

The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

EDITORIAL POLICY

THE FIBONACCI QUARTERLY seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

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The Fibonacci Quarterly

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*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

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A NOTE ON A GEOMETRICAL PROPERTY OF FIBONACCI NUMBERS

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(Submitted March 1993)

INTRODUCTION

In [2] the authors, amid a more extensive analysis, prove an interesting geometrical property of Fibonacci numbers. They adopt the unusual convention (see [1] for the usual convention) that the Fibonacci sequence is given by

$$f_0 = f_1 = 1, \quad f_{n+2} = f_{n+1} + f_n, \quad n \geq 0, \quad (1)$$

Let F_n be the point (f_{n-1}, f_n) in the coordinate plane; let $X_n = (f_{n-1}, 0)$, $Y_n = (0, f_n)$; and let p_n be the broken line from O to F_n consisting of the straight line segments $OF_1, F_1F_2, \dots, F_{n-1}F_n$. Then it is proved in [2] that p_n separates the rectangle $OX_nF_nY_n$ into two regions of equal area, provided that n is odd. Our main object in this note is to give an elementary geometrical proof of their quoted result, and then to give an elementary algebraic proof of a generalized version of this result.

PROOF WITHOUT WORDS

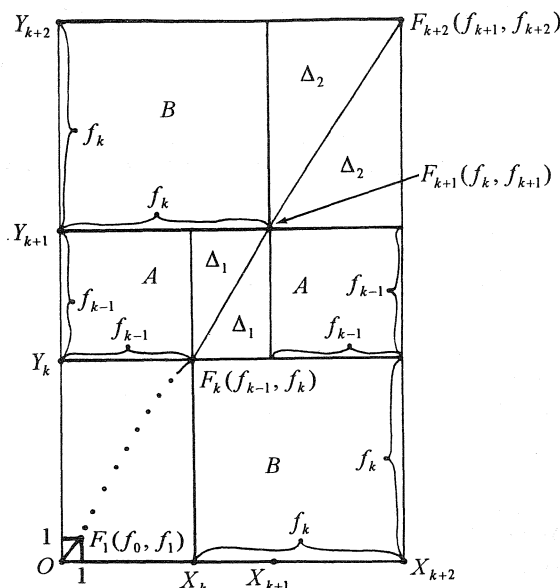


FIGURE 1

Figure 1 shows a path that begins at the origin and then progresses through the points $F_k(f_{k-1}, f_k)$, where the f_k are defined as in (1) above. We quote the first result of [2]:

... for each $n \geq 1$, the polygonal path $F_0 F_1 F_2 \dots F_{2n+1}$ splits the rectangle $F_0 X_{2n+1} F_{2n+1} Y_{2n+1}$ into two regions of equal area. (Note that, in [2], the origin is referred to as F_a .)

Inspection of Figure 1 (where congruent regions are labeled with the same symbol) reveals that the above result may be seen to be true by simply looking at the geometry of the suitably subdivided rectangle which evolves as a polygonal path passes from F_k through F_{k+1} to F_{k+2} . For Figure 1 clearly shows that, for *all* $k \geq 1$,

$$\text{area } Y_k F_k F_{k+1} F_{k+2} Y_{k+2} = \text{area } X_k F_k F_{k+1} F_{k+2} X_{k+2},$$

and hence it follows that, since the polygonal path from F_0 to F_1 obviously splits the rectangle $F_0 X_1 F_1 Y_1$ into two regions of equal area, then the polygonal path from F_0 to F_{2k+1} splits the rectangle $F_0 X_{2k+1} F_{2k+1} Y_{2k+1}$ into two regions of equal area. Notice that Figure 1 also tells us that the first line segment could have gone straight from F_0 to F_j , $j \geq 1$, and then the polygonal path from F_0 to F_{2k+j} would split the rectangle $F_0 X_{2k+j} F_{2k+j} Y_{2k+j}$ into two regions of equal area. Furthermore, since the calculation of the lengths of the sides of the squares in Figure 1 depends effectively only on the recurrence relation in (1), and not on the initial values, any sequence of positive numbers (the Lucas sequence, for example) satisfying (1) will produce a similar result.

THE THEOREM

We consider *any* sequence $\{u_n\}$ of nonnegative numbers satisfying the recurrence relation $u_{n+2} = u_{n+1} + u_n$; notice that, in particular, we might consider the Fibonacci sequence or the Lucas sequence starting at *any* place along the sequence. We proceed exactly as in the Introduction, replacing f_n by u_n , so that $U_n = (u_{n-1}, u_n)$, $X_n = (u_{n-1}, 0)$, $Y = (0, u_n)$, and the broken line $p_n = OU_1 U_2 \dots, U_n$ separates the rectangle $OX_n U_n Y_n$ into two regions.

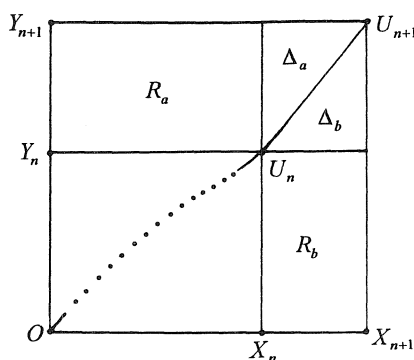


FIGURE 2

Theorem: The dotted line p_n separates the rectangle $OX_n U_n Y_n$ into two regions of equal area, provided that n is odd.

We need the following simple lemma.

Lemma: $u_n^2 - u_{n+1}u_{n-1} = -(u_{n-1}^2 - u_n u_{n-2})$.

Proof of Lemma: $u_n^2 - u_{n+1}u_{n-1} = (u_n^2 - u_n u_{n-1}) - (u_{n+1}u_{n-1} - u_n u_{n-1}) = u_n u_{n-2} - u_{n-1}^2$.

Proof of Theorem: We argue by induction on n , the case $n = 1$ being trivial. Consider the piece added on in passing from the rectangle $OX_nU_nY_n$ to the rectangle $OX_{n+1}U_{n+1}Y_{n+1}$. This may be subdivided, as in Figure 2, into a triangle Δ_a and a rectangle R_a above p_{n+1} , and a triangle Δ_b and a rectangle R_b below p_{n+1} . Obviously,

$$\begin{cases} \text{area } \Delta_a = \text{area } \Delta_b, \\ \text{area } R_a = u_{n-1}(u_{n+1} - u_n) = u_{n-1}^2, \\ \text{area } R_b = u_n(u_n - u_{n-1}) = u_n u_{n-2}. \end{cases} \quad (2)$$

Let A_n be the area of the region above p_n , and B_n the area of the region below p_n in the n^{th} -stage rectangle. We have proved that

$$A_{n+1} - B_{n+1} = A_n - B_n + D_n, \text{ where } D_n = u_{n-1}^2 - u_n u_{n-2}. \quad (3)$$

Now our Lemma asserts that

$$D_{n+1} = -D_n. \quad (4)$$

Thus, by (3) and (4),

$$A_{n+2} - B_{n+2} = A_n - B_n. \quad (5)$$

The equality (5) provides the inductive step to complete the proof.

REMARKS

(i) Equality (5) shows that, if n is even, the discrepancy $A_n - B_n$ is still independent of n ; it will, however, depend on our particular choice of sequence $\{u_n\}$ since it will equal $D_1 = u_0^2 - u_1 u_{-1} = u_0^2 - u_1(u_1 - u_0) = u_0^2 + u_0 u_1 - u_1^2$. Thus, the conclusion of our Theorem also holds if n is even, if and only if u_0, u_1 are related by $u_1 = \frac{\sqrt{5}+1}{2} u_0$.

(ii) Since our proof is purely algebraic, it remains valid even if we allow negative values of u_n , provided we interpret area correctly (i.e., allowing for sign). Thus, in particular, we could consider the Fibonacci and Lucas sequences starting with some negative subscript.

(iii) The case considered by Page & Sastry in [2], that is, $u_n = f_n$, does have a special feature of interest. For $f_0^2 + f_0 f_1 - f_1^2 = 1$, so that, in their case, with n even, the area of the region above p_n exceeds that of the region below p_n by exactly one unit. Of course, this phenomenon continues to hold if we take $u_k = f_{n+k}$ for any even k . If we take k odd, on the other hand, then, for even values of n , it is the area of the region below p_n which exceeds that of the region above p_n by one unit.

(iv) Readers will probably wish to refer to [2] for related results, including matrix-generated area-splitting paths.

REFERENCES

1. Walter Ledermann, ed. *Handbook of Applicable Mathematics*. Chichester and New York: John Wiley & Sons, 1980.
2. Warren Page & K. R. S. Sastry. "Area-Bisecting Polygonal Paths." *The Fibonacci Quarterly* **30.3** (1992):263-73.

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THE FIBONACCI KILLER*

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(Submitted March 1993)

1. INTRODUCTION

We consider the following stochastic process: Assume that a "player" is hit at any time x with probability p . However, he dies only after two consecutive hits. We might code this process by **0** and **1**, marking a hit, e.g., by a "1". Then the sequences associated with a player can be described by

$$\{0, 10\}^* \cdot 11.$$

The notation $\{0, 10\}^*$ denotes arbitrary sequences consisting of the blocks **0** and **10**, the block **11** are the fatal hits. Notice that $\{0, 10\}^*$ are exactly the admissible blocks in the Fibonacci expansion of integers (*Zeckendorf* expansion, cf. [13]). Accordingly, the generating function

$$\frac{p^2 z^2}{1 - qz - pqz^2} \quad (1.1)$$

has as the coefficient of z^x the probability $\mathbb{P}\{X = x\}$ that the lifetime X of a player is exactly x . The generating function (1.1) is known in the context of the Fibonacci distribution or geometric distribution of order 2, cf. [1], [3], [4], [7], [8], [10], [12].

Here, we are interested in n (independent) players subject to this game and ask when (in the sense of a mean value) the last player dies.

Without the "Fibonacci" restriction, i.e., the maximum of n (independent) geometric random variables, this problem has been studied previously and has some applications. (Compare [5], [11].)

We have obviously

$$\mathbb{P}\{\max\{X_1, \dots, X_n\} \leq x\} = (\mathbb{P}\{X \leq x\})^n. \quad (1.2)$$

The generating function of $\mathbb{P}\{X > x\}$ is given by

$$\frac{1 + pz}{1 - qz - pqz^2}.$$

We now factor the denominator of this function to obtain

$$1 - qz - pqz^2 = (1 - az)(1 - bz)$$

with

$$a = \frac{q + \sqrt{q^2 + 4pq}}{2} \quad \text{and} \quad b = \frac{q - \sqrt{q^2 + 4pq}}{2}.$$

*This research was supported by the Austrian-Hungarian cooperation project 10U3.

Performing the partial fraction decomposition and extracting coefficients yields

$$\mathbb{P}\{X > x\} = \frac{1}{\sqrt{q^2 + 4pq}} (a^x(a+p) - b^x(b+p)).$$

Using (1.2) we obtain the expectation for the maximum lifetime of n players:

$$\mathbb{E}_n = \mathbb{E} \max\{X_1, \dots, X_n\} = \sum_{x \geq 0} \left(1 - \left(1 - \frac{1}{\sqrt{q^2 + 4pq}} (a^x(a+p) - b^x(b+p)) \right)^n \right). \quad (1.3)$$

By the binomial theorem we obtain

$$\mathbb{E}_n = \sum_{m=1}^n (-1)^{m-1} \binom{n}{m} \sum_{x \geq 0} (Aa^x - Bb^x)^m, \quad (1.4)$$

where we use the notation

$$A = \frac{a+p}{\sqrt{q^2 + 4pq}} = \frac{a^2}{q\sqrt{q^2 + 4pq}} \quad \text{and} \quad B = \frac{b+p}{\sqrt{q^2 + 4pq}} = \frac{b^2}{q\sqrt{q^2 + 4pq}}.$$

For example, in the symmetric case $p = q = \frac{1}{2}$, we have $a = \frac{1+\sqrt{5}}{4}$, $b = \frac{1-\sqrt{5}}{4}$, $A = \frac{5+3\sqrt{5}}{10}$, $B = \frac{5-3\sqrt{5}}{10}$.

We will find that $\mathbb{E}_n \sim \log_{1/a} n$ and refer for the (technical) proof and a more precise statement to the next section.

2. ASYMPTOTIC ANALYSIS

In (1.4) we found the expression

$$\mathbb{E}_n = \sum_{m=1}^n (-1)^{m-1} \binom{n}{m} f(m), \quad (2.1)$$

containing the function

$$f(z) = \sum_{x \geq 0} (Aa^x - Bb^x)^z \quad \text{for } \Re z > 0.$$

For an expression of that type we can write a complex contour integral

$$\mathbb{E}_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{(-1)^n n!}{z(z-1) \cdots (z-n)} f(z) dz, \quad (2.2)$$

where \mathcal{C} is a positively oriented Jordan curve encircling the points $1, 2, \dots, n$ (and no other integer points); this can easily be checked by residue calculus.

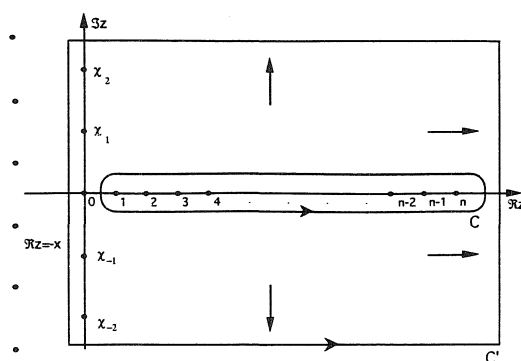
We will use Rice's method to obtain an asymptotic expansion for \mathbb{E}_n . For this we refer, e.g., to [2] and [6]. This method is based on a deformation of the contour of integration. For this purpose we need an analytic continuation of the function f to a region containing a half-plane $\Re z > -\varepsilon$ for $\varepsilon > 0$ (we actually give an analytic continuation to the whole complex plane).

Using the notation $C = B/A$ and $d = b/a$ (observe that $|C| < 1$ and $|d| < 1$) we obtain

$$\begin{aligned}
 f(z) &= A^z \sum_{x \geq 0} a^{xz} (1 - Cd^x)^z = A^z \sum_{x \geq 0} a^{xz} \sum_{\ell \geq 0} (-1)^\ell C^\ell d^{x\ell} \binom{z}{\ell} \\
 &= A^z \sum_{\ell \geq 0} (-1)^\ell C^\ell \binom{z}{\ell} \sum_{x \geq 0} (a^z d^\ell)^x = A^z \sum_{\ell \geq 0} \binom{z}{\ell} \frac{(-1)^\ell C^\ell}{1 - a^z d^\ell}
 \end{aligned} \tag{2.3}$$

where the reversion of the order of summation was justified because of the absolute convergence of the sum for $\Re z > 0$. The sum in the last line gives a valid expression for $f(z)$ for every complex number z which is not a solution of any of the equations $1 - a^z d^\ell = 0$. In the points $z_{\ell, x} = -\ell \frac{\log d}{\log a} + \frac{2x\pi i}{\log a}$ with $\ell = 0, 1, \dots$ and $x \in \mathbb{Z}$, there are simple poles with residue

$$A^{z_{\ell, x}} \binom{z_{\ell, x}}{\ell} \frac{(-1)^{\ell-1} C^\ell}{\log a}.$$



The Contours of Integration

In order to be able to deform the contour of integration, we need an estimate for $f(z)$ along the vertical line $\Re z = -u$. For this purpose, we write

$$f(z) - \frac{A^z}{1 - a^z} = \sum_{x \geq 0} A^z a^{xz} ((1 - Cd^x)^z - 1)$$

and observe the inequality $|(1 - Cd^x)^z - 1| \leq \min(2, |z|Cd^x)$. This yields

$$\left| f(z) - \frac{A^z}{1 - a^z} \right| \leq A^{-u} \left(\sum_{0 \leq x \leq \log|z|} 2|a|^{-xu} + |z| \sum_{x > \log|z|} a^{-xu} Cd^x \right) \ll |z|^\alpha \tag{2.4}$$

for $|d| < a^u < 1$ and $\alpha = -u \log a$.

We are now ready to start the deformation of the contour of integration: we take \mathcal{C}' as the new contour and write

$$\begin{aligned}
 &\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{(-1)^n n!}{z(z-1) \cdots (z-n)} f(z) dz \\
 &= \frac{1}{2\pi i} \oint_{\mathcal{C}'} \frac{(-1)^n n!}{z(z-1) \cdots (z-n)} f(z) dz - \sum_{z=z_i} \text{Res} \frac{(-1)^n n!}{z(z-1) \cdots (z-n)} f(z),
 \end{aligned} \tag{2.5}$$

Notice that there is a second-order pole at 0. Computation of residues yields (with $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$)

$$\begin{aligned} \operatorname{Res}_{z=0} \frac{(-1)^n n!}{z(z-1)\dots(z-n)} f(z) &= \frac{1}{\log a} H_n + \frac{\log A}{\log a} - \frac{1}{2}, \\ \operatorname{Res}_{z=\chi_x} \frac{(-1)^n n!}{z(z-1)\dots(z-n)} f(z) &= \frac{A^{\chi_x}}{\chi_x \log a} \frac{n! \Gamma(1-\chi_x)}{\Gamma(n+1-\chi_x)} \quad \text{for } x \neq 0, \end{aligned} \quad (2.6)$$

where $\chi_x = \frac{2x\pi i}{\log a} = z_{0,x}$.

Shifting the upper, the lower, and the right part of \mathcal{C}' (cf. the figure) to infinity and observing that the integrals over these parts of the contour vanish then yields

$$\begin{aligned} \mathbb{E}_n &= \frac{1}{\log \frac{1}{a}} H_n - \frac{\log A}{\log a} + \frac{1}{2} - \sum_{x \in \mathbb{Z} \setminus \{0\}} \frac{A^{\chi_x}}{\chi_x \log a} \frac{n! \Gamma(1-\chi_x)}{\Gamma(n+1-\chi_x)} \\ &\quad - \frac{1}{2\pi i} \int_{-u-i\infty}^{-u+i\infty} \frac{(-1)^n n!}{z(z-1)\dots(z-n)} f(z) dz. \end{aligned} \quad (2.7)$$

We now use the well-known asymptotic expansions

$$H_n = \log n + \gamma + O\left(\frac{1}{n}\right) \quad \text{and} \quad \frac{n!}{\Gamma(n+1-\chi_x)} = n^{\chi_x} \left(1 + O\left(\frac{x^2}{n}\right)\right)$$

(by Stirling's formula) to formulate our main result.

Theorem 1: The expected maximal lifetime \mathbb{E}_n of n independent players each of which has the Fibonacci distribution (or geometric distribution of order 2) fulfills, for $n \rightarrow \infty$,

$$\mathbb{E}_n = \log_{1/a} n - \frac{\gamma + \log A}{\log a} + \frac{1}{2} - \varphi(\log_{1/a} n) + O(n^{-u}), \quad (2.8)$$

for $0 < u < \min(1, \frac{\log |d|}{\log a})$, and φ denotes a continuous periodic function of period 1 and mean 0 given by the Fourier expansion

$$\varphi(t) = \frac{1}{\log a} \sum_{x \in \mathbb{Z} \setminus \{0\}} A^{\chi_x} \Gamma(-\chi_x) e^{2x\pi i t} = \frac{1}{\log a} \sum_{x \in \mathbb{Z} \setminus \{0\}} \Gamma(-\chi_x) e^{2x\pi i (t - \log_{1/a} A)}, \quad (2.9)$$

which is rapidly convergent due to the exponential decay of the Γ -function along vertical lines. The remainder term is obtained by a trivial estimate of the integral and the (uniform) O -terms in Stirling's formula.

3. EXTENSIONS

Here, we briefly sketch the more general case where k consecutive hits are necessary to kill a player. In this case, the probability $\mathbb{P}(X=x)$ was derived by Philippou and Muwafi [9] in terms of multinomial coefficients. As described in the introduction, there is a bijection to the sequences

$$\{0, 10, 110, \dots, 1^{k-1}0\} \cdot 1^k,$$

which yield the probability generating function

$$\frac{p^k z^k}{1 - qz - pqz^2 - \dots - p^{k-1} qz^k} = \frac{p^k z^k (1 - pz)}{1 - z + qp^k z^{k+1}} \quad (3.1)$$

for the lifetime of a player (cf. [1, pp. 299ff], [3, p. 428], [7, p. 207], [8]). Likewise, the generating function of $\mathbb{P}\{X > x\}$ is given by

$$\frac{1 - p^k z^k}{1 - z + qp^k z^{k+1}}. \quad (3.2)$$

Again we factor the polynomial in the denominator

$$1 - qz - pqz^2 - \dots - p^{k-1} qz^k = (1 - \alpha z)(1 - \alpha_2 z) \dots (1 - \alpha_k z)$$

with $|\alpha| > |\alpha_2| \geq \dots \geq |\alpha_k|$ ($\alpha > 0$). Then we have, by partial fraction decomposition and extracting coefficients,

$$\mathbb{P}\{X > x\} = A\alpha^x + A_2\alpha_2^x + \dots + A_k\alpha_k^x \quad (3.3)$$

with $A = \frac{\alpha(\alpha - p)}{q((k+1)\alpha - k)}$ and similar expressions for A_2, \dots, A_k .

For the expectation of the maximal lifetime of n players, we obtain

$$\mathbb{E}_{n,k} = \mathbb{E} \max\{X_1, \dots, X_n\} = \sum_{m=1}^n (-1)^{m-1} \binom{n}{m} g(m)$$

with

$$g(z) = \sum_{\ell \geq 0} (A\alpha^\ell + \dots + A_k\alpha_k^\ell) z^\ell \text{ for } \Re z > 0.$$

For the purpose of analytic continuation of g , we consider $g(z) - \frac{A^z}{1 - \alpha^z}$ and proceed as in (2.4) to obtain the continuation and a polynomial estimate for $g(z)$ along some vertical line $\Re z = -\varepsilon$ for sufficiently small $\varepsilon > 0$.

We are now ready to perform similar calculations as in Section 2. Thus, we obtain

Theorem 2: The expected maximal lifetime $\mathbb{E}_{n,k}$ of n players each of which has the geometric distribution of order k satisfies

$$\mathbb{E}_{n,k} = \log_{1/a} n - \frac{\gamma + \log A}{\log a} + \frac{1}{2} + \psi(\log_{1/a} n) + O(n^{-\varepsilon})$$

for $0 < \varepsilon < \min(1, \frac{\log|\alpha_2|}{\log \alpha})$ and a continuous periodic function ψ of period 1 and mean 0 whose Fourier expansion is given by

$$\psi(t) = \frac{1}{\log \alpha} \sum_{x \in \mathbb{Z} \setminus \{0\}} A^{\chi_x} \Gamma(-\chi_x) e^{2x\pi i t} = \frac{1}{\log \alpha} \sum_{x \in \mathbb{Z} \setminus \{0\}} \Gamma(-\chi_x) e^{2x\pi i (t - \log_{1/a} A)}$$

where $\chi_x = \frac{2x\pi i}{\log \alpha}$.

By *bootstrapping* we find that, for $k \rightarrow \infty$,

$$\alpha \sim 1 - qp^k + \dots \text{ and } A \sim 1 + kqp^k + \dots$$

REFERENCES

1. W. Feller. *An Introduction to Probability Theory and its Applications*. Vol. 1. New York: Wiley, 1957.
2. P. Flajolet R. Sedgewick. "Digital Search Trees Revisited." *SIAM J. Comput.* **15** (1986): 748-67.
3. N. L. Johnson, S. Kotz, & A. W. Kemp. *Discrete Univariate Distributions*. New York: Wiley, 1992.
4. A. W. Kemp. "Second-Order Linear Recurrence Distributions and Their Properties." Manuscript.
5. P. Kirschenhofer & H. Prodinger. "A Result in Order Statistics Related to Probabilistic Counting." *Computing* **51** (1993):15-27.
6. D. E. Knuth. *The Art of Computer Programming*. Vol. 3. Reading, Mass.: Addison-Wesley, 1973.
7. A. N. Philippou. "Distributions and Fibonacci Polynomials of Order k , Longest Runs, and Reliability of Consecutive- k out-of- n : F Systems." In *Fibonacci Numbers and Their Applications I* 2:203-227. Ed. A. N. Philippou, G. E. Bergum, & A. F. Horadam. Dordrecht, Reidel, 1986.
8. A. N. Philippou, C. Georgiou, & G. N. Philippou. "A Generalized Geometric Distribution and Some of Its Properties." *Stat. and Probab. Letters* **1** (1983):171-75.
9. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order k ." *The Fibonacci Quarterly* **20.1** (1982):28-32.
10. H. D. Shane. "A Fibonacci Probability Function." *The Fibonacci Quarterly* **11.5** (1973): 517-22.
11. W. Szpankowski & V. Rego. "Yet Another Application of a Binomial Recurrence: Order Statistics." *Computing* **43** (1990):401-10.
12. C. Tallie & G. P. Patil. "The Fibonacci Distribution Revisited." *Communications in Statistics—Theory and Methods* **15** (1986):951-59.
13. E. Zeckendorf. "Représentations des nombres naturels par une somme des nombres de Fibonacci ou des nombres de Lucas." *Bull. Soc. Royale Sci. Liège* **41** (1972):179-82.

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FIBONACCI, LUCAS AND CENTRAL FACTORIAL NUMBERS, AND π

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In [1], a solution of Problem B-705, the evaluation of the series $\sum_{n=1}^{\infty} a_{2n} / (n^2 \binom{2n}{n})$ for the Fibonacci numbers $a_n = F_n$ and the Lucas numbers $a_n = L_n$, proposed by H.-J. Seiffert, is given. The proof is essentially based on the power series expansion of $(\arcsin x)^2$. The same method yields, in the case $a_n = 1$, the Catalan-Apéry representation $\pi^2 = 18 \sum_{k=1}^{\infty} k^{-2} / \binom{2k}{k}$ (see [3]).

Now it is possible to deduce a more general formula by using the Taylor series expansion ([2])

$$\left(2 \arcsin \frac{x}{2}\right)^m = m! \sum_{k=m}^{\infty} \frac{|t(k, m)|}{k!} x^k, \quad |x| \leq 2, \quad m \in \mathbb{N}_0. \quad (1)$$

Here $t(k, m)$ denote the central factorial numbers of the first kind, which are defined by $(x^{[0]} := 1)$ (see [2], [4])

$$x^{[k]} := x \sum_{j=1}^{k-1} \left(x - \frac{k}{2} + j\right) = \sum_{m=0}^k t(k, m) x^m, \quad x \in \mathbb{R}.$$

Observing that $\arcsin(\alpha/2) = 3\pi/10$ and $\arcsin(\beta/2) = -\pi/10$, as well as Binet's formula $F_n = (\alpha^n - \beta^n) / \sqrt{5}$ and $L_n = \alpha^n + \beta^n$ [$\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$], and noting the representation (1), one can readily deduce

Theorem 1: For $m \in \mathbb{N}$, there hold

$$\begin{aligned} \pi^m &= \frac{5^m m! \sqrt{5}}{3^m + (-1)^{m+1}} \sum_{k=m}^{\infty} \frac{F_k |t(k, m)|}{k!}, \\ \pi^m &= \frac{5^m m!}{3^m + (-1)^m} \sum_{k=m}^{\infty} \frac{L_k |t(k, m)|}{k!}. \end{aligned} \quad (2)$$

The particular case $m = 1$ yields

$$\pi = \frac{5}{4} \sqrt{5} \sum_{k=0}^{\infty} \frac{F_{2k+1}}{2k+1} \frac{1}{16^k} \binom{2k}{k} = \frac{5}{2} \sum_{k=0}^{\infty} \frac{L_{2k+1}}{2k+1} \frac{1}{16^k} \binom{2k}{k}.$$

For $m = 2$, one obtains the solution of Problem B-705, and the case $m = 3$ in (2) gives, by formula (xxii) of [2],

$$\pi^3 = \frac{750}{7} \sqrt{5} \sum_{k=1}^{\infty} \frac{F_{2k+1}}{2k+1} \frac{1}{16^k} \binom{2k}{k} \sum_{j=1}^k \frac{1}{(2j-1)^2}.$$

Observe that, for large k , $\binom{2k}{k} / (16^k (2k+1)) \sim 4^{-k} k^{-3/2}$ (see [3]).

Vice versa, the Fibonacci and Lucas numbers can be expressed in terms of the central factorial numbers and π as follows.

Theorem 2: For $n \in \mathbb{N}$, there hold

$$F_n = \frac{(-1)^n n!}{\sqrt{5}} \sum_{k=n}^{\infty} i^{k+n} \frac{T(k, n)}{k!} \frac{\pi^k}{5^k} (3^k + (-1)^{k+1}),$$

$$L_n = (-1)^n n! \sum_{k=n}^{\infty} i^{k+n} \frac{T(k, n)}{k!} \frac{\pi^k}{5^k} (3^k + (-1)^k).$$

Here $T(k, n)$ are now the central factorial numbers of the second kind, given by

$$T(k, n) = \frac{1}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \left(\frac{n}{2} - j \right)^k, \quad n, k \in \mathbb{N}_0.$$

The central factorial numbers of the first and second kind are connected by the orthogonality formula $\sum_{k=0}^N t(n, k) T(k, m) = \delta_{n, m}$, $N := \max(n, m)$ (see [2]; [4], p. 213).

To prove Theorem 2, one inserts the values $x = 3\pi/5$ and $x = -\pi/5$ into the expansion (see [2])

$$\left(2 \sin \frac{x}{2} \right)^n = (-1)^n n! \sum_{k=n}^{\infty} i^{k+n} \frac{T(k, n)}{k!} x^k, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}_0,$$

and again uses the formula of Binet and $L_n = \alpha^n + \beta^n$, respectively.

REFERENCES

1. "Solution to Problem B-705: An Application of a Series Expansion for $(\arcsin x)^2$." (Proposed by H.-J. Seiffert.) *The Fibonacci Quarterly* **31** (1993):85-86.
2. P. L. Butzer, M. Schmidt, E. L. Stark, & L. Vogt. "Central Factorial Numbers: Their Main Properties and Some Applications." *Numer. Funct. Anal. Optim.* **10** (1989):419-88.
3. P. L. Butzer & M. Hauss. "Integral and Rapidly Converging Series Representations of the Dirichlet L -Functions $L_1(s)$ and $L_{-4}(s)$. *Atti Sem. Mat. Fis. Univ. Modena* **40** (1992):329-59.
4. J. Riordan. *An Introduction to Combinatorial Analysis*. New York: Wiley, 1958.

