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OF INTEGERS WITH SPECIAL PROPERTIES

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ON THE NUMBER OF INDEPENDENT SETS OF NODES IN A TREE

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1. INTRODUCTION

In [4] Wilf shows that the number of maximal independent sets of nodes (MIS) for a nonempty tree on n nodes is bounded above by

$$f(n)=\begin{cases}2^{n/2-1}+1 & \text{if } n \text{ is even,}\\ 2^{(n-1)/2} & \text{if } n \text{ is odd.}\end{cases}$$

For each value of n, he gives a tree, depending upon the parity of n, that attains these bounds. The two general forms are shown below in Figure 1.

(a) *n odd* (b) *n even*

FIGURE 1

Throughout, we assume nonempty trees and, following the notation in [4], let $\mu(T)$ be the number of MIS's in a tree T. We will derive lower and upper bounds on $\mu(T)$ in terms of $\beta_1(T)$, the maximum number of independent edges in T.

First observe that, in any graph, two degree-one nodes having a common neighbor occur in the same MIS's. Thus, the number of MIS's is unaffected by the removal of one of these nodes. Such "pruning" can be repeated, and we formalize this fact as a lemma. Although the lemma is stated here for trees, it is actually valid for arbitrary graphs, and demonstrates, in some sense, the independence between the number of nodes and the number of maximal independent sets of nodes.

Lemma 1: Let T be a tree and T' the tree obtained by removing all but one degree-one neighbor from every node having two or more such neighbors. Then $\mu(T)=\mu(T')$ and $\beta_1(T)=\beta_1(T')$.

Any tree with diameter d, $2\le d\le 4$, can be reduced by Lemma 1 to one of the forms in Figure 1. The n-even case arises from trees containing two degree-one nodes that are distance three from each other. Define T_e to be this set of trees and let T_o be the remaining trees with diameter between two and four. Notice that K_1 and K_2 are the only trees with diameter less than or equal to four that are not in $T_e \cup T_o$. Neither are they reducible to a tree of Figure 1. For these, though, we know that $\mu(K_1)=1$ and $\mu(K_2)=2$. We can determine exactly $\mu(T)$ for any tree T with diameter at most four.

Lemma 2: Let T be a tree with diameter at most four and $\beta_1 = \beta_1(T)$. Then

$$\mu(T) = \begin{cases} 2^{\beta_1 - 1} + 1 & \text{if } T \in T_e \cup \{K_2\}, \\ 2^{\beta_1} & \text{if } T \in T_o \cup \{K_1\}. \end{cases}$$

Proof: All trees in $T_e \cup T_o$ must, by the above discussion, reduce to either the n even (with $n = 2\beta_1$) or the n odd (with $n = 2\beta_1 + 1$) case in Figure 1. The result then follows from $f(n)$ given above. Finally, since $\beta_1(K_1) = 0$, $\mu(K_1) = 1$, $\beta_1(K_2) = 1$, and $\mu(K_2) = 2$, K_1 and K_2 also satisfy the lemma. □

The trees in $T_e \cup T_o$ will be called **terminal trees** or **terminal subtrees** when part of a larger tree, and will have an assigned root node u. With one exception, the root node must be selected from those nodes that, after pruning, would be nodes of maximum degree. The star $K_{1,n}$, the exception, must be rooted at a leaf node. The root of a terminal subtree S has a single neighbor not in S. The neighborhoods of all other nodes in S are a subset of S. In a pruned tree, a subtree whose removal would disconnect the graph or leave an isolated K_1 or K_2 is not a terminal subtree. The trees in Figure 1 are terminal trees. The tree T' in Figure 2 below is formed by removing a terminal subtree from T. All trees, other than K_1 and K_2, are either themselves terminal trees or contain at least two terminal subtrees. Thus, for any pruned tree T with diameter at least five, there exist adjacent nodes u and v permitting T to be drawn in one of the two forms of Figure 2, where u is the root of a terminal subtree and v is in the subtree T'.

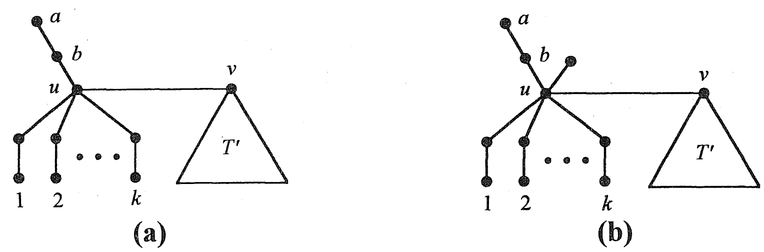

FIGURE 2

The structure of the graphs in Figure 2 corresponds to the structure in Figure 2 of Wilf's paper [4]. From this we see that Wilf's equation (2), a recursive equation solving $\mu(T)$, has a simpler form because of the pruning permitted by Lemma 2. We include it here, along with the conclusions of Lemma 2, where $\beta_1 = \beta_1(T)$ and k, a, and b are as in Figure 2.

$$\text{(1)} \qquad \mu(T) = \begin{cases} 2^{\beta_1 - 1} + 1 & \text{if } T \in T_e \cup \{K_2\}, \\ 2^{\beta_1} & \text{if } T \in T_o \cup \{K_1\}, \\ \mu(T - \{a,b\}) + 2^k \mu(T') & \text{otherwise.} \end{cases}$$

Part three applies when the diameter is at least five, and then $\beta_1(T - \{a,b\}) = \beta_1 - 1$ and $\beta_1(T')$ is either $\beta_1 - k - 2$ or $\beta_1 - k - 1$. In either case, the subtree T' has at least three nodes.

We use this result to obtain a lower bound on the number of MIS in a tree. Then we use another tree-reduction operation to determine new lower bounds, and also new upper bounds which normally improve those given by Wilf. Finally, we obtain bounds on the number of independent sets (including nonmaximal) of nodes in an arbitrary tree.

2. IMPROVED BOUNDS

Since $\mu(T)$ is essentially independent of the number of nodes in T, we look for bounds with respect to the edge independence number $\beta_1(T)$. The first, Theorem 1, is a lower bound for $\mu(T)$ that appeals to Lemma 1 and the Fibonacci numbers F_n.

Sanders [3] exhibits a tree on $2n$ nodes and proves it has F_{n+2} maximal independent sets of nodes. The tree, called an extended path, is formed by appending a single degree-one node to each node of a path on n nodes. In terms of their edge independence number, we have for such trees T that $\mu(T) = F_{\beta_1+2}$, where $\beta_1 = \beta_1(T) = n$. We show next that, for a given value of β_1, no tree T with $\beta_1(T) = \beta_1$ has a smaller number of maximal independent sets of nodes. Therefore, because of Lemma 1, extended paths actually represent, for each value of β_1, an infinite class of trees satisfying the bound.

Theorem 1: Let T be any tree with $\beta_1 = \beta_1(T)$. Then $\mu(T) \geq F_{\beta_1+2}$.

Proof: If $T \in T_e \cup T_o \cup \{K_1, K_2\}$, then the result follows because F_{β_1+2} is bounded above by the appropriate $2^{\beta_1-1}+1$ or 2^{β_1} value indicated in equation (1). Otherwise, we can use the recurrence formula in equation (1) inductively to conclude that $\mu(T) \geq F_{\beta_1+1} + 2^k F_{\beta_1-k}$. It is straightforward to show, by another induction argument, that $2^k F_{\beta_1-k} \geq F_{\beta_1}$. Therefore, $\mu(T) \geq F_{\beta_1+1} + F_{\beta_1} = F_{\beta_1+2}$. □

Terminal subtrees can be removed from a tree T, one at a time, until T is empty providing, in some sense, a count of the number of terminal subtrees in T. Since the order of removal is not unique, one might suspect that the subtrees obtained in such a removal scheme also may not be unique. This is indeed the case and can be verified by examining a few small examples. It also would seem the number found could vary depending upon the order of removal. We now show that this does not occur.

Lemma 3: For any tree, every order of terminal subtree removal results in the same number of removed subtrees.

Proof: Let $t_{\min}(T)$ and $t_{\max}(T)$ be the minimum and maximum number of terminal subtrees that can be removed from a tree T, under any order of removal. If T is itself a terminal tree, the result holds since there is no option but to remove the entire tree. This also implies $t_{\min}(T) = 2$ whenever $t_{\max}(T) = 2$. Now, letting $t_{\max}(T) = m \geq 3$, we show by induction that $t_{\min}(T)$ also must equal m. For some k, $2 \leq k \leq t_{\max}(T)$, there exist terminal subtrees $S_1, S_2, \ldots, S_k$ of T, any one of which can be an initial subtree removed from T. There exist indices i and j, $1 \leq i \neq j \leq k$, for which $t_{\max}(T - S_i) = t_{\max}(T) - 1 = m - 1$ and $t_{\min}(T - S_j) = t_{\min}(T) - 1 < m - 1$. By the induction hypothesis, terminal subtrees can be removed in any order from $T - S_i$ and $T - S_j$ without affecting the number of such removals. Furthermore, S_j is a terminal subtree of $T - S_i$ and S_i is one of $T - S_j$. Thus, $t_{\max}(T - S_i - S_j) = t_{\max}(T) - 2 = m - 2$ and $t_{\min}(T - S_j - S_i) = t_{\min}(T) - 2 < m - 2$, a contradiction implied by the induction hypothesis since $T - S_j - S_i = T - S_i - S_j$. Hence, $t_{\min}(T) = m$. □

In view of Lemma 3, it is now possible to define, for any tree T, a new invariant $t(T)$ to be the number of terminal subtrees removable from T. It is convenient to let $t(K_1) = t(K_2) = 0$.

Theorem 2: Let T be a tree with $\beta_1 = \beta_1(T)$ and $t = t(T)$. Then $2^{\beta_1-t} + 2^t - 1 \leq \mu(T) \leq 2^{\beta_1}$.

Proof: If $t(T) \leq 1$, then T is in $T_e \cup T_o \cup \{K_1, K_2\}$ and both bounds follow from the first two cases of equation (1). Now consider the case in which $\beta_1 = 2t$ Then the lower bound is $2^{t+1} - 1$. A straightforward induction argument shows $2^{t+1} - 1 \leq F_{2t+2}$; then Theorem 1 establishes that such trees satisfy the lower bound of the theorem. Now, assume T is a tree with $t(T) = t \geq 2$ and that the lower bound is satisfied by all trees with either fewer than t terminal subtrees or with t terminal subtrees and $2t$ independent edges. Then, we can invoke the third part of equation (1) inductively and have, referring to Figure 2,

$$\beta_1(T - \{a, b\}) = \beta_1 - 1;\ t - 1 \leq t(T - \{a, b\}) \leq t;$$
$$\beta_1 - k - 2 \leq \beta_1(T') \leq \beta_1 - k - 1;\ t(T') = t - 1.$$

The lower bound decreases as β_1 decreases and, when t decreases, the lower bound decreases if and only if $\beta_1 < 2t$ Thus, we must consider two cases:

Case 1. $\beta_1 < 2t$ and $t(T - \{a, b\}) = t - 1$. Then

$$\begin{aligned}\mu(T) &\geq \{2^{\beta_1 - t} + 2^{t-1} - 1\} + 2^k\{2^{\beta_1 - k - t - 1} + 2^{t-1} - 1\}\\ &= 2^{\beta_1 - t} + 2^{t-1} - 1 + 2^{\beta_1 - t - 1} + 2^k(2^{t-1} - 1) \geq 2^{\beta_1 - t} + 2^t - 1.\end{aligned}$$

Case 2. $\beta_1 > 2t$ and $t(T - \{a, b\}) = t$. The result when $\beta_1 = 2t$ has already been established. We again use the recursive part of equation (1), where $t(T - \{a,b\}) = t$ and $\beta_1(T - \{a,b\}) = \beta_1 - 1$, and proceed by induction on the value of β_1. It follows that

$$\begin{aligned}\mu(T) &\geq \{2^{\beta_1 - 1 - t} + 2^t - 1\} + 2^k\{2^{\beta_1 - k - t - 1} + 2^{t-1} - 1\}\\ &= 2^{\beta_1 - t - 1} + 2^t - 1 + 2^{\beta_1 - t - 1} + 2^k(2^{t-1} - 1) > 2^{\beta_1 - t} + 2^t - 1.\end{aligned}$$

establishes the lower bound.

To verify the right inequality, we again use equation (1) inductively. The result was shown above for all trees with $t(T) \leq 1$. Assume T is a tree with $t(T) \geq 2$ and that the result holds for all trees with edge independence number less than β_1. Then $\beta_1 = \beta_1(T) \geq 3$ and $\mu(T) \leq 2^{\beta_1 - 1} + 2^k 2^{\beta_1 - k - 1} = 2^{\beta_1}$. □

When $t(T) \leq 1$, regardless of the value of $\beta_1(T)$, equation (1) shows that equality holds on the right in the n odd cases of Figure 1 and on the left in the n even cases. Other trees can be obtained by appending an arbitrary number of degree-one neighbors to the degree-two nodes in either of the trees in Figure 1. This process produces all trees T for which $t(T) =$.

The upper bound also is achievable, for any β_1 and $t \geq 2$, by an infinite number of trees. Consider the tree in Figure 2(a). The recurrence in equation (1) can be iterated k times, on the first term, to give the equation $\mu(T) = \mu(T'') + (2^{k+1} - 1)\mu(T')$, where T' is the same as in Figure 2, and T'' is T' with node v having node u as a degree-one neighbor. We call this the **iterated recurrence formula**. From Lemma 1, if node v already has a degree-one neighbor, $\mu(T'') = \mu(T')$ and the recurrence formula simplifies to $\mu(T) = 2^{k+1}\mu(T')$. We now construct a tree T that has this property at each step of the iterated recurrence. Let T_1 be any tree in T_0. For $t \geq 2$, let S_t be any tree in T_0 with its identified root node u. Now, form T_t by adding an edge between node u in S_t and any node in T_{t-1} having a degree-one neighbor. Clearly, $\beta_1(T_t) = \beta_1(S_t) + \beta_1(T_{t-1})$, and an induction argument with $\mu(T_t) = 2^{k+1}\mu(T_{t-1})$ shows that $\mu(T_t) = 2^{\beta_1(T_t)}$. The lower bound is also achieved, when t = 2, by any tree that can be pruned to P_6, the path on six nodes.

We conclude this section with an upper bound on $\mu(T)$ for a restricted class of trees that will prove useful in the next section. Let T^* be the collection of trees with every node being a degree-one node or having a degree-one neighbor. First, an upper bound independent of $t(T)$ is given.

Theorem 3: Let $T \in T^*$ with $\beta_1 = \beta_1(T)$. Then $\mu(T) \le 2^{\beta_1 - 1} + 1$.

Proof: If $t(T) \le 1$, then $T \in T_e \cup \{K_2\}$ and equality follows from equation (1). Now, assume that $T \in T^*$, $t(T) \ge 2$, $\beta_1(T) = \beta_1 = m \ge 3$, and that the result holds for trees in T^* with fewer independent edges. Identify a terminal subtree, as in Figure 2(b), and use the recurrence in equation (1). We have $T - \{a, b\}$ and T' both in T^* and $\beta_1(T - \{a, b\}) = \beta_1 - 1$. Here, we are guaranteed that $\beta_1(T') = \beta_1 - k - 2$. Therefore, by the induction hypothesis,

$$\mu(T) \le 2^{\beta_1 - 2} + 1 + 2^k(2^{\beta_1 - k - 3} + 1) = 2^{\beta_1 - 2} + 1 + 2^{\beta_1 - 3} + 2^k.$$

Since $1 \le \beta_1(T') = \beta_1 - k - 2$, we have $k \le \beta_1 - 3$ and then $\mu(T) \le 2^{\beta_1 - 1} + 1$. □

Lemma 4: Let $T \in T^*$ with $\beta_1 = \beta_1(T)$ and $t = t(T)$. Then $\beta_1 \ge 2t$.

Proof: If $T = K_2$, then $t(T) = 0$ and the conclusion follows. If $t = 1$, then $T \in T_e$ and, for all such trees, $\beta_1(T) \ge 2 = 2t$. Assume $t > 1$ and that the result holds for all trees with fewer terminal subtrees. Now, let $T \in T^*$ with $t(T) = t$. From previous discussions and Figure 2(b), we know that $t(T') = t - 1$, $\beta_1(T') = \beta_1 - k - 2$, and $T' \in T^*$. Therefore, by the induction hypothesis,

$$\beta_1 - k - 2 \ge 2(t - 1).$$

Since $k \ge 0$, the result follows. □

The bound of Theorem 3 can be improved when $t(T)$ is known. We will again make use of the iterated form of the recurrence formula described after Theorem 2.

Theorem 4: Let $T \in T^* - \{K_2\}$ with $\beta_1 = \beta_1(T)$ and $t = t(T)$. Then $\mu(T) \le 3^{t-1} 2^{\beta_1 - 2t + 1} + 2^{t-1}$.

Proof: When $t = 1$, the right-hand side reduces to $2^{\beta_1 - 1} + 1$, the bound given in Theorem 3. Suppose $T \in T^*$ with $t(T) = t \ge 2$ and that the result holds for all trees with fewer terminal subtrees. The iterated form of the recurrence in equation (1) is $\mu(T) = \mu(T'') + (2^{k+1} - 1)\mu(T')$, where T'' is as described in the discussion following Theorem 2. Then $\beta_1(T'') = \beta_1 - k - 1$ and $t - 1 \le t(T'') \le t$ and, since the bound increases as t decreases, we have by the induction hypothesis that

$$\mu(T'') \le 3^{t-2} 2^{\beta_1 - k - 2t + 2} + 2^{t-2} \quad \text{and} \quad \mu(T') \le 3^{t-2} 2^{\beta_1 - k - 2t + 1} + 2^{t-2}.$$

This gives

$$\begin{aligned} \mu(T) &\le 3^{t-2} 2^{\beta_1 - k - 2t + 2} + 2^{t-2} + (2^{k+1} - 1)(3^{t-2} 2^{\beta_1 - k - 2t + 1} + 2^{t-2}) \\ &= 3^{t-2} 2^{\beta_1 - k - 2t + 2} + 3^{t-2} 2^{\beta_1 - 2t + 2} - 3^{t-2} 2^{\beta_1 - k - 2t + 1} + 2^{k+1} 2^{t-2} \\ &= 3^{t-2} 2^{\beta_1 - k - 2t + 1} + 3^{t-2} 2^{\beta_1 - 2t + 2} + 2^{k+1} 2^{t-2} \end{aligned}$$

Suppose this bound is greater than $3^{t-1} 2^{\beta_1 - 2t + 1} + 2^{t-1}$. Then we have

$$2^{k+1} 2^{t-2} - 2^{t-1} > 3^{t-1} 2^{\beta_1 - 2t + 1} - (3^{t-2} 2^{\beta_1 - 2t + 2} + 3^{t-2} 2^{\beta_1 - k - 2t + 1}) \text{ or}$$

$$2^{t-1}(2^k - 1) > 3^{t-2} 2^{\beta_1 - 2t + 1}(1 - 2^{-k}) \text{ or } 2^{t-1} 2^k > 3^{t-2} 2^{\beta_1 - 2t + 1}.$$

Since $k \le \beta_1 - 2t$, from the proof of Lemma 4, $2^{t-1} 2^{\beta_1 - 2t} > 3^{t-2} 2^{\beta_1 - 2t + 1}$ or $2^{t-2} > 3^{t-2}$, a contradiction. □

Trees achieving this bound are presented at the end of the next section.

3. NUMBER OF INDEPENDENT SETS

Now consider the number of independent sets of nodes in a tree. The counted sets must be distinct, but they need not be maximal, and we count the empty set. For example, the star $K_{1,n}$ has 2^n+1 independent sets of nodes. Denote this number by $\mu^*(T)$. Prodinger & Tichy [2] have shown, for an arbitrary tree on n nodes, that $F_{n+2} \le \mu^*(T) \le 2^{n-1}+1$. The left inequality holds for a path on n nodes and the right for the star $K_{1,n-1}$. We shall derive these bounds in a manner which also exhibits a relationship between this and the original problem of counting the number of maximal independent sets of nodes.

In this section, let T^* be the tree obtained from the tree T by appending a single pendant edge to each node of T.

Lemma 5: For any tree T, $\mu(T^*) = \mu^*(T)$.

Proof: Let T have nodes $V(T) = \{v_1, v_2, \ldots, v_n\}$ and T^* have additional nodes $\{w_1, w_2, \ldots, w_n\}$, where w_i has v_i as its only neighbor, for $1 \le i \le n$. For any set of nodes S, it is immediate that S is an independent set of nodes in T if and only if $S^* = S \cup \{w_i | v_i \notin S\}$ is a maximal independent set of nodes in T^*. □

If terminal subtrees are systematically removed from T^* until it is empty, one finds, as will be shown in Lemma 6, that the collection of identified root nodes forms a minimum node cover of the original tree T. The number of these covering nodes is equal to $\beta_1(T)$, a relationship that holds for any triangle-free graph [1, p. 171]. Let $\beta_0(T)$ be the node independence number of the tree T. Then, if T has n nodes, $n - \beta_0(T) = \beta_1(T)$ is the size of a smallest node cover of T.

Lemma 6: For any tree T, $t(T^*) = \beta_1(T)$.

Proof: Induct on the value of $t(T^*)$, and first consider the case in which $t(T^*) = 0$. Then $T^* = K_2$ and $T = K_1$, and the base case is established. Now, suppose T is a tree with $t(T^*) = m \ge 1$ and that the lemma holds for all similarly constructed trees T^* for which $t(T^*) < m$. Let S be any terminal subtree of T^*. Then $t(T^* - S) = t(T^*) - 1$ and, by the induction hypothesis,

$$t(T^*-S) = n - |S \cap T| - \beta_0(T - S \cap T) = n - \beta_0(T) - 1.$$

The result follows. □

The number of nodes in T is $\beta_1(T^*)$ and, from Lemma 6 and Theorems 1, 2, and 4, we have the following bounds on $\mu^*(T)$.

Theorem 5: Let T be any tree on $n \ge 2$ nodes with $\beta_1 = \beta_1(T)$. Then

$$\max\{F_{n+2}, 2^{n-\beta_1} + 2^{\beta_1} - 1\} \le \mu^*(T) \le 3^{\beta_1 - 1}2^{n-2\beta_1+1} + 2^{\beta_1 - 1}.$$

It is known [2] that $\mu^*(P_n) = F_{n+2}$, where P_n is the path on n nodes. Therefore, we have $\mu(T) = F_{\beta_1+2}$ for trees T constructed from a path on $\beta_1 = \beta_1(T)$ nodes with each node having one or more degree-one neighbors appended to it. These trees were introduced in the discussion prior to Theorem 1 and were shown to be a generalization of the extended paths given in [3]. The above has given an alternate proof for the number of MIS in such trees and reaffirms that they represent an infinite class of trees having the smallest number of MIS for a given number of maximum independent edges.

An infinite class of trees satissfying the bounds of Theorem 4 can be constructed with the aid of Lemmas 5 and 6. First, we will form a tree T for which $\mu^*(T) = 3^{\beta_1-1}2^{n-2\beta_1+1} + 2^{\beta_1-1}$, the upper bound of Theorem 5. For any positive integers t and β_1, $2t \le \beta_1$, construct a star on $\beta_1 - t + 1$ nodes. Next, append a degree-one node to $t - 1$ of the leaf nodes of the star, as in Figure 3, and let this tree be T. Observe that $\beta_1(T) = t$, $t(T) = 1$, and the number of nodes is β_1.

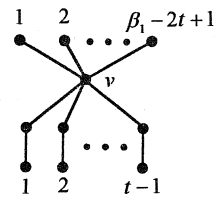

FIGURE 3

Now consider the number of independent sets of nodes in this tree. First, examine the independent sets of nodes not containing the center node v. Node v has $\beta_1 - 2t + 1$ degree-one neighbors that can be members of an independent set of nodes in 2^{β_1-2t+1} ways. It also has $t - 1$ degree-two neighbors, each with a degree-one neighbor. A degree-two node and its degree-one neighbor can contribute to an independent set of nodes in any of three ways: either node individually or neither node. Thus, there are 3^{t-1} ways to select independent sets of these nodes. Together, we have a total of $3^{t-1}2^{\beta_1-2t+1}$ ways to form independent sets of nodes not including node v. When node v is included, only the $t - 1$ nodes distance two from v can be used. There are 2^{t-1} such sets. The total now is $3^{t-1}2^{\beta_1-2t+1} + 2^{t-1}$ and, since $t = \beta_1(T)$ and β_1 is the number of nodes, T is a tree that leads to the upper bound of Theorem 5. Now, for any $n \ge 2\beta_1$, construct T^* by appending a degree-one node to every node of T. Then $\beta_1(T^*) = \beta_1$, $t(T^*) = t$, and Lemma 6 shows that the number of MIS in T^* is $3^{t-1}2^{\beta_1-2t+1} + 2^{t-1}$, the upper bound of Theorem 4. To obtain the desired number of nodes n, merely append a total of $n - 2\beta_1$ degree-one nodes to any node(s) already having at least one such neighbor.

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CARLITZ GENERALIZATIONS OF LUCAS AND LEHMER SEQUENCES

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1. INTRODUCTION

A Lucas fundamental sequence $\{u_n\}_{n=0}^{\infty}$ is a nondegenerate binary recurrence sequence with initial conditions $u_0 = 1$, $u_1 = P$ which satisfies the homogeneous second-order linear recurrence relation

$$u_n = Pu_{n-1} - Qu_{n-2}, \quad n \geq 2, \tag{1.1}$$

where P and Q are integers [12].

If the associated auxiliary equation

$$x^2 - Px + Q = 0 \tag{1.2}$$

has roots α, β, then

$$u_n = (\alpha^{n+1} - \beta^{n+1}) / (\alpha - \beta). \tag{1.3}$$

The Fibonacci, Mersenne, and Fermat numbers are all types of Lucas numbers. Their properties were studied extensively by Carmichael [5].

Many authors have generalized aspects of them by various alterations to the characteristic equations. Some of these may be found in Dickinson [6], Feinberg [7], Harris & Styles [8], Horadam [10], Miles [14], Raab [15], Williams [19], and Zeitlin [20]. Atanassov et al. [1] have coupled the recurrence relations in their generalizations.

Lehmer [11] generalized the results of Lucas on the divisibility properties of Lucas numbers to numbers

$$\ell_n = \begin{cases} (\alpha^n - \beta^n) / (\alpha - \beta), & \text{for } n \text{ odd} \\ (\alpha^n - \beta^n) / (\alpha^2 - \beta^2), & \text{for } n \text{ even.} \end{cases} \tag{1.4}$$

It is a generalization of these numbers that we wish to consider in this paper. It is of interest to note in passing that McDaniel has also recently studied analogies between the Lucas and Lehmer sequences [13].

2. DEFINITIONS

Following Carlitz [4], we define

$$f_n^{(r)} = (\alpha^{nk+k} - \beta^{nk+k}) / (\alpha^k - \beta^k) \tag{2.1}$$

and

$$g_n^{(r)} = (\alpha^{n+k} - \beta^{n+k}) / (\alpha^k - \beta^k) \tag{2.2}$$

which are not necessarily integers, where $k = r - 1$, and α and β are the roots of (1.2) as before. For example,

$$f_n^{(2)} = (\alpha^{n+1} - \beta^{n+1}) / (\alpha - \beta) = g_n^{(2)} = u_n$$

so that these numbers are generalizations of the Lucas numbers. They are also generalizations of the Lehmer numbers if we let

$$\ell_n = \begin{cases} f_{n-1}^{(2)}, & \text{for } n \text{ odd,} \\ g_{n-2}^{(3)}, & \text{for } n \text{ even,} \end{cases} \tag{2.3}$$

Carlitz [4] first defined the $f_n^{(r)}$ in another context and proved that

$$f_n^{(r)} = tr(A_{n+1}^r)$$

where

$$A_{n+1} = \left[\binom{t}{n-s}\right] \quad (t, s = 0, 1, \ldots n)$$

is a matrix of order $n + 1$.

Examples of $f_n^{(r)}$ and $g_n^{(r)}$ now follow. When $P = -Q = 1$ and $r = 2, 3, 4$, in turn, we have:

n	0	1	2	3	4	5	6	...
$f_n^{(2)}$	1	1	2	3	5	8	13	...
$f_n^{(3)}$	1	3	8	21	55	144	377	...
$f_n^{(4)}$	1	4	17	72	305	1292	5473	...
$g_n^{(2)}$	1	1	2	3	5	8	13	...
$g_n^{(3)}$	1	2	3	5	8	13	21	...
$g_n^{(4)}$	1	$\frac{3}{2}$	$\frac{5}{2}$	4	$\frac{13}{2}$	$\frac{21}{2}$	17	...

It can be seen from this table and the recurrence relations that other properties for these sequences could be developed by treating them as cases of Horadam's $Py_{k-1} + P^2 y_{k-3} = v_k \cdot \{w_n\}$ [10].

3. RECURRENCE RELATIONS

We also need the Lucas primordial sequence $\{v_n\}_{n=0}^{\infty}$ defined by the recurrence relation (1.1) with initial terms $v_0 = 2$ and $v_1 = P$, so that the general term is given by

$$v_n = \alpha^n + \beta^n. \tag{3.1}$$

We can show that

$$f_{n+1}^{(r)} = v_{r-1} f_n^{(r)} - Q^{r-1} f_{n-1}^{(r)} \tag{3.2}$$

and

$$g_{n+1}^{(r)} = v_1 g_n^{(r)} - Q g_{n-1}^{(r)}. \tag{3.3}$$

The latter is the same as (1.1) when $v = 2$ since $v_1 = P$.

