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PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

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In addition to the members of the Board of Directors and our Assistant Editors, the following mathematicians, engineers, and physicists have assisted *The Fibonacci Quarterly* by refereeing papers during the past year. Their special efforts are sincerely appreciated, and we apologize in advance for any names that inadvertently may have been overlooked or misspelled.

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THE ZECKENDORF ARRAY EQUALS THE WYTHOFF ARRAY

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1. INTRODUCTION

It is well known that every n in the set N of positive integers is uniquely a sum of non-consecutive Fibonacci numbers. This sum n is known as the *Zeckendorf representation of* n. We arrange these representations to form the *Zeckendorf array* $Z = Z(i, j)$ as follows: column j of Z is the increasing sequence of all n in whose Zeckendorf representation the least term is F_{j+1}. The first row of Z therefore consists of Fibonacci numbers:

$$z(1,1)=1=F_2 \quad z(1,2)=2=F_3 \quad z(1,3)=3=F_4 \quad \cdots \quad z(1,j)=F_{j+1}\ldots,$$

and the second row begins with the numbers $4=3+1, 7=5+2, 11=8+3$, and $18=13+5$. The reader is urged to write down several terms of the next two rows before reading further.

The Wythoff array, $W = W(i, j)$, partly shown in Table 1, was introduced by David R. Morrison [9], in connection with Wythoff pairs, which are the winning pairs of numbers in Wythoff's game. (See, for example, [2], [12]). Morrison proved several interesting things about W: every positive integer n occurs exactly once in W, as does every Wythoff pair; every row is a (generalized) Fibonacci sequence [i.e., $w(i,j) = w(i,j-1) + w(i,j-2)$ for every $i \geq 1$ and $j \geq 3$]. In fact, Morrison proved that, in a sense, *every* positive Fibonacci sequence of integers is a row of W.

TABLE 1. The Wythoff Array

1	2	3	5	8	13	21	34	55	89	144	...
4	7	11	18	29	47	76	123	199	322	521	
6	10	16	26	42	68	110	178	288	466	754	
9	15	24	39	63	102	165	267	432	699	1131	
12	20	32	52	84	136	220	356	576	932	1508	
14	23	37	60	97	157	254	411	665	1076	1741	
17	28	45	73	118	191	309	500	809	1309	2118	
19	31	50	81	131	212	343	555	898	1453	2351	
22	36	58	94	152	246	398	644	1042	1686	2728	
⋮											

Morrison also proved that the first column of W is given by $w(i,1) = [i\alpha]+i-1$, where $\alpha = (1+\sqrt{5})/2$. The rest of W is then given inductively by

$$w(i, j+1) = \begin{cases} [\alpha w(i,j)]+1 & \text{if } j \text{ is odd,} \\ [\alpha w(i,j)] & \text{if } j \text{ is even,} \end{cases} \quad \text{for } i = 1, 2, 3, \ldots . \tag{1}$$

2. SHIFTING SUBSCRIPTS: $F_{n+1} \rightarrow F_{n+2}$

We define a shift function $f: N \rightarrow N$ in terms of Zeckendorf representations:

$$\text{if } n = \sum_{h=1}^{\infty} c_h F_{h+1}, \text{ then } f(n) = \sum_{h=1}^{\infty} c_h F_{h+2}.$$

Lemma 1: The shift function f is a strictly increasing function.

We shall prove Lemma 1 in a more general form in Section 3.

Theorem 1: The first column of the Zeckendorf array Z determines all of Z by the recurrences

$$z(i, j+1) = f(z(i, j)) \tag{2}$$

for all $i \geq 1$ and $j \geq 1$.

Proof: We have $z(1, j) = F_{j+1}$ for all $j \geq 1$, so that row 1 of Z is determined by $z(1, 1) = 1$ and f. Assume $k \geq 1$ and that (2) holds for all $j \geq 1$, for all $i \leq k$. Write the Zeckendorf representation of $z(k+1, 1)$ as $z(k+1, 1) = \sum_{h=1}^{\infty} c_h F_{h+1}$, noting that the following conditions hold:

(i) $c_1 = 1$;
(ii) $c_h \in \{0, 1\}$ for every h in N;
(iii) for every h in N, if $c_h = 1$ then $c_{h+1} = 0$;
(iv) there exists n in N such that $c_h = 0$ for every $h \geq n$.

Let $m = f(z(k+1, 1))$. Then the representation $\sum_{h=1}^{\infty} c'_h F_{h+1}$, where $c'_1 = 0$ and $c'_h = c_{h-1}$ for all $h \geq 2$, is the Zeckendorf representation of m. Also, m is in column 2 of Z, since $c'_1 = 0$ and $c'_2 = 1$. By the induction hypothesis, $z(i, 2) = f(z(i, 1))$ for $i = 1, 2, \ldots, k$, and since column 2 is an increasing sequence, m must lie in a row numbered $\geq k+1$ by Lemma 1. We shall show that this row number cannot be $> k+1$.

Suppose $m > z(k+1, 2)$ and let the Zeckendorf representation for $z(k+1, 2)$ be $\sum_{h=1}^{\infty} d'_h F_{h+1}$. Then the number $q = \sum_{h=1}^{\infty} d_h F_{h+1}$, where $d_h = d'_{h+1}$ for $h \geq 1$, is the Zeckendorf representation for a number having $d_1 = 1$, so that this number lies in column 1 of Z. It is not one of the first k terms, and it is not $z(k+1, 1)$ since its sequel in row $k+1$ is not m. Therefore, $q = z(K, 1)$ for some $K \geq k+2$. We now have $z(k+1, 1) < q$ and $f(q) < f(z(k+1, 1))$, a contradiction to Lemma 1. Therefore, $z(k+1, 2) = f(z(k+1, 1))$.

Let $j \geq 2$ and suppose that $z(k+1, j) = f(z(k+1, j-1))$. The argument just used for $j = 2$ applies here in the same way, giving $z(k+1, j+1) = f(z(k+1, j))$. The induction on j is finished, so that (2) holds for all $j \geq 1$ for $i = k+1$. Consequently, the induction on k is finished, so that (2) holds throughout Z. □

Lemma 2:

$$f(n) = \begin{cases} [\alpha n] + 1 & \text{if } n \text{ is in an odd numbered column of } Z, \\ [\alpha n] & \text{if } n \text{ is in an even numbered column of } Z. \end{cases}$$

Proof: The fact that the continued fraction for α is [1, 1, 1, ...] leads as in [10, p. 10] to the well-known inequality

$$\frac{1}{F_{h+2}} < \left| \alpha F_h - F_{h+1} \right| < \frac{1}{F_{h+1}}$$

for $h = 1, 2, 3, \ldots$, and this in turn implies

$$\frac{1}{F_{h+2}} < \{\alpha F_h\} < \frac{1}{F_{h+1}} \text{ for odd } h \tag{3}$$

and

$$-\frac{1}{F_{h+1}} < \{\alpha F_h\} - 1 < -\frac{1}{F_{h+2}} \text{ for even } h. \tag{4}$$

Write the Zeckendorf representation of n as indicated by the sum

$$n = c_1F_2 + c_2F_3 + c_3F_4 + \cdots . \tag{5}$$

Then

$$f(n) = c_1F_3 + c_2F_4 + c_3F_5 + \cdots . \tag{6}$$

Also,

$$\begin{aligned} n\alpha &= c_1\alpha F_2 + c_2\alpha F_3 + c_3\alpha F_4 + \cdots \\ &= c_1(F_3 + \{\alpha F_2\} - 1) + c_2(F_4 + \{\alpha F_3\}) + c_3(F_5 + \{\alpha F_4\} - 1) + \cdots \\ &= f(n) + \mathcal{S}_1(n) + \mathcal{S}_2(n), \end{aligned}$$

where

$$\mathcal{S}_1(n) = c_1(\{\alpha F_2\} - 1) + c_3(\{\alpha F_4\} - 1) + c_5(\{\alpha F_6\} - 1) + \cdots$$

and

$$\mathcal{S}_2(n) = c_2\{\alpha F_3\} + c_4\{\alpha F_5\} + c_6\{\alpha F_7\} + \cdots .$$

<u>**Case 1**</u>**:** n is an even numbered column of Z. In this case, the least nonzero coefficient c_H in (5) has an even index H, so that

$$\begin{aligned} \mathcal{S}_1(n) + \mathcal{S}_2(n) &= \{\alpha F_{H+1}\} + \text{other terms} \\ &\le \{\alpha F_{H+1}\} + \{\alpha F_{H+3}\} + \cdots < \frac{1}{F_{H+2}} + \frac{1}{F_{H+4}} + \frac{1}{F_{H+6}} + \cdots \text{ by (3)} \\ &\le \frac{1}{2} + \frac{1}{5} + \frac{1}{13} + \cdots < \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1. \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{S}_1(n) + \mathcal{S}_2(n) &\ge \{\alpha F_{H+1}\} + (-1 + \{\alpha F_{H+4}\}) + (-1 + \{\alpha F_{H+6}\}) + \cdots \\ &> \frac{1}{F_{H+3}} - \frac{1}{F_{H+5}} - \frac{1}{F_{H+7}} - \cdots \text{ by (3) and (4)} \\ &\ge \frac{1}{F_{H+3}} - \frac{1}{F_{H+5}}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) \ge \frac{1}{F_{H+3}} - \frac{2}{F_{H+5}} > 0. \end{aligned}$$

The conclusion in Case 1 is that $f(n) = [n\alpha]$.

<u>**Case 2**</u>**:** n is an odd numbered column of Z. Then the least nonzero coefficient c_H in (5) has an odd index H, and

$$\begin{aligned} \mathcal{S}_1(n) + \mathcal{S}_2(n) &= -1 + \{\alpha F_{H+1}\} + \text{other terms} \\ &\le -1 + \{\alpha F_{H+1}\} + \{\alpha F_{H+4}\} + \{\alpha F_{H+6}\} + \cdots \\ &< -\frac{1}{F_{H+3}} + \frac{1}{F_{H+5}} + \frac{1}{F_{H+7}} + \cdots \text{ by (3) and (4)} \\ &\le -\frac{1}{F_{H+3}} + \frac{2}{F_{H+5}} < 0. \end{aligned}$$

Also

$$\begin{aligned}\mathcal{S}_1(n)+\mathcal{S}_2(n) &\geq -1+\{\alpha F_{H+1}\}+(-1+\{\alpha F_{H+3}\})+(-1+\{\alpha F_{H+5}\})+\cdots \\ &> -\frac{1}{F_{H+2}}-\frac{1}{F_{H+4}}-\frac{1}{F_{H+6}}-\cdots \text{ by (4)} \\ &> -\frac{2}{F_{H+2}} \geq -1.\end{aligned}$$

The conclusion in Case 2 is that $f(n)=[n\alpha]+1$. □

Theorem 2: The Zeckendorf array equals the Wythoff array.

Proof: Let C be the set of numbers in the first column of Z. Let S be the complement of C in the set of positive integers. Let $\{s_n\}$ be the sequence obtained by arranging the elements of S in increasing order. It is known [4] that this sequence of one-free Zeckendorf sums is given by $s_n=[(n+1)\alpha]-1$, $n=1,2,3,\ldots$. We shall apply Beatty's theorem (see [1], [11]) on complementary sequences of positive integers to prove that $z(i,1)=[i\alpha]+i-1$: first, $\frac{1}{\alpha}+\frac{1}{\alpha+1}=1$, so that, by Beatty's theorem, the sequences $[n\alpha]$ and $[i\alpha]+i$ are complementary; this implies that the sets $\{[(n+1)\alpha]\}$ and $\{[i\alpha]+i\}\cup\{1\}$ partition N, which in turn implies that the sequences s_n and $z(i,1)$ are complementary. Since $w(i,1)=[i\alpha]+i-1$, we have $z(i,1)=w(i,1)$. Now the recurrence (1) together with Theorem 1 and Lemma 2 imply that $Z=W$.

3. HIGHER-ORDER ZECKENDORF ARRAYS

Let m be an integer ≥ 2. Define a sequence $\{s_i\}$ inductively, as follows:

$$\begin{aligned} s_i &= 1 && \text{for } i=1,2,3,\ldots,m, \\ s_i &= s_{i-1}+s_{i-m} && \text{for } i=m+1,m+2,\ldots, \end{aligned}$$

and define *the Zeckendorf m-basis* as the sequence $\{b_j^{(m)}\}$, where $b_j^{(m)}=s_{m+j-1}$ for all j in N. It is proved in [5] and probably elsewhere that *every n in N is uniquely a sum*

$$b_{i_1}^{(m)}+b_{i_2}^{(m)}+\cdots+b_{i_v}^{(m)}, \text{ where } i_t-i_u\geq m \text{ whenever } t>u. \tag{7}$$

We call the sum in (7) the *m-order Zeckendorf representation* of n, and we define the *m-order Zeckendorf array* $Z^{(m)}=Z^{(m)}(i,j)$ as follows: column j of $Z^{(m)}$ is the increasing sequence of all n in whose m-order Zeckendorf representation the least term is $b_j^{(m)}$. The first row of $Z^{(m)}$ is the Zeckendorf m-basis. Of course, the Zeckendorf 2-basis is the Fibonacci sequence $(b_j^{(2)}=F_{j+1})$, and one may view the work in this section as an attempt to generalize the results in Section 2. Table 2 shows part of the 3-order Zeckendorf array.

Next, we generalize the shift function $f: N\to N$ as defined in Section 2. In terms of m-order Zeckendorf representations, the generalized function $f^{(m)}$ is given as follows:

$$\text{if } n=\sum_{h=1}^{\infty}c_h b_h^{(m)}, \text{ then } f^{(m)}(n)=\sum_{h=1}^{\infty}c_h b_{h+1}^{(m)}.$$

TABLE 2. The 3rd-Order Zeckendorf Array

1	2	3	4	6	9	13	19	28	41	60	
5	8	12	17	25	37	54	79	116	170	249	...
7	11	16	23	34	50	73	107	157	230	337	
10	15	22	32	47	69	101	148	217	318	466	
14	21	31	45	66	97	142	208	306	448	656	
18	27	40	58	85	125	183	268	393	576	844	
20	30	44	64	94	138	202	296	434	636	932	
24	36	53	77	113	166	243	356	522	765	1121	
26	39	57	83	122	179	262	384	563	825	1209	
⋮											

Lemma 1: The shift function $f^{(m)}$ is a strictly increasing function.

Proof: As a first inductive step, we have $2 = f^{(m)}(1) < 3 = f^{(m)}(2)$. Assume $K \geq 2$ and that for every $k_1 < K$ it is true that $f^{(m)}(k_1) < f^{(m)}(K)$. Let k be any positive integer satisfying $k \leq K$. Let

$$h_0 = \max_{b_j \leq K+1} \{j\} \text{ and } h_1 = \max_{b_j \leq k} \{j\}.$$

<u>Case 1</u>: $h_1 < h_0$. Let $x = b_{h_0-1}^{(m)} + b_{h_0-1-m}^{(m)} + b_{h_0-1-2m}^{(m)} + \cdots + b_s^{(m)}$, where $s = h_0 - 1 - [\frac{h_0-2}{m}]m$. Since $x+1 = b_{h_0}^{(m)}$, we have $k \leq x < K+1$ and

$$f^{(m)}(k) \leq f^{(m)}(x) = b_{h_0}^{(m)} + b_{h_0-m}^{(m)} + b_{h_0-2m}^{(m)} + \cdots + b_{s+1}^{(m)} < b_{h_0+1}^{(m)} \leq b_{h_1}^{(m)} \leq f^{(m)}(K+1).$$

<u>Case 2</u>: $h_1 = h_0$. Here, $k - b_{h_0}^{(m)} < K + 1 - b_{h_0}^{(m)}$. By the induction hypothesis,

$$f^{(m)}(k - b_{h_0}^{(m)}) < f^{(m)}(K + 1 - b_{h_0}^{(m)}).$$

Then

$$f^{(m)}(k) = f^{(m)}(k - b_{h_0}^{(m)}) + b_{h_0+1}^{(m)} < f^{(m)}(K + 1 - b_{h_0}^{(m)}) + b_{h_0+1}^{(m)} = f^{(m)}(K+1).$$

In both cases, $f^{(m)}(k) < f^{(m)}(K+1)$ for all $k < K+1$, so that we conclude that $f^{(m)}$ is strictly increasing. □

Theorem 3: The first column of the *m*-order Zeckendorf array determines all of the array by the recurrences (2) for all $i \geq 1$ and $j \geq 1$.

Proof: The proof is analogous to that of Theorem 1 and is omitted.

Theorem 4: For every $m \geq 2$, the *m*-order Zeckendorf array is an interspersion.

Proof: Of the four properties that define an interspersion (as introduced in [7]), it is clear that $Z^{(m)}$ satisfies the first three: every positive integer occurs exactly once in $Z^{(m)}$; every row of $Z^{(m)}$ is increasing; and every column of $Z^{(m)}$ is increasing. To prove the fourth property, suppose i, j, i', j' are indices for which

$$z(i, j) < z(i', j') < z(i, j+1).$$

Then, by Lemma 1,

$$f^{(m)}(z(i, j)) < f^{(m)}(z(i', j')) < f^{(m)}(z(i, j+1)),$$

so that, by Theorem 3,

$$z(i, j+1) < z(i', j'+1) < z(i, j+2),$$

as required. □

Consider the recurrence (1) which defines the Wythoff array W in terms of the golden mean, α. Since α is the real root of the characteristic polynomial $x^2 - x - 1$ of the recurrence relation for the row sequences of W, one must wonder if the real root $\alpha^{(m)}$ of $x^m - x^{m-1} - 1$ can, in some manner comparable to (1), be used to generate rows of the m-order Zeckendorf array. The answer seems to be no, although certain "higher-order" Wythoff-like arrays have been investigated (see [3], [6]).

However, Beatty's theorem leads to conjectures about column 1 of $Z^{(m)}$. It appears that each row of $Z^{(m)}$ has "slope" $\alpha^{(m)}$, so that the complement of column 1, ordered as an increasing sequence, is comparable to the set of numbers $[i\alpha^{(m)}]$. Beatty's theorem then suggests that column 1 is "close to" the sequence $\{c_i\}$ given by $c_i = \lfloor \frac{i\alpha^{(m)}}{\alpha^{(m)}-1} \rfloor$. For example, taking $m = 3$, let $s_i = \lfloor \frac{\alpha^{(3)}i}{\alpha^{(3)}-1} \rfloor - [\alpha^{(3)}]$. Let x_i denote the i^{th} number in column 1 of $Z^{(3)}$. We conjecture that $|z_i - s_i| \leq 1$ for all $i \geq 1$.

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ON THE (3, *F*) GENERALIZATIONS OF THE FIBONACCI SEQUENCE

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Recently the (2, *F*) generalized Fibonacci sequences were derived (see [1], [4], [5], [6]). The purpose of this paper is to derive formulas for all (3, *F*) generalized Fibonacci sequences as functions of the terms of seven sixth-order recursive sequences.

Let $S = \{a, b, c\}$ and S_c be the group of permutations on S. Let ι be the identity; and $\alpha = (a\ b)$, $\beta = (a\ c)$, $\gamma = (b\ c)$ be the two cycles; and $\delta = (a\ b\ c)$, $\varepsilon = (a\ c\ b)$ be the three cycles. Finally, let ϕ and τ be arbitrary permutations of S_c and $Y_1 = \{a_i, b_i, c_i\}$. Atanassov [2] considered the 36 possible systems of three second-order difference equations:

$$\iota Y_{n+2} = \phi Y_{n+1} + \tau Y_n, \quad n \geq 0, \tag{1}$$

with initial conditions $Y_0 = \{a_0, b_0, c_0\}$ and $Y_1 = \{a_1, b_1, c_1\}$, where $a_0, b_0, c_0, a_1, b_1,$ and c_1 are real numbers. Since the permutation in the left member of (1) is always the identity (ι), these systems can be represented by the ordered pair (ϕ, τ), where ϕ and τ are the permutations of the right member. Spickerman et al. [7] proved that the 36 systems are members of the eleven equivalence classes. The solutions of one of these classes is three generalized Fibonacci sequences. The solutions to three other classes consist of one generalized Fibonacci sequence and one (2, *F*) generalized Fibonacci sequence. The solutions to the other seven systems are the (3, *F*) generalized Fibonacci sequences. Atanassov [3] denoted each of these seven sequences by a number, as shown in Table 1. A notation in terms of ordered pairs of permutations of S_c is also given.

Considering each equivalence class, it follows that when the solution to one system in a class is known, the solutions to the other systems are permutations of the known solution. Atanassov et al. [3] proved

$$\iota Y^s_{n+2} = \phi Y^s_{n+1} + \tau Y^s_n, \quad s \in \{1, 2, 3, 4, 5, 6, 7\}, \tag{2}$$

with initial conditions

$$Y^s_0 = \{a^s_0, b^s_0, c^s_0\}, \quad Y_1 = \{a^s_1, b^s_1, c^s_1\},$$

can be replaced with seven sixth-order difference systems:

$$\sum_{i=0}^{6} k^s_i a^s_{n+6-i} = 0, \quad \sum_{i=0}^{6} k^s_i b^s_{n+6-i} = 0, \quad \sum_{i=0}^{6} k^s_i c^s_{n+6-i} = 0, \quad n \geq 0, \tag{3}$$

with initial conditions $\{a^s_i\}^5_0$, $\{b^s_i\}^5_0$, $\{c^s_i\}^5_0$, respectively. The values for k^s_i for $1 \leq s \leq 7$ are given in Table 2.

Let $p^s(x) = \sum_{i=0}^{6} k^s_i x^i$ and let $\{P^s_j\}^\infty_{j=0}$ be the recursive numbers (of order six) determined by $1/p^s(x)$. Then the seven recursion relations and first terms of the sequences are given in Table 3.

TABLE 1

Permutation Notation	Equivalence Class	Atanassov's Number
$[(\iota,\iota)]$	$\{(\iota,\iota)\}$	none*
$[(\alpha,\alpha)]$	$\{(\alpha,\alpha),(\beta,\beta),(\gamma,\gamma)\}$	none**
$[(\iota,\alpha)]$	$\{(\iota,\alpha),(\iota,\beta),(\iota,\gamma)\}$	none**
$[(\alpha,\iota)]$	$\{(\alpha,\iota),(\beta,\iota),(\gamma,\iota)\}$	none**
$[\iota,\delta]$	$\{(\iota,\delta),(\iota,\varepsilon)\}$	1
$[\delta,\iota]$	$\{(\delta,\iota),(\varepsilon,\iota)\}$	2
$[\delta,\delta]$	$\{(\delta,\delta),(\varepsilon,\varepsilon)\}$	3
$[\delta,\varepsilon]$	$\{(\delta,\varepsilon),(\varepsilon,\delta)\}$	4
$[\alpha,\beta]$	$\{(\alpha,\beta),(\alpha,\gamma),(\beta,\alpha),(\beta,\gamma),(\gamma,\alpha),(\gamma,\beta)\}$	5
$[\alpha,\delta]$	$\{(\alpha,\delta),(\alpha,\varepsilon),(\beta,\delta),(\beta,\varepsilon),(\gamma,\delta),(\gamma,\varepsilon)\}$	6
$[\delta,\alpha]$	$\{(\delta,\alpha),(\varepsilon,\alpha),(\delta,\beta),(\varepsilon,\beta),(\delta,\gamma),(\varepsilon,\gamma)\}$	7

*Solution is three generalized Fibonacci sequences.
**Solution is one generalized Fibonacci sequence and one (2, F) generalized Fibonacci sequence.

TABLE 2

	Values of k_i^S						
	i						
s	0	1	2	3	4	5	6
1	1	−3	3	−1	0	0	−1
2	1	0	−3	−1	3	0	−1
3	1	0	0	−1	−3	−3	−1
4	1	0	0	−4	0	0	−1
5	1	−1	−2	2	−1	0	1
6	1	−1	−1	0	1	−1	−1
7	1	0	−1	−2	−2	1	1

TABLE 3

s	Recursive Relation	First 7 Terms
1	$P_{n+6}=3P_{n+5}-3P_{n+4}+P_{n+3}+P_n$	1, 3, 6, 10, 15, 21, 29
2	$P_{n+6}=3P_{n+4}+P_{n+3}-3P_{n+2}+P_n$	1, 0, 3, 1, 6, 6, 11
3	$P_{n+6}=P_{n+3}+3P_{n+2}+3P_{n+1}+P_n$	1, 0, 0, 1, 3, 3, 2
4	$P_{n+6}=4P_{n+3}+P_n$	1, 0, 0, 4, 0, 0, 17
5	$P_{n+6}=P_{n+5}+2P_{n+4}-2P_{n+3}+P_{n+2}-P_n$	1, 1, 3, 3, 8, 9, 21
6	$P_{n+6}=P_{n+5}+P_{n+4}-P_{n+2}+P_{n+1}+P_n$	1, 1, 2, 3, 4, 7, 11
7	$P_{n+6}=P_{n+4}+2P_{n+3}+2P_{n+2}-P_{n+1}-P_n$	1, 0, 1, 2, 3, 3, 8

Let $f^s(x)$, $g^s(x)$, and $h^s(x)$ be the three solutions to the seven systems, and let

$$f^s(x)=\sum_{j=0}^{\infty}a_j^s x^i,\quad g^s(x)=\sum_{j=0}^{\infty}b_j^s x^i,\quad h^s(x)=\sum_{j=0}^{\infty}c_j^s x^j.$$

First, it follows that

$$\left(\sum_{i=0}^{6} k_i^s x^i\right) f^s(x) = \left(\sum_{i=0}^{6} k_i^s x^i\right)\left(\sum_{j=0}^{\infty} a_j^s x^j\right) = \sum_{j=0}^{\infty}\left[\sum_{i=0}^{n} k_i^s a_{j-i}^s\right] x^j \quad \begin{cases} n = j & \text{for } j \le 5, \\ n = 6 & \text{otherwise,} \end{cases}$$

$$= \sum_{j=0}^{5}\left[\sum_{i=0}^{j} k_i^s a_{j-i}^s\right] x^j + \sum_{j=6}^{\infty}\left[\sum_{i=0}^{6} k_i^s a_{j-i}^s\right] x^j.$$

In view of the difference systems (3), the last term is zero. Therefore,

$$\left(\sum_{i=0}^{6} k_i^s x^i\right) f^s(x) = \sum_{j=0}^{5}\left[\sum_{i=0}^{j} k_i^s a_{j-i}^s\right] x^j,$$

or

$$p^s(x) f^s(x) = \sum_{j=0}^{5}\left[\sum_{i=0}^{j} k_i^s a_{j-i}^s\right] x^j.$$

Let

$$q_j^s = \sum_{m=0}^{j} k_m^s a_{j-m}^s,$$

then

$$f^s(x) = \frac{\sum_{i=0}^{5} q_i^s x^i}{p^s(x)} = \left(\sum_{i=0}^{5} q_i^s x^i\right)\left(\frac{1}{p^s(x)}\right).$$

Consequently,

$$f^s(x) = \left(\sum_{i=0}^{5} q_i^s x^i\right)\left(\sum_{j=0}^{\infty} P_j^s x^j\right),$$

where P_j^s are from the sequences in Table 2. Expanding and collecting terms gives

$$f^s(x) = \sum_{j=0}^{\infty}\left[\sum_{i=0}^{m} q_i^s P_{j-i}^s\right] x^j \quad \begin{cases} m = j & \text{when } j < 5, \\ m = 5 & \text{otherwise,} \end{cases}$$

$$= \sum_{j=0}^{4}\left(\sum_{i=0}^{j} q_i^s P_{j-i}^s\right) x^j + \sum_{j=0}^{\infty}\sum_{i=0}^{5}\left(q_i^s P_{j-i}^s\right) x^j$$

for the generating function for $\{a_i^s\}_0^\infty$. The terms of the sequence are given by

$$a_j^s = \sum_{i=0}^{j} q_j^s P_{j-i}^s = \sum_{i=0}^{j}\left[\sum_{m=0}^{i} k_m^s a_{i-m}^s\right] P_{j-i}^s \qquad \text{for } j < 5,$$

and

$$a_j^s = \sum_{i=0}^{5} q_i^s P_{j-i}^s = \sum_{i=0}^{5}\left[\sum_{m=0}^{i} k_m^s a_{i-m}^s\right] P_{j-i}^s \qquad \text{for } j \ge 5.$$

The values of $a_i^s, 2 \le i \le 5$, are computed in terms of $a_0^s, a_1^s, b_0^s, b_1^s, c_0^s, c_1^s$ by use of equations (2). The sequences $\{b_i^s\}_0^\infty$ and $\{c_i^s\}_0^\infty$ have the same form.

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SOME INFINITE SERIES SUMMATIONS USING POWER SERIES EVALUATED AT A MATRIX

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1. INTRODUCTION

In the notation of Horadam [5], write

$$W_n = W_n(a, b; p, q), \tag{1.1}$$

so that

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a,\ W_1 = b,\ n \geq 2. \tag{1.2}$$

The sequence $\{W_n\}_{n=0}^{\infty}$ can be extended to negative subscripts by the use of (1.2) and, with this understanding, we simply write $\{W_n\}$.

The n^{th} terms of the well-known Fibonacci and Lucas sequences are then

$$\begin{cases} F_n = W_n(0, 1; 1, -1), \\ L_n = W_n(2, 1; 1, -1). \end{cases} \tag{1.3}$$

More generally, we write

$$\begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p; p, q), \end{cases} \tag{1.4}$$

where $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences, respectively, generated by (1.2). They have been studied extensively, particularly by Lucas [7].

The Binet forms for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.5}$$

$$V_n = \alpha^n + \beta^n, \tag{1.6}$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2} \tag{1.7}$$

are the roots, assumed distinct, of

$$x^2 - px + q = 0. \tag{1.8}$$

Write

$$\Delta = (\alpha - \beta)^2 = p^2 - 4q. \tag{1.9}$$

The Q-matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.10}$$

has been studied widely in connection with the Fibonacci numbers and has the property

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \ n \text{ an integer (see [4]).} \tag{1.11}$$

Filipponi and Horadam [2] considered the matrix

$$Q_{k,x} = xQ^k = \begin{pmatrix} xF_{k+1} & xF_k \\ xF_k & xF_{k-1} \end{pmatrix}, \tag{1.12}$$

where x is an arbitrary real number and k is a nonnegative integer, and noted that

$$Q_{k,x}^n = \begin{pmatrix} x^n F_{kn+1} & x^n F_{kn} \\ x^n F_{kn} & x^n F_{kn-1} \end{pmatrix}. \tag{1.13}$$

Then they evaluated certain power series at the matrix $Q_{k,x}$ to obtain summation identities involving the Fibonacci and Lucas numbers. The identities had the following forms:

$$\sum_{n=0}^{\infty} a_n x^n F_{kn+1} = \frac{\phi_1 f(x\phi_1^k) - \phi_2 f(x\phi_2^k)}{\sqrt{5}}, \tag{1.14}$$

$$\sum_{n=0}^{\infty} a_n x^n F_{kn} = \frac{f(x\phi_1^k) - f(x\phi_2^k)}{\sqrt{5}}, \tag{1.15}$$

$$\sum_{n=0}^{\infty} a_n x^n F_{kn-1} = \frac{\phi_1 f(x\phi_2^k) - \phi_2 f(x\phi_1^k)}{\sqrt{5}}, \tag{1.16}$$

$$\sum_{n=0}^{\infty} a_n x^n L_{kn} = f(x\phi_1^k) + f(x\phi_2^k), \tag{1.17}$$

where

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1.18}$$

and

$$\phi_1 = \frac{1+\sqrt{5}}{2}, \quad \phi_2 = \frac{1-\sqrt{5}}{2}.$$

They also indicated how their procedures could be generalized to apply to $W_n(0, 1; p, -1)$ and $W_n(2, p; p, -1)$.

