The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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VOLUME 32 JUNE-JULY 1994 NUMBER 3

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# The Fibonacci Quarterly

*Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)*
*and Br. Alfred Brousseau (1907-1988)*

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION*
*DEVOTED TO THE STUDY*
*OF INTEGERS WITH SPECIAL PROPERTIES*

# ON THE DIVISIBILITY BY 2 OF THE STIRLING NUMBERS OF THE SECOND KIND

**T. Lengyel**
Occidental College, 1600 Campus Road., Los Angeles, CA 90041
(*Submitted August 1992*)

## 1. INTRODUCTION

In this paper we characterize the divisibility by 2 of the Stirling numbers of the second kind, $S(n, k)$, where $n$ is a sufficiently high power of 2. Let $v_2(r)$ denote the highest power of 2 that divides $r$. We show that there exists a function $L(k)$ such that, for all $n \geq L(k)$, $v_2(k!S(2^n, k)) = k-1$ hold, independently from $n$. (The independence follows from the periodicity of the Stirling numbers *modulo* any prime power.) For $k \geq 5$, the function $L(k)$ can be chosen so that $L(k) \leq k-2$. We determine $v_2(k!S(2^n+u, k))$ for $k > u \geq 1$, in particular for $u = 1, 2, 3$, and 4. We show how to calculate it for negative values, in particular for $u = -1$. The characterization is generalized for $v_2(k!S(c \cdot 2^n + u, k))$, where $c > 0$ denotes an arbitrary odd integer.

## 2. PRELIMINARIES

The Stirling number of the second kind $S(n, k)$ is the number of partitions of $n$ distinct elements into $k$ nonempty subsets. The classical divisibility properties of the Stirling numbers are usually proved by combinatorial and number theoretical arguments. Here, we combine these approaches. Inductive proofs [1] and the generating function method [10] and [7] can also be used to prove congruences among combinatorial numbers. We note that Clarke [2] used an application of $p$-adic integers to obtain results on the divisibility of Stirling numbers.

We define the integer-valued *order* function, $v_a(r)$, for all positive integers $r$ and $a > 1$ by $v_a(r) = q$, where $a^q | r$ and $a^{q+1} \nmid r$, i.e., $v_a(r)$ denotes the highest power of $a$ that divides $r$. In this paper we are interested in characterizing $v_a(r)$, where $r = k!S(n, k)$ and $a = 2$. In a future paper, we plan to give a lower bound on $v_a(k!S(n, k))$ for $a \geq 3$.

Lundell [10] discussed the divisibility by powers of a prime of the greatest common divisor of the set $\{k!S(n, k), m \leq k \leq n\}$ for $1 \leq m \leq n$. Other divisibility properties have been found by Nijenhuis & Wilf [11], and recently these results have been improved by Howard [5]. Davis [3] gives a method to determine the highest power of 2 that divides $S(n, 5)$, i.e., $v_2(S(n, 5))$. A similar method can be applied for $S(n, 6)$ according to Davis.

We will use the well-known recurrence relation for $S(n, k)$, which can be proved by the inclusion-exclusion principle

$$k!S(n, k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n. \tag{1}$$

For each prime number $p$ and $1 \leq i \leq p-1$, $i^p \equiv i \pmod p$, by Fermat's theorem, and this implies [1] that, for $2 \leq k \leq p-1$, $S(p, k) \equiv 0 \pmod p$. We note that $S(p, 1) = S(p, p) = 1$.

Let $d(k)$ be the sum of the digits in the binary representation of $k$. Using a lemma by Legendre [9], we get $v_2(k!) = k - d(k)$.

Note that, for $1 \le k \le 4$, identity (1) implies that $\nu_2(S(2^n, k)) = d(k) - 1$. By other identities for Stirling numbers (cf. Comtet [1], p. 227), $\nu_2(S(2^n, k)) = d(k) - 1$ for $k$, $2^n - 3 \le k \le 2^n$.

Classical combinatorial quantities (factorials, Bell numbers, Fibonacci numbers, etc.) often form sequences that eventually become *periodic* modulo any integer, as pointed out by I. Gessel. The "vertical" sequence of the Stirling numbers of the second kind, $\{S(n, k) \pmod{p^N}\}_{n\ge 0}$ is periodic, i.e., there exist $n_0 \ge k$ and $\pi \ge 1$ such that $S(n+\pi, k) \equiv S(n, k) \pmod{p^N}$ for $n \ge n_0$.

For $N = 1$, the minimum period was given by Nijenhuis & Wilf [11], and this result was extended for $N > 1$ by Kwong ([7], Theorems 3.5-3.6). From now on, $\pi(k; p^N)$ denotes the minimum period of the sequence of Stirling numbers $\{S(n, k)\}_{n\ge k}$ modulo $p^N$, and $n_0(k, p^N) \ge k$ stands for the smallest number of nonrepeating terms. Clearly, $n_0(k, p^N) \le n_0(k, p^{N+1})$. Kwong proved

***Theorem A (Kwong [7]):*** For $k > \max\{4, p\}$, $\pi(k; p^N) = (p-1)p^{N+b(k)-2}$, where $p^{b(k)-1} < k \le p^{b(k)}$, i.e., $b(k) = \lceil \log_p k \rceil$.

From now on, we assume that $p = 2$, $n \ge 1$ and apply Theorem A for this case. Let $g(k) = d(k) + b(k) - 2$ and $c$ denote an odd integer. Identity (1) implies $\nu_2(S(c\cdot 2^n, k)) = d(k) - 1$ for $1 \le k \le \min\{4, c\cdot 2^n\}$ We also set $f(k) = f_c(k) = \max\{g(k), \lceil \log_2(n_0(k, 2^{d(k)})/c)\rceil\}$. Therefore, $c\cdot 2^{f(k)} \ge n_0(k, 2^{d(k)})$. We note that $g(k) \le 2\lceil \log_2 k\rceil - 2$. Lemma 3 in [8] yields $f(2^m) = m$ for $m \ge 1$ and $c = 1$.

In this paper we prove

***Theorem 1:*** For all positive integers $k$ and $n$ such that $n \ge f(k)$, we have $\nu_2(k!S(c\cdot 2^n, k)) = k - 1$ or, equivalently, $\nu_2(S(c\cdot 2^n, k)) = d(k) - 1$.

Numerical evidence suggests that the range might be extended for all $n$ provided $2^n \ge k$ and $c = 1$. For example, for $k = 7$, we get $g(7) = d(7) + b(7) - 2 = 4$ and $n_0(7, 2^3) = 7$; therefore, by Theorem 1, if $n \ge f(7) = 4$, then $\nu_2(S(2^n, 7)) = \nu_2(S(c\cdot 2^n, 7)) = 2$ for arbitrary positive integer $c$. Notice, however, that $\nu_2(S(8, 7)) = 2$ also. We make the following

***Conjecture:*** For all $k$ and $1 \le k \le 2^n$, we have $\nu_2(S(2^n, k)) = d(k) - 1$.

By Theorem 1, the Conjecture is true for all $k = 2^m$ with $m \le n$.

In section 3 we prove Theorem 2, which gives the exact order of $S(n, k)$ in a particular range for $k$ whose size depends on $\nu_2(n)$. Theorem 2 is the key tool in proving Theorem 1. Its proof makes use of the periodicity of the Stirling numbers. It would be interesting to determine the function $L(k)$, which is defined as the smallest integer $n'$ such that $\nu_2(S(c\cdot 2^n, k)) = d(k) - 1$ for all $n \ge n'$. By Theorem 2, we find that $L(k) \le k - 2$ and Theorem 1 improves the upper bound on $L(k)$ if $f(k) < k - 2$.

In section 4 we obtain some consequences of Theorem 2 by extending it for Stirling numbers of the form $S(c\cdot 2^n + u, k)$, where $u = 1, 2$, etc. We show how to calculate $\nu_2(S(c\cdot 2^n - 1, k))$. In neither case does the order of $S(c\cdot 2^n + u, k)$ depend on $n$ (if $n$ is sufficiently large), in agreement with Theorem A.

## 3. TOOLS AND PROOFS

We choose an integer $\ell$ such that $\ell \leq n$. We shall generalize identity (1) for any modulus of the form $2^\ell$. Observe that, for any $i$ even, $i^n \equiv 0 \pmod{2^\ell}$, and for all $i$ odd, $(-1)^{k-i}$ will have the same sign as $(-1)^{k-1}$. Therefore, by identity (1),

$$k!S(n,k) \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} i^n \pmod{2^\ell}. \tag{2}$$

The expression on the right-hand side of congruence (2) is called the *partial Stirling number* [10]. We explore identity (2) with different choices of $n$ in order to find $\nu_2(S(n,k))$.

We shall need the following

***Theorem 2:*** Let $c$ be an odd integer and let $n$ be a nonnegative integer. If $1 \leq k \leq n+2$, then $\nu_2(k!S(c\cdot 2^n, k)) = k-1$, i.e., $\nu_2(S(c\cdot 2^n, k)) = d(k)-1$.

Roughly speaking, Theorem 2 gives the exact value of $\nu_2(k!S(m,k))$, for $k \geq 2$, if $m$ is divisible by $2^{k-2}$. The higher the power of 2 that divides $m$, the larger the value of $k$ that can be used. We prove Theorem 1 and then return to the proof of Theorem 2.

***Proof of Theorem 1:*** Without loss of generality, we assume that $k > 4$. Observe that $\nu_2(S(c\cdot 2^n, k)) = d(k)-1$ is equivalent to

$$S(c\cdot 2^n, k) \equiv 0 \pmod{2^{d(k)-1}} \tag{3}$$

and

$$S(c\cdot 2^n, k) \not\equiv 0 \pmod{2^{d(k)}}. \tag{4}$$

The proof of identities (3) and (4) is by contradiction. To prove the former identity, we set $N = d(k)-1$, hence Theorem A yields

$$\pi(k; 2^N) = 2^{d(k)+b(k)-3} \tag{5}$$

where $d(k)+b(k)-3 < g(k) \leq f(k)$.

We assume, to the contrary of the claim, that $S(c\cdot 2^{f(k)}, k) \equiv a \not\equiv 0 \pmod{2^N}$. By Theorem A and the period given by (5), we obtain that, for every positive integer $m \geq c$, $S(m\cdot 2^{f(k)}, k) \equiv a \not\equiv 0 \pmod{2^N}$. This is a contradiction, for one can select $m$ so that $m\cdot 2^{f(k)}$ becomes $c\cdot 2^n$, with a large exponent $n$, and by Theorem 2, $S(c\cdot 2^n, k) \equiv 0 \pmod{2^N}$ should be for sufficiently large $n$. It follows that, in fact, $S(c\cdot 2^{f(k)}, k) \equiv 0 \pmod{2^N}$, and Theorem A implies $S(c\cdot 2^n, k) \equiv 0 \pmod{2^{d(k)-1}}$ for all $n \geq f(k)$.

To derive identity (4), we set $N = d(k)$. In order to obtain a contradiction, we assume that $S(c\cdot 2^{f(k)}, k) \equiv 0 \pmod{2^N}$. Now, by Theorem A, we get $\pi(k; 2^N) = 2^{d(k)+b(k)-2}$, where $d(k)+b(k)-2 = g(k) \leq f(k)$. We proceed in a manner similar to that used above by noting that the periodicity now yields $S(m\cdot 2^{f(k)}, k) \equiv 0 \pmod{2^N}$ for every positive integer $m \geq c$. It would imply that, for a sufficiently large $n$, $S(c\cdot 2^n, k) \equiv 0 \pmod{2^{d(k)}}$. However, this congruence contradicts Theorem 2. It follows that $S(c\cdot 2^n, k) \not\equiv 0 \pmod{2^{d(k)}}$ for $n \geq f(k)$, and the proof is now complete. □

***Proof of Theorem 2:*** We set $m = c \cdot 2^n$ and select an $\ell$ such that $1 \le \ell \le n+1$. By Euler's theorem, $\phi(2^\ell) = 2^{\ell-1}$; therefore, $i^m \equiv 1 \pmod{2^\ell}$ if $i$ is odd. By simple summation, identity (2) yields

$$k!\,S(m,k) \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} = (-2)^{k-1} \pmod{2^\ell}; \tag{6}$$

therefore, $v_2(k!\,S(m,k)) = k-1$, provided $0 \le k-1 < \ell$.

We have two cases if $k = n+2$. If $m$ is odd, then $n = 0$ and $k = 2$. The claim is true, since $S(m,2) = 2^{m-1} - 1$; therefore, $v_2(2!\,S(m,2)) = 1$. If $m$ is even, then we set $\ell = n+2 \ge 3$. By induction on $\ell \ge 3$, we can derive $i^{2^{\ell-2}} \equiv 1 \pmod{2^\ell}$ and identity (6) is verified again. □

***Remark:*** By setting $\ell = n+1$, identity (6) implies the lower bound $v_2(k!\,S(c \cdot 2^n, k)) \ge n+1$, for $k \ge n+2$.

## 4. RELATED RESULTS

We will use other special cases of identity (2). Similarly to the previous proof, we get that, for all $u \ge 0$, $n \ge \ell \ge 1$, and $k \le c \cdot 2^n + u$,

$$k!\,S(c \cdot 2^n + u, k) \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} i^{c \cdot 2^n + u} \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} i^u \pmod{2^{\ell+2}}. \tag{7}$$

We set

$$h(k,u) = (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} i^u.$$

By identity $x^u = \sum_{j=0}^{u} S(u,j) \binom{x}{j} j!$, we obtain

$$h(k,u) = (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} \sum_{j=0}^{u} S(u,j) \binom{i}{j} j! = (-1)^{k-1} \sum_{j=1}^{\min\{u,k\}} S(u,j) j! \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} \binom{i}{j}.$$

We focus on the case in which $k > u$ and derive

$$h(k,u) = (-1)^{k-1} \sum_{j=0}^{u} S(u,j) j! \binom{k}{j} \sum_{\substack{i=j \\ i \text{ odd}}}^{k} \binom{k-j}{i-j} = (-2)^{k-1} \sum_{j=0}^{u} \frac{S(u,j) j!}{2^j} \binom{k}{j}. \tag{8}$$

We introduce the notation $r(k,u) = v_2(h(k,u))$. Identity (8) implies that $r(k,u) \ge k-u-1$. Note that $|h(k,0)| = 2^{k-1}$ and, for $u \ge 1$,

$$|h(k,u)|/2^{k-u-1} \le \sum_{j=1}^{u} j^u 2^{u-j} k^j \le u(2u)^u (k/2)^u = u(uk)^u. \tag{9}$$

By identity (7), for $u \ge 0$ and any sufficiently large $\ell$ and $n \ge \ell$, we have $v_2(k!\,S(c \cdot 2^n + u, k)) = r(k,u)$. In fact, $n \ge \ell = r(k,u) - 1$ will suffice; for instance, $n \ge k-2$ will be large enough if $u = 0$ (Theorem 2). By identity (9), we derive that $r(k,u) \le k - u - 1 + u \log_2 k + (u+1)\log_2 u$; therefore, $k - u - 2 + \lceil u \log_2 k + (u+1)\log_2 u \rceil$ can be chosen for $n$ if $u > 0$. We note that, similarly to the proof of Theorem 1, this value might be decreased.

The values of $r(k,u)$ can be calculated by identity (8). For example, if $k>u\geq 0$, then

$$r(k,u)=\begin{cases} k-1, & \text{if } u=0,\\ k-2+v_2(k), & \text{if } u=1,\\ k-3+v_2(k)+v_2(k+1), & \text{if } u=2,\\ k-4+2v_2(k)+v_2(k+3), & \text{if } u=3,\\ k-5+v_2(k)+v_2(k+1)+v_2(k^2+5k-2), & \text{if } u=4. \end{cases} \tag{10}$$

We state two special cases that can be proved basically differently; although, in the second case, only a partial proof comes out by the applied recurrence relations.

***Theorem 3:*** For $k\geq 2$ and any sufficiently large $n$, $v_2(k!S(c\cdot 2^n+1,k))=k-2+v_2(k)$.

***Proof:*** The proof follows from Theorem 2 and using the recurrence relation $k!S(m,k)=k\{(k-1)!S(m-1,k-1)+k!S(m-1,k)\}$ with $m=c\cdot 2^n+1$. Notice that, by Theorem 1, $n\geq\max\{f(k),f(k-1)\}$ will be sufficiently large. □

***Theorem 4:*** For $k\geq 3$ and sufficiently large $n$, $v_2(k!S(c\cdot 2^n+2,k))=k-3+v_2(k)+v_2(k+1)$.

***Proof:*** By identity (10), we obtain $v_2(k!S(c\cdot 2^n+2,k))=r(k,2)=k-3+v_2(k)+v_2(k+1)$. Observe that $n\geq\max\{f(k),f(k-1),f(k-2)\}$ suffices. □

Notice that we could have used the expansion

$$k!S(c\cdot 2^n+2,k)=k\{(k-1)!S(c\cdot 2^n+1,k-1)+k!S(c\cdot 2^n+1,k)\}.$$

By Theorem 3, the first term of the second factor is divisible by a power of 2 with exponent $k-3+v_2(k-1)$, while the second term is divisible by 2 at exponent $k-2+v_2(k)$. The first factor contributes an additional exponent of $v_2(k)$ to the power of 2. We combine the two terms and find that there is always a unique term with the lowest exponent of 2 if $k\not\equiv 3 \pmod 4$. For $k\equiv 3 \pmod 4$, however, this argument falls short and we are able to obtain only the lower bound $k-1$ on $v_2(k!S(c\cdot 2^n+2,k))$.

It turns out that calculating $v_2(k!S(c\cdot 2^n+u,k))$ for negative integers $u$ is more difficult than for positive values. The periodicity guarantees that the order does not depend on $n$ (for sufficiently large $n$).

We extend the function $h(k,u)$ for negative integers $u$. We will choose an appropriate value $\ell\geq 1$ and then set $n$ so that it satisfies the inequality $c\cdot 2^n+u\geq 2^\ell$. We use the convenient notation $1/i$ for the unique integer solution $x$ of the congruence $i\cdot x\equiv 1 \pmod{2^{\ell+2}}$ if $i$ is odd. Similarly to identity (7), we obtain

$$k!S(c\cdot 2^n+u,k)\equiv(-1)^{k-1}\sum_{\substack{i=1\\ i\text{ odd}}}^{k}\binom{k}{i}\left(\frac{1}{i}\right)^{-u} \pmod{2^{\ell+2}}. \tag{11}$$

For $u<0$, we set

$$h(k,u)=(-1)^{k-1}\sum_{\substack{i=1\\ i\text{ odd}}}^{k}\binom{k}{i}\left(\frac{1}{i}\right)^{-u}$$

and express $h(k,u)$ as a fraction $\frac{p_k(u)}{q_k(u)}$ in lowest terms. Notice that $v_2(p_k(u)) \geq k - d(k)$ holds, since $k!$ divides both sides of (11) for any sufficiently large $\ell$. The order of $v_2(S(c \cdot 2^n + u, k))$ can be determined by choosing $\ell \geq v_2(p_k(u)) - 1$, and the *actual order* is $v_2(p_k(u)) - k + d(k)$. We remark that, for $c = 1$, the value of $n$ can be set to $v_2(p_k(u))$.

We focus on the case of $u = -1$. Let

$$a_k = \sum_{i=1}^{k} \binom{k}{i} \frac{1}{i}.$$

We get

$$a_s - a_{s-1} - \binom{s}{s}\frac{1}{s} = \sum_{i=1}^{s-1} \frac{1}{i}\left\{\binom{s}{i} - \binom{s-1}{i}\right\} = \sum_{i=1}^{s-1} \frac{1}{s}\binom{s}{i} = \frac{2^s - 2}{s} \quad (s \geq 2).$$

By summation, it follows that

$$a_k = \sum_{i=1}^{k} \frac{2^i}{i} - \sum_{i=1}^{k} \frac{1}{i}.$$

Similarly,

$$b_k = \sum_{i=1}^{k} \binom{k}{i} \frac{(-1)^{i+1}}{i} = \sum_{i=1}^{k} \frac{1}{i}$$

(cf. Hietala & Winter [4], or Solution to Problem E3052 in *Amer. Math. Monthly* **94.2** (1987): 185). Combining these two identities, we obtain

$$h(k,-1) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} \frac{1}{i} = \frac{1}{2} \sum_{i=1}^{k} \frac{2^i}{i} = \frac{p_k(-1)}{q_k(-1)}. \tag{12}$$

For example, for $k = 5$, we get $h(5,-1) = \frac{128}{15}$, $v_2(p_5(-1)) = 7$, and $n \geq 7$. E.g., $v_2(S(127,5)) = v_2(S(255,5)) = \cdots = 4$. We remark that $v_2(S(63,5)) = 4$ holds, too. Notice that the recurrence relation $S(N,K) = K \cdot S(N-1,K) + S(N-1,K-1)$ implies that $v_2(S(c \cdot 2^n - 1, 2^m - 1) = 0$ for every sufficiently large $n$. By the theory of $p$-adic numbers [6] and (12), we can derive that, for all sufficiently large $n$,

$$v_2(S(c \cdot 2^n - 1, k)) = v_2\left(\frac{1}{2}\sum_{i=1}^{k} \frac{2^i}{i}\right) - k + d(k) = v_2\left(\frac{1}{2}\sum_{i=k+1}^{\infty} \frac{2^i}{i}\right) - k + d(k),$$

where $v_2(a/b)$ is defined as $v_2(a) - v_2(b)$ if $a$ and $b$ are integers. This fact helps us to make observations for some special cases. For instance, if $n > m \geq 3$, then $v_2(S(c \cdot 2^n - 1, 2^m)) \geq 2$ holds and, therefore, $v_2(S(c \cdot 2^n - 1, 2^m + 1)) = 1$. Numerical evidence suggests that, for $n > m \geq 4$, $v_2(S(c \cdot 2^n - 1, 2^m)) = 2m - 2$, although we were unable to prove this.

We can determine $v_2(S(c \cdot 2^n - 1, k))$ for most of the odd values of $k$ by systematically evaluating $v_2(\sum_{i=1}^{k} \frac{2^i}{i})$, and obtain

***Theorem 5:*** For all sufficiently large $n$, $v_2(S(c \cdot 2^n - 1, k)) = d(k) - v_2(k+1)$, if $k \geq 1$ is odd and $k \not\equiv 5 \pmod 8$ and $k \not\equiv 59 \pmod{64}$ and $k \not\equiv 121 \pmod{128}$.

We leave the details of the proof to the reader.

We note that there is an alternative way of determining $p_k(-1)$. We set

$$I_{k-1} = \frac{k}{2^{k-1}} \frac{1}{2} \sum_{i=1}^{k} \frac{2^i}{i}.$$

One can prove that

$$I_k = \sum_{j=0}^{k} \frac{1}{\binom{k}{j}} \quad \text{and} \quad I_k = \frac{k+1}{2k} I_{k-1} + 1.$$

For other properties of $I_k$, see Comtet ([1], p. 294, Exercise 15). The latter recurrence relation simplifies the calculation of $\nu_2(S(c \cdot 2^n - 1, k))$ for large values of $k$.

We can use (7) in a slightly different way and gain information on the structure of the sequence $\{S(c \cdot 2^n + k, k), S(c \cdot 2^n + k + 1, k), \cdots, S((c+1) \cdot 2^n + k - 1, k) \pmod{2^q}\}$ for every $q$, $1 \le q \le d(k) - 1$ and sufficiently large $n$. We observe that the sequence always starts with a one and ends with at least $d(k) - q$ zeros. Note that, for every $\ell$ and $u$ such that $k > u \ge \ell > k - d(k)$,

$$0 = k! S(u, k) \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} \binom{k}{i} i^u \pmod{2^\ell}.$$

We set $q = \ell - k + d(k)$. Clearly, $1 \le q \le d(k) - 1$. By (7), we get that $k! S(c \cdot 2^n + u, k) \equiv 0$ (mod $2^t$) for all $n \ge \ell - 2 \ge 1$. This observation yields that the $d(k) - q$ consecutive terms,

$$S(c \cdot 2^n + u, k) \pmod{2^q}, \quad u = k - d(k) + q, k - d(k) + q + 1, \ldots, k - 1, \tag{13}$$

are all zeros. Similarly, we can derive $k! S(c \cdot 2^n + k, k) \equiv k! \not\equiv 0$ (mod $2^t$), i.e., $S(c \cdot 2^n + k, k) \equiv 1$ (mod $2^q$). Identities (8) and (10) imply that there might be many more zeros in the sequence at and after the term $S(c \cdot 2^n, k)$ (mod $2^q$).

For example, if $k = 7$ and $\ell = 5$, then $S(c \cdot 2^n + u, 7) \equiv 0$ (mod $2^1$) for $u = 5$ and 6 and for all $n \ge 3$. Similarly to the proof of Theorem 1, it follows that identity (13) holds if $n \ge f(k)$. For instance, if $k = 23$ and $\ell = 21$, then $S(c \cdot 2^n + u, 23) \equiv 0$ (mod $2^2$) for $u = 21$ and 22 provided $n \ge f(23) = 7$.

## ACKNOWLEDGMENT

The author would like to thank Professor Y. H. Harris Kwong for his helpful comments.

## REFERENCES

1. L. Comtet. *Advanced Combinatorics*. Dordrecht: Reidel, 1974.
2. F. Clarke. "Hensel's Lemma and the Divisibility by Primes of Stirling Numbers of the Second Kind." Manuscript, 1991.
3. D. M. Davis. "Divisibility by 2 of Stirling-Like Numbers." In *Proc. Amer. Math. Soc.* **110** (1990):597-600.
4. H. J. Hietala & B. B. Winter. "Note on a Combinatorial Identity." *Math. Magazine* **38** (1965):149-51.
5. F. T. Howard. "Congruences for the Stirling Numbers and Associated Stirling Numbers." *Acta Arithmetica* **55** (1990):29-41.
6. N. Koblitz. *p-adic Numbers, p-adic Analysis, and Zeta-Functions.* 2nd ed. New York: Springer-Verlag, 1984.

7. Y. H. Harris Kwong. "Minimum Periods of $S(n, k)$ Modulo $M$." *The Fibonacci Quarterly* **27.3** (1989):217-21.
8. Y. H. Harris Kwong. "Periodicities of a Class of Infinite Integer Sequences Modulo $M$." *J. Number Theory* **31** (1989):64-79.
9. A. M. Legendre. *Théorie des Nombres.* Vol. I. 3rd ed. Paris, 1830.
10. A. Lundell. "A Divisibility Property for Stirling Numbers." *J. Number Theory* **10** (1978):35-54.
11. A. Nijenhuis & H. S. Wilf. "Periodicities of Partition Functions and Stirling Numbers Modulo $p$." *J. Number Theory* **25** (1987):308-12.

AMS Classification Numbers: 11B73, 11B50, 11E95

# UNIQUE MINIMAL REPRESENTATION OF INTEGERS BY NEGATIVELY SUBSCRIPTED PELL NUMBERS


**A. F. Horadam**

The University of New England, Armidale, Australia 2351



(*Submitted September 1992*)


## 1. BACKGROUND

In this paper, we prove the following uniqueness and minimality result for Pell numbers $P_{-i}$ (see [3]):

***Theorem:*** The representation of **any** integer $N$ as

$$N = \sum_{i=1}^{\infty} a_i P_{-i} \tag{1.1}$$

where

$$\begin{cases} a_i = 0, 1, 2 \\ a_i = 2 \Rightarrow a_{i+1} = 0 \end{cases} \tag{1.2}$$

is unique and minimal.

**Pell numbers** $P_n$ are defined in [3] as members of the two-way infinite Pell sequence $\{P_n\}$ satisfying the recurrence

$$P_{n+1} = 2P_n + P_{n-1},\ P_0 = 0,\ P_1 = 1. \tag{1.3}$$

To compute terms of the sequence with positive subscripts, extend $(0, 1, \ldots)$ to the right using (1.3); to compute terms of the sequence with negative subscripts, extend $(\ldots, 0, 1)$ to the left using

$$P_{n-1} = -2P_n + P_{n+1}. \tag{1.4}$$

Induction may be used to establish that

$$P_{-n} = (-1)^{n+1} P_n. \tag{1.5}$$

Associated with $P_n$ are the numbers

$$q_n = P_n + P_{n-1}, \tag{1.6}$$

where $2q_n = Q_n$, the $n^{\text{th}}$ **Pell-Lucas number**.

From (1.3) and (1.6), it easily follows that

$$q_{n+1} = 2q_n + q_{n-1}. \tag{1.7}$$

Some of the smallest $P_n$ and $q_n$ are:

**TABLE 1. Values of $P_n$ and $q_n$**

| $n =$ | … | $-7$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | … |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $P_n =$ | … | 169 | $-70$ | 29 | $-12$ | 5 | $-2$ | 1 | 0 | 1 | 2 | 5 | 12 | 29 | 70 | 169 | … |
| $q_n =$ | … | … | … | … | … | … | … | … | 1 | 1 | 3 | 7 | 17 | 41 | 99 | 239 | … |

While values of $q_n$ can be readily extended through negative values of $n$, for our purposes we need only positive values of $n$. For negative subscripts, $q_{-n} = (-1)^n q_n$.

It is a straightforward exercise to establish the sums [3, Theorem 2]

$$\sum_{i=1}^{n} P_{-2i} = \frac{1 - P_{-(2n+1)}}{2} \tag{1.8}$$

and

$$\sum_{i=1}^{n} P_{-2i+1} = \frac{-P_{-2n}}{2}. \tag{1.9}$$

Our procedure in demonstrating the truth of the Theorem is to adapt and extend the technique used in [2] for positively subscripted Pell numbers.

Two important differences between the criteria (1.2) in our Theorem for $P_{-n}$ $(n > 0)$ and those in [2] for $P_n$ $(n > 0)$ must be noted:

(i) In [2], $\varepsilon_i = 2 \Rightarrow \varepsilon_{i-1} = 0$, while in (1.2), $a_i = 2 \Rightarrow a_{i+1} = 0$.

(ii) In [2], $\varepsilon_1 = 0$; $\varepsilon_i = 0, 1, 2$ $(i > 1)$, while in (1.2), $a_i = 0, 1, 2$ $(i \geq 1)$.

The restriction on $\varepsilon_1$ in (ii) arises from the fact that, for $n$ positive, a distinction has to be made between $P_2 = 2$ and $2P_1 = 2$ (the latter being excluded). No such difficulty occurs for negatively subscripted Pell numbers since $P_{-2} = -2$, $P_{-1} = 1$.

## 2. THE SEQUENCES $(a_1, a_2, \ldots, a_n)$

Let us now concentrate on the sequence of length $n \geq 1$,

$$(a_1, a_2, \ldots, a_n), \tag{2.1}$$

with conditions (1.2) attached. Write $S_n$ for the number of sequences (2.1) with (1.2). $S_0$ is not defined.

Omitting commas and brackets for convenience, we may enumerate several $S_n$ thus:

**TABLE 2. Sequences Counted by $S_n$ $(n = 1, 2, 3, 4)$**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $S_1$ | 0 | 1 | 2 | | | | |
| $S_2$ | 00 | 01 | 02 | 10 | 11 | 12 | 20 |
| $S_3$ | 000 | 001 | 002 | 010 | 011 | 012 | 020 |
| | 100 | 101 | 102 | 110 | 111 | 112 | 120 |
| | 200 | 201 | 202 | | | | |
| $S_4$ | 0000 | 0001 | 0002 | 0010 | 0011 | 0012 | 0020 |
| | 0100 | 0101 | 0102 | 0110 | 0111 | 0112 | 0120 |
| | 1000 | 1001 | 1002 | 1010 | 1011 | 1012 | 1020 |
| | 1100 | 1101 | 1102 | 1110 | 1111 | 1112 | 1120 |
| | 2000 | 2001 | 2002 | 2010 | 2011 | 2012 | 2020 |
| | 0200 | 0201 | 0202 | 1200 | 1201 | 1202 | |

Perusal of this tabulation reveals the methodical extension of the structure of the sequences of $S_n$ to those of $S_{n+1}$.

Some lemmas are needed for the proof of the Theorem.

***Lemma 1:*** $S_n = q_{n+1}$.

***Proof:*** This equality is easily checked in Table 2 for $n = 1, 2, 3, 4$ for which $q_{n+1} = 3, 7, 17, 41$, respectively.

Proceed by induction on $n$. Assume the lemma is true for $n = k > 4$; that is, assume that $S_k = q_{k+1}$ $(k > 4)$. Now, to generate $S_{k+1}$ from $S_k$,

***(i)*** prefix 0 and 1 separately to each of the $q_{k+1}$ sequences, and

***(ii)*** prefix 2 followed by 0, by (1.2), to each of the $q_k$ sequences.

Therefore, $S_{k+1} = 2q_{k+1} + q_k = q_{k+2}$ by (1.7). Thus, the Lemma is also valid for $n = k+1$ and the Lemma is proved.

Observe that $q_{n+1}$ here plays the role for $P_{-n}$ $(n > 0)$ which $P_{n+1}$ plays for $P_n$ $(n > 0)$ in [2].

Consider now

$$N = a_1P_{-1} + a_2P_{-2} + \cdots + a_nP_{-n}, \tag{2.2}$$

where $a_i$ satisfy (1.2), i.e., the integer $N$ is determined by the sequence (2.1).

***Lemma 2:***

***(i)*** $1 - P_{-n} \le N \le -P_{-(n+1)}$ $(n$ odd)

***(ii)*** $1 - P_{-(n+1)} \le N \le -P_{-n}$ $(n$ even).

***Proof:*** Clearly, the maximum integer $N$ generated by $(a_1, \dots, a_n)$ is given by

$$\begin{array}{ll} 20202\dots2 & (n \text{ odd}) \\ 20202\dots20 & (n \text{ even}) \end{array}$$

which are the same, whereas the minimum integer $N$ generated by $(a_1, \dots, a_n)$ is given by

$$\begin{array}{ll} 0202\dots0 & (n \text{ odd}) \\ 0202\dots02 & (n \text{ even}) \end{array}$$

which are different.

Appealing to (1.8) and (1.9), we derive (i) and (ii) immediately.

Notice that Lemma 2 can be recast as

***Lemma 2a:***

***(i)*** $-P_{-n} < N \le -P_{-(n+1)}$ $(n$ odd),

***(ii)*** $-P_{-(n+1)} < N - P_{-n}$ $(n$ even).

Next, we link Lemmas 1 and 2.

