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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

A JUXTAPOSITION PROPERTY FOR THE 4 × 4 MAGIC SQUARE

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(*Submitted December 1991*)

Consider the standard 4×4 magic square:

$$M = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}.$$

Now (using decimal notation) let us "juxtapose" the entries of the first two columns to form a single column of numbers and then compute the sum. Similarly, juxtapose the last two columns and consider that sum:

C1&C2	C3&C4
162	313
511	108
97	612
414	151
1184	1184

For the first operation here, we juxtapose Column 1 with Column 2, which we indicate by writing C1&C2. The other operation above is C3&C4. Curiously, these sums are equal.

Similarly, we can combine the other pairs of columns, with the extra condition that the entry "9" is viewed as the 2digit number "09":

C2&C1	C4&C3	C1&C3	C3&C1	C2&C4	C4&C2
216	133	163	316	213	132
115	810	510	105	118	811
709	126	96	609	712	127
144	115	415	154	141	114
1184	1184	1184	1184	1184	1184

Repeating this process with the other four possible choices, we obtain a different set of equal sums:

C1&C4	C4&C1	C2&C3	C3&C2
1613	1316	23	32
58	85	1110	1011
912	1209	76	67
41	14	1415	1514
2624	2624	2624	2624

Performing similar operations with the rows of M, we find that the sums do not behave quite so nicely, but still there are a number of equalities. For example:

R1&R2	R4&R3	R1&R4	R4&R1
165	409	164	416
211	147	214	142
310	156	315	153
138	112	131	113
824	824	824	824

The entry "9" was considered as a 2-digit number "09" throughout these operations. That was done to make the patterns of 1-digit numbers and 2-digit numbers in the square suitably symmetric. One possible way to avoid that device is to rewrite the square using "base nine" notation rather than the usual "base ten." The square can be written out in base nine as follows:

$$M_9 = \begin{bmatrix} 17 & 2 & 3 & 14 \\ 5 & 12 & 11 & 8 \\ 10 & 7 & 6 & 13 \\ 4 & 15 & 16 & 1 \end{bmatrix}.$$

This is the same square as the original, except that the numbers are written in another notation. Here, for example, $17_{\text{nine}} = 1*9^1 + 7*9^0 = 16_{\text{ten}}$. Then we note that the "juxtaposition property" works very well for M_9, since the 2-digit numbers are symmetrically distributed in this square. For example:

C1&C2	C3&C4
172	314
512	118
107	613
415	161
1317	1317
(Base nine)	(Base nine)

Here the numbers and additions are all done in base nine. For example "1317" equals 988 in base ten notation. The juxtaposition property can be shown to work just as well when base nine is used throughout the process.

It can be shown that the juxtaposition property is a consequence of the well-known 2×2 magic properties of M and the symmetries in the number of digits of the entries of M. According to the 2×2 magic property, M can be partitioned into four 2×2 squares, and the sum of the entries in each of these squares is again the magic constant 34. For example, the upper left corner is $\left[\begin{smallmatrix} 16 & 2 \\ 5 & 11 \end{smallmatrix}\right]$, which has the sum $16+2+5+11=34$. The same holds for all the corner squares, the inner central square, and the square formed by the four corner entries of M. If the inner columns are interchanged, or the inner two rows are interchanged, or if both the operations are performed together, the sum of the entries of all these squares remains 34. We can use these 2×2 properties to "explain" the patterns found on juxtaposition.

Consider the sum of the rows R1 and R3 to be R1 + R3 = (a, b, c, d). Since the sum of all four rows is (34, 34, 34, 34), we see that R2 + R4 = (a', b', c', d'), where $a' = 34 - a$, $b' = 34 - b$, $c' = 34 - c$, and $d' = 34 - d$. Note that $a + b$ is the sum of the entries of the upper left 2×2 square after interchanging the middle two rows and therefore equals 34; similarly, $c + d = a + c = b + d = 34$. Therefore, we see that

$$\text{R1} + \text{R3} = (a, a', a', a) \quad \text{and} \quad \text{R2} + \text{R4} = (a', a, a, a'), \tag{1}$$

where $a + a' = 34$.

For M, $a = 25$ and $a' = 9$. For M_9, $a = 27$ and $a' = 9$. Now the juxtaposition sum Cn&Cm can be written as

$$Cn\&Cm = 10^{d_{1m}} M_{1n} + M_{1m} + 10^{d_{2m}} M_{2n} + M_{2m} + 10^{d_{3m}} M_{3n} + M_{3m} + 10^{d_{4m}} M_{4n} + M_{4m}.$$

Here M_{kL} are the entries of the magic square in matrix notation, and d_{km} is the number of (base ten) digits in the number M_{km}. Since $M_{1m} + M_{2m} + M_{3m} + M_{4m} = 34$,

$$Cn\&Cm = 10^{d_{1m}} M_{1n} + 10^{d_{2m}} M_{2n} + 10^{d_{3m}} M_{3n} + 10^{d_{4m}} M_{4n} + 34.$$

Now certain symmetries can be observed in M (with 09 instead of 9):

$$d_{1m} = d_{3m} \quad \text{and} \quad d_{2m} = d_{4m} \quad \text{for all } m.$$

Therefore,

$$Cn\&Cm = 10\left(10^{(d_{1m}-1)}(\mathrm{R1}+\mathrm{R3})_n + 10^{(d_{2m}-1)}(\mathrm{R2}+\mathrm{R4})_n\right) + 34,$$

where the subscript n denotes the n^{th} element of the row sum.

Using the 2×2 magic properties (1),

$$Cn\&Cm = 10\left[10^{(d_{1m}-1)}a + 10^{(d_{2m}-1)}a'\right] + 34 \text{ for } n = 1 \text{ or } 4, \text{ and}$$
$$Cn\&Cm = 10\left[10^{(d_{1m}-1)}a' + 10^{(d_{2m}-1)}a\right] + 34 \text{ for } n = 2 \text{ or } 3.$$

It can be seen from M that:

(1) For $n = 1$ or 4 and $m = 2$ or 3, $d_{1m} = 1$ and $d_{2m} = 2$. Therefore,

$$\mathrm{C1\&C2 = C1\&C3 = C4\&C2 = C4\&C3} = 10(a + 10a') + 34.$$

(2) For $n = 2$ or 3 and $m = 1$ or 4, $d_{1m} = 2$ and $d_{2m} = 1$. Therefore,

$$\begin{aligned} \mathrm{C2\&C1 = C2\&C4 = C3\&C1 = C3\&C4} &= 10(a + 10a') + 34 \\ &= \mathrm{C1\&C2 = C1\&C3 = C4\&C2 = C4\&C3}. \end{aligned}$$

Since $a = 25$ and $a' = 9$, all the above are equal to 1184.

(3) For $n = 1$ and $m = 4$, and for $n = 4$ and $m = 1$, $d_{1m} = 2$ and $d_{2m} = 1$.

(4) For $n = 2$ and $m = 3$, and for $n = 3$ and $m = 2$, $d_{1m} = 1$ and $d_{2m} = 2$.

For all these cases, the juxtaposition sums turn out to be equal to $10(a' + 10a) + 34 = 2624$.

From the 2×2 magic properties similar behaviors to equation (1) can be found for columns also. However, the pattern of 1- and 2-digit numbers needed for the equalities of juxtaposition sums do not match up so nicely for the row juxtapositions as for the column juxtapositions. Therefore, the number of equalities are less for the former.

All the above relations hold for bases other than ten, provided the symmetries in the number of digits in the entries are satisfied.

In summary, we find that these "juxtaposition properties" of the 4×4 magic square can be seen as some of the well-known internal symmetries of M.

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SOME REMARKS ON $\sigma(\phi(n))$

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It was conjectured in 1964 by A. Makowski & A. Schinzel [4] that, for every natural number n,

$$\frac{\sigma(\phi(n))}{n} \geq \frac{1}{2}. \tag{1}$$

They remarked also that even the weaker result

$$\inf \frac{\sigma(\phi(n))}{n} > 0 \tag{2}$$

is still unproved. Carl Pomerance [5] gave a proof of (2). Also S. W. Graham et al. [2] stated in the abstract, their result,

$$.576 < \liminf \frac{\sigma(\phi(P_m))}{P_m} \leq \limsup \frac{\sigma(\phi(P_m))}{P_m} \leq 1,$$

where P_m is the product of the first m primes.

Notation: We use p and q to denote exclusively primes, $m|n$ to denote m dividing n, and $m\nmid n$ to denote m not dividing n. We use $p^a \| n$ to mean $p^a|n$ and $p^{a+1}\nmid n$. Also n is k-full means $p|n$ impli es $p^k|n$

First, we observe that validity of (1) for all $n \geq 1$ implies

$$\frac{\sigma(\phi(n))}{n} \geq 1, \tag{3}$$

for odd n. This can be seen easily from the fact that when n is odd, $\phi(2n) = \phi(n)$. On the other hand, (3) implies (1) can be seen with the help of (4) below. It also implies that (1) is a strict inequality if $4|n$.

As in [5], we factor $\frac{\sigma(\phi(n))}{n}$ and obtain

$$\begin{aligned}\frac{\sigma(\phi(n))}{n} &= \frac{\sigma(\phi(n))}{\phi n}\frac{\phi(n)}{n} \\ &= \prod_{p^a \| \phi(n)}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^a}\right)\prod_{p|n}\left(1-\frac{1}{p}\right) \qquad (4) \\ &= \prod_{p|n}\left(1-\frac{1}{p}\right)\prod_{p|\phi(n)}\left(1-\frac{1}{p}\right)^{-1}\prod_{p^a\|\phi(n)}\left(1-\frac{1}{p^{a+1}}\right) \qquad (5)\end{aligned}$$

and it follows that if n is k-full, $k \geq 2$, then

$$\frac{\sigma(\phi(n))}{n} \geq \frac{1}{\zeta(2)}\prod_{p|\phi(n);\, p\nmid n}\left(1-\frac{1}{p}\right)^{-1} \geq \frac{1}{\zeta(2)},$$

for any n. (Of course, for odd and k-full n or k-full n with particular prime factors, we get better bounds from here.) In Theorem 1 below, we improve this bound in the case of any k-full n, for $k \geq 3$.

We can see from (5) that the essential problem is to prove the inequality

$$\prod_{p|n}\left(1-\frac{1}{p}\right) \geq \prod_{q|\prod_{p|n}(p-1)}\left(1-\frac{1}{q}\right). \tag{6}$$

In fact, the conjecture for odd n (3), implies (6) for odd n. On the other hand, it is clear that, with the help of (5), (6) implies (1) with $1/(2\zeta(2))$ on the right side in place of $1/2$. Pomerance interprets (6) as follows: For odd $n \geq 1$, $\phi(n) \geq$ the geometric mean of n and $\phi(\phi(n))$.

We mention the following consequence of the conjecture. Call a set of primes $S = \{2 = q_1, q_2, \dots, q_t\}$ self-filled if, for any prime p, $p|\prod_{r=1}^{t}(q_r - 1)$ implies $p \in S$. The sets $\{2\}, \{2, 3, 7\}$ are, for example, self-filled sets. Let S as above be a self-filled set. Let $T = \{p_1, p_2, \cdots\}$ be the set of primes of the form $p_r = q_1^{a_1} q_2^{a_2} \cdots q_t^{a_t} + 1$ for $a_1 \geq 1$ and the other $a_r \geq 0$. Observe that, for $r \geq 2$, $q_r \in T$. Then the conjecture implies

$$\prod_{p\in T;\, p\notin S}\left(1-\frac{1}{p}\right) \geq \frac{1}{2}. \tag{7}$$

Indeed, assume the conjecture (3) holds for odd n. Let $n = \prod_{p\in T;\, p<x} p$. With the help of (5), we see that (3) implies (6) which, in turn, implies (7) since $q_r \in T$ for $r \geq 2$ and x is arbitrary. [Observe that, when x is large enough, the set of prime factors of $\prod_{p\in T;\, p<x}(p-1)$ is precisely S.] When $S = \{2\}$, the corresponding set $T = T_2$ is the set of Fermat primes for which (7) is valid. This is easily checked thus,

$$\prod_{p\in T_2;\, p\notin S}\left(1-\frac{1}{p}\right) \geq \lim_{t\to\infty}\prod_{r=0}^{t}\frac{2^{2^r}}{2^{2^r}+1} = \lim_{t\to\infty}\frac{1}{2}\left(1-\frac{1}{2^{2^{t+1}}}\right)^{-1} = \frac{1}{2},$$

which is (7). This, of course, verifies (6) also, when n is composed only of primes of the form $2^a + 1$ and, hence, implies (1) with $1/(2\zeta(2))$ on the right side instead of $1/2$ there for such n.

Theorem 1: Let $k \geq 2$. For k-full n, we have

$$\frac{\sigma(\phi(n))}{n} \geq \zeta^{-1}(k).$$

Theorem 2: We have, for infinitely many primes P,

$$\frac{\sigma(\phi(P))}{P} \geq (1+o(1))e^{\gamma}\log\log P \quad \text{as } P \to \infty.$$

Also, for all large n, we have

$$\frac{\sigma(\phi(n))}{n} \leq (1+o(1))e^{\gamma}\log\log n \quad \text{as } n \to \infty.$$

That is, the maximum order of $\frac{\sigma(\phi(n))}{n}$ is $e^{\gamma}\log\log n$.

Theorem 2 quantifies a result of Alaoglu & Erdös [1].

Proof of Theorem 1: Let $n = \prod_{r=1}^{k} p_r^{a_r}$. We note that $\prod_{r=1}^{k} p_r^{e_r}$ for $0 \le e_r \le a_r - 1$, $1 \le r \le k$, are different integers for different k-tuples $(e_1, e_2, \dots, e_k)$. Hence, the $a_1 a_2 \dots a_k$ integers $\prod_{r=1}^{k} p_r^{e_r}(p_r - 1)$ are all distinct. All these $a_1 a_2 \dots a_k$ integers are divisors of $\phi(n)$ as well. Therefore, we have $\sigma(\phi(n))$ at least as large as the sum of these divisors. That is,

$$\sigma(\phi(n)) \ge \prod_{r=1}^{k}\left((1 + p_r + \dots + p_r^{a_r - 1})(p_r - 1)\right)$$
$$\ge n \prod_{r=1}^{k}\left(1 - \frac{1}{p_r^{a_r}}\right) \ge n / \zeta(k),$$

since $a_r \ge k$ for all r, and the proof of Theorem 1 is complete.

Proof of Theorem 2: Let $2 = p_1$, $3 = p_2, \dots$ be the sequence of primes. Let $Q_k = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $a_r = 2\left[\frac{\log p_k}{\log p_r}\right] + 1$, so that

$$p_r^{a_r + 1} \ge p_k^2. \tag{8}$$

Let m be the least integer such that $P = P_k = Q_k m + 1$ is prime. We see that

$$Q_k \le \exp\left(\sum_{r=1}^{k} a_r \log p_r\right) \le p_k^{3k}$$

and hence, by the theorem on least primes in arithmetic progression (see, e.g., [3]), we obtain $P \le p_k^{30k}$ (we do not need the best exponent), and thus,

$$\log\log P \le (1 + o(1)) \log p_k \quad \text{as } k \to \infty. \tag{9}$$

Now, remembering that P is prime, we get, using (4), that

$$\frac{\sigma(\phi(P))}{P} = (1 - 1/P) \prod_{p^a \| \phi(P)} \left(1 + \frac{1}{p} + \cdots \frac{1}{p^a}\right)$$
$$\ge (1 + o(1)) \prod_{r=1}^{k}\left(1 + \frac{1}{p_r} + \dots + \frac{1}{p_r^{a_r}}\right)$$
$$\ge (1 + o(1)) \prod_{r=1}^{k}\left(1 + \frac{1}{-p_r}\right)^{-1} \prod_{r=1}^{k}\left(1 - \frac{1}{p_r^{a_r + 1}}\right)$$
$$\ge (1 + o(1)) e^{\gamma} \log p_k,$$

using (8), and Mertens' theorem and the lower bound in Theorem 2 now follows from (9).

It follows from (5) that, for any n,

$$\frac{\sigma(\phi(n))}{n} \le \prod_{p \mid \phi(n)}\left(1 - \frac{1}{p}\right)^{-1} \le (1 + o(1)) e^{\gamma} \log\log \phi(n),$$

and the proof of Theorem 2 is complete.

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ON THE GREATEST INTEGER FUNCTION AND LUCAS SEQUENCES

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In 1972, Anaya & Crump [1] proved, for the Fibonacci numbers F_n, that

$$\left[\alpha^k F_n + \frac{1}{2}\right] = F_{n+k},\ n \ge k > 1, \tag{1}$$

where $\alpha = (1+\sqrt{5})/2$ and $[x]$ denotes the greatest integer $\le x$. Carlitz [2] later proved, for the sequence of Lucas numbers L_n, that

$$\left[\alpha^k L_n + \frac{1}{2}\right] = L_{n+k},\ n \ge k+2,\ k \ge 2. \tag{2}$$

Let P and Q be relatively prime integers with $P > 0$ and $D = P^2 - 4Q > 0$. Let α and β, $\alpha > \beta$, be the roots of $x^2 - Px + Q = 0$; the Lucas sequences are defined, for $n \ge 0$, by

$$U_n = U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = V_n(P,Q) = \alpha^n + \beta^n.$$

In 1975, Everett [3] showed that, if $Q = -1$, then

$$\left[\alpha^k U_n + \frac{P}{P+1}\right] = U_{n+k} \text{ or } U_{n+k} + 1,\ n \ge k \ge 2,$$

with the latter value obtaining when n and k are odd and $1/(P+1) \le |\beta|^n U_k$.

The results of (1) and (2) can be extended to all Lucas sequences $\{U_n\}$ and $\{V_n\}$ with $Q = \pm 1$, and, interestingly, in view of Everett's result, with no restrictions on n or k for $n \ge k \ge 2$. It seems, also, not to have been recognized, even for the case where $P = 1, Q = -1$ (i.e., for the sequences of Fibonacci and Lucas numbers), that the existence of the relations for a given pair, P, Q, for the sequence $\{V_n\}$ implies the existence of the corresponding relations for the sequence $\{U_n\}$. We show this dependence and obtain the extension of (1) and (2) to all Lucas sequences with $Q = \pm 1$ and $n \ge k \ge 1$.

The proofs are straightforward. We recall that $[b] = a$ iff $0 \le b - a < 1$.

Lemma: Let k and n be integers, where $n \ge k \ge 1$, and let t be a real number, $0 \le t < \sqrt{D}/2$. If $[\alpha^k V_n + t] = V_{n+k}$, then $[\alpha^k U_n + 1/2] = U_{n+k}$.

Proof: Let $A = \alpha^k V_n - V_{n+k}$ and assume $[\alpha^k V_n + t] = V_{n+k}$. Then $0 \le \alpha^k V_n + t - V_{n+k} < 1$; that is, $-t \le A < 1 - t$. Now,

$$A = \alpha^k V_n - V_{n+k} = \alpha^k(\alpha^n + \beta^n) - (\alpha^{n+k} + \beta^{n+k}) = \beta^n(\alpha^k - \beta^k)$$

and

$$\alpha^k U_n - U_{n+k} = \alpha^k(\alpha^n - \beta^n)/\sqrt{D} - (\alpha^{n+k} - \beta^{n+k})/\sqrt{D} = \beta^n(\beta^k - \alpha^k)/\sqrt{D}.$$

Thus, $\alpha^k U_n - U_{n+k} = -A/\sqrt{D}$, and $-t \le A < 1-t$ implies

$$(t-1)/\sqrt{D} < -A/\sqrt{D} \le t/\sqrt{D}. \tag{3}$$

Noting that $D = P^2 \pm 4 \ge 5$, it follows from (3) that, if $0 \le t < \sqrt{D}/2$, then $-1/2 < -A/\sqrt{D} < 1/2$; hence, $0 < \alpha^k U_n + 1/2 - U_{n+k} < 1$, establishing the Lemma.

In the following theorem, values of t are given such that $[\alpha^k V_k + t] = V_{n+k}$ for all $n \ge k \ge 1$. With one exception, $(P, Q, k, n) = (1, -1, 1, 1)$, we have $0 \le t < \sqrt{D}/2$; we observe, in particular, in (f), $7/5 < \sqrt{8}/2 \le \sqrt{D}/2$ for $Q = -1$ and $P \ge 2$, and in (g), $1.1 < \sqrt{5}/2 = \sqrt{D}/2$ for $Q = -1$ and $P = 1$.

Theorem 1:

(a) $[\alpha^k V_n + 1/2] = V_{n+k}$ if $Q = \pm 1, n \ge k+2, k \ge 1$, and $(P, k, n) \ne (1, 1, 3)$;

(b) $[\alpha^k V_n + 1/2] = V_{n+k}$ if $Q = 1, n = k+1, k \ge 1$;

(c) $[\alpha^k V_n + 1] = V_{n+k}$ if $\begin{cases} (P, k, n) = (1, 1, 3), \text{ or} \\ Q = -1,\ n = k+1,\ n \text{ odd},\ k \ge 1; \end{cases}$

(d) $[\alpha^k V_n] = V_{n+k}$ if $Q = -1, n = k+1, n$ even, $k \ge 1$;

(e) $[\alpha^n V_n] = V_{2n}$ if $Q = 1$, or $Q = -1$ and n is even;

(f) $[\alpha^n V_n + 7/5] = V_{2n}$ if $Q = -1$ and n is odd;

(g) $[\alpha^n V_n + 1.1] = V_{2n}$ if $Q = -1$, $P = 1$, and n is odd, $n > 1$.

Proof: Let $Q = \pm 1$. Since $P > 0, D \ge 5$, and $1/\alpha = 2/(P + \sqrt{D})$, we have $0 < 1/\alpha \le 2/(1+\sqrt{5}) < .62$ for all P, and $1/\alpha < 2/(2+\sqrt{5}) < 1/2$ if $P \ge 2$. We show that the relation $[b] = a$ holds in each case by showing that $|b - a - 1/2| < 1/2$. For any t,

$$\left|\alpha^k V_n - V_{n+k} + t - \frac{1}{2}\right| = \left|\beta^n(\alpha^k - \beta^k) + t - \frac{1}{2}\right| = \left|Q^n(1/\alpha^{n-k} - Q^k/\alpha^{n+k}) + t - \frac{1}{2}\right|. \tag{4}$$

Case 1. $n \ge k+2, k \ge 1, t = 1/2, (P, k, n) \ne (1, 1, 3)$. By (4),

$$\left|\alpha^k V_n - V_{n+k} + t - \frac{1}{2}\right| = \left|Q^n(1/\alpha^{n-k} - Q^k/\alpha^{n+k})\right| \le \left|1/\alpha^{n-k}\right| + \left|1/\alpha^{n+k}\right|.$$

If $P \ge 2$, this sum is $< (1/2)^2 + (1/2)^3 < 1/2$, and if $P = 1$ and $n \ge 4$, the sum is $\le (.62)^2 + (.62)^5 < 1/2$; this proves (a).

Case 2. $n = k+1, k \ge 1$. If $Q = 1$ and $t = 1/2$, (4) equals $|1/\alpha - 1/\alpha^{2n-1}|$. Since $D = P^2 - 4 > 0$, $P \ge 3$, implying that $0 < 1/\alpha < 1/2$; hence, $|1/\alpha - 1/\alpha^{2n-1}| = 1/\alpha - 1/\alpha^{2n-1} < 1/\alpha < 1/2$, proving (b). If $(P, k, n) = (1, 1, 3)$, then $0 < P^2 - 4Q = 1 - 4Q$ implies $Q = -1$, and

$$\alpha^k V_n + 1 = \alpha^1 L_3 + 1 = 4 \cdot (1+\sqrt{5})/2 + 1 \approx 7.472;$$

thus, $[\alpha L_3 + 1] = 7 = V_4$. If $Q = -1, t = 1, n = k+1, k \ge 1$, and n is odd, (4) equals

$$\left|-1/\alpha+(-1)^k/\alpha^{2n-1}+\frac{1}{2}\right|=\left|1/\alpha-(-1)^k/\alpha^{2n-1}-\frac{1}{2}\right|.$$

Since $n\geq 3$, $0<1/\alpha\pm 1/\alpha^{2n-1}<.62+(.62)^5<1$, so $|1/\alpha-(-1)^k/\alpha^{2n-1}-1/2|<1/2$, proving (c). If $Q=-1, t=0$, and n is even, (4) equals$|1/\alpha-(-1)^k/\alpha^{2n-1}-1/2|$. Since $0<1/\alpha\pm 1/\alpha^{2n-1}<$ $.62+(.62)^3<1$, $|1/\alpha-(-1)^k/\alpha^{2n-1}-1/2|<1/2$, proving (d).

Case 3. $n=k$. In this case, (4) is $|Q^n(1-(Q/\alpha^2)^n)+t-1/2|$. If $Q=1$ and $t=0$, this equals $|1/2-(1/\alpha^2)^n|<1/2$, proving (e) for $Q=1$; if $Q=-1$, $t=0$, and n is even, (4) has exactly the same value as for $Q=1, t=0$, completing the proof of (e). If $Q=-1$, $t=7/5$, and n is odd, (4) equals

$$\left|-\left(1+\frac{1}{\alpha^{2n}}\right)+\frac{9}{10}\right|=\frac{1}{\alpha^{2n}}+\frac{1}{10}<(.62)^2+.10<\frac{1}{2},$$

proving (f). If $Q=-1, P=1, t=1.1$, and $n>1$ is odd, then (4) equals

$$\left|-\left(1+\frac{1}{\alpha^{2n}}\right)+\frac{11}{10}-\frac{1}{2}\right|=\left|-.40-\frac{1}{\alpha^{2n}}\right|=\frac{1}{\alpha^{2n}}+.40<(.62)^6+.40<\frac{1}{2},$$

establishing the last relation of the theorem.

As noted in the paragraph preceding Theorem 1, the hypothesis of the Lemma is satisfied for $n\geq k\geq 1$, with one exception, yielding the following theorem.

Theorem 2: If $Q=\pm 1$ and $n\geq k\geq 1$, then $[\alpha^k U_n+1/2]=U_{n+k}$ with the single exception $U_n=F_n$ with $n=k=1$.

It should perhaps be mentioned that the exception was properly excluded in (1) at the beginning of our paper, but that the case $n=k=1$ was mistakenly included in [1]. In the interest of completeness, we observe that $[\alpha F_1]=[(1+\sqrt{5})/2]=1=F_2$.

Example 1: Let $P=3, Q=-1, n=5, k=4$. The first ten terms of $\{U_n(3,-1)\}$ $(0\leq n\leq 9)$ are 0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970. Therefore, $U_9=12970$. Since $\alpha^2-P\alpha+Q=0$, $\alpha^2=3\alpha+1$, and $\alpha^4=9\alpha^2+6\alpha+1=33\alpha+10$. (It is easy to show, incidentally, that $\alpha^r=U_r\alpha-QU_{r-1}$ for $r>0$.) Hence,

$$\alpha^4 U_5+\frac{1}{2}=\left(33\left(\frac{3+\sqrt{13}}{2}\right)+10\right)109+\frac{1}{2}\approx 12970.58397,$$

showing that $[\alpha^4 U_5+1/2]=U_9$.

Example 2: Let $P=6, Q=1, n=k=4$. Using $\alpha^2=6\alpha-1$, we find that $\alpha^4 V_4=1331714.99^+$, implying $V_8=1331714$, by Theorem 1(e). This agrees with the result obtained using the well-known formula $V_{2n}=V_n^2-2Q^n$, recursively, for $n=1, 2$, and 4.

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THE FIRST COLUMN OF AN INTERSPERSION

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INTRODUCTION

In 1977, K B. Stolarsky [9] introduced an array of positive integers whose first row consists of the Fibonacci numbers $\{F_n : n \geq 2\} : 1\ 2\ 3\ 5\ 8\ 13 \ldots$ The subsequent rows are "generalized Fibonacci sequences." In fact, much more is true. The rows of the array are, in a sense, the set of *all* "positive Fibonacci sequences" of integers. This fact was proved by D. Morrison [7], who also introduced the Wythoff array and proved that it has many of the properties of the original Stolarsky array. In order to study from a general point of view the properties which the Stolarsky and Wythoff arrays have in common, the notion of *interspersion* was introduced in [4]. The name "interspersion" was chosen to match property I4 in the definition given below.

Much of the reason for interest in interspersions, especially those known as Stolarsky interspersions, lies with the first column of such an array: its high degree of regularity versus the possible unavailability of a nice formula for the n^{th} term. In the case of the original Stolarsky and Wythoff arrays, however, such formulas are known (see Section 5). From Example 1.1(i) of [7, p. 307] and these formulas, we find that the first columns of the Stolarsky and Wythoff arrays are uniformly distributed mod m for every positive integer m. In contrast to this, we construct in Section 4 a new Stolarsky interspersion for which every element of the first column, after the initial element 1, is *even*; we call it the *even first column array* (EFC).

1. WHAT IS AN INTERSPERSION?

Throughout this paper, the notation $A = A(i, j)$ denotes an array of distinct positive integers $a(i, j)$ with increasing first column. For such A, let $\hat{A} = \hat{A}(i, j)$ and $\check{A} = \check{A}(i, j)$ be the arrays of positive integers defined by

$$\hat{a}(i, j) = a(i, j+1) \text{ for } i \geq 1, j \geq 1,$$

and

$$\check{a}(i, j) = \text{the number of terms of } \hat{A} \text{ which are } \leq a(i, j+1),$$

respectively. Note that $\hat{A}$ is obtained from A by simply removing the first column of A. If the terms of $\hat{A}$ are then ordered as an increasing sequence, then $\check{a}(i, j)$ is simply the rank of $\hat{a}(i, j)$ in this sequence. (The reader is urged to write out several terms of A using the array in Table 1.) We call $\check{A}$ the *rank array of* A and prove in Theorem 1.1 that an array A is its own rank array iff A is an *interspersion*, as defined in [4] by the following properties:

I1. the rows of A comprise a partition of the positive integers;

I2. every row of A is an increasing sequence;

I3. every column of A is an increasing (possibly finite) sequence;

I4. if $\{u_j\}$ and $\{v_j\}$ are distinct rows of A, and p and q are indices for which $u_p < v_q < u_{p+1}$, then $u_{p+1} < v_{q+1} < u_{p+2}$.

Perhaps the simplest example of an interspersion is given by $a(i, j) = i + \frac{(i+j-1)(i+j-2)}{2}$.

Theorem 1.1: An array A is an interspersion iff $\check{A} = A$.

Proof: First, suppose A is an interspersion. Then, by Lemma 2 in [4],

$$a(i, j+1) = a(i, j) + C(a(i, j+1)),$$

where $C(m)$ denotes, for $m \geq 1$, the number of terms in the first column of A that are $\leq m$. Thus, $a(i, j)$ is the number of terms of A that are $\leq a(i, j+1)$ and are not in column 1. That is, $\check{a}(i, j) = a(i, j)$, as required.

For the converse, suppose $\check{a}(i, j) = a(i, j)$ for all i and j. Then property I1 must hold, since $\check{a}(i, j)$ ranges through all the positive integers without repetition.

Now, since $a(i, j)$ is the number of terms of $\hat{A}$ that are $\leq a(i, j+1)$ for all i and j, we have $a(i, j) \leq a(i, j+1)$, and this strengthens to $a(i, j) < a(i, j+1)$ since the terms of A are distinct; thus, property I2 holds.

By hypothesis, column 1 of A is increasing. Suppose for arbitrary $j \geq 1$ that column j is increasing, and suppose $i \geq 1$. The number of terms of $\hat{A}$ that are $\leq a(i+1, j+1)$ is $a(i+1, j)$, and this by the induction hypothesis exceeds $a(i, j)$, which is the number of terms of $\hat{A}$ that are $\leq a(i, j+1)$. Therefore, $a(i+1, j+1) > a(i, j+1)$, and property I3 holds.

Arrange the numbers in $\hat{A}$ in increasing order, forming a sequence s_n such that

$$\hat{a}(i, j) = s_{\hat{a}(i, j-1)} = s_{a(i, j)}.$$

If $u_p < v_q < u_{p+1}$, as in I4, then $s_{u_p} < s_{v_q} < s_{u_{p+1}}$, since s_n is an increasing sequence. That is, property I4 holds. □

To summarize, Theorem 1.1 shows that an interspersion is an array A whose characteristic property is that for any successive terms u and v in any row, v is the u^{th} term not in column 1.

2. STOLARSKY INTERSPERSIONS

Certain interspersions which have received much attention are the Stolarsky interspersions (e.g., [4], [5], [6], [8], [9]). These are shown in [6] to be in one-to-one correspondence with the set of all zero-one sequences $\{\delta_i\}$ that begin with 1. The correspondence is given as follows: for each row number i, the number $a(i, 2)$ in column 2 must be one of the two numbers $[\alpha a(i, 1) + \delta_i]$, where $\alpha = (1+\sqrt{5})/2$; thus, the numbers in column 2 depend on those in column 1 and, moreover, the numbers in columns numbered higher than 2 are determined by the recurrence

$$a(i, j) = a(i, j-1) + a(i, j-2), \quad j = 3, 4, 5, \ldots. \tag{1}$$

Accordingly, each row of a Stolarsky interspersion depends in a simple manner on whatever number occupies the first position in the row. This first number is always the least positive integer not appearing in any previous row. (See Tables 1-3.) We leave open the question of whether almost all Stolarsky interspersions have a uniformly distributed first column.