Proof of (3.2):

$$\begin{aligned} v_{r-1}f_n^{(r)} - Q^{r-1}f_{n-1}^{(r)} &= \left((\alpha^k+\beta^k)(\alpha^{nk+k}-\beta^{nk+k}) - (\alpha\beta)^k(\alpha^{nk}-\beta^{nk})\right)\Big/(\alpha^k-\beta^k) \\ &= \left(\alpha^{nk+2k}-\beta^{nk+2k} + (\alpha\beta)^k(\alpha^{nk}-\beta^{nk}) - (\alpha\beta)^k(\alpha^{nk}-\beta^{nk})\right)\Big/(\alpha^k-\beta^k) \\ &= (\alpha^{nk+2k}-\beta^{nk+2k})/(\alpha^k-\beta^k) \\ &= f_{n+1}^{(r)}, \text{ as required.} \end{aligned}$$

Proof of (3.3):

$$\begin{aligned} v_1 g_n^{(r)} - Q g_{n-1}^{(r)} &= \left((\alpha+\beta)(\alpha^{n+k}-\beta^{n+k}) - (\alpha\beta)(\alpha^{n+k-1}-\beta^{n+k-1})\right)\Big/(\alpha^k-\beta^k) \\ &= \left(\alpha^{n+k+1}-\beta^{n+k+1} + (\alpha\beta)(\alpha^{n+k-1}-\beta^{n+k-1}) - (\alpha\beta)(\alpha^{n+k-1}-\beta^{n+k-1})\right)\Big/(\alpha^k-\beta^k) \\ &= (\alpha^{n+k+1}-\beta^{n+k+1})/(\alpha^k-\beta^k) \\ &= g_{n+1}^{(r)}, \text{ as required.} \end{aligned}$$

Thus, the ordinary generating functions will be given (formally) by

$$\sum_{n=0}^{\infty} f_n^{(r)} x^n = 1/(1 - v_{r-1}x + Q^{r-1}x^2), \tag{3.4}$$

and

$$\sum_{n=0}^{\infty} g_n^{(r)} x^n = \left(1 - Q\left(\frac{u_{r-3}}{u_{r-2}}\right)x\right)\Big/(1 - v_1 x + Qx^2). \tag{3.5}$$

Proof of (3.4):

$$\begin{aligned} (1 - v_{r-1}x + Q^{r-1}x^2)\sum_{n=0}^{\infty} f_n^{(r)} x^n &= f_0^{(r)} + \left(f_1^{(r)} - f_0^{(r)} v_{r-1}\right)x && \text{[by (3.2)]} \\ &= 1 + \left(\frac{\alpha^{2k}-\beta^{2k}}{\alpha^k-\beta^k} - (\alpha^k+\beta^k)\right)x && \text{[by (2.1)]} \\ &= 1. \end{aligned}$$

Proof of (3.5):

$$\begin{aligned} (1 - v_1 x + Qx^2)\sum_{n=0}^{\infty} g_n^{(r)} x^n &= g_0^{(r)} + \left(g_1^{(r)} - g_0^{(r)} v_1\right)x \\ &= 1 + \left(\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha^k-\beta^k} - (\alpha+\beta)\right)x \\ &= 1 - Q\left(\frac{u_{k-2}}{u_{k-1}}\right)x, \text{ as required.} \end{aligned}$$

The $g_n^{(r)}$ are related to the $f_n^{(r)}$ by

$$g_{kn}^{(r)} = f_n^{(r)}, \tag{3.6}$$

and to the Lucas primordial numbers v_n by

(3.7) $$v_n = Pg_{n-1}^{(r)} - Qv_{r-2}g_{n-k-1}^{(r)}$$

and

(3.8) $$v_n = g_{2n-k}^{(r)} / g_{n-k}^{(r)}.$$

Proof of (3.7):

$$\begin{aligned} Pg_n^{(r)} - Qv_{r-2}g_{n-k}^{(r)} &= \big((\alpha+\beta)(\alpha^{n+k}-\beta^{n+k}) - (\alpha\beta)(\alpha^{k-1}+\beta^{k-1})(\alpha^n-\beta^n)\big)\big/(\alpha^k-\beta^k) \\ &= (\alpha^{n+k+1}-\beta^{n+k+1}+\alpha^k\beta^{n+1}-\alpha^{n+1}\beta^k)/(\alpha^k-\beta^k) \\ &= \big(\alpha^{n+1}(\alpha^k-\beta^k)+\beta^{n+1}(\alpha^k-\beta^k)\big)\big/(\alpha^k-\beta^k) \\ &= v_{n+1}, \text{ as required.} \end{aligned}$$

Proof of (3.8):

$$\begin{aligned} g_{n-k}^{(r)}v_n &= (\alpha^n-\beta^n)(\alpha^n+\beta^n)/(\alpha^k-\beta^k) \\ &= (\alpha^{2n}-\beta^{2n})/(\alpha^k-\beta^k) \\ &= g_{2n-k}^{(r)}, \text{ as required.} \end{aligned}$$

4. GENERALIZATIONS OF BARAKAT'S RESULTS

As an analog of Simson's relation, we have

$$(g_n^{(r)})^2 - g_{n-k}^{(r)}g_{n+k}^{(r)} = Q^n.$$

Proof: The numerator of the left-hand side reduces to

$$\begin{aligned} (\alpha\beta)^n\alpha^{2k} + (\alpha\beta)^n\beta^{2k} - 2(\alpha\beta)^n(\alpha\beta)^k &= (\alpha\beta)^n(\alpha^k-\beta^k)^2 \\ &= Q^n(\alpha^k-\beta^k)^2 \end{aligned}$$

which is Q^n times the denominator of the left-hand side.

When $P=-Q$, we are able to relate the $f_n^{(r)}$ to the ordinary Lucas fundamental numbers, u_n, by means of a generalization of a result of Barakat [2] for the ordinary Lucas fundamental numbers. Barakat proved

$$\begin{aligned} u_n &= \sum_{0\le 2m\le n}\binom{n-m}{m}P^{n-2m}(-Q)^m \\ &= \sum_{0\le 2m\le n}\binom{n-m}{m}P^{n-m} \text{ when } P=-Q. \end{aligned}$$

We define

$$x_n = u_n/P = (\alpha^{n+1}-\beta^{n+1})/(\alpha^2-\beta^2),$$

and, for notational convenience, set $y_n = x_{n+1}$. Thus, from Simson's relation, which can be expressed as $u_k^2 - u_{k-1}u_{k+1} = (-P)^k$, we have

(4.2) $$x_ky_{k-1} - x_{k-1}y_k = (-P)^{k-2}.$$

Proof: If we divide the left-hand side of Simson's relation by P^2, we get

$$x_k y_{k-1} - x_{k-1} y_k = (-P)^{k-2}$$

from which the result follows.

A variation of equation (3.7) is then

$$P y_{k-1} + P^2 y_{k-3} = v_k. \tag{4.3}$$

Proof: The numerator of the left-hand side is

$$\begin{aligned}
&(\alpha+\beta)(\alpha^{k+1}-\beta^{k+1}) + (-\alpha\beta)(\alpha+\beta)(\alpha^{k-1}-\beta^{k-1}) \\
&= \alpha^{k+2} - \beta^{k+2} - \alpha\beta^{k+1} + \alpha^{k+1}\beta - \alpha^{k+1}\beta + \alpha^2\beta^k - \alpha^k\beta^2 + \alpha\beta^{k+1} \\
&= (\alpha^k + \beta^k)(\alpha^2 - \beta^2) \\
&= v_k \cdot (\alpha^2 - \beta^2) \quad \text{which gives the result (4.3).}
\end{aligned}$$

We are now in a position to assert a property which relates these generalized Fibonacci numbers to ordinary Fibonacci numbers and at the same time gives an iterative formula for the general term. This formula generalizes Barakat [2] and Shannon [16].

$$f_n^{(r)} = \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} u_{k-2}^{m-s} u_{k-1}^{2s} u_k^{n-m-s} P^m.$$

Proof:

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n^{(r)} z^n &= (1 - v_k z + (-P)^k z^2)^{-1} && \text{[from (3.4)]} \\
&= (1 - (P^2 y_{k-3} + P y_{k-1})z + (-P)^k z^2)^{-1} && \text{[from (4.3)]} \\
&= (1 - (P^2 x_{k-2} + P y_{k-1})z + P^3 (x_{k-2} y_{k-1} - x_{k-1} y_{k-2}) z^2)^{-1} && \text{[from (4.2)]} \\
&= ((1 - P^2 x_{k-2} z)(1 - P y_{k-1} z) - P^3 x_{k-1} y_{k-2} z^2)^{-1} \\
&= \sum_{s=0}^{\infty} (1 - P^2 x_{k-2} z)^{-s-1} (1 - P y_{k-1} z)^{-s-1} x_{k-1}^s y_{k-2}^s P^{3s} z^{2s} \\
&= \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+s}{s} (1 - P y_{k-1} z)^{-s-1} x_{k-1}^s x_{k-2}^m y_{k-2}^s P^{3s+2m} z^{2s+m} \\
&= \sum_{m=0}^{\infty} \sum_{s=0}^{m} \binom{m}{s} (1 - P y_{k-1} z)^{-s-1} x_{k-1}^s x_{k-2}^{m-s} y_{k-2}^s P^{s+2m} z^{m+s} \\
&= \sum_{m=0}^{\infty} \sum_{s=0}^{m} \sum_{t=0}^{\infty} \binom{m}{s}\binom{t+s}{s} x_{k-1}^s x_{k-2}^{m-s} y_{k-1}^t y_{k-2}^s P^{s+2m+t} z^{m+s+t} \\
&= \sum_{n=0}^{\infty} \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} x_{k-1}^s x_{k-2}^{m-s} y_{k-1}^{n-m-s} y_{k-2}^s P^{n+m} z^n.
\end{aligned}$$

So by equating coefficients of z^n we find

$$f_n^{(r)} = \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} \frac{u_{k-1}^{2s} u_{k-2}^{m-s} u_k^{n-m-s} P^{n+m}}{P^{s+m-s+n-m-s+s}}$$

$$= \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} u_{k-2}^{m-s} u_{k-1}^{2s} u_k^{n-m-s} P^m, \quad \text{as required.}$$

For example, when $r = 2$ (and so $k = 1$), since $f_{-1}^{(2)} = 0$,

$$f_n^{(2)} = \sum_{0 \le 2m \le n} \binom{n-m}{m} P^{n-2m} P^m$$

$$= \sum_{0 \le 2m \le n} \binom{n-m}{m} P^{n-m},$$

which agrees with the result due to Barakat above. Note that Bruckman [3] has provided a neater proof for (4.3) in the case when $P = 1$.

5. CONCLUDING COMMENTS

Other properties can be readily developed to relate the $f_n^{(r)}$ and $g_n^{(r)}$ to other parts of the recurrence relation theory. For instance, we can prove that

$$\left\{ {n \atop j} \right\} \prod_{i=2}^{j+1} g_{n-2i+4}^{(i)} \tag{5.1}$$

where $\left\{ {n \atop j} \right\}$ is the analogue of the binomial coefficient used extensively in recurrence relation work (for example, Horadam [9]), and defined by

$$\left\{ {n \atop j} \right\} = \frac{u_n u_{n-1} \cdots u_{n-j-1}}{u_0 u_1 \cdots u_{j-1}}$$

Proof:

$$\left\{ {n \atop j} \right\} = \frac{(\alpha^{n+1} - \beta^{n+1})(\alpha^n - \beta^n) \cdots (\alpha^{n-j+2} - \beta^{n-j+2})}{(\alpha - \beta)(\alpha^2 - \beta^2) \cdots (\alpha^j - \beta^j)}$$

$$= g_n^{(2)} g_{n-2}^{(3)} \cdots g_{n-2j+2}^{(j+1)} \qquad \text{[by (2.2)]}$$

$$= \prod_{i=2}^{j+1} g_{n-2i+4}^{(i)}.$$

As another instance, consider

$$f_n^{(r)} = \beta^{kn} \left(\underline{(\alpha / \beta)^k} \right)_{n+1} \tag{5.2}$$

where $\underline{x}_n$ represents the n^{th} reduced Fermatian of index x as mentioned by Whitney [18] and utilized by Shannon [17]. It is defined formally by $\underline{x}_n = 1 + x + x^2 + \cdots + x^{n-1}$.

Proof of (5.2):

$$f_n^{(r)} = \frac{(\alpha^k)^{n+1} - (\beta^k)^{n+1}}{(\alpha^k) - (\beta^k)} = \beta^{kn}\left(1 + \left(\left(\frac{\alpha}{\beta}\right)^k\right) + \cdots + \left(\left(\frac{\alpha}{\beta}\right)^k\right)^n\right) = \beta^{kn}\left(\underline{(\alpha/\beta)}^k\right)_{n+1}.$$

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DETERMINING THE DIMENSION OF FRACTALS GENERATED BY PASCAL'S TRIANGLE

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1. INTRODUCTION

Pascal's triangle has long fascinated mathematicians with its intriguing number patterns. The triangle consists of the binomial coefficients of the expansion of $(x+y)^n$, where n is a nonnegative integer. When numbering the rows starting with 0 and the elements of each row starting with 0, the terms are $\binom{\text{row}}{\text{element}}$, where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$, assuming $k \le n$. The first eight rows are shown in Figure 1.

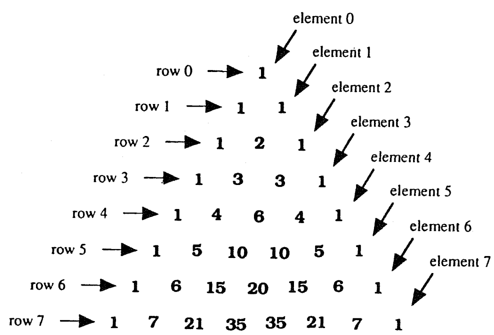


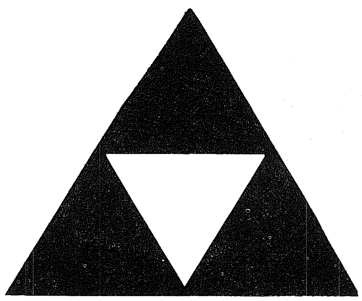
FIGURE 1. Eight rows of Pascal's triangle

A multidimensional pyramid of multinomial coefficients can be generalized from the definition for Pascal's triangle. Each entry is represented as

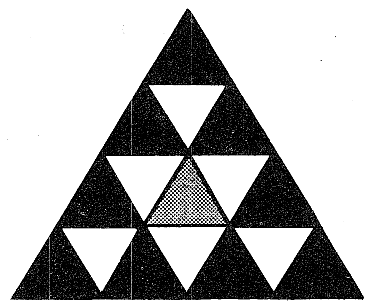
$$\binom{c}{a^1, a^2, a^3, \ldots, a^k} = \frac{c!}{a^1!a^2!a^3!\cdots a^k!},$$

where $c = a^1 + a^2 + a^3 + \cdots + a^k$. (Superscripts are used here to allow subscripts to take on a different meaning later in the paper.) This is the coefficient of the term $x_1^{a^1} x_2^{a^2} x_3^{a^3} \ldots x_k^{a^k}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_k)^c$, where a^i is the exponent of x_i, $i = 1, 2, \ldots, k$. In the case where $k = 3$, a triangular pyramid of integers is formed with each of the lateral faces duplicating Pascal's triangle. The apex of the pyramid is formed by a single 1, and each triangle below corresponds to a particular value of c. The vertices of each such triangle correspond to $a^1 = c$, $a^2 = c$, and $a^3 = c$.

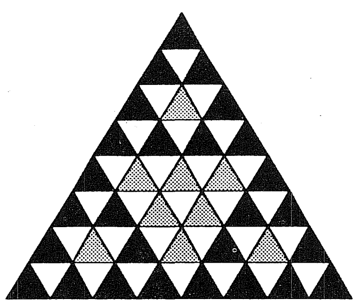
Consider the replacement of each element in Pascal's triangle by its remainder upon division by a prime p. This is called reducing to the least residue modulo p. The set of nonzero entries in this reduced triangle corresponds to a fractal according to the following construction. Consider the first p^n rows of Pascal's triangle, and call this set P_{p^n}. For each p^n, we construct a subset A_{p^n} of the triangle with vertices $(0,0)$, $(1,0)$, and $\left(\frac{1}{2},1\right)$. The fractal generated will lie in this triangle. Let $a_0 = (\frac{1}{2}, 1)$, and let $a_1, a_2, \ldots, a_{p^n-1}$ be equally spaced points of the segment joining $(\frac{1}{2}, 1)$ with $(0,0)$ such that a_{p^n} is $(0,0)$. Let $b_1, b_2, \ldots, b_{p^n-1}$ be equally spaced along the segment joining $(\frac{1}{2}, 1)$ with $(1,0)$. Finally, let $c_1, c_2, \ldots, c_{p^n-1}$ divide the segment $(0,0)$ to $(1,0)$ into p^n equal parts. Connect pairs of points of the form $a_i, b_i; b_i, c_i$; and a_i, c_{p^n-i}, to form p^{2n} triangular regions, $\frac{(p^n+1)p^n}{2}$ of them "pointing upwards" and $\frac{(p^n+1)p^n}{2}$ "pointing downwards." The first p^n rows of Pascal's triangle have a total of $\frac{(p^n+1)p^n}{2}$ entries. Now every integer in Pascal's triangle can be associated with a triangle which points up. Define the sets A_{p^n} as follows: $A_{p^n} = \{x | x$ belongs to a triangle associated with a nonzero entry in Pascal's triangle$\}$. The fractal associated with Pascal's triangle modulo a prime number is the limiting set A' as n goes to infinity. For $p = 2$, this set is the Sierpinski triangle.



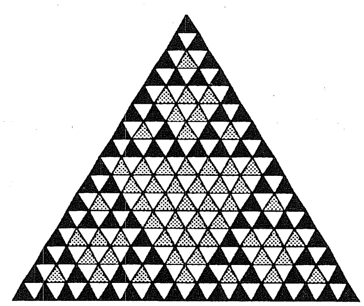
A_{2^1}: In 2^1 rows, there are 3 nonzero entries in $\frac{(2^1+1)2^1}{2} = 3$ upward triangles



A_{2^2}: In 2^2 rows there are 3^2 nonzero entries in $\frac{(2^2+1)2^2}{2} = 10$ upward triangles



A_{2^3}: In 2^3 rows, there are 3^3 nonzero entries in $\frac{(2^3+1)2^3}{2} = 36$ upward triangles.



A_{2^4}: In 2^4 rows, there are 3^4 nonzero entries in $\frac{(2^4+1)2^4}{2} = 136$ upward triangles.

FIGURE 2. Pascal's triangle reduced modulo 2: entries congruent to zero are shaded and nonzero entries are blackened

Figure 3 shows 256 rows of Pascal's triangle reduced modulo 2. Willson [1] showed that a cellular automaton with a linear transformation, that is, one in which each entry is determined by some linear combination of entries in the previous row, may have fractional fractal dimension. Pascal's triangle satisfies this criterion.

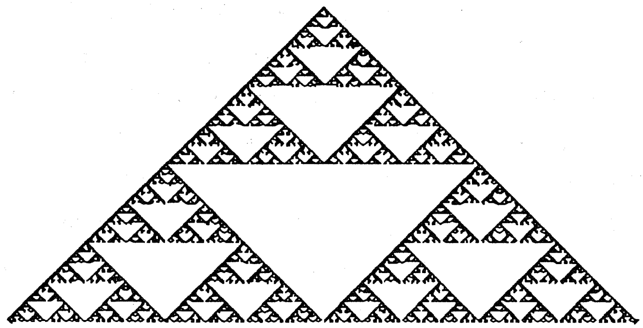


FIGURE 3. Pascal's triangle modulo 2

The fractals described by this construction have fractional fractal dimension. The fractal dimension D is defined as follows (see [2]): Let A be a compact subset of X, where (X,d) is a metric space. For each $\varepsilon > 0$, let $N(A, \varepsilon)$ be the minimum number of closed balls of radius ε which are needed to cover A. Then, the fractal dimension of A is given by

$$D = \lim_{\varepsilon\to 0} \frac{\ln(N(A,\varepsilon)}{\ln(1/\varepsilon)}.$$

By a slight variation of the Box Counting Theorem, the dimension of the fractals can be determined by using $\varepsilon = \frac{1}{p^n}$ and $N(A,\varepsilon) =$ number of triangles in the first p^n rows associated with nonzero entries in the reduced Pascal triangle. Then

$$D = \lim_{\varepsilon\to\infty} \frac{\ln(\#\text{ of nonzero entries in the first } p^n \text{ rows})}{\ln(p^n)}.$$

In section 2, a theorem about divisibility of multinomial coefficients by powers of primes is proven. This theorem is used to prove that the fractal dimension of Pascal's triangle modulo a prime p is

$$\frac{\ln[p(p+1)/2]}{\ln p}.$$

This determination is supported with computer results in section 3. Finally, in section 4, a generalization of this formula is proven for the analog of Pascal's triangle which contains multinomial coefficients reduced modulo p.

2. THEORETICAL DETERMINATION OF DIMENSION OF PASCAL'S TRIANGLE

The symbol $p^r|x$ means that p^r divides x with remainder zero. The symbol $p^r\|x$ means r is the largest integer for which $p^r|x$. Throughout the paper, p will refer to a prime.

First, we will work toward the dimension of Pascal's triangle reduced modulo p. The following lemma, proven by C. T. Long [3], will allow us to determine the requirements for divisibility of multinomial coefficients by a prime.

Long's Lemma: If p is prime, $n = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots + a_r p^r$ with $a_r \neq 0$ and $0 \leq a_i < p$ for each $i \leq r$, and $p^e \| n!$, then

$$e = \frac{n - (a_0 + a_1 + a_2 + a_3 + \cdots + a_r)}{p-1}.$$

Now we will apply this to determine the divisibility of multinomial coefficients. Let a^1, a^2, $a^3, \ldots, a^k$ be positive integers and let a_j^i denote the coefficient of p^j in the base p representation of a^i, so that if a^i has m_i digits in its base p representation, then

$$a^i = \sum_{j=0}^{m_i} a_j^i p^j.$$

The sum of these a^i is denoted by c, so that

$$c = \sum_{i=1}^{k} a^i = \sum_{j=0}^{m+s} c_j p^j,$$

where m is the maximum value of m_i. The c_j are the base p digits of c, and the additional s digits in c allow for large carries in the sum of the a^i.

Theorem 1—Multinomial Divisibility Theorem: For prime p, $p^r \| \binom{c}{a^1, a^2, a^3, \ldots, a^k}$, iff r is the sum of the carries made when adding the a^i in base p.

Proof: Let $d_0, d_1\, d_2, d_3, \ldots, d_{m+s-1}$ be the carries when adding the a^i, so that the sum of the digits in each position equals the digit for c in that position plus p times the carry to the next digit in c:

$$\sum_{i=1}^{k} a_0^i = c_0 + pd_0, \quad d_0 + \left(\sum_{i=1}^{k} a_1^i\right) = c_1 + pd_1, \quad d_i + \left(\sum_{i=1}^{k} a_2^i\right) = c_2 + pd_2, \ldots,$$

$$d_{m-1} + \left(\sum_{i=1}^{k} a_m^i\right) = c_m + pd_m, \qquad d_m = c_{m+1} + pd_{m+1},$$

$$d_{m+1} = c_{m+2} + pd_{m+2}, \ldots, \quad d_{m+s-2} = c_{m+s-1} + pd_{m+s-1}, \quad d_{m+s-1} = c_{m+s}.$$

Notice that extra digits beyond c_{m+1} in c occur if the carry from the m^{th} digit of c is greater than p.

Solving for the d_i, we have:

$$\frac{\left(\sum_{i=1}^{k} a_0^i\right) - c_0}{p} = d_0, \quad \frac{\left(\sum_{i=1}^{k} a_1^i\right) - c_1 + d_0}{p} = d_1, \ldots, \quad \frac{\left(\sum_{i=1}^{k} a_m^i\right) - c_m + d_{m-1}}{p} = d_m,$$

$$\frac{-c_{m+1} + d_m}{p} = d_{m+1}, \ldots, \quad \frac{-c_{m+s-1} + d_{m+s-2}}{p} = d_{m+s-1}, \quad d_{m+s-1} - c_{m+s} = 0.$$

The sum of the carries is $\sum_{i=0}^{m+s-1} d_i$, which equals

$$\frac{\left(\sum_{i=1}^{k} a_0^i\right)-c_0+\left(\sum_{i=1}^{k} a_1^i\right)-c_1+d_0+\cdots+\left(\left(\sum_{i=1}^{k} a_m^i\right)-c_m+d_{m-1}\right)+(-c_{m+1}+d_{m-1})+\cdots+(-c_{m+s-1}+d_{m+s-2})+(-c_{m+s}+d_{m+s-1})}{p}$$

$$=\frac{\left(\sum_{i=1}^{k} a_0^i\right)+\left(\sum_{i=1}^{k} a_1^i\right)+\left(\sum_{i=1}^{k} a_2^i\right)+\cdots+\left(\sum_{i=1}^{k} a_m^i\right)-(c_0+c_1+\cdots+c_{m+s})+(d_0+d_1+\cdots+d_{m+s-1})}{p}$$

$$=\frac{\left[\sum_{j=0}^{m}\sum_{i=1}^{k} a_j^i\right]-\left(\sum_{i=0}^{m+s} c_i\right)+\left(\sum_{i=0}^{m+s-1} d_i\right)}{p}=\sum_{i=0}^{m+s-1} d_i.$$

Multiplying by p on both sides,

$$p\left(\sum_{i=0}^{m+s-1} d_i\right)=\left[\sum_{j=0}^{m}\sum_{i=1}^{k} a_j^i\right]-\left(\sum_{i=0}^{m+s} c_i\right)+\left(\sum_{i=0}^{m+s-1} d_i\right).$$

Hence,

$$(p-1)\left(\sum_{i=0}^{m+s-1} d_i\right)=\left[\sum_{j=0}^{m}\sum_{i=1}^{k} a_j^i\right]-\left(\sum_{i=0}^{m+s} c_i\right).$$

Dividing by $p-1$, we get

$$\sum_{i=0}^{m+s-1} d_i=\frac{\left[\sum_{j=0}^{m}\sum_{i=1}^{k} a_j^i\right]-\left(\sum_{i=0}^{m+s} c_i\right)}{p-1}.$$

Since $c=\sum_{i=1}^{k} a^i$, we can add $c-\sum_{i=1}^{k} a^i$, so that

$$\sum_{i=0}^{m+s-1} d_i=\frac{\left[\sum_{j=0}^{m}\sum_{i=1}^{k} a_j^i\right]-\left(\sum_{i=0}^{m+s} c_i\right)+c-\left(\sum_{i=1}^{k} a^i\right)}{p-1}$$

$$=\frac{c-\left(\sum_{i=0}^{m+s} c_i\right)}{p-1}-\frac{\sum_{i=1}^{k}\left(a^i-\sum_{j=0}^{m} a_j^i\right)}{p-1}.$$

By Long's Lemma, $\left(c-\sum_{i=0}^{m+s} c_i\right)/(p-1)$ is the highest power of p which divides $c!$ Likewise, each $\left(a^i-\sum_{j=0}^{m} a_j^i\right)/(p-1)$ is the highest power of p which divides a^i. Thus, the previous expression simplifies to

$$\sum_{i=0}^{m+s-1} d_i=(\text{highest power of } p \text{ which divides } c!)-\sum_{i=1}^{k}(\text{highest power of } p \text{ which divides } a^i!)$$

The highest power of p which divides the multinomial coefficient $\frac{c!}{a^1!a^2!a^3!\ldots a^k!}$ is the highest power which will divide $c!$ minus the highest powers which divide each of the a^i. Therefore,

$$\sum_{i=0}^{m+s-1} d_i = \text{ highest power of } p \text{ which divides } \binom{c}{a^1, a^2, \ldots, a^k}.$$

This theorem can now be used to develop a more efficient method for determining entries in Pascal's triangle which are not divisible by p, in order to determine the dimension of Pascal's triangle. When computing the self-similarity dimension using p^m rows, each entry corresponds to a triangle of length $1/p^m$. If we consider covering the fractal with triangular boxes, the number of boxes needed to cover the fractal is equal to the number of entries which are not congruent to zero. The dimension is then

$$\lim_{\#\text{ rows}\to\infty} \frac{\ln(\#\text{ nonzero entries})}{\ln(\#\text{ rows considered})}.$$

Theorem 2—Dimension of Pascal's Triangle Modulo p: The fractal generated by Pascal's triangle modulo p has fractal dimension

$$\frac{\ln[p(p+1)/2]}{\ln p}.$$

Proof: Consider entries $\binom{A}{B}$ in Pascal's triangle, such that all $a_i \geq b_i$ in the base p representations :

$$A = a_0 p^0 + a_1 p^1 + a_2 p^2 + a_3 p^3 + \cdots + a_m p^m;$$
$$B = b_0 p^0 + b_1 p^1 + b_2 p^2 + b_3 p^3 + \cdots + b_m p^m.$$

We require that $a_m \neq 0$ so that m cannot be reduced, but it is not necessary that $b_m \neq 0$. Using the binomial case of Theorem 1, the highest power of p which divides the term $\binom{A}{B}$ is equal to the number of carries when $(A-B)$ is added to B in base p.

$$A - B = (a_0 - b_0)p^0 + (a_1 - b_1)p^1 + (a_2 - b_2)p^2 + \cdots + (a_m - b_m)p^m.$$

$$B + (A-B) = (b_0 + (a_0 - b_0))p^0 + (b_1 + (a_1 - b_1))p^1 + \cdots + (b_m + (a_m - b_m))p^m.$$

Since each $a_i \geq b_i$, and $a_i < p$, no carries will occur when adding $a_i - b_i$ and b_i. Conversely, if, for any i, $b_i > a_i$, then the sum $(a_i - b_i) + b_i$ will cause a carry so that $p|\binom{A}{B}$. Thus, in order to determine the entries which are not divisible by p, we need only that the $a_i \geq b_i$ for each digit in the base p representations.

The next step will be to determine the fractal dimension of Pascal's triangle modulo p. As discussed above, to find the dimension of this fractal, we find the number N of triangles of side length ε which correspond to nonzero entries. If we consider Pascal's triangle down to row p^m, scaled to have side length 1, then the triangles have side length $\varepsilon = 1/p^m$, such that each triangle corresponds to exactly one entry.

We are interested in determining how many entries $\binom{A}{B}$ in Pascal's triangle, through the first p^m rows, are *not* divisible by p. By the above argument, this is equal to the number of ways to choose A and B such that $0 \leq B \leq A < p^m$ where $0 \leq b_i \leq a_i < p$ for $i = 0, 1, \ldots, m$. The number of ways to choose the first such pair of base p digits a_0, b_0 is $p(p+1)/2$ by a simple counting argument. Therefore, the number of ways to choose $m+1$ such pairs independently is $[p(p+1)/2]^m$. The number of boxes of size $(1/p^m)$ needed to cover the p^m rows of the triangle is $[p(p+1)/2]^m$. Using the self-similarity definition of dimension, the fractal has dimension

$$\lim_{m\to\infty} \frac{\ln(\#\text{ nonzero entries})}{\ln(p^m\text{rows considered})} = \lim_{m\to\infty} \frac{\ln[p(p+1)/2]^m}{\ln(p^m)},$$

which simplifies to $\frac{\ln[p(p+1)/2]}{\ln(p)}$.

3. COMPUTER VERIFICATION OF THEORETICAL RESULTS

In 1989, N. S. Holter et al. [4] proposed without proof a dimension for Pascal's triangle modulo p. Their formula agrees with the one determined here. Their determination was based on a computer program which considers all elements whose distance form the top of the triangle is less than n and counts the number of elements x which are not divisible by the modulus. The values $D_n = \frac{\ln(x)}{\ln(n)}$ are approximations to the dimension, and $\lim_{n\to\infty} D_n$ is the fractal dimension. In their paper, they reported values of D_n for $n = 198$, 500, and 1000.

This experimental determination of dimension has two shortfalls. First, these cutoff values fall at different places in the approximations to the fractal, so that the figures cannot be rescaled to produce similar images. Second, since the determination is based on distance from the top rather than row numbers, the method sweeps out sectors rather than the triangular fractals studied here. These two problems make it difficult to determine the true limit, which is obscured by changes in marking places. Figure 4 illustrates these differences in the two determinations. (See, also, Table 1 on page 119.)

For this paper, a different experimental determination was performed using values of n which were powers of the modulus used. Also, triangles were used rather than sectors. Using this method and larger values of n, the values did approach the theoretically determined limit of

$$\frac{\ln 3}{\ln 2} = 1.58496\ldots.$$

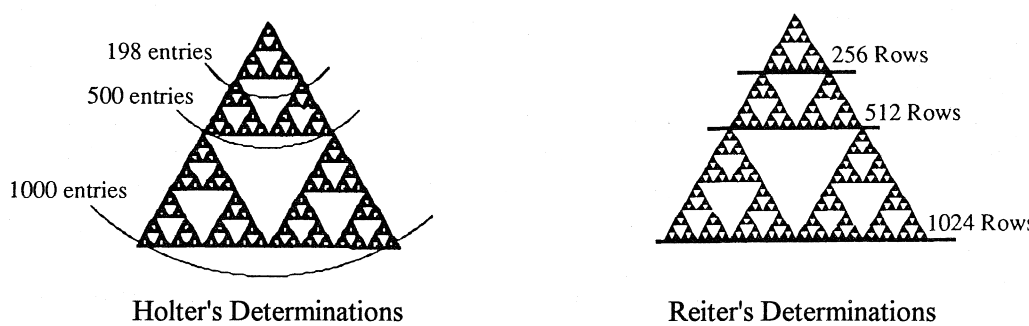


FIGURE 4. Diagrams of cutoff values in computer determinations of the dimension of Pascal's triangle modulo 2

TABLE 1. Data from computer determinations of the dimension of Pascal's triangle modulo 2 and 4

Holter's	# Rows in Pascal's Triangle	Reiter's
1.5681	198	
	256	**1.577785**
1.5716	500	
	512	**1.580738**
1.5738	1000	
	1024	**1.582439**
	2048	**1.583437**

4. GENERALIZATION TO MULTINOMIAL ANALOG OF PASCAL'S TRIANGLE

Now we will generalize to multinomial coefficients and the fractals generated by them. Using a method similar to that in Theorem 2, the dimension of the fractals generated by the multinomial coefficients modulo p will be determined and proved.

Theorem 3—Multinomial Dimension Theorem: Consider a prime p and a k-dimensional pyramid consisting of multinomial coefficients. The fractal formed when the entries which are not divisible by a particular prime p are shaded has fractal dimension equal to $\ln\binom{p-1+k}{k}/\ln p$.

