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TABLE OF CONTENTS

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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

FORMULAS FOR $a+a^2 2^p+a^3 3^p+\cdots+a^n n^p$

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(*Submitted June 1993*)

1. INTRODUCTION

Let $S_{a,p}(n)=a+a^2 2^p+a^3 3^p+\cdots+a^n n^p$, with $n\in N, p\in N$, and $a\in R$ $(a\neq 0, a\neq 1)$, where N and R are, respectively, the sets of positive integers and real numbers.

In [2] N. Gauthier used a calculus-based method to evaluate $S_{a,p}(n)$. He wrote $S_{a,p}(n)$ as a^n times a polynomial of degree p in n plus a term which is n-independent. The coefficients are then determined recursively.

In this paper methods similar to those used in [1] are employed to derive various formulas for $S_{a,p}(n)$. Recurrence formulas in terms of powers of n and of $n+1$ are given. Explicit expressions for $S_{a,p}(n)$ in determinant form in terms of n and of $n+1$ are then derived from these formulas. These determinants are finally used to write $S_{a,p}(n)$ in terms of polynomials of degree p in n and in $n+1$.

2. FORMULAS IN TERMS OF POWERS OF $n+1$

2.1 A Recurrence Formula

Let $n\in N$. For $k\in N$ and $a\in R$ $(a\neq 0, a\neq 1)$, let

$$S_{a,k}(n)=a+a^2 2^k+a^3 3^k+\cdots+a^n n^k=\sum_{r=0}^{n} a^r r^k$$

and take

$$S_{a,0}(n)=1+a+a^2+\cdots+a^n=\frac{a^{n+1}-1}{a-1}.$$

Then

$$\begin{aligned}a^{n+1}(n+1)^k &= S_{a,k}(n+1)-S_{a,k}(n)\\ &=\sum_{r=0}^{n} a^{r+1}(r+1)^k - S_{a,k}(n)\\ &=\sum_{r=0}^{n}\left(a^{r+1}\sum_{i=0}^{k}\binom{k}{i}r^i\right)-S_{a,k}(n)\\ &=a\sum_{i=0}^{k}\left(\binom{k}{i}\sum_{r=0}^{n}a^r r^i\right)-S_{a,k}(n)\\ &=a\sum_{i=0}^{k}\binom{k}{i}S_{a,i}(n)-S_{a,k}(n).\end{aligned}\tag{2.1.1}$$

The equation

$$a(S+1)^k-S^k=a^{n+1}(n+1)^k,\tag{2.1.2}$$

in which the binomial power on the left-hand side is expanded and S^i $(i=0,1,2,\dots,k)$ are then replaced by $S_{a,i}(n)$, provides a mnemonic for (2.1.1).

For example, for $k=1$, formula (2.1.2) gives

$$a(S+1)-S=a^{n+1}(n+1),$$

and so

$$(a-1)S_{a,1}(n)+a\left(\frac{a^{n+1}-1}{a-1}\right)=a^{n+1}(n+1).$$

Hence,

$$S_{a,1}(n)=\frac{a^{n+1}}{a-1}(n+1)-\frac{a}{(a-1)^2}(a^{n+1}-1). \tag{2.1.3}$$

Also, by (2.1.2), with $k=2$,

$$a^{n+1}(n+1)^2=a(S+1)^2-S^2=a(S^2+2S+1)-S^2,$$

which implies that

$$(a-1)S_{a,2}(n)+2aS_{a,1}(n)+a\left(\frac{a^{n+1}-1}{a-1}\right)=a^{n+1}(n+1)^2.$$

Thus, by (2.1.3),

$$S_{a,2}(n)=\frac{a^{n+1}}{a-1}(n+1)^2-\frac{2a^{n+2}}{(a-1)^2}(n+1)+\frac{a^{n+1}-1}{(a-1)^3}(a^2+a).$$

2.2 $S_{a,p}(n)$ as a Determinant

Let $p\in N$ and let $k=1,2,\dots,p$ in (2.1.1). It follows, applying Cramer's rule to these p equations together with the equation $S_{a,0}(n)=\frac{a^{n+1}-1}{a-1}$, that

$$S_{a,p}(n)=\frac{a^{n+p}}{(a-1)^p}\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & a^{-n}\left(\frac{a^{n+1}-1}{a-1}\right) \\ 1 & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0 & n+1 \\ 1 & \binom{2}{1} & \frac{a-1}{a} & 0 & \cdots & 0 & 0 & (n+1)^2 \\ \vdots & \vdots & \vdots & & & & & \vdots \\ 1 & \binom{p-1}{1} & \binom{p-1}{2} & & \cdots & \binom{p-1}{p-2} & \frac{a-1}{a} & (n+1)^{p-1} \\ 1 & \binom{p}{1} & \binom{p}{2} & & \cdots & \binom{p}{p-2} & \binom{p}{p-1} & (n+1)^p \end{vmatrix} \tag{2.2.1}$$

$$=\frac{a^{n+p}}{(a-1)^p}p!\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & a^{-n}\left(\frac{a^{n+1}-1}{a-1}\right) \\ \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0 & \frac{n+1}{1!} \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0 & \frac{(n+1)^2}{2!} \\ \vdots & \vdots & \vdots & & & & & \vdots \\ \frac{1}{(p-1)!} & \frac{1}{(p-2)!} & \frac{1}{(p-3)!} & & \cdots & \frac{1}{1!} & \frac{a-1}{a} & \frac{(n+1)^{p-1}}{(p-1)!} \\ \frac{1}{p!}, & \frac{1}{(p-1)!} & \frac{1}{(p-2)!} & & \cdots & \frac{1}{2!} & \frac{1}{1!} & \frac{(n+1)^p}{p!} \end{vmatrix}. \tag{2.2.2}$$

2.3 $S_{a,p}(n)$ in Terms of a Polynomial

By expanding the determinant (2.2.2) with respect to the last column,

$$S_{a,p}(n)=\sum_{r=0}^{p-1}\alpha_r(n+1)^{p-r}+\alpha_p a^{-(n+1)}(a^{n+1}-1), \tag{2.3.1}$$

with $\alpha_0=\frac{a^{n+1}}{a-1}$ and, for $r=1,2,\dots,p$,

$$\alpha_r=\binom{p}{r}\frac{a^{n+1}}{a-1}\frac{a^r}{(a-1)^r}r!(-1)^r\begin{vmatrix}\frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0\\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & & & & \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & & & \cdots & \frac{1}{1!} & \frac{a-1}{a}\\ \frac{1}{r!} & \frac{1}{(r-1)!} & & & \cdots & \frac{1}{2!} & \frac{1}{1!}\end{vmatrix}.$$

Now, let $f_0(a)=1$ and, for $r=1,2,3,\dots$,

$$f_r(a)=\frac{a^r}{(a-1)^r}r!(-1)^r\begin{vmatrix}\frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0\\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & & & & \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & & & \cdots & \frac{1}{1!} & \frac{a-1}{a}\\ \frac{1}{r!} & \frac{1}{(r-1)!} & & & \cdots & \frac{1}{2!} & \frac{1}{1!}\end{vmatrix}. \tag{2.3.2}$$