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CHARACTERIZING THE 2-ADIC ORDER OF THE LOGARITHM

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1. INTRODUCTION

We define $v_p(x)$ as the highest power of prime p which divides the integer x . The function $v_p(x)$ is often called the p -adic order of x . In this paper we characterize the divisibility by 2 of the series $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$ and $\sum_{k=1}^{\infty} \frac{x^k}{k}$, i.e., we determine their 2-adic orders. The characterization generalizes previously known results on 2-adic orders and is based on elementary proofs.

2. RESULTS

For an integer x , the p -adic order $v_p(x)$ of x is the highest power of prime p that divides x . We can think of the relations $p|x$ and $p \nmid x$ as $v_p(x) \geq 1$ and $v_p(x) = 0$, respectively.

We set $v_p(0) = \infty$ and $v_p(x/y) = v_p(x) - v_p(y)$ if both x and y are integers. Therefore, for all nonzero rational numbers, the order is defined to be a finite integer. From now on, all rational numbers will be meant in lowest terms.

For rational numbers a_k ($k \geq 0$) and rational x , the p -adic order, $v_p(\sum_{k=0}^{\infty} a_k x^k)$ of the series $\sum_{k=0}^{\infty} a_k x^k$ can be introduced as $\lim_{n \rightarrow \infty} v_p(\sum_{k=0}^n a_k x^k)$ if the limit exists, in which case there exists an n_0 such that $v_p(\sum_{k=0}^n a_k x^k) = v_p(\sum_{k=0}^{\infty} a_k x^k)$ for $n \geq n_0$. To illustrate this, we consider the series $\frac{x}{1-x} = x + x^2 + x^3 + \dots$. The reader can easily verify that $v_p(\frac{x}{1-x}) = v_p(x)$ if $v_p(x) \geq 1$ and the limit does not exist if $v_p(x) \leq 0$. Actually, $v_p(x + x^2 + x^3 + \dots + x^n) = nv_p(x)$ if $v_p(x) < 0$. Notice that if $v_2(x) = 0$ then $v_2(x + x^2 + x^3 + \dots + x^{2^{n+1}}) = 0$, while $v_2(x + x^2 + x^3 + \dots + x^{2^n}) \geq n$. Finding the p -adic order of functions helps in analyzing the divisibility property of the underlying or related functions. We note that Clarke [1] has recently studied the p -adic order of the logarithm by using p -adic arguments in order to characterize the divisibility properties of the *Stirling* and *partial Stirling numbers*. The interested reader should consult a book on p -adic metrics (e.g., [2]) for a general treatise of p -adic power series.

In this paper we consider the series $\log(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$ and $-\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ and determine their 2-adic orders by elementary arguments based on binomial expansion.

In most cases the p -adic order of $\log(1+x)$ can be derived by the well-known

Theorem A (Yu [4]): We have

$$v_p(\log(1+x)) = v_p\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}\right) = v_p(x) \text{ if } v_p(x) > \frac{1}{p-1},$$

and $v_p(\log(1+x))$ does not exist if $v_p(x) \leq 0$. In particular, for any integer x , $v_p(\log(1+x)) = v_p(x)$ if $p \geq 3$ and $p|x$, or if $p = 2$ and $4|x$, while for $p \nmid x$ the p -adic order $v_p(\log(1+x))$ does not exist.

In fact, Theorem A completely describes the p -adic order for $p \geq 3$. The purpose of this paper is to characterize the 2-adic orders of the two series in the case not covered by Theorem A, i.e., for every even integer x and $p = 2$. We note that the proof of Theorem A is based on the observation that under the conditions of Theorem A given for p and x , the p -adic order of the terms $(-1)^{k-1} \frac{x^k}{k}$, $k \geq 2$, of the infinite series $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$ exceeds that of the first term, x (cf. [2], p. 81).

If $p = 2$ and $x = 2$, then the following lemma (cf. [2], Ex. 7, p. 83) describes the 2-adic "behavior" of $\sum_{k=1}^n \frac{2^k}{k}$, i.e., the sum of the first n terms of the expansion $-\log(1-x)$.

Lemma B: The 2-adic order of the rational number $\sum_{k=1}^n \frac{2^k}{k}$ approaches infinity as n increases.

An elementary proof can be given based on the observation that

$$v_2\left(\sum_{k=n+1}^{\infty} \frac{2^k}{k}\right) \geq \min_{k \geq n+1} (k - v_2(k)),$$

which assures that $v_2(\sum_{k=n+1}^{\infty} \frac{2^k}{k})$ becomes arbitrarily large as $n \rightarrow \infty$. One can prove that

$$v_2\left(\sum_{k=1}^n \frac{2^k}{k}\right) \geq v_2\left(\sum_{k=n+1}^{\infty} \frac{2^k}{k}\right)$$

holds for infinitely many values n . In fact, a p -adic argument shows that equality holds for all n . We leave the details to the reader.

We set $v_p(\sum_{k=0}^n a_k x^k) = \infty$ if, for every integer $N \geq 1$, there exists an integer n_0 such that p^N divides $\sum_{k=0}^n a_k x^k$ for every $n \geq n_0$. In this case, $v_p(\sum_{k=0}^n a_k x^k) = v_p(\sum_{k=n+1}^{\infty} a_k x^k)$ holds. By the Lemma, we set $v_2(\sum_{k=1}^{\infty} \frac{2^k}{k}) = \infty$. We note that 0 and 2 play a special role in the 2-adic analysis of $\log(1-x)$ for these are the values for which $v_2(\log(1-x)) = \infty$ (cf. [2]). Our results are summarized in the following two theorems.

Theorem 1: For any even positive integer x ,

$$v_2\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}\right) = \begin{cases} 2, & \text{if } x = 2, \\ 2, & \text{if } x \equiv 2 \pmod{16}, \\ 2, & \text{if } x \equiv 4 \pmod{16}, \\ 3, & \text{if } x \equiv 6 \pmod{16}, \\ 3, & \text{if } x \equiv 8 \pmod{16}, \\ 2, & \text{if } x \equiv 10 \pmod{16}, \\ 2, & \text{if } x \equiv 12 \pmod{16}, \\ v_2(x+2), & \text{if } x \equiv 14 \pmod{16}, \\ v_2(x), & \text{if } x \equiv 0 \pmod{16}. \end{cases}$$

Theorem 2: For any even positive integer x ,

$$v_2\left(\sum_{k=1}^{\infty} \frac{x^k}{k}\right) = \begin{cases} \infty, & \text{if } x = 2, \\ v_2(x-2), & \text{if } x \equiv 2 \pmod{16}, \\ 2, & \text{if } x \equiv 4 \pmod{16}, \\ 2, & \text{if } x \equiv 6 \pmod{16}, \\ 3, & \text{if } x \equiv 8 \pmod{16}, \\ 3, & \text{if } x \equiv 10 \pmod{16}, \\ 2, & \text{if } x \equiv 12 \pmod{16}, \\ 2, & \text{if } x \equiv 14 \pmod{16}, \\ v_2(x), & \text{if } x \equiv 0 \pmod{16}. \end{cases}$$

Remark 1: The above theorems could be restated in a more compact form:

$$v_2\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}\right) = \begin{cases} v_2(x), & \text{if } x \equiv 0, 4, 8, 12 \pmod{16}, \\ v_2(x+2), & \text{if } x \equiv 2, 6, 10, 14 \pmod{16}, \end{cases}$$

and

$$v_2\left(\sum_{k=1}^{\infty} \frac{x^k}{k}\right) = \begin{cases} v_2(x), & \text{if } x \equiv 0, 4, 8, 12 \pmod{16}, \\ v_2(x-2), & \text{if } x \equiv 2, 6, 10, 14 \pmod{16}. \end{cases}$$

Notice the sharp contrast between $v_2(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^k}{k})$ and $v_2(\sum_{k=1}^{\infty} \frac{2^k}{k})$. We can combine the cases $x \neq 2$ of the two theorems by substituting $-x$ in place of x and carrying out the modular calculations.

For a rational $x = a/b$ with $v_2(x) = 1$ and $b > 1$, there exists a sufficiently large integer m such that $v_2(\log(1+x)) < m$. We set $x' = a * b^{-1}$, where b^{-1} is the unique solution to the equation $b * b^{-1} \equiv 1 \pmod{2^m}$ with $0 < b^{-1} < 2^m$. We can proceed to determine $v_2(\log(1+x'))$ by Theorem 1 and observing that $v_2(\log(1+x)) = v_2(\log(1+x'))$. If $x' \not\equiv 14 \pmod{16}$, then $m = 4$ is an appropriate choice. However, if it turns out that the remainder is 14, then one should check whether $v_2(x'+2) < m$ and try a larger m if it fails. A similar method works for determining $v_2(\log(1-x))$, too.

For example, if $x = 6/5$, then $v_2(\log(1-6/5)) = 2$ follows easily with $m = 4$. We use $m = 5$ and have $x' = 6 * 13 \equiv 14 \pmod{16}$ in order to obtain $v_2(\log(1+6/5)) = v_2(6 * 13 + 2) = 4$. For $x = 426/555$, we start with $m = 4$. Since $x' = 426 * 3 \equiv 14 \pmod{16}$ and $v_2(426 * 3 + 2) = 8$, we note that we need a larger m . By using $m = 10$, we obtain $x' = 426 * 131 \equiv 14 \pmod{16}$ and $v_2(\log(1+426/555)) = v_2(426 * 131 + 2) = 9$.

Remark 2: Similarly to the proof of Theorem A, we observe that $v_2(2^s) < v_2((2^s)^k / k)$ if $k \geq 2$ and $s \geq 2$. Therefore,

$$v_2\left(\sum_{k=1}^{\infty} \frac{(2^s)^k}{k}\right) = v_2\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{(2^s)^k}{k}\right) = v_2(2^s) = s \text{ if } s \geq 2.$$

3. PROOFS

Proof of Theorem 1: The case of $x = 2$ is easily verified by checking the first couple of terms of $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$. Indeed, $v_2(\sum_{k=1}^4 (-1)^{k-1} \frac{2^k}{k}) = 2$ and $v_2(2^k/k) > 2$ for $k \geq 5$.

If $x = 6$ or 10 , then by inspecting the sum of the first few terms we obtain, similarly to the case of $x = 2$, that the orders are 3 and 2, respectively.

We can extend these results for $x \equiv 2, 6, \text{ and } 10 \pmod{16}$. From now on a denotes an arbitrary integer while b is an arbitrary odd integer. The basic idea is that if $v_2(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}) = r < s$ then $v_2(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x+a2^s)^k}{k}) = r$, too, since $x^k \equiv (x+a2^s)^k \pmod{2^s}$. [Of course, the same applies if we omit the factors $(-1)^{k-1}$.] By the previous observations, we can set $s = 4$.

For $x \equiv 0, 4, 8, \text{ or } 12 \pmod{16}$, the statement follows from Theorem A which claims that the order must be $v_2(x)$.

Instead of simply proving the remaining case $x \equiv 14 \pmod{16}$, we combine the cases $x \equiv 2$ and $14 \pmod{16}$ to make this proof transparent to prove Theorem 2. Let $s = 4$. We calculate the 2-adic order of $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$ using the binomial expansion of the terms $x^k = (b2^s + 2c)^k$ where c is either 1 or -1 . The expansion yields

$$(b2^s + 2c)^k = (2(b2^{s-1} + c))^k = \sum_{\ell=0}^k 2^k \binom{k}{\ell} (b2^{s-1})^{\ell} c^{k-\ell}.$$

Note that the identity $\binom{k}{\ell} = \frac{k}{\ell} \binom{k-1}{\ell-1}$ implies that $\binom{k}{\ell}/k$ is an integer multiple of $1/\ell$. Consider the sum

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{(b2^s + 2c)^k}{k}$$

in three terms, one term for $\ell = 0$, another for $\ell = 1$, and the last one for all the remaining cases, $\ell \geq 2$. We get

$$\begin{aligned} & \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(b2^s + 2c)^k}{k} \\ &= -\sum_{k=1}^{\infty} \frac{(-2c)^k}{k} + \sum_{k=1}^{\infty} b2^{k+s-1} (-c)^{k-1} + \sum_{k=1}^{\infty} (-1)^{k-1} \sum_{\ell=2}^k \frac{\binom{k-1}{\ell-1} b^{\ell} 2^{\ell(s-1)+k} c^{k-\ell}}{\ell}. \end{aligned} \quad (1)$$

Obviously, the 2-adic order of the second term is s if $b \neq 0$. Notice that the third term is always divisible by 2^{s+1} for $s \geq 3$, since this condition implies that $\ell(s-1) + k - v_2(\ell) \geq \ell(s-1) + k - \log_2 \ell \geq s+1$. It turns out that the 2-adic order of the first term on the right side of identity (1) is 2 if $c = 1$ as we have seen it at the beginning of the proof. By Lemma B, the 2-adic order of the first term is ∞ if $c = -1$. It follows that $v_2(\sum_{k=1}^{\infty} (-1)^{k-1} (b2^s + 2c)^k / k) = s$ if $c = -1$ (and $b \neq 0$), while it is 2 if $c = 1$. \square

Proof of Theorem 2: Basically, the proof of Theorem 1 can be repeated here except for $x = 2$, which case is the content of Lemma B. Careful inspection reveals that the 2-adic orders are switched for $x \equiv 6$ and $10 \pmod{16}$.

Similarly to identity (1), we have

$$\sum_{k=1}^{\infty} \frac{(b2^s + 2c)^k}{k} = \sum_{k=1}^{\infty} \frac{(2c)^k}{k} + \sum_{k=1}^{\infty} b2^{k+s-1} c^{k-1} + \sum_{k=1}^{\infty} \sum_{\ell=2}^k \frac{\binom{k-1}{\ell-1} b^{\ell} 2^{\ell(s-1)+k} c^{k-\ell}}{\ell}, \quad (2)$$

where the last term is always divisible by 2^{s+1} for $s \geq 3$.

By simply switching the cases $c = 1$ and $c = -1$ in the previous proof and using identity (2), we derive that $v_2(\sum_{k=1}^{\infty} \frac{(b2^s + 2c)^k}{k}) = s$ if $c = 1$ (and $b \neq 0$), while it is 2 if $c = -1$. \square

We note that Clarke [1] has recently proved similar results by using p -adic arguments.

Lemma B points to the odd behavior of $v_2(\sum_{k=1}^n \frac{x^k}{k})$ at $x = 2$. Analysis of this behavior gives rise to the question on the rate at which $v_2(\sum_{k=1}^n \frac{2^k}{k})$ increases as n gets larger. We were unable to answer this question; however, numerical evidence suggests some pattern for the increase of the 2-adic order. The following conjecture has been proposed in [3], in the context of the divisibility by 2 of the Stirling numbers of the second kind, $S(a2^n - 1, 2^m)$, where $n > m \geq 4$ and a is a positive integer.

Conjecture 3: For $m \geq 4$, $v_2(\sum_{k=1}^{2^m} \frac{2^k}{k}) = 2^m + 2m - 2$.

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REFERENCES

1. F. Clarke. "Hensel's Lemma and the Divisibility by Primes of Stirling-Like Numbers." Pre-print, 1993.
2. N. Koblitz. *p-adic Numbers, p-adic Analysis, and Zeta-Functions*. 2nd ed. New York: Springer-Verlag, 1984.
3. T. Lengyel. "On the Divisibility by 2 of the Stirling Numbers of the Second Kind." (To appear in *The Fibonacci Quarterly*.)
4. K. Yu. "Linear-Forms in p -adic Logarithms." *Acta Arithmetica* **53** (1989):107-86.

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NUMBER OF MULTINOMIAL COEFFICIENTS NOT DIVISIBLE BY A PRIME

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We consider the n^{th} row of multinomial coefficients of the order ℓ :

$$(j_1, j_2, \dots, j_\ell) = \frac{n!}{j_1! j_2! \dots j_\ell!},$$

where $j_i \geq 0, i = 1, \dots, \ell$, and $n = j_1 + j_2 + \dots + j_\ell$.

The number of multinomial coefficients not divisible by p^N , where p is prime and N is a fixed whole integer, for various ℓ, p , and N was studied by L. Carlitz [1], [2], F. T. Howard [3], [4], [5], [12], R. J. Martin and G. L. Mullen [6], and the author [7]. Let $g(n, \ell, p^N)$ be a number of multinomial coefficients in the n^{th} row of order ℓ not divisible by p^N , and

$$G(n, \ell, p^N) = \sum_{k=1}^{n-1} g(k, \ell, p^N).$$

In the general case, an exact formula for $g(n, \ell, p^N)$ was obtained by F. T. Howard [5] for $N = 1, 2, 3$ and by the author [7] for $N = 1, 2$. It is not difficult to show that the behavior of $g(n, \ell, p^N)$ is very irregular and from that point of view it is better to study $G(n, \ell, p^N)$ which changes more regularly. The function $G(n, \ell, p^N)$ was studied by K. B. Stolarsky [8], [9] and H. Harborth [10] for $N = 1, \ell = p = 2$; by A. H. Stein [11] for $N = 1, \ell = 2$, and arbitrary p ; and by the author [7] for arbitrary ℓ and p .

More precisely, the following exact formula was obtained in [7]:

$$G(n, \ell, p) = \sum_{k=0}^m (\ell, p-1)^k \frac{a_k}{\ell} \prod_{i=k}^m (\ell-1, a_i), \quad (1)$$

where $n-1 = a_0 + a_1 p + \dots + a_m p^m$. It is not difficult to show that $G(n, \ell, p)$ is of the order n^θ , where $\theta = \log_p(\ell, p-1)$. The following theorem gives a more exact result.

Theorem 1: $\alpha \equiv \limsup_{n \rightarrow \infty} G(n, \ell, p) / n^\theta = 1$.

Unfortunately, there are no similar results for $\beta \equiv \liminf_{n \rightarrow \infty} G(n, \ell, p) / n^\theta$ even in particular cases.

In the general case, only the following elementary estimate is known: $\beta \geq (\ell, p-1)^{-1}$.

In the particular case $p = 2$ (following H. Harborth [10]), we are able to prove the following result.

* This research was undertaken while the author was in the Statistical Department of the University of Melbourne.

Theorem 2: If we consider the sequence $q_r = G(n_r, \ell, 2) / n_r^\theta$ with $n_r = 2n_{r-1} \pm 1, n_0 = 1$, where $+$ or $-$ is chosen so that q_r becomes minimal, then $\{q_r\}$ is strictly decreasing.

This theorem is a generalization of the lemma from [10] for the case of binomial coefficients to the case of multinomial coefficients. We should also note that the sequence $\{n_r\}$ is not the same for different ℓ . In Table 1 the values of n_r for various r and ℓ are given.

TABLE 1

ℓ	r											
	1	2	3	4	5	6	7	8	9	10	15	30
2	3	5	11	21	43	87	173	347	693	1387	44395	1454730075
3	3	7	13	27	53	107	215	429	859	1719	54999	1802202477
4	3	7	13	27	55	109	219	439	877	1755	56171	1840625371
5	3	7	13	27	55	109	219	439	877	1755	56173	1840700855
10	3	7	13	27	55	111	221	443	887	1775	56795	1861082589

In Table 2 we give values of n_r and $q_r = G(n_r, 2, 2) / n_r^\theta$.

TABLE 2

r	n_r	q_r	r	n_r	q_r
1	1	1.000000	26	45460315	0.812556563402
2	3	0.876497	27	90920629	0.812556561634
3	5	0.858126	28	181841259	0.812556559863
4	11	0.827243	29	363682519	0.812556559862
5	21	0.826359	30	727365037	0.812556559272
6	43	0.816719	31	1454730075	0.812556559174
7	87	0.815382	32	2909460149	0.812556559092
8	173	0.813788222	33	5818920299	0.8125565590457850017
9	347	0.813086063	34	11637840597	0.8125565590398820396
10	693	0.812934013	35	23275681195	0.8125565590234059925
11	1387	0.812675296	36	46551362391	0.8125565590216437317
12	2775	0.812657623	37	93102724781	0.8125565590182076960
13	5549	0.812592041	38	186205449563	0.8125565590170475496
14	11099	0.812575228	39	372410899125	0.8125565590166681715
15	22197	0.812567096	40	744821798251	0.8125565590162182798
16	44395	0.812560137	41	1489643596503	0.8125565590162065045
17	88789	0.812559941	42	2979287193005	0.8125565590160702999
18	177579	0.812557589	43	5958574386011	0.8125565590160436690
19	355159	0.812557229	44	11917148772021	0.8125565590160253147
20	710317	0.812556865	45	23834297544043	0.8125565590160134328
21	1420635	0.812556846	46	47668595088085	0.8125565590160123562
22	2841269	0.812556653	47	95337190176171	0.8125565590160082524
23	5682539	0.812556588	48	190674380352343	0.8125565590160076856
24	11365079	0.812556582	49	381348760704685	0.8125565590160069672
25	22730157	0.812556563	50	762697521409371	0.8125565590160066187

On the other hand, if we consider the case $\ell = 2, p = 3, 5, 7$, then there exist increasing sequences $\{n_r\}$ such that $G(n_r, 2, p) / n_r^\theta < G(n_{r-1}, 2, p) / n_{r-1}^\theta$. Calculations give us the following sequences:

$$\begin{aligned} n_0 &= 0, \quad n_r = 3n_{r-1} + 1, \quad \text{for } p = 3, \\ n_0 &= 0, \quad n_r = 5n_{r-1} + 2, \quad \text{for } p = 5, \\ n_0 &= 0, \quad n_r = 7n_{r-1} + 3, \quad \text{for } p = 7. \end{aligned}$$

If we denote $\beta_p = \liminf_{n \rightarrow \infty} G(n, 2, p) / n^\theta$ and $\hat{\beta}_p = \liminf_{r \rightarrow \infty} G(n_r, 2, p) / n_r^\theta$, then

$$\left. \begin{aligned} \hat{\beta}_3 &= 2^{\log_3 2 - 1} = 0.774281326315 \\ \hat{\beta}_5 &= 2^{\log_5 3 - 1} = 0.802518299262 \\ \hat{\beta}_7 &= 2^{\log_7 4 - 1} = 0.819271977267 \end{aligned} \right\} \quad (*)$$

Very probably $\beta_p = \hat{\beta}_p$, but at the present time we do not have complete proof of this fact. For that purpose it is necessary to show that the sequences $\{n_r\}$ which were defined earlier have the following property: $G(n_r, 2, p) / n_r^\theta < \min_{n_{r-1} < n < n_r} G(n, 2, p) / n^\theta$, $r = 1, 2, \dots$, for $p = 2, 3, 5$, and 7 .

Proof of Theorem 1: It follows from (1) that

$$G(p^m, \ell, p) / p^{m\theta} = (\ell, p-1)^m / p^{m\theta} = 1 \quad (2)$$

for all m , which gives us $\alpha \geq 1$.

Furthermore, we will show that

$$b_i \equiv G(ip^m, \ell, p) / (ip^m)^\theta \leq 1, \quad \text{when } 1 \leq i \leq p. \quad (3)$$

For this purpose, we consider the fraction $b_i / b_{i+1} \equiv c_i$ which, due to (1), is

$$c_i = \left(\frac{i+1}{i} \right)^\theta \left(\frac{i}{\ell+1} \right),$$

and we shall show that $c_1 = 2^\theta / (\ell+1) \geq 1$ or, in other words, that

$$(\ell, p-1) > (\ell+1)^{\log_2 p}. \quad (4)$$

Since

$$\frac{(\ell, p-1)(\ell+2)^{\log_2 p}}{(\ell+1)^{\log_2 p}(\ell+1, p-1)} = \frac{\ell+1}{\ell+p} \left(1 + \frac{1}{\ell+1} \right)^{\log_2 p},$$

we consider, under $t \geq 3$, the function

$$\varphi(p, t) = \frac{t}{t+p-1} \left(1 + \frac{1}{t} \right)^{\log_2 p}$$

as a function of p and, taking the derivitave, we find that

$$\varphi(p, t) = \frac{\varphi(p, t)}{p \ln 2} \left[\ln \left(1 + \frac{1}{t} \right) - \frac{p \ln 2}{t+p-1} \right] < \frac{\varphi(p, t)}{tp \ln 2} \left(1 - \frac{tp \ln 2}{t+p-1} \right) < 0$$

because $tp / (t+p-1)$ is increased either by p or by t , and

$$\left. \frac{tp \ln 2}{t+p-1} \right|_{t=3, p=2} = \frac{3}{2} \ln 2 > 1.$$

Hence, $\varphi(p, \ell + 1) < \varphi(2, \ell + 1) = 1$, and $(\ell, p - 1) / (\ell + 1)^{\log_2 p}$ is decreased in ℓ . So

$$(\ell, p - 1) / (\ell + 1)^{\log_2 p} < (3, p - 1) / p^2 \leq 1, \text{ when } \ell \geq 3,$$

which proves (4).

As the derivative of the function

$$\psi(x) = \left(\frac{x+1}{x} \right)^\theta \frac{x}{\ell+x}$$

is equal to

$$\psi(x) = \left(\frac{x+1}{x} \right)^\theta \frac{\ell(x+1) - \theta(\ell+x)}{(x+1)(\ell+x)^2},$$

then $\psi(x)$ has only one extreme and, as $c_1 > 1$, this extreme is the minimum. As $b_1 = b_p = 1$ for $2 \leq i \leq p-1$, we have $b_i \leq 1$.

From (1) it is easy to prove, for $0 \leq x \leq p^m$, that the following recurrent formula is valid:

$$G(a_m p^m + x, \ell, p) = G(a_m p^m, \ell, p) + (\ell - 1, a_m) G(x, \ell, p), \quad (5)$$

where $1 \leq a_m \leq p-1$. We show that

$$G(a_m p^m + x, \ell, p) / (a_m p^m + x)^\theta \leq 1 \text{ for all } x = 0, \dots, p^m - 1, m = 0, 1, \dots \quad (6)$$

is valid.

The inequality (6) is evident when $m = 0$. Let us suppose that (6) is valid in the case of all positive numbers less than m . Then we will have $G(x, \ell, p) \leq x^\theta$ for $0 \leq x \leq p^m$. Then, from (3) and (5), we have

$$\begin{aligned} G(a_m p^m + x, \ell, p) / (a_m p^m + x)^\theta &= [G(a_m p^m, \ell, p) + (\ell - 1, a_m) G(x, \ell, p)] / (a_m p^m + x)^\theta \\ &\leq [(a_m p^m)^\theta + (\ell - 1, a_m)^\theta] / (a_m p^m + x)^\theta \equiv f(x), \quad 0 \leq x < p^m. \end{aligned} \quad (7)$$

In the interval $[0, p^m]$ the function $f(x)$ has only one extreme, which is the minimum. So (5) is valid. From (3) and (6), we have $\alpha \leq 1$ and, when (2) is added, $\alpha = 1$. Theorem 1 is proved.

Proof of Theorem 2: We suppose

$$G(2n_r + 1, \ell, 1) / (2n_r + 1)^\theta \geq q_r \text{ and } G(2n_r - 1, \ell, 1) / (2n_r - 1)^\theta \geq q_r. \quad (8)$$

If we denote $a = 2n_r$ and $b = \ell^{t_r} / ((\ell + 1)G(n_r, \ell, 2))$, then, from the definition of q_r and assumptions (7) and (8), we have

$$1 + b \geq \left(1 + \frac{1}{a} \right)^\theta \text{ and } 1 - b \geq \left(1 - \frac{1}{a} \right)^\theta.$$

Addition of these two inequalities yields the contradiction $2 \geq 2 + \theta(\theta - 1) / a^2 + \dots > 2$. Thus, the inequalities (7) cannot both be true, which proves that the sequence $\{q_r\}$ is strictly decreasing. Theorem 2 is proved.

Returning to formulas (*), it is necessary to note that calculations for $p = 3, 5, 7$ are very simple, by using (1). We omit the proof.

REFERENCES

1. L. Carlitz. "The Number of Binomial Coefficients Divisible by a Fixed Power of a Prime." *Rend. Circ. Mat. Palermo* **16** (1967):299-320.
2. L. Carlitz. "Distribution of Binomial Coefficients." *Riv. Mat. Univ. Parma* **11** (1970):45-64.
3. F. T. Howard. "The Number of Binomial Coefficients Divisible by a Fixed Power of 2." *Proc. Amer. Math. Soc.* **29** (1971):236-42.
4. F. T. Howard. "The Number of Binomial Coefficients Divisible by a Fixed Power of a Prime." *Proc. Amer. Math. Soc.* **37** (1973):358-62.
5. F. T. Howard. "The Number of Multinomial Coefficients Divisible by a Fixed Power of a Prime." *Pacific J. Math.* **50** (1974):99-108.
6. R. J. Martin & G. L. Mullen. "Reducing Multinomial Coefficients Modulo a Prime Power." *Computers and Mathematics with Applications* **10** (1984):37-41.
7. N. A. Volodin. "Distribution of Polynomial Coefficients Congruent Modulo p^N ." *Mathematical Notes* **45** (1989):195-99.
8. K. B. Stolarsky. "Digital Sums and Binomial Coefficients." *Notices Amer. Math. Soc.* **22** (1975):A-669.
9. K. B. Stolarsky. "Power and Exponential Sums of Digital Sums Related to Binomial Coefficient Parity." *SIAM J. Appl. Math.* **32** (1977):717-30.
10. H. Harborth. "Number of Odd Coefficients." *Proc. Amer. Math. Soc.* **62** (1977):19-22.
11. A. H. Stein. "Binomial Coefficients Not Divisible by a Prime." *Number Theory*, (New York, 1985/1988), *Lecture Notes in Mathematics* **1383**:170-77 (Berlin-New York: Springer, 1989).
12. F. T. Howard. "Multinomial and Q -Binomial Coefficients Modulo 4 and Modulo p ." *The Fibonacci Quarterly* **31.1** (1993):53-64.

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A NOTE ON BROWN AND SHIUE'S PAPER ON A REMARK RELATED TO THE FROBENIUS PROBLEM

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Given relatively prime positive integers a, b , let NR denote the set of positive integers with no representation by the linear form $ax + by$ in nonnegative integers x, y . It is well known that the set NR is finite. For a nonnegative integer m , we put

$$S_m(a, b) = \sum_{n \in \text{NR}} n^m.$$

Sylvester [3] showed that $\#\text{NR} = S_0(a, b) = \frac{1}{2}(a-1)(b-1)$ and, recently, Brown and Shiue [1] found a similar closed form for $S_1(a, b)$. Brown and Shiue did this by determining a closed form for the generating function $f(x)$ of the characteristic function of the set NR and then computing $f'(1) = S_1(a, b)$. In this note we use a more direct approach, which gives us a closed form for $S_m(a, b)$ valid for every nonnegative integer m .