TABLE 1. The Original Stolarsky Array ([9], 1977)

1	2	3	5	8	13	21	34	55	89	144	...
4	6	10	16	26	42	68	110	178	288	466	
7	11	18	29	47	76	123	199	322	521	843	
9	15	24	39	63	102	165	267	432	699	1131	
12	19	31	50	81	131	212	343	555	898	1453	
14	23	37	60	97	157	254	411	665	1076	1741	
17	28	45	73	118	191	309	500	809	1309	2118	
20	32	52	84	136	220	356	576	932	1508	2440	
22	36	58	94	152	246	398	644	1042	1686	2728	
⋮											

TABLE 2. The Wythoff Array

1	2	3	5	8	13	21	34	55	89	144	...
4	7	11	18	29	47	76	123	199	322	521	
6	10	16	26	42	68	110	178	288	466	754	
9	15	24	39	63	102	165	267	432	699	1131	
12	20	32	52	84	136	220	356	576	932	1508	
14	23	37	60	97	157	254	411	665	1076	1741	
17	28	45	73	118	191	309	500	809	1309	2118	
19	31	50	81	131	212	343	555	898	1453	2351	
22	36	58	94	152	246	398	644	1042	1686	2728	
⋮											

Construction 2.1: Every Stolarsky interspersion can be constructed inductively using the rules described above: row 1 must be 1 2 3 5 8 13 21...; once k rows have been constructed, there are two and only two possibilities for row $k+1$. The first term u must be the least positive integer not already used in the first k rows. The second term can be either $[\alpha u]$ or $[\alpha u+1]$, and the remaining terms are determined by the recurrence (1).

For any given zero-one sequence δ with initial term 1, the corresponding Stolarsky interspersion $A(\delta)$ would be easy to write out if only the first column were not, generally speaking, so mysterious. It turns out to be somewhat surprising how nearly determined these mysterious numbers are. This section is devoted to such determinations. We begin with a restatement of Lemma 1.5 of [6].

Lemma 2.2: Suppose $\{r_j\}$ is a row of a Stolarsky interspersion. Then either

$$r_{2k}=[\alpha r_{2k-1}] \text{ and } r_{2k+1}=[\alpha r_{2k}+1] \text{ for all } k\geq 1, \text{ or else}$$
$$\mathrm{r}_{2k}=[\alpha r_{2k-1}+1] \text{ and } r_{2k+1}=[\alpha r_{2k}] \text{ for all } k\geq 1.$$

Lemma 2.3: Suppose u and v are adjacent terms in a row of a Stolarsky interspersion, and $u<v$. Then $u\in\left\{\left[\frac{v}{\alpha}\right],\left[\frac{v+1}{\alpha}\right]\right\}$.

Proof: By Lemma 2.2, $v\in\{[\alpha u],[\alpha u+1]\}$, It is easy to confirm that if $v=[\alpha u]$, then $u=\left[\frac{v+1}{\alpha}\right]$, and if $v=[\alpha u+1]$, then $u=\left[\frac{v}{\alpha}\right]$.

Theorem 2.4: Let A be a Stolarsky interspersion. Let $\{s_n\}$ be the ordered complement of the first column of A. Then $s_n\in\{[n\alpha],[n\alpha+1]\}$ for $n=1,2,3,\ldots$.

Proof: By Lemma 3 of [4], we have $s_{a(i,j)} = a(i, j+1)$ for all i, j. By Lemma 2.2,

$$a(i, j+1) \in \{[\alpha a(i, j)], [\alpha a(i, j)+1]\}.$$

Since $a(i, j)$ ranges through all the positive integers n, we therefore have

$$s_n \in \{[n\alpha], [n\alpha+1]\} \text{ for } n = 1, 2, 3, \ldots. \ \square$$

Lemma 2.5: Suppose $\{c_i\}$ and $\{s_i\}$ are infinite complementary sequences of positive integers, $m \geq 0$, and $\{\sigma_i\}$ is a zero-one sequence in which the maximal string length of ones is m. Let $s_i^* = s_i + \sigma_i$ and suppose $s_{i+1}^* \neq s_i^*$ for all i. Let $\{c_j^*\}$ be the ordered complement of $\{s_i^*\}$. Then $0 \leq c_j - c_j^* \leq m$ for all j.

Proof: The sequence of positive integers can be represented in increasing order as a sequence of strings of two types: S strings consisting of consecutive s_i's, and C strings consisting of consecutive c_j's. Each S string is followed by a C string, which is followed by an S string. Either $s_1 = 1$ or else $c_1 = 1$; we assume the former, noting that the proof in case $c_1 = 1$ can easily be obtained from what follows and is, therefore, omitted. Write the initial string as $S_1 = s_1, s_2, \ldots, s_{m_1}$ $(= 1, 2, \ldots, m_1$, where $m_1 \geq 1)$, and the initial C string as $C_1 = c_1, c_2, \ldots, c_{n_1}$ $(= m_1+1, \ldots, m_1+n_1$, where $n_1 \geq 1)$. Following C_1 is S_2, and so on, so that our representation of the positive integers is as a sequence of strings: $S_1C_1S_2C_2S_3\ldots$, where $S_i = s_{m_{i-1}+1}, \ldots, s_{m_i}$, $C_i = c_{n_{i-1}+1}, \ldots, c_{n_i}$, $m_0 = n_0 = 0$, $1 \leq m_1 < m_2 < \cdots$ and $1 \leq n_1 < n_2 < \cdots$.

Let N denote the null string. Each string S_i is a juxtaposition of two substrings, L_i and R_i, which satisfy the following conditions:

(i) If $L_i = \mathcal{N}$, then $R_i \neq \mathcal{N}$;

(ii) If $L_i \neq \mathcal{N}$, then L_i has the form $s_{m_{i-1}+1}, \ldots, s_{m_{i-1}+k_i}$ for some $k_i \geq 1$, and $s_\ell^* = s_\ell$ for $m_{i-1}+1 \leq \ell \leq m_{i-1}+k_i$;

(iii) If $R_i \neq \mathcal{N}$, then R_i has the form $s_{m_{i-1}+k_i+1}, \ldots, s_{m_i}$, and $s_\ell^* = s_\ell + 1$ for $m_{i-1}+k_i+1 \leq \ell \leq m_i$.

Consider an arbitrary triple $L_iR_iC_i$. If $R_i = \mathcal{N}$, then clearly $s_\ell^* = s_\ell$ for ℓ as in (ii) and $c_\ell^* = c_\ell$ for $n_{i-1}+1 \leq \ell \leq n_i$. Otherwise, we have $s_\ell^* = s_\ell$ for the terms of L_i and $s_\ell^* = s_\ell + 1$ for those of R_i, so that $c_{n_{i-1}+1}^* = s_{m_{i-1}+k_i+1}$, and $c_\ell^* = c_\ell$ for $\ell = n_{i-1}+2, \ldots, n_i$. Thus, $0 \leq c_\ell - c_\ell^* \leq m_i - k_i \leq m$ for $\ell = n_{i-1}+1, n_{i-1}+2, \ldots, n_i$. Now, putting the triples $L_iR_iC_i$ together in order, we conclude that $0 \leq c_\ell - c_\ell^* \leq m$ for all ℓ. $\square$

Lemma 2.6: Suppose A is a Stolarsky interspersion. Let $\{s_n^*\}$ be the ordered sequence of terms of A that are not in the first column of A. Let $s_n = [n\alpha]$. Let $\sigma_n = s_n^* - s_n$. Then $\{\sigma_n\}$ is a zero-one sequence, $s_{n+1}^* \neq s_n^*$ for all n, and the maximum string length of ones in $\{\sigma_n\}$ is 2.

Proof: By Theorem 2.4, $\{\sigma_n\}$ is a zero-one sequence. Also, $s_{n+1}^* \neq s_n^*$ for all n, since the terms of A are distinct. Now suppose n is a positive integer and write

$$n\alpha = [n\alpha] + \varepsilon_1, \ (n+2)\alpha = [n+2\alpha] + \varepsilon_2, \text{ where } 0 < \varepsilon_i < 1 \text{ for } i = 1, 2.$$

Then $[(n+2)\alpha] - [n\alpha] = 2\alpha - \varepsilon_2 + \varepsilon_1$, which, as an integer within 1 of 2α $(= 1+\sqrt{5})$, must be 3 or 4. Therefore, the three integers $[n\alpha]+1$, $[(n+1)\alpha]+1$, $[(n+2)\alpha]+1$ cannot be consecutive integers. Consequently, there is no string of ones of length ≥ 3 in $\{\sigma_n\}$.

Theorem 2.7: Let $u_i = a(i, 1)$, the i^{th} term of column 1 of a Stolarsky interspersion A. Then $[i\alpha]+i-2 \le u_i \le [i\alpha]+i$ for every i.

Proof: The ordered complement of $\{s_i\} = \{[i\alpha]\}$ is $\{c_i\} = \{[i\alpha]+i\}$, by the well-known Beatty theorem on complementary sequences (see Theorem XI in [1]). Lemmas 2.5 and 2.6 imply that $0 \le c_i - u_i \le 2$, from which the desired inequality immediately follows. □

Corollary 2.8: Let $u_i = a(i, 1)$, the i^{th} term of column 1 of a Stolarsky interspersion A. Let w_i be the i^{th} term of the first column of the Wythoff array (see Table 2). Then $-1 \le w_i - u_i \le 1$ for every i.

Proof: This follows immediately from Theorem 2.7 and the fact that $w_i = [i\alpha]+i-1$. □

Lemma 2.9: If u is a positive integer, then exactly one of the following statements is true:

(i) $\exists\, n \ni u = [n\alpha]$ and $[(n+1)\alpha] = u+1$;
(ii) $\exists\, n \ni u = [n\alpha]$ and $[(n+1)\alpha] = u+2$;
(iii) $\exists\, n \ni u = [n\alpha+1]$ and $[(n+1)\alpha] = u+1$.

Proof: If there exists n satisfying $u = [n\alpha]$, then clearly $[(n+1)\alpha]$ must equal $u+1$ or $u+2$, since $0 < \alpha < 1$. If u is not of the form $[m\alpha]$, then since $0 < \alpha < 2$, there must exist n satisfying $u = [n\alpha+1]$. Since $u \ne [(n+1)\alpha]$, we have $1 \le n\alpha + \alpha - u$. Also, $n\alpha + 1 - u < 1$, so that $n\alpha + \alpha - u < 2$. That is, $u+1 < n\alpha + \alpha < u+2$, so that $[(n+1)\alpha] = u+1$. □

Theorem 2.10: The first column of a Stolarsky interspersion does not contain two consecutive integers.

Proof: If u is a positive integer in column 1 of a Stolarsky array A, and u is as in (i) or (ii) in Lemma 2.9, then the immediate successor of n in a row of A is, by Lemma 2.2, $u+1 = [n\alpha+1]$, so that u + 1 is not in column 1.

By Lemma 2.9, the only remaining case is that $u = [n\alpha+1]$ and $u+1 = [(n+1)\alpha]$. Assume that both u and u + 1 lie in column 1 of A, and assume that u is the least such positive integer. By Lemma 2.2, the immediate successor of n in a row of A must then be $u-1$, and the immediate follower of $n+1$ must be $u+2$. Since $n < u$, at least one of the numbers n and $n+1$ does not lie in column 1. If n is not in column 1, then by (1), n is immediately preceded by $u-1-n$; and if $n+1$ is not in column 1, then $n+1$ is immediately preceded by $u+2-(n+1) = u+1-n$.

Now, $u = n\alpha + 1 - \varepsilon_1, 0 < \varepsilon_1 < 1$, so that $u\alpha - n\alpha^2 + \alpha - \alpha\varepsilon_1$. Since

$$\alpha^2 = \alpha + 1, \tag{2}$$

we have

$$u\alpha - n\alpha = n + \alpha(1-\varepsilon_1). \tag{3}$$

Also, $u+1 = n\alpha + \alpha - \varepsilon_2, 0 < \varepsilon_2 < 1$, so that $u\alpha + a = n\alpha^2 + \alpha^2 - \alpha\varepsilon_2$, which yields

$$u\alpha - n\alpha = n + 1 - \alpha\varepsilon_2. \tag{4}$$

Equations (3) and (4) show that $\alpha(1-\varepsilon_1) = 1 - \alpha\varepsilon_2 < 1$, so that (3) implies $n = [(u-n)\alpha]$. Now, by Lemma 2.2, in a row of A the integer $u-n$ must immediately precede n or n + 1, whichever of these is not in column 1. However, it has already been proved that the immediate predecessor of n, if there is one, is $u-n-1$, and the immediate predecessor of $n+1$, if there is one, is $u-n+1$. This contradiction shows that u and $u+1$ cannot both lie in column 1 of A. □

Lemma 2.11: If u is a positive integer, then exactly one of the following statements is true:

(i) $\exists n \ni u=[n\alpha+1]=[(n+1)\alpha]$;
(ii) $\exists n \ni u=[n\alpha+1]$ and $[(n+1)\alpha]=u+1$;
(iii) $\exists n \ni u=[n\alpha]$ and $[(n-1)\alpha+1]=u-1$.

The proof of Lemma 2.9 can serve as a guide for proving Lemma 2.11. We omit a proof but do pause to note that each of these two lemmas partitions the set of positive integers into three subsets that can be expressed in terms of fractional parts. These are, in the order (i), (ii), (iii), as follows:

$$\{u:\{u\alpha\}>4-2\alpha\},\ \{u:2-\alpha<\{u\alpha\}<4-2\alpha\},\text{ and }\{u:\{u\alpha\}<2-\alpha\}\text{ for Lemma 2.9,}$$

$$\{u:2-\alpha<\{u\alpha\}<\alpha-1\},\ \{u:\{u\alpha\}<2-\alpha\},\ \{u:\{u\alpha\}>\alpha-1\}\text{ for Lemma 2.11.}$$

Theorem 2.12: Suppose successive terms of column 1 of a Stolarsky interspersion differ by 2: $a(i+1,1)-a(i,1)=2$. Then the integer $a(i,1)+1$ lies in a column numbered greater than 2.

Proof: Let $u=a(i,1)+1$. By Theorem 2.10, u does not lie in column 1; suppose u lies in column 2. Let n be the immediate predecessor of u in a row of A. We shall see that n must be related to u as in one of the three cases in Lemma 2.11. The only possible exception would be if $u=[p\alpha]$ for some p and also $u=[q\alpha+1]$ for some q. It is easy to check here that $q=p-1$. To see that $n=p-1$, suppose to the contrary that $n=p$. Then $[(n-1)\alpha+1]=u$ and $[(n-1)\alpha]=u-1$; now $u-1$ is in column 1, so that the immediate follower of $n-1$ in a row of A must be u, by Lemma 2.2. However, this contradicts the hypothesis that u follows n.

In case (1), $u=[n\alpha+1]=[(n+1)\alpha]$. In A, the integer $n+1$ must, by Lemma 2.2, be followed by $[(n+1)\alpha]$ or $[(n+1)\alpha+1]$. The former is u, which follows n, not $n+1$, and the latter is $u+1$, which lies in column 1. For u as in (ii), a contradiction is similarly obtained.

In case (iii), $u<n\alpha$, so that $u\alpha<n\alpha^2=n\alpha+n$, and $u\alpha-n\alpha+1<n+1$. Also, $n\alpha-\alpha+1<u$, so that $n\alpha^2-\alpha^2+\alpha<u\alpha$, which yields $n<u\alpha-n\alpha+1$. Therefore, $[(u-n)\alpha+1]=n$. In a row of A, the term immediately following $u-n$ is not $[(u-n)\alpha+1]$, for this number, coming just before u, must lie in column 1 and, thus, has no immediate predecessor. Therefore, by Lemma 2.2, the follower must be $[(u-n)\alpha]$, which is $n-1$. By (1), the number $u-1=u-n+[(u-n)\alpha]$ must lie in column 3, contrary to the hypothesis that it lies in column 1. Therefore, if as in (iii), u cannot lie in column 2.

Since u does not lie in column 1 or column 2, it must, by property I1, lie elsewhere. □

3. TWO MORE THEOREMS ABOUT COLUMN 1

Following Construction 2.1, we indicated that it is a difficult problem to formulate the first column of a Stolarsky interspersion in terms of an arbitrary given classification sequence, but that, surprisingly, in view of this difficulty, these terms can be "almost formulated" without great difficulty. Theorem 2.7, especially, tells what we mean by "almost formulated," and in addition to it we present here two more theorems.

Let $S_i = \{k : \exists$ Stolarsky interspersion $A \ni k = a(i,1)$ for some $i\}$. Thus, S_i is the set of all possible values that can be taken by the i^{th} element of column 1 in a Stolarsky interspersion; e.g., $S_1 = \{1\}$, $S_2 = \{4\}$, $S_3 = \{6, 7\}$, $S_4 = \{9, 10\}$, $S_5 = \{11, 12\}$, $S_6 = \{14, 15\}$, and $S_7 = \{16, 17, 18\}$.

Theorem 3.1: The sets $\{S_i\}_{i=1}^{\infty}$ are pairwise disjoint.

Proof: Suppose two of the sets S_i and S_j, where $j > i$, have a common element. By Theorem 2.7, it is clear that j must be $i + 1$ and that the only number that S_i could possibly share with S_{i+1} is

$$[(i+1)\alpha + i - 1] = [i\alpha + i]. \tag{5}$$

Assuming this possibility, let B be a Stolarsky interspersion satisfying $b(i+1, 1) = [(i+1)\alpha + i - 1]$. Now, $b(i,1) \in \{[i\alpha + i - 2], [i\alpha + i - 1], [i\alpha + i]\}$, by Theorem 2.7. Since $b(i,1) \neq [i\alpha + i]$, by property I1, and $b(i,1) \neq [i\alpha + i - 1]$, by Theorem 2.10, we have $b(i,1) = [i\alpha + i - 2]$.

Let $\varepsilon = \{i\alpha\}$, the fractional part, $i\alpha - [i\alpha]$, of $i\alpha$. Then (5) can easily be proved equivalent to

$$0 < \varepsilon < 2 - \alpha. \tag{6}$$

Since $b(i+1,1) = b(i,1) + 2$, the position of the number $x = [i\alpha + i - 1]$ in B is, by Theorem 2.12, in a column numbered ≥ 3. Thus, the row of B containing x contains an immediate predecessor w of x and also an immediate predecessor v of w. Now x must be one of the numbers $[w\alpha]$ or $[w\alpha] + 1$, by Lemma 2.2. We consider these two cases separately.

Case 1: $x = [w\alpha]$. By Lemma 2.3, $w = \left[\frac{x}{\alpha} + 1\right] = [x(\alpha - 1) + 1] = [x\alpha] - x + 1$. Thus,

$$\begin{aligned}
w &= \left[\alpha[i\alpha + i - 1]\right] - [i\alpha + i - 1] + 1 \\
&= \left[((\alpha - 1)[i\alpha] + i) + 2 - \alpha\right] \\
&= [(\alpha - 1)(i\alpha - \varepsilon + i) + 2 - \alpha] \\
&= [i\alpha^2 - \alpha\varepsilon + \varepsilon - i + 2 - \alpha] \\
&= [i\alpha - \alpha\varepsilon + \varepsilon + 2 - \alpha] \\
&= \left[[i\alpha] + (1 + \varepsilon)(2 - \alpha)\right] \\
&= [i\alpha],
\end{aligned}$$

since $0 < (1 + \varepsilon)(2 - \alpha) < 1$. The equations $w = [i\alpha]$, $x = [i\alpha + i - 1]$, and $x = w + v$ imply $v = i - 1$. Then $[v\alpha + 2] = [i\alpha - \alpha - 2] = [[i\alpha] + \{i\alpha\} + 2 - \alpha]$, which by (6) equals w. Thus, neither $[\alpha v]$ nor $[\alpha v + 1]$ equals w. This contradiction to Lemma 2.2 completes the proof for Case 1.

Case 2: $x = [w\alpha + 1]$. By Lemma 2.3, $w = \left[\frac{x}{\alpha}\right] = [x(\alpha - 1)]$, so that

$$\begin{aligned}
w &= [(i\alpha - \varepsilon)(\alpha - 1) + (\alpha - 1)(i - 1)] \\
&= [i\alpha + (1 - \alpha)(1 + \varepsilon)] \\
&= [i\alpha - 1], \text{ since } -1 < (1 - \alpha)(1 + \varepsilon) < 0.
\end{aligned}$$

The equations $w = [i\alpha - 1]$, $x = [i\alpha + i - 1]$, and $x = w + v$ imply $v = i$. Then $w = [v\alpha - 1]$, contrary to Lemma 2.2. □

Theorem 3.2: Let $S = \bigcup_{i=1}^{\infty} S_i$. Let $\mathbb{F} = \{2, 3, 5, 8, 13, \ldots\}$, the set of Fibonacci numbers $F_3 = 2$, $F_4 = 3, \ldots, F_n = F_{n-1} + F_{n-2}$. Then S is the set of all positive integers not in $\mathbb{F}$.

Proof: Each number in $\mathbb{F}$ necessarily lies in row 1 and not in column 1. We shall show that, for any positive integer x other than these, there exists a Stolarsky interspersion containing x in its first column.

Let E_k be the statement that, for all $m \le k$ such that $m \notin \mathbb{F}$, there exists a Stolarsky interspersion in which m occurs in the first column. Clearly E_k is true for k = 1, 2, 3, 4. Assume for arbitrary $k \ge 4$ that E_k is true. If $k+1 \in \mathbb{F}$, then clearly E_{k+1} is true. Suppose $k+1 \notin \mathbb{F}$. Let $\delta = \left[\frac{k+2}{\alpha}\right] - \left[\frac{k+1}{\alpha}\right]$. Since $1 < \alpha < 2$, we have $\delta \in \{0, 1\}$. If $\delta = 1$, let $m = \left[\frac{k+2}{\alpha}\right]$ and obtain $k+1 = [m\alpha]$, but if $\delta = 0$, let $m = \left[\frac{k+1}{\alpha}\right]$ and obtain $k+1 = [m\alpha + 1]$.

<u>**Case 1:**</u> $m \notin \mathbb{F}$. Here, by the induction hypothesis, there exists a Stolarsky interspersion B containing m in its first column. Write $m = b(i_0, 1)$. We shall construct a new Stolarsky interspersion A as follows: Define $a(i, j) = b(i, j)$ for all $i \le i_0 - 1, j \ge 1$. Define $a(i_0, 1) = m$. If $k+1 = [m\alpha]$, then define $a(i_0, 2) = [m\alpha + 1]$, but if $k+1 = [m\alpha + 1]$, then define $a(i_0, 2) = [m\alpha]$. Define the rest of row i_0 recursively: $a(i_0, j) = a(i_0, j-1) + a(i_0, j-2)$. Then finish defining A as in Construction 2.1. By Theorem 5 of [6], A contains $k+1$ in its first column.

<u>**Case 2:**</u> $m = F_p$ for some p. Here, $\delta = 1$, $k+2 = F_{p+1} = [m\alpha + 1]$, and $k = [(m-1)\alpha]$. Since $m - 1 \notin \mathbb{F}$, there exists a Stolarsky interspersion B having $m-1$ in its first column. Necessarily, B contains $k+2$ in its first row, immediately following m. As in Case 1, we construct from B a Stolarsky interspersion A in which the immediate follower of $m-1$ is k. Now the only possible immediate predecessors of $k+1$ are m and $m-1$. Since neither of these is followed by $k+1$ in A, we conclude that $k+1$ lies in the first column of A. □

4. A NEW STOLARSKY INTERSPERSION: THE EVEN FIRST COLUMN ARRAY

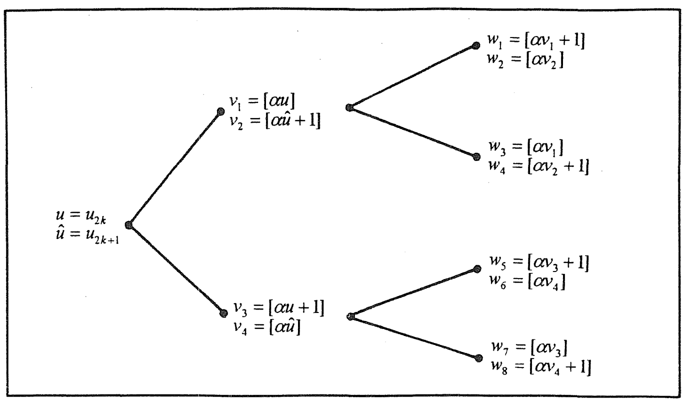
In addition to the two well-known Stolarsky interspersions of Tables 1 and 2 above, we introduce here a third, in which the only odd number in the first column is 1. Because of this property, we call this the *even first column array*, or *EFC array*. The array is defined by its classification sequence, namely, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, ... In section 2, we encountered the sense in which a classification sequence defines a Stolarsky interspersion: if the sequence is $\{\delta_i\}$, then the number $a(i, 2)$ in column 2 must be $[\alpha a(i, 1) + \delta_i]$. [Recall that $a(i, 1)$ is always the least positive integer not in any previous row, and $a(i, j)$ for $j \ge 3$ is determined by (1).] The main objective in this section is to prove that the first column does indeed consist solely of even integers except for the first one.

Throughout this section, let $E = E(i, j)$, denote the EFC array with terms $e(i, j)$, and let $u_i = e(i, 1)$. Table 3 shows that the first few u_i are 1, 4, 6, 10, 12, 14, 16. We shall deal with the u_i in pairs: 4, 6; 10, 12; 14, 16; etc. Each such pair u_{2k}, u_{2k+1} generates six terms u_m where $m \ge 2k$. To describe these "higher u_m's," we define the u_{2k}*-tree*, written as $T(u_{2k})$, as shown in Figure 1.

The classification sequence has $\delta_{2k} = 1$ and $\delta_{2k+1} = 0$, so that (7) shows that the numbers v_3 and v_4 must lie in column 2 of E, so that v_1 and v_2 must, by Lemma 2.2, be higher u_m's. We shall show below that v_1 and v_2 are, respectively, of the forms u_{2p+1} and u_{2q}. Assuming this for now, it follows by Lemma 2.2 that w_3, w_4, w_7, w_8 must lie in column 3 of E. Now w_1, w_2, w_5, w_6 must lie in E, but for each of these, its only possible immediate predecessor, as given by Lemma 2.3, is immediately followed by one of w_3, w_4, w_7, w_8. Therefore, w_1, w_2, w_5, w_6 are higher u_m's.

TABLE 3. The Even First Column Array

1	2	3	5	8	13	21	34	55	89	144	...
4	7	11	18	29	47	76	123	199	322	521	
6	9	15	24	39	63	102	165	267	432	699	
10	17	27	44	71	115	186	301	487	788	1275	
12	19	31	50	81	131	212	343	555	898	1453	
14	23	37	60	97	157	254	411	665	1076	1741	
16	25	41	66	107	173	280	453	733	1186	1919	
20	33	53	86	139	225	364	589	953	1542	2495	
22	35	57	92	149	241	390	631	1021	1652	2673	
26	43	69	112	181	293	474	767	1241	2008	3249	
28	45	73	118	191	309	500	809	1309	2118	3427	
30	49	79	128	207	335	542	877	1419	2296	3715	
32	51	83	134	217	351	568	919	1487	2406	3893	
36	59	95	154	249	403	652	1055	1707	2762	4469	
38	61	99	160	259	419	678	1097	1775	2872	4647	
40	65	105	170	275	445	720	1165	1885	3050	4935	
42	67	109	176	285	461	746	1207	1953	3160	5113	
⋮											



(7)

FIGURE 1. The Tree $T(u_{2k})$

Lemma 4.1: Suppose j and k are nonzero integers. Let $\alpha = (1+\sqrt{5})/2$. Then

$$\{[j\alpha]\alpha\} - \{[k\alpha]\alpha\} = (1-\alpha)(\{j\alpha\} - \{k\alpha\}). \tag{8}$$

Proof: For any nonzero integer k, we have

$$\begin{aligned} 1 &= \big\{-\{\alpha\}\{k\alpha\}\big\} + \{\alpha\}\{k\alpha\} = \big\{[k\alpha] - \{\alpha\}\{k\alpha\}\big\} + \{\alpha\}\{k\alpha\} \\ &= \big\{k\alpha + k - \{k\alpha\}\alpha\big\} + \{\alpha\}\{k\alpha\} = \big\{k\alpha^2 - \{k\alpha\}\alpha\big\} + \{k\alpha\}\{\alpha\}, \text{ by (2)} \\ &= \big\{(k\alpha - \{k\alpha\})\alpha\big\} + \{k\alpha\}\{\alpha\} \\ &= \{[k\alpha]\alpha\} + \{k\alpha\}\{\alpha\}. \end{aligned}$$

So, if j and k are nonzero integers, we have $\{[j\alpha]\alpha\}+\{j\alpha\}\{\alpha\}=\{[k\alpha]\alpha\}+\{k\alpha\}\{\alpha\}$, and (8) follows. □

Lemma 4.2: Suppose j and k are nonzero integers. Let $\alpha=(1+\sqrt{5})/2$. Then

$$\{[j\alpha^2]\alpha\}-\{[k\alpha^2]\alpha\}=(2-\alpha)(\{j\alpha\}-\{k\alpha\}). \tag{9}$$

Proof: For any nonzero integer k, we have $\{[k\alpha]\alpha\}+\{k\alpha\}\{\alpha\}=1$ from the proof of Lemma 4.1, so that $\{[k\alpha]\alpha\}+\{k\alpha\}>1$, a fact used below:

$$\begin{aligned}\{[k\alpha^2]\alpha\}&=\{[k\alpha+k]\alpha\}=\{[k\alpha]\alpha+k\alpha\}=\{[k\alpha]\alpha\}+\{k\alpha\}-1\\&=\{(k\alpha-\{k\alpha\})\alpha\}+\{k\alpha\}-1=\left\{k\alpha^2-\{k\alpha\}\alpha\right\}+\{k\alpha\}-1\\&=\left\{k\alpha-\{k\alpha\}\alpha\right\}+\{k\alpha\}-1=\left\{\{k\alpha\}-\{k\alpha\}\alpha\right\}+\{k\alpha\}-1\\&=1-(\alpha-1)\{k\alpha\}+\{k\alpha\}-1\\&=(2-\alpha)\{k\alpha\}.\end{aligned}$$

So, if j and k are nonzero integers, then (9) holds.

Lemma 4.3: An integer u is of the form $[j\alpha]$ for some integer j iff $\{u\alpha\}>2-\alpha$. Equivalently, an integer u is of the form $[j\alpha]+j$ for some integer j iff $\{u\alpha\}<2-\alpha$. (This inequality is stated without proof in [2].)

Proof: Lemma 4.1 implies that, for any integers j and k, we have $\{j\alpha\}>\{k\alpha\}$ iff $\{[j\alpha]\alpha\}<\{[k\alpha]\alpha\}$. The well-known fact that $\max\{\{j\alpha\}:1\le j\le F_{2n}\}=\{\alpha F_{2n}\}$ implies, therefore, that $\min\{\{[j\alpha]\alpha\}:1\le j\le F_{2n}\}=\{[\alpha F_{2n}]\alpha\}$. Since $\lim_{n\to\infty}\{[\alpha F_{2n}]\alpha\}=2-\alpha$, we have $\{[j\alpha]\alpha\}>2-\alpha$ for all positive integers j.

For the converse, Lemma 4.2 implies that, for any integers j and k, we have $\{j\alpha\}>\{k\alpha\}$ iff $\{[j\alpha^2]\alpha\}>\{[k\alpha^2]\alpha\}$. The fact that $\max\{\{j\alpha\}:1\le j\le F_{2n}\}=\{\alpha F_{2n}\}$ implies, therefore, that $\max\{\{[j\alpha^2]\alpha\}:1\le j\le F_{2n}\}=\{[\alpha^2F_{2n}]\alpha\}$. Since $\lim_{n\to\infty}\{[\alpha^2F_{2n}]\alpha\}=2-\alpha$, we have, for all positive integers j, $\{[j\alpha^2]\alpha\}<2-\alpha$. But, by Beatty's theorem, as j ranges through the positive integers, the numbers $[j\alpha^2]$ range through all the positive integers not of the form $[j\alpha]$. Since $[j\alpha^2]=[j\alpha]+j$, the proof is finished. □

Lemma 4.4: Suppose u has the form $2[n\alpha]+2n$ and $v=[u\alpha]$. Let $q=\left[\frac{u}{2\alpha}\right]$. Then

$$v=2[q\alpha]+2q+2 \quad\text{and}\quad u+v=[v\alpha+1].$$

Proof: We have $\frac{u}{2\alpha}-1<q<\frac{u}{2\alpha}$, so that $\frac{u}{2}-\alpha<q\alpha<\frac{u}{2}$. Thus, $q\alpha$ is strictly less than the integer $\frac{u}{2}$. so that $[q\alpha]=\frac{u}{2}-1$. Also $q=\left[\frac{u}{2\alpha}\right]=\left[\frac{u}{2}(\alpha-1)\right]=\left[\frac{u\alpha}{2}\right]-\frac{u}{2}$. Accordingly,

$$2[q\alpha]+2q+2=u-2+2\left[\frac{u\alpha}{2}\right]-u+2=2\left[\frac{u\alpha}{2}\right]=2[[n\alpha]\alpha+n\alpha].$$

By Lemma 4.3, $\{[n\alpha]\alpha+n\alpha\}<1/2$, and this implies $2[[n\alpha]\alpha+n\alpha]=[2[n\alpha]\alpha+2n\alpha]$, which is v. Next,

$$
\begin{aligned}
[v\alpha+1] &= \big[\alpha[u\alpha]\big]+1 = [\alpha(u\alpha-\varepsilon)]+1, \quad \text{where } \varepsilon=\{u\alpha\},\\
&= [u\alpha^2-\alpha\varepsilon]+1 = [u\alpha+u-\alpha\varepsilon]+1 = u\alpha-\alpha\varepsilon-\{u\alpha-\alpha\varepsilon\}+u+1\\
&= u\alpha-\alpha\varepsilon-\big\{u\alpha-\{u\alpha\}-\{\alpha\}\{u\alpha\}\big\}+u-1\\
&= u\alpha-\alpha\varepsilon-\big\{-\{\alpha\}\{u\alpha\}\big\}+u+1, \quad \text{since } u\alpha-\{u\alpha\} \text{ is an integer}\\
&= u\alpha-\alpha\{u\alpha\}+\{\alpha\}\{u\alpha\}+u = u\alpha-\{u\alpha\}+u\\
&= v+u
\end{aligned}
$$

Lemma 4.5: Suppose $\hat{u}$ has the form $2[n\alpha]+2n+2$ and $\hat{v}=[\hat{u}\alpha+1]$. Let $q=\left[\frac{\hat{u}}{2\alpha}+1\right]$. Then $\hat{v}=2[q\alpha]+2q$ and $\hat{u}+\hat{v}=[\hat{v}\alpha]$.

