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The primary function of THE FIBONACCI QUARTERLY is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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# A MATRIX METHOD TO SOLVE LINEAR RECURRENCES WITH CONSTANT COEFFICIENTS* 

## Bolian Liu

South China Normal University, Guangzhou, P.R. of China
(Submitted April 1989)
In this paper we provide a matrix method to solve linear recurrences with constant coefficients.

Consider the linear recurrence relation with constant coefficients

$$
\left\{\begin{array}{l}
u_{n+k}=\alpha_{1} u_{n+k-1}+\alpha_{2} u_{n+k-2}+\cdots+\alpha_{k} u_{n}+b_{n}  \tag{1}\\
u_{0}=c_{0}, u_{1}=c_{1}, \ldots, u_{k-1}=c_{k-1}
\end{array}\right.
$$

where $\alpha_{i}$ and $c_{i}$ are constants $(i=0,1,2, \ldots, k)$ and where $\left\langle b_{n}\right\rangle_{n \in N}$ is a given sequence.

In order to solve this recurrence relation generally, we first find the general solution $\left\langle\tilde{\mathcal{U}}_{m}\right\rangle_{m \in N}$ of the corresponding homogeneous relation

$$
\left\{\begin{array}{l}
u_{n+k}=\alpha_{1} u_{n+k-1}+\alpha_{2} u_{n+k-2}+\cdots+\alpha_{k} u_{n}  \tag{2}\\
u_{0}=c_{0}, u_{1}=c_{1}, \ldots, u_{k-1}=c_{k-1}
\end{array}\right.
$$

and then find a particular solution $\left\langle u_{m}^{\prime}\right\rangle_{m \in N}$ of (1) satisfying the initial conditions. Then $\left\langle\tilde{u}_{m}+u_{m}^{\prime}\right\rangle_{m \in N}$ is a solution of (1).

The general method (see [1]) for solving recurrence (2) requires, as a first step, solving the corresponding characteristic equation

$$
\begin{equation*}
\lambda^{k}-\alpha_{1} \lambda^{k-1}-\alpha_{2} \lambda^{k-2}-\cdots-\alpha_{k}=0 \tag{3}
\end{equation*}
$$

Generally, when $k \geq 3$, it is rather difficult to find the roots $\lambda_{i}$ of (3).
Now we construct a matrix $A$ such that (3) is the characteristic equation of $A$, and then obtain the general solution of (1) from $A^{m}$.

Let $A$ be the $k \times k$ companion matrix of the polynomial of (3):

$$
A=\left[\begin{array}{llllll}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
\alpha_{k} & \alpha_{k-1} & \alpha_{k-2} & & \alpha_{2} & \alpha_{1}
\end{array}\right]
$$

Then the characteristic equation of $A$ is (3) and, by the Hamilton-Cayley theorem,

$$
\begin{equation*}
A^{k}-\alpha_{1} A^{k-1}-\alpha_{2} A^{k-2}-\cdots-\alpha_{k} I=0 \tag{4}
\end{equation*}
$$

Consider the following $k \times 1$ matrices:

$$
C=\left(c_{0}, c_{1}, \ldots, c_{k-1}\right)^{t}, B_{j}=\left(0,0, \ldots, 0, b_{j}\right)^{t}, j=0,1, \ldots
$$

Let

$$
\begin{equation*}
A^{m} C+A^{m-1} B_{0}+A^{m-2} B_{1}+\cdots+A^{k-1} B_{m-k}=\left(\alpha^{(m)}, \ldots\right)^{t} \tag{5}
\end{equation*}
$$

We will prove that $\left\langle\alpha^{(m)}\right\rangle_{m \in N}$ satisfies (1). By equation (4),

[^0]Hence,

$$
A^{m} C=\sum_{i=1}^{k} \alpha_{i} A^{m-i} C, A^{m-j-1} B_{j}=\sum_{i=1}^{k} \alpha_{i} A^{m-j-1-i} B_{j}, j=0,1,2, \ldots
$$

$$
\begin{align*}
& \left(\alpha^{(n+k)}, \ldots\right)^{t}=A^{n+k} C+A^{n+k-1} B_{0}+A^{n+k-2} B_{1}+\cdots+A^{k} B_{n-1}+A^{k-1} B_{n}  \tag{6}\\
& =\sum_{i=1}^{k} \alpha_{i} A^{n+k-i} C+\sum_{i=1}^{k} \alpha_{i} A^{n+k-1-i_{B}} B_{0}+\cdots+\sum_{i=1}^{k} \alpha_{i} A^{k-i} B_{n-1}+A^{k-1} B_{n} \\
& =\alpha_{1}\left(A^{n+k-1} C+\sum_{i=1}^{n} A^{n+k-1-i_{B}}{ }_{i-1}\right)+\alpha_{2}\left(A^{n+k-2} C+\sum_{i=1}^{n-1} A^{n+k-2-i_{B}} B_{i-1}\right) \\
& \quad+\sum_{i=2}^{k} \alpha_{i} A^{k-i} B_{n-1}+\alpha_{3}\left(A^{n+k-3} C+\sum_{i=1}^{n-2} A^{n+k-3-i} B_{i-1}\right) \\
& \quad+\sum_{i=3}^{k} \alpha_{i} A^{k+1-i_{B}} B_{n-2}+\cdots+\alpha_{k}\left(A^{n} C+\sum_{i=1}^{n-k+1} A^{n-i} B_{i-1}\right) \\
& \quad+\alpha_{k} A^{k-2} B_{n-k+1}+A^{k-1} B_{n} .
\end{align*}
$$

Since

$$
\begin{aligned}
& (i+1) \\
& A^{i}=\left[\begin{array}{ccccccc}
0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
& \vdots & & & & \vdots &
\end{array}\right], i=0,1,2, \ldots, k-1 \text {, } \\
& A^{i} B_{j}=(0, \ldots)^{t} \text {, when } 0 \leq i \leq k-2 \text {, }
\end{aligned}
$$

and

$$
A^{k-1} B_{n}=\left(b_{n}, \ldots\right)^{t}
$$

Then, from (6), we have:

$$
\begin{aligned}
\left(\alpha^{(n+k)}, \ldots\right)^{t}= & \alpha_{1}\left(\alpha^{(n+k-1)}, \ldots\right)^{t} \\
+ & \alpha_{2}\left(\alpha^{(n+k-2)}, \ldots\right)^{t}+(0, \ldots)^{t} \\
+ & \alpha_{3}\left(\alpha^{(n+k-3)}, \ldots\right)^{t}+(0, \ldots)^{t} \\
& \vdots \\
+ & \alpha_{k}\left(\alpha^{(n)}, \ldots\right)^{t}+(0, \ldots)^{t}+\left(b_{n}, \ldots\right)^{t}
\end{aligned}
$$

This is

$$
a^{(n+k)}=\alpha_{1} \alpha^{(n+k-1)}+\alpha_{2} \alpha^{(n+k-2)}+\cdots+\alpha_{k} \alpha^{(n)}+b_{n}
$$

and (1.1) is satisfied.
By (5),

$$
\begin{aligned}
& \left(\alpha^{(0)}, \ldots\right)^{t}=A^{0} C=\left(c_{0}, \ldots\right)^{t} \\
& \left(\alpha^{(1)}, \ldots\right)^{t}=A C=\left(c_{1}, \ldots\right)^{t} \\
& \vdots \\
& \left(\alpha^{(k-1)}, \ldots\right)^{t}=A^{k-1} C=\left(c_{k-1}, \ldots\right)^{t}
\end{aligned}
$$

that is, $a^{(i)}=c_{i}, i=0,1,2, \ldots, k-1$, and (1.2) also holds. Thus,
(7) $\left\langle u_{m}\right\rangle_{m \in N}=\left\langle a^{(m)}\right\rangle_{m \in N}$
is a solution of (1). Now we find a combinatorial expression for $\alpha^{(m)}$. From formula (5),

$$
\begin{align*}
a^{(m)}=c_{0} a_{11}^{(m)}+c_{1} a_{12}^{(m)}+c_{2} a_{13}^{(m)} & +\cdots+c_{k-1} a_{1 k}^{(m)}+b_{0} a_{1 k}^{(m-1)}+b_{1} a_{1 k}^{(m-2)}  \tag{8}\\
& +\cdots+b_{m-k} a_{1 k}^{(k-1)}
\end{align*}
$$

We consider the associated directed graph $D$ of $A$ with weights $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}$ as drawn in Figure 1.


Figure 1
The Associated Digraph $D$ of $A$
(Arcs with no assigned weight have weight 1.)
The definition of $D$ is given as follows. If $A=\left[\alpha_{i j}\right]$, then $D$ is the digraph in which there is an arc ( $i, j$ ) with weight $\alpha_{i j}$ from $i$ to $j$ if and only if $\alpha_{i j}$ $\neq 0(i, j=1, \ldots, n)$. The weight of a walk in $D$ is defined to be the product of the weights of all of the arcs on the walk. $A_{i j}^{(m)}$ is the sum of weights of all walks with length $m$ from $i$ to $j$ (see [2]). We now have
Lemma 1: $\alpha_{1 j}^{(m)}=\alpha_{j j}^{(m+1-j)}$.
Proof: Consider the sum of weights of all walks with length $m$ from 1 to $j$ ( $j=1,2,3, \ldots, n$ ). For $1 \leq m \leq k-1$,

$$
\alpha_{1 j}^{(m)}= \begin{cases}1 & \text { if } m=j-1 \\ 0 & \text { otherwise } .\end{cases}
$$

Clearly,

$$
a_{j j}^{(m+1-j)}= \begin{cases}1 & \text { if } m=j-1 \\ 0 & \text { if } j \leq m \leq k-1\end{cases}
$$

Now let $m>k-1$. The walks of length $m$ from 1 to $j$ must be of the form

$$
1 \rightarrow 2 \rightarrow \ldots \rightarrow j \rightarrow \ldots \rightarrow k \rightarrow \ldots \rightarrow j
$$

Eliminating the path from 1 to $j$, we see that the preceding walks are in one-to-one correspondence with the walks of length $m-j+1$ from $j$ to $j$. Since the weight of the path $1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow j$ is 1 , we have

$$
\alpha_{1 j}^{(m)}=\alpha_{j j}^{(m+1-j)}
$$

Lemma 2: $a_{j j}^{(m)}=\sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-k+i-1)} \quad(j=1,2, \ldots, k-1, k)$,
where

$$
f^{(t)}=0(t<0), \quad f^{(0)}=1,
$$

and

$$
f^{(m)}=\sum_{\substack{s_{1}+2 s_{2}+\ldots+k s_{k}=m \\ s_{i} \geq 0(i=1,2, \ldots, k)}}\binom{s_{1}+s_{2}+\ldots+s_{k}}{s_{1}, s_{2}, \ldots, s_{k}} \alpha_{1}^{s_{1}} \alpha_{2}^{s_{2}} \ldots \alpha_{k}^{s_{k}}
$$

Proof: From the digraph $D$, it is not difficult to see that there are $k$ classes of circuits from vertex $k$ to $k$ in $D$ as given in the following table.

| NAME | CIRCUIT | LENGTH | WEIGHT |
| :---: | :---: | :---: | :---: |
| $C_{1}$ | $k \rightarrow k$ | 1 | $\alpha_{1}$ |
| $C_{2}$ | $k \rightarrow(k-1) \rightarrow k$ | 2 | $\alpha_{2}$ |
| $C_{3}$ | $k \rightarrow(k-2) \rightarrow(k-1) \rightarrow k$ | 3 | $\alpha_{3}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $C_{k}$ | $k \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow k$ | $k$ | $\alpha_{k}$ |

Hence, any walk with length $m$ from $k$ to $k$ must consist of $s_{1} C_{1}{ }^{\prime} s, s_{2} C_{2}$ 's, ..., $s_{k} C_{k}$ 's.

The walks with length $m$ from $j$ to $j, 1 \leq j \leq k-1$, have one of the $j$ following forms:

| NAME | CIRCUIT |
| :---: | :---: |
| Form (1) | $j \rightarrow \ldots \rightarrow k \rightarrow \ldots \rightarrow k \rightarrow 1 \rightarrow 2 \rightarrow \ldots \rightarrow j$ <br> where $k \rightarrow \cdots \rightarrow k$ means passing through many circuits |
| Form (2) | $j \rightarrow \ldots \rightarrow k \rightarrow \ldots \rightarrow k \rightarrow 2 \rightarrow 3 \rightarrow \ldots \rightarrow j$ |
| Form (3) | $j \rightarrow \ldots \rightarrow k \rightarrow \ldots \rightarrow k \rightarrow 3 \rightarrow 4 \rightarrow \ldots \rightarrow j$ |
| $\vdots$ |  |
| Form $(j)$ | $j \rightarrow \ldots \rightarrow k \rightarrow \ldots \rightarrow k \rightarrow j$ |

Clearly, the front path and the back path in form ( $i$ ), where $i=1,2, \ldots, j$, together give a circuit $C_{k-i+1}$. Namely, there must be a circuit of length $k-i+1$. Thus, for any fixed $i(1 \leq i \leq j)$,

$$
\begin{aligned}
& a_{j j}^{(m)}=\sum_{i=1}^{j} \sum_{\substack{s_{1}+2 s_{2}+\ldots+k s_{k}=m \\
s_{t} \geq 0, t \neq k-i+1 \\
s_{t} \geq 1, t=k-i+1}}\left(\begin{array}{l}
s_{1}+s_{2}+\cdots+\left(s_{k-i+1}-1\right)+\ldots+s_{k} \\
\left.s_{1}, s_{2}, \ldots,\left(s_{k-i+1}-1\right), \ldots, s_{k}^{s_{1}} \alpha_{1}^{s_{2}} \ldots \alpha_{k}^{s_{k}}\right]
\end{array}\right. \\
& =\sum_{i=1}^{j} \alpha_{k-i+1} \sum_{\substack{s_{1}+2 s_{2}+\ldots+k s_{k}=m-k+i-1 \\
s_{t} \geq 0}}\binom{s_{1}+s_{2}+\cdots+s_{k}}{s_{1}, s_{2}, \ldots, s_{k}} \alpha_{1}^{s_{1}} \alpha_{2}^{s_{2}} \ldots \alpha_{k}^{s_{k}}, \quad 1 \leq j \leq k .
\end{aligned}
$$

For convenience, let

$$
\left.\begin{array}{rl}
f^{(m)} & =f^{(m)}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}\right) \\
& =\sum_{\substack{s_{1}+2 s_{2}+\ldots+k s_{k}=m \\
s_{t} \geq 0}}\left(\begin{array}{l}
s_{1}+s_{2}+\ldots+s_{k} \\
s_{1},
\end{array} s_{2}, \ldots, s_{k}\right.
\end{array}\right) \alpha_{1}^{s_{1}} \alpha_{2}^{s_{2}} \ldots \alpha_{k}^{s_{k}} . \quad .
$$

Hence,

$$
\alpha_{j j}^{(m)}=\sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-k+i-1)}, 1 \leq j \leq k . \quad \text { 比 }
$$

Lemma 3: For $f^{(m)}$, we have the following recurrence:

$$
f^{(m)}=\alpha_{k k}^{(m)}=\sum_{i=1}^{k} \alpha_{k-i+1} f^{(m-k+i-1)}
$$

Proof: According to the preceding analysis,

$$
a_{k k}^{(m)}=\sum_{\substack{s_{1}+2 s_{2}+\cdots+k s_{k}=m \\
s \geq 0}}\left(\begin{array}{l}
s_{1}+s_{2}+\cdots+s_{k} \\
s_{1}, \\
\left.s_{2}, \ldots, k\right)
\end{array}\right) \alpha_{1}^{s_{1}} \alpha_{2}^{s_{2}} \ldots \alpha_{k}^{s_{k}}=f^{(m)}
$$

By Lemma 2,

$$
a_{k k}^{(m)}=\sum_{i=1}^{k} \alpha_{k-i+1} f^{(m-k+i-1)}
$$

Thus,

$$
f^{(m)}=\alpha_{k k}^{(m)}=\sum_{i=1}^{k} \alpha_{k-i+1} f^{(m-k+i-1)}
$$

Theorem: The solution of the recurrence relation (1) is

$$
\begin{align*}
& u_{m}=\sum_{j=1}^{k} c_{j-1} \sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-k-j+i)}+\sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)}  \tag{9}\\
& u_{m}=\sum_{j=1}^{k} c_{j-1} \sum_{i=1}^{j} \alpha_{k-i+1} \sum_{\substack{s_{1}+2 s_{2}+\ldots+k s_{k}=m-k+i-j \\
s_{t} \geq 0 \\
(t=1, \ldots, k)}}\binom{s_{1}+s_{2}+\cdots+s_{k}}{s_{1}, s_{2}, \ldots, s_{k}} \alpha_{1}^{s_{1}} \alpha_{2}^{s_{2}} \cdots \alpha_{k}^{s_{k}}
\end{align*}
$$

Proof: By (7) and (8),

$$
\begin{aligned}
u_{m}=a^{(m)} & =\sum_{j=1}^{k} c_{j-1} a_{1 j}^{(m)}+\sum_{j=1}^{m-k+1} b_{j-1} a_{1 k}^{(m-j)} \\
& =\sum_{j=1}^{k} c_{j-1} a_{j j}^{(m+1-j)}+\sum_{j=1}^{m-k+1} b_{j-1} a_{k k}^{(m-k+1-j)} \quad \text { (Lemma 1) } \\
& =\sum_{j=1}^{k} c_{j-1} \sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-j-k+i)}+\sum_{j=1}^{m-k+1} b_{j-1} f^{(m-k+1-j)} \quad \text { (Lemmas 2 }
\end{aligned}
$$

Corollary 1:

$$
u_{m}=\alpha_{k-1} f^{(m-k+1)}+\sum_{j=1}^{k-1} c_{j-1} \sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-k-j+i)}+\sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)}
$$

Proof: This formula follows by using Lemma 3 and (9).
Corollary 2: The homogeneous recurrence (1) with constant coefficient has the solution

$$
u_{m}=\alpha_{k-1} f^{(m-k+1)}+\sum_{j=1}^{k-1} c_{j-1} \sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-k-j+i)}
$$

Corollary 3: The recurrence relation

$$
\left\{\begin{array}{l}
u_{n+k}=\alpha u_{n+r}+\beta u_{n}+b_{n}  \tag{10}\\
u_{0}=c_{0}, u_{1}=c_{1}, \ldots, u_{k-1}=c_{k-1} \quad(1 \leq l \leq k-1)
\end{array}\right.
$$

has the solution

$$
u_{m}=\sum_{j=0}^{r-1} c_{j} \beta f^{(m-k-j)}+\sum_{j=r}^{k-1} c_{j} f^{(m-j)}+\sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)}
$$

where

$$
f^{(m)}=\sum_{\substack{(k-r) y=m \\ x, y \geq 0}}\binom{x+y}{y} \beta^{x} \alpha^{y} \quad(m \geq 0) .
$$

Proof: Let $\alpha_{k}=\beta, \alpha_{k-r}=\alpha$, and $\alpha_{i}=0$, otherwise, in (1). By (9),

$$
\begin{align*}
u_{m}= & \sum_{j=1}^{r} c_{j-1} \beta f^{(m-k-j+1)}+\sum_{j=r+1}^{k} c_{j-1}\left(\beta f^{(m-k-j+1)}+\alpha f^{(m-k-j+r+1)}\right) \\
& +\sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)} \\
= & \sum_{j=0}^{r} c_{j-1} \beta f^{(m-k-j+1)}+\sum_{j=r+1}^{k} c_{j-1} f^{(m-j+1)}+\sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)}  \tag{Lemma3}\\
= & \sum_{j=0}^{r-1} c_{j} \beta f^{(m-k-j)}+\sum_{j=r}^{k-1} c_{j} f^{(m-j)}+\sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)}
\end{align*}
$$

where

$$
f^{(m)}=\sum_{\substack{(k-r) y=m \\ x, y \geq 0}}\binom{x+y}{y} \beta^{x} \alpha^{y}
$$

When $b_{n}=0$ in (10), Corollary 3 coincides with a result in [3]. When $b_{n}=0$, $\alpha=\beta=1, l=1, k=2$, and $c_{0}=c_{1}=1$,

$$
\begin{aligned}
u_{m}=c_{0} f^{(m-2)}+c_{1} f^{(m-1)} & =f^{(m-2)}+f^{(m-1)} \\
& =f^{(m)}=\sum_{\substack{2 x+y=m \\
x, y \geq 0}}\binom{x+y}{y}=\sum_{k=0}^{[m / 2]}\binom{m-k}{k},
\end{aligned}
$$

which is the combinatorial expression of the Fibonacci series.
Example 1: $F_{n+5}=2 F_{n+4}+3 F_{n}+(2 n-1)$

$$
F_{0}=1, F_{1}=0, F_{2}=1, F_{3}=2, F_{4}=3
$$

Solution: $\quad k=5, l=4, \alpha=2, \beta=3, b_{n}=2 n-1$

$$
c_{0}=1, c_{1}=0, c_{2}=1, c_{3}=2, c_{4}=3
$$

By Formula (10), one easily finds

$$
\begin{aligned}
F_{n}= & 3 \sum_{x=0}^{[(n-5) / 5]}\binom{n-4 x-5}{x} 3^{x} 2^{n-5 x-5}+3 \sum_{x=0}^{[(n-7) / 5]}\binom{n-4 x-7}{x} 3^{x} 2^{n-5 x-7} \\
& +6 \sum_{x=0}^{[(n-8) / 5]}\binom{n-4 x-8}{x} 3^{x} 2^{n-5 x-8}+3 \sum_{x=0}^{[(n-4) / 5]}\binom{n-4 x-4}{x} 3^{x} 2^{n-5 x-4} \\
& +\sum_{j=1}^{n-4}(2 j-3) \sum_{x=0}^{[(n-4-j) / 5]}\binom{n-4 x-4-j}{x} 3^{x} 2^{n-5 x-4-j} .
\end{aligned}
$$

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# THE TETRANACCI SEQUENCE AND GENERALIZATIONS 

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## 1. Introduction

Many papers concerning a variety of generalizations of the Fibonacci sequence have appeared, primarily in The Fibonacci Quarterly, in recent years. Horadam [l] was one of the first to initiate this interest when he changed the two initial terms of the Fibonacci sequence from 0 , 1 to $H_{0}, H_{1}$, arbitrary integers, while maintaining the recurrence relation. He remarked in [1] that there are fundamentally two ways in which the Fibonacci sequence may be generalized; namely, either the recurrence relation can be changed or the initial terms can be altered. The two techniques can be combined, of course. Of the two alterations, a change in the recurrence relation seems to lead to greater complexity in the properties of the resulting sequence.

Some generalizations have been given names. The Tribonacci sequence, $\left\{T_{n}\right\}$, is defined by

$$
\begin{equation*}
T_{n}=T_{n-1}+T_{n-2}+T_{n-3} \quad(n \geq 3), \quad T_{0}=0, T_{1}=T_{2}=1 \tag{1}
\end{equation*}
$$

A generalized Tribonacci sequence results when the recurrence relation is the same and $T_{0}, T_{1}, T_{2}$ are arbitrary. The Tribonacci sequence and this particular generalization have been examined rather extensively in the literature. See, for example, [2], [3], [4], [5], [6], [7].

The Tetranacci sequence, $\left\{M_{n}\right\}$, is defined by

$$
\begin{equation*}
M_{n}=M_{n-1}+M_{n-2}+M_{n-3}+M_{n-4} \quad(n \geq 4), \quad M_{0}=M_{1}=0, \quad M_{2}=M_{3}=1 \tag{2}
\end{equation*}
$$

The first mention of the Tetranacci sequence seems to have occurred in [2], and it has received further brief attention or reference in [8], [9], [10], [11], [12]. Some writers have used the name "Quadranacci" (Latin) instead of "Tetranacci" (Greek). We use the latter, as in [2].

The characteristics and properties of the Tetranacci sequence apparently have not been examined in detail, and that, along with an examination of the generalization which occurs when the four initial terms are chosen as arbitrary integers, is the purpose of this paper.

As the recurrence relation and initial terms of Fibonacci-type sequences become more general, we quite naturally expect that the relationships among terms and the formal properties of the resulting sequences will become more complicated and complex, and this indeed is true. Nevertheless, by employing appropriate techniques, particularly by using vector and matrix methods, a number of properties of the Tetranacci sequence and generalizations and identities involving terms of these sequences are found and proved.

## 2. Fundamental Properties

As we begin an examination of the Tetranacci sequence and generalizations, two "companion" sequences emerge and are considered along with (2). These sequences are designated $\left\{N_{n}\right\}$ and $\left\{S_{n}\right\}$ and are defined as follows:

$$
\begin{array}{ll}
N_{n}=N_{n-1}+N_{n-2}+N_{n-3}+N_{n-4} & (n \geq 4), \quad N_{0}=N_{2}=0, \quad N_{1}=N_{3}=1 \\
S_{n}=S_{n-1}+S_{n-2}+S_{n-3}+S_{n-4} & (n \geq 4), \quad S_{0}=S_{3}=1, \quad S_{1}=S_{2}=0 \tag{4}
\end{array}
$$

The sequences $\left\{N_{n}\right\}$ and $\left\{S_{n}\right\}$ have the same recurrence relation as $\left\{M_{n}\right\}$ but different initial terms. The initial terms are, in fact, two distinct permutations of the four initial terms of $\left\{M_{n}\right\}$. It can be shown also that these two companion sequences are further related to $\left\{M_{n}\right\}$ by

$$
\begin{align*}
& N_{n}=M_{n-1}+M_{n-2}+M_{n-3} \quad(n \geq 3),  \tag{5}\\
& S_{n}=M_{n-1}+M_{n-2} \quad(n \geq 2)
\end{align*}
$$

We define the generalized Tetranacci sequence, $\left\{\mu_{n}\right\}$, as
(7) $\quad \mu_{n}=\mu_{n-1}+\mu_{n-2}+\mu_{n-3}+\mu_{n-4} \quad(n \geq 4)$
where $\mu_{0}, \mu_{1}, \mu_{2}, \mu_{3}$ are arbitrary integers.
The analogous generalized companion sequences, $\left\{\nu_{n}\right\}$ and $\left\{\sigma_{n}\right\}$, then become

$$
\begin{equation*}
v_{n}=v_{n-1}+v_{n-2}+v_{n-3}+v_{n-4} \quad(n \geq 4) \tag{8}
\end{equation*}
$$

or, alternately,

$$
\begin{equation*}
v_{n}=\mu_{n-1}+\mu_{n-2}+\mu_{n-3} \quad(n \geq 3) \tag{9}
\end{equation*}
$$

where $\nu_{0}=\mu_{1}-\mu_{0}, \nu_{1}=\mu_{2}-\mu_{1}, \nu_{2}=\mu_{3}-\mu_{2}, \nu_{3}=\mu_{2}+\mu_{1}+\mu_{0}$,
and

$$
\begin{equation*}
\sigma_{n}=\sigma_{n-1}+\sigma_{n-2}+\sigma_{n-3}+\sigma_{n-4} \quad(n \geq 4) \tag{10}
\end{equation*}
$$

or, alternately,

$$
\begin{equation*}
\sigma_{n}=\mu_{n-1}+\mu_{n-2} \quad(n \geq 2) \tag{11}
\end{equation*}
$$

where $\sigma_{0}=\mu_{2}-\mu_{1}-\mu_{0}, \sigma_{1}=\mu_{3}-\mu_{2}-\mu_{1}, \sigma_{2}=\mu_{1}+\mu_{0}, \sigma_{3}=\mu_{2}+\mu_{1}$.
The choice of the initial terms of $\left\{\nu_{n}\right\}$ and $\left\{\sigma_{n}\right\}$ is not arbitrary but is determined by their relationship to $\left\{\mu_{n}\right\}$.

The table below gives values of the three sequences $\left\{M_{n}\right\},\left\{N_{n}\right\}$, and $\left\{S_{n}\right\}$ for $n=0$ to 18 .

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $M_{n}$ | 0 | 0 | 1 | 1 | 2 | 4 | 8 | 15 | 29 | 56 | 108 | 208 | 401 | 773 | 1490 | 2872 | 5536 | 10,671 | 20,569 |
| $N_{n}$ | 0 | 1 | 0 | 1 | 2 | 4 | 7 | 14 | 27 | 52 | 100 | 193 | 372 | 717 | 1382 | 2664 | 5135 | 9,898 | 19,079 |
| $S_{n}$ | 1 | 0 | 0 | 1 | 2 | 3 | 6 | 12 | 23 | 44 | 85 | 164 | 316 | 609 | 1174 | 2263 | 4362 | 8,408 | 16,207 |

The analogue of Binet's formula for the Fibonacci sequence can be derived for $\left\{M_{n}\right\}$ and $\left\{\mu_{n}\right\}$. In [7] Spickerman and in [3] Waddill and Sacks derived the analogue of Binet's formula for the Tribonacci sequence and later in [8] Spickerman and Joyner generalized the result obtained in [7] to recursive sequences of order $K$. Since the Tetranacci sequence is a variation of the recursive sequence of order 4 in [8], the formula there may be adapted to give Binet's formula for the Tetranacci sequence; namely,

$$
\begin{equation*}
M_{n}=A_{1} r_{1}^{n}+A_{2} r_{2}^{n}+A_{3} r_{3}^{n}+A_{4} r_{4}^{n}, \tag{12}
\end{equation*}
$$

where $A_{i}$ are constants and $r_{i}$ are the four distinct roots of

$$
x^{4}-x^{3}-x^{2}-x-1=0
$$

Binet's formula for $\mu_{n}$ is the same as (12) except that the $A_{i}$ are functions of $\mu_{0}, \mu_{1}, \mu_{2}, \mu_{3}$. The $A_{i}$ and $r_{i}$ in (12) may be computed routinely but the resulting formula is long and cumbersome; hence, it is not written explicitly here nor used in the sequel.

A useful means of representing the recurrence relation of the Tetranacci sequence is by employing what we call the $T$-matrix, the analogue of the $Q-$ matrix [13] which has been widely used in establishing properties of the Fibonacci sequence.

The $T$-matrix is defined to be

$$
T=\left[\begin{array}{llll}
1 & 1 & 1 & 1  \tag{13}\\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]
$$

Induction proofs may be used to establish

$$
\left[\begin{array}{l}
M_{n}  \tag{14}\\
M_{n-1} \\
M_{n-2} \\
M_{n-3}
\end{array}\right]=\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]^{n-3}\left[\begin{array}{c}
M_{3} \\
M_{2} \\
M_{1} \\
M_{0}
\end{array}\right]
$$

$$
\left[\begin{array}{l}
\mu_{n} \\
\mu_{n-1} \\
\mu_{n-2} \\
\mu_{n-3}
\end{array}\right]=\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]^{n-3}\left[\begin{array}{l}
\mu_{3} \\
\mu_{2} \\
\mu_{1} \\
\mu_{0}
\end{array}\right]
$$

and

$$
\left[\begin{array}{llll}
1 & 1 & 1 & 1  \tag{16}\\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]^{n}=\left[\begin{array}{llll}
M_{n+2} & N_{n+2} & S_{n+2} & M_{n+1} \\
M_{n+1} & N_{n+1} & S_{n+1} & M_{n} \\
M_{n} & N_{n} & S_{n} & M_{n-1} \\
M_{n-1} & N_{n-1} & S_{n-1} & M_{n-2}
\end{array}\right]
$$

The right side of equation (16) indicates a reason for calling $\left\{N_{n}\right\}$ and $\left\{S_{n}\right\}$ "companion" sequences of $\left\{M_{n}\right\}$ : both occur naturally in successive powers of the $T$-matrix.

Although up to this point, we have restricted the subscripts of the Tetranacci sequence and generalizations to being nonnegative, we may remove that restriction and define $\left\{M_{n}\right\},\left\{N_{n}\right\},\left\{S_{n}\right\}$ and their corresponding generalizations for all $n$.

By writing the difference equation (2) as
(17) $\quad M_{n}=M_{n+4}-M_{n+3}-M_{n+2}-M_{n+1}$,
and choosing $n<0$, then $n+4, n+3, n+2$, and $n+1$ are all greater than $n$, which allows us to define $M_{n}$ by the four terms immediately following it. That is,

$$
\begin{aligned}
& M_{-1}=M_{3}-M_{2}-M_{1}-M_{0} \\
& M_{-2}=M_{2}-M_{1}-M_{0}-M_{-1}
\end{aligned}
$$

and so on.
We may obtain another useful definition of $M_{n}, n<0$, by using the $T$-matrix. We first write (14) as

$$
\left[\begin{array}{l}
M_{n}  \tag{18}\\
M_{n+1} \\
M_{n+2} \\
M_{n+3}
\end{array}\right]=\left[\begin{array}{llll}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1
\end{array}\right]^{n}\left[\begin{array}{l}
M_{0} \\
M_{1} \\
M_{2} \\
M_{3}
\end{array}\right]
$$

Now, in (18), if we replace $n$ by $-n$, we have, for $n>0$,

$$
\left[\begin{array}{l}
M_{-n}  \tag{19}\\
M_{-n+1} \\
M_{-n+2} \\
M_{-n+3}
\end{array}\right]=\left[\begin{array}{llll}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1
\end{array}\right]^{-n}\left[\begin{array}{l}
M_{0} \\
M_{1} \\
M_{2} \\
M_{3}
\end{array}\right]=\left[\begin{array}{rrrr}
-1 & -1 & -1 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]^{n}\left[\begin{array}{l}
M_{0} \\
M_{1} \\
M_{2} \\
M_{3}
\end{array}\right],
$$

which defines $M_{n}$ for $n<0$; and this definition using the $T$-matrix is equivalent to (17).

The sequences $\left\{N_{n}\right\},\left\{S_{n}\right\},\left\{\mu_{n}\right\},\left\{\nu_{n}\right\},\left\{\sigma_{n}\right\}$ may be defined for $n<0$ in like manner.

We now establish some interesting and useful identities. Using (15) and (16), we may write

$$
\begin{align*}
{\left[\begin{array}{l}
\mu_{n+p} \\
\mu_{n+p-1} \\
\mu_{n+p-2} \\
\mu_{n+p-3}
\end{array}\right] } & =\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]^{p}\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]^{n-3}\left[\begin{array}{l}
\mu_{3} \\
\mu_{2} \\
\mu_{1} \\
\mu_{0}
\end{array}\right]  \tag{20}\\
& =\left[\begin{array}{llll}
M_{p+2} & N_{p+2} & S_{p+2} & M_{p+1} \\
M_{p+1} & N_{p+1} & S_{p+1} & M_{p} \\
M_{p} & N_{p} & S_{p} & M_{p-1} \\
M_{p-1} & N_{p-1} & S_{p-1} & M_{p-2}
\end{array}\right]\left[\begin{array}{l}
\mu_{n} \\
\mu_{n-1} \\
\mu_{n-2} \\
\mu_{n-3}
\end{array}\right]
\end{align*}
$$

From which we conclude that

$$
\begin{equation*}
\mu_{n+p}=M_{p+2} \mu_{n}+N_{p+2} \mu_{n-1}+S_{p+2^{\mu-2}}^{\mu_{n-2}}+M_{p+1} \mu_{n-3} \tag{21}
\end{equation*}
$$

or

$$
\begin{equation*}
\mu_{n+p}=M_{n+2} \mu_{p}+N_{n+2} \mu_{p-1}+S_{n+2} \mu_{p-2}+M_{n+1} \mu_{p-3} \tag{22}
\end{equation*}
$$

By replacing $N_{p+2}$ and $S_{p+2}$ using (5) and (6), regrouping and then employing (9) and (11), we find that (21) and (22) may be written

$$
\begin{equation*}
\mu_{n+p}=M_{p+2} \mu_{n}+M_{p+1} \nu_{n}+M_{p} \sigma_{n}+M_{p-1} \mu_{n-1} \tag{23}
\end{equation*}
$$

or

$$
\begin{equation*}
\mu_{n+p}=M_{n+2} \mu_{p}+M_{n+1} \nu_{p}+M_{n} \sigma_{p}+M_{n-1} \mu_{p-1} \tag{24}
\end{equation*}
$$

As special cases of (21) and (23), respectively, when $p=0$, we have
or

$$
\mu_{n}=M_{n-1} \mu_{3}+N_{n-1} \mu_{2}+S_{n-1} \mu_{1}+M_{n-2} \mu_{0}
$$

$$
\mu_{n}=M_{n-1} \mu_{3}+M_{n-2} \nu_{3}+M_{n-3} \sigma_{3}+M_{n-4} \mu_{2}
$$

We next consider the sequence $\left\{R_{n}\right\}$ which is defined by

$$
R_{0}=M_{1}, R_{1}=S_{2}, R_{2}=N_{2}, R_{3}=M_{2}
$$

and

$$
\left[\begin{array}{l}
R_{3 n}  \tag{25}\\
R_{3 n-1} \\
R_{3 n-2} \\
R_{3 n-3}
\end{array}\right]=\left[\begin{array}{l}
M_{n+1} \\
N_{n+1} \\
S_{n+1} \\
M_{n}
\end{array}\right]=\left[\begin{array}{llll}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array}\right]^{n-1}\left[\begin{array}{l}
R_{3} \\
R_{2} \\
R_{1} \\
R_{0}
\end{array}\right]
$$

The generating matrix of $\left\{R_{n}\right\}$ is the transpose of the $T$-matrix, and the terms of $\left\{R_{n}\right\}$ are generated in groups of three rather than singularly as in (14). It is evident that the sequence $\left\{R_{n}\right\}$ is merely a meshing of the three sequences $\left\{M_{n}\right\},\left\{N_{n}\right\},\left\{S_{n}\right\}$, and, consequently, its terms are not as "spread out" as the terms of either of these sequences individually. This latter property become useful in establishing identities later on.

The generalized sequence for $\left\{R_{n}\right\}$ is designated $\left\{\rho_{n}\right\}$ and is defined as expected by

$$
\rho_{0}=\mu_{1}, \rho_{1}=\sigma_{2}, \rho_{2}=\nu_{2}, \rho_{3}=\mu_{2}
$$

and

$$
\begin{align*}
{\left[\begin{array}{l}
\rho_{3 n} \\
\rho_{3 n-1} \\
\rho_{3 n-2} \\
\rho_{3 n-3}
\end{array}\right] } & =\left[\begin{array}{l}
\mu_{n+1} \\
\nu_{n+1} \\
\sigma_{n+1} \\
\mu_{n}
\end{array}\right]=\left[\begin{array}{llll}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array}\right]^{n-1}\left[\begin{array}{l}
\rho_{3} \\
\rho_{2} \\
\rho_{1} \\
\rho_{0}
\end{array}\right]  \tag{26}\\
& =\left[\begin{array}{llll}
M_{n+1} & M_{n} & M_{n-1} & M_{n-2} \\
N_{n+1} & N_{n} & N_{n-1} & N_{n-2} \\
S_{n+1} & S_{n} & S_{n-1} & S_{n-2} \\
M_{n} & M_{n-1} & M_{n-2} & M_{n-3}
\end{array}\right]\left[\begin{array}{l}
\rho_{3} \\
\rho_{2} \\
\rho_{1} \\
\rho_{0}
\end{array}\right]
\end{align*}
$$

Identities analogous to (21) and (23) may now be written for the sequences $\left\{\nu_{n}\right\}$ and $\left\{\sigma_{n}\right\}$. Using (26) and writing

$$
\begin{align*}
{\left[\begin{array}{l}
\mu_{n+p} \\
\nu_{n+p} \\
\sigma_{n+p} \\
\mu_{n+p-1}
\end{array}\right] } & =\left[\begin{array}{llll}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array}\right]^{p}\left[\begin{array}{llll}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array}\right]^{n-3}\left[\begin{array}{l}
\mu_{3} \\
\nu_{3} \\
\sigma_{3} \\
\mu_{2}
\end{array}\right]  \tag{27}\\
& =\left[\begin{array}{llll}
M_{p+2} & M_{p+1} & M_{p} & M_{p-1} \\
N_{p+2} & N_{p+1} & N_{p} & N_{p-1} \\
S_{p+2} & S_{p+1} & S_{p} & S_{p-1} \\
M_{p+1} & M_{p} & M_{p-1} & M_{p-2}
\end{array}\right]\left[\begin{array}{l}
\mu_{n} \\
\nu_{n} \\
\sigma_{n} \\
\mu_{n-1}
\end{array}\right],
\end{align*}
$$

from (27) we conclude that

$$
\begin{align*}
& \nu_{n+p}=N_{p+2} \mu_{n}+N_{p+1} \nu_{n}+N_{p} \sigma_{n}+N_{p-1} \mu_{n-1},  \tag{28}\\
& v_{n+p}=N_{n+2} \mu_{p}+N_{n+1} \nu_{p}+N_{n} \sigma_{p}+N_{n-1} \mu_{p-1}, \tag{29}
\end{align*}
$$

or by (20) replacing $\mu_{i}$ with $\nu_{i}$, we have

$$
\begin{align*}
& \nu_{n+p}=M_{p+2} \nu_{n}+N_{p+2} \nu_{n-1}+S_{p+2} \nu_{n-2}+M_{p+1} \nu_{n-3},  \tag{30}\\
& \nu_{n+p}=M_{n+2} \nu_{p}+N_{n+2} \nu_{p-1}+S_{n+2} \nu_{p-2}+M_{n+1} \nu_{p-3} . \tag{31}
\end{align*}
$$

Similarly,

$$
\begin{align*}
& \sigma_{n+p}=S_{p+2} \mu_{n}+S_{p+1} \nu_{n}+S_{p} \sigma_{n}+S_{p-1} \mu_{n-1},  \tag{32}\\
& \sigma_{n+p}=S_{n+2} \mu_{p}+S_{n+1} \nu_{p}+S_{n} \sigma_{p}+S_{n-1} \mu_{p-1},  \tag{33}\\
& \sigma_{n+p}=M_{p+2} \sigma_{n}+N_{p+2} \sigma_{n-1}+S_{p+2} \sigma_{n-2}+M_{p+1} \sigma_{n-3},  \tag{34}\\
& \sigma_{n+p}=M_{n+2} \sigma_{p}+N_{n+2} \sigma_{p-1}+S_{n+2} \sigma_{p-2}+M_{n+1} \sigma_{p-3} . \tag{35}
\end{align*}
$$

We may further generalize (21) to read

$$
\begin{equation*}
\mu_{n+p}=M_{p+k+2} \mu_{n-k}+N_{p+k+2} \mu_{n-k-1}+S_{p+k+2} \mu_{n-k-2}+M_{p+k+1} \mu_{n-k-3} \tag{36}
\end{equation*}
$$

where $k$ is any integer. Since $\left\{\mu_{n}\right\}$ has been defined for all $n$, all terms in (36) are defined even if a chosen value of $k$ produces negative subscripts. Also equations (22)-(24) and (28)-(35) can be written in this more general way.