Proof: In entries $\binom{c}{a^1, a^2, \ldots, a^k}$, let c denote the sum of the $a^i, i = 1, 2, \ldots, k$. Let $c = c_0p^0 + c_1p^1 + c_2p^2 + c_3p^3 + \cdots + c_mp^m$, where $c_j \le p-1$. According to Theorem 1, $\binom{c}{a^1, a^2, \ldots, a^k}$ is divisible by p if and only if at least one carry occurs in the summing of the base p expressions of the a^i. In any set of a^i for which $\binom{c}{a^1, a^2, \ldots, a^k}$ is not divisible by p, there must not be a carry when adding the a^i in base p. If no carries occur, then $c_j = a_j^1 + a_j^2 + a_j^3 + \cdots + a_j^k$ for each j. Since $c_j \le p-1$, we can write $p-1 = a_j^1 + a_j^2 + a_j^3 + \cdots + a_j^k + z$, where $z = (p-1) - c_j$ is a non-negative integer. Thus, we are partitioning $p - 1$ units into $k + 1$ base p digits. These $k + 1$ digits are the k possible a^i and the z which "takes up the slack" in each digit. If we consider values of c which are only one digit in base p, then each a^i is only one digit, so there are $\binom{p-1+k}{k}$ choices for the set of a^i. This follows from the observation that there are $\binom{p-1+k}{k}$ solutions among nonnegative integers to the equation $x_1 + x_2 + \cdots + x_k + z = p-1$. For each increase by one in the number of digits in the base p expression of c, the number of entries which are not divisible by p increases by a factor of $\binom{p-1+k}{k}$. The digits are not interdependent because we know there are no carries. Increasing the number of digits increases the number of rows and rescales the image by a factor of p. Thus, the dimension of the fractal corresponding to the pyramid of multinomial reduced modulo p is equal to

$$\lim_{m\to\infty} \frac{\ln\binom{p-1+k}{k}^m}{\ln p^m} = \frac{\ln\binom{p-1+k}{k}}{\ln p}.$$

Notice that when $k = 2$, this agrees with the result in Theorem 2.

5. DISCUSSION

As early as 1972, W. A. Broomhead [4] noted the self-similar nature of Pascal's triangle reduced modulo a prime. A great deal of study has been done on the specific case of mod 2, which generates Sierpinski's triangle. No work has been done on the dimension of the multinomial coefficients as defined here.

There are many extensions of this work which deserve further study. When the entries are reduced to their least residue mod n, where n is an integer other than a prime, the result is a pattern with fractional dimension, but which is not strictly self-similar. The determination of the dimension of such a fractal is a natural extension. Because these fractals are the union of two fractals with different dimensions, they are not strictly self-similar. I conjecture that the dimension of such a fractal is equal to the dimension of the fractal corresponding to the largest prime factor of n. Recent work [5] done on the divisibility of entries in Pascal's triangle by products of primes could be the basis for rigorous proof. Other cellular automata and the fractals which they generate are also likely candidates for this type of dimensional study.

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TRIPLE FACTORIZATION OF SOME RIORDAN MATRICES

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1. INTRODUCTION

When examining a combinatorial sequence, generating functions are often useful. That is, if we are interested in analyzing the sequence $a_0, a_1, a_2, \ldots$, we investigate the formal power series

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots .$$

In a recent paper [2], techniques are discussed that assist in finding closed-form expressions for the formal power series for a select, but large, set of combinatorial sequences. The methods involve using infinite matrices and the Riordan group. The Riordan group is defined in section 2 of this paper. Each matrix, L, in the Riordan group is associated with a combinatorial sequence and with a matrix, S_L, called the Stieltjes matrix. S_L is defined in section 3. In this paper, we show that when S_L is tridiagonal, then $L = PCF$, where the first factor P is a Pascal-type matrix, the second factor C involves the generating function for the Catalan numbers, and the third factor F involves the Fibonacci generating function. The following is an example:

$$\begin{bmatrix} 1 &&&&&&& . \\ 3 & 1 &&&& \mathbf{0} && . \\ 9 & 5 & 1 &&&&& . \\ 27 & 18 & 7 & 1 &&&& . \\ 81 & 56 & 31 & 9 & 1 &&& . \\ 243 & 162 & 109 & 48 & 11 & 1 && . \\ 729 & 458 & 332 & 194 & 69 & 13 & 1 & . \\ . & . & . & . & . & . & . & . \end{bmatrix} =$$

$$\begin{bmatrix} 1 &&&&&&& . \\ 1 & 1 &&&& \mathbf{0} && . \\ 1 & 2 & 1 &&&&& . \\ 1 & 3 & 3 & 1 &&&& . \\ 1 & 4 & 6 & 4 & 1 &&& . \\ 1 & 5 & 10 & 10 & 5 & 1 && . \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 &&&&&&& . \\ 0 & 1 &&&& \mathbf{0} && . \\ -1 & 0 & 1 &&&&& . \\ 0 & -2 & 0 & 1 &&&& . \\ 2 & 0 & -3 & 0 & 1 &&& . \\ 0 & 5 & 0 & -4 & 0 & 1 && . \\ -5 & 0 & 9 & 0 & -5 & 0 & 1 & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 &&&&&&& . \\ 1 & 1 &&&& \mathbf{0} && . \\ 2 & 1 & 1 &&&&& . \\ 3 & 2 & 1 & 1 &&&& . \\ 5 & 3 & 2 & 1 & 1 &&& . \\ 8 & 5 & 3 & 2 & 1 & 1 && . \\ 13 & 8 & 5 & 3 & 2 & 1 & 1 & . \\ . & . & . & . & . & . & . & . \end{bmatrix}$$

The matrices in the Riordan group are infinite and lower triangular. So the example shows only the first seven rows. The first factor on the right is the Pascal matrix. The first column in the second factor has $C(-x^2)$ as generating function, where

$$C(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}x^n$$

is the generating function for the Catalan numbers. The third factor has the Fibonacci numbers in each column. See section 6 for further examples of this triple factorization.

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In section 2, we define the Riordan group R and list some properties that we use in the proofs of the propositions which are given in section 4. In section 3, we discuss the unique Stieltjes matrix S_L associated with each L in the Riordan group. In this paper, we concentrate on the subset of R given by $R_T = \{L \in R : S_L \text{ is tridiagonal}\}$. In section 5, we derive a recurrence relation for the sequence associated with each member of R_T, and we discuss the asymptotic behavior of these sequences. In section 6, we provide two examples involving well-known sequences. For each example, we give the triple factorization, the Stieltjes matrix, the recurrence relation and asymptotic behavior of the corresponding sequence.

2. THE RIORDAN GROUP

A detailed description of this group is given in [2]. Here we provide a brief summary.

Let $M = (m_{ij})_{i,j\geq 0}$ be an infinite matrix with elements from C, the set of complex numbers. Let $c_i(x)$ be the generating function of the i^{th} column of M. That is

$$c_i(x) = \sum_{n=0}^{\infty} m_{n,i} x^n.$$

We call M a Riordan matrix if $c_i(x) = g(x)[f(x)]^i$, where

$$g(x) = 1 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots, \quad \text{and} \quad f(x) = x + f_2 x^2 + f_3 x^3 + \cdots.$$

In this case, we write $M = (g(x), f(x))$. We denote by R the set of Riordan matrices. R is a group under matrix multiplication with the following properties:

(i) $(g(x), f(x)) * (h(x), \ell(x)) = (g(x)h(f(x)), \ell(f(x)))$.

(ii) $I = (1, x)$ is the identity element.

(iii) The inverse of M is given by

$$M^{-1} = \left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right),$$

where $\bar{f}$ is the compositional inverse of f.

(iv) If $(a_0, a_1, a_2, \ldots)^T$ is a column vector with generating function $A(x)$, then multiplying $M = (g(x), f(x))$ on the right by this column vector yields a column vector with generating function $B(x) = g(x)A(f(x))$.

3. STIELTJES MATRIX

Let L be Riordan and let $\bar{L}$ be the matrix obtained from L by deleting the first row. For example, if I is the identity, we have

$$\bar{I} = \begin{bmatrix} 0 & 1 & & & \mathbf{0} & \cdot \\ 0 & 0 & 1 & & & \cdot \\ 0 & 0 & 0 & 1 & & \cdot \\ 0 & 0 & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Observe that $\bar{L} = \bar{I}L$. There exists a unique matrix, S_L, such that $LS_L = \bar{L}$. We call this matrix the Stieltjes matrix of L.

Example: If

$$L = \left(\frac{1}{1-x}, \frac{x}{1-x}\right) = \begin{bmatrix} 1 & & & & \mathbf{0} & \cdot \\ 1 & 1 & & & & \cdot \\ 1 & 2 & 1 & & & \cdot \\ 1 & 3 & 3 & 1 & & \cdot \\ 1 & 4 & 6 & 4 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

then

$$S_L = \begin{bmatrix} 1 & 1 & & & \mathbf{0} & \cdot \\ 0 & 1 & 1 & & & \cdot \\ 0 & 0 & 1 & 1 & & \cdot \\ 0 & 0 & 0 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

4. PROPOSITIONS

Proposition 1: If $L = (g(x), f(x))$ is Riordan and S_L is tridiagonal, then

(a)

$$S_L = \begin{bmatrix} b_0 & 1 & & & & \mathbf{0} & \cdot \\ \lambda_1 & b & 1 & & & & \cdot \\ 0 & \lambda & b & 1 & & & \cdot \\ 0 & 0 & \lambda & b & 1 & & \cdot \\ 0 & 0 & 0 & \lambda & b & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

(b) $f = x(1 + bf + \lambda f^2)$ and $g = \dfrac{1}{1 - b_0 x - \lambda_1 x f}$ iff S_L is as in (a).

Proof: Let

$$S_L = \begin{bmatrix} b_0 & 1 & & & \mathbf{0} & \cdot \\ \lambda_1 & b_1 & 1 & & & \cdot \\ 0 & \lambda_2 & b_2 & 1 & & \cdot \\ 0 & 0 & \lambda_3 & b_3 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

With $c_i(x)$ the generating function for the i^{th} column of L, $i \geq 0$, we have $c_i = gf^i$. By looking at the first column of LS_L and $\bar{L}$, we obtain $b_0 xg + \lambda_1 xgf = g - 1$, i.e.,

$$g(x) = \frac{1}{1 - b_0 x - \lambda_1 xf}.$$

For $i \geq 1$, we obtain from $LS_L = \bar{L}$,

$$
\begin{aligned}
& c_i = x(c_{i-1} + b_i c_i + \lambda_{i+1} c_{i+1}). \\
\therefore \quad & gf^i = x(gf^{i-1} + b_i gf^i + \lambda_{i+1} gf^{i+1}) \\
\Leftrightarrow \quad & f = x(1 + b_i f + \lambda_{i+1} f^2) \\
\therefore \quad & 0 = (b_i - b_j) f + (\lambda_{i+1} - \lambda_{j+1}) f^2 \text{ for all } i \text{ and } j \geq 1.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad & b_i = b_j \text{ and } \lambda_{i+1} = \lambda_{j+1} \text{ for all } i \text{ and } j \geq 1. \\
\therefore \quad & \text{we can take } b = b_1 = b_2 = b_3 = \cdots \\
& \text{and } \lambda = \lambda_2 = \lambda_3 = \lambda_4 = \cdots. \\
\therefore \quad & f = x(1 + bf + \lambda f^2).
\end{aligned}
$$

Remark: If S_L is tridiagonal, it has the form in (a) and then either

(a) $\lambda = 0$ and $f = \dfrac{x}{1-bx}$ and $g = \dfrac{1-bx}{1-(b_1+b_0)x+(bb_0-\lambda_1)x^2}$ or

(b) $\lambda \neq 0$ and $f = \dfrac{1-bx-\sqrt{(b^2-4\lambda)x^2-2bx+1}}{2\lambda x}$ and $g = \dfrac{1}{1-b_0x-\lambda_1 xf}$.

Proposition 2: If $L = (g, f)$ is Riordan, then $S_L = S_{L^*} + bI$ if and only if $L = P^b L^*$, where

$$
P^b = \left(\frac{1}{1-bx}, \frac{x}{1-bx}\right) = \begin{bmatrix} 1 & & & & & \cdot \\ b & 1 & & \mathbf{0} & & \cdot \\ b^2 & 2b & 1 & & & \cdot \\ b^3 & 3b^2 & 3b & 1 & & \cdot \\ b^4 & 4b^3 & 6b^2 & 4b & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \text{(cf. [3], p. 171)}
$$

Proof: Note that

$$
S_{P^b} = \begin{bmatrix} b & 1 & & & & & \cdot \\ 0 & b & 1 & & \mathbf{0} & & \cdot \\ 0 & 0 & b & 1 & & & \cdot \\ 0 & 0 & 0 & b & 1 & & \cdot \\ 0 & 0 & 0 & 0 & b & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = bI + \bar{I}.
$$

So,

$$
\begin{aligned}
L = P^b L^* \Rightarrow \bar{I}L = \bar{I}P^b L^* & \\
\Rightarrow \bar{L} = \overline{P^b} L^* &= P^b S_{P^b} L^* = P^b(bI + \bar{I})L^* = bP^b L^* + P^b \bar{I} L^* \\
&= bL + P^b \bar{L}^* = bL + P^b L^* S_{L^*} = L(bI + S_{L^*}).
\end{aligned}
$$

Conversely, suppose $S_L = bI + S_{L^*}$. Then

$$
\begin{aligned}
P^b L^*(bI + S_{L^*}) &= \underline{bIP^b L^*} = P^b \bar{L}^* = \underline{P^b bIL^*} + P^b \bar{I} L^* = P^b(bI + \bar{I})L^* \\
&= \overline{P^b} L^* = \bar{I}(P^b L^*) = \overline{P^b L^*}.
\end{aligned}
$$

Proposition 3: If $L=(g,f)$ is Riordan and

$$S_L=\begin{bmatrix} b+\varepsilon & 1 & & & & & . \\ \lambda+\delta & b & 1 & & & & . \\ 0 & \lambda & b & 1 & & & . \\ 0 & 0 & \lambda & b & 1 & & . \\ 0 & 0 & 0 & \lambda & b & 1 & . \\ . & . & . & . & . & . & . \end{bmatrix},$$

then $L=P^b L_1$, where

$$S_{L_1}=\begin{bmatrix} \varepsilon & 1 & & & & . \\ \lambda+\delta & 0 & 1 & & & . \\ 0 & \lambda & 0 & 1 & & . \\ 0 & 0 & \lambda & 0 & 1 & . \\ 0 & 0 & 0 & \lambda & 0 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Proof: This follows immediately from Proposition 2.

Proposition 4 (PCF Factorization): In Proposition 3, $L_1=C_\lambda F_{\varepsilon,\delta}$, where

$$C_\lambda=\left(c(\lambda x^2),xc(\lambda x^2)\right)$$

with

$$c(x)=1+x[c(x)]^2=\frac{1-\sqrt{1-4x}}{2x}=\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n, \text{ and } F_{\varepsilon,\delta}=\left(\frac{1}{1-\varepsilon x-\delta x^2},x\right).$$

Proof: Let $L_1=(g_1,f_1)$. Then, from Proposition 1, we must have, when $\lambda\neq 0$,

$$f_1=\frac{1-\sqrt{1-4\lambda x^2}}{2\lambda x} \quad\text{and}\quad g_1=\frac{1}{1-\varepsilon x-(\lambda+\delta)xf_1}.$$

$$f_1=xc(\lambda x^2) \quad\text{and}\quad g_1=\frac{1}{1-\varepsilon x-(\lambda+\delta)x^2c(\lambda x^2)}.$$

Now, from section 2, property 1, we have

$$C_\lambda F_{\varepsilon,\delta}=\left(c(\lambda x^2),\,xc(\lambda x^2)\right)*\left(\frac{1}{1-\varepsilon x-\delta x^2},x\right)=\left(\frac{c(\lambda x^2)}{1-\varepsilon xc(\lambda x^2)-\delta[xc(\lambda x^2)]^2},xc(\lambda x^2)\right).$$

But

$$\frac{1}{1-\varepsilon x-(\lambda+\delta)x^2c(\lambda x^2)}=\frac{c(\lambda x^2)}{1-\varepsilon xc(\lambda x^2)-\delta x^2[c(\lambda x^2)]^2}$$
$$\Leftrightarrow 1-\varepsilon xc(\lambda x^2)-\delta x^2[c(\lambda x^2)]^2=c(\lambda x^2)-\varepsilon xc(\lambda x^2)-(\lambda+\delta)x^2[c(\lambda x^2)]^2$$
$$\Leftrightarrow 1-c(x^2)+\lambda x^2[c(\lambda x^2)]^2=0.$$

5. RECURRENCE RELATIONS AND ASYMPTOTICS

We have proved that when $L = (f(x), g(x))$ is Riordan and S_L is tridiagonal with the form

$$S_L = \begin{bmatrix} b_0 & 1 & & & & & . \\ \lambda_1 & b & 1 & & & & . \\ 0 & \lambda & b & 1 & & & . \\ 0 & 0 & \lambda & b & 1 & & . \\ 0 & 0 & 0 & \lambda & b & 1 & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

then

$$f(x) = \sum_{n=0}^{\infty} f_n x^n = \frac{1 - bx - \sqrt{(b^2 - 4\lambda)x^2 - 2bx + 1}}{2\lambda x}$$

and

$$g(x) = \sum_{n=0}^{\infty} g_n x^n = \frac{1}{1 - b_0 x - \lambda_1 x f(x)}.$$

Using the J.C. P. Miller formula (see Henrici [3]), we obtain for f_n the three-term recurrence

$$(n+2) f_{n+1} = (2n+1) b f_n + (1-n)(b^2 - 4\lambda) f_{n-1},$$

and for g_n the five-term recurrence

$$\begin{aligned} nAg_n = {} & [(2n-3)bA - nB] g_{n-1} + [(2n-3)bB + (3-n)(b^2 - 4\lambda)A - nC] g_{n-2} \\ & + [(2n-3)bC + (3-n)(b^2 - 4\lambda)B] g_{n-3} + [(3-n)(b^2 - 4\lambda)C] g_{n-4} \end{aligned}$$

where $A = \lambda - \lambda_1$, $B = \lambda_1 b + \lambda_1 b_0 - 2\lambda b_0$, $C = \lambda_1^2 - \lambda_1 b b_0 + \lambda b_0^2$. For the asymptotics, we use the methods described in Wilf [4, Ch. 5]. For large n, we obtain

$$f_n \sim \frac{(n+1)^{-3/2} (b + 2\sqrt{\lambda})^{n+1/2}}{2\lambda^{3/4}\sqrt{\pi}}$$

where $b^2 > 4\lambda > 0$.

Because there are too many cases to consider, we do not attempt to provide a general formula for the asymptotic value of g_n. However, the examples in section 6 illustrate the techniques involved.

6. EXAMPLES

Example 1—Big Schröder Numbers: If we take $\lambda = 2$, $b = 3$, $\lambda_1 = \lambda + \delta = 2$, and $b_0 = b + \varepsilon = 2$, then

$$f = \frac{1 - 3x - \sqrt{x^2 - 6x + 1}}{4x} \quad \text{and} \quad g = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x}.$$

g is the generating function for the Big Schröder numbers [1].

$$L = (g, f) \text{ with } S_L = \begin{bmatrix} 2 & 1 & & & & & . \\ 2 & 3 & 1 & & \mathbf{0} & & . \\ 0 & 2 & 3 & 1 & & & . \\ 0 & 0 & 2 & 3 & 1 & & . \\ 0 & 0 & 0 & 2 & 3 & 1 & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

and

$$L = P^3 C_2 F_{-1,0} = \left(\frac{1}{1-3x}, \frac{x}{1-3x}\right) * \left(c(2x^2), xc(2x^2)\right) * \left(\frac{1}{1+x}, x\right)$$

$$= \begin{bmatrix} 1 & & & & & . \\ 3 & 1 & & \mathbf{0} & & . \\ 9 & 6 & 1 & & & . \\ 27 & 27 & 9 & 1 & & . \\ 81 & 108 & 54 & 12 & 1 & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 & & & & & . \\ 0 & 1 & & \mathbf{0} & & . \\ 2 & 0 & 1 & & & . \\ 0 & 4 & 0 & 1 & & . \\ 8 & 0 & 6 & 0 & 1 & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 & & & & & . \\ -1 & 1 & & \mathbf{0} & & . \\ 1 & -1 & 1 & & & . \\ -1 & 1 & -1 & 1 & & . \\ 1 & -1 & 1 & -1 & 1 & . \\ . & . & . & . & . & . \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & & . \\ 2 & 1 & & \mathbf{0} & & . \\ 6 & 5 & 1 & & & . \\ 22 & 23 & 8 & 1 & & . \\ 90 & 107 & 49 & 11 & 1 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Recurrence Relations: Here $A = 0$, $B = 2$, $C = 0$, $(n+2)f_{n+1} = 3(2n+1)f_n + (1-n)f_{n-1}$ for $n \geq 1$. $f_0 = 0$, $f_1 = 1$. $ng_{n-1} = 3(2n-3)g_{n-2} + (3-n)g_{n-3}$, $(n+1)g_n = 3(2n-1)g_{n-1} + (2-n)g_{n-2}$, for $n \geq 2$. $g_0 = 1$, $g_1 = 2$.

Asymptotics:

$$f_n = [x^n]f(x) \sim \frac{(n+1)^{-3/2}(b+2\sqrt{\lambda})^{n+1/2}}{2\lambda^{3/4}\sqrt{\pi}} = \frac{(n+1)^{-3/2}(3+2\sqrt{2})^{n+1/2}}{2\cdot 2^{3/4}\cdot\sqrt{\pi}},$$

$$g_n = [x^n]g(x) = [x^n]\frac{1-x-\sqrt{x^2-6x+1}}{2x}.$$

For large n,

$$g_n = -\frac{1}{2}[x^{n+1}](x^2-6x+1)^{1/2} = 2f_n \sim \frac{(n+1)^{-3/2}(3+2\sqrt{2})^{n+1/2}}{2^{3/4}\sqrt{\pi}}.$$

Example 2—Legendre Polynomials: We require

$$g(x) = \frac{1}{\sqrt{x^2-2tx+1}}.$$

We take

$$\lambda = \frac{t^2-1}{4}, \quad b = t, \quad \lambda_1 = \lambda + \delta = \frac{t^2-1}{2}, \text{ and } b_0 = b + \varepsilon = t.$$

Triple Factorization:

$$L=\left(\frac{1}{\sqrt{x^2-2tx+1}},\frac{2(1-tx-\sqrt{x^2-2tx+1})}{(t^2-1)x}\right)=P^tC_{(t^2-1)/4}F_{0,(t^2-1)/4},$$

$$S_L=\begin{bmatrix} t & 1 & & & & & . \\ \frac{t^2-1}{2} & t & 1 & & & & . \\ 0 & \frac{t^2-1}{4} & t & 1 & & & . \\ 0 & 0 & \frac{t^2-1}{4} & t & 1 & & . \\ 0 & 0 & 0 & \frac{t^2-1}{4} & t & 1 & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

Recurrence Relations: Here $A=\frac{1-t^2}{4}$, $B=\frac{t(t^2-1)}{2}$, $C=\frac{1-t^2}{4}$. $(n+2)f_{n+1}=(2n+1)tf_n+(1-n)f_{n-1}$, for $n\geq 1$. $f_0=0, f_1=1$. $ng_n=(4n-3)tg_{n-1}+(3-2n)(1+2t^2)g_{n-2}+(4n-9)tg_{n-3}+(3-n)g_{n-4}$, for $n\geq 4$. $g_0=1, g_1=t, g_2=\frac{3}{2}t^2-\frac{1}{2}, g_3=\frac{5}{2}t^3-\frac{3t}{2}$.

Asymptotics: We assume that $t^2>1$, so that the roots of $x^2-2tx+1=0$ are real. Denote these roots by $\hat{r}$ and $\tilde{r}$ with $|\hat{r}|<|\tilde{r}|$. We obtain

$$\begin{aligned}[x^n]f(x)&=-\frac{2}{t^2-1}[x^{n+1}](x^2-2tx+1)^{1/2}\\ &\sim\left(\frac{-2}{t^2-1}\right)\frac{1}{(\tilde{r})^{n+1}}\frac{(n+1)^{-3/2}}{-2\sqrt{\pi}}\left(1-\frac{\hat{r}}{\tilde{r}}\right)^{1/2}\\ &=\left(\frac{1}{t^2-1}\right)\frac{(\tilde{r})^n}{n+1}\sqrt{\frac{(\tilde{r})^2-1}{n\pi}};\end{aligned}$$

$$[x^n]g(x)=[x^n](x^2-2tx+1)^{-1/2}\sim\frac{(\tilde{r})^{n+1}}{\sqrt{n\pi((\tilde{r})^2-1)}}.$$

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THREE-SQUARE THEOREM AS AN APPLICATION OF ANDREWS' IDENTITY

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1. INTRODUCTION

The representation of an integer n as a sum of k squares is one of the most beautiful problems in the theory of numbers. Such representations are useful in lattice point problems, crystallography, and certain problems in mechanics [6, pp. 1-4]. If $r_k(n)$ denotes the number of representations of an integer n as a sum of k squares, Jacobi's two- and four-square theorems [9] are:

(1) $$r_2(n) = 4[d_1(n) - d_3(n)]$$

and

(2) $$r_4(n) = 8 \sum_{\substack{d|n \\ d \not\equiv 0 \pmod 4}} d$$

where $d_i(n)$ denotes the number of divisors of n, $d \equiv i \pmod 4$. In literature there are several proofs of (1) and (2). For instance, M. D. Hirschhorn [7; 8] proved (1) and (2) using Jacobi's triple product identity. S. Bhargava & Chandrashekar Adiga [4] have proved (1) and (2) as a consequence of Ramanujan's ${}_1\Psi_1$ summation formula [10]. Recently R. Askey [2] has proved (1) and also derived a formula for the representation of an integer as a sum of a square and twice a square. The authors [5] have derived a formula for the representation of an integer as a sum of a square and thrice a square. These works of Askey [2] and the authors [5] also rely on Ramanujan's ${}_1\Psi_1$ summation [10].

In 1951 P. T. Bateman [3] obtained the following formula for $r_3(n)$:

(3) $$r_3(n) = \frac{16}{\pi}\sqrt{n}L(1,\chi)q(n)P(n),$$

where $$n = 4^a n_1, \quad 4 \nmid n_1,$$

$$q(n) = \begin{cases} 0 & \text{if } n_1 \equiv 7 \pmod 8, \\ 2^{-a} & \text{if } n_1 \equiv 3 \pmod 8, \\ 3\cdot 2^{-a-1} & \text{if } n_1 \equiv 1,2,5, \text{ or } 6 \pmod 8, \end{cases}$$

$$P(n) = \prod_{\substack{p^{2b}|n \\ p \text{ odd}}} \left[1 + \sum_{j=1}^{b-1} p^{-j} + p^{-b}\left(1 - \left[\frac{(-n/p^{2b})}{p}\right]\frac{1}{p}\right)^{-1}\right],$$

$$(P(n)=1 \text{ for square-free } n), \text{ and}$$

$$L(S,\chi)=\sum_{m=1}^{\infty}\chi(m)m^{-S} \text{ with } \chi(m), \text{ the Legendre-Jacobi-Kronecker symbol:}$$

$$\chi(m)=\left(\frac{-4}{m}\right)=\begin{cases}1 & \text{if } m\equiv 1 \pmod 4,\\ 0 & \text{if } m\equiv 0 \pmod 2,\\ -1 & \text{if } m\equiv 3 \pmod 4.\end{cases}$$

In this note we obtain an alternate formula (13) for $r_3(n)$ which involves only partition functions unlike Bateman's formula (3) which is expressed in terms of Dirichlet's series {6, pp. 54, 55]. To derive our formula (13) for $r_3(n)$, we employ G. E. Andrews' [1] generalization of Ramanujan's ${}_1\Psi_1$ summation:

$$\frac{(a^{-1}-b^{-1})(A)_\infty(B)_\infty(bq/a)_\infty(aq/b)_\infty(q)_\infty(AB/ab)_\infty}{(-b)_\infty(-a)_\infty(-A/b)_\infty(-A/a)_\infty(-B/b)_\infty(-B/a)_\infty} \tag{4}$$

$$=a^{-1}\sum_{m=0}^{\infty}\frac{(-q/a)_m(AB/ab)_m(-b)^m}{(-B/a)_{m+1}(-A/a)_{m+1}}-b^{-1}\sum_{m=0}^{\infty}\frac{(A)_m(-aq/B)_m(-B/b)^m}{(-a)_{m+1}(-A/b)_{m+1}},$$

where

$$(a)_\infty=(a;q)_\infty=\prod_{m=0}^{\infty}(1-aq^m)$$

and

$$(a)_m=(a;q)_m=\frac{(a;q)_\infty}{(aq^m;q)_\infty},\quad |q|<1.$$

2. THREE-SQUARE THEOREM

In this section we derive a formula for $r_3(n)$. the number of representations of an integer n as a sum of three squares. For convenience, we first transform Andrews' formula (4).

Lemma 2.1 (G. E. Andrews' [1]):

$$\frac{(A;q^2)_\infty(-A\beta/\alpha qz;q^2)_\infty(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}{(-A/\alpha qz;q^2)_\infty(A/\alpha q^2;q^2)_\infty(-\alpha qz;q^2)_\infty(-\beta q/z;q^2)_\infty(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty} \tag{5}$$

$$=\frac{1}{[1-(A/\alpha q^2)]}+\sum_{m=1}^{\infty}\frac{(1/\alpha;q^2)_m(-A\beta/\alpha qz;q^2)_m(-\alpha q)^m)}{(\beta q^2;q^2)_m(A/\alpha q^2;q^2)_{m+1}}z^m$$

$$+\sum_{m=1}^{\infty}\frac{(1/\beta;q^2)_m(A;q^2)_{m-1}(-\beta q)^m}{(\alpha q^2;q^2)_m(-A/\alpha qz;q^2)_m}z^{-m}$$

if $|\beta q|<|z|<1/|\alpha q|$ and $|q|<1$ with none of the factors in the denominators of (5) being 0.

Proof: Equation (4) is equivalent to

$$\frac{a^{-1}[1-(a/b)](A)_\infty(B)_\infty(bq/a)_\infty(aq/b)_\infty(q)_\infty(AB/ab)_\infty}{[1+(B/a)](-b)_\infty(-a)_\infty(-A/b)_\infty(-A/a)_\infty(-B/b)_\infty(Bq/a)_\infty}$$

$$=a^{-1}\frac{1}{[1+(B/a)]}\sum_{m=0}^{\infty}\frac{(-q/a)_m(AB/ab)_m(-b)^m}{(-Bq/a)_m(-A/a)_{m+1}}$$

$$-b^{-1}\frac{(-b/B)}{[1+(a/B)]}\sum_{m=0}^{\infty}\frac{(A)_m(-a/B)_{m+1}(-b/B)^{-m-1}}{(-a)_{m+1}(-A/b)_{m+1}}$$

which, in turn is equivalent to

$$\frac{(A)_\infty(B)_\infty(bq/a)_\infty(a/b)_\infty(q)_\infty(AB/ab)_\infty}{(-b)_\infty(-a)_\infty(-A/b)_\infty(-A/a)_\infty(-B/b)_\infty(-Bq/a)_\infty} \tag{6}$$

$$=\sum_{m=0}^{\infty}\frac{(-q/a)_m(AB/ab)_m(-b)^m}{(-Bq/a)_m(-A/a)_{m+1}}+\sum_{m=0}^{\infty}\frac{(A)_m(-a/B)_{m+1}(-b/B)^{-m-1}}{(-a)_{m+1}(-A/b)_{m+1}}.$$

Change b to $-z$, a to $-q/a'$, B to b'/a' in (6) to obtain

$$\frac{(A)_\infty(b'/a')_\infty(za')_\infty(q/a'z)_\infty(q)_\infty(Ab'/zq)_\infty}{(z)_\infty(q/a')_\infty(A/z)_\infty(Aa'/q)_\infty(b'/a'z)_\infty(b')_\infty} \tag{7}$$

$$=\sum_{m=0}^{\infty}\frac{(a')_m(Ab'/zq)_m z^m}{(b')_m(Aa'/q)_{m+1}}+\sum_{m=0}^{\infty}\frac{(A)_m(q/b')_{m+1}(a'z/b')^{-(m+1)}}{(q/a')_{m+1}(A/z)_{m+1}}.$$

Change q to q^2, a' to $1/\alpha$, b' to βq^2, and z to $-\alpha qz$ in (7) to obtain (5). Hence, the lemma.

