^{n} k \sum_{k_1=0}^{k} f_{k_1} = (n+1)f_{n+4} - f_{n+6} + 5 - n(n+1)/2$$

(17) $$\sum_{k=0}^{n} k^2 f_k = (n^2 + 2)f_{n+2} - (2n-3)f_{n+3} - 8$$

(18) $$\sum_{k=0}^{n} k^3 f_k = (n^3 + 6n - 12)f_{n+2} - (3n^2 - 9n + 19)f_{n+3} + 50$$

(19) $$\sum_{k=0}^{n} k^4 f_k = (n^4 + 12n^2 - 48n + 98)f_{n+2} - (4n^3 - 18n^2 + 76n - 159)f_{n+3} - 416$$

(20) $$\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$$

(21) $$\sum_{k=0}^{n} f_k^2 f_{k+1} = \frac{1}{2} f_{n+2} f_{n+1} f_n$$

(22) $$\sum_{k=0}^{n} f_k^2 f_{k+2} = \frac{1}{2} (f_{n+3} f_{n+1} f_n - (-1)^n f_{n-1} + 1)$$

(23) $$\sum_{k=0}^{n} f_k^3 = \frac{1}{2} (f_{n+1}^2 f_n - (-1)^n f_{n-1} + 1) \quad n \geq 1$$

(24) $$\sum_{k=0}^{n} kf_k^3 = \frac{n+1}{2} (f_{n+1}^2 f_n - (-1)^n f_{n-1}) - \frac{1}{4} f_{n+2} f_{n+1}^2 + \frac{(-1)^n}{4}(3f_n - 2f_{n-1}) + \frac{5}{4}$$

The well-known method of summation by parts is established from the identity

$$u_k \Delta v_k = \Delta (u_k v_k) - v_{k+1} \Delta u_k$$

On summing there results

$$\sum_{k=0}^{n} u_k \Delta v_k = u_k v_k \Big|_0^{n+1} - \sum_{k=0}^{n} v_{k+1} \Delta u_k$$

Of course, a suitable choice of u_k and Δv_k is essential just as it is in integration by parts. In order to find v_k from Δv_k results in [1] and [2] have been used when needed. Also, any constant term in v_k can be omitted in the two terms of the right member.

To prove (1), let $u_k = k$ and $\Delta v_k = f_k$. Then $\Delta u_k = 1$ and

$$v_k = \sum_{i=0}^{k-1} f_i = f_{k+1} - 1 .$$

Omitting the constant -1 from v_k, we find

$$\sum_{k=0}^{n} kf_k = kf_{k+1}\Big|_0^{n+1} - \sum_{k=0}^{n} 1\cdot f_{k+2} = (n+1)f_{n+2} - (f_{n+4} - 1 - f_1)$$

$$= nf_{n+2} - (f_{n+4} - f_{n+2}) + 2$$

$$= nf_{n+2} - f_{n+3} + 2$$

To prove (2), let $u_k = k$ and

$$\Delta v_k = (-1)^k f_k = \sum_{i=0}^{k} (-1)^i f_i - \sum_{i=0}^{k-1} (-1)^i f_i .$$

Then $\Delta u_k = 1$ and $v_k = (-1)^k f_{k-2} - 1$. Omitting the term -1 from v_k, with $k \geq 2$

$$\sum_{k=0}^{n} (-1)^k kf_k = \sum_{k=2}^{n} (-1)^k kf_k - 1 = k(-1)^{k-1} f_{k-2}\Big|_2^{n+1} - \sum_{k=2}^{n} (-1)^k f_{k-1} - 1$$

$$= (-1)^n (n+1) f_{n-1} + \sum_{k=1}^{n-1} (-1)^k f_k - 1$$

$$= (-1)^n (n+1) f_{n-1} + (-1)^{n-1} f_{n-2} - 2$$

To prove (3) and (4), together, write in (3) $u_k = (-1)^k$ and

$$\Delta v_k = \sum_{i=0}^{k} f_{2i} - \sum_{i=0}^{k-1} f_{2i}$$

So that $\Delta u_k = 2(-1)^{k-1}$ and $v_k = f_{2k-1}$. Then

$$A = \sum_{k=0}^{n} (-1)^k f_{2k} = \sum_{k=1}^{n} (-1)^k f_{2k} = (-1)^k f_{2k-1} \Big|_1^{n+1} - 2 \sum_{k=0}^{n} (-1)^{k+1} f_{2k+1} - 2$$

$$= (-1)^{n+1} f_{2n+1} - 1 + 2B,$$

where

$$B = \sum_{k=0}^{n} (-1)^k f_{2k+1}.$$

In (4) let $u_k = (-1)^k$ and $\Delta v_k = f_{2k+1}$ so $\Delta u_k = 2(-1)^{k+1}$ and $v_k = f_{2k}$. Then

$$B = \sum_{k=0}^{n} (-1)^k f_{2k+1} = (-1)^k f_{2k} \Big|_0^{n+1} - 2 \sum_{k=0}^{n} (-1)^{k+1} f_{2k+2}$$

$$= (-1)^{n+1} f_{2n+2} + 2(-1)^n f_{2n+2} - 2A$$

Solving gives the results.

To obtain (5) let $u_k = k$ and then $v_k = f_{2k-1}$. This gives

$$\sum_{k=0}^{n} kf_{2k} = kf_{2k-1} \Big|_0^{n+1} - \sum_{k=0}^{n} f_{2k+1} = (n+1)f_{2n+1} - f_{2n+2}$$

The others are proved similarly, except that (23) was obtained from (21) and (22). Note that the same method could be used to extend the results.

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1. K. Subba Rao, "Some Properties of Fibonacci Numbers," American Mathematical Monthly, Vol. 60, 1953, pp. 680-684.

