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# p-ADIC CONGRUENCES FOR GENERALIZED <br> FIBONACCI SEQUENCES 

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## 1. STATEMENT OF RESULTS

Let $\lambda, \mu \in \mathbb{Z}$ and define a sequence of integers $\left\{\gamma_{n}\right\}_{n \geq 0}$ by the binary linear recurrence

$$
\begin{equation*}
\gamma_{0}=0, \gamma_{1}=1 \text {, and } \gamma_{n+1}=\lambda \gamma_{n}+\mu \gamma_{n-1} \text { for } n>0 . \tag{1.1}
\end{equation*}
$$

It is well known [9] that the polynomial $P(t)=1-\lambda t-\mu t^{2}$ has the property that

$$
\begin{equation*}
P(t)^{-1}=\sum_{n=1}^{\infty} \gamma_{n} t^{n-1} \tag{1.2}
\end{equation*}
$$

is the ordinary formal power series generating function for the sequence $\left\{\gamma_{n+1}\right\}_{n \geq 0}$ (cf. [12]. Furthermore, it is easy to see [1] that when the discriminant $\Delta=\lambda^{2}+4 \mu$ of $P(t)$ is nonnegative and $\lambda \neq 0$, the ratios $\gamma_{n+1} / \gamma_{n}$ converge (in the usual archimedean metric on $\mathbb{R}$ ) to a reciprocal root $\alpha$ of $P(t)$. In this article we show that ratios of these $\gamma_{n}$ also exhibit rapid convergence properties relating to $P(t)$ in the $p$-adic metrics on $\mathbb{Q}$. Precisely, we prove that for all primes $p$ and all positive integers $m$ the ratios $\gamma_{m p^{r}} / \gamma_{m p^{r-1}}$ converge $p$-adically in $\mathbb{Z}$; this is shown via congruences that extend those predicted by the theory of formal group laws (cf. [2], [7], [10]) or the theory of $p$-adic hypergeometric functions (cf. [13]). When $p$ does not divide $\gamma_{m} \Delta$, these ratios converge to the quadratic character of $\Delta$ modulo $p$; otherwise, the limit is $p$ or zero. Moreover, when $p>3$ and $p$ divides $\Delta$, one obtains a supercongruence (cf. [2], [5], and eqs. (1.6), (3.8) below). These results are then used to give formal-group-law interpretations of some generalized Lucas sequences $\left\{\lambda_{n}\right\}=\left\{\gamma_{2 n} / \gamma_{n}\right\}$, and of the sequence $\left\{T_{n}\right\}=\left\{F_{5_{n}} /\left(5 F_{n}\right)\right\}$ (where $\left\{F_{n}\right\}$ is the familiar Fibonacci sequence associated to $\lambda=\mu=1$ ) which has been studied in [3]. The results are as follows.

Theorem 1: (i) If $p$ is a prime not dividing $\gamma_{m} \Delta$, then for all $r \in \mathbb{Z}^{+}$we have

$$
\begin{equation*}
\frac{\gamma_{m p^{r}}}{\gamma_{m p^{r-1}}} \equiv(\Delta \mid p)\left(\bmod p^{r} \mathbb{Z}\right) . \tag{1.3}
\end{equation*}
$$

(ii) If $p$ divides $\gamma_{m} \Delta$, then for all $r \in \mathbb{Z}^{+}$such that $\gamma_{m p^{r-1}} \neq 0$ we have

$$
\begin{equation*}
\frac{\gamma_{m p^{r}}}{\gamma_{m p^{r-1}}} \equiv L\left(\bmod p^{r} \mathbb{Z}\right) \tag{1.4}
\end{equation*}
$$

where $L=0$ or $L=p$ according to whether or not $p$ divides $\mu$.
(iii) The congruence (1.4) holds modulo $p^{r+1} \mathbb{Z}$ if $p>2$ and $p$ divides $\gamma_{m}$ but not $\Delta$; or if $(\Delta \mid p)=0$ and either $p>3$ or $p=3$ and $r>1$.

Corollary 1: (i) For all primes $p$ and all $m, r \in \mathbb{Z}^{+}$we have

$$
\begin{equation*}
\gamma_{m p^{r}} \equiv(\Delta \mid p) \gamma_{m p^{r-1}}\left(\bmod p^{r} \mathbb{Z}\right) . \tag{1.5}
\end{equation*}
$$

(ii) If $p$ divides $\gamma_{m}$ but not $\Delta$, then for all $r \in \mathbb{Z}^{+}$we have

$$
\begin{equation*}
\gamma_{m p^{r}} \equiv L \gamma_{m p^{r-1}}\left(\bmod p^{2 r} \mathbb{Z}\right) \tag{1.6}
\end{equation*}
$$

where $L=0$ or $L=p$ according to whether or not $p$ divides $\mu$.
Theorem 2: Suppose $\lambda=1$ and $\mu \neq-1$, and for $n>0$ set $\lambda_{\mathrm{n}}=\gamma_{2 n} / \gamma_{n}$. Then the formal power series

$$
\begin{equation*}
\ell(t)=\sum_{n=1}^{\infty} \lambda_{n} \frac{t^{n}}{n} \tag{1.7}
\end{equation*}
$$

is the logarithm of a one-dimensional formal group law over $\mathbb{Z}$ which is strictly isomorphic over $\mathbb{Z}$ to the formal multiplicative group law $\mathbb{G}_{m}(X, Y)=X+Y+X Y$.

Theorem 3: Let $\left\{F_{n}\right\}$ denote the usual Fibonacci sequence, i.e., the solution to (1.1) in the case $\lambda=\mu=1$, and for $n>0$ set $T_{n}=F_{5 n} /\left(5 F_{n}\right)$. Then the formal power series

$$
\begin{equation*}
\tau(t)=\sum_{n=1}^{\infty} T_{n} \frac{t^{n}}{n} \tag{1.8}
\end{equation*}
$$

is the logarithm of a one-dimensional formal group law over $\mathbb{Z}$ which is strictly isomorphic over $\mathbb{Z}$ to the formal multiplicative group law $\mathbb{G}_{m}(X, Y)=X+Y+X Y$.

## 2. PRELIMINARY RESULTS

The congruences (1.5) of Corollary 1(i) are typical of those obtained from the theory of formal group laws; in fact (1.5) implies (via [10], Theorem A.8) that the formal differential $\omega=P(t)^{-1} d t$ is the canonical invariant differential on a formal group law over the ring $\mathbb{Z}_{p}$ of $p$ adic integers when $(\Delta \mid p) \neq 0$ (cf. eqs. (3.6), (3.7) below). Hazewinkel's book [7] is an excellent reference on formal group laws; the aspects of the theory most relevant to the present article are also summarized nicely in ([2], pp. 143-45; [5], §2.3; [10], Appendix). Our proof of Theorem 1, however, uses only the elementary theory of finite and $p$-adic fields; for an exposition of these topics, the reader is referred to [8].

For $p$ a prime number, $\mathbb{Z}_{p}, \mathbb{Q}_{p}$, and $\mathbb{F}_{p^{d}}$ denote the ring of $p$-adic integers, the field of $p$-adic numbers, and the finite field of $p^{d}$ elements, respectively. We define $K=\mathbb{Q}_{p}(\sqrt{\Delta})$ if $p$ does not divide $\Delta$ and $K=\mathbb{Q}_{p}(\sqrt{\Delta}, \sqrt{p})$ if $p$ divides $\Delta$. We let $\mathfrak{D}_{K}$ denote the ring of algebraic integers of $K, \mathfrak{M}_{K}$ its unique maximal ideal, and $\bar{K}=\mathfrak{D}_{K} / \mathfrak{M}_{K}$ the residue-class field of $K$; for $x \in \mathfrak{D}_{K}$, $\bar{x}$ denotes its image in $\bar{K}$. Let the positive integer $d$ be defined so that $\bar{K} \cong \mathbb{F}_{p^{d}}$; then, if $x \in \mathfrak{D}_{K}$, the Teichmüller representative $\hat{x}$ of $x$ is the unique element of $\mathfrak{\Im}_{K}$ satisfying $\hat{x} \equiv x\left(\bmod \mathfrak{M}_{K}\right)$ and $\hat{x}^{p^{d}}=\hat{x}$. It is easily seen that $\hat{x}$ is given by the $p$-adic limit $\hat{x}=\lim _{r \rightarrow \infty} x^{p^{d r}}$.

If $p$ is an odd prime and $D$ is an integer, then $\sqrt{D} \in \mathbb{Z}_{p}$ if $(D \mid p)=1$ and $\sqrt{D} \notin \mathbb{Z}_{p}$ if $(D \mid p)=-1$; here $(\cdot \mid p)$ denotes the Legendre symbol. For ease of notation, we extend the definition of $(\Delta \mid p)$ to the case $p=2$ by

$$
(\Delta \mid 2)= \begin{cases}1, & \text { if } \Delta \equiv 1(\bmod 8),  \tag{2.1}\\ -1, & \text { if } \Delta \equiv 5(\bmod 8) \\ 0, & \text { if } \Delta \equiv 0(\bmod 4)\end{cases}
$$

This is analogous to the Legendre symbol in that $\sqrt{\Delta} \in \mathbb{Z}_{2}$ if $(\Delta \mid 2)=1$ and $\sqrt{\Delta} \notin \mathbb{Z}_{2}$ if $(\Delta \mid 2)=-1$.
If $\Delta \neq 0$ then $P(t)=(1-\alpha t)(1-\beta t)$, where $\alpha, \beta$ are distinct elements of $\mathfrak{S}_{K}$. It is well known, and easily computed from (1.2), that in this case we have the Binet form

$$
\begin{equation*}
\gamma_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta} \tag{2.2}
\end{equation*}
$$

for $\gamma_{n}$. It follows that, for all primes $p$ and all positive integers $m, r$ such that $\gamma_{m p^{r-1}} \neq 0$, we have

$$
\begin{equation*}
\frac{\gamma_{m p^{r}}}{\gamma_{m p^{r-1}}}=\frac{\alpha^{m p^{r}}-\beta^{m p^{r}}}{\alpha^{m p^{r-1}}-\beta^{m p^{r-1}}}=\Phi_{p}\left(\alpha^{m p^{r-1}}, \beta^{m p^{r-1}}\right) \tag{2.3}
\end{equation*}
$$

where $\Phi_{p}(X, Y)=X^{p-1}+X^{p-2} Y+\cdots+X Y^{p-2}+Y^{p-1}$ is the (two-variable) $p^{\text {th }}$ cyclotomic polynomial.

Considering $P(t) \in \mathbb{R}[t]$, if $\Delta>0$ then $\alpha, \beta \in \mathbb{R}$, and if $\lambda \neq 0$ then $\alpha \neq-\beta$; therefore, $\gamma_{n} \neq 0$ for all $n$ if $\Delta>0$ and $\lambda \neq 0$. However, when $\Delta<0$ one can have $\gamma_{n}=0$ in certain cases. We now show that this can only occur when $P(t)$ is equal to $1-t+t^{2}, 1-2 t+2 t^{2}, 1-3 t+3 t^{2}$, or one of these polynomials with $t$ replaced by $k t$ for some integer $k$. We state Proposition 1 explicitly as follows.

Proposition 1: Suppose $P(t)=1-\lambda t-\mu t^{2}=(1-\alpha t)(1-\beta t)$ with $\lambda, \mu \in \mathbb{Z}$, and let $n \in \mathbb{Z}^{+}$. Then the following are equivalent:
(A) $\alpha^{n}=\beta^{n}$.
(B) One of the following holds:
(i) $\Delta=0$;
(ii) $n$ is even and $\lambda=0$;
(iii) $n$ is divisible by 3 , and $\lambda=k, \mu=-k^{2}$ for some $k \in \mathbb{Z}$;
(iv) $n$ is divisible by 4 , and $\lambda=2 k, \mu=-2 k^{2}$ for some $k \in \mathbb{Z}$;
(v) $n$ is divisible by 6 , and $\lambda=3 k, \mu=-3 k^{2}$ for some $k \in \mathbb{Z}$.

Proof: Suppose $\alpha^{n}=\beta^{n}$. If $n=1$, then $\alpha=\beta$, so $\Delta=(\alpha-\beta)^{2}=0$, as in (i). Now suppose $\alpha \neq \beta$; therefore, $\alpha, \beta$, and $\Delta$ are all nonzero, so $\alpha^{n}=\beta^{n}$ implies $(\alpha / \beta)^{n}=1$.

Choose $m$ to be the minimal positive integer such that $(\alpha / \beta)^{m}=1$; then $m>1$ and $\alpha / \beta=\zeta_{m}$ is a primitive $m^{\text {th }}$ root of unity. It follows that $\alpha^{n}=\beta^{n}$ if and only if $n$ is a multiple of $m$. If $m=2$, then $\alpha^{2}=\beta^{2}$, so $\alpha=-\beta$, whence $\lambda=\alpha+\beta=0$, as in (ii).

We now suppose $m>2$; then $\zeta_{m}$ does not lie in $\mathbb{Q}$. The minimal polynomial of $\zeta_{m}$ over $\mathbb{Q}$ is the $m^{\text {th }}$ cyclotomic polynomial $\Phi_{m}(X, 1)$, which is irreducible of degree $\phi(m)$. [Here $\phi(m)$ denotes Euler's totient.] But $\zeta_{m}=\alpha / \beta$ lies in the quadratic field $\mathbb{Q}(\sqrt{\Delta})$, so the minimal polynomial of $\zeta_{m}$ has degree 2 over $\mathbb{Q}$. Thus, $\phi(m)=2$, which occurs precisely when $m=3,4$, or 6 .

For $m=3$ we have $\Phi_{3}(X, 1)=X^{2}+X+1$ and $\zeta_{m}=\alpha / \beta=(-1 \pm \sqrt{-3}) / 2$, so $\arg (\alpha / \beta)=$ $\pm 2 \pi / 3$. Since $\alpha$ and $\beta$ are complex conjugates, $\arg (\alpha / \beta)=2 \arg (\alpha)$, whence $\arg (\alpha)= \pm \pi / 3$ or $\pm 2 \pi / 3$. Therefore, $\alpha=k \cdot(1 \pm \sqrt{-3}) / 2$ for some real scalar $k$, whence $P(t)=1-k t+k^{2} t^{2}$. Since $P(t) \in \mathbb{Z}[t]$, we must have $k \in \mathbb{Z}$, precisely as in (iii). In this case, $\Delta=-3 k^{2}$.

For $m=4$, we have $\Phi_{4}(X, 1)=X^{2}+1$ and $\zeta_{m}=\alpha / \beta= \pm \sqrt{-1}$, so $\arg (\alpha / \beta)= \pm \pi / 2$. Thus, $\arg (\alpha)= \pm \pi / 4$ or $\pm 3 \pi / 4$, so $\alpha=k \cdot(1 \pm \sqrt{-1})$ for some real scalar $k$. Therefore, $P(t)=1-$ $2 k t+2 k^{2} t^{2}$, and since $P(t) \in \mathbb{Z}[t]$, we must have $k \in \mathbb{Z}$, precisely as in (iv). In this case, $\Delta=-4 k^{2}$.

For $m=6$, we have $\Phi_{6}(X, 1)=X^{2}-X+1$ and $\zeta_{m}=\alpha / \beta=(1 \pm \sqrt{-3}) / 2$, so $\arg (\alpha / \beta)=$ $\pm \pi / 3$. Thus, $\arg (\alpha)= \pm \pi / 6$ or $\pm 5 \pi / 6$, or $\alpha=k \cdot(3 \pm \sqrt{-3}) / 2$ for some real scalar $k$. Therefore, $P(t)=1-3 k t+3 k^{2} t^{2}$, and since $P(t) \in \mathbb{Z}[t]$, we must have $k \in \mathbb{Z}$, precisely as in (v). In this case, $\Delta=-3 k^{2}$.

We have shown that (A) implies (B). Using the above calculations, we find that (B) implies (A) by direct computation. This concludes the proof.

When $\gamma_{m} \neq 0$, it is also well known that $\varepsilon_{m}(n)=\lambda_{m n} / \lambda_{m}$ is an integer for all $n \in \mathbb{Z}^{+}$. In fact, it is easily seen from the Binet form (2.2) that $\varepsilon_{m}(n)$ satisfies the recursion (1.1) with $\lambda$ and $\mu$ replaced by $\lambda_{m}=\alpha^{m}+\beta^{m}$ and $(-1)^{m-1} \mu^{m}=-\alpha^{m} \beta^{m}$, respectively, and the parameters $\lambda_{m}=$ $\lambda \gamma_{m}+2 \mu \gamma_{m-1}$ and $(-1)^{m-1} \mu^{m}$ clearly lie in $\mathbb{Z}$. Our method will be to use (2.3) to deduce integral congruences for the integers $\gamma_{m p^{r}} / \gamma_{m p^{r-1}}$ from the following $p$-adic congruences for powers of $\alpha$ and $\beta$.

Proposition 2: Suppose $P(t)=1-\lambda t-\mu t^{2}=(1-\alpha t)(1-\beta t)$ with $\lambda, \mu \in \mathbb{Z}$.
(i) If $(\Delta \mid p)=1$, then $\alpha^{m p^{r}} \equiv \alpha^{m p^{r-1}}\left(\bmod p^{r} \mathbb{Z}_{p}\right)$;
(ii) If $(\Delta \mid p)=-1$, then $\alpha^{m p^{r}} \equiv \beta^{m p^{r-1}}\left(\bmod p^{r} \Im_{K}\right)$;
(iii) If $p>2$ and $(\Delta \mid p)=0$, then $\alpha^{m p^{r}} \equiv \alpha^{m p^{r-1}} \equiv \beta^{m p^{r-1}} \equiv \beta^{m p^{r}}\left(\bmod p^{r-1 / 2} \mathfrak{O}_{K}\right)$;
(iv) If $(\Delta \mid 2)=0$, then $\alpha^{m 2^{r-1}} \equiv \beta^{m 2^{r-1}}\left(\bmod 2^{r} \Im_{K}\right)$ and $\alpha^{m 2^{r-1}} \equiv \alpha^{m 2^{r}}\left(\bmod 2^{r-1} \mathfrak{O}_{K}\right)$.

Proof: If $x, y, p^{s} \in \mathfrak{D}_{K}$ and $x \equiv y\left(\bmod p^{s} \mathfrak{O}_{K}\right)$ write $x=y+z$ with $z \in p^{s} \mathfrak{D}_{K}$; then

$$
\begin{equation*}
x^{p}=y^{p}+\left(\sum_{k=1}^{p-1}\binom{p}{k} y^{p-k} z^{k}\right)+z^{p} \tag{2.4}
\end{equation*}
$$

and hence $x^{p} \equiv y^{p}\left(\bmod p^{s+1} \mathfrak{Q}_{K}\right)$ if $s p \geq s+1$. Thus, we need only prove these results in the case $r=1$ and in addition that $a^{2 m} \equiv a^{4 m}\left(\bmod 2 \Im_{K}\right)$ when $(\Delta \mid 2)=0$; we may also assume $m=1$ with no loss of generality.

If $(\Delta \mid p)=1$, then $d=1, K=\mathbb{Q}_{p}, \supseteq_{K}=\mathbb{Z}_{p}, \mathfrak{M}_{K}=p \mathbb{Z}_{p}$, and $\bar{K} \cong \mathbb{F}_{p}$. The statement $\alpha^{p} \equiv \alpha$ $\left(\bmod p \mathbb{Z}_{p}\right)$ is Fermat's little theorem, which proves (i) in the case $r=1$.

If $(\Delta \mid p)=-1$, then $d=2$ and $\alpha, \beta$ are conjugates in the unramified extension $K$ of $\mathbb{Q}_{p}$ (their minimal polynomial over $\mathbb{Q}_{p}$ is $\left.t^{2}+\lambda \mu^{-1} t-\mu^{-1}\right)$. We note that $p$ does not divide $\mu$, since if $p$ divides $\mu$ then $\Delta \equiv \lambda^{2}(\bmod 4 p \mathbb{Z})$ and then $(\Delta \mid p)=1$. Therefore, $\alpha, \beta$ are units in $\mathfrak{D}_{K}$ (since $\alpha \beta=-\mu$ ), and $\bar{\alpha}, \bar{\beta}$ are conjugates in $\bar{K}$ over $\mathbb{F}_{p}$ (their minimal polynomial being $t^{2}+\bar{\lambda} \bar{\mu}^{-1} t-$ $\bar{\mu}^{-1}$ ). Since $\bar{K} \cong \mathbb{F}_{p^{2}}$ and $x \mapsto x^{p}$ is the nontrivial automorphism of $\mathbb{F}_{p^{2}}$ over $\mathbb{F}_{p}$, we have $\bar{\alpha}^{p}=\bar{\beta}$ and $\bar{\beta}^{p}=\bar{\alpha}$; therefore, $\alpha^{p} \equiv \beta$ and $\beta^{p} \equiv \alpha$ modulo $\mathfrak{M}_{K}$. Since $K$ is unramified, we have $\mathfrak{M}_{K}=$ $p \mathfrak{૭}_{K}$, yielding the $r=1$ case of (ii).

If $(\Delta \mid p)=0$, then $q$ divides $\Delta=(\alpha-\beta)^{2}$, where $q=p$ if $p>2$ and $q=4$ if $p=2$. Therefore, $\alpha \equiv \beta\left(\bmod q^{1 / 2} \mathfrak{\Im}_{K}\right)$, giving the middle congruence of (iii) and the first part of (iv) in the case $r=1$. As in (i) and (ii) above, we have $\alpha^{p} \equiv \alpha$ or $\beta\left(\bmod \mathfrak{M}_{K}\right)$ according to whether $d=1$ or $d=2$, which completes (iii) for $r=1$, since $\mathcal{M}_{K}=p^{1 / 2} \mathfrak{D}_{K}$. Finally, if $(\Delta \mid 2)=0$, then 2 divides $\lambda$, and thus $\bar{\alpha}, \bar{\beta}$ are roots of $t^{2}-\bar{\mu}^{-1}$; this shows that $\bar{K} \cong \mathbb{F}_{2}$ and so $\alpha, \beta \equiv 0$ or $1(\bmod$ $2^{1 / 2} \mathfrak{O}_{K}$ ). Writing $\alpha=y+z$ with $z \in 2^{1 / 2} \mathfrak{D}_{K}$ and $y=0$ or 1 , we use (2.4) to check that $\alpha^{2} \in y+2 \mathfrak{N}_{K}$ and $\alpha^{4} \in y+4 \mathfrak{N}_{K}$, proving the $r=2$ case of the second statement of (iv).

Remarks: This proposition and its proof remain valid for $\lambda, \mu$ lying in $\mathbb{Z}_{p}$ (not just in $\mathbb{Z}$ ) provided one replaces the Legendre symbol with the Hilbert symbol. Furthermore, this proposition implies that, for each $m \in \mathbb{Z}^{+}$and each prime $p$, the sequence $\left\{\alpha^{m p^{d r}}\right\}$ is a $p$-adically Cauchy sequence in $\mathfrak{V}_{K}$; the limit is the Teichmüller representative $\hat{\alpha}^{m}$.

## 3. DEMONSTRATION OF THEOREMS

Proof of Theorem 1: From Proposition 2(i), (ii), we have

$$
\alpha^{m p^{r}} \equiv\left\{\begin{array}{ll}
\alpha^{m p^{r-1}}, & \text { if }(\Delta \mid p)=1,  \tag{3.1}\\
\beta^{m p^{r-1}}, & \text { if }(\Delta \mid p)=-1,
\end{array}\left(\bmod p^{r} \oiint_{K}\right)\right.
$$

and similarly for $\beta^{m p^{r}}$. Since $\Phi_{p} \in \mathbb{Z}[X, Y]$ and $\Phi_{p}(X, Y)=\Phi_{p}(Y, X)$, we have, in either case,

$$
\begin{equation*}
\frac{\gamma_{m p^{r}}}{\gamma_{m p^{r-1}}}=\Phi_{p}\left(\alpha^{m p^{r-1}}, \beta^{m p^{r-1}}\right) \equiv \Phi_{p}\left(\alpha^{m p^{r}}, \beta^{m p^{r}}\right) \equiv \cdots \equiv \Phi_{p}\left(\hat{\alpha}^{m}, \hat{\beta}^{m}\right)\left(\bmod p^{r} \oiint_{K}\right) \tag{3.2}
\end{equation*}
$$

provided $\gamma_{m p^{r-1}} \neq 0$. Evaluating $\lim _{r \rightarrow \infty} \alpha^{m p^{d r}}$ using (3.1), we find that

$$
\hat{\alpha}^{m p}= \begin{cases}\hat{\alpha}^{m}, & \text { if }(\Delta \mid p)=1,  \tag{3.3}\\ \hat{\beta}^{m}, & \text { if }(\Delta \mid p)=-1 .\end{cases}
$$

If $p$ does not divide $\gamma_{m} \Delta=(\alpha-\beta)\left(\alpha^{m}-\beta^{m}\right)$, then $\hat{\alpha}^{m} \neq \hat{\beta}^{m}$; therefore, $\gamma_{m p^{r-1}} \neq 0$ for all $r$. Thus, we have

$$
\begin{equation*}
\Phi_{p}\left(\hat{\alpha}^{m}, \hat{\beta}^{m}\right)=\frac{\hat{\alpha}^{m p}-\hat{\beta}^{m p}}{\hat{\alpha}^{m}-\hat{\beta}^{m}}=(\Delta \mid p) . \tag{3.4}
\end{equation*}
$$

Together with (3.2) this shows that $\gamma_{m p^{r}} / \gamma_{m p^{r-1}} \equiv(\Delta \mid p)\left(\bmod p^{r} \mathfrak{O}_{K}\right)$; since both sides of this congruence are integers, the congruence must hold modulo $p^{r} \mathbb{Z}$, completing the proof of (i).

As in (3.2), one can see from Proposition 2 that, provided $\gamma_{m p^{r-1}}$ is always nonzero, one has $\Phi_{p}\left(\hat{\alpha}^{m}, \hat{\beta}^{m}\right)$ as the $p$-adic limit of $\gamma_{m p^{r}} / \gamma_{m p^{r-1}}$, and thus determine the value $L$ as stated in part (ii) of the theorem. One may discover the stronger congruences of (iii) [which will be useful in the proofs of Corollary 1(ii) and Theorem 3], however, by making a simple algebraic manipulation.

Suppose that $p$ divides $\gamma_{m} \Delta$; then write $x_{r}=\alpha^{m p^{r-1}}, y_{r}=\beta^{m p^{r-1}}, z_{r}=x_{r}-y_{r}$, and

$$
\begin{equation*}
\frac{\gamma_{m p^{r}}}{\gamma_{m p^{r-1}}}=\frac{x_{r}^{p}-y_{r}^{p}}{x_{r}-y_{r}}=\frac{\left(y_{r}+z_{r}\right)^{p}-y_{r}^{p}}{z_{r}}=p y_{r}^{p-1}+\left(\sum_{k=2}^{p-1}(p) p_{r}^{p-k} \frac{z_{r}^{k-1}}{k}\right)+z_{r}^{p-1} . \tag{3.5}
\end{equation*}
$$

If $p$ divides $\gamma_{m}=\left(\alpha^{m}-\beta^{m}\right) /(\alpha-\beta)$ but not $\Delta=(\alpha-\beta)^{2}$, then $\alpha^{m} \equiv \beta^{m}\left(\bmod p \bigcirc_{K}\right)$; therefore, $\hat{\alpha}^{m}=\hat{\beta}^{m}$. Since $\left\{\bar{\alpha}^{p}, \bar{\beta}^{p}\right\}=\{\bar{\alpha}, \bar{\beta}\}$ and $\bar{\alpha}^{m}=\bar{\beta}^{m}$, we have $\bar{\alpha}^{m}=\bar{\beta}^{m} \in \mathbb{F}_{p}$; thus, $\hat{\alpha}^{m}=\hat{\beta}^{m} \in \mathbb{Z}_{p}$. Note that $\hat{\alpha}, \hat{\beta} \neq 0$ since $p$ does not divide $\Delta$; hence, $p$ does not divide $\mu=-\alpha \beta$, and by Fermat's little theorem, $\hat{\beta}^{m(p-1)}=1$. From Proposition 2(i), (ii), we have $\alpha^{m p^{r-1}} \equiv \hat{\alpha}^{m}=\hat{\beta}^{m} \equiv \beta^{m p^{r-1}}$ $\left(\bmod p^{r} \Im_{K}\right)$. Therefore, the term $p y_{r}^{p-1}$ in (3.5) is congruent to $p$ modulo $p^{r+1} \mathfrak{D}_{K}$. The final term $z_{r}^{p-1}$ is zero modulo $p^{r(p-1)} \mathfrak{O}_{K}$, which shows that $\gamma_{m p^{r}} / \gamma_{m p^{r-1}} \equiv p\left(\bmod p^{r} \Im_{K}\right)$; since both sides are integers, the congruence holds modulo $p^{r} \mathbb{Z}$, as asserted in (ii). In fact, since $r(p-1) \geq r+1$ for $p>2$ and $r>0$, we see that the congruence (1.3) holds modulo $p^{r-1} \mathbb{Z}$ when $p>2$ and $p$ divides $\gamma_{m}$ but not $\Delta$.

The case $(\Delta \mid p)=0, \Delta \neq 0$ is similar; using Proposition 2(iii) we find that for $p>2$ the term $p y_{r}^{p-1}$ in (3.5) is congruent to $p \hat{\beta}^{m(p-1)}$ modulo $p^{r+1 / 2} \Im_{K}$, all terms within the summation in (3.5) are zero modulo $p^{r+1 / 2} \mathfrak{D}_{K}$, and the final term $z_{r}^{p-1}$ is zero modulo $p^{(r-1 / 2)(p-1)} \mathfrak{D}_{K}$. Thus, for $p>2$, we have $\gamma_{m p^{r}} / \gamma_{m p^{r-1}} \equiv L\left(\bmod p^{r} \mathfrak{O}_{K}\right)$ and, therefore, modulo $p^{r} \mathbb{Z}$. In addition, since $(r-1 / 2)(p-1) \geq r+1 / 2$ for $p>3$ or for $p=3$ and $r>1$, in these cases the congruence (1.4) holds modulo $p^{r+1} \mathbb{Z}$, since it holds modulo $p^{r+1 / 2} \mathfrak{O}_{K}$ while both sides lie in $\mathbb{Z}$. If $(\Delta \mid 2)=0$, we find from Proposition 2 (iv) that $2 \alpha^{m 2^{r-1}} \equiv L\left(\bmod 2^{r} \mathfrak{S}_{K}\right)$ and $z_{r} \equiv 0\left(\bmod 2^{r} \mathfrak{O}_{K}\right)$, giving the result in that case.

Finally, if $\Delta=0$, then $P(t)=(1-\alpha t)^{2}$ for some $\alpha \in \mathbb{Z}$, and a quick computation from (1.2) yields $\gamma_{n}=n \alpha^{n-1}$. If $\lambda \neq 0$, then $\alpha \neq 0$; therefore, we have $\gamma_{m p^{r}} / \gamma_{m p^{r-1}}=p \alpha^{m p^{r-1}(p-1)} \in \mathbb{Z}$. As in Proposition 2(i), if $p$ does not divide $\alpha$ this lies in $p+p^{r+1} \mathbb{Z}$, whereas if $\alpha \in p \mathbb{Z}$, it is clearly congruent to zero modulo $p^{r+1} \mathbb{Z}$.

Proof of Corollary 1: We first treat the case where $\Delta>0$ and $\lambda \neq 0$ so that $\gamma_{n} \neq 0$ for all $n$. If $p$ does not divide $\gamma_{m} \Delta$, part (i) follows directly from (1.4) upon multiplication by $\gamma_{m p^{r-1}}$. From Theorem 1(ii), we find by induction on $r$ that $\gamma_{m p^{r}} \equiv 0\left(\bmod p^{r+1} \mathbb{Z}\right)$ if $p$ divides $\gamma_{m}$, and $\gamma_{m p^{r}} \equiv 0$ $\left(\bmod p^{r} \mathbb{Z}\right)$ if $p$ divides $\Delta$. It then follows that both sides of (1.5) are zero modulo $p^{r} \mathbb{Z}$ if $p$ divides $\gamma_{m} \Delta$.

For (ii), we recall from Theorem 1(iii) that the congruence (1.4) holds modulo $p^{r+1} \mathbb{Z}$ when $p>3$ and $p$ divides $\Delta$. In this case or in the case where $p$ divides $\gamma_{m}$, we obtain (ii) upon multiplication of (1.4) by $\gamma_{m p^{r-1}}$.

To extend these results to arbitrary $\Delta$ and $\lambda$, we observe that if $\lambda^{\prime}=\lambda+p^{N}$ and $\gamma_{n}^{\prime}$ is defined by $\gamma_{0}^{\prime}=0, \gamma_{1}^{\prime}=1$, and $\gamma_{n+1}^{\prime}=\lambda^{\prime} \gamma_{n}^{\prime}+\mu \gamma_{n-1}^{\prime}$, then $\gamma_{n}^{\prime} \equiv \gamma_{n}\left(\bmod p^{N} \mathbb{Z}\right)$ for all $n$. It is clear that we may choose $N$ large enough so that $N \geq 2 r, \Delta^{\prime}=\left(\lambda^{\prime}\right)^{2}+4 \mu>0$, and $\lambda^{\prime} \neq 0$. Since $\Delta^{\prime} \equiv$ $\Delta(\bmod p \mathbb{Z})$, the results for any $\Delta, \lambda$ follow from the results for $\Delta^{\prime}, \lambda^{\prime}$.

Remarks: One can easily determine from [4] with the aid of $\S 5.8$ in [7] that $\omega=P(t)^{-1} d t$ is the canonical invariant differential on the formal group law $F(X, Y)$ over $\mathbb{Z}$ given by the rational function

$$
\begin{equation*}
F(X, Y)=(X+Y-\lambda X Y) /(1+\mu X Y) \tag{3.6}
\end{equation*}
$$

(equivalently, $\sum_{n=1}^{\infty} \gamma_{n} T^{n} / n$ is the logarithm of this formal group law). From this, it follows ([2]; [10], Theorem A.8) that there exist congruences of the type

$$
\begin{equation*}
\gamma_{m p^{r}} \equiv H \gamma_{m p^{r-1}}\left(\bmod p^{r} \mathbb{Z}_{p}\right) \tag{3.7}
\end{equation*}
$$

for some $H \in \mathbb{Z}_{p}$, when $p$ does not divide $\gamma_{p}$ [which is equivalent, via Corollary $1(\mathrm{i})$, to the condition $(\Delta \mid p) \neq 0]$. What is surprising about Corollary 1 is that the congruences obtained also hold, and are in fact stronger, in the cases not predicted by the theory of formal group laws [i.e., when $(\Delta \mid p)=0$ ]. Other congruences of the type

$$
\begin{equation*}
c_{m p^{r}} \equiv H c_{m p^{r-1}}\left(\bmod p^{a r} \mathbb{Z}_{p}\right) \tag{3.8}
\end{equation*}
$$

with $a \geq 2$ (called "supercongruences") have also been observed involving binomial coefficients [6] and the Apéry numbers [2], and have been conjectured in [11].

Proof of Theorem 2: The statement that the formal power series (1.7) is the logarithm of a formal group law over $\mathbb{Z}$ which is strictly isomorphic over $\mathbb{Z}$ to $\mathbb{G}_{m}$ is equivalent to requiring that $\lambda_{n} \in \mathbb{Z}, \lambda_{1}=1$, and for all primes $p$ and all $m, r \in \mathbb{Z}^{+}$the congruences

$$
\begin{equation*}
\lambda_{m p^{r}} \equiv \lambda_{m p^{r-1}}\left(\bmod p^{r} \mathbb{Z}\right) \tag{3.9}
\end{equation*}
$$

(cf. [2], pp. 143-45; [10], Theorem A.9). Assuming $\lambda=1$ and $\mu \neq-1$, Proposition 1 tells us that $\gamma_{n}$ is never zero, so $\lambda_{n} \in \mathbb{Z}$ for $n>0$ and, from (2.3), we have $\lambda_{n}=\alpha^{n}+\beta^{n}$. We have $\lambda=\lambda_{1}=1$ and $\Delta=\lambda^{2}+4 \mu$ is odd, so it follows from Proposition 2(i), (ii), (iii), that the congruences (3.9) hold modulo $p^{r-1 / 2} \Im_{K}$, but both sides are integers, so the theorem follows.

Proof of Theorem 3: From [3] we know that $T_{n} \in \mathbb{Z}$ for all $n$, and it is clear that $T_{1}=1$. Therefore, as in Theorem 2, we must show that for all primes $p$ and all $m, r \in \mathbb{Z}^{+}$, we have

$$
\begin{equation*}
T_{m p^{r}} \equiv T_{m p^{r-1}}\left(\bmod p^{r} \mathbb{Z}\right) \tag{3.10}
\end{equation*}
$$

From the definition of $T_{n}$, one has

$$
\begin{equation*}
T_{n}=\frac{1}{5} \Phi_{5}\left(\alpha^{n}, \beta^{n}\right) \tag{3.11}
\end{equation*}
$$

where $\alpha, \beta$ are the reciprocal roots of the polynomial $P(t)=1-t-t^{2}$ associated to $\lambda=\mu=1$. Since $\Delta=5$, for all primes $p \neq 5$ these congruences follow directly from Proposition 2(i), (ii), as in (3.2). To complete the proof, we take advantage of the fact that

$$
\begin{equation*}
\frac{F_{m s^{r}}}{F_{m 5^{r-1}}} \equiv 5\left(\bmod 5^{r+1} \mathbb{Z}\right), \tag{3.12}
\end{equation*}
$$

which is a consequence of Theorem 1 (iii). Dividing by 5 , we obtain

$$
\begin{equation*}
T_{m 5^{r-1}}=\frac{F_{m s^{r}}}{5 F_{m s^{r-1}}} \equiv 1\left(\bmod 5^{r} \mathbb{Z}\right), \tag{3.13}
\end{equation*}
$$

which proves the congruence (3.10) in the case $p=5$, completing the proof.
Remark: The result (3.13) is not best possible; in fact, the congruence $T_{5^{r}} \equiv 1\left(\bmod 5^{2 r} \mathbb{Z}\right)$ has been shown in ([3], Lemma 2).

## 4. CONCLUDING REMARKS

In [3] it is noted that for $k \in \mathbb{Z}^{+}$the sequences $\{T(k, n)\}_{n>0}$ given by $T(k, n)=F_{k n} /\left(F_{k} F_{n}\right)$ are always integral in the three special cases $k=1[T(1, n)=1$ for all $n], k=2\left[T(2, n)=L_{n}\right.$, the $n^{\text {th }}$ Lucas number $]$, and $k=5\left[T(5, n)=T_{n}\right]$. Our Theorem 2 and Theorem 3 explain that all three of these sequences occur as the expansion coefficients for the logarithms of formal group laws over $\mathbb{Z}$ which are strictly isomorphic over $\mathbb{Z}$ to the same formal group law $\mathbb{G}_{m}$.

