The Fibonacci Quarterly

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

RECURSIONS AND PASCAL-TYPE TRIANGLES

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INTRODUCTION

Triangular arrays of numbers similar to or derived from Pascal's triangle frequently appear in the mathematical literature. (See, for example, [3], [5], and [8].) The purpose of this paper is to study a generalization of the array in [8]. In section 1, recursion formulas for the row and diagonal row sums are derived. In section 2, the determinants of a set of matrices associated with the triangular array of [8] are calculated.

1. GENERAL PROPERTIES OF THE ARRAYS

Consider a family of triangular arrays of numbers, indexed by the reals. For each $a \in \mathbf{R}$, the array is a doubly infinite set of numbers $d(a; n, k)$; $n, k \in \mathbf{Z}$, such that:

a. $d(a; n, k) = 0, \;\; n < 0;$
b. $d(a; n, k) = 0, \;\; k < 0$ or $k > n;$
c. $d(a; 0, 0) = a,$
d. $d(a; 1, 0) = d(a; 1, 1) = 1;$ and
e. $d(a; n, k) = d(a; n-2, k-1) + d(a; n-1, k-1) + d(a; n-1, k), \;\; n \geq 2.$

The triangular array studied by Wong & Maddocks [8] corresponds to the case $a = 1$. Their general term $M_{k,r}$ corresponds to the term $d(1; k+r, r)$ here. Tables 1, 2, and 3 contain the initial rows for the arrays $d(1; n, k)$, $d(0; n, k)$, and the general array $d(a; n, k)$, respectively. As mentioned above, Table 1 appears in [8]. It also appears in [1].

TABLE 1. $d(1; n, k)$

				1				
			1		1			
		1		3		3		
	1		5		5		1	
1		7		13		7		1

TABLE 2. $d(0; n, k)$

				0				
			1		1			
		1		2		1		
	1		4		4		1	
1		6		10		6		1

TABLE 3. $d(a; n, k)$

					a					
				1		1				
			1		$2+a$		1			
		1		$4+a$		$4+a$		1		
	1		$6+a$		$10+3a$		$6+a$		1	
1		$8+a$		$20+5a$		$20+5a$		$8+a$		1

An examination of these arrays reveals that, for $n \geq 2$,

$$d(a; n, k) = d(0; n, k) + a[d(1; n-2, k-2)].$$

Thus, calculations for any array $d(a; n, k)$ reduce to calculations on $d(0; n, k)$ and $d(1; n, k)$.

Definition 1: For fixed n, we call the sums

(1) $D(a; n) = \sum_{k=0}^{n} d(a; n, k)$; and

(2) $D^*(a; n) = \sum_{k=0}^{n} (-1)^k d(a; n, k)$

the *row sums* and the *alternating row sums*, respectively, of the array $d(a; n, k)$.

It is immediate that, for $n \geq 2$,

a. $D(a; n) = D(0; n) + a[D(1; n-2)]$; and

b. $D^*(a; n) = D^*(0; n) + (-a)[D^*(1; n-2)]$.

Theorem 1: The sequences $\{D(1; n)\}$ and $\{D(0; n)\}$ satisfy:

(a) $D(1; 0) = 1$; $D(1; 1) = 2$; and, for $n \geq 2$, $D(1; n) = 2D(1; n-1) + D(1; n-2)$;

(b) $D^*(1; n) = \begin{cases} 0, & n \text{ odd}, n > 0, \\ (-1)^m, & n = 2m, m \geq 0; \end{cases}$

(c) $D(0; 0) = 0$; $D(0; 1) = 2$; and, for $n \geq 1$, $D(0; n) = 2D(0; n-1) + D(0; n-2)$; and

(d) For $n \geq 0$, $D^*(0; n) = 0$.

Proof of (a): The proof is by induction. Obviously,

$$D(1; 0) = 1;\ D(1; 1) = 2; \text{ and } D(1; 2) = 2D(1; 1) + D(1; 0).$$

Assume the proposition is true for $2 \leq n < m$. For $n = m$,

$$\begin{aligned} D(1; m) &= \sum_{k=0}^{m} d(1; m, k) = \sum_{k=0}^{m} \{d(1; m-2, k-1) + d(1; m-1, k-1) + d(1; m-1, k)\} \\ &= \sum_{k=0}^{m} d(1; m-2, k-1) + \sum_{k=0}^{m} \{d(1; m-1, k-1) + d(1; m-1, k)\}. \end{aligned}$$

The first summation is $D(1; m-2)$. The second summation is

$$\begin{aligned} &\{d(1; m-1, -1) + d(1; m-1, 0)\} + \{d(1; m-1, 0) + d(1; m-1, 1)\} \\ &\quad + \{d(1; m-1, 1) + d(1; m-1, 2)\} + \cdots + \{d(1; m-1, m-2) \\ &\quad + d(1; m-1, m-1)\} + \{d(1; m-1, m-1) + d(1; m-1, m)\}. \end{aligned}$$

Recall that $d(1; m-1, -1) = d(1; m-1, m) = 0$. Regrouping, the summation becomes:

$$\begin{aligned} &2d(1; m-1, 0) + 2d(1; m-1, 1) + \cdots + 2d(1; m-1, m-2) \\ &\quad + 2d(1; m-1, m-1) = 2D(1; m-1). \end{aligned}$$

Thus, $D(1; m) = 2D(1; m-1) + D(1; m-2)$.

The proofs of (b), (c), and (d) are similar. □

The recursions (a) and (c) identify the sequences $\{D(1;n)\}$ and $\{D(0;n)\}$ as Pell sequences [2]. The initial terms of the $D(1;n)$ sequences are: 1, 2, 5, 12, 29, 70, 169, ... This sequence is number 552 in Sloane [6]. The $D(0;n)$ sequence starts: 0, 2, 4, 10, 24, 58, ... The terms are all even. Dividing by 2 yields: 0, 1, 2, 5, 12, 29, 70, 169, ..., which is again Sloane's sequence 552.

Given Definition 1 and Theorem 1, a simple calculation yields

Corollary 1: The sequences $\{D(a;n)\}$ and $\{D^*(a;n)\}$ satisfy:

(a) $D(a;0)=a;\ D(a;1)=2;\ D(a;n)=2D(a;n-1)+D(a;n-2),\ n\geq 2.$

(b) $D^*(a;n)=\begin{cases}0, & n \text{ odd},\\ a(-1)^m, & n=2m.\end{cases}$

Definition 2: Sums of the form

(1) $\partial(a;n)=d(a;n,0)+d(a;n-1,1)+d(a;n-2,2)+\cdots$, and

(2) $\partial^*(a;n)=d(a;n,0)-d(a;n-1,1)+d(a;n-2,2)-d(a;n-3,3)+\cdots$, will be called *diagonal sums* and *alternating diagonal sums*, respectively, for the array $d(a;n,k)$.

Theorem 2: The diagonal sums $\partial(1;n)$ and $\partial(0;n)$ satisfy:

(a) $\partial(1;0)=\partial(1;1)=1;\ \partial(1;2)=2;$
and $\partial(1;n)=\partial(1;n-1)+\partial(1;n-2)+\partial(1;n-3);\ n\geq 3;$

(b) $\partial(0;0)=0;\ \partial(0;1)=1;\ \partial(0;2)=2;$
and $\partial(0;n)=\partial(0;n-1)+\partial(0;n-2)+\partial(0;n-3);\ n\geq 3.$

Proof: ***(a)*** Proved in [1] and [8]; ***(b)*** Direct calculation. □

The initial terms of the $\partial(1;n)$ sequence are: 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, ... This is Sloane's sequence 406 [6]. This sequence appeared in [1], [4], and [7], where it is called the *Tribonacci sequence*. The terms of $\partial(0;n)$ are: 0, 1, 2, 3, 6, 11, 20, 37, ...; Sloane's sequence 296. Both sequences have a three-term recursion; i.e., for both sequences, the recursion is of the form $s(n)=s(n-1)+s(n-2)+s(n-3), n\geq 3$. The difference between the two sequences reqults from different initial terms. Sequences with a three-term recurrence have been studied previously, e.g., [4], [7]. The recursion relations for both $\partial(0;n)$ and $\partial(1;n)$ can be written in matrix form [7].

Theorem 3: The alternating diagonal sums $\partial^*(1;n)$ and $\partial^*(0;n)$ satisfy the relations:

(a) $\partial^*(1;0)=\partial^*(1;1)=1;\ \partial^*(1;2)=0;$ and
$\partial^*(1;n)=\partial^*(1;n-1)-\partial^*(1;n-2)-\partial^*(1;n-3),\ n\geq 3.$

(b) $\partial^*(0;0)=0;\ \partial^*(0;1)=1;\ \partial^*(0;2)=0;$ and
$\partial^*(0;n)=\partial^*(0;n-1)-\partial^*(0;n-2)-\partial^*(0;n-3),\ n\geq 3.$

Corollary 2: The diagonal sums $\partial(a; n, k)$ satisfy

(a) $\partial(a; 0) = a;\ \partial(a; 1) = 1;\ \partial(a; 2) = 2;$

(b) $\partial(a; n) = \partial(a; n-1) + \partial(a; n-2) + \partial(a; n-3);\ n \geq 3.$

The alternating diagonal sums $\partial^*(a; n)$ satisfy

(c) $\partial^*(a; 0) = a;\ \partial^*(a; 1) = 1;\ \partial^*(a; 2) = 0;$

(d) $\partial^*(a; n) = \partial^*(a; n-1) - \partial^*(a; n-2) - \partial^*(a; n-3);\ n \geq 3.$

2. THE ASSOCIATED MATRICES

Rotate the array $d(1; n, k)$ counterclockwise so that the diagonals become rows and columns to produce the following infinite matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 3 & 5 & 7 & 9 & \cdots \\ 1 & 5 & 13 & 25 & 41 & \cdots \\ 1 & 7 & 25 & 63 & 129 & \cdots \\ 1 & 9 & 41 & 129 & 321 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

The recursion relations for the triangle translate to the following relations for the terms $m_{i,j}$ of the matrix:

a. $m_{1,j} = m_{i,1} = 1$, for all i, j; and

b. $m_{i,j} = m_{i,j-1} + m_{i-1,j-1} + m_{i-1,j}, i > 1, j > 1.$

Let M_n be the $(n \times n)$-submatrix whose rows and columns are the first n rows and n columns of **M**, and $|M_n|$ the corresponding determinant.

Theorem 4: For $n \geq 1$, $|M_n| = 2^{n(n-1)/2}$.

Proof: By induction. For $n = 1$, the result is immediate.

For $k > 1$, the matrix can be changed by elementary row and column operations so that, in block form,

$$M_k = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 2M_{k-1} \end{array}\right]$$

The rest follows. □

ACKNOWLEDGMENT

The author wishes to acknowledge the referees comments which substantially improved the paper and provided references the author was unfamiliar with.

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AMS numbers: 11B83; 11B39

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May 10, 1993

Dear Editor:

May I inform you that I have just read with interest the paper "On Extended Generalized Stirling Pairs" by A. G. Kyriakoussis, which appeared in *The Fibonacci Quarterly* **31.1** (1993):44-52. I wish to mention that Kyriakoussis' "EGSP" ("extended generalized Stirling pair") is actually a particular case included in the second class of extended "GSN" pairs considered in my paper "Theory and Application of Generalized Stirling Number Pairs," *J. Math. Res. and Exposition* **9** (1989):211-20. His first characterization theorem for "EGSP" is a special case of my Theorem 6 (*loc. cit.*). In fact, a basic result corresponding with his case appeared much earlier in the paper by J. L. Fields & M. E. H. Ismail, entitled "Polynomial Expansions," *Math. Comp.* **29** (1975):894-902.

Thank you for your attention.

Yours sincerely,

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REDUCED AND AUGMENTED AMICABLE PAIRS TO 10^8

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1. PRELIMINARIES

A *reduced amicable pair* is a pair of natural numbers, m and n, such that

$$m = \sigma(n) - n - 1; \quad n = \sigma(m) - m - 1,$$

where σ is the sum of divisors function. Jerrard and Temperley [4] studied numbers k satisfying $k = \sigma(k) - k \pm 1$ which they named *almost perfect numbers.* Lal and Forbes [5] first studied reduced amicable pairs and discovered nine pairs with smaller number $\leq 10^5$. (They coined the name "reduced amicable pair.") In an earlier paper [1], we extended the search to pairs with smaller number $\leq 10^6$, finding six new pairs. Hagis and Lord [3] extended the list to 10^7, discovering thirty-one new pairs, including two missed in [1]. The present paper extends the listing to 10^8. The paper [1] included a study of pairs m and n satisfying

$$m = \sigma(n) - n + 1; \quad n = \sigma(m) - m + 1,$$

called *augmented amicable pairs* and listed all pairs with smaller number less than 10^6. There were nine plus two other pairs both of whose elements exceeded one million. These arose from iterating the function $s_+(n) = \sigma(n) - n + 1$ on integers less than one million. A computer search extended the list to all pairs with smaller number less than one hundred million. Table 2 lists the pairs with one element less than ten million, except for powers of 2. Powers of 2 are fixed points of s_+ and are not included here. A complete list of the 84 pairs up to 10^8 is available from either author. The searches were carried out on the CRAY Y-MP at the University of Illinois at Urbana-Champaign, on Sun 4 work stations at the University of Northern Iowa, and on NeXT and Macintosh IIci stations at California State University, Fresno. Over half the search was run twice, once each at the latter two institutions.

* The first author was partially supported by National Center for Supercomputing Applications under TRA910033N and utilized the CRAY Y-MP system at the National Center for Supercomputing Applications, University of Illinois at Urbana-Champaign.

2. TABLES OF REDUCED AND AUGMENTED AMICABLE PAIRS

The tables of reduced and augmented amicable pairs follow.

TABLE 1. Reduced Amicable Pairs from 10^7 to 10^8

1.	12146750 = 5(3).7.11.631;	16247745 = 3(2).5.127.2843
2.	12500865 = 3(3).5.13.17.419;	12900734 = 1.7.11.19.4409
3.	13922100 = 2(2).3(2).5(2).31.499;	31213899 = 3(2).1549.2239
4.	14371104 = 2(5).3.11.31.439;	28206815 = 5.7.13.47.1319
5.	22013334 = 2.3(2).7.17.43.239;	37291625 = 5(3).7.17.23.109
6.	22559060 = 2(2).5.47.103.233;	26502315 = 3.5.7.83.3041
7.	23379224 = 2(3).11.23.11551;	26525415 = 3.5.7(2).151.239
8.	23939685 = 3(3).5.7(3).11.47;	31356314 = 2.11.23.31.1999
9.	26409320 = 2(3).5.7.257.367;	41950359 = 3(3).11.127.1031
10.	27735704 = 2(3).17.109.1871;	27862695 = 3(2).5.7.197.449
11.	28219664 = 2(4).11.109.1471;	32014575 = 3(3).5(2).43.1103
12.	33299000 = 2(3).5(3).7.67.71;	58354119 = 3(2).29.47.67.71
13.	34093304 = 2(3).97.31.41.479;	43321095 = 3(3).5.223.1439
14.	37324584 = 2(3).3(3).11.23.683;	80870615 = 5.7.17.199.683
15.	40818855 = 3.5.7.11.59.599;	42125144 = 2(3).23.179.1279
16.	41137620 = 2(2).3.5.17.31.1301;	84854315 = 5.7.13.251.743
17.	49217084 = 2(2).7.47.149.251;	52389315 = 3(3).5.11.35279
18.	52026920 = 2(3).5.11.23.53.97;	85141719 = 3(3).13.107.2267
19.	52601360 = 2(4).5.7.29.41.79;	97389039 = 3.11.17.173599
20.	61423340 = 2(2).5.11.23.61.199;	88567059 = 2.7(3).17.61.83
21.	62252000 = 2(5).5(3).79.197;	93423519 = 3(2).7.107.13859
22.	64045904 = 2(4).13.367.839;	70112175 = 3.5(2).7.83.1609
23.	66086504 = 2(3).11.750983;	69090615 = 3(2).5.11.29.4813
24.	66275384 = 2(3).7.17.43.1619;	87689415 = 3.5.11.179.2969
25.	68337324 = 2(2).3(3).11.23.41.61;	141649235 = 5.7.13.419.743
26.	72917000 = 2(3).5(3).13.71.79;	115780599 = 3(2).11.47.149.167
27.	76011992 = 2(3).7.179.7583;	87802407 = 3(3).7.11.157.269
28.	77723360 = 2(5).5.511.13.43.79;	145810719 = 3(3).41.107.1231
29.	89446860 = 2(2).3(2).5.17.29231;	197845235 = 5.7.17.332513
30.	93993830 = 2.5.7.727.1847;	99735705 = 3(5).5.23.43.83
31.	94713300 = 2(2).3(4).5(2).11.1063;	240536075 = 5(2).13.37.83.241
32.	94970204 = 2(2).7.107.31699;	96751395 = 3(3).5.13.29.1901
33.	97797104 = 2(4).19.23.71.197;	114332175 = 3(3).5(2).107.1583

Conjecture 0: There are infinitely many reduced (augmented) amicable pairs.

All pairs found have opposite parity. Since $\sigma(n) = m+n\pm 1 = \sigma(m)$, m and n have the same parity iff $\sigma(n) = \sigma(m)$ are odd iff odd prime factors in m and n occur only in even powers. Thus, we have

Conjecture 1: The numbers in a reduced (augmented) amicable pair are of opposite parity.

For each pair, consider the ratio k of the larger number divided by the smaller. In Table 1 the ratios range from 1.0045786 to 2.53962; in Table 2 from 1.0011028 to 2.64749. Thus,

Conjecture 2: For any $\beta > 0$, no matter how small, there exists a reduced (augmented) amicable pair such that $1 < k < 1 + \beta$.

TABLE 2. Augmented Amicable Pairs to 10^7

1.	6160 = 2(4).5.7.11;	11697 = 3.7.557
2.	12220 = 2(2).5.13.47;	16005 = 3.5.11.97
3.	23500 = 2(2).5(3).47;	28917 = 3(5).7.17
4.	68908 = 2(2).7.23.107;	76245 = 3.5.13.17.23
5.	249424 = 2(4).7.17.131;	339825 = 3.5(2).23.197
6.	425500 = 2(2).5(3).23.37;	570405 = 2.5.11.3457
7.	434784 = 2(5).3.7.647;	871585 = 5.11.13.23.53
8.	649990 = 2.5.11.19.311;	697851 = 3(2).7.11.19.53
9.	660825 = 3(3).5(2).11.89;	678376 = 2(3).19.4463
10.	1017856 = 2(11).7.71;	1340865 = 3(2).5.83.359
11.	1077336 = 2(3).3(2).13.1151;	2067625 = 5(3).7.17.139
12.	1238380 = 2(2).5.11.13.433;	1823925 = 3.5(2).83.293
13.	1252216 = 2(3).7.59.379;	1483785 = 3(3).5.29.379
14.	1568260 = 2(2).5.19.4127;	1899261 = 3(3).7.13.773
15.	1754536 = 2(3).7.17.19.97;	2479065 = 3.5.29.41.139
16.	2166136 = 2(3).7.47.823;	2580105 = 3.5.11.19.823
17.	2362360 = 2(3).5.7.11.13.59;	4895241 = 3.13.31.4049
18.	2482536 = 2(3).3.7(2).2111;	4740505 = 5.7(2).11.1759
19.	2537220 = 2(2).3.5.7(2).863;	5736445 = 5.11.13.71.113
20.	2876445 = 3(3).5.11.13.149;	3171556 = 2(2).19.29.1439
21.	3957525 = 3(3).5(2).11.13.41;	4791916 = 2(2).41.61.479
22.	4177524 = 2(2).3.13.61.439;	6516237 = 3.7.13.23869
23.	4287825 = 3(2).5(2).17.19.59;	4416976 = 2(4).59.4679
24.	5224660 = 2(2).5.7.67.557;	7524525 = 3.5(2).41.2447
25.	5559510 = 2.3.5.11.17.991;	9868075 = 5(2).7.17.31.107
26.	5641552 = 2(4).7.17.2963;	7589745 = 3(2).5.227.743
27.	5654320 = 2(4).5.7.23.439;	10058961 = 3.11.19.61.263
28.	5917780 = 2(2).5.11.37.727;	8024877 = 3(2).7(2).31.587
29.	6224890 = 2.5.7.17.5231;	7336455 = 3.5.7.107.653
30.	6274180 = 2(2).5.11.19(2).79;	9087741 = 3(3).13.17.1523
31.	6711940 = 2(2).5.17.19.1039;	9012861 = 3(2).11.13.47.149
32.	7475325 = 3.5(2).11.13.17.41;	8273668 = 2(2).13.107.1487
33.	7626136 = 2(3).7.43.3167;	9100905 = 3.5.11.19.2903
34.	7851256 = 2(3).7.19.47.157;	10350345 = 3.5.19.23.1579
35.	7920136 = 2(3).7.233.607;	9152505 = 3(2).5.23.37.239
36.	9026235 = 3(5).5.17.19.23;	9843526 = 2.7.11.41.1559

3. THE UNITARY CASE

In [2] searches for the unitary analogues of reduced and augmented amicable pairs to 10^5 were reported. Except for trivial cases, none were found. The search has been extended to 10^6 with no new results.

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ON INDEPENDENT PYTHAGOREAN NUMBERS

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INTRODUCTION

In a paper of Sypriya Mohanty and S. P. Mohanty (refer to [1]), the notion of an independent Pythagorean number is introduced and discussed. Recall that any Pythagorean triple (x, y, z) may be represented by

$$x = 2uvt,\ \ y = t(u^2 - v^2),\ \ z = t(u^2 + v^2) \tag{1}$$

where u and v are relatively prime natural numbers of opposite parity, that is, $u + v \equiv 1 \pmod 2$, $(u, v) = 1$, $u > v$, and t some natural number.

In the same paper, Definition 1 (p. 31) calls the area of a Pythagorean triangle a "Pythagorean number." And that of a primitive Pythagorean triangle a " primitive Pythagorean number." Thus, a Pythagorean number is a positive integer of the form

$$A = \frac{1}{2}(2uvt)[t(u^2 - v^2)] = t^2uv(u^2 - v^2), \tag{2}$$

where the natural numbers u and v satisfy the above conditions.

When the Pythagorean triangle at hand is primitive, i.e., when $t = 1$, we obtain the general form of a primitive Pythagorean number described by

$$B = uv(u^2 - v^2). \tag{3}$$

The authors define the notion of an independent Pythagorean number and they prove that there exist infinitely many primitive Pythagorean numbers that are not independent (Theorem 10, p. 40). According to that definition (Definition 2, p. 40), a Pythagorean number is called independent if it cannot be obtained from another Pythagorean number by multiplying the latter by t^2, where t is a natural number > 1.

Note that if a Pythagorean number is independent, it must be primitive. The converse, of course, is false, as the authors have proved: there exist (infinitely many) primitive Pythagorean numbers that are not independent.

In this paper, we will address Problem 2 in the author's paper. Namely, find sufficient conditions for an integer B to be an independent Pythagorean number. We will find families of primitive Pythagorean numbers that are independent. First, we will state the two theorems of this paper, then their proofs.

Theorem 1: Let u and v be natural numbers such that $u + v \equiv 1 \pmod 2$, $(u, v) = 1$, and $u > v$. Assume that either

(a) all four numbers $u, v, u - v,$ and $u + v$ are squarefree (the case $v = 1$ included), or

(b) the three integers $u - v, u + v,$ and $\frac{uv}{4}$ are all squarefree and $\frac{uv}{4}$ odd (the case $v = 1$ included).

Then the primitive Pythagorean number $uv(u^2 - v^2)$ is independent.

Theorem 2: Let $p > 3$ be a prime and $v \geq 1, w \geq 3$ be odd squarefree natural numbers (the case $v = 1$ included) both of whose (distinct) prime divisors are all congruent to 1 mod p. Let n be a positive integer and r an odd prime distinct from p and the prime divisors of w. Assume that $(u, v) = 1$, where $u = 2^n \cdot r \cdot {}^2 w$. Furthermore, suppose that $u - v$ is a squarefree integer such that each of its prime divisors is congruent to 1 mod p and that $u + v$ is a squarefree integer containing exactly one prime divisor $q \not\equiv 1$ (mod p), while the rest of its prime divisors, if any, are all congruent to 1 modulo p. Assume that $n = 1$ or $n = 2$.

Then the primitive Pythagorean number $uv(u^2 - v^2)$ is independent.

Proof of Theorem 1: Suppose that

$$uv(u^2 - v^2) = t^2 \cdot b \tag{4}$$

where b is a Pythagorean number and t some positive integer. Since b is a Pythagorean number, according to (2), b must be of the form

$$b = T^2 \cdot U \cdot V(U^2 - V^2), \tag{5}$$

for some positive integers T, U and V where

$$U > V, U + V \equiv 1 \,(\text{mod}\, 2) \text{ and } (U, V) = 1. \tag{6}$$

Substituting for b in (4), we obtain

$$\begin{gathered} uv(u^2 - v^2) = t^2 \cdot T^2 \cdot U \cdot V \cdot (U^2 - V^2) \text{ or} \\ uv(u - v)(u + v) = t^2 \cdot T^2 \cdot U \cdot V \cdot (U^2 - V^2). \end{gathered} \tag{7}$$

If hypothesis (a) is satisfied, then the product $uv(u - v)(u + v)$ must be a squarefree integer, since each of the numbers $uv, u - v$, and $u + v$ is squarefree, and these three integers are mutually coprime in view of $(u, v) = 1$ and $u + v \equiv 1 \,(\text{mod}\, 2)$. Then (7) clearly implies $t^2T^2 = 1 \Rightarrow tT = 1 \Rightarrow t = T = 1$.

If hypothesis (b) is satisfied, 4 must exactly divide the left-hand side of (7). Since $uv \equiv 0$ and $u \pm v \equiv 1 \,(\text{mod}\, 2)$, t^2T^2 must be odd and $uv \equiv 0$ (mod 4). Dividing (7) by 4, we obtain

$$\frac{uv}{4} \cdot (u - v)(u + v) = t^2T^2 \cdot \frac{UV}{4} \cdot (U^2 - V^2) \tag{8}$$

Since the left-hand side of (8) is an odd squarefree integer, we have $t^2T^2 = 1 \Rightarrow tT = 1 \Rightarrow t = T = 1$. Hence, $uv(u^2 - v^2)$ is an independent Pythagorean number.

Proof of Theorem 2: Evidently, according to the hypothesis, the Pythagorean number $uv(u^2 - v^2)$ must be of the form

$$uv(u^2 - v^2) = uv(u - v)(u + v) = 2^n \cdot q \cdot r^2 \cdot p_1 \cdots p_m,,$$

where all the odd primes $q, r, p_1, \cdots, p_m$ are distinct and $p_1 \equiv \cdots \equiv p_m \equiv 1 \,(\text{mod}\, p)$. Suppose that

$$2^n \cdot q \cdot r^2 \cdot p_1 \cdots p_m = t^2 ab(a-b)(a+b), \tag{9}$$

where the positive integers a and b have opposite parity, $(a,b)=1$, and $a>b$. Assume that a is odd and b even (the case a even and b odd is treated in exactly the same way). We set $b=2^k \cdot B$, B odd, and $t=2^\delta T$ in (9) to obtain

$$2^n \cdot q \cdot r^2 \cdot p_1 \cdots p_m = T^2 \cdot 2^{2\delta+k} \cdot a \cdot B(a-2^k \cdot B)(a+2^k \cdot B), \tag{10}$$

which gives

$$q \cdot r^2 \cdot p_1 \cdots p_m = T^2 \cdot a \cdot B(a-2^k \cdot B)(a+2^k \cdot B), \tag{11}$$

since we must have $2\delta + k = n$; with $1 \le k \le n$, $\delta \ge 0$, and T odd.

First, we will prove that (11) cannot be satisfied for T odd and $T>1$. Let us assume to the contrary that (11) is satisfied for some $T>1$ and T odd. In view of the fact that the left-hand side of (11) represents the unique factorization of the right-hand side of (11) into powers of distinct primes and because r^2 is the only square of a prime, it is rather obvious that we must have $T=r$; hence, (11) implies

$$q \cdot p_1 \cdots p_m = a \cdot B(a-2^k \cdot B)(a+2^k \cdot B). \tag{12}$$

Since $p_1 \equiv \cdots \equiv p_m \equiv 1 \pmod p$, (12) clearly shows that if $q|aB$, then $a-2^kB \equiv 1$ and $a+2^kB \equiv 1$ (mod p); so $2a \equiv 2$ and $2^{k+1}B \equiv 0 \pmod p$; therefore if $q|aB$,

$$a \equiv 1 \text{ and } B \equiv 0 \pmod p, \tag{13}$$

which is a contradiction, since p as a divisor of B would divide the left-hand side of (12), contrary to the fact that p is distinct from $q, p_1, \cdots, p_m$. Next, suppose that $q|(a-2^k \cdot B)$ or that $q|(a+2^k \cdot B)$. Equation (12) clearly implies in such a case, $a \equiv B \equiv 1 \pmod p$. Also if $q|a-2^k \cdot B$, we must have $a-2^k \cdot B \equiv q \pmod p$; and since $a \equiv 1 \pmod p$, we end up with $2 \equiv q+1 \pmod p \Rightarrow q \equiv 1 \pmod p$, contradicting the hypothesis again [note that $a-2^kB \equiv q$ and $a+2^kB \equiv 1 \pmod p$ or vice versa].

Hence, we conclude that (11) is not possible with $T>1$. Consequently, $T=1$; thus, from $t=2^\delta \cdot T$, we obtain $t=2^\delta$. We will show that $\delta=0$. According to the hypothesis, $n=1$ or 2. If $n=1$, then, from $2\delta+k=n$ and $k \ge 1$, we immediately obtain $\delta=0$. For $n=2$, again we must have $\delta=0$, in view of $2\delta+k=n$ and $k \ge 1$. Therefore, $\delta=0$, and since we also have $T=1$, it follows that $t=2^\delta T \Rightarrow t=1$. The proof is complete.