Let integers n, r, s be connected by the relations

$$r \equiv n \pmod{a}, \quad 0 \leq r < a; \quad bs \equiv r \pmod{a}, \quad 0 \leq s < a.$$

We have that $n \in \text{NR}$ if and only if $n = -at + bs$ for some integer t in the interval $1 \leq t \leq \lfloor bs/a \rfloor$, that is, if and only if $n = ak + r$ for some integer k in the interval $0 \leq k \leq (bs-r)/a-1$. Hence,

$$S_m(a, b) = \sum_{r=0}^{a-1} \sum_{k=0}^{\frac{bs-r}{a}-1} (ak+r)^m.$$

For the exponential generating function of the sequence $\{S_m\}$, this gives

$$\begin{aligned} \sum_{m=0}^{\infty} S_m(a, b) \frac{z^m}{m!} &= \sum_{r=0}^{a-1} \sum_{k=0}^{\frac{bs-r}{a}-1} \sum_{m=0}^{\infty} (ak+r)^m \frac{z^m}{m!} \\ &= \sum_{r=0}^{a-1} \sum_{k=0}^{\frac{bs-r}{a}-1} e^{(ak+r)z} = \frac{1}{e^{az} - 1} \left(\sum_{r=0}^{a-1} e^{bsz} - \sum_{r=0}^{a-1} e^{rz} \right). \end{aligned}$$

As r runs through the set $\{0, 1, \dots, a-1\}$, so does s . Hence,

$$\sum_{r=0}^{a-1} e^{bsz} = \sum_{s=0}^{a-1} e^{bsz},$$

and we find that

$$\sum_{m=0}^{\infty} S_m(a, b) \frac{z^m}{m!} = \frac{e^{abz} - 1}{(e^{az} - 1)(e^{bz} - 1)} - \frac{1}{e^z - 1}.$$

Multiplying this relation by z gives

$$\sum_{m=1}^{\infty} m S_{m-1}(a, b) \frac{z^m}{m!} = \sum_{i=0}^{\infty} B_i a^i \frac{z^i}{i!} \sum_{j=0}^{\infty} B_j b^j \frac{z^j}{j!} \sum_{k=0}^{\infty} \frac{a^k b^k}{k+1} \cdot \frac{z^k}{k!} - \sum_{m=0}^{\infty} B_m \frac{z^m}{m!},$$

where $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, ... are the Bernoulli numbers; cf. formula (6.81) and section 7.6 in [2]. Equating coefficients of z^m now gives the

Theorem: For $m = 1, 2, \dots$, we have

$$S_{m-1}(a, b) = \frac{1}{m(m+1)} \sum_{i=0}^m \sum_{j=0}^{m-i} \binom{m+1}{i} \binom{m+1-i}{j} B_i B_j a^{m-j} b^{m-i} - \frac{1}{m} B_m.$$

It is not difficult to see that, considered as a polynomial in a and b , $S_m(a, b)$ has the algebraic factor $(a-1)(b-1)$. In addition, if m is even ≥ 2 , then $S_m(a, b)$ also has the factor $ab(ab-a-b)$.

Our theorem gives us, of course, Sylvester's result for S_0 and Brown and Shiue's formula [1],

$$S_1(a, b) = \frac{1}{12} (a-1)(b-1)(2ab-a-b-1).$$

Also, for S_2 , we obtain a rather simple formula:

$$S_2(a, b) = \frac{1}{12} (a-1)(b-1)ab(ab-a-b).$$

REFERENCES

1. T. C. Brown & P. J.-S. Shiue. "A Remark Related to the Frobenius Problem." *The Fibonacci Quarterly* **31.1** (1993):32-36.
2. R. L. Graham, D. E. Knuth, & O. Patashnik. *Concrete Mathematics*. Reading, Mass.: Addison-Wesley, 1990.
3. J. J. Sylvester. "Mathematical Questions with Their Solutions." *Educational Times* **41** (1884):21.

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AN ALTERNATIVE PROOF OF A UNIQUE REPRESENTATION THEOREM

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This note describes an alternative approach to the proof in [2] of a representation theorem involving negatively subscripted Pell numbers P_{-n} ($n > 0$), namely,

Theorem: The representation of any integer N as

$$N = \sum_{i=1}^{\infty} a_i P_{-i} \quad (1)$$

where $a_i = 0, 1, 2$ and $a_i = 2 \Rightarrow a_{i+1} = 0$, is unique and minimal.

To conserve space and avoid unnecessary repetition, we assume that the notation and results in [2] will be familiar to the reader. Our alternative treatment, however, requires the fresh result:

$$2 \sum_{i=1}^{n-1} (-1)^{i+1} P_{-i} = -1 + (-1)^n (P_{-n} + P_{-n-1}). \quad (2)$$

Repeated use of the recurrence relation for P_{-n} leads to (2). Observe [2] that in (2)

$$q_{-n} = P_{-n} + P_{-n-1} \quad (q_{-1} = -1, q_0 = 1, q_1 = 1). \quad (3)$$

Proof of the Theorem: Suppose there are two different representations

$$N = \sum_{i=1}^h a_i P_{-i}, \quad a_h \neq 0, a_i = 2 \Rightarrow a_{i+1} = 0 \quad (a_i = 0, 1, 2) \quad (4)$$

and

$$N = \sum_{i=1}^m b_i P_{-i}, \quad b_m \neq 0, b_i = 2 \Rightarrow b_{i+1} = 0 \quad (b_i = 0, 1, 2). \quad (5)$$

Case I. Assume $h = m$, so that the Pell numbers in (4) and (5) are the same, but the coefficients a_i, b_i are generally different. Write

$$c_i = a_i - b_i \quad (c_i = 0, \pm 1, \pm 2; i = 1, 2, \dots, m). \quad (6)$$

Subtract (5) from (4) to derive

$$\sum_{i=1}^m c_i P_{-i} = 0 \quad \text{by (6),} \quad (7)$$

that is,

$$c_m P_{-m} + \sum_{i=1}^{m-1} c_i P_{-i} = 0, \quad (8)$$

whence, by (2), for a maximum or minimum sum, i.e., $c_i = \pm 2$ ($i = 1, 2, \dots, m-1$),

$$c_m P_{-m} + (-1)^m (P_{-m} + P_{-m-1}) = 1. \quad (9)$$

[The notation of (3) may be used in (9).] We concentrate on $c_m P_{-m}$ since this term dominates the sums (7)-(9).

m even ($P_{-m} < 0$): Here (9) gives

$$(c_m + 1)P_{-m} + P_{-m-1} = 1. \quad (9a)$$

Now, in (9a),

$$\begin{aligned} \text{(i)} \quad c_m = 0 &\Rightarrow q_{-m} = 1 && \text{by (3)} \\ \text{(ii)} \quad c_m = 1 &\Rightarrow P_{-m+1} = 1 \\ \text{(iii)} \quad c_m = 2 &\Rightarrow q_{-m+1} = 1 && \text{by (3)} \end{aligned}$$

where in (ii) and (iii) the recurrence relation for Pell numbers [2] has been invoked.

m odd ($P_{-m} > 0$): Here (9) gives

$$(c_m - 1)P_{-m} - P_{-m-1} = 1. \quad (9b)$$

Next, in (9b),

$$\begin{aligned} \text{(iv)} \quad c_m = 0 &\Rightarrow -q_{-m} = 1 && \text{by (3)} \\ \text{(v)} \quad c_m = 1 &\Rightarrow -P_{-m-1} = 1 \\ \text{(vi)} \quad c_m = 2 &\Rightarrow P_{-m} - P_{-m-1} = 1. \end{aligned}$$

All the equations (i)-(vi) involve contradictions. Of these, perhaps (ii) is the least obvious. Let us therefore examine (ii), which is true for $m = 2$ (even) leading to $c_2 = 1, c_1 = 2$ from (ii) and (8). Now $c_2 = 1 = a_2 - b_2$ implies that $a_2 = 2$ ($b_2 = 1$) or $a_2 = 1$ ($b_2 = 0$), i.e., $a_2 \neq 0$, which contradicts $c_1 = 2 = a_1 - b_1$ since this means that $a_1 = 2$ ($b_1 = 0$) and, hence, $a_1 = 2 \Rightarrow a_2 = 0$ by (1). Thus, (i)-(vi) and, ultimately, (7) are impossible.

Similar reasoning applies when $c_m = -1, -2$. Consequently, the assumption in Case 1 is invalid.

Summary of Case I Results: If $h = m$, then $a_i = b_i$ ($i = 1, \dots, m$), i.e., the representations (4) and (5) are identical, so that the representation (4), or (1), is unique.

Case II: Assume $h > m$. Then four subcases exist, depending on the parity of h and m . From [2], with n standing for h and m , in turn,

$$-P_{-n} < N \leq -P_{-n-1} \quad n \text{ odd} \quad (10)$$

and

$$-P_{-n-1} < N \leq -P_{-n} \quad n \text{ even.} \quad (11)$$

These restrictions impose a range of values upon N for each integer $n > 0$, for example [2],

$$\begin{aligned} n = 1: & \quad 0 \leq N \leq 2 \\ n = 2: & \quad -4 \leq N \leq 2 \\ n = 3: & \quad -4 \leq N \leq 12 \\ n = 4: & \quad -28 \leq N \leq 12 \\ n = 5: & \quad -28 \leq N \leq 70, \end{aligned} \quad (12)$$

the number of integers [= sums (1)] being 3, 7, 17, 41, 99, in turn, which equal q_2, q_3, q_4, q_5, q_6 , respectively.

Results (10) and (11) reveal that each number N , as it occurs for the first time in the ranges (12), is represented uniquely and minimally. For instance,

$$-3 = 1 \cdot P_{-1} + 2 \cdot P_{-2} + 0 \cdot P_{-3} + 0 \cdot P_{-4} + 0 \cdot P_{-5} + \cdots$$

has unique and minimal representation $1 \cdot P_{-1} + 2 \cdot P_{-2}$. We conclude that $h \neq m$. Similarly, $h \neq m$. Therefore, $h = m$, and Case 1 and the Summary are true.

Combining all the preceding discussion, we argue that the validity of the Theorem has been justified.

See [2] for further relevant information and [1] for an analogous treatment of representations involving negatively subscripted Fibonacci numbers.

REFERENCES

1. M. W. Bunder. "Zeckendorf Representations Using Negative Fibonacci Numbers." *The Fibonacci Quarterly* **30.2** (1992):111-15.
2. A. F. Horadam. "Unique Minimal Representation of Integers by Negatively Subscripted Pell Numbers." *The Fibonacci Quarterly* **32.3** (1994):202-06.

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NEW EDITORIAL POLICIES

The Board of Directors of The Fibonacci Association during their last business meeting voted to incorporate the following two editorial policies effective January 1, 1995:

1. All articles submitted for publication in The Fibonacci Quarterly will be blind refereed.
2. In place of Assistant Editors, The Fibonacci Quarterly will change to utilization of an Editorial Board.

SOME INFORMATION ABOUT THE BINOMIAL TRANSFORM

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(Submitted April 1993)

A few days ago I saw the paper [4]. I think I can make some additional remarks that might not be totally useless for the Fibonacci Community!

Let (a_n) be a given sequence and $s_n = \sum_{k=0}^n \binom{n}{k} a_k$. Denoting the respective (ordinary) generating functions by $A(x)$ and $S(x)$, the paper in question mainly deals with the consequences of the formula

$$S(x) = \frac{1}{1-x} A\left(\frac{x}{1-x}\right). \quad (1)$$

Knuth [7] has introduced the *binomial transform* by

$$\hat{a}_n = \sum_{k=0}^n \binom{n}{k} (-1)^k a_k,$$

and it is clear that this is the situation from above. But Philippe Flajolet and the present writer agreed about ten years ago that there are just *exponential generating functions* hidden! They have a convolution formula

$$c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k},$$

and upon choosing the b_k 's to be equal to 1, we have the old situation. So, denoting the exponential generating functions by $\bar{A}(x)$ and $\bar{S}(x)$, we have the even simpler formula $\bar{S}(x) = e^x \bar{A}(x)$. This can readily be inverted as $\bar{A}(x) = e^{-x} \bar{S}(x)$, whence

$$a_n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} s_k.$$

These facts about exponential generating functions are of course folklore; one particular reference is [3].

Flajolet & Richmond [2], Schmid [8], and Kirschenhofer & Prodinger [6] all made heavy use of (1). Schmid observed (among other writers) that an exponential generating function will be transformed into an ordinary generating function by the *Borel transform*.

Now the *generalization*

$$s_n = \sum_{k=0}^n \binom{n}{k} b^{n-k} c^k a_k \quad \text{or} \quad S(x) = \frac{1}{1-bx} A\left(\frac{cx}{1-bx}\right)$$

translates into

$$\bar{S}(x) = e^{bx} \bar{A}(cx).$$

Since

$$\bar{A}(x) = e^{-\frac{b}{c}x} \bar{S}\left(\frac{x}{c}\right),$$

we find the inversion formula

$$\alpha_n = c^{-n} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} b^{n-k} s_k.$$

The discussion in Theorem 2 becomes quite transparent, considering exponential generating functions. It is asked whenever we have

$$F_{pn+r} = \sum_{k=0}^n \binom{n}{k} t^{n-k} s^k F_{qk+r},$$

where F_n denote Fibonacci numbers. The exponential generating function of the Fibonacci numbers F_n is

$$\frac{1}{\sqrt{5}}(e^{\alpha x} - e^{\beta x}),$$

with the usual $\alpha = (1 + \sqrt{5})/2$ and $\beta = -1/\alpha = (1 - \sqrt{5})/2$. More generally, the sequence F_{pn+r} leads to

$$\frac{1}{\sqrt{5}}(\alpha^r e^{\alpha^p x} - \beta^r e^{\beta^p x}) = e^{tx} \frac{1}{\sqrt{5}}(\alpha^r e^{\alpha^q s x} - \beta^r e^{\beta^q s x}),$$

from which we deduce the two equations,

$$\alpha^p = t + \alpha^q s \quad \text{and} \quad \beta^p = t + \beta^q s.$$

Subtracting them, we see that

$$s = \frac{\alpha^p - \beta^p}{\alpha^q - \beta^q} = \frac{F_p}{F_q}.$$

Further,

$$t = \alpha^p - \alpha^q \frac{\alpha^p - \beta^p}{\alpha^q - \beta^q} = (-1)^p \frac{\alpha^{q-p} - \beta^{q-p}}{\alpha^q - \beta^p} = (-1)^p \frac{F_{q-p}}{F_q}.$$

To justify this *equating of coefficients*, we note that the functions $e^{\lambda x}$ are linearly independent; and the other possibility of grouping terms from the left and the right side would lead to the impossible equation $\alpha^r = -\beta^r$.

In [4] there is also the modification: What are the coefficients of

$$T(x) = A\left(\frac{cx}{1-bx}\right)?$$

That means: What is the effect of deleting the first factor? We can answer this much more generally by considering (with an arbitrary complex parameter d),

$$T(x) = \frac{1}{(1-bx)^d} A\left(\frac{cx}{1-bx}\right).$$

In this derivation, we will use the concept of *residues*, interesting *per se*.

We are using the substitution $w = \frac{cx}{1-bx}$ or $x = \frac{w}{c+bw}$. Therefore, $1-bx = \frac{c}{c+bw}$ and $dx = \frac{c}{(c+bw)^2} dw$; thus,

$$\begin{aligned}
t_n &:= [x^n]T(x) = \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} T(x) \\
&= \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} \frac{(c+bw)^d}{c^d} A(w) \\
&= \frac{1}{2\pi i} \oint \frac{cdw}{(c+bw)^2} \frac{(c+bw)^{n+1}}{w^{n+1}} \frac{(c+bw)^d}{c^d} A(w) \\
&= c^{1-d} [w^n](c+bw)^{n+d-1} A(w) \\
&= \sum_{k=0}^n \binom{n+d-1}{n-k} b^{n-k} c^k a_k.
\end{aligned}$$

Since

$$A(w) = \left(\frac{c}{c+bw} \right)^d T\left(\frac{w}{c+bw} \right),$$

we find in a similar way the inversion formula

$$a_n = c^{-n} \sum_{k=0}^n \binom{n+d-1}{n-k} (-1)^{n-k} b^{n-k} t_k.$$

The formula (1) is also useful to deal with Knuth's sum [5, eq. (7.6)]

$$u_n = \sum_{k=0}^n \binom{n}{k} \left(-\frac{1}{2} \right)^k \binom{2k}{k}.$$

Since

$$f(x) := \sum_{k \geq 0} \left(-\frac{1}{2} \right)^k \binom{2k}{k} x^k = \sum_{k \geq 0} \left(-\frac{x}{2} \right)^k \binom{2k}{k} = \frac{1}{\sqrt{1+2x}},$$

the generating function of the sequence u_n turns out to be

$$\frac{1}{1-x} \frac{1}{\sqrt{1+2\left(\frac{x}{1-x}\right)}} = \frac{1}{\sqrt{1-x^2}} = \sum_{n \geq 0} x^{2n} \binom{2n}{n} 4^{-n}.$$

From this, we see that $u_n = 2^{-n} \binom{n}{n/2}$ if n is even, and $u_n = 0$ otherwise.

I communicated this idea to Knuth, and he reported that Herbert Wilf came to this (or a similar) approach independently.

Formula (1) also has a *combinatorial interpretation*. If, for example, $A(x)$ enumerates certain words, so that a_n is the number of words of length n with a certain property, and we perform the operation "fill-in a new letter where and as often as you want," then the new "language" has the generating function $S(x)$. For further details on such *combinatorial constructions*, we refer the reader to [1].

REFERENCES

1. P. Flajolet. "Mathematical Methods in the Analysis of Algorithms and Data Structures." In *Trends in Theoretical Computer Science*, pp. 225-304. Ed. E. Börger. Rockville, MD: Computer Science Press, 1988.
2. P. Flajolet & L. B. Richmond. "Generalized Digital Trees and Their Difference-Differential Equations." *Random Structures and Algorithms* **3** (1992):305-20.
3. H. W. Gould. "Series Transformations for Finding Recurrences for Sequences." *The Fibonacci Quarterly* **28.2** (1990):166-71.
4. P. Haukkanen. "Formal Power Series for Binomial Sums of Sequences of Numbers." *The Fibonacci Quarterly* **31.1** (1993):28-31.
5. A. Jonassen & D. E. Knuth. "A Trivial Algorithm Whose Analysis Isn't." *Journal of Computer and System Sciences* **16** (1978):301-22.
6. P. Kirschenhofer & H. Prodinger. "The Path Length of Random Skip Lists." Submitted for publication, 1993.
7. D. E. Knuth. *The Art of Computer Programming*. Vol. 3. Reading, MA: Addison-Wesley, 1973.
8. U. Schmid. "The Average CRI-Length of a Tree Collision Resolution Algorithm in Presence of Multiplicity-Dependent Capture Effects." *Lecture Notes in Computer Science* **623** (1992): 223-34.

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GENERALIZED PASCAL TRIANGLES AND PYRAMIDS: THEIR FRACTALS, GRAPHS, AND APPLICATIONS

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This monograph was first published in Russia in 1990 and consists of seven chapters, a list of 406 references, an appendix with another 126 references, many illustrations and specific examples. Fundamental results in the book are formulated as theorems and algorithms or as equations and formulas. For more details on the contents of the book, see *The Fibonacci Quarterly* 31.1 (1993):52.

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PIERCE EXPANSIONS AND RULES FOR THE DETERMINATION OF LEAP YEARS

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I. INTRODUCTION

The length of the physical year in days is not an integer. This simple fact has complicated efforts to make a calendar for thousands of years. Both the Julian and the Gregorian calendars use a scheme that involves the periodic insertion of extra days, or *intercalation*. A year with an intercalated day is called a *leap year*.

In the Julian calendar, an extra day was inserted every fourth year. In the Gregorian calendar (commonly in use today) an extra day is inserted every fourth year, exclusive of century years, which are leap years only if divisible by 400. From this, we see that the average length of the year in the Julian calendar was

$$365 + \frac{1}{4} = 365.25$$

days, while in the Gregorian calendar, the average length is

$$365 + \frac{1}{4} - \frac{1}{100} + \frac{1}{400} = 365 + \frac{1}{4} - \frac{1}{4 \cdot 25} + \frac{1}{4 \cdot 25 \cdot 4} = 365.2425$$

days. Both these numbers are approximations to the true length of the year, which is currently about 365.242191 days [1, p. C1].

In this note, we will examine a scheme for leap year determination which generalizes both the Julian and Gregorian calendars and includes the modifications of the Gregorian calendar suggested by McDonnell [2]. Although our results will be phrased in the language of the calendar, they are in fact purely number theoretical in nature.

II. THREE INTERCALATION SCHEMES FOR LEAP YEARS

An *intercalation scheme* describes when to insert extra days in a year to keep the calendar synchronized with the physical year. We assume that exactly 0 or 1 extra days are inserted each year. A year when one day is inserted is called a *leap year*.

Let the length of the year be $I + \beta$ days, where I is an integer and $0 \leq \beta < 1$. Let $L(N)$ count the number of years y in the range $1 \leq y \leq N$ which are declared to be leap years. A good intercalation scheme will certainly have $\lim_{N \rightarrow \infty} \frac{L(N)}{N} = \beta$. A much stronger condition is that $|L(N) - \beta N|$ should not be too large.

We now describe three intercalation schemes.

A Method Generalizing the Julian and Gregorian Calendars

Let a_1, a_2, \dots be a finite or infinite sequence of integers with $a_1 \geq 1$ and $a_i \geq 2$ for $i \geq 2$. We call such a sequence (a_i) an *intercalation sequence*.

We now say that N is a leap year if N is divisible by a_1 , unless N is also divisible by a_1a_2 , in which case it is not, unless N is also divisible by $a_1a_2a_3$, in which case it is, etc. More formally, define the year N to be a leap year if and only if

$$\sum_{k=1}^{\infty} (-1)^{k+1} \text{div}(N, a_1a_2 \dots a_k) = 1,$$

where the function $\text{div}(x, y)$ is defined as follows:

$$\text{div}(x, y) = \begin{cases} 1, & \text{if } y|x; \\ 0, & \text{otherwise.} \end{cases}$$

For the Julian calendar, the intercalation sequence is of length 1: $a_1 = 4$. The Gregorian calendar increased the length to 3: $a_1 = 4, a_2 = 25, a_3 = 4$. Herschel ([5], p. 55) proposed extending the Gregorian intercalation sequence by $a_4 = 10$, which results in the estimate $\beta = .24225$. McDonnell [2] has proposed

$$(a_1, a_2, \dots, a_5) = (4, 25, 4, 8, 27),$$

corresponding to the estimate $\beta = .242199$.

The method has the virtue that it is very easy to remember and is a simple generalization of existing rules. In section III of this paper we examine some of the consequences of this scheme.

An Exact Scheme

Suppose we say that year y is a leap year if and only if

$$\lfloor \beta y + 1/2 \rfloor - \lfloor \beta(y-1) + 1/2 \rfloor = 1.$$

Then $L(N) = \lfloor \beta N + 1/2 \rfloor$; in other words, $L(N)$ is the integer closest to βN . This is clearly the most accurate intercalation scheme possible. However, it suffers from two drawbacks: it is unwieldy for the average person to apply in practice, and β must be known explicitly.

This method can easily be modified to handle the case in which β varies slightly over time. Further, it works well when the fundamental unit is not the year but is, for example, the second. It then describes when to insert a "leap second." This method is essentially that used currently to make yearly corrections to the calendar.

A Method Based on Continued Fractions

We could also find good rational approximations to β using continued fractions. For example, using the approximation .242191 to the fractional part of the solar year, we find

$$.242191 = [0, 4, 7, 1, 3, 17, 5, 1, 1, 7, 1, 1, 2]$$

and the first four convergents are $1/4$, $7/29$, $8/33$, and $31/128$. The last convergent, for example, tells us to intercalate 31 days every 128 years. McDonnell notes [personal communication] that had binary arithmetic been in popular use then, Clavius would almost certainly have suggested an intercalation scheme based on this approximation.

The method suffers from the drawback that the method for actually designating the particular years to be leap years is not provided. For example, the third convergent tells us to intercalate 8

days in 33 years, but which of the 33 should be leap years? In 1079, Omar Khayyam suggested that years congruent to 0, 4, 8, 12, 16, 20, 24, and 28 (mod 33) should be leap years [5].

III. SOME THEOREMS

Given an intercalation sequence (a_1, a_2, \dots) , it is easy to compute $L(N)$ using the following theorem.

Theorem 1: Let $L(N)$ be the number of leap years occurring on or before year N , i.e.,

$$L(N) = \sum_{k=1}^N \sum_{i \geq 1} (-1)^{i+1} \text{div}(k, a_1 a_2 \dots a_i).$$

Then

$$L(N) = \sum_{i \geq 1} (-1)^{i+1} \left\lfloor \frac{N}{a_1 a_2 \dots a_i} \right\rfloor.$$

Proof: It is easy to see that, for $y \geq 1$, we have

$$\text{div}(x, y) = \left\lfloor \frac{x}{y} \right\rfloor - \left\lfloor \frac{x-1}{y} \right\rfloor.$$

Thus, we have

$$\begin{aligned} L(N) &= \sum_{k=1}^N \sum_{i \geq 1} (-1)^{i+1} \text{div}(k, a_1 a_2 \dots a_i) \\ &= \sum_{i \geq 1} (-1)^{i+1} \sum_{k=1}^N \text{div}(k, a_1 a_2 \dots a_i) \\ &= \sum_{i \geq 1} (-1)^{i+1} \sum_{k=1}^N \left(\left\lfloor \frac{k}{a_1 \dots a_i} \right\rfloor - \left\lfloor \frac{k-1}{a_1 \dots a_i} \right\rfloor \right) \\ &= \sum_{i \geq 1} (-1)^{i+1} \left\lfloor \frac{N}{a_1 \dots a_i} \right\rfloor, \end{aligned}$$

which completes the proof. \square

Theorem 1 explains several things. First of all, it gives the relationship between the intercalation sequence a_i and the length of the physical year in days. Write

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \dots.$$

Clearly we have

$$\lim_{N \rightarrow \infty} \frac{L(N)}{N} = \alpha.$$

Then if the length of the physical year is $I + \beta$ days, where $0 \leq \beta < 1$, we would like α to be as close as possible to β ; for, otherwise, the calendar will move more and more out of synchronization with the physical year.

Therefore, to minimize error, we can assume that the a_i have been chosen so that $\alpha = \beta$. It is somewhat surprising to note that even this choice will cause arbitrarily large differences between the calendar and the physical year; this in spite of the fact that the behavior *on average* will be correct.

Suppose a_1, a_2, \dots have been chosen such that

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \dots.$$

The next theorem estimates how far out of sync the calendar can be.

Theorem 2: Define $N_j = -1 + a_1 - a_1 a_2 + \dots + (-1)^{j+1} a_1 a_2 \dots a_j$. Then, for all $r \geq 0$,

$$N_{2r+1} \alpha - L(N_{2r+1}) \geq \sum_{j=1}^{r+1} \left(1 - \frac{1}{a_{2j-1}} \right) \left(1 - \frac{1}{a_{2j}} \right) \geq \frac{r}{4}.$$

Proof: It is easily verified that, if $i \leq j$, then

$$\frac{N_j}{a_1 \dots a_i} - \left\lfloor \frac{N_j}{a_1 \dots a_i} \right\rfloor = \begin{cases} \frac{N_{j-1}}{a_1 \dots a_i}, & \text{if } i \text{ is even;} \\ \frac{N_i}{a_1 \dots a_i}, & \text{if } i \text{ is odd.} \end{cases}$$

Thus, we find

$$\begin{aligned} N_{2r+1} \alpha - L(N_{2r+1}) &= \left(N_{2r+1} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{a_1 a_2 \dots a_i} \right) - \sum_{i=1}^{\infty} (-1)^{i+1} \left\lfloor \frac{N_{2r+1}}{a_1 \dots a_i} \right\rfloor \\ &= \sum_{i=1}^{2r+2} (-1)^{i+1} \left(\frac{N_{2r+1}}{a_1 \dots a_i} - \left\lfloor \frac{N_{2r+1}}{a_1 \dots a_i} \right\rfloor \right) + \sum_{i=2r+3}^{\infty} (-1)^{i+1} \frac{N_{2r+1}}{a_1 \dots a_i} \\ &\geq \sum_{i=1}^{2r+2} (-1)^{i+1} \left(\frac{N_{2r+1}}{a_1 \dots a_i} - \left\lfloor \frac{N_{2r+1}}{a_1 \dots a_i} \right\rfloor \right) \\ &= \sum_{j=1}^{r+1} \frac{N_{2j-1}}{a_1 \dots a_{2j-1}} - \sum_{j=1}^{r+1} \frac{N_{2j-1}}{a_1 \dots a_{2j}} \\ &= \sum_{j=1}^{r+1} \frac{N_{2j-1}}{a_1 \dots a_{2j-1}} \left(1 - \frac{1}{a_{2j}} \right). \end{aligned}$$

Now, if we observe that $N_{2j-1} \geq a_1 \dots a_{2j-1} - a_1 \dots a_{2j-2}$, then we find that

$$N_{2r+1} \alpha - L(N_{2r+1}) \geq \sum_{j=1}^{r+1} \left(1 - \frac{1}{a_{2j-1}} \right) \left(1 - \frac{1}{a_{2j}} \right) \geq \frac{r}{4},$$

which is the desired result. \square

Thus, the difference $N_{2r+1} \alpha - L(N_{2r+1})$ can be made as large as desired as $r \rightarrow \infty$. Therefore, if α is an irrational number, there is no way to avoid large swings of the calendar.

As an example, consider the Gregorian calendar with intercalation sequence $(a_1, a_2, a_3) = (4, 25, 4)$. Then $N_3 = 303$. For example, in the period from 1600 to 1903, we would expect to see $303 \cdot 2425 = 73.4775$ leap years (assuming the length of the year is precisely 365.2425 days), whereas the Gregorian scheme produces only 72 leap years.

We now assume that the fractional part of the year's length in days is an irrational number α . We also assume that the intercalation sequence a_k is that given by the *Pierce expansion* (see [3], [4], [6]) of α , i.e., the unique way to write

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \dots$$

such that the a_i are integers with $1 \leq a_1 < a_2 \dots$. It is known that the expansion terminates if and only if α is rational. For example,

$$.242191 = \frac{1}{4} - \frac{1}{4 \cdot 32} + \frac{1}{4 \cdot 32 \cdot 2232} - \frac{1}{4 \cdot 32 \cdot 2232 \cdot 15625}.$$

Then we will show that

Theorem 3: For almost all α , we have

$$\limsup_{N \rightarrow \infty} \frac{N\alpha - L(N)}{\sqrt{\log N}} = \frac{\sqrt{2}}{2}.$$

Proof: The proof is in two parts. First, we show that, for all $\varepsilon > 0$, there exists an integer N such that

$$\frac{N\alpha - L(N)}{\sqrt{\log N}} \geq \frac{\sqrt{2}}{2} (1 - 3\varepsilon).$$

Second, we show that, for all $\varepsilon > 0$, we have

$$\frac{N\alpha - L(N)}{\sqrt{\log N}} \leq \frac{\sqrt{2}}{2} (1 + 5\varepsilon)$$

for all N sufficiently large.