The object of this paper is to extend (1.14)-(1.17) to apply to the more general fundamental and primordial sequences of Lucas as defined in (1.4). Then, specializing to the Chebyshev polynomials of the first and second kinds, we obtain infinite series summations involving the sine and cosine functions that we believe are new.

2. THE MATRIX $A_{k,x}$

Define the matrix A by

$$A = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

Then it can be shown by induction that

$$A^n = \begin{pmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{pmatrix}, \quad n \geq 0. \tag{2.2}$$

Associated with A, we define the matrix $A_{k,x}$ by

$$A_{k,x} = xA^k = \begin{pmatrix} xU_{k+1} & -x^nqU_{kn} \\ xU_k & -xqU_{k-1} \end{pmatrix}, \tag{2.3}$$

where x is an arbitrary real number and k is a nonnegative integer.

To prove the following lemma, we need to note that

$$V_k = U_{k+1} - qU_{k-1}, \tag{2.4}$$

$$U_k^2 - U_{k+1}U_{k-1} = q^{k-1}. \tag{2.5}$$

Each can be proved using Binet forms, and (2.5) is in fact a generalization of Simson's identity for Fibonacci numbers.

Lemma 1: The eigenvalues of $A_{k,x}$ are $x\alpha^k$ and $x\beta^k$.

Proof: Using (2.4) and (2.5), we see that the characteristic equation of $A_{k,x}$ simplifies to

$$t^2 - xV_kt + x^2q^k = 0. \tag{2.6}$$

Recalling that $V_k = \alpha^k + \beta^k$ and $q = \alpha\beta$, we see, by substitution, that the eigenvalues are as stated. □

Another important property of $A_{k,x}$ is

$$A_{k,x}^n = (xA^k)^n = x^nA^{kn} = \begin{pmatrix} x^nU_{kn+1} & -xqU_{kn} \\ x^nU_{kn} & -x^nqU_{kn-1} \end{pmatrix}, \text{ by (2.2).} \tag{2.7}$$

The following is easily proved by induction:

$$\alpha^n = \alpha U_n - qU_{n-1}, \quad n \geq 0. \tag{2.8}$$

Of course, (2.8) remains valid if we replace α by β.

3. THE MAIN RESULT

Assuming that f as defined in (1.18) has a domain of convergence which includes $x\alpha^k$ and $x\beta^k$ we have, using (2.7),

$$f(A_{k,x}) = \sum_{n=0}^{\infty} a_nA_{k,x}^n = \begin{pmatrix} \sum_{n=0}^{\infty} a_nx^nU_{kn+1} & -q\sum_{n=0}^{\infty} a_nx^nU_{kn} \\ \sum_{n=0}^{\infty} a_nx^nU_{kn} & -q\sum_{n=0}^{\infty} a_nx^nU_{kn-1} \end{pmatrix}. \tag{3.1}$$

On the other hand, from the theory of matrices [3], it is known that $f(A_{k,x}) = c_0I + c_1A_{k,x}$, where I is the identity 2×2 matrix and where c_0 and c_1 can be obtained by solving

$$\begin{cases} c_0 + c_1 x\alpha^k = f(x\alpha^k), \\ c_0 + c_1 x\beta^k = f(x\beta^k). \end{cases}$$

That is,

$$f(A_{k,x}) = \left(\frac{x\alpha^k f(x\beta^k) - x\beta^k f(x\alpha^k)}{x(\alpha^k - \beta^k)} \right) I + \left(\frac{f(x\alpha^k) - f(x\beta^k)}{x(\alpha^k - \beta^k)} \right) A_{k,x}. \tag{3.2}$$

This is Sylvester's matrix interpolation formula [8]. Noting that $\alpha^k - \beta^k = \sqrt{\Delta}\, U_k$ and using (2.8), the right side of (3.2) can be simplified to yield

$$f(A_{k,x}) = \begin{pmatrix} \dfrac{\alpha f(x\alpha^k) - \beta f(x\beta^k)}{\sqrt{\Delta}} & \dfrac{q(f(x\beta^k) - f(x\alpha^k))}{\sqrt{\Delta}} \\ \dfrac{f(x\alpha^k) - f(x\beta^k)}{\sqrt{\Delta}} & \dfrac{\alpha f(x\beta^k) - \beta f(x\alpha^k)}{\sqrt{\Delta}} \end{pmatrix}. \tag{3.3}$$

These observations lead to our main result.

Theorem 1: If f as defined in (1.18) has a domain of convergence which includes $x\alpha^k$ and $x\beta^k$, then

$$\sum_{n=0}^{\infty} a_n x^n U_{kn+1} = \frac{\alpha f(x\alpha^k) - \beta f(x\beta^k)}{\sqrt{\Delta}}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn} = \frac{f(x\alpha^k) - f(x\beta^k)}{\sqrt{\Delta}}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} = \frac{\beta f(x\alpha^k) - \alpha f(x\beta^k)}{q\sqrt{\Delta}}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} a_n x^n V_{kn} = f(x\alpha^k) + f(x\beta^k). \tag{3.7}$$

We note that (3.4)-(3.6) are obtained by comparing (3.1) and (3.3). Identity (3.7) is obtained by using (2.4), (3.4), and (3.6).

It is easily seen that (3.4)-(3.7) generalize (1.14)-(1.17) and also (5.6)-(5.17) of [2]. In the next section we apply (3.5) and (3.7) to the Chebyshev polynomials and obtain infinite sums involving the sine and cosine functions.

4. APPLICATIONS

Let $\{T_n(x)\}_{n=0}^{\infty}$ and $\{S_n(x)\}_{n=0}^{\infty}$ denote the Chebyshev polynomials of the first and second kinds, respectively. Then

$$\left.\begin{aligned} S_n(x) &= \frac{\sin n\theta}{\sin\theta} \\ T_n(x) &= \cos n\theta \end{aligned}\right\}, \quad x = \cos\theta,\ n \ge 0.$$

Indeed $\{S_n(x)\}_{n=0}^{\infty}$ and $\{2T_n(x)\}_{n=0}^{\infty}$ are the fundamental and primordial sequences, respectively, generated by (1.2), where $p=2\cos\theta$, $q=1$. Thus,

$$\alpha=e^{i\theta} \text{ and } \beta=e^{-i\theta}, \tag{4.1}$$

which are obtained by solving $t^2-2\cos\theta t+1=0$. Further information about the Chebyshev polynomials can be found, for example, in [1] and [6].

To begin, we consider the following well-known power series each of which has the complex plane as its domain of convergence:

$$\sin z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}, \tag{4.2}$$

$$\cos z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}, \tag{4.3}$$

$$\sinh z=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}, \tag{4.4}$$

$$\cosh z=\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}. \tag{4.5}$$

Now in (3.5), taking $U_n=\frac{\sin n\theta}{\sin\theta}$ and replacing f by the functions in (4.2)-(4.5), we obtain, respectively,

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\sin k(2n+1)\theta}{(2n+1)!}=\cos(x\cos k\theta)\sinh(x\sin k\theta), \tag{4.6}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1} x^{2n}\sin 2kn\theta}{(2n)!}=\sin(x\cos k\theta)\sinh(x\sin k\theta), \tag{4.7}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}\sin k(2n+1)\theta}{(2n+1)!}=\sin(x\sin k\theta)\cosh(x\cos k\theta), \tag{4.8}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n}\sin 2kn\theta}{(2n)!}=\sin(x\sin k\theta)\sinh(x\cos k\theta). \tag{4.9}$$

In (3.7), taking $V_n=2\cos n\theta$ and replacing f by the functions in (4.2)-(4.5), we obtain, respectively,

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\cos k(2n+1)\theta}{(2n+1)!}=\sin(x\cos k\theta)\cosh(x\sin k\theta), \tag{4.10}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}\cos 2kn\theta}{(2n)!}=\cos(x\cos k\theta)\cosh(x\sin k\theta), \tag{4.11}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}\cos k(2n+1)\theta}{(2n+1)!}=\cos(x\sin k\theta)\sinh(x\cos k\theta), \tag{4.12}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n}\cos 2kn\theta}{(2n)!}=\cos(x\sin k\theta)\cosh(x\cos k\theta). \tag{4.13}$$

At this point, we note that (4.6), (4.7), (4.10), and (4.11) generalize (40), (42), (41), and (43), respectively, of Walton [9].

As an example of the method, we prove (4.11).

Proof of (4.11): In (3.7), taking $V_n = 2\cos n\theta$ and $f(x)=\cos x$ we have, using (4.1) and (4.3),

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n} 2\cos 2kn\theta}{(2n)!} &= \cos(xe^{ik\theta})+\cos(xe^{-ik\theta})\\ &= 2\cos\left(x\left(\frac{e^{ik\theta}+e^{-ik\theta}}{2}\right)\right)\cos\left(x\left(\frac{e^{ik\theta}-e^{-ik\theta}}{2}\right)\right)\\ &= 2\cos(x\cos k\theta)\cos(ix\sin k\theta)\\ &= 2\cos(x\cos k\theta)\cosh(x\sin k\theta),\end{aligned}$$

which yields the result. □

We now obtain further interesting sums by employing some power series which occur in [1]. We restate them here for easy reference:

$$\log_e\left(1+\frac{z}{m}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot\frac{z^n}{m^n},\quad |z|<|m|, \tag{4.14}$$

$$\tan^{-1}\left(\frac{z}{m}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\cdot\frac{z^{2n+1}}{m^{2n+1}},\quad |z|<|m|, \tag{4.15}$$

$$\sec\left(\frac{z}{m}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}}{(2n)!}\cdot\frac{z^{2n}}{m^{2n}},\quad |z|<\frac{\pi}{2}|m|, \tag{4.16}$$

$$\tan\left(\frac{z}{m}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}\cdot\frac{z^{2n-1}}{m^{2n-1}},\quad |z|<\frac{\pi}{2}|m|, \tag{4.17}$$

$$\operatorname{cosec}\left(\frac{z}{m}\right)-\frac{m}{z}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n}-2)B_{2n}}{(2n)!}\cdot\frac{z^{2n-1}}{m^{2n-1}},\quad 0<|z|<\pi|m|, \tag{4.18}$$

$$\cot\left(\frac{z}{m}\right)-\frac{m}{z}=\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n}B_{2n}}{(2n)!}\cdot\frac{z^{2n-1}}{m^{2n-1}},\quad 0<|z|<\pi|m|. \tag{4.19}$$

Here, B_n and E_n are the Bernoulli and Euler numbers, respectively.

In (3.5), taking $U_n=\frac{\sin n\theta}{\sin\theta}$ and replacing f by the functions in (4.14)-(4.19) we obtain, respectively,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n\sin kn\theta}{nm^n}=\frac{1}{2i}\log_e\left(\frac{m+xe^{ik\theta}}{m+xe^{-ik\theta}}\right),\quad |x|<|m|, \tag{4.20}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\sin k(2n+1)\theta}{(2n+1)m^{2n+1}}=\frac{1}{2}\tanh^{-1}\left(\frac{2mx\sin k\theta}{m^2+x^2}\right),\quad |x|<|m|, \tag{4.21}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}x^{2n}\sin 2kn\theta}{(2n)!m^{2n}}=\frac{2\sin\left(\frac{x\cos k\theta}{m}\right)\sinh\left(\frac{x\sin k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)+\cosh\left(\frac{2x\sin k\theta}{m}\right)},\quad |x|<\frac{\pi}{2}|m|, \tag{4.22}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}\sin k(2n-1)\theta}{(2n)!m^{2n-1}}\\&=\frac{\sinh\left(\frac{2x\sin k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)+\cosh\left(\frac{2x\sin k\theta}{m}\right)},\quad |x|<\frac{\pi}{2}|m|,\end{aligned} \tag{4.23}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n}-2)B_{2n}x^{2n-1}\sin k(2n-1)\theta}{(2n)!m^{2n-1}}\\&=\frac{2\cos\left(\frac{x\cos k\theta}{m}\right)\sinh\left(\frac{x\sin k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)-\cosh\left(\frac{2x\sin k\theta}{m}\right)}+\frac{m\sin k\theta}{x},\quad 0<|x|<\pi|m|,\end{aligned} \tag{4.24}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n}B_{2n}x^{2n-1}\sin k(2n-1)\theta}{(2n)!m^{2n-1}}\\&=\frac{\sinh\left(\frac{2x\sin k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)-\cosh\left(\frac{2x\sin k\theta}{m}\right)}+\frac{m\sin k\theta}{x},\quad 0<|x|<\pi|m|.\end{aligned} \tag{4.25}$$

As stated at the beginning of Section 3, the domains of validity are determined by the requirement that the eigenvalues, in this case $xe^{ik\theta}$ and $xe^{-ik\theta}$, must lie within the radius of convergence of the relevant power series. The proofs follow essentially the same lines as the proof of (4.11) demonstrated earlier, employing well-known properties of the relevant functions.

Finally in (3.7), taking $V_n=2\cos n\theta$ and replacing f by the functions in (4.14)-(4.19), we obtain, respectively,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n\cos kn\theta}{nm^n}=\frac{1}{2}\log_e\left(1+\frac{2x\cos k\theta}{m}+\frac{x^2}{m^2}\right),\quad |x|<|m|, \tag{4.26}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\cos k(2n+1)\theta}{(2n+1)m^{2n+1}}=\frac{1}{2}\tan^{-1}\left(\frac{2mx\cos k\theta}{m^2-x^2}\right),\quad |x|<|m|, \tag{4.27}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}x^{2n}\cos 2kn\theta}{(2n)!\,m^{2n}}=\frac{2\cos\left(\frac{x\cos k\theta}{m}\right)\cosh\left(\frac{x\sin k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)+\cosh\left(\frac{2x\sin k\theta}{m}\right)},\quad |x|<\frac{\pi}{2}|m|, \tag{4.28}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}\cos k(2n-1)\theta}{(2n)!\,m^{2n-1}}\\&=\frac{\sin\left(\frac{2x\cos k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)+\cosh\left(\frac{2x\sin k\theta}{m}\right)},\quad |x|<\frac{\pi}{2}|m|,\end{aligned} \tag{4.29}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n}-2)B_{2n}x^{2n-1}\cos k(2n-1)\theta}{(2n)!\,m^{2n-1}}\\&=\frac{2\sin\left(\frac{x\cos k\theta}{m}\right)\cosh\left(\frac{x\sin k\theta}{m}\right)}{\cosh\left(\frac{2x\sin k\theta}{m}\right)-\cos\left(\frac{2x\cos k\theta}{m}\right)}-\frac{m\cos k\theta}{x},\quad 0<|x|<\pi|m|,\end{aligned} \tag{4.30}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{n}2^{2n}B_{2n}x^{2n-1}\cos k(2n-1)\theta}{(2n)!\,m^{2n-1}}\\&=\frac{\sin\left(\frac{2x\cos k\theta}{m}\right)}{\cosh\left(\frac{2x\sin k\theta}{m}\right)-\cos\left(\frac{2x\cos k\theta}{m}\right)}-\frac{m\cos k\theta}{x},\quad 0<|x|<\pi|m|.\end{aligned} \tag{4.31}$$

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VARN CODES AND GENERALIZED FIBONACCI TREES

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INTRODUCTION AND BACKGROUND

Varn's [6] algorithm solves the problem of finding an optimal code tree, optimal in the sense of minimum average cost, when the code symbols are of unequal cost and the source symbols are equiprobable. He addresses both exhaustive codes, for which the code tree is a full tree, as well as nonexhaustive codes, but only the exhaustive case will be of concern here. In particular, for code symbol costs $c(1) \leq c(2) \leq \cdots \leq c(r)$ and a uniform source of size N, where $(N-1)/(r-1)$ is an integer, the Varn code tree is generated as follows. Start with an r-ary tree consisting of a root node from which descend r leaf nodes labeled from left to right by $c(1), c(2), \ldots, c(r)$, the costs associated with the corresponding code symbols. Select the lowest cost node, let c be its cost, and let descend from it r leaf nodes labeled from left to right by $c+c(1), c+c(2), \ldots, c+c(r)$. Continue, by selecting the lowest cost node from the new tree, until N leaf nodes have been created.

Horibe [3] studied a sequence of binary trees and showed that each tree in the sequence is a Varn code tree for $c(1)=1, c(2)=2$. In particular, the k^{th} tree has the $k-1^{\text{st}}$ tree as its left subtree and the $k-2^{\text{nd}}$ tree as its right subtree; for $k=1$ and $k=2$, the tree is only the root; $c(1)$ is associated with the left descendant of a node and $c(2)$ with the right descendant. These trees are called Fibonacci trees, and the number of leaves in the k^{th} tree is the k^{th} Fibonacci number. Note that some integers N are not equal to the k^{th} Fibonacci number for any k so that not every Varn code tree for $c(1)=1, c(2)=2$ is a Fibonacci tree.

Chang [1] studied a sequence of r-ary trees that reduces to Horibe's sequence of Fibonacci trees for $r=2$. In particular, the k^{th} tree has the $k-i^{\text{th}}$ tree as its i^{th} leftmost subtree, $i=1, \ldots, r$; for $k=1, \ldots, r$, the tree is only the root; and $c(i)=i, i=1, \ldots, r$ is associated with the descendants of a node in left to right order. For these particular costs, $c(i)=i, i=1, \ldots, r$, Chang's trees are Varn code trees, and the number of leaves in the k^{th} tree is determined according to an integer sequence that general-izes the Fibonacci sequence.

It is the purpose of this note to examine sequences of trees that are recursively constructed and are Varn code trees for integer costs $c(1), \ldots, c(r)$ whose greatest common divisor is 1. Since common factors shared by all costs do not affect Varn's algorithm, the costs considered here are essentially all rational costs or all sets of rational costs with a common irrational multiplier. Thus, previous work on recursive characterizations of Varn code trees for particular integer code symbol costs is extended to the case of arbitrary integer code symbol costs.

RECURSIVE CONSTRUCTION OF TREES

For fixed integer costs $c(1) \leq c(2) \leq \cdots \leq c(r)$ with greatest common divisor 1, we will have $c(r)$ "types" of leaf nodes denoted by $a_1, a_2, \ldots, a_{c(r)}$. The $k+1^{\text{st}}$ tree $T(k+1)$ is constructed from the previous tree $T(k)$ according to the following set of rules. A leaf node of type a_1 in $T(k)$ will be replaced by r descendant nodes of types $a_{c(1)}, a_{c(2)}, \ldots, a_{c(r)}$ in left to right order in $T(k+1)$. A node of type a_j in $T(k)$ will be replaced by a node of type a_{j-1} in $T(k+1)$, $j=2$, ...,

$c(r)$. The sequence of trees begins with $T(1)$, which consists of a single root node of type $a_{c(r)}$. This construction generalizes Horibe [3] and Chang [1].

An example of trees constructed in this fashion is given in Table 1 for the costs $c(1)=2$, $c(2)=3$, $c(3)=3$, $c(4)=5$. The trees are described using the following compact notation. Sibling nodes in left to right order are separated by + signs, and parentheses are used to indicate depth in tree from the root so that, for example, $((a_2+a_3+a_3+a_5)+a_1+a_1+a_3)$ denotes a 4-ary tree with 4 depth 2 leaves descending from the root through a common intermediate node and 3 depth 1 leaves descending from the root in left to right order and labeled according to type in left to right order as $a_2, a_3, a_3, a_5, a_1, a_1, a_3$, respectively.

TABLE 1. $T(k)$ for $c(1)=2, c(2)=3, c(3)=3, c(4)=5$

k	$T(k)$
1	a_5
2	a_4
3	a_3
4	a_2
5	a_1
6	$(a_2+a_3+a_3+a_5)$
7	$(a_1+a_2+a_2+a_4)$
8	$((a_2+a_3+a_3+a_5)+a_1+a_1+a_3)$
9	$((a_1+a_2+a_2+a_4)+(a_2+a_3+a_3+a_5)+(a_2+a_3+a_3+a_5)+a_2)$
10	$(((a_2+a_3+a_3+a_5)+a_1+a_1+a_3)+(a_1+a_2+a_2+a_4)+(a_1+a_2+a_2+a_4)+a_1)$
11	$(((a_1+a_2+a_2+a_4)+(a_2+a_3+a_3+a_5)+(a_2+a_3+a_3+a_5)+a_2)$ $+((a_2+a_3+a_3+a_5)+a_1+a_1+a_3)+((a_2+a_3+a_3+a_5)$ $+a_1+a_1+a_3)+(a_2+a_3+a_3+a_5))$
...	...

By induction, $T(k), k>c(r)$, has $T(k-c(i))$ as its i^{th} leftmost subtree, $i=1,\dots,r$. Because of the recursive tree construction, it is easy to give recurrence relations for the number of leaf nodes of each type in $T(k)$. Use $f^j(k)$ to denote the number of leaves of type a_j, $j=1,\dots,c(r)$, in $T(k)$. Then

$$f^j(k)=\sum_{1\le i\le r} f^j(k-c(i)), \tag{1}$$

where our initialization is $f^j(k)=1$ for $k+j=c(r)+1$, $1\le k\le c(r)$, and $f^j(k)=0$ for $k+j\ne c(r)+1$, $1\le k\le c(r)$. Clearly, the number of leaves in $T(k)$, $f(k)$, is given by

$$f(k)=\sum_{1\le j\le c(r)} f^j(k)=\sum_{1\le i\le r} f(k-c(i)). \tag{2}$$

VARN CODES FOR $N = f(k)$

The reason the recursive tree construction of the previous section is interesting is because the trees constructed are the minimum average codeword cost code trees for equiprobable sources of size $f(k)$, the Varn code trees for these source sizes. This is apparent because the construction rule splits the lowest cost leaf node at each stage, the node of type a_1, and the evolution of the node types from $T(k)$ to $T(k+1)$ keeps track of the relative node costs, that is, how many trees until that node type becomes the least cost node. Thus, the analysis of the average cost of $T(k)$, $C(T(k))$, is of interest.

To find $C(T(k))$, the assumption is that the tree is being used to encode an equiprobable source of $f(k)$ source symbols, and the costs of the codewords are the costs of the leaves of the tree. In $T(k)$, a leaf node of type a_j costs $k-(c(r)+1-j)$ by induction on k. Thus, $C(T(k))$ is given by

$$C(T(k)) = \sum_{1\le j\le c(r)} (k-(c(r)+1-j)) f^j(k)/f(k). \tag{3}$$

We now need to analyze these recurrence relations. By the method of generating functions (see, e.g., [2]), we have from (1) and its initialization that $f^j(k)$ satisfies

$$\sum_{1\le k\le\infty} f^j(k)x^k = x^{c(r)+1-j}\left(1-\sum_{1\le i\le r} I(c(i)<j)x^{c(i)}\right)\Bigg/\left(1-\sum_{1\le i\le r} x^{c(i)}\right),$$

where $I(c(i)<j)=1$ if $c(i)<j$ and 0 otherwise. The coefficients of x^k obtained from the right-hand side of this expression give $f^j(k)$.

For the example of Table 1 with $F^j(x)=\sum_{1\le k\le\infty} f^j(k)x^k$, we have

$$\begin{aligned}
F^1(x) &= x^5/(1-x^2-2x^3-x^5) = x^5+x^7+2x^8+x^9+5x^{10}+5x^{11}+\cdots,\\
F^2(x) &= x^4/(1-x^2-2x^3-x^5) = x^4+x^6+2x^7+x^8+5x^9+5x^{10}+8x^{11}+\cdots,\\
F^3(x) &= x^3(1-x^2)/(1-x^2-2x^3-x^5) = x^3+2x^6+3x^8+4x^9+3x^{10}+12x^{11}+\cdots,\\
F^4(x) &= x^2(1-x^2-2x^3)/(1-x^2-2x^3-x^5) = x^2+x^7+x^9+2x^{10}+x^{11}+\cdots,\\
F^5(x) &= x(1-x^2-2x^3)/(1-x^2-2x^3-x^5) = x^1+x^6+x^8+2x^9+x^{10}+5x^{11}+\cdots,
\end{aligned}$$

from which it can be observed that

$$\begin{aligned}
f^2(k) &= f^1(k+1),\\
f^3(k) &= f^1(k+2)-f^1(k),\\
f^4(k) &= f^1(k+3)-f^1(k+1)-2f^1(k),\\
f^5(k) &= f^1(k+4)-f^1(k+2)-2f^1(k+1).
\end{aligned}$$

Although we do not have a convenient closed form expression for $f^1(k)$ in terms of k, it is interesting to note that

$$f(k) = -2f^1(k)-2f^1(k+1)+f^1(k+3)+f^1(k+4).$$

From (2), we have $F(x)=\sum_{1\le j\le c(r)} F^j(x)$ which, for the example, becomes

$$\begin{aligned}
F(x) &= (x+x^2-2x^4-2x^5)/(1-x^2-2x^3-x^5)\\
&= x+x^2+x^3+x^4+x^5+4x^6+4x^7+7x^8+13x^9+16x^{10}+31x^{11}+\dots .
\end{aligned}$$

From (3), we have that the generating function for the unnormalized cost,

$$\sum_{1\le k\le\infty} f(k)C(T(k))x^k,$$

becomes

$$xdF(x)/dx - \sum_{1\le j\le c(r)} (c(r)+1-j)F^j(x).$$

For the example, this generating function becomes

$$(13x^6+13x^7+6x^8-2x^9-2x^{10})/(1-x^2-2x^3-x^5)^2$$
$$=13x^6+13x^7+32x^8+76x^9+101x^{10}+241x^{11}+\cdots,$$

so that the normalized costs $C(T(k))$ are as given in Table 2.

TABLE 2. $C(T(k))$ for $c(1)=2, c(2)=3, c(3)=3, c(4)=5$ and Its Entropy Lower Bound

k	$C(T(k))$	$-\log_t f(k)$
6	$13/4=3.25$	3.00
7	$13/4=3.25$	3.00
8	$32/7=4.57$	4.21
9	$76/13=5.85$	5.55
10	$101/16=6.31$	6.00
11	$241/31=7.77$	7.43
...	...	...

Performance bounds on the minimum expected cost of code trees for unequal costs are given in Krause [4] in terms of the source entropy base t, where t is the unique positive root of $1-\Sigma_{1\le i\le r} x^{c(i)}=0$. For $f(k)$ equiprobable source symbols, this entropy is $-\log_t f(k)$, and the bounds are

$$-\log_t f(k)\le C(T(k))\le -\log_t f(k)+c(r).$$

However, the code whose cost satisfies the upper bound is not necessarily exhaustive; thus, only the lower bound is relevant here. For the example used here, with $c(1)=2, c(2)=3, c(3)=3, c(4)=5, t\approx 0.63$, and the source entropy base t is also provided in Table 2 for comparison with $C(T(k))$. Also of interest in this connection are the new performance bounds due to Savari [5].

A few comments should be made about this approach to Varn codes. First, the indexing of trees in the order generated by the construction procedure is key; that is, the recurrence relations are elegant stated with this indexing but, possibly, disconcerting aspects of the indexing arise, such as $T(7)$ and $T(6)$ in the example being identical trees with respect to node costs (although different with respect to node types). Also, for some choices of costs, $c(1), c(2), \ldots, c(r)$, it is convenient to solve the recurrences explicitly, particularly when the roots of $1-\sum_{1\le i\le r} x^{c(i)}$ are easy to find, as in the case in which $r=2$.

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A COMBINATORIAL PROBLEM WITH A FIBONACCI SOLUTION

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1. THE PROBLEM

For many years I have enjoyed lecturing to groups of high school students about the excitement of mathematics. One diversion that never failed to capture their attention was as follows: Everyone was asked to *write down a three digit number* (*abc with* $a > c$), *reverse it* (*to form cba*), *find the difference* (*as a three-digit number*) *between the two numbers, and add the difference to its own reverse*. The amazed looks on the students' faces at discovering they had all reached the same *end number* 1089 was a sight to behold. The elementary algebra

$$\begin{array}{rccc} & a & b & c \\ - & c & b & a \\ \hline & a-c-1 & 9 & c-a+10 \\ \hline + & c-a+10 & 9 & a-c-1 \\ \hline & 10 & 8 & 9 \\ \hline \end{array}$$

quickly explained the surprise. Finally, I would tease that the number 1089 is interesting in itself, being the square of 33 and its reverse being the square of 99.

The origins of the diversion are unknown to me, I learned of it from Rouse Ball ([1], p. 9). The question arises: *Can the diversion be extended to numbers other than those having three digits*? For two-digit numbers, the answer is yes, the end number always being 99. For four- and five-digit numbers, a little effort shows that *three* end numbers are possible, although the three numbers are different in the two cases. More effort is required to show that six- and seven-digit numbers give rise to different sets of *eight* possible end numbers. Thus, the sequence of the numbers of possible end numbers, corresponding to initial numbers of 2, 3, 4, 5, 6, 7, ... digits begins 1, 1, 3, 3, 8, 8, ... No prizes for guessing how it continues! Our main result is that the number of possible end numbers corresponding to initial numbers of $n+1$ digits is the Fibonacci number $F_{2[\frac{n+1}{2}]}$.

What of the end numbers themselves? The unique end number generated by two-digit numbers is 99, which turns out to be a divisor of *all* end numbers. The unique end number generated by three-digit numbers is 1089, which is 99×11. The three end numbers generated by four-digit numbers are 9999, 10890, and 10989, which are, respectively, 99×101, 99×110, and 99×111. These examples illustrate the general situation. With each $n+1$-digit number $X = x_n \ldots x_0$ ($x_n > x_0$) we associate an n-digit number $X^\flat$ consisting of a string of 0s and 1s, which has the property that the end number generated by X is $99X^\flat$. We give a simple characterization of the numbers $X^\flat$, and hence of the end numbers themselves.

2. THE CODE OF A NUMBER

Throughout our discussion, nonnegative integers will be written in decimal form and T will denote the number 10. We write $X = x_n \dots x_0$, where $x_0, \dots, x_n$ are integers between 0 and 9 inclusive, to denote the $n+1$-digit number $\sum_{i=0}^{n} x_i T^i$. The $n+1$-digit number obtained by reversing the digits of X is called the *reverse of* X (in $n+1$-digit arithmetic) and is denoted by X', whence $X' = x_0 \dots x_n$. Suppose that $X = x_n \dots x_0$ is such that $x_n > x_0$. Write the number $X - X'$ as an $n+1$-digit number—this may necessitate including some zeros at the front of the standard decimal representation of $X - X'$. Now reverse the digits of the difference $X - X'$ to obtain the number $(X - X')'$. Finally, add the difference to its reverse to produce the number X^* defined by the equation

$$X^* = X - X' + (X - X')'.$$

We wish to find the number, denoted here by a_n, of different (end numbers) X^* that are possible as X ranges over all $n+1$-digit numbers $x_n \dots x_0$ $(x_n > x_0)$. The diversion that motivated our discussion depends on the fact that $a_2 = 1$, i.e., for three-digit numbers, only one (end number) X^* can occur.

The key to our analysis is the association with each $n+1$-digit number $X = x_n \dots x_0$ $(x_n > x_0)$ an $n+1$-digit number $X^\sharp$ called the *code* of X. This code $X^\sharp$ comprises a string of 0s and 1s, has leading digit 1, final digit 0, and encodes all the information needed to pass from X to the (end number) X^* to which it gives rise. We first explain the construction of $X^\sharp$ informally, leaving a formal definition until later.