In the vector on the left side of (15) the terms

$$
\mu_{n}, \mu_{n-1}, \mu_{n-2}, \mu_{n-3}
$$

are clearly adjacent terms of the sequence $\left\{\mu_{n}\right\}$. By using appropriate matrices we can write a vector in which the four terms are not adjacent but are "spread out" in a prescribed manner.

By (21) we have, for arbitrary integers $p, q$, and $r$,

$$
\left[\begin{array}{l}
\mu_{n+p}  \tag{37}\\
\mu_{n+q} \\
\mu_{n+r} \\
\mu_{n}
\end{array}\right]=\left[\begin{array}{llll}
M_{p+2} & N_{p+2} & S_{p+2} & M_{p+1} \\
M_{q+2} & N_{q+2} & S_{q+2} & M_{q+1} \\
M_{r+2} & N_{r+2} & S_{r+2} & M_{r+1} \\
1 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{l}
\mu_{n} \\
\mu_{n-1} \\
\mu_{n-2} \\
\mu_{n-3}
\end{array}\right]
$$

Using (23), (28), and (32), we conclude that

$$
\left[\begin{array}{l}
\mu_{n+p}  \tag{38}\\
\nu_{n+q} \\
\sigma_{n+r} \\
\mu_{n}
\end{array}\right]=\left[\begin{array}{llll}
M_{p+2} & M_{p+1} & M_{p} & M_{p-1} \\
N_{q+2} & N_{q+1} & N_{q} & N_{q-1} \\
S_{r+2} & S_{r+1} & S_{r} & S_{r-1} \\
1 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{l}
\mu_{n} \\
\nu_{n} \\
\sigma_{n} \\
\mu_{n-1}
\end{array}\right]
$$

Equations (37) and (38) will be used later on.

## 3. Linear Sums

A number of linear sum identities were discovered and proved. We give some of these and write them in terms of the generalized Tetranacci sequence, even though each has as a special case the corresponding identity for the Tetranacci sequence. All the listed identities may be proved by induction, but that method of proof gives no clue about their discovery. We give one proof to indicate how these identities, in general, were discovered.

We have

$$
\begin{align*}
& \sum_{i=0}^{n} \mu_{i}=\frac{1}{3}\left[\mu_{n+2}+2 \mu_{n}+\mu_{n-1}+2 \mu_{0}+\mu_{1}-\mu_{3}\right],  \tag{39}\\
& \sum_{i=0}^{n} \mu_{2 i+1}=\frac{1}{3}\left[2 \mu_{2 n+2}+\mu_{2 n}-\mu_{2 n-1}-2 \mu_{0}+2 \mu_{1}-3 \mu_{2}+\mu_{3}\right],  \tag{40}\\
& \sum_{i=0}^{n} \mu_{2 i}=\frac{1}{3}\left[2 \mu_{2 n+1}+\mu_{2 n-1}-\mu_{2 n-2}+4 \mu_{0}-\mu_{1}+3 \mu_{2}-2 \mu_{3}\right],  \tag{41}\\
& \sum_{i=0}^{n} \mu_{3 i}=\frac{1}{9}\left[4 \mu_{3 n+1}+3 \mu_{3 n}-\mu_{3 n-1}+\mu_{3 n-2}+5 \mu_{0}-5 \mu_{1}-3 \mu_{2}+2 \mu_{3}\right],  \tag{42}\\
& \sum_{i=0}^{n} \mu_{3 i+1}=\frac{1}{9}\left[4 \mu_{3 n+2}+3 \mu_{3 n+1}-\mu_{3 n}+\mu_{3 n-1}+2 \mu_{0}+7 \mu_{1}-3 \mu_{2}-\mu_{3}\right],  \tag{43}\\
& \sum_{i=0}^{n} \mu_{3 i+2}=\frac{1}{9}\left[4 \mu_{3 n+3}+3 \mu_{3 n+2}-\mu_{3 n+1}+\mu_{3 n}-\mu_{0}+\mu_{1}+6 \mu_{2}-4 \mu_{3}\right]  \tag{44}\\
& \sum_{i=1}^{n} \mu_{4 i}=\sum_{i=0}^{4 n-1} \mu_{i}=\frac{1}{3}\left[\mu_{4 n+1}+2 \mu_{4 n-1}+\mu_{4 n-2}+2 \mu_{0}+\mu_{1}-\mu_{3}\right],  \tag{45}\\
& \sum_{i=1}^{n} \mu_{4 i+1}=\sum_{i=1}^{4 n} \mu_{i}=\frac{1}{3}\left[\mu_{4 n+2}+2 \mu_{4 n}+\mu_{4 n-1}-\mu_{0}+\mu_{1}-\mu_{3}\right],  \tag{46}\\
& \sum_{i=1}^{n} \mu_{4 i+2}=\sum_{i=2}^{4 n+1} \mu_{i}=\frac{1}{3}\left[\mu_{4 n+3}+2 \mu_{4 n+1}+\mu_{4 n}-\mu_{0}-2 \mu_{1}-\mu_{3}\right],  \tag{47}\\
& \sum_{i=1}^{n} \mu_{4 i+3}=\sum_{i=3}^{4 n+2} \mu_{i}=\frac{1}{3}\left[\mu_{4 n+4}+2 \mu_{4 n+2}+\mu_{4 n+1}-\mu_{0}-2 \mu_{1}-3 \mu_{2}-\mu_{3}\right] . \tag{48}
\end{align*}
$$

Proof of (39) : We write the following obvious equations;

$$
\begin{aligned}
\mu_{0}+\mu_{1}+\mu_{2} & =\mu_{4}-\mu_{3} \\
\mu_{1}+\mu_{2}+\mu_{3} & =\mu_{5}-\mu_{4} \\
\mu_{2}+\mu_{3}+\mu_{4} & =\mu_{6}-\mu_{5} \\
\cdot & \cdot
\end{aligned} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \mu_{n} \cdot \mu_{n+3}-\mu_{n+2} .
$$

Now, adding these equations, we have

$$
\sum_{i=0}^{n} \mu_{i}+\sum_{i=0}^{n} \mu_{i}+\mu_{n+1}-\mu_{0}+\sum_{i=0}^{n} \mu_{i}+\mu_{n+1}+\mu_{n+2}-\mu_{0}-\mu_{1}=\mu_{n+4}-\mu_{3}
$$

or

$$
3 \sum_{i=0}^{n} \mu_{i}=\mu_{n+4}-2 \mu_{n+1}-\mu_{n+2}+2 \mu_{0}+\mu_{1}-\mu_{3}
$$

which may be reduced easily to (39) by using (7) and dividing both sides by 3 .
The remaining identities, (40)-(48), are derived using similar techniques.

## 4. Quadratic, Cubic, and Quartic Identities

An application of the $T$-matrix is in deriving and proving the quadratic identity

$$
\begin{equation*}
M_{n+1}^{2}+M_{n}^{2}+M_{n-1}^{2}+2 M\left(M_{n-1}+M_{n-2}\right)=M_{2 n} \tag{49}
\end{equation*}
$$

Proof of (49): By (16), we have

$$
T^{2 n}=\left[\begin{array}{llll}
M_{2 n+2} & N_{2 n+2} & S_{2 n+2} & M_{2 n-1}  \tag{50}\\
M_{2 n+1} & N_{2 n+1} & S_{2 n+1} & M_{2 n-2} \\
M_{2 n} & N_{2 n} & S_{2 n} & M_{2 n-3} \\
M_{2 n-1} & N_{2 n-1} & S_{2 n-1} & M_{2 n-4}
\end{array}\right]=\left[\begin{array}{llll}
M_{n+2} & N_{n+2} & S_{n+2} & M_{n+1} \\
M_{n+1} & N_{n+1} & S_{n+1} & M_{n} \\
M_{n} & N_{n} & S_{n} & M_{n-1} \\
M_{n-1} & N_{n-1} & S_{n-1} & M_{n-3}
\end{array}\right]^{2}
$$

Now we carry out the matrix multiplication on the right side of (50) and equate the elements in the third row, first column on both sides of (50) to obtain

$$
M_{n} M_{n+2}+N_{n} M_{n+1}+S_{n} M_{n}+M_{n-1}^{2}=M_{2 n}
$$

which is equivalent to (49).
By equating corresponding elements in the fourth row, first column of (50), we obtain

$$
\begin{equation*}
M_{n+2} M_{n}-M_{n}^{2}+M_{n} M_{n-3}+M_{n-1}^{2}+2 M_{n-1} M_{n-2}=M_{2 n-1} \tag{51}
\end{equation*}
$$

The generalized versions of (49) and (51) are, respectively,

$$
\begin{align*}
& \mu_{n+1}^{2}+\mu_{n}^{2}+\mu_{n-1}^{2}+2 \mu_{n}\left(\mu_{n-1}+\mu_{n-2}\right)  \tag{52}\\
& =\mu_{3} \mu_{2 n-1}+\mu_{2}\left(\mu_{2 n}-\mu_{2 n-1}\right)+\mu_{1}\left(\mu_{2 n-2}+\mu_{2 n-3}\right)+\mu_{0} \mu_{2 n-2}
\end{align*}
$$

and

$$
\begin{align*}
& \mu_{n+2} \mu_{n}-\mu_{n}^{2}+\mu_{n} \mu_{n-3}+\mu_{n-1}^{2}+2 \mu_{n-1} \mu_{n-2}  \tag{53}\\
& =\mu_{3} \mu_{2 n-2}+\mu_{2}\left(\mu_{2 n-2}-\mu_{2 n-6}\right)+\mu_{1}\left(\mu_{2 n-3}+\mu_{2 n-4}\right)+\mu_{0} \mu_{2 n-3}
\end{align*}
$$

In (52), if we let $\mu_{0}=\mu_{1}=0$ and $\mu_{2}=\mu_{3}=1$, we have

$$
M_{n+1}^{2}+M_{n}^{2}+M_{n-1}^{2}+2 M_{n}\left(M_{n-1}+M_{n-2}\right)=M_{2 n}
$$

which is (49). By letting $p=n-1, \mu_{0}=\mu_{1}=0, \mu_{2}=\mu_{3}=1$, and replacing $n$ by $n+1$ is (21), we obtain (49) also. However, (21) is not readily obtainable from (52) nor is (52) obtainable from (21).

The same technique used in the proof of (49) may be used to find and prove cubic identities. In this case, we use the fact that for the $T$-matrix,

$$
\begin{equation*}
T^{3 n-2}=T^{n-1} T^{n-1} T^{n} \tag{54}
\end{equation*}
$$

and again after expanding and equating appropriate corresponding terms on each side of (54), we obtain, for example,

$$
\begin{equation*}
M_{3 n}=M_{n+2}\left(R_{1} \cdot C_{1}\right)+M_{n+1}\left(R_{1} \cdot C_{2}\right)+M_{n}\left(R_{1} \cdot C_{3}\right)+M_{n-1}\left(R_{1} \cdot C_{4}\right) \tag{55}
\end{equation*}
$$

where $R_{1}$ is the first row of $T^{n-1}, C_{i}$ is the $i^{\text {th }}$ column of $T^{n-1}$ and is the usual dot product of two vectors. The right side of (55) is clearly a cubic which we do not expand completely because of its length.

The analogue of (55) for $\left\{\mu_{n}\right\}$ may be written in a manner similar to the way in which we wrote (52).

We may continue using the above technique to find quartic, quintic, and higher-ordered relations, but it is clear that one side (the side involving powers) of the equation becomes exceedingly long and complex.

One of the oldest and perhaps best known identities for the Fibonacci sequence is

$$
\begin{equation*}
F_{n+1} F_{n-1}-F_{n}^{2}=(-1)^{n+1} \tag{56}
\end{equation*}
$$

which was derived first by $R$. Simson [14]. In [3], the identity analogous to (56) was found for the Tribonacci sequence. We now pursue a like identity for the Tetranacci sequence. The simplest one may be obtained as in [3] by considering the determinants of both sides of (16) to obtain

$$
\begin{align*}
& \left|\begin{array}{llll}
M_{n+2} & M_{n+1} & M_{n} & M_{n-1} \\
M_{n+1} & M_{n} & M_{n-1} & M_{n-2} \\
M_{n} & M_{n-1} & M_{n-2} & M_{n-3} \\
M_{n-1} & M_{n-2} & M_{n-3} & M_{n-4}
\end{array}\right|=-\left|\begin{array}{llll}
M_{n+2} & M_{n-1} & M_{n} & M_{n+1} \\
M_{n+1} & M_{n-2} & M_{n-1} & M_{n} \\
M_{n} & M_{n-3} & M_{n-2} & M_{n-1} \\
M_{n-1} & M_{n-4} & M_{n-3} & M_{n-2}
\end{array}\right|  \tag{57}\\
& =-\left|\begin{array}{llll}
M_{n+2} & N_{n+2} & S_{n+2} & M_{n+1} \\
M_{n+1} & N_{n+1} & S_{n+1} & M \\
M_{n} & N_{n} & S_{n} & M_{n-1} \\
M_{n-1} & N_{n-1} & S_{n-1} & M_{n-2}
\end{array}\right|=-\left|\begin{array}{llll}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right|^{n}=(-1)^{n+1} .
\end{align*}
$$

We shall not expand the left side of (57), but it is clearly a quartic consisting of 24 terms.

We now consider some generalizations of (57). First, we rewrite (57) for the sequence $\left\{\mu_{n}\right\}$ to obtain

$$
\left|\begin{array}{llll}
\mu_{n+2} & \mu_{n+1} & \mu_{n} & \mu_{n-1}  \tag{58}\\
\mu_{n+1} & \mu_{n} & \mu_{n-1} & \mu_{n-2} \\
\mu_{n} & \mu_{n-1} & \mu_{n-2} & \mu_{n-3} \\
\mu_{n-1} & \mu_{n-2} & \mu_{n-3} & \mu_{n-4}
\end{array}\right|=(-1)^{n}\left|\begin{array}{llll}
\mu_{6} & \mu_{5} & \mu_{4} & \mu_{3} \\
\mu_{5} & \mu_{4} & \mu_{3} & \mu_{2} \\
\mu_{4} & \mu_{3} & \mu_{2} & \mu_{1} \\
\mu_{3} & \mu_{2} & \mu_{1} & \mu_{0}
\end{array}\right|
$$

a quartic expression independent of $n$ except for sign.
Proof of (58): By (15), we have the following matrix equation:

$$
\left[\begin{array}{llll}
\mu_{n+2} & \mu_{n+1} & \mu_{n} & \mu_{n-1}  \tag{59}\\
\mu_{n+1} & \mu_{n} & \mu_{n-1} & \mu_{n-2} \\
\mu_{n} & \mu_{n-1} & \mu_{n-2} & \mu_{n-3} \\
\mu_{n-1} & \mu_{n-2} & \mu_{n-3} & \mu_{n-4}
\end{array}\right]=\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right] n-4\left[\begin{array}{llll}
\mu_{6} & \mu_{5} & \mu_{4} & \mu_{3} \\
\mu_{5} & \mu_{4} & \mu_{3} & \mu_{2} \\
\mu_{4} & \mu_{3} & \mu_{2} & \mu_{1} \\
\mu_{3} & \mu_{2} & \mu_{1} & \mu_{0}
\end{array}\right]
$$

Now, by taking determinants of both sides of (59), we have (58).
As a special case of (58), consider the sequence $\left\{\alpha_{n}\right\}$ where $\alpha_{0}=\alpha_{1}=0$, $\alpha_{2}=1, \alpha_{4}=\alpha$, arbitrary. The, determinant on the right side of (58) then becomes

$$
\left|\begin{array}{cccc}
4(\alpha+1) & 2(\alpha+1) & \alpha+1 & \alpha  \tag{60}\\
2(\alpha+1) & (\alpha+1) & \alpha & 1 \\
(\alpha+1) & \alpha & 1 & 0 \\
\alpha & 1 & 0 & 0
\end{array}\right|,
$$

which is a quartic polynomial in $\alpha$. Consequently, an algebraic integer $\alpha=\beta$ exists, which makes the determinant (60) zero. Thus, for any $n$, the sequence $\left\{\alpha_{n}\right\}$ whose initial terms are $0,0,1, \beta$, where $\beta$ is chosen so as to make (60) equal 0, always results in

$$
\left|\begin{array}{llll}
\alpha_{n+2} & \alpha_{n+1} & \alpha_{n} & \alpha_{n-1} \\
\alpha_{n+1} & \alpha_{n} & \alpha_{n-1} & \alpha_{n-2} \\
\alpha_{n} & \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} \\
\alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \alpha_{n-4}
\end{array}\right|=0
$$

To obtain a more general form of (58), we first observe that the quartics on the left side of (57) and (58) involve seven adjacent terms in the sequences $\left\{M_{n}\right\}$ and $\left\{\mu_{n}\right\}$, respectively. We use the technique in the proof of (58) along with (37) to show that the terms of the quartic may be "spread out," so to speak, and that the number of terms involved may be as great as 16 . Specifically, we prove the following identity:

$$
\begin{align*}
& \left|\begin{array}{llll}
\mu_{n+m}+r & \mu_{n+p+r} & \mu_{n+q+r} & \mu_{n+r} \\
\mu_{n+m+s} & \mu_{n+p+s} & \mu_{n+q}+s & \mu_{n+s} \\
\mu_{n+m}+t & \mu_{n+p}+t & \mu_{n+q}+t & \mu_{n+t} \\
\mu_{n+m} & \mu_{n+p} & \mu_{n+q} & \mu_{n}
\end{array}\right|  \tag{61}\\
& =(-1)^{n-1}\left|\begin{array}{lll}
M_{r+1} & M_{r} & M_{r-1} \\
M_{s+1} & M_{s} & M_{s-1} \\
M_{t+1} & M_{t} & M_{t-1}
\end{array}\right|\left|\begin{array}{llll}
\mu_{m+3} & \mu_{p+3} & \mu_{q+3} & \mu_{3} \\
\mu_{m+2} & \mu_{p+2} & \mu_{q+2} & \mu_{2} \\
\mu_{m+1} & \mu_{p+1} & \mu_{q+1} & \mu_{1} \\
\mu_{m} & \mu_{p} & \mu_{q} & \mu_{0}
\end{array}\right|,
\end{align*}
$$

like (58) a quartic expression independent of $n$ except for sign.
Proof of (61): By (37) and (20), we have the following matrix equation:

$$
\begin{align*}
& {\left[\begin{array}{llll}
\mu_{n+m}+r & \mu_{n+p}+r & \mu_{n+q}+r & \mu_{n+r} \\
\mu_{n+m+s} & \mu_{n+p}+s & \mu_{n+q}+s & \mu_{n+s} \\
\mu_{n+m+t} & \mu_{n+p}+t & \mu_{n+q}+t & \mu_{n+t} \\
\mu_{n+m} & \mu_{n+p} & \mu_{n+q} & \mu_{n}
\end{array}\right]}  \tag{62}\\
& =\left[\begin{array}{llll}
M_{r+2} & N_{r+2} & S_{n+2} & M_{n+1} \\
M_{s+2} & N_{s+2} & S_{s+2} & M_{s+1} \\
M_{t+2} & N_{t+2} & S_{t+2} & M_{t+1} \\
1 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{lll}
\mu_{n+m} & \mu_{n+p} & \mu_{n+q} \\
\mu_{n+m-1} & \mu_{n+p-1} & \mu_{n+q-1} \\
\mu_{n+m-2} & \mu_{n+p-2} & \mu_{n-1} \\
\mu_{n+m-3} & \mu_{n+p-3} & \mu_{n+q-3} \\
\mu_{n-2} & \mu_{n-3}
\end{array}\right] \\
& =\left[\begin{array}{llll}
M_{r+2} & N_{r+2} & S_{r+2} & M_{r+1} \\
M_{s+2} & N_{s+2} & S_{s+2} & M_{s+1} \\
M_{t+2} & N_{t+2} & S_{t+2} & M_{t+1} \\
1 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]\left[\begin{array}{llll}
\mu_{m+3} & \mu_{p+3} & \mu_{q+3} & \mu_{3} \\
\mu_{m+2} & \mu_{p+2} & \mu_{q+2} & \mu_{2} \\
\mu_{m+1} & \mu_{p+1} & \mu_{q+1} & \mu_{1} \\
\mu_{m} & \mu_{p} & \mu_{q} & \mu_{0}
\end{array}\right]
\end{align*}
$$

We take determinants of both sides of (62) to obtain (61) since, by using (5) and (6) and well-known determinant properties, we can show that

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$$
\left|\begin{array}{llll}
M_{r+2} & N_{r+2} & S_{r+2} & M_{r+1} \\
M_{s+2} & N_{s+2} & S_{s+2} & M_{s+1} \\
M_{t+2} & N_{t+2} & S_{t+2} & M_{t+1} \\
1 & 0 & 0 & 0
\end{array}\right|=\left|\begin{array}{lll}
M_{r+1} & M_{r} & M_{r-1} \\
M_{s+1} & M_{s} & M_{s-1} \\
M_{t+1} & M_{t} & M_{t-1}
\end{array}\right|
$$

For the sequence $\left\{M_{n}\right\}$, (61) becomes.

$$
\left|\begin{array}{llll}
M_{n+m+r} & M_{n+p+r} & M_{n+q+r} & M_{n+r}  \tag{63}\\
M_{n+m+s} & M_{n+p+s} & M_{n+q+s} & M_{n+s} \\
M_{n+m+t} & M_{n+p+t} & M_{n+q+t} & M_{n+t} \\
M_{n+m} & M_{n+p} & M_{n+q} & M_{n}
\end{array}\right|=(-1)^{n-1}\left|\begin{array}{lll}
M_{r+1} & M_{r} & M_{r-1} \\
M_{s+1} & M_{s} & M_{s-1} \\
M_{t+1} & M_{t} & M_{t-1}
\end{array}\right|\left|\begin{array}{lll}
M_{m+1} & M_{m} & M_{m-1} \\
M_{p+1} & M_{p} & M_{p-1} \\
M_{q+1} & M_{q} & M_{q-1}
\end{array}\right|
$$

Several special cases of (61) are worth mentioning. First, let $q=t, s=$ $p=2 t, m=r=3 t, n$ arbitrary, to obtain

$$
\begin{align*}
& \left|\begin{array}{llll}
\mu_{n+6 t} & \mu_{n+5 t} & \mu_{n+4 t} & \mu_{n+3 t} \\
\mu_{n+5 t} & \mu_{n+4 t} & \mu_{n+3 t} & \mu_{n+2 t} \\
\mu_{n+4 t} & \mu_{n+3 t} & \mu_{n+2 t} & \mu_{n+t} \\
\mu_{n+3 t} & \mu_{n+2 t} & \mu_{n+t} & \mu_{n}
\end{array}\right|  \tag{64}\\
& =(-1)^{n-1}\left|\begin{array}{lll}
M_{3 t+1} & M_{2 t+1} & M_{t+1} \\
M_{3 t} & M_{2 t} & M_{t} \\
M_{3 t-1} & M_{2 t-1} & M_{t-1}
\end{array}\right|\left|\begin{array}{llll}
\mu_{3 t+3} & \mu_{2 t+3} & \mu_{t+3} & \mu_{3} \\
\mu_{3 t+2} & \mu_{2 t+2} & \mu_{t+2} & \mu_{2} \\
\mu_{3 t+1} & \mu_{2 t+1} & \mu_{t+1} & \mu_{1} \\
\mu_{3 t} & \mu_{2 t} & \mu_{t} & \mu_{0}
\end{array}\right|,
\end{align*}
$$

which displays an interesting symmetry.
Another special case of (61), which displays even greater symmetry, is obtained by letting $q=t=n, p=s=2 n, m=r=3 n$. We then have

$$
\begin{align*}
& \left|\begin{array}{llll}
\mu_{7 n} & \mu_{6 n} & \mu_{5 n} & \mu_{4 n} \\
\mu_{6 n} & \mu_{5 n} & \mu_{4 n} & \mu_{3 n} \\
\mu_{5 n} & \mu_{4 n} & \mu_{3 n} & \mu_{2 n} \\
\mu_{4 n} & \mu_{3 n} & \mu_{2 n} & \mu_{n}
\end{array}\right|  \tag{65}\\
& =(-1)^{n-1}\left|\begin{array}{lll}
M_{3 n+1} & M_{2 n+1} & M_{n+1} \\
M_{3 n} & M_{2 n} & M_{n} \\
M_{3 n-1} & M_{2 n-1} & M_{n-1}
\end{array}\right|\left|\begin{array}{llll}
\mu_{3 n+3} & \mu_{2 n+3} & \mu_{n+3} & \mu_{3} \\
\mu_{3 n+2} & \mu_{2 n+2} & \mu_{n+2} & \mu_{2} \\
\mu_{3 n+1} & \mu_{2 n+1} & \mu_{n+1} & \mu_{1} \\
\mu_{3 n} & \mu_{2 n} & \mu_{n} & \mu_{0}
\end{array}\right| .
\end{align*}
$$

Note how all terms in the determinant on the left of (65) are $n$ units apart, whereas those on the right occur contiguously in groups of three or four, and the groups are $n-3$ units apart.

## 5. Concluding remarks

Many number-theoretic properties for the Fibonacci sequence quite expectedly do not extend to the Tetranacci sequence. However, the following divisibility properties hold:

$$
\begin{align*}
& M_{5 n-1} \equiv M_{5 n} \equiv M_{5 n+1} \equiv 0(\bmod 2),  \tag{66}\\
& M_{5 n-2} \equiv M_{5 n+2} \equiv 1(\bmod 2),  \tag{67}\\
& M_{5 n} \equiv M_{5 n+1} \equiv 0(\bmod 4),  \tag{68}\\
& M_{5 n-2} \equiv 1(\bmod 4) \tag{69}
\end{align*}
$$

Proof of (66) and (67): We consider the sequence $\left\{M_{n}\right\}$ (mod 2) and display the results in the following table:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $M_{n}(\bmod 2)$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |

From the table, it is clear that $\left\{M_{n}\right\}$ (mod 2) starts to repeat after five terms and, since the pattern of zeros and ones will then continue to repeat in the same order, we have

$$
\begin{aligned}
& M_{4} \equiv M_{5 n-1} \equiv 0(\bmod 2), \quad M_{5} \equiv M_{5 n} \equiv 0(\bmod 2), \quad M_{6} \equiv M_{5 n+1} \equiv 0(\bmod 2), \\
& M_{3} \equiv M_{5 n-2} \equiv 1(\bmod 2), \quad M_{2} \equiv M_{5 n+2} \equiv 1(\bmod 2) .
\end{aligned}
$$

Since by (66), $M_{5 n-1}, M_{5 n}, M_{5 n+1}$ are even, it is clear that three arbitrary adjacent terms of the Tetranacci sequence may have greatest common divisor greater than one. However, we can show that the greatest common divisor of

$$
M_{n}, M_{n+1}, M_{n+2}, M_{n+3},
$$

any four consecutive terms of $\left\{M_{n}\right\}$, is one.
This paper, quite clearly, is not intended as an exhaustive treatment of properties of the Tetranacci sequence and generalizations. Some fundamental identities and sufficient other results and techniques for proving them are given to indicate the rich and remarkable nature of this sequence and generalizations.

## Acknowledgment

The author gratefully acknowledges the suggestion of the referee to use an alternate, more concise proof of (58) and (61).

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Announcement
FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

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University of St. Andrews
St. Andrews KY16 9SS
Fife, Scotland

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## CALL FOR PAPERS

The FIFTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of St. Andrews, St. Andrews, Scotland from July 20 to July 24, 1992. This Conference is sponsored jointly by the Fibonacci Association and The University of St. Andrews.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts are to be submitted by March 15, 1992. Manuscripts are due by May 30, 1992. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of The Fibonacci Quarterly to:

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The Fibonacci Quarterly
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# NEGATIVE ORDER GENOCCHI POLYNOMIALS 

## A. F. Horadam

University of New England, Armidale, Australia (Submitted January 1990)

## 1. Introduction

Elsewhere [2], I have investigated the properties of $G_{n}^{(k)}(x)$, the Genocchi polynomials of order $k(\geq 0)$, which were shown to be related to $E_{n}^{(k)}(x)$, the Euler polynomials of order $k$, and to $B_{n}^{(k)}(x)$, the Bernoulli polynomials of order k。

When $k=1$, we have the Genocchi polynomials of the first order, the simplest polynomials of Genocchi type.

If $x=0$, the Genocchi numbers arise.
Following Nörlund ([4] and [5]), who pioneered the study of $B_{n}^{(-k)}(x)$ and $E_{n}^{(-k)}(x)$, the Bernoulli and Euler polynomials, respectively, of negative order, I here offer some of the most important properties of $G_{n}^{(-k)}(x)$, the Genocchi polynomiats of order $-k(k>0, n \geq-k)$. So far as I am aware, the material in this contribution represents new information.

The justification for seeking knowledge about the negative order polynomials is stated by Nörlund [4]. After saying that there is advantage in extending to negative order the notion of functions of positive order, Nörlund continues: "On peut ainsi faire rentrer dans un même cadre des fonctions qui apparaissent jusqu'ici comme distinctes." [We can thus combine in one framework functions which up to now appear as distinct.]

Beyond this justification, I feel that the $G_{n}^{(-k)}(x)$ have a vitality of their own which deserves recognition.

## Euler and Bernoulli Polynomials of Negative Order

Nörlund ([4] and [5]) defines the Euler polynomials of negative order $-k$ by

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{t^{n}}{n!} E_{n}^{(-k)}\left(x \mid w_{1} \ldots w_{k}\right)=\frac{\left(e^{w_{1} t}+1\right) \cdots\left(e^{w_{k} t}+1\right) e^{t x}}{2^{k}} \tag{1.1}
\end{equation*}
$$

and the Bernoulli polynomials of negative order $-k$ by

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{t^{n}}{n!} B_{n}^{(-k)}\left(x \mid w_{1} \ldots w_{k}\right)=\frac{\left(e^{w_{1} t}-1\right) \ldots\left(e^{w_{k} t}-1\right) e^{t x}}{w_{1} \ldots w t} \tag{1.2}
\end{equation*}
$$

If $w_{1}=w_{2}=\ldots=w_{k}=1$, then (1.1) and (1.2) become
(1.1)' $\sum_{n=0}^{\infty} \frac{t^{n}}{n!} E_{n}^{(-k)}(x)=\left(\frac{e^{t}+1}{2}\right)^{k} e^{t x}$
and
$(1.2)^{\prime} \sum_{n=0}^{\infty} \frac{t^{n}}{n!} B_{n}^{(-k)}(x)=\left(\frac{e^{t}-1}{t}\right)^{k} e^{t x}$.
The definition to be given in (2.1) for Genocchi polynomials follows the modified forms (1.1)' and (1.2)', though an extension to the patterns in (1.1) and (1.2) could be adopted.

For subsequent comparison with corresponding forms for $G_{n}^{(-k)}(x)(k=1,2,3$, ...), the first few expressions for $E_{n}^{(-k)}(x)$ and $B_{n}^{(-k)}(x)$ are:

$$
\begin{align*}
& E_{0}^{(-k)}(x)=1  \tag{1.3}\\
& E_{1}^{(-k)}(x)=x+\frac{1}{2} k \\
& E_{2}^{(-k)}(x)=x^{2}+k x+\frac{k(k+1)}{4} \\
& E_{3}^{(-k)}(x)=x^{3}+\frac{3}{2} k x^{2}+\frac{3 k(k+1)}{4} x+\frac{k^{2}(k+3)}{8} \\
& E_{4}^{(-k)}(x)=x^{4}+2 k x^{3}+\frac{3 k(k+1)}{2} x^{2}+\frac{k^{2}(k+3)}{2} x+\frac{k(k+1)\left(k^{2}+5 k-2\right)}{16}
\end{align*}
$$

and

$$
\begin{align*}
& B_{0}^{(-k)}(x)=1  \tag{1.4}\\
& B_{1}^{(-k)}(x)=x+\frac{k}{2} \\
& B_{2}^{(-k)}(x)=x^{2}+k x+\frac{k(3 k+1)}{12} \\
& B_{3}^{(-k)}(x)=x^{3}+\frac{3}{2} k x^{2}+\frac{k(3 k+1)}{4} x+\frac{k^{2}(k+1)}{8} \\
& B_{4}^{(-k)}(x)=x^{4}+2 k x^{3}+\frac{k(3 k+1)}{2} x^{2}+\frac{k^{2}(k+1)}{2} x+\frac{k\left(15 k^{3}+30 k^{2}+5 k-2\right)}{240}
\end{align*}
$$

Putting $k=1$, we readily derive the table:

$$
\begin{array}{lcc} 
& E_{n}^{(-1)}(x) & B_{n}^{(-1)}(x)  \tag{1.5}\\
n=0 & 1 & 1 \\
n=1 & x+\frac{1}{2} & x+\frac{1}{2} \\
n=2 & x^{2}+x+\frac{1}{2} & x^{2}+x+\frac{1}{3} \\
n=3 & x^{3}+\frac{3}{2} x^{2}+\frac{3}{2} x+\frac{1}{2} & x^{3}+\frac{3}{2} x^{2}+x+\frac{1}{4} \\
n=4 & x^{4}+2 x^{3}+3 x^{2}+2 x+\frac{1}{2} & x^{4}+2 x^{3}+2 x^{2}+x+\frac{1}{5}
\end{array}
$$

## 2. Generalized Genocchi Polynomials of Negative Order

## Definition and Basic Properties

Define

$$
\begin{equation*}
\sum_{n=-k}^{\infty} G_{n}^{(-k)}(x) \frac{t^{n}}{|n|!}=\left(\frac{1+e^{t}}{2 t}\right)^{k} e^{t x} \quad(k=1,2,3, \ldots) \tag{2.1}
\end{equation*}
$$

whence
$(2.1)^{\prime} G_{n}^{(-k)}(x)$ is undefined when $n<-k$,
i.e., $n+k \geq 0$ is necessary for the existence of $G_{n}^{(-k)}(x)$.

Putting $k=0$ in (2.1) leads to the situation covered in [2] when $k=0$, so we exclude this repetition.

Calculation in (2.1) gives us the first few Genocchi polynomials:

$$
\begin{align*}
& G_{-k}^{(-k)}(x)=|-k|!  \tag{2.2}\\
& G_{-k+1}^{(-k)}(x)=|-k+1|!\left\{x+\frac{1}{2} k\right\} \\
& G_{-k+2}^{(-k)}(x)=\frac{|-k+2|!\left\{x^{2}+k x+\frac{k(k+1)}{2!}\right\}}{2}
\end{align*}
$$

$$
\begin{aligned}
& G_{-k+3}^{(-k)}(x)= \frac{|-k+3|!}{3!}\left\{x^{3}+\frac{3 k}{2} x^{2}+\frac{3 k(k+1)}{4} x+\frac{k^{2}(k+3)}{8}\right\} \\
& G_{-k+4}^{(-k)}(x)= \frac{|-k+4|!}{4!}\left\{x^{4}+2 k x^{3}\right. \\
&+\frac{3 k(k+1)}{2} x^{2}+\frac{k^{2}(k+3)}{2} x \\
&\left.+\frac{k(k+1)\left(k^{2}+5 k-2\right)}{16}\right\}
\end{aligned}
$$

In particular, when $k=1$ :
(2.3) $\quad G_{-1}^{(-1)}(x)=1$

$$
\begin{aligned}
G_{0}^{(-1)}(x) & =x+\frac{1}{2} \\
G_{1}^{(-1)}(x) & =\frac{1}{2}\left\{x^{2}+x+\frac{1}{2}\right\} \\
G_{2}^{(-1)}(x) & =\frac{1}{3}\left\{x^{3}+\frac{3}{2} x^{2}+\frac{3}{2} x+\frac{1}{2}\right\}=\frac{1}{3}\left(x+\frac{1}{2}\right)\left(x^{2}+x+1\right) \\
G_{3}^{(-1)}(x) & =\frac{1}{4}\left\{x^{4}+2 x^{3}+3 x^{2}+2 x+\frac{1}{2}\right\} \\
G_{4}^{(-1)}(x) & =\frac{1}{5}\left\{x^{5}+\frac{5}{2} x^{4}+5 x^{3}+5 x^{2}+\frac{5}{2} x+\frac{1}{2}\right\} \\
& =\frac{1}{5}\left(x+\frac{1}{2}\right)\left(x^{4}+2 x^{3}+4 x^{2}+3 x+1\right)
\end{aligned}
$$

The Genocchi numbers $G_{n}^{(-1)}(n \geq 0)$ thus form the sequence $(2.3), \frac{1}{2}\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$,
while
(2.3) ${ }^{\prime \prime} G_{n-1}^{(-1)} \div G_{n}^{(-1)}=\frac{n+1}{n} \rightarrow 1$ as $n \rightarrow \infty$.

Comparison of (2.1) with (1.1)' reveals that

$$
\begin{equation*}
G_{n}^{(-k)}(x)=\frac{|n|!}{(n+k)!} E_{n+k}^{(-k)}(x) \tag{2.4}
\end{equation*}
$$

Differentiating both sides of (2.1) w.r.t. $x$ leads to the Appell property
[2], $\quad \frac{d G_{n}^{(-k)}(x)}{d x}=n G_{n-1}^{(-k)}(x), n+k>1, n>0$,
$\begin{aligned} & \text { whence } \\ & \text { (2.6) }\end{aligned} \frac{d^{p} G_{n}^{(-k)}(x)}{d x^{p}}=n(n-1) \cdots(n-p+1) G_{n-p}^{(-k)}(x), \quad n-p \geq 0$,
so that, using (2.3), we have
(2.7) $\quad \frac{d^{n+1} G_{n}^{(-k)}(x)}{d x^{n+1}}=n$ !

Integration of (2.5) gives (with $n \rightarrow n+1$ ):

$$
\begin{equation*}
\int_{x}^{x+1} G_{n}^{(-k)}(x) d x=\frac{G_{n+1}^{(-k)}(1+x)-G_{n+1}^{(-k)}(x)}{n+1} \tag{2.8}
\end{equation*}
$$

## Summation Formula

Theorem 1:

$$
\begin{equation*}
G_{n}^{(-k)}(x+y)=\sum_{j=-k}^{n} \frac{|n|!}{(n-j)!|j|!} G_{j}^{(-k)}(x) y^{n-j} \tag{2.9}
\end{equation*}
$$

Proof:

$$
\begin{aligned}
& \sum_{n=-k}^{\infty} G_{n}^{(-k)}(x+y) \frac{t^{n}}{|n|!}=\left(\frac{1+e^{t}}{2 t}\right)^{k} e^{t x} e^{t y}=\sum_{r=-k}^{\infty} G_{r}^{(-k)}(x) \frac{t^{r}}{|r|!} \sum_{m=0}^{\infty} \frac{y^{m} t^{m}}{m!} \\
&= \sum_{n=-k}^{\infty} \sum_{j=-k}^{n} \frac{|n|!}{(n-j)!|j|!} G_{j}^{(-k)}(x) y^{n-j} \frac{t}{|n|!} \\
& \text { after rearranging the terms. }
\end{aligned}
$$

Equate coefficients of $t^{n} /|n|!$ and the result follows.
For example, if $k=n=y=2$, both sides of the formula (2.9) lead to the expression, also derivable from (2.2),

$$
G_{2}^{(-2)}(x+2)=\frac{1}{12} x^{4}+x^{3}+4 \frac{3}{4} x^{2}+10 \frac{1}{2} x+9 \frac{1}{24} .
$$

Furthermore, if $k=3, n=1, x=0$, and $y$ is replaced by $x$, then (2.9) gives

$$
G_{-1}^{(-3)}(x)=\frac{1}{2}\left(x^{2}+3 x+3\right)
$$

in conformity with (2.2).

