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The primary function of THE FIBONACCI QUARTERLY is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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# HOFSTADTER'S EXTRACTION CONJECTURE 

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Let $\alpha, 0<\alpha<1$, be irrational. For integer $n>0$, define $f(n)=[(n+1) \alpha]-[n \alpha]$. Define $g(n)=c$ if $f(n)=0$ and $g(n)=d$ if $f(n)=1$ and let $x=x(\alpha)$ be the infinite string whose $n^{\text {th }}$ element is $g(n)$.

Both the string $x$, and the three functions $f(n), g(n)$, and $[n \alpha]$ have been studied extensively. Classically, an astronomical problem of Bernoulli led Markov to prove results about the structure of $x$. A concise summary is presented in [12]. The results use continued fraction methods and the theory of semigroups.

Recent research connects $x$ with monoid homomorphisms (e.g., Fraenkel et al. [4]), outputs of automata (e.g., Shallit [9]), and general properties of strings (e.g., Mignosi [8]).

These functions and their related sequences have obvious recreational and experimental flavor, are noted for their exotic functional patterns (e.g., Doster [1]), and lend themselves readily to computer experiments (e.g., Engel [2], or Hofstadter [7]).

In this paper we study a problem first described by Hofstadter in an unpublished manuscript [6]:

But now I would like to give an example par excellence of horizontal properties, a property which I call "extraction." The idea is this. To begin with, write down $x$. Now choose some arbitrary term in it, called the "starting point." Beginning at the starting point, try to match $x$ term by term. Every time you find a match, circle that term. Soon you will come to a term which differs from $x$. When this happens, just skip over it without circling it, and look for the earliest match to the term of $x$ you are seeking. Continue this process indefinitely. In the end you have circled a great number of terms after the starting point, and left some uncircled. We are interested in the uncircled terms, which are now "extracted" from $x$. The interesting fact is that the extracted sequence is the subsequence of $x$ which begins two terms earlier than the starting point! To decrease confusion, I now show an example, where instead of circling I underline the terms which match $x$. In this example, $\alpha=(\sqrt{5}-1) / 2$.

I have chosen this " d " as the starting point:

```
    I
    |
\textrm{dcd}}\underline{\textrm{d}}\underline{\textrm{c}}\underline{\textrm{d}}c\underline{\textrm{d}}d\underline{\textrm{c}}\underline{\textrm{d}}d\underline{\textrm{c}}\underline{\textrm{d}}c\underline{\textrm{d}}d\underline{\textrm{c}}\underline{\textrm{d}}c\underline{\textrm{d}}d\underline{\textrm{c}}d\underline{\textrm{d}
```

The underlined sequence matches the full sequence, $x$, term by term. Now what is the extracted sequence? It is:

$$
c d d c d c d d c d d c d c d d c d \cdots
$$

And you will find that this matches with the sequence which begins two places earlier than the starting point. Carrying it further is tedious, and does nothing but confirm our observation. Why does this extraction-property hold? At this point, I must admit that I don't know. It is a curious property which needs further investigation.

[^0]To rigorously formulate this, we present the following definition.
Definition 1: Suppose $U=u_{1} \ldots u_{n}, V=v_{1} \ldots v_{m}$, and $E=e_{1} \ldots e_{p}$ with $u_{i}, v_{j}, e_{k} \in\{c, d\}, n, m>0$, and $n=m+p$. We say $U$ aligns (with) $V$ with extraction $E$ (notationally indicated by $U \supset V ; E$ ), if there exist integers $j(0), j(1), j(2), \ldots, j(p)$, such that

$$
\left.U=\left\{v_{1} \ldots v_{j(1)}\right\} e_{1}\left\{v_{j(1)+1} \ldots v_{j(2)}\right\} e_{2} \ldots e_{p}\left\{v_{j(p)+1} \ldots v_{m}\right\} \quad \text { (with }\left\{v_{a \ldots} v_{b}\right\} \text { empty if } b<a\right) \text {, }
$$

where
(i) $0=j(0) \leq j(1) \leq j(2) \leq \cdots \leq j(p)<m$,
(ii) $e_{i} \neq v_{j(i)+1}$, for $1 \leq i \leq p$.

For example, if $p=0, U \supset V$; $E$ with $U=V$ and $E$ the empty string. Throughout this paper we use the nonstandard symbol $\phi$ to denote the empty string. It is easy to see that $U \supset V ; \phi$ if and only if $U=V$.

If $U \supset V ; E$, then $U, V$, and $E$ are call the original, aligned, and extracted strings, respectively, and the relationship itself is called an alignment.

Remark: Define strings $U=d c d c d$ and $V=d d$. To clarify some subtleties in Definition 1, we explore the consequences of dropping requirements (i) or (ii).

If we drop the requirement of strict inequality, $j(p)<m$, in Definition 1(i), then we allow $U \supset V ; c c d$ with $j(1)=1, j(2)=j(3)=m=2$.

If we keep requirement (i) but drop requirement (ii), then we allow $U \supset V$; $c d c$, with $j(1)=$ $j(2)=j(3)=1, \quad m=2, e_{2}=v_{j(2)+1}$ and, similarly, we allow $U \supset V ; d c c$, with $j(1)=j(2)=0$, $j(3)=1, m=3, e_{1}=v_{j(1)+1}$.

Thus, for given original and aligned strings, without requirements (i) and (ii), the extracted string is not necessarily unique. However, with requirements (i) and (ii), we can prove the following lemma.

Lemma 1: For given strings $U$ and $V$, there is at most one string $E$ such that $U \supset V ; E$.
Proof: We suppose $U \supset V ; E, U \supset V ; E^{\prime}$, and $E \neq E^{\prime}$ and derive a contradiction.
By Definition 1, there are sequences $j(1), \ldots, j(p)$, and $j^{\prime}(1), \ldots, j^{\prime}(p)$ satisfying (i) and (ii) of Definition 1 and

$$
\begin{align*}
& U=\left\{v_{1 \ldots} v_{j(1)}\right\} e_{1}\left\{v_{j(1)+1} \ldots v_{j(2)}\right\} e_{2 \ldots} e_{p}\left\{v_{j(p)+1} \ldots v_{m}\right\},  \tag{*}\\
& U=\left\{v_{1 \ldots} \ldots v_{j^{\prime}(1)}\right\} e_{1}^{\prime}\left\{v_{j^{\prime}(1)+1} \ldots v_{j^{\prime}(2)}\right\} e_{2}^{\prime} \ldots e_{p}^{\prime}\left\{v_{j^{\prime}(p)+1} \ldots v_{m}\right\} . \tag{**}
\end{align*}
$$

Observe that, for $1 \leq r \leq p, e_{r}=$ the $\{j(r)+r\}^{\text {th }}$ element of $U$. Similarly, if $t$ is given such that either $j(r)+r<t<j(r+1)+(r+1)$ for some $r, 0 \leq r \leq p-1$, or $j(r)+r<t \leq m$ with $r=p$, then $v_{t-r}=$ the $t^{\text {th }}$ element of $U$.

Let $s$ be the largest integer such that $j(r)=j^{\prime}(r)$ for $0 \leq r<s$. Then $s$ exists and is positive because $j(0)=0=j^{\prime}(0)$. Since we assume $E \neq E^{\prime}, s \leq p$.

If we further suppose that $j(s)<j^{\prime}(s)$, then $j^{\prime}(s-1)+(s-1)<j(s)+s<j^{\prime}(s)+s$.

Therefore, by considering $(*)$ and $(* *)$, respectively, the $\{j(s)+s\}^{\text {st }}$ element of $U$ is, simultaneously, $e_{s}$ and $v_{j(s)+1}$, contradicting Definition 1(ii). A similar argument holds if $j^{\prime}(s)<j(s)$. These contradictions show that $E=E^{\prime}$ and complete the proof.

Recall that $u$ is a prefix (that is, left factor) of $v$ if there is a string $y$ such that $v=u y$. Similarly, $u$ is a suffix (or right factor) of $v$, if $v=y u$ for some string $y$. We say that the string $y$ is the limit of the sequence of strings $y(n), n=1,2,3, \ldots$, notationally indicated by $y=\lim y(n)$, if, for each positive integer $m$ less than or equal to the length of $y$, the left factors of length $m$ of $y(n)$ and $y$ are equal for all sufficiently large $n$.

Definition 2: Suppose $U, V$, and $E$ are (possibly infinite) strings. Suppose $U(n), V(n)$, and $E(n), n \geq 1$, are sequences of finite strings such that $U(n) \supset V(n) ; E(n)$, with $\lim U(n)=U$, $\lim V(n)=V$, and $\lim E(n)=E$. Then we say $U$ aligns $V$ with extraction $E$ and indicate this, notationally, by $U \supset V ; E$ (we do not require $E$ to be infinite).

Remark: By a proof similar to that of Lemma 1, it can be proved in the infinite case also that $E$ is (uniquely) functionally dependent on $U$ and $V$.

Let $x_{m}$ denote $x$ with the left factor of length $m$ deleted. We can now formulate the general Hofstadter conjecture as follows:

Hofstadter's Coniecture: For any $\alpha$ and any $m \geq 2$

$$
\begin{equation*}
x_{m} \supset x ; x_{m-2} \tag{1}
\end{equation*}
$$

Example 1: For the remainder of this paper we assume $\alpha=(\sqrt{5}-1) / 2$. In this case, the sequence

$$
x=d c d d c d c d d c d d c d c d d c d ~ c d ~ d c d ~ d c d ~ c d ~ d c d ~ d c d ~ c d ~ d c d ~ c d ~ d c d ~ d c d ~ c d ~ d c d \cdots
$$

has been described fairly thoroughly in the literature (see Tognetti et al. [11]). The sequence is referred to as the Golden sequence or, sometimes, the Fibonacci sequence. With

$$
\begin{aligned}
& x_{1}=c d d c d ~ c d d c d d c d c d d c d ~ c d d c d d c d c d d c d \cdots \\
& x_{3}=d c d ~ c d d c d d c d c d d c d ~ c d d c d d c d c d d c d \cdots
\end{aligned}
$$

Hofstadter's conjecture for $m=3$ asserts $x_{3} \supset x ; x_{1}$.
We define $c_{0}=c, c_{1}=d$,

$$
\begin{equation*}
c_{n}=c_{n-2} c_{n-1}, \quad n \geq 2 \tag{2}
\end{equation*}
$$

Then $c_{2}=c d, c_{3}=d c d, c_{4}=c d d c d, c_{5}=d c d ~ c d d c d$, and $c_{6}=c d d c d d c d c d d c d$.
The following result is well known [12].
Lemma 2: $x=c_{1} c_{2} \ldots$.
A crucial component of the proof of Hofstadter's conjecture is a concatenation lemma asserting that under approprite conditions the extractions of concatenated strings are the concatenations of their extractions.

## Lemma 3:

(i) Let $U, V, E$ and $U^{\prime}, V^{\prime}, E^{\prime}$ denote arbitrary strings of finite length. If $U \supset V ; E$ and $U^{\prime} \supset V^{\prime} ; E^{\prime}$, then $U U^{\prime} \supset V V^{\prime} ; E E^{\prime}$.
(ii) If $U_{i}, V_{i}$, and $E_{i}, 1 \leq i \leq m$, are arbitrary strings of finite lengths with $m$ some integer, and if $U_{i} \supset V_{i} ; E_{i}, 1 \leq i \leq m$, then $\Pi U_{i} \supset \Pi V_{i} ; \Pi E_{i}$ (with products denoting concatenation).

Proof: Part (ii) follows from part (i) by simple induction. To prove (i), we suppose, using Definition 1, that

$$
\begin{aligned}
U & =\left\{v_{1} \ldots v_{j(1)}\right\} e_{1}\left\{v_{j(1)+1} \ldots v_{j(2)}\right\} e_{2} \ldots e_{p}\left\{v_{j(p)+1} \ldots v_{m}\right\}, \\
U^{\prime} & =\left\{v_{1}^{\prime} \ldots v_{j^{\prime}(1)}^{\prime}\right\} e_{1}^{\prime}\left\{v_{j^{\prime}(1)+1}^{\prime} \ldots v_{j^{\prime}(2)}^{\prime}\right\} e_{2}^{\prime} \ldots e_{p^{\prime}}^{\prime}\left\{v_{j^{\prime}\left(p^{\prime}\right)+1}^{\prime} \ldots v_{m^{\prime}}^{\prime}\right\},
\end{aligned}
$$

for some sequences of integers, $0 \leq j(1) \leq \cdots \leq j(p)<m, 0 \leq j^{\prime}(1) \leq \cdots \leq j^{\prime}\left(p^{\prime}\right)<m^{\prime}$ with $V=v_{1} \cdots$ $v_{m}, V^{\prime}=v_{1}^{\prime} \ldots v_{m^{\prime}}^{\prime}, E=e_{1} \ldots e_{p}$, and $E^{\prime}=e_{1}^{\prime} \ldots e_{p^{\prime}}^{\prime}$. Then

$$
U U^{\prime}=\left\{v_{1 \ldots} v_{j(1)}\right\} e_{1}\left\{v_{j(1)+1 \ldots} v_{j(2)}\right\} e_{2 \ldots} e_{p}\left\{v_{j(p)+1} \ldots v_{m}\right\}\left\{v_{1}^{\prime} \ldots v_{j^{\prime}(1)}^{\prime}\right\} e_{1 \ldots}^{\prime} \ldots e_{p^{\prime}}^{\prime}\left\{v_{j^{\prime}\left(p^{\prime}\right)+1 \ldots}^{\prime} \ldots v_{m^{\prime}}^{\prime}\right\}
$$

To prove $U U^{\prime} \supset V V^{\prime} ; E E^{\prime}$, we verify that requirements (i) and (ii) of Definition 1 are satisfied by the sequence of integers $0 \leq j(1) \leq j(2) \leq \cdots \leq j(p)<m+j^{\prime}(1) \leq m+j^{\prime}(2) \leq \cdots \leq m+j^{\prime}\left(p^{\prime}\right)<$ $m+m^{\prime}$.

The applicability of Lemma 3 will be enhanced by developing a notation for products of $c_{n}$. Formally, for integers $k, p \geq 0, q \geq 1$, with $q$ dividing $(p-k)$, recursively define $P_{k, p ; q}=$ $P_{k, p-q ; q} c_{p}$ if $p>k$, and $P_{k, k ; q}=c_{k}$. If $p<k$, then $P_{k, p ; q}=\phi$. If $q=1$, then by abuse of notation we will drop $q$ and let $P_{k, p}=P_{k, p ; 1}$. Similarly, we let $P_{k}=\lim _{p \rightarrow \infty} P_{k, p ; 1}$. Using this notation, Lemma 2 reads $x=P_{1}$.
Lemma 4: $\quad P_{a+2, b} \supset P_{a+1, b-1} ; P_{a, b-2}, \quad$ for $a \geq 0, b \geq a+2$,

$$
\begin{aligned}
P_{a+2, b} & \supset P_{a, b-1} ; P_{a+1, b-2}, & & \text { for } a \geq 0, b \geq a+2, \\
P_{a, b} & \supset P_{a, b} ; \phi, & & \text { for } b \geq a \geq 0 .
\end{aligned}
$$

Proof: First, observe that $c_{2} \supset c_{1} ; c_{0}$ and $c_{3} \supset c_{2} ; c_{1}$. If, by an induction assumption, $c_{n-2} \supset$ $c_{n-3} ; c_{n-4}$ and $c_{n-1} \supset c_{n-2} ; c_{n-3}$, for some $n \geq 4$, then, by Lemma 3 and (2), $c_{n} \supset c_{n-1} ; c_{n-2}$. Consequently, applying concatenation (that is, Lemma 3) to the $b+1-(a+2)$ alignments, $c_{a+2+i} \supset$ $c_{a+1+i} ; c_{a+i}, 0 \leq i \leq b-(a+2)$, yields $P_{a+2, b} \supset P_{a+1, b-1} ; P_{a, b-2}$.

To prove the second assertion in Lemma 4, note that $c_{a+2} \supset c_{a} c_{a+1} ; \phi$, by (2). We then apply concatenation to this alignment and the alignments $c_{a+2+i} \supset c_{a+1+i} ; c_{a+i}, 1 \leq i \leq b-(a+2)$. Note that, if $b=a+2$, then $P_{a+1, b-2}=\phi$ and both the statement and the proof are still valid.

The last assertion in Lemma 4 is obvious.
Corollary: $P_{a+2} \supset P_{a+1} ; P_{a}, P_{a+2} \supset P_{a} ; P_{a+1}, P_{a} \supset P_{a} ; \phi$.
Proof: Let $b$ go to infinity in Lemma 4.
Examples: Using Lemmas 3 and 4 and the Corollary, we can explore Hofstadter's conjecture, (1), for $m=2,3,4$.
$\underline{m=2}$ : By applying concatenation to $d \supset d ; \phi$ and $P_{3} \supset P_{2} ; P_{1}$, we infer $x_{2} \supset x ; x$.
$\underline{m=3}$ : The assertion $P_{3} \supset P_{1} ; P_{2}$ is equivalent to $x_{3} \supset x ; x_{1}$.
$\underline{m=4}$ : Note that $x_{4}=c d P_{4}, x=d P_{2}$, and $x_{3}=P_{3}$. Therefore, applying concatenation to the alignments $c d \supset d ; c$ and $P_{4} \supset P_{2} ; P_{3}$ implies that $x_{4} \supset x ; c x_{3}$. Consequently, by Lemma 1, (1) cannot hold for $m=4$, since $x_{2}$ begins with a $d$. Similar reasoning shows that (1) is false for $m=9,12, \ldots$.

To generalize the $m=4$ case precisely, recall Zeckendorf's result that every integer $m$ can be represented uniquely as a sum of nonconsecutive Fibonacci numbers, $m=\sum_{i \geq 2} \varepsilon(i) F_{i}$, with $\varepsilon(i)$ in $\{0,1\}, \varepsilon(i)=0$ if $\varepsilon(\underset{i}{i}+1)=1$, and $\varepsilon(n)=1$ with $\varepsilon(i)=0$ for $i \geq n+1$, for some integer $n \geq 2$. The ascending set of $\varepsilon(i)$ is the Fibonacci representation of $m$ [9]. We define an injective map from nonnegative integers to finite binary strings, $m^{*}=s$, such that $s$ has length $n-1$ and the $i^{\text {th }}$ component of $s$ equals $\varepsilon(i+1)$ for $1 \leq i \leq n-1$.

We will use standard conventions about exponents and string concatenations. For example, $54^{*}=(01)^{4}$. In the sequel, in the proofs of Lemma 5 and Theorem 1, certain closed formulas will be given for $(m+1)^{*}$ and $(m-2)^{*}$. The relationships between $m^{*}$ and $(m \pm j)^{*}$ can be "translated" easily into well-known identities. For example, the assertion that, if $m^{*}=(10)^{k} 1$ for some $k \geq 0$, then $(m+1)^{*}=(00)^{k} 01$ is seen to correspond to the identity $F_{2}+F_{4}+\cdots+F_{2 k+2}=F_{2 k+3}-1$.

Therefore, in the proofs of Lemma 5 and Theorem 1, these closed formulas will simply be stated without further elaboration.

Some of the relationships between $m^{*}$ and the $m^{\text {th }}$ character of $x$ are explored in [3]. The examples for which (1) fails, $m=4,9,12,17,22,25,30,33, \ldots$, have Fibonacci representations beginning with a one followed by an odd number of zeros. This suggests the following modified Hofstadter's conjecture:

For all $m \geq 2$, if $m^{*}=10^{2 k+1} 1 s$, for some integer $k \geq 0$ and some binary string $s$, then

$$
\begin{equation*}
x_{m} \supset x ; c x_{m-1} . \tag{3}
\end{equation*}
$$

Otherwise, (1) holds.
Remark: By the examples presented after Lemma 4 and its corollary, the modified Hofstadter conjecture is true for $m=2,3,4$.

We now state all identities needed in the proofs of Lemma 5 and Theorem 1:

$$
\begin{gather*}
c_{1} P_{2,2 k ; 2}=c_{2 k+1}, \quad \text { for } k \geq 1,  \tag{4}\\
c_{2} P_{3,2 k-1 ; 2}=c_{2 k}, \quad \text { for } k \geq 1,  \tag{5}\\
P_{3,2 k+1 ; 2}=P_{1,2 k}, \quad \text { for } k \geq 1,  \tag{6}\\
P_{2,2 k ; 2}=c P_{1,2 k-1}, \quad \text { for } k \geq 1,  \tag{7}\\
c_{1} P_{4,2 k ; 2}=P_{1,2 k-1}, \quad \text { for } k \geq 1,  \tag{8}\\
P_{a+1, b-2} c_{b+1}=c_{a+1} P_{a+2, b}=P_{a+1, b} \text { if } a+1 \leq b-1 . \tag{9}
\end{gather*}
$$

For $t \geq 2$, and integers $K(i)$, with $K(i+1) \geq K(i)+2, j \leq i \leq t-1$, with $j$ in $\{0,1\}$,

$$
\begin{equation*}
P_{K(j)+1, K(j+1)-2} \ldots P_{K(t-1)+1, K(t)-2} c_{K(t)+1}=P_{K(j)+1, K(j+1)}\left\{P_{K(j+1)+2, K(j+2)} \ldots P_{K(t-1)+2, K(t)}\right\} \tag{10}
\end{equation*}
$$

the expression in braces being empty if $t<j+2$.
To prove (4), note that, if $k=1$, then $c_{1} c_{2}=c_{3}$ while, if $k>1$, then, by (2) and an induction assumption, $c_{2 k+1}=c_{2 k-1} c_{2 k}=c_{1} P_{2,2 k-2 ; 2} c_{2 k}=c_{1} P_{2,2 k ; 2}$. The proofs of (5)-(7) also follow from (2) and an induction assumption. Equation (8) follows from (7) by cancelling the leftmost $c$ on both sides of the equation.

To prove (9) note that, if $a+1 \leq b-2$, then, by (2), $P_{a+1, b-2} c_{b+1}=P_{a+1, b}=c_{a+1} P_{a+2, b}$ while, if $a+1=b-1$, then $P_{a+1, b-2}=\phi$, so that (9) becomes $c_{b+1}=c_{b-1} P_{b, b}=P_{b-1, b}$, which follows from (2). Note, however, that, if $a+1 \geq b$, (9) is false. Equation (10) follows from (9) by a straightforward induction.

Definition 3: Given an integer $m$, a strictly increasing function $f$ on the positive integers is said to be a representation of $x_{m}$ if $x_{m}=c_{f(1)} c_{f(2)} c_{f(3)} \cdots$.

To each integer $m \geq 1$ with Fibonacci representation, $\varepsilon(i), i \geq 2$, with $\varepsilon(n)=1, \varepsilon(i)=0$ for $i \geq n+1$, we associate a triple $\langle n, j, z\rangle$, where $n-j$ is the total number of ones in the Fibonacci representation $\varepsilon$ of $m$, and $z$ is a strictly increasing sequence, $z(1), z(2), \ldots, z(j)$ with $\varepsilon(z(i)+1)=0,1 \leq i \leq j$. As an example, if $m=54$, then $n=9, j=4$, and $z(i)=2 i-1$ for $i=1,2$, 3,4 . We now describe a canonical representation of $x_{m}$.

Lemma 5: Given an integer $m \geq 2$ and its associated triple, $\langle n, j, z\rangle$, the function $f$, defined by $f(i)=z(i), 1 \leq i \leq j, f(j+1+t)=n+t, t=0,1,2,3 \ldots$, is a representation of $x_{m}$.

Proof: To start an induction argument, we treat the case $m=2$. If $m=2$, then $m^{*}=01$, $n=3, j=1$, and $z(1)=1$. Clearly, $x_{m}=c_{1} P_{3}$ as required. The induction step has three cases.

Case $1-m^{*}=00 s$ with $s$ a binary string: Clearly $(m+1)^{*}=10 s$. By induction, we may assume that a representation $f$ of $x_{m}$ exists such that $f(i)=i, i=1,2$. Thus, $x_{m}=c_{1} c_{2} y$ for some infinite string $y$ and, consequently, $x_{m+1}=c_{2} y$ as required.

Case $2-m^{*}=(01)^{k} 00 s$ with $k \geq 1$ and $s$ a (possibly empty) binary string: Then ( $\left.m+1\right)^{*}$ $=(00)^{k} 10 s$. By induction, we may assume that there is a representation $f$ of $x_{m}$ such that, whether $s$ is empty or not, $f(i)=2 i-1,1 \leq i \leq k$, and $f(k+i)=2 k+i, i=1,2$. Thus, $x_{m}=$ $P_{1,2 k+1 ; 2} y$ for some infinite string $y$ and, therefore, by (6), $x_{m+1}=P_{3,2 k+1 ; 2} y=P_{1,2 k} y$ as required.

Case $3-\boldsymbol{m}^{*}=(10)^{k} 0 s$ with $k \geq \mathbb{1}$ and $\boldsymbol{s}$ a binary string: Then $(m+1)^{*}=(00)^{k-1} 010 s$. By induction, we may assume there is a representation $f$ of $x_{m}$ with $f(i)=2 i, 1 \leq i \leq k, f(k+1)=$ $2 k+1$. Thus, $x_{m}=P_{2,2 k ; 2} y$ for some infinite string $y$ and, consequently, by (8), $x_{m+1}=c_{1} P_{4,2 k ; 2} y$ $=P_{1,2 k-1} y$ as required.

Clearly, for each $m \geq 2$, one of these three cases must hold and, consequently, the proof is complete.

Theorem 1: The modified Hofstadter's conjecture is true for all $m \geq 2$.

Proof: The theorem has already been verified for $m=2,3,4$. If $m \geq 5$, then there exist integers $t \geq 1, k(1), k(2), \ldots k(t), k(i) \geq 1$, such that either

$$
\begin{equation*}
m^{*}=10^{k(1)} 1 \ldots 0^{k(t)} 1 \tag{11}
\end{equation*}
$$

or

$$
\begin{equation*}
m^{*}=0^{k(1)} 10^{k(2)} 1 \ldots 0^{k(t)} 1 . \tag{12}
\end{equation*}
$$

To prove the theorem, we need the Fibonacci representations for $(m-1)^{*}$ and $(m-2)^{*}$. There are now four cases- $1 \mathrm{~A}, 1 \mathrm{~B}, 1 \mathrm{C}$, and 1 D -depending on whether $m^{*}$ begins with a 1 or not and depending on whether $k(1)$ is even or odd.

Case 1A-(11) holds, with $\boldsymbol{k}(\mathbf{1})$ odd: Then, clearly, $(m-1)^{*}=0^{k(1)+1} 1\left\{0^{k+2)} 1 \ldots 0^{k(t)} 1\right\}$, the expression in braces being empty if $t=1$.

Define integers

$$
\begin{equation*}
K(0)=0, K(i+1)=K(i)+1+k(i+1), i=0,1, \ldots, t-1 . \tag{13}
\end{equation*}
$$

Clearly,

$$
\begin{equation*}
K(i+1) \geq K(i)+2, i=0,1, \ldots, t-1 . \tag{14}
\end{equation*}
$$

By Lemma 5,

$$
\begin{equation*}
x_{m}=P_{K(0)+2, K(1)} \ldots P_{K(t-1)+2, K(t)} P_{K(t)+2} \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{m-1}=P_{1, K(1)}\left\{P_{K(1)+2, K(2)} \ldots P_{K(t-1)+2, K(t)}\right\} P_{K(t)+2} . \tag{16}
\end{equation*}
$$

The expression in braces is empty if $t=1$.
Using Lemma 4 and its corollary, we apply concatenation to the alignments
and

$$
\begin{aligned}
P_{2, K(1)} & \supset P_{1, K(1)-1} ; P_{0, K(1)-2}, \\
P_{K(i)+2, K(i+1)} & \supset P_{K(i), K(i+1)-1} ; P_{K(i)+1, K(i+1)-2,}, 1 \leq i \leq t-1, \\
P_{K(t)+2} & \supset P_{K(t)} ; P_{K(t)+1},
\end{aligned}
$$

to obtain

$$
\begin{equation*}
x_{m} \supset x ; y \tag{17}
\end{equation*}
$$

with

$$
\begin{equation*}
y=c P_{K(0)+1, K(1)-2} \ldots P_{K(t-1)+1, K(t)-2} P_{K(t)+1} . \tag{18}
\end{equation*}
$$

Since $k(1)$ is odd, we must prove (3). By (17), to prove (3), it suffices to prove $y=c x_{m-1}$. Therefore, by (16) and (18), it suffices to prove

$$
P_{K(0)+1, K(1)-2} \ldots P_{K(t-1)+1, K(t)-2} c_{K(t)+1}=P_{K(0)+1, K(1)}\left\{P_{K(1)+2, K(2)} \ldots P_{K(t-1)+2, K(t)}\right\},
$$

which follows from (14) and (10).
Case 1B-(11) holds with

$$
\begin{equation*}
k(1)=2 k, k \geq 1: \tag{19}
\end{equation*}
$$

For notational reasons, it will be clearer in cases $1 \mathrm{~B}, 1 \mathrm{C}$, and 1 D to first assume that $t \geq 2$. The $t=1$ case can then be treated separately. If $t \geq 2$, then $(m-2)^{*}=(10)^{k} 100^{k(2)} 1 \ldots 0^{k(t)} 1$. Define $K(i)$ as in (13). Then (14) and (15) still hold. By Lemma 5, we have

$$
\begin{equation*}
x_{m-2}=P_{2,2 k ; 2} c_{2 k+2} P_{K(1)+2, K(2)} \ldots P_{K(t-1)+2, K(t)} P_{K(t)+2} \tag{20}
\end{equation*}
$$

Proceeding as in case 1A, we apply Lemmas 3 and 4. Equations (17) and (18) still hold.
Since $k(1)$ is even, we must prove (1) instead of (3). By (17), to prove (1) it suffices to prove $y=x_{m-2}$. Therefore, by (18) and (20), it suffices to prove

$$
\begin{equation*}
c P_{K(0)+1, K(1)-2} \ldots P_{K(t-1)+1, K(t)-2} c_{K(t)+1}=P_{2,2 k ; 2} c_{2 k+2} P_{K(1)+2, K(2)} \ldots P_{K(t-1)+2, K(t)} . \tag{21}
\end{equation*}
$$

By (19) and (13), $K(0)+1=1$ and $K(1)-2=\{k(1)+1\}-2=2 k-1$. Hence, by (7), proving (21) is equivalent to proving

$$
P_{2,2 k ; 2} P_{K(1)+1, K(2)-2} \ldots P_{K(t-1)+1, K(t)-2} c_{K(t)+1}=P_{2,2 k ; 2} c_{K(1)+1} P_{K(1)+2, K(2)} \ldots P_{K(t-1)+2, K(t)},
$$

which follows from (14) and (10).
To complete the proof of case 1 B , we treat the $t=1$ case: If $t=1$, then $m^{*},(m-2)^{*}, x_{m}$, and $x_{m-2}$ are $10^{2 k} 1,(10)^{k} 1, P_{2,2 k+1} P_{2 k+3}$, and $P_{2,2 k ; 2} P_{2 k+2}$, respectively. Using Lemma 4, we apply concatenation to the alignments $P_{2,2 k+1} \supset P_{1,2 k} ; c P_{1,2 k-1}$ and $P_{2 k+3} \supset P_{2 k+1} ; P_{2 k+2}$ to obtain (17) with $y=c P_{1,2 k-1} P_{2 k+2}$ To prove (1), it suffices to prove $x_{m-2}=y$, which follows from (7).

Case 1C-(12) holds with (19): For $t \geq 2$, we have $(m-2)^{*}=10(01)^{k-1} 00^{k(2)} 1 \ldots 0^{k(t)} 1$. Define

$$
\begin{equation*}
K(0)=0, K(1)=k(1), K(i+1)=K(i)+1+k(i+1), 1 \leq i \leq t-1 . \tag{22}
\end{equation*}
$$

Note that, by (19), (14) still holds. By Lemma 5,
and

$$
\begin{equation*}
x_{m}=P_{1, K(1)} P_{K(1)+2, K(2)} \ldots P_{K(t-1)+2, K(t)} P_{K(t)+2} \tag{23}
\end{equation*}
$$

$$
\begin{equation*}
x_{m-2}=c_{2} P_{3,2 k-1 ; 2} c_{2 k+1} P_{K(1)+2, K(2)} \ldots P_{K(t-1)+2, K(t)} P_{K(t)+2} \tag{24}
\end{equation*}
$$

Using Lemma 4 and its corollary, we apply concatenation to the alignments
and

$$
\begin{aligned}
P_{1, K(1)} & \supset P_{1, K(1)} ; \phi, \\
P_{K(1)+2, K(2)} & \supset P_{K(1)+1, K(2)-1} ; P_{K(1), K(2)-2}, \\
P_{K(i)+2, K(i+1)} & \supset P_{K(i), K(i+1)-1} ; P_{K(i)+1, K(i+1)-2}, 2 \leq i \leq t-1,
\end{aligned}
$$

to obtain (17) with

$$
\begin{equation*}
y=P_{K(1), K(2)-2}\left\{P_{K(2)+1, K(3)-2} \ldots P_{K(t-1)+1, K(t)-2}\right\} P_{K(t)+1}, \tag{25}
\end{equation*}
$$

the expression in braces being empty if $t=2$.
By (17), to prove (1) it suffices to prove $y=x_{m-2}$. Therefore, by (25) and (24), it suffices to prove

$$
\begin{equation*}
P_{K(1), K(2)-2}\left\{P_{K(2)+1, K(3)-2} \ldots P_{K(t-1)+1, K(t)-2}\right\} c_{K(t)+1}=c_{2} P_{3,2 k-1 ; 2} c_{2 k+1} P_{K(1)+2, K(2)} \ldots P_{K(t-1)+2, K(t)} . \tag{26}
\end{equation*}
$$

By (19) and (22), $K(1)=2 k$ so that, by (5), proof of (26) is reduced to proof of

$$
P_{K(1)+1, K(2)-2} \ldots P_{K(t-1)+1, K(t)-2} c_{K(t)+1}=P_{K(1)+1, K(2)}\left\{P_{K(2)+2, K(2)} \ldots P_{K(t-1)+2, K(t)}\right\},
$$

which follows from (10) and (14).

It remains to treat the case $t=1$. If $k=1$ also, then case 1 C reduces to (1) with $m=3$, which has already been treated. If $k>1$, then $m^{*},(m-2)^{*}, x_{m}$, and $x_{m-2}$ are $0^{2 k} 1,10(01)^{k-1}$, $P_{1,2 k} P_{2 k+2}$, and $c_{2} P_{3,2 k-1 ; 2} P_{2 k+1}$, respectively. By concatenating the alignments, $P_{1,2 k} \supset P_{1,2 k} ; \phi$ and $P_{2 k+2} \supset P_{2 k+1} ; P_{2 k}$, we derive (17) with $y=P_{2 k}=c_{2 k} P_{2 k+1}$. To prove (1), we must prove that $y=x_{m-2}$, which follows from (5).