We need the following two simple lemmas.

Lemma 4: For almost all α ,

$$\lim_{n \rightarrow \infty} \frac{\log(a_1 \dots a_n)}{n^2 / 2} = 1.$$

Proof: In [6] it is shown that, for almost all α ,

$$\lim_{n \rightarrow \infty} \frac{\log a_n}{n} = 1.$$

From this, the desired result follows easily. \square

Lemma 5: $\sum_{k=1}^{\infty} \frac{1}{a_k}$ converges for almost all α .

Proof: See Theorem 12 in [6]. \square

Now we can return to the proof of the first part of Theorem 3. Let α be chosen, and write

$$C = \frac{1}{a_1} + \frac{1}{a_2} + \dots$$

Let ε be given, and choose r_1 sufficiently large so

$$\frac{\log(a_1 \dots a_r)}{r^2/2} < \frac{1}{(1-\varepsilon)^2}$$

for all $r \geq r_1$. This can be done by Lemma 4. Also choose r sufficiently large so

$$\frac{C\sqrt{2}}{2r+1} < \varepsilon.$$

This can be done by Lemma 5.

Then we find

$$N_{2r+1}\alpha - L(N_{2r+1}) \geq \sum_{j=1}^{r+1} \left(1 - \frac{1}{a_{2j-1}}\right) \left(1 - \frac{1}{a_{2j}}\right) \geq r+1-C. \quad (1)$$

Now, from the definition of N_j , we have $N_{2r+1} \leq a_1 \dots a_{2r+1}$; therefore,

$$\sqrt{\log N_{2r+1}} \leq \frac{2r+1}{\sqrt{2}} \left(\frac{1}{1-\varepsilon} \right)$$

because we have chosen r sufficiently large.

Now, dividing both sides of (1) by $\sqrt{\log N_{2r+1}}$ and using the estimate just obtained, we see

$$\begin{aligned} \frac{N_{2r+1}\alpha - L(N_{2r+1})}{\sqrt{\log N_{2r+1}}} &\geq \frac{r+1-C}{2r+1} \sqrt{2}(1-\varepsilon) \\ &\geq \left(\frac{\sqrt{2}}{2} - \frac{C\sqrt{2}}{2r+1} \right) (1-\varepsilon) \geq \left(\frac{\sqrt{2}}{2} - \varepsilon \right) (1-\varepsilon) \geq \frac{\sqrt{2}}{2} (1-3\varepsilon), \end{aligned}$$

which completes the proof of the first part of Theorem 3.

Now let us complete the proof of Theorem 3 by showing that, for almost all α and all N sufficiently large,

$$\frac{N\alpha - L(N)}{\sqrt{\log N}} \leq \frac{\sqrt{2}}{2} (1+5\varepsilon).$$

We need the following simple lemma.

Lemma 6: $N\alpha - \sum_{i=1}^r (-1)^{i+1} \left\lfloor \frac{N}{a_1 \dots a_i} \right\rfloor \leq \frac{r+1}{2} + \frac{N}{a_1 \dots a_r (a_r + 1)}.$

Proof: Remez [4] has noted that

$$\alpha - \sum_{i=1}^r (-1)^{i+1} \frac{1}{a_1 \dots a_i} \leq \frac{1}{a_1 \dots a_r (a_r + 1)}.$$

Multiplying by N , we get

$$N\alpha - N \sum_{i=1}^r (-1)^{i+1} \frac{1}{a_1 \dots a_i} \leq \frac{N}{a_1 \dots a_r (a_r + 1)}. \quad (2)$$

Also we have

$$\left(N \sum_{i=1}^r (-1)^{i+1} \frac{1}{a_1 \dots a_i} \right) - \sum_{i=1}^r (-1)^{i+1} \left\lfloor \frac{N}{a_1 \dots a_i} \right\rfloor \leq \frac{r+1}{2}, \quad (3)$$

since, to maximize this difference, we let the odd-numbered terms equal 1. Adding (2) and (3), we get the desired result. \square

Now, given ε , choose N sufficiently large so

- (a) $\log N > \frac{1}{2\varepsilon^2}$;
- (b) $\frac{\log(a_1 \dots a_r)}{r^2/2} > \frac{1}{(1+\varepsilon)^2}$ for all $r \geq \sqrt{2 \log N}$.

By Lemma 4, this can be done for almost all α .

Now, from Lemma 6, we have

$$N\alpha - \sum_{i=1}^r (-1)^{i+1} \left\lfloor \frac{N}{a_1 \dots a_i} \right\rfloor \leq \frac{r+1}{2} + \frac{N}{a_1 \dots a_r (a_r + 1)}. \quad (4)$$

Put $r = \left\lceil \sqrt{2(\log N)(1+\varepsilon)^2} \right\rceil$. Then, from part (b) of the hypothesis on N , we have

$$\frac{\log(a_1 \dots a_r)}{(\log N)(1+\varepsilon)^2} > \frac{1}{(1+\varepsilon)^2},$$

and so $a_1 \dots a_r > N$. Therefore

$$\sum_{i=1}^r (-1)^{i+1} \left\lfloor \frac{N}{a_1 \dots a_i} \right\rfloor = L(N).$$

Hence, we can substitute in equation (4) to get

$$N\alpha - L(N) \leq \frac{r+1}{2} + \frac{1}{a_r + 1} \leq \frac{\sqrt{2(\log N)(1+\varepsilon)^2} + 2}{2} + \frac{1}{a_r + 1} \leq \frac{\sqrt{2}}{2} \sqrt{\log N} (1+\varepsilon) + 2,$$

since $a_r \geq 1$. Dividing both sides of this equation by $\sqrt{\log N}$, we see that

$$\frac{N\alpha - L(N)}{\sqrt{\log N}} \leq \frac{\sqrt{2}}{2} (1+\varepsilon) + 2\sqrt{2}\varepsilon \leq \frac{\sqrt{2}}{2} (1+5\varepsilon),$$

which completes the proof of Theorem 3. \square

In a similar fashion, we can show that

$$\liminf_{N \rightarrow \infty} \frac{N\alpha - L(N)}{\sqrt{\log N}} = -\frac{\sqrt{2}}{2}.$$

Roughly speaking, Theorem 3 states that we can expect fluctuations of approximately $\sqrt{\frac{\log N}{2}}$ days at year N of the calendar. Though this type of fluctuation can grow arbitrarily large, it is small for years of reasonable size. For example, for most α , we would have to wait until about the year $3.6 \cdot 10^{42}$ to see fluctuations on the order of a week in size.

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REFERENCES

1. *Astronomical Almanac for the Year 1989*. Washington, D.C.: U.S. Government Printing Office, 1988.
2. E. E. McDonnell. "Spirals and Time." *APL Quote Quad* **7.4** (Winter 1977):20-22.
3. T. A. Pierce. "On an Algorithm and Its Use in Approximating Roots of Algebraic Equations." *Amer. Math. Monthly* **36** (1929):523-25.
4. E. Ya. Remez. "On Series with Alternating Sign Which May Be Connected with Two Algorithms of M. V. Ostrogradskij for the Approximation of Irrational Numbers." *Uspekhi Mat. Nauk (N. S.)* **6.5** (1951):33-42.
5. V. F. Rickey. "Mathematics of the Gregorian Calendar." *Math. Intelligencer* **7.1** (1985):53-56.
6. J. O. Shallit. "Metric Theory of Pierce Expansions." *The Fibonacci Quarterly* **24.1** (1986):22-40.

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SOME CONGRUENCE PROPERTIES OF GENERALIZED SECOND-ORDER INTEGER SEQUENCES

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1. INTRODUCTION

Hoggatt and Bicknell [3] proved that for a prime p

$$L_{kp} \equiv L_k \pmod{p} \quad (1.1)$$

where $\{L_n\}$ is the Lucas sequence. Robbins [8] proved more general results for a broader class of integer sequences $\{U_n\}$ and $\{V_n\}$ which we soon define.

In the notation of Horadam [4] write

$$W_n = W_n(a, b; P, Q) \quad (1.2)$$

so that

$$W_n = PW_{n-1} - QW_{n-2}, \quad W_0 = a, \quad W_1 = b, \quad n \geq 2. \quad (1.3)$$

Then

$$\begin{cases} U_n = W_n(0, 1; P, Q) \\ V_n = W_n(2, P; P, Q) \end{cases} \quad (1.4)$$

Indeed, $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences generated by (1.3). They have been studied extensively, particularly by Lucas [7]. Further information can be found, for example, in [1], [4], and [6].

All sequences generated by (1.3) can be extended to negative subscripts using either the Binet form [4] or the recurrence relation (1.3). In all that follows, a , b , P , and Q are assumed to be integers. Robbins proved the following theorem.

Theorem 1: Let p be prime. If $\Delta = P^2 - 4Q$, then

$$V_{kp^n} \equiv V_{kp^{n-1}} \pmod{p^n}, \quad \text{all } p, \quad (1.5)$$

$$U_{kp^n} \equiv \left(\frac{\Delta}{p}\right) U_{kp^{n-1}} \pmod{p^n}, \quad \text{for } p \text{ odd and } p \nmid \Delta, \quad (1.6)$$

$$U_{k2^n} \equiv (-1)^Q U_{k2^{n-1}} \pmod{2^n}, \quad (1.7)$$

where $\left(\frac{\Delta}{p}\right)$ is the Legendre symbol.

Remark 1: Robbins proved Theorem 1 under two strong assumptions. Firstly he assumed that $(P, Q) = 1$ and secondly that $\Delta > 0$. The first of these assumptions was used by Lucas [7] in his study of the sequences (1.4) and need not be adhered to in all contexts. Indeed, Robbins' arguments do not make explicit use of it and so it may be dropped. The assumption that $\Delta > 0$ was apparently made to ensure that $\sqrt{\Delta}$, which appears in a key proof involving Binet forms (Lemma 2.12), is real. However, this proof remains valid for $\Delta < 0$. In work on second-order recurrences the assumption $\Delta \neq 0$ is usually made so that the Binet form does not degenerate. However, in this context, following convention and putting $(\frac{a}{p}) = 0$, the proofs of certain key results (Lemmas 2.3 and 2.13) are greatly simplified when $\Delta = 0$. This is because the Binet forms become

$$\begin{cases} U_n = nA^{n-1} \\ V_n = 2A^n \end{cases}$$

where A is an integer. Likewise, putting $(\frac{a}{p}) = 0$ when $p|\Delta$, the proof of Robbins' Lemma 2.14, another key result, becomes trivial.

With these observations, and following Robbins' arguments, Theorem 1 remains valid for all integers P and Q . Indeed, for p odd and $p|\Delta$, (1.6) becomes

$$U_{kp^n} \equiv 0 \pmod{p^n}. \quad (1.8)$$

The object of this paper is to generalize (1.5)-(1.8) to the sequence $W_n = W_n(a, b; P, Q)$.

2. PRELIMINARY RESULTS

We now state some identities which are used subsequently.

$$V_n = U_{n+1} - QU_{n-1}, \quad (2.1)$$

$$2U_{n+1} = V_n + PU_n, \quad (2.2)$$

$$-2QU_{n-1} = V_n - PU_n, \quad (2.3)$$

$$W_n = W_0U_{n+1} + (W_1 - PW_0)U_n, \quad (2.4)$$

$$W_n = -QW_0U_{n-1} + W_1U_n, \quad (2.5)$$

$$2W_n = W_0V_n + (2W_1 - PW_0)U_n, \quad (2.6)$$

$$2W_{m+n} = W_mV_n + (2W_{m+1} - PW_m)U_n, \quad (2.7)$$

$$W_mU_{n+1} - W_{m+1}U_n = Q^nW_{m-n}, \quad (2.8)$$

$$Q^nU_{-n} = -U_n, \quad (2.9)$$

$$Q^nV_{-n} = V_n. \quad (2.10)$$

Identity (2.1) is easily proved using Binet forms and (2.2) and (2.3) can be obtained from (2.1) by simple substitution using (1.3). However, we state (2.2) and (2.3) for easy reference subsequently. Identity (2.4) is essentially (2.14) in [4] where the initial terms of $\{U_n\}$ are shifted. Identity (2.5) is obtained from (2.4) using (1.3) and (2.6) is obtained by adding (2.4) and (2.5).

Identity (2.7) is obtained from (2.6) by shifting the initial terms of $\{W_n\}$ to W_m, W_{m+1} . Finally, (2.8)-(2.10) are easily obtained using Binet forms.

3. A RESULT FOR ODD PRIMES

We now state and prove a result which generalizes (1.5) and (1.6) for odd primes p to the sequence $\{W_n\}$. Throughout, Δ is as in Theorem 1.

Theorem 2: Let p be an odd prime and k and m be nonnegative integers. Then

$$W_{m+kp^n} \equiv \begin{cases} W_{m+kp^{n-1}} & (\text{mod } p^n) \text{ if } \left(\frac{\Delta}{p}\right) = 1, \\ Q^{kp^{n-1}} W_{m-kp^{n-1}} & (\text{mod } p^n) \text{ if } \left(\frac{\Delta}{p}\right) = -1. \end{cases} \quad (3.1)$$

Proof: Suppose $\left(\frac{\Delta}{p}\right) = 1$. Then in (2.7), if we replace n by kp^n and use (1.5) and (1.6), we obtain

$$2W_{m+kp^n} \equiv W_m V_{kp^{n-1}} + (2W_{m+1} - PW_m) U_{kp^{n-1}} \pmod{p^n}. \quad (3.2)$$

Using (2.7) to substitute for the right side gives

$$2W_{m+kp^n} \equiv 2W_{m+kp^{n-1}} \pmod{p^n}, \quad (3.3)$$

and since 2 has a multiplicative inverse modulo p^n , the first half of Theorem 2 follows.

If $\left(\frac{\Delta}{p}\right) = -1$, then in (2.7) we replace n by kp^n and use (1.5) and (1.6) to obtain

$$2W_{m+kp^n} \equiv W_m V_{kp^{n-1}} - (2W_{m+1} - PW_m) U_{kp^{n-1}} \pmod{p^n}, \quad (3.4)$$

and rearranging terms gives

$$2W_{m+kp^n} \equiv W_m (V_{kp^{n-1}} + PU_{kp^{n-1}}) - 2W_{m+1} U_{kp^{n-1}} \pmod{p^n}. \quad (3.5)$$

Now (2.2) reduces (3.5) to

$$2W_{m+kp^n} \equiv 2W_m U_{kp^{n-1}+1} - 2W_{m+1} U_{kp^{n-1}} \pmod{p^n}, \quad (3.6)$$

and making use of (2.8) completes the proof. \square

Using a similar argument, we see that if $p|\Delta$ then (1.8) generalizes to

$$W_{m+kp^n} \equiv ((p^n + 1)/2) W_m V_{kp^{n-1}} \pmod{p^n}. \quad (3.7)$$

Remark 2: If we take the case $m = 0$ and $\{W_n\} = \{U_n\}$, then (2.9) shows that Theorem 2 reduces to (1.6). If we take the case $m = 0$ and $\{W_n\} = \{V_n\}$, then (2.10) shows that Theorem 2 reduces to (1.5). Thus, for p odd Theorem 2 both unifies and generalizes Robbins' results.

4. A RESULT FOR THE PRIME $p = 2$

We now prove the following theorem.

Theorem 3: If k and m are nonnegative integers and W_m is even, then

$$W_{m+k2^n} \equiv \begin{cases} W_{m+k2^{n-1}} & (\text{mod } 2^n) \text{ if } Q \text{ is even,} \\ Q^{k2^{n-1}} W_{m-k2^{n-1}} & (\text{mod } 2^n) \text{ if } Q \text{ is odd.} \end{cases} \quad (4.1)$$

Proof: Putting $W_m = 2Q_m$, Q_m an integer, we use (2.7) to write

$$W_{m+n} = Q_m V_n + (W_{m+1} - PQ_m)U_n. \quad (4.2)$$

Now with $k2^n$ in place of n , (1.5) and (1.7) imply

$$W_{m+k2^n} \equiv Q_m V_{k2^{n-1}} + (-1)^Q (W_{m+1} - PQ_m)U_{k2^{n-1}} \pmod{2^n}. \quad (4.3)$$

If Q is even, (4.3) becomes

$$W_{m+k2^n} \equiv Q_m V_{k2^{n-1}} + (W_{m+1} - PQ_m)U_{k2^{n-1}} \pmod{2^n} \quad (4.4)$$

and the right side of (4.4) simplifies using (4.2) to prove the theorem for Q even.

If Q is odd, (4.3) becomes

$$W_{m+k2^n} \equiv Q_m V_{k2^{n-1}} - (W_{m+1} - PQ_m)U_{k2^{n-1}} \pmod{2^n}, \quad (4.5)$$

and rearranging terms gives

$$W_{m+k2^n} \equiv Q_m (V_{k2^{n-1}} + PU_{k2^{n-1}}) - W_{m+1}U_{k2^{n-1}} \pmod{2^n}. \quad (4.6)$$

Now using (2.2) and recalling that $W_m = 2Q_m$, (4.6) becomes

$$W_{m+k2^n} \equiv W_m U_{k2^{n-1}+1} - W_{m+1}U_{k2^{n-1}} \pmod{2^n}. \quad (4.7)$$

We now use (2.8) to simplify the right side of (4.7) and this completes the proof. \square

Remark 3: If we take $\{W_n\} = \{U_n\}$ and $m = 0$, then $U_0 = 0$ is even and we see with the aid of (2.9) that Theorem 3 reduces to (1.7). If we take $\{W_n\} = \{V_n\}$ and $m = 0$, then $V_0 = 2$ is even and we see with the aid of (2.10) that Theorem 3 reduces to (1.5) for the case $p = 2$.

Remark 4: Bisht [2] proved that (1.5) carries over to higher-order analogues of $\{V_n\}$. However, we have seen no results similar to (1.6) and (1.7) for higher-order analogues of $\{U_n\}$.

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REFERENCES

1. G. E. Bergum & V. E. Hoggatt, Jr. "Sums and Products for Recurring Sequences." *The Fibonacci Quarterly* **13.2** (1975):115-20.
2. C. S. Bisht. "Some Congruence Properties of Generalized Lucas Integral Sequences." *The Fibonacci Quarterly* **22.3** (1984):290-95.
3. V. E. Hoggatt, Jr., & M. Bicknell. "Some Congruences of the Fibonacci Numbers Modulo a Prime p ." *Mathematics Magazine* **47** (1974):210-14.
4. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.2** (1965):161-76.
5. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." *Duke Mathematical Journal* **32** (1965):437-46.
6. D. Jarden. *Recurring Sequences*. Jerusalem: Riveon Lematematika, 1966.
7. E. Lucas. *Théorie des nombres*. Paris: Albert Blanchard, 1961.
8. N. Robbins. "Some Congruence Properties of Binomial Coefficients and Linear Second Order Recurrences." *International Journal of Mathematics and Mathematical Sciences* **11** (1988):743-50.
9. C. W. Wall. "Some Congruences Involving Generalized Fibonacci Numbers." *The Fibonacci Quarterly* **17.1** (1979):29-33.

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PARTIAL SUMS FOR SECOND-ORDER RECURRENCE SEQUENCES

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(Submitted April 1993)

1. BACKGROUND MATERIAL

Motivation for this paper comes from a short article [4] in which some relations between a generalized Fibonacci sequence and the sequence of its partial sums were investigated. An opportunity was clearly provided for a deeper exploration of this theme.

Accordingly, the purpose of this paper is

- (a) to extend the relations in [4] to generalized Pell numbers with (i) positive and (ii) negative subscripts, and
- (b) as an addendum, to expand the results in [4] to generalized Fibonacci numbers having negative subscripts.

Consider the *generalized Pell sequence* $\{P_n\}$ defined for all integers n by

$$P_{n+2} = 2P_{n+1} + P_n \quad P_1 = a, P_2 = b \quad (P_0 = b - 2a). \quad (1.1)$$

When $a = 1$, $b = 2$, the ordinary Pell sequence $\{p_n\}$ is generated, while when $a = 1$, $b = 3$, we derive the sequence $\{q_n\}$ defined by

$$q_{n+2} = 2q_{n+1} + q_n \quad q_1 = 1, q_2 = 3 \quad (q_0 = 1) \quad (1.2)$$

so that $q_n = \frac{1}{2}Q_n$, the n^{th} Pell-Lucas number [2]. Thus, we have the tabulation:

$$\begin{array}{l} n: \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad \dots \\ p_n: \quad 0 \quad 1 \quad 2 \quad 5 \quad 12 \quad 29 \quad 70 \quad 169 \quad 408 \quad \dots \\ q_n: \quad 1 \quad 1 \quad 3 \quad 7 \quad 17 \quad 41 \quad 99 \quad 239 \quad 577 \quad \dots \end{array} \quad (1.3)$$

Observe that the numbers in $\{p_n\}$ are alternately even and odd, while those in $\{q_n\}$ are all odd.

The first few numbers in $\{P_n\}$ and the corresponding sums $S_n = \sum_{i=1}^n P_i$ are from (1.1) for $n = 1, 2, \dots, 10$, therefore:

$$\begin{array}{rcccl} n & & P_n & & S_n \\ 1 & & a & & a \\ 2 & & & b & a + b \\ 3 & & a & + & 2b \\ 4 & & 2a & + & 5b \\ 5 & & 5a & + & 12b \\ 6 & & 12a & + & 29b \\ 7 & & 29a & + & 70b \\ 8 & & 70a & + & 169b \\ 9 & & 169a & + & 408b \\ 10 & & 408a & + & 985b \end{array} \quad (1.4)$$

By standard techniques, e.g., use of (1.1) and induction, it is easy to establish that

$$P_n = ap_{n-2} + bp_{n-1} \quad (n \geq 1, p_{-1} = 1 \text{ [see (3.1)]}) \quad (1.5)$$

and

$$S_n = \frac{P_n + P_{n+1} + a - b}{2}, \quad (1.6)$$

whence we deduce the recurrence

$$S_{n+2} = 2S_{n+1} + S_n + b - a \quad (S_0 = 0 \neq P_0 \text{ [see (1.1)]}). \quad (1.7)$$

For subsequent calculations, we will need the *Binet forms*

$$p_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (1.8)$$

and

$$q_n = \frac{\alpha^n + \beta^n}{2}, \quad (1.9)$$

where

$$\alpha = 1 + \sqrt{2}, \quad \beta = 1 - \sqrt{2}, \quad \text{so } \alpha + \beta = 2, \quad \alpha\beta = -1, \quad \alpha - \beta = 2\sqrt{2}. \quad (1.10)$$

Use of (1.8)-(1.10) produces the *Simson formulas*

$$p_{n+1}p_{n-1} - (p_n)^2 = (-1)^n \quad (1.11)$$

and

$$q_{n+1}q_{n-1} - (q_n)^2 = (-1)^{n+1}2, \quad (1.12)$$

as well as

$$p_n = p_{n-1} + q_{n-1}, \quad (1.13)$$

$$q_n = p_n + p_{n-1}, \quad (1.14)$$

$$\frac{q_n}{p_n} \rightarrow \sqrt{2} \text{ as } n \rightarrow \infty, \quad (1.15)$$

and the Binet forms for P_n and S_n .

Repeated use of the recurrence relations (1.1) for $\{p_n\}$, where $a = 1, b = 2$, and (1.2) for $\{q_n\}$, where $a = 1, b = 3$, respectively, lead to

$$\sum_{i=1}^n p_i = \frac{p_n + p_{n+1} - 1}{2} = \frac{q_{n+1} - 1}{2} \text{ by (1.14)} \quad (1.16)$$

and

$$\sum_{i=1}^n q_i = p_{n+1} - 1. \quad (1.17)$$

After considerable laborious, but nonetheless satisfying, calculations involving the above equations as appropriate, we determine the Simson formulas for P_n and S_n from (1.5) and (1.6), namely,

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n (a^2 + 2ab - b^2) \quad (1.18)$$

and ($n \geq 1$)

$$S_{n+1}S_{n-1} - S_n^2 = \frac{1}{2} \{(-1)^n(a^2 + 2ab - b^2) + a^2q_{n-2} - b^2q_{n-1} + 2abp_{n-2}\}. \quad (1.19)$$

Accordingly, when $n = 5$ for instance, $S_6S_4 - S_5^2 = 3a^2 + 4ab - 8b^2$ from (1.19) or directly from (1.4), while $P_6P_4 - P_5^2 = -a^2 - 2ab + b^2$ from (1.18) or (1.4). [Who would ever have surmised anything like (1.19)?]

Important special cases of (1.19) arise when $a = 1, b = 2$ (for p_n), and $a = 1, b = 3$ (for q_n).

Generally, $S_0 = 0 \neq P_0 = b - 2a$, unless $b = 2a$. Expressed otherwise, P_0 is not part of the summation process.

2. PARTIAL SUMS: POSITIVE SUBSCRIPTS

A basic set of theorems on partial sums can now be established, of which only the first will show the detail.

Theorem 1: $S_{4n} = q_{2n}(aq_{2n-1} + bq_{2n}) + a - b$.

Proof:

$$\begin{aligned} S_{4n} &= \frac{P_{4n} + P_{4n+1} + a - b}{2} && \text{by (1.6)} \\ &= \frac{a(p_{4n-2} + p_{4n-1}) + b(p_{4n-1} + p_{4n}) + a - b}{2} && \text{by (1.5)} \\ &= \frac{a(\alpha^{4n-1} + \beta^{4n-1} - 2) + b(\alpha^{4n} + \beta^{4n} + 2)}{4} + a - b && \text{by (1.8), (1.10)} \\ &= aq_{2n}q_{2n-1} + b(q_{2n})^2 + a - b && \text{by (1.9)} \\ &= q_{2n}(aq_{2n-1} + bq_{2n}) + a - b. \end{aligned}$$

Likewise,

Theorem 2: $S_{4n-2} = q_{2n-1}(aq_{2n-2} + bq_{2n-1})$.

Theorem 3: $S_{4n+1} = q_{2n}(aq_{2n} + bq_{2n+1}) - b$.

Theorem 4: $S_{4n-1} = q_{2n}(aq_{2n-2} + bq_{2n-1}) - a$.

Special cases occur when $a = 1, b = 2$ (i.e., the Pell sequence $\{p_n\}$), namely, for $s_n = \sum_{i=1}^n p_i$,

$$s_{4n} = q_{2n}q_{2n-1} - 1, \quad (2.1)$$

$$s_{4n-2} = q_{2n}q_{2n-1}, \quad (2.2)$$

$$s_{4n+1} = (q_{2n+1})^2, \quad (2.3)$$

$$s_{4n-1} = (q_{2n})^2 - 1 = 2(p_{2n})^2. \quad (2.4)$$

All four formulas (2.1)-(2.4) may be incorporated into the one neat expression [see (1.16)],

$$s_n = \frac{q_{n+1} - 1}{2} \quad (s_0 = 0), \quad (2.5)$$

[where we have invoked (1.6), ($P_n \equiv p_n$ here), and (1.14)].

However, a virtue of the forms (2.1)-(2.4) is that they display various obvious divisibility properties. Thus, $q_{2n}|s_{4n-2}$, $q_{2n-1}|s_{4n-2}$, $p_{2n}|s_{4n-1}$; in particular, $n=3$ in (2.2) gives $4059 = 41 \cdot 99$, and $n=3$ in (2.4) gives $9800 = 2(70)^2$. As an example of (2.5), $s_8 = 696 = \frac{q_9-1}{2}$ from (1.3).

Observe also the important recurrence from (1.7),

$$s_{n+2} = 2s_{n+1} + s_n + 1 \quad (s_0 = 0). \quad (2.6)$$

Next, write $s'_n = \sum_{i=1}^n q_i$. Then $a=1$, $b=3$ (i.e., the sequence $\{q_n\}$) in (1.6) lead to

$$s'_n = p_{n+1} - 1 \quad (s'_0 = 0), \quad (2.7)$$

i.e. (1.17), since $p_{n+1} = \frac{q_n + q_{n+1}}{2}$ by (1.8) and (1.9), and in (1.7) lead to the recurrence

$$s'_{n+2} = 2s'_{n+1} + s'_n + 2 \quad (s'_0 = 0). \quad (2.8)$$

Let

$$\sigma_n = s'_n - s_n. \quad (2.9)$$

Then, from (2.5) and (2.7), it follows that the sequence $\{\sigma_n\}$ is

$$\begin{array}{cccccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \\ s'_n & 1 & 4 & 11 & 28 & 69 & 168 & 407 & 984 & \dots \\ s_n & 1 & 3 & 8 & 20 & 49 & 119 & 288 & 696 & \dots \\ \sigma_n & 0 & 1 & 3 & 8 & 20 & 49 & 119 & 288 & \dots \end{array} \quad (2.10)$$

from which, by (2.6), (2.8), and (2.9), we derive the recurrence [cf. (2.6)]

$$\sigma_{n+2} = 2\sigma_{n+1} + \sigma_n + 1 \quad (\sigma_0 = 0). \quad (2.11)$$

Reverting to (1.4), we notice that

$$S_n = a(\sigma_{n-1} + 1) + b\sigma_n. \quad (2.12)$$

From (1.3) and (2.10),

$$q_n = \sigma_{n+1} - \sigma_{n-1} \quad (2.13)$$

and

$$\sigma_n = \frac{q_n - 1}{2}, \quad (2.14)$$

while, from (1.12), we have the Simson formula for $\{\sigma_n\}$,

$$\sigma_{n+1}\sigma_{n-1} - \sigma_n^2 = \frac{1}{2} \{(-1)^{n+1} - q_{n-1}\}. \quad (2.15)$$

Other properties of the sequences which flow from the above data include

$$s_n = \sigma_{n+1}, \quad (2.16)$$

$$s'_n = \sigma_n + \sigma_{n+1} = p_{n+1} - 1, \quad (2.17)$$

$$s_n - s_{n-1} = p_n, \quad (2.18)$$

$$s'_n - s'_{n-1} = q_n, \quad (2.19)$$

$$s_n - s_{n-2} = q_n, \quad (2.20)$$

$$s'_n - s'_{n-2} = 2p_n. \quad (2.21)$$

Some of the above features are interrelated, e.g., (2.14) and (2.16) together confirm (2.5).