***Lemma 3:*** The $q_{n+1}$ integers are

$$\begin{cases} 1 - P_{n+1}, \dots, 0, \dots, P_n & (n \text{ even}) \\ 1 - P_n, \dots, 0, \dots, P_{n+1} & (n \text{ odd}). \end{cases}$$

***Proof:***

$$\begin{aligned} q_{n+1} &= P_{n+1} + P_n \quad \text{by (1.6)} \\ &= (\text{number of integers} \le 0) + (\text{number of integers} > 0), \end{aligned}$$

the order in the addition being determined by the parity of $n$.

Thus, for $n = 7$ (so $q_8 = 577$), the numbers are $-168, \ldots, 408$.

See Table 3 for numerical details for $n = 1, \ldots, 6$. (Cf. the result in [2] corresponding to Lemma 3.)

Calculation yields the following information about $S_n$:

**TABLE 3**

| $S_n$ | Integers Generated by $(a_1, \ldots, a_n)$ |
|---|---|
| $S_1 = q_2 = 3$ | $0, 1, 2$ |
| $S_2 = q_3 = 7$ | $-4, \ldots, -1, 0, 1, 2$ |
| $S_3 = q_4 = 17$ | $-4, \ldots, -1, 0, 1, \ldots, 12$ |
| $S_4 = q_5 = 41$ | $-28, \ldots\ldots, -1, 0, 1, \ldots, 12$ |
| $S_5 = q_6 = 99$ | $-28, \ldots\ldots, -1, 0, 1, \ldots\ldots, 70$ |
| $S_6 = q_7 = 239$ | $-168, \ldots\ldots\ldots, -1, 0, 1, \ldots\ldots, 70$ |

***Lemma 4:*** $n$ is uniquely determined by $N(a_n \ne 0)$.

***Proof:*** This follows from Lemma 2a.

***Lemma 5:*** $a_n$ $(\ne 0)$ is uniquely determined by $N$.

***Proof:*** Consider $N - a_n P_{-n}$, a specific integer in (2.2). The result follows.

***Examples:***

Lemma 2a: (i) $-P_{-7}\,(=-169) < 100 \le -P_{-8}\,(=408)$.
Therefore, $N = 100 \Rightarrow n = 7$ (Lemma 4).

(ii) $-P_{-9}\,(=-985) < -500 \le -P_{-8}\,(=408)$
Therefore, $N = -500 \Rightarrow n = 8$ (Lemma 4).

Lemma 5: Consider $N = P_{-1} + P_{-2} + P_{-4} + 2P_{-5} = 45$.

$$\text{Therefore, } \begin{cases} N - P_{-5} = 16 & \text{i.e., } a_5 = 1, \\ N - 2P_{-5} = -13 & \text{i.e., } a_5 = 2. \end{cases}$$

***Proof of the Theorem:*** Combining Lemmas 1, 2, 3, 4, and 5, we see that the representation (1.1) with (1.2) is unique and minimal.

Minimality occurs since a number given by $(a_1, \ldots, a_n)$ is identical with the numbers given by $(a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots)$ when we adjoin as many zeros as we wish.

The reader is referred to:

(a) [3] for an algorithm that generates minimal representations of integers by Pell numbers with negative subscripts, and

(b) [1] for similar work relating to Fibonacci numbers.

Another approach to the proof of the Theorem is to adapt the methods used in [1] for Fibonacci numbers. Basically, this alternative treatment assumes that there are two permissible representations of $N$ as a sum, and then demonstrates that this assumption leads to contradictions. To conserve space, we do not develop this complicated argument here, though it has some interesting ramifications. Inevitably, there will be some overlap of material in the two approaches.

### Note on Maximality

As indicated in [1] for the Fibonacci case, we likewise assert that there can be **no** maximal representation of an integer by means of $P_{-n}$. This conviction is easy to justify from the obvious fact that

$$\sum_{i=1}^{n} a_i P_{-i} = \sum_{i=1}^{n-1} a_i P_{-i} + a_n P_{-n},$$

where $a_n = 1$ or 2, and then from successive replacements of the last term.

For instance, with $n = 6$, i.e., $a_6 = 1$ or 2, we have (say)

$$\begin{aligned} -59 &= P_{-1} + 2P_{-3} + P_{-6} && (a_6 = 1) \\ &= P_{-1} + 2P_{-3} + \overbrace{2P_{-7} + P_{-8}} \\ &= P_{-1} + 2P_{-3} + 2P_{-7} + \overbrace{2P_{-9} + P_{-10}} && \text{and so on,} \end{aligned}$$

while

$$\begin{aligned} -129 &= P_{-1} + 2P_{-3} + 2P_{-6} && (a_6 = 2) \\ &= P_{-1} + 2P_{-3} + \overbrace{P_{-5} - P_{-7}} \\ &= P_{-1} + 2P_{-3} + P_{-5} - \overbrace{2P_{-8} - P_{-9}} && \text{and so on.} \end{aligned}$$

Clearly, the summations extend as far as we wish, so there is no maximal representation.

## REFERENCES


1. M. W. Bunder. "Zeckendorf Representations Using Negative Fibonacci Numbers." *The Fibonacci Quarterly* **30.2** (1992):111-15.
2. L. Carlitz, R. Scoville, & V. E. Hoggatt, Jr. "Pellian Representations." *The Fibonacci Quarterly* **10.5** (1972):449-88.
3. A. F. Horadam. "Zeckendorf Representations of Positive and Negative Integers by Pell Numbers." In *Applications of Fibonacci Numbers* **5**:305-16. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1993.


AMS Classification Numbers: 11B37, 11A67

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# POWERS OF DIGITAL SUMS

**Tom C. Brown**
Simon Fraser University, Burnaby, B.C., Canada
(*Submitted September 1992*)

## 1. INTRODUCTION

Let $s(n)$ denote the sum of the base 10 digits of the nonnegative integer $n$, and let $\log x$ denote the base 10 logarithm of $x$. R. E. Kennedy and C. Cooper have shown [1] that for any positive integer $k$,

$$\frac{1}{x}\sum_{n\le k} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O(\log^{k-\frac{1}{3}} x),$$

and they conjectured that for any positive integer $k$,

$$\frac{1}{x}\sum_{n\le k} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O(\log^{k-1} x).$$

Recently [2] the same authors have shown, providing some evidence for the truth of this conjecture, that for each fixed positive integer $k$,

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^k = \left(\frac{9}{2}\right)^k n^k + O(n^{k-1}).$$

In this note, we extend the result just mentioned. When $k$ is a fixed positive integer, we show that for each $m$ it is true that

$$\frac{1}{x}\sum_{n\le k} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O(\log^{k-1} x),$$

provided that $x$ is restricted to the set of those positive integers having at most $m$ nonzero digits in their base 10 representations. (Thus, the Kennedy & Cooper result is exactly the case $m = 1$.) We use the Kennedy & Cooper result in the course of our proof.

We state our result in the following form.

***Proposition:*** Let $m \ge 1$ and $k \ge 1$ be fixed integers. Then there is a constant $A = A(k, m)$ such that if $x$ is a positive integer with at most $m$ nonzero digits in its base 10 representation,

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k (\log x)^k\right| < A(\log x)^{k-1}.$$

## 2. REMARKS AND LEMMAS

***Remark 1:*** It is easy to check that if $m$, $k$ are fixed positive integers and

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k [\log x]^k\right| < c[\log x]^{k-1},$$

then

$$\left|\frac{1}{x}\sum_{i=0}^{x-1}s(i)^k-\left(\frac{9}{2}\right)^k(\log x)^k\right|<d(\log x)^{k-1},$$

where [ ] denotes the greatest integer function and $d$ is a constant that depends only on $c$ and $k$.

To see this, suppose $10^n \le x \le 10^{n+1}$, so that $n=[\log x]$. Let $\log x = n+\alpha$, where $0\le\alpha<1$. Then

$$\left|\frac{1}{x}\sum_{i=0}^{x-1}s(i)^k-\left(\frac{9}{2}\right)^k(n+\alpha)^k\right|<\left|\frac{1}{x}\sum_{i=0}^{x-1}s(i)^k-\left(\frac{9}{2}\right)^k n^k\right|+\left|\left(\frac{9}{2}\right)^k n^k-\left(\frac{9}{2}\right)^k(n+\alpha)^k\right|$$

$$=\left|\frac{1}{x}\sum_{i=0}^{x-1}s(i)^k-\left(\frac{9}{2}\right)^k(n+\alpha)^k\right|+\left(\frac{9}{2}\right)^k\left\{k\alpha n^{k-1}+\binom{k}{2}\alpha^2n^{k-2}+\cdots+\binom{k}{k}\alpha^k\right\}$$

$$<cn^{k-1}+c'n^{k-1}=dn^{k-1}\le d(n+\alpha)^{k-1}.$$

***Remark 2:*** In view of Remark 1, to prove the Proposition above, it is sufficient (and convenient) to prove the following statement, which will be done by induction on $m$.

For fixed positive integers $m$, $k$, there is an $A=A(k,m)$ such that if $10^n\le x<10^{n+1}$ and $x$ has at most $m$ nonzero digits in its base 10 representation, then

$$\left|\frac{1}{x}\sum_{i=0}^{x-1}s(i)^k-\left(\frac{9}{2}\right)^k n^k\right|<An^{k-1}.$$

In the following three lemmas, $k$, $n$, $p$, $y$, and $t$ all denote integers.

***Lemma 1:*** For each $k\ge 1$, there is an $A(k)$ such that

$$\frac{n\cdot 10^{p+1}}{10^n+10^{p+1}}\left(1-\left(\frac{p}{n}\right)^k\right)<A(k)\quad\text{for all } n, p \text{ with } 1\le p\le n.$$

***Proof:*** Let $s=n-p-1$. Then $-1\le s\le n-2$ and

$$\frac{n\cdot 10^{p+1}}{10^n+10^{p+1}}\left(1-\left(\frac{p}{n}\right)^k\right)=\frac{n}{10^s+1}\left(1-\left(1-\frac{s+1}{n}\right)^k\right)$$

$$=\frac{n}{10^s+1}\left(k\cdot\frac{s+1}{n}-\binom{k}{2}\frac{(s+1)^2}{n^2}+\binom{k}{3}\frac{(s+1)^3}{n^3}-\cdots-(-1)^k\binom{k}{k}\frac{(s+1)^k}{n^k}\right)$$

$$=\frac{k\cdot(s+1)}{10^s+1}+o(1)<A(k).$$

Note that for $i\ge 2$,

$$\frac{n}{10^s+1}\cdot\frac{(s+1)^i}{n^i}\le\left(\max_{-1\le s<\infty}\frac{(s+1)^i}{10^s+1}\right)\cdot\frac{1}{n^{i-1}}\to 0 \text{ as } n\to\infty.$$

***Lemma 2:*** Fix $k \geq 1$. Let $A(k)$ be as in Lemma 1. Then for any $y$ in the interval $10^p \leq y < 10^{p+1} \leq 10^n$ and any $t \geq 1$,

$$\left| \frac{t \cdot 10^n}{t \cdot 10^n + y} \left(\frac{9}{2}\right)^k n^k + \frac{y}{t \cdot 10^n + y} \left(\frac{9}{2}\right)^k p^k - \left(\frac{9}{2}\right)^k n^k \right| < \left(\frac{9}{2}\right)^k A(k) n^{k-1}.$$

***Proof:***

$$\left| \frac{t \cdot 10^n}{t \cdot 10^n + y} n^k + \frac{y}{t \cdot 10^n + y} p^k - n^k \right| = n^k \frac{y}{t \cdot 10^n + y} \left(1 - \left(\frac{p}{n}\right)^k\right)$$

$$< n^k \frac{10^{p+1}}{t \cdot 10^n + 10^{p+1}} \cdot \left(1 - \left(\frac{p}{n}\right)^k\right) \leq n^{k-1} \cdot \frac{n \cdot 10^{p+1}}{10^n + 10^{p+1}} \left(1 - \left(\frac{p}{n}\right)^k\right) < A(k) \cdot n^{k-1},$$

where the last inequality is given by Lemma 1.

***Lemma 3:*** Let $t \geq 1$ and $k \geq 1$ be fixed. Then

$$\frac{1}{t \cdot 10^n} \sum_{i=0}^{t \cdot 10^n - 1} s(i)^k = \left(\frac{9}{2}\right)^k n^k + O(n^{k-1}).$$

In other words, there is $B(t, k)$ with

$$\left| \frac{1}{t \cdot 10^n} \sum_{i=0}^{t \cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k n^k \right| < B(t, k) n^{k-1}.$$

To prove the Proposition, we will only need this lemma for $t = 1, 2, \ldots, 9$.

***Proof:*** We use induction on $t$. For $t = 1$, this is the result of Kennedy & Cooper mentioned above. Now fix $t \geq 1$ and assume the result for this $t$. Then, using the fact that $s(t \cdot 10^n + i) = s(t) + s(i)$, $0 \leq i \leq 10^n - 1$,

$$\left| \frac{1}{(t+1) \cdot 10^n} \sum_{i=0}^{(t+1) \cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k \right| \leq \left| \frac{t}{t+1} \left( \frac{1}{t \cdot 10^n} \sum_{i=0}^{t \cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k \right) \right|$$

$$+ \left| \frac{1}{t+1} \left( \frac{1}{10^n} \sum_{i=0}^{10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k \right) \right| + \frac{1}{t+1} \cdot \frac{1}{10^n} \sum_{i=0}^{10^n - 1} \left(c_1 s(i)^{k-1} + c_2 s(i)^{k-2} + \cdots + c_k\right)$$

$$< \frac{t}{t+1} B(t, k) n^{k-1} + \frac{1}{t+1} B(1, k) n^{k-1} + C n^{k-1} < B(t+1, k) n^{k-1}.$$

Note that we used the result of Kennedy & Cooper a second time. Here $c_1, \ldots, c_k$, $C$ are constants that depend only on $k$ and $t$.

## 3. PROOF OF THE PROPOSITION

According to Remark 2, we need to show that, for each $m \geq 1$ and $k \geq 1$, there is a constant $A(k, m)$ such that if $10^n \leq x < 10^{n+1}$ and $x$ has at most $m$ nonzero digits in its base 10 representation, then

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k n^k\right| < A(k,m)n^{k-1}.$$

If $m = 1$, this follows from Lemma 3 (for all $k$), since then $x = t\cdot 10^n$, $1\le t\le 9$.

Now assume the result for a given $m\ge 1$, and let $x$ have $m+1$ nonzero digits, say,

$$10^n \le x < 10^{n+1},\ x = t\cdot 10^n + y,\ 1\le t\le 9,\ 10^p \le y < 10^{p+1} \le 10^n,$$

where $y$ has $m$ nonzero digits. Then, using $s(t\cdot 10^n + i) = t + s(i)$,

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k\right| \le \left|\frac{t\cdot 10^n}{t\cdot 10^n + y}\left(\frac{1}{t\cdot 10^n}\cdot\sum_{i=0}^{t\cdot 10^n-1} s(i)^k - \left(\frac{9}{2}\right)^k\right)\right|$$

$$+\left|\frac{y}{t\cdot 10^n + y}\left(\frac{1}{y}\sum_{i=0}^{y-1} s(i)^k - \left(\frac{9}{2}\right)^k\right)\right| + \frac{y}{t\cdot 10^n + y}\cdot\frac{1}{y}\sum_{i=0}^{y-1}\left(c_1 s(i)^{k-1} + c_2 s(i)^{k-2} + \cdots + c_k\right)$$

$$< A(k,1)n^{k-1} + A(k,m)p^{k-1} + Dp^{k-1} < A(k,m+1)n^{k-1}.$$

Here $c_1, \ldots, c_k$, $D$ are constants that depend on $k$ and $t$, but since $1\le t\le 9$, they in fact depend only on $k$. For the second equality, we used Lemma 3 as well as the induction hypothesis.

## REFERENCES


1. C. Cooper & R. E. Kennedy. "Digit Sum Sums." *J. Inst. Math. Comp. Sci.* **5** (1992):45-49.
2. C. Cooper & R. E. Kennedy. "Sums of Powers of Digital Sums." *The Fibonacci Quarterly* **31.4** (1993):341-45.


AMS Classification Number: 11A25

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# THE LIMIT OF THE GOLDEN NUMBERS IS 3/2


**Gregory A. Moore**

West Shore Community College, Scottville, MI 49454-0277

gamoore@aol.com



(*Submitted September 1992*)


## 1. INTRODUCTION

The "Golden polynomials" $\{G_n(x)\}$ (defined in [2]) are Fibonacci polynomials satisfying

$$G_{n+2}(x) = x \cdot G_{n+1}(x) + G_n(x) \tag{1}$$

for $n \geq 0$, where $G_0(x) = -1$ and $G_1(x) = x - 1$. The maximal real root, $g_n$, of the function $G_n(x)$, can be considered to an $n^{\text{th}}$-dimensional golden ratio.

Our concern here is the study of the sequence $\{g_n\}$ of "golden numbers." A computer analysis of this sequence of roots indicated that the odd-indexed subsequence of $\{g_n\}$ was monotonically increasing and convergent to $3/2$ from below, while the even-indexed subsequence was monotoni-cally decreasing and convergent to $3/2$ from above.

In this paper, the implications of the computer analysis are proven correct. In the process, a number of lesser computational results are also developed. For example, the derivative of $G_n'(x)$ is bounded below by the Fibonacci number $F_{n+1}$ on the interval $[3/2, \infty)$.

## 2. EXISTENCE

We begin with a simple yet useful formula.

***Formula 2.1:*** $G_n\left(\frac{3}{2}\right) = -\left(\frac{1}{2}\right)^n$.

***Proof:*** The formula is readily verified for $n = 1$ and $n = 2$ by direct computation:

$$G_0\left(\frac{3}{2}\right) = -1 = -\left(-\frac{1}{2}\right)^0 \quad \text{and} \quad G_1\left(\frac{3}{2}\right) = \frac{3}{2} - 1 = \frac{1}{2} = -\left(-\frac{1}{2}\right)^1.$$

We proceed by induction assuming the proposition is true for all indices less than $n$:

$$\begin{aligned} G_n\left(\frac{3}{2}\right) &= \frac{3}{2}G_{n-1}\left(\frac{3}{2}\right) + G_{n-2}\left(\frac{3}{2}\right) = \frac{3}{2}\left(-\left(-\frac{1}{2}\right)^{n-1}\right) + \left(-\left(-\frac{1}{2}\right)^{n-2}\right) \\ &= \left(\frac{3(-1)^{n-1}}{2^n} + \frac{(-1)^{n-1}}{2^{n-2}}\right) = (-1)^n\left(\frac{3-2^2}{2^n}\right) - \left(-\frac{1}{2}\right)^n. \qquad \square \end{aligned}$$

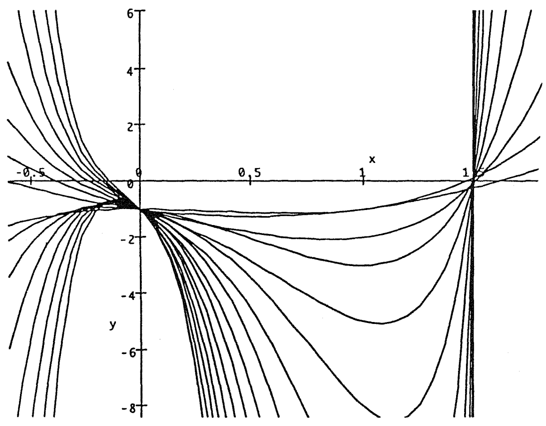
Incidentally, it is apparent from this formula that

$$\lim_{n\to\infty} G_n\left(\frac{3}{2}\right) = \lim_{n\to\infty} -\left(-\frac{1}{2}\right)^n = 0.$$

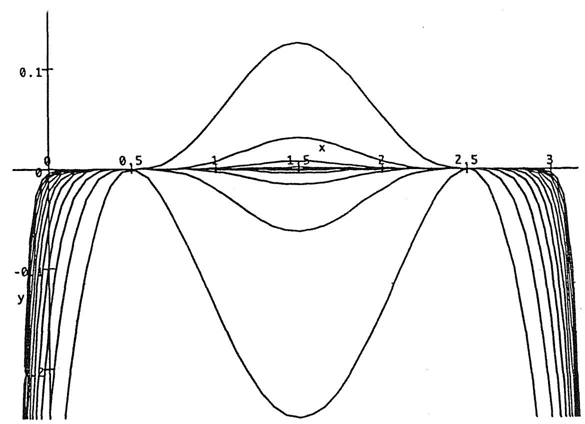
While suggestive, this is not sufficient to show the desired result about the convergence of the roots. For example, these same properties hold for the sequence of functions

$$f_n(x) = -\left(\frac{\left(x-\frac{3}{2}\right)^2 - 1}{2}\right)^n.$$

However, these roots remain at $\frac{1}{2}$ and $\frac{5}{2}$ for all values of $n$ and do not converge to $\frac{3}{2}$.




$G_n(x)$ **for** $n = 2, \ldots, 17$




$f_n(x)$ **for** $n = 2, \ldots, 12$

Throughout this paper we will limit our discussion to polynomial functions with positive leading coefficient. These functions have the following easily proven properties.

***Lemma 2.2:***

*A.* If $r$ is the maximal root of a function $f$, then $f(x) > 0$ for all $x > r$. Conversely, if $f(x) > 0$ for all $x \geq t$, then $r < t$. If $f(s) < 0$, then $s < r$.

*B.* Suppose $R$ is an upper bound for the roots of the functions $f_1(x), f_2(x), \ldots, f_n(x)$, and the functions $u_0(x), u_1(x), u_2(x), \ldots, u_n(x)$ have no positive real roots. Then $R$ is also an upper bound for the roots of the function $f(x)$ defined by

$$f(x) = f_n(x)\cdot u_n(x) + f_{n-1}(x)\cdot u_{n-1}(x) + \cdots + f_1(x)\cdot u_1(x) + u_0(x).$$

To demonstrate the existence of the sequence $\{g_n\}$, we will require two minor results from [1]. First, from Corollary 2.4, $G_n(1) = -F_{n-1}$ and $G_n(-1) = (-1)^n L_{n-1}$ [where $F_{n-1}$ is the $(n-1)^{\text{th}}$ Fibo-nacci number and $L_{n-1}$ is the $(n-1)^{\text{th}}$ Lucas number]. Second, from Corollary 4.3, each $G_n(x)$ is monic with constant term -1.

***Proposition 2.3:*** Existence of $\{g_n\}$

For each $n > 1$: *A.* $G_n(x)$ has a maximum root $g_n$ in the interval (1, 2).

*B.* $G_n(x)$ has no rational roots. In particular, each $g_n$ is irrational.

***Proof:***

Part A. Since each $G_n$ is monic and $G_n(1) = -F_{n-1} < 0$, then $G_n(x)$ must have a root larger than 1 (Lemma 2.2A). Since $g_n$ is the largest root by definition, we have $g_n > 1$.

By direct computation, $G_1(x)$ and $G_2(x)$ are strictly positive on the interval $[2, \infty)$. Using the recursive relation (1) and an inductive argument, it is easy to see that each $G_n(x)$ is strictly positive on $[2, \infty)$. Therefore, $g_n < 2$ (Lemma 2.2A).

Part B. Suppose $r$ is a rational root of $G_n(x)$, say $r = b/c$. Then $G_n$ would be divisible by a linear factor of the form $(cx - b)$. In this case, $b$ would divide the constant term of $-1$, and $c$ would divide the leading coefficient of $+1$. The only possibilities are $\pm(x-1)$ and $\pm(x+1)$, which indicate $G_n(x)$ has a root of $+1$ or $-1$, respectively. However, $G_n(1) = -F_{n-1}$ and $G_n(-1) = (-1)^n L_{n-1}$. Hence, $G_n(x)$ has no rational roots. □

## 3. EVEN/ODD DISTINCTIONS

It is useful to note that when $n$ is odd, $G_n(x)$ can be expressed entirely in terms of smaller odd-indexed functions and the seed function $G_0(x)$. Similarly, when $n$ is even, we can write $G_n(x)$ in terms of smaller even-indexed functions and the seed function $G_1(x)$. Specifically, by repeated substitution, we obtain

***Formula 3.1:***

*a.* $G_{2n+1}(x) = (x^2+1)G_{2n-1}(x) + x^2 G_{2n-3}(x) + \cdots + x^2 G_1(x) + xG_0(x)$.

*b.* $G_{2n}(x) = (x^2+1)G_{2n-2}(x) + x^2 G_{2n-4}(x) + \cdots + x^2 G_2(x) + xG_1(x)$.

We can now show that $3/2$ is an upper bound for all of the odd-indexed $g_n$ and a lower bound for the even-indexed $g_n$.

***Observation 3.2:*** $g_{2n-1} < 3/2 < g_{2n}$ for all $n > 0$.

***Proof:***

Case: Even Indices. $(3/2 < g_{2n})$

By Formula 2.1, $G_{2n}(3/2) = -2^{-2n} < 0$. Since $g_n$ is defined to be the largest root of $G_n(x)$, the result is indicated by Lemma 2.2A.

Case: Odd Indices. $(g_{2n-1} < 3/2)$

Note that $g_1 = 1 < 3/2$. Assume then that the proposition is true for $g_{2k-1}$ for $k < n$. Using Formula 3.1, we write

$$G_{2n+1}(x) = (x^2+1)\cdot G_{2n-1}(x) + x^2 \cdot G_{2n-3}(x) + \cdots + x^2 \cdot G_1(x) + x.$$

We can apply Lemma 2.2B because the functions $x, x^2$, and $(x^2+1)$ have no positive roots, and $3/2$ is an upper bound for the roots of the $G_n(x)$ on the right side. □

## 4. MONOTONICITY

***Formula 4.1:*** $G_{n+k}(g_n)=(-1)^{k+1}G_{n-k}(g_n)$.

***Proof:***

<u>$k=1$</u>. Write (1) in the form $G_{n+1}(x)=x\cdot G_n(x)+G_{n-1}(x)$, and evaluate at $x=g_n$, noting that $G_n(g_n)=0$.

<u>$k=2$</u>. Write (1) in the forms $G_n(x)=x\cdot G_{n-1}(x)+G_{n-2}(x)$ and $G_{n+2}(x)=x\cdot G_{n+1}(x)+G_n(x)$. Now plug in $x=g_n$ and note that $G_{n-1}(g_n)=G_{n+1}(g_n)$ (the case of $k=1$) to get $G_{n-2}(g_n)=-G_{n+2}(g_n)$.

<u>$k<j$</u>. Now assume the proposition is true for $k=1,2,\ldots,j-1$ (holding $n$ fixed) and define $A$ as the quotient

$$A=\frac{G_{n+k}(g_n)}{G_{n-k}(g_n)}.$$

We will show $A=(-1)^{j+1}$ to complete the proof. We can simplify $A$ using (1) for the numerator and (1) solved for the last term, $G_n=G_{n+2}-x\cdot G_{n+1}$, for the denominator:

$$A=\frac{g_nG_{n+j-1}(g_n)+G_{n+j-2}(g_n)}{G_{n-j+2}(g_n)-g_nG_{n-j+1}(g_n)}=\frac{g_nG_{n+(j-1)}(g_n)+G_{n+(j-2)}(g_n)}{G_{n-(j-2)}(g_n)-g_nG_{n-(j-1)}(g_n)}.$$

Also define $B$ and $C$ and simplify using the validity of the formula for smaller values of $k$.

$$B=G_{n+j-1}(g_n)=G_{n-(j-1)}(g_n)=(-1)^jG_{n-(j-1)}(g_n),$$
$$C=G_{n+j-2}(g_n)=G_{n-(j-2)}(g_n)=(-1)^{j-1}G_{n-(j-2)}(g_n).$$

Substituting $B$ and $C$ into the simplification of $A$, we get

$$A=\frac{g_nB+C}{(-1)^{j-1}C-g_n(-1)^jB}=\frac{g_nB+C}{(-1)^{j+1}(C+g_nB)}=(-1)^{j+1}.$$

This shows the formula to be valid for $k=j$. □

***Proposition 4.2:*** The subsequence of $\{g_n\}$ with odd indices is a monotonically increasing sequence; and the subsequence with even indices is monotonically decreasing.

***Proof:***

<u>Odd Indices</u>. By direct computation, $g_3>g_1=1$. Assume the proposition holds up to $g_{2k-1}$, that is, $g_1<g_3<\cdots<g_{2k-3}<g_{2k-1}$. Then $G_{2k-3}(g_{2k-1})>0$ (Lemma 2.2A). Using Formula 4.1,

$$G_{2k+1}(g_{2k-1})=G_{(2k-1)+2}(g_{2k-1})=-G_{(2k-1)-2}(g_{2k-1})=-G_{2k-3}(g_{2k-1})<0.$$

$G_{2k+1}$ must have a root greater than $g_{2k-1}$ by Lemma 2.2A. It follows that $g_{2k+1}>g_{2k-1}$.

<u>Even Indices</u>. Note first that $g_2={}^{(1+\sqrt{5})}\!/_2>{}^3\!/_2$. Since $g_{2n-1}<{}^3\!/_2$ (Observation 3.2), then $G_{2n-1}(x)>0$ on $[{}^3\!/_2,\infty)$. Rewriting (1), we have $G_{2n}-G_{2n-2}=x\cdot G_{2n-1}>0$ on $[{}^3\!/_2,\infty)$. Thus, $G_{2n}>G_{2n-2}$ for all $x\geq{}^3\!/_2$; and $G_{2n}$ has no root greater than $g_{2n-2}$. But $G_{2n}({}^3\!/_2)<0$ by Formula

2.1. By the intermediate value theorem, there must be a root between $\frac{3}{2}$ and $g_{2n-2}$. This root must be $g_{2n}$. □

## 5. THE ODD-INDEXED CONVERGENCE

We now know that the odd-indexed $\{g_n\}$ form a monotonically increasing sequence bounded above by $\frac{3}{2}$, and the even-indexed $\{g_n\}$ form a monotonically decreasing sequence bounded below by $\frac{3}{2}$. Thus, limits do exist for both subsequences. We need two additional lemmas.

***Lemma 5.1:*** The derivatives $G'_{2n-1}(x)$ are bounded below by $F_{2n}$ on the interval $(g_{2n-1}, \infty)$, where $F_{2n}$ is the $(2n)^{\text{th}}$ Fibonacci number.

***Proof:*** Substituting for both $G_{2n+1}(x)$ and $G_{2n-1}(x)$ in Formula 3.1, we obtain

$$\begin{aligned} G_{2n+1}(x)-G_{2n-1}(x) &= [(x^2+1)G_{2n-1}(x)+x^2G_{2n-3}(x)+\cdots+x^2G_1(x)+xG_0(x)] \\ &\quad -[(x^2+1)G_{2n-3}(x)+x^2G_{2n-5}(x)+\cdots+x^2G_1(x)+xG_0(x)] \\ &= (x^2+1)G_{2n-1}(x)-G_{2n-3}(x). \end{aligned}$$

Solving for $G_{2n+1}(x)$ gives us $G_{2n+1}(x)=(x^2+2)G_{2n-1}(x)-G_{2n-3}(x)$. Differentiating gives

$$G'_{2n+1}(x)=(x^2+2)G'_{2n-1}(x)-G'_{2n-3}(x)+2xG_{2n-1}(x).$$

For $x > g_{2n-1}$, the last term is positive; thus, for all $x > 1$,

$$G'_{2n+1}(x) > (x^2+2)G'_{2n-1}(x)-G'_{2n-3}(x) > 3\cdot G'_{2n-1}(x)-G'_{2n-3}(x). \tag{2}$$

We compute

$$\begin{aligned} &G'_1(x)=(x-1)'=1=F_2, \\ &G'_3(x)=(x^3-x^2-1)'=3x^2-2x>3=F_4 \ \ (\text{for } x\ge g_3>\sqrt{2}), \\ &G'_5(x)>3G'_3(x)>3(3)-1=8=F_6 \ \ (\text{for } x>g_3). \end{aligned}$$

Using induction and the Fibonacci identity $F_{2n}=3\cdot F_{2n-2}-F_{2n-4}$, (2) becomes

$$G'_{2n+1}(x) > F_{2n+2}. \ \square$$

Actually, the growth rates of these derivatives can easily be shown to be even greater, although they are adequate for our purposes here. We are ready to demonstrate that the odd-indexed roots converge to $\frac{3}{2}$, with the aid of the following simple lemma.

***Lemma 5.2:*** If polynomial functions $f(x)$ and $g(x)$ have the properties that $f(b)=g(b)>0$ and $f'(x)>g'(x)>0$ for all $x$ in $(a,b)$, then $f(x)<g(x)$ on $(a,b)$. Furthermore, if $g$ has a root $c$ in $(a,b)$, then $f(x)$ also has a root in $(c,b)$.

***Proof:*** Let $h(x)=f(x)-g(x)$. Then $h'(x)=f'(x)-g'(x)>0$, which implies $h(x)$ is increasing. Since $h(x)<h(b)=0$ for all $x$ in $(a,b)$, we have $f(x)-g(x)<0$ and the first result follows. If $g(c) = 0$, then $f(c) < g(c) = 0$. Since $f(b) = g(b) > 0$, $f$ must have a root in the interval $(c, b)$. $\square$

***Proposition 5.3:*** The odd-indexed subsequence of $\{g_n\}$ converges to $3/2$. That is,

$$\lim_{n\to\infty} g_{2n-1} = \frac{3}{2}.$$

***Proof:*** Because the odd-indexed subsequence $\{g_{2n-1}\}$ is monotonically increasing and bounded above by $3/2$, we know that the limit exists and is less than or equal to $3/2$. We need only show it is no less than $3/2$.

We apply Lemma 5.2, setting $f(x) = G_{2n-1}(x)$ and $g(x) = x - (3/2 - 2^{2n-1})$. We note that $f(3/2) = g(3/2) = 2^{-(2n-1)} > 0$ (Formula 2.1), and $f'(x) = G'_{2n-1}(x) > F_{2n} > 1 = g'(x)$ (Lemma 5.1). Since $g(x)$ has a root at $x = (3/2 - 2^{2n-1})$, it follows that $G_{2n-1}(x)$ has a root in the interval $(3/2 - 2^{2n-1}, 3/2)$. Thus, $3/2 > g_{2n-1} > 3/2 - 2^{2n-1}$ for all $n$. $\square$

## 6. THE EVEN-INDEXED SUBSEQUENCE

We now address the even-indexed subsequence in a somewhat analogous way.