Proof: The proof is similar to that of Lemma 4.4 and is omitted. □

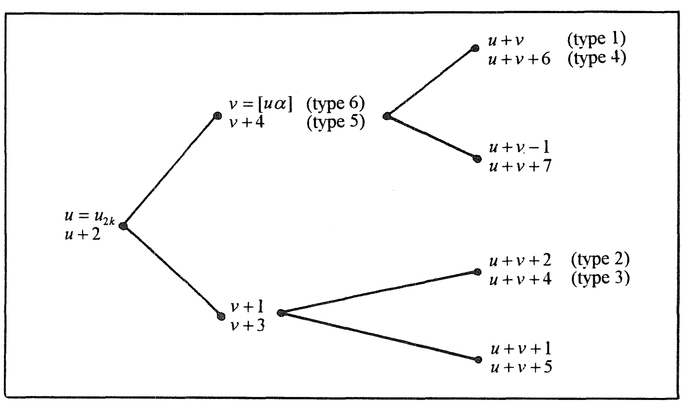
Lemma 4.6: Suppose u has the form $2[n\alpha]+2n$ and $\hat{u}=u+2$ in the u_{2k}-tree of Figure 1 (7). Then $v_2=v_1+4$.

Proof: By Beatty's theorem, $[n/\alpha]$ is not of the form $[w\alpha]$, so that, by Lemma 4.3, $\{\alpha[n/\alpha]\}<2-\alpha$. Substituting $\alpha+1$ for $1/\alpha$ and multiplying by 2 gives $2\{\alpha([n\alpha]+n)\}<4-2\alpha$. Then $\{\alpha([n\alpha]+n\}<1/2$ since $4-2\alpha<1$, so that $\{u\alpha\}=\{2\alpha([n\alpha]+n\}<4-2\alpha$. Since $\{2\alpha\}=2\alpha-3$, we have $\{u\alpha+2\alpha\}=\{u\alpha\}+\{2\alpha\}$, from which follows $[u\alpha+2\alpha]-[u\alpha]=3$. Equivalently, $v_2-v_1=[u\alpha+2\alpha+1]-[u\alpha]=4$. □

Lemma 4.7: In the u_{2k}-tree (7), suppose u has the form $2[n\alpha]+2n$ and $\hat{u}=u+2$. Let $v=v_1$. Then $v_2=v+4, v_3=v+1, v_4=v+3$. Moreover, $w_1=u+v, w_2=u+v+6, w_3=u+v+1$, and $w_4=u+v+7$; also, $w_5=u+v+2, w_6=u+v+2, w_7=u+v+1$, and $w_8=u+v+5$.

Proof: Clearly $v_3=v+1$. By Lemma 4.6, $v_2=v+4$, so that $v_4=v+3$. By Lemma 4.4, $w_1=u+v$, so that $w_3=u+v-1$. Now $w_2=[\alpha v_2]$, which by Lemma 4.5 equals $\hat{u}+\hat{v}$, which is $u+v+6$, and then $w_4=u+v+7$. By recurrence (1), $w_7=u+v+1$ and $w_8=u+v+5$, and from these follow $w_5=u+v+2$ and $w_6=u+v+4$. □

Under the assumption that u $(=u_{2k})$ is of the form $2[n\alpha]+2n$ and $\hat{u}=u+2$, we can summarize Lemma 4.7 by rewirting the tree $T(u_{2k})$ in (7) with new labels:



(10)

FIGURE 2. The Tree $T(u_{2k})$, Relabeled

Lemma 4.8: Suppose k is a positive integer, $u = 2[k\alpha]+2k$ and $\hat{u} = 2[k\alpha]+2k+2$. Then $\hat{u}-u = 4$ or $\hat{u}-u = 6$, according as $\{k\alpha\} < 2-\alpha$ or $\{k\alpha\} > 2-\alpha$.

Proof: The proof is easy and is omitted. □

Lemma 4.9: Let E be the EFC array. The numbers u_k in column 1 of E are given by $u_1 = 1$ and

$$u_{2n} = 2[n\alpha]+2n, \tag{11}$$

$$u_{2n+1} = 2[n\alpha]+2n+2 \tag{12}$$

for $n = 1, 2, 3, \ldots$.

Proof: It is easy to check that (11) and (12) hold for $1 \le n \le 3$ and that in the tree $T(u_2)$ we find $u_3 = 6$ of type 6, $u_4 = 10$ of type 5 (and also of type 1), $u_5 = 12$ of type 2, $u_6 = 14$ of type 3, $u_7 = 16$ of type 4. Suppose now that $m \ge 7$, and as an induction hypothesis, assume that for every h satisfying $3 \le h \le m$ the following conditions hold:

(i) there exists k such that $2k \le h-2$ and u_h is a vertex of tree $T(u_{2k})$, and in that tree, u_h is of one of the six types identified in (10);

(ii) in $T(u_{2k})$, if u_h is of type 1, 3, or 5, then h is even, and if $h = 2p$, then $u_h = 2[p\alpha]+2p$;

(iii) in $T(u_{2k})$, if u_h is of type 2, 4, or 6, then h is odd, and if $h = 2p+1$, then $u_h = 2[p\alpha]+2p+2$.

<u>**Case 1:**</u> u_m is of type 1 (or type 3) in a tree $T(u_{2k})$. By (ii), $u_m = 2[p\alpha]+2p$, where $m = 2p$. Theorem 2.10 and Lemma 4.7 then imply $u_{m+1} = 2[p\alpha]+2p+2$, so that u_{m+1} is of type 2 (or type 4) and satisfies (12).

<u>**Case 2:**</u> u_m is of type 2 in a tree $T(u_{2k})$. By (iii), $u_m = 2[p\alpha]+2p+2$, where $m = 2p+1$. Theorem 2.10 and Lemma 4.7 then imply $u_{m+1} = 2[p\alpha]+2p+4 = 2[(p+1)\alpha]+2(p+1)$, so that u_{m+1} is of type 3 and satisfies (12).

<u>**Case 3:**</u> u_m is of type 4 in a tree $T(u_{2k})$. As in the proof of Lemma 4.6, we have $\{\alpha u_{2k}\} < 4-2\alpha$, so that

$$\frac{4\alpha-7}{\alpha-1} < \{\alpha u_{2k}\} < \frac{4\alpha-6}{\alpha-1} = 4-2\alpha,$$

which implies $0 < 4\alpha-6+(1-\alpha)\{\alpha u_{2k}\} < 1$, so that $6 = [(1-\alpha)\{\alpha u_{2k}\}+4\alpha]$ and $6 = [\alpha u_{2k} - [\alpha u_{2k}] - \alpha\{\alpha u_{2k}\}+4\alpha]$. Adding $u_{2k}+[\alpha u_{2k}]$ to both sides and applying Lemma 4.7 give

$$\begin{aligned} u_m &= [\alpha u_{2k}+u_{2k}-\alpha\{\alpha u_{2k}\}+4\alpha] \\ &= [\alpha(\alpha u_{2k}-\{\alpha u_{2k}\}+4)], \text{ by (2)} \\ &= [\alpha([\alpha u_{2k}]+4)], \end{aligned}$$

which is the number of type 6 in tree $T([\alpha u_{2k}]+4)$. By Lemma 4.6, the number u_m+4 is of type 5 in tree $T([\alpha u_{2k}]+4)$.

By Theorem 2.7, $u_{m+1} \le [(m+1)\alpha]+m+1$ and $[(m-3)\alpha]+m-5 \le u_{m-3}$, so that

$$u_{m+1}-u_{m-3} \ge [(m+1)\alpha]-[(m-3)\alpha]+6 \ge 11.472,$$

but, since $u_{m+1}-u_{m-3}$ is an integer, we have

$$u_{m+1}-u_{m-3}\geq 12. \tag{13}$$

Since u_m is of type 4, the number u_{m-i} must, by the induction hypothesis, be of type $4-i$, for $i=1,2,3$, so that $u_m=u_{m-1}+2$, $u_{m-1}=u_{m-2}+2$, and $u_{m-2}=u_{m-3}+2$; these imply

$$u_m-u_{m-3}=8. \tag{14}$$

By Theorem 2.7, $u_{m+1}\in\{u_m+2, u_m+3, u_m+4\}$, so that (13) and (14) force u_{m+1} to be u_m+4.

By the induction hypothesis, $u_m=2[p\alpha]+2p+2$, where $m=2p+1$, $u_{m-1}=2[p\alpha]+2p$, and $u_{m-2}=2[(p-1)\alpha]+2(p-1)+2$. The equation $u_{m+1}-u_{m-2}=2$ therefore easily yields

$$[p\alpha]-[(p-1)\alpha]=1. \tag{15}$$

Now, if $[(p+1)\alpha]-[p\alpha]=1$, this and (15) imply $[(p+1)\alpha]-[(p-1)\alpha]=2$, which is easily seen to be impossible, since $1/2<\alpha<1$. Therefore, $[(p+1)\alpha]-[p\alpha]=2$, so that $u_{m+1}=u_m+4=2[p\alpha]+2p+6=2[(p+1)\alpha]+2(p+1)$, and (11) holds.

Case 4: u_m is of type 5 in a tree $T(u_{2k})$. Before breaking this into two subcases, we note that

$$\{\alpha u_{2k}\}=\{\alpha(2[k\alpha]+2k)\}=\{4k\alpha-2\alpha\{k\alpha\}\}=(4-2\alpha)\{k\alpha\}. \tag{16}$$

Case 4.1: $\{k\alpha\}>2-\alpha$. In this case, (16) implies $\{\alpha u_{2k}\}>(4-2\alpha)(2-\alpha)=2(5-3\alpha)$. The inequality $\{\alpha u_{2k}\}>5-3\alpha$ implies

$$5-3\alpha=\frac{2\alpha-3}{\alpha}<\{\alpha u_{2k}\}<1<2\alpha-2,$$

which implies $[2\alpha-\alpha\{\alpha u_{2k}\}]=2$, so that

$$\begin{aligned}[\alpha u_{2k}+4]&=[\alpha u_{2k}]+2+[2\alpha-\alpha\{\alpha u_{2k}\}]\\&=[\alpha u_{2k}-u_{2k}+2]+[\alpha^2u_{2k}-\alpha\{\alpha u_{2k}\}-\alpha u_{2k}+2\alpha]\\&=[\alpha u_{2k}-u_{2k}+2]+[\alpha[\alpha u_{2k}-u_{2k}+2]].\end{aligned}$$

This shows that the number u_m of type 5 in a tree $T(u_{2k})$, namely, $[\alpha u_{2k}+4]$, is the same as the number of type 1 in tree $T(\alpha u_{2k}-u_{2k}+2)$. It follows from Case 1 that u_{m+1} is of type 2 in tree $T(\alpha u_{2k}-u_{2k}+2)$ and satisfies the required conditions.

Case 4.2: $\{k\alpha\}>2-\alpha$. Again (16) applies, giving $\{\alpha u_{2k}\}<2(5-3\alpha)<7-4\alpha=1-\{4\alpha\}$, so that $\{\alpha u_{2k}\}+\{4\alpha\}<1$. Consequently, $\{\alpha u_{2k}+4\alpha\}-\{\alpha u_{2k}\}=4\alpha-6$, so that

$$[(u_{2k}+4)\alpha]=[\alpha u_{2k}+4]+2. \tag{17}$$

Since $m\geq 7$, we have $2k\leq m-2$, by hypothesis (i), so that Lemma 4.8 gives $u_{2k+2}=u_{2k}+4$. Then (17) implies that u_{m+1} is the number of type 6 in tree $T(u_{2k+2})$, and (12) holds.

Case 5: u_m is of type 6 in a tree $T(u_{2k})$. We already know by Lemma 4.6 that the number u_m+4 is of type 5 in tree $T(u_{2k})$. If $u_{m+1}=u_m+2$, then we would have $u_{m+1}-u_{m-3}=10$ and a contradiction as in the proof for Case 3. Moreover, $u_{m+1}-u_m$ cannot be 1 or 3, by Theorem 2.10. Therefore, $u_{m+1}=u_m+4$, and as in the proof for Case 3, we find that (11) holds.

We have now shown that the conditions (i), (ii), and (iii) stated in the induction hypothesis all hold for $h = m+1$. Therefore, equations (11) and (12) hold for all positive integers n. □

5. CONCLUSION

It is clear from the induction method of the proof of Theorem 4.9 that the EFC array is the only Stolarsky interspersion having only even numbers in the first column, except for the initial 1.

We recount the connections between certain classification sequences $\{\delta_i\}$ and the first columns of the associated Stolarsky interspersions $\{u_i\}$.

Wythoff Array (Table 2): $\delta_i = 1$ for all i, and $u_i = [i\alpha]+i-1$ for all i. In fact, *all* the terms $a(i, j)$ of the Wythoff array are conveniently expressible: $a(i, j) = [i\alpha]F_{j+1} + (i-1)F_j$. Corollary 2.8 shows that the Wythoff array is "central" among Stolarsky interspersions.

Dual of the Wythoff Array: $\delta_1 = 1$ and $\delta_i = 0$ for all $i \geq 2$, and

$$u_i = \begin{cases} [i\alpha]+i & \text{if } i \text{ is of the form } [k\alpha]+k+1, \\ [i\alpha]+i-1 & \text{otherwise.} \end{cases}$$

Stolarsky Array (Table 1): $u_i = \left[\left(i-\frac{1}{2}\right)\alpha\right]+i$. No convenient formula for δ_i has been found; the sequence begins like this: 1 0 0 1 0 1 1 0 1 0 1 1 0 1 0 0 1 0 1.

EFC Array (Table 3): $\delta_i = 1$, $\delta_{2k} = 1$, $\delta_{2k+1} = 0$ for all $k \geq 1$, and

$$u_i = \begin{cases} 2\left[\frac{i\alpha}{2}\right]+i & \text{if } i \text{ is even,} \\ 2\left[\frac{i-1}{2}\alpha\right]+i+1 & \text{otherwise,} \end{cases}$$

by Theorem 4.9.

ESC Array: Introduced here by its classification sequence, $\{\delta_i\} = \{1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots\}$. We conjecture that the second column of this array consists solely of even integers, beginning with 2, 6, 12, 14, 18, 24, 28, 32, 36, 40. Can someone figure out a formula for u_i?

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CONGRUENCES AND RECURRENCES FOR BERNOULLI NUMBERS OF HIGHER ORDER

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1. INTRODUCTION

The Bernoulli polynomials of order k, for any integer k, may be defined by (see [10], p. 145):

$$\frac{x^k e^{xz}}{(e^x-1)^k} = \sum_{n=0}^{\infty} B_n^{(k)}(z)\frac{x^n}{n!}. \tag{1.1}$$

In particular, $B_n^{(k)}(0) = B_n^{(k)}$, the Bernoulli number of order k, and $B_n^{(1)} = B_n$, the ordinary Bernoulli number. Note also that $B_n^{(0)} = 0$ for $n > 0$.

The polynomials $B_n^{(k)}(z)$ and the numbers $B_n^{(k)}$ were first defined and studied by Niels Nörlund in the 1920s; later they were the subject of many papers by L. Carlitz and others. For the past twenty-five years not much has been done with them, although recently the writer found an application for $B_n^{(k)}$ involving congruences for Stirling numbers (see [8]). For the writer, the higher-order Bernoulli polynomials and numbers are still of interest, and they are worthy of further investigation.

Apparently, not much is known about the divisibility properties of $B_n^{(k)}$ for general k. Carlitz [2] proved that if p is prime and

$$k = a_1 p^{k_1} + a_2 p^{k_2} + \cdots + a_r p^{k_r} \quad (0 \le k_1 < k_2 < \cdots k_r;\ 0 < a_j < p),$$

then $p^r B_n^{(k)}$ is integral (mod p) for all n. He (see [4], [5]) also proved the following congruences for primes $p > 3$:

$$B_p^{(p)} \equiv -\frac{1}{2}p^2(p-1)!\ (\text{mod}\ p^5), \tag{1.2}$$

$$B_{p+2}^{(p+1)} \equiv \frac{1}{6}p^3\ (\text{mod}\ p^4), \tag{1.3}$$

$$B_{p+2}^{(p)} \equiv \frac{1}{p+1}p^2 B_{p+1}\ (\text{mod}\ p^4), \tag{1.4}$$

where B_{p+1} is the ordinary Bernoulli number. F. R. Olson [11] was able to extend (1.2) and (1.3) slightly by proving congruences modulo p^6 and p^5, respectively. Carlitz [4] proved that $B_n^{(p)}$ is integral (mod p), $p \ge 3$, unless $n \equiv 0\ (\text{mod}\ p-1)$ and $n \equiv 0$ or $p-1\ (\text{mod}\ p)$, in which case $pB_n^{(p)}$ is integral. He also proved congruences for special cases of $B_n^{(p)}$.

The writer [8] examined the numbers $B_n^{(n)}$ and proved that, for p prime, $p > 3$, r odd, and $p+1 \ge r \ge 5$,

$$B_p^{(p)} = -\sum_{j=1}^{r-4} \frac{1}{j+1} s(p,j)p^{j+1}\ (\text{mod}\ p^r), \tag{1.5}$$

where $s(p, j)$ is the Stirling number of the first kind. (The Stirling numbers are defined in section 2.) This enables us to extend (1.2), theoretically, to any modulus p^r. Many other properties of $B_n^{(n)}$ are worked out in [8], and applications are given that involve new congruences for the Stirling numbers.

The purpose of the present paper is to examine the divisibility properties of $B_n^{(k)}$ for arbitrary n and k. We are able to extend congruences (1.3) and (1.4), and we also generalize many of the results in [8] and [10]. A summary of the main results follows.

1. We prove that the Bernoulli polynomials have the following property:

$$B_{n+k}^{(n)}\left(z+\frac{1}{2}n\right)=(-1)^{n+k}B_{n+k}^{(n)}\left(-z+\frac{1}{2}n\right).$$

To the writer's knowledge, this is a new result. It is very helpful in proving congruences (1.6)-(1.9) below.

2. We extend (1.3) and (1.4) by proving, for $p>5$:

$$B_{p+2}^{(p+1)}\equiv-\frac{1}{12}(p+2)!\,p^2 \pmod{p^6}, \tag{1.6}$$

$$B_{p+2}^{(p)}\equiv\frac{1}{24}p^2(p+2)!(p+12b_{p+1}) \pmod{p^7}, \tag{1.7}$$

$$B_{p+4}^{(p)}\equiv\frac{1}{12}p^2(p+4)!(3p+2)b_{p+3} \pmod{p^4}, \tag{1.8}$$

where b_n is the Bernoulli number of the second kind, defined and studied by Jordan [9], pp. 265-287 and by Carlitz [1]. The numbers b_n are also defined in section 2 of this paper, and we show in section 2 that $B_n^{(n-1)}=-(n-1)n!\,b_n$

3. Motivated by (1.6), we prove that if n is odd and composite, $n>9$, then

$$B_{n+2}^{(n+1)}\equiv 0 \pmod{n^4}. \tag{1.9}$$

4. For $k\geq 0$, we define

$$A_k(p;n)=\frac{(-1)^n p^{[n/(p-1)]}}{n!}B_n^{(n-k)},$$

and we prove that $A_k(p;n)$ is integral (mod p); in fact, if p does not divide k, then $\frac{1}{n-k}A_k(p;n)$ is integral (mod p). This improves results of Carlitz [2], [3].

5. With $A_k(p;n)$ as defined above, we prove

$$A_k(p;r(p-1))\equiv(-1)^r\binom{r+k}{k} \pmod{p}, \tag{1.10}$$

$$A_k(p;r(p-1)+1)\equiv\frac{1}{2}(-1)^{r-1}(r+k-1)\binom{r+k}{k} \pmod{p}\quad (p>2). \tag{1.11}$$

These congruences give us some insight into the highest power of p (especially $p=2$) dividing the denominator of $B_n^{(n-k)}$. This is discussed in sections 3 and 4.

6. We prove the following recurrence formulas, which generalize results of Nörlund [10], p. 150, for $k=0$. For $k \geq 0$,

$$\frac{B_n^{(n-k+1)}}{n!}=\sum_{r=0}^{n}\frac{(-1)^{n-r}}{n+1-r}\frac{B_r^{(r-k)}}{r!},$$

$$\frac{(-1)^{n+k}B_{n+k}^{(n)}}{(n+k)!}=\sum_{r=0}^{n}\binom{n}{r}\frac{B_{r+k}^{(r)}}{(r+k)!}.$$

These recurrences turn out to be helpful in proving (1.10), (1.11), and the fact that $A_k(p;n)$ is integral (mod p).

Section 2 is a preliminary section that includes the definitions and known results that we need. In section 3 we examine $B_n^{(k)}$ for arbitrary n and k, and we find new congruences, generating functions and recurrences. In section 4 we look at $B_n^{(n-1)}$ in more detail, and we find some additional properties.

Throughout the paper, the letter p designates a prime number and the letter n denotes a nonnegative integer.

2. PRELIMINARIES

We first note some special cases (see [4]). If $n<k$, then $B_n^{(k)}=\binom{k-1}{n}^{-1}s(k,k-n)$, where $s(k,k-n)$ is the Stirling number of the first kind, defined by

$$x(x-1)\cdots(x-n+1)=\sum_{k=0}^{n}s(n,k)x^k, \tag{2.1}$$

or by the generating function

$$\{\log(1+x)\}^k=k!\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!}.$$

If $k>0$, then $B_n^{(-k)}=\binom{n+k}{k}^{-1}S(n+k,k)$, where $S(n+k,k)$ is the Stirling number of the second kind, defined by

$$x^n=\sum_{k=0}^{n}S(n,k)x(x-1)\cdots(x-k+1),$$

or by the generating function

$$(e^x-1)^k=k!\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!}.$$

Since the Stirling numbers are well known and have been extensively studied (see, e.g., [6], ch. 5; [8]; and [9], ch. 4), in this paper we will concentrate on $B_n^{(k)}$ for $0\leq k\leq n$.

It follows from (1.1) that (see [10], p. 150):

$$B_n^{(k)}(x+y)=\sum_{j=0}^{n}\binom{n}{j}x(x-1)\cdots(x-j+1)B_{n-j}^{(k-j)}(y)=\sum_{j=0}^{n}\binom{n}{j}B_{n-j}^{(k)}(y)x^j. \tag{2.2}$$

$$\frac{d}{dz}B_n^{(k)}(z)=nB_{n-1}^{(k)}(z), \tag{2.3}$$

$$B_n^{(k)}(z+1)-B_n^{(k)}(z)=nB_{n-1}^{(k-1)}(z). \tag{2.4}$$

Nörlund [10], p. 145, proved

$$B_n^{(k+1)}(z) = \left(1 - \frac{n}{k}\right) B_n^{(k)}(z) + (z-k)\frac{n}{k} B_{n-1}^{(k)}(z),$$

so that

$$B_n^{(n-k)} = \frac{k-n}{k} B_n^{(n-k+1)} + \frac{(k-n)n}{k} B_{n-1}^{(n-k)}. \tag{2.5}$$

Nörlund [10], p. 148, also proved

$$B_n^{(k)} = k\binom{n}{k}\sum_{r=0}^{k-1}(-1)^{k-1-r} s(k, k-r)\frac{B_{n-r}}{n-r}, \tag{2.6}$$

which is the basis for some of the results of Carlitz [3], such as (1.2)-(1.4) and the congruences for $B_n^{(p)}$. In (2.6), B_{n-r} is the ordinary Bernoulli number.

Nörlund [10], p. 147, proved the following integration formulas

$$B_n^{(n)}(x) = \int_x^{x+1} (t-1)(t-2)\cdots(t-n)\,dt, \tag{2.7}$$

$$B_{n+1}^{(n)} = -n\int_0^1 t(t-1)\cdots(t-n)\,dt, \tag{2.8}$$

which, when compared with (2.1), indicate the close relationship between the Stirling numbers and the higher-order Bernoulli numbers. Nörlund [10], pp. 147 and 150, also gave the following generating functions:

$$\frac{x^k}{\{\log(1+x)\}^k} = -k \sum_{\substack{n=0 \\ (n\neq k)}}^{\infty} \frac{B_n^{(n-k)}}{n-k}\frac{x^n}{n!}, \tag{2.9}$$

$$\frac{x}{(1+x)\log(1+x)} = \sum_{n=0}^{\infty} B_n^{(n)}\frac{x^n}{n!}. \tag{2.10}$$

Jordan [9], pp. 265-87, defined and studied b_n, the Bernoulli number of the second kind. The generating function is

$$\frac{x}{\log(1+x)} = \sum_{n=0}^{\infty} b_n x^n. \tag{2.11}$$

Comparing (2.9), (2.10), and (2.11), we see that, for $n \neq 1$,

$$\frac{1}{1-n} B_n^{(n-1)} = n!\,b_n = B_n^{(n)} + nB_{n-1}^{(n-1)}. \tag{2.12}$$

The last equality also holds when $n = 1$. To the writer's knowledge, this relationship between b_n and $B_n^{(n-1)}$ has not been pointed out before.

Jordan [9], p. 265, defined the polynomial $\Psi_n(z)$, which has the generating function

$$\frac{x(1+x)^z}{\log(1+x)} = \sum_{n=0}^{\infty} \Psi_n(z) x^n, \tag{2.13}$$

and he proved

$$\Psi_n\left(z-1+\frac{1}{2}n\right)=(-1)^n\Psi_n\left(-z-1+\frac{1}{2}n\right). \tag{2.14}$$

Carlitz [1] extended (2.13) in the logical way by defining $\beta_n^{(k)}(z)$:

$$\frac{x^k(1+x)^z}{\{\log(1+x)\}^k}=\sum_{n=0}^{\infty}\beta_n^{(k)}(z)\frac{x^n}{n!}. \tag{2.15}$$

Thus, $\beta_n^{(k)}(z)$ is analogous to $B_n^{(k)}(z)$, and $\beta_n^{(1)}(z)=n!\Psi_n(z)$. Carlitz also proved the very useful result,

$$\beta_n^{(k+1)}(z-1)=B_n^{(n-k)}(z). \tag{2.16}$$

Note that by (2.9), (2.12), (2.15), and (2.16) we have

$$B_n^{(n-k)}(1)=\beta_n^{(k+1)}(0)=\frac{k+1}{k+1-n}B_n^{(n-k-1)};\ B_n^{(n)}(1)=n!b_n. \tag{2.17}$$

3. $B_n^{(k)}$ for $0\le k\le n$

We first prove a theorem that is the basis for many of our later results.

Theorem 3.1: For all nonnegative integers k,

$$B_{n+k}^{(n)}\left(z+\frac{1}{2}n\right)=(-1)^{n+k}B_{n+k}^{(n)}\left(-z+\frac{1}{2}n\right). \tag{3.1}$$

Proof: We use induction on k. The theorem is true for $k=0$, since by (2.14) and (2.16) we have

$$B_n^{(n)}\left(z+\frac{1}{2}n\right)=\beta_n^{(1)}\left(z-1+\frac{1}{2}n\right)=n!\Psi_n\left(z-1+\frac{1}{2}n\right)$$
$$=(-1)^n n!\Psi_n\left(-z-1+\frac{1}{2}n\right)=(-1)^n B_n^{(n)}\left(-z+\frac{1}{2}n\right).$$

Assume (3.1) holds for a fixed $k-1$, i.e.,

$$B_{n+k-1}^{(n)}\left(z+\frac{1}{2}n\right)=(-1)^{n+k-1}B_{n+k-1}^{(n)}\left(-z+\frac{1}{2}n\right).$$

Then, if $n+k$ is even, $B_{n+k-1}^{(n)}(z+\frac{1}{2}n)$ is an odd function of z. By (2.3), this implies $B_{n+k}^{(n)}(z+\frac{1}{2}n)$ is an even function of z. That is, (3.1) holds for $n+k$ even. If $n+k$ is odd, then $n+1+k$ is even, and we apply the operator Δ to both sides of

$$B_{n+1+k}^{(n+1)}\left(z+\frac{1}{2}+\frac{1}{2}n\right)=B_{n+1+k}^{(n+1)}\left(-z+\frac{1}{2}+\frac{1}{2}n\right)$$

to get, by (2.4),

$$B_{n+k}^{(n)}\left(z+\frac{1}{2}+\frac{1}{2}n\right)=-B_{n+k}^{(n)}\left(-z-1+\frac{1}{2}+\frac{1}{2}n\right).$$

Letting $y=z+\frac{1}{2}$, we obtain, for $n+k$ odd:

$$B_{n+k}^{(n)}\left(y+\frac{1}{2}n\right)=-B_{n+k}^{(n)}\left(-y+\frac{1}{2}n\right).$$

This completes the proof.

We note that Theorem 3.1 implies, for $k \geq 0$ and $n \geq 0$,

$$B_{n+k}^{(n)}(n)=(-1)^{n+k}B_{n+k}^{(n)}. \tag{3.2}$$

Now, since $B_n^{(n)}(z+1)=n!\Psi_n(z)$, and since Jordan [9], p. 265, has shown

$$\frac{d}{dz}\Psi_n(z)=\binom{z}{n-1}=\frac{1}{(n-1)!}\sum_{r=0}^{n-1}s(n-1,r)z^r,$$

it follows that

$$B_{n+1}^{(n+1)}(z)=(n+1)\sum_{r=0}^{n}\frac{1}{r+1}s(n,r)(z-1)^{r+1}+B_{n+1}^{(n+1)}(1). \tag{3.3}$$

Equation (3.3) was also proved in [8], with different notation. Integrating (3.3) k times, using (2.3), we obtain

$$\begin{aligned}B_{n+1+k}^{(n+1)}(z)&=\binom{n+k+1}{k+1}\sum_{r=0}^{n}\binom{r+k+1}{r}^{-1}s(n,r)(z-1)^{r+k+1}\\&+\sum_{r=0}^{k}\binom{n+k+1}{r}B_{n+k+1-r}^{(n+1)}(1)(z-1)^r.\end{aligned} \tag{3.4}$$

We now plug $z=n+1$ into (3.4). By (2.17) and (2.5), the first two terms in the last summation are

$$B_{n+k+1}^{(n+1)}(1)=(n+k+1)B_{n+k}^{(n)}+B_{n+k+1}^{(n+1)},$$

$$(n+k+1)nB_{n+k}^{(n+1)}(1)=-k(n+k+1)B_{n+k}^{(n)},$$

so by (3.2) we have, if $n+k$ is odd:

$$\begin{aligned}(k-1)(n+k+1)B_{n+k}^{(n)}&=\binom{n+k+1}{k+1}\sum_{r=1}^{n}\binom{r+k+1}{r}^{-1}s(n,r)n^{r+k+1}\\&+\sum_{r=2}^{k}\binom{n+k+1}{r}B_{n+k+1-r}^{(n+1)}(1)n^r,\end{aligned} \tag{3.5}$$

and if $n+k$ is even, we have

$$\begin{aligned}-2B_{n+k+1}^{(n+1)}&=\binom{n+k+1}{k+1}\sum_{r=1}^{n}\binom{r+k+1}{r}^{-1}s(n,r)n^{r+k+1}\\&+\sum_{r=2}^{k}\binom{n+k+1}{r}B_{n+k+1-r}^{(n+1)}(1)n^r+(n+k+1)(1-k)B_{n+k}^{(n)}.\end{aligned} \tag{3.6}$$

It is important to remember that (3.5) is valid when $n+k$ is odd, and (3.6) is valid when $n+k$ is even. We are now in a position to prove congruences (1.6)-(1.9).

Theorem 3.2: If p is prime, $p > 5$, then $B_{p+2}^{(p+1)} \equiv -\frac{1}{12}(p+2)!\,p^2 \pmod{p^6}$.

Proof: In (3.6), let $n = p$ and $k = 1$. Then we have

$$B_{p+2}^{(p+1)} \equiv -\frac{1}{2}(p+2)(p+1)\sum_{r=1}^{4}\frac{s(p,r)}{(r+1)(r+2)}p^{r+2} \pmod{p^6}.$$

It is well known [5], pp. 218 and 229, that

$$s(p,j) \equiv 0 \pmod{p} \quad (1 < j < p),$$
$$s(p,2j) \equiv 0 \pmod{p^2} \quad \left\{1 \le j \le \frac{1}{2}(p-3)\right\},$$

so we have $B_{p+2}^{(p+1)} \equiv -\frac{1}{12}(p+2)(p+1)s(p,1)p^3 \equiv -\frac{1}{12}(p+2)(p+1)(p-1)!\,p^3 \pmod{p^6}$. This completes the proof.

Theorem 3.2 extends Carlitz's congruence (1.3) and the work of Olson [11]. The motivation for (1.3) was evidently the congruence $B_{p+2}^{(p+1)} \equiv 0 \pmod{p^2}$, which was proved by S. Wachs [12] in 1947.

We will return to (3.5) later to prove congruences for $B_{p+2}^{(p)}$ and $B_{p+4}^{(p)}$. Next we prove two recurrence formulas that will be useful. Both formulas are given in [10], p. 150, for $k = 0$ only.