Observe, from (1.4), (2.10), and (2.12), that σ_n is the coefficient of b in S_n . Another way of arriving at this conclusion is to recall that in (2.9) $\alpha = 1$ for both $\{p_n\}$ and $\{q_n\}$ while $b = 3$ for $\{q_n\}$ but $b = 2$ for $\{p_n\}$, i.e., a "b" difference of $3 - 2 = 1$.

Similar remarks apply later in relation to (1.4a), (2.10a), and (2.12a).

3. PARTIAL SUMS: NEGATIVE SUBSCRIPTS

Corresponding to the results for positive subscripts in the previous section, we have, for negative subscripts,

$$\begin{array}{cccccccccc} n: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \\ p_{-n}: & 1 & -2 & 5 & -12 & 29 & -70 & 169 & -408 & \dots \\ q_{-n}: & -1 & 3 & -7 & 17 & -41 & 99 & -239 & 577 & \dots \end{array} \quad (1.3a)$$

since

$$p_{-n} = (-1)^{n+1} p_n \quad (3.1)$$

and

$$q_{-n} = (-1)^n q_n, \quad (3.2)$$

as may be readily demonstrated.

Tabulating the simplest expressions in the generalized Pell sequence $\{P_{-n}\}$, and the corresponding sequence of sums $\{S_{-n}\}$ which begins afresh with $S_{-1} = P_{-1}$, gives:

n	P_{-n}	S_{-n}
1	$5a - 2b$	$5a - 2b$
2	$-12a + 5b$	$-7a + 3b$
3	$29a - 12b$	$22a - 9b$
4	$-70a + 29b$	$-48a + 20b$
5	$169a - 70b$	$121a - 50b$
6	$-408a + 169b$	$-287a + 119b$
7	$985a - 408b$	$698a - 289b$
8	$-2378a + 985b$	$-1680a + 696b$

(1.4a)

Clearly,

$$P_{-n} = ap_{-n-2} + bp_{-n-1} \quad [P_0 = b - 2a \text{ as in (1.1)}]. \quad (1.5a)$$

Write $s_{-n} = \sum_{i=1}^n p_{-i}$. Then, as for (1.16), we obtain

$$s_{-n} = \frac{-P_{-n} - P_{-n-1} + 1}{2} = \frac{-q_{-n} + 1}{2} \quad (s_0 = 0) \quad (1.16a)$$

since, by (1.8) and (1.9),

$$q_{-n} = p_{-n} + p_{-n-1}. \quad (1.14a)$$

With a little effort, we derive

$$S_{-n} = \frac{-P_{-n} - P_{-n-1} + 3a - b}{2} = a(s_{-n-2} + 1) + b(s_{-n-1} - 1) \quad (S_0 = 0) \quad (1.6a)$$

and the recurrence

$$S_{-n+2} = 2S_{-n+1} + S_{-n} - 3a + b \quad (S_0 = 0). \quad (1.7a)$$

Paralleling the procedures in the previous section, we have the following four theorems.

Theorem 1a: $S_{-4n} = q_{2n}(-aq_{2n+2} + bq_{2n+1}) + 3a - b.$

Theorem 2a: $S_{-4n+2} = q_{2n}(-aq_{2n} + bq_{2n-1}) + 2a.$

Theorem 3a: $S_{-4n+1} = q_{2n}(aq_{2n+1} - bq_{2n}) + a.$

Theorem 4a: $S_{-4n-1} = q_{2n+1}(aq_{2n+2} - bq_{2n+1}) + 2a - b.$

Putting $a = 1, b = 2$, we have the Pell numbers results:

$$s_{-4n} = -(q_{2n})^2 + 1, \quad (2.1a)$$

$$s_{-4n+2} = -(q_{2n-1})^2, \quad (2.2a)$$

$$s_{-4n+1} = q_{2n}q_{2n-1} + 1, \quad (2.3a)$$

$$s_{-4n-1} = q_{2n}q_{2n+1}. \quad (2.4a)$$

Fortunately, (2.1a)-(2.4a) may be amalgamated into one pleasing form [cf. (1.16a)],

$$s_{-n} = \frac{-q_n + 1}{2}. \quad (2.5a)$$

Furthermore, from (1.7a),

$$s_{-n+2} = 2s_{-n+1} + s_{-n} - 1. \quad (2.6a)$$

Coming now to the special case $a = 1, b = 3$ again, we see that, denoting $s'_{-n} = \sum_{i=1}^n q_{-i}$,

$$s'_{-n} = -p_{-n} \quad (s'_0 = 0) \quad (2.7a)$$

and

$$s'_{-n+2} = 2s'_{-n+1} + s'_{-n}. \quad (2.8a)$$

Writing

$$\sigma_{-n} = s'_{-n} - s_{-n}, \quad (2.9a)$$

we may tabulate values of $\{\sigma_{-n}\}$ as in (2.10) with a recurrence corresponding to (2.11), thus,

$n =$	1	2	3	4	5	6	7	8	...
$s'_{-n} =$	-1	2	-5	12	-29	70	-169	408	...
$s_{-n} =$	1	-1	4	-8	21	-49	120	-288	...
$\sigma_{-n} =$	-2	3	-9	20	-50	119	-289	696	...

(2.10a)

whence

$$\sigma_{-n+2} = 2\sigma_{-n+1} + \sigma_{-n} + 1 \quad (\sigma_0 = 0). \quad (2.11a)$$

It follows from (1.4a) that

$$S_{-n} = a(\sigma_{-n-1} + 2) + b\sigma_{-n}. \quad (2.12a)$$

Furthermore,

$$q_{-n} = \sigma_{-n} - \sigma_{-n+2} \quad (2.13a)$$

while

$$\sigma_{-n} = \frac{-q_{-n-1} - 1}{2}. \quad (2.14a)$$

Additional results include

$$s_{-n} = \sigma_{-n+1} + 1, \quad (2.16a)$$

$$s'_{-n} = -\sigma_{-n+1} + \sigma_{-n+2}, \quad (2.17a)$$

$$s_{-n-1} - s_{-n} = p_{-n-1}, \quad (2.18a)$$

$$s'_{-n-1} - s'_{-n} = q_{-n-1}, \quad (2.19a)$$

$$s_{-n-2} - s_{-n} = q_{-n-1}, \quad (2.20a)$$

$$s'_{-n-2} - s'_{-n} = 2p_{-n-1}. \quad (2.21a)$$

One may also ascertain that

$$\begin{cases} \sigma_n + \sigma_{-n+1} = -1 & n \text{ even,} \\ \sigma_n - \sigma_{-n+1} = 0 & n \text{ odd.} \end{cases} \quad (3.3)$$

Properties of $\{\sigma_n\}$ are the subject of another paper, so we do not pursue the occurrence of it in this exposition.

Other facets of the patterns in $P_{\pm n}$ and $S_{\pm n}$ may be recorded:

$$P_n + (-1)^{n-1} P_{-n+2} = 2aq_{n-1}, \quad (3.4)$$

$$P_n + (-1)^n P_{-n+2} = 2(-a+b)p_{n-1}, \quad (3.5)$$

$$P_n + (-1)^n P_{-n+4} = 2bq_{n-2}, \quad (3.6)$$

$$P_n + (-1)^{n-1} P_{-n+4} = 2(a+b)p_{n-2}, \quad (3.7)$$

$$S_{2n} + S_{-2n+1} = 2ap_{2n} + 2a - b, \quad (3.8)$$

$$S_{2n} - S_{-2n+1} = (-a+b)q_{2n} - a, \quad (3.9)$$

$$S_{2n+1} + S_{-2n} = (-a+b)q_{2n+1} + 2a - b, \quad (3.10)$$

$$S_{2n+1} - S_{-2n} = 2ap_{2n+1} - a. \quad (3.11)$$

Simson formulas for P_{-n} and S_{-n} may be obtained in the manner used for (1.18) and (1.19). In the first instance,

$$P_{-n-1}P_{-n+1} - P_{-n}^2 = (-1)^n(a^2 + 2ab - b^2), \quad (1.18a)$$

i.e., (1.18) is valid for all n . Discovery of the negative-subscript Simson analogue of (1.19) (with specializations for s_{-n} and s'_{-n}) is left to the spirit of enquiry and adventure of the reader (to be attempted because it is **there!**).

4. THE FIBONACCI CASE

A more expansive treatment of [4] will now be outlined. Ordinary Fibonacci and Lucas numbers will be represented by f_n and ℓ_n , respectively, while the upper-case notation F_n for the *generalized Fibonacci number* will be retained. To avoid confusion, we will use $T_n = \sum_{i=1}^n F_i$. Basic properties of $\{f_n\}$ and $\{\ell_n\}$ will be assumed.

Mutatis mutandis, we have [4]

$$T_n = F_{n+2} - b = af_n + b(f_{n+1} - 1) \quad (T_0 = 0 \neq F_0 = -a + b) \quad (4.1)$$

with, in particular,

$$T_{4n} = \ell_{2n} F_{2n+2} - 2b \quad [= F_{4n+2} - b \text{ from (4.1)}] \quad (4.2)$$

and

$$T_{4n-2} = \ell_{2n-1} F_{2n+1} \quad [= F_{4n} - b \text{ from (4.1)}]. \quad (4.3)$$

Moreover [4], there is the recurrence

$$T_{n+2} = T_{n+1} + T_n + b. \quad (4.4)$$

If $a = 1, b = 1$, and if we write $t_n = \sum_{i=1}^n f_i$, then

$$t_n = f_{n+2} - 1, \quad (4.5)$$

$$t_{4n} = \ell_{2n} f_{2n+2} - 2 \quad [= f_{4n+2} - 1 \text{ from (4.5)}], \quad (4.6)$$

and

$$t_{4n-2} = \ell_{2n-1} f_{2n+1} \quad [= f_{4n} - 1 \text{ from (4.5)}], \quad (4.7)$$

so that $\ell_{2n-1} | t_{4n-2}, f_{2n+1} | t_{4n-2}$, e.g., for $n = 4, (\ell_7 = 29) | 986$ and $(f_9 = 34) | 986$. Furthermore, (4.4) yields the recurrence

$$t_{n+2} = t_{n+1} + t_n + 1 \quad (t_0 = 0). \quad (4.8)$$

Instead of focusing on f_n , suppose we put $a = 1, b = 3$ and write $t'_n = \sum_{i=1}^n \ell_i$. Then

$$t'_n = \ell_{n+2} - 3, \quad (4.9)$$

$$t'_{4n} = \ell_{2n} \ell_{2n+2} - 6 \quad [= \ell_{4n+2} - 3 \text{ from (4.9)}], \quad (4.10)$$

and

$$t'_{4n-2} = \ell_{2n-1} \ell_{2n+1} \quad [= \ell_{4n} - 3 \text{ from (4.9)}], \quad (4.11)$$

with the recurrence

$$t'_{n+2} = t'_{n+1} + t'_n + 3 \quad (t'_0 = 0). \quad (4.12)$$

Again, observe the factorization and divisibility in (4.11).

Table 1 lists values of T_n, t_n, t'_n [and τ_n (4.15)].

TABLE 1. Partial Sums for F_n ($n = 1, 2, \dots, 10$)

n	F_n		T_n		t_n	t'_n	τ_n
1	a		a		1	1	0
2		b	$a + b$		2	4	2
3	$a + b$		$2a + 2b$		4	8	4
4	$a + 2b$		$3a + 4b$		7	15	8
5	$2a + 3b$		$5a + 7b$		12	26	14
6	$3a + 5b$		$8a + 12b$		20	44	24
7	$5a + 8b$		$13a + 20b$		33	73	40
8	$8a + 13b$		$21a + 33b$		54	120	66
9	$13a + 21b$		$34a + 54b$		88	196	108
10	$21a + 34b$		$55a + 88b$		143	319	176

Negative subscripts are utilized to obtain results paralleling those above. First, however, we remark that [cf. (3.1), (3.2)]

$$f_{-n} = (-1)^{n+1} f_n \quad (4.13)$$

and

$$\ell_{-n} = (-1)^n \ell_n. \quad (4.14)$$

Readers are urged to construct appropriate tables of values for f_{-n} and ℓ_{-n} from (4.13) and (4.14). See Table 2 for $T_{-n} = \sum_{i=1}^n F_{-i}$ and hence for t_{-n} and t'_{-n} [and τ_{-n} (4.15a)].

TABLE 2. Partial Sums for F_{-n} ($n = 1, 2, \dots, 10$)

n	F_{-n}			T_{-n}			t_{-n}	t'_{-n}	τ_{-n}
1	$2a$	$-$	b	$2a$	$-$	b	1	-1	-2
2	$-3a$	$+$	$2b$	$-a$	$+$	b	0	2	2
3	$5a$	$-$	$3b$	$4a$	$-$	$2b$	2	-2	-4
4	$-8a$	$+$	$5b$	$-4a$	$+$	$3b$	-1	5	6
5	$13a$	$-$	$8b$	$9a$	$-$	$5b$	4	-6	-10
6	$-21a$	$+$	$13b$	$-12a$	$+$	$8b$	-4	12	16
7	$34a$	$-$	$21b$	$22a$	$-$	$13b$	9	-17	-26
8	$-55a$	$+$	$34b$	$-33a$	$+$	$21b$	-12	30	42
9	$89a$	$-$	$55b$	$56a$	$-$	$34b$	22	-46	-68
10	$-144a$	$+$	$89b$	$-88a$	$+$	$55b$	-33	77	110

Repeated application of the recurrence relation for $\{F_n\}$, with the initial conditions, yields

$$T_{-n} = -F_{-n+1} + a \quad (T_0 = 0). \quad (4.1a)$$

In particular,

$$T_{-10} = -F_{-9} + a = -88a + 55b = 11(-8a + 5b) = \ell_5 F_{-4} \quad (\text{i.e., } \ell_5 | T_{-10}, F_{-4} | T_{-10}).$$

Accordingly,

$$t_{-n} = -f_{-n+1} + 1 \quad (t_0 = 0), \quad (4.5a)$$

and

$$t'_{-n} = -\ell_{-n+1} + 1 \quad (t'_0 = 0). \quad (4.9a)$$

Setting

$$\tau_n = t'_n - t_n \quad (\tau_0 = 0), \quad (4.15)$$

we discover [cf. (2.11)-(2.15)] the following:

$$\tau_n = 2(f_{n+1} - 1) = 2t_{n-1}, \quad (4.16)$$

$$\tau_n - \tau_{n-2} = 2f_n, \quad (4.17)$$

$$\tau_{n+2} = \tau_{n+1} + \tau_n + 2, \quad (4.18)$$

$$\tau_{n+1}\tau_{n-1} - \tau_n^2 = 4\{(-1)^{n+1} - f_{n-2}\}. \quad (4.19)$$

Moreover,

$$T_n = a\left(\frac{\tau_{n-1}}{2} + 1\right) + b\frac{\tau_n}{2}. \quad (4.20)$$

Replacing n by $-n$ in (4.15) so that

$$\tau_{-n} = t'_{-n} - t_{-n} \quad (\tau_0 = 0), \quad (4.15a)$$

one may obtain a table of values of the numbers in the sequence $\{\tau_{-n}\}$, whence

$$\tau_{-n} = 2(-f_{-n}) = 2(t_{-n} - 1), \quad (4.16a)$$

$$\tau_{-n} - \tau_{-n+2} = 2f_{-n+1} \quad (n \geq 2), \quad (4.17a)$$

$$\tau_{-n+2} = \tau_{-n+1} + \tau_{-n}, \quad (4.18a)$$

$$\tau_{-n-1}\tau_{-n+1} - \tau_{-n}^2 = 4(-1)^n, \quad (4.19a)$$

$$T_{-n} = a \frac{\tau_{-n-2}}{2} + b \frac{\tau_{-n}}{2}. \quad (4.20a)$$

Note that $\frac{1}{2}\tau_n$ and $\frac{1}{2}\tau_{-n}$ in (4.20) and (4.20a) are the coefficients of b in T_n and T_{-n} , respectively. Also refer to Tables 1 and 2. The reason for this is that $a = 1$ for both $\{f_n\}$ and $\{\ell_n\}$, but $b = 3$ for $\{\ell_n\}$ and $b = 1$ for $\{f_n\}$, i.e., there is a "b" difference of $3 - 1 = 2$.

Going back now to $\{F_n\}$ and $\{T_n\}$, we discover [cf. (3.4)-(3.11)]:

$$F_n + (-1)^{n+1}F_{-n+2} = a\ell_{n-1} \quad (n \geq 2), \quad (4.21)$$

$$F_n + (-1)^nF_{-n+2} = (-1)^n(-a + 2b)f_{n-1} \quad (n \geq 2), \quad (4.22)$$

$$F_n + (-1)^{n-1}F_{-n+4} = (2a + b)f_{n-2} \quad (n \geq 4), \quad (4.23)$$

$$F_n + (-1)^nF_{-n+4} = b\ell_{n-2}, \quad (4.24)$$

$$T_{2n} + T_{-2n+1} = (2a + b)f_{2n} + a - b, \quad (4.25)$$

$$T_{2n} - T_{-2n+1} = b\ell_{2n} - (a + b), \quad (4.26)$$

$$T_{2n+1} + T_{-2n} = b\ell_{2n+1} + a - b, \quad (4.27)$$

$$T_{2n+1} - T_{-2n} = (2a + b)f_{2n+1} - (a + b). \quad (4.28)$$

No doubt further identities of this *genre* are discoverable.

Frequent comparison of corresponding outcomes for the Pell and Fibonacci cases is both desirable and instructive. In this context, discovery of the Simson formulas for F_n , T_n , t_n , and t'_n (for $n > 0$, $n < 0$)—some of them not a pretty sight!—might be undertaken.

The Additions $s_n + s'_n$ and $t_n + t'_n$

Instead of considering the differences $\sigma_n = s'_n - s_n$ and $\tau_n = t'_n - t_n$, suppose the additions $\kappa_n = s'_n + s_n$ and $\lambda_n = t'_n + t_n$ are examined.

Consider then Table 3,

TABLE 3. Addition of Partial Sums

	$n =$	1	2	3	4	5	6	7	8	...
$\kappa_n =$	$s'_n + s_n =$	2	7	19	48	118	287	695	1680	...
$\kappa_{-n} =$	$s'_{-n} + s_{-n} =$	0	1	-1	4	-8	21	-49	120	...
$\lambda_n =$	$t'_n + t_n =$	2	6	12	22	38	64	106	174	...
$\lambda_{-n} =$	$t'_{-n} + t_{-n} =$	0	2	0	4	-2	8	-8	18	...

in which

$$\kappa_0 = 0, \quad (4.29)$$

$$\lambda_0 = 0, \quad (4.30)$$

whence

$$\kappa_n = \sigma_{n+2} - 1 = s_{n+1} - 1, \quad (4.31)$$

$$\kappa_{n+2} = 2\kappa_{n+1} + \kappa_n + 3, \quad (4.32)$$

$$\kappa_{n+2} - \kappa_n = q_{n+3}, \quad (4.33)$$

$$\kappa_{n+1} - \kappa_n = p_{n+1}. \quad (4.34)$$

Moreover,

$$\kappa_{-n} = \sigma_{-n+2} + 1, \quad (4.31a)$$

$$\kappa_{-n+2} = 2\kappa_{-n+1} + \kappa_{-n} - 1. \quad (4.32a)$$

On the other hand,

$$\lambda_n = \tau_{n+2} - 2, \quad (4.35)$$

$$\lambda_{n+2} = \lambda_{n+1} + \lambda_n + 4, \quad (4.36)$$

$$\lambda_{n+2} - \lambda_n = 2f_{n+4}, \quad (4.37)$$

$$\lambda_{n+1} - \lambda_n = 2f_{n+2}, \quad (4.38)$$

while

$$\lambda_{-n} = \tau_{-n+2} + 2, \quad (4.35a)$$

$$\lambda_{-n+2} = \lambda_{-n+1} + \lambda_{-n} - 2. \quad (4.36a)$$

Aware of the opportunities offered by this amplification of our theory, we may develop properties corresponding to those for differences until satiated.

5. CONCLUDING REMARKS

Finally, there are a few thoughts worthy of further consideration.

- (a) Other pairs of sequences related like $\{f_n\}$ and $\{\ell_n\}$, and $\{p_n\}$ and $\{q_n\}$ exist. Our results above suggest analogous—if, perhaps, less interesting—properties for such pairs.
- (b) Sequences $\{\sigma_n\}$ and $\{\frac{1}{2}\tau_n\}$ ($n > 0$) occur naturally *inter alia* in the minimal and maximal representations of positive integers by Pell and Fibonacci numbers, respectively. The former sequence is part of the stimulus for a separate research program.
- (c) Recurrences of the form

$$R_{n+2} = kR_{n+1} + R_n + c \quad (k, c \text{ constants}) \quad (5.1)$$

appear in many guises in this paper, for example, when $R_n = S_n, s_n, s'_n, \sigma_n, \kappa_n, T_n, t_n, t'_n, \tau_n$, and λ_n , with extensions to negative subscripts. Such recurrences (5.1) arise in other circumstances, e.g., in a graph-theoretic context, and are the subject of a separate investigation.

- (d) Numbers q_n of the sequence $\{q_n = \frac{1}{2}Q_n\}$, where Q_n are the *Pell-Lucas numbers*, feature prominently in a variety of papers. They (and p_n) have been called the *Eudoxus numbers* [1], though their first "official" appearance, according to [5], seems to have been in [3] in 1916, while some of the properties of q_n in relation to p_n have been recorded in [6] in 1949.

Can anyone tell me if there is any justification for the name "Eudoxus numbers" to describe the members of these interesting sequences? After all, the life-span of the ancient Greek mathematical genius, Eudoxus (ca. 408-355 B.C.), is a very far off human event.

Many, indeed, have been the fascinating and pleasurable ramifications of our modest attempt to expand the brief material in [4]. Evidently, there is much scope for further exploration and discovery in this field. Mindful of our stated objectives, however, we rest our case at this point.

REFERENCES

1. F. J. Budden. *An Introduction to Number Scales and Computers*. London: Longmans, 1965.
2. A. F. Horadam & Br. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.
3. W. J. Miller, ed. *Mathematical Questions and Solutions from the Educational Times* **1** (1916):9.
4. M. Rumney & E. J. F. Primrose. "Relations between a Sequence of Fibonacci Type and a Sequence of its Partial Sums." *The Fibonacci Quarterly* **9.3** (1971):296-98.
5. N. J. A. Sloane. *A Handbook of Integer Sequences*. New York: Academic Press, 1973.
6. V. Thébault. "Concerning Two Classes of Remarkable Perfect Square Pairs." *Amer. Math. Monthly* **56** (1949):443-48.

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SEVENTH INTERNATIONAL RESEARCH CONFERENCE

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CYCLIC FIBONACCI ALGEBRAS

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0. INTRODUCTION

A *Fibonacci algebra* is a group equipped with a unary operation ϕ satisfying the laws

$$(xy)\phi = x\phi y\phi, \text{ and } xx\phi \dots x\phi^{m-1} = x\phi^m$$

for a fixed integer $m \geq 2$. If, in addition, the law

$$x\phi^n = x$$

holds for a fixed integer $n \geq 2$, the algebra is called *periodic*. The corresponding variety $\mathfrak{B}(m, n)$ has been studied by several authors (see [4] and the references cited there) and, in particular, it is known that the monogenic free object $A(m, n)$ is just the Fibonacci group

$$F(m, n) = \langle x_1 \dots x_n \mid x_i x_{i+1} \dots x_{i+m-1} = x_{i+m}, 1 \leq i \leq n, i \bmod n \rangle$$

made abelian.

It is also known [3] that $A(m, n)$ is always a finite group whose order $a_{m,n}$ is the resultant of the polynomials

$$f(x) = x^n - 1, \quad g(x) = 1 + x + \dots + x^{m-1} - x^m, \quad (1)$$

namely,

$$a_{m,n} = (m-1) \prod_{k=1}^{n-1} |g(\omega_k)|, \quad (2)$$

where the product is taken over all nontrivial n^{th} roots of unity, $\omega_k = e^{2\pi ki/n}$, $k = 1, 2, \dots, n-1$. It follows that, for any prime p dividing $a_{m,n}$, the highest common factor $(f(x), g(x))_p$ over the prime field $GF(p)$ has positive degree. It is shown in [5] that $A(m, n)$ is cyclic if and only if

$$\deg(f(x), g(x))_p = 1 \quad \forall p \mid a_{m,n}. \quad (3)$$

We shall apply this criterion to certain (classes of) values of m and n to determine when $A(m, n)$ is cyclic. It follows that, in these cases, the exponent of the free objects in $\mathfrak{B}(m, n)$ is just $a_{m,n}$. This reconfirms some of the results in [2], where a constructive approach is adopted to calculating exponents in $\mathfrak{B}(m, n)$. On the other hand, the case when $A(m, n)$ is noncyclic is also of interest, at least when $m = 2$. For then it follows from results in [1] that $F(2, n)$ maps homomorphically onto the free object of rank two in the variety of groups of exponent p and class four for some prime p .

In each of the ensuing sections, we consider the $A(m, n)$ with $m, n \geq 2$ and related as in the section heading. We fix the notation in (1) and (2) above along with

$$f_1 := f / (x-1) = 1 + x + \dots + x^{n-1},$$

and emphasize the fact that, throughout what follows, **we consider only primes p dividing $a_{m,n}$.**

1. $m \equiv -1 \pmod{n}$

Setting $m = qn - 1$, we see that $g = (1 + x^n + \cdots + x^{(q-1)n})f_1 - 2x^m$, so that $a_{m,n} = (m-1)2^{n-1}$. Also, for p odd, $(g, f_1)_p = (-2x^m, f_1)_p = 1$, so that $(f, g)_p = x - 1$ and (3) holds in this case.

When $p = 2$, however, $(g, f_1)_2 = f_1$, whence $(g/(x+1), f_1)_2 = f_1$ or $f_1/(x+1)$, which has degree ≥ 1 unless $f_1 = x+1$, that is, $n = 2$. In the case $p = n = 2$, $f = 1 + x^2$, $g = 1 + x + \cdots + x^{2q-1} = (1+x)(1+x^2 + \cdots + x^{2q-2})$, and $(f, g)_2 = 1 + x$ if and only if q is odd, that is, $m \equiv 1 \pmod{4}$.

Proposition 1: When $m \equiv 1 \pmod{n}$, $A(m, n)$ has order $(m-1)2^{n-1}$ and is cyclic if and only if $n = 2$ and $m \equiv 1 \pmod{4}$.

2. $m \equiv 0 \pmod{n}$

Here, the calculation is similar to (but much easier than) the above, and we obtain the following. We leave the proof as an exercise.

Proposition 2: When $m \equiv 0 \pmod{n}$, $A(m, n)$ has order $(m-1)$ and is cyclic.

3. $m \equiv 1 \pmod{n}$

Setting $m = qn + 1$, we see that

$$g = (1 + x^n + \cdots + x^{(q-1)n})f_1 + x^{m+1} - x^m,$$

so that $a_{m,n} = (m-1)n$ and we consider primes $p \mid (m-1)n$. It is clear that, over any field,

$$h_1 := (g, f_1) = (x^{m-1} - x^m, f_1) = (1 - x, f_1).$$

Now $f_1(1) = n$, so that for $p \nmid n$, this hcf is 1 and $(f, g)_p = x - 1$ satisfies (3).

But if $p \mid n$, then $h_1 = x - 1$ and $(f, g)_p = (x - 1)^2$ or $(x - 1)$ according as $x - 1$ divides

$$g_1 := g / (1 - x) = 1 + 2x + \cdots + (m-1)x^{m-2} + x^{m-1}$$

or not. But

$$g_1(1) = \frac{1}{2}m(m-1) + 1 = \frac{1}{2}(qn+1)qn + 1,$$

and for $p \mid n$ this is zero modulo p if and only if

$$p = 2, \quad q \text{ is odd, and } n \equiv 2 \pmod{4}.$$

Proposition 3: When $m \equiv 1 \pmod{n}$, $A(m, n)$ has order $(m-1)n$ and is cyclic except when $n \equiv 2 \equiv m-1 \pmod{4}$.

4. $m \equiv -2 \pmod{n}$

We let $m = qn - 2$ so that

$$g = (1 + x^n + \cdots + x^{(q-1)n})f_1 - x^m(2 + x),$$

and

$$\begin{aligned} a_{m,n} &= (m-1) \prod_{\omega^n=1 \neq \omega} |g(\omega)| \\ &= (m-1) \prod_{\omega^n=1 \neq \omega} |2+\omega| = (m-1) |f_1(-2)|. \end{aligned}$$

Moreover, $(g, f_1) = (2+x, f_1)$, and (g, f) is a divisor of $(x-1)(x+2)$. However, $|f_1(-2)| = (2^n - (-1)^n)/3$, and we distinguish four cases.

- (i) $p \nmid (2^n - (-1)^n)/3$, when $(g, f) = x-1$ and (3) holds
- (ii) $p \nmid (m-1)$, when $(g, f) = x+2$ and (3) holds.
- (iii) $p \mid (m-1, (2^n - (-1)^n)/3)$ and $p \neq 3$, when $(g, f) = (x-1)(x+2)$ and (3) fails.
- (iv) $p = 3 \mid (m-1, (2^n - (-1)^n)/3)$, when $-2 \equiv 1 \pmod{3}$ and

$$(g, f)_3 = (x-1)(1+x+\cdots+x^{n-1}, 1+2x+\cdots+(m-1)x^{m-2}+x^{m-1})_3.$$

But the second term in the hcf, evaluated at $x=1$, is $\frac{1}{2}m(m-1)+1 \equiv 1 \pmod{3}$, showing that $(g, f)_3 = x-1$ and (3) holds.

It follows that $A(m, n)$ is cyclic in this case except when case (iii) arises, that is, when there is a prime $p \neq 3$ such that $qn \equiv 3, (-2)^n \equiv 1 \pmod{p}$.

Proposition 4: When $m \equiv -2 \pmod{n}$, $A(m, n)$ has order $(m-1)(2^n - (-1)^n)/3$ and is cyclic unless there is a prime $p \neq 3$ such that

$$m \equiv 1 \pmod{p} \quad \text{and} \quad n = ka,$$

where a is the order of $-2 \pmod{p}$.

Thus, for example, we see that $A(6, 4)$ is noncyclic by taking $p = 5$.