2. David Zeitlin, "On Identities for Fibonacci Numbers," American Mathematical Monthly, Vol. 70, 1963, pp. 987-991.

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CONCERNING THE EUCLIDEAN ALGORITHM

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In most discussions of the integer solutions of the equation

$$(1) \qquad ax + by = 1, \qquad (a, b) = 1,$$

reference is made to the fact that an integer solution of (1) may be obtained by using the Euclidean algorithm. With the restriction that $a > b > 1$ we shall show that in the x-y plane the solution of (1) obtained by the Euclidean algorithm is the lattice point on the line (1) which is nearest the origin. This is probably not a new result, but we cannot find a reference to it in the literature. Dickson [1, pp. 41-52] gives other algorithms for solving (1) for which it is known that the algorithm yields the lattice point on (1) which is nearest the origin.

Suppose $a > b$, $(a, b) = 1$, and $a \not\equiv 1 \pmod{b}$ and consider the Euclidean algorithm applied to the integers a and b. One obtains the well-known sequence of equations:

$$\begin{array}{lllll}
a & = b & q_1 & + r_1, & 1 < r_1 < b \\
b & = r_1 & q_2 & + r_2, & 1 < r_2 < r_1 \\
r_1 & = r_2 & q_3 & + r_3, & 1 < r_3 < r_2 \\
 & \vdots & & & \\
r_{n-3} & = r_{n-2} & q_{n-1} & + r_{n-1}, & 1 < r_{n-1} < r_{n-2} \\
r_{n-2} & = r_{n-1} & q_n & + r_n &
\end{array}$$

with $r_n = 1$. The requirement that $a \not\equiv 1 \pmod{b}$ is equivalent to $r_1 > 1$, and hence the Euclidean algorithm will require at least a second step. Hence $n \geqq 2$ and $r_{n-1} \geqq 2$.

To obtain a solution of (1) one then derives the following sequence of equations in which, for notational convenience, $a = r_{-1}$ and $b = r_0$:

$$
\begin{aligned}
1 = r_n &= r_{n-2} - q_n r_{n-1} \\
&= -q_n r_{n-3} + (1 + q_n q_{n-1}) r_{n-2} \\
&\;\;\vdots \\
&= P_i r_{n-i-1} + Q_i r_{n-i} \qquad (3) \\
&\;\;\vdots \\
&= P_n r_{-1} + Q_n r_0 \; .
\end{aligned}
$$

The P_i and Q_i are polynomials in the q_i and the solution (P_n, Q_n) will be called the Euclidean algorithm solution of (1). It is determined uniquely by the algorithm described by the equations (2) and (3).

Lemma 1: $|P_n| < \frac{1}{2} b$ and $|Q_n| < \frac{1}{2} a$.

Proof: We first prove by induction

$$|P_i| \leqq \frac{1}{2} r_{n-i} \tag{4}$$

and

$$|Q_i| < \frac{1}{2} r_{n-i-1} \quad \text{for} \quad i = 1, \ldots, n, \tag{5}$$

with equality possible in (4) only if $i = 1$. We have

$$1 = P_i r_{n-i-1} + Q_i r_{n-i} ,$$

and since

$$r_{n-i-2} = r_{n-i-1} q_{n-i} + r_{n-i}$$

it follows that

$$1 = Q_i r_{n-i-2} + (P_i - q_{n-i} Q_i) r_{n-i-1}$$

and we have the recurrence relations

$$P_{i+1} = Q_i \tag{6}$$

and

(7) $$Q_{i+1} = P_i - q_{n-i}Q_i$$

with $P_1 = 1$ and $Q_1 = -q_n$. To prove that $|P_1| = 1 \leqq \frac{1}{2} r_{n-1}$ recall that $r_{n-1} \geqq 2$. Similarly,

$$|Q_1| = q_n = \left[\frac{r_{n-2}}{r_{n-1}}\right] < \frac{r_{n-2}}{r_{n-1}} < \frac{1}{2} r_{n-2} ,$$

From (6) it follows that $|P_2| < \frac{1}{2} r_{n-2}$, and from (7) $|Q_2| < \frac{1}{2} r_{n-3}$ since

$$\begin{aligned}|Q_2| = |P_1 - q_{n-1}Q_1| &\leqq |P_1| + q_{n-1} |Q_1| \\ &< \frac{1}{2} r_{n-1} + q_{n-1} \cdot \frac{1}{2} \cdot r_{n-2} \\ &= \frac{1}{2} r_{n-3} .\end{aligned}$$

Now suppose that

$$|P_k| < \frac{1}{2} r_{n-k} \quad \text{and} \quad |Q_k| < \frac{1}{2} r_{n-k-1}$$

for $k = 2, \ldots, i$. Then from (6)

$$|P_{k+1}| = |Q_k| < \frac{1}{2} r_{n-k-1} ,$$

and

$$\begin{aligned}|Q_{k+1}| = |P_k - q_{n-k}Q_k| &\leqq |P_k| + q_{n-k} |Q_k| \\ &< \frac{1}{2} r_{n-k} + q_{n-k} \left(\frac{1}{2} r_{n-k-1}\right) \\ &= \frac{1}{2} r_{n-k-2} .\end{aligned}$$

This completes the induction. Since $r_{-1} = a$ and $r_0 = b$, we have proved the lemma if we take $i = n$ in (4) and (5).

It seems intuitively clear that there cannot be two lattice points on (1) which are equidistant from the origin if $a \neq b$. The proof of this is straightforward but for completeness we give it here.

Lemma 2: If $a > b > 0$ and $(a, b) = 1$, there do not exist two distinct lattice points on $ax + by = 1$ which are equidistant from the origin.

Proof: Suppose (α, β) and (ξ, η) are distinct lattice points on the given line which are equidistant from the origin. Then

$$\alpha^2 + \beta^2 = \xi^2 + \eta^2 \tag{8}$$

and $a\alpha + b\beta = a\xi + b\eta = 1$. We solve for β in terms of α, η in terms of ξ, and substitute these in (8) to obtain

$$(\alpha^2 - \xi^2)b^2 = 2a(\alpha - \xi) - a^2(\alpha^2 - \xi^2). \tag{9}$$

Since $\alpha \neq \xi$ by hypothesis,

$$(\alpha + \xi)b^2 = 2a - a^2(\alpha + \xi). \tag{10}$$

But this implies that $a \mid (\alpha + \xi)$ since $(a, b) = 1$, and also that $(\alpha + \xi) \mid 2a$. Hence, $\alpha + \xi = \pm a$, or $\alpha + \xi = \pm 2a$. If $\alpha + \xi = \pm a$, then (10) implies the Diophantine equation $a^2 + b^2 = \pm 2$ which is impossible if $a \neq b$. If $\alpha + \xi = \pm 2a$, then $a^2 + b^2 = \pm 1$. Clearly there is no solution to this equation such that $a > b > 0$ and $(a, b) = 1$.