For $p \neq 2$ one may also approach these $p$-adic properties of the sequence $\left\{\gamma_{n}\right\}$ via its combinatorial form

$$
\begin{equation*}
\gamma_{n+1}=\sum_{k=0}^{[n / 2]}\binom{n-k}{k} \lambda^{n-2 k} \mu^{k} \tag{4.1}
\end{equation*}
$$

[9], which may be expressed in terms of hypergeometric functions as

$$
\gamma_{n+1}=\lambda^{n}{ }_{2} F_{1}\left(\begin{array}{c}
-n / 2,(1-n) / 2  \tag{4.2}\\
-n
\end{array}-4 \mu / \lambda^{2}\right)
$$

We sketch the method here: Taking $n+1=m p^{r}$ and letting $r \rightarrow \infty$, the parameters $-n / 2$, $(1-n) / 2$, and $-n$ converge $p$-adically to $1 / 2,1$, and 1 , respectively. Using a suitable modification of the argument in ([13], Theorem 4.1) one can show that when $p$ does not divide $\gamma_{p}$, the $p$-adic limit of $\gamma_{p^{r}} / \gamma_{p^{r-1}}$ is given by

$$
\lim _{r \rightarrow \infty} \frac{\gamma_{p^{r}}}{\gamma_{p^{r-1}}}={ }_{2} \mathscr{F}_{1}\left(\begin{array}{c}
\frac{1}{2}, 1  \tag{4.3}\\
1
\end{array} ;\left(-\widehat{4 \mu} / \lambda^{2}\right)\right),
$$

where (as in the notation of [13]) the symbol ${ }_{2} \widetilde{\mathscr{F}}_{1}(x)$ denotes the $p$-adic "analytic continuation" of ${ }_{2} F_{1}(x) /{ }_{2} F_{1}\left(x^{p}\right)$. Since ${ }_{2} F_{1}(1 / 2,1 ; 1 ; x)={ }_{1} F_{0}(1 / 2 ; ; x)=(1-x)^{-1 / 2}$, the same value for the $p$-adic limit in (4.3) is also obtained from $\lim _{r \rightarrow \infty}\left(c_{p^{r}} / c_{p^{r-1}}\right)$, where

But clearly $\lim _{r \rightarrow \infty}\left(c_{p^{r}} / c_{p^{r-1}}\right)=\lim _{r \rightarrow \infty} \Delta^{p^{r-1}(p-1) / 2}=\hat{\Delta}^{(p-1) / 2}$, which is seen to be precisely $(\Delta \mid p)$ from Euler's criterion

$$
\begin{equation*}
(\Delta \mid p) \equiv \Delta^{(p-1) / 2}(\bmod p \mathbb{Z}) \tag{4.5}
\end{equation*}
$$

and the fact that $(\widehat{ \pm 1})= \pm 1$. The point is that the sequences $\left\{\gamma_{n+1}\right\}$ and $\left\{\Delta^{n / 2}\right\}$ should have the same $p$-adic congruence behavior because they arise from hypergeometric functions that are $p$ adically proximate (when $n+1=m p^{r}$ ) So, if one is willing to appeal to the $p$-adic analytic properties of the combinatorial form (4.1), one may obtain a fair explanation for the occurrence of $(\Delta \mid p)$ in Theorem 1(i) when $(\Delta \mid p) \neq 0$. But again, Theorem 1 (ii) shows that the $p$-adic limit in (4.3) even exists when $(\Delta \mid p)=0$ [which is equivalent to $p$ dividing $\gamma_{p}$, by Corollary 1(i)], a fact that is not predicted by the theory of $p$-adic hypergeometric functions (cf. [13], Theorem 2.3).

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# ON A CONJECTURE OF PIERO FILIPPONI 

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## 1. INTRODUCTION

Let us define a generalized Lucas sequence $\left\{H_{n}(m)\right\}$ by

$$
\begin{equation*}
H_{n}(m)=H_{n-1}(m)+m H_{n-2}(m), H_{0}(m)=2, H_{1}(m)=1, \tag{1}
\end{equation*}
$$

where $m \geq 1$ is a natural number.
In a communication that appeared in a recent issue of this journal [1], P. Filipponi showed that

$$
\begin{equation*}
H_{p^{s}}(p) \equiv 1\left(\bmod p^{s}\right) \tag{2}
\end{equation*}
$$

where $p$ is an odd prime, and he proposed also the following Conjecture:

$$
\begin{equation*}
H_{p^{s}}(p-1) \equiv 1\left(\bmod p^{s}\right) \tag{3}
\end{equation*}
$$

where $p \geq 5$ is a prime number.
Following a method introduced by Lucas ([2], p. 209; [3]), we shall prove here generalizations of (2) and (3), namely,

Theorem 1: If $p \geq 1$ is a natural number, and if $m \equiv 0(\bmod p)$, then

$$
H_{p^{s}}(m) \equiv 1\left(\bmod p^{s+1}\right), s \geq 0 .
$$

Theorem 2: If $p \geq 5$ is a prime number and if $m \equiv-1(\bmod p)$, then

$$
H_{p^{s}}(m) \equiv 1\left(\bmod p^{s+1}\right), s \geq 0 .
$$

## 2. PRELIMINARIES

Let us recall Waring's formula

$$
x^{p}+y^{p}=(x+y)^{p}+p \sum_{k=1}^{[p / 2]}(-1)^{k} C_{p, k}(x y)^{k}(x+y)^{p-2 k},
$$

where $p$ is a natural integer, and

$$
C_{p, k}=\frac{1}{p-k}\binom{p-k}{k}=\frac{1}{k}\binom{p-k-1}{k-1}, \text { for } 1 \leq k \leq[p / 2] .
$$

In our proofs, we shall need the following three lemmas.
Lemma 1: (i) If $p$ is a natural integer, then $p, C_{p, k}$ is integral;
(ii) If $p$ is a prime, then $C_{p, k}$ is integral.

Proof: (i) The result follows from the relation

$$
p C_{p, k}=\binom{p-k}{k}+\binom{p-k-1}{k-1} .
$$

(ii) From the relation

$$
k\binom{p-k}{k}=(p-k)\binom{p-k-1}{k-1},
$$

and since $\operatorname{gcd}(k, p-k)=1$, it is clear that $k$ divides $\binom{p-k-1}{k-1}$.
Lemma 2: If $p \equiv \pm 1(\bmod 6)$ is a natural number, then $\sum_{k=1}^{[p / 2]}(-1)^{k} C_{p, k}=0$..
Proof: Let us put $x=e^{i \pi / 3}$ and $y=e^{-i \pi / 3}$ in Waring's formula to get

$$
2 \cos p \pi / 3=1+p \sum_{k=1}^{[p / 2]}(-1)^{k} C_{p, k}
$$

and the conclusion follows from this, since $2 \cos p \pi / 3=1$, when $p \equiv \pm 1(\bmod 6)$.
Lemma 3: If $p$ is an odd integer, then $(\ell p-1)^{p^{s}} \equiv-1\left(\bmod p^{s+1}\right), \ell \geq 0$.
Proof: The statement clearly holds for $s=0$. Supposing that $(\ell p-1)^{p^{s}}=-1+A p^{s+1}$, where $A$ is an integer, one can write

$$
\begin{aligned}
(\ell p-1)^{p+1} & =\left(-1+A p^{s+1}\right)^{p} \\
& =(-1)^{p}+\binom{p}{1}(-1)^{p-1} A p^{s+1}+\binom{p}{2}(-1)^{p-2} A^{2} p^{2 s+2}+\cdots+A^{p} p^{p(s+1)} \equiv-1\left(\bmod p^{s+2}\right),
\end{aligned}
$$

since $p$ is odd and $\binom{p}{1}=p$.
Let us return to the recurrence relation (1). We have $H_{n}(m)=\alpha_{m}^{n}+\beta_{m}^{n}$, where $\alpha_{m}$ and $\beta_{m}$ are the real numbers such that $\alpha_{m}+\beta_{m}=1$ and $\alpha_{m} \beta_{m}=-m$. Following Lucas ([2], p. 212), we replace $x$ (resp. $y$ ) by $\alpha_{m}^{p^{s}}$ (resp. $\beta_{m}^{p^{s}}$ ) in Waring's formula to get

$$
\begin{equation*}
H_{p^{s+1}}(m)=H_{p^{s}}^{p}(m)+p \sum_{k=1}^{[p / 2]}(-1)^{k\left(1+p^{s}\right)} C_{p, k} m^{k p^{s}} H_{p^{s}}^{p-2 k}(m), \tag{4}
\end{equation*}
$$

where $p$ is a natural number.

## 3. PROOF OF THEOREM 1

The case $p=1$ needs no comment, since $H_{1}=1$, so we suppose in the sequel that $p \geq 2$, and thus that $[p / 2] \geq 1$.

Let us write $H_{n}$ instead of $H_{n}(m)$ in (4), to get

$$
\begin{equation*}
H_{p^{s+1}}=H_{p^{s}}^{p}+(-1)^{1+p^{s}} p m^{p^{s}} H_{p^{s}}^{p-2}+\sum_{k=2}^{[p / 2]}(-1)^{k\left(1+p^{s}\right)} p C_{p, k} m^{k p^{s}} H_{p^{s}}^{p-2 k} \tag{5}
\end{equation*}
$$

since $C_{p, 1}=1$. Notice that the last sum is empty for $p=2$ and $p=3$ and that $p C_{p, k}$ is an integer, by Lemma 1(i).

We proceed by induction upon $s$. The statement clearly holds for $s=0$ since $H_{1}=1$.
Now, let us suppose that

$$
H_{p^{s}} \equiv 1\left(\bmod p^{s+1}\right) .
$$

By using an argument similar to the one used in Lemma 3, one can easily deduce from this that

$$
\begin{equation*}
H_{p^{s}}^{p} \equiv 1\left(\bmod p^{s+2}\right) . \tag{6}
\end{equation*}
$$

Next we have, for every $s \geq 0$ and every $p \geq 2, p^{s} \geq 2^{s} \geq s+1$, and thus
(a) $p m^{p^{s}} \equiv 0\left(\bmod p^{s+2}\right)$.

On the other hand we have, for every $k \geq 2, k p^{s} \geq 22^{s}=2^{s+1} \geq s+2$, and thus
(b) $m^{k p^{s}} \equiv 0\left(\bmod p^{s+2}\right)$.

Now, by using (6), (a), and (b) in (5), we have

$$
H_{p^{s+1}} \equiv 1\left(\bmod p^{s+2}\right) .
$$

This concludes the proof of Theorem 1.

## 4. PROOF OF THEOREM 2

We suppose now that $p \geq 5$ is a prime number, and thus that $p \equiv \pm 1(\bmod 6)$. Let us put $m=\ell p-1$ in (4) and write $H_{n}$ instead of $H_{n}(\ell p-1)$ to obtain

$$
\begin{equation*}
H_{p^{s+1}}=H_{p^{s}}^{p}+p \sum_{k=1}^{[p / 2]} C_{p, k}(\ell p-1)^{k p^{s}} H_{p^{s}}^{p-2 k} \tag{7}
\end{equation*}
$$

We proceed by induction on $s$. The statement clearly holds for $s=0$, since $H_{1}=1$. Supposing that $H_{p^{s}} \equiv 1\left(\bmod p^{s+1}\right)$, we obtain

$$
\begin{equation*}
H_{p^{s}}^{p-2 k} \equiv 1\left(\bmod p^{s+1}\right), \text { for } 1 \leq k \leq[p / 2] \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{p^{s}}^{p} \equiv 1\left(\bmod p^{s+2}\right) . \tag{9}
\end{equation*}
$$

On the other hand, we have, by Lemma 3,

$$
\begin{equation*}
(\ell p-1)^{k p^{s}} \equiv(-1)^{k}\left(\bmod p^{s+1}\right) \tag{10}
\end{equation*}
$$

By Lemma 1(ii), $C_{p, k}$ is an integer, and by (8), (10), and Lemma 2, we obtain

$$
\begin{equation*}
\sum_{k=1}^{[p / 2]} C_{p, k}(\ell p-1)^{k p^{s}} H_{p^{s}}^{p-2 k} \equiv \sum_{k=1}^{[p / 2]} C_{p, k}(-1)^{k} \equiv 0\left(\bmod p^{s+1}\right) \tag{11}
\end{equation*}
$$

Now, by (7), (9), and (11), it is clear that $H_{p^{s+1}} \equiv 1\left(\bmod p^{s+2}\right)$. This concludes the proof of Theorem 2.

## ACKNOWLEDGMENT

The author would like to thank the referee for his helpful and detailed comments.

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Announcement

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# ON FIBONACCI NUMBERS AND PRIMES OF THE FORM $4 k+1$ 

Neville Robbins

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(Submitted February 1992)
A well-known theorem of elementary number theory states:
There exist infinitely many primes $p$ such that $p \equiv 1(\bmod 4)$.
(See [1], p. 224.)
One can prove (I) constructively by generating an infinite sequence $\left\{p_{n}\right\}$ of distinct primes such that $p_{n} \equiv 1(\bmod 4)$ for all $n \geq 1$. To obtain the sequence $\left\{p_{n}\right\}$, let $\left\{u_{n}\right\}$ be a sequence of natural numbers such that:
(i) $u_{n}>1$ for all $n \geq 1$.
(ii) If $q$ is prime and $q \mid u_{n}$, then $q \equiv 1(\bmod 4)$.
(iii) $\left(u_{m}, u_{n}\right)=1$ for all $m \neq n$.

If we let $p_{n}$ be the least prime divisor of $u_{n}$ for all $n \geq 1$, then the sequence $\left\{p_{n}\right\}$ yields the desired result.

Let $u_{n}=a_{n}^{2}+b_{n}^{2}$ where $a_{n}, b_{n}$ are natural numbers such that $\left(a_{n}, b_{n}\right)=1$ and $a_{n} \neq b_{n}(\bmod 2)$. Then the sequence $\left\{u_{n}\right\}$ satisfies (i) and (ii). If (iii) also holds, then $\left\{u_{n}\right\}$ fulfills all our requirements.

Customarily, one lets $u_{n}=\phi_{n}=2^{2^{n}}+1$, the $n^{\text {th }}$ Fermat number. If $n \geq 1$, then

$$
\phi_{n}=\left(2^{2^{n-1}}\right)^{2}+1^{2}
$$

where $2^{2^{n-1}}$ and 1 are relatively prime and of opposite parity. Since it is also true that $\left(\phi_{m}, \phi_{n}\right)$ $=1$ for all $m \neq n$, we are done.

An alternative procedure utilizes the Fibonacci sequence $\left\{F_{n}\right\}$ or, more precisely, an infinite subsequence thereof. We need the following properties of Fibonacci numbers:

$$
\begin{gather*}
F_{2 k+1}=F_{k}^{2}+F_{k+1}^{2} .  \tag{1}\\
\left(F_{m}, F_{n}\right)=F_{(m, n)} .  \tag{2}\\
2 \mid F_{n} \text { iff } 3 \mid n . \tag{3}
\end{gather*}
$$

If $n \geq 3$, then $F_{n}>1$.
(See [2].)
Suppose we number the primes starting with 5 as follows: $q_{1}=5, q_{2}=7, q_{3}=11$, etc. Let $u_{n}=F_{q_{n}}$ for $n \geq 1$. Now (1) implies

$$
F_{q_{n}}=F_{1 / 2\left(q_{n}-1\right)}^{2}+F_{1 / 2\left(q_{n}+1\right)}^{2} \text { for all } n \geq 1
$$

Since $\left(1 / 2\left(q_{n}-1\right), 1 / 2\left(q_{n}+1\right)\right)=1$, (2) implies

$$
\left(F_{1 / 2\left(q_{n}-1\right)}, F_{1 / 2\left(q_{n}+1\right)}\right)=F_{1}=1 .
$$

Since $q_{n}>3$ and $q_{n}$ is prime by definition, (3) implies $2 \nmid F_{q_{n}}$, so

$$
F_{1 / 2\left(q_{n}-1\right)} \not \equiv F_{1 / 2\left(q_{n}+1\right)}(\bmod 2) .
$$

Finally, if $m \neq n$, then $q_{m} \neq q_{n}$, so $\left(q_{m}, q_{n}\right)=1$. Therefore, (2) implies $\left(F_{q_{m}}, F_{q_{n}}\right)=1$.
In summary, an infinitude of primes $p$ such that $p \equiv 1(\bmod 4)$ can be obtained by considering the least prime divisors of the various Fibonacci numbers $F_{q}$, where $q$ is prime and $q \geq 5$.

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# APPLICATIONS OF FIBONACCI NUMBERS 

VOLUME 4
New Publication
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## Edited by G. E. Bergum, A. N. Philippou, and A. F. Horadam

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# CROSS-JUMP NUMBERS 

Kanakku Puly<br>c/o B. Sury, 106 Panini, TIFR, Homi Bhabha Road, Bombey 400005, India<br>(Submitted March 1992)

Consider any $n$-digit integer expressed in the base $b$. Divide it into a right part of $r$ digits and a left part of $n-r$ digits. To the left part add a number $L<b$ and to the right part add some $R<b$. The addition is done modulo $b$ and the "carry-over" is ignored. Transfer the left part to the right of the right part and we again get an $n$-digit number. Apply this same process (which we call "cross-jumping") to the new number. Iterating this several times, we can ask if we get the original number back, and, if so, what is the least number $N$ of steps required? We prove that

$$
N=\frac{b n}{(b, L+R) \cdot(n, r)}
$$

where $(a, b)$ denotes the G.C.D. of two numbers $a$ and $b$. We first illustrate this by an example.
Example: We take $b=10, n=8, r=2, L=4, R=2$. Starting with the number 56240317, the iteration gives

| 56240317 | 26051556 | 07175426 |
| :--- | :--- | :--- |
| 19562407 | 58260519 | 28071758 |
| 09195628 | 11582609 | 50280711 |
| 20091950 | 01115820 | 13502801 |
| 52200913 | 22011152 | 03135022 |
| 15522003 | 54220115 | 24031354 |
| 05155224 | 17542205 | 56240317 |

which gives back the original number in the $20^{\text {th }}$ step.
Let us prove our claimed formula for $n$. We denote the positions of the $n$ digits from left to right by $1,2, \ldots, n$, respectively. The positions change as $a \rightarrow a+r \rightarrow a+2 r \ldots$ for each $a \leq n$, where + is addition modulo $n$. For repetition of the original number, we should have some $k>0$ so that $a+k r \equiv a \bmod n$. Clearly then, $k=n /(n, r)$ is the least such $k$. The choice of $k$ only ensures that the positions of the original digits are the same after every $k$ steps. Now, for any $m \leq$ $k=n /(n, k)$, there is a corresponding $a_{0}$ such that $a_{0}+m r=n$. We have

$$
a_{0} \rightarrow a_{0}+r \cdots \rightarrow a_{0}+(m-1) r=n-r \xrightarrow{L} a_{0}+m r=n \xrightarrow{R} a_{0}+(m+1) r \cdots a_{0}+k r=a_{0},
$$

where we have written $L, R$ over an arrow to indicate an increase in the value of that digit by $L$, $R$, etc. Thus, we have an increment of $L+R$ in the value of each digit for every $k$ steps. For repetition of the original number, this increment should be a multiple of $b$ and, therefore, $N$ must be a multiple of $k$ as well as of $k b /(L+R)$. This gives $N=$ L.C.M. of $k$ and $k b /(L+R)$, i.e.,

$$
N=\frac{b n}{(b, L+R) \cdot(n, r)} .
$$

In our example, $N=20$.

## ON SUMS OF RECIPROCALS OF FIBONACCI AND LUCAS NUMBERS

## Derek Jennings

University of Southampton, England
(Submitted March 1992)

In this paper we present some remarkable elementary identities for sums of powers of reciprocals of Fibonacci and Lucas numbers. The Fibonacci numbers are defined for all $n \geq 0$ by the recurrence relation $F_{n+1}=F_{n}+F_{n-1}$, where $F_{0}=0$ and $F_{1}=1$. The Lucas numbers $L_{n}$ are defined for all $n \geq 0$ by the same recurrence relation, where $L_{0}=2$ and $L_{1}=1$. The general theorems in this paper include as special cases the following results:

$$
\begin{align*}
& \sum_{n=1}^{\infty} \frac{1}{F_{4 n-2}^{3}}=\frac{5}{2} \sum_{n=1}^{\infty} \frac{n(n-1)}{F_{4 n-2}},  \tag{1}\\
& \sum_{n=1}^{\infty} \frac{1}{L_{2 n-1}^{3}}=\frac{1}{2} \sum_{n=1}^{\infty} \frac{n(n-1)}{L_{2 n-1}},  \tag{2}\\
& \sum_{n=1}^{\infty} \frac{1}{F_{2 n-1}^{2}}=\sqrt{5} \sum_{n=1}^{\infty}(-1)^{n+1} \frac{n}{F_{2 n}},  \tag{3}\\
& \sum_{n=1}^{\infty} \frac{1}{L_{2 n-1}^{2}}=\frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{n}{F_{2 n}} . \tag{4}
\end{align*}
$$

Identity (3) appears on page 98 of [1]. Identity (4) is really just the complementary result of (3). Identities (1) and (2) are believed to be new. The above four results are just the first cases of the following theorems.

Theorem 1: For $k=1,2,3, \ldots$, we have

$$
\frac{1}{5^{k}} \sum_{n=1}^{\infty} \frac{1}{F_{4 n-2}^{2 k+1}}=\frac{1}{(2 k)!} \sum_{n=1}^{\infty} \frac{(n-k)(n-k+1) \cdots(n+k-1)}{F_{4 n-2}} .
$$

Theorem 2: For $k=1,2,3, \ldots$, we have

$$
\sum_{n=1}^{\infty} \frac{1}{L_{2 n-1}^{2 k+1}}=\frac{1}{(2 k)!} \sum_{n=1}^{\infty} \frac{(n-k)(n-k+1) \cdots(n+k-1)}{L_{2 n-1}} .
$$

Theorem 3: For $k=1,2,3, \ldots$, we have

$$
\frac{1}{5^{k-1 / 2}} \sum_{n=1}^{\infty} \frac{1}{F_{2 n-1}^{2 k}}=\frac{(-1)^{k}}{(2 k-1)!} \sum_{n=1}^{\infty} \frac{(-1)^{n}(n-k+1)(n-k+2) \cdots(n+k-1)}{F_{2 n}} .
$$

Theorem 4: For $k=1,2,3, \ldots$, we have

$$
\sum_{n=1}^{\infty} \frac{1}{L_{2 n-1}^{2 k}}=\frac{1}{(2 k-1)!\sqrt{5}} \sum_{n=1}^{\infty} \frac{(n-k+1)(n-k+2) \cdots(n+k-1)}{F_{2 n}} .
$$

Theorems 1 and 2 are corollaries of the following Theorem 5. We note that $F_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}$, where $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$. So, if we let $q:=q^{2}$ in Theorem 5 , then put $q=\beta$, we have Theorem 1 . Similarly, setting $q=\beta$ in Theorem 5 gives Theorem 2 .

Theorem 5: For $|q|<1$ and $k=1,2,3, \ldots$, we have

$$
\sum_{n=1}^{\infty} \frac{q^{(2 k+1)(2 n-1)}}{\left(1-q^{4 n-2}\right)^{2 k+1}}=\frac{1}{(2 k)!} \sum_{n=1}^{\infty} \frac{(n-k)(n-k+1) \cdots(n+k-1) q^{2 n-1}}{1-q^{4 n-2}}
$$

Theorems 3 and 4 are corollaries of the following theorem.
Theorem 6: For $|q|<1$ and $k=1,2,3, \ldots$, we have

$$
\sum_{n=1}^{\infty} \frac{q^{k(2 n-1)}}{\left(1-q^{2 n-1}\right)^{2 k}}=\frac{1}{(2 k-1)!} \sum_{n=1}^{\infty} \frac{(n-k+1)(n-k+2) \cdots(n+k-1) q^{n}}{1-q^{2 n}}
$$

To derive Theorem 3 from Theorem 6, we let $q:=-q$. Then $q:=q^{2}$, and we set $q=\beta$ where $\beta=(1-\sqrt{5}) / 2$. Theorem 4 follows similarly by setting $q:=q^{2}$ then $q=\beta$.

Theorems 5 and 6 are proved in a similar way; therefore, we present only the proof of Theorem 5.

Proof of Theorem 5: For $|q|<1$ and $k=1,2,3, \ldots$, we have, by the binomial theorem,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{q^{(2 k+1)(2 n-1)}}{\left(1-q^{4 n-2}\right)^{2 k+1}} & =\sum_{n=1}^{\infty} q^{(2 k+1)(2 n-1)} \sum_{m=1}^{\infty} \frac{m(m+1) \cdots(m+2 k-1)}{(2 k)!} q^{(4 n-2)(m-1)} \\
& =\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m(m+1) \cdots(m+2 k-1)}{(2 k)!} q^{(2 m+2 k-1)(2 n-1)}
\end{aligned}
$$

with $m:=m-k$

$$
=\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(m-k)(m-k+1) \cdots(m+k-1)}{(2 k)!} q^{(2 m-1)(2 n-1)}
$$

which, on interchanging the order of summation,

$$
=\frac{1}{(2 k)!} \sum_{m=1}^{\infty} \frac{(m-k)(m-k+1) \cdots(m+k-1) q^{2 m-1}}{1-q^{4 m-2}}
$$

This completes the proof of Theorem 5 and, hence, that of Theorems 1 and 2.
In a similar way to Theorems 1-4, we can demonstrate the following results:

$$
\begin{gather*}
\sum_{n=1}^{\infty} \frac{n}{F_{2 n}^{3}}=\frac{5 \sqrt{5}}{2} \sum_{n=1}^{\infty} \frac{n(n-1)}{L_{2 n-1}^{2}}  \tag{5}\\
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{F_{2 n}^{3}}=\frac{\sqrt{5}}{2} \sum_{n=1}^{\infty} \frac{n(n-1)}{F_{2 n-1}^{2}} \tag{6}
\end{gather*}
$$

and

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{n}{F_{2 n}^{4}}=\frac{5}{6} \sum_{n=1}^{\infty} \frac{n\left(n^{2}-1\right)}{F_{2 n}^{2}} . \tag{7}
\end{equation*}
$$

The above results are special cases of the following theorems.
Theorem 7: For $k=0,1,2,3, \ldots$, we have

$$
\frac{1}{5^{k+1 / 2}} \sum_{n=1}^{\infty} \frac{n}{F_{2 n}^{2 k+1}}=\frac{1}{(2 k)!} \sum_{n=1}^{\infty} \frac{(n-k)(n-k+1) \cdots(n+k-1)}{L_{2 n-1}^{2}},
$$

where $(n-k)(n-k+1) \cdots(n+k-1)$ is taken to be 1 when $k=0$.
Theorem 8: For $k=0,1,2,3, \ldots$, we have

$$
\frac{1}{5^{k-1 / 2}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{F_{2 n}^{2 k+1}}=\frac{1}{(2 k)!} \sum_{n=1}^{\infty} \frac{(n-k)(n-k+1) \cdots(n+k-1)}{F_{2 n-1}^{2}}
$$

where $(n-k)(n-k+1) \cdots(n+k-1)$ is taken to be 1 when $k=0$.
Theorem 9: For $k=1,2,3, \ldots$, we have

$$
\frac{1}{5^{k-1}} \sum_{n=1}^{\infty} \frac{n}{F_{2 n}^{2 k}}=\frac{1}{(2 k-1)!} \sum_{n=1}^{\infty} \frac{(n-k+1)(n-k+2) \cdots(n+k-1)}{F_{2 n}^{2}} .
$$

Theorems 7-9 are corollaries of Theorems 10 and 11 below.
Theorem 10: For $|q|<1$ and $k=0,1,2,3, \ldots$, we have

$$
\sum_{n=1}^{\infty} \frac{n q^{n(2 k+1)}}{\left(1-q^{2 n}\right)^{2 k+1}}=\frac{1}{(2 k)!} \sum_{n=1}^{\infty} \frac{(n-k)(n-k+1) \cdots(n+k-1) q^{2 n-1}}{\left(1-q^{2 n-1}\right)^{2}},
$$

where $(n-k)(n-k+1) \cdots(n+k-1)$ is taken to be 1 when $k=0$.
Theorem 11: For $|q|<1$ and $k=1,2,3, \ldots$, we have

$$
\sum_{n=1}^{\infty} \frac{n q^{2 n k}}{\left(1-q^{2 n}\right)^{2 k}}=\frac{1}{(2 k-1)!} \sum_{n=1}^{\infty} \frac{(n-k+1)(n-k+2) \cdots(n+k-1) q^{2 n}}{\left(1-q^{2 n}\right)^{2}} .
$$

As with Theorems 5 and 6 , the proofs of Theorems 10 and 11 are very similar; thus, we present only the proof of Theorem 11.

Proof of Theorem 11: For $|q|<1$ and $k=1,2,3, \ldots$, we have, by the binomial theorem,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{n q^{2 n k}}{\left(1-q^{2 n}\right)^{2 k}} & =\sum_{n=1}^{\infty} n q^{2 n k} \sum_{m=1}^{\infty} \frac{m(m+1) \cdots(m+2 k-2)}{(2 k-1)!} q^{2 n(m-1)} \\
& =\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m(m+1) \cdots(m+2 k-2)}{(2 k-1)!} n q^{n(2 m+2 k-2)}
\end{aligned}
$$

with $m:=m-k+1$

$$
=\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(m-k+1)(m-k+2) \cdots(m+k-1)}{(2 k-1)!} n q^{2 m n}
$$

which, on interchanging the order of summation and summing $\sum_{n=1}^{\infty} n q^{2 m n}$,

$$
=\frac{1}{(2 k-1)!} \sum_{m=1}^{\infty} \frac{(m-k+1)(m-k+2) \cdots(m+k-1) q^{2 m}}{\left(1-q^{2 m}\right)^{2}} .
$$

This completes the proof of Theorem 11.
Theorem 7 follows by letting $q:=q^{2}$, then $q=\beta$ in Theorem 10. Theorem 8 follows by letting $q:=-q$, then $q=\beta$ in Theorem 10, and Theorem 9 follows by letting $q:=q^{2}$, then $q=\beta$ in Theorem 11.

## REFERENCE

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# FIBONACCI-TYPE SEQUENCES AND MINIMAL SOLUTIONS OF DISCRETE SILVERMAN GAMES 

Gerald A. Heuer<br>Department of Mathematics and Computer Science, Concordia College, Moorhead, MN 56562<br>Ulrike Leopold-Wildburger<br>Institut für Statistik, Ökonometrie und Operations Research, Karl-Franzens-Universität Graz, A-8010 Graz, Austria<br>(Submitted March 1992)

## 1. GAME THEORY BACKGROUND

While the principal results of this paper seem to us to be of interest in their own right, and can be understood with no reference to game theory, the problems addressed arose in a game theory setting, and their solution has important consequences for the analysis of Silverman games. It seems appropriate therefore to sketch briefly the game theory background. Silverman games are two-person, zero-sum games in which, roughly speaking, the higher bid wins, unless it is too much higher than the other, in which case it loses. More precisely, let $S_{\mathrm{I}}$ and $S_{\mathrm{II}}$ be sets of positive real numbers, and $T$ and $v$ be parameters with $T>1$ and $v>0$. The sets $S_{\mathrm{I}}$ and $S_{\mathrm{II}}$ are the pure strategy sets for Players I and II, respectively. Each player chooses a number from his strategy set, and the higher number wins 1 , unless it is at least $T$ times as large as the other, in which case it loses $v$. The parameters $T$ and $v$ are referred to as the threshold and the penalty, respectively. If $S_{\mathrm{I}}=S_{\mathrm{II}}$, the game is symmetric, and in this case, if optimal strategies exist they are the same for both players, and the game value is 0 .

The prototype games are attributed to David Silverman, although the earliest published mention of such a game of which we are aware is by Herstein and Kaplansky ([3], p. 212). The symmetric game on an open interval was analyzed by R. J. Evans [1] for arbitrary $T$ and $v$, and the symmetric game on discrete sets by Evans and Heuer [2]. An analogous symmetric game on [ $1, \infty$ ) is examined in [5]. Discrete games with $S_{\mathrm{I}} \cap S_{\mathrm{II}}=\varnothing$ are examined in [4] and [8]. In [6] it is shown that when $v \geq 1$ Silverman games reduce by dominance to games on bounded sets, and in [7] this and other types of dominance are used to reduce discrete games with $v \geq 1$ to finite games, and their payoff matrices have a simple characteristic form.

Many semi-reduced games can be further reduced in the sense that there still are proper subsets $W_{\mathrm{I}}$ and $W_{\mathrm{II}}$ of the strategy sets, with the property that optimal mixed strategies for the game on $W_{\mathrm{I}} \times W_{\text {II }}$ are optimal for the full game. This further reduction leads to games some of which are $2 \times 2$ and the rest of which fall into eight families, four of even-order games and four of oddorder games (see [7]). It was our conjecture that when $v>1$, no further reduction of any of these games is possible. This would mean that optimal mixed strategies for such a reduced game are minimal optimal strategies for the original game. We shall show here that, for the odd-order games, this is indeed the case, and using similar techniques we obtain explicitly the unique optimal mixed strategies and game values for these reduced games. The even-order cases will be treated in a forthcoming paper.

## 2. THE ASSOCIATED MATRICES

Let $B$ denote the payoff matrix of our reduced game and $V$ the game value. Then $B$ is always square, and as discussed in Section 13 of [7], the game is not further reducible if and only if there is a unique probability vector $P$, with all components positive, such that

$$
\begin{equation*}
P B=(V, V, \ldots, V) . \tag{2.1a}
\end{equation*}
$$

In this case there is also a unique probability vector $Q$ such that

$$
\begin{equation*}
B Q^{t}=(V, V, \ldots, V)^{t}, \tag{2.1b}
\end{equation*}
$$

and $P$ and $Q$ are the unique optimal mixed strategy vectors for the row player and column player, respectively. (We are writing vectors as row vectors.)

Let $B_{. j}$ denote the $j^{\text {th }}$ column of $B$. If $B$ is $2 n+1$ by $2 n+1$, then ( $2.1 a$ ) is equivalent to

$$
\begin{equation*}
P B_{. j}=V \text { for } j=1,2, \ldots, 2 n+1 . \tag{2.2}
\end{equation*}
$$

With the understanding that $P$ is to be a probability vector, this, in turn, is equivalent to

$$
\begin{equation*}
P\left(B_{. j}-B_{. j+1}\right)=0 \text { for } j=1,2, \ldots, 2 n, \text { and } \sum_{i=1}^{2 n+1} p_{i}=1, \text { with each } p_{i}>0 . \tag{2.3}
\end{equation*}
$$

Now let $A$ be the $2 n+1$ by $2 n+1$ matrix, the $i^{\text {th }}$ row of which is $\left(B_{i}-B_{i+1}\right)^{t}$ for $i=1,2, \ldots$, $2 n$, and the $(2 n+1)^{\text {th }}$ row of which is $(1,1, \ldots, 1)$. Then (2.3) is equivalent to

$$
\begin{equation*}
A P^{t}=(0,0, \ldots, 0,1)^{t}, \tag{2.4}
\end{equation*}
$$

which has a unique solution if and only if $A$ is nonsingular. Thus, it suffices to show that $A$ is nonsingular and that a probability vector $P$ with all components positive exists, satisfying (2.4).

The four families of odd-order payoff matrices $B$ and the associated matrices $A$ are illustrated below. The variable $x$ is $1+v$, and with $v>1$ we have $x>2$. Types (i), (ii), (iii), and (iv) here correspond to (8.0.5A), (8.0.5B), (8.0.5C), and (8.0.5D), respectively, in [7]. The main diagonal and first superdiagonal of $A$ consist entirely of 1s, with two exceptions. In column $a+1$, the pair $\binom{2}{0}$ occurs in place of $\binom{1}{1}$, and in column $n+a+2,\binom{0}{2}$ occurs. In general, the matrix $A$ of type (i) has $a$ columns preceding the first irregular one, then $d$ regular columns, a central column, $a$ regular columns, the second irregular one, and $d$ regular ones, for a total of $2 n+1=2 a+2 d+3$ columns.

$$
B=\left(\begin{array}{rrrrrrrrrrr}
0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v & v \\
1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & v & v & v \\
1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v \\
1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v \\
1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \\
-v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\
-v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\
-v & -v & -v & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\
-v & -v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right) ;
$$

$$
A=\left(\begin{array}{rrrrrrrrrrr}
1 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x \\
-x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & -x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right) .
$$

Type (i), parameters $a \geq 0, d \geq 0 ; n=a+d+1$. Illustrated with $a=2, d=2$.
In the matrix $A$ of type (ii), there are three irregular columns. The parameters here are $c$ and $d$, and the pattern is $c+1$ regular columns, the column with the $\left.{ }_{\left({ }^{2}\right)}^{0}\right), d$ regular columns, the central column, $c$ regular columns, two columns with $\binom{0}{2}$ in place of $\binom{1}{1}$, and $d$ regular columns. We illustrate it here with $c=1, d=2 ; n=c+d+2=5$, so again $B$ and $A$ are $11 \times 11$.

$$
\begin{aligned}
& B=\left(\begin{array}{rrrrrrrrrrr}
0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v & v \\
1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & v & v & v \\
1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v \\
1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v \\
1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \\
-v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\
-v & -v & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
-v & -v & -v & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\
-v & -v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right) ; \\
& A=\left(\begin{array}{rrrrrrrrrrr}
1 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x \\
-x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & -x & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -x & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right) .
\end{aligned}
$$

Type (ii), parameters $c \geq 0, d \geq 0 ; n=c+d+2$. Illustrated with $c=1, d=2$.
We illustrate type (iii) below.

$$
\begin{aligned}
& B=\left(\begin{array}{rrrrrrrrrrr}
0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v & v \\
1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & v & v & v \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & v & v \\
1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v \\
1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \\
-v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\
-v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\
-v & -v & -v & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\
-v & -v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right) ; \\
& A=\left(\begin{array}{rrrrrrrrrrr}
1 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -x & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -x & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x \\
-x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & -x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right) .
\end{aligned}
$$

Type (iii), parameters $a \geq 0, b \geq 0 ; n=a+b+2$. Illustrated with $a=2, b=1$.
In the matrix $A$ of type (iii), shown above, there are again three irregular columns. The parameters are $a$ and $b$, and the pattern of columns is: $a$ regular columns, two columns with $\binom{2}{0}$ in place of $\binom{1}{1}, b$ regular columns, the central column, $a$ regular, one with $\binom{0}{2}$ and $b+1$ regular.

Finally, in matrix $A$ of type (iv), there are two irregular columns. The parameters are denoted $c$ and $b$, and the pattern of columns is $c+1$ regular, one with $\binom{2}{0}, b$ regular, the central column, $c$ regular columns, one with $\binom{0}{2}$, and $b+1$ regular. We illustrate type (iv) below, with $c=2, b=1$; $n=c+b+2=5$.

$$
B=\left(\begin{array}{rrrrrrrrrrr}
0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v & v \\
1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v \\
1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v & v \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & v & v \\
1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v \\
1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \\
-v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\
-v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\
-v & -v & -v & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\
-v & -v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right) ;
$$

$$
A=\left(\begin{array}{rrrrrrrrrrr}
1 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & -x & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -x & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x \\
-x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & -x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right) .
$$

Type (iv), parameters $c \geq 0, b \geq 0 ; n=c+b+2$. Illustrated with $c=2, b=1$.
Main Theorem: For $x>2$, every matrix in these four two-parameter families is nonsingular, and the unique vector $P$ satisfying (2.4) has all components positive.