Write down the number $X = x_n \dots x_0$ $(x_n > x_0)$ and beneath it, its reverse $X' = x_0 \dots x_n$, as shown below:

$$\begin{array}{r} x_n \dots x_0 \\ - \underline{x_0 \dots x_n} \\ \underline{* \;\cdots\; *} \end{array}$$

Consider the role played by the i^{th} column from the right ($i = 0, \dots, n$) in the subtraction of X' from X. Define an integer z_i as follows: if a *ten* has to be *borrowed* from the $i+1^{\text{th}}$ column, z_i is 1; otherwise, it is 0. In this way we construct a string $z_0, \dots, z_n$ of 0s and 1s. The $n+1$-digit number $z_0 \dots z_n$ is called the *code* of X and is denoted by $X^\sharp$. Since we are assuming that $x_n > x_0$, $z_0 = 1$, and $z_n = 0$. The n-digit number $z_0 \dots z_{n-1}$ obtained by deleting the final 0 (z_n) from the code $z_0 \dots z_n$ of X is called the *truncated code* of X and is denoted by $X^\flat$.

To illustrate the above ideas, consider the six-digit number $X = 812311$. Subtracting X' from X, we find that

$$\begin{array}{r} 812311 \\ -\underline{113218} \\ \underline{699093} \end{array}$$

The columns for which a ten has to be borrowed from the adjacent column to the left are (labeling from the right) the $0^{\text{th}}, 1^{\text{st}}, 3^{\text{rd}}$, and 4^{th}, whence (using the above notation) $z_0 = 1, z_1 = 1$, $z_2 = 0$, $z_3 = 1$, $z_4 = 1$, and $z_5 = 0$. Hence, $X^\sharp = 110110$ and $X^\flat = 11011$. For this particular X, $X^* = 699093 + 390996 = 1090089 = 99 \times 11011 = 99X^\flat$. That this is no chance happening is shown in our first result.

Theorem 1: Let $X = x_n \dots x_0$ $(x_n > x_0)$ have truncated code $X^\flat = z_0 \dots z_{n-1}$. Then $X^* = 99X^\flat$.

Proof: Now

$$X = \sum_{i=0}^{n} x_i T^i \quad \text{and} \quad X' = \sum_{i=0}^{n} x_{n-i} T^i.$$

Suppose that $X^\sharp = z_0 \dots z_n$. Then the definitions of $z_0, \dots, z_n$ show that

$$X - X' = \sum_{i=0}^{n} (x_i + z_i T - x_{n-i} - z_{i-1}) T^i,$$

where we have written $z_{-1} = 0$. Hence,

$$\begin{aligned}
X^* &= X - X' + (X - X')' \\
&= \sum_{i=0}^{n} (x_i + z_i T - x_{n-i} - z_{i-1} + x_{n-i} + z_{n-i} T - x_i - z_{n-i-1}) T^i \\
&= \sum_{i=0}^{n} (z_i T - z_{i-1} _ z_{n-i} T - z_{n-i-1}) T^i \\
&= \sum_{i=0}^{n-1} z_i T^{i+1} - \sum_{i=0}^{n-1} z_i T^{i+1} + T^2 \sum_{i=1}^{n} z_{n-i} T^{i-1} - \sum_{i=1}^{n} z_{n-i} T^{i-1} \\
&= (T^2 - 1) z_0 \dots z_{n-1} \\
&= 99 X^b. \quad \square
\end{aligned}$$

Theorem 1 shows that the number a_n we seek is the same as the number of different truncated codes X^b or, equivalently, codes $X^\sharp$ there are as X ranges over all $n+1$-digit numbers $X = x_0 \dots x_n$ $(x_n > x_0)$. The idea of a truncated code was introduced to allow Theorem 1 to be stated effectively, and from now on only the codes themselves will be considered. To help calculate a_n. we need to reformulate and formalize the definition of $X^\sharp$ given earlier. Define the *code* $X^\sharp$ of $X = x_n \dots x_0$ $(x_n > x_0)$ to be the number $y_n \dots y_0$, where the $y_n, \dots, y_0$ are defined inductively as follows. Let $y_n = 1$. For $i = 1, \dots, n$, define y_{n-i} to be 1 if *either* $x_{n-i} > x_i$ *or* $x_{n-i} = x_i$ and $y_{n-i+1} = 1$, and to be 0 otherwise, i.e., if *either* $x_{n-i} < x_i$ *or* $x_{n-i} = x_i$ and $y_{n-i+1} = 0$. This definition clearly accords with that given previously.

Theorem 2: The $n+1$-digit number $y_n \dots y_0$ is the code of some $n+1$-digit number $x_n \dots x_0$ $(x_n > x_0)$ if and only if: (i) each of $y_0, \dots, y_n$ is 0 or 1 and $y_0 = 0, y_n = 1$; (ii) if, for some $i = 0$, ..., $n-1$, $y_{n-i} = 0$, and $y_{n-i-1} = 1$, then $y_{i+1} = 0$; (iii) if, for some $i = 0$, ..., $n-1$, $y_{n-i} = 1$ and $y_{n-i-1} = 0$, then $y_{i+1} = 1$.

Proof: The *only if* part of the assertion follows directly from the definition of code just given. To establish the *if* part, suppose that $y_0, \dots, y_n$ satisfy conditions (i)-(iii). Let $w_n \dots w_0$ be the code of $y_n \dots y_0$. Then (ii) shows that $w_n = y_n = 1$. Either y_{n-1} is 0 or 1. Suppose first that $y_{n-1} = 0$. Then (ii) shows that $y_1 = 1$, whence $w_{n-1} = y_{n-1} = 0$. Suppose next that $y_{n-1} = 1$. Since y_1 is 0 or 1 and $w_n = 1$, the definition of w_{n-1} shows that $w_{n-1} = y_{n-1} = 1$. Therefore, in all cases, $w_{n-1} = y_{n-1}$. Continuing in this way, it can be shown that $w_{n-2} = y_{n-2}, \dots, w_0 = y_0$, whence $w_n \dots w_0 = y_n \dots y_0$, i.e., $y_n \dots y_0$ is its own code. $\square$

3. THE CALCULATION OF a_n

We call any $n+1$-digit number $y_n \dots y_0$ satisfying conditions (i)-(iii) of Theorem 2 an $n+1$-digit code. Theorems 1 and 2 together show that a_n is simply the number of $n+1$-digit codes that there are, and it is this observation we use to calculate a_n. Trivially, the only two-digit code is 10 and the only three-digit code is 110. There are precisely three four-digit codes—1010, 1100, 1110—and three five-digit codes—10010, 11100, 11110. Thus, $a_1 = a_2 = 1$ and $a_3 = a_4 = 3$. It should be noted that, in each of the five-digit codes, the second and third digits are equal, and if the middle (i.e., third) digit is removed, then a four-digit code is obtained. Conversely, if each of the four-digit codes is extended by repeating its second digit, a five-digit code is obtained. These remarks explain why $a_4 = a_3$. We now extend these ideas.

Suppose that $y_{2n} \dots y_{n+1} y_n y_{n-1} \dots y_0$ is a $2n+1$-digit code ($n \geq 1$). Then conditions (ii) and (iii) of Theorem 2 show that $y_{n+1} = y_n$. It follows easily that $y_{2n} \dots y_{n+1} y_{n-1} \dots y_0$ is a $2n$-digit code. Conversely, if $z_{2n-1} \dots z_n z_{n-1} \dots z_0$ is a $2n$-digit code, then $z_{2n-1} \dots z_n z_n z_{n-1} \dots z_0$ is a $2n+1$-digit code. Hence, there is a bijection between the set of $2n+1$-digit codes and the set of $2n$-digit codes, whence $a_{2n} = a_{2n-1}$.

To help find a recurrence relation satisfied by the a_n, we consider, for each natural number n, the set $\mathcal{S}_n$ comprising all $n+1$-digit numbers $s_n \dots s_0$ satisfying: (a) each of $s_0, \dots, s_n$ is 0 or 1; (b) if, for some $i = 0, \dots, n-1$, $s_{n-i} = 0$ and $s_{n-i-1} = 1$, then $s_{i+1} = 0$; (c) if, for some $i = 0, \dots, n-1$, $s_{n-i} = 1$ and $s_{n-i-1} = 0$, then $s_{i+1} = 1$. Thus, a_n is the number of those elements $s_n \dots s_0$ in $\mathcal{S}_n$ for which $s_n = 1$ and $s_0 = 0$. If an element of $\mathcal{S}_n$ is taken, and each 0 in it is changed to 1, and each 1 to 0, then another element of $\mathcal{S}_n$ is obtained. Hence, the number of elements $s_n \dots s_0$ in $\mathcal{S}_n$ for which $s_n = 0$ and $s_0 = 1$ is also a_n. Similarly, the number of those elements $s_n \dots s_0$ in $\mathcal{S}_n$ for which $s_n = s_0 = 0$ is the same number as those for which $s_n = s_0 = 1$; we denote this common number by b_n.

The members $s_n \dots s_0$ of $\mathcal{S}_n$ ($n \geq 3$) for which $s_n = s_0 = 0$, other than the one comprising all zeros, have one of the forms,

$$0 \dots 0X0 \dots 0,$$

in which there are r initial zeros, r final zeros, and X is an $n-2r+1$-digit code, for some natural number r satisfying $2r \leq n-1$. Conversely, each $n+1$-digit number of the above form lies in $\mathcal{S}_n$ and has both its initial and final digits zero. Thus, for $n \geq 3$,

$$b_n = \begin{cases} a_{n-2} + \cdots + a_4 + a_2 + 1 & (n \text{ even}) \\ a_{n-2} + \cdots + a_3 + a_1 + 1 & (n \text{ odd}). \end{cases}$$

Trivially, $b_2 = b_1 = 1$.

Since $a_{2n} = a_{2n-1}$, we need only calculate a_{2n-1}. To this end, we note that every $2n+2$-digit code has one of the forms,

$$1X0, \ 1Y0, \ 1Z0,$$

where $X, Y, Z \in \mathcal{S}_{2n-1}$ are such that $X = s_{2n-1} \dots s_0$ satisfies $s_{2n-1} = 1, s_0 = 0$, $Y = s_{2n-1} \dots s_0$ satisfies $s_{2n-1} = 0, s_0 = 1$, and $Z = s_{2n-1} \dots s_0$ satisfies $s_{2n-1} = s_0 = 1$. Conversely, each such X, Y, Z gives rise, respectively, to a $2n+2$-digit code $1X0, 1Y0, 1Z0$. In view of our earlier remarks, the number of possible Xs is a_{2n-1}, the number of possible Ys is a_{2n-1}, and the number of possible Zs is b_{2n-1}. Hence, for $n \geq 2$,

$$\begin{aligned} a_{2n+1} &= a_{2n-1} + a_{2n-1} + b_{2n-1} \\ &= a_{2n-1} + a_{2n-1} + a_{2n-3} + \cdots a_3 + a_1 + 1 \\ &= 2a_{2n-1} + a_{2n-3} + \cdots + a_3 + a_1 + 1. \end{aligned}$$

This recurrence relation enables us to prove our main result.

Theorem 3: For each natural number n, $a_{2n} = a_{2n-1} = F_{2n}$, i.e., $a_n = F_{2[\frac{n+1}{2}]}$.

Proof: Since $a_{2n} = a_{2n-1}$, it remains only to prove that $a_{2n-1} = F_{2n}$. We do this by induction on n. The cases $a_1 = 1 = F_2$ and $a_3 = 3 = F_4$ have been established earlier. Suppose that $a_{2k-1} = F_{2k}$, for $k = 1, \ldots, n$, where $n \geq 2$. Then

$$\begin{aligned} a_{2n+1} &= 2a_{2n-1} + a_{2n-3} + \cdots + a_3 + a_1 + 1 \\ &= 2F_{2n} + F_{2n-2} + \cdots + F_4 + F_2 + 1 \\ &= F_{2n} + (F_{2n+1} - F_{2n-1}) + (F_{2n-1} - F_{2n-3}) + \cdots + (F_5 - F_3) + (F_3 - F_1) + 1 \\ &= F_{2n} + F_{2n+1} \\ &= F_{2n+2}. \end{aligned}$$

This completes the proof by induction. □

An easy exercise shows that, for $n \geq 2$,

$$b_{2n} = b_{2n-1} = F_{2n-2} + \cdots + F_4 + F_2 + 1 = F_{2n-1}.$$

Since $b_2 = b_1 = F_1$, the b_ns are the Fibonacci numbers with odd suffixes, in the same way that the a_ns are those with even suffixes.

4. CONCLUDING REMARKS

Our original problem extends in the obvious way to include as initial numbers every $n+1$-digit number X whose reverse X' satisfies $X' \leq X$. In this wider context, we ask: How many end numbers are now possible and what are they? The *extra* initial numbers that have to be considered either generate the end number 0 or have the form YXY', where Y is an r-digit number, X is an $n-2r+1$-digit number whose initial digit exceeds its final one, and the natural number r satisfies $2r \leq n-1$. This latter form gives rise to the a_{n-2r+1} end numbers $99(10^{r-1})$ code of X. Thus, the total number of end numbers now possible is:

$$\begin{cases} a_n + a_{n-2} + \cdots + a_2 + 1 = F_n + F_{n-2} + \cdots + F_2 + 1 = F_{n+1} & (n \text{ even}), \\ a_n + a_{n-2} + \cdots + a_1 + 1 = F_{n+1} + F_{n-1} + \cdots F_2 + 1 = F_{n+2} & (n \text{ odd}). \end{cases}$$

Denoting this latter number by α_n, we see that $\alpha_{2n} = \alpha_{2n-1} = F_{2n+1}$, i.e., $\alpha_n = F_{2[\frac{n+1}{2}]+1}$.

Although our discussion has been concerned exclusively with base 10 arithmetic, it generalizes, with only minor modifications, to an arbitrary base m. The main change required is that in Theorem 1 the number 99 has to be replaced by m^2-1. A propos the concluding remarks of the opening paragraph, the unique end number generated by a three-digit number abc $(a > c)$ to base m is the four-digit number $1\,0\,m-2\,m-1$, which equals $(m-1)(m+1)^2$ and is a square precisely when $m-1$ is; this is fortuitously so when $m=10$. On the other hand, the reverse of $1\,0\,m-2\,m-1$ is $m-1\,m-2\,0\,1$, which equals $(m^2-1)^2$ and is always square.

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INVERSE TRIGONOMETRIC AND HYPERBOLIC SUMMATION FORMULAS INVOLVING GENERALIZED FIBONACCI NUMBERS

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(*Submitted June 1993*)

1. INTRODUCTION

Define the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ for all integers n by

$$U_n = pU_{n-1} + U_{n-2},\ U_0 = 0,\ U_1 = 1,\ n \geq 2, \tag{1.1}$$

$$V_n = pV_{n-1} + V_{n-2},\ V_0 = 2,\ V_1 = p,\ n \geq 2. \tag{1.2}$$

Of course, these sequences can be extended to negative subscripts by the use of (1.1) and (1.2). The Binet forms for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.3}$$

and

$$V_n = \alpha^n + \beta^n, \tag{1.4}$$

where

$$\alpha = \frac{p + \sqrt{p^2+4}}{2},\quad \beta = \frac{p - \sqrt{p^2+4}}{2}. \tag{1.5}$$

Certain specializations of the parameter p produce sequences that are of interest here and Table 1 summarizes these.

TABLE 1

p	1	2	$2x$
U_n	F_n	P_n	$P_n(x)$
V_n	L_n	Q_n	$Q_n(x)$

Here $\{F_n\}$ and $\{L_n\}$ are the Fibonacci and Lucas sequences, respectively. The sequences $\{P_n\}$ and $\{Q_n\}$ are the Pell and Pell-Lucas numbers, respectively, and appear, for example, in [3], [5], [7], [11], and [17]. The sequences $\{P_n(x)\}$ and $\{Q_n(x)\}$ are the Pell and Pell-Lucas polynomials, respectively, and have been studied, for example, in [12], [14], [15], and [16].

Hoggatt and Ruggles [9] produced some summation identities for Fibonacci and Lucas numbers involving the arctan function. Their results are of the same type as the striking result of D. H. Lehmer,

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2i+1}}\right) = \frac{\pi}{4}, \tag{1.6}$$

to which reference is made in their above-mentioned article [9]. Mahon and Horadam [13] produced identities for Pell and Pell-Lucas polynomials leading to summation formulas for Pell and Pell-Lucas numbers similar to (1.6). For example,

$$\sum_{i=0}^{\infty} \tan^{-1}\left(\frac{2}{P_{2i+1}}\right) = \frac{\pi}{2}. \tag{1.7}$$

Here, we produce similar results involving the arctan function and terms from the sequences $\{U_n\}$ and $\{V_n\}$. Some of our results are equivalent to those obtained in [13] but most are new. We also obtain results involving the arctanh function, all of which we believe are new.

2. PRELIMINARY RESULTS

We make consistent use of the following results which appear in [1] and [6]:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy < 1, \tag{2.1}$$

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), \text{ if } xy > -1, \tag{2.2}$$

$$\tanh^{-1} x + \tanh^{-1} y = \tanh^{-1}\left(\frac{x+y}{1+xy}\right), \tag{2.3}$$

$$\tanh^{-1} x - \tanh^{-1} y = \tanh^{-1}\left(\frac{x-y}{1-xy}\right), \tag{2.4}$$

$$\tanh^{-1} x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right), \ |x| < 1, \tag{2.5}$$

$$\coth^{-1} x = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right), \ |x| > 1, \tag{2.6}$$

$$\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right), \tag{2.7}$$

$$\tanh^{-1} x = \coth^{-1}\left(\frac{1}{x}\right). \tag{2.8}$$

We note from (2.7) and (2.8) that all results obtained involving arctan (arctanh) can be expressed equivalently using arccot (arccoth).

If k and n are integers, and writing

$$\Delta = (\alpha - \beta)^2 = p^2 + 4, \tag{2.9}$$

we also have the following:

$$U_n^2 - U_{n+k}U_{n-k} = (-1)^{n+k}U_k^2, \tag{2.10}$$

$$V_{n+k}V_{n-k} - V_n^2 = \Delta(-1)^{n+k}U_k^2, \tag{2.11}$$

$$U_{n+k} - U_{n-k} = \begin{cases} U_kV_n, & k \text{ even,} \\ U_nV_k, & k \text{ odd,} \end{cases} \tag{2.12}$$

$$U_{n+k} + U_{n-k} = \begin{cases} U_nV_k, & k \text{ even,} \\ U_kV_n, & k \text{ odd,} \end{cases} \tag{2.13}$$

$$V_{n+k} - V_{n-k} = \begin{cases} \Delta U_kU_n, & k \text{ even,} \\ V_kV_n, & k \text{ odd,} \end{cases} \tag{2.14}$$

$$V_{n+k} + V_{n-k} = \begin{cases} V_kV_n, & k \text{ even,} \\ \Delta U_kU_n, & k \text{ odd.} \end{cases} \tag{2.15}$$

$$U_{n+2} + U_n = V_{n+1}, \tag{2.16}$$

$$U_nU_{n+2} + (-1)^n = U_{n+1}^2. \tag{2.17}$$

Identities (2.12)-(2.15) occur as (56)-(63) in [2], and the remainder can be proved using Binet forms. Indeed, (2.10) and (2.11) resemble the famous Catalan identity for Fibonacci numbers,

$$F_n^2 - F_{n+k}F_{n-k} = (-1)^{n-k}F_k^2. \tag{2.18}$$

We assume throughout that the parameter p is real and $|p| \geq 1$. If $p \geq 1$, then $\{U_n\}_{n=2}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ are increasing sequences. If $p \leq -1$, then $\{|U_n|\}_{n=2}^{\infty}$ and $\{|V_n|\}_{n=1}^{\infty}$ are increasing sequences and, if $n > 0$, then

$$\begin{cases} U_n < 0, & n \text{ even,} \\ U_n > 0, & n \text{ odd,} \\ V_n < 0, & n \text{ odd,} \\ V_n > 0, & n \text{ even.} \end{cases} \tag{2.19}$$

Furthermore, if $|p| \geq 1$, then using Binet forms it is seen that

$$\lim_{n\to\infty} \frac{U_{n+m}}{U_n} = \lim_{n\to\infty} \frac{V_{n+m}}{V_n} = \begin{cases} \delta^m, & m \text{ even or } p \geq 1, \\ -\delta^m, & m \text{ odd and } p \leq -1, \end{cases} \tag{2.20}$$

where

$$\delta = \frac{|p| + \sqrt{p^2+4}}{2}. \tag{2.21}$$

3. MAIN RESULTS

Theorem 1: If n is an integer, then

$$\tan^{-1} U_{n+2} - \tan^{-1} U_n = \tan^{-1}\left(\frac{p}{U_{n+1}}\right), \quad n \text{ even}, \tag{3.1}$$

$$\tan^{-1}\left(\frac{1}{U_n}\right) + \tan^{-1}\left(\frac{1}{U_{n+2}}\right) = \tan^{-1}\left(\frac{V_{n+1}}{U_{n+1}^2}\right), \quad n \text{ odd}, \ n \neq -1. \tag{3.2}$$

Proof:

$$\tan^{-1} U_{n+2} - \tan^{-1} U_n = \tan^{-1}\left(\frac{U_{n+2} - U_n}{1 + U_n U_{n+2}}\right) = \tan^{-1}\left(\frac{p}{U_{n+1}}\right),$$

where we have used (2.2), (1.1), and (2.17).

To prove (3.2), proceed similarly using (2.1), (2.16), and (2.17). □

Now, in (3.1), replacing n by 0, 2, ..., $2n-2$, we obtain a sum which telescopes to yield

$$\sum_{i=1}^{n} \tan^{-1}\left(\frac{p}{U_{2i-1}}\right) = \tan^{-1} U_{2n}. \tag{3.3}$$

Similarly, in (3.2), replacing n by 1, 3, ..., $2n-1$ yields

$$\sum_{i=1}^{n} (-1)^{i-1} \tan^{-1}\left(\frac{V_{2i}}{U_{2i}^2}\right) = \frac{\pi}{4} + (-1)^{n-1} \tan^{-1}\left(\frac{1}{U_{2n+1}}\right). \tag{3.4}$$

The corresponding limiting sums are

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{p}{U_{2i-1}}\right) = \begin{cases} \dfrac{\pi}{2}, & p \geq 1, \\ \dfrac{-\pi}{2}, & p \leq -1, \end{cases} \tag{3.5}$$

$$\sum_{i=1}^{\infty} (-1)^{i-1} \tan^{-1}\left(\frac{V_{2i}}{U_{2i}^2}\right) = \frac{\pi}{4}. \tag{3.6}$$

We note here that (3.3) and (3.4) were essentially obtained by Mahon and Horadam [13], (3.3) in a slightly different form. When $p = 1$, (3.5) reduces essentially to Lehmer's result (1.6) stated earlier.

Theorem 2: For positive integers k and n,

$$\tan^{-1}\left(\frac{U_n}{U_{n+k}}\right) - \tan^{-1}\left(\frac{U_{n-k}}{U_n}\right) = \begin{cases} \tan^{-1}\left(\dfrac{(-1)^n U_k^2}{V_k U_n^2}\right), & k \text{ even}, \\ \tan^{-1}\left(\dfrac{(-1)^{n-1} U_k}{U_{2n}}\right), & k \text{ odd}. \end{cases} \tag{3.7}$$

Proof: Use (2.2), (2.10), and (2.13). □

Now, in (3.7), replacing n by k, $2k$, ..., nk to form a telescoping sum yields

$$\sum_{i=1}^{n} \tan^{-1}\left(\frac{U_k^2}{V_k U_{ik}^2}\right) = \tan^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right), \quad k \text{ even} \tag{3.8}$$

$$\sum_{i=1}^{n} \tan^{-1}\left(\frac{(-1)^{i-1}U_k}{U_{2ik}}\right) = \tan^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right), \quad k \text{ odd.} \tag{3.9}$$

Using (2.20), the limiting sums are, respectively,

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{U_k^2}{V_k U_{ik}^2}\right) = \tan^{-1}(\delta^{-k}), \quad k \text{ even,} \tag{3.10}$$

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{(-1)^{i-1}U_k}{U_{2ik}}\right) = \begin{cases} \tan^{-1}(\delta^{-k}), & k \text{ odd, } p \geq 1, \\ -\tan^{-1}(\delta^{-k}), & k \text{ odd, } p \leq -1. \end{cases} \tag{3.11}$$

Theorem 3: For positive integers k and n,

$$\tan^{-1}\left(\frac{V_{n-k}}{V_n}\right) - \tan^{-1}\left(\frac{V_n}{V_{n+k}}\right) = \begin{cases} \tan^{-1}\left(\dfrac{\Delta(-1)^n U_k^2}{V_k V_n^2}\right), & k \text{ even,} \\[2ex] \tan^{-1}\left(\dfrac{(-1)^{n-1}U_k}{U_{2n}}\right), & k \text{ odd.} \end{cases} \tag{3.12}$$

Proof: Use (2.2), (2.11), and (2.15). □

Again in (3.12), replacing n by k, $2k$, ..., nk yields

$$\sum_{i=1}^{n} \tan^{-1}\left(\frac{\Delta U_k^2}{V_k V_{ik}^2}\right) = \tan^{-1}\left(\frac{2}{V_k}\right) - \tan^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right), \quad k \text{ even,} \tag{3.13}$$

$$\sum_{i=1}^{n} \tan^{-1}\left(\frac{(-1)^{i-1}U_k}{U_{2ik}}\right) = \tan^{-1}\left(\frac{2}{V_k}\right) - \tan^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right), \quad k \text{ odd.} \tag{3.14}$$

Since the left sides of (3.9) and (3.14) are the same, we can write

$$\tan^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right) + \tan^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right) = \tan^{-1}\left(\frac{2}{V_k}\right), \quad k \text{ odd,} \tag{3.15}$$

and taking limits using (2.20) gives

$$\tan^{-1}(\delta^{-k}) = \frac{1}{2}\tan^{-1}\left(\frac{2}{|V_k|}\right), \quad k \text{ odd.} \tag{3.16}$$

The limiting sum arising from (3.13) is

$$\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{\Delta U_k^2}{V_k V_{ik}^2}\right)=\tan^{-1}\left(\frac{2}{V_k}\right)-\tan^{-1}(\delta^{-k}),\ k \text{ even.} \tag{3.17}$$

Theorem 4: If $n > 2$, then

$$\tanh^{-1}\left(\frac{1}{U_n}\right)+\tanh^{-1}\left(\frac{1}{U_{n+2}}\right)=\tanh^{-1}\left(\frac{V_{n+1}}{U_{n+1}^2}\right),\ n \text{ even,} \tag{3.18}$$

$$\tanh^{-1}\left(\frac{1}{U_n}\right)-\tanh^{-1}\left(\frac{1}{U_{n+2}}\right)=\tanh^{-1}\left(\frac{p}{U_{n+1}}\right),\ n \text{ odd.} \tag{3.19}$$

Proof: To prove (3.18) use (2.3), (2.16), and (2.17); (3.19) is proved similarly. □

These results lead, respectively, to

$$\sum_{i=1}^{n}(-1)^{i-1}\tanh^{-1}\left(\frac{V_{2i+3}}{U_{2i+3}^2}\right)=\tanh^{-1}\left(\frac{1}{U_4}\right)+(-1)^{n-1}\tanh^{-1}\left(\frac{1}{U_{2n+4}}\right), \tag{3.20}$$

$$\sum_{i=1}^{n}\tanh^{-1}\left(\frac{p}{U_{2i+2}}\right)=\tanh^{-1}\left(\frac{1}{U_3}\right)-\tanh^{-1}\left(\frac{1}{U_{2n+3}}\right). \tag{3.21}$$

Note that in Theorem 4 our assumption that $n > 2$, together with our earlier assumption that $|p| \geq 1$, is necessary to ensure that the arctanh function is defined. The corresponding limiting sums are

$$\sum_{i=1}^{\infty}(-1)^{i-1}\tanh^{-1}\left(\frac{V_{2i+3}}{U_{2i+3}^2}\right)=\tanh^{-1}\left(\frac{1}{U_4}\right), \tag{3.22}$$

$$\sum_{i=1}^{\infty}\tanh^{-1}\left(\frac{p}{U_{2i+2}}\right)=\tanh^{-1}\left(\frac{1}{U_3}\right). \tag{3.23}$$

We refrain from giving proofs for the theorems that follow, since the proofs are similar to those already given.

Theorem 5: Let $n \geq k$ be positive integers where $(k, n) \neq (1, 1)$ if $p = \pm 1$. Then

$$\tanh^{-1}\left(\frac{U_n}{U_{n+k}}\right)-\tanh^{-1}\left(\frac{U_{n-k}}{U_n}\right)=\begin{cases}\tanh^{-1}\left(\dfrac{(-1)^n U_k}{U_{2n}}\right), & k \text{ even,}\\[2ex] \tanh^{-1}\left(\dfrac{(-1)^{n-1}U_k^2}{V_k U_n^2}\right), & k \text{ odd.}\end{cases} \tag{3.24}$$

This leads to

$$\sum_{i=1}^{n}\tanh^{-1}\left(\frac{U_k}{U_{2ik}}\right)=\tanh^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right),\ k \text{ even,} \tag{3.25}$$

$$\sum_{i=1}^{n} \tanh^{-1}\left(\frac{(-1)^{i-1}U_k^2}{V_k U_{ik}^2}\right) = \tanh^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right), \quad k \text{ odd.} \tag{3.26}$$

The corresponding limiting sums are

$$\sum_{i=1}^{\infty} \tanh^{-1}\left(\frac{U_k}{U_{2ik}}\right) = \tanh^{-1}(\delta^{-k}), \quad k \text{ even,} \tag{3.27}$$

$$\sum_{i=1}^{\infty} \tanh^{-1}\left(\frac{(-1)^{i-1}U_k^2}{V_k U_{ik}^2}\right) = \begin{cases} \tanh^{-1}(\delta^{-k}), & k \text{ odd, } p \ge 1, \\ -\tanh^{-1}(\delta^{-k}), & k \text{ odd, } p \le -1. \end{cases} \tag{3.28}$$

Theorem 6: Let $n \ge k$ be positive integers where $(k, n) \ne (1, 1)$ if $1 \le |p| \le 2$. Then

$$\tanh^{-1}\left(\frac{V_{n-k}}{V_n}\right) - \tanh^{-1}\left(\frac{V_n}{V_{n+k}}\right) = \begin{cases} \tanh^{-1}\left(\dfrac{(-1)^n U_k}{U_{2n}}\right), & k \text{ even,} \\ \tanh^{-1}\left(\dfrac{\Delta(-1)^{n-1}U_k^2}{V_k V_n^2}\right), & k \text{ odd.} \end{cases} \tag{3.29}$$

The resulting sums are

$$\sum_{i=1}^{n} \tanh^{-1}\left(\frac{U_k}{U_{2ik}}\right) = \tanh^{-1}\left(\frac{2}{V_k}\right) - \tanh^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right), \quad k \text{ even,} \tag{3.30}$$

$$\sum_{i=1}^{n} \tanh^{-1}\left(\frac{\Delta(-1)^{i-1}U_k^2}{V_k V_{ik}^2}\right) = \tanh^{-1}\left(\frac{2}{V_k}\right) - \tanh^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right), \quad k \text{ odd.} \tag{3.31}$$

Since the left sides of (3.25) and (3.30) are the same, we can write

$$\tanh^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right) + \tanh^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right) = \tanh^{-1}\left(\frac{2}{V_k}\right), \quad k \text{ even,} \tag{3.32}$$

and taking limits yields

$$\tanh^{-1}(\delta^{-k}) = \frac{1}{2}\tanh^{-1}\left(\frac{2}{V_k}\right), \quad k \text{ even.} \tag{3.33}$$

This should be compared with (3.16). The limiting sum arising from (3.31) is

$$\sum_{i=1}^{\infty} \tanh^{-1}\left(\frac{\Delta(-1)^{i-1}U_k^2}{V_k V_{ik}^2}\right) = \begin{cases} \tanh^{-1}\left(\dfrac{2}{V_k}\right) - \tanh^{-1}(\delta^{-k}), & k \text{ odd, } p \ge 1, \\ \tanh^{-1}\left(\dfrac{2}{V_k}\right) + \tanh^{-1}(\delta^{-k}), & k \text{ odd, } p \le -1. \end{cases} \tag{3.34}$$

At this point we remark that Mahon and Horadam [13] obtained results similar to our Theorems 2 and 3 and derived summation formulas from them. However, in our notation, they considered only the case k odd.