## Complementary Arguments

We say that $x$ and $-k-x$ are complementary arguments.
Theorem 2:

$$
\begin{equation*}
G_{n}^{(-k)}(-k-x)=(-1)^{n+k} G_{n}^{(-k)}(x) \tag{2.10}
\end{equation*}
$$

Proof:

$$
\begin{aligned}
\sum_{n=-k}^{\infty} G_{n}^{(-k)}(-k-x) \frac{t^{n}}{|n|!} & =\left(\frac{1+e^{t}}{2 t}\right)^{k} e^{(-k-x) t}=(-1)^{k}\left(\frac{1+e^{-t}}{2(-t)}\right)^{k} e^{-t x} \\
& =(-1)^{k} \sum_{n=-k}^{\infty}(-1)^{n} G_{n}^{(-k)}(x) \frac{t^{n}}{|n|!} \\
& =(-1)^{n+k} \sum_{n=-k}^{\infty} G_{n}^{(-k)}(x) \frac{t^{n}}{|n|!}
\end{aligned}
$$

Comparison of the coefficients of $t^{n} /|n|$ ! yields the result.
Corollary 1:

$$
G_{n}^{(-k)}(-k-x)= \begin{cases}G_{n}^{(-k)}(x) & \text { if } k+n \text { is even }  \tag{2.11}\\ -G_{n}^{(-k)}(x) & \text { if } k+n \text { is odd }\end{cases}
$$

Special cases of interest occur when $x=0$ and (equivalently) $x=-k$. In either of these instances, consider also $k=1$.
Corollary 2: In Theorem 2, replace $x$ by $x-(k / 2)$. Then

$$
\begin{align*}
& G_{n}^{(-k)}\left(-x-\frac{k}{2}\right)=(-1)^{n+k} G_{n}^{(-k)}\left(x-\frac{k}{2}\right)  \tag{2.12}\\
& \quad \text { If } x=0 \text { in Corollary } 2 \text { (or } x=-k / 2, k+n \text { odd, in Corollary } 1),
\end{align*}
$$

then

$$
\begin{equation*}
G_{n}^{(-k)}\left(-\frac{k}{2}\right)=0, \quad k+n \text { odd } \tag{2.13}
\end{equation*}
$$

i.e., $G_{n}^{(-k)}(x)$ has a zero when $x=-k / 2$ for $k+n$ odd.

Thus, in (2.2), $G_{-k+\ell}^{(-k)}(x)$ has a zero when $x=-k / 2$ for $\ell$ odd.

## Analogue of the Multiplication Theorem

More accurately, this analogue of the multiplication theorem [2] could be called a "division theorem" for negative first order Genocchi polynomials. As in [2], there are two cases to consider, one of which involves $B_{n}^{(-1)}(x)$. Unfortunately, as for $k>0$, this theorem does not extend beyond $k=-1$.

Case I: m odd
Theorem 3a:

$$
\begin{align*}
& G_{n}^{(-1)}\left(\frac{x-1}{m}\right)=-m^{-n-1} \sum_{s=-1}^{m-2}(-1)^{s} G_{n}^{(-1)}(x+s)  \tag{2.14}\\
& \sum_{n=-1}^{\infty} \frac{t^{n}}{|n|!} \sum_{s=-1}^{m-2}(-1)^{s} G_{n}^{(-1)}(x+s)=\sum_{s=-1}^{m-2} \frac{1+e^{t}}{2 t}(-1)^{s} e^{t x} e^{s t} \\
& =\frac{1+e^{t}}{2 t} e^{t x}\left(-e^{-t}+1-e^{t}+\cdots+(-1)^{m-2} e^{(m-2) t}\right) \\
& =\frac{1+e^{t}}{2 t} e^{t x}\left(-e^{-t}\right)\left(1-e^{t}+e^{2 t}-\cdots+(-1)^{m-1} e^{(m-1) t}\right) \\
& =-\frac{1+e^{t}}{2 t} e^{t(x-1)} \cdot \frac{1+e^{m t}}{1+e^{t}}, \text { since } m \text { is odd } \\
& =-m\left(\frac{1+e^{m t}}{2 m t}\right) e^{\frac{m t(x-1)}{m}}=-\sum_{n=-1}^{\infty} m \frac{(m t)^{n}}{|n|!} G_{n}^{(-1)}\left(\frac{x-1}{m}\right) .
\end{align*}
$$

Therefore,

$$
G_{n}^{(-1)}\left(\frac{x-1}{m}\right)=-m^{-n-1} \sum_{s=-1}^{m-2}(-1)^{s} G_{n}^{(-1)}(x+s), \quad m \text { odd }
$$

Case II: m even
Theorem 3b:

$$
\begin{equation*}
B^{(-1)}\left(\frac{x-1}{m}\right)=2 m^{-n-1} \sum_{s=-1}^{m-2}(-1)^{s} G_{n}^{(-1)}(x+s) \tag{2.15}
\end{equation*}
$$

Proof:

$$
\begin{aligned}
& \sum_{n=-1}^{\infty} \frac{t^{n}}{|n|!} \sum_{s=-1}^{m-2}(-1)^{s} G_{n}^{(-1)}(x+s) \\
& =\frac{1+e^{t}}{2 t} e^{t x} \cdot-e^{-t}\left(1-e^{t}+e^{2 t}-e^{3 t}+\cdots+(-1)^{m-1} e^{(m-1) t}\right) \text {, as in } \text { Theorem 3a } \\
& =-\frac{1+e^{t}}{2 t} e^{t(x-1)} \frac{1-e^{m t}}{1+e^{t}}, \quad \text { since } m \text { is even } \\
& =-\frac{e^{t(x-1)}}{2 t}\left(1-e^{m t}\right)=m \cdot \frac{1}{2} \cdot \frac{e^{m t}-1}{m t} \cdot e^{\frac{m t(x-1)}{m}} \\
& =\frac{m}{2} \sum_{n=0}^{\infty} \frac{(m t)^{n}}{n!} B_{n}^{(-1)}\left(\frac{x-1}{m}\right), \quad \text { on using }(1.2)^{\prime} \\
& =\frac{m^{n+1}}{2} \sum_{n=0}^{\infty} \frac{t^{n}}{n!} B_{n}^{(-1)}\left(\frac{x-1}{m}\right) .
\end{aligned}
$$

Equate corresponding coefficients of $t^{n} / n!$ and the result follows. It is to be noted that, in the left-hand side summation, $n=-1$ and $m$ even lead to the term

$$
\frac{1}{1} \cdot \frac{m}{n}(-1+1)=0
$$

## Relations between Polynomials of Successive Orders

Theorem 4:

$$
G_{n}^{(-1)}(x-1)+G_{n}^{(-1)}(x)= \begin{cases}2 n G_{n-1}^{(-2)}(x-1) & n=1,2,3, \ldots,  \tag{2.16}\\ \frac{2}{|n-1|!} G_{n-1}^{(-2)}(x-1) & n=-1,0\end{cases}
$$

Proof:

$$
\begin{aligned}
& \sum_{n=-1}^{\infty}\left[G_{n}^{(-1)}(x-1)+G_{n}^{(-1)}(x)\right] \frac{t^{n}}{|n|!}=\left(\frac{1+e^{t}}{2}\right) e^{t x-t}+\left(\frac{1+e^{t}}{2}\right) e^{t x} \\
& =\frac{1+e^{t}}{2 t} e^{t x}\left(1+e^{-t}\right)=2 t\left(\frac{1+e^{t}}{2 t}\right)^{2} e^{t(x-1)} \\
& =2 G_{-2}^{(-2)}(x-1) \frac{t^{-1}}{|-2|!}+2 G_{-1}^{(-2)}(x-1) \frac{t^{0}}{|-1|!}+\sum_{n=1}^{\infty} 2 n G_{n-1}^{(-2)}(x-1) \frac{t^{n}}{n!}
\end{aligned}
$$

Equate coefficients of $t^{n} /|n|!$ and the result follows.
Clearly, the result can be extended to $G_{n}^{(-k)}(x)$.
With $x \rightarrow x+1$ in Theorem 4 , we have
Theorem 5:

$$
G_{n}^{(-1)}(1+x)+G_{n}^{(-1)}(x)= \begin{cases}2 n G_{n-1}^{(-2)}(x) & n=1,2,3, \ldots,  \tag{2.17}\\ \frac{2}{|n-1|!} G_{n-1}^{(-2)}(x) & n=-1,0,\end{cases}
$$

with a straightforward extension to $n=-k$ if desired.
A companion result is
Theorem 6:

$$
\begin{equation*}
G_{n}^{(-1)}(1+x)-G_{n}^{(-1)}(x)=2^{n} B_{n}^{(-1)}\left(\frac{x}{2}\right), \quad(n \geq 0) . \tag{2.18}
\end{equation*}
$$

Proof:

$$
\begin{aligned}
{\left[\sum_{n=0}^{\infty} G_{n}^{(-1)}(1+x)-G_{n}^{(-1)}(x)\right] \frac{t^{n}}{n!} } & =\left(\frac{1+e^{t}}{2 t}\right)\left(e^{t}-1\right) e^{t x} \\
& =\frac{e^{2 t}-1}{2 t} e^{2 t \cdot \frac{x}{2}} \\
& =2^{n} \sum_{n=0}^{\infty} B_{n}^{(-1)}\left(\frac{x}{2}\right) \frac{t^{n}}{n!}, \text { on using (1.2)', }
\end{aligned}
$$

from which the formula follows.
To generalize Theorem 6, we need to expand ( $\left.e^{t}-1\right)^{k}$. After suitable algebraic manipulation, it ensues as in the proof of Theorem 6 that

$$
\begin{equation*}
\sum_{j=0}^{k}(-1)^{j-1}\binom{k}{j} G_{n}^{(-k)}(j+x)=(-1)^{k+1} 2^{n} B_{n}^{(-k)}\left(\frac{x}{2}\right) \quad(n \geq 0) \tag{2.19}
\end{equation*}
$$

Theorem 7:

$$
\begin{equation*}
(n+1) G_{n}^{(-1)}(x)=n(x+1) G_{n-1}^{(-1)}(x)-\frac{1}{2} G_{n}^{(0)}(x) \quad(n \geq 1) \tag{2.20}
\end{equation*}
$$

Proof: Differentiate both sides of (2.1) for $k=1$ w.r.t. $t$ partially, and then multiply by $t$. It follows that

$$
\begin{aligned}
-\frac{1}{t}+\sum_{n=1}^{\infty} G_{n}^{(-1)}(x) \frac{n t^{n}}{n!} & =\left(\frac{1+e^{t}}{2}\right) e^{t x} \cdot x t+\frac{e^{t x}}{2}\left(\frac{t e^{t}-\left(1+e^{t}\right)}{t}\right) \\
& =\left(\frac{1+e^{t}}{2 t}\right) e^{t x} \cdot x t-\left(\frac{1+e^{t}}{2 t}\right) e^{t x}+\frac{e^{t x}}{2 t} t\left(1+e^{t}-1\right) \\
& =\left(\frac{1+e^{t}}{2 t}\right) e^{t x} t(x+1)-\left(\frac{1+e^{t}}{2 t}\right) e^{t x}-\frac{e^{t x}}{2} .
\end{aligned}
$$

Equate coefficients of $t^{n} / n$ ! and the result follows. Observe (see [2]) that $G_{n}^{(0)}(x)=x^{n}$.

The $n=0$ term, being a constant, does not contribute to the summation on differentiation w.r.t. $t$ partially.

Proceeding in the same manner, we may establish the generalization

$$
\begin{equation*}
(n+k) G_{n}^{(-k)}(x)=n(k+x) G_{n-1}^{(-k)}(x)-\frac{k}{2} G_{n}^{-(k-1)}(x) \quad(n \geq 1) \tag{2.21}
\end{equation*}
$$

In particular, when $k=2$, the left-hand side of the first line of the proof in Theorem 7 (after partial differentiation and multiplication by $t$ ) becomes

$$
-\frac{2}{t^{2}}-\frac{(1+x)}{t}+0+\sum_{n=1}^{\infty} G_{n}^{(-2)}(x) \frac{n t}{n!}
$$

since the $n=0$ term does not contribute, being a constant as far as partial differentiation w.r.t. $t$ is concerned.
$G_{n}^{(-k)}(x)$ in Terms of $G_{m}^{(-1)}(f(x))$
Adopting a different technique, we are enabled to derive formulas connecting $G_{n}^{(-k)}(x)$ with negative first order Genocchi polynomials of appropriate functions $f(x)$ of $x$. When $k=2$, 3 , we have
Theorem 8: If $n \geq 0$,

$$
\begin{array}{ll}
2(n+1) G_{n}^{(-2)}(x) & =2\left\{2^{n+1} G_{n+1}^{(-1)}\left(\frac{x}{2}\right)+G_{n+1}^{(-1)}(x)\right\}-\frac{x^{n+2}}{n+2} \\
4(n+2)(n+1) G_{n}^{(-3)}(x) & =3\left\{3^{n+1} G_{n+2}^{(-1)}\left(\frac{x}{3}\right)+G_{n+2}^{(-1)}(x+1)\right\}
\end{array}
$$

Proof: Consider

$$
\begin{equation*}
\left(\frac{1+e^{t}}{2 t}\right)^{2} e^{t x}=\frac{2}{2 t}\left(\frac{1+e^{2 t}}{2 \cdot 2 t}\right) e^{2 t \cdot \frac{x}{2}}+\frac{2}{2 t}\left(\frac{1+e^{t}}{2 t}\right) e^{t x}-\frac{e^{t x}}{2 t^{2}} \tag{2.23}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\frac{1+e^{t}}{2 t}\right)^{3} e^{t x}=\frac{3}{4 t^{2}}\left(\frac{1+e^{3 t}}{2 \cdot 3 t}\right) e^{3 t \cdot \frac{x}{3}}+\frac{3}{4 t^{2}}\left(\frac{1+e^{t}}{2 t}\right)^{t(x+1)} \tag{2.24}
\end{equation*}
$$

Equate coefficients of $t^{n} / n!$ and the results follow. $\left(x^{n+2}=G_{n+2}^{(0)}(x)\right.$ by [2].)

Determination of the somewhat complicated extensions of (2.22) for general $k$ is left to the curiosity of the reader. Depending on the parity of $k$, we will obtain two separate expressions in the generalization. Nevertheless, there is a unifying principle in the proof, namely, the grouping of pairs of appropriate terms; when $k$ is even, there will be additionally a single unpaired term.

Similar kinds of results may be obtained for $E_{n}^{(-k)}(x)$ and $B_{n}^{(-k)}(x)$ on using (1.1)' and (1.2)'. However, in the case of Bernoulli polynomials we remark that, for $k$ even, $B_{n}^{(-k)}(x)$ is expandable in terms of Genocchi polynomials.
$G_{n}^{(-1)}(x)$ in Terms of $G_{m}^{(-1)}\left(\frac{1}{2}\right)$

## Theorem 9:

$$
\begin{equation*}
G_{n}^{(-1)}(x)=\sum_{r=-1}^{n} \frac{|n|!}{(n-r)!|r|!} G_{r}^{(-1)}\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^{n-r} . \tag{2.25}
\end{equation*}
$$

Proof:

$$
\sum_{n=-1}^{\infty} G_{n}^{(-1)}(x) \frac{t^{n}}{|n|!}=\left(\frac{1+e^{t}}{2 t}\right) e^{t x}=\left(\frac{1+e^{t}}{2 t}\right) e^{\frac{t}{2}} \cdot e^{\left(x-\frac{1}{2}\right) t}
$$

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$$
=\sum_{s=-1}^{\infty} \frac{G_{-1}^{(-1)}\left(\frac{1}{2}\right)}{|-1|!t} \cdot \frac{\left(x-\frac{1}{2}\right)^{s+1} t^{s+1}}{(s+1)!}+\left\{\sum_{r=0}^{\infty} G^{(-1)}\left(\frac{1}{2}\right) \frac{t}{r!}\right\}\left\{\sum_{m=0}^{\infty}\left(x-\frac{1}{2}\right)^{m} \frac{t^{m}}{m!}\right\} .
$$

Application of Cauchy's multiplication of power series and comparison of coefficients of $t^{n} / n$ ! yield the desired result.

Sums of Products
What happens if we square both sides of (2.1)? Clearly,

$$
\begin{align*}
\left(\sum_{n=-1}^{\infty} G_{n}^{(-1)}(x) \frac{t^{n}}{|n|!}\right)\left(\sum_{n=-1}^{\infty} G_{n}^{(-1)}(x) \frac{t^{n}}{|n|!}\right) & =\left(\frac{1+e^{t}}{2}\right)^{2} e^{t \cdot 2 x}  \tag{2.26}\\
& =\sum_{n=-2}^{\infty} G_{n}^{(-2)}(2 x) \frac{t^{n}}{|n|!}
\end{align*}
$$

Comparison of coefficients of $t^{n} /|n|!$ yields a set of sums of products, expressible in general form as
(2.27) $G_{n}^{(-2)}(2 x)= \begin{cases}2 \sum_{j=-1}^{\left[\frac{n}{2}\right]} G_{j}^{(-1)}(x) \frac{G_{n-j}^{(-1)}(x)}{|n-j|!} & n \text { odd, } \\ \left.2 \frac{n-1}{2}\right] \\ \sum_{j=-1}^{2} G_{j}^{(-1)}(x) \frac{G_{n-j}^{(-1)}(x)}{|n-j|!}+G_{n / 2}^{(-1)}(x) & n \text { even. }\end{cases}$

Furthermore, if we replace $t$ by $-t$ in one of the infinite sums in (2.26), we find

$$
\begin{align*}
\left(\sum_{n=-1}^{\infty} G_{n}^{(-1)}(x) \frac{t^{n}}{|n|!}\right)\left(\sum_{n=-1}^{\infty} G_{n}^{(-1)}(x) \frac{(-t)^{n}}{|n|!}\right) & =-\left(\frac{1+e^{t}}{2 t}\right)^{2} e^{-t}  \tag{2.28}\\
& =-\sum_{n=-2}^{\infty} G_{n}^{(-2)}(-1) \frac{t^{n}}{|n|!}
\end{align*}
$$

leading to formulas for $G_{n}^{(-2)}(-1)$ similar to those in (2.27). Observe that $G_{n}^{(-2)}(-1)=0$ when $n$ is odd, by (2.13).
Putting $x=-1 / 2$ in (2.27), we also obtain formulas for $G_{n}^{(-2)}(-1)$ in terms of $G_{m}^{(-1)}(-1 / 2)$.

Interested readers may wish to extend the above theory to unspecified $k$ in $G_{n}^{(-k)}(x)$. Additionally, one may determine results corresponding to those in (2.27) for Euler and Bernoulli polynomials.

## 3. Miscellaneous Theorems

## Use of Boole's Theorem

For a polynomial $P(x)$, Boole's theorem states that

$$
P(x+y)=\nabla P(x)+E_{1}(y) \nabla P^{\prime}(x)+\frac{1}{2!} E_{2}(y) \nabla P^{\prime \prime}(x)+\frac{1}{3!} E_{3}(y) \nabla P^{\prime \prime \prime}(x)+\cdots,
$$

where the symbol $\nabla$ ('nabla') represents the operation of the mean of the function (see [2]) and $E_{i}(x)(i=1,2,3, \ldots)$ are the Euler polynomials $E_{i}^{(1)}(x)$ obtained from (1.3) by replacing $k$ by -1 . Prime superscripts signify differentiation w.r.t. $x$.

Now

$$
\nabla G_{n}^{(-1)}(x)=\frac{1}{2}\left(G_{n}^{(-1)}(1+x)+G_{n}^{(-1)}(x)\right) \quad \text { by the definition of } \nabla
$$

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$$
=\left\{\begin{array}{ll}
n G_{n-1}^{(-2)}(x) & (n=1,2,3, \ldots) \\
\frac{1}{|n-1|!} G_{n-1}^{(-2)}(x) & (n=-1,0)
\end{array} \quad \text { by Theorem } 5\right.
$$

Put $y=0$ in Boole's theorem and take $P(x)=G_{n}^{(-1)}(x)$.
Then Boole's theorem becomes, for $n>0$ (2.5),

$$
G_{n}^{(-1)}(x)=\nabla G_{n}^{(-1)}(x)+E_{1}(0) \nabla G_{n}^{(-1)^{\prime}}(x)+\frac{1}{2!} E_{2}(0) \nabla G_{n}^{(-1)^{\prime \prime}}(x)+\cdots,
$$

that is,
Theorem 10: When $n=1,2,3, \ldots$,

$$
\begin{equation*}
G_{n}^{(-1)}=n G_{n-1}^{(-2)}(x)+E_{1}(0) \cdot n G_{n-1}^{(-2)^{\prime}}(x)+\frac{1}{2!} E_{2}(0) \cdot n G_{n-1}^{(-2)^{\prime \prime}}(x)+\cdots . \tag{3.1}
\end{equation*}
$$

For example, if $n=2$, the right-hand side reduces to

$$
\frac{1}{3}\left(x^{3}+\frac{3}{2} x^{2}+\frac{3}{2} x+\frac{1}{2}\right) \quad\left[=G_{2}^{(-1)}(x) \text { as in (2.3) }\right]
$$

## Genocchi Polynomials in Terms of Bernoulli Polynomials

The Euler-Maclaurin theorem (see [3]) states, in the case of polynomials $G_{n}^{(-1)}(x)$, that

$$
G_{n}^{(-1)^{\prime}}(x)=\Delta G_{n}^{(-1)}(0)+B_{1}(x) \Delta G_{n}^{(-1)^{\prime}}(x)+\frac{B_{2}(x)}{2!} \Delta G_{n}^{(-1)^{\prime \prime}}(0)+\cdots,
$$

where $B_{i}(x) \quad(i=1,2,3, \ldots)$ are the Bernoulli polynomials $B_{i}^{(1)}(x)$ obtained from (1.4) by replacing $k$ by -1 and $\Delta$ is the symbol for the operation of taking the difference.

Now, by (2.5),

$$
G_{n}^{(-1)^{\prime}}(x)=n G_{n-1}^{(-1)}(x) \quad(n>0)
$$

and, by the definition of $\Delta$,

$$
\begin{align*}
\Delta G_{n}^{(-1)}(x) & =G_{n}^{(-1)}(1+x)-G_{n}^{(-1)}(x)  \tag{3.2}\\
& =2^{n} B_{n}^{(-1)}\left(\frac{x}{2}\right) \quad \text { by Theorem } 6(n \geq 0) .
\end{align*}
$$

Then, by (2.5) and (3.2), the Euler-Maclaurin theorem leads to
Theorem 11:

$$
\begin{equation*}
n G_{n-1}^{(-1)}(x)=2^{n}\left\{B_{n}^{(-1)}(0)+B_{1}(x) B_{n}^{(-1)^{\prime}}(0)+\frac{B_{2}(x)}{2!} B_{n}^{(-1)^{\prime \prime}}(0)+\cdots\right\} \quad(n \geq 1) \tag{3.3}
\end{equation*}
$$

When $n=3$, the theorem reduces to

$$
3 G_{2}^{(-1)}(x)=x^{3}+\frac{3}{2} x^{2}+\frac{3 x}{2}+\frac{1}{2}
$$

which is true by (2.3). Theorem 11 enables us to display $G_{n}^{(-1)}(x)$ entirely by means of Bernoulli expressions. Both Theorems 10 and 11 (for $k=1$ ) may be extended to cover the case when $k$ is general.

Some 'Hybrid' Products
Let us write

$$
\left\{\begin{array}{l}
G \equiv \sum_{n=0}^{\infty} G_{n}^{(1)}(x) \frac{t^{n}}{n!}, \quad G_{-} \equiv \sum_{n=0}^{\infty} G_{n}^{(1)}(x) \frac{(-t)^{n}}{n!},  \tag{3.4}\\
G^{*} \equiv \sum_{n=-1}^{\infty} G_{n}^{(-1)}(x) \frac{t^{n}}{|n|!}, \quad G^{*} \equiv \sum_{n=-1}^{\infty} G_{n}^{(-1)}(x) \frac{(-t)^{n}}{|n|!}
\end{array}\right.
$$

where $G$ is as defined in [2], $G^{*}$ refers to (2.1) when $k=1$, and $G_{-}$, $G^{*}$ are obtained from $G, G^{*}$, respectively, by replacing $t$ by $-t$. Corresponding symbolism $E, \ldots, E_{\star}^{*}, B, \ldots, B^{\star}$ relates to Euler and Bernoulli polynomials, where $E$ and $B$ are also defined in [2].

Then, by [2] and (2.1)
(3.5) $G G^{*}=e^{2 t x}$
and
(3.6) $G G_{-}^{*}=-e^{-t}$.

Equating appropriate coefficients yields the hybrid results
(3.7) $\quad \sum_{j=1}^{n+1}\left(\frac{G_{j}^{(1)}(x)}{j!} \cdot \frac{G_{n-j}^{(-1)}(x)}{|n-j|!}\right)=\frac{(2 x)^{n}}{n!}$
and

$$
\begin{equation*}
\sum_{j=1}^{n+1}\left(\frac{G_{n}^{(1)}(x)}{j!} \frac{(-1)^{n-j} G_{n-j}^{(-1)}(x)}{|n-j|!}\right)=\frac{(-1)^{n-1}}{|n-1|!} \tag{3.8}
\end{equation*}
$$

Similarly,
(3.9) $G_{-} G^{*}=-e^{t}=\left(G G_{-}^{\stackrel{*}{*}}\right)^{-1}$
and
(3.10) $G_{-} G_{-}^{*}=e^{-2 t x}=\left(G G^{*}\right)^{-1}$,
yielding results corresponding to (3.7) and (3.8). The case $G^{*} G^{*}{ }_{1}$ has been covered in (2.28). In addition,

$$
G_{\underline{*}}^{*} G_{\underline{-}}^{*}=\left(\frac{1+e^{t}}{2 t}\right)^{2} e^{-t(2 x+2)}
$$

gives the summation (2.1) for $G_{n}^{(-2)}\{-(2 x+2)\}$.
Moreover,

$$
\left\{\begin{align*}
E E^{*} & =B B^{*}=e^{2 t x}  \tag{3.11}\\
E_{-} E^{*} & =B-B^{*}=e^{t} \\
G^{*} E & =\frac{1}{t} e^{2 t x} \\
G E^{*} & =t e^{2 t x} \\
G^{*} E^{*} & =t\left(\frac{1+e^{t}}{2 t}\right)^{2} e^{t \cdot 2 x} \\
G^{*} B^{*} & =\frac{1}{t} \frac{e^{2 t}-1}{2 t} e^{2 t \cdot \frac{x}{2}} \\
B^{*} E^{*} & =\frac{e^{2 t}-1}{2 t} e^{2 t\left(-\frac{1}{2}\right)}
\end{align*}\right.
$$

for example, among a variety of possible products. The last three equations in (3.11) give the summations (2.1) and (1.2)' for

$$
G_{n-1}^{(-2)}(2 x), \quad B_{n+1}^{(-1)}\left(\frac{x}{2}\right), \quad \text { and } B_{n}^{(-1)}\left(-\frac{1}{2}\right)
$$

respectively.
Our theory may be extended to values of $k>1$.
Products of powers of the $G, E$, and $B$ symbols give rise to an immense number of identities, for example

$$
\begin{cases}G G_{-} G^{*} G^{*} 1 & =1  \tag{3.12}\\ G E\left(E^{*}\right)^{2} & =t e^{4 t x} \\ G^{3} G_{-}^{2}\left(G^{*}\right)^{2} B_{-} B^{*}\left(E_{-}^{*}\right)^{3} & =t^{3}\end{cases}
$$

To avoid tedium, we leave the challenge of exploring such possibilities, which may be continued almost ad infinitum, ad nauseam!, to the ingenuity and perseverance of the reader.

## 4. Differential Equations

## Descending Diagonal Functions

Arrange the $G_{n}^{(-1)}(x)$ in (2.3) according to the following pattern:

$$
\begin{align*}
& G_{-1}^{(-1)}(x)=G_{-1}^{(-1)}  \tag{4.1}\\
& G_{0}^{(-1)}(x)=G_{0}^{(-1)}+x G_{-1}^{(-1)} \\
& G_{1}^{(-1)}(x)=G_{1}^{(-1)}+x G_{0}^{(-1)}+\frac{1}{2} x^{2} G_{-1}^{(-1)} \\
& G_{2}^{(-1)}(x)=G_{2}^{(-1)}+2 x G_{1}^{(-1)}+x^{2} G_{0}^{(-1)}+\frac{1}{3} x^{3} G_{-1}^{(-1)} \\
& G_{3}^{(-1)}(x)=G_{3}^{(-1)}+3 x G_{2}^{(-1)}+3 x^{2} G_{1}^{(-1)}+x^{3} G_{0}^{(-1)}+\frac{1}{4} x^{4} G_{-1}^{(-1)} \\
& G_{4}^{(-1)}(x)=G_{4}^{(-1)}+4 x G_{3}^{(-1)}+6 x^{2} G_{2}^{(-1)}+4 x^{3} G_{1}^{(-1)}+x^{4} G_{0}^{(-1)}+\frac{1}{5} x^{5} G_{-1}^{(-1)}
\end{align*}
$$

in which

$$
\begin{equation*}
G_{n}^{(-1)}(x)=\sum_{j=-1}^{n} \frac{|n|!}{(n-j)!|j|!} G_{j}^{(-1)} x^{n-j} \tag{4.2}
\end{equation*}
$$

as in [2] for $G_{n}^{(1)}(x)$.
Imagine now that the terms are considered to lie in an infinite set of downward slanting "parallel lines" to form the following set of descending diagonal functions $\left\{g_{n}^{(-1)}(x)\right\}(n=-1,0,1,2, \ldots)$ and their generating functions $(|x|<1)$ :

$$
\begin{align*}
& g_{-1}^{(-1)}(x)=G_{-1}^{(-1)}\left(1+x+\frac{1}{2} x^{2}+\frac{1}{3} x^{3}+\frac{1}{4} x^{4}+\ldots\right)=G_{-1}^{(-1)}(1-\log (1-x))  \tag{4.3}\\
& g_{0}^{(-1)}(x)=G_{0}^{(-1)}\left(1+x+x^{2}+x^{3}+x^{4}+\ldots\right)=G_{0}^{(-1)}(1-x)^{-1} \\
& g_{1}^{(-1)}(x)=G_{1}^{(-1)}\left(1+2 x+3 x^{2}+4 x^{3}+\ldots\right) \\
& g_{2}^{(-1)}(x)=G_{1}^{(-1)}(1-x)^{-2} \\
& \ldots \ldots \ldots . \ldots \ldots G_{2}^{(-1)}(1-\ldots \ldots)^{-3}
\end{align*}
$$

with, generally, as in [2] for $G_{n}^{(1)}(x)$,
(4.4) $\quad g_{n}^{(-1)}(x)=G_{n}^{(-1)}(1-x)^{-(n+1)}$.

Note that

$$
\left.\begin{array}{l}
\text { 5) }\left\{\begin{array}{rl}
g_{n}^{(-1)}(x)=G_{n}^{(-1)} \sum_{j=0}^{\infty}\binom{n+j}{j} x^{j} & n \geq 0 \\
g_{n}^{(-1)}(0)=G_{n}^{(-1)} & n \geq 0 \\
g_{n}^{(-1)}\left(\frac{1}{2}\right)=2^{n+1} G_{n}^{(-1)} & n \geq(1+\log 2) G_{-1}^{(-1)}
\end{array}\right. \\
=\left(\begin{array}{ll}
n=-1
\end{array}\right. \\
g_{n}^{(-1)}(1) \text { is not defined. }
\end{array}\right\} \begin{aligned}
& \text { Write }  \tag{4.6}\\
& \text { 6) } D \equiv D(x, y)=\sum_{n=1}^{\infty} g_{n-1}^{(-1)}(x) y^{n-1}=\sum_{n=1}^{\infty} G_{n-1}^{(-1)}(1-x)^{-n} y^{n-1}
\end{aligned}
$$

whence
(4.7) $n y \frac{\partial D}{\partial y}-(n-1)(1-x) \frac{\partial D}{\partial x}=0$,
while, from (4.5),

$$
\begin{equation*}
(1-x) \frac{d g_{n}^{(-1)}(x)}{d x}=(n+1) g_{n}^{(-1)}(x) \tag{4.8}
\end{equation*}
$$

Observe in (4.6) that $g_{-1}^{(-1)}(x)$ has been omitted.
Reverting now to (4.2), we may easily generalize this formula by replacing -1 by $-k$ (three times). For what follows, the reader may find it helpful to construct a partial table like (4.1) from (2.2). An analysis of the cases $k=2,3, \ldots$ then discloses the interesting nexus:

When $n<-1$, there is no such simple pattern as in (4.9) [though, exceptionally, $g_{-2}^{(-2)}(x)$ is expressible in terms of $\left.g_{-1}^{(-1)}(x)\right]$. This unstructured situation results from the somewhat wayward behavior, as $k$ varies, of

$$
\begin{equation*}
g_{-k}^{(-k)}(x)=G_{-k}^{(-k)}\left\{1+\frac{1}{|-k|!}\left(|-k+1|!x+\frac{|-k+2|!}{2!} x^{2}+\frac{|-k+3|!}{3!} x^{3}+\cdots\right)\right\} \tag{4.10}
\end{equation*}
$$

which is aberrant on account of the unusual presence of modulus factorials.
The repetitive nature of the $g_{n}^{(-k)}(x)$ is understood if we examine successive levels in the layout of

$$
G_{-k}^{(-k)}(x), \quad G_{-k+1}^{(-k)}(x), \quad G_{-k+2}^{(-k)}(x), \cdots
$$

corresponding to (4.1).
Consider, for example, the coefficients of $x$ in $G_{-k+3}^{(-k)}(x)$ and $G_{-k+4}^{(-k)}(x)$, i.e.,

$$
\frac{|-k+3|!}{|-k+2|!} G_{-k+2}^{(-k)} \quad \text { and } \quad \frac{|-k+4|!}{|-k+3|!} G_{-k+3}^{(-k)}
$$

respectively. Substituting $k=2$ in the first case and $k=3$ in the second, we have immediately $1 \cdot G_{0}^{(-2)}$ and $1 \cdot G_{0}^{(-3)}$, i.e., the coefficient 1 is repeated.

## Rising Diagonal Functions

Concentrate next on the infinite set of upward slanting "parallel lines" which form the following rising diagonal functions:

```
(4.11) \(h_{-1}^{(-1)}(x)=G_{-1}^{(-1)}\)
    \(h_{0}^{(-1)}(x)=G_{0}^{(-1)}\)
    \(h_{1}^{(-1)}(x)=x G_{-1}^{(-1)}+G_{1}^{(-1)}\)
    \(h_{2}^{(-1)}(x)=x G_{0}^{(-1)}+G_{2}^{(-1)}\)
    \(h_{3}^{(-1)}(x)=\frac{1}{2} x^{2} G_{-1}^{(-1)}+2 x G_{1}^{(-1)}+G_{3}^{(-1)}\)
    \(h_{4}^{(-1)}(x)=x^{2} G_{0}^{(-1)}+3 x G_{2}^{(-1)}+G_{4}^{(-1)}\)
    \(h_{5}^{(-1)}(x)=\frac{1}{3} x^{3} G_{-1}^{(-1)}+3 x^{2} G_{1}^{(-1)}+4 x G_{3}^{(-1)}+G_{5}^{(-1)}\)
    \(h_{6}^{(-1)}(x)=x^{3} G_{0}^{(-1)}+6 x^{2} G_{2}^{(-1)}+5 x G_{4}^{(-1)}+G_{6}^{(-1)}\)
    \(h_{7}^{(-1)}(x)=\frac{1}{4} x^{4} G_{-1}^{(-1)}+4 x^{3} G_{1}^{(-1)}+10 x^{2} G_{3}^{(-1)}+6 x G_{5}^{(-1)}+G_{7}^{(-1)}\)
    \(h_{8}^{(-1)}(x)=x^{4} G_{0}^{(-1)}+10 x^{3} G_{2}^{(-1)}+15 x^{2} G_{4}^{(-1)}+7 x G_{6}^{(-1)}+G_{8}^{(-1)}\)
```

$$
\begin{align*}
& \text { Generally, } \\
& \text { 12) } h_{n}^{(-1)}(x)=\sum_{j=0}^{\left[\frac{n+1}{2}\right]} \frac{|n-j|!}{j!|n-2 j|!} G_{n-2 j}^{(-1)} x^{j} . \tag{4.12}
\end{align*}
$$

Clearly,
(4.13) $h_{n}^{(-1)}(0)=G_{n}^{(-1)}=g_{n}^{(-1)}(0)$.

Consider
(4.14) $R \equiv R(x, y)=\sum_{n=1}^{\infty} h_{n-1}^{(-1)}(x) y^{n-1}$

$$
\begin{aligned}
=\left(1-x y^{2}\right)^{-1} G_{0}^{(-1)} & +y\left(1-x y^{2}\right)^{-2} G_{1}^{(-1)} \\
& +y^{2}(1-x y)^{-3} G_{2}^{(-1)}+\cdots .
\end{aligned}
$$

Writing
(4.15) $\quad \psi \equiv\left(1-x y^{2}\right)^{-2} G_{0}^{(-1)}+y\left(1-x y^{2}\right)^{-3} G_{1}^{(-1)}+y^{2}\left(1-x y^{2}\right)^{-4} G_{2}^{(-1)}+\cdots$
and
(4.16) $\quad \phi \equiv\left(1-x y^{2}\right)^{-2} G_{1}^{(-1)}+2 y\left(1-x y^{2}\right)^{-3} G_{2}^{(-1)}+3 y^{2}\left(1-x y^{2}\right)^{-4} G_{3}^{(-1)}+\cdots$ we readily obtain, as in [2], the partial differential equations (4.17) $\frac{\partial R}{\partial x}=y^{2} \psi$
and
(4.18) $\frac{\partial R}{\partial y}=2 x y \psi+\phi$,
leading to
(4.19) $\frac{\partial \phi}{\partial x}=y^{2} \frac{\partial \psi}{\partial y}-2 x y \frac{\partial \psi}{\partial x}$
on partially differentiating (4.17) w.r.t. $y$ and (4.18) w.r.t. $x$ and then applyign Bernoulli's theorem:

$$
\frac{\partial^{2} R}{\partial x \partial y}=\frac{\partial^{2} R}{\partial y \partial x} .
$$

Generally,

$$
\begin{equation*}
h_{n}^{(-k)}(x)=\sum_{j=0}^{\left[\frac{n+k}{2}\right]} \frac{|n-j|!}{j!|n-2 j|!} G_{n-2 j}^{(-k)} x^{j}, \tag{4.20}
\end{equation*}
$$

i.e., -1 in (4.12) has been replaced by $-k$ (three times), and an extended theory for differential equations may be pursued corresponding to that given in [2]. Observe that, whereas in (4.20) the number $G_{-1}^{(-1)}$ has been omitted, in the general case, the numbers $G_{-1}^{(-k)}, G_{-2}^{(-k)}, \ldots, G_{-k}^{(-k)}$ will be missing.
5. Concluding Remarks

Many other properties of $G_{n}^{(-k)}(x)$ may be developed, but it is hoped that this exposition will give a flavor of the basic ingredients of the mixture. Further extensions could, for instance, involve relationships with $B_{n}^{(-k)}(x)$ and $E_{n}^{(-k)}(x)$. As a guide to the possibilities, one might consult [2] for corresponding material relating to $G_{n}^{(-k)}(x)$, e.g., graphs, and for appropriate references.

In treating $G_{n}^{(-k)}(x)$, there is the obvious choice of deciding whether or not to exclude the cases $n=-k,-k+1, \ldots,-1$. Inclusion of these values does add to complications in the theory. Without them, one can sometimes proceed from results in [2] for $k \geq 0$ to those established here, simply by replacing $k$ by $-k$. This situation gives the continuity and unity mentioned by Nörlund (for Euler and Bernoulli polynomials) in the French quote in the Introduction.