It is well known that if (x_0, y_0) is any lattice point on (1) then all of the lattice points on (1) are given by the equations

$$x = x_0 - bt$$
$$y = y_0 + at$$

where t runs over the set of all integers. We can now prove our

Theorem. If $a > b > 1$ and $(a, b) = 1$ then the Euclidean algorithm solution of (1) is the lattice point on (1) which is nearest the origin.

Proof. First suppose that $a \not\equiv 1 \pmod{b}$. Denote the Euclidean algorithm solution of (1) by (P_n, Q_n). Clearly the set, S, of positive integers $(P_n - bt)^2 + (Q_n + at)^2$ has a smallest member. If $P_n^2 + Q_n^2$ is not the smallest number in S then there exists an integer $t \neq 0$ such that

$$P_n^2 + Q_n^2 > (P_n - bt)^2 + (Q_n + at)^2$$

or

$$0 < (a^2 + b^2)\ |t| < 2|P_n b - Q_n a|.$$

But from the lemma we have

$$0 < (a^2+b^2)\ |t| \leqq 2(|P_n|\ b + |Q_n|a) < a^2 + b^2.$$

This is impossible; hence $t = 0$ and (P_n, Q_n) is the smallest number in S.

The only remaining case is if $a \equiv 1 \pmod b$ and $a > b > 1$. Here the Euclidean algorithm is complete in one step and $P_1 = 1$ and $Q_1 = -q_1 = -(a-1)/b$. The expression $S(t) = (P_1 - bt)^2 + (Q_1 + at)^2$ can be rewritten

$$c\left[t - \frac{c-a}{bc}\right]^2 + \frac{1}{b^2}$$

where $c = a^2 + b^2$. Now $S(t)$ is a minimum for $t=t^*=(c-a)/bc$, but $b > 1$ and $c > a$ imply that $c(b-1) + a > 0$, or $0 < t^* < 1$. Therefore, the integer t for which $S(t)$ is a minimum is either 0 or 1. It is easy to show that $S(1) > S(0)$ if $(c-a)/bc < 1/2$. But

$$\frac{c-a}{bc} < \frac{1}{b} \text{ and } b \geqq 1;$$

hence (P_1, Q_1) is the point on $ax + by = 1$ which is nearest the origin. This completes the proof of the theorem.

It is an easy consequence of this theorem that if a and b are consecutive Fibonacci numbers, $a > b > 1$, then the lattice point P on the line $ax + by = 1$ which is nearest the origin has Fibonacci coordinates. In fact, if $a = F_{m+1}$, then P is $(F_{n-1}, -F_n)$ where n is the greatest even integer not exceeding m. This follows readily from the identity

$$F_{m+1} F_{n-1} - F_m F_n = (-1)^n F_{m-n+1}.$$

REFERENCES

1. Dickson, L. E., "History of the Theory of Numbers," Vol. 2, Chelsea, New York (1952).

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A STRIP METHOD OF SUMMING LINEAR FIBONACCI EXPRESSIONS

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Given a linear Fibonacci expression such as

$$362880\,F_{r+21} - 2177280\,F_{r+19} + 5594400\,F_{r+17} - 8013600\,F_{r+15}$$
$$+ 6972840\,F_{r+13} - 3759840\,F_{r+11} + 1225230\,F_{r+9} - 223290\,F_{r+7}$$
$$+ 19171\,F_{r+5} - 512\,F_{r+3} + F_{r+1}$$

we wish to express this, for example, as

$$A\,F_{r+11} + B\,F_{r+10}$$

The formulas for doing so are well known being

$$F_n = F_{k+1}F_{n-k} + F_k F_{n-k-1}$$

and

$$F_n = F_k F_{n+k+1} - F_{k+1} F_{n+k}$$

However, the direct process can be replaced by a strip method in which the given coefficients are arranged in descending order of F subscripts, one space being allowed for each subscript, even though certain subscripts may be missing in the given linear expression. This may be done conveniently on ruled paper, the strip employed having the same spacing in its rulings as the paper.

The strip consists of the Fibonacci numbers in descending order. To obtain the coefficient of the higher subscript Fibonacci number in the summation, place the 1 above the zero at the place of the higher subscript, multiply each number on the strip by the corresponding given coefficient and add the results. To find the coefficient of the lower subscript Fibonacci number, do likewise with the 1 below the zero opposite the position of the lower subscript Fibonacci number.

The work is shown for the example given at the beginning of this note.

UPPER SUBSCRIPT CALCULATION

STRIP	GIVEN COEFFICIENTS	PRODUCTS
89	362880	32296320
55		
34	-2177280	-74027520
21		
13	5594400	72727200
8		
5	-8013600	-40068000
3		
2	6972840	13945680
1		
1	-3759840 (F_{r+11})	-3759840
0		
1	1225230	1225230
-1		
2	-223290	-446580
-3		
5	19171	95855
-8		
13	-512	-6656
-21		
34	1	34
		1981723

LOWER SUBSCRIPT CALCULATIONS

STRIP	GIVEN COEFFICIENTS	PRODUCTS
55	362880	19958400
34		
21	-2177280	-45722880
13		
8	5594400	44755200
5		
3	-8013600	-24040800
2		
1	6972840	6972840
1		
0	-3759840 (F_{r+10})	0
1		
-1	1225230	-1225230
2		
-3	-223290	669870
5		
-8	19171	-153368
13		
-21	-512	10752
34		
-55	1	-55
		1224729

The final result would thus be

$$1981723\ F_{r+11} + 1224729\ F_{r+10}$$

In carrying out these calculations it goes without saying that the products need not be written out but may be cumulated on a calculator.