## Case 1D-(12) holds with

$$
\begin{equation*}
k(1)=2 k+1, k \geq 0: \tag{27}
\end{equation*}
$$

For $t \geq 2$, we have $(m-2)^{*}=0(01)^{k} 00^{k(2)} 1 \ldots 0^{k(t)} 1$. Define $K(i)$ by (22). Then (14) and (23) still hold. By Lemma 5,

$$
\begin{equation*}
x_{m-2}=c_{1} P_{2,2 k ; 2} c_{2 k+2} P_{K(1)+2, K(2)} \ldots P_{K(t-1)+2, K(t)} P_{K(t)+2} \tag{28}
\end{equation*}
$$

Proceeding as in case 1 C , we have (17) with (25). By (17), to prove (1) it suffices to show that $y=x_{m-2}$. Therefore, by (25) and (28), it suffices to show

$$
\begin{equation*}
c_{1} P_{2,2 k ; 2} c_{2 k+2} P_{K(1)+2, K(2)} \ldots P_{K(t-1)+2}, K(t)=c_{K(1)} P_{K(1)+1, K(2)-2} \ldots P_{K(t-1)+1, K(t)-2} c_{K(t)+1} \tag{29}
\end{equation*}
$$

By (27) and (22), $K(1)=2 k+1$; therefore, by (4), proof of (29) reduces to proof of

$$
c_{K(1)} c_{K(1)+1} P_{K(1)+2, K(2)} \ldots P_{K(t-1)+2, K(t)}=c_{K(1)} P_{K(1)+1, K(2)-2}\left\{P_{K(2)+1, K(3)-2} \ldots P_{K(t-1)+1, K(t)-2}\right\} c_{K(t)+1},
$$

which follows from (10) and (14).
The $t=1$ case is treated in a manner similar to the $t=1$ case in 1B and 1C. This completes the proof of Theorem 1.

The proof and formulation of a modified Hofstadter's conjecture for other irrationals remains an open and difficult problem. To generalize (3), it seems reasonable to conjecture that, for every irrational, there exists a finite set of strings and a finite set of integers such that, for every $m$, $x_{m} \supset x ; Q x_{m-n}$ with $Q$ and $n$ belonging to these finite sets. The authors announced a proof of the deceptively simple case $\alpha=\sqrt{2}-1$ with $m$ equal to sums of Pell numbers [5]. This proof required considerable alteration of Definition 1 and Lemma 3, as well as a more developed form of Lemma 4.

## ACKNOWLEDGMENTS

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AMS Classification Numbers: 68R15, 20M35, 20M05
$\% \%$

# AN UNEXPECTED ENCOUNTER WITH THE FIBONACCI NUMBERS 

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In this article, an incident is narrated whereby the author unexpectedly came across the Fibonacci numbers while solving a problem concerning arithmetic progressions. The incident occurred when the author observed that $3+4+5+6=18=3 \cdot 6$. That is, the sum of the elements in the arithmetic progression $3,4,5,6$ is equal to the product of the first and last terms of the progression. Its generalization can be stated as follows.

Problem: Find three positive integers $a, h$, and $n$ such that

$$
a+(a+h)+\cdots+(a+(n-1) h)=a(a+(n-1) h)
$$

Solution: First, note that since we have an arithmetic progression, we have a solution when

$$
\begin{equation*}
n a+\frac{n(n-1) h}{2}=a^{2}+a(n-1) h \tag{1}
\end{equation*}
$$

which on solving for $a$ becomes

$$
\begin{equation*}
a=\frac{n-(n-1) h+\sqrt{n^{2}+(n-1)^{2} h^{2}}}{2} \tag{2}
\end{equation*}
$$

Since $a$ is an integer, for a solution, there must be an integer $z$ such that $z^{2}=n^{2}+(n-1)^{2} h^{2}$ or such a triple $(n,(n-1) h, z)$ is a Pythagorean triple. Hence, by the well-known parametrization for Pythagorean triples, a solution must exist if and only if there exist integers $x$ and $y$ such that

$$
\begin{equation*}
2 x y=n, x^{2}-y^{2}=(n-1) h, x^{2}+y^{2}=z \tag{3}
\end{equation*}
$$

or

$$
\begin{equation*}
2 x y=(n-1) h, x^{2}-y^{2}=n, x^{2}+y^{2}=z \tag{4}
\end{equation*}
$$

Solving for $h$ in both (3) and (4) and then finding the value of $a$ in (2), we have a solution to (1) whenever there exists a pair of integers $x$ and $y$ such that

$$
\begin{equation*}
\frac{x^{2}-y^{2}}{2 x y-1}=h, a=y(x+y), n=2 x y \tag{5}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{2 x y}{x^{2}-y^{2}-1}=h, a=x(x-y), n=x^{2}-y^{2} \tag{6}
\end{equation*}
$$

A program in BASIC was written and run to obtain such pairs of integers $x$ and $y$. A careful analysis of the output revealed that many solutions were related to the Fibonacci numbers. That is

Theorem: Let $m$ be any positive integer, $F_{k}$ be the $k^{\text {th }}$ Fibonacci number, $n=F_{2 m-1} F_{2 m+2}$, $a=F_{2 m}^{2}+1$, and $h=2$. Then, we have a solution to (1).

Proof: First, we observe the well-known facts that

$$
n=F_{2 m-1} F_{2 m+2}=F_{2 m+1}^{2}-F_{2 m}^{2}
$$

and

$$
a=F_{2 m-1} F_{2 m+1}=F_{2 m}^{2}+1 .
$$

Now, using these identities with (1) and the definition of the Fibonacci numbers, we have

$$
\begin{aligned}
n(a+(n-1)) & =\left(F_{2 m+1}^{2}-F_{2 m}^{2}\right)\left(F_{2 m}^{2}+F_{2 m+1}^{2}-F_{2 m}^{2}\right) \\
& =\left(F_{2 m+1}^{2}-F_{2 m}^{2}\right) F_{2 m+1}^{2} \\
& =F_{2 m+1}^{2} F_{2 m-1} F_{2 m+2} \\
& =F_{2 m+1}^{2} F_{2 m-1}\left(2 F_{2 m+1}^{2}-F_{2 m+1} F_{2 m-1}\right) \\
& =F_{2 m+1} F_{2 m-1}\left(2 F_{2 m+1}^{2}-F_{2 m}^{2}-1\right) \\
& =F_{2 m+1} F_{2 m-1}\left(F_{2 m}^{2}+1+2 F_{2 m+1}^{2}-2 F_{2 m}^{2}-2\right) \\
& =a(a+2(n-1)) .
\end{aligned}
$$

Hence, there exist a countable infinite set of segments of arithmetic progressions with a common difference of 2 such that the sum of the elements in the segments is equal to the product of the first and last terms. Below, we give a few examples:

$$
\begin{array}{ll}
2,4,6 & (m=1), \\
10,12,14, \ldots, 40 & (m=2), \\
65,67,69, \ldots, 273 & (m=3), \\
442,444,446, \ldots, 1870 & (m=4) .
\end{array}
$$

The other values generated by the BASIC program did not appear to be related to the Fibonacci numbers.

The connection between the solution of the problem and elements of the Pell sequence is established by the author in an article which will appear in Math. Student 63 (1994).

## ACKNOWLEDGMENT

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AMS Classification Numbers: 11B39, 11B25, 11D09

## ADDENDUM TO

## "Second Derivative Sequences of Fibonacci and Lucas Polynomials" <br> by

## Piero Filipponi and Alwyn F. Horadam

In the above paper [1], the proof of Proposition 9 was inadvertently omitted. It reads as follows:

Proof of Proposition 9: From (1.8) we have

$$
\begin{align*}
C_{n} & =\sum_{i=0}^{n} F_{i}^{(1)} F_{n-i}=\frac{1}{5}\left(\sum_{i=0}^{n} i L_{i} F_{n-i}-\sum_{i=0}^{n} F_{i} F_{n-i}\right)  \tag{5.11}\\
& =\frac{1}{5} \sum_{i=0}^{n} i F_{n}+\frac{1}{5} \sum_{i=0}^{n} i(-1)^{i} F_{n-2 i}-\frac{1}{5} \sum_{i=0}^{n} F_{i} F_{n-i}
\end{align*}
$$

From (5.1) and (5.3), (5.11) can be rewritten as

$$
\begin{aligned}
C_{n} & =\frac{1}{10}\left[n(n+1) F_{n}\right]-\frac{1}{25}\left(n L_{n+1}+2 F_{n}\right)-\frac{1}{25}\left(n L_{n}-F_{n}\right) \\
& =\frac{1}{50}\left[5 n(n+1) F_{n}-2 F_{n}-2 n L_{n+2}\right]=\frac{1}{50}\left[\left(5 n^{2}-2\right) F_{n}+5 n F_{n}-2 n L_{n+2}\right] \\
& \left.=\frac{1}{50}\left[\left(5 n^{2}-2\right) F_{n}-n\left(2 L_{n+2}-5 F_{n}\right)\right]=\frac{1}{50}\left[5 n^{2}-2\right) F_{n}-3 n L_{n}\right]=F_{n}^{(2)} / 2
\end{aligned}
$$

Additional comment: With regard to Conjectures 1-7 in [1], some of which were known by us to be true, we wish to record that, in private correspondence with us, both Richard André-Jeannin and David Zeitlin have independently established the validity of these Conjectures.

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# THE FIBONACCI AND LUCAS TRIANGLES MODULO 2 

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## 1. INTRODUCTION

The Fibonacci and Lucas coefficients are defined as

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]_{\mathscr{F}}=\frac{F_{n} F_{n-1} \cdots F_{1}}{\left(F_{k} F_{k-1} \cdots F_{1}\right)\left(F_{n-k} F_{n-k-1} \cdots F_{1}\right)} \text {, and }\left[\begin{array}{l}
n \\
k
\end{array}\right]_{\mathscr{L}}=\frac{L_{n} L_{n-1} \cdots L_{1}}{\left(L_{k} L_{k-1} \cdots L_{1}\right)\left(L_{n-k} L_{n-k-1} \cdots L_{1}\right)} .
$$

These coefficients have been studied by several authors, [2], [8], [14], and [18]. Using these definitions, what we call the Fibonacci and Lucas triangles are formed in the same way as Pascal's Triangle is formed from ordinary binomial coefficients, that is, the $n^{\text {th }}$ row is $\left[\begin{array}{l}n \\ k\end{array}\right]$ for $0 \leq k \leq n$. Other authors, e.g., [3], [10], have also constructed such triangles in various ways. The ordinary binomial coefficients modulo 2 and Pascal's Triangle modulo 2 have been studied extensively in [4], [5], [6], [7], [11], [17], [20], [22], [23], and [25]. Among problems of interest have been the determination of the parity of binomial coefficients, the number of odd coefficients in the $n^{\text {th }}$ row of Pascal's Triangle, and the iterative structure of Pascal's Triangle modulo 2. We will extend these results to both the Fibonacci and Lucas coefficients modulo 2 in sections 2 and 3. In section 4 we also determine the relationship between the Fibonacci and Lucas coefficients.

Portions of these triangles, both the originals and their modulo 2 reductions, are shown below. Since the Lucas coefficients are not always integers, the symbol $\alpha$ will be used to denote those coefficients, $\left[\begin{array}{c}n \\ k\end{array}\right]_{\mathscr{Q}}$, that have a higher power of 2 in the denominator than in the numerator.
1
11
121
1331
14641
15101051
1615201561
172135352171
18285670562881

Pascal's Triangle


Fibonacci Triangle

$$
\begin{gathered}
1 \\
11 \\
101 \\
1111 \\
10001 \\
110011 \\
1010101 \\
11111111 \\
100000001
\end{gathered}
$$

## Pascal's Triangle Modulo 2

$1_{1}^{1}$<br>111<br>$\begin{array}{llll}1 & 0 & 0 & 1\end{array}$<br>110111<br>$\begin{array}{llllll}1 & 1 & 1 & 1 & 1\end{array}$<br>$\begin{array}{lllllll}1 & 0 & 0 & 0 & 0 & 0 & 1\end{array}$<br>$\begin{array}{llllllll}1 & 1 & 0 & 0 & 0 & 0 & 1 & 1\end{array}$<br>$\begin{array}{lllllllll}1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1\end{array}$

Fibonacci Triangle Modulo 2


Lucas Triangle

```
                        1
                        1 1
                        1 11
                        10
                        1 1 1 0 1 1
                            111111111
        1 0 0 a 0 0 1
1 1 0 \alpha \alpha 0 1 1
111 \alpha \alpha \alpha 1 1 1
```


## Lucas Triangle Modulo 2

We will need the following information about regularly divisible sequences, generalized bases, and a generalized form of Kummer's Theorem.

Divisibility questions about sequences, such as which terms are divisible by a given prime, have been investigated by several authors, e.g., [9], [13], [15]. A sequence $\left\{u_{n}\right\}$ is said to be strongly divisible provided

$$
\operatorname{gcd}\left(u_{m}, u_{n}\right)=u_{\operatorname{gcd}(m, n)} \quad \text { for all } m, n \geq 1
$$

The term regularly divisible by all primes is defined in [16] and is shown to be equivalent to that of strongly divisible. We use the following definition which defines the divisibility of the sequence for a set of primes rather than for all primes.

Definition: Let $\left\{A_{n}\right\}_{j=1}^{\infty}$ be a sequence of positive integers. We say that $\left\{A_{n}\right\}_{j=1}^{\infty}$ is regularly divisible with respect to a set of primes $\mathscr{G}=\left\{p_{1}, p_{2}, \ldots\right\}$, provided that, for each $p \in \mathscr{Y}, p^{i} \mid A_{j}$ if and only if $r\left(p^{i}\right) \mid j$, for all $i \geq 1$ and $j \geq 1$, where $r\left(p^{i}\right)$ is the rank of apparition of $p^{i}$, that is, $A_{r\left(p^{i}\right)}$ is the first term in the sequence divisible by $p^{i}$.

A sequence is said to be regularly divisible if it is regularly divisible by all primes. Since the Fibonacci sequence satisfies the requirements for strong divisibility [9], it is a regularly divisible sequence.

We will use $r=r(2)=3$ for the rank of apparition of 2 . That is, $F_{r}$ is the first term in the sequence that is divisible by 2 . For the rank of apparition of $2^{i}$, we will use $r\left(2^{i}\right)=r_{i}$..

We will use a generalized base for the positive integers. Since the Fibonacci sequence is regularly divisible by 2 , we have that $\frac{r_{i+1}}{r_{i}}$ is always an integer. Thus, a generalized base $\mathscr{P}=\{1, r$, $\left.r_{2}, \ldots, r_{i}, \ldots\right\}$ can be used [21] and the number $n$ can be uniquely expressed as

$$
n=\left(n_{t} n_{t-1} \cdots n_{1} n_{0}\right)_{\mathscr{P}}=n_{t} r_{t}+n_{t-1} r_{t-1}+\cdots+n_{1} r+n_{0} \text {, where } 0 \leq n_{i}<\frac{r_{i+1}}{r_{i}} \text {. }
$$

The version of Kummer's Theorem we need is that in [27]:
Kummer's Theorem for Generalized Binomial Coefficients: Let $\mathscr{A}=\left\{\mathscr{A}_{j}\right\}_{j=1}^{\infty}$ be a sequence of positive integers. If $\mathscr{A}$ is regularly divisible by $p$, then the highest power of $p$ that divides $\left[\begin{array}{c}m+n \\ m\end{array}\right]_{A}$ is the number of carries that occur when the integers $n$ and $m$ are added in base $\mathscr{P}$, where $\mathscr{P}=\left\{r_{j}\right\}_{j=0}^{\infty}$ with $r_{0}=1$ and $r_{i}=r\left(p^{i}\right)$, for all $i \geq 1$.

## 2. THE FIBONACCI TRIANGLE MODULO 2

One of the interesting results for Pascal's Triangle modulo 2 is that the number of coefficients in the $n^{\text {th }}$ row which are congruent to 1 modulo 2 , denoted $N[n, 2,1]$, is equal to $2^{t}$, where $t$ is the number of ones in $n$ 's base two representation [24]. A similar result follows for the Fibonacci triangle.

Theorem 1: For the Fibonacci triangle modulo 2, the number of coefficients in the $n^{\text {th }}$ row congruent to 1 modulo 2 is given by $N[n, 2,1]=2^{t} 3^{s}$, where $t=$ number of 1 's and $s=$ number of 2 's in $n$ 's base $\mathscr{P}$ representation.

Proof: The generalized base for the Fibonacci sequence is $\mathscr{P}=\{1,3,6,6,12, \ldots\}$. Since

$$
r_{i+1}=\left\{\begin{array}{l}
r_{i} \\
\text { or } \\
2 r_{i}
\end{array}\right.
$$

for $n=\left(\ldots n_{2} n_{1} n_{0}\right)_{\mathscr{P}}$ and $k=\left(\ldots k_{2} k_{1} k_{0}\right)_{\mathscr{P}}$, we have that $0 \leq n_{i}, k_{i}<2$ for $i \geq 1$ and $0 \leq n_{0}, k_{0}<3$. From Webb \& Wells [27], $N[n, 2,1]=\prod_{i \geq 0}\left(n_{i}+1\right)$. For no borrow to occur in the base $\mathscr{P}$ subtraction of $k$ from $n$, there are two choices for $k_{i}$ for each $n_{i}=1$, and one choice for each $n_{i}=0$. If $n_{0}=2$, there are three choices for $k_{0}$. Therefore, $N[n, 2,1]=2^{t} 3^{s}$ where $t=$ number of 1's and $s=$ number of 2 's in $n$ 's base $\mathscr{P}$ representation.

The following theorem, which is similar to Lucas's theorem for binomial coefficients, provides a way to investigate the iterative behavior of the Fibonacci triangle modulo 2.

Theorem 2: The Fibonacci coefficients satisfy

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]_{\mathscr{F}} \equiv \prod_{i \geq 1}\binom{n_{i}}{k_{i}}\left[\begin{array}{l}
n_{0} \\
k_{0}
\end{array}\right]_{\mathscr{F}}(\bmod 2)
$$

where

$$
\left[\begin{array}{l}
n_{0} \\
k_{0}
\end{array}\right]_{\mathscr{F}}=0 \text { for } k_{0}>n_{0} \text { and }\binom{n_{i}}{k_{i}}=0 \text { for } k_{i}>n_{i} .
$$

Proof: If a borrow occurs in the base $\mathscr{P}$ subtraction of $k$ from $n$, then $n_{i}<k_{i}$ for some $i$. Thus, either $\left[\begin{array}{c}n_{0} \\ k_{0}\end{array}\right]_{s}=0$ for $k_{0}>n_{0}$ or $\binom{n_{i}}{k_{i}}=0$ for some $i \geq 1$ and the result holds trivially.

If no borrow occurs, $0 \leq k_{i} \leq n_{i}<2$ for $i \geq 1$, so that

$$
\binom{n_{i}}{k_{i}} \equiv 1(\bmod 2) .
$$

For $i=0,0 \leq k_{0} \leq n_{0}<3$, and

$$
\left[\begin{array}{l}
n_{0} \\
k_{0}
\end{array}\right]_{\mathscr{F}} \equiv 1(\bmod 2)
$$

Thus,

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]_{\mathscr{F}} \equiv \prod_{i \geq 1}\binom{n_{i}}{k_{i}}\left[\begin{array}{l}
n_{0} \\
k_{0}
\end{array}\right]_{\mathscr{F}}(\bmod 2) .
$$

Corollary 2.1: For $n=3 h+n_{0}$ and $m=3 k+k_{0}$,

$$
\left[\begin{array}{l}
n \\
m
\end{array}\right]_{\mathscr{F}} \equiv\binom{h}{k}\left[\begin{array}{l}
n_{0} \\
m_{0}
\end{array}\right]_{\mathscr{F}}(\bmod 2) .
$$

Proof: Let

$$
n=n_{t} r_{t}+n_{t-1} r_{t-1}+\cdots+n_{1} r+n_{0}=3 h+n_{0}
$$

and

$$
m=n_{t} r_{t}+m_{t-1} r_{t-1}+\cdots+m_{1} r+m_{0}=3 k+m_{0} .
$$

Since

$$
r_{i+1}=\left\{\begin{array}{l}
r_{i} \\
o r \\
2 r_{i}
\end{array}\right.
$$

the coefficients in the ordinary base 2 expansion of $h$ and $k$ will be from the sets

$$
\left\{n_{t}, n_{t-1}, \ldots, n_{1}\right\} \text { and }\left\{m_{t}, m_{t-1}, \ldots, m_{1}\right\} .
$$

When $r_{i}=r_{i+1}$ for some $i$, such as $r_{2}=r_{3}=6$, the base $\mathscr{P}$ requires $n_{i}, m_{i}=0$. The base 2 coefficients of $h$ and $k$ will be $n_{i}$ and $m_{i}$ where $r_{i} \neq r_{i+1}$. Although the exact power of 2 associated with each coefficient can be determined only by looking at the relationship between all the elements in the base, $h$ and $k$ will still have an appropriate base 2 expansion. The residue of $\binom{h}{k}$ modulo 2 will be

$$
\binom{h}{k} \equiv \prod_{i \geq 1}\binom{n_{i}}{m_{i}}(\bmod 2) .
$$

The above corollary can be used to investigate the iterative behavior of the Fibonacci triangle modulo 2. To begin, we will use the notation of Long [20].

Theorem 3: Let $\Delta_{n, k}$ denote the following triangle,

$$
\begin{gathered}
{\left[\begin{array}{l}
3 n \\
3 k
\end{array}\right]_{\mathscr{F}}} \\
{\left[\begin{array}{c}
3 n+1 \\
3 k
\end{array}\right]_{\mathscr{F}}\left[\begin{array}{l}
3 n+1 \\
3 k+1
\end{array}\right]_{\mathscr{F}}} \\
{\left[\begin{array}{c}
3 n+2 \\
3 k
\end{array}\right]_{\mathscr{F}}\left[\begin{array}{l}
3 n+2 \\
3 k+1
\end{array}\right]_{\mathscr{F}}\left[\begin{array}{l}
3 n+2 \\
3 k+2
\end{array}\right]_{\mathscr{F}}}
\end{gathered}
$$

a. The entries in $\Delta_{n, k}$ will be either all congruent to 1 or all congruent to 0 modulo 2. The entries in the Fibonacci triangle not included in one of the triangles $\Delta_{n, k}$ are congruent to 0 .
b. The triangles satisfy an element-wise addition modulo $2, \Delta_{n-1, k-1}+\Delta_{n-1, k} \equiv \Delta_{n, k}(\bmod 2)$.
c. The Fibonacci triangle of $\Delta_{n, k}$ 's is in 1-1 correspondence with Pascal's Triangle modulo 2.

Proof: Since $[t]_{s} \equiv 1(\bmod 2)$ for $0 \leq s \leq t<3$, we have

$$
\left.\begin{array}{cc}
{\left[\begin{array}{l}
3 n \\
3 k
\end{array}\right]_{\mathscr{F}}} & \binom{n}{k}\left[\begin{array}{l}
0 \\
0
\end{array}\right]_{\mathscr{F}} \\
{\left[\begin{array}{c}
3 n+1 \\
3 k
\end{array}\right]_{\mathscr{F}}\left[\begin{array}{l}
3 n+1 \\
3 k+1
\end{array}\right]_{\mathscr{F}}} & \equiv \\
{\left[\begin{array}{c}
3 n+2 \\
3 k
\end{array}\right]_{\mathscr{F}}\left[\begin{array}{l}
n \\
k
\end{array}\right)\left[\begin{array}{l}
1 \\
0
\end{array}\right]_{\mathscr{F}}\binom{n}{k}\left[\begin{array}{l}
1 \\
1
\end{array}\right]_{\mathscr{F}}} \\
3 k+1
\end{array}\right]_{\mathscr{F}}\left[\begin{array}{l}
3 n+2 \\
3 k+2
\end{array}\right]_{\mathscr{F}} \quad\binom{n}{k}\left[\begin{array}{l}
2 \\
0
\end{array}\right]_{\mathscr{F}}\left[\begin{array}{l}
n \\
k
\end{array}\right)\left[\begin{array}{l}
2 \\
1
\end{array}\right]_{\mathscr{F}}\binom{n}{k}\left[\begin{array}{l}
2 \\
2
\end{array}\right]_{\mathscr{F}} .
$$

Therefore,

$$
\left.\begin{array}{c}
{\left[\begin{array}{l}
3 n \\
3 k
\end{array}\right]_{\mathscr{F}}} \\
{\left[\begin{array}{c}
3 n+1 \\
3 k
\end{array}\right]_{\mathscr{F}}\left[\begin{array}{l}
3 n+1 \\
3 k+1
\end{array}\right]_{\mathscr{F}}} \\
+2 \\
3 k
\end{array}\right]_{\mathscr{F}}\left[\begin{array} { c } 
{ 1 } \\
{ [ \begin{array} { l } 
{ 3 n + 2 } \\
{ 3 k + 1 }
\end{array} ] _ { \mathscr { F } } [ \begin{array} { l } 
{ 3 n + 2 } \\
{ 3 k + 2 }
\end{array} ] _ { \mathscr { F } } }
\end{array} \equiv \left\{\begin{array}{c}
T_{1}=11 \text { if }\binom{n}{111} \equiv 1 \\
0 \\
T_{0}=1 \text { if } \begin{array}{l}
n \\
000
\end{array}\binom{n}{k} \equiv 0
\end{array}\right.\right.
$$

The entries not included in one of these triangles are of the form $\left[\begin{array}{c}3 n+t \\ 3 k+s\end{array}\right]$ with $0 \leq t<s \leq 2$, and so are congruent to 0 modulo 2 .

From Corollary 2.1, we have that

$$
\left[\begin{array}{c}
3(n-1)+t \\
3(k-1)+s
\end{array}\right]_{\mathscr{F}}+\left[\begin{array}{c}
3(n-1)+t \\
3 k+s
\end{array}\right]_{\mathscr{F}} \equiv\binom{n-1}{k-1}\left[\begin{array}{c}
t \\
s
\end{array}\right]_{\mathscr{F}}+\binom{n-1}{k}\left[\begin{array}{l}
t \\
s
\end{array}\right]_{\mathscr{F}} \equiv\binom{n}{k}\left[\begin{array}{l}
t \\
s
\end{array}\right]_{\mathscr{F}} \equiv\left[\begin{array}{c}
3 n+t \\
3 k+s
\end{array}\right]_{\mathscr{F}} .
$$

Thus, there is an element-wise addition of triangles that satisfies

$$
\Delta_{n-1, k-1}+\Delta_{n-1, k} \equiv \Delta_{n, k}(\bmod 2)
$$

If the identification $T_{1} \leftrightarrow 1$ and $T_{0} \leftrightarrow 0$ is made, the Fibonacci triangle of $\Delta$ 's is in 1-1 correspondence with Pascal's Triangle modulo 2.


Fibonacci Triangle Modulo 2
Pascal's Triangle Modulo 2

With this theorem, once the identification with Pascal's Triangle is made, one can see that the pattern continues at all levels of $r\left(2^{t}\right)$. For example, at the level of $r\left(2^{2}\right)=6$, with

$$
T_{1}^{T_{1}} T_{1} \leftrightarrow 1 \quad \text { and } \quad T_{0} T_{0} T_{0} \leftrightarrow 0
$$

we.have the identification shown below


Fibonacci Triangle Modulo 2

## Pascal's Triangle Modulo 2

## 3. THE LUCAS TRIANGLE MODULO 2

Although the Lucas sequence is not regularly divisible, the structure of the triangle modulo 2 is still determined by the highest power of 2 that divides [ $n$ ]! defined below. To determine the residues of the coefficients modulo 2 , the following lemma will be needed. We will use the notation $2^{t} \| m$ to mean $2^{t} \mid m$ but that $2^{t+1} \mid m$.

Lemma 1: If [ $n$ ]! $=L_{n} L_{n-1} \ldots L_{2} L_{1}$, then

$$
2^{3 k} \|[n]!\text { for } 3(2 k) \leq n<3(2 k+1) \quad \text { and } 2^{3 k-1} \|[n]!\text { for } 3(2 k-1) \leq n<3(2 k) .
$$

Proof: For the Lucas sequence

$$
L_{12}=322 \equiv 2 \text { and } L_{13}=521 \equiv 1(\bmod 8) .
$$

Thus, the length of the period modulo 8 is 12 , because $L_{0}=2$ and $L_{1}=1$. Also since

$$
L_{n} \not \equiv 0 \text { for } 1 \leq n \leq 12(\bmod 8),
$$

we have that $8 \backslash L_{n}$ for any $n$.
Also, as above,

$$
L_{6}=18 \equiv 2 \text { and } L_{7}=29 \equiv 1(\bmod 4),
$$

so the length of the period modulo 4 is 6 . For $1 \leq n \leq 6, L_{n} \equiv 0(\bmod 4)$ only for $n=3$. Thus, $L_{n} \equiv 0(\bmod 4)$ for $n=3+6 k=3(2 k+1), k \geq 0$. For $1 \leq n \leq 6,, L_{n} \equiv 2(\bmod 4)$ only for $n=6$. So $2 \mid L_{n}$ and $4 \backslash L_{n}$ for $n=6 k$.

In $[n]$ ! there are $\left\lfloor\frac{n}{3}\right\rfloor$ factors that are divisible by 2 and $\left\lfloor\left.\frac{n}{6} \right\rvert\,\right.$ factors that are exactly divisible by 2. Thus, there are $\left\lfloor\frac{n}{3}\right\rfloor-\left\lfloor\frac{n}{6}\right\rfloor$ that are exactly divisible by $2^{2}$, and so
$2^{3 k} \|[n]!$ for $3(2 k-1) \leq n<3(2 k+1)$ and $2^{3 k-1} \|[n]!$ for $3(2 k-1) \leq n<3(2 k)$.
Theorem 4: The Lucas coefficients satisfy the following congruences.
For $0 \leq m \leq n$ and $0 \leq s \leq t \leq 2$,

$$
\left[\begin{array}{c}
3 n+t \\
3 m+s
\end{array}\right]_{\mathscr{L}} \equiv\left\{\begin{array}{ll}
\alpha & \text { for } n \text { even and } m \text { odd } \\
1 & \text { otherwise }
\end{array}(\bmod 2) .\right.
$$

For $0 \leq m \leq n$ and $0 \leq t<s \leq 2$,

$$
\left[\begin{array}{l}
3 n+t \\
3 m+s
\end{array}\right]_{\mathscr{L}} \equiv 0(\bmod 2) .
$$

Proof: Let $e$ be the highest power of 2 that exactly divides $\left[\begin{array}{c}3 n+t \\ 3 m+s\end{array}\right]$. Then $e=e_{1}-\left(e_{2}+e_{3}\right)$, where $2^{e_{1}} \mathrm{i}[3 n+t]!, 2^{e_{2}} i[3 m+s]$ ! and $2^{e_{3}} \mathrm{i}[3(n-m)+(t-s)]$ !.

By examining the different cases for $n$ and $m$ odd or even, and applying Lemma 1 , we obtain the following values for $e$.

For $0 \leq m \leq n$ and $0 \leq s \leq t \leq 2$,

$$
e= \begin{cases}-1 & \text { if } n \text { is even and } m \text { is odd; } \\ 0 & \text { if } n \text { is even and } m \text { is even; } \\ 0 & \text { if } n \text { is odd and } m \text { is odd; } \\ 0 & \text { if } n \text { is odd and } m \text { is even. }\end{cases}
$$

For $0 \leq m \leq n$ and $0 \leq t<s \leq 2$,

$$
e= \begin{cases}1 & \text { if } n \text { is even and } m \text { is odd; } \\ 1 & \text { if } n \text { is even and } m \text { is even; } \\ 1 & \text { if } n \text { is odd and } m \text { is odd; } \\ 2 & \text { if } n \text { is odd and } m \text { is even. }\end{cases}
$$

This theorem can be used to count the number of each of the residues modulo 2 in the $n^{\text {th }}$ row of the Lucas triangle and to investigate the iterative patterns in the triangle. The Lucas sequence has the same recurrence relation as the Fibonacci sequence and, like the Fibonacci sequence, satisfies $r(2)=3$, which is also equal to the period of 2 . In determining the number of each of the residues in the $n^{\text {th }}$ row of the Lucas triangle, we will use the generalized base corresponding to 2 for the Fibonacci sequence, $\mathscr{P}=\{1,3,6,6,12, \ldots\}$.

Theorem 5: Let $N[n, 2, a]$ be the number of Lucas coefficients in the $n^{\text {th }}$ row congruent to $a$. For $n=3 h+n_{0}, 0 \leq n_{0}<3$,

$$
N[n, 2,1]=\left\{\begin{array}{ll}
(h+1)\left(n_{0}+1\right) & \text { if } h \text { is odd, } \\
\left(\frac{h}{2}+1\right)\left(n_{0}+1\right) & \text { if } h \text { is even, }
\end{array} \quad N[n, 2, \alpha]= \begin{cases}0 & \text { if } h \text { is odd } \\
\left(\frac{h}{2}\right)\left(n_{0}+1\right) & \text { if } h \text { is even }\end{cases}\right.
$$

and

$$
N[n, 2,0]=h\left(2-n_{0}\right)
$$

Proof: For $n=3 h+n_{0}$ and $m=3 k+m_{0}$, if $h$ is odd, then

$$
\left[\begin{array}{l}
3 h+n_{0} \\
3 k+m_{0}
\end{array}\right]_{\mathscr{L}} \equiv 1(\bmod 2),
$$

provided $0 \leq m_{0} \leq n_{0}$. Therefore, there are $h+1$ choices for $k$, and there are $n_{0}+1$ choices for $m_{0}$. Thus, $N[n, 2,1]=(h+1)\left(n_{0}+1\right)$.

If $h$ is even, then

$$
\left[\begin{array}{l}
3 h+n_{0} \\
3 k+m_{0}
\end{array}\right]_{\mathscr{L}} \equiv \begin{cases}1 & \text { for } k \text { even, } \\
\alpha & \text { for } k \text { odd. }\end{cases}
$$

Thus, there are $\left(\frac{h}{2}+1\right)$ choices for $k$ to be even and $\left(\frac{h}{2}\right)$ choices for $k$ to be odd. There are still $\left(n_{0}+1\right)$ choices for $m_{0}$, so that

$$
N[n, 2,1]=\left(\frac{h}{2}+1\right)\left(n_{0}+1\right) \text { and } N[n, 2, \alpha]=\left(\frac{h}{2}\right)\left(n_{0}+1\right), \text { for } h \text { even. }
$$

If $0 \leq n_{0}<m_{0} \leq 2$, then

$$
\left[\begin{array}{l}
3 h+n_{0} \\
3 k+m_{0}
\end{array}\right]_{\mathscr{L}} \equiv 0(\bmod 2) .
$$

There are $h$ choices for $k$ and $\left(2-n_{0}\right)$ choices for $m_{0}$, so that $N[n, 2,0]=h\left(2-n_{0}\right)$.
Theorem 6: For $0 \leq m \leq n$, the entries in the Lucas triangle denoted $\Delta_{n, m}$,

$$
\begin{gathered}
{\left[\begin{array}{l}
3 n \\
3 m
\end{array}\right]_{\mathscr{L}}} \\
{\left[\begin{array}{c}
3 n+1 \\
3 m
\end{array}\right]_{\mathscr{L}}\left[\begin{array}{l}
3 n+1 \\
3 m+1
\end{array}\right]_{\mathscr{L}}} \\
{\left[\begin{array}{c}
3 n+2 \\
3 m
\end{array}\right]_{\mathscr{L}}\left[\begin{array}{l}
3 n+2 \\
3 m+1
\end{array}\right]_{\mathscr{L}}\left[\begin{array}{l}
3 n+2 \\
3 m+2
\end{array}\right]_{\mathscr{L}}}
\end{gathered}
$$

are either all congruent to one or all congruent to $\alpha$ modulo 2 . The entries not included in these triangles are congruent to zero modulo 2.