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A NOTE ON RATIONAL ARITHMETIC FUNCTIONS OF ORDER (2, 1)

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1. INTRODUCTION

In [4], among other things, the connection between specially multiplicative functions and generalized Fibonacci sequences is discussed. In this paper we shall discuss the similar connection that exists between rational arithmetic functions of order (2, 1) (to be defined in section 2) and generalized Fibonacci sequences. The generalized Fibonacci sequence studied in this paper is the sequence $\{w_n(a,b;c,d)\}$ or, briefly, $\{w_n\}$ of complex numbers, which is defined by

$$w_0 = a,\ w_1 = b,\ w_n = cw_{n-1} - dw_{n-2}\ \ (n \geq 2).$$

This sequence has been extensively studied by Horadam (e.g., [2]).

Section 2 motivates the study of rational arithmetic functions of order (2, 1), while section 3 considers the main theme of this paper, namely, the connection between rational arithmetic functions of order (2, 1) and the sequence $\{w_n\}$. Arising from this connection, identities are presented involving the sequences $\{w_n\}$ and $\{u_n\}$, where $u_n = u_n(c,d) = w_n(1,c;c,d)$. The sequence $\{u_n\}$ is particularly important as indicated in [4]. Finally, in section 4, an identity for rational arithmetic functions of order (2, 1) is proven with the aid of the identities of section 3.

For general background on arithmetic functions, reference is made to the books by Paul McCarthy [3] and Sivaramakrishnan [6]. The basic concepts used in this paper are reviewed here.

An arithmetic function f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$. If $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all m and n, then f is said to be completely multiplicative. An arithmetic function f is said to be quasi-multiplicative if $f(1) \neq 0$ and there exists a complex number q such that $qf(mn) = f(m)f(n)$ whenever $(m,n) = 1$. It follows immediately that $q = f(1)$. If $f(1) \neq 0$ and $f(1)f(mn) = f(m)f(n)$ for all m and n, then f is said to be a completely quasi-multiplicative function. It is clear that each (completely) multiplicative function is (completely) quasi-multiplicative.

For a prime number p, the generating series of a multiplicative arithmetic function f to the base p is defined by

$$f_p(x) = \sum_{n=0}^{\infty} f(p^n)x^n$$

(see [7]). Each multiplicative function is completely determined by its generating series (at all primes p). It is easy to see that generating series can also be used in the context of quasi-multiplicative functions.

The Dirichlet convolution $f * g$ of two arithmetic functions f and g is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

It is clear that, for all primes p, $(f * g)_p(x) = f_p(x)g_p(x)$, when f and g are multiplicative.

2. DEFINITION

The arithmetic function β introduced by S. S. Pillai [5] is given by

$$\beta(n) = \sum_{k=1}^{n} (k, n),$$

where (k, n) is the greatest common divisor of k and n. The structure of β is

$$\beta = I * I * e^{-1} = I * I * \mu, \tag{1}$$

where $I(n) = n, e(n) = 1\,(n \ge 1)$, and μ is the classical Möbius function (see [6, p. 8]). The arithmetic function β is an example of a rational arithmetic function of order (2, 1) in the terminology of Vaidynathaswamy [7], who called a multiplicative arithmetic function f a rational arithmetic function of order (r, s) if there exist nonnegative integers r, s and completely multiplicative functions $g_1, \ldots, g_r$, $h_1, \ldots, h_s$ such that

$$f = g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1}.$$

Conventionally, the identity function e_0 is a rational arithmetic function of order (0, 0), where $e_0(1) = 1$ and $e_0(n) = 0$ for $n > 1$.

By (1),

$$\sum_{d|n} \beta(d) = (I * I)(n) = n\tau(n),$$

where $\tau(n)$ is the number of positive divisors of n. The function $n\tau(n)$ is a quadratic function [7], that is, a rational arithmetic function of order (2, 0). A quadratic function is also called a specially multiplicative function in the literature (see, e.g., [4]). If g is specially multiplicative and $g = g_1 * g_2$, then

$$g(m)g(n) = \sum_{d|(m,n)} g(mn/d^2)(g_1 g_2)(d) \tag{2}$$

for all m and n, or, equivalently,

$$g(mn) = \sum_{d|(m,n)} g(m/d)g(n/d)\mu(d)(g_1 g_2)(d) \tag{3}$$

for all m and n (see, e.g., [3, Th. 1.12]). Section 3 includes generalizations of these identities in terms of the sequences $\{w_n\}$ and $\{u_n\}$.

A specially multiplicative function also satisfies

$$g(m)(g_1 g_2)(n) = \sum_{d|n} g(n/d)g(mnd)\mu(d)$$

for all m and n (see [1, Prob. 4, p. 139]). Examination of whether a similar identity holds for β shows that

$$\beta(m)n = \sum_{d|n} \tau(n/d)\beta(mnd)\mu(d)/d$$

for all m and n. Section 4 shows that a similar identity holds for all rational arithmetic functions of order (2, 1).

3. CONNECTIONS WITH GENERALIZED FIBONACCI SEQUENCES

Let g be a specially multiplicative function given by $g = g_1 * g_2$, where g_1 and g_2 are completely multiplicative functions, and let h be a completely quasi-multiplicative function. Let f be defined by $f = g * h^{-1}$. Then $f/f(1)$ is a rational arithmetic function of order (2, 1). Note that $1/f(1) = h(1)$. The generating series of f and g to the base p are

$$f_p(x) = \frac{\frac{1}{h(1)} - \frac{h(p)}{h(1)^2}x}{1 - g(p)x + (g_1g_2)(p)x^2}, \text{ and } g_p(x) = \frac{1}{1 - g(p)x + (g_1g_2)(p)x^2}.$$

The generating series of the sequences $\{w_n\}$ and $\{u_n\}$ are

$$w(x) \equiv \sum_{n=0}^{\infty} w_n x^n = \frac{a + (b - ca)x}{1 - cx + dx^2}, \text{ and } u(x) \equiv \sum_{n=0}^{\infty} u_n x^n = \frac{1}{1 - cx + dx^2}.$$

Thus, for each arithmetic function f given by $f = g * h^{-1}$, where g is a specially multiplicative function and h is a completely quasi-multiplicative function, we have

$$\{f(p^n)\} = \{w_n(f(1), f(p); g(p), (g_1g_2)(p))\}, \text{ and } \{g(p^n)\} = \{u_n(g(p), (g_1g_2)(p))\}.$$

Example 1: For all primes p,

$$\{\beta(p^n)\} = \{w_n(1, 2p-1; 2p, p^2)\}, \text{ and } \{(\beta * e)(p^n)\} = \{u_n(2p, p^2)\}.$$

Conversely, for each sequence $\{w_n\}$ with $a \neq 0$, we have

$$\{w_n(a, b; c, d)\} = \{f(p^n)\}, \text{ and } \{u_n(c, d)\} = \{g(p^n)\},$$

where $f = g * h^{-1}$, g being the specially multiplicative function given by $g(p) = c$, $(g_1g_2)(p) = d$, and h being the completely quasi-multiplicative function given by $h(1) = 1/a$, $h(p) = c/a - b/a^2$. Namely, the above generating series gives $1/h(1) = a, -h(p)/h(1)^2 = b - ca$.

Example 2: For all primes p,

$$\{w_n(2, 1; 1, -1)\} = \{L_n\} = \{f(p^n)\}, \text{ and } \{u_n(1, -1)\} = \{F_{n+1}\} = \{g(p^n)\},$$

where $h(1) = \frac{1}{2}, h(p) = \frac{1}{4}, g(p) = 1, (g_1g_2)(p) = -1$, and F_n, L_n are the Fibonacci and Lucas numbers, respectively.

Using the connection that $w_n = (g_1 * g_2 * h^{-1})(p^n)$ and $u_n = (g_1 * g_2)(p^n)$ it can be proved by some calculations that

$$w_{m+n} = u_m w_n - u_{m-1} w_{n-1} d \quad (m, n \geq 1), \tag{4}$$

and

$$u_m w_n = \sum_{i \leq m, n} w_{m+n-2i} d^i \quad (m, n \geq 1). \tag{5}$$

These identities may be considered as generalizations of the classical identities (2) and (3) for specially multiplicative functions.

There also exist identities that involve generalized Ramanujan sums in identities for specially multiplicative functions (see [6, Th. 124]). The following analogous identities are proposed for the sequences $\{w_n\}$ and $\{u_n\}$: Let $\{\alpha_i\}$ be a sequence of complex numbers, and k, q nonnegative integers. Let $\{\mu_n\}$ be the sequence given by $\mu_0 = 1, \mu_1 = -1, \mu_n = 0 \ (n \geq 2)$. Then we have

$$\sum_{i \leq m, n} d^i u_{m-i} w_{n-i} \sum_{\substack{j \leq i \\ jk \leq q}} \alpha_j \mu_{i-j} = \sum_{\substack{i \leq m, n \\ ik \leq q}} \alpha_i d^i w_{m+n-2i}, \tag{6}$$

$$\sum_{i \leq m, n} w_{m+n-2i} d^i \sum_{\substack{j \leq i \\ jk \leq q}} \alpha_j = \sum_{\substack{i \leq m, n \\ ik \leq q}} \alpha_i d^i u_{m-i} w_{n-i}. \tag{7}$$

Note that with $q = 0$ and $\alpha_0 = 1$, (6) and (7) reduce to (4) and (5), respectively.

4. AN IDENTITY

This section presents the identity for rational arithmetic functions of order (2, 1) mentioned at the end of section 2. Let f be a rational arithmetic function of order (2, 1) given by

$$f = g * h^{-1} = g_1 * g_2 * h^{-1},$$

where g_1, g_2, and h are completely multiplicative functions. Use is made of the identity (4) written in terms of f. For all primes p and positive integers r and s,

$$f(p^{r+s}) = g(p^r) f(p^s) - g(p^{r-1}) f(p^{s-1})(g_1 g_2)(p). \tag{8}$$

Theorem: If f is a rational arithmetic function of order (2, 1), then

$$f(m)(g_1 g_2)(n) = \sum_{d \mid n} g(n/d) f(mnd) \mu(d) \tag{9}$$

for all m and n.

Proof: By multiplicativity, it suffices to consider the case in which m and n are prime powers, say, $m = p^a, n = p^b$. If $b = 0$, both sides of (9) reduce to $f(p^a)$. If $a = 0, b = 1$, then (9) is obtained by (8) with $r = s = 1$. Assume that $a = 0, b > 1$, then the right-hand side of (9) is

$$g(p^b) f(p^b) - g(p^{b-1}) f(p^{b+1}).$$

By (8), this can be written as

$$g(p^b)[g(p^{b-1}) f(p) - g(p^{b-2})(g_1 g_2)(p)] - g(p^{b-1})[g(p^b) f(p) - g(p^{b-1})(g_1 g_2)(p)]$$

or, after simplification,

$$(g_1 g_2)(p)[g^2(p^{b-1}) - g(p^{b-2}) g(p^b)].$$

It can be verified that

$$g^2(p^{b-1}) - g(p^{b-2}) g(p^b) = (g_1 g_2)(p^{b-1})$$

(see [6, Lemma, p. 287]). Since g_1g_2 is completely multiplicative, the left-hand side of (9) is arrived at. The case $a, b > 0$ could be considered in a similar way. The details are not included here.

Remark: Identity (9) in terms of the sequences $\{w_n\}$ and $\{u_n\}$ is:

$$w_n d = u_n w_{m+n} - u_{n-1} w_{m+n+1} \quad (m \geq 0, n \geq 1).$$

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REAL FIBONACCI AND LUCAS NUMBERS WITH REAL SUBSCRIPTS

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1. INTRODUCTION

Several definitions of Fibonacci and Lucas numbers with real subscript are available in literature. In general, these definitions give complex quantities when the subscript is not an integer [1], [3], [8], [9].

In this paper we face, from a rather general point of view, the problem of defining numbers F_x and L_x which are *real* when subscript x is real. In this kind of definition, the minimum requirement is, obviously, that F_x and L_x and the usual Fibonacci numbers F_n and Lucas numbers L_n coincide when $x = n$ is an integer. Further, for all x, the fulfillment of some of the main properties possessed by F_n and L_n is desirable. Some of these definitions have already been given by other authors (e.g., [6], [10]).

Here, after a brief discussion on some general aspects of these definitions, we propose two distinct expressions for both F_x and L_x and study some of their properties. More precisely, in Section 2 we give an *exponential* representation for F_x and L_x, whereas in Section 3 we give a *polynomial* representation for these numbers. In spite of the fact that the numbers defined in the above said ways coincide only when x is an integer, they are denoted by the same symbol. Nevertheless, there is no danger of confusion since each definition applies only to the proper section.

We confine ourselves to consider only *nonnegative* values of the subscript, so that in all the statements involving numbers of the form F_{x-y} and L_{x-y} *it is understood that* $y \leq x$. The following notation is used throughout the paper:

$\lambda(x)$, the greatest integer not exceeding x,
$\mu(x)$, the smallest integer not less than x.

2. EXPONENTIAL REPRESENTATION OF F_x AND L_x

Keeping in mind the Binet forms for F_n and L_n leads, quite naturally to consideration of expressions of the following types:

$$F_x = [\alpha^x - f(x)\alpha^{-x}]/\sqrt{5} \tag{2.1}$$

and

$$L_x = \alpha^x + f(x)\alpha^{-x}, \tag{2.2}$$

where $\alpha = (1+\sqrt{5})/2$ is the positive root of the equation $z^2 - z - 1 = 0$, and $f(x)$ is a function of the real variable x such that

$$f(n) = (-1)^n \text{ for all integers } n. \tag{2.3}$$

It is plain that the numbers F_x and L_x defined by (2.1)-(2.3) and the usual Fibonacci numbers F_n and Lucas numbers L_n coincide whenever $x = n$ is an integer.

If we require that F_x and L_x enjoy some of the properties of F_n and L_n, we must require that $f(x)$ has some additional properties beyond that stated in (2.3).

Theorem 1: If, for all x,

$$f(x+1) = -f(x) \tag{2.4}$$

then the fundamental relations

$$F_{x+2} = F_{x+1} + F_x \tag{2.5}$$

and

$$L_{x+2} = L_{x+1} + L_x \tag{2.6}$$

are satisfied.

Proof: By (2.2) and (2.4), we can write

$$\begin{aligned} L_{x+1} + L_x &= \alpha^{x+1} + f(x+1)\alpha^{-x-1} + \alpha^x + f(x)\alpha^{-x} \\ &= \alpha^{x+1} + \alpha^x + f(x)(\alpha^{-x} - \alpha^{-x-1}). \end{aligned}$$

Since $\alpha^2 = \alpha + 1$ and $\alpha^{-2} = 1 - \alpha^{-1}$, we have $\alpha^{x+1} + \alpha^x = \alpha^{x+2}$ and $\alpha^{-x} - \alpha^{-x-1} = \alpha^{-x-2}$. Thus,

$$L_{x+1} + L_x = \alpha^{x+2} + f(x)\alpha^{-x-2} + f(x+2)\alpha^{-x-2} = L_{x+2}. \text{ Q.E.D.}$$

Theorem 2: If, for a particular x,

$$f^2(x) = f(2x), \tag{2.7}$$

then the identity

$$F_x L_x = F_{2x} \tag{2.8}$$

is satisfied.

Proof: By (2.1) and (2.2), after some simple manipulations, we get

$$F_x L_x = [\alpha^{2x} - f^2(x)\alpha^{-2x}] / \sqrt{5}. \text{ Q.E.D.}$$

Theorem 3: If the condition (2.4) is satisfied for all x, then the identity

$$L_x = F_{x-1} + F_{x+1} \tag{2.9}$$

holds.

The proof of Theorem 3 is analogous to that of Theorem 1 and is omitted for brevity.

Parker [10] used the function

$$f(x) = \cos(\pi x) \tag{2.10}$$

to obtain real Fibonacci and Lucas numbers with real subscripts. The function (2.10) satisfies (2.3) and (2.4) but does not satisfy (2.7). Other circular functions (or functions of circular functions) might be used as $f(x)$. For example, $f(x) = \cos^k(\pi x)$ and $f(x) = \cos^{1/k}(\pi x)$ (k an odd integer) satisfy the above properties as well. Further functions might be considered. For example, the function

$$f(x) = [a^2 - \sin^2(\pi x)]^{1/2} - a + \cos(\pi x) \quad (a \geq 1),$$

which describes the piston stroke as a function of the crank angle πx and the ratio a of the rod length to the crank radius, satisfies (2.3) but does not satisfy (2.4).

In my opinion, the simplest function $f(x)$ satisfying (2.3) and (2.4) is the function

$$f(x) = (-1)^{\lambda(x)} \tag{2.12}$$

which leads to the definitions

$$F_x = [\alpha^x - (-1)^{\lambda(x)}\alpha^{-x}]/\sqrt{5} \tag{2.13}$$

and

$$L_x = \alpha^x + (-1)^{\lambda(x)}\alpha^{-x}. \tag{2.14}$$

Observe that (2.12) can be viewed as a particular function of circular functions. In fact, this function and the special Fourier series

$$f(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin[(2k+1)\pi x]}{2k+1} \tag{2.12′}$$

coincide, except for the integral values of x.

As an illustration, the behavior of F_x vs x is shown in Figure 1 for $0 \leq x \leq 10$.

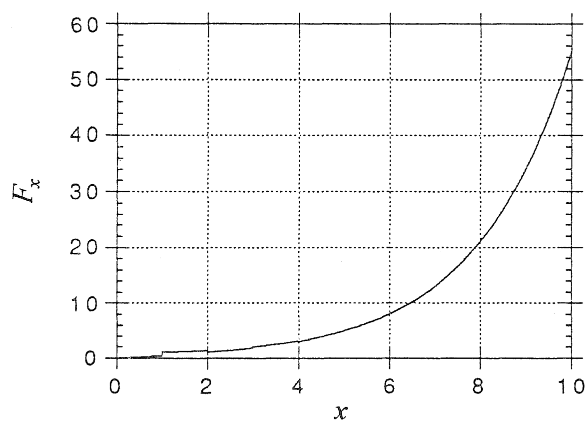

FIGURE 1. Behavior of F_x vs x for $0 \leq x \leq 10$

The discontinuities (observable for small values of x) connected with the integral values of x are obviously due to the greatest integer function inherent in the definition (2.13).

The numbers F_x and L_x defined by (2.13) and (2.14), respectively, enjoy several properties of the usual Fibonacci and Lucas numbers. For example, the following two propositions can be stated.

Proposition 1: $5F_x^2 = L_x^2 - 4(-1)^{\lambda(x)}$.

Proposition 2 (Simson formula analog): $F_{x-1}F_{x+1} - F_x^2 = (-1)^{\lambda(x)}$.

For the sake of brevity, we shall prove only Proposition 2.

Proof of Proposition 2: From (2.13), we can write

$$\begin{aligned} F_{x-1}F_{x+1}-F_x^2 &= [\alpha^{2x}-(-1)^{\lambda(x+1)}\alpha^{-2}-(-1)^{\lambda(x-1)}\alpha^2+(-1)^{\lambda(x-1)+\lambda(x+1)}\alpha^{-2x} \\ &\qquad -\alpha^{2x}-\alpha^{-2x}+2(-1)^{\lambda(x)}]/5 \\ &= [-(-1)^{\lambda(x)+1}\alpha^{-2}-(-1)^{\lambda(x)-1}\alpha^2+(-1)^{2\lambda(x)}\alpha^{-2x}-\alpha^{-2x}+2(-1)^{\lambda(x)}]/5 \\ &= [(-1)^{\lambda(x)}(\alpha^2+\alpha^{-2})+2(-1)^{\lambda(x)}]/5 \\ &= (-1)^{\lambda(x)}(L_2+2)/5=(-1)^{\lambda(x)}.\ \text{Q.E.D.} \end{aligned}$$

Let us conclude this section by offering the sums of some finite series involving the numbers F_x and L_x. These are

$$\sum_{k=0}^{n} T_{x+k} = T_{n+x+2} - T_{x+1}, \tag{2.15}$$

where T stands for both F and L, and

$$\sum_{k=1}^{n} F_{k/n} = 1+\frac{1}{\sqrt{5}(L_{1/n}-2)}-\frac{F_{(n-1)/n}+F_{1/n}}{L_{1/n}-2} \quad (n\geq 2), \tag{2.16}$$

$$\sum_{k=1}^{n} L_{k/n} = \frac{\sqrt{5}-L_{(n-1)/n}}{L_{1/n}-2} \quad (n\geq 2). \tag{2.17}$$

The proofs of (2.15)-(2.17) can be obtained from (2.13) and (2.14) with the aid of the geometric series formula. They are left to the interested reader.

3. POLYNOMIAL REPRESENTATION OF F_x AND L_x

Let us recall the well-known formula

$$F_n = \sum_{j=0}^{U}\binom{n-1-j}{j}, \tag{3.1}$$

where U is a suitable integral function of n, which gives the n^{th} Fibonacci number. It is also well known (see, e.g., [5, p. 48]) that the binomial coefficient defined as

$$\binom{a}{0}=1,\ \binom{a}{k}=\frac{a(a-1)\cdots(a-k+1)}{k!} \quad (k\geq 1 \text{ an integer}) \tag{3.2}$$

makes sense also if a is any real quantity.

In light of (3.2), some conditions must be imposed on the upper range indicator, U, for (3.1) to be efficient. In my opinion, the usual choice $U=\infty$ (see, e.g., [13, (54)]) is not correct. For example, for $n=5$ and $U=\infty$, we have the infinite series

$$F_5 = \binom{4}{0}+\binom{3}{1}+\binom{2}{2}+\binom{1}{3}+\binom{0}{4}+\binom{-1}{5}+\binom{-2}{6}+\binom{-3}{7}+\binom{-4}{8}+\cdots$$
$$= 1 + 3 + 1 + 0 + 0 - 1 + 7 - 36 + 165 - \cdots$$

the sum of which is clearly different from 5. It can be readily proved that formula (3.1) works correctly if the following inequalities are satisfied

$$\lambda[(n-1)/2] \le U \le n-1. \tag{3.3}$$

On the basis of (3.2) and (3.3), a polynomial representation of F_x can be obtained by simply replacing n by x in (3.1). Following the choice of Schroeder [12, p. 68] (i.e., $U = \lambda[(n-1)/2]$), we define the numbers F_x as

$$F_x = \sum_{j=0}^{\lambda[(x-1)/2]} \binom{x-1-j}{j} \tag{3.4}$$

Observe that, under the convention that a sum vanishes when the upper range indicator is smaller than the lower one and taking into account that $\lambda(-x) = -\mu(x)$, expression (3.4) allows us to obtain $F_0 = 0$.

Other choices of U are possible, within the interval (3.3). In a recent paper [1] André-Jeannin considered the numbers $G(x)$ ($x > 0$) obtained by replacing n by x and U by $m(x)$ in (3.1), $m(x)$ being the integer defined by $x/2-1 \le m(x) < x/2$. It is readily seen that $m(x) = \mu(x/2-1)$, and $m(x) = \lambda[(x-1)/2]$ when x is an integer. Moreover, we can see that F_x and $G(x)$ coincide for $2h-1 \le x \le 2h$ (h = 1, 2, ...), and both of them give the usual Fibonacci numbers F_n when $x = n$ is an integer.

As an illustration, we give the value of F_x for $0 \le x < 9$.

$$\begin{aligned}
&F_x = 0, \text{ for } 0 \le x < 1,\\
&F_x = 1, \text{ for } 1 \le x < 3,\\
&F_x = x-1, \text{ for } 3 \le x < 5,\\
&F_x = (x^2-5x+10)/2, \text{ for } 5 \le x < 7,\\
&F_x = (x^3-12x^2+59x-90)/6, \text{ for } 7 \le x < 9.
\end{aligned}$$

The behavior of F_x vs x for $0 \le x < 9$ is shown in Figure 2 below.

Replacing n by x in [4, (1.3)-(1.4)] leads to an analogous polynomial representation of L_x:

$$L_x = \sum_{j=0}^{\lambda(x/2)} \frac{x}{x-j}\binom{x-j}{j}. \tag{3.5}$$

Observe that, for x = 0, this definition gives the indeterminate form 0/0. So, $L_0 = 2$ cannot be defined by (3.5). As an illustration, we show the values of L_x for $0 < x < 8$.

$$\begin{aligned}
&L_x = 1, \text{ for } 0 < x < 2,\\
&L_x = x+1, \text{ for } 2 \le x < 4,\\
&L_x = (x^2-x+2)/2, \text{ for } 4 \le x < 6,\\
&L_x = (x^3-6x^2+17x+6)/6, \text{ for } 6 \le x < 8.
\end{aligned}$$

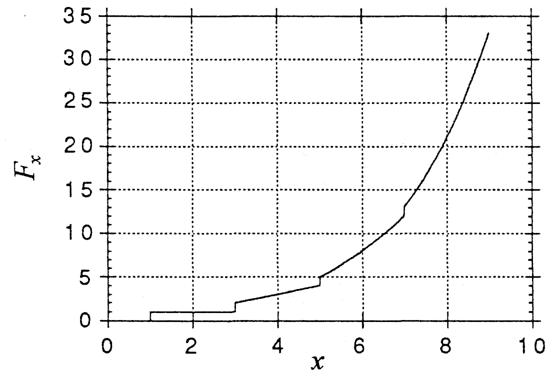

FIGURE 2. Behavior of F_x vs x for $0 \le x < 9$

Also the numbers F_x and L_x defined by (3.4) and (3.5), respectively, enjoy several properties of the usual Fibonacci and Lucas numbers. Sometimes these properties hold for all x, but, in most cases, their validity depends on the parity of $\lambda(x)$. We shall give an example for each case. The proof of the latter is omitted for brevity.

Proposition 3: $F_{x-1} + F_{x+1} = L_x$.

Proof: From (3.5), the binomial identity available in [11, p. 64], and (3.4), we can write

$$L_x = \sum_{j=0}^{\lambda(x/2)} \left[\binom{x-j}{j} + \binom{x-1-j}{j-1}\right] = F_{x+1} + \sum_{j=0}^{\lambda(x/2)} \binom{x-1-j}{j-1} = F_{x+1} + \sum_{j=-1}^{\lambda(x/2)-1} \binom{x-2-j}{j}.$$

By virtue of the assumption [5, p. 48],

$$\binom{x}{-k} = 0 \quad (k \ge 1 \text{ an integer}), \tag{3.6}$$

by using the equality

$$\lambda(x/2) - 1 = \lambda[(x-2)/2], \tag{3.7}$$

and definition (3.4), the previous expression becomes

$$L_x = F_{x+1} + \sum_{j=0}^{\lambda[(x-2)/2]} \binom{x-2-j}{j} = F_{x+1} + F_{x-1}. \quad \text{Q.E.D.}$$

Proposition 4:

$$F_x + F_{x+1} = \begin{cases} F_{x+2}, & \text{if } \lambda(x) \text{ is even,} \\ F_{x+2} - \binom{x-\lambda(x/2)-1}{\lambda(x/2)+1}, & \text{if } \lambda(x) \text{ is odd.} \end{cases}$$

Let us conclude this section by considering a special case [namely, $n = (2k+1)/2$] of the well-known identity $F_n L_n = F_{2n}$. The numerical evidence shows that

$$F_{(2k+1)/2} L_{(2k+1)/2} = F_{2k+1} - g(k). \tag{3.8}$$

The values of $g(k)$ for the first few values of k are shown below:

$$\begin{array}{ll} g(0)=1 & g(4)=2.9375 \\ g(1)=1 & g(5)=3.734375 \\ g(2)=1.5 & g(6)=6.4921875 \\ g(3)=1.75 & g(7)=8.57421875. \end{array}$$

I was able to find neither a closed form expression nor sufficiently narrow bounds for $g(k)$. Establishing an expression for this quantity is closely related to the problem of expressing F_x and L_x as functions of $F_{\lambda(x)}$ and $L_{\lambda(x)}$, respectively ($x = k + 1/2$ in the above case). This seems to be a challenging problem the solution of which would allow us to find many more identities involving the numbers F_x and L_x. Any contribution of the readers on this topic will be deeply appreciated.

4. CONCLUDING REMARKS

In this paper we have proposed an exponential representation and a polynomial representation for Fibonacci numbers F_x and Lucas numbers L_x that are real if x is real. Some of their properties have also been exhibited.

As for the polynomial representation, we point out that other sums, beyond (3.1) and [4, (1.3)-(1.4)] [see (3.5)], give the Fibonacci and Lucas numbers. These sums can be used to obtain further polynomial representations for F_x and L_x. For example, if we replace n by x in the expression for Fibonacci numbers available in [2], we have

$$F_x = \sum_{j=-\lambda(x/5)}^{\lambda[(x-1)/5]} (-1)^j \binom{x-1}{\lambda[(x-1-5j)/2]}. \tag{4.1}$$

Observe that (4.1) and (3.4) coincide for $0 \le x < 5$. Getting the polynomials in x given by (4.1) for higher values of x, requires a lot of tedious calculations. As an illustration, we give the value of F_x for $0 \le x < 8$.

$$\begin{aligned} &F_x = 0, \text{ for } 0 \le x < 1, \\ &F_x = 1, \text{ for } 1 \le x < 3, \\ &F_x = x-1, \text{ for } 3 \le x < 5, \\ &F_x = (-x^4+10x^3-23x^2+14x)/24, \text{ for } 5 \le x < 6, \\ &F_x = (-x^5+15x^4-85x^3+285x^2-454x+120)/120, \text{ for } 6 \le x < 7, \\ &F_x = (-x^5+15x^4-65x^3+105x^2-54x-120)/120, \text{ for } 7 \le x < 8. \end{aligned}$$

Plotting these values shows clearly that definition (4.1) is rather unsatisfactory if compared with definition (3.4). We reported definition (4.1) here for the sake of completeness and because it might be interesting *per se*.

