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The primary function of THE FIBONACCI QUARTERLY is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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# SECOND DERIVATIVE SEQUENCES OF FIBONACCI AND LUCAS POLYNOMIALS 

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## 1. INTRODUCTION AND GENERALITIES

Let us consider the Fibonacci polynomials $U_{n}(x)$ and the Lucas polynomials $V_{n}(x)$ (or simply $U_{n}$ and $V_{n}$, when no misunderstanding can arise) defined by the second-order linear recurrence relations

$$
\begin{equation*}
U_{n}=x U_{n-1}+U_{n-2}\left(U_{0}=0, U_{1}=1\right), \tag{1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{n}=x V_{n-1}+V_{n-2} \quad\left(V_{0}=2, V_{1}=x\right) \tag{1.2}
\end{equation*}
$$

where $x$ is an indeterminate. It is well known that the polynomials $U_{n}$ and $V_{n}$, can be expressed by means of the Binet forms

$$
\begin{equation*}
U_{n}=\left(\alpha^{n}-\beta^{n}\right) / \Delta \tag{1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{n}=\alpha^{n}+\beta^{n}, \tag{1.4}
\end{equation*}
$$

where

$$
\begin{align*}
& \Delta=\sqrt{x^{2}+4} \\
& \alpha=(x+\Delta) / 2  \tag{1.5}\\
& \beta=(x-\Delta) / 2=-1 / \alpha=x-\alpha
\end{align*}
$$

Recall that further expressions for $U_{n}$ and $V_{n}$, (e.g., see [1], [3]) are
and

$$
\begin{equation*}
U_{n}=\sum_{j=0}^{\lfloor(n-1) / 2\rfloor}\binom{n-1-j}{j} x^{n-1-2 j}(n \geq 1) \tag{1.6}
\end{equation*}
$$

$$
\begin{equation*}
V_{n}=\sum_{j=0}^{\lfloor n / 2\rfloor} \frac{n}{n-j}\binom{n-j}{j} x^{n-2 j} \quad(n \geq 1) \tag{1.7}
\end{equation*}
$$

where $\lfloor a\rfloor$ denotes the greatest integer not exceeding $a$.
In [4] we considered the numbers $F_{n}^{(1)}$ and $L_{n}^{(1)}$ obtainable by taking the first derivative of the polynomials (1.6) and (1.7) at $x=1$, and studied their properties. The basic results established in [4] are

$$
\begin{equation*}
F_{n}^{(1)}=\left[\frac{d}{d x} U_{n}(x)\right]_{x=1}=\left(n L_{n}-F_{n}\right) / 5 \tag{1.8}
\end{equation*}
$$

$$
\begin{equation*}
L_{n}^{(1)}=\left[\frac{d}{d x} V_{n}(x)\right]_{x=1}=n F_{n}, \tag{1.9}
\end{equation*}
$$

where $F_{n}$ and $L_{n}$ are the usual Fibonacci and Lucas numbers, respectively. Observe that the numbers $F_{n}^{(1)}$ and $L_{n}^{(1)}$ are, respectively, denoted by $F_{n}^{\prime}$ and $L_{n}^{\prime}$ in [4].

In this paper we consider the second derivative with respect to $x$ of the polynomials (1.6) and (1.7) and investigate some of their properties, thus keeping, in part, the promise made to the reader in section 4 of [4]. In the concluding section, we offer a brief glimpse of the implications of investigating the $k^{\text {th }}$ derivatives of $U_{n}(x)$ and $V_{n}(x)$

### 1.1 Definitions

Let us define the polynomials $U_{n}^{(2)}$ and $V_{n}^{(2)}$, which are also obtainable from (1.6) and (1.7), as

$$
\begin{equation*}
U_{n}^{(2)}=\frac{d^{2}}{d x^{2}} U_{n}=\sum_{j=0}^{\lfloor(n-3) / 2\rfloor}(n-1-2 j)(n-2-2 j)\binom{n-1-j}{j} x^{n-3-2 j} \quad(n \geq 1), \tag{1.10}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{n}^{(2)}=\frac{d^{2}}{d x^{2}} V_{n}=\sum_{j=0}^{\lfloor(n-2) / 2\rfloor} \frac{n(n-2 j)(n-1-2 j)}{n-j}\binom{n-j}{j} x^{n-2-2 j} \quad(n \geq 1) . \tag{1.11}
\end{equation*}
$$

Observe that

$$
\begin{equation*}
U_{0}^{(2)}=V_{0}^{(2)}=0[\text { from (1.1) and (1.2) }] \tag{1.12}
\end{equation*}
$$

and

$$
\begin{equation*}
U_{1}^{(2)}=U_{2}^{(2)}=V_{1}^{(2)}=0 \tag{1.12}
\end{equation*}
$$

according to the convention that a sum vanishes whenever the upper range indicator is less than the lower one. From (1.10)-(1.12) we can write the first few elements of the sequences $\left\{U_{n}^{(2)}\right\}_{0}^{\infty}$ and $\left\{V_{n}^{(2)}\right\}_{0}^{\infty}$, namely,

$$
\begin{array}{l|l}
U_{0}^{(2)}=U_{1}^{(2)}=U_{2}^{(2)}=0 & \begin{array}{l}
(2)=V_{1}^{(2)}=0 \\
U_{3}^{(2)}=2
\end{array} \\
U_{4}^{(2)}=6 x & V_{2}^{(2)}=2 \\
U_{3}^{(2)}=12 x^{2}+6 & V_{3}^{(2)}=6 x \\
U_{4}^{(2)}=120 x^{3}+24 x & V_{s}^{(2)}=20 x^{3}+30 x \\
U_{1}^{(2)}=30 x^{4}+60 x^{2}+12 & V_{6}^{(2)}=30 x^{4}+72 x^{2}+18 \\
U_{8}^{(2)}=42 x^{5}+120 x^{3}+60 x & V_{\substack{(2)} 42 x^{5}+140 x^{3}+84 x}^{U_{8}^{(2)}=56 x^{6}+210 x^{4}+180 x^{2}+20}  \tag{1.13}\\
U_{10}^{(2)}=72 x^{7}+336 x^{5}+420 x^{3}+120 x=56 x^{6}+240 x^{4}+240 x^{2}+32 \\
& V_{9}^{(2)}=72 x^{7}+378 x^{5}+540 x^{3}+180 x \\
V_{10}^{(2)}=90 x^{8}+560 x^{6}+1050 x^{4}+600 x^{2}+50 .
\end{array}
$$

In this paper we confine ourselves to studying some properties of the above sequences for the case $x=1$. Since, letting $x=1$ in (1.1)-(1.5), we have the usual Fibonacci and Lucas numbers, the sequences of integers $\left\{U_{n}^{(2)}(1)\right\}$ and $\left\{V_{n}^{(2)}(1)\right\}$ will be denoted by $\left\{F_{n}^{(2)}\right\}$ and $\left\{L_{n}^{(2)}\right\}$ and defined as Fibonacci and Lucas second derivative sequences, respectively.

From (1.13), the first few values of $F_{n}^{(2)}$ and $L_{n}^{(2)}$ are

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $F_{n}^{(2)}$ | 0 | 0 | 0 | 2 | 6 | 18 | 44 | 102 | 222 | 466 | 948 |
| $L_{n}^{(2)}$ | 0 | 0 | 2 | 6 | 20 | 50 | 120 | 266 | 568 | 1170 | 2350 |

A large number of relationships involving $F_{n}^{(2)}, L_{n}^{(2)}, F_{n}^{(1)}, L_{n}^{(1)}, F_{n}$ and $L_{n}$ will be exhibited in the following sections. Their proofs are not very complicated but they are rather lengthy, so, for the sake of brevity, only some of them will be given in full detail.

## 2. EXPRESSIONS FOR $F_{n}^{(2)}$ AND $L_{n}^{(2)}$ IN TERMS OF FIBONACCI AND LUCAS NUMBERS

Expressions for $F_{n}^{(2)}$ and $L_{n}^{(2)}$ in terms of $U_{n}$ and $V_{n}$ can be obtained from the definitions (1.10) and (1.11) and the Binet forms (1.3)-(1.5). Letting the bracketed superscript ( ${ }^{k}$ ) denote the $k^{\text {th }}$ derivative with respect to $x$ and taking into account the results established in section 2 of [4], we can write

$$
\begin{align*}
U_{n}^{(2)} & =\frac{d^{2}}{d x^{2}} \frac{\alpha^{n}-\beta^{n}}{\Delta}=\frac{d}{d x} U_{n}^{(1)}=\frac{d}{d x} \frac{n\left(\alpha^{n}+\beta^{n}\right) \Delta-x\left(\alpha^{n}-\beta^{n}\right)}{\Delta^{3}} \\
& =\frac{\left[n\left(\alpha^{n}+\beta^{n}\right) \Delta-x\left(\alpha^{n}-\beta^{n}\right)\right]^{(1)} \Delta^{3}-\left(\Delta^{3}\right)^{(1)}\left[n\left(\alpha^{n}+\beta^{n}\right) \Delta-x\left(\alpha^{n}-\beta^{n}\right)\right]}{\Delta^{6}} \\
& =\frac{\left[\left(n^{2}-1\right) \Delta U_{n}\right] \Delta^{3}-3 x \Delta\left[n \Delta V_{n}-x \Delta U_{n}\right]}{\Delta^{6}}=\frac{\left[\left(n^{2}-1\right) \Delta^{2}+3 x^{2}\right] U_{n}-3 n x V_{n}}{\Delta^{4}} . \tag{2.1}
\end{align*}
$$

Analogously, we have

$$
\begin{align*}
V_{n}^{(2)} & =\frac{d^{2}}{d x^{2}}\left(\alpha^{n}+\beta^{n}\right)=\frac{d}{d x} V_{n}^{(1)}=\frac{d}{d x} \frac{n\left(\alpha^{n}-\beta^{n}\right)}{\Delta} \\
& =n \frac{\left[\left(\alpha^{n}\right)^{(1)}-\left(\beta^{n}\right)^{(1)}\right] \Delta-\Delta^{(1)}\left(\alpha^{n}-\beta^{n}\right)}{\Delta^{2}} \\
& =n \frac{n \alpha^{n}+n \beta^{n}-x\left(\alpha^{n}-\beta^{n}\right) / \Delta}{\Delta^{2}}=\frac{n\left(n V_{n}-x U_{n}\right)}{\Delta^{2}} . \tag{2.2}
\end{align*}
$$

Letting $x=1$ in (2.1) and (2.2) yields

$$
\begin{equation*}
F_{n}^{(2)}=\frac{\left(5 n^{2}-2\right) F_{n}-3 n L_{n}}{25} \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{n}^{(2)}=\frac{n\left(n L_{n}-F_{n}\right)}{5} \tag{2.4}
\end{equation*}
$$

whence the expressions for negative-subscripted elements of the Fibonacci and Lucas second derivative sequences can be easily deduced, namely,

$$
\begin{equation*}
F_{-n}^{(2)}=(-1)^{n+1} F_{n}^{(2)} \tag{2.5}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{-n}^{(2)}=(-1)^{n} L_{n}^{(2)} . \tag{2.6}
\end{equation*}
$$

Observe that, from (1.8), (1.9), (2.3), and (2.4), we get the following equivalent expressions for $F_{n}^{(2)}$ and $L_{n}^{(2)}$ :

$$
\begin{equation*}
F_{n}^{(2)}=\left(n L_{n}^{(1)}-3 F_{n}^{(1)}-F_{n}\right) / 5, \tag{2.7}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{n}^{(2)}=n F_{n}^{(1)} . \tag{2.8}
\end{equation*}
$$

## 3. SOME IDENTITIES INVOLVING THE NUMBERS $F_{n}^{(2)}$ AND $L_{n}^{(2)}$

Some simple properties of the numbers $F_{n}^{(2)}$ and $L_{n}^{(2)}$ can be derived from (1.8), (1.9), and (2.3)-(2.8). First, let us state the following four identities.

Identity 1: $F_{n+m}^{(2)}+(-1)^{m} F_{n-m}^{(2)}=L_{m} F_{n}^{(2)}+F_{n} L_{m}^{(2)}+2 m F_{m} F_{n}^{(1)}$.
Identity 2: $F_{n+m}^{(2)}-(-1)^{m} F_{n-m}^{(2)}=F_{m} L_{n}^{(2)}+L_{n} F_{m}^{(2)}+2 n F_{n} F_{m}^{(1)}$.
Identity 3: $L_{n+m}^{(2)}+(-1)^{m} L_{n-m}^{(2)}=L_{m} L_{n}^{(2)}+L_{n} L_{m}^{(2)}+2 L_{n}^{(1)} L_{m}^{(1)}$.
Identity 4: $L_{n+m}^{(2)}-(-1)^{m} L_{n-m}^{(2)}=n L_{n} F_{m}^{(1)}+m L_{m} F_{n}^{(1)}+\left(n^{2}+m^{2}\right) F_{n} F_{m}$.
For the sake of brevity, we shall prove only Identity 1.
Proof of Identity 1: From (2.3) we write

$$
\begin{align*}
F_{n+m}^{(2)}+(-1)^{m} F_{n-m}^{(2)}= & \left\{\left[5(n+m)^{2}-2\right] F_{n+m}-3(n+m) L_{n+m}\right. \\
& \left.+(-1)^{m}\left[5(n-m)^{2}-2\right] F_{n-m}-3(-1)^{m}(n-m) L_{n-m}\right] / 25 \\
= & \left\{\left[5\left(n^{2}+m^{2}\right)-2\right]\left[F_{n+m}+(-1)^{m} F_{n-m}\right]+10 n m\left[F_{n+m}-(-1)^{m} F_{n-m}\right]\right. \\
& \left.-3 n\left[L_{n+m}+(-1)^{m} L_{n-m}\right]-3 m\left[L_{n+m}-(-1)^{m} L_{n-m}\right]\right\} / 25 . \tag{3.1}
\end{align*}
$$

After some manipulations involving the use of (2.3), (2.4), (1.8), and the identities $\mathrm{I}_{21}-\mathrm{I}_{24}$ [5, page 59] a compact form of which is

$$
\begin{aligned}
& F_{h+k}+(-1)^{k} F_{h-k}=F_{h} L_{k} \\
& F_{h+k}-(-1)^{k} F_{h-k}=L_{h} F_{k},
\end{aligned}
$$

the identity (3.1) can be rewritten as

$$
\begin{aligned}
F_{n+m}^{(2)}+(-1)^{m} F_{n-m}^{(2)} & =\left[5\left(n^{2}+m^{2}\right) F_{n} L_{m}+10 n m F_{m} L_{n}-2 F_{n} L_{m}-3 n L_{n} L_{m}-15 m F_{n} F_{m}\right] / 25 \\
& =L_{m}\left[\left(5 n^{2}-2\right) F_{n}-3 n L_{n}\right] / 25+m F_{n}\left(m L_{m}-3 F_{m}\right) / 5+2 n m F_{m} L_{n} / 5 \\
& =L_{m} F_{n}^{(2)}+m F_{n}\left(m L_{m}-F_{m}\right) / 5-2 m F_{n} F_{m} / 5+2 n m F_{m} L_{n} / 5 \\
& =L_{m} F_{n}^{(2)}+F_{n} L_{m}^{(2)}+2 m F_{m}\left(n L_{n}-F_{n}\right) / 5 \\
& =L_{m} F_{n}^{(2)}+F_{n} L_{m}^{(2)}+2 m F_{m} F_{n}^{(1)} .
\end{aligned}
$$

Particular cases of Identities 1-4 are
Identity 5 ( $m=1$ in Id. 2): $F_{n-1}^{(2)}+F_{n+1}^{(2)}=L_{n}^{(2)}$.
Identity $6\left(m=1\right.$ in Id. 4): $L_{n-1}^{(2)}+L_{n+1}^{(2)}=F_{n}^{(1)}+\left(n^{2}+1\right) F_{n}=F_{n}^{(1)}+n L_{n}^{(1)}+F_{n}$.
Identity $7\left(m=2\right.$ in Id. 2): $F_{n+2}^{(2)}-F_{n-2}^{(2)}=L_{n}^{(2)}+2 L_{n}^{(1)}$.
Identity 8 ( $n=m$ in Id. 2): $F_{2 m}^{(2)}=3 F_{m} L_{m}^{(2)}+L_{m} F_{m}^{(2)}$.
Identity $9\left(n=m\right.$ in Id. 3): $L_{2 m}^{(2)}=2\left[L_{m} L_{m}^{(2)}+\left(L_{m}^{(1)}\right)^{2}\right]$.
Identity $10\left(n=2 m\right.$ in Id. 2): $F_{3 m}^{(2)}=F_{m}\left[L_{2 m}^{(2)}+4 m L_{m} F_{m}^{(1)}\right]+\left[L_{2 m}+(-1)^{m}\right] F_{m}^{(2)}$.
Identity $11\left(n=2 m\right.$ in Id. 3): $L_{3 m}^{(2)}=3\left\{L_{m}^{(2)}\left[L_{2 m}+(-1)^{m}\right]+2 L_{m}\left(L_{m}^{(1)}\right)^{2}\right\}$.
Next, we derive
Identity 12: $F_{n}^{(1)} L_{n}^{(2)}-L_{n}^{(1)} F_{n}^{(2)}=\left[F_{n}\left(5 L_{n}^{(2)}+4 L_{n}^{(1)}\right)+4(-1)^{n} n^{3}\right] / 25$.
Proof: From (1.8), (1.9), (2.7), and (2.8), we have

$$
\begin{equation*}
F_{n}^{(1)} L_{n}^{(2)}-L_{n}^{(1)} F_{n}^{(2)}=\left[5 n\left(F_{n}^{(1)}\right)^{2}-n\left(L_{n}^{(1)}\right)^{2}+3 F_{n}^{(1)} L_{n}^{(1)}+F_{n} L_{n}^{(1)}\right] / 5 . \tag{3.2}
\end{equation*}
$$

Using the identities

$$
\begin{gather*}
\left(F_{n}^{(1)}\right)^{2}=\left(n^{2} L_{n}^{2}+F_{n}^{2}-2 n F_{2 n}\right) / 25,  \tag{3.3}\\
\left(L_{n}^{(1)}\right)^{2}=n^{2} F_{n}^{2},  \tag{3.4}\\
F_{n}^{(1)} L_{n}^{(1)}=n\left(n F_{2 n}-F_{n}^{2}\right) / 5,  \tag{3.5}\\
F_{n} L_{n}^{(1)}=n F_{n}^{2}, \tag{3.6}
\end{gather*}
$$

and the identity $\mathrm{I}_{12}$ [5, page 56] [namely, $\left.5 F_{k}^{2}=L_{k}^{2}-4(-1)^{k}\right]$, we find that (3.2) becomes

$$
\begin{aligned}
F_{n}^{(1)} L_{n}^{(2)}-L_{n}^{(1)} F_{n}^{(2)} & =\left(\frac{n^{3} L_{n}^{2}+n F_{n}^{2}-2 n^{2} F_{2 n}}{5}-n^{3} F_{n}^{2}+\frac{3 n^{3} F_{2 n}-3 n F_{n}^{2}}{5}+n F_{n}^{2}\right) / 5 \\
& =\left[n^{3}\left(L_{n}^{2}-5 F_{n}^{2}\right)+n^{2} F_{2 n}+3 n F_{n}^{2}\right] / 25=\left[4(-1)^{n} n^{3}+n^{2} F_{2 n}+3 n F_{n}^{2}\right] / 25 \\
& =\left[n F_{n}\left(n L_{n}+3 F_{n}\right)+4(-1)^{n} n^{3}\right] / 25=\left[5 F_{n} L_{n}^{(2)}+4 n F_{n}^{2}+4(-1)^{n} n^{3}\right] / 25 \\
& =\left[F_{n}\left(5 L_{n}^{(2)}+4 L_{n}^{(1)}\right)+4(-1)^{n} n^{3}\right] / 25 .
\end{aligned}
$$

Let us conclude this section by giving the Simson formula analogs for $F_{n}^{(2)}$ and $L_{n}^{(2)}$.
Identity 13: $\left(F_{n}^{(2)}\right)^{2}-F_{n-1}^{(2)} F_{n+1}^{(2)}=\frac{2 n^{2} L_{2 n}-6 n F_{2 n}+8 F_{n}^{2}-n^{2}(-1)^{n}\left(5 n^{2}-13\right)}{125}$.
Identity 14: $\left(L_{n}^{(2)}\right)^{2}-L_{n-1}^{(2)} L_{n+1}^{(2)}=\frac{2 n^{2} L_{2 n}-2 n F_{2 n}-4 F_{n}^{2}+5 n^{2}(-1)^{n}\left(n^{2}-1\right)}{25}$.

## SECOND DERIVATIVE SEQUENCES OF FIBONACCI AND LUCAS POLYNOMIALS

Proof of Identity 14: Using (2.4) and identities $\mathrm{I}_{19}, \mathrm{I}_{20}$ [5, page 59],

$$
\begin{aligned}
& F_{h-k} F_{h+k}-F_{n}^{2}=(-1)^{h+k+1} F_{k}^{2} \\
& L_{h-k} L_{h+k}-L_{h}^{2}=5(-1)^{h+k} F_{k}^{2}
\end{aligned}
$$

we can write

$$
\begin{align*}
\left(L_{n}^{(2)}\right)^{2}-L_{n-1}^{(2)} L_{n+1}^{(2)}= & n^{2}\left(n L_{n}-F_{n}\right)^{2} / 25-\left(n^{2}-1\right)\left[(n-1) L_{n-1}-F_{n-1}\right]\left[(n+1) L_{n+1}-F_{n+1}\right] / 25 \\
= & n^{2}\left(n^{2} L_{n}^{2}+F_{n}^{2}-2 n F_{2 n}\right) / 25-\left(n^{2}-1\right)\left\{\left(n^{2}-1\right)\left[L_{n}^{2}-5(-1)^{n}\right]\right. \\
& \left.-(n-1)\left[F_{2 n}-(-1)^{n}\right]-(n+1)\left[F_{2 n}+(-1)^{n}\right]+F_{n}^{2}+(-1)^{n}\right\} / 25 \tag{3.7}
\end{align*}
$$

After some manipulations involving the use of $\mathrm{I}_{12}$ [5, page 56$]$ and the identities $\mathrm{I}_{15}, \mathrm{I}_{18}$ [5, page 59] a compact form of which is $L_{2 h}+2(-1)^{h}=L_{h}^{2}$, the identity (3.7) can be rewritten as

$$
\begin{aligned}
\left(L_{n}^{(2)}\right)^{2}-L_{n-1}^{(2)} L_{n+1}^{(2)} & =\left[\left(2 n^{2}-1\right) L_{n}^{2}-2 n F_{2 n}+F_{n}^{2}+(-1)^{n}\left(5 n^{4}-9 n^{2}+4\right)\right] / 25 \\
& =\left[2 n^{2} L_{2 n}-2 n F_{2 n}+F_{n}^{2}-L_{n}^{2}+4(-1)^{n}+5 n^{2}(-1)^{n}\left(n^{2}-1\right)\right] / 25 \\
& =\left[2 n^{2} L_{2 n}-2 n F_{2 n}-4 F_{n}^{2}+5 n^{2}(-1)^{n}\left(n^{2}-1\right)\right] / 25 .
\end{aligned}
$$

Sirnson formula analogs for $U_{n}^{(2)}$ and $V_{n}^{(2)}$ may be obtained from (2.1) and (2.2), but their discovery is left to the perseverance of the reader.

## 4. SOME SIMPLE CONGRUENCE PROPERTIES OF $\boldsymbol{F}_{n}^{(2)}$ AND $\mathbb{L}_{n}^{(2)}$

Letting $m=1$ in Identity 1 and Identity 3, we obtain

$$
\begin{equation*}
F_{n+1}^{(2)}-F_{n-1}^{(2)}=F_{n}^{(2)}+2 F_{n}^{(1)} \tag{4.1}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{n+1}^{(2)}-L_{n-1}^{(2)}=L_{n}^{(2)}+2 L_{n}^{(1)} \tag{4.2}
\end{equation*}
$$

respectively. From (4.1) and (4.2), the recurrence relations

$$
\begin{equation*}
F_{n}^{(2)}=F_{n-1}^{(2)}+F_{n-2}^{(2)}+2 F_{n-1}^{(1)} \quad\left(F_{0}^{(2)}=F_{1}^{(2)}=0\right) \tag{4.3}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{n}^{(2)}=L_{n-1}^{(2)}+L_{n-2}^{(2)}+2 L_{n-1}^{(1)} \quad\left(L_{0}^{(2)}=L_{1}^{(2)}=0\right) \tag{4.4}
\end{equation*}
$$

can be readily obtained, where the initial conditions have been taken from (1.14). The relations (4.3) and (4.4) allow us to state the following proposition.

Proposition 1: $F_{n}^{(2)}$ and $L_{n}^{(2)}$ are even for all $n$.
Further congruence properties of $F_{n}^{(2)}$ and $L_{n}^{(2)}$ can be easily established.
Proposition 2: $F_{n}^{(2)} \equiv 0(\bmod 6)$ for $n \equiv 0, \pm 1, \pm 2, \pm 4, \pm 5(\bmod 12)$.
Proposition 3: $L_{n}^{(2)} \equiv 0(\bmod 6)$ for $n \equiv 0(\bmod 3)$ or $n \equiv \pm 1(\bmod 12)$.
Proposition 4: $L_{n}^{(2)} \equiv 0(\bmod 10)$ for $n \equiv 0, \pm 1(\bmod 5)$.

The proofs of Propositions 2-4 are similar, so, for the sake of brevity, we shall prove only Proposition 3.

Proof of Proposition 3: From (2.4) and Proposition 1, it is apparent that we have to find conditions for $n\left(n L_{n}-F_{n}\right)$ to be divisible by 3 . The first condition is trivial: $n \equiv 0(\bmod 3)$. The second condition is given by the solution of the congruence $n L_{n} \equiv F_{n}(\bmod 3)$. The repetition period of the sequences $\left\{\left\langle F_{n}\right\rangle_{3}\right\}$ and $\left\{\left\langle L_{n}\right\rangle_{3}\right\}$ (the Fibonacci and Lucas sequences reduced modulo 3 ) is 8 (see [2, page 55]), whereas the repetition period of the sequence of naturals reduced modulo 3 is 3 . Since l.c.m. $(3,8)=24$, we have to inspect the elements of the sequences $\left\{\left\langle n L_{n}\right\rangle_{3}\right\}_{0}^{23}$ and $\left\{\left\langle F_{n}\right\rangle_{3}\right\}_{0}^{23}$ and look for the equality

$$
\begin{equation*}
\left\langle n L_{n}\right\rangle_{3}=\left\langle F_{n}\right\rangle_{3} \tag{4.5}
\end{equation*}
$$

It is readily seen that $(4.5)$ is fulfilled for $n \equiv 0, \pm 1(\bmod 12)$.

## 5. EVALUATION OF SOME SERIES INVOLVING $F_{n}^{(2)}$ AND $\mathbb{L}_{n}^{(2)}$

In this section, several finite series involving $F_{n}^{(2)}$ and $L_{n}^{(2)}$ are considered and closed form expressions for their sums are exhibited. For the sake of brevity, only a few among them are proved in detail by using some results obtained in [4] and the further identities

$$
\begin{gather*}
\sum_{i=0}^{n} i(-1)^{i} F_{n-2 i}=-\left(n L_{n+1}+2 F_{n}\right) / 5=-F_{n+1}^{(1)}  \tag{5.1}\\
\sum_{i=0}^{n} i(-1)^{i} L_{n-2 i}=n F_{n+1}=L_{n+1}^{(1)}-F_{n+1}  \tag{5.2}\\
\sum_{i=0}^{n} F_{i} F_{n-i}=\left(n L_{n}-F_{n}\right) / 5=F_{n}^{(1)}  \tag{5.3}\\
\sum_{i=0}^{n} F_{i} L_{n-i}=(n+1) F_{n}=L_{n}^{(1)}+F_{n} \tag{5.4}
\end{gather*}
$$

The proofs of (5.1)-(5.4) can be carried out with the aid of the Binet forms (1.3)-(1.5) and [4, (3.1)]. Since they are rather tedious, they are omitted in this context.

### 5.1. Results

The following results have been obtained.
Proposition 5: $\sum_{i=0}^{n} F_{i}^{(2)}=F_{n+2}^{(2)}-2\left(F_{n+3}^{(1)}-F_{n+4}+1\right)$.
Proposition 6: $\sum_{i=0}^{n} L_{i}^{(2)}=L_{n+2}^{(2)}-2\left(L_{n+3}^{(1)}-L_{n+4}+2\right)$.
Proposition 7: $\sum_{i=0}^{n}\binom{n}{i} F_{i}^{(2)}=\left[5 n^{2} F_{2 n-2}-(3 n+2) F_{2 n}+n F_{2 n-7}\right] / 25$.

Proposition 8: $\sum_{i=0}^{n}\binom{n}{i} L_{i}^{(2)}=n\left[(n-1) L_{2 n-2}+2 F_{2 n-2}\right] / 5$.
We point out that several equivalent expressions for the above sums can be given. For example, we have
Proposition 8': $\sum_{i=0}^{n}\binom{n}{i} L_{i}^{(2)}=L_{n-1}\left(L_{n-1}^{(2)}+F_{n-1}^{(1)}\right)+n\left[3 F_{2 n-2}+2(n-1)(-1)^{n}\right] / 5$.
Finally, the following convolution identities have been established.
Proposition 9: $\sum_{i=0}^{n} F_{i}^{(1)} F_{n-i}=\frac{1}{2} F_{n}^{(2)}$.
Proposition 10: $\sum_{i=0}^{n} L_{i}^{(1)} F_{n-i}=\frac{1}{2} L_{n}^{(2)}$.
Proposition 11: $\sum_{i=0}^{n} F_{i}^{(1)} L_{n-i}=\frac{1}{2} L_{n}^{(2)}+F_{n}^{(1)}$.
Proposition 12: $\sum_{i=0}^{n} L_{i}^{(1)} L_{n-i}=\frac{5}{2} F_{n}^{(2)}+2 F_{n}^{(1)}+L_{n}^{(1)}+F_{n}$.

### 5.2 Proofs

Proof of Proposition 5: From (2.7), (1.8), and (1.9), we have

$$
\begin{equation*}
A_{n}=\sum_{i=0}^{n} F_{i}^{(2)}=\frac{1}{5}\left(\sum_{i=0}^{n} i L_{i}^{(1)}-3 \sum_{i=0}^{n} F_{i}^{(1)}-\sum_{i=0}^{n} F_{i}\right)=\frac{1}{5}\left(\sum_{i=0}^{n} i^{2} F_{i}-\frac{3}{5} \sum_{i=0}^{n} i L_{i}-\frac{2}{5} \sum_{i=0}^{n} F_{i}\right) \tag{5.5}
\end{equation*}
$$

Using the Binet forms (1.3)-(1.5) (with $x=1$ ), [4, (3.1) and (3.2)] and identity $\mathrm{I}_{1}$ [5, page 52]

$$
\sum_{i=1}^{k} F_{i}=F_{k+2}-1
$$

we find that (5.5) becomes

$$
\begin{aligned}
A_{n}= & \frac{1}{5}\left[\frac{1}{\sqrt{5}}\left(\sum_{i=0}^{n} i^{2} \alpha^{i}-\sum_{i=0}^{n} i^{2} \beta^{i}\right)-\frac{3}{5}\left(\sum_{i=0}^{n} i \alpha^{i}+\sum_{i=0}^{n} i \beta^{i}\right)-\frac{2}{5}\left(F_{n+2}-1\right)\right] \\
=\frac{1}{5} & {\left[\frac { 1 } { \sqrt { 5 } } \left(\frac{n^{2} \alpha^{n+3}-\left(2 n^{2}+2 n-1\right) \alpha^{n+2}+(n+1)^{2} \alpha^{n+1}-\alpha^{2}-\alpha}{-\beta^{3}}\right.\right.} \\
& \left.-\frac{n^{2} \beta^{n+3}-\left(2 n^{2}+2 n-1\right) \beta^{n+2}+(n+1)^{2} \beta^{n+1}-\beta^{2}-\beta}{-\alpha^{3}}\right) \\
& \left.-\frac{3}{5}\left(\frac{n \alpha^{n+2}-(n+1) \alpha^{n+1}+\alpha}{\beta^{2}}+\frac{n \beta^{n+2}-(n+1) \beta^{n+1}+\beta}{\alpha^{2}}\right)\right]
\end{aligned}
$$

$$
\begin{align*}
& =\frac{1}{5}\left\{n^{2} F_{n+6}-\left(2 n^{2}+2 n-1\right) F_{n+5}+(n+1)^{2} F_{n+4}-8-\frac{3}{5}\left[n L_{n+4}-(n+1) L_{n+3}+4\right]-\frac{2}{5}\left(F_{n+2}-1\right)\right\} \\
& =\frac{1}{5}\left[-n^{2} F_{n+3}-2 n F_{n+3}+F_{n+6}+n^{2} F_{n+4}-\frac{3}{5}\left(n L_{n+2}-L_{n+3}\right)-\frac{2}{5} F_{n+2}-10\right] \\
& =\frac{1}{5}\left[n^{2} F_{n+2}-2 n F_{n+3}+F_{n+6}-\frac{3}{5}\left(n L_{n+2}-2 L_{n+2}-F_{n+2}\right)-F_{n+2}+\frac{3}{5}\left(L_{n+3}+2 L_{n+2}\right)-10\right] \\
& =\frac{1}{5}\left\{\left(n^{2}-1\right) F_{n+2}-2 n F_{n+3}+F_{n+6}-\frac{3}{5}\left[(n+2) L_{n+2}-F_{n+2}\right]+3 F_{n+3}-10\right\} \\
& =\frac{1}{5}\left[\left(n^{2}-1\right) F_{n+2}-2 n F_{n+3}+3 F_{n+3}+F_{n+6}-3 F_{n+2}^{(1)}-10\right] \\
& =\frac{1}{25}\left[\left(5 n^{2}-2\right) F_{n+2}-3 n L_{n+2}-10 n F_{n+3}-6 L_{n+2}+5\left(3 F_{n+3}+F_{n+6}\right)-50\right] \tag{5.6}
\end{align*}
$$

The equality (5.6) can be rewritten as

$$
\begin{aligned}
A_{n} & =\frac{1}{25}\left\{\left[5(n+2)^{2}-2\right] F_{n+2}-3(n+2) L_{n+2}-20(n+1) F_{n+2}-10 n F_{n+3}+10 L_{n+4}-50\right\} \\
& =F_{n+2}^{(2)}-\frac{1}{25}\left[10 n\left(2 F_{n+2}+F_{n+3}\right)+10\left(2 F_{n+2}-L_{n+4}\right)+50\right] \\
& =F_{n+2}^{(2)}-\frac{1}{5}\left[2 n L_{n+3}-2\left(L_{n+4}-2 F_{n+2}\right)\right]-2=F_{n+2}^{(2)}+\frac{2}{5}\left(F_{n+5}-F_{n}-n L_{n+3}\right)-2 \\
& =F_{n+2}^{(2)}-2 F_{n+3}^{(1)}+\frac{2}{5}\left(F_{n+4}-F_{n}+3 L_{n+3}\right)-2=F_{n+2}^{(2)}-2 F_{n+3}^{(1)}+2 F_{n+4}-2 .
\end{aligned}
$$

Proof of Proposition 7: From (2.7), we can write

$$
\begin{equation*}
B_{n}=\sum_{i=0}^{n}\binom{n}{i} F_{i}^{(2)}=\frac{1}{5}\left[\sum_{i=0}^{n}\binom{n}{i} i L_{i}^{(1)}-3 \sum_{i=0}^{n}\binom{n}{i} F_{i}^{(1)}-\sum_{i=0}^{n}\binom{n}{i} F_{i}\right] \tag{5.7}
\end{equation*}
$$

Now, from [4, (3.5), (3.10), (3.3)], we have

$$
\begin{gather*}
\sum_{i=0}^{n}\binom{n}{i} i L_{i}^{(1)}=\sum_{i=0}^{n}\binom{n}{i} i^{2} F_{i}=n F_{2 n-1}+n(n-1) F_{2 n-2}  \tag{5.8}\\
\sum_{i=0}^{n}\binom{n}{i} F_{i}^{(1)}=F_{2 n-1}^{(1)} / 2=\frac{1}{10}\left[(2 n-1) L_{2 n-1}-F_{2 n-1}\right]  \tag{5.9}\\
\sum_{i=0}^{n}\binom{n}{i} F_{i}=F_{2 n} \tag{5.10}
\end{gather*}
$$

respectively. Therefore, from (5.8)-(5.10) and (1.8), (5.7) can be rewritten as

$$
\begin{aligned}
B_{n} & =\frac{1}{5}\left[n F_{2 n-1}+n(n-1) F_{2 n-2}-\frac{3(2 n-1) L_{2 n-1}-3 F_{2 n-1}}{10}-F_{2 n}\right] \\
& =\frac{1}{50}\left[10 n F_{2 n-1}+10 n^{2} F_{2 n-2}-10 n F_{2 n-2}-6 n L_{2 n-1}+3 L_{2 n-1}+3 F_{2 n-1}-10 F_{2 n}\right] \\
& =\frac{1}{25}\left[5 n^{2} F_{2 n-2}+n\left(5 F_{2 n-1}-5 F_{2 n-2}-3 L_{2 n-1}\right)-2 F_{2 n}\right] \\
& =\frac{1}{25}\left[5 n^{2} F_{2 n-2}+n\left(F_{2 n-7}-3 F_{2 n}\right)-2 F_{2 n}\right] .
\end{aligned}
$$

## 6. FURTHER RESEARCH

The first and the second derivatives of polynomials (1.6) and (1.7) have been considered in [4] and in this paper, respectively. More particularly, several properties of the sequences of integers obtainable by taking the above mentioned derivatives at $x=1$ have been investigated.