Proof: From Theorem 4, it follows directly that the entries in the initial triangles are all congruent to $\alpha$ modulo 2 if $n$ is even and $m$ is odd. Otherwise, all entries are congruent to 1 . The entries not included in these triangles are $\left[\begin{array}{c}3 n+t \\ 3 m+s]_{\varepsilon}\end{array}\right.$, where $^{2} \leq t<s \leq 2$, and so are congruent to zero modulo 2.

Theorem 7: For $r_{i}=2^{i-1}$, let $\Delta_{n, m}$ denote the following entries in the Lucas triangle,

$$
\begin{aligned}
& {\left[\begin{array}{l}
n r_{i} \\
m r_{i}
\end{array}\right]_{\mathscr{L}}} \\
& {\left[\begin{array}{c}
n r_{i}+1 \\
m r_{i}
\end{array}\right]_{\mathscr{L}}\left[\begin{array}{c}
n r_{i}+1 \\
m r_{i}+1
\end{array}\right]_{\mathscr{L}}} \\
& {\left[\begin{array}{c}
n r_{i}+r_{i}-1 \\
m r_{i}
\end{array}\right]_{\mathscr{L}} \quad\left[\begin{array}{c}
n r_{i}+r_{i}-1 \\
m r_{i}+r_{i}-1
\end{array}\right]_{\mathscr{L}},}
\end{aligned}
$$

and let $\nabla_{n, m}$ denote the entries not included in one of these triangles.
a. For $i=1$, the initial triangles, $\Delta_{n, m}, \Delta_{n, m+1}, \Delta_{n+1, m+1}$, do not satisfy an element-wise addition modulo 2 as in the Fibonacci triangle.
b. For $i>1$, the triangles satisfy

$$
\begin{gathered}
\Delta_{n, m} \equiv \Delta_{n, m+1} \equiv \Delta_{n+1, m+1} \equiv \Delta_{0,0} \\
\nabla_{n, m} \equiv \nabla_{n, m+1} \equiv \nabla_{n+1, m+1} .
\end{gathered}
$$

Proof: For $i=1$, from the Lucas triangle modulo 2, we can see that

$$
\begin{array}{ll}
\Delta_{1,0}+\Delta_{1,1} \not \equiv \Delta_{2,1} & (\bmod 2) \\
\Delta_{5,2}+\Delta_{5,3} \not \equiv \Delta_{6,3} & (\bmod 2) .
\end{array}
$$

Thus, the initial triangles do not satisfy an element-wise addition modulo 2 .
For $i>1$ and $0 \leq h, k \leq 2^{i-1}-1, h$ and $k$ determine whether $2^{i-1} n+h$ and $2^{i-1} m+k$ are odd or even, so that

$$
\left[\begin{array}{l}
n r_{i}+3 h+t \\
m r_{i}+3 k+s
\end{array}\right]_{\mathscr{L}}=\left[\begin{array}{l}
3\left(2^{i-1} n+h\right)+t \\
3\left(2^{i-1} m+k\right)+s
\end{array}\right]_{\mathscr{L}} \equiv\left[\begin{array}{l}
3 h+t \\
3 k+s
\end{array}\right]_{\mathscr{L}}(\bmod 2) .
$$

Thus,

$$
\left[\begin{array}{l}
n r_{i}+3 h+t \\
m r_{i}+3 k+s
\end{array}\right]_{\mathscr{L}} \equiv\left[\begin{array}{c}
n r_{i}+3 h+t \\
(m+1) r_{i}+3 k+s
\end{array}\right]_{\mathscr{L}} \equiv\left[\begin{array}{c}
(n+1) r_{i}+3 h+t \\
(m+1) r_{i}+3 k+s
\end{array}\right]_{\mathscr{L}} \equiv\left[\begin{array}{l}
3 h+t \\
3 k+s
\end{array}\right]_{\mathscr{L}}(\bmod 2) .
$$

Therefore,

$$
\Delta_{n, m} \equiv \Delta_{n, m+1} \equiv \Delta_{n+1, m+1} \equiv \Delta_{0,0} \quad \text { and } \quad \nabla_{n, m} \equiv \nabla_{n, m+1} \equiv \nabla_{n+1, m+1} .
$$

From Theorem 7, the Lucas triangle of $\Delta \mathrm{s}$ with $i=1$ has initial triangles

$$
\begin{array}{ccc}
1 & & \alpha \\
T_{1}=11 & \text { and } & T_{\alpha}=\alpha \alpha \\
111 & & \alpha \alpha \alpha
\end{array} .
$$

Using the identification $T_{1} \leftrightarrow 1$ and $T_{\alpha} \leftrightarrow \alpha$, the pattern in the Lucas triangle becomes more apparent.


## Lucas Triangle Modulo 2

Also from Theorem 7, we see that this pattern does not continue for $i>1$. For example, with $i=2$, if the following correspondence is made,

$$
T_{1}=\begin{gathered}
1 \\
11 \\
111 \\
111011 \\
111111
\end{gathered} \leftrightarrow 1
$$

then the Lucas triangle modulo 2 can be associated with a triangle of all ones. That is, the initial triangle will be the only triangle repeated.

$$
\begin{aligned}
& \begin{array}{cc}
1 \\
11 \\
111 \\
100
\end{array} \\
& \begin{array}{llll}
11 & 0 & 0 & 1 \\
111011
\end{array} \\
& 111111 \\
& 100 \alpha 001 \\
& 110 \alpha \alpha 011 \\
& \begin{array}{lll}
111 & \alpha \alpha \alpha 111 & \\
1001001001 & \quad & 1
\end{array} \\
& 11011011011 \\
& 1111111111111
\end{aligned}
$$

$$
\begin{aligned}
& 111 \alpha \alpha \alpha 111 \alpha \alpha \alpha 111 \\
& 1001001001001001 \\
& 11011011011011011 \\
& 11111111111111111111
\end{aligned}
$$

## Lucas Triangle Modulo 2

## 4. THE RELATIONSHIP BETWEEN THE FIBONACCI AND LUCAS TRIANGLES MODULO 2

We can use Theorem 2 and Theorem 4 to look at the relationship between the Fibonacci triangle and the Lucas triangle modulo 2.


## Lucas Triangle Modulo 2

Fibonacci Triangle Modulo 2

Theorem 8: The Fibonacci and Lucas coefficients satisfy the following relationships modulo 2:

$$
\begin{aligned}
& \text { If }\left[\begin{array}{c}
n \\
m
\end{array}\right]_{\mathscr{F}} \equiv 1 \text {, then }\left[\begin{array}{l}
n \\
m
\end{array}\right]_{\mathscr{Q}} \equiv 1 . \\
& \text { If }\left[\begin{array}{l}
n \\
m
\end{array}\right]_{\mathscr{F}} \equiv 0 \text {, then }\left[\begin{array}{l}
n \\
m
\end{array}\right]_{\mathscr{L}} \equiv \begin{cases}0 & \text { if a borrow occurs in the } n_{0} \text { position, } \\
1 & \text { if a borrow occurs in the } n_{1} \text { position, } \\
\text { If }\left[\begin{array}{l}
n \\
m
\end{array}\right]_{\mathscr{L}} & \equiv 0 \text {, then }\left[\begin{array}{l}
n \\
m
\end{array}\right]_{\mathscr{F}} \equiv 0 .\end{cases}
\end{aligned}
$$

If $\left[\begin{array}{l}n \\ m\end{array}\right]_{\mathscr{L}} \equiv \alpha$, then $\left[\begin{array}{l}n \\ m\end{array}\right]_{\mathscr{F}} \equiv 0$.
If $\left[\begin{array}{l}n \\ m\end{array}\right]_{\mathscr{L}} \equiv 1$, then $\left[\begin{array}{l}n \\ m\end{array}\right]_{\mathscr{F}} \equiv \begin{cases}0 & \text { if a borrow occurs, } \\ 1 & \text { if no borrow occurs. }\end{cases}$
Proof: For $n=3 h+n_{0}=\left(\ldots n_{2} n_{1} n_{0}\right)_{\mathscr{P}}$ and $m=3 k+m_{0}=\left(\ldots m_{2} m_{1} m_{0}\right)_{\mathscr{P}}$, if $\left[\begin{array}{c}n \\ m\end{array}\right]_{\mathscr{s}} \equiv 1(\bmod 2)$, then $m_{1} \leq n_{1}<2$ and $m_{0} \leq n_{0}<3$. Thus, if $n_{1}=1, h$ is odd and $\left[\begin{array}{l}n \\ m\end{array}\right]_{s} \equiv 1(\bmod 2)$.

If $n_{1}=0$, then $k_{1}=0$ and $h$ and $k$ are even, so that $\left[\begin{array}{l}n \\ m\end{array}\right]_{\Phi} \equiv 1(\bmod 2)$.
If $\left[\begin{array}{c}n \\ m\end{array}\right]_{g} \equiv 0(\bmod 2)$, then a borrow occurs. If the borrow occurs in the $n_{0}$ position, $\left[\begin{array}{l}n \\ m\end{array}\right]_{\Phi} \equiv 0$ $(\bmod 2)$. If the borrow occurs in the $n_{1}$ position, then $h$ is even and $k$ is odd. Thus, $\left[\begin{array}{c}n \\ m\end{array}\right]_{\mathscr{e}} \equiv \alpha(\bmod$ 2). For all other borrows, $\left[\begin{array}{c}n \\ {\left[\begin{array}{l}]_{9}\end{array} \equiv 1(\bmod 2) \text {. }\right.}\end{array}\right.$

If $\left[\begin{array}{l}n \\ m\end{array}\right]_{\mathscr{q}} \equiv 0(\bmod 2)$, then $0 \leq n_{0}<m_{0}<3$. Thus, a borrow occurs in the base $\mathscr{P}$ subtraction of $m$ from $n$. Therefore, $\left[\begin{array}{c}n \\ m\end{array}\right]_{s} \equiv 0(\bmod 2)$.

If $\left[\begin{array}{c}n \\ m\end{array}\right]_{9} \equiv \alpha(\bmod 2)$ implies $h$ is even and $k$ is odd, which occurs only if $n_{1}=0$ and $m_{1}=1$. This means a borrow will occur in the base $\mathscr{P}$ subtraction of $m$ from $n$ and $\left[\begin{array}{c}n \\ m\end{array}\right]_{s} \equiv 0(\bmod 2)$.

If $\left[\begin{array}{c}n \\ m\end{array}\right]_{q} \equiv 1(\bmod 2)$, then no borrow occurs in the $n_{0}$ or $n_{1}$ positions. However, a borrow may occur in other positions. Thus,

$$
\left[\begin{array}{l}
n \\
m
\end{array}\right]_{\mathscr{F}} \equiv \begin{cases}0 & \text { if a borrow occurs, } \\
1 & \text { if no borrow occurs. }\end{cases}
$$

## 5. CONCLUSION

The iterative patterns in the Fibonacci triangle and Pascal's Triangle modulo 2 are similar except for the initial triangles that are repeated in both. For the Fibonacci triangle, the initial triangle is

$$
\begin{array}{r}
1 \\
11 \\
T=111
\end{array}
$$

and for Pascal's Triangle, the initial triangle is

$$
T=\frac{1}{11} .
$$

These triangles arise because $r(2)=3$ for the Fibonacci case, which also equals the period modulo 2 for the Fibonacci sequence and $r(2)=2$ for the Pascal case, which also equals the period modulo 2 for the positive integers. If we look at all second-order sequences, $u_{n}=a u_{n-1}+b u_{n-2}$ with initial conditions $u_{0}=0$ and $u_{1}=1$, they can be categorized into four types.

1. For $a \equiv 0, b \equiv 1(\bmod 2), u_{n} \equiv u_{n-2}(\bmod 2)$, for $n \geq 2$ and $r(2)=2$ which equals the period of 2 .
2. For $a, b \equiv 1(\bmod 2), u_{n} \equiv u_{n-1}(\bmod 2)$, for $n \geq 2$ and $r(2)=3$ which equals the period of 2 .
3. For $a \equiv 1, b \equiv 0(\bmod 2), u_{n} \equiv u_{n-1}(\bmod 2)$, for $n \geq 2$. The prime 2 does not occur as a factor.
4. For $a, b \equiv 0(\bmod 2), u_{n} \equiv 0(\bmod 2)$, for $n \geq 2$. All terms are divisible by 2 .

This means there are only four distinct triangles modulo 2 formed by the generalized coefficients,

$$
\left[\begin{array}{c}
n \\
k
\end{array}\right]=\frac{u_{n} u_{n-1} \cdots u_{1}}{\left(u_{k} u_{k-1} \cdots u_{1}\right)\left(u_{n-k} u_{n-k-1} \cdots u_{1}\right)} .
$$

1. Pascal's Triangle comes from type 1 sequences: $\begin{array}{cc} & 1 \\ 11 \\ 111 \\ 1111 \\ 100011 \\ 110011 \\ 1010101 \\ 11111111\end{array}$.
2. The Fibonacci triangle comes from type 2 sequences: $\begin{gathered}11 \\ \\ \\ 111 \\ 1001 \\ \\ \\ \\ \\ 1101111 \\ 1000001 \\ 1100011 \\ \\ 11100011\end{gathered}$.
3. A triangle of 1's comes from type 3 sequences: $\begin{gathered}1 \\ 11 \\ 1111 \\ 11111 .\end{gathered}$

111111
11111111
4. A triangle of 0 's comes from type 4 sequences:
1
11
101
1001
10001
100001
1000001
10000001.

Thus, Pascal's Triangle and the Fibonacci triangle are the only two significant triangles modulo 2 . They only differ by the repetition of the initial triangle. When the initial conditions are changed, the sequence is no longer regularly divisible. The triangles of coefficients from these sequences, such as the Lucas triangle, do not have the same iterative behavior as Pascal's Triangle.

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# MODIFIED NUMERICAL TRIANGLE AND THE FIBONACCI SEQUENCE 

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## 1. INTRODUCTION

A study of the number $(1+\sqrt{5}) / 2 \cong 1.618 \ldots$ and of its Fibonacci derivation has received considerable attention not only in the field of pure mathematics but also in mathematical modeling and analysis of such physical plants as cascades of two-ports, hot mill metallurgical processes, multicomponent rectifications in distillation column, reactions in stirred tank reactors and batch reactors [6]. As the mathematical basis for solutions of various problems in these systems serves usually the theory of recurrence equations ([2], [3]) of the Fibonacci sequence and their generalizations ([1], [4]). Many problems concerning a variety of generalizations of the Fibonacci sequence have appeared, primarily in The Fibonacci Quarterly, in recent years.

We shall be concerned in this paper with the Fibonacci sequence introduced via a modified numerical triangle (MNT). We shall involve a generalized Pascal triangle (GGPT) and "shifted" form of the MNT (SMNT) and show how the MNT results from a suitable superposition of the generalized and shifted triangles. We shall also prove that a transfer ratio $T_{k}(k=0,1,2, \ldots, n)$ of the output to input-signal along an electrical ladder network is determined by polynomials with coefficients belonging to the MNT.

## 2. THE MODIFIED NUMERICAL TRIANGLE

The MNT is defined here in connection with studies of distributions of voltages and currents along an electrical ladder network with $n$ identical interacting cells [5]. One elementary section of such structures is characterized by a parameter $x$ determined by the product of impedance of a longitudinal branch and admittance of a transversal branch.

The transfer ratio $T_{k}(k=0,1,2, \ldots, n)$ of the output- to input-signal (voltage or current) along the network (Fig. 1) is determined by polynomials in $x$ of the corresponding degree. It can be determined from a solution of the following recurrence equation,

$$
\begin{equation*}
a_{k+1}-(2+x) a_{k}+a_{k-1}=0 \tag{1a}
\end{equation*}
$$

with

$$
\begin{equation*}
a_{1}=(1+x) a_{0} \tag{lb}
\end{equation*}
$$

where $a_{0}$ denotes a known signal at the input port of the first cell and $a_{k}$ is the corresponding signal at the $k$-port of the network (e.g., $a_{k}=V_{k}$ as shown in Fig. 1).


Figure 1

The ratio $T_{k}$ follows from the relation

$$
\begin{equation*}
T_{k}=\frac{a_{k}}{a_{0}}, k=0,1,2, \ldots, n \tag{2}
\end{equation*}
$$

It is easy to see that $T_{k}$ is determined by a polynomial in $x$ of the $k^{\text {th }}$ degree, so we can write

$$
\begin{equation*}
T_{k}=\sum_{m=0}^{k} p_{k, m} x^{m}, k=0,1,2, \ldots, n . \tag{3}
\end{equation*}
$$

From direct inspection of the above expression, we have that

$$
\begin{align*}
& T_{0}(x)=1, \\
& T_{1}(x)=1+x, \\
& T_{2}(x)=1+3 x+x^{2}, \\
& T_{3}(x)=1+6 x+5 x^{2}+x^{3},  \tag{4}\\
& T_{4}(x)=1+10 x+15 x^{2}+7 x^{3}+x^{4}, \\
& T_{5}(x)=1+15 x+35 x^{2}+28 x^{3}+9 x^{4}+x^{5},
\end{align*}
$$

The polynomial coefficients

$$
\begin{equation*}
p_{k, m}=\left.\frac{1}{m!} \frac{\partial^{m} T_{k}(x)}{\partial^{m} x}\right|_{x=0} \tag{5}
\end{equation*}
$$

belong to the MNT that takes the following form:

| $k$ | MNT |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 0 | 1 |  |  |  |  |  |
| 1 | 1 | 1 |  |  |  |  |
| 2 | 1 | 3 | 1 |  |  |  |
| 3 | 1 | 6 | 5 | 1 |  |  |
| 4 | 1 | 10 | 15 | 7 | 1 |  |
| 5 | 1 | 15 | 35 | 28 | 9 | 1 |

It must be noted that from the MNT it is easy to obtain the expression of the polynomial $T_{k}(x)$ for small values of $k$. To determine $T_{k}(x)$ for large values of $k$, we can use formulas (1) and (2).

Observe that formally the MNT is apparently similar to the classic Pascal Triangle. Note that the MNT coefficients cannot be evaluated directly by applying the rule corresponding to the classic Pascal Triangle. On the other hand, it is possible, by some appropriate modification of the Pascal Triangle, to establish a suitable recurrence rule for constructing the MNT coefficients. We will present a solution to this problem in the next section.

## 3. THE GENERALIZED AND SHIFTED TRIANGLES AND THEIR LINKS WITH THE MNT

By a slight modification of the MNT we can establish the so-called shifted modified numerical triangle (SMNT). We draw SMNT from MNT by shifting its rows and columns by two places in the bottom and then annihilating all coefficients in the first column. If we denote by $s_{k, m}$ a coefficient
for a node $(k, m), k=0,1,2, \ldots, n$ and $m=0,1, \ldots, k$, then the corresponding formula takes the following form:

$$
s_{k, m}= \begin{cases}0 & \text { for } m=0  \tag{6}\\ p_{k-2, m} & \text { for } 1 \leq m \leq k-2, \\ 0 & \text { for } m>k-2\end{cases}
$$

The resulting SMNT is demonstrated by the following construction:

| $k$ | SMNT |  |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 0 | 0 |  |  |  |  |  |  |
| 1 | 0 | 0 |  |  |  |  |  |
| 2 | 0 | 0 | 0 |  |  |  |  |
| 3 | 0 | 1 | 0 | 0 |  |  |  |
| 4 | 0 | 3 | 1 | 0 | 0 |  |  |
| 5 | 0 | 6 | 5 | 1 | 0 | 0 |  |
| 6 | 0 | 10 | 15 | 7 | 1 | 0 | 0 |
| 7 | 0 | 15 | 35 | 28 | 9 | 1 | 0 |
| $\ldots$ | 0 | 0 |  |  |  |  |  |

Applying the above rule, for instance for $s_{5,2}$, we obtain

$$
s_{5,2}=p_{3,2}=5
$$

The GGPT is constructed similarly to the usual Pascal Triangle (PT), with only two modifications. First, in evaluating a given node numerical element in the GGPT, its upper right-hand side node element is taken twice and the upper left-hand node coefficient is taken in the same way as in the classic PT. Second, before performing calculations for node coefficients in the $(k+1)$ row of the GGPT we must subtract the $k^{\text {th }}$ row of the SMNT from the $k^{\text {th }}$ row of the GPT. If we denote by $g_{k, m}$ the GGPT coefficient corresponding to the $(k, m)^{\text {th }}$ node, then the following rule,

$$
\begin{equation*}
g_{k, m}=g_{k-1, m-1}-s_{k-1, m-1}+2\left(g_{k-1, m}-s_{k-1, m}\right) \tag{7}
\end{equation*}
$$

holds for $k=0,1,2, \ldots, n$ and $m=0,1, \ldots, k$ with $g_{k, 0}=1$ and $g_{k-1, m}=0$ for $m>k-1$ and $g_{k-1, m-1}$ $=0$ for $m-1<0$. The above rule is illustrated by the following representation of the GGPT:

| $k$ | GGPT |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 0 | 1 |  |  |  |  |  |
| 1 | 1 | 1 |  |  |  |  |
| 2 | 1 | 3 | 1 |  |  |  |
| 3 | 1 | 7 | 5 | 1 |  |  |
| 4 | 1 | 13 | 16 | 7 | 1 |  |
| 5 | 1 | 21 | 40 | 29 | 9 | 1 |

Now we can show a link between the MNT and the generalized and shifted triangles. Applying successively, row-by-row, the rules corresponding to the GGPT and SMNT for the MNT and comparing coefficients that correspond to a given node in all three triangles it is easy to demonstrate that by formal notation we have

$$
\begin{equation*}
\mathrm{MNT}=(\mathrm{GGPT}-\mathrm{SMNT}) \tag{8}
\end{equation*}
$$

We must emphasize that in this expression the subtraction must be performed successively "row-by-row." This process can be represented by the following diagram:

$$
\begin{aligned}
& \text { Row } k \text { of MNT } \xrightarrow{\text { rule (6) }} \text { Row } k+2 \text { of SMNT } \\
& \longrightarrow \text { Row } \mathrm{k}+1 \text { of GGPT }- \text { Row } \mathrm{k}+1 \text { of } \operatorname{SMNT}=\text { Row } \mathrm{k}+1 \text { of } \mathrm{MNT} \text {. }
\end{aligned}
$$

An illustration of this procedure is shown in the following construction:

| $k$ | GGPT |  |  |  |  |  |  | $k$ | SMNT |  |  |  |  |  |  |  | $k$ | MNT |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 |  |  |  |  |  |  | 0 |  | 0 |  |  |  |  |  |  | 0 | 1 |  |  |  |  |  |  |
| 1 | 1 | 1 |  |  |  |  |  | 1 |  | 0 | 0 |  |  |  |  |  | 1 | 1 | 1 |  |  |  |  |  |
| 2 | 1 | 3 | 1 |  |  |  |  | 2 |  | 0 | 0 | 0 |  |  |  |  | 2 | 1 | 3 | 1 |  |  |  |  |
| 3 | 1 | 7 | 5 | 1 |  |  |  | 3 |  | 0 | 1 | 0 | 0 |  |  |  | $=3$ | 1 | 6 | 5 | 1 |  |  |  |
| 4 | 1 | 13 | 16 | 7 | 1 |  |  | 4 |  |  | 3 | 1 | 0 | 0 |  |  | 4 | 1 | 10 | 15 | 7 | 1 |  |  |
| 5 | 1 | 21 | 40 | 29 | 9 | 1 |  | 5 |  | 0 | 6 | 5 | 1 | 0 | 0 |  | 5 | 1 | 15 | 35 | 28 | 9 |  |  |
| 6 | 1 | 31 | 85 | 91 | 46 | 11 | 1 | 6 |  | 0 | 10 | 15 | 7 | 1 | 0 | 0 | 6 | 1 | 21 | 70 | 84 | 45 | 11 |  |

Studying the above construction, it is easy to prove that the following recurrence equation:

$$
\begin{equation*}
p_{k, m}=p_{k-1, m-1}+2 p_{k-1, m}-p_{k-2, m} \tag{9}
\end{equation*}
$$

holds for the coefficients of the MNT with $k=0,1,2, \ldots, n$ and $m=0,1, \ldots, k$, where $p_{r, s}=0$ if $r<0$ and/or $s>0$. For example, isf we fix $k=5$ and $m=3$, then we obtain

$$
p_{5,3}=p_{4,2}+2 p_{4,3}-p_{3,3}=28
$$

The above construction leads to important simplifications in determinating the transfer functions of a ladder network with a large number of interacting cells. Some other interesting results may be obtained by considering special diagonals of the usual Pascal Triangle or a particular direct formula for successive rows of the MNT. The work in this direction is under development, and further results will be published soon.

## 4. NUMBER OF TERMS IN THE TRANSFER FUNCTION AND THE FIBONACCI SEQUENCE

To each ladder network can be attributed a corresponding signal flow graph by virtue of which the transfer function from the source node to a sink node can be determined. In the signal flow graph of a ladder network, there are no loops and, consequently, the total transfer function simplifies to the form of expression (3). On the other hand, the signal flow graph of a ladder network can be represented by an oriented graph attributed to the MNT (see Fig, 2).


Figure 2
Each oriented branch of this graph is labeled by a transmittance equal to one. The resulting node coefficients correspond to the respective coefficients of the MNT and, simultaneously, to the
number of open paths in the signal flow graph counted from the source node (the top of the graph) to the sink node (the given node in the graph). The presented graph is very useful for determining all paths appearing in the total transfer function of a ladder network containing a large number of interacting cells as, for instance, a dozen or several dozen. Moreover, it gives a possibility to answer the following question, among others: How many different open paths and corresponding total transfer functions appear in the signal flow graph and in the ladder network, respectively? It must be noted that, in the case of a quite simple ladder network, the number of open paths in the total transmittance increases rapidly with the number of cells. It can be determined by suitable use of the Fibonacci sequence that

$$
\begin{equation*}
F_{k+1}=F_{k}+F_{k-1}, k=0,1,2, \ldots, n \tag{10}
\end{equation*}
$$

with $F_{0}=0$ and $F_{1}=1$.
From Binet's formula, we have

$$
\begin{equation*}
F_{k}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}-\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right], k-0,1,2, \ldots, n \tag{11}
\end{equation*}
$$

The form of this expression can be simplified by taking into consideration the Newton expansion of a power of a binomial in which the values $a=1$ and $b=+\sqrt{5}$ or $-\sqrt{5}$ are substituted. Finally, we obtain

$$
\begin{equation*}
F_{k}=\frac{1}{2^{k-1}}\left[\binom{k}{1}+5\binom{k}{3}+5^{2}\binom{k}{5}+\cdots 5^{r}\binom{k}{2 r+1}+\cdots+\right] . \tag{12}
\end{equation*}
$$

Note that the right-hand side of this expression vanishes for $2 r+1>k$. For instance, at $k=7$, the first vanishing term corresponds to $2 r+1>7$, i.e., $r \geq 4$. In this case, the value of $F_{k}$ amounts to $F_{7}=21$ and is composed of four terms. Moreover, a direct inspection of the oriented graph shown in Figure 2 points to a relationship between the number of open paths appearing in the total transmittance of a given ladder network. It is equal to the sum of all paths counted from the top node to all sink nodes attributed to a given level in the oriented graph representing the MNT. For a ladder network composed of $n$ cells, the total number of terms which determine the transfer function $T_{r}(x)$ at the input of the $r^{\text {th }}$ cell is equal to $F_{k}$ given by (12). If the voltage to voltage ratio is computed, then we must take $k=2 r+1$ and when the current to current ratio is determined, then $k=2 r$. For example, if a network consists of eight cells, then the total number of terms in the voltage transfer function $T_{v 8}(x)$ is equal to

$$
S_{v 8}=F_{17}=\frac{1}{2^{16}}\left[\binom{17}{1}+5\binom{17}{3}+5^{2}\binom{17}{5}+\cdots+5^{8}\binom{17}{17}\right]=1597 .
$$

This number determines, simultaneously, the sum of all coefficients in the MNT at the level $k=8$. The result can be easily checked by direct inspection of the MNT up to the $8^{\text {th }}$ level.

## 5. CONCLUSIONS

The Fibonacci sequence (10) has been effectively applied to the analysis of ladder networks consisting of identical interacting cells. It has been shown that the modified numerical triangle corresponds to the respective polynomials determining the transfer functions in the network. Mapping the MNT by an oriented graph gives a possibility to evaluate all coefficients in the transfer function and the total number of terms appearing in this function. This number is
expressed by the Fibonacci sequence $F_{2 k+1}$ for the voltage transfer function and by the Fibonacci sequence $F_{2 k}$ if the current transfer function is determined.

## ACKNOWLEDGMENT

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# INTEGRATION AND DERIVATIVE SEQUENCES FOR PELL AND PELL-LUCAS POLYNOMIALS 

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## 1. INTRODUCTION

Previously in [1] and [2], in which integration and first derivative sequences for Fibonacci and Lucas polynomials were introduced, it was suggested that these investigations could be extended to Pell and Pell-Lucas polynomials. Here, we explore some of their basic features in outline to obtain the flavor of their substance. Further details may be found in [5], with some variation in notation.

Pell polynomials $P_{n}(x)$ are defined by the recurrence relation

$$
\begin{equation*}
P_{n}(x)=2 x P_{n-1}(x)+P_{n-2}(x), \quad P_{0}(x)=0, P_{1}(x)=1, \tag{1.1}
\end{equation*}
$$

while the associated Pell-Lucas polynomials $Q_{n}(x)$ are defined by

$$
\begin{equation*}
Q_{n}(x)=2 x Q_{n-1}(x)+Q_{n-2}(x), \quad Q_{0}(x)=2, Q_{1}(x)=2 x . \tag{1.2}
\end{equation*}
$$

Standard procedures readily lead to the Binet forms

$$
\begin{equation*}
P_{n}(x)=\frac{\alpha^{n}(x)-\beta^{n}(x)}{2 \Delta(x)} \tag{1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
Q_{n}(x)=\alpha^{n}(x)+\beta^{n}(x), \tag{1.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta(x)=\sqrt{x^{2}+1}, \alpha(x)=x+\Delta(x), \beta(x)=x-\Delta(x) \tag{1.5}
\end{equation*}
$$

Properties of $P_{n}(x)$ and $Q_{n}(x)$ are given in [3] and [5].
Substitution of $x=1$ in (1.1) and (1.2) leads to the corresponding Pell numbers $P_{n}=P_{n}(1)$ and Pell-Lucas numbers $Q_{n}=Q_{n}(1)$. For reference, we tabulate some values of $P_{n}$ and $Q_{n}$ :

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $P_{n}$ | 0 | 1 | 2 | 5 | 12 | 29 | 70 | 169 | 408 | $\cdots$ |
| $Q_{n}$ | 2 | 2 | 6 | 14 | 34 | 82 | 198 | 478 | 1154 | $\cdots$ |

All the $Q_{n}$ are even numbers, as is manifest from (1.2). The $P_{n}$ are alternately odd and even.

## 2. PROPERTIES OF DERIVATIVE SEQUENCES

Using known [3] summation formulas for $P_{n}(x)$ and $Q_{n}(x)$, we derive the first derivative Pell sequence $\left\{P_{n}^{\prime}(x)\right\}$ given by

$$
\begin{equation*}
P_{n}^{\prime}(x)=2 \sum_{m=0}^{\left[\frac{n-1}{2}\right]}(n-2 m-1)\binom{n-m-1}{m}(2 x)^{n-2 m-2} \quad(n \geq 1) \tag{2.1}
\end{equation*}
$$

and the first derivative Pell-Lucas sequence $\left\{Q_{n}^{\prime}(x)\right\}$ for which

$$
\begin{equation*}
Q_{n}^{\prime}(x)=2 n \sum_{m=0}^{\left[\frac{n}{2}\right]}\binom{n-m-1}{m}(2 x)^{n-2 m-1} \quad(n \geq 1) \tag{2.2}
\end{equation*}
$$

where the dash denotes differentiation with respect to $x$ and the symbol [•] represents the greatest integer function.

From (1.1) and (1.2), we must have

$$
\begin{equation*}
P_{0}^{\prime}(x)=0 \text { and } Q_{0}^{\prime}(x)=0 . \tag{2.3}
\end{equation*}
$$

Expressions (2.1) and (2.2) yield the first few polynomials $P_{n}^{\prime}(x)$ and $Q_{n}^{\prime}(x)$ [5]:

$$
\begin{array}{ll}
P_{1}^{\prime}(x)=0 & Q_{1}^{\prime}(x)=2 \\
P_{2}^{\prime}(x)=2 & Q_{2}^{\prime}(x)=8 x \\
P_{3}^{\prime}(x)=8 x & Q_{3}^{\prime}(x)=24 x^{2}+6 \\
P_{4}^{\prime}(x)=24 x^{2}+4 & Q_{4}^{\prime}(x)=64 x^{3}+32 x \\
P_{5}^{\prime}(x)=64 x^{3}+24 x & Q_{5}^{\prime}(x)=160 x^{4}+120 x^{2}+10 \\
P_{6}^{\prime}(x)=160 x^{4}+96 x^{2}+6 & Q_{6}^{\prime}(x)=384 x^{5}+384 x^{3}+72 x \\
P_{7}^{\prime}(x)=384 x^{5}+320 x^{3}+48 x & Q_{7}^{\prime}(x)=896 x^{6}+1120 x^{4}+336 x^{2}+14 \\
P_{8}^{\prime}(x)=896 x^{6}+960 x^{4}+240 x^{2}+8 & Q_{8}^{\prime}(x)=2048 x^{7}+3072 x^{5}+1280 x^{3}+128 x .
\end{array}
$$

Putting $x=1$ in (2.4), we derive the corresponding first derivative Pell sequence numbers $\left\{P_{n}^{\prime}\right\}=\left\{P_{n}^{\prime}(1)\right\}$ and first derivative Pell-Lucas sequence numbers $\left\{Q_{n}^{\prime}\right\}=\left\{Q_{n}^{\prime}(1)\right\}$, tabulated thus [5]:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $P_{n}^{\prime}$ | 0 | 0 | 2 | 8 | 28 | 88 | 262 | 752 | 2104 | $\cdots$ |
| $Q_{n}^{\prime}$ | 0 | 2 | 8 | 30 | 96 | 290 | 840 | 2366 | 6528 | $\cdots$ |

All the numbers $P_{n}^{\prime}$ and $Q_{n}^{\prime}$ are even, by virtue of the factor 2 in (2.1) and (2.2).
Elementary calculations using (1.5) produce

$$
\begin{align*}
& \alpha^{\prime}(x)=\frac{\alpha(x)}{\Delta(x)},  \tag{2.6}\\
& \beta^{\prime}(x)=-\frac{\beta(x)}{\Delta(x)} \tag{2.7}
\end{align*}
$$

$$
\begin{align*}
& \left\{\alpha^{n}(x)\right\}^{\prime}=\frac{n \alpha^{n}(x)}{\Delta(x)},  \tag{2.8}\\
& \left\{\beta^{n}(x)\right\}^{\prime}=-\frac{n \beta^{n}(x)}{\Delta(x)}, \tag{2.9}
\end{align*}
$$

whence we derive, after a little calculation using (1.3) and (1.4),

$$
\begin{equation*}
P_{n}^{\prime}(x)=\frac{n Q_{n}(x)-2 x P_{n}(x)}{2 \Delta^{2}(x)} \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
Q_{n}^{\prime}(x)=2 n P_{n}(x) \tag{2.11}
\end{equation*}
$$

Taken in conjunction with (1.3) and (1.4), equations (2.10) and (2.11) allow us to express $P_{n}^{\prime}(x)$ and $Q_{n}^{\prime}(x)$ in their Binet forms.