Consideration of negative values of $n$ in $G_{n}^{(-k)}(x)$ adds much to the completeness of the theory and, despite the difficulties involved, enhances the enjoyment of the work.

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# ON SOME NUMBER SEQUENCES RELATED TO THE PARITY OF BINOMIAL COEFFICIENTS 

Bernard R. Hodgson<br>Université Laval, Québec G1K 7P4, Canada<br>(Submitted February 1990)<br>1. Pascal's Triangle Mod 2

It is well known that striking patterns of triangles can be produced from Pascal's triangle by replacing each binomial coefficient by its residue with respect to a certain modulus. The arrays thus produced were considered by various authors; see, for instance, Gould [5], Gardner [1], Long [10], or Sved [17]. For example, Pascal's triangle mod 2 (Fig. 1) is the array of zeros and ones obtained by considering the parity of each entry in the usual Pascal triangle. It can be readily constructed using the basic recursion formula

$$
\begin{equation*}
\binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r} \tag{1.1}
\end{equation*}
$$

together with the rules for addition mod 2. (In Fig. 1, this array is shown "right-justified" for convenience in further discussions, with all entries resulting from coefficients of the form $\binom{n}{n}$ aligned in the rightmost column. Furthermore, for the sake of clarity, groups of zeros in this figure have been enclosed within triangular shapes.)

We shall be concerned in this paper with some number sequences introduced via Pascal's triangle mod 2. Gould [5] has considered the sequence obtained by reading the rows of this array as base two representations of numbers. We shall introduce analogously other number sequences and show how certain regularities of such sequences follow directly from the patterns found within the array. It is our purpose in this paper to base our discussion essentially on the geometrical structure of Pascal's triangle mod 2. So we complete this introduction with a description of this geometrical structure.

We borrow the following terminology from Sved [16, 17], with the notation $\left|\begin{array}{l}n \\ r\end{array}\right|$ representing the residue of $\binom{n}{r} \bmod 2(0 \leq r \leq n)$. The cluster of order $h$, or h-cluster, is the portion of the array formed by all the residues $\left|\begin{array}{l}n \\ r\end{array}\right|$ for $0 \leq n<2^{h}$, and the zero-hole of order $h(h \neq 0)$, or $h$-hole, is the (inverted, left-justified) triangular array made of ( $2^{h}-1$ ) decreasing rows of zeros, with ( $2^{h}-1$ ) entries in the first row down to a single entry in its last row. [Anticipating the forthcoming geometrical characterization of Pascal's triangle mod 2, see the following paragraph, the $h$-hole thus corresponds to all residues $\left|\begin{array}{l}n \\ r\end{array}\right|$ with
(1.2) $2^{h} \leq n<2^{h+1}-1$ and $n-\left(2^{h}-1\right) \leq r<2^{h}$.]

For example, the clusters of orders 0,1 , and 2 are, respectively,
1
11
$1, \quad \begin{array}{llllll} & 1 \\ 1 & 1,\end{array} \quad \begin{array}{lll}1 & 0 & 1 \\ 1 & 1 & 1 \\ 1\end{array}$,
while the zero-holes of orders 1 and 2 are of the form
0 and $\quad \begin{array}{lll}0 & 0 & 0 . \\ 0 & 0 & \\ 0 & & \end{array}$

ON SOME NUMBER SEQUENCES RELATED TO THE PARITY OF BINOMIAL COEFFICIENTS


## Figure 1

The overall structure of Pascal's triangle mod 2 can be described as follows. Let us observe the array as it grows downward, thus producing successive (nested) clusters. Then the cluster of order $h$, consisting of rows 0 down to $2^{h}-1$, is made of three clusters of order (h - 1) surrounding a zero-hole of order (h - 1) (see Fig. 2, where the three ( $h-1$ )-clusters have been labeled, respectively, I, II, and III). A formal proof of this characterization can be given by induction, using the recursion formula (1.1) (see Sved [16]). The geometrical pattern of the array could become even more striking by replacing all zeros by blanks in Figure 1. Note that when extending the process to an infinite number of rows, the limiting pattern is found to be "self-similar" with fractal dimension $\log _{2} 3$, as discussed in Wolfram [19] (see also the "Sierpinski gasket" described in Mandelbrot [12, p. 142]).

This geometrical characterization of Pascal's triangle mod 2 allows us to state a few basic properties.
(1.i) Row $2^{h}-1$ consists of $2^{h}$ ones: 111...111 (2h $\left.1^{\prime \prime} s\right)$.
(1.ii) Row $2^{h}$ consists of two ones separated by $2^{h}-1$ zeros: 100...001 ( $2^{h}-10^{\prime} s$ ).
(1.iii) More generally, row $2^{h}+u, 0 \leq u<2^{h}$, consists of two copies of row $u$ separated by ( $2^{h}-1-u$ ) zeros.
Result (1.i) follows from the fact that row $2^{h}-1$, which is the bottom row of the $h$-cluster, is obtained by concatenating the bottom row of the (h - 1 )cluster with itself. Property (1.ii) is then an easy consequence of (1.i) using mod 2 addition. As for property (1.iii), it expresses the fact that row $2^{h}+u$, which is located in the $(h+1)$-cluster, is obtained by inserting, in
between two copies of row $u$ of the $h$-cluster, the $(u+1)$ th row of the $h$-hole [i.e., a sequence of ( $2^{h}-1-u$ ) zeros]. These results can be rephrased as follows in terms of the parity of binomial coefficients.
(1.i') All coefficients $\binom{2^{h}-1}{p}, 0 \leq r \leq 2^{h}-1$, are odd.
(1.ii') Coefficients $\binom{2^{h}}{r}$ are odd only for $r=0$ and $2^{h}$.
(1.iii') $\left|\begin{array}{c}2^{h}+u \\ r\end{array}\right|=\left|\begin{array}{c}2^{h}+u \\ 2^{h}+r\end{array}\right|=\left|\begin{array}{l}u \\ r\end{array}\right|$ for $0 \leq u<2^{h}$ and $0 \leq r \leq u$.


Figure 2. The cluster of order $h$
(The dotted line indicates the "principal diagonal" $\Delta_{2^{h}-1^{\circ}}$ )
As was observed by Kung [9] or Sved [17], results (1.i)-(1.iii') follow from a simple glance at the binomial array mod 2. However, the reader should note that all six of these properties can also be obtained as immediate consequences of certain well-known facts about binomial coefficients. For instance, by a result due to Kummer [8, p. 116], one has the following:
(1.iv) The exponent of 2 in the prime factorization of $\binom{n}{r}$ equals the number of borrows in the subtraction $n-r$ in base two.
(See Singmaster [14] or Goetgheluck [4] for recent proofs.) Hence $\left|\begin{array}{l}n \\ r\end{array}\right|=1$ if and only if there are no borrows in this subtraction. A direct algebraic proof of property (1.i) can be found in Vinogradov [18, p. 20]. Alternately, as observed by Roberts [13], (1.i) follows immediately from the fact that, for a fixed $n$, the number of odd binomial coefficients $\binom{n}{r}$ is given by $2^{\#_{1}(n)}$, where $\#_{1}(n)$ represents the number of 1 's appearing in the base two representation of $n$. This last result, stated in Glaisher [3], is easily justified using the following theorem of Lucas [11]:

$$
\binom{n}{p} \equiv\binom{n_{k}}{n_{k}}\binom{n_{k-1}}{n_{k-1}} \cdots\binom{n_{0}}{r_{0}} \quad(\bmod 2)
$$

where $\left(n_{k} n_{k-1} \ldots n_{0}\right)_{\text {two }}$ and $\left(r_{k} r_{k-1} \ldots r_{0}\right)$ two are the binary representations of $n$ and $r$, respectively. (This last result of Lucas plays a central role in the "masking" relation used by Jones \& Matijasevic [7] for encoding the history of calculations of a Turing machine. It is this latter work which has prompted the present author's interest in Pascal's triangle mod 2.)

## 2. Gould's Numbers

Let us now use the binomial array just discussed to define a sequence $\left\{G_{n}\right\}_{n \geq 0}$ of natural numbers as follows: $G_{n}$ is the number whose binary representation is given by the $n^{\text {th }}$ row of Pascal's triangle mod 2 . This sequence, which starts

$$
1,3,5,15,17,51,85,255,257,771,1285, \ldots,
$$

was considered by Gould [5] (see Sloane [15], sequence no. 988). We shall call the $G_{n}$ 's Gould's numbers.

We can use facts (1.i)-(1.iii) about Pascal's triangle mod 2 to deduce some basic relationships among Gould's numbers. For instance, we have

$$
\begin{equation*}
G_{2^{h}}=2^{2^{h}}+1=F_{h} \tag{2.1}
\end{equation*}
$$

where $F_{h}$ denotes the $h^{\text {th }}$ Fermat number. This stems immediately from the particular form of row $2^{h}$ [see (1.ii) above]. [Similarly, by (1.i), $G_{2^{h}-1}=2^{2^{h}}-1$.] It then readily follows that for an arbitrary $n=2^{h}+u, 0 \leq u<2^{h}$, we have

$$
\begin{equation*}
G_{2^{h}+u}=G_{2^{h}} \cdot G_{u} \tag{2.2}
\end{equation*}
$$

since the sequence of $1^{\prime} s$ and $0^{\prime} s$ forming row $2^{h}+u$, as described in (1.iii), can be directly seen as being the (binary) product of row $2^{h}$ and row $u$.

For $n$ having the binary representation $\left(n_{k} n_{k-1} \ldots n_{0}\right)$ two , one then deduces from (2.1) and (2.2) the remarkable relation

$$
\begin{equation*}
G_{n}=\prod_{i=0}^{k} E_{i}^{n_{i}} \tag{2.3}
\end{equation*}
$$

Indeed, writing $n$ as a sum of powers of two, the digits $n_{i}$ indicate the powers $2^{i}$ needed for expressing $n$. Result (2.3) was stated by Gould [5] [see formula (50)] and a proof was given by Hewgill [6]. (Gould [5] stated another remarkable relation about Gould's numbers, namely: $G_{2 n+1}=3 G_{2 n}$. This result is easily proved inductively from the geometrical pattern of the binomial array mod 2. A formal proof of the same result can be found in Garfinke1 \& Selkow [2].)

When (2.2) is rewritten in the form

$$
\begin{equation*}
G_{2^{h}+u}=G_{u} 2^{2^{h}}+G_{u} \tag{2.4}
\end{equation*}
$$

one obtains nice symmetrical representations. For instance, (2.4) yields the following for $h=3$ and $0 \leq u<8$ :
$G_{8}=257=1 \cdot 256+1$
$G_{9}=771=3 \cdot 256+3$
$G_{10}=1,285=5 \cdot 256+5$
$G_{11}=3,855=15 \cdot 256+15$
$G_{12}=4,369=17 \cdot 256+17$
$G_{13}=13,107=51 \cdot 256+51$
$G_{14}=21,845=85 \cdot 256+85$
$G_{15}=65,535=255 \cdot 256+255$

A suggestive interpretation of Gould's numbers can also be obtained from (2.4), using property (1.iii'). Let us recall that by definition the $G_{n}$ 's satisfy the equality

$$
G_{n}=\sum_{r=0}^{n}\left|\begin{array}{l}
n  \tag{2.5}\\
r
\end{array}\right| 2^{n-r}
$$

Then (2.4) says that for $n=2^{h}+u$ this sum can be seen as made of two parts corresponding, respectively, to the successive values $0,1, \ldots, u$ and $2^{h}, 2^{h}+1$, $\ldots, 2^{h}+u=n$ of the index $r$. In the former case one has, because of (1.iii'),

$$
\sum_{r=0}^{u}\left|\begin{array}{l}
n \\
r
\end{array}\right| 2^{n-r}=\sum_{r=0}^{u}\left|\begin{array}{l}
u \\
r
\end{array}\right| 2^{n-r}=\left(\sum_{r=0}^{u}\left|\begin{array}{l}
u \\
r
\end{array}\right| 2^{u-r}\right) \cdot 2^{2^{h}}
$$

so this partial sum corresponds to $G_{u}$ shifted by a factor of $2^{2^{h}}$. In the latter case, the terms sum directly to $G_{u}$ because, for $r=2^{h}+i, 0 \leq i \leq u$, one can write, again by (1.iii'),

$$
\left|\begin{array}{l}
n \\
r
\end{array}\right| 2^{n-r}=\left|\begin{array}{c}
u \\
i
\end{array}\right| 2^{u-i}
$$

Figure 1 nicely illustrates the situation, since it has been displayed in such
 $2^{2^{h}}$ in the binary expansion of $G_{2^{h}+u}$ is easily located.

## 3. Along Fibonacci Diagonals

We now want to use the binomial array mod 2 to introduce other sequences of numbers. Among remarkable lines in the (standard) Pascal triangle are the Fibonacci diagonals, i.e., those slant lines whose entries sum to consecutive terms of the Fibonacci sequence. When the binomial coefficients are displayed in the shape of a right-justified triangle, similar to Figure 1 , the $n{ }^{\text {th }}$ Fibonacci diagonal, starting at $\binom{n}{0}$, contains all those entries obtained by moving successively two columns to the right and one row up. By the basic formula (1.1), the $n^{\text {th }}$ Fibonacci number $f_{n}$ (where $f_{0}=f_{1}=1$ and $f_{n+2}=f_{n+1}+f_{n}$ ) is the sum of all coefficients thus obtained:

$$
f_{n}=\sum_{r=0}^{\left[\frac{n}{2}\right]}\binom{n-r}{r}
$$

with $[x]$ indicating the integer part of $x$. For instance,

$$
f_{9}=\binom{9}{0}+\binom{8}{1}+\binom{7}{2}+\binom{6}{3}+\binom{5}{4}=1+8+21+20+5=55
$$

In analogy with the way Gould's numbers were defined, we now want to introduce a sequence $\left\{H_{n}\right\}_{n \geq 0}$ of natural numbers whose binary representations are given by the Fibonacci diagonals in Pascal's triangle mod 2. Let us use $\Delta_{n}$, $n \geq 0$, to represent the string of digits found along the $n$th Fibonacci diagonal $\bmod 2$ (for instance, $\Delta_{g}=10101$ ). Then $H_{n}$ is the number represented in base two by $\Delta_{n}$. We thus have, analogously to (2.5),

$$
H_{n}=\sum_{r=0}^{\left[\frac{n}{2}\right]}\left|\begin{array}{c}
n-r  \tag{3.1}\\
r
\end{array}\right| 2^{\left[\frac{n}{2}\right]-r}
$$

so that, e.g., $H_{9}=(10101)_{\text {two }}=21$. The first values of the $H_{n}$-sequence are:

$$
1,1,3,2,7,5,13,8,29,21,55,34,115,81,209,128,465,337,883, \ldots .
$$

As might be expected, the $H_{n}$ 's satisfy some nice generation rules, and we will make use of the geometry of Pascal's triangle mod 2 to give proofs of these rules. Before doing so, however, it is interesting to redisplay the entries of Figure 1 so that the $n^{\text {th }}$ diagonal $\Delta_{n}$ becomes the $n^{\text {th }}$ row in the new array (see Fig. 3). Some striking patterns can be observed in this array.

For instance, it is readily seen, for those values listed, that $\Delta_{2}^{h}-1$ consists of a one followed by a string of $2^{h-1}-1$ zeros, so that

$$
H_{2^{h}-1}=2^{2^{h-1}-1}
$$

Also, the staircase pattern of Figure 3 appears to be made of symmetrical parts that could be directly described by introducing a terminology based on "clusters" and "holes," as was done for Figure 1. For example, rows 0 down to 14
can be seen as being separated in two layers by row 7; the lower layer (rows 8 to 14) is made of three sections, namely a copy of the upper "cluster" (rows 0 to 6) that has been shifted down by 8 rows and to the left by 4 columns, next to an upside down copy of this same part (i.e., a mirror image of rows 0-6 in row 7), the remaining entries in between these blocks being filled with a "hole" of zeros. Thus, $\Delta_{9}$ consists of a copy of $\Delta_{1}$ (1) and a copy of $\Delta_{5}$ (101) separated by one zero, while $\Delta_{13}$ consists, conversely, of $\Delta_{5}$ and $\Delta_{1}$ separated by three zeros.

| 0 : |  |  |  |  |  |  |  |  |  |  |  |  | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1: |  |  |  |  |  |  |  |  |  |  |  |  | 1 |
| 2: |  |  |  |  |  |  |  |  |  |  |  |  | 1 |
| 3: |  |  |  |  |  |  |  |  |  |  |  |  | 0 |
| 4: |  |  |  |  |  |  |  |  |  |  |  | 11 |  |
| $5:$ |  |  |  |  |  |  |  |  |  |  |  | 0 |  |
| 6: |  |  |  |  |  |  |  |  |  |  | 11 | 0 |  |
| 7: |  |  |  |  |  |  |  |  |  |  |  | 0 |  |
| 8 8: |  |  |  |  |  |  |  |  |  |  | 11 | 0 |  |
| 9: |  |  |  |  |  |  |  |  |  |  | 01 | 0 |  |
| 10: |  |  |  |  |  |  |  |  |  | 10 | 01 | 1 |  |
| 11: |  |  |  |  |  |  |  |  |  | 00 | 00 | 1 | 0 |
| 12: |  |  |  |  |  |  |  |  | 1 | 10 | 00 | 1 |  |
| 13: |  |  |  |  |  |  |  |  | 0 | 10 | 00 | 0 | 1 |
| 14: |  |  |  |  |  |  |  | 11 | 0 | 10 | 00 | 0 |  |
| 15: |  |  |  |  |  |  |  | 10 | 0 | 00 | 00 | 0 | 0 |
| 16: |  |  |  |  |  |  |  | 11 | 0 | 10 | 00 | 0 |  |
| 17: |  |  |  |  |  |  |  | 01 | 0 | 10 | 00 | 0 | 1 |
| 18: |  |  |  |  |  |  | 10 | 01 | 1 | 10 | 00 | 1 |  |
| 19: |  |  |  |  |  |  | 0 | 00 | 1 | 00 | 00 | 1 | 0 |
| 20: |  |  |  |  |  | 1 | 1 | 00 | 1 | 10 | 01 | 1 |  |
| 21: |  |  |  |  |  | 0 | 1 | 00 | 0 | 10 | 01 | 0 | 1 |
| 22: |  |  |  |  | 11 | 0 | 1 | 00 | 0 | 11 | 11 | 0 |  |
| 23: |  |  |  |  | 10 | 0 | 0 | 0 | 0 | 01 | 10 | 0 | 0 |
| 24: |  |  |  |  | 11 | 0 | 10 | 0 | 0 | 01 | 11 | 0 |  |
| 25: |  |  |  |  | 1 | 0 | 10 | 0 | 0 | 00 | 01 | 0 | 1 |
| 26: |  |  |  | 10 | 1 | 1 | 10 | 0 | 0 | 00 | 01 | 1 | 1 |
| 27: |  |  |  | 0 | 0 | 1 | 0 | 0 | 0 | 00 | 0 | 1 | 0 |
| 28: |  |  | 1 | 10 | 0 | 1 | 10 | 0 | 0 | 00 | 0 | 1 | 1 |
| 29: |  |  | 0 | 10 | 0 | 0 | 10 | 0 | 0 | 00 | 0 | 0 |  |
| 30: |  | 11 | 0 | 10 | 0 | 0 | 10 | 0 | 0 | 00 | 0 | 0 |  |
| 31: |  | 10 | 0 | 00 | 0 | 0 | 00 | 0 | 0 | 00 | 0 | 0 | 0 |
| 32: |  | 1 i | 0 | 10 | 0 | 0 | 10 | 0 | 0 | 00 | 0 | 0 |  |
| 33: |  | 01 | 0 | 10 | 0 | 0 | 10 | 0 | 0 | 00 | 0 | 0 |  |

Figure 3
Such observations about the geometrical pattern of Figure 3 will be made more precise in the following sections, where the main results of this paper will be established.

## 4. The Principal Diagonals

Going back to Pascal's triangle mod 2 (Fig. 1), we call the Fibonacci diagonal $\Delta_{2^{h}-1}$ the principal diagonal of the cluster of order $h$. It thus consists of all entries of the form

$$
\left|2^{h}-1-r\right| \text { for } 0 \leq r \leq 2^{h-1}-1
$$

We now prove a few basic properties of principal diagonals.
Lemma 4.1: $\quad \Delta_{2^{h}-1}=100 \ldots 0 \quad\left(2^{h-1}-1\right.$ zeros).
This result could be obtained directly from property (l.iv) it suffices to note that

$$
\left|2^{h}-1-r\right|=0, \text { un1ess } r=0
$$

since, for $r>0$, a borrow certainly occurs when subtracting $r$ from $2^{h}-1-r$ (in base two representations). The following is an alternate proof, based solely on the geometrical observations introduced above. We use the notation $\Delta_{n, r}$ for the $r^{\text {th }}$ element $\left|\begin{array}{c}n-r \\ r\end{array}\right|$ of string $\Delta_{n}, 0 \leq r \leq\left[\frac{n}{2}\right]$, so that

$$
\Delta_{n}=\Delta_{n, 0} \Delta_{n, 1} \cdots \Delta_{n,\left[\frac{n}{2}\right]}
$$

Proof: The elements $\Delta_{2} h-1, r, 0 \leq r \leq 2^{h-1}-1$, can be separated in two groups, according to the value of the index $r$.
(i) $0 \leq r \leq 2^{h-2}-1$. By property (1.iii'), we have

$$
\begin{aligned}
\Delta_{2^{h}-1, r}=\left|2^{h}-1-r\right| & =\left|2^{h-1}+\left(2^{h-1}-1-r\right)\right|=\left|2^{h-1}-1-r\right| \\
& =\Delta_{2^{h-1}-1, r}
\end{aligned}
$$

so that the portion of $\Delta_{2^{h}-1}$ corresponding to the given range of $r$ is identical to the principal diagonal of the (h-1)-cluster.
(ii) $2^{h-2} \leq r \leq 2^{h-1}-1$. It is readily checked that the bounding conditions (1.2), as modified for the zero-hole of order ( $h-1$ ), are satisfied by the two components of each entry $\Delta_{2}{ }^{h}-1, r$. Thus, this portion of $\Delta_{2}{ }^{h}-1$ is entirely included in the ( $h-1$ )-hole.

The proof of the Lemma can then be completed by an easy induction on $h$. The base case can be read directly from Figure 3 and the induction step follows from (i) and (ii): the first portion of $\Delta_{2}{ }^{h}-1$, which by the induction hypothesis is of the form $100 \ldots 0\left(2^{h-2}-1\right.$ zeros), gets juxtaposed to the second portion made of $2^{h-2}$ zeros, so to give the desired form for the principal diagonal of the $h$-cluster.

It follows from the preceding proof that the principal diagonal $\Delta_{2^{h-1}}$ of the cluster of order $h$ goes through this cluster in a very regular way. For instance, when $r=2^{h-2}$, we get the entry

$$
\left|2_{2^{h-2}}^{2^{h-2}-1}\right|:
$$

this tells us that the diagonal $\Delta_{2^{h}-1}$ "enters" the ( $h-1$ )-hole at the first entry of its middle row. Similarly, for $r=2^{h-1}-1$, we have the entry

$$
\left|\begin{array}{c}
2^{h-1} \\
2^{h-1}-1
\end{array}\right|
$$

so that the last element of $\Delta_{2^{h}-1}$ is found at the end of the first row of the ( $h-1$ )-hole. When combined with the fact that the principal diagonal starts at the leftmost entry of the cluster, this information on specific entries of $\Delta_{2}{ }^{h}-1$ leads to the dotted line of Figure 2, which represents this principal diagonal. We now want to make explicit certain types of symmetry within the $h$ cluster connected to the principal diagonal.

It is trivially true that each line of the Pascal triangle mod 2 is symmetrical (with respect to its middle), i.e., remains the same when inverted from left to right: this indeed is even true of the (standard) Pascal triangle itself, because of the basic relationship

$$
\binom{n}{p}=\binom{n}{n-p}
$$

We now want to prove a symmetry property concerning the columns. We show that the principal diagonal $\Delta_{2^{h}-1}$ can be seen as an "axis of vertical symmetry" for the portions of the columns determined by the $h$-cluster, in the sense that the entries above and under the principal diagonal, on each such portion of column,
are pairwise identical. For instance, the principal diagonal $\Delta_{15}$ of the 4cluster cuts it in such a way that the section of column 5 included in this cluster is separated into $110-011$ by $\Delta_{15}$, and that column 12 is separated into 100010-0-010001 (with the middle 0 belonging to $\Delta_{15}$ ).
Lemma 4.2: The cluster of order $h$ is "vertically symmetrical" with respect to its principal diagonal $\Delta_{2^{h}-1}$.
Proof: The proof is by induction on $h$, with basic cases being easily verified. Let us consider, for a given $r$ such that $0 \leq r \leq 2^{h}-1$, the portion of the $r^{\text {th }}$ column inside the $h$-cluster (which we shall call abusively the " $r^{\text {th }}$ column of the $h$-cluster"). There are then two possible cases:
(i) $0 \leq r \leq 2^{h-1}-1$. By the geometry of the binomial array mod 2 , column $r$, which is entirely included in the (h-1)-cluster II (see Fig. 2), is a copy of the analogous column of the ( $h-1$ )-cluster $I$, so the symmetry property follows from case (i) of the proof of Lemma 4.1 and from the induction hypothesis.
(ii) $2^{h-1} \leq r \leq 2^{h}-1$. Column $r$ then consists of three parts: a vertical string in the ( $h-1$ )-cluster $I$ and another one in III separated by a vertical section of the ( $h$ - 1)-hole (see Fig. 2). Clearly, again because of the geometry of the array, the parts in $I$ and III are identical and each is selfsymmetrical, by the induction hypothesis. It thus remains to show that $\Delta_{2^{h}-1}$ cuts the ( $h$ - 1)-hole symmetrically. But this is an easy consequence of the fact just mentioned above that $\Delta_{2^{h}-1}$ enters the ( $h-1$ )-hole at the first element of the middle row of this hole and ends at the last element of the first row.

A consequence of Lemma 4.2 is that the symmetry with respect to the principal diagonal $\Delta_{2^{h}-1}$ inside the $h$-cluster can also be seen as acting along diagonal lines, in the sense that two strings "paralle1" and "equidistant" to $\Delta_{2^{h}-1}$ will be identical. For instance, $\Delta_{12}=1110011$, whose first entry is $\left|\begin{array}{c}12 \\ 0\end{array}\right|$, is identical to the string determined by the line of the same slope starting at $\left|\begin{array}{c}15 \\ 3\end{array}\right|$. More generally, any entry $\left|\begin{array}{l}v \\ 0\end{array}\right|$, with $v<2^{h}-1$, which is located at the top of column $2^{h}-1-v$ of the $h$-cluster, is surely equal to the bottom entry on this same column inside the $h$-cluster, namely,

$$
\left|2^{2^{h}-1}-1-v\right| \cdot
$$

Now, if we issue from these two entries two lines parallel to the diagonal $\Delta_{2^{h}-1}$, we will obtain identical strings, because we then encounter pairs of entries, located on same columns, which are equidistant from the principal diagonal, hence equal by Lemma 4.2. We thus have
Lemma 4.3: The cluster of order $h$ is "obliquely symmetrical" with respect to its principal diagonal $\Delta_{2^{h}-1}$.

## 5. The Geometry of $\Delta_{n}$

Given $n=2^{h}+u$ with $0 \leq u<2^{h}-1$, we present in this section some rules for expressing the $n^{\text {th }}$ Fibonacci diagonal $\Delta_{n}$ in terms of diagonals depending on $u$ (note that for $u=2^{h}-1$, the rule for $\Delta_{n}$ is given by Lemma 4.1 above). This diagonal $\Delta_{2}{ }^{h}+u$, which contains the entries

$$
\Delta_{2^{h}+u, r}=\left|\begin{array}{c}
2^{h}+u-r \\
r
\end{array}\right|, \text { for } 0 \leq r \leq\left[\frac{2^{h}+u}{2}\right]=2^{h-1}+\left[\frac{u}{2}\right] \text {, }
$$

can be found in the cluster of order $(h+1)$, starting at entry $\left|\begin{array}{c}2^{h}+u \\ 0\end{array}\right|$ and moving upward diagonally. It thus consists of three parts corresponding, respectively, to the regions of the $(h+1)-c l u s t e r ~ b e i n g ~ c u t ~ b y ~ t h i s ~ d i a g o n a l ~(s e e ~$
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Fig. 4): the head of $\Delta_{2}{ }^{h}+u$ is made of all the entries $\Delta_{2^{h}}+u, r$ belonging to the $h$-cluster II, the body is the part included in the $h$-hole and the tail comes from the h-cluster I. We now prove some basic results about the Fibonacci diagonal $\Delta_{2^{h}+u}$.


Figure 4. The Fibonacci diagonal $\Delta_{2^{h}}+u$ in the $(h+1)$-cluster
Lemma 5.1: The head, the body, and the tail of $\Delta_{2^{h}+u}$ consist, respectively, of the entries $\Delta_{2^{h}}+u, r$ such that
$\begin{aligned} & & 0 & \leq r \leq\left[\frac{u}{2}\right] \\ & \text { a) head: } & & \\ \text { b) } & & {\left[\frac{u}{2}\right]+1 } & \leq r \leq u \\ & \text { c) tail: } & u+1 & \leq r \leq 2^{h-1}+\left[\frac{u}{2}\right] .\end{aligned}$
Proof: The verification involves routine calculations. For example, the range of $r$ for the head follows from the stipulation to stay inside II and from the slope of Fibonacci diagonals being $1 / 2$. For other cases, we need to identify the values of $r$ for which conditions (1.2) are satisfied. For instance, for $r$ $=[u / 2]+1$, we get

$$
\Delta_{2^{h}+u,\left[\frac{u}{2}\right]+1}=\left|\begin{array}{c}
2^{h}+u-\left[\frac{u}{2}\right]-1 \\
{\left[\frac{u}{2}\right]+1}
\end{array}\right|
$$

and (1.2) can easily be verified for $0<u<2^{h}-1$. The value of $r$ can be increased up to $u$ while remaining in the $h$-hole, and we then get

$$
\Delta_{2^{h}+u, u}=\left|\begin{array}{c}
2^{h} \\
u
\end{array}\right|
$$

which again satisfies (1.2). But, for $r=u+1$, we have

$$
\Delta_{2 h}+u, u+1=\left|\begin{array}{c}
2 h-1 \\
u+1
\end{array}\right|
$$

which is above the zero-hole of order $h$ and thus in the tail of $\Delta_{2} h+u$. To complete the proof, we just note that in the case $u=0$, the body is void since the head then consists of the single element $\Delta_{2}{ }^{h}, 0=1$, which is the first entry of row $2^{h}$, located at the apex of the $h$-cluster II, while the next element $\Delta_{2}{ }^{h}, 1=1$ is in $I$ and thus belongs to the tail.

Lemma 5.2: The head and the tail of $\Delta_{2^{h}+u}$ are, respectively, $\Delta_{u}$ and $\Delta_{2^{h}-2-u}$. Proof: For $r$ such that $0 \leq r \leq\left[\frac{u}{2}\right]$, we have

$$
\Delta_{2^{h}+u, r}=\left|\begin{array}{c}
2^{h}+u-r \\
r
\end{array}\right|=\left|\begin{array}{c}
u-r \\
r
\end{array}\right|=\Delta_{u, r},
$$

where the second equality is true by (1.iii'), since we have $0 \leq u-r \leq 2^{h}$ and $0 \leq r \leq u-r$. This shows that the head of $\Delta_{2^{h}+u}$ is $\Delta_{u}$. In the case of the tail, let us notice that its first entry, namely,

$$
\Delta_{2^{h}+u, u+1}=\left|\begin{array}{c}
2^{h}-1 \\
u+1
\end{array}\right|
$$

is located at the bottom of column $u+1$ inside the $h$-cluster I. From the discussion preceding Lemma 4.3, the top entry of this column is

$$
\left|\begin{array}{c}
2^{h}-2-u \\
0
\end{array}\right|
$$

i.e., the first element of $\Delta_{2^{h}-2-u}$. We then conclude by the "oblique symmetry" of Lemma 4.3.

It should be noted here that our assumption that $u<2^{h}-1$ ensures that $2^{h}-2-u \geq 0$.

Before closing this section, we would like to comment further on the relationship between the entries making up the tail of $\Delta_{2^{h}}+u$ and the diagonal $\Delta_{2^{h}-2-u}$. By Lemma 5.1(c), these entries are of the general form

$$
\Delta_{2^{h}+u, r}=\left|\begin{array}{c}
2^{h}+u-r  \tag{5.1}\\
r
\end{array}\right|, \text { for } r=u+1, \ldots,\left[\frac{2^{h}+u}{2}\right] .
$$

We just noted in the proof of Lemma 5.2 that for the first of these entries we can write, by "oblique symmetry,"

$$
\Delta_{2^{h}+u, u+1}=\left|\begin{array}{c}
2^{h}-1 \\
u+1
\end{array}\right|=\left|\begin{array}{c}
2^{h}-2-u \\
0
\end{array}\right|
$$

More generally, for $r$ ranging over the values indicated in (5.1), this same "oblique symmetry" described in Lemma 4.3 gives us

$$
\left|\begin{array}{c}
2^{h}+u-r  \tag{5.2}\\
r
\end{array}\right|=\left|\begin{array}{c}
2^{h}-2-u-t \\
t
\end{array}\right|,
$$

where $t=r-u-1$, i.e., $t$ takes on, successively, the values $0,1, \ldots$ up to

$$
\left[\frac{2^{h}+u}{2}\right]-u-1
$$

Another equivalent way of expressing this relationship is that the element

$$
\begin{equation*}
\left|2^{h}+u-r\right|, \text { with } u+1 \leq r \leq\left[\frac{2^{h}+u}{2}\right] \tag{5.3}
\end{equation*}
$$

can be directly rewritten, by a simple change of variable, as

$$
\left|\begin{array}{l}
2^{h}-1-s  \tag{5.4}\\
u+1+s
\end{array}\right| \text {, where } 0 \leq s \leq\left[\frac{2^{h}+u}{2}\right]-u-1
$$

(note that this last expression still represents an element of the diagonal $\Delta_{2^{h}+u}$ ). In turn, by the symmetry property of Lemma 4.3, entry (5.4) becomes an element of $\Delta_{2^{h}-2-u}$, namely,
(5.5) $\left|\begin{array}{c}2^{h}-2-u-t \\ t\end{array}\right|$, where, again, $0 \leq t \leq\left[\frac{2^{h}+u}{2}\right]-u-1$.
[Feb.

These remarks will be used in the proof of the next result.

$$
\text { 6. Calculating the } H_{n} \text { 's }
$$

We are now in position to give some calculation rules for the numbers $H_{n}$.
Proposition 6.1: (i) $H_{2^{h-1}}=2^{2^{h-1}-1}$.
(ii) $H_{2^{h}+u}=H_{u} \cdot 2^{2^{h-1}}+H_{2^{h}-2-u}$, for $0 \leq u<2^{h}-1$,

Proof: Case (i) follows immediately from Lemma 4.1. The idea behind the proof of case (ii) is that the value of $H_{2^{h}+u}$ can be obtained in three steps by looking consecutively at the head, body, and tail of $\Delta_{2^{h}+u}$ as described in Lemmas 5.1-5.2. Taking into consideration the shifting of $\Delta_{u}$ when it becomes the head of $\Delta_{2^{h}+u}$, the result is then transparent.

To be more precise, let us evaluate, for $n=2^{h}+u$, the three partial sums, $S_{1}, S_{2}$, and $S_{3}$, obtained from (3.1) according to the ranges of $r$ identified in Lemma 5.1. We first get:

$$
\begin{aligned}
S_{1} & =\sum_{r=0}^{\left[\frac{u}{2}\right]}|n-r| 2^{\left[\frac{n}{2}\right]-r}=\sum_{r=0}^{\left[\frac{u}{2}\right]}|u-r| 2^{2^{h-1}+\left[\frac{u}{2}\right]-r} \\
& =\left(\sum_{r=0}^{\left[\frac{u}{2}\right]}\left|\begin{array}{c}
u-r \\
r
\end{array}\right| 2^{\left[\frac{u}{2}\right]-r}\right) \cdot 2^{2^{h-1}}=H_{u} \cdot 2^{2^{h-1}},
\end{aligned}
$$

where the second equality follows from (l.iii'). This first partial sum thus corresponds to the shifting of $\Delta_{u}$ by $2^{h-l}$ positions in order to get the head of $\Delta_{2^{h}+u}$.

The second partial sum is

$$
S_{2}=\sum_{r=\left[\frac{u}{2}\right]+1}^{u}\left|\begin{array}{c}
n-r \\
r
\end{array}\right| 2^{\left[\frac{n}{2}\right]-r}
$$

It was already observed in the proof of Lemma 5.1 that, for the given values of $r$, we have

$$
\left|\begin{array}{c}
2^{h}+u-r \\
r
\end{array}\right|=0, \text { so that } S_{2}=0
$$

Finally, we turn to the last partial sum, $S_{3}$. From the discussion surrounding expression (5.2) [or equivalently (5.3)-(5.5)], we can write

$$
S_{3}=\sum_{r=u+1}^{\left[\frac{n}{2}\right]}\left|\begin{array}{c}
n-r \\
r
\end{array}\right|^{\left[\frac{n}{2}\right]-r}=\sum_{t=0}^{\left[\frac{n}{2}\right]-u-1}\left|\begin{array}{c}
2^{h}-2-u-t \\
t
\end{array}\right|^{\left[\frac{n}{2}\right]-u-1-t}=H_{2^{h}-2-u}
$$

The value of this partial sum thus corresponds to the tail of $\Delta_{2^{h}+u}$ being given, by "oblique symmetry," by $\Delta_{2^{h}-2-u}$.

Proposition 6.1, case (ii), provides us with nice symmetrical representations for the $H_{n}$ 's, as was the case with Gould's numbers. For instance, for $h=3$ and $0 \leq u<7$, we have the following expressions:

$$
\begin{array}{ll}
H_{8}=29=1 \cdot 16+13 & H_{12}=115=7 \cdot 16+3 \\
H_{9}=21=1 \cdot 16+5 & H_{13}=81=5 \cdot 16+1 \\
H_{10}=55=3 \cdot 16+7 & H_{14}=209=13 \cdot 16+1
\end{array}
$$

$$
H_{11}=34=2 \cdot 16+2
$$

By restricting the sequence $\left\{H_{n}\right\}_{n \geq 0}$, respectively, to elements of even and of odd ranks, we obtain the two subsequences
$1,3,7,13,29,55,115,209,465,883, \ldots$
and
$1,2,5,8,21,34,81,128,337,546, \ldots$.
To these sequences would correspond triangular arrays that could be obtained from Figure 3 by deleting appropriate alternate rows. The behavior of these new sequences is very close to that of the $H_{n}$ 's and it is possible to deduce for them results entirely analogous to those presented in Lemmas 5.1-5.2 and Proposition 6.1. We omit the details.

Using the tools developed above, we can now easily prove other properties of the $H_{n}$ 's. For example, we have the following two results.

Proposition 6.2: $H_{2 n}=H_{2 n-1}+H_{2 n+1}$.
Proof: From (3.1), we can write directly

$$
\begin{align*}
H_{2 n-1}+H_{2 n+1} & =\sum_{s=0}^{n-1}|2 n-1-s| 2^{n-1-s}+\sum_{r=0}^{n}\left|\begin{array}{c}
2 n+1-r \mid \\
s
\end{array}\right| 2^{n-r} \\
& =\left|\begin{array}{c}
2 n+1 \\
0
\end{array}\right| 2^{n}+\sum_{r=1}^{n}\left(\left|\begin{array}{c}
2 n-r \\
r-1
\end{array}\right|+\left|\begin{array}{c}
2 n+1-r \\
r
\end{array}\right|\right) 2^{n-r} . \tag{6.1}
\end{align*}
$$

Now let us observe that from the basic relation (1.1), it follows that

$$
\left|\begin{array}{c}
2 n+1-r \\
r
\end{array}\right| \equiv\left|\begin{array}{c}
2 n-r \\
r-1
\end{array}\right|+\left|\begin{array}{c}
2 n-r \\
r
\end{array}\right| \quad(\bmod 2) .
$$

So, by substitution of the right-hand side of this congruence into the coefficient of the large summand of equation (6.1), we obtain

$$
\left|\begin{array}{c}
2 n-r \\
r-1
\end{array}\right|+\left|\begin{array}{c}
2 n-r \\
r-1
\end{array}\right|+\left|\begin{array}{c}
2 n-r \\
r
\end{array}\right| \equiv\left|\begin{array}{c}
2 n-r \\
r
\end{array}\right| \quad(\bmod 2) .
$$

Moreover, the coefficient of the first term can be trivially replaced by $\left|\begin{array}{c}2 n \\ 0\end{array}\right|$. Hence, we finally get

$$
H_{2 n-1}+H_{2 n+1}=\sum_{r=0}^{n}\left|\begin{array}{c}
2 n-r \\
r
\end{array}\right| 2^{n-r}=H_{2 n}
$$

Proposition 6.3: $H_{2^{h}-2}+H_{2^{h}}=2 \cdot H_{2^{h}+1}$.
Proof: $H_{2^{h}-2}+H_{2^{h}}=\left(H_{2^{h}-3}+H_{2^{h}-1}\right)+\left(H_{2^{h-1}}+H_{2^{h}+1}\right)$ by Proposition 6.2
$=\left(H_{2^{h}-3}+2^{2^{h-1}}\right)+H_{2^{h}+1}$
$=\left(H_{1} \cdot 2^{2^{h-1}}+H_{2^{h}-3}\right)+H_{2^{h}+1}$
$=H_{2^{h}+1}+H_{2^{h}+1}$ by Proposition 6.1.

