POWER IDENTITIES FOR SEQUENCES DEFINED BY $W_{n+2} = dW_{n+1} - cW_n$

David Zeitlin
Minneapolis, Minnesota

1. INTRODUCTION

Let W_o, W_1, $c \neq 0$, and $d \neq 0$ be arbitrary real numbers, and define

$$(1.1) \qquad W_{n+2} = dW_{n+1} - cW_n \ , \ d^2 - 4c \neq 0 \ , \quad (n = 0, 1, \ldots),$$

$$(1.2) \qquad Z_n = (\alpha^n - \beta^n)/(\alpha - \beta) \qquad (n = 0, 1, \ldots),$$

$$(1.3) \qquad V_n = \alpha^n + \beta^n \qquad (n = 0, 1, \ldots),$$

where $\alpha \neq \beta$ are roots of $y^2 - dy + c = 0$. We shall define

$$(1.4) \qquad W_{-n} = (W_o V_n - W_n)/ c^n \quad (n = 1, 2, \ldots).$$

If $W_o = 0$ and $W_1 = 1$, then $W_n \equiv Z_n$, $n = 0, 1, \ldots$; and if $W_o = 2$ and $W_1 = d$, then $W_n \equiv V_n$, $n = 0, 1, \ldots$. The phrase, Lucas functions (of n) is often applied to Z_n and V_n, which may also be expressed in terms of Chebyshev polynomials (see (5.1) and (5.2)).

In this paper, general results (see section 3) have been obtained that yield new even power identities (Theorem 1) for sequences defined by (1.1). An additional result, Theorem 2, which contains Theorem 1 as a special case, yields identities whose typical term is the product of an even number of arbitrary terms taken from a given sequence defined by (1.1). Particular applications will be given for Fibonacci sequences and Chebyshev polynomials.

2. PRELIMINARIES

We shall need the following result:

Lemma 1. Let W_o, W_1, $c \neq 0$, and $d \neq 0$ be arbitrary real numbers, and let W_n, $n = 0, 1, \ldots$, satisfy (1.1). Let $m, p = 1, 2, \ldots$, and define

$$(2.1) \quad Q(n, p, m, i_1, \ldots, i_p) \equiv \prod_{s=1}^{p} W_{mn+i_s} = Q_n \quad (n = 0, 1, \ldots),$$

where i_s, $s = 1, 2, \ldots, p$, are positive integers or zero. Then Q_n satisfies a homogeneous, linear difference equation of order $p+1$ with real, constant coefficients whose characteristic equation is $g(y) = 0$, where

$$
(2.2)\quad g(y) \equiv \begin{cases} \displaystyle\prod_{j=0}^{(p-1)/2} (y^2 - c^{mj}V_{m(p-2j)}y + c^{mp}) & (p = 1, 3, 5, \ldots); \\ \displaystyle(y - c^{pm/2}) \prod_{j=0}^{(p-2)/2} (y^2 - c^{mj}V_{m(p-2j)}y + c^{mp}) & \\ & (p = 2, 4, 6, \ldots). \end{cases}
$$

<u>Proof</u>. Let A, B, and C_s, $s = 0, 1, \ldots, p$, denote arbitrary constants. If $\alpha \neq \beta$ denote the roots of $y^2 - dy + c = 0$, then

$$W_n = A\alpha^n + B\beta^n$$

and

$$W_{mn+i_s} = A\alpha^{i_s}\,\alpha^{mn} + B\beta^{i_s}\,\beta^{mn}.$$

Observing that

$$Q_n = \sum_{s=0}^{p} C_s\,(\alpha^{m(p-s)}\beta^{ms})^n\ , \quad n = 0, 1, \ldots,$$

we can now conclude that Q_n satisfies a homogeneous, linear difference equation of order $p+1$ with real, constant coefficients, and that $\alpha^{m(p-s)}\,\beta^{ms}$, $s = 0, 1, \ldots, p$, are the distinct roots of the corresponding characteristic equation $g(y) = 0$, where

$$g(y) \equiv \prod_{s=0}^{p} (y - \alpha^{m(p-s)}\beta^{ms}),$$

which simplifies to (2.2) as follows:

Let $R_s = \alpha^{m(p-s)}\,\beta^{ms}$, $s = 0, 1, \ldots, p$. If $p = 1, 3, 5, \ldots$, there is an even number of roots, R_s, and thus $(p+1)/2$ pairs, $(y-R_j)\cdot(y-R_{p-j})$, $j = 0, 1, \ldots, (p-1)/2$. Since $\alpha\beta = c$, $V_n = \alpha^n + \beta^n$, $n = 0, 1, \ldots$, we have $R_j + R_{p-j} = c^{mj}V_{m(p-2j)}$ $\qquad R_jR_{p-j} = c^{mp}$.

If $p = 2, 4, 6, \ldots$, there is an odd number of roots, R_s, and thus $p/2$ pairs, $(y-R)(y-R_{p-j})$, $j = 0, 1, \ldots, (p-2)/2$. The linear term, $y - R_{p/2} = y - c^{pm/2}$, accounts for the unpaired root, i.e., the middle root, $R_{p/2}$. This completes the proof of Lemma 1. Applications of (2.2) for $m = 1$ may be found in [1], [2], [3], and [4].

In terms of the translation operator, E, where $E^jQ_n = Q_{n+j}$, $j = 0, 1, \ldots,$ set

$$u_n \equiv \left[\prod_{j=0}^{(p-2)/2} (E^2 - c^{mj}V_{m(p-2j)}E + c^{mp})\right] Q_n \quad (p = 2, 4, 6, \ldots).$$

Then, from (2.2), since $g(E)Q_n = (E - c^{pm/2})u_n = 0$, we have

$$u_n \equiv u_o c^{mpn/2} \quad (n = 0, 1, \ldots; \; p = 2, 4, \ldots). \tag{2.3}$$

We now define

$$\sum_{k=0}^{p} h_k^{(p)}(d/(2\sqrt{c}))y^{p-k} = \prod_{j=0}^{(p-2)/2} (y^2 - c^{mj}V_{m(p-2j)}y + c^{mp}) \quad (p = 2, 4, \ldots). \tag{2.4}$$

The coefficients $h_k^{(p)}(d/(2\sqrt{c}))$, $k = 0, 1, \ldots, p$, are also dependent on m, which is notationally suppressed for simplicity. Using (2.4), we may now rewrite (2.3) as

$$\begin{aligned} &\sum_{k=0}^{p} h_k^{(p)}(d/(2\sqrt{c})) \prod_{s=1}^{p} W_{m(n+p-k)+i_s} \\ &\equiv c^{mpn/2} \sum_{k=0}^{p} h_k^{(p)}(d/(2\sqrt{c})) \prod_{s=1}^{p} W_{m(p-k)+i_s} \\ &\qquad (n = 0, 1, \ldots; \; p = 2, 4, \ldots). \end{aligned} \tag{2.5}$$

Let $p = 2q$, $q = 1, 2, \ldots$. Since $V_{2mk} = \alpha^{2mk} + \beta^{2mk}$ and $c = \alpha\beta$, we can write (2.4) as

$$\sum_{k=0}^{2q} h_{2q-k}^{(2q)}(d/(2\sqrt{c}))y^k = \prod_{k=1}^{q} (y^2 - c^{m(q-k)}V_{2mk}y + c^{2mq}) \tag{2.6}$$

$$= \prod_{k=1}^{q} (y - c^{m(q-k)}\alpha^{2mk})(y - c^{m(q-k)}\beta^{2mk})$$

$$= \prod_{k=1}^{q} \left[y - c^{mq}(\alpha/\beta)^{mk}\right] \left[y - c^{mq}(\alpha/\beta)^{-mk}\right] .$$

Set $y = c^{mq}x$ in (2.6), which now simplifies to

$$(2.7) \quad \sum_{k=0}^{2q} h_{2q-k}^{(2q)}(d/(2\sqrt{c}))c^{mqk}x^k = c^{2mq^2} \prod_{k=1}^{q} \left[x-(\alpha/\beta)^{mk}\right]\left[x-(\beta/\alpha)^{mk}\right]$$

We now define

$$(2.8) \quad b_k^{(2q)}(d/(2\sqrt{c})) \equiv c^{-mqk}h_k^{(2q)}(d/(2\sqrt{c})) \quad (k = 0, 1, \ldots, 2q) .$$

The, (2.7), with x replaced by y, now reads

$$(2.9) \quad \sum_{k=0}^{2q} b_k^{(2q)}(d/(2\sqrt{c}))y^{2q-k} \equiv \prod_{k=1}^{q} \left[y-(\alpha/\beta)^{mk}\right]\left[y-(\beta/\alpha)^{mk}\right]$$

$$= \prod_{k=1}^{q} (y^2 - c^{-mk}V_{2mk}y+1)$$

$$(m, q = 1, 2, \ldots).$$

If we replace y by $(1/y)$ in (2.9), we conclude that

$$(2.10) \quad b_k^{(2q)}(d/(2\sqrt{c})) = b_{2q-k}^{(2q)}(d/(2\sqrt{c})) \qquad (k = 0, 1, \ldots, 2q).$$

Our results will be expressed in terms of $b_k^{(2q)}(d/(2\sqrt{c}))$. Recalling (1.2) and that $c = \alpha\beta$, we obtain from (2.9) for $y = 1$

$$(2.11) \quad \sum_{k=0}^{2q} b_k^{(2q)}(d/(2\sqrt{c})) = (-1)^q c^{-mq(q+1)/2} \prod_{k=1}^{q} (\alpha^{mk} - \beta^{mk})^2$$

$$= (-1)^q(\alpha - \beta)^{2q} c^{-mq(q+1)/2} \prod_{k=1}^{q} Z_{mk}^2$$

$$= (4c-d^2)^q c^{-mq(q+1)/2} \prod_{k=1}^{q} Z_{mk}^2 ,$$

since

$$(-1)^q(\alpha - \beta)^{2q} = \left[2\alpha\beta - (\alpha^2 - \beta^2)\right]^q = \left[2c - V_2\right]^q,$$

and

$$V_2 = dV_1 - cV_o = d^2 - 2c.$$

We will use (2.11) in the proof of Theorems 1 and 2.

3. TWO THEOREMS

Our first general result is as follows:

Theorem 1. Let W_o, W_1, $c \neq 0$, and $d \neq 0$ be arbitrary real numbers, and define W_n by (1.1). Let $n_o = 0, 1, \ldots; m, q = 1, 2, \ldots;$ and $r = 0, 1, \ldots, q$. Then, for $n = 0, 1, \ldots,$ we have

$$(3.1) \qquad c^{-mrn} \sum_{k=0}^{2q} c^{mrk}\, b_k^{(2q)}(d/(2\sqrt{c}))\, W^{2r}_{m(n+2q-k)+n_o}$$

$$= c^{rn_o + (mq(4r-q-1)/2} \binom{2r}{r} (4c - d^2)^{q-r}$$

$$\cdot\, (W_1^2 - dW_oW_1 + cW_o^2)^r \prod_{k=1}^{q} Z^2_{mk} \quad ,$$

where $b_k^{(2q)}(d/(2\sqrt{c}))$, $k = 0, 1, \ldots, 2q$, are defined by (2.9).

Proof. Since $\alpha \neq \beta$, the general solution to (1.1) is $W_n = A\alpha^n + B\beta^n$, $n = 0, 1, \ldots,$ where A and B are arbitrary constants whose values satisfy $W_o = A + B$ and $W_1 = A\alpha + B\beta$. We readily find that

$$(3.2) \qquad (\beta-\alpha)A = W_o\beta - W_1 \quad , \quad (\beta-\alpha)B = W_1 - \alpha W_o \quad .$$

Since $\alpha + \beta = d$, $c = \alpha\beta$, and $(\beta-\alpha)^2 = d^2 - 4c$, we obtain from (3.2)

$$(3.3) \qquad (d^2 - 4c)AB = -(W_1^2 - dW_oW_1 + cW_o^2)$$

Using the binomial theorem and then interchanging summations, we obtain

$$(3.4) \qquad S \equiv c^{-mrn} \sum_{k=0}^{2q} c^{mr(2q-k)}\, b_{2q-k}^{(2q)}(d/(2\sqrt{c}))\, W^{2r}_{m(n+k)+n_o}$$

$$= c^{-mrn} \sum_{k=0}^{2q} (\alpha\beta)^{-mrk} b_{2q-k}^{(2q)}(d/(2\sqrt{c}))(A\alpha^{mn+mk+n_o} + B\beta^{mn+mk+n_o})^{2r}$$

$$= c^{mr(2q-n)} \sum_{s=0}^{2r} \binom{2r}{s} A^s B^{2r-s} (\alpha^s \beta^{2r-s})^{mn+n_o} G((\alpha/\beta)^{m(s-r)})$$

where, by (2.9) with $y = (\alpha/\beta)^{n(s-r)}$, we have

$$(3.5) \qquad G((\alpha/\beta)^{m(s-r)}) \equiv \sum_{k=0}^{2q} b_{2q-k}^{(2q)} (d/(2\sqrt{c})) \left[(\alpha/\beta)^{m(s-r)}\right]^k$$

$$= \prod_{k=1}^{q} \left[(\alpha/\beta)^{m(s-r)} - (\alpha/\beta)^{mk}\right] \cdot \left[(\alpha/\beta)^{m(s-r)} - (\alpha/\beta)^{-mk}\right] .$$

Since $0 \leq r \leq q$ and $0 \leq s \leq 2r$, we have $-q \leq s-r \leq q$. Thus, for $0 \leq s \leq 2r$, $s \neq r$, the sum in (3.5) vanishes; but for $s = r$, we obtain the non-zero term $G(1)$ (see (2.10), (2.11)). Thus, from (3.4), we obtain

$$(3.6) \qquad S = c^{2mrq+rn_o} \binom{2r}{r} (AB)^r \sum_{k=0}^{2q} b_k^{(2q)} (d/(2\sqrt{c})) ,$$

which yields the desired result with substitutions from (2.11) and (3.3)

The following general result yields Theorem 1 as an important special case:

<u>Theorem 2</u>. Let W_o, W_1, $c \neq 0$, and $d \neq 0$ be arbitrary real numbers and define W_n by (1.1). Let $m, q = 1, 2, \ldots,$ and $t_r = i_1 + i_2 + \ldots + i_{2r}$, where i_s, $s = 1, 2, \ldots, 2r$, $(r = 1, 2, \ldots, q)$, are positive integers or zero. Then, for $n = 0, 1, \ldots,$ we have

$$(3.7) \quad c^{-mrn} \sum_{k=0}^{2q} c^{mrk} b_k^{(2q)} (d/(2\sqrt{c})) \prod_{s=1}^{2r} W_{m(n+2q-k)+i_s}$$

$$= c^{mq(4r-q-1)/2} K_r (4c-d^2)^{q-r} (W_1^2 - dW_oW_1 + cW_o^2)^r \prod_{k=1}^{q} Z_{mk}^2 ,$$

with

$$(3.8)\qquad K_r = \sum_{j=1}^{\binom{2r-1}{r}} c^{\sigma(j,r)} V_{t_r - 2\sigma(j,r)} \qquad (r = 1, 2, \ldots, q)\ ,$$

$$(3.9)\quad \sigma(j,r) = i_1^{(j)} + i_2^{(j)} + i_3^{(j)} + \ldots + i_r^{(j)} \qquad (j = 1, 2, \ldots, \binom{2r-1}{r})\ ,$$

where, for each j, $\sigma(j,r)$, as the sum of r integers, $i_s^{(j)}$, $s = 1, 2, \ldots, r$, represents one of the $\binom{2r-1}{r}$ combinations obtained by choosing r numbers from the $2r-1$ numbers, $i_1, i_2, i_3, \ldots, i_{2r-1}$.

Proof. From Lemma 1, we have

$$(3.10)\qquad Q_n = \prod_{s=1}^{2r} W_{mn+i_s} = \sum_{s=0}^{2r} C_s(\beta^{m(2r-s)}\alpha^{ms})^n\ ,$$

where C_s, $s = 0, 1, \ldots, 2r$, are arbitrary constants independent of n. Recalling the proof of Theorem 1, we have (see (3.7))

$$(3.11)\ S \equiv c^{-mrn} \sum_{k=0}^{2q} c^{mr(2q-k)} b_{2q-k}^{(2q)}(d/(2\sqrt{c})) \sum_{s=0}^{2r} C_s(\beta^{m(2r-s)}\alpha^{ms})^{n+k}$$

$$= c^{-mrn+2mqr} \sum_{s=0}^{2r} C_s(\beta^{2r-s}\alpha^s)^{mn} \sum_{k=0}^{2q} b_{2q-k}^{(2q)}(d/(2\sqrt{c}))((\alpha/\beta)^{m(s-r)})^k$$

$$= c^{2mqr}\, C_r \sum_{k=0}^{2q} b_k^{(2q)}\,(d/(2\sqrt{c}))\ .$$

We proceed now to evaluate C_r. From (3.10), we have

$$(3.12)\qquad \prod_{s=1}^{2r} W_{mn+i_s} = \beta^{2mrn} \sum_{s=0}^{2r} C_s((\alpha/\beta)^{mn})^s\ ,$$

which is a polynomial in the variable $(\alpha/\beta)^{mn}$. Since $W_n = A\alpha^n + B\beta^n$, we have

$$W_{mn+i_s} = \left[A\alpha^{i_s}\,(\alpha/\beta)^{mn} + B\beta^{i_s}\right]\ ,$$

and thus

$$(3.13)\qquad \prod_{s=1}^{2r} W_{mn+i_s} = \beta^{2mrn} \prod_{s=1}^{2r} \left[A\alpha^{i_s}(\alpha/\beta)^{mn} + B\beta^{i_s}\right]$$

$$= \beta^{2mrn}A^{2r}\alpha^{t_r} \prod_{s=1}^{2r}\left[(\alpha/\beta)^{mn}+(B/A)(\beta/\alpha)^{i_s}\right] .$$

If we compare (3.12) and (3.13), and recall the definition of the elementary symmetric functions of the roots of a polynomial equation, we conclude that

$$(3.14)\qquad C_r = A^{2r}\alpha^{t_r}(-1)^r \sum_{k=1}^{\binom{2r}{r}} (-B/A)^r \prod_{s=1}^{r} (\beta/\alpha)^{i_{s,k}}$$

$$= (AB)^r \sum_{k=1}^{\binom{2r}{r}} \alpha^{\left(t_r - \sum_{s=1}^{r} i_{s,k}\right)} \beta^{\left(\sum_{s=1}^{r} i_{s,k}\right)} ,$$

where for each fixed k, $k = 1, 2, \ldots, \binom{2r}{r}$, each set of numbers, $i_{s,k}$, $s = 1, 2, \ldots, r$, is one of the $\binom{2r}{r}$ combinations obtained by choosing r numbers from the $2r$ numbers, i_s, $s = 1, 2, \ldots, 2r$. It should be noted that since (3.13) is a symmetric function in the variables i_s, $s = 1, 2, \ldots, 2r$, the role of i_{2r} in the definition of $\sigma(j, r)$ (see (3.9)) was a convenient choice. Since a choice of r numbers from a set of $2r$ numbers leaves another set of r numbers, we may pair off related terms of the sum in (3.14), noting our role assigned to i_{2r}. Thus, since $\binom{2r}{r} = 2\binom{2r-1}{r}$, and

$$\alpha^{t_r - \sigma(j,r)}\beta^{\sigma(j,r)} + \alpha^{\sigma(j,r)}\beta^{t_r - \sigma(j,r)} = c^{\sigma(j,r)}V_{t_r - 2\sigma(j,r)}$$

(see (1.3)), we have

$$(3.15)\qquad C_r = (AB)^r K_r \quad (r = 1, 2, \ldots, q) .$$

Recalling definitions (2.11) and (3.3), we obtain our desired result (3.7) from (3.11).

Remarks. For $r = 2$, we have $\sigma(1,2) = i_1 + i_2$, $\sigma(2,2) = i_1 + i_3$, and $\sigma(3,2) = i_2 + i_3$.