***Lemma 6.1:*** The derivative $G'_{2n}(x)$ is bounded below by the Fibonacci number $F_{2n+1}$ on $[3/2, \infty)$.

***Proof:*** For $x > 3/2$, $G'_2(x) = 2x - 1 > 2(3/2) - 1 = 2 = F_3$. Assume $G'_{2n-2}(x) > F_{2n-1}$. Differentiating (1) gives $G'_{2n}(x) = x \cdot G'_{2n-1}(x) + G'_{2n-2}(x) + G_{2n-1}(x)$.Keeping in mind that $G'_{2n-1}(x) > F_{2n}$ (Lemma 5.1) and $G_{2n-1}(3/2) \geq 2^{-n} > 0$ (Formula 2.1 and Lemma 5.1), we write

$$G'_{2n}(x) > (3/2) \cdot F_{2n} + F_{2n-1} + 2^{-2n-1} > F_{2n} + F_{2n-1} = F_{2n+1}. \quad \square$$

Combining Lemmas 5.1 and 6.1, we have the side result

***Corollary 6.2:*** $G'_n(x) > F_{n+1}$ on the interval $[3/2, \infty)$.

***Lemma 6.3:*** Suppose polynomial functions $f(x)$ and $g(x)$ have the properties $f(a) = g(a) < 0$ and $f'(x) > g'(x) > 0$ for all $x$ in $(a, b)$. Then $f(x) > g(x)$ on $(a, b)$. Furthermore, if $g(x)$ has a root $c$ in $(a, b)$, then $f(x)$ also has a root in $(a, c)$.

***Proof:*** Apply Lemma 5.2 to the functions $-f(a+b-x)$ and $-g(a+b-x)$. $\square$

We can now show that the even-indexed roots converge to $3/2$ from above.

***Proposition 6.4:*** The even-indexed subsequence of $\{g_n\}$ converges to $3/2$. That is,

$$\lim_{n\to\infty} g_{2n} = \frac{3}{2}.$$

***Proof:*** Because the sequence $\{g_{2n}\}$ is monotonically decreasing and bounded below by $3/2$, we know that the limit exists and is no less than $3/2$. We need only show that the limit is no more than $\frac{3}{2}$. Apply Lemma 6.3, letting $f(x) = G_{2n}(x)$ and $g(x) = x - (\frac{3}{2} + 2^{-2n})$. Then $f(\frac{3}{2}) = g(\frac{3}{2}) = -2^{-2n}$ and $f'(x) > F_{2n+1} > 1 = g(x)$ (Lemma 6.1). Thus, $f(x) = G_{2n}(x)$ has a root interval $(\frac{3}{2}, \frac{3}{2} + 2^{-2n})$. This means that $\frac{3}{2} < g_{2n} < \frac{3}{2} + 2^{2n-1}$ for all $n$. □

## 7. CONCLUDING REMARKS

While the golden numbers form an irrational sequence converging to $\frac{3}{2}$ with odd and even subsequences converging monotonically from below and above, respectively, there are other questions to consider. For example, computer analysis also yields the apparent approximation,

$$g_n \approx \frac{3}{2} + \Delta \cdot (-1)^{-n},$$

which could be explored. Also, it is quite likely that these results can be extended to other Fibonacci polynomial sequences. Many of the formulas and lemmas here relied only on the basic Fibonacci relationship (1) and not the specific definition of the particular functions $\{G_n\}$. Possibly there is a number like $\frac{3}{2}$ for each Fibonacci polynomial sequence.

## REFERENCES

1. V. E. Hoggatt, Jr. & M. Bicknell. "Roots of Fibonacci Polynomials." *The Fibonacci Quarterly* **11.3** (1973):271-74.
2. G. Moore. "A Fibonacci Polynomial Sequence Defined by Multidimensional Continued Fractions; and Higher-Order Golden Ratios." *The Fibonacci Quarterly* **31.4** (1993):354-64.

AMS Classification Numbers: 11B39, 11B37, 11C08

# THE MULTIPARAMETER NONCENTRAL STIRLING NUMBERS

**B. S. El-Desouky**
Department of Mathematics, Faculty of Science, Aswan, Egypt
(*Submitted September 1992*)

## 1. INTRODUCTION

The Sterling numbers of the first and second kind were introduced by Stirling in 1749 (see [9]). Recently, several generalizations and extensions of the Stirling numbers are given and many combinatorial, probabilistic, and statistical applications are discussed (see [1], [2], [3], [4], and [8]).

In a recent paper [6], Koutras defined $s(n,k;\alpha)$ and $S(n,k;\alpha)$ [we used these symbols instead of $s_\alpha(n,k)$ and $S_\alpha(n,k)$ to avoid ambiguity with Comtet's numbers], the noncentral Stirling numbers of the first and second kind, by

$$(t)_n = \sum_{k=0}^{n} s(n,k;\alpha)\,(t-\alpha)^k, \tag{1.1}$$

$$(t-\alpha)^n = \sum_{k=0}^{n} S(n,k;\alpha)\,(t)_k. \tag{1.2}$$

In this paper we use the following notations:

$$(t/\alpha)_n = \prod_{j=0}^{n-1}(t-\alpha_j),\ (t/\alpha)_0 = 1,\ \text{ and }\ (\alpha_k)_\ell = \prod_{\substack{j=0\\ j\neq k}}^{\ell}(\alpha_k-\alpha_j),\ k\le \ell.$$

Comtet [5] defined $s_\alpha(n,k)$ and $S_\alpha(n,k)$, the generalized Stirling numbers of the first and second kind associated with $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, by

$$(t/\alpha)_n = \sum_{k=0}^{n} s_\alpha(n,k)t^k, \tag{1.3}$$

$$t^n = \sum_{k=0}^{n} S_\alpha(n,k)\,(t/\alpha)_k. \tag{1.4}$$

The main purpose of this paper is to modify the noncentral Stirling numbers of the first and second kind.

In sections 2 and 3 we define $s(n,k;\overline{\alpha})$ and $S(n,k;\overline{\alpha})$, the multiparameter noncentral Stirling numbers of the first and second kind; recurrence relations, generating functions, and explicit forms are obtained.

Some special cases are discussed and a relation between the multiparameter noncentral Stirling numbers and other Stirling numbers are found. Finally, in section 4, some applications are derived.

## 2. THE MULTIPARAMETER NONCENTRAL STIRLING NUMBERS OF THE FIRST KIND

***Definition:*** Let $t$ be a real number, $n$ a nonnegative integer, and $\overline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ where $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1}$ are real numbers.

We define the multiparameter noncentral Stirling numbers of the first kind, $s(n, k;\ \alpha_0, a_1, \ldots, \alpha_{n-1})$, briefly denoted by $s(n, k; \overline{\alpha})$, with parameters $\overline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, by

$$(t)_n = \sum_{k=0}^{n} s(n, k; \overline{\alpha})\,(t / \alpha)_k, \tag{2.1}$$

where $s(0, 0; \overline{\alpha}) = 1$ and $s(n, k; \overline{\alpha}) = 0$ for $k > n$.

***Theorem 2.1:*** The multiparameter noncentral Stirling numbers of the first kind $s(n, k; \overline{\alpha})$ satisfy the recurrence relation

$$s(n+1, k; \overline{\alpha}) = s(n, k-1; \overline{\alpha}) + (\alpha_k - n)s(n, k; \overline{\alpha}) \quad \text{for } k \geq 1, \tag{2.2}$$

where $s(0, 0; \overline{\alpha}) = 1$ and $s(n, k; \overline{\alpha}) = 0$ for $k > n$ and

$$s(n, 0; \overline{\alpha}) = (\alpha_0 - n + 1)(\alpha_0 - n + 2) \cdots (\alpha_0 - 1)\alpha_0 = (\alpha_0)_n.$$

***Proof:*** Since $(t)_{n+1} = (t)_n[(t - \alpha_k) + (\alpha_k - n)]$, we have

$$\begin{aligned}\sum_{k=0}^{n+1} s(n+1, k; \overline{\alpha})\,(t / \alpha)_k &= (t - \alpha_k)\sum_{k=0}^{n} s(n, k; \overline{\alpha})\,(t / \alpha)_k + (\alpha_k - n)\sum_{k=0}^{n} s(n, k; \overline{\alpha})\,(t / \alpha)_k \\ &= \sum_{k=1}^{n+1} s(n, k-1; \overline{\alpha})\,(t / \alpha)_k + (\alpha_k - n)\sum_{k=0}^{n} s(n, k; \overline{\alpha})\,(t / \alpha)_k.\end{aligned}$$

Equating the coefficients of $(t / \alpha)_k$ on both sides, we get (2.2). For $k = 0$ we get $s(n+1, 0; \overline{\alpha}) = (\alpha_0 - n)s(n, 0; \overline{\alpha})$; therefore, $s(n, 0; \overline{\alpha}) = (\alpha_0)_n$ follows by induction.

***Remarks:*** We discuss the following special cases:

***i)*** If $\alpha_i = \alpha$, $i = 0, 1, \ldots, n-1$, then from (2.2) we have

$$s(n+1, k; \alpha) = s(n, k-1; \alpha) + (\alpha - n)s(n, k; \alpha),$$

where $s(n, k; \alpha)$ denotes the noncentral Stirling numbers of the first kind that is defined by Koutras [6].

***ii)*** If $\alpha_i = 0$, $i = 0, 1, \ldots, n-1$, then we have

$$s(n+1, k) = s(n, k-1) - ns(n, k),$$

where $s(n, k)$ denotes the usual Stirling numbers of the first kind [9].

***iii)*** If $\alpha_i = i$, $i = 0, 1, \ldots, n-1$, then $s(n, k; \overline{\alpha})$ reduces to the $C$-numbers, where $r = 1$, i.e., $C(n, k, 1)$ (see [3]).

***Theorem 2.2:*** The multiparameter noncentral Stirling numbers of the first kind have the exponential generating function

$$\phi_k(t;\overline{\alpha}) = \sum_{n=k}^{\infty} s(n,k;\overline{\alpha})\frac{t^n}{n!} = \sum_{j=0}^{k}\frac{(1+t)^{\alpha_j}}{(\alpha_j)_k}. \tag{2.3}$$

***Proof:*** Let $\phi_k(t,\overline{\alpha})$ be the exponential generating function of $s(n,k;\overline{\alpha})$, then

$$\begin{aligned}\phi_k(t;\overline{\alpha}) &= \sum_{n=0}^{\infty} s(n,k;\overline{\alpha})\frac{t^n}{n!}, \quad \text{where}\\ \phi_0(t;\overline{\alpha}) &= \sum_{n=0}^{\infty} s(n,0;\overline{\alpha})\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\alpha_0)_n\frac{t^n}{n!} = (1+t)^{\alpha_0}.\end{aligned} \tag{2.4}$$

Differentiating both sides of (2.4) with respect to $t$, we get

$$\phi_k'(t;\overline{\alpha}) = \sum_{n=k}^{\infty} s(n,k;\overline{\alpha})\frac{t^{n-1}}{(n-1)!},$$

and from (2.2) we get

$$\begin{aligned}\phi_k'(t;\overline{\alpha}) &= \sum_{n=k}^{\infty} s(n-1,k-1;\overline{\alpha})\frac{t^{n-1}}{(n-1)!} + \alpha_k\sum_{n=k+1}^{\infty} s(n-1,k;\overline{\alpha})\frac{t^{n-1}}{(n-1)!}\\ &\qquad - t\sum_{n=k+1}^{\infty} s(n-1,k;\overline{\alpha})\frac{t^{n-2}}{(n-2)!}\\ &= \phi_{k-1}(t;\overline{\alpha}) + \alpha_k\phi_k(t,\overline{\alpha}) - t\phi_k'(t;\overline{\alpha});\end{aligned}$$

hence,

$$\phi_k'(t;\overline{\alpha}) - \frac{\alpha_k}{(1+t)}\phi_k(t;\overline{\alpha}) = \frac{1}{1+t}\phi_{k-1}(t;\overline{\alpha}).$$

Solving this difference-differential equation, we obtain (2.3).

***Theorem 2.3:*** The numbers $s(n,k;\overline{\alpha})$ have the following explicit form:

$$s(n,k;\overline{\alpha}) = \sum_{r=k}^{n}\frac{n!}{r!}(-1)^{n-r}\sum\frac{1}{\ell_1\ell_2\ldots\ell_r}\sum_{j=0}^{k}\frac{\alpha_j^r}{(\alpha_j)_k}, \tag{2.5}$$

where, in the second sum, the summation extends over all ordered $n$-tuples of integers $(\ell_1, \ell_2, \ldots, \ell_r)$ satisfying the conditions $\ell_1+\ell_2+\cdots+\ell_r = n$ and $\ell_i \ge 1$, $i = 1, 2, \ldots, r$.

***Proof:*** From (2.3),

$$\begin{aligned}\phi_k(t;\overline{\alpha}) &= \sum_{j=0}^{k}\frac{(1+t)^{\alpha_j}}{(\alpha_j)_k} = \sum_{j=0}^{k}\frac{e^{\alpha_j\log(1+t)}}{(\alpha_j)_k}\\ &= \sum_{j=0}^{k}\frac{1}{(\alpha_j)_k}\sum_{r=0}^{\infty}\frac{(\alpha_j\log(1+t))^r}{r!} = \sum_{j=0}^{k}\frac{1}{(\alpha_j)_k}\sum_{r=0}^{\infty}\frac{\alpha_j^r}{r!}\left(\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{t^\ell}{\ell}\right)^r,\end{aligned}$$

and using Cauchy's rule of multiplication of infinite series, we get (2.5).

In the following, we find a relationship between $s(n, k)$ and $s(n, k; \overline{\alpha})$. From (2.1), we have

$$(t)_n = \sum_{k=0}^{n} s(n, k; \overline{\alpha})\,(t/\alpha)_k\,;$$

hence,

$$\sum_{i=0}^{n} s(n, i)t^i = \sum_{k=0}^{n} s(n, k; \overline{\alpha}) \sum_{i=0}^{k} s_\alpha(k, i)t^i$$
$$= \sum_{i=0}^{n} \left( \sum_{k=i}^{n} s(n, k; \overline{\alpha}) s_\alpha(k, i) \right) t^i.$$

Equating the coefficients of $t^i$ on both sides, we get

$$s(n, i) = \sum_{k=i}^{n} s(n, k; \overline{\alpha}) s_\alpha(k, i). \tag{2.6}$$

Similarly, we can express $s(n, k; \overline{\alpha})$ in terms of $s(n, k)$. Since

$$(t)_n = \sum_{k=0}^{n} s(n, k)t^k,$$

we have, from (1.4) and (2.1),

$$(t)_n = \sum_{k=0}^{n} s(n, k) \sum_{i=0}^{k} S_\alpha(k, i)\,(t/\alpha)_i\,;$$

therefore,

$$\sum_{i=0}^{n} s(n, i; \overline{\alpha})\,(t/\alpha)_i = \sum_{i=0}^{n} \left( \sum_{k=i}^{n} s(n, k) S_\alpha(k, i) \right) (t/\alpha)_i,$$

and hence,

$$s(n, i; \overline{\alpha}) = \sum_{k=i}^{n} s(n, k) S_\alpha(k, i). \tag{2.7}$$

Also we can express $S_\alpha(n, k)$ in terms of $s(n, k; \overline{\alpha})$. Since

$$t^n = \sum_{k=0}^{n} S(n, k)\,(t)_k = \sum_{k=0}^{n} S(n, k) \sum_{i=0}^{k} s(k, i; \overline{\alpha})\,(t/\alpha)_i,$$

we have

$$\sum_{i=0}^{n} S_\alpha(n, i)\,(t/\alpha)_i = \sum_{i=0}^{n} \left( \sum_{k=i}^{n} S(n, k) s(k, i; \overline{\alpha}) \right) (t/\alpha)_i\,;$$

hence,

$$S_\alpha(n, i) = \sum_{k=i}^{n} S(n, k) s(k, i; \overline{\alpha}). \tag{2.8}$$

Combining equations (2.6) and (2.7), we get an orthogonality relation of $s_\alpha(n, k)$ and $S_\alpha(n, k)$. Since

$$s(n,i)=\sum_{k=i}^{n}s_\alpha(k,i)\sum_{\ell=k}^{n}s(n,i)S_\alpha(\ell,k)=\sum_{\ell=i}^{n}\left(\sum_{k=i}^{\ell}s_\alpha(k,i)S_\alpha(\ell,k)\right)s(n,\ell);$$

hence,

$$\sum_{k=i}^{\ell}S_\alpha(\ell,k)s_\alpha(k,i)=\delta_{\ell i},$$

where $\delta_{\ell i}$ is Kronecker's delta.

## 3. THE MULTIPARAMETER NONCENTRAL STIRLING NUMBERS OF THE SECOND KIND

***Definition:*** Let $t$ be a real number, $n$ a nonnegative integer, and $\overline{\alpha}=(\alpha_0,\alpha_1,\ldots,\alpha_{n-1})$, where $\alpha_0<\alpha_1<\cdots<\alpha_{n-1}$ are real numbers.

We define $S(n,k;\alpha_0,\alpha_1,\ldots,\alpha_{n-1})$, briefly denoted by $S(n,k;\overline{\alpha})$, the multiparameter noncentral Stirling numbers of the second kind with parameters $\overline{\alpha}=(\alpha_0,\alpha_1,\ldots,\alpha_{n-1})$, by

$$(t/\alpha)_n=\sum_{k=0}^{n}S(n,k;\overline{\alpha})\,(t)_k, \tag{3.1}$$

where $S(0,0;\overline{\alpha})=1$ and $S(n,k;\overline{\alpha})=0$ for $k>n$.

***Theorem 3.1:*** The numbers $S(n,k;\overline{\alpha})$ satisfy the recurrence relation

$$S(n,k;\overline{\alpha})=S(n-1,k-1;\overline{\alpha})+(k-\alpha_{n-1})S(n-1,k;\overline{\alpha}). \tag{3.2}$$

***Proof:*** Since $(t/\alpha)_n=(t/\alpha)_{n-1}(t-\alpha_{n-1})=(t/\alpha)_{n-1}[(t-k)+(k-\alpha_{n-1})]$, we obtain, from (3.1),

$$\sum_{k=0}^{n}S(n,k;\overline{\alpha})\,(t)_k=(t-k)\sum_{k=0}^{n-1}S(n-1,k;\overline{\alpha})\,(t)_k+(k-\alpha_{n-1})\sum_{k=0}^{n-1}S(n-1,k;\overline{\alpha})\,(t)_k,$$

which gives us (3.2).

We discuss the following special cases:

***i)*** If $\alpha_i=\alpha$, $i=0,1,\ldots,n-1$, then from (3.2) we have

$$S(n,k;\alpha)=S(n-1,k-1;\alpha)+(k-\alpha)S(n-1,k;\alpha),$$

where $S(n,k;\alpha)$ denotes the noncentral Stirling numbers of the second kind as defined by Koutras [6].

***ii)*** If $\alpha_i=0$, $i=0,1,\ldots,n-1$, then from (3.2) we have

$$S(n,k)=S(n-1,k-1)+kS(n-1,k),$$

where $S(n,k)$ denotes the Stirling numbers of the second kind (see [9]).

***iii)*** If $\alpha_i=i$, $i=0,1,\ldots,n-1$, then $S(n,k;\overline{\alpha})$ reduces to the $C$-numbers, where $r=1$, i.e., $C(n,k;1)$ (see [2]).

In the following, we find a relationship between $s_\alpha(n,k)$ and $S(n,k;\overline{\alpha})$.

From (3.1) we have

$$(t/\alpha)_n = \sum_{k=0}^{n} S(n,k;\overline{\alpha})\,(t)_k = \sum_{k=0}^{n} S(n,k;\overline{\alpha}) \sum_{i=0}^{k} s(k,i) t^i;$$

hence,

$$\sum_{i=0}^{n} s_\alpha(n,i) t^i = \sum_{i=0}^{n} \left( \sum_{k=i}^{n} S(n,k;\overline{\alpha})\, s(k,i) \right) t^i,$$

and equating the coefficients of $t^i$ on both sides, we get

$$s_\alpha(n,i) = \sum_{k=i}^{n} S(n,k;\overline{\alpha})\, s(k,i). \tag{3.3}$$

Similarly, we have

$$(t/\alpha)_n = \sum_{k=0}^{n} s_\alpha(n,k) t^k = \sum_{k=0}^{n} s_\alpha(n,k) \sum_{i=0}^{k} S(k,i)\,(t)_i;$$

therefore,

$$\sum_{i=0}^{n} S(n,i;\overline{\alpha})\,(t)_i = \sum_{i=0}^{n} \left( \sum_{k=i}^{n} s_\alpha(n,k) S(k,i) \right) (t)_i,$$

and hence,

$$S(n,i;\overline{\alpha}) = \sum_{k=i}^{n} s_\alpha(n,k) S(k,i). \tag{3.4}$$

Also, we can express $S(n,k)$ in terms of $S(n,k;\overline{\alpha})$. It follows from (1.4) that

$$t^n = \sum_{k=0}^{n} S_\alpha(n,k)\,(t/\alpha)_k = \sum_{k=0}^{n} S_\alpha(n,k) \sum_{i=0}^{n} S(k,i;\overline{\alpha})\,(t)_i.$$

Thus,

$$\sum_{i=0}^{n} S(n,i)\,(t)_i = \sum_{i=0}^{n} \left( \sum_{k=i}^{n} S_\alpha(n,k) S(k,i,\overline{\alpha}) \right) (t)_i,$$

implying that

$$S(n,i) = \sum_{k=i}^{n} S_\alpha(n,k) S(k,i;\overline{\alpha}). \tag{3.5}$$

Moreover, we can find a relationship between $s(n,k;\overline{\alpha})$, $s_\alpha(n,k)$, and $s(n,k;\alpha)$, as follows. From (2.6) and (1.9) in [6], we get

$$\sum_{\ell=i}^{n} \binom{n}{\ell} (-\alpha)_{n-\ell}\, s(\ell,i;\alpha) = \sum_{\ell=i}^{n} s(n,\ell;\overline{\alpha}) s_\alpha(\ell,i);$$

hence,

$$\sum_{\ell=i}^{n} \left( s(n,\ell;\overline{\alpha}) s_\alpha(\ell,i) - \binom{n}{\ell} (-\alpha)_{n-\ell}\, s(n,i;\alpha) \right) = 0. \tag{3.6}$$

Similarly, from (2.5) and equation (2.5a) in [6], we get

$$\sum_{k=i}^{n} \left( S_\alpha(n,k) S(k,i;\overline{\alpha}) - \binom{n}{k} \alpha^{n-k} S(k,i;\alpha) \right) = 0. \tag{3.7}$$

## 4. APPLICATIONS

**i.** From (2.6), and since

$$s(n,i)=\sum_{k=i}^{n}(-1)^{i}L(n,k)s(k,i),$$

where $L(n,k)$ denotes the Lah numbers (see [3]); hence, we obtain the combinatorial identity

$$\sum_{k=i}^{n}\left((-1)^{i}L(n,k)s(k,i)-s(n,k;\overline{\alpha})s_{\alpha}(k,i)\right)=0. \tag{4.1}$$

Similarly, from (2.6), and since

$$s(n,i)=r^{-i}\sum_{k=i}^{n}C(n,k,r)s(k,i),$$

where $C(n,k,r)$ denotes the $C$-numbers (see [3]), we have the combinatorial identity

$$\sum_{k=i}^{n}\left(s(n,k;\alpha)s_{\alpha}(k,i)-r^{-i}C(n,k,r)s(k,i)\right)=0. \tag{4.2}$$

**ii.** We find an orthogonality relation of $s(n,k;\overline{\alpha})$ and $S(n,k;\overline{\alpha})$. From (2.1) and (3.1), we get

$$\begin{aligned}(t)_n&=\sum_{k=0}^{n}s(n,k;\overline{\alpha})\,(t/\alpha)_k=\sum_{k=0}^{n}s(n,k;\overline{\alpha})\left(\sum_{i=0}^{k}S(k,i;\overline{\alpha})\,(t)_i\right)\\&=\sum_{i=0}^{n}\left(\sum_{k=i}^{n}s(n,k;\overline{\alpha})S(k,i;\overline{\alpha})\right)(t)_i;\end{aligned}$$

hence,

$$\sum_{k=i}^{n}s(n,k;\overline{\alpha})S(k,i;\overline{\alpha})=\delta_{ni}, \tag{4.3}$$

where $\delta_{ni}$ is Kronecker's delta.

**iii.** Let $M_{j,k}(x)$ denote the $B$-spline of Curry Schoenberg with knots $\xi_j<\xi_{j+1}<\cdots<\xi_{j+k}$ $(j\in Z, k=1,2,\ldots)$ as defined in [7]. The moments $\mu_\ell(k,\xi)$ of the $B$-spline $M_{j,k}(x)$ when the index $j$ is equal to 0 is given by

$$\mu_\ell(k,\xi)=\int_{-\infty}^{\infty}x^{\ell}M_{0,k}(x)\,dx,\ \ \ell=0,1,\ldots;\ k=1,2,\ldots;\quad \xi=(\xi_0,\xi_1,\ldots,\xi_k).$$

From (3.3) and Proposition 3.1 in [7], we get

$$\mu_{n-i}(k,\xi)=\binom{n}{k}^{-1}\sum_{k=i}^{n}S(n,k;\overline{\alpha})s(k,i). \tag{4.4}$$

## REFERENCES


1. T. Cacoullous. "Multiparameter Stirling and $C$-Type Distributions." In *Statistical Distributions in Scientific Work*, Vol. I: *Models and Structures*, pp. 19-30. Ed. G. P. Patil et al. Dordrecht: Reidel, 1974.
2. Ch. Charalambides. "The Generalized Stirling and $C$-Numbers." *Sankhya (Ser A)* **36** (1974): 419-36.
3. Ch. Charalambides. "A New Kind of Numbers Appearing in the $n$-Fold Convolution of Truncated Binomial and Negative Binomial Distributions." *SIAM J. Appl. Math.* **33.2** (1977):279-88.
4. Ch. Charalambides & J. Singh. "A Review of the Stirling Numbers, Their Generalization and Statistical Applications." *Communstatist. —Theory Meth.* **17.8** (1988):2533-95.
5. M. Comtet. "Numbers de stirling generaux et fonctions symétriques." *C. R. Acad. Sc. Par. (Series A)* **275** (1972):747-50.
6. M. Koutras. "Non-Central Stirling Numbers and Some Applications." *Discrete Math.* **42** (1982):73-89.
7. E. Neuman. "Moments and Fourier Transforms of $B$-Spline." *J. Comput. Appl. Math.* **7.1** (1981):51-62.
8. J. Riordan. *An Introduction to Combinatorial Analysis*. New York: Wiley, 1958.
9. J. Stirling. *Methodus Differentialissme Tractus de Summatione et Interpolatione Serierum Infinitarum*. London, 1730. [English translation by F. Holliday with title *The Differential Method*, London, 1749.]


AMS Classification Numbers: 05A15, 05A19, 05A10

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# AN "ALL OR NONE" DIVISIBILITY PROPERTY FOR A CLASS OF FIBONACCI-LIKE SEQUENCES OF INTEGERS

**Juan Pla**
315 Rue de Belleville, 75019 Paris, France
(*Submitted September 1992*)

In this note, we prove the following theorem.

***Theorem:*** Let $u_n$ be the general term of a given sequence of integers such that $u_{n+2}=u_{n+1}+u_n$, where $u_0$ and $u_1$ are arbitrary integers. Let $x$ be an arbitrary integer other than $-2, -1, 0$, and $1$. Let $D$ be any divisor of $x^2+x-1$ other than 1. Then, the sequence $w_n = xu_{n+1}-u_n$, where $n \geq 0$, is such that:

***(a)*** $D$ divides every $w_n$;
***(b)*** $D$ divides no $w_n$.

***Proof:*** It is a well-known fact [1] that

$$u_{n+p+1} = F_{p+1}u_{n+1} + F_p u_n, \tag{1}$$

where $F_p$ is the $p^{\text{th}}$ Fibonacci number. Considering the following product of two polynomials in the variable $x$,

$$(x^2+x+1)\sum_{p=0}^{p=n} F_{p+1}x^p, \tag{2}$$

and taking advantage of the fundamental properties of the Fibonacci sequence, we can see that most of the terms in (2) vanish when we develop the product, to obtain

$$(x^2+x+1)\sum_{p=0}^{p=n} F_{p+1}x^p = -1 + x^{n+1}\big((1+x)F_{n+1}+F_n\big). \tag{3}$$

Since $x$ is an integer, the two integers $x^2+x-1$ and $k_n = (1+x)F_{n+1}+F_n$, by (3), cannot share any common divisor. That is,

$$(x^2+x-1, k_n) = 1, \ n \geq 0. \tag{4}$$

Letting

$$\begin{cases} k_p = (1+x)F_{p+1}+F_p, \\ k_{p-1} = F_{p+1}+xF_p, \end{cases} \tag{5}$$

we have a linear system whose determinant is $x^2+x-1$. Since we assume that $x$ is an integer, and that this polynomial has no integer as a root, this means that the system (5) has one solution, which can be expressed as

$$\begin{cases} (x^2+x-1)F_{p+1} = xk_p - k_{p-1}, \\ (x^2+x-1)F_p = (1+x)k_{p-1} - k_p. \end{cases} \tag{6}$$

If we substitute the values of $F_{p+1}$ and $F_p$ from (6) into (1), we have

$$\begin{aligned}(x^2+x-1)u_{n+p+1} &= (xk_p - k_{p-1})u_{n+1} + \big((1+x)k_{p-1} - k_p\big)u_n \\ &= (xu_{n+1} - u_n)k_p + \big((1+x)u_n - u_{n+1}\big)k_{p-1}.\end{aligned} \tag{7}$$

Recalling that $w_n = xu_{n+1} - u_n$ and that $u_{n+1} = u_n + u_{n-1}$, we can substitute these values into the right-hand side of (7) and by simplifying obtain

$$(x^2+x-1)u_{n+p+1} = w_n k_p + w_{n-1}k_{p-1}. \tag{8}$$

Now let $D$ be any divisor of $x^2+x-1$ (except 1) and assume $D$ divides $w_n$ for some $n$. Since, by (4), $D$ does not divide $k_p$, we see that $D$ divides $w_{n-1}$. It is now obvious, by induction, that all the terms of $\{w_n\}$ are divisible by $D$. Similarly, if there exists one $w_n$ that is not divisible by $D$, then there is no $w_n$ that is divisible by $D$.

***Examples:***

***a)*** The first interesting value is $x=2$, for which $x^2+x-1=5$, and

$$w_n = 2u_{n+1} - u_n = u_{n+1} + u_{n-1}.$$

Letting $u_n = F_n$, we have $w_n = L_n$, where $L_n$ is the $n^{\text{th}}$ Lucas number. Since 5 does not divide $L_0 = w_0$, we have established the well-known fact that no $L_n$ is divisible by 5. On the contrary, if we let $u_n = L_n$, then $w_n = L_{n+1} + L_{n-1}$. Here, all terms of $w_n$ are divisible by 5, since $w_1 = 5$.

***b)*** A consequence of this "all or none" property is that no Fibonacci-like sequence of integers $u_n$ exists such that $F_n = u_{n+1} + u_{n-1}$ for all $n$ because some of the Fibonacci numbers are divisible by 5 and some are not.

***c)*** When $x^2+x-1$ is composite, it is easy to build sequences displaying the "none" property for some of the divisors and the "all" property for the other ones. For instance, when $x=7$, $x^2+x-1=55=5*11$ and $w_n = 7u_{n+1} - u_n$. With $u_0 = 3$ and $u_1 = 2$, we get $w_0 = 11$, which means that $w_n$ displays the property "none" for 5, and the property "all" for 11.

## ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to the editor for his assitance in restructuring the original version of this paper and to the referee for his/her many valuable suggestions which also improved the quality of this article.

## REFERENCE

1. A. F. Horadam. "A Generalized Fibonacci Sequence." *Amer. Math. Monthly* **68.5** (1961): 455-59.

AMS Classification Numbers: 11B37, 11B39

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# A GENERALIZATION OF MORGAN-VOYCE POLYNOMIALS

**Richard André-Jeannin**
IUT GEA, Route de Romain, 54400 Longwy, France
(*Submitted September 1992*)

## 1. INTRODUCTION

Recently Ferri, Faccio, & D'Amico ([1], [2]) introduced and studied two numerical triangles, named the DFF and the DFFz triangles. In this note, we shall see that the polynomials generated by the rows of these triangles (see [1] and [2]) are the Morgan-Voyce polynomials, which are well known in the study of electrical networks (see [3], [4], [5], and [6]). We begin this note by a generalization of these polynomials.

## 2. THE GENERALIZED MORGAN-VOYCE POLYNOMIALS

Let us define a sequence of polynomials $\{P_n^{(r)}\}$ by the recurrence relation

$$P_n^{(r)}(x) = (x+2)P_{n-1}^{(r)}(x) - P_{n-2}^{(r)}(x), \ n \geq 2, \tag{1}$$

with $P_0^{(r)}(x) = 1$ and $P_1^{(r)}(x) = x + r + 1$.

Here and in the sequel, $r$ is a fixed real number. It is clear that

$$P_n^{(0)} = b_n \tag{2}$$

and that

$$P_n^{(1)} = B_n, \tag{3}$$

where $b_n$ and $B_n$ are the classical Morgan-Voyce polynomials (see [3], [4], [5], and [6]). We see by induction that there exists a sequence $\{a_{n,k}^{(r)}\}_{n\geq 0,\, k\geq 0}$ of numbers such that

$$P_n^{(r)}(x) = \sum_{k\geq 0} a_{n,k}^{(r)} x^k,$$

with $a_{n,k}^{(r)} = 0$ if $k > n$ and $a_{n,n}^{(r)} = 1$ if $n \geq 0$.

The sequence $a_{n,0}^{(r)} = P_n^{(r)}(0)$ satisfies the recurrence relation

$$a_{n,0}^{(r)} = 2a_{n-1,0}^{(r)} - a_{n-2,0}^{(r)}, \ n \geq 2,$$

with $a_{0,0}^{(r)} = 1$ and $a_{1,0}^{(r)} = 1 + r$.