Theorem 3.3: For $k \ge 0$,

$$\frac{B_n^{(n-k+1)}}{n!} = \sum_{r=0}^{n}\frac{(-1)^{n-r}}{n+1-r}\frac{B_r^{(r-k)}}{r!}.$$

Proof: In the first equation of (2.2), we replace n by $n+1$, we replace k by $n+1-k$, and we let $y = 0$. We then subtract $B_{n+1}^{(n+1-k)}$ from both sides and divide by x to obtain

$$\frac{B_{n+1}^{(n+1-k)}(x) - B_{n+1}^{(n+1-k)}}{x} = \sum_{j=1}^{n+1}\binom{n+1}{j}(x-1)(x-2)\cdots(x-j+1)B_{n+1-j}^{(n+1-k-j)}. \tag{3.7}$$

We now take the limit as $x \to 0$ of both sides of (3.7). The limit of the left side is

$$\lim_{x\to 0}\frac{d}{dx}B_{n+1}^{(n+1-k)}(x) = (n+1)B_n^{(n+1-k)}.$$

Thus, we have

$$(n+1)B_n^{(n+1-k)} = \sum_{j=1}^{n+1}\binom{n+1}{j}(-1)^{j-1}(j-1)!\,B_{n+1-j}^{(n+1-k-j)},$$

and Theorem 3.3 follows by dividing both sides by $(n+1)!$ and letting $r = n+1-j$. This completes the proof.

Theorem 3.4: For $k \ge 0$,

$$\frac{(-1)^{n+k}B_{n+k}^{(n)}}{(n+k)!} = \sum_{r=0}^{n}\binom{n}{r}\frac{B_{r+k}^{(r)}}{(r+k)!}.$$

Proof: In the first equation of (2.2), replace n by $n+k$, replace k by n, let $y=0$, and let $x=n$. Theorem 3.4 now follows from (3.2), and the proof is complete.

Now for $k \geq 0$, p prime, and $[x]$ the greatest integer function, define

$$A_k(p; n) = \frac{(-1)^n p^{[n/(p-1)]}}{n!} B_n^{(n-k)}. \tag{3.8}$$

It was proved in [8] that $A_0(p; n)$ is integral (mod p); we now show that $A_k(p; n)$ has that same property. We note that $A_k(p; 0) = 1$, by (2.9). Theorem 3.3 gives us

$$A_k(p; n) = A_{k-1}(p; n) - \sum_{r=0}^{n-1} \frac{p^{[n/(p-1)]-[r/(p-1)]}}{n+1-r} A_k(p; r). \tag{3.9}$$

It was proved in [8] that if $p^t | (n+1-r)$ then $[n/(p-1)]-[r/(p-1)] \geq t$. Therefore, we can use induction on k and on n in (3.9) to prove $A_k(p; n)$ is integral (mod p). In fact, it follows from (2.5) that

$$\frac{1}{n-k} A_k(p; n) = -\frac{1}{k} A_{k-1}(p; n) + \frac{1}{k} A_{k-1}(p; n-1) p^{[n/(p-1)]-[(n-1)/(p-1)]},$$

so if p does not divide k, we see that $\frac{1}{n-k} A_k(p; n)$ is integral (mod p). Before putting this information together in a theorem, we make the following definitions.

Let $\alpha_p(n; k)$ denote the exponent of the highest power of p dividing the denominator of $B_n^{(n-k)}$ and let $\upsilon_p(n)$ denote the exponent of the highest power of p dividing $n!$ It is well known that if

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_m p^m \quad (0 \leq n_i < p), \tag{3.10}$$

then $\upsilon_p(n) = \frac{1}{p-1}(n - n_0 - n_1 - \cdots - n_m)$.

We can now state the following theorem.

Theorem 3.5: Let p be prime and let $k \geq 0$. Let n have base p expansion (3.10) and let $\alpha_p(n; k)$ and $\upsilon_p(n)$ be as defined above. Then

$$\alpha_p(n; k) \leq \left[\frac{n}{p-1}\right] - \upsilon_p(n) = \left[\frac{n_0 + n_1 + \cdots + n_m}{p-1}\right].$$

If $p^j | (n-k)$ and p does not divide k, then

$$\alpha_p(n; k) \leq \left[\frac{n_0 + n_1 + \cdots + n_m}{p-1}\right] - j.$$

Corollary: Suppose n has base p expansion (3.10) and suppose $n_0 + n_1 + \cdots + n_m < p-1$. If $p^j | (n-k)$ and p does not divide k, then $B_n^{(n-k)} \equiv 0 \pmod{p^j}$. For example if $0 < k < p-2$ and $j \geq 1$, then $B_{p^j+k}^{(p^j)} \equiv 0 \pmod{p^j}$.

Theorem 3.6: Let $A_k(p;n)$ be defined by (3.8). Then, for $h \geq 0$ and p prime,

$$A_k(p;h(p-1)) \equiv (-1)^h \binom{h+k}{k} \pmod{p}, \tag{3.11}$$

$$A_k(p;h(p-1)+1) \equiv \frac{1}{2}(-1)^{h-1}(h+k-1)\binom{h+k}{k} \pmod{p} \quad (p>2). \tag{3.12}$$

Proof: We will use equation (3.9). It was proved in [8] that we can have

$$p^w|(n+1-r) \text{ and } \left[\frac{n}{p-1}\right]-\left[\frac{r}{p-1}\right]=w$$

only when $w=0$ or $w=1$. Thus, we have, for $0 \leq t < p-1$,

$$\begin{aligned}A_k(p;h(p-1)+t) &\equiv A_{k-1}(p;h(p-1)+t)-A_k(p;(h-1)(p-1)+t)\\ &-\sum_{i=0}^{t-1}\frac{1}{t-i+1}A_k(p;h(p-1)+i) \pmod{p}.\end{aligned}$$

In particular, for $t=0$, we have

$$A_k(p;h(p-1)) \equiv A_{k-1}(p;h(p-1))-A_k(p;(h-1)(p-1)) \pmod{p}. \tag{3.13}$$

In [8] it was proved that (3.11) is true for $k=0$. Also, $A_k(p;0)=1$. Thus, we can use induction on k and on h in (3.13) to prove that (3.11) is true for all k and h.

To prove (3.12), we first note that Theorem 3.4 tells us that if $n+k$ is odd, then

$$2A_k(p;n+k)=\sum_{r=0}^{n-1}\binom{n}{r}(-1)^{r+k}A_k(p;r+k)p^{[(n+k)/(p-1)]-[(r+k)/(p-1)]}.$$

Thus,

$$\begin{aligned}2A_k(p;h(p-1)+1) &\equiv (h(p-1)+1-k)A_k(p;h(p-1))\\ &\equiv -(h+k-1)(-1)^h\binom{h+k}{k} \pmod{p},\end{aligned}$$

and the proof is complete.

For certain values of n, Theorem 3.6 gives us the exact value of $\alpha_p(n;k)$. For example, suppose $p=2$ and

$$\begin{aligned}n &= n_0+n_1 2+n_2 2^2+\cdots+n_m 2^m \quad (0 \leq n_i \leq 1),\\ n+k &= t_0+t_1 2+t_2 2^2+\cdots+t_m 2^m \quad (0 \leq t_i \leq 1),\\ k &= k_0+k_1 2+k_2 2^2+\cdots+k_m t^m \quad (0 \leq k_i \leq 1).\end{aligned}$$

By Theorem 3.6, we see that if $k_i \leq t_i$ for all i, then

$$\alpha_2(n;k)=n-\upsilon_2(n)=n_0+n_1+\cdots+n_m. \tag{3.14}$$

In particular, if $n=2^j$, then $\alpha_2(n;k)=1$ for all $k \neq n$; that is, if n is a power of 2, then 2, but not 4, divides the denominator of $B_n^{(n-k)}$ for all k such that $0 \leq k < n$. More generally, if $2^j|n$ and $k<2^j$, then (3.14) holds.

Theorem 3.7: If $p > 5$, we have

$$B_{p+2}^{(p)} \equiv \frac{1}{24} p^2 (p+2)!(p+12b_{p+1}) \pmod{p^7}, \tag{3.15}$$

$$B_{p+4}^{(p)} \equiv \frac{1}{12} p^2 (p+4)!(3p+2)b_{p+3} \pmod{p^4}, \tag{3.16}$$

where b_n is the Bernoulli number of the second kind, defined by (2.11). In general,

$$B_{p+2k}^{(p)} \equiv 0 \pmod{p^2} \left\{k = 1, 2, \dots, \frac{1}{2}(p-3)\right\}.$$

Proof: In (3.5), let $n = p$ and let $k = 2$. Then we have

$$\begin{aligned} B_{p+2}^{(p)} &\equiv \frac{1}{24}(p+2)(p+1)s(p,1)p^4 + \frac{1}{2}(p+2)B_{p+1}^{(p+1)}(1)p^2 \\ &\equiv \frac{1}{24}(p+2)!\,p^3 + \frac{1}{2}(p+2)(p+1)!\,b_{p+1}p^2 \pmod{p^7}, \end{aligned}$$

and (3.15) is proved. Now in (3.5) we let $n = p$ and $k = 4$ to get

$$3(p+5)B_{p+4}^{(p)} \equiv \sum_{r=2}^{4} \binom{p+5}{r} B_{p+5-r}^{(p+1)}(1)p^r \pmod{p^4}.$$

By (2.17), Theorem 3.2, and (3.15), we see that

$$p^3 B_{p+2}^{(p+1)}(1) \equiv 0 \equiv p^4 B_{p+1}^{(p+1)}(1) \pmod{p^4}.$$

Thus, we have

$$B_{p+4}^{(p)} \equiv \frac{1}{6}(p+4)B_{p+3}^{(p+1)}(1)p^2 \pmod{p^4}. \tag{3.17}$$

By (2.5), (2.12), and Theorem 3.2,

$$B_{p+3}^{(p+1)}(1) \equiv -\frac{1}{2}(p+1)B_{p+3}^{(p+2)} \equiv \frac{1}{2}(p+1)(p+2)(p+3)!\,b_{p+3} \pmod{p^2}, \tag{3.18}$$

and we know $(p+3)!b_3$ is integral (mod p) by (2.12). The proof of (3.16) now follows immediately from (3.17) and (3.18). The last statement of Theorem 3.7 is clear from (3.5) and the proof is complete.

We next derive another formula like (3.4). By (2.7) we have

$$\frac{d}{dz} B_n^{(n)}(z) = n \sum_{r=1}^{n} s(n,r) z^{r-1}, \quad \text{so} \quad B_n^{(n)} = n \sum_{r=1}^{n} \frac{1}{r} s(n,r) z^r + B_n^{(n)}.$$

Integrating k times, using (2.3), we get

$$B_n^{(n-k)}(z) = \binom{n}{k+1} \sum_{r=1}^{n-k} \binom{r+k}{k+1}^{-1} s(n-k,r) z^{r+k} + \sum_{j=0}^{k} \binom{n}{j} B_{n-j}^{(n-k)} z^j. \tag{3.19}$$

Equation (3.19) also follows directly from the second equality of (2.2). By (3.2) and (3.19) we have, for $n+k$ odd,

$$-2B_{n+k}^{(n)}=\binom{n+k}{k+1}\sum_{r=1}^{n}\binom{r+k}{k+1}^{-1}s(n,r)n^{r+k}+\sum_{j=1}^{k}\binom{n+k}{j}B_{n+k-j}^{(n)}n^{j}. \tag{3.20}$$

Carlitz [4] proved that $B_m^{(p)}$ is integral (mod p), $p\geq 3$, unless $m\equiv 0\ (\text{mod}\ p-1)$ and $m\equiv 0$ or $p-1$ (mod p), in which case $pB_m^{(p)}$ is integral. We note that by (3.20), with $n+k=m$ and $n=p$, we can say: If m is odd, if $p|m$, and if $p-1$ does not divide $m-1$, then $B_m^{(p)}\equiv 0\ (\text{mod}\ p^2)$.

4. THE NUMBERS $B_n^{(n-1)}$

Because of their close relationship to the Bernoulli numbers of the second kind, that is, $B_n^{(n-1)}=(1-n)n!b_n$ (proved in section 2), the numbers $B_n^{(n-1)}$ deserve special consideration. We first note that, by (2.15) and (2.16), we have the generating function

$$\frac{x^2}{(1+x)\{\log(1+x)\}^2}=\sum_{n=0}^{\infty}B_n^{(n-1)}\frac{x^n}{n!}.$$

If we integrate the right side of (2.8) we have, for $n\geq 0$,

$$B_n^{(n-1)}=(1-n)\sum_{r=1}^{n}\frac{1}{r+1}s(n,r), \tag{4.1}$$

which provides a way of computing $B_n^{(n-1)}$ if a table of Stirling numbers is available. For example,

$$B_3^{(2)}=-2\left\{\frac{1}{2}s(3,1)+\frac{1}{3}s(3,2)+\frac{1}{4}s(3,3)\right\}=-2\left(\frac{1}{2}\cdot 2-\frac{1}{3}\cdot 3+\frac{1}{4}\right)=-\frac{1}{2}.$$

Equation (4.1) was also given in [9], p. 267, as a formula for b_n.

Another useful formula is the following: If n is odd, then

$$B_{n+2}^{(n+1)}=\binom{n+2}{2}\sum_{r=0}^{n+1}\frac{1}{r+1}s(n+1,r)n^{r+1}. \tag{4.2}$$

Equation (4.2) follows from [9], p. 267,

$$(n+1)!\Psi_{n+2}(z)=\sum_{r=0}^{n+1}\frac{1}{r+1}s(n+1,r)z^{r+1}+(n+1)!b_{n+2}, \tag{4.3}$$

where $\Psi_n(z)$ is defined by (2.13). If we plug $z=n$ into (4.3) and use $\Psi_{n+2}(n)=(-1)^n b_{n+2}$, which follows from (2.14), then (4.2) follows for odd n. We can now prove the following theorem.

Theorem 4.1: If n is odd and composite, $n>9$, then $B_{n+2}^{(n+1)}\equiv 0\ (\text{mod}\ n^4)$.

Proof: It was proved in [8] that if $r\geq 3$ and n is odd and composite, $n>9$, then $\frac{1}{r+1}n^{r+1}\equiv 0$ (mod n^4). Thus, by (4.2), we have

$$B_{n+2}^{(n+1)}\equiv\binom{n+2}{2}\left\{\frac{1}{2}s(n+1,1)n^2+\frac{1}{3}s(n+1,2)n^3\right\}\ (\text{mod}\ n^4). \tag{4.4}$$

Now for n composite and $n>9$ (see [6], p. 217),

$$s(n+1,1)=-n!\equiv 0 \pmod{n^2},$$
$$s(n+1,2)=n!\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\equiv 0 \pmod{n}.$$

Also, we can easily see that if $3^j|n$, then

$$s(n+1,2)\equiv 0 \pmod{3^{j+1}} \quad (j>2).$$

Thus, Theorem 4.1 follows from (4.4), and the proof is complete.

For convenience, we again use the notation

$$A_1(p;n)=\frac{(-1)^n p^{[n/(p-1)]}}{n!}B_n^{(n-1)}.$$

Because of (2.12), many properties of b_n and $B_n^{(n-1)}$ follow from properties of $B_n^{(n)}$. Using results in [8], we can write down the following:

$$\frac{1}{1-n}A_1(2;n)\equiv 1 \pmod 8 \quad (n\neq 1), \tag{4.5}$$

$$\frac{1}{2r-1}A_1(3;2r)\equiv(-1)^{r-1}(3r^3+3r+1) \pmod 9, \tag{4.6}$$

$$\frac{1}{2r}A_1(3;2r+1)\equiv(-1)^{r-1}(4r^3+3r^2+1) \pmod 9 \quad (r\geq 1). \tag{4.7}$$

Congruence (4.5) gives us $\alpha_2(n;1)$, the exact power of 2 dividing the denominator of $B_n^{(n-1)}$. Using the notation of section 3, we have $\alpha_2(n;1)=n-\upsilon_2(n)-j=n_0+n_1+\cdots+n_m-j$, where 2^j is the highest power of 2 dividing $n-1$, and $n_0,n_1,\ldots,n_m$ are the digits in the base 2 expansion of n. Similarly, if n is not an odd integer congruent to 2 (mod 3), then (4.6) and (4.7) give

$$\alpha_3(n,1)=\left[\frac{n}{2}\right]-\upsilon_3(n)-j=\left[\frac{n_0+n_1+\cdots+n_m}{2}\right]-j. \tag{4.8}$$

where 3^j is the highest power of 3 dividing $n-1$, and $n_0,n_1,\ldots,n_m$ are the digits in the base 3 expansion of n. If n is an odd integer congruent to 2 (mod 3), we must replace the first "equals" symbol in (4.8) by "$<$."

We know from section 3 that $\frac{1}{1-n}A_1(p;n)$ is integral (mod p) for any $n\neq 1$.

Jordan [9], p. 267, proved $(-1)^{n+1}b_n>0$ for $n>0$. Hence, we have $(-1)^n B_n^{(n-1)}>0\ (n>1)$. In general, the sign of $B_n^{(n-k)}$ is not known. It seems that the signs usually alternate when $n-k>0$, but there are exceptions. For example, $B_{16}^{(10)}$ and $B_{17}^{(11)}$ are both positive, $B_8^{(3)}$ and $B_9^{(4)}$ are both positive, $B_{10}^{(7)}$ and $B_{11}^{(8)}$ are both negative.

Nörlund [10], p. 461, gave a table of values for $B_n^{(n-1)}$ for $n=2,3,\ldots,12$, and Jordan [9], p. 266, listed b_n for $n=0,1,2,\ldots,10$. We give here the first fifteen values of $B_n^{(n-1)}$ with numerators and denominators factored.

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Table of the Numbers $B_n^{(n-1)}$

$B_0^{(-1)} = 1$	
$B_1^{(0)} = 0$	$B_8^{(7)} = \frac{7\cdot 19\cdot 1787}{2\cdot 3^2\cdot 5}$
$B_2^{(1)} = \frac{1}{2\cdot 3}$	$B_9^{(8)} = -\frac{2\cdot 7^3\cdot 167}{5}$
$B_3^{(2)} = -\frac{1}{2}$	$B_{10}^{(9)} = \frac{3\cdot 3250433}{2^2\cdot 11}$
$B_4^{(3)} = \frac{19}{2\cdot 5}$	$B_{11}^{(10)} = -\frac{3^7\cdot 5^2\cdot 173}{2^2}$
$B_5^{(4)} = -9$	$B_{12}^{(11)} = \frac{11\cdot 541\cdot 4801\cdot 5273}{2^2\cdot 3\cdot 5\cdot 7\cdot 13}$
$B_6^{(5)} = \frac{5\cdot 863}{2^2\cdot 3\cdot 7}$	$B_{13}^{(12)} = -\frac{11^3\cdot 2207\cdot 8329}{2\cdot 5\cdot 7}$
$B_7^{(6)} = -\frac{5^3\cdot 11}{2^2}$	$B_{14}^{(13)} = \frac{13\cdot 132282840127}{2^3\cdot 3^2\cdot 5}$

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FIBONACCI NETWORKS

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1. INTRODUCTION

Several interconnection networks have been proposed in literature for interconnecting computing elements. The interconnection network usually forms a *regular* pattern, which is exploited by the algorithms running on the network. Some of the commercially available networks are the hypercube, mesh, etc., which are highly regular. The advantage of using such regular networks is that the algorithms written for one network can be extended with minimal effort to larger versions of the same network. However, networks like the hypercube, mesh, etc, have one significant disadvantage; they do not scale in increments of one. A hypercube scales in exponents of two, and a mesh scales in order of n or k, in an $n \times k$ mesh.

A tree is the cheapest interconnection network but has unacceptably poor communication and fault-tolerant properties. On the other hand, the complete graph K_n is highly reliable but is extremely expensive. Some of the desirable properties of interconnection networks are high fault tolerance, small diameter, small degree, high connectivity, symmetry/regularity, etc. (most of which are conflicting properties).

A class of networks called *Iterative networks* were proposed to address some of the drawbacks of commercially available networks [3, 4, 7, 8, 12]. Iterative networks can be scaled in increments of one. In fact, they can scale by *any* k, where $k \geq 1$. Interconnection networks are often modeled as undirected graphs, where vertices correspond to processor-memory nodes, and edges represent full-duplex communication links between pairs of nodes. An iterative network of n nodes is a subgraph of the network with $n+1$ nodes. The algorithms running on iterative networks require minimal modifications when extended to scaled versions of the network. This is a significant advantage over networks like hypercube, mesh, etc.

Some of the proposed iterative networks that have appeared in literature are mentioned below. Stirling networks [3] are defined using Stirling numbers of the first kind. Rencontres networks [4] are defined based on rencontres numbers. Pascal networks [7] are defined using the Pascal triangle. Several others, like Steinhaus networks [12], Circulants [2], Topelitz networks [8], etc., have also been proposed in literature. All of these have some of the desirable properties of interconnection networks, but also have certain drawbacks. So the search for new interconnection networks for various classes of problems continues.

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In this paper we introduce a new class of iterative networks using Fibonacci numbers, which we call *Fibonacci networks*. We investigate their graph-theoretic properties and study their suitability for implementing multicomputer systems. The paper is organized as follows: In section 2 we show how Fibonacci networks are constructed. In section 3 we explore some of the properties of these interconnection networks. We show that Fibonacci networks have most of the properties desirable in an interconnection network except that it has too many links making it expensive. We then show how the number of links can be reduced while still maintaining the basic structure of the network. We also explore the properties of the modified network and show that it still retains most of the desired properties of interconnection networks. In section 4 we show that routing can be accomplished very efficiently in Fibonacci networks. In section 5 we show how other networks can be embedded onto Fibonacci networks of comparable size. In section 6 we design some of the basic algorithms, like finding a minimum spanning tree, that can be implemented on Fibonacci networks. Finally, we present some concluding remarks. We have used standard graph-theoretic notation throughout this paper [6]. All logarithms are with respect to base 2 unless specifically mentioned otherwise.

2. FIBONACCI NETWORKS

Fibonacci networks are a class of iterative/recursive networks constructed as described below. Let $fib(q)$ denote the q^{th} Fibonacci number F_q (0 and 1 being the 0^{th} and 1^{st} Fibonacci numbers, respectively). Let $FT(r,k)=fib(k+\sum_{i=0}^{r-1} i)$ for $0<k<r$. An $n\times n$ symmetric matrix is called a *Fibonacci Matrix* $FM^p(n)$ of order n if its main diagonal entries are all 0 and its lower triangular entries (and, therefore, upper also) consist of the $\{0, 1\}$ predicate values $(FT(n-1,k)$ $(\text{mod}\, p)\neq 0)$, where p is usually a small prime. (Later we will show how this definition can be extended when p is a set of primes.) Let

$$fm^p_{i,j} = (i,j)^{\text{th}} \text{ element of } FM^p(n)\in\{0,1\},$$

$$ft_{i,j} = (i,j)^{\text{th}} \text{ element of } FT(n,k)\in N.$$

Then, by definition,

$$fm^p_{i,j} = (ft_{i-1,j} \pmod p \neq 0),$$

and hence,

$$fm^p_{i,j} = \left(fib\left(j+\sum_{x=0}^{i-2} x \right) \pmod p \neq 0 \right),$$

or, alternately,

$$fm^p_{i,j} = \left(fib\left(\frac{(i-1)(i-2)}{2} + j \right) \pmod p \neq 0 \right),\ j=1,2,\dots,i-1.$$

An undirected simple (without parallel edges or self loops) graph that has $FM^p(n)$ as its adjacency matrix is called a *Fibonacci Graph* $FG^p(n)$ or order n. The vertices are numbered in the same order as the rows of $FM^p(n)$. Figure 1 depicts Fibonacci Graphs $FG^2(1)$ to $FG^2(8)$; Matrix 1 shows the matrix $FM^2(8)$.

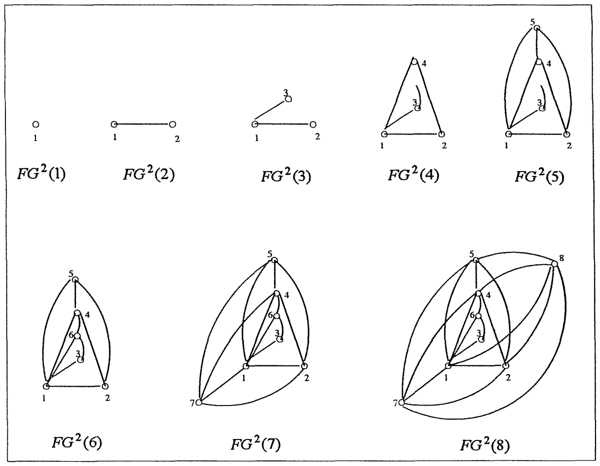

FIGURE 1. Fibonacci Graphs: $FG^2(1)-FG^2(8)$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

MATRIX 1: $FM^2(8)$

A *Fibonacci network* with n processors and prime p is a mapping of the graph $FG^p(n)$. The vertices of the graph correspond to the processors and will be called *nodes*. The edges correspond to the communication *links* between nodes. By definition, $FG^p(n)$ is a subgraph of $FG^p(n+1)$. Hence, Fibonacci networks can be constructed incrementally. Addition of a node causes new links to be added from the new node to some of the existing nodes. None of the existing links are deleted.

Below, we list some of the symbols that are used throughout this paper.

$v_i \to v_j$ = Node v_i is adjacent to node v_j.

$v_i \mapsto v_j$ = Node v_i is not adjacent to node v_j.

$fib(n)$ = The n^{th} Fibonacci number F_n. We redefine this notation for convenience.

v_i = Vertex i in $FG^p(n)$ or node i in the corresponding network.

$\langle d_k d_{k-1} \dots d_1 d_0 \rangle$ = Decimal representation of a positive integer. d_0 is the least significant digit and d_k is the most significant digit.

$Dia^p(n)$ = The diameter of the graph $FG^p(n)$.

$Dia_f^p(n)$ = The fault diameter of the graph $FG^p(n)$.

$deg^p(v_i) =$ The degree of vertex v_i.

$pkt_m =$ Message packet m

$dest(pkt_m) =$ Destination node of packet m.

$e^p(n) =$ The number of edges in the Fibonacci network $FG^p(n)$.

$s_p =$ The smallest integer greater than 0 such that p divides $fib(s_p)$.

Often we will omit the superscript p, in which case p is assumed to be 2.

It should be noted that this construction is different from the construction of Fibonacci Cubes [11] which also use Fibonacci numbers in their construction. However, Fibonacci Cubes are more like the hypercube and scale in increments equal to the Fibonacci numbers. The construction in [11] involves representing each node by a Fibonacci bit representation and determining adjacencies by differences in bit patterns.

3. PROPERTIES OF FIBONACCI NETWORKS

We first introduce some properties of Fibonacci numbers with respect to divisibility by primes and the degree of a vertex. The following lemma will be useful later.

Lemma 1: Prime p divides $fib(j \times s_p)$ for all $i > 0$.

Proof: We prove the lemma by induction on i. The base case is satisfied by definition of s_p. By hypothesis, let us assume that p divides $fib(j \times s_p)$ for some j. To prove that p divides $fib((j+1) \times s_p)$, we invoke the following [9]:

$$fib(n+k) = fib(k) \times fib(n+1) + fib(k-1) \times fib(n).$$

Substituting the above in $fib((j+1) \times s_p)$, we get

$$fib(j \times s_p + s_p) = fib(s_p) \times fib(j \times s_p + 1) + fib(s_p - 1) \times fib(j \times s_p).$$

Since p divides $fib(s_p)$ by base case, and p divides $fib(j \times s_p)$ by hypothesis, p divides $fib((j+1) \times s_p)$. A stronger property can be inferred immediately that p divides $fib(m)$ if and only if $m = j \times s_p$ for some integer j, since s_p is the smallest integer for which p divides $fib(s_p)$ □

We define yet another property of s_p.

Theorem 1: Let $p = (d_k d_{k-1} \dots d_1 d_0)$ be a prime less than 40; if p has t decimal digit representation, then $d_i = 0$ for all $i > (t-1)$.

$$s_p = \begin{cases} (p-1) & \text{if } d_0 = 1 \text{ or } (d_0 = 9 \text{ and } d_1 \text{ is odd}), \\ p & \text{if } d_0 = 5, \\ (p+1) & \text{if } (d_0 = 2) \text{ or } ((d_0 = 3 \text{ or } 7) \text{ and } d_i \text{ is even}), \\ (p+1)/2 & \text{if } ((d_0 = 3 \text{ or } 7) \text{ and } (d_1 \text{ is odd})), \\ (p-1)/2 & \text{if } ((d_0 = 9) \text{ and } (d_1 \text{ is even})). \end{cases}$$

Proof: We have verified the preceding relation for all primes less than 40 (using Mathematica). In this paper we will limit ourselves to primes $p < 10$ and will, therefore, assume this theorem to be true. □

Fibonacci Networks with $p = 2$

We now introduce some properties of Fibonacci networks when $p = 2$. From the previous section it is clear that $s_2 = 3$. We will assume the superscript to be 2 whenever omitted, for convenience in notation.

Proposition 1: The degree of a node v_k, $deg^2(v_k)$ in $FG^2(n)$, is given by

$$e(k) - e(k-1) + \sum_{i=k+1}^{n} \left(\left(\left(k + \sum_{j=0}^{i-2} j \right) \pmod 3 \right) \neq 0 \right).$$

Proof: $e(k) - e(k-1)$ sums all the "1" entries in row k from column 1 until the main diagonal of the adjacency matrix FM^2. The rest of the expression sums all the "1" entries in column k starting from row $k+1$ until row n of FM^2. Since $s_2 = 3$, we know that

$$e(k) = s(k) - \left\lfloor \frac{s(k)}{3} \right\rfloor, \text{ where } s(k) = (k \times (k-1))/2.$$

Substituting for $e(k)$ and $e(k-1)$ in the above equation and simplifying, we get

$$deg^2(v_k) = (k-1) - \left\lfloor \frac{k(k-1)}{6} \right\rfloor + \left\lfloor \frac{(k-1)(k-2)}{6} \right\rfloor + \sum_{i=k+1}^{n} \left(\left((i^2 - 3i + 2k + 2) \pmod 6 \right) \neq 0 \right). \ \square$$

For $k < i$, matrix entry $f_{i,k}^2$ is "1" if and only if

$$\left(\left((i^2 - 3i + 2 + 2k) \pmod 6 \right) \neq 0 \right).$$

We can now construct the following modulo 6 table for $k < i$.

TABLE 1. Connectivity of FM^2

i	i^2	$-3i$	$f_k(i) = i^2 - 3i + 2k + 2$
1	1	-3	$2k$
2	4	0	$2k$
3	3	-3	$2k+2$
4	4	0	$2k$
5	5	-3	$2k$
6	0	0	$2k+2$

Proposition 2: Node v_{3i+1} is adjacent to node v_j for all $j > (3i+1)$.

Proof: We first prove that $v_i \rightarrow v_j$ for all $j > 1$. Since $s_2 = 3$, the result then follows for all v_{3i+1}. By definition, $v_1 \rightarrow v_i$ if and only if $(fib(1 + \sum_{m=0}^{i-2} m) \pmod 2)) \neq 0$. Therefore,it suffices to prove that the value $val(i) = (1 + \sum_{m=0}^{i-2} m)$ is not divisible by $s_2 = 3$ for any $i > 1$. We prove this by contradiction. Let us assume that $(((i-1) \times (i-2))/2 + 1 + 1)$ is divisible by 3 for some i. Thus, $i^2 - 3i + 4$ must be divisible by 3 for some i. Clearly if, for some i, 3 divides $(i^2 + 1)$, then 3 cannot divide i^2 (hence, cannot divide i). Thus, we have $i^2 \equiv 1 \pmod 3$ for some i by Fermat's theorem. Therefore, $i^2+1 \equiv 2 \pmod 3$; hence, 3 does not divide (i^2+1) for any integer i. □

Proposition 3: Node v_{3i+1} is adjacent to node v_j for all j, if $j \pmod 3 \neq 0$.

Proof: For $j > (3i+1)$, the proof follows from Proposition 2. For $j < (3i+1)$, the entry $fm^2_{3i+1,\,j}$ is "1" if and only if $\left(\left(j+\frac{(3i-1)(3i)}{2}\right) \pmod 3 \neq 0\right)$; the proof follows. □

Proposition 4: Node v_{3k+2} is adjacent to node v_i if and only if $(i \pmod 3 \neq 0)$.

Proof: For entry $fm^2_{i,3k+2}$ to be "1," $\left(\left(3k+2+\frac{(i-2)(i-1)}{2}\right) \pmod 3 \neq 0\right)$. On simplifying, we need to prove that $((i^2-3i) \pmod 3 \neq 0)$. This is clearly true if and only if $(i \pmod 3 \neq 0)$. □

Proposition 5: Node v_{3k} is adjacent to node v_i (where $i > 3k$), if and only if $i = 3j$ for some $j > k$.

Proof: For node v_{3k} to be adjacent to node v_i, $\left(\left(3k+\frac{(i-2)(i-1)}{2}\right) \pmod 3 \neq 0\right)$, when $i = 3j$ for some $j > k$. On simplifying, we need to prove that $((i^2-3i+2) \pmod 6 \neq 0)$ for $i = 3j$. On substituting for $i = 1, 2, \ldots, 6$, we find that $v_i \to v_{3k}$ if and only if $i = 3j$ for some $j > k$. □

Proposition 6: Let $t(n)$ be the number of edges added to $FG(n-1)$ to get $FG(n)$. Then

$$t(n) = \begin{cases} 2\times\left(\frac{n}{3}\right)-1 & \text{if } n \pmod 3 = 0, \\ 2\times\left\lceil\left(\frac{n}{3}\right)\right\rceil-1 & \text{if } n \pmod 3 = 2, \\ 2\times\left\lceil\left(\frac{n}{3}\right)\right\rceil-2 & \text{if } n \pmod 3 = 1. \end{cases}$$

Proof: The proof follows directly from the definition of $FG^2(n)$ and Lemma 1. □

Proposition 7: The number of edges $e(n)$ in the Fibonacci network $FG^2(n)$ with n nodes is given by

$$e(n) = \begin{cases} \frac{n}{3}\times(n-1) & \text{if } n \pmod 3 = 0, \\ e(n-1)+t(n) & \text{if } n \pmod 3 = 1, \\ e(n-2)+t(n)+t(n-1) & \text{if } n \pmod 3 = 2. \end{cases}$$

Proof: From Proposition 6 and Proposition 1, we know that $e(3k) = e(3k-3)+6(k-1)+2$. Solving this recurrence we get $3(3k) = k\times(3k-1)$. The result follows from this equation and Proposition 6. □

Proposition 8: The maximum degree of a vertex in $FG(n)$ is $n-1$.