For $r = 3$, we have

$$\begin{aligned}
\sigma(1,3) &= i_1 + i_2 + i_3, & \sigma(6,3) &= i_1 + i_4 + i_5,\\
\sigma(2,3) &= i_1 + i_2 + i_4, & \sigma(7,3) &= i_2 + i_3 + i_4,\\
\sigma(3,3) &= i_1 + i_2 + i_5, & \sigma(8,3) &= i_2 + i_3 + i_5,\\
\sigma(4,3) &= i_1 + i_3 + i_4, & \sigma(9,3) &= i_2 + i_4 + i_5,\\
\sigma(5,3) &= i_1 + i_3 + i_5, & \sigma(10,3) &= i_3 + i_4 + i_5,
\end{aligned}$$

If $i_s = n_o$, $s = 1, 2, \ldots, 2r$, then $t_r - 2\sigma(j,r) = 2rn_o - 2rn_o = 0$, $V_o = 2$, and $K_r = c^{rn_o}\binom{2r}{r}$. Thus, (3.7) yields (3.1) as a special case. Indeed, using the binomial theorem on $W_{mn+n_o} = A\alpha^{n_o}\alpha^{mn} + B\beta^{n_o}\beta^{mn}$, we obtain

$$W_{mn+n_o}^{2r} = \sum_{s=0}^{2r} \binom{2r}{s} A^s B^{2r-s} (\alpha^s\beta^{2r-s})^{n_o} (\beta^{m(2r-s)}\alpha^{ms})^n ,$$

where, (see (3.10)) $C_s = \binom{2r}{s} A^s B^{2r-s}(\alpha^s\beta^{2r-s})^{n_o}$, $s = 0, 1, \ldots, 2r$, and thus $C_r = c^{rn_o}\binom{2r}{r}(AB)^r$.

Consider the special case $i_s = n_o$, $s = 1, 2, \ldots, 2r-1$, and $i_{2r} \neq n_o$. Then $\sigma(j,r) \equiv rn_o$, $t_r = (2r-1)n_o + i_{2r}$, and thus (see (3.8))

$$K_r = c^{rn_o}\binom{2r-1}{r} V_{-n_o + i_{2r}} .$$

Next, consider the special case $i_s = n_o$, $s = 1, 2, \ldots, 2r-2$; $i_{2r-1} \neq i_{2r} \neq n_o$. Of the set of $\binom{2r-1}{r}$ combinations for $\sigma(j,r)$, there are $\binom{2r-2}{r-1}$ combinations which contain i_{2r-1}. For these cases, $\sigma(j,r) \equiv (r-1)n_o + i_{2r-1}$; and for the remaining $\binom{2r-1}{r} - \binom{2r-2}{r-1} = \binom{2r-2}{r}$

combinations, we have $\sigma(j, r) \equiv rn_o$. Thus, from (3.8), with $t_r = (2r-2)n_o + i_{2r-1} + i_{2r}$, we obtain

$$K_r = c^{(r-1)n_o + i_{2r-1}} \binom{2r-2}{r-1} V_{i_{2r} - i_{2r-1}} + c^{rn_o} \binom{2r-2}{r} V_{i_{2r-1} + i_{2r} - 2n_o} . \tag{3.16}$$

4. IDENTITIES FOR FIBONACCI SEQUENCES

Generalized Fibonacci numbers, H_n, are defined by $H_{n+2} = H_{n+1} + H_n$, $n = 0, 1, \ldots,$ where H_o and H_1 are arbitrary integers. In the notation of (1.2) and (1.3), we have $Z_n = F_n$, and $V_n = L_n$, the Lucas numbers. The following result is an application of Theorem 1, where $d = -c = 1$:

Theorem 3. Define (see (2.9))

$$\sum_{k=0}^{2q} b_k^{(2q)}(-i/2)y^{2q-k} = \prod_{k=1}^{q} (y^2 - (-1)^{mk}L_{2mk}y + 1) \tag{4.1}$$

$$(m, q = 1, 2, \ldots).$$

Let $n_o = 0, 1, \ldots;$ $m, q = 1, 2, \ldots;$ and $r = 0, 1, \ldots, q$. Then, for $n = 0, 1, \ldots,$ we have

$$(-1)^{mrn} \sum_{k=0}^{2q} (-1)^{mrk} b_k^{(2q)} (-i/2) H^{2r}_{m(n+2q-k)+n_o} = (-1)^{rn_o + (mq(q+1)/2)} \binom{2r}{r} (-5)^{q-r} (H_1^2 - H_oH_1 - H_o^2)^r \prod_{k=1}^{q} F^2_{mk} , \tag{4.2}$$

$$(-1)^{mrn} \sum_{k=0}^{2q} (-1)^{mrk} b_k^{(2q)} (-i/2) F^{2r}_{m(n+2q-k)+n_o} = (-1)^{rn_o + (mq(q+1)/2)} \binom{2r}{r} (-5)^{q} \prod_{k=1}^{q} F^2_{mk} , \tag{4.3}$$

$$(4.4) \qquad (-1)^{mrn} \sum_{k=0}^{2q} (-1)^{mrk} b_k^{(2q)} (-i/2) L^{2r}_{m(n+2q-k)+n_o}$$

$$= (-1)^{rn_o+(mq(q+1)/2)} \binom{2r}{r} (-5)^q \prod_{k=1}^{q} F^2_{mk} \quad .$$

Remarks. For the same values of r, n_o, m, and q, the constant term on the right-hand side of (4.4) is $(-5)^r$ times as great as the constant term on the right-hand side of (4.3)

In the examples given below, valid for $n = 0, 1, \ldots$, we have set $D \equiv H_1^2 - H_oH_1 - H_o^2$. Applications of D in the ordering of Fibonacci sequences are given in [5].

$$(4.5) \qquad (-1)^{mn} (H^2_{m(n+2)+n_o} - L_{2m}H^2_{m(n+1)+n_o} + H^2_{mn+n_o})$$

$$= 2(-1)^{m+n_o} DF_m^2 \qquad (n_o = 0, 1, \ldots; \; m = 1, 2, \ldots) \quad ,$$

$$(4.6) \qquad H^4_{n+4} - 4H^4_{n+3} - 19H^4_{n+2} - 4H^4_{n+1} + H^4_n = -6D^2 \quad ,$$

$$(4.7) \qquad (-1)^n(H^2_{n+4} + 4H^2_{n+3} - 19H^2_{n+2} + 4H^2_{n+1} + H^2_n) = 10D \quad ,$$

$$(4.8) \quad H_{n+4}H^3_{n+5} - 4H_{n+3}H^3_{n+4} - 19H_{n+2}H^3_{n+3} + 4H_{n+1}H^3_{n+2} + H_nH^3_{n+1}$$

$$= 3D^2 \quad ,$$

$$(4.9) \; H^2_{n+4}H^2_{n+5} - 4H^2_{n+3}H^2_{n+4} - 19H^2_{n+2}H^2_{n+3} - 4H^2_{n+1}H^2_{n+2} + H^2_nH^2_{n+1} = D^2,$$

$$(4.10) \qquad (-1)^n(H^6_{n+6} - 14H^6_{n+5} - 90H^6_{n+4} + 350H^6_{n+3}$$

$$-90H^6_{n+2} - 14H^6_{n+1} + H^6_n) = 80D^3 \quad ,$$

(4.11) $$H_{n+6}^4 + 14H_{n+5}^4 - 90H_{n+4}^4 - 350H_{n+3}^4 - 90H_{n+2}^4 + 14H_{n+1}^4 + H_n^4 = -120D^2,$$

(4.12) $$(-1)^n(H_{n+6}^2 - 14H_{n+5}^2 - 90H_{n+4}^2 + 350H_{n+3}^2 - 90H_{n+2}^2 - 14H_{n+1}^2 + H_n^2) = 200D,$$

(4.13) $$H_{n+6}^5H_{n+7} - 14H_{n+5}^5H_{n+6} - 90H_{n+4}^5H_{n+5} + 350H_{n+3}^5H_{n+4} - 90H_{n+2}^5H_{n+3} - 14H_{n+1}^5H_{n+2} + H_n^5H_{n+1} = 40(-1)^nD^3,$$

(4.14) $$H_{n+6}^3H_{n+7}^3 - 14H_{n+5}^3H_{n+6}^3 - 90H_{n+4}^3H_{n+5}^3 + 350H_{n+3}^3H_{n+4}^3 - 90H_{n+2}^3H_{n+3}^3 - 14H_{n+1}^3H_{n+2}^3 + H_n^3H_{n+1}^3 = 20(-1)^{n+1}D^3,$$

(4.15) $$H_{n+8}^8 - 33H_{n+7}^8 - 747H_{n+6}^8 + 3894H_{n+5}^8 + 16270H_{n+4}^8 + 3894H_{n+3}^8 - 747H_{n+2}^8 - 33H_{n+1}^8 + H_n^8 = 2520D^4,$$

(4.16) $$H_{n+8}^6 + 33H_{n+7}^6 - 747H_{n+6}^6 - 3894H_{n+5}^6 + 16270H_{n+4}^6 - 3894H_{n+3}^6 - 747H_{n+2}^6 + 33H_{n+1}^6 + H_n^6 = 3600(-1)^{n+1}D^3.$$

Two identities, (4.6) and a special case of (4.5), with $m = 1$ and $n_o = 0$, have been given previously in [6].

5. IDENTITIES FOR CHEBYSHEV POLYNOMIALS

Chebyshev polynomials [7, pp. 183-187] of the first kind, $T_n(x)$, and of the second kind, $U_n(x)$, are solutions of (1.1) when $d = 2x$ and $c = 1$. Thus, $W_n \equiv T_n(x)$ for $W_o = 1$, $W_1 = x$; $W_n \equiv U_n(x)$ for $W_o = 1$, $W_1 = 2x$; $Z_n \equiv U_{n-1}(x)$; and $V_n \equiv 2T_n(x)$.

We will now show that the Lucas functions Z_n and V_n of (1.1), where $c \neq 0$ and $d \neq 0$ are arbitrary real numbers, can be expressed in terms of Chebyshev polynomials as follows:

$$Z_{n+1} = c^{n/2}\, U_n(d/(2\sqrt{c})) \quad (n = 0, 1, \ldots), \tag{5.1}$$

$$V_n = 2c^{n/2}\, T_n(d/(2\sqrt{c})) \quad (n = 0, 1, \ldots). \tag{5.2}$$

<u>Proof.</u> Since $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$, set $x = d/(2\sqrt{c})$ and then multiply both sides by $c^{(n+1)/2}$. Thus, using (5.1), we have $Z_o = 0$, $Z_1 = 1$, and $Z_{n+2} = dZ_{n+1} - cZ_n$, $n = 0, 1, \ldots$.

Since $T_{n+2}(x) = 2x\,T_{n+1}(x) - T_n(x)$, set $x = d/(2\sqrt{c})$ and then multiply both sides by $2c^{(n+2)/2}$. Thus, using (5.2), we have $V_o = 2$, $V_1 = d$, and $V_{n+2} = dV_{n+1} - cV_n$, $n = 0, 1, \ldots$.

The following result is an application of Theorem 1, where $d = 2x$ and $c = 1$:

<u>Theorem 4.</u> Define (see (2.9))

$$\sum_{k=0}^{2q} b_k^{(2q)}(x) y^{2q-k} = \prod_{k=1}^{q} (y^2 - 2T_{2mk}(x)y + 1) \quad (m, q = 1, 2, \ldots). \tag{5.3}$$

Let $n_o = 0, 1, \ldots$; $m, q = 1, 2, \ldots$; and $r = 0, 1, \ldots, q$. Then, for $n = 0, 1, \ldots$, we have

$$\sum_{k=0}^{2q} b_k^{(2q)}(x)\, T^{2r}_{m(n+2q-k)+n_o}(x) \tag{5.4}$$

$$= 4^{q-r} \binom{2r}{r} (1-x^2)^q \prod_{k=1}^{q} U^2_{mk-1}(x) \quad ,$$

$$(5.5)\quad \sum_{k=0}^{2q} b_k^{(2q)}(x)U^{2r}_{m(n+2q-k)+n_o}(x) = 4^{q-r}\binom{2r}{r}(1-x^2)^{q-r}\prod_{k=1}^{q} U^2_{mk-1}(x) \quad .$$

Remarks. Identities (5.4) and (5.5) yield trigonometric identities by recalling that if $x = \cos\theta$, then $T_n(\cos\theta) = \cos(n\theta)$ and $U_n(\cos\theta) = \sin(n+1)\theta/(\sin\theta)$. Since $\sin(i\theta) = i\sinh\theta$ and $\cos(i\theta) = \cosh\theta$, identities for the hyperbolic functions are then obtained from the corresponding trigonometric identities. Additional complicated identities can be obtained from (5.4) and (5.5) by differentiation with respect to x. Some sample identities, valid for $n = 0, 1, \ldots,$ are given below:

$$(5.6)\quad T^2_{m(n+2)+n_o}(x) - 2T_{2m}(x)T^2_{m(n+1)+n_o}(x) + T^2_{mn+n_o}(x)$$
$$= 2(1-x^2)U^2_{m-1}(x) \quad (m = 1, 2, \ldots;\ n_o = 0, 1, \ldots),$$

$$(5.7)\quad T^4_{n+4}(x) - (16x^4-12x^2)T^4_{n+3}(x) + (64x^6-96x^4+40x^2-2)T^4_{n+2}(x)$$
$$-(16x^4-12x^2)T^4_{n+1}(x) + T^4_n(x) = 24x^2(1-x^2)^2 \quad ,$$

$$(5.8)\quad T^3_{n+4}(x)T_{n+5}(x) - (16x^4-12x^2)T^3_{n+3}(x)T_{n+4}(x)$$
$$+(64x^6-96x^4+40x^2-2)T^3_{n+2}(x)T_{n+3}(x) - (16x^4-12x^2)T^3_{n+1}(x)T_{n+2}(x)$$
$$+ T^3_n(x)T_{n+1}(x) = 24x^3(1-x^2)^2 \quad .$$

Let

$$A_1(x) = 64x^6 - 80x^4 + 24x^2 - 2 \quad ,$$
$$A_2(x) = 1024x^{10} - 2304x^8 + 1792x^6 - 560x^4 + 64x^2 - 1 \quad ,$$
$$A_3(x) = 4096x^{12} - 12288x^{10} + 14080x^8 - 7552x^6 + 1856x^4 - 176x^2 + 4 \quad .$$

Then

$$(5.9)\quad T^6_{n+6}(x) - A_1(x)T^6_{n+5}(x) + A_2(x)T^6_{n+4}(x) - A_3(x)T^6_{n+3}(x)$$
$$+ A_2(x)T^6_{n+2}(x) - A_1(x)T^6_{n+1}(x) + T^6_n(x) = 80x^2(1-x^2)^3(4x^2-1)^2 \quad ,$$

(5.10) $$T_{n+6}^4(x) - A_1(x)T_{n+5}^4(x) + A_2(x)T_{n+4}^4(x) - A_3(x)T_{n+3}^4(x)$$

$$+ A_2(x)T_{n+2}^4(x) - A_1(x)T_{n+1}^4(x) + T_n^4(x) = 96x^2(1-x^2)^3(4x^2-1)^2 \quad ,$$

(5.11) $$T_{n+6}^3(x)T_{n+7}^3(x) - A_1(x)T_{n+5}^3(x)T_{n+6}^3(x) + A_2(x)T_{n+4}^3(x)T_{n+5}^3(x)$$

$$- A_3(x)T_{n+3}^3(x)T_{n+4}^3(x) + A_2(x)T_{n+2}^3(x)T_{n+3}^3(x) - A_1(x)T_{n+1}^3(x)T_{n+2}^3(x)$$

$$+ T_n^3(x)T_{n+1}^3(x) = 16x^3(2x^2+3)(1-x^2)^3(4x^2-1)^2 \quad .$$

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OMISSION AND INFORMATION

The "Factorization of 36 Fibonacci Numbers F_n with $n > 100$" by L. A. G. Dresel and D. E. Daykin should have included the following references.

1. Dov Jarden Recurring Sequences, Israel, 1958, contains many factorizations of first 385 L_n and F_n. This is being reissued soon and will be available again from the Fibonacci Association.

2. Brother U. Alfred and John Brillhart "Fibonacci Century Mark Reached" FQJ, Vol. I, No. 1, p. 45, Feb., 1963.

3. Brother U. Alfred "Fibonacci Discovery" contains factors of first 100 F_n and first 50 L_n. See ad this issue page 291.

The factors available now allows one to factor higher Fibonacci Numbers since $F_{2n} = L_n F_n$.

John Brillhart reports that in a short time he will have published a report containing all the prime factors less than 2^{30} of F_n for $n < 2000$ and of L_n for $n < 1000$. This is exciting news.

A VARIANT OF PASCAL'S TRIANGLE

H. W. Gould*
West Virginia University, Morgantown, W. Va.

1. INTRODUCTION

Professor Charles A. Halijak has called my attention to the following interesting variant of Pascal's triangle [7]

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 1 \quad 2 \quad 1 \\
1 \quad 1 \quad 3 \quad 2 \quad 1 \\
1 \quad 1 \quad 4 \quad 3 \quad 3 \quad 1 \\
1 \quad 1 \quad 5 \quad 4 \quad 6 \quad 3 \quad 1 \\
1 \quad 1 \quad 6 \quad 5 \quad 10 \quad 6 \quad 4 \quad 1 \\
1 \quad 1 \quad 7 \quad 6 \quad 15 \quad 10 \quad 10 \quad 4 \quad 1 \\
1 \quad 1 \quad 8 \quad 7 \quad 21 \quad 15 \quad 20 \quad 10 \quad 5 \quad 1
\end{array}
\tag{1.1}
$$

The law for formation is evident. One alternately adds together two elements or brings down a single element in order to obtain a new element in the next row. It appears that the elements turn out to be binomial coefficients. More interestingly, it appears that the elements in any row add to give a Fibonacci number: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, etc.

The object of the present note is to verify these observations and to develop some other relations suggested by the array of numbers.

2. RECURRENCE RELATIONS

We may symbolize the array (1.1) as follows:

$$
\begin{array}{c}
A_0^0 \\
A_0^1 \qquad A_1^1 \\
A_0^2 \qquad A_1^2 \qquad A_2^2 \\
\ldots \text{ etc.}
\end{array}
$$

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If we let A_j^n, $j = 0, 1, 2, \ldots, n$, designate an arbitrary element of the array then we may use the defining recurrence relation (law of formation) to give an inductive definition of the array (1.1). Indeed we may say that the conditions

$$A_{2k+1}^{n+1} = A_{2k}^n , \tag{2.1}$$

$$A_{2k}^{n+1} = A_{2k-1}^n + A_{2k}^n , \tag{2.2}$$

$$A_j^n = 0, \ j > n \text{ or } j < 0 , \tag{2.3}$$

$$A_0^n = 1, \quad n = 0, 1, 2, \ldots , \quad A_1^1 = 1, \tag{2.4}$$

are sufficient to define the array (1.1). We may combine (2.1) and (2.2) into a single recurrence relation

$$A_j^{n+1} = A_{j-1}^n + \frac{1 + (-1)^j}{2} A_j^n \tag{2.5}$$

if we desire.

It is not difficult to conjecture (and prove by induction) that

$$A_{2k}^n = \binom{n - k}{k} , \tag{2.6}$$

$$A_{2k+1}^n = \binom{n - 1 - k}{k} , \tag{2.7}$$

and, again, these may be expressed in the single formula

$$A_j^n = \binom{n - \left[\frac{1}{2}(j+1)\right]}{\left[\frac{1}{2}(j)\right]} \tag{2.8}$$

where $[x]$ would mean the integral part of x (the "greatest integer in x").

