The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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THE FIBONACCI QUARTERLY seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

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The Fibonacci Quarterly

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OF INTEGERS WITH SPECIAL PROPERTIES

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CYCLIC BINARY STRINGS WITHOUT LONG RUNS OF LIKE (ALTERNATING) BITS

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(Submitted February 1991)

1. A binary n-bit *cyclic* string (briefly n-CS) is a sequence of n 0's and 1's (the bits), with the first and last bits considered to be adjacent (i.e., the first bit follows the last bit). This condition is visible when the string is displayed in a circle with one bit "capped": the capped bit is the first bit and reading clockwise we see the second bit, the third bit, and so on to the n^{th} bit (the last bit). In an n-CS, a subsequence of consecutive bits is a *run*. Motivated by a problem of genetic information processing, Agur, Fraenkel, and Klein [1] derived formulas for the number of n-CSs with no runs 0 0 0 nor 1 1 1 (i.e., all runs of like bits have length ≤ 2) and for the number with no runs 0 1 0 nor 1 0 1 (i.e., all runs of alternating bits have length ≤ 2). These are the cases $w = 2$ of

$$L_{\le w}(n) = \text{the number of } n\text{-CSs in which all runs of like bits have length} \le w$$

and

$$A_{\le w}(n) = \text{the number of } n\text{-CSs in which all runs of alternating bits have length} \le w.$$

In this note we prove the

Theorem:

$$L_{\le w}(n) = \begin{cases} 2^n, & \text{if } 1 \le n \le w-1, \\ F_w(n) + D_w(n), & \text{if } n \ge w, \end{cases}$$

and

$$A_{\le w}(n) = \begin{cases} 2^n, & \text{if } 1 \le n \le w-1, \\ F_w(n) + (-1)^n D_w(n), & \text{if } n \ge w, \end{cases}$$

where

(1)
$$F_w(0) = w, \quad F_w(n) = 2^n - 1, \quad 1 \le n \le w-1,$$
$$F_w(n) = F_w(n-1) + F_w(n-2) + \cdots + F_w(n-w), \quad n \ge w,$$

and

(2)
$$D_w(n) = \begin{cases} w, & \text{if } n \ge 1 \text{ and } w+1 \mid n, \\ -1, & \text{if } n \ge 1 \text{ and } w+1 \nmid n. \end{cases}$$

Furthermore,

(3)
$$L_{\le w}(n) \sim A_{\le w}(n) \sim c\alpha^n$$

where c is a constant (which depends only on w) and

$$2 - \frac{2}{2^w} < \alpha < 2 - \frac{1}{2^w}.$$

2. Consider for any n-CS

$$x = x_1 x_2 x_3 \dots x_n, \quad x_i = 0, 1,$$

the n-CS

$$T(x) = y_1 y_2 \dots y_n, \quad y_i = \begin{cases} 0, & \text{if } x_i = x_{i-1}, \\ 1, & \text{if } x_i \neq x_{i-1}, \end{cases} \quad i = 1, 2, \dots, n \ (x_0 = x_n).$$

For example,

$$T(001110010110001111) = 101001011101001000$$
$$T(101100011101010111) = 011010010011111100$$
$$T(110011001100100000) = 101010101010110000$$

Thus, when passing over the bits of x, $T(x)$ records the changes (from 0 to 1 or from 1 to 0) by a 1, and records no change (from 0 to 0 or from 1 to 1) by a 0.

Of course

$$T(\tilde{x}) = T(x),$$

where $\tilde{x}$ is the complementary n-CS

$$\tilde{x} = y_1 y_2 y_3 \dots y_n, \quad y_i = \begin{cases} 1, & \text{if } x_i = 0, \\ 0, & \text{if } x_i = 1. \end{cases}$$

However, for any two <u>different</u> n-CSs u and v, <u>both with first bit 1</u>, $T(u) \neq T(v)$. Indeed, T is bijective between the set of 2^{n-1} n-CSs with first bit 1 and the set of 2^{n-1} n-CSs with an even number of 1's. Thus, an n-CS x with first bit 1 corresponds to an n-CS $T(x)$ with an even number of 1's, and then a run of w like bits in x corresponds to a run of $w-1$ 0's in $T(x)$, while a run of w alternating bits in x corresponds to a run of $w-1$ 1's in $T(x)$.

Hence,

x is an n-CS with first bit 1 and all runs of like bits have length $\leq w$
if and only if
$T(x)$ is an n-CS with an even number of 1's and all runs of 0's have length $\leq w-1$,

so we have

$$L_{\leq w}(n) = 2B^e_{w-1}(n), \quad n \geq 1,$$

where

$B^e_w(n)$ = the number of n-CSs with an even number of 1's and all runs of 0's have length $\leq w$.

Also,

x is an n-CS with first bit 1 and all alternating runs have length $\leq w$
if and only if
$T(x)$ is an n-CS with an even number of 1's and all runs of 1's have length $\leq w-1$
if and only if
$\widetilde{T(x)}$ is an n-CS with an even number of 0's and all runs of 0's have length $\leq w-1$
if and only if

n is even, $\widetilde{T(x)}$ has an even number of 1's and
all runs of 0's have length $\le w-1$
or
n is odd, $\widetilde{T(x)}$ has an odd number of 1's and
all runs of 0's have length $\le w-1$,

and we have

$$A_{\le w}(n)=\begin{cases}2B^e_{w-1}(n), & \text{if } n \text{ is even},\\ 2B^o_{w-1}(n), & \text{if } n \text{ is odd},\end{cases}$$

where

$B^o_w(n)$ = the number of n-CSs with an odd number of 1's and all runs of 0's have length $\le w$.

In terms of $B_w(n)=B^e_w(n)+B^o_w(n)$ and $C_w(n)=B^e_w(n)-B^o_w(n)$,

(4) $$L_{\le w}(n)=B_{w-1}(n)+C_{w-1}(n),\quad A_{\le w}(n)=B_{w-1}(n)+(-1)^n C_{w-1}(n),\quad n\ge 1.$$

In order to determine $B^e_w(n)$, $B^o_w(n)$, and $B_w(n)$, we naturally investigate (for $n\ge 1$, $k\ge 0$, $w\ge 1$) $(n:k)_w$ = the number of n-CSs with exactly k 1's ($n-k$ 0's) and all runs of 0's have length $\le w$. Clearly

(5) $$(n:0)_w=\begin{cases}1, & 1\le n\le w,\\ 0, & n\ge w+1,\end{cases}$$

and

(6) $$(n:k)_w=\begin{cases}\binom{n}{k}, & 1\le n\le w+k,\\ 0, & 1\le k,\ k(w+1)<n,\end{cases}$$

where

$$\binom{n}{k}=\begin{cases}n!/k!(n-k)!, & 0\le k\le n,\\ 0, & 0\le n<k.\end{cases}$$

Consider an n-CS counted in $(n:k)_w$, $n\ge w+2$, $k\ge 2$. If the first bit is 1 (i.e., capped bit is $\hat{1}$), and the last 1 is followed by exactly i 0's ($0\le i\le w$), delete this last 1 and the i 0's which follow it and then we have an $(n-1-i)$-CS with $k-1$ 1's, first bit 1, and every run of 0's has length $\le w$. If the first bit is 0, and the first 1 is followed by i 0's ($0\le i\le w$), delete this first 1 and the i 0's which follow it and we then have an $(n-1-i)$-CS with $k-1$ 1's, first bit 0, and every run of 0's has length $\le w$. Hence,

(7) $$(n:k)_w=(n-1:k-1)_w+(n-2:k-1)_w+\cdots+(n-1-w:k-1)_w,\ k\ge 2,\ n\ge w+2.$$

Of course,

$$B^e_w(n)=\sum_{k=0}(n:2k)_w,\quad B^o_w(n)=\sum_{k=1}(n:2k-1)_w.$$

From (5), (6), and (7), we deduce that

$$B_w^e(n) = \begin{cases} 2^{n-1}, & 1 \le n \le w, \\ 2^{n-1} - 1, & n = w+1, \\ B_w^o(n-1) + B_w^o(n-2) + \cdots + B_w^o(n-1-w), & n \ge w+2, \end{cases}$$

$$B_w^o(n) = \begin{cases} 2^{n-1}, & 1 \le n \le w+1, \\ B_w^e(n-1) + \cdots + B_w^e(n-1-w) + n - 2(w+1), & w+2 \le n \le 2w+1, \\ B_w^e(n-1) + B_w^e(n-2) + \cdots + B_w^e(n-1-w), & n \ge 2w+2, \end{cases}$$

$$\begin{aligned} B_w(n) &= B_w^e(n) + B_w^o(n) \\ &= \begin{cases} 2^n, & n = 1, 2, \ldots, w, \\ 2^n - 1, & n = w+1, \\ B_w(n-1) + \cdots + B_w(n-1-w) + n - 2(w+1), & w+2 \le n \le 2w+1, \\ B_w(n-1) + B_w(n-2) + \cdots + B_w(n-1-w), & n \ge 2w+2. \end{cases} \end{aligned}$$

Furthermore, the numbers $C_w(n) = B_w^e(n) - B_w^o(n)$ are seen to satisfy

$$C_w(n) = \begin{cases} 0, & 1 \le n \le w, \\ -1, & n = w+1, \\ w+1, & n = w+2, \\ -1, & w+3 \le n \le 2w+1, \\ -\{C_w(n-1) + C_w(n-2) + \cdots + C_w(n-1-w)\}, & n \ge 2w+2, \end{cases}$$

that is,

$$C_w(n) = \begin{cases} 0, & 1 \le n \le w, \\ w+1, & n \ge w+1,\ w+2 \mid n, \\ -1, & n \ge w+1,\ w+2 \nmid n. \end{cases}$$

Now it is easy to verify that

$$B_{w-1}(n) = F_w(n),\ n \ge w, \quad C_{w-1}(n) = D_w(n),\ n \ge w,$$

where $F_w(n)$ and $D_w(n)$ are defined by (1) and (2), and this with (4) completes the first part of the Theorem.

It is well known that any sequence $\{H_w(n)\}_{n=0}$ which satisfies

$$H_w(n) = H_w(n-1) + H_w(n-2) + \cdots + H_w(n-w),\ n \ge w,$$

can be written

$$H_w(n) = \sum_{i=1} c_i \alpha_i^n,\ n = 0, 1, 2, \ldots,$$

where $\alpha_i = \alpha_i^{(w)}$, $(i = 1, 2, \ldots, w)$ are the roots of

(8) $$z^w - z^{w-1} - z^{w-2} - \cdots - z - 1$$

and the $c_i = c_i^{(w)}$, $(i = 1, 2, \ldots, w)$ are determined by the w equations

$$c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_w\alpha_w^n = H_w(n),\ n = 0, 1, \ldots, w-1.$$

This depends on the fact that $\alpha_1, \alpha_2, \ldots, \alpha_w$ are distinct. This fact is easily proved: multiply (8) by $x-1$ to get $x^{w+1} - 2x^w + 1$ which has no multiple roots because it has no roots in common with its derivative.

Cappocelli and Cull [2] have shown that exactly one of the roots of (8) is real and positive, say $\alpha = \alpha_1$, and it satisfies

$$2 - \frac{2}{2^w} < \alpha < 2 - \frac{1}{2^w}, \tag{9}$$

while all the other roots satisfy

$$\frac{1}{\sqrt[w]{3}} < |\alpha_i| < 1, \quad i = 2, 3, \ldots, w.$$

It follows that $H_w(n) \sim c\alpha^n$. This leads to

$$L_{\le 2}(n) \sim A_{\le 2}(n) \sim \left(\frac{1+\sqrt{5}}{2}\right)^n,$$

and for $w \ge 2$, $L_{\le w}(n) \sim A_{\le w}(n) \sim c\alpha^n$, where c is a constant (which depends only on w) and α satisfies (9).

The following tables show $F_w(n)$, $D_w(n)$, $L_{\le w}(n)$, and $A_{\le w}(n)$ for $w = 2, 3, 4$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$F_2(n)$	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843
$D_2(n)$		−1	−1	2	−1	−1	2	−1	−1	2	−1	−1	2	−1	−1
$L_{\le 2}(n)$		2	2	6	6	10	20	28	46	78	122	198	324	520	842
$A_{\le 2}(n)$		2	4	2	6	12	20	30	46	74	122	200	324	522	842

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$F_3(n)$	3	1	3	7	11	21	39	71	131	241	443	815	1499	2757
$D_3(n)$		−1	−1	−1	3	−1	−1	−1	3	−1	−1	−1	3	−1
$L_{\le 3}(n)$		2	4	6	14	20	38	70	134	240	442	814	1502	2756
$A_{\le 3}(n)$		2	4	8	14	22	38	72	134	242	442	816	1502	2578

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$F_4(n)$	4	1	3	7	15	26	51	99	191	367	708	1365	2631	5071
$D_4(n)$		−1	−1	−1	−1	4	−1	−1	−1	−1	4	−1	−1	−1
$L_{\le 4}(n)$		2	4	8	14	30	50	98	190	366	712	1364	2630	5070
$A_{\le 4}(n)$		2	4	8	14	22	50	100	190	368	712	1366	2630	5072

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COMPLEX FIBONACCI AND LUCAS NUMBERS, CONTINUED FRACTIONS, AND THE SQUARE ROOT OF THE GOLDEN RATIO

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1. INTRODUCTION

The golden ratio $\phi = (\sqrt{5}+1)/2$ is mathematically ancient (see [3] for example), while both ϕ and its square root are of historical architectural significance,[1] and are therefore points of contact between the "two cultures." (Compare the cultural and historical approach to the theory of numbers in [12].) It is pleasing that ϕ and $\sqrt{\phi}$ have a further relationship in terms of continued fractions. The formula

$$(1) \qquad \phi = 1 + \frac{1}{1+}\ \frac{1}{1+}\cdots$$

is very familiar and it will be proved below that

$$(2) \qquad \sqrt{1+2i} = \sqrt{\phi} + i/\sqrt{\phi} = 1 + i + \frac{1}{2+2i+}\ \frac{1}{2+2i+}\ \frac{1}{2+2i+}\cdots$$

where $i = \sqrt{-1}$. A similar result is

$$(3) \qquad \sqrt{1+i/2} = \frac{1}{2}(\sqrt{5}+2)^{1/2} + \frac{1}{2}i(\sqrt{5}-2)^{1/2} = \frac{1}{2}(1+i) + \frac{1}{1+i+}\ \frac{1}{1+i+}\cdots$$

---

[1]The golden ratio, or as Kepler, following Luca Pacioli [11], called it, the "divine proportion," and also its square root, are related to two of the most famous buildings of all time, the Parthenon at Athens and the Great Pyramid, respectively.

The golden rectangle is exemplified by the face of the Parthenon ([7, pp. 62 and 63]; [13, p. 139]). The Parthenon was built only about half a century after the death of Pythagoras so the choice of ϕ, if it was deliberate, might well have been influenced by the Pythagorean philosophy, for ϕ occurs conspicuously in the theory of the pentagram, which was the badge of the Pythagoreans [7, p. 28].

Reference [4, p. 162], refers to the "perfect phi pyramid" whose square base is 2 by 2 units, the length of the apothem (the segment from the apex to the midpoint of a side of the base) is ϕ, and the height is $\sqrt{\phi}$. Gardner says "Herodotus [the 'father of history'] was the first to suggest (c. 500 B.C.) that the area of the face of the Great Pyramid [of King Khufu (also called Cheops) at al-Jiza (Giza)] is equal to the square of the pyramid's height." This is another way of suggesting that the Great Pyramid is a perfect phi pyramid. But Gardner has now informed me that Fischler (1991), in a forthcoming article, has argued that the source of the alleged interpretation of Herodotus's wording goes back only to 1859. Herodotus's wording was seemingly incorrect. Nevertheless, according to [8] the measurements of the base are, in feet, 755.43, 756.08, 755.08, and 755.77, with an average of 755.59, while the height is 481.4 feet, so the ratio of the height to half the base is close to 1.274, whereas $\sqrt{\phi}$ = 1.2720. The deviation from the perfect phi pyramid is much too small to be discernible by eye and small enough to be due to erosion. The Egyptian architect, two thousand years before Herodotus was born, might well have aimed at a perfect phi pyramid. Maybe the architect's plans will eventually be found entombed with his mummy.

The left sides of (3) can also be expressed as

(3a) $\frac{1}{2}\{(1+i)\sqrt{\phi}+(1-i)/\sqrt{\phi}\}$.

We will see, in a corollary to Theorem 3, that there is another close relationship between (2) and (3).

Some related "complex Fibonacci and Lucas numbers" will be investigated.[2]

A condensed version of this work was published in [5].

2. PROOFS OF THE CONTINUED FRACTIONS

To prove (1)-(3), and similar results, we can make use of the following special case of a theorem given, for example, by [6, p. 146].

The numerator p_n and denominator q_n of the n^{th} convergent (n = 1, 2, 3, ...) of

(4) $A+\frac{1}{A+}\ \frac{1}{A+}\cdots$

are given by

(5) $p_n = F_{n+2},\quad q_n = F_{n+1}$

where

(6) $F_n = (x^n - y^n)/(x-y)$

and

(7) $x=\frac{1}{2}\left(A+\sqrt{A^2+4}\right),\quad y=\frac{1}{2}\left(A-\sqrt{A^2+4}\right),$

which are the roots of $x^2 - Ax - 1 = 0$. Of course $xy = -1$.

Reference [6] assumes that A is an integer. But everything in the (inductive) proof of the theorem in [6] is also applicable if A is any nonzero real or complex number and, in particular, when $A = a+ib$ where a and b are integers, that is, when A is a *Gaussian integer*.

It follows from the theorem that the infinite continued fraction (4) is equal to x if $|x|>|y|$ and to y if $|y|>|x|$. If A is a positive integer, as in (1), then $|x|>|y|$ and the continued fraction converges to x. The convergence is fast when $|y/x|$ is small.

For the sake of simplicity, let us consider the special case where $A = a+ia$ where a is a positive integer. Then

(8) $x=\frac{1}{2}\left\{a+ia+\sqrt{4+2ia^2}\right\},\quad y=\frac{1}{2}\left\{a+ia-\sqrt{4+2ia^2}\right\},\quad xy=-1,$

[2]Complex continued fractions can be used to solve problems in the theory of Gaussian integers similar to those solved for integers by using ordinary continued fractions. For example, one can solve a complex form of Pell's equation at least sometimes. This is shown, among other things, in my Technical Report 91-2 which, however, contains several incomplete proofs and conjectures.

where $\surd$ denotes the complex square root with positive real part. By means of de Moivre's theorem (anticipated by Roger Cotes), and some trigonometry, we infer that

$$(9)\qquad \begin{aligned} 2x &= a+ia+\left\{\left(\sqrt{4+a^4}+2\right)^{1/2}+i\left(\sqrt{4+a^4}-2\right)^{1/2}\right\} \\ 2y &= a+ia-\left\{\left(\sqrt{4+a^4}+2\right)^{1/2}+i\left(\sqrt{4+a^4}-2\right)^{1/2}\right\} \end{aligned}$$

(Note the checks that $xy=-1$ and $(x-y)^2=4+2ia^2$.) It is straightforward to show that $|x|>|y|$ by calculating $|2x|^2-|2y|^2$. Therefore,

$$(10)\qquad \frac{1}{2}\left(\sqrt{4+a^4}+2\right)^{1/2}+\frac{i}{2}\left(\sqrt{4+a^4}-2\right)^{1/2}\frac{1}{2}(a+ia)+\frac{1}{a+ia+}\;\frac{1}{a+ia+}\cdots .$$

Equations (2) and (3) are the special cases $a=2$ and $a=1$.

3. COMPLEX AND GAUSSIAN FIBONACCI AND LUCAS NUMBERS

Let us write

$$(11)\qquad F_{\xi,n}=\frac{\xi^n-\eta^n}{\xi-\eta},\quad L_{\xi,n}=\xi^n+\eta^n\quad (\xi\neq\eta,\ \xi\eta=-1)$$

where n is any integer (not necessarily positive) and ξ and η might be complex (in which case we can think of $F_{\xi,n}$ and $L_{\xi,n}$ as *complex Fibonacci and Lucas numbers*). Note that

$$F_{\xi,-n}=(-1)^{n-1}F_{\xi,n},\quad L_{\xi,-n}=(-1)^n L_{\xi,n}.$$

The ordinary Fibonacci and Lucas numbers are $F_n=F_{\phi,n}$, and $L_n=L_{\phi,n}$.

Theorem 1: The sequences $\{F_{\xi,n}\}$ and $\{L_{\xi,n}\}$ $(n=\cdots-2,-1,0,1,2,\dots)$ satisfy the recurrence relations

$$(12)\qquad F_{\xi,n+1}=(\xi+\eta)F_{\xi,n}+F_{\xi,n-1}$$

and

$$(13)\qquad L_{\xi,n+1}=(\xi+\eta)L_{\xi,n}+L_{\xi,n-1}.$$

The proofs are left to the reader.

Vajda [13, pp. 176-84] lists 117 identities satisfied by the ordinary Fibonacci and Lucas numbers. Most of these identities apply equally to $F_{\xi,n}$ and $L_{\xi,n}$ and can be readily proved straight from the definitions (11).

Theorem 2: A necessary and sufficient condition for $F_{\xi,n}$ and $L_{\xi,n}$ to be Gaussian (or natural) integers for all n is that $\xi+\eta$ should be a Gaussian integer (or a natural integer, respectively).

Proof: That the condition is necessary is obvious because $\xi+\eta=L_{\xi,1}=F_{\xi,2}$. That the condition is also sufficient follows inductively, both for positive and negative n, from the recurrence relations (12) and (13) because $F_{\xi,0}$, $F_{\xi,1}$, $L_{\xi,0}$, and $L_{\xi,1}$ are Gaussian integers, namely, 0, 1, 2, and $\xi+\eta$, respectively, and because the recurrence relations (12) and (13) work backwards as well as forwards.

In this paper we will be mainly concerned with the case in which $\xi+\eta=a+ia$ where a is a positive integer, especially the cases $a = 1$ and $a = 2$ with which we began in the Introduction. Then $\xi=x$ and $\eta=y$ where x and y are defined by equations (8) or (9). We write $F_{x,n} = F_n^{(a)}$ and $L_{x,n} = L_n^{(a)}$, but when a is held fixed in some context *we usually abbreviate the notations* to F_n and L_n. We call $F_n^{(a)}$ and $L_n^{(a)}$ *Gaussian* Fibonacci and Lucas numbers. Also we write $F_n = f_n + if_n'$ and $L_n = \ell_n + i\ell'_n$ to show the real and imaginary parts Some numerical values are listed in Tables 1 and 2 for the cases $a = 1$ and $a = 2$. These tables can be generated from the recurrence relations

(14) $$F_0=0,\ F_1=1,\ F_{n+2}=(a+ia)F_{n+1}+F_n$$

and

(15) $$L_0=2,\ L_1=a+ia,\ L_{n+2}=(a+ia)L_{n+1}+L_n$$

where n is any integer, positive, negative, or zero.

Greater generality would be possible by writing $(a+ib)F_{n+1}+(c+id)F_n$ on the right of (14) where a, b, c, and d are integers [and similarly for (15)], but simplicity is also a virtue, and there is plenty to say about the special case of $F_n^{(a)}$ and $L_n^{(a)}$.

Individual values of F_n and L_n can be computed from the formulas

(16) $$F_n=\frac{r^n e^{in\theta-i\psi}-r^{-1}e^{-in(\theta+\pi)-i\psi}}{\sqrt{2}\left(4+a^4\right)^{1/4}},\ \text{and } L_n=r^n e^{in\theta}+r^{-n}e^{-in(\theta+\pi)},$$

where

$$2r=\left[(a+\gamma)^2+(a+\delta)^2\right]^{1/2},$$

$$\theta=\arctan\left(\frac{a+\delta}{a+\gamma}\right),\quad \psi=\arctan(\delta/\gamma),$$

where

$$\gamma=(\beta+2)^{1/2},\ \delta=(\beta-2)^{1/2},\ \beta=(4+a^4)^{1/2}.$$

The notations r, θ, ψ, β, γ, δ are provisional and are introduced here only to simplify the printing of the formulas for F_n and L_n. and to make them easier to program. (I used a hand-held calculator, an HP15C.) In spite of the heirarchy of square roots, F_n and L_n. are, of course, Gaussian integers, a fact that acts as an excellent check on computer programs.

The tables can be used for checking and guessing various properties of the Gaussian Fibonacci and Lucas numbers. In this section I give a small selection of the most easily proved properties.

The first few properties resemble formulas (10.14.16)–(10.14.9) of [6] and are almost as easy to prove as in the real case if one holds in mind that $xy = -1$ and, for (19) and (21), that $(x-y)^2=4+2ia^2$. We have

(17) $2F_{m+n} = F_m L_n + F_n L_m,$

and in particular,

(18) $F_{2n} = F_n L_n;$

(19) $L_n^2 - (4+2ia^2)F_n^2 = (-1)^n 4,$

(20) $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1},$

(20k) $F_n^2 - F_{n+k}F_{n-k} = (-1)^{n+k} F_k^2,$

(21) $L_n^2 - L_{n-1}L_{n+1} = (-1)^n (4+2ia^2).$

Two similar formulas (see [13, formulas (11) and (17c)]), convenient for "leaping ahead," are

(22) $F_{2n+1} = F_{n+1}^2 + F_n^2$

and

(23) $L_{2n} = L_n^2 + (-1)^{n-1}2.$

A couple of results, corresponding to Theorem 179 of [6], and which readily follow inductively from the recurrence relations (14) and (15), are

(24) $(F_n, F_{n+1}) = 1, \quad (L_n, L_{n-1}) = (2, a),$

meaning, for example, that F_n and F_{n+1} have no common factor other than the units ± 1 and $\pm i$; and, for all r and n,

(25) $F_n | F_{rn}$

(meaning that F_n "divides" F_{rn}). But the proof of (25) given by [6] for ordinary Fibonacci numbers does not extend so easily as for (17)–(24). Instead, the proof in [13, pp. 66 and 67] extends immediately, and has the merit of expressing F_{rn}/F_n explicitly in terms of Lucas numbers, in fact as a linear combination. For example,

(26) $F_{3n}/F_n = L_{2n} + (-1)^n, \quad F_{5n}/F_n = L_{4n} + (-1)^n L_{2n} + 1.$

Several surprising formulas can be obtained by the methods of [1]. For example,

(27) $$\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_8} + \cdots = y + 1 + \frac{2}{a+ia}$$

and

(28) $$\frac{L_1}{F_3} + \frac{L_3}{F_9} + \frac{L_9}{F_{27}} + \cdots = -y.$$

4. A RELATIONSHIP BETWEEN THE SEQUENCES $\{L_n^{(1)}\}$ AND $\{L_n^{(2)}\}$

Theorem 3:

(29) $L_n^{(2)} = i^n \overline{L_{2n}^{(1)}}$ for all n,

where the vinculum indicates complex conjugacy.

Proof: (29) is true when $n = 0$, and when $n = 1$, because

$$L_0^{(2)} = 2 \;\text{ while }\; \overline{L_0^{(1)}} = 2$$

$$L_1^{(2)} = 2+2i,\; L_2^{(1)} = (1+i)^2 + 2 = 2+2i,\; i\,\overline{L_2^{(1)}} = 2+2i = L_1^{(2)}.$$

So an inductive proof will follow if we can show that the sequences $\{L_n^{(2)}\}$ and $\{K_n\}$ satisfy the same recurrence relation, where $K_n = i^n\, \overline{L_{2n}^{(1)}}$ by definition

The recurrence relation satisfied by $L_n^{(2)}$ is, of course,

$$L_{n+1}^{(2)} = (2+2i)L_n^{(2)} + L_{n-1}^{(2)}.$$

To obtain the recurrence relation satisfied by $\{K_n\}$, note first that

$$L_{\xi,m+2} = L_{\xi,2}L_{\xi,m} - L_{\xi,m-2}$$

which follows readily from (11). It is stated in [13, formula (17a)] for ordinary Lucas numbers, but it is equally clear for complex Lucas numbers and, in particular, for Gaussian numbers. On putting $m = 2n$ we get, again in particular,

$$L_{2n+2}^{(1)} = L_2^{(1)}L_{2n}^{(1)} - L_{2n-2}^{(1)} = (2+2i)L_{2n}^{(1)} - L_{2n-2}^{(1)}.$$

Therefore,

$$\overline{L_{2n+2}^{(1)}} = (2-2i)\overline{L_{2n}^{(1)}} - \overline{L_{2n-2}^{(1)}}.$$

Multiply by i^{n+1} to get

$$\begin{aligned} K_{n+1} &= i(2-2i)K_n - i^2 K_{n-1} \\ &= (2+2i)K_n + K_{n-1} \end{aligned}$$

so the sequence $\{K_n\}$ does satisfy the same recurrence relation as $\{L_n^{(2)}\}$ as required.

A more direct but slightly messy proof can be obtained from equation (8). Note that $L_n^{(2a)} \neq i^n\, \overline{L_{2n}^{(a)}}$ unless $a = 1$.

Corollary to Theorem 3:

(30) $$i\left[1-i+\frac{1}{1-i+}\;\frac{1}{1-i+}\cdots\right]^2 = 2+2i+\frac{1}{2+2i+}\;\frac{1}{2+2i+}\cdots$$

Proof: From the theorem, we have

(31) $$i\,\overline{L_{2n+2}^{(1)}}\,/\,\overline{L_{2n}^{(1)}} = L_{n+1}^{(2)}\,/\,L_n^{(2)}.$$

Now, in the theorem at the start of Section 2, take $A = 2 + 2i$ [when x is given by (9) with $a = 2$]. Then the continued fraction (4) equals

$$\begin{aligned} \lim_{n\to\infty}(p_n/q_n) &= \lim(x^{n+2}-y^{n+2})/(x^{n+1}-y^{n+1}) = x \text{ (because } |x|>|y|) \\ &= \lim(x^{n+1}+y^{n+1})/(x^n+y^n) = \lim L_{n+1}^{(2)}/L_n^{(2)} \end{aligned}$$

so the right side of (31) tends to that of (30). Again, in (4), take $A = 1 + i$ to find that

$$1+i+\frac{1}{1+i+}\ \frac{1}{1+i+}\cdots=\lim(L_{n+1}^{(1)}/L_n^{(1)})=\lim(L_{n+2}^{(1)}/L_{n+1}^{(1)}).$$

Therefore,

$$\left(1+i+\frac{1}{1+i+}\ \frac{1}{1+i+}\cdots\right)^2=\lim\frac{L_{n+1}^{(1)}}{L_n^{(1)}}\cdot\frac{L_{n+2}^{(1)}}{L_{n+1}^{(1)}}=\lim\frac{L_{n+2}^{(1)}}{L_n^{(1)}}$$

and hence,

$$i\left(1-i+\frac{1}{1+i+}\ \frac{1}{1+i+}\cdots\right)^2=\lim i\,\overline{L_{n+2}^{(1)}}\,/\,\overline{L_n^{(1)}},$$

so the left side of (31) tends to that of (30) and this completes the proof.