The generalization to the analogous sequences $\left\{F_{n}^{(k)}\right\}$ and $\left\{L_{n}^{(k)}\right\}$, defined as

$$
\begin{equation*}
F_{n}^{(k)}=\left[\frac{d^{k}}{d x^{k}} U_{n}(x)\right]_{x=1}=\sum_{j=0}^{\lfloor(n-k-1) / 2\rfloor}\left[\binom{n-1-j}{j} \prod_{i=1}^{k}(n-i-2 j)\right] \quad(n \geq 1) \tag{6.1}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{n}^{(k)}=\left[\frac{d^{k}}{d x^{k}} V_{n}(x)\right]_{x=1}=\sum_{j=0}^{\lfloor(n-k) / 2\rfloor}\left[\frac{n}{n-j}\binom{n-j}{j} \prod_{i=1}^{k}(n-i+1-2 j)\right] \quad(n \geq 1) \tag{6.2}
\end{equation*}
$$

(with $F_{0}^{(k)}=0$ for $k \geq 0$ and $L_{0}^{(k)}=0$ for $k \geq 1$ ), seems to be very interesting and will be the goal of a future work. In this section we confine ourselves to offering some conjectures about the properties of these sequences.
Conjecture 1: $L_{n}^{(k)}=n F_{n}^{(k-1)}$.
Conjecture 2: $L_{n}^{(k)}=(n-k+1) L_{n}^{(k-1)}-2\left(L_{n-1}^{(k)}+F_{n-1}^{(k-1)}\right)$.
Conjecture 3: $F_{n}^{(k)}=F_{n-1}^{(k)}+F_{n-2}^{(k)}+k F_{n-1}^{(k-1)}$.
Conjecture 4: $L_{n}^{(k)}=L_{n-1}^{(k)}+L_{n-2}^{(k)}+k L_{n-1}^{(k-1)}$.
Conjecture 5: $\quad F_{n-1}^{(k)}+F_{n+1}^{(k)}=L_{n}^{(k)}$.
Conjecture 6: $\quad F_{n}^{(k)} \equiv L_{n}^{(k)} \equiv 0(\bmod 2)$ for $k \geq 2$.
Conjecture 7: $L_{n}^{(k)} \equiv 0(\bmod n)$ for $k \geq 1$.
Moreover, we leave to the reader the proof of the following:

$$
\begin{align*}
L_{n}^{(n)} & =L_{n}^{(n-1)}=n!\quad(n \geq 1),  \tag{6.3}\\
L_{n}^{(n-2)} & =\frac{n+1}{2(n-1)} n!\quad(n \geq 2),  \tag{6.4}\\
L_{n}^{(n-3)} & =\frac{n+5}{6(n-1)} n!\quad(n \geq 3), \tag{6.5}
\end{align*}
$$

$$
\begin{equation*}
L_{n}^{(n-4)}=\frac{n+10}{24(n-2)} n!\quad(n \geq 4) \tag{6.6}
\end{equation*}
$$

Observe that (6.3)-(6.6) hold also for the minimum admissible value $v$ of $n$, for which one has $L_{v}^{(0)}=L_{v}$. Analogous identities for $F_{n}^{(k)}$ can be stated whence the validity of Conjecture 1 can be checked. More generally, all the conjectures and results presented above can be checked against the numerical triangles shown in Figures 1 and 2, which have been obtained by (6.1) and (6.2), respectively. It must be noted that $F_{n}^{(k)}=0$ for $k>n-1$, whereas $L_{n}^{(k)}=0$ for $k>n$.

| $\lambda k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 |  |  |  |  |  |  |  |  | 2 |  |  |  |  |  |  |  |  |
| 1 | 1 | 0 |  |  |  |  |  |  |  | 1 | 1 |  |  |  |  |  |  |  |
| 2 | 1 | 1 | 0 |  |  |  |  |  |  | 3 | 2 | 2 |  |  |  |  |  |  |
| 3 | 2 | 2 | 2 | 0 |  |  |  |  |  | 4 | 6 | 6 | 6 |  |  |  |  |  |
| 4 | 3 | 5 | 6 | 6 | 0 |  |  |  |  | 7 | 12 | 20 | 24 | 24 |  |  |  |  |
| 5 | 5 | 10 | 18 | 24 | 24 | 0 |  |  |  | 11 | 25 | 50 | 90 | 120 | 120 |  |  |  |
| 6 | 8 | 20 | 44 | 84 | 120 | 120 | 0 |  |  | 18 | 48 | 120 | 264 | 504 | 720 | 720 |  |  |
| 7 | 13 | 38 | 102 | 240 | 480 | 720 | 720 | 0 |  | 29 | 91 | 266 | 714 | 1680 | 3360 | 5040 | 5040 |  |
| 8 | 21 | 71 | 222 | 630 | 1560 | 3240 | 5040 | 5040 | 0 | 47 | 168 | 568 | 1776 | 5040 | 12480 | 25920 | 40320 | 40320 |

Fig. 1. Triangle $F_{n}^{(k)}(0 \leq n, k \leq 8)$
Fig. 2. Triangle $L_{n}^{(k)}(0 \leq n, k \leq 8)$
As indicated at the end of [4], the theory in this paper can be extended to cover Pell polynomials and numbers, and Pell-Lucas polynomials and numbers. In this case, we first replace $x$ by $2 x$ in (1.1) and (1.2), differentiate, and then put $x=1$.

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# RATIONAL CHEBYSHEV APPROXIMATIONS OF ANALYTIC FUNCTIONS 

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## 1. RATIONAL CHEBYSHEV APPROXIMATIONS OF ANALYTIC FUNCTIONS

We proceed to establish the main result of this paper: a general procedure to obtain rational Chevyshev approximations of analytic functions. Let $f(z)$ be analytic at $z_{0}$. Then, by composition, $g(z)=f\left(\cos z+z_{0}-1\right)$ is analytic at the origin. Hence, we can write

$$
\begin{equation*}
g(z)=f\left(\cos z+z_{0}-1\right)=\sum_{n=0}^{\infty} g^{(2 n)}(0) \frac{z^{2 n}}{(2 n)!} . \tag{1.1}
\end{equation*}
$$

If an explicit expansion of $f\left(\cos z+z_{0}-1\right)$ is not available, then successive coefficients in (1.1) are found directly from the formula for Maclaurin expansions, i.e., by simply calculating successive derivatives of (1.1) and setting $z=0$. To wit,

$$
\begin{align*}
& g(0)=f\left(z_{0}\right),  \tag{1.2}\\
& g^{\prime \prime}(0)=-f^{\prime}\left(z_{0}\right),  \tag{1.3}\\
& g^{(\mathrm{iv})}(0)= 3 f^{\prime \prime}\left(z_{0}\right)+f^{\prime}\left(z_{0}\right),  \tag{1.4}\\
& g^{(\mathrm{vi})}(0)=-15 f^{\prime \prime \prime}\left(z_{0}\right)-15 f^{\prime \prime}\left(z_{0}\right)-f^{\prime}\left(z_{0}\right),  \tag{1.5}\\
& g^{\text {(viii) }}(0)= 105 f^{(\mathrm{iv})}\left(z_{0}\right)+210 f^{\prime \prime \prime}\left(z_{0}\right)+63 f^{\prime \prime}\left(z_{0}\right)+f^{\prime}\left(z_{0}\right),  \tag{1.6}\\
& g^{(\mathrm{x})}(0)=-945 f^{(\mathrm{v})}\left(z_{0}\right)-3150 f^{(\mathrm{iv})}\left(z_{0}\right)-2205 f^{\prime \prime \prime}\left(z_{0}\right)-255 f^{\prime \prime}\left(z_{0}\right)-f^{\prime}\left(z_{0}\right),  \tag{1.7}\\
& g^{(\text {xii) }}(0)= 10395 f^{(\mathrm{vi)}}\left(z_{0}\right)+51975 f^{(\mathrm{v})}\left(z_{0}\right)+65835 f^{(\mathrm{ivv})}\left(z_{0}\right)+21120 f^{\prime \prime \prime}\left(z_{0}\right) \\
&+1023 f^{\prime \prime}\left(z_{0}\right)+f^{\prime}\left(z_{0}\right),  \tag{1.8}\\
& g^{\text {(xiv) }}(0)=-135135 f^{\text {(vii) }}\left(z_{0}\right)-945945 f^{(\mathrm{vi})}\left(z_{0}\right)-1891890 f^{(\mathrm{v})}\left(z_{0}\right)-1201200 f^{\text {(iv) }}\left(z_{0}\right) \\
&-195195 f^{\prime \prime \prime}\left(z_{0}\right)-4095 f^{\prime \prime}\left(z_{0}\right)-f^{\prime}\left(z_{0}\right), \tag{1.9}
\end{align*}
$$

etc.; the derivatives of odd order at the origin being at zero, since $g(z)$ is an even function of $z$.
Now, consider the expression

$$
\begin{align*}
g(z) \approx & A_{1} \cos z-A_{2} g(z) \cos z+A_{3} \cos 2 z-A_{4} g(z) \cos 2 z+\cdots \\
& +A_{2 s-1} \cos s z-A_{2 s} g(z) \cos s z \tag{1.10}
\end{align*}
$$

where the $A_{k}$ 's are constants to be determined, and the $\approx$ in (1.10) is to be interpreted in the sense that the Maclaurin expansions of both sides agree through the first $2 s$ terms.

Note that both sides of (1.10) are, of course, even, as they should be.

Observe that the Cauchy product of $g(z)$ and $\cos m z$ is

$$
\begin{equation*}
g(z) \cos m z=\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{g^{(2 n-2 k)}(0)(-1)^{k} m^{2 k} z^{2 n}}{(2 n-2 k)!(2 k)!} . \tag{1.11}
\end{equation*}
$$

Since $\cos m z$ is entire, the above Cauchy product will have the same circle of convergence that equation (1.1) has (see [4]).

Using (1.11) to equate powers of $z$ in (1.10) we find, after multiplying through by $(-1)^{n}(2 n)$ !,

$$
\begin{align*}
(-1)^{n} g^{(2 n)}(0)= & A_{1}-A_{2} \sum_{k=0}^{n}(-1)^{n-k}\binom{2 n}{2 k} g^{(2 n-2 k)}(0)+2^{2 n} A_{3} \\
& -A_{4} \sum_{k=0}^{n}(-1)^{n-k} 2^{2 k}\binom{2 n}{2 k} g^{(2 n-2 k)}(0)+\cdots+s^{2 n} A_{2 s-1} \\
& -A_{2 s} \sum_{k=0}^{n}(-1)^{n-k} s^{2 k}\binom{2 n}{2 k} g^{(2 n-2 k)}(0), \tag{1.12}
\end{align*}
$$

where $\binom{n}{k}$ is the binomial coefficient.
Letting $n=0,1,2, \ldots, 2 s-1$ in (1.12), we find an algebraic system of $2 s$ equations with $2 s$ unknowns for the determination of the $A$ 's. Then, $g(z)$ is found as

$$
\begin{equation*}
g(z) \approx \frac{A_{1} \cos z+A_{3} \cos 2 z+\cdots+A_{2 s-1} \cos s z}{1+A_{2} \cos z+A_{4} \cos 2 z+\cdots+A_{2 s} \cos s z} . \tag{1.13}
\end{equation*}
$$

Now, in equation (1.13), replace the above $z$ by $\cos ^{-1}\left(z-z_{0}+1\right)$, and make use of the defining equation for Chebyshev polynomials of the first kind $T_{n}(z)=\cos \left(n \cos ^{-1} z\right)$, recalling the relation between $f(z)$ and $g(z)$ to obtain

$$
\begin{equation*}
f(z) \approx \frac{A_{1} T_{1}\left(z-z_{0}+1\right)+A_{3} T_{2}\left(z-z_{0}+1\right)+\cdots+A_{2 s-1} T_{s}\left(z-z_{0}+1\right)}{T_{0}\left(z-z_{0}+1\right)+A_{2} T_{1}\left(z-z_{0}+1\right)+\cdots+A_{2 s} T_{s}\left(z-z_{0}+1\right)}, \tag{1.14}
\end{equation*}
$$

which gives a rational Chebyshev approximation of $f(z)$ where the only restriction which has been assumed is analyticity of the function at $z_{0}$.

Power series of the form given in (1.1) are sometimes found Taylor-made in the literature. For instance, see [6],

$$
\begin{equation*}
\exp (\cos z-1)=1-\frac{1}{2} z^{2}+\frac{1}{6} z^{4}-\frac{31}{720} z^{6}+\cdots, \tag{1.15}
\end{equation*}
$$

where the general coefficient is

$$
\begin{equation*}
\frac{(-1)^{n} 2^{1-n}}{n!(2 n)!} \sum_{k=0}^{n-1}(-1)^{k} 2^{k}(-n)_{k} \sum_{r=0}^{n-k-1} \frac{(2 k-2 n)_{r}}{r!}(n-k-r)_{2 n}, \tag{1.16}
\end{equation*}
$$

where $(\alpha)_{n}=\alpha(\alpha+1)(\alpha+2) \cdots(\alpha+n-1),(\alpha)_{0}=1, \alpha \neq 0$, is Pochhammer's symbol. In series (1.15), $z_{0}=0$.

Also, see [5],

$$
\begin{equation*}
\log \cos z=\sum_{n=1}^{\infty}(-1)^{n}\left(2^{2 n}-1\right) 2^{2 n-1} B_{2 n^{2}} z^{2 n} /[n(2 n)!] \tag{1.17}
\end{equation*}
$$

where the $B_{2 n}$ are Bernoulli numbers (see [1]). In the series (1.17), $z_{0}=1$.
It will be noticed that the coefficient of $f^{(j)}\left(z_{0}\right)$ in the sum for $g^{(2 i)}(0),(i=1,2, \ldots, 2 s-1$, $j=1,2, \ldots, 2 s-1)$, exemplified in the list given at the beginning of this section, equations (1.2) through (1.9), is also the coefficient of $\cos j z$, evaluated at $z=0$, in

$$
\frac{d^{2 j}}{d z^{2 j}}(\exp (\cos z-1))
$$

This provides a simple computer algorithm for generating these coefficients. This observation is due to one of the authors (Rosenthal).

## 2. ADAPTING THE ALGORITHM FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION

The method we have developed enables us to find, in simple fashion, a rational Chebyshev approximation for the generalized hypergeometric function ${ }_{p} F_{q}(z)$ :

$$
\begin{equation*}
{ }_{P} F_{q}\left(a_{1}, a_{2}, \ldots, a_{p} ; b_{1}, b_{2}, \ldots, b_{q} ; z\right)=1+\sum_{n=1}^{\infty} \frac{\left(a_{1}\right)_{n}\left(a_{2}\right)_{n} \cdots\left(a_{p}\right)_{n} z^{n}}{\left(b_{1}\right)_{n}\left(b_{2}\right)_{n} \cdots\left(b_{q}\right)_{n} n!} \tag{2.1}
\end{equation*}
$$

where none of the $b$ 's is zero or a negative integer (see [14]).
The derivative of (2.1) is given by (see [14])

$$
\begin{equation*}
\frac{a_{1} a_{2} \cdots a_{p}}{b_{1} b_{2} \cdots b_{q}}{ }_{p} F_{q}\left(a_{1}+1, a_{2}+1, \ldots, a_{p}+1 ; b_{1}+1, b_{2}+1, \ldots, b_{q}+1 ; z\right) \tag{2.2}
\end{equation*}
$$

The value of the hypergeometric function at the origin is 1 . Hence, choosing $z_{0}=0$, it is quite simple to determine successive derivatives of the ${ }_{p} F_{q}(z)$ at the origin to find, with the aid of equations (1.2) through (1.9), the values of $g(z)$ and its derivatives at $z=0$.

Note that $g(0)$ and its derivatives at the origin will be given as rational functions of the coefficients of the ${ }_{p} F_{q}(z)$. In particular, if these coefficients are themselves rational, then the rational Chebyshev approximation will involve only rational coefficients.

As the reader no doubt knows, many known functions are special cases (at most with a multiplicative monomial) of the generalized hypergeometric function. We will choose Bessel functions,

$$
\begin{equation*}
J_{n}(z)=\frac{(z / 2)^{n}}{\Gamma(1+n)}{ }_{0} F_{1}\left(-; 1+n ;-\frac{1}{4} z^{2}\right) \tag{2.3}
\end{equation*}
$$

to illustrate the algorithm.
It will be recalled that we mentioned, following (2.2), that, if the parameters appearing in the hypergeometric function are rational numbers, then the $A$ 's, the solutions of the system of equations (1.12), are also rational numbers. This holds true in most of the important cases. For this reason, we found it desirable to make use of a program (we chose REDUCE [15]) that did not execute the operation of division, so that the $A$ 's would be given in fractional form.

We close this section by making a comment that is probably obvious to the reader. If one wishes to go from a given $s$, the highest order of the Chebyshev polynomials in (1.14), to $s+1$ in the system of equations (1.12), then the matrix of the coefficients for $s+1$ will be the same as that for $s$, except that two rows and two columns will be added. Hence, knowing the inverse of the $2 s \times 2 s$ matrix one can find the inverse of the $(2 s+2) \times(2 s+2)$ matrix by using the method of partitioning in the technique known as "inversion by bordering."

## 3. LLLUSTRATING THE ALGORITHM

We will now give some examples of rational Chebyshev approximations obtained by use of the procedure outlined in the previous section. To list the approximations, we will give them in the following format:

$$
\begin{equation*}
f(z) \approx \frac{a z^{k}\left(p_{0} z^{n}+p_{1} z^{n-1}+p_{2} z^{n-2}+\cdots+p_{n-1} z+p_{n}\right)}{b\left(q_{0} z^{m}+q_{1} z^{m-1}+q_{2} z^{m-2}+\cdots+q_{m-1} z+q_{m}\right)} \tag{3.1}
\end{equation*}
$$

where $k+n \leq s$, and $m \leq s$. For each $s$ we will simply list the coefficients in (3.1).

$$
\begin{gathered}
\underline{\underline{f(z)=J_{0}(z)}} \\
\underline{s=2} \\
a=4, k=0, n=2, p_{0}=2, p_{1}=0, p_{2}=-3 ; \\
b=1, m=2, q_{0}=5, q_{1}=0, q_{2}=-12 . \\
\underline{s=3} \\
a=-1, k=0, n=3, p_{0}=69, p_{1}=51, p_{2}=-368, p_{3}=-272 \\
b=1, m=3, q_{0}=23, q_{1}=17, q_{2}=368, q_{3}=272 . \\
\underline{s=6} \\
a=12, k=0, n=6, p_{0}=5776742, p_{1}=0, p_{2}=-183879735, \\
p_{3}=0, p_{4}=1007089152, p_{5}=0, p_{6}=-789561600 \\
\dot{b}=1, m=6, q_{0}=6035647, q_{1}=0, q_{2}=370582236, q_{3}=0 \\
q_{4}=9716385024, q_{5}=0, q_{6}=-9474739200
\end{gathered}
$$

The reader should observe that the magnitude of the coefficients increases quite rapidly with increasing $s$. We shall shortly see that the quality of the approximation also improves very rapidly as $s$ increases.

$$
\begin{array}{cc}
\underline{s}=10 \\
& \\
& \\
p_{0}=3=300, k=0, n=10, & \\
p_{2}=-428033754501951886781, & p_{1}=0, \\
p_{4}=28117868036658189018624, & p_{5}=0, \\
p_{6}=-619413498859286266377984, & p_{7}=0, \\
p_{8}=3132683622732366982938624, & p_{9}=0, \\
p_{10}=-2373905902961822921588736 ; &
\end{array}
$$

\[

\]

## 4. NUMERICAL VALUES AND GRAPHS OF SOME RATIONAL CHEBYSHEV APPROXIMATIONS

In this section we present the results of evaluating the rational forms given in section 3. The runs for different values of the parameter $s$ will be contrasted with the tabulated values given in [1]. The latter will be taken, for purposes of comparison, as exact values.

| $f(z)=J_{0}(z)$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Exact values |  | $s=2$ |  |  | $s=3$ |  |  |
| $z$ | $J_{0}(z)$ | $z$ |  | $J_{0}(z)$ | $z$ |  | $J_{0}(z)$ |
| 0.0 | 1.000000000000000 | 0.0 | 1.00000 | 0000000000 | 0.0 | 1.000000 | 0000000000 |
| 0.1 | 0.997501562066040 | 0.1 | 0.99748 | 9539748954 | 0.1 | 0.997501 | 1561524048 |
| 0.2 | 0.990024972239576 | 0.2 | 0.98983 | 0508474576 | 0.2 | 0.990024 | 4937655860 |
| 0.3 | 0.977626246538296 | 0.3 | 0.97662 | 3376623377 | 0.3 | 0.97762 | 5854568055 |
| 0.4 | 0.960398226659563 | 0.4 | 0.95714 | 2857142857 | 0.4 | 0.960396 | 6039603960 |
| 0.5 | 0.938469807240813 | 0.5 | 0.93023 | 2558139535 | 0.5 | 0.93846 | 1538461539 |
| 0.6 | 0.912004863497211 | 0.6 | 0.89411 | 7647058824 | 0.6 | 0.91198 | 0440097800 |
| 0.7 | 0.881200888607405 | 0.7 | 0.84607 | 3298429319 | 0.7 | 0.88114 | 0084899939 |
| 0.8 | 0.846287352750480 | 0.8 | 0.78181 | 8181818182 | 0.8 | 0.84615 | 3846153846 |
| 0.9 | 0.807523798122545 | 0.9 | 0.69433 | 9622641509 | 0.9 | 0.80725 | 7584770970 |
| 1.0 | 0.765197686557967 | 1.0 | 0.57142 | 8571428571 | 1.0 | 0.76470 | 5882352941 |
| 1.1 | 0.719622018527511 | 1.1 | 0.38991 | 5966386554 | 1.1 | 0.71876 | 8158047647 |
| 1.2 | 0.671132744264363 | 1.2 | 0.09999 | 9999999999 | 1.2 | 0.66972 | 4770642202 |
| 1.3 | 0.620085989561509 | 1.3 | -0.42816 | 9014084507 | 1.3 | 0.61786 | 3199547767 |
| 1.4 | 0.566855120374289 | 1.4 | -1.67272 | 7272727273 | 1.4 | 0.56347 | 4387527840 |
| 1.5 | 0.511827671735918 | 1.5 | -8.00000 | 0000000000 | 1.5 | 0.50684 | 9315068493 |
| 1.6 | 0.455402167639381 | 1.6 | 10.60000 | 0000000000 | 1.6 | 0.44827 | 5862068966 |
| 1.7 | 0.397984859446109 | 1.7 | 4.53877 | 5510204082 | 1.7 | 0.38803 | 5997882478 |
| 1.8 | 0.339986411042558 | 1.8 | 3.31428 | 5714285714 | 1.8 | 0.32640 | 3326403326 |
| 1.9 | 0.281818559374385 | 1.9 | 2.79008 | 2644628099 | 1.9 | 0.26364 | 0999490056 |
| 2.0 | 0.223890779141236 | 2.0 | 2.50000 | 0000000000 | 2.0 | 0.20000 | 0000000000 |
| 2.1 | 0.166606980331990 | 2.1 | 2.31641 | 7910447761 | 2.1 | 0.13571 | 7785399314 |
| 2.2 | 0.110362266922174 | 2.2 | 2.19016 | 3934426230 | 2.2 | 0.07101 | 7274472169 |
| 2.3 | 0.055539784445602 | 2.3 | 2.09826 | 9896193771 | 2.3 | 0.00610 | 6153123532 |
| 2.4 | 0.002507683297244 | 2.4 | 2.02857 | 1428571429 | 2.4 | -0.05882 | 3529411765 |
| 2.5 | -0.04838 3776468198 | 2.5 | 1.97402 | 5974025974 | 2.5 | -0.12359 | 5505617978 |
| Error at $z=0.1: 1.20 \mathrm{E}-05$ <br> Error at $z=1.5: 8.512$ <br> Error at $z=2.5:-2.022$ |  |  |  |  | $\begin{aligned} & \text { Error at } z=0.1: 5.42 \mathrm{E}-10 \\ & \text { Error at } z=1.5: 0.005 \\ & \text { Error at } z=2.5: 0.075 \end{aligned}$ |  |  |

RATIONAL CHEBYSHEV APPROXIMATIONS OF ANALYTIC FUNCTIONS

| $s=6$ |  |  |  | $s=10$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $z$ | $J_{0}(z)$ |  |  | $z$ | $J_{0}(z)$ |  |  |
| 0.0 | 1.00000 | 00000 | 00000 | 0.0 | 1.00000 | 00000 | 00000 |
| 0.1 | 0.99750 | 15620 | 66040 | 0.1 | 0.99750 | 15620 | 66040 |
| 0.2 | 0.99002 | 49722 | 39576 | 0.2 | 0.99002 | 49722 | 39576 |
| 0.3 | 0.97762 | 62465 | 38249 | 0.3 | 0.97762 | 62465 | 38296 |
| 0.4 | 0.96039 | 82266 | 57938 | 0.4 | 0.96039 | 82266 | 59564 |
| 0.5 | 0.93846 | 98072 | 14225 | 0.5 | 0.93846 | 98072 | 40813 |
| 0.6 | 0.91200 | 48632 | 17224 | 0.6 | 0.91200 | 48634 | 97211 |
| 0.7 | 0.88120 | 08863 | 32675 | 0.7 | 0.88120 | 08886 | 07405 |
| 0.8 | 0.84628 | 73359 | 87120 | 0.8 | 0.84628 | 73527 | 50480 |
| 0.9 | 0.80752 | 36414 | 25774 | 0.9 | 0.80752 | 37981 | 22545 |
| 1.0 | 0.76519 | 88991 | 81058 | 1.0 | 0.76519 | 76865 | 57967 |
| 1.1 | 0.71962 | 28437 | 77746 | 1.1 | 0.71962 | 20185 | 27512 |
| 1.2 | 0.67113 | 39900 | 87712 | 1.2 | 0.67113 | 27442 | 64364 |
| 1.3 | 0.62008 | 81243 | 85951 | 1.3 | 0.62008 | 59895 | 61514 |
| 1.4 | 0.56685 | 88740 | 92599 | 1.4 | 0.56685 | 51203 | 74305 |
| 1.5 | 0.51183 | 42373 | 87263 | 1.5 | 0.51182 | 76717 | 35967 |
| 1.6 | 0.45541 | 34601 | 77972 | 1.6 | 0.45540 | 21676 | 39523 |
| 1.7 | 0.39800 | 38749 | 36571 | 1.7 | 0.39798 | 48594 | 46502 |
| 1.8 | 0.34001 | 77192 | 20127 | 1.8 | 0.33998 | 64110 | 43589 |
| 1.9 | 0.28186 | 89650 | 63377 | 1.9 | 0.28181 | 85593 | 76972 |
| 2.0 | 0.22397 | 01919 | 55021 | 2.0 | 0.22389 | 07791 | 47447 |
| 2.1 | 0.16672 | 95358 | 25093 | 2.1 | 0.16660 | 69803 | 46316 |
| 2.2 | 0.11054 | 77454 | 23837 | 2.2 | 0.11036 | 22669 | 54003 |
| 2.3 | 0.05581 | 53758 | 52507 | 2.3 | 0.05553 | 97845 | 13916 |
| 2.4 | 0.00291 | 01468 | 75270 | 2.4 | 0.00250 | 76834 | 39234 |
| 2.5 | -0.04780 | 55089 | 54713 | 2.5 | -0.04838 | 37761 | 81732 |
| Error at $z=0.1: 0$ |  |  |  | Error at $z=0.1: 0$ |  |  |  |
| Err | at $z=1$ | 5:-6. | 57E-06 |  | at $z=1$ | 5:-4.9 | 0E-14 |
| Err | at $z=2$ | 5:-5. | 78E-04 | Err | at $z=2$ | 5:-2.8 | 6E-10 |

The algorithm is seen to be very stable. As the value of $s$ increases, the quality of the approximations improves notably. The last example above, $J_{0}(z)$ for $s=10$, gives remarkable agreement throughout the range $0 \leq|z| \leq 2.5$.

## 5. ZEROS OF THE DENOMINATOR POLYNOMIALS OF THE RATIONAL CHEBYSHEV APPROXIMATIONS

If in equation (1.10) we let $s$ increase without bound, then both sides will represent the same function since their Maclaurin expansions agree for all terms. In this case, equation (1.13) will have an infinite series in both the numerator and denominator. The values of $z$ for which the series in the denominator converges to zero will be singular points of $g(z)$, unless the series in the numerator also converges to zero there. As equation (1.13) stands, it being an approximate relation, it is conceivable that the right-hand side may have poles which are not singular points of the function $g(z)$. This implies, of course, that the right-hand side of equation (1.14) may also have poles which are not singular points of $f(z)$. These would be the so-called spurious poles. Let us look at this phenomenon somewhat more closely for the example given in Section 3.

The denominator polynomial of the rational Chebyshev approximation for the Bessel function $J_{0}(z)$ corresponding to $s=10$ has real zeros at the points

$$
z= \pm 0.957781276624968227260590945945 .
$$

Yet, the graph given in Figure 1, and the table of values of this function do not seem to indicate any abnormal behavior in the neighborhood of this point. However, if we analyze the rational approximation within $\pm \mathrm{E}-18$ of this point, then the rational form is seen to undergo marked oscil-
lations with nearly infinite slope. Nevertheless, as soon as we are within $\pm \mathrm{E}-17$ of the point in question, the erratic behavior disappears and the algorithm again represents the correct values of the Bessel function $J_{0}(z)$.


Figure 1
This figure shows the Bessel function of the first kind of order zero, $J_{0}(z)$ plotted against the rational Chebyshev approximation corresponding to $s=10$. After $z=9$, the Bessel function continues to oscillate, while the approximation separates from this behavior. The two functions move apart after $z=7$. The algorithm approximates the first zero of the Bessel function to be 2.40482 55580, and the second zero to be 5.5196087207 . These results compare favorably with the correct values 2.4048255577 and 5.5200781103 given in [1].

We shall now speak of the significance of these roots. The highly localized character of the oscillation indicates that the numerator polynomial also has zeros which are very close to the zeros of the denominator polynomial. This is indeed the case for all of the examples we studied. The numerator polynomial of the $s=10$ approximation of the Bessel function, for instance, has real zeros at the points

$$
z= \pm 0.957781276624968221503291384229
$$

which match the zeros of the denominator polynomial through seventeen decimal places. The oscillatory behavior is then simply a reflection of the computer's arithmetic inability to handle $0 / 0$. The algorithm, we see, is a self-correcting one that introduces zeros in the numerator and denominator polynomials in a way that ensures the correct approximation to the function for a given value of $s$.

In essence, our method provides a rational approximation $P_{s}(z) / Q_{s}(z)$ such that its Taylor expansion about the point $z_{0}$ agrees with the Taylor expansion of $f(z)$ through the first $2 s$ terms. This requirement may be written as

$$
Q_{s}(z) f(z)-P_{s}(z)=\left(z-z_{0}\right)^{2 s+1} \sum_{k=0}^{\infty} c_{k}\left(z-z_{0}\right)^{k}
$$

and it is equivalent to the criterion for choosing the $s^{\text {th }}$ diagonal entry in the Pade table for $z_{0}=0$.
Because of the proximity of the real zeros of the numerator and denominator polynomials of the Bessel function approximation corresponding to $s=10$, we chose to divide out the zeros and try out the outcome against the tabulated values given before. The resulting expression is:

$$
\begin{array}{cl}
a=300, k=0, n=8 & \\
p_{0}=2114635700054536614.00000000000, & p_{1}=0, \\
p_{2}=-426093904070975989175.46176795488, & p_{3}=0 \\
p_{4}=27726992935369126065928.27801886000, & p_{5}=0, \\
p_{6}=-593978281249957947189316.44693046000, & p_{7}=0 \\
p_{8}=2587800631849664222173861.61878680000, & \\
& b=1, m=8 \\
& \\
q_{0}=3272566141496807057.00000000000, & q_{1}=0 \\
q_{2}=987657023587922726257.68351763259, & q_{3}=0 \\
q_{4}=160673172249783603741999.24516095000, & q_{5}=0 \\
q_{6}=15891563013737422140956470.02415100000, & q_{7}=0 \\
q_{8}=776340189554899257318873022.34814000000 &
\end{array}
$$

The tabulated values resulting from this approximation are:

|  | $J_{0}(z)$ |  |  |  |
| :--- | :--- | :--- | :--- | :--- |
| $z$ |  |  |  |  |
| 0.0 | 1.00000 | 00000 | 00000 |  |
| 0.1 | 0.99750 | 15620 | 66040 |  |
| 0.2 | 0.99002 | 49722 | 39576 |  |
| 0.3 | 0.97762 | 62465 | 38296 |  |
| 0.4 | 0.96039 | 82266 | 59564 |  |
| 0.5 | 0.93846 | 98072 | 40813 |  |
| 0.6 | 0.91200 | 48634 | 97211 |  |
| 0.7 | 0.88120 | 08886 | 07405 |  |
| 0.8 | 0.84628 | 73527 | 50480 |  |
| 0.9 | 0.80752 | 37981 | 22545 |  |
| 1.0 | 0.76519 | 76865 | 57967 |  |
| 1.1 | 0.71962 | 20185 | 27512 |  |
| 1.2 | 0.67113 | 27442 | 64364 |  |
| 1.3 | 0.62008 | 59895 | 61514 |  |
| 1.4 | 0.56685 | 51203 | 74305 |  |
| 1.5 | 0.51182 | 76717 | 35967 |  |
| 1.6 | 0.45540 | 21676 | 39523 |  |
| 1.7 | 0.39798 | 48594 | 46502 |  |
| 1.8 | 0.33998 | 64110 | 43589 |  |
| 1.9 | 0.28181 | 85593 | 76972 |  |
| 2.0 | 0.22389 | 07791 | 47447 |  |
| 2.1 | 0.16660 | 69803 | 46316 |  |
| 2.2 | 0.11036 | 22669 | 54003 |  |
| 2.3 | 0.05553 | 97845 | 13916 |  |
| 2.4 | 0.00250 | 76834 | 39234 |  |
| 2.5 | -0.04838 | 37761 | 81732 |  |
|  |  |  |  |  |
| Error | at | $z=0.1: 0$ |  |  |
| Error | at | $z=1.5:-4.90 \mathrm{E}-14$ |  |  |
| Error | at | $z=2.5:-2.86 E-10$ |  |  |

These are exactly the same values, to fifteen-decimal accuracy, obtained with the $s=10$ approximation of the Bessel function $J_{0}(z)$ before the roots are divided out!-These results imply a substantial saving in computer time since the number of divisions required for a given approximation is reduced by two.

A comment is in order, though it is probably obvious to the reader. The results shown in the above table were obtained by dividing the numerator polynomial by its real roots, and the denominator polynomial by its corresponding real roots. Slightly better accuracy is obtained (though the
above table does not indicate it) if we divide both numerator and denominator polynomials by either the real roots of the numerator or the real roots of the denominator since, in this case, all we are doing is dividing numerator and denominator of the $s=10$ approximation by a common factor.

It is worth emphasizing that the rational Chebyshev approximations our algorithm provides are not optimal, in the sense that error does not remain constant within the range of approximation. Rather, error is least when one is sufficiently near the point $z_{0}$ and the quality of the approximation deteriorates as we move away from the point in question. The importance of the method lies, we believe, in the extreme simplicity with which it can provide rational Chebyshev approximations of any accuracy for a wide variety of functions. These nonoptimal approximations may easily be used to obtain optimal Chebyshev approximations. Several algorithms have been developed to this effect.

Let us speak now of the origin of the problem that has occupied us in the last five sections.