ACKNOWLEDGMENTS

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EDITOR ON LEAVE OF ABSENCE

The Editor has been asked to visit Yunnan Normal University in Kunming, China, for the Fall semester of 1993. This is an opportunity that the Editor and his wife feel cannot be turned down. They will be in China from August 1, 1993, until approximately January 10, 1994. The August and November issues of *The Fibonacci Quarterly* will be delivered to the printer early enough so that these two issues can be published while the Editor is out of the country. The Editor has also arranged for several individuals to send out articles to be refereed which have been submitted for publication in *The Fibonacci Quarterly* or submitted for presentation at the *Sixth International Conference on Fibonacci Numbers and Their Applications.* Things may be a little slower than normal, but every attempt will be made to insure that all goes as smoothly as possible while the Editor is on leave in China. PLEASE CONTINUE TO USE THE NORMAL ADDRESS FOR SUBMISSION OF PAPERS AND ALL OTHER CORRESPONDENCE.

FIBONACCI NUMBERS: REDUCTION FORMULAS AND SHORT PERIODS

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(Submitted November 1991)

Formulas for determining the Fibonacci numbers F_{2n} and F_{2n-1} in terms of F_n and F_{n-1} are well known as are some higher reduction formulas. For example, formulas for F_{3n} and F_{3n-1} are assigned as homework in Alfred [1], and in Chapter 17 of Dickson [3] there is a formula for F_{pn} when p is odd. This note describes a technique for constructing "simplified" formulas for F_{in} and F_{in-1} in terms of F_n and F_{n-1}. Two families of recursively defined polynomials can be used to parametrize these formulas. This parametrization can be applied to the study of the period of the Fibonacci sequence modulo m. These periods have been the subject of considerable study; see [4], [6], and [7] as well as [2] which contains generalizations to continued fractions. The period of the Fibonacci sequence modulo m is often close to the modulus in size, but Ehrlich [4] showed that the period of the Fibonacci sequence was surprisingly small for Fibonacci moduli and many other small periods do appear. His work utilized the reduction formulas for F_{2n} and F_{2n-1}. We can generalize this result using the simplified reduction formulas for F_{in} and F_{in-1} for each even multiplier i.

INTRODUCTION

It is well known that the Fibonacci numbers can be computed by taking powers of a matrix. Namely, if

$$T=\begin{pmatrix} F_0 & F_1 \\ F_1 & F_2 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \text{ then } T^n=\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

Consider the matrix U, given below, that captures the symmetry of T^n and the fact that the (2, 2)-entry is sum of the entries in the first row. Its powers, U^i, can be used to get information about T^{in}. In particular, when $a=F_{n-1}$ and $b=F_n$, the first row gives reduction formulas for F_{in-1} and F_{in} in terms of F_{n-1} and F_n.

$$U=\begin{pmatrix} a & b \\ b & a+b \end{pmatrix}, \quad U^2=\begin{pmatrix} a^2+b^2 & 2ab+b^2 \\ 2ab+b^2 & a^2+2ab+2b^2 \end{pmatrix}$$

$$U^3=\begin{pmatrix} a^3+3ab^2+b^3 & 3a^2b+3ab^2+2b^3 \\ 3a^2b+3ab^2+2b^3 & a^3+3a^2b+6ab^2+3b^3 \end{pmatrix}$$

The first row of U^2 gives the reduction formulas:

$$F_{2n-1}=F_{n-1}^2+F_n^2, \quad F_{2n}=2F_{n-1}F_n+F_n^2.$$

Those equations and simple variations are well known. The first row of U^3 gives additional, less well known reduction formulas:

$$F_{3n-1} = F_{n-1}^3 + 3F_{n-1}F_n^3, \quad F_{3n} = 3F_{n-1}^2 F_n + 3F_{n-1}F_n^2 + 2F_n^3.$$

Higher reduction formulas can be produced by computing higher powers of U. It is easy to see that the entries in U^i are homogeneous polynomials of degree i in the variables a and b. Many other formulas for F_{in} and F_{in-1} in terms of F_n and F_{n-1} are possible since

$$F_{n-1}^2 = F_n^2 - F_{n-1}F_n + (-1)^n.$$

In particular, consider simplifying the polynomials in U^2 and U^3 by the corresponding relation

$$a^2 = b^2 - ab + (-1)^n. \tag{$*$}$$

(One can think of this as a simplification that introduces a new formal parameter n, or as two separate simplifications, depending on whether n is even or odd.) The relation can be applied to a^i for all $i \geq 2$. The result can be simplified again and the process repeated until the variable a appears only linearly. We say that a polynomial that has been simplified in this way is *a-simplified*. For example, the a-simplified form of the first row of U^2 is

$$\left((-1)^n - ab + 2b^2,\ 2ab + b^2\right).$$

The a-simplified form of the first row of U^3 is

$$\left((-1)^n a - (-1)^n b + 5ab^2,\ 3(-1)^n b + 5b^3\right).$$

These give other reduction formulas for Fibonacci numbers:

$$F_{2n-1} = (-1)^n + F_n(2F_n - F_{n-1}), \quad F_{2n} = F_n(2F_{n-1} + F_n),$$
$$F_{3n-1} = (-1)^n(F_{n-1} - F_n) + 5F_{n-1}F_n^2, \quad F_{3n} = 3(-1)^n F_n + 5F_n^3.$$

These formulas are simpler because of the reduction that took place. In fact, since these a-simplified formulas have few multiplications, they are useful for very rapid computation of large Fibonacci numbers, see [5]. Consider one more example as a preview. The first row of U^6, a-simplified, then written in a special way, and with $n = 0$ is:

$$\left(1 + b(3+5b^2)\left(-(a(1+5b^2)) + b(7+10b^2)\right),\ b(2a+b)(1+5b^2)(3+5b^2)\right)$$

This is interesting because, when reduced modulo any factor of $b(3+5b^2)$, this is congruent to $(1, 0)$. This leads to repetition of the Fibonacci sequence at this stage modulo that factor.

These a-simplified formulas can be computed directly by raising U to the appropriate power and applying identity $(*)$ repeatedly, but in the next section we see that they can be computed quickly using simple recursive formulas. Properties of these a-simplified polynomials are established. In the last section, we use the special form of these a-simplified reduction formulas to see that for many infinite families of moduli, the Fibonacci sequence reduced by that modulus has a short period.

PARAMETRIZING THE a-SIMPLIFIED REDUCTION FORMULAS

We begin by defining the following intertwined polynomials in one variable b and with the parameter n giving a choice of sign. Only even indices are used for later convenience.

$$\begin{cases} R_0 = 0,\ R_2 = 1,\ R_{2j} = S_{2j-2} + (-1)^n R_{2j-4} & \text{for } j \ge 2, \\ S_0 = 2,\ S_2 = 1,\ S_{2j} = 5b^2 R_{2j-2} + (-1)^n S_{2j-4} & \text{for } j \ge 2. \end{cases} \tag{$**$}$$

Of course, this gives two sequences of polynomials, one sequence for odd n, the other for n even. Let R_{2j}^0 designate the sequence when n is even and R_{2j}^1 designate the sequence when n is odd.

Lemma 1:

(i) The polynomials R_{2j} and S_{2j} only include even degree terms.

(ii) $\deg(R_{4j-2}) = 2j-2,\ \deg(S_{4j-2}) = 2j-2,$

(iii) $\deg(R_{4j}) = 2j-2,\ \deg(S_{4j}) = 2j.$

(iv) The polynomial R_{2j}^0 has positive coefficients and R_{2j}^1 is identical except that every other even degree coefficient, beginning with the second highest, is the opposite of the corresponding coefficient of R_{2j}^0

(v) The polynomial S_{2j}^0 has positive coefficients and S_{2j}^1 is identical except that every other even degree coefficient, beginning with the second highest, is the opposite of the corresponding coefficient of S_{2j}^0 .

Proof: (i) This is true for $j = 0$ and $j = 1$ and is preserved by the recursive definitions in $(**)$.

(ii) and (iii) These are true for $j = 0, 1$. [Notice that $\deg(R_0) = -2$ is an acceptable convention since $R_0 = 0 = 0b^{-2}$.] Checking the induction step for the four cases is direct:

$$\begin{aligned} \deg(R_{4j+2}) &= \deg(S_{4j} + (-1)^n R_{4j-2}) = \max(2j, 2j-2) = 2j, \\ \deg(S_{4j+2}) &= \deg(5b^2 R_{4j} + (-1)^n S_{4j-2}) = \max(2+2j-2, 2j-2) = 2j, \\ \deg(R_{4j+4}) &= \deg(S_{4j+2} + (-1)^n R_{4j}) = \max(2j, 2j-2) = 2j, \\ \deg(S_{4j+4}) &= \deg(5b^2 R_{4j+2} + (-1)^n S_{4j}) = \max(2+2j, 2j) = 2j+2. \end{aligned}$$

Notice that in each case the highest-order term does not involve $(-1)^n$ so that the highest coefficients are positive and there is no possibility of cancellation.

(iv) and (v) First we claim that R_{2j} and S_{2j} are homogeneous in the expressions b^2 and $(-1)^n$. The claim is true when $j = 0$ and $j = 1$. By parts (ii) and (iii) $\deg(R_{2j}) = \deg(S_{2j-2})$ and $\deg(S_{2j}) = 2 + \deg(R_{2j-2})$, thus, this homogeneity is preserved by the recursive definitions in $(**)$; hence, the claim is true. As noted above, the highest terms of R_{2j} and S_{2j} do not involve any powers of $(-1)^n$; by the claim, each term with lower powers of b^2 will have complementary powers of $(-1)^n$; hence, the alternation of signs when n is odd. □

As an example, $S_{12} = 2(-1)^{3n} + 45(-1)^{2n} b^2 + 150(-1)^n b^4 + 125b^6$ has degree 6 and $S_{12}^0 = 2 + 45b^2 - 150b^4 + 125b^6$. Table 1 contains the first few R_{2j}^0 and S_{2j}^0 polynomials.

Lemma 2: For $j \geq 1$,

(i) $R_{2j+2}S_{2j-2} - R_{2j}S_{2j} = (-1)^{(j-1)n}$,

(ii) $R_{2j-2}S_{2j+2} - R_{2j}S_{2j} = -(-1)^{(j-1)n}$.

Proof: We prove (i) and (ii) simultaneously by induction. When $j = 1$, $R_4S_0 - R_2S_2 = 1\cdot 2 - 1\cdot 1 = (-1)^{0\cdot n}$ and $R_0S_4 - R_2S_2 = 0\cdot S_4 - 1\cdot 1 = -(-1)^{0\cdot n}$. Assuming (i) and (ii) hold for j, we see:

$$\begin{aligned} R_{2j+4}S_{2j} - R_{2j+2}S_{2j+2} &= (S_{2j+2} + (-1)^n R_{2j})S_{2j} - (S_{2j} + (-1)^n R_{2j-2})S_{2j+2}, \text{ by def.} \\ &= (-1)^n (R_{2j}S_{2j} - R_{2j-2}S_{2j+2}) = (-1)^{jn} \end{aligned}$$

using the induction hypothesis about part (ii). This completes the induction step of part (i). The induction step for part (ii) can be handled in a similar manner. □

TABLE 1. The Polynomials R_{2j}^0 and S_{2j}^0 for Small j

$R_0^0 = 0 = 0b^{-2}$	$S_0^0 = 2$
$R_2^0 = 1$	$S_2^0 = 1$
$R_4^0 = 1$	$S_4^0 = 2 + 5b^2$
$R_6^0 = 3 + 5b^2$	$S_6^0 = 1 + 5b^2$
$R_8^0 = 2 + 5b^2 + 25b^4$	$S_8^0 = 2 + 20b^2 + 25b^4$
$R_{10}^0 = 5 + 25b^2 = 5(1 + 5b^2 + 5b^4)$	$S_{10}^0 = 1 + 15b^2 + 25b^4$
$R_{12}^0 = 3 + 20b^2 + 25b^4 = (1 + 5b^2)(3 + 5b^2)$	$S_{12}^0 = 2 + 45b^2 + 150b^4 + 125b^6 = (2 + 5b^2)(1 + 20b^2 + 25b^4)$
$R_{14}^0 = 7 + 70b^2 + 125b^4 + 125b^6$	$S_{14}^0 = 1 + 30b^2 + 125b^4 + 125b^6$
$R_{16}^0 = 4 + 50b^2 + 150b^4 + 125b^6 = (2 + 5b^2)(2 + 20b^2 + 25b^4)$	$S_{16}^0 = 2 + 80b^2 + 500b^4 + 1000b^6 + 625b^8$
$R_{18}^0 = 9 + 150b^2 + 675b^4 + 1125b^6 + 625b^8$ $= (3 + 5b^2)(3 + 45b^2 + 150b^4 + 125b^6)$	$S_{18}^0 = 1 + 50b^2 + 375b^4 + 875b^6 + 625b^8$ $= (1 + 5b^2)(1 + 45b^2 + 150b^4 + 125b^6)$
$R_{20}^0 = 5 + 100b^2 + 525b^4 + 1000b^6 + 625b^8$ $= 5(1 + 5b^2 + 5b^4)(1 + 15b^2 + 25b^4)$	$S_{20}^0 = 2 + 125b^2 + 1250b^4 + 4375b^6 + 6250b^8 + 3125b^{10}$ $= (2 + 5b^2)(1 + 60b^2 + 475b^4 + 1000b^6 + 625b^8)$

We are now able to parametrize the a-simplified formulas for the powers of U in terms of these polynomials.

Theorem 3: For $j \geq 1$, define the following vector with entries that are polynomials in a and b (linear in a) and which includes the parity parameter n:

$$v(j) = \left((-1)^{jn} + bR_{2j}\left(-aS_{2j} + b(5(-1)^n R_{2j-2} + 2S_{2j})\right), b(2a + b)R_{2j}S_{2j}\right).$$

The first row of U^{2j} after being a-simplified is given by $v(j)$.

Proof:

$$\begin{aligned} v(1) &= \left((-1)^n + bR_2\left(-aS_2 + b(5(-1)^n R_0 + 2S_2)\right), b(2a + b)R_2S_2\right) \\ &= \left((-1)^n - ab + 2b^2, 2ab + b^2\right) \end{aligned}$$

as required.

Assuming this is true for j, we want to show it for $j+1$; i.e., we need to show that the a-simplified form of $v(j)U^2$ is $v(j+1)$.

The second component of the a-simplified form of $v(j)U^2$ is obtained by multiplying $v(j)$ times the a-simplified form of the second column of U^2:

$$\begin{aligned} v(j)\cdot\left(b(2a+b),(-1)^n+ab+3b^2\right) &= b(2a+b)\left((-1)^{jn}+5(-1)^n b^2 R_{2j}R_{2j-2}+(-1)^n R_{2j}S_{2j}+5b^2R_{2j}S_{2j}\right) \\ &= b(2a+b)\left(5b^2R_{2j}\left((-1)^n R_{2j-2}+S_{2j}\right)+(-1)^n R_{2j+2}S_{2j-2}\right) \end{aligned}$$

using $(-1)^{(j-1)n}+R_{2j}S_{2j}=R_{2j+2}S_{2j-2}$ from Lemma 2(i). Then using the recursive definitions of R_{2j+2} and then S_{2j+2}, we see the above is $b(2a+b)R_{2j+2}S_{2j+2}$ as required.

The first component of the a-simplified form of $v(j)U^2$ can be shown to be the first component of $v(j+1)$ in a straightforward, but more tedious, manner. However, it is convenient to first simplify the identity required for the first component using the identity obtained above for the second component. We leave the details for the reader. □

As an example, consider $j = 4$. By Theorem 3 we see the first row of U^8 after being a-simplified is:

$$\left((-1)^{4n}+bR_8\left((2b-a)S_8+5b(-1)^n R_6\right), b(2a+b)R_8S_8\right)$$

where $R_6=3(-1)^n+5b^2$, $R_8=2(-1)^n+5b^2$, and $S_8=2(-1)^{2n}+20(-1)^n b^2+25b^4$ as can be seen from Table 1 and Lemma 1. Now letting $a=F_{n-1}$ and $b=F_n$, we get

$$F_{8n-1}=1+F_n\left(2(-1)^n+5F_n^2\right)\left((2F_n-F_{n-1})\left(2+20(-1)^nF_n^2+25F_n^4\right)+5F_n(-1)^n\left(3(-1)^n+5F_n^2\right)\right)$$

and

$$F_{8n}=F_n(2F_{n-1}+F_n)\left(2(-1)^n+5F_n^2\right)\left(2+20(-1)^nF_n^2+25F_n^4\right).$$

In particular, when $n = 3$, we have $F_3=2$, $F_2=1$, so

$$F_{23}=1+2(-2+20)((4-1)(2-80+400)-10(-3+20))=28657$$

and

$$F_{24}=2(4)(-2+20)(2-80+400)=46368,$$

which are correct.

Corollary 4: Let $j\ge 1$. The first row of U^{2j+1} after being a-simplified is given by

$$\begin{aligned}&\left((-1)^{jn}a-(-1)^n bR_{2j}S_{2j}+5ab^2R_{2j}R_{2j+2},\right.\\ &\qquad\qquad\left.(-1)^{jn}b+2(-1)^n bR_{2j}S_{2j}+5b^3R_{2j}R_{2j+2}\right).\end{aligned}$$

Proof: Multiplying out $v(j)U$ and reducing a^2 by $(*)$ gives

$$\left((-1)^{jn}a-(-1)^n bR_{2j}S_{2j}+5(-1)^n ab^2R_{2j}R_{2j-2}+5ab^2R_{2j}S_{2j},\right.$$
$$\left.(-1)^{jn}b+2(-1)^n bR_{2j}S_{2j}+5(-1)^n b^3R_{2j}R_{2j-2}+5b^3R_{2j}S_{2j}\right).$$

The recursive definition for R_{2j+2} simplifies that into the desired result. □

Notice in particular that the second component depends on b but not on a. Thus, we get a formula for $F_{(2j+1)n}$ in terms of F_n alone. As an example, consider $j = 3$. Corollary 4 gives the a-simplified form of the first row of U^7 as:

$$\left(-(-1)^n bR_6S_6+a\left((-1)^{3n}+5b^2R_6R_8\right),(-1)^{3n}b+bR_6\left(2(-1)^nS_6+5b^2R_8\right)\right)$$
$$=\left(-3(-1)^{3n}b-20(-1)^{2n}b^3-25(-1)^nb^5+a\left((-1)^{3n}+30(-1)^{2n}b^2+125(-1)^nb^4+125b^6\right),\right.$$
$$\left.7(-1)^{3n}b+70(-1)^{2n}b^3+175(-1)^nb^5+125b^7\right).$$

So, if n is even, $a = F_{n-1}$, and $b = F_n$, we see:

$$F_{7n-1}=-3F_n-20F_n^3-25F_n^5+F_{n-1}(1+30F_n^2+125F_n^4+125F_n^6),$$
$$F_{7n}=7F_n+70F_n^3+175F_n^5+125F_n^7.$$

In particular, if $n = 2$, $F_{n-1}=F_1=1$ and $F_n=F_2=1$, so

$$F_{13}=-3-20-25+1+30+125+125=233$$

and

$$F_{14}=7+70+175+125=377$$

as is easy to check. Of course, there are similar formulas when n is odd.

SHORT PERIODS MODULO M

As noted earlier, it is well known that the Fibonacci sequence is purely periodic when reduced modulo an integer m. We write $k = k(m)$ to designate the period of the Fibonacci sequence modulo m. For example, consider the Fibonacci sequence and its residues modulo eight:

0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
0	1	1	2	3	5	0	5	5	2	7	1	0	1	1	2	3

The repetition of the 0–1 pair at F_{12}–F_{13} guarantees that the sequence modulo eight will repeat. Therefore, $k(8) = 12$. In general, we have

Lemma 5: The period $k = k(m)$ is the smallest positive number such that $F_k \equiv 0 \pmod{m}$ and $F_{k+1} \equiv 1 \pmod{m}$.

Proof: By definition, k is the smallest positive integer such that $F_{k+n} \equiv F_n$ for all $n \geq 0$. It is clear that this implies $F_k \equiv F_0 = 0$ and $F_{k+1} \equiv F_1 = 1$. If there is any other occurrence of these congruences, namely, $F_j \equiv 0 = F_0$ and $F_{j+1} \equiv 1 = F_1$, then by adding those equation we see $F_{j+2} \equiv F_2$ and by induction $F_{j+n} \equiv F_n$ for all $n \geq 0$. Thus, $j \geq k$ by the definition of k, and we see that k is the smallest positive number satisfying the desired congruences. □

Lemma 6: If $F_c \equiv 0$ and $F_{c+1} \equiv 1$ modulo m, then $k(m)|c$.

Proof: We can write $c = qk(m) + r$ where $0 \leq r < k(m)$. Now $F_{k+n} \equiv F_n$ modulo m implies that we can add multiples of $k(m)$ to the index and get a congruent number: $0 = F_c \equiv F_{c-qk(m)} = F_r$ and $1 = F_{c+1} \equiv F_{c+1-qk(m)} = F_{r+1}$. Since $r < k(m)$, we know by the previous lemma that $r = 0$. Hence, $c = qk(m)$ and so $k(m)$ divides c. □

The next theorems give techniques for generating many infinite families of moduli m with very small periods modulo m. The examples all have period bounded by a constant times the logarithm of the modulus. Ehrlich [4] showed that to be the case for the Fibonacci moduli; these would be given by the families below with trivial choice of $g(b) = b$.

Theorem 7: Let n be even and $g(b)$ be any polynomial that divides $bR_{2j}^0(b)$ and let $m = g(F_n)$ and $k = k(m)$. Then k divides $2jn$.

Proof: If we let $a = F_{n-1}$ and $b = F_n$ in Theorem 3, we see that since all the terms of $v(j)$ are divisible by $bR_{2j}^0(b)$ except the term $(-1)^{jn}$, we get

$$\begin{aligned}(F_{2jn-1}, F_{2jn}) = v(j) &\equiv \left((-1)^{jn}, 0\right) \mod m \\ &= (1, 0) \text{ since } n \text{ is even.}\end{aligned}$$

Thus, $F_{2jn} \equiv 0$ and $F_{2jn+1} \equiv 1$; thus, k divides $2jn$ by Lemma 6. □

Theorem 8: Let n be odd and $g(b)$ be any polynomial that divides $bR_{2j}^1(b)$ and let $m = g(F_n)$ and $k = k(m)$. Then

(i) if j is even, k divides $2jn$;

(ii) if j is odd, then k divides $4jn$.

Proof: Again we let $a = F_{n-1}$ and $b = F_n$ in Theorem 3 to get

$$(F_{2jn-1}, F_{2jn}) = v(j) \equiv \left((-1)^{jn}, 0\right) \mod m.$$

(i) If j is even, then this is (1, 0); thus, $F_{2jn} \equiv 0$ and $F_{2jn-1} \equiv F_{2jn+1} \equiv 1$; hence, k divides $2jn$ by Lemma 6.

(ii) If j is odd, then this is (-1, 0); thus, $F_{2jn} \equiv 0$ and $F_{2jn-1} \equiv 1$. In the first section we saw identities $F_{2s-1} = F_{s-1}^2 + F_s^2$ and $F_{2s} = 2F_{s-1}F_s + F_s^2$, with $s = 2jn$ we see $F_{4jn-1} \equiv (-1)^2 + 0^2 = 1$ and $F_{4jn} \equiv 0$. So $F_{4jn+1} \equiv 1$ and k divides $4jn$ by Lemma 6. □

Since m is exponential in n (becasue the Fibonacci numbers are), these theorems give examples where the periods are bounded above by a constant times the logarithm of the modulus. Lower bounds will be considered after considering some examples.

Table 2 shows the periods for moduli generated by taking $g(b) = bR_6^1(b)$ with n odd. Table 3 gives periods for even n for the corresponding polynomial.

Table 4 gives the periods for moduli near 196400. This gives some idea of how small the period $k(196418) = 108$ that also appears in Table 2 is relative to "typical" values.

TABLE 2. Periods for Moduli Generated with $g(b) = bR_6^1(b)$

n	F_n	$m = g(F_n)$	$k(m)$
1	1	2	3*
3	2	34	36
5	5	610	60
7	13	10946	84
9	34	196418	108
11	89	3524578	132
13	233	63245986	156
15	610	1134903170	180
17	1597	20365011074	204
19	4181	365435296162	228

* Period is less than the maximum allowed by the theorems.

TABLE 3. Periods for Moduli Generated with $g(b) = bR_6^0(b)$

n	F_n	$m = g(F_n)$	$k(m)$
2	1	8	12
4	3	144	24
6	8	2584	36
8	21	46368	48
10	55	832040	60
12	144	14930352	72
14	377	267914296	84
16	987	4807526976	96
18	2584	86267571272	108
20	6765	1548008755920	120

TABLE 4. Some Periods Near 196400

m	$k(m)$	m	$k(m)$
196400	29400	196413	352
196401	27720	196414	196416
196402	49416	196415	9840
196403	62028	196416	480
196404	105672	196417	364
196405	340	196418	108
196406	56112	196419	728
196407	43608	196420	240
196408	147300	196421	99216
196409	197604	196422	31032
196410	196440	196423	704
196411	12064	196424	25080
196412	98208	196425	264600

Table 5 gives periods for $g(b) = R_{10}^0(b)$ with n even. These moduli get large quickly while the periods stay small. Table 6 gives values for a nontrivial divisor of bR_{12}.

TABLE 5. Periods for Moduli Generated with $g(b) = R_{10}^{0}(b)$

n	F_n	$m = g(F_n)$	$k(m)$
2	1	55	20
4	3	2255	40
6	8	104005	60
8	21	4873055	80
10	55	228841255	100
12	89	10750060805	120
14	377	505019869255	140
16	987	23725155368255	160

TABLE 6. Periods for Moduli Generated with a Factor of $bR_{12}^{1}(b)$: $g(b) = -b + 5b^3$

n	F_n	$m = g(F_n)$	$k(m)$
1	1	4	6^*
3	2	38	18^*
5	5	620	60
7	13	10972	84
9	34	196486	108
11	89	3524756	132
13	233	63246452	156
15	610	1134904390	180

Notice that in these examples the periods $k(m)$ were exactly the quantity that Theorems 7 and 8 give as a multiple of the period except for a few small moduli in Table 2 and Table 6. In general, it appears that the bounds given in the theorems are met for sufficiently large n. While we cannot prove such a theorem, we can show that the periods cannot be much smaller than the given period for sufficiently large n for the full polynomial factors.

Lemma 9: Let τ be the golden ratio, then for $n \geq 1$ we have: $\tau^{n-2} \leq F_n \leq \tau^{n-1}$.

Proof: The theorem is true for $n = 1$ and $n = 2$ by direct computation: $\tau^{-1} < F_1 = 1 = \tau^0$, $\tau^0 = F_2 = 1 < \tau$. If it is true for n and $n - 1$, we can add inequalities to get:

$$\tau^{n-3} + \tau^{n-2} \leq F_{n-1} + F_n \leq \tau^{n-2} + \tau^{n-1}.$$

This simplifies to $\tau^{n-1} \leq F_{n+1} \leq \tau^n$, using $\tau^2 = \tau + 1$, completing the induction. □

In Lemma 9, notice that strict inequality must hold for $n \geq 3$ since F_n is an integer.

Lemma 10: Let $m \geq 2$ be a modulus and τ the golden ratio, then $k(m) > \frac{\log(m)}{\log(\tau)}$.

Proof: We can pick n so that $\tau^{n-1} < m < \tau^n$. Since $F_n < \tau^{n-1}$ it is not possible for F_j to be reduced to zero modulo m for any $j \leq n$. Therefore, $k(m) > n$. However, $m < \tau^n$ implies $n > \log(m)/\log(\tau)$ and, hence, the conclusion. □

While the upper bounds on $k(m)$ given in Theorems 7 and 8 are $2jn$ or $4jn$, we can show that for sufficiently large n that $k(m)$ is not many factors smaller than those bounds. However, we conjecture that equality holds for sufficiently large n.

Theorem 11: Suppose $g(b)$ is $R_{2j}^{e}(b)$ or $bR_{2j}^{e}(b)$ with $j \geq 3$ where e is 0 or 1. Also let $m = g(F_n)$ where n has the same parity as e and let $k = k(m)$. Then $k(m) > 0.3jn$ for sufficiently large n.

Proof: In Lemma 1 and Table 1 we see the highest-order term of $R_{2j}^{e}(b)$ is at least five times b^{j-1} or b^{j-2}. Therefore, for sufficiently large n, $m > F_n^{j-2}$. From that inequality and Lemmas 9 and 10, we see that, for sufficiently large n,

$$k(m) > \frac{\log(m)}{\log(\tau)} > \frac{(j-2)\log(F_n)}{\log(\tau)} > (j-2)(n-2) \geq \frac{3}{10}jn,$$

since $(j-2)/j \geq 1/3$ and $(n-2)/n \geq 18/20$ for $n \geq 20$. □

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SYLVESTER'S FORGOTTEN FORM OF THE RESULTANT

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1. INTRODUCTION

It is well known that Euclid's algorithm for computing the greatest common divisor (gcd) of two integer numbers is more than two thousand years old and, as it turns out, it is the oldest known algorithm. Interest in computing a gcd of two polynomials first appeared only in the sixteenth century and the problem was solved by Simon Stevin [13] simply by applying Euclid's algorithm (for integers) to polynomials with integer coefficients. However, from the computational point of view, Euclid's algorithm applied to polynomials with integer coefficients is very inefficient because of the growth of coefficients that takes place and the eventual slowdown of computations. This growth of coefficients is due to the fact that the ring $\mathbf{Z}[x]$ is not a Euclidean domain, and hence, divisions (as we know them) cannot always be performed.

For example, take the two polynomials $p_1(x) = x^3 - 7x + 7$ and $p_2(x) = 3x^2 - 7$ which have very small coefficients. Observe that, *over the integers*, we cannot divide $p_1(x)$ by $p_2(x)$ (since 3 does not divide 1) and, hence, we have to introduce the concept of *pseudo-division*, which always yields a pseudo-quotient and pseudo-remainder. According to this process, we have to premultiply $p_1(x)$ by the leading coefficient of $p_2(x)$ raised to the power 2 [that is, we premultiply $p_1(x)$ by $9 = 3^2$] and then apply our usual polynomial division algorithm. [Below we denote the leading coefficient (lc) of a polynomial $p(x)$ by $\mathrm{lc}(p(x))$ and its degree by $\deg(p(x))$.]