Equation (30) was discovered by the method shown above. A less interesting proof can be obtained from (2) and (3) or (3a).

5. CONGRUENCE PROPERTIES

Hardy & Wright [6, p. 149] prove that every ordinary (rational) prime divides some ordinary Fibonacci number (and therefore an infinity of them). To prove similar results we need to recall that a prime Gaussian integer $\alpha+i\beta$ (with $\alpha\beta\neq 0$) can be defined by the property that $\alpha^2+\beta^2$ is either 2 or an ordinary prime congruent to 1 modulo 4. For any such ordinary prime p, the corresponding Gaussian prime is unique up to conjugacy or multiplication by a unit (a power of i). This beautiful and famous theorem is proved, for example, in [10, p. 128]. Denote one of the Gaussian primes that corresponds to p by p_G and its complex conjugate by $\overline{p}_G$. Then, of course,

(32) $\quad p_G\overline{p}_G=p.$

Using this notation we have the following divisibility result, the proof being an elaboration of that of Theorem 180 in [6].

Definition: We call a number *pure* if it is either purely real or purely imaginary.

Theorem 4: Let a be fixed and let $p\equiv 1 \pmod 4$ be a rational prime, not a factor of $4+a^4$. Then,

(33) (i) $F_p^2\equiv 1 \pmod p$;

(This, of course, makes an assertion about both the real and imaginary parts of F_p^2.)

(34) (ii) F_p is pure, modulo p;

(35) (iii) $|F_p|^2\equiv\left(\frac{4+a^4}{p}\right)=\pm 1$

(the Legendre symbol is not 0 because p does not divide $4+a^4$);

(36) (iv) p divides $|F_{p-1}|^2$ or $|F_{p+1}|^2$ or both;

(37) (v) p_G (and $\overline{p}_G$) divides either F_{p-1} or F_{p+1} but not both apart from the uninteresting case in which p divides a.

Thus every Gaussian prime divides some Gaussian Fibonacci number [and therefore, by (25), an infinity of them].

(vi) For $n \geq 2$ we have $L_{2^n} \equiv 2 \pmod{2^n}$.

(vii) For $n \geq 2$ we have $F_{2^n} \equiv 2 \pmod{2^n}$

Before proving this theorem, let us look at some numerical examples deduced from Tables 1 and 2 combined with formulas (18), (22), and (23). These examples are shown in Table 3. Note, for example, that this table confirms Part (iii) of the theorem in that $4+1^4$ and $4+2^4$ are squares (quadratic residues) modulo 29 but not modulo 13, 17, or 37.

TABLE 1. Values of F_n and L_n when $a = 1$

$$F_n = f_n + if'_n \quad L_n = \ell_n + \ell'_n$$

n	f_n	f'_n	ℓ_n	ℓ'_n	n	f_n	f'_n	ℓ_n	ℓ'_n
−1	1	0	−1	−1	16	1728	1520	2818	3968
0	0	0	2	0	17	1513	3608	1361	8161
1	1	0	1	1	18	−367	6641	−3982	13,490
2	1	1	2	2	19	−5495	9882	−16,111	17,669
3	1	2	1	5	20	−15,744	11,028	−37,762	15,048
4	0	4	−2	8	21	−32,267	5166	−68,921	−5045
5	−3	6	−9	11	22	−53,177	−16,073	−101,638	−58,918
6	−9	7	−22	10	23	−69,371	−64,084	−111,641	−165,601
7	−19	4	−41	−1	24	−58,464	−149,528	−47,678	−336,160
8	−32	−8	−62	−32	25	21,693	−272,076	176,841	−549,439
9	−43	−36	−71	−95	26	235,305	−399,911	678,602	−708,758
10	−39	−87	−38	−198	27	656,909	−436,682	1,564,201	−579,595
11	5	−162	89	−331	28	1,328,896	−179,684	2,822,398	275,848
12	128	−244	382	−440	29	2,165,489	712,530	4,110,751	2,518,651
13	377	−278	911	−389	30	2,781,855	2,698,335	4,414,498	6,905,250
14	783	−145	1682	82	31	2,249,009	6,192,720	1,619,999	13,838,399
15	1305	360	2511	1375	32	−1,161,856	11,140,064	−7,803,902	22,363,648

TABLE 2. Values of F_n and L_n when $a = 2$

$$F_n = f_n + if'_n \quad L_n = \ell_n + i\ell'_n$$

n	f_n	f'_n	ℓ_n	ℓ'_n	n	f_n	f'_n	ℓ_n	ℓ'_n
−1	1	0	−2	−2	10	13,386	−2358	37,762	15,048
0	0	0	2	0	11	34,625	18,552	58,918	101,638
1	1	0	2	2	12	45,532	103,996	−47,678	336,160
2	2	2	2	8	13	−82,303	317,608	−708,758	678,602
3	1	8	−10	22	14	−754,290	574,606	−2,822,398	275,848
4	−12	20	−62	32	15	−2,740,095	−41,760	−6,905,250	−4,414,498
5	−63	24	−198	−38	16	−6,150,960	−4,989,104	−7,803,902	−22,363,648
6	−186	−58	−382	−440	17	−5,063,807	−22,321,888	22,214,242	−64,749,598
7	−319	−464	−82	−1682	18	28,365,202	−59,760,494		
8	104	−1624	2818	−3968					
9	3137	−3504	13,490	−3982	20	540,965,316	112,389,732		

TABLE 3(i), $a = 1$. Values modulo p

p	F_p	F_p^2	$\|F_p\|^2$	F_{p-1}	F_{p+1}
13	$-5i$	1	−1	$-2+3i$	$3-2i$
17	$4i$	1	−1	$(3+i)(4+i)$	$(3-i)(1+4i)$
29	1	1	1	0	$1+i$
37	$-6i$	1	−1	$(2i-3)(1+6i)$	$(4+3i)(1-6i)$

TABLE 3(ii), $a = 2$. Values modulo p

p	F_p	F_p^2	$\|F_p\|^2$	F_{p-1}	F_{p+1}
13	$5i$	1	−1	$3(2+3i)$	$-2(2-3i)$
17	$-4i$	1	−1	$-3(1+4i)$	$(1+i)(1-4i)$
29	1	1	1	0	$2(1+i)$
37	$6i$	1	−1	$3(-13+14i)(1+6i)$	$1-6i$

Proof of Theorem 4: From (7), where $A = a + ia$, we have (by the binomial theorem)

$$\begin{aligned} 2^{p-1}F_p &= 2^{p-1}(x^p - y^p)/(A^2+4)^{1/2} \\ &= pA^{p-1} + \binom{p}{3}A^{p-3}(A^2+4) + \binom{p}{5}A^{p-5}(A^2+4)^2 + \cdots + (A^2+4)^{(p-1)/2} \\ &\equiv (A^2+4)^{(p-1)/2} \pmod{p} \text{ since } p \text{ is prime.} \end{aligned}$$

Therefore,

$$F_p \equiv (A^2+4)^{(p-1)/2} \pmod p \text{ (by Fermat's theorem, not the "last" but not least)}$$
$$= (4+2ia^2)^{(p-1)/2} = 2^{(p-1)/2}(2+ia^2)^{(p-1)/2}.$$

Therefore,

$$(38) \qquad F_p^2 \equiv 2^{p-1}(2+ia^2)^{p-1} \equiv (2+ia^2)^{p-1} \pmod p.$$

and

$$(39) \qquad |F_p|^2 \equiv 2^{p-1}(4+a^4)^{(p-1)/2} \equiv (4+a^4)^{(p-1)/2} \pmod p,$$

again by Fermat's theorem. Part (iii) of the theorem now follows from (39) combined with Theorem 83 of [6].

From (38) we obtain

$$\begin{aligned}(2+ia^2)F_p^2 &\equiv (2+ia^2)^p \\ &\equiv 2^p + i^p a^{2p} \quad \text{(again by the binomial theorem)} \\ &\equiv 2+ia^2 \quad [\text{because } p \equiv 1 \pmod 4 \text{ and is prime}].\end{aligned}$$

But $(4+a^4,\, p)=1$ and therefore $(2+ia^2,\, p)=1$. Hence, $F_p^2 \equiv 1 \pmod p$, which is Part (i) of the theorem.

To prove Part (ii), assume that $F_p \equiv \alpha + i\beta \pmod p$. Thus $F_p^2 \equiv \alpha^2 - \beta^2 + 2i\alpha\beta$. But $F_p^2 \equiv 1$ and is therefore real, so $\alpha\beta = 0$. Thus $\alpha = 0$ or $\beta = 0$ so F_p is pure $\pmod p$.

Part (i) combined with (20) shows that $F_{p-1}F_{p+1} \equiv 0 \pmod p$. Since p is not a *Gaussian* prime, it does not follow that p divides either F_{p-1} or F_{p+1}, but Part (v) *does* follow because we must have $F_{p-1}F_{p+1} \equiv 0$ (mod p_G and also mod $\bar{p}_G$). [The recurrence relation (14), together with Part (i), shows that p_G cannot divide both F_{p-1} and F_{p+1} when p does not divide a. But sometimes both p_G and $\bar{p}_G$ divide F_{p-1}, or perhaps both divide F_{p+1}, and then p divides either F_{p-1} or F_{p+1} but not both.] Then Part (iv) follows from Part (v).

To prove Part (vi), note that

$$L_4 = 4(1-a^4+2ia^2)-2 \equiv -2 \pmod 4 \equiv 2 \pmod 4,$$

and the result then follows by induction from formula (23).

To prove Part (vii), we have

$$F_4 = 2a[(1-a^2)+(1+a^2)] \equiv 0 \pmod 4$$

whether a is even or odd. Then the result follows by induction from (18) combined with Part (vi).

Lemma: For any integer n, L_{2n} is of the form $2s+2ti$ and L_{2n+1} is of the form $a \pm ai + 2au + 2avi$ where s, t, u, and v are integers.

Proof: Note first that it is irrelevant whether we take the plus sign or the minus sign. Now $L_0 = 2$ and $L_1 = a+ai$, so we can "start" an inductive proof, and we can readily complete the induction by means of the recurrence relation (15).

Theorem 5: Let p be an odd (ordinary) prime. Then

$$(40)\qquad \begin{aligned} L_p &\equiv a+ia \pmod{p} \text{ if } p\equiv 1 \pmod 4 \\ L_p &\equiv a-ia \pmod{p} \text{ if } p\equiv 3 \pmod 4. \end{aligned}$$

More informatively,

$$(41)\qquad \begin{aligned} L_p &\equiv a+ia \pmod{2ap} \text{ if } p\equiv 1 \pmod 4 \\ L_p &\equiv a-ia \pmod{2ap} \text{ if } p\equiv 3 \pmod 4. \end{aligned}$$

Proof: We have $L_p = x^p + y^p$, so

$$2^{p-1}L_p = A^p + \binom{p}{2}A^{p-2}(A^2+4)+\cdots+pA(A^2+4)^{(p-1)/2}$$

Now $2^{p-1}\equiv 1 \pmod{p}$, by Fermat's theorem, so

$$\begin{aligned} L_p &\equiv A^p \pmod{p} = (a+ia)(2ia^2)^{(p-1)/2} \\ &= (a+ia)\left(\frac{2}{p}\right)i^{(p-1)/2}a^{p-1} \equiv (a+ia)\left(\frac{2}{p}\right)i^{(p-1)/2} \end{aligned}$$

again by Fermat's theorem. Formulas (40) are therefore proved if p divides a, so we shall now assume that it does not. Now, by [6, p. 75], we have

$$\left(\frac{2}{p}\right) = 1 \ \text{ if } p\equiv \pm 1 \pmod 8)$$

and equations (40) follow readily. But by the Lemma we have $L_p - a \mp ia = M(2a)$ and (41) follows at once because $(2a, p) = 1$. [$\lambda = M(\mu)$ means $\lambda \equiv 0 \pmod{\mu}$.]

Corollary: ***(i)*** If p is an odd prime, then

$$(42)\qquad |L_p|^2 \equiv 2a^2 \pmod{p}.$$

(ii) If p is an odd prime and a is not a multiple of p, then $|L_p|^2/(2a^2)$ is an integer and is congruent to 1 modulo p.

Proof: From (41), L_p is of the form $a+ai+2ap(s+it)$ where s and t are integers. Hence

$$\begin{aligned} |L_p|^2 &= (a+2aps)^2 + (a+2apt)^2 \\ &= 2a^2(1+2ps+2pt+2p^2s^2+2p^2t^2) \end{aligned}$$

and the Corollary follows at once.

Comment: If p is an odd number and fails to satisfy any of the conclusions in Theorems 4 and 5, then p is composite, and if it does satisfy the theorems it can perhaps be described as "probably" prime or at least as a new kind of "pseudoprime." For example, $L_n \not\equiv a \pm ia \pmod{n}$ for any composite n shown in Table 1 or 2.

Theorem 5 and its corollary are analogous to the theorem that the ordinary Lucas number $L_{\phi,p} \equiv 1 \pmod{p}$; see, for example, [13, p. 80], where it is mentioned, with a reference, that $L_{\phi,705} \equiv 1 \pmod{705}$ although 705 is composite. So the converse of our Theorem 5 is probably false. Anyway, the converse would be too good to be true. It would be interesting to know whether any parts of Theorems 4 and 5 have "modified converses."

Theorem 6: Every Gaussian number $G = g + ig'$ (not just the Gaussian primes) divides some Gaussian Fibonacci number (apart from F_0). (We are still regarding a as fixed.)

Proof: The sequence of Gaussian Fibonacci numbers (mod G) must be periodic with period no more than $(gg')^2 - 1$. This follows by the argument given, for example, in [13, p. 88] with a minor modification to allow for the complexity of G. But 0 is one of the Fibonacci numbers, and is divisible by G, and the result follows.

We next prove two congruence relations needed in Section 6. The first part sharpens Part (vi) of Theorem 4.

Theorem 7: ***(i)*** $L_{2^n} \equiv 2 \pmod{2^{n+2}}$ when $n \geq 3$, for all a.

(ii) $L_{2^n} \equiv 2 \pmod{2^{n+2}}$ when $n \geq 3$, while $1 \leq a \leq 5$.

[Part (i) can probably be sharpened, for example, $xL_{32}^{(1)} \equiv 2$ (mod 512), while Part (ii) might be true for all values of a.]

Proof: We have

$$L_2 = (a+ia)L_1 + L_0 = 2 + 2ia^2$$

so, by (23),

$$L_4 = L_2^2 - 2 = 2 - 4a^4 + 8ia^2 \equiv (-1)^a 2 \pmod 8.$$

Therefore, again by (23),

$$L_8 = L_4^2 - 1 = [M(8) + (-1)^a 2]^2 - 2 \equiv 2 \pmod{32}.$$

Therefore,

$$L_{16} = L_8^2 - 2 = [M(32) + 2]^2 - 2 \equiv 2 \pmod{64},$$

and so on, inductively, giving Part (i).

To prove Part (ii), note first that it is true for $n = 3$ and for $n = 4$ as we can see for $a = 1$ or 2 from Tables 1 and 2 and by calculations, not shown here, for $a = 3$, 4, and 5. We complete the proof inductively by noting first that

$$L_{\xi,2m+1} = L_{\xi,m}L_{\xi,m+1} - (-1)^m(\xi + \eta) \quad (m = 0, 1, 2, \dots)$$

as follows easily from (11). On putting $\xi = x$ and $m = 2^{n-1}$, we infer that

$$\begin{aligned} L_{2^n+1} &= L_{2^{n-1}}L_{2^{n-1}+1} - (a+ia) \quad (n \geq 2) \\ &\equiv 2L_{2^{n-1}+1} - (a+ia) \pmod{2^{n+1}, n \geq 4} \text{ [by Part (i)]} \\ &\equiv 2[M(2^{n-1}) + a + ia] - (a+ia) \pmod{2^{n+1}, n \geq 4} \\ &\quad \text{(by the inductive hypothesis)} \\ &\equiv a + ia \pmod{2^n, n \geq 4} \end{aligned}$$

and this completes the inductive proof.

6. A PERIODICITY PROPERTY

Periodicity properties, modulo a given integer, of ordinary Fibonacci and Lucas sequences, are surveyed by Vajda [13, Chapter VII]. Our final theorem reveals a very simple periodicity of the Gaussian Lucas sequences at least when $a < 6$.

Theorem 8: For $1 \le a \le 5$, the period of the sequence $\ldots L_{-2}, L_{-1}, L_0, L_1, L_2, \ldots \pmod{2^n, n \ge 3}$ is a power of 2 not exceeding 2^n.

In view of the recurrence relation, in order to prove that 2^n is *a* period it is sufficient to prove that $L_m \equiv L_{m+2^n}$ for two consecutive values of m. By Theorem 7 this is achieved by taking $m = 0$ and $m = 1$.

The period (that is, the shortest period) must divide any known period and must therefore be a power of 2. There seems to be a 'tendency' for the period to be 2^n, for example, when $a = 1$ or 3 the periods modulo 8, 16, and 32 are, respectively, also 8, 16, and 32. But, when $a = 2$, the period modulo 32 is only 8.

7. LOOSE ENDS

There are many loose ends in this work. For example, we wondered whether Part (ii) of Theorem 7 is true for all values of a, in which case the same is true for Theorem 8. As another example, if $p \equiv 1 \pmod 4$ and $p > 5$, is $|F^{(a)}_{(p-1)/2}|^2$ always congruent to 0 or ± 1 modulo p? I have verified this for $a = 1$ and 2 and $p < 113$, and for $a = 3$, $p < 61$. Note that $|L^{(1)}_p|^2/2$, where p is an odd prime, has a tendency to avoid having small factors, where the meaning of *small* increases when p increases. The values for $p = 3, 5, 7, 11, 13, 17$, and 19 are, respectively, 13, 101, 29^2, 58741 (prime), 53×9257, 34227121, and 185878941. Neither of the last two numbers has a factor less than 100. (Both are beyond the scope of [9].) It seems reasonable to conjecture that when the prime $p \to \infty$ and $a \ne M(p)$, then the smallest factor of $|L^{(1)}_p|^2/(2a^2)$ (which is an integer by the corollary of Theorem 5) also tends to infinity. The analogous property might be true also for the ordinary Lucas numbers $L_{\phi,p}$. It is possibly significant that 29^2 divides $|L^{(1)}_7|^2$, 29^3 divides $|F^{(2)}_7|^2$, and 89^2 divides $|L^{(2)}_{11}|^2$.

How much of the theory goes through if $a + ia$ is replaced by $a + ib$ throughout?

But the most interesting question is: Under what conditions is a "pseudoprime" a prime?

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CONNECTIONS IN MATHEMATICS: AN INTRODUCTION TO FIBONACCI VIA PYTHAGORAS

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(Submitted March 1991)

1. INTRODUCTION

We are all familiar with the traditional presentation of the Fibonacci sequence in classes designed for liberal arts majors. We "wow" the students with pinecones and pineapples! We talk to them about the wondrous "appearance" of Fibonacci numbers in **their** world. But can we use Fibonacci to bring these students into **our** world?

Is it possible for liberal arts majors to appreciate mathematics—apart from applications of the subject in art, nature, and other areas? Can we develop courses that instill in students a sense of excitement about making connections in mathematics, given the attitudes toward our subject that many of them bring to class? As Lynn Arthur Steen says: "For students in the arts and humanities, mathematics is an invisible culture—feared, avoided, and consequently misunderstood" (3).

Designing requisite[1] mathematics courses for liberal arts students is difficult. In attempting to give them a sense of mathematics, we often resort to overstressing its utility—reaching for areas outside of mathematics to validate the study of the subject. Or we assemble what appears to the students a collection of disjoint topics—offering little motivation for them to search for connections.

The challenge is to draw the students **into** mathematics—generating in them an excitement about making mathematical connections and an appreciation of fundamental interrelationships between topics. I believe I have met this challenge by introducing Fibonacci via Pythagoras, and I want to share the experience![2]

2. RATIONALE

Mathematics builds upon itself in a way that other sciences do not. Even topics developed in antiquity continue to be relevant today to mathematical growth—knowledge gives rise to new knowledge; problems generate new problems. One objective in teaching mathematical concepts is to give students a sense of how these ideas fit into the edifice we call mathematics. Examination of connections between mathematical topics is one way to achieve this goal.

The connection between Fibonacci numbers and Pythagorean triples is well known (cf. [6], [16], [17]). But this connection is not frequently used to *introduce* Fibonacci numbers. I propose a classroom lesson that involves students working with a familiar topic (Pythagorean triples) prior to connecting it to an unfamiliar one (Fibonacci numbers). In my experience, the preliminary work with triples motivates a discussion of this connection, and stimulates students to want to

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[1]In 1983, the largest public university system in the country—the California State University System—established a mathematics course as a graduation requirement for all students at any of its nineteen campuses.

[2]The appendix includes an annotated list of books, journal articles, and videotapes that I used as resources for classroom discussion and projects. All numbers in brackets [ ] refer to the resources listed in this appendix.

learn more about the Fibonacci numbers. Perhaps more importantly, it creates in the students a genuine excitement about making **mathematical** connections.

As an alternative to what is perhaps a traditional introduction to the Fibonacci numbers based on their "surprise" applications in nature—breeding rabbits, patterns in pinecones and pineapples, etc. (cf. [9], [12], [19]), my presentation enables students to experience this kind of "surprise" in connecting **mathematical** areas. The classroom experience begins with discussion of problems inspired by Pythagorean triples incorporating assignments and activities generating Pythagorean triples; then follows with an examination of connections between the products of Fibonacci and Pythagoras and an investigation of the historical and present-day significance of the Fibonacci numbers. This approach conforms to the goals expressed by Alvin White (1985) in his article "Beyond Behavioral Objectives":

> . . . our guidelines and teaching objectives should not have as their major target or focus the mastery of facts and techniques. Rather the facts and techniques should be the skeletal framework which supports our objective of imbuing our students with the spirit of mathematics and a sense of excitement about the historical development and the creative process. (3, p. 850)

3. CLASSROOM LESSONS

A. Pythagorean Triples

I begin by proposing that Pythagorean triples give evidence of how problems generate new problems in mathematics. The students are amazed to discover that such integers were known in ancient civilizations 1200 years before Pythagoras. I try to give them a sense that what perhaps is more exciting is that these triples have inspired interesting problems in mathematics since the time of Pythagoras. My students learn, by perusing the list of resources, that the generation of Pythagorean triples is a topic that still fascinates some contemporary mathematicians (cf. [2], [5], [6], [10], [11], [14], [16], [17], [20], [22], [23]).

I challenge the students to work in groups to discover common characteristics of Pythagorean triples with the goal of finding a generating form for them. I begin by asking the students to create triples, after we list the ones known to them. This induces a discussion of multiples of triples and a conjecture that multiples of triples are triples, motivating the need for a proof that this is indeed so. I form the class into small groups (four to five students per group) and ask them to form conjectures about characteristics of primitive Pythagorean triples [we had read and discussed Polya's heuristics for problem solving (1)]. The students work together, observing patterns and making guesses. For example, by looking at (3, 4, 5), (5, 12, 13), and (8, 5, 17), (12, 35, 37), they make the following conjectures: only one number of the triple could be even; when the smallest number of the triple is even, the difference between the two larger numbers is two; when the smallest number of the triple is odd, the difference between the two larger numbers is one; that five is a factor of some number of the triple; etc. Then, as a class, we discuss each group's conjectures, attempting to prove or disprove *their* hypotheses. Their conjectures introduce many interesting class discussions about numbers and their relationships. We explored, for example, questions of divisibility, prime factorization, what it means for integers to be relatively prime, etc. After the students play with the Pythagorean triples and examine some characteristics of these numbers, they are eager to find a systematic way to generate them.

Methods for obtaining Pythagorean triples cited in the Annotated List of Resources range from simple to sophisticated. I refer to several so the students can get a sense of the range of options (cf. [10], [14], [20], [23]). One they particularly like is Kalman's method (cf. [20]) for generating Pythagorean triples from proper fractions. Kalman starts with a right triangle with angle A, such that $\tan A$ = a proper fraction, say p/q. He then constructs another right triangle using $2A$ as one angle. Since $\tan 2A = 2\tan A / (1-\tan^2 A) = 2pq / (q^2 - p^2)$, the legs of the new triangle can be labeled $2pq$ and $(q^2 - p^2)$. Using the Pythagorean Theorem to determine the length of the hypotenuse will produce an integer. This proves that Kalman's procedure always produces a Pythagorean triple when $\tan A$ is rational. With the "hands-on" experience of generating triples and the knowledge gained from exploring other attempts at generating triples using familiar objects (like fractions), the students are ready to venture into unfamiliar territory.

B. Fibonacci Numbers

Since the students have an understanding of the difficulties involved in generating Pythagorean triples and a look at the diversity of methods for doing so, they are ready to learn about the connection between the mathematical products of Fibonacci and Pythagoras. We read and discuss several articles describing the use of Fibonacci numbers to produce Pythagorean triples (cf. [6], [16], [17]). The students are intrigued with the connection. The discussion of Fibonacci and Pythagoras provides a historical perspective and the use of Fibonacci's numbers to generate Pythagoras' numbers illustrates how mathematics builds upon itself using newer techniques to re-examine old problems.

The videotape *The Theorem of Pythagoras* (cf. [2]) shows dynamical versions of dissection proofs of the Pythagorean theorem. For a classroom activity, I organize students into small groups to "play with" a cardboard model of a dissection proof, asking them to assemble pre-cut pieces to illustrate the proof. This gives them a sense of what is involved in a dissection proof. I then follow with a classroom experiment based on the idea of dissection proof designed to show the students that evidence is different from proof and to prepare the way for a Fibonacci connection. I ask the students to construct an 8×8 square and calculate the area of 64. Then I direct them to dissect their model into a 5×13 figure as indicated:

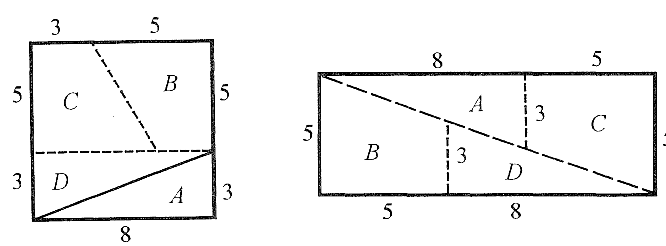

FIGURE 1

They quickly assume this figure is a rectangle, so when I ask them to calculate its area, they compute an area of 65. This seems to "prove" that 64 (the area of the original square) is equal to 65 (the area of the rectangle formed form the dissected pieces of the square). Our investigation of the new "rectangle" (via similar triangles) illustrates that the reconstructed figure is not truly a rectangle. This activity reinforces the necessity of rigorous proof in mathematics and alerts students to the dangers of accepting visual evidence as proof.

The culmination of this lesson is reading and discussing the article "Fibonacci Sequences and a Geometrical Paradox" (cf. [15]) in which Horadam shows how the Fibonacci numbers can be used to describe the area that appears to be gained in rearranging the parts of the square to form a rectangle. Using Horadam's article as a guide, we again analyze our 5×13 rectangle rearranged from the 8×8 square (Fig. 1). We observe that the one unit gain in area can be described by the relationship $5 \times 13 - 8^2 = 1$, a particular example of connecting three successive Fibonacci numbers (F_n, F_{n+1}, F_{n+2}) by the generalized formula $F_n F_{n+1} - F_{2n+1} = (-1)^{n+1}$. We then examine the relationship by considering the two cases, discovering: 1) when n is odd (as in our Fig. 1), the gain of one unit is the result of the appearance within the rectangle of a small parallelogram of unit area; 2) when n is even, the loss of one unit occurs because the unit parallelogram overlaps the dissected pieces. This gives the students visual evidence of how the Fibonacci numbers can be used to explain their dissection experiment, and how the results of their experiment can be expanded to include other cases. It is again, for them, yet another experience of utility **within** mathematics—the use of one mathematical topic to explain or clarify another.

Now the students, appreciating the use of Fibonacci numbers within mathematics, are ready to explore the many directions that Fibonacci numbers can inspire outside of mathematics —describing natural phenomena, determining outcomes of games, providing economic solutions for ecological problems, etc. (cf. List of Resources). Because the students were tuned in to the idea of connections, these discussions and activities were more meaningful than they had ever been in any previous classes I had taught.

4. CONCLUSION

My presentation of Fibonacci numbers via Pythagorean triples at the beginning of the course helps students to see that mathematical concepts often interrelate. The success of this classroom experiment lies in getting students to appreciate these interrelationships—enabling them to experience satisfaction in making mathematical connections. They learn to appreciate utility **within** mathematics as well as exterior to it. This experience set the tone for the entire course. One student wrote:

> I never learned interesting things like this in high school algebra. This topic contributed the most to my intellectual growth this semester, because it grabbed my attention, and allowed me to be open to other new concepts that we would study throughout the semester. The Fibonacci sequence opened the door to my mind, for it made me realize that math is going on all around me, and that it's important for me to understand why.

I encourage you to replicate this classroom experiment, and I welcome your reports about the results.

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2. Lynn Arthur Steen. "Restoring Scholarship to Collegiate Mathematics." *Focus* **6** (1986):1-7.
3. Alvin White. "Beyond Behavioral Objectives." *Amer. Math. Monthly* **82** (1985):849-51.

APPENDIX

ANNOTATED LIST OF RESOURCES

[1] Apostol, Tom. "The Pythagorean Theorem." In *MATHEMATICS!* California Institute of Technology, 1988.

This is an award-winning 20-minute computer-animated videotape, with accompanying workbook, on history, proofs, and applications of the theorem.

[2] Arpaia, P. J. "A Generating Property of Pythagorean Triples." *Mathematics Magazine* **44** (1971):26-27.

Based on the generating property of pairs of Pythagorean triples given by Courant and Robbins in *What Is Mathematics?*, this note establishes a generating property of any Pythagorean triple.

[3] Basin, S. L. "Generalized Fibonacci Sequences and Squared Rectangles." *American Mathematical Monthly* **70** (1963):372-79.

The author shows how generalized Fibonacci numbers can be used to generate squared rectangles (the problem of squaring a rectangle first appeared in the literature as a mathematical puzzle). The article concludes with an application (a ladder-network in communications systems) based on a model of the squaring of a rectangle of order n. This article is difficult, but not inaccessible to liberal arts majors.

[4] Beran, Ladislav. "Schemes Generating the Fibonacci Sequence." *Mathematical Gazette* **70** (1986):38-40.

The author shows that a resistance equation can be written in terms of the Fibonacci sequence and proves it by induction.