From this, we get that

$$a_{n,0}^{(r)} = 1 + nr, \ n \geq 0. \tag{4}$$

In particular, we have

$$a_{n,0}^{(0)} = 1, \ n \geq 0 \tag{5}$$

and

$$a_{n,0}^{(1)} = 1 + n, \ n \geq 0. \tag{6}$$

Following [1] and [2], one can display the sequence $\{a_{n,k}^{(r)}\}$ in a triangle:

| $n \backslash k$ | 0 | 1 | 2 | 3 | $\cdots$ |
|---|---|---|---|---|---|
| 0 | $1$ | | | | $\cdots$ |
| 1 | $1+r$ | $1$ | | | $\cdots$ |
| 2 | $1+2r$ | $3+r$ | $1$ | | $\cdots$ |
| 3 | $1+3r$ | $6+4r$ | $5+r$ | $1$ | $\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

Comparing the coefficient of $x^k$ in the two members of (1), we see that, for $n \geq 2$ and $k \geq 1$,

$$a_{n,k}^{(r)} = 2a_{n-1,k}^{(r)} - a_{n-2,k}^{(r)} + a_{n-1,k-1}^{(r)}. \tag{7}$$

By this, we can easily obtain another recurring relation

$$a_{n,k}^{(r)} = a_{n-1,k}^{(r)} + \sum_{\alpha=0}^{n-1} a_{\alpha,k-1}^{(r)}, \quad n \geq 1, k \geq 1. \tag{8}$$

In fact, (8) is clear for $n \leq 2$ by direct computation. Supposing that the relation is true for $n \geq 2$, we get, by (7), that

$$\begin{aligned} a_{n+1,k}^{(r)} &= a_{n,k}^{(r)} + \left(a_{n,k}^{(r)} - a_{n-1,k}^{(r)}\right) + a_{n,k-1}^{(r)} \\ &= a_{n,k}^{(r)} + \sum_{\alpha=0}^{n-1} a_{\alpha,k-1}^{(r)} + a_{n,k-1}^{(r)} = a_{n,k}^{(r)} + \sum_{\alpha=0}^{n} a_{\alpha,k-1}^{(r)}, \end{aligned}$$

and the proof is complete by induction.

We recognize in (8) the recursive definition of the DFF and DFFz triangles. Moreover, using (5) and (6), we see that the sequence $\{a_{n,k}^{(0)}\}$ (resp. $\{a_{n,k}^{(1)}\}$) is exactly the DFF (resp. the DFFz) triangle. Thus, by (2) and (3), the generating polynomial of the rows of the DFF (resp. the DFFz) triangle is the Morgan-Voyce polynomial $b_n$ (resp. $B_n$).

## 3. DETERMINATION OF THE $\{a_{n,k}^{(r)}\}$

In [1] and [2], the authors gave a very complicated formula for $\{a_{n,k}^{(0)}\}$ and $\{a_{n,k}^{(1)}\}$. We shall prove here a simpler formula that generalizes a known result [5] on the coefficients of Morgan-Voyce polynomials.

***Theorem:*** For any $n \geq 0$ and $k \geq 0$, we have

$$a_{n,k}^{(r)} = \binom{n+k}{2k} + r\binom{n+k}{2k+1}, \tag{9}$$

where $\binom{a}{b} = 0$ if $b > a$.

***Proof:*** If $k = 0$, the theorem is true by (4). Assume the theorem is true for $k-1$. We shall proceed by induction on $n$. Equality (9) holds for $n = 0$ and $n = 1$ by definition of the sequence

$\{a_{n,k}^{(r)}\}$. Assume that $n \geq 2$, and that (9) holds for the indices $n-2$ and $n-1$. By (7), we then have $a_{n,k}^{(r)} = 2a_{n-1,k}^{(r)} - a_{n-2,k}^{(r)} + a_{n-1,k-1}^{(r)} = X_{n,k} + rY_{n,k}$, where

$$X_{n,k} = 2\binom{n+k-1}{2k} - \binom{n+k-2}{2k} + \binom{n+k-2}{2k-2} \text{ and } Y_{n,k} = 2\binom{n+k-1}{2k+1} - \binom{n+k-2}{2k+1} + \binom{n+k-2}{2k-1}.$$

Recall that

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1} = \binom{a-2}{b} + 2\binom{a-2}{b-1} + \binom{a-2}{b-2}.$$

From this, we have

$$\begin{aligned} X_{n,k} &= 2\left(\binom{n+k-2}{2k} + \binom{n+k-2}{2k-1}\right) - \binom{n+k-2}{2k} + \binom{n+k-2}{2k-2} \\ &= \binom{n+k-2}{2k} + 2\binom{n+k-2}{2k-1} + \binom{n+k-2}{2k-2} = \binom{n+k}{2k}. \end{aligned}$$

In the same way, one can show that $Y_{n,k} = \binom{n+k}{2k+1}$; this completes the proof.

The following particular cases have been known for a long time (see [5]). If $r = 0$ (DFF triangle and Morgan-Voyce polynomial $b_n$), then

$$a_{n,k}^{(0)} = \binom{n+k}{2k}$$

and, if $r = 1$ (DFFz triangle and Morgan-Voyce polynomial $B_n$), then

$$a_{n,k}^{(1)} = \binom{n+k}{2k} + \binom{n+k}{2k+1} = \binom{n+k+1}{2k+1}.$$

***Remark:*** The sequence $w_n = P_n^{(r)}(1)$ satisfies the recurrence relation $w_n = 3w_{n-1} - w_{n-2}$. On the other hand, the sequence $\{F_{2n}\}$, where $F_n$ denotes the usual Fibonacci number, satisfies the same relation. From this, it is easily verified that

$$P_n^{(r)}(1) = F_{2n+2} + (r-1)F_{2n} = F_{2n+1} + rF_{2n}.$$

For instance, we have two known results (see [1] and [2]), $P_n^{(0)}(1) = F_{2n+1}$ and $P_n^{(1)}(1) = F_{2n+2}$. We also get a new result,

$$P_n^{(2)}(1) = F_{2n+2} + F_{2n} = L_{2n+1},$$

where $L_n$ is the usual Lucas number.

## 4. MORGAN-VOYCE AND CHEBYSHEV POLYNOMIALS

Let us recall that the Chebyshev polynomials of the *second* kind, $\{U_n(w)\}$, are defined by the recurrence relation

$$U_n(\omega) = 2\omega U_{n-1}(\omega) - U_{n-2}(\omega), \tag{10}$$

with initial conditions $U_0(\omega) = 0$ and $U_1(\omega) = 1$. It is clear that the sequence $\{P_n^{(r)}(2\omega - 2)\}$ satisfies (10). Comparing the initial conditions, we obtain

$$P_n^{(r)}(2\omega-2) = U_{n+1}(\omega) + (r-1)U_n(\omega).$$

If $\omega = \cos t,\ 0 < t < \pi$, it is well known that

$$U_n(\omega) = \frac{\sin(nt)}{\sin t}.$$

Thus, we have

$$P_n^{(r)}(2\omega-2) = \frac{\sin(n+1)t + (r-1)\sin nt}{\sin t}.$$

From this, we get the following formulas, where $\omega = \cos t = (x+2)/2$,

$$b_n(x) = P_n^{(0)}(x) = \frac{\cos(2n+1)t/2}{\cos t/2}, \tag{11}$$

$$B_n(x) = P_n^{(1)}(x) = \frac{\sin(n+1)t}{\sin t}. \tag{12}$$

Formulas (11) and (12) were first given by Swamy [6]. We also have a similar formula for $P_n^{(2)}(x)$, namely,

$$P_n^{(2)}(x) = \frac{\sin(2n+1)t/2}{\sin t/2}. \tag{13}$$

From (11) and (12), we see that the zeros $x_k$ (resp. $y_k$) of the polynomial $b_n$ (resp. $B_n$) are given by (see [6])

$$x_k = -4\sin^2\left(\frac{k\pi}{2n+2}\right), k = 1, 2, \ldots, n, \text{ and } y_k = -4\sin^2\left(\frac{(2k-1)\pi}{4n+2}\right), k = 1, 2, \ldots, n.$$

Similarly, the zeros $z_k$ of the polynomial $P_n^{(2)}(x)$ are given by

$$z_k = -4\sin^2\left(\frac{k\pi}{2n+1}\right), k = 1, 2, \ldots, n.$$

## REFERENCES


1. G. Ferri, M. Faccio, & A. D'Amico. "A New Numerical Triangle Showing Links with Fibonacci Numbers." *The Fibonacci Quarterly* **29.4** (1991):316-20.
2. G. Ferri, M. Faccio, & A. D'Amico. "Fibonacci Numbers and Ladder Network Impedance." *The Fibonacci Quarterly* **30.1** (1992):62-67.
3. J. Lahr. "Fibonacci and Lucas Numbers and the Morgan-Voyce Polynomials in Ladder Networks and in Electrical Line Theory." In *Fibonacci Numbers and Their Applications*, ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam, **I**:141-61. Dordrecht: Kluwer, 1986.
4. A. M. Morgan-Voyce. "Ladder Networks Analysis Using Fibonacci Numbers." *I.R.E. Trans. Circuit Theory* **6.3** (1959):321-22.
5. M. N. S. Swamy. "Properties of the Polynomial Defined by Morgan-Voyce." *The Fibonacci Quarterly* **4.1** (1966):73-81.
6. M. N. S. Swamy. "Further Properties of Morgan-Voyce Polynomials." *The Fibonacci Quarterly* **6.2** (1968):166-75.


AMS Classification Number: 11B39

# ANOTHER INSTANCE OF THE GOLDEN RIGHT TRIANGLE

**Ian Bruce**
Department of Physics and Math. Physics, University of Adelaide, Adelaide, S. Australia, P.C. 5001
(*Submitted October 1992*)

The golden ratio $\tau = (1+\sqrt{5})/2$, the positive root of $x^2 = x+1$, makes an unexpected appearance in [1], where a certain right triangle turns out to be a "Golden Right Triangle" (GRT), one having sides proportional to $(1, \tau^{1/2}, \tau)$. The author wonders about the existence of other sets of circumstances where the GRT makes an unexpected appearance. In this note, such an occasion arises.

Suppose you are given two line segments of length $a$ and $b$, then consider the problem in Euclidean geometry of constructing a right-angled triangle with hypotenuse of length equal to the arithmetic mean ($\overline{A}$) of $a$ and $b$, that is,

$$\overline{A} = \frac{a+b}{2},$$

and one other side equal to their geometric mean ($\overline{G}$), that is

$$\overline{G} = \sqrt{ab}.$$

This problem is readily solved, as indicated in Figure 1.

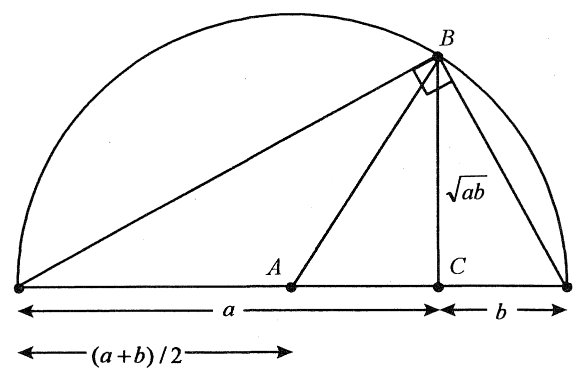


**Figure 1**

If we now demand that the shortest side $AC$ of $\triangle ABC$ in Figure 1 is the harmonic mean ($\overline{H}$) of $a$ and $b$, that is, $2/\overline{H} = 1/a + 1/b$, then we obtain the triangle with sides of length indicated in Figure 2. The problem now is to determine $a$ and $b$ so the lengths are proportional to $(\overline{A}, \overline{G}, \overline{H})$.

We can set $b=1$ without loss of generality, and apply Pythagoras' Theorem to obtain, after some algebra: $a^4 - 18a^2 + 1 = 0$, giving $a^2 = 9 \pm 4\sqrt{5} = \tau^6$ and $1/\tau^6$ for positive and negative signs, respectively.

For the positive sign, and the larger root, we have $a = \tau^3 = 2\tau + 1$. In this case the required triangle has sides of length

$$\left((\tau+1), (2\tau+1)^{1/2}, \frac{(2\tau+1)}{(\tau+1)}\right).$$

This can be written as $(\tau^2, \tau^{3/2}, \tau)$, which is proportional to $(\tau, \tau^{1/2}, 1)$, the GRT above.

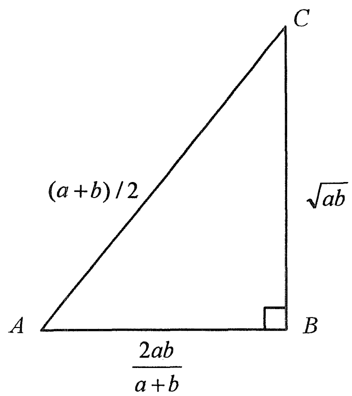


**Figure 2**

For the negative sign, and smaller root, we have $a = 1/\tau^3$ and $b = 1$. This results in a triangle with sides of length

$$\left(\frac{(\tau+1)}{(2\tau+1)}, \frac{1}{(2\tau+1)^{1/2}}, \frac{1}{(\tau+1)}\right).$$

Again, this is proportional to GRT, the side lengths being reciprocal to those above.

Here, then, is another situation in which the golden ratio makes an unexpected appearance.

## REFERENCE

1. Duane W. DeTemple. "The Triangle of Smallest Perimeter Which Circumscribes a Given Semicircle." *The Fibonacci Quarterly* **30.3** (1992):274.

AMS Classification Numbers: 51M04, 11B39

# INFINITE PRODUCTS AND FIBONACCI NUMBERS

**Don Redmond**
Department of Mathematics, Southern Illinois University, Carbondale, IL 62901-4408
*(Submitted October 1992)*

In this paper we wish to describe how certain identities for infinite products lead to some striking infinite products involving terms of binary recurrences.

## 1. INFINITE PRODUCTS

We begin with the result on infinite products.

***Theorem 1:*** If $|x|<1$, $S$ is a set of positive integers and $h$ and $g$ are functions such that $|g(x)|$, $|h(x)|< Cx^{\alpha}$ for all $x$, where $C>0$ and $\alpha \geq 0$ are constants, then

$$\prod_{k\in S}(1+x^k)^{g(k)/k}(1-x^k)^{h(k)/k} = \exp\left\{-\sum_{n=1}^{+\infty}\sum_{\substack{d\mid n\\ d\in S}}\left(h(d)+(-1)^{n/d}g(d)\right)\frac{x^n}{n}\right\}.$$

***Proof:*** Let

$$F(x)=\prod_{k\in S}(1+x^k)^{g(k)/k}(1-x^k)^{h(k)/k}.$$

Note that the infinite product converges absolutely for $|x|<1$. Then

$$\begin{aligned}\log F(x) &= \sum_{k\in S}\left\{\frac{g(k)}{k}\log(1+x^k)+\frac{h(k)}{k}\log(1-x^k)\right\}\\ &= \sum_{k\in S}\frac{g(k)}{k}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}x^{nk}}{n}-\sum_{k\in S}\frac{h(k)}{k}\sum_{n=1}^{+\infty}\frac{x^{nk}}{n}.\end{aligned}$$

Since $|x|<1$ and $g(k)$ and $h(k)$ are bounded by powers of $k$, we see that the two double series converge absolutely, and so we may interchange the order of summation. We obtain

$$\begin{aligned}\log F(x) &= -\sum_{n=1}^{+\infty}\frac{x^n}{n}\sum_{\substack{d\mid n\\ d\in S}}(-1)^{n/d}g(d)-\sum_{n=1}^{+\infty}\frac{x^n}{n}\sum_{\substack{d\mid n\\ d\in S}}h(d)\\ &= -\sum_{n=1}^{+\infty}\frac{x^n}{n}\sum_{\substack{d\mid n\\ d\in S}}\left(h(d)+(-1)^{n/d}g(d)\right).\end{aligned}$$

If we exponentiate, the result follows.

The following two corollaries are the results we will be using in what follows. In the first corollary, we take $S$ to be the set of odd integers and $g=-h=f$, where $f$ is any function that satisfies the order of magnitude bound on Theorem 1. In the second corollary, we take $S$ to be the set of natural numbers and $g=-h=f$ as before.

***Corollary 1.1:*** Under the hypotheses of Theorem 1, we have

$$\sum_{k=0}^{+\infty}\left(\frac{1+x^{2k+1}}{1-x^{2k+1}}\right)^{f(2k+1)/(2k+1)}=\exp\left\{2\sum_{k=0}^{+\infty}\left(\sum_{d\mid 2k+1}f(d)\right)\frac{x^{2k+1}}{2k+1}\right\}.$$

***Corollary 1.2:*** Under the hypotheses of Theorem 1, we have

$$\sum_{k=1}^{+\infty}\left(\frac{1+x^{k}}{1-x^{k}}\right)^{f(k)/k}=\exp\left\{\sum_{n=1}^{+\infty}\left(\sum_{d\mid n}f(d)\left(1-(-1)^{n/d}\right)\right)\frac{x^{n}}{n}\right\}.$$

## 2. BINARY RECURSIONS

Consider the binary recursion relation

$$u_{n+2}=au_{n+1}+bu_n,\ n\ge 0, \tag{1}$$

where $u_0$ and $u_1$ are some given values. Let $\alpha$ and $\beta$ be the roots of $x^2-ax-b=0$, where we take

$$\alpha=\frac{a+\sqrt{a^2+4b}}{2}\quad\text{and}\quad\beta=\frac{a-\sqrt{a^2+4b}}{2}.$$

If we assume $a>0$ and $a^2+4b>0$, then we have that

$$|\beta/\alpha|<1. \tag{2}$$

Let $\{P_n\}$ be the solution to the recursion (1) with initial conditions $P_0=0$ and $P_1=1$. Then it is well known that we may write

$$P_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}. \tag{3}$$

If we let $\{Q_n\}$ be the solution to (1) with $Q_0=2$ and $Q_1=a$, then we have

$$Q_n=\alpha^n+\beta^n. \tag{4}$$

The most well known of these sequences are the Fibonacci and Lucas numbers that satisfy (1) with $a=b=1$. In this case,

$$\alpha=\frac{1+\sqrt{5}}{2}\quad\text{and}\quad\beta=\frac{1-\sqrt{5}}{2}.$$

## 3. SOME ARITHMETIC FUNCTIONS

In our applications of Corollaries 1.1 and 1.2, we will take $f$ to be some well-known arithmetic functions, namely, the Euler function, $\varphi$, and the Möbius function, $\mu$. The reason for discussing these two function is that they have the following well-known properties:

$$\sum_{d\mid n}\varphi(d)=n \tag{5}$$

and

$$\sum_{d|n}\mu(d)=\begin{cases}1 & n=1,\\ 0 & n>1.\end{cases} \tag{6}$$

These two results allow us to easily sum the infinite series that appear on the right-hand sides of Corollaries 1.1 and 1.2. Unfortunately, not many other arithmetic functions have such simple sums as in (5) and (6).

A generalization of the Euler function, namely, the Jordan functions, $J_k$, satisfies

$$\sum_{d|n}J_k(d)=n^k,$$

but this leads us to sums of the form

$$\sum_{n=1}^{+\infty}n^{k-1}x^n,$$

which have closed-form expressions of the form

$$\frac{P_k(x)}{(1-x)^k},$$

where $P_k$ is a polynomial. For general $k$, the polynomial $P_k$ is not that tractable, and so we have chosen to go with just the Euler function.

A function that generalizes both the Euler function and the Möbius function is the Ramanujan sum, $c_n(m)$, which can be defined by

$$c_n(m)=\sum_{d|(n,m)}d\mu(u/d).$$

Then we have $c_n(1)=\mu(n)$ and $c_n(0)=\varphi(n)$. The Ramanujan sum has the nice property that

$$\sum_{d|n}c_d(m)=\begin{cases}n & n|m,\\ 0 & \text{otherwise.}\end{cases}$$

If we use this in the corollaries, we end up with sums of the form

$$\sum_{d|m}x^d,$$

which are easy to deal with for individual $m$, but not in general.

Therefore, in what follows, we shall restrict ourselves to the use of only the Euler and Möbius functions.

## 4. APPLICATION OF COROLLARY 1.1

If we let $f=\varphi$ or $\mu$, then, since $\varphi(n)\le n$ and $|\mu(n)|\le 1$, we see that we can use either of these choices in Corollary 1.1. If $|x|<1$, then we have, by (5),

$$\prod_{k=0}^{+\infty}\left(\frac{1+x^{2k+1}}{1-x^{2k+1}}\right)^{\varphi(2k+1)/(2k+1)}=\exp\left\{2\sum_{n=0}^{+\infty}\frac{x^{2n+1}}{2n+1}\sum_{d|2n+1}\varphi(d)\right\}=\exp\left\{2\sum_{n=0}^{+\infty}x^{2n+1}\right\}=\exp\left(\frac{2x}{1-x^2}\right). \tag{7}$$

Similarly, if we use (6), we obtain, for $|x|<1$,

$$\prod_{k=0}^{+\infty}\left(\frac{1+x^{2k+1}}{1-x^{2k+1}}\right)^{\mu(2k+1)/(2k+1)} = e^{2x}. \tag{8}$$

***Theorem 2:*** We have

$$\prod_{k=0}^{+\infty}\left(\frac{Q_{2k+1}}{(\alpha-\beta)P_{2k+1}}\right)^{\varphi(2k+1)/(2k+1)} = \exp\left(\frac{-2b}{a\sqrt{a^2+4b}}\right) \tag{9}$$

and

$$\prod_{k=0}^{+\infty}\left(\frac{Q_{2k+1}}{(\alpha-\beta)P_{2k+1}}\right)^{\mu(2k+1)/(2k+1)} = \exp\left(\frac{a^2+2b-a\sqrt{a^2+4b}}{-b}\right). \tag{10}$$

***Proof:*** Let $x=\beta/\alpha$. By (2), we see that $|x|<1$, and so we can use (7) and (8). We have

$$\prod_{k=0}^{+\infty}\left(\frac{1+(\beta/\alpha)^{2k+1}}{1-(\beta/\alpha)^{2k+1}}\right)^{f(2k+1)/(2k+1)} = \prod_{k=0}^{+\infty}\left(\frac{\alpha^{2k+1}+\beta^{2k+1}}{\alpha^{2k+1}-\beta^{2k+1}}\right)^{f(2k+1)/(2k+1)}$$

$$= \prod_{k=0}^{+\infty}\left(\frac{Q_{2k+1}}{(\alpha-\beta)P_{2k+1}}\right)^{f(2k+1)/(2k+1)}.$$

Taking $f=\varphi$ and $\mu$ gives the left-hand sides of (9) and (10), respectively.

If we put $x=\beta/\alpha$ into the right-hand side of (7), we obtain

$$\frac{2(\beta/\alpha)}{1-(\beta/\alpha)^2} = \frac{2\alpha\beta}{(\alpha^2-\beta^2)} = \frac{2(-b)}{(\alpha-\beta)P_2} = \frac{-2b/a}{\alpha-\beta} = \frac{-2b}{a\sqrt{a^2+4b}},$$

which completes the proof of (9).

To prove (10), we put $x=\beta/\alpha$ into the right-hand side of (8) and obtain

$$2\left(\frac{\beta}{\alpha}\right) = \frac{a^2+2b-a\sqrt{a^2+4b}}{-b},$$

which proves (10) and completes the proof of Theorem 2.

If we take $a=b=1$ to obtain the Fibonacci and Lucas sequences, we get the following corollary.

***Corollary 2.1:*** We have

$$\prod_{k=0}^{+\infty}\left(\frac{L_{2k+1}}{\sqrt{5}F_{2k+1}}\right)^{\varphi(2k+1)/(2k+1)} = e^{-2\sqrt{5}}$$

and

$$\prod_{k=0}^{+\infty}\left(\frac{L_{2k+1}}{\sqrt{5}F_{2k+1}}\right)^{\mu(2k+1)/(2k+1)} = e^{-3+\sqrt{5}}.$$

## 5. AN IDENTITY FOR MULTIPLICATIVE FUNCTIONS

***Theorem 3:*** Let $f$ be a multiplicative function.

***1)*** If $n$ is odd, then

$$\sum_{d|n}(-1)^{n/d}f(d)=-\sum_{d|n}f(d).$$

***2)*** If $n$ is even, $n=2^s m$, $s\ge 1$, and $m$ is odd, then

$$\sum_{d|n}(-1)^{n/d}f(d)=\sum_{d|n}f(d)-2f(2^s)\sum_{s|m}f(s).$$

***Proof:*** If $n$ is odd and $d|n$, then $n/d$ is also odd. Thus, if $n$ is odd, we have

$$\sum_{d|n}(-1)^{n/d}f(d)=\sum_{d|n}(-1)f(d)=-\sum_{d|n}f(d),$$

which proves 1).

Suppose $n$ is even and write $n=2^s m$, where $s\ge 1$ and $m$ is an odd integer. Then

$$\sum_{d|n}(-1)^{n/d}f(d)=\sum_{\substack{d|n\\ n/d\text{ even}}}f(d)-\sum_{\substack{d|n\\ n/d\text{ odd}}}f(d)=\sum_{d|n}f(d)-2\sum_{\substack{d|n\\ n/d\text{ odd}}}f(d).$$

Now if $d|n$ and $n/d$ is odd, we can write $d=2^s\delta$, where $\delta|m$. Thus,

$$\sum_{d|n}(-1)^{n/d}f(d)=\sum_{d|n}f(d)-2\sum_{\delta|m}f(2^s\delta).$$

Since $f$ is multiplicative, we can write $f(2^s\delta)=f(2^s)f(\delta)$ and this gives 2) and completes the proof of the theorem.

The following corollary is just a rewriting of Theorem 3 in a form applicable to Corollary 1.2.

***Corollary 3.1:*** Let $f$ be a multiplicative function. Then, with the notation of Theorem 3, we have

$$\sum_{d|n}f(d)\left(1-(-1)^{n/d}\right)=\begin{cases}2\sum_{d|n}f(d) & \text{if } n \text{ is odd,}\\ 2f(2^s)\sum_{d|m}f(d) & \text{if } n=2^s m \text{ is even.}\end{cases}$$

We now apply the corollary to our specific choices of function, namely, $\varphi(n)$ and $\mu(n)$. Since both of these are multiplicative, we can apply Corollary 3.1 to obtain the following result.

***Corollary 3.2:*** We have

$$\sum_{d|n}\varphi(d)\left(1-(-1)^{n/d}\right)=\begin{cases}2n & \text{if } n \text{ is odd,}\\ n & \text{if } n \text{ is even,}\end{cases}$$

and

$$\sum_{d|n}\mu(d)\left(1-(-1)^{n/d}\right)=\begin{cases}2 & \text{if } n=1,\\ -2 & \text{if } n=2,\\ 0 & \text{if } n>2.\end{cases}$$

***Proof:*** If $n$ is odd, then we have

$$2\sum_{d|n}\varphi(d)=2n,$$

and if $n=2^s m$ is even, with $s\geq 1$ and $m$ odd, then

$$2\varphi(2^s)\sum_{\delta|m}\varphi(\delta)=2\cdot 2^{s-1}\cdot m=2^s m=n.$$

This proves (11).

If $n$ is odd, then we have

$$2\sum_{d|n}\mu(d)=\begin{cases}2\cdot 1=2 & \text{if } n=1,\\ 0 & \text{if } n>1.\end{cases}$$

If $n=2^s m$ is even, then

$$2\mu(2^s)\sum_{\delta|m}\mu(\delta)=\begin{cases}2\mu(2^s) & \text{if } m=1,\\ 0 & \text{if } m>1,\end{cases}$$

and

$$\mu(2^s)=\begin{cases}-1 & \text{if } s=1,\\ 0 & \text{if } s>1.\end{cases}$$

If we combine these last two results, we see that

$$2\mu(2^s)\sum_{\delta|m}\mu(\delta)=\begin{cases}-2 & \text{if } s=1,\ m=1,\\ 0 & \text{otherwise.}\end{cases}$$

This proves (12) and completes the proof of the corollary.

## 6. APPLICATION OF COROLLARY 1.2

If we proceed as we did in section 4 and now apply Corollary 3.2, we obtain the following theorem and corollary.

***Theorem 4:*** We have

$$\prod_{k=1}^{+\infty}\left(\frac{Q_k}{(\alpha-\beta)P_k}\right)^{\varphi(k)/k}=\exp\left(\frac{\beta^2+2\alpha\beta}{\alpha^2-\beta^2}\right)\quad\text{and}\quad\prod_{k=1}^{+\infty}\left(\frac{Q_k}{(\alpha-\beta)P_k}\right)^{\mu(k)/k}=\exp\left(\frac{2\alpha\beta-\beta^2}{\alpha^2}\right).$$

***Corollary 4.1:*** We have

$$\prod_{k=1}^{+\infty}\left(\frac{L_k}{\sqrt{5}F_k}\right)^{\varphi(k)/k}=e^{-(1+\sqrt{5})/2\sqrt{5}}\quad\text{and}\quad\prod_{k=1}^{+\infty}\left(\frac{L_k}{\sqrt{5}F_k}\right)^{\mu(k)/k}=e^{(-13+5\sqrt{5})/2}.$$

AMS Classification Numbers: 11B37, 11B39, 11Y60

# MAXIMAL REPRESENTATIONS OF POSITIVE INTEGERS BY PELL NUMBERS

**A. F. Horadam**
The University of New England, Armidale, Australia 2351
(*Submitted October 1992*)

## 1. INTRODUCTION

In [4], the unique *Zeckendorf representations* of positive and negative integers by distinct Pell numbers was *minimal*, i.e., the number of terms in each representational sum was the least possible.

Here we show how to represent positive integers maximally by means of Pell numbers. That is, each positive integer is to be given as a sum in a *maximal* representation by using the greatest number of terms involving distinct Pell numbers (see Table 1).

Short tables for minimal and maximal representations of positive integers in terms of (i) Fibonacci numbers and (ii) Lucas numbers, are given in [3].

Our theory for Pell numbers will be analogous to that used for Fibonacci numbers in [1], where a "Dual-Zeckendorf theorem" is established. Enough variations and complications exist, however, to make this investigation worthwhile *per se*. (Theorems for Lucas numbers corresponding to those for Fibonacci numbers may be found in [2].)

Positive *Pell numbers* are defined by the recurrence

$$P_{n+2} = 2P_{n+1} + P_n, \; n \geq 0, \tag{1.1}$$

with

$$P_0 = 0, \; P_1 = 1. \tag{1.2}$$

Thus, the first few Pell numbers are

$$\begin{cases} n = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ P_n = & 0 & 1 & 2 & 5 & 12 & 29 & 70 & 169 & 408 & 985 & \cdots \end{cases} \tag{1.3}$$

Repeated use of (1.1) leads to

$$P_{k+1} = 2(P_k + P_{k-2} + P_{k-4} + \cdots + P_{k-2t+2}) + P_{k-2t+1}, \tag{1.4}$$

in which

$$\begin{cases} t = 1, 2, \ldots, \frac{k}{2} & k \text{ even}, \\ t = 1, 2, \ldots, \frac{k+1}{2} & k \text{ odd}. \end{cases} \tag{1.5}$$

Consequently,

$$P_{k+1} - 1 = 2(P_k + P_{k-2} + P_{k-4} + \cdots + P_4 + P_2) \quad k \text{ even} \tag{1.6}$$

and

$$P_{k+1} - 1 = 2(P_k + P_{k-2} + P_{k-4} + \cdots + P_1) - 1 \quad k \text{ odd}. \tag{1.7}$$

Also, by repeated use of (1.1),

$$\sum_{i=1}^{k-1} P_i = \begin{cases} \frac{P_k + P_{k-1} - 1}{2} & k \geq 2, \\ \frac{P_{k+1} - P_k - 1}{2} & k \geq 2. \end{cases} \tag{1.8}$$

## 2. MAXIMAL REPRESENTATION THEOREM

***Theorem:*** Every positive integer $n$ has a unique representation in the form

$$n = \sum_{i=1}^{k} \beta_i P_i \quad (\beta_i = 0, 1, \text{ or } 2), \tag{2.1}$$

where

$$\beta_i = 0 \Rightarrow \beta_{i-1} = 2 \quad (2 \leq i < k) \tag{2.2}$$

and

$$\beta_k = 1 \text{ or } 2. \tag{2.3}$$

For a given $n$ in (2.1), the unique integer $k$ satisfies

$$P_k + P_{k-1} - 1 < n \leq P_{k+1} + P_k - 1. \tag{2.4}$$

***Proof:***

***(i)*** *Maximality*. Suppose $n$ satisfies (2.4). Then, equivalently from (1.8),

$$2\sum_{i=1}^{k-1} P_i < n \leq 2\sum_{i=1}^{k} P_i. \tag{2.5}$$

But, by the Zeckendorf theorem [1] for a positive integer, namely, the left-hand side of (2.5), we have

$$2\sum_{i=1}^{k} P_i - n = \sum_{i=1}^{\infty} \alpha_i P_i \quad (\geq 0), \tag{2.6}$$

with

$$\alpha_i = 0, 1, \text{ or } 2 \text{ and } \alpha_i = 2 \Rightarrow \alpha_{i-1} = 0 \quad (i \geq 1). \tag{2.7}$$

Now, from (2.5),

$$2\sum_{i=1}^{k} P_i - n < 2\left(\sum_{i=1}^{k} P_i - \sum_{i=1}^{k-1} P_i\right) = 2P_k. \tag{2.8}$$

This implies $\alpha_i = 0$ in (2.6) for $i > k$, and $\alpha_k = 0$ or 1. Consequently, (2.6) can be rewritten as

$$2\sum_{i=1}^{k} P_i - n = \sum_{i=1}^{k} \alpha_i P_i \quad (\alpha_k = 0 \text{ or } 1) \tag{2.9}$$

which, in turn, may be expressed as

$$n = \sum_{i=1}^{k} (2 - \alpha_i) P_i = \sum_{i=1}^{k} \beta_i P_i \quad (\beta_k = 1 \text{ or } 2), \tag{2.10}$$

where

$$\beta_i = 2 - \alpha_i \quad (i = 1, 2, \ldots, k) \tag{2.11}$$

in accordance with the statements (2.1) and (2.3) of the Theorem.