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# NUMBERS WITHOUT ONES 

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## 1. Introduction

We report solutions to the following general problem:
Fix a base $b$ and a positive integer $k$. Does every set of positive integers $\left\{x_{1}, \ldots, x_{k}\right\}$ have an integer multiplier $m \geq 1$ such that none of $m x_{1}$, ..., $m x_{k}$ contains the digit 1 in various positions of its base $b$ representation?

It has been known for more than a century ([1], p. 454) that every positive integer $x$ has a multiple $m x$ consisting of repetitions of any prescribed string of digits followed perhaps by zeros. But the structure of a set of numbers $\left\{m x_{1}, \ldots, m x_{k}\right\}$ is not so easy to stipulate, even if we merely require that the digits differ from 1. Related questions are discussed in [1] Ch. XX, [2] Ch. IX, and, in connection with the generation of pseudo-random numbers, [3] Sec. 3.2 .

## 2. Summary of Results

Let the base $b$ be a positive integer $\geq 2$, and let the variables $k, m, n$, $x_{1}, \ldots, x_{k}$ denote positive integers. Our results are the following:
Result 1: (i) If $2^{k}<b$, then for any set $\left\{x_{1}, \ldots, x_{k}\right\}$ there is an $m$ such that none of $m x_{1}, \ldots, m x_{k}$ has leftmost digit 1.
(ii) If $2^{k} \geq b$, then there exist sets $\left\{x_{1}, \ldots, x_{k}\right\}$ such that for any $m$ at least one of $m x_{1}, \ldots, m x_{k}$ has leftmost digit 1 .
Result 2: (i) If $b$ is not a prime power, or if $b=p^{n}$ for some prime $p$ and $k<n\left(p^{n}-p^{n-1}\right)$, then for any set $\left\{x_{1}, \ldots, x_{k}\right\}$ there is an $m$ such that none of $m x_{1}, \ldots, m x_{k}$ has rightmost nonzero digit 1 .
(ii) If $b=p^{n}$ and $k \geq n\left(p^{n}-p^{n-1}\right)$, then there exist sets $\left\{x_{1}, \ldots\right.$, $\left.x_{k}\right\}$ such that for any $m$ at least one of $m x_{1}, \ldots, m x_{k}$ has rightmost nonzero digit 1.

Result 3: If $k \leq b-2$ when $b$ is prime, or $k \leq$ the smallest prime factor of $b$ when $b$ is not prime, then, for any $n$ and any set $\left\{x_{1}, \ldots, x_{k}\right\}$, there is an $m$ such that none of $m x_{1}, \ldots, m x_{k}$ has the digit 1 among its $n$ rightmost nonzero digits (a string of consecutive digits the last of which is the rightmost nonzero digit of the number).

## 3. The Leftmost Digit Case

Given a set of positive integers $x_{1}, \ldots, x_{k}$, we express them in scientific notation by $x_{i}=\alpha_{i} b^{k_{i}}$ with $k_{i}$ in $\{0,1,2, \ldots\}$ and $\alpha_{i}$ in $[1, b) \cap Q$, and order them so that $\alpha_{1} \leq \cdots \leq \alpha_{k}$.

Proposition 3.1: Let $b$ be $\geq 3$. The following are equivalent:

$$
\begin{aligned}
& \text { (3.1.1) for each integer } m \geq 1 \text { at least one of } m x_{1}, \ldots, m x_{k} \text { has leftmost } \\
& \text { digit equal to } 1 \text {; }
\end{aligned}
$$

and

$$
\begin{equation*}
\frac{b}{2} \leq \frac{a_{k}}{a_{1}} \quad \text { and } \quad \frac{a_{i+1}}{a_{i}} \leq 2, \quad \text { for a11 } i=1, \ldots, k-1 \tag{3.1.2}
\end{equation*}
$$

Proof: Suppose (3.1.1) fails for some $m$. Then each $m x_{i}$ has leftmost digit $\geq 2$. Choose $j$ such that $2 b^{j} \leq x_{1}<b^{j+1}$, and let $n=j-k_{1}$. If (3.1.2) is true, an induction shows that $m \alpha_{i}<b^{n+1}$ for each $i\left(m a_{i+1} \leq 2 m \alpha_{i}<2 b^{n+1}\right.$ implies that $\left.m a_{i+1}<b^{n+1}\right)$. This gives a contradiction since $2 b^{n+1} \leq m b \alpha_{1} \leq 2 m a_{k}<b^{n+1}$.

Conversely, suppose (3.1.2) fails. If $\alpha_{j+1}>2 \alpha_{j}$ for some $j$, set $m_{i}=k_{i}$ for $i \leq j$ and $m_{i}=k_{i}+1$ for $i>j$. Then it is straightforward to verify that the inequalities

$$
a_{1} \leq \cdots \leq a_{j}<\frac{a_{j+1}}{2} \leq \cdots \leq \frac{a_{k}}{2}
$$

can be rewritten as:

$$
\begin{equation*}
\max \left\{\frac{2 b^{m_{i}}}{x_{i}}: 1 \leq i \leq k\right\}<\min \left\{\frac{b^{m_{i}+1}}{x_{i}}: 1 \leq i \leq k\right\} . \tag{3.1.3}
\end{equation*}
$$

(3.1.3) is also true when $b / 2>\alpha_{k} / a_{1}$ provided $m_{i}=k_{i}$ for every $i$. Accordingly, we can find rational numbers of the form $m / b^{q}$ strictly between the two bounds in (3.1.3). Then

$$
2 b^{m_{i}+q}<m x_{i}<b^{m_{i}+q+1}
$$

for all $i$, and (3.1.1) fails.
Part (i) of Result 1 is an immediate consequence of Proposition 3.1 since (3.1.1) can only be true if

$$
\frac{b}{2} \leq \frac{a_{k}}{a_{1}}=\left(\frac{a_{2}}{a_{1}}\right) \cdot \ldots \cdot\left(\frac{a_{k}}{a_{k}-1}\right) \leq 2^{k-1}
$$

and this cannot occur if $2^{k}<b$.
A set $Y$ is called a multiple of $\left\{x_{1}, \ldots, x_{k}\right\}$ if and only if $Y=\left\{m x_{1}, \ldots\right.$, $\left.m x_{k}\right\}$ for some positive integer $m$. $Y$ is called a quasimultiple (in base b) of $\left\{x_{1}, \ldots, x_{k}\right\}$ if and only if

$$
Y=\left\{m^{\prime} \cdot x_{1} \cdot b^{n(1)}, \ldots, m^{\prime} \cdot x_{k} \cdot b^{n(k)}\right\}
$$

where $m^{\prime}$ is a positive integer and $n(1), \ldots, n(k)$ are nonnegative integers. For example, $\{6,9,15\}$ is a multiple of $\{2,3,5\}$, and $\{9,600,150\}$ is a quasimultiple (in base 10) of \{2, 3,5$\}$.

Part (ii) of Result 1 follows from the next proposition.
Proposition 3.2: Let $2^{k} \geq b$. Then every quasimultiple $\left\{x_{1}, \ldots, x_{k}\right\}$ of $\{1,2$, ..., $\left.2^{k-1}\right\}$ has property (3.1.1). There are other sets with this property if and only if $2^{k}>b$.
Proof: The set $T=\left\{1,2, \ldots, 2^{k-1}\right\}$ satisfies (3.1.2) if $2^{k} \geq b$. Hence, it satisfies (3.1.1). Since (3.1.1) is preserved under quasimultiplication (multiplication by powers of $b$ merely adjoins zeros on the right), quasimultiples of $T$ also satisfy (3.1.1).

If $\left\{x_{1}, \ldots, x_{k}\right\}$ has property (3.1.1) and is indexed as in Proposition 3.1, then (3.1.2) can be rewritten as
(3.2.1) $1 \leq \frac{a_{i+1}}{a_{i}} \leq 2, \frac{b}{2} \leq\left(\frac{a_{2}}{a_{1}}\right) \cdot \ldots \cdot\left(\frac{a_{k}}{a_{k-1}}\right) \leq 2^{k-1}$.

If $2^{k}=b, a_{i+1} / a_{i}$ must equal 2 for each $i$. Then

$$
x_{i}=2^{i-1} a_{1} b^{k_{i}} \text { for each } i
$$

where $a_{1}=x_{1} / b^{k_{1}}$ is a fraction of the form $m / 2^{q}$ with $m$ odd. It follows easily that $x_{i}=m \cdot 2^{m_{i}}$, where each $m_{i}$ is a distinct integer mod $k$. Hence $\left\{x_{1}, \ldots\right.$, $\left.x_{k}\right\}$ is a quasimultiple of $T$. On the other hand, if $2^{k}>b$, we can choose frac-
tions $r_{i}=\alpha_{i+1} / \alpha_{i}$ satisfying (3.2.1) with each inequality satisfied strictly and the product less than min $\left\{b, 2^{k-1}\right\}$. We also choose a fraction $a_{1} \geq 1$ such that $a_{1} \cdot r_{2} \cdot \cdots \cdot r_{k}<b$. All of these fractions can be taken to have the form $m / b^{q}$ with $m$ odd. Multiplying each $\alpha_{i}=a_{1} \cdot r_{2} \cdot \cdots \cdot r_{i}$ by the smallest power of $b$ that makes the product an integer, we get $x_{i}$ 's satisfying (3.1.2), and hence (3.1.1). Since each $x$ is odd, $\left\{x_{1}, \ldots, x_{k}\right\}$ cannot be a quasimultiple of $T$.

## 4. The Rightmost Digit Case

Proposition 4.1: Let $b$ be neither a prime nor a prime power. Then for any set of positive integers $x_{1}, \ldots, x_{k}$ there is an integer $m \geq 1$ such that none of $m x_{1}$, ..., $m x_{k}$ has rightmost nonzero digit 1.

Proof: Express $b$ as the product of two relatively prime integers $r$ and $s$ that are greater than l. Let $t$ be the highest power of $r$ that occurs in any of $x_{1}$, $\ldots, x_{k}$, and let $m=s^{t+1}$.

If for some $i$ the rightmost nonzero digit of $m x_{i}$ is 1 , then

$$
m x_{i}=s^{t+1} x_{i} \equiv b^{n-1} \quad\left(\bmod b^{n}\right)
$$

for some positive integer $n$. So $r^{n-1}$ divides $x_{i}$ and $n-1 \leq t$. Removing the common factor $s^{n-l}$ from the equation above, we conclude that $s$ divides a power of $r$. Since this is impossible, all of the integers $m x_{i}$ have rightmost nonzero digit distinct from 1.

Proposition 4.2: Let $b=p^{n}$ where $p$ is a prime. Then the following are equivalent:
(4.2.1) for each integer $m \geq 1$ at least one of $m x_{1}, \ldots, m x_{k}$ has rightmost nonzero digit 1 ,
and
(4.2.2) for each positive integer $c$ in $\{1, \ldots, b-1\}$ that is relatively prime to $p$ and each integer $i$ in $\{0, \ldots, n-1\}$, there is an $x$ in $\left\{x_{1}, \ldots, x_{k}\right\}$ such that $y \equiv c p^{i}\left(\bmod p^{n+i}\right)$ where $y$ is the quotient obtained by dividing $x$ by the highest power of $b$ in $x$.

Proof: Suppose that (4.2.2) holds. To establish (4.2.1), we assume without loss of generality that $m$ is a positive integer not divisible by $b$. Then $m=\alpha p^{s}$, where $\alpha$ is a positive integer not divisible by $p$, and $0 \leq s \leq n-1$. Because $a$ and $b$ are relatively prime, there are integers $c$ and $d$ such that $a c+$ $b d=1$ with $1 \leq c \leq b-1$. If $s=0$, let $i=0$ and choose $x, y$ as in (4.2.2) so that $y \equiv c(\bmod b)$. Then $m y \equiv m c \equiv 1(\bmod b)$. So my has rightmost digit 1 , and (4.2.1) holds for $m x$. If $s \geq 1$, let $i=n-s$ and choose $x, y$ as in (4.2.2) so that $y \equiv c p^{n-s}\left(\bmod p^{2 n-s}\right)$. Then
$m y \equiv a c p^{n}\left(\bmod p^{2 n}\right) \equiv p^{n}\left(\bmod p^{2 n}\right) \equiv b\left(\bmod b^{2}\right)$.
Thus, my has its two rightmost digits equal to $10, m x$ has rightmost nonzero digit 1 , and (4.2.1) holds.

Conversely, suppose (4.2.1) holds. Remove all powers of $b$ from each $x_{i}$, and the resulting set $\left\{y_{1}, \ldots, y_{k}\right\}$ still satisfies (4.2.1) with none of the $y_{i}$ 's divisible by $b$. Let $c$ be any integer from 1 to $b-1$ relatively prime to $p$. Choose integers $a$ and $d$ such that $a c+b d=1$ and $1 \leq a \leq b-1$. Let $m=a p^{n-i}$ with $0 \leq i \leq n-1$, and by (4.2.1) choose $y$ in $\left\{y_{1}, \ldots, y_{k}\right\}$ such that my has rightmost nonzero digit 1 . Then

$$
m y \equiv a p^{n-i} y \equiv b^{s}\left(\bmod b^{s+1}\right) \text { for some } s \geq 0
$$

Since $p$ does not divide $a$, and $b$ does not divide $y, s=1$ and $p^{i}$ divides $y$. Then

$$
\alpha y \equiv p^{i}\left(\bmod p^{n+i}\right) \text { and } y=(a c+b d) y \equiv c p^{i}\left(\bmod p^{n+i}\right),
$$

as required in (4.2.2).
Corollary 4.3: Let $b=p^{n}$. Then there exist sets $\left\{x_{1}, \ldots, x_{k}\right\}$ which satisfy (4.2.1) if and only if $k \geq n\left(p^{n}-p^{n-1}\right)$.

Proof: The number of positive integers $c$ in $\{1, \ldots, b-1\}$ relatively prime to $p$ is $p^{n}-p^{n-1}$, and the number of equations of the form $y \equiv c p^{i}\left(\bmod p^{n+i}\right)$, with $c$ as above and $0 \leq i \leq n-1$, is $n\left(p^{n}-p^{n-1}\right)$. It is easy to see that no integer $y$ satisfies two different equations of this form. Thus (4.2.2), and hence (4.2.1), can be satisfied precisely when $k \geq n\left(p^{n}-p^{n-1}\right)$.

Parts (i) and (ii) of Result 2 follow at once from Proposition 4.1 and Corollary 4.3.

## 5. Strings of Rightmost Digits

Lemma 5.1: Let $\left(z_{1}, \ldots, z_{k}\right)$ be an ordered $k$-tuple of positive integers satisfying

$$
\begin{equation*}
\sum_{i=1}^{k} \operatorname{gcd}\left(b, z_{i}\right) \leq b-2 \tag{5.1.1}
\end{equation*}
$$

Then, for every ordered $k$-tuple ( $y_{1}, \ldots, y_{k}$ ) of integers, there is an integer $m$ in $\{1, \ldots, b-1\}$ such that none of the equations

```
(5.1.2) mzi}\equiv\mp@code{yi (mod b), i = 1, ...,k,
```

is true.
If it is assumed that
(5.1.3) $\quad \sum_{i=1}^{k} \operatorname{gcd}\left(b, z_{i}\right) \leq b-1$,
the conclusion above holds for some integer $m$ in $\{0, \ldots, b-1\}$.
Proof: By elementary number theory ([4], p. 102), the equation $m z_{i} \equiv y_{i}(\bmod b)$ has a solution $m$ in the integers mod $b$ if and only if $y_{i}$ is divisible by gcd ( $b$, $z_{i}$ ). When such a solution exists, there are exactly $\operatorname{gcd}\left(b, z_{i}\right)$ of them. If we assume the worst, then equations (5.1.2) all have distinct solutions. This leaves

$$
\left[b-1-\sum_{i=1}^{k} \operatorname{gcd}\left(b, z_{i}\right)\right] \quad(>0)
$$

integers $m$ among the integers $1,2, \ldots, b-1$ to satisfy the conditions of the lemma.

The last statement is proved similarly.
The $n$ rightmost nonzero digits of $x$ refers to the string of $n$ successive digits in $x$ whose rightmost member is the rightmost nonzero digit of $x$. Thus, e.g., in base 10 the three rightmost nonzero digits of 740,500 are 4,0 , and 5 .

Proposition 5.2: Let $\left\{x_{1}, \ldots, x_{k}\right\}$ be a set of positive integers whose rightmost nonzero digits satisfy (5.1.1), and let $n$ be a positive integer. Then there exists an integer $m$ in $\left\{1, \ldots, b^{n}-1\right\}$ such that none of $m x_{1}, \ldots, m x_{k}$ has the digit 1 among its $n$ rightmost nonzero digits.
Proof: Let $z_{1}, \ldots, z_{k}$ be the rightmost digits of $x_{1}, \ldots, x_{k}$, none of them zero without loss of generality. By Lemma 5.1, choose $m_{0}$, the rightmost digit of $m$, in $\{1, \ldots, b-1\}$ so that the equations

```
mozi}\equiv1(\operatorname{mod}b) for i such that gcd(b, z zi) = 1
m0}\mp@subsup{z}{i}{}\equiv0(\operatorname{mod}b) for i such that gcd (b, zi ) > 1
```


## NUMBERS WITHOUT ONES

$(i=1, \ldots, k)$ are all false. Then $m_{0} z_{i}(\bmod b)$, the rightmost digit of $m x_{i}$, is in the set $\{2, \ldots, b-1\}$ for each $i$.

If the first $j$ digits of $m$ from right to left- $m_{0}, \ldots, m_{j-1}$-have already been chosen, then the $(j+1)^{\text {th }}$ digit of $m x_{i}$ will equal $m_{j} z_{i}+u_{i j}(\bmod b)$, where $u_{i j}$ is an integer depending on the first $j$ digits of $m$ and $x_{i}$ and the $(j+1)^{\text {th }}$ digit of $x_{i}$. By Lemma 5.1 , choose $m_{j}$ in $\{1, \ldots, b-1\}$ so that none of the equations

$$
m_{j} z_{i} \equiv 1-u_{i j}(\bmod b), \quad i=1, \ldots, k
$$

holds. For $j \geq n$, set $m_{j}=0$. Then $m$ is as required.
Proof of Result 3: Let $q$ be the smallest prime factor of $b$. The rightmost nonzero digits $z_{i}$ of $x_{i}$ satisfy $\operatorname{gcd}\left(b, z_{i}\right) \leq b / q$. If
(5.3.1) $k \cdot\left(\frac{b}{q}\right) \leq b-2$,
then (5.1.1) is true and Proposition 5.2 yields Result 3.
When $b$ is prime or $k \leq q-1$, the hypotheses in Result 3 ensure that (5.3.1) holds. Thus, we need only consider the case when $b$ is composite and $k=q$.

Suppose until further notice that $\operatorname{gcd}\left(b, z_{i}\right)$ is smaller than $b / q$ for at least one $i$. Then the left side of (5.1.1) is bounded above by $(q-1)(b / q)+$ $r^{\prime}$, where $r^{\prime}$ is the second largest factor of $b$, the largest being $r=b / q$. If $r^{\prime} \leq r-2$, (5.1.1) applies again. If not, $b=6$ or 4 .

If $b=6$, then $q=2$ and $\left\{z_{1}, z_{2}\right\}$ is a pair. Either $m_{0}=1$ fails to satisfy each of the two equations (5.2.1) or else one of $z_{1}$ and $z_{2}$ is 1 . In the latter case,

$$
\operatorname{gcd}\left(b, z_{1}\right)+\operatorname{gcd}\left(b, z_{2}\right) \leq 3+1 \leq 6-2,
$$

and (5.1.1) is fulfilled. In the former case, the induction in Proposition 5.2 can proceed using (5.1.3) since

$$
\operatorname{gcd}\left(b, z_{1}\right)+\operatorname{gcd}\left(b, z_{2}\right) \leq 3+2 \leq 6-1
$$

and $m_{j}$ can be chosen equal to zero if necessary (for $j \geq 1$ ).
If $b=4$, then $q=2$ again. An argument similar to the last one applies except when $\left\{z_{1}, z_{2}\right\}=\{1,2\}$. Then $m_{0}=3$ can be used to falsify equations (5.2.1), and $2+1 \leq 4-1$ ensures that (5.1.3) applies to the later digits of m.

Finally, suppose $\operatorname{gcd}\left(b, z_{i}\right)=r$ for each $i=1, \ldots, q$. Then each $x_{i}=$ $y_{i} r^{s}$ with $s \geq 1$, where $r^{s}$ is the largest power of $r$ dividing all $x_{i}$. Then $\left\{y_{1}\right.$, $\left.\ldots, y_{q}\right\}$ is covered by the earlier arguments since not all $y_{i}$ 's are divisible by $r$. If none of $m y_{l}, \ldots, m y_{q}$ has 1 's among the $n$ rightmost digits, the same is true for

$$
\left(q^{s} m\right) x_{1}, \ldots,\left(q^{s} m\right) x_{q}=m y_{1} b^{s}, \ldots, m y_{q} b^{s}
$$

## 6. Further Questions

1. To what extent do these results apply to other digits (or strings of digits) and other positions? For example, under what conditions can we ensure that the two rightmost nonzero digits differ from 1 ?
2. Can the hypotheses of Result 3 be weakened, or are they necessary as well as sufficient?
3. What are the smallest multipliers needed in Results 1,2 , and 3 ? The proofs provide upper bounds, but calculations suggest that much smaller multipliers will often suffice.
[^1]
# Applications of Fibonacci Numbers 

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# THE FIBONACCI AND PARACHUTE INEQUALITIES FOR $\ell_{1}$-METRICS 

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## 1. Introduction

Let $X$ be a finite set with $|X|=n$. A metric on $X$ is a real valued function $d$ defined on all pairs of points of $X$ and satisfying the triangle inequality:

$$
\begin{equation*}
d(i, j)+d(j, k) \geq d(i, k) \tag{1}
\end{equation*}
$$

for all triples $(i, j, k)$ of points of $X$. We allow $d(i, j)=0$ for some pairs ( $i, j$ ); so we use the term "metric" for denoting what is usually called semimetric. We set $d(i, j)=d(j, i)$ for all pairs $(i, j)$ and $d(i, i)=0$ for all points $i$ of $X$. The pair $(X, d)$ is called a metric space. The $\ell_{1}$-metric on $R^{m}$ is defined by:

$$
d(x, y)=\|x-y\|_{1}=\sum_{1 \leq i \leq m}\left|x_{i}-y_{i}\right|
$$

A metric space ( $X$, d) is isometrically $\ell_{1}$-embeddable if there exist points $x_{0}$, $x_{1}, \ldots, x_{n}$ in some space $R^{m}$ such that

$$
d(i, j)=\left\|x_{i}-x_{j}\right\|_{1} \text { for all } 0 \leq i<j \leq n .
$$

The family of all metrics $d$ on $X$ which are isometrically $\ell_{1}$-embeddable forms a cone: $C(X)=C_{n}$, called cut cone (or Hamming cone). The cut cone $C_{n}$ is generated by the cut metrics $d_{S}$ for subsets $S$ of $X$, where

$$
d_{S}(i, j)=1 \text { if }|S \cap\{i, j\}|=1 \text { and } d_{S}(i, j)=0 \text { otherwise. }
$$

Therefore, a metric $d$ on $X$ is isometrically $\ell_{1}$-embeddable if and only if $d$ is the conic hull of cut metrics:

$$
d=\sum_{S \subseteq X} \lambda_{S} d_{S}, \text { with } \lambda_{S} \geq 0
$$

The cut metrics $d_{S}$ correspond, in graph terminology, to the cuts $\delta(S)$; we shall use the latter terminology here. The study of the $\ell_{1}$-embeddable finite metric spaces, i.e., of the cut cone $C_{n}$, was started in 1960 in [5] and continued, e.g., in [1], [3], [6], [8], [9], and [10]. If $d$ is rational valued, then $d$ is isometrically $\ell_{1}$-embeddable if and only if $k d$ is isometrically embeddable into the vertex set of the hypercube of $R^{m}$ for some integers $k, m$ ([2]).

Given a vector $v=\left(v_{i j}\right)_{1 \leq i \leq j<n}$, the inequality $v . x \leq 0$ is called valid over the cut cone $C_{n}$ if it is satisfied by all points of $C_{n}$ or, in other words, by all metrics on $n$ points which are isometrically $\ell_{1}$-embeddable. The roots of inequality $v . x \leq 0$ are the cuts $\delta(S)$ satisfying equality: $v . \delta(S)=0$. The rank of inequality $v . x \leq 0$ is the rank of its set of roots. Geometrically, valid inequalities correspond to faces of the cone $C_{n}$ while valid inequalities of highest possible rank:

$$
\binom{n}{2}-1=n(n-1) / 2-1
$$

define facets of $C_{n}$.
[Feb.

Many examples of valid inequalities over the cut cone $C_{n}$ are known; for example,
(a) the hypermetric inequalities ([5], [11], [8]) of the form

$$
\sum_{1 \leq i<j \leq n} b_{i} b_{j} d(i, j) \leq 0
$$

where $b_{1}, \ldots, b_{n}$ are integers satisfying

$$
\sum_{1 \leq i \leq n} b_{i}=1
$$

including triangle inequality (1) as a special case for $b=(1,1,-1,0, \ldots, 0)$.
(b) the bicycle odd wheel inequality [4], defined on $2 k+3$ points $\left\{0,0^{\prime}\right.$, $1,2, \ldots, 2 k+1\}$ for $k \geq 1$ by

$$
\begin{equation*}
d\left(0,0^{\prime}\right)-\sum_{1 \leq i \leq 2 k+1}\left(d(0, i)+d\left(0^{\prime}, i\right)\right)+\sum_{(i, j) \in C} d(i, j) \leq 0 \tag{2}
\end{equation*}
$$

where $C$ denotes the cycle ( $1,2, \ldots, 2 k+1$ ).
(c) the parachute inequality [8], denoted as $\operatorname{Par}_{2 k+1}$, defined on the $2 k+1$ points $\left\{0,1,2, \ldots, k, 1^{\prime}, 2^{\prime}, \ldots, k^{\prime}\right\}$ by:

$$
\begin{align*}
\operatorname{Par}_{2 k+1} \cdot d=\sum_{(i, j) \in P} d(i, j) & -\sum_{1 \leq i \leq k-1}\left(d(0, i)+d\left(0, i^{\prime}\right)\right.  \tag{3}\\
& \left.+d\left(k, i^{\prime}\right)+d\left(k^{\prime}, i\right)\right)-d\left(k, k^{\prime}\right) \leq 0
\end{align*}
$$

where $P$ is the path $\left(k, k-1, \ldots, 2,1,1^{\prime}, 2^{\prime}, \ldots .,(k-1)^{\prime}, k^{\prime}\right)$.
(d) the Fibonacci inequality [10], denoted as Fib $2 k$, defined on the $2 k$ points $\left\{0,0^{\prime}, 1,1^{\prime}, 2,2^{\prime}, \ldots,(k-1),(k-1)^{\prime}\right\}$ by

$$
\begin{align*}
\mathrm{Fib}_{2 k} \cdot d= & \sum_{(i, j) \in Q} d(i, j)-  \tag{4}\\
& -\sum_{1 \leq i \leq k-1}\left(d(0, i)+d\left(0, i^{\prime}\right)\right) \\
& \sum_{1 \leq i \leq k-2}\left(d\left(0^{\prime}, i\right)+d\left(0^{\prime}, i^{\prime}\right)\right) \leq 0
\end{align*}
$$

where $Q$ is the path $\left(k-1, k-2, \ldots, 2,1,1^{\prime}, 2^{\prime}, \ldots,(k-2)^{\prime},(k-1)^{\prime}\right)$. We call the above inequality (4) the Fibonacci inequality, since its number of roots is related to the Fibonacci number $f_{k}\left(f_{1}=f_{2}=1, f_{k+2}=f_{k+1}+f_{k}\right)$.

See Figures $1-3$ for the graphic representation of inequalities (2) and (3) on seven points and inequality (4) on six points (a plain line indicates coefficient +1 and a dotted line indicates coefficient -1 ).


Figure 1
The Bicycle Odd Wheel Inequality on 7 Points


Figure 2
The Parachute Inequality $\mathrm{Par}_{7}$ on 7 Points


Figure 3
The Fibonacci Inequality $\mathrm{Fib}_{6}$ on 6 Points

In this note, we consider the Fibonacci and parachute inequalities, their number of roots in terms of Fibonacci numbers, their rank, their symmetries, and the connections with the bicycle odd wheel inequality (2).

## 2. Parachute and Fibonacci Inequalities: Equality Case

Given a path $A=(1,2, \ldots, n)$, a subset $S$ of $[1, n]$ is called altermated along path $A$ if

$$
|S \cap\{i, i+1\}| \leq 1 \text { for all } i=1, \ldots, n-1,
$$

and pseudo-aZternated along path $A$ if

$$
|S \cap\{i, i+1\}|=1 \text { for a11 } i=1, \ldots, j-1, j+1, \ldots, n-1
$$

and

$$
|S \cap\{j, j+1\}|=0 \text { or } 2 \text { for some } j \in[1, n-1] .
$$

One observes easily that, for $n$ even, the number of pseudo-alternated subsets $S$ along path $A=(1, \ldots, n)$ for which nodes $1, n$ belong to $S$ is equal to $n-1$; an easy induction on $n$ shows that the number of alternated subsets of $[1, n]$ along path $(1,2, \ldots, n)$ is equal to the Fibonacci number $f_{n+2}$ (where $[1, n]$ denotes the set of integers $1,2,3, \ldots, n)$.

Call a cut $\delta(S)$ symmetric if, for $i=1,2, \ldots, k$, $i$ belongs to $S$ if and only if $i^{\prime}$ belongs to $S$, i.e., the involution

$$
\alpha=\prod_{1 \leq i \leq k}\left(i i^{\prime}\right)
$$

belonging to the symmetric group $\operatorname{Sym}(2 k+1)$ leaves $S$ invariant.
We describe below the roots of the parachute inequality.
Proposition 1: The roots of the parachute inequality $\operatorname{Par}_{2 k+1}$ are the cuts $\delta(S)$ for which $S$ is a subset of $[1, k] \cup\left[1^{\prime}, K^{\prime}\right]$ of one of the following four types:

Type 1: nodes $k, K^{\prime}$ belong to $S$ and $S$ is pseudo-alternated along path $P$.
Type 2: nodes $k, K^{\prime}$ do not belong to $S$ and $S$ is alternated along path $Q$.
Type 3: for $k$ odd, node $k$ belongs to $S$, node $k^{\prime}$ does not belong to $S$ and (a) or (b) holds:
(a) $S=\left\{2^{\prime}, 4^{\prime}, \ldots,(k-1)^{\prime}, k\right\} \cup T$, where $T$ is a subset of $\{1$, $2, \ldots, k-2\}$ alternated along path $\{1,2, \ldots, k-2\} ;$
(b) $S=\left\{k, 1^{\prime},(k-1)^{\prime}\right\} \cup T \cup V$, where $T$ is a subset of $\{2,3, \ldots$, $k-2\}$ alternated along path (2, 3, ..., $k-2$ ) and $V$ is a subset of $\left\{2^{\prime}, 3^{\prime}, \ldots .,(k-2)^{\prime}\right\}$ such that $V \cup\left\{1{ }^{\prime},(k-1)^{\prime}\right\}$ is pseudo-alternated along path ( $\left.1^{\prime}, 2^{\prime}, \ldots,(k-1)^{\prime}\right)$.
Type $3^{\prime}$ : similar to type 3 , exchanging nodes $i$, $i^{\prime}$ for all $i=1,2, \ldots, k$.
There are $2 k-1$ roots of type 1 , all of them linearly independent and the only symmetric root among them is $\delta\left(\left\{1,3, \ldots, k, 1^{\prime}, 3^{\prime}, \ldots, k^{\prime}\right\}\right)$ for $k$ odd and $\delta\left(\left\{2,4, \ldots, k, 2^{\prime}, 4^{\prime}, \ldots, k^{\prime}\right\}\right)$ for $k$ even. There are $f_{2 k}$ roots of type 2 , their rank is $\left(\begin{array}{c}2 k-1\end{array}\right)-2 k+3$ and there are $f_{k}$ symmetric roots among them. The roots of type $3,3^{\prime}$ exist only for $k$ odd; there are altogether $2\left(f_{k}+\right.$ $(k-1) f_{k-1}$ ) such roots and there are no symmetric roots among them.
Proposition 2: (i) The number of roots [including zero, i.e., cut $\delta(\phi)$ ] of the parachute inequality $\operatorname{Par}_{2 k+1}$ is equal to $f_{2 k}+2 k f_{k-1}+2 f_{k-2}+2 k-1$ for $k$ odd and $f_{2 k}+2 k-1$ for $k$ even, while the number of (nonzero) symmetri_c roots is always the Fibonacci number $f_{k}$. (ii) The parachute inequality $P$ ar $2 k+1$ is facet inducing for $k$ odd; for $k$ even, it has rank $\left(2 k_{2}-1\right)+2$, but it is not valid.

Proof of Propositions 1 and 2: Given a subset $S$ of $[1, k] \cup\left[1^{\prime}, k^{\prime}\right]$, we set $s=|S \cap[1, k-1]|$ and $s^{\prime}=\left|S \cap\left[1^{\prime},(k-1)^{\prime}\right]\right|$.

In order to characterize which cuts $\delta(S)$ are roots of the parachute inequality $\operatorname{Par}_{2 k+1}$, we distinguish four cases:

Case 1: $k, k^{\prime} \in S$
Then $\delta(S)$ is a root of $\operatorname{Par}_{2 k+1}$ if and only if $|\delta(S) \cap P|=2 k-2$, i.e., all edges of $P$ but one are edges of $\delta(S)$, i.e., $S$ is pseudo-alternated along path $P$. So there are $2 k-1$ such roots, among them only one symmetric root:

$$
\delta\left(\left\{k, \ldots, 3,1,1^{\prime}, 3^{\prime}, \ldots, k^{\prime}\right\}\right) \text { for } k \text { odd }
$$

and

$$
\delta\left(\left\{k, \ldots, 4,2,2^{\prime}, 4^{\prime}, \ldots, k^{\prime}\right\}\right) \text { for } k \text { even. }
$$

Case 2: $k, k^{\prime} \notin S$
Then $\delta(S)$ is a root of $\operatorname{Par}_{2 k+1}$ if and only if $|\delta(S) \cap P|=2\left(s+s^{\prime}\right)=2|S|$, i.e., $S$ is alternated along the path ( $\left.k-1, \ldots, 1,1^{\prime}, \ldots,(k-1)^{\prime}\right)$ on $2 k-2$ nodes, so there are $f_{2 k}$ such roots. Among them, the number of symmetric roots (including zero) is equal to the number of alternated subsets along path (2, 3, ..., k - 1), i.e., to $f_{k}$.
Case 3: $k \in S, k^{\prime} \notin S$
Then $\delta(S)$ is a root of $\operatorname{Par}_{2 k+1}$ if and only if $|\delta(S) \cap P|=k+2 s$. Since

$$
|\delta(S) \cap P|=\left|\delta(S) \cap \operatorname{Path}\left(k, \ldots, 1,1^{\prime}\right)\right|+\left|\delta(S) \cap \operatorname{Path}\left(1^{\prime}, \ldots, k^{\prime}\right)\right|
$$

with the first term being less than or equal to $2 s+2$, we have to distinguish two cases.
Case 3a: $\left|\delta(S) \cap \operatorname{Path}\left(1^{\prime}, \ldots, k^{\prime}\right)\right|=k-1$
If $k$ is even, then, necessarily, $1^{\prime} \in S$, contradicting the fact that

$$
\left|\delta(S) \cap \operatorname{Path}\left(1,1^{\prime}, \ldots, k^{\prime}\right)\right|=2 s+1
$$

If $k$ is odd, then

$$
S \cap\left\{1^{\prime}, 2^{\prime}, \ldots, k^{\prime}\right\}=\left\{2^{\prime}, 4^{\prime}, \ldots,(k-1)^{\prime}\right\}
$$

and

$$
\left|\delta(S) \cap \operatorname{Path}\left(1^{\prime}, 1,2, \ldots, k\right)\right|=2 s+1
$$

i.e., $S$ is alternated along path (1, 2, ..., $k-2$ ); so there are $f_{k}$ such roots.

Case 3b: $\left|\delta(S) \cap \operatorname{Path}\left(1^{\prime}, \ldots, k^{\prime}\right)\right|=k-2$
If $k$ is even, then, necessarily, $1^{\prime} \notin S$, contradicting the fact that

$$
\left|\delta(S) \cap \operatorname{Path}\left(1^{\prime}, 1,2, \ldots, k\right)\right|=2 s+2
$$

If $k$ is odd, then, necessarily, $1^{\prime},(k-1)^{\prime} \in S$ and, since

$$
\left|\delta(S) \cap \operatorname{Path}\left(1^{\prime}, 1, \ldots, k\right)\right|=2 s+2
$$

$S$ is alternated along path (2, 3, ..., $K-2$ ), while $S$ is pseudo-alternated along path ( $1^{\prime}, \ldots, k^{\prime}$ ); so there are $(k-1) f_{k-1}$ such roots.
Case 4: Identical to Case 3, exchanging nodes $i$, $i^{\prime}$ for $i=1, \ldots, k$.
Hence, the total number of roots is:

$$
2 k-1+f_{2 k}+2 f_{k}+2(k-1) f_{k-1}
$$

$$
=2 k-1+f_{2 k}+2 k f_{k-1}+2 f_{k-2} \text { for } k \text { odd }
$$

and

$$
2 k-1+f_{2 k} \quad \text { for } k \text { even }
$$

while the number of nonzero symmetric roots is $f_{k}$, stating Proposition 2 (i).
We now prove Proposition 2(ii). It was proven in [8] that Par $2 k+1$ is facet inducing for $k$ odd and that it is not valid for $k$ even. We now consider $\operatorname{Par} 2 k+1$ for $k$ even; the set of its roots is $R_{1} \cup R_{2}$, where $R_{i}$ denotes the set of roots
of type $i$ (Proposition 1), for $i=1,2$. To facilitate the computation of the rank of the set of roots, we use the following notion of intersection vector: for a subset $S$ of $[1, k] \cup\left[1^{\prime}, k^{\prime}\right]$, define the vector $\pi(S)$ of $\{0,1\}^{k(2 k+1)}$ by

$$
\pi(S)_{i j}=1 \text { if } i, j \in S \text { and } \pi(S)_{i j}=0 \text { otherwise }
$$

for all. $i$, $j$ (not necessarily distinct) in [1, $k] \cup\left[1^{\prime}, k^{\prime}\right]$. Given a family of subsets $\left(S_{a}: a \in A\right)$ of $[1, k] \cup\left[1^{\prime}, k^{\prime}\right]$, the family of cut vectors ( $\delta\left(S_{a}\right): a \in A$ ) is linearly independent if and only if the family of intersection vectors ( $\pi\left(S_{a}\right): a \in A$ ) is linearly independent (see [8]).

First, we check that all roots in $R_{1}=\left\{\delta\left(S_{a}\right): \alpha \in A\right\}$ are linearly independent. For this, we take a linear combination of their intersection vectors:

$$
\sum_{a \in A} \lambda_{a} \pi\left(S_{a}\right)=0
$$

To verify that $\lambda_{a}=0$ for all $\alpha$, observe that, for each root $\delta\left(S_{a}\right)$ of $R_{1}$, one can find a pair ( $i, j$ ) such that $\{i, j\} \subseteq S_{a}$, while $\{i, j\} \nsubseteq S_{b}$ for the other roots $\delta\left(S_{b}\right)$ of $R_{1}$ [for instance, take the pair $(k-1, k)$ for the root $\delta(\{k$, $\left.\left.\left.k-1, k-3, \ldots, 2,1^{\prime}, 3^{\prime}, \ldots, k^{\prime}\right\}\right)\right]$.