[5] Bergum, Gerald, & Yocom, Ken. "Tchebysheff Polynomials and Primitive Pythagorean Triples.," In *Two Year College Mathematics Readings*, Washington, D.C: The Mathematical Association of America, 1981.

This note illustrates and proves how to produce primitive integer-sides of a right triangle with hypotenuse c^n when the set of integers $\{a, b, c\}$ is primitive.

[6] Boulger, William. "Pythagoras Meets Fibonacci." *Mathematics Teacher* **82.4** (1989):277-81.

Boulger makes a nice connection between Fibonacci numbers and Pythagorean triples and proves it. He also illustrates a relationship between Fibonacci numbers and the golden ratio.

[7] DeTemple, Duane. "A New Angle on the Geometry of the Fibonacci Numbers." *Fibonacci Quarterly* **19.1** (1981):35-39.

The author gives a very nice geometric visualization of the Fibonacci sequence by representing the Fibonacci numbers by gnomons (Pythagorean carpenter squares).

[8] Federico, P. J. "A Fibonacci Perfect Squared Square." *American Mathematical Monthly* **71** (April 1964):404-06.

The author (following on Basin's paper [3]) describes how to construct "Fibonacci perfect squared squares."

[9] Freeman, W. H. (publisher). "Scale and Form." In *For All Practical Purposes*, 1988.

This is a 30-minute videotape that discusses Fibonacci numbers and gives applications in nature. This videotape supplements the mathematics textbook *For All Practical Purposes.*

[10] Goodrich, Merton. "A Systematic Method of Finding Pythagorean Numbers." *Mathematics Magazine* **19** (1945):395-97.

The author constructs a table to accomplish just what the title suggests.

[11] Hildebrand, W. J. "Generalized Pythagorean Triples." *The College Mathematics Journal* **16.1** (1985):48-52.

This article presents an algorithm to generate all possible integer triples (a,b,c) that are the sides of a triangle that contains angle C whose cosine is the rational number p/q. It also gives applications. The article assumes a knowledge of calculus and elementary number theory.

[12] Honsberger, Ross. "A Second Look at the Fibonacci and Lucas Numbers." In *Mathematical Gems III.* Washington, D.C.: The Mathematical Association of America, 1985.

This wonderfully written chapter demonstrates novel applications of Fibonacci numbers, proofs of many results, and sets of exercises for students. The author discusses the genealogy of the male honeybees and connections between Fibonacci and Lucas numbers.

[13] Honsberger, Ross. "The Fibonacci Sequence." In *Mathematical Morsels.* Washington, D.C.: The Mathematical Association of America, 1978.

The author discusses the problem of determining how many terms of the Fibonacci sequence are less than or equal to a given natural number N.

[14] Honsberger, Ross. "Pythagorean Arithmetic." In *Ingenuity in Mathematics.* Washington, D.C.: The Mathematical Association of America, 1970.

This short essay describes the arithmetic methods of the Pythagoreans and develops a procedure for generating Pythagorean triples. It is interesting reading for high school and college students and a good source of problems for instructors.

[15] Horadam, A. F. "Fibonacci Sequences and a Geometrical Paradox." *Mathematics Magazine* **35** (1962):1-11.

The author recalls the geometric paradox of subdividing an 8 unit square with area 64 square units and rearranging it to form a 5×13 unit rectangle with area 65 square units. He notes how the relationship $5 \times 13 - 8^2 = 1$ is a particular example of a result connecting 3 successive Fibonacci numbers. In the article, he extends the paradox to cover all sets of three consecutive Fibonacci numbers and generalizes the paradox by means of a generalized Fibonacci sequence.

[16] Horadam, A. F. "On Khazanov's Formulae." *Mathematics Magazine* **36** (1963):219-20.

The author outlines Khazanov's method for finding Pythagorean triples, and his own method for finding triples using a generalized Fibonacci sequence, then illustrates the connection between the two.

[17] Horadam, A. F. "Fibonacci Number Triples." *American Mathematical Monthly* **68** (1961): 751-53.

The author explores the connection between generalized Fibonacci numbers and Pythagorean triples.

[18] Howlett, G. et al. "Consecutive Heads and Fibonacci." *Mathematical Gazette* **69** (1985): 208-11.

These authors discuss the use of Fibonacci to determine the expected number of tosses of a coin before two consecutive heads are obtained. They comment on the Roland article "Toss Fibonacci" (see below).

[19] Jean, Roger, & Johnson, Marjorie. "An Adventure into Applied Mathematics with Fibonacci Numbers." *School Science and Mathematics* **89.6** (1989):487-98.

In this article, the authors illustrate how Fibonacci numbers arise in concrete situations: the genealogy of drones (bees), growth of sunflowers and pinecones, reflections on glass plates, economic solutions for the treatment of sewage in towns along a river bank, etc. They include a short bibliography with other resources of this type.

[20] Kalman, Dan. "Angling for Pythagorean Triples." *The College Mathematics Journal* **17.2** (1986):167-68.

The author generates Pythagorean triples from common fractions. A knowledge of trigonometry is assumed.

[21] Kuipers, L. posed the problem "No Fibonacci Pythagorean Triples." Solution given by M. Bicknell-Johnson in *The Fibonacci Quarterly* **27.1** (1989):88.

The solver proves that there are no positive integers r, s, and t, such that (F_r, F_s, F_t) is a Pythagorean triple.

[22] Nishi, Akihiro. "A Method of Obtaining Pythagorean Triples." *American Mathematical Monthly* **94.4** (1987):869-72.

This article, which develops what the title indicates, assumes a knowledge of elementary number theory.

[23] Ore, Oystein. "The Pythagorean Problem." In *Invitation to Number Theory.* Washington, D.C.: the Mathematical Association of America, 1967.

This chapter introduces Pythagorean triples and develops the formula for generating them. It concludes with problems related to Pythagorean triangles. Mathematical skill at the level of intermediate algebra is sufficient to comprehend this presentation.

[24] Roland, Tim. "Toss Fibonacci." *Mathematical Gazette* **68** (1984):183-86.

The author gives some probability problems and makes what he calls "natural connections" with Fibonacci.

[25] Tomkins, A., & Pitt, D. "Runs and the Generalized Fibonacci Sequence." *Mathematical Gazette* **69** (1985):109-13.

This article provides a nice entry into statistics and recursive relationships. The authors tackle the question of winning in a gambling system by increasing the amount staked each time one loses. They ask: "In a given number of races, on how many occasions are we to expect a run of losers of a certain length?" Generalized Fibonacci sequences provide an answer.

[26] Vajda, S. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications* New York: Halsted Press, a division of John Wiley & Sons, 1989.

This book includes discussions in algebra, analysis, geometry, probability theory, and number theory. The author begins with a brief survey of problems that are solved by Fibonacci numbers, divisibility properties, generation of random numbers, game theory; the Golden Section and properties of platonic solids.

[27] Vorob'ev, N. N. *Fibonacci Numbers* (translation of *Chisla fibonachchi* [Moscow-Leningrad: Gostekhteoretizdat, 1951]). London: Pergamon Press, Ltd., 1961.

This book contains a set of problems that were the themes of several meetings of a mathematics club of Leningrad State University in 1949-1950. It is a wonderful resource for liberal arts classes as it rarely requires any knowledge beyond high school mathematics.

Subject Index

Mathematical Subject Classification

859-98 Mathematical education, collegiate

859-51 Geometry

859-01 History and Biography

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FORMAL POWER SERIES FOR BINOMIAL SUMS OF SEQUENCES OF NUMBERS

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1. INTRODUCTION

Let $\{A_n\}_{n=0}^{\infty}$ be a given sequence of numbers and let

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}A_k,\quad n=0,1,\ldots$$

Let $A(x)$ and $S(x)$ denote the formal power series determined by the sequences $\{A_n\}_{n=0}^{\infty}$ and $\{S(n)\}_{n=0}^{\infty}$, that is,

$$A(x)=\sum_{n=0}^{\infty}A_nx^n,\quad S(x)=\sum_{n=0}^{\infty}S(n)x^n.$$

Recently, H. W. Gould [2] pointed out that

$$S(x)=\frac{1}{1-x}A\left(\frac{x}{1-x}\right). \tag{1}$$

In this paper we shall give a straightforward generalization of (1) and an application and a modification of the generalization.

2. A GENERALIZATION

Let s, t be given complex numbers and let $\{A_n\}_{n=0}^{\infty}$ be a given sequence of numbers. Denote

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}t^{n-k}s^kA_k,\quad n=0,1,\ldots$$

Theorem 1: We have

$$S(x)=\frac{1}{1-tx}A\left(\frac{sx}{1-tx}\right).$$

Proof: The proof is similar to that of (1) given in [2]. In fact,

$$\begin{aligned}S(x)&=\sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}\binom{n}{k}t^{n-k}s^kA_k=\sum_{k=0}^{\infty}A_ks^kx^k\sum_{n=k}^{\infty}\binom{n}{k}t^{n-k}x^{n-k}\\&=\sum_{k=0}^{\infty}A_ks^kx^k(1-tx)^{-k-1}=(1-tx)^{-1}A\left(\frac{sx}{1-tx}\right).\end{aligned}$$

This completes the proof.

3. AN APPLICATION

Let F_n, $n = 0, 1, \ldots$, be the Fibonacci numbers, and take $F_{-n} = (-1)^{n-1} F_n$. Let p, q be fixed nonzero integers such that $p \neq q$, and let r be a fixed integer. L. Carlitz [1, Theorem 4] proved that

$$\lambda^n F_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk+r} \quad (n = 0, 1, \ldots)$$

if and only if

$$\lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}.$$

We shall apply Theorem 1 to give a proof for this result. This result is given in Theorem 2 and in a slightly different form.

Lemma 1: We have

$$\sum_{n=0}^{\infty} F_{pn+r} x^n = \frac{F_r + (-1)^r F_{p-r} x}{1 - L_p x + (-1)^p x^2},$$

where L_n is the n^{th} Lucas number and $L_{-n} = (-1)^n L_n$ for $n \geq 0$.

Lemma 2: We have

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k F_{qk+r} = \frac{F_r + ((-1)^r F_{q-r} s - F_q t)x}{1 - (2t + L_q s)x + (t^2 + L_q ts + (-1)^q s^2)x^2}.$$

Lemma 1 is the same as formula (6) of [3]. Lemma 2 follows from Theorem 1 and Lemma 1.

Theorem 2: We have

$$\sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k F_{qk+r} = F_{pn+r} \quad (n = 0, 1, \ldots) \tag{2}$$

if and only if

$$s = F_p / F_q, \quad t = (-1)^p F_{q-p} / F_q. \tag{3}$$

Proof: By Lemmata 1 and 2, (2) holds if and only if,

$$(-1)^r F_{q-r} s - F_q t = (-1)^r F_{p-r}, \tag{4}$$

$$2t + L_q s = L_p, \tag{5}$$

$$t^2 + L_q ts + (-1)^q s^2 = (-1)^p. \tag{6}$$

Solving (4) and (5) gives (3). It can be verified that (6) holds for those values of s and t. This completes the proof.

4. A MODIFICATION

An interesting problem is to find a sequence $\{T(n)\}_{n=0}^{\infty}$ such that

$$(7) \qquad T(x) = A\left(\frac{sx}{1-tx}\right).$$

The solution is simple. It is given in Theorem 3. Applications of (7) are given in Theorem 4 and Theorem 5.

Theorem 3: Let $T(0) = A_0$ and

$$(8) \qquad T(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} t^{n-k} s^k A_k, \quad n = 1, 2, \ldots$$

Then (7) holds.

Proof: We have

$$T(x) = (1-tx)S(x).$$

Thus, $T(0) = S(0) = A_0$ and for $n \geq 1$,

$$T(n) = S(n) - tS(n-1) = s^n A_n + \sum_{k=0}^{n-1}\left[\binom{n}{k} - \binom{n-1}{k}\right] t^{n-k} s^k A_k$$
$$= \sum_{k=1}^{n} \binom{n-1}{k-1} t^{n-k} s^k A_k.$$

This completes the proof.

Remark: Theorem 3 could also be proved in a similar way to Theorem 1.

Theorem 4: If $s \neq 0$ and $T(n)$, $n = 0, 1, \ldots$, is given by (8), then $A_0 = T(0)$ and

$$A_n = s^{-n} \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} t^{n-k} T(k), \quad n = 1, 2, \ldots .$$

Proof: By (7),

$$A(x) = T\left(\frac{x}{s+tx}\right) = T(0) + \sum_{n=0}^{\infty} x^n s^{-n} \sum_{k=1}^{n} \binom{n-1}{k-1} (-t)^{n-k} T(k).$$

This proves Theorem 4.

Let m be a nonnegative integer. Then we define $T_m(n)$, $n = 0, 1, \ldots$, inductively by

$$T_0(n) = A_n, \quad n = 0, 1, \ldots,$$
$$T_{m+1}(0) = A_0, \quad T_{m+1}(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} t^{n-k} s^k T_m(n), \quad n = 1, 2, \ldots$$

when $m \geq 0$.

Theorem 5: If $s \neq 1$, then

$$T_m(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} t^{n-k} \left(\frac{s^m - 1}{s-1} \right)^{n-k} s^{mk} A_k, \; n = 1, 2, \ldots .$$

Proof: Theorem 5 can be proved by applying the formula

$$T_{m+1}(x) = T_m\left(\frac{sx}{1-tx} \right).$$

Remark: The transformations T and T_m have their analogues in the theory of arithmetic functions (see [4]).

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A REMARK RELATED TO THE FROBENIUS PROBLEM

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The Frobenius problem [2; 3] is to find, for a given set $a_1, \ldots, a_m$ of relatively prime integers, the largest number which is **not** a linear combination of $a_1, \ldots, a_m$ with nonnegative integer coefficients.

Given a set $a_1, \ldots, a_m$ of relatively prime positive integers, let us agree to call a number *representable* if it is a linear combination of $a_1, \ldots, a_m$ with nonnegative integer coefficients; otherwise, we call it *nonrepresentable.*

The fact that every sufficiently large positive integer is representable (given relatively prime $a_1, \ldots, a_m$) has been rediscovered many times, and makes a good exercise in a course in elementary number theory.

The Frobenius problem has a long history. (See [7] for a list of references.) In 1884, J. J. Sylvester [8] completely solved the problem for $m = 2$. He found that if a and b are positive integers such that $(a, b) = 1$, then every $n \geq (a-1)(b-1)$ can be expressed in the form $n = xa + by$ where x, y are nonnegative integers, and $ab - a - b$ cannot be so expressed. That is, the largest nonrepresentable number in this case is $ab - a - b$. Sylvester also found that among the $(a-1)(b-1)$ numbers $0, 1, 2, \ldots, ab - a - b$, exactly half are representable and half are nonrepresentable.

When $m = 3$, no closed-form expression for the largest nonrepresentable number is known (except in some special cases), although there do exist explicit algorithms for calculating it.

In the general case, various upper bounds are known for the largest nonrepresentable number, and closed-form expressions are known in a few special cases, for example, in the case that $a_1, \ldots, a_m$ are in arithmetic progression [1; 6].

In this note we consider an aspect of the case $m = 2$ which seems not to have been examined previously. We start by defining two notations.

For given a, b with $(a, b) = 1$, let $\mathrm{NR}(a, b)$ denote the *set* of numbers nonrepresentable in terms of a, b. Thus, $\mathrm{NR}(a, b)$ is the set of all those nonnegative integers n which *cannot* be expressed in the form $n = xa + by$, where x, y are nonnegative integers. Let

$$S(a,b) = \Sigma\{n : n \in \mathrm{NR}(a,b)\}$$

equal the *sum* of the numbers nonrepresentable in terms of a and b.

Although Sylvester showed that exactly $\frac{1}{2}(a-1)(b-1)$ of the numbers 0, 1, 2, .., $ab - a - b$ are nonrepresentable, that is that

$$|\mathrm{NR}(a,b)| = \frac{1}{2}(a-1)(b-1),$$

additional information about the nonrepresentable numbers would be provided by an estimate of their sum $S(a,b) = \Sigma\{n : n \in \mathrm{NR}(a,b)\}$.

A crude upper bound for $S(a,b)$ is obtained by taking the sum of the $\frac{1}{2}(a-1)(b-1)$ largest integers in the interval $[0,\ ab-a-b]$. Similarly, a crude lower bound is obtained by taking the sum of the $\frac{1}{2}(a-1)(b-1)$ smallest integers in the interval $[0,\ ab-a-b]$. This gives

$$\frac{1}{8}(a-1)^2(b-1)^2 - \frac{1}{4}(a-1)(b-1) \le S(a,b) \le \frac{3}{8}(a-1)^2(b-1)^2 - \frac{1}{4}(a-1)(b-1),$$

an upper bound of order $\frac{3}{8}(ab)^2$ and a lower bound of order $\frac{1}{8}(ab)^2$.

C. W. Ho, J. L. Parish, & P. J. Shiue [4] found that if A is any finite set of nonnegative integers such that the complement of A (in the set of nonnegative integers) is closed under addition, then $\Sigma\{n : n \in A\} \le |A|^2$. Since the sum of two representable numbers is certainly a representable number, we can take $\mathrm{NR}(a,b)$ in the place of A, and obtain

$$S(a,b) = \Sigma\{n : n \in \mathrm{NR}(a,b)\} \le |\mathrm{NR}(a,b)|^2 = \frac{1}{4}(a-1)^2(b-1)^2$$

an upper bound for $S(a,b)$ of order $\frac{1}{4}(ab)^2$, which considerably improves the previous upper bound.

In this note we find that, in fact,

$$S(a,b) = \frac{1}{12}(a-1)(b-1)(2ab-a-b-1),$$

so that the exact order of $S(a,b)$ is $\frac{1}{6}(ab)^2$.

For example, when $a = 4$, $b = 7$, the nonrepresentable numbers are 1, 2, 3, 5, 6, 9, 10, 13, and 17, which sum to

$$S(4,7) = 66 = \frac{1}{12}(4-1)(7-1)(56-4-7-1).$$

For the remainder of the paper, a, b are fixed positive integers with $(a,b) = 1$.

To calculate $S(a,b)$, we use the following idea:

For each $n \ge 0$, let $r(n)$ be the *number* of representations of n in the form $n = sa + tb$, where s, t are nonnegative integers. That is, $r(n)$ is the number of ordered pairs (s,t) such that $n = sa + tb$.

For example, $r(ab) = 2$, since if $ab = sa + tb$, then $(a,b) = 1$ implies that a divides t and b divides s, so that the only possibilities for (s,t) are $(b,0)$ and $(0,a)$.

It is not hard to see that if $0 \le n \le ab-1$, then either $r(n) = 0$ or $r(n) = 1$. For suppose that $r(n) \ge 2$ and that $n = s_1a + t_1b = s_2a + t_2b$ where (without loss of generality) $s_1 > s_2$. Then $0 = (s_1 - s_2)a + (t_1 - t_2)b$. Therefore, b divides $s_1 - s_2$, so $s_1 \ge b$ and $n \ge ab$.

Now, we define

$$f(x) = \sum_{n=0}^{ab-a-b}[1 - r(n)]x^n.$$

Using the fact that $r(n) = 0$ or $r(n) = 1$ for $0 \le n \le ab-1$, we obtain

$$f'(1) = \sum_{n=1}^{ab-a-b} n[1-r(n)] = \Sigma\{n : 1 \le n \le ab-a-b \text{ and } r(n) = 0\}$$
$$= \Sigma\{n : n \in \mathrm{NR}(a,b)\} = S(a,b).$$

Thus, the problem of finding $S(a,b)$ has been reduced to calculating $f'(1)$. It will turn out that

$$f(x) = \frac{P(x)-1}{x-1}, \text{ where } P(x) = \frac{(x^{ab}-1)(x-1)}{(x^a-1)(x^b-1)}.$$

This remarkably simple formula for $f(x)$ was discovered by Ali Ozluk, and appears in a more general setting (using $a_1, \ldots, a_m$ instead of a,b) in a paper by Ozluk & Sertoz [5]. For our case of $m = 2$, the calculations can be done as follows.

Let

$$A(x) = 1/(1-x^a)(1-x^b) = \left(\sum_{n=0}^{\infty} x^{an}\right)\left(\sum_{n=0}^{\infty} x^{bn}\right) = \sum_{n=0}^{\infty} r(n)x^n.$$

Now, since $(a,b) = 1$, it follows that

$$P(x) = \frac{(x^{ab}-1)(x-1)}{(x^a-1)(x^b-1)}$$

is a polynomial, with leading coefficient 1. (This can be seen by factoring both the numerator and denominator into complex linear factors. Since $(a,b) = 1$, there are integers s, t such that $as+bt = 1$. Let ζ be any complex number such that both $\zeta^a = 1$ and $\zeta^b = 1$; then $\zeta = \zeta^1 = \zeta^{as+bt}$ $= (\zeta^a)^s(\zeta^b)^t = 1$. In other words, no linear factor [except for $(x-1)$] appears twice in the denominator of $P(x)$; hence, every linear factor in the denominator cancels against a linear factor in the numerator.)

Since $P(1) = 1$ by L'Hospital's rule, we have that $\frac{P(x)-1}{x-1}$ is also a polynomial, of degree $ab-a-b$, with leading coefficient 1.

Now we write,

$$\frac{P(x)-1}{x-1} = \frac{P(x)}{x-1} + \frac{1}{1-x} = (x^{ab}-1)A(x) + \frac{1}{1-x}$$
$$= \sum_{n=0}^{\infty} r(n)x^{ab+n} + \sum_{n=0}^{\infty}[1-r(n)]x^n$$
$$= \sum_{n=ab}^{\infty} [r(n-ab)+1-r(n)]x^n + \sum_{n=0}^{ab-1}[1-r(n)]x^n.$$

Since we know that this power series is really a polynomial of degree $ab-a-b$ with leading coefficient 1, we can deduce that the power series coefficient of the $(ab-a-b)^{\text{th}}$ term is 1, and all later power series coefficients are zero. [Therefore, in particular, $r(ab-a-b) = 0$, $r(n) = 1$ for $ab-a-b < n \le ab-1$, and $r(n) = r(n-ab)+1$ for $n \ge ab$, although we do not use these facts in what follows.]

Thus, we now have

$$\frac{P(x)-1}{x-1}=\sum_{n=0}^{ab-a-b}[1-r(n)]x^n=f(x)$$

We now proceed to calculate $f'(1)$. Let $y=x^a$. Then

$$P(x)=\frac{(x^{ab}-1)(x-1)}{(x^a-1)(x^b-1)}=\sum_{k=0}^{b-1}y^k\Big/\sum_{k=0}^{b-1}x^k$$

and

$$f(x)=\frac{P(x)-1}{x-1}=\frac{\sum_{k=0}^{b-1}y^k-\sum_{k=0}^{b-1}x^k}{(x-1)\sum_{k=0}^{b-1}x^k}=\frac{\sum_{k=1}^{b-1}\frac{y^k-x^k}{x-1}}{\sum_{k=0}^{b-1}x^k}=\frac{g(x)}{h(x)},$$

where

$$g(x)=\sum_{k=1}^{b-1}\frac{y^k-x^k}{x-1},\quad h(x)=\sum_{k=0}^{b-1}x^k.$$

Now we use

$$\frac{y^k-x^k}{x-1}=\frac{x^{ak}-x^k}{x-1}=(x^k+x^{k+1}+\cdots+x^{ak-1})$$

to get

$$g(x)=\sum_{k=1}^{b-1}(x^k+x^{k+1}+\cdots+x^{ak-1}).$$

Then,

$$g(1)=\sum_{k=1}^{b-1}(a-1)k=\frac{1}{2}(a-1)(b-1)b,\ h(1)=b,\ h'(1)=\frac{1}{2}b(b-1).$$

Using the fact that

$$k+(k+1)+\cdots+(ka-1)=\frac{1}{2}ka(ka-1)-\frac{1}{2}k(k-1)=\frac{1}{2}(k^2(a^2-1)-k(a-1)),$$

we get that

$$\begin{aligned}g'(1)&=\sum_{k=1}^{b-1}(k+(k+1)+\cdots+(ka-1))=\sum_{k=1}^{b-1}\frac{1}{2}(k^2(a^2-1)-k(a-1))\\&=\frac{1}{2}(a^2-1)\sum_{k=1}^{b-1}k^2-\frac{1}{2}(a-1)\sum_{k=1}^{b-1}k=\frac{1}{2}(a^2-1)\frac{1}{6}(b-1)b(2b-1)-\frac{1}{2}(a-1)\frac{1}{2}(b-1)b\\&=b(a-1)(b-1)\left(\frac{(a+1)(2b-1)}{12}-\frac{1}{4}\right).\end{aligned}$$

Finally, we get

$$S(a,b)=f'(1)=\frac{h(1)g'(1)-g(1)h'(1)}{(h(1))^2}=\frac{1}{12}(a-1)(b-1)(2ab-a-b-1).$$

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RATIONAL NUMBERS WITH NON-TERMINATING, NON-PERIODIC MODIFIED ENGEL-TYPE EXPANSIONS

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In a recent paper [3] Kalpazidou, Knopfmacher, & Knopfmacher discussed expansions for real numbers of the form

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_1+1}\cdot\frac{1}{a_2} + \frac{1}{(a_1+1)(a_2+1)}\cdot\frac{1}{a_3} - \cdots \tag{1}$$

which they called a "modified Engel-type" alternating expansion. Here a_0 is an integer and a_1 is a positive integer for $i \geq 1$. If $a_{i+1} \geq a_i$, this expansion is essentially unique. To save space, we will abbreviate (1) by $A = \{a_0, a_1, a_2, \ldots\}$.

They say, "The question of whether or not all rationals have a finite or recurring expansion has not been settled." (By "recurring" we understand "ultimately periodic.")

In this note, we prove that the rational numbers $\frac{2}{2r+1}$ (r an integer ≥ 2) have modified Engel-type expansions that are neither finite nor ultimately periodic.

Theorem: Let r be an integer ≥ 1. Then

$$\frac{2}{2r+1} = \{a_0, a_1, a_2, \ldots\}$$

where $a_0 = 0$, and $a_{2i-1} = b_i$, $a_{2i} = 2b_i - 1$ for $i \geq 1$, and $b_1 = r$, $b_{n+1} = 2b_n^2 - 1$ for $n \geq 1$.

Proof: As in [3], we have

$$a_0 = \lfloor A \rfloor,\ A_1 = A - a_0,\ a_n = \lfloor 1/A_n \rfloor \text{ for } n \geq 1, \text{ and}$$
$$A_{n+1} = (1/a_n - A_n)(a_n + 1) \text{ for } n \geq 1.$$

From this we see that $a_0 = \left\lfloor \frac{2}{2r+1} \right\rfloor = 0$.

We now prove the following four assertions by induction on n: (i) $A_{2n-1} = \frac{2}{2b_n+1}$; (ii) $a_{2n-1} = b_n$; (iii) $A_{2n} = \frac{b_n+1}{b_n(2b_n+1)}$; and (iv) $a_{2n} = 2b_n - 1$.

It is easy to verify these assertions for $n = 1$, as we find

(i) $A_1 = \frac{2}{2r+1} = \frac{2}{2b_1+1}$;

(ii) $a_1 = \left\lfloor \frac{1}{A_1} \right\rfloor = r = b_1$;

(iii) $A_2 = \left(\frac{1}{r} - \frac{2}{2r+1}\right)(r+1) = \frac{r+1}{r(2r+1)} = \frac{b_1+1}{b_1(2b_1+1)}$;

(iv) $a_2 = \left\lfloor \frac{1}{A_2} \right\rfloor = \left\lfloor \frac{r(2r+1)}{r+1} \right\rfloor = \left\lfloor 2r - 1 + \frac{1}{r+1} \right\rfloor = 2r - 1 = 2b_1 - 1$.

Now assume the result is true for all $i \le n$. We prove it for $n+1$:

(i) $$A_{2n+1} = \left(\frac{1}{a_{2n}} - A_{2n}\right)(a_{2n}+1) = \left(\frac{1}{2b_n - 1} - \frac{b_n + 1}{b_n(2b_n+1)}\right)(2b_n) = \frac{2}{4b_n^2 - 1} = \frac{2}{2b_{n+1}+1}.$$

(ii) $$a_{2n+1} = \left\lfloor \frac{1}{A_{2n+1}} \right\rfloor = \left\lfloor \frac{2b_{n+1}+1}{2} \right\rfloor = b_{n+1}.$$

(iii) $$A_{2n+2} = \left(\frac{1}{a_{2n+1}} - A_{2n+1}\right)(a_{2n+1}+1) = \left(\frac{1}{b_{n+1}} - \frac{2}{2b_{n+1}+1}\right)(b_{n+1}+1) = \frac{b_{n+1}+1}{b_{n+1}(2b_{n+1}+1)}.$$

(iv) $$a_{n+2} = \left\lfloor \frac{1}{A_{2n+2}} \right\rfloor = \left\lfloor \frac{b_{n+1}(2b_{n+1}+1)}{b_{n+1}+1} \right\rfloor = \left\lfloor 2b_{n+1} - 1 + \frac{1}{b_{n+1}+1} \right\rfloor = 2b_{n+1} - 1.$$

This completes the proof. □

Corollary: For $r \ge 2$, the rational numbers $\frac{2}{2r+1}$ have non-terminating, non-ultimately-periodic modified Engel-type expansions.

Additional Remarks:

- For $r = 1$, the theorem gives the ultimately periodic expansion

  $2/3 = \{0, 1, 1, 1, 1, \ldots\}$.

- For $r \ge 2$, the expansion is not ultimately periodic; e.g.,

  $2/5 = \{0, 2, 3, 7, 13, 97, 193, 18817, \ldots\}$.

  In this case, we have the following brief table:

n	a_n	b_n	A_n
1	2	2	2/5
2	3	7	3/10
3	7	97	2/15
4	13	18817	8/105
5	97	708158977	2/195
6	193	1002978273411373057	89/18915

- The sequence $b_1, b_2, \ldots = 2, 7, 97, 18817, 708158977, \ldots$, corresponding to $r = 2$, appears to have been discussed first by G. Cantor in 1869 [1], who gave the infinite product

  $$\sqrt{3} = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{7}\right)\left(1 + \frac{1}{97}\right)\cdots.$$

  For more on this product of Cantor, see Spiess [9], Sierpinski [7], Engel [2], Stratemeyer [10; 11], Ostrowski [6], and Mendès France & van der Poorten [5]. The sequence 2, 7, 97, 18817, ... was also discussed by Lucas [4]. It is sequence #720 in Sloane [8].

- The sequence $b_1, b_2, \ldots = 3, 17, 577, 665857, \ldots,$ corresponding to $r = 3$, was also discussed by Cantor [1], who gave the infinite product

$$\sqrt{2} = \left(1+\frac{1}{3}\right)\left(1+\frac{1}{17}\right)\left(1+\frac{1}{577}\right)\cdots.$$

Also see the papers mentioned above. The sequence was also discussed by Wilf [12], and it is sequence #1234 in Sloane [8].