When the diagonal of the payoff matrix $B$ consists entirely of zeros the game is symmetric, and has been shown in [2] to have a unique optimal mixed strategy. It follows in that case that the associated matrix of (2.4), which we denote $A^{*}$, is nonsingular. This matrix $A^{*}$ is like those in the four families above, but without the irregularities; i.e., the main diagonal and the first superdiagonal consist entirely of 1 s . We shall in each instance prove that $A$ is nonsingular by exhibiting a matrix $D$ such that $A D=A^{*}$, and prove that a completely mixed (all components positive) vector $P$ satisfying (2.4) exists by exhibiting it. The task of obtaining such a $D$ is lightened substantially by the observation that in each of the four classes, the matrix $A$ differs from $A^{*}$ in at most two columns. It suffices, therefore, to show that these columns of $A^{*}$ lie in the column space of $A$, and we accomplish this by producing columns $D_{. j}$ such that $A D_{. j}=A_{. j}^{*}$ for the appropriate $j$.

We illustrate here using the case $a=d=1$ in type (i). Then $n=3$, and the matrices $A$ and $A^{*}$ are $7 \times 7$.

$$
A=\left(\begin{array}{rrrrrrr}
1 & 2 & 0 & 0 & -x & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -x & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & -x \\
-x & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & -x & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -x & 0 & 0 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right) .
$$

This matrix differs from $A^{*}$ only in columns $2(=a+1)$ and $6(=n+a+2)$. The column $D_{a+1}$ is given by (4.0.2), and in this illustration it is

$$
D_{\cdot 2}=\left(\begin{array}{c}
-2(x+2) T_{0}+x(x+2) T_{1}+2 \\
x(x+2) E_{2}+1 \\
-2 x(x+2) E_{1}+x^{2}(x+2) E_{-1}+x \\
-2 x(x+2) E_{0}+x^{2}(x+2) E_{0}+x \\
-2 x(x+2) E_{-1}+x^{2}(x+2) E_{1}+x \\
-(x+2) T_{2}+1 \\
-2(x+2) T_{1}+x(x+2) T_{0}+2
\end{array}\right) .
$$

If $\Delta=x(x+2) R_{2}$ [see (4.0.1)], the reader may verify, using identities (3.2), (3.0.5), (3.0.3), (3.8), and (3.9), and particular values of $E_{n}$ and $T_{n}$ given by (3.0.10) and (3.0.13), that $A D_{.2}=\Delta A_{2}^{*}$, where $A_{2}^{*}=(1,1,0,0,-x, 0,1)^{t}$ (This is then a special case of Theorem 4.1.)

## 3. THE POLYNOMIAL SEQUENCES

We shall describe the matrix $D$ in terms of six Fibonacci-like sequences of polynomials, and use Fibonacci-like properties of these sequences to prove that $A D=A^{*}$. Each sequence is a particular solution to the recursion

$$
\begin{equation*}
Y_{m+1}=\left(x^{2}-2\right) Y_{m}-Y_{m-1}+C, \tag{3.0.1}
\end{equation*}
$$

where the constant $C$ is 0,1 , or 2 . For some earlier work on sequences generated by a recursion like (3.0.1) without the ( $x^{2}-2$ ) coefficient (see [9] and [10]).

Define polynomial sequences $E_{m}, R_{m}, G_{m}, T_{m}, H_{m}$, and $K_{m}$ as follows:

$$
\begin{gather*}
E_{0}=1, E_{1}=x^{2}-1, E_{m+1}=\left(x^{2}-2\right) E_{m}-E_{m-1}+1 .  \tag{3.0.2}\\
R_{m}=E_{m}-E_{m-1} .  \tag{3.0.3}\\
G_{m}=R_{m}-R_{m-1} .  \tag{3.0.4}\\
T_{m}=E_{m}+E_{m-1} .  \tag{3.0.5}\\
H_{m}=R_{m}+R_{m-1}=E_{m}-E_{m-2}=T_{m}-T_{m-1} .  \tag{3.0.6}\\
K_{m}=H_{m}-H_{m-1}=R_{m}-R_{m-2}=G_{m}+G_{m-1}=T_{m}-2 T_{m-1}+T_{m-2} . \tag{3.0.7}
\end{gather*}
$$

In (3.0.6) and (3.0.7) the first equality is to be understood as the definition; the others follow immediately. One sees further at once that

$$
\begin{equation*}
R_{m}, G_{m}, H_{m} \text {, and } K_{m} \text { satisfy (3.0.1) with } C=0 \text {, } \tag{3.0.8}
\end{equation*}
$$

and that

$$
\begin{equation*}
T_{m} \text { satisfies (3.0.1) with } C=2 . \tag{3.0.9}
\end{equation*}
$$

The recursion (3.0.1) can be used to extend the sequence in both directions, and we regard each of the sequences as being defined for all integers $m$. From the recursions, one finds readily the following:

$$
\begin{gather*}
E_{-1}=E_{-2}=0, E_{-3}=E_{0}-1, \text { and } E_{-m}=E_{m-3} .  \tag{3.0.10}\\
R_{0}=1, R_{-1}=0, R_{-2}=-1, \text { and } R_{-m}=-R_{m-2} .  \tag{3.0.11}\\
G_{0}=G_{-1}=1, \text { and } G_{-m}=G_{m-1} .  \tag{3.0.12}\\
T_{0}=1, T_{-1}=0, T_{-2}=1, \text { and } T_{-m}=T_{m-2} .  \tag{3.0.13}\\
H_{0}=1, H_{-1}=-1, \text { and } H_{-m}=-H_{m-1} .  \tag{3.0.14}\\
K_{1}=x^{2}-2, K_{0}=2, \text { and } K_{-m}=K_{m} . \tag{3.0.15}
\end{gather*}
$$

Theorem 3.1: Every polynomial $E_{m}$ with $m \geq 0$ takes only positive values for $x>2$. The same is true of each of the other sequences defined by (3.0.2) to (3.0.7).

Proof: It is a routine exercise to prove by induction that $E_{m+1} \geq E_{m} \geq 0$ for $x>2$ and all $m$. The same goes for each of the other sequences.

Following are some further properties of these polynomials that we will find useful.

$$
\begin{equation*}
x^{2} E_{m}=T_{m}+T_{m+1}-1 \tag{3.2}
\end{equation*}
$$

This is immediate from (3.0.2) and (3.0.5).
Similarly, from the recursion (3.0.8) for $G_{m}$ and (3.0.7), we have

$$
\begin{equation*}
x^{2} G_{m}=K_{m+1}+K_{m} \tag{3.3}
\end{equation*}
$$

and from the recursion (3.0.8) for $R_{m}$ and (3.0.6),

$$
\begin{equation*}
x^{2} R_{m}=H_{m+1}+H_{m} \tag{3.4}
\end{equation*}
$$

From (3.0.8) and (3.0.4) we obtain

$$
\begin{equation*}
\left(x^{2}-4\right) R_{m}=G_{m+1}-G_{m} \tag{3.5}
\end{equation*}
$$

and from (3.0.2), (3.0.3), and (3.0.4), we have

$$
\begin{equation*}
\left(x^{2}-4\right) E_{m}+1=G_{m+1} \tag{3.6}
\end{equation*}
$$

Similarly we obtain

$$
\begin{equation*}
\left(x^{2}-4\right) T_{m}+2=K_{m+1} \tag{3.7}
\end{equation*}
$$

From (3.0.9) we have that $\left(x^{2}-2\right) T_{i}-T_{i+1}-T_{i-1}=-2$. Upon summing this for $0 \leq i \leq m$, adding $T_{m+1}-T_{m}-1$ to both sides, and using (3.0.13), we obtain

$$
\begin{equation*}
\left(x^{2}-4\right) \sum_{i=0}^{m} T_{i}=T_{m+1}-T_{m}-2 m-3 \tag{3.8}
\end{equation*}
$$

In exactly the same way, using (3.0.2) and (3.0.10), we obtain

$$
\begin{equation*}
\left(x^{2}-4\right) \sum_{i=0}^{m} E_{i}=E_{m+1}-E_{m}-m-2 \tag{3.9}
\end{equation*}
$$

Theorem 3.10: For all integers $r$ and $m$,

$$
\begin{equation*}
G_{r} H_{m}+G_{m} H_{r}=2 R_{r+m} \tag{3.10.1}
\end{equation*}
$$

Proof: For fixed $r$, both members are sequences indexed by $m$ satisfying the homogeneous difference equation (3.0.1), as noted in (3.0.8). It will suffice, therefore, to show equality in (3.10.1) for $m=-1$ and $m=0$. But from (3.0.4) and (3.0.6) we have $-G_{r}+H_{r}=2 R_{r-1}$ and $G_{r}+H_{r}=2 R_{r}$, which, in view of (3.0.12) and (3.0.14), establishes (3.10.1) for $m=-1$ and $m=0$.

Theorem 3.11: For all integers $r$ and $m$,

$$
\begin{equation*}
G_{r} R_{m}+G_{m} R_{r-1}=R_{r+m} \tag{3.11.1}
\end{equation*}
$$

Proof: This is proved in the same way as (3.10), using (3.0.4).

In much the same way, one shows

$$
\begin{gather*}
G_{r} R_{m}-G_{r-1} R_{m-1}=G_{r+m},  \tag{3.12}\\
R_{r} H_{m}-R_{m-1} H_{r}=R_{r+m},  \tag{3.13}\\
K_{r+1} R_{m}-G_{r} H_{m}=G_{r+m+1},  \tag{3.14}\\
K_{r+1} R_{m}+K_{m+1} R_{r}=2 R_{r+m+1},  \tag{3.15}\\
K_{r+1} H_{m}+K_{m} H_{r}=x^{2} R_{r+m},  \tag{3.16}\\
G_{r} H_{m}-G_{m} H_{r}=2 R_{m-r-1},  \tag{3.17}\\
G_{r} R_{m}-G_{m+1} R_{r-1}=R_{m-r},  \tag{3.18}\\
G_{r} R_{m}-G_{r+1} R_{m-1}=G_{r-m},  \tag{3.19}\\
R_{r} G_{m}-R_{m} G_{r}=R_{r-m-1}, \tag{3.20}
\end{gather*}
$$

and

$$
\begin{equation*}
R_{r} K_{m+1}-R_{m} K_{r+1}=2 R_{r-m-1} . \tag{3.21}
\end{equation*}
$$

Many further identities of this type could be given, but these are the ones used in the remainder of the paper.

## 4. GAMES OF TYPE (i)

Suppose that $A$ is a matrix of type (i) with parameters $a$ and $d$. Then $A$ is $2 n+1 \times 2 n+1$, where $n=a+d+1$. To show that there is a matrix $D$ such that $A D=A^{*}$, as discussed in Section 2 , is equivalent to showing that each column of $A^{*}$ is in the column space of $A$. However, with the exception of the two irregular columns, every column of $A$ is itself a column of $A^{*}$, so we have only to show that columns $a+1$ and $n+a+2$ of $A^{*}$ are in the column space of $A$. Let $D_{. j}$ and $A_{. j}^{*}$ denote the $j^{\text {th }}$ column of $D$ and $A^{*}$, respectively. What we shall actually exhibit are columns $D_{. j}$ such that $A D_{. j}=\Delta A_{. j}^{*}$ for $j=a+1$ and $n+a+2$, where

$$
\begin{equation*}
\Delta=x(x+2) R_{n-1} \quad(n=a+d+1) . \tag{4.0.1}
\end{equation*}
$$

This suffices, in view of the fact that, by Theorem 3.1, $\Delta>0$ for $x>2$.
The column $D_{\cdot a+1}$ is defined as follows:

$$
\begin{align*}
d_{i, a+1} & =-2(x+2) T_{a-i}+x(x+2) T_{n-a+i-2}+2 & & \text { for } 1 \leq i \leq a ; \\
d_{a+1, a+1} & =x(x+2) E_{n-1}+1 ; & & \\
d_{i, a+1} & =-2 x(x+2) E_{n+a-i}+x^{2}(x+2) E_{i-a-3}+x & & \text { for } a+2 \leq i \leq n+a+1 ;  \tag{4.0.2}\\
d_{n+a+2, a+1} & =-(x+2) T_{n-1}+1 ; & & \\
d_{i, a+1} & =-2(x+2) T_{2 n+a+1-i}+x(x+2) T_{i-n-a-3}+2 & & \text { for } n+a+3 \leq i \leq 2 n+1 .
\end{align*}
$$

Theorem 4.1: Let $A$ be a matrix of type (i) as described in Section 2, with parameters $a$ and $d$. With $D_{a+1}, \Delta$, and $A^{*}$ as defined above, we have

$$
\begin{equation*}
A D_{\cdot a+1}=\Delta A_{\cdot a+1}^{*} . \tag{4.1.1}
\end{equation*}
$$

Proof: The column $A_{a+1}^{*}$ has 1 s in rows $a, a+1$, and $2 n+1,-x$ in row $n+a+1$, and all other elements are 0 . Thus, we need to show that the following equations are satisfied:

$$
\begin{array}{rlrl}
d_{i, a+1}+d_{i+1, a+1}-x d_{n+i+1, a+1} & =0 & & \text { for } 1 \leq i \leq a-1 ; \\
d_{a, a+1}+2 d_{a+1, a+1}-x d_{n+a+1, a+1} & =\Delta ; & & \\
d_{a+2, a+1}-x d_{n+a+2, a+1} & =\Delta ; & & \\
d_{i, a+1}+d_{i+1, a+1}-x d_{n+i+1, a+1} & =0 \quad \text { for } a+2 \leq i \leq n ; \\
-x d_{i, a+1}+d_{n+i, a+1}+d_{n+i+1, a+1} & =0 \quad \text { for } 1 \leq i \leq a ; \\
-x d_{a+1, a+1}+d_{n+a+1, a+1} & =-x \Delta ; & & \\
-x d_{a+2, a+1}+2 d_{n+a+2, a+1}+d_{n+a+3, a+1} & =0 ; & & \\
-x d_{i, a+1}+d_{n+i, a+1}+d_{n+i+1, a+1} & =0 & & \text { for } a+3 \leq i \leq n ; \\
\sum_{i=1}^{2 n+1} d_{i, a+1} & =\Delta . & & \tag{4.1.10}
\end{array}
$$

Since the second subscript is $a+1$ in every case, there should be no confusion if we drop it; i.e., we will write $d_{i}$ for $d_{i, a+1}$. To establish (4.1.2) note that, for $1 \leq i \leq a-1$, we have

$$
\begin{aligned}
d_{1}+d_{i+1}-x d_{n+i+1}= & -2(x+2) T_{a-i}+x(x+2) T_{n-a-2+i}+2-2(x+2) T_{a-i-1} \\
& +x(x+2) T_{n-a-1+i}+2+2 x^{2}(x+2) E_{a-i-1}-x^{3}(x+2) E_{n-a-2+i}-x^{2} \\
= & -2(x+2)\left(T_{a-i}+T_{a-i-1}-x^{2} E_{a-i-1}\right) \\
& +x(x+2)\left(T_{n-a-2+i}+T_{n-a-1+i}-x^{2} E_{n-a-2+i}\right)+4-x^{2} \\
= & (x-2)(x+2)+4-x^{2}=0, \quad \text { by }(3.2) .
\end{aligned}
$$

For (4.1.3), we have

$$
\begin{aligned}
d_{a}+2 d_{a+1}-x d_{n+a+1}= & -2(x+2) T_{0}+x(x+2) T_{n-2}+2+2 x(x+2) E_{n-1}+2 \\
& +2 x^{2}(x+2) E_{-1}-x^{3}(x+2) E_{n-2}-x^{2} \\
= & x(x+2)\left(T_{n-2}+2 E_{n-1}-x^{2} E_{n-2}-1\right), \text { by (3.0.10) and (3.0.13) } \\
= & \Delta, \text { by (3.2), (3.0.5), and (3.0.3). }
\end{aligned}
$$

For (4.1.4), note that

$$
d_{a+2}-x d_{n+a+2}=x(x+2)\left(T_{n-1}-2 E_{n-2}\right)=\Delta \text {, by (3.0.10), (3.0.5), and (3.0.3). }
$$

Both (4.1.5) and (4.1.6) are immediate from (3.0.5).
For (4.1.7), we have

$$
-x d_{a+1}+d_{n+a+1}=-x^{2}(x+2)\left(E_{n-1}-E_{n-2}\right)=-x \Delta, \quad \text { by (3.0.3) and (3.0.10). }
$$

For (4.1.8),

$$
-x d_{a+2}+2 d_{n+a+2}+d_{n+a+3}=2(x+2)\left(x^{2} E_{n-2}-T_{n-1}-T_{n-2}+1\right)=0 \text {, by (3.2). }
$$

For (4.1.9), we have, for $a+3 \leq i \leq n$, that

$$
\begin{aligned}
-x d_{i}+d_{n+i}+d_{n+i+1}= & 2(x+2)\left(x^{2} E_{n+a-i}-T_{n+a+1-i}-T_{n+a-i}\right) \\
& +x(x+2)\left(T_{i-a-3}+T_{i-a-2}-x^{2} E_{i-a-3}\right)+4-x^{2} \\
= & 0, \quad \text { by }(3.2) .
\end{aligned}
$$

Finally, for (4.1.10), we have

$$
\begin{aligned}
\sum_{i=1}^{2 n+1} d_{i}= & -2(x+2) \sum_{i=0}^{a-1} T_{i}+x(x+2) \sum_{i=n-a-1}^{n-2} T_{i}+2 a+x(x+2) E_{n-1}+1 \\
& -2 x(x+2) \sum_{i=-1}^{n-2} E_{i}+x^{2}(x+2) \sum_{i=-1}^{n-2} E_{i}+n x-(x+2) T_{n-1}+1 \\
& -2(x+2) \sum_{i=a}^{n-2} T_{i}+x(x+2) \sum_{i=0}^{n-a-2} T_{i}+2(n-a-1) \\
= & \left(x^{2}-4\right) \sum_{i=0}^{n-2} T_{i}+x\left(x^{2}-4\right) \sum_{i=0}^{n-2} E_{i}+x(x+2) E_{n-1}-(x+2) T_{n-1}+n(x+2) .
\end{aligned}
$$

With the use of (3.8) and (3.9) we obtain, upon simplification,

$$
\sum_{i=1}^{2 n+1} d_{i}=\left(1-T_{n-1}-T_{n-2}\right)+x\left(E_{n-1}-E_{n-2}-T_{n-1}\right)+\left(x^{2}+2 x\right) E_{n-1}
$$

Then, using (3.2), (3.0.5), and (3.0.3), we have

$$
\sum_{i=1}^{2 n+1} d_{i}=-x^{2} E_{n-2}-2 x E_{n-2}+\left(x^{2}+2 x\right) E_{n-1}=\left(x^{2}+2 x\right)\left(E_{n-1}-E_{n-2}\right)=\Delta
$$

and the proof is complete.
The column $D_{\cdot n+a+2}$ is defined as follows:

$$
\begin{align*}
d_{i, n+a+2} & =-2(x+2) T_{n-a+i-2}+x(x+2) T_{a-i}+2 & & \text { for } 1 \leq i \leq a ; \\
d_{a+1, n+a+2} & =-(x+2) T_{n-1}+1 ; & & \\
d_{i, n+a+2} & =-2 x(x+2) E_{i-a-3}+x^{2}(x+2) E_{n+a-i}+x & & \text { for } a+2 \leq i \leq n+a+1 ;  \tag{4.1.11}\\
d_{n+a+2, n+a+2} & =x(x+2) E_{n-1}+1 ; & & \\
d_{i, n+a+2} & =-2(x+2) T_{i-n-a-3}+x(x+2) T_{2 n+a+1-i}+2 & & \text { for } n+a+3 \leq i \leq 2 n+1 .
\end{align*}
$$

Theorem 4.2: With $A, A^{*}$, and $\Delta$ as in Theorem 4.1, and $D_{\cdot n+a+2}$ as defined in (4.1.11), we have

$$
\begin{equation*}
A D_{\cdot n+a+2}=\Delta A_{\cdot n+a+2}^{*} \tag{4.2.1}
\end{equation*}
$$

Proof: The column $A_{\cdot n+a+2}^{*}$ has $-x$ in row $a+1,1$ in rows $n+a+1, n+a+2$, and $2 n+1$, and 0 in each of the remaining rows. We need to show that the following equations are satisfied:

$$
\begin{align*}
d_{i, n+a+2}+d_{i+1, n+a+2}-x d_{n+i+1, n+a+2} & =0 \quad \text { for } 1 \leq i \leq a-1 ;  \tag{4.2.2}\\
d_{a, n+a+2}+2 d_{a+1, n+a+2}-x d_{n+a+1, n+a+2} & =0 ;  \tag{4.2.3}\\
d_{a+2, n+a+2}-x d_{n+a+2, n+a+2} & =-x \Delta ;  \tag{4.2.4}\\
d_{i, n+a+2}+d_{i+1, n+a+2}-x d_{n+i+1, n+a+2} & =0 \quad \text { for } a+2 \leq i \leq n ;  \tag{4.2.5}\\
-x d_{i, n+a+2}+d_{n+i, n+a+2}+d_{n+i+1, n+a+2} & =0 \quad \text { for } 1 \leq i \leq a ;  \tag{4.2.6}\\
-x d_{a+1, n+a+2}+d_{n+a+1, n+a+2} & =\Delta ;  \tag{4.2.7}\\
-x d_{a+2, n+a+2}+2 d_{n+a+2, n+a+2}+d_{n+a+3, n+a+2} & =\Delta ;  \tag{4.2.8}\\
-x d_{i, n+a+2}+d_{n+i, n+a+2}+d_{n+i+1, n+a+2} & =0 \quad \text { for } a+3 \leq i \leq n ;  \tag{4.2.9}\\
\sum_{i=1}^{2 n+1} d_{i, n+a+2} & =\Delta \tag{4.2.10}
\end{align*}
$$

Again we drop the second subscript, which is $n+a+2$ in every instance. Thus, we write $d_{i}$ for $d_{i, n+a+2}$. To show (4.2.2) we note that, for $1 \leq i \leq a-1$.

$$
\begin{aligned}
d_{i}+d_{i+1}-x d_{n+i+1}= & -2(x+2)\left(T_{n+i-a-2}+T_{n+i-a-1}-x^{2} E_{n+i-a-2}\right) \\
& +x(x+2)\left(T_{a-i}+T_{a-i-1}-x^{2} E_{a-i-1}\right)+4-x^{2} \\
= & 0, \text { by (3.2). }
\end{aligned}
$$

For (4.2.3),

$$
\begin{aligned}
d_{a}+2 d_{a+1}-x d_{n+a+1} & =-2(x+2)\left(T_{n-2}+T_{n-1}-x^{2} E_{n-2}\right)+x(x+2)+4-x^{2} \\
& =0, \quad \text { by }(3.2) .
\end{aligned}
$$

For (4.2.4), $d_{a+2}-x d_{n+a+2}=-x^{2}(x+2)\left(E_{n-1}-E_{n-2}\right)=-x \Delta$, by (3.0.3).
For (4.2.5), note that, for $a+2 \leq i \leq n$,

$$
\begin{aligned}
d_{i}+d_{i+1}-x d_{n+i+1} & =-2 x(x+2)\left(E_{i-a-3}+E_{i-a-2}-T_{i-a-2}\right)+x^{2}(x+2)\left(E_{n+a-i}+E_{n+a-i-1}-T_{n+a-i}\right) \\
& =0, \quad \text { by }(3.0 .5)
\end{aligned}
$$

For (4.2.6), we have, for $1 \leq i \leq a$, that

$$
\begin{aligned}
-x d_{i}+d_{n+i}+d_{n+i+1} & =2 x(x+2)\left(T_{n+i-a-2}-E_{n+i-a-3}-E_{n+i-a-2}\right)+x^{2}(x+2)\left(E_{a-i}+E_{a-i-1}-T_{a-i}\right) \\
& =0, \text { by }(3.0 .5) .
\end{aligned}
$$

For (4.2.7), $-x d_{a+1}+d_{n+a+1}=x(x+2)\left(T_{n-1}-2 E_{n-2}\right)=\Delta$, by (3.0.5) and (3.0.3).
For (4.2.8) we have, using (3.0.10) and (3.0.13),

$$
\begin{aligned}
-x d_{a+2}+2 d_{n+a+2}+d_{n+a+3} & =-x^{3}(x+2) E_{n-2}+2 x(x+2) E_{n-1}+x(x+2) T_{n-2}+4-x^{2}-2(x+2) \\
& =x(x+2)\left(-x^{2} E_{n-2}+2 E_{n-1}+T_{n-2}-1\right) \\
& =\Delta, \text { by; }(3.2),(3.0 .5), \text { and (3.0.3). }
\end{aligned}
$$

For (4.2.9), note that, for $a+3 \leq i \leq n$,

$$
\begin{aligned}
-x d_{i}+d_{n+i}+d_{n+i+1}= & 2(x+2)\left(x^{2} E_{i-a-3}-T_{i-a-3}-T_{i-a-2}\right) \\
& +x(x+2)\left(-x^{2} E_{n+a-i}+T_{n+a+1-i}+T_{n+a-i}\right)+4-x^{2} \\
= & 0, \quad \text { by }(3.2) .
\end{aligned}
$$

Finally, (4.2.10) follows from (4.1.10) since the elements of $D_{. n+a+2}$ are precisely those of $D_{. a+1}$ but reordered. This completes the proof.

We turn now to the solution of (2.4). Let $U$ be the column with components

$$
\begin{align*}
u_{i} & =G_{d} K_{a+1-i} & & \text { for } 1 \leq i \leq a ; \\
u_{a+1} & =G_{d} ; & & \\
u_{i} & =x G_{a} G_{i-a-2} & & \text { for } a+2 \leq i \leq n+1 ; \\
u_{i} & =x G_{n+a+1-i} G_{d} & & \text { for } n+1 \leq i \leq n+a+1 ;  \tag{4.2.11}\\
u_{n+a+2} & =G_{a} ; & & \\
u_{i} & =K_{i-n-a-2} G_{a} & & \text { for } n+a+3 \leq i \leq 2 n+1 .
\end{align*}
$$

(Note that $u_{n+1}$ occurs twice but that the two expressions agree.)

Theorem 4.3: With $U$ as defined by (4.2.11), the column $P^{t}=U /(x+2) R_{n-1}$ satisfies (2.4), namely $A P^{t}=(0,0, \ldots, 0,1)^{t}$, for the matrix $A$ of type (i), and has all components positive for $x>2$. The vector $P$ is thus the unique optimal strategy for the row player in the reduced game of type (i).

Proof: That all components are positive for $x>2$ is clear from Theorem 3.1. To prove that (2.4) is satisfied, we show that $A U=(0,0, \ldots, 0, \Delta)^{t}$, where $\Delta=(x+2) R_{n-1}$.

For $1 \leq i \leq a-1$, we have

$$
A_{i} U=u_{i}+u_{i+1}-x u_{n+i+1}=G_{d}\left(K_{a+1-i}+K_{a-i}-x^{2} G_{a-i}\right)=0, \quad \text { by }(3.3)
$$

Also,

$$
A_{a} \cdot U=u_{a}+2 u_{a+1}-x u_{n+a+1}=G_{d}\left(K_{1}+2-x^{2}\right)=0, \quad \text { by }(3.0 .15)
$$

and

$$
A_{a+1} U=u_{a+2}-x u_{n+a+2}=x G_{a} G_{0}-x G_{a}=0
$$

since $G_{0}=1$.
For $a+2 \leq i \leq n$,

$$
A_{i} U=u_{i}+u_{i+1}-x u_{n+i+1}=x G_{a}\left(G_{i-a-2}+G_{i-a-1}-K_{i-a-1}\right)
$$

and, for $n+1 \leq i \leq n+a$,

$$
A_{i} \cdot U=-x u_{i-n}+u_{i}+u_{i+1}=x G_{d}\left(-K_{n+a+1-i}+G_{n+a+1-i}+G_{n+a-i}\right)
$$

and both of these are 0 by (3.0.7).
Next,

$$
A_{n+a+1} U=-x u_{a+1}+u_{n+a+1}=-x G_{d}+x G_{d}=0
$$

and

$$
A_{n+a+2} \cdot U=-x u_{a+2}+2 u_{n+a+2}+u_{n+a+3}=G_{a}\left(-x^{2}+2+K_{1}\right)=0, \quad \text { by }(3.0 .15)
$$

For $n+a+3 \leq i \leq 2 n$, we have, by (3.3),

$$
A_{i} \cdot U=-x u_{i-n}+u_{i}+u_{i+1}=G_{a}\left(-x^{2} G_{i-n-a-2}+K_{i-n-a-2}+K_{i-n-a-1}\right)=0
$$

Finally, using (3.0.7), (3.0.4), (3.0.14), and (3.0.12), we have

$$
\begin{aligned}
A_{2 n+1} U & =\sum_{i=1}^{2 n+1} u_{i}=G_{d}\left(1+\sum_{i=1}^{a} K_{i}\right)+x G_{a} \sum_{i=0}^{d} G_{i}+x G_{d} \sum_{i=0}^{a-1} G_{i}+G_{a}\left(1+\sum_{i=1}^{d} K_{i}\right) \\
& =\left(G_{d} H_{a}+G_{a} H_{d}\right)+x\left(G_{a} R_{d}+G_{d} R_{a-1}\right)
\end{aligned}
$$

(recall that $d=n-a-1$ ), and in view of (3.10.1) and (3.11.1) this is equal to $(x+2) R_{n-1}$, as claimed. This completes the proof.

For the column player's optimal strategy, we use the vector $W=\left(w_{1}, w_{2}, \ldots, w_{2 n+1}\right)$ defined by (4.3.1) below:

$$
\begin{align*}
w_{i} & =x\left(x^{2}-4\right) R_{a-i} H_{d} & & \text { for } 1 \leq i \leq a ; \\
W_{a+1} & =2 H_{a}+x H_{d} ; & & \\
w_{i} & =\left(x^{2}-4\right) H_{a} H_{i-a-2} & & \text { for } a+2 \leq i \leq n+a+1 ;  \tag{4.3.1}\\
w_{i} & =\left(x^{2}-4\right) H_{n+a+1-i} H_{d} & & \text { for } n+1 \leq i \leq n+a+1 ; \\
w_{n+a+2} & =x H_{a}+2 H_{d} ; & & \\
w_{i} & =x\left(x^{2}-4\right) H_{a} R_{i-n-a-3} & & \text { for } n+a+3 \leq i \leq 2 n+1 .
\end{align*}
$$

Theorem 4.4: For $x>2$, the vector $Q=W / x(x+2) R_{n-1}$, where $W$ is defined by (4.3.1), has all components positive and satisfies (2.1b) for the batrix $B$ of type (i). This is therefore the unique optimal strategy for the column player.

Proof: The proof is very similar to that of the preceding theorem, and we omit the details.
The game value, $V_{(\mathrm{i})}$, for the reduced game of type (i) is now easily computed as well. It is given by the product $P B_{. j}$ for any column $B_{. j}$ of the payoff matrix. Using the middle column, we have

$$
V_{(\mathrm{i})}=P B_{\cdot n+1}=\left(-\sum_{i=1}^{n} u_{i}+\sum_{i=n+2}^{2 n+1} u_{i}\right) /(x+2) R_{n-1} \text {, }
$$

and with the use of (3.0.7), (3.0.4), (3.17), and (3.18), we obtain (4.5.1) below.
Theorem 4.5: For $x>2$, the game value $V_{(\mathrm{i})}$ for the reduced game of type (i) is given by

$$
\begin{equation*}
V_{(\mathrm{i})}=\frac{(x-2) R_{a-d-1}}{(x+2) R_{a+d}} . \tag{4.5.1}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
V_{(\mathrm{i})}>0, V_{(\mathrm{i})}=0 \text {, or } V_{(\mathrm{i})}<0 \text { according as } a>d, a=d \text {, or } a<d \text {. } \tag{4.5.2}
\end{equation*}
$$

Proof: The assertion (4.5.2) follows from Theorem 3.1 and (3.0.11).

## 5. GAMES OF TYPE (ii)

In a matrix $A$ of type (ii), only columns $c+2, n+c+2$, and $n+c+3$ differ from the corresponding columns of $A^{*}$, so to show nonsingularity of $A$ it would suffice to show that these three columns of $A^{*}$ lie in the column space of $A$. However, we can simplify the problem further by the observation that the type (ii) matrix $A$ with parameters $c, d$ differs from the type (i) matrix $A^{\prime}$ with parameters $a^{\prime}=c+1, d^{\prime}=d$ only in column $n+c+2=n+a^{\prime}+1$, and in this column, $A^{\prime}$ agrees with $A^{*}$. Thus, it suffices to show that $A_{\cdot n+c+2}^{*}$ lies in the column space of the type (ii) matrix $A$. To that end, we use the column $D$ defined by (5.0.1) below, and show that $A D=$ $x G_{n-1} A_{\cdot n+c+2}^{*}$, which suffices in view of Theorem 3.1.

$$
\begin{align*}
d_{i} & =-K_{n+i-c-2} & & \text { for } 1 \leq i \leq c+1 ; \\
d_{c+2} & =H_{n-1} ; & & \\
d_{i} & =-x G_{i-c-3} & & \text { for } c+3 \leq i \leq n+c+1 ;  \tag{5.0.1}\\
d_{n+c+2} & =x R_{n-1} ; & & \\
d_{n+c+3} & =-1 ; & & \\
d_{i} & =-K_{i-n-c-3} & & \text { for } n+c+4 \leq i \leq 2 n+1 .
\end{align*}
$$

Theorem 5.1: Let $A$ be a matrix of type (ii) with parameters $c$ and $d$, and let $A^{*}$ be the associated matrix of the same dimensions as $A$ as described in Section 2. With $D$ as defined in (5.0.1), we have

$$
\begin{equation*}
A D=x G_{n-1} A_{\cdot n+c+2}^{*} \tag{5.1.1}
\end{equation*}
$$

Proof: The column $A_{\cdot n+a+2}^{*}$ has $-x$ in row $c+1,1$ in rows $n+c+1, n+c+2$, and $2 n+1$, and 0 in each of the remaining rows. We need only show, therefore, that the following conditions are fulfilled:

$$
\begin{align*}
d_{i}+d_{i+1}-x d_{n+i+1} & =0 \quad \text { for } 1 \leq i \leq c  \tag{5.1.2}\\
d_{c+1}+2 d_{c+2}-x d_{n+c+2} & =-x^{2} G_{n-1}  \tag{5.1.3}\\
d_{c+3}-x d_{n+c+3} & =0 ;  \tag{5.1.4}\\
d_{i}+d_{i+1}-x d_{n+i+1} & =0 \quad \text { for } c+3 \leq i \leq n ;  \tag{5.1.5}\\
-x d_{i-n}+d_{i}+d_{i+1} & =0 \quad \text { for } n+1 \leq i \leq n+c  \tag{5.1.6}\\
-x d_{c+1}+d_{n+c+1} & =x G_{n-1}  \tag{5.1.7}\\
-x d_{c+2}+2 d_{n+c+2} & =x G_{n-1} ;  \tag{5.1.8}\\
-x d_{c+3}+2 d_{n+c+3}+d_{n+c+4} & =0 ;  \tag{5.1.9}\\
-x d_{i-n}+d_{i}+d_{i+1} & =0 \quad \text { for } n+c+4 \leq i \leq 2 n ;  \tag{5.1.10}\\
\sum_{i=1}^{2 n+1} d_{i} & =x G_{n-1} \tag{5.1.11}
\end{align*}
$$

For (5.1.2) we have, for $1 \leq i \leq c$,

$$
d_{i}+d_{i+1}-x d_{n+i+1}=-K_{n+i-c-2}-K_{n+i-c-1}+x^{2} G_{n+i-c-2}=0, \text { by }(3.3)
$$

For (5.1.3),

$$
\begin{aligned}
d_{c+1}-2 d_{c+2}-x d_{n+c+2} & =-K_{n-1}+2 H_{n-1}-x^{2} R_{n-1} \\
& =x^{2} G_{n-1}, \quad \text { by }(3.3),(3.4), \text { and }(3.0 .7)
\end{aligned}
$$

For (5.1.4), $d_{c+3}-x d_{n+c+3}=-x G_{0}+x=0, \quad$ by (3.0.12) .
For (5.1.5), note that, for $c+3 \leq i \leq n$,

$$
d_{i}+d_{i+1}-x d_{n+i+1}=-x\left(G_{i-c-3}+G_{i-c-2}-K_{i-c-2}\right)=0, \quad \text { by }(3.0 .7)
$$

For (5.1.6), we have, for $n+1 \leq i \leq n+c$,

$$
-x d_{i-n}+d_{i}+d_{i+1}=x\left(K_{i-c-2}-G_{i-c-3}-G_{i-c-2}\right)=0, \quad \text { by }(3.0 .7)
$$

For (5.1.7), $-x d_{c+1}+d_{n+c+1}=x\left(K_{n-1}-G_{n-2}\right)=x G_{n-1}$, by (3.0.7).
For (5.1.8), observe that

$$
-x d_{c+2}+2 d_{n+c+2}=x\left(-H_{n-1}+2 R_{n-1}\right)=x G_{n-1}, \quad \text { by (3.0.6) and (3.0.4) }
$$

For (5.1.9), we have

$$
-x d_{c+3}+2 d_{n+c+3}+d_{n+c+4}=x^{2} G_{0}-2-K_{1}=0, \quad \text { by }(3.0 .12) \text { and (3.0.15) }
$$

For (5.1.10), we note that, for $n+c+4 \leq i \leq 2 n$,

$$
-x d_{i-n}+d_{i}+d_{i+1}=x^{2} G_{i-n-c-3}-K_{i-n-c-3}-K_{i-n-c-2}=0, \quad \text { by (3.3). }
$$

Finally, for (5.1.11), we have

$$
\sum_{i=1}^{2 n+1} d_{i}=-\sum_{i=1}^{n-1} K_{i}-1+H_{n-1}-x \sum_{i=0}^{n-2} G_{i}+x R_{n-1} .
$$

From (3.0.7) and (3.0.15), we obtain

$$
\sum_{i=1}^{n-1} K_{i}=H_{n-1}-1
$$

and from (3.0.4) and (3.0.11),

$$
\sum_{i=0}^{n-2} G_{i}=R_{n-2}
$$

so that

$$
\sum_{i=1}^{2 n+1} d_{i}=x\left(R_{n-1}-R_{n-2}\right)=x G_{n-1}
$$

and the proof is complete.
We turn now to the solution of (2.4) for matrices $A$ of type (ii). Let $D$ be the column with components as given in (5.1.12) below.

$$
\begin{align*}
d_{i} & =\left(x^{2}-4\right) G_{d} H_{c+1-i} & & \text { for } 1 \leq i \leq c+1 ; \\
d_{c+2} & =2 G_{d} ; & & \cdot \\
d_{i} & =x\left(x^{2}-4\right) R_{c} G_{i-c-3} & & \text { for } c+3 \leq i \leq n+1 ; \\
d_{i} & =x\left(x^{2}-4\right) R_{n+1+c-i} G_{d} & & \text { for } n+1 \leq i \leq n+c+1 ;  \tag{5.1.12}\\
d_{n+c+2} & =x G_{d} ; & & \\
d_{n+c+3} & =\left(x^{2}-4\right) R_{c} ; & & \\
d_{i} & =\left(x^{2}-4\right) R_{c} K_{i-n-c-3} & & \text { for } n+c+4 \leq i \leq 2 n+1 .
\end{align*}
$$

Note again that the two expressions for $d_{n+1}$ agree.
Theorem 5.2: Let $A$ be the matrix of type (ii) with parameters $c$ and $d$. Let $P^{t}=D /(x+2) G_{n-1}$, where $D$ is as defined by (5.1.12). Then $P$ satisfies (2.4), namely $A P^{t}=(0,0, \ldots, 0,1)^{t}$, and has all components positive for $x>2$.