In the general case where $\deg(p_1(x)) = n$, and $\deg(p_2(x)) = m$, we premultiply $p_1(x)$ by $\mathrm{lc}(p_2(x))^{n-m+1}$. In this was we know for sure that all the polynomial divisions involved in the process of computing a greatest common divisor of $p_1(x)$ and $p_2(x)$ will be carried out in $\mathbf{Z}[x]$. That is, in general, we start with

$$\mathrm{lc}(p_2(x))^{n-m+1} p_1(x) = q_1(x) p_2(x) + p_3(x), \quad \deg(p_3(x)) < \deg(p_2(x)) \tag{1}$$

and applying the same process $p_2(x)$ and $p_3(x)$, and then to $p_3(x)$ and $p_4(x)$, etc. (Euclid's algorithm), we obtain a *polynomial remainder sequence* (prs)

$$p_1(x), p_2(x), p_3(x), \ldots, p_h(x), p_{h+1}(x) = 0,$$

where $p_h(x) \neq 0$ is a greatest common divisor of $p_1(x)$ and $p_2(x)$, denoted by $\gcd(p_1(x),\ p_2(x))$. The reader should compute the prs of the above example and verify that the coefficients grow rather rapidly (even when we start with such very small coefficients!!) **Answer:** $q_1(x) = 3x$, $p_3(x) = -42x + 63$, $q_2(x) = -126x - 189$, $p_4(x) = -441$, $q_3(x) = 18522x - 27783$, and $p_5(x) = 0$.

Note that we are dealing with exact integer computations and, for reasons that cannot be discussed here, the length of the integers involved is taken into consideration when we analyze the complexity of an algorithm. (Generally speaking, the complexity of an algorithm refers to the according to this number that the various algorithms are being compared for efficiency.) For an introduction to Computer Algebra, the area that deals with exact integer computations, see [3].

Therefore, the problem with the above approach is that the coefficients of the polynomials in the prs grow exponentially and, hence, slow down the computations. We wish to control this coefficient growth without having to compute gcd's of coefficients (because that in itself can be time consuming). In what follows, we use the following conventions: if $n_i = \deg(p_i(x))$ and we have $n_i - n_{i+1} = 1$, for all i, the prs is called *complete*, otherwise, it is called *incomplete*; moreover, a polynomial $p(x)$ is called *primitive* if its coefficients are relatively prime.

As we will see immediately below, using pseudo-divisions, the problem of controlling the coefficient growth was originally solved (at least partially) by Sylvester in his 1853 paper and fully by Habicht in 1948. Equivalently, as we will see in §2, the problem can be solved by triangularizing the matrix corresponding to what we call Sylvester's form of the resultant (and which form is different from the one people are used to), thus avoiding explicit polynomial pseudo-divisions. It turns out that Sylvester's paper of 1853 is the basis for both classical methods to restrict the coefficient growth (see Figure 1 below) and, thus, we have one more case indicating the importance of mathematics of the last century, and its connection with computational mathematics as done today.

$\leftarrow$ **Sylvester's paper of 1853** $\rightarrow$	
$\downarrow$	$\downarrow$
$\downarrow$	$\downarrow$
The pseudo-divisions subresultant prs method initiated by Sylvester in 1853 and completed by Habicht in 1948.	The matrix-triangularization subresultant prs method developed by the author in 1986.

FIGURE 1.

Overview of the historical development of the two classical subresultant prs methods for restricting the growth of coefficients. The method developed by Sylvester should be used only when the prs is complete, whereas the one by Habicht should be used when the prs is incomplete, something which we do not know apriori. (Actually, Habicht's method also can be used when the prs is complete, at additional computational cost.) The matrix-triangularization method can be identically used for both kinds of prs's.

To see how Sylvester's results can be applied in the pseudo-divisions method, observe that (1) can also be written, for any two successive polynomials $p_i(x)$ and $p_{i+1}(x)$ of the prs, as

$$\mathrm{lc}(p_{i+1}(x))^{n_i - n_{i+1}+1} p_i(x) = q_i(x)p_{i+1}(x) + \beta_i p_{i+2}(x), \ \deg(p_{i+2}(x)) < \deg(p_{i+1}(x)), \tag{2}$$

$i = 1, 2, \ldots, h-1$, where β_i is the integer which we want to divide out of the coefficients of $p_{i+2}(x)$. That is, if a method for choosing β_i is given, the above equation provides an algorithm for constructing a prs. The obvious choice $\beta_i = 1$, for all i, is called the *Euclidean prs*; it was described above and, as we saw, it leads to exponential growth of coefficients. Next, choosing β_i to be the greatest common divisor of the coefficients $p_{i+2}(x)$ results in the *primitive prs*, and it is the best that can be done to control the coefficient growth. (Notice that here we are dividing $p_{i+2}(x)$ by the greatest common divisor of its coefficients before we use it again.) However, as we indicated above, computing the gcd of the coefficients for each member of the prs (after the first two, of course) is an expensive operation and should be avoided.

So far, in order both to control the coefficient growth and to avoid the coefficient gcd computations, either the *reduced* or the (improved) *subresultant* prs have been used. In the reduced prs (developed by Sylvester) we choose

$$\beta_1 = 1 \text{ and } \beta_i = \mathrm{lc}(p_i(x))^{n_{i-1}-n_i+1}, \; i = 2, 3, \ldots, h-1, \tag{3}$$

whereas, in the subresultant prs (developed by Habicht) we have

$$\beta_1 = (-1)^{n_1-n_2+1} \text{ and } \beta_i = (-1)^{n_i-n_{i+1}+1}\mathrm{lc}(p_i(x))H_i^{n_i-n_{i+1}}, \; i = 2, 3, \ldots, h-1, \tag{4}$$

where

$$H_2 = \mathrm{lc}(p_2(x))^{n_1-n_2} \quad \text{and} \quad H_i = \mathrm{lc}(p_i(x))^{n_{i-1}-n_i} H_{i-1}^{1-(n_{i-1}-n_i)}, \; i = 3, 4, \ldots, h-1.$$

That is, in both cases above, we divide $p_{i+2}(x)$ by the corresponding β_i before we use it again.

Consider again the above-stated example where we are dealing with a complete prs and, hence, (3) and (4) yield exactly the same results [note that, using (4), we have to perform some extra computations]; the reader should verify that, in both cases, we obtain $\beta_1 = 1$ and, hence, $p_3(x) = -42x + 63$ whereas $\beta_2 = 9$ and, hence, $p_4(x) = -49\ (= 441/9)$ instead of $p_4(x) = -441$ obtained before. Note that, with this approach, we were able to reduce the coefficients of $p_4(x)$, but there is no way to reduce the coefficients of $p_3(x)$!

The reduced prs algorithm is recommended if the prs is complete, whereas if the prs is incomplete the subresultant prs algorithm is to be preferred. The proofs that the β_i's shown in (3) and (4) exactly divide $p_{i+2}(x)$ *were* very complicated [7] and have up to now obscured simple divisibility properties [13] (see also [5] and [6]). For a simple proof of the validity of the reduced prs, see [1]; analogous proof for the subresultant prs can be found in [10] and [3]. A very simple proof of Habicht's theorem can be found in the recent work of Gonzalez et al. [9]. For some interesting comments regarding priority rights for the development of these prs algorithms see [11] and Historical Notes to Chapter 5 in [3, p. 282].

In contrast to the above prs algorithms, the matrix-triangularization subresultant prs method avoids explicit polynomial divisions. In what follows, we present this method and show how to solve the example mentioned above.

2. SYLVESTER'S FORGOTTEN FORM OF THE RESULTANT

Consider the two polynomials in $\mathbf{Z}[x]$, $p_1(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ and $p_2(x) = d_m x^m + d_{m-1}x^{m-1} + \cdots + d_0, c_n \neq 0, d_m \neq 0, n \geq m$. In the literature, the most commonly encountered forms of the resultant of $p_1(x)$ and $p_2(x)$ (both known as "Sylvester's" forms) are:

$$\operatorname{res}_B(p_1(x), p_2(x)) = \begin{vmatrix} c_n & c_{n-1} & \cdots & & c_0 & 0 & \cdots & 0 \\ 0 & c_n & c_{n-1} & \cdots & & c_0 & \cdots & 0 \\ & & & & & \vdots & & \\ 0 & 0 & \cdots & & c_n & c_{n-1} & \cdots & c_0 \\ d_m & d_{m-1} & \cdots & d_0 & 0 & 0 & \cdots & 0 \\ 0 & d_m & d_{m-1} & \cdots & d_0 & 0 & \cdots & 0 \\ & & & & \vdots & & & \\ 0 & 0 & \cdots & d_m & d_{m-1} & & \cdots & d_0 \end{vmatrix}$$

and

$$\operatorname{res}_T(p_1(x), p_2(x)) = \begin{vmatrix} c_n & c_{n-1} & \cdots & & c_0 & 0 & \cdots & 0 \\ 0 & c_n & c_{n-1} & \cdots & & c_0 & \cdots & 0 \\ & & & & & \vdots & & \\ 0 & 0 & \cdots & & c_n & c_{n-1} & \cdots & c_0 \\ 0 & 0 & \cdots & d_m & d_{m-1} & & \cdots & d_0 \\ & & & & \vdots & & & \\ 0 & d_m & d_{m-1} & \cdots & d_0 & 0 & \cdots & 0 \\ d_m & d_{m-1} & \cdots & d_0 & 0 & 0 & \cdots & 0 \end{vmatrix}$$

where in both cases we have m rows of c's and n rows of d's; that is, the determinant is of order $m+n$. Contrary to established practice, we call the first di Bruno's and the second Trudi's form of the resultant [3] (di Bruno was sanctified by the Roman Catholic Church in the 1980s). Notice that $\operatorname{res}_B(p_1(x), p_2(x)) = (-1)^{n(n-1)/2} \operatorname{res}_T(p_1(x), p_2(x))$. For these two forms of the resultant, the following theorem holds.

Theorem 1 (Laidacker [12]): If we transform the matrix corresponding to $\operatorname{res}_B(p_1(x), p_2(x))$ into its upper triangular form $T_B(R)$ using row transformations only, then the last nonzero row of $T_B(R)$ gives the coefficients of a greatest common divisor off $p_1(x)$ and $p_2(x)$.

Theorem 1 indicates that using these forms of the resultant we can obtain only a greatest common divisor of $p_1(x)$ and $p_2(x)$ but, in general, none of the remainder polynomials.

In order to compute both a $\gcd(p_1(x), p_2(x))$ and all the polynomial remainders we have to use Sylvester's form of the resultant. We choose to call Sylvester's form the one described below; this form was "buried" in Sylvester's 1853 paper [14] and is only once mentioned in the literature in a paper by Van Vleck [15]. Sylvester indicates [14, p. 426] that he had produced this form in 1839 or 1840 and some years later Cayley unconsciously reproduced it as well. This form is of order $2n$ (as opposed to $n + m$ for the other two forms) and can be written as follows [$p_2(x)$ has now been transformed into a polynomial of degree n by introducing zero coefficients]:

$$\operatorname{res}_S(p_1(x), p_2(x)) = \begin{vmatrix} c_n & & c_{n-1} & \cdots & c_0 & 0 & 0 & \cdots & 0 \\ d_n & & d_{n-1} & \cdots & d_0 & 0 & 0 & \cdots & 0 \\ 0 & & c_n & \cdots & & c_0 & 0 & \cdots & 0 \\ 0 & & d_n & \cdots & & d_0 & 0 & \cdots & 0 \\ & & & \vdots & & & & & \\ 0 & \cdots & 0 & c_n & & c_{n-1} & & \cdots & c_0 \\ 0 & \cdots & 0 & d_n & & d_{n-1} & & \cdots & d_0 \end{vmatrix} \tag{S}$$

Sylvester obtained this form from the system of equations [14, pp. 427-28]

$$\begin{aligned} p_1(x) &= 0 \\ p_2(x) &= 0 \\ x \cdot p_1(x) &= 0 \\ x \cdot p_2(x) &= 0 \\ x^2 \cdot p_1(x) &= 0 \\ x^2 \cdot p_2(x) &= 0 \\ &\cdots \\ x^{n-1} \cdot p_1(x) &= 0 \\ x^{n-1} \cdot p_2(x) &= 0 \end{aligned}$$

and he indicated that if we take k pairs of the above equations, the highest power of x appearing in any of them will be x^{n+k-1}. Therefore, we shall be able to eliminate so many powers of x that x^{n-k} will be the highest power uneliminated and $n - k$ will be the degree of a member of the Sturmian polynomial remainder sequence generated by $p_1(x)$ and $p_2(x)$. Moreover, Sylvester showed that the polynomial remainders thus obtained are what he terms *simplified residues*; that is, the coefficients are the smallest possible obtained *without integer gcd computations and without introducing rationals.* Stated in Sylvester's words, the polynomial remainders have been freed from their corresponding *allotrious factors.*

It has been proved [15] that if we want to compute the *complete* polynomial remainder sequence $p_1(x), p_2(x), p_3(x), \ldots, p_h(x), \deg(p_1(x)) = n, \deg(p_2(x)) = m, n \geq m,$ we can obtain the (negated) coefficients of the $(i+1)^{\text{th}}$ member of the prs, $i = 0, 1, 2, \ldots, h-1$, as minors formed from the first $2i$ rows of (S) by successively associating with the first $2i - 1$ columns [of the $(2i)$ by $(2n)$ matrix] each succeeding column in turn.

However, instead of proceeding as in [15], and in order to handle incomplete prs's as well, we transform the matrix corresponding to the resultant (S) into its upper triangular form and obtain the members of the prs with the help of Theorem 2 below. We also use Dodgson's integer-preserving transformation algorithm [8], which works as follows: Suppose that

$$r_{ij}^{(0)} = r_{ij}, \qquad i, j = 1, \ldots, n$$

are the matrix elements at the beginning of the algorithm (0^{th} iteration). There are n iterations performed, and in the k^{th} one (indicated here) the following actions are taken: (a) the elements of the k^{th} column located below the k^{th} (diagonal) element are being turned to zero, (b) all the elements located in rows *and* columns greater than k get updated as shown below, and (c) all the other elements of the matrix remain unchanged. In this way, at the end of the process, all the elements of the matrix located below the diagonal are zero. That is, we have: let

$$r_{00}^{(-1)} = 1, \quad \text{and} \quad r_{ij}^{(0)} = r_{ij}, \quad i, j = 1, \ldots, n;$$

then for $k < i, j \leq n$,

$$r_{ij}^{(k)} := \left(1/r_{k-1,k-1}^{(k-2)}\right) \cdot \begin{vmatrix} r_{kk}^{(k-1)} & r_{kj}^{(k-1)} \\ r_{ik}^{(k-1)} & r_{ij}^{(k-1)} \end{vmatrix}. \tag{D}$$

Of particular importance in Dodgson's algorithm is the fact that the determinant of order 2 is divided *exactly* by $r_{k-1,k-1}^{(k-2)}$ (a very short and clear proof of (D) is described in Bareiss's paper [4]—see also the Historical note at the end of this paper) and that the resulting coefficients are the smallest that can be expected without coefficient gcd computations and without introducing rationals. Notice how all the complicated expressions for β_i in the reduced and subresultant prs algorithms are mapped to the simple factor $r_{k-1,k-1}^{(k-2)}$ of this method.

It should be pointed out that using Dodgson's algorithm (D) we will have to perform pivots (interchange two rows) which will result in a change of signs. We define the term *bubble* pivot as follows: if the diagonal element in row i is zero and the next nonzero element down the column is in row $i+j$, $j>1$, then row $i+j$ will beccome row i after pairwise interchanging it with the rows above it. (Note that, after a bubble pivot, ex-row i becomes row $i+1$, whereas with regular pivot it would have become row $i+j$.) Bubble pivot preserves the symmetry of the determinant.

The following theorem helps us locate the members of the (complete or incomplete) prs in the final, triangularized, matrix.

Theorem 2 ([2]): Let $p_1(x)$ and $p_2(x)$ be two polynomials of degrees n and m, respectively, $n \geq m$. Then, using Dodgson's algorithm (D), transform the matrix corresponding to $\operatorname{res}_S(p_1(x), p_2(x))$ into its upper triangular form $T_S(R)$; let n_i be the degree of the polynomial corresponding to the i^{th} row of $T_S(R)$, $i = 1, 2, \ldots, 2n$, and let $p_k(x)$, $k \geq 2$, be the k^{th} member of the (*complete or incomplete*) polynomial remainder sequence of $p_1(x)$ and $p_2(x)$. Then if $p_k(x)$ is in row i of $T_S(R)$, the coefficients of $p_{k+1}(x)$ (within sign) are obtained from row $i+j$ of $T_S(R)$, where j is the smallest integer such that $n_{i+j} < n_i$. [If $n = m$, associate both $p_1(x)$ and $p_2(x)$ with the first row of $T_S(R)$.]

Therefore, we see that, based on Theorem 2, we have a new method for computing the polynomial remainder sequence and a greatest common divisor of two polynomials. This new method uniformly treats both complete and incomplete prs's and provides the smallest coefficients that can be expected without coefficient gcd computation.

3. THE MATRIX-TRIANGULARIZATION SUBRESULTANT PRS METHOD

The inputs are two (primitive) polynomials in **Z**[x], $p_1(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ and $p_2(x) = d_m x^m + d_{m-1}x^{m-1} + \cdots + d_0, c_n \neq 0, d_m \neq 0, n \geq m$.

Step 1: Form the resultant (S), $\mathrm{res}_S(p_1(x), p_2(x))$, of the two polynomials $p_1(x)$ and $p_2(x)$.

Step 2: Using Dodgson's algorithm (D) (and bubble pivot), transform the matrix corresponding to the resultant (S) into its upper triangular form $T_S(R)$; then the coefficients of all the members of the polynomial remainder sequence of $p_1(x)$ and $p_2(x)$ are obtained from the rows of $T_S(R)$ with the help of Theorem 2.

The computing time of this method is given by the following theorem (see [2]).

Theorem 3: Let $p_1(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ and $p_2(x) = d_m x^m + d_{m-1}x^{m-1} + \cdots + d_0$, $c_n \neq 0$, $d_m \neq 0, n \geq m$, be two (primitive) polynomials in **Z**[x] and, for some polynomial $P(x)$ in **Z**[x] let $|P|_\infty$ represent its maximum coefficient in absolute value. Then the method described above computes a greatest common divisor of $p_1(x)$ and $p_2(x)$ along with all the polynomial remainders in time $0(n^5 L(|p|_\infty)^2)$ where $|p|_\infty = \max(|p_1|_\infty, |p_2|_\infty)$ and $L(|p|_\infty)$ is the length, in bits (or even the logarithm) of the maximum coefficient (of the two polynomials) in absolute value.

Proof: The result follows by combining (a) the well-known result that in the matrix-triangularization procedure there are performed $0(n^3)$ multiplications and (b) the fact that we are now using exact integer arithmetic and, hence, each multiplication is executed in time $0(n^2 L(|p|_\infty)^2)$ (see [2] and [3]). □

Below, we present the example stated in the introduction solved using this new approach; the reader should observe that the coefficients obtained for $p_3(x)$ are smaller than those obtained using the reduced (or the improved, for that matter) subresultant prs algorithm.

Example: Let us find the polynomial remainder sequence of the polynomials $p_1(x) = x^3 - 7x + 7$ and $p_2(x) = 3x^2 - 7$ using the matrix-triangularization procedure described above. Below, the matrix on the left side is the starting one, and the one on the right side is the final (transformed) one, obtained after application of Dodgson's method (D).

$$\begin{bmatrix} 1 & 0 & -7 & 7 & 0 & 0 \\ 0 & 3 & 0 & -7 & 0 & 0 \\ 0 & 1 & 0 & -7 & 7 & 0 \\ 0 & 0 & 3 & 0 & -7 & 0 \\ 0 & 0 & 1 & 0 & -7 & 7 \\ 0 & 0 & 0 & 3 & 0 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -7 & 7 & 0 & 0 \\ 0 & 3 & 0 & -7 & 0 & 0 \\ 0 & 0 & 9 & 0 & -21 & 0 \\ 0 & 0 & 0 & -42 & 63 & 0 \\ 0 & 0 & 0 & 0 & 196 & -294 \\ 0 & 0 & 0 & 0 & 0 & -49 \end{bmatrix} \begin{matrix} \\ \\ \\ * \\ \\ \\ \end{matrix}$$

The *-ed row indicates that a (normal) pivot was performed between the third and fourth rows. With the help of Theorem 2 we see, from the transformed matrix, that the polynomial remainders (within sign) are $p_3(x) = -42x + 63$ and $p_4(x) = -49$ (as obtained before); also note that, using this approach, there is no way for us to obtain the quotients. The smaller coefficients for $p_3(x)$ are obtained if we save the row before picot; in our example, the row before pivot was $p_3(x) = -14x+21$, which was then changed to $p_3(x) = -42x+63$. Thus, the remainder polynomials are $p_3(x) = -14x+21$ and $p_4(x) = -49$ and, in this case, we did manage to reduce the coefficients of $p_3(x)$!

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Historical Note: Note that we depart from established practice and give credit to Dodgson—and not to Bareiss [4]—for the integer-preserving transformations; see also the work of Waugh and Dwyer [16] where they use the same method as Bareiss, but 23 years earlier, and they name Dodgson as their source—differing from him only in the choice of the pivot element [16, p. 266]. Charles Lutwidge Dodgson (1832-1898) is the same person widely known for his writing *Alice in Wonderland* under the pseydonym Lewis Carroll.

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ON THE STRUCTURE OF THE SET OF DIFFERENCE SYSTEMS DEFINING (3, *F*) GENERALIZED FIBONACCI SEQUENCES

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The $(2, F)$ generalized Fibonacci sequences were defined in [1] and [2]. In [3] K. Atanassov extended the definition to the case of three sequences and listed thirty-six systems defining the $(3, F)$ generalized Fibonacci sequences. Ten of these thirty-six systems were discarded as trivial and the remaining twenty-six were placed in seven classes termed "groups." In this paper the structure of the systems of three second-order difference equations defining the $(3, F)$ generalized Fibonacci sequences is developed. This development is based on the following definitions of the permutations on the letters *a*, *b*, and *c*:

$$i:\ \begin{array}{l} a\to a\\ b\to b\\ c\to c\end{array}\qquad \alpha:\ \begin{array}{l} a\to b\\ b\to a=(ab)\\ c\to c\end{array}\qquad \beta:\ \begin{array}{l} a\to c\\ b\to b=(ac)\\ c\to a\end{array}$$

$$\gamma:\ \begin{array}{l} a\to a\\ b\to c=(bc)\\ c\to b\end{array}\qquad \delta:\ \begin{array}{l} a\to c\\ b\to a=(acb)\\ c\to b\end{array}\qquad \varepsilon:\ \begin{array}{l} a\to b\\ b\to c=(abc)\\ c\to a\end{array}$$

Note that $\delta=\alpha\beta$ where $\alpha\beta$ indicates that the permutation β is applied first, followed by α. Similarly, $\varepsilon=\beta\alpha$.

These definitions give rise to the following multiplication table for the six permutations:

$\cdot$	i	α	β	γ	δ	ε
i	i	α	β	γ	δ	ε
α	α	i	δ	ε	β	γ
β	β	ε	i	δ	γ	α
γ	γ	δ	ε	i	α	β
δ	δ	γ	α	β	ε	i
ε	ε	β	γ	α	i	δ

The six permutations of the letters *a*, *b*, and *c* form a group which is isomorphic to the symmetric group S_3. The group of permutations of the letters *a*, *b*, and *c* will be denoted by S_c.

Using these preliminaries, the $(3, F)$ generalizations of the Fibonacci sequence may be defined.

Definition: Let C_i, $1\le i\le 6$, be six real numbers; $X_0=\{a_0,b_0,c_0\}=\{C_1,C_2,C_3\}$; $X_1=\{a_1,b_1,c_1\}$ $=\{C_4,C_5,C_6\}$; and let ρ, σ, and τ be permutations of S_c. Then the solutions

$$X=\langle X_i\rangle_0^\infty=\langle\{a_i,b_i,c_i\}\rangle_0^\infty=\{\langle a_i\rangle_0^\infty,\langle b_i\rangle_0^\infty\ \langle c_i\rangle_0^\infty\}$$

of the difference system

$$\rho X_{i+2} = \sigma X_{i+1} + \tau X_i, \quad i \geq 0, \tag{1}$$

with initial conditions X_0, X_1, are the $(3, F)$ generalizations of the Fibonacci sequence. Since there are six permutations in S_c, there are a total of 216 systems of form (1). The systems of form (1) can be represented by ordered triples of permutations of S_c. Thus,

$$(\rho, \sigma, \tau) \text{ represents } \rho X_{i+2} = \sigma X_{i+1} + \tau X_i, \quad i \geq 0.$$

Consequently, the triple (i, δ, ε) represents the equations

$$\begin{aligned} a_{i+2} &= c_{i+1} + b_i \\ b_{i+2} &= a_{i+1} + c_i, \quad i \geq 0, \\ c_{i+2} &= b_{i+1} + a_i \end{aligned}$$

which is S_{30} in Atanassov [3]. Two different systems (ρ, σ, τ) and (ρ', σ', τ') may not define distinct $(3, F)$ sequences. For example, with given initial conditions X_0, X_1, the system

$$(\varepsilon, i, \delta) = \begin{cases} b_{i+2} = a_{i+1} + c_i \\ c_{i+2} = b_{i+1} + a_i, \quad i \geq 0, \\ a_{i+2} = c_{i+1} + b_i \end{cases}$$

defines the same sequence as $S_{30} = (i, \delta, \varepsilon)$ since the same equations determine the successive terms of the sequences. Observe that the two systems (i, δ, ε) and (ε, i, δ) are row equivalent. In general, two systems (ρ, σ, τ) and (ρ', σ', τ') are row equivalent if and only if one system can be obtained from the other by multiplication of the permutations of the other system by the same permutation. That is,

Definition: Let $\rho, \sigma, \tau, \rho', \sigma', \tau'$ be six permutations of S_c. Then the systems (ρ, σ, τ) and (ρ', σ', τ') are row equivalent if there exists a permutation η in S_c such that $\eta(\rho, \sigma, \tau) = (\eta\rho, \eta\sigma, \eta\tau) = (\rho', \sigma', \tau')$.

Since there are six permutations in S_c, there are six systems that are row equivalent to a given system (ρ, σ, τ). Thus, the 216 systems are partitioned into thirty-six equivalence classes of row equivalent systems which are the systems considered by Atanassov in [3]. For example, the systems S_{30} and S_{22} of Atanassov are

$$[S_{30}] = [(i, \delta, \varepsilon)] = \{(i, \delta, \varepsilon), (\varepsilon, i, \delta), (\alpha, \beta, \gamma), (\gamma, \alpha, \beta), (\beta, \gamma, \alpha), (\delta, \varepsilon, i)\}, \text{ and}$$

$$[S_{22}] = [(i, \alpha, \beta)] = \{(i, \alpha, \beta), (\alpha, i, \delta), (\beta, \varepsilon, i), (\gamma, \delta, \varepsilon), (\delta, \gamma, \alpha), (\varepsilon, \beta, \gamma)\},$$

where $[(\rho, \sigma, \tau)]$ indicates the equivalence class of (ρ, σ, τ). Since each equivalence class contains one system that has the identity as the first permutation, the classes may be uniquely represented by an ordered pair of permutations (ϕ, ψ) where ϕ and ψ are permutations of S_c.

A relation is now defined on the equivalence classes of row equivalent systems.

Definition: Let ϕ, ψ, ϕ', and ψ' be permutations of S_c. The ordered pair (ϕ, ψ) is equivalent to the ordered pair (ϕ', ψ'), written $(\phi, \psi) \equiv (\phi', \psi')$, if there exists a η in S_c such that $\phi' = \eta\phi\eta^{-1}$ and $\psi' = \eta\psi\eta^{-1}$.

Since $\phi = i\phi i^{-1}$ and $\psi = i\psi i^{-1}$, the relation is reflexive. Suppose $(\phi, \psi) \equiv (\phi', \psi')$. Then, for some μ and μ^{-1} in S_c, $\phi' = \mu\phi\mu^{-1}$ and $\psi' = \mu\psi\mu^{-1}$. Therefore, for $\eta = \mu^{-1}, \phi = \eta\phi'\eta^{-1}$ and $\psi = \eta\psi'\eta^{-1}$. Hence, the relation is symmetric. Suppose $(\phi, \psi) \equiv (\phi', \psi')$ and $(\phi', \psi') \equiv (\phi'', \psi'')$. Then, for some η and μ in S_c, $\phi' = \eta\phi\eta^{-1}, \psi' = \eta\psi\eta^{-1}, \phi'' = \mu\phi'\mu^{-1}$, and $\psi'' = \mu\psi'\mu^{-1}$. Consequently, $\phi'' = \mu\phi'\mu^{-1} = \mu\eta\phi\eta^{-1}\mu^{-1} = \rho\phi\rho^{-1}$ and $\psi'' = \mu\psi'\mu^{-1} = \mu\eta\psi'\eta^{-1}\mu^{-1} = \rho\psi\rho^{-1}$ for $\rho = \mu\eta$. Hence, by definition, $(\phi, \psi) \equiv (\phi'', \psi'')$, and the relation is transitive. Thus, the relation is an equivalence relation. The definition of equivalent systems requires that there exists η in S_c such that ϕ and ϕ', ψ and ψ' belong to the same conjugate classes for that η.