- It is easy to prove that $b_{n+1} = B_{2^n}$ where $B_0 = 1, B_1 = r$, and $B_n = 2rB_{n-1} - B_{n-2}$ for $n \geq 2$. This gives a closed form for the sequence (b_n):

$$b_{n+1} = \frac{\left(r+\sqrt{r^2-1}\right)^{2^n} + \left(r-\sqrt{r^2-1}\right)^{2^n}}{2}.$$

- 3/7 is the "simplest" rational for which no simple description of the terms in its modified Engel-type expansion is known. The first forty terms are as follows:

3/7 = {0, 2, 4, 5, 7, 8, 10, 25, 53, 62, 134, 574, 2431, 13147, 27167, 229073, 315416, 435474, 771789, 1522716, 3853889, 7878986, 7922488, 8844776, 9182596, 9388467, 14781524, 135097360,1374449987, 1561240840, 4408239956, 11166053604, 12014224315, 23110106464, 553192836372, 900447772231, 1189661630241, 2058097840143484, 6730348855426376, 12928512475357529, ...}.

More generally, it would be of interest to know whether it is possible to characterize the modified Engel expansion of every rational number.

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UNARY FIBONACCI NUMBERS ARE CONTEXT-SENSITIVE

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Moll and Venkatesan showed in [2] that the set of Fibonacci numbers is not context-free. Recall that a language is CF (context-free) if and only if there exists a context-free grammar generating it. It is only natural to ask where exactly in Chomsky's Hierarchy the Fibonacci numbers lie. By the Hierarchy Theorem (Theorem 9.9 of [1]), we have the following proper containments:

Regular sets $\subset$ CFL's $\subset$ CSL's $\subset$ RE's

RE's (recursively enumerable languages) are defined to be those sets generated by unrestricted grammars. Unrestricted grammars are simply grammars in which all the productions are of the form $\alpha \to \beta$, where α and β are arbitrary strings of grammar symbols, with $\alpha \neq \varepsilon$. By definition, CSL's (context-sensitive languages) are generated by CSG's (context-sensitive grammars). CSG's are very much like unrestricted grammars, with the added condition that for all productions $\alpha \to \beta$, $|\alpha| \leq |\beta|$.

In this paper we offer a CSG G generating the language of unary Fibonacci numbers, $L = \{0^i \mid i = F_n\}$, hence demonstrating the title claim. But before doing this, it will prove useful to construct an unrestricted grammar G' for L.

THE UNRESTRICTED GRAMMAR G'

Formally define $G' = (V', T, P', S)$, where $V' = \{S, A, B, C, D, E, F, G, H, J, K, L, M, N, P\}$, $T = \{0\}$, and P' is given by the list of productions:

1)	$S \to 0$	14)	$KC \to LC0$
2)	$S \to AE0B0CD$	15)	$0L \to L0$
3)	$AE \to AH$	16)	$BL \to BJ$
4)	$H0 \to F0$	17)	$BJC \to BM$
5)	$F00 \to 0F0$	18)	$M0 \to 0M$
6)	$F0B \to BF0$	19)	$MD \to NCD$
7)	$F0C \to GC0$	20)	$0N \to N0$
8)	$0G \to G0$	21)	$BN \to NB$
9)	$BG \to GB$	22)	$AN \to AE$
10)	$AG \to AH$	23)	$AE \to P$
11)	$AHB \to ABJ$	24)	$P0 \to 0P$
12)	$BJ0 \to 0BK$	25)	$PB \to P$
13)	$K0 \to 0K$	26)	$PCD \to \varepsilon$

Observe that there are two starting productions. Production 1 generates the nonrecursive base cases; production 2 generates all other Fibonacci numbers F_n, with $n > 2$. In general selection of production 3 eventually leads to a string of the form

$(*) \qquad AE0\ldots0B0\ldots0CD.$

The 0's between A and B represent unary F_{n-2}, while those between B and C represent F_{n-1}. Repeated selection of production 3 "increments" $(*)$, while choosing production 23 outputs F_n by eliminating the markers.

In summary, productions 1 and 2 enable us to generate either the base or recursive case. Productions 3 through 11 move F_{n-2} into the space between C and D; productions 12 through 22 perform the updating and restoration of the string to the form of (∗). Finally, productions 23 to 26 output the answer. It is easily verified that G' generates exactly L. □

Because G' is an unrestricted grammar that generates L, L is recursively-enumerable. Note that G' is not a CSG because the left-hand sides of productions 23, 25, and 26 are longer than their right-hand sides.

THE CONTEXT-SENSITIVE GRAMMAR *G*

We use the method of Example 9.5 of [1] to create a context-sensitive grammar G which mimics G'. Instead of the "single" variables of G', we use "composite" variables that combine 0 with each of its possible contexts. For example, the single nonterminal $[AE0]$ replaces the two variable string AE in a particular context.

Formally define $G=(V,T,P,[S])$, where $V=$ {$[S]$, $[AE0]$, $[B0CD]$, $[AH0]$, $[AF0]$, $[ABF0]$, $[0CD]$, $[AB0]$, $[F0CD]$, $[F0]$, $[A0]$, $[BF0]$, $[B0]$, $[GC0D]$, $[C0D]$, $[GC0]$, $[0D]$, $[ABG0]$, $[G0]$, $[BG0]$, $[GB0]$, $[AG0]$, $[C0]$, $[AGB0]$, $[AHB0]$, $[ABJ0]$, $[BKC0D]$, $[BK0]$, $[BJ0]$, $[BKC0]$, $[KC0D]$, $[K0]$, $[KC0]$, $[BLC0]$, $[LC0]$, $[BL0]$, $[L0]$, $[BJC0]$, $[BM0]$, $[M0D]$, $[0MD]$, $[M0]$, $[0NCD]$, $[N0CD]$, $[BN0]$, $[0CD]$, $[NB0]$, $[AN0]$, $[N0]$, $[P0]$, $[PB0]$, $[P0CD]$, $[0PCD]$}, and P is given by the following list of productions, which are grouped according to the production of G' they mimic:

1) $[S]\rightarrow 0$

2) $[S]\rightarrow [AE0][B0CD]$

3) $[AE0]\rightarrow [AH0]$

4) $[AH0]\rightarrow [AF0]$

5) $[ABF0][0CD]\rightarrow [AB0][F0CD]$
$[ABF0]0\rightarrow [AB0][F0]$
$[F0][0CD]\rightarrow 0[F0CD]$
$[AF0]0\rightarrow [A0][F0]$
$[BF0]0\rightarrow [B0][F0]$
$[F0]0\rightarrow 0[F0]$

6) $[AF0][B0CD]\rightarrow [ABF0][0CD]$
$[AF0][B0]\rightarrow [ABF0]0$
$[F0][B0]\rightarrow [BF0]0$

7) $[F0CD]\rightarrow [GC0D]$
$[F0][C0D]\rightarrow [GC0][0D]$

8) $[AB0][GC0D]\rightarrow [ABG0][C0D]$
$0[GC0D]\rightarrow [G0][C0D]$
$[AB0][G0]\rightarrow [ABG0]0$
$0[G0]\rightarrow [G0]0$
$[B0][G0]\rightarrow [BG0]0$
$[A0][GB0]\rightarrow [AG0][B0]$
$0[GC0]\rightarrow [G0][C0]$

14) $[BKC0D]\rightarrow [BLC0][0D]$
$[KC0D]\rightarrow [LC0][0D]$
$[BKC0]\rightarrow [BLC0]0$
$[KC0]\rightarrow [LC0]0$

15) $[B0][LC0]\rightarrow [BL0][C0]$
$0[LC0]\rightarrow [L0][C0]$
$[B0][L0]\rightarrow [BL0]0$

16) $[BLC0]\rightarrow [BJC0]$
$[BL0]\rightarrow [BJ0]$

17) $[BJC0]\rightarrow [BM0]$

18) $[BM0][0D]\rightarrow [B0][M0D]$
$[M0D]\rightarrow [0MD]$
$[BM0]0\rightarrow [B0][M0]$
$[M0][0D]\rightarrow 0[M0D]$
$[M0]0\rightarrow 0[M0]$

19) $[0MD]\rightarrow [0NCD]$

20) $[0NCD]\rightarrow [N0CD]$
$[B0][N0CD]\rightarrow [BN0][0CD]$
$[A0][NB0]\rightarrow [AN0][B0]$
$0[N0CD]\rightarrow [N0][0CD]$
$[B0][N0]\rightarrow [BN0]0$
$0[NB0]\rightarrow [N0][B0$
$[A0][N0]\rightarrow [AN0]0$
$0[N0]\rightarrow [N0]0$

9) $[ABG0] \to [AGB0]$
$[BG0] \to [GB0]$

10) $[AGB0] \to [AHB0]$
$[AG0] \to [AH0]$

11) $[AHB0] \to [ABJ0]$

12) $[ABJ0][C0D] \to [A0][BKC0D]$
$[ABJ0]0 \to [A0][BK0]$
$[A0][BJ0][C0] \to [A0]0[BKC0]$
$[BJ0]0 \to 0[BK0]$
$[BJ0][C0] \to 0[BKC0]$

13) $[BK0][C0D] \to [B0][KC0D]$
$[BK0]0 \to [B0][K0]$
$[K0][C0] \to 0[KC0]$
$[BK0][C0] \to [B0][KC0]$
$[K0]0 \to 0[K0]$

21) $[BN0] \to [NB0]$

22) $[AN0] \to [AE0]$

23) $[AE0] \to [P0]$

24) $[P0]0 \to 0[P0]$
$[P0][B0] \to 0[PB0]$
$[P0][0CD] \to 0[P0CD]$
$[P0CD] \to [0PCD]$

25) $[PB0] \to [P0]$

26) $[0PCD] \to 0$

It is straightforward to see that $S \overset{*}{\Rightarrow} \alpha'$ (i.e., a string α' is derived from S) through G' if and only if $[S] \overset{*}{\Rightarrow} \alpha$ with G, where α is formed from α' by grouping with a 0 all markers (i.e., elements of $V' - \{S\}$) appearing between it and the 0 to its left, and also by grouping the first 0 with any markers to its left and with the last 0 any markers to its right; e.g., if α' is $A00B0KC000D$, then α is $[A0]0[B0][KC0]0[0D]$. Observe that the right side of every production of G is at least as long as the left side. Clearly, G is a context-sensitive grammar. □

Thus, we have

Theorem: L is a context-sensitive language.

Proof: Immediate from construction of G. □

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ON EXTENDED GENERALIZED STIRLING PAIRS

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1. INTRODUCTION

Following Carlitz's terminology (see [2] and [3]), we define a generalized Stirling pair (GSP) as follows:

Definition 1: Let f and g belong to the commutative ring of formal power series with real or complex coefficients and let

(1) $$\sum_{n\geq 0} A_1(n,k)t^n / n! = (f(t))^k / k!$$

(2) $$\sum_{n\geq 0} A_2(n,k)t^n / n! = (g(t))^k / k!$$

Then $\{A_1(n,k),\ A_2(n,k)\}$ is called a GSP if and only if f and g are reciprocal (inverse) of each other, in the sense that

$$f(g(t)) = g(f(t)) = t \text{ with } f(0) = g(0) = 0.$$

From Carlitz [1], we have that $\{A_1(n,k),\ A_2(n,k)\}$ is a GSP if and only if the double sequences of numbers $A_1(n,k)$ and $A_2(n,k)$ satisfy the orthogonality relations

(3) $$\sum_{n=k}^{m} A_1(m,n)A_2(n,k) = \sum_{n=k}^{m} A_2(m,n)A_1(n,k) = \delta_{mk},$$

where δ_{mk} is the Kronecker symbol, or, equivalently, they satisfy the inverse relations

(4) $$a_n = \sum_{k=0}^{n} A_1(n,k)b_k, \quad b_k = \sum_{k=0}^{n} A_2(n,k)a_k,$$

where $n = 0, 1, 2, \ldots,$ and either $\{a_k\}$ or $\{b_k\}$ is given arbitrarily. That is, a GSP is characterized by a pair of orthogonality or inverse relations.

Note that the pair of Stirling numbers of the first and second kind is a special case of a GSP $[g(t) = e^t - 1,\ f(t) = \ln(1+t)]$.

In the present paper, we define an extended generalized Stirling pair, say $\{B_1(n,k),\ B_2(n,k)\}$, which covers in particular the above known results and other interesting pairs of numbers with combinatorial interpretations. Moreover, similar relations to the orthogonality ones are established which characterize this extended generalized Stirling pair.

Finally, recurrence and congruence relations concerning the numbers $B_i(n,k),\ i = 1,2,$ are obtained.

It is worth mentioning the following result, due to Carlitz (see [2]), which also leads to orthogonality relations. Let $\{f_k(z)\}$ denote a sequence of polynomials such that

$$\deg f_k(z) = k;\ f_k(0) = 0 \text{ for } k > 0;\ \left(1 + \sum_{n=1}^{\infty} c_n x^n / n!\right)^z = \sum_{k=0}^{\infty} f_k(z)x^k / k!$$

Put

$$F_1(n,n-k)=\binom{k-n}{k}f_k(n);\quad F(n,n-k)=\binom{n}{k}f_k(-n+k).$$

Then

$$\sum_{k=j}^{n}(-1)^{n-k}F_1(n,k)F(k,j)=\sum_{k=j}^{n}(-1)^{k-j}F(n,k)F_1(k,j)=\delta_{nj}.$$

The functions $F_1(z,z-k)$, $F(z,z-k)$ are called a Stirling pair, and a generalization is given in [3]. Note that for $f_k(z)$ the Norlund polynomial $B_k(z)$ defined by

$$\left(\frac{x}{e^x-1}\right)^z=\sum_{k=0}^{\infty}B_k(z)x^k/k!$$

where z is an arbitrary complex number (see [9, ch. 6]), the numbers $F_1(n,n-k)$ and $F(n,n-k)$ reduce to the ordinary Stirling numbers of the first kind (signless) and the ones of the second kind, respectively.

2. THE DEFINITION OF THE $\{B_1(n,k),\ B_2(n,k)\}$-SPECIAL CASES

We define an extended generalized Stirling pair (EGSP) as follows:

Definition 2: Let h, f, and g belong to Γ and let

$$\text{(5)}\qquad \sum_{n\geq 0}B_1(n,k)t^n/n!=h(t)(f(t))^k/k!,\quad h(t)\neq 0,$$

$$\text{(6)}\qquad \sum_{n\geq 0}B_2(n,k)t^n/n!=\frac{1}{h(g(t))}(g(t))^k/k!$$

Then $\{B_1(n,k),\ B_2(n,k)\}$ is called an extended, generalized Stirling pair (EGSP) if and only if f and g are reciprocal of each other.

Note that if $\{B_1(n,k),\ B_2(n,k)\}$ is an EGSP, then $B_i(n,k)=0,\ i=1,2$ if $k>n$. For $h(t)=1$, the pair $\{B_1(n,k),\ B_2(n,k)\}$ reduces to the one $\{A_1(n,k),\ A_2(n,k)\}$. Some interesting special cases of extended generalized Stirling pairs are given below.

1. For $h(t)=e^t,\ f(t)=t,\ g(t)=t$, we have the pair

$$\left\{\binom{n}{k},\ (-1)^{n-k}\binom{n}{k}\right\}.$$

2. For $h(t)=e^{\lambda t}$, λ a real number, $f(t)=e^t-1,\ g(t)=\ln(1+t)$, we have the pair

$$\{(-1)^{n-k}R_1(n,k,\lambda),\quad R(n,k,\lambda)\}$$

where $R_1(n,k,\lambda)$ and $R(n,k,\lambda)$ are the weighted Stirling numbers of the first and second kind, respectively (see [5]). Since $R_1(n,k,-\alpha)=(-1)^{n-k}s_\alpha(n,k)$ and $R(n,k,-\alpha)=S_\alpha(n,k)$ where $s_\alpha(n,k)$ and $S_\alpha(n,k)$ are the noncentral Stirling numbers of the first and second kind, respectively (see [8]), we have that $\{s_\alpha(n,k),\ \ S_\alpha(n,k)\}$ is also an EGSP.

3. For $h(t)=(1+t)^s$, s a real number, $f(t)=(1+t)^r-1$, r a real number, $r\neq 0$, $g(t)=(1+t)^{\frac{1}{r}}-1$, we have the pair $\{G(n,k,r,s),\ G(n,k,1/r,-s/r)\}$. When both r and s are positive or negative integers, we have that $\frac{k!}{n!}G(n,k,r,s)$ is the number of ways of putting n like balls into k different cells of r different compartments each and a (control) cell of s different compartments with limited or unlimited capacity (see [6]).

4. For $h(t)=(1-t)^{\theta-\lambda}$, θ,λ real numbers, $f(t)=[1-(1-t)^\theta]/\theta$ and $g(t)=(1+\theta t)^\mu-1$ where $\mu\theta=1$, we have the pair $\{(-1)^{n-k}S_1(n,k,\lambda+\theta|\theta),\ S(n,k,\lambda|\theta)\}$ where $S_1(n,k,\lambda|\theta)$ and $S(n,k,\lambda|\theta)$ are the degenerate weighted Stirling numbers of the first and second kind, respectively (see [7]).

   Letting $\lambda=0$, we see that the degenerate numbers of Carlitz [4]

$$\{(-1)^{n-k}S_1(n,k|\theta),\quad S(n,k|\theta)\}$$

form a GSP, since $S_1(n,k,\theta|\theta)=S_1(n,k|\theta)$.

Letting $\theta=0$, we see again that $\{(-1)^{n-k}R_1(n,k,\lambda),\quad R(n,k,\lambda)\}$ is an EGSP.

3. CHARACTERIZATIONS

An EGSP is characterized by a pair of orthogonality relations, as we show in what follows.

Theorem 1: The numbers $B_i(n,k)$, $i=1,2$, are given by the relations (5) and (6), respectively. Then, $\{B_1(n,k),\ B_2(n,k)\}$ is an EGSP if and only if the following orthogonality relations,

$$\sum_{n=k}^{m}B_2(m,n)B_1(n,k)=\sum_{n=k}^{m}B_1(m,n)B_2(n,k)=\delta_{mk}, \tag{7}$$

hold, where δ_{mk} is the Kronecker symbol.

Proof: Setting $t\to g(t)$ in (5) and using (6), we get

$$(f(g(t)))^k/k!=\sum_{n=k}^{\infty}B_1(n,k)\sum_{m=n}^{\infty}B_2(m,n)t^m/m! \tag{8}$$

$$=\sum_{m=k}^{\infty}\left\{\sum_{n=k}^{m}B_2(m,n)B_1(n,k)\right\}t^m/m!$$

Similarly, setting $t\to f(t)$ in (6) and using (5), we get

$$\frac{h(t)}{h(g(f(t)))}(g(f(t)))^k/k!=\sum_{m=k}^{\infty}\left\{\sum_{n=k}^{m}B_1(m,n)B_2(n,k)\right\}t^m/m! \tag{9}$$

The "if" part: Substituting the relations (7) into relations (8) and (9), we have

$$(f(g(t)))^k=t^k,\quad k=0,1,2,\dots,$$

and

$$\frac{h(t)}{h(g(f(t)))}(g(f(t)))^k=t^k,\ k=0,1,2,\dots,$$

from which we deduce that

$$f(g(t)) = g(f(t)) = t.$$

The "only if" part: Suppose that $f(g(t)) = g(f(t)) = t$ from relations (8) and (9), then we have

$$t^k / k! = \sum_{m=k}^{\infty}\left\{\sum_{n=k}^{m} B_2(m,n)B_1(n,k)\right\} t^m / m!$$

$$= \sum_{m=k}^{\infty}\left\{\sum_{n=k}^{m} B_1(m,n)B_2(n,k)\right\} t^m / m!$$

Equating coefficients of $t^m / m!$, we obtain the relations (7).

By the following theorem, we show that an EGSP is characterized by a pair of relations similar to the orthogonality relations.

Theorem 2: The numbers $B_i(n,k)$, $i = 1,2$, are given by (5) and (6), respectively, and the numbers $A_i(n,k)$ $i = 1,2$, are given by (1) and (2), respectively. Thus, $\{B_1(n,k),\ B_2(n,k)\}$ is an EGSP if and only if

$$\text{(10)} \qquad \sum_{n=k}^{m} B_1(m,n)A_2(n,k) = \binom{m}{k} h_{m-k}$$

where h_j, $j = 0, 1, 2, \ldots$, are the coefficients in the expansion

$$h(t) = \sum_{j=0}^{\infty} h_j t^j / j!$$

or

$$\text{(11)} \qquad \sum_{n=k}^{m} B_2(m,n)A_1(n,k) = \binom{m}{k} h^*_{m-k}$$

where h^*_j, $j = 0, 1, 2, \ldots$, are the coefficients in the expansion

$$\frac{1}{h(g(t))} = \sum_{j=0}^{\infty} h^*_j t^j / j!.$$

Proof: From relation (5), we get

$$\sum_{m=k}^{\infty} B_1(m,k)t^m / m! = \sum_{j=0}^{\infty} h_j t^j / j! \sum_{r=k}^{\infty} A_1(r,k)t^r / r!$$

$$= \sum_{m=k}^{\infty}\left\{\sum_{j=0}^{m-k}\binom{m}{j} h_j A_1(m-j,k)\right\} t^m / m!$$

Equating coefficients of $t^m / m!$, we obtain

$$B_1(m,k) = \sum_{j=k}^{m}\binom{m}{j} h_{m-j} A_1(j,k).$$

Multiplying both sides of the above relation by $A_2(n,k)$ and summing for all $n=k,\ k+1,\dots,m$, we obtain

$$\sum_{n=k}^{m} B_1(m,n)A_2(n,k)=\sum_{n=k}^{m}\sum_{j=n}^{m}\binom{m}{j}h_{m-j}A_1(j,n)A_2(n,k). \tag{12}$$

The "if" part: Comparing the relations (12) and (10), we get

$$\binom{m}{k}h_{m-k}=\sum_{j=k}^{m}\binom{m}{j}h_{m-j}\sum_{n=k}^{j}A_1(j,n)A_2(n,k).$$

Multiplying both sides by $t^m/m!$ and summing for all $m=k,\quad k+1,\dots,$ we have

$$\frac{t^k}{k!}h(t)=\sum_{m=k}^{\infty}\sum_{j=k}^{m}\sum_{n=k}^{j}\binom{m}{j}h_{m-j}A_1(j,n)A_2(n,k)t^m/m!$$
$$=\sum_{j=k}^{\infty}\sum_{m=j}^{\infty}\sum_{n=k}^{j}A_1(j,n)A_2(n,k)\binom{m}{j}h_{m-j}t^m/m!$$

or

$$\frac{t^k}{k!}=\sum_{j=k}^{\infty}\sum_{n=k}^{j}A_1(j,n)A_2(n,k)\frac{t^j}{j!}$$

from which we obtain

$$\sum_{n=k}^{j}A_2(j,n)A_1(n,k)=\delta_{jk}.$$

Consequently, $\{A_1(n,k),\quad A_2(n,k)\}$ is a GSP or, equivalently, $f(g(t))=g(f(t))=t$.

The "only if" part We have that $f(g(t))=g(f(t))=t$ or, equivalently, the relations (3) hold. Consequently, relation (12) becomes

$$\sum_{n=k}^{m}B_1(m,n)A_2(n,k)=\sum_{j=k}^{m}\binom{m}{j}h_{m-j}\delta_{jk}=\binom{m}{k}h_{m-k}.$$

Remark 1: The relations (10) and (11) lead to the inverse relations

$$A_2(m,k)=\sum_{n=k}^{m}\binom{n}{k}h_{n-k}B_1(m,n),$$
$$A_1(m,k)=\sum_{n=k}^{m}\binom{n}{k}h^*_{n-k}B_2(m,n).$$

4. RECURRENCES

Let $\{B_1(n,k),\ B_2(n,k)\}$ be an EGSP.

Differentiating both sides of relation (5) and of relation (6) with respect to t, we can easily obtain the following recurrences:

(13) $$B_1(n+1,k)=\sum_{j=0}^{n}\binom{n}{j}\alpha_{j+1}B_1(n-j,k)+\sum_{j=0}^{n}\binom{n}{j}f_{j+1}B_1(n-j,k-1)$$

where f_j, $j=0,1,\ldots$, and α_j, $j=0,1,\ldots$, are, respectively, the coefficients in the expansions

$$f(t)=\sum_{j=1}^{\infty}f_jt^j/j! \text{ and } h'(t)/h(t)=\sum_{j=0}^{\infty}\alpha_jt^j/j!,\ \ h'(t)=\frac{d}{dt}h(t)$$

and

(14) $$B_2(n+1,k)=\sum_{j=0}^{n}\binom{n}{j}\beta_{j+1}B_2(n-j,k)+\sum_{j=0}^{n}\binom{n}{j}g_{j+1}B_2(n-j,k-1)$$

where g_j, $j=0,1,\ldots$, and β_j, $j=0,1,\ldots$, are, respectively, the coefficients in the expansions

$$g(t)=\sum_{j=1}^{\infty}g_jt^j/j! \text{ and } \left(\frac{1}{\mathrm{h(g)(t))}}\right)'\Big/\left(\frac{1}{h(g(t))}\right)=\sum_{j=0}^{\infty}\beta_jt^j/j!.$$

Remark 2: Let $\{B_1(n,k),\ B_2(n,k)\}$ be an EGSP. From the definition of Bell polylnomials (cf. Riordan [10]),

$$Y_n(gf_1,gf_2,\ldots,gf_n)=\sum\left(\frac{n!g_k}{j_1!j_2!\ldots j_n!}\left(\frac{f_1}{1!}\right)^{j_1}\left(\frac{f_2}{2!}\right)^{j_2}\cdots\left(\frac{f_n}{n!}\right)^{j_n}\right)$$

where the sum extends over all n-tuples $(j_1,j_2,\ldots j_n)$ of nonnegative integers such that $j_1+j_2+\cdots+j_n=k$ and $j_1+2j_2+\cdots+nj_n=n$, one may obtain the following system of linear equations,

(15) $$y_n(gf_1,\ldots,gf_n)=\delta_{n1}\quad(n=1,2,\ldots),$$

from which we conclude that, for any given sequence $\{f_j\}$, the sequence $\{g_j\}$ can be determined.

We also have

$$\left(\frac{1}{h(g(t))}\right)'\Big/\left(\frac{1}{h(g(t))}\right)=-(h(g(t))'/h(g(t))=-g(t)\sum_{i\geq0}h_{i+1}\frac{(g(t))^i}{i!}\ \frac{1}{h(g(t))}$$
$$=-\sum_{s\geq0}g_{s+1}\frac{t^s}{s!}\sum_{i\geq0}h_{i+1}\sum_{r=i}^{\infty}B_2(r,i)\frac{t^r}{r!}$$
$$=-\sum_{j\geq0}\left\{\sum_{r=0}^{j}\sum_{i=0}^{r}h_{i+1}B_2(r,i)g_{j-r+1}\binom{j}{r}\right\}\frac{t^j}{j!}$$

from which we get

(16) $$\beta_j=\sum_{i=0}^{j}h_{i+1}\sum_{r=i}^{j}\binom{j}{r}g_{j-r+1}B_2(r,i),\quad j=0,1,\ldots\ .$$

From relations (15) and (16) we have that, for any given sequences $\{f_i\}$ and $\{h_j\}$, the sequences $\{g_j\}$ and $\{\beta_j\}$ can be determined.

Consequently, having the recurrence (13), we may conclude the one (14).

An interesting special case of the above situation is given by the following Proposition.

Proposition 1: Let $\{B_1(n,k),\ B_2(n,k)\}$ be an EGSP and the numbers $B_1(n,k)$ satisfy the triangular array recurrence relation

$$B_1(n+1,k)=(c_1n+c_2k+c_3)B_1(n,k)+c_4B_1(n,k-1) \tag{17}$$

where c_i, $i=1,2,3,4$ constants, $k=0,1,\dots,n+1$, $n=0,1,2,\dots$, with initial conditions

$$B_1(0,0)=1,\quad B_1(n,k)=0 \text{ if } n<k,$$

then the numbers $B_2(n,k)$ satisfy the triangular array recurrence relation

$$B_2(n+1,k)=\left(-\frac{c_2}{c_4}n-\frac{c_1}{c_4}k-\frac{c_3}{c_4}\right)B_2(n,k)+\frac{1}{c_4}B_2(n,k-1) \tag{18}$$

where $k=0,1,\dots,n+1$, $n=0,1,\dots$, with initial conditions

$$B_2(0,0)=1,\quad B_2(n,k)=0 \text{ if } n<k.$$

Proof: Multiplying both sides of (17) by $t^n/n!$, summing for all $n=0,1,\dots$, and using relation (5), we have that (17) holds if and only if

$$h'(t)=\frac{c_3h(t)}{1-c_1t} \quad\text{and}\quad f'(t)=\frac{c_2f(t)+c_4}{1-c_1(t)}. \tag{19}$$

Moreover,

$$\left(\frac{1}{h(g(t))}\right)'=\frac{-c_3(1/h(g(t)))}{c_2t+c_4}$$

and

$$g'(t)=1/f'(g(t))=\frac{(-c_1/c_4)g(t)+1/c_4}{(c_2/c_4)t+1},$$

which, on using (6) and (19), gives (18).

5. CONGRUENCES

Let $\{B_1(n,k),\ B_2(n,k)\}$ be an EGSP and $\{A_1(n,k),\ A_2(n,k)\}$ be the corresponding GSP given by Definitions 2 and 1, respectively. In this section we are interested in integer pairs. A question now arises: Under what conditions on $h(t)$ and $f(t)$ are the above numbers integers?

Supposing that $h(t)$ and $f(t)$ are Hurwitz series, in the sense that the coefficients h_j, $j=0,1,\dots$, and f_j, $j=0,1,\dots$, in their expansions are integers, and that $h(0)=f(0)=1$; it can easily be proved, on using Taylor's expansions and the relation $f(g(t))=t$, that $g(t)$ and $1/h(g(t))$ are also Hurwitz series. In this case, the numbers $A_i(n,k)$, $i=1,2$, and $B_i(n,k)$, $i=1,2$, are integers, as we can easily see from their definitions and the fact that if $f(0)=0$ and $f(t)$ is a Hurwitz series, then $(f(t))^k/k!$, $k=0,1,\dots$, is also a Hurwitz series, and that Hurwitz series are closed under multiplication.

We have already proved the following Proposition.