## 6. SOME HISTORY

About one hundred and twenty-five years ago, the Russian mathematician Pafnuty Lvovich Chebyshev (1821-1894) set himself the problem of finding the best rational approximation of a continuous function specified on an interval $[a, b]$. Specifically, he wanted to determine parameters $p_{0}, p_{1}, \ldots, p_{n} ; q_{0}, q_{1}, \ldots, q_{m}$ in the expression

$$
\begin{equation*}
Q(x)=s(x) \frac{p_{0} x^{n}+p_{1} x^{n-1}+\cdots+p_{n}}{q_{0} x^{m}+q_{1} x^{m-1}+\cdots+q_{m}} \tag{6.1}
\end{equation*}
$$

where $m$ and $n$ are given, and $s(x)$ is a function continuous on $[a, b]$, so that the deviation of $Q(x)$ from a chosen continuous function $f(x)$,

$$
\begin{equation*}
H_{Q}=\max _{a \leq x \leq b}|f(x)-Q(x)| \tag{6.2}
\end{equation*}
$$

shall be a minimum.
Chebyshev established the beautiful existence theorem [6;2]:
The function $P(x)$, which deviates least from the function $f(x)$ than does any other function of the type exemplified by equation (6.1) is completely characterized by the following property: If the function can be expressed in the form

$$
P(x)=s(x) \frac{a_{0} x^{n-\sigma}+a_{1} x^{n-\sigma-1}+\cdots+a_{n-\sigma}}{b_{0} x^{m-\tau}+b_{1} x^{m-\tau-1}+\cdots+b_{m-\tau}}=s(x) \frac{A(x)}{B(x)}
$$

where $0 \leq \sigma \leq n, 0 \leq \tau \leq m, b_{0} \neq 0$ and the fraction $\frac{A(x)}{B(x)}$ is irreducible, then the number $N$ of consecutive points of the interval $[a, b]$ at which the difference $f(x)-P(x)$, with alternate change of sign, takes on the value $H_{p}$, is not less than $m+n+2-d$, where $d=\min (\sigma, \tau)$; in case $P(x) \equiv 0$, then $N \geq n+2$.

Chebyshev did not provide a constructive approach to the problem of finding the rational approximations whose existence is guaranteed by the above theorem. He, and E. Solotarev did work out one example, based on the theory of Jacobian elliptic functions, that meets the require-
ments of the theorem [16]. Since that time, though, many people have sought to obtain an explicit method of attack for determining these rational approximations [ $8 ; 9 ; 10]$. The problem is especially complicated by the fact that the class of continuous functions is a very broad one. Most of the methods of attack that have been developed deal with a more restrictive class of functions: bounded variation, analytic, or the like.

A substantial advance was made by H. Padé in his now classic thesis of 1892 [13]. Padé's method, mentioned briefly at the end of the last section, yields excellent rational approximations of analytic functions by means of solutions of a system of linear algebraic equations [18]. The method is an extension of some earlier work of Frobenius [6]. However, it does not provide rational Chebyshev approximations. It is known that rational forms in Chebyshev polynomials yield better accuracy than ordinary rational forms [16].

Maehly gave a method for obtaining rational Chebyshev approximations of functions of bounded variation on the unit interval [12; 16]. It has the substantial disadvantage of requiring that the given function be first expanded in a series of Chebyshev polynomials. If the function is anywhere complicated, these expansions may be devilishly hard to obtain.

To the best of our knowledge, no method is known for obtaining rational Chebyshev approximations that is better, more direct, or more powerful than the one we have presented in this paper. The method was discovered by one of the authors (Castellanos) as a result of his work on formulas to approximate $\pi$ while in preparation of "The Ubiquitous $\pi$," Math. Magazine 61.2-3 (April-June 1988). The delicate and time-consuming task of carrying the algorithm into a working computer program was done by the other author (Rosenthal).

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## PALINDROMES

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## INTRODUCTION

A palindrome is a finite sequence of positive integers which is unchanged when written in reverse order. Sometimes such sequences are referred to as symmetric (see [3] and [5]). The objective of this paper is to show how some simple properties of palindromes can be used to obtain results in elementary number theory. We give new elementary proofs of known results and what appear to be some new results.

In §1 we prove some elementary properties of palindromes and their associated finite continued fractions. In §2 we apply the properties established in §1. The reader will note that the application of Proposition 4 of $\S 1$ constitutes a method for obtaining the results of $\S 2$.

## 1. ELEMENTARY PROPERTIES OF PALINDROMES

Let $n$ be a nonnegative integer. We call a sequence of positive integers $\alpha=\{\alpha(0), \alpha(1), \ldots$, $\alpha(n)\}$ of length $n+1$ a palindrome if $\alpha(i)=\alpha(n-i)$ for $0 \leq \mathrm{i} \leq \mathrm{n}$.

Example: Let $n$ be a nonnegative integer and define the sequence $\alpha$ by

$$
\alpha(i)=\binom{n}{i}=\frac{n!}{(n-i)!i!},
$$

for $0 \leq i \leq n$. The condition for $\alpha$ to be a palindrome is the well-known binomial coefficient identity $\left.\binom{n}{i}=\begin{array}{c}n \\ n-i\end{array}\right)$.

We are especially interested in sequences of positive integers generated by the division algorithm (see [1]). Explicitly, if $P$ and $Q$ are relatively prime integers such that $1<Q<P$. Then (see [1], p.325) $P$ and $Q$ uniquely determine two sequences of positive integers $\alpha$ and $r$ as follows:

$$
\begin{array}{rlrl}
P & =\alpha(0) Q+r(0), & & 0<r(0)<Q ; \\
Q & =\alpha(1) r(0)+r(1), & & 0<r(1)<r(0) ; \\
\vdots & & \\
r(i-2) & =\alpha(i) r(i-1)+r(i), & & 0<r(i)<r(i-1) ;  \tag{1}\\
\vdots & \vdots & & \\
r(n-3) & =\alpha(n-1) r(n-2)+r(n-1), & & 1=r(n-1)<r(n-2) ; \\
r(n-2) & =\alpha(n) r(n-1)=\alpha(n) . & &
\end{array}
$$

Since $\alpha(n)=r(n-2)>1$, we have $\alpha(n) \geq 2$. We call $\alpha$ the sequence of quotients and $r$ the sequence of remainders determined by the pair $(P, Q)$. For any integer $c$ we define

$$
A_{c}=\left(\begin{array}{ll}
c & 1 \\
1 & 0
\end{array}\right) .
$$

The equations in (1) are equivalent to

$$
A_{\alpha(i)}\binom{r(i-1)}{r(i)}=\binom{r(i-2)}{r(i-1)}, \text { for } 0 \leq i \leq n,
$$

where $r(-2)=P, r(-1)=Q$, and $r(n)=0$. Hence

$$
\begin{equation*}
A_{\alpha(0)} A_{\alpha(1)} \cdots A_{\alpha(n)}\binom{1}{0}=\binom{P}{Q} . \tag{2}
\end{equation*}
$$

Let $\alpha$ be any sequence of positive integers of length $n+1$. If we define

$$
\left(\begin{array}{cc}
P_{i} & Q_{i}  \tag{3}\\
P_{i-1} & Q_{i-1}
\end{array}\right)=A_{\alpha(i)} A_{\alpha(i-1)} \cdots A_{\alpha(0)},
$$

then it is known (see [6]) that

$$
\begin{equation*}
P_{i} / Q_{i}=[\alpha(0), \alpha(1), \ldots, \alpha(n)]=\alpha(0)+\frac{1}{\alpha(1)+\frac{1}{\alpha(2)+\ddots \frac{1}{\alpha(i-1)+\frac{1}{\alpha(i)}}}} \tag{4}
\end{equation*}
$$

a finite simple continued fraction. The elementary properties of continued fractions that we need can be found in [1]. In what follows we denote the greatest integer function by $\mathbb{\rrbracket}$.

Lemma 1: Let $\alpha=\{\alpha(0), \alpha(1), \ldots, \alpha(n)\}$ be a sequence of positive integers of length $n+1>2$. If

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P_{n} & Q_{n} \\
P_{n-1} & Q_{n-1}
\end{array}\right)
$$

then $P_{n} / Q_{n}$ is not an integer and $\llbracket P_{n} / Q_{n} \rrbracket=\alpha(0)$
Proof: Using (3) and (4), we have $P_{n} / Q_{n}=\alpha(0)+1 /[\alpha(1), \alpha(2), \ldots, \alpha(n)]$. So we need only show that $[\alpha(1), \alpha(2), \ldots, \alpha(n)]>1$ to obtain the result. To that end, we note that because $n+1>2,[\alpha(1), \alpha(2), \ldots, \alpha(n)] \geq \alpha(2) \geq 1$. Thus,

$$
[\alpha(1), \alpha(2), \ldots, \alpha(n)]=\alpha(1)+1 /[\alpha(2), \alpha(3), \ldots, \alpha(n)]>\alpha(1) .
$$

Now, since $\alpha(1) \geq 1$, the conclusion follows.
The following Lemma, accounting for a difference in notation, can be found as an exercise in [6, p. 251]. Since we use it in an essential way, we provide a proof for the sake of completeness.
Lemma 2: If $\alpha$ and $\beta$ are two sequences of positive integers of lengths $n+1$ and $m+1$, respectively, then

$$
\begin{equation*}
A_{\alpha(n)} A_{\alpha(n-1)} \ldots A_{\alpha(0)}=A_{\beta(m)} A_{\beta(m-1)} \ldots A_{\beta(0)} \tag{5}
\end{equation*}
$$

if and only if $n=m$ and $\alpha=\beta$.
Proof: We will proceed by induction on the length of the sequence $\alpha$. If $n+1=1$, then $n=0$ and

$$
\left(\begin{array}{cc}
\alpha(0) & 1 \\
1 & 0
\end{array}\right)=A_{\beta(m)} A_{\beta(m-1)} \cdots A_{\beta(0)}=\left(\begin{array}{cc}
P_{m} & Q_{m} \\
P_{m-1} & Q_{m-1}
\end{array}\right) .
$$

Thus, $\alpha(0)=P_{m} / Q_{m}$ is an integer. So, by Lemma $1, m \leq 1$. Now $\operatorname{det}\left(\left(\begin{array}{cc}c & 1 \\ 1 & 1\end{array}\right)\right)=-1$, for any integer $c$, where $\operatorname{det}()$ is the determinant. So, if $m=1$, we would have $-1=\operatorname{det}\left(A_{\alpha(0)}\right)=\operatorname{det}\left(A_{\beta(1)} A_{\beta(0)}\right)$ $=1$. Thus, $m=0$ and $\alpha(0)=\beta(0)$

Now assume our result is true when the length of $\alpha$ is less than $n+1$, with $n \geq 1$. We first note that $m \geq 1$. Because, if $m=0$, we argue as above, with the roles of $\alpha$ and $\beta$ interchanged, and conclude that $n=0$. Multiplying both sides of (5) on the left by $A_{1}$, we have

$$
A_{1} A_{\alpha(n)} \cdots A_{\alpha(0)}=A_{1} A_{\beta(m)} \cdots A_{\beta(0)}=\left(\begin{array}{cc}
P_{m+1} & Q_{m+1} \\
P_{m} & Q_{m}
\end{array}\right)
$$

Because $\{\alpha(0), \alpha(1), \ldots, \alpha(n), 1\}$ and $\{\beta(0), \beta(1), \ldots, \beta(m), 1\}$ both have length bigger than 2 , we have by Lemma 1 that $\alpha(0)=\llbracket P_{m+2} / Q_{m+2} \rrbracket=\beta(0)$. Finally, multiplying both sides of (5) on the right by the inverse of $A_{\alpha(0)}$, we have

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(1)}=A_{\beta(m)} A_{\beta(m-1)} \cdots A_{\beta(1)} .
$$

Hence, by the induction hypothesis, $\{\alpha(1), \alpha(2), \ldots, \alpha(n)\}=\{\beta(1), \beta(2), \ldots, \beta(m)\}$ and, thus, $n=m$ and $\alpha=\beta$.
Proposition 1: If $\alpha$ is a sequence of positive integers of length $n+1$, then $\alpha$ is a palindrome if and only if the matrix

$$
\begin{equation*}
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)} \tag{6}
\end{equation*}
$$

is symmetric.
Proof: Since each $A_{\alpha(i)}$ is symmetric, the transpose of (6) is

$$
A_{\alpha(0)} A_{\alpha(1)} \cdots A_{\alpha(n)}
$$

So by Lemma 2 the result follows.
Proposition 2: If $\alpha$ is a palindrome of length $n+1$ and

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P_{n} & Q_{n}  \tag{7}\\
P_{n-1} & Q_{n-1}
\end{array}\right) .
$$

Then

$$
Q_{n}^{2} \equiv(-1)^{n}\left(\bmod P_{n}\right)
$$

Proof: By Proposition 1, $P_{n-1}=Q_{n}$. Since the determinant of $A_{\alpha(i)}$ is -1 for all $i$, we have, by taking determinants in (7), $(-1)^{n+1}=P_{n} Q_{n-1}-Q_{n}^{2}$ and, thus, $Q_{n}^{2}=(-1)^{n}+P_{n} Q_{n-1}$.

Now we give an elementary proof of an easy extension of a result which can be found in [3].
Proposition 3: Let $P$ and $Q$ be integers such that $1<Q<P$ and $Q^{2} \equiv \pm 1(\bmod P)$. Then there exists a palindrome $\alpha$ of length $n+1$ with

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P & Q \\
Q & Q_{n-1}
\end{array}\right)
$$

where $Q^{2} \equiv(-1)^{n}(\bmod P)$. Further, $\alpha$ is uniquely determined by $P$ and $Q$.

Proof: Let $\alpha$ be the sequence of quotients in the division algorithm determined by the pair $(P, Q)$. Set

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P_{n} & Q_{n}  \tag{8}\\
P_{n-1} & Q_{n-1}
\end{array}\right) .
$$

Taking the transpose in (2) we have (1 0 ) $A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{ll}P & Q\end{array}\right)$. Thus, $P_{n}=P$ and $Q_{n}=Q$.

Taking determinants in (8), we have $(-1)^{n+1}=P_{n} Q_{n-1}-Q^{2}$ and, thus, $P_{n-1} Q=(-1)^{n}+P Q_{n-1}$. Further, because $Q^{2} \equiv \pm 1(\bmod P)$, we have

$$
\begin{equation*}
P_{n-1} \equiv-( \pm 1)(-1)^{n} Q(\bmod P) . \tag{9}
\end{equation*}
$$

Next, because

$$
A_{\alpha(0)} A_{\alpha(1)} \cdots A_{\alpha(n)}=\left(\begin{array}{cc}
P & P_{n-1} \\
Q & Q_{n-1}
\end{array}\right),
$$

we have $P / P_{n-1}=[\alpha(n), \alpha(n-1), \ldots, \alpha(0)] \geq \alpha(n)$. We know, from (1), that $\alpha(n) \geq 2$ and, thus, $P_{n-1} \leq P / 2$. Now, if $Q<P / 2$, then (9) implies that $Q=P_{n-1}$. So, by Proposition 1, $\alpha$ is a palindrome.

Suppose $P / 2<Q<P$. Then $1<P / Q<2$ and $\alpha(0)=\llbracket P / Q \rrbracket=1$. Next, if we multiply both sides of (8) on the left by

$$
A_{1} A_{\alpha(n)-1} A_{\alpha(n)}^{-1}=\left(\begin{array}{cc}
1 & 0 \\
1 & -1
\end{array}\right)
$$

we have

$$
A_{1} A_{\alpha(n)-1} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P & Q \\
P-P_{n-1} & Q-Q_{n-1}
\end{array}\right) \text {. }
$$

Taking the determinant, we have $P\left(Q-Q_{n-1}\right)-\left(P-P_{n-1}\right) Q=(-1)^{n}$ and, so,

$$
(P-Q) P_{n-1}-P\left(P_{n-1}-q_{n-1}\right)=(-1)^{n+1} .
$$

Hence, $(P-Q) P_{n-1} \equiv(-1)^{n+1}(\bmod P)$. Because $P-Q \equiv-Q(\bmod P)$, we have $(P-Q)^{2} \equiv \pm 1$ $(\bmod P)$ and, thus,

$$
\begin{equation*}
P_{n-1} \equiv-( \pm 1)(-1)^{n+1}(P-Q)(\bmod P) \tag{10}
\end{equation*}
$$

Since $P-Q<P / 2$ and $P_{n-1}<P / 2$, (10) implies that $P-Q=P_{n-1}$. That is, $Q=P-P_{n-1}$ and, so, by Proposition $1,\{\alpha(0), \alpha(1), \ldots, \alpha(n-1), \alpha(n)-1,1\}$ is a palindrome.

Now we prove uniqueness. Let $\alpha$ be the palindrome constructed above and $\beta$ another of length $m+1$ with

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P & Q \\
Q & Q_{n-1}
\end{array}\right) \text { and } A_{\beta(m)} A_{\beta(m-1)} \cdots A_{\beta(0)}=\left(\begin{array}{cc}
P & Q \\
Q & R
\end{array}\right) .
$$

Taking determinants, we have $P Q_{n-1}-Q^{2}=(-1)^{n+1}$ and $P R-Q^{2}=(-1)^{m+1}$. Thus, $Q^{2} \equiv(-1)^{n}$ $(\bmod P)$ where, because $P>2$, we must have $(-1)^{n}=(-1)^{m}$. Hence, $P\left(Q_{n-1}-R\right)=0$ and, thus, $R=Q_{n-1}$. Finally, by Lemma 2, $\alpha=\beta$.

Corollary 1: Let $P$ and $Q$ be integers such that $1<Q<P$ and $Q^{2} \equiv \pm 1(\bmod P)$. If $\alpha$ is the sequence of quotients in the division algorithm determined by the pair $(P, Q)$, then $\alpha$ or $\{\alpha(0)$, $\alpha(1), \ldots, \alpha(n-1), \alpha(n)-1,1\}$ is a palindrome.

Proof: It follows from (1) that $\alpha(n) \geq 2$. Further, the palindrome referred to in Propositioin 3 was shown to be either $\alpha$ or $\{\alpha(0), \alpha(1), \ldots, \alpha(n-1), \alpha(n)-1,1\}$.
Proposition 4: Let $\alpha$ be a sequence of positive integers of length $n+1$ and $n$ a nonnegative integer such that

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P & Q  \tag{11}\\
P_{n-1} & Q_{n-1}
\end{array}\right),
$$

where $Q^{2} \equiv \pm 1(\bmod P)$ and $1 \leq Q<P$, with $P>2$. Then we have exactly one of the following possibilities:
(a) $\alpha$ is a palindrome and $Q^{2} \equiv(-1)^{n}(\bmod P)$.
(b) $\alpha(n)=1,\{\alpha(0), \alpha(1), \ldots, \alpha(n-2), \alpha(n-1)+1\}$ is a palindrome, and $Q^{2} \equiv(-1)^{n+1}(\bmod P)$.
(c) $\alpha(n)>1,\{\alpha(0), \alpha(1), \ldots, \alpha(n-1), \alpha(n)-1,1\}$ is a palindrome, and $Q^{2} \equiv(-1)^{n+1}(\bmod P)$.

Proof: If $\alpha(n)>1$, then $\alpha$ is clearly the sequence of quotients in the division algorithm for the pair $(P, Q)$. Hence, by Corollary 1, either $\alpha$ or $\{\alpha(0), \alpha(1), \ldots, \alpha(n-1), \alpha(n)-1,1\}$ is a palindrome. Now, if $\alpha$ is a palindrome, then by Proposition $2, Q^{2} \equiv(-1)^{n}(\bmod P)$. Next, if $\{\alpha(0), \alpha(1), \ldots, \alpha(n-1), \alpha(n)-1,1\}$ is a palindrome, then multiplying both sides of $(11)$ by

$$
A_{1} A_{\alpha(n)-1}-1 A_{\alpha(n)}^{-1}=\left(\begin{array}{rr}
1 & 0 \\
1 & -1
\end{array}\right),
$$

we have

$$
A_{1} A_{\alpha(n)-1} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P & Q \\
P-P_{n-1} & Q-Q_{n-1}
\end{array}\right) .
$$

So, by Propositions 1 and $2, P-P_{n-1}=Q$ and $Q^{2} \equiv(-1)^{n+1}(\bmod P)$.
If $\alpha(n)=1$, then multiplying both sides of (11) on the left by

$$
A_{\alpha(n-1)+1} A_{\alpha(n-1)}^{-1} A_{1}^{-1}=\left(\begin{array}{rr}
1 & 0 \\
1 & -1
\end{array}\right)
$$

we have

$$
A_{\alpha(n-1)+1} A_{\alpha(n-2)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P & Q  \tag{12}\\
P-P_{n-1} & Q-Q_{n-1}
\end{array}\right) .
$$

So, again, $\{\alpha(0), \alpha(1), \ldots, \alpha(n-2), \alpha(n-1)+1\}$ is the sequence of quotients in the division algorithm for the pair $(P, Q)$. Hence, by Corollary $1,\{\alpha(0), \alpha(1), \ldots, \alpha(n-2), \alpha(n-1)+1\}$ or $\alpha$ is a palindrome. If $n=1$, we understand $\{\alpha(0), \alpha(1), \ldots, \alpha(n-2), \alpha(n-1)+1\}$ to be $\{\alpha(0)+1\}$. Now, if $\{\alpha(0), \alpha(1), \ldots, \alpha(n-2), \alpha(n-1)+1\}$ is a palindrome, then (12) and Proposition 2 give $Q^{2} \equiv(-1)^{n+1}(\bmod P)$.

Next, we show that two of these possibilities cannot hold at the same time. Clearly (b) and (c) cannot both be true. If (a) and (b) or (a) and (c) hold, then $Q^{2} \equiv(-1)^{n}(\bmod P)$ and $Q^{2} \equiv$ $(-1)^{n+1}(\bmod P)$. That is $1 \equiv-1(\bmod P)$, which is impossible since $P>2$.

If $\alpha$ is a sequence of positive integers of length $n+1$, we obtain a sequence $\alpha_{*}$ of length $n$ by deleting $\alpha(0)$. Specifically, $\alpha_{*}(i)=\alpha(i+1)$ for $0 \leq i \leq n-1$. That is, $\alpha_{*}=\{\alpha(1), \alpha(2), \ldots, \alpha(n)\}$. Further, if $\alpha(n)>1$, we form a sequence $\alpha^{*}$ of length $n+1$ by deleting $\alpha(0)$ and replacing $\alpha(n)$ by $\{\alpha(n)-1,1\}$. That is, $\alpha^{*}=\{\alpha(1), \ldots, \alpha(n-1), \alpha(n)-1,1\}$, where $\alpha^{*}(i)=\alpha(i+1)$ for $0 \leq i \leq$ $n-2, \alpha^{*}(n-1)=\alpha(n)-1$ and $\alpha^{*}(n)=1$.

Proposition 5: If $\alpha$ and $\alpha_{*}$ are both palindromes, then $\alpha(i)=\alpha(0)$ for $0 \leq i \leq n$.
Proof: If $0 \leq i \leq n-1$, then $\alpha(i+1)=\alpha_{*}(i)=\alpha_{*}(n-1-i)=\alpha(i)$.
Proposition 6: If $\alpha(n)>1$ and both $\alpha$ and $\alpha^{*}$ are palindromes, then we have two possibilities:
(a) If $n$ is odd, then $\alpha(0)=\alpha(n)=2$ and $\alpha(i)=1$ for $1 \leq i \leq n-1$.
(b) If $n$ is even, then $\alpha(0)=\alpha(n)=c>1$. Further, $\alpha(2 k-1)=1$ for $1 \leq k \leq n / 2$ and $\alpha(2 k)=$ $c-1$ for $1 \leq k<n / 2$.

Proof: If $0<i<n-2$, then $\alpha^{*}(i-1)=\alpha(i)=\alpha(n-i)=\alpha^{*}(n-i-1)=\alpha^{*}(i+1)=\alpha(i+2)$. Hence, $1=\alpha^{*}(n)=\alpha^{*}(0)=\alpha(1)=\alpha(2 k-1)$ for $1 \leq k \leq \llbracket n / 2 \rrbracket$. Further, $\alpha(2)=\alpha(2 k)$ for $1 \leq k \leq \llbracket n / 2 \rrbracket$, where $\alpha(2)=\alpha^{*}(1)=\alpha^{*}(n-1)=\alpha(n)-1=\alpha(0)-1$.

So, if $n$ is even, we have proved (b). If $n$ is odd, then $\alpha((n-1) / 2)=\alpha(n-(n-1) / 2)=$ $\alpha((n+1) / 2)$. Since one of $(n-1) / 2$ and $(\mathrm{n}+1) / 2$ is even and the other odd, we must have $\alpha(i)=1$ for $1 \leq i<n-1$. Further, since $1=\alpha(2)=\alpha(0)-1$, we also have $\alpha(0)=\alpha(n)=2$.

## 2. APPLICATIONS

In what follows, we sill prove four propositions. Propositions 7 and 10 are known results and we give new elementary proofs. Propositions 8 and 9 are of the same general type as Proposition 7 and are apparently new.

We define a sequence of polynomials as follows: $J_{1}(X)=0, J_{0}(X)=1$ and $J_{k+1}(X)=$ $X J_{k}(X)+J_{k-1}(X)$ for $k \geq 0$ or, equivalently, for $k \geq 1$,

$$
\left(\begin{array}{cc}
J_{k}(X) & J_{k-1}(X)  \tag{13}\\
J_{k-1}(X) & J_{k-2}(X)
\end{array}\right)=\left(\begin{array}{cc}
X & 1 \\
1 & 0
\end{array}\right)^{k} .
$$

Remark 2: It is easy to see that $\left\{F_{k}=J_{k-1}(1) \mid k \geq 0\right\}$ is the sequence of Fibonacci numbers.
The following is an elementary proof of a result of Owings (see [2]).
Proposition 7: If $P$ and $Q$ are integers with $1 \leq Q<P, Q^{2} \equiv-1(\bmod P)$ and $P^{2} \equiv-1(\bmod Q)$, then there exists an odd integer $k$ such that

$$
Q=F_{k} \quad \text { and } P=F_{k+2},
$$

where $F_{k}$ is the $k^{\text {th }}$ Fibonacci number.

Proof: If $Q=1$, then $Q^{2} \equiv-1(\bmod P)$ and $P>Q$ implies that $P=2$. Hence, $Q=F_{1}$ and $P=F_{3}$. Next, if $Q=2$, then $Q^{2} \equiv-1(\bmod P)$ implies that $P=5$. So $Q=F_{3}$ and $P=F_{5}$.

From now on, we will assume that $Q>2$. By Proposition 3 there is a palindrome $\alpha$ of even length $n+1$ such that

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P & Q \\
Q & Q_{n-1}
\end{array}\right)
$$

and $Q^{2} \equiv(-1)^{n}(\bmod P)$. We will prove that $\alpha^{*}$ is a palindrome. To that end, we note that

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(1)}=\left(\begin{array}{cc}
Q & P-\alpha(0) Q \\
Q_{n-1} & Q-\alpha(0) Q_{n-1}
\end{array}\right) .
$$

Since $P-\alpha(0) Q \equiv P(\bmod Q)$ and $P^{2} \equiv-1(\bmod Q)$, we have that $(P-\alpha(0) Q)^{2} \equiv-1(\bmod Q)$. Now, because $Q>2$, we have by Proposition 4 exactly one of the following possibilities:
(a) $\{\alpha(1), \alpha(2), \ldots, \alpha(n)\}$ is a palindrome with $(P-\alpha(0) Q)^{2} \equiv(-1)^{n-1}(\bmod Q)$;
(b) $\alpha(n)=1$ and $\{\alpha(1), \ldots, \alpha(n-2), \alpha(n-1)+1\}$ is a palindrome with $(P-\alpha(0) Q)^{2} \equiv(-1)^{n}$ $(\bmod Q)$;
(c) $\alpha(n)>1$ and $\alpha^{*}=\{\alpha(1), \ldots, \alpha(n-1), \alpha(n)-1,1\}$ is a palindrome with $(P-\alpha(0) Q)^{2} \equiv(-1)^{n}$ $(\bmod Q)$.

The case (a) cannot hold since $n-1$ is even, and the two congruences, $P-\alpha(0) Q \equiv P$ $(\bmod Q)$ and $P^{2} \equiv-1(\bmod Q)$, imply that $1 \equiv-1(\bmod Q)$. Contradicting that $Q>2$. Now, suppose $\alpha(n)=1$ and $\{\alpha(1), \ldots, \alpha(n-2), \alpha(n-1)+1\}$ is a palindrome. Since $n$ is odd, it follows that $n>2$ and, thus, $\alpha(n-1)=\alpha(1)=\alpha(n-1)+1$ yields a contradiction. Hence, we have shown that $\alpha(n)>1$ and $\alpha^{*}=\{\alpha(1), \ldots, \alpha(n-1), \alpha(n)-1,1\}$ is a palindrome.

Because $\alpha$ and $\alpha^{*}$ are both palindromes and $n$ is odd, we have, by Proposition 5, that $\alpha(0)=\alpha(n)=2$ and $\alpha(i)=1$ for $1 \leq i \leq n-1$. Hence,

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(1)}=A_{2} A_{1}^{n-1} A_{2}
$$

But, from (13) we have

$$
A_{1}^{n-1}=\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)^{n-1}=\left(\begin{array}{ll}
J_{n-1}(1) & J_{n-2}(1) \\
J_{n-2}(1) & J_{n-3}(1)
\end{array}\right) .
$$

Hence,

$$
A_{2} A_{1}^{n-1} A_{2}=A_{2}\left(\begin{array}{ll}
J_{n-1}(1) & J_{n-2}(1) \\
J_{n-2}(1) & J_{n-3}(1)
\end{array}\right) A_{2}=\left(\begin{array}{ll}
J_{n+3}(1) & J_{n+1}(1) \\
J_{n+1}(1) & J_{n-1}(1)
\end{array}\right) .
$$

We have already observed that $F_{i+1}=J_{i}(1)$ for $i \geq 0$. Thus, the result is established.
Proposition 8: If $P$ and $Q$ are integers with $1 \leq Q<P, P^{2} \equiv \pm 1(\bmod Q)$ and $Q^{2} \equiv-( \pm 1)(\bmod$ $P)$, then there exist integers $k \geq 0$ and $c \geq 1$ such that $J_{k}(c)=Q$ and $J_{k+1}(c)=P$.

Proof: If $Q=1$ then, for any $P>1$, we have $J_{0}(P)=Q$ and $J_{1}(P)=P$. Next, if $Q=2$, then $Q^{2} \equiv-( \pm 1)(\bmod P)$ implies that $P=3$ or $P=5$. If $P=3$, we have $J_{2}(1)=2$ and $J_{3}(1)=3$. For the case $P=5$, we have $J_{1}(2)=2$ and $J_{2}(2)=5$.

From now on, we will assume that $Q>2$. By Proposition 3, there is a palindrome $\alpha$ of length $n+1$ such that

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P & Q \\
Q & Q_{n-1}
\end{array}\right)
$$

where $Q^{2} \equiv(-1)^{n}(\bmod P)$. We will show that $\alpha^{*}$ is a palindrome. To that end, we note that

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(1)}=\left(\begin{array}{cc}
Q & P-\alpha(0) Q \\
Q_{n-1} & Q-\alpha(0) Q_{n-1}
\end{array}\right)
$$

Now, since $P-\alpha(0) Q \equiv P(\bmod Q)$, it follows that $(P-\alpha(0) Q)^{2} \equiv-(-1)^{n}(\bmod Q)$. Therefore, because $Q>2$, we have, by Proposition 4, exactly one of the following possibilities:
(a) $\alpha_{*}=\{\alpha(1), \alpha(2), \ldots, \alpha(n)\}$ is a palindrome with $(P-\alpha(0) Q)^{2} \equiv(-1)^{n-1}(\bmod Q)$;
(b) $\alpha(n)=1$ and $\{\alpha(1), \ldots, \alpha(n-2), \alpha(n-1)+1\}$ is a palindrome with $(P-\alpha(0) Q)^{2} \equiv(-1)^{n}$ $(\bmod Q) ;$
(c) $\alpha(n)>1$ and $\{\alpha(1), \ldots, \alpha(n-1), \alpha(n)-1,1\}$ is a palindrome with $(P-\alpha(0) Q)^{2} \equiv(-1)^{n}$ $(\bmod Q)$.
If either (b) or (c) holds, it follows, by Proposition 3, that $(P-\alpha(0) Q)^{2} \equiv(-1)^{n}(\bmod Q)$. Since $P-a(0) Q \equiv P(\bmod Q)$ and $P^{2} \equiv-(-1)^{n}(\bmod Q)$, we have $(-1)^{n} \equiv-(-1)^{n}(\bmod Q)$, but $Q>2$ makes this impossible. So $\alpha_{*}$ is indeed a palindrome.

By Proposition 5, $\alpha(i)=\alpha(0)=c$ for $0 \leq i \leq n$. That is, $\alpha=\{c, c, \ldots, c\}$. Thus,

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{ll}
c & 1 \\
1 & 0
\end{array}\right)^{n+1}=\left(\begin{array}{cc}
J_{n+1}(c) & J_{n}(c) \\
J_{n}(c) & J_{n-1}(c)
\end{array}\right)
$$

and hence our result.
We need another sequence of polynomials as follows: $H_{-1}(X)=0, H_{0}(X)=1$ and, for $k \geq 0$, $H_{k+1}(X)=(X+1) H_{k}(X)-H_{k-1}(X)$. Equivalently, for $k \geq 1$,

$$
\left(\begin{array}{cc}
X+1 & -1 \\
1 & 0
\end{array}\right)^{k}=\left(\begin{array}{cc}
H_{k}(X) & -H_{k-1}(X) \\
H_{k-1}(X) & -H_{k-2}(X)
\end{array}\right)
$$

Proposition 9: Suppose $P$ and $Q$ are integers with $1<Q<P, Q^{2} \equiv 1(\bmod P), P^{2} \equiv 1(\bmod Q)$, and $P \neq Q+1$. Then there exist integers $k$ and $c$ such that $H_{k}(c)=Q$ and $H_{k+1}(c)=P$.

Proof: By Proposition 3, there exists a Palindrome $\alpha$ of odd length $n+1$ such that

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P & Q \\
Q & Q_{n-1}
\end{array}\right)
$$

We will prove first that $\alpha^{*}$ is also a palindrome. To that end, we observe that

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(1)}=\left(\begin{array}{cc}
Q & P-\alpha(0) Q \\
Q_{n-1} & Q-\alpha(0) Q_{n-1}
\end{array}\right) .
$$

Now $Q>2$ since, otherwise, $Q^{2} \equiv 1(\bmod P)$ implies that $P=3$. That is, $P=Q+1$, a case we have excluded. Next, $P-\alpha(0) Q \equiv P(\bmod Q)$ gives $(P-\alpha(0) Q)^{2} \equiv 1(\bmod Q)$. Now, since $Q>2$ and $(P-\alpha(0) Q)^{2} \equiv 1(\bmod Q)$ we have, by Proposition 4, exactly one of the following possibilities:
(a) $\{\alpha(1), \alpha(2), \ldots, \alpha(n)\}$ is a palindrome and $(P-\alpha(0) Q)^{2} \equiv(-1)^{n-1}(\bmod Q)$;
(b) $\alpha(n)=1,\{\alpha(1), \ldots, \alpha(n-2), \alpha(n-1)+1\}$ is a palindrome and $(P-\alpha(0) Q)^{2} \equiv(-1)^{n}(\bmod$ $Q$ );
(c) $\alpha(n)>1, \alpha^{*}=\{\alpha(1), \ldots, \alpha(n-1), \alpha(n)-1,1\}$ is a palindrome and $(P-\alpha(0) Q)^{2} \equiv(-1)^{n}$ $(\bmod Q)$.
If (a) were true, we would have $(P-\alpha(0) Q)^{2} \equiv-1(\bmod Q)$, since $n-1$ is odd. However, $(P-\alpha(0) Q) \equiv P(\bmod Q)$ and $P^{2} \equiv 1(\bmod Q)$ give $1 \equiv-1(\bmod Q)$, contradicting the conclusion that $Q>2$.

Next, if (b) is true and $n>2$, with $\alpha$ and $\{\alpha(1), \ldots, \alpha(n-2), \alpha(n-1)+1\}$ both palindromes, implies that $\alpha(n-1)=\alpha(1)=\alpha(n-1)+1$, which is clearly impossible. So, if (b) is true, we have $n=2$ and, thus, $\alpha=\{1, \alpha(1), 1\}$. However, in this case, $Q=\alpha(1)+1$ and $P=\alpha(1)+2$, a case we have excluded. Hence, $\alpha(n)>1$ and $\alpha^{*}$ is a palindrome.

Since $\alpha$ and $\alpha^{*}$ are both palindromes and $n$ is even, we have, by Proposition 6, that $\alpha(0)=$ $\alpha(n)=c>1,1=\alpha(1)=\alpha(2 k-1)$ for $1 \leq k \leq n / 2$ and $\alpha(2 k)=c-1$ for $1 \leq k<n / 2$. If $n=2$, then

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=A_{c} A_{1} A_{c}=\left(\begin{array}{cc}
c+1 & -1 \\
1 & 0
\end{array}\right)^{2}\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

An easy induction on $n$ gives, in general, that

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
c+1 & -1 \\
1 & 0
\end{array}\right)^{(n+2) / 2}\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

Now, recalling that $\left(\begin{array}{cc}X+1 & -1 \\ 1 & 0\end{array}\right)^{k}=\left(\begin{array}{cc}H_{k}(X) & -H_{k-1}(X) \\ H_{k-1}(X) & -H_{k-2}(X)\end{array}\right)$ we have

$$
A_{\alpha(n)} A_{\alpha(n-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
H_{n / 2+1}(X) & -H_{n / 2}(X) \\
H_{n / 2}(X) & -H_{n / 2-1}(X)
\end{array}\right)\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)=\left(\begin{array}{cc}
H_{n / 2+1}(c) & H_{n / 2}(c) \\
H_{n / 2}(c) & H_{n / 2-1}(c)
\end{array}\right)
$$

and thus our result.
Remark 3: In the above result $c=\llbracket P / Q \rrbracket$. Furthermore, by Leme's Theorem (see [4]), $n<5 \log _{10}(Q)$, and so $k=n / 2<(5 / 2) \log _{10}(Q)<(5 / 2) \log _{10}(P / 2)$.