3. FIBONACCI NUMBERS

The Fibonacci numbers, F_n, may be defined by the conditions $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$. Explicitly it is easy to show that

$$(3.1) \qquad F_{n+1} = \sum_{k=0}^{[n/2]} \binom{n-k}{k} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} \ ,$$

and this well-known formula provides the clue to our next results. We have

Theorem 1. For the array (1.1) we have

$$(3.2) \qquad \sum_{j=0}^{n} A_j^n = F_{n+2} \ , \quad n \geq 0.$$

Proof.

$$\sum_{j=0}^{n} A_j^n = \sum_{k=0}^{[n/2]} A_{2k}^n + \sum_{k=0}^{[(n-1)/2]} A_{2k+1}^n$$

$$= F_{n+1} + F_n = F_{n+2}$$

as desired to show.

Next we may establish

Theorem 2. For the array (1.1) we have

$$(3.3) \qquad \sum_{j=0}^{n} (-1)^j A_j^n = F_{n-1} \ , \quad n \geq 1.$$

This would also be true for $n = 0$ if we extend the Fibonacci sequence backwards as is usually done. As for the proof, the same steps as used for Theorem 1 give us at once $F_{n+1} - F_n$ or F_{n-1} as claimed.

4. A GENERAL POLYNOMIAL

We now define the polynomial $A_n(x)$ by

$$(4.1) \qquad A_n(x) = \sum_{j=0}^{n} A_j^n x^j \ .$$

In view of (2.6) and (2.7) we have

$$(4.2)\quad A_n(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n-1-k}{k} x^{2k+1} .$$

The polynomial $A_n(x)$ satisfies a simple recurrence relation which we may find as follows. By means of (2.5) we have

$$\sum_{j=1}^{n+1} A_j^{n+1} x^j = \sum_{j=1}^{n+1} A_{j-1}^n x^j + \frac{1}{2} \sum_{j=1}^{n+1} A_j^n x^j + \frac{1}{2} \sum_{j=1}^{n+1} (-1)^j A_j^n x^j ,$$

or

$$\sum_{j=0}^{n+1} A_j^{n+1} x^j = \sum_{j=0}^{n} A_j^n x^{j+1} + \frac{1}{2} \sum_{j=0}^{n} A_j^n x^j + \frac{1}{2} \sum_{j=0}^{n} A_j^n (-x)^j ,$$

or therefore

$$(4.3)\qquad 2\,A_{n+1}(x) = (2x+1)A_n(x) + A_n(-x).$$

It would be possible to set down a closed expression for $A_n(x)$ by means of the summation formula

$$(4.4)\qquad \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k = \frac{u^{n+1} - 1}{(u-1)(1+u)^n} , \quad x = \frac{-u}{(1+u)^2} ,$$

but this does not seem to simplify very nicely. It would be of interest to evaluate $A_n(x)$ for values of x other than $x = 1$ and $x = -1$, however. We remark that (4.4) may be written in the alternative form

$$(4.5)\quad \sum_{k=0}^{[n/2]} \binom{n-k}{k} 2^{n-2k} x^k = \frac{u^{n+1} - v^{n+1}}{u - v} , \quad \begin{cases} u = 1 + \sqrt{x+1} , \\ v = 1 - \sqrt{x+1} . \end{cases}$$

5. LUCAS NUMBER VARIANT OF PASCAL'S TRIANGLE

Using the same law of formation as we imposed to generate rows in (1.1) we may form the array

$$\tag{5.1}\begin{array}{c}
1\\
1 \quad 2\\
1 \quad 1 \quad 2\\
1 \quad 1 \quad 3 \quad 2\\
1 \quad 1 \quad 4 \quad 3 \quad 2\\
1 \quad 1 \quad 5 \quad 4 \quad 5 \quad 2\\
1 \quad 1 \quad 6 \quad 5 \quad 9 \quad 5 \quad 2\\
1 \quad 1 \quad 7 \quad 6 \quad 14 \quad 9 \quad 7 \quad 2\\
1 \quad 1 \quad 8 \quad 7 \quad 20 \quad 14 \quad 16 \quad 7 \quad 2\\
1 \quad 1 \quad 9 \quad 8 \quad 27 \quad 20 \quad 30 \quad 16 \quad 9 \quad 2
\end{array}$$

where the only difference is that we use a different initial value in the second spot on the second row. Let us symbolize the array by using the notation B_j^n in the same way we discussed A_j^n. We first observe that the rows add to give the Lucas numbers: 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, etc. In other words, we have, evidently, the two relations

$$\tag{5.2}\sum_{j=0}^{n} B_j^n = L_{n+1} ,$$

and

$$\tag{5.3}\sum_{j=0}^{n} (-1)^j B_j^n = L_{n-2} ,$$

where the Lucas numbers are defined by

$$L_0 = 2, \ L_1 = 1, \ L_{n+1} = L_n + L_{n-1} .$$

Explicitly, we have

$$\tag{5.4} L_n = \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n} .$$

The array (5.1) may be specified by the conditions

(5.5) $$B_{2k+1}^{n+1} = B_{2k}^{n},$$

(5.6) $$B_{2k}^{n+1} = B_{2k-1}^{n} + B_{2k}^{n},$$

(5.7) $$B_j^n = 0, \; j > n \text{ or } j < 0,$$

(5.8) $$B_0^n = 1, \; n = 0, 1, 2, \ldots, \quad B_1^1 = 2,$$

We may combine (5.5) and (5.6) by writing

(5.9) $$B_j^{n+1} = B_{j-1}^{n} + \frac{1 + (-1)^j}{2} B_j^n,$$

and we conjecture on the basis of (5.4) and the above that

(5.10) $$B_{2k}^{n} = \frac{n}{n-k}\binom{n-k}{k},$$

and

(5.11) $$B_{2k+1}^{n} = \frac{n-1}{n-1-k}\binom{n-1-k}{k}, \quad B_1^1 = 2.$$

The two relations could be combined into a single expression, however, the result is not as simple as was the case with (2.8).

Associated with the Lucas variant of Pascal's triangle we may consider the polynomial

(5.12) $$B_n(x) = \sum_{j=0}^{n} B_j^n x^j.$$

In view of the recurrence (5.9), just as in the case of (2.5), we may show that the companion relation to (4.3) is

(5.13) $$2B_{n+1}(x) = (2x + 1)B_n(x) + B_n(-x).$$

The formula

(5.14) $$\sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} 2^{n-2k} x^k = 2\,\frac{u^n + v^n}{u + v},$$

where

$$u = 1 + \sqrt{x+1}\ , \quad v = 1 - \sqrt{x+1}\ ,$$

could be used to give a closed form for (5.12).

6. GENERALIZATION

A general array suggested by the two cases we have discussed may be set down as follows:

$$\begin{array}{c}
a \\
a \qquad b \\
a \qquad a \qquad b \\
a \qquad a \qquad a+b \qquad b \\
a \qquad a \qquad 2a+b \qquad a+b \qquad b \\
a \qquad a \qquad 3a+b \qquad 2a+b \qquad a+2b \qquad b \\
a \qquad a \qquad 4a+b \qquad 3a+b \qquad 3a+3b \qquad a+2b \qquad b \\
a \qquad a \qquad 5a+b \qquad 4a+b \qquad 6a+4b \qquad 3a+3b \qquad a+3b \qquad b \\
a \qquad a \qquad 6a+b \qquad 5a+b \qquad 10a+5b \qquad 6a+4b \qquad 4a+6b \qquad a+3b \qquad b
\end{array} \tag{6.1}$$

We may define the array by the following conditions:

$$C_0^0 = 0_0^1 = a, \quad C_1^1 = b, \tag{6.2}$$

$$C_j^n = 0, \text{ if } j > n \text{ or } j < 0, \tag{6.3}$$

$$C_j^{n+1} = C_{j-1}^n + \frac{1 + (-1)^j}{2} C_j^n\ , \quad n \geq 1,\ j \geq 0. \tag{6.4}$$

For the recurrence (6.4) we have imposed the condition that $n \geq 1$. We do this for the following reason. Choose $C_0^0 = a$. Then, by (6.4), we have $C_0^1 = C_{-1}^0 + C_0^0 = C_0^0$ provided we impose (6.3). But then we have $C_1^1 = C_0^0 + 0 = a$, not b. To avoid this difficulty we may define $C_1^1 = b$. For the next row we have then

$$C_0^2 = C_{-1}^1 + C_0^1 = 0 + a = a,$$

$$C_1^2 = C_0^1 + 0 = a,$$

$$C_2^2 = C_1^1 + C_2^1 = b + 0 = b.$$

Thus a simple condition to attach to the recurrence is that $n \geq 1$. Another way to proceed would be to define $C_0^0 = b$ and $C_0^1 = a$. Everything would be the same except the topmost element, and the recurrence would hold in all cases. However, then the niceness of the array (5.1) would suffer by having $B_0^0 = 2$ which would not fit so well with the Lucas numbers. There is a certain arbitrariness in combining the various properties which seem to be of interest. Because of this, the reader may find it instructive to examine other possible definitions.

From our definition it is easy to show that the row-sums are given by

$$(6.5) \qquad S_n(a, b) = \sum_{j=0}^{n} C_j^n = aF_{n+1} + bF_n, \quad n \geq 0,$$

in terms of the Fibonacci numbers. Thus we find $S_n(1, 1) = F_{n+1} + F_n = F_{n+2}$ as before. Also, $S_n(1, 2) = F_{n+1} + 2F_n = F_{n+1} + F_n + F_n = F_{n+2} + F_n = L_{n+1}$ as before. (It is easily proved that $L_n = F_{n+1} + F_{n-1}$.)

The arbitrariness involved in the first two rows, however, shows up again when we consider the alternating row-sums. We find these are

$$T_n(a, b) = \sum_{j=0}^{n} (-1)^j C_j^n = b,\ a - b,\ b,\ a,\ a + b,\ 2a + b,\ 3a + 2b,\ \ldots$$

and, except for the first such sum, we can show that

$$(6.6) \qquad T_n(a, b) = \sum_{j=0}^{n} (-1)^j C_j^n = aF_{n-2} + bF_{n-3}, \quad n \geq 1.$$

Remark: The usual definition of Fibonacci numbers with negative index is

$$F_{-n} = (-1)^{n-1} F_n$$

so that the doubly infinite sequence of Fibonacci numbers is

$$\ldots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \ldots$$

In view of this, the formula (6.7) breaks down for $n = 0$ as it then gives the value $-a + 2b$ instead of the value b. However, for $n \geq 1$ agreement is found. In particular, when $a = 1 = b$, we have $T_n(1,1) = F_{n-2} + F_{n-3} = F_{n-1}$ as in (3.3) A similar result holds for the Lucas number variant (5.1).

7. FURTHER RELATIONS FOR THE POLYNOMIAL $A_n(x)$

By means of relation (4.2) we may show readily that $A_n(x)$ satisfies the second-order recurrence relation

$$A_{n+2}(x) = A_{n+1}(x) + x^2 A_n(x) . \tag{7.1}$$

In fact we have

$$A_{n+1}(x) = \sum_{0 \leq k \leq \frac{n+1}{2}} \binom{n+1-k}{k} x^{2k} + \sum_{0 \leq k \leq \frac{n}{2}} \binom{n-k}{k} x^{2k+1},$$

and

$$x^2 A_n(x) = \sum_{0 \leq k \leq \frac{n}{2}} \binom{n-k}{k} x^{2k+2} + \sum_{0 \leq k \leq \frac{n-1}{2}} \binom{n-1-k}{k} x^{2k+3}$$

$$= \sum_{1 \leq k \leq \frac{n+2}{2}} \binom{n+1-k}{k-1} x^{2k} + \sum_{1 \leq k \leq \frac{n+1}{2}} \binom{n-k}{k-1} x^{2k+1}$$

Using the fact that

$$\binom{p-k}{k} + \binom{p-k}{k-1} = \binom{p+1-k}{k},$$

it then readily follows that

$$A_{n+1}(x) + x^2 A_n(x) = \sum_{0 \le k \le \frac{n+2}{2}} \binom{n+2-k}{k} x^{2k} + \sum_{0 \le k \le \frac{n+1}{2}} \binom{n+1-k}{k} x^{2k+1}$$

$$= A_{n+2}(x).$$

Associated with $A_n(x)$ we may next introduce a related polynomial $K_n(x)$ defined by

$$K_n(x) = x^n A_n(\tfrac{1}{x}) = \sum_{j=0}^{n} A_j^n x^{n-j} \quad . \tag{7.2}$$

Relation (7.1) then becomes

$$K_{n+2}(x) = xK_{n+1}(x) + K_n(x), \text{ with } K_0(x) = 1,\ K_1(x) = x + 1 \ . \tag{7.3}$$

This recurrence relation is of the same form as one studied by Catalan [4]. This is mentioned by Byrd [3].

It may be of interest to indicate how the Q-matrix technique [1] may be applied to a study of $K_n(x)$. Define

$$Q = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} . \tag{7.4}$$

Then

$$Q^n = \begin{pmatrix} f_{n+1}(x) & f_n(x) \\ f_n(x) & f_{n-1}(x) \end{pmatrix}, \quad n \ge 1, \tag{7.5}$$

where the f's are Fibonacci polynomials defined by

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x), \quad f_0(x) = 0, \ f_1(x) = 1. \tag{7.6}$$

It is easily shown that

$$K_n(x) = f_{n+1}(x) + f_n(x) \ . \tag{7.7}$$

From (7.7) we have next

$$(-1)^{j+1} K_j(x) = (-1)^{j+1} f_{j+1}(x) - (-1)^j f_j(x)$$

whence

$$\sum_{j=0}^{n} (-1)^j K_j(x) = (-1)^n f_{n+1}(x) \quad , \tag{7.8}$$

so that the Fibonacci polynomials $f_n(x)$ may be expressed in terms of the K or A polynomials very easily.

We next observe that (7.5) and (7.7) yield

$$Q^n + Q^{n-1} = \begin{pmatrix} K_n(x) & K_{n-1}(x) \\ K_{n-1}(x) & K_{n-2}(x) \end{pmatrix} . \tag{7.9}$$

From this result it is possible to evaluate the determinant of the K's as follows. To begin with, $|Q^n| = |Q|^n = (-1)^n$. Then we find that

$$\begin{aligned} \begin{vmatrix} K_n(x) & K_{n-1}(x) \\ K_{n-1}(x) & K_{n-2}(x) \end{vmatrix} = |Q^n + Q^{n-1}| &= |Q^{n-1}(Q + I)| , \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \\ &= |Q^{n-1}| \cdot |Q + I| \\ &= (-1)^{n-1} x \ . \end{aligned}$$

We may state the result more elegantly in the form

$$\begin{vmatrix} K_{n+1}(x) & K_n(x) \\ K_n(x) & K_{n-1}(x) \end{vmatrix} = (-1)^n x \ . \tag{7.10}$$

This may be compared with the relation

$$\begin{vmatrix} F_{n+a} & F_{n+a+b} \\ F_n & F_{n+b} \end{vmatrix} = (-1)^n F_a F_b \tag{7.11}$$

for the ordinary Fibonacci numbers ($F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$) which was posed as a problem in the American Mathematical Monthly[8]. In particular, this raises the question about a similar generalization of the determinant (7.10). Indeed, we shall now prove by induction that

$$(7.12)\qquad \begin{vmatrix} K_{n+a}(x) & K_{n+a+b}(x) \\ K_n(x) & K_{n+b}(x) \end{vmatrix} = (-1)^n \begin{vmatrix} K_a & K_{a+b} \\ K_0 & K_b \end{vmatrix} = (-1)^n(K_aK_b - K_{a+b}).$$

This will be true for all integers if we define

$$(7.13)\qquad K_{-n}(x) = K_{n-1}(-x)$$

as is suggested by recurrence relation (7.3).

As for the proof of (7.12), we may first show that (as is obvious for $n = 0$)

$$(7.14)\qquad K_{n+1}K_{n+b} + K_nK_{n+b+1} = (-1)^n(K_1K_b - K_0K_{1+b}) ,$$

where, for brevity, we omit writing x which will remain unchanged. Now, in fact, by means of (7.3) we have

$$\begin{aligned}(-1)^n\left[K_{n+1}K_{n+b} - K_nK_{n+b+1}\right] &= (-1)^n\left[K_{n+1}(K_{n+b+2} - xK_{n+b+1}) - K_nK_{n+b+1}\right] \\ &= (-1)^n\left[K_{n+1}K_{n+b+2} - (xK_{n+1} + K_n)K_{n+b+1}\right] \\ &= (-1)^{n+1}\left[K_{n+2}K_{n+b+1} - K_{n+1}K_{n+b+2}\right] ,\end{aligned}$$

so that the expression is unchanged when n is replaced by $n + 1$. By induction, then, relation (7.14) follows.

In the same way, we could show that (7.12) holds for $a = 2$, that is,

$$(7.15)\qquad K_{n+2}K_{n+b} - K_nK_{n+b+2} = (-1)^n(K_2K_b - K_0K_{2+b}) .$$

We may complete the argument by an induction on a. Suppose that (7.12) holds for fixed n, b and up to a certain value of $a (\geq 1)$. Then

$$K_{n+a}K_{n+b} - K_nK_{n+a+b} = (-1)^n\ K_aK_b - K_0K_{a+b}$$

and

$$K_{n+a-1}K_{n+b} - K_nK_{n+a-1+b} = (-1)_n\ K_{a-1}K_b - K_0K_{a-1+b} ,$$

and if we multiply the first of these by x, add to the second, and recall the basic recurrence relation (7.3), we obtain precisely

$$K_{n+a+1}K_{n+b} - K_nK_{n+a+1+b} = (-1)^n \; K_{a+1}K_b - K_0K_{a+1+b} \quad ,$$

so that the induction goes through. This proof is nothing more than a variant of a similar proof for Problem E 1396, relation (7.11) above, suggested by Mr. John H. Biggs who was then a graduate student at West Virginia University. Clearly the same technique may be used in other cases where a recurrence relation of a suitable sort is presupposed. Thus (7.12) also holds for $f_n(x)$ in place of $K_n(x)$.

We should like to mention still another interesting relation involving the polynomial $K_n(x)$. The reader may find it worthwhile to carry out an inductive proof that

$$K_n(x) + (-1)^a K_{n+2a}(x) + xK_{n+a}(x) = 0 \quad . \tag{7.16}$$

When $a = 1$ this becomes again (7.3). It is possible to base a proof of (7.12) on this relation. The idea traces back as far as George Boole [2], and may have further unsuspected possibilities. Under miscellaneous propositions, in Chapter XII, pp. 229-231, Boole uses an invariance technique which may be of interest. By (7.16) we have (omitting x for brevity)

$$K_n + (-1)^a K_{n+2a} = -xK_{n+a} \quad .$$

This relation being true for all integers n, a, we next replace n by $n + b$, and we have, for arbitrary n, a, b,

$$K_{n+b} + (-1)^a K_{n+2a+b} = -xK_{n+a+b} \quad .$$

Here, $-x$ plays the part of the number p in Boole's argument. We may eliminate $-x$ from the last two relations by multiplying the former by K_{n+a+b}, the latter by K_{n+a}, and equating the resulting left-hand members. This yields

$$K_{n+a}K_{n+b} + (-1)^a K_{n+a}K_{n+2a+b} = (-1)^a K_{n+2a}K_{n+a+b} + K_nK_{n+a+b} \quad .$$

Multiplying through by $(-1)^n$ we have, transposing terms,

$$(-1)^n\left[K_{n+a}K_{n+b} - K_nK_{n+a+b}\right] = (-1)^{n+a}\left[K_{n+2a}K_{n+a-b} - K_{n+a}K_{n+2a+b}\right]. \tag{7.17}$$

Call the left-hand member of this $F(n)$. Then the crux of Boole's argument would be that (7.17) asserts that $F(n) = F(n + a)$. This being so for a perfectly arbitrary integer a, as we supposed to begin with, then it follows that $F(n)$ is invariant with respect to n. Hence we have only to set $n = 0$, and we find that

$$F(n) = F(0) = K_a K_b - K_0 K_{a+b}$$

and this of course is precisely what we claimed in relation (7.12).

The beauty of Boole's method is that one may oftentimes begin with a non-linear recurrence relation (difference equation), such as (7.12) is indeed, and relate this back to a linear relation, as (7.16) actually is. The method is especially useful in the study of determinants of polynomials which satisfy suitable recurrence relations.