Next, we check that the rank of the family $R_{2}$ is

$$
\binom{2 k-1}{2}-2 k+3
$$

For this, observe first that the subfamily $R_{2}^{\prime}$ of $R_{2}$ consisting of all possible singletons and pairs of $[1, k-1] \cup\left[1^{\prime},(k-1)^{\prime}\right]$ has full rank equal to

$$
2 k-2+\binom{2 k-2}{2}-(2 k-3)=\binom{2 k-1}{2}-(2 k-3)
$$

(easy if one considers the intersection vectors). Then, note that, for every cut $\delta(S)$ of $R_{2}$, nodes $k, k^{\prime}$ do not belong to $S$ and $S$ is alternated along $Q$, implying that

$$
x_{k k}=x_{k^{\prime} k^{\prime}}=x_{k k^{\prime}}=x_{k i}=x_{k^{\prime} i}=x_{i, i+1}=0
$$

for $i \in[1, k-1] \cup\left[1^{\prime},(k-1)^{\prime}\right]$, where $x=\pi(S)$ for $\delta(S) \in R_{2}$. Therefore, we deduce that the rank of $R_{2}$ is less than or equal to

$$
\binom{2 k+1}{2}-(6 k-4)=\binom{2 k-1}{2}-(2 k-3)
$$

Finally, we verify that the family $R_{1} \cup R_{2}^{\prime}$ is linearly independent, thus stating that the rank of face $\operatorname{Par}_{2 k+1}$ for $k$ even is

$$
2 k-1+\binom{2 k-1}{2}-(2 k-3)=\binom{2 k-1}{2}+2
$$

Again, we take a linear combination of the intersection vectors

$$
\sum \lambda_{a} \pi\left(S_{a}\right)+\sum \mu_{c} \pi\left(T_{c}\right)=0
$$

where the first sum is over the intersection vectors corresponding to cuts in $R_{1}$ and the second one corresponds to cuts in $R_{2}^{\prime}$. It is enough to show that $\lambda_{a}=0$ for all $a$. For this, for the roots $\delta\left(S_{a}\right)$ of $R_{1}$ having $\{i, i+1\} \subseteq S_{a}$ for some $i$, by looking at the coordinate ( $i, i+1$ ) in the above linear combination we obviously obtain that $\lambda_{a}=0$. For remaining roots $\delta\left(S_{a}\right)$ of $R_{1}$, looking at coordinate ( $k, i$ ) with $i \in S_{a}$ also yields $\lambda_{a}=0$.

Given a vector $v=\left(v_{i j}\right)_{1 \leq i<j \leq n}$ and two points, say 1 and $n$, the vector obtained from $v$ by collapsing points 1 and $n$ into the single point 1 is the vector $v^{\prime}=\left(v_{i j}^{\prime}\right)_{1 \leq i<j \leq n-1}$ defined by

$$
v_{1 i}^{\prime}=v_{1 i}+v_{i n} \text { for } 2 \leq i \leq n-1 \text { and } v_{i j}^{\prime}=v_{i j} \text { for } 2 \leq i<j \leq n-1
$$

The Fibonacci inequality $\mathrm{Fib}_{2 k}$ can be obtained precisely by collapsing points $k$, $k^{\prime}$ into a single point $0^{\prime}$ in the parachute inequality $\operatorname{Par}_{2 k+1}$. Using this observation, the roots of $\mathrm{Fib}_{2 k}$ correspond to the roots of $\operatorname{Par}_{2 k+1}$ of types 1 and 2. So, $\mathrm{Fib}_{2 k}$ and $\operatorname{Par}_{2 k+1}$, for $k$ even, have the same rank, but $\mathrm{Fib}_{2 k}$ is valid while $\operatorname{Par}_{2 k+1}$ is not. Observe also that $\mathrm{Fib}_{2 k}$ coincides (up to renumerotation of the points) with the inequality obtained by collapsing in the bicycle odd wheel inequality (2) point 0 and one point of cycle C. From the above two facts follows the next result.
Proposition 3: The Fibonacci inequality $\mathrm{Fib}_{2 k}$ is valid over the cut cone for any $k \geq 3$ and its rank is

$$
\binom{2 k-1}{2}+2=\binom{2 k}{2}-2 k+3
$$

Its roots are the cuts $\delta\left(S-\left\{k, k^{\prime}\right\}+\left\{0^{\prime}\right\}\right)$ for $S$ of type 1 and $\delta(S)$ for $S$ of type 2. So, $\mathrm{Fib}_{2 k}$ has $2 k-1+f_{2 k}$ roots and $f_{k}$ nonzero symmetric roots.

## 3. Symmetries of the Parachute Inequality

The following two operations on facets of the cut cone $C_{n}$ are given in [8]: (a) permutation-given a vector $v=\left(v_{i j}\right)_{1 \leq i<j \leq n}$ and a permutation $\sigma$ of $\operatorname{Sym}(n)$, set $v_{i j}^{\sigma}=v_{\sigma(i) \sigma(j)}$ for $1 \leq i<j \leq n$; then, inequality $v^{\sigma} . x \leq 0$ is said to be permutation equivalent to $v . x \leq 0$. (b) switching-given vector $v$ and a root $\delta(S)$ of inequality $v . x \leq 0$, set $v_{i j}^{S}=-v_{i j}$ if $|S \cap\{i, j\}|=1$ and $v_{i j}^{S}=v_{i j}$ otherwise; then, inequality $v^{S} . x \leq 0$ is said to be switching equivalent to $v . x \leq 0$. If inequality $v . x \leq 0$ is valid (resp. facet inducing) over the cut cone $C_{n}$, then both inequalities $v^{\sigma} . x \leq 0, v^{S} . x \leq 0$ are valid (resp. facet inducing) over $C_{n}$. In [7] it is shown that permutation and switching (by any cut) are the only symmetries of the cut polytope. The automorphism group Aut (v) of inequality $v . x \leq 0$ is the group $\left\{\sigma \in \operatorname{Sym}(n): v^{\sigma}=v\right\}$ and its group PS $(v)$ of double symmetries is the group $\left\{\sigma \in \operatorname{Sym}(n): v^{\sigma}=v^{S}\right.$ for some root $\delta(S)$ of $v . x \leq 0\}$; so $\operatorname{Aut}(v) \subseteq \operatorname{PS}(v)$ and $\operatorname{PS}(v)$ is the group of permutations which act simultaneously as switchings. So any facet yields many equivalent ones by switching and permutation. For instance, facet Par 7 yields precisely 7560 equivalent facets of $C_{7}$.

The example of facet $\mathrm{Par}_{7}$ presents a lot of beautiful symmetries that we describe in more detail. The automorphism group of $\operatorname{Par}_{7}$ is the subgroup of Sym(7) generated by the involution $\alpha=\left(11^{\prime}\right)\left(22^{\prime}\right)\left(33^{\prime}\right)$, so it is isomorphic to Sym(2). The group PS $\left(\operatorname{Par}_{7}\right)$ of double symmetries of $\operatorname{Par}_{7}$ is the dihedral group $D_{7}$ 。

Facet $\mathrm{Par}_{7}$ has 21 roots (so it is a simplicial facet) partitioned into 3 classes:
$R_{a}=\left\{\delta\left(a_{i}\right): i \in[0,6]\right\}, R_{b}=\left\{\delta\left(b_{i}\right): i \in[0,6]\right\}$,
and $\quad R_{c}=\left\{\delta\left(c_{i}\right): i \in[0,6]\right\}$,
where $\alpha_{i}$ for $i=0,1, \ldots, 6$ denote, respectively, the sets
$\phi,\{2\},\left\{2^{\prime}\right\},\left\{1,3,2^{\prime}\right\},\left\{1^{\prime}, 3^{\prime}, 2\right\},\left\{2,1^{\prime}\right\},\left\{2^{\prime}, 1\right\}$,
$b_{i}$ for $i=0,1, \ldots, 6$ denote, respectively, the sets
$\left\{2,2^{\prime}\right\},\left\{1^{\prime}\right\},\{1\},\left\{2,3,1^{\prime}, 3^{\prime}\right\},\left\{2^{\prime}, 3^{\prime}, 1,3\right\},\left\{2,3^{\prime}\right\},\left\{2^{\prime}, 3\right\}$,
and $c_{i}$ for $i=0,1, \ldots, 6$ denote, respectively, the sets
$\left\{1,3,1^{\prime}, 3^{\prime}\right\},\left\{1^{\prime}, 3\right\},\left\{1,3^{\prime}\right\},\left\{1^{\prime}, 3^{\prime}, 3\right\},\left\{1,3,3^{\prime}\right\}$,
$\left\{1,2,3^{\prime}\right\},\left\{1^{\prime}, 2^{\prime}, 3\right\}$.

Each class $R_{a}, R_{b}, R_{c}$ is the union of four orbits of Aut ( $\operatorname{Par}_{7}$ ) (one of size 1 for the symmetric root and three of size 2). Denote by $F_{a}=\operatorname{Par}_{7}, F_{b}, F_{c}$ the facets obtained by switching $\operatorname{Par}_{7}$ by the symmetrical roots $a_{0}, b_{0}, c_{0}$, respectively. The facets $F_{a}, F_{b}, F_{c}$ are not permutation equivalent; however, they have the same automorphism group: \{id, $\alpha\}$.

We consider the following involutions:

$$
\begin{aligned}
& \pi_{1}=(03)\left(13^{\prime}\right)\left(1^{\prime} 2^{\prime}\right), \pi_{2}=\alpha \pi_{1} \alpha, \pi_{3}=(02)\left(1^{\prime} 3^{\prime}\right)\left(32^{\prime}\right), \\
& \pi_{4}=\alpha \pi_{3} \alpha, \pi_{5}=(01)\left(21^{\prime}\right)\left(2^{\prime} 3^{\prime}\right), \pi_{6}=\alpha \pi_{5} \alpha .
\end{aligned}
$$

Then, it turns out that, for $i \in[1,6]$, the facet obtained by switching of Par ${ }_{7}$ by root $\delta\left(a_{i}\right)$ [resp. $\left.\delta\left(b_{i}\right), \delta\left(c_{i}\right)\right]$ coincides with the facet obtained by permutation of $\mathrm{Par}_{7}$ by $\pi_{i}$. Therefore, $\mathrm{Par}_{7}$ has three nonpermutation equivalent switchings. Its group of double symmetries is the dihedral group $D_{7}$ with generators $\alpha, \pi_{i}$ for $1 \leq i \leq 6$.

Finally, we mention two more curiosities on the roots of $\mathrm{Par}_{7}$ :
(a) all subsets of $\left\{1,1^{\prime}, 2,2^{\prime}, 3,3^{\prime}\right\}$ can be generated by taking symmetric differences of members of the set $\left\{a_{i}: i \in[1,6]\right\}$, or of $\left\{b_{i}: i \in[1,6]\right\}$, or of $\left\{c_{i}: i \in[1,6]\right\}$.
(b) $c_{0}$ is the complement of $b_{0} \Delta\{0\}, c_{i}=b_{i} \Delta\{x\}$ with $x=3,3^{\prime}, 2,2^{\prime}, 1$, $1^{\prime}$, for $i=1,2,3,4,5,6$, respectively.

Most of the above symmetries are lost for the parachute facet $\operatorname{Par}_{2 k+1}$ with $k \geq 5, k$ odd. The automorphism group of $\operatorname{Par}_{2 k+1}$ is still the group of order 2 generated by the involution

$$
\prod_{1 \leq i \leq k}\left(i i^{\prime}\right)
$$

The number of orbits of the set of roots of $\operatorname{Par}_{2 k+1}$ is:

$$
3 f_{k} / 2+f_{2 k} / 2+(k-1) f_{k-1}+k
$$

(number of symmetric roots plus one-half of number of nonsymmetric roots). It is known that the number of orbits of the set of roots is an upper bound for the number of nonpermutation equivalent switchings (see [7]); we conjecture that equality holds for $\operatorname{Par}_{2 k+1}, k$ odd, $k \geq 5$ (but equality does not hold for $\operatorname{Par}_{7}$ ).

## 4. Concluding Remarks

It turns out that both the parachute inequality and the bicycle odd wheel inequality can be decomposed as integer combination of triangle inequalities with all coefficients +1 except one coefficient -1 . For instance, the parachute facet $\operatorname{Par}_{2 k+1}$ for odd $k$ can be decomposed as follows:

$$
\begin{aligned}
\operatorname{Par}_{2 k+1} \cdot x=\sum_{1 \leq i \leq k-1}\left(T\left(\alpha_{i^{\prime}}, i, i+1\right)\right. & \left.+T\left(\alpha_{i}, i^{\prime},(i+1)^{\prime}\right)\right) \\
& +T\left(0,1,1^{\prime}\right)-T\left(0, k, k^{\prime}\right)
\end{aligned}
$$

where $\alpha_{i}=k$ for $i$ odd and $\alpha_{i}=\alpha_{i^{\prime}}=0$ for $i$ even, and

$$
T(a, b, c)=x_{b c}-x_{a b}-x_{a c}
$$

denotes the left-hand side of the triangle inequality on nodes $\alpha, b, c$. A nice property of inequalities $v . x \leq 0$ which can be "triangulated" is that $v . \delta(S)$ is even for all cuts $\delta(S)$. On the other hand, the Fibonacci face $\mathrm{Fib}_{2 k}$ is the sum of triangles; for instance, for $k$ even, we have:

$$
\begin{aligned}
\operatorname{Fib}_{2 k} \cdot x=\sum_{1 \leq i \leq(k-2) / 2}(T(0,2 i, & 2 i+1)+T\left(0,(2 i)^{\prime},(2 i+1)^{\prime}\right) \\
& +T\left(0^{\prime},(2 i-1)^{\prime},(2 i)^{\prime}\right)+T\left(0,1,1^{\prime}\right)
\end{aligned}
$$

Furthermore, we checked that any parachute, Fibonacci, or bicycle odd wheel inequality reduces, by consecutive collapsing, to some triangle inequality (the same holds for their switchings, see [6]).

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## Author and Title Indexed for The Fibonacci Quarterly

Currently, Dr. Charles K. Cook of the University of South Carolina at Sumter is working on an AUTHOR index, TITLE index and PROBLEM index for The Fibonacci Quarterly. In fact, the three indices are already completed. We hope to publish these indices in 1993 which is the 30th anniversary of The Fibonacci Quarterly. Dr. Cook and I feel that it would be very helpful if the publication of the indices also had AMS classification numbers for all articles published in The Fibonacci Quarterly. We would deeply appreciate it if all authors of articles published in The Fibonacci Quarterly would take a few minutes of their time and send a list of articles with primary and secondary classification numbers to

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# FIBONACCI NUMBERS AND LADDER NETWORK IMPEDANCE 

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## 0. Introduction

This work can be considered the natural extension of a previous study about the same subject. In fact, the authors have studied [4], from a mathematical point of view, a particular numerical triangle, called DFF, characterizing the transfer function of an electrical ladder network formed by a cascade of $N$ identical coupled cells.

The present paper deals with the study of another new triangle named DFFz from the authors' initials and from the fact that it characterizes the equivalent impedances determination of the same type of electrical network. In particular, this triangle is strictly related to Thevenin's equivalent impedance which can be expressed by the ratio of two polynomials: the one related to DFFz and the other to DFF triangle.

The DFFz triangle is shown to have a noteworthy interest from the mathematical point of view, because some of its properties are connected with Fibonacci numbers.

## 1. The Generating Polynomials

The DFFz triangle can be formed in the following manner (with $\alpha_{n, k}$ being the general coefficient).

We define [3]:

| (1.1) | $a_{n, k}=0$ |
| :--- | :--- |$\quad$ if $n<k$

while the other elements of the triangle can be derived from the recursive formula
(1.4) $\quad a_{n, k}=a_{n-1, k}+\sum_{\alpha=0}^{n-1} a_{\alpha, k-1} \quad$ if $n>k$.

In this manner, we have the DFFz triangle for values of $\alpha_{n, k}$ :
$\left.\begin{array}{l|rrrrrrrr}n k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ \hline 0 & 1 & & & & & & & \\ 1 & 2 & 1 & & & & & & \\ 2 & 3 & 4 & 1 & & & & & \\ 3 & 4 & 10 & 6 & 1 & & & & \\ 4 & 5 & 20 & 21 & 8 & 1 & & & \\ 5 & 6 & 35 & 56 & 36 & 10 & 1 & & \\ 6 & 7 & 56 & 126 & 120 & 55 & 12 & 1 & \\ \ldots & . & . & . & . & . & . & . & .\end{array}\right) . .$.

Thus, for example, $\alpha_{2,1}=4$ and $\alpha_{6,2}=126$.
The generating polynomial $P_{n}(x)$ is defined [1] as

$$
\begin{equation*}
P_{n}(x)=\sum_{k=0}^{n} a_{n, k} x^{k}, \tag{1.5}
\end{equation*}
$$

where
(1.6) $\quad a_{n, k}=\left.\frac{D^{k} P_{n}(x)}{k!}\right|_{x=0}$

From the DFFz triangle, it is possible to obtain the expressions of the polynomial for small values of $n$, namely,

$$
\begin{align*}
& P_{0}(x)=1 \\
& P_{1}(x)=2+x \\
& P_{2}(x)=3+4 x+x^{2}  \tag{1.7}\\
& P_{3}(x)=4+10 x+6 x^{2}+x^{3}
\end{align*}
$$

and so on.
From (1.1), (1.2), (1.3), (1.4), and (1.5), we have

$$
\begin{equation*}
\sum_{k=1}^{n} a_{n, k} x^{k}=\sum_{k=1}^{n} a_{n-1}, k^{x^{k}}+\sum_{k=1}^{n} \sum_{\alpha=0}^{n-1} a_{\alpha, k-1} x^{k} \tag{1.8}
\end{equation*}
$$

From (1.8), using (1.1), (1.2), (1.3), and (1.5), we have

$$
\begin{equation*}
P_{n}(x)-a_{n, 0}=P_{n-1}(x)-a_{n-1,0}+x \sum_{\alpha=0}^{n-1} \sum_{k=1}^{\alpha+1} a_{\alpha, k-1} x^{k-1} \tag{1.9}
\end{equation*}
$$

(1.10) $P_{n}(x)-(n+1)=P_{n-1}(x)-n+x \sum_{\alpha=0}^{n-1} P_{\alpha}(x)$
(1.11) $\quad P_{n}(x)=1+P_{n-1}(x)+x \sum_{\alpha=0}^{n-1} P_{\alpha}(x)$
which is the recursive formula for the polynomials.
With the initial condition $P_{0}(x)=1$, it is easy to obtain the polynomials (1.7). Furthermore, we can also use (1.6) to find the triangle coefficients.

In order to find the polynomials, we must apply the previous method. Let
(1.12) $f(x, t)=\sum_{n=1}^{\infty} P_{n}(x) t^{n}$.

Then
(1.13) $\quad P_{n}(x)=\left.\frac{D^{n}[f(x, t)]}{n!}\right|_{t=0}$

From (1.11) and (1.12), we have

$$
\begin{align*}
f(x, t) & =\sum_{n=1}^{\infty}\left[1+P_{n-1}(x)\right] t^{n}+x \sum_{n=1}^{\infty} \sum_{\alpha=0}^{n-1} P_{\alpha}(x) t^{n}  \tag{1.14}\\
& =t \sum_{n=1}^{\infty}\left[1+P_{n-1}(x)\right] t^{n-1}+x \sum_{n=1}^{\infty} t^{n}\left[P_{0}(x)+P_{1}(x)+\cdots+P_{n-1}(x)\right] \\
& =t \sum_{n=1}^{\infty} P_{n-1}(x) t^{n-1}+\sum_{n=1}^{\infty} t^{n}+x[1+f(x, t)] \frac{t}{1-t} \\
& =t[1+f(x, t)]+\frac{t}{1-t}+x[1+f(x, t)] \frac{t}{1-t} \\
& =\frac{-t^{2}+t(2+x)}{t^{2}-t(2+x)+1} .
\end{align*}
$$

If we develop the denominator in (1.14) in partial fractions, we obtain
(1.15) $f(x, t)=\frac{1 / y(x)}{t-b(x) / 2}+\frac{-1 / y(x)}{t-c(x) / 2}-1$,
where

$$
y(x)=\left(x^{2}+4 x\right)^{1 / 2}, b(x)=2+x+y, \text { and } c(x)=2+x-y
$$

From the binomial expansion in (1.15), and after simplification, we also have

$$
\text { (1.16) } \begin{aligned}
f(x, t) & =\frac{-2}{y b(x)}\left\{1+\sum_{n \geq 1}\left[\frac{t}{b(x) / 2}\right]^{n}\right\}+\frac{2}{y c(x)}\left\{1+\sum_{n \geq 1}\left[\frac{t}{c(x) / 2}\right]^{n}\right\}-1 \\
& =\sum_{n \geq 1}\left\{\frac{2}{y c(x)}\left[\frac{t}{c(x) / 2}\right]^{n}-\frac{2}{y b(x)}\left[\frac{t}{b(x) / 2}\right]^{n}\right\} \\
& =\sum_{n \geq 1} \frac{2^{n+1}}{y(x)}\left[\frac{1}{[c(x)]^{n+1}}-\frac{1}{[b(x)]^{n+1}}\right] t^{n}
\end{aligned}
$$

from which we have, using (1.12):

$$
\begin{equation*}
P_{n}(x)=\frac{2^{n+1}}{\sqrt{x^{2}+4 x}}\left\{\frac{1}{\left[2+x-\sqrt{x^{2}+4 x}\right]^{n+1}}-\frac{1}{\left[2+x+\sqrt{x^{2}+4 x}\right]^{n+1}}\right\} . \tag{1.17}
\end{equation*}
$$

Furthermore, considering the binomial expansion, we are able to put $P_{n}(x)$ in the following better way:

$$
\begin{align*}
P_{n}(x)=\frac{1}{2^{n+1}}\left\{\left(\frac{2+x}{\sqrt{x^{2}+4 x}}\right.\right. & +1) \sum_{h=0}^{n}\binom{n}{h}(x+2)^{n-h} y^{h}  \tag{1.18}\\
& \left.-\left(\frac{2+x}{\sqrt{x^{2}+4 x}}-1\right) \sum_{h=0}^{n}(-1)^{n}\binom{n}{h}(x+2)^{n-h} y^{n}\right\}
\end{align*}
$$

From this equation, on distinguishing the case of odd $h$ from that of even $h$, we can write

$$
\begin{gather*}
P_{n}(x)=\frac{1}{2^{n}}\left[\sum_{h \equiv 0}^{n}\left(\begin{array}{l}
n \\
h \neq d) \\
h
\end{array}\right)(x+2)^{n-h} x^{h / 2}(x+4)^{n / 2}\right.  \tag{1.19}\\
+\sum_{h \equiv 1}^{\left.\sum_{(\bmod 2)}^{n}\binom{n}{h}(x+2)^{n-h+1} x^{(h-1) / 2}(x+4)^{(h-1) / 2}\right] .} \\
2 . \text { Determination of } a_{n, k}
\end{gather*}
$$

From (1.6), we have

$$
\begin{align*}
a_{n, k}= & \frac{1}{k!2^{n}}\left[\sum _ { h \equiv 0 } ^ { n } \left(\begin{array}{l}
n \bmod 2)
\end{array}\binom{n}{h} \sum_{j=0}^{k}\binom{k}{j} D^{j}\left[x^{h / 2}(x+4)^{h / 2}\right]\right.\right.  \tag{2.1}\\
& \cdot D^{k-j}[x+2]^{n-h}+\sum_{h \equiv 1}^{n}(\bmod 2) \\
& \cdot\binom{n}{h} \sum_{j=0}^{k}\binom{k}{j} D^{j}\left[x^{(h-1) / 2}(x+4)^{(h-1) / 2}\right] \\
& \left.D^{k-j}[x+2]^{n-h+1}\right]_{x=0} .
\end{align*}
$$

Considering Leibniz's formula, we may write

$$
\begin{align*}
D^{j}\left[x^{h / 2}(x+4)^{h / 2}\right]= & \sum_{m=0}^{j}\binom{j}{m}\binom{h / 2}{m} m!x^{(h / 2)-m}  \tag{2.2}\\
& \cdot\binom{h / 2}{j-m}(j-m)!(x+4)^{(h / 2)-j+m}
\end{align*}
$$

$$
\begin{align*}
& D^{j}\left[x^{(h-1) / 2}(x+4)^{(h-1) / 2}\right]=\sum_{m=0}^{j}\binom{j}{m}\binom{(h-1) / 2}{m} m!x^{((h-1) / 2)-m}  \tag{2.3}\\
& \quad \cdot\binom{(h-1) / 2}{j-m}(j-m)!(x+4)^{((h-1) / 2)-j+m} \\
& D^{k-j}\left[(x+2)^{n-h}\right]=\binom{n-h}{k-j}(k-j)!(x+2)^{n-h-k+j}  \tag{2.4}\\
& D^{k-j}\left[(x+2)^{n-h+1}\right]=\binom{n-h+1}{k-j}(k-j)!(x+2)^{n-h+1-k+j} \tag{2.5}
\end{align*}
$$

From equations (2.3), (2.4), and (2.5), and from the properties of binomial coefficients, (2.2) becomes

$$
\begin{align*}
& \alpha_{n, k}= \frac{1}{2^{n}}\left\{\begin{array}{l}
\sum_{h \equiv 0}^{n}\binom{n}{h} \sum_{j=0}^{k} \sum_{m=0}^{j} x^{(h / 2)}-m\binom{n-h}{k-j}(x+2)^{n-h-k+j} \\
\end{array}\right.  \tag{2.6}\\
&+\binom{h / 2}{m}\binom{h / 2}{j-m}(x+4)^{(h / 2)-j+m} \\
& \quad \sum_{(m \equiv 1}^{n}\binom{n}{h} \sum_{j=0}^{k} \sum_{m=0}^{j}\binom{(h-1) / 2}{m}\binom{(h-1) / 2}{j-m}(x+4)^{((h-1) / 2)-j+m} \\
& \bullet x^{\left.((h-1) / 2)-m\binom{n-h+1}{k-j}(x+2)^{n-h+1-k+j}\right\}_{x=0}}
\end{align*}
$$

When $x=0$, the $m$-sums (which contain the $x$-term) exist if and only if $m=(h-1) / 2$ and $m=h / 2$, respectively. So we can write

$$
\left.\begin{array}{l}
\text { (2.7) } \quad a_{n, k}=\sum_{h \equiv 0}^{n}\binom{n}{h} \sum_{j=0}^{k}\binom{h / 2}{j-h / 2}\binom{n-h}{k-j} 2^{h-k-j} \\
\quad+\sum_{h \equiv 1(\bmod 2)}^{n}\binom{n}{h} \sum_{j=0}^{k}(j-(h-1) / 2 \\
j-(h-1) / 2
\end{array}\right)\binom{n-h+1}{k-j} 2^{h-k-j-1} .
$$

Equation (2.7) is the wanted expression which permits us to determine $a_{n, k}$ by substituting for $n$ and $k$.

## 3. The Properties of $a_{n, k}$

### 3.1 The row sums of the triangle are equal to

Fibonacci numbers with even subscripts
From the expression of $P_{n}(x)$, when $x=1$, we have

$$
\begin{equation*}
P_{n}(1)=\frac{1}{2^{n+1} \sqrt{5}}\left[(3+\sqrt{5})^{n+1}-(3-\sqrt{5})^{n+1}\right] \tag{3.1}
\end{equation*}
$$

From Binet's formula, we have
(3.2) $\quad F_{2 n+2}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2 n+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{2 n+2}\right]$.

If we notice that
(3.3) $\left(\frac{1 \pm \sqrt{5}}{2}\right)^{2}=\frac{3 \pm \sqrt{5}}{2}$,
then (3.2) becomes
(3.4) $\quad E_{2 n+2}=\frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{3-\sqrt{5}}{2}\right)^{n+1}\right]$.

In this manner, we have

$$
P_{n}(1)=F_{2 n+2},
$$

where $F_{0}=0, F_{2}=1, F_{4}=3, \ldots$.

### 3.2 The sums of the triangle diagonals

give the powers of 2

From a direct inspection of DFFz triang1e and (1.4), we have that the sum of the elements of an upward-slanting diagonal is equal to the sum of all elements which are above this diagonal PLUS ONE and, consequently, it is equal to the sum of all superior upward-slanting diagonals plus one. This sum value is a power of 2 .

In fact, if we define

$$
\sum^{n}=\sum_{r=0}^{n} a_{n-r, r},
$$

it is possible to write

$$
\begin{aligned}
\sum^{n} & =\sum^{n-1}+\sum^{n-2}+\cdots+\sum^{1}+1+1 \\
& =2\left(\sum^{n-2}+\sum^{n-3}+\cdots+\sum^{1}+2\right) \\
& =\cdots \\
& =2^{n-2}\left(\sum^{1}+2\right) \\
& =2^{n}
\end{aligned}
$$

since $\quad \sum^{l}=2$.

## 4. Conclusions

The principal aim of this paper has been the determination of a closed expression of the general coefficient $a_{n, k}$ of a new numerical triangle, named DFFz, which characterizes Thevenin's equivalent impedance of a ladder network whose elementary cells are directly coupled. Moreover, the authors present some of the interesting mathematical properties of the triangle, one of which is connected with Fibonacci numbers.

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## FIBONACCI WORDS

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## 1. Introduction

In this paper, the notion of Fibonacci word is introduced and the structure of these words is investigated.

Let $\mathscr{A}$ be a nonempty set and let $x$ and $y$ be two words in the alphabet $\mathscr{A}$. A Fibonacci sequence of words derived from $x$ and $y$ is a sequence of words $w_{1}, w_{2}$, $\omega_{3}$, ... with the property that

$$
w_{1}=x, w_{2}=y, w_{n+1}=w_{n} w_{n-1} \text { or } w_{n-1} w_{n}
$$

The pair $x$ and $y$ are called the initial words of the Fibonacci sequence of words; the words $w_{n} w_{n-1}$ and $w_{n-1} w_{n}$ are called the immediate successors of $w_{n}$ and $w_{n}$ is their immediate predecessor. We remark that the Fibonacci sequence of words considered by Knuth [3, p. 85] is the one obtained as above by letting

$$
w_{n+1}=w_{n} w_{n-1} \text { for a11 } n \geq 3
$$

and the one considered by Higgins [1] and Turner [4, 5] is obtained by letting
$w_{n+1}=w_{n-1} w_{n}$ for all $n \geq 3$.
Let $\tilde{\mathscr{S}}$ be the set of all such sequences of words derived from $x$ and $y$; let $\mathscr{S}_{n}$ be the collection of words which happen to be the $n^{\text {th }}$ term of some members of $\tilde{\mathscr{S}}$. For example,

$$
\left.\mathscr{S}_{1}=\{x\}, \mathscr{S}_{2}=\{y\}, \mathscr{H}_{3}=\{y x, x y\}, \mathscr{L}_{4}=\{y x y, y y x, x y\}\right\}
$$

Denote the union of $\mathscr{S}_{n}(n=1,2, \ldots)$ by $\mathscr{S}$. Members of $\mathscr{S}_{n}$ (resp. $\mathscr{S}$ ) are called the $n$th Fibonacci words (resp. Fibonacci words). Note that each word has an obvious representation in terms of $x$ and $y$. Throughout this paper, we consider only such a representation.
Lemma 1: Let $w$ be a Fibonacci word. Then the following statements are true.
(i) If $w$ starts (resp. ends) with an $x$, then $w$ cannot end (resp. start) with an $x$.
(ii) If $w$ starts (resp. ends) with a $y$, then $w$ cannot end (resp. start) with a $y y$.
(iii) There cannot be three or more consecutive occurrences of $y$ and there cannot be two or more consecutive occurrences of $x$ in $w$.
Proof: The result is proved by mathematical induction.
Let $\mathscr{S}_{n}(x, \cdot)$ [resp. $\left.\mathscr{S}_{n}(\cdot, y)\right]$ denote those $n^{\text {th }}$ Fibonacci words which start with an $x$ (resp. end with a $y$ ) and let

$$
\mathscr{S}_{n}(x, y)=\mathscr{S}_{n}(x, \cdot) \cap \mathscr{S}_{n}(\cdot, y)
$$

Define $\mathscr{S}(x, \cdot), \mathscr{S}(\cdot, y)$, etc., in a similar way.

## Corollary 2:

$$
\begin{aligned}
\mathscr{S}_{n} & =\mathscr{S}_{n}(x, y) \cup \mathscr{S}_{n}(y, y) \cup \mathscr{S}_{n}(y, x) \quad \text { for all } n ; \\
\mathscr{S} & =\mathscr{S}(x, y) \cup \mathscr{S}(y, y) \cup \mathscr{S}(y, x)
\end{aligned}
$$

Using finite binary sequences, let us label the Fibonacci words as follows:

and, in general, we have

$$
w_{n+2}^{r_{1} r_{2} \ldots r_{n-1} r_{n}}= \begin{cases}w_{n+1}^{r_{1} r_{2} \ldots r_{n-1}} w_{n}^{r_{1} r_{2} \ldots r_{n-2}} & \text { if } r_{n}=0  \tag{1.2}\\ w_{n}^{r_{1} r_{2} \ldots r_{n-2}} w_{n+1}^{r_{1} r_{2} \ldots r_{n-1}} & \text { if } r_{n}=1\end{cases}
$$

 indicate the initial words. For simplicity we write $w_{n}^{0}$ (resp. $w_{n}^{1}$ ) if $n>3$ and $r_{1}=r_{2}=\ldots=r_{n-2}=0$ (resp. 1). We sometimes write $\omega_{1}^{0}$ and $w_{2}^{0}$ for $w_{1}$ and $w_{2}$, respectively. Note that
(i) the superscript $r_{1} r_{2} \ldots r_{n-2}$ indicates how the Fibonacci word $w_{n}^{r_{1} r_{2} \ldots r_{n-2}}$ is obtained from $x$ and $y$;
(ii) the Fibonacci word $w_{n+1}^{r_{1} r_{2} \cdots r_{n-1}}$ is always an immediate predecessor of the Fibonacci word $w_{n+2}^{r_{1} r_{2} \ldots r_{n}}$;
(iii) the same Fibonacci word may have several different labels;
(iv) Knuth's Fibonacci sequence of words is $\left\{w_{n}^{0}\right\}$ while Higgins' and Turner's is $w_{1}, w_{2}, w_{3}^{1}, \ldots, w_{n}^{1}, \ldots$.
Define the reverse operation $R$ by setting $R\left(x_{1} x_{2} \ldots x_{m}\right)=x_{m} \ldots x_{2} x_{1}$, where $x_{1}, \ldots, x_{m} \in\{x, y\}$. A word $w=x_{1} x_{2} \ldots x_{m}$ is said to be symmetric if $R(w)=w$. For example, the words $y x y$ and $x y y x$ are symmetric.

Theorem 3:
(i) If $w \in \mathscr{S}_{n}$, then $R(w) \in \mathscr{S}_{n}$. Moreover, if $n \geq 3$ and $w=w_{n}^{r_{1} r_{2} \ldots r_{n-2}}$ where the $r^{\prime}$ s are 0 or 1 , then $R(w)=w_{n}^{s_{1} s_{2} \ldots s_{n-2}}$ where $s_{j}=1-r_{j}, j=1,2$, ..., $n-2$.
(ii) If $v$ is an immediate predecessor of $w$, then $R(v)$ is an immediate predecessor of $R(w)$.
Proof: Suppose that the results are true for all positive integers less than $n$. Let $w=w_{n}^{r_{1} r_{2} \ldots r_{n-2}}$ where $r_{1}, r_{2}, \ldots, r_{n-2} \in\{0,1\}$. If $r_{n-2}=0$, then $w=v u$ where
$v=w_{n-1}^{r_{1} r_{2} \ldots r_{n-3}} \in \mathscr{S}_{n-1}$ is an immediate predecessor of $w$ $u=\omega_{n-2}^{r_{1} r_{2} \ldots r_{n-4} \in \mathscr{S}_{n-2} \text { is an immediate predecessor of } v . ~ . ~ . ~}$
Clearly $R(w)=R(u) R(v)$. By the induction hypothesis,

$$
R(\mathcal{u})=w_{n-2}^{s_{1} s_{2} \cdots s_{n-4} \in \mathscr{S}_{n-2}}
$$

is an immediate predecessor of

$$
R(v)=w_{n-1}^{s_{1} s_{2} \cdots s_{n-3} \in \mathscr{S}_{n-1}, ~}
$$

where $s_{j}=1-r_{j}, j=1,2, \ldots, n-3$. Hence, $R(v)$ is an immediate predecessor of $R(w)$ and

$$
\begin{aligned}
R(w)=R(u) R(v) & =w_{n-2}^{s_{1}} s_{2} \cdots s_{n-4} \omega_{n-1}^{s_{1} s_{2}} \ldots s_{n-3} \\
& =w_{n}^{s_{1} s_{2} \ldots s_{n-3} s_{n-2} \in \mathscr{S}_{n},}
\end{aligned}
$$

where $s_{n-2}=1$. The case $r_{n-2}=1$ is proved similarly.

## 2. Factorization of $w_{n}^{0}$ into a Product of Symmetric Factors

Let $v_{5}=y, u_{5}=x y y x, v_{6}=y x y, u_{6}=y x y x y$. For $n \geq 7$, put

$$
\begin{aligned}
& u_{n}=R\left(c_{n}\right) v_{n-1} c_{n} \\
& v_{n}=v_{n-2} u_{n-2} v_{n-2}
\end{aligned}
$$

where $c_{n}$ equals $x y$ if $n$ is even and equals $y x$ if $n$ is odd. We sometimes write $u_{n}(x, y)$ and $v_{n}(x, y)$ for $u_{n}$ and $v_{n}$, respectively. Plainly, all $u_{n}^{\prime} s$ and $v_{n}^{\prime} s$ are symmetric.

Theorem 4: For $n \geq 5$, we have
(i) $w_{n}^{0}=v_{n} u_{n}$;
(ii) $v_{n} c_{n-1}=w_{n-1}^{0}$;
(iii) $u_{n}=c_{n-1} w_{n-2}^{0}$;
(iv) $w_{n}^{1} \stackrel{n}{=} u_{n} v_{n}$.

Proof: Clearly the results are true for $n=5$ and 6 . Suppose $n>6$ and that the results hold for all integers less than $n$. Then

$$
\begin{aligned}
v_{n} c_{n-1} & =v_{n-2} u_{n-2} v_{n-2} c_{n-1}=w_{n-2}^{0} w_{n-3}^{0}=w_{n-1}^{0} ; \\
u_{n} & =R\left(c_{n}\right)\left(v_{n-1} c_{n}\right)=c_{n-1} w_{n-2}^{0} ; \\
w_{n}^{0} & =w_{n-1}^{0} w_{n-2}^{0}=v_{n} c_{n-1} w_{n-2}^{0}=v_{n} u_{n} .
\end{aligned}
$$

This proves (i)-(iii). Assertion (iv) is a consequence of Theorem 3 and the fact that $u_{n}$ and $v_{n}$ are symmetric.

Let $w$ be a word in the alphabet $\mathscr{A}$. Designate the length of $w$ by $l(w)$. In the following lemma we compute the length of the words $w_{n}^{0}, u_{n}$, and $v_{n}$.
Lemma 5: For $n \geq 3$, we have

$$
\begin{align*}
& l\left(w_{n}^{0}\right)=l\left(w_{n-1}^{0}\right)+l\left(w_{n-2}^{0}\right) ;  \tag{i}\\
& l\left(w_{n}^{0}\right)=F_{n-2} l(x)+F_{n-1} l(y)=\sum_{j=1}^{n-2} l\left(w_{j}^{0}(x, y)\right)+l(y) .
\end{align*}
$$

For $n \geq 5$, we have

$$
\begin{align*}
& l\left(u_{n}(x, y)\right)=l\left(w_{n-2}^{0}\right)+l(x)+l(y)=\left(F_{n-4}+1\right) l(x)+\left(F_{n-3}+1\right) l(y) ;  \tag{iii}\\
& l\left(v_{n}(x, y)\right)=l\left(w_{n-1}^{0}\right)-l(x)-l(y)=\left(F_{n-3}-1\right) l(x)+\left(F_{n-2}-1\right) l(y) . \tag{iv}
\end{align*}
$$

Proof: Both (i) and (ii) are proved by induction on $n$ and both (iii) and (iv) follow from (i), (ii), and Theorem 4.