It is well known that the conjugate classes of S_3 are the permutations with the same cycle structure (see [6]). Since S_c is isomorphic to S_3, the conjugate classes of S_c are: $C\{i\} = \{i\}$, $C\{\alpha\} = \{\alpha, \beta, \gamma\}$, $C\{\delta\} = \{\delta, \varepsilon\}$, where $C\{\sigma\}$ denotes the conjugate class of σ. Let $\overline{(\phi, \psi)}$ denote the equivalence class of (ϕ, ψ). If ϕ and ψ belong to different conjugate classes, then the recursion systems in the equivalence class $\overline{(\phi, \psi)}$ are the ordered pairs with ϕ a member of $C\{\phi\}$ and ψ a member of $C\{\psi\}$. Thus, there is one equivalence class for each pair of conjugate classes in S_c. The classes and the corresponding schemes of Atanassov [3] are:

$$\overline{(i, \alpha)} = \{(i, \alpha), (i, \beta), (i, \gamma)\} = \{S_5, S_{10}, S_2\},$$
$$\overline{(i, \delta)} = \{(i, \delta), (i, \varepsilon)\} = \{S_9, S_6\},$$
$$\overline{(\alpha, i)} = \{(\alpha, i), (\beta, i), (\gamma, i)\} = \{S_{13}, S_{27}, S_3\},$$
$$\overline{(\delta, i)} = \{(\delta, i), (\varepsilon, i)\} = \{S_{25}, S_{15}\},$$
$$\overline{(\alpha, \delta)} = \{(\alpha, \delta), (\beta, \delta), (\gamma, \delta), (\beta, \varepsilon), (\gamma, \varepsilon)\} = \{S_{21}, S_{35}, S_{11}, S_{18}, S_{32}, S_8\},$$
$$\overline{(\delta, \alpha)} = \{(\delta, \alpha), (\delta, \beta), (\delta, \gamma), (\varepsilon, \alpha), (\varepsilon, \beta), (\varepsilon, \gamma)\} = \{S_{29}, S_{34}, S_{26}, S_{19}, S_{24}, S_{16}\}.$$

If ϕ and ψ belong to the same conjugate class of S_c, and $\phi = \psi$, then ϕ' and ψ' must also belong to the same conjugate class and $\phi' = \psi'$. Consequently, there are as many classes of this type as there are conjugate classes in S_c, namely, three. Moreover, there are as many systems in each class as there are permutations in $C\{\phi\}$. The classes of this type are:

$$\overline{(i, i)} = \{(i, i)\} = \{S_1\},$$
$$\overline{(\alpha, \alpha)} = (\alpha, \alpha), (\beta, \beta), (\gamma, \gamma)\} = \{S_{17}, S_{36}, S_4\},$$
$$\overline{(\delta, \delta)} = \{(\delta, \delta), (\varepsilon, \varepsilon)\} = \{S_{33}, S_{20}\}.$$

If ϕ and ψ belong to the same conjugate class, but $\phi \neq \psi$, then ϕ' and ψ' are distinct and also belong to the same conjugate class. There are as many equivalence classes of this type as there are conjugate classes in S_c which contain at least two distinct permutations, namely, two. There are as many systems in each class as there are combinations of distinct permutations in one conjugate class. The systems of this type are:

$$\overline{(\alpha,\beta)} = \{(\alpha,\beta),(\alpha,\gamma),(\beta,\alpha),(\beta,\gamma),(\gamma,\alpha),(\gamma,\beta)\} = \{S_{22}, S_{14}, S_{31}, S_{28}, S_7, S_{12}\},$$
$$\overline{(\delta,\varepsilon)} = \{(\delta,\varepsilon),(\varepsilon,\delta)] = \{S_{30}, S_{23}\}.$$

Hence, the thirty-six systems defined by Atanassov [3] belong to eleven equivalence classes as listed above.

Theorem: Let (ϕ,ψ) and (ϕ',ψ') be two systems, let $\langle X_i\rangle_0^\infty$ and $\langle Z_i\rangle_0^\infty$ be solutions to (ϕ,ψ) and (ϕ',ψ'), respectively, and let $\eta X_0 = Z_0$ and $\eta X_1 = Z_1$. then (ϕ,ψ) and (ϕ',ψ') are equivalent systems if and only if $\langle \eta X_i\rangle_0^\infty = \langle Z_i\rangle_0^\infty$.

Proof: Suppose (ϕ,ψ) and (ϕ',ψ') are equivalent systems. Then $Z_2 = \phi' Z_1 + \psi' Z_0$. Since the systems are equivalent, for some η in S_c,

$$\begin{aligned} Z_2 &= \eta\phi\eta^{-1}Z_1 + \eta\psi\eta^{-1}Z_0 = \eta\phi X_1 + \eta\psi X_0 \\ &= \eta(\phi X_1 + \psi X_0) = \eta X_2. \end{aligned}$$

The theorem is true for $i = 2$. Assume that it is true for all $k \le n$ for some integer $n \ge 2$: $Z_{n+1} = \phi' Z_n + \psi' Z_{n-1}$. So again, since the systems are equivalent,

$$\begin{aligned} Z_{n+1} &= \eta\phi'\eta^{-1}Z_n + \eta\psi'\eta^{-1}Z_{n-1} \\ &= \eta\phi X_n + \eta\psi X_{n-1} = \eta X_{n+1}, \end{aligned}$$

the theorem holds for all $i \ge 0$.

Now assume that $\langle \eta X_i\rangle_0^\infty = \langle Z_i\rangle_0^\infty$. Then

$$X_{i+2} = \phi X_{i+1} + \psi X_i, \quad i \ge 0.$$

Since $\eta^{-1}Z_i = X_i$ for all a $i \ge 0$,

$$\eta^{-1}Z_{i+2} = \phi\eta^{-1}Z_{i+1} + \psi\eta^{-1}Z_i,$$

which is row equivalent to

$$\eta\eta^{-1}Z_{i+2} = \eta\phi\eta^{-1}Z_{i+1} + \eta\phi\eta^{-1}Z_i = Z_{i+2}, \quad i \ge 0.$$

But. $Z_{i+2} = \phi' Z_{i+1} + \psi' Z_i$ for $i \ge 0$. Therefore, $\phi' = \eta\phi\eta^{-1}$ and $\psi' = \eta\psi\eta^{-1}$, and the systems are equivalent. Thus, the theorem holds. As a result of the above theorem, only one system in each equivalence class need be solved since the solutions to the systems in an equivalence class are related to each other by a permutation of S_c. Therefore, all $(3, F)$ generalized Fibonacci sequences are determined by solving eleven systems. Furthermore, four of these systems, namely, $\overline{(i,i)}, \overline{(i,\alpha)}, \overline{(\alpha,i)}$, and $\overline{(\alpha,\alpha)}$, can be written in terms of generalized Fibonacci sequences and

$(2, F)$ generalized Fibonacci sequences. Generalized Fibonacci sequences are discussed in [4] and the $(2, F)$ generalized Fibonacci sequences are developed in [1], [2], [5], and [7]. Consequently, only seven new systems need to be solved in order to generate the solutions to all eleven equivalence classes.

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SUMS OF UNIT FRACTIONS HAVING LONG CONTINUED FRACTIONS

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Let

$$[x, y, \ldots, z] = x + \cfrac{1}{y + \ddots + \cfrac{1}{z}}$$

be a simple continued fraction. In the representation of an element of $\mathbb{Q}\setminus\mathbb{Z}$ as a simple continued fraction, we normalize by $z \geq 2$. A "unit fraction" $\frac{1}{a} = [0, a]$ for $a \geq 2$ is a very short, simple continued fraction. One may ask if the sum, difference of two unit fractions (with relatively prime denominators) can have arbitrarily many terms in its simple continued fraction. We use the Fibonacci numbers to show that the answer is "yes."

Letting $\frac{1}{F_n} + \frac{1}{F_{n+1}^2} = [0, F_n - 1, \rho_n]$ for $n > 2$, we find $\rho_n \to \phi = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \ldots]$ as $n \to \infty$. Letting $\frac{1}{F_n} - \frac{1}{F_{n+1}^2} = [0, F_n, \sigma_n]$ for $n > 1$, we find that $\sigma_n \to 1 + \phi$ as $n \to \infty$.
The rates of convergence can easily be estimated. For this, instead of using $\frac{1}{a} \pm \frac{1}{b^2}$, it is better to use

$$\frac{1}{a} - \frac{1}{b^2 + a} = \left[0, a, \frac{b^2}{a^2}\right] \text{ and } \frac{1}{a} + \frac{1}{b^2 + a^2 - a} = \left[0, a-1, 1, \frac{b^2}{a^2}\right] \tag{1}$$

where $2 \leq a < b$. Starting with $\frac{F_3}{F_2} = 2 = [2]$ and $\frac{F_4}{F_3} = \frac{3}{2} = [1, 2]$, it is easy to show by induction that

$$\frac{F_{n+1}}{F_n} = [\underbrace{1, \ldots, 1}_{n-2}, 2] \text{ for } n > 1. \tag{2}$$

Lemma: For $n \geq 4$, we have

$$\frac{F_{n+1}^2}{F_n^2} = \left[\underbrace{2, 1, \ldots, 1}_{n-3}, 3, \frac{F_{n-1}}{F_{n-2}}\right]. \tag{3}$$

Proof: Subtracting 2 from both sides of (3) gives, equivalently,

$$\frac{F_n^2}{F_{n+1}^2 - 2F_n^2} = [\underbrace{1, \ldots, 1}_{n-3}, B \text{ where } B = \left[3, \frac{F_{n-1}}{F_{n-2}}\right].$$

But

$$[\underbrace{1, \ldots, 1}_{n-3}, B = \frac{BF_{n-2} + F_{n-3}}{BF_{n-3} + F_{n-4}}.$$

Multiplying the right-hand side by $\frac{F_{n-1}}{F_{n-1}}$ and substituting for B, the numerator turns out to be F_n^2 and the denominator is $F_{n+1}^2 - 2F_n^2$, as it should be.

Letting $a = F_n, b = F_{n+1}$, we have $(a, b^2 + a) = 1$ and $(a, b^2 + a^2 - a) = 1$. Using (3), (2), and (1), we have

$$\frac{1}{F_n} - \frac{1}{F_{n+1}^2 + F_n} = [0, F_n, 2, \underbrace{1, \ldots, 1}_{n-3}, 3, \underbrace{1, \ldots, 1}_{n-4}, 2] \text{ for } n > 3 \tag{4}$$

and

$$\frac{1}{F_n} + \frac{1}{F_{n+1}^2 + F_n^2 - F_n} = [0, F_n - 1, 1, 2, \underbrace{1, \ldots, 1}_{n-3}, 3, \underbrace{1, \ldots, 1}_{n-4}, 2] \text{ for } n > 3. \tag{5}$$

For any value of $n \geq 4$, (4) has $2n - 2$ terms and (5) has $2n - 1$ terms. Instead of (1), we could use [Dr. Göttsch, private communication]

$$\frac{1}{b} - \frac{1}{b^2 + b + a^2} = \left[0, b, 1, \frac{b^2}{a^2}\right] \text{ and } \frac{1}{b} + \frac{1}{2b^2 - b + a^2} = \left[0, b - 1, 1, 1, \frac{b^2}{a^2}\right].$$

This means one additional term 1 each. Letting $a = F_n, b = F_{n+1}$, we have $(b, b^2 + b + a^2) = 1$ and $(b, 2b^2 - b + a^2) = 1$. In the analogues of (4) and (5), we also have one additional term 1 each, namely,

$$\frac{1}{F_{n+1}} - \frac{1}{F_{n+1}^2 + F_{n+1} + F_n^2} = [0, F_{n+1}, 1, 2, \underbrace{1, \ldots, 1}_{n-3}, 3, \underbrace{1, \ldots, 1}_{n-4}, 2] \text{ for } n > 3$$

and

$$\frac{1}{F_{n+1}} + \frac{1}{2F_{n+1}^2 - F_{n+1} + F_n^2} = [0, F_{n+1} - 1, 1, 1, 2, \underbrace{1, \ldots, 1}_{n-3}, 3, \underbrace{1, \ldots, 1}_{n-4}, 2] \text{ for } n > 3.$$

This proves

Theorem: For every integer $m > 5$, resp. $m > 6$, there exist integers $b_m > a_m > 1$ with $(b_m, a_m) = 1$, resp. $d_m > c_m > 1$ with $(d_m, c_m) = 1$, such that the simple continued fraction of $\frac{1}{a_m} - \frac{1}{b_m}$, resp. $\frac{1}{c_m} + \frac{1}{d_m}$, has exactly m terms.

By $\frac{1}{2} \mp \frac{1}{3}$, $\frac{1}{2} \mp \frac{1}{5}$, $\frac{1}{2} \mp \frac{1}{7}$, $\frac{1}{2} \mp \frac{1}{9}$, Theorem 1 holds for $m > 1$ and $m > 2$. We have

$$\phi_n = \frac{F_{n+2}}{F_{n+1}} = [\underbrace{1, \ldots, 1}_{n+1}] \text{ for } n \geq 0$$

without normalization. For every real $\tilde{\mu}$ between ϕ_{n-1} and ϕ_n, we have

$$\tilde{\mu} = [\underbrace{1, \ldots, 1}_{n}, \ldots] \text{ for } n > 0. \tag{6}$$

We also have

$$\phi-\phi_n=\frac{(-1)^n}{F_{n+1}(\phi F_{n+1}+F_n)},\quad |\phi-\phi_n|>\frac{1}{2\phi F_{n+1}^2}>\phi^{-2n-2}\ \text{ for } n>0.$$

Trivially, we observe that every real $\widetilde{\mu}$ with

$$\phi<\widetilde{\mu}<\phi+\phi^{-2n-2} \tag{7}$$

or with

$$\phi-\phi^{-2n-2}<\widetilde{\mu}<\phi \tag{8}$$

satisfies (6).

For primes p, q, let $q>p^2+p$, $\mu=\frac{q-p}{p^2}$. Then we have $\mu>1$, $\frac{1}{p}-\frac{1}{q}=[0,p,\mu]$. $\widetilde{\mu}=\mu$ should satisfy (7), which means $\phi p^2+p<q<\phi p^2+p+\phi^{-2n-2}p^2$. For $x>x_0$, there exist primes q with $x<q<x+x^{2/3}$, by Hoheisel (see [1]) and others. We use this with $x=\phi p^2+p$ and choose $p>x_0$ so that

$$\phi^{-2n-2}p^2\ge(\phi p^2+p)^{2/3};$$

by $\phi^2p^2>\phi p^2+p$, the choice $p>x_0+\phi^{3n+5}$ is sufficient. By the "Bertrand postulate" (and especially by Hoheisel), $p<2(x_0+\phi^{3n+5})$ can be satisfied. This proves

$$\underset{C\in\mathbb{R}_{>1}}{\exists}\ \underset{n>0}{\forall}\ \underset{\substack{p_n,\,q_n\in\mathbb{P}\\ p_n<q_n<C^n}}{\exists}\ \frac{1}{p_n}-\frac{1}{q_n}=[0,p_n,\underbrace{1,\dots,1}_{n},\dots]. \tag{9}$$

For primes p, q, let $q>p^2-p$, $\lambda=\frac{p+q}{p+q-p^2}$. Then we have $\lambda>1$, $\frac{1}{p}+\frac{1}{q}=[0,p-1,\lambda]$. $\widetilde{\mu}=\lambda$ should satisfy (8), which means (after rewriting)

$$\phi^2p^2-p<q<\phi^2p^2-p+\phi^{-2n-1}(p+q-p^2). \tag{10}$$

Since $p+q-p^2>p+(\phi^2p^2-p)-p^2=\phi p^2$, the condition $\phi^2p^2-p<q<\phi^2p^2-p+\phi^{-2n}p^2$ is sufficient for (10). As above, we apply Hoheisel. This proves

$$\underset{C\in\mathbb{R}_{>1}}{\exists}\ \underset{n>0}{\forall}\ \underset{\substack{p_n,\,q_n\in\mathbb{P}\\ p_n<q_n<C^n}}{\exists}\ \frac{1}{p_n}+\frac{1}{q_n}=[0,p_n-1,\underbrace{1,\dots,1}_{n},\dots]. \tag{11}$$

In (9) and in (11), we have $q_n>F_n$.

On examining the argument, we see that p and q in (9) and also in (11) can be taken from arbitrary sets $\subset\mathbb{N}$ which satisfy conditions of types Bertrand and Hoheisel, respectively.

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SUMS OF POWERS OF DIGITAL SUMS

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1. INTRODUCTION

In a recent article [2], the authors showed that, for any positive integer k,

$$\frac{1}{x}\sum_{n\le x} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O\left(\log^{k-\frac{1}{3}} x\right),$$

where $s(n)$ denotes the digital sum of the nonnegative integer n and $\log x$ denotes the base 10 logarithm of x. It was conjectured that, for any positive integer k,

$$\frac{1}{x}\sum_{n\le x} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O\left(\log^{k-1} x\right).$$

During a presentation of this result, Carl Pomerance asked if there was any evidence for this better "big-oh" term. At the time, the conjecture was based solely on two results, one by Cheo and Yien [1] which states that

$$\frac{1}{x}\sum_{n\le x} s(n) = \frac{9}{2}\log x + O(1)$$

and the other by Kennedy and Cooper [3] which states that

$$\frac{1}{x}\sum_{n\le x} s(n)^2 = \left(\frac{9}{2}\right)^2 \log^2 x + O(\log x).$$

In this article we will show that

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^k = \left(\frac{9}{2}\right)^k n^k + O\left(n^{k-1}\right).$$

This provides more evidence for this better "big-oh" term.

In [1], Cheo and Yien found that

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i) = \frac{9}{2}n.$$

In a similar manner, Kennedy and Cooper [3] showed that

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^2 = \frac{81}{4}n^2 + \frac{33}{4}n.$$

Furthermore, using MAPLE, the following formulas were calculated:

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^3 = \frac{729}{8}n^3 + \frac{891}{8}n^2,$$

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^4 = \frac{6561}{16}n^4 + \frac{8019}{8}n^3 + \frac{3267}{16}n^2 - \frac{3333}{40}n,$$

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^5 = \frac{59049}{32}n^5 + \frac{120285}{16}n^4 + \frac{147015}{32}n^3 - \frac{29997}{16}n^2,$$

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^6 = \frac{531441}{64}n^6 + \frac{3247695}{64}n^5 + \frac{3969405}{64}n^4 - \frac{1080783}{64}n^3 - \frac{329967}{32}n^2 + \frac{15873}{4}n,$$

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^7 = \frac{4782969}{128}n^7 + \frac{40920957}{128}n^6 + \frac{83357505}{128}n^5 - \frac{56133}{128}n^4 - \frac{20787921}{64}n^3 + \frac{999999}{8}n^2,$$

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^8 = \frac{43046721}{256}n^8 + \frac{122762871}{64}n^7 + \frac{750217545}{128}n^6 + \frac{76284747}{32}n^5 - \frac{1372208607}{256}n^4 + \frac{67777479}{64}n^3 + \frac{371095263}{320}n^2 - \frac{33333333}{80}n.$$

These results were obtained by initially considering the function

$$f(x) = (1 + x + x^2 + \cdots + x^9)^n.$$

We then repeatedly differentiated, multiplied by x, and substituted $x = 1$. However, when an exponent of 9 was used, the computation became too big for the memory of the computer. Nevertheless, these results reinforced our belief that the conjecture is true. We proceeded to delve more deeply into the generating function.

2. HIGHER DERIVATIVES

Because of the form of the function which was initially differentiated, i.e.,

$$(1 + x + x^2 + \cdots + x^9)^n,$$

we set out to find a formula for

$$\frac{d^m}{dx^m} g(x)^n,$$

where n and m are positive integers and g is an arbitrary, continuously differentiable function. After investigating the situation using the computer algebra system DERIVE, we noticed the following pattern.

Lemma 1: Let n and m be positive integers and g be a continuously differentiable function. Then

$$\frac{d^m}{dx^m} g^n = \sum_{n_1+2n_2+\cdots+mn_m=m} n(n-1)\cdots(n-n_1-\cdots-n_m+1)g^{n-n_1-\cdots-n_m} \cdot \frac{m!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(m!)^{n_m}n_m!}(g^{(1)})^{n_1}(g^{(2)})^{n_2}\cdots(g^{(m)})^{n_m},$$

where $n_1, n_2, \ldots, n_m$ are nonnegative integers.

The proof of this result is by induction on m. However, it might be noted here that Lemma 1 is just a special case of Faá di Bruno's formula [4] which states that if $f(x)$ and $g(x)$ are functions for which all the necessary derivatives are defined and m is a positive integer, then

$$\frac{d^m}{dx^m} f(g(x)) = \sum_{n_1+2n_2+\cdots+mn_m=m} \frac{m!}{n_1!\cdots n_m!}\left(\frac{d^{n_1+\cdots+n_m}}{dx^{n_1+\cdots+n_m}} f\right)(g(x)) \cdot \left(\frac{\frac{d}{dx}g(x)}{1!}\right)^{n_1} \cdots \left(\frac{\frac{d^m}{dx^m}g(x)}{m!}\right)^{n_m},$$

where $n_1, n_2, \ldots, n_m$ are nonnegative integers.

3. MAIN RESULT

We will need one final lemma before we can state and prove the main result. To do this, we let

$$f_0(x) = (1+x+x^2+\cdots+x^9)^n$$

and for any positive integer k,

$$f_k(x) = x \cdot f'_{k-1}(x).$$

Using f_k, we have the identity

$$\sum_{i=0}^{10^n-1} s(i)^k = f_k(1).$$

With these definitions in mind, we can state the following lemma.

Lemma 2: For any positive integer m,

$$f_m(x) = \sum_{i=1}^{m} \left\{ m \atop i \right\} x^i f_0^{(i)}(x),$$

where $\{\cdot\}$ denotes a Stirling number of the second kind.

Proof: We shall prove this result by induction on m. The result is clearly true for $m = 1$. Now assume that the result is true for any positive integer $m \geq 1$. By the definition of f_{m+1} and the induction hypothesis, we have

$$f_{m+1}(x) = x \cdot f'_m(x) = x \cdot \frac{d}{dx}\left(\sum_{i=1}^{m} \left\{ m \atop i \right\} x^i f_0^{(i)}(x)\right).$$

Next, by the product rule and simplification, we have

$$x \cdot \frac{d}{dx}\left(\sum_{i=1}^{m} \left\{ m \atop i \right\} x^i f_0^{(i)}(x)\right) = x \cdot \sum_{i=1}^{m} \left\{ m \atop i \right\}\left(x^i f_0^{(i+1)}(x) + f_0^{(i)}(x) \cdot ix^{i-1}\right)$$

$$= \sum_{i=1}^{m}\left(\left\{ m \atop i \right\} x^{i+1} f_0^{(i+1)}(x) + \left\{ m \atop i \right\} \cdot ix^i f_0^{(i)}(x)\right).$$

Finally, by simplification and the fact that

$$\left\{ {m \atop i-1} \right\} + i \left\{ {m \atop i} \right\} = \left\{ {m+1 \atop i} \right\},$$

we have that

$$\sum_{i=1}^{m} \left(\left\{ {m \atop i} \right\} x^{i+1} f_0^{(i+1)}(x) + \left\{ {m \atop i} \right\} \cdot i x^i f_0^{(i)}(x) \right)$$

$$= \left\{ {m \atop 1} \right\} x f_0^{(1)}(x) + \sum_{i=2}^{m} \left(\left\{ {m \atop i-1} \right\} + i \left\{ {m \atop i} \right\} \right) x^i f_0^{(i)}(x) + \left\{ {m \atop m} \right\} x^{m+1} f_0^{(m+1)}(x)$$

$$= \sum_{i=1}^{m+1} \left\{ {m+1 \atop i} \right\} x^i f_0^{(i)}(x).$$

Thus, the result is true for $m + 1$. Therefore, by induction, Lemma 2 is true for any positive integer m.

Finally, we have the main theorem.

Theorem: For all positive integers n and k,

$$\sum_{i=0}^{10^n-1} s(i)^k = \left(\frac{9}{2}\right)^k n^k 10^n + O(n^{k-1} 10^n).$$

Proof: We first use Lemma 2 to obtain

$$\sum_{i=0}^{10^n-1} s(i)^k = f_k(1) = \sum_{i=1}^{k} \left\{ {k \atop i} \right\} 1^i f_0^{(i)}(1).$$

Next, by Lemma with $g^n = f_0$ and the fact that $f_0(x) = (1 + x + x^2 + \cdots + x^9)^n$, we have that

$$\sum_{i=1}^{k} \left\{ {k \atop i} \right\} 1^i f_0^{(i)}(1) = n^k 10^{n-k} 45^k + O(n^{k-1} 10^n) = \left(\frac{9}{2}\right)^k n^k 10^n + O(n^{k-1} 10^n).$$

This proves our main result.

4. QUESTIONS

We conclude this paper with some open questions:

Can we find an exact formula for

$$\frac{1}{10^n} \sum_{i=0}^{10^n-1} s(i)^9$$

and is there a general exact formula for

$$\frac{1}{10^n} \sum_{i=0}^{10^n-1} s(i)^k$$

for all positive integers n and k? Finally, despite the fact that we now have more compelling evidence, we still have not established the conjecture that, for any positive integer k,

$$\frac{1}{x}\sum_{n\leq x} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O(\log^{k-1} x).$$

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RETROGRADE RENEGADES AND THE PASCAL CONNECTION II: REPEATING DECIMALS REPRESENTED BY SEQUENCES OF DIAGONAL SUMS OF GENERALIZED PASCAL TRIANGLES APPEARING FROM RIGHT TO LEFT

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INTRODUCTION

Repeating decimals containing the Fibonacci and Lucas numbers when their repetends are viewed in retrograde fashion, reading from the rightmost digit of the repeating cycle toward the left, have been explored in [1], [2], [3], [4], and [5]. Here, the sequences of generalized Fibonacci numbers $u(n; p, q)$ which can be interpreted as sums along diagonals in Pascal's binomial coefficient triangle [6] and extended to multinomial coefficient arrays [7] are found within repetends, both as read left to right and as read right to left.

1. BINOMIAL DIAGONAL SUMS

Let $u(n; p, q)$ be the sum of terms found along the rising diagonals of Pascal's binomial coefficient array written in left-justified form,

$$\begin{array}{cccccc} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \tag{1.1}$$

Call the top row the zero$^{\text{th}}$ row and the left-most column the zero$^{\text{th}}$ column. Then $u(n; p, q)$ is the sum of those elements found by beginning in the zero$^{\text{th}}$ column and n^{th} row and taking steps p units up and q units right throughout the left-justified array. Note that $u(n; 1, 1) = F_{n+1}$, the $(n+1)^{\text{st}}$ Fibonacci number. The sequence $u(n; p, 1)$ has the generating function [7]

$$\frac{1}{1-x-x^{p+1}} = \sum_{n=0}^{\infty} u(n; p, 1)x^n \tag{1.2}$$

which converges for $|x| < 1/2$. From the generating function, the recursion for the $u(n; p, 1)$ is

$$u(n; p, 1) = u(n-1; p, 1) + u(n-1-p; p, 1), \quad n \geq p+1,$$

where $u(n; p, 1) = 1$ for $n = 0, 1, \ldots, p$.

Then, taking $x = 1/10$ in (1.2), the decimal representation of the fraction

$$\frac{10^{p+1}}{10^{p+1} - 10^p - 1}$$

has successive terms $u(n; p, 1)$ appearing as successive digits in its repetend until carrying disguises the pattern. When $p = 1$, we display the Fibonacci numbers in the well known

$$
\begin{array}{l}
100/89 = 1.12358 \\
\qquad\qquad\quad\;\; 13 \\
\qquad\qquad\quad\;\;\; 21\ldots
\end{array}
$$

where the decimal is moved from the usual 1/89 so that the left-most digit is $u(0; 1, 1) = F_1$. We also have

$$
\begin{array}{l}
1.00 \\
\;.11 \\
\;.0121 \\
\;.001331\ldots
\end{array}
$$

or $1 + 11/10^2 + 11^2/10^4 + \cdots = 10^2/(10^2 - 11) = 100/89$ by summing the geometric series. Similarly, for $u(n; p, 1)$, since $(10^p + 1)^k$ displays the coefficients of the k^{th} row of Pascal's triangle interspersed by $(p-1)$ zeros, we can sum elements that are p units up and 1 unit over by summing the geometric series

$$1 + (10^p + 1)/10^{p+1} + (10^p + 1)^2/10^{2p+2} + \cdots = 10^{p+1}/\left(10^{p+1} - (10^p + 1)\right).$$

From [1], since $10(10^p + 1) - 1 = 10^{p+1} + 9$, the repetend of the fraction $1/(10^{p+1} + 9)$ ends in powers of $(10^p + 1)$, and thus gives $u(n; p, 1)$ reading from right to left in the repetend. Again, we have the symmetric coefficients of Pascal's triangle interspersed with $(p-1)$ zeros, so, for example, for $p = 2$, powers of 101 appear from the right as

$$
\begin{array}{r}
1 \\
101 \\
10201 \\
1030301 \\
104060401
\end{array}
$$

making as a sum $\ldots 6432111$ where $u(n; 2, 1)$: $1, 1, 1, 2, 3, 4, 6, \ldots$. Notice that we are summing elements that are up 2 and over 1 in the Pascal array (1.1), applying the Pascal connection of [1].

So far, the sequences $u(n; p, 1)$ mimic the Fibonacci sequence in these applications. However, $u(n; 1, q)$ is more challenging.

Start with $u(n; 1, 2)$: $1, 1, 1, 2, 4, 7, 12, 21, \ldots$, which has zero column elements in its definition. Then $u(n; 1, 2)$ has the associated sequence $v(n; 1, 2)$: $0, 1, 2, 3, 5, 9, \ldots$, $n = 0, 1, 2, \ldots$, formed by summing up 1 and over q throughout array (1.1) but starting with column one instead of column zero. Consider

$$
\begin{array}{l}
10.00 \\
\;\;\;.11 \\
\;\;\;.00121 \\
\;\;\;.00001331 \\
\;\;\;.00000014641\ldots
\end{array}
$$

which is $1/10^{-1} + 11/10^2 + 11^2/10^5 + \cdots = 10^4/(10^3 - 11) = 10000/989$. The coefficients of successive powers of ten appearing are $1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, \ldots$, where the odd terms give $u(n; 1, 2)$ and the even terms give $v(n; 1, 2)$, and we see powers of 11, and $989 = 10^3 - 11$.