Then, by (2.3.1),

$$S_{a,p}(n)=\frac{a^{n+1}}{a-1}\sum_{r=0}^{p-1}\binom{p}{r}f_r(a)(n+1)^{p-r}+f_p(a)\left(\frac{a^{n+1}-1}{a-1}\right). \tag{2.3.3}$$

The real numbers $f_r(a)$, $r=1,2,3,\dots$, can also be calculated recursively in the following way. Consider, for $r\in N$,

$$a^n f_r(a)=\frac{a^{n+r}}{(a-1)^r}r!(-1)^r\begin{vmatrix}\frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0\\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & & & & \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & & & \cdots & \frac{1}{1!} & \frac{a-1}{a}\\ \frac{1}{r!} & \frac{1}{(r-1)!} & & & \cdots & \frac{1}{2!} & \frac{1}{1!}\end{vmatrix}$$

$$=\frac{a^{n+r}}{(a-1)^r}r!\begin{vmatrix}1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1\\ \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0 & 0\\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & & & & & & \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & & & \cdots & \frac{1}{1!} & \frac{a-1}{a} & 0\\ \frac{1}{r!} & \frac{1}{(r-1)!} & & & \cdots & \frac{1}{2!} & \frac{1}{1!} & 0\end{vmatrix}.$$

Observe that the last determinant differs from that of $S_{a,r}(n)$, as obtained by setting $p=r$ in (2.2.2), only with respect to the last column. It follows [cf. (2.1.1)] that $f_0(a), f_1(a), f_2(a), \ldots$ satisfy the recurrence formula

$$f_0(a)=1,\ a\sum_{i=0}^{r}\binom{r}{i}f_i(a)-f_r(a)=0\ (r=1,2,3,\ldots). \tag{2.3.4}$$

Here the equation

$$a(f+1)^r-f^r=0, \tag{2.3.5}$$

in which the binomial power is expanded and f^r $(r=0,1,2,3,\ldots)$ are then replaced by $f_r(a)$, provides a mnemonic for (2.3.4).

Note that (2.3.4), with $a=1$, is the well-known recurrence formula for the Bernoulli numbers. The real numbers $f_r(a)$, $r=0,1,2,3,\ldots$, could therefore be called the a-Bernoulli numbers. For example, by (2.3.2) or, recursively, by (2.3.5),

$$f_0(a)=1,\ f_1(a)=\frac{-a}{a-1},\ f_2(a)=\frac{a+a^2}{(a-1)^2},\ \text{and } f_3(a)=\frac{-(a+4a^2+a^3)}{(a-1)^3}. \tag{2.3.6}$$

Hence, $1, -2, 6, -26$ are the first four 2-Bernoulli numbers.

3. FORMULAS IN TERMS OF POWERS OF n

Let $n\in N$. For $k\in N$ and $a\in R$ $(a\neq 0, a\neq 1)$, let

$$S_{a,k}(n)=a+a^2 2^k+a^3 3^k+\cdots+a^n n^k=\sum_{r=1}^{n}a^r r^k,\ \ S_{a,k}(0)=0,$$

and take

$$S_{a,0}(n)=a+a^2+\cdots+a^n=\frac{a^{n+1}-a}{a-1}.$$

Then, arguing as in Section 2.1,

$$a^n n^k=S_{a,k}(n)-S_{a,k}(n-1)=S_{a,k}(n)-\frac{1}{a}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}S_{a,k}(n).$$

Hence,

$$a^{n+1}n^k=aS_{a,k}(n)-\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}S_{a,i}(n). \tag{3.1}$$

The equation

$$aS^k-(S-1)^k=a^{n+1}n^k,$$

in which the binomial power on the left-hand side is expanded and S^i $(i=0,1,2,\ldots,k)$ are then replaced by $S_{a,i}(n)$, provides a mnemonic for (3.1).

Furthermore, methods similar to those employed in Sections 2.2 and 2.3 can be used to derive the following results from (3.1).

$$S_{a,p}(n)=\frac{a^{n+1}}{(a-1)^p}p!\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & a^{-(n+1)}\left(\frac{a^{n+1}-a}{a-1}\right) \\ \frac{1}{1!} & a-1 & 0 & 0 & \cdots & 0 & 0 & \frac{n}{1!} \\ -\frac{1}{2!} & \frac{1}{1!} & a-1 & 0 & \cdots & 0 & 0 & \frac{n^2}{2!} \\ \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & a-1 & \cdots & 0 & 0 & \frac{n^3}{3!} \\ \vdots & \vdots & & & & & & \vdots \\ \frac{(-1)^p}{(p-1)!} & \frac{(-1)^{p+1}}{(p-2)!} & & & \cdots & \frac{1}{1!} & a-1 & \frac{n^{p-1}}{(p-1)!} \\ \frac{(-1)^{p+1}}{p!} & \frac{(-1)^{p+2}}{(p-1)!} & & & \cdots & -\frac{1}{2!} & \frac{1}{1!} & \frac{n^p}{p!} \end{vmatrix},$$

and

$$S_{a,p}(n)=\frac{a^{n+1}}{a-1}\sum_{r=0}^{p-1}\binom{p}{r}g_r(a)n^{p-r}+g_p(a)\left(\frac{a^{n+1}-a}{a-1}\right), \tag{3.2}$$

with $g_0(a)=1$ and, for $r=1,2,3,\ldots,$

$$g_r(a)=\frac{r!(-1)^r}{(a-1)^r}\begin{vmatrix} \frac{1}{1!} & a-1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{2!} & \frac{1}{1!} & a-1 & 0 & \cdots & 0 & 0 \\ \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & a-1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \\ \frac{(-1)^r}{(r-1)!} & \frac{(-1)^{r+1}}{(r-2)!} & & & \cdots & \frac{1}{1!} & a-1 \\ \frac{(-1)^{r+1}}{r!} & \frac{(-1)^{r+2}}{(r-1)!} & & & \cdots & -\frac{1}{2!} & \frac{1}{1!} \end{vmatrix}.$$

The real numbers $g_r(a)$, $r=1,2,3,\ldots,$ can also be calculated recursively in a similar way as it is done in the case of $f_r(a)$, $r=1,2,3,\ldots,$ in Section 2.3. However, it is easier to observe that, by (2.3.3) and (3.2) (comparing n-free terms), $f_r(a)=ag_r(a)$ for each $r\in N$. Hence, by (3.2),

$$S_{a,p}(n)=\frac{a^{n+1}}{a-1}n^p+\frac{a^n}{a-1}\sum_{r=1}^{p-1}\binom{p}{r}f_r(a)n^{p-r}+f_p(a)\left(\frac{a^n-1}{a-1}\right),\ \text{for } p>1. \tag{3.3}$$

For example, let $p=2$ in (3.3). Then, by (2.3.6),

$$\begin{aligned} S_{a,2}(n)&=\frac{a^{n+1}}{a-1}n^2+\frac{2a^n}{a-1}f_1(a)n+f_2(a)\left(\frac{a^n-1}{a-1}\right) \\ &=\frac{a^{n+1}}{a-1}n^2-\frac{2a^{n+1}}{(a-1)^2}n+\frac{(a+a^2)(a^n-1)}{(a-1)^3}. \end{aligned}$$

In particular,

$$S_{3,2}(n)=\sum_{r=1}^{n}3^r r^2=\frac{3^{n+1}}{2}n^2-\frac{3^{n+1}}{2}n+\frac{3^{n+1}}{2}-\frac{3}{2}.$$

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GENERATING FIBONACCI WORDS

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INTRODUCTION

A word w is called an n^{th}*-order Fibonacci word* derived from two distinct letters a and b if there exists a finite sequence $w_1, w_2, \ldots, w_n$ of words with $w_1 = a$, $w_2 = b$, $w_n = w$ and each w_k equals $w_{k-1}w_{k-2}$ or $w_{k-2}w_{k-1}$, $3 \le k \le n$. The basic structure of Fibonacci words has been studied in [2]. In this paper we discuss various methods of generating Fibonacci words.