## 3. Cyclic Shift on Fibonacci Words

In this section, our main result states that every $n^{\text {th }}$ Fibonacci word is a cyclic shift of every other $n^{\text {th }}$ Fibonacci word. A cyclic shift operation $T_{n}$ acting on words in the alphabet $\mathscr{A}$ that have lengths $n$ is given by

$$
T_{n}\left(c_{1} c_{2} \ldots c_{n}\right)=c_{2} c_{3} \ldots c_{n} c_{1}
$$

where $c_{1}, c_{2}, \ldots, c_{n} \in \mathscr{A}$.
Theorem 6: Every Fibonacci word in $\mathscr{S}_{n}$ is a cyclic shift of $\omega_{n}^{0}$. More precise$1 y$, for $n \geq 3$, we have

$$
w_{n}^{r_{1} r_{2} \cdots r_{n-2}}=T^{k}\left(w_{n}^{0}\right),
$$

[Feb.
where $T=T_{l\left(\omega_{n}^{0}\right)}$ and

$$
k=\sum_{j=1}^{n-2} l\left(w_{j+1}^{0}\right) r_{j}=\sum_{j=1}^{n-2}\left(F_{j-1} l(x)+F_{j} l(y)\right) r_{j}
$$

Proof: The result is trivial for $n=3$. Suppose that $n>3$ and the result is true for all integers less than $n$ and greater than or equal to 3 .

$$
r_{1}=0:
$$

$$
\begin{aligned}
\omega_{n}^{r_{1} r_{2} \ldots r_{n-2}}(x, y) & =w_{n-1}^{r_{2} r_{3} \ldots r_{n-2}}(y, y x) \\
& =T^{k_{1}}\left(w_{n-1}^{0}(y, y x)\right)=T^{k}\left(w_{n}^{0}(x, y)\right)
\end{aligned}
$$

where

$$
k_{1}=\sum_{j=2}^{n-2} l\left(w_{j}^{0}(y, y x)\right) r_{j}=\sum_{j=2}^{n-2} l\left(w_{j+1}^{0}(x, y)\right) r_{j}=\sum_{j=1}^{n-2} l\left(w_{j+1}^{0}(x, y)\right) r_{j}=k .
$$

$$
r_{1}=1:
$$

$$
\begin{align*}
w_{n}^{r_{1} r_{2} \cdots r_{n-2}}(x, y) & =w_{n-1}^{r_{2} r_{3} \cdots r_{n-2}}(y, x y)  \tag{3.1}\\
& =T^{k_{1}}\left(w_{n-1}^{0}(y, x y)\right)=T^{k_{1}}\left(w_{n}^{10} \cdots{ }^{0}(x, y)\right)
\end{align*}
$$

where

$$
k_{1}=\sum_{j=2}^{n-2} l\left(w_{j}^{0}(y, x y)\right) r_{j}=\sum_{j=2}^{n-2} l\left(w_{j+1}^{10 \ldots 0}(x, y)\right) r_{j}=\sum_{j=2}^{n-2} l\left(w_{j+1}^{0}(x, y)\right) r_{j}
$$

By Theorem 4 and (iv) of Lemma 5, we have

$$
\begin{aligned}
& w_{n}^{1}(x, y)=u_{n}(x, y) v_{n}(x, y)=T l\left(v_{n}\right)\left(w_{n}^{0}(x, y)\right) \\
&=T l\left(w_{n-1}^{0}(x, y)\right)-l(x)-\ell(y) \\
&\left(w_{n}^{0}(x, y)\right)
\end{aligned}
$$

With $r_{1}=r_{2}=\cdots=r_{n-2}=1$ in (1), we have

$$
w_{n}^{1}(x, y)=T^{k_{2}}\left(w_{n}^{10 \cdots 0}(x, y)\right)
$$

where
so that:

$$
k_{2}=\sum_{j=2}^{n-2} l\left(w_{j+1}^{0}(x, y)\right)
$$

$$
w^{10 \cdots 0}(x, y)=T^{k_{3}}\left(w_{n}^{0}(x, y)\right)
$$

where

$$
\begin{align*}
k_{3} & =-k_{2}+l\left(w_{n-1}^{0}(x, y)\right)-l(x)-l(y)  \tag{3.2}\\
& =-\sum_{j=1}^{n-2} l\left(w_{j}^{0}(x, y)\right)=-l\left(w_{n}^{0}(x, y)\right)+l(y) \\
& =-l\left(w_{n}^{0}(x, y)\right)+l\left(w_{2}^{0}(x, y)\right),
\end{align*}
$$

in view of (ii) of Lemma 5. Combining (3.1) and (3.2), we have the desired result.

In the case that $x$ and $y$ are distinct alphabets, it turns out that $\mathscr{S}_{n}$ consists of all the cyclic shifts of $w_{n}^{0}$.
Theorem 7: Let $x$ and $y$ be distinct alphabets $a$ and $b$, respectively. Then a word $w$ in the alphabet $\{a, b\}$ is an $n^{\text {th }}$ Fibonacci word if and only if $w$ is a cyclic shift of $\omega_{n}^{0}$.
Proof: The "only if" part is contained in Theorem 6. The "if" part is a consequence of the following Lemma (about Fibonacci numbers) whose proof is easy and is therefore omitted.

Lemma 8: Let $n \geq 3$. For all $0 \leq r \leq F_{n}-1$, the equation

$$
\sum_{j=1}^{n-2} F_{j+1} r_{j}=r
$$

has at least one solution $r_{1}, r_{2}, \ldots, r_{n-2}$ in $\{0,1\}$.
We remark that this lemma also leads to the known representation theorem which states that every positive integer can be represented as a sum of a finite number of Fibonacci numbers in which each Fibonacci number occurs at most once.

## 4. The Case $x$ and $y$ Are Alphabets

As in the last theorem of Section 3, let $x$ and $y$ be distinct alphabets $a$ and $b$, respectively. Let $q_{n}=w_{n}^{l 010 l \ldots} \quad(n=1,2, \ldots)$. In this section, we locate the $\alpha^{\prime}$ s in $q_{n}$ and show that all the shifts of $q_{n}$ (resp. $w_{n}^{0}$ ) are distinct and hence that $\mathscr{I}_{n}$ consists of precisely $F_{n}$ Fibonacci words. The main result is based on the following two lemmas.
Lemma 9: Let $n \geq 3$. Then $j F_{n-1}$ (resp. $j F_{n-2}$ ), $0 \leq j \leq F_{n}-1$, is a complete residue system modulo $F_{n}$.

Lemma 10: (a) Let $n$ be an odd integer greater than 4.
(i) For $1 \leq j \leq F_{n-4}$, let $k$ be the unique number such that
(4.1) $1 \leq k \leq F_{n-2}$ and $j F_{n-3} \equiv k\left(\bmod F_{n-2}\right)$.

Then there exists a unique $r_{j}$ such that

$$
\begin{equation*}
1 \leq r_{j} \leq F_{n-2} \text { and } k \equiv r_{j} F_{n-1}\left(\bmod F_{n}\right) \tag{4.2}
\end{equation*}
$$

(ii) For $1 \leq i \leq F_{n-3}$, let $k$ be the unique number such that

$$
\begin{equation*}
F_{n-2}+1 \leq k \leq F_{n} \quad \text { and } \quad i F_{n-3} \equiv k-F_{n-2}\left(\bmod F_{n-1}\right) \tag{4.3}
\end{equation*}
$$

Then there exists a unique $t_{i}$ such that
(4.4) $1 \leq t_{i} \leq F_{n-2}$ and $k \equiv t_{i} F_{n-1}\left(\bmod F_{n}\right)$.

Furthermore,
(4.5) $\quad\left\{r_{j}: 1 \leq j \leq F_{n-4}\right\} \cup\left\{t_{i}: 1 \leq i \leq F_{n-3}\right\}=\left\{1,2, \ldots, F_{n-2}\right\}$.
(b) Let $n$ be an even integer greater than 4.
(iii) For $1 \leq j \leq F_{n-3}$, let $k$ be the unique number such that

$$
1 \leq k \leq F_{n-1} \quad \text { and } \quad j F_{n-2} \equiv k\left(\bmod F_{n-1}\right)
$$

Then there exists a unique $r_{j}$ such that

$$
1 \leq r_{j} \leq F_{n-2} \text { and } k \equiv r_{j} F_{n-2}\left(\bmod F_{n}\right)
$$

(iv) For $1 \leq i \leq F_{n-4}$, let $k$ be the unique number such that

$$
F_{n-1}+1 \leq k \leq F_{n} \quad \text { and } \quad i F_{n-4} \equiv k-F_{n-1}\left(\bmod F_{n-2}\right)
$$

Then there exists a unique $t_{i}$ such that

$$
1 \leq t_{i} \leq F_{n-2} \text { and } k \equiv t_{i} F_{n-2} \quad\left(\bmod F_{n}\right)
$$

Furthermore,

$$
\left\{r_{j}: 1 \leq j \leq F_{n-3}\right\} \cup\left\{t_{i}: 1 \leq i \leq F_{n-4}\right\}=\left\{1,2, \ldots, F_{n-2}\right\}
$$

Proof: We prove (a) only.
(i) Let $j$ and $k$ satisfy condition (4.1). We show that (4.2) holds. Write $k=j F_{n-3}-s F_{n-2} \quad$ where $s$ is an integer.

Since

$$
-F_{n-2} \leq-k<j F_{n-3}-k=s F_{n-2} \leq j F_{n-3}-1 \leq F_{n-4} F_{n-3}-1=F_{n-5} F_{n-2}
$$

we see that $0 \leq s \leq F_{n-5}$. Thus,

$$
k=j F_{n-3}-s F_{n-2}=(2 j+s) F_{n-1}-(j+s) F_{n} \equiv r F_{n-1}\left(\bmod F_{n}\right)
$$

where $1 \leq r=2 j+s \leq 2 F_{n-4}+F_{n-5}=F_{n-2}$. This proves (4.2).
(ii) Let $i$ and $k$ satisfy condition (4.3). We show that (4.4) holds. Write

$$
\begin{align*}
k=i F_{n-3}+F_{n-2}-s F_{n-1} & =(2 i-s-1) F_{n-1}-(i-1) F_{n}  \tag{4.6}\\
& \equiv t F_{n-1}\left(\bmod F_{n}\right)
\end{align*}
$$

where $s$ is an integer and $t=2 i-s-1$. From (4.6), we have

$$
\begin{aligned}
F_{n-2}+1 \leq k \leq t F_{n-1} & =k+(i-1) F_{n} \leq F_{n}+(i-1) F_{n} \\
& =i F_{n} \leq F_{n-3} F_{n}=F_{n-1} F_{n-2}-1<F_{n-1} F_{n-2}
\end{aligned}
$$

so that $1 \leq t<F_{n-2}$. This proves (4.4).
Now we prove (4.5). It is clear that the sets

$$
A=\left\{r_{j}: 1 \leq j \leq F_{n-4}\right\} \quad \text { and } \quad B=\left\{t_{i}: 1 \leq i \leq F_{n-3}\right\}
$$

are contained in $\left\{1,2, \ldots, F_{n-2}\right\}$. To prove equality in (4.5), we show that $A$ has $F_{n-4}$ elements, $B$ has $F_{n-3}$ elements, and that $A$ and $B$ are disjoint.
(a) If $r_{j_{1}}=r_{j_{2}}$, where $j_{1}$ and $j_{2}$ lie between 1 and $F_{n-4}$, then $k_{j_{1}} \equiv r_{j_{1}} F_{n-1}=r_{j_{2}} F_{n-1} \equiv k_{j_{2}}\left(\bmod F_{n}\right)$.
Since both $k_{j_{1}}$ and $k_{j_{2}}$ lie between 1 and $F_{n-2}$, this implies that $k_{j_{1}}=k_{j_{2}}$ and so

$$
j_{1} F_{n-3} \equiv j_{2} F_{n-3}\left(\bmod F_{n-2}\right)
$$

Since $F_{n-2}$ and $F_{n-3}$ are relatively prime, we have $j_{1}=j_{2}$. Hence, all the p's are distinct.
(b) A similar proof shows that all the $t^{\prime}$ 's are distinct.
(c) If $r_{j} \in A, t_{i} \in B$, and $r_{j}=t_{i}$, then $k \equiv k^{\prime}\left(\bmod F_{n}\right)$, where $r_{j} F_{n-1} \equiv k$ and $t_{i} F_{n-1} \equiv \mathcal{K}^{\prime}\left(\bmod F_{n}\right)$, and both $k$ and $k^{\prime}$ lie between 1 and $F_{n}$. Therefore, we have $k=K^{\prime}$. But this is impossible because $k \geq F_{n-2}+1>K^{\prime}$. Thus, $A$ and $B$ are disjoint.

This proves (4.5), and the proof is complete.
In part (a) of Lemma 10, two injective mappings

$$
\begin{aligned}
& r: j \in\left\{1,2, \ldots, F_{n-4}\right\} \mapsto r_{j} \in\left\{1,2, \ldots, F_{n-2}\right\} \\
& t: i \in\left\{1,2, \ldots, F_{n-3}\right\} \mapsto t_{i} \in\left\{1,2, \ldots, F_{n-2}\right\}
\end{aligned}
$$

are defined by (4.1) and (4.2) and by (4.3) and (4.4), respectively. The disjoint union of their ranges gives the whole of $\left\{1,2, \ldots, F_{n-2}\right\}$. Part (b) of Lemma 10 has an analogous meaning.

Now write $q_{n}=a_{1} a_{2} \ldots a_{F_{n}}$ where $a_{j} \in\{a, b\}$.
Theorem 11: Let $n$ be a positive integer greater than 3 . Let $t=F_{n-1}$ if $n$ is odd and $t=F_{n-2}$ if $n$ is even. Then $\alpha_{k}=\alpha$ if and only if $k \equiv j t\left(m o d F_{n}\right)$ for some $1 \leq j \leq E_{n-2}$.
Proof: The results are clearly true for $n<7$. Now suppose that $n \geq 7$ and $n$ is odd. Then $q_{n}=q_{n-2} q_{n-1}$ where

$$
q_{n-2}=\alpha_{1} a_{2} \ldots a_{F_{n-2}} \text { and } q_{n-1}=a_{F_{n-2}+1} \ldots \alpha_{F_{n}}
$$

## FIBONACCI WORDS

By the induction hypothesis, the following statements are true:
(i) For $1 \leq k \leq F_{n-2}$, we have

$$
\begin{aligned}
& \alpha_{k}=a \text { if and only if } k \equiv j F_{n-3}\left(\bmod F_{n-2}\right) \text { for some } 1 \leq j \leq F_{n-4} \\
& \text { (ii) For } F_{n-2}+1 \leq k \leq F_{n} \text {, we have } \\
& \alpha_{k}=\alpha \text { if and only if } k-F_{n-2} \equiv j F_{n-3}\left(\bmod F_{n-1}\right) \text { for some } 1 \leq j \leq F_{n-3}
\end{aligned}
$$ The result now follows from Lemmas 9 and 10 . For even $n$ the proof is similar.

Let $w=c_{1} c_{2} \ldots c_{n}$ where $c_{j}$ equals $a$ or $b$. We designate by $S(w)$ the sum $(\bmod n)$ of the indices $j$ for which $c_{j}=a$.
Corollary 12: Let $n$ be a positive integer greater than 2 . For odd $n$, let

$$
s=F_{n-2} \quad \text { and } \quad t=F_{n-1}
$$

for even $n$, let

$$
s=F_{n-1} \quad \text { and } \quad t=F_{n-2}
$$

Suppose that $1 \leq j \leq F_{n}-1$ and $T^{j s} q_{n}=c_{1} c_{2} \ldots c_{F_{n}}$ where $c_{k} \in\{\alpha \quad b\}$ and $T=$ $T_{F_{n}}$. Then
(i) $c_{k}=a$ if and only if $k \equiv(j+r) t\left(\bmod F_{n}\right)$ for some $1 \leq r \leq F_{n-2}$.
(ii) $S\left(T^{j s} q_{n}\right)-S\left(T^{(j-1) s} q_{n}\right) \equiv 1\left(\bmod F_{n}\right)$, and $S\left(T^{j s} q_{n}\right) \equiv S\left(q_{n}\right)+j\left(\bmod F_{n}\right)$.
(iv)

Proof:
(i) By Theorem 11, we have

$$
\begin{gathered}
c_{k}=a \Leftrightarrow k+j s \equiv r t\left(\bmod F_{n}\right) \text { for some } 1 \leq r \leq F_{n-2} \\
\leftrightarrow k \equiv(j+r) t\left(\bmod F_{n}\right) \text { for some } 1 \leq r \leq F_{n-2} \\
S\left(T^{j s} q_{n}\right)-S\left(T^{(j-1) s} q_{n}\right)
\end{gathered} \begin{gathered}
\equiv \sum_{r=1}^{F_{n-2}}(j+r) t-\sum_{r=1}^{F_{n-2}}(j+r-1) t \\
\equiv F_{n-2} t \equiv 1\left(\bmod F_{n}\right)
\end{gathered}
$$

(ii)

Statement (iii) follows from (ii); statement (iv) is a consequence of (iii) and Lemma 9.
Corollary 13: Let $n, s$, and $t$ be the same as in Corollary 12.
(i) If $0 \leq j \leq F_{n-2}-1$, then $T^{j s} q_{n}$ starts with an $\alpha$.
(ii) If $F_{n-2} \leq j \leq 2 F_{n-2}-1$, then $T^{j s} q_{n}$ starts with a $b a$.
(iii) If $2 F_{n-2} \leq j \leq F_{n}-1$, then $T^{j s} q_{n}$ starts with a $b b a$.
(iv) If $F_{n-2} \leq j \leq F_{n-1}-1$, then $T^{j s} q_{n}$ starts with a $b$ and ends with a $b$.

Proof: Write $T^{j s} q_{n}=c_{1} c_{2} \ldots c_{F_{n}}$ where $c_{k} \in\{a, b\}$. We shall use Lemma 1, (i) of Corollary 12, and the fact that $i \equiv i F_{n-2} t\left(\bmod F_{n}\right)$ where $i=1$, 2, and 3 .
(i) If $0 \leq j \leq F_{n-2}-1$, then $c_{1}=\alpha$ because $j+1 \leq F_{n-2} \leq j+F_{n-2}$.
(ii) If $F_{n-2} \leq j \leq 2 F_{n-2}-1$, then the inequalities $j+1 \leq 2 F_{n-2} \leq j+F_{n-2}$
imply that $c_{2}=a$ and hence $c_{1}=b$, according to Lemma 1 .
(iii) If $2 F_{n-2} \leq j \leq F_{n}-1$, then the inequalities

$$
j+1 \leq F_{n} \leq 3 F_{n-2} \leq j+F_{n-2}
$$

imply that $c_{3}=\alpha$ and $c_{F_{n}}=\alpha$; hence $c_{1}=c_{2}=b$.

$$
\begin{aligned}
& \text { (iv) If } F_{n-2} \leq j \leq F_{n-1}-1 \text {, then } c_{1}=b \text {, by (ii), and since } \\
& F_{n}-1 \equiv-F_{n-2} t \equiv F_{n-1} t \text { and } j+1 \leq F_{n-1} \leq 2 F_{n-2} \leq j+F_{n-2},
\end{aligned}
$$

we have $c_{F_{n}-1}=a$, so that $c_{F_{n}}=b$.
Theorem 14: Let $n$ be a positive integer. Then

$$
\begin{aligned}
& \left|\mathscr{S}_{n}\right|=F_{n} ;\left|\mathscr{S}_{n}(a, b)\right|=F_{n-2}=\left|\mathscr{S}_{n}(b, a)\right| ; \\
& \left|\mathscr{S}_{n}(b, b)\right|=F_{n-3} ;\left|\mathscr{S}_{n}(b, \cdot)\right|=\left|\mathscr{S}_{n}(\cdot, b)\right|=F_{n-1} .
\end{aligned}
$$

Proof: The results follow from Theorem 7 and Corollaries 12 and 13.

## 5. Two Algorithms

In this section, the initial words are again taken to be alphabets $\alpha$ and $b$. Two algorithms will be given. Algorithm A constructs the Fibonacci word for which the multiplications involved are preassigned by means of a finite binary sequence as in (3.1) and (3.2). Algorithm B tests whether a given word in the alphabet $a$ and $b$ is a•Fibonacci word or not.

For simplicity, we replace $a$ by 1 and $b$ by 0 in both algorithms so that Fibonacci words are represented by binary sequences.

Since

$$
w=\omega_{n}^{r_{1} r_{2} \ldots r_{n-2}=T^{k_{1}}\left(\omega_{n}^{0}\right), ~ ; ~}
$$

where

$$
k_{1}=\sum_{i=1}^{n-2} F_{i+1} r_{i},
$$

$$
q_{n}= \begin{cases}T^{F_{n}-1}\left(\omega_{n}^{0}\right) & \text { if } n \text { is odd } \\ T^{F_{n-1}-1}\left(\omega_{n}^{0}\right) & \text { if } n \text { is even }\end{cases}
$$

it follows that $w=T^{j s} q_{n}$ where

$$
\begin{aligned}
& s= \begin{cases}F_{n-2} \text { if } n \text { is odd } \\
F_{n-1} \text { if } n \text { is even, }\end{cases} \\
& j \equiv \begin{cases}k F_{n-1}\left(\bmod F_{n}\right) & \text { if } n \text { is odd } \\
k F_{n-1}-1 & \text { if } n \text { is even, }\end{cases}
\end{aligned}
$$

and $k=k_{1}+1$. Thus, the positions of the 1 's in $w$ can be determined by Corollary 12.
Algorithm A: Input a positive integer $n$ and a binary sequence $r_{1}, r_{2}, \ldots$, $r_{n-2}$. This algorithm constructs the Fibonacci word $w=w_{n}^{r_{1} r_{2} \ldots r_{n-2}}$.

1) Compute $t=\left\{\begin{array}{ll}F_{n-1} & \text { if } n \text { is odd } \\ F_{n-2} & \text { if } n \text { is even, }\end{array} \quad k=\sum_{i=1}^{n-2} F_{i+1} r_{i}+1\right.$,
and $j$ satisfying

$$
j \equiv \begin{cases}k F_{n-1} & \left(\bmod F_{n}\right) \\ k F_{n-1}-1 & \text { if } n \text { is odd } \\ \text { if } n \text { is even }\end{cases}
$$

and $1 \leq j \leq F_{n}$.

```
    2) For }r=1,2,\ldots,\mp@subsup{F}{n-2}{},\mathrm{ let }\mp@subsup{c}{m}{}=1\mathrm{ if }m\equiv(j+r)t(mod F F ) and 1\leq
\leq En; let cm
    3) w = c
    We now turn to the identification of Fibonacci words. First, observe that
n=[(ln}(\sqrt{}{5}(\mp@subsup{F}{n}{}+1/2)))/\operatorname{ln}(\alpha)],\mathrm{ where }\alpha=(1+\sqrt{}{5})/2
```

Algorithm $B$ : Input a positive integer $h$ and a binary sequence $w=c_{1}, c_{2}, \ldots$, $c_{h}$. This algorithm tests whether or not $\omega$ is a Fibonacci word.

1) Let $n=[(\ln (\sqrt{5}(h+1 / 2))) / \ln (\alpha)]$.
2) If $h \neq F_{n}$, then $\omega \notin \mathscr{S}$.
3) If $h=F_{n}^{\prime}$, let $t= \begin{cases}F_{n-1} & \text { if } n \text { is odd } \\ F_{n-2} & \text { if } n \text { is even. }\end{cases}$
4) Compute the sum $S$ of all indices $i$ such that $c_{i}=1$ and count the number $m$ of 1 's in $\omega$.
5) If $m \neq F_{n-2}$, then $w \notin \mathscr{S}$.
6) If $m=F_{n-2}$, let $j$ be such that $1 \leq j \leq F_{n}$ and $j \equiv S-F_{n-2}\left(F_{n-2}+1\right) t / 2\left(\bmod F_{n}\right)$.
7) For $r=1,2, \ldots, F_{n-2}$, let $k$ be such that $1 \leq k \leq F_{n}$ and $k \equiv(j+r) t$ $\left(\bmod F_{n}\right)$. If $c_{k} \neq 1$ for some $r$, then $w \notin \mathscr{S}$; otherwise $w \in \mathscr{S}$.

Note that in step $6, j \equiv S(w)-S\left(q_{n}\right)\left(\bmod F_{n}\right)$ and so either

$$
w=T^{j s}\left(q_{n}\right) \in \mathscr{f}, \text { where } s= \begin{cases}F_{n-2} & \text { if } n \text { is odd } \quad \text { (the latter case } \\ F_{n-1} & \text { if } n \text { is even, } \\ \text { in step 7) }\end{cases}
$$

or $\quad \omega \notin \mathscr{S}$ (the former case in step 7).

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# RECURRENT FORMULAS OF THE GENERALIZED FIBONACCI AND TRIBONACCI SEQUENCES 

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In [l], it was shown that there are 36 different schemes of generalization for the Fibonacci sequence in the case of three sequences. Ten of these are trivial and the other 26 are grouped into seven classes. The elements of each class are equivalent with exactness up to a substitution of their members. Thus, for every class, we shall give the recurrent formulas of the members of one of its schemes, using the notation in [1].

Everywhere, 1et

$$
a_{0}=C_{1}, b_{0}=C_{2}, c_{0}=C_{3}, a_{1}=C_{4}, b_{2}=C_{5}, c_{2}=C_{6}
$$

and assume that $n \geq 0$ is a natural number where $C_{1}, C_{2}, \ldots, C_{6}$ are given constants and $x$ is one of the symbols $a, b$, and $c$. Class $I$ contains the schemes $S_{6}$ and $S_{9}$, where

$$
s_{6}:\left\{\begin{array}{l}
a_{n+1}=a_{n+2}+b_{n} \\
b_{n+2}=b_{n+1}+c_{n} \\
c_{n+2}=c_{n+1}+a_{n}
\end{array}\right.
$$

The recurrent formula for this scheme is

$$
x_{n+6}=3 x_{n+5}-3 x_{n+4}+x_{n+3}+x_{n}
$$

that is,

$$
\begin{aligned}
& a_{n+6}=3 a_{n+5}-3 a_{n+4}+a_{n+3}+a_{n} \\
& b_{n+6}=3 b_{n+5}-3 b_{n+4}+b_{n+3}+b_{n}, \\
& c_{n+6}=3 c_{n+5}-3 c_{n+4}+c_{n+3}+c_{n} .
\end{aligned}
$$

Class II contains the schemes $S_{15}$ and $S_{25}$, where

$$
S_{15}:\left\{\begin{array}{l}
a_{n+2}=b_{n+1}+a_{n} \\
b_{n+2}=c_{n+1}+b_{n} \\
c_{n+2}=a_{n+1}+c_{n}
\end{array}\right.
$$

The recurrent formula for this scheme is

$$
x_{n+5}=3 x_{n+4}+x_{n+3}-3 x_{n+2}+x_{n}
$$

Class III contains the schemes $S_{20}$ and $S_{33}$, where

$$
S_{20}:\left\{\begin{array}{l}
a_{n+2}=b_{n+1}+b_{n} \\
b_{n+2}=c_{n+1}+c_{n} \\
c_{n+2}=a_{n+1}+a_{n}
\end{array}\right.
$$

The recurrent formula for this scheme is

$$
x_{n+6}=x_{n+3}+3 x_{n+2}+3 x_{n+1}+x_{n}
$$

Class IV contains the schemes $S_{23}$ and $S_{33}$, where

$$
S_{23}:\left\{\begin{array}{l}
a_{n+2}=b_{n+1}+c_{n} \\
b_{n+2}=c_{n+1}+a_{n} \\
c_{n+2}=a_{n+1}+b_{n}
\end{array} \quad\right. \text { (cf. [3]) }
$$

The recurrent formula for this scheme is

$$
x_{n+6}=4 x_{n+3}+x_{n} \text {. }
$$

Class $V$ contains the schemes $S_{7}, S_{12}, S_{14}, S_{22}, S_{28}$, and $S_{31}$, where

$$
s_{7}:\left\{\begin{array}{l}
a_{n+2}=a_{n+1}+b_{n} \\
b_{n+2}=c_{n+1}+a_{n} \\
c_{n+2}=b_{n+1}+c_{n}
\end{array}\right.
$$

The recurrent formula for this scheme is

$$
x_{n+6}=x_{n+5}+2 x_{n+4}-2 x_{n+3}+x_{n+2}-x_{n}
$$

Class VI contains the schemes $S_{8}, S_{11}, S_{18}, S_{21}, S_{32}$, and $S_{35}$, where

$$
s_{8}:\left\{\begin{array}{l}
a_{n+2}=a_{n+1}+b_{n} \\
b_{n+2}=c_{n+1}+c_{n} \\
c_{n+2}=b_{n+1}+a_{n}
\end{array}\right.
$$

The recurrent formula for this scheme is

$$
x_{n+6}=x_{n+5}+x_{n+4}=x_{n+2}+x_{n+1}+x_{n}
$$

Class VII contains the schemes $S_{16}, S_{19}, S_{24}, S_{26}, S_{29}$, and $S_{34}$, where

$$
S_{16}:\left\{\begin{array}{l}
a_{n+2}=b_{n+1}+a_{n} \\
b_{n+2}=c_{n+1}+c_{n} \\
c_{n+2}=a_{n+1}+b_{n}
\end{array}\right.
$$

The recurrent formula for this scheme is

$$
x_{n+6}=x_{n+4}+2 x_{n+3}+2 x_{n+2}-x_{n+1}-x_{n}
$$

Using the data given above and some ideas from [3], we can construct eight different schemes of generalized Tribonacci sequences in the case of two sequences. We introduce their recurrent formulas below.

Everywhere let

$$
a_{0}=C_{1}, b_{0}=C_{2}, a_{1}=C_{3}, b_{1}=C_{4}, a_{2}=C_{5}, b_{2}=C_{6}
$$

and assume that $n \geq 0$ is a natural number, where $C_{1}, C_{2}, \ldots, C_{6}$ are given constants and $x$ is one of the symbols $a$ or $b$.

The different schemes are as follows:

$$
\begin{aligned}
& T_{1}:\left\{\begin{array}{l}
a_{n+3}=a_{n+2}+a_{n+1}+a_{n} \\
b_{n+3}=b_{n+2}+b_{n+1}+b_{n}
\end{array},\right. \\
& T_{3}:\left\{\begin{array}{l}
a_{n+3}=a_{n+2}+b_{n+1}+a_{n} \\
b_{n+3}=b_{n+2}+a_{n+1}+b_{n}
\end{array},\left\{\begin{array}{l}
a_{n+3}=a_{n+2}+a_{n+1}+b_{n} \\
b_{n+3}=b_{n+2}+b_{n+1}+a_{n}
\end{array}\right.\right. \\
& T_{5}:\left\{\begin{array}{l}
r_{4}:\left\{\begin{array}{l}
a_{n+3}=a_{n+2}+b_{n+1}+b_{n} \\
a_{n+3}=b_{n+2}+a_{n+1}+a_{n} \\
b_{n+3}=a_{n+2}+b_{n+1}+b_{n}
\end{array},\right. \\
T_{7}:\left\{\begin{array}{l}
a_{n+3}=b_{n+2}+b_{n+1}+a_{n} \\
a_{n+3}=a_{n} \\
b_{n+3}=a_{n+2}+a_{n+1}+b_{n}
\end{array},\left\{\begin{array}{l}
a_{n+3}=b_{n+2}+a_{n+1}+b_{n} \\
b_{n+3}=a_{n+2}+b_{n+1}+a_{n}
\end{array},\right.\right.
\end{array} \quad T_{8}:\left\{\begin{array}{l}
a_{n+3}=b_{n+2}+b_{n+1}+b_{n} \\
b_{n+3}=a_{n+2}+a_{n+1}+a_{n}
\end{array}\right.\right.
\end{aligned}
$$

The first scheme is trivial. All of the others are nontrivial; they have the following recurrent formulas for $n \geq 0$ :

- for $T_{2}: x_{n+6}=2 x_{n+5}+x_{n+4}-2 x_{n+3}-x_{n+2}+x_{n}$,
- for $T_{3}: x_{n+6}=2 x_{n+5}-x_{n+4}+2 x_{n+3}-x_{n+2}-x_{n}$;
- for $T_{4}: x_{n+6}=2 x_{n+5}-x_{n+4}+x_{n+2}+2 x_{n+1}+x_{n}$;
- for $T_{5}: x_{n+6}=3 x_{n+4}+2 x_{n+3}-x_{n+2}-2 x_{n+1}-x_{n}$;
- for $T_{6}: x_{n+6}=3 x_{n+4}+x_{n+2}+x_{n}$;
- for $T_{7}: x_{n+6}=x_{n+4}+4 x_{n+3}+x_{n+2}-x_{n}$;
- for $T_{8}: x_{n+6}=x_{n+4}+2 x_{n+3}+3 x_{n+2}+2 x_{n+1}+x_{n}$.

The proofs for these facts can be shown by induction, using methods similar to those in [2] or [3].

An open problem is the construction of an explicit formula for each of the schemes given above.

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2. K. Atanassov. "On a Generalization of the Fibonacci Sequence in the Case of Three Sequences." Fibonacci Quarterly 27.1 (1989):7-10.
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# ELEMENTARY PROPERTIES OF THE SUBTRACTIVE EUCLIDEAN ALGORITHM 

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## 1. Introduction

In a recent article in this journal, T. Moore [4] used a microcomputer to make a study of the length of the Euclidean algorithm in determining the greatest common divisor of two nonzero integers $m$, $n$. Our intention is to make a similar study of the lengths of the subtractive Euclidean algorithm. Recall that to determine for example $\operatorname{gcd}(11,3)$ by the subtractive algorithm we perform the operations:

$$
11-3=8,8-3=5,5-3=2,3-2=1,2-1=1,1-1=0 \text {; }
$$

a total of six steps. By contrast, the ordinary Euclidean algorithm yields

$$
\begin{aligned}
11 & =(3)(3)+2 \\
3 & =(1)(2)+1 \\
2 & =(2)(1),
\end{aligned}
$$

in only three steps. However, if we use the Euclidean algorithm to express $11 / 3$ as a regular continued fraction,

$$
\frac{11}{3}=3+\frac{1}{1+\frac{1}{2}} \equiv[3,1,2],
$$

we notice that the partial quotient 3 corresponds to the number of subtractions of 3 above, etc. In general, it is easy to see that the length of the subtractive Euclidean algorithm for ( $m, n$ ) is equal to the sum of the partial quotients in the regular continued fraction expansion of $\mathrm{m} / \mathrm{n}$.

## 2. Analysis

Following the approach of T. Moore [4], we begin our investigation by representing the pair of integers $m, n$ as a lattice point ( $m, n$ ) in the plane and plotting this point only if it has a subtractive length equal to the fixed value in which we are interested. In view of the equivalence of the subtractive length to the sum of the continued fraction partial quotients for ( $m, n$ ), we can implement the subtractive algorithm computations merely by changing line 280 in the basic program given by Moore [4, fig. 1] for the Euclidean algorithm to read
$280 D C=D C+\operatorname{INT}(N 1 / M 1)$.
It is also necessary to swap $m$ and $n$ in the case $m>n$. The graphic results from four different choices of subtractive lengths are shown in Figure 1 below for all pairs ( $m, n$ ) belonging to the range $-320 \leq m \leq 320,-175 \leq n \leq 175$. The range of coordinates here is a consequence of the EGA resolution of an IBM compatible computer.


Figure 1
Screendumps showing all integer pairs ( $m, n$ ) in the range $-320 \leq m \leq 320$, $-175 \leq n \leq 175$, whose gcds are obtained in exactly $k$ steps of the subtractive Euclidean algorithm

In contrast to the output from the ordinary Euclidean algorithm [4, fig. 2], the patterns shown here are surprisingly regular. In each case, the pairs ( $m, n$ ) having subtractive length $k$ are seen to lie on $2^{k}$ straight lines. Since $\operatorname{gcd}( \pm m, \pm n)=\operatorname{gcd}(m, n)=\operatorname{gcd}(n, m)$,
we can restrict our attention to the pairs $m \geq 1, n \geq 1$, and $m \geq n$. A mathematical description of these pictures is then given by the following theorem.

Theorem 1: For any fixed integer $k \geq 1$, the pair of coprime positive integers ( $m, n$ ) with $m \geq n$ has subtractive length $k$ iff $m / n=\left[\alpha_{1}, a_{2}, \ldots, a_{r}\right]$, where

$$
\alpha_{1}+\alpha_{2}+\cdots+\alpha_{r}=k, a_{i} \geq 1,1 \leq i \leq r-1 \text { and } a_{r} \geq 2
$$

Furthermore, there are $2^{k-2}$ such coprime pairs which, together with their multiples and symmetry, make up the lattice points lying on the $2^{k}$ lines in the corresponding diagram.

Proof: If ( $m, n$ ) has subtractive length $k$, then, by previous remarks,
$\frac{m}{n}=\left[a_{1}, a_{2}, \ldots, a_{r}\right]$ where $a_{1}+a_{2}+\ldots+a_{r}=k, a_{i} \geq 1,1 \leq i \leq r$.
In addition, $\alpha_{r} \geq 2$ since a value $a_{r}=1$ corresponds to the final step

$$
r_{k-1}=(1) \times r_{k}+0
$$

in the Euclidean algorithm, which contradicts $0<r_{k}<r_{k-1}$. Since

$$
\frac{j m}{j n}=\frac{m}{n}, \quad j \geq 1
$$

we can restrict our attention to $m$, $n$ coprime; the multiples $j m$, $j n$ thus give the other integer points lying on the line determined by ( $m, n$ ). The number of lines is therefore determined by the number of solutions in positive integers, $a_{1}, \ldots, a_{r}$ of the equation $\alpha_{1}+\ldots+a_{r}=k$ where $a_{r} \geq 2$. Since any solution with $\alpha_{r}=1$ can be paired uniquely to a solution

$$
a_{1}+\cdots+\left(a_{r-1}+1\right)=k \text {, with } a_{r-1}+1 \geq 2,
$$

the quantity we require is half of the total number of solutions to $\alpha_{1}+\ldots+$ $a_{r}=k$ in positive integers. Now, for fixed $r$, the number of such solutions is $\binom{k-1}{r-1}$ (see, e.g., Brualdi [1, p. 38]). Since $r$ can take on any value from 1 to $k$, the total number of solutions is

$$
\sum_{r=1}^{k}\binom{k-1}{r-1}=\sum_{r=0}^{k-1}\binom{k-1}{r}=2^{k-1}
$$

and the result follows.
It is interesting to note that for each $k$ various integer pairs consisting of Fibonacci and Lucas numbers occur among the coprime pairs with subtractive length $k$. In particular, the pairs

$$
\left(F_{k+1}, F_{k}\right),\left(F_{k+1}, F_{k-1}\right),\left(L_{k-1}, L_{k-2}\right),\left(L_{k-1}, L_{k-3}\right),\left(L_{k-1}, F_{k}\right)
$$

are included in the set. To see this, we can use the recurrence relationships for the Fibonacci and Lucas numbers to derive the following continued fraction expansions, the partial quotients of which sum in each case to $k$ :

$$
\begin{aligned}
& \frac{F_{k+1}}{F_{k}}=[1, \ldots, 1,2] \quad(k-2 \text { ones }), \quad k \geq 2 \\
& \frac{F_{k+1}}{F_{k-1}}=[2,1, \ldots, 1,2] \quad(k-4 \text { ones }), \quad k \geq 4 \\
& \frac{L_{k-1}}{L_{k-2}}=[1, \ldots, 1,3](k-3 \text { ones }), \quad k \geq 3 \\
& \frac{L_{k-1}}{L_{k-3}}=[2,1, \ldots, 1,3](k-5 \text { ones }), k \geq 5 \\
& \frac{L_{k-1}}{F_{k}}=[1,2,1, \ldots, 1,2](k-5 \text { consecutive ones }), k \geq 5
\end{aligned}
$$

In addition, it is well known that among the pairs ( $m, n$ ), with $m \geq n$, that require $k$ steps of the ordinary Euclidean algorithm, the Fibonacci pair ( $F_{k+1}$, $F_{k}$ ) is the smallest. By contrast, we show ( $F_{k+1}, F_{k}$ ) is the largest coprime pair that has subtractive length $k$. [The smallest such pair is, of course, ( $k, 1$ ).]