4. APPLICATIONS

We now use some of our results to obtain identities for the Fibonacci and Lucas numbers. From (3.22) and (3.23), we have

$$\sum_{i=1}^{\infty}(-1)^{i-1}\tanh^{-1}\left(\frac{L_{2i+3}}{F_{2i+3}^2}\right)=\frac{1}{2}\log_e 2, \tag{4.1}$$

$$\sum_{i=1}^{\infty}\tanh^{-1}\left(\frac{1}{F_{2i+2}}\right)=\frac{1}{2}\log_e 3. \tag{4.2}$$

In terms of infinite products, these become, respectively,

$$\prod_{i=1}^{\infty}\frac{F_{2i+3}^2+(-1)^{i-1}L_{2i+3}}{F_{2i+3}^2+(-1)^{i}L_{2i+3}}=2, \tag{4.3}$$

$$\prod_{i=1}^{\infty}\frac{F_{2i+2}+1}{F_{2i+2}-1}=3. \tag{4.4}$$

In (3.28), keeping in mind the constraints in the statement of Theorem 5 and taking $k=3$, we obtain

$$\sum_{i=1}^{\infty}(-1)^{i-1}\tanh^{-1}\left(\frac{1}{F_{3i}^2}\right)=\frac{1}{2}\log_e \phi, \tag{4.5}$$

or

$$\prod_{i=1}^{\infty}\frac{F_{3i}^2+(-1)^{i-1}}{F_{3i}^2+(-1)^{i}}=\phi, \tag{4.6}$$

where $\phi=\frac{1+\sqrt{5}}{2}$ is the Golden Ratio.

Finally, (3.34) yields, after simplifying the right side,

$$\sum_{i=1}^{\infty}(-1)^{i-1}\tanh^{-1}\left(\frac{5}{L_{3i}^2}\right)=\frac{1}{2}\log_e(3(\phi-1)), \tag{4.7}$$

or

$$\prod_{i=1}^{\infty}\frac{L_{3i}^2+(-1)^{i-1}5}{L_{3i}^2+(-1)^{i}5}=3(\phi-1). \tag{4.8}$$

Of course, many other examples can be given by varying the parameter k.

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SEQUENCES RELATED TO AN INFINITE PRODUCT EXPANSION FOR THE SQUARE ROOT AND CUBE ROOT FUNCTIONS

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In the first section we shall consider three sequences associated with the square root function. In the second section we shall consider three sequences associated with the cube root function. In the third section, after considering three different sequences associated with the square root function, we make comparisons with the hope (unfulfilled) of a possible generalization.

1. THE SQUARE ROOT FUNCTION

In [1], Eric Wingler showed that repeated use of the identity

$$\sqrt{1+r}=\frac{2r+2}{r+2}\sqrt{1+\frac{r^2}{4r+4}}$$

leads to an infinite product expansion of $\sqrt{1+r}$ in the following manner: For $a_1>-1$ and n a positive integer, defining

$$a_{n+1}=\frac{a_n^2}{4a_n+4}\quad\text{and}\quad b_n=\frac{2a_n+2}{a_n+2}$$

implies $\sqrt{1+a_1}=\prod_{i=1}^{\infty}b_i$.

In the sequel, n will always denote a positive integer and, *a propos* the preceding product, for $n\geq 1$, define $c_n=b_1b_2b_3\dots b_n$.

In Definition 1 we shall define three sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$, which will depend on a_1 and which are related to $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. These definitions are motivated by our desire to have, when a_1 is a positive integer, x_n, y_n, and z_n be integers such that $c_n=x_n/y_n$, $(x_n,y_n)=1$, and z_n is the numerator of a_{n+1} when it is written as a reduced fraction with positive numerator. As can be seen from Theorem 2 and Lemma 3, these definitions will give us even more than what we desire.

Definition 1: Define the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ as follows:

For $2|a_1$, define

$$x_1=a_1+1,\ y_1=\frac{1}{2}a_1+1,\ \text{and}\ z_1=\left(\frac{a_1}{2}\right)^2;$$

otherwise,

$$x_1=2a_1+2,\ y_1=a_1+2,\ \text{and}\ z_1=a_1^2.$$

For $4|a_1$ and $n\geq 1$, define

$$x_{n+1}=x_ny_n,\quad y_{n+1}=y_n^2-\frac{z_n}{2},\ \text{and}\ z_{n+1}=\left(\frac{z_n}{2}\right)^2;$$

otherwise,

$$x_{n+1} = 2x_n y_n, \quad y_{n+1} = 2y_n^2 - z_n, \quad \text{and} \quad z_{n+1} = z_n^2.$$

As an example, for $a_1 = 6$, we have that the first five terms of each of our six sequences are:

$$\begin{array}{lllll}
a_1 = 6 & a_2 = \frac{9}{7} & a_3 = \frac{81}{448} & a_4 = \frac{6561}{947968} & a_5 = \frac{43046721}{3619451788288} \\
b_1 = \frac{7}{4} & b_2 = \frac{32}{23} & b_3 = \frac{1058}{977} & b_4 = \frac{1909058}{1902497} & b_5 = \frac{7238989670018}{7238946623297} \\
c_1 = \frac{7}{4} & c_2 = \frac{56}{23} & c_3 = \frac{2576}{977} & c_4 = \frac{5033504}{1902497} & c_5 = \frac{19152452518976}{7238946623297} \\
x_1 = 7 & x_2 = 56 & x_3 = 2576 & x_4 = 5033504 & x_5 = 19152452518976 \\
y_1 = 4 & y_2 = 23 & y_3 = 977 & y_4 = 1902497 & y_5 = 7238946623297 \\
z_1 = 9 & z_2 = 81 & z_3 = 6561 & z_4 = 43046721 & z_5 = 1853020188851841.
\end{array}$$

We also have that $a_6 = \frac{1853020188851841}{52402348213090018234298368}$.

By Definition 1, for a_1 not an integer, the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are defined by:

$$x_1 = 2a_1 + 2, \quad y_1 = a_1 + 2, \quad \text{and} \quad z_1 = a_1^2$$

and, for $n \ge 1$,

$$x_{n+1} = 2x_n y_n, \quad y_{n+1} = 2y_n^2 - z_n, \quad \text{and} \quad z_{n+1} = z_n^2.$$

The main results, namely, Lemmas 3-6 and Corollary 7, do not require a_1 to be an integer. In fact, the only results for the square root function that do not hold when a_1 is not an integer are, not surprisingly, the ones relating to x_n, y_n, and z_n being relatively prime (Lemmas 8-10).

In Theorem 2 we shall state our results concerning the square root function. These results relate the six sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$.

Theorem 2: Let a_1 and n be integers such that $n \ge 1$ and $a_1 > -1$. We have that

$$a_{n+1} = \frac{z_n}{y_n^2 - z_n}, \quad b_{n+1} = \frac{x_{n+1} y_n}{x_n y_{n+1}}, \quad \text{and} \quad c_n = \frac{x_n}{y_n}.$$

In addition, depending on whether $4|a_1$ or not,

$$b_{n+1} = \frac{y_n^2}{y_{n+1}} \quad \text{or} \quad b_{n+1} = \frac{2y_n^2}{y_{n+1}}.$$

For a_1 an integer, we also have that

$$(z_n, y_n^2 - z_n) = 1, \quad (x_n, y_n) = 1, \quad \text{and} \quad (2y_n^2, y_{n+1}) = 1.$$

With Definition 1 as made, the proof of Theorem 2 is fairly straightforward and follows from Lemmas 3-6 and 8-10.

Lemma 3: For $n \ge 1$, $x_n^2 - (a_1 + 1)y_n^2 = -(a_1 + 1)z_n$.

Proof: This result is easily shown to be true for $n = 1$. Thus, assume this result is true for $n = k$, where $k \ge 1$. We shall prove this result is true for $n = k + 1$ in the case where 4 does not divide a_1. The proof is similar for $4|a_1$.

We have that

$$\begin{aligned}x_{k+1}^2-(a_1+1)y_{k+1}^2&=(2x_ky_k)^2-(a_1+1)(2y_k^2-z_k)^2\\&=4x_k^2y_k^2-4(a_1+1)y_k^4+4(a_1+1)y_k^2z_k-(a_1+1)z_k^2\\&=4y_k^2[x_k^2-(a_1+1)y_k^2]+4(a_1+1)y_k^2z_k-(a_1+1)z_k^2\\&=-4y_k^2(a_1+1)z_k+4(a_1+1)y_k^2z_k-(a_1+1)z_k^2=-(a_1+1)z_{k+1}.\quad\square\end{aligned}$$

Comment: Let a_1 be an integer such that a_1+1 is a perfect square. Since, by Definition 1, z_n is also a perfect square, we can let

$$k_n^2=(a_1+1)\frac{x_n^2}{(a_1+1)^2}=\frac{x_n^2}{a_1+1}\quad\text{and}\quad p_n^2=z_n.$$

Thus, by Lemma 3, $y_n^2=p_n^2+k_n^2$.

For $a_1=8$ and $n=1$, 2, 3, and 4, the identity $y_n^2=p_n^2+k_n^2$ gives us

$$\begin{aligned}5^2&=4^2+3^2\\17^2&=8^2+15^2\\257^2&=32^2+255^2\\65537^2&=512^2+65535^2.\end{aligned}$$

In this example, y_n is the n^{th} Fermat number.

Lemma 4: For $n\geq 1$ and $a_1>-1$, we have that $a_{n+1}=z_n/(y_n^2-z_n)$.

Proof: This result is easily shown to be true for $n=1$. Assume $a_{k+1}=z_k/(y_k^2-z_k)$, where $k\geq 1$. We shall prove this result is true for $n=k+1$ in the case where 4 does not divide a_1. The proof is similar for $4|a_1$.

Since

$$(y_k^2-z_k)^2a_{k+1}^2=z_k^2=z_{k+1}$$

and

$$\begin{aligned}(y_k^2-z_k)^2(4a_{k+1}+4)&=4(y_k^2-z_k)(y_k^2-z_k)(a_{k+1}+1)\\&=4(y_k^2-z_k)[z_k+(y_k^2-z_k)]\\&=4(y_k^2-z_k)y_k^2\\&=4y_k^4-4y_k^2z_k+z_k^2-z_k^2\\&=(2y_k^2-z_k)^2-z_k^2\\&=y_{k+1}^2-z_{k+1}\end{aligned}$$

we see that

$$a_{k+2}=\frac{a_{k+1}^2}{4a_{k+1}+4}=\frac{(y_k^2-z_k)^2a_{k+1}^2}{(y_k^2-z_k)^2(4a_{k+1}+4)}=\frac{z_{k+1}}{y_{k+1}^2-z_{k+1}}.\quad\square$$

Lemma 5: For $n\geq 1$ and $a_1>-1$, we have that $b_{n+1}=x_{n+1}y_n/x_ny_{n+1}$. Also, for $4|a_1$, we have $b_{n+1}=y_n^2/y_{n+1}$; otherwise, $b_{n+1}=2y_n^2/y_{n+1}$.

Proof: By Lemma 4

$$b_{n+1}=\frac{2a_{n+1}+2}{a_{n+1}+2}=\frac{2y_n^2}{2y_n^2-z_n}.$$

Thus, for $4|a_1$,

$$b_{n+1}=\frac{2y_n^2}{2y_n^2-z_n}=\frac{y_n^2}{y_{n+1}}=\frac{x_ny_n^2}{x_ny_{n+1}}=\frac{x_{n+1}y_n}{x_ny_{n+1}};$$

otherwise,

$$b_{n+1}=\frac{2y_n^2}{2y_n^2-z_n}=\frac{2y_n^2}{y_{n+1}}=\frac{2x_ny_n^2}{x_ny_{n+1}}=\frac{x_{n+1}y_n}{x_ny_{n+1}}. \quad \square$$

Lemma 6: For $n\geq 1$ and $a_1>-1$, we have that $c_n=x_n/y_n$.

Proof: We easily see that

$$c_1=b_1=\frac{2a_1+2}{a_1+2}=\frac{x_1}{y_1}.$$

Now assume, for $k\geq 1$, that $c_k=x_k/y_k$. Thus, by Lemma 5,

$$c_{k+1}=c_kb_{k+1}=\frac{x_k}{y_k}\cdot\frac{x_{k+1}y_k}{x_ky_{k+1}}=\frac{x_{k+1}}{y_{k+1}}. \quad \square$$

As a corollary to Lemmas 4, 3, and 6, we have

Corollary 7: For $n\geq 1$ and $a_1>-1$, we have that $a_{n+1}=\frac{a_1+1}{c_n^2}-1$.

Proof: We have that

$$a_{n+1}=\frac{z_n}{y_n^2-z_n}=\frac{(a_1+1)z_n}{(a_1+1)(y_n^2-z_n)}=\frac{(a_1+1)y_n^2-x_n^2}{x_n^2}$$
$$=(a_1+1)\left(\frac{y_n}{x_n}\right)^2-1=\frac{a_1+1}{c_n^2}-1. \quad \square$$

The next lemma follows directly from Definition 1.

Lemma 8: For a_1 and n integers such that $n\geq 1$, exactly one of x_n, y_n, and z_n is even. More explicitly, we have that

when $a_1\equiv 0 \pmod 4$, z_n is even,
when $a_1\equiv 2 \pmod 4$, y_1 is even and, for $n\geq 2$, x_n is even,
when $a_1\equiv 1 \pmod 2$, x_n is even.

Lemma 9: For a_1 and n integers with $n\geq 1$, each of (y_n,z_n), (y_n,y_{n+1}), and (x_n,y_n) is a power of 2.

Proof: By Definition 1, $(y_1,z_1)=1=2^0$. We shall complete the proof by mathematical induction; thus, we shall also assume (y_k,z_k) is a power of 2, where $k\geq 1$. Also assume there is an odd prime p that divides (y_{k+1}, z_{k+1}). Since p divides z_{k+1} and $z_{k+1}|z_k^2$, we must have $p|z_k$. Now either

$$2y_{k+1} = 2y_k^2 - z_k \text{ or } y_{k+1} = 2y_k^2 - z_k.$$

Hence, since p is an odd prime such that $p|y_{k+1}$, and $p|z_k$, we see that $p|y_k$. Thus, p divides (y_k, z_k). This contradicts (y_k, z_k) being a power of 2.

Using the fact that, for $n \geq 1$, (y_n, z_n) is a power of 2, we shall now give indirect proofs that (y_n, y_{n+1}) and (x_n, y_n) are also powers of 2.

Thus, assume p is an odd prime that divides (y_n, y_{n+1}). Now

$$2y_{n+1} - 2y_n^2 = -z_n \text{ or } y_{n+1} - 2y_n^2 = -z_n.$$

In either case, $p|z_n$. Thus, p is an odd prime dividing (y_n, z_n); this is a contradiction.

Finally, assume p is an odd prime dividing (x_n, y_n). Thus, by Lemma 3, p divides

$$x_n\left(\frac{x_n}{a_1+1}\right) - y_n^2 = -z_n.$$

Thus, p is an odd prime dividing (y_n, z_n); this is a contradiction. □

Lemma 10: For a_1 and n integers such that $n \geq 1$, we have that

$$(z_n, y_n^2 - z_n) = 1,\ (2y_n^2, y_{n+1}) = 1, \text{ and } (x_n, y_n) = 1.$$

Proof: First notice that, by the preceding two lemmas,

$$(y_n, z_n) = 1,\ (y_n, y_{n+1}) = 1, \text{ and } (x_n, y_n) = 1.$$

Thus,

$$(z_n, y_n^2 - z_n) = (z_n, y_n^2) = 1$$

and, since y_{n+1} is an odd integer,

$$(2y_n^2, y_{n+1}) = (y_n^2, y_{n+1}) = 1. \ \square$$

2. THE CUBE ROOT FUNCTION

In [1], Eric Wingler also showed that repeated use of the identity

$$\sqrt[3]{1+s} = \frac{2s+3}{s+3}\sqrt[3]{1+\frac{2s^3+s^4}{(2s+3)^3}}$$

leads to an infinite product expansion of $\sqrt[3]{1+s}$ in the following manner: For $a_1 > 0$ and n a positive integer, defining

$$d_1 = a_1,\ d_{n+1} = \frac{2d_n^3 + d_n^4}{(2d_n+3)^3}, \text{ and } e_n = \frac{2d_n+3}{d_n+3},$$

implies $\sqrt[3]{1+d_1} = \prod_{i=1}^{\infty} e_i$.

A propos the preceding product, for $n \geq 1$, let $f_n = e_1 e_2 e_3 \ldots e_n$.

In Definition 11, we shall define three sequences, $\{u_n\}, \{v_n\}$, and $\{w_n\}$, which will depend on a_1 and which are related to $\{d_n\}, \{e_n\}$, and $\{f_n\}$. These definitions are motivated by our desire to have, when a_1 is a positive integer, u_n, v_n, and w_n be integers such that $f_n = u_n / v_n$ and w_n can be a numerator of d_{n+1} when it is written as a fraction; we do not require the fractions to be written in lowest terms. As can be seen in Theorem 12, which does not require a_1 to be an integer, the definitions in Definition 11 will give us even more than we desire.

Definition 11: Define the sequences $\{u_n\}, \{v_n\}$, and $\{w_n\}$ as follows:

$$u_1 = 2a_1 + 3, \quad v_1 = a_1 + 3, \quad \text{and} \quad w_1 = a_1^4 + 2a_1^3,$$

and, for $n \geq 1$, define

$$u_{n+1} = u_n(3u_n^3 + 2w_n), \quad v_{n+1} = v_n(3u_n^3 + w_n), \quad \text{and} \quad w_{n+1} = w_n^3(2u_n^3 + w_n).$$

For a_1 an integer, the sequences $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are integer sequences.

In Theorem 12, we shall state our results concerning the cube root function. These results relate the six sequences $\{d_n\}, \{e_n\}, \{f_n\}, \{u_n\}, \{v_n\}$, and $\{w_n\}$.

Theorem 12: For $n \geq 1$,

$$d_{n+1} = \frac{w_n}{u_n^3}, \quad e_{n+1} = \frac{u_{n+1}v_n}{u_n v_{n+1}}, \quad \text{and} \quad f_n = \frac{u_n}{v_n}.$$

We also have that

$$e_{n+1} = \frac{3u_n^3 + 2w_n}{3u_n^3 + w_n}.$$

We shall now prove four lemmas and a corollary. These five results are analogous (also see the comment at the beginning of Section 3) to Lemmas 3-6 and Corollary 7. The four lemmas will provide a proof of Theorem 12.

Lemma 13: For $n \geq 1$, $u_n^3 - (a_1 + 1)v_n^3 = -w_n$.

Proof: This lemma is true for $n = 1$. Assuming this lemma is true for $n = k$, we see that

$$\begin{aligned} u_{k+1}^3 - (a_1+1)v_{k+1}^3 &= u_k^3(3u_k^3 + 2w_k)^3 - (a_1+1)v_k^3(3u_k^3 + w_k)^3 \\ &= u_k^3(3u_k^3 + 2w_k)^3 - (u_k^3 + w_k)(3u_k^3 + w_k)^3 \\ &= u_k^3(27u_k^9 + 54u_k^6 w_k + 36u_k^3 w_k^2 + 8w_k^3) \\ &\quad - (u_k^3 + w_k)(27u_k^9 + 27u_k^6 w_k + 9u_k^3 w_k^2 + w_k^3) \\ &= u_k^3(27u_k^6 w_k + 27u_k^3 w_k^2 + 7w_k^3) - w_k(27u_k^9 + 27u_k^6 w_k + 9u_k^3 w_k^2 + w_k^3) \\ &= -2u_k^3 w_k^3 - w_k^4 = -w_{k+1}. \quad \square \end{aligned}$$

Lemma 14: For $n \geq 1$ and $a_1 > -3/2$, $d_{n+1} = w_n / u_n^3$.

Proof: This result is easily seen to be true for $n = 1$. Thus, assume that, for $k \geq 1$, $d_{k+1} = w_k / u_k^3$. Since

$$2d_{k+1}^3 + d_{k+1}^4 = d_{k+1}^3(d_{k+1}+2) = \frac{w_k^3}{u_k^9} \cdot \frac{2u_k^3 + w_k}{u_k^3} = \frac{w_{k+1}}{u_k^{12}}$$

and

$$2d_{k+1} + 3 = \frac{3u_k^3 + 2w_k}{u_k^3} = \frac{u_k(3u_k^3 + 2w_k)}{u_k^4} = \frac{u_{k+1}}{u_k^4},$$

we have that

$$d_{k+2} = \frac{2d_{k+1}^3 + d_{k+1}^4}{(2d_{k+1}+3)^3} = \frac{w_{k+1}}{u_k^{12}} \cdot \frac{u_k^{12}}{u_{k+1}^3} = \frac{w_{k+1}}{u_{k+1}^3}. \quad \square$$

Lemma 15: For $n \geq 1$ and $a_1 > -3/2$,

$$\frac{3u_n^3 + 2w_n}{3u_n^3 + w_n} = e_{n+1} = \frac{u_{n+1}v_n}{u_n v_{n+1}}.$$

Proof: Let $n \geq 1$. By Lemma 14,

$$e_{n+1} = \frac{2d_{n+1}+3}{d_{n+1}+3} = \frac{3u_n^3 + 2w_n}{u_n^3} \cdot \frac{u_n^3}{3u_n^3 + w_n} = \frac{3u_n^3 + 2w_n}{3u_n^3 + w_n}.$$

By Definition 11, this implies that

$$e_{n+1} = \frac{u_n v_n (3u_n^3 + 2w_n)}{u_n v_n (3u_n^3 + w_n)} = \frac{u_{n+1}v_n}{u_n v_{n+1}}. \quad \square$$

Lemma 16: For $n \geq 1$ and $a_1 > -3/2$, $f_n = u_n / v_n$.

Proof: Since $u_1 = 2d_1 + 3$ and $v_1 = d_1 + 3$,

$$f_1 = e_1 = \frac{2d_1+3}{d_1+3} = \frac{2a_1+3}{a_1+3} = \frac{u_1}{v_1}.$$

Now assume that, for $k \geq 1$, $f_k = u_k / v_k$. Thus,

$$f_{k+1} = f_k e_{k+1} = \frac{u_k}{v_k} \cdot \frac{u_{k+1}v_k}{u_k v_{k+1}} = \frac{u_{k+1}}{v_{k+1}}. \quad \square$$

Corollary 17: For $n \geq 1$ and $a_1 > -3/2$, we have that

$$d_{n+1} = \frac{a_1+1}{f_n^3} - 1.$$

Proof: We have, by Lemmas 14, 13, and 16,

$$d_{n+1} = \frac{w_n}{u_n^3} = \frac{(a_1+1)v_n^3 - u_n^3}{u_n^3} = (a_1+1)\left(\frac{v_n}{u_n}\right)^3 - 1 = \frac{a_1+1}{f_n^3} - 1. \quad \square$$

3. COMPARING THE SEQUENCES ASSOCIATED WITH THE SQUARE ROOT AND CUBE ROOT FUNCTIONS

Comparing Definition 1 with a_1 not being an even integer and Definition 11, we have, for $n \geq 1$,

$$x_{n+1} = 2x_n y_n,\ y_{n+1} = 2y_n^2 - z_n, \text{ and } z_{n+1} = z_n^2,$$

but

$$u_{n+1} = u_n(3u_n^3 + 2w_n),\ v_{n+1} = v_n(3u_n^3 + w_n), \text{ and } w_{n+1} = w_n^3(2u_n^3 + w_n).$$

This does not lead to any obvious generalization.

Recall that one of the reasons for our choice of the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ was to have $(x_n, y_n) = 1$. When choosing the sequences $\{u_n\}, \{v_n\}$, and $\{w_n\}$, to make our task less difficult, we did not require that $(u_n, v_n) = 1$. If, for the square root function, we relax the relatively prime requirement, we can define three sequences that are associated with the square root function (compare Lemmas 3-6 with Lemmas 19-22) and which show more similarities with the three sequences we defined for the cube root function. We shall now define these three different sequences for the square root case.

Definition 18: Define the sequences $\{g_n\}, \{h_n\}$, and $\{j_n\}$ as follows:

$$g_1 = 2a_1 + 2,\ h_1 = a_1 + 2, \text{ and } j_1 = a_1^2(a_1 + 1),$$

and define, for $n \geq 1$,

$$g_{n+1} = g_n(2g_n^2 + 2j_n) = 2g_n(g_n^2 + j_n),\ h_{n+1} = h_n(2g_n^2 + j_n),\ j_{n+1} = j_n^2(g_n^2 + j_n).$$

We shall now verify four lemmas similar to Lemmas 3-6.

Lemma 19: For $n \geq 1$, $g_n^2 - (a_1 + 1)h_n^2 = -j_n$.

Proof: This result is easily shown to be true for $n = 1$. Thus, assume this result is true for $n = k$, where $k \geq 1$. We shall prove this result is true for $n = k + 1$. We have that

$$\begin{aligned} g_{k+1}^2 - (a_1 + 1)h_{k+1}^2 &= 4g_k^2(g_k^2 + j_k)^2 - (a_1 + 1)h_k^2(2g_k^2 - j_k)^2 \\ &= 4g_k^4[g_k^2 - (a_1 + 1)h_k^2] + 4g_k^4 j_k + 4g_k^2 j_k[g_k^2 - (a_1 + 1)h_k^2] \\ &\quad + 4g_k^2 j_k^2 - (a_1 + 1)h_k^2 j_k^2 \\ &= -4g_k^4 j_k + 4g_k^4 j_k - 4g_k^2 j_k^2 + 4g_k^2 j_k^2 - j_k^2(a_1 + 1)h_k^2 \\ &= -j_k^2(g_k^2 + j_k) = -j_{k+1}. \ \square \end{aligned}$$

Lemma 20: For $n \geq 1$ and $a_1 > -1$, we have that $a_{n+1} = j_n / g_n^2$.

Proof: This result is easily shown to be true for $n = 1$. Assume $a_{k+1} = j_k / g_k^2$, where $k \geq 1$. Now

$$a_{k+2} = \frac{a_{k+1}^2}{4a_{k+1} + 4} = \frac{j_k^2}{g_k^4} \cdot \frac{g_k^2}{4(g_k^2 + j_k)} = \frac{j_k^2}{4g_k^2(g_k^2 + j_k)} = \frac{j_k^2(g_k^2 + j_k)}{4g_k^2(g_k^2 + j_k)^2} = \frac{j_{k+1}}{g_{k+1}^2}. \ \square$$

Lemma 21: For $n \geq 1$ and $a_1 > -1$, we have that

$$\frac{2g_n^2 + 2j_n}{2g_n^2 + j_n} = b_{n+1} = \frac{g_{n+1}h_n}{g_n h_{n+1}}.$$

Proof: By Lemma 20,

$$b_{n+1} = \frac{2a_{n+1} + 2}{a_{n+1} + 2} = \frac{2(g_n^2 + j_n)}{g_n^2} \cdot \frac{g_n^2}{2g_n^2 + j_n} = \frac{2(g_n^2 + j_n)}{2g_n^2 + j_n} = \frac{2g_n(g_n^2 + j_n)h_n}{g_n h_n (2g_n^2 + j_n)} = \frac{g_{n+1}h_n}{g_n h_{n+1}}. \quad \square$$

Lemma 22: For $n \geq 1$ and $a_1 > -1$, we have that $c_n = g_n / h_n$.

Proof: This result is easily shown to be true for $n = 1$. Assume $c_k = g_k / h_k$. Thus, by Lemma 21,

$$c_{k+1} = c_k b_{k+1} = \frac{g_k}{h_k} \cdot \frac{g_{k+1}h_k}{g_k h_{k+1}} = \frac{g_{k+1}}{h_{k+1}}. \quad \square$$

Comparing Definitions 18 and 11 and Lemmas 19-22 with Lemmas 13-16, we see a very close connection between the square root function and the cube root function:

- $g_1 = 2a_1 + 2, \quad h_1 = a_1 + 2, \quad j_1 = a_1^2(a_1 + 1)$, and
  $u_1 = 2a_1 + 3, \quad v_1 = a_1 + 3, \quad w_1 = a_1^3(a_1 + 2)$

and, for $n \geq 1$ and $a_1 > -1$,

- $g_{n+1} = g_n(2g_n^2 + 2j_n), \quad h_{n+1} = h_n(2g_n^2 + j_n), \quad j_{n+1} = j_n^2(g_n^2 + j_n)$, and
  $u_{n+1} = u_n(3u_n^3 + 2w_n), \quad v_{n+1} = v_n(3u_n^3 + w_n), \quad w_{n+1} = w_n^3(2u_n^3 + w_n)$,
- $g_n^2 - (a_1 + 1)h_n^2 = -j_n$ and $u_n^3 - (a_1 + 1)v_n^3 = -w_n$,
- $a_{n+1} = \dfrac{j_n}{g_n^2}$ and $d_{n+1} = \dfrac{w_n}{u_n^3}$,
- $\dfrac{2g_n^2 + 2j_n}{2g_n^2 + j_n} = b_{n+1} = \dfrac{g_{n+1}h_n}{g_n h_{n+1}}$ and $\dfrac{3u_n^3 + 2w_n}{3u_n^3 + w_n} = e_{n+1} = \dfrac{u_{n+1}v_n}{u_n v_{n+1}}$,
- $c_n = \dfrac{g_n}{h_n}$ and $f_n = \dfrac{u_n}{v_n}$.

Sometimes the correct generalization, if any, and the obvious generalization, if any, are not quite exactly the same.

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GREATEST INTEGER IDENTITIES FOR GENERALIZED FIBONACCI SEQUENCES $\{H_n\}$, WHERE $H_n = H_{n-1} + H_{n-2}$

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The generalized Fibonacci sequence $\{H_n\}$ where $H_n = H_{n-1} + H_{n-2}$, $H_1 = A, H_2 = B$, A and B integers, has been studied in the classic paper by Horadam [6] and by Hoggatt [4] and Brousseau [1], among others. Here we develop ten greatest integer identities for $\{H_n\}$. Rather than establishing these identities "for n sufficiently large," we show exact lower boundaries for subscript n dependent upon the subscript of F_k, the k^{th} Fibonacci number.

Let A, B be positive integers with $A \le B$ and define H_n $(= H_n(A, B))$ by

$$H_1 = A,\ H_2 = B,\quad H_n = H_{n-1} + H_{n-2} \text{ for } n \ge 3.$$

It is not difficult to see that in the sequence $B, A, B-A, 2A-B, 2B-3A, 5A-3B, \dots$ there is a leftmost term the double of which is less than or equal to the preceding term; otherwise, the rational number A/B would satisfy $F_{2n}/F_{2n+1} < A/B < F_{2n+1}/F_{2n+2}$ for all n. Consequently, every sequence $H_n(A, B)$ agrees, except for some initial finite set of terms, with a sequence $H_n(A', B')$, where A' and B' are positive integers with $A' = B'$ or $2A' < B'$. Then, without loss of generality, we take $A = B$ or $2A < B$ to standardize the subscripts of $\{H_n\}$ so that $H_n \ge 0$ for all $n \ge 0$ where we take $H_0 = B - A$. (The term $2A - B$ preceding H_0 will be negative when $A \ne B$.)