Proposition 2: Let $\{B_1(n,k),\ B_2(n,k)\}$ be an EGSP and $\{A_1(n,k),\ A_2(n,k)\}$ be the corresponding GSP. If $h(t)$ and $f(t)$ are Hurwitz series with $h(0)=f'(0)=1$, then $B_i(n,k),\ i=1,2$, and $A_i(n,k),\ i=1,2$, are integers.

Now, from the proof of Theorem 2, we have

$$B_1(m,k)=\sum_{j=k}^{m}\binom{m}{j}h_{m-j}A_1(j,k),\ k=1,2,\ldots,m \tag{20}$$

and, similarly,

$$B_2(m,k)=\sum_{j=k}^{m}\binom{m}{j}h^*_{m-j}A_2(j,k),\ k=1,2,\ldots,m. \tag{21}$$

Using (20) and (21) and the fact that $\binom{p}{j}\equiv 0 \pmod p$ for each prime p, except $\binom{p}{0}=\binom{p}{p}=1$, we obtain the following congruence:

$$B_i(p,k)\equiv A_i(p,k) \pmod p,\quad i=1,2, \tag{22}$$

for each prime p, $k=1,2,\ldots,p$, while $B_1(p,0)=h_p$, $B_2(p,0)=h^*_p$ and $A_i(p,0)=0$, $i=1,2$.

Also, using relations (7) and

$$B_1(m,m)=B_2(m,m)=1,$$

we obtain

$$B_2(m,k)=-\sum_{j=k}^{m-1}B_1(m,j)B_2(j,k)$$

and

$$B_1(m,k)=-\sum_{j=k}^{m-1}B_2(m,j)B_1(j,k)$$

from which, on using (22) and (10), we get

$$B_i(p,k)\equiv A_i(p,k)\equiv 0 \pmod p,\quad i=1,2,$$

for each prime p, $k=1,2,\ldots,p-1$.

As examples of integer EGSP's we give, using Proposition 2, the special cases of EGSP's referred to in the present work (§2) for λ, s, θ integers and $r=\pm 1$.

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GENERALIZED PASCAL TRIANGLES AND PYRAMIDS THEIR FRACTALS, GRAPHS, AND APPLICATIONS

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As stated by the author in his preface, this monograph is devoted to the more profound questions connected with the study of the Pascal triangle, and its planar as well as spatial analogs. It also contains an extensive discussion of the divisibility of the binomial, trinomial, and multinomial coefficients by a prime p, as well as the distributions of these coefficients with respect to the modulus p, or p^s, in corresponding arithmetic triangles, pyramids and hyperpyramids. Particular attention is also given to the subject of fractals obtained from the Pascal triangle and other arithmetic triangles. The author also constructs and investigates matrices and determinants whose elements may be binomial, generalized binomial and trinomial coefficients, and other special values. Furthermore, the author pays particular attention to the development of effective combinatorial methods and algorithms for the construction of basis systems of polynomial solutions of partial differential equations, including equations of higher order and with mixed derivatives. The algorithms he proposes are invariant with respect to the order, and the iteration, of operators arising in connection with the differential equations. Finally, the author also discusses non-orthogonal polynomials of binomial type, and polynomials whose coefficients may be Fibonacci, Lucas, Catalan, and other special numbers.

The monograph first published in Russia in 1990 consists of seven chapters, a list of 406 references, an appendix with another 126 references, many illustrations and specific examples. Fundamental results in the book are formulated as theorems and algorithms or as equations and formulas.

The intention of the translator is to make the work of Dr. Bondarenko widely accessible because he feels that Dr. Bondarenko has done the mathematical community a valuable service by writing a useful and interesting compendium of results on Pascal's triangle as well as its ramifications.

The translation of the book is being reproduced and sold with the permission of the author, the translator and the "FAN" Edition of the Academy of Science of the Republic of Uzbekistan. The book, which contains approximately 250 pages, is a paper back with a plastic spiral binding. The cost of the book is $31.00 plus postage and handling where postage and handling will be $6.00 if mailed to anywhere in the United State or Canada, $9.00 by surface mail or $16.00 by airmail to anywhere else. A copy of the book can be purchased by sending a check made out to **THE FIBONACCI ASSOCIATION** for the appropriate amount along with a letter requesting a copy of the book to: **RICHARD VINE, SUBSCRIPTION MANAGER, THE FIBONACCI ASSOCIATION, SANTA CLARA UNIVERSITY, SANTA CLARA, CA 95053.**

MULTINOMIAL AND Q-BINOMIAL COEFFICIENTS MODULO 4 AND MODULO P

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1. INTRODUCTION

Hexel and Sachs [3] examined the n^{th} row of Pascal's triangle and worked out formulas for the number of occurrences of each residue modulo p, where p is any prime. For $p > 3$ the formulas are very involved. Davis & Webb [1] recently considered the same problem modulo 4, and they pointed out that a composite modulus requires an approach different from the one in [3]. To date, 4 is the only composite modulus for which formulas for the number of occurrences of each residue have been obtained. It appears to be very difficult to find results of this type for arbitrary composite moduli.

The purpose of the present paper is to extend the results of [1] and [3] to multinomial and q-binomial coefficients. Thus, in section 3 we examine the q-binomial coefficients $\left[{n \atop r}\right]$, $0 \le r \le n$, and determine the number of occurrences of each residue modulo 4. In section 4 we consider the same problem modulo p, and we obtain explicit formulas for $p = 3$. For $p \ge 3$, we show how formulas can be worked out in terms of the results of [3]. Similarly, in section 6 we examine the multinomial coefficients $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$, and we find formulas that enable us to compute the number of occurrences of each residue modulo 4. In section 7 we consider the same problem modulo p. An explicit formula for $p = 3$ is determined, and formulas for $p \ge 3$ are found in terms of the results of [3]. In sections 2 and 5 we state the basic properties of the q-binomial and multinomial coefficients that we need, and we also explain the notation used in this paper.

2. Q-BINOMIAL COEFFICIENTS: PRELIMINARIES

The q-binomial coefficient is defined by

$$(2.1)\qquad \begin{bmatrix} n \\ r \end{bmatrix} = \prod_{j=1}^{r} \frac{q^{n-j+1}-1}{q^j - 1}$$

for q an indeterminate and n a nonnegative integer. When considering $\left[{n \atop r}\right]$ modulo j, for any j, unless otherwise stated q will always be a rational number, $q = u/v$, with $\gcd(u,v) = \gcd(u,j) = \gcd(v,j) = 1$ The q-binomial coefficient is a polynomial in q, and for $q = 1$ it reduces to the ordinary binomial coefficient. It is clear from (2.1) that, for $r > 0$,

$$(2.2)\qquad \begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix},$$

$$(2.3)\qquad \begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ n-r \end{bmatrix}.$$

As much as possible, we shall use the notation of [1]. Thus, if

$$n=\sum_{i=0}^{k} a_i 2^i \quad (\text{each } a_i = 0 \text{ or } 1), \tag{2.4}$$

we define

$$B(n)=\sum_{i=0}^{k} a_i .$$

Similarly, we define

$$C(n)=\sum_{i=0}^{k} c_i, \quad D(n)=\sum_{i=0}^{k} d_i,$$

where $c_i = 1$ if and only if $a_{i+1}=1$, $a_i=0$, and $d_i=(a_{i+1})(a_i)$. That is, $C(n)$ is the number of "10" blocks and $D(n)$ is the number of "11" blocks in the base 2 representation of n. The same notation was used in [1].

We shall also use the notation

$$\begin{Bmatrix} n \\ r \end{Bmatrix}_w = j \text{ if and only if } \begin{bmatrix} n \\ r \end{bmatrix} \equiv j \pmod{w} \quad (0 \le j \le w-1),$$

and $N_1^{(w)}(q;n)$ is the number of ones, $N_2^{(w)}(q;n)$ is the number of twos, $N_3^{(w)}(q;n)$ is the number of threes, etc., in the set

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix}_w, \begin{Bmatrix} n \\ 1 \end{Bmatrix}_w, \ldots, \begin{Bmatrix} n \\ n \end{Bmatrix}_w .$$

In [2] Fray proved a rule for finding the highest power of a prime p dividing $\begin{bmatrix} n \\ r \end{bmatrix}$. The following lemma is a special case of that rule.

Lemma 2.1: Let p be a prime number and let e be the smallest positive integer such that $q^e \equiv 1 \pmod{p}$. Write n, r, and $n-r$ uniquely as

$$n = a_{-1} + e\cdot a = a_{-1} + e\sum_{i=0}^{k} a_i p^i \quad (0 \le a_{-1} < e,\ 0 \le a_i < p), \tag{2.5}$$

$$r = b_{-1} + e\cdot b = b_{-1} + e\sum_{i=0}^{k} b_i p^i \quad (0 \le b_{-1} < e,\ 0 \le b_i < p), \tag{2.6}$$

$$n-r = w_{-1} + e\sum_{i=0}^{k} w_i p^i \quad (0 \le w_{-1} < e,\ 0 \le w_i < p). \tag{2.7}$$

We can write

$$\begin{aligned} b_{-1}+w_{-1} &= e\varepsilon_0 + a_{-1}, \\ \varepsilon_0 + b_0 + w_0 &= p\varepsilon_1 + a_0, \\ &\cdots \\ \varepsilon_{k-1} + b_{k-1} + w_{k-1} &= p\varepsilon_k + a_{k-1}, \\ \varepsilon_k + b_k + w_k &= a_k, \end{aligned}$$

with each $\varepsilon_i = 0$ or 1. Then $\begin{bmatrix} n \\ r \end{bmatrix}$ is relatively prime to p if and only if $\varepsilon_i = 0$ for each i.

Note that (2.5)-(2.7) are possible by the division algorithm. Also note that when $p = 2$, we have $e = 1$ and $a_{-1} = b_{-1} = w_{-1} = 0$.

Fray [2] also proved the following useful lemma.

Lemma 2.2: Let n and r have expansions (2.5) and (2.6). Then

$$\begin{bmatrix} n \\ r \end{bmatrix} \equiv \begin{bmatrix} a_{-1} \\ b_{-1} \end{bmatrix}\binom{a}{b} \equiv \begin{bmatrix} a_{-1} \\ b_{-1} \end{bmatrix}\binom{a_o}{b_0}\binom{a_1}{b_1}\cdots\binom{a_k}{b_k} \pmod{p}.$$

The second congruence of Lemma 2.2 follows from a well-known theorem of Lucas.

Let $\alpha_j(p;n)$ denote the number of q-binomial coefficients $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$, $r = 0, 1, \ldots, n$, divisible by exactly p^j (that is, divisible by p^j but not by p^{j+1}). Fray [2] proved that if n has expansion (2.5), then

$$\alpha_0(p;n) = (a_{-1}+1)(a_0+1)\cdots(a_k+1).$$

In particular, for $p = 2$, let

$$\alpha_j(n) = \alpha_j(2;n),$$

so

$$\alpha_0(n) = 2^{B(n)}.$$

The writer [4] proved that

$$\alpha_1(n) = \begin{cases} C\left(\dfrac{n-a_0}{2}\right)2^{B(n)-1} & \text{if } q \equiv 3 \pmod{4}, \\ C(n)2^{B(n)-1} & \text{if } q \equiv 1 \pmod{4}. \end{cases} \tag{2.8}$$

3. Q-BINOMIAL COEFFICIENTS MODULO 4

We shall use the notation of section 2, and for convenience we let

$$\begin{Bmatrix} n \\ r \end{Bmatrix} = \begin{Bmatrix} n \\ r \end{Bmatrix}_4,$$

$$N_j(n) = N_4^{(j)}(q;n).$$

Also define

$$N(n) = (N_1(n), N_2(n), N_3(n)).$$

First we take care of some trivial cases. If $q \equiv 0 \pmod{4}$, we see from (2.1) that

$$\begin{Bmatrix} n \\ r \end{Bmatrix} = 1 \quad (r = 0, 1, \ldots, n)$$

If $q \equiv 2 \pmod{4}$, we see from (2.1) that

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{Bmatrix} n \\ n \end{Bmatrix} = 1, \quad \begin{Bmatrix} n \\ r \end{Bmatrix} = 3 \quad (r = 1, \ldots, n-1)$$

If $q \equiv 1 \pmod 4$, then

$$\left\{ {n \atop r} \right\} \equiv \binom{n}{r} \pmod 4,$$

and the results of [1] can be used.

In the remainder of this section we shall assume $q \equiv 3 \pmod 4$. We shall also use the notation of section 2.

We know from (2.3) that

$$N_2(n) = C\left(\frac{n-a_0}{2}\right)2^{B(n)-1}.$$

Note that

$$C\left(\frac{n-a_0}{2}\right) = \begin{cases} C(n) & \text{if } n \not\equiv 2 \pmod 4, \\ C(n)-1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

It is clear from (2.2) that

$$\left\{ {n \atop r} \right\} \equiv \left\{ {n-1 \atop r-1} \right\} + (-1)^r \left\{ {n-1 \atop r} \right\} \pmod 4, \tag{3.1}$$

and the following lemmas are clear from (2.1), (2.5), and Lemma 2.1.

Lemma 3.1: When $k > 1$, $N(2^k) = (2,1,0)$.

Lemma 3.2: Let $n = 2^k + L$, where $0 < L < 2^k$. Then

$$\left\{ {n \atop r} \right\} \equiv 0 \pmod 2 \quad (L < r < 2^k).$$

The analogous results for ordinary binomial coefficients are proved in [1]. By (3.1) and Lemmas 3.1 and 3.2, we see that the q-binomial Pascal triangle modulo 4 for $q \equiv 3 \pmod 4$ has the following form:

$$\begin{array}{lc} & 1 \\ & 1\ 1 \\ & 1\ 0\ 1 \\ & 1\ 1\ 1\ 1 \\ & 1\ 0\ 2\ 0\ 1 \\ & \cdots \\ (2^k \text{ row}) & 1\ 0 \ldots 0\ 2\ 0 \ldots 0\ 1 \\ (2^k+1 \text{ row}) & 1\ 1\ 0 \ldots 0\ 2\ 2\ 0 \ldots 0\ 1\ 1 \end{array}$$

By using (3.1) and comparing this triangle with Pascal's triangle modulo 4 (see [1]), we see that the two triangles satisfy the same recursive relations. That is, in Part 1 and Part 2 of [1], we can replace $\langle \cdots \rangle$ by $\{\cdots\}$. We shall not reproduce all those relations here, but we note the following.

Lemma 3.3: Suppose $n = 2^k + L,\ 0 \le L < 2^k$

(a) If $L < 2^{k-1}$, then

$$\left\{ {n \atop r} \right\} \text{ is } \begin{cases} = \left\{ {L \atop r} \right\} & \text{if } 0 \le r \le L, \\ \equiv 0 \pmod 2 & \text{if } L < r < 2^k. \end{cases}$$

(b) If $2^{k-1} \le L < 2^k$, then

$$\left\{ {n \atop r} \right\} \text{ is } \begin{cases} = \left\{ {L \atop r} \right\} & \text{if } 0 \le r < 2^{k-1}, \\ \equiv \left\{ {L \atop r} \right\} + 2\left\{ {L \atop r - 2^{k-1}} \right\} \pmod 4 & \text{if } 2^{k-1} \le r \le L, \\ \equiv 0 \pmod 2 & \text{if } L < r < 2^k. \end{cases}$$

Because of the symmetry of the triangle, i.e., property (2.3), we now have all the information we need.

Recall that $D(n) > 0$ if and only if the base 2 representation of n has a "11" block.

Theorem 3.1 If $D(n) = 0$, or $n = 3 + 8m$ with $D(m) = 0$, then $N_1(n) = 2^{B(n)}$ and $N_3(n) = 0$.

Proof: We use induction on n. The theorem is true for $n \le 3$; assume it is true for all nonnegative integers less than n. If n satisfies the hypotheses of the theorem, then $n = 2^k + L$ with $L < 2^{k-1}$, and either $D(L) = 0$ or $L = 3 + 8y$ with $D(y) = 0$. Thus,

$$N_1(L) = 2^{B(L)} \text{ and } N_3(L) = 0.$$

Note that $B(n) = B(L) + 1$. We know

$$\left\{ {n \atop r} \right\} \text{ is } \begin{cases} = \left\{ {L \atop r} \right\} & \text{if } 0 \le r \le L, \\ \equiv 0 \pmod 2 & \text{if } L < r < 2^k. \end{cases}$$

Since $\left\{ {n \atop r} \right\} = \left\{ {n \atop n-r} \right\}$ and $2^k > n/2$, we have

$$N_1(n) = 2N_1(L) = 2^{B(n)} \text{ and } N_3(n) = 0.$$

This completes the proof.

Theorem 3.2: If $D(n) > 0$ and $n \ne 3 + 8m$ with $D(m) = 0$, then $N_1(n) = N_3(n) = 2^{B(n)-1}$.

The proof of Theorem 3.2 is the same as the proof of Theorem 6 in [1], with $\langle \cdots \rangle$ replaced by $\{\cdots\}$. We shall not reproduce it here.

In summary we have:

- If $D(n) = 0$ or $n = 3 + 8m$ with $D(m) = 0$, then

$$N(n) = \left(2^{B(n)},\ C\left(\frac{n - a_0}{2}\right) 2^{B(n)-1},\ 0 \right).$$

- If $D(n) > 0$ and $n \neq 3+8m$ with $D(m) = 0$, then

$$N(n) = \left(2^{B(n)-1},\ C\left(\frac{n-a_0}{2}\right)2^{B(n)-1},\ 2^{B(n)-1}\right).$$

4. Q-BINOMIAL COEFFICIENTS MODULO P

In this section we assume p is an odd prime and n has expansion (2.5). We shall use the following notation:

Let A_j be the number of coefficients a_i $(-1 \le i \le k)$ in (2.5) that are equal to j.

Let t be the order of 2 modulo p, and for m > 0 let $t(m)$ be the smallest nonnegative solution x to $2^x \equiv m \pmod p$, if one exists.

Recall that $N_m^{(p)}(q;n)$ is the number of q-binomial coefficients $\left[{n \atop r}\right]$ congruent to m modulo p.

Theorem 4.1 Suppose n has expansion (2.5) with $0 \le a_{-1} \le 1$ and $0 \le a_i \le 2$ for each $i \ge 0$.

(a) If $2^x \equiv m \pmod p$ has no solutions x, then $N_m^{(p)}(q;n) = 0$.

(b) If $2^x \equiv m \pmod p$ has solutions, then

$$N_m^{(p)}(q;n) = 2^{A_1}\sum_{j=0}^{s}\binom{A_2}{t(m)+jt}2^{A_2-t(m)-jt}, \tag{4.1}$$

where $t(m)+st \le A_2 < t(m)+(s+1)t$.

Proof: We see from Lemma 2.2 that to have $\left[{n \atop r}\right] \equiv m \pmod p$ we must have h integers i such that

$$\binom{a_i}{b_i} = \binom{2}{1} \text{ and } 2^h \equiv m \pmod p.$$

Thus, part (a) is clear. Now $2^h \equiv 2^{t(m)} \pmod p$ implies $2h \equiv t(m) \pmod t$, so $h = t(m)+jt$ for some j. There are $\binom{A_2}{h}$ ways to have h terms $\binom{2}{1}$; there are two choices for each of the remaining $A_2 - h$ terms $\binom{2}{b_i}$, namely, $b_i = 0$ or $b_i = 2$; there are two choices for each of the A_1 terms $\binom{1}{b_i}$, namely, $b_i = 0$ or $b_i = 1$. Thus, we have (4.1), and the proof is complete.

Corollary: If $p = 3$ [and thus $q \equiv \pm 1 \pmod 3$], we have

$$N_1^{(3)}(q;n) = \frac{1}{2}\cdot 2^{A_1}(3^{A_2}+1) \quad\text{and}\quad N_2^{(3)}(q;n) = \frac{1}{2}\cdot 2^{A_1}(3^{A_2}-1).$$

Proof: Since $p = 3$, we have all the hypotheses of Theorem 4.1 with $t(1) = 0$, $t(2) = 1$, and $t = 2$. Thus,

$$N_1^{(3)}(q;n) = 2^{A_1}\sum_{j=0}^{s}\binom{A_2}{2j}2^{A_2-2j} = \frac{1}{2}\cdot 2^{A_1}\left[(2+1)^{A_2}+(2-1)^{A_2}\right].$$

The formula for $N_2^{(3)}(q;n)$ is proved in a similar way, thus completing the proof.

By Lemma 2.2, it is clear that

$$N_m^{(p)}(q;n)=\begin{cases} N_m^{(p)}(1;a) & \text{if } a_{-1}=0,\\ 2N_m^{(p)}(1;a) & \text{if } a_{-1}=1,\end{cases}$$

where a is defined by (2.5) and $N_m^{(p)}(1;a)$ is the number of binomial coefficients $\binom{a}{r}$ that are congruent to m modulo p. Thus, when $a_{-1}=0$ or 1, the formulas of [3] can be used to evaluate $N_m^{(p)}(q;n)$. More generally, define $y(r)$ to be the smallest nonnegative solution to

$$\begin{bmatrix} a_{-1} \\ r \end{bmatrix} x \equiv m \pmod{p}.$$

Then the following theorem is clear from Lemma 2.2.

Theorem 4.2: If n has expansion (2.5) and $y(r)$ is defined as above, then

$$N_m^{(p)}(q;n)=\sum_{r=0}^{a_{-1}} N_{y(r)}^{(p)}(1;a).$$

For example, let $p=5$ and $q\equiv 3 \pmod 5$, so $e=4$. We have

$$N_1^{(5)}(3;n)=\begin{cases} N_1^{(5)}(1;a) & \text{if } n\equiv 0 \pmod 4,\\ 2N_1^{(5)}(1;a) & \text{if } n\equiv 1 \pmod 4,\\ 2N_1^{(5)}(1;a)+N_4^{(5)}(1;a) & \text{if } n\equiv 2 \pmod 4,\\ 2N_1^{(5)}(1;a)+2N_2^{(5)}(1;a) & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

Theoretically, then, we can evaluate $N_m^{(p)}(q;n)$ for any q by using Theorem 4.2 and the formulas of [3].

For completeness, we note that for $p=2$ and $q\not\equiv 0 \pmod 2$ we have

$$N_1^{(2)}(q;n)=(a_0+1)(a_1+1)\cdots(a_k+1)=2^{B(n)},$$

where n has expansion (2.4).

5. MULTINOMIAL COEFFICIENTS: PRELIMINARIES

The multinomial coefficient is defined by

$$(5.1)\qquad (n_1,n_2,\ldots,n_r)=\frac{n!}{n_1!n_2!\cdots n_r!}\quad (n_1+\cdots+n_r=n).$$

Obviously (5.1) reduces to the ordinary binomial coefficient for $r=2$. In this paper we consider (5.1) for all compositions (ordered partitions) of n into r parts. The order of the terms $n_1,\ldots,n_r$ is important; we are distinguishing between (0, 0, 1, 2) and (1, 0, 0, 2), for example. Note that 0 can be one or more of the parts. It is well known that the number of compositions of n into r parts is $\binom{n+r-1}{n}$.

Fray [2] proved the following rule for determining the highest power of a prime p dividing $(n_1,\ldots,n_r)$.

Lemma 5.1: Let n have base p representation

$$(5.2) \qquad n=\sum_{j=0}^{k} a_j p^j \quad (0 \le a_j < p)$$

and let $n = n_1 + n_2 + \cdots + n_r$. For $i = 1, \ldots, r$, let $n_i = \sum_{j=0}^{k} a_{i,j} p^j \quad (0 \le a_{i,j} < p)$. If

$$\begin{aligned} a_{1,0} + \cdots + a_{r,0} &= p\varepsilon_0 + a_0, \\ \varepsilon_0 + a_{1,1} + \cdots + a_{r,1} &= p\varepsilon_1 + a_1, \\ &\vdots \\ \varepsilon_{k-1} + a_{1,k} + \cdots + a_{r,k} &= a_k, \end{aligned}$$

where each $\varepsilon_i = 0, 1, \ldots,$ or $r-1$. Then the highest power of p dividing $(n_1, n_2, \ldots, n_r)$ is p^s, where $s = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{k-1}$.

We shall use the notation $B(n)$, $C(n)$, and $D(n)$ given in section 2.

Let n have base p expansion (5.2) and let $\theta_j^{(p)}(r; n)$ be the number of multinomial coefficients $(n_1, n_2, \ldots, n_r)$ divisible by exactly p^j. The writer [5] proved

$$\theta_0^{(p)}(r;n) = \binom{a_0 + r - 1}{r-1}\binom{a_1 + r - 1}{r-1}\cdots\binom{a_k + r - 1}{r-1}.$$

For $p = 2$, we have

$$(5.3) \qquad \theta_0^{(2)}(r;n) = r^{B(n)},$$

and the writer [4] proved

$$(5.4) \qquad \theta_1^{(2)}(r;n) = C(n)\binom{r}{2} r^{B(n)-1} + D(n)\binom{r}{3} r^{B(n)-2}.$$

For $w = 0, 1, \ldots, j-1$, we define $N_{r,w}^{(j)}(n)$ as the number of multinomial coefficients $(n_1, n_2, \ldots, n_r)$ such that $(n_1, n_2, \ldots, n_r) \equiv w \pmod{j}$.

6. MULTINOMIAL COEFFICIENTS MODULO 4

The notation for this section comes from sections 2 and 5. For convenience we shall use

$$N_{r,w}(n) = N_{r,w}^{(4)}(n),$$

so that

$$(6.1) \qquad N_{r,2}(n) = \theta_1^{(2)}(r;n)$$

and

$$N_{r,0}(n) = \binom{n+r-1}{n} - N_{r,1}(n) - N_{r,2}(n) - N_{r,3}(n).$$

We also define

$$N_r(n) = \left(N_{r,1}(n), N_{r,2}(n), N_{r,3}(n)\right).$$

By (5.4) and (6.1), the only problem, then, is to find $N_{r,1}(n)$ and $N_{r,3}(n)$.

Theorem 6.1: If $D(n) = 0$, then $N_{r,1}(n) = r^{B(n)}$ and $N_{r,3}(n) = 0$.

Proof: Suppose $D(n)=0$ and $n=n_1+n_2+\cdots+n_r$. If $(n_1,n_2,\ldots,n_r)\not\equiv 0 \pmod 2$, then by Lemma 5.1 we know if $a_j=0$ then $a_{i,j}=0$ for $i=1,\ldots,r$. Also, if $a_j=1$ then $a_{i,j}=1$ for exactly one i. Since

$$(6.2)\qquad (n_1,n_2,\ldots,n_r)=\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots,$$

and since $D(n-n_1-\cdots-n_j)=0$ for $j=1,\ldots,r$, we see that none of the binomial coefficients on the right side of (6.2) is congruent to 3 modulo 4. Thus, $(n_1,n_2,\ldots,n_r)\equiv 1 \pmod 4$, and the proof is complete.

Hence, if $D(n)=0$, we have

$$N_r(n)=\left(r^{B(n)},\,C(n)\binom{r}{2}r^{B(n)-1},\,0\right).$$

The situation is much harder if $D(n)>0$. We shall use the following notation:

$$f_j(n,i)=\begin{cases}1 & \text{if } \binom{n}{i}\equiv j \pmod 4,\\ 0 & \text{otherwise.}\end{cases}$$

Since

$$(n_1,n_2,\ldots,n_r)=\binom{n}{n_1}(n_2,\ldots,n_r),$$

we have

$$N_{r,1}(n)=\sum_{i=0}^{n}\left[f_1(n,i)\cdot N_{r-1,1}(n-i)+f_3(n,i)\cdot N_{r-1,3}(n-i)\right].$$

We refine this in the next theorem by using the facts that if $(n_1,n_2,\ldots,n_r)\not\equiv 0$ (mod 2), then in Lemma 5.1 each $\varepsilon_i=0$, and

$$N_{r,1}(n)+N_{r,3}(n)=r^{B(n)}.$$

Theorem 6.2: Let n have base 2 representation (2.4). Then, for $r\geq 3$,

$$N_{r,1}(n)=N_{r-1,1}(n)+1+\sum_i (r-1)^{B(n-i)}f_3(n,i)+\sum_i\left[f_1(n,i)-f_3(n,i)\right]N_{r-1,1}(n-i),$$

where each sum is over all integers i such that $0<i<n$ and

$$i=\sum_{j=0}^{k}e_j2^j\quad (0\leq e_j\leq a_j)$$

To illustrate Theorem 6.2, suppose

$$(6.3)\qquad n=2^{k-1}+2^k.$$

Then, using Theorem 6.1 and the results of [1], we know $f_3(n,i)=1$ for $i=2^{k-1}$ and $i=2^k$, and

$$N_{r-1,1}(2^{k-1})=N_{r-1,1}(2^k)=r-1.$$

Theorem 6.2 gives us

$$N_{r,1}(n) = N_{r-1,1}(n) + 1 + 2(r-1) - 2(r-1) = N_{r-1,1}(n) + 1.$$

Thus, if n is given by (6.3), we have

$$(6.4) \qquad N_{r,1}(n) = r, \quad N_{r,3}(n) = r^2 - r.$$

Now consider

$$(6.5) \qquad n = 2^s + 2^{k-1} + 2^k \quad (0 \le s < k-2).$$

Then, using [1] and (6.4), we have

$$\begin{aligned} f_1(n,i) &= 1 && \text{for } i = 2^s \text{ and } i = 2^{k-1} + 2^k, \\ f_3(n,i) &= 1 && \text{for } i = 2^{k-1}, 2^k, 2^s + 2^{k-1}, \text{ and } 2^s + 2^k, \\ N_{r-1,1}(x) &= \begin{cases} (r-1)^2 & \text{for } x = 2^s + 2^{k-1} \text{ and } x = 2^s + 2^k, \\ r-1 & \text{for } x = 2^{k-1} + 2^k, 2^s, 2^{k-1}, \text{ and } 2^k. \end{cases} \end{aligned}$$

Theorem 6.2 gives us $N_{r,1}(n) = N_{r-1,1}(n) + 2r - 1$. Thus, if n is given by (6.5), we have

$$N_{r,1}(n) = r^2, \quad N_{r,3}(n) = r^3 - r^2.$$

Using this method on the other cases of $B(n) = 3$, we can prove the following.

Theorem 6.3: Suppose $B(n) = 3$ and $D(n) > 0$. Then

$$N_r(n) = \begin{cases} \left(r^2, C(n)\binom{r}{2}r^2 + \binom{r}{3}r, \; r^3 - r^2\right) & \text{if } D(n) = 1, \\ \left(r^3 - 2r^2 + 2r, C(n)\binom{r}{2}r^2 + 2\binom{r}{3}r, \; 2r^2 - 2r\right) & \text{if } D(n) = 2. \end{cases}$$

We could next look at the case $B(n) = 4$ and get similar results. In general, after examining the case $B(n) = j$, we can move to the case $B(n) = j + 1$. As j increases, the formulas become much more complicated.