Lastly, by (2.11), the characteristic Zeckendorf condition (2.7) for Pell numbers becomes $\beta_i = 0 \Rightarrow \beta_{i-1} = 2$, which confirms the requirement (2.2) in the enunciation of our Theorem.

***(ii)*** *Uniqueness.* Assume that the positive integer $n$ has two different representations

$$n = \sum_{i=1}^{m} \beta_i P_i = \sum_{i=1}^{m'} \beta_i' P_i, \tag{2.12}$$

where $\beta_m, \beta'_{m'} = 1$ or 2 and $\beta_i = 0 \Rightarrow \beta_{i-1} = 2$ for $i = 2, 3, \ldots, m-1$ and $\beta_i' = 0 \Rightarrow \beta_{i-1}' = 2$ for $i = 2, 3, \ldots, m' - 1$.

Suppose $m > m'$. Now

$$\begin{aligned} \sum_{i=1}^{m} \beta_i P_i &\geq \begin{cases} 2(P_m + P_{m-2} + \cdots + P_2) & m \text{ even} \\ 2(P_m + P_{m-2} + \cdots + P_1) - 1 & m \text{ odd} \end{cases} \\ &\geq P_{m+1} - 1, \end{aligned} \tag{2.13}$$

by (1.6) and (1.7), whereas

$$\begin{aligned} \sum_{i=1}^{m'} \beta_i' P_i \leq 2\sum_{i=1}^{m'} P_i \leq 2\sum_{i=1}^{m-1} P_i &= P_{m+1} - P_m - 1 \quad \text{by (1.8)} \\ &< P_{m+1} - 1. \end{aligned} \tag{2.14}$$

Conclusions (2.13) and (2.14) involve a contradiction. Similarly for $m < m'$.

Hence, $m' = m$.

If $\alpha_i = 2 - \beta_i$, $\alpha_i' = 2 - \beta_i'$ $(i = 1, 2, \ldots, m)$, then (2.12) leads to

$$\sum_{i=1}^{m} (2 - \alpha_i) P_i = \sum_{i=1}^{m} (2 - \alpha_i') P_i, \tag{2.15}$$

where $\alpha_i = 2 \Rightarrow \alpha_{i-1} = 0$, $\alpha_i' = 2 \Rightarrow \alpha_{i-1}' = 0$, whence

$$\sum_{i=1}^{m} \alpha_i P = \sum_{i=1}^{m} \alpha_i' P_i. \tag{2.16}$$

Both sides of (2.16) are Zeckendorf representations of positive integers by Pell numbers, the uniqueness of which [4] yields $\alpha_i' = \alpha_i$.

Thus, $\beta_i' = \beta_i$.

Consequently, the uniqueness of (2.1) with (2.2) and (2.3) is demonstrated.

***Remarks:***

***(a)*** Implications (2.2) and (2.7), which characterize the representations, are one-way only.

***(b)*** As a numerical illustration of (2.4), take $n = 25$. Then

$$P_4 + P_3 - 1\,(=16) < 25 \leq P_5 + P_4 - 1\,(=40)$$

so that $k = 4$ here. Likewise, when $n = 999$, then $k = 8$.

*(c)* Integers having identical maximal and minimal representations are worthy of a separate investigation. Please see the Concluding Remarks.

## 3. CONCLUDING REMARKS

In Table 1, which may be extended indefinitely, the pattern of digits 0 (blank space), 1, and 2 reveals the visible mechanism of the representation. Two successive zeros do not occur in this table. Observe that in the maximal representations (Table 1) we write $2\ (= P_2) = 2P_1\ (= 2P_1 + P_0)$, whereas in the minimal representations (Table 2) we retain $2 = P_2$.

For the *Pell-Lucas numbers* $Q_n$, defined by the recurrence relation

$$Q_{n+2} = 2Q_{n+1} + Q_n \quad (n \geq 0) \tag{3.1}$$

with

$$Q_0 = 2,\ Q_1 = 2, \tag{3.2}$$

we observe that they are all even. It follows that there can be no representation of integers, maximal or minimal, involving Pell-Lucas numbers, since odd integers would necessarily be excluded.

**TABLE 1. Maximal Representations of Positive Integers by Sums of Pell Numbers**

| $n^+$ | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | 26 | 2 | 1 | 2 | 1 | | 51 | 1 | 2 | 1 | 1 | 1 |
| 2 | 2 | | | | | 27 | 1 | 2 | 2 | 1 | | 52 | 2 | 2 | 1 | 1 | 1 |
| 3 | 1 | 1 | | | | 28 | 2 | 2 | 2 | 1 | | 53 | 2 | | 2 | 1 | 1 |
| 4 | 2 | 1 | | | | 29 | 1 | 2 | | 2 | | 54 | 1 | 1 | 2 | 1 | 1 |
| 5 | 1 | 2 | | | | 30 | 2 | 2 | | 2 | | 55 | 2 | 1 | 2 | 1 | 1 |
| 6 | 2 | 2 | | | | 31 | 2 | | 1 | 2 | | 56 | 1 | 2 | 2 | 1 | 1 |
| 7 | 2 | | 1 | | | 32 | 1 | 1 | 1 | 2 | | 57 | 2 | 2 | 2 | 1 | 1 |
| 8 | 1 | 1 | 1 | | | 33 | 2 | 1 | 1 | 2 | | 58 | 1 | 2 | | 2 | 1 |
| 9 | 2 | 1 | 1 | | | 34 | 1 | 2 | 1 | 2 | | 59 | 2 | 2 | | 2 | 1 |
| 10 | 1 | 2 | 1 | | | 35 | 2 | 2 | 1 | 2 | | 60 | 2 | | 1 | 2 | 1 |
| 11 | 2 | 2 | 1 | | | 36 | 2 | | 2 | 2 | | 61 | 1 | 1 | 1 | 2 | 1 |
| 12 | 2 | | 2 | | | 37 | 1 | 1 | 2 | 2 | | 62 | 2 | 1 | 1 | 2 | 1 |
| 13 | 1 | 1 | 2 | | | 38 | 2 | 1 | 2 | 2 | | 63 | 1 | 2 | 1 | 2 | 1 |
| 14 | 2 | 1 | 2 | | | 39 | 1 | 2 | 2 | 2 | | 64 | 2 | 2 | 1 | 2 | 1 |
| 15 | 1 | 2 | 2 | | | 40 | 2 | 2 | 2 | 2 | | 65 | 2 | | 2 | 2 | 1 |
| 16 | 2 | 2 | 2 | | | 41 | 2 | | 2 | | 1 | 66 | 1 | 1 | 2 | 2 | 1 |
| 17 | 1 | 2 | | 1 | | 42 | 1 | 1 | 2 | | 1 | 67 | 2 | 1 | 2 | 2 | 1 |
| 18 | 2 | 2 | | 1 | | 43 | 2 | 1 | 2 | | 1 | 68 | 1 | 2 | 2 | 2 | 1 |
| 19 | 2 | | 1 | 1 | | 44 | 1 | 2 | 2 | | 1 | 69 | 2 | 2 | 2 | 2 | 1 |
| 20 | 1 | 1 | 1 | 1 | | 45 | 2 | 2 | 2 | | 1 | 70 | 2 | | 2 | | 2 |
| 21 | 2 | 1 | 1 | 1 | | 46 | 1 | 2 | | 1 | 1 | 71 | 1 | 1 | 2 | | 2 |
| 22 | 1 | 2 | 1 | 1 | | 47 | 2 | 2 | | 1 | 1 | 72 | 2 | 1 | 2 | | 2 |
| 23 | 2 | 2 | 1 | 1 | | 48 | 2 | | 1 | 1 | 1 | 73 | 1 | 2 | 2 | | 2 |
| 24 | 2 | | 2 | 1 | | 49 | 1 | 1 | 1 | 1 | 1 | 74 | 2 | 2 | 2 | | 2 |
| 25 | 1 | 1 | 2 | 1 | | 50 | 2 | 1 | 1 | 1 | 1 | 75 | 1 | 2 | | 1 | 2 |

**TABLE 2. Minimal Representations of Positive Integers by Sums of Pell Numbers**

| $n^+$ | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ | | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ | | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | 26 | | 1 | | 2 | | | 51 | | | 2 | 1 | 1 | |
| 2 | | 1 | | | | | 27 | 1 | 1 | | 2 | | | 52 | 1 | | 2 | 1 | 1 | |
| 3 | 1 | 1 | | | | | 28 | | 2 | | 2 | | | 53 | | | | 2 | 1 | |
| 4 | | 2 | | | | | 29 | | | | | 1 | | 54 | 1 | | | 2 | 1 | |
| 5 | | | 1 | | | | 30 | 1 | | | | 1 | | 55 | | 1 | | 2 | 1 | |
| 6 | 1 | | 1 | | | | 31 | | 1 | | | 1 | | 56 | 1 | 1 | | 2 | 1 | |
| 7 | | 1 | 1 | | | | 32 | 1 | 1 | | | 1 | | 57 | | 2 | | 2 | 1 | |
| 8 | 1 | 1 | 1 | | | | 33 | | 2 | | | 1 | | 58 | | | | | 2 | |
| 9 | | 2 | 1 | | | | 34 | | | 1 | | 1 | | 59 | 1 | | | | 2 | |
| 10 | | | 2 | | | | 35 | 1 | | 1 | | 1 | | 60 | | 1 | | | 2 | |
| 11 | 1 | | 2 | | | | 36 | | 1 | 1 | | 1 | | 61 | 1 | 1 | | | 2 | |
| 12 | | | | 1 | | | 37 | 1 | 1 | 1 | | 1 | | 62 | | 2 | | | 2 | |
| 13 | 1 | | | 1 | | | 38 | | 2 | 1 | | 1 | | 63 | | | 1 | | 2 | |
| 14 | | 1 | | 1 | | | 39 | | | 2 | | 1 | | 64 | 1 | | 1 | | 2 | |
| 15 | 1 | 1 | | 1 | | | 40 | 1 | | 2 | | 1 | | 65 | | 1 | 1 | | 2 | |
| 16 | | 2 | | 1 | | | 41 | | | | 1 | 1 | | 66 | 1 | 1 | 1 | | 2 | |
| 17 | | | 1 | 1 | | | 42 | 1 | | | 1 | 1 | | 67 | | 2 | 1 | | 2 | |
| 18 | 1 | | 1 | 1 | | | 43 | | 1 | | 1 | 1 | | 68 | | | 2 | | 2 | |
| 19 | | 1 | 1 | 1 | | | 44 | 1 | 1 | | 1 | 1 | | 69 | 1 | | 2 | | 2 | |
| 20 | 1 | 1 | 1 | 1 | | | 45 | | 2 | | 1 | 1 | | 70 | | | | | | 1 |
| 21 | | 2 | 1 | 1 | | | 46 | | | 1 | 1 | 1 | | 71 | 1 | | | | | 1 |
| 22 | | | 2 | 1 | | | 47 | 1 | | 1 | 1 | 1 | | 72 | | 1 | | | | 1 |
| 23 | 1 | | 2 | 1 | | | 48 | | 1 | 1 | 1 | 1 | | 73 | 1 | 1 | | | | 1 |
| 24 | | | | 2 | | | 49 | 1 | 1 | 1 | 1 | 1 | | 74 | | 2 | | | | 1 |
| 25 | 1 | | | 2 | | | 50 | | 2 | 1 | 1 | 1 | | 75 | | | 1 | | | 1 |

Further references to Zeckendorf representations may be found in [4].

Finally, a natural question to ask is this: Are there any numbers for which the maximal and minimal representations are the same? Examination of Tables 1 and 2 leads us to the reasonable conviction that this situation arises only when all the coefficients of the Pell numbers in the summations are unity. That is, the required numbers are $\sum_{i=1}^{k-1} P_i$ for $k \geq 2$ [see (1.8)], namely, 1, 3, 8, 20, 49, 119, ... . Compare this with the corresponding situation for Fibonacci numbers in [3] and [5].

Properties of the sequence of numbers, 1, 3, 8, 20, 49, 119, ..., are the subject of a further research article.

## REFERENCES


1. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *The Fibonacci Quarterly* **3.1** (1965):1-8.
2. J. L. Brown, Jr. "Unique Representations of Integers as Sums of Distinct Lucas Numbers." *The Fibonacci Quarterly* **7.3** (1969):243-52.
3. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.
4. A. F. Horadam. "Zeckendorf Representations of Positive and Negative Integers by Pell Numbers." In *Applications of Fibonacci Numbers* **5**:305-16. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1993.
5. D. A. Klarner. "Representation of $N$ as a Sum of Distinct Elements from Special Sequences." *The Fibonacci Quarterly* **4.4** (1966):289-306.
6. J. Pihko. "On Fibonacci and Lucas Representations and a Theorem of Lekkerkerker." *The Fibonacci Quarterly* **26.3** (1988):256-61.


AMS Classification Numbers: 11B37, 11A67

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# ON THE INTEGRITY OF CERTAIN FIBONACCI SUMS

**Piero Filipponi and Marco Bucci**
Fondazione Ugo Bordoni, I-00142 Roma, Italy
(*Submitted October 1992*)

## 1. AIM OF THE PAPER

Some years ago we were rather surprised at the integrity of the infinite sum

$$\sum_{i=0}^{\infty} F_i / 2^i = 2 \quad (F_i \text{ the } i^{\text{th}} \text{ Fibonacci number}) \tag{1.1}$$

which was obtained in [2] as a by-product result. Our mathematical curiosity led us to investigate (see [1] and [3]) the rational values (in particular, the integral values) of $r$ for which the sum

$$\sum_{i=0}^{\infty} F_i / r^i \tag{1.2}$$

gives a positive integer.

The aim of this paper is to extend the results established in [1] and [3] by finding the sef of *all rational* values of $r$ for which the sum

$$S(r,n) = \sum_{i=0}^{\infty} F_{ni} / r^i \quad (r \neq 0) \tag{1.3}$$

[$n$ is an arbitrary natural number, $r$ is an arbitrary (nonzero) real quantity) gives a *positive* integer $k$. Since both $r$ and $k$ turn out to be Fibonacci number ratios, the results established in this paper can be viewed as a particular kind of Fibonacci identities that are believed to be new [see (4.7) and (4.8)].

Throughout the paper we shall make use of the following properties of the Fibonacci numbers and of the Lucas numbers $L_n$ which are either available in [5] and [11] or can be readily derived by using the Binet forms for $F_n$ and $L_n$:

$$F_{2n} = F_n L_n, \tag{1.4}$$

$$5F_n^2 = L_n^2 - 4(-1)^n, \tag{1.5}$$

$$L_{2n} - 2(-1)^n = 5F_n^2, \tag{1.6}$$

$$F_n \text{ divides } F_k \text{ iff } n \text{ divides } k \text{ (for } n \geq 3\text{)}, \tag{1.7}$$

$$L_n \equiv L_k \pmod 5 \text{ iff } n \equiv k \pmod 4, \tag{1.8}$$

$$L_{n+k} - (-1)^k L_{n-k} = 5F_n F_k, \tag{1.9}$$

$$L_{n+k} + (-1)^k L_{n-k} = L_n L_k. \tag{1.10}$$

## 2. THE VALUES OF *r* FOR WHICH $S(r, n)$ IS A POSITIVE INTEGER

The closed-form expression

$$S(r,n)=\frac{rF_n}{r^2-rL_n+(-1)^n} \tag{2.1}$$

which is valid if and only if the inequality

$$|r|>\alpha^n=[(1+\sqrt{5})/2]^n \tag{2.2}$$

is satisfied, can be obtained as a particular case of formula (5.2) in [6]. On the other hand, (2.1) and (2.2) can be obtained with the aid of the Binet form and the geometric series formula. If (2.2) is not satisfied, then $S(r,n)$ diverges. Now let us ask ourselves the following question:

"For which values $r_k$ of $r$ does $S(r,n)$ equal a positive integer $k$?"

To answer this question, let us equate the right-hand side of (2.1) to $k$, thus obtaining the second-degree equation

$$kr^2-(F_n+kL_n)r+k(-1)^n=0 \tag{2.3}$$

in the unknown $r$, the roots of which are

$$r_1=\frac{F_n+kL_n+\sqrt{D}}{2k}, \qquad r_2=\frac{F_n+kL_n-\sqrt{D}}{2k}, \tag{2.4}$$

where

$$D=(F_n+kL_n)^2-4k^2(-1)^n. \tag{2.5}$$

Observe that, by (1.4) and (1.5), $D$ can be equivalently expressed as

$$D=(5k^2+1)F_n^2+2kF_{2n}. \tag{2.6}$$

After some tedious manipulations involving the use of the Binet forms, it is seen that, for $k,n\geq 1$,

$$\begin{cases} r_1>\alpha^n, \\ r_1=(-1)^n/r_2. \end{cases} \tag{2.7}$$

From (2.7), we get the inequality $|r_2|<\alpha^n$, so that only the "plus" sign must be considered in (2.4) [see (2.2)]. It follows that $S(r,n)$ equals a positive integer $k$ iff

$$r\equiv r_1\stackrel{\text{def}}{=}r(n)=\frac{F_n+kL_n+\sqrt{D}}{2k}. \tag{2.8}$$

## 3. THE RATIONAL VALUES OF *r* FOR WHICH $S(r, n)$ IS A POSITIVE INTEGER

Since the numbers $r(n)$ defined by (2.8) are, in general, irrational, let us ask ourselves whether or not there exist rational values of them. This is equivalent to asking whether there exist positive integers $k$ for which $D$ is the square of an integer: the answer is in the affirmative, as we shall see in the sequel.

In [1] it has been proved that the set of rational numbers $r$ for which $S(r,1)$ is a positive integer is

$$\{F_{2h+1}/F_{2h} | h=1,2,\ldots\}; \tag{3.1}$$

moreover,

$$S(F_{2h+1}/F_{2h},1)=F_{2h}F_{2h+1}. \tag{3.2}$$

For the general case (i.e., $n \geq 1$), we state the following

***Theorem 1 (Main Result):*** Let $S(r,n)=\sum_{i=0}^{\infty} F_{ni}/r^i$.

***(i)*** If $n$ is even, then the set of all rational numbers $r$ for which $S(r,n)$ is a positive integer is

$$\{F_{(h+1)n}/F_{hn} | h=1,2,\ldots\}; \tag{3.3}$$

moreover,

$$S(F_{(h+1)n}/F_{hn},n)=F_{(h+1)n}F_{hn}/F_n. \tag{3.4}$$

***(ii)*** If $n$ is odd, then the set of all rational numbers $r$ for which $S(r,n)$ is a positive integer is

$$\{F_{(2h+1)n}/F_{2hn} | h=1,2,\ldots\}; \tag{3.5}$$

moreover,

$$S(F_{(2h+1)n}/F_{2hn},n)=F_{(2h+1)n}F_{2hn}/F_n. \tag{3.6}$$

By means of formula (11) in [4], it can be proved that

$$\frac{F_{(h+1)n}F_{hn}}{F_n}=\sum_{i=1}^{h} F_{2ni} \quad (n \text{ even}) \tag{3.7}$$

and

$$\frac{F_{(2h+1)n}F_{2hn}}{F_n}=\sum_{i=1}^{h}\sum_{j=1}^{n} F_{4ni-2(n-j)-1} \quad (n \text{ odd}). \tag{3.7'}$$

Since (3.7) and (3.7′) are nothing but marginal results, their detailed proofs are omitted.

To prove Theorem 1 we have to prove the following two theorems.

***Theorem 2:***

***(i)*** If $n$ is even, the discriminant $D=(5k^2+1)F_n^2+2kF_{2n}$ [see (2.6)] is the square of an integer iff

$$k=F_{(h+1)n}F_{hn}/F_n. \tag{3.8}$$

***(ii)*** If $n$ is odd, the discriminant $D$ is the square of an integer iff

$$k=F_{(2h+1)n}F_{2hn}/F_n. \tag{3.9}$$

***Theorem 3:***

***(i)*** If $n$ is even and (3.8) holds, then [cf. (2.8)] $r(n)=F_{(h+1)n}/F_{hn}$.

***(ii)*** If $n$ is odd and (3.9) holds, then $r(n)=F_{(2h+1)n}/F_{2hn}$.

***Proof of Theorem 2:*** We shall prove that, if $D$ is the square of a generic integer, then $k$ must necessarily be either of the form (3.8) (if $n$ is even) or of the form (3.9) (if $n$ is odd). Let us suppose that $D = X^2$ ($X \in \mathbb{N}$). From (2.6) we can write

$$5k^2F_n^2 + 2kF_{2n} + F_n^2 - X^2 = 0, \tag{3.10}$$

whence we have

$$k = [-F_{2n} \pm \sqrt{F_{2n}^2 - 5F_n^2(F_n^2 - X^2)}]/(5F_n^2). \tag{3.11}$$

After some simple manipulations involving the use of (1.4), and taking into account that $k$ must be positive (by hypothesis), (3.11) can be rewritten as

$$k = [-L_n + \sqrt{L_n^2 - 5F_n^2 + 5X^2}]/(5F_n). \tag{3.12}$$

Now let us distinguish two cases according to the parity of $n$.

**Case 1:** $n$ is even.

From (1.5), (3.12) becomes

$$k = [-L_n + \sqrt{5X^2 + 4}]/(5F_n). \tag{3.13}$$

For $k$ to be an integer, at least we must have that

$$5X^2 + 4 = Q^2 \quad (Q \in \mathbb{N}). \tag{3.14}$$

The solution in integers of the above *Pell equation* is (e.g., see Lemma 1 in [7] or formulas (3.7)-(3.8) in [1])

$$Q = L_{2s}, \; X = F_{2s} \quad (s = 0, 1, 2, \ldots), \tag{3.15}$$

so that, from (3.13)-(3.15), we have

$$k = (L_{2s} - L_n)/(5F_n). \tag{3.16}$$

Now, for $k$ to be a positive integer, both the inequality

$$2s > n, \tag{3.17}$$

and the congruences

$$L_{2s} - L_n \equiv 0 \pmod 5, \tag{3.18}$$

$$L_{2s} - L_n \equiv 0 \pmod{F_n} \tag{3.19}$$

must simultaneously hold. Let us find conditions on $s$ for (3.18) and (3.19) to be satisfied. From (3.18) and (1.8), we see that the congruence

$$2s \equiv n \pmod 4 \tag{3.20}$$

must hold. Now let us rewrite the numerator of (3.16) as

$$L_{2s} - L_n = L_{(2s+n)/2+(2s-n)/2} - L_{(2s+n)/2-(2s-n)/2} \tag{3.21}$$

and observe that, in virtue of (3.20), the integer $(2s-n)/2$ must be even. Under this condition, we can use (1.9) to obtain

$$L_{2s} - L_n = 5F_{(2s-n)/2}F_{(2s+n)/2}. \tag{3.22}$$

First, let us consider the case $n=2$. From (3.16) and (3.22), we obtain the equality $k = F_{s-1}F_{s+1}$, where, from (3.17) and (3.20), $s$ ranges over all odd integers greater than 1. It follows that the above equality can be rewritten as $k = F_{2h}F_{2(h+1)}$ $(h = 1, 2, \ldots)$ [cf. (3.8) for $n=2$ and take into account that $F_2 = 1$].

For $n \geq 4$, the equality (3.22) shows clearly that (3.19) is satisfied iff [see (1.7)]

$$\frac{2s-n}{2} \left(\text{or } \frac{2s+n}{2}\right) \equiv 0 \pmod{n}.$$

Taking (3.17) into account, the above congruence can be written as

$$\frac{2s-n}{2} = hn \quad (h = 1, 2, \ldots). \tag{3.23}$$

From (3.23) we have

$$\frac{2s+n}{2} = (h+1)n \quad (h = 1, 2, \ldots). \tag{3.24}$$

Finally, from (3.16) and (3.22)-(3.24), we obtain the desired result

$$k = \frac{5F_{hn}F_{(h+1)n}}{5F_n} = \frac{F_{hn}F_{(h+1)n}}{F_n} \quad (h = 1, 2, \ldots).$$

**Case 2:** $n$ is odd.

The proof is analogous to that of Case 1, so it is simply sketched. From (1.5), the equality (3.12) and the Pell equation (3.14) become

$$k = [-L_n + \sqrt{5X^2 - 4}]/(5F_n) \tag{3.13$'$}$$

and

$$5X^2 - 4 = Q^2 \quad (Q \in \mathbb{N}), \tag{3.14$'$}$$

respectively. The solution in integers of (3.14′) is (see Lemma 2 in [7])

$$Q = L_{2s+1}, \quad X = F_{2s+1} \quad (s = 0, 1, 2, \ldots). \tag{3.15$'$}$$

Therefore, by means of the same argument as that of Case 1, we get the following relations:

$$k = (L_{2s+1} - L_n)/(5F_n), \tag{3.16$'$}$$

$$2s+1 > n, \tag{3.17$'$}$$

$$2s+1 \equiv n \pmod{4}, \tag{3.20$'$}$$

$$L_{2s+1} - L_n = 5F_{(2s+1-n)/2}F_{(2s+1+n)/2}. \tag{3.22$'$}$$

Taking (3.17′) into account, and recalling that $n$ is odd and $(2s+1-n)/2$ must be even [in virtue of (3.20′)], we can write [see (1.7)]

$$\frac{2s+1-n}{2} = 2hn \quad (h = 1, 2, \ldots). \tag{3.23$'$}$$

From (3.23′) we have

$$\frac{2s+1+n}{2} = (2h+1)n \quad (h = 1, 2, \ldots). \tag{3.24′}$$

Finally, from (3.16′) and (3.22′)-(3.24′), we obtain

$$k = \frac{F_{2hn}F_{(2h+1)n}}{F_n} \quad (h = 1, 2, \ldots) \quad \text{Q.E.D.}$$

***Proof of Theorem 3:*** Let us distinguish two cases according to the parity of $n$.

**Case 1:** $n$ is even.

First, let us replace $k$ by the right-hand side of (3.8) in (2.6), thus obtaining

$$D = 5F_{(h+1)n}^2 F_{hn}^2 + F_n^2 + 2F_{(h+1)n}F_{hn}L_n \stackrel{\text{def}}{=} D(n), \tag{3.25}$$

where (1.4) has been invoked. With the aid of (1.9) and (1.5), the relation (3.25) can be rewritten as

$$\begin{aligned} D(n) &= 5[L_{(2h+1)n} - L_n / 5]^2 + F_n^2 + 2L_n(L_{(2h+1)n} - L_n)/5 \\ &= (L_{(2h+1)n}^2 - L_n^2)/5 + F_n^2 = (L_{(2h+1)n}^2 - L_n^2)/5 + (L_n^2 - 4)/5 \\ &= (L_{(2h+1)n}^2 - 4)/5 = F_{(2h+1)n}^2. \end{aligned} \tag{3.26}$$

Then, let us replace $k$ by the right-hand side of (3.8) and $D$ by $D(n)$ in (2.8), thus obtaining

$$r(n) = \frac{F_n^2 + L_nF_{(h+1)n}F_{hn} + F_nF_{(2h+1)n}}{2F_{(h+1)n}F_{hn}} \stackrel{\text{def}}{=} \frac{N_1}{N_2}. \tag{3.27}$$

Now, it is plain that, in order to prove the theorem, it is sufficient [cf. (3.3)] to prove that $N_1 = 2F_{(h+1)n}^2$. In fact, using (1.9), we get, from (3.27), the equality

$$N_1 = F_n^2 + L_n(L_{(2h+1)n} - L_n)/5 + (L_{2(h+1)n} - L_{2hn})/5,$$

whence, using (1.5) and (1.10), we have

$$\begin{aligned} 5N_1 &= -4 + L_nL_{(2h+1)n} + L_{2(h+1)n} - L_{2hn} \\ &= -4 + L_{2(h+1)n} + L_{2hn} + L_{2(h+1)n} - L_{2hn} = 2(L_{2(h+1)n} - 2). \end{aligned} \tag{3.28}$$

Finally, using (1.6), equality (3.28) becomes $5N_1 = 10F_{(h+1)n}^2$, whence, as desired, we obtain $N_1 = 2F_{(h+1)n}^2$.

**Case 2:** $n$ odd.

The proof is obtained by replacing $k$ by the right-hand side of (3.9) and by using the same properties of Fibonacci numbers as those used in Case 1. Thus, the proof is omitted for the sake of brevity. We confine ourselves to putting into evidence that, in this case, we have

$$D(n) = F_{(4h+1)n}^2 \tag{3.26′}$$

and

$$N_1 = F_n^2 + L_n F_{(2h+1)n} F_{2hn} + F_n F_{(4h+1)n} = 2F_{(2h+1)n}^2. \quad \text{Q.E.D.}$$

## 4. CONCLUDING REMARKS

The Fibonacci-type sum $S(r, n)$ has been investigated and the rational values of $r$ for which this sum is a positive integer have been determined. We can observe that, as required [see (2.2)],

$$\frac{F_{(h+1)n}}{F_{hn}}, \frac{F_{(2h+1)n}}{F_{2hn}} > \alpha^n. \tag{4.1}$$

More particularly, with the aid of the Binet form, we can see that the two quantities on the left-hand side of (4.1) tend to $\alpha^n$ as $h$ tends to infinity.

***Remark 1:*** Let us answer the question of whether or not there exist *integral* values of $r$ for which $S(r, n)$ is a positive integer. From (1.7), (3.3), and (3.5), and taking into account that $F_2 = 1$ divides $F_k$ for all $k$, it follows that the only integral values of $r$ for which $S(r, n)$ is a positive integer are

$$r = F_{2n} / F_n = L_n \quad (n = 2, 4, \ldots) \tag{4.2}$$

and

$$r = F_3 / F_2 = 2 \quad [\text{cf. } (1.1)]. \tag{4.3}$$

Recalling that $L_0 = 2$, it is apparent that the set of such values of $r$ is constituted by all the even-subscripted Lucas numbers.

***Remark 2:*** The *generalized Fibonacci numbers* $U_i(m)$ have been considered in [1], [3], [6], [9], and [10]. These numbers are defined by

$$U_0(m) = 0, \; U_1(m) = 1, \; U_i(m) = mU_{i-1}(m) + U_{i-2}(m) \; \text{ if } i > 1, \tag{4.4}$$

where $m$ is an arbitrary natural number. They give the Fibonacci numbers and the Pell numbers when $m = 1$ and 2, respectively. Once $F$ has been replaced by $U$ in (1.3), the solution in integers of the Pell equations (e.g., see [8], pp. 305-09)

$$(m^2 + 4)X^2 \pm 4 = Q^2 \tag{4.5}$$

allows to prove that the results established in Theorem 1 apply to the numbers $U_i(m)$ as well, provided the inequality $|r| > [(m + \sqrt{m^2 + 4}) / 2]^n$ is satisfied.

Finally, we point out that the results established in this paper give rise to the following Fibonacci identities which we hope will be of some interest to the reader:

$$\sum_{i=0}^{\infty} \frac{F_{ni} F_{hn}^i}{F_{(h+1)n}^i} = \frac{F_{(h+1)n} F_{hn}}{F_n} \quad (n \geq 2 \text{ even}, \; h \geq 1), \tag{4.6}$$

$$\sum_{i=0}^{\infty} \frac{F_{ni} F_{2hn}^i}{F_{(2h+1)n}^i} = \frac{F_{(2h+1)n} F_{2hn}}{F_n} \quad (n \geq 1 \text{ odd}, \; h \geq 1). \tag{4.7}$$

Observe that the right-hand sides of (4.6) and (4.7) can be replaced by those of (3.7) and (3.7′) according to the parity of $n$. As particular instances, letting $h=1$ in (4.6) yields

$$\sum_{i=0}^{\infty} \frac{F_{ni}}{L_n^i} = F_{2n} \quad (n \geq 2 \text{ even}), \tag{4.6′}$$

whereas letting $n=h=1$ in (4.7) yields (1.1).

## ACKNOWLEDGMENTS

This work has been carried out in the framework of an agreement between the Italian PT Administration and the Fondazione Ugo Bordoni. The authors wish to thank the anonymous referee for his valuable suggestions which led to a substantial improvement of the paper.

## REFERENCES

1. O. Brugia, A. Di Porto, & P. Filipponi. "On Certain Fibonacci-Type Sums." *Int. J. Math. Educ. Sci. Technol.* **22.4** (1991):609-13.
2. P. Filipponi & A. F. Horadam. "A Matrix Approach to Certain Identities." *The Fibonacci Quarterly* **26.2** (1988):115-26.
3. P. Filipponi. "A Note on Fridy and Roberts' Paper on Fibonacci-Type Sums." *Gazette Australian Math. Soc.* **18.1** (1991):8-10.
4. H. T. Freitag & P. Filipponi. "On the Representation of Integral Sequences $\{F_n / d\}$ and $\{L_n / d\}$ as Sums of Fibonacci Numbers and as Sums of Lucas Numbers." In *Applications of Fibonacci Numbers*, Vol. 2, pp. 97-112. Dordrecht: Kluwer, 1988.
5. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton Mifflin, 1969.
6. A. F. Horadam & P. Filipponi. "Cholesky Algorithm Matrices of Fibonacci Type and Properties of Generalized Sequences." *The Fibonacci Quarterly* **29.2** (1991):164-73.
7. C. T. Long & J. H. Jordan. "A Limited Arithmetic on Simple Continued Fractions." *The Fibonacci Quarterly* **5.2** (1967):113-28.
8. W. Sierpinski. *Elementary Theory of Numbers.* Warsaw: Panstwowe Wydawnictwo Naukowe, 1964.
9. G. J. Tee. "Proof of Filipponi's Conjecture on Non-Integer Sums of Fibonacci-Type Series." *Gazette Australian Math. Soc.* **18** (1991):161-64.
10. G. J. Tee. "Integer Sums of Recurring Series." *New Zealand J. Math.* **22** (1993):85-100.
11. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section.* Chichester: Ellis Horwood, 1989.