Substituting $x=1$ in (2.10) and (2.11), we have immediately

$$
\begin{equation*}
P_{n}^{\prime}=\frac{n Q_{n}-2 P_{n}}{4} \tag{2.12}
\end{equation*}
$$

and

$$
\begin{equation*}
Q_{n}^{\prime}=2 n P_{n} . \tag{2.13}
\end{equation*}
$$

For example, $P_{6}^{\prime}=262=\frac{6 \cdot 198-2 \cdot 70}{4}=\frac{69_{6}-2 P_{6}}{4}$ by (2.12) and (1.6).
Other basic results are [5]:

$$
\begin{align*}
& \left.\begin{array}{l}
P_{n}^{\prime}=2 P_{n-1}^{\prime}+P_{n-2}^{\prime}+2 P_{n-1} \\
Q_{n}^{\prime}=2 Q_{n-1}^{\prime}+Q_{n-2}^{\prime}+2 Q_{n-1}
\end{array}\right\} \text { recurrence relations },  \tag{2.14}\\
& P_{n+1}^{\prime}+P_{n-1}^{\prime}=Q_{n}^{\prime},  \tag{2.16}\\
& Q_{n+1}^{\prime}+Q_{n-1}^{\prime}=2 n Q_{n}+4 P_{n},  \tag{2.17}\\
& \left.P_{n+1}^{\prime} P_{n-1}^{\prime}-\left(P_{n}^{\prime}\right)^{2}=\frac{8 n^{2}(-1)^{n+1}+4(-1)^{n}-Q_{n}^{2}}{16} \text { (Simson's formula }\right), \tag{2.18}
\end{align*}
$$

and

$$
\begin{equation*}
Q_{n+1}^{\prime} Q_{n-1}^{\prime}-\left(Q_{n}^{\prime}\right)^{2}=4\left\{(-1)^{n}\left(n^{2}-1\right)-P_{n}^{2}\right\} \quad \text { (Simson's formula) } \tag{2.19}
\end{equation*}
$$

To obtain these results, we use (2.12) and (2.13) as well as properties of $P_{n}$ and $Q_{n}$ (1.6). Proof of Simson's formula (2.18) requires much careful calculation though (2.19) follows readily from (2.13) and Simson's formula for $P_{n}$. One may note en passant that (2.16) is analogous to the well-known relations between $P_{n}$ and $Q_{n}$, and $F_{n}$ and $L_{n}$ (Fibonacci and Lucas numbers).

Numerical illustrations of (2.14), (2.17), and (2.18) are, by (1.6) and (2.5), respectively,

$$
\begin{array}{ll}
n=5: & 2 P_{4}^{\prime}+P_{3}^{\prime}+2 P_{4}=56+8+24=88=P_{5}^{\prime}, \\
n=5: & Q_{6}^{\prime}+Q_{4}^{\prime}=840+96=936=10 \cdot 82+4 \cdot 29=10 Q_{5}+4 P_{5}, \\
n=5: & \left\{\begin{array}{l}
P_{6}^{\prime} P_{4}^{\prime}-\left(P_{5}\right)^{2}=262 \cdot 28-88^{2}=-408, \\
\frac{8 \cdot 5^{2}(-1)^{5+1}+4(-1)^{5}-Q_{5}^{2}}{16}=\frac{200-4-6724}{16}=\frac{-6528}{16}=-408 .
\end{array}\right.
\end{array}
$$

Analogues of Simson's formulas (2.18) and (2.19) can be obtained for $P_{n}^{\prime}(x)$ and $Q_{n}^{\prime}(x)$.

## 3. INTEGRATION SEQUENCES

Consider, in a new notation, the integrals [5]

$$
\begin{equation*}
' P_{n}(x)=\int_{0}^{x} P_{n}(s) d s \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
' Q_{n}(x)=\int_{0}^{x} Q_{n}(s) d s \tag{3.2}
\end{equation*}
$$

where the pre-symbol dash represents integration.
Using the summation formulas for $P_{n}(x)$ and $Q_{n}(x)$ [3], we readily obtain

$$
\begin{equation*}
{ }^{\prime} P_{n}(x)=\sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{2^{n-2 m-1}}{n-2 m}\binom{n-1-m}{m} x^{n-2 m} \quad(n \geq 1) \tag{3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
Q_{n}(x)=\sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{n 2^{n-2 m}}{(n-m)(n-2 m+1)}\binom{n-m}{m} x^{n-2 m+1} \quad(n \geq 1) . \tag{3.4}
\end{equation*}
$$

From (1.1), (1.2), (3.1), and (3.2), we deduce that

$$
\begin{equation*}
' P_{0}(x)=0, \quad ' Q_{0}(x)=2 x \tag{3.5}
\end{equation*}
$$

Sequences $\left\{{ }^{\prime} P_{n}(x)\right\}$ and $\left\{{ }^{\prime} Q_{n}(x)\right\}$ may be called the Pell integration sequence and the PellLucas integration sequence, respectively. Their first few expressions, obtained from (3.3) and (3.4), are [5]:

$$
\begin{array}{ll}
' P_{1}(x)=x & ' Q_{1}(x)=x^{2} \\
{ }^{\prime} P_{2}(x)=x^{2} & ' Q_{2}(x)=\frac{4}{3} x^{3}+2 x \\
{ }^{\prime} P_{3}(x)=\frac{4}{3} x^{3}+x & ' Q_{3}(x)=2 x^{4}+3 x^{2} \\
{ }^{\prime} P_{4}(x)=2 x^{4}+2 x^{2} & ' Q_{4}(x)=\frac{16}{5} x^{5}+\frac{16}{3} x^{3}+2 x \\
{ }^{\prime} P_{5}(x)=\frac{16}{5} x^{5}+4 x^{3}+x & ' Q_{5}(x)=\frac{16}{3} x^{6}+10 x^{4}+5 x^{2}  \tag{3.6}\\
{ }^{\prime} P_{6}(x)=\frac{16}{3} x^{6}+8 x^{4}+3 x^{2} & \prime Q_{6}(x)=\frac{64}{7} x^{7}+\frac{96}{5} x^{5}+12 x^{3}+2 x \\
{ }^{\prime} P_{7}(x)=\frac{64}{7} x^{7}+16 x^{5}+8 x^{3}+x & \prime Q_{7}(x)=16 x^{8}+\frac{112}{3} x^{6}+28 x^{4}+7 x^{2} \\
{ }^{\prime} P_{8}(x)=16 x^{8}+32 x^{6}+20 x^{4}+4 x^{2} & ' Q_{8}(x)=\frac{256}{9} x^{9}+\frac{512}{7} x^{7}+64 x^{5}+\frac{64}{3} x^{3}+2 x .
\end{array}
$$

Putting $x=1$ in (3.6), we obtain the Pell integration sequence numbers $\left\{{ }^{\prime} P_{n}(1)\right\}=\left\{{ }^{\prime} P_{n}\right\}$, and the Pell-Lucas integration sequence numbers $\left\{{ }^{\prime} Q_{n}(1)\right\}=\left\{{ }^{\prime} Q_{n}\right\}$, respectively, of which the first few members are [5]:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| ${ }^{\prime} P_{n}$ | 0 | 1 | 1 | $\frac{7}{3}$ | 4 | $\frac{41}{5}$ | $\frac{49}{3}$ | $\frac{239}{7}$ | 72 | $\cdots$ |
| ${ }^{\prime} Q_{n}$ | 2 | 1 | $\frac{10}{3}$ | 5 | $\frac{158}{15}$ | $\frac{61}{3}$ | $\frac{1482}{35}$ | $\frac{265}{3}$ | $\frac{11902}{63}$ | $\cdots$ |

Two elementary properties of $\left\{{ }^{\prime} P_{n}\right\}$ and $\left\{{ }^{\prime} Q_{n}\right\}$ are

$$
' P_{n}= \begin{cases}\frac{Q_{n}}{2 n} & (n>0, \text { odd })  \tag{3.8}\\ \frac{Q_{n}-2}{2 n} & (n>0, \text { even })\end{cases}
$$

and

$$
' Q_{n}= \begin{cases}\frac{2 n\left(2 P_{n}-1\right)-Q_{n}}{n^{2}-1} & (n>1, \text { odd })  \tag{3.9}\\ \frac{4 n P_{n}-Q_{n}}{n^{2}-1} & (n>1, \text { even })\end{cases}
$$

Proofs of these [5] are lengthy but of a relatively elementary nature and are omitted to conserve space. The procedure is to begin with (3.1), (3.2), then integrate with the aid of (1.3)-(1.5), and eventually set $x=1$, taking into account the values of $P_{n}(0)$ and $Q_{n}(0)$ for $n$ even and $n$ odd.

Complicated Binet forms of ${ }^{\prime} P_{n}(x),{ }^{\prime} Q_{n}(x),{ }^{\prime} P_{n}$, and ' $Q_{n}$ are obtainable on applying the corresponding Binet forms for the undashed symbols from (1.3) and (1.4).

From (3.6) and (3.7), we may obtain

$$
\begin{equation*}
' P_{n+1}+{ }^{\prime} P_{n-1}={ }^{\prime} Q_{n} \tag{3.10}
\end{equation*}
$$

and

$$
\begin{equation*}
' P_{n+1}-{ }^{\prime} P_{n-1}=\frac{Q_{n}-Q_{n}}{n} \tag{3.11}
\end{equation*}
$$

Once again, it is worth commenting on the fundamental nature of property (3.10) [cf. (2.16)].
Numerical illustrations of (3.10) and (3.11) are, respectively,

$$
\begin{aligned}
& n=4: \quad{ }^{\prime} P_{5}+{ }^{\prime} P_{3}=\frac{41}{5}+\frac{7}{3}=\frac{158}{15}={ }^{\prime} Q_{4}, \\
& n=6: \quad{ }^{\prime} P_{7}-{ }^{\prime} P_{5}=\frac{239}{7}-\frac{41}{5}=\frac{908}{35}=\frac{198-\frac{1482}{35}}{6}=\frac{Q_{6}-Q_{6}}{6} .
\end{aligned}
$$

The Simson formula analogue for ' $P_{n}$ takes two forms, depending on whether $n$ is odd or even. From (3.8), and invoking the use of Simson's formula for $Q_{n}$, we obtain

$$
' P_{n+1}^{\prime} P_{n-1}-\left({ }^{\prime} P_{n}\right)^{2}= \begin{cases}\frac{8(-1)^{n+1} n^{2}+4 n^{2}+Q_{n}^{2}-16 n^{2} P_{n}}{4 n^{2}\left(n^{2}-1\right)} & (n \text { odd })  \tag{3.12}\\ \frac{8(-1)^{n+1} n^{2}+Q_{n}^{2}+4\left(n^{2}-1\right)\left(Q_{n}-1\right)}{4 n^{2}\left(n^{2}-1\right)} & (n \text { even }) .\end{cases}
$$

As an example, when $n=5$, both sides of (3.12) equal $-\frac{143}{75}$, whereas, if $n=4$, both sides reduce to $\frac{47}{15}$.

From (3.9), a Simson formula analogue for ' $Q_{n}$ is clearly obtainable but its form is left to the curiosity of the reader. Corresponding analogues also exist for ' $P_{n}(x)$ and ' $Q_{n}(x)$.

To check the consistency of the results, one might establish that ${ }^{\prime}\left(P_{n}^{\prime}(x)\right)=\left({ }^{\prime} P_{n}(x)\right)^{\prime}=P_{n}(x)$ and similarly for $Q_{n}(x)$.

## 4. CONCLUDING REMARKS

## Extensions:

Two observations on the foregoing material are relevant:
(i) clearly, the procedures for obtaining integration and first derivative sequences for Fibonacci and Lucas polynomials as in [1] and [2], and for Pell and Pell-Lucas polynomials as herein, can be made more general to embody multiple integration sequences and $n^{\text {th }}$-order derivative sequences, and
(ii) the ideas delineated here are applicable to the generalized recurrence-generated polynomials for which the coefficient $2 x$ in (1.1) and (1.2) is replaced by $k x$, with appropriate initial conditions.

## Simson v Simpson:

Occurrences of analogues to Simson's original formula in 1753 for Fibonacci numbers [4], and the frequent misspellings of Simson's name, prompt us to offer a brief, if only peripherally relevant, historical explanation to clarify the situation. The formula is due to the distinguished Scot, Robert Simson (1687-1768), who was also the author of a highly successful text-book on Euclidean geometry. He is not to be confused with his able contemporary English mathematician, Thomas Simpson (1710-1761), whose name is associated with the rule for approximate quadratures by means of parabolic arcs. Our man is Robert Simson.

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# THE ORDER OF A PERFECT $\boldsymbol{k}$-SHUFFLE 

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(Submitted June 1992)
When you break open a new deck of 52 cards you might wonder how many times you would have to "perfectly" shuffle the cards to return the deck to its original configuration. Our curiosity about this led ultimately to the contents of this paper. By a perfect shuffle here we mean separate the cards into two piles of 26 cards each, then reorder the cards by alternately taking a card from the first pile then one from the other. We call this a perfect 2 -shuffle, which is the same as the out Faro shuffle mentioned in [2], [4], [5], [6], and [9]. The answer to the question above is 8, i.e., the order of a perfect 2 -shuffle on 52 cards is 8 .

As in [4] and [7], we will generalize the idea of a 2 -shuffle to that of a $k$-shuffle. We will then proceed to the main goal of this paper, which is to produce necessary and sufficient conditions under which the order is large in comparison with the number of cards. We will also give a lower bound for the order. The results, embodied in Theorems 1,4 , and 6 , imply certain properties of the graph obtained when one plots order versus deck size. This will, in turn, shed light on question 5 in [9], which asks for reasons for the appearance of such a graph. See also the figures accompanying this paper. In these graphs, it appears as if sets of points line up in straight lines all passing through a common point with more irregularly positioned points above or below these lines. Our concluding remarks indicate how this behavior and much more can be explained. Following these remarks will be found a short description of how we discovered the results communicated in this paper.

Definition 1: Let $k$ and $s$ be integers greater than 1. Take $n=k s$ cards numbered in order 1 through $n$. Place the cards in $k$ piles of $s$ each, the first pile containing, in order, cards 1 through $s$, the second pile containing cards $s+1$ through $2 s$, etc., with the last pile containing cards $(k-1) s+1$ through $k s$. Now, in order, pick up the first card in each pile, then the second, etc., ending with the last card in each pile. The result we call a "perfect $k$-shuffle." The order of this $k$-shuffle, $d_{k}(n)$, will be the minimum number of times the $k$-shuffle needs to be repeated to return the cards to their original configuration. The order of a card will be the minimum number of $k$-shuffles needed to return that card to its original position.

In [4] Medvedoff and Morrison show that $d_{k}(n)$ is the order of $k(\bmod (n-1))$, i.e., the minimum positive integer $r$ such that $k^{r} \equiv 1(\bmod (n-1))$. The key to the proof is in demonstrating that if the cards are numbered 0 through $n-1$, rather than 1 through $n$, then the card numbered $x \neq 0$ or $n-1$, i.e., an interior card, ends up after one perfect $k$-shuffle in the position formerly occupied by the card numbered $k x(\bmod (n-1))$. The first and last cards obviously remain unchanged. The fact that $d_{k}(n) \leq n-2$ is easily deduced from properties of the Euler $\phi$-function. It is also not hard to see that the order of a card divides $d_{k}(n)$ and is the length of the cycle that it is in when the $k$-shuffle is represented as a member of the permutation group on $n$ objects. It is also true that $d_{k}(n)$ is the length of the longest cycle and card 2 will always be in such a cycle.

As an example, consider the case $k=3$ and $s=5$. The three piles

| 1 | 6 | 11 |  | 1 | 12 | 9 |
| ---: | ---: | ---: | :--- | ---: | ---: | ---: |
| 2 | 7 | 12 |  | 6 | 3 | 14 |
| 3 | 8 | 13 | become | 11 | 8 | 5 |
| 4 | 9 | 14 |  | 2 | 13 | 10 |
| 5 | 10 | 15 |  | 7 | 4 | 15 |

after one 3 -shuffle. The permutation representation is

$$
(1)(2,4,10,14,12,6)(8)(3,7,5,13,9,11)(15)
$$

and $d_{3}(15)=6$.
We now produce the promised lower bound for $d_{k}(n)$. This lower bound is related to Theorem 2 on page 9 in [4].

Theorem 1 If $n=k^{t}$, then $d_{k}(n)=t$. Furthermore, if $k^{t}<n<k^{t+1}$, then $d_{k}(n)>t+1$. Hence, $d_{k}(n)=\log _{k}(n)$ if $n=k^{t}$ and $d_{k}(n)>\log _{k}(n)+1$ if $k^{t}<n<k^{t+1}$.

Proof: If $n=k^{t}$, then $k^{t} \equiv 1(\bmod (n-1))$ and $k^{r} \equiv 1(\bmod (n-1))$ with $r<t$ is not possible since $k^{r}-1<n-1$. So, $d_{k}(n)=t$. If $n>k^{t}$, then $u \leq t$ implies $k^{u}-1 \leq k^{t}-1<n-1$ and $k^{u} \neq 1$ $(\bmod (n-1))$. Thus, $d_{k}(n)>t$. Assume $d_{k}(n)=t+1$ for $k^{t}<n<k^{t+1}$. Then $k^{t+1}-1=m(n-1)$ for some $m>1$. Then $k^{t+1}=m k s-(m-1)$, so that $k \mid(m-1), k \leq m-1, k<m$. We also have

$$
n=\frac{k^{t+1}-1}{m}+1<\frac{k^{t+1}-1}{k}+1=k^{t}+\frac{k-1}{k}<k^{t}+1,
$$

a contradiction. Thus, $d_{k}(n)>t+1$ for $k^{t}<n<k^{t+1}$.
The fact that $d_{2}(22)=6$ and $d_{3}(21)=4$ shows that $d_{k}(n)=t+2$ is possible when $k^{t}<n<$ $k^{t+1}$.

Let us now define what we mean by $d_{k}(n)$ being large in comparison with $n$.
Definition 2: If $d_{k}(n)=n-2$, we say that the $k$-shuffle is full. If $d_{k}(n)>(n-2) / 2$, we say that it is over half full.

We are interested in circumstances under which the $k$-shuffle is over half full. The following two theorems follow from the fact that $d_{k}(n)$ is the order of $k(\bmod (n-1))$, the proof of that fact, and elementary number theoretic ideas.

Theorem 2: If the $k$-shuffle is full, then $p=n-1$ is prime.
Theorem 3: If $p=n-1$ is prime, then all interior cards have order $d_{k}(n)=(n-2) / c$, where $c$ is the largest positive integer such that $c \mid(n-2)$ and there is an $x$ such that $x^{c} \equiv k(\bmod p)$, i.e., $c$ is the largest divisor of $n-2$ such that $k$ is a $c$-residue modulo $p$.

The fact that $d_{5}(110)=27$ shows that the converse of Theorem 2 does not hold. On the other hand, Theorem 3 is illustrated by the fact that the 108 interior cards appear in four cycles of 27 each, i.e., each interior card has order 27. Furthermore, $28^{4} \equiv 5(\bmod 109)$ while 5 is not a $c$ residue modulo 109 where $4<c \mid 108$. Although $3^{16} \equiv 5(\bmod 109), 16$ does not divide 108.

The fact that $d_{2}(2048)=11$ and $2047=23.89$ shows that all interior cards can have the same order without $n-1$ being prime. On the other hand, the fact that $d_{2}(10)=6$ and cards 4 and 7 have order 2 shows that, in general, not all interior cards have the same order.

Theorems 2 and 3 together yield the following necessary and sufficient conditions for a $k$-shuffle to be full. Recall that $a$ is a primitive root of $m$ if $(a, m)=1$ and $a$ is of order $\phi(m)$ modulo $m$, where $\phi$ is the Euler $\phi$-function.

Theorem 4: A perfect $k$-shuffle is full, i.e., $d_{k}(n)=n-2$ if and only if $p=n-1$ is a prime (odd) and $k$ is a primitive root of $p$.

Since $d_{2}(20)=18$, for example, 2 must be a primitive root of 19 . From Theorem 3, we see further that, if $p=n-1$ is prime and $k$ is not a primitive root of $p$, then $d_{k}(n) \leq(n-2) / 2$ and the $k$-shuffle is not over half full.

It is interesting that, for some $k$, there can be no full shuffles. Using quadratic reciprocity, we can show that, if $k \equiv 0$ or $1(\bmod 4)$ and $n-1$ is prime, then $k$ is a quadratic residue modulo $p$. Thus, Theorems 2 and 3 show lack of fullness. See also Lemma 2 on page 5 of [4]. A computer check suggests the conjecture that, if $k \equiv 2$ or $3(\bmod 4)$, then there is an $n=k s$ such that $d_{k}(n)=n-2$, i.e., $k$ and $s$ are primitive roots of a prime $p=n-1$. This is similar to Artin's conjecture that, if $k$ is a positive integer that is not a perfect square, then $k$ is a primitive root of infinitely many primes (see [8], p. 81). Not surprisingly, we have made no headway in proving or disproving our conjecture. We can rule out certain cases. Again, using quadratic reciprocity, we can show that, if $k=4 j+2$ and

$$
\begin{aligned}
& n-1=k s-1=p \equiv \pm 1(\bmod 8) \text { and } j \text { or } s \text { is even or } \\
& n-1=k s-1=p \equiv \pm 3(\bmod 8) \text { and } j \text { and } s \text { are odd, }
\end{aligned}
$$

then $k$ is a quadratic residue modulo $p$ and the $k$-shuffle is not full. Examples include

$$
d_{10}(80)=13, d_{14}(168)=83, d_{14}(182)=45 .
$$

Furthermore, if $k=4 j+3, n-1=k s-1=p$, and $s=4 w$, the $k$-shuffle cannot be full. An example is $d_{11}(44)=7$. But note that $d_{2}(44)=14$, where $20^{3} \equiv 2(\bmod 43)$ and 2 is not a $c$-residue modulo 43 for $3<c \mid 42$, is not covered by any of the above cases, all of which employ quadratic residues, while this example involves a cubic residue.

We now turn to necessary and sufficient conditions for which a $k$-shuffle is over half full but not full. From the preceding, it is clear that $n-1$ cannot be prime. We can, in fact, say much more about necessary conditions.

Theorem 5: If $\frac{n-2}{2}<d_{k}(n)<n-2$, then $n-1=p^{a}$, where $p$ is an odd prime and $a \geq 2$.
Before we prove this theorem, we need the following easily verified lemma.
Lemma 1: If $k$ is odd and $a \geq 3$, then $k^{2^{a-2}} \equiv 1\left(\bmod 2^{a}\right)$.
A proof of Theorem 5 is as follows: Suppose $n-1=h g=k s-1$ with $(h, g)=1$. In the case in which $h, g>2$, we have $\phi(h), \phi(g)$ even and

$$
\left(k^{\frac{\phi(h)}{2}}\right)^{\phi(g)} \equiv 1(\bmod g), \quad\left(k^{\frac{\phi(g)}{2}}\right)^{\phi(h)} \equiv 1(\bmod h),
$$

$$
k^{\frac{g(h) \rho(g)}{2}} \equiv 1(\bmod h g) \equiv 1(\bmod (n-1))
$$

and

$$
d_{k}(n) \leq \frac{\phi(h) \phi(g)}{2} \leq \frac{(h-1)(g-1)}{2}=\frac{h g-h-g+1}{2}<\frac{h g-5}{2}=\frac{n-6}{2}<\frac{n-2}{2} .
$$

In the case $h=2, g>2$, we have $k$ odd,

$$
k^{\phi(g)} \equiv 1(\bmod g), \quad k^{\phi(g)} \equiv 1(\bmod h) \quad k^{\phi(g)} \equiv 1(\bmod h g)
$$

and

$$
\phi(g) \leq g-1<\frac{n-1}{2},
$$

so that

$$
d_{k}(n) \leq \phi(g)<\frac{n-1}{2}-\frac{1}{2}=\frac{n-2}{2} .
$$

Thus, $n-1=p^{a}$, where $p$ is a prime and $a \geq 2$. Since $p=a=2$ is impossible, consider $p=2$, $a \geq 3$. Then $n=k s$ is odd, $k$ is odd, and Lemma 1 shows that

$$
d_{k}(n) \leq 2^{a-2}=\frac{2^{a}-2^{a-1}}{2}<\frac{2^{a}-1}{2}=\frac{n-2}{2} .
$$

Thus, $p$ is odd.
To produce sufficient conditions we utilize a lemma (see [8], pp. 98-99).
Lemma 2: Let $p$ be a prime. Then $k$ is a primitive root of $p^{2}$ if and only if $k$ is a primitive root of $p^{a}$ for all $a \geq 1$.

We can now state and prove the theorem we have been aiming for.
Theorem 6: A perfect $k$-shuffle is over half full, but not full, i.e., $\frac{n-2}{2}<d_{k}(n)<n-2$, if and only if $n-1=p^{a}$, where $p$ is an odd prime, $a \geq 2$, and $k$ is a primitive root of $p^{2}$.

Proof: Assume $\frac{n-2}{2}<d_{k}(n)<n-2$. By Theorem $5, n-1=p^{a}$, where $p$ is an odd prime and $a \geq 2$ is necessary. If $k$ is not a primitive root of $p^{2}$ then, by Lemma $2, k$ is not a primitive root of $p^{a}$. Thus,

$$
d_{k}(n)=d_{k}\left(p^{a}+1\right) \leq \frac{(p-1) p^{a-1}}{2}=\frac{p^{a}-p^{a-1}}{2} \leq \frac{p^{a}+1-4}{2}=\frac{n-4}{2}<\frac{n-2}{2} .
$$

Thus, $k$ must be a primitive root of $p^{2}$.
Conversely, assume that $n-1=p^{a}$, where $p$ is an odd prime and $a \geq 2$ and $k$ is a primitive root of $p^{2}$. Then, by Lemma $2, k$ is a primitive root of $p^{a}$. Thus,

$$
d_{k}(n)=d_{k}\left(p^{a}+1\right)=(p-1) p^{a-1}>(p-1)\left(p^{a-1}-\frac{1}{p}\right)=\frac{p-1}{p}\left(p^{a}-1\right)=\frac{p-1}{p}(n-2) .
$$

Since $p \geq 3, \frac{p-1}{p} \geq \frac{2}{3}>\frac{1}{2}$, and we are done.
We can draw the following interesting facts from the proof of Theorem 6.

Corollary 1: A $k$-shuffle is over half full if and only if it is over two-thirds full, i.e.,

$$
d_{k}(n)>\frac{n-2}{2} \text { if and only if } d_{k}(n)>\frac{2}{3}(n-2) .
$$

Furthermore, if the conditions of Theorem 6 hold and $d_{k}(n)=m(n-2)$, then $m$ increases to 1 as $p$ increases and decreases to $\frac{p-1}{p}$ as $a$ increases.

Before we illustrate Theorem 6, note the following, the proofs of which we leave as a challenge to the reader.

Lemma 3: Let $p$ be prime. If the order of $k\left(\bmod p^{j}\right)=b$, then the order of $k\left(\bmod p^{j+1}\right)=b$ or $b p$ and in the latter case the order of $k\left(\bmod p^{r}\right)=b p^{r-j}$ for all $r \geq j$.

Lemma 4: Let $p$ be prime. If $p^{t} \equiv 1(\bmod k)$ and $k \mid\left(p^{j}+1\right)$, then $k \mid p^{t r+j}+1$ for all $r \geq 0$.
Lemma 4 is useful in generating sequences of $k$-shuffles.
Consider the following six examples, each with a slightly different flavor. Notice the relevance of Corollary 1 and Lemmas 3 and 4.
(1) The order of $2\left(\bmod 3^{2}\right)=2 \cdot 3$. Thus, $d_{2}\left(3^{a}+1\right)=2 \cdot 3^{a-1}$. Thus, a 2 -shuffle on $3^{a}+1$ cards is over half full, over $2 / 3$ full in fact. It is full if $a=1$, not full if $a \geq 2$. As $a$ increases, the ratio decreases to $2 / 3$.
(2) The order of $2(\bmod 7)=3$ and the order of $2\left(\bmod 7^{2}\right)=3 \cdot 7 \neq 6 \cdot 7$. Thus, $d_{2}\left(7^{a}+1\right)=$ $3 \cdot 7^{a-1}$. Thus, a 2 -shuffle on $7^{a}+1$ cards is half full if $a=1$ and less than half full if $a \geq 2$.
(3) The order of $3\left(\bmod 5^{2}\right)=4 \cdot 5$. Thus, $d_{3}\left(5^{2 r+1}+1\right)=4 \cdot 5^{2 r}$. Thus, a 3 -shuffle on $5^{2 r+1}+1$ cards is over half full, over $4 / 5$ full in fact. It is full if $r=0$, not full if $r \geq 1$. As $r$ increases, the ratio decreases to $4 / 5$.
(4) The order of $3(\bmod 11)=5$, the order of $3\left(\bmod 11^{2}\right)=5$ and the order of $3\left(\bmod 11^{3}\right)=$ $5 \cdot 11 \neq 10 \cdot 11^{2}$. Thus, $d_{3}(11+1)=5$ and $d_{3}\left(11^{2 r+1}+1\right)=5 \cdot 11^{2 r-1}$ for $r \geq 1$. Thus, a 3-shuffle on $11+1$ cards if half full and a 3 -shuffle on $11^{2 r+1}+1$ cards, $r \geq 1$, is less than half full, much less.
(5) The order of $5\left(\bmod 7^{2}\right)=6.7$. Thus, $d_{5}\left(7^{4 r+2}+1\right)=6 \cdot 7^{4 r+1}$ for $r \geq 0$. Thus, a 5 -shuffle on $7^{4 r+2}+1$ cards is over half full, over $6 / 7$ full in fact, with the ratio decreasing to $6 / 7$ as $r$ increases.
(6) The order of $10(\bmod 487)=486$, and the order of $10\left(\bmod 487^{2}\right)=486 \neq 486 \cdot 487$. Thus, a 10 -shuffle on $487^{4 r+2}+1$ cards where $r \geq 0$ is less than half full, much less.
Example (6) was found on page 102 in [8] and shows that $k$ a primitive root of $p^{2}$ in Theorem 6, cannot be replaced by $k$ a primitive root of $p$.

Remarks: If one were to graph the function $y=d_{k}(n), k$ a constant, plotting $y$ versus $n$, every time $n=p+1, p$ a prime, by Theorems 3 and 4 the points would lie on one of the lines $y=\frac{n-2}{c}$ where $c \mid n-2$ and $c=1$ when $k$ is a primitive root of $p$. See Figure 1 for $k=2$ and recall examples (1) and (2) above. See Figure 2 for $k=3$ and recall examples (3) and (4) above. More irregularly positioned points above or below and sometimes on the lines $y=\frac{n-2}{c}$ are supplied
examples (1) and (2) above. See Figure 2 for $k=3$ and recall examples (3) and (4) above. More irregularly positioned points above or below and sometimes on the lines $y=\frac{n-2}{c}$ are supplied when $n-1$ is composite. In order for points to lie between $y=n-2$ and $y=\frac{n-2}{2}, n$ would have to be a $p^{a}+1$ with $p$ an odd prime, $a \geq 2$, and $k$ a primitive root of $p^{2}$. This is rare and, in fact, sometimes cannot happen, for example, when $k$ is a perfect square. See Figure 3 for $k=4$. Clearly, no point can be above $y=n-2$. If $k \equiv 0$ or $1(\bmod 4)$, no point will lie on $y=n-2$. See Figure 3 again for $k=4$. See Figure 4 for $k=5$ and recall example (5) above. In Figure 1 those points above $y=\frac{n-2}{2}$ are all above $y=\frac{2}{3}(n-2)$ and those near $y=n-2$ are due to large $p$. In Figure 2 those points above $y=\frac{n-2}{2}$ are all above $y=\frac{4}{5}(n-2)$. In Figure 3 those points just below $y=\frac{n-2}{2}$ are all above $y=\frac{n-2}{3}$. In Figure 4 those points above $y=\frac{n-2}{2}$ are all above $y=\frac{2}{3}(n-2)$ and those near $y=n-2$ are due to large $p$. By Theorem 1 all points are on or above $y=\log _{k}(n)$.

We thus have at least a partial explanation for the appearance of the graph in Figure 1 ([9], p. 145 ) which is for in-shuffles with $k=2$ but is similar to a graph for out-shuffles talked about in this paper. Since the order of an in-shuffle on $n$ cards is the order of $k(\bmod (n+1))$ as opposed to the order of $k(\bmod (n-1))$ for the order of an out-shuffle on $n$ cards (see [4], p. 6), the lines in Figure 1 ([9], p. 145) are $y=n / c$. In fact that graph is just a translation of the graph in Figure 1 of this paper.


FIGURE 1. Graph of Order vs Deck Size, $k=2$


FIGURE 2. Graph of Order vs Deck Size, $k=3$


FIGURE 3. Graph of Order vs Deck Size, $\boldsymbol{k}=4$


FIGURE 4. Graph of Order vs Deck Size, $k=5$

## DISCOVERY

After our initial curiosity was aroused, we wrote out a few shuffle permutations by hand for small $n$. It was not long before we had discovered and proved correct the formula for $d_{k}(n)$. It was a shock to later see this as Proposition 1 in [4]. A simple program in BASIC produced printouts of $d_{k}(n)$ using a PC. When we saw what ideas seemed to play significant rules, modifications in the program checked $n-1$ for being prime, $d_{k}(n)$ for being $n-2$, and $d_{k}(n)$ for dividing $n-2$. Essentially every result in this paper represents the successful justification of conjectures suggested by the printouts. Early success with techniques from elementary number theory prompted us to continue in that direction.

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AMS Classification Numbers: 10A10

# GENERALIZED PASCAL TRIANGLES AND PYRAMIDS: THEIR FRACTALS, GRAPHS, AND APPLICATIONS <br> by Dr. Boris A. Bondarenko <br> Associate member of the Academy of Sciences of the Republic of Uzbekistan, Tashkent 

## Translated by Professor Richard C. Bollinger <br> Penn State at Erie, The Behrend College

This monograph was first published in Russia in 1990 and consists of seven chapters, a list of 406 references, an appendix with another 126 references, many illustrations and specific examples. Fundamental results in the book are formulated as theorems and algorithms or as equations and formulas. For more details on the contents of the book, are The Fibonacci Quarterly 31.1 (1993):52.

The translation of the book is being reproduced and sold with the permission of the author, the translator, and the "FAN" Edition of the Academy of Sciences of the Republic of Uzbekistan. The book, which consists approximately 250 pages, is a paperback with a plastic spiral binding. The price of the book is $\$ 31.00$ plus postage and handling where postage and handling will be $\$ 6.00$ if mailed anywhere in the United States or Canada, $\$ 9.00$ by surface mail or $\$ 16.00$ by airmail elsewhere. A copy of the book can be purchased by sending a check made out to THE FIBONACCI ASSOCIATION for the appropriate amount along with a letter requesting a copy of the book to: RICHARD VINE, SUBSCRIPTION MANAGER, THE FIBONACCI ASSOCIATION, SANTA CLARA UNIVERSITY, SANTA CLARA, CA. 95053.