Throughout this paper, let Q_n denote the set of all n^{th}-order Fibonacci words derived from distinct letters a and b. Some of these methods generate all the Fibonacci words in Q_n from any given u in Q_n without repetitions and some of them generate Q_n from Q_{n-1}.

1. BINARY TREES

Let $X = \{a, b\}$ be an alphabet of two letters and let X^* be the free monoid generated by X. Elements of X^* are called words. For any word $w = a_1a_2 \cdots a_n \in X^*$, define $f(w)$ [resp. $g(w)$] to be the word in X^* obtained by replacing each a in w by b and each b in w by ba (resp., by ab). Also define $T(w) = a_2 \cdots a_n a_1$ and $R(w) = a_n \cdots a_2 a_1$. A word w is called a *symmetric word* or a *palindrome* if $R(w) = w$.

Associated with each finite binary sequence $r_1, r_2, \ldots, r_{n-2}$ there are four words in X^*,

$$w_n^{r_1r_2\cdots r_{n-2}}, \quad w_n^{(r_1r_2\cdots r_{n-2})}, \quad w_n^{[r_1r_2\cdots r_{n-2}]}, \quad w_n^{\{r_1r_2\cdots r_{n-2}\}},$$

defined as follows:

$$w_1 = a, \qquad w_2 = b,$$

$$w_m^{r_1r_2\cdots r_{n-2}} = \begin{cases} w_{n-1}^{r_1r_2\cdots r_{n-3}} w_{n-2}^{r_1r_2\cdots r_{n-4}}, & \text{if } r_{n-2} = 0, \\ w_{n-2}^{r_1r_2\cdots r_{n-4}} w_{n-1}^{r_1r_2\cdots r_{n-3}}, & \text{if } r_{n-2} = 1; \end{cases}$$

$$w_n^{(r_1r_2\cdots r_{n-2})} = \begin{cases} R\left(w_{n-1}^{(r_1r_2\cdots r_{n-3})}\right) w_{n-2}^{(r_1r_2\cdots r_{n-4})}, & \text{if } r_{n-2} = 0, \\ w_{n-2}^{(r_1r_2\cdots r_{n-4})} R\left(w_{n-1}^{(r_1r_2\cdots r_{n-3})}\right), & \text{if } r_{n-2} = 1; \end{cases}$$

$$w_n^{[r_1r_2\cdots r_{n-2}]} = \begin{cases} f\left(w_{n-1}^{[r_1r_2\cdots r_{n-3}]}\right), & \text{if } r_{n-2} = 0, \\ g\left(w_{n-1}^{[r_1r_2\cdots r_{n-3}]}\right), & \text{if } r_{n-2} = 1; \end{cases}$$

and

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$$w_n^{\{r_1r_2\cdots r_{n-2}\}} = \begin{cases} f\left(w_{n-1}^{\{r_1r_2\cdots r_{n-3}\}}\right), & \text{if either } r_{n-2}=0 \text{ and } n \text{ is odd} \\ & \text{or } r_{n-2}=1 \text{ and } n \text{ is even,} \\ g\left(w_{n-1}^{\{r_1r_2\cdots r_{n-3}\}}\right) & \text{if either } r_{n-2}=1 \text{ and } n \text{ is odd} \\ & \text{or } r_{n-2}=0 \text{ and } n \text{ is even,} \end{cases}$$

$n \geq 3$. The superscript does not appear if the subscript is less than or equal to 2. For simplicity, we denote $w_n^{00\ldots0}$ (resp. $w_n^{(00\ldots0)}$, $w_n^{[00\ldots0]}$, $w_n^{\{00\ldots0\}}$) by w_n^0 (resp. $w_n^{(0)}$, $w_n^{[0]}$, $w_n^{\{0\}}$).

The word $w_n^{r_1r_2\cdots r_{n-2}}$ [or, more precisely, $w_n^{r_1r_2\cdots r_{n-2}}(a,b)$] is an n^{th}-order Fibonacci word derived from the pair of initial letters (a, b). More generally, we can define n^{th}-order Fibonacci words derived from a pair of initial words (x, y) (see [2]).

Now we have four binary trees whose nodes are words. We shall prove in Theorem 1 that each level of these trees consists of the n^{th}-order Fibonacci words with repetitions. More precisely, the words in each level of each tree is just a permutation of the words of the same level of any other tree, with the number of repetitions of each word unchanged. The relations between the Fibonacci words $w_n^{r_1r_2\cdots r_{n-2}}$, $w_n^{(r_1r_2\cdots r_{n-2})}$, $w_n^{[r_1r_2\cdots r_{n-2}]}$, and $w_n^{\{r_1r_2\cdots r_{n-2}\}}$ tell us how a particular Fibonacci word can be generated in different ways.

Theorem 1: Let $n \geq 3$, $r_1, r_2, \cdots, r_{n-2}$ be a binary sequence and let $s_i = 1 - r_i$, $1 \leq i \leq n-2$. Then

(a) $R(w_n^{r_1r_2\cdots r_{n-2}}) = w_n^{s_1s_2\cdots s_{n-2}}$.

Similar results hold for $w_n^{(r_1r_2\cdots r_{n-2})}$, $w_n^{[r_1r_2\cdots r_{n-2}]}$, and $w_n^{\{r_1r_2\cdots r_{n-2}\}}$.

(b) $w_n^{[r_1r_2\cdots r_{n-2}]} = w_n^{r_{n-2}\cdots r_2r_1}$.

(c) $$w_n^{(r_1r_2\cdots r_{n-2})} = \begin{cases} w_n^{r_1s_2r_3\cdots s_{n-3}r_{n-2}} & (n \text{ odd}), \\ w_n^{s_1r_2\cdots s_{n-3}r_{n-2}} & (n \text{ even}). \end{cases}$$

(d) $$w_n^{\{r_1r_2\cdots r_{n-2}\}} = w_n^{(r_{n-2}\cdots r_2r_1)} = \begin{cases} w_n^{r_{n-2}s_{n-3}r_{n-4}\cdots s_2r_1} & (n \text{ odd}), \\ w_n^{s_{n-2}r_{n-3}\cdots s_2r_1} & (n \text{ even}), \end{cases}$$

$$= \begin{cases} w_n^{[r_1s_2r_3\cdots s_{n-3}r_{n-2}]} & (n \text{ odd}), \\ w_n^{[r_1s_2\cdots r_{n-3}s_{n-2}]} & (n \text{ even}). \end{cases}$$

Proof: First, note that part 1 of (a) has been proved in [2]. Part 3 (resp. part 2) of (a) follows from (b) [resp. (c)] and part 1 of (a).