5. $n = 2m$

In this case

$$\begin{aligned} a_{m,n} &= (m-1) \prod_{\omega^n=1 \neq \omega} |(1+\omega+\cdots+\omega^{m-1})-\omega^m| \\ &= (m-1) \prod_{\omega^m=-1} \left| 1 + \frac{1-\omega^m}{1-\omega} \right| = (m-1) \prod_{\omega^m=-1} \left| \frac{3-\omega}{1-\omega} \right| \\ &= (m-1)(1+3^m)/2. \end{aligned}$$

As usual, let $p \mid a_{m,n}$ and assume first that p is odd. Then $f = x^{2m} - 1$ is the product of co-prime polynomials $x^m - 1$ and $x^m + 1$ and we compute $(f, g)_p$ in two stages. Firstly,

$$((x^m - 1)/(x - 1), g) = ((x^m - 1)/(x - 1), x^m) = 1,$$

so that $((x^m - 1), g)_p = x - 1$ or 1 according as $p \mid (m-1)$ or not. Secondly,

$$(1+x^m, (1-x)g) = (1+x^m, 1-2x^m+x^{m+1}) = (1+x^m, 3-x),$$

which is $x-3$ or 1 according as $p|(1+3^m)$ or not, and since p is odd this is also the hcf of $1+x^m$ and g . Thus, for p odd, $(f, g)_p$ is linear unless p divides both $m-1$ and 3^m+1 .

Now let $p=2$ so that m must be odd, $2k+1$ say, and a simple calculation shows that $(f, g)_2 = x+1$ or x^2+1 according as k is even or odd.

Proposition 5: When $n=2m$, $A(m, n)$ has the order $(m-1)(1+3^m)/2$ and is cyclic unless either $m \equiv 3 \pmod{p}$ or there is an odd prime p such that $m \equiv 1 \pmod{p}$ and $3^m \equiv -1 \pmod{p}$.

D. A. Burgess has pointed out that these equations certainly have a solution in the case in which $p \equiv 6 \pm 1 \pmod{12}$.

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REFERENCES

1. H. Aydin & G. C. Smith. "Finite p -Quotients of Some Cyclically-Presented Groups." To appear in *J. London Math. Soc.*
2. M. W. Bunder, D. L. Johnson, & A. C. Kim. "On the Variety of Algebras $V(m, n)$." Preprint.
3. D. L. Johnson. "A Note on the Fibonacci Groups." *Israel J. Math.* **17** (1974):277-82.
4. D. L. Johnson & A. C. Kim. "Periodic Fibonacci Algebras." *Proc. Edinburgh Math. Soc.* **35** (1992):169-72.
5. D. L. Johnson & R. W. K. Odoni. "Some Results on Symmetrically-Presented Groups." To appear in *Proc. Edinburgh Math. Soc.*

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A NOTE ON A GENERAL CLASS OF POLYNOMIALS

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1. INTRODUCTION

We consider polynomials $\{U_n(p, q; x)\}$ such that

$$U_n(p, q; x) = (x + p)U_{n-1}(p, q; x) - qU_{n-2}(p, q; x), \quad n \geq 2 \quad (1)$$

with $U_0(p, q; x) = 0$ and $U_1(p, q; x) = 1$.

The parameters p and q are arbitrary real numbers (with $q \neq 0$), and we denote by α, β the numbers such that $\alpha + \beta = p$ and $\alpha\beta = q$.

We see by induction that there exists a sequence $\{c_{n,k}(p, q)\}_{n \geq 0, k \geq 0}$ of numbers such that

$$U_{n+1}(p, q; x) = \sum_{k \geq 0} c_{n,k}(p, q) x^k, \quad (2)$$

with

$$c_{n,k}(p, q) = 0 \text{ if } k > n \text{ and } c_{n,n}(p, q) = 1, \quad n \geq 0.$$

The first few terms of the sequence $\{U_n(p, q; x)\}$ are

$$\begin{cases} U_2(p, q; x) = p + x \\ U_3(p, q; x) = (p^2 - q) + 2px + x^2 \\ U_4(p, q; x) = (p^3 - 2pq) + (3p^2 - 2q)x + 3px^2 + x^3. \end{cases}$$

Particular cases of $U_n(p, q; x)$ are the Fibonacci polynomials $F_n(x)$, the Pell polynomials $P_n(x)$ [4], the first Fermat polynomials $\Phi_n(x)$ [5], the Morgan-Voyce polynomials of the second kind $B_n(x)$ ([3], [6], [8], [9]), and the Chebyshev polynomials of the second kind $S_n(x)$ given by

$$\begin{aligned} U_n(0, -1; x) &= F_n(x), \\ U_n(0, -1; 2x) &= P_n(x), \\ U_n(0, 2; x) &= \Phi_n(x), \\ U_{n+1}(2, 1; x) &= B_n(x), \\ U_n(0, 1; 2x) &= S_n(x). \end{aligned}$$

We have used S_n in place of the customary U_n since U_n has been used in a different way in the present paper. For particular values of the variable x , one can obtain some interesting sequences of numbers.

(i) The sequence $\{U_n(p, q; -p)\}$ satisfies the recurrence

$$U_n(p, q; -p) = -qU_{n-2}(p, q; -p), \quad n \geq 2;$$

thus,

$$U_{2n}(p, q; -p) = 0 \quad \text{and} \quad U_{2n+1}(p, q; -p) = (-q)^n.$$

By (2), these can also be written

$$\sum_{k=0}^{2n-1} (-1)^k p^k c_{2n-1,k}(p, q) = 0 \quad (3)$$

and

$$\sum_{k=0}^{2n} (-1)^k p^k c_{2n,k}(p, q) = (-1)^n q^n. \quad (4)$$

(ii) It follows at once that the sequence $\{U_n(p, q; 0)\}$ is the generalized Fibonacci sequence defined by

$$U_n(p, q; 0) = pU_{n-1}(p, q; 0) - qU_{n-2}(p, q; 0),$$

with $U_0(p, q; 0) = 0$ and $U_1(p, q; 0) = 1$. Therefore,

$$U_{n+1}(p, q; 0) = \sum_{i+j=n} \alpha^i \beta^j = \begin{cases} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} & \text{if } \alpha \neq \beta, \\ (n+1)\alpha^n & \text{if } \alpha = \beta. \end{cases}$$

By (2), notice that

$$c_{n,0}(p, q) = U_{n+1}(p, q; 0) = \sum_{i+j=n} \alpha^i \beta^j. \quad (5)$$

More generally, our aim is to express the coefficient $c_{n,k}(p, q)$ as a polynomials in (α, β) and as a polynomial in (p, q) .

2. THE TRIANGLE OF COEFFICIENTS

One can display the sequence $\{c_{n,k}(p, q)\}$ in a triangle, thus:

$n \backslash k$	0	1	2	3	...
0	1	0	0	0	
1	p	1	0	0	
2	$p^2 - q$	$2p$	1	0	
3	$p^3 - 2pq$	$3p^2 - 2q$	$3p$	1	
\vdots					

Comparing the coefficients of x^k in the two members of (1), we see by (2) that, for $n \geq 2$ and $k \geq 1$,

$$\begin{aligned} c_{n,k}(p, q) &= c_{n-1,k-1}(p, q) + pc_{n-1,k}(p, q) - qc_{n-2,k}(p, q) \\ &= c_{n-1,k-1} + \beta c_{n-1,k} + \alpha(c_{n-1,k} - \beta c_{n-2,k}) \\ &= c_{n-1,k-1} + \alpha c_{n-1,k} + \beta(c_{n-1,k} - \alpha c_{n-1,k}), \end{aligned} \quad (6)$$

where, for brevity, we put $c_{n,k}$ for $c_{n,k}(p, q)$. From this, one can easily obtain another recurrence relation.

Theorem 1: For every $n \geq 1$ and $k \geq 1$, we have

$$\begin{aligned} c_{n,k} &= \beta c_{n-1,k} + \sum_{i=0}^{n-1} \alpha^{n-1-i} c_{i,k-1} \\ &= \alpha c_{n-1,k} + \sum_{i=0}^{n-1} \beta^{n-1-i} c_{i,k-1}. \end{aligned} \quad (7)$$

Proof: In fact, (7) is clear by direct computation for $n \leq 2$ (recall that $\alpha + \beta = p$). Supposing that the relation is true for $n \geq 2$, then we have by (6) that

$$\begin{aligned} c_{n+1,k} &= \beta c_{n,k} + \alpha(c_{n,k} - \beta c_{n-1,k}) + c_{n,k-1} \\ &= \beta c_{n,k} + \alpha \sum_{i=0}^{n-1} \alpha^{n-1-i} c_{i,k-1} + c_{n,k-1} \\ &= \beta c_{n,k} + \sum_{i=0}^n \alpha^{n-i} c_{i,k-1}. \end{aligned}$$

This concludes the proof, and the other formula can be proved in the same way.

Let us examine some particular cases.

(i) Fibonacci polynomials. In this case we have $p = 0, q = -1$, and $\alpha = -\beta = 1$. From this, (7) becomes

$$\begin{aligned} c_{n,k} &= -c_{n-1,k} + \sum_{i=0}^{n-1} c_{i,k-1} \\ &= c_{n-1,k} + \sum_{i=0}^{n-1} (-1)^{n-1-i} c_{i,k-1}. \end{aligned}$$

(ii) Morgan–Voyce polynomials of the second kind. In this case, we have $p = 2, q = 1$, and $\alpha = \beta = 1$. Thus, (7) becomes

$$c_{n,k} = c_{n-1,k} + \sum_{i=0}^{n-1} c_{i,k-1},$$

which is the recursive definition of the DFFz triangle [2], known to be the triangle of coefficients of Morgan–Voyce polynomials ([1], [3]).

3. DETERMINATION OF $c_{n,k}(p, q)$ AS A POLYNOMIAL IN (α, β)

In our proof we shall need the following lemma.

Lemma: For every $k \geq 0$, we have

$$\frac{1}{(1 - pt + qt^2)^{k+1}} = \sum_{n \geq 0} d_{n,k} t^n, \quad (8)$$

with

$$d_{n,k} = \sum_{i+j=n} \binom{k+i}{k} \binom{k+j}{k} \alpha^i \beta^j.$$

Proof: Recall that

$$\phi_r(t) = \frac{1}{(1-rt)^{k+1}} = \sum_{n \geq 0} \binom{k+n}{k} r^n t^n,$$

where r is a real or complex parameter and $|rt| < 1$. Thus, we have

$$\begin{aligned} \frac{1}{(1-pt+qt^2)^{k+1}} &= \frac{1}{(1-\alpha t)^{k+1}(1-\beta t)^{k+1}} \\ &= \sum_{n \geq 0} \binom{k+n}{k} \alpha^n t^n \cdot \sum_{n \geq 0} \binom{k+n}{k} \beta^n t^n \\ &= \sum_{n \geq 0} d_{n,k} t^n, \end{aligned}$$

where

$$d_{n,k} = \sum_{i+j=n} \binom{k+i}{k} \binom{k+j}{k} \alpha^i \beta^j,$$

by application of Cauchy's rule for multiplying power series. Q.E.D.

Theorem 2: For every $n \geq 0$ and $k \geq 0$, we have

$$c_{n,k}(p, q) = \sum_{i+j=n-k} \binom{k+i}{k} \binom{k+j}{k} \alpha^i \beta^j, \quad (9)$$

where we have used the convention $\sum_{i+j=s} a_{i,j} = 0$, if $s < 0$.

Proof: For brevity, we put $U_n(p, q; x) = U_n(x)$ and $c_{n,k}(p, q) = c_{n,k}$. Let us define the generating function of the sequence $\{U_n(x)\}$ by

$$f(x, t) = \sum_{n \geq 0} U_{n+1}(x) t^n.$$

By (1), we get

$$f(x, t) - 1 = \sum_{n \geq 1} U_{n+1}(x) t^n = t(x+p) \sum_{n \geq 1} U_n(x) t^{n-1} - qt^2 \sum_{n \geq 1} U_{n-1}(x) t^{n-2}.$$

The last sum can be written as $\sum_{n \geq 2} U_{n-2}(x) t^{n-2}$, since $U_0(x) = 0$. It follows from this that

$$f(x, t) - 1 = t(x+p)f(x, t) - qt^2 f(x, t).$$

Thus,

$$f(x, t) = \frac{1}{1 - (x+p)t + qt^2}. \quad (10)$$

We deduce from (10) that

$$\begin{aligned} \frac{k! t^k}{(1 - (x+p)t + qt^2)^{k+1}} &= \frac{\partial^k}{\partial x^k} f(x, t) = \sum_{n \geq 0} U_{n+1}^{(k)}(x) t^n \\ &= \sum_{n \geq k} U_{n+1}^{(k)}(x) t^n = \sum_{n \geq 0} U_{n+k+1}^{(k)}(x) t^{n+k}, \end{aligned}$$

since $U_{n+1}(x)$ is a polynomial of degree n .

Put $x = 0$ in the last formula and recall that

$$c_{n+k,k} = \frac{U_{n+k+1}^{(k)}(0)}{k!},$$

by Taylor's formula, to obtain

$$\frac{1}{(1-pt+qt^2)^{k+1}} = \sum_{n \geq 0} c_{n+k,k} t^n. \quad (11)$$

Comparing this formula with (8), we see that

$$c_{n+k,k} = d_{n,k} = \sum_{i+j=n} \binom{k+i}{k} \binom{k+j}{k} \alpha^i \beta^j.$$

This concludes the proof.

Remarks: (i) If $k = 0$, then (9) reduces to the classical formula (5).

(ii) Notice that (11) is the generating function of the k^{th} column of the triangle of coefficients $c_{n,k}$. If $k = 0$, we obtain in particular the well-known generating function of the generalized Fibonacci sequence, namely,

$$\frac{1}{1-pt+qt^2} = \sum_{n \geq 0} U_{n+1}(p, q; 0) t^n. \quad (12)$$

(iii) Using (6), one can obtain, by induction and with a little manipulation, another proof of Theorem 2.

Corollary 1: For every $n \geq 0$ and $k \geq 0$, we have

$$c_{n,k}(-p, q) = (-1)^{n-k} c_{n,k}(p, q).$$

Proof: The result follows immediately from (9) and the fact that $(-\alpha) + (-\beta) = -p$ and $(-\alpha)(-\beta) = q$.

4. SOME PARTICULAR CASES

The general formula (9) can be simplified in two cases:

(i) Supposing that $p^2 = 4q$, we have $\alpha = \beta$ and (8) becomes

$$\frac{1}{(1-pt+qt^2)^{k+1}} = \frac{1}{(1-\alpha t)^{2k+2}} = \sum_{n \geq 0} \binom{n+2k+1}{2k+1} \alpha^n t^n.$$

Hence, by (11), $c_{n,k} = c_{n,k}(p, q)$ takes the simpler form

$$c_{n,k} = \binom{n+k+1}{2k+1} \alpha^{n-k} = \binom{n+k+1}{2k+1} (p/2)^{n-k}.$$

If $p = 2$ and $q = 1$ (Morgan–Voyce polynomials of the second kind), we obtain the known relation [8]

$$B_n(x) = \sum_{k=0}^n \binom{n+k+1}{2k+1} x^n.$$

(ii) Supposing that $p = 0$, we have $\alpha = -\beta$ and (8) becomes

$$\frac{1}{(1-pt+qt^2)^{k+1}} = \frac{1}{(1+qt^2)^{k+1}} = \sum_{n \geq 0} (-1)^n \binom{n+k}{k} q^n t^{2n}.$$

Thus, by (11),

$$c_{2n+k,k} = (-1)^n \binom{n+k}{n} q^n \quad \text{and} \quad c_{2n+k+1,k} = 0 \quad \text{for } n \geq 0 \text{ and } k \geq 0.$$

This can be written

$$c_{2k+n,n} = (-1)^k \binom{k+n}{k} q^k \quad \text{and} \quad c_{2k+n+1,n} = 0.$$

Hence,

$$c_{n,n-2k} = (-1)^k \binom{n-k}{k} q^k, \quad \text{for } n-2k \geq 0 \quad \text{and} \quad c_{n,n-2k-1} = 0, \quad \text{for } n-2k-1 \geq 0.$$

Now, by (2),

$$U_{n+1}(0, q; x) = \sum_{k=0}^n c_{n,k}(0, q) x^k = \sum_{k=0}^n c_{n,n-k}(0, q) x^{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,n-2k}(0, q) x^{n-2k}.$$

Thus, we get the simplified formula

$$U_{n+1}(0, q; x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} q^k x^{n-2k}. \quad (13)$$

If $p = 0$ and $q = -1$, we obtain the known decomposition of Fibonacci polynomials

$$F_{n+1}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k},$$

and if $p = 0$ and $q = 1$, we have the similar expression of Chebyshev polynomials of the second kind

$$S_{n+1}(x) = U_{n+1}(0, 1; 2x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}.$$

5. DETERMINATION OF $c_{n,k}(p, q)$ AS A POLYNOMIAL IN (p, q)

Theorem 3: For every $n \geq 0$ and $k \geq 0$, we have

$$c_{n,k}(p, q) = \sum_{r=0}^{\lfloor (n-k)/2 \rfloor} (-1)^r \binom{n-r}{r} \binom{n-2r}{k} q^r p^{n-2r-k}. \quad (14)$$

Proof: It is clear that $U_{n+1}(p, q; x) = U_{n+1}(0, q; x+p)$. Thus,

$$c_{n,k}(p, q) = \frac{U_{n+1}^{(k)}(p, q; 0)}{k!} = \frac{U_{n+1}^{(k)}(0, q; p)}{k!}.$$

By (13), one can express the last member as

$$\begin{aligned} & \sum_{r=0}^{[n/2]} (-1)^r \binom{n-r}{r} \frac{(n-2r) \cdots (n-2r-k+1)}{k!} q^r p^{n-2r-k} \\ &= \sum_{r=0}^{[(n-k)/2]} (-1)^r \binom{n-r}{r} \binom{n-2r}{k} q^r p^{n-2r-k} \end{aligned}$$

This completes the proof of Theorem 3.

If $k = 0$, we get the formula known by Lucas ([7], p. 207), namely,

$$U_{n+1}(p, q; 0) = \sum_{r=0}^{[n/2]} (-1)^r \binom{n-r}{r} q^r p^{n-2r}. \quad (15)$$

6. RISING DIAGONAL FUNCTIONS

Let us define the rising diagonal functions $\{\Psi_n(p, q; x)\}$ of the sequence $\{c_{n,k}(p, q)\}$ —see the table—by $\Psi_0(p, q; x) = 0$ and

$$\Psi_{n+1}(p, q; x) = \sum_{k=0}^n c_{n-k,k}(p, q) x^k = \sum_{k=0}^{[n/2]} c_{n-k,k}(p, q) x^k, \text{ for } n \geq 0. \quad (16)$$

Notice that, from the table,

$$\Psi_1(p, q; x) = 1, \Psi_2(p, q; x) = p, \text{ and } \Psi_3(p, q; x) = p^2 - q + x. \quad (17)$$

Theorem 4: For every $n \geq 2$, we have

$$\Psi_n(p, q; x) = p\Psi_{n-1}(p, q; x) + (x - q)\Psi_{n-2}(p, q; x). \quad (18)$$

Proof: For brevity, we put $\Psi_n(p, q; x) = \Psi_n(x)$ and $c_{n,k}(p, q) = c_{n,k}$. By (17), the statement holds for $n = 2$ and $n = 3$. Supposing that (18) is true for $n \geq 3$, then we get, by (16),

$$\Psi_{n+1}(x) = c_{n,0} + \sum_{k=1}^{[n/2]} c_{n-k,k} x^k.$$

Recall from (5) that $c_{n,0} = U_{n+1}(0) = pc_{n-1,0} - qc_{n-2,0}$, and notice that $n - k \geq n - [n/2] \geq 2$, since $n \geq 3$. By these remarks and (6), one can write

$$\begin{aligned} \Psi_{n+1}(x) &= pc_{n-1,0} - qc_{n-2,0} + \sum_{k=1}^{[n/2]} (c_{n-1-k,k-1} + pc_{n-1-k,k} - qc_{n-2-k,k}) x^k \\ &= p \sum_{k=0}^{[n/2]} c_{n-1-k,k} x^k - q \sum_{k=0}^{[n/2]} c_{n-2-k,k} x^k + x \sum_{k=0}^{[n/2]-1} c_{n-2-k,k} x^k \\ &= p\Psi_n(x) + (x - q)\Psi_{n-1}(x), \text{ since } [n/2] - 1 = [(n-2)/2]. \end{aligned}$$

This concludes the proof.

Corollary 2: For every $n \geq 0$, we have

$$\Psi_{n+1}(p, q; x) = \sum_{r=0}^{[n/2]} \binom{n-r}{r} p^{n-2r} (x-q)^r. \quad (19)$$

Proof: By Theorem 4, and since $\Psi_0(x) = 0$, $\Psi_1(x) = 1$, it is clear that

$$\Psi_n(p, q; x) = U_n(p, q-x; 0),$$

and the result follows by (15).

Let us examine some particular cases.

(i) Put $x = q$ in (19) to get, by (16),

$$\sum_{k=0}^{[n/2]} q^k c_{n-k, k}(p, q) = p^n.$$

If $p = 1$ and $q = 1$, we get a known identity on the coefficients of the Morgan-Voyce polynomial of the second kind B_n , first noticed by Ferri, Faccio and D'Amico ([2], [3]), namely,

$$\sum_{k=0}^{[n/2]} c_{n-k, k}(2, 1) = 2^n.$$

(ii) Put $x = 1$ in (19) to get, by (16),

$$\sum_{k=0}^{[n/2]} c_{n-k, k}(p, q) = \sum_{r=0}^{[n/2]} \binom{n-r}{r} p^{n-2r} (1-q)^r,$$

which is more general than the above result.

(iii) If $p = 0$, then Corollary 2 implies by (16) that

$$\sum_{k=0}^n c_{2n-k, k}(0, q) x^k = (x-q)^n.$$

If $q = 1$ (Chebyshev polynomials of the second kind), or $q = 2$ (first Fermat polynomials), this identity was first noticed by Horadam [5] with slightly different notations.

7. THE ORTHOGONALITY OF THE SEQUENCE $\{U_n(p, q; x)\}$

In this paragraph we shall suppose that $q > 0$. Consider the sequence $\{R_n(p, q; x)\}$ defined by

$$R_n(p, q; x) = q^{(n-1)/2} S_n\left(\frac{x+p}{2\sqrt{q}}\right), \quad (20)$$

where $S_n(x)$ is the n^{th} Chebyshev polynomial of the second kind. Let us determine the recurrence satisfied by the sequence $\{R_n(p, q; x)\}$. One can write

$$R_n(p, q; x) = q^{(n-1)/2} \left[\left(\frac{x+p}{\sqrt{q}} \right) S_{n-1}\left(\frac{x+p}{2\sqrt{q}}\right) - S_{n-2}\left(\frac{x+p}{2\sqrt{q}}\right) \right]$$

$$\begin{aligned}
 &= (x+p)1^{(n-2)/2}S_{n-1}\left(\frac{x+p}{2\sqrt{q}}\right) - q q^{(n-3)/2}S_{n-2}\left(\frac{x+p}{2\sqrt{q}}\right) \\
 &= (x+p)R_{n-1}(p, q; x) - qR_{n-2}(p, q; x).
 \end{aligned}$$

Observe that the sequence $\{R_n(p, q; x)\}$ satisfies the recurrence (1) with $R_0(p, q; x) = 0$ and $R_1(p, q; x) = 1$, so that

$$R_n(p, q; x) = U_n(p, q; x). \quad (21)$$

Recalling that the sequence $\{S_n(x)\}$ is orthogonal over $[-1, 1]$ with respect to the weight $\sqrt{1-x^2}$, we deduce that the sequence $\{U_n(p, q; x)\}$ is orthogonal over $[-p-2\sqrt{q}, -p+2\sqrt{q}]$ with respect to the weight $w(x) = \sqrt{-x^2 - 2px - \Delta}$, where $\Delta = p^2 - 4q$.

In fact, for $n \neq m$, we have

$$\begin{aligned}
 \int_{-p-2\sqrt{q}}^{-p+2\sqrt{q}} U_n(x)U_m(x)w(x)dx &= q^{((n+m)/2)-1} \int_{-p-2\sqrt{q}}^{-p+2\sqrt{q}} S_n\left(\frac{x+p}{2\sqrt{q}}\right)S_m\left(\frac{x+p}{2\sqrt{q}}\right)w(x)dx \\
 &= 4q^{(n+m)/2} \int_{-1}^{+1} S_n(\omega)S_m(\omega)\sqrt{1-\omega^2} d\omega = 0,
 \end{aligned}$$

where $\omega = \frac{x+p}{2\sqrt{q}}$. In the case of the Morgan-Voyce polynomial of the second kind, $B_n(x)$, this orthogonality result was first given by Swamy [8].

If $\omega = \cos t$ ($0 < t < \pi$), it is well known that $S_n(\omega) = \frac{\sin nt}{\sin t}$. Thus, by (20) and (21), we have

$$U_n(p, q; -p+2\omega\sqrt{q}) = q^{(n-1)/2}S_n(\omega) = q^{(n-1)/2} \frac{\sin nt}{\sin t}.$$

From this, we see that the roots of $U_n(p, q; x)$ are given by

$$x_k = -p + 2\sqrt{q} \cos(k\pi/n), \quad k = 1, \dots, (n-1).$$

For instance, the roots of the Morgan-Voyce polynomial of the second kind, $B_n(x) = U_{n+1}(2, 1; x)$, are (see [9])

$$x_k = -2 + 2 \cos\left(\frac{kn}{n+1}\right) = -4 \sin^2\left(\frac{k\pi}{2n+2}\right), \quad k = 1, \dots, (n-1).$$

Under the hypothesis $q > 0$, we deduce from the general expression for x_k that the generalized Fibonacci sequences $U_n(p, q; 0)$ vanish if and only if there exists an integer k ($1 \leq k \leq n-1$) such that $\cos(k\pi/n) = p/2\sqrt{q}$.

8. CONCLUDING REMARK

In a future paper, we shall investigate the sequence $\{V_n(p, q; x)\}$ of polynomials, defined by

$$V_n(p, q; x) = (x+p)V_{n-1}(p, q; x) - qV_{n-2}(p, q; x), \quad n \geq 2,$$

with $V_0(p, q; x) = 2$ and $V_1(p, q; x) = x + p$.

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REFERENCES

1. R. André-Jeannin. "A Generalization of Morgan–Voyce Polynomials." *The Fibonacci Quarterly* (to appear).
2. G. Ferri, M. Faccio, & A. D'Amico. "Fibonacci Numbers and Ladder Network Impedance." *The Fibonacci Quarterly* **30.1** (1992):62-67.
3. G. Ferri, M. Faccio, & A. D'Amico. "The DFF and DFFz Triangles and Their Mathematical Properties." *Proceedings of the Fifth International Conference on Fibonacci Numbers and Their Applications*. (to appear).
4. A. F. Horadam & Br. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.2** (1985):7-20.
5. A. F. Horadam. "Chebyshev and Fermat Polynomials for Diagonal Functions." *The Fibonacci Quarterly* **19.4** (1979):328-333.
6. J. Lahr. "Fibonacci and Lucas Numbers and the Morgan–Voyce Polynomials in Ladder Networks and in Electrical Line Theory." In *Fibonacci Numbers and Their Applications* 1:141-61. Ed. A. N. Philippou, G. E. Bergum, & A. F. Horadam. Dordrecht: Kluwer, 1986.
7. E. Lucas. "Théorie des fonctions numériques simplement périodiques." *Amer. J. Math.* **1** (1878):184-220, 289-321.
8. M. N. S. Swamy. "Properties of the Polynomials Defined by Morgan–Voyce." *The Fibonacci Quarterly* **4.1** (1966):73-81.
9. M. N. S. Swamy. "Further Properties of Morgan–Voyce Polynomials." *The Fibonacci Quarterly* **6.2** (1968):167-75.

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EXTENDED DICKSON POLYNOMIALS

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1. PRELIMINARIES

The polynomials $p_n(x, c)$ defined by

$$p_n(x, c) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} (-c)^i x^{n-2i} \quad (n > 0), \quad (1.1)$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function and x is an indeterminate, are commonly referred to as *Dickson polynomials* (e.g., see [6]). These polynomials have been studied in the past years, both from the point of view of their theoretical properties [2], [6], and [14], and from that of their practical applications [7], [9], [10], and [13]. In particular, their relevance to public-key cryptosystems has been pointed out in [8], [11], [12], and [16]. As is shown, e.g., in [14], the coefficients of $p_n(x, c)$ are integers for any positive integer n and $c \in \mathbb{Z}$. It is also evident that

$$p_n(x, -1) = V_n(x), \quad (1.2)$$

where $V_n(x) = xV_{n-1}(x) + V_{n-2}(x)$ [$V_0(x) = 2, V_1(x) = x$] are the *Lucas polynomials* considered in [3] and [5]. In particular, we have

$$p_n(1, -1) = L_n, \quad (1.3)$$

where L_n is the n^{th} Lucas number.

In this paper, we consider the *extended Dickson polynomials* $p_n(x, c, U)$ defined in the next section.

2. INTRODUCTION AND DEFINITIONS

Let us define the extended Dickson polynomials $p_n(x, c, U)$ as the polynomials obtainable by replacing the upper range indicator in the sum (1.1) by a positive integer $U > \lfloor n/2 \rfloor$. This paper is essentially dedicated to the study of the case $x = -c = 1$.

By (1.1) we have

$$p_n(1, -1, U) \stackrel{\text{def}}{=} T_n(U) = \sum_{i=0}^U \frac{n}{n-i} \binom{n-i}{i} \quad (n > 0). \quad (2.1)$$

If $\lfloor n/2 \rfloor \leq U \leq n-1$, the sum (2.1) gives L_n as the binomial coefficient vanishes when $\lfloor n/2 \rfloor + 1 \leq i \leq n-1$. For example, if $n=5$ (so $U=2, 3$, or 4), then $T_5(U) = L_5 = 11$. If $U \geq n$, the upper argument of the binomial coefficient becomes negative for $i \geq n+1$, and the (nonzero) value of the

binomial coefficient can be obtained by (2.6). For $i = n$, the argument of the sum (2.1) assumes the indeterminate form $0 \cdot n / 0$ which will be settled in the sequel.