# ON SOME PROPERTIES OF FIBONACCI DIAGONALS IN PASCAL'S TRIANGLE 

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## 1. INTRODUCTION

Although it has been studied extensively, Pascal's triangle remains fascinating to explore and there always seems to be some new aspects that are revealed by looking at it closely. In this paper we shall examine a few nice properties of the so-called Fibonacci diagonals, that is, those slant lines whose entries sum to consecutive terms of the Fibonacci sequence. We adopt throughout our text the convention that the $n^{\text {th }}$ Fibonacci diagonal is the one that contains the binomial coefficients

$$
\binom{n-1}{0}\binom{n-2}{1}\binom{n-3}{2} \ldots
$$

With that notation, the first diagonal contains only $\binom{0}{0}$, the second one contains only $\binom{1}{0}$, the third one contains $\binom{2}{0}$ and $\binom{1}{1}$, and so on. Addition of the terms of the $n^{\text {th }}$ Fibonacci diagonal gives the $n^{\text {th }}$ term of the Fibonacci sequence

$$
1,1,2,3,5,8,13,21,34,55, \ldots
$$

For instance, the terms of the $10^{\text {th }}$ Fibonacci diagonal sum to

$$
\binom{9}{0}+\binom{8}{1}+\binom{7}{2}+\binom{6}{3}+\binom{5}{4}=1+8+21+20+5=55
$$

We shall also be interested in the corresponding diagonals in Pascal's triangle mod 2, that is, the triangle in which the entry $\binom{n}{k}$ is replaced by $\left|\begin{array}{l}n \\ k\end{array}\right|$, its residue $\bmod 2$.

Throughout our discussion, it will be convenient to consider the rows or diagonals of Pascal's triangle as vectors with integer components. For instance, the $n^{\text {th }}$ horizontal row, $n \geq 0$, will be seen as the vector $\vec{X}_{n}$ in $\mathbf{Z}^{n+1}$ defined by

$$
\vec{X}_{n}=\left(\binom{n}{0},\binom{n}{1},\binom{n}{2}, \ldots,\binom{n}{n}\right) .
$$

Various well-known operations on these rows or diagonals can be seen as the scalar product of such vectors with $\vec{B}_{n}=\left(b^{n}, b^{n-1}, \ldots, b, 1\right) \in \mathbf{Z}^{n+1}$ for some $b \in \mathbf{N}$. Let us give some examples involving the above $\vec{X}_{n}$. We shall use the notation $\pi_{b} \vec{X}_{n}$ to designate the scalar product

$$
\vec{X}_{n} \cdot \vec{B}_{n}=\sum_{k=0}^{n}\binom{n}{k} b^{n-k}
$$

(this notation is motivated by the fact that in some sense the vector $\vec{X}_{n}$ is being "projected" on the powers of $b$ ).

By the Binomial Theorem,

$$
\begin{equation*}
\pi_{b} \vec{X}_{n}=(b+1)^{n} \tag{*}
\end{equation*}
$$

In particular, for $b=1$, one gets $\pi_{1} \vec{X}_{n}=2^{n}$, i.e., the terms of the $n^{\text {th }}$ row of Pascal's triangle sum to $2^{n}$. And for $b=10$, one gets $\pi_{10} \vec{X}_{n}=(11)^{n}$. This last equality can be interpreted as follows (see Gardner [1]): when the entries of the rows of Pascal's triangle are considered as the values of a place-value, base-ten numeral, the numbers obtained are the successive powers of 11 . We could of course have a similar interpretation by replacing base-ten numeral by base- $b$ numeral and then the powers of 11 by the powers of $(b+1)$.

Note that (*) can be rewritten as

$$
\pi_{b} \vec{X}_{n}=\pi_{b+1} \overrightarrow{1}_{n}
$$

where $\overrightarrow{1}_{n}=(1,0,0, \ldots, 0) \in \mathbf{Z}^{n+1}$, with a projection appearing on both sides of the equality sign, but with different bases. Such a "change of base" phenomenon will be encountered again in section 2.

A similar discussion can also be undertaken considering the rows of Pascal's triangle mod 2. The $n^{\text {th }}$ row will now be interpreted as the vector

$$
\vec{Y}_{n}=\left(\left|\begin{array}{l}
n \\
0
\end{array}\right|\left|\begin{array}{l}
n \\
1
\end{array}\right|,\left|\begin{array}{l}
n \\
2
\end{array}\right|, \ldots,\left|\begin{array}{l}
n \\
n
\end{array}\right|\right)
$$

in $\mathbb{Z}^{n+1}$ with components 0's and 1's. It was shown by Glaisher [2] that the projection

$$
\pi_{1} \vec{Y}_{n}=\sum_{k=0}^{n}\left|\begin{array}{l}
n \\
k
\end{array}\right|
$$

i.e., the number of odd binomial coefficients $\binom{n}{k}$ for a given $n$, is again a power of 2 , namely $2^{\#(n)}$, where \#(n) represents the number of 1's in the base-two representation of $n$. For instance, the $5^{\text {th }}$ row vector is $\vec{Y}_{5}=(1,1,0,0,1,1)$ so that $\pi_{1} \vec{Y}_{5}=4=2^{\#(5)}$, which corresponds to the fact that 5 is written as 101 in base two with the digit 1 appearing twice. When $b=2$, the projection

$$
\pi_{2} \vec{Y}_{n}=\sum_{k=0}^{n}\left|\begin{array}{l}
n \\
k
\end{array}\right| 2^{n-k}
$$

gives Gould's numbers. These numbers were introduced in Gould [3], where a recursion formula was given for them and a relationship with Fermat's primes was obtained (see also Hodgson [4] for details).

We shall be concerned in this paper with the study of analogous results obtained when Fibonacci diagonals are considered instead of horizontal rows, both in Pascal's triangle and in Pascal's triangle $\bmod 2$.

## 2. FIBONACCI DIAGONALS IN THE STANDARD PASCAL TRIANGLE

Recall that Fibonacci diagonals are numbered starting with $n=1$. For further reference, we list the first twelve vectors thus obtained:

$$
\begin{array}{ll}
\vec{S}_{1}=(1) & \vec{S}_{7}=(1,5,6,1) \\
\vec{S}_{2}=(1) & \vec{S}_{8}=(1,6,10,4) \\
\vec{S}_{3}=(1,1) & \vec{S}_{9}=(1,7,15,10,1) \\
\vec{S}_{4}=(1,2) & \vec{S}_{10}=(1,8,21,20,5) \\
\vec{S}_{5}=(1,3,1) & \vec{S}_{11}=(1,9,28,35,15,1) \\
\vec{S}_{6}=(1,4,3) & \vec{S}_{12}=(1,10,36,56,35,6)
\end{array}
$$

Clearly $\pi_{1} \vec{S}_{n}$ gives the $n^{\text {th }}$ term of the Fibonacci sequence. We now study the projections $\pi_{b} \vec{S}_{n}$ for $b \in \mathbf{N}$.

We first note that, for all $n \geq 1, \vec{S}_{2 n-1}$ and $\vec{S}_{2 n}$ are both vectors in $\mathbf{Z}^{n}$. The following notation will be convenient in the sequel. For $\vec{S}_{n}=\left(a_{1}, a_{2}, a_{3}, \ldots\right)$, we say that $i_{n} \vec{S}_{n}=\left(a_{1}, 0,-a_{2}, 0, a_{3}, \ldots\right)$ is the image of $\vec{S}_{n}$ in $\mathbb{Z}^{n}$ and that $i_{n+1} \vec{S}_{n}=\left(0, a_{1}, 0,-a_{2}, 0, a_{3}, \ldots\right)$ is the image of $\vec{S}_{n}$ in $\mathbb{Z}^{n+1}$ (note that these image vectors are obtained by assigning in turn + and - signs to the components of $\vec{S}_{n}$ and then inserting 0 's in between those entries).

Before stating the general result, it is instructive to look at a few examples. Let us first consider the vector $\vec{S}_{8}=(1,6,10,4) \in \mathbf{Z}^{4}$; clearly $\pi_{10} \vec{S}_{8}=1 \cdot 10^{3}+6 \cdot 10^{2}+10 \cdot 10^{1}+4 \cdot 10^{0}=1704$. It can also be checked that 1704 can be given by a simple expression involving only the entries of $\vec{S}_{4}=(1,2)$, namely, $1704=1 \cdot 12^{3}-2 \cdot 12^{1}$; we can thus write $\pi_{10} \vec{S}_{8}=\pi_{12} i_{4} \vec{S}_{4}$, where $i_{4} \vec{S}_{4}=(1,0$, $-2,0$ ).

For $\vec{S}_{10}=(1,8,21,20,5)$, we find $\pi_{10} \vec{S}_{10}=20305$; since $i_{5} \vec{S}_{5}=(1,0,-3,0,1)$, we obtain similarly

$$
\pi_{12} i_{5} \vec{S}_{5}=1 \cdot 12^{4}-3 \cdot 12^{2}+1 \cdot 12^{0}=20736-432+1=20305=\pi_{10} \vec{S}_{10} .
$$

On the other hand, for $\vec{S}_{9}=(1,7,15,10,1)$, we have $\pi_{10} \vec{S}_{9}=18601$; introducing the two image vectors $i_{5} \vec{S}_{5}=(1,0,-3,0,1)$ and $i_{5} \vec{S}_{4}=(0,1,0,-2,0)$, it is easily checked that

$$
\pi_{12} i_{5} \vec{S}_{5}-\pi_{12} i_{5} \vec{S}_{4}=18601=\pi_{10} \vec{S}_{9}
$$

The use of base $b=10$ was by no means essential in the above examples, as we shall now show.

## Theorem:

a) $\pi_{b} \vec{S}_{2 n}=\pi_{b+2} i_{n} \vec{S}_{n}, n \geq 1$.
b) $\pi_{b} \vec{S}_{2 n-1}=\pi_{b+2} i_{n} \vec{S}_{n}-\pi_{b+2} i_{n} \vec{S}_{n-1}, \quad n \geq 2$.

Proof: Using the basic recursion formula

$$
\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}
$$

it is easily seen that

$$
\pi_{b} \vec{S}_{2 n-1}+\pi_{b} \vec{S}_{2 n-2}=\pi_{b} \vec{S}_{2 n}, \text { that is, } \pi_{b} \vec{S}_{2 n-1}=\pi_{b} \vec{S}_{2 n}-\pi_{b} \vec{S}_{2 n-2}
$$

It is thus sufficient to prove a), since b) then follows at once.

Proof of a): Let us expand both sides of the required equality. One must thus establish that

$$
\sum_{k=0}^{n-1}\binom{2 n-1-k}{k} b^{n-1-k}=\sum_{t=0}^{\left[\frac{n-1}{2}\right]}(-1)^{t}\binom{n-1-t}{t}(b+2)^{n-1-2 t}
$$

where $[x]$ denotes the integer part of $x$. This can of course be done using the techniques of generating functions. We prefer, however, to give a proof based on a common combinatorial interpretation of both sides.

We first use the Binomial Theorem to replace the last factor in the right-hand side of the above, thus getting

$$
\sum_{k=0}^{n-1}\binom{2 n-1-k}{k} b^{n-1-k}=\sum_{t=0}^{\left[\frac{n-1}{2}\right]}(-1)^{t}\binom{n-1-t}{t}\left[\sum_{u=0}^{n-1-2 t}\binom{n-1-2 t}{u} b^{n-1-2 t-u} 2^{u}\right] .
$$

We now need to expand the right-hand side of this inequality as a polynomial in $b$ and then compare the coefficients of the powers of $b$ with those occurring on the left-hand side. For a fixed $k$, we are thus interested in values of $t$ and $u$ such that $u=k-2 t$, since only these terms will contribute to the coefficient of $b^{n-1-k}$. One is then lead to prove that

$$
\binom{2 n-1-k}{k}=\sum_{j=0}^{\left[\frac{k}{2}\right]}(-1)^{j} 2^{k-2 j}\binom{n-1-j}{j}\binom{n-1-2 j}{k-2 j}
$$

or, equivalently, that

$$
\begin{equation*}
\binom{2 n-1-k}{k}=2^{k} \cdot\binom{n-1}{k}-\sum_{j=1}^{\left[\frac{k}{2}\right]}(-1)^{j+1} 2^{k-2 j}\binom{n-1-j}{j}\binom{n-1-2 j}{k-2 j} \tag{**}
\end{equation*}
$$

for $k \leq n-1$.
It is easily verified, for instance by induction on $k$, that $\binom{m+1-k}{k}$ can be interpreted as the number of ways of selecting $k$ integers among $1,2,3, \ldots, m$ in such a way that no two of them are consecutive. The left-hand side of (**) can thus be seen as the number of ways of picking $k$ integers among $1,2,3, \ldots, 2 n-2$, no two of them being consecutive.

We want to show, of course, that the right-hand side of (**) counts exactly the same number. Let us first observe that the first term, $2^{k} \cdot\binom{n-1}{k}$, can be seen as the number of ways of picking $k$ integers among $1,2,3, \ldots, 2 n-2$ by the following two-step procedure:

Step 1: Select $k$ pairs of integers of the form $\{2 s-1,2 s\}$ among $1,2,3, \ldots, 2 n-2$. This can be done in $\binom{n-1}{k}$ ways.
Step 2: Pick an integer in each of the $k$ pairs selected. This can be done in $2^{k}$ ways.
While this procedure clearly generates any set of $k$ integers chosen among $1,2,3, \ldots, 2 n-2$ in such a way that no two of them are consecutive, it does, however, also allow picking both integers $2 i$ and $2 i+1$. When this happens, we shall say that the event $A_{i}$ has occurred. Note that, in such a case, the index $i$ can take the values $1,2,3, \ldots, n-2$. Also, when both events $A_{i_{1}}$ and $A_{i_{2}}$ ( $i_{1} \neq i_{2}$ ) occur within a given selection of integers, the indices $i_{1}$ and $i_{2}$ are not consecutive.

It thus remains to show that the number of elements corresponding to the event $A_{1} \cup A_{2} \cup \ldots$ $\cup A_{n-2}$ is given exactly by the subtrahend on the right-hand side of ( $* *$ ). Such a proof follows directly from the usual "inclusion-exclusion technique" for counting the elements in a union of events: one first $(j=1)$ adds up the counts in each $A_{i}$, one then $(j=2)$ subtracts the counts in each $A_{i_{1}} \cap A_{i_{2}}\left(i_{1}<i_{2}\right)$, then $(j=3)$ one adds the counts in each $A_{i_{1}} \cap A_{i_{2}} \cap A_{i_{3}}\left(i_{1}<i_{2}<i_{3}\right)$, etc.

Let us consider, for instance, the case $j=1 .\binom{n-1-1}{1}$ is the number of ways of selecting an index $i$ (that is, two integers) so that the event $A_{i}$ has occurred. In order to complete a choice of $k$ integers, one first selects $k-2$ pairs among the remaining integers [Step 1-this can be done in $\binom{n-1-2}{k-2}$ ways], and then -Step 2-picks one integer from each of these pairs (which can be done in $2^{k-2}$ ways).

A similar argument applies generally for any $j>1$. One must first note that $\binom{n-1-j}{j}$ is the number of ways of selecting the indices $i_{1}<i_{2}<\cdots<i_{j}$ in such a way that no two of them are consecutive ( $2 j$ integers are thus chosen through this stage). Then, as above, $\binom{n-1-2 j}{k-2 j}$ counts the number of ways of selecting $k-2 j$ pairs among the remaining integers-Step 1 -and $2^{k-2 j}$ is the number of ways of performing Step 2.

The theorem is thus proven.
Taking $b=1$ in the above theorem, we have the following equalities:
a) $\pi_{1} \vec{S}_{2 n}=\pi_{3} i_{n} \vec{S}_{n}, n \geq 1$.
b) $\pi_{1} \vec{S}_{2 n-1}=\pi_{3} i_{n} \vec{S}_{n}-\pi_{3} i_{n} \vec{S}_{n-1}, \quad n \geq 2$.

Hence, the $(2 n)^{\text {th }}$ Fibonacci number can be calculated by using a base-three interpretation of the $n^{\text {th }}$ Fibonacci diagonal, whereas the $(2 n-1)^{\text {th }}$ Fibonacci number can be calculated via a basethree interpretation of both the $n^{\text {th }}$ and the $(n-1)^{\text {th }}$ Fibonacci diagonals. For instance, the $6^{\text {th }}$ Fibonacci number is 8 and it can be obtained from $\vec{S}_{3}=(1,1)$ as $1 \cdot 3^{2}-1 \cdot 3^{0}$. The $11^{\text {th }}$ Fibonacci number is 89 , which can be obtained via the diagonals $\vec{S}_{6}=(1,4,3)$ and $\vec{S}_{5}=(1,3,1)$ : one has here

$$
\pi_{3} i_{6} \vec{S}_{6}=1 \cdot 3^{5}-4 \cdot 3^{3}+3 \cdot 3^{1}=144
$$

and

$$
\pi_{3} i_{6} \vec{S}_{5}=1 \cdot 3^{4}-3 \cdot 3^{2}+1 \cdot 3^{0}=55
$$

## 3. FIBONACCI DIAGONALS IN PASCAL'S TRIANGLE MOD 2

The Theorem of section 2 tells us how certain computations regarding Fibonacci diagonals can be "lifted" to computations done just half-way down Pascal's triangle. Such a theorem is in the same spirit as the results presented in Hodgson [4] with respect to Pascal's triangle mod 2. We now briefly recall these results.

Let us denote by $\vec{T}_{n}, n \geq 1$, the vector representing the $n^{\text {th }}$ Fibonacci diagonal mod 2. These diagonals have already been studied in Hodgson [4] where numbers $H_{n}$, analogous to Gould's numbers, have been introduced. In our notation, we have $H_{n}=\pi_{2} \vec{T}_{n}$. The following calculation rules for $H_{n}$ were proven in Hodgson [4] (see Proposition 6.1 therein):

$$
\begin{equation*}
H_{2^{h}}=2^{2^{2^{-1}-1}} . \tag{i}
\end{equation*}
$$

$$
\begin{equation*}
H_{2^{h}+u}=H_{u} \cdot 2^{2^{h-1}}+H_{2^{h}-u} \text { for } 1 \leq u<2^{h} . \tag{ii}
\end{equation*}
$$

(The reader should be aware that the slightly different form of those rules in [4] is due to the numbering of diagonals there starting with $n=0$.) The proof of these recursion formulas is essentially based on an algebraic translation of the "geometry" of Pascal's triangle mod 2, that is, the very interesting way in which the 0 's and the 1 's are distributed (the reader should write down the first $n$ rows of that triangle and observe the nice pattern obtained).

We now end this paper by describing techniques that allow the computation of both $\pi_{2} \vec{T}_{n}$ and $\pi_{1} \vec{T}_{n}$ in a most direct fashion. In opposition to the above formulas that relate the value of a certain $H_{n}$ to powers of 2 and previous $H_{i}$ 's, the procedures below give the value of both $H_{n}=\pi_{2} \vec{T}_{n}$ and $\pi_{1} \vec{T}_{n}$ by working directly on the index $n$. Figures 2 and 3 illustrate the simplicity of these methods, whose validity is a consequence of the following discussion.

For convenience, let us introduce the notation $t_{n}$ to represent the base-two representation of $H_{n}$. (Note that $t_{n}$ can be simply seen as the vector $\vec{T}_{n}$ with the commas removed.) Formulas (i) and (ii) now become

$$
\begin{gather*}
t_{2^{h}}=1000 \ldots 0 \quad\left(2^{h-1}-1 \text { zeros }\right),  \tag{i'}\\
t_{2^{h}+u}=t_{u} 000 \ldots 0 t_{2^{h}-u} \quad \text { for } 1 \leq u<2^{h}, \tag{ii'}
\end{gather*}
$$

where the number of intermediate zeros is such that the string $000 \ldots 0 t_{2^{h}-u}$ is made of exactly $2^{h-1}$ digits. As an example, let us compute $t_{29}$. Since it is trivially verified that $t_{3}=11$, we thus have

$$
\begin{aligned}
t_{29} & =t_{13} 000000 t_{3} \\
& =t_{5} 00 t_{3} 00000011 \\
& =t_{1} t_{3} 001100000011 \\
& =111001100000011
\end{aligned}
$$

[the number of intermediate zeros introduced at each computation step follows from (ii)].
The preceding calculations can also be conveniently displayed as in the tableau of Figure 1. In general, given $n=2^{h}+u$, we shall need a tableau made of $h$ rows, each one containing $2^{h}$ positions to be ultimately filled at the last step of the procedure. Rows are indexed by decreasing powers of two that serve to split each number appearing on the preceding row. At the row of index $2^{k}$, any number (from row $2^{k+1}$ ) of the form $2^{k}+v$ becomes split into $v$ and $2^{k}-v$, while any number $w \leq 2^{k}$ splits into 0 and $w$. This procedure may be better grasped by displaying the entries as in the tree diagram given in Figure 2. (For odd $n$, this algorithm directly gives $t_{n}$ at its last step of computation. However, because of parity considerations, the last row will, for even $n$, always contain 0 's and 2 's: we note that $t_{n}$ can then be obtained by merely replacing each digit 2 by a 1 .)

We finally present a technique for the computation of $\pi_{1} \vec{T}_{n}$ (compare with Glaisher's rule for the calculation of $\pi_{1} \vec{Y}_{n}$ mentioned in the Introduction). Note that we are now interested solely in the total number of 1's, and no longer in their exact position. All amounts to finding how one can build $n$ using only powers of two-or, if one prefers, to what extent $n$ is "far" from being itself a power of two. For this purpose, we introduce a notion of weight. The diagram of Figure 3 (for $n=29$ ) helps to clarify the discussion. Let us read that diagram from the bottom up. Powers of two (here, 16 and 32 ) are considered to be of weight 1 . Then 24 , being halfway between powers
of two, is of weight $2(=1+1)$. Since 28 is halfway between 24 and a power of two, it is given weight $2+1=3$. Continuing in this manner, 29 receives a weight of $3+4=7$ : this weight is also the value of $\pi_{1} \vec{T}_{29}$, the total number of 1 's appearing in $t_{29}$. (It is usually more convenient to consider Figure 3 as being built from the top down, with the weights being incorporated into the diagram at the end of the process.) The general validity of this procedure follows from recursive applications of formulas (i') and (ii) above.


Figure 1

16
\&

4

2


Figure 2


Figure 3

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# ON THE INFINITUDE OF LUCAS PSEUDOPRIMES 

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The following properties of certain positive integers $n$ are set forth:

$$
\begin{gather*}
F_{n-(5 / n)} \equiv 0, \text { where } \operatorname{gcd}(n, 10)=1  \tag{1}\\
\text { and }(5 / n) \text { is a Jacobi symbol; } \\
L_{n} \equiv 1(\bmod n) \tag{2}
\end{gather*}
$$

It is well-known that properties (1) and (2) are satisfied if $n$ is prime. If (1) is satisfied for some composite $n$, then $n$ is called a Fibonacci pseudoprime (or FPP). If (2) is satisfied for some composite $n$, then $n$ is called a Lucas pseudoprime (or LPP). Let $U$ and $V$ denote the sets of FPP's and LPP's, respectively.

It must be remarked that the above terminology is different from that used by many other authors; frequently, the term "Fibonacci pseudoprime" is used to describe numbers that satisfy (2), and/or "Lucas pseudoprime" sometimes is used to describe numbers that satisfy (1). There are, no doubt, some very good reasons for describing such numbers by one term versus another. In most papers that this author has seen, the subject matter is only one of the types of numbers here described, which tends to minimize confusion. When both types of numbers are being discussed, as is the case in this paper, it seems preferable to adopt the terminology defined above. Readers of this journal may tend to be more sympathetic to this usage, for obvious reasons. Apologies are made here and now to those readers who may take exception to the nomenclature adopted here.

In a 1964 paper [2], E. Lehmer showed that $U$ is an infinite set, specifically by proving that $n=F_{2 p}$ satisfies (1) for any prime $p>5$. In a 1970 paper [3], E. A. Parberry proved some interesting results related to those of Lehmer, indirectly commenting on the infinitude of $U$, by a different approach. It is informative to paraphrase that portion of Parberry's results that touches on the subject of this paper; we state this as a theorem.

Theorem 1: (*) If $\operatorname{gcd}(n, 30)=1$ and $n$ is a FPP, then $F_{2 n}$ has these same properties.
Note that if $\operatorname{gcd}(n, 30)=1$, then it is also true that $\operatorname{gcd}\left(F_{2 n}, 30\right)=1$. Theorem 1 implies that, beginning with any FPP $n$ with $\operatorname{gcd}(n, 30)=1$ (e.g., $n=323=17 \cdot 19$, which is the smallest element of $U$ ), we may form the infinite sequence:

$$
\begin{equation*}
n, F_{2 n}, F_{2 F_{2 n}}, F_{2 F_{2 F_{2 n}}} \text {, etc., each element of which is a FPP. } \tag{3}
\end{equation*}
$$

We have, therefore, another demonstration (distinct from Lehmer's) that $U$ is an infinite set.
In a 1986 paper [1], P. Kiss, B. M. Phong, \& E. Lieuwens showed, along with other important results, that there exist infinitely many numbers $n$ that are simultaneously FPP's and LPP's (i.e., the set $U \cap V$ is infinite). Actually, this is a corollary of their more general results. By the way, we remark that the smallest element of $U \cap V$ is $4181=37 \cdot 113=F_{19}$.

In light of these results, it may seem redundant to prove once again that $V$ is infinite, as the title of this paper implies. Nevertheless, the approach used below differs from that of Kiss,

Phong, \& Lieuwens, and is worthy of mention. Moreover, it displays a kind of symmetry in relation to Theorem 1, providing as it does the "Lucas" counterpart of that theorem. This is stated as follows.

Theorem 2: (**) If $\operatorname{gcd}(n, 6)=1$ and $n$ is a LPP, then $L_{n}$ has these same properties.
Proof: Let $u=L_{n}, v=\frac{1}{2}(u-1)$. Note that $u$ must be odd, since $\operatorname{gcd}(n, 3)=1$, hence $v$ is an integer. We consider three possibilities:
(a) $n \equiv 1(\bmod 12)$ : then $u \equiv 1(\bmod 8)$, hence $v \equiv 0(\bmod 4)$. Let $v=2^{r} w$, where $r \geq 2$ and $w$ is odd. Since $L_{n} \equiv 1(\bmod n)$, thus $n \mid 2 v$. However, $n$ is odd, so $n \mid w$. Then $L_{n} \mid L_{w}$. Now $F_{v}=F_{w} L_{w} L_{2 w} L_{4 w} \ldots L_{2^{r-1} w}$, which shows that $u \mid F_{v}$. Also, since $v$ is even, the following identity is satisfied: $L_{u}-1=5 F_{v} F_{v+1}$. Therefore, $L_{u} \equiv 1(\bmod u)$.
(b) $n \equiv 7(\bmod 12)$ : then $u \equiv 5(\bmod 8)$, hence $v \equiv 2(\bmod 4)$. In this case, $v=2 w$, where $w$ is odd. As in (a), $u \mid F_{v}$ and $L_{u}-1=5 F_{v} F_{v+1}$, so $L_{u} \equiv 1(\bmod u)$.
(c) $n \equiv 5$ or $11(\bmod 12)$ : then $u \equiv 3$ or $7(\bmod 8)$, hence $v$ is odd. As above, we have $n \mid 2 v \Rightarrow$ $n|v \Rightarrow u| L_{v}$. Now, however, $L_{u}-1=L_{v} L_{v+1}$. Thus, $L_{u} \equiv 1(\bmod u)$.
In all cases, $L_{u} \equiv 1(\bmod u)$. It only remains to show that $u=L_{n}$ is composite; however, this follows immediately from the fact that $n$ is odd and composite, since $L_{p} \mid u$ for any prime divisor $p$ of $n$. Thus, $L_{n}$ is a LPP and $\operatorname{gcd}\left(L_{n}, 6\right)=1$, proving the theorem.

The smallest LPP not divisible by 2 or 3 is $m=2465=5 \cdot 17 \cdot 29$ (in fact, no LPP is even, as Di Porto and this author have independently shown). Beginning with $m$, for example (or any other LPP not divisible by 3 ), we may form the infinite sequence:

$$
\begin{equation*}
m, L_{m}, L_{L_{m}}, L_{L_{L_{m}}} \text {, etc., each element of which is a LPP. } \tag{4}
\end{equation*}
$$

Therefore, $V$ is infinite.
Clearly, the sequences indicated in (3) and (4) increase extremely rapidly, an observation that may have some applications in primality testing. This aspect is left for other researchers. Also, the focus of this paper has been on the so-called "Fibonacci pseudoprimes" and "Lucas pseudoprimes," rather than on any of the many generalizations of these numbers studied by other writers. No doubt, such generalizations may be readily found; however, this was not explored here, and is left for future research.

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# LUCAS PSEUDOPRIMES ARE ODD 

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It is well known that the congruence

$$
\begin{equation*}
L_{n} \equiv 1(\bmod n) \tag{1}
\end{equation*}
$$

is satisfied by all prime $n$. However, there are also many composite $n$ satisfying (1), the smallest example being $n=705=3 \cdot 5 \cdot 47$ (indeed, there are infinitely many such $n$ ). The term "Lucas pseudoprime" (or LPP) appears to be appropriate to describe such composite $n$. It must be mentioned, however, that there is little uniformity in the literature regarding this subject. An alternative term which is frequently used is "Fibonacci pseudoprime"; however, since this term has occasionally been used to describe those composite $n$ which satisfy the following congruence:

$$
\begin{align*}
F_{n-(5 / n)} \equiv 0(\bmod n), & \text { where } \operatorname{gcd}(n, 10)=1  \tag{2}\\
& \text { and }(5 / n) \text { is a Jacobi symbol, },
\end{align*}
$$

it was felt advisable to avoid the latter term in this article. Accordingly, we adopt the term "Lucas pseudoprimes" (or LPP's) to describe those composite $n$ satisfying (1). Incidentally, if $U$ and $V$ represent the sets of composite integers satisfying (2) and (1), respectively, it is known that $U, V$, and $U \cap V$ are infinite sets.

Di Porto \& Filipponi [3] indicated the values of all LPP's $<10^{6}$ (a total of 86 values). Also, in private correspondence [4], Filipponi provided the author with a table of 852 LPP's, which are all the LPP's $<10^{8}$. On the basis of the values obtained, Di Porto \& Filipponi proposed several conjectures. We are concerned here only with proving one of these conjectures, namely, that all LPP's are odd.

As it turns out, Di Porto (one of the proposers of this conjecture) has recently proven her own conjecture independently (see [5]); moreover, it came to the author's attention that a much earlier proof of this result had been given by White, Hunt, \& Dresel [7] no later than 1977. The author was made aware of these revelations only after this paper was originally submitted for publication. The author publicly acknowledges the priority of these earlier efforts, and also gives Di Porto credit for her independently derived proof, which predates this paper. Developments such as these give an indication of the rapid rate of growth of knowledge in this fascinating field.

In spite of the earlier proofs, it does not seem amiss to present another proof of the statement that all LPP's are odd; this is particularly true since the proof given here differs from the earlier proofs in several particulars.

Our proof depends, in part, on some results obtained in [3], namely, that the existence of any even LPP, which we denote by $n$, implies that $n \equiv \pm 2(\bmod 12)$, and that $n \neq 2 p$, where $p$ is prime. We also require a result which we state as a lemma, without proof, the reader is referred to [1] for a proof.

## Lemma 1:

$$
\begin{equation*}
L_{s^{r}} \equiv L_{s^{r-1}}\left(\bmod 5^{r}\right), r=1,2, \ldots \tag{3}
\end{equation*}
$$

In addition, we will need some basic results concerning the Fibonacci rank of appearance, or entry-point. We recall that, for a given $n \geq 1$, the rank of appearance (or entry-point) of $n$, which we denote as $Z(n)$, is defined to be the smallest positive integer $t$ such that $n \mid F_{t}$. Various other terms and/or notation have been used by other authors, again pointing to a dearth of uniformity in the literature. One frequently used term, namely, "rank of apparition," is particularly odious to this writer, and shall be avoided steadfastly. As has been pointed out by Ribenboim [6], the latter term stems from a bad translation of the French loi d'apparition, which means "law of appearance," not "law of apparition"; in all English dictionaries, "apparition" means "ghost."

The following properties are well known and stated without further comment:
(i) $Z(n)$ exists for all $n \geq 1$;
(ii) $Z(m) \mid n$ iff $m \mid F_{n}$;
(iii) $Z(m) \mid Z(n)$ iff $m \mid n$ iff $F_{m} \mid F_{n}$;
(iv) If $n=\prod_{i=1}^{\omega} p_{i}^{e_{i}}$, then $Z(n)=\operatorname{lcm}\left[Z\left(p_{1}^{e_{1}}\right), Z\left(p_{2}^{e_{2}}\right), \ldots, Z\left(p_{\omega}^{e_{\omega}}\right)\right]$;
(v) $Z\left(p^{e}\right)=p^{f} Z(p)$, where $0 \leq f<e$.

Finally, we require another result, also stated without proof as a lemma; refer to [2] for a proof.

Lemma 2:

$$
\begin{equation*}
n=Z(n) \text { iff } n=5^{u} \text { or } n=12 \cdot 5^{u}, u \geq 0 \tag{5}
\end{equation*}
$$

With these tools, the proof of the oddness of LPP's is surprisingly elementary. Now for our proof!

Suppose, to the contrary that $2 n$ is a LPP. Thus, we assume that

$$
\begin{equation*}
L_{2 n} \equiv 1(\bmod 2 n) \tag{6}
\end{equation*}
$$

where $n$ is composite and $\operatorname{gcd}(n, 6)=1$, using Di Porto \& Filipponi's results in [3]. The following simple identities are readily verifiable: $L_{2 n}-1=F_{3 n} / F_{n}$ and $L_{n}^{2}=5 F_{n}^{2}-4=L_{2 n}-2$. Along with (6), these imply the congruences:

$$
\begin{equation*}
\text { (i) } L_{n}^{2} \equiv-1(\bmod 2 n) \text {; } \tag{7}
\end{equation*}
$$

(ii) $5 F_{n}^{2} \equiv 3(\bmod 2 n)$;
(iii) $\quad F_{3 n} \equiv 0(\bmod 2 n)$.

From (7)(i) and (ii), we see that $L_{n} \not \equiv 0, F_{n} \not \equiv 0(\bmod 2 n)$. Thus, $F_{m} \not \equiv 0(\bmod 2 n)$ for all $m$ dividing $n$, since $F_{m} \mid F_{n}$. From (7)(iii), it follows that

$$
\begin{equation*}
Z(2 n)=3 n \tag{8}
\end{equation*}
$$

Now $n$, and thus $Z(2 n)$, are odd. Also, $Z(2 n)=\operatorname{lcm}[Z(2), Z(n)]$, or

$$
\begin{equation*}
Z(2 n)=3 n=\operatorname{lcm}[3, Z(n)] \tag{9}
\end{equation*}
$$

Since $\operatorname{gcd}(3, n)=1$, we see from (9) that $3^{e} \| Z(n) \Rightarrow e=0$ or 1 . We consider these two possibilities as separate cases.