Assertions (b), (c), and (d) are proved by induction.

We illustrate the theorem with the following examples.

Example 1: $\{w_n^0\}$ and $\{w_n^1\}$ are well-known sequences of Fibonacci words (see [4]). Recently they are used by Hendel and Monteferrante [6] and by Chuan [5] to solve an extraction problem of the golden sequence posed by Hofstadter [7].

By Theorem 1,

$$w_n^{[0]} = w_n^0 = \begin{cases} w_n^{(010\ldots10)} & (n \text{ odd}), \\ w_n^{(10\ldots10)} = R(w_n^{(010\ldots101)}) & (n \text{ even}), \end{cases} \quad (n \geq 3).$$

The first equality means that the sequence given by $w_1 = a$, $w_2 = b$, and $w_n = w_{n-1}w_{n-2}$ $(n \geq 3)$ is precisely the sequence $\{w_n\}$ where $w_1 = a$, $w_2 = b$, and w_n is obtained from w_{n-1} ;by replacing each a in w_{n-1} by b and each b in w_{n-1} by ba. The second equality means that, if $q_1 = a$, $q_2 = b$, and

$$q_n = \begin{cases} R(q_{n-1})q_{n-2} & (n \text{ odd}), \\ q_{n-2}R(q_{n-1}) & (n \text{ even}), \end{cases} \quad (n \geq 3),$$

then $w_n = q_n$ if n is odd and $w_n = R(q_n)$ if n is even. A similar result holds for w_n^1.

Example 2: By Theorem 1,

$$w_n^{[0101\ldots]} = \begin{cases} w_n^{010\ldots10} & (n \text{ odd}), \\ w_n^{10\ldots10} = R(w_n^{010\ldots101}) & (n \text{ even}), \end{cases}$$

$$= w_n^{(00\ldots0)} = T^{F_{n-1}-1}(w_n^0) = T(w_n^1).$$

See [2] for the last two equalities. Again, the sequence $\{w_n^{(0)}\}$ can be generated by three different methods. This is also observed by Anderson [1].

Example 3: Let $v_1 = a$, $v_2 = b$, and $v_n = v_{n-2}R(v_{n-1})$ $(n \geq 3)$. Then $v_n = R(w_n)$ where w_n is as in Example 2. This is because

$$v_n = w_n^{(11\ldots1)} = R(w_n^{(00\ldots0)}) = R(w_n).$$

Example 4: Let $w_1 = a$, $w_2 = b$, and $w_n = w_{n-1}R(w_{n-2})$ $(n \geq 3)$. Then

(a) $w_n = w_n^{r_1r_2\cdots r_{n-2}}$ where

$$r_i = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

(b) w_n is symmetric $\Leftrightarrow n \not\equiv 0 \pmod 3$; hence, $\{w_n\}$ contains all the symmetric Fibonacci words (see [3]).

(c) $w_{3k+2} = w_{3k+1}R(w_{3k}) = w_{3k}w_{3k+1}$, $k \geq 1$.

(d) $w_n = \begin{cases} R(w_{n-1})w_{n-2}, & \text{if } n \equiv 0 \pmod 3, \\ w_{n-2}R(w_{n-1}), & \text{otherwise.} \end{cases}$

(e) $w_n = w_n^{(t_1t_2\cdots t_{n-2})}$ where $t_i = \begin{cases} 0, & \text{if } i \equiv 1 \pmod 3, \\ 1, & \text{otherwise.} \end{cases}$

2. LOCATING THE LETTERS

For $n > 2$, let

$$s = \begin{cases} F_{n-1} & (n \text{ even}), \\ F_{n-2} & (n \text{ odd}); \end{cases}$$

and

$$t = \begin{cases} F_{n-2} & (n \text{ even}), \\ F_{n-1} & (n \text{ odd}). \end{cases}$$

Theorem 2: Let $n > 2$, $q_n = w_n^{10101\ldots}$, $T^{js}(q_n) = c_1 c_2 \cdots c_{F_n}$ where $c_i \in \{a, b\}$. Then

$$\begin{aligned} c_k = a &\Leftrightarrow k \equiv (r+j)t \pmod{F_n} && \text{for some } 1 \le r \le F_{n-2} \\ &\Leftrightarrow k \equiv (r-j)s \pmod{F_n} && \text{for some } F_{n-1} \le r \le F_n - 1 \\ &\Leftrightarrow k \equiv 1+(r-j)s \pmod{F_n} && \text{for some } 0 \le r \le F_{n-2} - 1 \\ &\Leftrightarrow k \equiv 1+(r+j)t \pmod{F_n} && \text{for some } F_{n-1}+1 \le r \le F_n. \end{aligned} \tag{1}$$

$$\begin{aligned} c_k = b &\Leftrightarrow k \equiv (r+j)t \pmod{F_n} && \text{for some } F_{n-2}+1 \le r \le F_n \\ &\Leftrightarrow k \equiv (r-j)s \pmod{F_n} && \text{for some } 0 \le r \le F_{n-1} - 1 \\ &\Leftrightarrow k \equiv 1+(r-j)s \pmod{F_n} && \text{for some } F_{n-2} \le r \le F_n - 1 \\ &\Leftrightarrow k \equiv 1+(r+j)t \pmod{F_n} && \text{for some } 1 \le r \le F_{n-1}. \end{aligned}$$

Proof: The case where $j = 0$ in (1) has been proved in [2] and the other results follow easily from (1).

Given $r_1, r_2, \ldots, r_{n-2}$, to generate the Fibonacci word $w = w_n^{r_1 r_2 \cdots r_{n-2}}$, we first compute $k = \sum_{i=1}^{n-2} F_{i+1} r_i + 1$ and j satisfying

$$j \equiv \begin{cases} kF_{n-1} \pmod{F_n} & (n \text{ odd}), \\ kF_{n-1} - 1 \pmod{F_n} & (n \text{ even}), \end{cases}$$

and $1 \le j \le F_n$. Then $w = T^{js}(q_n)$ (see [2]); thus, any one of the first four conditions in Theorem 2 gives precisely the positions of the letter "a" in w. Hence, w can be constructed easily.

Besides using congruences, other methods of locating the letters are discussed in [4]; for example, using Zeckendorf representations and the golden ratio.

3. SHIFT OPERATION

It has been shown in [2] that Q_n consists of F_n distinct elements and, for any $w \in Q_n$, w, $T(w)$, ..., $T^{F_n - 1}(w)$ is a list of all these elements. In this way, every n^{th}-order Fibonacci word is a generator of Q_n.

4. ADJACENT TRANSPOSITION AND MINIMUM SUM

Let q_n, $n = 3, 4, \ldots, s, t$ be as in section 2. For $w = c_1 c_2 \cdots c_m$ where c_j equals a or b, we designate by $S(w)$ the sum of the indices j for which $c_j = a$ and, for $1 \le k \le m$, we put

$$h_k(w) = d_1 d_2 \cdots d_m$$

where $d_k = c_{k+1}$, $d_{k+1} = c_k$, with subscripts modulo m, and $d_j = c_j$, otherwise.