Proof: That all components are positive for $x>2$ is clear from Theorem 3.1. To prove that (2.4) is satisfied, we show that $A D=(0,0, \ldots, 0, \Delta)$, where $\Delta=(x+2) G_{n-1}$. Let $A_{i}$. denote the $i^{\text {th }}$ row of $A$.

For $1 \leq i \leq c$,

$$
A_{i} . D=d_{i}+d_{i+1}-x d_{n+i+1}=\left(x^{2}-4\right) G_{d}\left(H_{c+1-i}+H_{c-i}-x^{2} R_{c-i}\right)=0 \text {, by (3.4). }
$$

Also,

$$
A_{c+1} \cdot D=d_{c+1}+2 d_{c+2}-x d_{n+c+2}=\left(x^{2}-4\right) G_{d}-x^{2} G_{d}=0 .
$$

Next,

$$
A_{c+2} \cdot D=d_{c+3}-x d_{n+c+3}=x\left(x^{2}-4\right) R_{c} G_{0}-x\left(x^{2}-4\right) R_{c}=0
$$

For $c+3 \leq i \leq n$,

$$
A_{i} . D=d_{i}+d_{i+1}-x d_{n+i+1}=x\left(x^{2}-4\right) R_{c}\left(G_{i-c-3}+G_{i-c-2}-K_{i-c-2}\right)=0, \quad \text { by }(3.0 .7)
$$

For $n+1 \leq i \leq n+c$,

$$
A_{i} \cdot D=-x d_{i-n}+d_{i}+d_{i+1}=x\left(x^{2}-4\right)\left(-H_{n+c+1-i}+R_{n+c+1-i}+R_{n+c-i}\right)=0, \quad \text { by }(3.0 .6)
$$

We have

$$
\begin{aligned}
& A_{n+c+1} \cdot D=-x d_{c+1}+d_{n+c+1}=x\left(x^{2}-4\right) G_{d}\left(-H_{0}+R_{0}\right)=0 \\
& A_{n+c+2} \cdot D=-x d_{c+2}+2 d_{n+c+2}=0
\end{aligned}
$$

and

$$
\begin{aligned}
A_{n+c+3} \cdot D & =-x d_{c+3}+2 d_{n+c+3}+d_{n+c+4}=\left(x^{2}-4\right) R_{c}\left(-x^{2} G_{0}+2+K_{1}\right) \\
& =0, \text { by }(3.0 .12) \text { and }(3.0 .15)
\end{aligned}
$$

For $n+c+4 \leq i \leq 2 n$,

$$
A_{i} \cdot D=-x d_{i-n}+d_{i}+d_{i+1}=\left(x^{2}-4\right) R_{c}\left(-x^{2} G_{i-n-c-3}+K_{i-n-c-3}+K_{i-n-c-2}\right)=0, \quad \text { by }(3.3)
$$

Finally,

$$
\begin{aligned}
A_{2 n+1} D= & \sum_{i=1}^{2 n+1} d_{i} \\
= & \left(x^{2}-4\right) G_{d} \sum_{i=0}^{c} H_{i}+2 G_{d}+x\left(x^{2}-4\right) R_{c} \sum_{i=0}^{d} G_{i}+x\left(x^{2}-4\right) G_{d} \sum_{i=0}^{c-1} R_{i} \\
& +x G_{d}+\left(x^{2}-4\right) R_{c}\left(1+\sum_{i=1}^{d} K_{i}\right) \\
= & \left(x^{2}-4\right) G_{d} T_{c}+(x+2) G_{d}+x\left(x^{2}-4\right) R_{c} R_{d}+x\left(x^{2}-4\right) G_{d} E_{c-1}+\left(x^{2}-4\right) R_{c} H_{d}
\end{aligned}
$$

using, in turn, (3.0.6), (3.0.13), (3.0.11), (3.0.3), (3.0.10), (3.0.7), and (3.0.14). Upon factoring out $(x+2)$ and separating into even and odd parts, we obtain

$$
\begin{aligned}
\frac{1}{x+2} \sum_{i=1}^{2 n+1} d_{i}= & \left(-2\left(G_{d} T_{c}+H_{d} R_{c}\right)+G_{d}+x^{2}\left(R_{c} R_{d}+E_{c-1} G_{d}\right)\right) \\
& +x\left(\left(G_{d} T_{c}+H_{d} R_{c}\right)-2\left(R_{c} R_{d}+E_{c-1} G_{d}\right)\right)
\end{aligned}
$$

The odd part is 0 , since $G_{d}\left(T_{c}-2 E_{c-1}\right)+R_{c}\left(H_{d}-2 R_{d}\right)=G_{d} R_{c}-R_{c} G_{d}$, by (3.0.5), (3.0.3), (3.0.6), and (3.0.4). Thus, we have

$$
\begin{aligned}
& \frac{1}{x+2} \sum_{i=1}^{2 n+1} d_{i}=\left(x^{2}-4\right)\left(R_{c} R_{d}+E_{c-1} G_{d}\right)+G_{d} \\
& =\left(G_{c+1}-G_{c}\right) R_{d}+G_{c} G_{d}, \quad \text { by (3.5) and (3.6), } \\
& =G_{c+1} R_{d}-G_{c} R_{d-1} \\
& =G_{c+d+1}=G_{n-1}, \quad \text { by }(3.12)
\end{aligned}
$$

This completes the proof.

For the optimal strategy for the column player, we use the vector $W$ with components as given in (5.2.1) below.

$$
\begin{align*}
w_{i} & =x\left(x^{2}-4\right) G_{c+1-i} H_{d} & & \text { for } 1 \leq i \leq c+1 ; \\
w_{c+2} & =2 K_{c+1} ; & & \\
w_{i} & =\left(x^{2}-4\right) K_{c+1} H_{i-c-3} & & \text { for } c+3 \leq i \leq n+1 ; \\
w_{i} & =\left(x^{2}-4\right) K_{n+c+2-i} H_{d} & & \text { for } n+1 \leq i \leq n+c+1 ;  \tag{5.2.1}\\
w_{n+c+2} & =\left(x^{2}-4\right) H_{d} ; & & \\
w_{n+c+3} & =x K_{c+1} ; & & \\
w_{i} & =x\left(x^{2}-4\right) K_{c+1} R_{i-n-c-4} & & \text { for } n+c+4 \leq i \leq 2 n+1 .
\end{align*}
$$

Theorem 5.3: For $x>2$, the vector $Q=W / x(x+2) G_{n-1}$, where $W$ is the vector defined by (5.2.1), has all components positive, and satisfies (2.1b) for the matrix $B$ of type (ii). Therefore, this is the unique optimal strategy for the column player in the game with payoff matrix $B$.

The proof is similar to the proof of the preceding theorem, and we omit the details.
The middle column, $B_{\cdot n+1}$ is the same for all four types of reduced matrix, and we use it again to compute the game value, $V_{\text {(ii) }}$. With $D$ as given by (5.1.12), we have

$$
V_{\text {(ii) }}=\left(-\sum_{i=1}^{n} d_{i}+\sum_{i=n+2}^{2 n+1} d_{i}\right) /(x+2) G_{n-1},
$$

and with the use of (3.0.6), (3.0.4), (3.0.3), (3.0.7), (3.5), (3.6), (3.7), (3.17), and (3.19), we obtain (5.4.1) below.

Theorem 5.4: For $x>2$, the game value, $V_{\text {(ii) }}$, for the reduced game of type (ii) is given by

$$
\begin{equation*}
V_{\text {(ii) }}=\frac{(x-2) G_{c-d}}{(x+2) G_{c+d+1}}, \tag{5.4.1}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{\text {(ii) }}>0 \text { for all } c \text { and } d \text {. } \tag{5.4.2}
\end{equation*}
$$

Proof: The assertion (5.4.2) follows from (3.0.12) and Theorem 3.1.

## 6. GAMES OF TYPE (iii)

The payoff matrix for a game of type (iii) is sufficiently closely related to that for a game of type (ii) that we may use our results from Section 5 to obtain the corresponding theorems here. The key observation is the following.

Remark 6.1: Let $B$ be the payoff matrix for a game of type (iii) with parameters $a$ and $b$, and let $B^{\prime}$ be the payoff matrix for a game of type (ii) with parameters $c^{\prime}=b$ and $d^{\prime}=a$. If we change all signs in $B$, transpose about the main diagonal, and then transpose about the lower left to upper right diagonal, we obtain the matrix $B^{\prime}$.

The matrix $-B^{t}$ obtained after the first two steps in Remark 6.1 is the payoff matrix of the game $B$ with the roles of the players reversed. The third step obviously also preserves rank, so uniqueness of solutions $P, Q$, and $V$ to

$$
\begin{equation*}
P B=(V, V, \ldots, V) \tag{6.1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
B Q^{t}=(V, V, \ldots, V)^{t} \tag{6.1.2}
\end{equation*}
$$

follow from uniqueness of solutions to

$$
\begin{equation*}
P^{\prime} B^{\prime}=\left(V^{\prime}, V^{\prime}, \ldots, V^{\prime}\right) \tag{6.1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
B^{\prime} Q^{\prime t}=\left(V^{\prime}, V^{\prime}, \ldots, V^{\prime}\right) \tag{6.1.4}
\end{equation*}
$$

Moreover, the transposition of $-B^{t}$ about its counterdiagonal sends row $i$ of $-B^{t}$ to column $2 n+1-i$ of $B^{\prime}$, and column $j$ of $-B^{t}$ to row $2 n+1-j$ of $B^{\prime}$. Thus we see that, if $P^{\prime}, Q^{\prime}$, and $V^{\prime}$ satisfy (6.1.3) and (6.1.4), and we define $P$ to be the vector $Q^{\prime}$ with the order of the elements reversed, $Q$ to be $P^{\prime}$ reversed, and $V=-V^{\prime}$, then $P, Q$, and $V$ satisfy (6.1.1) and (6.1.2). We summarize this in the next theorem.

Theorem 6.2: Let $B$ be the payoff matrix of a game of type (iii), with $x>2$ and $B^{\prime}$ the associated payoff matrix of type (ii) as described above. Let $P^{\prime}, Q^{\prime}$, and $V^{\prime}$ be, respectively, the optimal strategy for the row player, the optimal strategy for the column player, and the game value for $B^{\prime}$, and let $P$ and $Q$ be, respectively, $Q^{\prime}$ reversed and $P^{\prime}$ reversed. Then $P$ and $Q$ are the optimal strategies for the row and column players, respectively, for the game $B$, and the game value, $V_{\text {(iii) }}$, is given by

$$
\begin{equation*}
V_{\text {(iii) }}=-V^{\prime}=-\frac{(x-2) G_{b-a}}{(x+2) G_{b+a+1}} \text {. } \tag{6.2.1}
\end{equation*}
$$

The game value is negative for all values of $b$ and $a$.

## 7. GAMES OF TYPE (iv)

The type (iv) matrix $A$, with parameters $c$ and $b$, is a $2 n+1 \times 2 n+1$ matrix, where $n=c+$ $b+2$. In this matrix $A$, only column $c+1$ differs from the corresponding column of $A^{\prime}$, where $A^{\prime}$ is the type (iii) matrix with parameters $a^{\prime}=c$ and $b^{\prime}=b$. We shall establish nonsingularity of $A$ by exhibiting a column $D$ such that

$$
\begin{equation*}
A D=A_{\cdot c+1}^{\prime} \Delta \tag{7.0.1}
\end{equation*}
$$

where $A_{c+1}^{\prime}$ is column $c+1$ of $A^{\prime}$ and $\Delta=x(x+2) R_{n-1}$. The column $D$ is defined by (7.0.2) below.

$$
\begin{align*}
d_{i} & =-x(x+2) H_{b+i} & & \text { for } 1 \leq i \leq c ; \\
d_{c+1} & =x(x+2) G_{n-1} ; & & \\
d_{c+2} & =2 x(x+2) E_{n-2}-x ; & & \\
d_{i} & =-x^{2}(x+2) R_{i-c-3} & & \text { for } c+3 \leq i \leq n+c+1 ;  \tag{7.0.2}\\
d_{n+c+2} & =x^{2}(x+2) E_{n-2}+x ; & & \\
d_{i} & =-x(x+2) H_{i-n-c-3} & & \text { for } n+c+3 \leq i \leq 2 n+1 .
\end{align*}
$$

Theorem 7.1: Let $A$ be a matrix of type (iv) with parameters $c$ and $b$, and $x>2$. Let $A^{\prime}$ be the matrix of type (iii) with parameters $a^{\prime}=c$ and $b^{\prime}=b$. Then the column $D$ defined by (7.0.2) satisfies (7.0.1) and thus $A$ is nonsingular.

Proof: The column $A_{c+1}^{\prime}$ has a 2 in row $c$ (if $c>0$ ), $-x$ in row $n+c+1,1$ in the last row, and 0 in all other rows.

For $1 \leq i \leq c, A_{i} \cdot D=d_{i}+d_{i+1}-x d_{n+i+1}$. If $i<c$, this is $x(x+2)\left(-H_{b+i}-H_{b+i+1}+x^{2} R_{b+i}\right)$, which is 0 by (3.4). If $i=c$, we have $A_{i} D=x(x+2)\left(-H_{n-2}+G_{n-1}-x^{2} R_{n-2}\right)=2 x(x+2) R_{n-1}=$ $2 \Delta$, by (3.4), (3.0.4), and (3.0.6).

For rows $c+1$ and $c+2$, we have

$$
A_{c+1} \cdot D=d_{c+1}+2 d_{c+2}-x d_{n+c+2}=x(x+2)\left(G_{n-1}+4 E_{n-2}-x^{2} E_{n-2}-1\right)=0, \quad \text { by }(3.6),
$$

and

$$
A_{c+2} \cdot D=d_{c+3}-x d_{n+c+3}=x^{2}(x+2)\left(-R_{0}+H_{0}\right)=0 .
$$

For $c+4 \leq i \leq n$,

$$
A_{i} . D=d_{i}+d_{i+1}-x d_{n+i+1}=x^{2}(x+2)\left(-R_{i-c-3}-R_{i-c-2}+H_{i-c-2}\right)=0 \text {, by (3.0.6). }
$$

For $n+1 \leq i \leq n+c$,

$$
A_{i} \cdot D=-x d_{i-n}+d_{i}+d_{i+1}=x^{2}(x+2)\left(H_{i+b-n}-R_{i-c-3}-R_{i-c-2}\right)=0 \text {, by (3.0.6); }
$$

since $n=b+c+2$.
With $i=n+c+1$, we have

$$
A_{n+c+1} \cdot D=-x d_{c+1}+d_{n+c+1}=-x^{2}(x+2)\left(G_{n-1}+R_{n-2}\right)=-x \Delta \text {, by (3.0.4), }
$$

and

$$
A_{n+c+2} \cdot D=-x d_{c+2}+2 d_{n+c+2}+d_{n+c+3}=x^{2}+2 x-x(x+2) H_{0}=0 .
$$

For $n+c+3 \leq i \leq 2 n$,

$$
A_{i} \cdot D=-x d_{i-n}+d_{i}+d_{i+1}=x(x+2)\left(-x^{2} R_{i-n-c-3}+H_{i-n-c-3}+H_{i-n-c-2}\right)=0, \quad \text { by (3.4). }
$$

Finally,

$$
A_{2 n+1} D=\sum_{i=1}^{2 n=1} d_{i}=x(x+2)\left(-\sum_{i=0}^{n-2} H_{i}+G_{n-1}+(x+2) E_{n-2}-x \sum_{i=0}^{n-2} R_{i}\right) .
$$

By (3.0.3) and (3.0.10), $\sum_{i=0}^{n-2} R_{i}=E_{n-2}$, and by (3.0.6) and (3.0.13), $\sum_{i=0}^{n-2} H_{i}=T_{n-2}$. Thus,

$$
\sum_{i=1}^{2 n+1} d_{i}=x(x+2)\left(-T_{n-2}+G_{n-1}+2 E_{n-2}\right)
$$

and with the help of (3.0.5), (3.0.4), and (3.0.3), this is easily seen to be equal to $\Delta$. This completes the proof.

We turn now to the solution of (2.4) for the matrix $A$ of type (iv). Let $D$ be the column with components as defined in (7.1.1) below.

$$
\begin{align*}
d_{i} & =\left(x^{2}-4\right) H_{c+1-i} R_{b} & & \text { for } 1 \leq i \leq c+1 ; \\
d_{c+2} & =x R_{c}+2 R_{b} ; & & \\
d_{i} & =x\left(x^{2}-4\right) R_{c} R_{i-c-3} & & \text { for } c+3 \leq i \leq n+1 ; \\
d_{i} & =x\left(x^{2}-4\right) R_{n+1+c-i} R_{b} & & \text { for } n+1 \leq i \leq n+c+1 ;  \tag{7.1.1}\\
d_{n+c+2} & =2 R_{c}+x R_{b} ; & & \\
d_{i} & =\left(x^{2}-4\right) H_{i-n-c-3} R_{c} & & \text { for } n+c+3 \leq i \leq 2 n+1 .
\end{align*}
$$

Theorem 7.2: Let $A$ be the matrix of type (iv) with parameters $c$ and $b$. Let $P^{t}=D /(x+2) R_{n-1}$, where $D$ is defined by (7.1.1). Then $P$ satisfies (2.4) and has all components positive for $x>2$. Thus, $P$ is the unique optimal strategy vector for the row player in the game of type (iv).

Proof: That all components are positive is clear from Theorem 3.1. To prove that (2.4) is satisfied, we show that $A D=(0,0, \ldots, 0, \Delta)$, where $\Delta=(x+2) R_{n-1}$.

For $1 \leq i \leq c$,

$$
A_{i} . D=d_{i}+d_{i+1}-x d_{n+i+1}=\left(x^{2}-4\right) R_{b}\left(H_{c+1-i}+H_{c-i}-x^{2} R_{c-i}\right)=0 \text {, by (3.4). }
$$

For rows $c+1$ and $c+2$, we have

$$
A_{c+1} \cdot D=d_{c+1}+2 d_{c+2}-x d_{n+c+2}=\left(x^{2}-4\right) H_{0} R_{b}+2 x R_{c}+4 R_{b}-2 x R_{c}-x^{2} R_{b}=0,
$$

and

$$
A_{c+2} \cdot D=d_{c+3}-x d_{n+c+3}=x\left(x^{2}-4\right) R_{c}\left(R_{0}-H_{0}\right)=0,
$$

since $H_{0}=R_{0}=1$ by; (3.0.11) and (3.0.14).
For $c+3 \leq i \leq n$,

$$
A_{i} . D=d_{i}+d_{i+1}-x d_{n+i+1}=x\left(x^{2}-4\right) R_{c}\left(R_{i-c-3}+R_{i-c-2}-H_{i-c-2}=0\right. \text {, by (3.0.6). }
$$

For $n+1 \leq i \leq n+c$,

$$
A_{i} \cdot D=-x d_{i-n}+d_{i}+d_{i+1}=x\left(x^{2}-4\right) R_{b}\left(-H_{n+c+1-i}+R_{n+c+1-i}+R_{n+c-i}\right)=0, \quad \text { by (3.0.6). }
$$

For the next two rows, we have

$$
A_{n+c+1} D=x d_{c+1}+d_{n+c+1}=x\left(x^{2}-4\right) R_{b}\left(H_{0}-R_{0}\right)=0,
$$

and

$$
A_{n+c+2} \cdot D=-x d_{c+2}+2 d_{n+c+2}+d_{n+c+3}=\left(-x^{2}+4\right) R_{c}+(-2 x+2 x) R_{b}+\left(x^{2}-4\right) H_{0} R_{c}=0 .
$$

For $n+c+3 \leq i \leq 2 n$,

$$
A_{i} \cdot D=-x d_{i-n}+d_{i}+d_{i+1}=\left(x^{2}-4\right) R_{c}\left(-x^{2} R_{i-n-c-3}+H_{i-n-c-3}+H_{i-n-c-2}\right)=0 \text {, by (3.4). }
$$

Finally, using (3.0.6), (3.0.13), (3.0.3), and (3.0.10), we have

$$
\begin{aligned}
A_{2 n+1} D= & \sum_{i=1}^{2 n+1} d_{i}=\left(x^{2}-4\right) R_{b} \sum_{i=0}^{c} H_{i}+(x+2)\left(R_{c}+R_{b}\right) \\
& +x\left(x^{2}-4\right) R_{c} \sum_{i=0}^{b} R_{i}+x\left(x^{2}-4\right) R_{b} \sum_{i=0}^{c-1} R_{i}+\left(x^{2}-4\right) R_{c} \sum_{i=0}^{b} H_{i} \\
= & \left(x^{2}-4\right)\left(R_{b} T_{c}+R_{c} T_{b}\right)+(x+2)\left(R_{c}+R_{b}\right)+x\left(x^{2}-4\right)\left(R_{c} E_{b}+R_{b} E_{c-1}\right) .
\end{aligned}
$$

Upon factoring out $(x+2)$ and separating into even and odd parts, we obtain

$$
\frac{1}{x+2} \sum_{i=1}^{2 n+1} d_{i}=\left(R_{b}\left(x^{2} E_{c-1}-2 T_{c}+1\right)+R_{c}\left(x^{2} E_{b}-2 T_{b}+1\right)\right)+x\left(R_{b}\left(T_{c}-2 E_{c-1}\right)+R_{c}\left(T_{b}-2 E_{b}\right)\right) .
$$

The odd part is easily seen to be 0 using (3.0.5) and (3.0.3), and with the help of (3.2), (3.0.6), and (3.13), we see that the even part is $R_{b+c+1}$. Since $n=b+c+2$, we have

$$
A_{2 n+1} \cdot D=(x+2) R_{n-1}
$$

and the proof is complete.
To describe the optimal strategy for the column player, we use the vector $W$ defined in (7.2.1) below.

$$
\begin{align*}
w_{i} & =x G_{c+1-i} K_{b+1} & & \text { for } 1 \leq i \leq c+1 ; \\
w_{c+2} & =K_{c+1} ; & & \\
w_{i} & =K_{c+1} K_{i-c-2} & & \text { for } c+3 \leq i \leq n+1 ;  \tag{7.2.1}\\
w_{i} & =K_{n+c+2-i} K_{b+1} & & \text { for } n+1 \leq i \leq n+c+1 ; \\
w_{n+c+2} & =K_{b+1} ; & & \\
w_{i} & =x K_{c+1} G_{i-n-c-3} & & \text { for } n+c+3 \leq i \leq 2 n+1 .
\end{align*}
$$

Theorem 7.3: The vector $Q=W / x(x+2) R_{n-1}$, where $W$ is defined by (7.2.1), has all components positive for $x>2$, and satisfies $(2.1 b)$ for the matrix $B$ of type (iv). Therefore, this is the unique optimal strategy for the column player in the game with payoff matrix $B$.

The proof is straightforward and is left to the reader.
With $D$ as given by (7.1.1), we again express the game value $V_{\text {(iv) }}$ in the form

$$
V_{\mathrm{(iv)}}=\left(-\sum_{i=1}^{n} d_{i}+\sum_{i=n+2}^{2 n+1} d_{i}\right) /(x+2) R_{n-1}
$$

and using (3.0.6), (3.0.3), (3.7), (3.6), (3.21), and (3.20), we obtain (7.4.1) below.
Theorem 7.4: The game value $V_{(\text {(iv) }}$ for the reduced game of type (iv) with $x>2$ is given by

$$
\begin{equation*}
V_{(\mathrm{iv})}=\frac{(x-2) R_{b-c-1}}{(x+2) R_{b+c+1}} . \tag{7.4.1}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
V_{(\mathrm{iv})}>0, V_{(\mathrm{iv})}=0 \text {, or } V_{\text {(iv) }}<0 \text { according as } b>c, b=c \text {, or } b<c . \tag{7.4.2}
\end{equation*}
$$

With the theorems of Sections 4-7 we have now established the irreducibility of the Silverman games in the four classes of odd order games which arise in Chapter 8 of [7], and have given game values and optimal strategies explicitly in terms of the various parameters involved.

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## $\% \%$

# GENERALIZED PASCAL TRIANGLES AND PYRAMIDS: THEIR FRACTALS, GRAPHS, AND APPLICATIONS <br> by Dr. Boris A. Bondarenko <br> Associate member of the Academy of Sciences of the Republic of Uzbekistan, Tashkent <br> Translated by Professor Richard C. Bollinger <br> Penn State at Erie, The Behrend College 

This monograph was first published in Russia in 1990 and consists of seven chapters, a list of 406 references, an appendix with another 126 references, many illustration and specific examples. Fundamental results in the book are formulated as theorems and algorithms or as equations and formulas. For more details on the contents of the book, see The Fibonacci Quarterly 31.1 (1993):52.

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# EULERIAN NUMBERS ASSOCIATED WITH SEQUENCES OF POLYNOMIALS 

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## 1. INTRODUCTION

The well-known Eulerian numbers may be defined either by their generating function or as the coefficients of $\binom{t+n-k}{n}, k=0,1, \ldots, n$, in the factorial expansion of $t^{n}$. During their long history they were extensively studied (and frequently rediscovered) especially with respect to their number-theoretic properties and their connection with certain combinatorial problems (see [2], [3], [18], [19], and references therein). In the last decades, several interesting extensions and modifications were considered along with related combinatorial, probabilistic, and statistical applications ([4]-[8], [10], [12], [13], [15]).

The present paper was motivated by the problem of providing a unified approach to the study of Eulerian-related numbers, which on one hand will be general enough to cover the majority of the known cases and give rise to new sequences of numbers, but on the other will show up the common mathematical properties of the quantities under investigation.

In Section 2 we consider the expansion of a polynomial $p_{n}(t)$ in a series of factorials of order $n$ and introduce the notion of $p_{n}$-associated Eulerian numbers and polynomials. Explicit expressions, recurrence relations, generating functions, and connection to other types of numbers are discussed. In Section 3 we first indicate how well-known results can be directly deduced through the general formulation and in the sequel discuss some additional interesting special cases. Section 4 deals with several statistical and mathematical applications. Finally, in Section 6, we proceed with a further generalization through exponential generating function considerations. A brief study of the most important properties of the generalized quantities is also included.

## 2. THE $\boldsymbol{p}_{\boldsymbol{n}}$-ASSOCIATED EULERIAN NUMBERS AND POLYNOMIALS

Let $\left\{p_{n}(t), n=0,1, \ldots\right\}$ be a class of polynomials with the degree of $p_{n}(t)$ being $n$ and $p_{0}(t)=1$. The coefficients $A_{n, k}$ of the expansion of $p_{n}(t)$ in a series of factorials of degree $n$, namely

$$
\begin{equation*}
p_{n}(t)=\sum_{k=0}^{n} A_{n, k}\binom{t+n-k}{n} \tag{2.1}
\end{equation*}
$$

will be called the $\boldsymbol{p}_{\boldsymbol{n}}$-associated Eulerian numbers.
The respective polynomial

$$
\begin{equation*}
A_{n}(t)=\sum_{k=0}^{n} A_{n, k} t^{k} \tag{2.2}
\end{equation*}
$$

will be referred to as the $\boldsymbol{p}_{\boldsymbol{n}}$-associated Eulerian polynomial.
In Proposition 2.1 we provide an expression for $A_{n, k}$ and $A_{n}(t)$ through the polynomials $p_{n}(t)$.

Proposition 2.1:
a. $\quad A_{n, k}=\sum_{j=0}^{k}(-1)^{j}\binom{n+1}{j} p_{n}(k-j)$.
b. $\quad A_{n}(t)=(1-t)^{n+1} \sum_{j=0}^{\infty} p_{n}(j) t^{j}$.

Proof: Making use of expansion (2.1) for $t=k-j$ and interchanging the order of summation, we obtain

$$
\sum_{j=0}^{k}(-1)^{j}\binom{n+1}{j} p_{n}(k-j)=\sum_{r=0}^{n} A_{n, r} \sum_{j=0}^{k}(-1)^{j}\binom{n+1}{j}\binom{n+k-j-r}{n} .
$$

By virtue of Cauchy's formula, we have

$$
\sum_{j=0}^{k}(-1)^{j}\binom{n+1}{j}\binom{n+k-j-r}{n}=(-1)^{k-r} \sum_{j=0}^{k-r}\binom{n+1}{j}\binom{-n-1}{k-r-j}=(-1)^{k-r} \delta_{k r}
$$

and the first part of the proposition follows immediately. Finally, substituting the explicit expression of $A_{n, k}$ in $A_{n}(t)$ yields

$$
A_{n}(t)=\left\{\sum_{j=0}^{\infty}(-1)^{j}\binom{n+1}{j} t^{j}\right\}\left\{\sum_{j=0}^{\infty} p_{n}(j) t^{j}\right\}=(1-t)^{n+1} \sum_{j=0}^{\infty} p_{n}(j) t^{j} .
$$

It is worth mentioning that the numbers $A_{n, k}$ can be expressed through finite difference operators as follows: If $E$ is the displacement operator, $\nabla=1-E^{-1}$ and

$$
\underline{p_{n, k}}(t)= \begin{cases}p_{n}(t) & \text { if } 0 \leq t \leq k \\ 0 & \text { otherwise }\end{cases}
$$

then

$$
A_{n, k}=\left[\nabla^{n+1} \underline{E^{k} p_{n}}(t)\right]_{t=0}
$$

A lot of numbers used in Combinatorial Analysis can be defined as coefficients of the expansion of a polynomial in a series of factorials. Well-known cases are the (usual and noncentral) Stirling numbers of the second kind, the Lah numbers ([16], [18], [19], and references therein), and the Gould-Hopper numbers ([9], [14]). The author [17] stated some general results for the numbers $P_{n, k}$ appearing in the expansion of an arbitrary polynomial $p_{n}(t)$ in a series of factorials, i.e.,

$$
\begin{equation*}
p_{n}(t)=\sum_{k=0}^{n} P_{n, k}(t)_{k} . \tag{2.3}
\end{equation*}
$$

The next proposition furnishes the connection between the two double sequences of numbers $A_{n, k}$ and $P_{n, k}$.

Proposition 2.2: The $p_{n}$-associated Eulerian numbers $A_{n, k}$ are related to the numbers $P_{n, k}$ defined by (2.3), by

$$
\begin{gather*}
A_{n, k}=\sum_{j=0}^{k}(-1)^{k-j} j!\binom{n-j}{k-j} P_{n, j},  \tag{2.4}\\
P_{n, k}=\frac{1}{k!} \sum_{j=0}^{k}\binom{n-j}{k-j} A_{n, j} . \tag{2.5}
\end{gather*}
$$

Proof: Proposition 2.1b, by virtue of (2.3) yields

$$
A_{n}(t)=(1-t)^{n+1} \sum_{j=0}^{n} P_{n, j} \sum_{i=j}^{\infty}(i)_{j} t^{i}=\sum_{j=0}^{n} j!P_{n, j} t^{j}(1-t)^{n-j},
$$

and expanding $(1-t)^{n-j}$ we deduce that

$$
A_{n}(t)=\sum_{j=0}^{n} j!P_{n, j} \sum_{k=j}^{n}\binom{n-j}{k-j}(-1)^{k-j} t^{k}=\sum_{k=0}^{n}\left\{\sum_{j=0}^{k}(-1)^{k-j} j!\binom{n-j}{k-j} P_{n, j}\right\} t^{k} .
$$

Comparing the last expression to (2.2), we immediately derive equality (2.4). The truth of (2.5) can be easily verified by inverting relation (2.4).

We are now going to prove a result referring to recurrence relations satisfied by the numbers $A_{n, k}$ when a certain recurrence holds true for the polynomials $p_{n}(t)$. More specifically, we have

Proposition 2.3: If there exists a relation of the form

$$
\begin{equation*}
p_{n+1}(t)=\left(\alpha_{n} t+\beta_{n}\right) p_{n}(t)+\left(\gamma_{n} t+\delta_{n}\right) p_{n-1}(t), \quad n \geq 1, \tag{2.6}
\end{equation*}
$$

connecting three polynomials with consecutive indices, then the numbers $A_{n, k}$ satisfy the next recurrence relation

$$
\begin{align*}
A_{n+1, k}=\left(k \alpha_{n}\right. & \left.+\beta_{n}\right) A_{n, k}+\left[\alpha_{n}(n-k+2)-\beta_{n}\right] A_{n, k-1}+\left(k \gamma_{n}+\delta_{n}\right) A_{n-1, k}  \tag{2.7}\\
& +\left[\gamma_{n}(n-2 k+2)-2 \delta_{n}\right] A_{n-1, k-1}+\left[-\gamma_{n}(n-k+2)+\delta_{n}\right] A_{n-1, k-2}, \quad n \geq 1 .
\end{align*}
$$

Proof: Employing Proposition 2.1a and replacing $p_{n+1}(k-j)$ by virtue of (2.6), we obtain

$$
\begin{aligned}
A_{n+1, k}=\left(\alpha_{n} k\right. & \left.+\beta_{n}\right) \sum_{j=0}^{k}(-1)^{j}\binom{n+2}{j} p_{n}(k-j)-\alpha_{n} \sum_{j=1}^{k}(-1)^{j} j\binom{n+2}{j} p_{n}(k-j) \\
& +\left(\gamma_{n} k+\delta_{n}\right) \sum_{j=0}^{k}(-1)^{j}\binom{n+2}{j} p_{n-1}(k-j)-\gamma_{n} \sum_{j=1}^{k}(-1)^{j} j\binom{n+2}{j} p_{n-1}(k-j) .
\end{aligned}
$$

Recurrence (2.7) is easily deduced by introducing the expressions

$$
\begin{array}{ll}
\binom{n+2}{j}=\binom{n+1}{j}+\binom{n+1}{j-1}, & j\binom{n+2}{j}=(n+2)\binom{n+1}{j-1}, \\
\binom{n+2}{j}=\binom{n}{j}+2\binom{n}{j-1}+\binom{n}{j-2}, & j\binom{n+2}{j}=(n+2)\left\{\binom{n}{j-1}+\binom{n}{j-2}\right\}
\end{array}
$$

in the four summands appearing above, and making repeated use of Proposition 2.1a.

It is worth mentioning that, in the special case $\gamma_{n}=\delta_{n}=0$ [i.e., when the polynomials $p_{n}(t)$ have real roots], the resulting numbers $A_{n, k}$ consistitute a triangular array of numbers.

In the remainder of this section we shall establish a connection between the exponential generating function (egf) of the polynomials $p_{n}(t)$ and the respective egf of the $p_{n}$-associated Eulerian polynomials. The basic assumption made here is that the egf

$$
P(t, u)=\sum_{n=0}^{\infty} p_{n}(t) \frac{u^{n}}{n!}
$$

of the sequence of polynomials $p_{n}(t), n=0,1, \ldots$, can be expressed in the form

$$
\begin{equation*}
P(t, u)=g(u) \exp [t(F(u)-F(0))] \tag{2.8}
\end{equation*}
$$

with $g(0)=1$. This setting is general enough to include a lot of important special cases with diverse applications to combinatorics, physics, and mathematical analysis itself, as will be indicated in the next section. We mention here in brief that the special case $g(u)=1$ leads to the well-known exponential Bell polynomials which have been studied in great detail (see [1], [18], [19]).

Proposition 2.4: If the polynomials $p_{n}(t), n=0,1, \ldots$, have egf of the form (2.8), then the egf of the $p_{n}$-associated Eulerian polynomials

$$
A(t, u)=\sum_{n=0}^{\infty} A_{n}(t) \frac{u^{n}}{n!}
$$

is given by

$$
\begin{equation*}
A(t, u)=g((1-t) u) \frac{1-t}{1-t f((1-t) u)} \tag{2.9}
\end{equation*}
$$

where $f(u)=\exp [F(u)-F(0)]$.
Proof: By virtue of Proposition 2.1b, we find that

$$
A(t, u)=(1-t) \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{\infty} p_{n}(j) t^{j}\right\} \frac{[(1-t) u]^{n}}{n!}=(1-t) \sum_{j=0}^{\infty} P(j,(1-t) u) t^{j}
$$

and on making use of (2.8) we easily deduce the desired expression (2.9).
It is easily seen that $A(t, u)$ is the double generating function of the numbers $A_{n, k}$ and writing

$$
A(t, u)=\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} A_{n, k} \frac{u^{n}}{n!}\right\} t^{k}
$$

we conclude that the (single) egf of the numbers $A_{n, k}, n=k, k+1, \ldots$, may be obtained by computing the coefficients of $t^{k}$ in the power series expansion of $A(t, u)$ with respect to $t$. This is, in general, a difficult task.