The relations (7.11) and (7.12) may be called Turán relations, and the reader is referred to [5, 6] for pertinent journal references and some variations. A detailed bibliography on the Turán expressions (and Turán inequalities) would contain over 110 references to journal articles and books according to the author's current file on the literature.

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ON A GENERATING FUNCTION ASSOCIATED WITH GENERALIZED FIBONACCI SEQUENCES

I. I. Kolodner(1)(2)
Carnegie Institute of Technology, Pittsburgh, Pennsylvania

Let p, q, a and b be complex numbers and assume that $q \neq 0$ and $q \neq p^2$. Let the sequence $u_n(p, q; a, b)$ be the solution of the recurrence relation

(1) $$u_n = 2pu_{n-1} - qu_{n-2}, \qquad n \geq 2,$$

with the "initial condition"

(2) $$u_o = a, \qquad u_1 = b + pa.$$

Here and below we omit arguments whenever they are obvious.

If $p = 1/2$ and $q = -1$, the above sequence reduces to $u_n(1/2, -1; a, b) = H_n$, the generalized Fibonacci sequence. Further, $u_n(1/2, -1; 0, 1) = F_n$, the Fibonacci sequence, and $u_n(1/2, -1; 2, 0) = L_n$, the Lucas sequence.

For any integer k, define the function $x \longrightarrow u^{(k)}(x; p, q; a, b) = u^{(k)}(x)$ by the formula

(3) $$u^{(k)}(x) = \sum_{0 \leq n \leq \infty} (u_n)^k x^n.$$

Since, as is easily verified, $u_n \leq A\, s^n$ where $s = |p| + \sqrt{|p|^2 + |q|}$, the series in (3) converges at least for $x < s^{-k}$. A few years ago Carlitz [1] summed the series for $u^{(k)}$ in special cases when $a = 1$, $b = p$ (using the present notation) and $a = 2$, $b = 0$. For related results see also the papers by Gould [2] and Riordan [3]. A. F. Horadam recently studies (3) the generating functions $u^{(k)}$ and indicated that they can be summed by using methods analogous to those employed by Carlitz, which are rather complicated. The objective of this paper is to give a straightforward derivation of a formula for $u^{(k)}$ with any a and b.

<u>Theorem</u>. <u>If</u> $u_n(p, q; a, b)$ <u>is defined by (1-2) and</u> $u^{(k)}$ <u>is defined by (3), then</u>

$$u^{(k)}(x) = \sum_{0 \le \gamma < k/2} \frac{A_{\gamma,k} + q^{\gamma}[B_{\gamma,k}u_{k-2\gamma}(p,q;0,1) - A_{\gamma,k}u_{k-2\gamma}(p,q;1,0)]x}{1 - 2q^{\gamma}u_{k-2\gamma}(p,q;1,0)x + q^k x^2}$$

$$+ \frac{(1 + (-1)^k)A_{k/2,k}}{4\,[1 - q^{k/2}x]},$$

where $A_{\gamma,k}$ and $B_{\gamma,k}$ with $\gamma \le [k/2]$ are homogeneous forms in a, and b of degree k defined by

$$A_{\gamma,k} = 2^{1-k}\binom{k}{\gamma}(a^2 - \beta)^{\gamma} \sum_{0 \le 2j \le k-2\gamma} \binom{k-2\gamma}{2j} a^{k-2\gamma-2j}\beta^j .$$

$$B_{\gamma,k} = 2^{1-k}\binom{k}{\gamma}(a^2 - \beta)^{\gamma}\, b \sum_{0 \le 2j+1 \le k-2\gamma} \binom{k-2\gamma}{2j+1} a^{k-2\gamma-2j-1}\beta^j ,$$

with $\beta = b^2/(p^2 - q)$.

Note that in the last term in the formula for $u^{(k)}(x)$ the first factor is zero if k is odd so that we should not be concerned by the fact that $A_{k/2,k}$ is not defined when k is odd.

Our proof exploits the fact that the zeros of $z^2 - 2z\cos\theta + 1$, with any θ real or complex, are $e^{i\theta}$ and $e^{-i\theta}$ whose powers are easily managed. Let α and θ be such that

$$\text{(4)} \qquad \alpha^2 = q, \quad p = \alpha\cos\theta .$$

Since $q \ne 0$ and $p \ne q^2$, $\alpha \ne 0$ and $\cos\theta \ne \pm 1$. Since the function $z \longrightarrow \cos z$ is onto the complex plane, a number θ satisfying (4) exists; it may be, or course, a complex number. Note that $\alpha^2\sin^2\theta = \alpha^2 - \alpha^2\cos^2\theta = q - p^2 \ne 0$.

Set $u_n = \alpha^n v_n$. Then $v_n = 2(\cos\theta)v_{n-1} - v_{n-2}$ $(n \ge 2)$ from which it follows, by using well known results for linear recurrences with constant coefficients, that $v_n = s\,e^{in\theta} + t\,e^{-in\theta}$ with some s and t which are determined by the "initial conditions" (2). We now conclude that

$$u_n(p, q; a, b) = \alpha^n(s\, e^{in\theta} + t\, e^{-in\theta}) \quad . \tag{5}$$

Setting $n = 0, 1$ in succession we get

$$s + t = a \tag{6'}$$

and $\alpha(\cos\theta)(s + t) + i\alpha(\sin\theta)(s - t) = b + pa$, whence it follows, on using (6') and (4), that

$$s - t = b/(i\alpha \sin\theta). \tag{6''}$$

The expressions for s and t may be easily obtained but will not be needed here. On the other hand we note that if $s = t = 1/2$ then $a = 1$ and $b = 0$, while if $s = -t = 1/2$ then $a = 0$ and $b = i\alpha \sin\theta$. Thus it follows from (5) that

$$\begin{cases} \alpha^n \cos n\theta = u_n(p, q; 1, 0), \\ \alpha^n \sin n\theta = \alpha(\sin\theta) u_n(p, q; 0, 1), \end{cases} \tag{7}$$

identifications which will be used in the sequel. [4]

We are now ready for the evaluation of $u^{(k)}(x)$. Using the binomial theorem, we get

$$\begin{aligned} (8)\ (s\, e^{in\theta} + t\, e^{-in\theta})^k &= \sum_{0 \le \gamma \le k} \binom{k}{\gamma} s^\gamma t^{k-\gamma} e^{in(2\gamma - k)\theta} \\ &= \sum_{0 \le \gamma \le k/2} \binom{k}{\gamma} (st)^\gamma (s^{k-2\gamma} e^{in(k-2\gamma)\theta} + t^{k-2\gamma} e^{-in(k-2\gamma)\theta}) \\ &\quad + 2^{-1}(1 + (-1)^k)\binom{k}{k/2}(st)^{k/2} , \end{aligned}$$

where the last equality follows by pairing off terms with γ and $k - \gamma$, and where the last term is to be set equal to zero if k is odd. On substituting (5) in (3), using (8), interchanging the order of summation, and finally summing geometric series we obtain

(9)
$$u^{(k)}(x) = \sum_{0 \le \gamma < k/2} \binom{k}{\gamma}(st)^{\gamma} \sum_{n=0}^{\infty} [s^{k-2\gamma}(x\alpha^k e^{i(k-2\gamma)\theta})^n$$

$$+ t^{k-2\gamma}(x\alpha^k e^{-(k-2\gamma)\theta})^n]$$

$$+ 2^{-1}(1 + (-1)^k)\binom{k}{k/2}(st)^{k/2} \sum_{n=0}^{\infty} (x\alpha^k)^n$$

$$= \sum_{0 \le \gamma < k/2} \binom{k}{\gamma}(st)^{\gamma}\left[\frac{s^{k-2\gamma}}{1 - x\alpha^k e^{i(k-2\gamma)\theta}} + \frac{t^{k-2\gamma}}{1 - x\alpha^k e^{-i(k-2\gamma)\theta}}\right]$$

$$+ 2^{-1}(1 + (-1)^k)\binom{k}{k/2} \frac{(st)^{k/2}}{1 - x\alpha^k} .$$

Observing that $\alpha^{2k} = q^k$, $\alpha^k = q^{k/2}$ if k is even, $\alpha^k\cos(k-2\gamma)\theta = q^{\gamma}u_{k-2\gamma}(p, q; 1, 0)$ and $\alpha^k\sin(k-2\gamma) = q^{\gamma}\alpha(\sin\ \theta)u_{k-2\gamma}(p, q; 0, 1)$ if $2\gamma < k$, see formulae (7), the form for $u^{(k)}(x)$ asserted in the theorem follows from (9) if we define

(10)
$$A_{\gamma, k} = \binom{k}{\gamma}(st)^{\gamma}[s^{k-2\gamma} + t^{k-2\gamma}], \qquad 2\gamma \le k,$$

$$B_{\gamma, k} = i\binom{k}{\gamma}(st)^{\gamma}[s^{k-2\gamma} - t^{k-2\gamma}]\alpha\ \sin\ \theta, \qquad 2\gamma < k.$$

It remains to evaluate $A_{\gamma, k}$ and $B_{\gamma, k}$ in terms of a and b. Let $\beta = [b/(i\alpha\ \cos\ \theta)]^2 = b^2/(p^2 - q)$. From (6') and (6'') we get:

$$st = (a^2 - \beta)/4,$$

whence $(st)^{\gamma} = 2^{-2\gamma}(a^2 - \beta)^{\gamma}$,

$$s^m + t^m = 2^{-m}([(s+t)+(s-t)]^m + [(s+t)-(s-t)]^m) = 2^{1-m} \sum_{0 \le 2j \le m} \binom{m}{2j} a^{m-2j}\beta^j,$$

$$s^m - t^m = 2^{-m}([(s+t)+(s-t)]^m - [(s+t)-(s-t)]^m)$$

$$= 2^{1-m} \frac{b}{i\alpha \sin\theta} \sum_{0 \le 2j+1 \le m} \binom{m}{2j+1} a^{m-2j-1} \beta^j .$$

Substituting these in (10) we get the stated result. This completes the proof of the theorem.

It might not be superfluous to point out some special cases which may be obtained from the theorem. If $p = 1/2$ and $q = -1$, then $u^{(k)}(x;1/2,-1;a,b) = H^{(k)}(x;a,b)$, the generating function for k^{th} powers of the generalized Fibonacci sequence $H_n(a,b)$. In this case the formulae for $A_{\gamma,k}$ and $B_{\gamma,k}$ do not simplify appreciably except that we have now $\beta = 4b^2/5$, while $u_n(1/2,-1;0,1) = F_n$ and $u_n(1/2,-1;1,0) = L_n/2$. Furthermore, if also $a = 0$ and $b = 1$, then $A_{\gamma,k} = 0$ if k is odd and $B_{\gamma,k} = 0$ if k is even, while $B_{\gamma,k} = (-1) \binom{k}{\gamma} 5^{(1-k)/2}$ if k is odd and $A_{\gamma,k} = 2(-1) \binom{k}{\gamma} 5^{-k/2}$ if k is even. The theorem then yields the well known formulae

$$(11)\begin{cases} F^{(k)}(x) = 5^{(1-k)/2} x \displaystyle\sum_{0 \le \gamma \le (k-1)/2} \frac{\binom{k}{\gamma} F_{k-2\gamma}}{1-(-1)^{\gamma} L_{k-2\gamma}x - x^2}, & \text{if } k \text{ is odd,} \\ F^{(k)}(x) = 5^{-k/2} \Bigg[\displaystyle\sum_{0 \le \gamma < k/2} \frac{\binom{k}{\gamma}(2(-1)^{\gamma} - L_{k-2\gamma}x)}{1-(-1)^{\gamma} L_{k-2\gamma}x + x^2} + \frac{\binom{k}{k/2}}{(-1)^{k/2} - x} \Bigg], & \text{if } k \text{ is even,} \end{cases}$$

Lastly, if $p = 1/2$, $q = -1$, $a = 2$ and $b = 0$, we get $A_{\gamma,k} = 2\binom{k}{\gamma}$ and $B_{\gamma,k} = 0$ whence

(12) $$L^{(k)}(x) = \sum_{0 \le \gamma < k/2} \frac{\binom{k}{\gamma}(2 - (-1)^{\gamma} L_{k-2\gamma}x)}{1-(-1)^{\gamma} L_{k-2\gamma}x + (-1)^k x^2}$$

$$+ \frac{\binom{k}{k/2}}{1-(-1)^{k/2}x} \cdot \frac{1+(-1)^k}{2} .$$

In conclusion we note that by squaring the two equalities in (7) and adding we get the identity

(13) $$q^n = [u_n(p, q; 1, 0)]^2 + (q-p^2)[u_n(p, q; 0, 1)]^2$$

If $p = 1/2$ and $q = -1$, the identity (13) simplifies to the well-known identity

(14) $$4(-1)^n = L_n^2 - 5F_n^2 .$$

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UNIVERSITY OF NEW MEXICO

FOOTNOTES

(1) Present address: Carnegie Institute of Technology

(2) The author wishes to thank the referee for most scholarly work in evaluating this paper and for really helpful suggestions.

(3) Oral communication.

(4) Formulae (7) show the connection between u_n and the Chebyshev polynomials. For example, $u_n(p, q; 1, 0) = \alpha^n T_n(p/\alpha)$, where $\alpha^2 = q$.

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A PERMUTATIVE PROPERTY OF CERTAIN MULTIPLES OF THE NATURAL NUMBERS

W. D. Skees
Digital Computer Laboratory
University of Illinois

1. INTRODUCTION

In number theory one encounters such numbers as

$$.105263157894736842$$

(the period of 2/19) and .102564 (the period of 4/39) one of whose very interesting properties will be treated here. If the terminal digit be removed from the end of the number and placed at the beginning, the result is the product of that digit and the original number.

Examples:

$$\begin{array}{r} .10526315789473684\underline{2} \\ \times\ 2 \\ \hline .\underline{2}10526315789473684 \end{array} \quad \text{and} \quad \begin{array}{r} .10256\underline{4} \\ \times\ 4 \\ \hline .\underline{4}10256 \end{array}$$

The purpose of this paper will be to investigate the existence and characteristics of such numbers.

2. DEFINITIONS

A positive number G will be called a gauntlet if it has a cyclic permutation with the property that, when the natural number g making up its last n digits be moved to the first n digits' positions of the number, then the result is exactly the product gG. When such a number G exists for a natural number g we will occasionally write G(g) for emphasis. The product gG is called the second order gauntlet, written $G^{(2)}$.

We also define the function D whose value D(x) is the number of digits in x. It follows from the above definitions that $D(G) = D(G^{(2)})$.

3. FAMILY OF GAUNTLETS

The question arises: are there many gauntlets for a single natural number? We answer with a theorem.

Theorem 1. Each natural number for which a gauntlet exists has infinitely many gauntlets each consisting of a number of sets of the same period.

Proof. Let $.p_1p_2\cdots p_{D(G)}$ be a digit-wise representation of G, a gauntlet of the natural number g. We observe that gG is of the form $.q_1q_2\cdots q_{D(g)}$ because $D(G^{(2)}) = D(G)$. This means there is no carry on the left after multiplication of G by g. This implies

$$g\cdot(.p_1\cdots p_{D(G)}p_1\cdots p_{D(G)}) = .q_1\cdots q_{D(G)}q_1\cdots q_{D(G)}$$

and the theorem follows by induction.

Example:

$$g = 4$$

$$G_1(g) = .10256\underline{4}$$

$$G_1^{(2)}(g) = .\underline{4}10256$$

$$G_2(g) = .10256410256\underline{4}$$

$$G_2^{(2)} = .\underline{4}10256410256$$

Let us call numbers which are gauntlets for the same natural number and whose digits are repetitions of the digits of a simpler gauntlet members of the family of that gauntlet. Similarly we define a family of second order gauntlets. Hereafter unless otherwise stated G and $G^{(2)}$ will be understood to be the least positive gauntlets of their families.

4. DIGITS COMMON TO ALL GAUNTLETS

Theorem 2. The leading non-zero digit of a gauntlet is 1.

Proof. Let g be represented by the digit-wise expansion $c_1c_2\cdots c_{D(g)}$. Then $G^{(2)} = .c_1c_2\cdots c_{D(g)}x_1x_2\cdots x_{D(G)-D(g)}$. Now

$$\text{(1)} \qquad c_1\cdots c_{D(g)}\overset{.0\ \ldots 0 \qquad 1 \qquad \ldots}{\overline{\big).\,c_1\cdots c_{D(g)-1}c_{D(g)}x_1x_2\cdots x_{D(G)-D(g)}}}$$

and by definition the quotient must be G.

Corollary 2. A gauntlet of the natural number g has exactly $D(g)-1$ leading zeros.

Proof. Count the leading zeros of the quotient of (1).

Note. The leading zeros are part of the repeating set of digits in the family of a gauntlet.

Theorem 3. For g not a power of 10 there are exactly $2D(g)-1$ zeros to the immediate right of the leading non-zero digit 1 of G.

Proof. From (1)

$$G = .0_1 \dots 0_{D(g)-1} 1 x_1 \dots x_{D(G)-D(g)}$$

(the x_i are now the unknown digits of the numerator) where

$$x_{D(G)-2D(g)+1} \dots x_{D(G)-D(g)} = c_1 \dots c_{D(g)} = g.$$

Whence

$$G^{(2)} = .c_1 \dots c_{D(g)} 0_1 \dots 0_{D(g)-1} 1\, x_1 x_2 \dots x_{D(G)-2D(g)}.$$

Then by definition

$$\begin{array}{l} \quad .0_1 \dots 0_{D(g)-1} 1 \quad 0_1 \dots 0_{D(g)-1} 0_{D(g)} \; \dots \\ g\,\overline{\big)\, .c_1 \dots c_{D(g)-1} c_{D(g)} 0_1 \dots 0_{D(g)-1} 1 \quad x_1 x_2 \dots x_{D(G)-2D(g)}} \end{array}$$

which implies

$$G = .0_1 \dots 0_{D(g)-1} 1\; 0_1 \dots 0_{D(g)-1} 0_{D(g)} \dots$$

This means that

$$\begin{array}{l} \quad .0_1 \dots 0_{D(g)-1} 1 \quad 0_1 \dots 0_{D(g)-1} 0_{D(g)} 0_{D(g)+1} \dots 0_{2D(g)-1} x \dots \\ g\,\overline{\big)\, .c_1 \dots c_{D(g)-1} c_{D(g)} 0_1 \dots 0_{D(g)-1} 1 \quad 0_1 \;\dots\; 0_{D(g)-1}\; 0_{D(g)} \dots} \end{array}$$

(and x is non-zero because $10_1 \dots 0_{D(g)}$ is greater that g) which proves the theorem.

Corollary 3. The gauntlet of a natural number g which is a power of 10 is exactly $.0_1 \dots 0_{D(g)-1} 1\; 0_1 \dots 0_{D(g)-1}$.

Proof. That $g=10^n$ implies $D(g)=n+1$. That is to say $g=10_1 \dots 0_n = 10_1 \dots 0_{D(g)-1}$, the terminal $D(g)$ digits of

$$.0_1 \dots 0_{D(g)-1} 1\; 0_1 \dots 0_{D(g)-1},$$

and

$$\begin{array}{r} .0_1\cdots 0_{D(g)-1}1\ 0_1\cdots 0_{D(g)-1} \\ \times\ 1\ 0_1\cdots 0_{D(g)-1} \\ \hline .10_1\cdots 0_{D(g)-1}0_1\cdots 0_{D(g)-1} \end{array} \qquad \text{Q. E. D.}$$

Exceptions must always be made in the following discussion for $g=10^n$ because only with such a g are the $D(g)$ initial digits of g^2 the digits of g itself.

Examples for the corollary.

$$G(1) = .1$$

$$G(10) = .010$$

It should be obvious by now that it is largely inconsequential whether we consider gauntlets as integers or decimals, because whether the number is 010 or .010 the digits are the same and our primary concern is which leading or trailing zeros are part of the number, not where the decimal point goes. It is more amenable to the notion of families to use decimals because of the obvious similarity to periodic decimals. However, in a following theorem (Theorem 5) the proof is expedited by reference to gauntlets as integers.