Theorem 3: For $1 \le j \le F_n$, let $k_j \equiv jt \pmod{F_n}$ and $1 \le k_j \le F_n$. Then

$$h_{k_j}(T^{(j-1)s}(q_n)) = T^{js}(q_n),\ 1 \le j \le F_n.$$

Proof: By Theorem 2, the positions of the letter "a" in $T^{(j-1)s}(q_n)$, $h_{k_j}(T^{(j-1)s}(q_n))$, $T^{js}(q_n)$ are, respectively,

$$jt,\ (j+1)t, \ldots, (j+F_{n-2}-1)t, \tag{2}$$

$$jt+1,\ (j+1)t, \ldots, (j+F_{n-2}-1)t, \tag{3}$$

$$(j+1)t, \ldots, (j+F_{n-2}-1)t,\ (j+F_{n-2})t, \tag{4}$$

modulo F_n. Since $(j+F_{n-2})t \equiv jt+1 \pmod{F_n}$, it follows that $h_{k_j}(T^{(j-1)s}(q_n)) = T^{js}(q_n)$.

Corollary 1: Let $u^{(0)} = q_n$, $u^{(j)} = h_{k_j}(u^{(j-1)})$, $1 \le j \le F_n - 1$. Then the sequence $u^{(0)}, u^{(1)}, \ldots, u^{(F_n-1)}$ is precisely the sequence $q_n, T^s(q_n), \ldots, T^{(F_n-1)s}(q_n)$ and consists of all n^{th}-order Fibonacci words.

More generally, given a word $w \in Q_n$, let $0 \le j \le F_n - 1$ be such that

$$j \equiv S(w) - S(q_n) \equiv S(w) - F_{n-2}(F_{n-2}+1)t/2 \pmod{F_n}.$$

[The last congruence follows from (4).] Then $w = T^{js}(q_n)$, so the sequence

$$v^{(0)} = w,\ v^{(r)} = h_{k_{j+r}}(v^{(r-1)}),\ 1 \le r \le F_n - 1 \tag{5}$$

(with subscript $j+r$ modulo F_n) coincides with the sequence

$$T^{js}(q_n),\ T^{(j+1)s}(q_n), \ldots,\ T^{(j+F_n-1)s}(q_n)$$

and consists of all the n^{th}-order Fibonacci words. The importance of this method is that, in the sequence (5), any two successive Fibonacci words differ only by a pair of consecutive letters (the first and the last letter in a word are considered as consecutive letters). This gives a simple way of generating all the n^{th}-order Fibonacci words from any given n^{th}-order Fibonacci word.

For example, with $n = 6$ and $w = bababbab$, we have $j \equiv 3$, and the sequence $v^{(r)}$ in (5) is given as follows:

r	$j+r \pmod{F_n}$	k_{j+r}	$v^{(r)}$
0	3		$bab\mathbf{ab}bab$
1	4	4	$babbab\mathbf{ab}$
2	5	7	$b\mathbf{ab}babba$
3	6	2	$bbab\mathbf{ab}ba$
4	7	5	$\mathbf{b}babbab\mathbf{a}$
5	8	8	$ab\mathbf{ab}babb$
6	1	3	$abbab\mathbf{ab}b$
7	2	6	$abbabbab$

When the "**ab**" in bold face in each word in the last column is replaced by "*ba*," the next word is obtained. Note also that, in view of Corollary 1, the same list of Fibonacci words can be obtained by shifting the letters in the Fibonacci word five places to the left in each step.

Corollary 2: $S(T^{js}(q_n)) - S(T^{(j-1)s}(q_n)) = 1,\ 1 \le j \le F_n - 1.$

Proof: If $1 \le j \le F_n - 1$, then $k_j \not\equiv F_n$; thus,

$$S(T^{js}(q_n)) = S(h_{k_j}(T^{(j-1)s}(q_n))) = S(T^{(j-1)s}(q_n)) + 1$$

according to (2) and (3).

We have seen in [3] that $T^{(F_n-1)s}(q_n) = R(q_n)$. Therefore, we obtain the following corollary.

Corollary 3: $S(q_n) = \min\{S(w): w \in Q_n\}$; $S(R(q_n)) = \max\{S(w): w \in Q_n\}$.

Finally, it is easy to see that $S(q_n)$ and $S(w_n^0)$ satisfy, respectively, the following recursive relations:

$$S(q_n) = \begin{cases} S(q_{n-1}) + S(q_{n-2}) + F_{n-4}F_{n-1}, & \text{if } n \text{ is even,} \\ S(q_{n-1}) + S(q_{n-2}) + F_{n-3}F_{n-2}, & \text{if } n \text{ is odd,} \end{cases}$$

$$= \begin{cases} S(q_{n-1}) + S(q_{n-2}) + F_{n-3}F_{n-2} - 1, & \text{if } n \text{ is even,} \\ S(q_{n-1}) + S(q_{n-2}) + F_{n-3}F_{n-2}, & \text{if } n \text{ is odd,} \end{cases}$$

$$S(w_n^0) = S(w_{n-1}^0) + S(w_{n-2}^0) + F_{n-4}F_{n-1},$$

$n \ge 5$, and $S(q_3) = S(q_4) = 1$, $S(w_3^0) = S(w_4^0) = 2$. Also, we have $S(q_n) \equiv F_{n-2}(F_{n-2}+1)t/2$ (mod F_n) according to (4).

5. FIBONACCI WORD PATTERNS

The *Fibonacci word patterns* $F^0(a, b)$ and $F^1(a, b)$ are defined by

$$F^0(a, b) = w_1 w_2 w_3^0 w_4^0 \ldots w_n^0 \ldots,$$

$$F^1(a, b) = w_1 w_2 w_3^1 w_4^1 \ldots w_n^1 \ldots,$$

where $w_1 = a, w_2 = b$. $F^1(a, b)$ has been studied by Turner ([8], [9]), and $F^1(b, ab)$ is a golden sequence.

The following embedding theorem has been proved in [4]. The notation $u[p:q]$ means the subword $a_p a_{p+1} \ldots a_q$ of the infinite word $u = a_1 a_2 a_3 \ldots$ where each a_n, $n \ge 1$, is a letter.

Theorem 4 (Embedding Theorem):

(a) Let all the Fibonacci words be listed in the following order:

$$w_1, w_2, w_3^0, T(w_3^0), \ldots, w_n^0, T(w_n^0), \ldots, T^{F_n-1}(w_n^0), \ldots.$$

Then the j^{th} Fibonacci word in the above list is $T^i(w_n^0)$ where n is the largest positive integer such that $F_{n+1} \le j$ and $i = j - F_{n+1}$. This Fibonacci word is precisely $F^0(a, b)[j : j + F_n - 1]$.