## 3. SPECIAL CASES

In this section we shall treat some important special cases of $p_{n}$-associated Eulerian numbers, obtained by making certain choices of the polynomials $p_{n}(t)$.
a. If $p_{n}(t)=(t+r)^{n}$, then the expansion formula (2.1) indicates that $A_{n, k}$ are the cumulative numbers used by Dwyer ([12], [13]) to express the ordinary moments of a frequency distribution in terms of the cumulative totals. Now we have

$$
P(t, u)=e^{(t+r) u}, F(u)=u, f(u)=e^{u}, g(u)=e^{r u}, \alpha_{n}=1, \beta_{n}=r, \gamma_{n}=0, \delta_{n}=0,
$$

and applying Propositions 2.1-2.4, we deduce that

$$
\begin{gathered}
A_{n, k}=\sum_{j=0}^{k}(-1)^{j}\binom{n+1}{j}(k+r-j)^{n} \quad \text { (see Dwyer [12], Theorem III, p. 292), } \\
A_{n}(t)=(1-t)^{n+1} \sum_{j=0}^{\infty}(j+r)^{n} t^{j}, \\
A_{n, k}=\sum_{j=0}^{k}(-1)^{k-j} j!\binom{n-j}{k-j} S_{-r}(n, j), \quad S_{-r}(n, k)=\frac{1}{k!} \sum_{j=0}^{k}\binom{n-j}{k-j} A_{n, j},
\end{gathered}
$$

where $S_{-r}(n, k)$ are the non-central Stirling numbers of the second kind (see [16]),

$$
\begin{gathered}
A_{n+1, k}=(k+r) A_{n, k}+(n-k+2-r) A_{n, k-1}(\text { Dwyer }[12], \mathrm{p} .294), \\
A(t, u)=\exp [r(1-t) u] \frac{1-t}{1-t \exp [(1-t) u]} .
\end{gathered}
$$

We mention here that the corresponding $p_{n}$-associated Eulerian polynomials are closely related to the quantities $H_{n}(r \mid t)$, which were studied in detail by Carlitz [2]. Note also that, for $r=0$, the numbers $A_{n, k}$ coincide with the usual Eulerian numbers (see [2], [18], [19])

$$
\begin{equation*}
A_{n, k}=\sum_{j=0}^{k}(-1)^{j}\binom{n+1}{j}(k-j)^{n} \tag{3.1}
\end{equation*}
$$

while $S_{0}(n, k)=S(n, k)$ are the Stirling numbers of the second kind.
b. If $p_{n}(t)=(s t+r)_{n}$, then Proposition 2.1a yields

$$
A_{n, k}=\sum_{j=0}^{k}(-1)^{j}\binom{n+1}{j}(s(k-j)+r)_{n}=n!A(n, k, s, r)
$$

where $A(n, k, s, r)$ are the composition numbers. These numbers, as pointed out in [7], have many applications in combinatorics and statistics. It is obvious that to comply with our general setting, we must take

$$
\begin{aligned}
P(t, u) & =(1+u)^{s t+r}, F(u)=s \log (1+u), f(u)=(1+u)^{s}, g(u)=(1+u)^{r}, \\
\alpha_{n} & =s, \beta_{n}=r-n, \gamma_{n}=\delta_{n}=0,
\end{aligned}
$$

and applying Propositions 2.1-2.4 we may derive the explicit expressions, recurrence relations, and egf of $A_{n, k}$ given by Charalambides [7]. Note that the numbers $P_{n, k}=G(n, k ; s, r)$ of Proposition 2.2 which appear in the expansion

$$
(s t+r)_{n}=\sum_{k=0}^{n} G(n, k ; s, r)(t)_{k}
$$

are the so-called Gould-Hopper numbers (see [9], [14]). Note also that the limit $\lim _{s \rightarrow \pm \infty} s^{-n} A_{n, k}$ yields the Dwyer numbers mentioned in a. We finally mention that the special case $r=0$ corresponds to the numbers

$$
\begin{equation*}
A_{n, k}=\sum_{j=0}^{k}(-1)^{j}\binom{n+1}{j}(s(k-j))_{n}=s^{n} A_{n, k}\left(s^{-1}\right) \tag{3.2}
\end{equation*}
$$

where $A_{n, k}(\cdot)$ are the polynomials studied by Carlitz ([4], Ch. 7). As Carlitz, Roselle, \& Scoville [5] pointed out, for $s<0$, the number of ordered sets $\left(i_{1}, i_{2}, \ldots, i_{n}\right)$ with $i_{j} \in\{1,2, \ldots,|s|\}$ and showing exactly $k$ increases between adjacent elements, is equal to

$$
\left|A_{n, k}\right| / n!=(-1)^{n} A_{n, k} / n!.
$$

c. The author [17], motivated by the problem of providing explicit expressions for the distribution of two-sample sums from Poisson and binomial distributions, one of which is lefttruncated, introduced the $r-q$ polynomials

$$
r_{n}(t ; s, r)=\frac{d^{n}}{d x^{n}}\left[x^{s t} e^{-r(x-1)}\right]_{x=1}, \quad q_{n}(t ; r)=\frac{d^{n}}{d x^{n}}\left[x^{r} e^{-t(x-1)}\right]_{x=1} .
$$

Both sequences of polynomials comply with the restrictions set in the general context and give rise to two double sequences of numbers which, to our knowledge have not appeared in the literature yet. More specifically, we have
(i) The polynomials $r_{n}(t ; s, r)$ satisfy the recurrence

$$
\begin{aligned}
r_{n+1}(t ; s, r) & =(r+s t-n) r_{n}(t ; s, r)+r n r_{n-1}(t ; s, r), \quad n \geq 1, \\
r_{0}(t ; s, r) & =1, \quad r_{1}(t ; s, r)=s t+r
\end{aligned}
$$

and have egf

$$
r(t, u ; s, r)=\sum_{n=0}^{\infty} r_{n}(t ; s, r) \frac{u^{n}}{n!}=(1+u)^{s t} e^{r u} .
$$

Therefore,

$$
F(u)=s \log (1+u), f(u)=(1+u)^{s}, g(u)=e^{r u}, \alpha_{n}=s, \beta_{n}=r-n, \gamma_{n}=0, \delta_{n}=r n,
$$

and applying Propositions 2.2-2.4 for the numbers defined by the expansion

$$
r_{n}(t ; s, r)=\sum_{k=0}^{n} A_{n, k}\binom{t+n-k}{n},
$$

we conclude that

$$
\begin{aligned}
A_{n+1, k} & =(s k+r-n) A_{n, k}+[s(n-k+2)+n-r] A_{n, k-1}+r n\left[A_{n-1, k}-2 A_{n-1, k-1}+A_{n-1, k-2}\right], \quad n \geq 1, \\
A_{00} & =1, A_{10}=r, A_{11}=s-r,
\end{aligned}
$$

$$
\begin{aligned}
A(t, u) & =\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n, k} t^{k} \frac{u^{n}}{n!}=e^{r(1-t) u} \frac{1-t}{1-t[1+(1-t) u]^{s}}, \\
A_{n, k} & =\sum_{j=0}^{k}(-1)^{k-j} j!\binom{n-j}{k-j} R(n, j ; s, r),
\end{aligned}
$$

where $R(n, k ; s, r)$ are the numbers appearing in the convolution of two-sample sums from a binomial and a zero-truncated Poisson distribution (see [17]). Note that the numbers $A_{n, k}$ defined above give, in particular for $r=0$, the quantities (3.2).
(ii) For the polynomials $q_{n}(t ; r)$, we have

$$
\begin{aligned}
q_{n+1}(t ; r) & =(r+t-n) q_{n}(t ; r)+n t q_{n-1}(t), \quad n \geq 1, \\
q_{0}(t ; r) & =1, q_{1}(t ; r)=t+r, \\
q(t, u ; r) & =\sum_{n=0}^{\infty} q_{n}(t ; r) \frac{u^{n}}{n!}=(1+u)^{r} e^{t u} .
\end{aligned}
$$

Hence,

$$
F(u)=u, f(u)=e^{u}, g(u)=(1+u)^{r}, \alpha_{n}=1, \beta_{n}=r-n, \gamma_{n}=n, \delta_{n}=0,
$$

and applying Propositions 2.2-2.4 for the numbers defined by the expansion

$$
q_{n}(t ; r)=\sum_{k=0}^{n} A_{n, k}\binom{t+n-k}{n}
$$

we obtain

$$
\begin{align*}
& A_{n+1, k}=(r-n+k) A_{n, k}+(2 n-k-r+2) A_{n, k-1} \\
&+n\left\{k A_{n-1, k}+(n-2 k+2) A_{n-1, k-1}-(n-k+2) A_{n-1, k-2}\right\}, \quad n \geq 1, \\
& A_{00}=1, A_{10}=r, A_{11}=1-r, \\
& A(t, u)= \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n, k} t^{k} \frac{u^{n}}{n!}=[1+(1-t) u]^{r} \frac{1-t}{1-t \exp [(1-t) u]},  \tag{3.3}\\
& A_{n, k}= \sum_{j=0}^{k}(-1)^{k-j} j!\binom{n-j}{k-j} Q(n, j ; r),
\end{align*}
$$

where

$$
Q(n, k ; r)=\sum_{j=k}^{n}\binom{n}{j}(r)_{n-j} S(j, k)
$$

are the numbers appearing in the convolution of two-sample sums from a Poisson and a zerotruncated binomial distribution (see [17]). As can easily be verified from egf (3.3), the special case $r=0$ yields the usual Eulerian numbers (3.1).
d. The Hermite Polynomials

$$
H_{n}(t)=(-1)^{n} e^{t^{2}} \frac{d^{n} e^{t^{2}}}{d t^{n}}
$$

satisfy the recurrence

$$
\begin{aligned}
H_{n+1}(t) & =2 t H_{n}(t)-2 n H_{n-1}(t), \quad n \geq 1 \\
H_{0}(t) & =1, H_{1}(t)=2 t
\end{aligned}
$$

while their egf is

$$
H(t, u)=\sum_{n=0}^{\infty} H_{n}(t) \frac{u^{n}}{n!}=e^{-u^{2}+2 u t}
$$

Thus,

$$
F(u)=2 u, f(u)=e^{2 u}, g(u)=e^{-u^{2}}, \alpha_{n}=2, \beta_{n}=\gamma_{n}=0, \delta_{n}=-2 n
$$

and the next results for the Hermite-associated Eulerian numbers $A_{n, k}$ are immediate consequences of Propositions 2.1-2.4:

$$
\begin{gather*}
A_{n, k}=\sum_{j=0}^{k}(-1)^{j}\binom{n+1}{j} H_{n}(k-j), \\
H_{n}(t)=\sum_{k=0}^{n} P_{n, k}(t)_{k}, \text { where } P_{n, k}=\frac{1}{k!} \sum_{j=0}^{k}\binom{n-j}{k-j} A_{n, j}, \\
A_{n+1, k}=2 k A_{n, k}+2(n-k+2) A_{n, k-1}-2 n\left\{A_{n-1, k}-2 A_{n-1, k-1}+A_{n-1, k-2}\right\}, \quad n \geq 1,  \tag{3.4}\\
A_{00}=1, A_{10}=0, A_{11}=2, \\
A(t, u)=\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n, k} t^{k} \frac{u^{n}}{n!}=\exp \left[-(1-t)^{2} u^{2}\right] \frac{1-t}{1-t \exp [2(1-t) u]} .
\end{gather*}
$$

e. Another important class of polynomials encountered in several applications, especially in mathematical physics, consists of the (generalized) Laguerre polynomials $L_{n}^{(p)}(t)$, defined by

$$
L_{n}^{(p)}(t)=\frac{1}{n!} e^{t} t^{-p} \frac{d^{n}}{d t^{n}}\left[e^{-t} t^{n+p}\right], \quad n=0,1, \ldots, p>-1
$$

Considering the polynomials

$$
L_{n}(t ; p)=n!L_{n}^{(p)}(t)=e^{t} t^{-p} \frac{d^{n}}{d t^{n}}\left[e^{-t} t^{n+p}\right], \quad n \geq 0, p>-1
$$

we get [making use of the respective results on $L_{n}^{(p)}(t)$ ]

$$
\begin{aligned}
L_{n+1}(t ; p) & =(2 n+p+1-t) L_{n}(t ; p)-n(n+p) L_{n-1}(t ; p), \quad n \geq 1 \\
L_{0}(t ; p) & =1, L_{1}(t ; p)=-t+p+1 \\
L(t, u) & =\sum_{n=0}^{\infty} L_{n}(t ; p) \frac{u^{n}}{n!}=\sum_{n=0}^{\infty} L_{n}^{(p)}(t) u^{n}=(1-u)^{-p-1} \exp \left\{-\frac{t u}{1-u}\right\},|t|<1 .
\end{aligned}
$$

Therefore,

$$
\begin{gathered}
F(u)=\frac{u}{u-1}, f(u)=\exp \left\{\frac{u}{u-1}\right\}, g(u)=(1-u)^{-p-1} \\
\alpha_{n}=-1, \beta_{n}=2 n+p+1, \gamma_{n}=0, \delta_{n}=-n(n+p)
\end{gathered}
$$

and applying Propositions 2.1-2.4, we deduce the following properties of the Laguerreassociated Eulerian numbers $A_{n, k}$ :

$$
\begin{aligned}
& A_{n, k}=\sum_{j=0}^{k}(-1)^{j}\binom{n+1}{j} L_{n}(k-j ; p), \\
& L_{n}(t)=\sum_{k=0}^{n} P_{n, k}(t)_{k}, \text { where } P_{n, k}=\frac{1}{k!} \sum_{j=0}^{k}\binom{n-j}{k-j} A_{n, j}, \\
& A_{n+1, k}= \\
& \quad(2 n+p+1-k) A_{n, k}-(3 n-k+p+3) A_{n, k-1} \\
& \\
& \quad-n(n+p)\left\{A_{n-1, k}-2 A_{n-1, k-1}+A_{n-1, k-2}\right\}, n \geq 1, \\
& A_{00}= \\
& 1, A_{10}=p+1, A_{11}=-p-2, \\
& A(t, u)=\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n, k} t^{k} \frac{u^{n}}{n!}=\frac{1}{[1-(1-t) u]^{p+1}} \cdot \frac{1-t}{1-t \exp \{(1-t) u}\left(\frac{(1-t) u-1}{}\right\}
\end{aligned} .
$$

## 4. APPLICATIONS

In the present section we consider a number of applications involving the $p_{n}$-associated Eulerian numbers and polynomials.

The first application refers to the computation of the mean value of polynomial functions of logarithmic random variables. More specifically, consider a random variable $X$ with the logarithmic series distribution

$$
P[X=x]=-\frac{1}{\log (1-\theta)} \cdot \frac{\theta^{x}}{x}, x=1,2, \ldots, 0<\theta<1,
$$

and let $p_{n}(\cdot)$ be a polynomial of degree $n$. Then

$$
v_{n}=E\left[X p_{n}(X)\right]=c\left\{\sum_{x=0}^{\infty} p_{n}(x) \theta^{x}-p_{n}(0)\right\}, c=-1 / \log (1-\theta),
$$

and employing the $p_{n}$-associated Eulerian polynomials $A_{n}(t)$, we may write, by virtue of Proposition 2.1b,

$$
\begin{equation*}
v_{n}=c\left\{(1-\theta)^{-n-1} A_{n}(\theta)-p_{n}(0)\right\} . \tag{4.1}
\end{equation*}
$$

Formula (4.1) is useful for the derivation of recurrence relations for the quantities $v_{n}$ [mean value of an $(n+1)$-degree polynomial with no constant term] by making use of the respective recurrence relations of the Eulerian polynomials $A_{n}(\theta)$. Note also that, under the assumptions made in Proposition 2.4, the egf of $v_{n}, n=0,1, \ldots$, is given by

$$
\sum_{n=0}^{\infty} v_{n} \frac{u^{n}}{n!}=c g(u) \frac{\theta f(u)}{1-\theta f(u)} .
$$

We mention in particular (see Section 3, cases a and b) that

$$
\begin{aligned}
& \sum_{n=0}^{\infty} E\left[X(X+r)^{n}\right] \frac{u^{n}}{n!}=c \cdot \frac{\theta e^{(r+1) u}}{1-\theta e^{u}} \\
& \sum_{n=0}^{\infty} E\left[X(s X+r)_{n}\right] \frac{u^{n}}{n!}=c \cdot \frac{\theta(1+u)^{r+s}}{1-\theta(1+u)^{s}} .
\end{aligned}
$$

The second statistical application of the $p_{n}$-associated Eulerian numbers is in the computation of the polynomial mean of a frequency distribution with the use of cumulative totals. This method was used by Dwyer [12], [13] for the computation of the moments and by Charalambides [7] for the factorial moments. The main advantage of the method lies in the fact that the many multiplications involved in the usual computation process are replaced by additions. Since the generalization presented here is rather straightforward, we omit the details and state only the results. Let $f_{x}$ denote a frequency distribution and

$$
C f_{x}=\sum_{j \geq x} f_{j}, \quad C^{m+1} f_{x}=C\left(C^{m} f_{x}\right), \quad m=1,2, \ldots,
$$

the successive frequency cumulations. Then, employing Dwyer's successive cumulation theorem, we may easily deduce that for any polynomial $p_{n}(\cdot)$,

$$
E\left[p_{n}(X)\right]=\sum_{j \geq 0} p_{n}(j) f_{j}=\sum_{k=0}^{n} A_{n, k} C^{n+1} f_{k},
$$

where $A_{n, k}$ are the Eulerian numbers corresponding to $p_{n}(\cdot)$.
As a last application, we consider the problem of evaluating the sum of the values of a polynomial $p_{n}(\cdot)$ over the first $m+1$ nonnegative integers, namely, $S=\sum_{x=0}^{m} p_{n}(x)$. Because of (2.1) we may write

$$
S=\sum_{x=0}^{m} \sum_{k=0}^{n} A_{n, k}\binom{x+n-k}{n}=\sum_{k=0}^{n} A_{n, k} \sum_{x=0}^{m}\binom{x+n-k}{n},
$$

and since the inner sum equals $\binom{m+n-k+1}{n+1}$, it follows that

$$
\begin{equation*}
\sum_{x=0}^{m} p_{n}(x)=\sum_{k=0}^{n} A_{n, k}\binom{m+n-k+1}{n+1} . \tag{4.2}
\end{equation*}
$$

Consider in particular the next two special cases:
(i) Let $p_{n}(t)=p_{3}(t)=(s t)_{3}$. Then, by virtue of (3.2) (or employing the respective recurrence relation for $A_{n, k}$ ) we get $A_{30}=1, A_{31}=(s)_{3}, A_{32}=4(s+1)_{3}, A_{33}=(s+2)_{3}$, and (4.2) yields

$$
\sum_{x=0}^{m}(s x)_{3}=\binom{m+4}{4}+(s)_{3}\binom{m+3}{4}+4(s+1)_{3}\binom{m+2}{4}+(s+2)_{3}\binom{m+1}{4} .
$$

(ii) Let $p_{n}(t)=H_{2}(t)$ denote the Hermite polynomial of degree 2. Recurrence (3.4) yields $A_{20}=-2, A_{21}=8, A_{22}=2$, and, therefore,

$$
\sum_{x=0}^{m} H_{2}(x)=-2\binom{m+3}{3}+8\binom{m+2}{3}+2\binom{m+1}{3}=\frac{2(m+1)}{3}\left\{2 m^{2}+m-3\right\} .
$$

## 5. THE GENERALIZED $p_{n}$-ASSOCIATED EULERIAN NUMERS AND POLYNOMIALS

Carlitz \& Scoville [6] introduced the generalized Eulerian numbers in connection with the problem of enumerating ( $a, b$ )-sequences (generalized permutations). These numbers, which are also related to Janardan's [15] generalized Eulerian numbers (used for the statistical analysis of an interesting ecology model), were extensively studied by Charalambides [8]. Recently, Charalambides \& Koutras [10] considered an alternative ecology model and introduced a double sequence of numbers that are asymptotically connected with the numbers of Carlitz \& Scoville.

In the present section we provide a unified approach to generalizations of this kind, bringing into focus the common properties of them and supplying the means for further extensions.

Let $\left\{p_{n}(t), n=0,1, \ldots\right\}$ be a class of polynomials with egf given by (2.8). Then, the numbers $A_{n, k}(a, b)$ with egf

$$
\begin{equation*}
A(t, u ; a, b)=\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n, k}(a, b) t^{k} \frac{u^{n}}{n!}=g((1-t) u) f^{a}((1-t) u)\left\{\frac{1-t}{1-t f((1-t) u)}\right\}^{a+b} \tag{5.1}
\end{equation*}
$$

will be called generalized $p_{n}$-associated Eulerian numbers. Similarly, the polynomials

$$
A_{n}(t ; a, b)=\sum_{k=0}^{n} A_{n, k}(a, b) t^{k}
$$

will be named generalized $p_{n}$-associated Eulerian polynomials. It is evident that

$$
A_{n, k}(0,1)=A_{n, k}, \quad A_{n}(t ; 0,1)=A_{n}(t) .
$$

Proposition 5.1:
a. $\quad A_{n}(t ; a, b)=(1-t)^{n+a+b} \sum_{j=0}^{\infty}\binom{a+b+j-1}{j} t^{j} p_{n}(a+j)$.
b. $\quad A_{n, k}(a, b)=\sum_{j=0}^{k}(-1)^{j}\binom{n+a+b}{j}\binom{a+b+k-j-1}{k-j} p_{n}(a+k-j)$.

Proof: Expanding the term $[1-t f((1-t) u)]^{-(a+b)}$ of (5.1) yields

$$
A(t, u ; a, b)=(1-t)^{a+b} \sum_{j=0}^{\infty}\binom{a+b+j-1}{j}^{j}\left\{g((1-t) u) f^{a+j}((1-t) u)\right\},
$$

and applying (2.8) on the extreme right term, we obtain

$$
A(t, u ; a, b)=(1-t)^{a+b} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty}\binom{a+b+j-1}{j} t^{j} p_{n}(a+j) \frac{[(1-t) u]^{n}}{n!} .
$$

The first part of the proposition is readily established by interchanging the order of summation and considering the coefficient of $u^{n} / n!$ in the resulting power series. The second part follows immediately from $a$ by expanding $(1-t)^{n+a+b}$ and performing the multiplication of the two series.

We note that, in particular, for $p_{n}(t)=t^{n}$ and $p_{n}(t)=(s t)_{n}$, the numbers appearing in [6], [8], [15], and [10], respectively are obtained.

Taking the limit as $t \rightarrow 1$ in (5.1), it follows that

$$
\lim _{t \rightarrow 1} A(t, u ; a, b)=\left[1-f^{\prime}(0) u\right]^{-a-b}=\sum_{n=0}^{\infty}\binom{a+b+n-1}{n}\left[f^{\prime}(0)\right]^{n} u^{n}
$$

implying

$$
\begin{equation*}
A_{n}(1 ; a, b)=\sum_{k=0}^{n} A_{n, k}(a, b)=(a+b+n-1)_{n}\left[f^{\prime}(0)\right]^{n} \tag{5.2}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
f_{n}(x ; a, b)=P[X=x]=\frac{A_{n, x}(a, b)}{(a+b+n-1)_{n}\left[f^{\prime}(0)\right]^{n}}, x=0,1,2, \ldots, n \tag{5.3}
\end{equation*}
$$

defines a legitimate probability function (provided that the ratios are nonnegative for all $x=0,1$, $\ldots, n$ ) which will be called generalized $p_{n}$-associated Eulerian distribution. It is straightforward that the probability generating function of (5.3) can be expressed as

$$
E\left[t^{X}\right]=\sum_{x=0}^{n} f(x ; a, b) t^{x}=\frac{A_{n}(t ; a, b)}{(a+b+n-1)_{n}\left[f^{\prime}(0)\right]^{n}}
$$

while the factorial moment generating function is expressed as

$$
\sum_{r=0}^{n} E\left[(X)_{r}\right] \frac{t^{r}}{r!}=\frac{A_{n}(t+1 ; a, b)}{(a+b+n-1)_{n}\left[f^{\prime}(0)\right]^{n}}
$$

The next proposition provides recurrence relations for the numbers $A_{n, k}(a, b)$ and is useful for tabulation purposes [we recall also formula (5.2), which can be employed as a convenient check].

Proposition 5.2: Under the assumption that (2.6) is true, the numbers $A_{n, k}(a, b)$ satisfy the next recurrence relation:

$$
\begin{align*}
A_{n+1, k}(a, b)= & {\left[(a+k) \alpha_{n}+\beta_{n}\right] A_{n, k}(a, b)+\left[\alpha_{n}(n+b-k+1)-\beta_{n}\right] A_{n, k-1}(a, b) } \\
& +\left[(a+k) \gamma_{n}+\delta_{n}\right] A_{n-1, k}(a, b)+\left[\gamma_{n}(n+b-2 k-a+1)-2 \delta_{n}\right] A_{n-1, k-1}(a, b)  \tag{5.4}\\
& +\left[-\gamma_{n}(n+b-k+1)+\delta_{n}\right] A_{n-1, k-2}(a, b), \quad n \geq 1 .
\end{align*}
$$

Proof: It is not difficult to verify that the auxiliary functions

$$
C_{n}(t ; a, b)=\sum_{j=0}^{\infty}\binom{a+b+j-1}{j} t^{a+j} p_{n}(a+j)=t^{a}(1-t)^{-(n+a+b)} A_{n}(t ; a, b)
$$

satisfy the difference-differential equation

$$
C_{n+1}(t ; a, b)=t \frac{d}{d t}\left\{\alpha_{n} C_{n}(t ; a, b)+\gamma_{n} C_{n-1}(t ; a, b)\right\}+\beta_{n} C_{n}(t ; a, b)+\delta_{n} C_{n-1}(t ; a, b)
$$

Replacing $C_{n}(t ; a, b)$ in terms of $A_{n}(t ; a, b)$, we obtain a difference-differential equation for $A_{n}(t ; a, b)$, and (5.4) is finally obtained after some lengthy but rather straightforward calculations. We mention that a proof similar to the one used in Proposition 2.3 could also be established; however, it is much more complicated.

In the remainder of this section we are going to state some interesting results for generalized $p_{n}$-associated Eulerian numbers whose generating polynomials have real roots, i.e.,

$$
\begin{align*}
p_{n+1}(t) & =\prod_{k=0}^{n}\left(\alpha_{k} t+\beta_{k}\right)=\left(\alpha_{n} t+\beta_{n}\right) p_{n}(t), \quad n \geq 0,  \tag{5.5}\\
p_{0}(t) & =1 .
\end{align*}
$$

In this case we have:

1. The numbers $A_{n, k}(a, b)$ satisfy the triangular recurrence relation

$$
A_{n+1, k}(a, b)=\left[(a+k) \alpha_{n}+\beta_{n}\right] A_{n, k}(a, b)+\left[\alpha_{n}(n+b-k+1)-\beta_{n}\right] A_{n, k-1}(a, b) .
$$

2. The probability function $f(x ; a, b)$ and the respective factorial moments $u_{(r)}(n ; a, b)=$ $E\left[(X)_{r}\right], r=0,1, \ldots$, satisfy the recurrences

$$
\begin{gathered}
f_{n+1}(x ; a, b)=\frac{(a+k) \alpha_{n}+\beta_{n}}{(a+b+n) f^{\prime}(0)} f_{n}(x ; a, b)+\frac{\alpha_{n}(n+b-k+1)-\beta_{n}}{(a+b+n) f^{\prime}(0)} f_{n}(x-1 ; a, b) ; \\
\mu_{(r)}(n+1 ; a, b)=\frac{r\left[\alpha_{n}(n+b-r+1)-\beta_{n}\right]}{(a+b+n) f^{\prime}(0)} \mu_{(r-1)}(n ; a, b)+\frac{\alpha_{n}(a+b+n-r)}{(a+b+n) f^{\prime}(0)} \mu_{(r)}(n ; a, b) .
\end{gathered}
$$

3. If $a \alpha_{n}+\beta_{n} \neq 0$ for all $n=0,1, \ldots$, then the polynomials $A_{n}(t ; a, b)$ have $n$ distinct real nonpositive roots (an easy way to prove this is to verify first that

$$
E_{n}(t ; a, b)=(1-t)^{-(n+a+b)} t^{\beta_{n} / \alpha_{n}+a} A_{n}(t ; a, b)
$$

satisfies a difference-differential equation of the form

$$
\left.\alpha_{n} t^{s} \frac{d}{d t} E_{n}(t ; a, b)=E_{n+1}(t ; a, b)\right) .
$$

Hence:
(a) $A_{n, k}(a, b)$ is a strictly concave function of $k$;
(b) the distribution $\left\{f_{n}(x ; a, b), x=0,1, \ldots, n\right\}$ is unimodal either with a peak or with a plateau of two points (see [11]);
(c) Any random variable $X$ obeying (5.3) can be expressed as a sum of $n$ independent zeroone random variables.

We recall that the generalized Eulerian numbers studied in [6], [8], [10], [15], along with their generalizations produced by the choices $p_{n}(t)=(t+r)^{n}, p_{n}(t)=(s t+r)_{n}$ (see Section 3, cases a and b), own the properties 1-3 above, since they are generated by polynomials of the form (5.5).

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# A LUCAS-TYPE THEOREM FOR FIBONOMIAL-COEFFICIENT RESIDUES 

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## 1. INTRODUCTION

A remarkable theorem of Lucas ([8], pp. 229-30) states that the value of the binomial coefficient $\binom{n}{k}$ is congruent, modulo a prime $p$, to the product of the binomial coefficients of the respective base- $p$ digits of $n$ and $k$. In other words, if

$$
n=\sum n_{j} p^{j}, \text { where } 0 \leq n_{j}<p \text { for each } j
$$

and

$$
k=\sum k_{j} p^{j}, \quad \text { where } 0 \leq k_{j}<p \text { for each } j,
$$

then

$$
\begin{equation*}
\binom{n}{k} \equiv \Pi\binom{n_{j}}{k_{j}} \quad(\bmod p) \tag{1}
\end{equation*}
$$

For example, since $2280=(6435)_{7}$ and $1823=(5213)_{7}$, we have

$$
\binom{2280}{1823} \equiv\binom{6}{5}\binom{4}{2}\binom{3}{1}\binom{5}{3} \equiv 6 \cdot 6 \cdot 3 \cdot 3 \equiv 2(\bmod 7)
$$

Formula (1) is equivalent to Lucas's earlier generalization of an 1869 result of H . Anton ([1], pp. 303-06; [7], p. 52; [2], p. 271):

$$
\begin{equation*}
\binom{n}{k} \equiv\binom{n \operatorname{div} p}{k \operatorname{div} p}\binom{n \bmod p}{k \bmod p}(\bmod p) \tag{2}
\end{equation*}
$$

where $n$ div $p$ denotes the integer quotient of $n$ by $p$, and $n \bmod p$ its remainder. For short proofs, see [3] and [9]. For our purposes, it is better to reformulate this theorem in terms of

$$
B(m, n):=\binom{m+n}{m}=\binom{m+n}{n}
$$

$$
\begin{equation*}
B(m, n) \equiv B(m \operatorname{div} p, n \operatorname{div} p) B(m \bmod p, n \bmod p)(\bmod p) \tag{3}
\end{equation*}
$$

[If $(m+n) \operatorname{div} p=m \operatorname{div} p+n \operatorname{div} p$ and $(m+n) \bmod p=m \bmod p+n \bmod p$, then this just reexpresses (2); if not, then, again by (2), both sides may be shown to be congruent to 0 .] Repeated application of (3) yields the following counterpart of (1):

$$
\begin{equation*}
B(m, n) \equiv \Pi B\left(m_{j}, n_{j}\right)(\bmod p) \tag{4}
\end{equation*}
$$

where $m_{j}$ and $n_{j}$ are the base- $p$ digits of $m$ and $n$, respectively. Our goal is to obtain formulas corresponding to (3) and (4) for Fibonomial coefficients.

In analogy with the usual definition of binomial coefficients

$$
B(m, n)=\binom{m+n}{m}=\prod_{j=0}^{m-1} \frac{m+n-j}{m-j} \quad(m, n \geq 0)
$$

we define the Fibonomial coefficients by

$$
C(m, n)=\left[\begin{array}{c}
m+n  \tag{5}\\
m
\end{array}\right]=\prod_{j=0}^{m-1} \frac{F_{m+n-j}}{F_{m-j}} \quad(m, n \geq 0)
$$

where $F_{k}$ denotes the $k^{\text {th }}$ Fibonacci number, and an empty product is taken to be 1 (see [8], §9; also [4] and [5]). Some values of $C(m, n)$ are tabulated in Table 1; there $C(0,0)$ appears at the upper left corner. We note that, for $m, n \geq 0$,

$$
C(m, 0)=1, C(0, n)=1, \text { and } C(m, n)=C(n, m) .
$$

## TABLE 1: Fibonomial Coefficients

| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 |
| 1 | 2 | 6 | 15 | 40 | 104 | 273 | 714 |
| 1 | 3 | 15 | 60 | 260 | 1092 | 4641 | 19635 |
| 1 | 5 | 40 | 260 | 1820 | 12376 | 85085 | 582505 |
| 1 | 8 | 104 | 1092 | 12376 | 136136 | 1514513 | 16776144 |
| 1 | 13 | 273 | 4641 | 85085 | 1514513 | 27261234 | 488605194 |
| 1 | 21 | 714 | 19635 | 582505 | 16776144 | 488605194 | 14169550626 |

Using the identity

$$
\begin{equation*}
F_{m+n}=F_{m+1} F_{n}+F_{m} F_{n-1} \quad(m, n \geq 0) \tag{6}
\end{equation*}
$$

and the definition, (5), one may deduce (see [4]) the key recurrence formula for $m, n \geq 1$ :

$$
\begin{equation*}
C(m, n)=F_{m+1} C(m, n-1)+F_{n-1} C(m-1, n) . \tag{7}
\end{equation*}
$$

This is the Fibonomial counterpart of the Pascal triangle recurrence,

$$
B(m, n)=B(m, n-1)+B(m-1, n) .
$$

[Alternatively, by symmetry, we also have

$$
C(m, n)=F_{m-1} C(m, n-1)+F_{n+1} C(m-1, n) .
$$

Then, in terms of the Lucas numbers $L_{k}=F_{k-1}+F_{k+1}$, we have, by addition, the symmetric recurrence formula

$$
\left.2 C(m, n)=L_{m} C(m, n-1)+L_{n} C(m-1, n) .\right]
$$

From (7) it follows that the Fibonomial coefficients must be integers ([8], p. 203).

## 2. COMPUTING FIBONOMIAL COEFFICIENTS MODULO A PRIME

To state our theorem, we need to introduce

$$
r=r(p):=\min \left\{k>0: p \mid F_{k}\right\},
$$

the rank of apparition of $p$ in the Fibonacci sequence, and

$$
t=t(p):=\text { the period of }\left(F_{k} \bmod p\right) .
$$

It is known [10] that, for any prime $p, t / r=1,2$, or 4 .
Theorem: Assume $p$ is a prime number $\neq 5$. Let $m^{\prime}=m \operatorname{div} r, m^{\prime \prime}=(m \bmod t) \operatorname{div} r, m^{*}=m$ $\bmod t$, and similarly for $n$. Then

$$
C(m, n) \equiv B\left(m^{\prime}, n^{\prime}\right)\left\{B\left(m^{\prime \prime}, n^{\prime \prime}\right)^{-1} \bmod p\right\} C\left(m^{*}, n^{*}\right)(\bmod p),
$$

where the term in braces is the modulo- $p$ multiplicative inverse of $B\left(m^{\prime \prime}, n^{\prime \prime}\right)$.
Notice that the first factor here is a binomial coefficient and is the same as the first factor in (3) except that $r$ replaces $p$. The last factor is a Fibonomial coefficient from the initial $t \times t$ (instead of $p \times p$ ) block of Fibonomial coefficients. We observe that the peculiar middle factor can only be: 1 if $t / r=1 ; 1$ or $2^{-1} \bmod p$, if $t / r=2$; and the $\bmod -p$ inverse of $1,2,3,4,6,10$, or 20 , if $t / r=4$. The omitted prime, $p=5$, can be handled by the proposition we shall give later, from which we shall derive Theorem 1.

By repeated application of Lucas's theorem, we get our counterpart of formula (4). It is not so tidy as the binomial case, depending as it does on the use of two mixed-radix representations:

$$
m=m_{k} p^{k-1} r+m_{k-1} p^{k-2} r+\cdots+m_{1} p^{0} r+m_{0},
$$

where

$$
0 \leq m_{0}<r \text { and } 0 \leq m_{j}<p \text { for } j \geq 1,
$$

and

$$
m=m^{\prime \prime \prime} t+m^{\prime \prime} r+m_{0},
$$

where

$$
0 \leq m^{\prime \prime}<t / r, \quad 0 \leq m^{\prime \prime \prime}<\infty, \text { and } m^{*}=m^{\prime \prime} r+m_{0},
$$

and similarly for $n$. Then, for a prime $p \neq 5$, we have our main formula:

$$
\begin{equation*}
C(m, n) \equiv \prod_{j \geq 1} B\left(m_{j}, n_{j}\right)\left\{B\left(m^{\prime \prime}, n^{\prime \prime}\right)^{-1} \bmod p\right\} C\left(m^{*}, n^{*}\right)(\bmod p) . \tag{8}
\end{equation*}
$$

As an example, let us compute $C(23,12) \bmod 3$. Here $p=3, r=4, t=8$,
and

$$
\begin{aligned}
m & =\underline{1} \cdot 3^{1} \cdot 4+\underline{2} \cdot 3^{0} \cdot 4+\underline{3} \\
& =\underline{2} \cdot 8+\underline{1} \cdot 4+\underline{3}
\end{aligned}
$$

$$
n=\underline{1} \cdot 3^{1} \cdot 4+\underline{0} \cdot 3^{0} \cdot 4+\underline{0}
$$

$$
=\underline{1} \cdot 8+\underline{1} \cdot 4+\underline{0}
$$

So

$$
\begin{align*}
C(23,12) & \equiv B(1,1) B(2,0)\left\{B(1,1)^{-1} \bmod 3\right\} C(7,4) & & (\bmod 3) \\
& \equiv 2 \cdot 1 \cdot\left\{2^{-1} \bmod 3\right\} \cdot 1 & & (\bmod 3) \\
& \equiv 2 \cdot 1 \cdot 2 \cdot 1 \equiv 1 & & (\bmod 3) .
\end{align*}
$$

The value for $C(7,4)$ mod 3 was obtained from Table 2 , which was generated by means of the basic recurrence formula. It also includes enough additional values to corroborate our answer for $C(23,12)$.

## TABLE 2. Fibonomials mod 3

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 2 | 0 | 2 | 2 | 1 | 0 | 1 | 1 | 2 | 0 | 2 | 2 | 1 | 0 |
| 2 | 1 | 2 | 0 | 0 | 1 | 2 | 0 | 0 | 1 | 2 | 0 | 0 | 1 | 2 | 0 | 0 |
| 3 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 |
| 4 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 1 | 2 | 1 | 2 |
| 5 | 1 | 2 | 2 | 0 | 1 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 1 | 0 |
| 6 | 1 | 1 | 0 | 0 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 7 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 |
| 8 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 9 | 1 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 1 | 0 |
| 10 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 0 | 0 |
| 11 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 |
| 12 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 1 |
| 13 | 1 | 2 | 2 | 0 | 2 | 1 | 1 | 0 | 1 | 2 | 2 | 0 | 1 | 2 | 2 | 0 |
| 14 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 2 | 2 | 0 | 0 |
| 15 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 16 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 2 |
| 17 | 1 | 1 | 2 | 0 | 1 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 0 |
| 18 | 1 | 2 | 0 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 0 | 0 |
| 19 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 20 | 1 | 2 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 2 | 1 |
| 21 | 1 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 2 | 0 |
| 22 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 0 | 0 |
| 23 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |

Exercises for the Reader: (a) Find $C(7,4) \bmod 2 ; \quad$ (b) find $C(1759,984) \bmod 7$.
[Answers: (a) $B(1,0) B(0,1)\left\{B(0,0)^{-1} \bmod 2\right\} C(1,1) \equiv 1(\bmod 2)$; cf. $C(7,4)=582505$ from Table 1; (b) $B(4,2) B(3,3) B(2,4)\left\{B(1,1)^{-1} \bmod 7\right\} C(15,8) \equiv 1(\bmod 7)$; using Table 3 below.]

## 3. DEDUCING THE RESIDUES OF THE FIBONOMIALS MOD $p$

Let $p$ be a fixed prime. Let $r, t, m^{\prime}, n^{\prime}, m^{\prime \prime}, n^{\prime \prime}, m^{*}$, and $n^{*}$ be as in the Theorem. Also, let $m_{0}=m \bmod r$ and $n_{0}=n \bmod r$.

We shall deduce the residues of $C(m, n) \bmod p$ in the following steps:
Step 1: Show $C(m, n) \equiv 0(\bmod p)$ for $(m, n)$ in the $(r-1) \times(r-1)$ triangles where $m_{0}+n_{0} \geq r$.
Step 2: Calculate $C\left(m^{\prime} r, n^{\prime} r\right) \bmod p\left(m^{\prime}, n^{\prime}=0,1,2, \ldots\right)$.
Step 3: Determine the remaining values $\bmod p$ from the basic recurrence relation (7).