Now, $10^2 \cdot 11 - 1 = 1099$ gives powers of 11 shifted in groups of 2 to make the same sum from the right in 1/1099, which ends in

$$\begin{array}{r} 1211101 \\ 1331 \\ 14641 \\ \dots \end{array}$$

which sums to $\dots 975432211101$, where the $u(n; 1, 2)$ and the $v(n; 1, 2)$ are interleaved.

Now, $u(n; 1, 3)$ begins on column zero, and has two related sequences $v(n; 1, 3)$ and $w(n; 1, 3)$ that begin with columns one and two in array (1.1):

$$\begin{array}{l} u(n; 1, 3)\text{: } 1, 1, 1, 1, 2, 5, 11, \dots \\ v(n; 1, 3)\text{: } 0, 1, 2, 3, 4, 6, 11, \dots \\ w(n; 1, 3)\text{: } 0, 0, 1, 3, 6, 10, \dots . \end{array}$$

Then $1/10^{-2} + 11/10^2 + 11^2/10^6 + \dots = 10^6/(10^4 - 11) = 1000000/9989 = 100.11012\dots$, where the coefficients of 10^k are the three sequences interleaved with $u(n; 1, 3)$ appearing as every third term. That is,

$$\begin{array}{l} 100.00 \\ .11 \\ .000121 \\ .0000001331 \\ .00000000014641\dots \end{array}$$

which sums with coefficients

$$\underline{1}, 0, 0, \underline{1}, 1, 0, \underline{1}, 2, 1, \underline{1}, 3, 3, \underline{2}, 4, 6, \underline{5}, 6, 10, \underline{11}, 11, \dots .$$

Now, $9989 = 10^4 - 11$, and $10^3 \cdot 11 - 1 = 10999$. The three sequences $u(n; 1, 3)$, $v(n; 1, 3)$, and $w(n; 1, 3)$ are interleaved from right to left in the decimal repetend of 1/10999.

In general, $u(n; 1, q)$ appears as one of q sequences that interleave from left to right in $10^{2q}/(10^{q+1} - 11)$ and from right to left in the repetend of $1/(10^q \cdot 11 - 1)$. The q sequences are formed by summing up 1 and over q throughout array (1.1), beginning with column k, $k = 0, 1, \dots, q - 1$.

Things get more peculiar if we take $q \neq 1$. $p \neq 1$. Take $p = 2$, $q = 2$, and let $v(n; 2, 2)$ be the related sèquence beginning at column one:

$$\begin{array}{l} u(n; 2, 2)\text{: } 1, 1, 1, 1, 2, 4, 7, 11, 17, 27, 44, 72, 117, 189, \dots \\ v(n; 2, 2)\text{: } 0, 1, 2, 3, 4, 6, 10, 17, 28, 45, 72, 116, \dots . \end{array}$$

We have to split Pascal's triangle into even and odd rows:

$$\begin{array}{l} 10.000 \\ .121 \\ .0014641 \\ .000016(15)(20)(15)61 \\ .000000\ \ 1\ \ \ \ 8\ \ \ \dots \end{array}$$

which is $1/10^{-1}+11^2/10^3+11^4/10^7+\cdots=100000/9879=10^5/(10^4-11^2)$, and which has for coefficients of successive powers of 10 from left to right

$$\underline{1}, 0, \underline{1}, 2, \underline{2}, 4, \underline{7}, 10, \underline{17}, 28, \ldots$$

while

.11
.001331
.000015(10)(10)51
.000000 1 7 ...

has sum $11/10^2+11^3/10^6+11^5/10^{10}+\cdots=1100/9879$ with coefficients of successive powers of 10 given by

$$\underline{1}, 1, \underline{1}, 3, \underline{4}, 6, \underline{11}, 17, \underline{27}, 45, \ldots,$$

where we see in both sequences that every second term of $u(n; 2, 2)$ is interleaved with every second term of $v(n; 2, 2)$. Now, $10^4-11^2=9879$ and $11^2\cdot10^2-1=12099$ so 1/12099 has powers of 11^2 in groups of 2 digits to give the same interleaved sequence from right to left as in the even split above.

If we take $p = 3$ and $q = 2$,

$$u(n; 3, 2)\text{: } 1, 1, 1, 1, 1, 2, 4, 7, 11, 16, 23, 34, 52, \ldots$$
$$v(n; 3, 2)\text{: } 0, 1, 2, 3, 4, 5, 7, 11, 18, 29, 45, 68, \ldots$$

then $1/10^{-1}+11^3/10^4+11^6/10^9+11^9/10^{14}+\cdots=10^6/(10^5-11^3)$ has as coefficients of 10^k from left to right

$$\underline{1}, 0, \underline{1}, 3, \underline{4}, 7, \underline{16}, 29, \underline{52}, \ldots,$$

where every second term comes from every third term in $u(n; 3, 2)$ and $v(n; 3, 2)$. There are three similar cases, where the other two come from $11^2/10^3+11^5/10^8+11^8/10^{13}+\cdots=11^2\cdot10^2/(10^5-11^3)$ and $11/10^2+11^4/10^7+11^7/10^{12}+\cdots=11\cdot10^3/(10^5-11^3)$. Now, $10^3\cdot11^3-1=1330999$, and the repetend of 1/1330999 has powers of 11^3 appearing in groups of 3 from right to left, and has the primary interleaved sequence appearing from right to left.

In general, for $u(n; p, q)$, $q\geq1$, $p\geq1$, the primary case of q sequences interleaved where every q^{th} term is every p^{th} one in the q sequences, appears from left to right in the coefficients of 10^k in the decimal expansion of the fraction $10^{p+2q-1}/(10^{p+q}-11^p)$ while the repetend has the primary case appearing right to left in the repetend of $1/(10^q\cdot11^p-1)$, where powers of 11^p appear in groups of q from right to left. If we take $q = 1$ in the formula for $u(n; p, q)$, we get $10^{p+1}/(10^{p+1}-11^p)$, which makes every p^{th} term of $u(n; p, 1)$ appear, in contrast to $10^{p+1}/(10^{p+1}-10^p-1)$, which makes all terms of $u(n; p, 1)$ appear.

These representations of $u(n; p, q)$ come from summing the geometric series

$$\frac{1}{10^{1-q}}+\frac{11^p}{10^{p+1}}+\frac{11^{2p}}{10^{2p+q+1}}+\cdots=\frac{10^{p+2q-1}}{10^{p+q}-11^p},$$

where 11^p gives coefficients of every p^{th} row of Pascal's triangle, 10^{p+1} gives a separate place value for each coefficient, the ratio $11^p/10^{p+q}$ moves p rows up and q columns over in the array (1.1), $1/10^{1-q}$ puts all zero terms of the q sequences to the left of the decimal point, and $u(1; p, q)$ is the coefficient of 1/10 in the decimal expansion. Summing all columns down catches all summands in the infinite sum, and makes q sequences interleaved. The repetend of $1/(10^q \cdot 11^p - 1)$, read from right to left, ends in p^{th} powers of 11 moved over q columns, again giving q interleaved sequences.

It is possible to make decimals for $u(n; p, q)$ that list every term of the q interleaved sequences if $(p, q) = 1$. If we sum

$$\frac{1}{10^{1-q}}+\frac{10^p+1}{10^{p+1}}+\frac{10^p+1}{10^{p+q+1}}+\cdots=\frac{10^{p+2q-1}}{10^{p+q}-10^p-1},$$

we have lined up the array to give successive terms of $u(n; p, q)$, $n = 0, 1, 2, \ldots$, interleaved with the successive terms of the other $q-1$ related sequences. Note that $10^p+1 = 10\ldots01, (p-1)$ zeros, will give coefficients of rows of Pascal's triangle interspersed with $(p-1)$ zeros, when raised to powers. The ratio $(10^p+1)/10^{p+q}$ gives successive rows shifted p units over and q units up to line up coefficients for summing. Then, $u(1; p, q)$ is the coefficient of 1/10 and $u(0; p, q)$ appears to the left of the decimal point, as do the zero terms of the other $(q-1)$ sequences. The terms of the sequence $u(n; p, q)$ are interspersed with the terms of the q related sequences as before. However, if $(p, q) \neq 1$, coefficients will not line up for proper summing to make $u(n; p, q)$. If $p = q$, we get $u(n; p, q)$ as given by the fraction

$$\frac{10^{3p-1}}{10^{2p}-10^p-1}=\frac{10^{p-1}\cdot(10^p)^2}{(10^p)^2-(10^p)^1-1}$$

where $u(n; 1, 1)$ is given by $10^2(10^2-10^1-1)$ from our earlier fraction for $u(n; p, 1)$. If we replace 10 by 10^p, we write a fraction where $u(n; 1, 1)$ appears as every p^{th} term, interspersed by $(p-1)$ zeros, and we get the fraction for $u(n; p, p)$ except for a shift of $(p-1)$ places in the decimal point. We also line up previously derived sequences whenever $(p, q) \neq 1$. Let $(p, q) = d$. Then the fraction for $u(n; p, q)$ gives the sequences $u(n; p/d, q/d)$ as every d^{th} coefficient, interspersed so that the q/d sequences are interleaved, but the decimal point is moved $(d-1)$ places to the right. When $(p, q) = 1$, $u(n; p, q)$ is given from the right in the repetend of the fraction $1/[10^q \cdot (10^p+1)-1]$ appearing as part of the q interleaved sequences.

Of course, [7] gives the generating function for $u(n; p, q)$ as

$$\frac{(1-x)^{q-1}}{(1-x)^q-x^{p+q}}=\sum_{n=0}^{\infty}u(n; p, q)x^n$$

which converges for $|x|<1/2$. Taking x = 1/10 and simplifying, the decimal expansion of $9^{q-1}\cdot 10^{p+2q-1}/(9^q\cdot 10^p-1)$ had $u(n; p, q)$ appearing as coefficients of 10^k from left to right but carrying makes the pattern disappear quickly. The pattern continues longer if we use $x = 1/10^k$, $k > 1$, and look at groups of k digits. This representation, however, does not lead to the same sequences being found in patterns from right to left except when $q = 1$.

2. TRINOMIAL DIAGONAL SUMS

The coefficients appearing in expansions of the trinomial $(1+x+x^2)^n, n = 0, 1, 2, \ldots$, written in left-justified form, are

$$\begin{array}{ccccccccccc} 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 3 & 6 & 7 & 6 & 3 & 1 \\ 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\ 1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \tag{2.1}$$

Call the top row the "zero$^{\text{th}}$ row" and the left column the "zero$^{\text{th}}$ column." Let $u(n; p, q)$ be the sum of the term in the left column and the n^{th} row and the terms obtained by taking steps p units up and q units right throughout the array. Then, from [7],

$$\frac{1}{1-x-x^{p+1}-x^{2p+1}} = \sum_{n=1}^{\infty} u(n; p, 1)x^n. \tag{2.2}$$

As in §1, the decimal expansion of $10^{2p+1}/(10^{2p+1}-(10^{2p}+10^p+1))$ has $u(n; p, 1)$ as the coefficients of successive powers of 1/10 where $u(0; p, 1)$ appears left of the decimal point, and powers of $(10^{2p}+10^p+1)$ appear as

$$1+\frac{(10^{2p}+10^p+1)}{10^{2p+1}}+\frac{(10^{2p}+10^p+1)^2}{10^{4p+2}}+\cdots = \frac{10^{2p+1}}{10^{2p+1}-(10^{2p}+10^p+1)}.$$

Since $10(10^{2p}+10^p+1)-1 = 10^{2p+1}+10^{p+1}+9$, the repetend of $1/(10^{2p+1}+10^{p+1}+9)$ ends in $u(n; p, 1)$ as in the binomial case. Note that $p = 1$ gives the Tribonacci case reported in [1]. As before, the case for $u(n; p, 1)$ is simple because the known generating function almost takes care of it, but we are on our own when $q > 1$.

Now, suppose that $p = 1$, and consider $u(n; 1, q)$, $q \geq 1$. Then the decimal expansion of $10^{2q+1}/(10^{q+2}-111)$ gives q interleaved sequences as in the binomial case from left to right, while the repetend of $1/(10^q \cdot 111-1)$ gives q interleaved sequences from right to left.

The case for $u(n; p, q)$ is given from left to right by $10^{2p+2q-1}/(10^{2p+q}-111^p)$, which generates every p^{th} term of q interleaved sequences. Each of the q sequences is generated by starting in the k^{th} column, n^{th} row, and summing elements found by taking steps of up p, right q, throughout array (2.1), for $k = 0, 1, \ldots, q-1$. Similarly to the binomial case, we sum

$1/10^{1-q}$ + $111^p/10^{2p+1} + 111^{2p}/10^{4p+q+1}$ ..., where the geometric ratio is $111^p/10^{2p+q}$ to select every p^{th} row and move q units right in the array, 111^p contains $2p + 1$ terms, and the zero term for each sequence appears to the left of the decimal point. The repetend of the fraction $1/(10^q \cdot 111^p - 1)$ will have the same interleaved sequences appearing from right to left and will show powers of 111^p diagonalized from the right.

Similarly to the binomial, we can write every term of the q interleaved sequences for $u(n; p, q)$, $(p, q) = 1$ from left to right by summing

$$\frac{1}{10^{1-q}} + \frac{10^{2p} + 10^p + 1}{10^{2p+1}} + \frac{(10^{2p} + 10^p + 1)^2}{10^{4p+q+1}} + \cdots = \frac{10^{2p+2q-1}}{10^{2p+q} - 10^{2p} - 10^p - 1}$$

and the same q sequences appear from right to left in the repetend of $1/(10^q \cdot (10^{2p} + 10^p + 1) - 1)$.

3. MULTINOMIAL DIAGONAL SUMS

Write the coefficients appearing in expansions of the multinomial $(1 + x + x^2 + \cdots + x^m)^n$, $n =$ 0, 1, 2, ..., in left-justified form. Call the top row the zero$^{\text{th}}$ row and the left column the zero$^{\text{th}}$ column. Let $u(n; p, q)$ be the sum of the term in the zero$^{\text{th}}$ column and n^{th} row and the terms obtained by taking steps p units up and q units right throughout the array. Then, from [7],

$$\frac{1}{1 - \mathrm{x} - \mathrm{x}^{\mathrm{p}+1} - x^{2p+1} - \cdots - x^{mp+1}} = \sum_{n=0}^{\infty} u(n; p, 1)x^n.$$

Thus, the decimal expansion of $10^{mp+1}/(10^{mp+1} - (10^{mp} + 10^{(m-1)p} + \cdots + 10 + 1))$ has $u(n; p, 1)$ appearing as coefficients of successive powers of 1/10, where $u(0; p, 1)$ appears left of the decimal. The repetend of the fraction $1/(10^{mp+1} + 10^{(m-1)p+1} + 10^{(m-2)p+1} + \cdots + 10^{p+1} + 9)$ has $u(n; p, 1)$ appearing from right to left as before. We expect that the repetend $1/(10^q \cdot 11 \ldots 1^p - 1)$, where $(m + 1)$ 1's appear in the multiplier of 10^q, would generate the p^{th} terms of q interleaved sequences related to $u(n; p, q)$ from right to left as before, and that the repetend of the fraction $10^{mp+2q-1}/(10^{mp+q} - (11 \ldots 1)^p)$ would generate those same interleaved sequences from left to right because we still have a "Pascal connection" available. The $(mp + 1)$ coefficients of the p^{th} row are generated by $(11 \ldots 1)^P$ (there are m + 1 1's), and the geometric ratio is $(11 \ldots 1)^p/10^{mp+q}$ to select every p^{th} row and move q units right, so we sum $1/10^{1-q} + (11 \ldots 1)^p/10^{mp+1} +$ $(11 \ldots 1)^{2p}/10^{2mp+q+1} + \cdots$ to form $10^{mp+2q-1}/(10^{mp+q} - (11 \ldots 1)^p)$. As before, we can write all the terms of $u(n; p, q)$, $(p, q) = 1$, interleaved as part of the q sequences, left to right by

$$\frac{10^{mp+2q-1}}{10^{mp+q} - (10^{mp} + 10^{(m-1)p} + \cdots 10^p + 1}$$

and from right to left in the repetend of

$$\frac{1}{10^q \cdot (10^{mp} + 10^{(m-1)p} + \cdots + 10^p + 1) - 1}.$$

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A FIBONACCI POLYNOMIAL SEQUENCE DEFINED BY MULTIDIMENSIONAL CONTINUED FRACTIONS; AND HIGHER-ORDER GOLDEN RATIOS

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INTRODUCTION

A sequence of polynomials is a *Fibonacci Sequence* if it satisfies the recursion:

$$f_{n+2}(x) = x \cdot f_{n+1}(x) + f_n(x) \quad \text{for } n \geq 0. \tag{1}$$

Two well-known Fibonacci sequences are the *Fibonacci Polynomials*, $\{F_n(x)\}$, defined using (1) with $F_1(x) = 1$ and $F_2(x) = x$, and the *Lucas Polynommials*, $\{L_n(x)\}$, defined using (1) with $L_1(x) = 2$ and $L_2(x) = x$ ([1], [3], [4], [11], [12], [22], [23]). In addition to being Fibonacci sequences, these polynomials produce Fibonacci and Lucas numbers, respectively, when evaluated at $+1$.

Here we examine a sequence of polynomials $\{G_n(x)\}$ originating from multidimensional continued fractions with all one's. The golden ratio is a root of the quadratic polynomial in this sequence; hence, there is justification to consider the roots of the other polynomials in this sequence to be higher-order golden ratios. Surprisingly, these polynomials also form a Fibonacci sequence, and Fibonacci and Lucas numbers result when evaluated at $+1$ and -1, respectively.

It turns out that the Fibonacci and Lucas polynomials, as well as this new sequence are examples of a larger class of Fibonacci polynomial sequences. We develop an explicit formula for this class and show specifically how the Fibonacci numbers are involved when evaluated at ± 1.

1. DEFINITION OF THE GOLDEN POLYNOMIALS $\{G_n(x)\}$

The continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} \tag{2}$$

satisfies the equation

$$x = 1 + \frac{1}{x}$$

which is readily converted to the polynomial equation $x^2 - x - 1 = 0$. We define $G_2(x)$ to be this second-degree polynomial, and denote its positive root as g_2. This root is the value of the continued fraction in (2), namely, the golden ratio

$$g_2 = \frac{1+\sqrt{5}}{2}.$$

Now consider a continued fraction of the form

$$1+\cfrac{1}{\left(1+\cfrac{1}{1+\cdots}\right)+\cfrac{1}{\left(1+\cfrac{1}{1+\cdots}\right)}}. \tag{3}$$

Whereas the sequence of denominators in the continued fraction in (2) could be written in a list, the denominators of (3) would require a binary tree of all 1's. This continued fraction can be written as

$$x=1+\cfrac{1}{x+\cfrac{1}{x}}$$

or as the polynomial equation $G_3(x)=x^3-x^2-1=0$, which has the value of (3) as a solution. Analogously, we designate this single positive root by g_3 as an indication that it is a root of the third-degree polynomial $G_3(x)$.

This process can be extended. Consider the family of recursive equations of the form

$$\underbrace{x=1+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{\cdots+\cfrac{1}{x}}}}}}_{n-1\ x\text{'s}} \tag{4}$$

These equations represent multidimensional continued fractions of all 1's that have n – 1 branches at each level. For each n, this equation can be transformed into an n^{th} degree polynomial equation $G_n(x)=0$. (For n = 1, there are no x's on the right side, so it is natural to define $G_1(x)=x-1$.)

In this way, we get a sequence of functions $\{G_n(x)\}$. Since each function $G_n(x)$ has a positive maximal root g_n [17], we also obtain a sequence of positive numbers $\{g_n\}$. In Section 5, we will see that there is justification to consider these roots to be higher-order golden ratios. Because of this, we will refer to these polynomials $\{G_n(x)\}$ as the "Golden Polynomials."

For example, we can write the coefficients for some of these polynomials as shown in Figure 1. Whereas the sum of the coefficients in the n^{th} row of Pascal's triangle is 2^n, the sum shown here is $-F_{n-1}$ (proven in Corollary 2.4 below). For other approaches to the generalization of the continued fraction algorithm, the reader is referred to Bernstein [2] and Szerkeres [21].

$n=1$	1	−1									
$n=2$	1	−1	−1								
$n=3$	1	−1	0	−1							
$n=4$	1	−1	1	−2	−1						
$n=5$	1	−1	2	−3	−1	−1					
$n=6$	1	−1	3	−4	0	−3	−1				
$n=7$	1	−1	4	−5	2	−6	−2	−1			
$n=8$	1	−1	5	−6	5	−10	−2	−4	−1		
$n=9$	1	−1	6	−7	9	−15	0	−10	−3	−1	
$n=10$	1	−1	7	−8	14	−21	5	−20	−5	−5	−1

FIGURE 1

2. FIBONACCI POLYNOMIAL SEQUENCES

A convenient way to express the Fibonacci recursion in (1) is to define the functional

$$\Phi(f,g)=x\cdot f(x)+g(x).$$

Similarly, we can represent a Fibonacci sequence generated by this functional by

$$\Phi\{f_1,f_0\}=\{f_n \mid f_{n+2}=\Phi(f_{n+1},f_n) \text{ for } n\geq 0\}.$$

This notation emphasizes that the entire sequence depends only on the two seed functions.

All Fibonacci sequences can be represented in this way. For example,

$$\Phi\{x,1\}=\{1,x,x^2+1,\dots\}=\{F_n(x)\}=\text{ the Fibonacci Polynomials;}$$
$$\Phi\{x,2\}=\{2,x,x^2+2,\dots\}=\{L_n(x)\}=\text{ the Lucas Polynomials.}$$

It is clear that there are many such sequences, and we let $\mathcal{F}$ denote the set of all sequences generated in this way. Other approaches to the structure of Fibonacci-type polynomials have been pursued in Horadam [14], Shannon [19], and Dilcher [9].

A number of simple properties are evident.

Observation 2.1:

a. $\Phi(c\cdot f, c\cdot g)=c\cdot\Phi(f,g)$ for any constant c.

b. $\Phi(f_1,g_1)+\Phi(f_2,g_2)=\Phi(f_1+f_2,g_1+g_2)$.

c. $\{f_n\}\in\mathcal{F}\Leftrightarrow\{-f_n\}\in\mathcal{F}$.

d. $\{f_n\},\{g_n\}\in\mathcal{F}\Rightarrow\{f_n+g_n\}\in\mathcal{F}$.

e. If $h_n=\Phi(f_n,g_n)$, where $\{f_n\},\{g_n\}\in\mathcal{F}$, then $\{h_n\}\in\mathcal{F}$.

To show that $\{G_n(x)\}$ is a Fibonacci sequence, we will need the following lemma.

Lemma 2.2: The polynomial numerator and denominator obtained by simplifying the expression

$$\underbrace{x+\cfrac{1}{x+\cfrac{1}{x+\cdots}}}_{n\ x\text{'s}}$$

are $F_{n+1}(x)$ and $F_n(x)$ respectively.

Proof: The lemma is easily verified for $n = 1$ and $n = 2$. Now assume the lemma holds for all $k < n$. We can then write the expression with n x's as follows:

$$\underbrace{\left(x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{\cdots}}}\right)}_{n\ x\text{'s}} = x+\cfrac{1}{\underbrace{\left(x+\cfrac{1}{x+\cfrac{1}{\cdots}}\right)}_{n-1\ x\text{'s}}} = x+\cfrac{1}{\left(\cfrac{F_n(x)}{F_{n-1}(x)}\right)} = \frac{xF_n(x)+F_{n-1}(x)}{F_n(x)} = \frac{F_{n+1}(x)}{F_n(x)}.$$

Noting that the consecutive Fibonacci polynomials share no common factors, and that $\{F_n(x)\} = \Phi\{x, 1\} \in \mathcal{F}$, this completes the proof. $\square$

We now show that the sequence of Golden Polynomials $\{G_n(x)\}$ is a Fibonacci sequence.

Theorem 2.3: $\{G_n(x)\} \in \mathcal{F}$.

Proof: Substituting $F_n(x)$ and $F_{n-1}(x)$ into (4) gives

$$x = 1+\cfrac{1}{\underbrace{\left(x+\cfrac{1}{x+\cfrac{1}{\cdots}}\right)}_{n-1\ x\text{'s}}} = 1+\cfrac{1}{\left(\cfrac{F_n(x)}{F_{n-1}(x)}\right)} = \frac{F_n(x)+F_{n-1}(x)}{F_n(x)}.$$

Simplifying, we have

$$G_n(x) = xF_n(x) - (F_n(x)+F_{n-1}(x)) = 0. \tag{5}$$

By Observations 2.1.c and 2.1.d and Lemma 2.2, we have $\{-(F_n(x)+F_{n-1}(x))\} \in \mathcal{F}$. By Observation 2.1.e, it follows that $\{G_n(x)\} \in \mathcal{F}$. $\square$

The Golden Polynomials are easily seen to be

$$\{G_n(x)\} = \Phi\{x-1, -1\}.$$

During the proof we discovered a relationship between these polynomials $\{G_n(x)\}$ and the Fibonacci Polynomials $\{F_n(x)\}$. Rewriting (5), we have $G_n(x) = (x-1)F_n(x) - F_{n-1}(x)$. Evaluating at ± 1 gives

Corollary 2.4:

a. $G_n(1) = -F_{n-1}$.

b. $G_n(-1) = (-1)^n L_{n-1}$.

This establishes another connection between continued fractions and the Fibonacci and Lucas numbers.

3. A SIMPLE GENERALIZATION

A familiar method of generalizing the golden ratio (Coleman [7], Raab [18], Bicknell & Hoggatt [4]) is to define "silver" and various other metallic ratios by forming a rectangle of dimensions 1 by x and removing c unit squares (see Figure 2 below). If the remaining rectangle is similar to the original, then x is called a "generalized golden ratio."

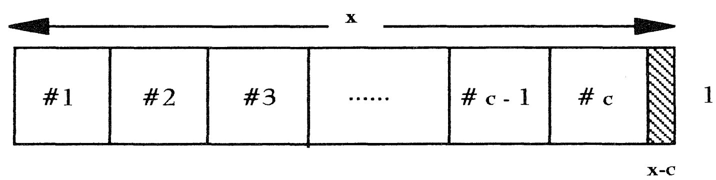

FIGURE 2

It is easily demonstrated that these numbers are precisely those that are expressed by continued fractions of period 1. If we then consider multidimensional continued fractions of period 1 and write them recursively as before, we would have a sequence of polynomials corresponding to each positive integer. For example, the cubic continued fraction of period 1, with c as the constant in the denominators, is

$$x = c + \cfrac{1}{\left((c+\cdots)+\cfrac{1}{(c+\cdots)}\right)+\cfrac{1}{\left((c+\cdots)+\cfrac{1}{(c+\cdots)}\right)}} = c + \cfrac{1}{x+\cfrac{1}{x}}.$$

Simplifying gives the third-degree polynomial $H_3(x,c) = x^3 - cx^2 - c$. In this way, we obtain the sequence $\{H_n(x,c)\} = \Phi\{x-c,-c\}$ where the coefficients of $H_n(x,c)$ are the same as those $G_n(x)$ with every other coefficient having an additional factor of c. In fact, these polynomials satisfy the relation $H_n(x,c) = (1/2)\{(1+c)G(x)+(1-c)G(-x)]$. Theorem 2.3 is easily extended to these polynomial sequences as well, i.e., $\{H_n(x,c)\} \in \mathcal{F}$.

4. AN EXPLICIT FORMULA FOR FIBONACCI SEQUENCES

Consider the Fibonacci sequences of the form $\Phi\{ax+b,c\}$. This class includes both the Fibonacci and Lucas Polynomials as well as the Golden Polynomials $\{G_n(x)\}$ and generalized Golden Polynomials $\{H_n(x;c)\}$. Here we examine an explicit formula for the functions in such a sequence.