7. MULTINOMIAL COEFFICIENTS MODULO P

Let p be an odd prime and recall that $N_{r,m}^{(p)}(n)$ is the number of multinomial coefficients $(n_1, n_2, \dots, n_r)$ such that $(n_1, n_2, \dots, n_r) \equiv m \pmod{p}$.

Let n have base p expansion (5.2) and let A_j be the number of coefficients a_i $(0 \le a_i \le k)$ that are equal to j.

We shall use the definitions of t and $t(m)$ given at the beginning of section 4.

Theorem 7.1: Let n have expansion (5.2) and suppose that $0 \le a_i \le 2$ for each a_i. Let m be a positive integer.

(a) If there are no solutions to $2^x \equiv m \pmod{p}$, then $N_{r,m}^{(p)}(n) = 0$.

(b) If there are solutions to $2^x \equiv m \pmod{p}$, then

(7.1) $$N_{r,m}^{(p)}(n) = r^{A_1}\sum_{j=0}^{s}\binom{A_2}{t(m)+jt}\binom{r}{2}^{t(m)+jt} r^{A_2-t(m)-jt},$$

where $t(m)+st \le A_2 < t(m)+(s+1)t$.

Proof: Suppose $(n_1, n_2, \dots, n_r) \equiv m \pmod{p}$. Since $m > 0$, in Lemma 5.1 we must have $\varepsilon_i = 0$ for $i = 0, 1, \dots, k-1$. In (6.2) we see that each binomial coefficient on the right side will be congruent to 2^w modulo p for some $w \ge 0$, and we must have

(7.2) $h = \Sigma(w)$ and $2^h \equiv m \pmod{p}$.

Thus, part (a) is clear. We now count the number of ways (7.2) can happen. Pick h of the A_2 rows adding up to 2, and pick two positions in each of these rows for 1's. There are $\binom{A_2}{h}\binom{r}{2}^h$ ways of doing this. In the remaining $A_2 - h$ rows, pick one position in each row for a 2. There are r^{A_2-h} ways of doing this. We see from the last part of Lemma 2.2 that when the binomial coefficients on the right side of (6.2) are broken down in terms of their coefficients modulo p, then we have

$$(n_1, n_2, \dots, n_r) \equiv 2^h \equiv m \pmod{p}.$$

As we saw in the proof of Theorem 4.1, $h = t(m) + jt$ for some j, and (7.1) follows. This completes the proof.

Corollary: Let $p = 3$. Then

$$N_{r,1}^{(3)}(n) = \frac{1}{2}\cdot r^{A_1}\left[\left(r+\binom{r}{2}\right)^{A_2} + \left(r-\binom{r}{2}\right)^{A_2}\right],$$

$$N_{r,2}^{(3)}(n) = \frac{1}{2}\cdot r^{A_1}\left[\left(r+\binom{r}{2}\right)^{A_2} - \left(r-\binom{r}{2}\right)^{A_2}\right].$$

We now prove a theorem analogous to Theorem 6.2. It follows immediately from

$$(n_1, n_2, \dots, n_r) = \binom{n}{n_1}(n_2, \dots, n_r).$$

Theorem 7.2 let m be a positive integer and suppose $\binom{n}{j} \not\equiv 0 \pmod{p}$. Let $g(j)$ be the smallest positive integer such that $\binom{n}{j}\cdot g(j) \equiv m \pmod{p}$. Then

$$N_{r,m}^{(p)}(n) = \sum_j N_{r-1,g(j)}^{(p)}(n-j),$$

where the sum is over all j such that $0 \le j \le n$ and $\binom{n}{j} \not\equiv 0 \pmod{p}$.

If n has the base p expansion (5.2), then in Theorem 7.2 the sum is over all j such that

$$j = \sum_{i=0}^{k} e_i p^i \quad (0 \le e_i \le a_i).$$

For example, let $p=5$, $r=3$, $n=11=1+2\cdot 5$. Then

$$\begin{aligned} N_{3,1}^{(5)}(11) &= N_{2,1}^{(5)}(11)+N_{2,1}^{(5)}(10)+N_{2,3}^{(5)}(6)+N_{2,3}^{(5)}(5)+N_{2,1}^{(5)}(1)+N_{2,1}^{(5)}(0) \\ &= 4+2+0+0+2+1=9. \end{aligned}$$

Similarly, we can show that $N_{3,2}^{(5)}(11)=9$.

Theoretically, then, if we know the values of $N_{2,m}^{(p)}(n)$, we can use Theorem 7.2 to find $N_{r,m}^{(p)}(n)$ for any r.

For completeness, we can use (5.3) to obtain $N_{r,m}^{(2)}(1)=r^{B(n)}$.

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FIBONACCI CUBES—A CLASS OF SELF-SIMILAR GRAPHS

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1. INTRODUCTION

The Fibonacci cube [6] is a new class of graphs that are inspired by the famous numbers. Because of the rich properties of the Fibonacci numbers [1], the graph also shows interesting properties. For a graph with N nodes, it is known [6] that the diameter, the edge connectivity, and the node connectivity of the Fibonacci cube are in the order of $O(\log N)$, which are similar to the Boolean cube (or hypercube, n-cube, cosmic cube [9]). A possible application of the Fibonacci cube is in the interconnection of large-scale multi-computers or distributed networks. Here we show that the Fibonacci cube has attractive recurrent structures (called *self-similarity*, §2) in the following sense:

1. A Fibonacci cube can be decomposed into subgraphs which are also Fibonacci cubes by themselves;
2. By suitably defining equivalence classes of vertices in the Fibonacci cube and merging the edges between vertices in different classes in a natural fashion, the resulting graph (of the equivalence classes) is again a Fibonacci cube.

This structural recurrence is useful to derive (divide-and-conquer) algorithms for a parallel computer based on the Fibonacci cube [6]. It is also useful to derive the embeddings of other types of graphs [8]. (See also §4 for discussions.)

This paper is organized as follows. Section 2 defines the Fibonacci cube based on the Fibonacci representation of integers. Section 3 provides a characterization of the new graph and discusses various decompositions. Section 4 briefly summarizes the results that are presented and discusses possible applications. The rest of Section 1 lists notations to be used throughout this paper.

A graph G is a pair (V, E), where V denotes the set of *vertices* (or, alternatively, *nodes*) and E the set of edges. The following terminology and notations will be used [3]:

- We write $G_2 \subseteq G_1$ (or, alternatively, $G_1 \supseteq G_2$) if G_2 is a subgraph of G_1. Write $G_1 \cong G_2$ if the two graphs are *isomorphic*.
- A subgraph of a graph $G = (V, E)$ *induced* by a subset of its vertices, $V' \subseteq V$, is the graph (V', E'). where $E' = \{(i, j) \in E : i, j \in V'\}$
- We write $G_1 \cup G_2$ to denote the graph $(V_1 \cup V_2, E_1 \cup E_2)$, and $G_1 \cap G_2$ to denote $(V_1 \cap V_2, E_1 \cap E_2)$, and $\bigcup_{1=i}^{m} G_i = G_1 \cup G_2 \cup \cdots \cup G_m$.
- If $G_2 \cap G_{3'} = (\phi, \phi)$, i.e., they are disjoint, then we write $G_1 = G_2 \uplus G_3$ instead of $G_2 \cup G_3$) to emphasize that G_1 consists of two disjoint subgraphs. Also, for convenience, write $\biguplus_{1=i}^{m} G_1 = m \cdot G$ if the graphs are all isomorphic, i.e., $G_i \cong G$ for $1 \le i \le m$.

2. DEFINITION OF FIBONACCI CUBE

The Fibonacci cube can be defined by using the Fibonacci representation of integers.

Definition: Assume that i is an integer, and $0 \le i < F_n$, where $n \ge 3$. The *order-n Fibonacci code* (or, simply, *Fibonacci code*, if n is implicit) of i is a sequence of $n-2$ binary digits $(b_{n-1}, \ldots, b_3, b_2)_F$, where

1. $b_j \cdot b_{j+1} = 0$ for $2 \le j \le (n-2)$, and

2. $i = \sum_{j=2}^{n-1} b_j \cdot F_j$.

Example: By *Zeckendorf's theorem* [10], any natural number can be uniquely represented in its Fibonacci code. The Fibonacci representation of an integer $N > 0$ can be obtained by using the following *greedy* approach [4]. First find the greatest F_k that is less than or equal to N, assign a "1" to the bit that corresponds to F_k, then proceed recursively for $N - F_k$ until the remainder is 0. The unassigned bits are 0's. Here the integers from 1 to 20 are given in this notation:

$0 = (000000)_F$, $1 = (000001)_F$, $2 = (000010)_F$, $3 = (000100)_F$, $4 = (000101)_F$, $5 = (001000)_F$, $6 = (001001)_F$, $7 = (001010)_F$, $8 = (010000)_F$, $9 = (010001)_F$, $10 = (010010)_F$, $11 = (010100)_F$, $12 = (010101)_F$, $13 = (100000)_F$, $14 = (100001)_F$, $15 = (100010)_F$, $16 = (100100)_F$, $17 = (100101)_F$, $18 = (101000)_F$, $19 = (101001)_F$, $20 = (101010)_F$.

Remarks: Notice that in the Fibonacci code, the rightmost bit corresponds to F_2. rather than F_1. Note also that no consecutive 1's appeared in the Fibonacci codes; to represent a number between 0 and $F_n - 1$ requires $n-2$ bits. Therefore, to represent the number $21 = (1000000)_F$ requires an additional bit (cf. the preceding example). □

Let $I = (b_{n-1}, \ldots, b_3, b_2)$ and $J = (c_{n-1}, \ldots, c_3, c_2)$ denote two sequences of 0's and 1's. The *Hamming distance* between I and J, denoted by $H(I, J)$, is the number of bits where the two sequences differ.

Definition [Fibonacci Cube of Order n]: Let $F(i)$ denote the Fibonacci code of i. The *Fibonacci cube of order n*, denoted by Γ_n, is a graph (V_n, E_n), where $V_n = \{0, 1, \ldots, F_n - 1\}$ and $E_n = \{(i, j) : H(F(i), F(j)) = 1,\ 0 \le i, j \le F_n - 1\}$. Define $\Gamma_0 = (\phi, \phi)$. □

Figure 1 shows the Fibonacci cubes Γ_i for $1 \le i \le 7$.

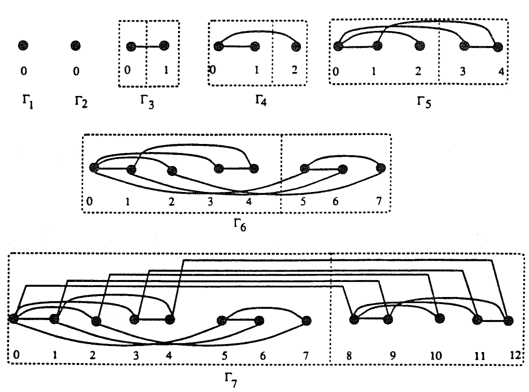


FIGURE 1. Fibonacci Cubes

Remarks:

1. The definition of Fibonacci cube parallels that of Boolean cube (hypercube). Specifically, the *Boolean cube of dimension n*, denoted by B_n, is a graph (V_n, E_n), where $V_n = \{0, 1, \ldots, 2^n - 1\}$ and $(i, j) \in E_n$ if and only if $H(I_B, J_B) = 1$, where I_B and J_B denote the (ordinary) binary representation of i and j, $0 \le i,\ j \le 2^n - 1$ (Fig. 2).

2. The preceding definition of the Fibonacci cube can be modified to accommodate a Fibonacci cube of *size* (i.e., number of nodes) N for an arbitrary integer $N \ge 1$ [6]. However, as we will see, when the size of the cube is a Fibonacci number, the Fibonacci cube has a recurrent structure and hence is more desirable.

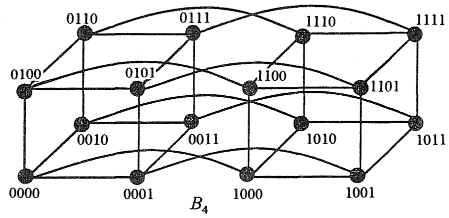


FIGURE 2. A Boolean Cube

3. RECURSIVE DECOMPOSITIONS OF THE FIBONACCI CUBE

In [6] it is shown that the *Fibonacci cube of order n*, where $n \ge 2$, contains two disjoint subgraphs that are isomorphic to Γ_{n-1} and Γ_{n-2}, respectively (with proper renaming of the vertices in Γ_{n-2}); moreover, there are exactly F_{n-2} edges linking the two subgraphs together.

Theorem 1 (Characterization of the Fibonacci Cube): Let $\Gamma_n = (V_n, E_n)$ denote the Fibonacci cube of order n, where $n \ge 2$. Let LOW(n) (resp., HIGH (n)) denote the subgraph induced by the set of nodes in $\{0, 1, \ldots, F_{n-1} - 1\}$ (resp., $\{F_{n-1}, \ldots, F_n - 1]$). Then

1. $\mathrm{LOW}(n) \cong \Gamma_{n-1}$ and $HIGH(n) \cong \Gamma_{n-2}$;

2. Let $0 \le i \le F_{n-1} - 1$ and $F_{n-1} \le j \le F_n - 1$. $(i, j) \in \Gamma_n$ if and only if $j - i = F_{n-1}$.

Proof: (We refer to [6].) □

Example: (Fig. 1). Γ_6 can be decomposed into two subgraphs that are isomorphic to Γ_5 and Γ_4, respectively. There are $F_4 = 3$ edges connecting the two subgraphs.

The above characterization can be expressed in terms of Fibonacci codes.

Corollary 1: Assume that $n \ge 2$. Let G_0 (resp., G_1) denote the subgraph of Γ_n induced by the set of vertices $\{i : i = (0b_{n-2}b_{n-3} \ldots b_2)_F\}$ (resp., $\{j : j = (1b_{n-2}b_{n-3} \ldots b_2)_F\}$). Then

1. $G_0 \cong \Gamma_{n-1}$ and $G_1 \cong \Gamma_{n-2}$.

2. Let $i = (0b_{n-2}b_{n-3} \ldots b_2)_F \in G_0$ and $j = (1b'_{n-1}b'_{n-2} \ldots b'_2)_F \in G_1$. $(i, j) \in \Gamma_n$ if and only if $b_k = b'_k$ for $n - 2 \ge k \ge 2$.

Proof (outlines):

Statement 1: Let $i = (b_{n-1} \ldots b_2)_F$. There are two cases:

1. $0 \le i < F_{n-1}$, in this case $b_{n-1} = 0$.

2. $F_{n-1} \le i < F_n$, in this case $b_{n-1} = 1$.

The result then follows by observing that $G_0 = \text{LOW}(n)$ and $G_1 = \text{HIGH}(n)$.

Statement 2 follows from Statement 2 of Theorem 1. □

3.1 A Generalization

A generalization of Theorem 1 can be obtained by applying the decomposition recursively. Recall that $\biguplus_{1=i}^{m} G_i = m \cdot G$ if $G_i \cong G$ for all $1 \le i \le m$.

Example: In Figure 1 we see that Γ_6 contains a subgraph Γ_5 and a subgraph Γ_4 (after renaming vertices). Since Γ_5 can be decomposed into a subgraph Γ_4 and a subgraph Γ_3, so Γ_6 contains two disjoint Γ_4 and one Γ_3. Using the notations introduced, we will write $\Gamma_6 \supseteq (2 \cdot \Gamma_4 \uplus \Gamma_3)$.

Theorem 2: Assume that $2 \le k \le n$. The Fibonacci cube of order n (Γ_n) admits the following decompositions:

(a) $\Gamma_n \supseteq (F_k \cdot \Gamma_{n-k+1} \uplus F_{k-1} \cdot \Gamma_{n-k})$;

(b) $\Gamma_n \supseteq (F_{n-k+1} \cdot \Gamma_k \uplus F_{n-k} \cdot \Gamma_{k-1})$.

Proof: We will prove Statement (a) by induction on n.

(Basis) If $n = 2$, then $k = 2$ and the statement can be easily verified.

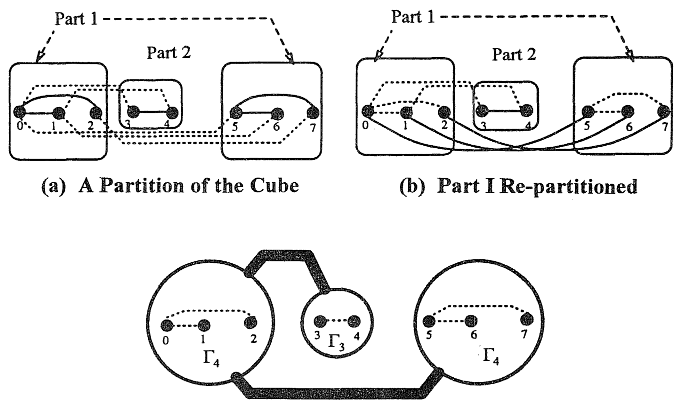
(Hypothesis) Assume that the statement is true for $n \le N$.

(Induction) Consider the case $n = N+1$. By Theorem 1, Γ_{N+1} consists of one Γ_N and one Γ_{N-1}. By hypothesis, for any k between 2 and $N-1$, Γ_N (resp., Γ_{N-1}) may be divided into F_k copies of Γ_{N-k+1} and F_{k-1} copies of Γ_{N-k} (resp., F_{k-1} copies of $\Gamma_{(N-1)-(k-1)+1}$ and F_{k-2} copies of $\Gamma_{(N-1)-(k-1)}$). Together, the number of copies of $\Gamma_{(N+1)-(k+1)+1}$ is $F_k + F_{k-1} = F_{k+1}$ and the number of copies of $\Gamma_{(N+1)-(k+1)}$ is $F_{k-1} + F_{k-2} = F_k = F_{(k+1)-1}$, which completes the proof in the case $3 \le k \le N$. The case $k = 2$, $N+1$ can be easily verified.

Statement (b) can be proved similarly [6]. □

Remarks: Note that the decompositions listed in the preceding lemma are based on the following property of Fibonacci numbers: $F_n = F_k F_{n-k+1} + F_{k-1} \cdot F_{n-k}$, which holds true for *all* integers k and n [4]. In Theorem 2, the first term $F_k \cdot F_{n-k+1}$ corresponds to a subgraph of Γ_n which is either divided into (i) F_k copies of Γ_{n-k+1} or (ii) F_{n-k+1} copies of Γ_k. (The second term $F_{k-1} \cdot F_{n-k}$ also suggests two possible decompositions.) Note that the decomposition in (ii) can be derived from (i) as follows. [Constructing (i) from (ii) is similar.] Each subgraph Γ_k. of (ii) is essentially constructed by taking one node from each of the F_k copies of Γ_{n-k+1} in (i). By construction, no two subgraphs from the two decompositions in (i) and (ii) share more than one node. Such decompositions will be referred to as *orthogonal decompositions.* □

Example: Take Γ_6 (Fig. 3) for instance. Let $k = 3$ and note that $8 = F_6 = F_3 \cdot F_4 + F_2 \cdot F_3 = 2 \cdot 3 + 1 \cdot 2$. By Theorem 2, Γ_6 can be decomposed into (Part 1) two copies of Γ_4 and (Part 2) one copy of Γ_3. In Figure 3(a), Part 1 consists of two subgraphs whose vertex sets are, respectively, $\{0, 1, 2\}$ and $\{5, 6, 7\}$. Note that, by Theorem 2, an alternative (and orthogonal) decomposition of Part 1 would be to divide the same set of nodes into three copies of Γ_3, where each Γ_3 is formed by taking one node from each copy of Γ_4. In Figure 3(b), for example, the nodes in Part 1 are re-partitioned into the following sets $\{0,5\}$, $\{1,6\}$, and $\{2,7\}$. Notice that no two subgraphs from the first partition and the second partition share more than one common node. Thus, the two partitions of Part 1 are orthogonal. Similarly, nodes in Part 2 can be redivided into two copies of Γ_1. □



(a) A Partition of the Cube

(b) Part I Re-partitioned

(c) Deriving a Quotient Fibonacci Cube

FIGURE 3. Decomposition of a Fibonacci Cube

In terms of the Fibonacci codes, we have the following corollary.

Corollary 2: Assume that $n \geq 2$. Let k and d denote two integers, where $1 \leq k \leq n-2$ and $0 \leq d \leq F_{n-k}-1$. Let $G_n(k,d)$ denote the subgraph of Γ_n induced by the set of vertices in $\{(b_{n-1}b_{n-2}\ldots b_2)_F : (b_{n-1}b_{n-2}\ldots b_{n-k})_F = d\}$. Then

1. $G_n(k,d) \cong \Gamma_{n-k}$ if $b_{n-k}=0$, and

2. $G_n(k,d) \cong \Gamma_{n-k-1}$ if $b_{n-k}=1$.

Proof: The argument parallels that of Theorem 2 (replace all instances of Theorem 1 with Corollary 1) [8]. □

3.2 Quotient Fibonacci Cubes

We will identify another level of recurrence with the decompositions of the Fibonacci cube in which the graph Γ_n is scaled down to a smaller Fibonacci cube. In fact, for any Γ_n and any integer k, where $1 \leq k \leq n-2$, we can define a *Quotient Fibonacci Cube* Γ_n / k as described in the following. (See [2]; cf. Theorem 2.)

We describe the idea in intuitive terms followed by a formal definition. Consider the first decomposition [i.e., Decomposition (a)] listed in Theorem 2. Let each of the $F_{k+1}=F_k+F_{k-1}$ subgraphs (Γ_{n-k+1} or Γ_{n-k}) be considered as an equivalence class. Then each node v in Γ_n / k corresponds to such an equivalence class. The edges between two equivalence classes v_1 and v_2 are given by $(v_1, v_2) \in \Gamma_n / k$ if and only if $\{(v_1, v_2): v_1 \in v_1 \text{ and } v_2 \in v_2\} \neq \phi$. (In other words, the edges connecting nodes in two subgraphs are merged into one.) Then the resulting graph Γ_n / k of the equivalence classes is itself a Fibonacci cube (as will be proved). A similar observation applies to Decomposition (b).

Example: Consider Γ_6 again [cf. Fig. 3(c)]. By considering each of the two copies of Γ_4 (indicated as Part 1 in Fig. 3) and the Γ_3 (Part 2) as a single node (an equivalence class) then merging the edges connecting them (as described in the preceding remarks), the resulting graph is isomorphic to Γ_4.

Definition Assume that $1 \le k \le n-2$ and $0 \le d \le F_{k+2}-1$. Let $G_n(k,d)$ denote the subgraph of Γ_n induced by the set of vertices in $\{(b_{n-1}b_{n-2}\ldots b_2)_F : (b_{n-1}b_{n-2}\ldots b_{n-k})_F = d\}$. Then the *Quotient Fibonacci Cube* $\Gamma_n/k = (V_n/k, E_n/k)$ is given by:

1. $(V_n/k = \{G_n(k,d) : 0 \le d \le F_{k+2}-1\}$, and

2. $(G_n(k,d),\ G_n(k,d')) \in E_n/k$ if and only if $d \ne d'$ and $\{(v_1, v_2) \in \Gamma_n : v_1 \in G_n(k,d),\ v_2 \in G_n(k,d')\} \ne \phi$

Theorem 3: Let Γ_n/k be the quotient Fibonacci cube as defined before, where $1 \le k \le n-2$. Then $\Gamma_n/k \cong \Gamma_{k+2}$.

Proof (outlines): It is straightforward to verify the theorem for $k = 1$. For example, in Theorem 1 (which corresponds to the case in which $k = 1$), the vertices in LOW(n) [resp., HIGH (n)] can be taken as an equivalence class v_1 (resp., v_2), and edges connecting LOW(n) and HIGH(n) can be taken as a single edge (v_1, v_2). The resulting graph $\Gamma_n/1 = (\{v_1, v_2\}, \{(v_1, v_2)\})$ is isomorphic to Γ_3.

The general case can be proved inductively by noting that each of the subgraphs can be decomposed recursively and there are links between these subgraphs (Theorem 1). □

Example: Figure 4 shows that $\Gamma_n/4$ can be derived from Γ_n in four refining steps. In the first step (when $k = 1$) decomposing Γ_n into (a) Γ_{n-1} and (b) Γ_{n-2}. By interpreting the edges between Γ_{n-1} and Γ_{n-2} as a single edge, the resulting graph is $\Gamma_n/1$, which is isomorphic to Γ_3 (cf. Fig. 1). In the subsequent steps (k = 2, 3, 4), Part (a) and Part (b) are recursively decomposed. The resulting graph $\Gamma_n/4$ is isomorphic to Γ_6 (cf. Fig. 1).

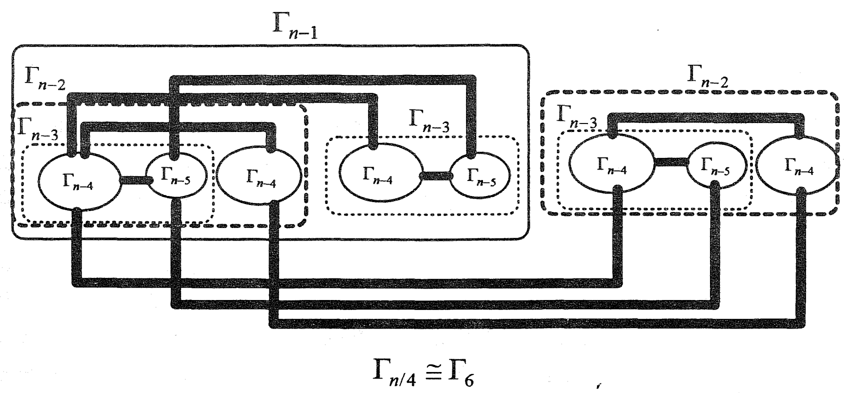


FIGURE 4. A Quotient Fibonacci Cube

3.3 Other Recursive Decompositions of Fibonacci Cubes

There are other conceivable ways to decompose the Fibonacci cube. We list the following while omitting the details of their proofs.

Lemma 1: Let Γ_n denote the Fibonacci cube of order n. Assume that $n \geq 1$. Then

(a) $\Gamma_{2n} \supseteq (G'_1 \uplus G'_3 \uplus G'_5 \uplus \cdots \uplus G'_{2n-1})$, where $G'_k \cong \Gamma_k$ for $k = 1, 3, 5, \ldots, (2n-1)$.

(b) $\Gamma_{n+2} \supseteq (G'_1 \uplus G'_2 \uplus G'_3 \uplus \cdots \uplus G'_n)$, where $G'_k \cong \Gamma_k$ for $k = 1, 2, 3, \ldots, n$.

Proof (outlines): Part (a) of this lemma is inspired by the known recurrence of Fibonacci numbers: $F_{2n} = \sum_{1 \leq k \leq n} F_{2k-1}$. Specifically, the graph Γ_{2n} can be decomposed into a copy of Γ_{2n-1} and a copy of Γ_{2n-2}. The latter can be decomposed further into Γ_{2n-3} and Γ_{2n-4}. Decompose Γ_{2n-4} again and we have Γ_{2n-5} and so on.

Part (b) is based on $F_{n+2} = \sum_{1 \leq k \leq n} F_k + 1$. □

Example Again consider Γ_6. Since $6 = 2 \cdot 3$, by Lemma 1(a), it can be decomposed into three subgraphs: Γ_1, Γ_3, and Γ_5. Also, since $6 = 4 + 2$, by Lemma 1(b), it can be decomposed into one Γ_1, one Γ_2, one Γ_3, and one Γ_4, and all of the subgraphs are disjoint.

4. DISCUSSION AND CONCLUSION

A possible application of the Fibonacci cube is in the interconnection of large-scale multicomputers, where a node corresponds to a processor and an edge to a communication link. In [6], it is shown that the Fibonacci cube contains about 1/5 fewer edges than the Boolean cube for the same number of vertices. Considering the relative sparsity in connections and the asymmetry in structure, it may well be expected that the Fibonacci cube cannot be as flexible as the Boolean cube, and certain functionality may be lost. For example, in the context of interconnection networks, the communication delays may become greater than that based on the Boolean cube, and the power of *embedding* (i.e., emulating other types of graphs) may be inferior to the Boolean cube. Nevertheless, because of the rich properties of Fibonacci numbers, we have been able to show here that the Fibonacci cubes can be flexibly decomposed into subgraphs of same kind (we are tempted to call this property *self-similarity*). In [8], by using these recursive decompositions, it is shown that the Fibonacci cube is flexible enough to embed common graphs such as linear arrays, rings, certain kinds of meshes, tori (mesh with wraparound), and trees, all with perfect dilation and expansion.

The recursive nature of the Fibonacci cube also has implications to the design and analysis of algorithms for parallel computers that are based on the Fibonacci cube. For example, to find the sum (product, maximum, and other associative operations) of a sequence of numbers, the data items can be distributed on the nodes (processors) of the Fibonacci cube. The sum can be found in a divide-and-conquer fashion, which matches well with the recursive decomposition of the graph. In [6], by using this approach, several routing algorithms have been designed for computer architectures based on the Fibonacci cube.

Perhaps the most interesting (and plausible) application of the self-similarity is in fault-tolerant computing. Again consider a parallel computer based on the Fibonacci cube. When some links or nodes of the computer fail, other functioning links and nodes may still be reconfigured to a smaller (but similar) graph and continue to operate (albeit with a degraded performance). In a multiple-processor system, one can also take advantage of this self-similarity to allocate processing resources to multiple users (each user could be assigned a subcube of some size).

We call for further investigation of this new class of graphs.

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AMS Classification Numbers: 68M10, 68Q22, 90B12

❖❖❖

A NEW BOOK ON LIBER ABACI

The editor has recently been informed that a new book on Fibonacci's *Liber Abaci* has appeared in Germany. The editor has been told that the book was written by Professor Heinz Lüneburg, a mathematics professor at the University of Kaiserslautern. The book's title was said to be LEONARDI PISANI LIBER ABACI ODER LESEVERGNÜGEN EINES MATHEMATIKERS. The publisher was reported as BI Verlag, Mannheim, and the cost was said to be 68 Deutsch Marks.

CONGRUENCE PROBLEMS INVOLVING STIRLING NUMBERS OF THE FIRST KIND

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1. INTRODUCTION AND SUMMARY OF RESULTS

It is by now well known that the parities of the binomial coefficients show a fractal-like appearance when plotted in the *x-y* plane. Similarly, if $f(n,k)$ is some counting sequence and p is a prime, we can plot an asterisk at (n,k) if $f(n,k)\not\equiv 0 \pmod p$, and a blank otherwise, to get other complex, and often interesting, patterns.

For the ordinary and Gaussian binomial coefficients and for the Stirling numbers of the second kind, formulas for the number of asterisks in each column are known ([12], [2], [4], [1]). Moreover, in each row the pattern is periodic, and formulas for the minimum period have been bound ([2], [3], [6], [7], [12], [13], [9], [15]) in all three cases.