TABLE 3. Fibonomials mod 7

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 2 | 3 | 5 | 1 | 6 | 0 | 6 | 6 | 5 | 4 | 2 | 6 | 1 | 0 |
| 2 | 1 | 2 | 6 | 1 | 5 | 6 | 0 | 0 | 1 | 2 | 6 | 1 | 5 | 6 | 0 | 0 |
| 3 | 1 | 3 | 1 | 4 | 1 | 0 | 0 | 0 | 6 | 4 | 6 | 3 | 6 | 0 | 0 | 0 |
| 4 | 1 | 5 | 5 | 1 | 0 | 0 | 0 | 0 | 1 | 5 | 5 | 1 | 0 | 0 | 0 | 0 |
| 5 | 1 | 1 | 6 | 0 | 0 | 0 | 0 | 0 | 6 | 6 | 1 | 0 | 0 | 0 | 0 | 0 |
| 6 | 1 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 1 | 6 | 1 | 6 | 1 | 6 | 1 | 6 | 2 | 5 | 2 | 5 | 2 | 5 | 2 | 5 |
| 9 | 1 | 6 | 2 | 4 | 5 | 6 | 6 | 0 | 5 | 2 | 3 | 6 | 4 | 2 | 2 | 0 |
| 10 | 1 | 5 | 6 | 6 | 5 | 1 | 0 | 0 | 2 | 3 | 5 | 5 | 3 | 2 | 0 | 0 |
| 11 | 1 | 4 | 1 | 3 | 1 | 0 | 0 | 0 | 5 | 6 | 5 | 1 | 5 | 0 | 0 | 0 |
| 12 | 1 | 2 | 5 | 6 | 0 | 0 | 0 | 0 | 2 | 4 | 3 | 5 | 0 | 0 | 0 | 0 |
| 13 | 1 | 6 | 6 | 0 | 0 | 0 | 0 | 0 | 5 | 2 | 2 | 0 | 0 | 0 | 0 | 0 |
| 14 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

To get started, we note that, for binomial coefficients, we have $B(m, n) \equiv 0(\bmod p)$ if $m \bmod$ $p+n \bmod p \geq p$. Similarly, for Fibonomial coefficients, we have

Lemma 1: $C(m, n) \equiv 0(\bmod p)$ if $m_{0}+n_{0} \geq r$.
Proof \#1: It follows from Knuth \& Wilf's extension of Kummer's theorem to Fibonomial coefficients ([6], Theorem 2) that $p \mid C(m, n)$ if and only if there is at least one carry across or to the left of the radix point when $m / r$ and $n / r$ are added in base $p$.

If $m \bmod r+n \bmod r \geq r$, then there will be a carry across the radix point.
Proof \#2: This time we appeal to another theorem of Lucas ([8], p. 206):

$$
\operatorname{gcd}\left(F_{m}, F_{n}\right)=F_{\operatorname{gcd}(m, n)}
$$

It follows from this theorem that all the Fibonacci numbers divisible by any prime power $p^{s}$ have indices of the form $\operatorname{kr}\left(p^{s}\right)$, where $r\left(p^{s}\right)$ is the rank of apparition of $p^{s}$. Now consider $C(m, n)=$ $C\left(m^{\prime} r+m_{0}, n^{\prime} r+n_{0}\right)$ :

$$
C(m, n)=\prod_{j=0}^{m^{\prime} r+m_{0}-1} F_{\left(m^{\prime}+n^{\prime}\right) r+m_{0}+n_{0}-j} / F_{m^{\prime} r+m_{0}-j}
$$

Our hypothesis is that $m_{0}+n_{0} \geq r$. Therefore, $F_{\left(m^{\prime}+n^{\prime}+1\right) r}$ is a numerator factor, and so the factors that are divisible by $p$ are the $m^{\prime}+1$ numerator factors

$$
F_{\left(m^{\prime}+n^{\prime}+1\right) r}, F_{\left(m^{\prime}+n^{\prime}\right) r}, \ldots, F_{\left(n^{\prime}+1\right) r}
$$

and the $m^{\prime}$ denominator factors

$$
F_{m^{\prime} r}, F_{\left(m^{\prime}-1\right) r}, \ldots, F_{r}
$$

Furthermore, by the consequence of Lucas's theorem noted above, every factor $F_{k r\left(p^{s}\right)}$ in the denominator is matched by such a factor in the numerator, without using the extra numerator factor $F_{\left(m^{\prime}+n^{\prime}+1\right) r}$. So $p \mid C(m, n)$.

In preparation for the next step, we note the following formula:

$$
\begin{equation*}
F_{k r+1} \equiv F_{k r-1} \equiv F_{r-1}^{k}(\bmod p) \quad(k \geq 0) . \tag{9}
\end{equation*}
$$

Since $F_{k r} \equiv 0(\bmod p)$, the first congruence is clear. The second then follows by applying identity (6) with $n=r$ and $m=r-1,2 r-1, \ldots,(k-1) r-1$.

Lemma 2: $C\left(m^{\prime} r, n^{\prime} r\right) \equiv B\left(m^{\prime}, n^{\prime}\right) F_{r-1}^{r m^{\prime} n^{\prime}}(\bmod p)$.
Proof: To simplify the notation, let us suppress the primes on $m$ and $n$ during this proof. If $m=0$ or $n=0$, then

$$
C(m r, n r)=1 \text { and } B(m, n) F_{r-1}^{r m n}=1 \cdot F_{r-1}^{0}=1 .
$$

Now assume $m \geq 1$ and $n \geq 1$. Applying the basic Fibonomial recurrence (7) and Lemma 1 repeatedly, we get

$$
\begin{aligned}
& C(m r,(n-1) r+1)=F_{m r+1} C(m r,(n-1) r)+F_{(n-1) r} C(m r-1,(n-1) r+1) \\
& \equiv F_{m r+1} C(m r,(n-1) r)(\bmod p) ; \\
& C(m r,(n-1) r+2)= F_{m r+1} C(m r,(n-1) r+1)+F_{(n-1) r+1} C(m r-1,(n-1) r+2) \\
& \equiv F_{m r+1} C(m r,(n-1) r+1)(\bmod p) \\
& \equiv F_{m r+1}^{2} C(m r,(n-1) r)(\bmod p) ; \\
& \cdots \\
& C(m r,(n-1) r+r-1)=F_{m r+1} C(m r,(n-1) r+r-2)+F_{(n-1) r+r-2} C(m r-1,(n-1) r+r-1) \\
& \equiv F_{m r+1} C(m r,(n-1) r+r-2)(\bmod p) \\
& \equiv F_{m r+1} F_{m r+1}^{r-2} C(m r,(n-1) r)(\bmod p) \\
&=F_{m r+1}^{r-1} C(m r,(n-1) r)(\bmod p) .
\end{aligned}
$$

Similarly,

$$
C((m-1) r+r-1, n r) \equiv F_{n r-1}^{r-1} C((m-1) r, n r)(\bmod p) .
$$

Then

$$
\begin{aligned}
C(m r, n r) & =F_{m r+1} C(m r, n r-1)+F_{n r-1} C(m r-1, n r) \\
& \equiv F_{m r+1}^{r} C(m r,(n-1) r)+F_{n r-1}^{r} C((m-1) r, n r)(\bmod p) .
\end{aligned}
$$

$\operatorname{By}(9), F_{m r+1}^{r} \equiv F_{m r-1}^{r} \equiv F_{r-1}^{r m}(\bmod p)$ and $F_{n r-1}^{r} \equiv F_{r-1}^{r n}(\bmod p)$. So, for $m, n \geq 1$,

$$
\begin{equation*}
C(m r, n r) \equiv F_{r-1}^{r m} C(m r,(n-1) r)+F_{r-1}^{r n} C((m-1) r, n r)(\bmod p) . \tag{10}
\end{equation*}
$$

Let $C^{\prime}(m, n):=C(m r, n r)$. Then (10) becomes

$$
\begin{equation*}
C^{\prime}(m, n) \equiv F_{r-1}^{r m} C^{\prime}(m, n-1)+F_{r-1}^{r n} C^{\prime}(m-1, n)(\bmod p), \tag{11}
\end{equation*}
$$

a recurrence formula that uniquely determines the values of $C^{\prime}(m, n)$ for $m, n \geq 1$, given the boundary conditions

$$
\begin{equation*}
C^{\prime}(m, 0)=1 \text { and } C^{\prime}(0, n)=1 \quad(m, n \geq 1) \tag{12}
\end{equation*}
$$

Hence, to complete the proof, we need only verify that $C^{\prime \prime}(m, n):=B(m, n) F_{r-1}^{r m n} \bmod p$ satisfies (11) and (12). The boundary conditions (12) are readily verified. Modulo $p$ we have

$$
\begin{aligned}
& F_{r-1}^{r m} C^{\prime \prime}(m, n-1)+F_{r-1}^{r n} C^{\prime \prime}(m-1, n) \equiv F_{r-1}^{r m} B(m, n-1) F_{r-1}^{r m(n-1)}+F_{r-1}^{r n} B(m-1, n) F_{r-1}^{r(m-1) n} \\
& \equiv F_{r-1}^{r m n} B(m, n-1)+F_{r-1}^{r m n} B(m-1, n) \\
& \equiv F_{r-1}^{r m n} B(m, n) \quad[\text { by the Pascal triangle rule }] \\
& \equiv C^{\prime \prime}(m, n),
\end{aligned}
$$

showing that (11) is satisfied.
We can refine Lemma 2 a little. By (9),

$$
F_{r-1}^{r m^{\prime} n^{\prime}} \equiv F_{r^{2} m^{\prime} n^{\prime}-1}(\bmod p)
$$

Here

$$
r m^{\prime}=r(m \operatorname{div} r)=m-m \bmod r \equiv(m \bmod t-m \bmod r)(\bmod t)=m^{\prime \prime} r,
$$

where $m^{\prime \prime}=(m \bmod t) \operatorname{div} r$. Because $t$ is the period of the Fibonacci sequence modulo $p$,

$$
F_{r m^{\prime} r n^{\prime}-1} \equiv F_{r m^{\prime \prime} r n^{\prime \prime}-1}(\bmod p),
$$

and so Lemma 2 becomes

$$
\begin{equation*}
C\left(m^{\prime} r, n^{\prime} r\right) \equiv B\left(m^{\prime}, n^{\prime}\right) F_{r^{2} m^{\prime \prime} n^{\prime \prime}-1}(\bmod p) . \tag{13}
\end{equation*}
$$

We shall complete our determination of the Fibonomial coefficient residues by applying the basic recurrence formula (7), $C(m, n)=F_{m+1} C(m, n-1)+F_{n-1} C(m-1, n)$, to the determination of $C\left(m^{\prime} r+m_{0}, n^{\prime} r+n_{0}\right) \bmod p$ from $C\left(m^{\prime} r, n^{\prime} r\right)$. By Lemma 1 we have

$$
\begin{equation*}
C\left(m^{\prime} r+m_{0}, n^{\prime} r-1\right) \equiv 0(\bmod p)\left(1 \leq m_{0}<r\right) \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
C\left(m^{\prime} r-1, n^{\prime} r+n_{0}\right) \equiv 0(\bmod p)\left(1 \leq n_{0}<r\right) \tag{15}
\end{equation*}
$$

and by Lemma 2 we know $C\left(m^{\prime} r, n^{\prime} r\right) \bmod p$. We observe that application of the basic recurrence formula (7) with these boundary conditions will uniquely determine $C\left(m^{\prime} r+m_{0}, n^{\prime} r+n_{0}\right)$ for $0 \leq m_{0}, n_{0}<r$, and that this solution matrix is proportional to the value $C\left(m^{\prime} r, n^{\prime} r\right)$. Also, the solution matrix depends on the coefficients used, namely, $F_{m^{\prime} r+1}, \ldots, F_{m^{\prime} r+r-1}$ and $F_{n^{\prime} r-1}, \ldots$, $F_{n^{\prime} r+r-2}$. Accordingly, we may make this

Definition: Let $A\left(m^{\prime}, n^{\prime} ; m_{0}, n_{0}\right)$ be the solution $C\left(m^{\prime} r+m_{0}, n^{\prime} r+n_{0}\right)$ of the basic recurrence formula (7) satisfying the boundary conditions (14), (15), and (the possibly contrary-to-fact condition) $C\left(m^{\prime} r, n^{\prime} r\right)=1$.

Since the coefficients $F_{k} \bmod p$ have period $t$, and since $m^{\prime} r+m_{0} \equiv m^{\prime \prime} r+m_{0}$ and $n^{\prime} r+n_{0} \equiv$ $n^{\prime \prime} r+n_{0}(\bmod t)$, we have $A\left(m^{\prime}, n^{\prime} ; m_{0}, n_{0}\right) \equiv A\left(m^{\prime \prime}, n^{\prime \prime} ; m_{0}, n_{0}\right)(\bmod p)$. Thus, we have proved

Lemma 3: $C(m, n) \equiv C\left(m^{\prime} r, n^{\prime} r\right) A\left(m^{\prime \prime}, n^{\prime \prime} ; m_{0}, n_{0}\right)(\bmod p)$.
By (13) and Lemma 3, we now have our general proposition.

Proposition: $C(m, n) \equiv B\left(m^{\prime}, n^{\prime}\right) F_{r^{2} m^{\prime \prime} n^{\prime \prime}-1} A\left(m^{\prime \prime}, n^{\prime \prime} ; m_{0}, n_{0}\right)(\bmod p)$.
As an example, let us determine $C(437,151) \bmod 5$. Here $p=5, r=5$, and $t=20$.

$$
\begin{aligned}
& m^{\prime}=437 \operatorname{div} 5=87=(322)_{5} ; n^{\prime}=151 \operatorname{div} 5=30=(110)_{5} ; \\
& m_{0}=437 \bmod 5=2 ; n_{0}=151 \bmod 5=1 ; \\
& m^{\prime \prime}=437 \bmod 20 \operatorname{div} 5=17 \operatorname{div} 5=3 ; n^{\prime \prime}=151 \bmod 20 \operatorname{div} 5=11 \operatorname{div} 5=2
\end{aligned}
$$

So $C(437,151) \equiv B(3,1) B(2,1) B(2,0) F_{5^{2} \cdot 3 \cdot 2-1} A(3,2 ; 2,1) \equiv 4 \cdot 3 \cdot 1 \cdot 4 \cdot 4 \equiv 2(\bmod 5)$. (We looked up the last factor in Table 4.)

TABLE 4. $A\left(m^{\prime \prime}, n^{\prime \prime} ; m_{0}, n_{0}\right)$ for $p=5$

|  | $n^{\prime \prime}$ |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | 0 |  |  |  |  | 1 |  |  |  |  |  | 2 |  |  |  |  |  | 3 |  |
|  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 11 |
|  | 1 | 1 | 2 | 3 | 0 | 3 | 3 | 1 | 4 | 0 |  | 4 | 4 | 3 | 2 | 0 | 2 | 2 | 2 | 4 | 10 |
| 0 | 1 | 2 | 1 | 0 | 0 | 4 | 3 | 4 | 0 | 0 |  | 1 | 2 | 1 | 0 | 0 | 4 | 3 | 3 | 4 | 00 |
|  | 1 | 3 | 0 | 0 | 0 | 2 | 1 | 0 | 0 | 0 |  | 4 | 2 | 0 | 0 | 0 | 3 | 4 | 4 | 0 | 0 0 |
|  | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |  | 1 | 0 | 0 | 0 | 0 |  | 0 | 0 | 0 | $0 \quad 0$ |
|  | 1 | 3 | 4 | 2 | 1 | 1 | 3 | 4 | 2 | 1 |  | 1 | 3 | 4 | 2 | 1 | 1 | 3 | 3 | 4 | 21 |
|  | 1 | 3 | 3 | 1 | 0 | 3 | 4 | 4 | 3 | 0 |  | 4 | 2 | 2 | 4 | 0 | 2 |  | 1 | 1 | 20 |
| 1 | 1 | 1 | 4 | 0 | 0 | 4 | 4 | 1 | 0 | 0 |  | 1 | 1 | 4 | 0 | 0 | 4 | 4 | 4 | 1 | $0 \quad 0$ |
|  | 1 | 4 | 0 | 0 | 0 | 2 | 3 | 0 | 0 | 0 |  | 4 | 1 | 0 | 0 | 0 | 3 | 2 | 2 | 0 | 00 |
|  | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |  | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | $0 \quad 0$ |
| $m^{\prime \prime}$ |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  | 1 | 4 | 1 | 4 | 1 | 1 | 4 | 1 | 4 | 1 |  | 1 | 4 | 1 | 4 | 1 | 1 | 4 | 4 | 1 | $4 \quad 1$ |
|  | 1 | 4 | 2 | 2 | 0 | 3 | 2 | 1 | 1 | 0 |  | 4 | 1 | 3 | 3 | 0 | 2 | 3 | 3 | 4 | 40 |
| 2 | 1 | 3 | 1 | 0 | 0 | 4 | 2 | 4 | 0 | 0 |  | 1 | 3 | 1 | 0 | 0 | 4 | 2 | 2 | 4 | $0 \quad 0$ |
|  | 1 | 2 | 0 | 0 | 0 | 2 | 4 | 0 | 0 | 0 |  | 4 | 3 | 0 | 0 | 0 | 3 |  | 1 | 0 | $0 \quad 0$ |
|  | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |  | 1 | 0 | 0 | 0 | 0 | 1 |  | 0 | 0 | 00 |
|  | 1 | 2 | 4 | 3 | 1 | 1 | 2 | 4 | 3 | 1 |  | 1 | 2 | 4 | 3 | 1 | 1 |  | 2 | 4 | 31 |
|  | 1 | 2 | 3 | 4 | 0 | 3 | 1 | 4 | 2 | 0 |  | 4 | 3 | 2 | 1 | 0 | 2 |  | 4 | 1 | 30 |
| 3 | 1 | 4 | 4 | 0 | 0 | 4 | 1 | 1 | 0 | 0 |  | 1 | 4 | 4 | 0 | 0 | 4 |  | 1 | 1 | $0 \quad 0$ |
|  | 1 | 1 | 0 | 0 | 0 | 2 | 2 | 0 | 0 | 0 |  | 4 | 4 | 0 | 0 | 0 |  |  | 3 | 0 | 0 0 |
|  |  | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |  | 1 | 0 | 0 | 0 | 0 |  |  | 0 | 0 | $0 \quad 0$ |

Finally, we get the formula stated in our Theorem by observing that in most cases we can find the $r \times r A$-blocks hidden in the initial $t \times t C$-block. In this block

$$
C\left(m^{\prime \prime} r+m_{0}, n^{\prime \prime} r+n_{0}\right) \equiv B\left(m^{\prime}, n^{\prime}\right) F_{r^{2} m^{\prime \prime} n^{\prime \prime}-1} A\left(m^{\prime \prime}, n^{\prime \prime} ; m_{0}, n_{0}\right)(\bmod p)
$$

and $m^{\prime}=m^{\prime \prime}$ and $n^{\prime}=n^{\prime \prime}$. So, if $B\left(m^{\prime \prime}, n^{\prime \prime}\right) \not \equiv 0(\bmod p)$, then

$$
\begin{equation*}
F_{r^{2} m^{\prime \prime} n^{\prime \prime}-1} A\left(m^{\prime \prime}, n^{\prime \prime} ; m_{0}, n_{0}\right) \equiv B\left(m^{\prime \prime}, n^{\prime \prime}\right)^{-1} C\left(m^{\prime \prime} r+m_{0}, n^{\prime \prime} r+n_{0}\right)(\bmod p) \tag{16}
\end{equation*}
$$

Here $0 \leq m^{\prime \prime}, n^{\prime \prime}<t / r$. Since $t / r \leq 4$, the possible values of $B\left(m^{\prime \prime}, n^{\prime \prime}\right)$ are $1,2,3,4,6,10$, and 20. The only case where some value of $B\left(m^{\prime \prime}, n^{\prime \prime}\right) \equiv 0(\bmod p)$ is $p=5$; then $t / r=4$, and $B(1,2)=B(2,1)=10$ and $B(2,2)=20$ are not invertible $\bmod 5$. So, if $p \neq 5$, we may use (16) in the Proposition to determine the residue modulo $p$ of the Fibonomial coefficient $C(m, n)$ in terms
of the binomial coefficients $B\left(m^{\prime}, n^{\prime}\right)$ and $B\left(m^{\prime \prime}, n^{\prime \prime}\right)$ and the Fibonomial coefficient $C\left(\dot{m}^{\prime \prime} r+\right.$ $\left.m_{0}, n^{\prime \prime} r+n_{0}\right)=C\left(m^{*}, n^{*}\right)$, thus proving our Lucas-type theorem for Fibonomial-coefficient residues.

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# FIBONACCI NUMBERS AND FRACTIONAL DOMINATION OF $\boldsymbol{P}_{\boldsymbol{m}} \times \boldsymbol{P}_{\boldsymbol{n}}$ 

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## 1. INTRODUCTION

The product of two paths, $P_{m} \times P_{n}$, is also known as the $m \times n$ complete grid graph, $G_{m, n}$, having vertex set $Z_{m} \times Z_{n}$, where $Z_{k}$ denotes the set $\{1,2, \ldots, k\}$. Two vertices, $(i, j)$ and $(r, s)$, are adjacent when $|i-r|+|j-s|=1$. Thus, $|V|=m n$ and $|E|=2 m n-(m+n)$.

Let $G=(V, E)$ be a graph and $v \in V(G)$. Then the closed neighborhood of $v$, denoted $N[v]$, is the set $\{v\} \cup\{u \in V(G) \mid u v \in E(G)\}$.

The definition of fractional domination, as introduced by Hedetniemi et al. [3] is as follows: If $g$ is a function mapping the vertex set, $V(G)$, into some set of real numbers, then for $S$ a subset of $V(G)$, let $g(S)=\sum g(v)$ over all $v \in S$. Let $|g|=g(V(G))=g\left(v_{1}\right)+g\left(v_{2}\right)+\cdots+g\left(v_{n}\right)$. A realvalued function $g: V(G) \rightarrow[0,1]$ is a fractional dominating function if for every $v \in V(G)$, $g(N[v]) \geq 1$. A dominating function is minimal if for every $v \in V(G)$ with $g(v)>0$, there exists a vertex $u \in N[v]$ such that $g(N[u])=1$. The fractional domination number of $G$, denoted $\gamma_{f}(G)$, is the minimum, $|g|$, over all minimal dominating functions $g$.

A real-valued function $g: V(G) \rightarrow[0,1]$ is a packing function if, for every $v \in V(G)$ with $g(v)<1$, there exists a vertex $u \in N[v]$ where $g(N[u])=1$. Then the (upper) fractional packing number of $G$, denoted $P_{f}(G)$, is the maximum $|g|$ such that $g$ is a maximal packing function.

The fractional parameters are related by the following.
Proposition 1.1: For every graph $G, P_{f}(G)=\gamma_{f}(G)$ (Domke [1]).
The formula of Proposition 1.2 computes the fractional domination number for $P_{2} \times P_{n}$. No general formula is known for $\gamma_{f}\left(P_{m} \times P_{n}\right)$, for $m>2$, but fractional domination numbers for any graph may be computed using linear programming.

Proposition 1.2: $\gamma_{f}\left(P_{2} \times P_{n}\right)=(n+1) / 2+(\lceil n / 2\rceil-\lfloor n / 2\rfloor-1) /(2 n+2)=n / 2+\lceil n / 2\rceil /(n+1)$.
Proof: It has been shown that $\gamma_{f}\left(P_{2} \times P_{n}\right)=\lceil n / 2\rceil=(n+1) / 2$ when $n \equiv 1(\bmod 2)$, and that $\gamma_{f}\left(P_{2} \times P_{n}\right)=\left(n^{2}+2 n\right) / 2(n+1)$ when $n \equiv 0(\bmod 2)$ (Hare [4]).

Values of fractional domination numbers for $P_{m} \times P_{n}$ for several small ( $m, n$ ) pairs may also be found in [4] and [5]. It would be interesting if a formula could be found for the arbitrary $m \times n$ complete grid graphs, as has been found for the 2 -packing number [2]. In the remainder of this paper we develop upper and lower bounds for the fractional domination number of $P_{m} \times P_{n}$.

## 2. BOUNDS FOR THE FRACTIONAL DOMINATION NUMBER

Let $D_{m}=3 F_{1} F_{m}+F_{2} F_{m-1}=3 F_{m}+F_{m-1}$, where $D$ stands for "denominator." We denote a vertex in the $i^{\text {th }}$ row and $j^{\text {th }}$ column of $G_{m, n}\left(=P_{m} \times P_{n}\right)$ by $v_{i, j}$. The following develops upper and lower bounds for $\gamma_{f}\left(P_{m} \times P_{n}\right)$ which depend only on $m$.

Proposition 2.3: Let $m>2, n>2,1<j<n$, and $g\left(v_{i, j}\right)=F_{i} F_{m-i+1} / D_{m}$. Then

$$
g\left(N\left[v_{1, j}\right]\right)=1=g\left(N\left[v_{2, j}\right]\right)
$$

Proof:

$$
\begin{aligned}
g\left(N\left[v_{1, j}\right]\right) & =g\left(v_{1, j-1}\right)+g\left(v_{1, j}\right)+g\left(v_{1, j+1}\right)+g\left(v_{2, j}\right) \\
& =3 F_{1} F_{m} / D_{m}+F_{2} F_{m-1} / D_{m}=1 . \\
g\left(N\left[v_{2, j}\right]\right)= & g\left(v_{1, j}\right)+g\left(v_{2, j-1}\right)+g\left(v_{2, j}\right)+g\left(v_{2, j+1}\right)+g\left(v_{3, j}\right) \\
= & \left(F_{1} F_{m}+3 F_{2} F_{m-1}+F_{3} F_{m-2}\right) / D_{m} .
\end{aligned}
$$

Since $3 F_{1} F_{m}+F_{2} F_{m-1}=F_{1} F_{m}+3 F_{2} F_{m-1}+F_{3} F_{m-2}$, it follows that $g\left(N\left[v_{2, j}\right]\right)=1$. By symmetry,

$$
g\left(N\left[v_{m, j}\right]\right)=g\left(N\left[v_{m-1, j}\right]\right)=1 .
$$

Proposition 2.4: Let $m>3, n>2,1<i<m-1,1<j<n$, and $g\left(v_{i, j}\right)=F_{i} F_{m-i+1} / D_{m}$. Then, $g\left(N\left[v_{1, j}\right]\right)=g\left(N\left[v_{i, j}\right]\right)=1$.

Proof:

$$
\begin{aligned}
g\left(N\left[v_{i, j}\right]\right) & =g\left(v_{i-1, j}\right)+g\left(v_{i, j-1}\right)+g\left(v_{i, j}\right)+g\left(v_{i, j+1}\right)+g\left(v_{i+1, j}\right) \\
& =\left(F_{i-1} F_{m-i+2}+3 F_{i} F_{m-i+1}+F_{i+1} F_{m-i}\right) / D_{m} . \\
g\left(N\left[v_{i+1, j}\right]\right) & =g\left(v_{i, j}\right)+g\left(v_{i+1, j-1}\right)+g\left(v_{i+1, j}\right)+g\left(v_{i+1, j+1}\right)+g\left(v_{i+2, j}\right) \\
& =\left(F_{i} F_{m-i+1}+3 F_{i+1} F_{m-1}+F_{i+2} F_{m-i-1}\right) / D_{m} .
\end{aligned}
$$

Since $F_{i-1} F_{m-i+2}+3 F_{i} F_{m-i+1}+F_{i+1} F_{m-i}=F_{i} F_{m-i+1}+3 F_{i+1} F_{m-i}+F_{i+2} F_{m-i-1}$, it follows that

$$
g\left(N\left[v_{i, j}\right]\right)=g\left(N\left[v_{i+1, j}\right]\right)
$$

From Proposition 2.3, $g\left(N\left[v_{2, j}\right]\right)=1$, so $g\left(N\left[v_{i, j}\right]\right)=1$ for all $i, 1 \leq i \leq m$.
Theorem 2.5: Let $C_{m}=g\left(v_{1, j}\right)+g\left(v_{2, j}\right)+\cdots+g\left(v_{m, j}\right)$ where $g\left(v_{i, j}\right)=F_{i} F_{m-i+1} / D_{m}$. Then, when $m \geq 3$, the sum of the function values over all vertices in column $j$ is given by $C_{m} / D_{m}$ where $C_{m}=\sum_{i=1, m}\left(F_{i} F_{m-i+1}\right) D_{m}$ and $\gamma_{f}\left(P_{m} \times P_{n}\right) \leq n C_{m}+c_{\gamma}$, where $c_{\gamma} \leq 2[m / 3] F_{m} / D_{m}$.

Proof: Since $g\left(N\left[v_{i, j}\right]\right)=1$ for $2 \leq j \leq n-1$, all vertices in columns 2 through $n-1$ are dominated. In order to dominate column 1, let $g\left(v_{i, 1}\right)$ be modified as follows:

For $1<i<m$, let $\sigma=\max \left\{g\left(v_{i-1, j}\right), g\left(v_{i, j}\right), g\left(v_{i+1, j}\right)\right\}$.
Case 1. $m \equiv 0 \bmod 3$.
If $\boldsymbol{i} \equiv 2 \bmod 3$, then $g\left(v_{i, 1}\right)=F_{i} F_{m-i+1} / D_{m}+\sigma$.
Case 2. $m \equiv 1 \bmod 3$.
If $2 i=(m+1)$, then $g\left(v_{i, 1}\right)=2 F_{i} F_{m-i+1} / D_{m}$.
Else If $[(i \equiv 2 \bmod 3)$ and $(2 i<m+1)]$ or $[(i \equiv 0 \bmod 3)$ and $(2 i>m+1)]$, then $g\left(v_{i, 1}\right)=F_{i} F_{m-i+1} / D_{m}+\sigma$.

Case 3. $m \equiv 2 \bmod 3$.
If $2 i=m$, then $g\left(v_{i, 1}\right)=2 F_{i} F_{m-i+1} / D_{m}$.
Else If $[(i \equiv 2 \bmod 3)$ and $(2 i<m)]$ or $[(i \equiv 1 \bmod 3)$ and $(2 i-2>m)]$, then $g\left(v_{i, 1}\right)=F_{i} F_{m-i+1} / D_{m}+\sigma$.

Observe that this assignment produces $g\left(N\left[v_{i, 1}\right]\right) \geq 1$ for all vertices in column 1. To show that $g$ is minimal, observe that $g\left(N\left[v_{i, j}\right]\right)=1$ for $1 \leq i \leq m$ and $2 \leq j \leq n-1$, except when $g\left(v_{1, j}\right) \neq F_{i} F_{m-i+1} / D_{m}$. Thus, only the case when $g\left(v_{i, 1}\right) \neq F_{i} F_{m-i+1} / D_{m}$ must be examined. In the above procedure, each modification produces an assignment such that $g\left(N\left[v_{i-1,1}\right]\right)=1$, $g\left(N\left[v_{i, 1}\right]\right)=1$, or $g\left(N\left[v_{i+1,1}\right]\right)=1$. Thus, $g$ is minimal.

To also dominate the vertices of column $n$, let $c_{\gamma}$ be twice the functional value added to column 1 by the above modification. It is straightforward to show by induction on $i, 1<i<m-1$, that $F_{m}=F_{i+1} F_{m-i}+F_{i} F_{m-i-1}$. Thus, $F_{m}>F_{i+1} F_{m-i}$. Let $j=i+1$. Then $F_{m}>F_{j} F_{m-j+1}$, which yields $2[m / 3]\left(F_{m} / D_{m}\right) \geq c_{\gamma}$.

Such a minimal dominating function is given for $P_{3} \times P_{n}$ by:

$$
\begin{aligned}
& g\left(v_{i, j}\right)=g\left(v_{3, j}\right)=2 / 7, \text { for } 1 \leq j \leq n, \\
& g\left(v_{2, j}\right)=1 / 7, \text { for } 1<j<n, \text { and } \\
& g\left(v_{2,1}\right)=g\left(v_{2, n}\right)=3 / 7
\end{aligned}
$$

Thus, $\gamma_{f}\left(P_{3} \times P_{n}\right) \leq n(5 / 7)+4 / 7$.

## 3. BOUNDS FOR THE FRACTIONAL PACKING NUMBER

From Proppositions 2.3 and 2.4 and the definition of fractional packing, it is clear that when $g\left(v_{i, j}\right)=F_{i} F_{m-i+1} / D_{m}$ for all $i$ and $j$, then $g$ is a maximal packing function and $|g|=n C_{m}$. However, the following improved bounds are easily obtained.

## Proposition 3.6:

$$
\begin{array}{ll}
P_{f}\left(P_{3} \times P_{n}\right) \geq n C_{3}+2 / 7, & \text { for } n \geq 3, C_{3}=5 / 7 \\
P_{f}\left(P_{4} \times P_{n}\right) \geq n C_{4}+4 / 11, & \text { for } n \geq 4, C_{4}=10 / 11 \\
P_{f}\left(P_{5} \times P_{n}\right) \geq n C_{5}+8 / 18, & \text { for } n \geq 5, C_{5}=20 / 18 \\
P_{f}\left(P_{6} \times P_{n}\right) \geq n C_{6}+18 / 29, & \text { for } n \geq 6, C_{6}=38 / 29 .
\end{array}
$$

Proof: The following assignments of $g$ produce maximal packing functions.
For $P_{3} \times P_{n}$ :

$$
\begin{aligned}
& g\left(v_{1,1}\right)=g\left(v_{1, n}\right)=g\left(v_{3,1}\right)=g\left(v_{3, n}\right)=3 / 7=F_{4} / D_{3}, \\
& g\left(v_{2,2}\right)=g\left(v_{2, n-1}\right)=0, \text { and } \\
& g\left(v_{i, j}\right)=F_{i} F_{m-i+1} / D_{3}, \text { otherwise. } \\
& \text { Thus, } P_{f}\left(P_{3} \times P_{n}\right) \geq n(5 / 7)+2 / 7 .
\end{aligned}
$$

For every vertex in rows 1 and $3, g\left[N\left(v_{i, j}\right)\right]=1$, except for columns 1 and $n$. However, $g\left[N\left(v_{2,1}\right)\right]=g\left[N\left(v_{2, n}\right)=1\right.$, so $g$ is maximal.

For $P_{4} \times P_{n}$ :

$$
\begin{aligned}
& g\left(v_{1,1}\right)=g\left(v_{1, n}\right)=g\left(v_{4,1}\right)=g\left(v_{4, n}\right)=5 / 11=F_{5} / D_{4}, \\
& g\left(v_{2,1}\right)=g\left(v_{2, n}\right)=g\left(v_{3,1}\right)=g\left(v_{3, n}\right)=3 / 11=F_{4} / D_{4}, \\
& g\left(v_{2,2}\right)=g\left(v_{3,2}\right)=g\left(v_{2, n-1}\right)=g\left(v_{3, n-1}\right)=0, \text { and } \\
& g\left(v_{i, j}\right)=F_{i} F_{m-i+1} / D_{4}, \text { otherwise. }
\end{aligned}
$$

For every vertex in rows 1 and $4, g\left[N\left(v_{i, j}\right)\right]=1$, so $g$ is maximal.
For $P_{5} \times P_{n}$ :

$$
\begin{aligned}
& g\left(v_{1,1}\right)=g\left(v_{1, n}\right)=g\left(v_{5,1}\right)=g\left(v_{5, n}\right)=8 / 18=F_{6} / D_{5} \\
& g\left(v_{2,1}\right)=g\left(v_{2, n}\right)=g\left(v_{4,1}\right)=g\left(v_{4, n}\right)=5 / 18=F_{5} / D_{5} \\
& g\left(v_{2,2}\right)=g\left(v_{2, n-1}\right)=g\left(v_{4,2}\right)=g\left(v_{4, n-1}\right)=0, \text { and } \\
& g\left(v_{i, j}\right)=F_{i} F_{m-i+1} / D_{5}, \text { otherwise. }
\end{aligned}
$$

For every vertex in rows 1,3 , and 5 except vertices $v_{3,2}$ and $v_{3, n-1}, g\left[N\left(v_{i, j}\right)\right]=1$, so $g$ is maximal.

For $P_{6} \times P_{n}$ :

$$
\begin{aligned}
& g\left(v_{1,1}\right)=g\left(v_{1, n}\right)=g\left(v_{6,1}\right)=g\left(v_{6, n}\right)=13 / 29=F_{7} / D_{6}, \\
& g\left(v_{2,1}\right)=g\left(v_{2, n}\right)=g\left(v_{4,1}\right)=g\left(v_{4, n}\right)=8 / 29=F_{6} / D_{6}, \\
& g\left(v_{2,2}\right)=g\left(v_{2, n-1}\right)=g\left(v_{4,2}\right)=g\left(v_{4, n-1}\right)=0, \\
& g\left(v_{3,1}\right)=g\left(v_{3, n}\right)=8 / 29, \\
& g\left(v_{4,1}\right)=g\left(v_{4, n}\right)=7 / 29, \text { and } \\
& g\left(v_{i, j}\right)=F_{i} F_{m-i+1} / D_{6}, \text { otherwise } .
\end{aligned}
$$

For every vertex in rows $1,3,4$, and 6 except $v_{3,2}, v_{4,2}, v_{3, n-1}$, and $v_{4, n-1}, g\left[N\left(v_{i, j}\right)\right]=1$, so $g$ is maximal.

Theorem 3. 7: When $m>6, n \geq m, P_{f}\left(P_{m} \times P_{n}\right) \geq n C_{m}+4\left(F_{m-1} / D_{m}\right)$.
Proof: For $P_{m} \times P_{n}$ :

$$
\begin{aligned}
& g\left(v_{1,1}\right)=g\left(v_{1, n}\right)=g\left(v_{m, 1}\right)=g\left(v_{m, n}\right)=F_{m+1} / D_{m}, \\
& g\left(v_{2,1}\right)=g\left(v_{2, n}\right)=g\left(v_{m-1,1}\right)=g\left(v_{m-1, n}\right)=F_{m} / D_{m}, \\
& g\left(v_{3,1}\right)=g\left(v_{3, n}\right)=g\left(v_{m-2,1}\right)=g\left(v_{m-2, n}\right)=F_{m} / D_{m}, \\
& g\left(v_{2,2}\right)=g\left(v_{m-1,2}\right)=g\left(v_{2, n-1}\right)=g\left(v_{m-1, n-1}\right)=0, \text { and } \\
& g\left(v_{i, j}\right)=F_{i} F_{m-i+1} / D_{m}, \text { otherwise }
\end{aligned}
$$

In column 1, $g\left[N\left(v_{1,1}\right)\right]=g\left[N\left(v_{2,1}\right)\right]=g\left[N\left(v_{m, 1}\right)\right]=g\left[N\left(v_{m-1,1}\right)\right]=1$. For all vertices in column 2 except $v_{2,2}, v_{3,2}, v_{m-1,2}$, and $v_{m-2,2}, g\left[N\left(v_{i, 2}\right)\right]=1$. For all vertices in colums 3 through $n-3$, $g\left[N\left(v_{i, j}\right)\right]=1$. Thus, every vertex is adjacent to some vertex (possibly itself) with $g\left[N\left(v_{i, j}\right]=1\right.$ and $g$ is maximal. Column summations yield a net gain of $4 F_{m-1} / D_{m}$.