If $D$ is a nonsquare, positive integer, then it is known (see [1]) that $\theta=[\sqrt{D}]+\sqrt{D}$ has an infinite purely periodic continued fraction expansion. Let $p$ be the smallest period of $\theta$. Our notation is $\theta=\left[\overline{a_{0}, a_{1}, \ldots, a_{p-1}}\right]$. We now give an elementary proof of a known result (see [5]).

Proposition 10: We claim that $\left\{a_{1}, a_{2}, \ldots, a_{p-1}\right\}$ is a palindrome.
Proof: Set $\alpha=\{\alpha(0), \alpha(1), \ldots, \alpha(p)\}$, where $\alpha(i)=a_{1}$ for $0 \leq i \leq p-1$ and $\alpha(p)=a_{0}$. Setting

$$
A_{\alpha(p-1)} A_{\alpha(p-2)} \cdots A_{\alpha(0)}=\left(\begin{array}{ll}
P_{p-1} & Q_{p-1} \\
P_{p-2} & Q_{p-2}
\end{array}\right)
$$

we have (see [1], p. 329)

$$
\theta=[\alpha(0), \alpha(1), \ldots, \alpha(p-1), \theta]=\frac{\theta P_{p-1}+P_{p-2}}{\theta Q_{p-1}+Q_{p-2}}
$$

Thus, $\theta$ is a root of the quadratic polynomial equation

$$
f(X)=Q_{p-1} X^{2}+\left(Q_{p-2}-P_{p-1}\right) X-P_{p-2}=0 .
$$

However, the minimal polynomial of $\theta$ over the rational numbers is

$$
\left.m(X)=X^{2}-\alpha(0) X+\left(\alpha(0)^{2}-4 D\right)^{2}-4 D\right) / 4
$$

Because $m(X)$ divides the polynomial $f(X)$, we have $Q_{p-2}-P_{p-1}=-\alpha(0) Q_{p-1}$. That is,

$$
P_{p-1}=\alpha(0) q_{p-1}+Q_{p-2}=Q_{p},
$$

where

$$
A_{\alpha(p)} A_{\alpha(p-1)} \cdots A_{\alpha(0)}=\left(\begin{array}{cc}
P_{p} & Q_{p} \\
P_{p-1} & Q_{p-1}
\end{array}\right) .
$$

So, by Proposition 1, $\alpha=\left\{a_{0}, a_{1}, \ldots, a_{p-1}, a_{0}\right\}$ is a palindrome. Thus, it follows that $\left\{a_{1}, \ldots, a_{p-1}\right\}$ is a palindrome.

Remark 4: If $P$ is a positive integer such that $P>1$ and $P$ is a product of primes congruent to 1 modulo 4 or twice such a product, then there exists an integer $Q$ with $1 \leq Q \leq P / 2$ and $Q^{2} \equiv-1$ $(\bmod P)$. By Proposition 3, there is a palindrome $\alpha_{Q}=\{\alpha(0), \alpha(1), \ldots, \alpha(n)\}$ of even length $L_{Q}=n+1$ such that $P / Q=[\alpha(0), \alpha(1), \ldots, \alpha(n)]$. We define the index of $P$ by

$$
I(P)=\min \left\{L_{Q} \mid Q\right\} .
$$

It is clear that for any integer of our type, $I(P)=2$ if and only if there is a positive integer $m$ such that $P=m^{2}+1$. The following seem to be natural questions:
(1) Are there infinitely many integers $P$ of index $i$, for $i$ an even integer bigger than 4 ?
(2) Let $M$ be a positive integer such that $M \geq 2$. Are there infinitely many primes $P$, with $P \equiv 1(\bmod 4)$ and $I(P) \leq M$ ?

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In (1) we have restricted ourselves to $I(P)>4$, because the curious reader will find it easy to produce an infinite number of $P$ with $I(P)=4$. Further, (2) simply generalizes the question: "Are there infinitely many primes of the form $m^{2}+1$ ?

Remark 5: In Proposition 7 we describe all pairs of positive integers $P$ and $Q$ with $P^{2} \equiv 1$ $(\bmod Q)$ and $Q^{2} \equiv 1(\bmod P)$. This problem was posed by Tom Cusick of the University of Buffalo at a meeting of the Seaway Number Theory Conference in May 1991. We understand that he also has a description by a different method.

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# GENERALIZED PASCAL TRIANGLES AND PYRAMIDS: THEIR FRACTALS, GRAPHS, AND APPLICATIONS 

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This monograph was first published in Russia in 1990 and consists of seven chapters, a list of 406 references, an appendix with another 126 references, many illustration and specific examples. Fundamental results in the book are formulated as theorems and algorithms or as equations and formulas. For more details on the contents of the book, see The Fibonacci Quarterly 31.1 (1993):52.

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# FIBONACCI TRANSMISSION LINES 

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## 1. INTRODUCTION

The Fibonacci sequence applies to many diverse areas in science and technology [1, 2]. In a book review by Brother Alfred Brousseau, a significant observation was made which will surely be true for all time: "Enter the magic door which leads to the wonderful world of Fibonacci" [3]. The author found the magic door and is overwhelmed at the beauty of the landscape. This paper will present those findings that helped the author locate the "magic door" and to be fascinated by what is inside. Many other investigators have significantly helped light the way for these findings [5, 6, 7, 8].

## 2. PRELIMINARIES

In Figure 1(a), a two wire transmission line having a characteristic impedance of $Z_{0}$ ohms is shown. The input terminals are marked $a-b$. If a resistive load whose value is chosen equal to the characteristic impedance is placed at a quarter or odd quarter wavelength from the input terminals, the input impedance will be equal to the characteristic impedance and will result in a "matched" line condition. In fact, as long as the load is matched to the characteristic impedance of the line, it can be placed anywhere along the line without changing the input impedance. From an ideal point of view, this is a desired condition; but, it is not achieved in practice whenever two or more loads are connected to the line. At this point, it will be practically advantageous to normalize all connected loads to the characteristic impedance of the line. All connected loads equal to $Z_{0}$ will have the normalized value of 1 or unity and will be referred to as unit loads. If an actual load value is needed, the normalized value can be multiplied by $Z_{0}$ ohms. The next step, as well as succeeding steps, will be to periodically "load" the line at quarter wavelength $\lambda / 4$ or odd quarter wavelength intervals with unit loads and to determine for each load the resultant input impedance. Figure 1(b) shows two loads connected across the line. The second load and associated quarter wavelength section of line places in parallel with the first unit load another unit load which when combined on a parallel resistor basis results in an equivalent load of one half unit. This equivalent load at the input terminals produces a value of two unit loads because of the inversion properties of a quarter wavelength section of line. The two unit loads and the input value of two are coincidental. If a third unit load is connected across the line at a quarter wavelength from load 2 , a total of three loads are connected and the length of the line is three quarter wavelength long relative to the input terminals $a-b$. This is shown schematically in Figure 1(c). Since the previous results showed that the input impedance is equivalent to two unit loads, this places two unit loads in parallel with one unit load which results in an equivalent load of $2 / 3$. At the input terminals, the input impedance becomes $3 / 2$. If this process is continued for " $n$ " sections, it is found that the normalized input impedance of a periodically loaded transmission line is equal to the ratio of two Fibonacci numbers, namely, $F_{n+1} / F_{n}$. This is shown in Figure 1(d). Such a line will be referred to as a "Fibonacci Transmission Line." And if this line is extended in the limit to
large values of " $n$," the normalized input impedance is found to be the "golden ratio," namely, 1.618... [8].

(d)

FIGURE 1
Periodically Loaded Transmission Line

## 3. ANALYSIS

There are many important parameters associated with every transmission line. Some of these parameters, such as the normalized input impedance, reflection coefficient, along with voltage and power ratios are shown in Table 1. Importantly, all parameters are functions of the Fibonacci numbers and related functions.

TABLE 1
Fibonacci Transmission Line Parameters

| $n$ | 1 | 2 | 3 | 4 | 5 | $n$ | $n \rightarrow \infty$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{Z_{I N}}{Z_{0}}$ | 1 | 2 | $\frac{3}{2}$ | $\frac{5}{3}$ | $\frac{8}{5}$ | $\frac{F_{n+1}}{F_{n}}$ | $1.6180 \ldots$ |
| $\Gamma$ | 0 | $-\frac{1}{3}$ | $-\frac{1}{5}$ | $-\frac{2}{8}$ | $-\frac{3}{13}$ | $-\frac{F_{n-1}}{F_{n+2}}$ | $-0.2360 \ldots$ |
| $V S W R$ | 1 | 2 | $\frac{3}{2}$ | $\frac{5}{3}$ | $\frac{8}{5}$ | $\frac{F_{n+1}}{F_{n}}$ | $1.6180 \ldots$ |
| $\frac{V_{I N}}{V_{s}}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{3}{5}$ | $\frac{5}{8}$ | $\frac{8}{13}$ | $\frac{F_{n+1}}{F_{n+2}}$ | $0.6180 \ldots$ |
| $\frac{P_{N}}{P_{A}}$ | 1 | $4 \cdot \frac{2}{9}$ | $4 \cdot \frac{6}{25}$ | $4 \cdot \frac{15}{64}$ | $4 \cdot \frac{40}{169}$ | $4 \cdot \frac{F_{n+1} F_{n}}{\left(F_{n+2}\right)^{2}}$ | $0.9442 \ldots$ |

The input impedance for a section of lossless transmission line is given by (see [9]),

$$
\begin{equation*}
Z_{1 N}=\frac{Z_{0}\left(Z_{L}+j Z_{0} \tan b 1\right)}{\left(Z_{0}+j Z_{L} \tan b 1\right)} \tag{1}
\end{equation*}
$$

where

$$
\begin{aligned}
& Z_{0}=\text { characteristic impedance } \\
& Z_{L}=\text { load impedance } \\
& b=\text { phase constant }=2 \Pi / \lambda \\
& \lambda=\text { wavelength }=v / f \\
& v=\text { wave velocity along the line } \\
& f=\text { frequency of voltage and current waves on the line } \\
& l=\text { length along the line } \\
& j=\sqrt{-1}
\end{aligned}
$$

For the special case when the line is equal to a quarter wavelength, equation (1) becomes

$$
\begin{equation*}
Z_{I N}=\frac{\left(Z_{0}\right)^{2}}{Z_{L}} \tag{2}
\end{equation*}
$$

In Figure 1(b), the input impedance of the second load and line is $Z_{0}$ ohms. This equivalent impedance when combined with load \#1 becomes

$$
\begin{equation*}
Z_{L}=\frac{Z_{0} Z_{0}}{\left(Z_{0}+Z_{0}\right)}=\frac{Z_{0}}{2} \tag{3}
\end{equation*}
$$

When the value of $Z_{L}$ is used in equation (2), the input impedance becomes

$$
\begin{equation*}
Z_{I N}=2 Z_{0} \tag{4}
\end{equation*}
$$

If another cell, as shown in Figure 1(c), is connected to the first two cells, the equivalent load can be determined by combining the resistive loads,

$$
\begin{equation*}
Z_{L}=\frac{2 Z_{0} Z_{0}}{\left(2 Z_{0}+Z_{0}\right)}=\frac{2 Z_{0}}{3}, \tag{5}
\end{equation*}
$$

using equation (2), the input impedance becomes

$$
\begin{equation*}
Z_{I N}=\left(\frac{3}{2}\right) Z_{0} \tag{6}
\end{equation*}
$$

For Figure 1(d), the input impedance for $n$ sections is given by

$$
\begin{equation*}
Z_{I N}=\left(\frac{F_{n+1}}{F_{n}}\right) Z_{0} ; n \geq 1, \tag{7}
\end{equation*}
$$

where $F_{n+1}$ and $F_{n}$ are two consecutive Fibonacci numbers. For large $n$, equation (7) becomes

$$
\begin{equation*}
Z_{I N}=\lim _{n \rightarrow \infty}\left(\frac{F_{n+1}}{F_{n}}\right) Z_{0}=(1.61803 \ldots) Z_{0} \tag{8}
\end{equation*}
$$

The reflection coefficient and voltage standing wave ratio are important parameters that describe the behavior of transmission lines in relation to another connected transmission line or to a connected load. The reflection coefficient is defined as the ratio of reflected to incident voltage or current wave amplitudes. In general, it will be a complex quantity having amplitude and angle values. In terms of a connected load, it is determined by

$$
\begin{equation*}
\Gamma=\frac{Z_{L}-Z_{0}}{Z_{L}+Z_{0}} \tag{9}
\end{equation*}
$$

For a Fibonacci transmission line, (9) becomes

$$
\begin{equation*}
\Gamma=-\frac{F_{n-1}}{F_{n+2}} ; n \geq 1 \tag{10}
\end{equation*}
$$

The voltage standing wave ratio is determined by the ratio of maximum to minimum voltage amplitudes along the line. In terms of the reflection coefficient,

$$
\begin{equation*}
V S W R=\frac{1+|\Gamma|}{1-|\Gamma|} \tag{11}
\end{equation*}
$$

Using (10), the VSWR is

$$
\begin{equation*}
V S W R=\frac{F_{n+1}}{F_{n}} ; n \geq 1 \tag{12}
\end{equation*}
$$

The circuit shown in Figure 2 will be used to determine the input voltage to a Fibonacci transmission line. The generator impedance is made equal to $Z_{0}$ for convenience. By using the voltage divider rule, the input voltage can be written as

$$
\begin{gather*}
V_{I N}=\frac{Z_{I N} V_{s}}{Z_{I N}+Z_{0}}  \tag{13}\\
\frac{V_{I N}}{V_{s}}=\frac{F_{n+1}}{F_{n+2}} ; n \geq 1 . \tag{14}
\end{gather*}
$$

The last parameter considered is the ratio of input power, $P_{I N}$, to available power, $P_{A}$ :

$$
\begin{gather*}
P_{I N}=\frac{V_{I N}^{2}}{Z_{I N}} ;  \tag{15}\\
P_{A}=\frac{V_{s}^{2}}{4 Z_{0}} ;  \tag{16}\\
\frac{P_{I N}}{P_{A}}=\frac{4 F_{n+1} F_{n}}{\left(F_{n+2}\right)^{2}}=\frac{\left(F_{n+2}\right)^{2}-\left(F_{n-1}\right)^{2}}{\left(F_{n+2}\right)^{2}}=1-\left(\frac{F_{n-1}}{F_{n+2}}\right)^{2}=1-\Gamma^{2} .  \tag{17}\\
\end{gather*}
$$

FIGURE 2

## Fibonacci Transmission Line Circuit

It is interesting to consider what if situations for Fibonacci Transmission Lines (FTL) and ladder-type electrical networks. First, for an $n$ loaded FTL: What is the resultant input impedance of an FTL if each of the $n$ loads of $Z_{0}$ ohms is replaced by another FTL which has $m-Z_{0}$ ohm loaded sections? A schematic of the what if FTL is shown in Figure 3.


FIGURE 3
Fibonacci Transmission Line of Fibonacci Transmission Lines

From basic transmission line theory, if an impedance $Z_{1}$ is connected as a load in a $\lambda / 4$ section of line having a characteristic impedance of $Z_{0}$, as shown in Figure 4(a), the input impedance is

$$
\begin{equation*}
Z_{N_{1}}=\frac{Z_{0}^{2}}{Z_{1}} \tag{18}
\end{equation*}
$$

If another identical load, $Z_{1}$, is connected, as shown in Figure 4(b), the input impedance is

$$
\begin{equation*}
Z_{I N_{2}}=\frac{Z_{1}^{2}+Z_{0}^{2}}{Z_{1}} \tag{19}
\end{equation*}
$$

If a third load, $Z_{1}$, is connected, as shown in Figure 4(c), the input impedance is

$$
\begin{equation*}
Z_{I N_{3}}=\frac{Z_{0}^{2}\left(2 Z_{1}^{2}+Z_{0}^{2}\right)}{\left(Z_{1}^{2}+Z_{0}^{2}\right) Z_{1}} . \tag{20}
\end{equation*}
$$

If a fourth load, $Z_{1}$, is connected, as shown in Figure 4(d), is

$$
\begin{equation*}
Z_{I N_{4}}=\frac{Z_{1}^{4}+3 Z_{1}^{2} Z_{0}^{2}+Z_{0}^{4}}{\left(2 Z_{1}^{2}+Z_{0}^{2}\right) Z_{1}} \tag{21}
\end{equation*}
$$

If a fifth load, $Z_{1}$, is connected, as shown in Figure 4(e), the input impedance is

$$
\begin{equation*}
Z_{I N_{\mathrm{s}}}=\frac{Z_{0}^{2}\left(3 Z_{1}^{4}+4 Z_{1}^{2} Z_{0}^{2}+Z_{0}^{4}\right)}{\left(Z_{1}^{4}+3 Z_{1}^{2} Z_{0}^{2}+Z_{0}^{4}\right) Z_{1}} \tag{22}
\end{equation*}
$$

Let $a_{1}$ be the parameter for the normalized $Z_{1}$ impedance, $a_{1}=\frac{Z_{1}}{Z_{0}}=\frac{F_{m+1}}{F_{m}}$, then:

$$
\begin{gather*}
Z_{I N_{1}}=\frac{Z_{0}^{2}}{Z_{1}}=\frac{Z_{0}}{\frac{Z_{1}}{Z_{0}}}=\frac{Z_{0}}{a_{1}} ;  \tag{23}\\
Z_{I N_{2}}=\frac{Z_{1}^{2}+Z_{0}^{2}}{Z_{1}}=\frac{Z_{0}}{a_{1}}\left(1+a_{1}^{2}\right) ;  \tag{24}\\
Z_{I N_{3}}=\frac{Z_{0}^{2}}{Z_{1}}\left(\frac{2 Z_{1}^{2}+Z_{0}^{2}}{Z_{1}^{2}+Z_{0}^{2}}\right)=\frac{Z_{0}}{a_{1}}\left(\frac{1+2 a_{1}^{2}}{1+a_{1}^{2}}\right) ;  \tag{25}\\
Z_{I N_{4}}=\frac{Z_{0}}{a_{1}} \cdot \frac{\left(1+3 a_{1}^{2}+a_{1}^{2}\right)}{\left(1+2 a_{1}^{2}\right)} ;  \tag{26}\\
Z_{I N_{5}}=\frac{Z_{0}}{a_{1}} \cdot \frac{\left(1+4 a_{1}^{2}+3 a_{1}^{4}\right)}{\left(1+3 a_{1}^{2}+a_{1}^{4}\right)} . \tag{27}
\end{gather*}
$$

## FIBONACCI TRANSMISSION LINES

The polynomials in the numerator and denominator are Jacobsthal polynomials (see [10, 11]).

$$
\begin{equation*}
J_{n}(x)=J_{n-1}(x)+x J_{n-2}(x) \tag{28}
\end{equation*}
$$

with $J_{1}(x)=J_{2}(x)=1$.
In the Fibonacci transmission line structure,

$$
\begin{equation*}
x=a_{1}^{2} \tag{29}
\end{equation*}
$$



FIGURE 4
Periodically Loaded Transmission Lines

Using the Jacobsthal polynomials, the input impedances can be rewritten as:

$$
\begin{gather*}
Z_{I N_{1}}=\frac{Z_{0}}{a_{1}} \cdot \frac{J_{2}}{J_{1}} ;  \tag{30}\\
Z_{I N_{2}}=\frac{Z_{0}}{a_{1}}\left(1+a_{1}^{2}\right)=\frac{Z_{0}}{a_{1}} \cdot \frac{J_{3}}{J_{2}} ;  \tag{3}\\
Z_{I N_{3}}=\frac{Z_{0}}{a_{1}}\left(\frac{1+2 a_{1}^{2}}{1+a_{1}^{2}}\right)=\frac{Z_{0}}{a_{1}} \cdot \frac{J_{4}}{J_{3}} . \tag{32}
\end{gather*}
$$

Finally, for $n$ connected $Z_{1}$ loads,

$$
\begin{equation*}
Z_{I N_{n}}=\frac{Z_{0}}{a_{1}} \cdot \frac{J_{n+1}\left(a_{1}\right)}{J_{n}\left(a_{1}\right)} . \tag{33}
\end{equation*}
$$

The general term of the Jacobsthal polynomials is given by

$$
\begin{equation*}
J_{n}\left(a_{1}\right)=\frac{1}{\sqrt{1+4 a_{1}^{2}}}\left[\left(\frac{1+\sqrt{1+4 a_{1}^{2}}}{2}\right)^{n}-\left(\frac{1-\sqrt{1+4 a_{1}^{2}}}{2}\right)^{n}\right] . \tag{34}
\end{equation*}
$$

For the case $a_{1}=1$, the Jacobsthal sequence is the Fibonacci sequence. Other expressions for Jacobsthal polynomials are:

$$
\begin{align*}
& J_{n}=\frac{2 \sqrt{a_{1}^{2 n}}}{\sqrt{1+4 a_{1}^{2}}} \cdot \sinh \left\{n\left[\ln \left(\frac{1+\sqrt{1+4 a_{1}^{2}}}{2 \sqrt{a_{1}^{2}}}\right)\right]\right\} ;  \tag{35}\\
& J_{n}=\frac{2 \sqrt{a_{1}^{2 n}}}{\sqrt{1+4 a_{1}^{2}}} \cdot \cosh \left\{n\left[\ln \left(\frac{1+\sqrt{1+4 a_{1}^{2}}}{2 \sqrt{a_{1}^{2}}}\right)\right]\right\} . \tag{36}
\end{align*}
$$

If each matched load in a Fibonacci transmission line is replaced by another Fibonacci transmission line, as shown in Figure 3, the resultant input impedance is given by

$$
\begin{equation*}
Z_{I N}^{(1)}=Z_{0}\left(\frac{F_{m}}{F_{m+1}}\right)\left(\frac{J_{n+1}}{J_{n}}\right), \tag{37}
\end{equation*}
$$

where the superscript number in parentheses represents the first replacement of each connected load by an $m$-loaded FTL. In the special case $m=n, Z_{I N}^{(1)}$ becomes

$$
\begin{equation*}
Z_{I N}^{(1)}=Z_{0}\left(\frac{F_{n}}{F_{n+1}}\right)\left(\frac{J_{n+1}\left(a_{1}\right)}{J_{n}\left(a_{1}\right)}\right) . \tag{38}
\end{equation*}
$$

If a second replacement of each $Z_{0}$ in the first replacement transmission line is made, the input impedance becomes

$$
\begin{equation*}
Z_{I N}^{(2)}=Z_{0}\left(\frac{F_{n+1}}{F_{n}}\right)\left[\frac{J_{n}\left(a_{1}\right)}{J_{n+1}\left(a_{1}\right)}\right]\left[\frac{J_{n+1}\left(a_{2}\right)}{J_{n}\left(a_{2}\right)}\right], \tag{39}
\end{equation*}
$$

where

$$
a_{1}=\frac{F_{n+1}}{F_{n}} \text { and } a_{2}=\frac{F_{n}}{F_{n+1}}\left[\frac{J_{n+1}\left(a_{1}\right)}{J_{n}\left(a_{1}\right)}\right] .
$$

If a third replacement is made, the input impedance becomes

$$
\begin{equation*}
Z_{I N}^{(3)}=Z_{0}\left(\frac{F_{n}}{F_{n+1}}\right)\left[\frac{J_{n+1}\left(a_{1}\right)}{J_{n}\left(a_{1}\right)}\right]\left[\frac{J_{n}\left(a_{2}\right)}{J_{n+1}\left(a_{2}\right)}\right]\left[\frac{J_{n+1}\left(a_{3}\right)}{J_{n}\left(a_{3}\right)}\right], \tag{40}
\end{equation*}
$$

where

$$
a_{3}=\frac{F_{n+1}}{F_{n}}\left[\frac{J_{n}\left(a_{1}\right)}{J_{n+1}\left(a_{1}\right)}\right]\left[\frac{J_{n+1}\left(a_{2}\right)}{J_{n}\left(a_{2}\right)}\right] .
$$

Next, for ladder electrical networks, the input resistance for $m$ half- $T$ sections is given by equation (9) in reference [8]. Rewriting the reference equation,

$$
\begin{equation*}
Z_{I N}=\left(\frac{F_{2 m+1}}{F_{2 m}}\right) R ; m \geq 1, \tag{41}
\end{equation*}
$$

where $R$ is the value in ohms of each resistor in the ladder network. The ladder network is shown schematically in Figure 5(a). Like the FTL, if each resistor $R$ in the ladder is replaced by $n$ half- $T$ sections in a ladder configuration with individual input impedance of

$$
\begin{equation*}
Z_{I N}=\left(\frac{F_{2 n+1}}{F_{2 n}}\right) R ; n \geq 1, \tag{42}
\end{equation*}
$$

the resultant input impedance of a ladder of ladders is

$$
\begin{equation*}
Z_{I N}=\left(\frac{F_{2 m+1}}{F_{2 m}}\right)\left(\frac{F_{2 n+1}}{F_{2 n}}\right) R ; m \geq 1 \text { and } n \geq 1 . \tag{43}
\end{equation*}
$$

For the special case $m=n$, the input resistance is

$$
\begin{equation*}
Z_{I N}=\left(\frac{F_{2 n+1}}{F_{2 n}}\right)^{2} R ; n \geq 1 . \tag{44}
\end{equation*}
$$

An implementation of a ladder of ladders is shown in Figure 5(b).
To conclude this development, the FTL and ladder will be extended to include $K$ and $M$ replacements or iterations of the basic symmetrical networks, respectively. After $K$ replacements, the input impedance, $Z_{I N}^{(K)}$, can be written as

$$
\begin{equation*}
Z_{I N}^{(K)}=Z_{0}\left(\frac{F_{n+1}}{F_{n}}\right)^{(-1)^{K}} \prod_{i=1}^{K}\left[\frac{J_{n+1}\left(a_{i}\right)}{J_{n}\left(a_{i}\right)}\right]^{(-1)^{K-1}} ; K \geq 1, \tag{45}
\end{equation*}
$$

and, for the symmetrical ladder network, the input impedance is

$$
\begin{equation*}
Z_{I N}=\left(\frac{F_{2 n+1}}{F_{2 n}}\right)^{M+1} R ; n \geq 1 \text { and } M=0, \pm 1, \pm 2, \ldots \tag{46}
\end{equation*}
$$

A brief look inside the $M$ door shows that the input impedance ranges from an open circuit to a short circuit as $M$ increases positively or negatively, respectively. Interestingly, for ladder networks, the equivalent resistance of each element increases for positive $M$ and decreases for negative $M$. This suggests series paths for positive $M$ and parallel paths for negative $M$. Figure 6 shows a ladder of ladders for different values of $M$.

(a) $m$ HALF-T SECTION LADDER NETWORK

(b) $m$ HALF-T SECTION LADDER OF LADDERS

FIGURE 5
Ladder of Ladders Network


## 4. CONCLUSIONS

The results of this investigation shows that the Fibonacci sequence and related functions can be used to analyze periodically loaded wave transmission structures. This is an important result that opens new doors to a variety of transmission systems investigations. For example, these results can be used to analyze local area networks (LAN) that use transmission lines to tie computers togather or for array-type antennas excited by transmission lines and used for either reception or transmission. Another important finding of this investigation is the extension of

Fibonacci transmission lines and ladder networks to higher-order structures by an iterative process. Importantly, the results presented in this paper open many new doors which lead to new doors and more doors or doors \{doors[doors(doors)]...\}. In conclusion, the world of Fibonacci provides many opportunities for new and exciting discoveries.

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# ON POINTS WHOSE COORDINATES ARE TERMS OF A LINEAR RECURRENCE* 

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## 1. INTRODUCTION

Let $R=\left\{R_{n}\right\}_{n=0}^{\infty}$ be a second-order recurrent sequence (generalized Fibonacci sequence) of integers defined by

$$
R_{n}=A R_{n-1}-B R_{n-2} \quad(\text { for } n>1),
$$

where the initial terms are $R_{0}=0, R_{1}=1$, and $A$ and $B$ are fixed nonzero integers. Let $\alpha$ and $\beta$ be the roots of the characteristic polynomial $x^{2}-A x+B$. We will assume that the discriminant $D=A^{2}-4 B>0$ and $D$ is not a perfect square. From this, it follows that the sequence $R$ is not degenerate, i.e., $\alpha / \beta$ is not a root of unity. In this case, $\alpha$ and $\beta$ are two irrational real numbers and $|\alpha| \neq|\beta|$, so we can suppose that $|\alpha|>|\beta|$. Also, $0<\beta$ iff $0<A \cdot B$. And $0<\beta<1$ holds iff $0<B(A-B-1)$.

It is well known that the terms of $R$ can be given by

$$
\begin{equation*}
R_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}=\frac{\alpha^{n}-\beta^{n}}{\sqrt{D}} . \tag{1}
\end{equation*}
$$

Furthermore

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{R_{n+1}}{R_{n}}=\alpha \tag{2}
\end{equation*}
$$

(see, e.g., [3] or [6]).
Limit (2) implies that $\alpha$ can be approximated by the rational numbers $R_{n+1} / R_{n}$. The second author, P. Kiss [5], proved that when $B=1$ this approximation is good in the sense that

$$
\left|\alpha-\frac{R_{n+1}}{R_{n}}\right|<\frac{1}{c \cdot R_{n}^{2}}
$$

holds for some $c$ and infinitely many $n$.
It was also proved in [5] that this inequality holds for infinitely many $n$ only when $|B|=1$.
In this paper the points $P_{n}=\left(R_{n}, R_{n+1}\right)$ will be considered from a geometric point of view, as points on the Euclidean plane. G. E. Bergum [1] and A. F. Horadam [2] showed that the points $P_{n}=(x, y)$ lie on the conic section $B x^{2}-A x y+y^{2}+e B^{n}=0$, where $e=A R_{0} R_{1}-B R_{0}^{2}-R_{1}^{2}$, and

[^0]the initial terms $R_{0}$ and $R_{1}$ are not necessarily 0 and 1 . In their treatment of this equation, they showed that in the case $|B|=1$, when the conic is a hyperbola, the asymptotes of the hyperbola are the lines $y=\alpha x$ and $y=\beta x$. This corresponds to limit (2). For the Fibonacci sequence, when $A=1$ and $B=-1$, C. Kimberling [4] characterized those conics satisfied by infinitely many Fibonacci lattice points $(x, y)=\left(F_{m}, F_{n}\right)$.

In this paper we again investigate the geometric properties of $P_{n}$ in both the two- and threedimensional cases.

## 2. THE TWO-DIMENSIONAL CASE

Let us consider the points $P_{n}=\left(R_{n}, R_{n+1}\right), n=0,1,2, \ldots$, on the plane whose coordinates are consecutive elements of the sequence $R$. Then (2) shows that the slope of the vector $O P_{n}$ tends to $\alpha$. But it is not obvious that the points $P_{n}$ approach the line $y=\alpha x$, as $n \rightarrow \infty$. The following theorem shows that this is the case, however.

Theorem 1: Let $d_{n}$ denote the distance from the point $P_{n}=\left(R_{n}, R_{n+1}\right)$ to the line $y=\alpha x$. Then $\lim _{n \rightarrow \infty} d_{n}=0$ if and only if $|\beta|<1$.

Proof: The distance $d_{x_{0}, y_{0}}$ from a point $\left(x_{0}, y_{0}\right)$ to the line $y=\alpha x$ is given by

$$
\begin{equation*}
d_{x_{0}, y_{0}}=\left|\frac{\alpha x_{0}-y_{0}}{\sqrt{\alpha^{2}+1}}\right| \text {, } \tag{3}
\end{equation*}
$$

so, using (1), we have

$$
\begin{equation*}
d_{n}=\left|\frac{\alpha R_{n}-R_{n+1}}{\sqrt{\alpha^{2}+1}}\right|=\left|\frac{\frac{\alpha^{n+1}-\beta^{n} \alpha}{\alpha-\beta}-\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}}{\sqrt{\alpha^{2}+1}}\right|=\frac{|\beta|^{n}}{\sqrt{\alpha^{2}+1}} \tag{4}
\end{equation*}
$$

from which the theorem follows.
Remark: $|\beta|<1$ holds when $|B+1|<|A|$.
This theorem implies that the points $P_{n}$ converge to the line $y=\alpha x$ if $|\beta|<1$, but not necessarily that these lattice points $P_{n}$ are the nearest (in the sense of Theorem 2) lattice points to $y=\alpha x$ in all cases. For, let $d_{x, y}$ denote the distance between the lattice point $(x, y)$ and the line $y=\alpha x$, and let $d_{n}$ be the distance mentioned in the theorem. We prove

Theorem 2: For integers $u, v$, denote by $d_{u, v}$ the distance from the lattice point $(u, v)$ to the line $y=\alpha x$ and let $d_{n}$ be the distance defined in Theorem 1. Then when $|B|=1$, there is no lattice point ( $x, y$ ) such that $d_{x, y} \leq d_{n}$ and $|x|<\left|R_{n}\right|$. Furthermore, for sufficiently large $n$, this holds if and only if $|B|=1$.

Proof: First suppose $|B|=1$. In this case, obviously, $|\beta|<1$ and $\alpha$ is irrational. Assume that for some $n$ there is a lattice point $(x, y)$ such that $d_{x, y} \leq d_{n}$ and $|x|<\left|R_{n}\right|$. Then, by (3) and
(4), $|\alpha x-y| \leq|\beta|^{n}$ follows. From this, using (1) and the fact that $|\alpha \beta|=|B|=1$, we obtain the inequalities

$$
\begin{equation*}
\left|\alpha-\frac{y}{x}\right| \leq \frac{|\beta|^{n}}{|x|}=\frac{1}{|\alpha|^{n}|x|}=\frac{\left|1-(\beta / \alpha)^{n}\right|}{\sqrt{D} \cdot\left|R_{n} x\right|}<\frac{\left|1-(\beta / \alpha)^{n}\right|}{\sqrt{D} \cdot x^{2}} . \tag{5}
\end{equation*}
$$

In [5] it was proved that if $|B|=1$, and $p, q$ are integers such that $(p, q)=1$ and

$$
\left|\alpha-\frac{p}{q}\right|<\frac{1}{\sqrt{D} \cdot q^{2}}
$$

then $p / q$ has the form $p / q=R_{i+1} / R_{i}$ for some $i$. The proof also shows that (5) holds only if $x=R_{i}$ and $y=R_{i+1}$ for some $i$, if $n$ is large enough. So $x=R_{i}$ is a term of the sequence $R$. The sequence $R$ is a nondegenerate one with $D>0$ and $|B|=1$. So it can easily be seen that $\left|R_{t}\right|$, $\left|R_{t+1}\right|, \ldots$, is an increasing sequence if $t$ is sufficiently large. Furthermore, by (4), $d_{k}>d_{j}$ if $k<j$. Thus, $i<n$ and $d_{i}>d_{n}$ follows, which contradicts $d_{i}=d_{x, y} \leq d_{n}$. So the first part of the theorem is proved.

To complete the proof, we have to prove that if $|B|>1$, then there are infinitely many pairs of lattice points $(x, y)$ such that $d_{x, y}<d_{n}$ and $|x|<\left|R_{n}\right|$ for any sufficiently large $n$.

Suppose $|B|>1$. If $|\beta|<1$, then, by (4), $d_{n} \rightarrow \infty$ as $n \rightarrow \infty$, so there are lattice points $(x, y)$ such that $d_{x, y}<d_{n}$ and $|x|<\left|R_{n}\right|$ for any sufficiently large $n$.

If $|\beta|=1$, then $d_{n}$ is a constant and there are infinitely many points $(x, y)$ such that $d_{x, y} \leq d_{n}$ and $|x|<\left|R_{n}\right|$ for some $n$, since $\left|R_{n}\right| \rightarrow \infty$ as $n \rightarrow \infty$.