If $\left[{n \atop k}\right]$, the signless Stirling number of the first kind, denotes, as usual, the number of permutations of n letters that have k cycles, then for fixed k and p we will show that there are only finitely many n for which $\left[{n \atop k}\right]\not\equiv 0 \pmod p$, i.e., there are only finitely many asterisks in each row of the pattern. Let $v(n,k)$ be the number of these.

To describe the generating function of the $v(n,k)$ we first need to define a *special integer modulo p*. We say that a nonnegative integer n is special modulo p if

$$n \bmod p + \left\lfloor \frac{n}{p} \right\rfloor \le p-1.$$

This means that n is a 1- or 2-digit p-ary integer and, in the addition of n to its digit reversal, there is no carry out of the units place. We denote by N_p the (finite) set of all special integers modulo p, and we write $N_p(x)$ for the polynomial $\sum_{n\in N_p} x^n$; e.g., $N_3(x)=1+x+x^2+x^3+x^4+x^6$.

Finally, we denote by c_p the finite sequence that is defined by $c_p(0)=1$ and

$$c_p(i)=\left|\left\{n\le p-1:\left[{n \atop i}\right]\not\equiv 0 \pmod p\right\}\right| \quad (1\le i\le p-1).$$

We write $C_p(x)$ for the polynomial $\sum_{0\le i\le p-1} c_p(i)x^i$; e.g., $C_3(x)=1+2x+x^2$.

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Theorem A: The number $v(k,p)$ of $n \geq 0$ such that $\left[{n \atop k}\right] \not\equiv 0 \pmod{p}$ is given by

$$\sum_{k\geq 0} v(k,p)x^k = C_p(x)\prod_{j\geq 0} N_p(x^{p^j})$$

As an illustration of Theorem A, take $p=3$. Then $C_3(x) = 1+2x+x^2$, and so

$$\begin{aligned}\sum_{k\geq 0} v(k,3)x^k &= (1+2x+x^2)\prod_{j\geq 0} N_3(x^{3^j}) \\ &= 1+3x+4x^2+5x^3+7x^4+7x^5+7x^6+9x^7+8x^8+12x^{10} \\ &\quad +12x^{11}+12x^{12}+16x^{13}+15x^{14}+13x^{15}+15x^{16}+12x^{17}+\cdots.\end{aligned}$$

Thus there are, for example, 12 values of n for which $\left[{n \atop 17}\right]$ is not a multiple of 3. In Corollary 2.2 below we will see that the largest of these is $\left[{kp \atop k}\right] = \left[{51 \atop 17}\right]$ (which has 59 decimal digits).

Along the way to proving Theorem A we will find the following result, which seems to be of independent interest.

Theorem B: Let $k \geq 0$ and p be a fixed integer and prime, respectively. Then the following two sets are equinumerous:

- The set of all $j \leq k/(p-1)$ for which the binomial coefficient $\binom{k-(p-1)j}{j} \not\equiv 0 \pmod{p}$, and
- The set of all partitions of the integer k into parts that are powers of p, and in which the multiplicity of each part is special modulo p.

Although we find Theorem B by means of generating functions, the form of the result suggests that there may be a natural bijection between the two sets. We will give such a bijection also.

Our results dualize, in an interesting way, results of Carlitz [1]. He studied similar questions for the Stirling numbers of the second kind. More precisely, he studied the number of k for which $\left\{{n \atop k}\right\}$ is not divisible by p and deduced infinite product-generating functions for these numbers that are quite similar to ours. His results are complete if p = 2, 3, 5, but only partial for other values of p. The duality of the questions and the similarity of the answers are arresting.

2. AN ANALOG OF LUCAS'S CONGRUENCE

Lucas's congruence ([2], [8]) for binomial coefficients asserts that

$$\binom{n'p+n_0}{k'p+k_0} \equiv \binom{n'}{k'}\binom{n_0}{k_0} \pmod{p}$$

if n', k', n_0, and k_0 are nonnegative integers with n_0 and k_0 less than p. It is easily proved by viewing $(x+1)^{n'p+n_0} = (x+1)^{n'p}(x+1)^{n_0}$ as an identity over $GF(p)[x]$ and using the "freshman" $((x+1)^p \equiv x^p+1)$ and binomial theorems. By imitating this proof we can get a somewhat similar congruence for the residue modulo p of $\left[{n \atop k}\right]$.

Recall that

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \text{ for } n, k > 0,$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 0 \text{ for } n > 0, \begin{bmatrix} 0 \\ k \end{bmatrix} = 0 \text{ for } k > 0, \text{ and } \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1.$$

Further recall that the ordinary generating function for $\left\{\begin{bmatrix} n \\ \cdot \end{bmatrix}\right\}$ is

$$s_n(x) = x(x+1)(x+2)\cdots(x+n-1).$$

For the remainder of this section, all of our computations will take place in the polynomial ring $GF(p)[x]$. To begin with, note that $s_0(x) = 1$ is the only $s_n(x)$ with a constant term, and that

(1) $$s_p(x) = x^p - x$$

in $GF(p)[x]$ since both sides of (1) are polynomials of degree p with simple roots $r = 0, 1, \ldots, p-1$ and leading coefficient 1.

Lemma 1: For all n,

(2) $$s_n(x) \equiv x^{n'}(x^{p-1}-1)^{n'} s_{n_0}(x) \pmod p$$

where $n' = \lfloor n/p \rfloor$, $n_0 = n \bmod p$.

Proof: We have $n = n'p + n_0$ with $0 \le n_0 < p$. Then

$$\begin{aligned} s_n(x) &= \prod_{t=0}^{n'-1}(x+tp)(x+tp+1)\cdots(x+tp+p-1)\cdot\prod_{u=0}^{n_0-1}(x+n'p+u) \\ &= \prod_{t=0}^{n'-1} x(x+1)\cdots(x+p-1)\cdot\prod_{u=0}^{n_0-1}(x+u) = s_p(x)^{n'} s_{n_0}(x) \\ &= (x^p - x)^{n'}\cdot s_{n_0}(x) \end{aligned}$$

where empty products are interpreted as 1. □

If we simply equate coefficients of like powers of x on both sides of (2), we obtain the known

Proposition 2.1: Let p be prime and let n and k be integers with $1 \le k \le n$. Let $n' = \lfloor n/p \rfloor$, $n_0 = n \bmod p$. Define integers i and j as follows:

(3) $$k - n' = j(p-1) + i \quad \begin{array}{l}(0 \le i < p-1 \text{ if } n_0 = 0; \\ 0 < i \le p-1 \text{ if } n_0 > 0).\end{array}$$

Then

(4) $$\begin{bmatrix} n \\ k \end{bmatrix} \equiv (-1)^{n'-j}\binom{n'}{j}\begin{bmatrix} n_0 \\ i \end{bmatrix} \pmod p.$$

Corollary 2.2: For a fixed k, the set $\left\{n : \begin{bmatrix} n \\ k \end{bmatrix} \not\equiv 0 \pmod p\right\}$ is finite. Its largest element is pk and its smallest is k.

Proof: If $n > pk$, then in (3) above $j < 0$ and so $\left[{n \atop k}\right] = 0 \pmod{p}$. If $n = pk$, then $n' = n_0 = i = j = 0$ and $\left[{pk \atop k}\right] = (-1)^k \pmod{p}$. If $n = k$ then $n' = j$, $n_0 = i$, and $\left[{k \atop k}\right] = 1 \pmod{p}$. If $n < k$, then either $n' < j$ or $n_0 < i$. In either case, $\left[{n \atop k}\right] = 0 \pmod{p}$. □

Comments: There are other approaches to Proposition 2.1. One [5] uses a double induction, first on k with $\lfloor n/p \rfloor = 1$, and then on $\lfloor n/p \rfloor$. Another [11] uses abelian group actions to prove an additive congruence for $\left[{n+p \atop k}\right]$, and then induction on n.

To obtain a more explicit form of the congruence (4) for $k \le n \le pk$, we can use the following iterated form of Lucas's congruence: If $(n'_s, n'_{s-1}, \dots, n'_0)_p$ and $(j_s, j_{s-1}, \dots, j_0)_p$ are the p-ary representations of the nonnegative integers n' and j, then

$$\binom{n'}{j} = \binom{n'_s}{j_s}\binom{n'_{s-1}}{j_{s-1}} \cdots \binom{n'_0}{j_0} \pmod{p}.$$

Therefore,

$$\left[{n \atop k}\right] \equiv (-1)^{n'-j}\binom{n'_s}{j_s}\binom{n'_{s-1}}{j_{s-1}} \cdots \binom{n'_0}{j_0}\left[{n_0 \atop i}\right] \pmod{p} \tag{5}$$

where $k = ej + i$, $0 \le i < e$; $n = en' + n_0 < e$, and $(n'_s, n'_{s-1}, \dots, n'_0)_p$ and $(j_s, j_{s-1}, \dots, j_0)_p$ are the p-ary expansions of n' and j as before.

3. EVALUATION OF $\left|\left\{k : \left[{n \atop k}\right] \not\equiv 0 \pmod{p}\right\}\right|$

For completeness, and since its proof is now quite simple, we recall the following result [10].

Theorem 3.1: For n fixed, the number $h(n,p)$ of Stirling numbers of the first kind that are not multiples of the prime p is given by

$$h(n,p) = (n_s+1)(n_{s-1}+1)\cdots(n_1+1)h(n_0,p)$$

where $(n_s, n_{s-1}, \dots, n_0)_p$ is the p-ary representation of n.

Proof: We count the nonzero terms of the polynomial $s_n(x) \in GF(p)[x]$, making use of (2). The number of nonzero terms in $s_{n_0}(x) \in GF(p)[x]$ is by definition $h(n_0,p)$, and the degrees of any two of its nonzero terms differ by less than $p-1$. To count the nonzero terms in $(x^{p-1}-1)^{n'}$ $GF(p)[x]$, we make use of the following well-known [2] corollary of Lucas's congruence (in iterated form): If $(m_s, m_{s-1}, \dots, m_0)_p$ is the p-ary representation of m, then the number of nonzero terms of $(a+b)^m \in GF(p)[a,b]$ is $\prod_{j=0}^{s}(m_j+1)$. Since $(n_s, n_{s-1}, \dots, n_1)_p$ is the p-ary representation of n', it follows that $(x^{p-1}-1)^{n'} \in GF(p)[x]$ has $(n_s+1)(n_{s-1}+1)\cdots(n_1+1)$ nonzero terms, the degrees of any two of which differ by at least $p-1$. Therefore, $s_n(x) \in GF(p)[x]$ has $(n_s+1)(n_{s-1}+1)\cdots(n_1+1)h(n_0,p)$ nonzero terms.

4. EVALUATION OF $\left|\left\{n: \left[{n \atop k}\right] \not\equiv 0 \pmod{p}\right\}\right|$

We now consider the more difficult problem of determining for fixed k the number of n such that $\left[{n \atop k}\right]$ is a nonmultiple of p. In this section we will reduce this to a problem concerning binomial coefficients, and in the next section we will solve the latter problem. For p prime and $k \geq 0$ define

$$v(k,p) = \left|\left\{n : \left[{n \atop k}\right] \not\equiv 0 \pmod{p}\right\}\right|;$$

$$b(k,p) = \left|\left\{j : k-(p-1)j \geq 0 \text{ and } \binom{k-(p-1)j}{j} \not\equiv 0 \mod p\right\}\right|.$$

Theorem 4.1 Let $c_p(0) = 1$ and

$$c_p(i) = \left|\left\{n \leq p-1 : \left[{n \atop i}\right] \not\equiv 0 \pmod{p}\right\}\right| \quad (1 \leq i \leq p-1).$$

Then for all $k \geq 0$,

(6) $\qquad v(k,p) = c_p(0)b(k,p) + c_p(1)b(k-1,p) + \cdots + c_p(p-1)b(k-p+1,p).$

Proof: For k and p fixed, let each n such that $k \leq n \leq pk$ determine n_0, n', i and j as in Proposition 2.1. Then

$$\begin{aligned} v(k,p) &= \left|\left\{n : k \leq n \leq kp \text{ and } \binom{k-i-(p-1)j}{j}\left[{n_0 \atop i}\right] \not\equiv 0 \pmod{p}\right\}\right| \\ &= \sum_{i=0}^{p-1}\sum_{n_0=i}^{p-1}\left|\left\{j : k-i-(p-1)j \geq 0 \text{ and } \binom{k-i-(p-1)j}{j}\left[{n_0 \atop i}\right] \not\equiv 0 \pmod{p}\right\}\right| \\ &= \sum_{i=0}^{p-1} c_p(i)\left|\left\{j : k-i-(p-1)j \geq 0 \text{ and } \binom{k-i-(p-1)j}{j} \not\equiv 0 \pmod{p}\right\}\right| \\ &= \sum_{i=0}^{p-1} c_p(i)b(k-i,p). \quad \square \end{aligned}$$

Comment: The first few coefficient sequences c_p are:

$c_2 : (1,1);\ c_3 : (1,2,1);\ c_5 : (1,4,3,2,1);\ c_7 : (1,6,5,3,3,2,1);$
$c_{11} : (1,10,7,8,7,6,5,4,3,2,1).$

5. DETERMINATION OF THE $b(k,p)$

Theorem 5.1: Let $k = pm + r$ with $0 \leq r < p$ and $m \geq 0$. Then

(7) $\qquad b(k,p) = b(mp+r,p) = b(m,p) + b(m-1,p) + \cdots + b(m-p+r+1,p).$

Proof: In the following computation, all binomial coefficients $\binom{a}{b}$ mentioned are implicitly assumed to be "classical" (i.e., a and b are nonnegative integers), and since the argument hinges on Lucas's congruence, one of the details that should be checked is that if they start out as such, then they remain so throughout the computation.

$$
\begin{aligned}
b(pm+r,p) &= \left|\left\{t : \binom{pm+r-(p-1)t}{t} \not\equiv 0 \pmod p\right\}\right| \\
&= \sum_{j=0}^{p-1}\left|\left\{s : \binom{pm+r-(p-1)(ps+j)}{ps+j} \not\equiv 0 \pmod p\right\}\right| \\
&= \sum_{j=0}^{p-r-1}\left|\left\{s : \binom{p(m-(p-1)s-j)+r+j}{ps+j} \not\equiv 0 \pmod p\right\}\right| \\
&\qquad + \sum_{j=p-r}^{p-1}\left|\left\{s : \binom{p(m-(p-1)s-j+1)+r+j-p}{ps+j} \not\equiv 0 \pmod p\right\}\right|
\end{aligned}
$$

where the last sum is the empty sum if $r = 0$. Applying Lucas's congruence to the last two sums, we get

$$
\begin{aligned}
b(pm+r,p) &= \sum_{j=0}^{p-r-1}\left|\left\{s : \binom{m-(p-1)s-j}{s}\binom{r+j}{j} \not\equiv 0 \pmod p\right\}\right| \\
&\quad + \sum_{j=p-r}^{p-1}\left|\left\{s : \binom{m-(p-1)s-j+1}{s}\binom{r+j-p}{j} \not\equiv 0 \pmod p\right\}\right| \\
&= \sum_{j=0}^{p-r-1}\left|\left\{s : \binom{m-j-(p-1)s}{s} \not\equiv 0 \pmod p\right\}\right| = \sum_{j=0}^{p-r-1} b(m-j,p). \qquad \square
\end{aligned}
$$

6. PROOFS OF THEOREMS A AND B

From the recurrence relation of Theorem 5.1 it is easy to obtain the generating function of the $\{b(\cdot, p)\}$, as follows. Define $B_p(x) = \sum_{k\ge 0} b(k,p)x^k$. Then multiply both sides of (7) by x^{mp+r} and sum over $m \ge 0$, $0 \le r \le p-1$. There results

$$
\begin{aligned}
B_p(x) &= \sum_{m\ge 0}\sum_{r=0}^{p-1} x^{mp+r}\sum_{j=0}^{p-r-1} b(m-j,p) \\
&= \sum_{r=0}^{p-1} x^r \sum_{j=0}^{p-r-1} x^{jp} \sum_{m\ge 0} b(m-j,p)x^{p(m-j)} = B_p(x^p)\left\{\sum_{r=0}^{p-1} x^r \sum_{j=0}^{p-r-1} x^{jp}\right\}.
\end{aligned}
$$

The quantity in curly braces in the rightmost member is exactly the generating function $N_p(x)$ that was defined in section 1 above. Hence, $B_p(x) = N_p(x)B_p(x^p)$ and, therefore, by iteration, the generating function of the $b(\cdot, p)$ is

$$
B_p(x) = \prod_{j\ge 0} N_p(x^{p^j}) = \sum_{k\ge 0} b(k,p)x^k. \tag{8}
$$

The infinite product that occurs in (8) is well known (e.g., [14, Eq. (3.16.4)]) to generate the number of partitions of the integer k into powers of p, each taken with a multiplicity that is special modulo p, as defined in section 1 above, and the proof of Theorem B is complete.

Theorem A now follows from this result and Theorem 4.1. Indeed, equation (6) above, when translated into generating function terms, states that the generating function of the $\{\nu(k,p)\}$ is $C_p(x)B_p(x)$, as claimed. □

7. BIJECTIVE PROOF OF THEOREM B

As promised in the introduction, we will give here a bijective proof of Theorem B. We let $\Pi(n,\mathcal{P},\mathcal{M})$ denote the set of all partitions of the integer n into parts that all belong to $\mathcal{P}$, each having a multiplicity that belongs to $\mathcal{M}$. Set $\mathcal{P}_p=\{p^i : i\geq 0\}$ and $\mathcal{M}_p\{a : a \text{ is special for } p\}$.

We will exhibit an explicit bijection between the sets

$$A_{n,p}=\left\{j\in\{0,1,\dots,\lfloor n/p\rfloor\}:\binom{n-(p-1)j}{j}\not\equiv 0 \pmod{p}\right\}$$
$$B_{n,p}=\Pi(n,\mathcal{P}_p,\mathcal{M}_p).$$

To do this, note first that Lucas's theorem gives a simple criterion for deciding whether a given j belongs to $A_{n,p}$; it says that

$$\binom{n-(p-1)j}{j}\equiv\binom{b_k}{a_k}\binom{b_{k-1}}{a_{k-1}}\cdots\binom{b_0}{a_0} \pmod{p},$$

where the b's and the a's are the p-ary digits of $n-(p-1)j$ and of j, respectively. In particular, the left side is not congruent to 0 provided that $a_i\leq b_i$ for all i.

We will now define a mapping $\phi:\{0,1,\dots,\lfloor n/p\rfloor\}\to\Pi(n,\mathcal{P}_p,\mathbf{N})$ as follows: $\phi(j)$ is that partition of the integer n in which, for all i, the part p^i occurs with multiplicity $b_i+(p-1)a_i$, and no other parts occur. Since Σa_ip^i and $n-(p-1)j=\Sigma b_ip^i$, it is clear that $\phi(j)$ is indeed a partition of n into powers of p.

To show that ϕ restricts to a bijection between $A_{n,p}$ and $B_{n,p}$, it is necessary to check that for $j\in A_{n,p}$ the image $\phi(j)$ is in $B_{n,p}$, and that the restriction is invertible. Consider then a $j\in A_{n,p}$, i.e., a j for which $a_i\leq b_i$ for all i. Now the number of parts of size p^i in $\phi(j)$ is

$$b_i+(p-1)a_i=pa_i+(b_i-a_i).$$

Since $0\leq a_i\leq b_i\leq p-1$, the multiplicity of p^i is $(a_i(b_i-a_i))_p$ in p-ary notation, and thus belongs to $\mathcal{M}_p$. We have shown that $\phi(A_{n,p})\subset B_{n,p}$.

Finally, if $\pi\in B_{n,p}$ is given, define $\psi(\pi)\in A_{n,p}$ by its p-ary expansion—the i^{th} digit of $\psi(\pi)$ is a_i, where $m_i=(a_i,c_i)_p$ is the multiplicity of p^i in π. So

$$\psi(\pi)=(a_k,a_{k-1},\dots,a_0)_p \quad\text{and}\quad n-p\psi(\pi)=(c_k,c_{k-1},\dots,c_0)_p.$$

Further, $n-(p-1)\psi(\pi)=((a_k+c_k),(a_{k-1}+c_{k-1}),\dots,(a_0+c_o))_p$. The last expression is a legitimate p-ary expansion because each m_i is special for p, and moreover it shows that

$$\binom{n-(p-1)\psi(\pi)}{\psi(\pi)}\not\equiv 0 \pmod{p}.$$

It is clear that ψ and ϕ are inverses.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3513@compuserve.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-730 ***Proposed by Herta Freitag, Roanoke, VA***

For $n \geq 0$, express the larger root of $x^2 - L_n x + (-1)^n = 0$ in terms of α, the larger root of $x^2 - x - |(-1)^n| = 0$.

B-731 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Evaluate the determinant:

$$\begin{vmatrix} F_0 & F_1 & F_2 & F_3 & F_4 \\ F_1 & F_0 & F_1 & F_2 & F_3 \\ F_2 & F_1 & F_0 & F_1 & F_2 \\ F_3 & F_2 & F_1 & F_0 & F_1 \\ F_4 & F_3 & F_2 & F_1 & F_0 \end{vmatrix}$$

Generalize.

B-732 ***Proposed by Richard André-Jeannin, Longwy, France***

Dedicated to Dr. A. P. Hillman

Let (w_n) be any sequence of integers that satisfies the recurrence

$$w_n = pw_{n-1} - qw_{n-2}$$

where p and q are odd integers. Prove that, for all n,

$$w_{n+6} \equiv w_n \pmod 4.$$

B-733 ***Proposed by Piero Filipponi, Rome, Italy***

Write down the Pell sequence, defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Form a difference triangle by writing down the successive differences in rows below it. For example,

$$\begin{array}{ccccccccccccccccc}
0 & & 1 & & 2 & & 5 & & 12 & & 29 & & 70 & & 169 & & \cdots \\
& 1 & & 1 & & 3 & & 7 & & 17 & & 41 & & 99 & & & \\
& & 0 & & 2 & & 4 & & 10 & & 24 & & 58 & & & & \\
& & & 2 & & 2 & & 6 & & 14 & & 34 & & & & & \\
& & & & 0 & & 4 & & 8 & & 20 & & & & & & \\
& & & & & 4 & & 4 & & 12 & & & & & & & \\
& & & & & & 0 & & 8 & & & & & & & & \\
& & & & & & & 8 & & & & & & & & & \\
& & & & & & & & \ddots & & & & & & & &
\end{array}$$

Identify the pattern that emerges down the left side and prove that this pattern continues.

B-734 ***Proposed by Paul S. Bruckman, Edmonds, WA***

If r is a positive integer, prove that

$$L_{5^r} \equiv L_{5^{r-1}} \pmod{5^r}.$$

B-735 ***Proposed by Curtis Cooper & Robert E. Kennedy, Central Missouri State Asylum for Crazed Mathematicians, Warrensburg, MO***

Let the sequence (y_n) be defined by the recurrence

$$\begin{aligned} y_{n+1} = {} & 8y_n + 22y_{n-1} - 190y_{n-2} + 28y_{n-3} + 987y_{n-4} - 700y_{n-5} - 1652y_{n-6} + 1652y_{n-7} \\ & + 700y_{n-8} - 987y_{n-9} - 28y_{n-10} + 190y_{n-11} - 22y_{n-12} - 8y_{n-13} + y_{n-14} \end{aligned}$$

for $n \geq 15$ with initial conditions given by the table:

n	y_n
1	1
2	1
3	25
4	121
5	1296
6	9025
7	78961
8	609961
9	5040025
10	40144896
11	326199721
12	2621952025
13	21199651201
14	170859049201
15	1379450250000

Prove that y_n is a perfect square for all positive integers n.

SOLUTIONS

A Sum Involving $F_{2^k}^4$

B-703 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Prove that for all positive integers n,

$$\sum_{k=1}^{n} 4^{n-k} F_{2^k}^4 = \frac{F_{2^{n+1}}^2 - 4^n}{5}.$$

Solution by Bob Prielipp, University of Wisconsin, Oshkosh, WI

Our solution will use the following known result (see Identity I_{39} on page 59 of [1]):

$$(*) \qquad F_m^4 = \frac{F_{2m}^2 - 4(-1)^m F_m^2}{5}.$$

To establish the desired result, it is sufficient to show that

$$\sum_{k=1}^{n} \frac{F_{2^k}^4}{4^k} = \frac{F_{2^{n+1}}^2 - 4^n}{5 \cdot 4^n}.$$

From $(*)$, we have

$$\sum_{k=1}^{n} \frac{F_{2^k}^4}{4^k} = \sum_{k=1}^{n} \frac{F_{2^{k+1}}^2 - 4F_{2^k}^2}{5 \cdot 4^k} = \frac{1}{5}\sum_{k=1}^{n}\left(\frac{F_{2^{k+1}}^2}{4^k} - \frac{F_{2^k}^2}{4^{k-1}}\right)$$

$$= \frac{1}{5}\left(\frac{F_{2^{n+1}}^2}{4^n} - F_2^2\right) \text{ (by telescoping) } = \frac{F_{2^{n+1}}^2 - 4^n}{5 \cdot 4^n}.$$

The proposer gave the generalization:

$$\sum_{k=1}^{n} 4^{n-k} F_{m2^k}^4 = \frac{F_{m2^{n+1}}^2 - 4^n F_{2m}^2}{5}$$

for all positive integers m and n. The proof is similar. No reader gave any generalizations involving $L_{2^k}^4$. Apparently there is no closed form for $\sum_{k=1}^{n} F_{2^k}^4$ or even $\sum_{k=1}^{n} F_{2^k}$. For which constants a, c, r can $\sum_{k=1}^{n} c^k F_{a^k}^r$ be expressed in closed form?

Reference:

1. Verner E. Hoggatt, Jr., *Fibonacci and Lucas Numbers* (Santa Clara, CA: The Fibonacci Association, 1979).

Also solved by Paul S. Bruckman, Herta T. Freitag, C. Georghiou, Russell Jay Hendel, Hans Kappus, Graham Lord, Ray Melham, Blagoj S. Popov, Sahib Singh, and the proposer.

Products of Terms of the Form $ax^2 + by^2$

B-704 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let a and b be fixed integers. Show that if three integers are of the form $ax^2 + by^2$ for some integers x and y, then their product is also of this form.

Solution by Ray Melham, University of Technology, Sydney, Australia

By expanding both sides, it is seen that

$$(ax_1^2+by_1^2)(ax_2^2+by_2^2)(ax_3^2+by_3^2)$$
$$=a(ax_1x_2x_3+bx_1y_2y_3+by_1x_2y_3-by_1y_2x_3)^2+b(ax_1x_2y_3-ax_1y_2x_3-ay_1x_2x_3-by_1y_2y_3)^2.$$

This proves the result.

Flanigan notes that the above identity holds in any commutative ring with identity. The proposer showed that the product of two integers of the form ax^2+by^2 can be written in the form X^2+abY^2 by means of the identity

$$(ax_1^2+by_1^2)(ax_2^2+by_2^2)=(ax_1x_2+by_1y_2)^2+ab(x_1y_2-x_2y_1)^2.$$

He then showed that the product of a number of the form (ax^2+by^2) *and a number of the form X^2+abY^2 can be written in the form (ar^2+bs^2) by means of the identity*

$$(au_1^2+bv_1^2)(u_2^2+abv_2^2)=a(u_1u_2+bv_1v_2)^2+b(u_2v_1-au_1v_2)^2.$$

Also solved by F. J. Flanigan, C. Georghiou, Russell Jay Hendel, Hans Kappus, H.-J. Seiffert, and the proposer. Most of the solutions were similar to that given above.

An Application of a Series Expansion for $(\arcsin x)^2$

B-705 ***Proposed by H.-J. Seiffert, Berlin, Germany***

(a) Prove that $\displaystyle\sum_{n=1}^{\infty}\frac{L_{2n}}{n^2\binom{2n}{n}}=\frac{\pi^2}{5}$.

(b) Find the value of $\displaystyle\sum_{n=1}^{\infty}\frac{F_{2n}}{n^2\binom{2n}{n}}$.

Nearly identical solutions by Russell Euler, Northwest Missouri State University, Maryville, MO; C. Ceorghiou, University of Patras, Patras, Greece; Hans Kappus, Rodersdorf, Switzerland; and Bob Prielipp, University of Wisconsin, Oshkosh, WI.

We start with the known result (see [1], [2], or [3]):

$$\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n^2\binom{2n}{n}}=2(\arcsin x)^2$$

which converges for $|x|\le 1$. In particular, for $x=\alpha/2$ and $x=\beta/2$, we have

$$\sum_{n=1}^{\infty}\frac{\alpha^{2n}}{n^2\binom{2n}{n}}=2\left(\arcsin\frac{\alpha}{2}\right)^2 \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{\beta^{2n}}{n^2\binom{2n}{n}}=2\left(\arcsin\frac{\beta}{2}\right)^2.$$

Now, from problem B-674 [*FQ* **29.3** (1991):280], we know that $\cos\pi/5=\alpha/2$ and $\cos 3\pi/5=\beta/2$. This implies that

$$\sin\frac{3\pi}{10} = \sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right) = \cos\frac{3\pi}{5} = \frac{\alpha}{2} \quad\text{and}\quad \sin\left(-\frac{\pi}{10}\right) = \sin\left(\frac{\pi}{2}-\frac{3\pi}{5}\right) = \cos\frac{3\pi}{5} = \frac{\beta}{2}.$$

Thus,

$$\arcsin\frac{\alpha}{2} = \frac{3\pi}{10} \quad\text{and}\quad \arcsin\frac{\beta}{2} = -\frac{\pi}{10}.$$

Therefore,

$$\text{(a)}\qquad \sum_{n=1}^{\infty}\frac{L_{2n}}{n^2\binom{2n}{n}} = 2\left[\left(\frac{3\pi}{10}\right)^2 + \left(-\frac{\pi}{10}\right)^2\right] = \frac{\pi^2}{5}$$

and

$$\text{(b)}\qquad \sum_{n=1}^{\infty}\frac{F_{2n}}{n^2\binom{2n}{n}} = \frac{2}{\sqrt{5}}\left[\left(\frac{3\pi}{10}\right)^2 - \left(-\frac{\pi}{10}\right)^2\right] = \frac{4\sqrt{5}\pi^2}{125}.$$

References:

1. Bruce C. Berndt, *Ramanujan's Notebooks*, Part 1 (New York: Springer Verlag, 1985, p. 262.
2. I. S. Gradshteyn & I. M. Ryzhik, *Tables of Integrals, Series and Products* (New York: Academic Press, 1980), p. 52.
3. L. B. W. Jolley, *Summation of Series*, 2nd ed. rev. (New York: Dover, 1961), p. 146, series 778.