Corollary 3.8: When $m>6, n \geq m$, then $P_{f}\left(P_{m} \times P_{n}\right) \geq m n / 5+(2 n / 5)\left(F_{m} / D_{m}\right)+4\left(F_{m-1} / D_{m}\right)$.

$$
\text { FIBONACCI NUMBERS AND FRACTIONAL DOMINATION OF } P_{m} \times P_{n}
$$

Proof: It is well known that, for $m \geq 4$,

$$
C_{m}=\sum_{i=1, m}\left(F_{i} F_{m-i+1}\right) / D_{m}=\left((3 m+2) F_{m}+m F_{m-1}\right) / 5=\left(m\left(3 F_{m}+F_{m-1}\right)+2 F_{m}\right) / 5 .
$$

Then

$$
\begin{aligned}
P_{f}\left(P_{m} \times P_{n}\right) & \geq n C_{m}+4\left(F_{m-1} / D_{m}\right) \\
& =m n / 5+(2 n / 5)\left(F_{m} / D_{m}\right)+4\left(F_{m-1} / D_{m}\right) .
\end{aligned}
$$

The recurrence $C_{m}=F_{m} / D_{m}+C_{m-1}+C_{m-2}$ follows immediately and, for large $m, C_{m}$ is approximately $m / 5+0.145$.

## 4. CONCLUDING REMARKS

It has been shown in this paper that

$$
\begin{aligned}
n(5 / 7)+2 / 7 \leq \gamma_{f}\left(P_{3} \times P_{n}\right) & \leq n(5 / 7)+4 / 7, \\
n(10 / 11)+4 / 11 \leq \gamma_{f}\left(P_{4} \times P_{n}\right) & \leq n(10 / 11)+12 / 11, \\
n(20 / 18)+8 / 18 \leq \gamma_{f}\left(P_{5} \times P_{n}\right) & \leq n(20 / 18)+20 / 18, \\
n(38 / 29)+18 / 29 \leq \gamma_{f}\left(P_{6} \times P_{n}\right) & \leq n(38 / 29)+32 / 29
\end{aligned}
$$

and, for $m>6, n \geq m$,

$$
n C_{m}+4\left(F_{m-1} / D_{m}\right) \leq \gamma_{f}\left(P_{m} \times P_{n}\right) \leq n C_{m}+2\lceil m / 3\rceil\left(F_{m} / D_{m}\right),
$$

where $C_{m}=\sum_{i=1, m}\left(F_{i} F_{m-i+1}\right) / D_{m}$ and $D_{m}=3 F_{m}+F_{m-1}$.
Although the methods of linear programming can be used to calculate $\gamma_{f}$ for individual graphs, no exact construction is known for $\gamma_{f}\left(P_{m} \times P_{n}\right)$ for $m>2$ Thus, the bounds presented in this paper provide a useful addition to our knowledge of domination parameters on grid graphs.

## ACKNOWLEDGMENT

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# CYCLES IN DOUBLING DIAGRAMS MOD $m$ 

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## 1. INTRODUCTION, NOTATIONS, AND A THEOREM

By "doubling diagram mod $m$ " we mean the directed graph whose vertices are 0 and the natural numbers less than $m$, with directed arcs (arrows) from each vertex $x$ to $2 x$ reduced modulo $m$.

$m=2$


$m=4$

$m=5$

$m=6$


I gave pages with the diagrams for $m$ 's from 2 to 31 to a group of students, and asked them to find regularities. This took place at the School of Education of Tel Aviv University, in an elective course for non-mathematicians, intended to improve their ideas about mathematics. The students recognized some known phenomena (see [1]), including the fact that, for an odd $m$, all the vertices are arranged in cycles. Suzanah Erseven, a prospective English teacher, examined the numbers of cycles in the diagrams for the odd $m$ 's, and found that the sequence of these numbers consists of two even numbers and two odd numbers, alternately. Her discovery is reformulated here as Theorem 1. Its proof is the central topic of this paper.

Notations: In the following, the variable $m$ will denote the modulus of the diagram, and will be limited to odd numbers.
$L(m)$ is the number of vertices in the cycle of 1,
$C(m)$ is the number of cycles with vertices that are relatively prime to $m$, and
$T(m)$ is the total number of cycles in the doubling diagram modulo $m$, including the cycle of 0 .
$\varphi(n)$ is the Euler function (number of natural numbers less than and relatively prime to $n$ ).
A number $m$ will be called "O.K." if it agrees with the following theorem.
Theorem 1: $m \equiv \pm 1(\bmod 8) \Rightarrow T(m)$ is an odd number.
$m \equiv \pm 3(\bmod 8) \Rightarrow T(m)$ is an even number.

## 2. PROOF OF THEOREM 1

The proof of Theorem 1 will emerge from some propositions and results. Let us start with these.

Since a chain of $n$ arrows leads from $x$ to $x$ iff $x \cdot 2^{n} \equiv x(\bmod m)$, that is, iff $m \mid x\left(2^{n}-1\right)$, it follows that
I. $L(m)$ is the minimal $n$ such that $m \mid 2^{n}-1$, and it divides any other $n$ with this property.
II. If $x$ is prime to $m$, then its cycle is also of length $L(m)$ (and all the vertices in this cycle are relatively prime to $m$ ).
III. The length of any cycle in the doubling diagram $\bmod m$ divides $L(m)$. $\left[m \mid 2^{L(m)}-1\right.$ hence $m \mid x\left(2^{L(m)}-1\right)$ hence $L(m)$ arrows from $x$ end at $x$.]
IV. $C(m) \cdot L(m)=\varphi(m)$. (This follows from II.)

Since $2 x \equiv y(\bmod m) \Leftrightarrow 2 k x \equiv k y(\bmod k m)$, it follows that
V . If we multiply by $k$ the values of the vertices of a cycle in the doubling diagram $\bmod m$, we get a cycle in the doubling diagram $\bmod \mathrm{km}$.
VI. $L(m) \mid L(k m)$. (This results from III and V.)

Proposition 1: For every prime number $p \neq 2, L\left(p^{n+1}\right)$ is equal to either $p \cdot L\left(p^{n}\right)$ or $L\left(p^{n}\right)$.
Proof: Denote $L\left(p^{n}\right)$ by $\lambda$. Then $\left.p^{n}\right|^{\lambda}-1$, that is, $2^{\lambda} \equiv 1\left(\bmod p^{n}\right)$, hence $2^{\lambda} \equiv 1(\bmod p)$, and so are all of the powers of $2^{\lambda}$. From this, it follows that $1+2^{\lambda}+2^{2 \lambda}+\cdots+2^{(p-1) \lambda}$ is divisible by $p$, and therefore $2^{\lambda p}-1$, which is the product of this sum and $2^{\lambda}-1$, is divisible by $p^{n+1}$. By this and by II we have that $L\left(p^{n+1}\right) \mid \lambda p$, and by VI we have that $\lambda \mid L\left(p^{n+1}\right)$.

Remark a: Let $k=\left(2^{\lambda}-1\right) / p^{n}$. Then

$$
\begin{aligned}
1+2^{\lambda}+2^{2 \lambda}+\cdots+2^{(p-1) \lambda} & =1+\left(k p^{n}+1\right)+\left(k p^{n}+1\right)^{2}+\cdots+\left(k p^{n}+1\right)^{(p-1)} \\
& =p+k p^{n} \cdot(1+2+\cdots+(p-1))+p^{2 n} \cdot(\cdots) \\
& =p+k p^{n} \cdot p \cdot(p-1) / 2+p^{2 n} \cdot(\cdots) .
\end{aligned}
$$

The second term on the extreme right side is divisible by $p^{2}$ (even for $n=1$, since $p \neq 2$ ), so the total sum is not a multiple of $p^{2}$. Therefore, if $2^{\lambda}-1$ is divisible by $p^{n}$ but not by $p^{n+1}$, then $2^{\lambda p}-1$ is divisible by $p^{n+1}$ but not by $p^{n+2}$. From this one gets that if, for some $n, L\left(p^{n+1}\right) \neq$ $L\left(p^{n}\right)$, then $L\left(p^{n+1}\right) \neq L\left(p^{n}\right)$ for all bigger $n$ 's.

Remark b: Computer runs show that, for all prime numbers up to 100,000 , there are just two cases where $L\left(p^{n+1}\right)=L\left(p^{n}\right)$. These are $L\left(1093^{2}\right)=L(1093)=364$ and $L\left(3511^{2}\right)=L(3511)=$ 1755.

Remark c: A theorem similar to Proposition 1 together with Remark a, but (still?) without examples as in Remark b, was proved by Wall [2] for the length of the period of the Fibonacci series reduced $\bmod m$.

Lemma 1: If $m=p$ is O.K., so is $m=p^{n}$.
Proof: $\varphi\left(p^{n+1}\right)=p^{n+1}-p^{n}=p \cdot \varphi\left(p^{n}\right)$. From this, together with IV and Proposition 1, it follows that $C\left(p^{n+1}\right)$ is either equal to $C\left(p^{n}\right)$ or else $p$ times greater. In any case, they are both even or both odd numbers.

By V , the vertices in the doubling diagram mod $p^{n+1}$ that are the multiples of $p$ form a subdiagram congruent to the diagram $\bmod p^{n}$, that is, they form $T\left(p^{n}\right)$ cycles. So $T\left(p^{n+1}\right)=$ $T\left(p^{n}\right)+C\left(p^{n+1}\right)$.

If $p \equiv \pm 1(\bmod 8)$, then so is every $p^{n}$. In this case $T(p)$ is an odd number (we've assumed $m=p$ is $\mathrm{O} . \mathrm{K}$.), so $C(p)$ is an even number (they differ just by the cycle of 0 ); hence, all the $C\left(p^{n}\right)$ 's are even numbers and, therefore, all the $T\left(p^{n}\right)$ 's are odd numbers.

If $p \equiv \pm 3(\bmod 8)$, then $p^{2} \equiv 1, p^{3} \equiv \pm 3, p^{4} \equiv 1$, and so on. In this case $T(p)$ is an even number, so $C(p)$ is an odd number; hence, all the $C\left(p^{n}\right)$ 's are odd numbers and, therefore, the $T\left(p^{n}\right)^{\prime} \mathrm{s}$ are even numbers and odd numbers, alternately.

Proposition 2: If $m_{1}$ and $m_{2}$ are relatively prime to each other and to 2 , then

$$
L\left(m_{1} \cdot m_{2}\right)=1 . \mathrm{c} . \mathrm{m} .\left(L\left(m_{1}\right), L\left(m_{2}\right)\right) .
$$

Proof: $2^{x}-12^{x y}-1$ (the quotient is a sum of a geometric sequence). By I it follows that both $m_{1}$ and $m_{2}$ divide $2^{\text {1.c.m. }\left(L\left(m_{1}\right), L\left(m_{2}\right)\right)}-1$, and so does their product. Hence, $L\left(m_{1} \cdot m_{2}\right)$ divides 1.c.m. $\left(L\left(m_{1}\right), L\left(m_{2}\right)\right)$. Their equality follows from VI.

Remark: Wall [2] proves a similar theorem for the lengths of periods of the Fibonacci sequence reduced $\bmod m$ (not limited to odd numbers). But this does not point at a special similarity between the Fibonacci sequence and the geometric sequence $1,2,4, \ldots$. In [3] I suggested a generalization of both Wall's theorem and Proposition 2. Let $a(i)$ be any sequence such that reducing its elements modulo $m$ gives, for some $m$ 's, a periodic sequence (the period does not have to start at the very beginning). Let $P(m)$ be the length of the period, and let $m_{1}$ and $m_{2}$ be any two numbers for which $P$ is defined. Then $P\left(1 . \mathrm{c.m} .\left(m_{1}, m_{2}\right)\right)=1$.c.m. $\left(P\left(m_{1}\right), P\left(m_{2}\right)\right)$. This result, like Theorem 1, emerged from a suggestion of a student of mine (in a mathematics club for high school students).

Proposition 3: If $m_{1}$ and $m_{2}$ are prime to each other and to 2 and different from 1 , then $C\left(m_{1} \cdot m_{2}\right)$ is an even number.

Proof: Let us recall two properties of the Euler $\varphi$ function: (a) If $n_{1}$ and $n_{2}$ are relatively prime, then $\varphi\left(n_{1} \cdot n_{2}\right)=\varphi\left(n_{1}\right) \cdot \varphi\left(n_{2}\right)$. (b) If $n \neq 2$, then $\varphi(n)$ is an even number.

Now,

$$
\begin{aligned}
C\left(m_{1} \cdot m_{2}\right) & =\varphi\left(m_{1} \cdot m_{2}\right) / L\left(m_{1} \cdot m_{2}\right) \\
& =\varphi\left(m_{1}\right) \cdot \varphi\left(m_{2}\right) / \text {.. c. m. }\left(L\left(m_{1}\right), L\left(m_{2}\right)\right) \\
& =\varphi\left(m_{1}\right) / L\left(m_{1}\right) \cdot \varphi\left(m_{2}\right) / L\left(m_{2}\right) \cdot \text { g.c. d. }\left(L\left(m_{1}\right), L\left(m_{2}\right)\right) \\
& =C\left(m_{1}\right) \cdot C\left(m_{2}\right) \cdot \text { g.c.d. }\left(L\left(m_{1}\right), L\left(m_{2}\right)\right) .
\end{aligned}
$$

At least one of the last three factors is an even number since, if $C(m)$ is an odd number, then $L(m)$, which equals $\varphi(m) / C(m)$, is an even number.

Lemma 2: If $m_{1}$ and $m_{2}$ are as in Proposition 3 and are both O.K., then so is $m=m_{1} \cdot m_{2}$.
Proof: By V, those vertices in the diagram mod $m$ that are multiples of $m_{1}$ form $T\left(m_{2}\right)$ cycles, and the multiples of $m_{2}$ form $T\left(m_{1}\right)$ cycles. Together they form $T\left(m_{1}\right)+T\left(m_{2}\right)-1$ cycles, since the cycle of 0 is the only one that is counted both in $T\left(m_{1}\right)$ and in $T\left(m_{2}\right)$.

Let us partition the other vertices into classes in the following way: For each pair $d_{1}$ and $d_{2}$ that are proper divisors of $m_{1}$ and $m_{2}$, respectively, let us form the class of all the vertices that are
multiples of $d_{1} \cdot d_{2}$ but not of any greater factor of $m$. We are going to show that the elements of such a class form an even number of cycles. Indeed, if we divide the elements of the class by $d_{1} \cdot d_{2}$, we get the vertices of the doubling diagram modulo $m_{1} / d_{1}$ and $m_{2} / d_{2}$ satisfy the conditions of Proposition 3.

It follows that $T(m)$ is an even number $\Leftrightarrow T\left(m_{1}\right)+T\left(m_{2}\right)-1$ is an even number $\Leftrightarrow$ just one of $T\left(m_{1}\right), T\left(m_{2}\right)$ is an even number $\Leftrightarrow$ just one of $m_{1}, m_{2}$ is $\equiv \pm 3(\bmod 8) \Leftrightarrow m \equiv \pm 3(\bmod 8)$.

Lemma 3: Every prime number $p \neq 2$ is O.K.
Proof: $p$ divides $2^{p-1}-1$. Therefore, it divides either $2^{(p-1) / 2}-1$ or $2^{(p-1) / 2}+1$.
If $p$ divides $2^{(p-1) / 2}+1$, then $(p-1) / 2$ arrows of the diagram $\bmod p$ lead from 1 to -1 (more precisely, to $p-1$ ). In one turn around the cycle of 1 , the number of arrows from 1 to -1 is equal to the number of arrows from -1 to 1 , so $(p-1) / 2$ arrows make an odd number of half-turns around this cycle [that is, $(p-1) / 2=$ an odd number $\cdot L(p) / 2$ ]. Since $C(p)=(p-1) / L(p)=$ $((p-1) / 2) /(L(p) / 2)$, it is an odd number and since, for a prime $p, T(p)=C(p)+1$, it follows that in our case $T(p)$ is an even number.

If $p$ divides $2^{(p-1) / 2}-1$, then $(p-1) / 2$ arrows lead from 1 to 1 , hence $(p-1) / 2$ is a multiple of $L(p)$, hence $C(p)=(p-1) / L(p)$ is an even number, so $T(p)$ is an odd number.

To complete our proof, we have to show that $\left.p\right|^{(p-1) / 2}-1 \Leftrightarrow p \equiv \pm 1(\bmod 8)$.
Corollary 2.28 (or Theorem 3.1a) in Niven-Zuckerman [4], with $a=2$, says that $p$ divides $2^{(p-1) / 2}-1$ iff there is a solution for $x^{2} \equiv 2(\bmod p)$. Problem 10 on page 73 (solved by the last part of Theorem 3.3$)$ says that $x^{2} \equiv 2(\bmod p)$ has a solution iff $p \equiv \pm 1(\bmod 8)$.

Proof of Theorem 1: By Lemma 3, Lemma 1, and Lemma 2.

## 3. ANOTHER POINT OF VIEW AND ANOTHER THEOREM

An exercise in long division in base 2 will show that $L(m)$ is the length of the period of the binacy fraction for $1 / m$. Moreover, $C(m)$ is the number of classes of fractions-in-lower-terms with the denominator $m$ and with binary expansions whose periods are equal to each other up to a cyclic permutation, while $T(m)$ may be described in the same way, omitting the words "in-lowerterms."

The analog of Theorem 1 for the base 10 is the following:
Theorem 2: Let $m$ be relatively prime to 10 , and consider the number of different periods in the decimal expansions of fractions with the denominator $m$. This number is an odd number iff $m \equiv \pm 1$ or $\pm 3$ or $\pm 9$ or $\pm 27(\bmod 40)$.

The proof is similar to that of Theorem 1 with some self-evident modifications, but two additional lemmas are needed. For convenience, I am going to write " $m$ is like 1 " for $m \equiv \pm 1$ or $\pm 3$ or $\pm 9$ or $\pm 27(\bmod 40)$, and $" m$ is like 7 " for $m \equiv \pm 7$ or $\pm 11$ or $\pm 17$ or $\pm 19(\bmod 40)$.

Lemma 4: The product of two numbers like 1 and the product of two numbers like 7 are like 1 ; the product of a number like 1 and a number like 7 is like 7 .

Proof: By checking the different cases.

The next lemma is needed for the last half of the proof of the base- 10 version of Lemma 1.
Lemma 5: Let $m$ be a natural number prime to 10. For each integer $i$ from 0 to ( $m-1$ )/2, let us write $r_{i}$ for the residue of $10 i$ when reduced $\bmod m$, and let $n$ be the number of $r_{i}$ 's that are greater than $m / 2$. With this notation, $n$ is an even number iff $m$ is like 1 .

Proof: Numerical checks show that the lemma holds for every $m<50$. We have to demonstrate that, if the lemma holds for some $m>10$, then it holds also for $m+40$. Let us assume $m \geq 11$.

Consider the sequence $0,10,20, \ldots, 5 m-5$. Reducing its elements modulo $m$ to get their $r_{i}$ 's consists of five stages: In the first stage we subtract $0 \cdot m$, in the second stage $1 \cdot m$, and so on until the fifth stage, where we subtract $4 m$ 's. Each stage starts by yielding an $r_{i}$ of one digit, followed by all the other numbers less than $m$, which end with that digit. (The fifth stage is not interrupted by the end of the sequence, since adding 10 to the last element gives a number $>5 \mathrm{~m}$.)

The $r_{i}$ 's we get in this way are different from each other, since $m$ is relatively prime to 10 , so they consist of all the integers from 0 to $m-1$, having one of certain five digits for their last digit. Consequently, every ten successive integers in $[0, m-1]$ include exactly five $r_{i}$ 's.

Replacing $m$ by $m^{\prime}=m+40$ does not change the above-mentioned set of five digits since, if $10 i-j m=r$ with $1 \leq(m-1) / 2$ and $j \leq 4$, then $10(i+4 j)-j m^{\prime}=r$ and $i+4 j \leq\left(m^{\prime}-1\right) / 2$. The set of the $r_{i}$ 's associated with $m^{\prime}$ that are greater than $m^{\prime} / 2$ include the old $r_{i}$ 's that are greater than $m / 2$, plus twenty new $r_{i}$ 's bigger than $m-1$, less ten $r_{i}$ 's that are between $m / 2$ and $m^{\prime} / 2$.

It follows that the $n$ associated with $m^{\prime}$ is an even number iff the $n$ associated with $m$ is an even number.

This lemma, together with Theorem 3.2 of [4] (a lemma of Gauss), are used instead of Problem 10 at the end of the proof of the base-10 version of Lemma 3. Theorem 3.2 says, for $a=10$, that if $m$ is a prime number different from 2 and from 5 , then the congruence $x^{2} \equiv 10$ $(\bmod m)$ has a solution iff the $n$ we have defined in Lemma 2 is an even number. Lemma 5 itself now completes the proof.

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# CONGRUENCES FOR A WIDE CLASS OF INTEGERS BY USING GESSEL'S METHOD 

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## 1. INTRODUCTION AND PREPARATORY RESULTS

Let $P_{n}, n=0,1,2, \ldots$, be a sequence of integers that is defined by its exponential generating function $f(x)$ as

$$
\begin{equation*}
\sum_{n=0}^{\infty} P_{n} x^{n} / n!=f(x) \tag{1}
\end{equation*}
$$

That is $f(x)$ is a Hurwitz series in $x$.
As regards Bell numbers $[f(x)=\exp \{\exp \{x\}-1\}]$, Lunnon, Pleasants, \& Stephens [4] and Gessel [1] showed that, for each positive integer $n$, there exist integers $a_{0}, a_{1}, \ldots, a_{n-1}$ such that, for all $m \geq 0, n \geq 0$,

$$
\begin{equation*}
P_{m+n}+a_{n-1} P_{m+n-1}+\cdots+a_{0} P_{m} \equiv 0(\bmod n!) \tag{2}
\end{equation*}
$$

Also, as regards tangent numbers $[f(x)=\tan x]$, Ira Gessel [1] showed that, for each positive integer $n$, there exist integers $b_{1}, b_{2}, \ldots, b_{n-1}$ such that, for all $m \geq 0, n \geq 1$,

$$
P_{m+n}+b_{n-1} P_{m+n-1}+\cdots+b_{1} P_{m+1} \equiv 0(\bmod (n-1)!n!)
$$

In the same paper, congruences similar to the above are obtained concerning the derangement num-bers and the numbers defined by $f(x)=(2-\exp \{x\})^{-1}$ and $f(x)=\exp \left\{x+x^{2} / 2\right\}$. In the same area of research, Kyriakoussis [3] proved the congruence (2) in the case in which

$$
f(x)=\exp \{g(x)\}, \text { for } g(x)=\sum_{j=1}^{\infty} c_{j} x^{j} / j
$$

where the $c_{j}, j=1,2, \ldots$, are integers. In [1], Gessel obtained the above congruence by introducing the following method:

Using Taylor's theorem and (1), we have

$$
\begin{equation*}
f(x+y)=\sum_{k=0}^{\infty} f^{(k)}(x) y^{k} / k!, \quad f^{(k)}(x)=\frac{d^{k} f(x)}{d x^{k}} \tag{3}
\end{equation*}
$$

Setting $y=S(z)$ in (3), where the function $S(z)$ is a Hurwitz series in $z$ with $S(0)=0$ and $S^{\prime}(0)=1$ and multiplying both sides by some Hurwitz series $H(z)$ with $H(0)=1$, we get

$$
H(z) f(x+S(z))=\sum_{k=0}^{\infty} f^{(k)}(x) H(z)(S(z))^{k} / k!
$$

If the functions $H(z)$ and $S(z)$ are chosen appropriately, the coefficients of $\frac{x^{m}}{m!} z^{n}$ on the left will be integral. Then the coefficients of $\frac{x^{m}}{m!} \frac{z^{n}}{n!}$ on the right is divisible by $n!$.

In other words, Gessel's method can be applied to a given Hurwitz series $f(x)$ if and only if there exist Hurwitz series $S(z)$ and $H(z)$ with $S(0)=0, S^{\prime}(0)=1$, and $H(0)=1$, such that, for all integers $m$ and $n$, the coefficients of $\frac{x^{m}}{m!} z^{n}$ in $H(z) f(x+S(z))$ is an integer. That is,

$$
\begin{equation*}
H(z) f(x+S(z))=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Q(m, n) \frac{x^{m}}{m!} z^{n}, \tag{4}
\end{equation*}
$$

where the numbers $Q(m, n)$ are integers for all $m$ and $n$.
In this paper we establish a necessary and sufficient condition on the function $f(x)$, given by (1), for Gessel's method to be applied, and we show the corresponding congruence concerning the numbers $P_{n}, n=0,1,2, \ldots$. Moreover, we consider a wide class of functions $f(x)$ to which Gessel's method can be applied.

It is well known that Hurwitz series are closed under multiplication and that, if $f(x)$ and $g(x)$ are Hurwitz series with $g(0)=0$, then the composition $(f \circ g)(x)$ is also a Hurwitz series. In particular, $(g(x))^{k} / k$ ! is a Hurwitz series for any nonnegative integer $k$.

Hurwitz series in two variables are of the form

$$
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m n} \frac{x^{m}}{m!} \frac{y^{n}}{n!},
$$

where the numbers $a_{m n}$ are integers. The properties of these series we will need follow from those for Hurwitz series in one variable.

We also need the following results:
a. Let $s^{-1}(x)$ be the inverse function of the Hurwitz series $s(x)$ with $s(0)=0$. Then $s^{-1}(x)$ is also a Hurwitz series with $s^{-1}(0)=0$, if $\left.\frac{d}{d x} s(x)\right|_{x=0}=s^{\prime}(0)=1$.
b. Let $h(x)$ be a Hurwitz series. Then the function $\frac{1}{h(x)}=(h(x))^{-1}$ is a Hurwitz series if and only if $h(0)=1$.

## 2. THE MAIN RESULTS

A necessary and sufficient condition for Gessel's method to be applied is given by the following theorem.

Theorem 1: Let $f(x)$ be the exponential generating function of the integers $P_{n}, n=0,1,2, \ldots$, as given by (1). Then Gessel's method can be applied to the Hurwitz series $f(x)$ if and only if there exist Hurwitz series $s(y)$ and $h(y)$ with $s(0)=0, s^{\prime}(0)=1$, and $h(0)=1$, such that

$$
\begin{equation*}
f(x+y)=h(y)\left[\sum_{n=0}^{\infty} G_{n}(x)(s(y))^{n}\right], \tag{5}
\end{equation*}
$$

where the functions $G_{n}(x), n=0,1,2, \ldots$, are Hurwitz series in $x$.
Proof: From relation (4) we can easily obtain relation (5), setting $z=s(y)$ where $s$ is the inverse function of $S,[H(s(y))]^{-1}=h(y)$ and $\sum_{m=0}^{\infty} Q(m, n) x^{m} / m!=G_{n}(x)$

From our comments in section 1, we can easily see that $s(y)$ and $h(y)$ are Hurwitz series in $y$ with $s(0)=0, s^{\prime}(0)=1$, and $h(0)=1$. Conversely, from relation (5) we obtain, in the same way, relation (4).

Example 1: $f(x)=\tan x$ and we have

$$
f(x+y)=\tan x+(\sec x)^{2} \sum_{n=1}^{\infty}(\tan x)^{n-1}(\tan y)^{n}
$$

Consequently, $h(y)=1, G_{0}(x)=\tan x, G_{n}(x)=\sec ^{2} x(\tan x)^{n-1}, n=1,2, \ldots, s(y)=\tan y, s^{-1}(z)=$ $\arctan z$, and Theorem 1 can be applied.

Now we show the corresponding congruence concerning the numbers $P_{n}, n=0,1,2, \ldots$.
From relations (1) and (3), we obtain

$$
\begin{equation*}
f(x+y)=\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} P_{m+k} \frac{x^{m}}{m!} \frac{y^{k}}{k!} \tag{6}
\end{equation*}
$$

Comparing relations (5) and (6), we obtain

$$
\begin{equation*}
\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} P_{m+k} \frac{x^{m}}{m!} \frac{y^{k}}{k!}=h(y)\left[\sum_{n=0}^{\infty} G_{n}(x)(s(y))^{n}\right] \tag{7}
\end{equation*}
$$

Setting $y=s^{-1}(z)$ in (7) and multiplying both sides by $\left(h\left(s^{-1}(z)\right)\right)^{-1}$, we obtain

$$
\begin{align*}
\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} P_{m+k} \frac{x^{m}}{m!}\left(h\left(s^{-1}(z)\right)\right)^{-1} \frac{\left(s^{-1}(z)\right)^{k}}{k!} & =\sum_{n=0}^{\infty} G_{n}(x) z^{n} \\
& =\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} n!Q(m, n) \frac{x^{m}}{m!} \frac{z^{n}}{n!} \tag{8}
\end{align*}
$$

where the integers $Q(m, n)$ are given by the relation

$$
\begin{equation*}
\sum_{m=0}^{\infty} Q(m, n) x^{m} / m!=G_{n}(x) \tag{9}
\end{equation*}
$$

From our comments in section 1 , we can define the integers $D(n, k), k=0,1, \ldots, n, n=0,1,2, \ldots$, by

$$
\begin{equation*}
\sum_{n=k}^{\infty} D(n, k) z^{n} / n!=\left(h\left(s^{-1}(z)\right)\right)^{-1} \frac{\left(s^{-1}(z)\right)^{k}}{k!} \tag{10}
\end{equation*}
$$

Substituting (10) into (8) we get, on using the relation $D(0,0)=1$,

$$
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} D(n, k) P_{m+k}\right) \frac{x^{m}}{m!} \frac{z^{n}}{n!}=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} n!Q(m, n) \frac{x^{m}}{m!} \frac{z^{n}}{n!}
$$

Equating coefficients of $\frac{x^{m}}{m!} \frac{z^{n}}{n!}$, we get

$$
\begin{equation*}
\sum_{k=0}^{n} D(n, k) P_{m+k}=n!Q(m, n) \tag{11}
\end{equation*}
$$

Now we consider a wide class of Hurwitz series $f(x)$ to which Gessel's method can be applied, by the following theorems.

Theorem 2: Gessel's method can be applied to the Hurwitz series $f(x)$, if

$$
\begin{equation*}
f(x)=(1+\beta g(x))^{\alpha} e^{\gamma x} \tag{12}
\end{equation*}
$$

where the constants $\alpha, \beta$, and $\gamma$ are integers and the function $g(x)$ is a Hurwitz series such that

$$
\begin{equation*}
g(x+y)=\sum_{n=0}^{\infty} H_{n}(x)(s(y))^{n}, H_{0}(0)=0, \tag{13}
\end{equation*}
$$

where the function $s(y)$ is a Hurwitz series in $y$ with $s^{\prime}(0)=1, s(0)=0$, and the functions $H_{n}(x)$, $n=0,1,2, \ldots$, are Hurwitz series in $x$.

Proof: From relation (12) we have, on using (13) and some well-known rules of multiplication of series

$$
\begin{aligned}
f(x+y) & =\left[1+\beta g(x+y)^{\alpha} e^{\gamma(x+y)}\right]=\left[1+\beta H_{0}(x)+\beta \sum_{n=1}^{\infty} H_{n}(x)(s(y))^{n}\right]^{\alpha} e^{\gamma(x+y)} \\
& =\left[1+\sum_{n=1}^{\infty} H_{n}^{*}(x)(s(y))^{n}\right]^{\alpha}\left[1+\beta H_{0}(x)\right]^{\alpha} e^{\gamma(x+y)},
\end{aligned}
$$

where $H_{n}^{*}(x)=\beta H_{n}(x) /\left[1+\beta H_{0}(x)\right]$ or

$$
\begin{aligned}
f(x+y) & =\left[1+\beta H_{0}(x)\right]^{\alpha} e^{\gamma(x+y)} \sum_{j=0}^{\infty}\binom{\alpha}{j}\left[\sum_{n=1}^{\infty} H_{n}^{*}(x)(s(y))^{n}\right]^{j} \\
& =\left[1+\beta H_{0}(x)\right]^{\alpha} e^{\gamma(x+y)}\left\{1+\sum_{j=1}^{\infty}\binom{\alpha}{j}\left[\sum_{m=j}^{\infty}(s(y))^{m} \sum H_{n_{1}}^{*}(x) H_{n_{2}}^{*}(x) \cdots H_{n_{j}}^{*}(x)\right]\right\},
\end{aligned}
$$

where the inner sum is extended over all ordered $j$-tuples $\left(n_{1}, n_{2}, \ldots, n_{j}\right)$ of positive integers such that $n_{1}+n_{2}+\cdots+n_{j}=m$ or

$$
f(x+y)=e^{p y}\left[1+\beta H_{0}(x)\right]^{\alpha} e^{\gamma x}\left\{1+\sum_{m=1}^{\infty}\left[\sum_{j=1}^{m}\binom{\alpha}{j} \sum H_{n_{1}}^{*}(x) \cdots H_{n_{j}}^{*}(x)\right](s(y))^{m}\right\}
$$

or

$$
\begin{equation*}
f(x+y)=h(y) \sum_{m=0}^{\infty} G_{m}(x)(s(y))^{m}, \tag{14}
\end{equation*}
$$

where $h(y)=e^{\eta y}, G_{0}(x)=\left[1+\beta H_{0}(x)\right]^{\alpha} e^{\gamma x}$ and

$$
G_{m}(x)=\left[1+\beta H_{0}(x)\right]^{\alpha} e^{\jmath x}\left[\sum_{j=1}^{m}\binom{\alpha}{j} \sum H_{n_{1}}^{*}(x) \cdots H_{n_{j}}^{*}(x)\right], m=1,2, \ldots
$$

(the inner sum is extended as before).

## CONGRUENCES FOR A WIDE CLASS OF INTEGERS BY USING GESSEL'S METHOD

Since $G_{m}(x), m=0,1,2, \ldots$, are Hurwitz series in $x$ and $s(y), h(y)$ are Hurwitz series in $y$ with $s(0)=0, s^{\prime}(0)=1$, and $h(0)=1$, we have, on using relation (14) and Theorem 1 , that Gessel's method can be applied.

Example 2: $f(x)=(1+\beta \tan x)^{\alpha}, \alpha$ an integer. We have $\gamma=0, g(x)=\tan x$, and

$$
g(x+y)=\tan x+\left(1+\tan ^{2} x\right) \sum_{n=1}^{\infty}(\tan x)^{n-1}(\tan y)^{n}
$$

Consequently, $s(y)=\tan y, H_{0}(x)=\tan x, H_{n}(x)=\left(1+\tan ^{2} x\right)(\tan x)^{n-1}, n=1,2, \ldots$, and Theorem 2 can be applied.

Example 3: $f(x)=e^{\imath x}\left(1-\beta\left(e^{x}-1\right)\right)^{-\alpha}$, where $\alpha, \beta, \gamma$ are integers. We have $g(x)=-\left(e^{x}-1\right)$ and $g(x+y)=-\left(e^{x}-1\right)-e^{x}\left(e^{y}-1\right)$ Hence, $s(y)=e^{y}-1, H_{0}(x)=-\left(e^{x}-1\right), H_{1}(x)=-e^{x}$, and Theorem 2 can be applied.

Note that the above function $f(x)$ is the exponential generating function for the moments for the Meixner polynomials.

Theorem 3: Gessel's method can be applied to the Hurwitz series $f(x)$ if

$$
\begin{equation*}
f(x)=\exp \{F(x)\} \tag{15}
\end{equation*}
$$

where $F(x)$ is a Hurwitz series in $x$ such that

$$
\begin{equation*}
F(x+y)=L(x)+\sum_{j=0}^{\infty} R_{j}(y)(r(x))^{j} / j! \tag{16}
\end{equation*}
$$

where $L(x)$ is a Hurwitz series in $x$ with $L(0)=0, R_{0}(y)$ is a Hurwitz series in $y$ with $R_{0}(0)=0$, $R_{j}(y), j=1,2, \ldots$, are power series in $s(y)$ with integer coefficients, $s(y)$ is a Hurwitz series in $y$ with $s(0)=0, s^{\prime}(0)=1$, and $r(x)$ is a Hurwitz series in $x$ with $r(0)=0$.

Proof: Introducing the exponential Bell polyomials $B_{n}=B_{n}\left(b_{1}, b_{2}, \ldots, b_{n}\right), n=0,1,2, \ldots$, that may be defined by their exponential generating function as

$$
\sum_{n=0}^{\infty} B_{n} t^{n} / n!=\exp \{\phi(t)\}
$$

where $\phi(t)=\sum_{j=1}^{\infty} b_{j} t^{j} / j$ !, we get

$$
\begin{equation*}
\exp \left\{\sum_{j=1}^{\infty} R_{j}(y)(r(x))^{j} / j!\right\}=\sum_{n=0}^{\infty} B_{n}\left(R_{1}(y), \ldots, R_{n}(y)\right)(r(x))^{n} / n! \tag{17}
\end{equation*}
$$

Explicit expressions for $B_{n}=B_{n}\left(b_{1}, b_{2}, \ldots, b_{n}\right)$ as functions of $b_{1}, b_{2}, \ldots, b_{n}$ are given in Kendall \& Stuart ([2], p. 69).

Since $R_{j}(y), j=1,2, \ldots$, are power series in $s(y)$, we have that $B_{n}, n=1,2, \ldots$, are also power series in $s(y)$. Therefore,

$$
\begin{equation*}
B_{n}\left(R_{1}(y), \ldots, R_{n}(y)\right)=\sum_{i=0}^{\infty} \alpha_{n, i}(s(y))^{i}, n=1,2, \ldots \tag{18}
\end{equation*}
$$

where the numbers $\alpha_{n, i}, i=0,1,2, \ldots$, are integers.

From relation (15) we have, on using relations (18), (17), and (16),

$$
\begin{equation*}
f(x+y)=h(y) \sum_{i=0}^{\infty} G_{i}(x)(s(y))^{i} \tag{19}
\end{equation*}
$$

where $h(y)=\exp \left[R_{0}(y)\right]$ and $G_{i}(x)=\{\exp [L(x)]\} \sum_{n=0}^{\infty} \alpha_{n, i}(r(x))^{n} / n!, i=0,1,2, \ldots$. Since $R_{0}(0)=L(0)=r(0)=0$, we have that $h(y)$ is a Hurwitz series in $y$ with $h(0)=1$ and $G_{i}(x), i=0,1$, $2, \ldots$, are Hurwitz series in $x$. We also have $s(0)=0$ and $s^{\prime}(0)=1$. Consequently, using Theorem 1, we conclude that Gessel's method can be applied.

Example 4: $f(x)=\exp \left\{\sum_{i=1}^{\infty} c_{i} x^{i} / i\right\}, c_{i}, i=1,2, \ldots$, integers. We have $F(x)=\sum_{i=1}^{\infty} c_{i} x^{i} / i$ and

$$
\begin{aligned}
F(x+y) & =\sum_{i=1}^{\infty} c_{i}(x+y)^{i} / i=\sum_{i=1}^{\infty}\left(c_{i} / i\right) \sum_{j=0}^{i}\binom{i}{j} x^{j} y^{i-j} \\
& =F(x)+F(y)+\sum_{i=2}^{\infty}\left(c_{i} / i\right) \sum_{j=1}^{i-1}\binom{i}{j} x^{j} y^{i-j}=F(x)+F(y)+\sum_{j=1}^{\infty} R_{j}(y) x^{j} / j!,
\end{aligned}
$$

where

$$
R_{j}(y)=\sum_{i=j+1}^{\infty} c_{i} \frac{(i-1)!}{(i-j)!} y^{i-j}=\sum_{i=1}^{\infty}\left[c_{i+j}(i+j-1)(j-1)!\right] y^{i}, j=1,2, \ldots .
$$

Thus, $L(x)=F(x), R_{0}(y)=F(y), R_{j}(y), j=1,2, \ldots$, are power series in $y$ with integer coefficients, $s(y)=y, r(x)=x$, and Theorem 3 can be applied.