5. GENERATION OF A GAUNTLET IN SETS OF DIGITS

Let us now examine the interrelationships of the digits within a gauntlet and the way in which a natural number generates its own gauntlet.

Remark. The following discussion develops an algorithm which finds G for $g \neq 10^n$. Corollary 3 found G for every $g=10^n$, and it may be readily verified that the algorithm of this section finds a larger member of the family of $G(10^n)$.

The terminal $D(g)$ digits of G make up g itself. Consequently the terminal $D(g)$ digits of $G^{(2)}$ must be the terminal $D(g)$ digits of g^2 which are also the $D(g)+1$st through the $2D(g)$th digits of G, counting from the righthand side. That is,

$$G = .x_{D(G)}\cdots x_{2D(g)+1}d_{2D(g)}\cdots d_{D(g)+1}c_{D(g)}\cdots c_1$$

where the d's are the D(g) terminal digits of g^2 and of $gG=G^{(2)}$. Moving leftward along G we see that the next set of D(g) x's must represent the terminal D(g) digits of the sum of the leading digits of g^2 not included in the set $d_{D(g)}\dots d_1$ and $g\cdot(d_{D(g)}\dots d_1)$. So is the next set of D(g) digits related to those to the right of it. To restate symbolically what we have just verbalized, the ith set of D(g) digits (counting from the right where the a's are the sets) may be written

$$(2) \qquad a_i = ga_{i-1}+r_{i-1} - \left[\frac{ga_{i-1}+r_{i-1}}{10^{D(g)}}\right] \cdot 10^{D(g)}$$

where

$$(3) \qquad r_i = \left[\frac{a_{i-1}g}{10^{D(g)}}\right], \quad a_1=g, \quad \text{and} \quad r_1=0.$$

(Brackets indicate greatest integer division.)

These equations, which follow directly from the definitions, constitute an algorithm which, depending upon g alone, inevitably produces G(g) if it exists. Since the algorithm generates only sets of D(g) digits each we may conclude D(g) divides D(G) and when G exists it has a left-most set a_j whose digit-wise representation is $0\dots01$ and that $r_{j+1}=0$. These conditions provide criteria for stopping the algorithm at a_j.

Remark. The single exception to the rule "D(g) divides D(G)" is for $g=10^n$. The reason is that the two a_i of $G(10^n)$ share the common digit 1. However, the algorithm will find a $G'(10^n) > G$ such that D(g) divides D(G'). That $G(10^n)$ is the only possible exception for the success of the algorithm may be readily verified.

Theorem 4. If G exists for a given g the algorithm (given above) generates G, and the condition $a_j=1$ and $r_{j+1}=0$ is sufficient to terminate the algorithm.

Proof. That the algorithm generates G follows from the preceding remarks in this section. If $a_j=1$ and $r_{j+1}=0$ the algorithm begins to repeat the digits of G because $a_{j+1}=g\cdot 1+0-0=g$, and $r_{j+2}=0$. This is identically the situation at the beginning of the algorithm, which means from this point it would regenerate the same digits. Hence if a_j is the first set equal to 1 and such that $r_{j+1}=0$ then the digits generated up to that point make up the least positive member of the family, that is G.

Remark. An algorithm mentioned by Johnson [2] will find the period of the reciprocal of 10m-1 (where m is a natural number), but the result does not have the combined multiplicative and permutative property, which is the subject of this paper, for m of more than one digit.

Example. The period .10027, a cyclic permutation of that found for m=37 by Johnson's method, has not the same property as has the number found by my method for m=37, namely

$$\begin{array}{r} .01000\ 27034\ 33360\ 36766\ 69\underline{37} \\ \times\ 37 \\ \hline .\underline{37}010\ 00270\ 34333\ 60367\ 6669 \end{array}$$

6. THE EXISTENCE THEOREM

Theorem 5. For every natural number there exists at least one gauntlet and hence one family of the gauntlet.

Proof. That $G(10^n)$ exists follows from Corollary 3. Assume $g \neq 10^n$. As usual we assume G is the smallest positive member of its family. We recall that D counts all the digits in a number which are part of that number. This includes leading zeros. Let G be considered an integer. The relationship between g and G, from the definitions, is

$$\frac{G-g}{10^{D(g)}} + g10^{D(G)-D(g)} = gG = g^{(2)}$$

which simplifies thus:

$$G-g + 10^{D(G)}g = 10^{D(g)}gG$$

$$G(1-10^{D(g)}g) + g(10^{D(G)}-1) = 0$$

$$G = \frac{g(10^{D(G)}-1)}{10^{D(g)}g-1}$$

Now we require that G be an integer, which is true if and only if $g(10^{D(G)}-1)$ is congruent to 0 modulo $10^{D(g)}g-1$. This means

$$10^{D(G)}g \equiv g \bmod(10^{D(g)}g-1).$$

Since $10^{D(g)}g-1$ and g are relatively prime

$$10^{D(G)} \equiv 1 \bmod(10^{D(g)}g-1).$$

Now

$$10^{x} \equiv 1 \bmod(10^{D(g)}g-1) \tag{4}$$

has a solution $x = \phi(10^{D(g)}g-1)$ by Fermat's theorem because 10 and $10^{D(g)}g-1$ are relatively prime. That is to say

$$10^{x}g \equiv g \bmod(10^{D(g)}g-1) \tag{5}$$

has a solution which means there exists an integer K such that

$$K = \frac{g(10^{x}-1)}{10^{D(g)}g-1} \tag{6}$$

for a given integer g.

All solutions to (4) may be found in the following way. We divide successively increasing powers of 10 by $10^{D(g)}g-1$ until finally we are left with a remainder of 1. This implies the solution to (5) may be found similarly. We divide the product of g and successively increasing powers of 10 by $10^{D(g)}g-1$ until finally there is a remainder of g. The number of zeros we use is the solution x.

Now (6) has a least positive solution x_0. Let the numerator (7) of the following expression be the least positive such numerator, that is let the appearance of g as a remainder be the first such appearance of g. If we can show that (7) is G we are finished since $D((7))$ which is x_0 will also be $D(G)$, and x_0 is known to be the least positive solution of (6) such that K is the least positive integer, and G is assumed to be the least positive gauntlet of g.

$$
\begin{array}{lr}
(7) & \qquad \qquad .p_1p_2\cdots p_{x_0} \\
 & 10^{D(g)}g-1 \;\overline{)g.0\;0\;\ldots\;0} \\
 & \vdots \\
(8) & \overline{q_1\ldots 0} \\
(9) & \ldots \\
(10) & \overline{\qquad g}
\end{array}
$$

For keeping track of our zeros we will revert to the use of decimals. Adding terminal zeros to 1.000... is simplified by the nature of the number (i.e. 1.0 followed by infinitely many zeros is equivalent to 1.0). We find ourselves studying

$$\frac{1}{g10^{D(g)}-1}$$

or, equivalently,

$$\frac{g}{g10^{D(g)}-1}$$

as far as x_0 is concerned, rather than

$$\frac{10^{x_0}}{g10^{D(g)}-1} \quad \text{or} \quad \frac{g10^{x_0}}{g10^{D(g)}-1} \; .$$

Let g be expanded digitwise as $c_1\ldots c_{D(g)}$. Since $10^{D(g)}g-1$ ends in 9, and (8) ends in 0 while g ends in $c_{D(g)}$, then p_{x_0} can only be $c_{D(g)}$. We rewrite (8), (9) and (10) as (13), (14) and (15) below:

$$
\begin{array}{ll}
(11) & \qquad \ldots \\
(12) & \overline{\qquad \ldots \qquad} \\
(13) & q_1 \;\; \ldots \;\; 0 \\
(14) & c_{D(g)}(g10^{D(g)}-1) \\
(15) & \overline{c_1 \;\; \ldots \;\; c_{D(g)}} \; .
\end{array}
$$

We introduce the convention of braces about the digit-wise expansion of a number to clarify arithmetic expressions. Then we may write (14) as

$$\{c_1\ldots c_{D(g)}\} \cdot 10^{D(g)}c_{D(g)} - c_{D(g)} \; .$$

Adding g we have (13):

$$\{c_1 \cdots c_{D(g)}\} \cdot c_{D(g)} 10^{D(g)} - c_{D(g)} + \{c_1 \cdots c_{D(g)}\}$$

which reduces to

$$\{c_1 \cdots c_{D(g)}\} \cdot c_{D(g)} 10^{D(g)} + \{c_1 \cdots c_{D(g)-1}\} \cdot 10.$$

But (13) without the suffixed 0 is

$$\{c_1 \cdots c_{D(g)}\} \cdot c_{D(g)} 10^{D(g)-1} + \{c_1 \cdots c_{D(g)-1}\}$$

which terminates in $c_{D(g)-1}$. This means that

$$p_{x_0 - 1} = c_{D(g)-1}, \text{ whence (12) is } (g10^{D(g)} - 1) \cdot c_{D(g)-1}.$$

This implies that (11) is

$$\{c_1 \cdots c_{D(g)}\} \cdot c_{D(g)-1} 10^{D(g)} - c_{D(g)-1}$$
$$+ \{c_1 \cdots c_{D(g)}\} \cdot c_{D(g)} 10^{D(g)-1} + \{c_1 \cdots c_{D(g)-1}\} .$$

Redcuing as before and removing the suffixed 0 we have for (11)

$$\{c_1 \cdots c_{D(g)}\} \cdot \{c_{D(g)-1} c_{D(g)}\} \cdot 10^{D(g)-2} + \{c_1 \cdots c_{D(g)-2}\} .$$

By induction after $D(g)$ such steps the remainder is

(16) $$\{c_1 \cdots c_{D(g)}\} \quad \{c_1 \cdots c_{D(g)}\} \cdot 10^0 + \{0\} .$$

At each step the terminal digit in the remainder was a c_i. This implies

$$p_{x_{0-D(g)+1}} \cdots p_{x_0} = c_1 \cdots c_{D(g)}.$$

At this point the remainder ends in $\langle g^2 \rangle$. (The new notation means the last digit of.) This means

$$p_{x_0 - D(g)} = \langle g^2 \rangle .$$

This seems to indicate generation of the same digits of the algorithm of section 5. Indeed they are identical because the minuend producing the remainder (16) is

$$\{c_1 \ldots c_{D(g)}\} \cdot 10^{D(g)} \langle g^2 \rangle - \langle g^2 \rangle + g^2$$

which after removal of the suffixed zero is

$$\{c_1 \ldots c_{D(g)}\} \quad \langle g^2 \rangle \; 10^{D(g)-1} + \frac{g^2 - \langle g^2 \rangle}{10}$$

which ends in $\left\langle g^2 - \langle g^2 \rangle \right\rangle$, and we see we must exhaust $D(g)$ powers of 10 again, thereby setting $p_{x_0 - 2D(g)+1} \ldots p_{x_0 - D(g)}$ equal to the terminal $D(g)$ digits of g^2.

Alternatively we must, every $D(g)$ steps, exhaust the $D(g)$ digits of a set which corresponds to some a_i of the algorithm. Therefore by Theorem 4 the numerator is G if its first $D(g)$ digits are $0_1 \ldots 0_{D(g)-1} 1$ and its next $D(g)$ digits are 0. This latter condition is sufficient to make $r_{i+1} = 0$.

We write the initial situation in the division process as

$$\begin{array}{r} .\,0_1 \ldots 0_{D(g)-1} 1 \;\; \ldots \\ \{c_1 \ldots c_{D(g)}\} \cdot 10^{D(g)} - 1 \,\big)\overline{c_1 \ldots c_{D(g)} \cdot 0_1 \ldots 0_{D(g)-1} 0_{D(g)} \ldots} \\ \{c_1 \ldots c_{D(g)} \; 0_1 \; \cdot \;\; \cdot \;\; 0_{D(g)}\} - 1 \\ \hline 1 \end{array}$$

because

$$\{c_1 \ldots c_{D(g)}\} \cdot 10^{D(g)} = c_1 \ldots c_{D(g)} 0_1 \ldots 0_{D(g)}$$

and since

$$10^{2D(g)-1} \le g 10^{D(g)} - 1 < 10^{2D(g)}$$

we have

$$\begin{array}{r} .\,0_1 \ldots 0_{D(g)-1} 1 \quad 0_1 \ldots 0_{2D(g)-1} \ldots \\ \{c_1 \ldots c_{D(g)}\} \cdot 10^{D(g)} - 1 \,\big)\overline{c_1 \ldots c_{D(g)} \cdot 0_1 \ldots 0_{D(g)-1} 0_{D(g)} 0_{D(g)+1} \ldots} \;. \end{array}$$

Q. E. D.

Corollary 5. For every natural number there is only one family of gauntlets and only one G, the least positive gauntlet.

Proof. The uniqueness of the algorithmic process and also of the division in the previous theorem.

7. ADDITIONAL THEOREMS

The following theorems, which may be easily verified, are submitted without proof.

Theorem 6. The period of $n/(n10^{D(n)}-1)$ where n is any positive integer is the same as the period of the reciprocal of $n10^{D(n)}-1$.

Theorem 7. Each digit of the period on $n/(n10^{D(n)}-1)$ appears in succession as the terminal digit of a remainder when decimal division is carried out.

Example:

$$g = 4$$
$$D(g) = .10256\underline{4}$$
$$g10^{D(g)}-1 = 39$$

```
          .102564
     39 )4.000000
         39
         --
          10
          00
          --
          100
           78
          ---
           220
           195
           ---
            250
            234
            ---
             160
             154
             ---
               4
```

Theorem 8. The digits of the period of $1/(n10^{D(n)}-1)$ are a cyclic permutation leftward $D(g)$ places of those of $n/(n10^{D(n)}-1)$ where n is any natural number, and theorem 7 holds for $1/(n10^{D(n)}-1)$.

Theorem 9. For G the gauntlet of a given g, the following relation holds, $2D(g10^{D(g)}-1) \leq D(G) \leq g10^{D(g)}-2$.

Theorem 10. $D(g)$ divides the period of $g/(g10^{D(g)}-1)$ and hence of $1/(g10^{D(g)}-1)$, provided $g \neq 10^n$, and, for $g=10^n$, then $D(G) = 2D(g)-1$.

8. PARTIAL TABLE OF THE FIRST 100 GAUNTLETS

g	G	D(G)	The Period of a permutation of	of
1	.1	1	$\frac{1}{9}$	$\frac{1}{9}$
2	.10526 31578 94736 842	18	$\frac{2}{19}$	$\frac{1}{19}$
3	.10344 82758 62068 96551 72413 793	28	$\frac{3}{29}$	$\frac{1}{29}$
4	.10256 4	6	$\frac{4}{39}$	$\frac{1}{39}$
7	.10144 92753 62318 84057 97	22	$\frac{7}{69}$	$\frac{1}{69}$
34	.01000 29420 41776 99323 33039 12915 5634	34	$\frac{34}{3399}$	$\frac{1}{3399}$
37	.01000 27034 33360 36766 6937	24	$\frac{37}{3699}$	$\frac{1}{3699}$
100	.00100	5	$\frac{100}{99999}$	$\frac{1}{99999}$

9. APPENDIX

An interesting question is, are there any more integers, g, such as 1 and 34, where D(G) = g?

ACKNOWLEDGMENTS

P. M. Weichsel, Ph. D., for encouragement and lectures on number theory. P. T. Bateman, Ph. D., for bibliographical recommendations.

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2. W. W. Johnson, "On a Method of Forming the Periods of Circulating Decimals, Messenger of Math., 14(1884-1885), 14-18.

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The appearance of a booklet entitled: "Introduction to Fibonacci Discovery" by Brother U. Alfred, Managing Editor of the Fibonacci Quarterly. As the title implies the aim of this publication is to provide the reader with the opportunity to work out various facets of the Fibonacci numbers by himself. At the same time, there is sufficient help in the form of answers and explanations to reassure him regarding the correctness of his work.

The treatment is relatively brief, there being some sixty pages in all. The material was set up by typewriter and subsequently lithographed. The books have a paper cover and are held together by glue binding. Price per copy is $1.50 with a quantity price of $1.25 when four or more copies are ordered at once. The following topics are treated:

Discovering Fibonacci Formulas
Proof of Formulas by Mathematical Induction
The Fibonacci Shift Formulas
Explicit Formulas for the Fibonacci and Lucas Sequences
Division Properties of Fibonacci Numbers
General Fibonacci Sequences
The Associated "Lucas" Sequence
The Fibonacci Sequence and Pascal's Triangle
The Golden Section
Matrices and Fibonacci Numbers
Continued Fractions and Fibonacci Numbers

This booklet should provide the means of becoming acquainted with Fibonacci numbers and some of their main ramifications. It should serve as a useful reference for readers of the Fibonacci Quarterly who wish to learn about the main aspects of Fibonacci numbers. It should also prove of value to groups of competent high school or college students. While not recommended for the "pro", it might be a useful reference to have on hand to loan to students or fellow faculty members who want to know something about Fibonacci numbers.

These booklets are now available for purchase. Send all orders to: Brother U. Alfred, Managing Editor, St. Mary's College, Calif. 94575 (Note. This address is sufficient, since St. Mary's College is a post office.)

Fibonacci Discovery	$1.50
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ON A CLASS OF NONLINEAR BINOMIAL SUMS

D. A. Lind
University of Virginia, Charlottesville, Virginia

It is known ([2] ; [4] ; [5]) that the Fibonacci numbers may be formed by adding the binomial terms on diagonals of Pascal's Triangle. Recently in this Quarterly V. C. Harris and Carolyn C. Styles [3] generalized the Fibonacci sequence by extending their consideration to sums along straight diagonals of any positive "slope" originating in the first column. As they noted, those sums are special cases of the linear binomial sums investigated by Dickinson [1] . Here we consider a nonlinear generalization of this connection, in which each sum contains a "dogleg" of binomial terms. We then note that these sums obey the same difference equation as the binomial coefficients. From this equation and a set of auxiliary numbers we derive some arithmetic properties, including connections with the Fibonacci numbers, and develop some general recurrences. Because of this close connection with the binomial coefficients, it is not surprising that most of the properties given here stem from corresponding properties of the binomial coefficients.

We define L(n, r), the r-th order nonlinear binomial sum, as the sum of the first r terms of the (n-1)-th row of Pascal's Triangle plus the terms on the rising stairstep diagonal originating at the r-th term. Thus

$$L(n, r) = \sum_{i=0}^{r-1} \binom{n-1}{i} + \sum_{j=1}^{[\frac{n-r}{2}]} \binom{n-j-1}{j+r-1}, \tag{1}$$

where [ ] denotes greatest integer, and the right-most sums is zero if $[\frac{n-r}{2}] < 1$. The sums $L(n, 1) = L(n-1, 2) = F_n$, the n-th Fibonacci number, are those previously considered in [2] , [4] , and [5] . For r = 3 we obtain the following series.

								1
1								2
1	1							4
1	2	1						7, 12
1	3	3	1					20, 33
1	4	6	4	1				54, 88
1	5	10	10	5	1			.
1	6	15	20	15	6	1		.
1	7	21	35	35	21	. . .		.
1	8	28	56	70	. . .			

Thus $L(1, 3) = 1$, $L(2, 3) = 2$, $L(3, 3) = 4$, etc. The 4-th order sequence is 1, 2, 4, 8, 15, 27, 47, 80, 134, $\cdots$.