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THE FIBONACCI ASSOCIATION ANNOUNCES.......

The appearance of a booklet entitled: "Introduction to Fibonacci Discovery" by Brother U. Alfred, Managing Editor of the Fibonacci Quarterly. As the title implies the aim of this publication is to provide the reader with the opportunity to work out various facets of the Fibonacci numbers by himself. At the same time, there is sufficient help in the form of answers and explanations to reassure him regarding the correctness of his work.

The treatment is relatively brief, there being some sixty pages in all. The material was set up by typewriter and subsequently lithographed. The books have a paper cover and are held together by glue binding. Price per copy is $1.50 with a quantity price of $1.25 when four or more copies are ordered at once. The following topics are treated:

Discovering Fibonacci Formulas
Proof of Formulas by Mathematical Induction
The Fibonacci Shift Formulas
Explicit Formulas for the Fibonacci and Lucas Sequences
Division Properties of Fibonacci Numbers
General Fibonacci Sequences
The Associated "Lucas" Sequence
The Fibonacci Sequence and Pascal's Triangle
The Golden Section
Matrices and Fibonacci Numbers
Continued Fractions and Fibonacci Numbers

This booklet should provide the means of becoming acquainted with Fibonacci numbers and some of their main ramifications. It should serve as a useful reference for readers of the Fibonacci Quarterly who wish to learn about the main aspects of Fibonacci numbers. It should also prove of value to groups of competent high school or college students. While not recommended for the "pro", it might be a useful reference to have on hand to loan to students or fellow faculty members who want to know something about Fibonacci numbers.

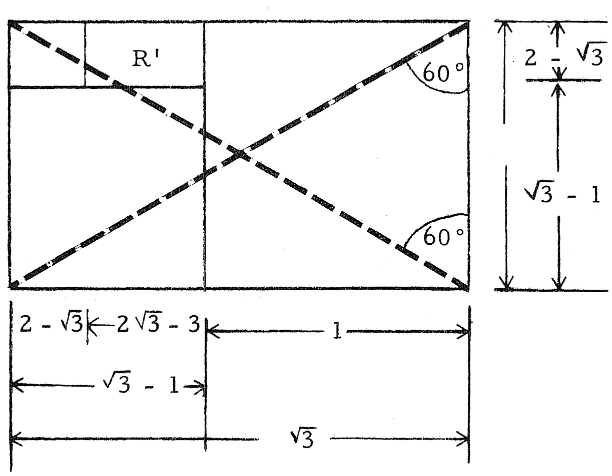
The booklet is now available for purchase. Send all orders to:
Brother U. Alfred, Managing Editor, St. Mary's College, Calif.
(Note. This address is sufficient, since St. Mary's College <u>is</u> a post office.)

A NEAR-GOLDEN RECTANGLE AND RELATED RECURSIVE SERIES

MARJORIE BICKNELL AND JAMES LEISSNER
Adrian C. Wilcox High School, Santa Clara, California
Thomas Jefferson High School, San Antonio, Texas

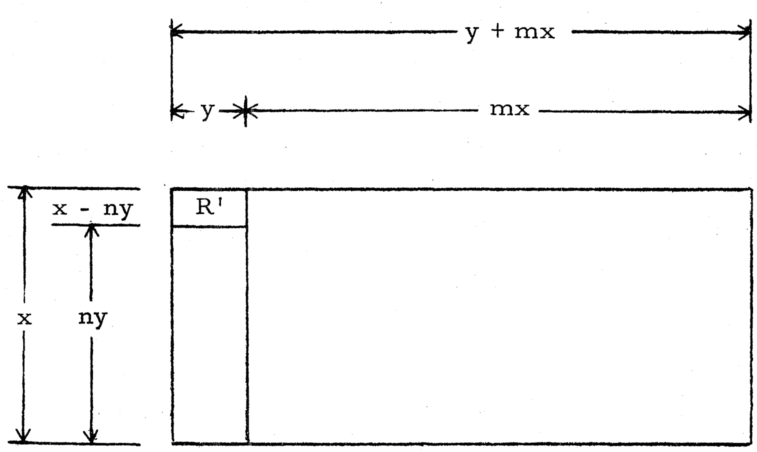
The rectangle whose diagonals form equilateral triangles with its widths has some surprising properties, including a related Fibonacci-like series of integers. Before discussing this rectangle, for later comparison, we call to mind another rectangle. The famous Golden Rectangle has the property that when a full-width square is cut from one end, the remaining part has the same proportions as the original rectangle, the ratio of length to width being $(1 + \sqrt{5})/2$. Joseph Raab discussed other golden-type rectangles [1], which have the property that when an integral number k of full-width squares are cut from one end, the remaining part has the same proportions as the original rectangle. These golden-type rectangles also have related series of integers.

In the rectangle whose diagonals form equilateral triangles with its widths, the ratio of length to width is $\sqrt{3}$, certainly not "golden." But after cutting a full-width square from one end, there appears a glitter as the ratio of length to width becomes $(1 + \sqrt{3})/2$. Operating similarly on this rectangle, the ratio becomes $\sqrt{3} + 1$, and repeating the process one last time makes the ratio of length to width again $\sqrt{3}$.



Some more "near-golden" rectangles appear as more general cases of removing squares of the width in a rectangle to obtain rectangles similar to the original. To simplify the discussion, we will designate a rectangle by a capital letter and its ratio of length to width by the corresponding small letter.

From a rectangle R with width x and length $y + mx$, remove the total number m of full-width squares contained in R to obtain rectangle P. From P, remove the total number n of full-width squares contained in P to form rectangle R'.



If R' is similar to R, then $r' = r$ so that $y/(x - ny) = (y + mx)/x$. Solving for x/y 1 p, we find

$$r' = r = (mn + \sqrt{m^2n^2 + 4mn})/2n,$$

$$p = (mn + \sqrt{m^2n^2 + 4mn})/2m,$$

(Note that $R:R' = rp$, and that $m = n = 1$ yields the Golden Rectangle.)