Note that, for $c_{i}=0, i=3,4, \ldots$, the above $f(x)$ is the exponential generating function for the moments for the Hermite polynomials.
Example 5: $f(x)=\exp \left\{\alpha\left(e^{x}-1\right)-\beta x\right\}, \alpha$ and $\beta$ integers. We have $F(x)=\alpha\left(e^{x}-1\right)+\beta x$ and $F(x+y)=\alpha\left(e^{x+y}-1\right)+\beta(x+y)=F(x)+F(y)+\left(e^{y}-1\right) \alpha\left(e^{x}-1\right)$. Consequently, $L(x)=F(x)$, $R_{0}(y)=F(y), R_{1}(y)=e^{y}-1, s(y)=e^{y}-1, r(x)=\alpha\left(e^{x}-1\right)$, and Theorem 3 can be applied.

Note that, for $\beta=0$, the above $f(x)$ is the generating function for the moments for the Charlier polynomials.

## ACKNOWLEDGMENT

The author wishes to thank the referee for helpful suggestions.

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2. M. G. Kendall \& A. Stuart. The Advanced Theory of Statistics. Vol. I: Distribution Theory. London: Griffin, 1969.
3. A. Kyriakoussis. "A Congruence for a Class of Exponential Numbers." The Fibonacci Quarterly 23.1 (1985):45-48.
4. W. F. Lunnon, P. A. B. Pleasants, \& N. M. Stephens. "Arithmetic Properties of Bell Numbers to a Composite Modulus I." Acta Arith. 35 (1979):1-16.
AMS Classification Numbers: 11A07, 05A15

# ELEMENTARY PROBLEMS AND SOLUTIONS 

## Edited by

## Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to $72717.3515 @$ compuserve.com on Internet. All correspondence will be acknowledged.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

## BASIC FORMULAS

The Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ satisfy

$$
\begin{aligned}
& F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1 ; \\
& L_{n+2}=L_{n+1}+L_{n}, L_{0}=2, L_{1}=1 .
\end{aligned}
$$

Also, $\alpha=(1+\sqrt{5}) / 2, \beta=(1-\sqrt{5}) / 2, F_{n}=\left(\alpha^{n}-\beta^{n}\right) / \sqrt{5}$, and $L_{n}=\alpha^{n}+\beta^{n}$.
The Pell numbers $P_{n}$ and their associated numbers $Q_{n}$ satisfy

$$
\begin{aligned}
& P_{n+2}=2 P_{n+1}+P_{n}, P_{0}=0, P_{1}=1 ; \\
& Q_{n+2}=2 Q_{n+1}+Q_{n}, Q_{0}=1, Q_{1}=1
\end{aligned}
$$

If $p=1+\sqrt{2}$ and $q=1-\sqrt{2}$, then $P_{n}=\left(p^{n}-q^{n}\right) / \sqrt{8}$ and $Q_{n}=\left(p^{n}+q^{n}\right) / 2$. The Pell-Lucas numbers, $R_{n}$, are given by $R_{n}=2 Q_{n}$. For more information about Pell numbers, see Marjorie Bicknell, "A Primer on the Pell Sequences and Related Sequences," The Fibonacci Quarterly 13.4 (1975):345-49.

## PROBLEMS PROPOSED IN THIS ISSUE

The problems in this issue all involve Pell numbers. See the basic formulas above for definitions.

## B-754 Proposed by Joseph J. Kostal, University of Illinois at Chicago, IL

Find closed form expressions for

$$
\sum_{k=1}^{n} P_{k} \quad \text { and } \quad \sum_{k=1}^{n} Q_{k} .
$$

## B-755 Proposed by Russell Jay Hendel, Morris College, Sumter, SC

Find all nonnegative integers $m$ and $n$ such that $P_{n}=Q_{m}$.

## B-756 Proposed by the editor

Find a formula expressing $P_{n}$ in terms of Fibonacci and/or Lucas numbers.

## B-757 Proposed by H.-J. Seiffert, Berlin, Germany

Show that for $n>0$,
(a)

$$
P_{3 n-1} \equiv F_{n+2}(\bmod 13)
$$

(b)

$$
P_{3 n+1} \equiv(-1)^{\lfloor(n+1) / 2\rfloor} F_{4 n-1}(\bmod 7)
$$

B-758 Proposed by Russell Euler, Northwest Missouri State University, Maryville, MO
Evaluate

$$
\sum_{k=0}^{\infty} \frac{k 2^{k} Q_{k}}{5^{k}}
$$

## B-759 Proposed by H.-J. Seiffert, Berlin, Germany

Show that for all positive integers $k$ and all nonnegative integers $n$,

$$
\sum_{j=0}^{n} F_{k(j+1)} P_{k(n-j+1)}=\frac{F_{k} P_{k(n+2)}-P_{k} F_{k(n+2)}}{2 Q_{k}-L_{k}}
$$

## SOLUTIONS

## A 7-Term Arithmetic Progression

## B-724 Proposed by Larry Taylor, Rego Park, NY

(Vol. 30, no. 4, November 1992)
Let $n$ be a positive integer. Prove that the numbers $L_{n-1} L_{n+1}, 5 F_{n}^{2}, L_{3 n} / L_{n}, L_{2 n}, F_{3 n} / F_{n}, L_{n}^{2}$, $5 F_{n-1} F_{n+1}$ are in arithmetic progression and find the common difference.

## Solution by Y. H. Harris Kwong, SUNY at Fredonia, NY

Using the basic formulas $F_{n}=\left(\alpha^{n}-\beta^{n}\right) / \sqrt{5}, L_{n}=\alpha^{n}+\beta^{n}$ and the identities $\alpha^{2}+\beta^{2}=3$, $\alpha \beta=-1$, it is easy to show that the seven numbers form the arithmetic progression $L_{2 n}+k(-1)^{n}$, $k=-3,-2, \ldots, 2,3$, with common difference $(-1)^{n}$.

For example,

$$
L_{n-1} L_{n+1}=\left(\alpha^{n-1}+\beta^{n-1}\right)\left(\alpha^{n+1}+\beta^{n+1}\right)=\alpha^{2 n}+\beta^{2 n}+(\alpha \beta)^{n-1}\left(\alpha^{2}+\beta^{2}\right)=L_{2 n}-3(-1)^{n}
$$

The other parts follow in a similar manner.
Most solutions were similar. The arithmetic progression can also be expressed as $L_{n}^{2}+k(-1)^{n}$, $k=-5,-4,-3, \ldots, 1$.

Also solved by M. A. Ballieu, Seung-Jin Bang, Brian D. Beasley, Scott H. Brown, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, C. Georghiou, Pentti Haukkanen, John Ivie, Russell Jay Hendel, Joseph J. Kostal, Carl Libis, Graham Lord, Igor Ol. Popov, Bob Prielipp, H.-J. Seiffert, A. G. Shannon, Sahib Singh, Lawrence Somer, Ralph Thomas, and the proposer.

## An Infinite Set of Right Triangles

B-725 Proposed by Russell Jay Hendel, Patchogue, NY and Herta T. Freitag, Roanoke, VA (Vol. 30, no. 4, November 1992)
(a) Find an infinite set of right triangles each of which has a hypotenuse whose length is a Fibonacci number and an area that is the product of four Fibonacci numbers.
(b) Find an infinite set of right triangles each of which has a hypotenuse whose length is the product of two Fibonacci numbers and an area that is the product of four Lucas numbers.

## Solution by the proposers

Recall that $A=x^{2}-y^{2}, B=2 x y$, and $C=x^{2}+y^{2}$ form a Pythagorean triangle with area $x y(x-y)(x+y)$.
(a) Let $x=F_{n}, y=F_{n-1}$, and use the fact that $F_{n}^{2}+F_{n-1}^{2}=F_{2 n-1}$ (see [1]).
(b) Let $x=L_{n+1}, y=L_{n}$, and use the fact that $L_{n}^{2}+L_{n+1}^{2}=F_{5} F_{2 n+1}$ (see [1]).

## Reference:

1. S. Vajda. Fibonacci \& Lucas Numbers, and the Golden Section: Theory and Applications. Chichester: Ellis Horwood Ltd., 1989, p. 29.

The proposers also found an infinite set of right triangles whose hypotenuse is a Pell number and whose area is the product of four Pell numbers. Shannon noted that if $\left(H_{n}\right)$ is any sequence that satisfies the recurrence $H_{n}=H_{n-1}+H_{n-2}$, then the triangle with sides $H_{n} H_{n+3}, 2 H_{n+1} H_{n+2}$, and $2 H_{n+1} H_{n+2}+H_{n}^{2}$ is a Pythagorean triangle with area $H_{n} H_{n+1} H_{n+2} H_{n+3}$. However, he was unable to put the length of the hypotenuse, $2 H_{n+1} H_{n+2}+H_{n}^{2}$, into a simpler form.
Also solved by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Daniel C. Fielder \& Cecil O. Alford, C. Georghiou, Igor Ol. Popov, Bob Prielipp, H.-J. Seiffert, A. G. Shannon, Sahib Singh, and Lawrence Somer.

## A Diverting Sum

## B-726 Proposed by Florentin Smarandache, Phoenix, $A Z$

(Vol. 30, no. 4, November 1992)
Let $d_{n}=P_{n+1}-P_{n}, n=1,2,3, \ldots$, where $P_{n}$ is the $n^{\text {th }}$ prime. Does the series

$$
\sum_{n=1}^{\infty} \frac{1}{d_{n}}
$$

converge?
Solution by C. Georghiou, University of Patras, Greece
The series diverges! This can be seen by noticing that

$$
d_{n}=P_{n+1}-P_{n}<P_{n+1} .
$$

We use the well-known fact ([1], p. 17) that the series of the reciprocals of the prime numbers diverges and the standard Comparison Test ([2], p. 777) which says that if $\sum a_{k}$ diverges and $b_{k}>a_{k}>0$ for all $k$, then $\sum b_{k}$ diverges.

## References:

1. G. H. Hardy \& E. M. Wright. An Introduction to the Theory of Numbers. 5th ed. Oxford: Oxford University Press, 1979.
2. George B. Thomas. Calculus and Analytic Geometry. 3rd ed. Reading, MA.: AddisonWesley, 1960.

Several solvers invoked Bertrand's Postulate ([1], p. 343). Seiffert asks if the series

$$
\sum_{n=1}^{\infty} \frac{(-1)^{n}}{d_{n}}
$$

diverges. He notes that this would follow from the twin prime conjecture ([1], p. 5).
Also solved by Charles E. Chace \& Russell Jay Hendel, Leonard A. G. Dresel, Piero Filipponi, H.-J. Seiffert, Sahib Singh, and the proposer.

## It's a Tanh

## B-727 Proposed by Ioan Sadoveanu, Ellensburg, WA (Vol. 30, no. 4, November 1992)

Find the general term of the sequence $\left(a_{n}\right)$ defined by the recurrence

$$
a_{n+2}=\frac{a_{n+1}+a_{n}}{1+a_{n+1} a_{n}}
$$

with initial values $a_{0}=0$ and $a_{1}=\left(e^{2}-1\right) /\left(e^{2}+1\right)$, where $e$ is the base of natural logarithms.

## Solution by C. Georghiou, University of Patras, Greece

Let $b_{n}$ be defined by $a_{n}=\tanh b_{n}$. This is possible because the hyperbolic tangent defined on $[0, \infty)$ and valued in $[0,1)$ is a one-to-one function. Note that $a_{1}=\left(e^{1}-e^{-1}\right) /\left(e^{1}+e^{-1}\right)=\tanh 1$ from the formula $\tanh x=\left(e^{x}-e^{-x}\right) /\left(e^{x}+e^{-x}\right)$. From the well-known formula

$$
\tanh (x+y)=\frac{\tanh x+\tanh y}{1+\tanh x \tanh y},
$$

we get $b_{n+2}=b_{n+1}+b_{n}$ with $b_{0}=0$ and $b_{1}=1$. Therefore, $b_{n}=F_{n}$, and the answer to the problem is $a_{n}=\tanh F_{n}$. (The $\tanh$ formulas can be found on page 24 in [1].)

## Reference:

1. I. S. Gradshteyn \& I. M. Ryzhik. Tables of Integrals, Series and Products. San Diego, CA: Academic Press, 1980.
Several solvers gave the equivalent answer $a_{n}=\left(e^{2 F_{n}}-1\right) /\left(e^{2 F_{n}}+1\right)$. In this form, Lord notes that " $e$ " could be any constant. The proposer solved the problem for $a_{0}$ and $a_{1}$ being arbitrary constants in $(-1,1)$, but the answer is a complicated expression.
Also solved by Tareq Alnaffouri, Richard Andró-Jeannin, Seung-Jin Bang, Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Jay Hendel, Y. H. Harris Kwong, Carl Libis, Graham Lord, Samih A. Obaid, Igor Ol. Popov, H.-J. Seiffert, A. G. Shannon, Sahib Singh, Ralph Thomas, and the proposer.

## When Does Mod $p$ Imply Mod $p^{2}$ ?

## B-728 Proposed by Leonard A. G. Dresel, Reading, England

 (Vol. 30, no. 4, November 1992)If $p>5$ is a prime and $n$ is an even integer, prove that
(a) if $L_{n} \equiv 2(\bmod p)$, then $L_{n} \equiv 2\left(\bmod p^{2}\right)$;
(b) if $L_{n} \equiv-2(\bmod p)$, then $L_{n} \equiv-2\left(\bmod p^{2}\right)$.

Solution by A. G. Shannon, University of Technology, Sydney, Australia
We have $L_{n}=\alpha^{n}+\beta^{n}$ (with $\alpha \beta=-1$ ) and $n=2 k$. Let $x=L_{n} \pm 2$. We want to show that if $p \mid x$, then $p^{2} \mid x$. Note that $x=L_{n} \pm 2=\left(\alpha^{k} \pm(-1)^{k} \beta^{k}\right)^{2}$. If $p \mid x$, then $p \mid L_{k}^{2}$ or $p \mid 5 F_{k}^{2}$. In either case, since $p$ is a prime larger than 5 , we must have $p^{2} \mid x$.
Several solvers noted that p could be any prime not equal to 5 .
Also solved by Richard André-Jeannin, Paul S. Bruckman, Russell Jay Hendel, Y. H. Harris Kwong, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

## Binet to the Rescue Again

B-729 Proposed by Lawrence Somer, Catholic University of America, Washington, DC (Vol. 30, no. 4, November 1992)

Let $\left(H_{n}\right)$ denote the second-order linear recurrence defined by $H_{n+2}=a H_{n+1}+b H_{n}$, where $H_{0}=0, H_{1}=1$, and $a$ and $b$ are integers. Let $p$ be a prime such that $p \nmid b$. Let $k$ be the least positive integer such that $H_{k} \equiv 0(\bmod p)$. (It is well known that $k$ exists.) If $H_{n} \not \equiv 0(\bmod p)$, let $R_{n} \equiv H_{n+1} H_{n}^{-1}(\bmod p)$.
(a) Show that $R_{n}+R_{k-n} \equiv a(\bmod p)$ for $1 \leq n \leq k-1$.
(b) Show that $R_{n} R_{k-n-1} \equiv-b(\bmod p)$ for $1 \leq n \leq k-2$.

Solution by A. G. Shannon, University of Technology, Sydney, Australia, and by Y. H. Harris Kwong, SUNY at Fredonia, NY (independently).

We will first prove the identity

$$
\begin{equation*}
H_{n+1} H_{k-n}+H_{n} H_{k-n+1}=H_{k}+a H_{n} H_{k-n} \tag{*}
\end{equation*}
$$

which is valid for all integers $n$ and $k$. The Binet form [1] giving the explicit value for $H_{n}$ is

$$
H_{n}=\frac{A^{n}-B^{n}}{A-B}
$$

where $A=\left(a+\sqrt{a^{2}+4 b}\right) / 2$ and $B=\left(a-\sqrt{a^{2}+4 b}\right) / 2$ are the roots of $x^{2}=a x+b$. Straightforward algebra allows us to check the identity

$$
\begin{aligned}
& \left(A^{n+1}-B^{n+1}\right)\left(A^{k-n}-B^{k-n}\right)+\left(A^{n}-B^{n}\right)\left(A^{k-n+1}-B^{k-n+1}\right) \\
& =(A-B)\left(A^{k}-B^{k}\right)+(A+B)\left(A^{n}-B^{n}\right)\left(A^{k-n}-B^{k-n}\right)
\end{aligned}
$$

from which (*) follows since $A+B=a$.
(a) From the definition of $R_{n}$, we have $H_{n} R_{n} \equiv H_{n+1}(\bmod p)$. From (*) and the fact that $H_{k} \equiv 0(\bmod p)$, we get

$$
\begin{aligned}
a H_{n} H_{k-n} & \equiv H_{n+1} H_{k-n}+H_{n} H_{k-n+1} & & (\bmod p) \\
& \equiv H_{n} H_{k-n} R_{n}+H_{n} H_{k-n} R_{k-n} & & (\bmod p)
\end{aligned}
$$

and the result follows since $H_{n} \not \equiv 0(\bmod p)$ for $1 \leq n \leq k-1$.
(b) Using part (a) and the definition of $R_{n}$ gives

$$
\begin{array}{rlrl}
H_{n} H_{k-n-1} R_{n} R_{k-n-1} & \equiv H_{n+1} H_{k-n} & & (\bmod p) \\
& \equiv H_{n}\left(a H_{k-n}-H_{k-n+1}\right) & (\bmod p) \\
& \equiv-b H_{n} H_{k-n-1} & & (\bmod p)
\end{array}
$$

and again the result follows for primes $p$ that do not divide $b$.

## Reference:

1. Ivan Niven, Herbert S. Zuckerman, \& Hugh L. Montgomery. An Introduction to the Theory of Numbers. 5th ed. New York: Wiley \& Sons, 1991, p. 199, Th. 4.10.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, H.-J. Seiffert, and the proposer.

## A Golden Quadratic

## B-730 Proposed by Herta T. Freitag, Roanoke, VA

(Vol. 31, no. 1, February 1993)
For $n \geq 0$, express the larger root of $x^{2}-L_{n} x+(-1)^{n}=0$ in terms of $\alpha$, the larger root of $x^{2}-x-\left|(-1)^{n}\right|=0$.

Solution by F. J. Flanigan, San Jose State University, San Jose, CA; Sahib Singh, Clarion University of Pennsylvania, Clarion, PA; and A. N. 't Woord, Eindhoven University of Technology, the Netherlands (independently)

From $L_{n}=\alpha^{n}+\beta^{n}$ and $\alpha \beta=-1$, we have

$$
x^{2}-L_{n} x+(-1)^{n}=x^{2}-\left(\alpha^{n}+\beta^{n}\right) x+(\alpha \beta)^{n}=\left(x-\alpha^{n}\right)\left(x-\beta^{n}\right)
$$

and since $\alpha>|\beta|>0$, the largest root is $\alpha^{n}$.
Haukkanen notes that the roots of $x^{2}+L_{n} x+(-1)^{n}=0$ are $x=-\beta^{n}$ and $x=-\alpha^{n}$; the roots of $x^{2}-\sqrt{5} F_{n} x-(-1)^{n}=0$ are $x=\alpha^{n}$ and $x=-\beta^{n}$; and the roots of $x^{2}+\sqrt{5} F_{n} x-(-1)^{n}=0$ are $x=\beta^{n}$ and $x=-\alpha^{n}$.
Also solved by Richard André-Jeannin, M. A. Ballieu, Seung-Jin Bang, Brian D. Beasley, Paul S. Bruckman, Joseph E. Chance, the Con Amore Problem Group, Elizabeth Desautel \& Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Pentti Haukkanen, Russell Jay Hendel, John Ivie, Ed Kornt-ved, Carl Libis, Don Redmond, H.-J. Seiffert, Lawrence Somer, J. Suck, Ralph Thomas, and the proposer.

Errata: Russell Jay Hendel was inadvertently omitted as a solver for Problems B-718, B-719, B-720 and B-722.

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by<br>Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

## PROBLEMS PROPOSED IN THIS ISSUE

## H-483 Proposed by James Nicholas Boots (deceased) \& Lawrence Somer, The Catholic

 University of America, Washington, D.C.Let $m \geq 2$ be an integer such that

$$
\begin{equation*}
L_{m} \equiv 1(\bmod m) \tag{1}
\end{equation*}
$$

It is well known (see [1], p. 44) that if $m$ is a prime, then (1) holds. It has been proved by H. J. A. Duparc [3] that there exist infinitely many composite integers, called Fibonacci pseudoprimes, such that (1) holds. It has also been proved in [2] and [4] that every Fibonacci pseudoprime is odd.
(i) Prove that

$$
L_{m-1}^{2}+L_{m-1}-6 \equiv 0(\bmod m)
$$

In particular, conclude that if $m$ is prime, then $L_{m-1} \equiv 2$ or $-3(\bmod m)$.
(ii) Prove that

$$
F_{m-2}-L_{m-1} F_{m-1} \equiv 1(\bmod m)
$$

## References

1. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $\alpha^{n} \pm \beta^{n}$." Ann. Math., Second Series 15 (1913):30-70.
2. A. Di Porto. "Nonexistence of Even Fibonacci Pseudoprimes of the $1^{\text {st }}$ Kind." The Fiboncci Quarterly 31.2 (1993):173-77.
3. H. J. A. Duparc. "On Almost Primes of the Second Order," pp. 1-13. Amsterdam: Rapport ZW, 1955-013, Math. Center, 1955.
4. D. J. White, J. N. Hunt, \& L. A. G. Dresel. "Uniform Huffman Sequences Do Not Exist." Bull. London Math. Soc. 9 (1977):193-98.

## H-484 Proposed by J. Rodriguez, Sonora, Mexico

Find a strictly increasing infinite series of integer numbers such that, for any consecutive three of them, the Smarandache Function is neither increasing nor decreasing.
*Find the largest strictly increasing series of integer numbers for which the Smarandache Function is strictly decreasing.

## H-485 Proposed by Paul S. Bruckman, Everett, WA

If $x$ is an unspecified large positive real number, obtain an asymptotic evaluation for the sum $S(x)$, where

$$
\begin{equation*}
S(x)=\sum_{p \leq x}(-1)^{Z(p)} ; \tag{1}
\end{equation*}
$$

here, the $p$ 's are prime and $Z(p)$ is the Fibonacci entry-point of $p$ (the smallest positive $n$ such that $p \mid F_{n}$ ).

## SOLUTIONS

## Sum Problem

H-469 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 30, no. 3, August 1992)
Define the Fibonacci polynomials by

$$
F_{0}(x)=0, F_{1}(x)=1, F_{n}(x)=x F_{n-1}(x)+F_{n-2}(x), \text { for } n \geq 2 .
$$

Show that for all positive integers $n$ and all positive reals $x$ :
(a)

$$
\frac{1}{F_{2 n-1}(x)}=\frac{x^{2}+4}{2 n-1} \sum_{k=0}^{2 n-2}(-1)^{k+n+1} \frac{\cos \frac{k \pi}{2 n-1}}{x^{2}+4 \cos ^{2} \frac{k \pi}{2 n-1}} ;
$$

$$
\begin{equation*}
\frac{1}{F_{2 n}(x)}=\frac{x\left(x^{2}+4\right)}{4 n} \sum_{k=0}^{2 n-1} \frac{(-1)^{k+n}}{x^{2}+4 \cos ^{2} \frac{k \pi}{2 n}} \tag{b}
\end{equation*}
$$

## Solution by Paul S. Bruckman, Everett, WA

From the given recurrence relation and the initial conditions, we readily establish that $F_{n}(x)$ is a monic polynomial in $x$ of degree $n-1$. In particular,

$$
\begin{equation*}
F_{n}(x)=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta} \tag{1}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha=\alpha(x)=\frac{1}{2}\left(x+\sqrt{x^{2}+4}\right), \beta=\beta(x)=\frac{1}{2}\left(x-\sqrt{x^{2}+4}\right) . \tag{2}
\end{equation*}
$$

If we make the substitution

$$
\begin{equation*}
x=2 \sinh \theta, \tag{3}
\end{equation*}
$$

we obtain $\alpha=e^{\theta}, \beta=-e^{-\theta}, 2 \cosh \theta=\sqrt{x^{2}+4}$. This leads to the alternative formulation:

$$
\begin{equation*}
F_{2 n-1}(x)=\frac{\cosh (2 n-1) \theta}{\cosh \theta} \tag{4}
\end{equation*}
$$

$$
\begin{equation*}
F_{2 n}(x)=\frac{\sinh 2 n \theta}{\cosh \theta} . \tag{5}
\end{equation*}
$$

Proof of Part (a): We readily find the zeros of $F_{2 n-1}(x)$ from (4) (2n-2 in number); we shall suppose that $n>1$ initially.

Denoting these by $x_{k}$, we obtain:

$$
\begin{array}{r}
x_{k}=y_{k} \text { or }-y_{k}=\bar{y}_{k}, \text { where } y_{k}=2 \sinh \frac{(2 k-1) i \pi}{2(2 n-1)}=2 i \sin \psi_{k}, \\
\text { and } \psi_{k}=\frac{(2 k-1) \pi}{2(2 n-1)}, k=1,2, \ldots, n-1 . \tag{6}
\end{array}
$$

Note that the $\psi_{k}$ 's are distinct and $0<\psi_{k}<\frac{1}{2} \pi$ for each $k$, thus, the $x_{k}$ 's are distinct. Therefore, the $x_{k}$ 's are simple poles of the function $1 / F_{2 n-1}(x)$. By the residue theory, we may find constants $A_{k}$ such that

$$
\begin{equation*}
1 / F_{2 n-1}(x)=\sum_{k=1}^{n-1}\left(\frac{A_{k}}{x-y_{k}}+\frac{\bar{A}_{k}}{x+y_{k}}\right) . \tag{7}
\end{equation*}
$$

In fact, the $A_{k}$ 's are determined from the following:

$$
\begin{equation*}
A_{k}=\lim _{x \rightarrow y_{k}} \frac{x-y_{k}}{F_{2 n-1}(x)} . \tag{8}
\end{equation*}
$$

Then, applying L'Hopitâl's Rule,

$$
\frac{1}{A_{k}}=\left.\frac{d}{d x} F_{2 n-1}(x)\right|_{x=y_{k}}=F_{2 n-1}^{\prime}\left(y_{k}\right) .
$$

Now $d x / d \theta=2 \cosh \theta$, which implies that $d \theta / d x=\frac{1}{2} \cosh \theta$. Hence, by (4), we obtain:

$$
F_{2 n-1}^{\prime}(x)=\frac{1}{2}(\cosh \theta)^{-3}[(2 n-1) \cosh \theta \sinh (2 n-1) \theta-\sinh \theta \cosh (2 n-1) \theta] .
$$

Then $\sinh (2 n-1) i \psi_{k}=i \sin \left(k-\frac{1}{2}\right) \pi=-i(-1)^{k}$, and $\cosh (2 n-1) i \psi_{k}=\cos \left(k-\frac{1}{2}\right) \pi=0$. Thus,

$$
F_{2 n-1}^{\prime}\left(y_{k}\right)=-i(-1)^{k}(2 n-1) / 2 \cos ^{-2} \psi_{k} \text {, and } A_{k}=\frac{2 i(-1)^{k}}{2 n-1} \cos ^{2} \psi_{k} \text {. }
$$

Then, using (7), we obtain:

$$
\begin{aligned}
1 / F_{2 n-1}(x) & =\frac{2 i}{2 n-1} \sum_{k=1}^{n-1}(-1)^{k} \cos ^{2} \psi_{k}\left[\left(x-y_{k}\right)^{-1}-\left(x+y_{k}\right)^{-1}\right] \\
& =\frac{2 i}{2 n-1} \sum_{k=1}^{n-1}(-1)^{k} \cos ^{2} \psi_{k} \cdot \frac{4 i \sin \psi_{k}}{x^{2}+4 \sin ^{2} \psi_{k}}=\frac{8}{2 n-1} \sum_{k=1}^{n-1} \frac{(-1)^{k+1} \cos ^{2} \psi_{k} \sin \psi_{k}}{x^{2}+4 \sin ^{2} \psi_{k}}
\end{aligned}
$$

Substituting $n-k$ for $k$ yields:

$$
1 / F_{2 n-1}(x)=\frac{8}{2 n-1} \sum_{k=1}^{n-1} \frac{(-1)^{n+k+1} \sin ^{2} \varphi_{k} \cos \varphi_{k}}{x^{2}+4 \cos ^{2} \varphi_{k}} \text {, where } \varphi_{k}=\frac{k \pi}{2 n-1} \text {. }
$$

Now, substituting $2 n-1-k$ for $k$ yields:

$$
1 / F_{2 n-1}(x)=\frac{8}{2 n-1} \sum_{k=n}^{2 n-2} \frac{(-1)^{k+n+1} \sin ^{2} \varphi_{k} \cos \varphi_{k}}{x^{2}+4 \cos ^{2} \varphi_{k}} .
$$

By addition, we obtain:

$$
\begin{equation*}
1 / F_{2 n-1}(x)=\frac{4}{2 n-1} \sum_{k=1}^{2 n-2} \frac{(-1)^{k+n+1} \sin ^{2} \varphi_{k} \cos \varphi_{k}}{x^{2}+4 \cos ^{2} \varphi_{k}} . \tag{9}
\end{equation*}
$$

We may also include the term for $k=0$ in the sum indicated in (9), since this term vanishes. Note that we have the following series manipulation:

$$
1 / F_{2 n-1}(x)=(2 n-1)^{-1} \sum_{k=0}^{2 n-2}(-1)^{k+n+1} \cos \varphi_{k} \cdot \frac{x^{2}+4-x^{2}-4 \cos ^{2} \varphi_{k}}{x^{2}+4 \cos ^{2} \varphi_{k}},
$$

or

$$
\begin{equation*}
1 / F_{2 n-1}(x)=\frac{x^{2}+4}{2 n-1} \sum_{k=0}^{2 n-2}(-1)^{k+n+1} \frac{\cos \varphi_{k}}{x^{2}+4 \cos ^{2} \varphi_{k}}+\frac{(-1)^{n}}{2 n-1} S_{n} \tag{10}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{n}=\sum_{k=0}^{2 n-2}(-1)^{k} \cos \varphi_{k} . \tag{11}
\end{equation*}
$$

Comparing (10) with the desired answer to part (a), we see that it only remains to show that $S_{n}=0$. This is readily determined as follows:

$$
\begin{aligned}
S_{n} & =\operatorname{Re} \sum_{k=0}^{2 n-2}(-1)^{k} \exp (i k \pi /(2 n-1)) \\
& =\operatorname{Re}\left\{\frac{1-\left(-\exp (i \pi /(2 n-1))^{2 n-1}\right.}{1+\exp (i \pi /(2 n-1))}\right\}=\operatorname{Re}\left\{\frac{1+\exp (i \pi)}{1+\exp (i \pi /(2 n-1))}\right\}=0 .
\end{aligned}
$$

Thus, part (a) is proved for $n>1$. Also, we see that the indicated formula gives the correct expression for $n=1$. This completes the proof of part (a).

Proof of Part (b): We suppose $n>0$. From (5), we find that $F_{2 n}(x)$ has $2 n-1$ simple zeros, given by $z_{0}=0, z_{k}$ or $-z_{k}=\bar{z}_{k}$, where $z_{k}=2 \sinh (k i \pi / 2 n)=2 i \sinh \xi_{k}$, and $\xi_{k}=k \pi / 2 n$, $k=1,2, \ldots, n-1$. As before, we find that

$$
\begin{equation*}
1 / F_{2 n}(x)=B_{0} / x+\sum_{k=1}^{n-1}\left(\frac{B_{k}}{x-z_{k}}+\frac{\bar{B}_{k}}{x+z_{k}}\right), \tag{12}
\end{equation*}
$$

where

$$
\begin{equation*}
B_{k}=\lim _{x \rightarrow z_{k}} \frac{x-z_{k}}{F_{2 n}(x)}=1 / F_{2 n}^{\prime}\left(z_{k}\right), k=0,1, \ldots, n-1 . \tag{13}
\end{equation*}
$$

We find that $F_{2 n}^{\prime}(x)=\frac{1}{2}(\cosh \theta)^{-3}[2 n \cosh \theta \cosh 2 n \theta-\sinh \theta \sinh 2 n \theta]$, using (5). Then $\cosh 2 i n \xi_{k}=\cos k \pi=(-1)^{k}$ and $\sinh 2 i n \xi_{k}=i \sin k \pi=0$; hence, $F_{2 n}^{\prime}\left(z_{k}\right)=n(-1)^{k} \cos ^{-2} \xi_{k}$, and $B_{k}=\frac{1}{n}(-1)^{k} \cos ^{2} \xi_{k}$ (note that $B_{0}=1 / n$ ). Then

$$
\begin{aligned}
1 / F_{2 n}(x) & =\frac{1}{n x}+\frac{1}{n} \sum_{k=1}^{n-1}(-1)^{k} \cos ^{2} \xi_{k}\left[\left(x-2 i \sin \xi_{k}\right)^{-1}+\left(x+2 i \sin \xi_{k}\right)^{-1}\right] \\
& =\frac{1}{n x}+\frac{2 x}{n} \sum_{k=1}^{n-1} \frac{(-1)^{k} \cos ^{2} \xi_{k}}{x^{2}+4 \sin ^{2} \xi_{k}}
\end{aligned}
$$

Replacing $k$ by $n-k$ yields:

$$
1 / F_{2 n}(x)=\frac{1}{n x}+\frac{2 x}{n} \sum_{k=1}^{n-1}(-1)^{n+k} \frac{\sin ^{2} \xi_{k}}{x^{2}+4 \cos ^{2} \xi_{k}} .
$$

Now, replacing $k$ by $2 n-k$ also yields:

$$
1 / F_{2 n}(x)=\frac{1}{n x}+\frac{2 x}{n} \sum_{k=n+1}^{2 n-1}(-1)^{n+k} \frac{\sin ^{2} \xi_{k}}{x^{2}+4 \cos ^{2} \xi_{k}} .
$$

Then, adding the last two expressions, we obtain:

$$
2 / F_{2 n}(x)=\frac{2}{n x}+\frac{2 x}{n} \sum_{k=0}^{2 n-1} U_{k}-\frac{2 x}{n} \cdot x^{-2},
$$

where

$$
U_{k}=(-1)^{k+n} \frac{\sin ^{2} \xi_{k}}{x^{2}+4 \cos ^{2} \xi_{k}} .
$$

Thus, we find that

$$
1 / F_{2 n}(x)=\frac{x}{4 n} \sum_{k=0}^{2 n-1}(-1)^{k+n} \frac{\left(x^{2}+4-x^{2}-4 \cos ^{2} \xi_{k}\right)}{x^{2}+4 \cos ^{2} \xi_{k}}=\frac{x\left(x^{2}+4\right)}{4 n} \sum_{k=0}^{2 n-1} V_{k}-(-1)^{n} \frac{x}{4 n} T_{n},
$$

where

$$
V_{k}=(-1)^{k+n}\left(x^{2}+4 \cos ^{2} k \pi / 2 n\right)^{-1}, \text { and } T_{n}=\sum_{k=0}^{2 n-1}(-1)^{k}
$$

Clearly, $T_{n}=0$. Therefore, the last result reduces to the expression given in part (b). Q.E.D.

## Also solved by Hans Kappus and the proposer.

## Characteristically Common

## H-470 Proposed by Paul S. Bruckman, Everett, WA

(Vol. 30, no. 3, August 1992)
Please see the issue of The Fibonacci Quarterly shown above for a complete presentation of this lengthy problem proposal.

## Solution by the proposer

Proof of Part (A): We begin with the definition of $p_{r}(z)$, namely, $p_{r}(z)=\left|z I_{r}-U_{1}^{(r)}\right|$, where $I_{r}$ is the $r \times r$ identity matrix. Thus,

$$
p_{r}(z)=\left|\begin{array}{ccccc}
z-a_{0} & -a_{1} & -a_{2} & \cdots & -a_{r-1} \\
-1 & z & 0 & \cdots & 0 \\
0 & -1 & z & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & z
\end{array}\right|
$$

Expanding along the last column, we obtain: $p_{r}(z)=(-1)^{r} a_{r-1} A_{r-1}(z)+z p_{r-1}(z)$, where $A_{r-1}(z)$ is the determinant of the $(r-1) \times(r-1)$ matrix whose elements $a_{i j}$ are defined by: $a_{i j}=z \delta_{i+1, j}-$ $\delta_{i, j}$; thus, this matrix is upper triangular, and so $A_{r-1}(x)=(-1)^{r-1}$, the product of the diagonal entries. Hence, $p_{r}(z)=z p_{r-1}(z)-a_{r-1}$. We note that $p_{1}(z)=z-a_{0}$; thus, $p_{2}(z)=z\left(z-a_{0}\right)-a_{1}=$ $z^{2}-a_{0} z-a_{1} ; p_{3}(z)=z\left(z^{2}-a_{0} z-a_{1}\right)-a_{2}=z^{3}-a_{0} z^{2}-a_{1} z-a_{2}$; and we see, in general, that we have:

$$
\begin{equation*}
p_{r}(z)=G_{r}(z) \tag{*}
\end{equation*}
$$

Proof of Part (C): We suppose $r>1$. Clearly, the desired relation is valid for $n=1$. Suppose it is valid for some value of $n \geq 1$. Then $\left(U_{1}^{(r)}\right)^{n} H_{1}^{(r)}=U_{1}^{(r)}\left(U_{1}^{(r)}\right)^{n-1} H_{1}^{(r)}$ or, by the inductive hypothesis:

$$
\begin{equation*}
\left.\left(U_{1}^{(r)}\right)^{n} H_{1}^{(r)}=U_{1}^{(r)} H_{n}^{(r)} \text { (for this special value of } n\right) \tag{6}
\end{equation*}
$$

Premultiplication of the $j^{\text {th }}$ column of $H_{n}^{(r)}$ by the $i^{\text {th }}$ row of $U_{1}^{(r)}$ replaces $H_{n+r-i, j}^{(r)}$ by $H_{n+r-i+1, j}^{(r)}$, which is clear, using (5), if $i>1$; however, this is also true for $i=1$, since $H_{n+r-1, j}^{(r)}$ is then transformed to $\sum_{k=0}^{r-1} a_{k} H_{n+r-1-k, j}^{(r)}$, which is equal to $H_{n+r, j}^{(r)}$, by the recurrence $G_{r}(E)\left(H_{n, j}^{(r)}\right)=0$. Thus, we see that $U_{1}^{(r)} H_{n}^{(r)}=H_{n+1}^{(r)}$; it follows from (6) that $\left(U_{1}^{(r)}\right)^{n} H_{1}^{(r)}=H_{n+1}^{(r)}$, which is the statement of part (C) for $n+1$. The result then follows by induction.

The proof of part (B) will appear in the May 1994 issue of this Quarterly.

## SUSTAINING MEMBERS



## BOOKS AVAILABLE THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.
A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci’s Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.
Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.
Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence-18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Applications of Fibonacci Numbers, Volumes 1-5. Edited by G.E. Bergum, A.F. Horadam and A.N. Philippou

Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger, FA, 1993.

Please write to the Fibonacci Association, Santa Clara University, Santa Clara CA 95053, U.S.A., for more information and current prices.