Corollary 2.1:

$$\left(\sum_{n=-\infty}^{\infty}q^{n^2}\right)^3=\frac{(-q;q^2)_\infty^3(q^2;q^2)_\infty^3}{(q;q^2)_\infty^3(-q^2;q^2)_\infty^3} \tag{8}$$

$$=1+2\sum_{m=1}^{\infty}\frac{(-q;q^2)_m q^m}{(1+q^{2m})(-q^2;q^2)_m}+4\sum_{m=1}^{\infty}\frac{(q^2;q^2)_{m-1}q^m}{(1+q^{2m})(q;q^2)_m}.$$

Proof: Putting $\alpha=\beta=-1$, $z=1$, and $A=q^2$ in (5), we have the second of the equations (8), the first being a well-known theta-function identity [10]. In fact, put $z=1$, $A=\alpha=\beta=0$ in (5) and use the easily verified Euler identity

$$(-q;q^2)_\infty=1/(q;q^2)_\infty(-q^2;q^2)_\infty.$$

Before stating the main theorem of this section, we introduce two partition-counting functions $p_m(n)$ and $q_m(n)$.

Definition 2.1: Given a partition π, let $e(\pi)$ denote the number of even parts in π. Define $P_m(n)$ to be the set of partitions of n in which odd parts are distinct and all parts are less than or equal to $2m$, $Q_m(n)$ to be the set of partitions of n in which even parts are distinct and all parts are less than or equal to $2m-1$. We define

$$p_m(n) = \sum_{\pi \in P_m(n)} (-1)^{e(\pi)}, \tag{9}$$

$$q_m(n) = \sum_{\pi \in Q_m(n)} (-1)^{e(\pi)}, \tag{10}$$

so that

$$\frac{(-q;q^2)_m}{(-q^2;q^2)_m} = \sum_{n=0}^{\infty} p_m(n)q^n, \tag{11}$$

$$\frac{(q^2;q^2)_{m-1}}{(q;q^2)_m} = \sum_{n=0}^{\infty} q_m(n)q^n. \tag{12}$$

Theorem 2.1: If $r_3(n)$ is the number of representations of n as sum of three squares and if $p_m(n)$ and $q_m(n)$ are as defined by (9)-(10), then

$$r_3(n) = \sum_{m=1}^{n} \sum_{0 \le i \le (n-m)/2m} (-1)^i [2p_m(n-2im-m) + 4q_m(n-2im-m)]. \tag{13}$$

Proof: Employing (11), (12), and the fact that

$$\frac{q^m}{1+q^{2m}} = \sum_{i=0}^{\infty} (-1)^i q^{2im+m}$$

in (8), we immediately have (13).

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NOTICE OF NOVEMBER 1992 VOLUME INDEX CORRECTION

- K. Atanassov's name was inadvertently omitted from the list of authors.
- K. Atanassov's coauthored article "Recurrent Formulas of the Generalized Fibonacci and Tribonacci Sequences" was incorrectly credited to Richard André-Jeannin.

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SOME POLYNOMIAL IDENTITIES FOR THE FIBONACCI AND LUCAS NUMBERS

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It is well known that

(a) $F_{3n} = F_n\{5F_n^2 + 3(-1)^n\}$

Less well known are:

(b) $F_{5n} = F_n\{25F_n^4 + 25(-1)^n F_n^2 + 5\}$;

(c) $F_{7n} = F_n\{125F_n^6 + 175(-1)^n F_n^4 + 70F_n^2 + 7(-1)^n\}$.

In this paper we are concerned with proving a general formula which encompasses the above identities. That is, expresses F_{mn} as a polynomial in F_n for odd m. Also we prove two additional formulas which express F_{mn} / F_n as a polynomial in the Lucas numbers L_n. Our first theorem is

Theorem 1:

$$F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q+1}{q+k+1} 5^k \binom{q+k+1}{2k+1} F_n^{2k}, \quad n, q \geq 0.$$

Taking $q = 1$, 2, and 3, respectively in Theorem 1 gives us (a), (b), and (c) above. From Theorem 1 a couple of well-known results follow as corollaries.

Corollary 1.1: For $n \geq 0, p$ prime, we have

$$F_{pn} \equiv \left(\frac{5}{p}\right) F_n \pmod{p}.$$

Proof: Take $p = 2q+1, p$ prime, in Theorem 1, and by Euler's criterion, we have

$$5^{\frac{(p-1)}{2}} \equiv \left(\frac{5}{p}\right) \pmod{p}. \quad \square$$

Corollary 1.2: For prime p and q, we have $F_{pq} \equiv F_p F_q \pmod{pq}$.

Proof: From Corollary 1.1 with $n = 1$ and $n = q$, we have

$$F_p \equiv \left(\frac{5}{p}\right) \pmod{p} \text{ and } F_{pq} \equiv \left(\frac{5}{p}\right) F_q \pmod{p}, \text{ respectively.}$$

Hence, $F_{pq} \equiv F_p F_q \pmod{p}$.

Similarly, $F_{pq} \equiv F_p F_q \pmod{q}$. $\square$

Proof of Theorem 1: First we need two lemmas.

Lemma (i):

$$\left(x^{2m}+\frac{1}{x^{2m}}\right)+\left(x^{2m-2}+\frac{1}{x^{2m-2}}\right)+\cdots+\left(x^{2}+\frac{1}{x^{2}}\right)+1=\sum_{k=0}^{m}\frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}\left(x-\frac{1}{x}\right)^{2k}$$

Lemma (ii):

$$\left(x^{2m}+\frac{1}{x^{2m}}\right)-\left(x^{2m-2}+\frac{1}{x^{2m-2}}\right)+\cdots+(-1)^{m+1}\left(x^{2}+\frac{1}{x^{2}}\right)+(-1)^{m}$$
$$=\sum_{k=0}^{m}(-1)^{m+k}\frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}\left(x+\frac{1}{x}\right)^{2k}.$$

Now, from $F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, where $\alpha+\beta=1$ and $\alpha\beta=-1$, we have for integer $p\geq 1$, $n\geq 1$,

$$\text{(1.1)}\qquad \frac{F_{pn}}{F_n}=\frac{\alpha^{pn}-\beta^{pn}}{\alpha_n-\beta_n}=x^{p-1}+x^{p-2}y+x^{p-3}y^2+\cdots+xy^{p-2}+y^{p-1},$$

where $x=\alpha^n$, $y=\beta^n=(-1)^n/x$.

Now, for odd p, the RHS of (1.1) is

$$\left(x^{p-1}+\frac{1}{x^{p-1}}\right)+(-1)^n\left(x^{p-3}+\frac{1}{x^{p-3}}\right)+\cdots+\left(x^{2}+\frac{1}{x^{2}}\right)+(-1)^n,\quad p\equiv 3 \pmod 4,$$

$$\left(x^{p-1}+\frac{1}{x^{p-1}}\right)+(-1)^n\left(x^{p-3}+\frac{1}{x^{p-3}}\right)+\cdots+(-1)^n\left(x^{2}+\frac{1}{x^{2}}\right)+1,\quad p\equiv 1 \pmod 4,$$

and $x+\frac{1}{x}=\alpha^n+\frac{1}{\alpha^n}=\alpha^n+(-1)^n\beta^n$. So that

$$\text{(1.2)}\qquad x+\frac{1}{x}=(\alpha-\beta)F_n\quad\text{for odd } n.$$

$$\text{(1.3)}\qquad x-\frac{1}{x}=(\alpha-\beta)F_n\quad\text{for even } n.$$

Since $\alpha-\beta=\sqrt{5}$, we have, from (1.2) and (1.3),

$$\text{(1.4)}\qquad \left(x+\frac{1}{x}\right)^2=5F_n^2\quad\text{for odd } n,$$

$$\text{(1.5)}\qquad \left(x-\frac{1}{x}\right)^2=5F_n^2\quad\text{for even } n.$$

So if we take $p = 2m+1$, and assume n is even, we have, from (1.1),

$$\frac{F_{pn}}{F_n} = \left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1.$$

Now apply Lemma (i) and use (1.5) to give Theorem 1 for even n. Similarly, setting $p = 2m+1$ and assuming n is odd, we have, from (1.1),

$$\frac{F_{pn}}{F_n} = \left(x^{2m} + \frac{1}{x^{2m}}\right) - \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + (-1)^{m+1}\left(x^2 + \frac{1}{x^2}\right) + (-1)^m.$$

Now apply Lemma (ii) and use (1.4) to give Theorem 1 for odd n. To complete the proof of Theorem 1, it only remains to prove Lemmas (i) and (ii). These can be proved by induction. For example, to prove Lemma (i), we set

$$P_m(x) = \left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1$$

and use

$$\left(x^2 + \frac{1}{x^2}\right)\left(x^{2m} + \frac{1}{x^{2m}}\right) = \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \left(x^{2m+2} + \frac{1}{x^{2m+2}}\right)$$

to give

$$\left(x^2 + \frac{1}{x^2}\right)P_m(x) = P_{m+1}(x) + P_{m-1}(x).$$

Hence,

(1.6) $$P_{m+1}(x) = \left\{\left(x - \frac{1}{x}\right)^2 + 2\right\}P_m(x) - P_{m-1}(x).$$

Then substitute the summation on the RHS of the identity in Lemma (i) for $P_m(x)$ and $P_{m-1}(x)$ in (1.6). Some careful work then gives $P_{m+1}(x)$ in the same form as the summation in Lemma (i). This proves Lemma (i). Lemma (ii) is proved in a similar manner, and this completes the proof of Theorem 1. □

For our next theorem we need some additional lemmas; these can be proved by induction in a way similar to that used to prove Lemma (i).

Lemma (iii):

$$\left(x^{2m} + \frac{1}{x^{2m}}\right) - \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + (-1)^{m+1}\left(x^2 + \frac{1}{x^2}\right) + (-1)^m = \sum_{k=0}^{m} \binom{m+k}{2k}\left(x - \frac{1}{x}\right)^{2k}.$$

Lemma (iv):

$$\left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1 = \sum_{k=0}^{m} (-1)^{m+k}\binom{m+k}{2k}\left(x + \frac{1}{x}\right)^{2k}.$$

Again from (1.1) with $p = 2m+1$, and noting that

(1.7) $$x - \frac{1}{x} = \alpha^n + \beta^n = L_n \quad \text{for odd } n,$$

(1.8) $$x + \frac{1}{x} = \alpha^n + \beta^n = L_n \quad \text{for even } n.$$

we have, from (1.7), (1.8), and Lemmas (iii) and (iv),

Theorem 2:

$$F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \binom{q+k}{2k} L_n^{2k}, \quad n, q \geq 0.$$

A well-known formula follows as a corollary by taking $n = 1$; since $L_1 = 1$, we have

Corollary 2.1:

$$F_{2q+1} = \sum_{k=0}^{q} \binom{q+k}{2k}.$$

Our final theorem is similarly derived from the following two lemmas.

Lemma (v):

$$\left(x^{2m-1} - \frac{1}{x^{2m-1}}\right) - \left(x^{2m-3} - \frac{1}{x^{2m-3}}\right) + \cdots + (-1)^m \left(x^3 - \frac{1}{x^3}\right) + (-1)^{m-1}\left(x - \frac{1}{x}\right) = \sum_{k=1}^{m} \binom{m+k-1}{2k-1} \left(x - \frac{1}{x}\right)^{2k-1}.$$

Lemma (vi):

$$\left(x^{2m-1} + \frac{1}{x^{2m-1}}\right) + \left(x^{2m-3} + \frac{1}{x^{2m-3}}\right) + \cdots + \left(x^3 + \frac{1}{x^3}\right) + \left(x + \frac{1}{x}\right) = \sum_{k=1}^{m} (-1)^{m+k} \binom{m+k-1}{2k-1} \left(x + \frac{1}{x}\right)^{2k-1}.$$

Using Lemmas (v) and (vi) along with (1.1) gives

Theorem 3:

$$F_{2qn} = F_n \sum_{k=1}^{q} (-1)^{(n+1)(q+k)} \binom{q+k-1}{2k-1} L_n^{2k-1}, \quad n \geq 0, q \geq 1.$$

Again taking $n = 1$ gives us a well-known formula as a corollary.

Corollary 3.1:

$$F_{2q} = \sum_{k=1}^{q} \binom{q+k-1}{2k-1}.$$

The reader may notice that we appear to have one theorem missing. Namely, a theorem that expresses F_{2qn} as a polynomial F_n. However, to obtain such a formula we would need to be able to express the LHS of Lemma (v) exactly in powers of $\left(x+\frac{1}{x}\right)$ for odd n in (1.1), and the LHS of Lemma (vi) exactly in powers of $\left(x-\frac{1}{x}\right)$ for even n in (1.1), neither of which is possible.

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A VARIATION ON THE TWO-DIGIT KAPREKAR ROUTINE

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In 1949 the Indian mathematician D. R. Kaprekar discovered a curious relationship between the number 6174 and other 4-digit numbers. For any 4-digit number n, whose digits are not all the same, let n' and n'' be the numbers formed by arranging the digits of n in descending and ascending order, respectively. Find the difference of these two numbers: $T(n) = n' - n''$. Repeat this process, known as the *Kaprekar routine*, on $T(n)$. In 7 or fewer steps, the number 6174 will occur. Moreover, 6174 is invariant; that is, $T(6174) = 6174$.

In the literature it is common to generalize the Kaprekar routine and apply it to any k-digit number in base g. Since there are only a finite number of k-digit numbers, repeated applications of T always become periodic. The result is not necessarily a single invariant; more frequently one or more cycles occur. The characterization of such cycles is a difficult problem which has not been completely solved. Among the questions studied are the following: Given k, for what value(s) of g does the Kaprekar routine produce a single invariant? When nontrivial cycles arise for a given g and k, how many cycles are there and what are their lengths? This author, among others, has studied these problems as well as many other fascinating questions associated with the above procedure. (See [1]-[12].)

Recently I was describing the Kaprekar routine to faculty colleagues. To demonstrate that not all k-digit numbers in base 10 give rise to a constant, I chose to illustrate the routine for 2-digit numbers. In that case, either one or two applications of T yields one of the numbers in the cycle

$$63 \to 27 \to 45 \to 09 \to 81 \to 63.$$

Embarrassingly, I made an arithmetic mistake, writing $T(96) = 96 - 69 = 37$ instead of $T(96) = 27$. Arleigh Bell, Associate Professor of Economics, asked what would happen if 10 or any other number r were always added to $T(n)$. What would the cycles look like in that case? Could there be a Kaprekar constant for some number r? This paper is an answer to his questions.

As is the usual practice, we will consider Bell's questions for a general base g. We will represent a 2-digit, base g number $n = a'g + a$, $0 \le a'$, $a < g$, by $n = [a' \;\; a]$. The Bell modification of the Kaprekar routine is a function $K_{[r' \;\, r]}(n)$ defined in the following manner. Let $[r' \;\; r]$ be a fixed 2-digit, base g number less than $[1 \;\; g-1]$; that is, $r' = 0$ or 1, $0 \le r \le g-1$ if $r' = 0$, and $0 \le r \le g-2$ if $r' = 1$. Then, for $n = [a' \;\; a]$

$$K_{[r' \;\, r]}(n) = |[a' \;\; a] - [a \;\; a']| + [r' \;\; r].$$

When the context is clear, we will omit the subscript and simply write $K(n)$. To see why we require $[r' \;\; r] < [1 \;\; g-1]$, note that

$$|[a' \;\; a] - [a \;\; a']| = [|a' - a| - 1 \;\; g - |a' - a|].$$

Now $|a' - a| - 1 \le g - 2$, so $|[a' \;\; a] - [a \;\; a']| \le [g-2 \;\; 1]$. Thus, the restriction $[r' \;\; r] < [1 \;\; g-1]$ insures that $K(n)$ is a 2-digit number.

Since there are only a finite number of 2-digit, base g numbers, the sequence

$$n,\ K(n),\ K^2(n),\ K^3(n), \ldots$$

must eventually repeat. If, for a given n, $K^i(n)=n$, where i is as small as possible, then we say that n is in a cycle of length i. We will denote a K-cycle by $\langle n_1, n_2, \ldots, n_i\rangle$, where $n_{j+1}=K(n_j)$ for $1 \le j \le i-1$ and $n_1 = K(n_i)$ We wish to characterize those n which are in cycles and to determine the lengths of these cycles.

For $n=[a' \ \ a]$, $d=|a'-a|$ is called the digit difference of n. Observe that, if $0<d$, then

$$|[a' \ \ a]-[a \ \ a']| = [d-1 \ \ g-d].$$

Thus, if $n=[a' \ \ a]$, $m=[b' \ \ b]$, and $d=|a'-a|=|b'-b|$, then $K(n)=K(m)$. In particular, if n is a 2-digit number whose digit difference is d, $K(n)$ equals

$$(1) \qquad \begin{array}{lll} [d & r-d] & \text{if } r'=0 \text{ and } 0 \le d \le r \\ [d-1 & g-(d-r)] & \text{if } r'=0 \text{ and } r<d<g \\ [d+1 & r-d] & \text{if } r'=1 \text{ and } 0 \le d \le r \\ [d & g-(d-r)] & \text{if } r'=1 \text{ and } r<d<g. \end{array}$$

Using (1), it is easy to see that the digit difference of $K(n)$ is

$$(2) \qquad \begin{array}{ll} |r-2d| & \text{if } r'=0 \text{ and } 0 \le d \le r \\ |g+r+1-2d| & \text{if } r'=0 \text{ and } r<d<g \\ |r-1-2d| & \text{if } r'=1 \text{ and } 0 \le d \le r \\ |g+r-2d| & \text{if } r'=1 \text{ and } r<d<g. \end{array}$$

We will denote the digit difference of $K_{[r' \ r]}(n)$ in (2) by $D_{[r' \ r]}(d)$ or $D(d)$. Note that, if $K([a' \ \ a])=[b' \ \ b]$, then $D(|a'-a|)=|b'-b|$. Thus, each $K_{[r' \ r]}$-cycle gives rise to a $D_{[r' \ r]}$-cycle of the same length. If we can characterize the D-cycles, then we will have made substantial progress in characterizing the K-cycles. That is, we will know how many such cycles there are and the length of each one.

As an example, let $g=10$, $r'=0$, and $r=7$. That is, we wish to apply the routine to base 10 numbers with 7, the added term. Using (2), we find

$$\begin{array}{lllll} D(0)=7 & D(1)=5 & D(2)=3 & D(3)=1 & D(4)=1 \\ D(5)=3 & D(6)=5 & D(7)=7 & D(8)=2 & D(9)=0. \end{array}$$

Thus, the D_7-cycles are $\langle 1,5,3\rangle$ and $\langle 7\rangle$. From these, it is easy to determine that the K_7-cycles are $\langle 34,16,52\rangle$ and $\langle 70\rangle$.

Examination of (2) shows that $D(d)$ always has the form $|s-2d|$ for some s. Consequently, we will first study a function based on this observation. In particular, let s be a fixed positive integer. For d with $0 \le d \le s$, define $F_s(d)=|s-2d|$. Since $0 \le F_s(d) \le s$, cycles must occur. The following observations about F, collected in a single theorem for convenience, are obvious.

Theorem 1: Let s and i be positive integers and let d be an integer satisfying $0 \le d \le s$. Then

- ***(a)*** $F_s(s)=s$, so $\langle s\rangle$ is an F_s-cycle of length 1.
- ***(b)*** $F_{is}(id)=iF_s(d)$
- ***(c)*** d is in an F_s-cycle if and only if id is in an F_{is}-cycle. In particular, $\langle d_1, d_2, \ldots, d_n\rangle$ is an F_s-cycle if and only if $\langle id_1, id_2, \ldots, id_n\rangle$ is an F_{is}-cycle.
- ***(d)*** $F_s^i(d)$ is congruent to either $2^i d$ or $-2^i d$ modulo s. □

For convenience, we will use the notation $F_s^i(d) \equiv \pm 2^i d \pmod{s}$ to represent statement (d) in Theorem 1.

Theorem 2: Suppose $2^k \| s$. Let d be an integer satisfying $0 \le d \le s$. If d is in an F_s-cycle, then $2^k \| d$.

Proof: Since $2^k \| s$, $s = 2^k t$ where $0 \le k$ and t is an odd positive integer. Write $d = 2^i w$ with $0 \le i$ and w odd. Then $F(d) = F(2^i w) = |2^k t - 2^{i+1} w|$. So

$$\begin{array}{ll} 2^{i+1} \| F(d) & \text{if } 0 \le i < k-1 \\ 2^{k+1} | F(d) & \text{if } i = k-1 \\ 2^k \| F(d) & \text{if } i > k-1. \end{array}$$

Thus, $2^j | F^j(d)$ for $j \le k-1$ and $2^k \| F^j(d)$ for $k+1 \le j$. Consequently, d is in a cycle only if $2^k \| d$. □

Corollary 1: Suppose $2^k \| s$. Let d be an integer satisfying $0 \le d \le s$. Then d is in an F_s-cycle if and only if $2^k \| d$.

Proof: First, suppose s is odd. By Theorem 2, it is sufficient to show that if d is odd, then it is in a cycle. Since s and d are odd, $(s+d)/2$ and $(s-d)/2$ are both nonnegative integers less than or equal to s. One of these numbers is odd and the other is even. Moreover, $F((s+d)/2) = d$ and $F((s-d)/2) = d$. Consequently, d has an odd predecessor. Since this is true for all odd integers between 0 and s, d must be in a cycle.

The case when s is even follows immediately using Theorem 2 and Theorem 1(c). □

Corollary 2: An integer s has only one F_s-cycle, namely $\langle s \rangle$, if and only if $s = 2^k$ for some k.

Proof: The proof is immediate using Theorem 1(a) and Corollary 1. □

By the results above, to characterize F-cycles it is sufficient to determine cycles for odd s. Additionally, we need only consider those d which are odd, have $\gcd(d, s) = 1$ and satisfy $1 \le d \le s-2$. We will call cycles containing such d nontrivial. All other cycles are trivial since they may be obtained using (a) and (c) of Theorem 1.

We will illustrate the comments above by finding the F-cycles for $s = 33$. By Corollary 1, only odd integers are in a cycle. Nontrivial F-cycles for $s = 3$ and $s = 11$ are $\langle 1 \rangle$ and $\langle 1, 9, 7, 3, 5 \rangle$, respectively. Thus, by Theorem 1(c),

$$\langle 11 \rangle,\ \langle 3, 27, 21, 9, 15 \rangle,\ \langle 33 \rangle \tag{3}$$

are trivial F-cycles for $s = 33$. We now want to calculate the nontrivial F-cycles. An efficient method, described for the general case and then applied to $s = 33$, is as follows. By Theorem 1(d), $F(d)$ is congruent to either $2d$ or $-2d$ modulo s. For s and d odd, exactly one of the numbers $2d$ or $-2d$ is congruent modulo s to an odd positive integer less than s. So to compute the cycle containing d, it is sufficient to calculate $\pm 2F^i(d) \equiv \pm 2F(F^{i-1}(d))$, choosing the appropriate sign so that the result modulo s is an odd integer. Applying this to our example $s = 33$ with $d = 1$ gives $1, -2 \equiv 31,\ 62 \equiv 29,\ 58 \equiv 25,\ 50 \equiv 17,\ 34 \equiv 1$, which yields the F-cycle

$$\langle 1, 31, 29, 25, 17 \rangle. \tag{4}$$

At this point we check to see if all odd integers d, $1 \le d \le s$, are accounted for. If not, we repeat the above procedure. In the present example, $d = 5$ is not contained in any of the cycles in (3) or (4). So we consider $5,\ -10 \equiv 23,\ 46 \equiv 13,\ -26 \equiv 7,\ -14 \equiv 19,\ 38 \equiv 5$, which gives

$$\langle 5, 23, 13, 7, 19 \rangle. \tag{5}$$

Thus, there are five F_{33}-cycles which are given in (3), (4), and (5).

For future reference, we record the F_s-cycles for $0 \le s \le 15$:

s	F_s - cycles	s	F_s - cycles
0	$\langle 0 \rangle$	8	$\langle 8 \rangle$
1	$\langle 1 \rangle$	9	$\langle 1, 7, 5 \rangle$, $\langle 3 \rangle$, $\langle 9 \rangle$
2	$\langle 2 \rangle$	10	$\langle 2, 6 \rangle$, $\langle 10 \rangle$
3	$\langle 1 \rangle$, $\langle 3 \rangle$	11	$\langle 1, 9, 7, 3, 5 \rangle$, $\langle 11 \rangle$
4	$\langle 4 \rangle$	12	$\langle 4 \rangle$, $\langle 12 \rangle$
5	$\langle 1, 3 \rangle$, $\langle 5 \rangle$	13	$\langle 1, 11, 9, 5, 3, 7 \rangle$, $\langle 13 \rangle$
6	$\langle 2 \rangle$, $\langle 6 \rangle$	14	$\langle 2, 10, 6 \rangle$, $\langle 14 \rangle$
7	$\langle 1, 5, 3 \rangle$, $\langle 7 \rangle$	15	$\langle 1, 13, 11, 7 \rangle$, $\langle 3, 9 \rangle$, $\langle 5 \rangle$, $\langle 15 \rangle$

Theorem 3: Let s be an odd positive integer and let m be the smallest integer such that $2^m \equiv \pm 1 \pmod{s}$. Then each nontrivial F_s-cycle is of length m and there are $\phi(s)/2m$ such cycles, where $\phi(s)$ is the Euler phi function.

Proof: As before, we write ± 1 to indicate that 2^m is congruent modulo s to either 1 or –1. Suppose d is odd with $\gcd(d, s) = 1$ and i is the smallest integer such that $F^i(d) = d$. That is, we assume that d is a nontrivial cycle of length i. By Theorem 1(d), $F^i(d) \equiv \pm 2^i d \pmod{s}$, so $\pm 2^i d \equiv d \pmod{s}$. Since $\gcd(d, s) = 1$, $2^i \equiv \pm 1 \pmod{s}$. Consequently, each cycle has length $i = m$. There are $\phi(s)/2$ odd positive integers less than s which are relatively prime to s. Therefore, there are $\phi(s)/2m$ nontrivial F-cycles. □

The smallest positive integer k such that $2^k \equiv 1 \pmod{s}$ is called the order of 2 modulo s and is denoted by $\mathrm{ord}_s 2$.

Corollary 3: Let s be an odd positive integer and let m be the smallest integer such that $2^m \equiv \pm 1 \pmod{s}$. If $2^m \equiv +1 \pmod{s}$, then each nontrivial F_s-cycle has length equal to $\mathrm{ord}_s 2$; otherwise, the length equals $(\mathrm{ord}_s 2)/2$.

Proof: If $2^m \equiv +1 \pmod{s}$, then $\mathrm{ord}_s 2 = m$ and the result follows immediately from Theorem 3.

If $2^m \equiv -1 \pmod{s}$, then $2^{2m} \equiv +1 \pmod{s}$. By a well-known theorem from Number Theory, $k | 2m$ where $k = \mathrm{ord}_2 2$. If k is odd, then $k | m$ and $m = kq$ for some q. But this implies that $2^m \equiv (2^k)^q \equiv 1 \pmod{s}$, which is a contradiction. Thus, it must be the case that k is even and $(k/2) | m$. If $(k/2) < m$, then $m = (k/2)q$ with $1 < q$. But then $2^{(k/2)2} \equiv 1 \pmod{s}$, which contradicts the choice of m. Thus, $m = k/2 = (\mathrm{ord}_s 2)/2$. □

Corollary 4: Let p be an odd prime. Then the length of each nontrivial F_p-cycle equals

$$m = \mathrm{ord}_p 2 / \gcd(\mathrm{ord}_p 2, 2).$$

Proof: Let m be the smallest integer such that $2^m \equiv \pm 1 \pmod{p}$. The proof of Corollary 3 shows that if $2^m \equiv -1 \pmod{p}$, then $\mathrm{ord}_p 2$ is even and $m = \mathrm{ord}_p 2 / 2 = \mathrm{ord}_p 2 / \gcd(\mathrm{ord}_p 2, 2)$.

If $2^m \equiv 1 \pmod p$ with $m = \mathrm{ord}_p 2$, then m must be odd. For if m were even, then $(2^{m/2})^2 \equiv 1 \pmod p$. Since p is prime, $2^{m/2} \equiv \pm 1 \pmod p$, which is a contradiction to the choice of m. Thus, $m = \mathrm{ord}_p 2 = \mathrm{ord}_p 2 / \gcd(\mathrm{ord}_p 2, 2)$. □

Corollary 5: Let s be an odd positive integer and suppose 2 is a primitive root of s. Then s has only one nontrivial F_s-cycle.

Proof: Since 2 is a primitive root of s, $\mathrm{ord}_s 2 = \phi(s)$. Moreover, there exists a unique positive integer i less than $\phi(s)$ such that $2^i \equiv -1 \pmod s$. By Corollary 3, the length of each nontrivial cycle is $\phi(s)/2$. Consequently, by Theorem 3, there is only one such cycle. □

We now state and prove three technical lemmas which will be useful when we apply this work to D-cycles.

Lemma 1: Let $s = g + r + 1$ and d be an integer satisfying $r < d < g$ and $r < F(d) < g$. Then $r < g/2$.

Proof: Suppose, to the contrary, that $g/2 \le r$. Since, by assumption, $r < d$, $g/2 < d$, which implies $g + r + 1 - 2d \le r$. Also, $d < g \le g/2 + r$ so that $2d - (g + r + 1) < r$. Thus,

$$F_s(d) = |g + r + 1 - 2d| \le r,$$

which is a contradiction to the hypothesis. □

Lemma 2: Let $s = g + r$ and d be an integer satisfying $r \le d < g$ and $r < F(d) < g$. Then $r < g/2$.

Proof: The proof is similar to that of Lemma 1. □

Lemma 3: Let $s = g + r$. If r has a predecessor under F_s, then $2|g$.

Proof: Suppose there exists d such that $F_s(d) = r$. Then either $g + r - 2d = r$ or $2d - (g + r) = r$. So either d equals $g/2$ or $r + g/2$. In either case, $2|g$. □

We are now in a position to characterize D-cycles.

Theorem 4: Let g be a positive integer and r an integer satisfying $0 \le r \le g - 1$. All F_r-cycles will be $D_{[0\ r]}$-cycles. If $r < g/2$ and there exists a d such that $r < F^i_{g+r+1}(d) < g$ for $0 \le i$, then this F_{g+r+1}-cycle is also a $D_{[0\ r]}$-cycle.

Proof: Since the added term is $[0\ r]$, the first two lines of (2) apply. From the first line, we see that all F_r-cycles will be $D_{[0\ r]}$-cycles. In order for the second line to give $D_{[0\ r]}$-cycles, it must be the case that all d in an F_{g+r+1}-cycle satisfy $r < d < g$. By Lemma 1, such cycles can occur only when $r < g/2$. □

As a consequence of Theorem 4, in order to find all $D_{[0\ r]}$-cycles for a given g, it is sufficient to examine all F_s-cycles for $0 \le s \le g + [(g+1)/2]$. For example, using (6), it is easy to find the $D_{[0\ r]}$-cycles for $g = 10$. These, as well as the corresponding $K_{[0\ r]}$-cycles, are:

(7)

r	$g+r+1$	$D_{[0\ r]}$ - cycles	$K_{[0\ r]}$ - cycles
0	11	$\langle 0\rangle$, $\langle 1,9,7,3,5\rangle$	$\langle 0\rangle$, $\langle 45,9,81,63,27\rangle$
1	12	$\langle 1\rangle$, $\langle 4\rangle$	$\langle 10\rangle$, $\langle 37\rangle$
2	13	$\langle 2\rangle$	$\langle 20\rangle$
3	14	$\langle 1\rangle$, $\langle 3\rangle$	$\langle 12\rangle$, $\langle 30\rangle$
4	15	$\langle 4\rangle$, $\langle 5\rangle$	$\langle 40\rangle$, $\langle 49\rangle$
5		$\langle 1,3\rangle$, $\langle 5\rangle$	$\langle 32,14\rangle$, $\langle 50\rangle$
6		$\langle 2\rangle$, $\langle 6\rangle$	$\langle 24\rangle$, $\langle 60\rangle$
7		$\langle 1,5,3\rangle$, $\langle 7\rangle$	$\langle 34,16,52\rangle$, $\langle 70\rangle$
8		$\langle 8\rangle$	$\langle 80\rangle$
9		$\langle 1,7,5\rangle$, $\langle 3\rangle$, $\langle 9\rangle$	$\langle 54,18,72\rangle$, $\langle 36\rangle$, $\langle 90\rangle$

Theorem 5: Let g be a positive integer and let r be an integer satisfying $0 \le r \le g-2$. All F_{r-1}-cycles will be $D_{[1\ r]}$-cycles. If $r < g/2$ and there exists a d such that $r < F^i_{g+r}(d) < g$ for $0 \le i$, then this F_{g+r}-cycle is also a $D_{[1\ r]}$-cycle. If $2|g$, and if $F^j_{g+r}(r+1) = r$ for some j, then r is in a $D_{[1\ r]}$-cycle.