If $|\boldsymbol{\beta}|<1$, then by (4) and $|B|>1$, we have

$$
\begin{equation*}
\left|\alpha-\frac{R_{n+1}}{R_{n}}\right|=\frac{|\beta|^{n}}{\left|R_{n}\right|}=\frac{|B|^{n}\left|1-(\beta / \alpha)^{n}\right|}{\sqrt{D} \cdot R_{n}^{2}}>\frac{Q}{R_{n}^{2}} \tag{6}
\end{equation*}
$$

for any fixed $Q>0$ if $n$ is sufficiently large. In this case, the roots $\alpha, \beta$ are irrational numbers since, if the roots of the polynomial $x^{2}-A x+B$ are rational, then they are integers; so $0<|\beta|<1$ would be impossible. It is known that if $r_{k}=y / x$ is a convergent of the continued fraction expansion of $\alpha$, then

$$
\begin{equation*}
\left|\alpha-\frac{y}{x}\right|<\frac{1}{2|x|^{2}} . \tag{7}
\end{equation*}
$$

Let $y$, and hence $x$, be large enough and let the index $n$ be defined by $\left|R_{n-1}\right| \leq|x|<\left|R_{n}\right|$. From (3), (4), (6), and (7), we obtain the inequalities

$$
d_{n}>\frac{Q}{\left|R_{n}\right| \cdot \sqrt{\alpha^{2}+1}} \text { and } d_{x, y}<\frac{1}{2|x| \sqrt{\alpha^{2}+1}} .
$$

But, by (1),

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$$
\frac{Q}{\left|R_{n}\right|}=\frac{Q}{\left|R_{n-1} \alpha\right|\left(1-(\beta / \alpha)^{n}\right) /\left(1-(\beta / \alpha)^{n-1}\right)}>\frac{1}{2\left|R_{n-1}\right|} \geq \frac{1}{2|x|}
$$

and so $d_{x, y}<d_{n}$ with $|x|<\left|R_{n}\right|$, which completes the proof of the theorem.
Lastly, we give equations that are satisfied by the lattice points ( $R_{n}, R_{n+1}$ ).
Theorem 3: All lattice points $(x, y)=\left(R_{n}, R_{n+1}\right)$ satisfy one of the equations

$$
\text { (i) } y=\alpha x+c(x) \cdot|x|^{\delta} \text { or (ii) } y=\alpha x-c(x) \cdot|x|^{\delta} \text {, }
$$

where $\delta=\log |\beta| / \log |\alpha|$ and $c(x)$ is a function such that $\lim _{x \rightarrow \infty} c(x)=\sqrt{D}^{\delta}$.
Remark: This shows that the sequence of lattice points $\left(R_{n}, R_{n+1}\right)$ tends to the line $y=\alpha x$ only if $\delta<0$, i.e., iff $|\boldsymbol{\beta}|<1$, as proved in Theorem 1.

Proof: By (1), we have

$$
\begin{equation*}
R_{n+1}=\alpha \frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}+\frac{\alpha \beta^{n}-\beta^{n+1}}{\alpha-\beta}=\alpha R_{n}+\beta^{n} \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|R_{n}\right|=\frac{|\alpha|^{n}}{\sqrt{D}}\left(1-(\beta / \alpha)^{n}\right) . \tag{9}
\end{equation*}
$$

From (9), we have $n=\frac{\log \left|R_{n}\right|+\log \sqrt{D}-\varepsilon_{n}}{\log |\alpha|}$ where $\varepsilon_{n}=\log \left(1-(\beta / \alpha)^{n}\right)$ and $\varepsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$ since $|\beta / \alpha|<1$. This implies that

$$
\begin{equation*}
\beta^{n}= \pm \exp \left\{\frac{\log |\beta| \cdot \log \left|R_{n}\right|}{\log |\alpha|}+\frac{\log |\beta| \cdot \log \sqrt{D}}{\log |\alpha|}-\frac{\varepsilon_{n} \cdot \log |\beta|}{\log |\alpha|}\right\}= \pm\left|R_{n}\right|^{\delta} \cdot \sqrt{D}^{\delta_{n}} \tag{10}
\end{equation*}
$$

where $\delta=\log |\beta| / \log |\alpha|$ and

$$
\begin{equation*}
\delta_{n}=\frac{\log |\beta|}{\log |\alpha|}-\frac{\varepsilon_{n} \cdot \log |\beta|}{\log \sqrt{D} \cdot \log |\alpha|} \rightarrow \delta \text { as } n \rightarrow \infty, \tag{11}
\end{equation*}
$$

since $\varepsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$.
From (8), (10), and (11), the theorem follows.
Remark: The lattice points ( $R_{n}, R_{n+1}$ ) safisfy (i) for every $n$ if $\beta>0$. If $\beta<0$, then the lattice points satisfy alternately (i) and (ii).

## 3. THE THREE-DIMENSIONAL CASE

Now we consider the three-dimensional vectors ( $R_{n}, R_{n+1}, R_{n+2}$ ). Since by (1),

$$
\frac{R_{n+2}}{R_{n}}=\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}=\alpha^{2} \frac{\left(1-(\beta / \alpha)^{n+1}\right)}{1-(\beta / \alpha)^{n}} \rightarrow \alpha^{2}, \text { as } n \rightarrow \infty,
$$

$R_{n+1} / R_{n} \rightarrow \alpha$, as $n \rightarrow \infty$, by (2), and

$$
\left(R_{n}, R_{n+1}, R_{n+2}\right)=R_{n}\left(1, \frac{R_{n+1}}{R_{n}}, \frac{R_{n+2}}{R_{n}}\right)
$$

That is, the direction of the vectors $\left(R_{n}, R_{n+1}, R_{n+2}\right)$ tends to the direction of the vector $\left(1, \alpha, \alpha^{2}\right)$. However, the sequence of the lattice points $P_{n}=\left(R_{n}, R_{n+1}, R_{n+2}\right)$ does not always tend to the line passing through the origin and parallel to the vector ( $1, \alpha, \alpha^{2}$ ). We will prove the analog of Theorem 1.

Theorem 4: let $L$ be a line defined by $x=t, y=\alpha t, z=\alpha^{2} t, t \in \mathbb{R}$. Furthermore, let $d_{n}$ be the distance from the point $\left(R_{n}, R_{n+1}, R_{n+2}\right), n=0,1,2, \ldots$, to the line $L$. Then $\lim _{n \rightarrow \infty} d_{n}=0$ if and only if $|\beta|<1$.

Proof: It is not difficult to show that the distance from the point $\left(x_{0}, y_{0}, z_{0}\right)$ to the line $L$ is

$$
\begin{equation*}
d_{x_{0}, y_{0}, z_{0}}=\sqrt{\frac{\left(x_{0} \alpha^{2}-z_{0}\right)^{2}+\left(x_{0} \alpha-y_{0}\right)^{2}+\left(y_{0} \alpha^{2}-z_{0} \alpha\right)^{2}}{1+\alpha^{2}+\alpha^{4}}} \tag{12}
\end{equation*}
$$

This notation is necessary for Theorem 5 .
By (12) and (1), we have

$$
\begin{align*}
d_{n} & =\sqrt{\frac{\left(\beta^{n+2}-\alpha^{2} \beta^{n}\right)^{2}+\left(\beta^{n+1}-\alpha \beta^{n}\right)^{2}+\left(\alpha \beta^{n+2}-\alpha^{2} \beta^{n+1}\right)^{2}}{(\alpha-\beta)^{2}\left(1+\alpha^{2}+\alpha^{4}\right)}} \\
& =|\beta|^{n} \sqrt{\frac{(\alpha+\beta)^{2}+1+(\alpha \beta)^{2}}{1+\alpha^{2}+\alpha^{4}}}=|\beta|^{n} \sqrt{\frac{A^{2}+B^{2}+1}{1+\alpha^{2}+\alpha^{4}}} \tag{13}
\end{align*}
$$

where we have used $\alpha+\beta=A$ and $\alpha \beta=B$ since $\alpha$ and $\beta$ are the zeros of the polynomial $x^{2}-A x+B$. From this, the theorem follows.

Theorem 2 can also be generalized to the three-dimensional case, i.e., to state that the lattice points $\left(R_{n}, R_{n+1}, R_{n+2}\right)$ are the nearest lattice points to the line $L$ iff $|B|=1$.

Theorem 5: Let $L$ be the line defined in Theorem 4. Let $d_{n}$ and $d_{x, y, z}$ be the distances defined in Theorem 4 and its proof. Then, for sufficiently large $n$, there is no lattice point $(x, y, z)$ such that $d_{x, y, z} \leq d_{n}$ and $|x|<\left|R_{n}\right|$ if and only if $|B|=1$.

Proof: Suppose $|B|=1$. Then, since $0<D=A^{2}-4 B, \alpha$ is irrational because $A^{2} \pm 4$ is not a perfect square.

Let $(x, y, z)$ be a lattice point such that

$$
\begin{equation*}
d_{x, y, z} \leq d_{n} \tag{14}
\end{equation*}
$$

for some $n$ and $|x|<\left|R_{n}\right|$. By Theorem $4 d_{x, y, z}<\varepsilon$ follows for any $\varepsilon>0$ if $n$ is sufficiently large. But then, by (12), $\left|x \alpha^{2}-z\right|,|x \alpha-y|$, and $\left|y \alpha^{2}-z \alpha\right|$ are sufficiently small. If $|x \alpha-y|$ is a small number then, since $\alpha^{2}=A \alpha-B,\left|x \alpha^{2}-z\right|=|A x \alpha-(z+B x)|$ can be small only if $z+B x=A y$, i.e., only if $z=A y-B x$. In this case

$$
\left|x \alpha^{2}-z\right|=A \cdot|x \alpha-y| \text { and }\left|y \alpha^{2}-z \alpha\right|=|(z-A y) \alpha+B y|=|B| \cdot|x \alpha-y|
$$

are also small. Thus, from (12), (13), and (14),

$$
d_{x, y, z}=\sqrt{\frac{A^{2}+B^{2}+1}{1+\alpha^{2}+\alpha^{4}}} \cdot|x \alpha-y| \leq|\beta|^{n} \sqrt{\frac{A^{2}+B^{2}+1}{1+\alpha^{2}+\alpha^{4}}}
$$

and so, using $|x|<\left|R_{n}\right|$ and $|\alpha \beta|=|B|=1$, we get

$$
\left|\alpha-\frac{y}{x}\right| \leq \frac{|\beta|^{n}}{|x|}=\frac{1}{|\alpha|^{n}|x|}=\frac{1-(\beta / \alpha)^{n}}{\left|R_{n}\right| \cdot \sqrt{D} \cdot|x|}<\frac{1-(\beta / \alpha)^{n}}{\sqrt{D} \cdot|x|^{2}} .
$$

From this, as above, we obtain $x=R_{i}, y=R_{i+1}$, and $z=A y-B x=R_{i+2}$ for some natural number $i$, if $n$ is sufficiently large. Thus, $d_{x, y, z}=d_{i}$. But by (13), $d_{k}<d_{n}$ only if $k>n$, so $i \geq n$ and $|x|=$ $\left|R_{i}\right| \geq\left|R_{n}\right|$, which contradicts the assumption $|x|<\left|R_{n}\right|$, since the sequence $\left|R_{n}\right|$ is ultimately increasing.

To complete the proof, we have to show that in the case $|\beta|<1$ there are infinitely many lattice points $(x, y, z)$ for which $d_{x, y, z} \leq d_{n}$ and $|x|<\left|R_{n}\right|$ for some $n$. Such points trivially exist by (13), when $|\beta|>1$ or when $|\beta|=1$, so we can suppose that $|\beta|<1$.

Suppose $|B|>1$ and $|\beta|<1$. In this case $\alpha$ is irrational. Let $r=y / x$ be a convergent of the continued fraction expansion of $\alpha$ and let $z$ be an integer defined by $z=A y-B x$. Then, by the elementary properties of continued fraction expansions of irrational numbers, using also the fact that $\alpha^{2}=A \alpha-B$, we have

$$
\begin{aligned}
& |x \alpha-y|=x\left|\alpha-\frac{y}{x}\right|<\frac{2}{2|x|}, \\
& \left|x \alpha^{2}-z\right|=|A x \alpha-(z+B x)|=|A x \alpha-A y|=|A x| \cdot\left|\alpha-\frac{y}{x}\right|<\frac{|A| \mid}{2|x|},
\end{aligned}
$$

and

$$
\left|y \alpha^{2}-z \alpha\right|=|(z-A y) \alpha+B y|=|B x| \cdot\left|\alpha-\frac{y}{x}\right|<\frac{|B|}{2|x|} .
$$

This, together with (12), implies the inequality

$$
\begin{equation*}
d_{x, y, z}<\frac{1}{2|x|} \cdot \sqrt{\frac{A^{2}+B^{2}+1}{1+\alpha^{2}+\alpha^{4}}}=\frac{c}{2|x|}\left(\text { for } c=\sqrt{\frac{A^{2}+B^{2}+1}{1+\alpha^{2}+\alpha^{4}}}\right) . \tag{15}
\end{equation*}
$$

Let $n$ be a natural number defined by $\left|R_{n-1}\right| \leq|x|<\left|R_{n}\right|$, For this $n$, by (13) and (15), we have

$$
\begin{aligned}
d_{n} & =|\beta|^{n} \sqrt{\frac{A^{2}+b^{2}+1}{1+\alpha^{2}+\alpha^{4}}}=\frac{|B|^{n}}{|\alpha|^{n}} \cdot c \\
& =\frac{|B|^{n}}{|\alpha|^{n-1}} \cdot \frac{c}{|\alpha|}=\frac{1}{\left|R_{n-1}\right|} \cdot \frac{c\left(1-(\beta / \alpha)^{n-1}\right)|B|^{n}}{|\alpha| \cdot \sqrt{D}}>\frac{c}{2|x|}>d_{x, y, z}
\end{aligned}
$$

if $x$ and hence $n$ is large enough, since $|B|>1$. This shows that, for any lattice point $(x, y, z)$ defined as above, there is an $n$ such that $d_{x, y, z}<d_{n}$ and $|x|<\left|R_{n}\right|$. This completes the proof.

Lastly we prove the three-dimensional analog of Theorem 3.
Theorem 6: The coordinates of the lattice points $(x, y, z)=\left(R_{n}, R_{n+1}, R_{n+2}\right)$ satisfy the system of equations

$$
\begin{aligned}
& x=t \\
& y=\alpha t+c(t)|t|^{\delta} \quad \text { or } \quad y=\alpha t-c(t)|t|^{\delta} \\
& z=\alpha^{2} t+A \cdot c(t)|t|^{\delta} \quad \text { or } \quad z=\alpha^{2} t-A \cdot c(t)|t|^{\delta}
\end{aligned}
$$

where $\delta=\log |\beta| / \log |\alpha|$ and $c(t)$ is a real-valued function for which $\lim _{t \rightarrow \infty} c(t)=\sqrt{D}^{\delta}$.
Proof: By (1), it can easily be shown that

$$
\begin{equation*}
R_{n+2}=\alpha^{2} R_{n}+(\alpha+\beta) \beta^{n}=\alpha^{2} R_{n}+A \beta^{n} . \tag{16}
\end{equation*}
$$

From (8), (10), (11), and (16), the theorem follows.

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# ON THE RECIPROCALS OF THE FIBONACCI NUMBERS 

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A well-known result concerning partial sums of the reciprocals of the natural numbers $1+1 / 2+1 / 3+\cdots+1 / n$, is that they never equal an integer (for $n>1$ ). A similar result concerning partial sums of the Fibonacci numbers, $F_{1}=1, F_{2}=1, F_{n}=F_{n-1}+F_{n-2}(n \geq 3)$, is trivial because

$$
3<\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{8}+\frac{1}{13}+\frac{1}{21}+\cdots<4 .
$$

However, some interesting questions arise if we consider integer multiples of the reciprocals. Specifically, since $F_{m+1} / F_{m} \geq 1$, the "integer status" of $F_{2} / F_{1}+F_{3} / F_{2}+\cdots+F_{n+1} / F_{n}$ is worth investigating ( $n \geq 3$ ).

Since $\left(F_{n}, F_{m}\right)=F_{(n, m)}[1 ; \mathrm{Th} . \mathrm{VI}]$, the following result tells us that $F_{2} / F_{1}+F_{3} / F_{2}+\cdots$ $+F_{n+1} / F_{n}$ is never an integer for $n \geq 3$.

Theorem 1: If $\left\{c_{j}\right\}$ is an arbitrary sequence of integers for which $F_{q} \nmid c_{q}$ whenever $q$ is an odd prime, then the sum $c_{1} / F_{1}+c_{2} / F_{2}+\cdots+c_{n} / F_{n}$ can never be an integer for $n \geq 3$.

Proof: If $n \geq 3$, then, by Bertrand's Postulate [2; p. 343], there is at least one odd prime number $p$ in the interval $] n / 2, n]$. For $1 \leq i \leq n$, let $\widetilde{F}_{i}=\left(F_{1} F_{2} \cdots F_{n}\right) / F_{i}$. We then have

$$
\left(F_{p}, \widetilde{F}_{i}\right)= \begin{cases}F_{p} & \text { if } i \neq p \\ 1 & \text { if } i=p\end{cases}
$$

because $\left(F_{p}, F_{j}\right)=F_{(p, j)}=F_{1}=1$ for $j \neq p$ and $1 \leq j \leq n$. Now

$$
\frac{c_{1}}{F_{1}}+\frac{c_{2}}{F_{2}}+\cdots+\frac{c_{n}}{F_{n}}=\frac{c_{1} \widetilde{F}_{1}+c_{2} \widetilde{F}_{2}+\cdots+c_{n} \widetilde{F}_{n}}{F_{1} F_{2} \cdots F_{n}} .
$$

Since $F_{p}\left|F_{1} F_{2} \cdots F_{n}, F_{p}\right| c_{i} \widetilde{F}_{i}$ for $i \neq p$, and $F_{p} \backslash c_{p} \widetilde{F}_{p}$ [by hypothesis and $\left.\left(F_{p}, \widetilde{F}_{p}\right)=1\right]$, it follows that

$$
\frac{q_{1} \widetilde{F}_{1}+c_{2} \widetilde{F}_{2}+\cdots+c_{n} \widetilde{F}_{n}}{F_{1} F_{2} \cdots F_{n}}
$$

can never be an integer.
Theorem 1 is a special case of a result that will be stated shortly. Theorem 1 was singled out because it is easily digested and its proof also works in a more general setting.

Let $P$ and $Q$ be relatively prime integers, and let $U_{n}$ and $V_{n}$ be the generalized Fibonacci and Lucas sequences, respectively, defined by (see [1] for information on these sequences):

$$
U_{n}=P U_{n-1}-Q U_{n-2}, U_{0}=0, U_{1}=1 \text { and } V_{n}=P V_{n-1}-Q V_{n-2}, V_{0}=2, V_{1}=P .
$$

Since $\left(U_{n}, U_{m}\right)=U_{(n, m)}[1 ; \mathrm{Th} . \mathrm{VI}]$, it seems that we should be able to replace the $F$ s by the $U$ 's in Theorem 1 and its proof and have a more general result. This is not the case, however, because $U_{m}=0$ and $U_{j}= \pm 1$ are possibilities for values of $m, j \geq 2$. If we require $P \neq Q$, so that $P=1=Q$ and $P=-1=Q$ are eliminated, then the discussion following Theorem I in [1] tells us that $U_{n}$ and $V_{n}$ are nonzero for $n \geq 1$. Thus, we require that $P \neq Q$.

The "revised proof" of Theorem 1 would be invalid if $U_{p}= \pm 1$. This can happen. In fact, if $P=2$ and $Q=3$, then it is easily seen that $U_{3}=1$. Certainly, if $P>0$ and $Q<0$, then $U_{n}>1$ and $V_{n}>1$ for $n>1$. For other values of $P$ and $Q$, the situation is not easily resolved; thus, we reflect this in the statement of the general result.

Theorem 2: Let $P$ and $Q$ be chosen so that $\left|U_{q}\right|>1$ for all odd primes $q$. If $\left\{c_{j}\right\}$ is an arbitrary sequence of integers for which $U_{q} \nmid c_{q}$ whenever $q$ is an odd prime, then the sum $c_{1} / U_{1}+c_{2} / U_{2}+\cdots+c_{n} / U_{n}$ can never be an integer for $n \geq 3$.

Proof: If we replace $F^{\prime}$ 's by $U$ 's, $\widetilde{F}$ 's by $\widetilde{U}$ 's, etc., in the proof of Theorem 1, then we get a proof of the fact that, for $n \geq 3, c_{1} / U_{1}+c_{2} / U_{2}+\cdots+c_{n} / U_{n}$ is never an integer.

The situation is more complicated for the $V_{i}$ 's. For example, if $P=4$ and $Q=7$, then $V_{1}=4$, $V_{2}=2$, and $V_{3}=-20$, so $1 / V_{1}+1 / V_{2}+(-5) / V_{3}=1$. The following results reveal the source of the complication and a condition that eliminates it.

Recall that $V_{n}=P V_{n-1}-Q V_{n-2}, V_{0}=2, V_{1}=P$, and $(P, Q)=1$.
Lemma 1: If $i$ is a natural number, then $\left(V_{i}, P\right)=P$ when $i$ is odd and $\left(V_{i}, P\right)=(2, P)$ when $i$ is even. Furthermore, if $m$ is odd and $j$ is a natural number that is relatively prime to $m$, then $\left(V_{m}, V_{j}\right)=P$ when $j$ is odd, $\left(V_{m}, V_{j}\right)=(2, P)$ when $j$ is even, and $\left(P^{-1} V_{m}, V_{j}\right)=1$ when $j$ is even.

Proof: $\quad\left(V_{i}, P\right)=\left(P V_{i-1}-Q V_{i-2}, P\right)=\left(-Q V_{i-2}, P\right)=\left(V_{i-2}, P\right) \quad$ [since $\left.\quad(P, Q)=1\right]$. This implies that $\left(V_{i}, P\right)=\left(V_{i-2}, P\right)=\left(V_{i-4}, P\right)=\cdots=\left(V_{1}, P\right)=P$ when $i$ is odd and $\left(V_{i}, P\right)=\left(V_{0}, P\right)=$ $(2, P)$ when $i$ is even.

We now consider natural numbers $m$ and $j$ where $m$ is odd and $j$ is relatively prime to $m$. Since $\left(U_{2 m}, U_{2 j}\right)=U_{(2 m, 2 j)}=U_{2}=P$ and $U_{2 n}=U_{n} V_{n}$ for any natural number $n$, it follows that $P=\left(U_{m} V_{m}, U_{j} V_{j}\right)$. This shows that $\left(V_{m}, V_{j}\right) \mid P$. This and the facts that $\left(V_{i}, P\right)=P$ when $i$ is odd and $\left(V_{i}, P\right)=(2, P)$ when $i$ is even imply that $\left(V_{m}, V_{j}\right)=P$ when $j$ is odd and $\left(V_{m}, V_{j}\right)=(2, P)$ when $j$ is even. Since $(2, P)=1$ if $P$ is odd, it follows that $\left(P^{-1} V_{m}, V_{j}\right)=1$ when $P$ is odd and $j$ is even. If $P$ is even, then

$$
\begin{aligned}
\left(P^{-1} V_{m}, 2\right) & =\left(P^{-1}\left(P V_{m-1}-Q V_{m-2}\right), 2\right)=\left(V_{m-1}-P^{-1} Q V_{m-2}, 2\right) \\
& =\left(-P^{-1} Q V_{m-2}, 2\right)\left[\text { since }\left(V_{m-1}, 2\right)=(P, 2)=2\right] \\
& =\left(P^{-1} V_{m-2}, 2\right)[(Q, 2)=1 \text { since }(Q, P)=1] .
\end{aligned}
$$

This implies that $\left(P^{-1} V_{m}, 2\right)=\left(P^{-1} V_{1}, 2\right)=1$. That is, $P^{-1} V_{m}$ is odd. Thus, $\left(P^{-1} V_{m}, V_{j}\right)=1$ also when $P$ is even and $j$ is even.

Remark 1: It is not always true that $\left(P^{-1} V_{m}, V_{j}\right)=1$ when $j$ is odd [again, $m$ is an odd natural number and $(m, j)=1]$. For example, if $P=6$ and $Q=1$, then $V_{0}=2, V_{1}=6, V_{2}=34, V_{3}=198$, and $\left(6^{-1} V_{3}, V_{1}\right)=(33,6)=3$. Actually, one can prove by mathematical induction that there exist integers $k_{n}$ and $r_{n}$ such that

$$
V_{n}= \begin{cases}k_{n} P^{3}+n P(-Q)^{(n-1) / 2} & \text { if } n \text { is odd }, \\ r_{n} P^{2}+2(-Q)^{n / 2} & \text { if } n \text { is even. }\end{cases}
$$

This form of $V_{n}$ shows that $\quad$ and hence $\left(P^{-1} V_{m}, V_{j}\right)=(m, P)$.
Theorem 3: Let $P$ and $Q$ be chosen so that $\left|P^{-1} V_{q}\right|>1$ for all odd primes $q$. If $\left\{c_{j}\right\}$ is an arbitrary sequence of integers for which $P^{-1} V_{q} \nmid c_{q}$ whenever $q$ is an odd prime, then the sum $c_{1} / V_{1}+c_{2} / V_{2}+\cdots+c_{n} / V_{n}$ can never be an integer for $n \geq 3$.

Proof: Let $p$ be an odd prime number in the interval $] n / 2, n]$ and let

$$
\widetilde{V}_{i}=\frac{V_{1} V_{2} \cdots V_{n}}{V_{i}} \text { for } 1 \leq i \leq n
$$

Since there are at least $[(n-3) / 2]$ odd numbers in the set $\{1,2, \ldots, i-1, i+1, \ldots, p-1, p+1, \ldots, n\}$ and $\left(V_{k}, P\right)=P$ when $k$ is odd, it follows that

$$
V_{p} P^{[(n-3) / 2]} \mid c_{i} \widetilde{V}_{i} \text { for } i \neq p
$$

This is not the case for $c_{p} \widetilde{V}_{p}$, as we now demonstrate.

$$
\begin{aligned}
V_{p} P^{[(n-3) / 2]} \mid c_{p} \tilde{V}_{p} & \left.\Leftrightarrow P^{-1} V_{p} P^{[(n-3) / 2]} \left\lvert\, c_{p} \frac{V_{2} V_{3} \cdots V_{n}}{V_{p}}\right. \text { (since } V_{1}=P\right) \\
& \Leftrightarrow P^{-1} V_{p} \left\lvert\, c_{p} \frac{P^{-[(n-3) / 2]} V_{2} V_{3} \cdots V_{n}}{V_{p}}\right.
\end{aligned}
$$

Since there are exactly $[(n-3) / 2]$ odd numbers in the set $\{2,3, \ldots, p-1, p+1, \ldots, n\}$,

$$
c_{p} \frac{P^{-[(n-3) / 2]} V_{2} V_{3} \cdots V_{n}}{V_{p}}=c_{p}\left(\prod_{i=1}^{[n / 2]} V_{2 i}\right)\left(\prod_{\substack{j=1 \\ j \neq(p-1) / 2}}^{[n / 2]} P^{-1} V_{2 j+1}\right) .
$$

By hypothesis, $P^{-1} V_{p} \nmid c_{p}$, and by Lemma 1, $\left(P^{-1} V_{p}, V_{2 i}\right)=1$ and $\left(P^{-1} V_{p}, P^{-1} V_{2 j+1}\right)=1$ (since $2 j+1$ is not divisible by $p$ ). This implies $V_{p} P^{[(n-3) / 2]} \nmid c_{p} \widetilde{V}_{p}$. Thus, as in the proof of Theorem 1 , we conclude that $c_{1} / V_{1}+c_{2} / V_{2}+\cdots+c_{n} / V_{n}$ can never be an integer for $n \geq 3$.

Corollary 1: If $P$ and $Q$ are chosen so that $\left|U_{q}\right|>1\left[\left|P^{-1} V_{q}\right|>1\right]$ for all odd primes $q$, then the sum

$$
\frac{U_{2}}{U_{1}}+\frac{U_{3}}{U_{2}}+\cdots+\frac{U_{n+1}}{U_{n}}\left[\frac{V_{2}}{V_{1}}+\frac{V_{3}}{V_{2}}+\cdots+\frac{V_{n+1}}{V_{n}}\right]
$$

can never be an integer for $n \geq 3$.
Proof: If $q$ is an odd prime, then $\left(U_{q+1}, U_{q}\right)=U_{1}=1\left[\left(V_{q+1}, P^{-1} V_{q}\right)=1\right.$ by Lemma 1].
Corollary 2: Let $k$ be a fixed positive integer. Let $P$ and $Q$ be chosen so that $\left|U_{q k}\right|>\left|U_{k}\right|$ for all odd primes $q$. If $\left\{c_{j}\right\}$ is an arbitrary sequence of integers for which $U_{q k} U_{k}^{-1} \nmid c_{q}$ whenever $q$ is an odd prime, then the sum $c_{1} / U_{k}+c_{2} / U_{2 k}+\cdots+c_{n} / U_{n k}$ can never be an integer for $n \geq 3$.
Remark 2: If $\alpha$ and $\beta$ are the roots of $x^{2}-P x+Q=0$, then it is well known that

$$
U_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta} \text { and } V_{n}=\alpha^{n}+\beta^{n}
$$

These forms establish the well-known facts that

$$
U_{n k}=V_{k} U_{(n-1) k}-Q^{k} U_{(n-2) k} \text { and } V_{n k}=V_{k} V_{(n-1) k}-Q^{k} V_{(n-2) k}
$$

Furthermore, using $V_{n k}=V_{k} V_{(n-1) k}-Q^{k} V_{(n-2) k}$ and mathematical induction, it is easy to see that $V_{k} \mid V_{(2 i+1) k}$ whenever $i$ is a positive integer. Also, for $k=2,3,4, \ldots$,

$$
\left(V_{k}, Q\right)=\left(P V_{k-1}-Q V_{k-2}, Q\right)=\left(P V_{k-1}, Q\right)=\left(V_{k-1}, Q\right)=\cdots=\left(V_{1}, Q\right)=1 .
$$

Proof of Corollary 2: If $\hat{U}_{n}=U_{n k} U_{k}^{-1}$, then $\hat{U}_{n}=V_{k} \hat{U}_{n-1}-Q^{k} \hat{U}_{n-2}$, a generalized Fibonacci sequence, and $\left|\hat{U}_{q}\right|>1$ for all odd primes $q$. It then follows from Theorem 2 that, if $\hat{U}_{q} \nmid c_{q}$ whenever $q$ is an odd prime, then $c_{1} / \hat{U}_{1}+c_{2} / \hat{U}_{2}+\cdots+c_{n} / \hat{U}_{n}$ is never an integer for $n \geq 3$. Thus, if $U_{q k} U_{k}^{-1} \nmid c_{q}$ whenever $q$ is an odd prime, then $U_{k}\left(c_{1} / U_{k}+c_{2} / U_{2 k}+\cdots+c_{n} / U_{n k}\right)$ is never an integer for $n \geq 3$, and consequently, $c_{1} / U_{k}+c_{2} / U_{2 k}+\cdots+c_{n} / U_{n k}$ is never an integer for $n \geq 3$.

Corollary 3: Let $k$ be a fixed positive integer. Let $P$ and $Q$ be chosen so that $\left|P^{-1} V_{q k}\right|>1$ for all odd primes $q$. If $\left\{c_{j}\right\}$ is an arbitrary sequence of integers for which $P^{-1} V_{q k}\left\{c_{q}\right.$ whenever $q$ is an odd prime, then the sum $c_{1} / V_{k}+c_{2} / V_{2 k}+\cdots+c_{n} / V_{n k}$ can never be an integer for $n \geq 3$.

Proof: If $\hat{V}_{n}=V_{n k}$, then $\hat{V}_{n}=V_{k} \hat{V}_{n-1}-Q^{k} \hat{V}_{n-2}$, a generalized Lucas sequence, and $\left|P^{-1} \hat{V}_{q}\right|>1$ for all odd primes $q$. Since $P^{-1} \hat{V}_{q} \nmid c_{q}$, the result follows from Theorem 3.

Corollary 4: If $\left\{c_{j}\right\}$ is an arbitrary sequence of integers for which $q \nmid c_{q}$ whenever $q$ is prime, then the sum $c_{1} / 1+c_{2} / 2+\cdots+c_{n} / n$ can never be an integer for $n \geq 2$.

Proof: If $U_{n}=n$, then $U_{n}=2 U_{n-1}-U_{n-2}$. That is, $\{n\}$ is a generalized Fibonacci sequence for which Theorem 2 applies.

Corollary 5: Let $P$ and $Q$ be chosen so that $\left|U_{q}\right|>1\left[\left|P^{-1} V_{q}\right|>1\right]$ for all odd primes $q$. If $\left\{c_{j}\right\}$ is an arbitrary sequence of integers for which $U_{q} \nmid c_{q}\left[P^{-1} V_{q} \nmid c_{q}\right]$ whenever $q$ is an odd prime, then the sum $c_{1} / U_{1}+c_{2} / U_{3}+\cdots+c_{n} / U_{2 n-1}\left[c_{1} / V_{1}+c_{2} / V_{3}+\cdots+c_{n} / V_{2 n-1}\right]$ can never be an integer for $n \geq 2$.

Proof: Consider the statement of Theorem 2 [Theorem 3] and just take $c_{2 j}=U_{2 j}$ [ $c_{2 j}=V_{2 j}$ ].

Remark 3: Results for $U^{\prime} \mathrm{s}$ and $V^{\prime} \mathrm{s}$ with even subscripts are special cases of Corollaries 2 and 3.

## ACKNOWLEDGMENT

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# SYMMETRIC FIBONACCI WORDS* 

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In [1] the author studied Fibonacci words; the study was motivated by the consideration of Fibonacci strings and Fibonacci word patterns by Knuth [5] and Turner [6, 7], respectively. It was shown in [1] that all the $n^{\text {th }}$ Fibonacci words can be obtained from any particular $n^{\text {th }}$ Fibonacci word, for example $w_{n}^{0}$, by shifting in a cyclic way the letters in it. Also it was shown that each of the Fibonacci words $w_{n}^{0}(n \geq 3)$ has a representation as a product of two symmetric words. In this paper, we show that every Fibonacci word has such a representation and that this representation is unique (Theorem 3). Furthermore, we prove that, for each positive integer $n$ that is not a multiple of 3 , there is precisely one symmetric Fibonacci word of length $F_{n}$, where $F_{n}$ denotes the $n^{\text {th }}$ Fibonacci number, while there are no symmetric Fibonacci words of length $F_{n}$ if $n$ is a multiple of 3 (Theorem 7).

Let $X$ be an alphabet and let $X^{*}$ be a free monoid of words over $X$ with identity 1. Denote by $\ell(w)$ the length of a word $w$. Define the reverse $R$ and the shift $T$ on $X^{*} /\{1\}$ by

$$
\begin{aligned}
& R\left(a_{1} a_{2} \ldots a_{n}\right)=a_{n} a_{n-1} \ldots a_{1}, \\
& T\left(a_{1} a_{2} \ldots a_{n}\right)=a_{2} \ldots a_{n} a_{1},
\end{aligned}
$$

where $a_{i} \in X, 1 \leq i \leq n$.
A word $w \in X^{*}$ is said to be symmetric if $w=1$ or $R(w)=w$. Let $\mathscr{S}$ denote the set of all symmetric words over $X$ and $\mathscr{S}^{2}=\{u v: u, v \in \mathscr{Y}\} \backslash\{1\}$. The representations $u v$ and $v u$ where $u, v \in \mathscr{Y}$, are considered to be the same if $v=1$.

Fibonacci words are defined recursively as follows. Fix two distinct letters $a$ and $b$ and put

$$
\begin{aligned}
& w_{1}=a, \\
& w_{2}=b, \\
& w_{3}^{0}=b a, w_{3}^{1}=a b, \\
& w_{4}^{00}=b a b, w_{4}^{01}=b b a, w_{4}^{10}=a b b, w_{4}^{11}=b a b .
\end{aligned}
$$

In general, suppose that $n \geq 5, r_{1}, r_{2}, \ldots, r_{n}$ is a finite binary sequence and that the words

$$
w_{n-2}^{r_{2}^{\prime} r_{2}, \ldots r_{n-4}}, w_{n-1}^{r_{n}^{\prime} r_{2}, r_{n-3}}
$$

have been defined. Then set

For simplicity, we write $w_{n}^{0}$ if $n>3$ and $r_{1}=r_{2}=\cdots r_{n-2}=0$. Each $w_{n}^{r_{2} r_{2} r_{n-2}}$ is called an $n^{\text {th }}$ Fibonacci word derived from the initial letters $a$ and $b$ and is known to have length $F_{n}$.

[^1]Among all the Fibonacci words, some of them are symmetric but some of them are not. For example, the Fibonacci words $b a b, b a b a b, b a b a b b a b b a b a b$ are symmetric while the Fibonacci words $a b b, b b a, a b a b b$ are not. Nevertheless, it turns out that each Fibonacci word is a unique product of two symmetric words. To prove this unique representation theorem (Theorem 3 below), we need some known results about Fibonacci words (see [1]) and products of two symmetric words (see [2]). The proof of Lemma 1 can be found in [1].

Lemma 1 (Theorems 4 and 7 and Corollary 12(iv) of [1]):
(a) Each $w_{n}^{0}(n \geq 1)$ is a product of two symmetric words, that is $w_{n}^{0} \in \mathscr{S}^{2}$.
(b) There are exactly $F_{n}$ distinct Fibonacci words of length $F_{n}$, namely, $T^{j}\left(w_{n}^{0}\right), 0 \leq j \leq$ $F_{n}-1$
In Theorem 2.4 of [2] it was proved that a word has more than one representation as a product of two symmetric words if and only if it is a power of another word which is itself a product of two symmetric words. The following lemma contains Theorem 2.1 of [2] and only part of the result just mentioned because we do not need to use the full power of it to prove the unique representation theorem. For completeness, we include a proof.