The connection between the Fibonacci numbers and binomial coefficients previously mentioned may be written as

$$F_n = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-i-1}{i} .$$

The difference between the nonlinear binomial sums and Fibonacci numbers is therefore

$$F_{n+r-1} - L(n, r) = \sum_{i=0}^{r-1} \binom{n+r-2-i}{i} - \sum_{i=0}^{r-1} \binom{n-1}{i} ,$$

which is a polynomial in n of degree $r-3$ for $r \geq 3$. By evaluating the right side of this equation for small values of r, we find, in addition to $L(n, 1) = L(n-1, 2) = F_n$, that

(2a) $$L(n, 3) = F_{n+2} - 1 ,$$

(2b) $$L(n, 4) = F_{n+3} - n - 1 .$$

Also, since

$$\sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1} ,$$

we see from the definition that

$$L(n, r) = 2^{n-1} \quad (n \leq r) .$$

Let the difference operator $\underset{n}{\Delta}$ be defined by $\underset{n}{\Delta} f(n) = f(n+1) - f(n)$. Then the recurrence relation for the binomial coefficients may be represented as

$$\underset{n}{\Delta} \binom{n}{r} = \binom{n}{r-1} . \tag{3}$$

From this and the explicit representation in (1), the important difference equation follows that

$$\underset{n}{\Delta} L(n, r) = L(n, r-1) . \tag{4}$$

Defining the iterated difference operator $\underset{n}{\Delta}^k$ by $\underset{n}{\Delta}^1 f(n) = \underset{n}{\Delta} f(n)$, $\underset{n}{\Delta}^k f(n) = \underset{n}{\Delta} [\underset{n}{\Delta}^{k-1} f(n)]$ for $k > 1$, it is of interest to note that

$$\underset{n}{\Delta}^{r-2} L(n, r) = L(n, 2) = F_{n+1} ,$$

$$\underset{n}{\Delta}^{r-1} L(n, r) = L(n, 1) = F_n .$$

It is apparent that (4) is indeed the same difference equation satisfied by the binomial coefficients in (3), the only change being in the initial values. Thus by using (4) and the easily determined boundary conditions

$$L(n, 1) = F_n , \quad L(1, r) = 1 ,$$

we may construct a table of $L(n, r)$ in which each term is the sum of the term above it and the term above and to the left.

Since the sequence $L(n, 2) = F_{n+1}$ satisfies the recurrence relation

$$L(n+2, 2) = L(n+1, 2) + L(n, 2) ,$$

we take the anti-difference of this $r-2$ times and obtain the general recurrence relation for r-th order terms as

(5) $$L(n+2, r) = L(n+1, r) + L(n, r) + A(n, r) ,$$

Table of $L(n, r)$

n \ r	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2
3	2	3	4	4	4	4	4	4	4	4
4	3	5	7	8	8	8	8	8	8	8
5	5	8	12	15	16	16	16	16	16	16
6	8	13	20	27	31	32	32	32	32	32
7	13	21	33	47	58	63	64	64	64	64
8	21	34	54	80	105	121	127	128	128	128
9	34	55	88	134	185	226	248	255	256	256
10	55	89	143	222	319	411	474	503	511	512

where the auxiliary numbers $A(n, r)$ obey

$$\underset{n}{\Delta}^{r-2} A(n, r) = 0$$

with the initial conditions

$$A(n, 1) = A(n, 2) = 0 \quad (n \geq 1); \; A(1, r) = 1 \quad (r \geq 3) \quad .$$

These numbers also obey the binomial recurrence

$$\underset{n}{\Delta} A(n, r) = A(n, r-1) ,$$

so that we may easily construct a table of $A(n, r)$ from the initial conditions using the same rule of formation as that for $L(n, r)$. It appears from this table that while $L(n, 1)$ and $L(n, 2)$ are sequence of the Fibonacci type, the next two obey the slightly more complicated recurrences

$$L(n+2, 3) = L(n+1, 3) + L(n, 3) + 1 ,$$

$$L(n+2, 4) = L(n+1, 4) + L(n, 4) + n .$$

These are readily proved by using equations (2a) and (2b).

Table of A(n, r)

n \ r	1	2	3	4	5	6	7	8	9	10
1	0	0	1	1	1	1	1	1	1	1
2	0	0	1	2	2	2	2	2	2	2
3	0	0	1	3	4	4	4	4	4	4
4	0	0	1	4	7	8	8	8	8	8
5	0	0	1	5	11	15	16	16	16	16
6	0	0	1	6	16	26	31	32	32	32
7	0	0	1	7	22	42	57	63	64	64
8	0	0	1	8	29	64	99	120	127	128
9	0	0	1	9	37	93	163	219	247	255
10	0	0	1	10	46	140	256	382	466	502

We may establish from (4) and (5) that the recurrence formula with respect to r is

$$L(n, r) = L(n, r-1) + L(n, r-2) - A(n, r) .$$

From this, with (4) again, it follows that

$$A(n, r) = L(n+1, r-1) - L(n, r) .$$

This last equation may be used to establish that

$$L(n, r) + \sum_{i=0}^{r-1} A(n+i, r-1) = F_{n+r-1} .$$

Taking $n = 1$, we see that the slant sums of the $A(n, r)$ are Fibonacci numbers diminished by a unity, i. e.

$$\sum_{i=1}^{r} A(i, r-i+1) = F_r - 1 .$$

It is also interesting to note that the $A(n, r)$ obey the curious diagonal recurrence

$$A(n+1, r+1) = 2\,A(n, r) + \binom{n-1}{r-2} .$$

The recurrence (5) may be easily extended by induction to

$$L(n, r) = F_{k+1}\,L(n-k, r) + F_k\,L(n-k-1, r) + \sum_{i=1}^{k} F_i\,A(n-i-1, r) \quad (0 \le k < n),$$

and the analogous extended recurrence with respect to r is

$$L(n, r) = F_{k+1}\,L(n, r-k) + F_k\,L(n, r-k-1) - \sum_{i=1}^{k} F_i\,A(n, r-i+1) \quad (0 \le k < n).$$

We remark that setting $r = 1$ in the former recurrence gives the familiar Fibonacci identity

$$F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1} .$$

We may prove by induction that for $r > 1$

$$\sum_{i=1}^{k} L(i, r) = L(k+1, r+1) - 1 ,$$

$$\sum_{i=1}^{k} A(i, r) = A(k+1, r+1) - 1 ,$$

which together imply

$$\sum_{i=1}^{n} [L(i, r) + A(i, r)] = L(n+2, r) - 2 \quad (r > 1) .$$

Finally, we extend the definition of $L(n, r)$ to negative r from (4) by putting

$$L(n, r) = F_{n+r-1} \quad (r \le 0) .$$

With this extension, the readily proved formula

$$L(n, r) = \sum_{i=0}^{k} \binom{k}{i} L(n-k, r-i)$$

if valid for all k such that $0 \le k < n$.

The author would like to thank Vincent C. Harris for his helpful suggestions.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by Verner E. Hoggatt, Jr.
San Jose State College
San Jose, California

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-70 *Proposed by C. A. Church, Jr., West Virginia University, Morgantown, West, Va.*

For $n = 2m$ show that the total number of k-combinations of the first n natural numbers such that no two elements i and $i + 2$ appear together in the same selection is F^2_{m+2}, and if $n = 2m+1$, the total is $F_{m+2}F_{m+3}$.

H-71 *Proposed by John L. Brown, Jr., Penn State University, State College, Pennsylvania*

Show

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^{k-1} L_k = 5^n$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^{k-1} F_k = 0 \quad .$$

H-72 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let $u_n = F_{nk}$, where F_m is the mth Fibonacci number, and k is any positive integer; and let

$$\begin{bmatrix} m \\ 0 \end{bmatrix} = \begin{bmatrix} m \\ m \end{bmatrix} = 1, \; \begin{bmatrix} m \\ n \end{bmatrix} = \frac{u_m \cdots u_1}{u_n u_{n-1} \cdots u_1 \, u_{m-n} u_{m-n-1} \cdots u_1} \quad ,$$

then show

$$2\begin{bmatrix} m \\ n \end{bmatrix} = L_{nk}\begin{bmatrix} m-1 \\ n \end{bmatrix} + L_{(m-n)k}\begin{bmatrix} m-1 \\ n-1 \end{bmatrix} .$$

H-73 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let $f_o(x) = 0$, $f_1(x) = 1$, and

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x) \qquad n \geq 0$$

and let $b_n(x)$ and $B_n(x)$ be the polynomials in H-69, show

$$f_{2n+2}(x) = x\,B_n(x^2) ,$$

and

$$f_{2n+1}(x) = b_n(x^2) ,$$

thus there is an intimate relationship between the Fibonacci polynomials, $f_n(x)$, and the Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$.

H-74 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Let f(n) denote the number of positive Fibonacci numbers not greater than a specified integer n. Show that for $n > 1$

$$f(n) = \left[K \ln(n\sqrt{5} + \tfrac{1}{2})\right] ,$$

where [x] denotes greatest integer not exceeding x, and **K** is a constant nearly equal to 2.078086943. (See H. W. Gould's Non-Fibonacci Numbers, Oct., 1965, FQJ).

H-75 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Show that the number of sets of distinct integers with one element n, all other elements less than n and not less than k, and such that no two consecutive integers appear in the set is F_{n-k+1}.

H-76 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California*

It is well known that the Fibonacci numbers are sums of the rising diagonals of Pascal's triangle. Find a recurrence relation for the rising diagonals for the Fibonomial triangle:

$$\begin{array}{ccccccccccc}
 & & & & u_1 & & u_2 & & u_3 & & u_4 \quad \cdots \\
 & & & & 1 & & & & & & \\
 & & & 1 & & 1 & & & & & \\
 & & 1 & & 1 & & 1 & & & & \\
 & 1 & & 2 & & 2 & & 1 & & & \\
1 & & 3 & & 6 & & 3 & & 1 & & \text{----LEVEL 4}
\end{array}$$

$$2\begin{bmatrix} m \\ n \end{bmatrix} = L_n \begin{bmatrix} m-1 \\ n \end{bmatrix} + L_{m-n}\begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \qquad \begin{bmatrix} m \\ o \end{bmatrix} = \begin{bmatrix} m \\ m \end{bmatrix} = 1$$

$u_1 = 1$, $u_2 = 1$, $u_3 = 2$, $u_4 = 2$, $u_5 = 4$, $u_6 = 6$ etc. See H-63 April 1965 FQJ p. 116 and H-72 this issue.

H-77 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{2k+2j+1} = 5^n L_{2n+2k+2}$$

for all integers k. Set $k = -(n+1)$ and derive

$$\sum_{j=0}^{n} \binom{2n+1}{n-j} F_{2j+1} = 5^n ,$$

a result* of S. G. Guba Problem #174 Issue #4 July-August 1965 p. 73 of Matematika V Škole.

AN ALTERNATE FORM

H-49 *Proposed by C. R. Wall, Texas Christian University, Ft. Worth, Texas*

Show that, for $n > 0$,

$$2^n F_{n+1} = \sum_{m=0}^{n} \frac{5^{[m/2]}\, n^{(m)}}{m!}$$

*Reported by H. W. Gould.

where $[x]$ denotes the integral part of x, and $x^{(n)} = x(x-1)\ldots(x-n+1)$.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

Solution: We first note $\binom{n}{m} = n^{(m)}/m!$ Horner ("Fibonacci and Pascal," Fibonacci Quarterly, Vol. 2, No. 3, p. 228) has given equivalently

$$2^n F_{n+1} = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \binom{n+1}{2k+1} 5^k,$$

so that

$$2^n F_{n+1} = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \left\{\binom{n}{2k} + \binom{n}{2k+1}\right\} 5^k$$

$$= \sum_{m=0}^{n} 5^{[m/2]} \binom{n}{m},$$

the desired result.

OOPS!

H-26 was finally solved by Douglas Lind and the solution appeared in the last issue.

PROBLEMS AND PAPERS

H-46 *Proposed by F. D. Parker, SUNY at Buffalo, Buffalo, New York*

Prove

$$D_n = |a_{ij}| = (-1)^n K,$$

where $a_{ij} = F^4_{n+1+j-2}$ (i, j = 1, 2, 3, 4, 5) and find the value of K. This problem and its generalizations will be discussed in separate papers by D. Klarner and L. Carlitz to appear later in the Quarterly.

NON-HOMOGENEOUS FIBONACCI

H-48 *Proposed by J. A. H. Hunter, Toronto, Ontario, Canada*

Solve the non-homogeneous difference equation

$$C_{n+2} = C_{n+1} + C_n + m^n ,$$

where C_1 and C_2 are arbitrary and m is a fixed positive integer.

Solution by Raymond E. Whitney, Lock Haven State College Lock Haven, Pennsylvania

Using the standard technique of converting the difference equation to a differential equation with the transform

$$Y(t) = \sum_{o}^{\infty} C_i t^i / i! \qquad (C_o \equiv C_2 - C_1 - 1),$$

we obtain

$$Y''(t) = Y'(t) + Y(t) + e^{mt}$$

Thus

$$Y(t) = A_e \left[(1 + \sqrt{5})/2\right] t + B_e \left[(1 - \sqrt{5})/2\right] t + \left[1/(m^2 - m - 1)\right] e^{mt}.$$

Hence

$$C_n = Y^{(n)}(o)$$

$$= A\left[(1 + \sqrt{5})/2\right]^n + B\left[(1 - \sqrt{5})/2\right]^n + m^n/(m^2 - m - 1),$$

where A, B are determined via boundary conditions $\left[C_o, C_1\right]$.

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CONTINUED FRACTION CONVERGENTS AS A SOURCE OF FIBONACCI AND LUCAS IDENTITIES

Clyde A Bridger and Marjorie Bicknell
Springfield, Illinois
Adrian Wilcox High School, Santa Clara, California

Properties of the convergents of continued fractions can be used to develop a number of Fibonacci and Lucas identities. Since references for continued fractions are so commonly available, only those properties of continued fractions necessary to the development of this paper are presented.

Let $\{a_i, b_i\}$ be a sequence of real numbers where $a_0 = 1$, b_0 may be zero, and all the other a_i and b_i are not zero. Then, the continued fraction is given by

$$X = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots}}}} \tag{1}$$

The convergent to X after i terms is given by

$$\frac{A_i}{B_i} = \frac{b_i A_{i-1} + a_i A_{i-2}}{b_i B_{i-1} + a_i B_{i-2}} \tag{2}$$

for $i = 2, 3, 4, \ldots$, $A_0 = b_0$, $B_0 = 1$, $B_1 = b_1$, and $A_1 = b_0 b_1 + a_1$. (In the special case that $a_i = b_i = 1$ for all i, $A_n = F_{n+2}$ and $B_n = F_{n+1}$, where F_n is the nth Fibonacci number.)

It is known that the difference between two successive convergents is

$$\frac{A_i}{B_i} - \frac{A_{i-1}}{B_{i-1}} = \frac{(-1)^{i-1} a_1 a_2 \ldots a_i}{B_i B_{i-1}} . \tag{3}$$

Next, let $S = u_0 + u_1 + u_2 + \ldots$ be any series with partial sums $S_0 = u_0$, $S_1 = u_0 + u_1$, $\ldots$, $S_i = u_0 + u_1 + u_2 + \ldots + u_i$, and set $S_i = A_i/B_i$ for all i. Since $S_i - S_{i-1} = u_i$, from Equation (3), $u_i = (-1)^{i-1} a_1 a_2 \ldots a_i/B_i B_{i-1}$, yielding $a_i = -B_i u_i/B_{i-2} u_{i-1}$ and $b_i = (u_{i-1} + u_i)B_i/u_{i-1}B_{i-1}$. Substituting these values for a_i and b_i into (1) will give the continued fraction representation of S_i below, but the result is very cumbersome to evaluate. The partial sum S_i can be written in the simple form

$$(4) \qquad S_i = u_0 + \cfrac{u_1}{1 - \cfrac{u_2}{(u_1 + u_2) - \cfrac{u_1 u_3}{(u_2 + u_3) - \cdots \; \cdots - \cfrac{u_{i-2}u_i}{(u_{i-1} + u_i)}}}} \; .$$

The development thus far is found in various standard sources dealing with continued fractions. At last, we have reached the point of departure for the promised Fibonacci and Lucas number representations.

Set $u_i = F_i$, the i-th Fibonacci number defined by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$. Then, since $F_{i+2} = 1 + (F_1 + F_2 + F_3 + \ldots + F_i) = 1 + S_i$,

$$(5) \qquad F_{i+2} = F_2 + \cfrac{F_1}{F_2 - \cfrac{F_2}{F_3 - \cfrac{F_1 F_3}{F_4 - \cdots \; \cdots - \cfrac{F_{i-2}F_i}{(F_{i-1} + F_i)}}}}$$

For example,

$$F_6 = F_2 + \cfrac{F_1}{F_2 - \cfrac{F_2}{F_3 - \cfrac{F_1F_3}{F_4 - \cfrac{F_2F_4}{F_5}}}} = 1 + \cfrac{1}{1 - \cfrac{1}{2 - \cfrac{1\cdot 2}{3 - \cfrac{1\cdot 3}{5}}}} = 8.$$

Similarily, if we set $u_i = L_i$, the i-th Lucas number defined by $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_{n-1} + L_n$, we can write an analogous expression by replacing each F with an L in the above continued fraction representation.

Equation (2) provides

$$b_i = (A_iB_{i-2} - B_iA_{i-2})/(A_{i-1}B_{i-2} - B_{i-1}A_{i-2}) \tag{6}$$

$$= \left(\frac{A_i}{B_i} - \frac{A_{i-2}}{B_{i-2}}\right)\left(\frac{B_i}{B_{i-1}}\right)\Bigg/\left(\frac{A_{i-1}}{B_{i-1}} - \frac{A_{i-2}}{B_{i-2}}\right) .$$

As above, let $u_i = F_i$ so that $S_i = A_i/B_i = F_{i+2} - F_2$, and comparing Equations (1) and (5) observe that $b_i = F_{i+1}$. Then, from (6),

$$F_{i+1} = \left[(F_{i+2} - F_2) - (F_i - F_2)\right]\cdot B_i/B_{i-1}\cdot\left[(F_{i+1} - F_2) - (F_i - F_2)\right]$$

which reduces at once to $B_i = B_{i-1}F_{i-1}$. Then, the equation above can be written as

$$F_{i+1} = (F_{i+2} - F_i)F_{i-1}/(F_{i+1} - F_i)$$

which becomes

$$F_{i+2}F_{i-1} = F_{i+1}^2 - F_i^2$$

or

$$F_{i+1}^2 - F_{i+1}F_i - (F_{i+2} - F_i)F_{i-1} = 0.$$

The second form has solution

$$2F_{i+1} = F_i \pm \sqrt{F_i^2 + 4\,F_{i-1}, \,(F_{i+2} - F_i)} \tag{7}$$

where obviously the radicand must be the square of a positive integer. Taking trial values $i = 5$ and $i = 6$ leads to $11^2 = L_5^2$ and $18^2 = L_6^2$, and suggests

$$F_i^2 + 4\,F_{i-1}\,F_{i+1} = L_i^2, \tag{8}$$

which can be established by mathematical induction. Taking the positive sign in (7) gives

$$2\,F_{i+1} = F_i + L_i \quad \text{or} \quad L_i = F_{i+1} + F_{i-1},$$

a well-known result.

A parallel development can be used for the Lucas numbers leading to

$$L_{i+2}\,L_{i-1} = L_{i+1}^2 - L_i^2,$$

$$L_{i+1}^2 - L_{i+1}L_i - (L_{i+2} - L_i)L_{i-1} = 0$$

with solution

$$2\,L_{i+1} = L_i \pm \sqrt{L_i^2 + 4\,L_{i-1}(L_{i+2} - L_i)}\ . \tag{9}$$

By using the identity $L_i = F_{i+1} + F_{i-1}$, the radicand can be reduced to $25F_i^2$, leading to the parallel of Equation (8),

$$L_i^2 + 4\,L_{i-1}\,L_{i+1} = 25F_i^2. \tag{10}$$

As a side benefit, combining Equations (8) and (10) gives us

$$6\,F_i^2 = L_{i-1}L_{i+1} + F_{i-1}F_{i+1},$$

and substituting $25F_i^2$ for the radicand in Equation (9) yields

$$5F_i = L_{i-1} + L_{i+1}.$$

Returning to Equation (3) and solving for a_i, we have

$$-a_i = (A_iB_{i-1} - B_iA_{i-1})/(A_{i-1}B_{i-2} - B_{i-1}A_{i-2})$$

$$= \left(\frac{A_i}{B_i} - \frac{A_{i-1}}{B_{i-1}}\right)\left(\frac{B_i}{B_{i-2}}\right) \Bigg/ \left(\frac{A_{i-1}}{B_{i-1}} - \frac{A_{i-2}}{B_{i-2}}\right) .$$

Comparing Equations (1) and (5) shows $-a_i = F_iF_{i-2}$, so that

$$(11)\quad F_iF_{i-2} = \left[(F_{i+2} - F_2) - (F_{i+1} - F_2)\right]B_i/B_{i-2}\left[(F_{i+1} - F_2) - (F_i - F_2)\right]$$

$$= (F_{i+2} - F_{i+1})(F_{i-1}F_{i-2})/(F_{i+1} - F_i).$$

Simplifying, we have

$$F_i^2 - F_i\,F_{i+1} + (F_{i+2} - F_{i+1})F_{i-1} = 0$$

with solution

$$2\,F_i = F_{i+1} \pm \sqrt{F_{i+1}^2 - 4\,F_{i-1}(F_{i+2} - F_{i+1})}$$

$$= F_{i+1} \pm \sqrt{F_{i+1}^2 - 4\,F_{i-1}(F_i)} \quad .$$

Replacing F_{i+1}^2 by $(F_i + F_{i-1})^2$ leads to

$$F_{i+1}^2 - 4\,F_iF_{i-1} = F_{i-2}^2$$

so that the equation above becomes

$$2\,F_i = F_{i+1} + F_{i-2}.$$

The Lucas number equivalents are found by replacing each F by an L from Equation (11) onwards.