Case I. $\operatorname{gcd}(3, Z(n))=1$
By (9), $Z(2 n)=3 Z(n)=3 n$, so $n=Z(n)$. Using Lemma 2 and the fact that $\operatorname{gcd}(6, n)=1$ and $n$ is composite, we see that $n=5^{u}, u \geq 2$. Let $n=5 m$, where $m=5^{u-1}$. Now (7)(i) implies that $L_{n}^{2} \equiv-1(\bmod n)$, and $L_{n} \equiv L_{m}(\bmod n)$, by Lemma 1; hence, $L_{m}^{2} \equiv-1(\bmod n) \Rightarrow L_{2 m} \equiv 1(\bmod n)$ $\Rightarrow L_{2 m} \equiv 1(\bmod m)$. Also, since $\operatorname{gcd}(3,2 m)=1, L_{2 m}$ is odd (another well-known fact). Therefore, $L_{2 m} \equiv 1(\bmod 2 m)$. This is equivalent to the statement that $2 m$ is a LPP, provided $m$ is composite. By an easy inductive process, we see that $2 n, 2 n / 5,2 n / 5^{2}, \ldots, 2 \cdot 5^{2}=2 n / 5^{u-2}$ are all LPP's. However, as we may readily verify from a table of Lucas numbers, $L_{50} \equiv 23 \equiv 1(\mathrm{mod}$ 50 ), so 50 is not a LPP. The contradiction eliminates this possibility.

Case III. $3^{1} \| Z(n)$
By (9), $Z(2 n)=Z(n)=3 n$. Also, $Z(12 n)=\operatorname{lcm}[Z(12), Z(n)]=1 \mathrm{~cm}[12,3 n]=12 n$. Again using Lemma 2 and the fact that $\operatorname{gcd}(6, n)=1$, we reach a contradiction, as in Case I.

We conclude that our original assumption is faulty and, therefore, that all LPP's are odd.

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# ON A CONJECTURE OF DI PORTO AND FILIPPONI 

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We begin by describing the following two properties of certain natural numbers $n$ :

$$
\begin{gather*}
F_{n-(5 / n)} \equiv 0(\bmod n), \begin{array}{c}
\text { where } \operatorname{gcd}(n, 10)=1, \\
\text { and }(5 / n) \text { is a Jacobi symbol; } \\
L_{n} \equiv 1(\bmod n) .
\end{array} . \tag{1}
\end{gather*}
$$

As is well known, properties (1) and (2) are satisfied if $n$ is prime. More interestingly, there are infinitely many composite numbers $n$ which satisfy (1) and/or (2). We call these $n$ "Fibonacci pseudoprimes" (or FPP's) if they satisfy (1), and "Lucas pseudoprimes" (or LPP's) if they satisfy (2). As has been remarked elsewhere [1], this nomenclature is not standard, but should be acceptable to most readers of this quarterly.

These numbers, and their generalizations, have been extensively studied by other writers. It is not our aim here to outline all the various results currently available, or in progress; suffice it to say that interest in these numbers is relatively recent, and known results are correspondingly scarce. Much of the interest in these numbers, in recent years, centers around their application in primality testing and public-key cryptography; however, it is beyond the scope of this paper to delve into this fascinating topic.

We also mention the work of Kiss, Phong, \& Lieuwens [4] which showed, among other things, that there exist infinitely many numbers $n$ that are simultaneously FPP's and LPP's. For the sake of our discussion, we shall term such numbers "Fibonacci-Lucas pseudoprimes" (or FLPP's).

In a 1989 paper [3], Di Porto \& Filipponi asked the following question (which we paraphrase here, to conform with our nomenclature): "Are all the composite Fibonacci and Lucas numbers with prime subscript LPP's?"

As we shall show, the answer to this question is affirmative, if we exclude the subscript 3 (a minor oversight which Di Porto \& Filipponi undoubtedly intended to account for). However, more is true: we shall, in fact, prove the following symmetric results.

Theorem 1: Given $n=F_{p}$, where $p$ is a prime $>5$, then $n$ is a FLPP if and only if $n$ is composite.
Theorem 2: Given $n=L_{p}$, where $p$ is a prime $>5$, then $n$ is a FLPP if and only if $n$ is composite.
Proof of Theorem 1: Note that $\operatorname{gcd}(n, 30)=1$. Let $m=\frac{1}{2}(n-1)$. We consider two possibilities:
(a) $p \equiv \pm 1$ or $\pm 11(\bmod 30)$ : then $n \equiv 1$ or $9(\bmod 20),(5 / p)=(5 / n)=1$, and $m$ is even. Also, $F_{p} \equiv(5 / p) \equiv 1(\bmod p)$, so $p \mid 2 m$. Since $p$ is odd, thus $p \mid m$, which implies $n \mid F_{m}$. As we may readily verify, $F_{n}-1=F_{m} L_{m+1}$; hence, $F_{n} \equiv 1(\bmod n)$. Also, $F_{n-1}=F_{2 m}=F_{m} L_{m} \equiv 0(\bmod$ $n$ ); therefore, $n$ satisfies property (1), and must either be prime or a FPP. Also, $L_{n}=F_{n-1}+$ $F_{n+1}=2 F_{n-1}+F_{n} \equiv 1(\bmod n)$, which shows that $n$ satisfies (2) as well. Thus, $n$ is either prime or a LPP. The conclusion of the theorem follows.
(b) $p \equiv \pm 7$ or $\pm 13(\bmod 30)$ : then $n \equiv 13$ or $17(\bmod 20),(5 / p)=(5 / n)=-1$, and $m$ is even. Also, $F_{p} \equiv(5 / p) \equiv-1(\bmod p)$, so $p \mid(2 m+2)$. Since $p$ is odd, thus $p \mid(m+1)$, which implies
$n \mid F_{m+1}$. As we may readily verify, $F_{n}+1=F_{m+1} L_{m}$; hence, $F_{n} \equiv-1(\bmod n)$. Also, $F_{n+1}=$ $F_{2 m+2}=F_{m+1} L_{m+1} \equiv 0(\bmod n)$; therefore, $n$ satisfies property (1), and must either be prime or a FPP. Also, $L_{n}=F_{n+1}+F_{n-1}=2 F_{n+1}-F_{n} \equiv 1(\bmod n)$, which shows that $n$ satisfies (2) as well. Thus, $n$ is either prime or a LPP. The conclusion of the theorem follows.
We may remark that $p=19$ is the smallest prime for which $F_{p}$ is composite; thus, $F_{19}=4181=37 \cdot 113$ is the smallest FLPP provided by the theorem.

Proof of Theorem 2: Note that $n \equiv \pm 1(\bmod 10)$, so $(5 / n)=1$. Let $m=\frac{1}{2}(n-1)$. Also, note that $L_{p} \equiv 1(\bmod p)$; hence, $p \mid 2 m$. Since $p$ is odd, thus $p \mid m$. We consider two possibilities:
(a) $n \equiv 1(\bmod 4)$ : then $m$ is even. Suppose $m=2^{r} d$, where $r \geq 1$ and $d$ is odd. Since $p$ is odd and $p \mid m$, thus $p \mid d$, which implies that $n \mid L_{d}$. Now $F_{2 m}=F_{d} L_{d} L_{2 d} L_{4 d} \ldots L_{2^{r} d}$; hence, $n \mid F_{2 m}$, i.e., $n \mid F_{n-1}$. Thus, $n$ satisfies (1). Also $L_{n}=1+5 F_{m} F_{m+1}$, as readily verified. Since $n \mid L_{d}$, it follows (as above) that $n \mid F_{m}$. Thus, $n$ satisfies (2) as well.
(b) $n \equiv 3(\bmod 4)$ : then $m$ is odd. Thus, $L_{p} \mid L_{m}$, i.e., $n \mid L_{m}$. Then $n \mid F_{2 m}=F_{m} L_{m}$, or $n \mid F_{n-1}$. Hence, $n$ satisfies (1). Also, $L_{n}=1+L_{m} L_{m+1}$, as is readily verified. Thus, $n \mid L_{m}$ implies (2).
In either case, $n$ satisfies both (1) and (2). The conclusion of the theorem now follows.
We may remark that $p=23$ is the smallest prime for which $L_{p}$ is composite; therefore, $L_{23}=64079=139.461$ is the smallest FLPP provided by the theorem.

It was brought to the author's attention by the referee that the question proposed by Di Porto \& Filipponi [3] (mentioned earlier) was answered affirmatively by the proposers in a paper [2] which, as fortune would have it, was presented at Eurocrypt ' 88 and was published before [3]. In [2], Di Porto \& Filipponi also generalized their results to more general types of sequences, but only dealt with LPP's (or their generalizations) and not with FPP's. One of their more interesting corollaries ([2], Corollary 3) is that $L_{2^{n}}$ is a LPP, if composite (paraphrasing to employ the nomenclature introduced here); the smallest such composite $L_{m}$ is $L_{32}=4870847=1087.4481$.

We close by remarking that the results derived in this paper may be generalized in various ways to yield comparable results for more general second-order sequences (as Di Porto and Filipponi, among others, have done); the Fibonacci and Lucas sequences are special cases of these more general types of sequences. No attempt at such generalization was made here, although it is likely that this would not present major difficulties.

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# EXTENSIONS TO THE GCD STAR OF DAVID THEOREM 

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## 1. INTRODUCTION

The GCD Star of David Theorem asserts that

$$
\begin{equation*}
\operatorname{gcd}\left\{\binom{n-1}{k},\binom{n}{k-1},\binom{n+1}{k+1}\right\}=\operatorname{gcd}\left\{\binom{n-1}{k-1},\binom{n}{k+1},\binom{n+1}{k}\right\} . \tag{1.1}
\end{equation*}
$$

It was first conjectured by H. W. Gould [2] in 1972.
It has been proven and/or generalized by Hillman \& Hoggatt [3, 4], Strauss [11], Hitotumatu \& Sato [5], Ando [1], Singmaster [10], and Long \& Ando [7, 8].

In this paper we will show some figures other than the hexagon described by the binomial coefficients in (1.1) that also have a gcd property. We also give a method whereby a new polygon with a gcd property can be constructed from known polygons with that property.

## 2. TERMINOLOGY

By a polygonal figure $P$ in Pascal's triangle, we mean a simple closed polygonal curve whose vertices are entries of Pascal's triangle. We also use the same symbol to represent the set of entries on the curve. The six binomial coefficients in (1.1) form a hexagon with $\binom{n}{k}$ at its center. This hexagon will be called a fundamental hexagon.

Following Long [6], we say $P$ is tiled by fundamental hexagons if $P$ is "covered" by a set $\mathfrak{F}$ of fundamental hexagons $F$ in such a way that
(1) The vertices of each $F$ in $\mathfrak{F}$ are coefficients in $P$ or interior of $P$.
(2) Each boundary coefficient of $P$ is a vertex of precisely one $F$ in $\mathfrak{F}$.
(3) Each interior coefficient of $P$ is interior to some $F$ in $\mathfrak{F}$ or is a vertex shared by precisely two elements in $\mathfrak{F}$.

If, in addition, the tiling has the property
(4) For all $F_{1}$ and $F_{2}$ in $\mathfrak{F}, F_{1}$ and $F_{2}$ have at most one vertex in common.
we say $P$ has a restricted tiling. The three polygons in Figure 1 illustrate the three possibilities. The upper left figure has no tiling. The bottom figure has a restricted tiling.

Let $P$ be some configuration of binomial coefficients in Pascal's triangle. Suppose $P=X \cup Y$. If $\operatorname{gcd} X=\operatorname{gcd} Y$ independent of the placement of $P$ in Pascal's triangle, then $P$ is said to have the gcd property with respect to $X$ and $Y$. The fundamental hexagon has this property with respect to the two sets of three coefficients on the alternate vertices of the hexagon.

An $m \times n \times k$ hexagon is a hexagon oriented along the rows and main diagonals of Pascal's triangle with $m, n, k, m, n$, and $k$ entries per side starting with $m$ entries on top and going clockwise around the hexagon.


## FIGURE 1. Some Polygons with Their Tilings

## 3. THE GENERAL METHOD

If we know that polygons $P_{1}$ and $P_{2}$ have the gcd property with respect to certain sets, we can construct a larger polygon with a gcd property by using the following theorem.

Theorem 1: Let $P_{1}$ and $P_{2}$ be two configurations. Suppose $P_{1}$ has the gcd property with respect to $X$ and $Y \cup S$, and suppose $P_{2}$ has the gcd property with respect to $U$ and $V \cup S$. Then $P=X \cup V \cup Y \cup U$ has the gcd property with respect to $X \cup V$ and $U \cup Y$.

Proof: $\operatorname{gcd} X \cup V=\operatorname{gcd}(Y \cup S) \cup V=\operatorname{gcd} Y \cup(S \cup V)=\operatorname{gcd} Y \cup U$.
Some figures satisfy the hypotheses of Theorem 1 in a very obvious way. We have the following corollary to Theorem 1.

Corollary 1: Let $P$ be a $2 \times 2 \times 2 n, 2 \times 2 n \times 2$, or $2 n \times 2 \times 2$ hexagon in Pascal's triangle. Label the elements along the boundary $a_{1}, a_{2}, a_{3}, \ldots, a_{4 n+2}$ in sequence. Let $X=\left\{a_{1}, a_{3}, \ldots, a_{4 n+1}\right\}$ and $Y=\left\{a_{2}, a_{4}, \ldots, a_{4 n+2}\right\}$. Then $\operatorname{gcd} X=\operatorname{gcd} Y$.

Each of the hexagons described above admits a restricted tiling by $n$ fundamental hexagons. The corollary is easily proved by induction on $n$ with Theorem 1 providing the inductive step.

## 4. OTHER POLYGONS WITH THE GCD PROPERTY

In what follows, we will label polygons in the following manner unless otherwise noted. The left-most vertex on the top row will be labeled $a_{1}$. Then as we travel clockwise along the boundary of the hexagon we label the coefficients $a_{2}, a_{3}, a_{4}, \ldots$.

We will show that any polygon with a restricted tiling of fewer than five fundamental hexagons has the gcd property with respect to the sets $\left\{a_{i} \mid i\right.$ odd $\}$ and $\left\{a_{i} \mid i\right.$ even $\}$.

First, a polygon $P$ that has such a tiling by one fundamental hexagon must be a fundamental hexagon. The GCD Star of David Theorem gives the desired result here.

Suppose $P$ has a restricted tiling by two fundamental hexagons. Then $P$ is either the disjoint union of two fundamental hexagons or is a $2 \times 2 \times 4,2 \times 4 \times 2$, or a $4 \times 2 \times 2$ hexagon. In the former case, each fundamental hexagon has the required gcd property and thus their disjoint union will also. The hexagons described in the latter case were shown in Corollary 1 to have the desired gcd property.

At this point, we drop from consideration the polygons that are comprised of two or more components, since their gcd properties are inherited from the separate components.

Now, let $P$ have a restricted tiling by three fundamental hexagons. There are two cases. Either some fundamental hexagon intersects only one of the other fundamental hexagons or each fundamental hexagon intersects both of the remaining two fundamental hexagons.

The polygons in the first case have the desired gcd property as a result of Theorem 1; the polygons described in the second case are either $2 \times 4 \times 2 \times 4 \times 2 \times 4$ or $4 \times 2 \times 4 \times 2 \times 4 \times 2$ hexagons.

The former is shown with its restricted tiling in Figure 2. We will show that each of the hexagons has the gcd property with respect to the sets $\left\{a_{1}, a_{3}, \ldots, a_{11}\right\}$ and $\left\{a_{2}, a_{4}, \ldots, a_{12}\right\}$.


FIGURE 2. A Diamond Formed by Three Fundamental Hexagons
We start with the hexagon in Figure 2. First, we prove the following lemma.
Lemma 1: With the notation of Figure 2,

$$
\begin{equation*}
\operatorname{gcd}\left\{y, a_{1}, a_{3}, \ldots, a_{11}\right\}=\operatorname{gcd}\left\{y, a_{2}, a_{4}, \ldots, a_{12}\right\} \tag{4.1}
\end{equation*}
$$

Proof: Applying Theorem 1 for

$$
\begin{aligned}
& X=\left\{a_{2}, a_{4}, a_{6}, a_{12}, y\right\}, \quad Y=\left\{a_{1}, a_{3}, a_{5}, a_{7}\right\}, \\
& U=\left\{y, a_{9}, a_{11}\right\}, \quad V=\left\{a_{8}, a_{10}\right\}, \quad S=\left\{x_{2}\right\},
\end{aligned}
$$

(4.1) holds as claimed.

We show that the element $y$ is superfluous in this lemma.
Theorem 2: With the notation of Figure 2,

$$
\begin{equation*}
\operatorname{gcd}\left\{a_{1}, a_{3}, \ldots, a_{11}\right\}=\operatorname{gcd}\left\{a_{2}, a_{4}, \ldots, a_{12}\right\} \tag{4.2}
\end{equation*}
$$

Proof: We will make use of the notation $v_{p}(n)=e$. By this we mean that $p^{e} \| n$. We will drop the subscript $p$ when no confusion arises about which base $p$ is to be used. Also, the notation $v_{p}(X)=e$ will imply that $p^{e} \| \operatorname{gcd} X$.

Now suppose that $h=\operatorname{gcd}\left\{a_{1}, a_{3}, \ldots, a_{11}\right\}$ and $g=\operatorname{gcd}\left\{a_{2}, a_{4}, \ldots, a_{12}\right\}$ and that $h>g$. Then there exists a prime $p$ for which $v_{p}(h)=e>v_{p}(g)$.

If $v\left(a_{6}\right) \geq e$, then $v\left(x_{5}\right)=v\left(a_{6}-a_{5}\right) \geq e$, which implies $v(y)=v\left(a_{7}-x_{5}\right) \geq e$. Then, from Lemma $1, v\left(\left\{y, a_{2}, a_{4}, \ldots, a_{12}\right\}\right) \geq e$. Thus, $v(g) \geq e$, and this is a contradiction. Similarly, if $v\left(a_{10}\right) \geq e$ or $v\left(a_{2}\right) \geq e$, then using Lemma 1 with $y$ replaced by $x_{2}$ or $x_{4}$, respectively, we have the same conclusion. Therefore, $v\left(a_{2}\right)<e, v\left(a_{6}\right)<e$, and $v\left(a_{10}\right)<e$.

Now, $a_{1} a_{5} a_{9}=a_{2} a_{6} a_{10}$. (See [9].) We know that $v\left(a_{1} a_{5} a_{9}\right) \geq 3 e$; thus, $v\left(a_{2} a_{6} a_{10}\right) \geq 3 e$, and this is a contradiction. Hence, $h \leq g$. Similarly, $g \leq h$. Therefore, $g=h$ and the theorem is proved.

To show that the $4 \times 2 \times 4 \times 2 \times 4 \times 2$ hexagon has the gcd property with respest to the same two sets, we first prove a lemma. To help with this lemma, we label a polygon $P$ a little differently. Assume one of the boundaries of $P$ falls along a row or main diagonal of Pascal's triangle and that the boundary consists of four or more consecutive binomial coefficients. We wish to adjoin a fundamental hexagon $H$ to $P$ to the right of the diagonal $n-k=c$, to the left of the diagonal $k=c$, or below the row $n=c$. This is illustrated in Figure 3. We label $P$ so that $a_{1}, a_{2}$, $a_{3}$, and $a_{4}$ are labeled as in Figure 3 and then continue around $P$ in the direction indicated labeling the coefficients $a_{5}, a_{6}, \ldots, a_{2 n}$.



## FIGURE 3. Adding a Fundamental Hexagon to a Polygon

Using this convention, we are now prepared to prove the following lemma.
Lemma 2: Let $P$ be a polygon in Pascal's triangle as labeled above. Suppose $P$ has the gcd property with respect to $S=\left\{a_{2 i-1} \mid i=1,2, \ldots, n\right\}$ and $T=\left\{a_{2 i} \mid i=1,2, \ldots, n\right\}$; that is, $\operatorname{gcd}\left\{a_{i} \mid i\right.$ odd $\}=$ $\operatorname{gcd}\left\{a_{i} \mid i\right.$ even $\}$. Let $H$ be the fundamental hexagon $\left\{a_{2}, a_{3}, x_{1}, x_{2}, x_{3}, x_{4}\right\}$ as in Figure 3. Then the polygon formed by $\left\{a_{1}, a_{4}, a_{5}, a_{6}, \ldots, a_{2 n}, x_{1}, \ldots, x_{4}\right\}$ has the gcd property with respect to $X=\left\{a_{1}, a_{5}, a_{7}, a_{9}, \ldots, a_{2 n-1}, x_{2}, x_{4}\right\}$ and $Y=\left\{a_{4}, a_{6}, \ldots, a_{2 n}, x_{1}, x_{3}\right\}$.

Proof: Suppose $p$ is a prime for which $e=v_{p}(X)>v_{p}(Y)$. Then we have $s=v\left(a_{3}\right)<e$.
If $t=v\left(a_{2}\right) \geq e$, then $v\left(\left\{x_{1}, x_{3}, a_{3}\right\}\right)=v\left(\left\{a_{2}, x_{2}, x_{4}\right\}\right) \geq e$. This is a contradiction. Hence, we have $v\left(a_{2}\right)<e$.

From this and $v(X)=e$, we have

$$
v\left(x_{1}\right)=v\left(a_{2}\right)=v\left(x_{5}\right)=v\left(x_{3}\right)=t<e .
$$

Thus, $v\left(a_{2} x_{2} x_{4}\right) \geq 2 e+t$ and $v\left(x_{1} a_{3} x_{3}\right)=2 t+s$. We have $2 t+s \geq 2 e+t$, since $a_{2} x_{2} x_{4}=x_{1} a_{3} x_{3}$. This reduces to $t+s \geq 2 e$, which is a contradiction.

Hence, $v(X) \leq v(Y)$ for any prime $p$. The argument to show that $v(Y) \leq v(X)$ is similar, and is omitted here. From this, it follows that $\operatorname{gcd} X=\operatorname{gcd} Y$.

Theorem 3: Using the notation given in Figure 4 below,

$$
\begin{equation*}
\operatorname{gcd}\left\{a_{1}, a_{3}, \ldots, a_{11}\right\}=\operatorname{gcd}\left\{a_{2}, a_{4}, \ldots, a_{12}\right\} \tag{4.3}
\end{equation*}
$$



FIGURE 4. The $4 \times 2 \times 4 \times 2 \times 4 \times 2$ Hexagon
Proof: The $4 \times 2 \times 2$ hexagon at the top has the gcd property with respect to the sets $\left\{a_{1}, a_{3}, a_{5}, x_{5}, a_{11}\right\}$ and $\left\{a_{2}, a_{4}, a_{6}, x_{4}, a_{12}\right\}$. Lemma 2 applies with $P$ as the $4 \times 2 \times 2$ hexagon and $H$ as the fundamental hexagon centered at $x_{6}$. This gives the desired result in (4.3).

Now let $P$ be any polygon that has a restricted tiling of four fundamental hexagons. If there is a fundamental hexagon in the tiling that intersects only one of the other fundamental hexagons in the tiling, the polygon $P$ will have the desired gcd property. Theorem 1 would give the result using the fundamental hexagon as one polygon and the other component as the second polygon.

Suppose then that each fundamental hexagon intersects at least two of the other fundamental hexagons. The only possibilities are shown with their tilings in Figure 5. They are the $2 \times 4 \times 4$, $4 \times 2 \times 4$, and $4 \times 4 \times 2$ hexagons.

Each of these hexagons has the desired gcd property. Each of these three cases follows from Theorem 2 and Lemma 2. The fundamental hexagon has been placed on the upper right, bottom, and upper left, respectively, to obtain the three hexagons.


FIGURE 5. Hexagons Tiled by Four Fundamental Hexagons
Therefore, we have the following theorem.
Theorem 4: Lete $P$ be any polygon in Pascal's triangle that admits a restricted tiling of four or fewer fundamental hexagons. Then $P$ has the gcd property with respect to the sets $\left\{a_{i} \mid i\right.$ odd $\}$ and $\left\{a_{i} \mid i\right.$ even $\}$.

This can also be extended to the $4 \times 2 \times 6 \times 2 \times 4 \times 4$, the $4 \times 4 \times 4 \times 2 \times 6 \times 2$, and the $6 \times 2 \times 4 \times 4 \times 4 \times 2$ hexagons using Theorem 4 and Lemma 2 .

Consider the polygon in Figure 6 below. The $4 \times 4 \times 4$ hexagon $P$ defined by $\left\{a_{1}, x_{1}, x_{2}, a_{6}\right.$, $\left.a_{7}, a_{8}, \ldots, a_{20}\right\}$ has been shown by Long $\&$ Ando [8] to have the gcd property with respect to $\left\{a_{1}, x_{2}, a_{7}, a_{9}, \ldots, a_{19}\right\}$ and $\left\{x_{1}, a_{6}, a_{8}, \ldots, a_{20}\right\}$. There is a fundamental hexagon $H$ centered at $x_{3}$.

Apply Lemma 2 with this $P$ and $H$. We see that the polygon of Figure 6 has the gcd property with respect to $\left\{a_{1}, a_{3}, \ldots, a_{19}\right\}$ and $\left\{a_{2}, a_{4}, \ldots, a_{20}\right\}$. This polygon has no tiling, restricted or otherwise, of fundamental hexagons.


FIGURE 6. A Polygon that Has No Tiling
Consider the polygon of Figure 7, which can be tiled by fundamental hexagons as illustrated. It does not have a restricted tiling. If $X=\left\{a_{1}, a_{3}, a_{5}, a_{7}\right\}$ and $Y=\left\{a_{2}, a_{4}, a_{6}, a_{8}\right\}$, we do not have $\operatorname{gcd} X=\operatorname{gcd} Y$. If we place the octagon so that $a_{1}=\binom{22}{11}$, we have $\operatorname{gcd} X=1292$ and $\operatorname{gcd} Y=646$.


## FIGURE 7. An Octagon

We close with the following observation. A polygon possessing a restricted tiling seems to have the desired gcd property. It is by no means a necessary condition, as the polygon in Figure 6 shows. However, possessing a tiling that is not a restricted tiling is not sufficient to guarantee the desired gcd property, as the octagon of Figure 7 shows.

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## Author and Title Index

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# FIBONACCI NUMBERS AND A CHAOTIC PIECEWISE LINEAR FUNCTION 

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## INTRODUCTION

The continuous piecewise linear function defined by

$$
g(x)= \begin{cases}x+1 / 2 & \text { for } x \text { in } H=[0,1 / 2] \\ 2(1-x) & \text { for } x \text { in } I=[1 / 2,1]\end{cases}
$$

was displayed by Xun Cheng Huang in [1, p. 97] as an example of a function having periodic points of every finite order $n$ under iteration by $g$ where $g^{n}(x)=g\left(g^{n-1}(x)\right)$, with $g^{0}(x)=x$. We shall examine the iterates of $g$, and show that there are $F_{n+2}$ subintervals of $U=[0,1]$ on which $g^{n}$ is linear, of which $F_{n}$ lie in $H$ and $F_{n+1}$ lie in $I$. Of the $F_{n}$ intervals in $H, F_{n-2}$ are mapped by $g^{n}$ onto $I$ and $F_{n-1}$ onto $U$; of the $F_{n+1}$ in $I, F_{n-1}$ are mapped onto $I$ and $F_{n}$ onto $U$; by $g^{n}$. Furthermore, the number of points in $U$ whose period is a factor of $n$ under iteration by $g$ is the Lucas number $L_{n}=F_{n-1}+F_{n+1}$. Finally, we examine the cycles in which rational numbers in $U$ with any given odd denominator appear under iteration by $g$.

## BUNS AND BINS, BUNKS AND BINKS

We shall call an interval mapped bijectively on $U$ by $g^{n}$ a "bun" and an interval mapped bijectively on $I=[1 / 2,1]$-but not in a bun-a "bin." "Bunks and "binks" are buns and bins of a fixed width $2^{-k}$. Each of the $F_{n+1}$ buns in $U$ and $F_{n-1}$ bins in $I$ contain a periodic point $x$ such that $g^{n}(x)=x$, so there are $L_{n}$ points $x$ in $U$ whose period under $g$ is a factor of $n$.

We denote by $H_{m, k}$ or $I_{m, k}$ a bin or a bun of width $2^{-k}$ having one endpoint $x=m / 2^{k}$ such that $g^{n}(x)=1$. If $m$ is odd, the bink $H_{m, k}$ is adjacent to the bunk $I_{m, k}$, preceding it for odd $k$, or following it for even $k$. There are $F_{n}$ such pairs. There are no bins with even $m$. Of the $F_{n-1}$ buns with even $m$ one is $I_{0, k}$ if $2 n=3 k+1$, or one is $I_{m, k}, m=2^{k}$, if $2 n=3 k$. The remaining buns are adjacent pairs, one twice as wide as the other, such as $I_{12,4}$ and $I_{6,3}$ for $n=5$ that have the common endpoint $x=12 / 16=6 / 8$.

When $n=1$, the $F_{3}=2$ intervals are $H=H_{1,1}=[0,1 / 2]$; and $I=I_{1,1}=[1 / 2,1]$. The $F_{n+2}=3,5$, and 8 intervals for $n=2,3$, and 4 are

$$
\begin{array}{ll}
I_{01} ; I_{32} H_{32} & n=2, F_{4}=3 \\
I_{12} H_{12} ; H_{53} I_{53}, I_{42} & n=3, F_{5}=5 \\
H_{13} I_{13} I_{22} ; I_{43}, I_{11,4} H_{11,4}, H_{73} I_{73} & n=4, F_{6}=8
\end{array}
$$

We separate by a semicolon the buns and bins in $H$ from those in $I$. To proceed from one $n$ to the next, we first list the intervals in $I$ for $n-1$ as the intervals in $H$ for $n$ with the same $k$, but with $m$ replaced by $m-2^{k-1}$ (or $x$ by $x-1 / 2$ ). Then, after a semicolon, we list all the intervals for $n-1$
in reverse order as intervals in $I$ for $n$, but with $k$ replaced by $k+1$ and $m$ by $2^{k+1}-m$, thus replacing $x=m / 2 k$ by $y=1-x / 2$, since $g$ replaces $y>1 / 2$ by $g(y)=2(1-y)=x$.

We assume as induction hypothesis that, for $n=N-1$ there are $F_{n-2}$ bins and $F_{n-1}$ buns in $H$, and $F_{n-1}$ bins and $F_{n}$ buns in $I$, for a total of $F_{n}$ bins and $F_{n+1}$ buns in $U$, of which $F_{n}$ intervals are in $H$ and $F_{n+1}$ in $I$. We verify this for $n=2$ and 3 . Then, since $F_{n-1}+F_{n}=F_{n+1}$, the construction given above shows that the same is true for $n=N$, proving the hypothesis for all $n>2$.

For $n=5$ we list the $F_{7}=13$ buns and bins as follows:

$$
I_{0,3}, I_{3,4} H_{3,4}, H_{3,3} I_{3,3}, I_{9,4} H_{9,4}, H_{21,5} I_{21,5}, I_{12,4} I_{6,3}, I_{15,4} H_{15,4}
$$

Next we classify the binks and bunks for fixed $n$ and $k$, and count them using binomial coefficients $b_{n, k}$ defined by

$$
b_{n, k}=\binom{k-1}{n-k}=\binom{k-1}{2 k-n-1}=\binom{k}{n-k}-\binom{k-1}{n-k-1}=b_{n+1, k+1}-b_{n-1, k}
$$

assuming $0 \leq n-k \leq k$. The sum over $k$ of $b_{n, k}$ is $F_{n}$.
For $2<n \geq 2 k$ the distributions are found to be as follows:

|  | In $H$ | In $I$ | In $U$ |  |
| :--- | :---: | :---: | :--- | :--- |
| Binks ( $m$ odd) | $b_{n-2, k-1}$ | $b_{n-1, k-1}$ | $b_{n, k}$ | $n>1$ |
| Bunks ( $m$ odd) | $b_{n-2, k-1}$ | $b_{n-1, k-1}$ | $b_{n, k}$ | $n>1$ |
| Bunks ( $m$ even) | $b_{n-3, k-1}$ | $b_{n-2, k-1}$ | $b_{n-1, k}$ | $n>2$ |
| Bunks (all $m$ ) | $b_{n-1, k}$ | $b_{n, k}$ | $b_{n+1, k+1}$ |  |

Summing over $k$, we replace $b_{n-i, k-j}$ by $F_{n-i}$, since

$$
\sum_{k} b_{n, k}=\sum_{k}\binom{k-1}{n-k}=F_{n} .
$$

For $n>2$ we prove this count by induction, first checking its validity for $n=3$ and 4. Bink and bunk counts for $g^{n}$ in $H$ are those for $g^{n-1}$ in $I$, with $n$ replaced by $n-1$. Bink and bunk counts for $g^{n}$ in $I$ are those for $g^{n-1}$ in $U$, with $n$ and $k$ replaced by $n-1$ and $k-1$, since $g$ doubles widths of intervals in $I$ mapped on $U$. Thus, the counts are valid for $n>2$.

## PERIODIC POINTS

A periodic point $x$ such that $g^{n}(x)=x$ is contained in each of the $F_{n+1}$ intervals $I_{m, k}$ for $g^{n}$ that map onto $U$, but only in the $F_{n-1}$ intervals $H_{m, k}$ in $I$, since $g^{n}$ maps $H_{m, k}$ intervals in $H$ onto $I$ without overlap. Thus, the number of periodic points in $U$ whose periods divide $n$ is

$$
L_{n}=F_{n-1}+F_{n+1}=\tau^{n}+(-\tau)^{-n}, \text { where } \tau=\left(5^{1 / 2}+1\right) / 2 .
$$

The coordinate $x$ of the periodic point in an $I_{m, k}$ interval is

$$
x=\left(m+(-1)^{k}(x-1)\right) / 2^{k}=\left(m-(-1)^{k}\right) /\left(2^{k}-(-1)^{k}\right) .
$$

The coordinate $x$ of the periodic point in an $H_{m, k}$ interval is

$$
x=1-y / 2=\left(m+(-1)^{k} y\right) / 2^{k}=\left(m+2(-1)^{k}\right) /\left(2^{k}+2(-1)^{k}\right) .
$$

For $n=5$ the 11 intervals and periodic points for $g$ are

$$
I_{0,3}, I_{3,4} I_{3,3} ; I_{9,4} H_{9,4}, H_{21,5} I_{21,5}, I_{12,4} I_{6,3}, I_{15,4} H_{15,4}
$$

Note that $H_{m, k}$ intervals yield even denominators. The point $22 / 33=2 / 3$ with $k=n$ is a fixed point of $g$. The others form two period 5 cycles with $k=3$ and 4 , respectively:

$$
(1 / 9,11 / 18,7 / 9,4 / 9,17 / 18),(2 / 15,19 / 30,11 / 15,8 / 15,14 / 15) .
$$

Each of the $\phi(b)$ rational numbers $x=a / b$ in $U$ with $b$ odd and $(a, b)=1$ is periodic under iterations of $g$. For $x$ in $I$ we have $b g(x)=2 b(1-a / b) \equiv-2 a(\bmod b)$, whereas for $x$ in $H$ we have $g(x)=a / b+1 / 2, b g^{2}(x)=2 b(1 / 2-a / b) \equiv-2 a(\bmod b)$. If $t$ is the exponent of $-2(\bmod$ $b$ ), there are $t$ fractions $j / b$ in the cycle with $a / b$, such that $0<j<b$. These $j$ form a coset of the subgroup generated by $b-2$ in the group $\phi(b)$ residues relatively prime to $b$. If -2 is a quadratic residue of $b$, then $t$ divides $\phi(b) / 2$. If $h$ is the number of the $j / b$ in the cycle with $a / b$ that lie in $H$, then the cycle contains $h$ fractions with denominator $2 b$, and has length $n=t+h$. The cycle containing $1-a / b$ has $t-h$ denominators $2 b$ and length $2 t-h$. There are a total of $\phi(b) / t$ cycles containing the $\phi(b)$ fractions $j / b$ and $\phi(b) / 2$ fractions $(2 j+b) / 2 b<1$.