For these reasons, in the following we confine our attention to the two cases (a) $A = B$; and (b) $2A < B$. We call these, respectively, the *Fibonacci case* and the *Lucas case.* Throughout, we put $H_0 = B - A$ and define k by $F_{k-1} < H_0 \le F_k$ for $k \ge 3$ in the Lucas case and by $F_k \le A < F_{k+1}$ for $k \ge 2$ in the Fibonacci case.

In the following we prove ten identities for the general sequences $\{H_n\}, 0 < A \le B$. In Sections 2 and 3 we give ten greatest integer properties of $\{H_n\}$ in the Fibonacci and Lucas cases and, finally, in Section 4 we give these ten properties in a form which includes both cases.

1. PROPERTIES OF $\{H_n\}$ WHERE $H_n = H_{n-1} + H_{n-2}$

The following identities needed for our development are true for all $\{H_n\}$, $H_n = H_{n-1} + H_{n-2}$, $0 < A \le B$, and are given in [1], [4], or [6] or else are proved here.

$$H_n = F_{n-2}A + F_{n-1}B. \tag{1.1}$$

$$F_n = (\alpha^n - \beta^n)/\sqrt{5}, \text{ where } \alpha = (1+\sqrt{5})/2, \beta = (1-\sqrt{5})/2$$
$$\text{are the roots of } x^2 - x - 1 = 0 \text{ and } \alpha\beta = -1, \alpha + \beta = 1 \tag{1.2}$$

$$H_n = c\alpha^n + d\beta^n \text{ for suitable } c \text{ and } d. \tag{1.3}$$

From (1.1) and (1.2),

$$\begin{aligned}\sqrt{5}\,H_n &= A(\alpha^{n-2}-\beta^{n-2})+B(\alpha^{n-1}-\beta^{n-1})\\ &= \alpha^{n-1}(B-\beta A)-\beta^{n-1}(B-\alpha A)\\ &= \alpha^{n}(\beta A-B)\beta+\beta^{n}(B-\alpha A)\alpha\\ &= \alpha^{n}(A-\beta(B-A))+\beta^{n}(\alpha(B-A)-A),\end{aligned}$$

so that one choice for c and d, where $A=H_1$ and $B-A=H_0$, is

$$c=(A-\beta(B-A))/\sqrt{5} \quad\text{and}\quad d=(\alpha(B-A)-A)/\sqrt{5}. \tag{1.4}$$

Identities (1.5) and (1.6) are easily established by mathematical induction:

$$\alpha^{k-2}<F_k<\alpha^{k-1},\ k\ge 3; \tag{1.5}$$

$$1/2^n<|\beta|^n<1/2,\ n\ge 2,\quad |\beta|^n<1/4,\ n\ge 3. \tag{1.6}$$

Lemma 1.7: There exists an expression $K(m)$ such that

$$\alpha^m F_n = F_{n+m}+\beta^{n-m}K(m)$$

where $|K(m)|<1, m\ge 1,$ and $K(m)<0$ if m is even while $K(m)>0$ if m is odd.

Proof: Multiply by α^m in (1.2) to write

$$\alpha^m F_n=\alpha^m(\alpha^n-\beta^n)/\sqrt{5}=(\alpha^{n+m}-\beta^{n+m}+\beta^{n+m}+(-1)^{m+1}\beta^{n-m})/\sqrt{5}$$

$$\alpha^m F_n=(\alpha^{n+m}-\beta^{n+m})/\sqrt{5}+\beta^{n-m}(\beta^{2m}+(-1)^{m+1})/\sqrt{5}, \tag{1.7}$$

which will verify Lemma 1.7. □

Lemma 1.8: There exists an expression $K^*(m), 0<K^*(m)<1$, such that

$$F_n/\alpha^m=F_{n-m}+\beta^{n-m}K^*(m),\ m\ge 1.$$

Proof: Multiplying by $1/\alpha^m$ in (1.2) yields

$$F_n/\alpha^m=(\alpha^n-\beta^n)/\alpha^m\sqrt{5}=(\alpha^{n-m}-\beta^{n-m})/\sqrt{5}+(\beta^{n-m}+(-1)^{m+1}\beta^{n+m})/\sqrt{5}$$

$$F_n/\alpha^m=F_{n-m}+\beta^{n-m}(1+(-1)^{m+1}\beta^{2m})/\sqrt{5}, \tag{1.8}$$

which will verify Lemma 1.8. □

The characteristic number D for $\{H_n\}$ is defined as $D=B^2-AB-A^2$ in [1] and [6], and

$$H_n^2-H_{n-1}H_{n+1}=(-1)^n D, \tag{1.9}$$

where $D>0$ in the Lucas case where $2A<B$, while $D=-1$ for the Fibonacci numbers,

$$F_n^2-F_{n-1}F_{n+1}=(-1)^{n+1}. \tag{1.10}$$

Identities (1.9) and (1.10) show a subtle but important difference in parity between the Lucas and Fibonacci cases, since n even in (1.9) gives a positive value while n even in (1.10) gives a negative value. The difference in parity causes us to consider the Fibonacci and Lucas cases separately.

2. THE FIBONACCI CASE: THE SEQUENCES $\{H_n\}$ WHERE $A = B$

Consider the Fibonacci case for $\{H_n\}$ where $A = B$. Then $H_n = AF_n$, $A \ge 1$. We write ten greatest integer identities which are true for $\{H_n\}$ when $H_n = AF_n$, and hence for $\{F_n\}$, since the Fibonacci sequence is the special case $A = B = 1$. We write $[x]$ to denote the greatest integer contained in x, and in every case, we determine k by $F_k \le A < F_{k+1}$, $k \ge 2$.

Theorem 2.1: $[\alpha AF_n] = AF_{n+1}$, $\quad n$ odd, $\; n \ge k, k \ge 2, A \ge 1$;
$[\alpha AF_n] = AF_{n+1} - 1$ $\quad n$ even, $\; n \ge k, k \ge 2, A \ge 1$.

Proof: Let $m = 1$ in (1.7) to write

$$\alpha F_n = F_{n+1} + \beta^{n-1}(\beta^2 + 1)/\sqrt{5}.$$

Multiplying by A and computing $(\beta^2 + 1)/\sqrt{5} = -\beta$,

$$\alpha AF_n = AF_{n+1} + (-A\beta^n). \tag{2.1}$$

If $A < F_{k+1}$, we have $A < \alpha^k$ by (1.5), and

$$|-A\beta^n| < |\alpha^k \beta^n| = |\beta^{n-k}| < 1$$

for $n \ge k, k \ge 2$, by (1.6). If n is odd, $0 < -A\beta^n < 1$, while if n is even, $0 > -A\beta^n > -1$, giving Theorem 2.1, for $n \ge k, k \ge 2$. □

Theorem 2.2: $[\alpha AF_n + 1/2] = AF_{n+1}$, $n \ge k+2$.

Proof: Since $|-A\beta^n| < |\alpha^k \beta^{n-k}| = |\beta^{n-k}| < 1/2$ if $n \ge k+2$, adding 1/2 to each side of (2.1) will ultimately yield Theorem 2.2. □

Theorem 2.3: $[AF_n/\alpha] = AF_{n-1}$, $\quad n$ odd, $\; n \ge k, k \ge 2, A \ge 1$;
$[AF_n/\alpha] = AF_{n-1} - 1$, $\quad n$ even, $\; n \ge k, k \ge 2, A \ge 1$.

Proof: By taking $m = 1$ in (1.8) and multiplying by A,

$$AF_n/\alpha = AF_{n-1} + (-A\beta^n). \tag{2.3}$$

The proof is finished by analyzing $|-A\beta^n|$ as in the proof of Theorem 2.1. □

As in Theorem 2.2, variations of Equation (2.3) will lead to Theorems 2.4 and 2.5; the proofs are omitted.

Theorem 2.4: $[AF_n/\alpha + 1/2] = AF_{n-1}$, $n \ge k+2$.

Theorem 2.5: $[(AF_n + 1)/\alpha] = AF_{n-1}$, $n \ge k+2$.

Theorem 2.6: $[\alpha^m AF_n] = AF_{n+m}$, $\quad n$ odd, $\; n \ge m+k$;
$[\alpha^m AF_n] = AF_{n+m} - 1$, $\quad n$ even, $\; n \ge m+k$.

Proof: Multiply by A in Lemma 1.7 to write

$$\alpha^m AF_n = AF_{n+m} + A\beta^{n-m}K(m) \tag{2.6}$$

where $|K(m)| < 1, m \geq 1,$ and $K(m) < 0$ if m is even while $K(m) > 0$ if m is odd. Since also $A < \alpha^k$ when $n \geq m+k$,

$$|A\beta^{n-m}K(m)| < |\alpha^k \beta^{n-m}K(m)| < |\beta^{n-m-k}| < 1.$$

If n is odd, and m even, $K(m) < 0, \beta^{n-m} < 0,$ and $0 < A\beta^{n-m}K(m) < 1,$ while m odd makes the same result from $K(m) > 0$ and $\beta^{n-m} > 0$. Thus, if n is odd, $[\alpha^m AF_n] = AF_{n+m}$..

If n is even, m odd makes $K(m) > 0,$ $\beta^{n-m} < 0,$ so that $0 > A\beta^{n-m}K(m) > -1,$ while m even gives the same result from $K(m) < 0$ and $\beta^{n-m} > 0$. If n is even, $[\alpha^m AF_n] = AF_{m+n} - 1$. □

Adding 1/2 to each side of (2.6) will ultimately yield Theorem 2.7.

Theorem 2.7: $[\alpha^m AF_n + 1/2] = AF_{n+m},\ n \geq m+k+2.$

Theorem 2.8 $[AF_n / \alpha^m] = AF_{n-m},\quad n-m$ even, $n \geq m+k;$
$[AF_n / \alpha^m] = AF_{n-m} - 1,\quad n-m$ odd, $n \geq m+k.$

Proof: Refer to Lemma 1.8 to write

$$AF_n / \alpha^m = AF_{n-m} + A\beta^{n-m}K^*(m) \tag{2.8}$$

where $0 < K^*(m) < 1, m \geq 1,$ and $A < \alpha^k$.

If $n-m$ is even and $n-m \geq k$,

$$0 < A\beta^{n-m}K^*(m) < \alpha^k \beta^{n-m}K^*(m) < 1.$$

If $n-m$ is odd, $\beta^{n-m} < 0$ while $A \geq 1$ and, if $n \geq m$,

$$0 > A\beta^{n-m}K^*(m) > \beta^{n-m}K^*(m) > -|\beta|^{n-m} > -1,$$

finishing the proof. □

Theorem 2.9: $[AF_n / \alpha^m + 1/2] = AF_{n-m},\ n \geq m+k+2.$

Proof: Add 1/2 to each side of (2.8), and analyze the resulting expressions for $n-m$ even, and for $n-m$ odd. □

Theorem 2.10: $AF_n = [A\alpha^n / \sqrt{5} + 1/2],\ n \geq k, k \geq 2, A \geq 1.$

Proof:

$$\begin{aligned} A\alpha^n / \sqrt{5} + 1/2 &= A(\alpha^n / \sqrt{5} - \beta^n / \sqrt{5}) + A\beta^n / \sqrt{5} + 1/2 \\ &= AF_n + A\beta^n / \sqrt{5} + 1/2 \end{aligned}$$

where

$$|A\beta^n / \sqrt{5} + 1/2| < |\alpha^k \beta^n / \sqrt{5} + 1/2| = |\beta^{n-k} / \sqrt{5} + 1/2| < 1$$

for $n \geq k$ and $k \geq 2$. □

If $A=1$, we have, of course, the Fibonacci numbers $\{F_n\}$. Theorems 2.1 and 2.6 for $\{F_n\}$ appear in [5], and Theorems 2.2 and 2.10 in [4], for $A=1$ and $n\geq 2$. By taking $A=1$ in the proof of Theorems 2.2, 2.4, and 2.5, we find that in the special case $AH_n=F_n$ all three are true for $n\geq 2$.

If $\{H_n\}$ contains $H_n=kF_n$ but $H_{n-1}\neq KF_{n-1}$, then we have the Lucas case $A\neq B$ of the next section.

3. THE LUCAS CASE: THE SEQUENCES $\{H_n\}$ WHERE $0<2A<B$

Let $H_n=H_{n-1}+H_{n-2}$ where $H_1=A$, $H_2=B$, and $0<2A<B$. We prove ten greatest integer identities as before, but we define k by

$$F_{k-1}<B-A\leq F_k,\ k\geq 3.$$

Referring to (1.5), we can combine inequalities to write

$$B-A<\alpha^{k-1},\ k\geq 3;\quad \text{and } 1\leq A. \tag{3.01}$$

By applying (1.4) and (3.01) and making careful analysis of signs, we next establish

$$|\sqrt{5}d\beta^n|<|\beta|^{n-k}-|\beta|^n,\ n\geq k, \tag{3.02}$$

where $d=(\alpha(B-A)-A)/\sqrt{5}>0$.

If n is even, $\beta^n>0$, and

$$\begin{aligned}0<\sqrt{5}\,d\beta^n&=(\alpha(B-A)-A)\beta^n<(\alpha\alpha^{k-1}-1)\beta^n\\&=(-1)^k\beta^{n-k}-\beta^n\\&=|\beta|^{n-k}-|\beta|^n.\end{aligned}$$

If n is odd, $-\beta^n>0$, and

$$\begin{aligned}0>\sqrt{5}\,d\beta^n&=(A-\alpha(B-A))(-\beta^n)>(1-\alpha\alpha^{k-1})(-\beta^n)\\&=-\beta^n+(-1)^k\beta^{n-k}\\&=|\beta|^n-|\beta|^{n-k}\end{aligned}$$

which establishes (3.02) and will allow us to write several identities for $\{H_n\}$, in the Lucas case.

Theorem 3.1: $[\alpha H_n]=H_{n+1}$, $\quad n$ even, $n\geq k$;
$[\alpha H_n]=H_{n+1}-1$, $\quad n$ odd, $n\geq k$.

Proof:

$$\begin{aligned}\alpha H_n&=\alpha(c\alpha^n+d\beta^n)\\&=c\alpha^{n+1}+d\beta^{n+1}-d\beta^{n+1}-d\beta^{n-1}\\&=H_{n+1}-d\beta^{n-1}(\beta^2+1)\end{aligned}$$

$$\alpha H_n=H_{n+1}+\sqrt{5}\,d\beta^n. \tag{3.1}$$

By (3.02), $|\sqrt{5}\,d\beta^n|<|\beta|^{n-k}-|\beta|^n<1-1/2^n, n\geq k$, which establishes Theorem 3.1 by considering the cases n even and n odd. □

Theorem 3.2: $[\alpha H_n + 1/2] = H_{n+1}$, $n \geq k+2$.

Proof: Add 1/2 to each side of (3.1) and use (3.02) to analyze the result. □

Theorem 3.3: $[H_n / \alpha] = H_{n-1}$, $\quad n$ even, $n \geq k$;
$[H_n / \alpha] = H_{n-1} - 1$, $\quad n$ odd, $n \geq k$.

Proof:

$$H_n / \alpha = (c\alpha^n + d\beta^n)/\alpha = c\alpha^{n-1} + d\beta^{n-1} - d\beta^{n-1} - d\beta^{n+1}$$

$$H_n / \alpha = H_{n-1} + \sqrt{5}\, d\beta^n \tag{3.3}$$

where we note the same fractional expression $\sqrt{5}\, d\beta^n$ as in Theorem 3.1. □

Theorem 3.3 corrects a proof of a theorem of Cohn [2; p. 31], in which he gives the next lower term to N as $[N/\alpha]$, which is true when n is even but not when n is odd. Dr. Cohn has acknowledged the error in a private correspondence with one of the authors.

Theorem 3.4: $[H_n / \alpha + 1/2] = H_{n-1}$, $n \geq k+2$.

The proof is identical to that of Theorem 3.2, but using (3.3). □

Theorem 3.5: $[(H_n + 1)/\alpha] = H_{n-1}$, $n \geq k+3$.

Proof: From (3.3),

$$(H_n + 1)/\alpha = H_{n-1} + \sqrt{5}\, d\beta^n + 1/\alpha.$$

By (3.02), $|\sqrt{5}\, d\beta^n| < 1/4 - 1/2^n$ for $n \geq k+3$. Adding $1/\alpha$ to each term of the inequality for the case n even, and then for the case n odd, we find that in either case, we obtain $0 \leq \sqrt{5}\, d\beta^n + 1/\alpha < 1$. □

Theorem 3.6: $[\alpha^m H_n] = H_{n+m}$, $\quad n$ even, $n \geq m+k, m \geq 2$;
$[\alpha^m H_n] = H_{n+m} - 1$, $\quad n$ odd, $n \geq m+k, m \geq 2$.

Proof: Since $1/\alpha^m = (-1)^m \beta^m$,

$$\begin{aligned} \alpha^m H_n &= \alpha^m (c\alpha^n + d\beta^n) \\ &= c\alpha^{n+m} + d\beta^{n+m} - d\beta^{n+m} + (-1)^m d\beta^{n-m} \\ &= H_{n+m} + \sqrt{5}\, d\beta^{n-m}((-1)^m - \beta^{2m})/\sqrt{5} \end{aligned}$$

$$\alpha^m H_n = H_{n+m} + \sqrt{5}\, d\beta^{n-m} M(m) \tag{3.6}$$

where $|M(m)| < 1$ for $m \geq 1$. By (3.02),

$$|\sqrt{5}\, d\beta^{n-m} M(m)| \leq |\sqrt{5}\, d\beta^{n-m}| < |\beta|^{n-m-k} - |\beta|^{n-m} < 1 - 1/2^{n-m}$$

for $n - m \geq k$. Consider the signs carefully. For n odd, m odd, $\beta^{n-m} > 0$ and $M(m) < 0$, while for n odd, m even, $\beta^{n-m} < 0$ and $M(m) > 0$, so whenever n is odd,

$$0 > \sqrt{5}\, d\beta^{n-m} M(m) > 1/2^{n-m} - 1 > -1,$$

so that $[\alpha^m H_n] = H_{n+m} - 1$. For n even, m odd, $\beta^{n-m} < 0$ and $M(m) < 0$, while for n even, m even, $\beta^{n-m} > 0$ and $M(m) > 0$, so whenever n is even,

$$0 < \sqrt{5}\, d\beta^{n-m} M(m) < 1 - 1/2^{n-m} < 1,$$

so that $[\alpha^m H_n] = H_{n+m}$. □

Theorem 3.7: $[\alpha^m H_n + 1/2] = H_{n+m}$, $n \geq m + k + 2$.

Proof: By (3.6),

$$\alpha^m H_n + 1/2 = H_{n+m} + \sqrt{5}\, d\beta^{n-m} M(m) + 1/2$$

where $|\sqrt{5}\, d\beta^{n-m} M(m)| < 1/2 - 1/2^{n-m}$ for $n - m \geq k + 2$. Add 1/2 to each member of the inequalities for the even and odd cases as in the proof of Theorem 3.2. □

Theorem 3.8: $[H_n / \alpha^m] = H_{n-m}$, $\quad n - m$ odd, $\quad n > m, n - m \geq k$;
$[H_n / \alpha^m] = H_{n-m} - 1$, $\quad n - m$ even, $\quad n > m, n - m \geq k$.

Proof: Since $1/\alpha^m = (-1)^m \beta^m$,

$$\begin{aligned} H_n / \alpha^m &= (c\alpha^n + d\beta^n)/\alpha^m \\ &= c\alpha^{n-m} + d\beta^{n-m} - d\beta^{n-m} + d(-1)^m \beta^{n+m} \\ &= H_{n-m} + d\beta^{n-m}(-1 + (-1)^{m+1}\beta^{2m}) \\ &= H_{n-m} + \sqrt{5}\, d\beta^{n-m}((-1 + (-1)^{m+1}\beta^{2m})\sqrt{5}) \end{aligned}$$

$$H_n / \alpha^m = H_{n-m} + \sqrt{5}\, d\beta^{n-m} J(m) \tag{3.8}$$

where $|J(m)| < 1$ for $m \geq 1$ but $J(m) < 0$ for $m \geq 1$. From (3.02) we have the same results as in the proof of Theorem 3.6 except for the signs:

$$|\sqrt{5}\, d\beta^{n-m} J(m)| < |\beta|^{n-m-k} - |\beta|^{n-m}.$$

For n odd and m odd, or for n even and m even, $\beta^{n-m} > 0$ and $J(m) < 0$, and we have

$$0 > \sqrt{5}\, d\beta^{n-m} J(m) > 1/2^{n-m} - 1 > -1$$

for $n - m \geq k, n - m$ even, making $|H_n / \alpha^m] = H_{n-m} - 1$.

For n even and m odd, or for n odd and m even, $\beta^{n-m} < 0$ and $J(m) < 0$;

$$0 < \sqrt{5}\, d\beta^{n-m} J(m) < 1 - 1/2^{n-m} < 1$$

for $n-m$ odd, $n - m \geq k$, and $[H_n / \alpha^m] = H_{n-m}$, finishing the proof. □

Theorem 3.9: $[H_n / \alpha^m + 1/2] = H_{n-m}, n > m, n - m \geq k + 2$.

Theorem 3.9 is proved by using the methods of Theorems 3.2 and 3.7 to operate on (3.8).

Theorem 3.10: $H_n = [c\alpha^n + 1/2]$, $n \geq k$, where $c = (H_1 - \beta H_0)/\sqrt{5}$.

Proof: By (1.3) and (1.4),

$$c\alpha^n + 1/2 = c\alpha^n + d\beta^n - d\beta^n + 1/2 = H_n - d\beta^n + 1/2.$$

Divide each term of inequality (3.2) by $\sqrt{5}$ to write

$$|d\beta^n| < (|\beta|^{n-k} - |\beta|^n)/\sqrt{5} < (1 - 1/2^n)/\sqrt{5} < 1/2, \ n \ge k.$$

If n is even, then $\beta^n > 0$, and $0 > d\beta^n > -1/2$. Add 1/2 to each term to determine that $1 > 1/2 - d\beta^n > 0$. If n is odd, then $\beta^n < 0$, and $0 < -d\beta^n < 1/2$ gives $0 < 1/2 - d\beta^n < 1$ upon adding 1/2 to each term. In either case, $H_n = [c\alpha^n + 1/2]$. □

Corollary 3.10: $L_n = [\alpha^n + 1/2]$ for the Lucas numbers $(L_n), n \ge 2$.

Corollary 3.10 appears in [4].

4. THE GENERAL CASE: (H_n) WHERE A = B OR 0 < 2A < B

In comparing the ten theorems of Sections 2 and 3, notice close agreement except for whether subscripts are odd or even, as expected from (1.9) and (1.10). The following results are true for both the Fibonacci and Lucas cases, and hence for all $\{H_n\}$, where we take k from $F_{k-1} < H_0 = B - A < F_{k+1}$ if $A \ne B$, and from $F_k \le A < F_{k+1}$ if $A = B$.

Theorem 4.1: $[\alpha H_n] = H_{n+1}$ or $H_{n+1} - 1, \ n \ge k$.

Theorem 4.2: $[\alpha H_n + 1/2] = H_{n+1}, \ n \ge k+2$.

Theorem 4.3: $[H_n/\alpha] = H_{n-1}$ or $H_{n-1} - 1, \ n \ge k$.

Theorem 4.4: $[H_n/\alpha + 1/2] = H_{n-1}, \ n \ge k+2$.

Theorem 4.5: $[(H_n + 1)/\alpha] = H_{n-1}, \ n \ge k+3$.

Theorem 4.6: $[\alpha^m H_n] = H_{n+m}$ or $H_{n+m} - 1, \ n \ge m+k+2, m \ge 2$.

Theorem 4.7: $[\alpha^m H_n + 1/2] = H_{n+m}, \ n \ge m+k+2, m \ge 2$.

Theorem 4.8 $[H_n/\alpha^m] = H_{n-m}$ or $H_{n-m} - 1, \ n \ge m+k, m \ge 2$.

Theorem 4.9: $[H_n/\alpha^m + 1/2] = H_{n-m}, \ n \ge m+k+2, m \ge 2$.

Theorem 4.10: $[c\alpha^n + 1/2] = H_n, \ c = (H_1 - \beta H_0)/\sqrt{5}, \ n \ge k$.

We can extend Theorems 4.1 through 4.10 for negative subscripts. Since $(-1)^{n+1}F_{-n} = F_n$, $|F_{-n}| = F_n$, Theorems 2.1 through 2.10 apply for sequences having $|F_{-n}|$ or $|AF_{-n}|$ as the n^{th} term. We can apply Theorems 3.1 through 3.10 for $\{H^*_n\}$ where $H^*_n = |H_{-n}|$ as well if we extend the definition of $\{H_n\}$ for negative subscripts so that (1.1) becomes

$$H_{-n} = AF_{-n-2} + BF_{-n-1} = A(-1)^{n+3}F_{n+2} + B(-1)^{n+2}F_{n+1},$$

$$H_{-n} = (-1)^n(BF_{n+1} - AF_{n+2}) = (-1)^n H^*_n, \tag{4.1}$$

where $\{H^*_n\}$ is the conjugate sequence [3] for $\{H_n\}$, $H^*_n = H^*_{n-1} + H^*_{n-2}$, $H^*_0 = B - A$, $H^*_1 = B - 2A = A^*$, $H^*_2 = 2B - 3A = B^*$. Notice that $|H_{-n}| = (-A^*)F_{n-2} + B^*F_{n-1} = H^*_n$, where $\{H^*_n\}$ is one of the sequences $\{H_n\}$ with positive subscripts. Thus, Theorems 3.1 through 3.10 and 4.1 through 4.10 can be extended to $\{H_n\}$ with negative subscripts by taking $|H_n| = H^*_n$ in all the theorem statements.

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Additional references for this paper were found upon reading the August 1994 issue of this quarterly, where Wayne L. McDaniel ("On the Greatest Integer Function and Lucas Sequences" **32.4** [1994]:297-300) gives related but not identical results to those appearing in this paper. In earlier issues of *The Fibonacci Quarterly*, Robert Anaya and Janice Crump ("A Generalized Greatest Integer Function Theorem" **10.2** [1972]:207-12) proved a special case of our Theorem 2.7, and L. Carlitz ("A Conjecture Concerning Lucas Numbers" **10.5** [1972]:526) proved a special case of our Theorem 3.7.

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DIOPHANTINE REPRESENTATION OF LUCAS SEQUENCES

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1. INTRODUCTION

The Lucas sequences $\{U_n(P,Q)\}$, with parameters P and Q, are defined by $U_0(P,Q)=0$, $U_1(P,Q)=1$, and

$$U_n(P,Q)=PU_{n-1}(P,Q)-QU_{n-2}(P,Q) \quad \text{for } n\ge 2,$$

and the "associated" Lucas sequences $\{V_n(P,Q)\}$ are defined similarly with initial terms equal to 2 and P, for $n=0$ and 1, respectively. The sequences of Fibonacci numbers and Lucas numbers are, of course, $\{F_n\}=\{U_n(1,-1)\}$ and $\{L_n\}=\{V_n(1,-1)\}$.

Several authors (e.g.,[3], [1], [6]) have discussed the conics whose equations are satisfied by pairs of successive terms of the Lucas sequences. In particular, it has been shown that $(x,y)=(w_n,w_{n+1})$ satisfies $y^2-Pxy+Qx^2+eQ^n=0$, where $w_n=U_n(P,Q)$ if $e=-1$ and $w_n=V_n(P,Q)$ if $e=P^2-4Q$. It has apparently not been recognized that the hyperbolas $y^2-Pxy+Qx^2+eR=0$, where $R=1$ if $Q=1$ and $R=\pm 1$ if $Q=-1$ *characterize* the Lucas sequences when $e=-1$, and the associated Lucas sequences when $e=P^2-4Q$ is square-free; that is, the set of lattice points on these conics is precisely the set of pairs of consecutive terms of $\{U_n(P,\pm 1)\}$ if $e=-1$, and of $\{V_n(P,\pm 1)\}$ if $e=P^2-4Q$ is square-free. Accordingly, we shall prove the converse of the results of [3] and [1] by showing that no lattice points exist for the above hyperbolas if $Q=\pm 1$ other than (w_n,w_{n+1}) [provided that when $w_n=V_n(P,Q)$, the discriminant D is square-free].

Using the above results, we then construct, for each of the sequences $\{U_n(P,-1)\}$, $\{U_n(P,1)\}$, and $\{V_n(P,1)\}$, a polynomial in two variables of degree 5, and a polynomial of degree 9 for $\{V_n(P,-1)\}$ whose positive values, for positive integral values of the variables, are precisely the terms of the sequence. This extends the results of Jones [4] and [5], who obtained a fifth-degree polynomial whose positive values are the Fibonacci numbers and a ninth-degree polynomial whose positive values are the Lucas numbers.

2. CONICS CHARACTERIZING THE LUCAS SEQUENCES

Assume $P>0$. To simplify notation, we let $U_n=U_n(P,-1)$, $V_n=V_n(P,-1)$, $u_n=U_n(P,1)$, and $v_n=V_n(P,1)$. A proof of the sufficiency in our theorems occurs as a general result in [3]; however, we include an alternate inductive proof in Theorem 1 for completeness.

Theorem 1: Let x and y be positive integers. The pair (x,y) is a solution of

$$y^2-Pxy-x^2=\pm 1 \tag{1}$$

iff there exists a positive integer n such that $x=U_n$ and $y=U_{n+1}$.

Proof: We show, first, that $U_{n+1}^2-PU_{n+1}U_n-U_n^2=(-1)^n$, by induction.

If $n=1$, $U_1=1$ and $U_2=P$ and the result clearly holds. Assume $U_n^2-PU_nU_{n-1}-U_{n-1}^2=(-1)^{n-1}$. Then

$$\begin{aligned} U_{n+1}^2 - PU_{n+1}U_n - U_n^2 &= (PU_n + U_{n-1})^2 - P(PU_n + U_{n-1})U_n - U_n^2 \\ &= U_n^2(P^2 - P^2 - 1) + PU_nU_{n-1}(2-1) + U_{n-1}^2 \\ &= -1(U_n^2 - PU_nU_{n-1} - U_{n-1}^2) = (-1)^n. \end{aligned}$$

To see that there are no other solutions of (1) in positive integers, suppose there exist solutions not of the form (U_n, U_{n+1}). Let x be the least positive integer such that, for some positive integer y, (x, y) is a solution of (1) and $(x, y) \neq (U_n, U_{n+1})$ for any positive integer n. Since $(1, P) = (U_1, U_2)$ satisfies (1), $x > 1$. Let $x_0 = y - Px$ and $y_0 = x$. We show that $0 < x_0 < x$ and that (x_0, y_0) satisfies (1). Since $x > 1$, $0 = y^2 - Pxy - x^2 \pm 1 = y(y - Px) - x^2 \pm 1 = yx_0 - x^2 \pm 1$ implies $x_0 > 0$, and from $yx_0 \pm 1 = x^2$, we have $(Px + x_0)x_0 \pm 1 = x^2$, i.e., $Pxx_0 \pm 1 = x^2 - x_0^2$, implying that $x_0 < x$. Now,

$$y_0^2 - Py_0x_0 - x_0^2 = x^2 - Px(y - Px) - (y - Px)^2 = x^2 + Pxy - y^2 = -(\pm 1).$$

Thus, (x_0, y_0) is a solution. By the induction hypothesis, there exists an n such that $x_0 = U_n$ and $y_0 = U_{n+1}$. Then $x = y_0 = U_{n+1}$ and

$$y = Px + x_0 = Py_0 + x_0 = PU_{n+1} + U_n = U_{n+2},$$

contradicting our assumption concerning (x, y).

According to Dickson ([2], Vol. 1, p. 405), Lucas [7] proved that, if x and y are consecutive Fibonacci numbers, then (x, y) is a lattice point on one of the hyperbolas $y^2 - xy - x^2 = \pm 1$, and J. Wasteels [12] proved the converse in 1902.