(b) Let all the Fibonacci words be listed in the following order:

$$w_1, w_2, T(w_3^1), T^2(w_3^1), \ldots, T(w_n^1), T^2(w_n^1), \ldots, T^{F_n}(w_n^1), \ldots.$$

Then the j^{th} Fibonacci word in the above list is $T^i(w_n^1)$ where n is the largest positive integer such that $F_{n+1} \le j$ and $i = j - F_{n+1} + 1$. This Fibonacci word is precisely $F^1(a, b)[j - F_n + 1 : j]$.

In other words, all the Fibonacci words are embedded in the Fibonacci word patterns $F^0(a, b)$ and $F^1(a, b)$ in the above sense.

6. GENERATION WITHOUT REPETITIONS

Besides those methods described in Sections 3-5, we shall develop two additional methods of generating all the n^{th}-order Fibonacci words without repetitions.

Let R be the set of all words in $X^* \setminus \{1\}$ that contain no consecutive letters "a." As before, the first and the last letter in a word are considered as consecutive letters. Clearly, each Q_n is a subset of R. For $w \in R$, let $h(w)$ be the word obtained from w by wrapping w around then replacing each ba in w by ab and then unwrapping it. For example,

$$h(\mathbf{ba}bb\mathbf{ba}bb) = abbabbb,$$
$$h(\mathbf{a}bb\mathbf{ba}b\mathbf{b}) = bbabba.$$

Only the letters in bold face have to be replaced.

Lemma 1: $h(w) = T(w)$ for all $w \in R$.

Proof: Let $w \in R$. Write

$$w = a_1a_2 \cdots a_n$$
$$h(w) = c_1c_2 \cdots c_n.$$

From the definition of h, we have

$$c_i = \begin{cases} b, & \text{if } a_ia_{i+1} = bb, \\ a, & \text{if } a_ia_{i+1} = ba, \\ b, & \text{if } a_ia_{i+1} = ab, \end{cases}$$

$1 \le i \le n$, with subscripts modulo n. Hence, $c_i = a_{i+1}$, $1 \le i \le n$, with subscripts modulo n. Therefore, $h(w) = T(w)$.

Theorem 5: Let $w \in Q_n$. Then the sequence

$$u^{(0)} = w,\ u^{(j)} = h(u^{(j-1)}),\quad j = 1, 2, \ldots, F_n - 1,$$

is precisely the sequence $w, T(w), \ldots, T^{F_n - 1}(w)$ and consists of all the n^{th}-order Fibonacci words.

Next we turn to a result that is related to the operations f and g defined in Section 1.

Lemma 2: Let $w \in X^* \setminus \{1\}$. Then

(a) $bg(w) = f(w)b$.

(b) $f(T(w)) = \begin{cases} g(w), & \text{if } w \text{ begins with an "}a\text{,"} \\ T(g(w)), & \text{if } w \text{ begins with a "}b\text{."} \end{cases}$

(c) $T(f(w)) = g(w)$.

Proof:

(a) We prove the result by induction on the length m of w. Clearly, the result holds for $m = 1$. Now assume that the result is true for some $m \geq 1$. Let $w \in X^* \setminus \{1\}$ have length m. Then

$$bg(aw) = bbg(w) = bf(w)b = f(aw)b,$$
$$bg(bw) = babg(w) = baf(w)b = f(bw)b,$$

by the induction hypothesis.

(b) By part (a), we have, for any $u \in X^*$,

$$f(T(au)) = f(ua) = f(u)b = bg(u) = g(au),$$
$$f(T(bu)) = f(ub) = f(u)ba = bg(u)a = T(abg(u)) = T(g(bu)).$$

Therefore, (b) holds.

(c) Clearly, this holds for w having length 1. Assume that w has length ≥ 1. Then

$$T(f(aw)) = T(bf(w)) = f(w)b = bg(w) = g(aw),$$
$$T(f(bw)) = T(baf(w)) = af(w)b = abg(w) = g(bw),$$

by part (a). Therefore, (c) follows.

With this lemma, we now have a method of generating Q_{n+1}, without repetition, from Q_n by means of f and g.

Let $n \geq 3$. List the images of the sequence $w_n^0, T(w_n^0), \ldots, T^{F_n - 1}(w_n^0)$ under f and g in the following order:

$$f(w_n^0), g(w_n^0), \ldots, f(T^i(w_n^0)), g(T^i(w_n^0)), \ldots, f(T^{F_n - 1}(w_n^0)), g(T^{F_n - 1}(w_n^0))$$

Then take away $g(T^i(w_n^0))$ from the list if $T^i(w_n^0)$ begins with an "a" because, in this case, $g(T^i(w_n^0)) = f(T^{i+1}(w_n^0))$ according to Lemma 2(b). Since there are F_{n-2} n^{th}-order Fibonacci words beginning with an "a" (see [2]), it follows that there are F_{n+1} words left in the list. Now, according to Lemma 2, we see that the resulting sequence coincides with the sequence

$$w_{n+1}^0, T(w_{n+1}^0), \ldots, T^{F_n - 1}(w_{n+1}^0).$$

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EXTRACTION PROPERTY OF THE GOLDEN SEQUENCE

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Let a and b be two distinct letters and let $\tau = (\sqrt{5}-1)/2$. Let x be the infinite string whose n^{th} term is "a" if $[(n+1)\tau]-[n\tau]=0$ and is "b" if $[(n+1)\tau]-[n\tau]=1$. Let s_m be the left factor of x of length m and let x_m be the corresponding right factor of x. Note that $x = x_0$ is the golden sequence. It is known that

$$x = c_1c_2c_3c_4\dots \tag{1}$$

where $c_0 = a$, $c_1 = b$, and $c_{n+1} = c_{n-1}c_n$ $(n \geq 1)$. In the notation of [1]-[3], $x = F^1(b, ab)$, $c_n = w_{n+1}^1$ and $s_{F_n} = w_n^0$ $(n \leq 1)$, where F_n denotes the n^{th} Fibonacci number.

Hofstadter [6] formulated the concept of aligning two strings. By way of illustration, we present the procedure by which x_m is aligned with $x = x_0$.

Starting from the $(m+1)^{\text{st}}$ term in x, an attempt is made to match each term in x with a term in x_m. After a term in x is matched with a term in x_m, one looks for the earliest match to the next term in x. Those terms in x_m that are skipped over form the extracted string $y_{m,0}$. For example, when $m = 4$,

$$\begin{array}{lccccccccccccccccccc}
x_4: & a & b & a & b & b & a & b & b & a & b & a & b & b & a & b & a & b & b & \cdots \\
 & & | & | & | & | & | & | & & | & | & & | & & | & | & & | & & \\
x: & & b & a & b & b & a & b & & a & b & & b & & a & b & & b & & \dots \\
y_{4,0}: & a & & & & & & & b & & & a & & b & & & a & & b & \dots
\end{array}$$

It was Hendel and Monteferrante [4] who first reformulated Hofstadter's alignment concept in terms of a formal relation on strings. If x_m aligns with x_n with extraction $y_{m,n}$, then we notationally indicate this by

$$x_m \supset x_n;\ y_{m,n}. \tag{2}$$

[4] also introduced the idea of representing x_m as a product of c_α with specific properties by using a canonical representation $x_m = c_{\alpha(1)}c_{\alpha(2)}\cdots$ where $\alpha(k)$ is an increasing function on the positive integers that can be derived from the Zeckendorf representation of m as a sum of Fibonacci numbers. Using this, they were able to completely determine $y_{m,0}$ for all positive integers m.