We will use the following notational conventions:

a. Binomial Coefficients: $C_{n,k} = \binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & \text{for } 0 \le k \le n, \\ 0, & \text{for } k < 0 \text{ or } k > n. \end{cases}$

b. Greatest Integer Function: $\lfloor x \rfloor = k$, for the greatest integer function.

c. Parity Function: $\delta_k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$

Theorem 4.1: For $f_n(x) \in \Phi\{ax+b, c\}$,

$$f_n(x) = \sum_{k=0}^{n} R_{n,k} x^{n-k},$$

where
$$R_{n,k} = S_{n,k} \cdot (a \cdot \delta_k + b \cdot (1-\delta_k)) + S_{n-1,k-2} \cdot (c \cdot \delta_k)$$

and
$$S_{n,k} = \binom{n - \lfloor \frac{k}{2} \rfloor - 1}{\lfloor \frac{k}{2} \rfloor}.$$

Proof: The formula is verified by direct computation for $n = 1$ and $n = 2$:

$f_1(x) = (a \cdot 1 + b \cdot 0 + c \cdot 0)x + (a \cdot 0 + b \cdot 1 + c \cdot 0) = ax + b$, and

$f_2(x) = (a \cdot 1 + b \cdot 0 + c \cdot 0)x^2 + (a \cdot 0 + b \cdot 1 + c \cdot 0)x + (a \cdot 0 + b \cdot 0 + c \cdot 1) = ax^2 + bx + c.$

Proceeding by induction, we write

$$\begin{aligned} f_{n+2}(x) = xf_{n+1}(x) + f_n(x) &= x\sum_{k=0}^{n+1} R_{n+1,k} x^{n+1-k} + \sum_{k=0}^{n} R_{n,k} x^{n-k} \\ &= \sum_{k=0}^{n+1} R_{n+1,k} x^{n+2-k} + \sum_{k=0}^{n} R_{n,k} x^{n-k} \\ &= R_{n+1,0} x^{n+2} + R_{n+1,1} x^{n+1} + \sum_{k=0}^{n-1} \left((R_{n+1,k+2} + R_{n,k}) x^{n-k} \right) + R_{n,n}. \end{aligned}$$

It now becomes a matter of verifying that the coefficients are correct. The first two terms are:

$$\begin{aligned} R_{n+1,0} &= S_{n+1,0}(a \cdot 1 + b \cdot 1) + 0 \cdot c(1) = a \cdot S_{n+1,0} \\ &= a \cdot C_{n,0} = a \cdot 1 = a \cdot C_{n+1,0} = a \cdot S_{n+2,0} = R_{n+2,0} \\ R_{n+1,1} &= S_{n+1,1}(a \cdot 0 + b \cdot 1) + 0 \cdot c(0) = b \cdot S_{n+1,1} \\ &= b \cdot C_{n,0} = b \cdot 1 = b \cdot C_{n+1,0} = b \cdot S_{n+2,1} = R_{n+2,1}. \end{aligned}$$

Now consider the constant term

$$R_{n,n} = S_{n,n}(a \cdot \delta_k + b \cdot (1-\delta_k)) + S_{n-1,n-2} c \cdot \delta_k.$$

If n is odd, $n = 2m + 1$, and we have

$$R_{n,n} = b \cdot S_{n,n} = b \cdot C_{m,m} = b \cdot 1 = b \cdot C_{m+1,\,m+1} = b \cdot S_{n+2,\,n+2} = R_{n+2,\,n+2}.$$

If n is even, $n = 2m$, and then

$$S_{n,n} = C_{m-1,\,m} = 0 = C_{m+1,\,m+2} = S_{n+2,\,n+2};$$
$$S_{n-1,\,n-2} = C_{m-1,\,m-1} = 1 = C_{m+1,\,m+1} = S_{n+1,\,n}.$$

Substituting these in, we have

$$\begin{aligned} R_{n,n} &= a \cdot S_{n,n} + c \cdot S_{n-1,\,n-2} = a \cdot 0 + c \cdot 1 \\ &= a \cdot S_{n+2,\,n+2} + c \cdot S_{n+1,\,n} = R_{n+2,\,n+2}. \end{aligned}$$

All of the other coefficients are of the form:

$$R_{n+1,\,k} + R_{n,\,k-2} = [S_{n-1,\,k} + S_{n,\,k-2}](a \cdot \delta_k + b \cdot (1 - \delta_k)) + [S_{(n+1)-1,\,k-2} + S_{n-1,\,(k-2)-2}]c \cdot \delta_k).$$

It will suffice to show that $S_{n+1,\,k} + S_{n,\,k-2} = S_{n+2,\,k}$. Writing $j = [k/2]$, we have

$$S_{n+1,\,k} + S_{n,\,k-2} = C_{n-j,\,j} + C_{n-j,\,j-1} = C_{n+1-j,\,j} = S_{n+2,\,k}$$

by the well-known additive relationship of Pascal's triangle. □

Noting that $\{G_n(x)\} = \Phi\{x-1, -1\}$, we have an explicit formula for the Golden Polynomials.

Corollary 4.2: $G_n(x) = \sum_{k=1}^{n} \big((S_{n,\,k} - S_{n-1,\,k-2})\delta_k - S_{n,\,k}(1-\delta_k)\big)x^{n-k}$.

We can also make a number of simple observations about this type of sequence.

Corollary 4.3: For each $f_n \in \Phi\{ax + b, c\}$,

a. (the leading coefficient of $f_n(x)$) = (the leading coefficient of $f_1(x)$) = a.

b. (the trace of $f_n(x)$) = $(-1)^n \cdot$ (the trace of $f_0(x)$) = $(-1)^n \cdot c$.

c. (the norm of $f_n(x)$) = (the norm of $f_1(x)$) = b.

d. $f_{2n}(0) = f_0(0) = c$ and $f_{2n-1}(0) = f_1(0) = b$.

We can now see how Fibonacci numbers are present in all sequences of this type.

Corollary 4.4: For each $f_n \in \Phi\{ax + b, c\}$,

$$f_n(1) = a \cdot F_n + b \cdot F_n + c \cdot F_{n-1} \text{ and } f_n(-1) = (-1)^n(a \cdot F_n - b \cdot F_n + c \cdot F_{n-1}).$$

Proof: Theorem 4.1 can be expressed more conveniently as

$$f_n(x) = a\sum_{\substack{k=0\\k\text{ even}}}^{n} S_{n,k}x^{n-k} + b\sum_{\substack{k=0\\k\text{ odd}}}^{n} S_{n,k}x^{n-k} + c\sum_{\substack{k=0\\k\text{ even}}}^{n} S_{n-1,k-2}x^{n-k}.$$

Evaluating at 1 gives

$$f_n(1) = a\sum_{\substack{k=0\\k\text{ even}}}^{n} S_{n,k} + b\sum_{\substack{k=0\\k\text{ odd}}}^{n} S_{n,k} + c\sum_{\substack{k=0\\k\text{ even}}}^{n} S_{n-1,k-2}. \tag{6}$$

Evaluating at -1 gives

$$f_n(-1) = \begin{cases} a\sum\limits_{\substack{k=0\\k\text{ even}}}^{n} S_{n,k} - b\sum\limits_{\substack{k=0\\k\text{ odd}}}^{n} S_{n,k} + c\sum\limits_{\substack{k=0\\k\text{ even}}}^{n} S_{n-1,k-2} & \text{for } n \text{ even}, \quad (7)\\ -a\sum\limits_{\substack{k=0\\k\text{ even}}}^{n} S_{n,k} + b\sum\limits_{\substack{k=0\\k\text{ odd}}}^{n} S_{n,k} - c\sum\limits_{\substack{k=0\\k\text{ even}}}^{n} S_{n-1,k-2} & \text{for } n \text{ odd}. \quad (8)\end{cases}$$

We simplify these sums using the Fibonacci identity

$$F_{n+1} = \sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n-j}{j},$$

which can be found in [4]. Applying this to the first sum in each of (6)-(8), we have

$$\sum_{\substack{k=0\\k\text{ even}}}^{n} S_{n,k} = \sum_{\substack{k=0\\k\text{ even}}}^{k\le n}\binom{n-\left[\frac{k}{2}\right]-1}{\left[\frac{k}{2}\right]} = \sum_{j=0}^{\left[\frac{n-1}{2}\right]}\binom{(n-1)-j}{j} = F_n.$$

Similarly, the second and third sums become

$$\sum_{\substack{k=0\\k\text{ odd}}}^{n} S_{n,k} = F_n \quad\text{and}\quad \sum_{\substack{k=0\\k\text{ even}}}^{n} S_{n-1,k-2} = F_{n-1}.$$

Substituting these into equations (6)-(8) gives the results. □

5. HIGHER-ORDER GOLDEN RATIOS

The applications of the golden ratio to geometry and the Fibonacci numbers are well documented ([5], [15]). Since the root g_2 has the value of the golden ratio, it is natural to ask if the other maximal roots $\{g_n\}$ have similar properties. It appears that this is the case. In the four examples that follow, we examine how the $\{g_n\}$ can be considered generalizations of the golden ratio to higher dimensions.

5.1 Geometric Properties

Consider a square of side x (labeled "square A" in the diagram below), containing a unit square (square B). Extending the sides of the unit square forms a third square of side $x - 1$ (square C).

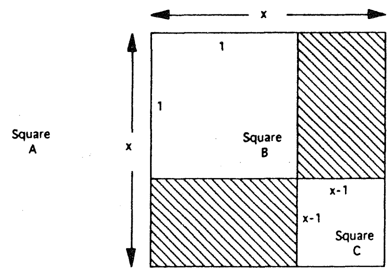

FIGURE 3

Note that the ratio of (the side of A) to (the side of B) is equal to the ratio of (the side of B) to (the side of C) only if x is the *golden ratio*, g_2. That is $x/1 = 1/(x-1)$. Note also, however, that the ratio of (the area of A) to (the side of B) is equal to the ratio of (the area of B) to (the side of C) only if x is g_3. That is $x^2/1 = 1^2(x-1)$.

A golden cuboid is a solid of unit volume having sides in the ratio of $g_2 : 1 : 1/g_2$ (Huntley [15]). It has the property that removing a slice off the top of dimensions $1/g_2 : 1 : 1/g_2$ leaves a smaller solid with the ratio of the volumes being g_2. We can analogously define a "platinum cuboid" of dimensions $g_3 : 1 : 1/g_3$. If instead of *removing* a slab of dimensions $1/g_3 : 1 : 1/g_3$, we *add* such a slab, the resulting ratio of volumes is g_3 (see Fig. 4).

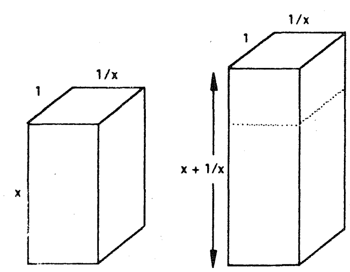

FIGURE 4

5.2 Continued Fractions and Continued Radicals

By definition, the $\{g_n'\}$ are precisely those numbers that can be expressed by multidimensional continued fractions using all 1's. This is perhaps the strongest argument to consider these numbers as higher-order golden ratios.

It is perhaps worth noting, since 1993 is the 400th anniversary of Vieta's continued radical expression for π (Smith [20]), that continued radicals were used extensively in past centuries (Cohen [6] and Shannon [19]). The golden ratio can also be expressed by continued radicals. That is,

$$g_2 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}} = \sqrt{1+g_2}.$$

Similarly, g_3 can be expressed with continued cube roots as

$$g_3 = \sqrt[3]{1+(1+(1+(1+\cdots)^{\frac{2}{3}})^{\frac{2}{3}})^{\frac{2}{3}}} = \sqrt[3]{1+(g_3)^{\frac{2}{3}}}.$$

5.3 Rational Sequences

The golden ratio, g_2, is the limit of consecutive Fibonacci numbers. This can be expressed as

$$g_2 = \lim_{k\to\infty} \frac{p_k}{q_k} \quad \text{where} \begin{cases} p_1 = q_1 = 1, \\ q_k = p_{k-1}, \\ p_k = p_{k-1} + q_{k-1}. \end{cases}$$

Similarly, there is a rational sequence that converges to g_3 defined by

$$g_3 = \lim_{k\to\infty} \frac{p_k}{q_k} \quad \text{where} \begin{cases} p_1 = q_1 = 1, \\ q_k = p_{k-1}^2 + q_{k-1}^2 & = q_{k-1}^2 F_2\left(\frac{p_{k-1}}{q_{k-1}}\right), \\ p_k = p_{k-1}^2 + p_{k-1}q_{k-1} + q_{k-1}^2 & = q_{k-1}^2\left(F_2\left(\frac{p_{k-1}}{q_{k-1}}\right) + F_1\left(\frac{p_{k-1}}{q_{k-1}}\right)\right). \end{cases}$$

Instead of the Fibonacci numbers, the convergents are 1/1, 3/2, 19/13, 797/550, ..., etc. In fact, a rational sequence can be constructed for each g_n using the Fibonacci Polynomials $F_{n-1}(x)$ and $F_{n-2}(x)$. Specifically, for a sequence that converges to g_{n+1}, begin with $p_1 = q_1 = 1$, then continue

$$\frac{p_{k+1}}{q_{k+1}} = \frac{q_k^n\left(F_n\left(\frac{p_k}{q_k}\right) + F_{n-1}\left(\frac{p_k}{q_k}\right)\right)}{q_k^n F_n\left(\frac{p_k}{q_k}\right)} = 1 + \frac{F_{n-1}\left(\frac{p_k}{q_k}\right)}{F_n\left(\frac{p_k}{q_k}\right)}.$$

5.4 Generated Integer Sequences

The Fibonacci and Lucas numbers are integer sequences generated by the golden ratio and its real conjugate using the Binet forms. In a similar way, we can define the sequence g_3 by

$$u_n = \frac{g_3^n + h_3^n + \bar{h}_3^n}{g_3 + h_3 + \bar{h}_3},$$

where h_3 and $\bar{h}_3$ are the complex conjugate roots of G_3. It can be shown that $\{u_n\}$ is the integer sequence defined by the recursive formula $u_{n+3} = u_{n+2} + u_n$ with initial values $u_0 = 3, u_1 = 1$, and $u_2 = 1$. This gives a "delayed" Fibonacci-type sequence (3), 1, 1, 4, 5, 6, 10, 15, 21, 31, 46, 67, 98, ..., etc. See [8] for additional information on this.

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AMS number: 11, number theory

REDUCED ϕ-PARTITIONS OF POSITIVE INTEGERS*

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(Submitted January 1992)

1. INTRODUCTION

As a generalization of the equation $\phi(x)+\phi(k)=\phi(x+k)$, ϕ-partitions and reduced ϕ-partitions and reduced ϕ-partitions of positive integers were considered by Patricia Jones [1]. That is, $n=a_1+\cdots+a_i$ is a ϕ-partition if $i>1$ and $\phi(n)=\phi(a_1)+\cdots+\phi(a_i)$, where ϕ is Euler's totient function. Furthermore, a ϕ-partition is reduced if each of its summands is simple, where a simple number is known as 1 or a product of the first primes.

In [1] the author conjectured that every nonsimple number has exactly one reduced ϕ-partition. Here, we show that the conjecture is false. In fact, we will see that the positive integers satisfying the conjecture are quite rare. The main purpose of this paper is to give a complete characterization of positive integers that have exactly one reduced ϕ-partition.

Throughout the paper, let p and q denote distinct primes, especially, p_i denote the i^{th} prime, and $A_0=1$, $A_i=\prod_{p\le p_i} p$ be the i^{th} simple number.

It is shown in [1] that every simple number has no ϕ-partitions and every nonsimple number has a ϕ-partition as follows:

(I) $\quad n=\underbrace{p^{\alpha-1}t+\cdots+p^{\alpha-1}t}_{p}$ if $n=p^{\alpha}t$ for $\alpha>1$ and $p\nmid t$;

(II) $\quad n=\underbrace{j+\cdots+j}_{p-q}+qj$ if $n=pj$ where p and q do not divide j and $q<p$.

This gives algorithms from which we can obtain at least one reduced ϕ-partition of any nonsimple number.

A nonsimple number is called semisimple if it has exactly one reduced ϕ-partition.

Our main result is the following:

Theorem: Let n be nonsimple. Then n is semisimple if and only if

(i) $\quad n$ is a prime or $n=3^2$, or

(ii) $\quad n=aq_1\cdots q_kA_i$ with $a(q_1-p_{i+1})\cdots(q_k-p_{i+1})<p_{i+1}$, where $i\gtrless 1$, $k\ge 0$, $q_1>q_2>\cdots>q_k>p_{i+1}$ are primes and a is a positive integer.

We will present the proof of the Theorem in Section 3.

It can be seen from the Theorem that $(p_{i+1}-1)A_i$ and $p_{i+2}A_i$ are semisimple. For $k\ge 2$, the smallest semisimple number is $2\times3\times5\times7\times11\times13\times19\times23=19\times23\times A_6$.

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2. LEMMAS

First, we state without proof a basic and simple lemma.

Lemma 1: Let n be semisimple and $n = a_1 + \cdots + a_i$ be any of its ϕ-partitions. Then every a_i is simple or semisimple.

Lemma 2: Let n be odd. Then n is not semisimple except $n = p$ or 3^2.

Proof: Using the algorithms (I, II), we know that one of pq and p^α ($\alpha > 1$ and $p^\alpha > 3^2$) equals n, or a summand of some ϕ-partition of n. We have the reduced ϕ-partitions of pq and p^α as follows:

$$pq = \underbrace{1 + \cdots + 1}_{(p-2)(q-2)-2} + \underbrace{2 + \cdots + 2}_{p+q-1} = \underbrace{1 + \cdots + 1}_{(p-2)(q-2)} + \underbrace{2 + \cdots + 2}_{p+q-5} + 6,$$

$$p^\alpha = \underbrace{1 + \cdots + 1}_{p^{\alpha-1}(p-2)} + \underbrace{2 + \cdots 2}_{p^{\alpha-1}} = \underbrace{1 + \cdots + 1}_{p^{\alpha-1}(p-2)+2} + \underbrace{2 + \cdots + 2}_{p^{\alpha-1}-4} + 6.$$

Now the result follows from Lemma 1. □

Lemma 3: Suppose

$$n = \underbrace{1 + \cdots + 1}_{x_0} + \underbrace{A_1 + \cdots + A_1}_{x_1} + \cdots \underbrace{A_i + \cdots + A_i}_{x_i}$$

is a ϕ-partition. Then n is not semisimple if $x_j \geq p_{j+1} + 1$ for some $1 \leq j \leq i$.

Proof: It is sufficient to show that

$$(p_{j+1} + 1)A_j = \underbrace{A_j + \cdots + A_j}_{p_j+1}$$

is not the only reduced ϕ-partition of $(p_{j+1} + 1)A_j$.

Since $A_j / 2$ is not simple, it has a reduced ϕ-partition

$$A_j / 2 = \underbrace{1 + \cdots + 1}_{y_0} + \underbrace{A_1 + \cdots + A_1}_{y_1} + \cdots + \underbrace{A_{j-1} + \cdots + A_{j-1}}_{y_{j-1}}$$

which is obtained by algorithm (II). (Notice that $y_\ell \neq 0$ for $0 \leq \ell \leq j-1$). Hence,

$$\phi(A_j) = \phi(A_j / 2) = y_0 + y_1\phi(A_1) + \cdots + y_{j-1}\phi(A_{j-1}).$$

It follows that

$$(p_{j+1} + 1)A_j = \underbrace{1 + \cdots + 1}_{2y_0} + \underbrace{A_1 + \cdots + A_1}_{2y_i} + \cdots + \underbrace{A_{j-1} + \cdots + A_{j-1}}_{2y_{j-1}} + A_{j+1} \tag{1}$$

is a reduced ϕ-partition. □

Lemma 4: Let $n = mA_i$ with $i > 1$, $p_{i+1} \nmid m$ and $p_{i+j}^2 | m$ for some $j > 1$. Then n is not semisimple.

Proof: Put $m' = m / p_{i+j}$. Then

$$n = \underbrace{m'A_i + \cdots + m'A_i}_{p_{i-j}}$$

is a ϕ-partition. Hence, if the reduced ϕ-partition

$$n = \underbrace{A_i + \cdots + A_i}_{x_i} + \underbrace{A_{i+1} + \cdots + A_{i+1}}_{x_{i+1}} + \cdots + \underbrace{A_{i+t} + \cdots + A_{i+t}}_{x_{i+t}}$$

is obtained by following the algorithms (I, II), then $x_i \geq p_{i+j} > p_{i+1}$. Thus, by Lemma 3, n is not semisimple. □

3. PROOF OF THE THEOREM

It is evident that primes and 3^2 are all semisimple. By Lemma 2 and Lemma 4, we need to consider only $n = aq_1 \cdots q_k A_i$ as given in the Theorem.

Write $q_j - p_{i+1} = \alpha_j$ for $i \leq j \leq k$ and $p_{i+2} - p_{i+1} = \beta$. Then $\alpha_1 > \alpha_2 > \cdots .\alpha_k$ and $\alpha_j > \beta$ for $1 \leq j \leq k-1$.

It is easy to see from the definition that n has a reduced ϕ-partition if and only if there are nonnegative integers $x_0, x_1, \ldots, x_\ell$ such that

$$\begin{cases} n = x_0 + x_1 A_1 + \cdots + x_\ell A_\ell, \\ \phi(n) = x_0 + x_1 \phi(A_1) + \cdots + x_\ell \phi(A_\ell). \end{cases} \tag{2}$$

Further, n is semisimple if $(x_0, x_1, \ldots, x_\ell)$ is unique.

For $n = aq_1 \cdots q_k A_i$, we have a reduced ϕ-partition

$$\begin{cases} n = a_i A_i + \cdots + a_{i+k} A_{i+k}, \\ \phi(n) = a_i \phi(A_1) + \cdots + a_{i+k} \phi(A_{i+k}), \end{cases} \tag{3}$$

which is obtained by the algorithm (II). On the other hand, we have the ϕ-partition

$$n = aq_1 \cdots q_k A_i = \underbrace{aq_1 \cdots q_{k-1} A_i + \cdots + aq_1 \cdots q_{k-1} A_i}_{\alpha_k} + aq_1 \cdots q_{k-1} A_{i+1}.$$

Let the reduced ϕ-partitions

$$\begin{cases} q_1 \cdots q_{k-1} A_i = b_i A_i + \cdots + b_{i+k-1} A_{i+k-1}, \\ \phi(q_1 \cdots q_{k-1} A_i) = b_i \phi(A_i) + \cdots + b_{i+k-1} \phi(A_{i+k-1}), \end{cases} \tag{4}$$

and

$$\begin{cases} q_1 \cdots q_{k-1} A_{i+1} = c_{i+1} A_{i+1} + \cdots + c_{i+k} A_{i+k}, \\ \phi(q_1 \cdots q_{k-1} A_{i+1}) = c_{i+1} \phi(A_{i+1}) + \cdots + c_{i+k} \phi(A_{i+k}), \end{cases} \tag{5}$$

be obtained by the algorithm (II). Then $a_i = ab_i \alpha_k, a_{i+j} = a(b_{i+j}\alpha_k + c_{i+j})$ for $1 \leq j \leq k-1$ and $a_{i+k} = ac_{i+k}$. It is not difficult to show by induction on k that

$$a_i = a\alpha_1 \cdots \alpha_k, b_i = \alpha_1 \cdots \alpha_{k-1} \text{ and } c_{i+1} = (\alpha_1 - \beta) \cdots (\alpha_{k-1} - \beta).$$

We now proceed by induction on k to prove that $a_i > a_{i+1} > \cdots > a_{i+k}$. When $k = 0$, there is nothing to show. Suppose that $k > 0$ and the conclusion holds for $k - 1$. From this, we can assume that

$$b_i > b_{i+1} > \cdots > b_{i+k-1} \text{ and } c_{i+1} > \cdots > c_{i+k}.$$

Thus,

$$a_{i+j} - a_{i+j+1} = a[(b_{i+j} - b_{i+j-1})\alpha_k + c_{i+j} - c_{i+j+1}] > 0 \text{ for } 1 \le j \le k-1.$$

It remains to show that $a_i > a_{i+1}$. We claim that $a_i = \beta a_{i+1} + a(\alpha_1 - \beta)\cdots(\alpha_k - \beta)$ which implies the conclusion. In fact, it is obvious for $k = 1$. Assume it holds for $k-1>0$. From this, it follows that $b_i = \beta b_{i+1} + (\alpha_1 - \beta)\cdots(\alpha_{k-1} - \beta) = \beta b_{i+1} + c_{i+1}$. Thus, $a_i = ab_i\alpha_k = a(\beta b_{i+1} + c_{i+1})\alpha_k$ $= a(\beta b_{i+1}\alpha_k + \beta c_{i+1}) + ac_{i+1}(\alpha_k - \beta)$ $= \beta a_{i+1} + a(\alpha_1 - \beta)\cdots(\alpha_k - \beta)$. Recall that $a_i < p_{i+1}$.

Set

$$S = S(n) = \{\underline{x} = (x_0, x_1, \ldots, x_{i+k}) \mid \underline{x} \text{ satisfies } (2)\}.$$

Then $\underline{a} = (a_0, \ldots, a_{i-1}, a_i, \ldots, a_{i+k}) \in S$, where $a_0 = \cdots a_{i-1} = 0$ and $a_i, \ldots, a_{i+k}$ are as in (3). Define on S an order "$\succ$" as $\underline{x} \succ \underline{x'}$ if $x_j > x'_j$, for some $j \ge 0$, and $x_{j+\ell} \ge x'_{j+\ell}$ for $\ell \ge 0$. Since

$$\begin{aligned} n = \sum_{j=i}^{i+k} a_j A_j &\le \sum_{j=i}^{i+k-1} (p_{j+1} - 1)A_j + a_{i+k}A_{i+k} \\ &= -A_i + (a_{i+k}+1)A_{i+k} < (a_{i+k}+1)A_{i+k} < A_{i+k+1}, \end{aligned}$$

every solution of (2) is contained in S, and similarly, we can show that $\underline{a}$ is the maximal element of the totally ordered set $(S, \succ)$. If $S \ne \{\underline{a}\}$, we let $\underline{b}$ be the maximal element of $(S \setminus \{\underline{a}\}, \succ)$ and distinquish two cases as follows:

(i) $b_j > p_{j+1}$ for some $1 \le j \le i+k$. Put

$$\underline{t} = (b_0 + y_0, b_1 + y_1, \ldots, b_{j-1} + y_{j-1}, b_j, b_{j+1} + 1, \ldots, b_{i+k})$$

where $y_0, y_1, \ldots, y_{j-1}$ are as in (1). Then it follows that $\underline{t} \in S$. Since $\underline{t} > \underline{b}$, then $\underline{t} = \underline{a}$. In fact, this is impossible since, in formula (1), $y_\ell \ne 0, \ell = 0, 1, \ldots, j-1$, always holds. This contradicts $a_0 = 0$.

(ii) $b_j \le p_{j+1}, j = 1, 3, \ldots, i+k$. Since $\underline{a} \succ \underline{b}$, there is an $\ell, i \le \ell \le i+k$, such that $a_\ell > x_\ell$ and $a_{\ell+j} = b_{\ell+j}$ for $j > 0$. Write $c = a_\ell - x_\ell^0$ and $c_j = x_j^0 - a_j, j = 0, 1, \ldots, \ell-1$. Then

$$cA_\ell = \sum_{j=0}^{\ell-1} c_j A_j \text{ and } c\phi(A_\ell) = \sum_{j=0}^{\ell-1} c_j \phi(A_j).$$

Thus,

$$c(A_\ell - \phi(A_t)) = \sum_{j=1}^{\ell-1} c_j(A_j - \phi(A_j)).$$

Set $\sigma_j = \phi(A_j)/A_j$. Then $\sigma_j > \sigma_{j+1}$ for $j \ge 1$, and $0 < (1-\sigma_j)/(1-\sigma_\ell) < 1$ for $1 \le j < \ell$. Put $\tau_j = (1-\sigma_j)/(1-\sigma_\ell)$. Then

$$cA_\ell = \sum_{j=1}^{\ell-1} c_j A_j \tau_j \le \sum_{j=1}^{\ell-1} |c_j| A_j \tau_j < \sum_{j=1}^{\ell-1} |c_j| A_j .$$

If $\ell = i$ (when $k = 0$ this is always the case), then $c_j = x_j^0$ for $0 \le j < \ell$. In this case,

$$cA_\ell < \sum_{j=1}^{\ell-1} |c_j| A_j = \sum_{j=1}^{\ell-1} c_j A_j \le cA_\ell,$$

which is a contradiction. If $\ell > i$, then $a_{\ell-1} > a_\ell \ge 1$, and

$$cA_\ell < \sum_{j=1}^{\ell-1} |c_j| A_j \le (p_\ell - 2) A_{\ell-1} + \sum_{j=1}^{\ell-2} p_{j+1} A_j = A_\ell - A_{\ell-1} + A_{\ell-2} + \cdots + A_2 < A_\ell$$

which again yields a contradiction. By the preceding discussion, we have shown $S = \{\underline{a}\}$, i.e., $\underline{a}$ is unique. The proof is complete. □

4. CONCLUDING REMARKS

We mention here that it would be interesting to find the set $S(n)$ for any nonsemisimple number n. We guess that there is a unique $\underline{x} = (x_0, x_1, \dots)$ in $S(n)$ such that $o \le x_j \le p_{j+1}$ for $j \ge 1$. In this case, $S(n)$ can be derived exclusively by using the algorithms (I, II) and formula (1).

ACKNOWLEDGMENT

I wish to thank the referee for a shorter and simpler proof of Lemma 3 and for many valuable comments.

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Announcement

SIXTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3515@compuserve.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-748 ***Proposed by Herta T. Freitag, Roanoke, VA***

Let $u_k = F_{kn}/F_n$ for some fixed positive integer n. Find a recurrence satisfied by the sequence (u_k).

B-749 ***Proposed by Richard André-Jeannin, Longwy, France***

For n a positive integer, define the polynomial $P_n(x)$ by $P_n(x) = x^{n+2} - x^{n+1} - F_n x - F_{n-1}$. Find the quotient and remainder when $P_n(x)$ is divided by $x^2 - x - 1$.

B-750 ***Proposed by Seung-Jin Bang, Albany, CA***

Find a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(F_n, L_n) = (F_{n+1}, L_{n+1})$.

B-751 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. College of Science, Gujarat State, India***

Prove that $6L_{n+3}L_{3n+4} + 7$ and $6L_n L_{3n+5} - 7$ are divisible by 25.

B-752 ***Proposed by Richard André-Jeannin, Longwy, France***

Consider the sequences (U_n) and (V_n) defined by the recurrences $U_n = PU_{n-1} - QU_{n-2}$, $n \geq 2$, with $U_0 = 0, U_1 = 1$, and $V_n = PV_{n-1} - QV_{n-2}$, $n \geq 2$, with $V_0 = 2, V_1 = P$, where P and Q are real numbers with $P > 0$ and $\Delta = P^2 - 4Q > 0$. Show that for $n \geq 0, U_{n+1} \geq (P/2)U_n$ and $V_{n+1} \geq (P/2)V_n$.

B-753 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. College of Science, Gujarat State, India***

Prove that, for all positive integers n,

$$\begin{vmatrix} F_{n-1}^3 & F_n^3 & F_{n+1}^3 & F_{n+2}^3 \\ F_n^3 & F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 \\ F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 \\ F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 & F_{n+5}^3 \end{vmatrix} = 36.$$

SOLUTIONS

Convolution Solution

B-720 ***Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy (Vol. 30, no. 3, August 1992)***

Find a closed form expression for $S_n = \sum_{h+k=2n} F_h F_k$, where the sum is taken over all pairs of positive integers (h, k) such that $h+k=2n$ and $h \le k$.