To see this, suppose inductively that $F_{k} / F_{k-1}$ is the largest pair requiring $k-1$ subtractive steps. Now to each of the $2^{k-3}$ positive integer pairs $\left(c_{k-1}\right.$, $d_{k-1}$ ), with $c_{k-1} \geq d_{k-1}$ with subtractive length $k-1$, we can associate two of the $2^{k-2}$ pairs of subtractive length $k$, namely, $\left(A_{k}, B_{k}\right)$ where

$$
\frac{A_{k}}{B_{k}}=1+\frac{c_{k-1}}{d_{k-1}}=\frac{c_{k-1}+d_{k-1}}{d_{k-1}}
$$

and ( $A_{k}^{\prime}, B_{k}^{\prime}$ ) where

$$
\frac{A_{k}^{\prime}}{B_{k}^{\prime}}=1+\frac{1}{c_{k-1} / d_{k-1}}=\frac{c_{k-1}+d_{k-1}}{c_{k-1}}
$$

By our inductive hypothesis, the largest pairs of the forms ( $A_{k}, B_{k}$ ) and ( $A_{k}^{\prime}$, $B_{k}^{\prime}$ ) will be $F_{k+1} / F_{k-1}$ and $F_{k+1} / F_{k}$, respectively. The latter pair gives the result.

## 3. Estimates for Almost All Pairs

We can use the above results to derive some elementary bounds for lengths of the subtractive Euclidean algorithm valid for almost all pairs ( $m$, $n$ ) with $1 \leq n \leq x, 1 \leq m \leq x$, as $x \rightarrow \infty$. For convenience, we denote the subtractive length for the pair $(m, n)$ by $L(m, n)$ and the set of all $x^{2}$ pairs ( $m, n$ ) with $1 \leq m \leq x, 1 \leq n \leq x$ by $S(x)$. We first show that the proportion of pairs in $S(x)$ for which $c \log _{2} x<L(m, n) \leq x, \rightarrow 1$ as $x \rightarrow \infty$ for any $0<c<1$.

For any fixed positive integer $k$, the pairs in $S(x)$ with subtractive length $k$ lie on at most $2^{k-l}$ straight lines. Each such line contains at most $x$ such pairs. It follows that for any $m \in \mathbb{N}$, the number of pairs in $S(x)$ with subtractive length $\leq m$ is not greater than $\sum_{k=1}^{m} 2^{k-1} x=x\left(2^{m}-1\right)$. Thus, the proportion of pairs with subtractive length $\leq m$ is bounded above by $2^{m} / x$. This tends to zero as $x \rightarrow \infty$ provided $m<c \log _{2} x$, for any $0<c<1$.

If we consider only coprime pairs in $S(x)$, then the corresponding result is as follows: The proportion of coprime pairs in $S(x)$ for which $c \log _{2} x<L(m, n)$ $<x, \rightarrow 1$ as $x \rightarrow \infty$ for any $0<c<2$. In this case, the number of coprime pairs with subtractive length $\leq m$ is at most $\sum_{k=1}^{m} 2^{k-1}=2^{m}-1$. Now, by Theorem 330 of Hardy \& Wright [2], the number of coprime pairs in $S(x)$ is asymptotically

$$
\frac{6 x^{2}}{\pi^{2}}+0\left(x \log ^{6} x\right) \text { as } x \rightarrow \infty
$$

Hence, the proportion of coprime pairs in $S(x)$ with subtractive length $\leq m$ is bounded above by

$$
\frac{\pi^{2}}{6} \frac{2^{m}}{x^{2}}+0\left(\frac{\log x}{x^{3}}\right) \text { as } x \rightarrow \infty,
$$

which tends to zero, provided $m<c \log _{2} x$ for any $0<c<2$.

## 4. Final Remarks

A graphical representation led us to various observations as well as estimates for the length of the subtractive Euglidean algorithm by elementary means. By a much deeper analytical approach, Knuth \& Yao [3] have shown that for fixed $m$ the average length of the subtractive algorithm over all pairs ( $m$, $n$ ) with $1 \leq n \leq m$ is

$$
6 \pi^{-2}(\ln m)^{2}+0\left(\ln m(\ln \ln m)^{2}\right) .
$$

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## REFEREES

In addition to the members of the Board of Directors and our Assistant Editors, the following mathematicians, engineers, and physicists have assisted THE FIBONACCI QUARTERLY by refereeing papers during the past year. Their special efforts are sincerely appreciated, and we apologize for any names that have inadvertently been overlooked or misspelled.

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## ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by<br>Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3515@compuserve.com on Internet. All correspondence will be acknowledged.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

Dedication. This year's column is dedicated to Dr. A. P. Hillman in recognition of his 27 years of devoted service as editor of the Elementary Problems Section. Devotees of this column are invited to thank Abe by dedicating their next proposed probleff to Dr. Hillman.

## BASIC FORMULAS

The Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ satisfy
$F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1 ;$ $L_{n+2}=L_{n+1}+L_{n}, L_{0}=2, L_{1}=1$ 。
A1so, $\alpha=(1+\sqrt{5}) / 2, \beta=(1-\sqrt{5}) / 2, F_{n}=\left(\alpha^{n}-\beta^{n}\right) / \sqrt{5}$, and $L_{n}=\alpha^{n}+\beta^{n}$.
PROBLEMS PROPOSED IN THIS ISSUE
B-706 Proposed by K. T. Atanassov, Sofia, Bulgaria
Prove that for $n \geq 0$,

$$
\left(\frac{\pi e}{\pi+e}\right)^{1.4 n}>F_{n}
$$

B-707 Proposed by Herta T. Freitag, Roanoke, VA
Consider a Pythagorean triple $(a, b, c)$ such that

$$
a=2 \sum_{i=1}^{n} F_{i}^{2} \quad \text { and } \quad c=F_{2 n+1}, \quad n \geq 2
$$

Prove or disprove that $b$ is the product of two Fibonacci numbers.
B-708 Proposed by Joseph J. Kostal, University of Illinois at Chicago, IL
Find the sum of the series

$$
\sum_{k=1}^{\infty} \frac{3^{k} F_{k}-2^{k} L_{k}}{6^{k}}
$$

B-709 Proposed by Alejando Necochea, Pan American University, Edinburg, TX
Express $\frac{1}{n!} \frac{d^{n}}{d t^{n}}\left[\frac{t}{1-t-t^{2}}\right]_{t=0}$ in terms of Fibonacci numbers.

B-710 Proposed by H.-J. Seiffert, Berlin, Germany
Let $P_{n}$ be the $n{ }^{\text {th }}$ Pell number, defined by $P_{0}=0, P_{1}=1, P_{n+2}=2 P_{n+1}+P_{n}$ for $n \geq 0$. Prove that
(a) $P_{3 n+1} \equiv L_{3 n+1}(\bmod 5)$,
(b) $P_{3 n+2} \equiv-L_{3 n+2}(\bmod 5)$.
(c) Find similar congruences relating Pell numbers and Fibonacci numbers.

B-711 Proposed by Mihály Bencze, Sacele, Romania
Let $r$ be a natural number. Find a closed form expression for

$$
\prod_{k=1}^{\infty}\left(1-\frac{L_{4 r}}{k^{4}}+\frac{1}{k^{8}}\right)
$$

## SOLUTIONS

## Edited by A. P. Hillman <br> Fibonacci Analogues

B-680 Proposed by Russell Jay Hendel \& Sandra A. Monteferrante, Dowling College, Oakdale, NY

For an integer $a \geq 0$, define a sequence, $x_{0}, x_{1}, \ldots$ by $x_{0}=0, x_{1}=1$, and $x_{n+2}=a x_{n+1}+x_{n}$ for $n \geq 0$. Let $d=\left(\alpha^{2}+4\right) 1 / 2$. For $n \geq 2$, what is the nearest integer to $d x_{n}$ ?

Solution by H.-J. Seiffert, Berlin, Germany
If $\alpha=0$, then $x_{n}$ is 0 for $n$ even and is 1 for $n$ odd. If $\alpha=1$, then $\left(x_{n}\right)$ is the sequence of Fibonacci numbers. 2 is the nearest integer to $\sqrt{5} F_{n}=d x_{n}$ for $n=2$ and $x_{n-1}+x_{n+1}=F_{n-1}+F_{n+1}=L_{n}$ is the nearest integer to $\sqrt{5 F_{n}}=$ $d x_{n}$ for $n \geq 3$ (see B-659). Now let $\alpha \geq 2$ and $n \geq 2$. It is well known that $x_{n}$ $=\left(b^{n}-c^{n}\right) / d$ and $x_{n-1}+x_{n+1}=b^{n}+c^{n}$, where $b=(\alpha+d) / 2$ and $c=(\alpha-d) / 2$. $a \geq 2$ implies the inequalities

$$
\begin{equation*}
|c|<1 \quad \text { and } \quad-1 / 4<c^{2}=a c+1<1 / 4 \tag{1}
\end{equation*}
$$

Thus, we have

$$
\begin{aligned}
\left|d x_{n}-\left(x_{n-1}+x_{n+1}\right)\right| & =\left|b^{n}-c^{n}-\left(b^{n}+c^{n}\right)\right| \\
& =2|c|^{n} \leq 2|c|^{2}=2|a c+1|<1 / 2
\end{aligned}
$$

This shows that $x_{n-1}+x_{n+1}$ is the nearest integer to $d x_{n}$.
Also solved by R. André-Jeannin, Paul S. Bruckman, Guo-Gang Gao, Lawrence Somer, and the proposers.

$$
\text { Straight Line Separating }\left(F_{n}, F_{n+1}\right) \text { from }\left(F_{n+1}, F_{n+2}\right)
$$

B-684 Proposed by L. Kuipers, Sierre, Switzerland
(a) Find a straight line in the Cartesian plane such that ( $F_{n}, F_{n+1}$ ) and $\left(F_{n+1}, F_{n+2}\right)$ are on opposite sides of the line for all positive integers $n$.
(b) Is the line unique?
[Feb.

Solution by Paul S. Bruckman, Edmonds, WA
Let $P_{n}=\left(F_{n}, F_{n+1}\right), n=1,2, \ldots$, denote points in the Cartesian plane. We use the identity
(1) $\quad F_{n+1}=\alpha F_{n}+\beta^{n}, n=1,2, \ldots$.

Since $\beta^{n}$ and $\beta^{n+1}$ have opposite signs (and are necessarily nonzero), we see that $\left(F_{n+1}-\alpha F_{n}\right)$ and $\left(F_{n+2}-\alpha F_{n+1}\right)$ have opposite signs. This implies that $P_{n}$ and $P_{n+1}$ lie on opposite sides of the line $L$, defined by

$$
\begin{equation*}
L=\{(x, y): y=\alpha x\} \tag{2}
\end{equation*}
$$

Therefore, $L$ satisfies the conditions of part (a).
Suppose $L^{\prime}=\{(x, y): y=r x+s\}$ is any line with this same property, where $r$ and $s$ are real. Since $L^{\prime}$ must intersect the segment $P_{n} P_{n+1}$ for each $n$, the following inequalities must hold:

$$
\begin{equation*}
r F_{n}+s>F_{n+1}, \quad r F_{n+1}+s<F_{n+2}, \quad n=1,3,5, \ldots . \tag{3}
\end{equation*}
$$

Then, using (1), $(\alpha-p) F_{n}+\beta^{n}<s<(\alpha-r) F_{n+1}+\beta^{n+1}$, or

$$
\begin{equation*}
\frac{s-\beta^{n+1}}{F_{n+1}}<\alpha-r<\frac{s-\beta^{n}}{F_{n}}, \quad n=1,3,5, \ldots . \tag{4}
\end{equation*}
$$

Taking limits in (4) as $n \rightarrow \infty$, we see that either end of the inequalities approaches 0 ; therefore, $r=\alpha$. Moreover, we must have

$$
\begin{equation*}
\beta^{n}<s<\beta^{n+1}, \quad n=1,3,5, \ldots \tag{5}
\end{equation*}
$$

Again taking limits as $n \rightarrow \infty$ in (5), we conclude that $s=0$. Thus, we conclude that $L^{\prime}=L$, i.e., the desired line $L$ is uniquely described by (2).

Also solved by Charles Ashbacher, Piero Filipponi, C. Georghiou, Russell Jay Hendel, Hans Kappus, Y. H. Harris Kwong, Mohammad Parvez Shaikh, Lawrence Somer, and the proposer.

Approximation to $k$ as a Function of $F_{k}$
B-685 Proposed by Stanley Rabinowitz, Westford, MA, and Gareth Griffith, University of Saskatchewan, Saskatoon, Saskatchewan, Canada

For integers $n \geq 2$, find $k$ as a function of $n$ such that

$$
F_{k-1} \leq n<F_{k}
$$

Solution by Lawrence Somer, Washington, D.C.
It follows from the Binet formula for $F_{k}$ that

$$
\sqrt{5} F_{k}+\beta^{k}=\alpha^{k}
$$

Note that $0<\beta^{k}<.5$ if $k \geq 2$ is even and $-.5<\beta^{k}<0$ if $k \geq 3$ is odd. Thus, it follows that if $k \geq 2$ is even, then $F_{k}$ is the largest integer $m$ such that
(1) $\quad \alpha^{k-1}<\sqrt{5} m<\alpha^{k}$.

It also follows that if $k \geq 3$ is odd, then $F_{k}$ is the smallest integer $m$ such that
(2) $\quad \alpha^{k}<\sqrt{5} m<\alpha^{k+1}$.

Using (1) and (2) and taking logarithms to the base $\alpha$, we have that

$$
k=\left[\log _{\alpha}(\sqrt{5} n+.5)\right]+1
$$

where $[x]$ denotes the greatest integer less than or equal to $x$.
Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, Russell Jay Hendel, and the proposers.

## Some Nearly Geometric Progressions

B-686 Proposed by Jeffrey Shallit, U. of Waterloo, Ontario, Canada
Let $a$ and $b$ be integers with $0<a \leq b$. Set $c_{0}=a, c_{1}=b$, and for $n \geq 2$ define $c_{n}$ to be the least integer with $c_{n} / c_{n-1}>c_{n-1} / c_{n-2}$. Find a closed form for $c_{n}$ in the cases:
(a) $a=1, b=2 ;$
(b) $a=2, b=3$.

Solution by C. Georghiou, University of Patras, Patras, Greece
(a) We will show that $c_{n}=F_{2 n+1}$. Suppose that it is true for $n=k-1$, and $n=k$. Then, from the definition of $c_{n}$ we have

$$
c_{k+1}=\left[c_{k}^{2} / c_{k-1}\right]+1
$$

where, as usual, $[x]$ denotes the greatest integer less than or equal to $x$.
Using the known identity $F_{2 k+3} F_{2 k-1}-F_{2 k+1}^{2}=1$, we get

$$
c_{k}^{2} / c_{k-1}=F_{2 k+1}^{2} / F_{2 k-1}=F_{2 k+3}-1 / F_{2 k-1}
$$

from which it follows that $c_{k+1}=F_{2 k+3}$. The proof is completed by noting that the assertion is true for $n=0$ and $n=1$.
(b) In a similar way we show that $c_{n}=2^{n}+1$. Assuming that

$$
c_{k-1}=2^{k-1}+1 \quad \text { and } \quad c_{k}=2^{k}+1
$$

we get

$$
c_{k+1}=\left[c_{k}^{2} / c_{k-1}\right]+1=\left[\left(2^{2 k}+2^{k+1}+1\right) /\left(2^{k-1}+1\right)\right]+1=2^{k+1}+1
$$

and at the same time

$$
c_{0}=2^{0}+1 \quad \text { and } \quad c_{1}=2^{1}+1
$$

Also solved by Paul S. Bruckman, Russell Euler, Herta Freitag, Russell Jay Hendel, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

B-687 Proposed by Jeffrey Shallit, University of Waterloo, Ontario, Canada
Let $c_{n}$ be as in Problem B-686. Find a closed form for $c_{n}$ in the case with $a=1$ and $b$ an integer greater than 1 .

Solution by Lawrence Somer, Washington, D.C.
Let $\left\{H_{n}\right\}$ denote the second-order linear recurrence which has initial terms $H_{0}=1, H_{1}=b$ and satisfies the recursion relation
$H_{n+2}=(b+1) H_{n+1}-(b-1) H_{n}$.
We claim that $c_{n}=H_{n}$. Clearly, $c_{0}=H_{0}$ and $c_{1}=H_{1}$. To prove our result, it suffices to show that
(1)

$$
H_{n+2} / H_{n+1}>H_{n+1} / H_{n}
$$

and
(2) $\quad\left(H_{n+2}-1\right) / H_{n+1} \leq H_{n+1} / H_{n}$.

Thus, it suffices to prove that
(3) $\quad H_{n+2} H_{n}>H_{n+1}^{2}$
and
(4) $\quad H_{n+2} H_{n} \leq H_{n+1}^{2}+H_{n}$.

One can easily show by induction that
(5) $\quad H_{n} \geq b^{n}$
and
(6) $\quad H_{n+2} H_{n}-H_{n+1}^{2}=(b-1)^{n}$.

Thus, from (6), we have that
(7)

$$
H_{n+2} H_{n}=H_{n+1}^{2}+(b-1)^{n}>H_{n+1}^{2}
$$

which establishes in@quality (3). From (5) and (7), we obtain

$$
H_{n+2} H_{n}=H_{n+1}^{2}+(b-1)^{n} \leq H_{n+1}^{2}+b^{n} \leq H_{n+1}^{2}+H_{n},
$$

which establishes inequality (4). Hence $c_{n}=H_{n}$. The closed form for $H_{n}$ is obtained using standard recursion theory.

Also solved by Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, and the proposer.

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by
Raymond E. Whitney
Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

## PROBLEMS PROPOSED IN THIS ISSUE

H-462 Proposed by Ioan Sadoveanuv, Ellensburg, WA
Let $G(x)=x^{k}+\alpha_{1} x^{k-1}+\ldots+\alpha_{k}$ be a polynomial with $c$ a root of order $p$. If $G^{(p)}(x)$ denotes the $p^{\text {th }}$ derivative of $G(x)$, show that $\left\{n^{p} c^{\left.n-p / G^{(p)}(c)\right\}}\right.$ is a solution of the recurrence $u_{n}=c^{n-k}-\alpha_{1} u_{n-1}-\alpha_{2} u_{n-2}-\cdots-\alpha_{k} u_{n-k}$.

H-463 Proposed by Paul Bruckman, Edmonds, WA
Establish the identity

$$
\begin{equation*}
\sum_{n=1}^{\infty} \Phi(n)\left(\frac{z^{n}}{1-z^{2 n}}\right)=\frac{z\left(1+z+z^{2}\right)}{\left(1-z^{2}\right)^{2}}, z \in \mathbb{C},|z|<1 \tag{1}
\end{equation*}
$$ and $\Phi$ is the Euler (totient) function.

As special cases of (1), obtain the following identities:

$$
\begin{equation*}
\sum_{n=1}^{\infty} \Phi(2 n) / F_{2 n s}=\sqrt{5} / L_{s}^{2}, s=1,3,5, \ldots ; \tag{2}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{n=1}^{\infty} \Phi(2 n-1) / L_{(2 n-1)_{s}}=F_{s} \sqrt{5} / L_{s}^{2}, s=1,3,5, \ldots ; \tag{3}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{n=1}^{\infty} \Phi(n) / F_{n s}=\left(L_{s}+1\right) / F_{s}^{2} \sqrt{5}, s=2,4,6, \ldots \tag{4}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{n=1}^{\infty}(-1)^{n-1} \Phi(n) / F_{n s}=\left(L_{s}-1\right) / F_{s}^{2} \sqrt{5}, s=2,4,6, \ldots \tag{5}
\end{equation*}
$$

$$
\sum_{n=1}^{\infty}(-1)^{n-1} 1_{\Phi(2 n) / F_{2 n s}}=\left\{\begin{array}{l}
1 / F_{s}^{2} \sqrt{5}, s=1,3,5, \ldots ;  \tag{6}\\
\sqrt{5} / L_{s}^{2}, s=2,4,6, \ldots ;
\end{array}\right.
$$

$$
\begin{align*}
& \sum_{n=1}^{\infty}(-1)^{n-1} \Phi(2 n-1) / F_{(2 n-1)_{s}}=L_{s} / F_{s}^{2} \sqrt{5}, s=1,3,5, \ldots  \tag{7}\\
& \sum_{n=1}^{\infty}(-1)^{n-1} \Phi(2 n-1) / L_{(2 n-1)_{s}}=F_{s} \sqrt{5} / L_{s}^{2}, s=2,4,6, \ldots . \tag{8}
\end{align*}
$$

H-464 Proposed by H.-J. Seiffert, Berlin, Germany
Show that
where

$$
\sum_{k=0}^{[n / 2]}\binom{n}{k} A_{n-2 k}=F_{n},
$$

$A_{j}=(-1)^{[(j+2) / 5]}-\left((-1)^{[j / 5]}+(-1)^{[(j+4) / 5]}\right) / 2$.
[ ] denotes the greatest integer function.
H-465 Proposed by Richard André-Jeannin, Tunisia
Let $p$ be a prime number, and let $r_{1}, r_{2}, \ldots, r_{s}$ be natural integers such that $s \geq 2, r_{1}<p$, and

$$
\sum_{k=1}^{k=s} k r_{k}=p .
$$

Show that the number

$$
B_{r_{1}, r_{2}, \ldots, r_{s}} \cdot=\frac{1}{r_{1}+r_{2}+\cdots+r_{s}} \frac{\left(r_{1}+r_{2}+\cdots+r_{s}\right)!}{r_{1}!r_{2}!\cdots r_{s}!}
$$

is an integer.

## SOLUTIONS

An Odd Problem
H-442 Proposed by Piero Filipponi, Rome, Italy
(Vol. 28, no. 2, May 1990)
Prove that the congruence
$\prod_{i=1}^{(d-3) / 2}(2 i+1)^{2} \equiv\left\{\begin{aligned} 1(\bmod d) & \text { if }(d+1) / 2 \text { is even } \\ -1(\bmod d) & \text { if }(d+1) / 2 \text { is odd }\end{aligned}\right.$
holds if and only if $d$ is an odd prime.

Solution by the proposer
Let $n$ be an even integer. The equality

$$
\begin{equation*}
n!=\frac{1}{2^{n}} \prod_{i=0}^{(n-2) / 2}[n(n+2)-4 i(i+1)] \tag{1}
\end{equation*}
$$

can be proved readily by writing $n=2 m$ and rewriting (1) as

$$
(2 m)!=\frac{1}{2^{2 m}} \prod_{i=0}^{m-1}[4 m(m+1)-4 i(i+1)]=\prod_{i=0}^{m-1}(m-i)(m+i+1) .
$$

Let $d$ be an odd integer. By (1) we have

$$
\begin{equation*}
(d-1)!=\frac{1}{2^{d-1}} \prod_{i=0}^{(d-3) / 2}\left[d^{2}-1-4 i(i+1)\right] . \tag{2}
\end{equation*}
$$

If $d$ is a prime, by using Fermat's little theorem, we obtain the congruence

$$
\begin{equation*}
(d-1)!\equiv \prod_{i=0}^{(d-3) / 2}\left(-1-4 i^{2}-4 i\right)=\prod_{i=0}^{(d-3) / 2}\left[-(2 i+1)^{2}\right] \quad(\bmod d) \tag{3}
\end{equation*}
$$

By using Wilson's theorem, ( $(d-1)!\equiv-1$ (mod $d)$ iff $d$ is prime); thus, by (3) we get

$$
\prod_{i=0}^{(d-3) / 2}\left[-(2 i+1)^{2}\right]=(-1)^{(d-1) / 2} \prod_{i=1}^{(d-3) / 2}(2 i+1)^{2} \equiv-1 \quad(\bmod d)
$$

iff $d$ is prime, that is,

$$
\prod_{i=1}^{(d-3) / 2}(2 i+1)^{2} \equiv(-1)^{(d+1) / 2}(\bmod d) \text { iff } d \text { is prime }
$$

Also solved by $P$. Bruckman, R. Hendel, and L. Somer.

## Another Odd One

H-443 Proposed by Richard André-Jeannin, Tunisia (Vol. 28, no. 3, August 1990)

Let us consider the recurrence

$$
w_{n}=m w_{n-1}+w_{n-2}
$$

where $m>0$ is an integer and $U_{n}, V_{n}$ the solutions defined by

$$
U_{0}=0, U_{1}=1 ; V_{0}=2, V_{1}=m
$$

Show that, if $q$ is an odd divisor of $m^{2}+1$, then

$$
V_{q} \equiv m(\bmod q)
$$

Solution by H.-J. Seiffert, Berlin, Germany

First, we prove that

$$
\begin{equation*}
V_{q} \equiv m^{k} V_{q-3 k}(\bmod q), k=0, \ldots,[q / 3] \tag{1}
\end{equation*}
$$

where [ ] denotes the greatest integer function.
Obviously, (1) is true for $k=0$. Assuming that it holds for $k$, where

$$
0 \leq k<[q / 3]
$$

we obtain

$$
\begin{aligned}
& V_{q} \equiv m^{k} V_{q-3 k} \\
&=m^{k}\left(m V_{q-3 k-1}+V_{q-3 k-2}\right) \\
&=m^{k}\left(m^{2}+1\right) V_{q-3 k-2}+m^{k+1} V_{q-3 k-3} \\
& \equiv m^{k+1} V_{q-3(k+1)} \quad(\bmod q)
\end{aligned}
$$

This completes the induction proof of (1).
For any odd prime divisor $p$ of $q$, the congruence $m^{2} \equiv-1$ (mod $p$ ) shows that -1 is a quadratic residue mod $p$; hence (see T. M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, 1976, Theorem 9.4, p. 181), $p \equiv 1$ (mod 4). This holds for any odd prime divisor $p$ of $q$. Since $q$ is odd, we also have $q \equiv 1(\bmod 4)$. In (1), we take $k=[q / 3]$. Hence, we have $0 \leq q-3 k \leq 2$. The case $q=3 k$ would imply $m^{2} \equiv-1$ (mod 3), which contradicts Fermat's little theorem. If $q=3 k+1$, then $q \equiv 1$ (mod 4) implies that $k$ is divisible by 4 . From $m^{2}=-1(\bmod q)$ and (1), we get

$$
V_{q} \equiv m^{k} V_{1}=m^{k+1} \equiv(-1)^{k / 2} m=m \quad(\bmod q)
$$

[Feb.

If $q=3 k+2$, then $q$ odd and $q \equiv 1(\bmod 4)$ yield $k \equiv 1(\bmod 4)$. Now $m^{2} \equiv-1$ $(\bmod q)$ and (1) give

$$
\begin{aligned}
V_{q} & \equiv m^{k} V_{2}=m^{k}\left(m^{2}+2\right)=m^{k}\left(m^{2}+1\right)+m^{k} \equiv m^{k} \\
& \equiv(-1)^{(k-1) / 2} m=m \quad(\bmod q)
\end{aligned}
$$

This completes the solution. Finally, it should be noted that (1) also follows from the identity

$$
V_{q}=\left(m^{2}+1\right) \sum_{j=0}^{k-1} m^{j} V_{q-3 j-2}+m^{k} V_{q-3 k}
$$

valid for $k=0, \ldots,[q / 3]$.

Also solved by P. Bruckman, F. Howard, L. Somer, and the proposer.

## Summing It Up

H-444 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 28, no. 3, August 1990)

Show that, for $=0,1,2, \ldots$,

$$
F_{n}=\sum_{\substack{k=0 \\(5, n-2 k)=1}}^{[n / 2]}(-1)^{[(n-2 k+2) / 5]}\binom{n}{k}
$$

where $(r, s)$ denotes the greatest common divisor of $r$ and $s$ and [ ] the greatest integer function.

Solution by Paul Bruckman, Edmonds, WA

We employ a generating function technique to prove what appears to be a very remarkable identity. Define

$$
\begin{equation*}
G_{n}=\sum_{\substack{k=0 \\(5, n-2 k)=1}}^{\left[\frac{1}{2} n\right]}(-1)^{\left[\frac{1}{5}(n-2 k+2)\right]}\binom{n}{k}, \quad n=0,1,2, \ldots, \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
g(x)=\sum_{n=0}^{\infty} G_{n} x^{n} \tag{2}
\end{equation*}
$$

Then (formally, at least),

Now

$$
\begin{aligned}
& g(x)= \sum_{\substack{n, k=0 \\
(n, 5)=1}}^{\infty}(-1)\left[\frac{1}{5}(n+2)\right] \\
& x^{n+2 k}\binom{n+2 k}{k} \\
& \sum_{k=0}^{\infty}\binom{n+2 k}{k} x^{2 k}=\sum_{k=0}^{\infty}\binom{n+2 k}{2 k}\binom{2 k}{k} /\binom{n+k}{k} x^{2 k}=\sum_{k=0}^{\infty} \frac{(n+1)_{2 k}}{(n+1)_{k}} \cdot \frac{x^{2 k}}{k!} \\
&=\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}(n+1)\right)_{k}\left(\frac{1}{2}(n+2)\right)_{k}}{(n+1)_{k}} \cdot \frac{(2 x)^{2 k}}{k!} \\
&={ }_{2} F_{I}\left[\frac{n+1}{2}, \frac{n+2}{2} ; 4 x^{2}\right]
\end{aligned}
$$

(the "standard" hypergeometric function). Hence,

$$
g(x)=\sum_{\substack{n=0  \tag{3}\\
(n, 5)=1}}^{\infty}(-1)^{\left[\frac{1}{5}(n+2)\right]} x^{n} \cdot{ }_{2} F_{1}\left[\begin{array}{c}
\left.\frac{n+1}{2}, \frac{n+2}{2} ; 4 x^{2}\right] . . ~ \\
n+1
\end{array}\right]
$$

We will use the following known transformations of the hypergeometric function:

$$
{ }_{2} F_{1}\left[\begin{array}{cc}
a, & b  \tag{4}\\
c & ; w
\end{array}\right]=(1-w)^{-\alpha}{ }_{2} F_{1}\left[\begin{array}{cc}
a, c-b \\
c & ; \frac{-w}{1-w}
\end{array}\right],
$$

(Theorem 20, p. 60, of [1])

$$
{ }_{2} F_{1}\left[\begin{array}{l}
a, b  \tag{5}\\
a+b+\frac{1}{2}
\end{array} ; 4 z(1-z)\right]={ }_{2} F_{1}\left[\begin{array}{l}
2 a, 2 b \\
a+b+\frac{1}{2}
\end{array}\right] z .
$$

(Theorem 25, p. 67, of [1])
In (4), let

$$
a=\frac{1}{2}(n+1), b=\frac{1}{2}(n+2), c=n+1, w=4 x^{2} .
$$

We thus obtain

$$
{ }_{2} F_{1}\left[\begin{array}{c}
\frac{n+1}{2}, \frac{n+2}{2} ; 4 x^{2}  \tag{6}\\
n+1
\end{array}\right]=\left(1-4 x^{2}\right)^{-\frac{1}{2}(n+1)} \cdot 2_{1} F_{1}\left[\begin{array}{c}
\frac{n+1}{2}, \frac{n}{2} ; \frac{-4 x^{2}}{1-4 x^{2}}
\end{array}\right]
$$

In (5), let

$$
a=\frac{1}{2}(n+1), \quad b=\frac{1}{2} n, z=\frac{1}{2}(1-\theta),
$$

where $\theta=\theta(x)=\left(1-4 x^{2}\right)^{-\frac{1}{2}}$; note that

$$
4 z(1-z)=(1-\theta)(1+\theta)=\frac{-4 x^{2}}{1-4 x^{2}}
$$

Then

$$
\begin{aligned}
{ }_{2} F_{1}\left[\begin{array}{c}
\left.\frac{n+1}{2}, \frac{n}{2} ; \frac{-4 x^{2}}{1-4 x^{2}}\right]
\end{array}\right. & ={ }_{2} F_{1}\left[\begin{array}{c}
n+1, n \\
n+1
\end{array} ; z\right]={ }_{1} F_{0}\left[\begin{array}{c}
n \\
-
\end{array}\right] \\
& =\sum_{k=0}^{\infty} \frac{(n)_{k}}{k!} z^{k}=\sum_{k=0}^{\infty}\binom{-n}{k}(-z)^{k}=(1-z)^{-n} .
\end{aligned}
$$

Therefore, using (3) and (6):

$$
g(x)=\sum_{\substack{n=0 \\(n, 5)=1}}^{\infty}(-1)^{\left[\frac{1}{5}(n+2)\right.} x_{x^{n}(1-z)^{-n} \theta^{n+1}}
$$

We now make the substitution, $y=\theta x(1-z)^{-1}$. Thus,

$$
y=2 x \theta(1+\theta)^{-1}=2 x \theta(1-\theta)\left(1-\theta^{2}\right)^{-1}=\frac{-2 x\left(1-\theta^{-1}\right)}{-4 x^{2}},
$$

or

$$
\begin{equation*}
y=\frac{1-\left(1-4 x^{2}\right)^{\frac{1}{2}}}{2 x} \tag{7}
\end{equation*}
$$

We then obtain

$$
\begin{equation*}
g(x)=\theta \sum_{\substack{n=0 \\(n, 5)=1}}^{\infty}(-1)^{\left[\frac{1}{5}(n+2)\right]} y^{n}, \text { where } y \text { is given by (7). } \tag{8}
\end{equation*}
$$

Next, we obtain a closed form for $g(x)$, as follows:

$$
\begin{aligned}
g(x) & =\theta \sum_{m=0}^{\infty} \sum_{r=1}^{4}(-1)^{\left[\frac{1}{5}(5 m+r+2)\right]} y^{5 m+r}=\theta \sum_{m=0}^{\infty}(-1)^{m} y^{5 m} \sum_{r=1}^{4}(-1)^{\left[\frac{1}{5}(r+2)\right]} y^{r} \\
& =\theta \sum_{m=0}^{\infty}\left(-y^{5}\right)^{m}\left(y+y^{2}-y^{3}-y^{4}\right)=\frac{\theta y(1+y)\left(1-y^{2}\right)}{1+y^{5}}
\end{aligned}
$$

hence,
(9)

$$
g(x)=\theta y\left(1-y^{2}\right)\left(1-y+y^{2}-y^{3}+y^{4}\right)^{-1}
$$

To evaluate $g(x)$ as an explicit function of $x$, we employ the readily verifiable result:
(10) $\quad y^{2}=y / x-1$ :

Using (10), we obtain

$$
\begin{aligned}
& y^{3}=y^{2} / x-y=-y+\frac{1}{x}(y / x-1)=\frac{-1}{x}+\left(1 / x^{2}-1\right) y \\
& y^{4}=-y / x+\left(1 / x^{2}-1\right) y^{2}=-y / x+\left(1 / x^{2}-1\right)(y / x-1) \\
&=1-x^{-2}+\frac{y}{x}\left(x^{-2}-2\right)
\end{aligned}
$$

Therefore,

$$
\left(1-y+y^{2}-y^{3}+y^{4}\right)=1+x^{-1}-x^{-2}-y\left(x^{-1}+x^{-2}-x^{-3}\right)
$$

after simplification, or

$$
\begin{equation*}
1-y+y^{2}-y^{3}+y^{4}=x^{-3}(y-x)\left(1-x-x^{2}\right) \tag{11}
\end{equation*}
$$

Also,

$$
\theta y\left(1-y^{2}\right)=\theta y+\theta\left(x^{-1}+\left(1-x^{-2}\right) y\right)=-\theta(y-x) x^{-2}+2 \theta y
$$

Therefore,

$$
\begin{equation*}
g(x)=x \theta\left(\frac{2 x^{2} y}{y-x}-1\right)\left(1-x-x^{2}\right)^{-1} \tag{12}
\end{equation*}
$$

$$
\begin{aligned}
& \text { From (10), } x y=y-x ; \text { therefore, } \\
& \qquad \frac{2 x^{2} y}{y-x}-1=\frac{2 x^{2} y}{x y^{2}}-1=\frac{2 x-y}{y}=\frac{4 x^{2}-1+\left(1-4 x^{2}\right)^{\frac{1}{2}}}{1-\left(1-4 x^{2}\right)^{\frac{1}{2}}}=\left(1-4 x^{2}\right)^{\frac{1}{2}}=\theta^{-1}
\end{aligned}
$$

Hence, we finally obtain
(13) $\quad g(x)=x\left(1-x-x^{2}\right)^{-1}$.

We recognize $g(x)$ in (13) as the generating function of the Fibonacci numbers; more specifically,

$$
\begin{equation*}
g(x)=\sum_{n=0}^{\infty} F_{n} x^{n} \tag{14}
\end{equation*}
$$

Comparison with (2) yields the desired result:

$$
\begin{equation*}
G_{n}=F_{n}, n=0,1,2, \ldots \cdot \text { Q.E.D. } \tag{15}
\end{equation*}
$$

## Reference:

1. E. D. Rainville. Special Functions. New York: Chelsea, 1960.

Also solved by $S$. Rizavi and the proposer.
Mu-ve Over
H-445 Proposed by Paul S. Bruckman, Edmonds, WA (Vol. 28, no. 3, August 1990)

Please refer to the volume of The Fibonacci Quarterly cited above for a complete statement of this problem.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY
For $|z|<1$, we have

$$
\begin{equation*}
\sum_{n=1}^{\infty} \mu(n)\left(\frac{z^{n}}{1-z^{2 n}}\right)=\sum_{n=1}^{\infty} \mu(n) \sum_{k=1}^{\infty} z^{(2 k-1) n}=\sum_{n=1}^{\infty}\left(\sum_{\substack{d \mid n \\ d \text { odd }}} \mu(n / d)\right) z^{n} \tag{*}
\end{equation*}
$$

Let $n=2^{s} t$, where $t$ is odd and $s \geq 0$. Then the set of odd divisors of $n$ is precisely the set of divisors of $t$. Thus,

$$
\sum_{d \mid n} \mu(n / d)=\mu\left(2^{s}\right) \sum_{d \mid t} \mu(t / d)=\left\{\begin{aligned}
1 & \text { if } s=0 \text { and } t=1 \\
-1 & \text { if } s=t=1 \\
0 & \text { otherwise }
\end{aligned}\right.
$$

Furthermore, ( $2 k-1$ ) $n$ in (*) is odd iff $n$ is odd. Hence, we conclude that for $|z|<1$,

$$
\sum_{n \text { odd }} \mu(n)\left(\frac{z^{n}}{1-z^{2 n}}\right)=z \quad \text { and } \quad \sum_{n \text { even }} \mu(n)\left(\frac{z^{n}}{1-z^{2 n}}\right)=-z^{2}
$$

which lead to (1). For (2)-(7), we shall use the identities:

$$
\frac{1}{\sqrt{5}} \frac{1}{E_{m s}}=\frac{\beta^{m s}}{(-1)^{m s}-\beta^{2 m s}} \quad \text { and } \quad \frac{1}{L_{m s}}=\frac{\beta^{m s}}{(-1)^{m s}+\beta^{2 m s}}
$$

(2) For $s=1,2,3, \ldots, \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{\mu(2 n)}{F_{2 n s}}=\sum_{m \text { even }} \mu(m)\left(\frac{\beta^{m s}}{1-\beta^{2 m s}}\right)=-\beta^{2 s}$.
(3) For $s=1,3,5, \ldots, \sum_{n=1}^{\infty} \frac{\mu(2 n-1)}{L_{(2 n-1) s}}=-\sum_{m \text { odd }} \mu(m)\left(\frac{\beta^{m s}}{1-\beta^{2 m s}}\right)=-\beta^{s}$.
(4) For $s=2,4,6, \ldots, \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{\mu(n)}{F_{n s}}=\sum_{n=1}^{\infty} \mu(n)\left(\frac{\beta^{n s}}{1-\beta^{2 n s}}\right)=\beta^{s}-\beta^{2 s}$.
(5) For $s=2,4,6, \ldots, \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \mu(n)}{F_{n s}}=-\sum_{n=1}^{\infty} \mu(n)\left(\frac{\left(-\beta^{s}\right)^{n}}{1-\left(-\beta^{s}\right)^{2 n}}\right)=\beta^{s}+\beta^{2 s}$.
(6) For $s=1,3,5, \ldots, \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \mu(2 n-1)}{F_{(2 n-1)_{s}}}=-\frac{1}{i} \sum_{m \text { odd }} \mu(m)\left(\frac{\left(i \beta^{s}\right)^{m}}{1-\left(i \beta^{s}\right)^{2 m}}\right)=-\beta^{s}$.
(7) For $s=2,4,6, \ldots, \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \mu(2 n-1)}{L_{(2 n-1) s}}=\frac{1}{i} \sum_{m \text { odd }} \mu(m)\left(\frac{\left(i \beta^{s}\right)^{m}}{1-\left(i \beta^{s}\right)^{2 m}}\right)=\beta^{s}$.

Also solved by C. Georghiou, H.-J. Seiffert, and the proposer.

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[^0]:    *This paper was written while the author was a visiting scholar at the University of Wisconsin.

[^1]:    4. Under what conditions can $l^{\prime}$ 's be eliminated in every position? Result 1 shows that $2^{k}<b$ is a necessary condition. However, even the following elementary question remains unanswered for bases $>4$ : Are there numbers $x$ and $y$ such that for every $m$ at least one of $m x$ or $m y$ contains the digit 1 ?

    Acknowledgment

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