Also solved by Paul S. Bruckman and the proposer.

An Exponential Inequality

B-706 ***Proposed by K. T. Atanassov, Sofia Bulgaria***

Prove that for $n \geq 0$, $\left(\dfrac{\pi e}{\pi + e}\right)^{1.4n} > F_n$.

Solution by Wray Brady, Chapala, Jalisco, Mexico

Let

$$k = \left(\frac{\pi e}{\pi + e}\right)^{1.4}.$$

We note that $\alpha \approx 1.618$ and $k \approx 1.694$, so that $\alpha < k$. Furthermore, since $\alpha > 1$ and $-1 < \beta < 0$, we have $|\beta^n| \leq 1 \leq \alpha^n$ for $n \geq 0$. Thus,

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \leq \frac{\alpha^n + \alpha^n}{\sqrt{5}} < \frac{2\alpha^n}{2} = \alpha^n < k^n.$$

The proposer also sent in several other inequalities involving Euler's constant and Catalan's constant; however, they were all of the form $k^n > F_n$ where k was some constant larger than α. The conclusion then follows similarly from the fact that $F_n < \alpha^n$. Gilbert showed by taking limits that α is the smallest number with this property. In other words, if $F_n < k^n$ for all $n \geq 0$, then $k \geq \alpha$. Several respondents noted the stronger inequality, $F_n \leq \alpha^{n-1}$ (see page 57 of [1]).

Reference:

1. S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section—Theory and Applications* (Chichester: Ellis Horwood Ltd., 1989).

Also solved by Charles Ashbacher, Glenn Bookhout, Paul S. Bruckman, Joseph E. Chance, C. Georghiou, Peter Gilbert, Pentti Haukkanen, Douglas E. Iannucci, Russell Jay Hendel, Bob Prielipp, Mike Rubenstein, H.-J. Seiffert, Lawrence Somer, Ralph Thomas, and the proposer.

Simple Pythagorean Triple

B-707 ***Proposed by Herta T. Freitag, Roanoke, VA***

Consider a Pythagorean triple (a, b, c) such that

$$a = 2\sum_{i=1}^{n} F_i^2 \quad \text{and} \quad c = F_{2n+1},\ n \geq 2.$$

Prove or disprove that b is the product of two Fibonacci numbers.

Solution by H.-J. Seiffert, Berlin, Germany

From equations (I_3) and (I_{11}) of [1], we have $a = 2F_nF_{n+1}$ and $c = F_{n+1}^2 + F_n^2$. Since, in a Pythagorean triple, $b^2 = c^2 - a^2$, we find that

$$b = F_{n+1}^2 - F_n^2 = (F_{n+1} - F_n)(F_{n+1} + F_n) = F_{n-1}F_{n+2},$$

which shows that b is always the product of two Fibonacci numbers.

Reference:

1. Verner E. Hoggatt, Jr., *Fibonacci and Lucas Numbers* (Santa Clara, CA: The Fibonacci Association, 1979).

Also solved by Charles Ashbacher, M. A. Ballieu, Wray Brady, Scott H. Brown, Paul S. Bruckman, Joseph E. Chance, C. Georghiou, Russell Jay Hendel, Joseph J. Kostal, Bob Prielipp, Sahib Singh, Lawrence Somer, Ralph Thomas, and the proposer. Many of the solutions were similar to the featured solution. One solution was received that did not contain the solver's name.

Exponential Summation

B-708 ***Proposed by Joseph J. Kostal, University of Illinois at Chicago, IL***

Find the sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{3^k F_k - 2^k L_k}{6^k}$.

Solution 1 by Glenn Bookhout, North Carolina Wesleyan College, Rocky Mount, NC

We use the well-known generating functions for F_n and L_n (see page 53 of [1]). They are given by the equations

$$\text{(1)} \qquad \sum_{k=0}^{\infty} F_k t^k = \frac{t}{1-t-t^2}$$

and

(2) $$\sum_{k=0}^{\infty} L_k t^k = \frac{2-t}{1-t-t^2}.$$

Since

$$\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \alpha$$

by formula (101) of [1], the power series (1) converges for $|t| < 1/\alpha$ by the Ratio Test. Similarly, since

$$\lim_{k\to\infty} \frac{L_{k+1}}{L_k} = \alpha$$

the power series (2) also converges for $|t| < 1/\alpha$.

Substituting 1/2 for t in power series (1) gives

(3) $$\sum_{k=1}^{\infty} \frac{F_k}{2^k} = 2.$$

Substituting 1/3 for t in power series (2) gives $\Sigma_{k=0}^{\infty}(L_k / 3^k) = 3$ so

(4) $$\sum_{k=1}^{\infty} \frac{L_k}{3^k} = 1.$$

It follows from equations (3) and (4) that

$$\sum_{k=1}^{\infty} \frac{3^k F_k - 2^k L_k}{6^k} = 1.$$

Seiffert and Bruckman proceeded similarly, but used the power series

$$\sum_{k=1}^{\infty} L_k t^k = \frac{t(1+2t)}{1-t-t^2}, \quad |t| < \alpha^{-1}.$$

Several readers blindly substituted values into equations (1) and (2) without first noting the radius of convergence of these series.

Reference:

1. S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section—Theory and Applications* (Chichester: Ellis Horwood Ltd., 1989).

Solution 2 by C. Georghiou, University of Patras, Greece

We have the following (converging) geometrical series:

$$\sum_{k=1}^{\infty} \frac{3^k \alpha^k}{6^k} = \frac{\alpha}{2-\alpha} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{3^k \beta^k}{6^k} = \frac{\beta}{2-\beta}.$$

Using the Binet form, $F_k = (\alpha^k - \beta^k)/(\alpha - \beta)$, we get

$$\sum_{k=1}^{\infty} \frac{3^k F_k}{6^k} = \frac{1}{\alpha-\beta}\left[\frac{\alpha}{2-\alpha} - \frac{\beta}{2-\beta}\right] = 2$$

where we have simplified by using the identities $\alpha + \beta = 1$ and $\alpha\beta = -1$.

In the same way, from

$$\sum_{k=1}^{\infty} \frac{2^k \alpha^k}{6^k} = \frac{\alpha}{3-\alpha} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{2^k \beta^k}{6^k} = \frac{\beta}{3-\beta}$$

and the Binet form, $L_k = \alpha^k + \beta^k$, we get

$$\sum_{k=1}^{\infty} \frac{2^k L_k}{6^k} = \frac{\alpha}{3-\alpha} + \frac{\beta}{3-\beta} = 1.$$

Therefore, the given sum evaluates to $2 - 1 = 1$.

Solution 3 by W. R. Spickerman, R. N. Joyner, & R. L. Creech (jointly), East Carolina University, Greenville, NC

Let $S_1 = \sum_{k=1}^{\infty} \frac{F_k}{2^k}$ and $S_2 = \sum_{k=1}^{\infty} \frac{L_k}{3^k}$.

Both series are seen to converge by the Ratio Test. Hence, the series consisting of the differences of successive terms of these series converges to $S_1 - S_2$. That is,

$$\sum_{k=1}^{\infty} \frac{3^k F_k - 2^k L_k}{6^k} = \sum_{k=1}^{\infty} \frac{F_k}{2^k} - \sum_{k=1}^{\infty} \frac{L_k}{3^k} = S_1 - S_2.$$

Multiplying the series for S_1 by 1, 1/2, and 1/4, respectively, we find that

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) S_1 = \frac{F_1}{2} + \frac{F_2}{4} - \frac{F_1}{4} + \sum_{k=3}^{\infty} \frac{1}{2^k}(F_k - F_{k-1} - F_{k-2}).$$

Since the Fibonacci sequence satisfies the recurrence $F_k = F_{k-1} + F_{k-2}$, the summation in this last equation is 0. Therefore,

$$\frac{1}{4} S_1 = \frac{F_1 + F_2}{4} = \frac{2}{4},$$

so $S_1 = 2$. Similarly,

$$\frac{5}{9} S_2 = \frac{2L_1 + L_2}{9} = \frac{5}{9},$$

so $S_2 = 1$. Hence, the desired sum is $S_1 - S_2 = 2 - 1 = 1$.

Redmond generalized by showing that for sequences defined by $P_n = aP_{n-1} - bP_{n-2}$ and $Q_n = aQ_{n-1} - bQ_{n-2}$ (with $a^2 \neq 4b$), and real numbers A, B, and C, we have

$$\sum_{k=0}^{\infty} \frac{A^k P_k + B^k Q_k}{C^k} = C\left[\frac{c_0(C - A\beta) + c_1(C - A\alpha)}{(C - A\alpha)(C - A\beta)} + \frac{d_0(C - B\beta) + d_1(C - B\alpha)}{(C - B\alpha)(C - B\beta)}\right]$$

where α and β are the roots of the characteristic equation, $x^2 - ax + b = 0$, chosen so that $\alpha - \beta = \sqrt{a^2 - 4b}$ and with initial conditions such that the Binet forms are $P_n = c_0\alpha^n + c_1\beta^n$ and $Q_n = d_0\alpha^n + d_1\beta^n$. The series converges if $\max(|A\alpha/C|, |A\beta/C|, |B\alpha/C|, B\beta/C|) < 1$.

Also solved by Wray Brady, Scott H. Brown, Paul S. Bruckman, Joseph E. Chance, Russell Euler, Herta T. Freitag (2 solutions), Douglas E. Iannucci, Russell Jay Hendel, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Ralph Thomas (2 solutions), and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems

PROBLEM PROPOSED IN THIS ISSUE

H-474 ***Proposed by R. André-Jeannin, Longwy, France***

Let us define the sequence $\{U_n\}$ by

$$U_0 = 0,\ U_1 = 1,\ U_n = PU_{n-1} - QU_{n-2},\ n \in Z,$$

where P and Q are nonzero integers. Assuming that $U_k \neq 0$, the matrix M_k is defined by

$$M_k = \frac{1}{U_k}\begin{pmatrix} U_{k+1} & iQ^{k/2} \\ iQ^{k/2} & -Q^k U_{1-k} \end{pmatrix},\quad k \geq 1,$$

where $i = \sqrt{(-1)}$.

Express in a closed form the matrix M_k^n, for $n \geq 0$.

Reference: A. F. Horadam & P. Filipponi. "Choleski Algorithm Matrices of Fibonacci Type and Properties of Generalized Sequences." *Fibonacci Quarterly* **29.2** (1991):164-73.

SOLUTIONS

How Many?

H-456 ***Proposed by David Singmaster, Polytechnic of the South Bank, London, England*** *(Vol. 29, no. 3, August 1991)*

Among the Fibonacci numbers, F_n, it is known that: 0, 1, 144 are the only squares; 0, 1, 8 are the only cubes; 0, 1, 3, 21, 55 are the only triangular numbers. [See Luo Ming's article in *The Fibonacci Quarterly* **27.2** (1989):98-108.]

A. Let $p(m)$ be a polynomial of degree at least 2 in m. Is it true that $p(m) = F_n$ has only finitely many solutions?

B. If we replace F_n by an arbitrary recurrent sequence f_n, we cannot expect a similar result, since f_n, can easily be a polynomial in n. Even if we assume the auxiliary equation of our recurrence has no repeated roots, we still cannot expect such a result. For example, if

$$f_n = 6f_{n-1} - 8f_{n-2},\ f_0 = 2,\ f_1 = 6,$$

then

$$f_n = 2^n + 4^n,$$

so every f_n is of the form $p(m) = m^2 + m$. What restriction(s) on f_n, is(are) needed to make $f_n = p(m)$ have only finitely many solutions?

Comments: The results quoted have been difficult to establish, so part A is likely to be quite hard and, hence, part B may well be extremely hard.

Solution by Paul S. Bruckman, Edmonds WA

To simplify the problem somewhat, we assume that $(f_n)_{n=1}^{\infty}$ is an increasing sequence of positive integers, and that the f_n's satisfy a homogeneous linear recurrence of order d $(d \ge 2)$. Furthermore, we assume that the roots of the characteristic equation of f_n are distinct. Let these roots be denoted by z_j, $j = 1, 2, \ldots, d$, with $|z_1| \le |z_2| \le \cdots \le |z_d|$. Then constants a_j exist such that

$$(1) \qquad f_n = \sum_{j=1}^{d} a_j z_j^n, \ n = 0, 1, 2, \ldots .$$

We shall also suppose that the sequence $(p(n))_{n=1}^{\infty}$ is an increasing sequence of positive integers from some point on. Let e denote the degree of p $(e \ge 2)$. Then constants b_j exist such that

$$(2) \qquad p(z) = \sum_{j=0}^{e} b_j z^j .$$

Under these assumptions, we shall prove the following

Theorem: $f_n = p(m_n)$ for infinitely many n, where the m_n's are positive integers, if and only if $f_n = p(z_1^n)$ for all n. If these conditions are met, we must also have:

(i) $p(0) = 0$;	(iii) z_1 is an integer > 1;
(ii) $d = e$;	(iv) $z_j = z_1^j$, $j = 1, 2, \ldots, d$.

Proof: If $f_n = p(z_1^n)$ for all n, clearly $f_n = p(m_n)$ for infinitely many n, with $m_n = z_1^n$. Conditions (i), (ii), (iii), and (iv) must then follow.

Conversely, suppose $f_n = p(m_n)$ for infinitely many n, for some sequence $(m_n)_{n=1}^{\infty}$ of positive integers. Then, for some subsequence $(n_k)_{k=1}^{\infty}$ of positive integers, we must have

$$(3) \qquad f_{n_k} = p(m_{n_k}), \ k = 1, 2, \ldots .$$

Given any $e+2$ consecutive elements $n_{1+t}, n_{2+t}, \ldots, n_{e+2+t}$ $(t = 0, 1, 2, \ldots)$, we may form the $(e+1)^{\text{th}}$ divided difference of p with respect to $m_{n_{1+t}}, m_{n_{2+t}}, \ldots, m_{n_{e+2+t}}$. Since p is a polynomial, this expression must vanish. Thus $\Delta^{e+1} m_{n_{1+t}}, m_{n_{2+t}}, \ldots, m_{n_{e+2+t}}(p) = 0$, or

$$(4) \qquad \sum_{k=1}^{e+2} c_{k,t}\, p\left(m_{n_{k+t}}\right) = 0, \ t = 0, 1, 2, \ldots,$$

where

$$(5) \qquad c_{k,t} = \prod_{\substack{j=1\\ j\neq k}}^{e+2} \left(m_{n_{k+t}} - m_{n_{j+t}}\right)^{-1}.$$

Then,

$$\sum_{k=1}^{e+2} c_{k,t} \sum_{j=0}^{e} b_j \left(m_{n_{k+t}}\right)^j = \sum_{j=0}^{e} b_j \sum_{k=1}^{e+2} c_{k,t} \left(m_{n_{k+t}}\right)^j = 0.$$

Since this is true for all $t \geq 0$, and the b_j's are assumed not to all equal 0, it follows that

$$(6) \qquad \sum_{k=1}^{e+2} c_{k,t} \left(m_{n_{k+t}}\right)^j = 0, \quad t = 0, 1, 2, \ldots, \quad j = 0, 1, \ldots, e.$$

On the other hand, due to (3), we also have the following:

$$\sum_{k=1}^{e+2} c_{k,t} f_{n_{k+t}} = 0, \text{ or } \sum_{k=1}^{e+2} c_{k,t} \sum_{j=1}^{d} a_j z_j^{n_{k+t}} = \sum_{j=1}^{d} a_j \sum_{k=1}^{e+2} c_{k,t} z_j^{n_{k+t}} = 0.$$

Again, since this is true for all $t \geq 0$, and all the a_j's are assumed not all equal to 0, we must have:

$$(7) \qquad \sum_{k=1}^{e+2} c_{k,t} z_j^{n_{k+t}} = 0, \quad t = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots, d.$$

Comparing (6) and (7), since these are true for all $t \geq 0$, the two expressions must be **identically** equal. Therefore, the following is implied:

$$(8) \qquad b_0 = 0; \ d = e; \ \left(m_{n_{k+t}}\right)^j = z_j^{n_{k+t}}, \quad t = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots, d.$$

We see that (8) implies conditions (i)-(iv) of the Theorem. As a result, we have:

$$(9) \qquad f_{n_k} = p\left(z_1^{n_k}\right), \quad k = 1, 2, \ldots .$$

Thus,

$$\sum_{j=1}^{d} a_j z_1^{jn_k} = \sum_{j=1}^{d} b_j \left(m_{n_k}\right)^j = \sum_{j=1}^{d} b_j z_1^{jn_k}.$$

Using the same argument as before (with k replacing t), it follows that

$$(10) \qquad a_j = b_j, \quad j = 1, 2, \ldots, d.$$

Therefore, for **all** n,

$$f_n = \sum_{j=1}^{d} a_j z_1^{nj} = \sum_{j=1}^{d} b_j z_1^{nj},$$

or

$$(11) \qquad f_n = p(z_1^n), \quad n = 0, 1, 2, \ldots .$$

Note that $m_n = z_1^n$ for all n; since the m_n's are to be integers, it must follows that z_1 is an integer. Also, since $(m_n)_{n=1}^{\infty}$ is increasing from some point on, we must have $z_1 > 1$; in fact, $(m_n)_{n=1}^{\infty}$ is increasing for **all** n. This completes the proof of the Theorem.

We can now readily dispose of the problem. Since $F_n = 5^{-\frac{1}{2}}(\alpha^n - \beta^n) = 5^{-\frac{1}{2}}[(-1)^n\beta^{-n} - \beta^n]$, we see that F_n cannot be expressed as a polynomial in β^n (nor, indeed, is β greater than 1, must less an integer). Therefore, the equation

(12) $\quad F_n = p(m_n)$, where $\deg(p) \geq 2$,

necessarily has only a finite number of solutions, for all acceptable given polynomials p.

The conditions sought for part B of the problem are those imposed by the conditions of the Theorem. Unless $f_n = p(z_1^n)$ for all n, where z_1 is an integer greater than 1, the equation $f_n = p(m_n)$ must have a finite number of solutions.

Note that the conditions of the Theorem are satisfied by the example cited in part B, with $m_n = 2^n$, $z_1 = 2$, $p(z) = z^2 + z$.

True or Not?

H-457 ***Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy***
(Vol. 29, no. 3, August 1991)

Let $f(N)$ denote the number of addends in the Zeckendorf decomposition of N. The numerical evidence resulting from a computer experiment suggests the following two conjectures. Can they be proved?

Conjecture 1: For given positive integers k and n, there exists a positive integer n_k (depending on k) such that $f(kF_n)$ has a constant value for $n \geq n_k$.

For example,

$$24F_n = F_{n+6} + F_{n+3} + F_{n+1} + F_{n-4} + F_{n-6} \text{ for } n \geq 8.$$

By inspection, we see that $n_1 = 1$, $n_k = 2$ for $k = 2$ or 3, $n_4 = 4$ and $n_k = 5$ for $5 \leq k \leq 8$.

Conjecture 2: For $k \geq 6$, let us define (i) μ, the subscript of the smallest odd-subscripted Lucas number such that $k \leq L_\mu$, and (ii) ν, the subscript of the largest Fibonacci number such that $k > F_\nu + F_{\nu-6}$. Then, $n_k = \max(\mu, \nu)$.

Solution by Paul S. Bruckman, Edmonds, WA

We suppose $n \geq 2$. As we know, any natural number u has a unique Zeckendorf representation (Z-rep. for short) which is given by:

(1) $\quad u = \sum_{j=2}^{r} \theta_j F_j$, where $\theta_j = 0$ or 1, $\theta_j\theta_{j+1} = 0$, $j = 2, 3, \ldots, r-1$, and $\theta_r = 1$.

We shall show that Conjecture 1 is true, Conjecture 2 false. Moreover, the following "observations" are the correct ones for n_k: $n_1 = 2$, $n_k = 4$ for $2 \leq k \leq 4$, $n_k = 6$ for $5 \leq k \leq 11$, $n_k = 8$ for $12 \leq k \leq 29$, etc.; in general:

(2) $\quad n_k = 2m + 2$, where m is determined by $L_{2m-1} < k \leq L_{2m+1}$, $m = 1, 2, \ldots$.

Therefore,

(3) $\quad n_k = 1 + \mu$, where μ is as defined by the proposer.

To prove the assertions in (2) and (3), it will suffice to prove (4) and (5) below.

(4) Given k such that $L_{2m-1} < k < L_{2m}$, then for all $n \geq 2m+2$ there exists a Z-rep. for kF_n given by:

$$kF_n = \sum_{j=-2m}^{2m-1} \theta_J^{(k)} F_{n+j}, \text{ where } \theta_{-2m}^{(k)} = \theta_{2m-1}^{(k)} = 1.$$

(5) Given k such that $L_{2m} \leq k \leq L_{2m+1}$, then for all $n \geq 2m+2$ there exists a Z-rep. for kF_n given by:

$$kF_n = \sum_{j=-2m}^{2m} \theta_j^{(k)} F_{n+j}, \text{ where } \theta_{-2m}^{(k)} = \theta_{2m}^{(k)} = 1.$$

In these expressions, the $\theta_j^{(k)}$'s are dependent on k but not on n. In the sequel, we shall frequently employ sums of the type

$$\sum_{j=r}^{s} \theta_j^{(k)} F_{n+j}.$$

For brevity, we shall denote such a sum by $S(r, s)$, If we wish to emphasize that $\theta_s^{(k)} = 1$, we shall use the notation $S(r, \underline{s})$; similar notation makes the symbols $S(\underline{r}, s)$ and $S(\underline{r}, \underline{s})$ self-explanatory. Of course, all such sums are understood to be Z-reps. Some preliminary lemmas are needed to prove (4) and (5).

Lemma 1:

(6) (i) $2F_n = F_{n+1} + F_{n-2}$; (ii) $3F_n = F_{n+2} + F_{n-2}$; (iii) $4F_n = F_{n+2} + F_n + F_{n-2}$.

We omit the proof, as this is readily verified. Note that the right member of the expressions in (i)-(iii) are Z-reps., with r = -2, and are therefore valid for all $n \geq 4$. Since $f(kF_n) = 2$, $k = 2, 3$, and $f(4F_n) = 3$ for all $n \geq 4$, it follows that $n_k = 4$ for $k = 2, 3, 4$. Of course, $F_n = F_n$ for all $n \geq 2$, so $n_1 = 2$.

Lemma 2:

(7) $L_{2m}F_n = F_{n+2m} + F_{n-2m}$.

This is also readily verified. Note that the right member of (7) is of the form $S(-\underline{2m}, \underline{2m})$, and is in fact the *unique* $S(-\underline{2m}, \underline{2m})$ of minimum length. Thus, $f(L_{2m}F_n) = 2$ for all $n \geq 2m+2$; hence, $n_{L_{2m}} = 2m+2$.

Lemma 3:

(8) $$L_{2m+1}F_n = \sum_{j=-m}^{m} F_{n+2j} = F_{n+2m+2} + F_{n-2m-2} - F_{n+2m} - F_{n-2m}.$$

We omit the proof, leaving it as an exercise. Note that $L_{2m+1} = L_{2m+2} - L_{2m}$, which leads to the second relation in (8), using Lemma 2. The sum in (8) is a Z-rep. of the form $S(-\underline{2m}, \underline{2m})$, valid for all $n \geq 2m+2$. Hence, $f(L_{2m+1}F_n) = 2m+1$ for all $n \geq 2m+2$, and $n_{L_{2m+1}} = 2m+2$.

We now proceed to the proof of (4) and (5), by induction on m. Let T denote the set of all positive integers m for which (4) and (5) are both true. (4) is true for $m=1$ ($k=2$), and (5) is true for $m=1$ ($k=3,4$), by Lemma 1. Therefore, $1 \in T$. Suppose $1, 2, \ldots, m \in T$ (the inductive hypothesis). We break up our proof into six subcases:

Case 1. Suppose $5F_{2m} < k < L_{2m+2}$. Then $L_{2m-1} < k - L_{2m+1} < L_{2m}$. Using (4) (supposed true for m), we have:

$$(k - L_{2m+1})F_n = \sum_{j=-2m}^{2m-1} \theta_j^{(k)} F_{n+j} \text{ for all } n \geq 2m+2.$$

Then, by Lemma 3,

$$\begin{aligned} kF_n &= S(-\underline{2m}, \underline{2m-1}) + F_{n+2m+2} + F_{n-2m-2} - F_{n+2m} - F_{n-2m} \\ &= S(-\underline{2m+2}, \underline{2m-1}) + F_{n+2m+1} + F_{n-2m-2} \\ &= S(-\underline{2m-2}, \underline{2m+1}), \text{ for all } n \geq 2m+4, \end{aligned}$$

which is the statement of (4) for $m + 1$.

Case 2. Suppose $2L_{2m} \leq k \leq 5F_{2m}$. Then $L_{2m-2} \leq k - L_{2m+1} \leq L_{2m-1}$. Using (5) for $m-1$,

$$(k - L_{2m+1})F_n = S(-\underline{2m+2}, \underline{2m-2}) \text{ for all } n \geq 2m.$$

Then, by Lemma 3,

$$\begin{aligned} kF_n &= S(-\underline{2m+2}, \underline{2m-2}) + F_{n+2m+2} + F_{n-2m-2} - F_{n+2m} - F_{n-2m} \\ &= S(-\underline{2m+1}, \underline{2m-2}) + F_{n+2m+1} + F_{n-2m-2} \\ &= S(-\underline{2m-2}, \underline{2m+1}), \text{ for all } n \geq 2m+4, \end{aligned}$$

which is the statement of (4) for $m + 1$.

Case 3. Suppose $L_{2m+1} < k < 2L_{2m}$. Then $L_{2m-1} < k - L_{2m} < L_{2m}$. By (4), for m,

$$(k - L_{2m})F_n = S(-\underline{2m}, \underline{2m-1}) \text{ for all } n \geq 2m+2.$$

Then, by Lemma 2,

$$\begin{aligned} kF_n &= S(-\underline{2m}, \underline{2m-1}) + F_{n+2m} + F_{n-2m} \\ &= S(-2m+2, 2m-3) + 2F_{n-2m} + F_{n+2m-1} + F_{n+2m} \\ &= S(-2m+2, 2m-3) + F_{n-2m+1} + F_{n-2m-2} + F_{n+2m+1}. \end{aligned}$$

If $\theta_{-2m+2}^{(k)} = 0$, then

$$kF_n = S(-\underline{2m+1}, \underline{2m+1}) + F_{n-2m-2} = S(-\underline{2m-2}, \underline{2m+1}).$$

If $\theta_{-2m+2}^{(k)} = 1$, then $\theta_{-2m+3}^{(k)} = 0$, and

$$\begin{aligned} kF_n &= S(-2m+4, 2m-3) + F_{n-2m+2} + F_{n-2m+1} + F_{n-2m-2} + F_{n+2m+1} \\ &= S(-2m+4, \underline{2m+1}) + F_{n-2m-2} + F_{n-2m+3}. \end{aligned}$$

Since $\theta_{2m+1}^{(k)} = \theta_{2m-3}^{(k)} = 1$, we must have $\theta_{2j}^{(k)} = 0$ for at least one j with $-m+2 \le j \le m-3$, and certainly $\theta_{2m-4}^{(k)} = \theta_{2m-2}^{(k)} = \theta_{2m}^{(k)} = 0$. Thus, $kF_n = S(-\underline{2m+2r+1}, \underline{2m+1}) + F_{n-2m-2}$ for some $r \ge 0$, which implies $kF_n = S(-\underline{2m-2}, \underline{2m+1})$ for all $n \ge 2m+4$. This is the statement of (4) for $m+1$.

Combining cases 1, 2, and 3, we see that if $L_{2m+1} < k < L_{2m+2}$, then the assertion of (4) for $m+1$ is valid. Thus, $m \in T$ implies (4) for $m+1$.

Case 4. Suppose $L_{2m+2} \le k \le 2L_{2m+1}$. Then $L_{2m} \le k - L_{2m+1} \le L_{2m+1}$. By (5), for m, $(k - L_{2m+1})F_n = S(-\underline{2m}, \underline{2m})$ for all $n \ge 2m+2$. Then, by Lemma 3,

$$\begin{aligned} kF_n &= S(-\underline{2m}, \underline{2m}) + F_{n+2m+2} + F_{n-2m-2} - F_{n+2m} - F_{n-2m} \\ &= S(-\underline{2m+2}, \underline{2m-2}) + F_{n+2m+2} + F_{n-2m-2} \\ &= S(-\underline{2m-2}, \underline{2m+2}) \text{ for all } n \ge 2m+4, \end{aligned}$$

which is the statement of (5) for $m+1$.

Case 5. Suppose $2L_{2m+1} < k < 5F_{2m+1}$. Then $L_{2m-1} < k - L_{2m+2} < L_{2m}$. By (4), for m, $(k - L_{2m+2})F_n = S(-\underline{2m}, \underline{2m-1})$ for all $n \ge 2m+2$. Then, by Lemma 2,

$$\begin{aligned} kF_n &= S(-\underline{2m}, \underline{2m-1}) + F_{n+2m+2} + F_{n-2m-2} \\ &= S(-\underline{2m-2}, \underline{2m+2}) \text{ for all } n \ge 2m+4, \end{aligned}$$

which is the statement of (5) for $m+1$.

Case 6. Suppose $5F_{2m+1} \le k \le L_{2m+3}$. Then $L_{2m} \le k - L_{2m+2} \le L_{2m+1}$. Then, using (5), for m, $(k - L_{2m+2})F_n = S(-\underline{2m}, \underline{2m})$ for all $n \ge 2m+2$. Then, by Lemma 2,

$$\begin{aligned} kF_n &= S(-\underline{2m}, \underline{2m}) + F_{n+2m+2} + F_{n-2m-2} \\ &= S(-\underline{2m-2}, \underline{2m+2}) \text{ for all } n \ge 2m+4. \end{aligned}$$

This is the statement of (5) for $m+1$.

Combining cases 4, 5, and 6, we see that if $L_{2m+2} \le k \le L_{2m+3}$ and $m \in T$ is assumed, then (5) holds for $m+1$. Combining this conclusion with the conclusion of case 3, we see that $m \in T$ implies $(m+1) \in T$. Since $1 \in T$, the proof of (4) and (5) by induction is complete.

These relations, in turn, imply the truth of the original assertions [(2) and (3)]. For (4) and (5) they may be combined as follows:

(9) Given k such that $L_{2m-1} < k \le L_{2m+1}$, then for all $n \ge 2m+2$,

$$kF_n = S(-\underline{2m}, 2m), \text{ and } \theta_{2m}^{(k)} + \theta_{2m-1}^{(k)} = 1.$$

We see from (9) that $n_k = 2m+2$, where $2m+1 = \mu$, as defined by the proposer. This proves (3). Q.E.D.

Editorial Note: Russell Hendel's name was omitted from the list of solvers of H-453.

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A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

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