Lemma 2 (Theorems 2.1 and 2.2 of [2]):
(a) $\mathscr{S}^{2}$ is invariant under $T$, that is, $T\left(\mathscr{S}^{2}\right) \subset \mathscr{S}^{2}$.
(b) If a word has more than one representation as a product of two symmetric words, then it is a power of another word. More precisely, if $p, r, m$ are positive integers such that $r<p \leq m$ and if, in the word $w=a_{1} a_{2} \ldots a_{m}$, the subwords

$$
\begin{align*}
& a_{1} a_{2} \ldots a_{p}, a_{p+1} \ldots a_{m} \\
& a_{1} a_{2} \ldots a_{r}, a_{r+1} \ldots a_{m} \tag{1}
\end{align*}
$$

are symmetric words, then $w=\left(a_{1} a_{2} \ldots a_{d}\right)^{m / d}$ where $d=(p-r, m)$.
Proof: (a) If $w=a_{1} a_{2} \ldots a_{m}$ is a symmetric word, then

$$
T w= \begin{cases}a_{1} & m=1 \\ a_{2} a_{1} & m=2 \\ \left(a_{2} \ldots a_{m-1}\right)\left(a_{m} a_{1}\right) & m>2\end{cases}
$$

If $w=\left(a_{1} a_{2} \ldots a_{p}\right)\left(a_{p+1} \ldots a_{m}\right)$ where $p$ is a positive integer less than $m$, and the words $a_{1} a_{2} \ldots a_{p}$ and $a_{p+1} \ldots a_{m}$ are symmetric, then

$$
T w= \begin{cases}\left(a_{2} \ldots a_{m}\right) a_{1} & p=1 \\ a_{2} a_{3} \ldots a_{m} a_{1} & p=2 \\ \left(a_{2} \ldots a_{p-1}\right)\left(a_{p} a_{p+1} \ldots a_{m} a_{1}\right) & p>2\end{cases}
$$

Therefore, (a) follows.
(b) First, note that since the subwords in (1) are symmetric, we have

$$
a_{k}=a_{p+1-k}=a_{r+1-k}(k=1,2, \ldots, m)
$$

[AUG.
with indices modulo $m$. Hence

$$
\begin{equation*}
a_{k}=a_{p-r+k}(k=1,2, \ldots, m) \tag{2}
\end{equation*}
$$

with indices modulo $m$. Now choose positive integers $i$ and $j$ such that $i(p-r)-j m=d$. Then, according to (2), we have

$$
a_{k}=a_{i(p-r)+k}=a_{j m+d+k}=a_{d+k}(k=1,2, \ldots, m)
$$

with indices modulo $m$. This proves (b).
Theorem 3 (Unique representation theorem): Every Fibonacci word has a unique representation as a product of two symmetric words.

Proof: Lemma 1 and Lemma 2(a) imply that every Fibonacci word belongs to $\mathscr{S}^{2}$. Suppose that some Fibonacci word $w$ has more than one representation as a product of two symmetric words. Then, Lemma 2(b) implies that $w=u^{c}$ for some word $u$ and $c \geq 2$. But then $T^{\ell(u)} w=w$. Since $1 \leq \ell(u)<\ell(w)$, this contradicts Lemma 1(b). This proves the theorem.

Now we determine all the symmetric Fibonacci words. Let

$$
s_{n}= \begin{cases}1 & \text { if } n \text { is a multiple of } 3, \\ 0 & \text { otherwise }\end{cases}
$$

and let

$$
t_{n}= \begin{cases}1 & \text { if } n \text { is odd } \\ 0 & \text { if } n \text { is even. }\end{cases}
$$

Let $p_{1}=a, p_{2}=b, p_{n}=w_{n}^{s_{n}^{s} s_{2} \ldots s_{n-2}}$, for $n \geq 3$, and let $q_{n}=w_{n}^{t_{1} t_{2} \ldots t_{n-2}}$, for $n \geq 3$. For odd $n$, let $s=F_{n-2}$ and $t=F_{n-1}$; for even $n$, let $s=F_{n-1}$ and $t=F_{n-2}$.

For $n>2$, let us list the $F_{n}$ Fibonacci words of length $F_{n}$ in the following order (Corollary 12(iv) of [1]):

$$
\begin{equation*}
T^{0} q_{n}, T^{s} q_{n}, \ldots, T^{\left(F_{n}-1\right) s} q_{n} \tag{3}
\end{equation*}
$$

If $n$ is a multiple of 3 , then the number of terms in (3) is even, it will be shown in Theorem 7 that there are no symmetric words in the list; however, if $n$ is not a multiple of 3 , the number of terms in (3) is odd and, again, it will be shown in Theorem 7 that only the middle term of (3) is a symmetric Fibonacci word.

Lemma 4: If $n>2$ is not a multiple of 3 , then $p_{n}=T^{j s} q_{n}$ where $j=\left(F_{n}-1\right) / 2$. In other words, $p_{n}$ is the middle term of the sequence (3).

Proof: As was proved in section 5 of [1], $p_{n}=T^{j s} q_{n}$ where

$$
j \equiv \begin{cases}m F_{n-1} & \text { if } n \text { is odd }  \tag{4}\\ m F_{n-1}-1 & \left(\bmod F_{n}\right) \\ \text { if } n \text { is even }\end{cases}
$$

where $m=1+\sum_{i=1}^{n-2} F_{i+1} s_{i}$. It follows from the identity $F_{1}+F_{4}+F_{7}+\cdots F_{3 k-2}=F_{3 k} / 2(k \geq 1)$ that

$$
m= \begin{cases}\frac{1}{2} F_{n-1} & \text { if } n \equiv 1(\bmod 3) \\ \frac{1}{2} F_{n+1} & \text { if } n \equiv 2(\bmod 3)\end{cases}
$$

Thus, if $n \equiv 1(\bmod 3)$, then

$$
j \equiv\left(F_{n-2} F_{n}-1\right) / 2 \equiv F_{n}\left(F_{n-2}-1\right) / 2+\left(F_{n}-1\right) / 2 \equiv\left(F_{n}-1\right) / 2\left(\bmod F_{n}\right) ;
$$

if $n \equiv 2(\bmod 3)$, then

$$
j \equiv\left(F_{n}^{2}-1\right) / 2 \equiv F_{n}\left(F_{n}-1\right) / 2+\left(F_{n}-1\right) / 2 \equiv\left(F_{n}-1\right) / 2\left(\bmod F_{n}\right) .
$$

This proves the lemma.
Lemma 5 (Corollary 12(i) of [1]): Let $n$ be a positive integer greater than 2 and $1 \leq j \leq F_{n}-1$. Then the $k^{\text {th }}$ letter in $T^{J s} q_{n}$ is an " $a$ " if and only if $k \equiv(j+r) t\left(\bmod F_{n}\right)$ for some $1 \leq r \leq F_{n-2}$.

Lemma 6: If $n$ is a positive integer greater than 2, then $R\left(T^{j s} q_{n}\right)=T^{\left(F_{n}-1-j\right) s} q_{n}$, for all $0 \leq j \leq$ $F_{n}-1$.

Proof: Let $0 \leq j \leq F_{n}-1$. Suppose that the $k^{\text {th }}$ letter in $T^{j s} q_{n}$ is an " $a$ ". Then, by Lemma $5, k \equiv(j+r) t\left(\bmod F_{n}\right)$ for some $1 \leq r \leq F_{n-2}$. Therefore, $1 \leq F_{n-2}+1-r \leq F_{n-2}$ and

$$
\begin{aligned}
\left(\left(F_{n}-1-j\right)+\left(F_{n-2}+1-r\right)\right) t & \equiv F_{n-2} t-(j+r) t \\
& \equiv F_{n-2} t-k \equiv F_{n}+1-k\left(\bmod F_{n}\right) .
\end{aligned}
$$

This proves that $\left(F_{n}+1-k\right)^{\text {th }}$ letter in $T^{\left(F_{n}-1-j\right) s} q_{n}$ is also an " $a$ ", again by Lemma 5. Consequently, the result holds.

The above lemma can also be proved by observing that $w_{n}^{r_{2} r_{2} \ldots r_{n-2}}=T^{j s} q_{n}$ where $j$ satisfies (4) with $m=1+\sum_{i=1}^{n-2} F_{i+1} r_{i}$ (section 5 of [1]) and that $R\left(w_{n}^{r_{1}^{\prime}, \ldots r_{n-2}}\right)=w_{n}^{v_{1} v_{2}, \ldots v_{n-2}}$, where $v_{i}=1-r_{i}$, $1 \leq i \leq n-2$ (Theorem 3(i) of [1]).

Theorem 7: Let $n$ be a positive integer greater than 2 .
(a) If $n$ is not a multiple of 3 , then $p_{n}$ is the only symmetric Fibonacci word of length $F_{n}$.
(b) If $n$ is a multiple of 3 , then no Fibonacci word of length $F_{n}$ is symmetric.

Proof: Let $0 \leq j \leq F_{n}-1$. Since $F_{n}-1-j=j \Leftrightarrow j=\frac{1}{2}\left(F_{n}-1\right)$, we see from Lemma 6 that

$$
\begin{equation*}
R\left(T^{j s} q_{n}\right)=T^{j s} q_{n} \Leftrightarrow j=\frac{1}{2}\left(F_{n}-1\right) . \tag{5}
\end{equation*}
$$

(a) If $n$ is not a multiple of 3 , then $F_{n}$ is odd; thus, among the Fibonacci words in (3), $p_{n}=T^{\frac{1}{2}\left(F_{n}-1\right) s} q_{n}$ is the only symmetric one, according to (5) and Lemma 4.
(b) If $n$ is a multiple of 3 , then, clearly, (5) implies that $T^{j s} q_{n}$ is not symmetric for all $0 \leq j \leq F_{n}-1$.

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Announcement

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# A SUMMATION RULE USING STIRLING NUMBERS OF THE SECOND KIND 

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## 1. A SUMMATION RULE

Recall that Stirling numbers of the second kind may be expressed as follows (cf., e.g., [1], [2]):

$$
S(m, j)=\frac{1}{j!} \Delta^{j} 0^{m}=\frac{1}{j!} \sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i} i^{m},
$$

where $\Delta^{j} 0^{m}$ is the $j^{\text {th }}$ difference of $x^{m}$ at $x=0$ so that $S(m, j)=0$ for $j>m, S(m, 0)=0$ for $m \geq 1$ and $S(0,0)=1$.

Summation Rule: Let $F(n, k)$ be a bivariate function defined for integers $n, k \geq 0$. If there can be found a summation formula or a combinatorial identity such as

$$
\begin{equation*}
\sum_{k=j}^{n} F(n, k)\binom{k}{j}=\phi(n, j) \quad(j \geq 0), \tag{1}
\end{equation*}
$$

then for every given $m \geq 0$ we have a summation formula or a combinatorial identity such as

$$
\begin{equation*}
\sum_{k=0}^{n} F(n, k) k^{m}=\sum_{j=0}^{m} \phi(n, j) j!S(m, j) \tag{2}
\end{equation*}
$$

which may be called a companion formula of (1).
Generally, (2) would be practically useful when $n$ is much bigger than $m$.
Proof: It is known that Stirling numbers of the second kind satisfy the following basic relation [which is often taken as a definition of $S(n, k)$ ]:

$$
\begin{equation*}
x^{m}=\sum_{j=0}^{m} S(m, j)(x)_{j}, \tag{3}
\end{equation*}
$$

where $(x)_{j}: x(x-1) \ldots(x-j+1)(j \geq 1)$ is the falling factorial with $(x)_{0}:=1$. Now, substituting (3) into the left-hand side of (2), changing the order of summation, and using (1), we easily obtain

$$
\sum_{k=0}^{n} F\left(n^{\prime}, k\right) k^{m}=\sum_{j=0}^{m} S(m, j) \sum_{k=0}^{n} F(n, k)(k)_{j}=\sum_{j=0}^{m} j!S(m, j) \phi(n, j) .
$$

Notice that the special case for $m=0$ is also true. Hence, (2) holds for every $m \geq 0$.
Remark Sometimes in applications of the rule function $F(n, k)$ may involve some independent parameters. Moreover, for the particular case in which $F(n, k)>0$, so that $\phi(n, 0)>0$, the lefthand side of (2) divided by $\phi(n, 0)$ may be considered as the $m^{\text {th }}$ moment (about the origin) of a
discrete random variable $X$ that may take possible values $0,1,2, \ldots, n$. This means that (2) may sometimes be used for computing moments whenever $F(n, k) / \phi(n, 0)$ just stands for probabilities $(0 \leq k \leq n)$, and the factorial moments $\phi(n, j) / \phi(n, 0)$ are easily found via (1) (cf. David and Barton [3]).

## 2. VARIOUS EXAMPLES

For the simplest case $F(n, k) \equiv 1$, we have

$$
\phi(n, j) \equiv \sum_{k=j}^{n}\binom{k}{j}=\binom{n+1}{j+1} .
$$

This leads to the familiar formula

$$
\begin{equation*}
\sum_{k=1}^{n} k^{m}=\sum_{j=1}^{m}\binom{n+1}{j+1} j!S(m, j) . \tag{4}
\end{equation*}
$$

Actually there are many known identities of type (1) in which $F(n, k)$ may consist of a binomial coefficient or a product of binomial coefficients. See, e.g., Egorychev [4], Gould [5], and Riordan [8]. Consequently, we may find various special summation formulas via (2). We now list a dozen formulas, as follows:

$$
\begin{equation*}
\sum_{k=1}^{n} k^{m}\binom{n}{k} p^{k} q^{n-k}=\sum_{j=1}^{m}\binom{n}{j} p^{j} j!S(m, j) \tag{5}
\end{equation*}
$$

where $p+q=1$ and $p>0$.

$$
\begin{gather*}
\sum_{k=0}^{[n / 2]}\binom{n}{2 k} k^{m}=\sum_{j=0}^{m} 2^{n-2 j-1}\binom{n-j}{j} \frac{n}{n-j} j!S(m, j),  \tag{6}\\
\sum_{k=0}^{[n / 2]}\binom{n+1}{2 k+1} k^{m}=\sum_{j=0}^{m} 2^{n-2 j}\binom{n-j}{j} j!S(m, j),  \tag{7}\\
\sum_{k=0}^{n}\binom{n-k}{s} k^{m}=\sum_{j=0}^{m}\binom{n+1}{s+j+1} j!S(m, j),  \tag{8}\\
\sum_{k=0}^{n}\binom{s+k}{s} k^{m}=\sum_{j=0}^{m}\binom{n+1}{j}\binom{n+1+s}{s} \frac{n+1-j}{s+1+j} j!S(m, j),  \tag{9}\\
\sum_{k=0}^{n}(-4)^{k}\binom{n+k}{2 k} k^{m}=\sum_{j=0}^{m}(-1)^{n} 2^{2 j}\binom{n+j}{2 j} \frac{2 n+1}{2 j+1} j!S(m, j),  \tag{10}\\
\sum_{k=0}^{n}(-4)^{k}\binom{n+k}{2 k} \frac{n}{n+k} k^{m}=\sum_{j=0}^{m}(-1)^{n} 2^{2 j}\binom{n+j}{2 j} \frac{n}{n+j} j!S(m, j), \tag{11}
\end{gather*}
$$

$$
\begin{gather*}
\sum_{k=0}^{[n / 2]}(-1)^{k}\binom{n-k}{k} 2^{n-2 k} k^{m}=\sum_{j=0}^{m}(-1)^{j}\binom{n+1}{2 j+1} j!S(m, j),  \tag{12}\\
\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k} k^{m}=\sum_{j=0}^{m}\binom{\alpha}{j}\binom{\alpha+\beta-j}{n-j} j!S(m, j), \tag{13}
\end{gather*}
$$

where $\alpha$ and $\beta$ are real parameters.

$$
\begin{gather*}
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{2 n-k}{n} k^{m}=\sum_{j=0}^{m}(-1)^{j}\binom{n}{j}^{2} j!S(m, j),  \tag{14}\\
\sum_{k=0}^{[n / 2]}\binom{n}{2 k}\binom{2 k}{k} 2^{n-2 k} k^{m}=\sum_{j=0}^{m}\binom{2 n-2 j}{n}\binom{n}{j} j!S(m, j),  \tag{15}\\
\sum_{k=1}^{n} k^{m} H_{k}=\sum_{j=1}^{m}\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right) j!S(m, j), \tag{16}
\end{gather*}
$$

where $H_{k}:=1+\frac{1}{2}+\cdots+\frac{1}{k},(k \geq 1)$, are harmonic numbers.
Though most of the above formulas [except (5)] appear unfamiliar, or are difficult to find in the literature, they are actually companion formulas of some known identities. In fact, (5) is known as the $m^{\text {th }}$ moment of the binomial distribution of a discrete random variable. Formulas (6) and (7) represent companion formulas of the pair of Moriarty identities (cf. [4, (2.73) and (2.74)]; [5, (3.120) and (3.121)]). Also, (9) and (12) are just companion formulas of the following identities:

$$
\sum_{k=j}^{n}\binom{k+s}{s}\binom{k}{j}=\binom{n+1}{j}\binom{n+1+s}{s} \frac{n+1-j}{s+1+j}
$$

and

$$
\sum_{k=j}^{[n / 2]}(-1)^{k}\binom{n-k}{k}\binom{k}{j} 2^{n-2 k}=(-1)^{j}\binom{n+1}{2 j+1}
$$

due to Knuth and Marcia Ascher, respectively (cf. [5, (3.155) and (3.179)]). Moreover, (16) may be inferred from the known relation (cf., e.g., [1, pp. 98-99]).

$$
\begin{equation*}
\sum_{k=j}^{n}\binom{k}{j} H_{k}=\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right) \tag{17}
\end{equation*}
$$

The verification of the rest of the formulas is left to the interested reader.
Evidently, both (8) and (9) imply (4) with $s=0$, and (13) yields the Vandermonde convolution identity when $m=0$. Moreover, it is easily found that (16) leads to an asymptotic relation, for $n \rightarrow \infty$, of the following,

$$
\sum_{k=1}^{n} k^{m} H_{k} \sim \frac{n^{m+1}}{m+1}\left(\log n+\gamma-\frac{1}{m+1}\right),
$$

where $\gamma:=\lim _{n}\left(H_{n}-\log n\right)=0.5772 \ldots$ is Euler's constant.

## 3. AN EXTENSION OF THE SUMMATION RULE

In what follows, we will adopt the notations:

$$
\begin{gathered}
(x \mid h)_{n}:=x(x-h)(x-2 h) \cdots(x-n h+h),(x \mid h)_{0}=1 \\
\binom{x}{n}_{h}:=(x \mid h)_{n} / n!,\binom{x}{n}_{1}=\binom{x}{n}=(x)_{n} / n!,\binom{x}{n}_{0}=x^{n} / n!
\end{gathered}
$$

Here $\binom{x}{n}$ is known as the generalized binomial coefficient (cf. Jordan [7, ch. 2, §22). Now, suppose that $\alpha$ and $\beta$ are two distinct real numbers. Consider the following pair of expressions for polynomials $(x \mid \alpha)_{n}$ and $(x \mid \beta)_{n}$ :

$$
\begin{align*}
& (x \mid \alpha)_{n}=\sum_{k=0}^{n} S_{\alpha}(n, k \mid \beta)(x \mid \beta)_{k}  \tag{18}\\
& (x \mid \beta)_{n}=\sum_{k=0}^{n} S_{\beta}(n, k \mid \alpha)(x \mid \alpha)_{k} \tag{19}
\end{align*}
$$

The coefficients $S_{\alpha}(n, k \mid \beta)$ and $S_{\beta}(n, k \mid \alpha)$ involved in (18) and (19) are uniquely determined, and they may be called a pair of symmetrically generalized Stirling numbers associated with the number pair $(\alpha, \beta)$. Consequently, the ordinary Stirling numbers of the first and second kinds are associated with the number pair $(1,0)$, and are usually denoted by the following:

$$
S_{1}(n, k) \equiv s(n, k):=S_{1}(n, k \mid 0), S_{2}(n, k) \equiv S(n, k):=S_{0}(n, k \mid 1)
$$

Certainly, all the well-known properties enjoyed by the ordinary Stirling numbers, e.g., recurrence relations, orthogonality relations, and inversion formulas, etc., can be readily extended to these generalized Stirling numbers. For example, a simple recurrence relation may be deduced from (19), namely

$$
\begin{equation*}
S_{\beta}(n, k \mid \alpha)=S_{\beta}(n-1, k-1 \mid \alpha)+(k \alpha-n \beta+\beta) S_{\beta}(n-1, k \mid \alpha),(k \geq 1) \tag{20}
\end{equation*}
$$

Recall that there is a general form of Newton's expansion for a polynomial $f(x)$ of degree $n$, viz.,

$$
\begin{equation*}
f(x)=\sum_{k=0}^{n} \frac{(x \mid \alpha)_{k}}{k!\alpha^{k}} \Delta_{\alpha}^{k} f(0) \tag{21}
\end{equation*}
$$

where $\Delta_{\alpha}^{k} f(0)$ is the $k^{\text {th }}$ difference (with increment $\alpha$ ) of $f(x)$ at $x=0$. Thus, comparing (21) with (19) and (18), we find (with $\alpha \beta \neq 0$ ),

$$
\begin{align*}
& S_{\alpha}(n, k \mid \alpha)=\left.\frac{1}{k!\alpha^{k}} \Delta_{\alpha}^{k}(x \mid \beta)_{n}\right|_{x=0}  \tag{22}\\
& S_{\alpha}(n, k \mid \beta)=\left.\frac{1}{k!\beta^{k}} \Delta_{\beta}^{k}(x \mid \alpha)_{n}\right|_{x=0} \tag{23}
\end{align*}
$$

Here, it is easily observed that $S_{\beta}(n, k \mid \alpha)=0$ for $k>n$, and $S_{\beta}(0,0 \mid \alpha)=S_{\beta}(n, n \mid \alpha)=1$. Moreover, notice that for $\beta=0$ (23) should be replaced by

$$
S_{\alpha}(n, k \mid 0)=\frac{1}{\left.k!\lim _{\beta \rightarrow 0} \frac{1}{\beta^{k}} \Delta_{\beta}^{k}(x \mid \alpha)_{n}\right|_{x=0}=\left.\frac{1}{k!}\left(\frac{d}{d x}\right)^{k}(x \mid \alpha)_{n}\right|_{x=0} . . . . . . .}
$$

Extended Summation Rule: Let $F(n, k)$ be defined for integers $n, k \geq 0$. If there can be found a summation formula such as

$$
\begin{equation*}
\sum_{k=0}^{n} F(n, k)\binom{k}{j}_{\alpha}=G(n, j),(j \geq 0) \tag{24}
\end{equation*}
$$

then for every $m \geq 0$ we have a summation formula of the form

$$
\begin{equation*}
\sum_{k=0}^{n} F(n, k)\binom{k}{m}_{\beta}=\sum_{j=0}^{m} G(n, j) \frac{j!}{m!} S_{\beta}(m, j \mid \alpha) . \tag{25}
\end{equation*}
$$

Also, suppose that the following series is convergent to $g(j)$ for every $j \geq 0$ :

$$
\begin{equation*}
\sum_{k=0}^{\infty} f(k)\binom{k}{j}_{\alpha}=g(j) \tag{26}
\end{equation*}
$$

Then we have a summation formula, as follows:

$$
\begin{equation*}
\sum_{k=0}^{\infty} f(k)\binom{k}{m}_{\beta}=\sum_{j=0}^{m} g(j) \frac{j!}{m!} S_{\beta}(m, j \mid \alpha) \tag{27}
\end{equation*}
$$

Proof: Notice that (19) implies

$$
\begin{equation*}
\binom{x}{m}_{\beta}=\frac{1}{m!} \sum_{j=0}^{m} j!S_{\beta}(m, j \mid \alpha)\binom{x}{j}_{\alpha} . \tag{28}
\end{equation*}
$$

Thus, both of the implications $(24) \Rightarrow(25)$ and $(26) \Rightarrow(27)$ can be verified in a manner similar to that used to prove (1) $\Rightarrow(2)$. In fact, the verification of (27) can be accomplished by substituting (28) into the left-hand side of (27) and by using (26), in which the change of order of summation is justified by the convergence of the series (26). Moreover, it is evident that

$$
S_{\alpha}(n, k \mid \alpha)= \begin{cases}1 & \text { for } k=n \\ 0 & \text { for } k<n\end{cases}
$$

so that (25) and (27) will transform back to (24) and (26), respectively, when $\beta=\alpha$. Hence, (27) holds for every real number $\beta$.
Examples: For the case $\alpha=1$, we may write

$$
\begin{equation*}
S_{\beta}(m, j \mid 1)=\left.\frac{1}{j!} \Delta^{j}(x \mid \beta)_{m}\right|_{x=0} . \tag{29}
\end{equation*}
$$

In particular, we have

$$
S_{0}(m, j \mid 1)=S(m, j), S_{-1}(m, j \mid 1)=\frac{m!}{j!}\binom{m-1}{j-1}
$$

where $S_{-1}(m, j \mid 1)(-1)^{m}$ is known as Lah's number.

Making use of the rule (24) $\Rightarrow(25)$ (with $\alpha=1$ ), it is readily seen that each of the formulas from (5) through (16) may be generalized to the form in which $k^{m}$ is replaced by $\binom{k}{m}_{\beta}$ and $S(m, j)$ by the following: $S_{\beta}(m, j \mid 1) / m!$. Thus, for instance, (13) and (16) may be replaced, respectively, by:

$$
\begin{align*}
& \sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}\binom{k}{m}_{\beta}=\sum_{j=0}^{m}\binom{x+y-j}{n-j} \frac{(x)_{j}}{m!} S_{\beta}(m, j \mid 1),  \tag{30}\\
& \sum_{k=1}^{n}\binom{k}{m}_{\beta} H_{k}=\sum_{j=1}^{m}\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right) \frac{j!}{m!} S_{\beta}(m, j \mid 1) . \tag{31}
\end{align*}
$$

In particular, for $\beta=0,1,-1$, we have $\binom{k}{m}_{0}=k^{m} / m!\binom{k}{m}_{1}=\binom{k}{m}$, and $\binom{k}{m}_{-1}=\binom{k+m-1}{m}$, so that either (30) or (31) may yield at least three special identities of some interest. Indeed, (31) implies (16), (17), and the identity

$$
\sum_{k=1}^{n}\binom{k+m-1}{m} H_{k}=\sum_{j=1}^{m}\binom{n+1}{j+1}\binom{m-1}{j-1}\left(H_{n+1}-\frac{1}{j+1}\right)
$$

Moreover, as a simple consequence of (30), one may take $x=y=n$ and $\beta=0$ to get

$$
\sum_{k=0}^{n}\binom{n}{k}^{2} k^{m}=\sum_{j=0}^{m}\binom{2 n-j}{n}(n)_{j} S(m, j) .
$$

This is an example mentioned in Comtet [2, ch. 5, p. 225].
To indicate an application of the rule $(26) \Rightarrow(27)$, let us consider the simple example with $f(k)=q^{k}:$

$$
\sum_{k=0}^{\infty}\binom{k}{j} q^{k}=q^{j}(1-q)^{-j-1},(|q|<1) .
$$

Consequently, we obtain

$$
\begin{equation*}
\sum_{k=0}^{\infty}\binom{k}{m}_{\beta} q^{k}=\sum_{j=0}^{m} \frac{q^{j} \cdot j!}{(1-q)^{j+1} m!} S_{\beta}(m, j \mid 1) . \tag{32}
\end{equation*}
$$

This may be used to evaluate an infinite series involving both generalized binomial coefficients and Fibonacci numbers. Denote $a=1 / 2(1+\sqrt{5}), b=1 / 2(1-\sqrt{5})$, and let $\rho>a$. Then the following series,

$$
S=\sum_{k=0}^{\infty}\binom{k}{m}_{\beta} \rho^{-k} F_{k},
$$

is obviously convergent for every $m \geq 0$, where $f_{k}=\left(a^{k+1}-b^{k+1}\right) / \sqrt{5}$. Certainly one may compute the series by means of (32) as follows:

$$
S=\frac{\rho}{\sqrt{5}} \sum_{k=0}^{\infty}\binom{k}{m}_{\beta}\left[(a / \rho)^{k+1}-(b / \rho)^{k+1}\right]=\frac{\rho}{\sqrt{5}} \sum_{j=0}^{m}\left[\left(\frac{a}{\rho-a}\right)^{j+1}-\left(\frac{b}{\rho-b}\right)^{j+1}\right] \frac{j!}{m!} S_{\beta}(m, j \mid 1) .
$$

In particular, we have

$$
\sum_{k=0}^{\infty} k^{m} \rho^{-k} F_{k}=\frac{\rho}{\sqrt{5}} \sum_{j=0}^{m}\left[\left(\frac{a}{\rho-a}\right)^{j+1}-\left(\frac{b}{\rho-b}\right)^{j+1}\right] j!S(m, j)
$$

Finally, it may be worthy of mention that, for the case $\alpha=1$, relation (26), apart from the factor $(-1)^{j}$ just stands for the $\delta^{*}$-transformation of the given sequence $\{f(k)\}$, which is connected with quasi-Hausdorff transformations (cf. Hardy [6, §11.19]). Moreover, it may be remarked that the rule $(24) \Rightarrow(25)$ can still be generalized. Let the functions $h(x, m)$ and $g(x, j)$ be related by

$$
\begin{equation*}
h(x, m)=\sum_{j=0}^{m} t(m, j) g(x, j) \tag{33}
\end{equation*}
$$

where the $t(m, j)$ are complex numbers. Define

$$
\begin{equation*}
\sum_{k=0}^{n} F(n, k) g(k, j)=\phi(n, j) \tag{34}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\sum_{k=0}^{n} F(n, k) h(k, m)=\sum_{j=0}^{m} \phi(n, j) t(m, j) \tag{35}
\end{equation*}
$$

This extended rule $(34) \Rightarrow(35)$ may even be used to obtain some interesting formulas involving Comtet's generalized Stirling numbers whose definitions may be found in [9]. However, we will omit the details here.

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# MANDELBROT'S FUNCTIONAL ITERATION AND CONTINUED FRACTIONS 

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Functional iteration which gives rise to the Mandelbrot set is concerned with functions of the form $g(x)=x^{2}+c$, where $c$ is a point in the complex plane. This paper provides an algorithm which uses Newton's method and Mandelbrot-type functional iteration to produce a sequence of rational numbers that converges quadratically to the square-root of any given positive integer, and has a best approximation property. The algorithm is then modified so that convergence can be accelerated to any power of 2 order.

## NEWTON'S METHOD VERSUS CONTINUED FRACTIONS

Let $n$ be any given positive integer that is not a perfect square. We can find $\sqrt{n}$ by using Newton's method with the equation $f(x)=x^{2}-n=0$. For an arbitrary choice of positive $x_{0}$, the sequence

$$
\begin{equation*}
x_{k+1}=x_{k}-\frac{x_{k}^{2}-n}{2 x_{k}}, \text { or } x_{k+1}=\frac{x_{k}^{2}+n}{2 x_{k}}, k \geq 0, \tag{1}
\end{equation*}
$$

will converge quadratically to the square-root of $n$. We say the sequence $\left\{t_{n}\right\}$ converges linearly or quadratically ( $\alpha=1$ or 2 ) to $t$ if:
(i) $\lim _{n \rightarrow \infty} t_{n}=t$,
(ii) $\lim _{n \rightarrow \infty} \frac{\left|t_{n+1}-t\right|}{\left|t_{n}-t\right|^{\alpha}}=\lambda$, where $0<\lambda<\infty$. [4]

On the other hand, using continued fractions, we can obtain a sequence of rational numbers, $p_{k} / q_{k}$, which converges linearly to the square-root of $n$. Each of these methods of obtaining $\sqrt{ } n$ has its advantages. Newton's method converges faster than does the continued fraction method, but continued fractions have the best possible approximation property. That is,

$$
\left|\sqrt{n}-\frac{p}{q}\right|<\left|\sqrt{n}-\frac{p_{k}}{q_{k}}\right| \text { implies that } q>q_{k} .
$$

What we seek to do is find a method that has the advantages of both Newton's method and the continued fractions method.

## NEWTON'S METHOD AND MANDELBROT ITERATION

Let $y_{k}=x_{k}^{2}-n$, then (1) implies

$$
y_{k+1}=x_{k+1}^{2}-n=\left(\frac{x_{k}^{2}+n}{2 x_{k}}\right)^{2}-n=\left(\frac{x_{k}^{2}-n}{2 x_{k}}\right)^{2}=\frac{y_{k}^{2}}{4\left(y_{k}+n\right)} .
$$

Let $y_{k}=1 / z_{k}$ and invert the above equation to obtain $z_{k+1}=4 z_{k}+4 n z_{k}^{2}$, which becomes

$$
\frac{w_{k+1}}{4 n}=\frac{4 w_{k}}{4 n}+4 n \frac{w_{k}^{2}}{16 n^{2}}, \text { if we let } z_{k}=\frac{w_{k}}{4 n} .
$$

Finally, setting $w_{k}=v_{k}-2$, the equation above becomes

$$
\begin{equation*}
v_{k+1}=v_{k}^{2}-2 \tag{2}
\end{equation*}
$$

But this is just Mandelbrot-type iteration with functions of the form $g(x)=x^{2}+c$, where in this case $c=-2$.

## RESULTS FROM THE THEORY OF CONTINUED FRACTIONS

The following points are known from the theory of continued fractions [2]:
(i) The continued fraction expansion of $\sqrt{n}=\left\langle a_{0}, \overline{a_{1}, a_{2}, \ldots, a_{r-1}, 2 a_{0}}\right\rangle$.
(ii) There exists a smallest subscript, $s$, such that the convergent $p_{s} / q_{s}$ has the property

$$
\begin{equation*}
p_{s}^{2}-n q_{s}^{2}=1 \tag{4}
\end{equation*}
$$

(iii) $s=(1+r \bmod 2) r-1$.
(iv) If $p, q$ are any integers that satisfy the equation $p^{2}-n q^{2}=1$, then $p / q$ is actually a continued fraction convergent to $\sqrt{n}$, say $p=p_{j}$ and $q=q_{j}$.
(v) If $t$ is a positive integer such that $j=s+(1+r \bmod 2)(t-1) r$, then

$$
\begin{equation*}
p_{j}+\sqrt{n} q_{j}=\left(p_{s}+\sqrt{n} q_{s}\right)^{t}, \text { and conversely. } \tag{5}
\end{equation*}
$$

(vi) For all positive integers $m$, if $j=s(1+r \bmod 2) m r$, then the continued convergent $p_{j} / q_{j}$ satisfies (4).

## A NEW SEQUENCE

Newton's method, (1), allows us arbitrary choice of $x_{0}$. This gives rise to an arbitrary choice of $v_{0}$ in (2). We seek to choose $v_{0}$ so that the sequence (2) is closely related to the sequence, $p_{k}$, given by the continued fraction expansion of $\sqrt{n}$.

Let us define the sequence $\left\{v_{k}\right\}, k=1,2, \ldots$ by letting

$$
\begin{equation*}
v_{0}=2 p_{s} \text { and } v_{k+1}=v_{k}^{2}-2, \text { for } k \geq 0 . \tag{6}
\end{equation*}
$$

Notice that this definition is the same as (2). This leads us to the following theorem.
Theorem 1: The sequence $\left\{v_{k}\right\}, k \geq 0$, is the same as the sequence $\left\{2 p_{j_{k}}\right\}$, where

$$
\begin{equation*}
j_{k}=s+(1+r \bmod 2)\left(2^{k}-1\right) r . \tag{7}
\end{equation*}
$$

Before we prove Theorem 1, we state the following
Lemma: If $j_{k}=s+(1+r \bmod 2)\left(2^{k}-1\right) r, k \geq 0$, then

$$
\begin{equation*}
p_{j_{k+1}}=p_{j_{k}}^{2}+n q_{j_{k}}^{2} . \tag{8}
\end{equation*}
$$

Proof: From (v),

$$
p_{j_{k+1}}+\sqrt{n} q_{j_{k+1}}=\left(p_{s}+\sqrt{n} q_{s}\right)^{2^{k+1}} \text { and } p_{j_{k}}+\sqrt{n} q_{j_{k}}=\left(p_{s}+\sqrt{n} q_{s}\right)^{2^{k}}
$$

So,

$$
\begin{aligned}
p_{j_{k+1}}+\sqrt{n} q_{j_{k+1}} & =\left(p_{j_{k}}+\sqrt{n} q_{j_{k}}\right)^{2} \\
& =p_{j_{k}}^{2}+n q_{j_{k}}^{2}+2 p_{j_{k}} q_{j_{k}} \sqrt{n} .
\end{aligned}
$$

We obtain $p_{j_{k+1}}=p_{j_{k}}^{2}+n q_{j_{k}}^{2}$, by equating the rational parts of both sides of the previous equation.
We return to the proof of Theorem 1. We use induction to prove our result. When $k=0$, $j_{0}=s$, so $2 p_{j 0}=2 p_{s}=v_{0}$, by definition 6 . Let us assume the result holds for some positive integer $k$. Then

$$
\begin{equation*}
v_{k+1}=v_{k}^{2}-2=\left(2 \dot{p}_{j_{k}}\right)^{2}-2=2\left(2 p_{j_{k}}^{2}-1\right) . \tag{9}
\end{equation*}
$$

From (vi), we conclude that

$$
2 p_{j_{k}}^{2}-1=p_{j_{k}}^{2}+n q_{j_{k}}^{2}=p_{j_{k+1}}, \text { using (8). }
$$

Putting this into (9), it follows that $v_{k+1}=2 p_{j_{k+1}}$, and the theorem is proved.