Theorem 2: Let x and y be positive integers, $x < y$. The pair (x, y) is a solution of

$$y^2 - Pyx + x^2 = 1, \quad P > 2, \tag{2}$$

iff there exists a positive integer n such that $x = u_n$ and $y = u_{n+1}$.

Proof: We note that, because of the symmetry, the assumption that $x < y$ is made without loss of generality. The proof parallels that of Theorem 1. (In proving the necessity, one lets $x_0 = Px - y$ and $y_0 = x$, and easily obtains $x_0 < x$, and $x_0y = x^2 - 1 < xy \Rightarrow x_0 < x$.)

It is known that, if $D = P^2 + 4$, the general solution in positive integers of $y^2 - Dx^2 = \pm 4$ is $(x, y) = (U_n, V_n)$, and if $D = P^2 - 4$, the general solution of $y^2 - Dx^2 = 4$ is (u_n, v_n). This may be shown using the known general solutions in terms of the fundamental solutions (for example, from $(x_n + y_n\sqrt{D})/2 = [(x_0 + y_0\sqrt{D})/2]^n$ for $x^2 - Dy^2 = 4$; see Mordell [9, p. 55], and Dickson [2, Ch. XII]). Using Theorems 1 and 2, we provide an alternate derivation of the general solution in terms of Lucas sequences of these Fermat-Pell equations.

Corollary 1: The solutions of $s^2 - Dt^2 = \pm 4$ for $D = P^2 + 4$ and of $s^2 - Dt^2 = 4$ for $D = P^2 - 4$ are precisely the pairs $(t, s) = (U_n, V_n)$ and (u_n, v_n), respectively.

Proof: It is well known that $V_n^2(P, Q) - D \cdot U_n^2(P, Q) = 4Q^n$ [11, p. 44]. Suppose (s, t) is any solution of $s^2 - Dt^2 = \pm 4$ $(D = P^2 + 4)$, i.e., of $s^2 - P^2t^2 = \pm 4 + 4t^2$. It is clear that s and Pt have the same parity, so $y = (s + Pt)/2$ is an integer. Upon substituting for s,

$$(2y-Pt)^2 - P^2t^2 = \pm 4 + 4t^2 \Rightarrow 4y^2 - 4Pty = \pm 4 + 4t^2.$$

That is, $y^2 - Pyt - t^2 = \pm 1$. By Theorem 1, $y = U_{n+1}$ and $t = U_n$ for some n. Now it is known that $V_n(P,Q) = 2U_{n+1}(P,Q) - PU_n(P,Q)$ [11, p. 44], implying that $s = V_n$.

The proof of the necessity for $s^2 - Dt^2 = 4$, $D = P^2 - 4$ is similar.

3. CONICS CHARACTERIZING THE ASSOCIATED LUCAS SEQUENCES

It is interesting that the solutions of the hyperbolas $y^2 - Pxy - x^2 = \pm D$, for $D = P^2 + 4$, include (V_n, V_{n+1}) for $n \geq 0$, and the solutions of $y^2 - Pxy + x^2 = -D$, for $D = P^2 - 4$, include (v_n, v_{n+1}) for $n \geq 0$ [3], but that there may be, in general, additional pairs of integral solutions. A case in point: $y^2 - 4xy - x^2 = 20$ has $(x, y) = (1, 7)$ as a solution (but $V_n \neq 1$ for any $n \geq 0$). It may be shown, however, that there are no additional solutions if D is square-free.

Theorem 3: Let $P^2 + 4 = D = a^2d$, d square-free. The set of lattice points with positive coordinates on the hyperbolas

$$y^2 - Pyx - x^2 = \pm D \tag{3}$$

is precisely the set $\{(V_n, V_{n+1})\}$ $(n \geq 0)$ iff the sets of x-coordinates of the solution sets of $x^2 - Dy^2 = \pm 4$ and $x^2 - dz^2 = \pm 4$ are equal.

Proof: As remarked above, (V_n, V_{n+1}) satisfies (3) for all $n \geq 0$. Assume that $x, y > 0$ and (x, y) is a solution of (3). Now, since P and D have the same parity, (3) implies that

$$y = \left[Px + \sqrt{D(x^2 \pm 4)}\right]\Big/2 = \left[Px + a\sqrt{d(x^2 \pm 4)}\right]\Big/2$$

is an integer iff $d(x^2 \pm 4)$ is a square; that is, iff, for some integer z, $x^2 \pm 4 = dz^2$, i.e., $x^2 - dz^2 = \pm 4$. Thus, the set of lattice points on (3) is precisely the set $\{V_n, V_{n+1}\}$ iff $x = V_n$ for some $n \geq 0$. By Corollary 1, on the other hand, the pair (x, y) is a solution of $x^2 - Dy^2 = \pm 4$ iff $x = V_n$ for some $n \geq 0$. This proves the theorem.

If D is square-free, then $d = D$, and we immediately have

Corollary 2: Let x and y be positive integers, and $D = P^2 + 4$ be square-free. The pair (x, y) is a solution of $y^2 - Pxy - x^2 = \pm D$ iff there exists a nonnegative integer n such that $x = V_n$ and $y = V_{n+1}$.

We note that the equations $x^2 - Dy^2 = \pm 4$ and $x^2 - dz^2 = \pm 4$ of Theorem 3 may have solution sets having identical x-coordinates when $D \neq d$. For example, if $D = 4d$ and $d \equiv 2$ or 3 (mod 4), since in these cases z must be even.

We may establish, in exactly the same way as for Theorem 3, the corresponding theorem for $y^2 - Pyx + x^2 = -D$, with $D = P^2 - 4$. We state only the analogous corollary.

Corollary 3: Let $D = P^2 - 4$ be square-free and x and y be positive integers. The pair (x, y) is a solution of

$$y^2 - Pyx + x^2 = -D \tag{4}$$

if there exists a nonnegative integer n such that $x = v_n$ and $y = v_{n+1}$.

4. DIOPHANTINE REPRESENTATION OF THE SEQUENCES

The set of terms of any Lucas sequence is a recursively enumerable set, and such sets have been shown to be Diophantine [8]. That is, for each recursively enumerable set S, there exists a polynomial $\mathcal{P}$ with integral coefficients, in variables $x_1, \ldots, x_n$, such that $x \in S$ iff there exist positive integers $y_1, \ldots, y_{n-1}$ such that $\mathcal{P}(x, y_1, \ldots, y_{n-1}) = 0$. As a consequence, it is possible to construct a polynomial whose positive values are precisely the elements of S. The construction is due to Putnam [10], who observed that $x(1-\mathcal{P}^2)$ has the desired property. Using equations (1), (2), (3), (4), and Corollary 1, we now obtain such polynomials for the set of terms of the sequences $\{U_n(P,-1)\}$, $\{U_n(P,1)\}$, $\{V_n(P,-1)\}$, and $\{V_n(P,1)\}$.

Theorem 5: Let $\mathcal{U}(P,Q)$ denote the set of terms of the sequence $\{U_n(P,Q)\}$, and $\mathcal{V}(P,Q)$ denote the set of terms of the sequence $\{V_n(P,Q)\}$. Then, if x and y assume all positive integral values, the set S is identical to the set of positive values of the polynomial

(i) $x[2-(y^2-Pxy-x^2)^2]$ if $S = \mathcal{U}(P,-1)$,

(ii) $x[2-(y^2-Pxy+x^2)^2]$ if $S = \mathcal{U}(P,1)$, $P>2$,

(iii) $y[1-((y^2-Dx^2)^2-16)^2]$ if $S = \mathcal{V}(P,-1)$, $D = P^2+4$,

(iv) $y[1-((y^2-Dx^2)-4)^2]$ if $S = \mathcal{V}(P,1)$, $D = P^2-4$.

Proof: In view of Theorems 1 and 2 and Corollary 1, the proof is obvious, provided we show that $y^2-Pxy-x^2$ and $y^2-Pxy+x^2$ $(P>2)$ are never 0 for x and y integers. However, if either equals 0, then

$$y = \frac{Px \pm x\sqrt{P^2+4}}{2} \quad \text{or} \quad y = \frac{Px \pm x\sqrt{P^2-4}}{2}, \quad (P>2);$$

clearly, since $D = P^2 \pm 4$ is not a square, y is irrational for all integral x values.

By Corollary 1, the polynomials in (i) and (ii) may be given, alternatively, as

$$x\left[1-\left((y^2-Dx^2)^2-16\right)^2\right], \text{for } D = P^2+4,$$

and

$$x\left[1-\left((y^2-Dx^2)-4\right)^2\right], \text{ for } D = P^2-4,$$

respectively. And, by Corollaries 2 and 3, the polynomials in (iii) and (iv) may be given, alternatively, if D is square-free, as

$$x\left[1-\left((y^2-Pyx-x^2)^2-(P^2+4)^2\right)^2\right]$$

and

$$x\left[1-(y^2-Pxy+x^2+P^2-4)^2\right],$$

respectively; however, in case (i) of the theorem, the degree of the alternative is higher.

For a summary of results on polynomials representing various additional sets, we refer the reader to [11, Ch. 3, III].

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NEW EDITORIAL POLICIES

The Board of Directors of The Fibonacci Association during their last business meeting voted to incorporate the following two editorial policies effective January 1, 1995.

1. All articles submitted for publication in *The Fibonacci Quarterly* will be blind refereed.
2. In place of Assistant Editors, *The Fibonacci Quarterly* will change to utilization of an Editorial Board.

SOME SUMMATION IDENTITIES USING GENERALIZED Q-MATRICES

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(*Submitted June 1993*)

1. INTRODUCTION

In a belated acknowledgment, Hoggatt [3] states:

> The first use of the Q-matrix to generate the Fibonacci numbers appears in an abstract of a paper by Professor J. L. Brenner by the title "Lucas' Matrix." This abstract appeared in the March 1951 *American Mathematical Monthly* on pages 221 and 222. The basic exploitation of the Q-matrix appeared in 1960 in the San Jose State College Master's thesis of Charles H. King with the title "Some Further Properties of the Fibonacci Numbers." Further utilization of the Q-matrix appears in the *Fibonacci Primer* sequence parts I-V.

For a comprehensive history of the Q-matrix, see Gould [2]. Numerous analogs of the Q-matrix relating to third-order recurrences have been used. See, for instance, Waddill and Sacks [13], Shannon and Horadam [10], and Waddill [11]. Mahon [8] has made extensive use of matrices to study his third-order diagonal functions of the Pell polynomials. Recently, Waddill [12] considered a general Q-matrix. He defined and used the $k \times k$ matrix

$$R = \begin{pmatrix} r_0 & r_1 & \cdots & r_{k-1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots 1 & 0 \end{pmatrix}$$

in relation to a k-order linear recursive sequence $\{V_n\}$, where

$$V_n = \sum_{i=0}^{k-1} r_i V_{n-1-i}, \quad n \geq k.$$

The matrix R generalized the matrix Q_r of Ivie [5].

In the notation of Horadam [4], write

$$W_n = W_n(a, b; p, q) \tag{1.1}$$

so that

$$W_n = pW_{n-1} - qW_{n-2}, \; W_0 = a, \; W_1 = b, \quad n \geq 2. \tag{1.2}$$

With this notation, define

$$\begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p; p, q). \end{cases} \tag{1.3}$$

Indeed, $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences generated by (1.2). They have been studied extensively, particularly by Lucas [7]. Further information can be found in [1], [4], and [6].

The most commonly used matrix in relation to the recurrence relation (1.2) is

$$M=\begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}, \tag{1.4}$$

which, for $p=-q=1$, reduces to the ordinary Q-matrix. In this paper we define a more general matrix $M_{k,m}$ parametrized by k and m and reducing to M for $k=m=1$. We use $M_{k,m}$ to develop various summation identities involving terms from the sequences $\{U_n\}$ and $\{V_n\}$.

Our work is a generalization of the work of Mahon and Horadam [9] who used several pairs of 2×2 matrices to generate summation identities involving terms from the Pell polynomial sequences

$$\begin{cases} P_n = W_n(0, 1; 2x, -1), \\ Q_n = W_n(2, 2x; 2x, -1). \end{cases} \tag{1.5}$$

We generalize their work in two ways. First, we consider sequences generated by a more general recurrence relation. Second, our parametrization of the matrix $M_{k,m}$ includes all the matrices considered by Mahon and Horadam as special cases.

2. THE MATRIX $M_{k,m}$

Before proceeding, we state some results which are used subsequently. None of these is new and each can be proved using Binet forms. If

$$\Delta = p^2-4q, \tag{2.1}$$

then

$$U_{n+1}-qU_{n-1}=V_n, \tag{2.2}$$

$$V_{n+1}-qV_{n-1}=\Delta U_n, \tag{2.3}$$

$$V_{2k}-2q^k=\Delta U_k^2, \tag{2.4}$$

$$U_{k+m}-q^mU_{k-m}=U_mV_k, \tag{2.5}$$

$$V_{k+m}-q^mV_{k-m}=\Delta U_kU_m \tag{2.6}$$

$$U_{k+m}U_{k-m}-U_k^2=-q^{k-m}U_m^2, \tag{2.7}$$

$$V_{k+m}V_{k-m}-V_k^2=\Delta q^{k-m}U_m^2, \tag{2.8}$$

$$U_{n+m}U_{n_1+m}-q^mU_nU_{n_1}=U_mU_{n+n_1+m}. \tag{2.9}$$

By induction it can be proved that, for the matrix M in (1.4),

$$M^n=\begin{pmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{pmatrix}, \tag{2.10}$$

where n is an integer.

We now give a generalization of the matrix M. Associated with the recurrence (1.2) and with $\{U_n\}$ as in (1.3), define

$$M_{k,m} = \begin{pmatrix} U_{k+m} & -q^m U_k \\ U_k & -q^m U_{k-m} \end{pmatrix}, \tag{2.11}$$

where k and m are integers. By induction and making use of (2.9), it can be shown that, for all integral n,

$$M_{k,m}^n = U_m^{n-1} \begin{pmatrix} U_{nk+m} & -q^m U_{nk} \\ U_{nk} & -q^m U_{nk-m} \end{pmatrix}. \tag{2.12}$$

When $k = m = 1$, we see that $M_{k,m}$ reduces to M and $M_{k,m}^n$ reduces to M^n.

3. SUMMATION IDENTITIES

We now use the matrix $M_{k,m}$ to produce summation identities involving terms from $\{U_n\}$ and $\{V_n\}$. Using (2.5) and (2.7), we find that the characteristic equation of $M_{k,m}$ is

$$\lambda^2 - U_m V_k \lambda + q^k U_m^2 = 0 \tag{3.1}$$

and, by the Cayley-Hamilton theorem,

$$M_{k,m}^2 - U_m V_k M_{k,m} + q^k U_m^2 I = 0, \tag{3.2}$$

where I is the 2×2 unit matrix. From (3.2), we have

$$(U_m V_k M_{k,m} - q^k U_m^2 I)^n M_{k,m}^j = M_{k,m}^{2n+j}, \tag{3.3}$$

and expanding yields

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} q^{k(n-i)} U_m^{2n-i} V_k^i M_{k,m}^{i+j} = M_{k,m}^{2n+j}. \tag{3.4}$$

Using (2.12) to equate upper left entries gives

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} q^{k(n-i)} V_k^i U_{(i+j)k+m} = U_{(2n+j)k+m}. \tag{3.5}$$

Again from (3.2),

$$(M_{k,m}^2 + q^k U_m^2 I)^n = U_m^n V_k^n M_{k,m}^n, \tag{3.6}$$

and expanding we have

$$\sum_{i=0}^{n} \binom{n}{i} q^{k(n-i)} U_m^{2(n-i)} M_{k,m}^{2i} = U_m^n V_k^n M_{k,m}^n. \tag{3.7}$$

Using (2.12) to equate upper left entries gives

$$\sum_{i=0}^{n} \binom{n}{i} q^{k(n-i)} U_{2ik+m} = V_k^n U_{nk+m}. \tag{3.8}$$

Once again, from (3.2),

$$(M_{2k,m}-q^kU_mI)^2=U_m(V_{2k}-2q^k)M_{2k,m}=\Delta U_mU_k^2M_{2k,m}, \tag{3.9}$$

and expanding, after taking n^{th} powers, we have

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i q^{k(2n-i)}U_m^{2n-i}M_{2k,m}^i=\Delta^nU_m^nU_k^{2n}M_{2k,m}^n. \tag{3.10}$$

Equating upper left entries yields

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i q^{k(2n-i)}U_{2ik+m}=\Delta^nU_k^{2n}U_{2nk+m}. \tag{3.11}$$

From (3.9),

$$(M_{2k,m}-q^kU_mI)^{2n+1}=\Delta^nU_m^nU_k^{2n}(M_{2k,m}^{n+1}-q^kU_mM_{2k,m}^n). \tag{3.12}$$

Equating upper left entries yields, after simplifying,

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1}q^{k(2n+1-i)}U_{2ik+m}=\Delta^nU_k^{2n}(U_{2(n+1)k+m}-q^kU_{2nk+m}), \tag{3.13}$$

and using (2.5) to simplify the right side gives

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1}q^{k(2n+1-i)}U_{2ik+m}=\Delta^nU_k^{2n+1}V_{(2n+1)k+m}. \tag{3.14}$$

This should be compared to (3.11).

Manipulating the characteristic equation (3.1), we have $(2\lambda-U_mV_k)^2=\Delta U_m^2V_k^2$, so that

$$(2M_{k,m}-U_mV_kI)^{2n}=\Delta^nU_m^{2n}U_k^{2n}I. \tag{3.15}$$

Expanding gives

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i2^iU_m^{2n-i}V_k^{2n-i}M_{k,m}^i=\Delta^nU_m^{2n}U_k^{2n}I. \tag{3.16}$$

Equating upper left entries and also lower left entries yields, respectively,

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i2^iV_k^{2n-i}U_{ik+m}=\Delta^nU_k^{2n}U_m, \tag{3.17}$$

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i2^iV_k^{2n-i}U_{ik}=0. \tag{3.18}$$

We note that (3.17) reduces to (3.18) when $m=0$.

Multiplying both sides of (3.15) by $(2M_{k,m}-U_mV_kI)$ and expanding gives

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1}2^iU_m^{2n+1-i}V_k^{2n+1-i}M_{k,m}^i=\Delta^nU_m^{2n}U_k^{2n}(2M_{k,m}-U_mV_kI). \tag{3.19}$$

Equating upper left entries yields

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1}2^i V_k^{2n+1-i}U_{ik+m} = \Delta^n U_k^{2n+1}V_m, \tag{3.20}$$

which should be compared to (3.17).

Now, using (3.5), we have

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}q^{k(n-i)}V_k^i(U_{(i+j)k+m+1}-qU_{(i+j)k+m-1}) = U_{(2n+j)k+m+1}-qU_{(2n+j)k+m-1},$$

and (2.2) shows that this simplifies to

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}q^{k(n-i)}V_k^iV_{(i+j)k+m} = V_{(2n+j)k+m}. \tag{3.21}$$

Making use of (2.2) and (2.3) and working in the same manner with identities (3.8), (3.11), (3.14), (3.17), and (3.20) yields, respectively,

$$\sum_{i=0}^{n}\binom{n}{i}q^{k(n-i)}V_{2ik+m} = V_k^nV_{nk+m}, \tag{3.22}$$

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i q^{k(2n-i)}V_{2ik+m} = \Delta^n U_k^{2n}V_{2nk+m}, \tag{3.23}$$

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1}q^{k(2n+1-i)}V_{2ik+m} = \Delta^{n+1}U_k^{2n+1}U_{(2n+1)k+m}, \tag{3.24}$$

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i 2^i V_k^{2n-i}V_{ik+m} = \Delta^n U_k^{2n}V_m, \tag{3.25}$$

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1}2^i V_k^{2n+1-i}V_{ik+m} = \Delta^{n+1}U_k^{2n+1}U_m. \tag{3.26}$$

In what follows, we make use of the following result:

$$M_{k,m}^n M_{k_1,m}^{n_1} = U_m^{n+n_1-1}\begin{pmatrix} U_{nk+n_1k_1+m} & -q^mU_{nk+n_1k_1} \\ U_{nk+n_1k_1} & -q^mU_{nk+n_1k_1-m}\end{pmatrix}. \tag{3.27}$$

This is proved by multiplying the matrices on the left and using (2.9).

Consider now the special case of (3.2), where $k=m$. Then, using (2.5),

$$M_{k,k}^2 = U_{2k}M_{k,k} - q^kU_k^2I. \tag{3.28}$$

Using (3.28) and (2.9), we can show by induction that, for $n\geq 2$,

$$M_{k,k}^n = U_k^{n-2}(U_{nk}M_{k,k} - q^kU_kU_{(n-1)k}I). \tag{3.29}$$

The binomial theorem applied to (3.29) gives

$$U_k^{(n-2)s}\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}q^{k(s-i)}U_k^{s-i}U_{(n-1)k}^{s-i}U_{nk}^{i}M_{k,k}^{i+j}=M_{k,k}^{ns+j} \tag{3.30}$$

Equating lower left entries of the relevant matrices then yields

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}q^{k(s-i)}U_{(n-1)k}^{s-i}U_{nk}^{i}U_{(i+j)k}=U_k^sU_{(ns+j)k}. \tag{3.31}$$

Multiplying both sides of (3.30) by $M_{k_1,k}$ and using (3.27) to equate lower left entries gives

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}q^{k(s-i)}U_{(n-1)k}^{s-i}U_{nk}^{i}U_{(i+j)k+k_1}=U_k^sU_{(ns+j)k+k_1}, \tag{3.32}$$

which generalizes (3.31).

Again from (3.29), after transposing terms and raising to a power s, we obtain

$$\sum_{i=0}^{s}\binom{s}{i}q^{k(s-i)}U_k^{(n-1)(s-i)}U_{(n-1)k}^{s-i}M_{k,k}^{ni}=U_k^{(n-2)s}U_{nk}^{s}M_{k,k}^{s}, \tag{3.33}$$

which yields

$$\sum_{i=0}^{s}\binom{s}{i}q^{k(s-i)}U_k^iU_{(n-1)k}^{s-i}U_{nik}=U_{nk}^sU_{sk}. \tag{3.34}$$

Multiplying both sides of (3.33) by $M_{k_1,k}$ and using (3.27) to equate lower left entries gives

$$\sum_{i=0}^{s}\binom{s}{i}q^{k(s-i)}U_k^iU_{(n-1)k}^{s-i}U_{nik+k_1}=U_{nk}^sU_{sk+k_1}, \tag{3.35}$$

which generalizes (3.34).

Continuing in this manner after yet again transposing terms in (3.29) and raising to a power s, we obtain

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^iU_k^{(n-2)(s-i)}U_{nk}^{s-i}M_{k,k}^{(n-1)i+s}=q^{ks}U_k^{(n-1)s}U_{(n-1)k}^{s}I. \tag{3.36}$$

Equating upper left entries and lower left entries yields, respectively,

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^iU_k^iU_{nk}^{s-i}U_{((n-1)i+s+1)k}=q^{ks}U_{(n-1)k}^{s}U_k, \tag{3.37}$$

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^iU_k^iU_{nk}^{s-i}U_{((n-1)i+s)k}=0. \tag{3.38}$$

Multiplying (3.36) by $M_{k_1,k}$ and equating lower left entries yields

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^iU_k^iU_{nk}^{s-i}U_{((n-1)i+s)k+k_1}=q^{ks}U_{(n-1)k}^{s}U_{k_1}. \tag{3.39}$$

We note that, when $k_1 = k$, (3.39) reduces to (3.37) and when $k_1 = 0$, (3.39) reduces to (3.38).

Now, manipulating (3.32), (3.35), and (3.39) in the same way that (3.5) was manipulated to yield (3.21), we obtain, respectively,

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}q^{k(s-i)}U_{(n-1)k}^{s-i}U_{nk}^{i}V_{(i+j)k+k_1} = U_k^s V_{(ns+j)k+k_1}, \tag{3.40}$$

$$\sum_{i=0}^{s}\binom{s}{i}q^{k(s-i)}U_k^i U_{(n-1)k}^{s-i}V_{nik+k_1} = U_{nk}^s V_{sk+k_1}, \tag{3.41}$$

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^i U_k^i U_{nk}^{s-i}V_{((n-1)i+s)k+k_1} = q^{ks}U_{(n-1)k}^s V_{k_1}. \tag{3.42}$$

4. THE MATRIX X_k

We have found a matrix having the property of generating terms from $\{U_n\}$ and $\{V_n\}$ simultaneously. It is a generalization of the matrix W introduced by Mahon and Horadam [9]. Define

$$X_k = \begin{pmatrix} V_k & U_k \\ \Delta U_k & V_k \end{pmatrix}, \ k \text{ an integer.} \tag{4.1}$$

Then by induction we have, for integral n,

$$X_k^n = 2^{n-1}\begin{pmatrix} V_{nk} & U_{nk} \\ \Delta U_{nk} & V_{nk} \end{pmatrix}. \tag{4.2}$$

Noting that $X_1^{m+n} = X_1^m \cdot X_1^n$ produces the well-known identities

$$2V_{m+n} = V_m V_n + \Delta U_m U_n, \tag{4.3}$$

$$2U_{m+n} = V_m U_n + U_m V_n. \tag{4.4}$$

The characteristic equation for X_k is

$$\lambda^2 - 2V_k\lambda + 4q^k = 0 \tag{4.5}$$

and so, by the Cayley-Hamilton theorem

$$X_k^2 - 2V_k X_k + 4q^k I = 0. \tag{4.6}$$

Using (4.3) and (4.4), we see that

$$X_k^n X_{k_1} = 2^n \begin{pmatrix} V_{nk+k_1} & U_{nk+k_1} \\ \Delta U_{nk+k_1} & V_{nk+k_1} \end{pmatrix}. \tag{4.7}$$

Considering the case $k = 1$, we can show by induction, with the aid of (4.6), that

$$X_1^n = 2^{n-1}(U_n X_1 - 2qU_{n-1}I), \ \ n \geq 2, \tag{4.8}$$

which is analogous to (3.29).

It is interesting to note that the methods applied to $M_{k,m}$ when applied to X_k produce most of the summation identities that we have obtained so far. The exceptions are the identities that arose by using (3.29). The analogous procedure for X_k is to use (4.8), but the identities that arise are less general. For example, (4.8) produces

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}q^{s-i}U_{n-1}^{s-i}U_n^iU_{i+j+k_1}=U_{ns+j+k_1}, \tag{4.9}$$

which is a special case of (3.32).

5. THE MATRIX $N_{k,m}$

We have found yet another matrix defined in a similar manner to $M_{k,m}$ whose powers also generate terms of the sequences $\{U_n\}$ and $\{V_n\}$. Define

$$N_{k,m}=\begin{pmatrix} V_{k+m} & -q^mV_k \\ V_k & -q^mV_{k-m}\end{pmatrix}. \tag{5.1}$$

Then for all integral n,

$$N_{k,m}^{2n}=U_m^{2n-1}\Delta^n\begin{pmatrix} U_{2nk+m} & -q^mU_{2nk} \\ U_{2nk} & -q^mU_{2nk-m}\end{pmatrix}, \tag{5.2}$$

$$N_{k,m}^{2n-1}=U_m^{2n-2}\Delta^{n-1}\begin{pmatrix} V_{(2n-1)k+m} & -q^mV_{(2n-1)k} \\ V_{(2n-1)k} & -q^mV_{(2n-1)k-m}\end{pmatrix}. \tag{5.3}$$

The characteristic equation of $N_{k,m}$ is

$$\lambda^2-\Delta U_kU_m\lambda-\Delta q^kU_m^2=0, \tag{5.4}$$

and so

$$N_{k,m}^2-\Delta U_kU_mN_{k,m}-\Delta q^kU_m^2I=0. \tag{5.5}$$

Using the previous techniques and due to the manner in which powers of $N_{k,m}$ are defined, we have found some interesting summation identities. We note, however, that some of the methods applied to $M_{k,m}$ do not apply to $N_{k,m}$. For example, we could find no succinct counterpart to (3.29). We state only the essential details and omit summation identities that we have obtained previously.

Manipulating (5.5), we can write

$$\Delta U_m(U_kN_{k,m}+q^kU_mI)=N_{k,m}^2 \tag{5.6}$$

and

$$(2N_{k,m}-\Delta U_kU_mI)^2=\Delta U_m^2V_k^2I. \tag{5.7}$$

From (5.6) and (5.7), we have

$$\Delta^nU_m^n(U_kN_{k,m}+q^kU_mI)^n=N_{k,m}^{2n}, \tag{5.8}$$

$$(2N_{k,m} - \Delta U_k U_m I)^{2n} = \Delta^n U_m^{2n} V_k^{2n} I, \tag{5.9}$$

$$(2N_{k,m} - \Delta U_k U_m I)^{2n+1} = \Delta^n U_m^{2n} V_k^{2n} (2N_{k,m} - \Delta U_k U_m I). \tag{5.10}$$

Now expanding each of (5.8)-(5.10) and equating upper left entries of the relevant matrices leads, respectively, to

$$\sum_{\substack{i=0 \\ i \text{ even}}}^{n} \binom{n}{i} q^{k(n-i)} \Delta^{\frac{i}{2}} U_k^i U_{ik+m} + \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \binom{n}{i} q^{k(n-i)} \Delta^{\frac{i-1}{2}} U_k^i V_{ik+m} = U_{2nk+m}, \tag{5.11}$$

$$\sum_{\substack{i=0 \\ i \text{ even}}}^{2n} \binom{2n}{i} 2^i \Delta^{\frac{2n-i}{2}} U_k^{2n-i} U_{ik+m} - \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-1} \binom{2n}{i} 2^i \Delta^{\frac{2n-1-i}{2}} U_k^{2n-i} V_{ik+m} = V_k^{2n} U_m, \tag{5.12}$$

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{2n+1} \binom{2n+1}{i} 2^i \Delta^{\frac{2n+1-i}{2}} U_k^{2n+1-i} V_{ik+m} - \sum_{\substack{i=0 \\ i \text{ even}}}^{2n} \binom{2n+1}{i} 2^i \Delta^{\frac{2n+2-i}{2}} U_k^{2n+1-i} U_{ik+m} = V_k^{2n+1} V_m. \tag{5.13}$$

Finally, making use of (2.2) and (2.3) and applying to (5.11)-(5.13) the same technique used to obtain (3.21), we have

$$\sum_{\substack{i=0 \\ i \text{ even}}}^{n} \binom{n}{i} q^{k(n-i)} \Delta^{\frac{i}{2}} U_k^i V_{ik+m} + \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \binom{n}{i} q^{k(n-i)} \Delta^{\frac{i+1}{2}} U_k^i U_{ik+m} = V_{2nk+m}, \tag{5.14}$$

$$\sum_{\substack{i=0 \\ i \text{ even}}}^{2n} \binom{2n}{i} 2^i \Delta^{\frac{2n-i}{2}} U_k^{2n-i} V_{ik+m} - \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-1} \binom{2n}{i} 2^i \Delta^{\frac{2n+1-i}{2}} U_k^{2n-i} U_{ik+m} = V_k^{2n} V_m, \tag{5.15}$$

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{2n+1} \binom{2n+1}{i} 2^i \Delta^{\frac{2n+3-i}{2}} U_k^{2n+1-i} U_{ik+m} - \sum_{\substack{i=0 \\ i \text{ even}}}^{2n} \binom{2n+1}{i} 2^i \Delta^{\frac{2n+2-i}{2}} U_k^{2n+1-i} V_{ik+m} = \Delta V_k^{2n+1} U_m. \tag{5.16}$$

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FASTER MULTIPLICATION OF MEDIUM LARGE NUMBERS VIA THE ZECKENDORF REPRESENTATION

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INTRODUCTION

Multiplication of two integers is a fundamental computational problem. Various authors have found nearly linear-time algorithms for integer multiplication; the best such result is that of Schonhage and Strassen (in [1]), who showed that the product of two n-bit numbers may be computed in $O(n \log n \log\log n)$ steps. Their algorithm involves a recursive application of the Fast Fourier Transform (FFT) and is quite intricate. However, even the simpler multiplication algorithms based on the FFT are not used in practice, unless enormous numbers are involved.