Solution by Russell Euler, Northwest Missouri State University, Maryville, MO

Using the Binet formula, we have

$$S_n = \sum_{h=1}^{n} F_h F_{2n-h} = \frac{1}{5}\sum_{h=1}^{n}\left[\alpha^{2n} + \beta^{2n} - \beta^{2n}\left(\frac{\alpha}{\beta}\right)^h - \alpha^{2n}\left(\frac{\beta}{\alpha}\right)^h\right]$$

$$= \frac{1}{5}\left[nL_{2n} - \beta^{2n}\frac{(\alpha/\beta)-(\alpha/\beta)^{n+1}}{1-\alpha/\beta} - \alpha^{2n}\frac{(\beta/\alpha)-(\beta/\alpha)^{n+1}}{1-\beta/\alpha}\right].$$

The sum was evaluated by the standard formula for the sum of a geometric progression:

$$\sum_{h=1}^{n} r^h = \frac{r-r^{n+1}}{1-r}.$$

Upon simplifying, we find that

$$S_n = \frac{1}{5}\left[nL_{2n} + F_{2n-1} - (-1)^n\right].$$

Most sums of this form can be found by the same method. Other, equivalent formulas found by solvers were: $(nL_{2n} + F_n L_{n-1})/5$, $nF_n^2 + [F_{2n-1} + (-1)^n(2n-1)]/5$, $[(n+1)L_{2n} - L_n F_{n+1}]/5$, $(n+1)F_n^2 - [F_{2n+1} - (2n+1)(-1)^n]/5$, and $[(5n+1)L_{2n} + L_{2n-2} - 5(-1)^n]/25$.

Seiffert mentions the related convolution ([1], p. 118):

$$\sum_{j=1}^{2n-1} F_j F_{2n-j} = \frac{1}{5}\left[(2n-1)F_{2n+1} + (2n+1)F_{2n-1}\right].$$

Reference:

1. V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson. "Fibonacci Convolution Sequences." *The Fibonacci Quarterly* **15.2** (1977):117-22.

Also solved by Paul S. Bruckman, Russell Euler, Graham Lord, Dorka Ol. Popova, Bob Prielipp, H.-J. Seiffert, Tony Shannon, Sahib Singh, and the proposer.

Brittany Climbs Some Stairs

B-721 ***Proposed by Russell Jay Hendel, Dowling College, Oakdale, NY (Vol. 30, no. 3, August 1992)***

Brittany is going to ascend an m step staircase. At any time she is just as likely to stride up one step as two steps. For a positive integer k, find the probability that she ascends the whole staircase in k strides.

Editor's Comment: *Only one correct solution was received. We therefore begin by analyzing where most solvers went wrong.*

Those "solvers" fell into two camps: Camp A believes the answer is $\binom{k}{m-k}/2^k$; Camp B believes the answer is $\binom{k}{m-k}/F_{m+1}$. Both camps agree that the number of distinct ascents with k strides is $\binom{k}{m-k}$ and that the total number of different ways of climbing the stairs is F_{m+1} (see [2], p. 10).

Let us look at a staircase with 3 steps (the case $m = 3$). Camps A and B would have us believe the probabilities as shown in the corresponding tables below. In these tables, $p(k)$ denotes the probability that Brittany ascends in k strides.

k	1	2	3
$p(k)$	0	$\frac{2}{4}$	$\frac{1}{8}$

Camp A

k	1	2	3
$p(k)$	0	$\frac{2}{3}$	$\frac{1}{3}$

Camp B

Camp A cannot be correct because their probabilities do not add up to 1.

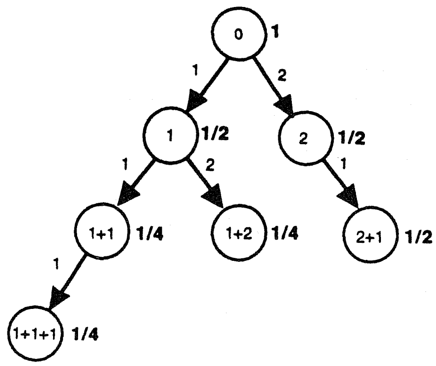
Camp B notes that there are 3 types of ascents, $2+1, 1+2$, and $1+1+1$. They assume each method of ascent is equally likely. Since there are 2 ascents of length 2 and 1 ascent of length 3, this determines the probabilities shown in their table above. The probabilities add up to 1. However, Camp B cannot be correct because they believe ascents that begin with a stride of 1 step ($1+2$ and $1+1+1$) occur twice as often as ascents that begin with a stride of 2 steps ($2+1$). Yet we know that on Brittany's first stride, she is just as likely to stride up 1 step as 2 steps.

To try to settle this discrepancy, I decided to watch Brittany during the past year and record data about her ascents. Fortunately, she climbs the Tower of London frequently, and I was able to record data on 4000 ascents of 3-step staircases. It turns out that 3015 ascents were of length 2 and 985 were of length 3. This suggests that the correct probabilities are those given in the table below:

k	1	2	3
$p(k)$	0	$\frac{3}{4}$	$\frac{1}{4}$

Observed Probabilities

These probabilities can be confirmed by the following transition tree. Each node represents a state during the ascent. The edges show whether Brittany makes a stride of 1 or 2 steps from each state. The probability of reaching each state is given to the right of that state. We assume that all paths out of a state are equally likely, since at any time, Brittany is just as likely to stride up 1 step as 2 steps. Of course, when Brittany is one step away from her goal, she is forced to make a final stride of 1 step.



One can see from this transition tree that the probability of a 3-stride ascent is 1/4. Next, we move on to the general solution.

Solution by Peter Griffin, California State University, Sacramento, CA

There are two mutually exclusive ways to ascend m steps in k strides.

If the final stride was a double-step, then Brittany made $m-k-1$ double-steps in her first $k-1$ strides. The number of ways this could happen is $\binom{k-1}{m-k-1}$. Each of the k strides in the complete ascent occurred with probability 1/2.

If the final stride was a single step, then the last stride was forced and thus was taken with probability 1. In her first $k-1$ strides, Brittany must have made $m-k$ double-steps (each with probability 1/2). The number of ways this could happen is $\binom{k-1}{m-k}$.

Thus, the probability of ascending the whole staircase in k strides is

$$\frac{\binom{k-1}{m-k-1}}{2^k}+\frac{\binom{k-1}{m-k}}{2^{k-1}}=\frac{3k-m}{k}\binom{k}{m-k}\Big/2^k.$$

The proposer indicated that his proposal generalizes Problem 10 on page 407 of [1] and that this problem is a natural example of a discrete probability space that can be represented by a tree whose paths are not all the same length.

References:

1. Billstein, Libeskind, & Lott. *Mathematics for Elementary School Teachers: A Problem Solving Approach.* 4th ed. Redwood City, CA: Benjamin/Cummings, 1990.
2. S. Vajda. *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Four incorrect solutions were received.

Fibonacci Integrand

B-722 ***Proposed by H.-J. Seiffert, Berlin Germany (Vol. 30, no. 3, August 1992)***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, and $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n\geq 2$. Show that for all nonnegative integers n,

$$\int_0^\infty \frac{dx}{(x^2+1)F_{2n+1}(2x)}=\frac{\pi}{4n+2}.$$

Solution by Hans Kappus, Rodersdorf, Switzerland

The Binet formula for the Fibonacci polynomials ([2], p. 99) is

$$F_n(x)=\frac{1}{\sqrt{x^2+4}}\left[\left(\frac{x+\sqrt{x^2+4}}{2}\right)^n-\left(\frac{x-\sqrt{x^2+4}}{2}\right)^n\right].$$

Thus, the integral turns out to be

$$I_n=\int_0^\infty \frac{2dx}{\sqrt{x^2+1}[(x+\sqrt{x^2+1})^{2n+1}-(x-\sqrt{x^2+1})^{2n+1}]}.$$

The substitution $x=\sinh t$ gives

$$I_n=\int_0^\infty \frac{dt}{\cosh[(2n+1)t]}.$$

Finally, the substitution $t=\theta/(2n+1)$ gives

$$I_n=\frac{1}{2n+1}\int_0^\infty \operatorname{sech}\theta\,d\theta=\frac{1}{2n+1}[\arctan(\sinh\theta)]_0^\infty=\frac{\pi}{4n+2}.$$

We have used the following well-known results about hyperbolic functions (see §4.5 of [1]):

$$\frac{d\sinh z}{dz}=\cosh z,\ \ \cosh^2 z-\sinh^2 z=1,\ \ (\cosh z+\sinh z)^n=\cosh nz+\sinh nz,$$

and

$$\int \operatorname{sech} z\,dz=\arctan(\sinh z).$$

References:

1. Milton Abramowitz & Irene A. Stegun. *Handbook of Mathematical Functions.* Washington, D.C.: National Bureau of Standards, 1964.
2. P. Filipponi & A. F. Horadam. "Derivative Sequences of Fibonacci and Lucas Polynomials." In *Applications of Fibonacci Numbers.* Vol. 4, pp. 99-108. Dordrecht: Kluwer, 1991.

Also solved by Seung-Jin Bang, Paul S. Bruckman, Piero Filipponi, Igor Ol. Popov, and the proposer.

The Great Divide

B-723 ***Proposed by Bruce Dearden & Jerry Metzger, U. of North Dakota, Grand Forks, ND (Vol. 30, no. 3, August 1992)***

(a) Show that, for $n \equiv 2 \pmod 4$, $F_{n+1}(F_n^2 + F_n - 1)$ divides $F_n^n(F_n^2 + F_{n+1}) - 1$.

(b) What is the analog of (a) for $n \equiv 0 \pmod 4$?

Solution by H.-J. Seiffert, Berlin, Germany

It is easily verified that, for all positive integers k,

$$\sum_{j=0}^{2k-1} F_{j+1}x^j = \frac{F_{2k}x^{2k+1} + F_{2k+1}x^{2k} - 1}{x^2 + x - 1}, \tag{1}$$

$$\sum_{r=0}^{k-1} (-1)^r F_{2r+1} = \frac{2 - (-1)^k L_{2k}}{5}, \tag{2}$$

$$\sum_{r=0}^{k-1} (-1)^r F_{2r+2} = \frac{1 - (-1)^k L_{2k+1}}{5}, \tag{3}$$

$$2 + F_{4k+2} + L_{4k+2} + F_{4k+2}L_{4k+3} = 5F_{4k+3}F_{2k+1}^2, \tag{4}$$

and

$$F_n^2 = F_{n-1}F_{n+1} - (-1)^n. \tag{5}$$

(a) If n is a positive integer with $n \equiv 2 \pmod 4$, then there exists a nonnegative integer k, such that $n = 4k+2$. Now in formula (1), replace k by $2k+1$ and set $x = F_{4k+2}$ to obtain

$$Q = \frac{F_{4k+2}^{4k+2}(F_{4k+2}^2 + F_{4k+3}) - 1}{F_{4k+2}^2 + F_{4k+2} - 1} = \sum_{j=0}^{4k+1} F_{j+1}F_{4k+2}^j.$$

Note that the left side of this equation, Q, is an integer, showing that $F_n^n(F_n^2 + F_{n+1}) - 1$ is divisible by $F_n^2 + F_n - 1$. It remains to show that Q is divisible by F_{n+1}.

Using result (5) with $n = 4k+2$, we find

$$Q = \sum_{r=0}^{2k} F_{2r+1}F_{4k+2}^{2r} + \sum_{r=0}^{2k} F_{2r+2}F_{4k+2}^{2r+1} \equiv \sum_{r=0}^{2k} (-1)^r F_{2r+1} + F_{4k+2}\sum_{r=0}^{2k} (-1)^r F_{2r+2} \pmod{F_{4k+3}}.$$

Applying results (2) and (3) followed by (4) gives

$$Q \equiv \frac{2 + F_{4k+2} + L_{4k+2} + F_{4k+2}L_{4k+3}}{5} \equiv F_{4k+3}F_{2k+1}^2 \equiv 0 \pmod{F_{4k+3}}.$$

Thus, Q is divisible by $F_{n+1} = F_{4k+3}$.

(b) By the same method, one can prove that, for $n \equiv 0 \pmod 4$,

$$F_{n+1}(F_n^2 - F_n - 1) \text{ divides } F_n^n(F_n^2 - F_{n+1}) + 1.$$

Also solved by Paul S. Bruckman and the proposers.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-481 ***Proposed by Richard André-Jeannin, Longwy, France***

Let $\phi(x)$ be the function defined by

$$\phi(x)=\sum_{n\geq 0}\frac{x^n}{F_{r^n}},$$

where $r\geq 2$ is a natural integer. Show that $\phi(x)$ is an irrational number, if x is a nonzero rational number.

H-482 ***Proposed by Larry Taylor, Rego Park, NY***

Let j, k, m, and n be integers. Let $A_n(m)=B_n(m-1)+4A_n(m-1)$ and $B_n(m)=4B_n(m-1)+5A_n(m-1)$ with initial values $A_n(0)=F_n$, $B_n(0)=L_n$.

(A) Generalize the numbers (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2) to form an eleven-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

(B) Generalize the numbers (3, 3, 3, 3, 3, 3, 3, 3, 3, 3) to form a ten-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

(C) Generalize the numbers (1, 1, 1, 1, 1, 1, 1, 1) to form an eight-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

Hint: $A_n(1)=-11(-1)^n A_{-n}(-1)$.

Reference: L. Taylor. Problem H-422. *The Fibonacci Quarterly* **28.3** (1990):285-87.

SOLUTIONS

A ... Periodic

H-464 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 30, no. 1, February 1992)

Show that $\sum_{k=1}^{[n/2]}\binom{n}{k}A_{n-2k}=F_n$, where $A_j=(-1)^{[(j+2)/5]}-\left((-1)^{[j/5]}+(-1)^{[(j+4)/5]}\right)/2$. [ ] denotes the greatest integer function.

Solution by C. Georghiou, University of Patras, Patras, Greece

First, note that A_j is periodic with period 10 and with $A_0 = A_5 = 0$, $A_1 = A_2 = A_8 = A_9 = 1$, and $A_3 = A_4 = A_6 = A_7 = -1$. Its (ordinary) generating function is

$$\begin{aligned} g(z) &= (z+z^2-z^3-z^4-z^6-z^7+z^8+z^9)/(1-z^{10}) \\ &= (z+z^2-z^3-z^4)/(1+z^5) = z(1-z)(1+z)/(1-z+z^2-z^3+z^4) \\ &= \frac{z^{-1}-z}{z^2-z+1-z^{-1}+z^2}. \end{aligned}$$

Second, let

$$f(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]}\binom{n}{k}A_{n-2k}\right)x^n = \sum_{n,k=0}^{\infty}\binom{n+2k}{k}A_n x^{n+2k} = \sum_{n=0}^{\infty} A_n x^n \, {}_2F_1\left[\begin{matrix} n/2+1/2, n/2+1 \\ n+1 \end{matrix}; 4x^2\right],$$

where ${}_2F_1[\ \ ]$ is the Gauss hypergeometric series (see solution of H-444). But

$${}_2F_1\left[\begin{matrix} a,\ a+1/2 \\ 2a \end{matrix}; z\right] = 2^{2a-1}(1-z)^{-1/2}\left[1+(1-z)^{1/2}\right]^{1-2a},$$

(see M. Abramowitz & I. A. Stegun, *Handbook of Mathematical Functions* [New York: Dover, 1965] Entry 15.1.14, p. 556), and therefore, by setting $\partial = (1-4x^2)^{1/2}$ we obtain

$$f(x) = \frac{1}{\partial}\sum_{n=0}^{\infty} A_n\left(\frac{2x}{1+\partial}\right)^n = \frac{1}{\partial}g\left(\frac{2x}{1+\partial}\right).$$

Now

$$\frac{1+\partial}{2x} - \frac{2x}{1+\partial} = \partial/x \qquad \frac{1+\partial}{2x} + \frac{2x}{1+\partial} = 1/x$$

and

$$\left(\frac{1+\partial}{2x}\right)^2 + \left(\frac{2x}{1+\partial}\right)^2 = 1/x^2 - 2.$$

Therefore

$$f(x) = \frac{x}{1-x-x^2}$$

which is the generating function of F_n, and the assertion follows readily. Note that the problem is the same as H-444.

Also solved by P. Bruckman and the proposer.

<u>B Good</u>

<u>H-465</u> ***Proposed by Richard André-Jeannin, Tunisia***
(Vol. 30, no. 1, February 1992)

Let p be a prime number, and let $r_1, r_2, \ldots, r_s$ be natural integers such that $s \geq 2, r_1 < p$, and $\sum_{k=1}^{k=s} kr_k = p$. Show that the number

$$B_{r_1,r_2,\dots,r_s} = \frac{1}{r_1+r_2+\cdots+r_s}\,\frac{(r_1+r_2+\cdots+r_s)!}{r_1!r_2!\cdots r_s!}$$

is an integer.

Solution by Paul S. Bruckman, Edmonds, WA

Let $B_s \equiv B_{r_1,r_2,\dots,r_s}$ for brevity. Let N denote the set of positive integers. We may express B_s as follows:

$$B_s = \frac{(r_1+r_2+\cdots+r_s-1)!}{r_1!r_2!\cdots r_s!}. \tag{1}$$

From the condition $\sum_{k=1}^{s} kr_k = p$, with $1 \le r_k$, $k = 1, 2, \dots, s$, it follows that $r_k < p$. Then, we see from (1) that $B_s = A/B$, say, where $\gcd(B, p) = 1$.

Also, there are s distinct ways to express B_s, as follows:

$$B_s = U_k / r_k,\ \ k = 1, 2, \dots, s, \tag{2}$$

where U_k is the multinomial coefficient defined as follows:

$$U_k = \frac{(r_1+r_2+\cdots+r_s-1)!}{r_1!r_2!\cdots(r_k-1)!\cdots r_s!}. \tag{3}$$

As we know, the U_k's are positive integers. Therefore, $r_kB_s \in N$. Therefore, $B_s\sum_{k=1}^{s} kr_k = pB_s \in N$. This implies that either $r_kB_s \in N$, or else $B_s = A/p$ for some integer A; however, as we have seen, this latter contingency is impossible, so we are done.

Also solved (partially) by the proposer.

A Unique Answer

H-466 ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 30, no. 2, May 1992)

Let p be a prime of the form ax^2+by^2, where a and b are relatively prime natural numbers neither of which is divisible by p; x and y are integers. Prove that x and y are uniquely determined, except for trivial variations of sign.

Solution by Don Redmond, Southern Illinois University, Carbondale, IL

Suppose that there are two representations, say, $p = ax^2+by^2$ and $p = ar^2+bs^2$, where we may assume that x, y, r, and s are natural numbers. Then $(x, y) = (r, s) = 1$. If we eliminate b between the two representations, we have $p(y^2-s^2) = a(r^2y^2-s^2x^2)$.

Since $p\nmid a$, we see that $p|(r^2y^2-s^2x^2)$, and so, for some choice of sign, we have

$$ry \equiv \pm sx \pmod{p}. \tag{1}$$

Also, the two representations give

$$p^2 = (ax^2+by^2)(ar^2+bs^2) = (axr \pm bys)^2 + ab(ry \mp sx)^2. \tag{2}$$

If $ry = sx$, then $(x, y) = 1 = (r, s)$ implies that $r = x$ and $s = y$.

If $ry \neq sx$, then (1) and (2) imply that $|ry \pm sx| = p, a = b = 1$, and $axr \pm bys = 0$. This implies, since $x^2 + y^2 = r^2 + s^2 = p$, that $x = s$ and $y = r$.

Thus p has essentially only one representation. □

Also solved by R. Isreal and the proposer.

Many Congruences

H-467 ***Proposed by Larry Taylor, Rego Park, NY (Vol. 30, no. 2, May 1992)***

Let (a_n, b_n, c_n) be a primitive Pythagorean triple for n = 1, 2, 3, 4, where a_n, b_n, c_n are positive integers and b_n is even. Let $p \equiv 1 \pmod 8$ be prime; $r^2 + s^2 \equiv t^2 \pmod p$, where the Legendre symbol $\left(\frac{(t+r)/2}{p}\right) = 1$.

Solve the following twelve simultaneous congruences:

$$\begin{aligned}(a_1, b_1, c_1) &\equiv (r, s, t),\\ (a_2, b_2, c_2) &\equiv (r, s, -t),\\ (a_3, b_3, c_3) &\equiv (s, r, t),\\ (a_4, b_4, c_4) &\equiv (s, r, -t) \pmod p.\end{aligned}$$

For example, if $(r, s, t) \equiv (3, 4, 5) \pmod{17}$,

$$\begin{aligned}(a_1, b_1, c_1) &= (3, 4, 5),\\ (a_2, b_2, c_2) &= (105, 208, 233),\\ (a_3, b_3, c_3) &= (667, 156, 685),\\ (a_4, b_4, c_4) &= (21, 20, 29).\end{aligned}$$

Solution by Paul S. Bruckman, Edmonds, WA

All congruences are assumed to be (mod p), unless otherwise specified. Some definitions and notational remarks are in order. A pair of integers (u, v) is said to be a *generator* of the primitive Pythagorean triple (p.P.t.) (a, b, c) if the following conditions hold:

$$u > v > 0;\ u \not\equiv v \pmod 2;\ \gcd(u, v) = 1. \tag{1}$$

In that event, we have

$$a = u^2 - v^2;\ b = 2uv;\ c = u^2 + v^2. \tag{2}$$

We also write $(u, v) \in G(a, b, c)$, meaning that (u, v) satisfies (1), and (2) holds.

The hypothesis implies that r and t have the same parity, since $\left(\frac{\frac{1}{2}(t+r)}{p}\right) = 1$ is a stronger statement than $\left(\frac{2^{-1}(t+r)}{p}\right) = 1$; also, it is implied that s is even. Since $\left(\frac{1}{2}s\right)^2 \equiv \left[\frac{1}{2}(t+r)\right]\left[\frac{1}{2}(t-r)\right]$, it follows that $\left(\frac{\frac{1}{2}(t-r)}{p}\right) = 1$. Therefore, there exist integers u' and v' such that

$$(u')^2 \equiv \tfrac{1}{2}(t+r),\quad (v')^2 \equiv \tfrac{1}{2}(t-r). \tag{3}$$

By adding or subtracting the congruences in (3), we obtain

$$t \equiv (u')^2 + (v')^2,\quad r \equiv (u')^2 - (v')^2. \tag{4}$$

Also, $4(u'v')^2 \equiv t^2 - r^2 \equiv s^2$; thus, by an appropriate choice of signs for u' and / or v', we have

$$s \equiv 2u'v'. \tag{5}$$

There is nothing in the hypotheses to suggest that (r, s, t) is a p.P.t., even though $(r, s, t) = (3, 4, 5)$ in the example, which is indeed a p.P.t.; we could just as well have been given $(r, s, t) = (37, -30, 73)$, which also satisfies the hypotheses for $p = 17$, yet $37^2 + 30^2 \neq 73^2$. Nor is it likely that our initial choice of u' and v' satisfying (3) and (5) satisfy (1). However, we see that by adding suitable multiples of p to u' and / or v', we do obtain a new pair (u_1, v_1) that satisfies (1). It is then true that $(u_1, v_1) \in G(a_1, b_1, c_1)$, where $(a_1, b_1, c_1) \equiv (r, s, t)$. To use the data of the example, we may take $(u_1, v_1) = (2, 1)$ as the solution of (3) and (5), with $p = 17$, $(r, s, t) = (3, 4, 5)$, also satisfying (1), since $(2, 1) \in G(3, 4, 5)$.

Next, we observe that since $p \equiv 1 \pmod 8$, there exist solutions i and j of the following congruences:

$$i^2 \equiv -1, \quad j^2 \equiv 2^{-1}. \tag{6}$$

In fact, there are *two* solutions for each congruence in (6). We will need to choose the signs of i and j such that appropriate generators (u_n, v_n) may be found for (a_n, b_n, c_n), $n = 2, 3, 4$. Thus, for $n = 2$, and for an appropriate solution i of (6), we claim that (u_2, v_2) is found from the following:

$$u_2 \equiv iv_1, \quad v_2 \equiv -iu_1. \tag{7}$$

Proof: Given (7), then $u_2^2 - v_2^2 \equiv i^2(v_1^2 - u_1^2) \equiv u_1^2 - v_1^2 \equiv r$; $2u_2v_2 \equiv -2i^2u_1v_1 \equiv 2u_1v_1 \equiv s$; and $u_2^2 + v_2^2 \equiv i^2(u_1^2 + v_1^2) \equiv -u_1^2 - v_1^2 \equiv -t$. Also, we determine u_2 and v_2 that satisfy (1). It then follows that $(u_2, v_2) \in G(a_2, b_2, c_2)$, with $(a_2, b_2, c_2) \equiv (r, s, -t)$. In this example, we take $i \equiv -4$, $u_2 \equiv -4 \cdot 1$, $v_2 \equiv 4 \cdot 2$. We find that we may take $(u_2, v_2) = (13, 8)$, and that $(13, 8) \in G(105, 208, 233)$; also, $(105, 208, 233) \equiv (3, 4, -5)$.

Next, we claim that, by an appropriate choice of j, we have:

$$u_3 \equiv j(u_1 + v_1), \quad v_3 \equiv j(u_1 - v_1). \tag{8}$$

Proof: $u_3^2 - v_3^2 \equiv j^2 \cdot 4u_1v_1 \equiv 2u_1v_1 \equiv s$; $2u_3v_3 \equiv 2j^2(u_1^2 - v_1^2) \equiv u_1^2 - v_1^2 \equiv r$; and $u_3^2 + v_3^2 \equiv 2j^2(u_1^2 + v_1^2) \equiv u_1^2 + v_1^2 \equiv t$. In the example, $j \equiv 3$; then, $u_3 \equiv 3 \cdot 3, v_3 \equiv 3 \cdot 1$. We may take $(u_3, v_3) = (26, 3)$, and we find that this pair generates $(a_3, b_3, c_3) = (667, 156, 685) \equiv (4, 3, 5)$.

Finally, we claim that, for appropriate i and j, we have

$$u_4 \equiv ij(u_1 - v_1), \quad v_4 \equiv -ij(u_1 + v_1); \tag{9}$$

equivalently,

$$u_4 \equiv -iv_3, \quad v_4 \equiv iu_3. \tag{10}$$

Proof: $u_4^2 - v_4^2 \equiv i^2(v_3^2 - u_3^2) \equiv u_3^2 - v_3^2 \equiv s$; $2u_4v_4 \equiv -2i^2u_3v_3 \equiv 2u_3v_3 \equiv r$; and $u_4^2 + v_4^2 \equiv -i^2(v_3^2 + u_3^2) \equiv -u_3^2 - v_3^2 \equiv -t$. In this example, take $i \equiv 4$. Then $u_4 \equiv -4 \cdot 3 \equiv 5$ and $v_4 \equiv 4 \cdot 26 \equiv 2$. We find that $(5, 2) \in G(21, 20, 29)$, where $(21, 20, 29) \equiv (4, 3, -5)$.

To summarize, $(u_n, v_n) \in G(a_n, b_n, c_n)$, $n = 1, 2, 3, 4$, where

$$\begin{aligned} &u_1^2 \equiv 2^{-1}(t+r),\ v_1^2 \equiv 2^{-1}(t-r);\ \ u_2 \equiv iv_1,\ v_2 \equiv -iu_1; \\ &u_3 \equiv j(u_1+v_1),\ v_3 \equiv j(u_1-v_1);\ \ u_4 \equiv -iv_3,\ v_4 \equiv iu_3; \end{aligned} \tag{11}$$

(u_1, v_1) and the values of i and j are obtained as appropriately chosen solutions of (3), (5), and (6), so as to satisfy (1) for each (u_n, v_n).

Also solved by the proposer.

A Very Odd Problem

H-468 ***Proposed by Lawrence Somer, Washington, D.C.***
(Vol. 30, no. 2, May 1992)

Let $\{v_n\}_{0 \le n < \infty}$ be a Lucas sequence of the second kind satisfying the recursion relation $v_{n+2} = av_{n+1} + bv_n$, where a and b are positive odd integers and $v_0 = 2, v_1 = a$. Show that v_{2n} has an odd prime divisor $p \equiv 3 \pmod 4$ for $n \ge 1$.

Solution by Russell Jay Hendel, Patchogue, NY

If a is odd, then $a^2 \equiv 1 \pmod 4$ and $2a \equiv 2 \pmod 4$. It follows that the congruence classes modulo 4 of the sequence $v_0, v_1, v_2, \ldots,$ are $2, a, 3, a(3+b), 3, 3ab, 2, a, \ldots$. Since this sequence has period 6, $v_{6n\pm 2} \equiv 3 \pmod 4$, implying that at least one of the prime factors of $v_{6n\pm 2}$ is congruent to 3 modulo 4.

v_{2n} is either of the form v_{6n} or $v_{6n\pm 2}$. Therefore, we have to deal with the case v_{2n}. First we note that $v_n | v_{nk}$ for any odd integer k. This follows because the Binet form of v_n is $\gamma^n + \delta^n$ with $\gamma = (a + \sqrt{\{a^2 + 4b\}})/2, \gamma + \delta = a. \gamma\delta = b$. Therefore, if k is an odd integer, the formula $x^k + y^k = (x+y)\{x^{k-1} + y^{k-1} - xy(x^{k-2} + y^{k-2}) + (xy)^2(x^{k-3} + y^{k-3}) \cdots \pm (xy)^{(k-1)/2}\}$ implies, with $x = \gamma^n$, $y = \delta^n$, that $v_n | v_{nk}$.

Proceeding as in [1], for each integer n, $6n = 2^m(6n' + 3)$, for some integers m and n'. Since $2^m \equiv \pm 2 \pmod 6$, there is a prime $p \equiv 3 \pmod 4$ such that p divides v_{2^m}. Since $6n/2^m$ is odd, p also divides v_{2n} and the proof is complete.

Reference:

1. Sahib Singh. "Thoro's Conjecture and Allied Divisibility Property of Lucas Numbers." *The Fibonacci Quarterly* **18.2** (1980):135.

Also solved by P. Bruckman, R. André-Jeannin, and the proposer.

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