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FIBONACCI SUMMATION ECONOMICS PART II*

Albert J. Faulconbridge
Chicago, Illinois

Elliot's observations have alerted us to the possible existence of untouched fields of Fibonacci summation principles relating to economic prediction. Dependence upon coincidence and seemingly unrelated facts invites error. An attempt to induce orderliness into the investigation will be made. When reasoning has gone as far as possible at the moment a working model consistent with the products of the fledgling reasoning will be constructed which model, if it works, provides some evidence that the reasoning might not be sufficiently erroneous to discard. To induce this orderliness parallel topics will be developed to a degree such that they can later be fused to purpose. These topics are:

1. What cycles have been observed?
2. What relationships, if any, do these cycles have to the Fibonacci sequences?
3. What other apparent co-incidences exist that might be related to the problem?

1. WHAT CYCLES HAVE BEEN OBSERVED?

The source for these cycles is relatively incomplete for at the time this article will appear in print there will have been published a complete compendium of cycles identified to date by the Foundation for the Study of Cycles. Partially completed data shows a list of cycles in many phenomena which on superficial examination bears no relation to Fibonacci series but when arranged into subgroups consistent within themselves do in fact show some tendency toward summation relationships. Some larger cycle groups show two or more summation relationships but with different time periods.

Notably there are $17\frac{1}{2}$ week cycles in industrial stocks and electrical potential of trees, 5.9, 12 and 13 month cycles in industrial stocks; a 12 month or 13 lunar month cycle in many phenomena including industrial stocks, some commodity prices, ratio of male to female

*Refer to Part I, Fibonacci Quarterly, December 1964, page 320.

conceptions, sleep characteristics, beef cattle prices, egg laying of domestic fowl, incidence of puerperal sepsis; a 17.8 month cycle in industrial stock prices, a 21 month cycle in rainfall in the Great Lakes region; a two year cycle in industrial stock prices, Great Lakes rainfall, sunspot numbers and Nile River floods; a grouping around 34 lunar months of such cycles as residential construction contracts, copper commodity prices, pig iron prices, automobile factory sales, Canadian Pacific Railway revenue ton-miles, Great Lakes rainfall, rayon production, some individual company sales, motor car and truck sales, department store sales and copper company share prices; a 3 year cycle in factory sales of passenger cars; a 3.2 to 3.4 year grouping in stock prices, bank clearings, copper share prices, general business conditions, pig iron production, factory sales of cars, atmospheric electricity, business failures, cocoa bean prices, and the solar constant; there is a 4.2 to 4.4 year or roughly a 55 lunar month cycle in company sales, temperature, industrial common stock, railroad stock prices, advertising effectiveness, European wheat prices, pig iron prices and Great Lakes rainfall.

There is an impressive group of cycles clustered within the 5.90 to 5.96 year period including the one fifth sidereal period of Saturn, liabilities of business failures, railroad stock prices, sunspots with alternate cycles reversed, sunspots, the combined index of stock prices, copper, cotton and pig iron prices, coal stocks, tree ring size, wheat prices and barometric pressure.

There is an 8 year cycle in cotton prices, cigarette production, lynx abundance, pig iron prices, rail stock prices, crop yields, bird abundance, industrial sales, rainfall, wholesale price index, steel ingot production, sunspots with alternate cycles reversed, wheat prices, an 8.8 to 9.6 year cycle in sunspots, widths of pre-glacial tree rings, pig iron prices, numbers of cattle raised, wholesale commodity prices, various stock price categories, grasshopper abundance, auto production, British Consol prices, business activity, copper prices, industrial, railroad and combined stock prices, liabilities of commercial and financial failures, manufacturing production, new members of Protestant churches, pig iron prices, manufacturing sales, current tree ring widths, wool prices, business failures, patents issued, cotton prices, abundance of marten, rabbits, lynx, foxes, ticks, wolves, acreage planted to wheat, Atlantic salmon and other fish abundance, human heart disease incidence, India rainfall, lunar cycle, ozone at London and Paris and tent catepillars; an 11.4 to 11.8 year cycle in rainfall, twin and genius births, infectious disease incidence, sex ratio of male to female births, slenderness of newborn, sunspots, and in the number of international battles.

There is an immense group of cycles whose periods lie between 17.0 and 18.3 years and some of whose exact length has been worked out accurate to two decimal places. This group includes the Smithsonian solar constant, rainfall figures, cattle prices, mean temperatures, building construction, real estate activity, population, common stock prices, wheat prices, sunspot numbers reversed, Nile floods, earthquakes, pig iron prices, cycle in the variable star Scorpius V, in war incidence, Arizona tree ring, business failures, cotton prices, international battles, and in civil war, sunspots with alternate cycles reversed, advances and recessions of glaciers, immigration, Java tree rings, sales of a public utility company, common stock lows, Canadian Pacific Railroad freight traffic, furniture production, loans and discounts, lumber production, financial panics, pig iron production, marriages, sunspots with alternate cycles reversed, and wheat acreage inverted.

There is a 34 to 36 year grouping which includes cycles in European harvests, U.S.A. Immigration, plant and tree growth in Europe, European tree ring thickness, lynx abundance, earthquakes in China, European weather, frequency of the Aurora Borealis, European barometric pressure, manufacturing production of the U.S.A., prices of British Consols, and European wheat prices.

A 42 year cycle exists in agriculture and related phenomena including tree ring widths, cotton prices, wheat prices, and sunspots.

A 55 year cycle exists in industrial and related phenomena including German coal production, various English, French and U.S. industrial statistics, worldwide pig iron and coal production and railroad stock prices. There is also some reflection in European wheat prices and tree ring widths. Finally there is a 67 and 144 year cycle in international battles.

When individual phenomena are investigated a number of cycles have already been identified. Railroad stock prices show cycles of 4.4, 5.9, 5.92, 6.4, 7.95, 8.39, and three cycles of 9.18, 9.20, and 9.30 years, $18\frac{1}{3}$ years and 55 years. Pig iron prices show 2.7, 4.4, 5.91, 6.3 to 6.5, 8, 8.9, 9.0, to 9.3, 9.2, 17.69, 17.75 years. Copper commodity prices show 2.7, 5.91, 9.0 to 9.3 year cycles. Coal production and coal stocks show 5.91, 8.0, 17.75 and 55 year cycles. Sunspot numbers with alternate cycles reversed show 5.9, 5.91, 8, 17, 17.3, 17.66, 17.75, 18.33 year cycles. Factory sales of cars show 2, 2.72, 2.75, 3, 3.4 year cycles and 6.3 to 6.5 year cycles. Cotton prices show 2, 5.91, 6.3 to 6.5, 6.9, 7.44(89 mos.), 7.88, 7.91, 7.95, 8.42, 9.47, 9.65, 11.3, 12.7 to 12.9, 14.27, 17.25, 17.75 and others peaking around 21, 42 and 89 years. Industrial stocks show cycle groups averaging 21, 55, and 89 weeks, 3, and 3.4 years, 42 and 55 months, 5.91, 5.90, 9.0, 9.2, 9.3 years, 17.2, 17.3, 17.7 years and three cycles of 18.33 years in length. Copper share prices show roughly a 34 and 42 month cycle. Sunspot numbers show a 2.0, 5.91, 8.76, 8.8, 8.94, 9.0 to 9.3, 11, 11.5, 18.2, 22, 22.75 and 42 years. Wheat prices show a 55 lunar month cycle, 5.96 year, 8, 9.0 to 9.3, 17.3, 34 to 36, 42, 42.5, 54, 55 year cycles. Tree rings show 5.91, 5.93, 6.3 to 6.5, 9.0 to 9.3, 17.75, 18.2, 35, 42 and roughly 55 year cycles. Great Lakes rainfall shows a two year group, and also a 21, 34, 42 month average group, and a 55, 89 and 144 month cycles. Earthquakes show 17.5 and 35 year cycles.

2. WHAT RELATIONSHIP IF ANY DO THESE CYCLES HAVE TO FIBONACCI SEQUENCES?

At first glance, none. On closer examination the 1, 2, 3, 8, 21, 34, 55, 89 and 144 unit length cycles speak for themselves, but the apparent anomales of 4.5, 5.91, 9, $17\frac{1}{2}$ to 18 and 42 unit length cycles must be explained. No shortage of reasonable explanations exists yet to be sure of the right one implies more understanding of the underlying principles then we have at the moment.

Should these cycles interact with one another they might summate at their mean. The 5.90 to 5.96 year cycle is roughly 72 months which is the mean between the 55 and 89 month cycles. The 4.5 year cycle is both half of the 9 year cycle and nearly equal to a 55 month cycle. The 5.91 year cycle is roughly both 1.618 times the 42 month cycle and 0.618 times the 9.3 year cycle. Interestingly, if the cycles are plotted from the same starting point on a graph in the form of sinusoidal waves it is noted that the 13 and 21 unit and the 34 and 55 unit cycles will summate to zero every $17\frac{1}{2}$ and 42 units respectively. A 21 and a 34 unit cycle never do summate to zero, whereas a 21 and a 55 unit cycle will maximize at 5.9 and summate to zero at $17\frac{1}{2}$ units. In addition, the 17+ unit cycle might be half of a 34+ unit cycle. The explanation for the 9+ year group is not so simple. It might be half of an 18+ unit cycle, such as that of the solar constant or it might be double the 55 month or $4\frac{1}{2}$ year cycle, or as mentioned above 1.6 times the 5.91 year cycle. Investigation of these anomales can become complex but seem to retain internal consistency. For example there have been identified 6, 12 and 13 month cycles in stock prices. If these were set in motion to summate we would have

A) 6+13=19 19+13=32 32+19=51 51+32=83 months

B) 12+12=24 24+12=36 36+24=60 months.

Now 83 and 60 months averaged together amount to $71\frac{1}{2}$ months, or 5.91 years.

3. WHAT OTHER APPARENT COINCIDENCES EXIST THAT MIGHT BE RELATED TO THE PROBLEM?

Wesley Mitchell in his book Business Cycles: The Problem and its Setting came to the conclusion after compiling an exhaustive correlation of business cycles with every conceivable proposed cause that the only phenomenon with which there is any reasonable correlation is that of sunspots. The similarity between sunspot cycle length and the length of a number of economic cycles has at the least called our attention to the possibility of such a relationship being in some way involved with celestial mechanics. If event groupings which are cycles were bunched together by energy due to planet polarity interaction it would not pay to reject the possibility out of hand before the following was considered. Angular momentum of a polarized planet rotating within a solar magnetic field could like a generator produce predictable amounts of energy to affect any number of factors including sunspots and weather which correlate well with cycles observed. Concerning unit energy production by rotational angular momentum through the celestial field it is to be noted that the angular momentum involved in the diurnal earth rotation is

$$5.91 \times 10^{40} \ \frac{\text{gm. cm.}^2}{\text{sec.}} ,$$

the angular momentum of the earth moon system rotation is

$$34.4 \times 10^{40} \ \frac{\text{gm. cm.}^2}{\text{sec.}}$$

and that of the earth's orbital motion around the sun is

$$42.31 \times 10^{45} \ \frac{\text{gm. cm.}^2}{\text{sec.}} .$$

There may be exposed now the starting point for an orderly investigation of a new system of quantitative economic prediction. The purpose in presenting the information herein is to enlist the aid of investigators trained in a different discipline than the author's, and it is felt that the Fibonacci Quarterly Journal is an ideal means of communicating with them.

On that account enough material has been presented initially to provide some guidelines concerning where solutions may lie while for the moment restricting description of the author's approaches which might prejudice an independent and more systematic start by others.

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GAUSSIAN FIBONACCI AND LUCAS NUMBERS

J. H. Jordan
Washington State University
Pullman, Washington

Recently A. F. Horadam [2] introduced the concept of the complex Fibonacci numbers and established some quite general identities concerning them. It is the purpose of this paper to consider merely two of the Complex Fibonacci sequences and extend some relationships which are known about the common Fibonacci sequences to the Complex Fibonaccies.

Def. 1: The Gaussian Fibonacci sequence is $GF_0 = i$; $GF_1 = 1$; $GF_n = GF_{n-1} + GF_{n-2}$ for $n > 1$. It is easy to see that $GF_n = F_n + F_{n-1}\, i$.

Def. 2: The Gaussian Lucas sequence is $GL_0 = 2-i$; $GL_1 = 1+2i$; $GL_2 = 3+i$ $GL_n = GL_{n-1} + GL_{n-2}$ for $n > 2$. It is easy to see that $GL_n = L_n + L_{n-1}i$.

Analogous to the usual identities stated by S. L. Basin and V. E. Hoggatt, Jr. [1], the following identities are easily attainable.

For $n \geq 2$

(1) $$\sum_{j=o}^{n} GF_j = GF_{n+2^{-1}}$$

(2) $$\sum_{j=o}^{n} GL_j = GL_{n+2} - (1 + 2i)$$

(3) $$GF_{n+1}\, GF_{n-1} - GF_n^{\,2} = (-1)^n(2-i)$$

(4) $$GL_{n+1} GL_{n-1} - GL_n^{\,2} = (-1)^{n+1}\, 5(2-i)$$

(5) $$GL_n = GF_{n+1} + GF_{n-1}$$

(6) $$GF_{n+1}^{\,2} + GF_n^{\,2} = F_{2n}(1 + 2i)$$

(7) $$GF_{n+1}^{\,2} - GF_{n-1}^{\,2} = F_{2n-1}(1 + 2i)$$

(8) $$GF_n GL_n = F_{2n-1}(1 + 2i)$$

(9) $$GF_{n+1} GF_{p+1} + GF_n\, GF_p = F_{n+p}(1 + 2i)$$

(10) $$\sum_{j=1}^{n} GF_j^{\,2} = F_n^{\,2}(1 + 2i) + (-1)^n i - i$$

(11) $$GL_n^{\,2} - 5\, GF_n^{\,2} = (-1)^n 4(2-i)$$

(12) $$GF_{-n} = iGF_n = i(F_n - F_{n-1} i)$$

Corollary to (11): GL_n is composite for $n \geq 2$.

The occurence of $1 + 2i$, $2 + i$, $(1-2i)$, and $(2-i)$ seems poetic in these formulae in view of the fact they are factors of 5. Some of the usual results mentioned in Vorob'ev [5] can be extended yielding

$$\sum_{j=1}^{n} GF_{2j-1} = GF_{2n} - i$$

$$\sum_{j=1}^{n} GF_{2j} = GF_{2n+1} - 1$$

$$\sum_{j=1}^{2n} (-1)^j\, GF_j = GF_{2n-1} - 1 + i$$

$$\sum_{j=1}^{n} (-1)^j\, GF_j = (-1)^{j+1} \quad GF_n - 1 + i$$

The norm of the Gaussian Fibonacci is $N(GF_n) = F_n^2 + F_{n-1}^2 = F_{2n-1}$,

A well known theorem mentioned in Hardy and Wright [3] is

Theorem A: For $n \geqq 2$, $F_n \mid F_m$ if and only if $n \mid m$

And a theorem mentioned recently by G. Michael [4] is

Theorem B: $(F_n, F_m) = F_{(n,m)}$.

The corresponding result for Theorem A with Gaussian Fibonacci numbers is

Theorem 1: For $n > 2$, $GF_n \mid FG_m$ if and only if $2n-1 \mid 2m-1$, divisibility in the sense of Gaussian Integers.

We start with the following preliminary.

Lemma: If $2n-1 \mid 2m-1$ then $2n-1 \mid m+n-1$.

Proof: It follows that if $2n-1 \mid 2m-1$ then $2n-1 \mid 2m-1-(2n-1) = 2m-2n$. Now $(2, 2n-1) = 1$ since $2n-1$ is odd therefore $2n-1 \mid m-n$. It now follows that $2n-1 \mid (2m-1)-(m-n) = m+n-1$.

Proof of the Theorem 1: A necessary condition for $GF_n \mid GF_m$ is that $N(GF_n) \mid N(GF_m)$. But this happens only when $F_{2n-1} \mid F_{2m-1}$ or by Theorem A only when $2n-1 \mid 2m-1$. Therefore one concludes that a necessary condition for $GF_n \mid GF_m$ is that $2n-1 \mid 2m-1$.

On the other hand if $2n-1 \mid 2m-1$ then $N(GF_n) = F_{2n-1} \mid F_{2m-1} = N(GF_m)$. This means that $N(GF_m/GF_n)$ is a positive integer. Now

$$\frac{GF_m}{GF_n} = \frac{F_m + F_{m-1}\, i}{F_n + F_{n-1}\, i}$$

$$= \frac{F_m F_n + F_{m-1} F_{n-1} + (F_{m-1} F_n - F_{n-1} F_m)\, i}{F_n^2 + F_{n-1}^2}$$

$$= \frac{F_m F_n + F_{m-1} F_{n-1}}{F_{2n-1}} + \frac{F_{m-1} F_n - F_{n-1} F_m\, i}{F_{2n-1}}$$

$$= \frac{F_{m+n-1}}{F_{2n-1}} + \frac{F_{m-1} F_n - F_{n-1} F_m\, i}{F_{2n-1}}$$

But by the lemma and Theorem A it follows that $F_{2n-1} \mid F_{m+n-1}$.

Hence F_{m+n-1} / F_{2n-1} is an integer a. It follows that

$$\frac{F_{m-1} F_n - F_{n-1} F_m}{F_{2n-1}}$$

must also be an integer, b, since the norm is an integer. Therefore $GF_m/GF_n = a+b\,i$. Q. E. D.

The following interesting by-product has been established.

Corollary: For $n \geq 2$, $F_{2n-1} \mid F_{m-1} F_n - F_{n-1} F_m$ if and only if $2n-1 \mid 2m-1$.

Def. 3: If z and w are Gaussian Integers and the greatest common divisor of z and w is that Gaussian Integer y such that $y \mid z$ and $y \mid w$ and if $t \mid z$ and $t \mid w$ then $N(t) \leq N(y)$. Notationwise $(z, w) = y$.

The analogy to Theorem B is as follows:

Theorem 2: $(GF_m, GF_n) = GF_k$ where $2k-1 = (2m-1, 2n-1)$.

Proof: Since $2k-1$ divides $2m-1$ and also $2n-1$ it follows from Theorem 1 that $GF_k \mid GF_m$ and $GF_k \mid GF_n$: If $H \mid GF_m$ and $H \mid GF_n$ then $N(H) \mid N(GF_m) = F_{2m-1}$ and $N(H) \mid N(GF_n) = G_{2n-1}$. Now by Theorem B $(F_{2m-1}, F_{2n-1}) = F_{(2m-1,\ 2n-1)} = F_{2k-1}$. Now $N(H) \mid F_{2k-1} = N(GF_k)$ hence $N(H) \leq N(GF_k)$. Q. E. D.