AMS Classification Numbers: 26C15, 11B39, 42A20

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# THE USE OF A SECOND-ORDER RECURRENCE RELATION IN THE DIAGNOSIS OF BREAST CANCER

**W. T. Hung, A. G. Shannon, and B. S. Thornton**
University of Technology, Sydney, NSW 2007, Australia
(*Submitted October 1992*)

## 1. INTRODUCTION

The purpose of this paper is to outline an application of number theory in medicine. More specifically, a linear second-order recurrence relation is utilized in a technique for the diagnosis of breast cancer. This continues a tradition in this journal of applications of second-order recurrences. Indeed, the very first issue contained an article by Basin [1] on the Fibonacci sequence in art and nature, and the tradition has been maintained over the years by such authors as Botten [2] who applied the more general sequence of Horadam to a problem in optics. Number theorists, while rightly valuing their work for its beauty and intrinsic worth, are not always aware of the extensive application of the elegant techniques they develop.

In this paper we develop an inhomogeneous linear second-order recurrence relation of the form

$$S_t - B_1 S_{t-1} - B_2 S_{t-2} = B_3 \tag{1.1}$$

for $S_t$, the relative thermal energy lost by the skin during ultrasonography. (Ultrasonography is a process of visualization of deep structures of the body by recording the echoes of pulses of ultrasonic waves directed into the tissues.) The solutions of (1.1) are then used to distinguish benign and malignant lesions.

The single recurrence relation of the form (1.1) was derived from three interrelated difference equations in a diagnostic model of a breast screening aid (Thornton, Hung, & Hirst [9]). It described the temporal energy changes $S(t)$ ($S_t = S(t)/S(t_0)$) in infrared response of the breast surface when ultrasound is applied to a suspect lesion for an extended time and the results used to evaluate successively the dependent biophysical variables of metabolic energy generated $M(t)$ and the blood perfusion $P(t)$ at each time period. (Perfusion refers to the passage of blood through vessels of a specific organ.) In the present paper an alternative use is made of the three basic difference equations to establish a matrix method which allows $S$, $M$, and $P$ to be evaluated at any subsequent time period in one set of matrix operations from the curve fitting to a set of experimental data. This avoids the need for the previous successive dependent calculations at each stage. The biophysical model [9] and clinical background to the project are summarized below in order to appreciate the manner in which the equations arise.

There is a need to minimize biopsies for benign impalpable lesions—those unable to be felt by touch—detected in breast cancer screening programs for healthy women (Hirst & Kearsley [4]). It is the purpose of this project to help reduce unnecessary and potentially harmful interventions into the lives of healthy women yet not miss any malignant cases.

Mammography is currently the only reliable means of detecting breast cancer before a mass can be felt by the act of physical breast examination. More sensitive diagnostic techniques used at early stages of breast cancer, as well as improved management of the disease itself, are now saving more than half of the women in whom breast cancer is detected at its early stage (Henderson [3]). However, because of the nonspecificity of the mammographic appearance of many malignant lesions, false positives can occur: that is, they are positive on screening but cancer is not subsequently diagnosed. Ultrasonography is used as a complement to mammography because the ultrasound characteristics of malignant lesions are often highlighted in dense parenchyma (the functional elements of an organ) and cystic lesions usually can be differentiated from solid masses.

## 2. THE BIOPHYSICAL BASIS AND MATHEMATICAL MODEL

Human skin emits infrared radiation, and the total radiated power per unit area, $W_T$ (watts per meter squared per second), is given by

$$W_T = \varepsilon\sigma T^4, \tag{2.1}$$

where $\varepsilon$ is the skin's emissivity and $T$ (°$K$) is the temperature of the skin area concerned. (The emissivity is a measure of how well a body can radiate energy; it has a value between 0 and 1.) $\sigma$ is Stefan's constant, which comes into many biomathematical applications (Reuben & Shannon [8]). The emissivity is approximately unity throughout the spectral region used in infrared thermographic studies. For a local change in skin temperature from $T(t_0)$ at time $t_0$ to $T(t)$ at time $t$,

$$W_{T(t)} / W_{T(t_0)} = S(t) / S(t_0) = (T(t) / T(t_0))^4. \tag{2.2}$$

Therefore, within a specific spectral range such as the small changes in the breast skin response during sonification of a suspect lesion we can plot the observed values of $T(t) / T(t_0)$ as a convenient basic parameter of thermal energy transfer which permits direct comparisons with $S_t$ calculated from the difference equations of the model described below.

Breast tissue is glandular, fibrous, and fatty, the last of which is the main bulk of the breast. Let $U$, $M$, $P$, and $S$ be, respectively, the ultrasound energy transmitted, the metabolic energy generated, the thermal energy carried away by perfusion and the thermal energy lost by emission from the skin. Figure 1 shows the energy distribution for these variables, which are all functions of time, when diagnostic ultrasound is directed on to the skin in the direction of the suspected lesion.

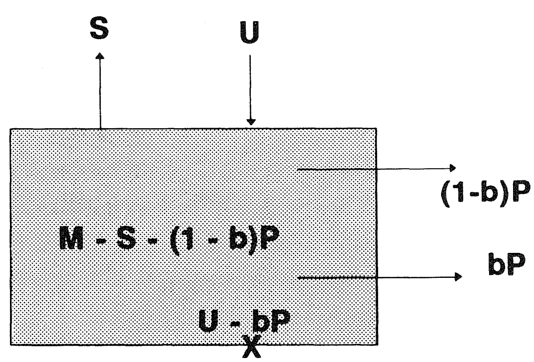


**FIGURE 1. The Energy Transfer Diagram**

Diagnostic ultrasonography utilizes a frequency range between 1 and 10 million hertz ($1 \times 10^6$ cycles per second). Such sound waves can only be transmitted in solids and liquids. By way of comparison, a frequency range of 20 to 20,000 cycles per second provides the stimulus for the subjective sensation of hearing [10]. When an ultrasound beam passes through tissue, energy is partly absorbed and converted to heat. This causes a rise in tissue temperature which depends upon several factors such as the heat conduction and transport by blood flow from the exposed tissue into surrounding regions. Figure 2 is a flow diagram to link these energy components.

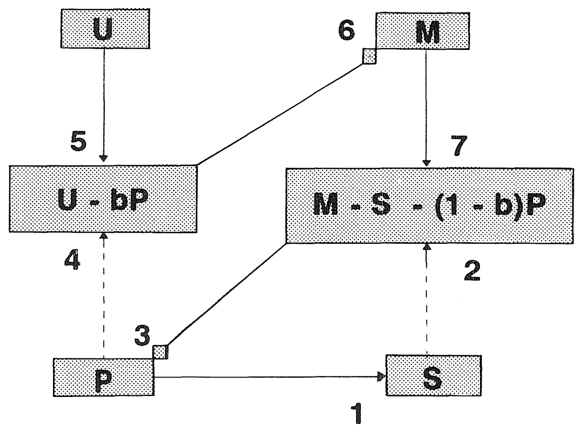


**FIGURE 2. The Flow Diagram**

If $bP$ represents that part of the ultrasound energy which is absorbed and carried away by perfusion ($0 \le b \le 1$), then $U - bP$ is the ultrasound energy which reaches the lesion. The perfusion factor $b$ is typically about 0.85 in this sort of work. Since it is generally recognized that there is increased metabolic activity within breast tumors, we can assume that the ultrasound energy received on the lesion will increase the local metabolic activity as formulated in

$$M(t) - M(t-1) = \mu[U(t-1) - bP(t-1)]. \tag{2.3}$$

The metabolic energy that remains after deducting part of it due to the energy lost from the skin and perfusion is $M - S - (1-b)P$. Since increased blood flow is associated with increased metabolic activity (Love, [7]), the increase in perfusion rate is associated with the increase in this remaining metabolic energy as expressed in

$$P(t) - P(t-1) = \lambda[M(t-1) - S(t-1) - (1-b)P(t-1)]. \tag{2.4}$$

Furthermore, skin temperature results primarily from blood perfusion to the tissues and the blood flow in the superficial veins (Love, [7]), as represented by

$$S(t) = aP(t). \tag{2.5}$$

There are two negative feedback loops shown in Figure 2, in which a line with an arrowhead represents a proportional effect and a line with a spearhead represents an integral effect. The proportional effect occurs when a high level of one variable leads to a high level (positive effect indicated by solid line) or low level (negative effect indicated by dotted line) of another variable.

An integral effect is one in which the rate of increase of one variable depends upon the level of another variable. For example, in the loop formed by the lines marked 1, 2, and 3, if $P$ is very high, then $S$ will be very high, but high $S$ will lead to a low value of $M-S-(1-b)P$, which in turn will cause a decrease in $P$. The second negative feedback loop is formed by lines with heads marked 4, 6, 7, and 3. If $P$ is very high, then $U-bP$ will be low, which will cause a low $M$ and, hence, a low $M-S-(1-b)P$, which in turn will cause a decrease in $P$.

## 3. THE DIFFERENCE EQUATION

The three equations (2.3), (2.4), and (2.5) can be combined as follows [9]:

$$\begin{aligned}
a\lambda\mu U(t-2) &= a\lambda M(t-1) - a\lambda M(t-2) + a\lambda\mu bP(t-2) && \text{[from (2.3)]}\\
&= aP(t) - aP(t-1) + a\lambda S(t-1) + a\lambda(1-b)P(t-1)\\
&\quad - aP(t-1) + aP(t-2) - a\lambda S(t-2) - a\lambda(1-b)P(t-2)\\
&\quad + a\lambda\mu bP(t-2) && \text{[from (2.4)]}\\
&= S(t) - S(t-1) + a\lambda S(t-1) + \lambda(1-b)S(t-1)\\
&\quad - S(t-1) + S(t-2) - a\lambda S(t-2) - \lambda(1-b)S(t-2)\\
&\quad + \lambda\mu bS(t-2) && \text{[from (2.5)]}\\
&= S(t) - (2 - a\lambda - \lambda(1-b))S(t-1) + (1 - a\lambda - \lambda(1-b) + \lambda\mu b)S(t-2).
\end{aligned}$$

Since the ultrasound energy applied at the surface is constant, we set $k = U(t-2)$. For scaling convenience we express $S(t)/S(t_0)$ as $S_t$, so that we can rewrite the second-order inhomogeneous linear difference equation as

$$S_t - [2 - \lambda a - \lambda(1-b)]S_{t-1} - [\lambda a - 1 + \lambda(1-b) - \lambda\mu b]S_{t-2} = a\lambda\mu k / S(t_0). \tag{3.1}$$

The characteristic equation of this is

$$r^2 - [2 - \lambda a - \lambda(1-b)]r - [\lambda a - 1 + \lambda(1-b) - \lambda\mu b] = 0, \tag{3.2}$$

from which we get

$$r = (2 - \lambda a - \lambda(1-b) \pm \sqrt{D})/2, \tag{3.3}$$

where $D = \lambda^2(a-b+1)^2 - 4\lambda\mu b$. The solutions of the homogeneous part of (3.1) are of the form

$$S_t^{(H)} = \begin{cases} C_1 r_1^t + C_2 r_2^t & \text{if } D > 0,\\ C_1 r^t + C_2 t r^t & \text{if } D = 0,\\ C_1 R^t \cos(\theta t) + C_2 R^t \sin(\theta t) & \text{if } D < 0. \end{cases} \tag{3.4}$$

In the context of the present paper, we note that equations (2.3), (2.4), and (2.5) can also be expressed in matrix form:

$$\begin{bmatrix} 1 & -a & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} S(t)\\ P(t)\\ M(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0\\ -\lambda & 1-\lambda(1-b) & \lambda\\ 0 & -b\mu & 1 \end{bmatrix}\begin{bmatrix} S(t-1)\\ P(t-1)\\ M(t-1) \end{bmatrix} + \begin{bmatrix} 0\\ 0\\ \mu U(t-1) \end{bmatrix}. \tag{3.5}$$

Now

$$\begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so if we let

$$V(t) = [S(t), \quad P(t), \quad M(t)]^T$$

and

$$L = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -\lambda & 1-\lambda(1-b) & \lambda \\ 0 & -b\mu & 1 \end{bmatrix} = \begin{bmatrix} -\lambda a & a(1-\lambda(1-b)) & a\lambda \\ -\lambda & 1-\lambda(1-b) & \lambda \\ 0 & -b\mu & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mu U(t-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mu U(t-1) \end{bmatrix},$$

then

$$\begin{aligned} V(t) &= LV(t-1) + C \\ &= L^t V(0) + (L^{t-1} + L^{t-2} + \cdots + L + I)C \\ &= L^t V(0) + (L^t - I)(L-I)^{-1} C. \end{aligned}$$

Although this latter expression is algebraically tedious, it can be numerically useful as follows: The first step of fitting $S_t$ to a set of experimental data yields the parameters $a$, $\lambda$ and $\mu$ (as in the example of Table 1). These parameters can then be used in the above matrix equation to evaluate $S$, $M$, and $P$ at any subsequent time period in one set of matrix operations rather than carry out a series of successively interdependent calculations.

## 4. SOLUTIONS

Horadam and Shannon [5] expound a method for solving equations of the form (3.1). For notational convenience, we let

$$\begin{aligned} B_1 &= 2 - \lambda a - \lambda(1-b), \\ B_2 &= \lambda a - 1 + \lambda(1-b) - \lambda\mu b, \\ B_3 &= a\lambda\mu k / S(t_0), \end{aligned}$$

so that the recurrence relation (3.1) can be rewritten as

$$S_t = \sum_{j=1}^{3} B_j S_{t-j},$$

in which $S_{t-3}$ is treated as though it were unity and $t = 0, 1, \ldots, n$, where $n+1$ is the number of data points. Suppose $S_t$ is the experimental data. The method of least squares is employed to estimate $B_j$ (the estimate of $B_j$ is denoted as $\hat{B}_j$). The sum of squares of errors, *SSE*, has the form of

$$SSE = \sum_{t=2}^{n} \left( S_t - \sum_{j=1}^{3} B_j S_{t-j} \right)^2. \tag{4.2}$$

By differentiating equation (4.2) with respect to $B_i$ $(i = 1, 2, 3)$ and equating each of them to zero, three normal equations will be obtained, that is,

$$S_{11}\hat{B}_1 + S_{12}\hat{B}_2 + S_{13}\hat{B}_3 = E_1$$
$$S_{21}\hat{B}_1 + S_{22}\hat{B}_2 + S_{23}\hat{B}_3 = E_2$$
$$S_{31}\hat{B}_1 + S_{32}\hat{B}_2 + S_{33}\hat{B}_3 = E_3$$

or

$$S\hat{B} = E, \tag{4.3}$$

where

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \end{bmatrix}, \quad \text{and} \quad E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix},$$

in which

$$S_{ij} = \sum_{t=2}^{n} S_{t-i}S_{t-j} \quad \text{and} \quad E_i = \sum_{t=2}^{n} S_t S_{t-i} \quad (i, j = 1, 2, 3).$$

Therefore,

$$\hat{B} = S^{-1}E. \tag{4.4}$$

The matrix $S$ is symmetric, so the Choleski-Turing method can find the inverse in an efficient manner (Irving & Mullineux [6]). There are five different situations that can occur depending on the values of $D$ and $r$ in an equation (3.3) and details are given in [9]. As described there, the parameters $a$, $\lambda$, and $\mu$ were computed from the equations in Section 4 by fitting the model to the thermal data. The values presented in Table 1 are for several cases for a value of $b = 0.85$ which corresponds to perfusion conditions for a lesion of approximately 5cm below the skin surface.

**TABLE 1. Results of Fitting the Model to the Experimental Data Using $b$ = 0.85**

| Patient | Remarks | $D$ | $a$ | $\lambda$ | $\mu$ |
|---|---|---|---|---|---|
| A | Benign | 1.9828 | 0.8050 | 1.6222 | 0.0757 |
| B | Benign | 2.3857 | 0.8144 | 2.2098 | 0.2869 |
| C | Malignant | −1.4888 | 0.8491 | 2.7175 | 0.9589 |
| D | Malignant | −1.2115 | 0.8409 | 1.8230 | 0.7220 |

## 5. CONCLUSION

Results from the project suggest that an infrared temporal response measured over an interval of several minutes with simultaneously applied ultrasound stimulation of the suspect region can provide additional information which may help to distinguish between benign and malignant lesions. Where a malignant process is present, a differential cooling pattern occurs in the local skin surface zone prior to recovery to the initial temperature at the skin [9]. Different responses (no recovery) were observed in benign lesions. From the experimental observations so far, it seems that, if the response curve shows an initial cooling and the fitting of data gives $D < 0$ and $\mu > 0.7$, then it indicates a malignant lesion. The second-order difference equation of the original model [9] reasonably accounts for the thermal changes observed on the skin of the breast, and the matrix method presented here permits improved computational convenience in determining the response, metabolic energy, and perfusion in the successive time periods.

## REFERENCES


1. S. L. Basin. "The Fibonacci Sequence as It Appears in Nature." *The Fibonacci Quarterly* **1.1** (1963):53-57.
2. L. C. Botten. "On the Use of Fibonacci Recurrence Relations in the Design of Long Wavelength Filters and Interferometers." *The Fibonacci Quarterly* **20.1** (1982):1-6.
3. J. C. Henderson. "Breast Cancer Therapy—the Price of Success." *The New England Journal of Medicine* **326.26** (1992):1774-75.
4. C. Hirst & J. H. Kearsley. "Breast Cancer Screening: 'One Swallow Doth Not a Summer Make.'" *The Medical Journal of Australia* **154** (1991):76-78.
5. A. F. Horadam & A. G. Shannon. "Asveld's Polynomials $p_j(n)$." In *Applications of Fibonacci Numbers*, Vol. 2, pp. 163-76, ed. A. N. Philippou, A. F. Horadam, & G. E. Bergum. Dordrecht: Kluwer, 1988.
6. J. Irving & N. Mullineux. *Mathematics in Physics and Engineering*, pp. 270-79. New York: Academic Press, 1964.
7. T. J. Love. "Thermography as an Indicator of Blood Perfusion." *Annals of the New York Academy of Sciences* (1980):429-37.
8. A. J. Reuben & A. G. Shannon. "Some Problems in the Mathematical Modelling of Erythrocyte Sedimentation." *IMA Journal of Mathematics Applied in Medicine and Biology* **7** (1990):145-56.
9. B. S. Thornton, W. T. Hung, & C. Hirst. "Diagnostic Model for Local Temporal Thermal Change at the Skin of the Breast during Extended Application of Diagnostic Ultrasound." *IMA Journal of Mathematics Applied in Medicine and Biology* **9** (1992):161-75.
10. G. J. Tortora N. P. Anagnostakos. *Principles of Anatomy and Physiology*, 5th ed., p. 390. New York: Harper & Row, 1987.


AMS Classification Numbers: 11B37, 92C50

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# DISTRIBUTION OF TWO-TERM RECURRENCE SEQUENCES MOD $P^e$


**Dana Carroll**

Department of Mathemetics, Ohio University, Athens, OH 45701

**Eliot Jacobson**

Department of Mathematics, Ohio University, Athens, OH 45701

**Lawrence Somer**

Department of Mathematics, Catholic University of America, Washington, D.C. 20064



(*Submitted November 1992*)


## INTRODUCTION

In [8], A. Schinzel studied the distribution of the residues of certain two-term recurrence sequences modulo a prime $p$, and classified the sets of distribution frequencies that occur according to the length of a full period. In the present work, we demonstrate a kind of stability that arises in one case of Schinzel's work, which allows an extension of his classification to prime powers. We conclude by giving some examples that show his results do not extend as naturally in the other cases. Related results concerning distribution questions for recurrence sequences can be found in [1]-[7] and [10]-[13].

## DEFINITIONS AND NOTATION

Define the two-term recurrence relation

$$u_0 = 0,\ u_1 = 1,\ u_n = Au_{n-1} + u_{n-2} \text{ for } n > 1,$$

where $A \neq 0$ is a fixed rational integer. Let $p > 7$ be prime, $p \nmid A(A^2+4)$. Let $\xi$ be a real root of $f(x) = x^2 - Ax - 1$ in its splitting field $K$ over $\mathbb{Q}$, and let $\mathcal{R}$ denote the ring of integers in $K$. Let $\mathcal{P}$ be a prime ideal of $\mathcal{R}$ lying over $(p)$ in $\mathbb{Z}$. By assumption on $p$, we do not incur any ramification. It will be clear during our discourse that any splitting that may occur is not a problem. Let $0 < e \in \mathbb{Z}$, and let $\delta(p^e)$ denote the order of $\xi + \mathcal{P}^e$ in $\mathcal{R}/\mathcal{P}^e$. Note that since $\xi$ divides 1 in $\mathcal{R}$, $\delta(p^e)$ exists for all $e$. For notational ease, for $x \in \mathcal{R}$ we denote $x + \mathcal{P}^e$ by $\bar{x}$. Define $k(p^e)$ to be the length of a shortest period of $\bar{u}_n$ and $S(p^e)$ to be the set of residue frequencies within any full period of $\bar{u}_n$. Note that since $u_n$ is a rational integer for all $n$, studying $u_n \pmod{\mathcal{P}^e}$ is equivalent to studying $u_n \pmod{p^e}$.

We prove the following theorem.

## MAIN THEOREM

Let $p > 7$ be prime and $e \geq 1$. If $k(p) \equiv 4 \pmod 8$, then $S(p^e) = \{0, 2, 4\}$.

We need some results from [8] and [16], which are stated here for the reader's convenience.

**Ward [16, pp. 619-20].** Let $t$ be the largest integer with $k(p) = k(p^t)$. Then $k(p^e) = pk(p^{e-1})$ for $e > t$.

In fact, Wall [15] conjectured that $k(p) \neq k(p^2)$ for every $p$ in the special case of the Fibonacci sequence, but this remains a difficult and open problem.

**Schinzel [8, Theorem 1].** For $p > 7$ prime, and $p \nmid A(A^2+4)$,

**(1)** if $k(p) \equiv 4 \pmod 8$, then $S(p) = \{0, 2, 4\}$;

**(2)** if $k(p) \equiv 0 \pmod 8$, then $S(p) = \{0, 1, 2\}$ or $\{0, 2, 3\}$ or $\{0, 1, 2, 4\}$ or $\{0, 2, 3, 4\}$;

**(3)** if $k(p) \not\equiv 0 \pmod 4$, then $S(p) = \{0, 1, 2\}$ or $\{0, 1, 2, 3\}$.

The proof of the Main Theorem will proceed by induction on $e$, after some preliminary lemmas.

***Lemma 1.*** The Binet formula

$$u_n = \frac{\xi^n - (-\xi^{-1})^n}{\xi + \xi^{-1}}$$

holds in $K$, and in $\mathcal{R}/\mathcal{P}^e$, for $e \geq 1$.

***Proof:*** Observe that $\xi$ and $-\xi^{-1}$ are the distinct roots of $f(x)$, hence

$$\begin{aligned}(\xi+\xi^{-1})^2 &= \xi^2 + 2 + \xi^{-2} \\ &= A\xi + 1 + 2 - A\xi^{-1} + 1 \\ &= A(\xi - \xi^{-1}) + 4 \\ &= A^2 + 4,\end{aligned}$$

which is nonzero in $K$, and hence $\xi + \xi^{-1}$ is a unit in $K$. The condition that $p \nmid A(A^2+4)$ ensures that $\xi + \xi^{-1}$ is a unit mod $\mathcal{P}^e$. Lemma 1 now follows easily by induction on $n$. $\square$

For the rest of the paper, we assume additionally that $k(p) \equiv 4 \pmod 8$. Hence, Ward's result gives immediately that $k(p^e) \equiv 4 \pmod 8$ for every $e \geq 1$.

***Lemma 2.*** For every $e \geq 1$, $k(p^e) = \delta(p^e)$.

***Proof:*** Set $k = k(p^e)$ and $\delta = \delta(p^e)$. Since $k$ is even, and $\overline{u}_k = \overline{0}$, $\overline{u}_{k+1} = \overline{1}$, it follows from Lemma 1 that $\overline{\xi}^k - \overline{\xi}^{-k} = \overline{0}$ and $\overline{\xi}^{k+1} + \overline{\xi}^{-k-1} = \overline{\xi} + \overline{\xi}^{-1}$. Thus,

$$\begin{aligned}\overline{\xi}^{k+1} + \overline{\xi}^{k-1} = \overline{\xi} + \overline{\xi}^{-1} &\Rightarrow (\overline{\xi}^k - \overline{1})(\overline{\xi} + \overline{\xi}^{-1}) = \overline{0} \\ &\Rightarrow \overline{\xi}^k = \overline{1}.\end{aligned}$$

Hence $\delta \mid k$.

Since $\delta(p) \mid \delta(p^e)$, it will follow that $\delta(p^e)$ is even if we can show that $\delta(p)$ is even. But this follows directly from [8, Lemma 1] and the fact that $k(p) \equiv 4 \pmod 8$, so that $\overline{u}_\delta = \overline{0}$ and $\overline{u}_{\delta+1} = \overline{1}$, and thus $k \leq \delta$. $\square$

***Definition:*** Let $n_{p^e}$ denote the smallest positive integer $n$ such that $p^e \mid u_n$, called the *rank of apparition of* $p^e$.

***Lemma 3:*** For every $e \geq 1$, $\overline{u}_n = \overline{0}$ if and only if $n \equiv 0 \left(\bmod \frac{k(p^e)}{4}\right)$, that is, $n_{p^e} = k(p^e)/4$.

***Proof:*** First note that $\overline{\xi}(\overline{\xi}-\overline{A})=\overline{1}$, so $\overline{\xi}$ is a unit. Thus,

$$\begin{aligned}\overline{u}_n=\overline{0} &\Leftrightarrow \overline{\xi}^n-(-\overline{\xi}^{-1})^n=\overline{0}\\ &\Leftrightarrow \overline{\xi}^n=(-\overline{\xi}^{-1})^n\\ &\Leftrightarrow \overline{\xi}^{2n}=(-\overline{1})^n.\end{aligned}$$

If $n$ is odd, then since $\delta(p^e)\equiv 4 \pmod 8$, $\overline{\xi}^{2n}=-\overline{1}\Leftrightarrow\overline{\xi}^{4n}=\overline{1}\Leftrightarrow\delta(p^e)=k(p^e)|4n$.

If $n$ is even, then since $k(p^e)/4$ is odd, $\overline{\xi}^{2n}=\overline{1}\Leftrightarrow\delta(p^e)|2n\Leftrightarrow k(p^e)/4|n$. □

***Lemma 4:*** For all $n$, $h\geq 0$,

$$u_{n+h}-u_n=(\xi^h-1)u_n+(-\xi^{-1})^n u_h.$$

***Proof:*** By Lemma 1,

$$\begin{aligned}(\xi^h-1)u_n+(-\xi^{-1})^n u_h &= \frac{1}{\xi+\xi^{-1}}\left((\xi^h-1)(\xi^n-(-\xi^{-1})^n)+(-\xi^{-1})^n(\xi^h-(-\xi^{-1})^h)\right)\\ &=\frac{1}{\xi+\xi^{-1}}\left(\xi^{n+h}-\xi^n+(-\xi^{-1})^n-(-\xi^{-1})^{n+h}\right)\\ &=u_{n+h}-u_n.\ \square\end{aligned}$$

***Lemma 5:*** Let $A(d;p^e)$ denote the number of times the residue $d$ appears within a full period of $\{u_n\}$ (mod $p^e$). If $k(p^e)\equiv 4 \pmod 8$, then $A(d;p^e)$ is even.

***Proof:*** Denote $k=k(p^e)$. First, if $n$ is even, then by Lemmas 1 and 2,

$$\begin{aligned}\overline{u}_{k/2-n} &= \frac{\xi^{k/2-n}-(-\xi^{-1})^{k/2-n}}{\xi+\xi^{-1}}\\ &=\frac{\xi^{k/2}\xi^{-n}-(-\xi^{-1})^{k/2}(-\xi^{-1})^{-n}}{\xi+\xi^{-1}}\\ &=\frac{-\xi^{-n}+\xi^n}{\xi+\xi^{-1}}\\ &=\overline{u}_n.\end{aligned}$$

Similarly, if $n$ is odd, then $\overline{u}_{k-n}=\overline{u}_n$. Since $k\equiv 4 \pmod 8$, the result follows. □

For the rest of the paper, assume $e>t$, where $t$ is the largest integer with $k(p)=k(p^t)$, and let $k=k(p^{e-1})$. Define the $p\times k$ integer matrix $T$ by setting $T_{ij}\equiv u_{(i-1)k+j-1} \pmod{p^e}$, where $0\leq T_{ij}<p^e$. Then each row of $T$ is congruent to a full period modulo $p^{e-1}$, and the rows laid end to end correspond to a full period modulo $p^e$. We will show that the entries in any column of $T$ are distinct.

***Lemma 6:*** The first column of $T$ has distinct entries.

***Proof:*** Assume that $\overline{u}_{ik} = \overline{u}_{tk}$ for some $0 \le i < t \le p-1$. By Lemma 4,

$$\overline{u}_{tk} - \overline{u}_{ik} = \overline{0} = (\overline{\xi}^{(t-i)k} - 1)\overline{u}_{ik} + (-\overline{\xi}^{-1})^{ik}\,\overline{u}_{(t-i)k}.$$

Since $\xi^k \equiv 1 \pmod{p^{e-1}}$ by Lemma 2, we have

$$\xi^{(t-i)k} - 1 \in \mathcal{P}^{e-1}. \tag{1}$$

Clearly $u_{ik} \in \mathcal{P}^{e-1}$, therefore, $(\xi^{(t-i)k} - 1)u_{ik} \in \mathcal{P}^{2e-2} \subseteq \mathcal{P}^e$ and, hence, $u_{(t-i)k} \in \mathcal{P}^e$ also. Thus, $n_{p^e} | (t-i)k$. But $n_{p^e} = pn_{p^{e-1}}$ and $k = 4n_{p^{e-1}}$, so $p | 4(t-i)$, a contradiction. □

***Lemma 7:*** Every column of $T$ has distinct entries.

***Proof:*** Assume that $\overline{u}_{ik+j} = \overline{u}_{tk+j}$ for some $0 < j \le k-1$, $0 \le i < t \le p-1$. By Lemma 4,

$$u_{tk+j} - u_{tk} = (\xi^j - 1)u_{tk} + (-\xi^{-1})^{tk} u_j,$$

and

$$u_{ik+j} - u_{ik} = (\xi^j - 1)u_{ik} + (-\xi^{-1})^{ik} u_j.$$

Subtracting these equations, and using the assumption,

$$\overline{u}_{ik} - \overline{u}_{tk} = (\overline{\xi}^j - 1)(\overline{u}_{tk} - \overline{u}_{ik}) + \overline{u}_j\left((-\overline{\xi}^{-1})^{tk} - (-\overline{\xi}^{-1})^{ik}\right)$$

so that

$$\overline{\xi}^j(\overline{u}_{tk} - \overline{u}_{ik}) = -\overline{u}_j\left((-\overline{\xi}^{-1})^{tk} - (-\overline{\xi}^{-1})^{ik}\right).$$

By Lemma 6, $u_{tk} - u_{ik} \in \mathcal{P}^{e-1} \setminus \mathcal{P}^e$ and hence

$$-u_j(-\xi^{-1})^{ik}\left((-\xi^{-1})^{(t-i)k} - 1\right) \in \mathcal{P}^{e-1} \setminus \mathcal{P}^e. \tag{2}$$

By Lemma 1, setting $n = tk + j$ and $m = ik + j$, and noting that $n + m$ is even,

$$\begin{aligned}\overline{0} &= (\overline{u}_n - \overline{u}_m)(\overline{\xi} + \overline{\xi}^{-1}) \\ &= \overline{\xi}^n - (-\overline{\xi}^{-1})^n - \overline{\xi}^m + (-\overline{\xi}^{-1})^m \\ &= \overline{\xi}^m(\overline{\xi}^{n-m} - \overline{1})\left(\overline{1} + (-1)^n(-\overline{\xi}^{-1})^{n+m}\right).\end{aligned}$$

Since $p$ does not divide $t-i$, it follows that $\xi^{n-m} - 1 = \xi^{(t-i)k} - 1 \in \mathcal{P}^{e-1} \setminus \mathcal{P}^e$. Therefore, $1 + (-1)^n(-\xi^{-1})^{n+m} \in \mathcal{P}$ and thus $\xi^{2(n+m)} - 1 \in \mathcal{P}$. Then $k(p) | 2(n+m) = 2(t+i)k + 4j$. Since $k(p) | k$, we get $k(p) | 4j$ and so $p | u_j$. Finally, this gives $u_j \in \mathcal{P}$ and hence $((-\xi^{-1})^{(t-i)k} - 1) \notin \mathcal{P}^{e-1}$ by (2), which contradicts (1). □

## PROOF OF MAIN THEOREM

Assume $p > 7$ is a prime with $p \nmid A(A^2 + 4)$ and $k(p) \equiv 4 \pmod 8$. The case $e = 1$ is just Schinzel's result.

As before, let $t$ be the largest integer such that $k(p) = k(p^t)$. It is easy to see that $\{|u_n|\}$ is a strictly increasing sequence for $n \ge 2$. Since $u_1 = 1$ and $u_2 = A$, it follows that $t$ exists. We now consider the case in which $t > 1$. Let $1 < e \le t$. Let $A(d; p^e)$ be as in Lemma 5. Clearly, $A(d; p^e) \le A(d; p)$. Since $\{0\} \subset S(p)$, it follows that $0 \in S(p^e)$. By Lemma 3, $k(p^e) = 4n_{p^e}$.

Thus, $A(0; p^e) = 4$ and $4 \in S(pe)$. By Lemma 5, $A(d; p^e)$ is even for every residue $d$. Since $2 \in S(p)$, there is a residue $d$ such that $A(d; p) = 2$. Let $u_n$ be such that $u_n \equiv d \pmod{p}$, and suppose $u_n \equiv d' \pmod{p^e}$. Since $A(d'; p^e)$ is even, $A(d'; p^e) \geq 1$, and $A(d'; p^e) \leq A(d; p) = 2$, we must have $A(d'; p^e) = 2$. Thus, $S(p^e) = \{0, 2, 4\}$

We now proceed by induction on $e$. Assume the theorem is true for $e-1$, $e \geq t+1$. By Ward's theorem, $k(p^e) \equiv 4 \pmod{8}$.

Let $x$ be any residue modulo $p^e$ appearing in $T$. Let $j$ be the least positive integer such that $u_j \equiv x \pmod{p^e}$, and let $0 \leq y < p^{e-1}$ satisfy $u_j \equiv y \pmod{p^{e-1}}$. By hypothesis, $y$ occurs either two or four times in any full period modulo $p^{e-1}$.