Proof: Since the added term is $[1\ r]$, the third and fourth lines of (2) apply. From the third, we see that all F_{r-1}-cycles will be $D_{[1\ r]}$-cycles. In order for the fourth to give $D_{[1\ r]}$-cycles, it must be the case that all d in an F_{g+r}-cycle satisfy $r < d < g$. By Lemma 2, cycles such as these can occur only when $r < g/2$. There is one more way in which $D_{[1\ r]}$-cycles can arise. Note that $D_{[1\ r]}(r) = r+1$ and $D^i_{[1\ r]}(r) = F^{i-1}_{g+r}(r+1)$ for $2 \le i$. So if, for some j, $F^j_{g+r}(r+1) = r$, then r will be in a $D_{[1\ r]}$-cycle even though it may not be in an F_{g+r}-cycle. By Lemma 3, in order for r to have an F_{g+r} predecessor, g must be even. □

Finding $D_{[1\ r]}$-cycles which do not contain r is similar to finding $D_{[0\ r]}$-cycles. In particular, we examing F_s-cycles for $1 \le s \le g-3$ and $g \le s \le g+[(g-1)/2]$. For example, again using (6), it is easy to find these cycles for $g = 10$.

(8)

r	$r-1$	$g+r$	$D_{[1\ r]}$ - cycles	$K_{[1\ r]}$ - cycles
0		10	$\langle 2,6\rangle$	$\langle 64,28\rangle$
1	0	11	$\langle 0\rangle$	$\langle 11\rangle$
2	1	12	$\langle 1\rangle$, $\langle 4\rangle$	$\langle 21\rangle$, $\langle 48\rangle$
3	2	13	$\langle 2\rangle$	$\langle 31\rangle$
4	3	14	$\langle 1\rangle$, $\langle 3\rangle$	$\langle 23\rangle$, $\langle 41\rangle$
5	4		$\langle 4\rangle$	$\langle 51\rangle$
6	5		$\langle 1,3\rangle$, $\langle 5\rangle$	$\langle 43,25\rangle$, $\langle 61\rangle$
7	6		$\langle 2\rangle$, $\langle 6\rangle$	$\langle 35\rangle$, $\langle 71\rangle$
8	7		$\langle 1,5,3\rangle$, $\langle 7\rangle$	$\langle 45,27,63\rangle$, $\langle 81\rangle$

Missing from (8) are those $D_{[1\ r]}$-cycles which contain r. The final theorems address this special case.

Theorem 6: Let g be an even positive integer. When r equals 1, $g/2-2$ or $g/2-1$, then

$$\text{(9)} \qquad \begin{array}{l} \langle 2, g-3, \ldots, F_{g+1}^{i}(1), \ldots, 1\rangle \text{ with } 2 \le i \\ \langle g/2-2, g/2-1, g/2\rangle \\ \langle g/2-1, g/2\rangle \end{array}$$

are $D_{[1\ r]}$-cycles, respectively.

Proof: The last two cases are easily verified. For the first, by Corollary 1, 1 is in an F_{g+1}-cycle; in particular

$$\langle 1, g-1, g-3, \ldots, F_{g+1}^{i}(1), \ldots, 1\rangle.$$

Since $D_{[1\ r]}(1) = 2$ and $D_{[1\ r]}(2) = g-3$, applying the $D_{[1\ r]}$-algorithm gives

$$\langle 2, g-3, \ldots, F_{g+1}^{i}(1), \ldots, 1\rangle. \ \square$$

Theorem 7: Let g and r be positive integers. If r is in an $D_{[1\ r]}$-cycle different from those in (9), then $r \le g/4-1$.

Proof: By Theorem 5, since r is in an $D_{[1\ r]}$-cycle, $F_{g+r}^{j}(r+1) = r$ for some $0 < j$. If $j = 1$, then $r = g/2-$, contradicting the hypothesis. Thus, $2 \le j$. Now,

$$D_{[1\ r]}^{3}(r) = F_{g+r}^{2}(r+1) = F_{g+r}(g-r-2) = |g-3r-4|.$$

By Lemma 2 and Theorem 6, $1 < r < g/2-2$ so that $D_{[1\ r]}^{3}(r) = g-3r-4$. If r is in an $D_{[1\ r]}$-cycle, then $r \le D_{[1\ r]}^{3}(r)$. This implies $r \le g/4-1$. $\square$

For $g = 10$, by Theorem 6, the following $D_{[1\ r]}$-cycles may be added to the list in (8):

(10)

r	$D_{[1\ r]}$-cycles	$K_{[1\ r]}$-cycles
1	$\langle 1, 2, 7, 3, 5\rangle$	$\langle 56, 20, 29, 74, 38\rangle$
3	$\langle 3, 4, 5\rangle$	$\langle 58, 40, 49\rangle$
4	$\langle 4, 5\rangle$	$\langle 59, 50\rangle$

By Theorem 7, these are the only $D_{[1\ r]}$-cycles that contain r. Thus, (7), (8), and (10) comprise a complete list of all $D_{[r'\ r]}$-cycles for $g = 10$.

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Announcement

SIXTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Tentatively set for July 11-15, 1994
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ON CONSECUTIVE NIVEN NUMBERS

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INTRODUCTION

In [1] the concept of a Niven number was introduced with the following definition.

Definition: A positive integer is called a Niven number if it is divisible by its digital sum.

Various articles have appeared concerning digital sums and properties of the set of Niven numbers. In particular, it was shown in [2] that no more than 21 consecutive Niven numbers is possible. Here, we will show, in fact, that no more than 20 consecutive Niven numbers is possible and give an infinite number of examples of such sequences.

DIGITAL SUMS AND CARRIES

In what follows, $s(n)$ will denote the digital sum of the positive integer n. The formula

$$s(n) = n - 9\sum_{t\ge 1}\left[\frac{n}{10^t}\right],$$

where the square brackets represent the greatest integer function, is well known and easily derived. Note that the sum has only a finite number of terms since $\left[\frac{n}{10^t}\right] = 0$ where $t > [\log n]$

For integers m and n, we let $c(m+n)$ denote the sum of the "carries" which occur when calculating the sum $m+n$. The following Lemma gives the relationship between $s(m+n)$ and $c(m+n)$.

Lemma: Let m, n be positive integers. Then

$$s(m+n) = s(m) + s(n) - 9c(m+n).$$

Proof: Since

$$s(m) = m - 9\sum_{t\ge 1}\left[\frac{m}{10^t}\right] \quad \text{and} \quad s(n) = n - 9\sum_{t\ge 1}\left[\frac{n}{10^t}\right],$$

it follows that

$$\begin{aligned} s(m)+s(n) &= m+n-9\sum_{t\ge 1}\left(\left[\frac{m}{10^t}\right]+\left[\frac{n}{10^t}\right]\right) \\ &= s(m+n)+9\sum_{t\ge 1}\left(\left[\frac{m+n}{10^t}\right]-\left[\frac{m}{10^t}\right]-\left[\frac{n}{10^t}\right]\right). \end{aligned}$$

Noting that the expression

$$\left[\frac{m+n}{10^t}\right]-\left[\frac{m}{10^t}\right]-\left[\frac{n}{10^t}\right]$$

is the carry that occurs when the $(t-1)^{\text{st}}$ right-most digit of n are added, the equality $s(m+n) = s(m)+s(n)-9c(m+n)$ follows.

In passing, the resder might be interested in proving that $s(mn) = s(m)s(n) - 9c(mn)$ where $c(mn)$ is the sum of the carries that occur in calculating the product of m and n by the usual multiplication algorithm. Here, however, we are concerned with sequences of consecutive Niven numbers.

CONSECUTIVE NIVEN NUMBERS

To discuss consecutive Niven numbers, we will introduce the idea of a decade and a century of numbers. A decade is a set of numbers

$$\{10n, 10n+1, \ldots, 10n+9\}$$

for any nonnegative integer n and a century is a set of numbers

$$\{100n, 100n+1, \ldots, 100n+99\}$$

for any nonnegative integer n. We first observe that in a given decade, either all the odd numbers have an even digital sum or all the odd numbers have an odd digital sum. To make the next observation, let E denote the statement "odd numbers which have an even digital sum" and O denote the statement "odd numbers which have an odd digital sum." We then note that the ten decades in a century alternate either O, E, O, E, O, E, O, E, O, E or E, O, E, O, E, O, E, O, E, O. Finally, we remark that in an E decade, none of the odd numbers can be Niven since their digital sum is even. Thus, the only way to get more than 11 consecutive Niven numbers is to cross a century boundary where the decades between centuries would be

$$\ldots, \text{E}, \text{O}, \text{E}, \text{O} \mid \text{O}, \text{E}, \text{O}, \text{E}, \ldots .$$

Hence, we cannot have more than 21 consecutive Niven numbers and if a list of 21 consecutive Niven numbers exists, it would have to commence with an even Niven number of the form

$$10^t n + 9_{t-1}0$$

where d_r denotes the concatenation of r d's in the decimal representation of an integer. For example,

$$89_5(24)_2 0_3 7 = 89999924240007.$$

Note that d does not have to be a digit. This notation will facilitate an efficient representation for certain large integers.

It is not difficult to find sequences of consecutive Niven numbers. For example, the sequence 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 is an example of 10 consecutive Niven numbers. It is, of course, the smallest such sequence. Other sequences of 10 consecutive Niven numbers can be found, but if a sequence of 21 consecutive Niven numbers could be found, we would have an example of every possible sequence of k consecutive Niven numbers for k = 1, 2, 3, ..., 21. As suggested in the introduction, however, it will be shown that k cannot be larger than 20, and an infinite number of examples with k = 20 will be given. Determining an example with k = 20 involves working with large integers, solving systems of linear congruences, choosing integers with "good" digital sums, a lot of adjusting partial results, and a lot of luck and intuition. Without the use of a computer capable of manipulating large numbers, we could not have found the following sequence in a reasonable length of time.

Let

$$\begin{aligned} a = {} & 4090669070187777592348077471447408839621564801 \\ & 2007115516094806249015486761744582584646124234 \\ & 1540855543641742325745294115007591954820126570 \\ & 087071005523266064292043054902370439430_{1120} \end{aligned}$$

and

$$\begin{aligned} b = {} & 2846362190166818294716429619770154544233311863 \\ & 4187301827478422658543387589306681088151446703 \\ & 2759507916140833155837906335537198825206802774 \\ & 843028314975502097292745955936059236215 69_{1119}0. \end{aligned}$$

Then a has 1296 digits, b has 1298 digits, $s(a) = 720$, and $s(b) = 10870$. Also note that each of

$$\begin{array}{c} 2464645030 \\ 2464645031 \\ \vdots \\ 2464645039 \\ 2464634960 \\ 2464634961 \\ \vdots \\ 2464634969 \end{array}$$

is a factor of a, and

$$\begin{array}{l} 2464645030 \text{ divides } b \\ 2464645031 \text{ divides } b+1 \\ \qquad\qquad\quad \vdots \\ 2464645039 \text{ divides } b+9 \\ 2464634960 \text{ divides } b+10 \\ 2464634961 \text{ divides } b+11 \\ \qquad\qquad\quad \vdots \\ 2464634969 \text{ divides } b+19. \end{array}$$

Now let m be any nonnegative integer and consider

$$x = a_{3423103}0_m b.$$

Then x has $44363342786 + m$ digits, and is a Niven number with $s(x) = 2464645030$. Furthermore, by construction, each of $x+1, x+2, x+3, \ldots, x+19$ is also a Niven number, and a sequence of 20 consecutive Niven numbers has been constructed. Also, since m is an arbitrary nonnegative integer, we have demonstrated an infinite number of such sequences. However, that the methods used in finding such a sequence cannot be used to find 21 consecutive Niven numbers, is revealed by the following discussion.

Suppose that there exists a sequence $x, x+1, x+2, \ldots, x+19, x+20$ of Niven numbers. Then $x \equiv 9_{t-1}0 \pmod{10^t}$ where we may assume that the $(t+1)^{\text{st}}$ right-most digit of x is not a 9. Thus,

(1)	$x \equiv 0 \pmod{s(x)}$	(5)	$x \equiv -4 \pmod{s(x)+4}$
(2)	$x \equiv -1 \pmod{s(x)+1}$	(6)	$x \equiv -5 \pmod{s(x)+5}$
(3)	$x \equiv -2 \pmod{s(x)+2}$	(7)	$x \equiv -6 \pmod{s(x)+6}$
(4)	$x \equiv -3 \pmod{s(x)+3}$	(8)	$x \equiv -7 \pmod{s(x)+7}$

(9)	$x \equiv -8 \pmod{s(x)+8}$	(16)	$x \equiv -15 \pmod{s(x)+15-9t}$
(10)	$x \equiv -9 \pmod{s(x)+9}$	(17)	$x \equiv -16 \pmod{s(x)+16-9t}$
(11)	$x \equiv -10 \pmod{s(x)+10-9t}$	(18)	$x \equiv -17 \pmod{s(x)+17-9t}$
(12)	$x \equiv -11 \pmod{s(x)+11-9t}$	(19)	$x \equiv -18 \pmod{s(x)+18-9t}$
(13)	$x \equiv -12 \pmod{s(x)+12-9t}$	(20)	$x \equiv -19 \pmod{s(x)+19-9t}$
(14)	$x \equiv -13 \pmod{s(x)+13-9t}$	(21)	$x \equiv -20 \pmod{s(x)+11-9t}$
(15)	$x \equiv -14 \pmod{s(x)+14-9t}$		

It should be pointed out that the form of the moduli in the above list follow by the Lemma. That is,

$$\begin{aligned} s(x+k) &= s(x)+s(k)-9c(x+k) \\ &= s(x)+s(k)-9(t-1). \end{aligned}$$

For example,

$$s(x+19) = s(x)+10-9(t-1) = s(x)+19-9t$$

and so the congruence

$$x+19 \equiv 0 \pmod{s(x+19)}$$

may be written as

$$x \equiv -19 \pmod{s(x)+19-9t}.$$

Since

$$x \equiv -20 \pmod{s(x)+11-9t},$$

and

$$x \equiv -11 \pmod{s(x)+11-9t},$$

we immediately have that

$$9 \equiv 0 \pmod{s(x)+11-9t},$$

and so, $s(x)+11-9t = 1, 3,$ or 9. Thus, $9t-s(x) = 2, 8,$ or 10. However, since

$$x \equiv 9_{t-1}0 \pmod{10^t},$$

we see that $s(x) \geq 9t-9$, and it follows that $9t-s(x) = 2$ or 8.

Suppose that $9t-s(x) = 8$. Then, by congruences (11), (12), (14), (16), and (20), we have the system

$$\begin{aligned} x &\equiv 0 \pmod{2} \\ x &\equiv 1 \pmod{3} \\ x &\equiv 2 \pmod{5} \\ x &\equiv 6 \pmod{7} \\ x &\equiv 3 \pmod{11} \end{aligned}$$

which, by the Chinese Remainder Theorem, has the solution

$$x \equiv 6922 \pmod{2310}.$$

But, since $x \equiv 9_{t-1}0 \pmod{10^t}$, it follows that 5 is a factor of x. This cannot be the case if $x \equiv 6922 \pmod{2310}$. Hence, we must conclude that $9t-s(x) \neq 8$.

Now suppose that $9t - s(x) = 2$. Then the congruences (1) through (21) may be rewritten as:

(1)	$x \equiv 0 \pmod{9t-2}$	(12)	$x \equiv 7 \pmod{9}$
(2)	$x \equiv -1 \pmod{9t-1}$	(13)	$x \equiv 8 \pmod{10}$
(3)	$x \equiv -2 \pmod{9t}$	(14)	$x \equiv 9 \pmod{11}$
(4)	$x \equiv -3 \pmod{9t+1}$	(15)	$x \equiv 10 \pmod{12}$
(5)	$x \equiv -4 \pmod{9t+2}$	(16)	$x \equiv 11 \pmod{13}$
(6)	$x \equiv -5 \pmod{9t+3}$	(17)	$x \equiv 12 \pmod{14}$
(7)	$x \equiv -6 \pmod{9t+4}$	(18)	$x \equiv 13 \pmod{15}$
(8)	$x \equiv -7 \pmod{9t+5}$	(19)	$x \equiv 14 \pmod{16}$
(9)	$x \equiv -8 \pmod{9t+6}$	(20)	$x \equiv 15 \pmod{17}$
(10)	$x \equiv -9 \pmod{9t+7}$	(21)	$x \equiv 7 \pmod{9}$,
(11)	$x \equiv 6 \pmod{8}$		

respectively.

Recall that if the system

$$\begin{aligned} x &\equiv r \pmod{m} \\ x &\equiv s \pmod{n} \end{aligned}$$

has a solution, then $\gcd(m, n)$ is a factor of $r-s$. See, for example, [3, Th. 5-11]. Thus, by use of the pairings

(4) with (13)
(6) with (13)
(7) with (13)
(10) with (13),

we have that

$$\begin{aligned} &\gcd(10, 9t+1) = 1 \\ &\gcd(10, 9t+3) = 1 \\ &\gcd(15, 9t+4) = 1 \\ &\gcd(10, 9t+7) = 1, \end{aligned}$$

respectively. The fact that x is even together with congruence (2), imply that t is even. But

$$\begin{aligned} t &\equiv 2 \pmod{10} \text{ implies that } \gcd(10, 9t+7) \neq 1, \\ t &\equiv 4 \pmod{10} \text{ implies that } \gcd(15, 9t+4) \neq 1, \\ t &\equiv 6 \pmod{10} \text{ implies that } \gcd(10, 9t+1) \neq 1, \\ t &\equiv 8 \pmod{10} \text{ implies that } \gcd(10, 9t+3) \neq 1, \end{aligned}$$

which contradict the above. So, it follows that

$$\begin{aligned} t &\not\equiv 2 \pmod{10} \\ t &\not\equiv 4 \pmod{10} \\ t &\not\equiv 6 \pmod{10} \\ t &\not\equiv 8 \pmod{10}. \end{aligned}$$

In addition, the pair of congruences $x \equiv 9_{t-1}0 \pmod{10^t}$ and $x \equiv -7 \pmod{9t+5}$ imply that $\gcd(10^t, 9t+5)$ divides $9_{t-1}7$, from which it follows that 5 cannot be a factor of $\gcd(10^t, 9t+5)$

and so we have that $t \not\equiv 0 \pmod{10}$. Hence, by assuming that $9t - s(x) = 2$, the fact that t is even is contradicted, and we conclude that $9t - s(x) \neq 2$. So, the only two possibilities for $9t - s(x)$ (by assuming that a sequence of 21 consecutive Niven numbers exists) are eliminated. We have, then, the following theorem.

Theorem: There does not exist a sequence of 21 consecutive Niven numbers.

CONCLUSION

Finally, we must admit that we do not know whether or not the sequence of 20 consecutive Niven numbers given here, with $m = 0$, is the smallest such sequence. That is, whether or not the integer $a_{3423103}b$ is the smallest integer that a sequence of 20 consecutive Niven numbers can commence. During the construction of this integer, many alternate possibilities presented themselves, and as mentioned, much intuition and luck were involved. We would, therefore, like to challenge the reader to find the least integer that is the first term in a sequence of 20 consecutive Niven numbers.

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MIXED FERMAT CONVOLUTIONS

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1. INTRODUCTION

The k^{th} convolution sequences for Fermat polynomials of the first kind $(a_{n,m}^{(k)}(x))$ and the second kind $(b_{n,m}^{(k)}(x))$ are defined in this paper. Generating functions, recurrence relations, and explicit representations are given for these polynomials. A differential equation that corresponds to polynomials of type $(a_{n,m}^{(k)}(x))$ is presented. Finally, k^{th} convolutions of mixed Fermat polyno-mials of $(c_{n,m}^{(s,r)}(x))$ are defined. In some special cases, polynomials of $(c_{n,m}^{(s,r)}(x))$ are transformed into already known polynomials of $(a_{n,m}^{(k)}(x))$ and of $(b_{n,m}^{(k)}(x))$.

2. POLYNOMIALS $a_{n,m}^{(k)}(x)$

A. F. Horadam [2] defined Fermat polynomials of the first kind $A_n(x)$ and the second kind $B_n(x)$ by

(2.1) $$A_n(x) = xA_{n-1}(x) - 2A_{n-2}(x),\ A_{-1}(x) = 0,\ A_0(x) = 1,$$

and

(2.2) $$B_n(x) = xB_{n-1}(x) - 2b_{n-2}(x),\ B_0(x) = 2,\ B_1(x) = x.$$

Their generating functions are

(2.3) $$(1 - xt + 2t^2)^{-1} = \sum_{n=0}^{\infty} A_n(x)t^n$$

and

(2.4) $$\frac{1-2t^2}{1-xt+2t^2} = \sum_{n=0}^{\infty} B_n(x)t^n.$$

From (2.1) and (2.2), we can find a few members of the sequence of polynomials $A_n(x)$ and $B_n(x)$:

$$A_1(x) = x,\ A_2(x) = x^2 - 2,\ A_3(x) = x^3 - 4x,\ A_4(x) = x^4 - 6x^2 + 4,$$

and

$$B_2(x) = x^2 - 4,\ B_3(x) = x^3 - 6x,\ B_4(x) = x^4 - 8x^2 + 8.$$

H. W. Gould [1] studied a class of generalized Humbert polynomials $P_n(m, x, y, p, C)$ defined by

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)t^n,$$

where $m \geq 1$ is integer and the other parameters are unrestricted in general. The recurrence relation for the generalized Humbert polynomials is

$$CnP_n - m(n-1-p)xP_{n-1} + (n-m-mp)yP_{n-m} = 0, \ n \geq m \geq 1,$$

where we put $P_n = P_n(m, x, y, p, C)$.

In this paper we consider the polynomials $\left(a_{n,m}^{(k)}(x)\right)$ defined by

$$a_{n,m}^{(k)}(x) = P_n(m, x/m, 2, -(k+1), 1).$$

Their generating function is given by

$$F(x,t) = (1 - xt + 2t^m)^{-(k+1)} = \sum_{n=0}^{\infty} a_{n,m}^{(k)}(x)t^n. \tag{2.5}$$

Comparing (2.3) to (2.5), we can conclude that

$$a_{n,2}^{(0)}(x) = A_n(x) \quad \text{[Fermat polynomials (2.1)]}.$$

Development of the function (2.5) gives

$$\sum_{n=0}^{\infty} a_{n,m}^{(k)}(x)t^n = \sum_{n=0}^{\infty} \frac{(k+1)_n}{n!} t^n (x - 2t^{m-1})^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\left[\frac{n}{m}\right]} (-2)^i \frac{(k+1)_{n-(m-1)i}}{i!(n-mi)!} x^{n-mi} \right) t^n.$$

Comparison of coefficients of t^n in the last equation shows that polynomials $\left(a_{n,m}^{(k)}(x)\right)$ possess explicit representation as follows:

$$a_{n,m}^{(k)}(x) = \sum_{i=0}^{\left[\frac{n}{m}\right]} (-2)^i \frac{(k+1)_{n-(m-1)i}}{i!(n-mi)!} x^{n-mi}. \tag{2.6}$$

If we differentiate the function $F(x,t)$ (2.5) with respect to t, and compare coefficients of t^n, we get the three-term recurrence relation

$$na_{n,m}^{(k)}(x) = x(n+k)a_{n-1,m}^{(k)}(x) - 2(n+mk)a_{n-m,m}^{(k)}(x), \ n \geq m.$$

The initial starting polynomials are

$$a_{0,m}^{(k)}(x) = 0, \quad a_{n,m}^{(k)}(x) = \frac{(k+1)_n}{n!} x^n, \quad n = 1, 2, \ldots, m-1.$$

Then, if we differentiate the polynomials $a_{n,m}^{(k)}(x)$ (2.6) s times, term by term, we get the equality [1]:

$$D^s a_{n,m}^{(k)}(x) = (k+1)_s a_{n-s,m}^{(k+s)}(x), \ n \geq s.$$

Let the sequence $(f_r)_{n=0}^{n}$ be given by $f_r = f(r)$, where

$$f(t) = (n-t)\left(\frac{n-t+m(k+1+t)}{m}\right)_{m-1}.$$

Let Δ be the standard difference operator defined by $\Delta f_r = f_{r+1} - f_r$, and its power by

$$\Delta^0 f_r = f_r, \quad \Delta^k f_r = \Delta\left(\Delta^{k-1} f_r\right).$$

We find that the next property of $a_{n,m}^{(k)}(x)$ is very interesting.

The polynomial $a_{n,m}^{(k)}(x)$ is a particular solution of the linear homogeneous differential equation of the m^{th} order [4],

$$y^{(m)} + \sum_{s=0}^{m} a_s x^s y^{(s)} = 0, \tag{2.7}$$

with coefficients a_s $(s = 0, 1, \ldots, m)$ given by

$$a_s = \frac{1}{2ms!}\Delta^s f_0. \tag{2.8}$$

From (2.8), we get

$$a_0 = \frac{1}{2m} n \binom{\frac{n+m(k+1)}{m}}{m-1}$$

$$a_1 = \frac{1}{2m}\left((n-1)\binom{\frac{n-1+m(k+2)}{m}}{m-1} - n\binom{\frac{n+m(k+1)}{m}}{m-1}\right).$$

Since

$$f(t) = -\left(\frac{m-1}{m}\right)^{m-1} t^m \text{ + term of lower degree,}$$

we see that

$$a_m = -\frac{1}{2m}\left(\frac{m-1}{m}\right)^{m-1}.$$

For $m = 2$, the differential equation (2.7) is

$$\left(1 - \frac{1}{8}x^2\right)y'' - \frac{2k = 3}{8}xy' + \frac{n}{8}(n + 2k + 2)y = 0,$$

and it corresponds to the polynomials $a_{n,2}^{(k)}(x)$.

For $m = 2$ and $k = 0$, the equation (2.7) is

$$\left(1 - \frac{1}{8}x^2\right)y'' - \frac{3}{8}xy' + \frac{n}{8}(n + 2)y = 0,$$

and it corresponds to Fermat polynomials of the first kind $A_n(x)$.

3. POLYNOMIALS $b_{n,m}^{(k)}(x)$

In this section we introduce a class of polynomials $\left(b_{n,m}^{(k)}(x)\right)$, $k \in N$.

Definition 3.1: The polynomials $b_{n,m}^{(k)}(x)$ are defined by

$$F(x, t) = \left(\frac{1-2t^m}{1-xt+2t^m} \right)^{k+1} = \sum_{n=0}^{\infty} b_{n,m}^{(k)}(x) t^n. \tag{3.1}$$

Comparing (2.4) to (3.1), we can see that

$$b_{n,2}^{(0)}(x) = B_n(x) \quad \text{[Fermat polynomials (2.2)]}.$$

Expanding the left-hand side of (3.1), we obtain the explicit formula

$$b_{n,m}^{(k)}(x) = \sum_{i=0}^{k+1} (-2)^i \binom{k+1}{i} a_{n-mi,m}^{(k)}(x). \tag{3.2}$$

For $m = 2$ and $k = 0$, the formula (3.2) is

$$b_{n,2}^{(0)}(x) = a_{n,2}^{(0)}(x) - 2a_{n-2,2}^{(0)}(x).$$

That is,

$$B_n(x) = A_n(x) - 2A_{n-2}(x).$$

and it corresponds to the known relation between the Fermat polynomials $A_n(x)$ and $B_n(x)$.

4. MIXED FERMAT CONVOLUTIONS

A. F. Horadam and J. M. Mahon [3] studied a class of polynomials $(\pi_n^{(a,b)}(x))$, mixed Pell polynomials. Similarly, we define and then carefully study polynomials $(c_{n,m}^{(s,r)}(x))$, mixed Fermat convolutions, where all parameters are natural numbers.

Definition 4.1: The polynomials $(c_{n,m}^{(s,r)}(x))$ are given by

$$F(x, t) = \frac{(1-2t^m)^r}{(1-xt+2t^m)^{r+s}} = \sum_{n=0}^{\infty} c_{n,m}^{(s,r)}(x) t^n, \tag{4.1}$$

on condition that $s + r \geq 1$.

The polynomials $(c_{n,m}^{(s,r)}(x))$ have some interesting characteristics, some of which are described in the results that follow.

Theorem 4.1: The polynomials $(c_{n,m}^{(s,r)}(x))$ have the representation

$$c_{n,m}^{(s,r)}(x) = \sum_{i=0}^{r-j} (-2)^i \binom{r-j}{i} c_{n-mi,m}^{(r+s-j,j)}(x). \tag{4.2}$$

Proof: By using (4.1), we obtain

$$\sum_{n=0}^{\infty} c_{n,m}^{(s,r)}(x) t^n = (1-2t^m)^{r-j} \cdot \frac{1}{(1-xt+2t^m)^{r+s-j}} \cdot \left(\frac{1-2t^m}{1-xt+2t^m} \right)^j$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{r-j} (-2)^i \binom{r-j}{i} c_{n-mi,m}^{(r+s-j,j)}(x) t^n.$$

If we compare coefficients of t^n in the last equality, we have (4.2). Using (4.1) again, we obtain the following representation:

$$c_{n,m}^{(s,r)}(x) = \sum_{k=0}^{\infty} a_{n-k,m}^{(s-1)}(x) b_{k,m}^{(r-1)}(x).$$

Also, we see that

$$F(x,t) = \frac{(1-2t^m)^r}{(1-xt+2t^m)^{r+s}} = (1-2t^m)^r \sum_{n=0}^{\infty} a_{n,m}^{(r+s-1)}(x) t^m = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{r} (-2)^i \binom{r}{i} a_{n-mi,m}^{(r+s-1)}(x) \right) t^n.$$

From the last equality, we can conclude that

$$c_{n,m}^{(s,r)}(x) = \sum_{i=0}^{r} (-2)^i \binom{r}{i} a_{n-mi,m}^{(r+s-1)}(x).$$

The Fermat polynomials of the first and of the second kind satisfy a three-term recurrence relation. But, mixed Fermat convolutions satisfy a four-term recurrence relation of unstandard form, which we prove in the following result.

Theorem 4.2: The polynomials $c_{n,m}^{(s,r)}(x)$ satisfy the recurrence relation

$$nc_{n,m}^{(s,r)}(x) = -2mrc_{n-m,m}^{(s+1,r-1)}(x) + x(r+s)c_{n-1,m}^{(s+1,r)}(x) - 2m(r+s)c_{n-m,m}^{(s+1,r)}(x), \; n \ge m. \tag{4.3}$$

Proof: If we differentiate $F(x,t)$, (4.1), with respect to t, we get

$$\sum_{n=1}^{\infty} nc_{n,m}^{(s,r)}(x) t^{n-1} = -2mrt^{m-1} \sum_{n=0}^{\infty} c_{n,m}^{(s+1,r-1)}(x) t^n + (r+s)(x - 2mt^{m-1}) \sum_{n=0}^{\infty} c_{n,m}^{(s+1,r)}(x) t^n.$$

Comparing coefficients of t^n in the last equality, we have (4.3).