The goal of this paper is to determine the remaining cases of $y_{m,n}$. In Section 2, $y_{0,m}$ is found to be precisely the reverse $R(s_m)$ of the left factor s_m of x of length m.

Here the *reverse operation* R is defined by

$$R(a_1a_2\dots a_k) = a_k\dots a_2a_1,$$

where $a_1, a_2, \dots, a_k$ are letters. The importance of the reversal operation in studying x was first observed by Higgins [5]. In Section 4, it is shown that $y_{m,n}$ and $y_{m-1,n-1}$ differ by at most the first letter. From this, $y_{m,n}$ can easily be determined by $y_{m-n,0}$ (if $m > n$) or $y_{0,n-m}$ (if $n > m$).

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1. BASIC LEMMAS AND DEFINITIONS ON EXTRACTION

The following definitions come from [4, Definitions 1 and 2]. Suppose that $U = u_1 \dots u_n$, $V = v_1 \dots v_m$, and $E = e_1 \dots e_p$ with $u_i, v_j, e_k \in \{a, b\}, n, m > 0, p \geq 0$, and $n = m + p$. We say that U *aligns* (with) V with *extraction* E if there exist integers $j(0), j(1), j(2), \dots, j(p)$ such that

$$U = (v_1 \dots v_{j(1)})e_1(v_{j(1)+1} \dots v_{j(2)})e_2 \dots e_p(v_{j(p)+1} \dots v_m),$$

with $v_i \dots v_k$ empty if $k < i$ and

(i) $0 = j(0) \leq j(1) \leq j(2) \leq \dots \leq j(p) < m$,

(ii) $e_i \neq v_{j(i)+1}$, for $1 \leq i \leq p$.

This relationship is called an *alignment* and is denoted by $U \supset V; E$. The strings U, V, and E are called the *original*, *aligned*, and *extracted strings*, respectively. If $U = V$, we write $U \supset V; 1$, where 1 denotes the empty string.

Suppose that U, V, and E are (possibly infinite) strings. Suppose that $U(n)$, $V(n)$, and $E(n)$, $n \geq 1$, are sequences of finite strings such that $U(n) \supset V(n);\ E(n)$, $\lim U(n) = U$, $\lim V(n) = V$, and $\lim E(n) = E$. Then we say that U *aligns* V with *extraction* E. This alignment is also denoted by $U \supset V; E$.

Lemma 1.1 [4, Lemmas 1 and 3]:

(a) (Uniqueness of extracted string) For given strings U and V, there is at most one string E such that $U \supset V; E$.

(b) (Concatenation) If U_i, V_i, and E_i, $1 \leq i \leq m$, are strings of finite lengths and if $U_i \supset V_i; E_i$, $1 \leq i \leq m$, then

$$U_1U_2 \dots U_m \supset V_1V_2 \dots V_m; E_1E_2 \dots E_m.$$

Lemma 1.2:

(i) $c_n \supset c_{n-1};\ c_{n-2},\ n \geq 2$.

(ii) $c_n \supset c_n;\ 1,\ n \geq 1$.

(iii) $c_n = c_{n-2}c_{n-1},\ n \geq 2$.

(iv) $c_nc_{n+1} \dots c_p \supset c_{n+1} \dots c_p;\ c_n,\ 1 \leq n < p$.

(v) $c_nc_{n+2} \supset c_{n+2};\ c_n,\ n \geq 1$.

(vi) $c_nc_n \supset c_{n+1};\ c_{n-2},\ n \geq 2$.

(vii) $c_nc_{n+3} \supset c_{n+1}c_{n+2};\ c_n,\ n \geq 0$.

Proof: Part (i) has been proved in [4] by induction. Parts (ii) and (iii) are trivial. According to (i) and (ii), we have

$$c_nc_{n+1} \supset c_{n+1};\ c_n$$
$$c_{n+i} \supset c_{n+i};\ 1,\ 2 \leq i \leq p-n.$$

Part (iv) now follows by concatenation [Lemma 1.1(b)]. The proofs of (v)-(vii) are similar to (iv).

Lemma 1.3: Let $t \geq 1$. Let $\gamma(0) = 0$ and let $\gamma(1), \ldots, \gamma(t)$ be positive integers such that $\gamma(i) + 2 \leq \gamma(i+1)$, $1 \leq i \leq t-1$. Let

$$U = c_1 c_2 \ldots c_{\gamma(t)} c_{\gamma(t)+1}$$

$$V = \begin{cases} c_1 c_2 \ldots c_{\gamma(1)-1} c_{\gamma(1)+1}, & \text{if } t = 1, \\ (c_1 c_2 \ldots c_{\gamma(1)-1})(c_{\gamma(1)+1} \ldots c_{\gamma(2)-1}) \cdots (c_{\gamma(t-1)+1} \ldots c_{\gamma(t)-1}) c_{\gamma(t)+1}, & \text{otherwise} \end{cases}$$

$$E = c_{\gamma(1)} c_{\gamma(2)} \ldots c_{\gamma(t)},$$

where the factor $c_1 c_2 \ldots c_{\gamma(1)-1}$ does not appear if $\gamma(1) = 1$. Then $U \supset V;\ E$.

Proof: By Lemma 1.2, we have

$$\begin{aligned} &c_1 c_2 \ldots c_{\gamma(1)-1} \supset c_1 c_2 \ldots c_{\gamma(1)-1};\ 1, && \text{if } \gamma(1) > 1, \\ &c_{\gamma(i)} c_{\gamma(i)+1} \ldots c_{\gamma(i+1)-1} \supset c_{\gamma(i)+1} \ldots c_{\gamma(i+1)-1};\ c_{\gamma(i)}, && 1 \leq i \leq t-1, \\ &c_{\gamma(t)} c_{\gamma(t)+1} \supset c_{\gamma(t)+1};\ c_{\gamma(t)}. \end{aligned}$$

The result now follows by concatenation.

Lemma 1.4 [4, Lemma 5]: Let $m \geq 1$ have Zeckendorf representation

$$m = F_{k(1)} + F_{k(2)} + \cdots + F_{k(t)} \tag{3}$$

with $k(1) \geq 2$, $k(i) + 2 \leq k(i+1)$, $i = 1, \ldots, t-1$. Let $\gamma(i) = k(i) - 1$, $1 \leq i \leq t$, and let V be as in Lemma 1.3. Then

$$x_m = V c_{\gamma(t)+2} c_{\gamma(t)+3} \cdots . \tag{4}$$

The ordered collection of indices $1, 2, \ldots, \gamma(1)-1, \gamma(1)+1, \ldots, \gamma(2)-1, \ldots, \gamma(t-1)+1, \ldots, \gamma(t)-1, \gamma(t)+1, \gamma(t)+2, \ldots$ is called the *canonical representation of* x_m. Actually [4, Definition 3] uses the term "canonical representation" to refer to the function of the positive integers enumerating this ordered collection. However, in the sequel, if there is no ambiguity, we will simply, by abuse of language, call (4) the canonical representation of x_m.