When we cut full-width squares from P, if we remove an integral number n less than the total number of full-width squares available, and if R' and R are similar,

$$r = (\sqrt{(m+n)^2 + 4} + m - n)/2,$$

$$p = (\sqrt{(m+n)^2 + 4} + m + n)/2.$$

(Note again the Golden Rectangle for $m = 1$ and $n = 0$, when $P = R'$.)

Suppose that we remove the full amount of available full-width squares in forming P and R', but R' and R are not similar. If a rectangle T, similar to R, can be obtained from R' by the removal of an integral number q of squares of the width of R', then

$$r = t = (\sqrt{n^2(m+q)^2 + 4n(m+q)} + n(m - q))/2n,$$

$$p = (\sqrt{n^2(m+q)^2 + 4n(m+q)} + n(m + q))/2(m + q),$$

$$r' = (\sqrt{n^2(m+q)^2 + 4n(m+q)} + n(m + q)/2n.$$

Again, $q = 0$ and $m = 1$ yields the Golden Rectangle, with $r = p = r' = (1 + \sqrt{5})/2$. Also, $q = m = n = 1$ yields (for R and T) the rectangle with diagonals forming equilateral triangles with its widths, with $p = (1 + \sqrt{3})/2$.

The similarity of form between the ratio $(1 + \sqrt{3})/2$, hereafter called θ, and the golden ratio given above, suggests that we seek a Fibonacci-type series associated with powers of θ. Consider the following:

$$\begin{aligned}
\theta &= (1 + \sqrt{3})/2 &&= (1)\theta + 0\\
\theta^2 &= (2 + \sqrt{3})/2 &&= (1)\theta + 1/2\\
\theta^3 &= (5 + 3\sqrt{3})/4 &&= (3/2)\theta + 1/2\\
\theta^4 &= (7 + 4\sqrt{3})/4 &&= (4/2)\theta + 3/4\\
\theta^5 &= (19+11\sqrt{3})/8 &&= (11/4)\theta + (4/4)\\
\theta^6 &= (26+15\sqrt{3})/8 &&= (15/4)\theta + (11/8).
\end{aligned}$$

The numerators of either the coefficients of θ or the constant addends and the coefficients of $\sqrt{3}$ form the following series: 1, 1, 3, 4, 11, 15, 41, 56, ... It can be proved by induction that this series is defined by

$$\begin{aligned}
P_{2n} &= P_{2n-1} + P_{2n-2}\\
P_{2n+1} &= 2P_{2n} + P_{2n-1}, \quad n = 1, 2, \ldots,
\end{aligned}$$

where $P_1 = P_2 = 1$. A second series: 1, 2, 5, 7, 19, 26, ..., having the same recursion formulas as the above, appears in the computation of powers of θ. We shall call the nth term in the second series R_n.

If $\theta = (1 + \sqrt{3})/2$ and $\phi = (1 - \sqrt{3})/2$, it is not difficult to show by induction that

$$P_n = (\theta^n - \phi^n)/\sqrt{3}\cdot 2^{[1-n/2]},$$

$$R_n = (\theta^n + \phi^n)/2^{[1-n/2]}, \quad n = 1, 2, 3, \ldots,$$

where $[x]$ is the largest integer in x. The series just defined bear a striking resemblance to the Fibonacci and Lucas series as defined by the Binet formula in terms of the golden ratio, where the nth Fibonacci and nth Lucas number are given respectively by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad L_n = \alpha^n + \beta^n \text{ for } \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

Use of the above form for P_n and R_n and standard limit theorems leads to

$$\lim_{n \to \infty} P_{2n}/P_{2n-1} = \theta \text{ and } \lim_{n \to \infty} R_{2n}/R_{2n-1} = \phi;$$

$$\lim_{n \to \infty} P_{2n+1}/P_{2n} = 2\theta \text{ and } \lim_{n \to \infty} R_{2n+1}/R_{2n} = 2\phi.$$

Finally, as n increases, R_n/P_n oscillates about its limit, $\sqrt{3}$.

Also established by induction are forms for powers of θ.

$$\theta^n = (P_n\theta)/2^{[(n-1)/2]} + P_{n-1}/2^{[n/2]} = (R_n + P_n\sqrt{3})/2^{[(n+1)/2]}$$

and

$$\theta^{-n} = (-2)^n \left(P_{n+1}/2^{[n/2]} - P_n\theta/2^{[(n-1)/2]}\right).$$

For comparison, if

$$\frac{1 + \sqrt{5}}{2} = \alpha, \text{ then } \alpha^n = (L_n + F_n\sqrt{5})/2,$$

where F_n is the nth Fibonacci number and L_n the nth Lucas number.

Other theorems, also possible to establish by induction, are:

$$\sum_{i=1}^{2n} P_i = P_{2n+1} - (P_{2n-1} + 1)/2,$$

$$\sum_{i=1}^{2n+1} P_i = (P_{2n+3} - 1)/2,$$

$$\sum_{i=1}^{2(2n-1)} P_i = P_{2n}P_{2n+1} - P_{2n-1}^2$$

$$P_nP_{n+3} - P_{n+1}P_{n+2} = (-1)^{n+1}.$$

Considering the even ordered elements and the odd ordered elements of the series separately leads to

$$P_{2n} = 4P_{2n-2} - P_{2n-4}$$

$$P_{2n+1} = 4P_{2n-1} - P_{2n-3},$$

which in turn can be used to prove the following relationships between R_n and P_n, and summation formulas for even or odd elements of the series P_n:

$$R_{2n} = P_{2n-1} + P_{2n},$$

$$3P_{2n} = R_{2n-1} + R_{2n};$$

$$\sum_{i=1}^{n} P_{2i} = (P_{2n+1} - 1)/2 = (3P_{2n} - P_{2n-2} - 1)/2,$$

and

$$\sum_{i=1}^{n} P_{2i-1} = P_{2n} = (P_{2n+3} - P_{2n+1})/2.$$

REFERENCES

1. Joseph A. Raab, "A Generalization of the Connection Between the Fibonacci Sequence and Pascal's Triangle," Fibonacci Quarterly, 3:1, Oct., 1963, pp. 21-32.