By (2.1) we can write

$$T_n(U) = L_n + H_n(k) \quad (k = U - n \geq 0), \quad (2.2)$$

where

$$H_n(k) = \sum_{i=n}^{n+k} \frac{n}{n-i} \binom{n-i}{i} = H_n(0) + \sum_{i=n+1}^{n+k} \frac{n}{n-i} \binom{n-i}{i} \quad (n > 0). \quad (2.3)$$

The quantity $H_n(0)$ in (2.3) is clearly given by the expression

$$H_n(0) = \sum_{i=n}^n \frac{n}{n-i} \binom{n-i}{i} \quad (n > 0), \quad (2.4)$$

which has the above said indeterminate form. In order to remove this obstacle, we use the combinatorial identities

$$\frac{h}{h-m} \binom{h-m}{m} = \binom{h-m}{m} + \binom{h-m-1}{m-1}, \quad (2.5)$$

$$\binom{-h}{m} = (-1)^m \binom{m+h-1}{h-1} = (-1)^m \binom{m+h-1}{m} \quad (2.6)$$

(available in [12], pp. 64 and 1, respectively), and rewrite (2.4) as

$$\begin{aligned} H_n(0) &= \sum_{i=n}^n \left[\binom{n-i}{i} + \binom{n-1-i}{i-1} \right] = \binom{0}{n} + \binom{-1}{n-1} \\ &= 0 + (-1)^{n-1} \binom{n-1}{n-1} = (-1)^{n-1} \quad (n > 0). \end{aligned} \quad (2.7)$$

For the sake of consistency, let us assume that the above result is valid also for $n = 0$, so

$$H_0(0) \stackrel{\text{def}}{=} (-1)^{-1} = -1. \quad (2.8)$$

On the basis of (2.3), (2.7), and (2.8), for given *nonnegative* integers n and k , let us define

$$H_n(k) \stackrel{\text{def}}{=} (-1)^{n-1} + \sum_{i=n+1}^{n+k} \frac{n}{n-i} \binom{n-i}{i} \quad (n, k \geq 0), \quad (2.9)$$

where the usual convention that

$$\sum_{i=a}^b f(i) = 0 \quad \text{for } b < a \quad (2.10)$$

has to be invoked for obtaining $H_0(0) = -1$.

The numbers $H_n(k)$ defined by (2.9) are the *companions* of the numbers

$$G_n(k) \stackrel{\text{def}}{=} \sum_{i=n}^{n+k} \binom{n-1-i}{i} = (-1)^n \sum_{j=0}^k (-1)^j \binom{n+2j}{j} \quad (2.11)$$

which have been thoroughly investigated in [4]. The numbers $G_n(k)$ arise from the incorrect use of a combinatorial formula for generating the Fibonacci numbers F_n , whereas the numbers $H_n(k)$

result from an analogous use of the combinatorial formula (2.1) which (under appropriate constraints on U) generates the Lucas numbers (compare (2.2) with [4, (1.7)]) and are the fruit of our mathematical curiosity. The principal aim of this paper is to give alternative expressions of the numbers $H_n(k)$ (Section 3), to find connections between these numbers and their companions $G_n(k)$, and to give a brief account of their properties (Sections 4 and 5). A glimpse of the application of the above argument to the Dickson polynomials (1.2) is caught in Section 6, where the polynomials $H_n(k, x)$ are considered.

3. THE NUMBERS $H_n(k)$

Letting $i = n + j$ in (2.9) yields

$$H_n(k) = (-1)^{n-1} + \sum_{j=1}^k \frac{n}{-j} \binom{-j}{n+j} \quad (3.1)$$

whence, by using the identity (2.6), we obtain the definition

$$H_n(k) = (-1)^{n-1} - (-1)^n \sum_{j=1}^k (-1)^j \frac{n}{j} \binom{n-1+2j}{j-1} \quad (3.2)$$

which can be rewritten as

$$H_n(k) = (-1)^{n-1} + (-1)^n \sum_{j=0}^{k-1} (-1)^j \frac{n}{j+1} \binom{n+1+2j}{j}. \quad (3.3)$$

By using (2.3), (2.5), and (2.6), the following equivalent definitions can be obtained, the proof of which are left as an exercise to the interested reader:

$$H_n(k) = (-1)^n \sum_{j=0}^k (-1)^j \left[\binom{n-1+2j}{j-1} - \binom{n-1+2j}{j} \right] \quad (3.4)$$

$$= (-1)^{n+1} \sum_{j=0}^{k-1} (-1)^j \binom{n+1+2j}{j} + (-1)^{n-1} \sum_{j=0}^k (-1)^j \binom{n-1+2j}{j}. \quad (3.5)$$

Definitions (3.4) and (3.5) show clearly that the numbers $H_n(k)$ are integers. Observe that $H_0(0) = -1$ results from (3.5) by invoking (2.10), and from (3.4) by assuming that

$$\binom{h}{-m} = 0 \quad (m \geq 1, h \text{ arbitrary}) \quad [12, \text{p. 2}]. \quad (3.6)$$

Some particular cases, beyond $H_n(0)$ given by (2.7) and (2.8), are

$$H_n(1) = (-1)^n (n-1), \quad (3.7)$$

$$H_n(2) = (-1)^{n-1} (n^2 + n + 2) / 2, \quad (3.8)$$

and

$$H_0(k) = -1 \quad \forall k, \quad (3.9)$$

which are readily obtainable by (3.2)–(3.5). The numbers $H_n(k)$ are shown in Table 1 for the first few values of n and k .

TABLE 1. The Numbers $H_n(k)$ for $0 \leq n, k \leq 5$

$k \backslash n$	0	1	2	3	4	5
0	-1	1	-1	1	-1	1
1	-1	0	1	-2	3	-4
2	-1	2	-4	7	-11	16
3	-1	-3	10	-21	37	-59
4	-1	11	-32	69	-128	216
5	-1	-31	100	-228	444	-785

4. SOME IDENTITIES INVOLVING THE NUMBERS $H_n(k)$ AND $G_n(k)$

First of all, we give a relation between the numbers $H_n(k)$ and their companions $G_n(k)$ [see (2.11)].

Proposition 1: $H_n(k) = G_{n-1}(k) + G_{n+1}(k-1)$ ($n, k \geq 0$).

Proof: For $n, k \geq 1$, the above identity readily follows from the definitions (2.11) and (3.5). For n and/or $k = 0$, let us use the expressions of $G_{-n}(k)$ and $G_n(-k)$ established in [4, §4].

Case 1: $n \geq 1$ and $k = 0$.

By [4, (4.1)], (2.11), and (2.7), we get

$$G_{n-1}(0) + G_{n+1}(-1) = G_{n-1}(0) + 0 = (-1)^{n-1} = H_n(0).$$

Case 2: $n = 0$ and $k \geq 1$.

By [4, (4.9)] and (3.9), we get

$$G_{-1}(k) + G_1(k-1) = -[F_1 + G_1(k-1)] + G_1(k-1) = -1 = H_0(k).$$

Case 3: $n = k = 0$.

By [4, (4.1) and (4.8)] and (2.8), we get

$$G_{-1}(0) + G_1(-1) = G_{-1}(0) + 0 = -F_1 = -1 = H_0(0). \quad \square$$

Proposition 1 together with some properties of the numbers $G_n(k)$ found in [4] will play a crucial role in establishing several properties of the numbers $H_n(k)$. A further connection between $H_n(k)$ and $G_n(k)$ is stated in the following proposition.

Proposition 2: $H_n(k) = G_{n+2}(k-2) - G_{n-2}(k)$ ($n, k \geq 0$).

Proof: By using the recurrence [4, (3.1)], namely,

$$G_{n+2}(k-1) = G_{n+1}(k) + G_n(k), \quad (4.1)$$

we can write

$$\begin{aligned} G_{n+2}(k-2) - G_{n-2}(k) &= G_{n+1}(k-1) + G_n(k-1) - G_{n-2}(k) \\ &= G_{n+1}(k-1) + G_n(k-1) - [G_n(k-1) - G_{n-1}(k)] \\ &= G_{n+1}(k-1) + G_{n-1}(k) = H_n(k) \quad (\text{by Proposition 1}). \quad \square \end{aligned}$$

Then, we establish a recurrence relation for the numbers $H_n(k)$.

Proposition 3: $H_{n+2}(k-1) = H_{n+1}(k) + H_n(k) \quad (k \geq 1)$.

Proof: $H_{n+1}(k) + H_n(k)$

$$\begin{aligned} &= G_n(k) + G_{n+2}(k-1) + G_{n-1}(k) + G_{n+1}(k-1) \quad (\text{by Proposition 1}) \\ &= G_n(k) + G_{n+3}(k-2) - G_{n+1}(k-1) + G_{n-1}(k) + G_{n+1}(k-1) \quad [\text{by (4.1)}] \\ &= G_{n+3}(k-2) + [G_n(k) + G_{n-1}(k) - G_{n+1}(k-1)] + G_{n+1}(k-1). \end{aligned}$$

Observing that the expression within square brackets vanishes in virtue of (4.1), we can write

$$H_{n+1}(k) + H_n(k) = G_{n+3}(k-2) + G_{n+1}(k-1) = H_{n+2}(k-1) \quad (\text{by Proposition 1}). \quad \square$$

As a direct consequence of Proposition 3, we can state the following proposition, the proof of which is omitted because of its triviality.

Proposition 4: $\sum_{n=s}^{s+2h-1} H_n(k) = \sum_{n=1}^h H_{2n+s}(k-1) \quad (k \geq 1)$.

Also, the curious identity

$$H_n(n) - H_n(n-1) = -\binom{3n-1}{2n} \quad (n \geq 1) \quad [\text{so } H_1(1) - H_1(0) = -1] \quad (4.2)$$

can be readily proved.

Proof of (4.2): By (3.3), we immediately obtain the recurrence relation

$$H_n(k+1) = H_n(k) + (-1)^{n+k} \frac{n}{k+1} \binom{n+1+2k}{k}. \quad (4.3)$$

Replace k by $n-1$ in (4.3) and use [12, (iii), p. 3] to obtain (4.2). \square

Let us conclude this section by proving a noteworthy property of the numbers $H_n(k)$.

Proposition 5: $R_n(h, k) \stackrel{\text{def}}{=} \sum_{i=0}^h \binom{h}{i} H_{n+i}(k) = \begin{cases} H_{n+2h}(k-h) & \text{if } k \geq h, \\ 0 & \text{if } k < h. \end{cases}$

Proof: Use Proposition 1 to write

$$R_n(h, k) = \sum_{i=0}^h \binom{h}{i} G_{n-1+i}(k) + \sum_{i=0}^h \binom{h}{i} G_{n+1+i}(k-1),$$

whence

$$\begin{aligned} R_n(h, k) &= G_{n-1+2h}(k-h) + G_{n+1+2h}(k-1-h) \quad (\text{by [4, Proposition 3]}) \\ &= \begin{cases} H_{n+2h}(k-h) & \text{if } k \geq h \quad (\text{by Proposition 1}) \\ 0 & \text{if } k < h \quad (\text{since } G_n(-k) = 0 \forall n, [4, (4.1)]). \end{cases} \quad \square \end{aligned}$$

Remark: The proof of Proposition 5 in the case $k < h$ can also be obtained by using double induction (on k and m) to prove that

$$\sum_{i=0}^{k+m} \binom{k+m}{i} H_{n+i}(k) = 0 \quad \text{if } m \geq 1. \quad (4.4)$$

This alternative and more direct proof is not difficult but it is rather lengthy and tedious, so it is omitted to save space.

5. SOME SIMPLE CONGRUENCE PROPERTIES OF $H_n(k)$

In this section we are concerned with some aspects of the parity of $H_n(k)$, and with a congruence property of these numbers that is valid for all prime values of the subscript n .

Proposition 6: $H_n(k) \equiv G_n(k) \pmod{2}$.

Proof: By Proposition 1 and (4.1), we can write

$$\begin{aligned} H_n(k) &= G_{n-1}(k) + G_{n+1}(k-1) = G_{n-1}(k) + G_n(k) + G_{n-1}(k) \\ &= G_n(k) + 2G_{n-1}(k) \equiv G_n(k) \pmod{2}. \quad \square \end{aligned}$$

The general solution of the problem of establishing the parity of $G_n(k)$ [and hence that of $H_n(k)$] seems to be rather difficult. On the basis of some partial results obtained in [4, §3.1], we show the solution for the particular cases $n = 3$ and 2^h . Namely, we have

$$H_3(k) \text{ is even iff } k = 2^h - 3 \quad (h \geq 2) \quad (5.1)$$

and

$$H_{2^n}(k) \text{ is odd iff } 2^{2^{h+n-2}} - 2^n \leq k \leq 2^{2^{h+n-1}} - 2^n - 1 \quad (n \geq 0; h \geq 1). \quad (5.2)$$

Proposition 7: If p is a prime and m is a nonnegative integer, then

$$(i) \quad H_p(mp) \equiv \sum_{j=0}^m (-1)^j C_j \pmod{p},$$

where $C_j = \frac{1}{j+1} \binom{2j}{j}$ is the j^{th} Catalan number, and

$$(ii) \quad H_p(k) \equiv H_p(mp) \pmod{p} \quad \text{if } mp+1 \leq k \leq (m+1)p-1.$$

Proof of Part (i): For $n = p$, consider the absolute value of the generic addend of the sum in (3.2), namely,

$$\frac{p}{j} \binom{p-1+2j}{j-1} \stackrel{\text{def}}{=} A_p(j) \quad (j = 1, 2, \dots, k). \quad (5.3)$$

By virtue of the integrality of $H_n(k)$ [see Definition (3.4) or (3.5)] and the replacement of k by $k-1$ in the recurrence (4.3), it is readily seen that $A_p(j)$ is an integer. If $j \not\equiv 0 \pmod{p}$, this quantity is clearly divisible by p . If $p > 2$, by (3.2) we can write

$$H_p(mp) \equiv 1 + \sum_{\substack{i=1 \\ i \equiv 0 \pmod{p}}}^{mp} (-1)^i A_p(i) = 1 + \sum_{j=1}^m (-1)^{jp} \frac{p}{jp} \binom{p-1+2jp}{jp-1} =$$

$$= 1 + \sum_{j=1}^m (-1)^j \frac{1}{j} \binom{2jp+p-1}{(j-1)p+p-1} \pmod{p}, \quad (5.4)$$

whence, by using Lucas' Theorem (e.g., see [1, Theorem 1.1]), we obtain

$$H_p(mp) \equiv 1 + \sum_{j=1}^m (-1)^j \frac{1}{j} \binom{2j}{j-1} = 1 + \sum_{j=1}^m (-1)^j \frac{1}{j+1} \binom{2j}{j} = \sum_{j=0}^m (-1)^j C_j \pmod{p}.$$

When $p = 2$, we have

$$H_2(2m) \equiv -1 + \sum_{j=1}^m C_j \pmod{2}. \quad (5.5)$$

Since $-1 \equiv 1 \pmod{2}$, the congruence (5.5) is clearly equivalent to (i).

Proof of Part (ii): For $mp+1 \leq k \leq (m+1)p-1$ [i.e., for $k \not\equiv 0 \pmod{p}$], rewrite (3.2) as

$$H_p(k) = (-1)^{p-1} - (-1)^p \sum_{j=1}^{mp} (-1)^j A_p(j) - (-1)^p \sum_{j=mp+1}^k (-1)^j A_p(j). \quad (5.6)$$

By (5.6), Proposition 7(i), and since $A_p(j) \equiv 0 \pmod{p}$ whenever $j \not\equiv 0 \pmod{p}$, we get the congruence

$$H_p(k) \equiv \sum_{j=0}^m (-1)^j C_j - 0 \equiv H_p(mp) \pmod{p}. \quad \square$$

Particular instances of Proposition 7 are:

$$H_p(k) \equiv 1 \pmod{p} \quad \text{if } 0 \leq k \leq p-1, \quad (5.7)$$

$$H_p(p) \equiv 0 \pmod{p}, \quad (5.8)$$

$$H_p(2p) \equiv 2 \pmod{p}, \quad (5.9)$$

$$H_p(3p) \equiv -3 \pmod{p}, \quad (5.10)$$

$$H_p(4p) \equiv 11 \pmod{p}, \quad (5.11)$$

and

$$H_p(5p) \equiv -31 \pmod{p}. \quad (5.12)$$

Proof of (5.7): Put $m = 0$ in Proposition 7(ii), thus getting the congruence $H_p(k) \equiv H_p(0) \pmod{p}$, if $1 \leq k \leq p-1$. Since $H_p(0) \equiv 1 \pmod{p} \forall p$ ($p = 2$ inclusive), the above congruence clearly can be rewritten as (5.7). \square

6. THE POLYNOMIALS $H_n(k, x)$

Let us consider the special Dickson polynomials $p_n(x, -1) = V_n(x)$ [see (1.2)]. Paralleling the argument of Section 2 leads us to define the polynomials [cf. (3.2)]

$$H_n(k, x) = \frac{(-1)^{n-1}}{x^n} \left[1 + \sum_{j=1}^k (-1)^j \frac{n}{j} \binom{n-1+2j}{j-1} \frac{1}{x^{2j}} \right] \quad (x \neq 0), \quad (6.1)$$

where x is a nonzero indeterminate. These polynomials are the companions of the polynomials

$$G_n(k, x) = \frac{(-1)^n}{x^{n+1}} \sum_{j=0}^k (-1)^j \binom{n+2j}{j} \frac{1}{x^{2j}} \quad (x \neq 0), \quad (6.2)$$

considered in [4, §5]. By using the identity (2.5), it can be readily proved that

$$H_n(k, x) = G_{n-1}(k, x) + G_{n+1}(k-1, x). \quad (6.3)$$

Observe that identity (6.3) generalizes Proposition 1.

We believe that the polynomials $H_n(k, x)$ are worthy of a deep investigation. Nevertheless, in this paper we confine ourselves to making nothing but a couple of observations on them.

Observation 1 [on the integrality of $H_n(k, x)$]

$H_n(k, x)$ is evidently an integer whenever x equals the reciprocal of an integer (say, $x = 1/h$). This fact does not exclude the existence of irrational (or complex) values of x for which $H_n(k, x)$ is an integer. For example, if x equals any of the roots of the third-degree equation $hx^3 - x^2 + 1 = 0$, then $H_1(1, x) = h$. Apart from the trivial case

$$H_0(k, x) = -1 \quad \forall k \text{ and } x, \quad (6.4)$$

the problem of the existence of *rational* values of $x \neq 1/h$ such that, for particular values of n and k , $H_n(k, x)$ is an integer is an open problem.

Observation 2 [on a limit concerning $H_n(k, x)$]

Consider the limit

$$\begin{aligned} \lim_{k \rightarrow \infty} H_n(k, x) &\stackrel{\text{def}}{=} H_n(\infty, x) \\ &= \frac{(-1)^{n-1}}{x^n} \left[1 + \sum_{j=1}^{\infty} (-1)^j \frac{n}{j} \binom{n-1+2j}{j-1} \frac{1}{x^{2j}} \right] \quad (x \neq 0) \quad [\text{by (6.1)}]. \end{aligned} \quad (6.5)$$

The results presented in the sequel can be readily deduced from the analogous results on $G_n(k, x)$ established in [4, §5]. First, observe that by (6.1) we can write

$$H_n(\infty, -|x|) = (-1)^n H_n(\infty, |x|), \quad (6.6)$$

so, for the sake of brevity, we shall consider only positive values of x . Then, let us state the following two propositions concerning a closed-form expression and a recurrence relation for $H_n(\infty, x)$, respectively.

Proposition 8: If $x > 2$, then $H_n(\infty, x) = -\left(\frac{x-\Delta}{2}\right)^n$, where $\Delta = \sqrt{x^2 + 4}$.

Proof: By (6.3) we have

$$H_n(\infty, x) = G_{n-1}(\infty, x) + G_{n+1}(\infty, x), \quad (6.7)$$

so that, by [4, (5.11)], namely,

$$G_n(\infty, x) = \frac{(x - \Delta)^n}{2^n \Delta} \quad (x > 2) \quad (6.8)$$

(although the above quantity unfortunately has been denoted in [4] by the symbol $H_n(x)$, it is only marginally related to the quantities denoted by $H_n(k)$ and $H_n(k, x)$ in this paper), we can write

$$H_n(\infty, x) = \frac{(x - \Delta)^{n-1}}{2^{n-1} \Delta} + \frac{(x - \Delta)^{n+1}}{2^{n+1} \Delta},$$

whence, after some simple manipulations, we obtain the desired result,

$$H_n(\infty, x) = -\left(\frac{x - \Delta}{2}\right)^n = -\Delta G_n(\infty, x).$$

We draw attention to the fact that, for $x < 2$, the series (6.5) diverges (see (6.7) and [4, (5.7)]), whereas nothing can be said when $x = 2$, although computer experiments suggest the conjecture $H_n(\infty, 2) \stackrel{c}{=} -(1 - \sqrt{2})^n$. Observe that $1 - \sqrt{2}$ is one of the roots of the characteristic equation for the Pell recurrence relation. \square

We point out that, since

$$-1 < \frac{x - \Delta}{2} < 0 \quad (0 < x < \infty), \quad (6.9)$$

there do not exist real values of x for which $H_n(\infty, x)$ is an integer.

Proposition 9: The numbers $H_n(\infty, x)$ obey the second-order recurrence relation

$$H_n(\infty, x) = xH_{n-1}(\infty, x) + H_{n-2}(\infty, x) \quad (n \geq 2) \quad (6.10)$$

with initial conditions

$$H_0(\infty, x) = -1 \quad \text{and} \quad H_1(\infty, x) = (\Delta - x)/2. \quad (6.10')$$

Proof: The proof can be obtained readily by (6.7), [4, Proposition 10], and Proposition 8. \square

Let us conclude Observation 2 and the paper by showing the set of all rational values r of x for which $H_n(\infty, r)$ is a rational number. On the basis of the results established in [4, §5.1], we see that this set can be generated by the formula

$$r = \frac{U^2 - V^2}{UV}, \quad (6.11)$$

where U and V range over the set of all positive integers and are subject to the condition

$$U > (1 + \sqrt{2})V. \quad (6.12)$$

The fulfillment of inequality (6.12) is necessary to satisfy the inequality $r > 2$ which, in turn, is required for the convergence of the series (6.5). It can be proved readily that the condition $\text{g.c.d.}(U, V) = 1$ must be imposed to obtain all *distinct* values of r .

ACKNOWLEDGMENT

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REFERENCES

1. D. F. Bailey. "More Binomial Coefficient Congruences." *The Fibonacci Quarterly* **30.2** (1992):121-25.
2. L. Dickson. "The Analytic Representation of Substitutions on a Power of a Prime Number of Letters with a Discussion of Linear Group I." *Ann. Math.* **11** (1896):65-120.
3. P. Filipponi & A. F. Horadam. "Derivative Sequences of Fibonacci and Lucas Polynomials." In *Applications of Fibonacci Numbers 4*:99-108. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1991.
4. P. Filipponi, O. Brugia, & A. F. Horadam. "A Note on the Improper Use of a Formula for Fibonacci Numbers." *Int. J. Educ. Sci. Technol.* **24.1** (1993):9-21.
5. A. F. Horadam & P. Filipponi. "Integration Sequences of Fibonacci and Lucas Polynomials." In *Applications of Fibonacci Numbers 5*:317-30. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1993.
6. H. Lausch & W. Nöbauer. *Algebra of Polynomials*. Amsterdam: North Holland, 1973.
7. Da-Xing Li. "Cryptanalysis of Public-Key Distribution Systems Based on Dickson Polynomials." *Electronic Letters* **27.3** (1991):228-29.
8. R. Lidl & W. B. Müller. "Permutation Polynomials in RSA-Cryptosystems." In *Advances in Cryptology, Proc. of CRYPTO '83*, pp. 293-301. Ed. D. Chaum. New York: Plenum, 1984.
9. R. Lidl, W. B. Müller, & A. Oswald. "Some Remarks on Strong Fibonacci Pseudoprimes." *Applicable Algebra in Eng. Comm. and Comp.* **1.1** (1990):59-65.
10. R. Lidl & W. B. Müller. "Generalization of the Fibonacci Pseudoprimes Test." *Discrete Mathematics* **92** (1991):211-20.
11. W. B. Müller & W. Nöbauer. "Some Remarks on Public-Key Cryptosystems." *Studia Sci. Math. Hungar.* **16** (1981):71-76.
12. W. B. Müller & W. Nöbauer. "Crypanalysis of the Dickson-Scheme." In *Lecture Notes in Computer Science* **219**:71-76. Berlin: Springer-Verlag, 1985.
13. W. B. Müller & A. Oswald. "Generalized Fibonacci Pseudoprimes and Probable Primes." In *Applications of Fibonacci Numbers 5*:459-64. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1993.
14. L. Rédei. *Algebra*. Vol. I. New York: Pergamon Press, 1967.
15. J. Riordan. *Combinatorial Identities*. New York: Wiley, 1968.
16. P. Smith. "LUC Public-Key Encryption." *Dr. Dobbs's Journal* (January 1993):44-49, 90-92.

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THE FIBONACCI CONFERENCE IN PULLMAN

Herta T. Freitag

Sponsored and supported by the Office of the President, of the Provost, of the Vice Provost for Research and Dean of the Graduate School, the Office of the Dean of the College of Science, and the Department of Pure and Applied Mathematics, the Sixth International Research Conference on Fibonacci Numbers and Their Applications convened at Washington State University from July 18-22, 1994.

We had our UPS and DOWNS. But they were due solely to the contours of the beautiful campus of Washington State University as we walked between the buildings.

How richly international we were! We had the pleasure of hearing 55 papers, 24 of them presented by mathematicians from America; Australia, Italy, and Japan tied by each having five representatives, closely followed by Germany's four. As usual, two New Zealanders enriched our sessions. There was one speaker from each of the other countries, one of them even coming from Brunei, almost all of them traveling long distances to serve the magnet of Fibonacci-related mathematics. Seven speakers were female.

The papers themselves were as remarkably diverse as the nationalities of the group, attesting to the richness of our discipline and the creative imagination of mathematicians. Those who had the misfortune of being unable to attend the Conference will concur in this estimate by studying *The Proceedings*. We did work hard. On our full-day sessions we heard 13 papers, and—on one of them—even 14. On the last day there were nine. Even with a shortened program on the day of the excursion we were yet enchanted by six papers.

The planned trip was wedged into our schedule in the middle of our sessions to provide an "intermission" in our work. Not only did it deepen the "up-and-down-skills" of the Conference participants, it also gave them beautiful vistas of the three waterfalls at the Elk River. The resulting ferocious appetites were befittingly satisfied by a romantic dinner. It was such by virtue of being in the midst of tall, densely-needled trees with the sun saying farewell for the day.

In our sessions, the atmosphere was scholarly and excitedly tense. The common magnetism of our Fibonacci specialty forged—as always—an international union. Mindstretching, indeed, was the Conference, but it was even more than that. "Heart-warming" would be my description, as friendships were deepened, and new ones developed. Indeed, many of the papers resulted from mathematicians infecting each other with ideas and collaborating as a result. To create such an atmosphere cannot be attributed to random constellations. It was indeed promoted by those outstanding and delightful Committee members, under the remarkable leadership of Calvin T. Long and William A. Webb, co-chairmen of the Local Committee, and A. F. Horadam (Australia) and A. M. Philippou (Cyprus), co-chairmen of the International Committee. We cannot help but think, too, of Verner E. Hoggatt, Jr., the founder of The Fibonacci Association and of *The Fibonacci Quarterly*, and to realize that it was Andreas N. Philippou, at the time Rector at Patras University, Greece, who gave birth to the idea of an international Fibonacci-related research conference. And we all deeply appreciate our highly esteemed and affectionately treasured editor, the mind and soul of our Conferences, Gerald E. Bergum.

However, the arts were represented, too, and to paraphrase E. T. Bell's words: It all goes to show that mathematicians are also human beings, sometimes DELIGHTFULLY more so! We were charmed by the artistic renderings of finite parts of hyperbolic tessellations (Heike Harborth), heard Fibonacci music (Peter G. Anderson), and, yes, we co-felt deeply with George M. Philipps when he gave us his own version of words to music by Leonard Bernstein:

North West Story

Everything's nice here in Pullman,
We can keep cool in the pool, man.
Even more cool is the math here—
Old Fibonacci makes us cheer!

Thanks to Bill Webb and dear Cal Long
To whom we dedicate this song
And to our friend Jerry Bergum
All of whom make us so welcome!

It was hard to say good-bye at our final get-together, the beautiful banquet at the Compton Union Building, but now it is

"Auf Wiedersehen"

in Graz, Austria (!) in 1996.

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stanley@MathPro.com on Internet. All correspondence will be acknowledged.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original, and require that they do not submit the problem elsewhere while it is under consideration for publication herein.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

Nary a month goes by without my receiving a problem proposal from the inveterate problemist, Herta Taussig Freitag. So, as a tribute to Herta, and to reduce my backlog, all of the problems in this issue come from her. As usual, generalizations are always welcome.

B-772 *Proposed by Herta T. Freitag, Roanoke, Virginia*

Prove that

$$\frac{L_n^2 + L_{n+a}^2}{F_n^2 + F_{n+a}^2}$$

is always an integer if a is odd. How should this problem be modified if a is even?

B-773 *Proposed by Herta T. Freitag, Roanoke, Virginia*

Find the number of terms in the Zeckendorf representation of $\sum_{i=1}^n F_i^2$.

[The Zeckendorf representation of an integer expresses that integer as a sum of distinct non-consecutive Fibonacci numbers.]

B-774 *Proposed by Herta T. Freitag, Roanoke, Virginia*

Let $\langle H_n \rangle$ be any sequence of integers such that $H_{n+2} = H_{n+1} + H_n$ for all n . Let p and m be positive integers such that $H_{n+p} \equiv H_n \pmod{m}$ for all integers n . Prove that the sum of any p consecutive terms of the sequence is divisible by m .

B-775 *Proposed by Herta T. Freitag, Roanoke, Virginia*

Let $g = \alpha + 2$. Express g^{17} in the form $p\alpha + q$ where p and q are integers.

B-776 *Proposed by Herta T. Freitag, Roanoke, Virginia*

Find all values of n for which $\sum_{k=1}^n kF_k$ is even.

B-777 *Proposed by Herta T. Freitag, Roanoke, Virginia*

Find all integers a such that $n \equiv a \pmod{4}$ if and only if $L_n \equiv a \pmod{5}$.

SOLUTIONS**Fibonacci Fractions****B-739** *Proposed by Ralph Thomas, University of Chicago, Dundee, Illinois
(Vol. 31, no. 2, May 1993)*

Let $S = \left\{ \frac{F_i}{F_j} \mid i \geq 0, j > 0 \right\}$. Is S dense in the set of nonnegative real numbers?

Solution by Margherita Barile, Universität Essen, Germany

The answer is no, since one has the following two facts:

- (a) for all $i \geq 2$ and $j \geq i + 2$, $\frac{F_i}{F_j} \leq \frac{F_i}{F_{i+2}} < \frac{1}{2}$;
 (b) for all $i \geq 5$ and $0 < j \leq i + 1$, $\frac{F_i}{F_j} \geq \frac{F_i}{F_{i+1}} > \frac{3}{5}$.

Hence, if $i \geq 5$, the fraction F_i / F_j cannot lie in the closed interval $[1/2, 3/5]$. This interval therefore only contains a finite number of elements of S and so S cannot be dense in that interval.