To illustrate the theory, we give some examples:
(a) If $b=23,-2$ is a quadratic nonresidue $(\bmod b)$, to $t=22$ and $h=11$. Since $23 \times 89=2^{11}-1$, 23 divdides $2^{t}-(-1)^{t}$.
(b) If $b=19,-2=6^{2}(\bmod b$, so $t$ divides 9. Powers of $-2(\bmod 19)$ are congruent to $-2,4,-8$, $-3,6,7,5,9,1$, so $h=6$ of these nine are between 0 and 19/2. Thus, $1 / 19$ and 18/19 are in cycles of $n=9+6$ and $9+3$. Since $513=27 \times 19, b$ divides $2^{9}+1$.
(c) If $b=33$, the powers of $-2(\bmod 33)$ are $-2,4,-8,16,1$, so $(t, h)=(5,3)$ and $(5,2)$ for cycles with $a / b=1 / 33$ and $32 / 33$. Since $\phi(b)=20$, there are two other cycles like these.

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# GENERATING SOLUTIONS FOR A SPECIAL CLASS OF DIOPHANTINE EQUATIONS 

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Let $p=p\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be a polynomial with positive integer coefficients. In this paper we shall discuss some methods for generating solutions for the equation

$$
\begin{equation*}
p+y^{2}=z^{2} . \tag{1}
\end{equation*}
$$

The approach we use is to start with a method for generating solutions for the equaiton

$$
\begin{equation*}
x^{2}+y^{2}=z^{2}, \tag{2}
\end{equation*}
$$

and show how the method is extended to equation (1) or to special cases of (1).

## 1. THE RULE OF PYTHAGORAS AND THE RULE OF PLATO

According to Dickson [1], it was Pythagoras who showed that, if we start with the odd integer $a$, let $b=\frac{1}{2}\left(a^{2}-1\right)$ and $c=b+1$, then $(a, b, c)$ is a solution of (2).

Again, according to Dickson [1], it was Plato who showed that, if we start with the even integer $a$, let $b=\frac{1}{4} a^{2}-1$ and $c=b+2$, then ( $a, b, c$ ) is also a solution of (2).

The methods of Pythagoras and Plato are extended to (1) by the following proposition.
Proposition 1: Let $a_{1}, a_{2}, \ldots, a_{n}$ be positive integers and let $a=p\left(a_{1}, a_{2}, \ldots, a_{n}\right)$.
i. If $a$ is odd, let $b=\frac{1}{2}(a-1)$ and $c=b+1$, then $\left(a_{1}, a_{2}, \ldots, a_{n}, b, c\right)$ is a solution of (1).
ii. If $a \equiv 0(\bmod 4)$, let $b=\frac{1}{4} a-1$ and $c \equiv b+2$, then $\left(a_{1}, a_{2}, \ldots, a_{n}, b, c\right)$ is a solution of (1).
iii. If $a \equiv 2(\bmod 4)$, then it is impossible to find integers $b$ and $c$ such that $\left(a_{1}, a_{2}, \ldots, a_{n}, b, c\right)$ is a solution of (1).

Proof: For i and ii, write $c^{2}-b^{2}$ as $(c-b)(c+b)$, substitute and simplify. If $a \equiv 2(\bmod 4)$, then, for integers $b$ and $c, a+b^{2} \equiv 2$ or $3(\bmod 4)$ depending on whether $b$ is even or odd, respectively, but $c^{2} \equiv 0$ or $1(\bmod 4)$ depending on whether $c$ is even or odd, respectively.

## 2. THE METHOD OF RECURSION

Let ( $a, b, c$ ) be a solution of (2). Let $d=c-b, a_{1}=a+d, b_{1}=a+b+\frac{d}{2}$, and $c_{1}=b_{1}+d$ In [2] I showed that ( $a_{1}, b_{1}, c_{1}$ ) is also a solution of (2). Let us call this method the "method of recursion." The following proposition extends the method of recursion to the equation

$$
\begin{equation*}
k_{1} x_{1}^{2}+k_{2} x_{2}^{2}+\cdots+k_{n} x_{n}^{2}+m+y^{2}=z^{2} . \tag{3}
\end{equation*}
$$

Proposition 2: Let $\left(a_{1}, a_{2}, \ldots, a_{n}, b, c\right)$ be a solution of equation (3) and let $d=c-b$. For $i=1$ to $n$ define

$$
a_{i}^{\prime}=a_{i}+d, b^{\prime}=\Sigma k_{i} a_{i}+b+\frac{d \Sigma k_{i}}{2}, \text { and } c^{\prime}=b^{\prime}+d .
$$

Then ( $a_{1}^{\prime}, a_{2}^{\prime}, \ldots, a_{n}^{\prime}, b^{\prime}, c^{\prime}$ ) is also a solution of (3).
Proof: Substitute $a_{i}+d$ for $a_{i}^{\prime}$ and simplify to obtain

$$
\Sigma k_{i}\left(a_{i}^{\prime}\right)^{2}=\Sigma k_{i}\left(a_{i}+d\right)^{2}=\Sigma k_{i} a_{i}^{2}+2 d \Sigma k_{i} a_{i}+d^{2} \Sigma k_{i} .
$$

Substitute $c^{2}-b^{2}-m$ for $\sum k_{i} a_{i}^{2}$, write $c^{2}-b^{2}$ as $d(c+b)$, and factor out $d$ to obtain

$$
d\left(c+b+2 \Sigma k_{i} a_{i}+d \Sigma k_{i}\right)-m .
$$

Substitute $2 b^{\prime}-2 b$ for $2 \Sigma k_{i} a_{i}+d \Sigma k_{i}$ to obtain

$$
d\left(c+b+2 \Sigma k_{i} a_{i}+d \Sigma k_{i}\right)-m=d\left(c-b+2 b^{\prime}\right)-m .
$$

And since $c-b=c^{\prime}-b^{\prime}=d$, we obtain

$$
d\left(c-b+2 b^{\prime}\right)-m=\left(c^{\prime}\right)^{2}-\left(b^{\prime}\right)^{2}-m .
$$

Note that when $d \Sigma k_{i}$ is odd we do not obtain integer solutions (see Example 1 below). In this case, apply the recursion twice to obtain the following corollary.

Corollary Let $\left(a_{1}, a_{2}, \ldots, a_{n}, b, c\right)$ be a solution of equation (3) and let $d=c-b$. For $i=1$ to $n$ define

$$
a_{i}^{\prime}=a_{i}+2 d, b^{\prime}=2 \Sigma k_{i}\left(a_{i}+d\right)+b, \text { and } c^{\prime}=b^{\prime}+d .
$$

Then ( $a_{1}^{\prime}, a_{2}^{\prime}, \ldots, a_{n}^{\prime}, b^{\prime}, c^{\prime}$ ) is also a solution of (3).
The following example illustrates the use of Proposition 1, Proposition 2, and its Corollary.
Example 1: Suppose we begin with the equation

$$
\begin{equation*}
2 x_{1}^{2}+x_{2}^{2}+2 x_{3}^{2}+4+y^{2}=z^{2} . \tag{4}
\end{equation*}
$$

If we let $x_{1}=x_{3}=1$ and $x_{2}=2$, then, by Proposition $1,(1,2,1,2,4)$ is a solution of (4). Here, $d=4-2=2$. Applying Proposition 2, we have

$$
\begin{aligned}
& a_{1}^{\prime}=3, \quad a_{2}^{\prime}=4, \quad a_{3}^{\prime}=3, \\
& b^{\prime}=2 \cdot 1+1 \cdot 2+2 \cdot 1+2+\frac{2(2+1+2)}{2}=13, \\
& c^{\prime}=15 .
\end{aligned}
$$

Hence, $(3,4,3,13,15)$ is also a solution of (4).
If we let $x_{1}=x_{2}=x_{3}=1$, then, by Proposition $1,(1,1,1,4,5)$ is a solution of (4). Here, $d=5-4=1$. Applying Proposition 2, we have

$$
\begin{aligned}
& a_{i}^{\prime}=2, \quad a_{2}^{\prime}=2, \quad a_{3}^{\prime}=2, \\
& b^{\prime}=2 \cdot 1+1 \cdot 1+2 \cdot 1+4+\frac{(2+1+2)}{2}=\frac{23}{2}, \\
& c^{\prime}=\frac{25}{2} .
\end{aligned}
$$

Hence, $\left(2,2,2, \frac{23}{2}, \frac{25}{2}\right)$ is also a solution of (4).

In this case, the solution is not an integer solution. However, if we apply the Corollary to Proposition 2, we obtain

$$
\begin{aligned}
& a_{1}^{\prime}=3, \quad a_{2}^{\prime}=3, \quad a_{3}^{\prime}=3, \\
& b^{\prime}=2(2 \cdot 2+1 \cdot 2+2 \cdot 2)+4=24, \\
& c^{\prime}=25 .
\end{aligned}
$$

Hence, (3, 3, 3, 24, 25) is also a solution of (4).

## 3. THE METHOD OF MATRICES

In [3], Hall showed that, if we mutliply a solution $(a, b, c)$ of (2) by any of the following three matrices, the product is also a solution of (2).

$$
\left[\begin{array}{lll}
1 & -2 & 2 \\
2 & -1 & 2 \\
2 & -2 & 3
\end{array}\right] \quad\left[\begin{array}{lll}
1 & 2 & 2 \\
2 & 1 & 2 \\
2 & 2 & 3
\end{array}\right] \quad\left[\begin{array}{lll}
-1 & 2 & 2 \\
-2 & 1 & 2 \\
-2 & 2 & 3
\end{array}\right]
$$

Let us call this method the "method of matrices." The following proposition extends the method of matrices to the equation

$$
\begin{equation*}
n x^{2}+y^{2}+m=z^{2} . \tag{5}
\end{equation*}
$$

Proposition 3: Let ( $a, b, c$ ) be a solution of equation (5).
i. If $n=2 k$, the product of $(a, b, c)$ and any of the following three matrices is also a solution of (5).

$$
\left[\begin{array}{ccc}
1 & -1 & 1 \\
2 k & 1-k & k \\
2 k & -k & k+1
\end{array}\right]\left[\begin{array}{ccc}
1 & 1 & 1 \\
2 k & k-1 & k \\
2 k & k & k+1
\end{array}\right]\left[\begin{array}{ccc}
-1 & 1 & 1 \\
-2 k & k-1 & k \\
-2 k & k & k+1
\end{array}\right]
$$

ii. If $n=2 k+1$, the product of $(a, b, c)$ and any of the following three matrices is also a solution of (5)

$$
\left[\begin{array}{ccc}
1 & -2 & 2 \\
2 n & 1-2 n & 2 n \\
2 n & -2 n & 2 n+1
\end{array}\right]\left[\begin{array}{ccc}
1 & 2 & 2 \\
2 n & 2 n-1 & 2 n \\
2 n & 2 n & 2 n+1
\end{array}\right]\left[\begin{array}{ccc}
-1 & 2 & 2 \\
-2 n & 2 n-1 & 2 n \\
-2 n & 2 n & 2 n+1
\end{array}\right]
$$

(Note that when $n=1$ we obtain Hall's matrices stated above.)
Proof: Equation (5) is a special case of equation (3). By Proposition 2, with $k_{1}=n$,

$$
a^{\prime}=a+d, b^{\prime}=n a+b+\frac{n d}{2}, \text { and } c^{\prime}=b^{\prime}+d
$$

is also solution of (5). Let $n=2 k$, substitute $c-b$ for $d$, and simplify to obtain

$$
\begin{aligned}
a^{\prime} & =a-b+c, \\
b^{\prime} & =2 k a+(1-k) b+k c, \\
c^{\prime} & =2 k a-k b+(k+1) c .
\end{aligned}
$$

In matrix form, this becomes

$$
\left[\begin{array}{ccc}
1 & -1 & 1 \\
2 k & 1-k & k \\
2 k & -k & k+1
\end{array}\right]\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right]
$$

To obtain the second matrix, note that, if $(a, b, c)$ is a solution, then so is $(a,-b, c)$. Hence

$$
\left[\begin{array}{ccc}
1 & -1 & 1 \\
2 k & 1-k & k \\
2 k & -k & k+1
\end{array}\right]\left[\begin{array}{c}
a \\
-b \\
c
\end{array}\right]
$$

is also a solution. But

$$
\left[\begin{array}{ccc}
1 & -1 & 1 \\
2 k & 1-k & k \\
2 k & -k & k+1
\end{array}\right]\left[\begin{array}{c}
a \\
-b \\
c
\end{array}\right]=\left[\begin{array}{ccc}
1 & -1 & 1 \\
2 k & 1-k & k \\
2 k & -k & k+1
\end{array}\right]\left[\begin{array}{rrr}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right] .
$$

The third matrix is obtained similarly.
When $n=2 k+1$, we use the Corollary to Proposition 2.
The following example illustrates the use of Proposition 1 and Proposition 3.
Example 2: Suppose we begin with the equation

$$
\begin{equation*}
2 x^{2}+y^{2}=z^{2} . \tag{6}
\end{equation*}
$$

By Proposition 1, (2, 1, 3) is a solution of equation (6). Since $n$ is even, by Proposition 3 the matrices

$$
\left[\begin{array}{rrr}
1 & -1 & 1 \\
2 & 0 & 1 \\
2 & -1 & 2
\end{array}\right] \quad\left[\begin{array}{lll}
1 & 1 & 1 \\
2 & 0 & 1 \\
2 & 1 & 2
\end{array}\right] \quad\left[\begin{array}{lll}
-1 & 1 & 1 \\
-2 & 0 & 1 \\
-2 & 1 & 2
\end{array}\right]
$$

and the triple $(2,1,3)$ will generate the solutions $(4,7,9),(6,7,11)$, and $(2,-1,3)$, respectively.
If we begin with the equation

$$
\begin{equation*}
3 x^{2}+y^{2}=z^{2} \tag{7}
\end{equation*}
$$

then, by Proposition 1, (1, 1, 2) is a solution of equation (7). Since $n$ is odd, by Proposition (3) the matrices

$$
\left[\begin{array}{lll}
1 & -2 & 2 \\
6 & -5 & 6 \\
6 & -6 & 7
\end{array}\right] \quad\left[\begin{array}{lll}
1 & 2 & 2 \\
6 & 5 & 6 \\
6 & 6 & 7
\end{array}\right] \quad\left[\begin{array}{lll}
-1 & 2 & 2 \\
-6 & 5 & 6 \\
-6 & 6 & 7
\end{array}\right]
$$

and the triple $(1,1,2)$ will generate the solutions $(3,13,14),(7,23,26)$, and $(5,11,14)$, respectively.

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# SEQUENCES OF CONSECUTIVE $\boldsymbol{n}$-NIVEN NUMBERS* 

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(Submitted August 1992)
A Niven number [2] is a positive integer that is divisible by the sum of its digits. In 1982, Kennedy [3] showed that there do not exist sequences of more than 21 consecutive Niven numbers. In 1992, Cooper \& Kennedy [1] improved this result by proving that there does not exist a sequence of more than 20 consecutive Niven numbers. They also proved that this bound is the best possible by producing an infinite family of sequences of 20 consecutive Niven numbers.

For any positive integer $n \geq 2$, define an $n$-Niven number to be a positive integer that is divisible by the sum of the digits in its base $n$ expansion. This paper examines the maximal possible lengths of sequences of consecutive $n$-Niven numbers. The main result is given in the following theorem.

Main Theorem: For any $n \geq 2$, there does not exist a sequence of more than $2 n$ consecutive $n$-Niven numbers.

Note that this result is entirely general and gives Cooper \& Kennedy's result as a special case.
Fix any $n \geq 2$. The following notation will be used throughout this paper. A number written in base $n$ will be subscripted with ( $n$ ). For example, $12=22_{(5)}$. When a string of (nonboldface) variables is subscripted, it is assumed that each variable represents a single digit in the given base. For example, $i j_{(n)}$ represents a number that can be expressed in two base $n$ digits, $0 \leq i, j<n$. (Note that $i=0$ is allowed.) A boldface variable in such a string represents a (possibly empty) string of digits in the given base. For example, a22 ${ }_{(5)}$ represents a number congruent to 12 modulo 25 . Let $s(\mathbf{a})$ be the sum of the digits in the string a.

Lemma 1: Suppose that

$$
\mathbf{a} 0_{(n)}, \mathbf{a} 1_{(n)}, \ldots, \mathbf{a}(n-1)_{(n)}
$$

is a sequence of $n$ consecutive $n$-Niven numbers. Then $n$ divides $s(\mathbf{a})$.
Proof: Let $s=s(\mathbf{a})$. The base $n$ digit sums of the numbers $\mathbf{a} 0_{(n)}, \ldots, \mathbf{a}(n-1)_{(n)}$ are $s, s+1$, $\ldots, s+n-1$. Exactly one of the digit sums is divisible by $n$. The corresponding $n$-Niven number must also be divisible by $n$ and thus must be $\mathbf{a} 0_{(n)}$. Hence, $n$ divides the digit sum of $\mathbf{a} 0_{(n)}$, i.e., $n$ divides $s$.

Lemma 2: The $n+1$ consecutive numbers $\mathbf{a} 00_{(n)}, \ldots$, a $10_{(n)}$ are not all $n$-Niven numbers.
Procf: Suppose to the contrary. Since $n$-Niven numbers are by definition positive; $s=$ $s(\mathbf{a})>0$. Further, by Lemma $1, n$ divides $s$. Thus, $n \leq s$. The base $n$ digit sum of both $\mathbf{a} 01_{(n)}$ and

[^1]$\mathbf{a l o}_{(n)}$ is $s+1$. Since $s+1$ divides each, it must divide their difference, $n-1$. So $s+1 \leq n-1<s$. Contradiction.

Lemma 3: If $i \neq n-1$ and $s(\mathbf{a})+i>0$, then $\mathbf{a} i(n-1)_{(n)}$ and $\mathbf{a}(i+1)(n-2)_{(n)}$ are not both $n$-Niven numbers.

Proof: Let $s=s(\mathbf{a})$. The base $n$ digit sum of both $\mathbf{a} i(n-1)_{(n)}$ and $\mathbf{a}(i+1)(n-2)_{(n)}$ is $s+i+n-1$. If both are $n$-Niven numbers, then $s+i+n-1$ must also divide their difference, $n-1$. But $s+i>0$ implies that $s+i+n-1>n-1$. Contradiction.

Theorem 4: If $\mathbf{a} j_{(n)}$ is the first term in a sequence of length at least $2 n$ of consecutive $n$-Niven numbers, then $i=n-1$ and $j=0$.

Proof: Let $\mathbf{a} i j_{(n)}$ be the first term in such a sequence. Let $s=s(\mathbf{a})$. Suppose $i \neq n-1$. Then

$$
\mathbf{a}(i+1) 0_{(n)}, \mathbf{a}(i+1) 1_{(n)}, \ldots, \mathbf{a}(i+1)(n-1)_{(n)}
$$

is a subsequence of consecutive $n$-Niven numbers. By Lemma $1, n$ divides $s+i+1$, and so $s+1>0$. Further, both $\mathbf{a} i(n-1)_{(n)}$ and $\mathbf{a}(i+1)(n-2)_{(n)}$ are $n$-Niven numbers. But this is impossible by Lemma 3.

Similarly, if $i=n-1$ and $j \neq 0$, then the sequence contains the subsequence

$$
(\mathbf{a}+1) 00_{(n)}, \ldots,(\mathbf{a}+1) 10
$$

which cannot all be $n$-Niven numbers by Lemma 2 .
We now prove the Main Theorem as an easy corollary to Theorem 4.
Proof of the Main Theorem: Suppose $x_{1}, x_{2}, \ldots$ is a sequence of more than $2 n$ consecutive $n$-Niven numbers. By Theorem 4, both $x_{1}$ and $x_{2}$ end in zero when written in base $n$. This is clearly impossible.

It is not known whether this bound is the best possible. The earlier results show that it is the best possible for $n=10$ and computer calculations have verified that it is optimal for a number of other small values of $n$. A general proof, however, that applies to all values of $n$ has yet to be found.

Conjecture 5: For each $n \geq 2$, there exists a sequence of consecutive $n$-Niven numbers of length $2 n$.

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# A BRACKET FUNCTION TRANSFORM AND ITS INVERSE 

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(Submitted September 1992)
The object of this paper is to present a bracket function transform together with its inverse and some applications. The transform is the analogue of the binomial coefficient transform discussed in [2]. The inverse form will be used to give a short proof of an explicit formula in [1] for $R_{k}(n)$, the number of compositions of $n$ into exactly $k$ relatively prime summands.

Theorem 1-Bracket Function Transform: Define

$$
\begin{gather*}
S(n)=\sum_{k=1}^{n}\left[\frac{n}{k}\right] A_{k}=\sum_{j=1}^{n} \sum_{d \mid j} A_{d},  \tag{1}\\
\mathscr{A}(x)=\sum_{n=1}^{\infty} x^{n} A_{n}, \tag{2}
\end{gather*}
$$

and

$$
\begin{equation*}
\mathscr{S}(x)=\sum_{n=1}^{\infty} x^{n} S_{n} . \tag{3}
\end{equation*}
$$

Then

$$
\begin{equation*}
\mathscr{S}(x)=\frac{1}{1-x} \sum_{n=1}^{\infty} A_{n} \frac{x^{n}}{1-x^{n}} . \tag{4}
\end{equation*}
$$

Proof: We need the fact that

$$
\begin{equation*}
\sum_{n=k}^{\infty}\left[\frac{n}{k}\right] x^{n-k}=\frac{1}{(1-x)\left(1-x^{k}\right)}, k \geq 1,|x|<1, \tag{5}
\end{equation*}
$$

which is easily proved and is the bracket function analogue of the binomial series

$$
\begin{equation*}
\sum_{n=k}^{\infty}\binom{n}{k} x^{n-k}=\frac{1}{(1-x)(1-x)^{k}}, k \geq 1,|x|<1 . \tag{6}
\end{equation*}
$$

Relations (5) and (6) were exhibited and applied in [1] for the purpose of establishing some number theoretic congruences.

By means of (5) we may obtain the proof of (4) as follows:

$$
\begin{aligned}
\sum_{n=1}^{\infty} x^{n} \sum_{k=1}^{n}\left[\frac{n}{k}\right] A_{k} & =\sum_{k=1}^{\infty} A_{k} \sum_{n=k}^{\infty}\left[\frac{n}{k}\right] x^{n} \\
& =\sum_{k=1}^{\infty} A_{k} x^{k} \sum_{n=k}^{\infty}\left[\frac{n}{k}\right] x^{n-k}=\sum_{k=1}^{\infty} x^{k} A_{k} \frac{1}{(1-x)\left(1-x^{k}\right)},
\end{aligned}
$$

which completes the proof.
Note that (4) does not turn out as nicely as the corresponding result in [2] because we now have $1-x^{k}$ instead of $(1-x)^{k}$, which is the striking difference between (5) and (6). As a result,
we are not able to express $\mathscr{S}(x)$ as some function multiplied times $\mathscr{A}(x)$ as we did in [2]. Nevertheless, the result does express $\mathscr{G}$ in terms of $A$ instead of $S$.

Transform (1) may next be inverted by use of the Möbius inversion theorem, but this requires some care. Here is how we do it:

$$
S(n)-S(n-1)=\sum_{k=1}^{n}\left[\frac{n}{k}\right] A_{k}-\sum_{k=1}^{n-1}\left[\frac{n-1}{k}\right] A_{k},
$$

or just

$$
\begin{equation*}
S(n)-S(n-1)=\sum_{k=1}^{n}\left\{\left[\frac{n}{k}\right]-\left[\frac{n-1}{k}\right]\right\} A_{k} . \tag{7}
\end{equation*}
$$

However,

$$
\left[\frac{n}{k}\right]-\left[\frac{n-1}{k}\right]= \begin{cases}1 & \text { if } k \mid n, \\ 0 & \text { if } k \nmid n,\end{cases}
$$

so that we find the relation

$$
\begin{equation*}
S(n)-S(n-1)=\sum_{d \mid n} A_{d}, \tag{8}
\end{equation*}
$$

which may be inverted at once by the standard Möbius theorem to get

$$
\begin{equation*}
A(n)=\sum_{d \mid n} \mu\left(\frac{n}{d}\right)\{S(d)-S(d-1)\} \tag{9}
\end{equation*}
$$

It is easy to see that the steps may be reversed and we may, therefore, enunciate the bracket function inversion pair as

## Theorem 2-Bracket Function Inverse Pair:

$$
\begin{equation*}
S(n)=\sum_{k=1}^{n}\left[\frac{n}{k}\right] A_{k}=\sum_{j=1}^{n} \sum_{d \mid j} A_{d} \tag{10}
\end{equation*}
$$

if and only if

$$
\begin{equation*}
A(n)=\sum_{d \mid n} \mu\left(\frac{n}{d}\right)\{S(d)-S(d-1)\} \tag{11}
\end{equation*}
$$

This inversion pair is the dual of the familiar binomial coefficient pair

$$
\begin{equation*}
S(n)=\sum_{k=0}^{n}\binom{n}{k} A_{k} \tag{12}
\end{equation*}
$$

if and only if

$$
\begin{equation*}
A_{n}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} S(k) . \tag{13}
\end{equation*}
$$

Sometimes it will be convenient to restate the pair (10)-(11) as
Theorem 3:

$$
f(n, k)=\sum_{j=1}^{n}\left[\frac{n}{j}\right] g(j, k)=\sum_{j=1}^{n} \sum_{d \mid j} g(d, k)
$$

if and only if

$$
\begin{equation*}
g(n, k)=\sum_{d \mid n} \mu\left(\frac{n}{d}\right)\{f(d, k)-f(d-1, k)\} \tag{15}
\end{equation*}
$$

We will apply this form of our inversion theorem to give a short proof of a formula in [1]. In that paper the expansion

$$
\begin{equation*}
\binom{n}{k}=\sum_{j=1}^{n}\left[\frac{n}{j}\right] R_{k}(j)=\sum_{j=1}^{n} \sum_{\substack{d \mid j \\ d \geq k}} R_{k}(d) \tag{16}
\end{equation*}
$$

was first proved, where $R_{k}(n)=$ the number of compositions of $n$ into exactly $k$ relatively prime positive summands, i.e., the number of solutions of the Diophantine equation $n=a_{1}+a_{2}+a_{3}+$ $\cdots+a_{k}$ where $1 \leq a_{i} \leq n$ and $\left(a_{1}, a_{2}, a_{3}, \ldots, a_{k}\right)=1$.

Applying (14)-(15) to this, we obtain

$$
R_{k}(n)=\sum_{d \mid n} \mu\left(\frac{n}{d}\right)\left\{\binom{d}{k}-\binom{d-1}{k}\right\}=\sum_{d \mid n} \mu\left(\frac{n}{d}\right)\binom{d-1}{k-1}
$$

which proves the desired formula for $R_{k}(n)$.
The series (11) may be restated in the form

$$
\begin{equation*}
A_{n}=\sum_{k=1}^{n} H_{k}^{n} S(k) \tag{17}
\end{equation*}
$$

but it is awkward to give a succinct expression for the $H_{k}^{n}$ coefficients. To obtain these numbers, however, we may proceed as follows. From (11), we have

$$
\begin{aligned}
A(n) & =\sum_{d \mid n} \mu\left(\frac{n}{d}\right) S(d)-\sum_{d \mid n} \mu\left(\frac{n}{d}\right) S(d-1)=\sum_{d \mid n} \mu\left(\frac{n}{d}\right) S(d)-\sum_{(d+1) \mid n} \mu\left(\frac{n}{d+1}\right) S(d) \\
& =\sum_{k=1}^{n}\left\{\left[\frac{n}{d}\right]-\left[\frac{n-1}{k}\right]\right\} \mu\left(\frac{n}{k}\right) S(k)-\sum_{k=1}^{n}\left\{\left[\frac{n}{k+1}\right]-\left[\frac{n-1}{k+1}\right]\right\} \mu\left(\frac{n}{k+1}\right) S(k)-\mu(n) S(0)
\end{aligned}
$$

so that we have the following explicit formula for the $H$ coefficients:

$$
\begin{equation*}
H_{k}^{n}=\left\{\left[\frac{n}{k}\right]-\left[\frac{n-1}{k}\right]\right\} \mu\left(\frac{n}{k}\right)-\left\{\left[\frac{n}{k+1}\right]-\left[\frac{n-1}{k+1}\right]\right\} \mu\left(\frac{n}{k+1}\right) \text { for } 1 \leq k \leq n \tag{18}
\end{equation*}
$$

Ordinarily, $S(0)$ from (1) has the value 0 ; however, it is often convenient to modify (1) and define

$$
\begin{equation*}
S(n)=1+\sum_{k=1}^{n}\left[\frac{n}{k}\right] A_{k} \tag{19}
\end{equation*}
$$

so that $S(0)=1$. With this train of thought in mind, we present a table of $H_{k}^{n}$ for $0 \leq k \leq n, n=$ $0(1) 18$, so that the table may be used for either situation. Thus, the 0 -column in the array will be given by $-\mu(n)$, but with $H_{0}^{0}=1$.

A way to check the rows in the table of values of $H_{k}^{n}$ is by the formula

$$
\begin{equation*}
\sum_{k=1}^{n} H_{k}^{n}=\mu(n) \text { for all } n \geq 1 \tag{20}
\end{equation*}
$$

which, in a sense, gives a new representation of the Möbius function. The proof is very easy. In expression (11) of Theorem 2, just choose $S(n)=1$ for all $n \geq 1$. This makes $A(n)=\mu(n)$ for all $n \geq 0$. But then, by relation (17), we have result (20) immediately.

A Table of the Numbers $H_{k}^{n}$ for $0 \leq k \leq n, n=0(1) 18$


If we adopt the convention that $H_{0}^{n}=-\mu(n)$, but with $H_{0}^{0}=1$, then (20) may be reformulated to say that

$$
\begin{equation*}
\sum_{k=0}^{n} H_{k}^{n}=0, \text { for all } n \geq 1 \tag{21}
\end{equation*}
$$

The author wishes to acknowledge helpful comments by the referee, especially some corrections to the table of values of the $H$ coefficients.

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# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by<br>Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to $72717.3515 @$ compuserve.com on Internet. All correspondence will be acknowledged.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

## BASIC FORMULAS

The Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ satisfy

$$
\begin{aligned}
& F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1 ; \\
& L_{n+2}=L_{n+1}+L_{n}, L_{0}=2, L_{1}=1 .
\end{aligned}
$$

Also, $\alpha=(1+\sqrt{5}) / 2, \beta=(1-\sqrt{5}) / 2, F_{n}=\left(\alpha^{n}-\beta^{n}\right) / \sqrt{5}$, and $L_{n}=\alpha^{n}+\beta^{n}$.

## PROBLEMS PROPOSED IN THIS ISSUE

## B-760 Proposed by Russell Euler, Northwest Missouri State University, Maryville, MO

Prove that $F_{n+1}^{2} \geq F_{2 n}$ for all $n \geq 0$.

## B-761 Proposed by Richard André-Jeannin, Longwy, France

Evaluate the determinants

$$
\left|\begin{array}{lllll}
L_{0} & L_{1} & L_{2} & L_{3} & L_{4} \\
L_{1} & L_{0} & L_{1} & L_{2} & L_{3} \\
L_{2} & L_{1} & L_{0} & L_{1} & L_{2} \\
L_{3} & L_{2} & L_{1} & L_{0} & L_{1} \\
L_{4} & L_{3} & L_{2} & L_{1} & L_{0}
\end{array}\right| \text { and }\left|\begin{array}{ccccc}
L_{0}^{2} & L_{1}^{2} & L_{2}^{2} & L_{3}^{2} & L_{4}^{2} \\
L_{1}^{2} & L_{0}^{2} & L_{1}^{2} & L_{2}^{2} & L_{3}^{2} \\
L_{2}^{2} & L_{1}^{2} & L_{0}^{2} & L_{1}^{2} & L_{2}^{2} \\
L_{3}^{2} & L_{2}^{2} & L_{1}^{2} & L_{0}^{2} & L_{1}^{2} \\
L_{4}^{2} & L_{3}^{2} & L_{2}^{2} & L_{1}^{2} & L_{0}^{2}
\end{array}\right| \text {. }
$$

## B-762 Proposed by Larry Taylor, Rego Park, NY

Let $n$ be an integer.
(a) Generalize the numbers $(2,2,2)$ to form three three-term arithmetic progressions of integral multiples of Fibonacci and/or Lucas numbers with common differences $3 F_{n}, 5 F_{n}$, and $3 F_{n}$.
(b) Generalize the numbers $(4,4,4)$ to form two such arithmetic progressions with common differences $F_{n}$ and $F_{n}$.
(c) Generalize the numbers $(6,6,6)$ to form four such arithmetic progressions with common differences $F_{n}, 5 F_{n}, 7 F_{n}$, and $F_{n}$.

## B-763 Proposed by Juan Pla, Paris, France

Let $A=\left(\begin{array}{cc}e^{i \pi / 3} & \sqrt{2} \\ \sqrt{2} & e^{-i \pi / 3}\end{array}\right)$. Express $A^{n}$ in terms of Fibonacci and/or Lucas numbers.

## B-764 Proposed by Mark Bowron, Tucson, $A Z$

Consider row $n$ of Pascal's triangle, where $n$ is a fixed positive integer. Let $S_{k}$ denote the sum of every fifth entry, beginning with the $k^{\text {th }}$ entry, $\binom{n}{k}$. If $0 \leq i<j<5$, show that $\left|S_{i}-S_{j}\right|$ is always a Fibonacci number.

For example, row 10 of Pascal's triangle is $1,10,45,120,210,252,210,120,45,10,1$. Thus, $S_{0}=1+252+1=254, S_{1}=10+210=220$, and $254-220=34=F_{9}$.

## B-765 Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN

Let $m$ and $n$ be positive integers greater than 1 , and let $x=F_{m n} /\left(F_{m} F_{n}\right)$. What famous constant is represented by

$$
\left[\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} \frac{1}{j!}\right)\left(\frac{1}{x^{i}}-\frac{1}{x^{i+1}}\right)\right]^{x} ?
$$

## SOLUTIONS

## The Determination

## B-731 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 31, no. 1, February 1993)

Evaluate the determinant:

$$
\left|\begin{array}{lllll}
F_{0} & F_{1} & F_{2} & F_{3} & F_{4} \\
F_{1} & F_{0} & F_{1} & F_{2} & F_{3} \\
F_{2} & F_{1} & F_{0} & F_{1} & F_{2} \\
F_{3} & F_{2} & F_{1} & F_{0} & F_{1} \\
F_{4} & F_{3} & F_{2} & F_{1} & F_{0}
\end{array}\right| .
$$

Generalize.