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FACTORIZATION OF 36 FIBONACCI NUMBERS F_n WITH $n > 100$

L. A. G. DRESEL AND D. E. DAYKIN
Reading University, England

The Fibonacci numbers F_n are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n > 1$. We present below the factorization of certain F_n with $n > 100$. The factors of F_n before the double asterisk are improper factors of F_n (they divide F_m with $m < n$), and those behind the double asterisk are proper factors of F_n. All the factors shown are believed to be primes. We obtained the results on the Elliott 803 computer at Reading University, and we hope to discuss our methods and extend the table in a later paper.

F102 = 2^3 * 1597 * 3571 * 6376021 ** 919 * 3469
F104 = 3 * 7 * 233 * 521 * 90481 ** 103 * 102193207
F105 = 2 * 5 * 13 * 61 * 421 * 141961 ** 8288823481
F106 = 953 * 55945741 ** 119218851371
F108 = 2^4 * 3^4 * 17 * 19 * 53 * 107 * 109 * 5779 ** 11128427
F110 = 5 * 11^2 * 89 * 199 * 661 * 474541 ** 331 * 39161
F112 = 3 * 7^2 * 13 * 29 * 47 * 281 * 14503 ** 10745088481
F114 = 2^3 * 37 * 113 * 797 * 9349 * 54833 ** 229 * 95419
F116 = 3 * 59 * 19489 * 514229 ** 347 * 1270083883
F117 = 2 * 17 * 233 * 135721 ** 29717 * 39589685693
F118 = 353 * 2710260697 ** 709 * 8969 * 336419
F120 = 2^5 * 3^2 * 5 * 7 * 11 * 23 * 31 * 41 * 61 * 2161 * 2521 ** 241 * 20641
F126 = 2^3 * 13 * 17 * 19 * 29 * 211 * 421 * 35239681 ** 1009 * 31249
F128 = 3 * 7 * 47 * 1087 * 2207 * 4481 ** 127 * 186812208641
F129 = 2 * 433494437 ** 257 * 5417 * 8513 * 39639893
F130 = 5 * 11 * 233 * 521 * 14736206161 ** 131 * 2081 * 24571
F132 = 2^4 * 3^2 * 43 * 89 * 199 * 307 * 9901 * 19801 ** 261399601
F134 = 269 * 116849 * 1429913 ** 4021 * 24994118449
F138 = 2^3 * 137 * 139 * 461 * 829 * 18077 * 28657 ** 691 * 1485571
F140 = 3 * 5 * 11 * 13 * 29 * 41 * 71 * 281 * 911 * 141961 ** 12317523121
F144 = 2^6 * 3^3 * 7 * 17 * 19 * 23 * 47 * 107 * 1103 * 103681 ** 10749957121
F147 = 2 * 13 * 97 * 421 * 6168709 ** 293 * 3529 * 347502052673
F148 = 3 * 73 * 149 * 2221 * 54018521 ** 11987 * 81143477963

F150 = 2^3 * 5^2 * 11 * 31 * 61 * 101 * 151 * 3001 * 230686501 ** 12301 * 18451

F156 = 2^4 * 3^2 * 79 * 233 * 521 * 859 * 90481 * 135721 ** 12280217041

F162 = 2^3 * 17 * 19 * 53 * 109 * 2269 * 4373 * 5779 * 19441 ** 3079 * 62650261

F165 = 2 * 5 * 61 * 89 * 661 * 19801 * 474541 ** 86461 * 518101 * 900241

F168 = 2^5 * 3^2 * 7^2 * 13 * 23 * 29 * 83 * 211 * 281 * 421 * 1427 * 14503 ** 167 * 65740583

F174 = 2^3 * 59 * 173 * 19489 * 514229 * 3821263937 ** 349 * 947104099

F180 = 2^4 * 3^3 * 5 * 11 * 17 * 19 * 31 * 41 * 61 * 107 * 181 * 541 * 2521 * 109441 ** 10783342081

F190 = 5 * 11 * 37 * 113 * 761 * 9349 * 29641 * 67735001 ** 191 * 41611 * 87382901

F198 = 2^3 * 17 * 19 * 89 * 197 * 199 * 9901 * 19801 * 18546805133 ** 991 * 2179 * 1513909

F204 = 2^4 * 3^2 * 67 * 919 * 1597 * 3469 * 3571 * 63443 * 6376021 ** 409 * 66265118449

F210 = 2^3 * 5 * 11 * 13 * 29 * 31 * 61 * 71 * 211 * 421 * 911 * 141961 * 8288823481 ** 21211 * 767131

F216 = 2^5 * 3^4 * 7 * 17 * 19 * 23 * 53 * 107 * 109 * 5779 * 103681 * 11128427 ** 6263 * 177962167367

F228 = 2^4 * 3^2 * 37 * 113 * 229 * 797 * 9349 * 54833 * 95419 * 29134601 ** 227 * 26449 * 212067587

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LETTER TO THE EDITOR

ERIC HALSEY
Redlands, California

Re: My article The Fibonacci Number F_u where u is not an integer in issue number 2 of the current volume of the Quarterly. I have discovered that, due to excessive haste and timidity on my part, I placed undue restrictions on the letter u. This variable can assume not only all rational values, as stated in the article, but all real values as well. Obviously, only for rational values can a complete numerical expression of F_u be obtained.

COMMENTS ON "THE GENERATED, COMPOSITIONS YIELD FIBONACCI NUMBERS"

HENRY WINTHROP
University of South Florida, Tampa, Florida

The following explanations will serve to round out the paper mentioned above which appears in The Fibonacci Quarterly (Vol. 3, No. 2, 131-4).

The expression $F(h_i, \phi)$ of model display (1) designates the partitions of the integer, i, in which the partitions are expressed as functions in ϕ and in which the coefficient of each partition represents the number of possible permutations of that partition.

The general term of model display (3) can be given as

$$(1) \qquad = ik + \frac{[(i-1)+2]!\,k^2}{(i-2)!\,3!} + \frac{[(i-2)+4]!\,k^3}{(i-3)!\,5!} + \ldots$$

$$+ \frac{[i-(n-1)+2(n-1)]!\,k^n}{(i-n)!\,(2n-1)!} + \ldots + \frac{k^i}{(i-n)!\,(2n-1)!}$$

where the coefficient of k^n is the n-th term of the 2r-th order of the figurate numbers.