Proof: The proof follows from Proposition 2. The degree of vertex v_1 is $n-1$. □

Proposition 9: The diameter $Dia(n) = 2$.

Proof: The $deg(v_1) = n-1$. This means that the diameter of $FG(n)$ is ≤ 2. □

Proposition 10: Node v_3 has minimum degree in $FG(n)$ for $n \geq 3$.

Proof: From Proposition 3, we know that the degree of nodes v_{3i+1} increases by at least 2 for every 3 nodes added to the network. From Proposition 4, the degree of nodes v_{3i+2} increases by 2 for every 3 nodes added in the network. From Proposition 5, the degree of nodes v_{3i} increases by only 1 for every 3 nodes added in the network, when the nodes have numbers greater than $3i$ and increases by 2 otherwise. So the node with minimum degree is the smallest v_{3i} node, which is v_3 and its degree is $\lfloor \frac{n}{3} \rfloor$. □

Proposition 11: The node connectivity of $FG(n) = deg(v_3)$.

Proof: To prove that there are at least $\lfloor \frac{n}{3} \rfloor$ node-disjoint paths between any 2 nodes (let us say, v_i and v_j) of $FG(n)$, we show that both v_i and v_j can reach all the nodes v_{3q+1}, where $q \leq (deg(v_3)-1)$, either directly or through a nonintersecting set of node paths or $v_i \rightarrow v_j$. Let $deg(v_3)-1 = k$.

Case 1. Integers i and j are both greater than $3k+1$. By Proposition 2, both v_i and v_j are adjacent to v_{3q+1} for $q \in \{0, 1, \ldots, k\}$. Therefore, there are at least $k+1$ node-disjoint paths between v_i and v_j.

Case 2. Integers i and j are both less than $3k+1$. We have three subcases to prove.

a. If $i = 3r+1$ or $i = 3r+2$, then, from Propositions 3 and 4, we know that $v_i \rightarrow v_{3q+1}$ for $q = \{0, 1, \ldots, k\}$.

b. If $i = 3r$ and v_j is not adjacent to v_{3q+1} for some $q \leq k$, then, from Proposition 5, $v_i \rightarrow v_{3q+3}$ and $v_{3q+3} \rightarrow v_{3q+1}$. Likewise, we can prove for v_j.

c. If v_j and v_i are not adjacent to the same v_{3q+1} for some q, then, by Proposition 5, $v_i \rightarrow v_j$.

Case 3. One of i or j is less than $3k+1$ and the other is greater. This is just a subcase of Cases 1 and 2. If the node number is less than v_{3k+1}, then Case 1 holds, and if it is greater, Case 2 holds.

The proof follows from these three cases. □

Proposition 12: The fault diameter $Dia_f(n) = Dia(n)+1$.

Proof: First, we show that the network remains connected in the event of $deg(v_3)-1$ node failures. We then show that the diameter of the fault-free network increases by at most 1. In the worst case, nodes v_{3q+1} all fail where $q = \{0, 1, \ldots, deg(v_3)-2\}$ since they are the nodes with maximum degree. We show that every node v_i can reach v_{3x+1} where $x = deg(v_3)-1$. We have two cases to consider:

Case 1. $i > (3x+1)$. In this case we know, from Proposition 2, that $v_i \rightarrow v_{3x+1}$.

Case 2. $i < (3x+1)$. In this case, if $i = 3r+1$ or $3r+2$ for some $r > 0$, then, from Propositions 3 and 4, $v_i \rightarrow v_{3x+1}$. If $i = 3r$ for some $r > 0$ and $j = 3y+2$ for some $y > 0$ as shown in the connectivity proof.

Thus, the fault diameter $Dia_f(n) = Dia(n)+1$. □

Proposition 13: $FG(n)$ is nonplanar for all $n \geq 7$.

Proof: The graph $FG(7)$ has K_5 as a subgraph (nodes: v_1, v_2, v_4, v_5, and v_7). Therefore, by Kuratowski's theorem, the proof follows since $FG(n+1)$ is a subgraph of $FG(n)$ for all integers $n > 7$. □

Fibonacci Networks with $p = 2$ and 3

When $p = 2$ as shown in Proposition 7, the number of links in the network is very high (order n^2). A simple way to reduce the number of links while still retaining most of the properties of the network is to modify the definition of $FM^p(n)$, where p is a set of primes $\{p_1, p_2, \ldots, p_k\}$. So, in this case, we have

$$X = fib\left(j + \sum_{m=0}^{i-2} m\right)$$

and

$$fmp_{i,j}^p = (X \pmod{p_1} \neq 0) \times (\bmod p_2) \neq 0) \times \cdots \times (X \pmod{p_k} \neq 0), \text{ for } j < i.$$

The rest of the definitions remain the same. The above construction deletes some of the links in the original network. $FM^p(n)$ is a symmetric $n \times n$ matrix whose main diagonal entries are all 0, and its lower triangle (and, therefore, upper also) consists of entries $fmp_{i,j}^p$ The graph which has $FM^p(n)$ as its adjacency matrix is represented by $FG^p(n)$. The graphs $FG^{\{2,3\}}(1)$ through $FG^{\{2,3\}}(10)$ are shown in Figure 2 and the matrix $FM^{\{2,3\}}(10)$ is shown in Matrix 2. For all $n > 0$, $FG^{\{2,3\}}(n)$ is a subgraph of $FG^2(n)$.

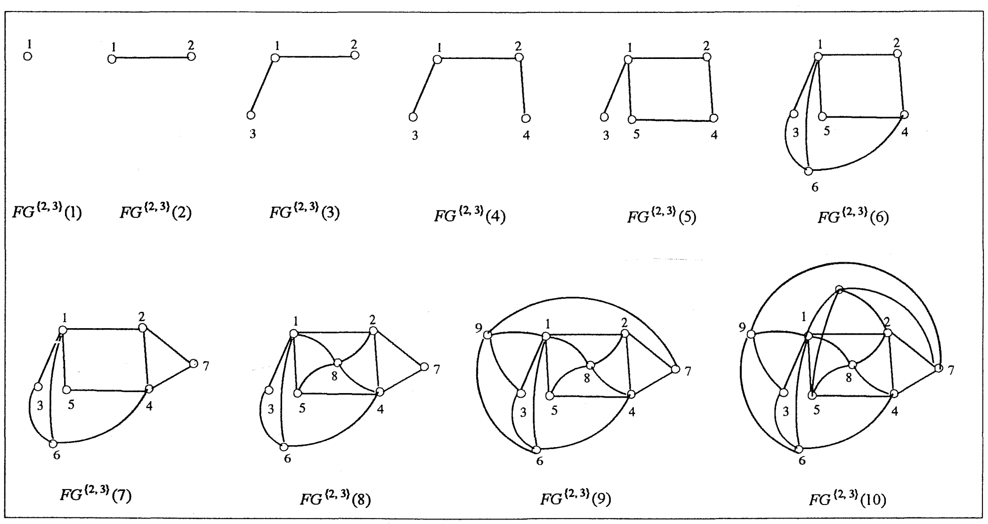

FIGURE 2. Fibonacci23 Graphs: $FG^{\{2,3\}}(1) - FG^{\{2,3\}}(10)$

$$\begin{pmatrix} 0&1&1&0&1&1&0&1&1&1\\ 1&0&0&1&0&0&1&1&0&1\\ 1&0&0&0&0&1&0&0&1&0\\ 0&1&0&0&1&1&1&1&0&0\\ 1&0&0&1&0&0&0&1&0&1\\ 1&0&1&1&0&0&0&0&1&0\\ 0&1&0&1&0&0&0&0&1&1\\ 1&1&0&1&1&0&0&0&0&0\\ 1&0&1&0&0&1&1&0&0&0\\ 1&1&0&0&1&0&1&0&0&0 \end{pmatrix}$$

Matrix 2: $FM^{2,3}$(10)

For the remainder of this subsection, the superscript {2, 3} is assumed and is omitted for the sake of clarity. From Theorem 1, we know that $s_2 = 3$ and $s_3 = 4$. Before exploring the connectivity of this modified network, we first prove a lemma that will be useful in later proofs.

Lemma 2: $fib(n)$ is divisible by 12 if and only if n is divisible by 12.

Proof: We prove the lemma by induction. The base case is clearly true since $fib(12) = 144$. By hypothesis, let $fib(12 \times k)$ be divisible by 12. We must prove that 12 divides $fib(12 \times (k+1))$. But from [9] we have $fib(12 \times (k+1)) = 144 \times fib(12 \times k + 1) + 89 \times fib(12 \times k)$. Since 12 divides $fib(12 \times k)$ by hypothesis, the lemma follows. □

Proposition 14 Let $s(n) = \sum_{i=0}^{n-1} i$, then

$$e(n) = s(n) - \left\lfloor \frac{s(n)}{3} \right\rfloor - \left\lfloor \frac{s(n)}{4} \right\rfloor + \left\lfloor \frac{s(n)}{12} \right\rfloor.$$

Proof: The total number of edges is equal to the number of "1" entries in the lower triangle of $FM^{\{2,3\}}(n)$. Since $s_2 = 3$ and $s_3 = 4$, the above expression follows from the principle of inclusion and exclusion. □

Proposition 15: The degree of a node v_k, $deg^{\{2,3\}}(v_k)$ in a network $FG^{\{2,3\}}(n)$, is given by

$$e(k) - e(k-1) + \sum_{i=k+1}^{n} ((X \pmod 3 \neq 0) \,\&\, (X \pmod 4 \neq 0)),$$

where $X = \left(k + \sum_{j=0}^{i-2} j\right)$.

Proof: This follows using the same outline as shown in the proof of Proposition 1. □

For $k < i$, the matrix entry $fmp_{i,k}^{2,3}$ is "1" if and only if

$$\left(((i^2 - 3i + 2 + 2k) \pmod 6 \neq 0) \,\&\, ((i^2 - 3i + 2 + 2k) \pmod 8 \neq 0)\right).$$

The expression inside the summation forms a field modulo 24 and the degree of nodes increases symmetrically with the addition of every 24 nodes (see Table A-1 in the Appendix).

Proposition 16: If a node $v_j \mapsto v_{3i+1}$ for some $j > (3i+1)$, then $v_j \mapsto v_{3i+4}$ and vice versa.

Proof: Let $X = \left((3i+1) + \Sigma_{k=1}^{j-2} k\right)$. We need to prove that $X+3$ is not divisible by 4 if $v_j \mapsto v_{3i+1}$. We know that if $v_j \mapsto v_{3i+1}$ for some $j > (3i+1)$, then, by Proposition 2, X must be divisible by 4. Since X is divisible by 4, $X+3$ cannot be divisible by 4. Hence, $v_j \mapsto v_{3i+4}$ if $v_j \mapsto v_{3i+1}$. The vice versa proof follows similarly. □

Proposition 17: The maximum degree of a node in $FG(n) = deg(v_1) = deg(v_4)$.

Proof: This follows from Proposition 15. The degree of node v_1 for every 24 nodes is $deg(v_1) = deg(v_4) = 17k$, where $k = \left\lfloor \frac{n}{24} \right\rfloor$. □

Proposition 18: The diameter of $FG(n)$, $Dia(n) = 3$.

Proof: The proof follows from Proposition 16.

Proposition 19: The minimum degree of a node in $FG(n) = deg(v_3)$.

Proof: This follows from Proposition 15, using the same argument as in the proof of Proposition 10. The degree of node v_3 for every 24 nodes is $deg(v_3) = deg(v_9) = 1 + 6k$, where $\left\lfloor \frac{n}{24} \right\rfloor$. □

4. ROUTING

Routing in Fibonacci networks can be preformed very efficiently because of their high connectivity. We consider the case in which $p = 2$. We exploit the fact that nodes $v_{3i+1} \mapsto v_j$ for all $j > (3i+1)$.

Input: A one-to-one permutation showing source and destination nodes.

Output: A path for each packet to be routed.

Step 1. Each node v_j routes its packet to node v_i, where $((i = \max(3k+1)) \leq j)$.

Step 2. Each v_i that receives a packet in Step 1 routes the packet pkt_m to v_ℓ such that $((\ell = \max(3r+1)) \leq dest(pkt_m))$.

Step 3. Each v_i that receives a packet in Step 2 routes the packet pkt_m to $dest(pkt_m)$.

The algorithm clearly runs in constant time. The number of packets at any node at any given instance of time is at most 3, assuming that each processor node works in synchronous lock step.

When $p = \{2, 3\}$, the routing algorithm requires only a minor modification, as shown below. If $v_j \mapsto v_i$, where $((i = \max(3k+1)) \leq j)$, then, by Proposition 12, $v_j \mapsto v_{3k+4}$. So, if $v_j \mapsto v_{3k=1}$ for some v_j in the previous algorithm, it reroutes through v_{3k+4}. This increases the routing complexity by 2 steps for certain packets and the maximum number of packets queued at any node at any given time is at most 6. The algorithm still runs in constant time, with constant queue lengths. We have shown that a network with $p = 2$ can be simulated by a network with $p = \{2, 3\}$ with a loss of speed by a constant factor only.

5. REDUCING THE TOTAL NUMBER OF LINKS

In Section 3 we showed how we could reduce the total number of links by using a higher prime number to prune some of the links. By using prime '3,' the number of links was reduced by 17%. In this section we describe three methods of further reducing the total number of links while maintaining the basic structure of the network.

1. Using higher primes: We follow the same technique as described in the construction of Fibonacci networks with primes 2 and 3. The following table shows the effect of using higher primes on the total number of links.

TABLE 2. Effect of Using Larger Primes

Primes Used	*Percentage of Links Pruned*	*New Diameter*
3	17	3
3, 5	27	4
3, 5, 13	32	6
3, 5, 13, 7	35	7

The number of links reduces by 35% from *FN* by using four more primes. The number of links pruned is computed using the principle of inclusion and exclusion, as shown in the proof of Proposition 14. The diameter results follow, using the argument given in the proof of Proposition 18. It should be noted that the primes p were selected based on the smallest s_p values ($s_{13} < s_7 < s_{11}$). The diameter, which reflects the slow-down in the routing time, increases almost linearly with the number of primes used. Therefore, the routing time slows down by a factor of 7, while 35% of the links in *FN* have been pruned. We observe that using primes higher than 13 results in diminishing returns.

2. Bounding maximum degree to log(n): The second technique that can be used is to bound the maximum degree of each node to $\log(n)$ (or any predefined constant) for $n > c$ (where c is a suitable constant). Therefore, for a network of size less than c, the network is identical to *FN*. For $n > c$ "1" entries in the matrix are set to "0" if the degree of the corresponding node has already reached $\log(n)$. It can easily be shown that the diameter of this network is $O(\log(n))$ and the total number of links is $e(n) < n \times \log(n)$. This network is quite similar to the hypercube. However, this technique does not preserve the basic structure of the Fibonacci network. The routing algorithm will have to be appropriately modified.

3. Cube connected Fibonacci network: The third technique is to replace each node in *FN* by a cycle of length equal to the degree of the node (just as is done in Cube Connected Cycles). This will increase the diameter of the network while reducing the overall degree. However, a problem with this approach is that the network is no longer scalable by one node.

6. EMBEDDING OF VARIOUS TOPOLOGIES

Claim 1: A complete binary tree of k levels (containing $2^k - 1$ nodes) is a subgraph of $FG(3 \times (2^{k-1} - 1))$.

Proof: We show that a complete binary tree of k levels can be mapped on $FG(3\times(2^{k-1}-1))$. From Proposition 2, we know that $v_{3i+1}\to v_j$ for all $j>(3i+1)$. As shown in Figure 3, we assign the nodes of level 1 through level $k-1$ processor nodes v_{3j+1} in order, where $j\in\{0,1,\ldots,(2^{k-1}-2)\}$ Each node v_{3j+1} in level $k-1$ is adjacent to nodes v_{3j+2} and v_{3j+3}, which form the leaf nodes. The number of processors required up to $k-1$ levels is $3\times(2^{k-1}-1)-2$. Therefore, the last leaf node processor required is $3\times(2^{k-1}-1)$

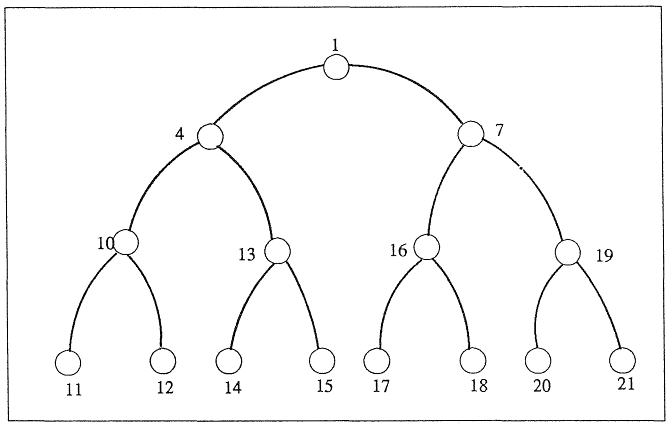

FIGURE 3. Embedding a Complete Binary Tree on *FN*

Claim 2: A complete ringed binary tree of k levels (containing 2^k-1 nodes) is a subgraph of $FG(3\times(2^k-1)-2)$.

Proof: We follow the same outline as in the previous proof. We construct all k levels the same way as we construct $k-1$ levels in the previous proof. From Proposition 2, we know that $v_{3j+1}\to v_j$ for all $j>(3i+1)$. As shown in Figure 4, we assign the nodes of level 1 through level k, processor nodes v_{3j+1} in order, where $j=\{0,1,\ldots,(2^k-2)\}$.

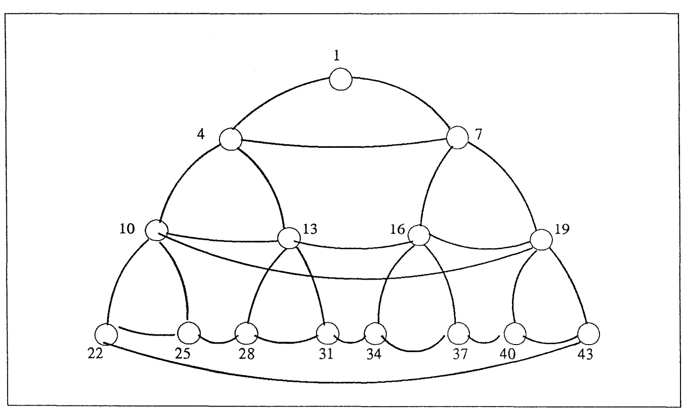

FIGURE 4. Embedding a Ringed Binary Tree of *FN*

Claim 3: A rectangular mesh of size $\ell + k$ is a subgraph of $FG\left(\left\lceil \frac{3(\ell \times k)}{2} \right\rceil\right)$.

Proof: We show how the mesh can be embedded on $FG(n)$. All nodes v_i such that $i \pmod 3 \neq 0$ can be arranged in increasing order, row-wise. The horizontal adjacencies are guaranteed by Propositions 2 and 5 above, and the vertical adjacencies are guaranteed by Proposition 2. Only every third node v_{3j} is not used in the embedding. Therefore, the number of nodes used is $\left\lceil \frac{3(\ell \times k)}{2} \right\rceil$. An example embedding of a 4×4 mesh is shown in Figure 5.

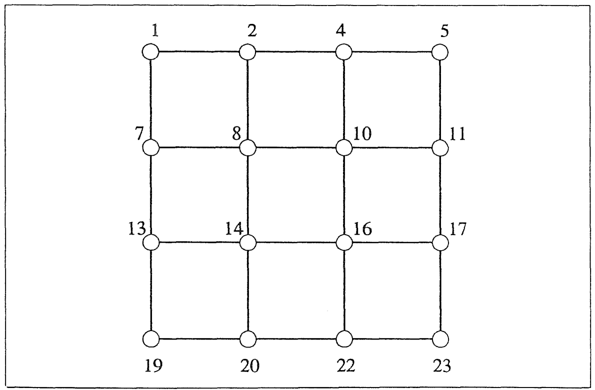

FIGURE 5. Embedding a 4 × 4 Mesh on *FN*

Claim 4: A complete bipartite graph $K_{n,n}$ is a subgraph of $FG(n)$.

Proof: We show how the complete bipartite graph can be embedded on $FG(n)$. We group the nodes v_i, where $i \neq 3k$ into two halves such that the lower half of the processor nodes are in one group and the upper half of the processors are in the other group. Each processor node in one group is adjacent to each processor node in the second group by Propositions 2 and 5.

Claim 5: An n-cube is a subgraph of $FG(3 \times 2^{n-1})$.

Proof: This follows immediately from the previous embedding proof. An example embedding of Q_3 (3-cube) is shown in Figure 6.

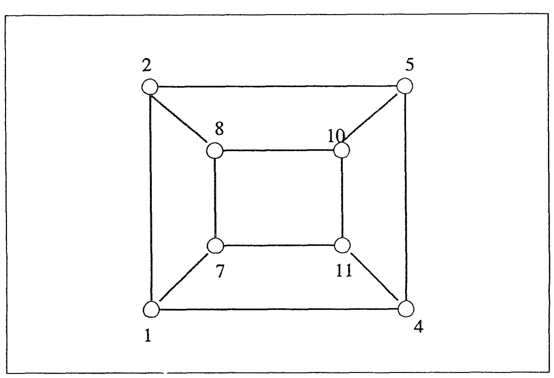

FIGURE 6. Embedding a Hypercube Q_3 on *FN*

7. IMPLEMENTATION OF DISTRIBUTED ALGORITHMS

In the previous section we showed how some of the common topologies can be mapped onto $FG(n)$. The algorithms that run on various topologies can be implemented on $FG(n)$ with minor modifications. Below, we show how a minimum weight spanning tree can be computed on *FN*.

Minimum Weight Spanning Tree

The problem is to find a spanning tree with minimum sum of edge weights in a given undirected, connected, weighted graph *G*, with *N* nodes. We show how this problem can be implemented efficiently on $FG(n)$. We implement Prim-Dijkstra's algorithm on $FG(n)$. A set *T* contains the set of nodes currently in the spanning tree, and a set *E* contains the set of edges currently in the spanning tree. We adapt the procedure outlined in [3], as follows:

Input: A graph *G* with *N* nodes and an adjacency matrix.

Output: A set of edges marked as belonging to the minimum weight spanning tree.

Step 1. $T \leftarrow \phi$. $E \leftarrow \phi$.

Step 2. Partition the nodes of *G* equally among the *n* processor nodes of $FG(n)$ so that each processor node is responsible for $\lceil N/n \rceil$ nodes.

Step 3. $T \leftarrow$ *vertex one* of G.

Step 4 Each processor examines its subset of nodes not in *T* and selects closest neighbor to *T* (*closest in terms of edge weight*).

Step 5. Processor P_1 finds the globally closest neighbor, say v_k.

Step 6. $T \leftarrow T \cup v_k$. $E \leftarrow E \cup edge(T, v_k)$.

Step 7. Processor P_1 broadcasts v_k to all processors.

Step 8. Each processor updates its closest neighbor information.

Repeat Steps 4 through 8 until all nodes have been included in *T*.

Steps 1, 2, and 3 require one time unit. Step 4 requires $O(\lceil N/n \rceil)$ time units in parallel. Step 5 requires $O(\lceil N/n \rceil)$ time units by processor one. Steps 6, 7, and 8 require one time unit. Steps 4 through 8 are repeated *N* times. The overall complexity of the algorithm is $O(N^2/n)$ The sequential algorithm takes $O(N^2)$; hence, this algorithm is optimal.

8. COMPARISON WITH OTHER ITERATIVE NETWORKS

We computed various structural properties of known iterative networks from *Path* to *Complete* networks of 35 nodes. Table 3 below shows these properties. Let

tot-deg = The total number of links in the network.
non-plan = smallest network size for which the network is nonplanar.
min-node = The node with the minimum number of links
min-deg = The degree of *min-node*.
max-node = The node with the maximum number of links.
max-deg = The degree of *max-node*.
inc-deg = Increase in degree with the addition of a node.
dia = The diameter of the network
f-dia = The fault fiameter of the network

inf = Disconnected network
$SG(n)$ = Stirling network of n nodes.
$PG(n)$ = Pascal network of n nodes.
$RG(n)$ = Rencontres network of n nodes.

TABLE 3. Comparison of Iterative Networks

Network	*tot-deg*	*non-plan*	*min-node*	*min-deg*	*max-node*	*max-deg*	*dia*	*f-dia*
Path	34	*inf*	1	1	2	2	34	*inf*
Stirling	169	8	1	2	31	17	6	9
Rencontres	166	7	34	2	2	18	3	3
Pascal	291	7	26	7	1	34	2	3
Fibonacci23	298	10	3	9	1	26	3	4
Fibonacci	397	7	3	11	1	34	2	3
Complete	595	5	1	34	1	34	1	2

The Path network has very low connectivity and is not fault-tolerant. The number of links in the Rencontres network, the Stirling network, and the Pascal network does not scale uniformly. These networks are not symmetric either. Fibonacci networks have too many links, making them prohibitively expensive.

In [5] it was shown that a *full ringed binary tree* with 2^k-1 nodes is a subgraph of $SG(2^k-1)$ for $k \geq 2$, a *full ringed tree machine* of $3(n/4)-2$ nodes when $n=2^k-1$ for $k \geq 3$ is contained in $SG(n)$ for any $\ell \leq k$, a *rectangular mesh* of size $2^\ell \times 2^{k-\ell}$ is embedded in a sub-network induced by the nodes 2^k through 2^{k+1} of $SG(n)$, and a *binary hypercube* is a homeomorphic subgraph of $SG(2^{t+1}-1)$ for $t \geq 3$.

Embeddability of the Rencontres network and the Pascal network have not been studied extensively. However, in [4] it was shown that $RG(n)$ contains a Hamiltonian circuit of n nodes and the *Complete bipartite network* $K_{n,n}$ is a subgraph of $RG(2^n)$. In [7] it was shown that $PG(n)$ contains a *startree* for all $n \geq 1$, that $PG(n)$ contains a Hamiltonian circuit $[1, 2, \ldots, n-1, n, 1]$, and that $PG(n)$ contains $W_n - x$ (wheel of order n minus an edge).

In section 6 we showed that various popular topologies can be embedded onto *FN*. It is clear from the above that we need to be able to fine-tune a network design that has characteristics almost midway between the Path networks and the Complete networks.

9. CONCLUDING REMARKS

Fibonacci networks have many properties desirable in interconnection networks. They have a small diameter, high fault tolerance, rich connectivity, small fault diameter, simple and fast routing, etc. A major disadvantage of the network is its high coast because of the large number of links $(O(n^2))$. We have suggested several ways of reducing the number of links symmetrically so that the basic structure of the network is still maintained. This method of reduction has been shown to cause only constant factor loss of speedup (especially in routing). Broadcasting can be accomplished in constant time assuming that node v_1 has enough buffer space to queue messages. Yet another method of reduction which could be used is to prune links least used by the routing algorithm. Several basic algorithms can be mapped onto Fibonacci networks. We are currently working on embedding other interconnection networks on Fibonacci networks and improving the efficiency of some basic algorithms running on Fibonacci networks.

10. APPENDIX: TABLES A-1 AND A-2

Let

k_{num} = the number of nonzero entries in f^1 and f^2 in Table A-1.
$rk_{num}(j)$ = the number of nonzero entries in the first j entries in Table A-1.
rem = $n \pmod{24}$.

The expression for degree of a node v_k in $FG^{2,3}(n)$ is given by

$$deg(v_k) = k_{num} \times \left\lfloor \frac{n}{24} \right\rfloor + rk_{num}(rem).$$

It should be noted that Table A-1 can be used only for the construction of the lower triangle of the adjacency matrix. Therefore, Table A-1 is true *only* when $k < i$. Since the adjacency matrix is symmetric, the upper tirangle is just a copy of the lower triangle.

TABLE A-1. Connectivity in $FM^{\{2,3\}}$

i	$f_k^1(i) = (i^2 - 3i + 2 + 2k) \pmod 6$	$f_k^2(i) = (i^2 - 3i + 2 + 2k) \pmod 8$
1	$2k$	$2k$
2	$2k$	$2k$
3	$2 + 2k$	$2 + 2k$
4	$2k$	$2k - 2$
5	$2k$	$2k - 4$
6	$2 + 2k$	$4 + 2k$
7	$2k$	$2k - 2$
8	$2k$	$2 + 2k$
9	$2 + 2k$	$2k$
10	$2k$	$2k$
11	$2k$	$2 + 2k$
12	$2 + 2k$	$2k - 2$
13	$2k$	$2k - 4$
14	$2k$	$4 + 2k$
15	$2 + 2k$	$2k - 2$
16	$2k$	$2 + 2k$
17	$2k$	$2k$
18	$2 + 2k$	$2k$
19	$2k$	$2 + 2k$
20	$2k$	$2k - 2$
21	$2 + 2k$	$2k - 4$
22	$2k$	$4 + 2k$
23	$2k$	$2k - 2$
24	$2 + 2k$	$2 + 2k$

The degree of a node v_k increases as follows (see Table A-2) for every 24 nodes added to the network.

TABLE A-2. Increase in Degree of Nodes for Every 24 Nodes

rem	1	2	3	4	5	6	7	8	9	10	11	12
inc. in deg.	17	11	7	17	11	8	14	12	7	14	12	8
rem	13	14	15	16	17	18	19	20	21	22	23	24
inc. in deg.	14	12	8	14	10	9	14	11	11	13	11	11

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❖❖❖

THE IRRATIONALITY OF CERTAIN SERIES WHOSE TERMS ARE RECIPROCALS OF LUCAS SEQUENCE TERMS

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1. INTRODUCTION

Let $(P, Q) = 1$ and α and β $(\alpha > \beta)$ be the roots of $x^2 - Px + Q = 0$. The Lucas sequence $U_n = U_n(P, Q)$ and "associated" Lucas sequence $V_n = V_n(P, Q)$ are defined, respectively, by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \; n \geq 0. \tag{0}$$

In 1878 Lucas ([10], p. 225) obtained the following formula:

$$\sum_{n=1}^{\infty} Q^{2^{n-1}r} / U_{2^n r} = \beta^r / U_r, \; r \geq 1.$$

Setting $Q = \pm 1$, it is seen immediately that, if $P^2 - 4Q > 0$, then $\sum_{n=1}^{\infty} 1/U_{2^n r}$ is irrational, since U_r and V_r are integers, $\alpha - \beta$ is irrational, and [from (0)] $\beta^r = (V_r - U_r(\alpha - \beta))/2$ is irrational. Special cases of this result were re-discovered in the mid-1970s for $F_n = U_n(1, -1)$ [6], [7], [9] (see [8] for a number of different methods for summing $\sum_{n=0}^{\infty} 1/F_{2^n}$)

It was now known until recently whether $\sum_{n=1}^{\infty} 1/U_{g(n)}$ is irrational for any values of the parameters P and Q if $g(n) \neq 2^n r$. Then, in 1987, Badea [3] answered a question posed by Erdös and Graham [5] when he proved that $\sum_{n=0}^{\infty} 1/F_{2^n+1}$ is irrational. André-Jeannin [2] has shown that, if $P > 0$ and $Q = \pm 1$, $\sum_{n=1}^{\infty} 1/U_n$ is irrational, and in a recent work [4], Badea proved that $\sum_{n=1}^{\infty} 1/U_{g(n)}$ is irrational for $P > 0$ and $Q < 0$ if $g(n+1) \geq 2g(n) - 1$ for all sufficiently large n.

In this paper we show that, for *all* Lucas sequences with $P > 0$, $(P, Q) = 1$, and $P^2 - 4Q > 0$, $\sum_{n=1}^{\infty} 1/U_{g(n)}$ is irrational if $g(n+1) \geq 2g(n)$ for all sufficiently large n, and show that if $g(n+1) \geq 2g(n) - 1$ for all sufficiently large n and $g(n)$ is even, the result holds for all such positive parameters P and Q. We obtain similar results for $\sum_{n=1}^{\infty} 1/V_{g(n)}$

Let $\sum_{k=1}^{\infty} 1/a_k$ be a series such that $a_{k+1} \geq a_k^2 > 1$ for $k \geq 1$, and denote the partial sum $\sum_{k=1}^{n} 1/a_k$ by x_n / y_n, where x_n and $y_n = a_1 \ldots a_n$ are natural numbers. If, now, $\sum_{k=1}^{\infty} 1/a_k = a/b$, a and b natural numbers, then $a/b = x_n / y_n + \sum_{k=1}^{\infty} 1/a_{n+k}$; that is,

$$0 < ay_n - bx_n = b \cdot \sum_{k=1}^{\infty} \frac{a_1 \ldots a_n}{a_{n+k}}.$$

The sequence $\left\{\frac{a_1 \ldots a_n}{a_{n+k}}\right\}_{n=1}^{\infty}$ is decreasing if $k = 1$ and strictly decreasing if $k > 1$ (implying $\sum_{k=1}^{\infty} \frac{a_1 \ldots a_n}{a_{n+k}}$ is a strictly decreasing function of n), since the ratio of the n^{th} and $(n+1)^{\text{st}}$ term is

$$\frac{a_{n+k+1}}{a_{n+1} \cdot a_{n+k}} \geq \frac{a_{n+k}}{a_{n+1}}$$

which equals 1 if $k=1$ and is >1 if $k>1$. But this implies $\{ay_n - bx_n\}_{n=1}^{\infty}$ is a strictly decreasing sequence of natural numbers, which is impossible; hence, $\sum_{k=1}^{\infty} 1/a_k$ is irrational. We thus have

Theorem A: Let $n \ge 0$. If $\{a_n\}$ is a sequence of integers, except for at most a finite number of terms that are noninteger rationals, and $a_{n+1} \ge a_n^2 > 1$ for all large n, then the series $\sum_{n=0}^{\infty} 1/a_n$ is an irrational number.