Another multiplication method, published by Karatsuba and Ofman (in [1]) uses $O(n^{1.585})$ operations and outperforms classical multiplication when n exceeds 1200 (i.e., about 360 decimal digits).

In 1972 Zeckendorf [8] introduced a representation of the integers as a sum of generalized Fibonacci numbers defined by the relation

$$\begin{aligned} &F_0^{(r)} = 0,\ F_1^{(r)} = 1,\ F_j^{(r)} = 2^{j-2}, && j = 2, 3, \ldots, r-1, \\ &F_i^{(r)} = F_{i-1}^{(r)} + F_{i-2}^{(r)} + \cdots + F_{i-r}^{(r)}, && i \ge r. \end{aligned} \tag{1}$$

The Fibonacci, Tribonacci [5], [7], and Quadranacci [6] numbers arise as a special case of (1) by letting $r = 2$, $r = 3$, and $r = 4$, respectively. Capocelli [3] gives an efficient algorithm for deriving the Zeckendorf representation of integers.

This paper compares the classical multiplication, Karatsuba-Ofman, and Schonhage-Strassen algorithms and multiplication with the Zeckendorf representation, and shows that medium sized numbers can be multiplied (on average) more quickly using the Zeckendorf Quadranacci representation.

ZECKENDORF REPRESENTATION OF THE INTEGERS

Recently, the Zeckendorf representation of the integers has been shown to be a useful alternative to the binary representation. Each nonnegative integer N has the following unique Zeckendorf representation in terms of Fibonacci numbers of degree r (see [7], [8]):

$$N = \alpha_2 F_2^{(r)} + \alpha_3 F_3^{(r)} + \cdots + \alpha_j F_j^{(r)}, \tag{2}$$

where $\alpha_i \in \{0, 1\}$ and $\alpha_i \alpha_{i-1} \alpha_{i-2} \alpha_{i-3} \ldots \alpha_{i-r+1} = 0$ (no r consecutive α's are 1).

Like the binary representation of integers, the Zeckendorf representation can be written as a string of 0's and 1's, i.e., $\alpha_j \alpha_{j-1} \alpha_{j-2} \ldots \alpha_3 \alpha_2$.

As was proved by Borel [2], almost all numbers have an equal number of zeros and ones in their standard binary representation. More generally we have that, if g is an integer greater than one, then

$$t = \frac{w_1(t)}{g} + \frac{w_2(t)}{g^2} + \cdots, \quad 0 \le t \le 1,$$

where every digit $w_i(t)$ is in $\{0, 1, \ldots, g-1\}$. Borel's theorem states: For almost all t ($0 \le t \le 1$),

$$\lim_{n\to\infty} \frac{F_n^{(k)}(t)}{n} = \frac{1}{g},$$

where $F_n^{(k)}$ denotes the number of those w from the first n, which are equal to k, $0 \le k \le g-1$.

Such a property is true for the binary representation of integers, that is, the proportions of 0's in strings of length n and the proportion of 1's in strings of length n are both equal to 1/2. In the Zeckendorf representation, this rule does not hold. From [4], we have the following result on the asymptotic proportion of ones.

Theorem 1: The proportion of 1's in the Zeckendorf representation of integers is

$$A_n^{(r)} = \frac{1}{\omega^{(r)}} - \frac{r}{(\omega^{(r)})^{r+1}} \cdot \frac{\omega^{(r)} - 1}{[(r+1)\omega^{(r)} - 2r]} + O(1/n), \tag{3}$$

which tends to 1/2 as r increases. $\omega^{(r)}$ is a real root of the equation

$$x^r - x^{r-1} - \cdots - 1 = 0.$$

This root lies between 1 and 2.

In Table 1 some values for $A_\infty^{(r)}$ are presented (see [4]).

TABLE 1. Asymptotic Values of $A_\infty^{(r)}$

r	2	3	4	5	6	7	8
$A_\infty^{(r)}$	0.2764	0.3816	0.4337	0.4621	0.4782	0.4875	0.4929

The roots $\omega^{(r)}$ form a strictly increasing sequence. That is,

$$1.618\ldots < \omega^{(2)} < \omega^{(3)} < \cdots < 2.$$

Zeckendorf representation of integers requires more space than the binary representation, see Table 2.

TABLE 2. Zeckendorf Space/Binary Space

r	2	3	4	5	6	7
$\log_{\omega^{(r)}} 2$	1.44	1.13	1.05	1.02	1.01	1.005

Let us fix the dynamic range of the input data to be n-bits in the binary number system (BNS). The number of one's for n-bit BNS numbers in the Zeckendorf representation will have an average at

$$N_{\text{ones}}^{(r)} = \log_{\omega^{(r)}} 2 \cdot A_{\infty}^{(r)} \cdot n.$$

The initial values of the function $Z(r) = N_{\text{ones}}^{(r)} / n$ are printed in Table 3.

TABLE 3. Average Proportion of One's for *n*-bit BNS Numbers in the Proposed Number System

r	2	3	4	5	6	7	8	9
$Z(r)$	0.398	0.434	0.458	0.474	0.484	0.494	0.497	0.499

It is clear that the representation using classical Fibonacci numbers requires 20% fewer 1's in comparison with BNS, which can be employed in many practical situations.

MULTIPLICATION OF THE NUMBERS IN ZECKENDORF REPRESENTATION

Let us consider the multiplication of two integers having a Zeckendorf representation. The multiplier may have only $A_{\infty}^{(r)}$ of its digits equal to 1, but it has $\log_{\omega^{(r)}} 2$ more digits. Hence, multiplication using Zeckendorf representation involves $A_{\infty}^{(r)} \cdot \log_{\omega^{(r)}} 2$ more additions than in the BNS case. Therefore, there are $A_{\infty}^{(r)} \cdot (\log_{\omega^{(r)}} 2)^2$ times as many digit operations. Because the final result may have more than r consecutive ones, it must be transformed into normal form. That is, every string ...01...10... must be replaced by 10...0. This transformation can be accomplished in $2 \cdot \log_{\omega^{(r)}} 2 \cdot n$ steps. Hence, using the Zeckendorf representation will require, on average,

$$S_n^{(r)} = \log_{\omega^{(r)}}^2 2 \cdot A_{\infty}^{(r)} \cdot n^2 + 2 \cdot \log_{\omega^{(r)}} 2 \cdot n \approx H(r) \cdot n^2$$

bit operations to perform multiplication, if the classical algorithm is used. In Table 4 the initial values for the function $H(r) = \log_{\omega^{(r)}}^2 2 \cdot A_{\infty}^{(r)}$ are tabulated.

TABLE 4. Initial Values of the Function *H*(*r*)

r	2	3	4	5	6	7	8	9
$H(r)$	0.574	0.494	0.484	0.486	0.490	0.493	0.497	0.499

$H(r)$ attains its minimum when $r = 4$. Thus, the Quadranacci number system seems to be faster than other generalized Fibonacci number systems and faster than the BNS from a multiplicative complexity point of view.

If the time for transformation to normal form was included, it was computed that Quadranacci multiplication outperformed binary multiplication when the number of bits exceeded 130 (about 43 decimal digits). The last conclusion follows from the solution of the inequality

$S_n^{(4)} < 0 \cdot 5n^2$. In Table 5 we printed the values for the dynamic range and the corresponding fastest algorithm for multiplication.

TABLE 5. Comparison among Different Algorithms for Multiplication

Range (bits)	0-130	131-1200	1201-4096	4097-∞
Algorithm	Standard	Zeckendorf	Karatsuba-Ofman	Schonhage-Strassen

CONCLUSIONS

A comparison between well-known algorithms for standard binary multiplication and multiplication using the Zeckendorf representation has been considered. It was shown that some of the proposed number systems (Fibonacci, Tribonacci, Quadranacci) possess advantages for performing multiplication. The hybrid between the classical multiplication algorithm and the above non-standard number systems can be used for fast multiplication of medium large integers.

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QUADRATIC RECIPROCITY VIA LUCAS SEQUENCES

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1. INTRODUCTION

Given $\lambda, \mu \in \mathbb{Z}$, the associated Lucas sequence $\{\gamma_n\}_{n\geq 0}$ is defined by the binary linear recurrence

$$\gamma_0 = 0, \; \gamma_1 = 1, \; \text{and} \; \gamma_{n+1} = \lambda\gamma_n + \mu\gamma_{n-1} \; \text{for} \; n > 0. \tag{1.1}$$

In this article we will show how these sequences may be used to give new proofs of the quadratic reciprocity theorem. It is well known that these sequences have the ordinary formal power series generating functions

$$P(t)^{-1} = \sum_{n=1}^{\infty} \gamma_n t^{n-1}, \tag{1.2}$$

where $P(t) = 1 - \lambda t - \mu t^2$. The reciprocity law follows from certain integrality relations in the formal power series ring $\mathbb{Q}[[t]]$ between these generating functions and a generating function for the quadratic character modulo the discriminant of $P(t)$. The only other tools needed are the elementary properties of quadratic Gauss sums.

2. LUCAS SEQUENCES AND THE LEGENDRE SYMBOL

The following formal power series identity expresses an interesting relation between the sequences $\{\gamma_n\}$ and the Legendre symbol $(n|q)$, where $|q|$ is the discriminant of $P(t)$.

Theorem: Let q be an odd prime and set $D = (-1|q)q$. Choose any integers λ, μ such that $\lambda^2 + 4\mu = D$, and define the sequence $\{\gamma_n\}$ by the recursion (1.1). Then there is a unique formal power series ϕ with integer coefficients and constant term zero such that

$$\sum_{n=1}^{\infty} \gamma_n \frac{\phi(t)^n}{n} = \sum_{n=1}^{\infty} \left(\frac{n}{q}\right) \frac{t^n}{n} \tag{2.1}$$

holds as an equality of formal power series.

Proof: Let ζ be any fixed primitive q^{th} root of unity. We define the quadratic Gauss sums $\tau(n)$ modulo q by

$$\tau(n) = \sum_{a=1}^{q-1} \left(\frac{a}{q}\right) \zeta^{na}. \tag{2.2}$$

It is an elementary property of these sums ([1], Theorem 9.13) that $\tau(1)^2 = D$ and, therefore, $\tau(1)$ is a square root of D. Hereafter, we dispense with the ambiguity in sign and simply *define* $\sqrt{D}$ to be $\tau(1)$. Now, since $\sum_{a=0}^{q-1} \zeta^a = 0$, we have

$$\frac{1}{2} \sum_{a=1}^{q-1} \left(\left(\frac{a}{q}\right) - 1\right) \zeta^a = \frac{1+\sqrt{D}}{2}, \tag{2.3}$$

which shows that $(1+\sqrt{D})/2$ lies in the ring $\mathbb{Z}[\zeta]$ and, therefore, $\mathcal{O}_D = \mathbb{Z}[(1+\sqrt{D})/2] \subseteq \mathbb{Z}[\zeta]$. We also recall the separability property $\tau(n) = (n|q)\sqrt{D}$ for every integer n ([1], p. 192, eq. (17)).

Define the rational function f by

$$f(t) = \prod_{a=1}^{q-1} (1-\zeta^a t)^{-(a|q)}. \tag{2.4}$$

It is readily seen that as a formal power series in t, the coefficients of f lie in $\mathbb{Z}[\zeta]$. Set $P(t) = 1-\lambda t - \mu t^2 = (1-\alpha t)(1-\beta t)$, where the reciprocal roots α, β are chosen so that $\alpha - \beta = \sqrt{D}$. Then define the rational function ϕ by

$$\phi(t) = \frac{f(t)-1}{\alpha f(t) - \beta}. \tag{2.5}$$

We claim this function ϕ, as a formal power series in t, satisfies the conditions of the theorem.

We first show that ϕ satisfies the equality (2.1). We compute that as formal power series,

$$\begin{aligned} \log f(t) &= \log\left(\prod_{a=1}^{q-1}(1-\zeta^a t)^{-(a|q)}\right) = -\sum_{a=1}^{q-1}\left(\frac{a}{q}\right)\log(1-\zeta^a t) \\ &= \sum_{a=1}^{q-1}\left(\frac{a}{q}\right)\sum_{n=1}^{\infty}\zeta^{an}\frac{t^n}{n} = \sum_{n=1}^{\infty}\tau(n)\frac{t^n}{n} = \sqrt{D}\sum_{n=1}^{\infty}\left(\frac{n}{q}\right)\frac{t^n}{n}. \end{aligned} \tag{2.6}$$

On the other hand, solving (2.5) for f yields

$$f(t) = \frac{1-\beta\phi(t)}{1-\alpha\phi(t)}. \tag{2.7}$$

Since $f(0) = 1$, we have $\phi(0) = 0$; therefore, we may also compute that as formal power series,

$$\begin{aligned} \log f(t) &= \log\left(\frac{1-\beta\phi(t)}{1-\alpha\phi(t)}\right) = \log(1-\beta\phi(t)) - \log(1-\alpha\phi(t)) \\ &= \sum_{n=1}^{\infty}(\alpha^n - \beta^n)\frac{\phi(t)^n}{n} = \sqrt{D}\sum_{n=1}^{\infty}\gamma_n\frac{\phi(t)^n}{n}, \end{aligned} \tag{2.8}$$

using the well-known Binet formula

$$\gamma_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{2.9}$$

(Note that expressions such as $\sum \gamma_n \phi^n / n$ make sense as formal power series in t, since the constant term of ϕ is zero.) Now, comparing the two expressions (2.6) and (2.8) shows that ϕ satisfies (2.1).

Turning now to the coefficients of ϕ, we write $\phi(t) = \sum_{n=1}^{\infty} a_n t^n$. Equating coefficients of t in (2.1) yields $a_1 = 1$; equating coefficients of t^n yields a recursion for a_n in terms of $a_1, \ldots, a_{n-1}$, demonstrating the uniqueness of ϕ. We first show that the coefficients of ϕ are rational: Suppose not, and let k be minimal such that $a_k \notin \mathbb{Q}$. For $1 \le j \le k$, let b_j denote the coefficient of t^k in $\phi(t)^j$; then $b_1 = a_k \notin \mathbb{Q}$, while $b_j \in \mathbb{Q}$ for $1 < j \le k$. Equating coefficients of t^k in (2.1) yields

$$b_1 + \sum_{j=2}^{k} \gamma_j \frac{b_j}{j} = \left(\frac{k}{q}\right)\frac{1}{k}, \tag{2.10}$$

which is impossible, since $b_1 \notin \mathbb{Q}$ while all other terms in (2.10) lie in $\mathbb{Q}$.

Now we show that the coefficients of ϕ are integers: Suppose not, and let k be minimal such that $a_k \notin \mathbb{Z}$. Again let b_j denote the coefficient of t^k in $\phi(t)^j$ for $1 \le j \le k$; then $b_j \in \mathbb{Z}$ for $1 < j \le k$, while $b_1 = a_k = r/s$ for some coprime integers r, s with $|s| > 1$. Expanding (2.7) formally yields

$$f(t) = (1 - \beta\phi(t))\left(\sum_{n=0}^{\infty} \alpha^n \phi(t)^n\right) = 1 + \sqrt{D}\sum_{n=1}^{\infty} \alpha^{n-1}\phi(t)^n, \tag{2.11}$$

and therefore the coefficient of t^k in f is

$$\sqrt{D}(b_1 + \alpha b_2 + \cdots + \alpha^{k-1} b_k). \tag{2.12}$$

We know from (2.4) that this coefficient lies in $\mathbb{Z}[\zeta]$, and we observe that $\sqrt{D}(\alpha b_2 + \cdots + \alpha^{k-1} b_k)$ lies in the subring $\mathcal{O}_D$, since $\alpha = (\lambda + \sqrt{D})/2$. So we must have $b_1\sqrt{D} \in \mathbb{Z}[\zeta]$, and therefore $(b_1\sqrt{D})^2 = r^2 D / s^2 \in \mathbb{Z}[\zeta]$. This is a contradiction, since $r^2 D/s^2 \in \mathbb{Q} \setminus \mathbb{Z}$, whereas $\mathbb{Z}[\zeta] \cap \mathbb{Q} = \mathbb{Z}$. This proves the theorem, and in passing also shows via (2.12) that f has coefficients in $\mathcal{O}_D$.

3. THE LAW OF QUADRATIC RECIPROCITY

Theorem (Gauss): Let p and q be distinct odd primes, and set $D = (-1|q)q$. Then

$$\left(\frac{p}{q}\right) = \left(\frac{D}{p}\right). \tag{3.1}$$

Proof: Choose any integers λ, μ that satisfy $\lambda^2 + 4\mu = D$, and let $P(t) = 1 - \lambda t - \mu t^2 = (1 - \alpha t)(1 - \beta t)$ and ϕ be as in the above theorem. For $1 \le k \le p$, let b_k denote the coefficient of t^p in $\phi(t)^k$. Equating the coefficients of t^p in (2.1) yields

$$\frac{\gamma_p}{p} + \sum_{k=1}^{p-1} \gamma_k \frac{b_k}{k} = \frac{\left(\frac{p}{q}\right)}{p}, \tag{3.2}$$

so that

$$\left(\frac{p}{q}\right) - \gamma_p = p\sum_{k=1}^{p-1} \gamma_k \frac{b_k}{k}. \tag{3.3}$$

Therefore, the sum $\sum_{k=1}^{p-1} \gamma_k b_k / k$ lies in $(1/p)\mathbb{Z}$; but the least common denominator of the terms is relatively prime to p, since each γ_k and b_k lies in $\mathbb{Z}$. So this sum must be an integer; thus

$$\gamma_p \equiv \left(\frac{p}{q}\right) \pmod{p\mathbb{Z}}. \tag{3.4}$$

On the other hand, we may easily compute (cf. [5], Corollary 1(i) with $m = r = 1$)

$$\gamma_p = \frac{\alpha^p - \beta^p}{\alpha - \beta} \equiv \frac{(\alpha - \beta)^p}{\alpha - \beta} = (\sqrt{D})^{p-1} = D^{(p-1)/2} \equiv \left(\frac{D}{p}\right) \pmod{p\mathbb{Z}}; \tag{3.5}$$

the first congruence holds modulo $p\mathcal{O}_D$, but both members are integers, so it holds modulo $p\mathbb{Z}$. Thus, $(p|q) \equiv (D|p) \pmod{p}$, but both are ± 1, so they must be equal.

4. CONCLUDING REMARKS

The quadratic Gauss sums have played a role in many quadratic reciprocity proofs, reaching back to Gauss's sixth proof published in 1818 (cf. [3]). Although our approach has features in common with other proofs of the reciprocity law, it does exhibit an unusual flexibility by giving, for fixed p and q, an infinite family of proofs corresponding to the variety of choices for λ and μ.

In [5], we employed elementary p-adic methods to prove congruences relating the ratios $\gamma_{mp^r}/\gamma_{mp^{r-1}}$ to the Legendre symbol $(D|p)$. In the language of formal group laws, these congruences imply that the formal differential $\omega = P(t)^{-1}\,dt$ is the canonical invariant differential on a formal group law defined over $\mathbb{Z}$, which is isomorphic over $\mathbb{Z}$ to the formal group law attached to the Dirichlet L-series $L(s,\chi)$ for the Dirichlet character χ of conductor $|D|$ associated to the quadratic field $K = \mathbb{Q}(\sqrt{D})$. Formally differentiating both sides of (2.1) and using (1.2) gives

$$P(\phi)^{-1}d\phi = \sum_{n=1}^{\infty}\left(\frac{n}{q}\right)t^n\frac{dt}{t}, \tag{4.1}$$

which implies that the power series ϕ defined in §2 actually *is* the isomorphism between these two formal group laws; however, we have used no formal group techniques in the construction of ϕ. The above theorem says that the differential equation (4.1) has a rather surprising property, namely, that of possessing a solution $\phi(t)$ at $t=0$, which is a rational function whose Maclaurin series has integer coefficients. It may be interesting to know the coefficients of ϕ more explicitly.

The use of formal group techniques to prove reciprocity laws originated with T. Honda [2], who gave a proof of quadratic reciprocity using formal group laws and Gauss sums. However, Honda used a formal group law defined over $\mathcal{O}_D$ rather than over $\mathbb{Z}$, and used the Galois theory of the extension $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{D}) \subseteq \mathbb{Q}(\zeta)$ to prove $\mathcal{O}_D$-integrality, whereas the present argument requires no such techniques.

It does not appear that our method readily proves the auxiliary result $(2|q) = (-1)^{(q^2-1)/8}$, which amounts to a congruence for q modulo 8. But it is easy to determine from (2.1) that $a_2 = ((2|q)-\lambda)/2$, and one may also note that

$$\left(\frac{2}{q}\right) = 1 \Leftrightarrow q \equiv \pm 1 \pmod{8} \Leftrightarrow D \equiv 1 \pmod{8} \Leftrightarrow \mu \equiv 0 \pmod{2}. \tag{4.2}$$

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A GENERALIZATION OF THE CATALAN IDENTITY AND SOME CONSEQUENCES

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(*Submitted June 1993*)

1. INTRODUCTION

The Catalan identity

$$F_{n-r}F_{n+r}-F_n^2=(-1)^{n-r+1}F_r^2 \tag{1.1}$$

has several generalizations. Here we obtain a new generalization and use it to generalize the Gelin-Cesàro identity

$$F_n^4-F_{n-2}F_{n-1}F_{n+1}F_{n+2}=1, \tag{1.2}$$

which was stated by Gelin and proved by Cesàro (see [1], p. 401). Furthermore, we establish that a certain expression arising from three-term recurrence relations is a perfect square, and this generalizes previous work.

Using the notation of Horadam [2], let

$$W_n=W_n(a,b;p,q) \tag{1.3}$$

so that

$$W_n=pW_{n-1}-qW_{n-2},\ W_0=a,\ W_1=b,\ n\geq 2. \tag{1.4}$$

If α,β, assumed distinct, are the roots of

$$\lambda^2-p\lambda+q=0, \tag{1.5}$$

we have the Binet form [2]

$$W_n=\frac{A\alpha^n-B\beta^n}{\alpha-\beta}, \tag{1.6}$$

in which

$$\begin{cases}A=b-a\beta\\ B=b-a\alpha.\end{cases} \tag{1.7}$$

Write

$$e=pab-qa^2-b^2=-AB. \tag{1.8}$$

As usual, $U_n=W_n(0,1;p,q)$ is the fundamental sequence of Lucas [4].

2. THE MAIN RESULT

We now generalize the Catalan identity and obtain some consequences.

Theorem: For $W_n=W_n(a,b;p,q)$ and $Y_n=W_n(a_1,b_1;p,q)$,

$$W_nY_{n+r+s}-W_{n+r}Y_{n+s}=\Psi(s)q^nU_r, \tag{2.1}$$

where

$$\Psi(s)=(pa_1b-qaa_1-bb_1)U_s+(ab_1-a_1b)U_{s+1}.$$

Proof: Using the Binet forms for W_n and Y_n we obtain, after some algebra,

$$W_nY_{n+r+s}-W_{n+r}Y_{n+s}=\frac{(AB_1\beta^s-A_1B\alpha^s)q^nU_r}{\alpha-\beta},$$

where, in the Binet form for Y_n,

$$\begin{cases}A_1=b_1-a_1\beta,\\ B_1=b_1-a_1\alpha.\end{cases} \tag{2.2}$$

Now, using (1.7) and (2.2) we see, after simplifying, that $\frac{AB_1\beta^s-A_1B\alpha^s}{\alpha-\beta}$ reduces to $\Psi(s)$. □

In (2.1), replacing n by $n-r$ and s by r gives

$$W_{n-r}Y_{n+r}-W_nY_n=\Psi(r)q^{n-r}U_r. \tag{2.3}$$

Replacing r by $r+1$ in (2.3), we have

$$W_{n-r-1}Y_{n+r+1}-W_nY_n=\Psi(r+1)q^{n-r-1}U_{r+1}. \tag{2.4}$$

Adding (2.3) and (2.4) gives

$$W_{n-r}Y_{n+r}+W_{n-r-1}Y_{n+r+1}=2W_nY_n+\Psi(r)q^{n-r}U_r+\Psi(r+1)q^{n-r-1}U_{r+1}. \tag{2.5}$$

Subtracting (2.4) from (2.3) gives

$$W_{n-r}Y_{n+r}-W_{n-r-1}Y_{n+r+1}=\Psi(r)q^{n-r}U_r-\Psi(r+1)q^{n-r-1}U_{r+1}. \tag{2.6}$$

Squaring (2.5) and subtracting the square of (2.6), we obtain

$$\begin{aligned}W_{n-r-1}W_{n-r}Y_{n+r}Y_{n+r+1}=W_n^2Y_n^2&+W_nY_nq^{n-r-1}(q\Psi(r)U_r+\Psi(r+1)U_{r+1})\\&+\Psi(r)\Psi(r+1)q^{2n-2r-1}U_rU_{r+1}.\end{aligned} \tag{2.7}$$

Putting $r=1$ in (2.7) yields

$$W_{n-2}W_{n-1}Y_{n+1}Y_{n+2}=W_n^2Y_n^2+W_nY_nq^{n-2}(q\Psi(1)+p\Psi(2))+p\Psi(1)\Psi(2)q^{2n-3}. \tag{2.8}$$

In (2.1), substituting $r=-1$, $s=m-n+1$ and noting that $U_{-1}=-q^{-1}$, we obtain

$$W_nY_m-W_{n-1}Y_{m+1}=-\Psi(m-n+1)q^{n-1}. \tag{2.9}$$

Furthermore, if $n=m-1$, then (2.9) yields

$$W_{m-1}Y_m-W_{m-2}Y_{m+1}=-\Psi(2)q^{m-2}. \tag{2.10}$$

Finally, from (2.1), it follows that

$$(W_nY_{n+r+s}-W_{n+r}Y_{n+s})^2=\Psi^2(s)q^{2n}U_r^2,$$

so that

$$4W_nW_{n+r}Y_{n+s}Y_{n+r+s}+\Psi^2(s)q^{2n}U_r^2=(W_nY_{n+r+s}+W_{n+r}Y_{n+s})^2,$$

thus establishing that

$$4W_nW_{n+r}Y_{n+s}Y_{n+r+s}+\Psi^2(s)q^{2n}U_r^2 \tag{2.11}$$

is a perfect square for nonnegative integers n, r, s and integers a, b, a_1, b_1, p, q.

3. RELATION TO OTHER GENERALIZATIONS

The results of the previous section generalize results of Horadam and Shannon [3] who, in turn, generalized work of Morgado [5] on the Fibonacci numbers. It suffices then to indicate how our work generalizes that of Horadam and Shannon.

In (2.1), when $(a_1, b_1)=(a, b)$, we have $\{W_n\}=\{Y_n\}$ and $\Psi(s)=eU_s$, so that (2.1) becomes

$$W_nW_{n+r+s}-W_{n+r}W_{n+s}=eq^nU_rU_s,$$

which Horadam and Shannon gave as a generalization of the Catalan identity. Under the same circumstances, noting that $\Psi(1)=e$ and $\Psi(2)=ep$, (2.8) reduces to

$$W_{n-2}W_{n-1}W_{n+1}W_{n+2}=W_n^4+W_n^2eq^{n-2}(p^2+q)+e^2q^{2n-3}p^2,$$

which Horadam and Shannon gave as a generalization of the Gelin-Cesàro identity.

Similarly, (2.9) and (2.10) reduce, respectively, to

$$W_nW_m-W_{n-1}W_{m+1}=-eq^{n-1}U_{m-n+1}$$

and

$$W_nW_{n-1}-W_{n-2}W_{n+1}=-epq^{n-2},$$

which are generalizations of results for Fibonacci numbers due to D'Ocagne (see [1], p. 402).

Finally, the expression (2.11) reduces to

$$4W_nW_{n+r}W_{n+s}W_{n+r+s}+e^2q^{2n}U_r^2U_s^2,$$

which was proved by Horadam and Shannon to be a perfect square.

ACKNOWLEDGMENT

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-778 ***Proposed by Eliot Jacobson, Ohio University, Athens, OH***

While paging through an old text in our library, I found a tattered and yellowed page, clearly out of place, as if it had been torn from a book, long forgotten. After months of tedious work, I have completed the translation of the scribbled markings on that page. In the margin was noted:

> *I have found a truly wondrous demonstration of the following theorem; unfortunately the margin of this page is too small to contain it.*

And then followed:

> **Fibonacci's Last Theorem:** The equation $x^n + y^n = z^n$ has no nontrivial solutions consisting entirely of Fibonacci numbers, for $n \geq 2$.

Can you supply the missing proof?

B-779 ***Proposed by Andrew Cusumano, Great Neck, NY***

Find integers a, b, c, and d (with $1 < a < b < c < d$) that make the following an identity:

$$F_n = F_{n-a} + 6F_{n-b} + F_{n-c} + F_{n-d}.$$

B-780 ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia***

Prove that:

(a) $F_1 \cdot F_2 \cdot F_3 \cdot \dots \cdot F_n \leq \exp(F_{n+2} - n - 1)$;

(b) $F_1 \cdot F_3 \cdot F_5 \cdot \dots \cdot F_{2n-1} \le \exp(F_{2n} - n)$;

(c) $F_2 \cdot F_4 \cdot F_6 \cdot \dots \cdot F_{2n} \le \exp(F_{2n+1} - n - 1)$.

B-781 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Let $F(j) = F_j$. Find a closed form for

$$\sum_{k=0}^{n} F\left(k - \lfloor \sqrt{k} \rfloor^2\right).$$

(The notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

B-782 ***Proposed by László Cseh, Stuttgart, Germany, & Imre Merény, Budapest, Hungary***

Express $(F_{n+h}^2 + F_n^2 + F_h^2)(F_{n+h+k}^2 + F_{n+k}^2 + F_k^2)$ as the sum of three squares.

B-783 ***Proposed by David Zeitlin, Minneapolis, MN***

Find a rational function $P(x, y)$ such that

$$P(F_n, F_{2n}) = \frac{105n^5 - 1365n^3 + 1764n}{25n^6 + 175n^4 - 5600n^2 + 5904}$$

for $n = 0, 1, 2, 3, 4, 5, 6$.

SOLUTIONS

Fun with Unit Fractions

B-745 ***Proposed by Richard André-Jeannin, Longwy, France (Vol. 31, no. 3, August 1993)***

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = 1 + \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}F_{2n}F_{2n+1}}$.

Solution by Paul S. Bruckman, Everett, WA

Let $S = \sum_{n=1}^{\infty} 1/F_{2n}$, $D_n = F_{2n-1}F_{2n}F_{2n+1}$, and $T = \sum_{n=1}^{\infty} 1/D_n$. We want to show that $S = 1 + T$. Clearly, the sums defining S and T are absolutely convergent, which justifies the following manipulations:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{F_{2n-1}F_{2n+1}}{D_n} = \sum_{n=1}^{\infty} \frac{F_{2n}^2 + 1}{D_n} = T + \sum_{n=1}^{\infty} \frac{F_{2n}}{F_{2n-1}F_{2n+1}} \\ &= T + \sum_{n=1}^{\infty} \frac{F_{2n+1} - F_{2n-1}}{F_{2n-1}F_{2n+1}} = T + \sum_{n=1}^{\infty} \left(\frac{1}{F_{2n-1}} - \frac{1}{F_{2n+1}} \right) \\ &= T + \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} - \sum_{n=2}^{\infty} \frac{1}{F_{2n-1}} = T + \frac{1}{F_1} = T + 1. \end{aligned}$$

Also solved by Leonard A. G. Dresel, Piero Filipponi, Russell Jay Hendel, Norbert Jensen, Murray S. Klamkin, Joseph J. Kostal, Bob Prielipp, Almas Rumov, H.-J. Seiffert, J. Suck, A. N. 't Woord, David Zeitlin, and the proposer.