## QUADRATIC CONVERGENCE

It is also known from the theory of continued fractions [1] that, if $p / q$ is a convergent for $\sqrt{n}$, then

$$
\frac{c(n)}{q^{2}}<\left|\sqrt{n}-\frac{p}{q}\right| \leq \frac{1}{q^{2}}, \text { where } c(n)>0 .
$$

That is, the error in estimation of $\sqrt{n}$ by $p / q$ is of the order $1 / q^{2}$. Now, for the sequence of approximations $p_{j_{k}} / q_{j_{k}}$ (which are also continued fraction convergents), we have

$$
\begin{equation*}
\frac{\text { (error at stage }(k+1))}{(\text { error at stage } k)^{2}} \cong \frac{\frac{1}{q_{j_{k+1}}^{2}}}{\left(\frac{1}{q_{j_{k}}^{2}}\right)^{2}} \cong \frac{1}{4 n} \frac{\left(v_{k+1}-2\right)^{2}}{v_{k+1}^{2}-4} \text {, using (2), (4), and Theorem } 1 . \tag{10}
\end{equation*}
$$

Since $v_{k} \rightarrow \infty$ as $k \rightarrow \infty$, the right-hand side of the last equation converges to $1 / 4 n$; this confirms that $p_{j_{k}} / q_{j_{k}}$ converges to $\sqrt{n}$ quadratically.

## THE ALGORITHM

1. Let $n$ be given, $n$ a positive integer that is not a perfect square.
2. Use the continued fractions algorithm for $\sqrt{n}$ [3] to find
(i) $\sqrt{n}=\left\langle a_{0}, \overline{a_{1}, \ldots, a_{r-1}, 2 a_{0}}\right\rangle$, and
(ii) $p_{s}, q_{s}$, where $s$ is the smallest positive integer such that $p_{s}^{2}-n q_{s}^{2}=1$ $(s=(1+r \bmod 2) r-1)$.
3. $v_{0}=2 p_{s}$, and $v_{k+1}=v_{k}^{2}-2$, for $k \geq 0$.
4. Define $j_{k}=s+(1+r \bmod 2)\left(2^{k}-1\right) r$, then

$$
p_{j_{k}}=\frac{v_{k}}{2}, \text { and } q_{j_{k}}=\sqrt{\frac{p_{j_{k}}^{2}-1}{n}}
$$

are such that $p_{j_{k}} / q_{j_{k}}$ is a continued fraction convergent to $\sqrt{n}$, and the sequence $p_{j_{k}} / q_{j_{k}}$ converges quadratically to $\sqrt{n}$.

## AN EXAMPLE

Let us consider an example of the algorithm where $n=19$. Then we can use the continued fraction algorithm to find:

$$
\sqrt{19}=\langle 4, \overline{2,1,3,1,2,8}\rangle
$$

In this case, we notice that $r=6$, and that $s=(1+r \bmod 2) r-1=5$. Thus, again using the continued fraction algorithm for convergents, we obtain:

| $j$ |  |  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $\cdots$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a$ |  |  | 4 | 2 | 1 | 3 | 1 | 2 | 8 |  |
| $p$ | 0 | 1 | 4 | 9 | 13 | 48 | 61 | 170 | 1421 | $\cdots$ |
| $q$ | 1 | 0 | 1 | 2 | 3 | 11 | 14 | 39 | 326 |  |

Since $s=5$, the convergent $p_{5} / q_{5}=170 / 39$ satisfies (4). Thus, $v_{0}=340$, and $v_{1}=g\left(v_{0}\right)=$ $340^{2}-2=11598$, which implies that

$$
p_{j_{1}}=57799 \text { and } q_{j_{1}}=\sqrt{\frac{p_{j_{1}}^{2}-1}{19}}=13260 .
$$

Similarly, $v_{2}=g(115598)=115598^{2}-2=13362897602$, and

$$
p_{j_{2}}=\frac{v_{2}}{2}=6681448801 \text { and } q_{j_{2}}=\sqrt{\frac{p_{j_{1}}^{2}-1}{19}}=1532829480
$$

Continuing in this manner, we obtain sequences $\left\{p_{j_{k}}\right\}$ and $\left\{q_{j_{k}}\right\}$ of integers such that the sequence $\left\{p_{j_{k}} / q_{j_{k}}\right\}$ is a sequence of continued fraction convergents which converge to $\sqrt{19}$ with quadratic order.

## CONCLUSION

We note that we obtained the sequence, $v_{k+1}=v_{k}^{2}-2, k \geq 0$, by iteration of the function $g(x)=x^{2}-2$ and starting at $v_{0}=2 p_{s}$. We can ask, "What happens if we iterate $g$ twice?" That is, let us define the sequence

$$
u_{k+1}=g^{2}\left(u_{k}\right)=g\left(g\left(u_{k}\right)\right)=u_{k}^{4}-4 u_{k}^{2}+2, k \geq 0
$$

with $u_{0}=2 p_{s}$. Using arguments similar to the ones given above, we can show that $u_{k}=2 p_{\ell_{k}}$, where

$$
\ell_{k}=s+(1+r \bmod 2)\left(2^{2 k}-1\right) r .
$$

Furthermore, if $q_{\ell_{k}}$ is obtained from $p_{\ell_{k}}$ by the use of (4), then $p_{\ell_{k}} / q_{\ell_{k}}$ define a sequence of continued fraction convergents which converge to $n$ with order $2^{2}$.

These methods generalize to prove
Theorem 2: Let $m$ be a positive integer, and let us define the sequence $\left\{t_{k}\right\}, k \geq 0$, as follows:

$$
t_{0}=2 p_{s}, \quad t_{k+1}=g^{m}\left(t_{k}\right),
$$

where $g(x)=x^{2}-2$ and $g^{m}$ is the $m$-fold iteration of $g$. Then $t_{k}=2 p_{h_{k}}$, where

$$
h_{k}=s+(1+r \bmod 2)\left(2^{m k}-1\right) r .
$$

Also, if

$$
q_{h_{k}}=\sqrt{\frac{p_{h_{k}}^{2}-1}{n}},
$$

then the sequence

$$
\frac{p_{h_{k}}}{q_{h_{k}}}, k \geq 0
$$

is a sequence of continued fraction convergents which converge to $\sqrt{n}$ with order of convergence equal to $2^{m}$.

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## EDITOR ON LEAVE OF ABSENCE

The Editor has been asked to visit Yunnan Normal University in Kunming, China, for the Fall semester of 1993. This is an opportunity that the Editor and his wife feel cannot be turned down. They will be in China from August 1, 1993, until approximately January 10, 1994. The August and November issues of The Fibonacci Quarterly will be delivered to the printer early enough so that these two issues can be published while the Editor is out of the country. The Editor has also arranged for several individuals to send out articles to be refereed which have been submitted for publication in The Fibonacci Quarterly or submitted for presentation at the Sixth International Conference on Fibonacci Numbers and Their Applications. Things may be a little slower than normal, but every attempt will be made to insure that all goes as smoothly as possible while the Editor is on leave in China. PLEASE CONTINUE TO USE THE NORMAL ADDRESS FOR SUBMISSION OF PAPERS AND ALL OTHER CORRESPONDENCE.

# THE RABBIT PROBLEM REVISITED 

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## INTRODUCTION

In Liber Abaci (1202), Leonardo da Pisa posed and solved the following problem.
A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

The sequence obtained to solve this problem-the celebrated Fibonacci sequence 1, 1, 2, 3, 5, 8, $13,21, \ldots$-appears in a large number of ratural phenomena (see [2], [6]) and has natural applications in computer science (see [1]).

Here we reformulate the rabbit problem to recover two generalizations of the Fibonacci sequence presented elsewhere (see [7], [8]). Then, using a fixed-point technique, we present an elementary proof of the convergence of the sequences of ratios of two successive generalized Fibonacci numbers. The limits of these sequences will be called here generalized golden numbers. Finally, we reconsider electrical schemes to generate these ratios (see also [3]).

## 1. THE RABBIT PROBLEM REVISITED

The modifications to the rabbit problem we would like to consider here are the possibility that the mature rabbits produce more than one new pair of rabbits, and also the possibility of an increase in the productivity.during the first few months. These two considerations lead to the following reformulation of the rabbit problem.

A certain man put a pair of newborn male-female rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that each month
(a) a $i$-month old pair of male-female rabbits gives birth to $(i-1) s$ pair(s) of male-female rabbits until it is $r$-months old, and
(b) a more than $r$-month old pair of male-female rabbits continues to give birth to $(r-1) s$ pairs of male-female rabbits?
In this formulation it is assumed that $s$ is a positive integer.
Let $u_{n}$ be the total number of pairs of male-female rabbits at the $n^{\text {th }}$ month, and $v_{n}^{i}$ be the number of $i$-month old pairs of male-female rabbits at the $n^{\text {th }}$ month. Since $v_{n}^{0}$ is the number of newborn pairs of male-female rabbits at the $n^{\text {th }}$ month, we have

Then

$$
\begin{gather*}
u_{n}=u_{n-1}+v_{n}^{0} \text { and } v_{n}^{i}=v_{n-i}^{0}  \tag{1}\\
v_{n}^{0}=0 \text { for } n=-1,-2,-3, \ldots  \tag{2a}\\
v_{0}^{0}=1  \tag{2b}\\
v_{n}^{0}=\sum_{i=1}^{r}(i-1) s v_{n}^{i}+\sum_{i=r+1}^{+\infty}(r-1) s v_{n}^{i} \text { for } n=1,2,3, \ldots \tag{2c}
\end{gather*}
$$

Using (1), (2c) becomes

$$
v_{n}^{0}=s \sum_{i=2}^{r} u_{n-i},
$$

and it follows that

$$
\begin{gather*}
u_{n}=0 \text { for } n=-1,-2,-3, \ldots  \tag{3a}\\
u_{0}=1  \tag{3b}\\
u_{n}=u_{n-1}+s \sum_{i=2}^{r} u_{n-i} \text { for } n=1,2,3, \ldots . \tag{3c}
\end{gather*}
$$

Remark 1: For $r=2$ we have the multi-nacci sequence of order $s$ recently considered by Levine [7]. One interesting property of these sequences is

$$
u_{n}^{2}-u_{n+1} u_{n-1}=(-s)^{n}
$$

Remark 2: For $s=1$ we have the $r$-generalized Fibonacci sequence introduced by Miles [8] and also studied by Flores [4] and Dubeau [3].

From these two remarks, we can call a $r$-generalized multi-nacci sequence of order $s$ the sequence of $u_{n}$ 's generated by (3).

## 2. CONVERGENCE OF RATIOS

In this section, we extend the method presented in [3] and [5] to obtain the limit of the sequence of ratios $t_{n}=u_{n} / u_{n-1}(n=1,2,3, \ldots)$. Since the $u_{n}$ 's form an increasing sequence, we have $t_{n} \geq 1$ for $n=1,2,3, \ldots$. From (3c) we have

$$
t_{n}=1+s \sum_{i=2}^{r} \frac{u_{n-i}}{u_{n-1}}(n \geq 1)
$$

and using the definition of $t_{n}$, we obtain

$$
t_{n}=\left\{\begin{array}{l}
1+s \sum_{i=2}^{n} \frac{1}{\prod_{j=1}^{i-1} t_{n-j}} n=1, \ldots, r-1, \\
1+s \sum_{i=2}^{r} \frac{1}{\prod_{j=1}^{i-1} t_{n-j}} n=r, r+1, r+2, \ldots
\end{array}\right.
$$

The results of this section are then mainly based on the following two remarks.
Remark 3: $t_{n}$ depends only on the preceding $r-1$ values $t_{n-1}, t_{n-2}, \ldots, t_{n-(r-1)}$, and we can write $t_{n}=f\left(t_{n-1}, \ldots, t_{n-(r-1)}\right)$.
Remark 4: If $t_{n-1}, \ldots, t_{n-(r-1)}$ are all greater than or equal to $b>0$, then $t_{n} \leq f(b, \ldots, b)$ and if $t_{n-1}, \ldots, t_{n-(r-1)}$ are all less than or equal to $b>0$, then $t_{n} \geq f(b, \ldots, b)$.

Let us use the function $f(;, \ldots, \cdot)$ to define another function $F(\cdot)$ as follows: $F(x)=$ $f(x, \ldots, x)$ or, explicitly,

$$
\begin{equation*}
F(x)=1+s \sum_{i=2}^{r} \frac{1}{x^{i-1}} \text { for } x \neq 0 \tag{4}
\end{equation*}
$$

The convergence result we look for will be obtained from the study of the function $F(\cdot)$. The next lemma summarizes the main properties of $F(\cdot)$.

Lemma 1: Let $s>0, r \in\{2,3,4, \ldots\}$ and $x \neq 0$. Then
(a) $F(\dot{x})= \begin{cases}1+s(r-1) & \text { if } x=1, \\ 1+\frac{s}{x^{r-1}} \frac{\left(x^{r-1}-1\right)}{(x-1)} & \text { if } x \neq 1 ;\end{cases}$
(b) $F(\cdot)$ is a strictly decreasing continuous convex function for $x>0$;
(c) $\lim _{x \rightarrow 0^{+}} F(x)=+\infty$ and $\lim _{x \rightarrow+\infty} F(x)=1$;
(d) the equation $x=F(x)$ has a unique solution $\tau$ in the interval $(0,+\infty)$ and $\tau$ is the unique positive root of the polynomial

$$
p(x)=x^{r}-x^{r-1}-s \sum_{i=2}^{r} x^{r-i} .
$$

Remarks 3 and 4 and the fact that $t_{k} \geq 1(k \geq 1)$ suggest the construction of a sequence $\left\{b_{\ell}\right\}_{\ell=1}^{+\infty}$ such that

$$
\begin{array}{lll}
b_{1}=1 & \leq t_{k} & \text { for } k \geq 1, \\
t_{k}
\end{array} \leq F\left(b_{1}\right)=b_{2} \begin{aligned}
& \text { for } k \geq 1+(r-1), \\
& b_{3}=F\left(b_{2}\right) \leq t_{k} \\
& t_{k}
\end{aligned} \leq F\left(b_{3}\right)=b_{4} \text { for } k \geq 1+2(r-1), ~ \begin{array}{ll}
\text { for } k \geq 1+3(r-1), \\
b_{5}=F\left(b_{4}\right) \leq \begin{array}{l}
t_{k} \\
\text { etc. }
\end{array} & \text { for } k \geq 1+4(r-1),
\end{array}
$$

We have the following results about the sequence $\left\{b_{\ell}\right\}_{\ell=1}^{+\infty}$.
Lemma 2: Let $\left\{b_{\ell}\right\}_{\ell=1}^{+\infty}$ such that $b_{1}=1$ and $b_{\ell+1}=F\left(b_{\ell}\right)$ for $\ell=1,2,3, \ldots$, then
(a) the subsequence $\left\{b_{2 \ell-1}\right\}_{\ell=1}^{+\infty}$ is strictly increasing and the subsequence $\left\{b_{2 \ell}\right\}_{\ell=1}^{+\infty}$ is strictly decreasing;
(b) for all $i$ and $j \geqq 1$, we have $b_{2 i-1}<b_{2 j}$;
(c) there exists a positive constant $\beta<1$ such that $0<b_{2 \ell+2}-b_{2 \ell+1}<\beta^{2 \ell} s(r-1)$ for $\ell=1,2$, $3, \ldots$;
(d) the sequence $\left\{b_{\ell}\right\}_{\ell=1}^{+\infty}$ converges to the unique positive real root of the polynomial

$$
p(x)=x^{r}-x^{r-1}-s \sum_{i=2}^{r} x^{r-i} .
$$

Proof: (a) and (b) follow from $1=b_{1}<b_{2}=F\left(b_{1}\right)$, and if $0<\alpha<\beta$, then $1<F(\beta)<F(\alpha)$. To prove (c) we use (4) and consider

$$
\begin{aligned}
0<b_{2 \ell+2}-b_{2 \ell+1} & =F\left(b_{2 \ell+1}\right)-F\left(b_{2 \ell}\right)=s \sum_{i=2}^{r}\left(\frac{1}{b_{2 \ell+1}^{i-1}}-\frac{1}{b_{2 \ell}^{i-1}}\right) \\
& =s \frac{\left(b_{2 \ell}-b_{2 \ell+1}\right)}{b_{2 \ell} b_{2 \ell+1}} \sum_{i=0}^{r-2} \sum_{j=0}^{i} \frac{1}{b_{2 \ell}^{j} b_{2 \ell+1}^{i-j}} \leq s \frac{\left(b_{2 \ell}-b_{2 \ell+1}\right)}{b_{2 \ell} b_{2 \ell+1}}\left(\sum_{i=0}^{r-2} \frac{1}{b_{2 \ell}^{i}}\right)\left(\sum_{i=0}^{r-2} \frac{1}{b_{2 \ell+1}^{i}}\right) .
\end{aligned}
$$

But

$$
\sum_{i=0}^{r-2} \frac{1}{b_{2 \ell}^{i}}=\frac{\left(b_{2 \ell+1}-1\right)}{s} b_{2 \ell} \text { and } \sum_{i=0}^{r-2} \frac{1}{b_{2 \ell+1}^{i}}=\frac{\left(b_{2 \ell+1}^{r-1}-1\right)}{b_{2 \ell+1}^{r-2}\left(b_{2 \ell+1}-1\right)}
$$

then

$$
0<b_{2 \ell+2}-b_{2 \ell+1} \leq\left(b_{2 \ell}-b_{2 \ell+1}\right)\left(1-\frac{1}{b_{2 \ell+1}^{r-1}}\right) .
$$

Also, $1 \leq b_{k} \leq 1+s(r-1)$, then $1 \leq b_{k}^{r-1} \leq[1+s(r-1)]^{r-1}$, and it follows that

$$
0 \leq 1-\frac{1}{b_{k}^{r-1}}<1-\frac{1}{[1+s(r-1)]^{r-1}}=\beta<1 .
$$

Hence, $0 \leq b_{2 \ell+2}-b_{2 \ell+1} \leq\left(b_{2 \ell}-b_{2 \ell+1}\right) \beta$. Similarly, we can prove $0<b_{2 \ell}-b_{2 \ell+1} \leq\left(b_{2 \ell}-b_{2 \ell-1}\right) \beta$, and we can conclude that

$$
0<b_{2 \ell+2}-b_{2 \ell+1} \leq \beta^{2}\left(b_{2 \ell}-b_{2 \ell-1}\right) \leq \cdots \leq \beta^{2 \ell}\left(b_{2}-b_{1}\right) .
$$

But $b_{2}-b_{1}=s(r-1)$, and the result follows. Finally, from (c), the upperbounded incieasing subsequence $\left\{b_{2 \ell+1}\right\}_{\ell=1}^{+\infty}$ and the lower bounded decreasing subsequence $\left\{b_{2 \ell}\right\}_{\ell=1}^{+\infty}$ both converge to the value $\tau$ defined in Lemma 1 .

Figure 1, on the following page, describes the construction and the convergence of the sequence $\left\{b_{\ell}\right\}_{\ell=0}^{+\infty}$.

We are now ready to prove the main results.
Theorem 1: Let $s>0, r \in\{2,3,4,5, \ldots\}, u_{n}$ as given by (3), and $t_{n}=u_{n} / u_{n-1}$ for $n \geq 1$. The sequence $\left\{t_{n}\right\}_{n=1}^{+\infty}$ converges to the unique positive root $\tau$ of the polynomial

$$
p(x)=x^{r}-x^{r-1}-s \sum_{i=2}^{r} x^{r-i} .
$$

Proof: From the way the sequences $\left\{t_{k}\right\}_{k=1}^{+\infty}$ and $\left\{b_{\ell}\right\}_{\ell=1}^{+\infty}$ are generated, we have

$$
t_{k} \geq b_{2 \ell+1} \text { for } k \geq 1+2 \ell(r-1) \text { and } t_{k} \leq b_{2 \ell} \text { for } k \geq 1+(2 \ell+1)(r-1) .
$$

The result follows from Lemma 2.


FIGURE 1. Graph of $\boldsymbol{y}=\boldsymbol{F}(\boldsymbol{x})$
Theorem 2: Let $\tau$ be considered as a function of $\tau$ and $s$. Then
(a) for any fixed $s>0$, we have
(i) $\tau=\frac{1+\sqrt{1+4 s}}{2}$ for $r=2$,
(ii) $\tau$ increases as $r$ increases,
and
(iii) $\lim _{r \rightarrow+\infty}=1+\sqrt{s}$,
(b) for any fixed $\tau, \tau \sim \sqrt{s}$ for large $s$.

Proof: For $r=2, \tau$ is the unique positive root of $p(x)=x^{2}-x-s$, which corresponds to the given formula. Because $F(x)$ increases as $r$ increases for fixed $x$ and $s, \tau$ increases as $r$ increases. Also

$$
F(1+\sqrt{s})=1+\sqrt{s}-\frac{\sqrt{s}}{(1+\sqrt{s})^{r-1}}<1+\sqrt{s},
$$

then

$$
1+\sqrt{s}-\frac{\sqrt{s}}{(1+\sqrt{s})^{r-1}}<\tau<1+\sqrt{s}
$$

and $\lim _{r \rightarrow+\infty} \tau=1+\sqrt{s}$. Also, from those formulas and inequalities, we obtain $\tau \sim \sqrt{s}$ when $s$ is large.

The table below presents values of $\tau$ for some $r$ and $s$. The last line of this table for $r=+\infty$ indicates $\lim _{r \rightarrow+\infty} \tau=1+\sqrt{s}$.

When $r=2$ and $s=1, \tau$ corresponds to the golden number. For $s=1$ and $r \in\{2,3, \ldots\}, \tau$ has been called the $r$-generalized golden number. Hence, for $s>0$ and $r \in\{2,3, \ldots\}$, we could call $\tau$ the $r$-generalized golden number of order $s$.

Table of $\tau$ Values for Given $r$ and $s$

|  | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 1.6180340 | 2.0000000 | 2.3027756 | 2.5615528 | 2.7912878 |
| 3 | 1.8392868 | 2.2695308 | 2.5986745 | 2.8751298 | 3.1179423 |
| 4 | 1.9275620 | 2.3593041 | 2.6868102 | 2.9611061 | 3.2017404 |
| 5 | 1.9659482 | 2.3924637 | 2.7160633 | 2.9874051 | 3.2257176 |
| 6 | 1.9835828 | 24054051 | 2.7262912 | 2.9958519 | 32328999 |
| 7 | 1.9919642 | 2.4106054 | 27299574 | 2.9986240 | 3.2350925 |
| 8 | 1.9960312 | 2.4127271 | 2.7312869 | 2.9995122 | 3.2357669 |
| 9 | 1.9980295 | 2.4135994 | 2.7317715 | 2.9998475 | 3.2359750 |
| 10 | 1.9990186 | 2.4139595 | 2.7319486 | 2.9999492 | 3.2360392 |
| 11 | 1.9995104 | 2.4141084 | 2.7320134 | 2.9999831 | 3.2360591 |
| 12 | 1.9997555 | 2.4141700 | 2.7320371 | 2.9999944 | 3.2360652 |
| 13 | 1.9998778 | 2.4141955 | 2.7320458 | 2.9999981 | 3.2360671 |
| 14 | 1.9999389 | 2.4142061 | 2.7320490 | 2.9999994 | 3.2360677 |
| 15 | 1.9999695 | 2.4142105 | 2.7320501 | 2.9999998 | 3.2360679 |
| 16 | 1.9999847 | 2.4142123 | 2.7320506 | 2.9999999 | 3.2360680 |
| 17 | 1.9999924 | 2.4142130 | 2.7320507 | 3.0000000 | 3.2360680 |
| 18 | 1.9999962 | 2.4142133 | 2.7320508 | 3.0000000 | 3.2360680 |
| 19 | 1.9999981 | 2.4142135 | 2.7320508 | 3.0000000 | 3.2360680 |
| 20 | 1.9999990 | 2.4142135 | 2.7320508 | 3.0000000 | 3.2360680 |
| 21 | 1.9999995 | 2.4142135 | 2.7320508 | 3.0000000 | 3.2360680 |
| 22 | 1.9999998 | 2.4142136 | 2.7320508 | 3.0000000 | 3.2360680 |
| 23 | 1.9999999 | 2.4142136 | 2.7320508 | 3.0000000 | 3.2360680 |
| 24 | 1.9999999 | 2.4142136 | 2.7320508 | 3.0000000 | 3.2360680 |
| 25 | 2.0000000 | 2.4142136 | 2.7320508 | 3.0000000 | 3.2360680 |
| + | 2. | 2.4142136 | 2.7320508 | 3. | 3.2360680 |

## 3. ELECTRICAL SCHEMES

The method presented in [3] to generate the sequences of ratios $\left\{u_{n} / u_{n-1}\right\}_{n=1}^{+\infty}$ using electrical schemes can also be used here. Indeed, if

$$
\begin{equation*}
\Omega_{j, i}=\frac{u_{j+i}}{u_{j}}=\frac{u_{j+i-1}+s \sum_{k=2}^{r} u_{j+i-k}}{u_{j}}=\Omega_{j, i-1}+s \sum_{k=2}^{r} \Omega_{j, i-k}, \tag{5}
\end{equation*}
$$

which correspond to $1+s(r-1)$ resistances connected in series. Also

$$
\begin{equation*}
\Omega_{j, i}=\frac{u_{j+i}}{u_{j}}=\frac{u_{j+i}}{u_{j-1}+s \sum_{k=2}^{r} u_{j-k}}=\frac{1}{\frac{1}{u_{j+i} / u_{j-1}}+s \sum_{k=2}^{r} \frac{1}{u_{j+i} / u_{j-k}}}=\frac{1}{\frac{1}{\Omega_{j-1, i+1}}+s \sum_{k=2}^{r} \frac{1}{\Omega_{j-k, i+k}}}, \tag{6}
\end{equation*}
$$

which correspond to $1+s(r-1)$ resistances connected in parallel. Here, again, it is assumed that $s$ is a positive integer.

Those two formulas, (5) and (6), suggest the following process to generate the resistances $\Omega_{j, i}(j=0,1,2, \ldots$, and $i=-(r-1, \ldots,-1,0,1, \ldots, r-1)$ :
(a) generate $\Omega_{j, i}(i=-(r-1), \ldots,-1)$ using (6) with $\Omega_{j-1, i+1}$ and $s$ of each $\Omega_{j-k, i+k}$ for $k=$ $2, \ldots, r$;
(b) $\Omega_{j, 0}=1$;
(c) gernerate $\Omega_{j, i}(i=1,2,3, \ldots, r-1)$ using (5) with $\Omega_{j, i-1}$ and $s$ of each $\Omega_{j, i-k}$ for $k=2$, ..., $r$.

Note that the ratios we are interested in correspond to $\Omega_{j, 1}(j=0,1,2,3, \ldots)$.

## REFERENCES

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AMS numbers: 11B39, 05A10, 94C05

## ERRATUM FOR "COMPLEX FIBONACCI AND LUCAS NUMBERS, CONTINUED FRACTIONS, AND THE SQUARE ROOT OF THE GOLDEN RATIO"

## The Fibonacci Quarterly 31.1 (1993):7-20

It has been pointed out to me by a correspondent who wished to remain anonymous that the number 185878941, which was printed in the "loose ends" Section 7 on page 19 of the paper, has a factor 3. This, however, was a misprint for 285878941 , which is $\left(\ell_{19}^{2}+\ell_{19}^{\prime 2}\right) / 2$, and the same correspondent has checked that this is a prime by using Mathematica. The misprint was important because it appeared to undermine one of the interesting conjectures on that page (and incidentally calls into question my ability to "cast out 3 s "!). The same correspondent pointed out that $34227121=137 \times 249833$.

# CIRCULAR SUBSETS WITHOUT $q$-SEPARATION AND POWERS OF LUCAS NUMBERS 

John Konvalina and Yi-Hsin Liu<br>Department of Mathematics and Computer Science, University of Nebraska at Omaha, Omaha, NE 68182-0243<br>(Submitted September 1991)

Let $n, q, k$ be integers, $n \geq 1, q \geq 1, k \geq 0$. Consider $1,2, \ldots, n$ displayed in a circle so that $n$ follows 1. Then the integers $i, j(1 \leq j<j \leq n)$ are said to be (circular) $q$-separate if $i+q=j$ or $j+q-n=i$. Let $C_{q}(n, k)$ denote the number of $k$-subsets of $\{1,2, \ldots, n\}$ without $q$-separation (no two integers in the subset are $q$-separate). The (total) number of subsets without $q$-separation is $C_{q}(n)=\sum_{k \geq 0} C_{q}(n, k)$. In this note we prove that

$$
\begin{equation*}
C_{q}(n)=L_{m}^{d}, \text { where } d=\operatorname{gcd}(n, q), m=n / d \tag{1}
\end{equation*}
$$

as follows. Partition the cycle $\{1,2, \ldots, n\}$ into $d$ disjoint cycles $S_{i}$ (reduced modulo $n$ ):

$$
\begin{equation*}
S_{i}=\{i, i+q, i+2 q, \ldots, i+(m-1) q\}, 1 \leq i \leq d . \tag{2}
\end{equation*}
$$

The cardinality of each $S_{i}$ is $m$, and $C_{q}(n)$ is equal to the product of the number of subsets of each $S_{i}$ not containing a pair of consecutive elements. Thus, $C_{q}(n)=\left(C_{1}(m)\right)^{\mu}$. But it is an old result that $C_{1}(n)=L_{n}$, since $C_{1}(n)$ can also be interpreted as the number of circular subsets without adjacencies ( 1 and $n$ are adjacent).

The case $q=2$ of (1) is

$$
C_{2}(n)=\left\{\begin{array}{ll}
L_{n / 2}^{2} & \text { if } n \text { is even, } \\
L_{n} & \text { if } n \text { is odd, }
\end{array} \quad\right. \text { given in [2]. }
$$

It should be noted that (1) is the special case $x=1$ of the polynomial identity

$$
\begin{gather*}
\sum_{k \geq 0} C_{q}(n, k) x^{k}=\left((\alpha(x))^{m}+(\beta(x))^{m}\right)^{d}  \tag{3}\\
d=\operatorname{gcd}(n, q), m=n / d, \alpha(x)+\beta(x)=1, g a(x) \beta(x)=-x
\end{gather*}
$$

established in [2], where the proof involves the same partitioning (2). In the special case $x=2$,
(3) becomes $\Sigma^{k \geq 0} C_{q}(n, k) 2^{k}=\left(2^{m}+(-1)^{m}\right)^{d}, d=\operatorname{gcd}(n, q), m=n / d$.

This has a pleasing combinatorial interpretation, namely, it is the number of 2 -colored circular subsets of $\{1,2, \ldots, n\}$ without $q$-separation.

## REFERENCES

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2. W. O. J. Moser. "The Number of Subsets without a Fixed Circular Distance." J. Combin. Theory A 43 (1986):130-32.
AMS number: 05A15

# THE CONNECTIVITY OF A PARTICULAR GRAPH 

Marc S. Ordower*<br>University of Waterloo, Waterloo, Ontario, Canada N2L 3G1<br>(Submitted November 1991)

Let $G$ be a graph with vertex set $N=\{1,2,3, \ldots\}$ and edge set $E$ where $\{a, b\} \in E$ if and only if $a^{2}+b^{2}=c^{2}$ for some $c$ in $N$. From the standard parameterization of Pythagorean triples, it is easy to deduce that 1 and 2 are isolated vertices and that 3 and 4 together comprise a connected component of $G$. Our result concerns the connectivity of the rest of the graph.

Theorem: $N \backslash\{1,2,3,4\}$ is connected in the graph $G$.
Proof: One may verify that $8,15,20,21,72,30,16$ is a path in $G$ between 8 and 16 . Note also that $\{a, b\} \in E$ implies that $\{c a, c b\} \in E$ for all $c \in N$. Therefore, by multiplying the elements in the above path by the appropriate power of 2 , we find a path in $G$ between $2^{k}$ and $2^{k+1}$ for all $k \geq 3$.

Next, given $n \geq 5$, we recursively find a path $P_{n}: n=n_{0}, n_{1}, \ldots, n_{r}=2^{k}$ for some $k \geq 3$ according to the following algorithm: factor $n_{i}=p_{i} m_{i}$ where $p_{i}$ is the largest prime factor of $n_{i}$; if $p_{i}=2$ then we are done; otherwise, set $n_{i+1}=\frac{p_{i}^{2}-1}{2} \cdot m_{i}$.

We make two observations to verify that this algorithm generates the desired path. First, note that

$$
n_{i}^{2}+n_{i+1}^{2}=\left(p_{i} m_{i}\right)^{2}+\left(\frac{p_{i}^{2}-1}{2} \cdot m_{i}\right)^{2}=\left(\frac{p_{i}^{2}+1}{2} \cdot m_{i}\right)^{2}
$$

implies that $\left\{n_{i}, n_{i+1}\right\} \in E$.
Second, note that all prime factors of $\frac{p^{2}-1}{2}(p$ an odd prime) are strictly less than $p$. Hence, for all $i \in\{1,2, \ldots, r-1\}$, if $n_{i}=p_{i}^{s_{i}} m_{i}^{\prime}$ where $\operatorname{gcd}\left(p_{i}, m_{i}^{\prime}\right)=1$, then $p_{i,}=p_{i+1}=\cdots p_{i+s_{i}-1}>p_{i+s_{i}}$. Therefore, the algorithm terminates after a finite number of steps.

Corollary: If $H$ is the graph with vertex set $N$ and edge set $E^{\prime}$ where $\{a, b\} \in E^{\prime}, a>b$ if and only if $a^{2}-b^{2}=c^{2}$ for some $c \in N$, then $N \backslash\{1,2\}$ is connected.

Proof: One notes that, for all $\{a, b\} \in E$, there exists a $c \in N$ such that $\{a, c\},\{b, c\} \in E^{\prime}$. Also note that $\{3,5\},\{4,5\} \in E^{\prime}$.

## REFERENCE

1. J. A. Bondy \& U. S. R. Murty. Graph Theory with Applications. New York: Elsevier; London: Macmillan, 1976.
AMS number: 11D16
[^2]
# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by<br>Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3515@compuserve.com on Internet. All correspondence will be acknowledged.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

## BASIC FORMULAS

The Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ satisfy

$$
\begin{aligned}
& F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1 \\
& L_{n+2}=L_{n+1}+L_{n}, L_{0}=2, L_{1}=1
\end{aligned}
$$

Also, $\alpha=(1+\sqrt{5}) / 2, \beta=(1-\sqrt{5}) / 2, F_{n}=\left(\alpha^{n}-\beta^{n}\right) / \sqrt{5}$, and $L_{n}=\alpha^{n}+\beta^{n}$.

## PROBLEMS PROPOSED IN THIS ISSUE

## B-742 Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University

 Warrensburg, MOPell numbers are defined by $P_{0}=0, P_{1}=1$, and $P_{n+1}=2 P_{n}+P_{n-1}$, for $n \geq 1$. Show that

$$
P_{23}=2^{11} \prod_{j=1}^{11}\left(3+\cos \frac{2 j \pi}{23}\right) .
$$

## B-743 Proposed by Richard André-Jeannin, Longwy, France

Find the modulus and the argument of the complex numbers

$$
u=\frac{\beta+i \sqrt{\alpha+2}}{2} \text { and } v=\frac{\alpha+i \sqrt{\beta+2}}{2} .
$$

## B-744 Proposed by Herta T. Freitag, Roanoke, VA

Let $n$ and $k$ be even positive integers. Prove that $L_{2 n}+L_{4 n}+L_{6 n}+\cdots+L_{2 k n}$ is divisible by $L_{n}$.

## B-745 Proposed by Richard André-Jeannin, Longwy, France

Show that

$$
\sum_{n=1}^{\infty} \frac{1}{F_{2 n}}=1+\sum_{n=1}^{\infty} \frac{1}{F_{2 n-1} F_{2 n} F_{2 n+1}}
$$

## B-746 Proposed by Seung-Jin Bang, Albany, CA

Solve the recurrence equation $a_{n+1}=4 a_{n}^{3}+3 a_{n}, n \geq 0$, with initial condition $a_{0}=1 / 2$.

## B-747 Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

Let

$$
S_{1}=\sum_{n=2}^{\infty} \frac{1}{(-1)^{n} L_{2 n-1}-1} \text { and } S_{2}=\sum_{n=2}^{\infty} \frac{1}{(-1)^{n} L_{2 n-1}+1}
$$

Prove that $S_{1} / S_{2}=\sqrt{5}$.

## SOLUTIONS

## Recurrence with a Twist

## B-714 Proposed by J. R. Goggins, Whiteinch, Glasgow, Scotland

(Vol. 30, no. 2, May 1992)
Define a sequence $G_{n}$ by $G_{0}=0, G_{1}=4$, and $G_{n+2}=3 G_{n+1}-G_{n}-2$ for $n \geq 0$. Express $G_{n}$ in terms of Fibonacci and/or Lucas numbers.

## Solution by Graham Lord, Stanford CA

We claim that $G_{n}=2 L_{2 n-1}+2$. To see this, note that

$$
L_{2 n+3}=L_{2 n+2}+L_{2 n+1}=2 L_{2 n+1}+L_{2 n}=2 L_{2 n+1}+\left(L_{2 n+1}-L_{2 n-1}\right)=3 L_{2 n+1}-L_{2 n-1} .
$$

Doubling and adding 2 to both sides gives

$$
2 L_{2 n+3}+2=3\left(2 L_{2 n+1}+2\right)-\left(2 L_{2 n-1}+2\right)-2
$$

Thus, $G_{n}$ and $2 L_{2 n-1}+2$ both satisfy the same recurrence. Since they also have the same initial values, they must represent the same sequence.