This result will suffice to prove Theorems 1, 2, and 4, and all but part (ii) of Theorem 3; for the latter, we require the following stronger criterion due to Badea [2] (rephrased to apply to sequences containing some negative and/or noninteger terms):

Theorem B: Let $n \ge 0$. If $\{a_n\}$ is a sequence of integers, except for at most a finite number of terms that are noninteger rationals, and $a_{n+1} > a_n^2 - a_n + 1 > 0$ for all large n, then the series $\sum_{n=0}^{\infty} 1/a_n$ is an irrational number.

The meanings of U_n and V_n are extended to negative subscripts by defining $U_{-n} = -U_n / Q^n$ and $V_{-n} = V_n / Q^n$. With these definitions, the following known relations hold for all integers m [proofs are readily obtained from (0)].

$$U_{2m} = U_m V_m, \tag{1}$$

$$U_{2m+1} = U_{m+1}^2 - QU_m^2, \tag{2}$$

$$V_{2m} = V_m^2 - 2Q^m, \tag{3}$$

$$V_m > U_m. \tag{4}$$

2. THE THEOREMS

We assume that $Q \ne 0$, $P \ge 1$, and the discriminant $D = P^2 - 2Q > 0$. It is known—and easily shown from (0)—that this assumption assures that $\{U_n\}$ and $\{V_n\}$ are increasing sequences of positive integers.

The proof of the following theorem, for $Q < 0$, is given in [4], but is included here for completeness.

Theorem 1: Let g be an integer-valued function such that $g(n+1) \ge 2g(n) - 1 > 1$ for all large n. The series $\sum_{n=0}^{\infty} 1/U_{g(n)}$ is irrational except possibly when $Q > 0$ and $g(n)$ is odd for infinitely many values of n.

Proof: Let $a_n = U_{g(n)}$ for all $n \ge 0$ and let N be such that $g(n+1) \ge 2g(n) - 1 > 1$ for $n > N$. Assume now that $n > N$.

Case 1. $g(n+1)$ odd. Let $m = m(n)$ be such that $g(n+1) = 2m+1$. Assume Q is negative. By (2),

$$a_{n+1} = U_{g(n+1)} = U_{2m+1} = U_{m+1}^2 - QU_m^2 > U_{m+1}^2.$$

Then $2m+1 = g(n+1)$ implies $m+1 = [g(n+1)+1]/2 \ge g(n)$, so $U_{m+1}^2 \ge U_{g(n)}^2$. Hence,

$$a_{n+1} > U_{g(n)}^2 = a_n^2.$$

Case 2. $g(n+1)$ even. Since $g(n+1) \geq 2g(n)-1$ and $g(n+1)$ is an even integer, $g(n+1) \geq 2g(n)$. Let $g(n+1) = 2m$. By (1) and (4),

$$a_{n+1} = U_{g(n+1)} = U_{2m} = U_m V_m > U_m^2.$$

Since $m = g(n+1)/2 \geq g(n)$, we again have $a_{n+1} > a_n^2$. Hence, by Theorem A, $\Sigma_{n=0}^{\infty} 1/U_{g(n)}$ is irrational in each case.

Theorem 2: The series $\Sigma_{n=0}^{\infty} 1/U_{g(n)}$ is irrational if g is an integer-valued function such that $g(n+1) \geq 2g(n) > 1$ for all sufficiently large n.

Proof: Assume that $N > 1$ is such that $g(n+1) \geq 2g(n) > 1$ for all $n > N$, and let $n > N$. By Theorem 1, the theorem is true if $g(n+1)$ is even. Let $g(n+1) = 2m+1$ and let $a_n = U_{g(n)}$. Since $m = [g(n+1)-1]/2 \geq g(n) - 1/2$ is an integer, $m \geq g(n)$. By (1) and (4),

$$a_{n+1} = U_{2m+1} > U_{2m} = U_m V_m > U_m^2 \geq U_{g(n)}^2 = a_n^2,$$

proving the theorem.

We now prove similar theorems for the series $\Sigma_{n=0}^{\infty} 1/V_{g(n)}$. In 1987 Badea [3] proved that $\Sigma_{n=0}^{\infty} 1/L_{2^n}$ is irrational (using Theorem B), and, more generally (in [4]), that $\Sigma_{n=0}^{\infty} 1/V_{g(n)}$ is irrational if $Q=-1$ and $g(n+1) \geq 2g(n)$. André-Jeannin [1] gave a direct proof that, for all positive integers k, $\Sigma_{n=0}^{\infty} (\pm 1)^n / V_{k2^n}$ is irrational, and (in [2]) proved that $\Sigma_{n=0}^{\infty} 1/V_n$ is irrational. Our Theorem 3 includes Badea's results and, for $P > |Q+1|$, André-Jennin's result that $\Sigma_{n=0}^{\infty} 1/V_{k2^n}$ is irrational.

Lemma 1: Let k be a positive integer. If $P > |Q+1|$, then, for all sufficiently large integers m, $kQ^m < V_m - 1$.

Proof: It is easily seen that $|\beta| = |(P-\sqrt{D})/2| < 1$ if and only if $P > |Q+1|$, and that $\alpha > 1$ for all P and Q. Hence, there exists an integer M such that, if $m > M$, then $|\beta|^m < 1/2k$ and $\alpha^m > 4$. It follows that

$$kQ^m = k\alpha^m \beta^m \leq k\alpha^m |\beta|^m < \alpha^m/2 < \alpha^m + \beta^m - 1 = V_m - 1.$$

It is readily shown that $\lim_{n\to\infty} V_{2n+1}/V_n^2 = \alpha > 1$, and this result is sufficient to prove part (i) of Theorem 3. However, it is of interest that $V_{2n+1} > V_n^2$ for *all n*, with one exception.

Lemma 2: If $n > 0$, then $V_{2n+1} \geq V_n^2$, with equality holding only when $(P, Q, n) = (3, 2, 1)$.

Proof: Let $r = \beta/\alpha$ and let

$$f(x) = \frac{\alpha^{2x+1} + \beta^{2x+1}}{(\alpha^x + \beta^x)^2} - 1 = \frac{\alpha(1+r^{2x+1})}{(1+r^x)^2} - 1, \; x \text{ real.}$$

We first observe that $f(1) \geq 0$. Now, since $P^2 - 4Q > 0$,

$$f(1) = V_3/V_1^2 - 1 = (P^2 - 3Q)/P - 1 > P/4 - 1,$$

so $f(1) > 0$ if $P > 4$, or if $Q < 0$. Since $P^2 - 4Q > 0$ implies $Q < 0$ for $P = 1$ or 2, $f(1) \le 0$ only if $P = 3$ or 4 and $Q > 0$. The reader may readily determine that, if $P = 3$ or 4, $f(1) \ge 0$ with equality holding only when $P = 3$ and $Q = 2$.

Case 1. $\beta > 0$. Then $0 < r < 1$. Now,

$$f'(x) = \frac{2\alpha r^x (r^{x+1} - 1)\ln r}{(1 + r^x)^3} > 0,$$

implying that f is a strictly increasing function of x; since $f(n) = V_{2n=1} / V_n^2 - 1$ and $f(1) \ge 0$, we conclude that $V_{2n=1} > V_n^2$.

Case 2. $\beta < 0$. If n is odd, by (3),

$$V_n^2 = V_{2n} + 2Q^n = V_{2n} + 2(\alpha\beta)^n < V_{2n};$$

hence, $V_{2n+1} - V_n^2 > V_{2n+1} - V_{2n} > 0$. Assume now that n is even. We let $t = -\beta / \alpha$ (so $0 < t < 1$), define

$$g(x) = \frac{\alpha(1 - t^{2x+1})}{(1 + t^x)^2} - 1,$$

find that g is a strictly increasing function of x, and conclude, since $g(n) = f(n)$ with $t = -r$, that $V_{2n+1} > V_n^2$ in this case, as well.

Theorem 3: Let g be an integer-valued function such that $g(n+1) \ge 2g(n) > 1$ for all large n. Then $\sum_{n=0}^{\infty} 1 / V_{g(n)}$ is irrational

(i) if $g(n)$ is an odd integer for all large n, or
(ii) if $P > |Q + 1|$.

Proof: Let $a_n = V_{g(n)}$ for all $n \ge 0$ and let $N > 1$ be such that $g(n+1) \ge 2g(n) > 1$ for $n > N$. Assume now that $n > N$.

(i) Assume that $g(n+1)$ is odd and let $g(n+1) = 2m+1$; since $m = [g(n+1) - 1] / 2 \ge g(n) - 1/2$ is an integer, $m \ge g(n)$. Then, by Lemma 2,

$$a_{n+1} = V_{g(n+1)} = V_{2m+1} > V_m^2 \ge V_{g(n)} = a_n^2,$$

proving (i).

(ii) Assume that $P > |Q + 1|$. We make the additional assumption that, if $r \ge g(n)$, then $V_r - 1 > 2Q^r$ (possible by Lemma 1). By part (i), we may assume that $g(n+1)$ is even; let $g(n+1) = 2m$. Then, by (3),

$$a_{n+1} = V_{g(n+1)} = V_{2m} = V_m^2 - 2Q^m.$$

By Lemma 1, $2Q^m < V_m - 1$ and, since $m \ge g(n), V_m \ge V_{g(n)}$, from which it follows that

$$a_{n+1} = V_m^2 - 2Q^m > V_m^2 - V_m + 1 = V_m(V_m - 1) + 1 > a_n^2 - a_n + 1.$$

This proves part (ii), by Theorem B.

Theorem 4: The series $\sum_{n=0}^{\infty} 1/V_{g(n)}$ is irrational if g is an integral-valued function such that $g(n+1) \geq 2g(n)+1 > 1$ for all sufficiently large n.

Proof: Assume that $g(n+1) \geq 2g(n)+1 > 1$ for all $n >$ some integer $N > 1$, and let $a_n = V_{g(n)}$. If $n > N$ and $g(n+1)$ is odd, then $a_{n+1} > a_n^2$ by Theorem 3. Assume $g(n+1)$ is even and let $g(n+1) = 2m$; then, since $m \geq g(n)+1/2$ is an integer, $m \geq g(n)+1$, i.e., $m-1 \geq g(n)$. By Lemma 2,

$$a_{n+1} = V_{g(n+1)} = V_{2m} > V_{2m-1} = V_{2(m-1)+1} > V_{m-1}^2 \geq V_{g(n)}^2 = a_n^2,$$

proving the result by Theorem A.

Examples: Since $F_n = U_n(1, -1)$, it is apparent that

$$\sum_{n=0}^{\infty} 1/F_{2^n}, \quad \sum_{n=0}^{\infty} 1/F_{2^n k}, \quad \text{and} \quad \sum_{n=0}^{\infty} 1/F_{2^n+1}$$

are special cases of Theorem 1. Other examples of series whose sum is irrational are

$$\sum_{n=0}^{\infty} 1/U_{cb^n} \quad (c \geq 1 \text{ and } b \geq 2) \quad \text{and} \quad \sum_{n=0}^{\infty} 1/U_{2^n-k}, \; k \geq -1.$$

In fact, it is readily seen that, for $\{U_n\}$ any Lucas sequence, $\sum_{n=0}^{\infty} 1/U_{g(n)}$ is irrational if $g(n) = cb^n - f(n)$, where $c \geq 1$, $b \geq 2$, and f is an integer-valued function such that $f(n+1) \leq 2f(n)$ for all large n, provided $g(n) > 1$ for all large n (f could be, for example, any polynomial in n with positive leading coefficient). Similar examples illustrating Theorems 3 and 4 are readily obtained.

It is interesting that the sum of the series $\sum_{n=0}^{\infty} 1/U_{2^n r}$, $r \geq 1$, found by Lucas for $Q = \pm 1$ is not known for any other value of Q. Also, the sum of $\sum_{n=0}^{\infty} 1/V_{2^n r}$ is not known (however, see [1]), for any value of Q, nor is the sum of any of the other series whose irrationality we have shown in this paper.

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A NOTE ON CONSECUTIVE PRIME NUMBERS

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1. INTRODUCTION

Let p_n denote the n^{th} prime and $d(n) = p_{n+1} - p_n$. Cramer [4], using a probabilistic argument, conjectured that $d(n) = 0((\log(p_n))^2)$. There have been several papers showing that $d(n) = 0(p_n^{\theta})$ (e.g., [7], [8], [9], [10], [12]), for which the value of θ has been reduced to $\frac{11}{20} - \frac{1}{384}$. These papers naturally used sophisticated techniques.

By using the Riemann hypothesis and other properties, one can show $d(n) = 0(p_n^{1/2}(\log(n))^c)$ for some $c > 0$; for example, using the Riemann hypothesis in connection with other assumptions, Heath-Brown & Goldston [6] show that $p_{n+1} - p_n = o(p_n^{1/2}(\log(p_n))^{1/2})$.

As usual, the phrase "almost all n" means that the number of $n \leq X$ for which the statement is false is $o(X)$. Now if one is willing to give up the principle of having $d(n) = 0(f(p_n))$ and, instead, demand $d(n) < f(p_n)$ for almost all n, then, as Montgomery showed in [11], for almost all n, the interval $[n, n + n^{1/5+\varepsilon}]$ contains a prime. Harman [5] showed that, for almost all n, the interval $[n, n + n^{1/10+\varepsilon}]$ contains a prime. Once again, sophisticated techniques are used. Better results can be achieved for these types of problems if one can incorporate the moment method found in the papers written by Cheer & Goldston [2], [3].

In this paper we will show that, if $\varepsilon > 0$, $K > 1$, and x is sufficiently large, then the number of indices $n < x$ for which $d(n) \geq K(\log(n))^{1+\varepsilon}$ is less than $x / ((K-1)(\log(x))^{\varepsilon})$. Professor Erdös informs me that, if one incorporates Brun's method along with the Prime Number Theorem, then one can establish that the number of indices $n < x$ for which $d(n) > K(\log(n))^{1+\varepsilon}$ is less than $(1-\varepsilon)x / ((K(\log(x)^{\varepsilon}))$. The theorems in this paper, though weaker, are elementary and do not depend on Brun's method. We need the following definitions and results. Let $M(x)$ be the set of all positive integers $6 \leq n \leq x$ for which $d(n) < K(\log(n))^{1+\varepsilon}$ does not hold and let $|M(x)|$ be the cardinality of $M(x)$.

$$\sum_{n \leq x} d_n \leq p_{x+1} - 2 \tag{1.1}$$

$$p_n \leq n(\log(n) + \log\log(n)), \quad n \geq 6. \tag{1.2}$$

It is obvious that (1.1) is a telescoping series. Rosser & Schoenfeld [14] proved (1.2).

2. THEOREMS, LEMMA, AND THEIR PROOFS

Lemma 1: Let $\varepsilon > 0$ and let $M(x)$ be the set of all positive integers $3 \leq n \leq x$ for which $d(n) < K(\log(n))^{1+\varepsilon}$ does not hold. Let $|M(x)|$ be the cardinality of $M(x)$. Then

$$\sum_{n \in M(x)} (\log(n))^{1+\varepsilon} \geq \int_3^{|M(x)|} (\log(t))^{1+\varepsilon}\, dt > t(\log(t))^{1+\varepsilon} - (1+\varepsilon)t(\log(t))^{\varepsilon} \Big|_3^{|M(x)|}.$$

Theorem 1: Let $\varepsilon > 0$ and

$$K > (x+1)(\log(x+1)+\log\log(x+1))\left[x\left(1-\frac{\log\log^{\varepsilon} x}{\log x}\right)^{\varepsilon}\left(\log(x)\left(1-\frac{\log\log^{\varepsilon} x}{\log x}\right)-1-\varepsilon\right)\right]^{-1}.$$

Let $M(x)$ = the set of positive integers $6 \le n \le x$ such that $d(n)/(\log(n))^{1+\varepsilon} < K$ does not hold. Then $|M(x)| < x/(\log(x))^{\varepsilon}$.

Proof: Let $f(n) = (\log(n))^{1+\varepsilon}$ and let $M'(x) = \{n \ge 6 : n \notin M\}$, that is, the complement of $M(x)$. We have

$$\sum_{6\le n\le x} (f(n)-d(n)) = \sum_{n\in M'(x)} (f(n)-d(n)) + \sum_{n\in M(x)} f(n)(1-d(n)/f(n)). \tag{2.1}$$

If $n \in M(x)$, then $d(n)/f(n) \ge K$, and using this we see that (2.1) becomes

$$\sum_{6\le n\le x} (f(n)-d(n)) \le \sum_{n\in M'(x)} (f(n)-d(n)) + (1-K)\sum_{n\in M(x)} f(n). \tag{2.2}$$

After several manipulations, (2.2) becomes

$$\sum_{n\in M'(x)} d(n) + K\sum_{n\in M(x)} f(n) \le \sum_{6\le n\le x} d(n). \tag{2.3}$$

Dropping the first term on the left-hand side of (2.3) and using (1.1) and (1.2), we now see that (2.3) becomes

$$K\sum_{n\in M(x)} f(n) \le (x+1)(\log(x+1)+\log\log(x+1)). \tag{2.4}$$

Applying Lemma 1 to the left-hand side of (2.4) gives

$$K\int_6^{|M(x)|} (\log(t))^{1+\varepsilon}\,dt \le (x+1)(\log(x+1)+\log\log(x+1)). \tag{2.5}$$

From (2.5), we get a contradiction if $|M(x)| \ge x/(\log(x))^{\varepsilon}$. Thus, $|M(x)| < x/(\log(x))^{\varepsilon}$.

Theorem 2: Let $\varepsilon > 0$ and $K > 1$. Let $M(x)$ = the set of positive integers $6 \le n \le x$ such that $d(n)/(\log(n))^{1+\varepsilon} < K$ does not hold. Then, for x sufficiently large, we have

$$|M(x)| < x/((K-1)(\log(x))^{\varepsilon}).$$

Proof: The proof is the same as Theorem 1 up to (2.5). Now

$$K\int_6^{|M(x)|} (\log(t))^{1+\varepsilon}\,dt \le (x+1)(\log(x+1)+\log\log(x+1)). \tag{2.6}$$

From (2.6), we get a contradiction if $|M(x)| \ge x/((K-1)(\log(x))^{\varepsilon})$.

3. CONCLUSION

We can now determine that Theorem 2 almost proves Cramer's Conjecture. Let $K > 1$, $\varepsilon = 1$, then for x sufficiently large, by Theorem 2, we have that the number of indices $n < x$ for which $d(n)/(\log(n))^2 < K$ is at least $x - x/((K-1)\log(x))$.

It is also possible to get weaker results without using (1.2). From Ribenboim [13], p. 160, we have $.92129x/\log(x) < \pi(x)$ for $x \geq 30$. If we incorporate this into Theorem 4.7 of Apostol [1], making some minor modifications, we have

$$p_n < 1.62815n(\log(n)) + 0.13347n, \text{ for } p_n \geq 100.$$

Then the following revisions of Theorem 1 and Theorem 2, though not as strong, do not depend on the Prime Number Theorem. The weaker form of Theorem 1 is: suppose $\varepsilon > 0$, $x \geq 100$, and

$$K > 1.62815(x+1)(\log(x+1) + 0.13347)\left[x\left(1 - \frac{\log\log^{\varepsilon} x}{\log x}\right)^{\varepsilon}\left(\log(x)\left(1 - \frac{\log\log^{\varepsilon} x}{\log x}\right) - 1 - \varepsilon\right)\right]^{-1}.$$

Let $M(x)$ = the set of positive integers $6 \leq n \leq x$ such that $d(n)/(\log(n))^{1+\varepsilon} < K$ does not hold. Then $|M(x)| < x/(\log(x))^{\varepsilon}$. The weaker form of Theorem 2 is: suppose $\varepsilon > 0$ and $K > 1$. Let $M(x)$ = the set of positive integers $6 \leq n \leq x$ such that $d(n)/(\log(n))^{1+\varepsilon} < K$ does not hold. Then, for x sufficiently large, we have

$$|M(x)| < 1.6282x/((K-1)(\log(x))^{\varepsilon}).$$

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FIBONACCI AUTOCORRELATION SEQUENCES

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1. INTRODUCTION AND GENERALITIES

For given nonnegative integers τ and n, let us define the elements $\Gamma_n(S_i, \tau)$ of the *Autocorrelation Sequences* of any sequence of numbers $\{S_i\}_0^\infty$.

Definition:

$$\Gamma_n(S_i, \tau) \stackrel{\text{def}}{=} \sum_{i=0}^{n} S_i S_{i+\tau} \quad (0 \le \tau \le n), \tag{1.1}$$

where the subscript $i+\tau$ must be considered as reduced modulo $n+1$.

Observe that Definition (1.1) differs from the definition of the *Cyclic Autocorrelation Function* for periodic sequences with period $n+1$, by the factor $1/(n+1)$ (e.g., see [2], p. 25).

It can readily be seen that Definition (1.1) can be written in the equivalent form

$$\Gamma_n(S_i, \tau) = \sum_{i=0}^{n-\tau} S_i S_{i+\tau} + \sum_{i=0}^{\tau-1} S_{i+n-\tau+1} S_i, \tag{1.1'}$$

where the second sum vanishes for $\tau = 0$. Moreover, we point out that the numbers $\Gamma_n(S_i, \tau)$ enjoy the following symmetry property

$$\Gamma_n(S_i, \tau) = \Gamma_n(S_i, n-\tau+1) \quad (0 < \tau \le n). \tag{1.2}$$

A numerical example will better clarify the above statements.

Example:

$$\begin{aligned}
\Gamma_5(S_i, 4) &= S_0S_4 + S_1S_5 + S_2S_0 + S_3S_1 + S_4S_2 + S_5S_3 && \text{[from (1.1)]} \\
&= (S_0S_4 + S_1S_5) + (S_0S_2 + S_1S_3 + S_2S_4 + S_3S_5) && \text{[from (1.1')]} \\
&= \Gamma_5(S_i, 2) = S_0S_2 + S_1S_3 + S_2S_4 + S_3S_5 + S_4S_0 + S_5S_1 && \text{[from (1.2)].}
\end{aligned}$$

For particular sequences S_i, a closed-form expression for $\Gamma_n(S_i, \tau)$ can readily be found. For example, if $S_i = i$ (the sequence of nonnegative integers), we have

$$\Gamma_n(i, \tau) = \{2n^3 - 3(\tau-1)(n^2-\tau) + n[3\tau(\tau-2)+1]\}/6. \tag{1.3}$$

Observe that, when $\tau = 0$, the identity (1.3) reduces to the well-known formula that gives the sum of the squares of the first n integers.

The aim of this paper is to establish closed-form expressions for the elements of the *Fibonacci Autocorrelation Sequences* $\{\Gamma_n(\tau)\}_\tau^\infty$ defined as

$$\Gamma_n(\tau) \stackrel{\text{def}}{=} \Gamma_n(F_i, \tau), \tag{1.4}$$

and to discover some properties of these integers (sections 2 and 3). In this paper, F_i and L_i will denote, as usual, the i^{th} Fibonacci number and Lucas number, respectively. The proofs of the obtained results are, in general, very lengthy and rather cumbersome and, in most cases, they must be split into four subcases according to the parity of τ and n. Sometimes the residue of n modulo 4 must also be taken into account. To save space, only one subcase for each proposition will be proved in full detail (section 4). The parity of $\Gamma_n(\tau)$ is also discussed (section 5), and a glimpse of possible future work concludes the paper (section 6).

The following Fibonacci identities will be used widely throughout the proofs:

$$\begin{cases} F_{h+k} + (-1)^k F_{h-k} = F_h L_k \\ F_{h+k} - (-1)^k F_{h-k} = L_h F_k \end{cases} \quad [3, \mathrm{I}_{21} - \mathrm{I}_{24},\ \text{p. } 59], \tag{1.5}$$

$$\begin{cases} L_{h+k} + (-1)^k L_{h-k} = L_h L_k \\ L_{h+k} - (-1)^k L_{h-k} = 5 F_h F_k \end{cases} \quad [4,\ \text{(17a) and (17b)}], \tag{1.6}$$

$$\sum_{j=1}^{k} F_{mj+n} = \frac{F_{m(k+1)+n} - (-1)^m F_{mk+n} - F_{m+n} + (-1)^m F_n}{L_m - (-1)^m - 1} \quad [1,\ (11)], \tag{1.7}$$

$$\sum_{j=1}^{k} L_{mj+n} = \frac{L_{m(k+1)+n} - (-1)^m L_{mk+n} - L_{m+n} + (-1)^m L_n}{L_m - (-1)^m - 1} \quad [1,\ (15)]. \tag{1.8}$$

2. CLOSED-FORM EXPRESSIONS FOR $\Gamma_n(\tau)$

In this section closed-form expressions for $\Gamma_n(\tau)$ are established and some particular cases are discussed. First of all, we show in Table 1 the integers $\Gamma_n(\tau)$ for the first few values of τ and n. The results presented in this section and in the rest of the paper can be readily checked against this table.

TABLE 1. The Numbers $\Gamma_n(\tau)$ for $0 \le \tau,\ n \le 10$

$\tau \rightarrow$

	0										
	1	0									
	2	1	1								
	6	3	4	3							
	15	9	8	8	9						
n	40	24	20	16	20	24					
$\downarrow$	104	64	47	37	37	47	64				
	273	168	117	84	78	84	117	168			
	714	441	293	202	165	165	202	293	441		
	1870	1155	748	495	374	330	374	495	748	1155	
	4895	3025	1924	1244	877	707	707	877	1244	1924	3025

By (1.1′), the numbers $\Gamma_n(\tau)$ can be expressed as

$$\Gamma_n(\tau) = \sum_{i=0}^{n-\tau} F_i F_{i+\tau} + \sum_{i=0}^{\tau-1} F_{i+n-\tau+1} F_i. \tag{2.1}$$

With the aid of the Binet form for F_i, (2.1) becomes

$$\begin{aligned}\Gamma_n(\tau) &= \frac{1}{5}\sum_{i=0}^{n-\tau}\left(\alpha^{2i+\tau} + \beta^{2i+\tau} - \alpha^i\beta^{i+\tau} - \beta^i\alpha^{i+\tau}\right) \\ &+ \frac{1}{5}\sum_{i=0}^{\tau-1}\left(\alpha^{2i+n-\tau+1} + \beta^{2i+n-\tau+1} - \alpha^i\beta^{i+n-\tau+1} - \beta^i\alpha^{i+n-\tau+1}\right),\end{aligned} \tag{2.2}$$

where $\alpha = 1-\beta = (1+\sqrt{5})/2$. By (2.2), using the Binet form for L_i, yields

$$\Gamma_n(\tau) = \frac{1}{5}\sum_{i=0}^{n-\tau} L_{2i+\tau} - \frac{1}{5}\sum_{i=0}^{n-\tau}(-1)^i L_\tau + \frac{1}{5}\sum_{i=0}^{\tau-1} L_{2i+n-\tau+1} - \frac{1}{5}\sum_{i=0}^{\tau-1}(-1)^i L_{n-\tau+1},$$

whence, by means of (1.8), we obtain

$$\Gamma_n(\tau) = \frac{1}{5}\left[L_{2n-\tau+1} - L_{\tau-1} + L_{n+\tau} - L_{n-\tau} - X(n,\tau)\right], \tag{2.3}$$

where

$$X(n,\tau) = \begin{cases} L_\tau & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases} \quad (\tau \text{ even}), \tag{2.4}$$

and

$$X(n,\tau) = \begin{cases} L_{n-\tau+1} & (n \text{ even}) \\ L_\tau + L_{n-\tau+1} & (n \text{ odd}) \end{cases} \quad (\tau \text{ odd}). \tag{2.4′}$$

Now, by virtue of (2.3)-(2.4′), (1.5), and (1.6), after some manipulations, we get

$$\Gamma_n(\tau) = \begin{cases} F_{n+1}F_{n-\tau} + F_nF_\tau & (n \text{ even}) \quad (2.5) \\ F_n(F_{n-\tau+1} + F_\tau) & (n \text{ odd}) \quad (2.5′) \end{cases} \quad (\tau \text{ even}),$$

and

$$\Gamma_n(\tau) = \begin{cases} F_nF_{n-\tau+1} + F_{n+1}F_{\tau-1} & (n \text{ even}) \quad (2.5″) \\ F_{n+1}(F_{n-\tau} + F_{\tau-1}) & (n \text{ odd}) \quad (2.5‴) \end{cases} \quad (\tau \text{ odd}).$$

The proofs of (2.5) - (2.5‴) are similar; thus, for the sake of brevity, we give only the proof of (2.5).

Proof of (2.5): By (2.3) and (2.4), we can write

$$\begin{aligned}\Gamma_n(\tau) &= \frac{1}{5}(L_{2n-\tau+1} - L_{\tau+1} + L_{n+\tau} - L_{n-\tau}) \\ &= \frac{1}{5}[L_{n+1+(n-\tau)} - L_{n+1-(n-\tau)} + L_{n+\tau} - L_{n-\tau}],\end{aligned}$$

whence, by (1.6), we get (2.5). □

The complete factorization of $\Gamma_n(\tau)$ in terms of Fibonacci and Lucas numbers can be obtained only for the cases (2.5′) and (2.5‴). We have

$$\Gamma_n(\tau) = \begin{cases} F_n F_{(n+1)/2-\tau} L_{(n+1)/2} & (n \equiv 1 \pmod 4) \\ F_n F_{(n+1)/2} L_{(n+1)/2-\tau} & (n \equiv 3 \pmod 4) \end{cases} \quad (\tau \text{ even}) \tag{2.6}$$

and

$$\Gamma_n(\tau) = \begin{cases} F_{n+1} F_{(n-1)/2} L_{(n+1)/2-\tau} & (n \equiv 1 \pmod 4) \\ F_{n+1} F_{(n+1)/2-\tau} L_{(n-1)/2} & (n \equiv 3 \pmod 4) \end{cases} \quad (\tau \text{ odd}). \tag{2.6′}$$

Proof of (2.6): Let us rewrite (2.5′) as

$$\Gamma_n(\tau) = F_n\left[F_{(n+1)/2+[(n+1)/2-\tau]} + F_{(n+1)/2-[(n+1)/2-\tau]}\right]. \tag{2.7}$$

Recalling that

$$\frac{n+1}{2} \text{ is } \begin{cases} \text{even if } n \equiv 3 \pmod 4 \\ \text{odd if } n \equiv 1 \pmod 4, \end{cases} \tag{2.8}$$

and taking into account that τ is even, use (2.7) and (1.5) to obtain (2.6). □

An analogous argument leads to the proof of (2.6′).

2.1 Particular Cases

By (2.5)-(2.5‴), simplified expressions of $\Gamma_n(\tau)$ can be obtained for some particular values of τ. In light of (1.2), we confine ourselves to considering values of τ less than or equal to $(n+1)/2$. The following results have been obtained.

$$\Gamma_n(0) = F_n F_{n+1} \quad (\text{cf. } [3, I_3]), \tag{2.9}$$

$$\Gamma_n(1) = \begin{cases} F_n^2 & (n \text{ even}) \\ F_n^2 - 1 & (n \text{ odd}) \quad (\text{by using the } \textit{Simson formula } [3, I_{13}]), \end{cases} \tag{2.10}$$

$$\Gamma_n(2) = \begin{cases} F_{n+1}F_{n-2} + F_n & (n \text{ even}) \\ F_n(F_{n-1}+1) & (n \text{ odd}), \end{cases} \tag{2.11}$$

$$\Gamma_n(3) = \begin{cases} F_nF_{n-2} + F_{n+1} & (n \text{ even}) \\ F_{n+1}(F_{n-3}+1) & (n \text{ odd}), \end{cases} \tag{2.12}$$

and

$$\Gamma_n\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) = \begin{cases} F_{n+2}F_{n/2} & (n \equiv 0 \pmod 4) & (2.13) \\ 2F_{n+1}F_{(n-1)/2} & (n \equiv 1 \pmod 4) & (2.13′) \\ F_nF_{n/2+1} + F_{n+1}F_{n/2-1} & (n \equiv 2 \pmod 4) & (2.13″) \\ 2F_nF_{(n+1)/2} & (n \equiv 3 \pmod 4), & (2.13‴) \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function. The algebraic manipulations necessary to obtain (2.9)-(2.13‴) from (2.5)-(2.5‴) are not difficult and are omitted for brevity. Observe that, by using the Binet form, the identity (2.13″) can be restated in the equivalent form

$$\Gamma_n(n/2) = (L_{3n/2+2} - 2L_{n/2+1})/5 \quad (n \equiv 2 \pmod 4)), \tag{2.14}$$

and that identities (2.13′) and (2.13‴) can be obtained immediately by the upper identity in (2.6′) and by the lower identity in (2.6), respectively, taking (2.8) into account.

3. SOME IDENTITIES INVOLVING THE NUMBERS $\Gamma_n(\tau)$

In this section we present some identities involving the numbers $\Gamma_n(\tau)$. The proofs of these results will be partially given in the next section.