L_{3^n} Recurs

B-746 *Proposed by Seung-Jin Bang, Albany, CA*
(Vol. 31, no. 3, August 1993)

Solve the recurrence equation $a_{n+1} = 4a_n^3 + 3a_n$, $n \geq 0$, with initial condition $a_0 = 1/2$.

Solution by Chris Long, Bridgewater, New Jersey

I claim that $a_n = \frac{1}{2}L_{3^n}$. Indeed, using the Binet form and the fact that $\alpha\beta = -1$, it follows that $L_n^3 = L_{3n} - 3L_n$. Thus, $L_{3n} = L_n^3 + 3L_n$, which implies that $L_{3n}/2 = 4 \cdot L_n/2^3 + 3 \cdot L_n/2$. The result follows, since $a_0 = 1/2 = L_1/2$.

The proposer stated that this problem was inspired by Problem 1809 in Crux Mathematicorum 19 (1993):16, proposed by David Doster.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Piero Filipponi, F. J. Flanigan, Norbert Jensen, Hans Kappus, Murray S. Klamkin, Juan Pla, Bob Prielipp, Almas Rumov, H.-J. Seiffert, J. Suck, David Zeitlin, and the proposer.

Great Sums from Partial Sums

B-747 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*
(Vol. 31, no. 3, August 1993)

Let

$$S_1 = \sum_{n=2}^{\infty} \frac{1}{(-1)^n L_{2n-1} - 1} \quad \text{and} \quad S_2 = \sum_{n=2}^{\infty} \frac{1}{(-1)^n L_{2n-1} + 1}.$$

Prove that $S_1/S_2 = \sqrt{5}$.

Solution by Hans Kappus, Rodersdorf, Switzerland

Consider the partial sums

$$s_1(n) = \sum_{k=2}^{n+1} \frac{1}{(-1)^k L_{2k-1} - 1} \quad \text{and} \quad s_2(n) = \sum_{k=2}^{n+1} \frac{1}{(-1)^k L_{2k-1} + 1}.$$

We shall prove that

$$s_1(n) = F_n / L_{n+1} \quad \text{and} \quad s_2(n) = F_n / (5F_{n+1}). \tag{1}$$

From (1), it follows easily that

$$S_1 = \lim_{n\to\infty} s_1(n) = \frac{1}{\sqrt{5}\alpha} = \frac{5-\sqrt{5}}{10} \quad \text{and} \quad S_2 = \lim_{n\to\infty} s_2(n) = \frac{1}{5\alpha} = \frac{\sqrt{5}-1}{10}.$$

Hence, $S_1/S_2 = \sqrt{5}$.

Proof of (1): In the known relations (see [1], p. 177)

$$L_{k+m} + (-1)^m L_{k-m} = L_m L_k \quad \text{and} \quad L_{k+m} - (-1)^m L_{k-m} = F_m F_k,$$

we put $m = k-1$. We then have

$$s_1(n)=\sum_{k=2}^{n+1}\frac{(-1)^k}{L_{k-1}L_k}=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{L_kL_{k+1}}\quad\text{and}\quad s_2(n)=\sum_{k=2}^{n+1}\frac{(-1)^k}{F_{k-1}F_k}=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{F_kF_{k+1}}.$$

The latter expressions are special cases of sums considered in Problem B-697 (see [2]), and it is readily seen that their closed forms are just those given by (1).

References

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood ltd., 1989.
2. Richard André-Jeannin. Problem B-697. *The Fibonacci Quarterly* **30.3** (1992):280.

Also solved by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Jay Hendel, Norbert Jensen, Bob Prielipp, H.-J. Seiffert, J. Suck, and the proposer.

A Recurrence for F_{kn}

B-748 ***Proposed by Herta T. Freitag, Roanoke, VA***
(Vol. 31, no. 4, November 1993)

Let $u_k = F_{kn}/F_n$ for some fixed positive integer n. Find a recurrence satisfied by the sequence $\langle u_k\rangle$.

Solution by Tony Shannon, University of Technology, Sydney, Australia

We have

$$u_{k+1}=\frac{\alpha^{kn}\alpha^n-\beta^{kn}\beta^n}{\alpha^n-\beta^n}=\alpha^n u_k+\beta^n u_k+\frac{\alpha^n\beta^{kn}-\beta^n\alpha^{kn}}{\alpha^n-\beta^n}=L_nu_k-(-1)^n u_{k-1}.$$

Haukkanen noted that for any function f(n), the sequence $\langle u_k\rangle$, given by $u_k = F_{kn}f(n)$, satisfies the recurrence $u_{k+2} = L_nu_{k+1}-(-1)^nu_k$. Libis expressed the recurrence neatly as $u_{k+2}=u_2u_{k+1}-(-1)^nu_k$. Kostal found the recurrence $u_k = L_{(k-1)n}+(-1)^nu_{k-2}$. Ballieu found the recurrence $u_k = \alpha^nu_{k-1}+(\beta^n)^{k-1}$. Somer reported that Lehmer found that the recurrence $u_{k+2}=L_nu_{k+1}-(-1)^nu_k$ is satisfied by the more general sequence defined by $u_k = W_{kn}/W_n$, where n is a fixed positive integer, $W_0=0$, $W_1=1$, and $W_{i+2}=\sqrt{R}\,W_{i+1}-QW_i$, where R and Q are relatively prime integers. See page 437 in D. H. Lehmer, "An Extended Theory of Lucas' Functions," Annals of Mathematics, Series 2, 31 (1930):419-448. The proposer stated that this problem was inspired by David Englund.

Also solved by Michel Ballieu, Paul S. Bruckman, Leonard A. G. Dresel, Steve Edwards, C. Georghiou, Pentti Haukkanen (two solutions), Russell Jay Hendel, Norbert Jensen, Joseph J. Kostal, Harris Kwong, Carl Libis, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, J. Suck, David C. Terr, A. N. 't Woord, David Zeitlin, and the proposer.

No Remainder

B-749 ***Proposed by Richard André-Jeannin, Longwy, France***
(Vol. 31, no. 4, November 1993)

For n a positive integer, define the polynomial $P_n(x)$ by $P_n(x)=x^{n+2}-x^{n+1}-F_nx-F_{n-1}$. Find the quotient and remainder when $P_n(x)$ is divided by x^2-x-1.

Solution by H. K. Krishnapriyan, Drake University, Des Moines, IA, and by Joseph J. Kostal, Chicago, IL (independently)

Direct multiplication confirms that

$$P_n(x) = (F_{n-1} + F_{n-2}x + F_{n-3}x^2 + \cdots + F_0 x^{n-1} + x^n)(x^2 - x - 1).$$

Thus, the quotient is $\sum_{k=0}^{n} F_{n-k-1}x^k$ and the remainder is 0.

Beasley found the analog for Lucas numbers: $x^{n+2} + x^{n+1} - 2x^n - L_n x - L_{n-1}$ *is divisible by* $x^2 - x - 1$. *Redmond found that if r and s are distinct roots of* $x^2 - ax - b = 0$ *and* $u_n = \frac{r^n - s^n}{r-s}$ *then* $x^{n+2} - ax^{n+1} + bu_n x - b^2 u_{n-1}$ *is divisible by* $x^2 - ax + b$. *Suck showed that if* f_n *satisfies the recurrence* $a_0 f_n + a_1 f_{n+1} + \cdots + a_r f_{n+r} = 0$ *then, for* $n \ge r - 1$,

$$\sum_{i=0}^{r-1}\sum_{j=0}^{i} a_j f_{m+n-i+j} x^i + \sum_{i=1}^{r}\sum_{j=i}^{r} a_j f_{m-i+j} x^{n+i}$$

is divisible by $a_0 + a_1 x + \cdots + a_r x^r$. *The given problem is the special case* $f_n = F_n$, $r = 2$, $a_0 = a_1 = -1$, $a_2 = 1$, *and* $m = -1$. *These solvers found the quotient in each case as well. Zeitlin found that* $x^{n+1} - x^{n-1} - F_n x - F_{n-1}$ *is also divisible by* $x^2 - x - 1$.

Also solved by Charles Ashbacher, Brian D. Beasley, Paul S. Bruckman, Leonard A. G. Dresel, Steve Edwards, F. J. Flanigan, Herta Freitag, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Hans Kappus, Harris Kwong, Carl Libis, Bob Prielipp, Don Redmond, H.-J. Seiffert, Tony Shannon, J. Suck, A. N. 't Woord, David Zeitlin, and the proposer.

A Linear Transformation that Shifts

B-750 ***Proposed by Seung-Jin Bang, Albany, CA (Vol. 31, no. 4, November 1993)***

Find a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(F_n, L_n) = (F_{n+1}, L_{n+1})$.

Solution by Leonard A. G. Dresel, Reading, England

Adding the identities $F_n = F_{n+1} - F_{n-1}$ and $L_n = F_{n+1} + F_{n-1}$, we obtain $F_n + L_n = 2F_{n+1}$. Similarly, adding the identities $L_n = L_{n+1} - L_{n-1}$ and $5F_n = L_{n+1} + L_{n-1}$, we obtain $5F_n + L_n = 2L_{n+1}$.

Hence, the required transformation is

$$T(x, y) = \left(\frac{x+y}{2}, \frac{5x+y}{2}\right).$$

This can also be written as

$$\begin{pmatrix} F_{n+1} \\ L_{n+1} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 5/2 & 1/2 \end{pmatrix}\begin{pmatrix} F_n \\ L_n \end{pmatrix}.$$

Redmond generalized to the sequences defined by $u_n = (r^n - s^n)/(r-s)$ and $v_n = r^n + s^n$, *where r and s are the distinct roots of* $x^2 - ax + b = 0$. *In this case, he found that if*

$$T(x, y) = \left(\frac{1}{2}xv_k + \frac{1}{2}yu_k,\ \frac{1}{2}x(r-s)^2 u_k + \frac{1}{2}yv_k\right)$$

then $T(u_n, v_n) = (u_{n+k}, v_{n+k})$. *The given problem is the special case where* $a = 1, b = -1,$ and $k = 1$. *Several solvers found a transformation such as* $T(x, y) = (y - xF_{n-1}/F_n,\ 5x - yL_{n-1}/L_n)$ *which, for a given fixed n, is a linear transformation. However, these solutions are not as elegant as the featured solution in which the linear transformation found is independent of n.*

Also solved by Charles Ashbacher, Michel Ballieu, Paul S. Bruckman, Charles K. Cook, Steve Edwards, F. J. Flanigan, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Hans Kappus, Joseph J. Kostal, H. K. Krishnapriyan, Harris Kwong, Stanley Wu-Wei Liu, Bob Prielipp, Don Redmond, H.-J. Seiffert, Lawrence Somer, J. Suck, David C. Terr, A. N. 't Woord, and the proposer.

Divisibility by 25

B-751 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. Col. Sci., Gujarat State, India (Vol. 31, no. 4, November 1993)***

Prove that $6L_{n+3}L_{3n+4} + 7$ and $6L_nL_{3n+5} - 7$ are divisible by 25.

Solution by Russell Jay Hendel, University of Louisville, Louisville, KY

This and similar problems can always be solved swiftly using periodicity properties.

Looking at L_n (mod 25), namely, 2, 1, 3, 4, 7, 11, 18, 4, 22, 1, -2, -1, ... shows that the function L_n modulo 25 has period 20. It immediately follows that the functions L_{n+3}, L_{3n+4}, L_n, and L_{3n+5} all have period 20 modulo 25. To prove the given assertions, it therefore suffices to check that the given two functions when calculated modulo 25 on $n = 0, 1, 2, \ldots, 19$ all equal 0.

As an example, if $n = 2$ then, modulo 25, we find $L_{n+3} \equiv 11$ and $L_{3n+4} = L_{10} \equiv 23$, so

$$6L_{n+3}L_{3n+4} + 7 \equiv 6(11)(-2) + 7 \equiv 0.$$

The editor found the explicit representations

$$6L_{n+3}L_{3n+4} + 7 = 25[7 + 69(-1)^n F_n^2 + 87F_n^4 + 15(-1)^n F_{2n} + 39F_n^3 L_n]$$

and

$$6L_nL_{3n+5} - 7 = 25[5 + 33(-1)^n F_n^2 + 33F_n^4 + 3(-1)^n F_{2n} + 3F_{4n}],$$

directly showing that these expressions are divisible by 25. He wonders if explicit representations can be found for all similar divisibility problems. (See, for example, the solution to Problem B-741 in the previous issue.)

Also solved by Charles Ashbacher, Paul S. Bruckman, Leonard A. G. Dresel, C. Georghiou, Norbert Jensen, H.-J. Seiffert, J. Suck, David Zeitlin, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-493 ***Proposed by Stefano Mascella & Piero Filipponi, Rome, Italy***

Let $P_k(d)$ denote the probability that the k^{th} digit (from left) of an ℓ digit ($\ell \geq k$) Fibonacci number F_n (expressed in base 10) whose subscript is randomly chosen within a *large* interval $[n_1, n_2]$ ($n_2 \gg n_1$) is d.

That the sequence $\{F_n\}$ obeys Benford's law is a well-known fact (e.g., see [1] and [2]). In other words, it is well known that $P_1(d) = \log_{10}(1+1/d)$.

Find an expression for $P_2(d)$.

References

1. P. Filipponi. "Some Probabilistic Aspects of the Terminal Digits of Fibonacci Numbers." *The Fibonacci Quarterly* (to appear).
2. L. C. Washington. "Benford's Law for Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **19.2** (1981):175-77.

H-494 ***Proposed by David M. Bloom, Brooklyn College, New York, NY***

It is well known that if $P(p)$ is the Fibonacci entry point ("rank of apparition") of the odd prime $p \neq 5$, then $P(p)$ divides $p+e$ where $e = \pm 1$. In [1] it is stated without proof [Theorem 5(b)] that the integer $(p+e)/P(p)$ has the same parity as $(p-1)/2$. Give a proof.

Reference

1. D. Bloom. "On Periodicity in Generalized Fibonacci Sequences." *Amer. Math. Monthly* **72** (1965):856-61.

H-495 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let p be a prime $\neq 2, 5$, and let $Z(p)$ denote the *Fibonacci entry-point* of p (i.e., the smallest positive integer m such that $p | F_m$). Prove the following "Parity Theorem" for the Fibonacci entry-point:

A. If $p \equiv 11$ or $19 \pmod{20}$, then $Z(p) \equiv 2 \pmod 4$;
B. if $p \equiv 13$ or $17 \pmod{20}$, then $Z(p)$ is odd;
C. if $p \equiv 3$ or $7 \pmod{20}$, then $4 | Z(p)$.

SOLUTIONS

Irrational Behavior

H-481 ***Proposed by Richard André-Jeannin, Longwy, France (Vol. 31, no. 4, November 1993)***

Let $\phi(x)$ be the function defined by

$$\phi(x) = \sum_{n\geq 0} \frac{x^n}{F_{r^n}}$$

where $r \geq 2$ is a natural integer. Show that $\phi(x)$ is an irrational number if x is a nonzero rational number.

Solution by Norbert Jensen, Kiel, Germany

Let $x \in \mathbb{Q} \setminus \{0\}$. We have to show the irrationality of $\phi(x)$.

The proof is similar to the well-known proof of the irrationality of e. Note that the series $\sum_{n=m}^{\infty} x^n / F_{r^n}$ and $\sum_{n=m}^{\infty} x^n / \alpha^{r^n}$ converge for all $m \in \mathbb{N}_0$. This can be proved by the ratio test. For the second series, the proof is obvious. Applying the test to the first series, one can use the following step (0).

Step (0): $F_{r^n} / F_{r^{n+m}} \leq 8\alpha^{r^n(1-r^m)}$ for all $n, m \in \mathbb{N}_0$.

Proof: $F_{r^n} / F_{r^{n+m}} = (\alpha^{r^n} - \beta^{r^n}) / (\alpha^{r^{n+m}} - \beta^{r^{n+m}}) \leq (\alpha^{r^n} + 1) / (\alpha^{r^{n+m}} - 1) \leq 2\alpha^{r^n} / (1/4)\alpha^{r^{n+m}} = 8\alpha^{r^n(1-r^m)}$. Q.E.D. [(0)]

Let $\rho_m = \sum_{n=m}^{\infty} x^n / F_{r^n}$ for all $m \in \mathbb{N}$.

Step (1): For an appropriate positive constant $c \in \mathbb{R}$, we have $|\rho_m| \leq c|x|^m / F_{r^m}$ for all $m \in \mathbb{N}$. c depends only on $|x|$.

Proof: From (0), we derive

$$|\rho_m| \leq \left(\sum_{n=m}^{\infty} F_{r^m} |x|^{n-m} / F_{r^n} \right) |x|^m / F_{r^m} \leq \left(8\alpha \sum_{n=m}^{\infty} |x|^{n-m} / \alpha^{r^{n-m}} \right) |x|^m / F_{r^m} = \left(8\alpha \sum_{n=0}^{\infty} |x|^n / \alpha^{r^n} \right) |x|^m / F_{r^m}.$$

Q.E.D. [(1)]

Step (2): Let $z \in \mathbb{N}$. Then $|z^{m-1} F_{r^{m-1}} \rho_m| < 1$ for all sufficiently large $m \in \mathbb{N}$.

Proof: $|z^{m-1} F_{r^{m-1}} \rho_m| \leq c z^{m-1} F_{r^{m-1}} |x|^m / F_{r^m} = c|x|(z|x|)^{m-1} F_{r^{m-1}} / F_{r^m} \leq d(z|x|)^{m-1} / \alpha^{r^{m-1}}$ by (1), (0), where d is an appropriate positive constant depending only on $|x|$. Since $\sum_{n=0}^{\infty} (z|x|)^n / \alpha^{r^n}$ converges, the last term tends to 0 as m tends to infinity. Q.E.D. [(2)]

Step (3): There is an $m_0 \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$, $m \geq m_0 : \rho_m \neq 0$.

Proof: **Case 1**—$x > 0$. The assertion follows because $\rho_m \geq x^m / F_{r^m} > 0$.

Case 2—$x < 0$. Let $m_0 \in \mathbb{N}$ such that $\alpha^{r^{m_0}(r-1)} > 8|x|$. Let $m, n \in \mathbb{N}$, $n \geq m \geq m_0$. Then $\alpha^{r^n(r-1)} \geq \alpha^{r^{m_0}(r-1)} > 8|x|$. Therefore, by (0): $F_{r^{n+1}} / F_{r^n} > |x|$ and $1 / F_{r^n} > |x| / F_{r^{n+1}}$, whence $|x|^n / F_{r^n} > |x|^{n+1} / F_{r^{n+1}}$.

If m is even, it follows that

$$\rho_m = \sum_{k=m/2}^{\infty} \left(|x|^{2k}/F_{r^{2k}} - |x|^{2k+1}/F_{r^{2k+1}} \right) \geq |x|^m/F_{r^m} - |x|^{m+1}/F_{r^{m+1}} > 0.$$

If m is odd, an analogous argument shows that $\rho_m < 0$. Q.E.D. [(3)]

Step (4): $\phi(x)$ is irrational.

Proof: Let $p, q \in \mathbb{Z}, q > 0$, such that $x = p/q$. Suppose on the contrary that $\phi(x)$ is rational. Then there are $a, b \in \mathbb{Z}, b > 0$, such that $\phi(x) = a/b$. Thus, $b\phi(x) \in \mathbb{Z}$.

According to (2) and (3), there is an $m \in \mathbb{N}, m \geq 2$, such that $|(bq)^{m-1} F_{r^{m-1}} \rho_m| < 1$ and $\rho_m \neq 0$. Let $\sigma_m := \sum_{n=0}^{m-1} x^n / F_{r^n}$. Now

$$(bq)^{m-1} F_{r^{m-1}} \phi(x) = (bq)^{m-1} F_{r^{m-1}} (\sigma_m + \rho_m) = (bq)^{m-1} F_{r^{m-1}} \sigma_m + (bq)^{m-1} F_{r^{m-1}} \rho_m \in \mathbb{Z}$$

and

$$(bq)^{m-1} F_{r^{m-1}} \sigma_m \in \mathbb{Z},$$

since F_{r^j} divides $F_{r^{m-1}}$ for $j = 0, 1, \ldots, m-1$. But $|(bq)^{m-1} F_{r^{m-1}} \rho_m| < 1$; hence, $(bq)^{m-1} F_{r^{m-1}} \rho_m = 0$, $\rho_m = 0$, a contradiction. Q.E.D.

Also solved by P. Bruckman, H.-J. Seiffert, and the proposer.

Generalize

H-482 ***Proposed by Larry Taylor, Rego Park, NY***
(Vol. 31, no. 4, November 1993)

Let j, k, m, and n be integers. Let $A_n(m) = B_n(m-1) + 4A_n(m-1)$ and $B_n(m) = 4B_n(m-1) + 5A_n(m-1)$ with initial values $A_n(0) = F_n$, $B_n(0) = L_n$.

(A) Generalize the numbers $(2,2,2,2,2,2,2,2,2,2,2)$ to form an eleven-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

(B) Generalize the numbers $(3,3,3,3,3,3,3,3,3,3)$ to form a ten-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

(C) Generalize the numbers $(1,1,1,1,1,1,1,1)$ to form an eight-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

Hint: $A_n(1) = -11(-1)^n A_{-n}(-1)$.

Reference: L. Taylor. Problem H-422. *The Fibonacci Quarterly* **28.3** (1990):285-87.

Solution by Paul S. Bruckman, Edmonds, WA

The recurrence defining the $A_n(m)$'s and $B_n(m)$'s may be put into matrix form

$$C\underline{x}_n(m-1) = \underline{x}_n(m), \tag{1}$$

where

$$C = \begin{pmatrix} 4 & 1 \\ 5 & 4 \end{pmatrix}, \tag{2}$$

$$\underline{x}_n(j) = \begin{pmatrix} A_n(j) & B_n(j) \end{pmatrix}^T. \tag{3}$$

We may invert the recurrence in the matrix form

$$\underline{x}_n(m-1) = C^{-1}\underline{x}_n(m), \quad \text{where } C^{-1} = \frac{1}{11}\begin{pmatrix} 4 & -1 \\ -5 & 4 \end{pmatrix}.$$

This yields the relations:

$$A_n(m-1) = \frac{1}{11}(4A_n(m) - B_n(m)), \quad B_n(m-1) = \frac{1}{11}(-5A_n(m) + 4B_n(m)). \tag{4}$$

By repeated application of (1) (in either direction), we obtain

$$C^m \underline{x}_n(0) = \underline{x}_n(m), \quad \text{with } \underline{x}_n(0) = (F_n \quad L_n)^T. \tag{5}$$

We may show that there exist two sequences of rationals (ρ_m) and (σ_m) (integers for $m \ge 0$), such that

$$C^m = \begin{pmatrix} \rho_m & \sigma_m \\ 5\sigma_m & \rho_m \end{pmatrix}. \tag{6}$$

We have no need to investigate further into these sequences, except to note that they are functions solely of m, and not of n. The relevant observation from (5)-(6) is the following:

$$A_n(m) = \rho_m F_n + \sigma_m L_n, \quad B_n(m) = 5\sigma_m F_n + \rho_m L_n. \tag{7}$$

Now using the identities $L_n = F_n + 2F_{n-1}, 5F_n = L_n + 2L_{n-1}$, and making the substitutions $\rho_m + \sigma_m = r_m, 2\sigma_m = s_m$, (7) is transformed to the following:

$$A_n(m) = r_m F_n + s_m F_{n-1}, \quad B_n(m) = r_m L_n + s_m L_{n-1}. \tag{8}$$

In this form, we see that $A_n(m)$ and $B_n(m)$ are generalized Fibonacci and Lucas numbers, respectively, as these were defined in part (B) of the published solution to H-422 (see reference [1]). Here, the $A_n(m), B_n(m), r_m$, and s_m replace the U_n, V_n, r, and s, respectively, as such were introduced in [1]. Note that the $A_n(m)$ and $B_n(m)$, for fixed m, satisfy the same linear recurrences as are satisfied by F_n and L_n; in the sequel, we shall tacitly use these without comment [e.g., $A_{n+2}(m) = A_{n+1}(m) + A_n(m), B_n(m) = A_{n+1(m)} + A_{n-1}(m)$, etc.]. Also, in the sequel, we will write (for brevity) $A_n \equiv A_n(m), \overline{A}_n \equiv A_n(m+1), \underline{A}_n \equiv A_n(m-1)$, with similar notation for the $B_n(m)$'s; however, in the final solution of each part, we will revert to the unabridged notation.

Solution of Part (A)

Using parts (B) and (A1) of [1], the following 7-term arithmetic progression (A.P.) is found, whose common difference (c.d.) is equal to A_n, and whose terms consist of integral multiples of A_{n+k} and/or B_{n+k}:

$$(-2A_{n-2}, A_{n-3}, 2A_{n-1}, B_n, 2A_{n+1}, A_{n+3}, 2A_{n+2}). \tag{9}$$

Our goal, if possible, is to affix four additional terms to the A.P. above (at one or both ends), such that these terms are of the form required in the statement of the problem, such that the c.d. for all 11 terms remains A_n, and such that, for some fixed m and n, all 11 terms equal 2. We require a few additional identities:

$$2A_{n+2} + A_n = \overline{A}_n. \tag{10}$$

Proof: Replacing m by $m+1$ in the original recurrence, we have:

$$\overline{A}_n = 4A_n + B_n = 4A_n + A_{n+1} + A_{n-1} = 4A_n + A_{n+1} + A_{n+1} - A_n = 3A_n + 2(A_{n+2} - A_n) = 2A_{n+2} + A_n.$$

$$\overline{A}_n + A_n = 2B_{n+1}. \tag{11}$$

Proof: Using (10), $\overline{A}_n + A_n = 2A_{n+2} + 2A_n = 2B_{n+1}$.

$$2A_{n-2} + A_n = 11\underline{A}_n. \tag{12}$$

Proof: From (4),

$$\begin{aligned} 11\underline{A}_n &= 4A_n - B_n = 4A_n - (A_{n+1} + A_{n-1}) = 4A_n - (A_n + 2A_{n-1}) \\ &= 3A_n - 2(A_n - A_{n-2}) = 2A_{n-2} + A_n. \end{aligned}$$

$$11\underline{A}_n + A_n = 2B_{n-1}. \tag{13}$$

Proof: Again using (4), $11\underline{A}_n + A_n = 5A_n - B_n = -B_n + B_{n+1} + B_{n-1} = 2B_{n-1}$.

Now, by inspection of (9)-(13), we see that the following is an 11-term A.P. of the required form, with c.d. = $A_n = A_n(m)$:

$$\begin{gathered} (-2B_{n-1}(m), -11A_n(m-1), -2A_{n-2}(m), A_{n-3}(m), 2A_{n-1}(m), B_n(m), 2A_{n+1}(m), \\ A_{n+3}(m), 2A_{n+2}(m), A_n(m+1), 2B_{n+1}(m)). \end{gathered} \tag{14}$$

It only remains to show that, for some fixed m and n, this A.P. reduces to an 11-tuple of 2's. We find that setting $m = n = 0$ accomplishes this; for, in that case, the c.d. is $A_0(0) = F_0 = 0$, and one term, e.g., $B_0(0)$, is equal to $L_0 = 2$. Therefore, (14) is a valid solution of part (A).

Solution of Part (B)

Using parts (B), (A4)(iii), and (A4)(i) of [1], the following pair of 4-term A.P.'s are found, with c.d. = A_n and with terms of the required form:

$$(-3A_{n-2}, -A_{n-4}, B_{n-2}, 3A_{n-1}); \tag{15}$$

$$(3A_{n+1}, B_{n+2}, A_{n+4}, 3A_{n+2}). \tag{16}$$

Our goal, if possible, is to affix two additional terms of the required form between the two 4-term A.P.'s above, thereby forming a 10-term A.P. which satisfies the condition that, for some fixed m and n, all 10 terms equal 3. We require a few additional identities:

$$3A_{n-1} + A_n = 11\underline{A}_{n+1}. \tag{17}$$

Proof: Replacing n by $n+1$ in (12), we have:

$$11\underline{A}_{n+1} = 2A_{n-1} + A_{n+1} = 2A_{n-1} + A_n + A_{n-1} = A_n + 3A_{n-1}.$$

$$3A_{n+1} - A_n = \overline{A}_{n-1}. \tag{18}$$

Proof: Replacing n by $n-1$ in (10), we have:

$$\overline{A}_{n-1} = 2A_{n+1} + A_{n-1} = 2A_{n+1} + A_{n+1} - A_n = 3A_{n+1} - A_n.$$

$$\overline{A}_{n-1} - A_n = 11\underline{A}_{n+1}. \tag{19}$$

Proof: By (18),

$$\overline{A}_{n-1} - A_n = 3A_{n+1} - 2A_n = 3A_{n+1} - 2(A_{n+1} - A_{n-1}) = A_{n+1} + 2A_{n-1} = 11\underline{A}_{n+1}$$

[using (12), with $n+1$ replacing n]. By inspection of (15)-(19), we see that we have "bridged the gap" between the two 4-term A.P.'s, as required, producing an A.P. of 10 terms of the required form, with c.d. = A_n; this is given as follows:

$$\begin{aligned}(-3A_{n-2}(m), -A_{n-4}(m), B_{n-2}(m), 3A_{n-1}(m), 11A_{n+1}(m-1),\\ A_{n-1}(m+1), 3A_{n+1}(m), B_{n+2}(m), A_{n+4}(m), 3A_{n+2}(m)).\end{aligned} \tag{20}$$

Again setting $m=n=0$, the c.d. is 0 in this case, and one term, e.g., $3A_1(0)=3F_1=3$; thus, in this case, we have a 10-tuple of 3's, as required. This shows that the 10-tuple in (20) provides a solution to part (B).

Solution of Part (C)

Using parts (B) and (A2) of [1], we find the following 6-term A.P. of the required form, with c.d. = A_n:

$$(-B_{n-1}, -A_{n-2}, A_{n-1}, A_{n+1}, A_{n+2}, B_{n+1}). \tag{21}$$

Our goal, if possible, is to affix two terms to this A.P. (at either end or at each end), which are of the required form and satisfy the desired conditions. We require two additional identities:

$$B_{n+1}+A_n = 11\underline{A}_{n+2}. \tag{22}$$

Proof: Replacing n by $n+2$ in (12), we have:

$$11\underline{A}_{n+2} = 2A_n + A_{n+2} = A_n + (A_n + A_{n+2}) = A_n + B_{n+1}.$$

$$B_{n-1}+A_n = \overline{A}_{n-2}. \tag{23}$$

Proof: Replacing n by $n-2$ in (10), we have:

$$\overline{A}_{n-2} = 2A_n + A_{n-2} = A_n + (A_n + A_{n-2}) = A_n + B_{n-1}.$$

By inspection of (21)-(23), we see that the following 8-term A.P. has c.d. = A_n:

$$(-A_{n-2}(m+1), -B_{n-1}(m), -A_{n-2}(m), A_{n-1}(m), A_{n+1}(m), A_{n+2}(m), B_{n+1}(m), 11A_{n+2}(m-1)). \tag{24}$$

Again setting $m=n=0$, we see that the c.d. = 0 and each term, e.g., $A_1(0)=F_1=1$ for this case; thus, for this case, we obtain an 8-tuple of 1's. This shows that (24) yields a solution to part (C) and we are done.

Also solved by the proposer.

❖❖❖

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