Notice any two entries in the same column of $T$ are congruent modulo $p^{e-1}$, since their subscripts differ by a multiple of $k(p^{e-1})$. Hence, $y$ will occur in either two columns or four columns of $T$. Since $x = ap^{e-1} + y$ for some $0 \leq a < p$, $x$ must occur once in each of the same columns, and nowhere else, so $x$ will occur in $T$ either two or four times. Thus, $S(p^e) \subseteq \{0, 2, 4\}$. Since there is at least one residue modulo $p$ that does not occur in $T$, there will also be at least one residue modulo $p^e$ not occurring in $T$, so $S(p^e) = \{0, 2, 4\}$. $\square$

***Remark:*** It follows by the proof of the Main Theorem that if $e > t$, then $A(d; p^e) = A(d; p^t)$.

***Examples:*** We have shown that in the case $k(p) \equiv 4 \pmod{8}$, Schinzel's result holds for any power of $p$; that is, $S(p^e) = \{0, 2, 4\}$ for all $e \geq 1$. We give examples here to show that an analogous generalization does not hold in the other cases of Schinzel's result.

First, we consider the case $k \not\equiv 0 \pmod{4}$. There are two subcases to consider:

***(1)*** $S(p) = \{0, 1, 2, 3\}$. If $A = 1$ and $p = 11$, then $S(p^2) = \{0, 1, 2, 3, 11\}$.
***(2)*** $S(p) = \{0, 1, 2\}$. If $A = 4$ and $p = 19$, then $S(p^2) = \{0, 1, 2, 19\}$.

Next, we consider $k \equiv 0 \pmod{8}$. There are four subcases to consider:

***(1)*** $S(p) = \{0, 1, 2, 4\}$. If $A = 1$ and $p = 23$, then $S(p^2) = \{0, 2, 4, 23\}$.
***(2)*** $S(p) = \{0, 1, 2\}$. If $A = 3$ and $p = 11$, then $S(p^2) = \{0, 2, 11\}$.
***(3)*** $S(p) = \{0, 2, 3\}$. If $A = 2$ and $p = 17$, then $S(p^2) = \{0, 2, 19\}$.
***(4)*** $S(p) = \{0, 2, 3, 4\}$. If $A = 2$ and $p = 11$, then $S(p^2) = \{0, 2, 4, 13\}$.

## REFERENCES

1. G. Bruckner. "Fibonacci Sequences Modulo a Prime $p \equiv 3 \pmod{4}$." *The Fibonacci Quarterly* **8.3** (1970):217-20.
2. S. A. Burr. "On Moduli for Which the Fibonacci Sequence Contains a Complete System of Residues." *The Fibonacci Quarterly* **9.5** (1971):497-504.
3. E. T. Jacobson. "A Brief Survey on Distribution Questions for Second Order Linear Recurrences." In *Proceedings of the First Meeting of the Canadian Number Theory Association*, pp. 249-54. Ed. Richard A. Mollin. Walter de Gruyter, 1990.
4. E. T. Jacobson. "Distribution of the Fibonacci Numbers Mod $2^k$." *The Fibonacci Quarterly* **30.3** (1992):211-15.
5. H. Niederreiter, A. Schinzel, & L. Somer. "Maximal Frequencies of Elements in Second-Order Recurrences over a Finite Field." *Elem. Math.* **46** (1991):139-43.

6. J. Pihko. "A Note on a Theorem of Schinzel." *The Fibonacci Quarterly* **29.4** (1991):333-38.
7. A. P. Shah. "Fibonacci Sequences Modulo *m*." *The Fibonacci Quarterly* **6.2** (1968):139-41.
8. A. Schinzel. "Special Lucas Sequences, Including the Fibonacci Sequence, Modulo a Prime." In *A Tribute to Paul Erdös*, pp. 349-57. Ed. A. Baker, B. Bollobás, & A. Hajnal. Cambridge: Cambridge University Press, 1990.
9. L. Somer. "The Divisibility Properties of Primary Lucas Recurrences with Respect to Primes." *The Fibonacci Quarterly* **18.4** (1980):316-34.
10. L. Somer. "Primes Having an Incomplete System of Residues for a Class of Second-Order Linear Recurrences." In *Applications of Fibonacci Nubmers* **2**:113-41. Ed. A. N. Philippou, A. F. Horadam, & G. E. Bergum. Dordrecht: Kluwer, 1988.
11. L. Somer. "Distribution of Residues of Certain Second-Order Linear Recurrences Modulo *p*." In *Applications of Fibonacci Numbers* **3**:311-24. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1990.
12. L. Somer. "Distribution of Residues of Certain Second-Order Linear Recurrences Modulo *p*—II." *The Fibonacci Quarterly* **29.1** (1991):72-78.
13. L. Somer. "Upper Bounds for Frequencies of Elements in Second-Order Recurrences over a Finite Field." In *Applications of Fibonacci Numbers* **5** (to appear).
14. J. Vinson. "The Relation of the Period Modulo *m* to the Rank of Apparition of *m* in the Fibonacci Sequence." *The Fibonacci Quarterly* **1.1** (1963):37-45.
15. D. D. Wall. "Fibonacci Series Modulo *m*." *Amer. Math. Monthly* **67** (1960):525-32.
16. M. Ward. "The Arithmetical Theory of Linear Recurring Sequences." *Trans. Amer. Math. Soc.* **35** (1933):600-28.

AMS Classification Numbers: 11B50, 11K36

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# A PERFECT CUBOID IN GAUSSIAN INTEGERS

**W. J. A. Colman**
Department of Mathematical and Physical Sciences, University of East London
(*Submitted November 1992*)

**1.** A perfect cuboid (if such exists) has rational integral sides $x, y,$ and $z$, with $xyz \neq 0$, such that the four equations

$$x^2+y^2=u^2,\ x^2+z^2=v^2,\ y^2+z^2=w^2, \text{ and } x^2+y^2+z^2=\ell^2 \tag{1.1}$$

are satisfied for rational integers $u, v, w,$ and $\ell$. No such perfect cuboids are known, but their nonexistence has not been demonstrated. It is known that any six of the quantities $x, y, z, u, v, w,$ and $\ell$ can be integral and that, in this case, an infinity of solutions exist (see [1] and [2]). We shall use the word "cuboid" in this case even when any square quantity is negative, and refer to the cuboid as nonreal, following Leech [2]. For example:

$$x=63,\ y=60,\ z^2=-3344,\ u=87,\ v=25,\ w=16, \text{ and } \ell=65.$$

In this paper, a parametric solution will be determined that has two integral sides $x$ and $y$ (say), integral face diagonals $u$, $v$, and $w$, and integral internal diagonal $\ell$. The third side $z$ will, in general, be irrational or complex. However, by a suitable choice of the parameters, a perfect cuboid in Gaussian integers results that satisfies the requirement that $xyz \neq 0$.

**2.** From the equations above, we have that

$$2(x^2+y^2+z^2)=u^2+v^2+w^2=2\ell^2. \tag{2.1}$$

The equation $u^2+v^2+w^2=2\ell^2$ has the four-parameter solution

$$\begin{aligned}
u &= 2(mt+mn+st-sn),\\
v &= 2ms+2nt+n^2+s^2-m^2-t^2,\\
w &= 2ms-2nt+n^2-s^2+m^2-t^2,\\
\ell &= m^2+n^2+s^2+t^2.
\end{aligned}$$

Substituting these values into equations (1.1) gives

$$\begin{aligned}
x^2 &= (m^2+n^2+s^2+t^2)^2-(2ms-2nt+n^2-s^2+m^2-t^2)^2,\\
y^2 &= (m^2+n^2+s^2+t^2)^2-(2ms+2nt+n^2+s^2-m^2-t^2)^2,\\
z^2 &= (m^2+n^2+s^2+t^2)^2-(2(mt+mn+st-sn))^2.
\end{aligned}$$

The first two equations give

$$\begin{aligned}
x^2 &= 4(m^2+n^2+ms-nt)(s^2+t^2-ms+nt),\\
y^2 &= 4(n^2+s^2+ms+nt)(m^2+t^2-ms-nt).
\end{aligned}$$

Let us put $m=ab, n=ac, s=-cd,$ and $t=bd$, then $ms+nt=0$ and

$$y^2=4(a^2c^2+c^2d^2)(a^2b^2+b^2d^2)=4c^2b^2(a^2+d^2)^2.$$

Hence, $y = 2bc(a^2 + d^2)$ and

$$x^2 = 4(a^2b^2 - 2abcd + a^2c^2)(c^2d^2 + 2abcd + b^2d^2)$$
$$= 4a^2d^2\left(b^2 - \frac{2bcd}{a} + c^2\right)\left(b^2 + \frac{2abc}{d} + c^2\right).$$

Write

$$b^2 - \frac{2bcd}{a} + c^2 = e^2 \tag{2.2}$$

and

$$b^2 + \frac{2abc}{d} + c^2 = f^2. \tag{2.3}$$

Putting $b^2 = 2bcd / a$ or $ab = 2cd$ in (2.2) and substituting in (2.3) gives $b^2 + 5c^2 = f^2$. In which case, $x = 2adcf$ and $z^2 = (a^2b^2 + c^2d^2 + a^2c^2 + b^2d^2)^2 - 4(ab(ac+bd) + cd(ac-bd))^2$. Therefore, we have the following parametric solution in which $x$, $y$, $u$, $v$, $w$, and $d$ are all integral:

$$x = 2adcf,$$
$$y = 2bc(a^2 + d^2),$$
$$z^2 = ((a^2 + d^2)(b^2 + c^2))^2 - 4(ab(ac+bd) + cd(ac-bd))^2,$$

where $b^2 + 5c^2 = f^2$ and $ab = 2cd$ with $a \neq d$; otherwise, $z^2 = 0$.

We can tidy up this solution as follows: The equation $b^2 + 5c^2 = f^2$ has the solution

$$b = 5\alpha^2 - \beta^2,\ c = 2\alpha\beta, \text{ and } f = 5\alpha^2 + \beta^2.$$

The equation $ab = 2cd$ or $a(5\alpha^2 - \beta^2) = 4\alpha\beta d$ can be satisfied if $a = 4\alpha\beta$ and $d = 5\alpha^2 - \beta^2$. The solution can now be written as

$$\begin{aligned} x &= 16\alpha^2\beta^2(25\alpha^4 - \beta^4), \\ y &= 4\alpha\beta(5\alpha^2 - \beta^2)(25\alpha^4 + 6\alpha^2\beta^2 + \beta^4), \\ z^2 &= (25\alpha^4 + 6\alpha^2\beta^2 + \beta^4)^2(25\alpha^4 - 6\alpha^2\beta^2 + \beta^4)^2 \\ &\quad - 16\alpha^2\beta^2(5\alpha^2 - \beta^2)^2(25\alpha^4 + 14\alpha^2\beta^2 + \beta^4)^2. \end{aligned} \tag{2.4}$$

If $\alpha = 1$ and $\beta = 2$, we have

$$x = 576,\quad y = 520,\quad z^2 = 618849,$$

which is the smallest real cuboid with one irrational edge (see [2]).

If $\alpha = 1$ and $\beta = 3$, we have

$$x = 63,\quad y = 60,\quad z^2 = -3344,$$

which is the smallest cuboid (nonreal) in this category, according to Leech [2].

**3.** Looking at the form for $z^2$ in (2.4), we see that we cannot choose positive integral $\alpha$ and $\beta$ to make

$$16\alpha^2\beta^2(5\alpha^2 - \beta^2)^2(25\alpha^4 + 14\alpha^2\beta^2 + \beta^4)^2 \tag{3.1}$$

zero. But we can put $25\alpha^4 - 6\alpha^2\beta^2 + \beta^4 = 0$ (say) to give

$$\frac{\alpha^2}{\beta^2} = \frac{3 \pm 4i}{25}.$$

Putting $\alpha^2 = 3 \pm 4i$ and $\beta^2 = 25$, we get $\alpha = 2 \pm i$ and $\beta = 5$. This gives, after cancelling common real factors

$$\begin{aligned} x &= 96 \pm 28i = 4(24 \pm 7i), \\ y &= 72 \pm 21i = 3(24 \pm 7i), \\ z &= 35 \mp 120i = 5(7 \mp 24i), \end{aligned}$$

and we have

$$\begin{aligned} &x = 4, \qquad y = 3, \qquad z = \mp 5i, \\ &x^2 + y^2 = (5)^2, \\ &x^2 + z^2 = (3i)^2, \\ &y^2 + z^2 = (4i)^2, \text{ and} \\ &x^2 + y^2 + z^2 = (0)^2. \end{aligned}$$

This is clearly so for the following Pythagorean values

$$x = 2pq, \quad y = p^2 - q^2, \quad \text{and} \quad z = i(p^2 + q^2).$$

Hence, according to the original definition, since $xyz \neq 0$, we have a perfect cuboid in Gaussian integers.

It would be interesting to know it if is possible to have a solution in Gaussian integers such that $xyzuvw\ell \neq 0$.

## REFERENCES

1. W. J. A. Colman. "On Certain Semi-Perfect Cuboids." *The Fibonacci Quarterly* **26.2** (1988):54-57.
2. J. Leech. "The Rational Cuboid Revisited." *Amer. Math. Monthly* **84** (1977):518-33.

AMS Classification Numbers: 11D09

# A NOTE ON THE NEGATIVE PASCAL TRIANGLE

**Jack Y. Lee**
Brooklyn, NY 11209
(*Submitted November 1992*)

We arrange the rising diagonals of Pascal's triangle in vertical columns so that the column sums form a Fibonacci sequence (see [1]). Let us arrange the coefficients of the expansion of $(1-x)^n$ symmetrically to the Pascal triangle, then in the resulting triangle, the negative Pascal triangle, the sums of the columns form the negative branch of the Fibonacci sequence. This is displayed in Table 1, where the $u_n$'s stand for Fibonacci numbers.

**TABLE 1. Pascal's Array and Corresponding Fibonacci Numbers**

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| −1 | | | | | | | | | | | | | | | | | | | | 1 |
| | 1 | | | | | | | | | | | | | | | | | | 1 | |
| −8 | | −1 | | | | | | | | | | | | | | | | 1 | | 8 |
| | 7 | | 1 | | | | | | | | | | | | | | 1 | | 7 | |
| −21 | | −6 | | −1 | | | | | | | | | | | | 1 | | 6 | | 21 |
| | 15 | | 5 | | 1 | | | | | | | | | | 1 | | 5 | | 15 | |
| −20 | | −10 | | −4 | | −1 | | | | | | | | 1 | | 4 | | 10 | | 20 |
| | 10 | | 6 | | 3 | | 1 | | | | | | 1 | | 3 | | 6 | | 10 | |
| −5 | | −4 | | −3 | | −2 | | −1 | | | | 1 | | 2 | | 3 | | 4 | | 5 |
| | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | |
| −55 | 34 | −21 | 13 | −8 | 5 | −3 | 2 | −1 | 1 | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 |
| | ... | | $u_{-7}$ | $u_{-6}$ | $u_{-5}$ | $u_{-4}$ | $u_{-3}$ | $u_{-2}$ | $u_{-1}$ | $u_0$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ | | ... | |

The fact that the sum of numbers in a column (diagonal) in the positive Pascal triangle is a Fibonacci number is well known. It is clear that the same holds for the negative Pascal triangle by its construction and by the relation $u_{-n} = (-1)^{n-1}u_n$.

To see that this extension of Pascal's triangle is made in a natural way, read the sequences parallel to the main diagonal from bottom right to upper left in Table 1. The sequences in the negative triangle constitute the coefficients of the expansion of $(1+x)^{-n}$, since the negative Pascal triangle in Table 2(a) is also expressed as (b) by means of the relation

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}.$$

This enables us to redefine the negative Pascal triangle as the binomial coefficients of negative exponents. Similarly, the sequences parallel to the sequence 1, 2, 3, ... consist of the coefficients of the expansion of $(1-x)^{-n}$ in the extended Pascal triangle.

The array corresponding to the general second-order recurrence $u_n = cu_{n-2} + bu_{n-1}$, where $b$ and $c$ are nonzero integers, is given in Table 3. In this case, the sequences parallel to the main diagonal are generated by the function $(c+bx)^n$ for any integer $n$, and the sequences parallel to the sequence 1, $b$, $b^2$, ... are generated by the function $c^{n-1}(1-bx)^{-n}$ for any integer $n$.

**TABLE 2. Two Expressions of the Negative Pascal Triangle**

$$
\begin{array}{ccccccc}
\binom{6}{6} & & & & & & \\
 & -\binom{5}{5} & & & & & \\
\binom{5}{4} & & \binom{4}{4} & & & & \\
 & -\binom{4}{3} & & -\binom{3}{3} & & & \\
\binom{4}{2} & & \binom{3}{2} & & \binom{2}{2} & & \\
 & -\binom{3}{1} & & -\binom{2}{1} & & -\binom{1}{1} & \\
\binom{3}{0} & & \binom{2}{0} & & \binom{1}{0} & & \binom{0}{0}
\end{array}
$$

**(a)**

$$
\begin{array}{ccccccc}
\binom{-1}{6} & & & & & & \\
 & \binom{-1}{5} & & & & & \\
\binom{-2}{4} & & \binom{-1}{4} & & & & \\
 & \binom{-2}{3} & & \binom{-1}{3} & & & \\
\binom{-3}{2} & & \binom{-2}{2} & & \binom{-1}{2} & & \\
 & \binom{-3}{1} & & \binom{-2}{1} & & \binom{-1}{1} & \\
\binom{-4}{0} & & \binom{-3}{0} & & \binom{-2}{0} & & \binom{-1}{0}
\end{array}
$$

**(b)**

**TABLE 3. The General Second-Order Array**

$$
\begin{array}{ccccccccccc}
b^4/c^5 & & & & & & & & & & b^4 \\
 & -b^3/c^4 & & & & & & & & b^3 & \\
3b^2/c^4 & & b^2/c^3 & & & & & & b^2 & & 3cb^2 \\
 & -2b/c^3 & & -b/c^2 & & & & b & & 2cb & \\
1/c & & 1/c^2 & & 1/c & & 1 & & c & & c^2 \\
\hdashline
u_{-5} & u_{-4} & u_{-3} & u_{-2} & u_{-1} & u_0 & u_1 & u_2 & u_3 & u_4 & u_5
\end{array}
$$

## REFERENCE

1. N. N. Vorob'ev. *Fibonacci Numbers.* New York: Blaisdell, 1961, p. 13.

AMS Classification Number: 05A10

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# FORMULAS FOR $1+2^p+3^p+\cdots+n^p$

**G. F. C. de Bruyn and J. M. de Villiers**
Department of Mathematics, University of Stellenbosch, Stellenbosch, South Africa
*(Submitted December 1992)*

## 1. INTRODUCTION

Let $S_p(n) = 1+2^p+3^p+\cdots+n^p$, with $n$ and $p$ positive integers. In [3] R. A. Khan, using the binomial theorem and a definite integral, gave a proof of the general recurrence formula for $S_1(n), S_2(n), S_3(n), \ldots$ in terms of powers of $n$. A matrix formula for $S_p(n)$, obtained by solving a difference equation by matrix methods, is given in [2].

In this note the recurrence formulas in terms of powers of $n$ and of $n+1$ are given in symbolic form, only using the binomial theorem. These formulas are both easily remembered and applied.

These recurrence formulas are then used to establish, employing Cramer's rule, explicit expressions for $S_p(n)$ in determinant form.

Finally, the usual formulas for $S_p(n)$ as polynomials of degree $p+1$ in $n$ and $n+1$, with coefficients in terms of the Bernoulli numbers, are derived from these determinants. It is noted that it is possible to do this without prior knowledge of the Bernoulli numbers.

## 2. FORMULAS IN TERMS OF POWERS OF $n$

### 2.1 A Recurrence Formula

Let $n \in N$, with $N$ the set of positive integers. For $k \in N$, let

$$S_k(n) = 1+2^k+3^k+\cdots+n^k = \sum_{r=1}^{n} r^k,$$

$S_k(0)=0$, and take $S_0(n)=n$. Then

$$\begin{aligned}
n^k &= S_k(n) - S_k(n-1) \\
&= S_k(n) - \sum_{r=1}^{n}(r-1)^k \\
&= S_k(n) - \sum_{r=1}^{n}\left(\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}r^i\right) \\
&= S_k(n) - \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}S_i(n) && (2.1.1) \\
&= -\sum_{i=0}^{k-1}\binom{k}{i}(-1)^{k-i}S_i(n). && (2.1.2)
\end{aligned}$$

The equation

$$S^k-(S-1)^k=n^k, \tag{2.1.3}$$

in which the binomial power is expanded and $S^i$ $(i=0,1,2,\dots,k)$ are then replaced by $S_i(n)$, provides a mnemonic for (2.1.1).

For example, for $k=2$, formula (2.1.3) yields

$$S^2-(S^2-2S+1)=n^2,$$

and thus

$$2S_1(n)-n=n^2,$$

giving the well-known result

$$S_1(n)=\frac{n}{2}(n+1). \tag{2.1.4}$$

Next, for $k=3$, it similarly follows that

$$3S_2(n)-3S_1(n)+n=n^3,$$

so that substitution of (2.1.4) leads to the formula

$$S_2(n)=\frac{n}{6}(n+1)(2n+1).$$

## 2.2 $S_p(n)$ as a Determinant

Let $p\in N$ and let $k=1,2,\dots,p,p+1$ in (2.1.2). Then, solving for $S_p(n)$ in the resulting $(p+1)\times(p+1)$ lower triangular linear system by means of Cramer's rule, the determinant representation

$$S_p(n)=\frac{1}{(p+1)!}\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & n \\ -1 & \binom{2}{1} & 0 & 0 & \cdots & 0 & n^2 \\ 1 & -\binom{3}{1} & \binom{3}{2} & 0 & \cdots & 0 & n^3 \\ \vdots & \vdots & \vdots & & & & \vdots \\ (-1)^{p-1} & (-1)^p\binom{p}{1} & (-1)^{p+1}\binom{p}{2} & & \cdots & \binom{p}{p-1} & n^p \\ (-1)^p & (-1)^{p+1}\binom{p+1}{1} & (-1)^{p+2}\binom{p+1}{2} & & \cdots & -\binom{p+1}{p-1} & n^{p+1} \end{vmatrix} \tag{2.2.1}$$

$$=p!\begin{vmatrix} \frac{1}{1!} & 0 & 0 & 0 & \cdots & 0 & \frac{n}{1!} \\ -\frac{1}{2!} & \frac{1}{1!} & 0 & 0 & \cdots & 0 & \frac{n^2}{2!} \\ \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 & \frac{n^3}{3!} \\ \vdots & \vdots & \vdots & & & & \vdots \\ \frac{(-1)^{p-1}}{p!} & \frac{(-1)^p}{(p-1)!} & \frac{(-1)^{p+1}}{(p-2)!} & & \cdots & \frac{1}{1!} & \frac{n^p}{p!} \\ \frac{(-1)^p}{(p+1)!} & \frac{(-1)^{p+1}}{p!} & \frac{(-1)^{p+2}}{(p-1)!} & & \cdots & -\frac{1}{2!} & \frac{n^{p+1}}{(p+1)!} \end{vmatrix} \tag{2.2.2}$$

is obtained. The step from (2.2.1) to (2.2.2) follows by first multiplying the $i^{\text{th}}$ row of the determinant by $1/i!$ for $i=1,2,\dots,p+1$, and then multiplying the $j^{\text{th}}$ column of the resulting determinant by $(j-1)!$ for $j=2,3,\dots,p$.

### 2.3 $S_p(n)$ as a Polynomial

By expanding the determinant (2.2.2) with respect to the last column,

$$S_p(n)=\sum_{r=0}^{p} a_{p+1-r}n^{p+1-r}, \tag{2.3.1}$$

with $a_{p+1}=\frac{1}{p+1}$ and, for $r=1,2,\dots,p$,

$$a_{p+1-r}=\frac{(-1)^r p!}{(p+1-r)!}\begin{vmatrix} -\frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 \\ \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{(-1)^{r+1}}{r!} & \frac{(-1)^r}{(r-1)!} & \frac{(-1)^{r-1}}{(r-2)!} & \cdots & \frac{1}{1!} \\ \frac{(-1)^{r+2}}{(r+1)!} & \frac{(-1)^{r+1}}{r!} & \frac{(-1)^r}{(r-1)!} & \cdots & -\frac{1}{2!} \end{vmatrix}$$

$$=\frac{p!}{(p+1-r)!}\begin{vmatrix} \frac{1}{2!} & -\frac{1}{1!} & 0 & \cdots & 0 \\ -\frac{1}{3!} & \frac{1}{2!} & -\frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{(-1)^{r+2}}{r!} & \frac{(-1)^{r+1}}{(r-1)!} & \frac{(-1)^r}{(r-2)!} & \cdots & -\frac{1}{1!} \\ \frac{(-1)^{r+3}}{(r+1)!} & \frac{(-1)^{r+2}}{r!} & \frac{(-1)^{r+1}}{(r-1)!} & \cdots & \frac{1}{2!} \end{vmatrix}. \tag{2.3.2}$$

Now multiply the $2^{\text{nd}}$, $4^{\text{th}}$, ... columns of the determinant in (2.3.2) by $-1$ and then multiply the $2^{\text{nd}}$, $4^{\text{th}}$, ... rows of the resulting determinant by $-1$. Then

$$a_{p+1-r}=\frac{p!}{(p+1-r)!}\begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(r+1)!} & \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \frac{1}{2!} \end{vmatrix}. \tag{2.3.3}$$

Next, recall (see, e.g., [4], p. 323) that the Bernoulli numbers $B_j$, $j=1,2,\dots$, can be represented by the determinants

$$B_j=(-1)^j j!\begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{j!} & \frac{1}{(j-1)!} & \frac{1}{(j-2)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(j+1)!} & \frac{1}{j!} & \frac{1}{(j-1)!} & \cdots & \frac{1}{2!} \end{vmatrix}. \tag{2.3.4}$$

Hence, by (2.3.3) and (2.3.4),

$$a_{p+1-r} = \frac{1}{p+1}(-1)^r \binom{p+1}{r} B_r, \; r = 1, 2, \ldots, p.$$

Thus, by (2.3.1),

$$S_p(n) = \frac{1}{p+1} \sum_{r=0}^{p} (-1)^r \binom{p+1}{r} B_r n^{p+1-r}. \tag{2.3.5}$$

Since $B_{2r+1} = 0 \; (r \in N)$,

$$S_p(n) = \frac{1}{p+1} n^{p+1} + \frac{1}{2} n^p + \frac{1}{2}\binom{p}{1} B_2 n^{p-1} + \frac{1}{4}\binom{p}{3} B_4 n^{p-3} + \cdots,$$

with the last term either containing $n$ or $n^2$. This is the form in which $S_p(n)$ was given by Jacques Bernoulli in [1].

## 3. FORMULAS IN TERMS OF POWERS OF $n+1$

### 3.1 A Recurrence Formula

Let $n \in N$. For $k \in N$, let $S_k(n) = 1 + 2^k + 3^k + \cdots + n^k = \Sigma_{r=0}^{n} r^k$ and take $S_0(n) = n+1$. Then, arguing as in the steps leading to (2.1.1) and (2.1.2),

$$\begin{aligned} (n+1)^k &= S_k(n+1) - S_k(n) \\ &= \sum_{i=0}^{k} \binom{k}{i} S_i(n) - S_k(n) \qquad (3.1.1) \\ &= \sum_{i=0}^{k-1} \binom{k}{i} S_i(n). \qquad (3.1.2) \end{aligned}$$

The equation

$$(S+1)^k - S^k = (n+1)^k, \tag{3.1.3}$$

in which the binomial power is expanded and $S^i$ $(i = 0, 1, 2, \ldots, k)$ are then replaced by $S_i(n)$, provides a mnemonic for (3.1.1). Note in particular that (3.1.3) can be obtained from (2.1.3) by merely increasing the values of $S$, $S-1$, and $n$ by one.

For example, let $k = 2$ in (3.1.3). Then $1 + 2S = (n+1)^2$ and thus $n + 1 + 2S_1(n) = (n+1)^2$, again yielding (2.1.4).

### 3.2 $S_p(n)$ as a Determinant

Let $p \in N$ and let $k = 1, 2, \ldots, p, p+1$ in (3.1.2). It follows as in section 2.2, with $(-1)^r$ replaced by 1, that

$$S_p(n)=\frac{1}{(p+1)!}\begin{vmatrix}1 & 0 & 0 & 0 & \cdots & 0 & n+1\\ 1 & \binom{2}{1} & 0 & 0 & \cdots & 0 & (n+1)^2\\ 1 & \binom{3}{1} & \binom{3}{2} & 0 & \cdots & 0 & (n+1)^3\\ \vdots & \vdots & \vdots & & & & \vdots\\ 1 & \binom{p}{1} & \binom{p}{2} & & \cdots & \binom{p}{p-1} & (n+1)^p\\ 1 & \binom{p+1}{1} & \binom{p+1}{2} & & \cdots & \binom{p+1}{p-1} & (n+1)^{p+1}\end{vmatrix} \tag{3.2.1}$$

or, alternatively,

$$S_p(n)=p!\begin{vmatrix}\frac{1}{1!} & 0 & 0 & 0 & \cdots & 0 & \frac{n+1}{1!}\\ \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & \cdots & 0 & \frac{(n+1)^2}{2!}\\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 & \frac{(n+1)^3}{3!}\\ \vdots & \vdots & \vdots & & & & \vdots\\ \frac{1}{p!} & \frac{1}{(p-1)!} & \frac{1}{(p-2)!} & & \cdots & \frac{1}{1!} & \frac{(n+1)^p}{p!}\\ \frac{1}{(p+1)!} & \frac{1}{p!} & \frac{1}{(p-1)!} & & \cdots & \frac{1}{2!} & \frac{(n+1)^{p+1}}{(p+1)!}\end{vmatrix}. \tag{3.2.2}$$

Note in particular that the determinants in (3.2.1) and (3.2.2) can be obtained from their counterparts in (2.2.1) and (2.2.2) by merely replacing $n$ by $n+1$ in the last column, and replacing all negative entries by their absolute values.

### 3.3 $S_p(n)$ as a Polynomial

Proceeding as in section 2.3, (3.2.2) can now be employed to establish the formula

$$S_p(n)=\sum_{r=0}^{p} c_{p+1-r}(n+1)^{p+1-r}, \tag{3.3.1}$$

with $c_{p+1}=\frac{1}{p+1}$ and, for $r=1,2,\ldots,p$,

$$c_{p+1-r}=\frac{(-1)^r p!}{(p+1-r)!}\begin{vmatrix}\frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0\\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \frac{1}{1!}\\ \frac{1}{(r+1)!} & \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \frac{1}{2!}\end{vmatrix}. \tag{3.3.2}$$

Hence, by (2.3.4) and (3.3.2),

$$c_{p+1-r}=\frac{1}{p+1}\binom{p+1}{r}B_r,\ r=1,2,\ldots,p.$$

Thus, by (3.3.1),

$$S_p(n)=\frac{1}{p+1}\sum_{r=0}^{p}\binom{p+1}{r}B_r(n+1)^{p+1-r}. \tag{3.3.3}$$

This standard form of $S_p(n)$ is usually established with the aid of the generating function $xe^{tx}/(e^x-1)$ of the Bernoulli polynomials.

The question arises if this method could have led to formulas (2.3.5) and (3.3.3) with $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_3=0$, $B_4=-\frac{1}{30}, \ldots$ without prior knowledge of the Bernoulli numbers. This indeed is the case. First note that, by (2.3.1) and (2.3.3), and (3.3.1) and (3.3.2),

$$S_p(n)=\frac{1}{p+1}\sum_{r=0}^{p}(-1)^r\binom{p+1}{r}b_r n^{p+1-r},$$

and

$$S_p(n)=\frac{1}{p+1}\sum_{r=0}^{p}\binom{p+1}{r}b_r(n+1)^{p+1-r},$$

with $b_0=1$ and, for $r=1,2,\ldots,p$,

$$b_r=(-1)^r r!\begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(r+1)!} & \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \frac{1}{2!} \end{vmatrix}$$

$$=r!\begin{vmatrix} \frac{1}{1!} & 0 & 0 & \cdots & 0 & 1 \\ \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \frac{1}{1!} & 0 \\ \frac{1}{(r+1)!} & \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \frac{1}{2!} & 0 \end{vmatrix}.$$

Now observe that the last determinant differs from that of $S_r(n)$, as obtained by setting $p=r$ in (3.2.2), only with respect to the last column—the entries of $b_r$, from top to bottom, are 1, 0, ..., 0 while those of $S_r(n)$ are $\frac{n+1}{1!}, \frac{(n+1)^2}{2!}, \ldots, \frac{(n+1)^{r+1}}{(r+1)!}$. It follows [cf. (3.1.2)] that $b_0, b_1, b_2, \ldots$ satisfy the recurrence formula

$$b_0=1,\quad \sum_{i=0}^{r-1}\binom{r}{i}b_i=0 \quad (r=2,3,4,\ldots),$$

which generates the numbers $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, \ldots$, i.e., the Bernoulli numbers.

## REFERENCES

1. J. Bernoulli. *Ars conjectandi*. 1713.
2. D. Kalman. "Sums of Powers by Matrix Methods." *The Fibonacci Quarterly* **28.1** (1990): 60-71.
3. R. A. Khan. "A Simple Derivation of a Formula for $\Sigma_{k=1}^{n} k^r$." *The Fibonacci Quarterly* **19.2** (1981):177-80.
4. L. A. Lyusternik & A. R. Yanpol'skii, eds. *Mathematical Analysis—Functions, Limits, Series, Continued Fractions*. New York: Pergamon Press, 1965.

AMS Classification Numbers: 05A10, 05A19

# PRIME POWERS OF ZEROS OF MONIC POLYNOMIALS WITH INTEGER COEFFICIENTS

**Garry J. Tee**
Department of Mathematics, University of Auckland, Auckland, New Zealand
(*Submitted December 1992*)

## 1. INTRODUCTION

For a monic polynomial with integer coefficients $x^d - a_1 x^{d-1} - \cdots - a_d$, the sum $S_k$ of the $k^{\text{th}}$ powers of the zeros is an integer, for positive integer $k$. For prime $p$, $S_p \equiv a_1 \pmod p$; and hence, if $a_1 = 0$ then $p|S_p$. If $a_d = \pm 1$, then similar congruences hold for sums of negative powers of the zeros. Illustrations are given for various types of Chebyshev polynomials with integer argument.