Both the claims (a) and (b) can be proved by induction in i . For claim (a), first note that

$$\frac{F_2}{F_4} = \frac{1}{3} < \frac{1}{2} \quad \text{and} \quad \frac{F_3}{F_5} = \frac{2}{5} < \frac{1}{2}.$$

Then take $i > 3$ and suppose the claim is true for $i - 1$ and $i - 2$. Then by induction,

$$\frac{F_i}{F_{i+2}} = \frac{F_{i-1} + F_{i-2}}{F_{i+1} + F_i} < \frac{\frac{1}{2}F_{i+1} + \frac{1}{2}F_i}{F_{i+1} + F_i} = \frac{1}{2}.$$

The proof of claim (b) proceeds similarly.

Several solvers stated that the set of limit points of S is $\{\alpha^p \mid p \in \mathbb{Z}\} \cup \{0\}$.

Also solved by Paul S. Bruckman, Russell Jay Hendel, H.-J. Seiffert, J. Suck, and the proposer.

Smarandache in Reverse**B-740** *Proposed by Thomas Martin, Phoenix, Arizona
(Vol. 31, no. 2, May 1993)*

Find all positive integers x such that 10 is the smallest integer, n , such that $n!$ is divisible by x .

Solution by Jane Friedman, University of San Diego, California

We are looking for all integers x such that $x|10!$ but $x \nmid n!$ for any $n < 10$. Let T be the set of all such integers. Since $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$, such integers must be of the form $2^a \cdot 3^b \cdot 5^c \cdot 7^d$ with a, b, c , and d nonnegative integers such that $a \leq 8, b \leq 4, c \leq 2$, and $d \leq 1$. But $9! = 2^7 \cdot 3^4 \cdot 5^1 \cdot 7^1$, so we have the additional constraint that $a = 8$ or $c = 2$ or both. Thus,

$$T = \{x | x = 2^8 \cdot 3^b \cdot 5^c \cdot 7^d, 0 \leq b \leq 4, 0 \leq c \leq 2, 0 \leq d \leq 1\} \\ \cup \{x | x = 2^a \cdot 3^b \cdot 5^2 \cdot 7^d, 0 \leq a \leq 8, 0 \leq b \leq 4, 0 \leq d \leq 1\}.$$

There are 110 such integers, so I will not list them all explicitly.

The proposer remarks that this problem is concerned with the inverse of the Smarandache Function $S(n)$, which is defined to be the smallest integer such that $S(n)!$ is divisible by n . For another problem about the Smarandache Function, see problem H-490 in this issue. For more information about the Smarandache Function, consult the "Smarandache Function Journal." Information about this journal can be obtained from its editor, Dr. R. Muller, at P.O. Box 10163, Glendale, AZ 85318-0163, U.S.A. For another solution to this problem, see "Elemente der Mathematik" 49 (1994):127.

Also solved by Charles Ashbacher, Margherita Barile, Paul S. Bruckman, Pentti Haukkanen, Russell Jay Hendel, Joseph J. Kostal, H.-J. Seiffert, Sahib Singh, Lawrence Somer, J. Suck, Ralph Thomas, and the proposer.

Factor 54 Where Are You?

B-741 Proposed by Jayantibhai M. Patel, Bhavan's R. A. College of Science, Gujarat, India (Vol. 31, no. 2, May 1993)

Prove that $S_n = F_{n+8}^4 + 331F_{n+4}^4 + F_n^4$ is always divisible by 54.

Solution by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

Using the known identities

$$F_n^4 = \frac{L_{4n} - 4(-1)^n L_{2n} + 6}{25} \quad \text{and} \quad L_{n+k} + L_{n-k} = L_n L_k, \quad k \text{ even}$$

(identities 81 and 17a in [1]), we get

$$S_n = \frac{(L_{16} + 331)L_{4n+16} - 4(-1)^n(L_8 + 331)L_{2n+8} + 1998}{25} = \frac{54}{25}[47L_{4n+16} - 28(-1)^n L_{2n+8} + 37].$$

Since 54 and 25 have no common factor, it follows that S_n is divisible by 54.

Dresel expressed S_n as $54[47F_{n+4}^4 + 32(-1)^n F_{n+4}^2 + 3]$, also elegantly showing that 54 is an explicit factor.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Chichester: Halsted, 1989.
- Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Russell Jay Hendel, Bob Prielipp, H.-J. Seiffert, Sahib Singh, J. Suck, Ralph Thomas, David Zeitlin, and the proposer.*

Pell's Triggy Product

B-742 *Proposed by Curtis Cooper & Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri (Vol. 31, no. 3, August 1993)*

Pell numbers are defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+1} = 2P_n + P_{n-1}$, for $n \geq 1$. Show that

$$P_{23} = 2^{11} \prod_{j=1}^{11} \left(3 + \cos \frac{2j\pi}{23} \right).$$

Solution by Lou Shapiro, Silver Spring, Maryland

The Binet form [3] for the Pell numbers is $P_n = \frac{1}{2\sqrt{2}}(p^n - q^n)$ where $p = 1 + \sqrt{2}$ and $q = 1 - \sqrt{2}$. A form of the cyclotomic identity is

$$z^n - y^n = \prod_{j=0}^{n-1} (z - w^j y)$$

where $w = e^{2\pi i/n}$ is a primitive n^{th} root of unity. If n is odd, we have [1]

$$z^n - y^n = (z - y) \prod_{j=1}^{\frac{n-1}{2}} (z - w^j y)(z - w^{-j} y) = (z - y) \prod_{j=1}^{\frac{n-1}{2}} \left(z^2 - 2 \cos \frac{2j\pi}{n} zy + y^2 \right).$$

Now let $z = p = 1 + \sqrt{2}$ and $y = q = 1 - \sqrt{2}$, and note that $p - q = 2\sqrt{2}$, $p^2 + q^2 = 6$, and $pq = -1$. Thus, we have

$$p^n - q^n = 2\sqrt{2} \prod_{j=1}^{\frac{n-1}{2}} \left(6 + 2 \cos \frac{2j\pi}{n} \right)$$

and, therefore,

$$P_n = 2^{(n-1)/2} \prod_{j=1}^{\frac{n-1}{2}} \left(3 + \cos \frac{2j\pi}{n} \right).$$

Letting $n = 23$ finishes the problem.

A similar result can be obtained when n is even. Combining these two results gives the general formula

$$P_n = 2^{\lfloor n/2 \rfloor} \prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(3 + \cos \frac{2j\pi}{n} \right) \quad (*)$$

which is true for all positive integers n .

Note that this same method gives an elementary proof of problems H-93 [2] and H-466 [4] which state that

$$F_n = \prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(3 + 2 \cos \frac{2j\pi}{n} \right).$$

Suck refers to problem H-64 [5] where it is shown that $F_n = \prod_{j=1}^{n-1} (1 - 2i \cos \frac{j\pi}{n})$. Zeitlin mentions that in his solution to H-64 [5], he showed that if Z_n satisfies the recurrence $Z_{n+2} = dZ_{n+1} - cZ_n$, with $Z_0 = 0$ and $Z_1 = 1$, then

$$Z_n = c^{(n-1)/2} \prod_{j=1}^{n-1} \left(\frac{d}{\sqrt{c}} - 2 \cos \frac{j\pi}{n} \right).$$

Hendel and Cook find recurrences for expressions similar to (*) where "3" is replaced by a fixed constant m . Seiffert shows that for all complex z ,

$$f_{2n-1}(z) = \prod_{j=1}^{n-1} \left(z^2 + 2 + 2 \cos \frac{2j\pi}{2n-1} \right) \quad \text{and} \quad f_{2n}(z) = z \prod_{j=1}^{n-1} \left(z^2 + 2 + 2 \cos \frac{j\pi}{n} \right)$$

where $\{f_n(x)\}$ are the Fibonacci polynomials defined by $f_{n+1}(x) = xf_n(x) + f_{n-1}(x)$ with $f_0(x) = 0$ and $f_1(x) = 1$.

References

1. I. S. Gradshteyn & I. M. Ryzhik. *Table of Integrals, Series, and Products* (corrected and enlarged edition), p. 34, formula 1.396.2. San Diego, Calif.: Academic Press, 1980.
2. Douglas Lind. "Problem H-93." *The Fibonacci Quarterly* 6.2 (1968):145-48.
3. Neville Robbins. *Beginning Number Theory*, p. 193. Dubuque, Iowa: Wm. C. Brown, 1993.
4. J. A. Sjogren. "Problem H-466: A Triggy Problem." *The Fibonacci Quarterly* 30.2 (1992): 188.
5. David Zeitlin. "Solution to Problem H-64." *The Fibonacci Quarterly* 5.1 (1967):74-75.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Russell Jay Hendel, Norbert Jensen, Hans Kappus, Joseph J. Kostal, Almas Rumov, H.-J. Seiffert, J. Suck, David Zeitlin, and the proposers.

Golden Argument of Tenth Roots of Unity

B-743 Proposed by Richard André-Jeannin, Longwy, France
(Vol. 31, no. 3, August 1993)

Find the modulus and the argument of the complex numbers

$$u = \frac{\beta + i\sqrt{\alpha+2}}{2} \quad \text{and} \quad v = \frac{\alpha + i\sqrt{\beta+2}}{2}.$$

Solution by H.-J. Seiffert, Berlin, Germany

From Problem B-674 (proposed by Richard André-Jeannin in *The Fibonacci Quarterly* 29.3 [1991]:280), we know that $\cos(\pi/5) = \alpha/2$ and $\cos(3\pi/5) = \beta/2$. Since $\sin(\pi/5) > 0$ and $\sin(3\pi/5) > 0$, we find

$$\sin \frac{\pi}{5} = \sqrt{1 - \cos^2 \frac{\pi}{5}} = \frac{\sqrt{4 - \alpha^2}}{2} = \frac{\sqrt{3 - \alpha}}{2} = \frac{\sqrt{\beta + 2}}{2},$$

where we have used $\alpha^2 = \alpha + 1$ and $\alpha = 1 - \beta$. Similarly, we find $\sin(3\pi/5) = \sqrt{\alpha + 2}/2$. Therefore, we have

$$u = \cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5} \quad \text{and} \quad v = \cos \frac{\pi}{5} + i \sin \frac{\pi}{5}.$$

This shows that u and v both have modulus 1, and the argument of u and v is $3\pi/5$ and $\pi/5$, respectively.

Flanigan notes that u and v are primitive tenth roots of unity.

Also solved by M. A. Ballieu, Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, F. J. Flanigan, Pentti Haukkanen, Russell Jay Hendel, Norbert Jensen, Hans Kappus, Carl Libis, Bob Prielipp, J. Suck, and the proposer.

A Sum Divisible

B-744 *Proposed by Herta T. Freitag, Roanoke, Virginia*
(Vol. 31, no. 3, August 1993)

Let n and k be even positive integers. Prove that $L_{2n} + L_{4n} + L_{6n} + \cdots + L_{2kn}$ is divisible by L_n .

Solution 1 by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

Let $S_k(n) = \sum_{j=1}^k L_{2nj}$ with even k . We shall prove that $S_k(n) \equiv 0 \pmod{L_n}$ if n is even and $S_k(n) \equiv 0 \pmod{5F_n}$ if n is odd.

Rewrite $S_k(n)$ as

$$S_k(n) = \sum_{j=1}^{k/2} [L_{(4j-1)n-n} + L_{(4j-1)n+n}]$$

and use the identities

$$L_{n+p} + L_{n-p} = \begin{cases} L_n L_p, & p \text{ even,} \\ 5F_n F_p, & p \text{ odd} \end{cases}$$

(formulas 17a and 17b from [2]) to obtain

$$S_k(n) = \begin{cases} L_n \sum_{j=1}^{k/2} L_{(4j-1)n}, & n \text{ even,} \\ 5F_n \sum_{j=1}^{k/2} L_{(4j-1)n}, & n \text{ odd.} \end{cases}$$

Solution 2 by Norbert Jensen, Kiel, Germany

We prove the stronger result that $L_{2n} + L_{4n} + L_{6n} + \cdots + L_{2kn}$ is divisible by L_n^2 when k and n are even.

Pairing the terms up two at a time, we find that in each pair, with j odd,

$$L_{2jn} + L_{2(j+1)n} = [\alpha^{nj} + \beta^{nj}]^2 + [\alpha^{n(j+1)} - \beta^{n(j+1)}]^2 = L_{nj}^2 + 5F_{n(j+1)}^2.$$

Since, when j is odd, L_n divides L_{nj} and L_n divides $F_{n(j+1)}$ ([1], Theorems 4 and 5, p. 40), we see that each pair is divisible by L_n^2 and so is the entire sum.

Bruckman notes that $S_k(n) = L_{n(k+1)} F_{nk} / F_n$ for all k and n .

References

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969; rpt. Santa Clara, Calif: The Fibonacci Association, 1979.
2. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Chichester: Halsted, 1989.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, F. J. Flanigan, Russell Jay Hendel, Chris Long, Richard McGuffin, Bob Prielipp, Almas Rumov, H.-J. Seiffert, Lawrence Somer, J. Suck, and the proposer.

Erratum: Paul S. Bruckman was inadvertently omitted as a solver of Problem B-726.



ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-490 *Proposed by A. Stuparu, Vâlcea, Romania*

Prove that the equation $S(x) = p$, where p is a given prime number, has just 2^{p-2} solutions, all of them in between p and $p!$. [$S(n)$ is the Smarandache Function: the smallest integer such that $S(n)!$ is divisible by n .]

H-491 *Proposed by Paul S. Bruckman, Highwood, Illinois*

Prove the following identities:

$$F_{2n} = 2 \binom{2n}{n}^{-1} \sum_{k=0}^{n-1} \binom{n-\frac{1}{2}}{k} \binom{n-\frac{1}{2}}{n-1-k} 5^k, \quad n = 1, 2, \dots; \quad (a)$$

$$F_{2n+1} = \binom{2n}{n}^{-1} \sum_{k=0}^n \binom{n-\frac{1}{2}}{k} \binom{n+\frac{1}{2}}{n-k} 5^k, \quad n = 0, 1, 2, \dots \quad (b)$$

H-492 *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$, $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, for $n \geq 2$. Show that, for all complex numbers x and y and all nonnegative integers n ,

$$\sum_{k=0}^{[n/2]} \binom{n}{k} F_{n-2k}(x) F_{n-2k}(y) = z^{n-1} F_n(xy/z), \quad (1)$$

where $z = (x^2 + y^2 + 4)^{1/2}$. $[]$ denotes the greatest integer function.

As special cases of (1), obtain the following identities:

$$\sum_{k=0}^{[n/2]} \binom{n}{k} F_{n-2k}^2 = (3^n - (-2)^n) / 5, \quad (2)$$

$$\sum_{k=0}^n \binom{2n+1}{n-k} F_{2k+1} = 5^n, \quad (3)$$

$$\sum_{k=0}^n \binom{2n}{n-k} F_{2k} F_{4k} = 5^{n-1} (4^n - 1) \quad (4)$$

$$\sum_{k=0}^n \binom{2n+1}{n-k} F_{2k+1} L_{4k+2} = 5^n (2^{2n+1} + 1), \quad (5)$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} F_{2n-4k} P_{n-2k} = F_n(6), \quad (6)$$

where $P_j = F_j(2)$ is the j^{th} Pell number,

$$\sum_{\substack{k=0 \\ (5, n-2k)=1}}^{\lfloor n/2 \rfloor} (-1)^{\lfloor (n-2k+2)/5 \rfloor} \binom{n}{k} = F_n. \quad (7)$$

The latter equation is the one given in H-444.

SOLUTIONS

Sum Problem

H-477 *Proposed by Paul S. Bruckman, Edmonds, Washington
(Vol. 31, no. 2, May 1993)*

Let

$$F_r(x) = z^r - \sum_{k=0}^{r-1} a_k z^{r-1-k}, \quad (1)$$

where $r \geq 1$, and the a_k 's are integers.

Suppose F_r has distinct zeros θ_k , $k = 1, 2, \dots, r$, and let

$$V_n = \sum_{k=1}^r \theta_k^n, \quad n = 0, 1, 2, \dots \quad (2)$$

Prove that, for all primes p ,

$$V_p \equiv a_0 \pmod{p}. \quad (3)$$

Solution by H.-J. Seiffert, Berlin, Germany

From (1), it follows that

$$(-1)^k a_k = (-1)^k a_k(\theta_1, \dots, \theta_r) = \sum_{1 \leq i_1 < \dots < i_{k+1} \leq r} \theta_{i_1} \dots \theta_{i_{k+1}}, \quad (4)$$

for $k = 0, \dots, r-1$, is the $(k+1)^{\text{th}}$ elementary symmetric polynomial. Let S_r denote the set of all permutations of $\{1, \dots, r\}$. For the r -tuple (j_1, \dots, j_r) , where $0 \leq j_1 \leq \dots \leq j_r < p$ and $j_1 + \dots + j_r = p$, we define an equivalence relation on S_r by $\pi \sim \sigma$ if and only if $(j_{\pi(1)}, \dots, j_{\pi(r)}) = (j_{\sigma(1)}, \dots, j_{\sigma(r)})$. Let A_1, \dots, A_m denote the equivalence classes with respect to this equivalence relation. For each $n \in \{1, \dots, m\}$, we choose a permutation $\pi_n \in A_n$. Then the polynomial

$$P_{j_1, \dots, j_r}(\theta_1, \dots, \theta_r) = \sum_{n=1}^m \theta_1^{j_{\pi(n)}} \dots \theta_r^{j_{\pi(r)}}$$

is symmetric. By the fundamental theorem on symmetric polynomials (see A. I. Kostrikin, *Introduction to Algebra* [Springer-Verlag, 1982], pp. 281-84), there exists a polynomial Q_{j_1, \dots, j_r} having integer coefficients such that [see (4)]

$$P_{j_1, \dots, j_r}(\theta_1, \dots, \theta_r) = Q_{j_1, \dots, j_r}(a_0, \dots, a_{r-1}). \quad (5)$$

The multinomial theorem gives

$$\left(\sum_{k=1}^r \theta_k\right)^p = \sum_{k=1}^r \theta_k^p + \sum_{\substack{0 \leq j_1, \dots, j_r < p \\ j_1 + \dots + j_r = p}} \binom{p}{j_1, \dots, j_r} \theta_1^{j_1} \dots \theta_r^{j_r},$$

or, in view of (2) and after a little sorting,

$$a_0^p = V_p + \sum_{\substack{0 \leq j_1 \leq \dots \leq j_r < p \\ j_1 + \dots + j_r = p}} \binom{p}{j_1, \dots, j_r} P_{j_1, \dots, j_r}(\theta_1, \dots, \theta_r). \quad (6)$$

Equations (5) and (6) show that V_p is indeed an integer. It is well known that

$$\binom{p}{j_1, \dots, j_r} \equiv 0 \pmod{p} \quad (7)$$

for all primes p and r -tuples (j_1, \dots, j_r) with $0 \leq j_1, \dots, j_r < p$ and $j_1 + \dots + j_r = p$. (5), (6), and (7) imply $V_p \equiv a_0^p \pmod{p}$. Using Fermat's little theorem, we obtain $V_p \equiv a_0 \pmod{p}$, the desired result. Finally, we note that the result remains true, if the zeros $\theta_1, \dots, \theta_r$ of F_r are not distinct. In such cases, each zero of F_r must occur in the definition of V_n respecting its multiplicity.

Comment on H-477: Using the result of H-477 (including my final remark), it is very easy to solve the following problem (O. Šuch, Problem 10268, *Amer. Math. Monthly* **99.10** [1992]:958).

Define a sequence (V_n) by

$$V_0 = 3, V_1 = 0, V_2 = 2, V_{n+3} = V_{n+1} + V_n, \text{ for all } n \geq 0.$$

If p is a prime, show that $p|V_p$.

According to the result of H-477, we only have to show that

$$V_n = \theta_1^n + \theta_2^n + \theta_3^n, \quad n \in N_0, \quad (8)$$

where

$$(z - \theta_1)(z - \theta_2)(z - \theta_3) = z^3 - z - 1. \quad (9)$$

To do so, it suffices to show that (8) holds for $n = 0, 1, 2$. For $n = 0$, (8) is true, since $V_0 = 3$. For $n = 1$, it follows from (9) and $V_1 = 0$. From (9), we get $\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1 = -1$. Hence,

$$0 = V_1^2 = (\theta_1 + \theta_2 + \theta_3)^2 = \theta_1^2 + \theta_2^2 + \theta_3^2 + 2(\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1) = \theta_1^2 + \theta_2^2 + \theta_3^2 - 2$$

implies $V_2 = 2 = \theta_1^2 + \theta_2^2 + \theta_3^2$.

Also solved by A. G. Dresel, F. J. Flanigan, L. Somer, L. Van Hamme, and the proposer.

String Along

H-478 *Proposed by Gino Taddei, Rome, Italy*
(Vol. 31, no. 3, August 1993)

Consider a string constituted by h labeled cells c_1, c_2, \dots, c_h . Fill these cells with the natural numbers $1, 2, \dots, h$ according to the following rule: 1 in c_1 , 2 in c_2 , 3 in c_4 , 4 in c_7 , 5 in c_{11} , and so on. Obviously, whenever the subscript j of c_j exceeds h , it must be considered as reduced modulo h . In other words, the integer n ($1 \leq n \leq h$) enters the cell $c_{j(n,h)}$, where

$$j(n, h) = \left\langle \frac{n^2 - n + 2}{2} \right\rangle_h,$$

and the symbol $\langle a \rangle_b$ denotes a if $a \leq b$, and the remainder of a divided by b if $a > b$.

Determine the set of all values of h for which, at the end of the procedure, each cell has been entered by exactly one number.

Solution by Paul S. Bruckman, Highwood, Illinois

Let $U(h) = \bigcup_{n=1}^h \{j(n, h)\}$ and $V(h) = \{1, 2, \dots, h\}$. We seek to characterize the set

$$S = \{h \in \mathbb{Z}^+ : U(h) = V(h)\}.$$

Clearly, $1 \in S, 2 \in S$.

First, we show that, if $h \in S, h > 1$, then h must be even. Suppose $h > 1$ is odd. Clearly, $j(1, h) = 1$ for all h . Also, $j(h, h) = \langle h \cdot \frac{1}{2}(h-1) + 1 \rangle_h = 1$, since $\frac{1}{2}(h-1)$ is an integer. Since $h > 1$, c_1 and c_h are distinct cells; however, they are both occupied by the number 1, which shows that $h \notin S$ if h is odd and $h > 1$.

Suppose $h \equiv 2^r \pmod{2^{r+1}}$, $r = 0, 1, 2, \dots$. Then

$$j(h \cdot 2^{-r}, h) = \left\langle \frac{h}{2^{r+1}} \left(\frac{h}{2^r} - 1 \right) + 1 \right\rangle_h = \left\langle h \cdot \left(\frac{h - 2^r}{2^{2r+1}} \right) + 1 \right\rangle_h = 1.$$

Also, $j(1, h) = 1$. The only way for cells c_1 and $c_{h \cdot 2^{-r}}$ to be identical is for $h = 2^r$; otherwise, $h \notin S$. In other words, all elements of S must be powers of 2.

Define the ordered h -tuple $W(h) = (j(1, h), j(2, h), \dots, j(h, h)) = (1, 2, 4, \dots)$, which orders the elements of $U(h)$ according to the cell numbers. We first show that, for all h ,

$$W(2h) \equiv (W(h), W^*(h)) \pmod{h} \text{ where } W^*(h) \text{ denotes the transpose of } W(h) [W(h) \text{ in reversed order}]. \quad (1)$$

Proof of (1): We first observe that, if $1 \leq n \leq h$,

$$j(n, 2h) \equiv j(n, h) \pmod{h}. \quad (2)$$

Also, $j(2h+1-n, 2h) = \langle \frac{1}{2}(2h+1-n)(2h-n) + 1 \rangle_{2h} = \langle 2h^2 - 2nh + h + \frac{1}{2}(n^2 - n) + 1 \rangle_{2h} = \langle h + \frac{1}{2}(n^2 - n) + 1 \rangle_{2h} \equiv \langle \frac{1}{2}(n^2 - n) + 1 + h \rangle_{2h} \equiv \langle \frac{1}{2}(n^2 - n) + 1 \rangle_h \pmod{h}$, or

$$j(2h+1-n, 2h) \equiv j(n, h) \pmod{h}, \quad 1 \leq n \leq h. \quad (3)$$

We see that (1) is a consequence of (2) and (3).

Suppose now that $h = 2^r, r \geq 2$. Then $j(n+h, 2h) = \langle \frac{1}{2}(n+h)(n+h-1)+1 \rangle_{2h} = \langle \frac{1}{2}(n^2-n) + \frac{1}{2}h(2n-1) + \frac{1}{2}h^2+1 \rangle_{2h} = \langle \frac{1}{2}(n^2-n) + 1 + \frac{1}{2}(h^2-h) + nh \rangle_{2h}$. If n is even, then $\frac{1}{2}(h^2-h) + nh \equiv \frac{1}{2}(h^2-h) \equiv 2^{r-1}(2^r-1) \pmod{2^{r+1}} \equiv -2^{r-1} \equiv -\frac{1}{2}h \pmod{2h}$; if n is odd, then $\frac{1}{2}(h^2-h) + nh \equiv \frac{1}{2}(h^2+h) \equiv 2^{r-1}(2^r+1) \pmod{2^{r+1}} \equiv 2^{r-1} \equiv \frac{1}{2}h \pmod{2h}$. In either case, we see that, if $h = 2^r, r \geq 2$, then $j(n, 2h) = j(n+h, 2h) + \frac{1}{2}h(-1)^n$, so

$$j(n, 2h) \neq j(n+h, 2h), \quad 1 \leq n \leq h. \quad (4)$$

We may now complete the proof of the desired result, namely,

$$S = \{1, 2, 2^2, 2^3, \dots\} = \text{the set of all nonnegative powers of } 2. \quad (5)$$

Our proof is by induction (on r). We suppose $h = 2^r, r \geq 0$, and $h \in S$. (Indeed, we already know that $1 \in S, 2 \in S$). Then the elements of $W(h)$ are distinct (mod h) and, *a fortiori*, (mod $2h$). Also, (1) holds. Therefore, the first (and also the last) h elements of $W(2h)$ are distinct. Moreover, it follows from (4) that the elements of the first half of $W(2h)$ are distinct from the elements of the second half of $W(2h)$. We conclude that $2h \in S$ as a consequence of $h \in S$. Since $W(4) = \{1, 2, 4, 3\}$, thus $4 \in S$. Then, by induction, (5) is established.

Also solved by P. G. Anderson, M. Barile, P. Filipponi, J. Hendel, N. Jensen, and A. N. 't Woord.

Close Ranks

H-479 Proposed by Richard André-Jeannin, Longwy, France
(Vol. 31, no. 3, August 1993)

Let $\{V_n\}$ be the sequence defined by $V_0 = 2, V_1 = P$, and $V_n = PV_{n-1} = QV_{n-2}$ for $n \geq 2$, where P and Q are real or complex parameters. Find a closed form for the sum

$$\sum_{k=1}^n \binom{2n-k-1}{n-1} P^k Q^{n-k} V_k.$$

Solution by Paul S. Bruckman, Everett, Washington

Let

$$S_n = \sum_{k=1}^n \binom{2n-k-1}{n-1} P^k Q^{n-k} V_k, \quad n = 1, 2, \dots \quad (1)$$

Replacing k by $n-k$ yields

$$S_n = \sum_{k=0}^{n-1} \binom{n-1+k}{n-1} P^{n-k} Q^k V_{n-k}. \quad (2)$$

We seek to prove the following:

$$S_n = P^{2n}, \quad n = 1, 2, \dots \quad (3)$$

Toward this end, let

$$D_n = S_{n+1} - P^2 S_n, \quad n = 1, 2, \dots \quad (4)$$

We may proceed to evaluate D_n in a straightforward manner, though not without some useful "tricks." Thus:

$$\begin{aligned}
 D_n &= \sum_{k=0}^n \binom{n+k}{n} P^{n+1-k} Q^k V_{n+1-k} - \sum_{k=0}^{n-1} \binom{n-1+k}{n-1} P^{n+2-k} Q^k V_{n-k} \\
 &= \sum_{k=1}^{n+1} \binom{n+k-1}{n} P^{n+2-k} Q^{k-1} V_{n+2-k} - \sum_{k=0}^{n-1} \binom{n+k-1}{n-1} P^{n+2-k} Q^k V_{n-k} \\
 &= \binom{2n}{n} P Q^n V_1 + \binom{2n-1}{n} P^2 Q^{n-1} V_2 - P^{n+2} V_n \\
 &\quad + \sum_{k=1}^{n-1} P^{n+2-k} Q^{k-1} \left[\binom{n+k-1}{n} V_{n+2-k} - \binom{n+k-1}{n-1} Q V_{n-k} \right] \\
 &= 2 \binom{2n-1}{n} P^2 Q^n + \binom{2n-1}{n} P^2 Q^{n-1} (P^2 - 2Q) - P^{n+2} V_n \\
 &\quad + \sum_{k=1}^{n-1} P^{n+2-k} Q^{k-1} \left[\binom{n+k-1}{n} (P V_{n+1-k} - Q V_{n-k}) - \binom{n+k-1}{n-1} Q V_{n-k} \right] \\
 &= \binom{2n-1}{n} P^4 Q^{n-1} - P^{n+2} V_n + \sum_{k=0}^{n-2} \binom{n+k}{n} P^{n+2-k} Q^k V_{n-k} \\
 &\quad - \sum_{k=1}^{n-1} \left[\binom{n+k-1}{n} + \binom{n+k-1}{n-1} \right] P^{n+2-k} Q^k V_{n-k} \\
 &= \binom{2n-1}{n} P^4 Q^{n-1} - P^{n+2} V_n + \sum_{k=0}^{n-2} \binom{n+k}{n} P^{n+2-k} Q^k V_{n-k} - \sum_{k=1}^{n-1} \binom{n+k}{n} P^{n+2-k} Q^k V_{n-k} \\
 &= \sum_{k=0}^{n-1} \binom{n+k}{n} P^{n+2-k} Q^k V_{n-k} - \sum_{k=0}^{n-1} \binom{n+k}{n} P^{n+2-k} Q^k V_{n-k} = 0.
 \end{aligned}$$

We have tacitly assumed that $n \geq 2$ in the above development; it is a trivial exercise to verify that $S_1 = P^2$, $S_2 = P^4$. Therefore, by an easy induction, since $D_n = 0$ for all $n \geq 1$, (3) is established.

Also solved by P. Filipponi, N. Jensen, H.-J. Seiffert, A. Shannon, and the proposer.



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