If we differentiate $F(x,t)$, (4.1), with respect to x, k times, term by term, we find that the polynomials $c_{n,m}^{(s,r)}(x)$ satisfy the equality

$$D^k c_{n,m}^{(s,r)}(x) = (r+s)_k c_{n-k,m}^{(s+k,r)}(x) \quad (n \ge k). \tag{4.4}$$

Special Cases

Starting with the equality

$$\frac{(1-2t^m)^{r+s}}{(1-xt+2t^m)^{2r+2s}} = \frac{(1-2t^m)^r}{(1-xt+2t^m)^{r+s}} \cdot \frac{(1-2t^m)^s}{(1-xt+2t^m)^{s+r}},$$

we get

$$\sum_{n=0}^{\infty} c_{n,m}^{(s+r,s+r)}(x) t^n = \left(\sum_{n=0}^{\infty} c_{n,m}^{(s,r)}(x) t^n \right) \left(\sum_{n=0}^{\infty} c_{n,m}^{(r,s)}(x) t^n \right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_{n-k,m}^{(s,r)}(x) c_{k,m}^{(r,s)}(x) \right) t^n.$$

From the last equality, we obtain

$$c_{n,m}^{(s+r,s+r)}(x) = \sum_{k=0}^{n} c_{n-k,m}^{(s,r)}(x) c_{k,m}^{(r,s)}(x). \tag{4.5}$$

For $r = s$, the equality (4.5) is

$$c_{n,m}^{(2s,2s)}(x) = \sum_{k=0}^{n} c_{n-k,m}^{(s,s)}(x) c_{k,m}^{(s,s)}(x).$$

From the equalities (2.5), (3.1), and (4.1), we obtain:

$$c_{n,m}^{(s,0)}(x) = a_{n,m}^{(s-1)}(x), \text{ for } r = 0 \tag{4.6}$$

and

$$c_{n,m}^{(0,r)}(x) = b_{n,m}^{(r-1)}(x), \text{ for } s = 0. \tag{4.7}$$

According to (4.4), (4.6), and (4.7), we get the inequalities

$$D^k a_{n,m}^{(s-1)}(x) = (s)_k a_{n-k,m}^{(s+k-1)}(x), \text{ for } r = 0$$

and

$$D^k b_{n,m}^{(r-1)}(x) = (r)_k c_{n-k,m}^{(k,r)}(x), \text{ for } s = 0.$$

For $r = 0$, the equality (4.5) becomes

$$c_{n,m}^{(s,s)}(x) = \sum_{k=0}^{n} a_{n-k,m}^{(s-1)}(x) b_{k,m}^{(s-1)}(x).$$

According to (4.3) and (4.5), we have

$$n\sum_{k=0}^{\infty} c_{n-k,m}^{(s,0)}(x) c_{k,m}^{(0,s)}(x) = -2msc_{n-m,m}^{(s+1,s-1)}(x) + 2xsc_{n-1,m}^{(s+1,s)}(x) - 4msc_{n-m,m}^{(s+1,s)}(x), \quad n \geq m.$$

From the equalities (4.2) and (4.7), for $j = s = 0$, $r = k+1$, it follows that

$$b_{n,m}^{(k)}(x) = \sum_{i=0}^{k+1} (-2)^i \binom{k+1}{i} a_{n-mi,m}^{(k)}(x).$$

Finally, from the equalities (4.2) and (4.6), for $j = r = 0$, $s = k+1$, we see that

$$a_{n,m}^{(k)}(x) = \sum_{i=0}^{k+1} (-2)^i \binom{k+1}{i} a_{n-mi,m}^{(k)}(x).$$

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HYPERSPACES AND FIBONACCI NUMBERS

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Let $(\mathcal{F}([0,1]), \tau_V)$ be the space of all closed non-empty subsets of the closed interval $[0,1]$ equipped with the Vietoris topology. The basic open subsets of τ_V are given by

$$\mathcal{F}_{G_1,\ldots,G_n} = \{A \in [0,1];\ A \neq \varnothing \text{ and closed such that } A \subseteq \cup_{i=1}^n G_i,\ G_i \cap A \neq \varnothing \forall i\}$$

for any collection $G_1, \ldots, G_n$ of nonempty open subsets of $[0,1]$.

Now let $G_1, \ldots, G_m$ $(m \in N)$ be open intervals of $0, 1]$ satisfying

(1) $$[0,1] = \cup_{i=1}^m G_i$$

(2) $$G_i \cap G_{i+1} \neq \varnothing \forall i = 1, \ldots, m-1$$

(3) $$G_i \cap G_j = \varnothing \text{ for } |i-j| \geq 2.$$

The main purpose of this paper is to calculate the number

(4) $$n(G_1, \ldots, G_m) = \max_A \left|\left\{\mathcal{F}_{(G_i)_{i\in I}};\ A \in \mathcal{F}_{(G_i)_{i\in I}},\ I \subseteq \{1, \ldots, m\}\right\}\right|$$

where A ranges over $\mathcal{F}([0,1])$ and where each $\mathcal{F}_{(G_i)_{i\in I}}$ is a basic open set of τ_V. Obviously, $n(G_1) = 1$; we shall investigate the case where $m \geq 2$. This problem was raised by the attempt to find a Hausdorff function h with zero local measure [2, 3], but non-σ-finite Hausdorff measure [1] for $\mathcal{F}([0,1])$. The calculation uses Fibonacci numbers [4].

Lemma 1: There exists $A_0 \in \mathcal{F}([0,1])$ which gives a maximum for (4) such that

(5) $$A_0 \subseteq \cup_{i\neq j}(G_i \cap G_j).$$

Proof: Without loss of generality, we may assume

$$|A_0 \cap (G_i \cap G_{i+1})| \leq 1 \text{ for } i = 1, \ldots, m-1$$

and

$$|A_0 \cap G_i \setminus (G_{i-1} \cup G_{i+1})| \leq 1 \text{ for } i = 2, \ldots, m-1$$

as well as

$$|A_0 \cap (G_1 \setminus G_2)| \leq 1 \text{ and } |A_0 \cap (G_m \setminus G_{m-1})| \leq 1.$$

Now let $i_0 \in \{2, \ldots, m-1\}$ (the cases $i_0 = 1$ and $i_0 = m$ can be handled in the same way) and suppose that

$$\{x\} = A_0 \cap G_{i_0} \setminus (G_{i_0-1} \cup G_{i_0+1})$$

for a set A_0 giving the maximum for (4). If $A_0 \in \mathcal{F}_{(G_i)_{i\in I}}$, then $i_0 \in I$. Hence, $A_0 \setminus \{x\}$ would also give the maximum. Deleting all such points, we obtain the desired set. □

Now let $x_i \in G_i \cap G_{i+1}$ for $i = 1, \ldots, m-1$ and $E = \{x_i; i = 1, \ldots, m-1\}$.

Lemma 2: For $m \geq 2$,

$$\left|\left\{\mathscr{F}_{(G_i)_{i\in I}};\ E \in \mathscr{F}_{(G_i)_{i\in I}},\ I \subseteq \{1, \dots, m\}\right\}\right| = u_{m+2}$$

where u_{m+2} is the $(\mathrm{m}+2)^{\text{th}}$ Fibonacci number, i.e., $u_1 = u_2 = 1$ and $u_{m+1} = u_m + u_{m-1}$ for $m \geq 2$ (see [4]).

Proof: Clearly $E \in \mathscr{F}_{(G_i)_{i\in I}}$ for $I \subseteq \{1, \dots, m\}$ if and only if

(6) $$i \notin I,\ i \notin \{1, \dots, m-1\} \text{ implies } i+1 \in I.$$

Let us now consider the hypergraphs

$$\mathscr{A}_m = \{I \subseteq \{1, \dots,\};\ I \text{ satisfies } (6)\}$$

and

$$\mathscr{B}_m = \{I \subseteq \{1, \dots, m\};\ m \in I \text{ and } I \text{ satisfies } (6)\}.$$

We see that

(7) $$\mathscr{A}_{m+1} = \{I \cup \{m+1\};\ I \in \mathscr{A}_m\} \cup \mathscr{B}_m.$$

Since $\{I \cup \{m+1\};\ I \in \mathscr{A}_m\} \cap \mathscr{B}_m = \varnothing$, it follows that

(8) $$|\mathscr{A}_{m+1}| = |\mathscr{A}_m| + |\mathscr{B}_m|$$

using the fact that $|\{I \cup \{m+1\};\ I \in \mathscr{A}_m\}| = |\mathscr{A}_m|$. We partition $\mathscr{B}_{m+1}$ as follows:

(9) $$\mathscr{B}_{m+1} = \{I \in \mathscr{B}_{m+1};\ m \in I\} \cup \{I \in \mathscr{B}_{m+1};\ m \notin I\}.$$

It is

(10) $$|\{I \in \mathscr{B}_{m+1};\ m \in I\}| = |\mathscr{B}_m|.$$

If $I \in \mathscr{B}_{m+1}$ and $m \notin I$, then $m-1 \in I$, implying that

(11) $$|\{I \in \mathscr{B}_{m+1};\ m \notin I\}| = |\mathscr{B}_{m+1}|.$$

Because of $\{I \in \mathscr{B}_{m+1};\ m \in I\} \cap \{I \in \mathscr{B}_{m+1};\ m \notin I\} = \varnothing$, we obtain

(12) $$|\mathscr{B}_{m+1}| = |\mathscr{B}_m| + |\mathscr{B}_{m+1}|.$$

We conclude, with $|\mathscr{B}_1| = 1$ and $|\mathscr{B}_2| = 2$, that

(13) $$|\mathscr{A}_{m+1}| = 1 + \sum_{k=1}^{m} |\mathscr{B}_k|$$

for $m \geq 1$ using (8). This gives, together with (12), that

(14) $$|\mathscr{A}_{m+1}| = |\mathscr{B}_{m+2}|.$$

Let $(u_m)_{m\in N}$ be the sequence of Fibonacci numbers with $u_1 = u_2 = 1$ and $u_{m+2} = u_{m+1} + u_m$, then it is easy to see that

(15) $$|\mathscr{A}_m| = u_{m+2}. \ \square$$

Lemma 3:

(16) $$u_{m+k+2} \le u_{m+2}u_{k+2} \text{ for } m \ge 2 \text{ and } k \ge 1.$$

In particular, $u_{k+4} < 3u_{k+2}$ for $k \ge 1$.

Proof: Using the well-known relation $u_{i+j} = u_{i-1}u_j + u_i u_{j+1}$ [4] with $i = m+1,\ j = k+1$ we obtain from $u_{k+1} < u_{k+2}$ that

$$u_m u_{k+1} < u_m u_{k+2}$$
$$u_m u_{k+1} < (u_{m+2} - u_{m+1})u_{k+2}$$
$$u_m u_{k+1} + u_{m+1}u_{k+2} < u_{m+2}u_{k+2}$$
$$u_{m+k+2} < u_{m+2}u_{k+2}. \quad \square$$

Theorem 1: For $m \ge 2$,

$$n(G_1, \ldots, G_m) = \begin{cases} 3^{\frac{m}{2}} & \text{if } m \text{ is even} \\ 5(3^{\frac{m-3}{2}}) & \text{if } m \text{ is odd.} \end{cases}$$

Proof: Let $A_0 \subseteq [0, 1]$ be a finite set with property (5) giving maximum. Then

(17) $$A_0 = \textstyle\bigcup_{j=1}^{\ell} A_{0,j} \quad (\ell < m)$$

with $A_{0,j} = \{x_{i_j}, x_{i_j+1}, \ldots, x_{i_j+k_j}\}$ such that

(18) $$i_j + k_j + 1 < i_{j+1} \text{ for } 1 \le j \le \mathrm{l} - 1.$$

Let $n_{A_0} = n(G_1, \ldots, G_m)$ and $n_{A_{0,j}}$ the number of $\mathcal{F}_{(G_i)_{i\in I}}$ which contain $A_{0,j}$. The condition (18) guarantees that a set $\mathcal{F}_{(G_i)_{i\in I}}$ containing $A_{0,j}$ cannot contain another $A_{0,j'}$ with $j \ne j'$. Now, let

$$\left(\mathcal{F}_{(G_i^{(j)})_{i\in I_j}}\right)_{j=1}^{\ell}$$

be any collection which contains $A_{0,1}, \ldots, A_{0,\mathrm{l}}$; then $\mathcal{F}_{(G_i^{(j)})_{i\in I_j}},\ j \in \{1, \ldots, \ell\}$ contains A_0. Conversely, we can split the collection $(G_i)_{i\in I}$ if $\mathcal{F}_{(G_i)_{i\in I}}$ contains A_0 into l subcollections giving families of sets $\mathcal{F}_{(G_i^{(j)})_{i\in I_j}}$ for $j \in \{1, \ldots, \ell\}$ which contain $A_{0,j}$ by using (18) again. Thus, we conclude that

$$n_{A_0} = \prod_{j=1}^{\ell} n_{A_{0,j}} \quad (\text{where } n_{A_{0,j}} = u_{k_j+4}).$$

We claim now that $|A_{0,j}| =$ with the possible exception of one index j for which $|A_{0,j}| = 2$. We claim, moreover, that there is a gap of one point x_i between $A_{0,j}$ and $A_{0,j+1}$ for $j = 1, \ldots, \ell - 1$. First it should be clear that for a set A_0 giving maximum for $n(G_1, \ldots, G_m)$ the gap between its components $A_{0,j}$ and $A_{0,j+1}$ is at most one point. If there is a component $A_{0,j}$ with $|A_{0,j}| \ge 3$ then we could delete, say x_{i_j+1} and the set $A_0 \backslash \{x_{i_j+1}\}$ is contained in

$$\frac{n(G_1, \ldots, G_m)}{u_{k_j+4}}(3u_{k_j+2})$$

sets $\mathscr{F}_{(G_i)_{i\in I}}$ which contradicts the assumption that A_0 already gives the maximum by using Lemma 3. Hence $|A_{0,j}| = 1$ or 2 for all j. Assuming that $|A_{0,j}| = |A_{0,j+1}| = 2$, then

$$(A_0 \setminus \{x_{i_j+1}, x_{i_{j+1}}\}) \cup \{x_{i_j+2}\}$$

is contained in more sets $\mathscr{F}_{(G_i)_{i\in I}}$ than A_0 since $A_{0,j}$ and $A_{0,j+1}$ give the factor 5 for the resulting product $n(G_1, \ldots, G_m)$. but three points give the factor 3 and 25 < 27. Furthermore, one can change the order of the $A'_{0,j}$'s to a consecutive one for the sets $A_{0,j}$ with $|A_{0,j}| = 2$. Thus, there exists at most one $A_{0,j}$ with $|A_{0,j}| = 2$. If m is even, then $|A_0| = \frac{m}{2}$ with $\frac{m-2}{2}$ gap points, i.e., $|A_{0,j}| = 1$ for all components of A_0, but the last component contains two points. This gives the announced result for $n(G_1, \ldots, G_m)$. □

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Editor on Leave of Absence

The Editor has been asked to visit Yunnan Normal University in Kunming, China for the Fall semester of 1993. This is an opportunity which the Editor and his wife feel cannot be turned down. They will be in China from August 1, 1993 until around January 10, 1994. The August and November issues of *The Fibonacci Quarterly* will be delivered to the printer early enough so that these two issues can still be published while the Editor is gone. The Editor has also arranged for several individuals to send out articles to be refereed which have been submitted for publication in *The Fibonacci Quarterly* or submitted for presentation at the ***Sixth International Conference on Fibonacci Numbers and Their Applications.*** Things may be a little slower than normal but every attempt will be made to insure that things go as smoothly as possible while the Editor is on leave in China. PLEASE CONTINUE TO USE THE NORMAL ADDRESS FOR SUBMISSION OF PAPERS AND ALL OTHER CORRESPONDENCE.

A DISJOINT SYSTEM OF LINEAR RECURRING SEQUENCES GENERATED BY $u_{n+2} = u_{n+1} + u_n$ WHICH CONTAINS EVERY NATURAL NUMBER

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(Submitted July 1991)

Burke and Bergum [1] called a (finite or infinite) family of n^{th}-order linear recurring sequences a (finite or infinite) *regular covering* if every natural number is contained in at least one of these sequences. If every natural number is contained in exactly one of these sequences, they called the family a (finite or infinite) *disjoint covering.* They gave examples of finite and infinite disjoint coverings generated by linear recurrences of every order n. In the case of the Fibonacci recurrence $u_{n+2} = u_{n+1} + u_n$, they constructed a regular covering which is not disjoint and asked whether a disjoint covering in this case exists as well. The following theorem answers this question.

Theorem: There is an infinite disjoint covering generated by the linear recurrence $u_{n+2} = u_{n+1} + u_n$.

We first state some easy properties of the Fibonacci numbers, $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n = 1, 2, \ldots$. Let $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$. We have

$$\alpha < 1, \quad -1 < \beta < 0 \tag{1}$$

and

$$\alpha|\beta| = 1. \tag{2}$$

For the Fibonacci numbers, the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad (n \in \mathbf{N}) \tag{3}$$

holds.

For all $i \in \mathbf{N}$, let $u_{i,1}, u_{i,2} \in \mathbf{N}$ and the sequences $(u_{i,n})_{n\in\mathbf{N}}$ be defined by

$$u_{i,n+2} = u_{i,n+1} + u_{i,n}. \tag{4}$$

Then we have

$$u_{i,n} = F_{n-1}u_{i,2} + F_{n-2}u_{i,1} \tag{5}$$

and

$$u_{i,n+1} = \alpha u_{i,n} + \beta^{n-2}(\beta u_{i,2} + u_{i,1}) \tag{6}$$

for all $i, n \in \mathbf{N}, n \geq 2$.

Proof of the Theorem: We will construct sequences $(u_{i,n})_{n\in\mathbf{N}}$ of natural numbers for all $i \in \mathbf{N}$ generated by (4).

Start with $(u_{i,n})_{n\in\mathbf{N}} = (F_{n+1})_{n\in\mathbf{N}}$ and assume that $(u_{i,n})_{n\in\mathbf{N}}$ has been constructed for $i = 1, 2, \ldots, k-1$ for some $k \in \mathbf{N}$, $k \geq 2$, and that $u_{i,n} = u_{j,m}$ if and only if $m = n$ and $i = j$ $(i < k, j < k)$.

Now we construct $(u_{k,n})_{n\in\mathbf{N}}$ with the same property. Let $V_i = \{u_{j,n} | n \in \mathbf{N}, j = 1, 2, \ldots, i\}$.

By (1), (3), and (4), we have $\mathbf{N} \setminus V_{k-1} \neq \varnothing$. Thus, we can choose

$$(7) \qquad u_{k,1} = \min(\mathbf{N} \setminus V_{k-1}).$$

We will show that there are $u_{k,2} \in \mathbf{N}$ with

$$(8) \qquad u_{k,2} > u_{k,1}$$

and

$$(9) \qquad u_{k,2} > \max\{u_{i,2} | i = 1, 2, \ldots, k-1\},$$

such that the sequence $(u_{k,n})_{n\in\mathbf{N}}$ generated by (4) has the following property P:

(P) If $i < k$, then $u_{k,n} \neq u_{i,m}$ for all $n, m \in \mathbf{N}$.

Let $M_k = \max\{u_{k,1}, u_{1,2}, u_{2,2}, \ldots, u_{k-1,2}\}$. Then $u_{k,2} > M_k$ is equivalent to (8) and (9).

Let $S_k \in \mathbf{R}$ be sufficiently large. More precisely

$$(10) \qquad S_k \geq 4\alpha^{-1} u_{k,1} \ (>1)$$

and

$$(11) \qquad S_k > 5(k-1)\big((\log 4S_k)/\log\alpha\big)^2 + M_k$$

$$(\text{e.g.: } S_k = \big((5(k-1)/\log^2\alpha)^2 + 1)M_k\big).$$

To prove the existence of $u_{k,2} \in (M_k, S_k] \cap \mathbf{N}$ such that $(u_{k,n})_{n\in\mathbf{N}}$ has property (P), we first count the number of those integers $u_{k,2} \in (M_k, S_k] \cap \mathbf{N}$ such that $(u_{k,n})_{n\in\mathbf{N}}$ does not have property (P). For these $u_{k,2}$, there are $m, n \in \mathbf{N}$ and $i \in \{1, 2, \ldots, k-1\}$ with

$$(12) \qquad u_{k,n} = u_{i,m}.$$

From (7), (8), and (9), we get $n \geq 2$ and $m \geq 3$. By (5) we can write (12) as follows:

$$F_{n-1}u_{k,2} + F_{n-2}u_{k,1} = F_{m-1}u_{i,2} + F_{m-2}u_{i,1}.$$

We obtain

$$(13) \qquad n < m$$

and by (3) also

$$\frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{5}}u_{k,2} + \frac{\alpha^{n-2}-\beta^{n-2}}{\sqrt{5}}u_{k,1} = \frac{\alpha^{m-1}-\beta^{m-1}}{\sqrt{5}}u_{i,2} + \frac{\alpha^{m-2}-\beta^{m-2}}{\sqrt{5}}u_{i,1}.$$

Since $u_{k,2} \leq S_k$ and $|\beta| < \alpha/2$, we get

$$S_k \geq u_{k,2} \geq \frac{\alpha^{m-1}-|\beta|^{m-1}}{\alpha^{n-1}+|\beta|^{n-1}}u_{i,2} + \frac{\alpha^{m-2}-|\beta|^{m-2}}{\alpha^{n-1}+|\beta|^{n-1}}u_{i,1} - \frac{\alpha^{n-2}+|\beta|^{n-2}}{\alpha^{n-1}-|\beta|^{n-1}}u_{k,1}$$

$$\geq \frac{\frac{1}{2}\alpha^{m-1}}{2\alpha^{n-1}}u_{i,2} + \frac{\frac{1}{2}\alpha^{m-2}}{2\alpha^{n-1}}u_{i,1} - 4\alpha^{-1}u_{k,1}.$$

Observing (10), this implies

$$8S_k + 4(S_k + 4\alpha^{-1}u_{k,1}) \ge \alpha^{m-n-1}(\alpha u_{i,2} + u_{i,1}) > 4g\alpha^{m-n-1}$$

$$2S_k > \alpha^{m-n-1}$$

$$\frac{\log 2S_k}{\log\alpha} > m-n-1. \tag{14}$$

We have $u_{k,n+1} \ne u_{i,m+1}$. Otherwise we would get from (12) and (4) that $u_{k,\ell} = u_{i,m-n+\ell}$ for all $\ell \in \mathbf{N}$. In particular, $u_{k,1} = u_{i,m-n+1}$ would contradict (7).

Using this and (6), (12), (1), (13), (8), (9), (2), and $u_{k,2} \le S_k$, we get

$$\begin{aligned}
1 &\le |u_{k,n+1} - u_{i,m+1}| \\
&= |\alpha u_{k,n} + \beta^{n-2}(\beta u_{k,2} + u_{k,1}) - \alpha u_{i,m} - \beta^{m-2}(\beta u_{i,2} + u_{i,1})| \\
&\le |\beta|^{n-2}\,|\beta u_{k,2} + u_{k,1}| + |\beta|^{m-2}\,|\beta u_{i,2} + u_{i,1}| \\
&\le |\beta|^{n-2}\,(|\beta u_{k,2}| + |u_{k,1}| + |\beta u_{i,2}| + |u_{i,1}|) \\
&\le |\beta|^{n-2} 4u_{k,2} \le \alpha^{-(n-2)} 4S_k
\end{aligned}$$

$$\alpha^{n-2} \le 4S_k$$

$$n \le \frac{\log 4S_k}{\log\alpha} + 2.$$

Combining this with (14), we obtain

$$m < \frac{\log 2S_k}{\log\alpha} + n + 1 \le \frac{\log 2S_k}{\log\alpha} + \frac{\log 4S_k}{\log\alpha} + 3 \le 3\frac{\log 4S_k}{\log\alpha}. \tag{15}$$

Now we will give an upper bound for the number of triples (n, m, i) such that $u_{k,n} = u_{i,m}$, $1 \le i \le k-1$. In this case (15) holds. First, fix *i* and *m*.

Since $2 \le n < m$, there are at most $m-2$ possible values for *n*. Since

$$3 \le m < (3\log 4S_k)/\log\alpha, \qquad \text{for fixed } i,$$

there are at most

$$\frac{1}{2}\left(\frac{3\log 4S_k}{\log\alpha} - 1\right)\left(\frac{3\log 4S_k}{\log\alpha} - 2\right) \le 5\left(\frac{\log 4S_k}{\log\alpha}\right)^2$$

possible pairs (m, n).

Finally, since $1 \le i \le k-1$, there are at most

$$5(k-1)\left(\frac{\log 4S_k}{\log\alpha}\right)^2$$

possible triples (n, m, i). To each triple such that $u_{k,n} = u_{i,m}$, $1 \le i \le k-1$ belongs exactly one $u_{k,2} \in (M_k, S_k] \cap \mathbf{N}$, because for two different values of $u_{k,2}$ and the fixed value of $u_{k,1}$, the recurrence (4) would give two different values of $u_{k,n}$, both of which cannot be equal to $u_{i,n}$. Consequently, there are at most

$$5(k-1)\left(\frac{\log 4S_k}{\log\alpha}\right)^2$$

values of $u_{k,2} \in (M_k, S_k] \cap \mathbb{N}$ such that $u_{k,n} = u_{i,m}$ for some $n, m, 1 \le i \le k-1$. Therefore, the number of values $u_{k,2} \in (M_k, S_k] \cap \mathbb{N}$ such that $u_{k,n} \ne u_{i,m}$ for all $n, m, 1 \le i \le k-1$ is at least

$$S_k - M_k - 5(k-1)\left(\frac{\log 4S_k}{\log\alpha}\right)^2,$$

which is positive by (11), and hence the choice of such an $u_{k,2}$ is possible.

This induction on k shows that there are infinitely many sequences $(u_{k,n})_{n\in\mathbb{N}}$. Every natural number occurs in one of these sequences by (7). It occurs exactly once by property (P) which holds for these sequences.

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GENERALIZED PASCAL TRIANGLES AND PYRAMIDS
THEIR FRACTALS, GRAPHS, AND APPLICATIONS

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This monograph was first published in Russia in 1990 and consists of seven chapters, a list of 406 references, an appendix with another 126 references, many illustrations and specific examples. Fundamental results in the book are formulated as theorems and algorithms or as equations and formulas. For more details on the contents of the book see *The Fibonacci Quarterly*, Volume 31.1, page 52.
The translation of the book is being reproduced and sold with the permission of the author, the translator and the "FAN" Edition of the Academy of Science of the Republic of Uzbekistan. The book, which contains approximately 250 pages, is a paper back with a plastic spiral binding. The price of the book is $31.00 plus postage and handling where postage and handling will be $6.00 if mailed to anywhere in the United States or Canada, $9.00 by surface mail or $16.00 by airmail to anywhere else. A copy of the book can be purchased by sending a check made out to **THE FIBONACCI ASSOCIATION** for the appropriate amount along with a letter requesting a copy of the book to: **RICHARD VINE, SUBSCRIPTION MANAGER, THE FIBONACCI ASSOCIATION, SANTA CLARA UNIVERSITY, SANTA CLARA, CA 95053.**

ASSOCIATED SEQUENCES OF GENERAL ORDER

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1. INTRODUCTION

The purpose of this article is to extend an idea in [1] to the generalized sequence $\{W_n\}$ defined [2] for all integers n by the recurrence relation

$$W_{n+2} = pW_{n+1} + qW_n \tag{1.1}$$

in which

$$W_0 = a, \ W_1 = b \tag{1.2}$$

where a, b, p, q are arbitrary integers.

The explicit *Binet form* is

$$W_n = \frac{(b-a\beta)\alpha^n - (b-a\alpha)\beta^n}{\alpha - \beta} \tag{1.3}$$

where

$$\alpha = \frac{p+\sqrt{p^2+4q}}{2}, \quad \beta = \frac{p-\sqrt{p^2+4q}}{2} \tag{1.4}$$

are the roots of

$$x^2 - px - q = 0. \tag{1.5}$$

From (1.4), or (1.5), we deduce that

$$\alpha+\beta = p, \ \alpha\beta = -q, \ \alpha-\beta = \sqrt{p^2+4q} = \Delta \tag{1.6}$$

and

$$\begin{cases} \frac{p\alpha}{2} + q = \frac{\Delta\alpha}{2}, & \frac{p\beta}{2} + q = -\frac{\Delta\beta}{2} \\ \alpha^2 + q = \Delta\alpha, & \beta^2 + q = -\Delta\beta \\ \alpha^2 - q = p\alpha, & \beta^2 - q = p\beta. \end{cases} \tag{1.7}$$

Special cases of $\{W_n\}$ which interest us are

$$\begin{cases} \text{the Fibonacci sequence } \{F_n\}: & p=1, q=1, a=0, b=1 \\ \text{the Lucas sequence } \{L_n\}: & p=1, q=1, a=2, b=1 \\ \text{the Pell sequence } \{P_n\}: & p=2, q=1, a=0, b=1 \\ \text{the Pell-Lucas sequence } \{Q_n\}: & p=2, q=1, a=2, b=2. \end{cases} \tag{1.8}$$

2. THE ASSOCIATED SEQUENCE $\left\{W_n^{(k)}\right\}$

Define the *first associated sequence* $\{W_n^{(1)}\}$ of $\{W_n\}$ from (1.1) by

$$W_n^{(1)} = pW_{n+1} + qW_{n-1}. \tag{2.1}$$

Repeat the operation in (2.1) to obtain

$$W_n^{(2)} = pW_{n+1}^{(1)} + qW_{n-1}^{(1)} = \frac{(b-a\beta)(p\alpha^2+q)^2\alpha^{n-2} - (b-a\alpha)(p\beta^2+q)^2\beta^{n-2}}{\Delta} \tag{2.2}$$

on using (1.3) and (2.1).

Generally,

$$W_n^{(k)} = pW_{n+1}^{(k-1)} + qW_{n-1}^{(k-1)} \tag{2.3}$$

where $W_n^{(k)}$ $(k = 1, 2, 3, \ldots)$ represents the k^{th} repetition of the association process (2.1). We may call $\{W_n^{(k)}\}$ the *associated sequence of* $\{W_n\}$ *of order k.*

(2.3)′ **Convention:** $W_n^{(0)} = W_n$.

Induction, with appeal to (2.3), may be used to establish the Binet form

$$W_n^{(k)} = \frac{A\alpha^{n-k} - B\beta^{n-k}}{\Delta} \tag{2.4}$$

in which

$$\begin{cases} A = (b-a\beta)(p\alpha^2+q)^k \\ B = (b-a\alpha)(p\beta^2+q)^k \end{cases} \tag{2.5}$$

As we expect, $\{W_n^{(k)}\}$ is a recurrence sequence like $\{W_n\}$, for, by (2.4),

$$\begin{aligned} pW_{n+1}^{(k)} + qW_n^{(k)} &= \frac{1}{\Delta}\{p(A\alpha^{n+1-k} - B\beta^{n+1-k}) + q(A\alpha^{n-k} - B\beta^{n-k})\} \\ &= \frac{1}{\Delta}\{(p\alpha+q)A\alpha^{n-k} - (p\beta+q)B\beta^{n-k}\} \\ &= W_{n+2}^{(k)} \end{aligned} \tag{2.6}$$

on putting $x = \alpha, x = \beta$ in turn in (1.5). Thus, (1.1) is valid for $\{W_n^{(k)}\}$. Consequently, (1.5) is also true for $\{W_n^{(k)}\}$.

Next, we define $\{\mathcal{W}_n^{(k)}\}$, the *conjugate sequence of* $\{W_n^{(k)}\}$, by

$$\mathcal{W}_n^{(k)} = A\alpha^{n-k} + B\beta^{n-k}. \tag{2.7}$$

It readily follows, on using (1.7), that

$$\mathcal{W}_n^{(k)} = p\mathcal{W}_{n+1}^{(k-1)} + q\mathcal{W}_{n-1}^{(k-1)} \tag{2.8}$$

and on using (1.5), that

$$\mathcal{W}_{n+2}^{(k)} = p\mathcal{W}_{n+1}^{(k)} + q\mathcal{W}_n^{(k)}. \tag{2.9}$$

That is, both the association and recurrence properties which are features of $\{W_n^{(k)}\}$ apply equally well to $\{\mathcal{W}_n^{(k)}\}$.

3. PROPERTIES OF $\{W_n^{(k)}\}$

Some consequences of our definitions and ideas now follow. Proofs of these results, obtainable from the preceding information, are left for the pleasure of the reader [employing (1.7) and (2.4)].