A discussion of figurate numbers of various orders will be found in Higher Algebra by Hall and Knight (Macmillan, 1936, 4th edition), pp. 319-22.

The following additional references to the paper in question will be found of value by the reader.

1. H. Winthrop, "Mathematics In The Social Sciences," School Science and Mathematics, 1957, Vol. 57, pp. 9-16.
2. H. Winthrop, "On The Use of Difference Equations In Behavioral Diffusion Theory," School Science and Mathematics, 1958, Vol. 58, pp. 1-6.
3. H. Winthrop, "A Kinetic Theory Of Socio-Psychological Diffusion," Journal of Social Psychology, 1945, Vol. 22, 31-60.
4. H. Winthrop, "Experimental Results In Relations To A Mathematical Theory Of Behavioral Diffusion," Journal of Social Psychology, 1958, Vol. 47, 85-99.

Continued on page 240.

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A. P. HILLMAN
University of Santa Clara, Santa Clara, California

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within two months of publication.

B-70 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Denote x^a by ex(a). Show that the following expression, containing n integrals,

$$\int_0^1 \mathrm{ex}\,(\int_0^1 \mathrm{ex}(\int_0^1 \mathrm{ex}(\dots\int_0^1 \mathrm{ex}(\int_0^1 x\,dx)dx)\dots dx)dx)dx$$

equals F_{n+1} / F_{n+2}, where F_n is the n-th Fibonacci number.

B-71 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Find $a^{-2} + a^{-3} + a^{-4} + \dots$, where $a = (1 + \quad 5)/2$.

B-72 *Proposed by J. A. H. Hunter, Toronto, Canada*

Each distinct letter in this simple alphametic stands for a particular and different digit. We all know how rabbits link up with the Fibonacci series, so now evaluate our RABBITS.

$$\begin{array}{r} \mathrm{R\ A\ B\ B\ I\ T\ S} \\ \mathrm{B\ E\ A\ R} \\ \mathrm{R\ A\ B\ B\ I\ T\ S} \\ \mathrm{A\ S} \\ \hline \mathrm{A\ S\ E\ R\ I\ E\ S} \end{array}$$

B-73 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Prove that

$$\sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k} \binom{k+r-j-1}{j} = 1 + \sum_{m=0}^{2n+r-2} \sum_{p=0}^{m} \binom{m-p-1}{p} ,$$

where $\binom{n}{r} = 0$ for $n < r$.

B-74 *Proposed by M. N. S. Swamy, University of Saskatchewan, Regina, Canada*

The Fibonacci polynomial $f_n(x)$ is defined by $f_1 = 1$, $f_2 = x$, and $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$ for $n > 2$. Show the following:

(a) $$x \sum_{r=1}^{n} f_r(x) = f_{n+1} + f_n - 1 .$$

(b) $$f_{m+n+1} = f_{m+1} f_{n+1} + f_m f_n .$$

(c) $$f_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1} ,$$

where $[k]$ is the greatest integer not exceeding k. Hence show that the n-th Fibonacci number

$$F_n = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} .$$

B-75 *Proposed by M. N. S. Swamy, University of Saskatchewan, Regina, Canada*

Let $f_n(x)$ be as defined in B-74. Show that the derivative

$$f_n'(x) = \sum_{r=1}^{n-1} f_r(x)\, f_{n-r}(x) \text{ for } n > 1.$$

SOLUTIONS

ONE, TWO, THREE — OUT

B-58 *Proposed by Sidney Kravitz, Dover, New Jersey*

Show that no Fibonacci number other than 1, 2, or 3 is equal to a Lucas number.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

Since $L_k = F_{k-1} + F_{k+1}$, the assertion is equivalent to

(1) $$F_n = F_{k-1} + F_{k+1}.$$

If $k \geq 3$, the $n > k+1$ and (1) is clearly impossible since

$$F_{k-1} + F_{k+1} < F_k + F_{k+1} = F_{k+2} \leq F_n.$$

Impossibility for $k \geq 3$ implies impossibility for $k \leq -3$ since only signs are different. For $-3 < k < 3$ we find $F_{-2} = L_{-1} = 1$, $F_3 = l_0 = 2$, $F_1 = L_1 = 1$, and $F_4 = L_2 = 3$, corresponding to $k = -1$, 0, 1, and 2 respectively. Hence these are the only solutions. (The crux of this problem is solved in the discussion of equation (12) in Carlitz' "A Note on Fibonacci Numbers," this Quarterly 1 (1964) No. 2 pp. 15-28).

Also solved by J. L. Brown, Jr.; Gary C. McDonald; C. B. A. Peck; and the proposer.

B-59 *Proposed by Brother U. Alfred, St. Mary's College, California*

Show that the volume of a truncated right circular cone of slant height F_n with F_{n-1} and F_{n+1} the diameters of the bases is

$$\sqrt{3}\,\pi(F_{n+1}^3 - F_{n-1}^3)/24.$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

It is well-known that if h is the height of the frustrum of a right circular cone, s the slant height, and r_1 and r_2 the radii of the bases, then the volume V is

$$V = (\pi h/3)(r_1^2 + r_1 r_2 + r_2^2)$$

$$= (\pi/3)\sqrt{s^2 - (r_2 - r_1)^2}\,(r_1^2 + r_1 r_2 + r_2^2).$$

For this problem $r_1 = F_{n-1}/2$, $r_2 = F_{n+1}/2$ and $s = F_n$, so that

$$V = \frac{\pi}{3}\sqrt{F_n^2 - (F_{n+1} - F_{n-1})^2/4}\,(F_{n-1}^2 + F_{n-1}F_{n+1} + F_{n+1}^2)/4$$

$$= \pi\sqrt{F_n^2 - F_n^2/4}\,(F_{n-1}^2 + F_{n-1}F_{n+1} + F_{n+1}^2)/12$$

$$= \sqrt{3}\pi\, F_n\,(F_{n-1}^2 + F_{n-1}\,F_{n+1} + F_{n+1}^2)/24$$

$$= \sqrt{3}\pi(F_{n+1} - F_{n-1})(F_{n-1}^2 + F_{n-1}F_{n+1} + F_{n+1}^2)/24$$

$$= \sqrt{3}\pi(F_{n+1}^3 - F_{n-1}^3)/24.$$

We remark that the area A of the curved surface of the frustrum is

$$A = \pi F_n(F_{n+1} + F_{n-1})/2 = (\pi/2)F_nL_n.$$

Also solved by Carole Bania, Gary C. McDonald, Kenneth E. Newcomer, C. B. A. Peck, M. N. S. Swamy, Howard L. Walton, John Wessner, Charles Ziegenfus, and the proposer. McDonald also added the formula for the curved surface.