First, let us state the recurrence relations

$$\Gamma_{n+1}(\tau) = \Gamma_n(\tau) + \Gamma_{n-1}(\tau) + \frac{1}{5}(2L_{2n-\tau} + L_{\tau+(-1)^{n+\tau}}), \tag{3.1}$$

$$\Gamma_n(\tau+1) = \Gamma_n(\tau-1) - \Gamma_n(\tau) - F_\tau + \frac{1}{5}(2L_{n+\tau} + (-1)^\tau L_{n-\tau+1}), \tag{3.2}$$

and

$$\Gamma_n(\tau) = \begin{cases} \Gamma_{n-1}(\tau) + \Gamma_{n-1}(\tau-1) + F_{n-\tau} + F_{\tau-1}[1-(-1)^n]/2 & (\tau \text{ even}) \\ \Gamma_{n-1}(\tau) + \Gamma_{n-1}(\tau-1) + F_{\tau-1}[1+(-1)^n]/2 & (\tau \text{ odd}). \end{cases} \tag{3.3}$$

Remark 1: Observe that, since $\Gamma_n(\tau)$ has not been defined for $\tau > n$, the recurrence relations (3.1)-(3.3) make sense only for $0 \le \tau \le n-1$, due to the presence of the quantities $\Gamma_{n-1}(\tau)$ and $\Gamma_n(\tau+1)$.

Then, let us consider the sums along the rows, the columns, and the rising diagonals of the triangular array shown in Table 1. Define

$$R_n \stackrel{\text{def}}{=} \sum_{\tau=0}^{n} \Gamma_n(\tau), \tag{3.4}$$

$$C_k(\tau) \stackrel{\text{def}}{=} \sum_{n=\tau}^{k} \Gamma_n(\tau), \ (k \ge \tau), \tag{3.5}$$

and

$$D_n \stackrel{\text{def}}{=} \sum_{\tau=0}^{\lfloor n/2 \rfloor} \Gamma_{n-\tau}(\tau), \tag{3.6}$$

and state the following propositions.

Proposition 1: $R_n = (F_{n+2} - 1)^2$.

By Proposition 1 and [3, I_3 and I_1], it can readily be seen that the sum of all elements of Table 1 from the 0^{th} row to the k^{th} row (inclusive) is

$$\sum_{n=0}^{k} R_n = \sum_{\tau=0}^{k} C_k(\tau) = F_{k+2}F_{k+3} - 2F_{k+4} + k + 4. \tag{3.7}$$

Proposition 2: If τ is even,

$$C_k(\tau) = \begin{cases} \Gamma_k(\tau) + \Gamma_k(\tau+1) - F_\tau[F_{\tau+1} + (k-\tau)/2] & (k \text{ even}) \\ \Gamma_{k+1}(\tau+1) - F_\tau[F_{\tau+1} + (k-\tau+1)/2] & (k \text{ odd}), \end{cases}$$

whereas, if τ is odd,

$$C_k(\tau) = \begin{cases} \Gamma_{k+1}(\tau) + F_{\tau-1}(F_{k+1} - F_{\tau+2}) - F_k F_{k-\tau+1} - F_\tau(k-\tau+1)/2 & (k \text{ even}) \\ \Gamma_k(\tau) + F_{\tau-1}(F_{k+2} - F_{\tau+2}) + F_k F_{k-\tau} - F_\tau(k-\tau)/2 & (k \text{ odd}). \end{cases}$$

Remark 2: For the same reason as that mentioned in Remark 1, the expression of $C_k(\tau)$ stated in Proposition 2 does not apply when $k = \tau$ is even, due to the presence of the addend $\Gamma_k(\tau+1)$. Of course, in this case, we have $C_k(k) = \Gamma_k(k)$.

Proposition 3:

$$D_n = \begin{cases} \dfrac{1}{5}\left[\dfrac{L_{2n+3} + nL_n - 5F_{(n+6)/2}}{2} - 3(F_n - 1)\right] & (n \text{ even}) \\ \dfrac{1}{5}\left[\dfrac{L_{2n+3} + (n-1)L_n - 5F_{(n+3)/2} - r + 1}{2} - 3F_n\right] + 1 & (n \text{ odd}), \end{cases}$$

where r denotes the residue of n modulo 4.

Finally, the following sums are considered:

$$A_n \stackrel{\text{def}}{=} \sum_{\tau=0}^{n} (-1)^\tau \Gamma_n(\tau), \tag{3.8}$$

$$B_n \stackrel{\text{def}}{=} \sum_{\tau=0}^{n} \binom{n}{\tau} \Gamma_n(\tau). \tag{3.9}$$

Proposition 4: $A_n = \begin{cases} F_n F_{n+1} & (n \text{ even}) \\ (F_{n-1} + 1)^2 & (n \text{ odd}). \end{cases}$

Proposition 5: $B_n = \begin{cases} \dfrac{1}{5}\left(L_{3n+2} - \dfrac{5F_{2n+2} + L_{n+1}}{2}\right) & (n \text{ even}) \\ \dfrac{1}{5}\left[L_{3n+2} - \dfrac{L_{n+1}(5F_{n+1} - 1)}{2}\right] & (n \text{ odd}). \end{cases}$

4. PROOFS

As mentioned in the Introduction, to save space, the identities stated in section 3 will be proved only for one case of the parity of τ and n. The interested reader can complete the proofs as an exercise.

Proof of (3.1): (τ and n even). By (2.3)-(2.4′), we can write

$$\begin{aligned}\Gamma_n(\tau)+\Gamma_{n-1}(\tau) &= \frac{1}{5}(L_{2n-\tau+1}+L_{2n-\tau-1}-2L_{\tau-1}+L_{n+\tau+1}-L_{n-\tau+1}-L_\tau-0)\\ &= \frac{1}{5}(L_{2n-\tau+3}-2L_{2n-\tau}-2L_{\tau-1}+L_{n+\tau+1}-L_{n-\tau+1}-L_\tau)\\ &= \Gamma_{n+1}(\tau)-\frac{1}{5}(2L_{2n-\tau}+L_{\tau-1}+L_\tau)\\ &= \Gamma_{n+1}(\tau)-\frac{1}{5}(2L_{2n-\tau}+L_{\tau+1}),\end{aligned}$$

whence the recurrence (3.1). □

Proof of (3.2): (τ even and n odd). By (2.3)-(2.4′) and (1.6), we can write

$$\begin{aligned}\Gamma_n(\tau)+\Gamma_n(\tau+1) &= \frac{1}{5}(L_{2n-\tau+2}-L_{\tau+1}+L_{n+\tau+2}-L_{n-\tau+1}-0-L_{\tau+1}-L_{n-\tau})\\ &= \frac{1}{5}[L_{2n-\tau+2}-L_{n-\tau+1}+(L_{n+\tau+2}-2L_{\tau+1}-L_{n-\tau})+(L_{n+\tau-1}-L_{n-\tau+2}-L_\tau)\\ &\qquad -(L_{n+\tau-1}-L_{n-\tau+2}-L_\tau)]\\ &= \Gamma_n(\tau-1)+\frac{1}{5}(L_{n+\tau+2}-2L_{\tau+1}-L_{n-\tau}-L_{n+\tau-1}+L_{n-\tau+2}+L_\tau)\\ &= \Gamma_n(\tau-1)+\frac{1}{5}[L_{n+\tau+2}-L_{n-\tau}-L_{n+\tau-1}+L_{n-\tau+2}-(L_{\tau+1}+L_{\tau-1})]\\ &= \Gamma_n(\tau-1)-F_\tau+\frac{1}{5}(L_{n+\tau+2}-L_{n+\tau-1}+L_{n-\tau+1})\\ &= \Gamma_n(\tau-1)-F_\tau+\frac{1}{5}(2L_{n+\tau}+L_{n-\tau+1}),\end{aligned}$$

whence the recurrence (3.2). □

Proof of (3.3): (τ and n even). By (2.5′) and (2.5‴), the right-hand side of (3.3) can be rewritten as

$$\begin{aligned}F_{n-1}(F_{n-\tau}+F_\tau)+F_n(F_{n-\tau}+F_{\tau-2})+F_{n-\tau} &= F_{n+1}F_{n-\tau}+F_{n-1}F_\tau+F_nF_{\tau-2}+F_{n-\tau}\\ &= F_{n+1}F_{n-\tau}+(F_n-F_{n-2})F_\tau+F_nF_{\tau-2}+F_{n-\tau}\\ &= (F_{n+1}F_{n-\tau}+F_nF_\tau)-F_{n-2}F_\tau+F_nF_{\tau-2}+F_{n-\tau}\\ &= \Gamma_n(\tau)-F_{n-2}F_\tau+F_nF_{\tau-2}+F_{n-\tau}\ \text{[by (2.5)]}.\end{aligned}$$

Now, it is sufficient to prove that $F_{n-\tau}+F_nF_{\tau-2}-F_{n-2}F_\tau=0$, that is

$$F_nF_{\tau-2}-F_{n-2}F_\tau=-F_{n-\tau}. \tag{4.1}$$

To do this, consider the Fibonacci identity

$$F_kF_h-F_{k-a}F_{h+a}=(-1)^{h+1}F_{k-h-a}F_a, \tag{4.2}$$

which can readily be proved by using the Binet form, and put $k=n, h=\tau-2$ (even, by hypothesis), and $a=2$ in (4.2) to obtain (4.1). □

Proof of Proposition 1: By (2.3) and [3, I_2],

$$5R_n = L_{2n+4} + 3 - 2L_{n+3} - \sum_{\tau=0}^{n} X(n, \tau). \tag{4.3}$$

If n is even, then by (2.4), (2.4′), and (1.8),

$$\sum_{\tau=0}^{n} X(n, \tau) = 2L_{n+1}$$

and Proposition 1 holds by [3, I_{16}]. If n is odd, then by (2.4), (2.4′), and (1.8),

$$\sum_{\tau=0}^{n} X(n, \tau) = 2(L_{n+1} - 2)$$

and Proposition 1 holds by [3, I_{17}]. □

Proof of Proposition 2: (τ even and k odd). Put $n = j + \tau - 1$ in (3.5), thus getting

$$C_k(\tau) = \sum_{j=1}^{k-\tau+1} \Gamma_{j+\tau-1}(\tau) = \sum_{j=1}^{(k-\tau+1)/2} [\Gamma_{2j+\tau-2}(\tau) + \Gamma_{2j+\tau-1}(\tau)]. \tag{4.4}$$

By (4.4), (2.3), and (2.4), we obtain

$$C_k(\tau) = \frac{1}{5} \sum_{j=1}^{(k-\tau+1)/2} (L_{4j+\tau-3} + L_{4j+\tau-1} - 2L_{\tau-1} + L_{2j+2\tau-2} + L_{2j+2\tau-1} - L_{2j-2} - L_{2j-1} - L_\tau),$$

whence, by (1.6),

$$\begin{aligned} C_k(\tau) &= \frac{1}{5} \sum_{j=1}^{(k-\tau+1)/2} (5F_{4j+\tau-2} - 5F_\tau + L_{2j+2\tau} - L_{2j}) \\ &= \sum_{j=1}^{(k-\tau+1)/2} F_{4j+\tau-2} - \frac{k-\tau+1}{2} F_\tau + \frac{1}{5} \sum_{j=1}^{(k-\tau+1)/2} (L_{2j+2\tau} - L_{2j}). \end{aligned} \tag{4.5}$$

By (4.5), using (1.7) and (1.8) yields

$$\begin{aligned} C_k(\tau) = &\frac{1}{5}(F_{2k-\tau+4} - F_{2k-\tau} - F_{\tau+2} + F_{\tau-2}) - F_\tau(k-\tau+1)/2 \\ &+ \frac{1}{5}(L_{k+\tau+3} - L_{k+\tau+1} - L_{2\tau+2} + L_{2\tau} - L_{k-\tau+3} + L_{k-\tau+1} + 1), \end{aligned}$$

whence, by (1.5),

$$C_k(\tau) = \frac{1}{5}(L_{2k-\tau+2} - L_\tau) - F_\tau(k-\tau+1)/2 + \frac{1}{5}(L_{k+\tau+2} - L_{2\tau+1} - L_{k-\tau+2} + 1)$$

and, by (1.6) (recalling that, since τ is even by hypothesis, $L_{-\tau} = L_\tau$),

$$\begin{aligned} C_k(\tau) &= F_{k+1}F_{k+1-\tau} - F_\tau(k-\tau+1)/2 + F_{k+2}F_\tau - F_\tau F_{\tau+1} \\ &= F_{k+1}F_{k+1-\tau} + F_{k+2}F_\tau - F_\tau[F_{\tau+1} + (k-\tau+1)/2]. \end{aligned} \tag{4.6}$$

By (4.6) and (2.5″), we obtain the desired result,

$$C_k(\tau) = \Gamma_{k+1}(\tau+1) - F_\tau[F_{\tau+1} + (k-\tau+1)/2]. \quad \square$$

Proof of Proposition 3: $[n \equiv 1 \,(\mathrm{mod}\, 4)]$. By (3.6), let us write

$$D_n = \sum_{\tau=0}^{(n-1)/4} \Gamma_{n-2\tau}(2\tau) + \sum_{\tau=1}^{(n-1)/4} \Gamma_{n-2\tau+1}(2\tau-1).$$

By (2.3)-(2.4′), and considering that $L_{-1} = -1$, the above expression becomes

$$\begin{aligned} D_n &= \frac{1}{5}\sum_{\tau=1}^{(n-1)/4} (L_{2n-6\tau+1} - L_{2\tau-1} + L_n - L_{n-4\tau}) + \frac{1}{5}(L_{2n}+1) \\ &\quad + \frac{1}{5}\sum_{\tau=1}^{(n-1)/4} (L_{2n-6\tau+4} - L_{2\tau-2} + L_n - L_{n-4\tau+2} - L_{n-4\tau+3}) \\ &= \frac{1}{5}\sum_{\tau=1}^{(n-1)/4} (L_{2n-6\tau+1} + L_{2n-6\tau+4} - L_{2\tau} + 2L_n - L_{n-4\tau+4} - L_{n-4\tau}) + \frac{1}{5}(L_{2n}+1), \end{aligned}$$

whence, by (1.6),

$$D_n = \frac{1}{5}\sum_{\tau=1}^{(n-1)/4} (2L_{2n-6\tau+3} - 3L_{n-4\tau+2} - L_{2\tau} + 2L_n) + \frac{1}{5}(L_{2n}+1). \tag{4.7}$$

By using (1.8), the identity (4.7) can be rewritten as

$$\begin{aligned} D_n = \frac{1}{5}\Bigg[& 2\frac{L_{(n-3)/2} - L_{(n+9)/2} - L_{2n-3} + L_{2n+3}}{16} - 3\frac{-1-4-L_{n-2}+L_{n+2}}{5} \\ & -(L_{(n+3)/2} - L_{(n-1)/2} - 1) + \frac{n-1}{2}L_n + L_{2n+1} + 1\Bigg]. \end{aligned} \tag{4.8}$$

Now, after some formal manipulations in the subscripts of the Lucas numbers in (4.8) (e.g., rewrite $L_{(n+9)/2} - L_{(n-3)/2}$ as $L_{[(n-3)/2+3]+3} - L_{[(n-3)/2+3]-3}$), use (1.6) once again to obtain

$$\begin{aligned} D_n &= \frac{1}{5}\left[\frac{L_{2n} - L_{(n+3)/2}}{2} - 3(F_n - 1) - L_{(n+1)/2} + \frac{n-1}{2}L_n + L_{2n+1} + 2\right] \\ &= \frac{1}{5}\left[\frac{L_{2n} + (n-1)L_n - 5F_{(n+3)/2}}{2} - 3(F_n-1) - L_{(n+1)/2} + 2\right] \\ &= \frac{1}{5}\left[\frac{2L_{2n+1} + L_{2n} + (n-1)L_n - 5F_{(n+3)/2}}{2} - 3F_n\right] + 1 \\ &= \frac{1}{5}\left[\frac{L_{2n+3} + (n-1)L_n - 5F_{(n+3)/2}}{2} - 3F_n\right] + 1. \quad \square \end{aligned}$$

Proof of Proposition 4: If n is even, by (3.8), (1.2), and (2.9), we have $A_n = \Gamma_n(0) = F_n F_{n+1}$. If n is odd, by (3.8), (2.9), (2.5′), and (2.5‴), we can write

$$A_n = \Gamma_n(0) + \sum_{\tau=1}^{(n-1)/2} \Gamma_n(2\tau) - \sum_{\tau=1}^{(n+1)/2} \Gamma_n(2\tau-1)$$
$$= F_n F_{n+1} + F_n \sum_{\tau=1}^{(n-1)/2} (F_{n-2\tau+1} + F_{2\tau}) - F_{n+1} \sum_{\tau=1}^{(n+1)/2} (F_{n-2\tau+1} + F_{2\tau-2}),$$

whence, by (1.7),

$$\begin{aligned} A_n &= F_n F_{n+1} + 2F_n(F_{n+1} - F_{n-1} - 1) - 2F_{n+1}(F_{n+1} - F_{n-1} - 1) \\ &= F_n F_{n+1} - 2F_{n-1}(F_n - 1) \\ &= F_n(F_{n+1} - 2F_{n-1}) + 2F_{n-1} \\ &= F_n F_{n-2} + 2F_{n-1}. \end{aligned} \tag{4.9}$$

By virtue of the identity [3, I_{19}], (4.9) becomes

$$A_n = F_{n-1}^2 + (-1)^{n-1} F_1 + 2F_{n-1} = F_{n-1}^2 + 1 + 2F_{n-1} = (F_{n-1} + 1)^2. \quad \square$$

The proof of Proposition 5 concludes this section. Here, we need the following four Lucas identities whose proofs can be obtained with the aid of the Binet form and the binomial formula:

$$\sum_{i=0}^{m} \binom{m}{i} L_{k+i} = L_{2m+k}, \tag{4.10}$$

$$\sum_{i=0}^{m} \binom{m}{i} L_{k-i} = L_{m+k}, \tag{4.11}$$

$$\sum_{i=0}^{m/2} \binom{m}{2i} L_{2i} = (L_{2m} + L_m)/2 \quad (m \text{ even}), \tag{4.12}$$

$$\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2i+1} L_{m-2i} = [L_{2m+1} + (-1)^m L_{m-1}]/2. \tag{4.13}$$

Proof of Proposition 5: (n even). By (3.9), (2.3), (4.10), and (4.11), we readily obtain

$$B_n = \frac{1}{5}\left[L_{3n+2} - L_{2n+1} - \sum_{\tau=0}^{n} \binom{n}{\tau} X(n, \tau) \right]$$

and, by (2.4) and (2.4′),

$$B_n = \frac{1}{5}\left\{ L_{3n+2} - L_{2n+1} - \left[\sum_{\tau=0}^{n/2} \binom{n}{2\tau} L_{2\tau} + \sum_{\tau=0}^{n/2-1} \binom{n}{2\tau+1} L_{n-2\tau} \right] \right\}. \tag{4.14}$$

Using (4.12) and (4.13), the equality (4.14) can be rewritten as

$$\begin{aligned} B_n &= \frac{1}{5}[L_{3n+2} - L_{2n+1} - (L_{2n} + L_n + L_{2n+1} + L_{n-1})/2] \\ &= \frac{1}{5}[L_{3n+2} - (2L_{2n+1} + L_{2n+2} + L_{n+1})/2] = \frac{1}{5}[L_{3n+2} - (L_{2n+1} + L_{2n+3} + L_{n+1})/2]. \end{aligned} \tag{4.15}$$

Rewrite (4.15) as

$$B_n = \frac{1}{5}[L_{3n+2} - (L_{(2n+2)+1} + L_{(2n+2)-1} + L_{n+1})/2],$$

and use (1.6) to obtain the desired result,

$$B_n = \frac{1}{5}[L_{3n+2} - (5F_{2n+2} + L_{n+1})/2]. \ \square$$

5. ON THE PARITY OF $\Gamma_n(\tau)$

The problem of establishing necessary and sufficient conditions for $\Gamma_n(\tau)$ to be divisible by a given integer k is believed to deserve a thorough investigation. Nevertheless, the general solution (if any) of this problem is beyond the scope of this paper. In this section we confine ourselves to solving the case $k=2$. The proofs of the results shown in the sequel are based on the well-known fact that

$$F_m \text{ is even if and only if } m \equiv 0 \pmod 3. \tag{5.1}$$

5.1 Results

The integer $\Gamma_n(\tau)$ is even if and only if

(i) n and τ even

$$n \equiv \begin{cases} 0 \\ 1 \\ 2 \end{cases} \pmod 3 \quad \text{and} \quad \tau \equiv \begin{cases} 0 \\ 2 \\ 0 \end{cases} \pmod 3,$$

(ii) n even and τ odd

$$n \equiv \begin{cases} 0 \\ 1 \\ 2 \end{cases} \pmod 3 \quad \text{and} \quad \tau \equiv \begin{cases} 1 \\ 0 \\ 0 \end{cases} \pmod 3,$$

(iii) n odd and τ even

$$n \equiv \begin{cases} 0 \\ 1 \\ 2 \end{cases} \pmod 3 \quad \text{and} \quad \tau \equiv \begin{cases} 0, 1, \text{ or } 2 \\ 1 \\ 0, 1, \text{ or } 2 \end{cases} \pmod 3,$$

(iv) n and τ odd

$$n \equiv \begin{cases} 0 \\ 1 \\ 2 \end{cases} \pmod 3 \quad \text{and} \quad \tau \equiv \begin{cases} 2 \\ 0, 1, \text{ or } 2 \\ 0, 1, \text{ or } 2 \end{cases} \pmod 3.$$

The above conditions on the parity of $\Gamma_n(\tau)$ are presented, in a more compact form, in Table 2, where h and k denote all nonnegative integers such that $0 \le \tau \le n$.

TABLE 2. Forms of n and τ for $\Gamma_n(\tau)$ To Be Even

n	τ		
$6h$	$6k$	or	$6k+1$
$6h+1$	$2k+1$	or	$6k+4$
$6h+2$	$6k$	or	$6k+3$
$6h+3$	$2k$	or	$6k+5$
$6h+4$	$6k+2$	or	$6k+3$
$6h+5$	k		

5.2 Proofs

The proofs of (iii) and (iv) are quite easy. The proofs of (i) and (ii) are similar, so we give only the latter in detail.

Proof of (ii): (n even and τ odd). By (2.5″) we see that $\Gamma_n(\tau)$ is even if and only if

$$\text{either } \begin{cases} A = F_n F_{n-\tau+1} & \text{is even} \\ B = F_{n+1} F_{\tau-1} & \text{is even} \end{cases} \quad \text{(Case 1)}, \quad \text{or } \begin{cases} A & \text{is odd} \\ B & \text{is odd.} \end{cases} \quad \text{(Case 2)}.$$

Case 1. A is even if and only if [see (5.1)]

$$\text{either } n \equiv 0 \pmod 3 \quad \text{or } n \equiv \tau - 1 \pmod 3,$$

whereas B is even if and only if

$$\text{either } n \equiv 2 \pmod 3 \quad \text{or } \tau \equiv 1 \pmod 3.$$

It follows that Case 1 occurs if and only if

$$n \equiv \begin{cases} 0 \\ 2 \end{cases} \pmod 3 \quad \text{and} \quad \tau \equiv \begin{cases} 1 \\ 0 \end{cases} \pmod 3. \tag{5.2}$$

Case 2. A is odd if and only if

$$n \not\equiv 0 \pmod 3 \quad \text{and} \quad n \not\equiv \tau - 1 \pmod 3,$$

whereas B is odd if and only if

$$n \not\equiv 2 \pmod 3 \quad \text{and} \quad \tau \not\equiv 1 \pmod 3.$$

It follows that Case 2 occurs if and only if

$$n \equiv 1 \pmod 3 \quad \text{and} \quad \tau \equiv 0 \pmod 3. \tag{5.3}$$

Combining (5.2) and (5.3) gives (ii). □

6. FURTHER WORK

Flowing from our development, there seem to be other possibilities for investigation. The main one among them consists of applying the operator Γ defined by (1.1) to other second-order recurring sequences, such as the Lucas sequence, the Pell sequence, and so on. As for the former, we obtained the identity

$$\Gamma_n(L_i, \tau) = 5\Gamma_n(\tau) + 2X(n, \tau) \quad \text{[cf. (2.4) and (2.4′)]}. \tag{6.1}$$

On the other hand, we believe that our investigation of the numbers $\Gamma_n(\tau)$ deserves some further deepening. For example, on the bases of (2.5)-(2.5‴) and the identity $F_{-n} = (-1)^{n+1}F_n$, we can generalize these numbers to *any* integer value of the parameters τ and n (i.e., $\tau > n$ and n and/or $\tau < 0$). As a minor instance, it can be shown that

$$\Gamma_n(-n) = F_n(F_{n+1}L_n - F_n) \quad (n \text{ even}). \tag{6.2}$$

Moreover, the results presented in section 5 could be extended to the divisibility of $\Gamma_n(\tau)$ by $k > 2$. In particular, a study on the primality of these numbers should be undertaken. Early responses to this effort allow us to state the following necessary conditions for $\Gamma_n(\tau)$ to be a prime:

$$\begin{array}{l} n \text{ must be even} \\ \;[\text{with the unique exception } \Gamma_3(1) = \Gamma_3(3) = F_4(F_2 + F_0) = 3], \end{array} \tag{6.3}$$

$$\gcd(n-\tau, n) \le 2 \quad (\tau \text{ even}) \tag{6.4}$$

and

$$\begin{cases} \gcd(n, \tau-1) \le 2 \\ \gcd(n-\tau+1, n+1) \le 2 \end{cases} (\tau \text{ odd}). \tag{6.5}$$

In passing, we observed that $\Gamma_n(0)$ is composite [except for $\Gamma_2(0) = F_2F_3 = 2$] and that $\Gamma_n(1)$ is composite as well, except for $\Gamma_2(1) = 1$.

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ROOTS OF SEQUENCES UNDER CONVOLUTIONS

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1. INTRODUCTION

The usual convolution of the sequences $\{r_n\}$ and $\{s_n\}$ is defined to be the sequence $\{t_n\}$ given by $t_n = \sum_{i=0}^{n} r_i s_{n-i}$ $(n \geq 0)$. The usual convolution comes out naturally from the product of the generating functions of the sequences $\{r_n\}$ and $\{s_n\}$:

$$\left(\sum_{n=0}^{\infty} r_n x^n\right)\left(\sum_{n=0}^{\infty} s_n x^n\right) = \sum_{n=0}^{\infty} t_n x^n.$$

This "usual" convolution is also called the Cauchy product. We define the k^{th} power $\{r_n^{(k)}\}$ of the sequence $\{r_n\}$ under the usual convolution as follows:

$$r_n^{(1)} = r_n; \quad r_n^{(k)} = \sum_{i=0}^{n} r_i r_{n-i}^{(k-1)} \quad (k \geq 2).$$

In other words, $r_n^{(k)} = \sum_{i_1+\cdots+i_k=n} r_{i_1} r_{i_2} \cdots r_{i_k}$.

Using the terminology of [6], the k^{th} power under the usual convolution is the $(k-1)^{\text{th}}$ iterated convolution.

The binomial convolution ([2], §7.6) of the sequences $\{r_n\}$ and $\{s_n\}$ is defined to be the sequence $\{u_n\}$ given by

$$u_n = \sum_{i=0}^{n} \binom{n}{i} r_i s_{n-i} \quad (n \geq 0).$$

This convolution arises from the product of the exponential generating functions. Namely,

$$\left(\sum_{n=0}^{\infty} r_n \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} s_n \frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} u_n \frac{x^n}{n!}.$$

We define the k^{th} power $\{r_n^{[k]}\}$ under the binomial convolution of $\{r_n\}$ naturally as follows:

$$r_n^{[1]} = r_n; \quad r_n^{[k]} = \sum_{i=0}^{n} \binom{n}{i} r_i r_{n-i}^{[k-1]} \quad (k \geq 2).$$

Thus,

$$r_n^{[k]} = \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1}\binom{n-i_1}{i_2}\cdots\binom{n-i_1-\cdots-i_{k-2}}{i_{k-1}} r_{i_1} r_{i_2} \cdots r_{i_k} = \sum_{i_1+\cdots+i_k=n} \frac{n!}{i_1!\,i_2!\cdots i_k!} r_{i_1} r_{i_2} \cdots r_{i_k}.$$

In this paper we shall study solutions of the equations $\{r_n^{(k)}\} = \{s_n\}$ and $\{r_n^{[k]}\} = \{s_n\}$ in $\{r_n\}$, where $\{s_n\}$ is a fixed sequence (see Sections 2 and 3). The solutions can be referred to as the k^{th}

roots of $\{s_n\}$ under the usual and the binomial convolution, respectively. In Section 4, roots of sequences under a general weighted convolution are briefly considered.

If $s_n = 0$ for all n, then $r_n = 0$ for all n is the only solution for the equations. Therefore, we may confine ourselves to sequences $\{s_n\}$ such that $s_n \neq 0$ for some n. The least n with $s_n \neq 0$ will be denoted by $\chi(s_n)$.

Since an arithmetic function $f(n)$ is uniquely determined by the corresponding sequence $\{f(1), f(2), f(3), \dots\}$, it follows that the study of the roots of sequences considered here is similar to the study of roots of arithmetic functions already made in papers [1] and [7] under Dirichlet convolution and in paper [3] under "exponential Narkiewicz" convolution. In [4], roots of arithmetic functions under a generalized Dirichlet convolution are studied.

2. ROOTS OF SEQUENCES UNDER THE USUAL CONVOLUTION

Theorem 1: Let $\{s_n\}$ be a given sequence such that $s_n \neq 0$ for some n. Then the equation $\{r_n^{(k)}\} = \{s_n\}$ has a solution in $\{r_n\}$ if and only if $\chi(s_n)$ is the k^{th} multiple of a nonnegative integer. In this case the equation has exactly k solutions, which can be written as

$$\{r_n\} = \{w_i \rho_n\}, \quad i = 1, \dots, k, \tag{1}$$

where $\{\rho_n\}$ is one solution and $w_1, \dots, w_k$ are the k^{th} roots of unity.

Proof: If $\{r_n^{(k)}\} = \{s_n\}$ has a solution, then $k\chi(r_n) = \chi(s_n)$; hence $\chi(s_n)$ is the k^{th} multiple of a nonnegative integer. Conversely, suppose that $\chi(s_n) = km$ for some nonnegative integer m. Then the solutions of $\{r_n^{(k)}\} = \{s_n\}$ can be found as follows. Since $r_n^{(k)} = 0$ for $n < km$, we have $r_n = 0$ for $n < m$. Further, $r_{km}^{(k)} = (r_m)^k$; hence $r_m = (s_{km})^{1/k}$. Finally, the values r_{m+n} $(n \geq 1)$ can be found inductively by using the equations $r_{km+n}^{(k)} = s_{km+n}$ $(n \geq 1)$, whereby it can also be verified that (1) holds. This completes the proof.

For certain sequences $\{s_n\}$, the use of generating functions is a very helpful method of solving the equation $\{r_n^{(k)}\} = \{s_n\}$. Namely, if $r(x)$ and $s(x)$ denote the generating functions of $\{r_n\}$ and $\{s_n\}$, respectively, then $r(x)^k = s(x)$, and hence $r(x) = s(x)^{1/k}$.

We shall illustrate this method in the following examples. For background information on generating functions we refer to [2], [5], and [8].

Example 1: Consider the equation $\{r_n^{(k)}\} = \{a^n\}$, where a is a constant. Then $r(x) = (1-ax)^{-1/k}$ and therefore one solution for the equation is

$$\rho_n = (-1)^n \binom{-1/k}{n} a^n.$$

All solutions can be found by (1). Note that for each integer m, $\rho_n^{(m)} = (-1)^n \binom{-m/k}{n} a^n$. This can be referred to as an $(m/k)^{\text{th}}$ power of the sequence $\{a^n\}$ under the usual convolution.

Example 2: Consider the equation $\{r_n^{(k)}\} = \{s_n\}$, where $\{s_n\}$ is the usual convolution of the sequences $\{a^n\}$ and $\{b^n\}$ with a and b constants. Then $r(x) = (1-ax)^{-1/k}(1-bx)^{-1/k}$ and therefore one solution for the equation is

$$\rho_n = (-1)^n \sum_{i=0}^{n} \binom{-1/k}{i} \binom{-1/k}{n-i} a^i b^{n-i}.$$

All solutions can be found by (1). With $a=(1+\sqrt{5})/2$, $b=(1-\sqrt{5})/2$, this gives the solutions for the equation $\{r_n^{(k)}\}=\{F_{n+1}\}$. Also note that

$$\rho_n^{(m)}=(-1)^n\sum_{i=0}^{n}\binom{-m/k}{i}\binom{-m/k}{n-i}a^i b^{n-i}$$

gives an $(m/k)^{\text{th}}$ power of $\{s_n\}$ under the usual convolution.

Example 3: Let a, b, and c be constants, and $\{\mu_n\}$ the sequence defined by $\mu_0=1$, $\mu_1=-1$, $\mu_n=0$ $(n\ge 2)$. Then $\{\mu_n\}$ is the inverse of the sequence $\equiv 1$, and the sequence $\{\mu_n c^n\}$ is the inverse of the sequence $\{c^n\}$. Consider the equation $\{r_n^{(k)}\}=\{s_n\}$, where $\{s_n\}$ is the usual convolution of the three sequences $\{a^n\}$, $\{b^n\}$, and $\{\mu_n c^n\}$. Then

$$r(x)=(1-ax)^{-1/k}(1-bx)^{-1/k}(1-cx)^{1/k}.$$

Therefore, one solution is the usual convolution of the three sequences

$$\left\{(-1)^n\binom{-1/k}{n}a^n\right\},\ \left\{(-1)^n\binom{-1/k}{n}b^n\right\},\ \text{and}\ \left\{(-1)^n\binom{1/k}{n}c^n\right\}.$$

That is, one solution is given by

$$\rho_n=(-1)^n\sum_{i_1+i_2+i_3=n}\binom{-1/k}{i_1}\binom{-1/k}{i_2}\binom{1/k}{i_3}a^{i_1}b^{i_2}c^{i_3}.$$

All solutions can be found by (1). With $a=(1+\sqrt{5})/2, b=(1-\sqrt{5})/2, c=1/2$, we obtain the solutions of the equation $\{r_n^{(k)}\}=\{L_n/2\}$. Further, multiplying these solutions by $2^{1/k}$ we obtain the solutions for the equation $\{r_n^{(k)}\}=\{L_n\}$.