## Solution 1 by Leonard A. G. Dresel, Reading, England

Let $M_{n}$ denote the $n \times n$ matrix ( $m_{i j}$ ) where $m_{i j}=F_{i-j \mid}$ so that the given determinant is $\operatorname{det}\left(M_{5}\right)$. For $n \geq 3, \operatorname{det}\left(M_{n}\right)$ remains unchanged if we subtract the sum of the second and third rows from the first row. This modified first row becomes $(-2,0,0, \ldots)$ because $F_{0}-F_{1}-F_{2}=-2$, $F_{1}-F_{0}-F_{1}=0$, and the remaining elements in the first row vanish because of the recurrence for $F_{j}$.

Hence, for $n \geq 3$, we have $\operatorname{det}\left(M_{n}\right)=(-2) \operatorname{det}\left(M_{n-1}\right)$. But, for $n=2$, we have $\operatorname{det}\left(M_{2}\right)=$ $F_{0}^{2}-F_{1}^{2}=-1$. Hence, we have by induction that, for $n \geq 2$, $\operatorname{det}\left(M_{n}\right)=-(-2)^{n-2}$. In particular, $\operatorname{det}\left(M_{5}\right)=8$.

## Solution 2 by Pentti Haukkanen, University of Tampere, Tampere, Finland

Replace the Fibonacci sequence $\left(F_{n}\right)$ by a sequence ( $w_{n}$ ) that satisfies $w_{n}=p w_{n-1}+w_{n-2}$ $(n \geq 2)$ with initial condition $w_{0}=0$, where $p$ and $w_{1}$ are arbitrary constants. Then, as in solution 1 , subtract $p$ times the second row plus the third row from row 1 . We find that the first row becomes $\left(-p w_{1}-w_{2}, 0,0, \ldots\right)$. By induction, the value of the $n \times n$ determinant is $\left(-p w_{1}-w_{2}\right)^{n-2}\left(-w_{1}^{2}\right)$. Since $w_{2}=p w_{1}$, this can be written as $(-1)^{n-1}(2 p)^{n-2} w_{1}^{n}$.
The proposer submitted the general case for an $n \times n$ matrix. This generalization was found by all solvers. In addition, Suck found the same generalization as Huakkanen. Bruckman obtained the result for a generalized Fibonacci sequence with two arbitrary initial conditions, but the result is a bit messy. See problem B-761 in this issue for a related problem.
Also solved by Richard André-Jeannin, Seung-Jin Bang, Glenn Bookhout, Scott H. Brown, Paul S. Bruckman, the Con Amore Problem Group, Russell Jay Hendel, Ed Korntved, Harris Kwong, Carl Libis, Graham Lord, Bob Prielipp, Sahib Singh, J. Suck, Ralph Thomas, A. N.'t Woord, and the proposer.

## The Mod Squad

## B-732 Proposed by Richard André-Jeannin, Longwy, France

(Vol. 31, no. 1, February 1993)
Let $\left(w_{n}\right)$ be any sequence of integers that satisfies the recurrence $w_{n}=p w_{n-1}-q w_{n-2}$ where $p$ and $q$ are odd integers. Prove that, for all $n, w_{n+6} \equiv w_{n}(\bmod 4)$.

## Solution by Ed Korntved, Morehead, KY

Since $p$ and $q$ are odd integers, both are congruent to $\pm 1$ modulo 4 . The sequence is completely determined by the first two elements of the sequence. There are four cases.
Case 1: $p \equiv 1, q \equiv 1(\bmod 4)$. The recurrence becomes $w_{n} \equiv w_{n-1}-w_{n-2}(\bmod 4) . ~ S t a r t i n g ~ w i t h ~$ $w_{1}=a$ and $w_{2}=b$, the sequence, modulo 4 , would be $a, b, b-a,-a,-b,-b+a, a, b, \ldots$, which is easily seen to have a period of 6 modulo 4 .
Case 2: $\quad p \equiv 1, q \equiv-1(\bmod 4)$. The recurrence becomes $w_{n} \equiv w_{n-1}+w_{n-2}(\bmod 4)$. The sequence would now be $a, b, a+b, a+2 b, 2 a+3 b, 3 a+b, a, b, \ldots$, which also has a period of 6 modulo 4 .
Case 3: $\quad p \equiv-1, q \equiv-1(\bmod 4)$ yields the sequence $a, b,-b+a, 2 b-a,-3 b+2 a, b-3 a, a, b, \ldots$, which has period 6 .
Case 4: $p \equiv-1, q \equiv 1(\bmod 4)$ yields the sequence $a, b,-a-b, a, b,-a-b, a, b, \ldots$, which also repeats every 6 terms.
Seiffert showed that $w_{n+12} \equiv w_{n}(\bmod 8)$. Is this the beginning of a trend?
Also solved by Seung-Jin Bang, Paul S. Bruckman, Joseph E. Chance, the Con Amore Problem Group, Charles K. Cook, Leonard A. G. Dresel, Herta T. Freitag, Jane E. Friedman, Russell Jay Hendel, Harris Kwong, Carl Libis, Don Redmond, H.-J. Seiffert, Sahib Singh, Lawrence Somer, J. Suck, Ralph Thomas, A. N.'t Woord, and the proposer. One incorrect solution was received.

## B-733 Proposed by Piero Filipponi, Rome, Italy

(Vol. 31, no. 1, February 1993)
Write down the Pell sequence, defined by $P_{0}=0, P_{1}=1$, and $P_{n+2}=2 P_{n+1}+P_{n}$ for $n \geq 0$. Form a difference triangle by writing down the successive differences in rows below it. For example,


Identify the pattern that emerges down the left side and prove that this pattern continues.

## Solution by Russell Jay Hendel, Morris College, Sumter, SC

It is straightforward to show that, if any row satisfies a linear recurrence with constant coefficients, then the difference row below it also satisfies the same recurrence. Thus, each row of Pell's difference triangle satisfies the Pell recurrence.

Now if $a$ and $b$ are any two successive terms in some row, then we have the following subtriangle:

$$
{ }^{a} \quad \begin{array}{cc}
b-a & \\
& \\
2 a
\end{array} \quad b+a{ }^{2 b+a}
$$

Thus, any element is twice the one two rows back (along the diagonal). Since the leftmost diagonal begins with a 0 and then a 1 , it follows that every second element along the diagonal is a 0 and that the intervening elements are successive powers of 2 .
Luchins and Hendel have found the pattern down the leftmost diagonal for the difference triangle of an arbitrary linear recurrence with constant coefficients. Their result is announced in [1]. See the previous issue of this Quarterly for more fun with Pell numbers.

## Reference

1. Edith H. Luchins \& Russell J. Hendel. Abstract 883-11-193: "Operators that Take Sequences to Diagonals of Their Difference Triangles." Abstracts of the American Mathematical Society 14 (1993):461.
Also solved by Richard André-Jeannin, Seung-Jin Bang, Paul S. Bruckman, Joseph E. Chance, the Con Amore Problem Group, Leonard A. G. Dresel, Russell Euler, Herta T. Freitag, Harris Kwong, H.-J. Seiffert, Tony Shannon, Sahib Singh, J. Suck, Ralph Thomas, David Tuller, A. N.'t Woord, and the proposer.

## Powers of 5

## B-734 Proposed by Paul S. Bruckman, Edmonds, WA

(Vol. 31, no. 1, February 1993)
If $r$ is a positive integer, prove that $L_{s^{r}} \equiv L_{s^{r-1}}\left(\bmod 5^{r}\right)$.

## Solution 1 by Leonard A. G. Dresel, Reading, England

From identity \#83 of [3], we have

$$
\begin{equation*}
F_{5 n}=F_{n}\left[25 F_{n}^{4}+25(-1)^{n} F_{n}^{2}+5\right] . \tag{1}
\end{equation*}
$$

Similarly, it is straightforward to show that

$$
\begin{equation*}
L_{5 n}=L_{n}\left[25 F_{n}^{4}+15(-1)^{n} F_{n}^{2}+1\right] . \tag{2}
\end{equation*}
$$

From equations (1) and (2), we see that

$$
\begin{equation*}
F_{5 n} \equiv 0\left(\bmod 5 F_{n}\right) \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{5 n} \equiv L_{n}\left(\bmod 5 F_{n}^{2}\right) \tag{4}
\end{equation*}
$$

Equation (3) can be written as $5 F_{n} \mid F_{5 n}$. Since $F_{1}=1$, it follows by induction that

$$
\begin{equation*}
5^{r} \mid F_{5^{r}} \tag{5}
\end{equation*}
$$

From equation (4) with $n=5^{r-1}$, we have $5 F_{n}^{2} \mid\left(L_{5 n}-L_{n}\right)$, so $5\left(5^{r-1}\right)^{2} \mid\left(L_{5 n}-L_{n}\right)$ or $L_{5^{r}} \equiv L_{5^{r-1}}$ $\left(\bmod 5^{2 r-1}\right)$, which generalizes the proposer's problem.

Singh, Somer, Suck, and Woord also found this generalization. Singh notes that since $L_{5^{r}}$ is always odd [this follows from the identity $L_{5 n}=L_{n}^{5}-5(-1)^{n} L_{n}^{3}+5 L_{n}$ ], we have the even stronger congruence: $L_{5^{r}} \equiv L_{5^{r-1}}\left(\bmod 2 \cdot 5^{2 r-1}\right)$. Prielipp points out that property (5) is given in [1]. Singh found it in [2]. Somer found it in [4].

## References

1. V. E. Hoggatt, Jr. Problem B-248. The Fibonacci Quarterly 11 (1973):553.
2. Verner E. Hoggatt, Jr., \& Gerald E. Bergum. "Divisibility and Congruence Relations." The Fibonacci Quarterly 12 (1974):189-95.
3. S. Vajda. Fibonacci \& Lucas Numbers, and the Golden Section: Theory and Applications. Chichester: Ellis Horwood Ltd., 1989.
4. D. D. Wall. "Fibonacci Series Modulo m." Amer. Math. Monthly 67 (1960):525-32.

## Solution 2 by Don Redmond, Southern Illinois University, Carbondale, IL

Let $\left(V_{n}\right)$ be a sequence defined by $V_{n+2}=a V_{n+1}+V_{n}$ with initial conditions $V_{0}=2$ and $V_{1}=a$, where $a$ is a positive integer. If $p$ is an odd prime and $r$ is a positive integer, we will show that

$$
\begin{equation*}
V_{p^{r}} \equiv V_{p^{r-1}}\left(\bmod p^{r}\right) \tag{1}
\end{equation*}
$$

In particular, if $a=1$, then $V_{n}$ becomes $L_{n}$ and we have

$$
\begin{equation*}
L_{p^{r}} \equiv L_{p^{r-1}}\left(\bmod p^{r}\right) \tag{2}
\end{equation*}
$$

This generalizes the current proposal, for which $p=5$.
To prove (1), we will use the fact that, if $n$ is odd, then

$$
\begin{equation*}
V_{p n}=V_{n}^{p}+\sum_{k=1}^{(p-1) / 2} p b_{k} V_{n}^{p-2 k} \tag{3}
\end{equation*}
$$

where $b_{k}=\frac{1}{k} \cdot\binom{p-k-1}{k-1}$. This was proven by Lucas in 1878 (see [2], p. 38). Since $c_{k}=p b_{k}$ can also be written as $\binom{p-k-1}{k}+.2\binom{p-k-1}{k-1}$, it is clear that $c_{k}$ is an integer. Furthermore, since $p$ is prime and $k<p$, we see that $c_{k}$ is divisible by $p$, so $b_{k}$ is an integer for all $k$.

We will proceed to prove equation (1) by induction. If equation (1) is true for some $r$, then we would have $V_{p^{r}}=V_{p^{r-1}}+p^{r} T$ for some integer $T$. Then by identity (3) with $n=p^{r}$ we find

$$
\begin{aligned}
V_{p^{r+1}} & =\left(V_{p^{r-1}}+p^{r} T\right)^{p}+\sum_{k=1}^{(p-1) / 2} p b_{k}\left(V_{p^{r-1}}+p^{r} T\right)^{p-2 k} \\
& =V_{p^{r-1}}^{p}+\sum_{k=1}^{(p-1) / 2} p b_{k} V_{p^{r-1}}^{p-2 k}+\text { a multiple of } p^{r+1} \\
& =V_{p^{r}}+\text { a multiple of } p^{r+1}
\end{aligned}
$$

which shows that equation (1) is true for $r+1$.
But equation (1) is true for $r=1$, since $V_{p} \equiv V_{1}^{p}=a^{p}(\bmod p)$ follows from identity (3) when $n=1$; and $a^{p} \equiv a(\bmod p)$ is true by Fermat's Little Theorem, since $p$ is a prime.

Thus, the induction is complete.
André-Jeannin also proved equation (2) and Seiffert found equation (2) in [1], p. 111.

## References

1. D. Jarden. Recurring Sequences, 3rd ed. Jerusalem: Riveon Lematematika, 1973.
2. Edouard Lucas. The Theory of Simply Periodic Numerical Functions. Santa Clara, Calif.: The Fibonacci Association, 1969.

Also solved by Richard André-Jeannin, Seung-Jin Bang, the Con Amore Problem Group, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, J. Suck, Ralph Thomas, A. N.'t Woord, and the proposer.

## Square Root of a Recurrence

## B-735 Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State Asylum for Crazed Mathematicians, Warrensburg, MO (Vol. 31, no. 1, February 1993)

Let the sequence $\left(y_{n}\right)$ be defined by the recurrence

$$
\begin{aligned}
y_{n+1}= & 8 y_{n}+22 y_{n-1}-190 y_{n-2}+28 y_{n-3}+987 y_{n-4}-700 y_{n-5}-1652 y_{n-6}+1652 y_{n-7} \\
& +700 y_{n-8}-987 y_{n-9}-28 y_{n-10}+190 y_{n-11}-22 y_{n-12}-8 y_{n-13}+y_{n-14}
\end{aligned}
$$

for $n \geq 15$ with initial conditions given by the table:

| $n$ | $y_{n}$ |
| :---: | :---: |
| 1 | 1 |
| 2 | 1 |
| 3 | 25 |
| 4 | 121 |
| 5 | 1296 |
| 6 | 9025 |
| 7 | 78961 |
| 8 | 609961 |
| 9 | 5040025 |
| 10 | 40144896 |
| 11 | 326199721 |
| 12 | 2621952025 |
| 13 | 21199651201 |
| 14 | 170859049201 |
| 15 | 1379450250000 |

Prove that $y_{n}$ is a perfect square for all positive integers $n$.

Editorial Note: For many years, back almost to the dawn of time, Paul S. Bruckman has been solving every single problem proposed in this column. When this "insane" problem came in, I jumped with the thought: "Aha! Now I can stump Bruckman." The proposers' solution involves pulling the recurrence $x_{n+1}=x_{n}+5 x_{n-1}+x_{n-2}-x_{n-3}$ out of a hat. It is then not too hard to show that the squares of the elements of this recurrence satisfy the original recurrence. My feeling was that there was no way anyone could find this "rabbit," and yet the proposers' solution was so simple that I could claim this problem was suitable for the Elementary Problem Column. Anyway, less than a week after the journal hit the newsstands, much to my chagrin, I received a letter from Bruckman containing a solution!

Several other readers also pulled the same rabbit out of the hat. They must be commended.

## Solution 1 by the proposers

Let $x_{n+1}=x_{n}+5 x_{n-1}+x_{n-2}-x_{n-3}$, for $n \geq 4$ with initial conditions $x_{1}=x_{2}=1, x_{3}=5$, and $x_{4}=11$.

We will show, by induction on $n$, that $y_{n}=x_{n}^{2}$ for $n \geq 1$.
The result is numerically true for $n=1,2, \ldots, 15$. Suppose the result is true for all $k<n$ where $n \geq 16$. Then, by the induction hypothesis,

$$
\begin{aligned}
y_{n+1}-x_{n+1}^{2}= & 8 x_{n}^{2}+22 x_{n-1}^{2}-190 x_{n-2}^{2}+28 x_{n-3}^{2}+987 x_{n-4}^{2}-700 x_{n-5}^{2}-1652 x_{n-6}^{2} \\
& +1652 x_{n-7}^{2}+700 x_{n-8}^{2}-987 x_{n-9}^{2}-28 x_{n-10}^{2}+190_{n-11}^{2}-22 x_{n-12}^{2} \\
& -8 x_{n-13}^{2}+x_{n-14}^{2}-\left(x_{n}+5 x_{n-1}+x_{n-2}-x_{n-3}\right)^{2} .
\end{aligned}
$$

In the right-hand side, make the substitutions $x_{k}=x_{k-1}+5 x_{k-2}+x_{k-3}-x_{k-4}$, for $k=n, n-1$, $\ldots, n-10$. A mere few hours of algebraic simplification then reveals that the right-hand side is identically 0 . Thus, $y_{n+1}=x_{n+1}^{2}$ and the induction is complete.

All other solvers found that $y_{n}$ satisfies the simpler recurrence,

$$
y_{n+1}=5 y_{n}+35 y_{n-1}-67 y_{n-2}-145 y_{n-3}+145 y_{n-4}+67 y_{n-5}-35 y_{n-6}-5 y_{n-7}+y_{n-8} .
$$

One of their solutions will be printed in a future issue, if space permits.
Also solved by Paul S. Bruckman, the Con Amore Problem Group, Leonard A. G. Dresel, H.-J. Seiffert, and A. N.'t Woord.

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by<br>Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

## FROBLEMS PROPOSED IN THIS ISSUE

## H-486 Proposed by Piero Filipponi, Rome, Italy

Let the terms of the sequence $\left\{Q_{k}\right\}$ be defined by the second-order recurrence relation $Q_{k}=$ $2 Q_{k-1}+Q_{k-2}$ with initial conditions $Q_{0}=Q_{1}=1$. Find restrictions on the positive integers $n$ and $m$ for

$$
T(n, m)=\sum_{k=1}^{\infty} \frac{k^{2} n^{k} Q_{k}}{m^{k}}
$$

to converge, and, under these restrictions, evaluate this sum. Moreover, find the set of all couples ( $n_{i}, m_{i}$ ) for which $T\left(n_{i}, m_{i}\right)$ is an integer.

## H-487 Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA

Suppose $H_{n}$ satisfies a second-order linear recurrence with constant coefficients. Let $\left\{a_{i}\right\}$ and $\left\{b_{i}\right\}, i=1,2, \ldots, r$, be integer constants and let $f\left(x_{0}, x_{1}, x_{2}, \ldots, x_{r}\right)$ be a polynomial with integer coefficients. If the expression

$$
f\left((-1)^{n}, H_{a_{1} n+b_{1}}, H_{a_{2} n+b_{2}}, \ldots, H_{a_{r} n+b_{r}}\right)
$$

vanishes for all integers $n>N$, prove that the expression vanishes for all integral $n$.
[As a special case, if an identity involving Fibonacci and Lucas numbers is true for all positive subscripts, then it must also be true for all negative subscripts as well.]

## SOLUTIONS

## Characteristically Common

## H-470 Proposed by Paul S. Bruckman, Everett, WA (Vol. 30, no. 3, August 1992)

Please see the issue of The Fibonacci Quarterly shown above for a complete presentation of this lengthy problem proposal.

Solution by the proposer (continued from Vol. 32, no. 1)
Proofs of parts (A) and (C) were given in the above issue of this Quarterly.

Proof of Part (B): We see that $U_{1}^{(r)}$ is a special case of $H_{1}^{(r)}$. Making the substitution $H_{1}^{(r)} \equiv U_{1}^{(r)}$ into part (C), the result follows at once.

Note: Although not required in the problem, we may obtain some interesting identities by taking determinants in the foregoing results. Moreover, special cases of $G_{r}(z)$ yield identities for the Fibonacci, Pell, Tribonacci, and Quadranacci numbers, some of which have already been studied extensively. For example, if $G_{2}(z)=z^{2}-z-1$, we obtain

$$
U_{1}^{(2)}=\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right) \quad \text { and } \quad U_{n}^{(2)}=\left(\begin{array}{cc}
F_{n+1} & F_{n} \\
F_{n} & F_{n-1}
\end{array}\right)
$$

In the general case, $p_{r}(0)=\left|-U_{1}^{(r)}\right|=(-1)^{r^{2}}\left|U_{1}^{(r)}\right|=(-1)^{r}\left|U_{1}^{(r)}\right|=G_{r}(0)=-a_{r-1}$, whence the result:

$$
\begin{equation*}
\left|U_{1}^{(r)}\right|=(-1)^{r-1} a_{r-1} \tag{**}
\end{equation*}
$$

Taking determinants in parts $(\mathrm{B})$ and $(\mathrm{C})$, we obtain

$$
\begin{gather*}
\left|U_{n}^{(r)}\right|=(-1)^{n(r-1)}\left(a_{r-1}\right)^{n}  \tag{***}\\
\left|H_{n}^{(r)}\right|=(-1)^{(n-1)(r-1)}\left(a_{r-1}\right)^{n-1}\left|H_{1}^{(r)}\right| \tag{****}
\end{gather*}
$$

Of course, $(* * * *)$ is a generalization of $(* * *)$. Again, special cases of $(* * * *)$ yield some wellknown results, e.g., $F_{n+1} F_{n-1}-F_{n}^{2}=(-1)^{n}$.

It is informative to apply the foregoing results for a special case which, however, has not been studied extensively. We will take $r=3$ and $G_{3}(z)=z^{3}-2 z^{2}-z-1$. We will choose $H_{n, j}^{(3)}$ 's so that, for $j=1,2,3, n=0,1,2$, we have $U_{n, j}^{(3)}+H_{n, j}^{(3)}=1$. We form the following brief table of values:

| $n$ | $U_{n, 1}$ | $U_{n, 2}$ | $U_{n, 3}$ | $H_{n, 1}$ | $H_{n, 2}$ | $H_{n, 3}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 2 | 1 | 0 | 0 | 0 | 1 | 1 |
| 3 | 2 | 1 | 1 | 2 | 3 | 3 |
| 4 | 5 | 3 | 2 | 5 | 7 | 8 |
| 5 | 13 | 7 | 5 | 12 | 18 | 20 |
| 6 | 33 | 18 | 13 | 31 | 46 | 51 |
| 7 | 84 | 46 | 33 | 79 | 117 | 130 |
| 8 | 214 | 117 | 84 | 201 | 298 | 331 |
| 9 | 545 | 298 | 214 | 512 | 759 | 843 |
| 10 | 1388 | 759 | 545 | 1304 | 1933 | 2147 |

We omit the superscript "(3)" for brevity. Then $\left|U_{1}\right|=1$ and $\left|H_{1}\right|=2$, as we may verify; hence, $\left|H_{n}\right|=2$ for all $n$. This may be left in determinant form or expanded into a sum of terms

$$
(-1)^{s-1} H_{n, i} H_{n+1, j} H_{n+2, k}
$$

where $i, j, k=1,2,3$ in some order, and $s$ is the minimum number of binary interchanges of digits required to obtain the triplet $(i, j, k)$ from the initial triplet $(1,2,3)$. This sum then must equal 2 . Clearly, many such identities may be devised.

## X It

## H-471 Proposed by Andrew Cusumano \& Marty; Samberg, Great Neck, NY

(Vol. 30, no. 4, November 1992)
Starting with a sequence of four ones, build a sequence of finite differences where the number of finite differences taken at each step is the term of the sequence. That is,

| $S_{1}$ | $S_{2}$ | $S_{3}$ |
| :---: | :---: | :---: |
| 1111 | 1111 | 1111 |
| 12345 | 12345 | 12345 |
|  | 12471116 | 12471116 |
|  |  | 1248152642 |

Now, reverse the procedure but start with the powers of the last row of differences and continue until differences are constant. For example, if the power is two, we have
1491625 3579 222
141649121256 etc.
$\begin{array}{lllll}3 & 12 & 33 & 72 & 135\end{array}$ 9213963
121824 66

The sequence of constants obtained when the power is two is $2,6,20,70, \ldots$, while the sequence of constants when the power is three is $6,90,1680,34650, \ldots$.

Let $N$ be the number of the term in the original difference sequence and $M$ be the power used in forming the reversed sequence. Show that the constant term is

$$
X(N, M)=\frac{(N \cdot M)!}{(N!)^{M}}, N=1,23, \ldots, M=2,3,4, \ldots
$$

For example, $X(2,3)=\frac{6!}{2^{3}}=90$.

## Solution by Paul S. Bruckman, Edmonds, WA

Let $\theta_{k, N}$ denote the $k^{\text {th }}$ term of row $S_{N}(k=1,2,3, \ldots)$. For example, $S_{2}=(1,2,4,7,11$, $16, \ldots$ ) and $\theta_{4,2}=7$. By definition, we are to have:

$$
\begin{align*}
& \theta_{k+1, N}-\theta_{k, N}=\theta_{k, N-1},  \tag{1}\\
& \theta_{k, 0}=1 \text { for all } k, \text { and }  \tag{2}\\
& \theta_{1, N}=1, N=1,2,3, \ldots . \tag{3}
\end{align*}
$$

Successive "finite integration" of (1), beginning with (2) and using (3), yields

$$
\begin{equation*}
\theta_{k, N}=\sum_{j=0}^{\left[\frac{1}{2} N\right]}\binom{k}{N-2 j} \tag{4}
\end{equation*}
$$

We note that $\theta_{k, N}$ is a polynomial in $k$ of degree $N$, whose leading term is equal to $k^{N} / N!$. Then $\left(\theta_{k, N}\right)^{M}$ is a polynomial in $k$ of degree $M N$, with leading term $k^{M N} /(N!)^{M}$. We then observe that $X(N, M)=\Delta^{M N}\left(\theta_{k, N}\right)^{M}=\Delta^{M N}\left(k^{M N} /(N!)^{M}\right)$, which yields

$$
X(N, M)=\frac{(M N)!}{(N!)^{M}}
$$

Note: Four "1's" in the original sequence will no longer suffice to display the constant term $X(N, M)$; the minimum number of "1's" required is $M N-N+1$. As expected, we find that $X(N, 1)=1$ for all $N$.

## Also solved by M. Deshpande.

## An Entry Level Job

## H-472 Proposed by Paul S. Bruckman, Edmonds, WA

(Vol. 30, no. 4, November 1992)
Let $Z(n)$ denote the Fibonacci entry point of the natural number $n$, that is, the smallest positive index $t$ such that $n \mid F_{t}$. Prove that $n=Z(n)$ iff $n=5^{u}$ or $n=12 \cdot 5^{u}$, for some $u \geq 0$.

## Solution by the proposer

Proof: We make use of the following special values:

$$
\begin{align*}
& Z\left(2^{r}\right)=1 \text { if } r=0,3 \cdot 2^{r-1} \text { if } r=1 \text { or } 2,3 \cdot 2^{r-2} \text { if } r \geq 3  \tag{1}\\
& Z\left(3^{s}\right)=1 \text { if } s=0,4 \cdot 3^{s-1} \text { if } s \geq 1  \tag{2}\\
& Z\left(5^{t}\right)=5^{t}, t \geq 0 \tag{3}
\end{align*}
$$

We will also make use of the following known facts regarding Fibonacci entry points:

$$
\begin{align*}
& Z\left(p^{e}\right)=p^{f} Z(p) \text {, for all primes } p \text {, where } 0 \leq f<e ;  \tag{4}\\
& \text { If } n=p_{1}^{e_{1}} p_{2}^{e_{2}} \cdots p_{\xi}^{e_{\xi}}, \text { then } Z(n)=\operatorname{LCM}\left[Z\left(p_{1}^{e_{1}}\right), Z\left(p_{2}^{e_{2}}\right), \ldots, Z\left(p_{\xi}^{e_{\xi}}\right)\right] . \tag{5}
\end{align*}
$$

Since $Z(12)=12$ and $Z\left(5^{u}\right)=5^{u}$, we see from (5) that $Z(n)=n$ if $n=5^{u}$ or $n=12 \cdot 5^{u}$.
Conversely, first suppose that $n=P$, where $P=2^{r} 3^{s}, r, s \geq 0$. If $r \geq 3$ and $s \geq 1$, then $Z(P)=2^{\max (r-2,2)} 3^{\max (s-1,1)}$; we see by inspecting the exponent of 2 that $z(P)=P$ is impossible. We may enumerate the remaining possibilities for $r$ and $s$ in the following table:

| $P$ | $Z(P)$ |
| :--- | :---: |
| $2^{0} 3^{0}=1$ | 1 |
| $2^{0} 3^{s}, s \geq 1$ | $2^{2} 3^{s-1}$ |
| $2^{1} 3^{0}=2$ | 3 |
| $2^{1} 3^{1}=6$ | 12 |
| $2^{1} 3^{s}, s \geq 2$ | $2^{2} 3^{s-1}$ |
| $2^{2} 3^{0}=4$ | 6 |
| $2^{2} 3^{1}=12$ | 12 |
| $2^{2} 3^{s}, s \geq 2$ | $2^{2} 3^{s-1}$ |
| $2^{r} 3^{0}, r \geq 3$ | $2^{r-2} 3^{1}$ |

We see that $n=P=Z(P)$ only if $n=1$ or 12 . Moreover, if we assume that $n=P \cdot 5^{u}$, we see that $n=Z(n)$ only if $n=5^{u}$ or $12 \cdot 5^{u}, u \geq 0$.

Next, we suppose that $n=P \cdot 5^{u} \cdot Q$, where $\operatorname{gcd}(Q, 30)=1$ and $Q>1$. Suppose $Q$ has the prime factorization: $Q=\prod_{i=1}^{\omega} q_{i}^{e_{i}}$; let $q=\max \left(q_{1}, q_{2}, \ldots, q_{\omega}\right)$ and $q^{e} \| Q$. Now $Z\left(q^{e}\right)=q^{f} Z(q)$,
where $0 \leq f<e$ and $Z(q)$ is divisible only by primes smaller than $q$ [since $Z(q) \leq q+1$ and $\operatorname{gcd}(q, Z(q))=1$; also, $q+1$ is even]. A fortiori, the same is true for the other $Z\left(q_{i}\right)$ 's. We therefore see that $q^{e} \| n$ and $q^{f} \| Z(n)$, which shows that $n \neq Z(n)$.

This exhausts the possibilities, and the problem is solved.

## Another Equivalence

## H-473 Proposed by A. G. Schaake \& J. C. Turner, Hamilton, New Zealand (Vol. 30, no. 4, November 1992)

Show that the following (see [1], p. 98) is equivalent to Fermat's Last Theorem: "For $n>2$ there does not exist a positive integer triple ( $a, b, c$ ) such that the two rational numbers $\frac{r}{s}, \frac{p}{q}$, with

$$
\begin{array}{ll}
r=c-a . & p=b-1, \\
s=\sum_{i=1}^{n} b^{n-i} & q=\sum_{i=1}^{n} a^{i-1} c^{n-i},
\end{array}
$$

are penultimate and final convergents, respectively, of the simple continued fraction (having an odd number of terms) for $\frac{p}{q}$."

## Reference

1. A. G. Schaake \& J. C. Turner. New Methods for Solving Quadratic Diophantine Equations (Part I and Part II). Research Report No. 192. Department of Mathematics and Statistics, University of Waikato, New Zealand, 1989.

## Solution by Paul S. Bruckman, Edmonds, WA

Suppose there exists a positive integer triple ( $a, b, c$ ) such that if $p, q, r$, and $s$ are as defined in the statement of the problem then

$$
\begin{equation*}
\frac{r}{s}=\left[\theta_{1}, \theta_{2}, \ldots, \theta_{m-1}\right] \text { and } \frac{p}{q}=\left[\theta_{1}, \theta_{2}, \ldots, \theta_{m-1}, \theta_{m}\right] \tag{1}
\end{equation*}
$$

for some sequence $\theta_{1}, \theta_{2}, \ldots, \theta_{m}$ of positive integers, where $m \geq 3$ is odd. The notation $\left[\theta_{1}, \theta_{2}, \ldots\right.$, $\left.\theta_{k}\right]$ represents the value of the simple continued fraction (s.c.f.) $=\theta_{1}+1 / \theta_{2}+1 / \theta_{3}+\cdots+1 / \theta_{k}$, $k=1,2, \ldots, m$, also known as the $k^{\text {th }}$ convergent of the s.c.f. for $p / q$.

Since $r / s$ and $p / q$ are supposed to be finite rationals, we require that $s>0, q>0$; moreover, we are interested only in positive rationals, so we require that $r>0$ and $p>0$. Hence, we suppose

$$
\begin{equation*}
b>1, c>a . \tag{2}
\end{equation*}
$$

Since $r / s$ and $p / q$ are successive convergents of a s.c.f., and since $m$ is odd, we must have

$$
\begin{equation*}
r q-p s=1 \tag{3}
\end{equation*}
$$

We now note that

$$
\begin{equation*}
s=\frac{b^{n}-1}{b-1}, q=\frac{c^{n}-a^{n}}{c-a} \quad(\text { for some } n>2) . \tag{4}
\end{equation*}
$$

Then $r q-p s=c^{n}-a^{n}-\left(b^{n}-1\right)=1$, which implies

$$
\begin{equation*}
c^{n}=a^{n}+b^{n} \tag{5}
\end{equation*}
$$

Thus, our assumption implies that ( $a, b, c$ ) satisfies Fermat's Last Theorem.
Conversely, suppose that there exists a positive integer triplet ( $a, b, c$ ) which satisfies (5) for some $n>2$, i.e., suppose Fermat's Last Theorem is false. If $b=a$, then $c / a=2^{1 / n}$, which is patently impossible; thus, without loss of generality, we may suppose $b<a$. Let $p, q, r$, and $s$ be as defined in the statement of the problem. We seek to prove that $p, q, r$, and $s$ satisfy (1) for some $m \geq 3$ odd, and some sequence $\theta_{1}, \theta_{2}, \ldots, \theta_{m}$ of natural numbers. We note that $r q-p s=$ $c^{n}-a^{n}-\left(b^{n}-1\right)=\left(c^{n}-a^{n}-b^{n}\right)+1$, which is the statement of (3). Hence,

$$
\begin{equation*}
r / s-p / q=1 / q s \tag{6}
\end{equation*}
$$

According to a well-known theorem of continued fraction theory (e.g., Theorem 184 in [1]), if $r / s-p / q<1 / 2 s^{2}$, then (1) holds. Therefore, in order to establish (1), it suffices to show that $1 / q s<1 / 2 s^{2}$, or

$$
\begin{equation*}
q>2 s . \tag{7}
\end{equation*}
$$

Now we note that

$$
q=\sum_{i=1}^{n} a^{i-1} c^{n-i}>\sum_{i=1}^{n} a^{i-1} a^{n-i}=n a^{n-1} .
$$

Also

$$
\left.s=\sum_{i=1}^{n} b^{n-i}<\sum_{i=1}^{n} a^{n-i}=\frac{a^{n}-1}{a-1}<\frac{a^{n}}{a-1} \quad \text { (using the assumption that } b<a\right) .
$$

Thus,

$$
\begin{equation*}
q / s>\frac{n(a-1)}{a} . \tag{8}
\end{equation*}
$$

Since $a>b>1$, thus $a \geq 3$. Also, $n \geq 3$. Thus, $q / s>3 \cdot 2 / 3$ or

$$
\begin{equation*}
q / s>2 . \tag{9}
\end{equation*}
$$

This establishes (1).
Thus, the negative of Fermat's Last Theorem is equivalent to the negative of the statement of the problem. It follows that Fermat's Last Theorem is equivalent to the statement of the problem.

## Reference

1. G. H. Hardy \& E. M. Wright. An Introduction to the Theory of Numbers. 4th ed. Oxford: The Clarendon Press, 1960.

## Also solved by the proposers.



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# BOOKS AVAILABLE through the fibonacci association 

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.
A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

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Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger, FA, 1993.

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