Firstly, we have the *Simson formula*

Theorem 1:

$$W_{n+1}^{(k)}W_{n-1}^{(k)} - \left[W_n^{(k)}\right]^2 = -(-q)^{n-1-k}AB. \tag{3.1}$$

More generally,

$$W_{n+r}^{(k)}W_{n-r}^{(k)} - [W_n^{(k)}]^2 = \frac{-(-q)^{n-r-k}AB(\alpha^r - \beta^r)^2}{\Delta^2}. \tag{3.1$'$}$$

Theorem 2:

$$\sum_{i=0}^{n}\binom{n-k}{i}q^i\left(\frac{p}{2}\right)^{n-k-i}W_{i-1}^{(k)} = \begin{cases}\left(\frac{\Delta}{2}\right)^{n-k}W_n^{(k)} & (n-k \text{ even})\\ \frac{\Delta^{n-k-1}}{2^{n-k}}\mathcal{W}_n^{(k)} & (n-k \text{ odd}).\end{cases} \tag{3.2}$$

Theorem 3:

$$[W_{n+1}^{(k)}]^2 + q[W_n^{(k)}]^2 = bW_{2n+1}^{(2k)} + aqW_{2n}^{(2k)}. \tag{3.3}$$

Theorem 4:

$$[W_{n+1}^{(k)}]^2 - q[W_n^{(k)}]^2 = \frac{p}{\Delta^2}\left(b\mathcal{W}_{2n+1}^{(2k)} + aq\mathcal{W}_{2n}^{(2k)}\right) + (-1)^{n-k}q^{n+1-k}4\frac{AB}{\Delta^2}. \tag{3.4}$$

(Not a pretty sight!)

Theorem 5:

$$[W_{n+2}^{(k)}]^3 - p^3[W_{n+1}^{(k)}]^3 - q^3[W_n^{(k)}]^3 = 3pqW_{n+1}^{(k)}W_n^{(k)}W_{n-1}^{(k)}. \tag{3.5}$$

This neat cubic property is derivable directly from (2.6), or, with more effort, from (2.4).

Theorem 6:

$$[W_{n+2}^{(k)}]^2 - p^2[W_{n+1}^{(k)}]^2 - q^2[W_n^{(k)}]^2 = 2pqW_{n+1}^{(k)}W_n^{(k)}. \tag{3.6}$$

This quadratic property which is easily deducible from (2.6) may be employed to produce a somewhat unattractive expression for $2pq\sum_{i=0}^{n}W_i^{(k)}W_{i+1}^{(k)}$.

All the above results (linear and nonlinear) for $W_n^{(k)}$ may be echoed for $\mathcal{W}_n^{(k)}$. Just one illustration (namely, the corresponding Simson formula) should suffice. Remaining results could be paralleled by the interested reader.

Theorem 1a:

$$\mathcal{W}_{n+1}^{(k)}\mathcal{W}_{n-1}^{(k)} - [\mathcal{W}_n^{(k)}]^2 = (-q)^{n-1-k}AB\Delta^2. \tag{3.1a}$$

This theorem can be extended as in (3.1)$'$ for $W_n^{(k)}$.

Some hybrid results involving mathematical cross-fertilization of $W_n^{(k)}$ and $\mathcal{W}_n^{(k)}$ are worth mentioning. For example,

Theorem 7:

(3.7)
$$W_n^{(k)}\mathcal{W}_n^{(k)} = bW_{2n}^{(2k)} + aqW_{2n-1}^{(2k)}.$$

When $a = 0, b = 1, q = 1$, we see that (3.7) leads to $F_n^{(k)}L_n^{(k)} = F_{2n}^{(2k)}$ and $P_n^{(k)}Q_n^{(k)} = P_{2n}^{(2k)}$ [cf. (1.8)] which are outgrowths of well-known results occurring for $k = 0$. [See (4.18).] In particular, $P_3^{(2)}Q_3^{(2)} = 137 \times 386 = 52,882 = P_6^{(4)}$ [see (4.9) and (4.10)].

Another interesting result is

Theorem 8:

(3.8)
$$\mathcal{W}_{n+1}^{(k)} + q\mathcal{W}_{n-1}^{(k)} = \Delta^2 W_n^{(k)}.$$

Thus, when $k = 0, q = 1$ we have (1.8)

$$L_{n+1} + L_{n-1} = 5F_n, \;\; Q_{n+1} + Q_{n-1} = 8P_n.$$

Observe that

$$F_n^{(1)} = L_n, \; F_n^{(2)} = 5F_n = L_n^{(1)}, \; F_n^{(3)} = 5F_n^{(1)} = 5L_n, \; F_n^{(4)} = 5L_n^{(1)} = 5^2 F_n, \ldots.$$

Generally,

(3.9)
$$\begin{cases} F_n^{(2m)} = 5^m F_n \\ F_n^{(2m+1)} = 5^m L_n \\ L_n^{(2m)} = 5^m L_n \\ L_n^{(2m-1)} = 5^m F_n. \end{cases}$$

Consequently, the association process effectively stops after two operations on F_n or L_n. However, for the Pell and Pell-Lucas numbers, for which $p = 2$, the association process is never-ending. [For P_n' to equal Q_n, we would require $P_n' = P_{n+1} + P_{n-1}$, which is contrary to (4.1).]

4. ASSOCIATED PELL SEQUENCE

As our special application of the general theory, we consider the Pell sequence $\{P_n\}$ defined in (1.8) by the recurrence relation

(4.1)
$$P_{n+2} = 2P_{n+1} + P_n$$

in which

(4.2)
$$P_0 = 0, \;\; P_1 = 1$$

with Binet form

(4.3)
$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$$

where

$$\alpha = 1+\sqrt{2},\ \beta = 1-\sqrt{2} \tag{4.4}$$

are the roots of

$$x^2 - 2x - 1 = 0. \tag{4.5}$$

From (4.4)

$$\alpha+\beta = 2,\ \alpha\beta = -1,\ \alpha-\beta = 2\sqrt{2}. \tag{4.6}$$

The *associated Pell sequence of order k*, $\{P_n^{(k)}\}$, is given by

$$P_n^{(k)} = 2P_{n+1}^{(k-1)} + P_{n-1}^{(k-1)} \quad (k \ge 1) \tag{4.7}$$

for which the Binet form is

$$P_n^{(k)} = \frac{(7+4\sqrt{2})^k \alpha^{n-k} - (7-4\sqrt{2})^k \beta^{n-k}}{2\sqrt{2}}. \tag{4.8}$$

Some elements of the first few sequences $\{P_n^{(k)}\}$ are:

(4.9)

$k \backslash n$	-1	0	1	2	3	4	5	6	$\cdots$
0	1	0	1	2	5	12	29	70	$\cdots$
1	-2	3	4	11	26	63	152	367	$\cdots$
2	13	6	25	56	137	330	797	1924	$\cdots$
3	-8	63	118	299	716	1731	4178	10087	$\cdots$
4	205	228	661	1550	3761	9072	21905	52882	$\cdots$

The conjugate sequence $\{Q_n^{(k)}\}$ for which

$$Q_n^{(k)} = P_{n+1}^{(k)} + P_{n-1}^{(k)} \tag{4.10}$$

has the Binet form

$$Q_n^{(k)} = (7+4\sqrt{2})^k \alpha^{n-k} + (7-4\sqrt{2})^k \beta^{n-k}. \tag{4.11}$$

One has also

$$Q_n^{(k)} = 2Q_{n+1}^{(k-1)} + Q_{n-1}^{(k-1)}. \tag{4.12}$$

Interested readers might wish to list the first few members of some of the $\{Q_n^{(k)}\}$, as was done for $\{P_n^{(k)}\}$ in (4.9). For example

$$Q_3^{(2)} = 386 = 56+330 = P_2^{(2)} + P_4^{(2)}.$$

Paralleling results in section 3, we have, for instance

Theorem 1':

$$P_{n+1}^{(k)} P_{n-1}^{(k)} - [P_n^{(k)}]^2 = (-1)^{n+k} 17^k \tag{4.13}$$

since $AB = 17^k$ in the case of Pell numbers, by (1.4), (1.8), and (2.5). There is an obvious extension analogous to that in (3.1)$'$.

Theorem 2':

$$\sum_{i=0}^{n}\binom{n-k}{i}P_{n-i}^{(k)}=\begin{cases}(\sqrt{2})^{n-k}P_n^{(k)} & (n-k\text{ even})\\ \frac{(\sqrt{2})^{n-k}}{2\sqrt{2}}Q_n^{(k)} & (n-k\text{ odd}).\end{cases} \tag{4.14}$$

Theorem 3':

$$[P_{n+1}^{(k)}]^2+[P_n^k]^2=P_{2n+1}^{(2k)}. \tag{4.15}$$

Theorem 4':

$$[P_{n+1}^{(k)}]^2-[P_n^k]^2=\frac{Q_{2n+1}^{(2k)}+(-1)^{n-k}\cdot 2\cdot 17^k}{4}. \tag{4.16}$$

Other special cases for Pell numbers, as in Theorems 5 and 6 and (2.6), follow.

Results involving $Q_n^{(k)}$ include (say)

Theorem 1a':

$$Q_{n+1}^{(k)}Q_{n-1}^{(k)}-[Q_n^{(k)}]^2=(-1)^{n+1+k}\cdot 8\cdot 17. \tag{4.17}$$

Theorem 7':

$$P_n^{(k)}Q_n^{(k)}=P_{2n}^{(2k)} \tag{4.18}$$

[already noted after (3.7)].

Reverting momentarily to $\{W_n^{(k)}\}$, we can use previously applied techniques to demonstrate that

$$bW_{n+2}^{(k)}+aqW_{n+1}^{(k)}=W_{n+1}W_n^{(k)}+qW_nW_1^{(k)}. \tag{4.19}$$

From (1.8), it follows that

$$P_{n+2}^{(k)}=P_{n+1}P_2^{(k)}+P_nP_1^{(k)}, \tag{4.20}$$

which expresses the $(n+2)^{\text{th}}$ term of the associated sequence in terms of the $(n+1)^{\text{th}}$ and n^{th} Pell numbers. When $k=4, n=3$, for example,

$$P_5^{(4)}=P_4P_2^{(4)}+P_3P_1^{(4)}=12\times 1550+5\times 661=21,905.$$

If $k=0$, then (4.20) leads directly to (4.1). Thus, in a pleasing way, (4.20) appears as a mathematical offspring of the definition of Pell numbers.

Equation (4.19) in conjunction with (1.8) also yields a result for Fibonacci numbers similar to (4.20), namely,

$$F_{n+2}^{(k)}=F_{n+1}F_2^{(k)}+F_nF_1^{(k)}. \tag{4.21}$$

5. NEGATIVE VALUES OF k AND n

As $\{W_n\}$ was defined in (1.1) for all integers n, the results we have obtained for $\{W_n^{(k)}\}$ apply irrespective of whether n is positive or negative. Indeed, the tabulation in (4.9) gives a brief indication of this aspect which could be extended to other negative subscript values.

But what happens if k is negative?

The Binet form for $\{W_n^{(-k)}\}$ is readily written down from (2.4) by replacing k by $-k$, and the theory for negative subscript k follows as for the case of k positive. Unhappily, computation does not always produce pleasing formulas. Indeed, the calculation of, say, $W_{-n}^{(k)}[W_n^{(-k)}]^{-1}$ leads to some unlovely algebra.

However, things are easier if we consider a particular instance of the general sequence, say, the Pell sequence. Calculation using (4.8) leads to

$$P_{-n}^{(k)}[P_n^{(-k)}]^{-1} = (1)^{n+k+1} \cdot (17)^k. \tag{5.1}$$

Application of (5.1) with the assistance of (4.8) allows us to compute numerical values of $P_n^{(-k)}$ for particular values of n and k to our heart's content. A short tabulation of $P_n^{(k)}$ for $k < 0$, $= 0$, > 0 is [cf. (4.9)]:

(5.2)

$k \backslash n$	-2	-1	0	1	2	$\cdots$
-2	$\frac{-56}{17^2}$	$\frac{25}{17^2}$	$\frac{-6}{17^2}$	$\frac{13}{17^2}$	$\frac{20}{17^2}$	$\cdots$
-1	$\frac{11}{17}$	$\frac{-4}{17}$	$\frac{3}{17}$	$\frac{2}{17}$	$\frac{7}{17}$	$\cdots$
0	-2	1	0	1	2	$\cdots$
1	7	-2	3	4	11	$\cdots$
2	-20	13	6	25	56	$\cdots$

Calculation gives [see (3.1)′ and (4.13)]

$$P_{n+r}^{(-k)} P_{n-r}^{(-k)} - \left[P_n^{(-k)}\right]^2 = (-1)^{n+r+k-1} \frac{(\alpha^r - \beta^r)^2}{8} (17)^{-k}. \tag{5.3}$$

Thus,

$$P_2^{(-1)} P_{-2}^{(-1)} - \left[P_0^{(-1)}\right]^2 = \frac{4}{17}.$$

One may readily reinterpret the earlier theory for $k \geq 0$ in terms of $k < 0$.

Similar procedures apply to $Q_n^{(-k)}$ on use of (4.11).

Likewise, the elementary properties of $F_n^{(-k)}$ and $L_n^{(-k)}$ can be established.

Discovery of other formulas pertinent to associated sequences is offered to the curiosity of the reader. While this brief exposition is only an introduction to the topic, it does allow us to savor something of the flow of ideas from definition (2.1).

Finally, it may be mentioned that, in analyzing the nature of associated sequences, the author first examined $\{P_n^{(k)}\}$ before proceeding to the generalization. This approach was helpful in investigating some of the more awkward features of the general theory.

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NONEXISTENCE OF EVEN FIBONACCI PSEUDOPRIMES OF THE 1ST KIND*

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1. INTRODUCTION AND PRELIMINARIES

Fibonacci pseudoprimes of the 1st *kind* (1-F.Psps.) have been defined [6] as composite integers n for which the Lucas congruence $L_n \equiv 1 \pmod{n}$ is satisfied.

The aim of this paper is to establish the following

Theorem: There do not exist *even Fibonacci pseudoprimes of the* 1st *kind.*

With regard to this problem, Di Porto and Filipponi, in [4], conjectured that there are no even-Fibonacci pseudoprimes of the 1st kind, providing some constraints are placed on their existence, and Somer, in [12], extends these constraints by stating some very interesting theorems. Moreover, in [1], a solution has been found for a similar problem, that is, for the sequence $\{V_n(2,1)\}$, defined by $V_0(2,1)=2$, $V_1(2,1)=3$, $V_n(2,1)=3V_{n-1}(2,1)-2V_{n-2}(2,1)=2^n+1$. Actu-ally Beeger, in [1], shows the existence of infinitely many even pseudoprimes n, that is, even n such that $2^n \equiv 2 \pmod{n} \Leftrightarrow V_n(2,1) \equiv 2+1 = V_1(2,1) \pmod{n}$.

After defining (in this section) the *generalized Lucas numbers*, $V_n(m)$, governed by the positive integral parameter m, and after giving some properties of the period of the sequences $\{V_n(m)\}$ reduced modulo a positive integer t, we define in section 2 the *Fibonacci pseudoprimes of the* m^{th} *kind* (m-F.Psps.) and we give some propositions. Finally, in section 3, we demonstrate the above theorem.

Throughout this paper, p will denote an odd prime and $V_n(m)$ will denote the *generalized Lucas numbers* (see [2], [7]), defined by the second-order linear recurrence relation

$$V_n(m) = mV_{n-1}(m) + V_{n-2}(m);\ V_0(m)=2,\ V_1(m)=m, \tag{1.1}$$

m being an arbitrary natural number. It can be noted that, letting $m=1$ in (1.1), the usual Lucas numbers L_n are obtained.

The period of the sequence $\{V_n(m)\}$ reduced modulo an integer $t>1$ will be denoted by $\mathrm{P}_{(t)}\{V_n(m)\}$. For the period of the sequence $\{V_n(m)\}$ reduced modulo p, it has been established (see [8], [13]) that

$$\text{if } J(m^2+4,p)=1, \text{ then } \mathrm{P}_{(p)}\{V_n(m)\} \mid (p-1), \tag{1.2}$$

$$\text{if } J(m^2+4,p)=-1, \text{ then } \mathrm{P}_{(p)}\{V_n(m)\} \mid 2(p+1), \tag{1.3}$$

where $J(a,n)$ is the Jacobi symbol (see [3], [10], [14]) of a with respect to n, and $x \mid y$ indicates that x divides y.

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Moreover, it can be immediately seen that

(1.4) $$\text{if } \gcd(m^2+4, p) = p, \text{ [i. e., } m^2 \equiv -4 \pmod{p}\text{], then } \mathrm{P}_{(p)}\{V_n(m)\} = 4,$$

and, if m is an odd positive integer,

(1.5) $$\mathrm{P}_{(2)}\{V_n(m)\} = 3;\ V_n(m) \equiv 0 \pmod 2 \text{ iff } n \equiv 0 \pmod 3.$$

Note that, according to (1.2), (1.3), and (1.4), the period of any generalized Lucas sequence reduced modulo a prime p is a divisor of $\Lambda(p) = \mathrm{lcm}(p-1, 2(p+1))$, that is,

(1.6) $$\mathrm{P}_{(p)}\{V_n(m)\} \mid \Lambda(p).$$

Finally, observe that, if m is a positive integer such that $m^2 \equiv -1 \pmod t$, then t is of the form

(1.7) $$t = 2^k \prod_j p_j^{k_j},$$

where p_j are odd rational primes of the form (see [8], [14])

$$p_j = 4h_j + 1,\ k \in \{0, 1\} \text{ and } k_j \geq 0.$$

In this case, it follows that

(1.8) $$\mathrm{P}_{(t)}\{V_n(m)\} = 12 \text{ and } V_1(m) \equiv V_5(m) \equiv m \pmod t.$$

2. THE FIBONACCI PSEUDOPRIMES: DEFINITION AND SOME PROPOSITIONS

The following *fundamental property* of the numbers $V_n(m)$ has been established [11]: If n is prime, then, for all m,

(2.1) $$V_n(m) \equiv m \pmod n.$$

The composite numbers n for which the congruence (2.1) holds are called *Fibonacci pseudoprimes of the* m$^{\text{th}}$ *kind* (m-F.Psps.) [6].

First, let us give some well-known results (see [5], [9]) that will be needed for our further work. Let d be an odd positive integer.

(2.2) $$V_{2d}(m) = [V_d(m)]^2 + 2,$$

(2.3) $$V_{2^k d}(m) = \left[V_{2^{k-1} d}(m)\right]^2 - 2;\ k > 1,$$

(2.4) $$V_{hd}(m) = V_h(V_d(m));\ h \geq 1.$$

To establish the theorem enounced in section 1, we state the following propositions.

Proposition 1: Let $m = 2r+1$ be an odd positive integer.

If $n = 2^k(2s+1)$, $(k \geq 1, s \geq 1)$, is an even composite integer such that $n \equiv 0 \pmod 3$, then n is not an m-F.Psp., that is,

(2.5) $$\text{If } n \equiv 0 \pmod 6, \text{ then } V_n(m) \not\equiv m \pmod n.$$

Proposition 2: Let $m = 2r+1$ be an odd positive integer.

(2.6) $$\text{If } n = 2^k,\ k \geq 1, \text{ then } V_{2^k}(m) \equiv -1 \pmod{2^k}.$$

From this proposition, it follows that

(2.7) $$\text{If } k > 1, \text{ then } 2^k \text{ is a } (2^k - 1)\text{-F.Psp.}$$

Proposition 3: Let $m = 2r+1$ be an odd positive integer.

(2.8) $$\text{If } n = 2^k(2s+1) \not\equiv 0 \pmod 3,\ k \geq 1, s \geq 2, \text{ then } V_n(m) \equiv -1 \pmod{2^k}.$$

Proof of Proposition 1: If $n \equiv 0 \pmod 6$, from (1.5) we have

(2.9) $$V_n(m) \equiv 0 \pmod 2,$$

whence we obtain

(2.10) $$V_n(m) \equiv 0 \not\equiv m = 2r+1 \pmod 2,$$

which implies that

(2.11) $$V_n(m) \not\equiv m \pmod{2^k} \Rightarrow V_n(m) \not\equiv m \pmod n. \text{ Q.E.D.}$$

Proof of Proposition 2 (by induction on k): The statement is clearly true for $k=1$. Let us suppose that the congruence

(2.12) $$V_{2^{k-1}}(m) \equiv -1 \pmod{2^{k-1}},\ k > 1$$

holds. Observing that (2.12) implies $[V_{2^{k-1}}(m)]^2 \equiv 1 \pmod{2^k}$ and, according to (2.3), we can write

(2.13) $$V_{2^k}(m) = [V_{2^{k-1}}(m)]^2 - 2 \equiv -1 \pmod{2^k}. \text{ Q.E.D.}$$

Notice that, with the same argument, it is also possible to state that

(2.14) $$\text{If } m = (2r+1), \text{ then } V_{2^k}(m) \equiv -1 \pmod{2^{k+1}} \text{ and } V_{2^k}(m) \not\equiv -1 \pmod{2^{k+2}}.$$

Proof of Proposition 3: If $n = 2^k(2s+1)$, from (2.4) we can write

(2.15) $$V_n(m) = V_{2^k}(V_{2s+1}(m));$$

moreover, if $n \not\equiv 0 \pmod 3$, we have [see (1.5)]

(2.16) $$V_{2s+1}(m) \equiv 1 \pmod 2 \Rightarrow V_{2s+1}(m) = 2h+1,\ h \geq 0,$$

whence, according to Proposition 2, we obtain

(2.17) $$V_{2^k}(V_{2s+1}(m)) = V_{2^k}(2h+1) \equiv -1 \pmod{2^k}. \text{ Q.E.D.}$$

3. THE MAIN THEOREM

Let n be an even composite number. First, observe that $1 \not\equiv -1 \pmod{2^k}$ for all $k > 1$. Propositions 1, 2, and 3 and the above obvious remark allow us to assert:

(a) If $n \equiv 0 \pmod 3$, then n is not an 1-F.Psp., according to Proposition 1;

(b) $n = 2^k$, $(k > 1)$, is not an 1-F.Psp., according to Proposition 2;

(c) $n = 2^k(2s+1) \not\equiv 0 \pmod 3$, $(k > 1, s \geq 2)$, is not an 1-F.Psp., according to Proposition 3.

Therefore, in order to demonstrate the Theorem, "There do not exist even 1-F.Psps.," it remains to prove the following

Proposition 4: Let

(3.1) $$d \not\equiv 0 \pmod 3, d > 1$$

be an odd integer, $d > 1$. If $n = 2d$ is an even composite integer, then $L_n \not\equiv 1 \pmod n$, that is, $n = 2d$ is not an 1-F.Psp.

Proof (ab absurdo): Let us suppose that

(3.2) $$L_n = L_{2d} \equiv 1 \pmod{2d} \Rightarrow L_{2d} \equiv 1 \pmod d;$$

by (2.2) we obtain

(3.3) $$[L_d]^2 = L_{2d} - 2 \equiv 1 - 2 \equiv -1 \pmod d$$

which implies [see (1.7), sec. 1]

(3.4) $$d = \prod_j p_j^{k_j},\ p_j = 4h_j + 1,\ k_j \geq 0.$$

Notice that (3.4) makes the $d \not\equiv 0 \pmod 3$ hypothesis unnecessary.

Under the conditions (3.1) and (3.4), we have

(3.5) $$d \equiv 1 \pmod{12} \text{ or } d \equiv 5 \pmod{12},$$

and we can find a positive integer m such that

(3.6) $$m^2 \equiv -1 \pmod d;$$

then, from (1.8) and (3.5), we can write the congruence

(3.7) $$V_d(m) \equiv m \pmod d,$$

which implies

(3.8) $$[V_d(m)]^2 \equiv m^2 \equiv -1 \pmod d.$$

Therefore, by (3.3) and (3.8), we obtain the congruence

(3.9) $$[L_d]^2 \equiv [V_d(m)]^2 \pmod d,$$

and, in particular, if p is the smallest prime factor of d, we can write

(3.10) $$[L_d]^2 \equiv [V_d(m)]^2 \pmod p \Rightarrow L_d \equiv \pm V_d(m) \pmod p.$$

First, observe that $\gcd(d, \Lambda(p)) = 1$, then we can find an odd positive integer d' such that

(3.11) $$d \cdot d' \equiv 1 \pmod{\Lambda(p)};$$

taking into account the equality (2.4), from (1.6), (3.10), and (3.11), we obtain

$$V_{d'}(L_d) = L_{d'd} \equiv 1 \equiv V_{d'}(\pm V_d(m)) = \pm V_{d'd}(m) \equiv \pm m \pmod p, \tag{3.12}$$

whence we obtain the congruence

$$m \equiv \pm 1 \pmod p$$

which contradicts the assumption

$$m^2 \equiv -1 \pmod d \Rightarrow m^2 \equiv -1 \pmod p. \text{ Q.E.D.}$$

ADDENDUM

About six months after this paper had been accepted for publication, I became aware of the fact that an alternative proof of the nonexistence of even 1-F.Psps. has been given by D. J. White, J. N. Hunt, and L. A. G. Dresel in their paper "Uniform Huffman Sequences Do Not Exist," published in *Bull. London Math. Soc.* **9** (1977):193-98.

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AMS Classification numbers: 11B39, 11B50, 11A07

THE MOMENT GENERATING FUNCTION OF THE GEOMETRIC DISTRIBUTION OF ORDER k

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(Submitted August 1991)

Let X be the random variable denoting the number of trials until the occurrence of the k^{th} consecutive success; the trials are independent with constant success probability p $(0<p<1)$. The probability density function f of X has been determined by Philippou and Muwafi [3]. (See also Philippou, Georghiou, and Philippou [4].) In this note we show that the moment generating function of X exists, and we determine a formula for it by means of the following recurrence. For two other recursive formulas of f, see [1] and [2].

Proposition: The probability density f of X satisfies the following relations. (Here, $q=1-p$.)

(a) $f(k)=p^k$.
(b) $f(n)=qp^k$ if $k+1\le n\le 2k-1$.
(c) $f(n)=qf(n-1)+qpf(n-2)+qp^2f(n-3)+\cdots+qp^{k-1}f(n-k)$ if $n\ge 2k$.

Hence, for $n\ge 2k$, the terms $f(n)$ satisfy a linear recursive relation of order k whose auxiliary equation is $x^k-qx^{k-1}-qpx^{k-2}-\cdots-qp^{k-1}=0$.

Proof: Clearly the formula holds for $n=k$. Suppose now that $k+1\le n\le 2k-1$. The first run of k consecutive successes ends on the n^{th} trial. These k successes are preceded by a failure, which in turn is preceded by any sequence of $n-k-1$ outcomes. Thus, $f(n)=qp^k$. Now let $n\ge 2k$, and consider a sequence of n Bernoulli trials where the first run of k consecutive successes ends on the n^{th} trial. The first failure must occur on or before the k^{th} trial and may occur on any of the first k trials. For $1\le j\le k$, let E_j be the event that the first run of k consecutive successes occurs on the n^{th} trial and that the first failure occurs on the j^{th} trial. Clearly $f(n)$ equals the sum of the probabilities of the E_j. We claim that the probability of E_j is $qp^{j-1}f(n-j)$. Points in E_j consist of $j-1$ successes, followed by a failure, followed by any sequence of $n-j$ outcomes consistent with the first run of k consecutive successes ending on the n^{th} trial, and so by independence the probability of E_j is as claimed.

Having established these properties for f, we proceed to our main result.

Theorem: The moment generating function $M(t)$ of X exists on some open interval containing 0 and is given by

$$M(t)=\frac{p^ke^{kt}}{1-qe^t-qpe^{2t}-\cdots-qp^{k-1}e^{kt}}=\frac{p^ke^{kt}(1-pe^t)}{1-e^t+(1-p)p^ke^{(k+1)t}}.$$

The proof of the theorem will be given after establishing the following lemma.

Lemma: The roots of the auxiliary equation are distinct and have absolute value less than 1.

Proof: We have seen that, for $n\ge 2k$, the terms $f(n)$ satisfy a linear recursive relation of order k whose auxiliary equation is

$$x^k-qx^{k-1}-qpx^{k-2}-\cdots-qp^{k-1}=0.$$

We now investigate this equation. Let $e(x)$ denote the polynomial

$$x^k - qx^{k-1} - qpk^{k-2} - \cdots - qp^{k-1} \text{ in } x,$$

and let $f(x) = (x-p)e(x) = x^{k+1} - x^k + qp^k$. Now

$$f'(x) = (k+1)x^k - kx^{k-1} = (k+1)x^{k-1}\left(x - \frac{k}{k+1}\right)$$

has roots 0 and $\frac{k}{k+1}$. Since 0 is not a root of f, f has a repeated root if and only if $\frac{k}{k+1}$ is a root of f. But

$$f\left(\frac{k}{k+1}\right) = \left(\frac{k}{k+1}\right)^{k+1} - \left(\frac{k}{k+1}\right)^k + qp^k = \frac{-1}{k+1}\left(\frac{k}{k+1}\right)^k + qp^k.$$

Since

$$(1-x)x^k - \frac{1}{k+1}\left(\frac{k}{k+1}\right)^k \le 0$$

on [0, 1] with equality if and only if $x = \frac{k}{k+1}$, we see that $\frac{k}{k+1}$ is a root of f if and only if $p = \frac{k}{k+1}$. Thus, f has a repeated root (of order 2) if and only if $p = \frac{k}{k+1}$. Hence, the roots of e are distinct.

We turn now to the absolute values of the roots of $e(x)$. We will show that if z is a (complex) number with $|z| \ge 1$, then z is not a root of the equation $e(x) = 0$.

$$\begin{aligned} |z^k - qz^{k-1} - qpz^{k-2} - \cdots - qp^{k-1}| &\ge |z|^k - q|z|^{k-1} - qp|z|^{k-2} - \cdots - qp^{k-1} \\ &\ge |z|^k - q|z|^k - qp|z|^k - \cdots - qp^{k-1}|z|^k \\ &\ge |z|^k - q|z|^k \frac{1-p^k}{1-p} \\ &= |z|^k - |z|^k(1-p^k) = p^k|z|^k > 0. \end{aligned}$$

Thus, all roots of the equation $e(x) = 0$ have absolute value less than 1.

Proof of the Theorem: Let $z_1, z_2, \ldots, z_k$ be the distinct roots of the auxiliary equation; then, from the theory of difference equations, we know that there exist (complex) constants $c_1, c_2, \ldots, c_k$ such that

$$f(n) = c_1 z_1^n + c_2 z_2^n + \cdots + c_k z_k^n \text{ if } n \ge k.$$

Now the series $\sum_{n=k}^{\infty} c_i z_i^n e^{nt} = c_i \sum_{n=k}^{\infty} (z_i e^t)^n$ converges to $\frac{c_i (z_i e^t)^k}{1 - z_i e^t}$ if $|z_i e^t| < 1$, that is, if $t < -\ln|z_i|$. Let $m = \min\{-\ln|z_1|, -\ln|z_2|, \ldots, -\ln|z_k|\}$ Then the moment generating function

$$M(t) = \sum_{n=k}^{\infty} e^{nt} f(n)$$

exists on the interval $(-\infty, m)$. The proof of the theorem now follows by substituting e^t for s in the formula of the probability generating function $\gamma_k(s)$ of [4, Lemma 2.3]. Alternatively, recasting the proposition above, we have

$$(*) \qquad f(n+k) = qf(n+k-1) + qp(n+k-2) + \cdots + qp^{k-1} f(n), \; n \ge 1,$$

with $f(1)=f(2)=\cdots=f(k-1)=0$ and $f(k)=p^k$. Therefore,

$$\begin{aligned}M(t)&=\sum_{n=k}^{\infty}e^{nt}f(n)=e^{kt}f(k)+\sum_{n=1}^{\infty}e^{(n+k)t}f(n+k)\\&=e^{kt}p^k+q\sum_{n=1}^{\infty}e^{(n+k)t}f(n+k-1)+qp\sum_{n=1}^{\infty}e^{(n+k)t}f(n+k-2)+\cdots+qp^{k-1}\sum_{n=1}^{\infty}e^{(n+k)t}f(n),\text{ by }(*),\\&=e^{kt}p^k+qe^t\sum_{n=1}^{\infty}e^{(n+k-1)t}f(n+k-1)+qpe^{2t}\sum_{n=1}^{\infty}e^{(n+k-2)t}f(n+k-2)+\cdots+qp^{k-1}e^{kt}\sum_{n=1}^{\infty}e^{nt}f(n)\\&=e^{kt}p^k+qe^tM(t)+qpe^{2t}M(t)+\cdots+qp^{k-1}e^{kt}M(t),\end{aligned}$$

from which the proof follows.