B-60 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $L_{2n}L_{2n+2} + 5F_{2n+1}^2 = 1$, where F_n and L_n are the n-th Fibonacci number and Lucas number, respectively.

Solution by 2nd Lt. Charles R. Wall, U. S. Army, A. P. O., San Francisco, Calif.

Using my second answer to B-22 (Vol. 2, No. 1, p. 78),

$$\begin{aligned} L_{2(n+1)}\, L_{2n} &= 5F_{(n+1)+n}^2 + L_{(n+1)-n}^2 \\ &= 5F_{2n+1}^2 + L_1^2 \\ &= 5F_{2n+1}^2 + 1 \end{aligned}$$

Thus

$$L_{2n+2}L_{2n} - 5F_{2n+1}^2 = 1.$$

Also solved by J. L. Brown, Jr.; J. A. H. Hunter; Douglas Lind, Kathleen Marafino, Gary C. McDonald, C. B. A. Peck, Benjamin Sharpe, M. N. S. Swamy, Howard L. Walton, John Wessner, Kathleen M. Wickett, David Zeitlin, Charles Ziegenfus, and the proposer. Also by David Klarner.

MODULO THREE

B-61 *Proposed by J. A. H. Hunter, Toronto, Ontario*

Define a sequence $U_1, U_2, \ldots$ by $U_1 = 3$ and

$$U_n = U_{n-1} + n^2 + n + 1 \quad \text{for} \quad n > 1 \ .$$

Prove that $U_n \equiv 0 \pmod{n}$ if $n \not\equiv 0 \pmod{3}$.

Solution by John Wessner, Melbourne, Florida

An alternative representation for U_n is

$$U_n = \sum_{k=1}^{n} (k^2+k+1).$$

Upon expanding the individual sums involved we obtain

$$U_n = [n(2n+1)(n+1)/6] + [n(n+1)/2] + n = (n/3)\ [(n+2)(n+1)+3] .$$

Hence, $U_n \equiv 0 \pmod{n}$ if and only if $(n+1)(n+2) \equiv 0 \pmod{3}$. This condition obtains if and only if $n \not\equiv 0 \pmod{3}$.

Also solved by Robert J. Hursey, Jr., Douglas Lind, Gary C. McDonald, Robert McGee, C. B. A. Peck, Charles R. Wall, David Zeitlin, and the proposer.

UNIQUE SUM OF SQUARES

B-62 *Proposed by Brother U. Alfred, St. Mary's College, California*

Prove that a Fibonacci number with odd subscript cannot be represented as the sum of squares of two Fibonacci numbers in more than one way.

Solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pa.

From the identity $F_{2n+1} = F_n^2 + F_{n+1}^2$, $(n \geq 1)$ it follows that $F_{2n+1} < (F_n + F_{n+1})^2 = F_{n+2}^2$. Therefore, any representation, $F_{2n+1} = F_k^2 + F_m^2$ $(k \leq m)$ must have both k and $m \leq n+1$. Then $k \geq n$ (otherwise $F_k^2 + F_m^2 < F_n^2 + F_{n+1}^2 = F_{2n+1}$ for $k > 2$).

Also solved by Douglas Lind, Joseph A. Orjechouski and Robert McGee (jointly), C. B. A. Peck, and the proposer.

AN ISOSCELES TRIANGLE

B-63 *An old problem whose source is unknown, suggested by Sidney Kravitz, Dover, New Jersey.*

In ΔABC let sides AB and AC be equal. Let there be a point D on side AB such that $AD = CD = BC$. Show that

$$2\cos \sphericalangle A = AB/BC = (1 + \sqrt{5})/2 \ ,$$

the golden mean.

Solution by John Wessner, Melbourne, Florida

By inspection of the figure and the law of cosines

$$AD^2 = CD^2 + AC^2 - 2CD \cdot AC \cos \sphericalangle A.$$

Since $AD = CD = BC$ and $AB = AC$, it follows immediately that

$$2 \cos \measuredangle A = AC/CD = AB/BC.$$

The second result comes from the fact that

$$\measuredangle B = \measuredangle BDC = \measuredangle A + \measuredangle DCA = 2 \measuredangle A$$

and hence

$$\measuredangle A = 36^\circ \text{ and } 2 \cos A = (1 + \sqrt{5})/2.$$

(See N. N. Vorobyov: The Fibonacci Numbers (New York, (1961) p. 56.)

Also solved by Herta Taussig Freitag, Cheryl Hendrix, Kathleen Marafino, and Carol Barrington (jointly), J. A. H. Hunter, Douglas Lind, James Leissner, C. B. A. Peck, Kathleen M. Wickett, and the proposer.

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Continued from page 234.

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6. H. Winthrop, "A Mathematical Model For The Study Of The Propagation Of Novel Social Behavior," Indian Sociological Bulletin, July 1965, Vol. II. (In Press)
7. H. Winthrop, "Some Generalizations Of The Dying Rabbit Problem," (In Preparation).
8. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell Publishing Company, New York, 1961.

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ASSOCIATION PUBLISHES BOOKLET

Brother U. Alfred has just completed a new booklet entitled: Introduction to Fibonacci Discovery. This booklet for teachers, researchers, and bright students can be secured for \$1.50 each or 4 copies for \$5.00 from Brother U. Alfred, St. Mary's College, Calif.