Solvers submitted many other correct solutions, including $F_{2 n+2}+F_{2 n-4}+2, L_{2 n}+L_{2 n-3}+2$, $L_{2 n}+F_{2 n-2}+F_{2 n-4}+2,5 F_{2 n}-L_{2 n}+2, L_{n-1} L_{n}+5 F_{n-1} F_{n}+2$, and $6 F_{2 n}-2 F_{2 n+1}+2$.
Also solved by Richard André-Jeannin, Mohammad K. Azarian, Seung-Jin Bang, Brian D. Beasley, Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, Herta T. Freitag; Jane Friedman, Marquis Griffith, Ryan Jackson \& Mika Wheeler (jointly); Russell Jay Hendel, Christos. Kavuklis, Harris Kwong, Carl Libis, Dorka Ol. Popova, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, and Ralph Thomas.

## Divisibility by Fibonacci Squares

## B-715 Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

(Vol. 30, no. 2, May 1992)
Prove that, if $s>2$,

$$
F_{m} \equiv 0\left(\bmod F_{s}^{2}\right) \text { if and only if } m \equiv 0\left(\bmod s F_{s}\right) .
$$

Solution by Bob Prielipp, University of Wisconsin, Oshkosh, WI
Our solution will use the following known results (where $u$ is an integer larger than 2):
(1) $F_{u} \mid F_{v}$ if and only if $u \mid v . \quad$ (For a proof, see [1], p. 39.)
(2) $F_{u}^{2} \mid F_{u r}$ if and only if $F_{u} \mid r \quad$ (For a proof, see [2], p. 3.)

Let $s$ be an integer larger than 2.
If $F_{m} \equiv 0\left(\bmod F_{s}^{2}\right)$, then $F_{s}^{2} \mid F_{m}$. By result (1) we have $s \mid m$. Thus, $m=j s$ for some integer $j$. Hence, $F_{s}^{2} \mid F_{j s}$ so $F_{s} \mid j$ by result (2). Therefore, $j=k F_{s}$ for some integer $k$. Thus, $m=j s=$ $k s F_{s}$, making $m \equiv 0\left(\bmod s F_{s}\right)$.

Conversely, if $m \equiv 0\left(\bmod s F_{s}\right)$, then $m=k s F_{s}$ for some integer $k$. Since $F_{s} \mid k F_{s}$, by result (2) we have $F_{s}^{2} \mid F_{k s F_{s}}$, so $F_{s}^{2} \mid F_{m}$. Hence, $F_{m} \equiv 0\left(\bmod F_{s}^{2}\right)$.

Somer proved that, if $k \geq 2$ and $s>2$, then

$$
F_{m} \equiv 0\left(\bmod F_{s}^{k}\right) \text { if and only if } m \equiv 0\left(\bmod \frac{s}{d} F_{s}^{k-1}\right)
$$

where $d=2$ if both $k \geq 3$ and $s \equiv 3(\bmod 6)$ and $d=1$ otherwise.
Seiffert gave an analog for Lucas numbers if $s>1: L_{m} \equiv 0\left(\bmod L_{s}^{2}\right)$ if and only if $m \equiv 0$ $\left(\bmod s L_{s}\right)$ and $m / s$ is odd.

## References:

1. Verner E. Hoggatt, Jr. Fibonacci and Lucas Numbers. Santa Clara, CA: The Fibonacci Association, 1979.
2. V. E. Hoggatt, Jr., \& Marjorie Bicknell-Johnson. "Divisibility by Fibonacci and Lucas Squares." The Fibonacci Quarterly 15 (1977):3-8.
Also solved by Paul S. Bruckman, Leonard A. G. Dresel, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

## The Sum of Two Lucas Numbers

## B-716 Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA

(Vol. 30, no.2, May 1992)
If $a$ and $b$ have the same parity, prove that $L_{a}+L_{b}$ cannot be a prime larger than 5 .

## Solution by Russell Jay Hendel, Patchogue, NY

The problem tacitly assumes that $a, b \geq 0$ since, if we allow negative subscripts, then $a=5$ and $b=-3$ have the same parity, but $L_{5}+L_{-3}=11+(-4)=7$, a prime larger than 5. Accordingly, assume $a, b \geq 0$.

Without loss of generality, further assume that $a \geq b$. Let $n=(a+b) / 2$ and $m=(a-b) / 2$. Since $a$ and $b$ have the same parity, $m$ and $n$ are integers and $0 \leq m \leq n$.

We make use of the following well-known formulas (see [1], p. 177):

$$
\begin{align*}
& L_{n+m}+(-1)^{m} L_{n-m}=L_{m} L_{n},  \tag{1}\\
& L_{n+m}-(-1)^{m} L_{n-m}=5 F_{m} F_{n} . \tag{2}
\end{align*}
$$

If $m$ is even, then by result (1) we have $L_{a}+L_{b}=L_{n+m}+L_{n-m}=L_{m} L_{n}$ and this product is composite unless $n=1$ and $m=0$, in which case $L_{a}+L_{b}=2$, which is not larger than 5 .

If $m$ is odd, then by result (2) we have $L_{a}+L_{b}=L_{n+m}+L_{n-m}=5 F_{m} F_{n}$ and this product is composite unless $F_{m}=F_{n}=1$, in which case $L_{a}+L_{b}=5$, which is not larger than 5 .

## Reference:

1. S. Vajda. Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications. Chichester: Ellis Horwood Ltd., 1989.
Also solved by Glenn Bookhout, Paul S. Bruckman, Leonard A. G. Dresel, Herta T. Freitag, Harris Kwong, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Ralph Thomas, and the proposer. A partial solution was submitted by Charles Ashbacher.

## Expanding arctan as a Lucas Series

## B-717 Proposed by L. Kuipers, Sierre, Switzerland

(Vol. 30, no. 2, May 1992)
Show that

$$
\arctan \frac{2}{5}=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2 n+1} \cdot \frac{L_{2 n+1}}{2^{2 n+1}} .
$$

Composite solution by Bob Prielipp, University of Wisconsin, Oshkosh, WI and Graham Lord, Stanford, CA

We use the following well-known facts:
If $\sum a_{n}$ converges to $A$ and $\sum b_{n}$ converges to $B$, then $\sum\left(a_{n}+b_{n}\right)$ converges to $A+B$,

$$
\begin{gather*}
\arctan x=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2 k+1} x^{2 k+1}, \quad|x| \leq 1,  \tag{2}\\
\arctan x+\arctan y=\arctan \frac{x+y}{1-x y}, \quad x y<1 .
\end{gather*}
$$

[For (1), see [1], p. 376. For (2), see [2], p. 51. For (3), which is related to the familiar formula $\tan (x+y)=(\tan x+\tan y) /(1-\tan x \tan y)$, see [2], p. 49.]

We will also use the facts that $L_{n}=\alpha^{n}+\beta^{n}, \alpha+\beta=1, \alpha \beta=-1$ and note that $|\beta|<|\alpha|<2$.
Then, if $|z| \geq|\alpha|$,

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2 n+1} \cdot \frac{L_{2 n+1}}{z^{2 n+1}} & =\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2 n+1}\left(\frac{\alpha}{z}\right)^{2 n+1}+\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2 n+1}\left(\frac{\beta}{z}\right)^{2 n+1} \\
& =\arctan \left(\frac{\alpha}{z}\right)+\arctan \left(\frac{\beta}{z}\right)=\arctan \frac{(\alpha+\beta) / z}{1-\alpha \beta / z^{2}}=\arctan \frac{z}{z^{2}+1} .
\end{aligned}
$$

The original proposal is a special case of this result, with $z=2$.
Bruckman showed that

$$
\arctan \frac{2 x}{5}=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2 n+1} \cdot \frac{L_{2 n+1}(x)}{2^{2 n+1}},
$$

where $L_{m}(x)=\alpha(x)^{m}+\beta(x)^{m}, \alpha(x)=\left(x+\sqrt{x^{2}+4}\right) / 2$ and $\beta(x)=\left(x-\sqrt{x^{2}+4}\right) / 2$.

Seiffert showed that

$$
\frac{1}{\sqrt{5}} \arctan \frac{2 \sqrt{5}}{3}=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2 n+1} \cdot \frac{F_{2 n+1}}{2^{2 n+1}}
$$

and, if $p$ and $q$ are natural numbers of different parity with $q \geq p+2$, then

$$
\arctan \frac{L_{p} F_{q}}{F_{q-1} F_{q+1}}=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2 n+1} \cdot \frac{L_{p(2 n+1)}}{F_{q}^{2 n+1}} .
$$

Redmond showed that if $P_{n}=c_{0} \alpha^{n}+c_{1} \beta^{n}$. where $\alpha, \beta, c_{0}$ and $c_{1}$ are arbitrary real numbers, then

$$
\sum_{n=0}^{\infty} \frac{(-1)^{n}}{a n+b} \frac{P_{a n+b}}{x^{a n+b}}=c_{0} \int_{0}^{\alpha / x} \frac{t^{b-1}}{1+t^{a}} d t+c_{1} \int_{0}^{\beta / x} \frac{t^{b-1}}{1+t^{a}} d t
$$

for $|x|>\max (|\alpha|,|\beta|)$. He used this to obtain some interesting results, such as

$$
\sum_{n=0}^{\infty} \frac{(-1)^{n}}{3 n+1} \frac{L_{3 n+1}}{2^{3 n+1}}=\frac{1}{6} \log \frac{25}{19}-\frac{\sqrt{3}}{3} \arctan \frac{\sqrt{3}}{4}+\frac{\pi \sqrt{3}}{9} .
$$

## References:

1. R. Courant. Differential and Integral Calculus. Vol. I. London: Blackie \& Son, Ltd., 1937.
2. I. S. Gradshteyn \& I. M. Ryzhik. Tables of Integrals, Series and Products. San Diego, CA: Academic Press, Inc., 1980.

Also solved by Richard André-Jeannin, Seung-Jin Bang, Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, Russell Jay Hendel, Harris Kwong, Igor Ol. Popov, Don Redmond, H.-J. Seiffert, Ralph Thomas, and the proposer.

## Golden Power

## B-718 Proposed by Herta T. Freitag, Roanoke, VA

(Vol. 30, no. 3, August 1992)
Prove that $\left[\left(F_{n}+L_{n}\right) \alpha+\left(F_{n-1}+L_{n-1}\right)\right] / 2$ is a power of the golden ratio, $\alpha$.
Solution by John Ivie, Saratoga, CA
This follows from the two well-known identities:

$$
\begin{equation*}
F_{n}+L_{n}=2 F_{n+1} \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\alpha^{n}=F_{n} \alpha+F_{n-1}, \tag{2}
\end{equation*}
$$

which can easily be proved by means of the Binet formulas.
We thus have that

$$
\frac{\left(F_{n}+L_{n}\right) \alpha+\left(F_{n-1}+L_{n-1}\right)}{2}=\frac{2 F_{n+1} \alpha+2 F_{n}}{2}=F_{n+1} \alpha+F_{n}=\alpha^{n+1} .
$$

Also solved by Charles Ashbacher, Michel Ballieu, Seung-Jin Bang, Brian D. Beasley, Scott H. Brown, Paul S. Bruckman, Charles K. Cook, Russell Euler, Jane Friedman, Pentti Haukkanen, Hans Kappus, Joseph J. Kostal, Graham Lord, Dorka Ol. Popova, Bob Prielipp,

## H.-J. Seiffert, Tony Shannon, Sahib Singh, Lawrence Somer, Ralph Thomas, and the

 proposer.
## A Pell Factorization

## B-719 Proposed by Herta T. Freitag, Roanoke, VA

 (Vol. 30, no. 3, August 1992)Let $P_{n}$ be the $n^{\text {th }}$ Pell number (defined by $P_{0}=0, P_{1}=1$, and $P_{n+2}=2 P_{n+1}+P_{n}$ for $n \geq 0$ ). Let $a$ be an odd integer. Show how to factor $P_{n+a}^{2}+P_{n}^{2}$ into a product of Pell numbers.

How should this problem be modified if $a$ is even?

## Solution by Paul S. Bruckman, Edmonds, WA

We establish the following identity, valid for all $n$ and $a$ :

$$
P_{n+a}^{2}-(-1)^{a} P_{n}^{2}=P_{a} P_{2 n+a} .
$$

Proof: We employ the Binet formula: $P_{m}=\left(u^{m}-v^{m}\right) / \sqrt{8}$, where $u=1+\sqrt{2}$ and $v=1-\sqrt{2}$. Note that $u v=-1$. Then

$$
\begin{aligned}
P_{n+a}^{2}-(-1)^{a} P_{n}^{2} & =\frac{1}{8}\left[u^{2 n+2 a}-2(-1)^{n+a}+v^{2 n+2 a}-(-1)^{a}\left(u^{2 n}-2(-1)^{n}+v^{2 n}\right)\right] \\
& =\frac{1}{8}\left[u^{2 n+2 a}+v^{2 n+2 a}-(-1)^{a}\left(u^{2 n}+v^{2 n}\right)\right] \\
& =\frac{1}{8} u^{2 n+a}\left(u^{a}-v^{a}\right)+\frac{1}{8} v^{2 n+a}\left(v^{a}-u^{a}\right) \\
& =\frac{1}{8}\left(u^{a}-v^{a}\right)\left(u^{2 n+a}-v^{2 n+a}\right)=P_{a} P_{2 n+a} .
\end{aligned}
$$

Therefore,

$$
P_{a} P_{2 n+a}= \begin{cases}P_{n+a}^{2}+P_{n}^{2}, & \text { if } a \text { is odd; } \\ P_{n+a}^{2}-P_{n}^{2}, & \text { if } a \text { is even }\end{cases}
$$

Pla and Somer note that the result is valid not only for Pell numbers, but more generally for any sequence that satisfies the recurrence relation $u_{n+2}=k u_{n+1}+u_{n}$ with $u_{0}=0$ and $u_{1}=1$.

Popova shows, by induction, the more general result

$$
\sum_{k=0}^{2 m-1}(-1)^{(a-1)(k-1)} P_{n+k a}^{2}=P_{a} P_{2 m a} P_{2 n+(2 m-1) a} / P_{2 a},
$$

where $a$ and $m$ are arbitrary positive integers.
Also solved by Charles Ashbacher, M. A. Ballieu, Russell Euler, Hans Kappus, Juan Pla, Dorka Ol. Popova, Bob Prielipp, H.-J. Seiffert, Tony Shannon, Lawrence Somer, and the proposer.

Errata: The name of the second proposer of Problem B-738 (Vol. 31, no. 2, 1993) should be Cecil O. Alford.
Brian D. Beasley was inadvertently omitted as a solver for Problems B-712 and B-713.

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by<br>Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

## PROBLEMS PROPOSED IN THIS ISSUE

## H-478 Proposed by Gino Taddei, Rome, Italy

Consider a string constituted by $h$ labelled cells $c_{1}, c_{2}, \ldots, c_{h}$. Fill these cells with the natural numbers $1,2, \ldots, h$ according to the following rule: 1 in $c_{1}, 2$ in $c_{2}, 3$ in $c_{4}, 4$ in $c_{7}, 5$ in $c_{11}$, and so on. Obviously, whenever the subscript $j$ of $c_{j}$ exceeds $h$, it must be considered as reduced modulo $h$. In other words, the integer $n(1 \leq n \leq h)$ enters the cell $c_{j(n, h)}$, where

$$
j(n, h)=\left\langle\frac{n^{2}-n+2}{2}\right\rangle_{h}
$$

and the symbol $\langle a\rangle_{b}$ denotes $a$ if $a \leq b$, and the remainder of $a$ divided by $b$ if $a>b$.
Determine the set of all values of $h$ for which, at the end of the procedure, each cell has been entered by exactly one number.

## H-479 Proposed by Richard André-Jeannin, Longwy, France

Let $\left\{V_{n}\right\}$ be the sequence defined by

$$
V_{0}=2, V_{1}=P, \text { and } V_{n}=P V_{n-1}-Q V_{n-2} \text { for } n \geq 2
$$

where $P$ and $Q$ are real or complex parameters. Find a closed form for the sum

$$
\sum_{k=1}^{n}\binom{2 n-k-1}{n-1} P^{k} Q^{n-k} V_{k}
$$

## H-480 Proposed by Paul S. Bruckman, Edmonds, WA

Let $p$ denote a prime $\equiv 1(\bmod 10)$.
(a) Prove that, for all $p \not \equiv 1(\bmod 1260)$, there exist positive integers $k, u$, and $v$ such that
(i) $k \mid u^{2}$;
(ii) $p+5 k=(5 u-1)(5 v-1)$.
(b) Prove or disprove the conjecture that the restriction $p \not \equiv 1(\bmod 1260)$ in part (1) may be removed, i.e., part $(\mathrm{a})$ is true for all $p \equiv 1(\bmod 10)$.

## SOLUTIONS

## Bunches of Recurrences

## H-461 Proposed by Lawrence Somer, Washington, D.C.

(Vol. 29, no. 4, November 1991)
Let $\left\{u_{n}\right\}=u(a, b)$ denote the Lucas sequence of the first kind satisfying the recursion relation $u_{n+2}=a u_{n+1}+b u_{n}$, where $a$ and $b$ are nonzero integers and the initial terms are $u_{0}=0$ and $u_{1}=1$. The prime $p$ is a primitive divisor of $u_{n}$ if $p \mid u_{n}$ but $p \nmid u_{m}$ for $1 \leq m \leq n-1$. It is known (see [1], p . 200) for the Fibonacci sequence $\left\{F_{n}\right\}=u(1,1)$ that, if $p$ is an odd prime divisor of $F_{2 n+1}$, where $n \geq 1$, then $p \equiv 1(\bmod 4)$.
(i) Find an infinite number of recurrences $u(a, b)$ such that every odd primitive prime divisor $p$ of any term of the form $u_{2 n+1}$ or $u_{4 n}$ satisfies $p \equiv 1(\bmod 4)$, where $n \geq 1$.
(ii) Find an infinite number of recurrences $u(a, b)$ such that every odd primitive prime divisor $p$ of any term of the form $u_{4 n}$ or $u_{4 n+2}$ satisfies $p \equiv 1(\bmod 4)$, where $n \geq 1$.

## Reference

1. E. Lucas. "Theorie des fonctions numeriques simplement périodiques." Amer. J. Math. 1 (1878):184-240, 289-321.

## Solution by Paul S. Bruckman, Edmonds, WA

We write $P \in P D\left(u_{n}\right)$ if $p$ is an odd primitive prime divisor of $u_{n}$. The following well-known result is stated in the form of a lemma.

Lemma: Suppose $m=x^{2}+y^{2}$, where $x, y \in Z^{+}$. If $p$ is any odd prime divisor of $m$, such that $p \nmid \operatorname{gcd}(x, y)$, then $p \equiv 1(\bmod 4)$.

Next, we indicate some easily-derived results for a (generalized) Lucas sequence of the first kind:

$$
\begin{equation*}
u_{n}=\frac{r^{n}-s^{n}}{r-s}, n=0,1,2, \ldots \tag{1}
\end{equation*}
$$

where

$$
\begin{equation*}
r=\frac{1}{2}(a+\theta), s=\frac{1}{2}(a-\theta), \theta=\left(a^{2}+4 b\right)^{\frac{1}{2}} . \tag{2}
\end{equation*}
$$

Note that

$$
\begin{equation*}
r+s=a, r-s=\theta, r s=-b \tag{3}
\end{equation*}
$$

Also, define the (generalized) Lucas sequence of the second kind as follows:

$$
\begin{equation*}
v_{n}=r^{n}+s^{n}, n=0,1,2, \ldots \tag{4}
\end{equation*}
$$

As we may readily verify:

$$
\begin{gather*}
u_{2 n}=u_{n} v_{n}  \tag{5}\\
u_{2 n+1}=b u_{n}^{2}+u_{n+1}^{2} . \tag{6}
\end{gather*}
$$

Also, it is clear that the $u_{n}$ 's and $v_{n}$ 's are integers for all $n$.
We will establish the following result, solving part (i) of the problem:

If $a=i^{2}-j^{2}, b=1^{2} j^{2}$, where $i, j \in Z^{+}, \operatorname{gcd}(i, j)=1$, then $p \equiv 1(\bmod 4)$ for all prime $p$ such that $p \in P D\left(u_{2 n+1}\right)$ or $p \in P D\left(u_{4 n}\right), n \geq 1$.

Proof of (*): We note that $\theta^{2}=a^{2}+4 b=\left(i^{2}-j^{2}\right)^{2}+4 i^{2} j^{2}=\left(i^{2}+j^{2}\right)^{2}$, so $\theta=i^{2}+j^{2}$. Also, $r=i^{2}, s=-j^{2}$. We see from (6) that $u_{2 n+1}=X^{2}+Y^{2}$, where $X=i j u_{n}, Y=u_{n+1}$. Also, from (4), $v_{2 n}=X_{1}^{2}+Y_{1}^{2}$, where $X_{i}=i^{2 n}, Y_{1}=j^{2 n}$. If $p \in P D\left(u_{2 n+1}\right), n \geq 1$, then $p \mid u_{2 n+1}, p \nmid u_{n}$, $p \nmid u_{n+1}$. We cannot have $p \mid i j$, for otherwise, $p|X \Rightarrow p| Y=u_{n+1}$, a contradiction. Therefore, $p \nmid X, p \nmid Y$. Then, by the lemma, $p \equiv 1(\bmod 4)$.

If $p \in P D\left(u_{4 n}\right), n \geq 1$, then $p \mid u_{4 n}, p \nmid u_{2 n}$. Note that $u_{4 n}=u_{2 n} v_{2 n}$ by (5). Thus, $p \mid v_{2 n}=$ $X_{1}^{2}+Y_{1}^{2}$. Since $\operatorname{gcd}(i, j)=1$, also $\operatorname{gcd}\left(X_{1}, Y_{1}\right)=1$. By the Lemma, $p \equiv 1(\bmod 4)$. This completes the proof of (*).

Also, we shall prove the following result, which solves part (ii):

$$
\begin{align*}
& \text { If } a=i^{2}+j^{2}, b=-i^{2} j^{2} \text {, where } i, j \in Z^{+}, \operatorname{gcd}(i, j)=1, i>j \text {, then }  \tag{**}\\
& p \equiv 1(\bmod 4) \text { for all prime } p \text { such that } p \in P D\left(u_{4 n}\right) \text { or } p \in P D\left(u_{4 n+2}\right), n \geq 1 .
\end{align*}
$$

Proof of (**): We note that $\theta^{2}=a^{2}+4 b=\left(i^{2}+j^{2}\right)^{2}-4 i^{2} j^{2}=\left(i^{2}-j^{2}\right)^{2}$, so $\theta=i^{2}-j^{2}$. Also, $r=i^{2}, s=j^{2}$, and so $v_{n}=X_{2}^{2}+Y_{2}^{2}$, where $X_{2}=i^{n}, Y_{2}=j^{n}$. If $p \in P D\left(u_{2 n}\right), n \geq 1$, then $p \mid u_{2 n}, p \nmid u_{n}$.. Using (5), $p \mid v_{n}=X_{2}^{2}+Y_{2}^{2}$. Since $\operatorname{gcd}(i, j)=1$ also $\operatorname{gcd}\left(X_{2}, Y_{2}\right)=1$. By the Lemma, $p \equiv 1(\bmod 4)$. Since $2 n=4 n^{\prime}$ or $4 n^{\prime}+2$, we see that $(* *)$ is proven.

In summary, we that $i$ and $j$ in (*) and (**) are arbitrary natural numbers, subject only to the condition that $\operatorname{gcd}(i, j)=1$ [and $i>j$ in $(* *)]$. Hence, there are infinitely many sequences $u(a, b)$, with $a$ and $b$ as given in (*) and $(* *)$, that provide solutions to the two parts of the problem.

Also solved by the proposer.

## Root of the Problem

## H-462 Proposed by Ioan Sadoveaanuv, Ellensburg, WA (Vol. 30, no. 1, February 1992)

Let $G(x)=x^{k}+a_{1} x^{k-1}+\cdots+a_{k}$ be a polynomial with $c$ a root of order $p$. If $G^{(p)}(x)$ denotes the $p^{\text {th }}$ derivative of $G(x)$, show that $\left\{n^{p} c^{n-p} / G^{(p)}(c)\right\}$ is a solution of the recurrence $u_{n}=c^{n-k}-a_{1} u_{n-1}-a_{2} u_{n-2}-\cdots-a_{k} u_{n-k}$.

## Solution by C. Georghiou, University of Patras, Patras, Greece

We will use the operator method of Difference Calculus (see, e.g., Marray R. Spiegel, Calculus of Finite Differences and Difference Equations [New York: McGraw-Hill, 1971], p. 156). Let $G(x)=(x-c)^{p} g(x)$. Then $g(c)=G^{(p)}(c) / p!(\neq 0)$. The given recurrence is written as $G(E) u_{n}=c^{n}$, where $E$ is the shift operator, i.e., $E u_{n}=u_{n+1}$. Therefore, the solution is

$$
u_{n}=\frac{1}{G(E)} c^{n}=\frac{1}{(E-c)^{p} g(E)} c^{n}=\frac{1}{(E-c)^{p}} \frac{c^{n}}{g(c)}=\frac{p!}{G^{(p)}(c)} c^{n} \frac{1}{(c E-c)^{p}} 1=\frac{p!c^{n-p}}{G^{(p)}(c)} \frac{1}{\Delta^{p}} 1 .
$$

Now, from the Summation Calculus, we have

$$
\begin{equation*}
\Delta^{-p} 1=\frac{n^{(p)}}{p!}+\sum_{k=1}^{p} A_{k} \frac{n^{(p-k)}}{(p-k)!} \tag{1}
\end{equation*}
$$

where, as usual, $n^{(k)}=n(n-1) \ldots(n-k+1)$ is the factorial function, and $A_{1}, A_{2}, \ldots, A_{k}$ are arbitrary constants. But it is known that

$$
\begin{equation*}
n^{p}=n^{(p)}+\sum_{k=0}^{p-1} S_{p}^{(k)} n^{(k)} \tag{2}
\end{equation*}
$$

where $S_{p}^{(k)}$ are the Stirling Numbers of the Second Kind. If we choose $A_{p-k}=k!S_{p}^{(k)} / p$ ! then (1), in view of (2), becomes $\Delta^{-p} 1=n^{p} / p!$ and the assertion follows readily.

Also solved by P. Bruckman and F. Flanigan.

## Fee Fi Fo Fum

## H-463 Proposed by Paul S. Bruckman, Edmonds, WA

(Vol. 30, no. 1, February 1992)
Establish the identity: $\quad \sum_{n=1}^{\infty} \Phi(n)\left(\frac{z^{n}}{1-z^{2 n}}\right)=\frac{z\left(1+z+z^{2}\right)}{\left(1-z^{2}\right)^{2}}$,
where $z \in C,|z|<1$, and $\Phi$ is the Euler totient function. As special cases of (1), obtain the following identities:

$$
\begin{gather*}
\sum_{n=1}^{\infty} \Phi(2 n) / F_{2 n s}=\sqrt{5} / L_{s}^{2}, s=1,3,5, \ldots ;  \tag{2}\\
\sum_{n=1}^{\infty} \Phi(2 n-1) / L_{(2 n-1) s}=F_{s} \sqrt{5} / L_{s}^{2}, s=1,3,5, \ldots ;  \tag{3}\\
\sum_{n=1}^{\infty} \Phi(n) / F_{n s}=\left(L_{s}+1\right) / F_{s}^{2} \sqrt{5}, s=2,4,6, \ldots ;  \tag{4}\\
\sum_{n=1}^{\infty}(-1)^{n-1} \Phi(n) / F_{n s}=\left(L_{s}-1\right) / F_{s}^{2} \sqrt{5}, s=2,4,6, \ldots ;  \tag{5}\\
\sum_{n=1}^{\infty}(-1)^{n-1} \Phi(2 n) / F_{2 n s}= \begin{cases}1 / F_{s}^{2} \sqrt{5}, & s=1,3,5, \ldots ; \\
\sqrt{5} / L_{s}^{2}, & s=2,4,6, \ldots ;\end{cases}  \tag{6}\\
\sum_{n=1}^{\infty}(-1)^{n-1} \Phi(2 n-1) / F_{(2 n-1) s}=L_{s} / F_{s}^{2} \sqrt{5}, s=1,3,5, \ldots ;  \tag{7}\\
\sum_{n=1}^{\infty}(-1)^{n-1} \Phi(2 n-1) / L_{(2 n-1) s}=F_{s} \sqrt{5} / L_{s}^{2}, s=2,4,6, \ldots \tag{8}
\end{gather*}
$$

## Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

For $|z|<1$,

$$
\sum_{n=1}^{\infty} \Phi(n) \frac{z^{n}}{1-z^{2 n}}=\sum_{n=1}^{\infty} \sum_{q \text { odd }} \Phi(n) z^{q n}
$$

For odd $t$ and $s \geq 0$, the coefficient of $z^{k}$, where $k=2^{s} t$, is

$$
\sum_{d \mid t} \Phi\left(2^{s} d\right)=\Phi\left(2^{s}\right) \sum_{d \mid t} \Phi(d)=\Phi\left(2^{s}\right) \cdot t= \begin{cases}2^{s-1} t & \text { if } s>0 \\ t & \text { if } s=0\end{cases}
$$

Therefore,

$$
\begin{equation*}
\sum_{n \text { odd }} \Phi(n) \frac{z^{n}}{1-z^{2 n}}=\sum_{n=1}^{\infty}(2 n+1) z^{2 n+1}=\frac{z\left(1+z^{2}\right)}{\left(1-z^{2}\right)^{2}} \tag{*}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{n \text { even }} \Phi(n) \frac{z^{n}}{1-z^{2 n}}=\sum_{n=1}^{\infty} n z^{2 n}=\frac{z^{2}}{\left(1-z^{2}\right)^{2}}, \tag{**}
\end{equation*}
$$

which prove (1). Letting $\alpha=(1+\sqrt{5}) / 2$ and $\beta=(1-\sqrt{5}) / 2$, we have $\alpha \beta=-1$ and the identities

$$
\begin{gather*}
\frac{1}{\sqrt{5}} \frac{1}{F_{n s}}=\frac{1}{\alpha^{n s}-\beta^{n s}}=\frac{\beta^{n s}}{(-1)^{n s}-\beta^{2 n s} .}  \tag{A}\\
\frac{1}{L_{n s}}=\frac{1}{\alpha^{n s}+\beta^{n s}}=\frac{\beta^{n s}}{(-1)^{n s}+\beta^{2 n s} .}  \tag{B}\\
\frac{\beta^{2 s}}{\left(1-\beta^{2 s}\right)^{2}}=\frac{(\alpha \beta)^{2 s}}{\left[\alpha^{s}-(\alpha \beta)^{s} \beta^{s}\right]^{2}}= \begin{cases}1 / L_{s}^{2} & \text { if } s \text { is odd } \\
1 / 5 F_{s}^{2} & \text { if } s \text { is even. }\end{cases}  \tag{C}\\
\frac{\beta^{s}\left(1+\beta^{2 s}\right)}{\left(1-\beta^{2 s}\right)^{2}}=\frac{(\alpha \beta)^{s}\left[\alpha^{s}+(\alpha \beta)^{s} \beta^{s}\right]}{\left[\alpha^{s}-(\alpha \beta)^{s} \beta^{s}\right]^{2}}= \begin{cases}-F_{s} \sqrt{5} / L_{s}^{2} & \text { if } s \text { is odd, } \\
L_{s} / 5 F_{s}^{2} & \text { if } s \text { is even. }\end{cases}  \tag{D}\\
\frac{\beta^{2 s}}{\left(1+\beta^{2 s}\right)^{2}}=\frac{(\alpha \beta)^{2 s}}{\left[\alpha^{s}+(\alpha \beta)^{s} \beta^{s}\right]^{2}}= \begin{cases}1 / 5 F_{s}^{2} & \text { if } s \text { is odd, } \\
1 / L_{s}^{2} & \text { if } s \text { is even. }\end{cases}  \tag{E}\\
\frac{\beta^{s}\left(1-\beta^{2 s}\right)}{\left(1+\beta^{2 s}\right)^{2}}=\frac{(\alpha \beta)^{s}\left[\alpha^{s}-(\alpha \beta)^{s} \beta^{s}\right]}{\left[\alpha^{s}+(\alpha \beta)^{s} \beta^{s}\right]^{2}}= \begin{cases}-L_{s} / 5 F_{s}^{2} & \text { if } s \text { is odd }, \\
F_{s} \sqrt{5} / L_{s}^{2} & \text { if } s \text { is even. } .\end{cases} \tag{F}
\end{gather*}
$$

To prove (2)-(8), proceed as follows:
(2) For odd $s$, it follows from (A), (**), and (C) that

$$
\frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{\Phi(2 n)}{F_{2 n s}}=\sum_{n \text { even }} \Phi(n) \frac{\beta^{n s}}{1-\beta^{2 n s}}=\frac{\beta^{2 s}}{\left(1-\beta^{2 s}\right)^{2}}=\frac{1}{L_{s}^{2}} .
$$

(3) For even $s$, it follows from (B), (*), and (D) that

$$
\sum_{n=1}^{\infty} \frac{\Phi(2 n-1)}{L_{(2 n-1) s}}=-\sum_{n \text { odd }} \Phi(n) \frac{\beta^{n s}}{1-\beta^{2 n s}}=-\frac{\beta^{s}\left(1+\beta^{s}\right)}{\left(1-\beta^{2 s}\right)^{2}}=\frac{F_{s} \sqrt{5}}{L_{s}^{2}} .
$$

(4) For even $s$, it follows from (A), (1), (C), and (D) that

$$
\frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{\Phi(n)}{F_{n s}}=\sum_{n=1}^{\infty} \Phi(n) \frac{\beta^{n s}}{1-\beta^{2 n s}}=\frac{\beta^{s}\left(1+\beta^{2 s}\right)+\beta^{2 s}}{\left(1-\beta^{2 s}\right)^{2}}=\frac{L_{s}+1}{5 F_{s}^{2}}
$$

(5) For even $s$, it follows from (A), (1), (C), and (D) that

$$
\frac{1}{\sqrt{5}} \sum_{n=1}^{\infty}(-1)^{n-1} \frac{\Phi(n)}{F_{n s}}=-\sum_{n=1}^{\infty} \Phi(n) \frac{\left(-\beta^{s}\right)^{n}}{1-\left(-\beta^{s}\right)^{2 n}}=\frac{\beta^{s}\left(1+\beta^{2 s}\right)-\beta^{2 s}}{\left(1-\beta^{2 s}\right)^{2}}=\frac{L_{s}-1}{5 F_{s}^{2}}
$$

(6) It follows from (A), $(* *)$, and (E) that

$$
\frac{1}{\sqrt{5}} \sum_{n=1}^{\infty}(-1)^{n-1} \frac{\Phi(2 n)}{F_{2 n s}}=-\sum_{n \text { even }} \Phi(n) \frac{\left(i \beta^{s}\right)^{n}}{1-\left(i \beta^{s}\right)^{2 n}}=\frac{\beta^{2 s}}{\left(1+\beta^{2 s}\right)^{2}}= \begin{cases}1 / 5 F_{s}^{2} & \text { if } s \text { is odd } \\ 1 / L_{s}^{2} & \text { if } s \text { is even }\end{cases}
$$

(7) For odd $s$, it follows from (A), (*), and (F) that

$$
\frac{1}{\sqrt{5}} \sum_{n=1}^{\infty}(-1)^{n-1} \frac{\Phi(2 n-1)}{F_{(2 n-1) s}}=-\frac{1}{i} \sum_{n \text { odd }} \Phi(n) \frac{\left(i \beta^{s}\right)^{n}}{1-\left(i \beta^{s}\right)^{2 n}}=-\frac{\beta^{s}\left(1-\beta^{2 s}\right)}{\left(1+\beta^{2 s}\right)^{2}}=\frac{L_{s}}{5 F_{s}^{2}}
$$

(8) For even $s$, it follows from (A), (*), and (F) that

$$
\sum_{n=1}^{\infty}(-1)^{n-1} \frac{\Phi(2 n-1)}{L_{(2 n-1) s}}=\frac{1}{i} \sum_{n \text { odd }} \Phi(n) \frac{\left(i \beta^{s}\right)^{n}}{1-\left(i \beta^{s}\right)^{2 n}}=\frac{\beta^{s}\left(1-\beta^{2 s}\right)}{\left(1+\beta^{2 s}\right)^{2}}=\frac{F_{s} \sqrt{5}}{L_{s}^{2}}
$$

Also solved by C. Georghiou, P. Haukkanen, R. Hendel, and the proposer.

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VOLUME 4

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## BOOKS AVAILABLE through the fibonacci association

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.
A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.
Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.
Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence-18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Applications of Fibonacci Numbers, Volumes 1-4. Edited by G.E. Bergum, A.F. Horadam and A.N. Philippou

Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger, FA, 1993.

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[^3]:    Please write to the Fibonacci Association, Santa Clara University, Santa Clara CA 95053, U.S.A., for more information and current prices.