Final Comment: From the moment generating function, one can calculate all the moments that are of interest. For example, when $p=\frac{1}{2}$, the mean of X is given by $\mu=2(2^k-1)$, and the variance of X by $\sigma^2=4(2^k-1)^2-(4k-6)(2^k-1)-4k$; the following table displays the skewness factor α_3 and the kurtosis factor α_4 for k = 1, ..., 10. Note that as k increases, α_3 and α_4 approach the skewness factor 2 and the kurtosis factor 9, respectively, of the Exponential Distribution.

k	α_3	α_4
1	2.211320344	9.5
2	2.035097747	9.144628099
3	2.010489423	9.042749454
4	2.003133201	9.012677353
5	2.000918388	9.003699063
6	2.000262261	9.00105334
7	2.000072886	9.00029223
8	2.000019756	9.00007913
9	2.000005243	9.000020986
10	2.000001368	9.000005473

ACKNOWLEDGMENT

The authors with to thank the referee for drawing their attention to two references, and for shortening their computation of the moment generating function.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3515@compuserve.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-736 ***Proposed by Herta T. Freitag, Roanoke, VA***

Prove that $(2L_n + L_{n-3})/5$ is a Fibonacci number for all n.

B-737 ***Proposed by Herta T. Freitag, Roanoke, VA***

A right triangle, one of whose legs is twice as long as the other leg, has a hypotenuse that is one unit longer than the longer leg. Let r be the inradius of this triangle (radius of inscribed circle) and let r_a, r_b, r_c be the exradii (radii of circles outside the triangle that are tangent to all three sides).

Express r, r_a, r_b, and r_c in terms of the golden ratio, α.

B-738 ***Proposed by Daniel C. Fielder & Ceceil O. Alford, Georgia Institute of Technology, Atlanta, GA***

Find a polynomial $f(w, x, y, z)$ such that

$$f(L_n, L_{n+1}, L_{n+2}, L_{n+3}) = 25f(F_n, F_{n+1}, F_{n+2}, F_{n+3})$$

is an identity.

B-739 ***Proposed by Ralph Thomas, University of Chicago, Dundee, IL***

Let $S = \left\{\frac{F_i}{F_j} \middle| i \geq 0, j > 0\right\}$. Is S dense in the set of nonnegative real numbers?

B-740 ***Proposed by Thomas Martin, Phoenix, AZ***

Find all positive integers x such that 10 is the smallest integer, n, such that $n!$ is divisible by x.

B-741 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. College of Science, Gujarat, India***

Prove that $F_{n+8}^4 + 331F_{n+4}^4 + F_n^4$ is always divisible by 54.

SOLUTIONS

Coefficients of a Maclaurin Series

B-709 ***Proposed by Alejando Necochea, Pan American University, Edinburg, TX***

Express

$$\frac{1}{n!}\frac{d^n}{dt^n}\left[\frac{t}{1-t-t^2}\right]_{t=0}$$

in terms of Fibonacci numbers.

Solution by Douglas E. Iannucci, Riverside, RI

Since $t/(1-t-t^2)$ is the generating function for the Fibonacci sequence, we have

$$\sum_{k=0}^{\infty} F_k t^k = \frac{t}{1-t-t^2}$$

(see [2], pp. 52-53). Thus

$$\frac{1}{n!}\frac{d^n}{dt^n}\left[\frac{t}{1-t-t^2}\right]_{t=0} = \frac{1}{n!}\frac{d^n}{dt^n}\left[\sum_{k=0}^{\infty} F_k t^k\right]_{t=0}$$

$$= \frac{1}{n!}\left[\sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)F_k t^{k-n}\right]_{t=0} = \left[\sum_{k=n}^{\infty}\binom{k}{n}F_k t^{k-n}\right]_{t=0} = F_n.$$

Several solvers blithely proceeded to differentiate the power series for $t/(1-t-t^2)$ term by term (as above) without justifying that this produces correct results. Several solvers quoted Taylor's Theorem, but this did not convince me. The procedure is valid by the following ([1], p. 26):

Theorem: In the interior of its circle of convergence, a power series may be differentiated term by term. The derived series converges and represents the derivative of the sum of the original power series. Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence $R > 0$, then $f(z)$ has derivatives of all orders; and for $|z| < R$, we have

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n z^{n-k}.$$

References:

1. Einar Hille. *Analytic Function Theory.* Vol. 1. New York: Blaisdell, 1959.
2. S. Vajda. *Fibonacci and Lucas Numbers, and the Golden Section—Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Also solved by Glenn Bookhout, Wray Brady, Paul S. Bruckman, Joseph E. Chance, Russell Euler, C. Georghiou, Russell Jay Hendel, Joseph J. Kostal, Igor Ol. Popov, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Ralph Thomas, and the proposer.

Pell-Lucas Congruences

B-710 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let P_n be the n^{th} Pell number, defined by $P_0 = 0, P_1 = 1, P_{n+2} = 2P_{n+1} + P_n$, for $n \geq 0$. Prove that

(a) $P_{3n+1} \equiv L_{3n+1} \pmod 5$,

(b) $P_{3n+2} \equiv -L_{3n+2} \pmod 5$.

(c) Find similar congruences relating Pell numbers and Fibonacci numbers.

Solution by Paul S. Bruckman, Edmonds, WA

We may readily form the following short table of P_n and L_n modulo 5:

n	$P_n \pmod 5$	$L_n \pmod 5$
0	0	2
1	1	1
2	2	−2
3	0	−1
4	2	2
5	−1	1
6	0	−2
7	−1	−1
8	−2	2
9	0	1
10	−2	−2
11	1	−1
12	0	2
13	1	1

Inspection of the foregoing table shows that P_n repeats every 12 terms and L_n repeats every 4 terms. Thus, to discover the relations between the P_n's and L_n's (mod 5), it suffices to consider the first 12 terms of the sequences involved. Parts (a) and (b) of the problem then follow immediately by inspecting this table and confirming the congruences.

To find relations between the P_n's and F_n's, we form a similar array tabulating P_n and F_n modulo 5. We find that F_n repeats every 20 terms. Thus, it suffices to consider the first 60 terms of the sequences involved (since lcm[12, 20] = 60). We omit the tabulation, but the 60-line table is straightforward to create. Inspecting that table, we find

$$\begin{aligned}
P_{15n+a} &\equiv F_{15n+a} \pmod 5 &\quad& \text{for } a \in \{-1, 0, 1\}\\
P_{15n+a} &\equiv -F_{15n+a} \pmod 5 && \text{for } a \in \{-4, 0, 4\}\\
2P_{15n+a} &\equiv -F_{15n+a} \pmod 5 && \text{for } a \in \{-2, 0, 2\}\\
2P_{15n+a} &\equiv F_{15n+a} \pmod 5 && \text{for } a \in \{-7, 0, 7\}.
\end{aligned}$$

This last set of congruences repeat every 15 entries (rather than every 60) because of the fact that $P_{n+15} \equiv 2P_n \pmod 5$ and $F_{n+15} \equiv 2F_n \pmod 5$.

Georghiou, Prielipp, Somer, and Thomas found the following congruences modulo 3:

$$P_{2n} \equiv -F_{2n} \pmod 3$$
$$P_{2n+1} \equiv F_{2n+1} \pmod 3$$

which can also be expressed as $P_n \equiv (-1)^{n+1} F_n \pmod 3$.

Also solved by Charles Ashbacher (parts a and b), Herta T. Freitag, C. Georghiou, Russell Jay Hendel, Joseph J. Kostal, Bob Prielipp, Stanley Rabinowitz, Lawrence Somer, Ralph Thomas, and the proposer.

Cosh, What a Product!

B-711 ***Proposed by Mihály Bencze, Sacele, Romania***

Let r be a natural number. Find a closed form expression for

$$\prod_{k=1}^{\infty}\left(1-\frac{L_{4r}}{k^4}+\frac{1}{k^8}\right).$$

Solution by H.-J. Seiffert, Berlin, Germany

It is known that $\sin \pi z$ and $\sinh \pi z$ have the following product expansions:

$$\sin \pi z = \pi z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right) \tag{1}$$

and

$$\sinh \pi z = \pi z\prod_{k=1}^{\infty}\left(1+\frac{z^2}{k^2}\right) \tag{2}$$

which are valid for all complex z (see [1], series 4.3.89 and 4.5.68; [2], p. 350; [3], p. 37, section 1.431; or [4], series 1016 and 1078).

Thus,

$$\begin{aligned}
P &= \prod_{k=1}^{\infty}\left(1-\frac{L_{4r}}{k^4}+\frac{1}{k^8}\right)=\prod_{k=1}^{\infty}\left(1-\frac{\alpha^{4r}}{k^4}\right)\left(1-\frac{\beta^{4r}}{k^4}\right)\\
&= \prod_{k=1}^{\infty}\left(1-\frac{\alpha^{2r}}{k^2}\right)\left(1+\frac{\alpha^{2r}}{k^2}\right)\left(1-\frac{\beta^{2r}}{k^2}\right)\left(1+\frac{\beta^{2r}}{k^2}\right)\\
&= \prod_{k=1}^{\infty}\left(1-\frac{\alpha^{2r}}{k^2}\right)\prod_{k=1}^{\infty}\left(1+\frac{\alpha^{2r}}{k^2}\right)\prod_{k=1}^{\infty}\left(1-\frac{\beta^{2r}}{k^2}\right)\prod_{k=1}^{\infty}\left(1+\frac{\beta^{2r}}{k^2}\right)\\
&= \frac{\sin(\pi\alpha^r)}{\pi\alpha^r}\,\frac{\sinh(\pi\alpha^r)}{\pi\alpha^r}\,\frac{\sin(\pi\beta^r)}{\pi\beta^r}\,\frac{\sinh(\pi\beta^r)}{\pi\beta^r}\\
&= \frac{1}{\pi^4}\sin(\pi\alpha^r)\sin(\pi\beta^r)\sinh(\pi\alpha^r)\sinh(\pi\beta^r).
\end{aligned}$$

Editorial note: *The step wherein we pass to a product of four infinite products needs some justification. The infinite product $\prod_{n=1}^{\infty}(1+x_n)$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty}|x_n|$ is convergent (see [2], p. 159). All the infinite products used above are absolutely convergent. It is known that the factors of an absolutely convergent infinite product may be rearranged arbitrarily without affecting the convergence of the product (see [5], p. 530). Thus, we are permitted to equate $\prod a_k b_k$ and $\prod a_k \prod b_k$, which justifies the above procedures.*

The result can be simplified further. Using the formulas

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)] \quad \text{and} \quad \sinh x \sinh y = \frac{1}{2}[\cosh(x+y) - \cosh(x-y)]$$

(see [1], formula 4.5.38), we find that

$$P = \frac{1}{4\pi^4}\left[\cos\left(\pi F_r\sqrt{5}\right) - \cos\left(\pi L_r\right)\right]\left[\cosh\left(\pi L_r\right) - \cosh\left(\pi F_r\sqrt{5}\right)\right].$$

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Also solved by Paul S. Bruckman, C. Georghiou, Igor Ol. Popov, and the proposer.

Another Lucas Number

B-712 ***Proposed by Herta T. Freitag, Roanoke, VA***

Prove that for all positive integers n, $\alpha(\sqrt{5}\alpha^n - L_{n+1})$ is a Lucas number.

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Since $\sqrt{5} = \alpha - \beta$ and $\alpha\beta = -1$, we have

$$\begin{aligned}\alpha(\sqrt{5}\alpha^n - L_{n+1}) &= \alpha[(\alpha-\beta)\alpha^n - (\alpha^{n+1} + \beta^{n+1})] \\ &= \alpha^{n+2} - \beta\alpha^{n+1} - \alpha^{n+2} - \alpha\beta^{n+1} \\ &= \alpha^n + \beta^n = L_n.\end{aligned}$$

Most solutions were similar. Redmond found an analog for Fibonacci numbers: $\alpha(\alpha^n - F_{n+1}) = F_n$. *Haukkanen found this too, as well as* $-\beta(\sqrt{5}\beta^n + L_{n+1}) = L_n$ and $\beta(\beta^n - F_{n+1}) = F_n$. *Redmond also generalized to arbitrary second-order linear recurrences.*

Also solved by Richard André-Jeannin, Charles Ashbacher, Mohammad K. Azarian, M. A. Ballieu, Seung-Jin Bang, Glenn Bookhout, Scott H. Brown, Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, Jane Friedman, Pentti Haukkenen, Russell Jay Hendel, Carl Libis, Graham Lord, Dorka Ol. Popova, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Ralph Thomas, and the proposer.

Complex Pythagorean Triple

B-713 ***Proposed by Herta T. Freitag, Roanoke, VA***

(a) Let S be a set of three consecutive Fibonacci numbers. In a Pythagorean triple, (a, b, c), a is the product of the elements in S; b is the product of two Fibonacci numbers (both larger than

1), one of them occurring in S; and c is the sum of the squares of two members of S. Determine the Pythagorean triple and prove that the area of the corresponding Pythagorean triangle is the product of four consecutive Fibonacci numbers.

(b) Same problem as part (a) except that Fibonacci numbers are replaced by Lucas numbers.

Solution by Paul S. Bruckman, Edmonds, WA

Part (a): Let $S=(F_{n-1},F_n,F_{n+1})$. Since we require $a=F_{n-1}F_nF_{n+1}$ to be the side of a triangle, we must have $n\geq 2$. Also, if $n=2$, then $a=2$, which cannot be the side of a Pythagorean triangle. Thus, $n\geq 3$. Since the sequence $(F_n)_{n=2}^{\infty}$ is increasing, the hypothesis implies that $c\leq F_n^2+F_{n+1}^2=F_{2n+1}$. Also, we must have $c>a$. Thus, we are to have $F_{n-1}F_nF_{n+1}<F_{2n+1}$. This inequality can be satisfied only for a finite number of n. In fact, it holds only for $n\leq 4$. Thus, the only possible solutions are generated by $n=3$ or $n=4$.

If $n=3$, we obtain the value $a=F_4F_3F_2=3\cdot 2\cdot 1=6$. Since $c>a$, we must have $c=3^2+2^2$ $=13$ or $c=3^2+1^2=10$. However, $13^2-6^2=133$, which is not a perfect square; so we reject that possibility. Since $10^2-6^2=8^2$, we try $b=8$. However, $8=8\cdot 1=F_6F_2$, which is the only factorization of 8 into two factors that are Fibonacci numbers. Since $F_2=1$, we must also reject this possibility.

If $n=4$, we obtain $a=F_5F_4F_3=5\cdot 3\cdot 2=30$. The only possible value for c is $F_5^2+F_4^2=$ $5^2+3^2=34$. This yields $b^2=34^2-30^2=16^2$, so $b=16$. We can factor 16 as a product of two Fibonacci numbers in only one way; namely, $16=8\cdot 2=F_6F_3$, and both factors are larger than 1. Moreover, F_3 divides a. Therefore, this is a valid solution and is, indeed, the only solution:

$$(a,b,c)=(F_5F_4F_3,\,F_6F_3,\,F_5^2+F_4^2)=(5\cdot 3\cdot 2,\,8\cdot 2,\,5^2+3^2)=(30,16,34).$$

For this unique solution, the area of the triangle formed by sides of length a, b, and c is equal to $\frac{1}{2}\cdot 30\cdot 16=240=2\cdot 3\cdot 5\cdot 8=F_3F_4F_5F_6$, as required.

Part (b): Bruckman's analysis of part (b) was similar, yielding the unique solution

$$(a,b,c)=(L_0L_1L_2,\,L_0L_3,\,L_1^2+L_2^2)=(2\cdot 1\cdot 3,\,2\cdot 4,\,1^2+3^2)=(6,8,10).$$

In this case, the area of the triangle formed by a, b, and c is equal to $\frac{1}{2}\cdot 6\cdot 8=24=2\cdot 1\cdot 3\cdot 4=$ $L_0L_1L_2L_3$, the product of four consecutive Lucas numbers, as required.

Some solvers found a *solution but did not prove that it was unique. Thus, they did not technically prove that the area of the triangle* must *be the product of four consecutive Fibonacci numbers.*

Also solved by Charles Ashbacher, Leonard A. G. Dresel; Jane Friedman, Marquis Griffith & Ryan Jackson (jointly); Russell Jay Hendel, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Editorial Note: *Problems B-707 and B-708 were also solved by Igor Ol. Popov.*

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems

PROBLEM PROPOSED IN THIS ISSUE

H-475 *Proposed by Larry Taylor, Rego Park, NY*

Professional chess players today use the algebraic chess notation. This is based upon the algebraic numbering of the chessboard. The eight letters a through h and the eight digits 1 through 8 are used to form sixty-four combinations of a letter and a digit which are called "symbol pairs." Those sixty-four symbol pairs are used to represent the sixty-four squares of the chess-board.

Develop a viable arithmetic numbering of the chessboard, as follows:

(a) Use twenty-five letters of the alphabet (all except U) and nine decimal digits (all except zero) to form 225 symbol pairs; choose sixty-four of those symbol pairs to represent the sixty-four squares of the chessboard.

(b) There are thirty-six squares from which a King can move to eight other squares. Let the nine symbol pairs representing the location of the King and the squares to which it can move contain all nine decimal digits.

(c) There are sixteen squares from which a Knight can move to eight other squares. A Queen located on one of those sixteen squares, moving one or two squares, can go to sixteen other squares. Let the twenty-five symbol pairs representing the location of the Knight or the Queen and the squares to which the Knight or the Queen can move contain all twenty-five letters of the alphabet.

(d) Let the algebraic Square $a8$ (the original location of Black's Queen Rook) correspond to the arithmetic Square $A1$; let the algebraic Square $h1$ (the original location of White's King Rook) correspond to the arithmetic Square $Z9$.

H-476 *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the Pell numbers by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$. Show that, for all positive integers n,

$$P_n = \sum_{\substack{k=0 \\ 4 \nmid 2n+k}}^{n-1} (-1)^{[(3k+3-2n)/4]} 2^{[3k/2]} \binom{n+k}{2k+1},$$

where [ ] denotes the greatest integer function.

__H-477__ ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let

$$F_r(z) = z^r - \sum_{k=0}^{r-1} a_k z^{r-1-k}, \tag{1}$$

where $r \geq 1$, and the a_k's are integers. Suppose F_r has distinct zeros θ_k, $k = 1, 2, ..., r$, and let

$$V_n = \sum_{k=1}^{r} \theta_k^n, \; n = 0, 1, 2, \ldots. \tag{2}$$

Prove that, for all primes p,

$$V_p \equiv a_0 \pmod{p}. \tag{3}$$

SOLUTIONS

Editorial Notes: *Paul S. Bruckman's name was omitted as a solver of H-435.*

A number of readers pointed out that exponent "u" was missing in two places in H-472.

Larry Taylor feels that the solution of H-454 as published was not complete, or at least was not what was intended. We therefore offer Mr. Taylor's solution here.

__Mix and Match__

__H-454__ ***Proposed by Larry Taylor, Rego Park, NY***
(Vol. 29, no. 2, May 1991)

Construct six distinct Fibonacci-Lucas identities such that

(a) Each identity consists of three terms;
(b) Each term is the product of two Fibonacci numbers;
(c) Each subscript is either a Fibonacci or a Lucas number.

Solution by the proposer

Let j, k, n, and t be integers. It is known that $F_jF_{n+k} = F_kF_{n+j} - F_{k-j}F_n(-1)^j$.

(1) Let $j = F_t, k = F_{t+1}, n = F_{t-1}$;
(2) Let $j = F_t, k = L_{t+1}, n = F_{t-1}$;
(3) Let $j = F_{t-1}, k = F_t, n = F_{t+1}$;
(4) Let $j = F_{t-1}, k = L_t, n = L_{t+1}$;
(5) Let $j = F_{t-1}, k = F_{t+1}, n = F_t$;
(6) Let $j = F_{t-2}, k = F_{t+2}, n = F_t$.

In each of the six identities, each of $n+k, n+j, k-j$ is either a Fibonacci or a Lucas number.

__Simply Wonderful__

__H-458__ ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 29, no. 3, August 1991)

Given an integer $m \geq 0$ and a sequence of natural numbers $a_0, a_1, \ldots, a_m$, form the periodic simple continued fraction (s.c.f.) given by

$$\theta = [a_0; \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}]. \tag{1}$$

The period is symmetric, except for the final term $2a_0$, and may or may not contain a central term [that is, a_m occurs either once or twice in (1)]. Evaluate θ in terms of nonperiodic s.c.f.'s.

Solution by Russell Jay Hendel, Dowling College, Oakdale, NY

Let n denote the length of the period of θ. We claim

Theorem: $\theta^2 = a_0^2 + 2Ma_0 + N$, where, for $n \geq 5$,

$$M = [0; a_1, a_2, \ldots, a_2, a_1]$$
$$N / M = [0; a_1, a_2, \ldots, a_2].$$

Remark 1: For $n \leq 4$, we have:

$n = 4$:	$M = [0; a_1, a_2, a_1]$	$N / M = [0; a_1, a_2]$
$n = 3$:	$M = [0; a_1, a_1]$	$N / M = [0; a_1]$
$n = 2$:	$M = [0; a_1]$	$N / M = [0]$
$n = 1$:	$M = [0]$	$N = 1.$

These initial cases were verified using DERIVE. The study of the initial cases also aided discovery of the general pattern.

Remark 2: Results on the continued fractions of quadratic irrationals are well known. Some standard references are [2; pp. 310-88] or [3; pp. 197-204]. Standard textbook exercises study partial quotients of continued fraction expansions of θ for small n (e.g., [2; p. 388, Probs. 4-7] or [3; p. 204, Probs. 1 and 2]). Note that older notations, e.g., [3], sometimes differ from modern ones by starting the continued fraction with a_1 instead of a_0

Remark 3: Let $C_k = p_k / q_k$ denote the k^{th} convergent of $(\theta - a_0)^{-1}$ for $k = 0, 1, 2, \ldots$. In particular, (1) implies $p_{n-1} / q_{n-1} = [a_1, a_2, \ldots, a_2, a_1, 2a_0]$. The following facts, used in the sequel, are well known (see [1; CF4 and CF1] and [2; p. 385, Eq. 10.17]).

$$p_{n-1} = (2a_0)p_{n-2} + q_{n-2}; \quad q_{n-1} = (2a_0)q_{n-2} + q_{n-3}. \tag{2}$$

$$1 / M = [a_1, a_2, \ldots, a_2, a_1] = p_{n-2} / q_{n-2} > 1; \quad M / N = [a_1, a_2, \ldots, a_2] = q_{n-2} / q_{n-3} > 1. \tag{3}$$

A real

$$x > 1 \tag{4}$$

satisfies the quadratic equation

$$q_{n-1}x^2 + \{q_{n-2} - p_{n-1}\}x - p_{n-2} = 0 \tag{5}$$

if and only if

$$x = (\theta - a_0)^{-1}. \tag{6}$$

Proof of the Theorem: Substitution, using (2), transforms the theorem assertion into the following equivalent claim, which we will prove:

$$\theta = \sqrt{a_0^2 + 2\frac{q_{n-2}}{p_{n-2}}a_0 + \frac{q_{n-3}}{p_{n-2}}}.$$

Let

$$x = \frac{1}{\sqrt{a_0^2 + 2\frac{q_{n-2}}{p_{n-2}}a_0 + \frac{q_{n-3}}{p_{n-2}}} - a_0}.$$

Then (3) implies (4) and straightforward expansion using (2) demonstrates (5). Equations (4) and (5) imply (6) and the result immediately follows.

References:

1. Attila Petho. "Simple Continued Fractions for the Fredholm Numbers." *J. Number Theory* **14** (1982):232-36.
2. Kenneth H. Rosen. *Elementary Number Theory and Its Applications.* Reading, Mass.: Addison Wesley, 1984.
3. James E. Shockley. *Introduction to Number Theory.* New York & Chicago: Holt, Rinehart, and Winston, 1967.

Also solved by the proposer.

Kind of Triggy

H-460 ***Proposed by H.-J. Seiffert, Berlin Germany***
(Vol. 29, no. 4, November 1991)

Define the Fibonacci polynomials by $F_0(x) = 0, F_1(x) = 1, F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$. Show that, for all positive reals x,.

(a) $$\sum_{k=1}^{n-1} 1\Big/\left(x^2 + \sin^2\frac{k\pi}{2n}\right) = \frac{(2n-1)F_{2n+1}(2x) + (2n+1)F_{2n-1}(2x)}{4x(x^2+1)F_{2n}(2x)} - \frac{1}{2x^2},$$

(b) $$\sum_{k=1}^{n-1} 1\Big/\left(x^2 + \sin^2\frac{k\pi}{2n}\right) \sim n/\left(x\sqrt{x^2+1}\right), \text{ as } n \to \infty,$$

(c) $$\sum_{k=1}^{n-1} 1\Big/\sin^2\frac{k\pi}{2n} = 2(n^2-1)/3.$$

Solution by Paul S. Bruckman, Edmonds, WA

The auxiliary equation for $F_n(2x)$ is given by

$$z^2 - 2xz - 1 = 0, \tag{1}$$

whose roots r and s are given by

$$r = x + y, \; s = x - y, \; \text{where } y = (x^2+1)^{1/2}. \tag{2}$$

If we set

$$x > \sinh\theta, \; \theta = 0, \tag{3}$$

we obtain

$$r = e^{\theta},\ s = -e^{-\theta}, \tag{4}$$

and

$$y = \cosh\theta. \tag{5}$$

Moreover, $F_n(2x) = (r^n - s^n)/(r-s)$, from which we obtain

$$F_{2n}(2x) = \sinh 2n\theta / \cosh\theta; \tag{6}$$

$$F_{2n+1}(2x) = \cosh(2n+1)\theta / \cosh\theta. \tag{7}$$

We may also easily verify the following identities:

$$F_{2n+1}(2x) + F_{2n-1}(2x) = 2\cosh 2n\theta; \tag{8}$$

$$F_{2n+1}(2x) - F_{2n-1}(2x) = 2xF_{2n}(2x). \tag{9}$$

From the recurrence relation defining the F_n's, it readily follows that the leading term of $F_{2n}(2x)$ is $(2x)^{2n-1}$. Moreover, we see from (6) that $F_{2n}(2x) = 0$ if and only if $2n\theta = ki\pi$, $k = 0, \pm 1, \pm 2, \ldots, \pm(n-1)$, or, equivalently, $2n\theta = \pm ki\pi$, $k = 0, 1, \ldots, n-1$. Thus, $F_{2n}(2x) = 0$ if and only if $x = \sinh\theta = \sinh(\pm ki\pi/2n) = \pm i \sin(k\pi/2n)$. From this, we obtain the factorization:

$$F_{2n}(2x) = 2^{2n-1}x\prod_{k=1}^{n-1}(x^2 + \sin^2 k\pi/2n). \tag{10}$$

Taking the logarithm and derivative in (10), we obtain

$$\frac{F'_{2n}(2x)}{xF_{2n}(2x)} - \frac{1}{2x^2} = \sum_{k=1}^{n-1}(x^2 + \sin^2 k\pi/2n)^{-1} \equiv S_n(x), \text{ say.} \tag{11}$$

Here and in the sequel, the prime symbol denotes differentiation with respect to x.

On the other hand, we may differentiate in (6), using the useful results:

$$y' = x/y = \coth\theta; \tag{12}$$

$$\theta' = 1/y = \operatorname{sech}\theta. \tag{13}$$

Then

$$\begin{aligned}2F'_{2n}(2x) &= \operatorname{sech}^2\theta[\cosh\theta\cdot 2n\cosh 2n\theta \operatorname{sech}\theta - \sinh 2n\theta \cdot \sinh\theta \operatorname{sech}\theta] \\ &= (x^2+1)^{-1}[2n\cosh 2n\theta - \tanh\theta\sinh 2n\theta] \\ &= (x^2+1)^{-1}[F_{2n+1}(2x) + F_{2n-1}(2x)]n - (x^2+1)^{-1}(x/y)yF_{2n}(2x).\end{aligned}$$

Thus,

$$F'_{2n}(2x)/xF_{2n}(2x) - 1/2x^2 = \frac{2n\left(F_{2n+1}(2x) + F_{2n-1}(2x)\right)}{4x(x^2+1)F_{2n}(2x)} - \frac{1}{2(x^2+1)} - \frac{1}{2x^2}$$

$$= \frac{(2n-1)F_{2n+1}(2x) + (2n+1)F_{2n-1}(2x)}{4x(x^2+1)F_{2n}(2x)} + \frac{F_{2n+1}(2x) - F_{2n-1}(2x)}{4x(x^2+1)F_{2n}(2x)} - \frac{1}{2(x^2+1)} - \frac{1}{2x^2};$$

using (9), this simplifies to:

$$\frac{F'_{2n}(2x)}{xF_{2n}(2x)}-\frac{1}{2x^2}=\frac{(2n-1)F_{2n+1}(2x)+(2n+1)F_{2n-1}(2x)}{4x(x^2+1)F_{2n}(2x)}-\frac{1}{2x^2}\equiv T_n(x),\text{ say.} \tag{14}$$

Comparison of (11) and (14) establishes part (a) of the problem, i.e., $S_n(x)=T_n(x)$.

Also, we may express $T_n(x)$ in the following form:

$$T_n(x)=\frac{2n\cosh 2n\theta-\tanh\theta\sinh 2n\theta}{2(x/y)(x^2+1)\sinh 2n\theta}-\frac{1}{2x^2}$$
$$=\frac{ny\coth 2n\theta}{xy^2}-\frac{1}{2(x^2+1)}-\frac{1}{2x^2}=\frac{n\coth 2n\theta}{xy}-\frac{2x^2+1}{2x^2(x^2+1)},$$

or, since $2x^2+1=2\sinh^2\theta=\cosh 2\theta$ and $4x^2(x^2+1)=4x^2y^2=(2\sinh\theta\cosh\theta)^2=\sinh^2 2\theta$, we obtain

$$S_n(x)=T_n(x)=\frac{n}{x\sqrt{x^2+1}}\coth 2n\theta-\frac{2\cosh 2\theta}{\sinh^2 2\theta}. \tag{15}$$

Now $\lim_{n\to\infty}\coth 2n\theta=\lim_{n\to\infty}\frac{\exp(4n\theta)+1}{\exp(4n\theta)-1}=1.$ Therefore, it follows from (15) that

$$S_n(x)\sim\frac{n}{x(x^2+1)^{1/2}}\quad\text{as } n\to\infty, \tag{16}$$

which is part (b) of the problem.

We see from (15) that

$$S_n(x)=2n\coth 2n\theta\cdot\operatorname{csch}2\theta-2\cosh 2\theta\cdot\operatorname{csch}^2 2\theta=U_n(\theta),\text{ say.} \tag{17}$$

From the definition in (3), it follows that $\lim_{x\to 0}S_n(x)=S_n(0)=U_n(0)=\lim_{\theta\to\infty}U_n(\theta)$, provided that either limit exists. Also, it appears easier to evaluate this limit by expansion, rather than attempt to apply L'Hôpital's Rule. Toward this end, we require the following expansions:

$$\coth z=z^{-1}(1+z^2/3+0(z^4));\quad \operatorname{csch}z=z^{-1}(1-z^2/6+0(z^4));$$
$$\cosh z=1+\tfrac{1}{2}z^2+0(z^4);\quad \operatorname{csch}^2 z=z^{-2}(1-z^2/3+0(z^4)).$$

Here, "big-O" functions are defined as $z\to 0$. Then

$$U_n(\theta)=\frac{2n}{2n\theta}(1+4n^2\theta^2/3+\cdots)(1/2\theta)(1-2\theta^2/3+\cdots)-2(1+2\theta^2+\cdots)(1/4\theta^2)(1-4\theta^2/3+\cdots),$$

where the "..." notation refers to terms that are $0(\theta^4)$. Then

$$U_n(\theta)=\frac{1}{2\theta^2}\left[1+\frac{2}{3}(2n^2-1)\theta^2-1-\frac{2}{3}\theta^2\right]+0(\theta^2)$$

or

$$U_n(\theta)=\frac{2}{3}(n^2-1)+0(\theta^2). \tag{18}$$

It follows from (18) that $U_n(0)=S_n(0)$ (the limit exists) $=\frac{2}{3}(n^2-1)$, which is part (c).

Also solved by the proposer.

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THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

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The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

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Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Applications of Fibonacci Numbers, Volumes 1-4. Edited by G.E. Bergum, A.F. Horadam and A.N. Philippou

Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger, FA, 1993.

Please write to the Fibonacci Association, Santa Clara University, Santa Clara CA 95053, U.S.A., for more information and current prices.