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EXPLORING GENERALIZED FIBONACCI-LUCAS RELATIONS

Brother U. Alfred
St. Mary's College, California

A generalized Fibonacci sequence with positive terms can be formed by taking any two positive integers and then applying the law of formation of Fibonacci sequences which states that each term is the sum of the two preceding terms. As a further refinement, one might number the terms of the sequence according to the scheme set up in [1]. In this arrangement, if f_i is the term of a generalized Fibonacci sequence, then f_1 if characterized by the fact that $f_1 < f_2/2$. (Note. This manner of notation does NOT apply to the Fibonacci sequence: 1, 1, 2, 3, 5, 8. . . . as usually numbered.) Then the characteristic number of the sequence which we have denoted D (see ref. 1) is given by:

$$D = f_1^2 - f_0 f_2$$

We now associated with this generalized Fibonacci sequence a Lucas sequence whose terms g_n are defined by:

$$g_n = f_{n-1} + f_{n+1}$$

It can be shown that this is also a Fibonacci sequence and that the characteristic number of the sequence is numerically equal to 5D.

A. F. Horadam has worked out and reported a large number of relations that apply to generalized Fibonacci sequences [2]. The present exploration is concerned with relations involving both f and g. A few samples are:

$$g_{2n}^2 - g_0^2 = 5\,(f_{2n}^2 - f_0^2)$$

$$g_{n+1}\, g_{p+1} + g_n\, g_p = 5(f_{n+1}\, f_{p+1} + f_n\, f_p)$$

$$f_{2n+1} = F_n\, g_{n+1} + (-1)^n f_1$$

where F_n is a member of the Fibonacci sequence properly so-called.

We would urge readers to report any and all relations of the above type that they may find, whether their work is extensive and formal or whether it is in the nature of a particular note. Proofs of results are also in order, but their absence should not prevent reporting a known relation.

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(2) A. F. Horadam, A Generalized Fibonacci Sequence, Amer. Math. Monthly, May 1961, pp. 455-459.

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A NOTE ON FIBONACCI SUBSEQUENCES

John H. Halton
Brookhaven National Laboratory
Upton, New York

The question has been raised, whether certain subsequences of the Fibonacci sequence

$$F_0 = 0, \; F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}, \tag{1}$$

can themselves be obtained directly from a recurrence-relation.

First, consider a <u>periodic subsequence</u>, $P_n = F_{nq+r}$, of every q-th Fibonacci number, starting with F_r. It is known (see, e.g., D. Ruggles, Fibonacci Quarterly 1(1963)2:77) that

$$F_{p+q} = L_q F_p + (-1)^{q-1} F_{p-q}, \tag{2}$$

Putting $p = nq + r$ and substituting the appropriate P_n, we obtain the hoped-for relation,

$$P_0 = F_r, \; P_1 = F_{q+r}, \quad P_{n+1} = L_q P_n + (-1)^{q-1} P_{n-1}. \tag{3}$$

On the other hand, we may wish to consider the <u>complementary sequence</u> of those F_i which are not of the form P_n. If these are written Q_k, it is easy to see that, after an initial $(r - 1)$ terms, this sequence comes in cycles of $(q - 1)$ consecutive F_i, and that

$$Q_1 = F_1, \; Q_2 = F_2, \; \ldots, \; Q_{r-1} = F_{r-1}; \; Q_r = F_{r+1}, \; \ldots,$$

$$Q_{n(q-1)+r} = F_{nq+r+1}, \; \ldots, \; Q_{n(q-1)+r+q-2} = F_{nq+r+q-1}, \; \ldots$$

Thus, $Q_{k+1} = Q_k + Q_{k-1}$, except when a P_n intervenes. If $q = 2$, we have the special situation, that there is a P_n between each adjacent pair of Q_k, and the complementary sequence is itself periodic and satisfies the relation (3):

$$Q_{k+1} = L_2 Q_k - Q_{k-1} = 3Q_k - Q_{k-1}. \tag{4}$$

if $q \geq 3$, at most one P_n can intervene between Q_{k-1} and Q_{k+1}. This occurs if $k = n(q-1)+r-1$, so that the remainder R_k when $(k-r+1)$

is divided by $(q - 1)$ is 0, when $Q_{k+1} = F_{nq+r} + Q_k = 2Q_k + Q_{k-1}$; and if $k = n(q-1)+r$, so that $R_k = 1$, when $Q_{k+1} = F_{nq+r} + Q_k = 2Q_k + Q_{k-1}$, and if $k = n(q-1)+r$, so that $R_k = 1$, when $Q_{k+1} = Q_k + F_{nq+r} = 2Qk - Q_{k-1}$.

If $q = 3$, R_k can only be 0 or 1, and we get the rather simple relation

$$Q_{k+1} = 2Q_k + (-1)^{R_k} Q_{k-1} = 2Q_k + (-1)^{k-r+1} Q_{k-1} \; ; \tag{5}$$

but if $q \geq 4$, the neatest formula I could find was to define

$$S_k = \max(2 + R_k - R_k^2,\ 1), \quad T_k = \min(R_k,\ 2),$$

when

$$Q_{k+1} = S_k Q_k + (-1)^{T_k} Q_{k-1} \, . \tag{6}$$

Alternatively, in terms of Kronecker's δ,

$$Q_{k+1} = \left\{1 + \delta_{OR_k} + \delta_{1R_k}\right\} Q_k + \left\{1 - 2\delta_{1R_k}\right\} Q_{k-1} \; . \tag{7}$$

An investigation of subsequences of the forms $X_n = F_{n^2}$ and $X_n = F_{2^n}$, for example, strongly suggests that only periodic sequences of the form P_n yield linear recurrence-relations with constant coefficients.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A. P. Hillman
University of New Mexico, Albuquerque, New Mexico

Send all communications concerning Elementary Problems and Soltuions to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within two months of publication.

B-76 *(Originally P-1 of this Quarterly, Vol. 1, No. 2, p. 74)*
Proposed by James A. Jeske, San Jose State College, San Jose, California.

The recurrence relation for the sequence of Lucas numbers is $L_{n+2} - L_{n+1} - L_n = 0$ with $L_1 = 1$, $L_2 = 3$.
Find the transformed equation, the exponential generating function, and the general solution.

B-77 *(Originally P-2 of this Quarterly, Vol. 1, No. 2, p. 74)*
Proposed by James A. Jeske, San Jose State College, San Jose, California.

Find the general solution and the exponential generating function for the recurrence relation

$$y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4y_n = 0,$$

with $y_0 = 0$, $y_1 = 0$, and $y_2 = -1$.

B-78 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Show that

$$F_n = L_{n-2} + L_{n-6} + \cdots + L_{n-2-4m} + e_n, \quad n > 2,$$

where m is the greatest integer in $(n - 3)/4$, and $e_n = 0$ if $n \equiv 0 \pmod 4$, $e_n = 1$ if $n \not\equiv 0 \pmod 4$.

B-79 *Proposed by Brother U. Alfred, St. Mary's College, St. Mary's College, California*

Let $a = (1 + \sqrt{5})/2$. Determine a closed expression for

$$X_n = [a] + [a^2] + \ldots + [a^n]$$

where the square brackets mean "greatest integer in."

B-80 *Proposed by Maxey Brooke, Sweeny, Texas*

Solve the division alphametic

$$\begin{array}{r} \text{PISA} \\ \text{FIB}\,\overline{)\,\text{XONACCI}} \end{array}$$

where each letter represents one of the nine digits $1, 2, \ldots, 9$ and two letters may represent the same digit.

B-81 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Prove that only one of the Fibonacci numbers $1, 2, 3, 5, \ldots$ is a prime in the ring of Gaussian integers.

SOLUTIONS

A LUCAS NUMBERS IDENTITY

B-64 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $L_n L_{n+1} = L_{2n+1} + (-1)^n$, where L_n is the n-th Lucas number defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$.

Solution by John Allen Fuchs, University of Santa Clara, Santa Clara, California

By the Binet formula

$$L_n = a^n + b^n$$

where $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$ and $ab = -1$. Then

$$L_n L_{n+1} = (a^n + b^n)(a^{n+1} + b^{n+1}) = a^{2n+1} + a^n b^{n+1} + a^{n+1} b^n + b^{2n+1}$$

$$= a^{2n+1} + b^{2n+1} + (ab)^n (a + b) = L_{2n+1} + (-1)^n .$$

Also solved by John E. Homer, Jr.; Douglas Lind; Benjamin Sharpe; M. N. Srikanta Swamy; John Wessner; and the Proposer

OPERATORS

B-65 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let u_n and v_n be sequences satisfying $u_{n+2}+au_{n+1}+bu_n = 0$ and $v_{n+2}+cv_{n+1}+dv_n = 0$ where a, b, c, and d are constants and let $(E^2+aE+b)(E^2+cE+d) = E^4+pE^3+qE^2+rE+s$. Show that $y_n = u_n + v_n$ satisfies

$$y_{n+4}+py_{n+3}+qy_{n+2}+ry_{n+1}+sy_n = 0 .$$

Solution by David Zeitlin, Minneapolis, Minnesota

Let $P(E) = E^2 + aE + b$ and $Q(E) = E^2 + cE + d$, where $P(E)u_n = 0$, $Q(E)v_n = 0$, $P(E)0 = 0$, and $Q(E)0 = 0$. Since $P(E)Q(E) \equiv Q(E)P(E)$ we have

$$P(E)Q(E)(u_n + v_n) = Q(E)\,[P(E)u_n] + P(E)0 = Q(E)0 = 0,$$

which is the desired result.

Also solved by Douglas Lind; M.N.S. Swamy; and the proposer

B-66 *Proposed by D. G. Mead, University of Santa Clara, Santa Clara, California*

Find constants p, q, r, and s such that

$$y_{n+4}+py_{n+3}+qy_{n+2}+ry_{n+1}+sy_n = 0$$

is a 4th order recursion relation for the term-by-term products $y_n = u_n v_n$ of solutions of $u_{n+2}-u_{n+1}-u_n = 0$ and $v_{n+2}-2v_{n+1}-v_n = 0$.

Solution by Jeremy C. Pond, Sussex, England

$u_n = Aa^n + Bb^n$ where a, b are the roots of $x^2 - x - 1 = 0$ and $v_n = Cc^n + Dd^n$ where c, d are the roots of $x^2 - 2x - 1 = 0$. Thus $y_n = AC(ac)^n + AD(ad)^n + BC(bc)^n + BD(bd)^n$, and so ac, ad, bc, bd are the solutions of

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

i.e.,

$$p = -(a + b)(c + d) = -2$$

$$q = b^2cd + abd^2 + 2abcd + abc^2 + a^2cd$$

$$= (a + b)^2cd + (c + d)^2ab - 2abcd = -1 -4 -2 = -7$$

$$r = -abcd(bd + bc + ad + ac) = -abcd(a + b)(c + d) = -2$$

$$s = (abcd)^2 = 1.$$

Summarizing: $p = -2$; $q = -7$; $r = -2$; $s = 1$.

Also solved by Douglas Lind; M.N.S. Swamy, David Zeitlin; and the proposer

B-67 *Proposed by D. G. Mead, University of Santa Clara, Santa Clara, California*

Find the sum $1\cdot 1+1\cdot 2+2\cdot 5+3\cdot 12+\ldots+F_nG_n$, where $F_{n+2}=F_{n+1}+F_n$ and $G_{n+2}=2G_{n+1}+G_n$.

Solution by M.N.S. Swamy, University of Saskatchewan, Regina, Canada

Using the result of Problem B-66, we have the recurrence relation,

$$y_{n+4} - 2y_{n+3} - 7y_{n+2} - 2y_{n+1} + y_n = 0 \qquad (1)$$

where, $y_n = F_nG_n$.

Substituting successively $1, 2, \ldots, n$ for n in (1) and adding we get

$$(y_n + y_2 + \ldots + y_n) - 2y_2 - 9y_3 - 11y_4 - 10(y_5 + \ldots + y_{n+1})$$

$$-8y_{n+2} - y_{n+3} + y_{n+4} = 0$$

or

$$9 \sum_1^n y_r = (10y_1 + 8y_2 + y_3 - y_4) - 10y_{n+1} - 8y_{n+2} - y_{n+3} + y_{n+4} \quad .$$

Now, $10y_1 + 8y_2 + y_3 - y_4 = 10 + 8\cdot 1\cdot 2 + 2\cdot 5 - 3\cdot 12 = 0.$

Hence,

$$9 \sum_1^n y_r = -10y_{n+1} - 8y_{n+2} - y_{n+3} + y_{n+4} \quad .$$

Substituting for y_{n+4} from (1), the above equation reduces to

$$9 \sum_1^n y_r = y_{n+3} - y_{n+2} - 8y_{n+1} - yn.$$

Again using (1), this may to reduced to

$$9 \sum_1^n y_r = y_{n+2} - y_{n+1} + y_n - y_{n-1}.$$

Therefore we have

$$1\cdot 1 + 1\cdot 2 + 2\cdot 5 + 3\cdot 12 + \ldots + F_n \cdot G_n$$

$$= (F_{n+2}G_{n+2} - F_{n+1}G_{n+1} + F_nG_n - F_{n-1}G_{n-1})/9.$$

Also solved by Douglas Lind, Jeremy C. Pond, David Zeitlin, and the proposer. Pond and Zeitlin simplified the sum to the form $(F_{n+1}G_n + F_nG_{n+1})/3$.

FIBONACCI DIMENSIONS FOR PARALLELEPIPEDS

B-68 *Proposed by Walter W. Horner, Pittsburgh, Pennsylvania*

Find expressions in terms of Fibonacci numbers which will generate integers for the dimensions and diagonal of a rectangular parallelepiped, i.e., solutions of

$$a^2+b^2+c^2 = d^2 \quad .$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

Let F_r and F_s be any two Fibonacci numbers of opposite parity. Then

$$F_r^2 + F_s^2 = 2k + 1 = (k + 1)^2 - k^2.$$

Since $k = \frac{1}{2}(F_r^2 + F_s^2 - 1)$, an expression of the desired type is

$$F_r^2 + F_s^2 + \left(\frac{F_r^2 + F_s^2 - 1}{2}\right)^2 = \left(\frac{F_r^2 + F_s^2 + 1}{2}\right)^2 .$$

Also solved by the proposer

SIMULTANEOUS EQUATIONS

B-69 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Solve the system of simultaneous equations:

$$xF_{n+1} + yF_n = x^2 + y^2$$

$$xF_{n+2} + yF_{n+1} = x^2 + 2xy$$

where F_n is the n-th Fibonacci number.

Solution by Jeremy C. Pond, Sussex, England

It is easy to check two solutions:

(a) $x = 0$ and $y = 0$

(b) $x = F_{n+1}$ and $y = F_n$.

Now from the second equation: $y = x(x - F_{n+2})/(F_{n+1} - 2x)$ unless $F_{n+1} = 2x$. This special case leads us to (a) and (b) with $n = -1$.

Substitute this expression for y in the first equation and multiply by $(F_{n+1} - 2x)^2$. This leads to

$$x(F_{n+1} - x)(F_{n+1} - 2x)^2 = x(x - F_{n+2})(x^2 - xF_{n+2} - F_nF_{n+1} + 2xF_n) .$$

One solution is $x = 0$ and the others satisfy:

$$(x - F_{n+1})(F_{n+1} - 2x)^2 + (x - F_{n+2})(x^2 - xF_{n-1} - F_n F_{n+1}) = 0 \ .$$

This is a cubic with three solutions. It is easy to verify that the sum of these two roots is $2F_{n+1}$ and the product is $(-1)^n F_{n+1}/5$.

We know that one of these solutions is F_{n+1} so the other two have sum F_{n+1} and products $(-1)^n/5$; i.e. they are:

$$(F_{n+1} \pm \sqrt{F_{n+1}^2 + [4(-1)^{n+1}/5]})/2 = \frac{\alpha^{n+1}}{\sqrt{5}}, -\frac{\beta^{n+1}}{\sqrt{5}}$$

Thus the complete solution of the system of equations is

(a) $x = 0$; $y = 0$

(b) $x = F_{n+1}$; $y = F_n$

(c) and (d)

$$x = (F_{n+1} \pm \sqrt{F_{n+1}^2 + [4(-1)^{n+1}/5]})/2 = \frac{\alpha^{n+1}}{\sqrt{5}}, -\frac{\beta^{n+1}}{\sqrt{5}}$$

$$v = \frac{\alpha^n}{\sqrt{5}}, -\frac{\beta^n}{\sqrt{5}}$$

Also solved by M. N. S. Swamy and the proposer

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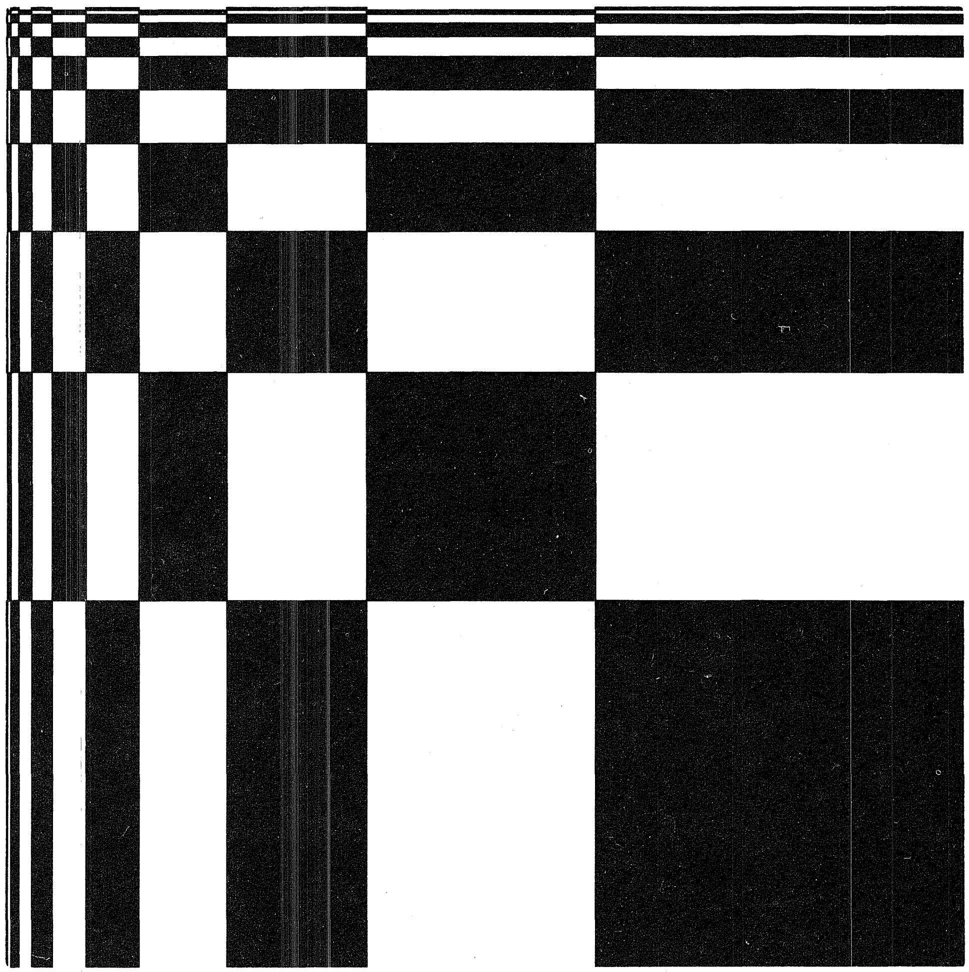

OP ART By Mrs. Betty Naysmith

The Fibonacci sequence used in the "OP ART" above started with one represented by $\frac{1}{32}$ of an inch so that the biggest rectangle is (55/32) x (89/32). The rectangles adjacent to the main diagonal would approach the Golden Rectangle as the Op Art is extended downward and to the right. Mrs. Naysmith is a student of Professor Ruth Ballard of University of Illinois.

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L. A. G. Dresel and D. E. Daykin, Factorization of 36 Fibonacci Numbers F_n With $n > 100$, FQ, Oct., 1965, p. 232

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