## 2. SYMMETRIC FUNCTIONS OF ROOTS

Consider the monic polynomial equation with complex (or real) coefficients

$$x^d - a_1 d^{d-1} - a_2 x^{d-2} - \cdots - a_d = 0. \tag{1}$$

The roots of equation (1) will be denoted by $\alpha, \beta, \gamma, \ldots, \psi, \omega$, and those symmetric functions of the roots that are called *sigma functions* will be denoted thus:

$$\begin{aligned}
\Sigma\alpha &\stackrel{\text{def}}{=} \alpha + \beta + \cdots + \omega,\\
\Sigma\alpha\beta &\stackrel{\text{def}}{=} \alpha\beta + \alpha\gamma + \cdots + \alpha\omega + \beta\gamma + \cdots + \beta\omega + \cdots + \psi\omega,\\
\Sigma\alpha^3\beta^2 &\stackrel{\text{def}}{=} \alpha^3\beta^2 + \alpha^3\gamma^2 + \cdots + \alpha^3\omega^2 + \beta^3\gamma^2 + \cdots + \beta^3\omega^2 + \cdots + \psi^3\omega^2\\
&\quad + \beta^3\alpha^2 + \gamma^3\alpha^2 + \cdots + \omega^3\alpha^2 + \gamma^3\beta^2 + \cdots + \omega^3\beta^2 + \cdots + \omega^3\psi^2,\\
&\qquad \textit{et cetera}.
\end{aligned} \tag{2}$$

The sigma functions $\Sigma\alpha, \Sigma\alpha\beta, \Sigma\alpha\beta\gamma, \ldots, \Sigma\alpha\beta\gamma\ldots\omega$ are called the *elementary symmetric functions* of $\alpha, \beta, \gamma, \ldots., \omega$, and Vieta's Rule expresses them in terms of the coefficients of the polynomial (1):

$$\begin{gathered}
\Sigma\alpha = a_1, \quad \Sigma\alpha\beta = -a_2, \quad \Sigma\alpha\beta\gamma = a_3,\\
\ldots, \ \Sigma\alpha\beta\gamma\ldots\omega = \alpha\beta\gamma\ldots\omega = (-1)^{d-1}a_d.
\end{gathered} \tag{3}$$

Each symmetric polynomial with integer coefficients can be expressed as a polynomial in the elementary symmetric functions, with integer coefficients ([1], p. 67).

Therefore, if all coefficients $a_1, \ldots, a_d$ of the monic polynomial (1) are integers (positive, negative, or zero), each symmetric polynomial [in the roots of (1)] with integer coefficients has integer value. In particular, each sigma function then has integer value.

For integer $k$, denote the sum of the $k^{\text{th}}$ powers of the roots as

$$S_k \stackrel{\text{def}}{=} \Sigma\alpha^k = \alpha^k + \beta^k + \cdots + \omega^k, \tag{4}$$

which is a sigma function if $k > 0$. The initial values $S_1, S_2, \ldots, S_d$ may be computed successively by Newton's Rule:

$$S_k = a_1 S_{k-1} + a_2 S_{k-2} + \cdots + a_{k-2} S_2 + a_{k-1} S_1 + k \cdot a_k \quad (k = 1, 2, \ldots, d), \tag{5}$$

and for $k > d$, Newton's Rule becomes the recurrence relation

$$S_k = a_1 S_{k-1} + a_2 S_{k-2} + \cdots + a_d S_{k-d} \quad (k = d+1, d+2, d+3, \ldots), \tag{6}$$

by which $S_{d+1}, S_{d+2}, S_{d+3}, \ldots$ may be computed successively.

If the coefficients $a_1, \ldots, a_d$ are integers, then $S_k$ has integer value for all positive integers $k$, by the general result cited above for symmetric polynomials with integer coefficients. But for the $S_k$, it is simpler to note [from (5)] that $S_1 = a_1$, and the result then follows from (5) and (6) by induction on $k$.

From Newton's Rule, the sums of powers of roots can be expressed in terms of the coefficients of the monic polynomial (1). For example,

$$\begin{aligned}
&S_1 = a_1, \quad S_2 = a_1^2 + 2a_2, \quad S_3 = a_1^3 + 3(a_1 a_2 + a_3),\\
&S_4 = a_1^4 + 4a_1^2 a_2 + 4a_1 a_3 + 2a_2^2 + 4a_4,\\
&S_5 = a_1^5 + 5(a_1^3 a_2 + a_1^2 a_3 + a_1(a_2^2 + a_4) + a_2 a_3 + a_5),\\
&S_6 = a_1^6 + 6a_1^4 a_2 + 6a_1^3 a_3 + a_1^2(9a_2^2 + 6a_4) + a_1(12a_2 a_3 + 6a_5)\\
&\qquad + 2a_2^3 + 18a_2 a_4 + 3a_3^2 + 6a_6,\\
&S_7 = a_1^7 + 7(a_1^5 a_2 + a_1^4 a_3 + a_1^3(2a_2^2 + a_4) + a_1^2(3a_2 a_3 + a_5)\\
&\qquad + a_1(a_2^3 + 2a_2 a_4 + a_3^2 + a_6) + a_2^2 a_3 + a_2 a_5 + a_3 a_4 + a_7),
\end{aligned} \tag{7}$$

where $a_j$ is taken as 0 if $j > d$.

Waring's formula (of 1762) expresses $S_k$ explicitly ([1], p. 72) in terms of the coefficients of the monic polynomial (1):

$$S_k = \Sigma \frac{k \cdot (r_1 + r_2 + \cdots + r_d - 1)!}{r_1! r_2! \ldots r_d!} a_1^{r_1} a_2^{r_2} \ldots a_d^{r_d}, \tag{8}$$

where the sum extends over all sets of nonnegative integers $r_1, r_2, \ldots, r_d$ for which

$$r_1 + 2r_2 + 3r_3 + \cdots + dr_d = k. \tag{9}$$

The expressions (7) for $S_1, \ldots, S_7$ suggest that $S_k$ has some interesting divisibility properties for prime $k$.

## 3. DIVISIBILITY OF SUMS OF PRIME POWERS OF ROOTS

Hereinafter, the polynomial coefficients $a_1, \ldots, a_d$ are taken to be integers, except where otherwise stated.

***Theorem 1:*** For all primes $p$, $S_p \equiv a_1 \pmod{p}$.

***Proof:*** If all roots are integers, then by Fermat's Little Theorem,

$$S_p = \alpha^p + \beta^p + \cdots + \omega^p \equiv \alpha + \beta + \cdots + \omega \equiv a_1 \pmod{p}. \tag{10}$$

In the general case, when the roots are algebraic numbers, expand $S_1^k$ by the Multinomial Theorem:

$$\begin{aligned} S_1^k &= (\alpha+\beta+\gamma+\cdots+\omega)^k \\ &= \alpha^k+\beta^k+\gamma^k+\cdots+\omega^k+\sum_{q+\cdots+v=k}\frac{k!}{q!r!s!\ldots v!}\sum\alpha^q\beta^r\gamma^s\ldots\omega^v, \end{aligned} \tag{11}$$

where at least two of the indices $q$, $r$, ..., $v$ are positive integers, and the others equal zero. This may be rewritten as:

$$a_1^k = S_k+\sum_{q+\cdots+v=k}\frac{k!}{q!r!s!\ldots v!}\sum\alpha^q\beta^r\gamma^s\ldots\omega^v. \tag{12}$$

Each multinomial coefficient is an integer; hence, the denominator $q!r!s! \ldots v!$ divides the numerator $k! = k(k-1)!$. Every factor in the denominator is strictly less than $k$; and hence, if $k$ is prime the denominator and $k$ are coprime, so the denominator must then divide the other factor $(k-1)!$ in the numerator. Therefore, if $k$ is prime then each such multinomial coefficient is an integer multiple of $k$.

But we have seen that, if all coefficients $a_1, \ldots, a_d$ are integers, then each of the sigma functions in (12) has integer value. Thus, if $k$ is any prime $p$, then it follows from (12) that

$$a_1^p = S_p + pF_p, \tag{13}$$

where $F_p$ is an integer* which depends on $p$ (and also on $a_1, a_2, \ldots, a_d$). Therefore,

$$S_p \equiv a_1^p \equiv a_1 \pmod{p}, \tag{14}$$

by Fermat's Little Theorem. □

***Corollary 1.1:*** If $p$ is prime, then $p|S_p \Leftrightarrow p|a_1$.

***Corollary 1.2:*** If $a_1 = \pm 1$, then $S_p$ is not a multiple of $p$ for any prime $p$.

***Corollary 1.3:*** If $a_1 = \pm q^e$, where $q$ is prime and $e \geq 1$, then $q$ is the only prime $p$ for which $p|S_p$.

It was shown above that, if $k$ is prime, then each such multinomial coefficient is an integer multiple of $k$. However, the converse does not hold. For example, $k!/(1!)^k = k(k-1)!$ for all $k \geq 2$; $k!/(2!(1!)^{k-1}) = k\times((k-1)(k-2)\ldots 3)$ for all $k \geq 3$; $8!/(2!)^4 = 8\times(7\times 5\times 3^2)$, and so on.

***Theorem 2:*** $S_p$ is an integer multiple of $p$ for all primes $p$, if and only if $a_1 = 0$.

***Proof:*** If $a_1 = 0$, then equation (13) reduces to $S_p = -pF_p$, and hence $p|S_p$.**

If $p|S_p$ then (by Theorem 1, Corollary 1), $p|a_1$ and, if this holds for infinitely many primes $p$, then $a_1 = 0$. □

The converse does not hold, since examples exist with $k|S_k$ where $k$ is composite. For example (see [2]), take $d = 3$ with roots 1, 1, –2 (with $\sum\alpha = a_1 = 0$), for which the characteristic

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* The proof given in Theorem 1 of [2] for this result is valid only for the case in which all roots $\alpha, \beta, \ldots$ are integers.

** This is Theorem 2 in [2].

polynomial is $(x-1)^2(x+2) = x^3-3x+2$ and $S_k = 2+(-2)^k$. In this case, $S_6 = 66$ so that $6|S_6$, and 6 is composite.

***Lemma:*** If $a_d = \pm 1$, then $S_k$ has integer values for all integers $k$—positive, zero, and negative.

***Proof:*** For general complex coefficients $a_1, \ldots, a_d$, if $a_d \neq 0$, then $\alpha\beta\gamma \ldots \omega = (-1)^{d-1}a_d \neq 0$, so that no root equals 0; hence, $S_0$ exists:

$$S_0 = \alpha^0 + \beta^0 + \cdots + \omega^0 = 1+1+\cdots+1 = d. \tag{15}$$

The monic polynomial equation inverse to (1),

$$z^d + \frac{a_{d-1}}{a_d}z^{d-1} + \frac{a_{d-2}}{a_d}z^{d-2} + \cdots + \frac{a_1}{a_d}z - \frac{1}{a_d} = 0, \tag{16}$$

has roots $\alpha^{-1}, \beta^{-1}, \ldots, \omega^{-1}$, including multiplicity. Accordingly, for $k \le -1$, $S_k$ can be constructed by Newton's Rule from the coefficients in (16), similarly to (5) and (6).

If all coefficients $a_1, \ldots, a_d$ in (1) are integers and $a_d = \pm 1$, then all coefficients of the monic polynomial (16) are integers. It follows as in (5) and (6) that $S_k$ has integer value for all integers $k \le -1$. Combining these results with the previous result for $k \ge 1$, we get that $S_k$ has integer value for all integers $k$. □

***Theorem 3:*** If $p$ is prime, $S_{-p} \equiv -a_{d-1} \pmod{p}$ if $a_d = 1$, and $S_{-p} \equiv a_{d-1} \pmod{p}$ if $a_d = -1$.

***Proof:*** Apply Theorem 1 to the inverse polynomial equation (13), which is now

$$\begin{cases} z^d + a_{d-1}z^{d-1} + a_{d-2}z^{d-2} + \cdots + a_1 z - 1 = 0 & \text{if } a_d = +1, \\ z^d - a_{d-1}z^{d-1} - a_{d-2}z^{d-2} - \cdots - a_1 z + 1 = 0 & \text{if } a_d = -1. \end{cases} \quad \square \tag{17}$$

Note that this result holds for a more general polynomial with integer coefficients, with leading term $-a_0x^d$ rather than $x^d$ as in (1).

***Corollary 3.1:*** If $a_d = \pm 1$ and $p$ is prime, then $p|S_{-p} \Leftrightarrow p|a_{d-1}$.

***Corollary 3.2:*** If $a_d = \pm 1$ and $a_{d-1} = \pm 1$, then $S_{-p}$ is not a multiple of $p$ for any prime $p$.

***Corollary 3.3:*** If $a_d = \pm 1$ and $a_1 = \pm 1$ and $a_{d-1} = \pm 1$, then, for all primes $p$, $p \nmid S_p$ and $p \nmid S_{-p}$.

***Corollary 3.4*** If $a_d = \pm 1$ and $a_{d-1} = \pm q^f$, where $q$ is prime and $f \ge 1$, then $q$ is the only prime $p$ for which $p|S_{-p}$.

***Corollary 3.5:*** If $a_d = \pm 1$ and $a_1 = \pm q^e$ and $a_{d-1} = \pm q^f$, where $q$ is prime and $e \ge 1$ and $f \ge 1$, then $q$ is the only prime $p$ for which $p|S_p$, and also $q$ is the only prime $p$ for which $p|S_{-p}$.

***Corollary 3.6:*** If $a_d = \pm 1$, then there is no prime $p$ that divides both $S_p$ and $S_{-p}$ if and only if $a_1$ and $a_{d-1}$ are coprime.

***Corollary 3.7:*** If $a_d = \pm 1$ and if $a_1$ and $a_{d-1}$ have the same set of prime divisors and if $p$ is prime, then $p|S_p \Leftrightarrow p|a_1 \Leftrightarrow p|a_{d-1} \Leftrightarrow p|S_{-p}$.

Note that $a_1$ and $a_{d-1}$ may have different signs, and they may have different exponents for their prime factors.

***Theorem 4:*** If $a_d = \pm 1$, then $S_{-p}$ is an integer multiple of $p$ for all primes $p$ if and only if $a_{d-1} = 0$.

***Proof:*** Apply Theorem 2 to the inverse polynomial (17). $\square$

***Theorem 5:*** For all polynomial equations of the form

$$x^d - a_2 x^{d-2} - a_3 x^{d-3} - \cdots - a_{d-3} x^3 - a_{d-2} x^2 \pm 1 = 0, \tag{18}$$

with integer coefficients, both $S_p$ and $S_{-p}$ are integer multiples of $p$ for all primes $p$.

***Proof:*** By Theorem 2, $p|S_p$ since $a_1 = 0$, and by Theorem 4, $p|S_{-p}$ since $a_d = \pm 1$ and $a_{d-1} = 0$. $\square$

## 4. APPLICATION TO CHEBYSHEV POLYNOMIALS

The Chebyshev polynomials of the first kind are defined by the initial values:

$$T_0(y) \stackrel{\text{def}}{=} 1, \quad T_1(y) \stackrel{\text{def}}{=} y; \tag{19}$$

with the recurrence relation

$$T_n(y) = 2yT_{n-1}(y) - T_{n-2}(y), \quad (n = 2, 3, \ldots). \tag{20}$$

In terms of the modified Chebyshev polynomial of the first kind,

$$C_n(z) \stackrel{\text{def}}{=} 2T_n\left(\frac{z}{2}\right), \tag{21}$$

the initial values are

$$C_0(z) \stackrel{\text{def}}{=} 2, \quad C_1(z) \stackrel{\text{def}}{=} z, \tag{22}$$

and the recurrence relation is

$$C_n(z) = zC_{n-1}(z) - C_{n-2}(z), \quad (n = 2, 3, \ldots). \tag{23}$$

The characteristic polynomial for $T_n(y)$ is

$$P(x) = x^2 - 2xy + 1. \tag{24}$$

In terms of the roots of the characteristic equation,

$$\alpha = y + \sqrt{y^2 - 1}, \quad \beta = y - \sqrt{y^2 - 1}, \tag{25}$$

(22) becomes

$$C_0(2y) = 2 = \alpha^0 + \beta^0 = S_0, \quad C_1(2y) = 2y = \alpha + \beta = S_1, \tag{26}$$

and it follows from (23) by induction on $n$ that

$$C_k(2y) = 2T_k(y) = \alpha^k + \beta^k = S_k \quad (k = 0, 1, 2, \ldots). \tag{27}$$

***Theorem 6:*** For integer $j$, $T_p(j) \equiv j \pmod p$ for all odd primes $p$, and $2T_p\left(j + \frac{1}{2}\right) \equiv (2j+1) \pmod p$ for all primes $p$.

***Proof:*** If $m = 2y$ is any integer, then it follows from (22) and (23) by induction on $n$ that $S_k = C_k(m) = 2T_k\left(\frac{m}{2}\right)$ is an integer for all integers $k \geq 0$, and Theorem 1 shows that, for every prime $p$,

$$2T_p\left(\frac{m}{2}\right) = S_p \equiv m \pmod{p}. \tag{28}$$

Therefore, if $y = j$ is any integer and $p$ is prime,

$$2T_p(j) \equiv 2j \pmod{p}; \tag{29}$$

and hence, for every integer $j$ and every odd prime $p$,

$$T_p(j) \equiv j \pmod{p}. \tag{30}$$

For $p = 2$,

$$T_2(j) = 2j^2 - 1, \tag{31}$$

so that (30) holds only for odd $j$.

If $2y = m = 2j + 1$ is odd, then, for every prime $p$, (28) becomes

$$2T_p\left(j + \frac{1}{2}\right) \equiv (2j+1) \pmod{p} \tag{32}$$

for all integers $j$. □

***Theorem 7:*** For odd prime $p$, $T_p(j) \equiv j$ (mod $jp$) for all integers $j$ except multiples of $p$, and if $j$ is odd (and not a multiple of $p$) then $T_p(j) \equiv j$ (mod $2jp$).

***Proof:*** For integer $j$ and odd prime $p$,

$$T_p(j) = j + ep, \tag{33}$$

where $e$ is an integer, in view of Theorem 6.

From the initial values (19), it follows from (20) by induction on $n$ that $T_n(y) = 2^{n-1}y^n - \cdots$ is a polynomial in $y$ of degree $n$ with integer coefficients, and that $T_n(y)$ is an even polynomial in $y$ if $n$ is even and $T_n(y)$ is an odd polynomial in $y$ if $n$ is odd. Hence, if $j$ is an integer and $n$ is odd, then $j \mid T_n(j)$. Thus, for all odd primes $p$,

$$j + ep = T_p(j) = jb \tag{34}$$

for some integer $b$.

If $j$ is an even integer then $jb$ is even; and hence $ep$ is even, so that $e = 2f$ for some integer $f$.

If $j$ is an odd integer then $T_0(j)$ and $T_1(j)$ are odd [from (19)], and it follows from (20) by induction on $n$ that $T_n(j)$ is odd for all $n \geq 0$. Thus, both $j$ and $T_p(j)$ in (33) are odd; hence, $ep$ is even, so that $e = 2f$.

Therefore, for all integers $j$ and odd prime $p$,

$$j + 2fp = T_p(j) = jb, \tag{35}$$

so that, if $j$ is not a multiple of $p$, then $j \mid (2f)$ and if $j$ is also odd then $j \mid f$. □

***Theorem 8:*** For prime $p \geq 5$ and odd integer $m$, $2T_p\left(\frac{m}{2}\right) \equiv m$ (mod $2p$), and if $m$ is not a multiple of $p$ then $2T_p\left(\frac{m}{2}\right) \equiv m$ (mod $2mp$).

***Proof:*** From (22) we get $C_0(m)=2$, which is even, and $C_1(m)=m$, which is odd; and from (23) we get $C_2(m)=m^2-2$, which is odd. It follows from (23), by induction on $n$, that $C_n(m)$ is even if and only if $3|n$. From (31),

$$C_p(m)=2T_p\left(\frac{m}{2}\right)=m+ep, \tag{36}$$

where $e$ is an integer; hence, for all primes $p\neq 3$, we must have $ep$ even. Thus, for all odd integers $m$ and for all primes $p\geq 5$, $e$ must be even $e=2f$; therefore,

$$2T_p\left(\frac{m}{2}\right)=m+2fp\equiv m \pmod{2p} \quad (p\geq 5). \tag{37}$$

From the initial values (19), it follows from (23) by induction on $n$ that $C_n(z)=z^n-\cdots$ is a monic polynomial in $z$ of degree $n$ with integer coefficients, and that $C_n(z)$ is an even polynomial in $z$ if $n$ is even and $C_n(z)$ is an odd polynomial in $z$ if $n$ is odd. Hence, if $j$ is an integer and $n$ is odd, then $j|C_n(j)$, so that for all odd primes $p$,

$$C_p(j)=jb, \tag{38}$$

where $b$ is an integer, and if $j=m$ is an odd integer and $p\geq 5$, then

$$m+2fp=C_p(m)=mb. \tag{39}$$

Therefore, if $m$ is not a multiple of $p$, then $m|(2f)$, and since $m$ is odd then $m|f$, so that

$$C_p(m)=2T_p\left(\frac{m}{2}\right)\equiv m \pmod{2mp}. \ \square \tag{40}$$

## REFERENCES


1. Leonard Eugene Dickson. *Elementary Theory of Equations*. New York: Wiley; London: Chapman & Hall, 1914.
2. B. H. Neumann & L. G. Wilson. "Some Sequences Like Fibonacci's." *The Fibonacci Quarterly* **17.1** (1979):80-83. (Rpt. in *Selected Works of B. H. Neumann and Hanna Neumann*, 6 vols [Winnipeg: The Charles Babbage Research Centre, 1988], **1**:131-34.)
3. B. H. Neumann & L. G. Wilson. "Corrigenda to 'Some Sequences Like Fibonacci's.'" *The Fibonacci Quarterly* **21.3** (1983):229. (Rpt. in *Selected Works of B. H. Neumann and Hanna Neumann*, 6 vols [Winnipeg: The Charles Babbage Research Centre, 1988], **1**:135.)


AMS Classification Numbers: 11A07, 11A41, 11C08

# A SYMMETRY PROPERTY OF ALTERNATING SUMS OF PRODUCTS OF RECIPROCALS

**I. J. Good**
Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0439
*(Submitted January 1993)*

Consider the homogeneous linear recurrence relation

$$G_{n+2}=aG_{n+1}+G_n \quad (n=0,\pm 1,\pm 2,\ldots) \tag{1}$$

where $a$ is a nonzero real or complex constant. The equation is especially familiar in relation to the theory of simple continued fractions, and in relation to the theory of numbers when $a$ is a natural integer. (For the case where $a$ is a Gaussian integer, see Good [4] and [5].) A solution $(G_n)$ can be regarded as a vector of a countably infinite number of components or elements, and which is completely determined "in both directions" by any two consecutive components. The general solution is a linear combination of any pair of linearly independent solutions. Two solutions are linearly independent under the nonvanishing of the 2-by-2 determinant consisting of two consecutive elements of one solution and the corresponding two elements of the other solution. Perhaps the simplest pair of independent solutions is given by

$$F_n=F_n(\xi)=\frac{\xi^n-\eta^n}{\xi-\eta}, \quad L_n=L_n(\xi)=\xi^n+\eta^n \quad (n=0,\pm 1,\pm 2,\ldots) \tag{2}$$

where

$$\xi=\frac{a+\sqrt{a^2+4}}{2}, \quad \eta=\frac{a-\sqrt{a^2+4}}{2} \quad (a\neq 2i). \tag{3}$$

Note that $|\xi|>|\eta|$ when $a$ is real and positive; also that $\xi=\eta=i$ if $a=2i$, and then $F_n=F_n(i)$ must be defined as $ni^{n-1}$ while $L_n(i)=2i^n$. The numbers $\xi$ and $\eta$ are the roots of the quadratic equation

$$x^2-ax-1=0, \tag{4}$$

and, of course,

$$\xi+\eta=a, \quad \xi\eta=-1. \tag{5}$$

In particular, when $a=1$, in which case $\xi$ is the golden ratio, $F_n$ and $L_n$ reduce to the Fibonacci and Lucas numbers. We write the general solution of (1) as

$$G_n=G_n(\xi)=\lambda F_n(\xi)+\mu L_n(\xi) \tag{6}$$

where $\lambda$ and $\mu$ are not necessarily real.

We shall prove the following symmetry property:

***Theorem:*** We have

$$F_k\sum_{n=1}^{m}\frac{(-1)^n}{G_nG_{n+k}}=F_m\sum_{n=1}^{k}\frac{(-1)^n}{G_nG_{n+m}}, \tag{7}$$

where $k$ and $m$ are nonnegative integers, and where we assume further that all the numbers $G_1, G_2, \ldots, G_{m+k}$ are nonzero.

***Comment (i):*** It follows from equations (2) and (3) that the nonzero condition is certainly true when $a$ is real and $(G_n)$ is either $(F_n)$ or $(L_n)$, that is, when $\lambda = 1$ and $\mu = 0$ or when $\lambda = 0$ and $\mu = 1$.

***Comment (ii):*** The theorem is presumably new even when $(G_n)$ reduces to the ordinary Fibonacci or Lucas sequence, that is, when $a = 1$ and $\lambda = 1$, $\mu = 0$ or $\lambda = 0$, $\mu = 1$.

***Comment (iii):*** If empty sums are regarded as vanishing, the theorem is true but uninformative when $k$ or $m$ is zero. It is also uninformative when $k = m$.

***Comment (iv):*** Even in the simple case $F_n = ni^{n-1}$, $L_n = 2i^n$, the identity (7) is not entirely obvious when $G_n$ is defined by (6).

***Corollary:*** When $|\xi| > |\eta|$ we have

$$F_k \sum_{n=1}^{\infty} \frac{(-1)^n}{G_n G_{n+k}} = \frac{1}{\lambda + \mu\sqrt{(a^2+4)}} \sum_{n=1}^{k} \frac{\eta^n}{G_n}. \tag{8}$$

***Comment (v):*** In the very special case $a = 1$, $k = 1$, $\mu = 0$, (8) reduces to formula (102) of Vajda [8]; and when $a = 1$, $k = 2$, $\mu = 0$, the evaluation of the left side of (8) was proposed as a problem by Clark [2]. The right side of (8) solves a much more general problem.

***Proof of the Theorem:*** Without loss of generality, we assume $m \geq k$. The proof depends on a double induction, beginning with an induction with respect to $m$. We first note that the result is obvious when $m = k$, so we can proceed at once to the body of the induction. For this we need to show that

$$\sum_{n=1}^{k} \frac{(-1)^n}{G_n}\left(\frac{F_{m+1}}{G_{m+n+1}} - \frac{F_m}{G_{m+n}}\right) = \frac{(-1)^{m+1}F_k}{G_{m+1}G_{m+k+1}}. \tag{9}$$

Now, by means of some straightforward algebra it can be shown from (2), and generalizing the case $h = 1$ of formulas (19b) and (20a) of Vajda [8], that

$$F_{m+1}F_{m+n} - F_m F_{m+n+1} = (-1)^m F_n \tag{10}$$

and that

$$F_{m+1}L_{m+n} - F_m L_{m+n+1} = (-1)^m L_n \tag{11}$$

and hence that

$$F_{m+1}G_{m+n} - F_m G_{m+n+1} = (-1)^m G_n. \tag{12}$$

Therefore, (9) is equivalent to the identity

$$\sum_{n=1}^{k} \frac{(-1)^n}{G_{m+n}G_{m+n+1}} = \frac{F_k}{G_{m+1}G_{m+k+1}}. \tag{13}$$

To prove this identity, we perform an induction, this time with respect to $k$, noting first that it is trivially true when $k = 1$. So we now want to prove that

$$\frac{(-1)^{k+1}}{G_{m+k+1}G_{m+k+2}} = \frac{-F_{k+1}}{G_{m+1}G_{m+k+2}} + \frac{F_k}{G_{m+1}G_{m+k+1}}, \tag{14}$$

that is, we want

$$F_{k+1}G_{m+k+1} - F_k G_{m+k+2} = (-1)^k G_{m+1}. \tag{15}$$

But this identity is equivalent to 12) with a change of notation in the subscripts. Hence, in turn, we have proved (14), (13), (9), and (7), the statement of the theorem.

We could reverse the steps of the argument to prove each statement in turn, but the order used here shows the motivation at each step and also shows the way that the *proof* was discovered. It is more difficult to describe, or even to recall, how the theorem itself was discovered except that naturally it depended in part on guesswork and on numerical experimentation. (Many nonmathematicians don't know that pure mathematics is an experimental science.) For an alternative proof, see the Appendix.

**Corresponding Trigonometrical Identities**

Corresponding to many identities involving ordinary Fibonacci and Lucas numbers, there are "parental" (more general) identities obtained by replacing the golden ratio, and minus its reciprocal, by $\xi$ and by $\eta = -1/\xi$, respectively. (See Lucas [7] and [3].) Our theorem and corollary have exemplified this procedure. We can then come down to "siblingal" formulas by giving $\xi$ special values. As mentioned earlier, the results are number theoretic when $\xi + \eta$ is a natural or Gaussian integer. But if we let $\xi = ie^{ix}$, $\eta = ie^{-ix}$, where $x$ is real, we obtain trigonometrical identities (Lucas [7]), for in this case we have

$$F_n(ie^{ix}) = i^{n-1}\sin nx / \sin x \tag{16}$$

when $x$ is not a multiple of $\pi$, and

$$L_n(ie^{ix}) = 2i^n \cos nx. \tag{17}$$

The trigonometrical "siblings," so to speak, of the "Fibonacci" and "Lucas" cases of (7) are

$$\sin kx \sum_{n=1}^{m} \operatorname{cosec} nx \operatorname{cosec}(n+k)x = \sin mx \sum_{n=1}^{k} \operatorname{cosec} nx \operatorname{cosec}(n+m)x \tag{18}$$

and

$$\sin kx \sum_{n=1}^{m} \sec nx \sec(n+k)x = \sin mx \sum_{n=1}^{k} \sec nx \sec(n+m)x \tag{19}$$

where $k$ and $m$ are positive integers and $k$, $m$, and $x$ are such that no infinities occur. No infinite terms will occur if $x$ is not a rational multiple of $\pi$ but the series on the left and the sequence on the right won't converge when $m \to \infty$ because arbitrarily large terms will occur. (The summations, for finite $k$ and $m$, can be numerically highly ill-conditioned.)

"Parents" and trigonometrical "siblings" can be written down corresponding to the vast majority of the identities on pages 176-183 of Vajda [8] where $\sqrt{5}$ is to be generalized to $\xi - \eta$. Some of these trigonometrical identities are familiar. Conversely, parents and Fibonaccian siblings can be obtained for many of the trigonometrical identities in, say, Hobson [6]. To carry out this program in detail would be straightforward but would occupy a lot of space.

Again, trigonometrical identities can be derived from identities given by Bruckman and Good [1], in addition to the Fibonaccian identities given there.

## APPENDIX

L. A. G. Dresel, on trying out the reverse argument, found the following more direct way of proving the identity (7).

On putting $m = t-1$ in (12) and dividing by $G_n G_{n+t-1} G_{n+t}$, we have

$$\frac{F_t}{G_n G_{n+t}} - \frac{F_{t-1}}{G_n G_{n+t-1}} = \frac{(-1)^{t-1}}{G_{n+t} G_{n+t-1}}. \tag{A.1}$$

Summing for $t = 1$ to $k$, we find that almost all of the terms on the left cancel in pairs, and since $F_0 = 0$ we have

$$\frac{F_k}{G_n G_{n+k}} = \sum_{t=1}^{k} \frac{(-1)^{t-1}}{G_{n+t} G_{n+t-1}}. \tag{A.2}$$

[This is the same as (13), with a change of notation in the subscripts, but is now *proved.*]

Multiplying by $(-1)^n$ and summing for $n = 1$ to $m$ gives

$$F_k \sum_{n=1}^{m} \frac{(-1)^n}{G_n G_{n+k}} = \sum_{n=1}^{m} \sum_{t=1}^{k} \frac{(-1)^{n+t-1}}{G_{n+t} G_{n+t-1}}. \tag{A.3}$$

Similarly, interchanging the roles of $k$ and $m$, we have

$$F_m \sum_{n=1}^{k} \frac{(-1)^n}{G_n G_{n+m}} = \sum_{n=1}^{k} \sum_{t=1}^{m} \frac{(-1)^{n+t-1}}{G_{n+t} G_{n+t-1}}. \tag{A.4}$$

But the double summations on the right of (A.3) and (A.4) are equal, as the summand is symmetrical in $n$ and $t$ and the order of summation is immaterial. Hence the left sides are equal, which proves the theorem.

## REFERENCES

1. P. S. Bruckman & I. J. Good. "A Generalization of a Series of de Morgan, with Applications of Fibonacci Type." *The Fibonacci Quarterly* **14.2** (1976):193-96.
2. Dean Clark. "Problem No. 10262. *Amer. Math. Monthly* **99** (1992):873.
3. Fibonacci Association. A translation by Sidney Kravitz of Lucas (1878), ed. Douglas Lind. Santa Clara, Calif.: The Fibonacci Association, 1969.
4. I. J. Good. "Complex Fibonacci and Lucas Numbers, Continued Fractions, and the Square Root of the Golden Ratio (condensed version)." *J. Opl. Res. Soc.* **43.8** (a special issue in honor of Steven Vajda, 1992):837-42.
5. I. J. Good. "Complex Fibonacci and Lucas Numbers, Continued Fractions, and the Square Root of the Golden Ratio." *The Fibonacci Quarterly* **31.1** (1993):7-20.
6. E. W. Hobson. *A Treatise on Plane and Advanced Trigonometry.* 7th ed. New York: Dover Publications, 1957. Originally published by Cambridge University Press with the title *A Treatise on Plane Trigonometry.*
7. Edouard Lucas. "Théorie des fonctions numériques simplement périodiques." *Amer. J. Math.* **1** (1878):184-240.
8. S. Vajda. *Fibonacci and Lucas Numbers, and the Golden Section.* Chichester: Ellis Horwood, 1989.

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