Example 4: Since $\chi(F_n)=1$, we see by Theorem 1 that the equation $\{r_n^{(k)}\}=\{F_n\}$ does not have a solution, except for the trivial case $k=1$.

3. ROOTS OF SEQUENCES UNDER THE BINOMIAL CONVOLUTION

Theorem 2: Let $\{s_n\}$ be a given sequence such that $s_n\ne 0$ for some n. Then the equation $\{r_n^{[k]}\}=\{s_n\}$ has a solution in $\{r_n\}$ if and only if $\chi(s_n)$ is the k^{th} multiple of a nonnegative integer. In this case the equation has exactly k solutions, which can be written as

$$\{r_n\}=\{w_i\rho_n\},\quad i=1,\dots,k, \tag{2}$$

where $\{\rho_n\}$ is one solution and $w_1,\dots,w_k$ are the k^{th} roots of unity.

Theorem 2 is similar to Theorem 1 in character. Also, Theorem 2 can be proved in a similar way to Theorem 1 and therefore we omit the proof.

The use of exponential generating functions is a helpful method of solving certain equations $\{r_n^{[k]}\}=\{s_n\}$. The following examples will illustrate this method. Here $r_E(x)$ denotes the exponential generating function of $\{r_n\}$.

Example 5: Consider the equation $\{r_n^{[k]}\}=\{\alpha^n\}$. Then $r_E(x)=e^{ax/k}$ and therefore one solution is given by $\rho_n=(a/k)^n$. All solutions can be found by (2).

Example 6: Consider the equation $\{r_n^{[k]}\} = \{(n+1)a^n\}$. Then $r_E(x) = (1+ax)^{1/k} e^{ax/k}$ and therefore one solution is the binomial convolution of the sequences

$$\left\{\frac{1}{k}\left(\frac{1}{k}-1\right)\cdots\left(\frac{1}{k}-(n-1)\right)\right\} \quad \text{and} \quad \left\{\left(\frac{a}{k}\right)^n\right\}.$$

All solutions can be found by (2).

4. A GENERALIZATION

The general weighted convolution of the sequences $\{r_n\}$ and $\{s_n\}$ is defined by

$$\sum_{i=0}^{n} f(n,i) r_i s_{n-i} \quad (n \ge 0),$$

where the weight function $f(n,i)$ is defined for $n \ge 0$ and $0 \le i \le n$. If the weight function satisfies the condition

$$f(n,i)f(i,j) = f(n,j)f(n-j,i-j) \tag{3}$$

for all n, i, j with $0 \le i \le n$, $0 \le j \le i$, then the weighted convolution is associative and we could define powers of sequences under this convolution. We could also consider roots of sequences, and assuming $f(n,i) \ne 0$ for all n and $0 \le i \le n$ we could verify that the result of Theorems 1 and 2 also holds with respect to the weighted convolution. We omit the details.

It is easy to see that both the usual and the binomial convolution are special cases of the weighted convolution satisfying (3).

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3515@compuserve.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-766 ***Proposed by R. André-Jeannin, Longwy, France***

Let n be an even positive integer such that $L_n \equiv 2 \pmod{p}$, where p is an odd prime. Prove that

$$L_{n+1} \equiv 1 \pmod{p}.$$

B-767 ***Proposed by James L. Hein, Portland State University, Portland, OR***

Consider the following two mutual recurrences:

$$G_1 = 1; \quad G_n = F_{n+1}G_{n-1} + F_nH_{n-2}, \ n \ge 2$$

and

$$H_0 = 0; \quad H_n = F_{n+1}G_n + F_nH_{n-1}, \ n \ge 1.$$

Prove that H_{n-1} and G_n are consecutive Fibonacci numbers for all $n \ge 1$.

B-768 ***Proposed by Juan Pla, Paris, France***

Let $u_n, v_n,$ and w_n be sequences defined by $u_1 = 1/2$, $v_1 = \sqrt{2}$, and $w_1 = (1/2)\sqrt{3}$; $u_{n+1} = u_n^2 + v_n^2 - w_n^2$, $v_{n+1} = 2u_nv_n$, and $w_{n+1} = 2u_nw_n$. Express $u_n, v_n,$ and w_n in terms of Fibonacci and/or Lucas numbers.

B-769 ***Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy***

Solve the recurrence

$$a_{n+1} = 5a_n^3 - 3a_n, \ n \ge 0$$

with initial condition $a_0 = 1$.

B-770 ***Proposed by Andrew Cusumano, Great Neck, NY***

Let $U(x)$ denote the unit's digit of x when written in base 10. Let H_n be any generalized Fibonacci sequence that satisfies the recurrence $H_n = H_{n-1} + H_{n-2}$. Prove that, for all n,

$$\begin{array}{ll} U(H_n + H_{n+4}) = U(H_{n+47}), & U(H_n + H_{n+17}) = U(H_{n+34}), \\ U(H_n + H_{n+5}) = U(H_{n+10}), & U(H_n + H_{n+19}) = U(H_{n+41}), \\ U(H_n + H_{n+7}) = U(H_{n+53}), & U(H_n + H_{n+20}) = U(H_{n+55}), \\ U(H_n + H_{n+8}) = U(H_{n+19}), & U(H_n + H_{n+23}) = U(H_{n+37}), \\ U(H_n + H_{n+11}) = U(H_{n+49}), & U(H_n + H_{n+25}) = U(H_{n+50}), \\ U(H_n + H_{n+13}) = U(H_{n+26}), & U(H_n + H_{n+28}) = U(H_{n+59}), \\ U(H_n + H_{n+16}) = U(H_{n+23}), & U(H_n + H_{n+29}) = U(H_{n+58}). \end{array}$$

B-771 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)F_n}{2^n n(n+1)} = \ln 4$.

SOLUTIONS

Square Root of a Recurrence

B-735 ***Proposed by Curtis Cooper & Robert E. Kennedy, Central Missouri State University, Warrensburg, MO***
(Vol. 31, no. 1, February 1993)

Let the sequence (y_n) be defined by the recurrence

$$\begin{aligned} y_{n+1} &= 8y_n + 22y_{n-1} - 190y_{n-2} + 28y_{n-3} + 987y_{n-4} - 700y_{n-5} - 1652y_{n-6} + 1652y_{n-7} \\ &\quad + 700y_{n-8} - 987y_{n-9} - 28y_{n-10} + 190y_{n-11} - 22y_{n-12} - 8y_{n-13} + y_{n-14} \end{aligned}$$

for $n \geq 15$ with initial conditions given by the table on page 185 of the May 1994 issue of this *Quarterly*. Prove that y_n is a perfect square for all positive integers n.

Solution 2 by Leonard A. G. Dresel, Reading, England, and the Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark (independently)

"Though this be madness, yet there is method in it."
—Shakespeare, *Hamlet*, Act 2, Scene 2

Let $x_n = \sqrt{y_n}$. We find, for $n = 1, 2, \ldots, 15$, that y_n is a perfect square, and $x_n = 1, 1, 5, 11, 36, 95, 281, 781, 2245, 6336, 18061, 51205, 145601, \ldots$. We will show that x_n satisfies the recurrence

$$x_{n+1} = x_n + 5x_{n-1} + x_{n-2} - x_{n-3}. \tag{1}$$

To do this, we consider the sequence (x_n) defined by recurrence (1) with initial conditions $x_1 = x_2 = 1$, $x_3 = 5$, and $x_4 = 11$. Then $x_n^2 = y_n$ for $1 \leq n \leq 15$, and we need to show that $x_n^2 = y_n$ for all n.

The characteristic polynomial for recurrence (1) is

$$x^4 - x^3 - 5x^2 - x + 1. \tag{2}$$

Writing this as $(x^2 - px + 1)(x^2 - qx + 1)$, we find that

$$x^4 - (p+q)x^3 + (pq+2)x^2 - (p+q)x + 1 = x^4 - x^3 - 5x^2 - x + 1.$$

This is an identity in x if $p+q=1$ and $pq=-7$, i.e., if p and q are the roots of the equation $w^2-w-7=0$. We thus find $p=(1+\sqrt{29})/2$ and $q=(1-\sqrt{29})/2$. The zeros of polynomial (2) are therefore a, b, a^{-1}, and b^{-1}, where

$$a+a^{-1}=\frac{1+\sqrt{29}}{2},\ b+b^{-1}=\frac{1-\sqrt{29}}{2},\ a^2+a^{-2}=\frac{11+\sqrt{29}}{2},\ \text{and } b^2+b^{-2}=\frac{11-\sqrt{29}}{2}.$$

The Binet form for recurrence (1) is

$$x_n=\frac{a^n+a^{-n}-b^n-b^{-n}}{\sqrt{29}}.$$

The recurrence whose elements are x_n^2 will have the form

$$y_n=\frac{(a^n+a^{-n}-b^n-b^{-n})^2}{29} \tag{3}$$

and we need to show that y_n satisfies the recurrence given in the problem statement, i.e., that it satisfies the characteristic polynomial

$$\begin{aligned}y^{15}-8y^{14}-22y^{13}+190y^{12}-28y^{11}-987y^{10}+700y^9+1652y^8\\-1652y^7-700y^6+987y^5+28y^4-190y^3+22y^2+8y-1.\end{aligned} \tag{4}$$

Expanding equation (3) shows that the characteristic polynomial for y_n is

$$(y-1)(y-a^2)(y-a^{-2})(y-b^2)(y-b^{-2})(y-ab)(y-a^{-1}b)(y-ab^{-1})(y-a^{-1}b^{-1}). \tag{5}$$

Breaking this up into parts, we find

$$\begin{aligned}(y-a^2)(y-a^{-2})(y-b^2)(y-b^{-2})&=\left(y^2-\frac{11+\sqrt{29}}{2}y+1\right)\left(y^2-\frac{11-\sqrt{29}}{2}y+1\right)\\&=y^4-11y^3+25y^2-11y+1.\end{aligned}$$

Another factor is $(y-ab)(y-a^{-1}b)(y-ab^{-1})(y-a^{-1}b^{-1})$. This polynomial must be symmetrical since its roots are reciprocal in pairs. Since

$$ab+a^{-1}b^{-1}+a^{-1}b+ab^{-1}=(a+a^{-1})(b+b^{-1})=\frac{1+\sqrt{29}}{2}\cdot\frac{1-\sqrt{29}}{2}\doteq-7$$

and

$$\begin{aligned}&ab\cdot a^{-1}b^{-1}+ab\cdot a^{-1}b+ab\cdot ab^{-1}+a^{-1}b^{-1}\cdot a^{-1}b+a^{-1}b^{-1}\cdot ab^{-1}+a^{-1}b\cdot ab^{-1}\\&=1+b^2+a^2+a^{-2}+b^{-2}+1=2+\frac{11+\sqrt{29}}{2}+\frac{11-\sqrt{29}}{2}=13,\end{aligned}$$

this polynomial must be $y^4+7y^3+13y^2+7y+1$. Thus, the characteristic polynomial (5) is

$$\begin{aligned}&(y-1)(y^4-11y^3+25y^2-11y+1)(y^4+7y^3+13y^2+7y+1)\\&=y^9-5y^8-35y^7+67y^6+145y^5-145y^4-67y^3+35y^2+5y-1.\end{aligned}$$

Since this polynomial divides the polynomial (4), we see that the squares of the x_n satisfy the original recurrence and hence every element of that recurrence is a perfect square.

Note: We also see that the original sequence satisfies the simpler recurrence

$$y_{n+1}=5y_n+35y_{n-1}-67y_{n-2}-145y_{n-3}+145y_{n-4}+67y_{n-5}-35y_{n-6}-5y_{n-7}+y_{n-8}.$$

B-736 ***Proposed by Herta T. Freitag, Roanoke, VA***
(Vol. 31, no. 2, May 1993)

Prove that $(2L_n + L_{n-3}) / 5$ is a Fibonacci number for all n.

Solution by Graham Lord, Mathtech, Inc., Princeton, NJ

If $A_n = (2L_n + L_{n-3}) / 5$, then $A_1 = 1$ and $A_2 = 1$. Furthermore, $A_{n-1} + A_n = A_{n+1}$ from the recursive property of Lucas numbers. Therefore, $A_n = F_n$.

Haukkanen found the corresponding result for Fibonacci numbers which is that $2F_n + F_{n-3}$ is a Lucas number for all n.

Also solved by Miguel Amengual Covas, Charles Ashbacher, M. A. Ballieu, Seung-Jin Bang, Margherita Barile, Glenn Bookhout, Scott H. Brown, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, Jane Friedman, Pentti Haukkanen, Russell Jay Hendel, Joe Howard, John Ivie, Joseph J. Kostal, Carl Libis, Bob Prielipp, H.-J. Seiffert, A. G. Shannon, Sahib Singh, Lawrence Somer, J. Suck, Ralph Thomas, and the proposer.

Golden Radii

B-737 ***Proposed by Herta T. Freitag, Roanoke, VA***
(Vol. 31, no. 2, May 1993)

A right triangle, one of whose legs is twice as long as the other leg, has a hypotenuse that is one unit longer than the longer leg. Let r be the inradius of this triangle (radius of inscribed circle) and let r_a, r_b, r_c be the exradii (radii of circles outside the triangle that are tangent to all three sides). Express r, r_a, r_b, and r_c in terms of the golden ratio, α.

Solution by Sahib Singh, Clarion University, Clarion, PA

Let the three sides of the right triangle be $x, 2x$, and $2x+1$. Thus, x is the positive root of $(2x+1)^2 = x^2 + 4x^2$ by the Pythagorean Theorem. This yields $x = 2 + \sqrt{5} = 2\alpha + 1$. Consequently, the sides of the triangle are $2\alpha + 1, 4\alpha + 2$, and $4\alpha + 3$. If A is the area of the triangle, we have $A = (2\alpha + 1)^2 = 4\alpha^2 + 4\alpha + 1 = 8\alpha + 5$. The semiperimeter, s, of the triangle is $(a+b+c)/2 = 5\alpha + 3$.

Using well-known formulas for the inradius and exradii [1], we find:

$$r = \frac{A}{s} = \frac{8\alpha+5}{5\alpha+3} = \alpha;$$

$$r_a = \frac{A}{s-a} = \frac{8\alpha+5}{(5\alpha+3)-(2\alpha+1)} = \frac{8\alpha+5}{3\alpha+2} = \alpha + 1 = \alpha^2;$$

$$r_b = \frac{A}{s-b} = \frac{8\alpha+5}{(5\alpha+3)-(4\alpha+2)} = \frac{8\alpha+5}{\alpha+1} = 3\alpha + 2 = \alpha^4;$$

$$r_c = \frac{A}{s-c} = \frac{8\alpha+5}{(5\alpha+3)-(4\alpha+3)} = \frac{8\alpha+5}{\alpha} = 5\alpha + 3 = \alpha^5.$$

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1. E. W. Hobson. *A Treatise on Plane and Advanced Trigonometry*. New York: Dover, 1957, p. 193.

Also solved by Miguel Amengual Covas, Seung-Jin Bang, Margherita Barile, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Bob Prielipp, H.-J. Seiffert, J. Suck, and the proposer.

A Dozen Identities

B-738 ***Proposed by Daniel C. Fielder & Cecil O. Alford, Georgia Institute of Technology, Atlanta, GA***
(Vol. 31, no. 2, May 1993)

Find a polynomial $f(w, x, y, z)$ such that

$$f(L_n, L_{n+1}, L_{n+2}, L_{n+3}) = 25f(F_n, F_{n+1}, F_{n+2}, F_{n+3})$$

is an identity.

Solutions 1-12 by many readers

Joseph J. Kostal:	$(xy-wz)^2$
A. N. 't Woord:	$(w^2+x^2+y^2+z^2)^2$
H.-J. Seiffert:	$(w^2+x^2)^2+(y^2+z^2)^2$
Leonard A. G. Dresel:	$(w^2+x^2)(y^2+z^2)$
J. Suck:	$(y^2-w^2)(z^2-x^2)$
Margherita Barile:	$[(x+z)^2+(y+w)^2]^2$
Paul S. Bruckman:	$(x^2+xy-y^2)^2$
Herta T. Freitag:	$(wz)^2+4(xy)^2$
Shannon/Hendel/et al.:	$(x^2-wy)(y^2-xz)$
Ralph Thomas:	$wxyz+\frac{3}{5}wz(xz-y^2)$
Paul S. Bruckman:	$w^4+(y+z)^4-4x^4-19y^4-4z^4$
David Zeitlin:	$y^4-wxz(y+z)$

Solution 13 by H.-J. Seiffert, Berlin, Germany

More generally, we show that if p is a natural number, then $f_p(w,x,y,z)=(w^2+x^2)^p+(y^2+z^2)^p$ is a polynomial such that $f_p(L_n, L_{n+1}, L_{n+2}, L_{n+3}) = 5^p f_p(F_n, F_{n+1}, F_{n+2}, F_{n+3})$ is an identity. Using equation (I_{12}) of [1], $L_k^2 = 5F_k^2+4(-1)^k$, we obtain

$$\begin{aligned}
f_p(L_n, L_{n+1}, L_{n+2}, L_{n+3}) &= (L_n^2+L_{n+1}^2)^p + (L_{n+2}^2+L_{n+3}^2)^p \\
&= (5F_n^2+4(-1)^n+5F_{n+1}^2+4(-1)^{n+1})^p \\
&\quad +(5F_{n+2}^2+4(-1)^{n+2}+5F_{n+3}^2+4(-1)^{n+3})^p \\
&= 5^p(F_n^2+F_{n+1}^2)^p+5^p(F_{n+2}^2+F_{n+3}^2)^p \\
&= 5^p f_p(F_n, F_{n+1}, F_{n+2}, F_{n+3}).
\end{aligned}$$

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1979.

Solution 14 by David Zeitlin, Minneapolis, MN

Let $f(w,x,y,z) = y^4 - wxz(y+z)$. If (H_n) is any sequence that satisfies the recurrence $H_{n+2} = H_{n+1} + H_n$, then

$$f(H_n, H_{n+1}, H_{n+2}, H_{n+3}) = (H_2^4 - H_0H_1H_3H_4)f(F_n, F_{n+1}, F_{n+2}, F_{n+3}).$$

This follows from [1], where it is shown that, for all nonnegative integers n,

$$H_{n+2}^4 - H_nH_{n+1}H_{n+3}H_{n+4} = H_2^4 - H_0H_1H_3H_4.$$

Note that for the Fibonacci sequence, the value of $H_2^4 - H_0H_1H_3H_4$ is 1, and for the Lucas sequence, the value is 25.

See also [2] for related identities.

References

1. David Zeitlin. "Generating Functions for Products of Recursive Sequences." *Transactions of the American Mathematical Society* **116** (1965):300-15.
2. David Zeitlin. "Power Identities for Sequences Defined by $W_{n+2} = dW_{n+1} - cW_n$." *The Fibonacci Quarterly* **3.3** (1965):241-56.

Thankfully, no solver submitted the "trivial" solution: $f(w,x,y,z) = x+y-z$. Zeitlin points out that the solutions given are not independent. If f_1 and f_2 are correct solutions, then so are f_1+f_2 and f_1-f_2. Thus, for example, Seiffert's solution plus 2 times Dresel's solution yields Woord's solution. The identities $w+x=y$ and $x+y=z$ can also be used to transform one solution into another valid solution.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-488 ***Proposed by Paul S. Bruckman, Highwood, IL***

The *Fibonacci pseudoprimes* (or FPP's) are those composite integers n with $\gcd(n, 10) = 1$ and satisfying the following congruence:

$$F_{n-\varepsilon_n} \equiv 0 \pmod{n}, \tag{i}$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{10}, \\ -1 & \text{if } n \equiv \pm 3 \pmod{10}. \end{cases}$$

$$\left[\text{Thus, } \varepsilon_n = \left(\tfrac{5}{n}\right), \text{ a Jacobi symbol.}\right]$$

Given a prime $p > 5$, prove that $u = \frac{1}{3}L_{2p}$ is a FPP if u is composite.

The *Lucas pseudoprimes* (or LPP's) are those composite positive integers n satisfying the following congruence:

$$L_n \equiv 1 \pmod{n}. \tag{ii}$$

Given a prime $p > 5$, prove that $u = \frac{1}{3}L_{2p}$ is a LPP if u is composite.

H-489 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the sequences of Pell numbers and Pell-Lucas numbers by

$$P_0 = 0,\ P_1 = 1,\ P_{k+2} = 2P_{k+1} + P_k,$$
$$Q_0 = 2,\ Q_1 = 2,\ Q_{k+2} = 2Q_{k+1} + Q_k.$$

Show that

(a) $$\sum_{n=1}^{\infty} \frac{F_{2^n}Q_{2^n}}{8(L_{2^n}P_{2^n})^2 - 5(F_{2^n}Q_{2^n})^2} = \frac{1}{12},$$

(b) $$\sum_{n=1}^{\infty} \frac{L_{2^n}P_{2^n}}{8(L_{2^n}P_{2^n})^2 - 5(F_{2^n}Q_{2^n})^2} = \frac{8-3\sqrt{2}}{48}.$$

SOLUTIONS

A Soft Matrix

H-474 ***Proposed by R. Andre-Jeannin, Longwy, France***
(Vol. 31, no. 1, February 1993)

Let us define the sequence $\{U_n\}$ by

$$U_0 = 0,\ U_1 = 1, \text{ and } U_n = PU_{n-1} - QU_{n-2},\ n \in \mathbb{Z},$$

where P and Q are nonzero integers. Assuming that $U_k \neq 0$, the matrix M_k is defined by

$$M_k = \frac{1}{U_k}\begin{pmatrix} U_{k+1} & iQ^{k/2} \\ iQ^{k/2} & -Q^k U_{1-k} \end{pmatrix},\ k \geq 1,$$

where $i = \sqrt{(-1)}$.

Express in a closed form the matrix M_k^n, for $n \geq 0$.

Reference: A. F. Horadam & P. Filipponi, "Choleski Algorithm Matrices of Fibonacci Type and Properties of Generalized Sequences," *The Fibonacci Quarterly* **29.2** (1991):164-73.

Solution by H.-J. Seiffert, Berlin, Germany

First, we prove that for all integers m, h, and j,

$$U_{m+h}U_{m+j} - U_m U_{m+h+j} = Q^m U_h U_j. \tag{1}$$

We consider the Fibonacci polynomials defined by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ n \in \mathbb{Z}.$$

It is easily seen that

$$U_n = (-Q)^{(n-1)/2} F_n(x),\ n \in \mathbb{Z}, \tag{2}$$

where $x = P/\sqrt{-Q}$. Multiplying the well-known equation [see A. F. Horadam & Bro. J. M. Mahon, "Pell and Pell-Lucas Polynomials," *The Fibonacci Quarterly* **23.1** (1985):12, formula (3.32), where the polynomials $P_k(x) = F_k(2x)$ are considered]

$$F_{m+h}(x)F_{m+j}(x) - F_m(x)F_{m+h+j}(x) = (-1)^m F_h(x)F_j(x)$$

by $(-Q)^{m-1+(h+j)/2}$ and regarding (2), we obtain (1). From (2), it also follows that

$$U_{-n} = -Q^{-n}U_n,\ n \in \mathbb{Z}. \tag{3}$$

For $m = k$, $h = 1$, and $j = n$, (1) yields

$$U_{k+1}U_{k+n} - Q^k U_n = U_k U_{k+n+1}. \tag{4}$$

Similarly, with $m = k$, $h = n - k$, and $j = 1$,

$$U_n U_{k+1} - Q^k U_{n-k} = U_k U_{n+1}, \tag{5}$$

with $m = k$, $h = n$, and $j = 1 - k$,

$$U_{k+n} - Q^k U_n U_{1-k} = U_k U_{n+1}, \tag{6}$$

and finally, with $m=n$, $h=1-k$, and $j=k-n$, (1) gives

$$U_n + Q^n U_{1-k} U_{k-n} = U_k U_{n+1-k}$$

or, by (3),

$$U_n - Q^k U_{1-k} U_{n-k} = U_k U_{n+1-k}. \tag{7}$$

With the help of (4)-(7), it is easily proved by induction on n that

$$M_k^n = \frac{1}{U_k}\begin{pmatrix} U_{k+n} & iQ^{k/2}U_n \\ iQ^{k/2}U_n & -Q^k U_{n-k} \end{pmatrix}, \quad n \geq 1.$$

Using (3), it is easily seen that this equation also holds for $n=0$.

Also solved by P. Bruckman, A. G. Shannon, and the proposer.

Get It off Your Chess

H-475 *Proposed by Larry Taylor, Rego Park, NY*
(Vol. 31, no. 2, May 1993)

Professional chess players today use the algebraic chess notation. This is based upon the algebraic numbering of the chessboard. The eight letters *a* through *h* and the eight digits 1 through 8 are used to form sixty-four combinations of a letter and a digit which are called "symbol pairs." Those sixty-four symbol pairs are used to represent the sixty-four squares of the chessboard.

Develop a viable arithmetic numbering of the chessboard, as follows:

(a) Use twenty-five letters of the alphabet (all except *U*) and nine decimal digits (all except zero) to form 225 symbol pairs; choose sixty-four of those symbol pairs to represent the sixty-four squares of the chessboard.

(b) There are thirty-six squares from which a King can move to eight other squares. Let the nine symbol pairs representing the location of the King and the squares to which it can move contain all nine decimal digits.

(c) There are sixteen squares from which a Knight can move to eight other squares. A Queen located on one of those sixteen squares, moving one or two squares, can go to sixteen other squares. Let the twenty-five symbol pairs representing the location of the Knight or the Queen and the squares to which the Knight or the Queen can move contain all twenty-five letters of the alphabet.

(d) Let the algebraic Square *a*8 (the original location of Black's Queen Rook) correspond to the arithmetic Square *A*1; let the algebraic Square *h*1 (the original location of White's King Rook) correspond to the arithmetic Square *Z*9.

Solution by Leonard A. G. Dresel, Reading, England

Consider the basic 3×3 and 5×5 patterns given by:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5^* & 6 \\ 7 & 8 & 9^* \end{matrix} \quad \text{and} \quad \begin{matrix} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M^* & N & O \\ P & Q & R & S & T \\ V & W & X & Y & Z^* \end{matrix}$$

Repeating these patterns across the 8×8 board, left to right and then top to bottom, and superposing them, we can satisfy conditions (b) and (c). To satisfy (d) and obtain $Z9$ in the bottom right corner, we exchange 5 with 9 and M with Z in the basic patterns. Thus, we arrive at a viable numbering given by:

A1	B2	C3	D1	E2	A3	B1	C2
F4	G9	H6	I4	J9	F6	G4	H9
K7	L8	Z5	N7	O8	K5	L7	Z8
P1	Q2	R3	S1	T2	P3	Q1	R2
V4	W9	X6	Y4	M9	V6	W4	X9
A7	B8	C5	D7	E8	A5	B7	C8
F1	G2	H3	I1	J2	F3	G1	H2
K4	L9	Z6	N4	O9	K6	L4	Z9

Since 3 and 5 are co-prime, the repeating patterns ensure that no alpha-numeric combination occurs more than once.

The solution is not unique, as we can choose the modified basic patterns in $(7!) \times (23!)$ ways to satisfy condition (d).

Also solved by P. Bruckman, J. Hendel, and the proposer.

Pell-Mell

H-476 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 31, no. 2, May 1993)

Define the Pell numbers by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$. Show that, for all positive integers n,

$$P_n = \sum_{\substack{k=0 \\ 4 \nmid 2n+k}}^{n-1} (-1)^{[(3k+3-2n)/4]} 2^{[3k/2]} \binom{n+k}{2k+1},$$

where [ ] denotes the greatest integer function.

Solution by Paul S. Bruckman, Highwood, IL

Let S_n denote the sum given in the statement of the problem. It is easily shown that

$$\frac{x}{f(x)} = \sum_{n=1}^{\infty} P_n x^n, \; |x| < \sqrt{2} - 1, \tag{1}$$

where

$$f(x) = 1 - 2x - x^2. \tag{2}$$

To prove that $S_n = P_n$, $n = 1, 2, \ldots$, it will suffice to show that $g(x) = \frac{x}{f(x)}$, where

$$g(x) = \sum_{n=1}^{\infty} S_n x^n; \tag{3}$$

presumably, this is to be valid for all x with $|x| < \sqrt{2} - 1$.

As usual with generating function techniques, we will ignore questions of convergence (which *should* be considered, *a posteriori*). Then

$$g(x) = \sum_{\substack{k, m \geq 0 \\ 4 \nmid 2m+3k+2}} x^{m+k+1} (-1)^{\left[\frac{1}{4}(k+1-2m)\right]} 2^{[3k/2]} \binom{m+2k+1}{2k+1}.$$

Letting $m = 2u$ or $m = 2u+1$, we obtain

$$\begin{aligned}
g(x) &= \sum_{\substack{k,u\ge 0\\ 4\nmid 3k+2}} x^{2u+k+1}(-1)^{[(k+1)/4]+u}2^{[3k/2]}\binom{2u+2k+1}{2u} \\
&\quad + \sum_{\substack{k,u\ge 0\\ 4\nmid 3k}} x^{2u+k+2}(-1)^{[(k-1)/4]+u}2^{[3k/2]}\binom{2u+1+2k+1}{2u+1} \\
&= \sum_{\substack{j\ge 1\\ 4\nmid j+1}} x^j(-1)^{[j/4]}2^{\left[\frac{3}{2}(j-1)\right]}\sum_{u\ge 0}(-1)^u x^{2u}\binom{-2j}{2u} \\
&\quad - \sum_{\substack{j\ge 1\\ 4\nmid j-1}} x^j(-1)^{\left[\frac{1}{4}(j-2)\right]}2^{\left[\frac{3}{2}(j-1)\right]}\sum_{u\ge 0}(-1)^u x^{2u+1}\binom{-2j}{2u+1}.
\end{aligned}$$

Now

$$\sum_{u\ge 0}(-1)^u x^{2u}\binom{-2j}{2u} = \sum_{v\ge 0}(ix)^v e_v\binom{-2j}{v},$$

where $e_v = \frac{1}{2}(1+(-1)^v)$, which equals $\frac{1}{2}(\theta^{-2j}+\overline{\theta}^{-2j}) = \operatorname{Re}(\theta^{-2j})$, with $\theta \equiv 1+ix$. Likewise,

$$\sum_{u\ge 0}(-1)^u x^{2u+1}\binom{-2j}{2u+1} = -i\sum_{v\ge 0}(ix)^v o_v\binom{-2j}{v},$$

where $o_v = \frac{1}{2}(1-(-1)^v)$, which equals $\frac{1}{2i}(\theta^{-2j}-\overline{\theta}^{-2j}) = \operatorname{Im}(\theta^{-2j})$. Therefore,

$$g(x) = \operatorname{Re}(U(x)+iV(x)), \tag{4}$$

where

$$U(x) = \sum_{\substack{j\ge 1\\ 4\nmid j+1}} x^j(-1)^{\left[\frac{1}{4}j\right]}2^{\left[\frac{3}{2}(j-1)\right]}\theta^{-2j}, \tag{5}$$

$$V(x) = \sum_{\substack{j\ge 1\\ 4\nmid j-1}} x^j(-1)^{\left[\frac{1}{4}(j-2)\right]}2^{\left[\frac{3}{2}(j-1)\right]}\theta^{-2j}. \tag{6}$$

Making the substitutions $j = 4i+r$, where $i \ge 0$ and $r = 1, 2,$ or 4 in (5), $r = 2, 3, 4$ in (6), we find that

$$U(x) = (x/\theta^2 + 2x^2/\theta^4 - 16x^4/\theta^8)\cdot h(x), \tag{7}$$

$$V(x) = (2x^2/\theta^4 + 8x^3/\theta^6 + 16x^4/\theta^8)\cdot h(x), \tag{8}$$

where

$$h(x) = \sum_{i=0}^{\infty}(-1)^i 2^{6i}x^{4i}\theta^{-8i}. \tag{9}$$

Thus,

$$h(x) = (1+64x^4/\theta^8)^{-1} = \frac{\theta^8}{\theta^8+64x^4},$$

from which we obtain

$$U(x) = \frac{x}{\theta^8+64x^4}\cdot(\theta^6+2x\theta^4-16x^3), \tag{10}$$

$$V(x)=\frac{2x^2}{\theta^8+64x^4}\cdot(\theta^4+4x\theta^2+8x^2). \tag{11}$$

As we may verify, $\theta^8+64x^4=(\theta^4+4x\theta^2+8x^2)(\theta^4-4x\theta^2+8x^2)$ and $\theta^6+2x\theta^4-16x^3=(\theta^4+4x\theta^2+8x^2)(\theta^2-2x)$. Thus,

$$U(x)=\frac{x(\theta^2-2x)}{\theta^4-4x\theta^2+8x^2},\quad V(x)=\frac{2x^2}{\theta^4-4x\theta^2+8x^2}. \tag{12}$$

Next, we observe that $\theta^2=1+2ix-x^2=1-2x-x^2+2x(1+i)=f(x)+2x(1+i)$. Also, we have $\theta^4=(f(x))^2+4xf(x)\cdot(1+i)+4x^2\cdot 2i=f(x)[f(x)+4ix]+4xf(x)+8ix^2$, from which it follows that $\theta^4-4x\theta^2+8x^2=f(x)[f(x)+4ix]$. Then

$$U(x)+iV(x)=\frac{x(\theta^2-2x+2ix)}{f(x)[f(x)+4ix]}=\frac{x[f(x)+4ix]}{f(x)[f(x)+4ix]}=\frac{x}{f(x)}.$$

Hence, we see that $U(x)+iV(x)$ is real, so that

$$\operatorname{Re}(U(x)+iV(x))=U(x)+iV(x)=g(x)=\frac{x}{f(x)}.\quad \text{Q.E.D.}$$

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