Corollary 1.5: Let $x_m = c_{\alpha(1)} c_{\alpha(2)} \ldots$ be a canonical representation. Then

(i) $(\alpha(1), \alpha(2)) \in \{(1,2), (1,3), (2,3), (2,4)\}$.

(ii) $\alpha(k+1) \in \{\alpha(k)+1, \alpha(k)+2\}$, for all $k \geq 1$.

(iii) There exists a positive integer r such that $\alpha(k+1) = \alpha(k) + 1$ for all $k \geq r$.

2. THE ALIGNMENTS $x \supset x_m;\ y_{0,m}$ AND $x_m \supset x;\ y_{m,0}$

We now express the extraction $y_{0,m}$ in terms of the c_i.

Lemma 2.1: For $m \geq 1$, let m have Zeckendorf representation (3). Let $\gamma(i) = k(i) - 1$, $1 \leq i \leq t$. Then

$$y_{0,m} = c_{\gamma(1)} \cdots c_{\gamma(t)},$$

where $y_{0,m}$ is defined by (2).

Proof: The result follows from (1), (4), Lemma 1.3, and Lemma 1.2(ii) by concatenation.

Next, we look at the left factors of the golden sequence. Let

$$w_1 = a,\ x_2 = b,\ w_{n+1} = w_n w_{n-1},\ n \geq 2.$$

In the notation of [1]-[3], $w_n = w_n^0,\ n \geq 1$.

Lemma 2.2: Let $n \geq 4$. Then $w_n w_n$ is a left factor of x.

Proof: First, observe that

$$\begin{aligned} w_{n+2} &= w_{n+1}w_n \\ &= (w_n w_{n-1})(w_{n-1}w_{n-2}) \\ &= w_n w_{n-1} w_{n-2} w_{n-3} w_{n-2} \\ &= w_n w_n w_{n-3} w_{n-2}. \end{aligned}$$

By Lemma 1.4 of [3], w_{n+2} is a left factor of x, for all $n \geq 4$. The result immediately follows.

Lemma 2.3: Let $m \geq 1$ have Zeckendorf representation (3). Then

$$s_m = w_{k(t)} \dots x_{k(2)} w_{k(1)}.$$

Proof: The result clearly holds for $m = 1, 2, 3$. Suppose $m \geq 4$ and that the result is true for all positive integers less than m.

First, suppose $t = 1$ so that, by (3), $m = F_n$ for some n. By Lemma 2.2, w_n is a left factor of x. By definition, s_m is also a left factor of x. Since both these left factors of x have the same length F_n, they are both equal.

Next, suppose that t > 1. Then, by (3),

$$F_{k(t)} < m < F_{k(t)+1} \leq 2F_{k(t)}.$$

Note that $s_{F_{k(t)}} = w_{k(t)}$ since they are both left factors of x of the same length. let

$$s_m = s_{F_{k(t)}} s = w_{k(t)} s,$$

where s has length $m - F_{k(t)}$. By Lemma 2.2, $w_{k(t)} w_{k(t)}$ is a left factor of x. Since $s_m = w_{k(t)} s$ is also a left factor of x, it follows that s is a left factor of $w_{k(t)}$. Therefore, $s = s_{m-F_{k(t)}}$; hence,

$$s_m = w_{k(t)} s_{m-F_{k(t)}}.$$

By the induction hypothesis, the Zeckendorf representation

$$m - F_{k(t)} = F_{k(t-1)} + \cdots + F_{k(2)} + F_{k(1)}$$

gives the factorization

$$s_{m-F_{k(t)}} = w_{k(t-1)} \cdots w_{k(2)} w_{k(1)}.$$

Consequently, s_m has the desired factorization.

Theorem 2.4: For $m \geq 1$,

$$y_{0,m} = R(s_m).$$

Proof: We have

$R(s_m) = R(w_{k(1)})R(w_{k(2)}) \cdots R(w_{k(t)})$, by Lemma 2.3,

$= c_{k(1)-1}c_{k(2)-1} \cdots c_{k(t)-1}$, by the result, $R(w_n) = c_{n-1}$, of Theorem 3 in [1],

$= c_{\gamma(1)} \cdots c_{\gamma(t)}$, using the notations of Lemma 1.4,

$= y_{0,m}$, by Lemma 2.1.

Theorem 2.5 (Modified Hofstadter's conjecture [4]): Let $m \geq 2$ have Zeckendorf representation (3). Then

$$x_m \supset x;\ ax_{m-1}, \quad \text{if } k(1) = 2 \text{ and } k(2) \text{ is even;}$$
$$x_m \supset x;\ x_{m-2}, \quad \text{otherwise.}$$

In other words, $y_{m,0} = ax_{m-1}$ in the first case (this is also true when $m = 1$) and $y_{m,0} = x_{m-2}$ in the second case.

3. SOME LEMMAS

The goal of this section is to prove that, under appropriate conditions, if $s \supset t;\ u$, then $c_p s \supset t;\ c_p u$. The precise statement and conditions are set forth in Lemma 3.5. The major tool in proving Lemma 3.5 will be Lemma 3.1, which considers three cases.

Throughout this section, we let $p \geq 2$ and we let

$$s = c_{\alpha(1)}c_{\alpha(2)} \cdots$$
$$t = c_{\beta(1)}c_{\beta(2)} \cdots$$

with

$$\alpha(1) = p+2 \text{ or } p+3, \qquad \beta(1) = p+1 \text{ or } p+2.$$

We suppose that r is a positive integer such that, for $k < r$, we have

$$\alpha(k+1) \in \{\alpha(k)+1,\ \alpha(k)+2\}, \qquad \beta(k+1) \in \{\beta(k)+1,\ \beta(k)+2\},$$

while, for $k \geq r$, we have

$$\alpha(k+1) = \alpha(k)+1, \qquad \beta(k+1) = \beta(k)+1.$$

Lemma 3.1: There is some k such that either cases (i) and (ii) listed below hold, or else case (iii) below holds for all k.

Case (i). There exists a string u_k such that

$$c_{\alpha(1)} \cdots c_{\alpha(k)} \supset c_{\beta(1)} \cdots c_{\beta(k)};\ u_k, \tag{5}$$

$$c_p c_{\alpha(1)} \cdots c_{\alpha(k)} \supset c_{\beta(1)} \cdots c_{\beta(k)};\ c_p u_k. \tag{6}$$

Case (ii). There exists a string u_k such that

$$c_{\alpha(1)} \cdots c_{\alpha(k-1)} c_{\alpha(k)-2} \supset c_{\beta(1)} \cdots c_{\beta(k)};\ u_k, \tag{7}$$

$$c_p c_{\alpha(1)} \cdots c_{\alpha(k-1)} c_{\alpha(k)-2} \supset c_{\beta(1)} \cdots c_{\beta(k)};\ c_p u_k. \tag{8}$$

Case (iii).

$$\beta(k) = \alpha(k) - 1, \tag{9}$$

and there exist strings u_k and v_k such that

$$v_k c_{\alpha(k)-1} = c_p u_k, \tag{10}$$

$$c_{\alpha(1)} \dots c_{\alpha(k)} \supset c_{\beta(1)} \dots c_{\beta(k)};\ u_k, \tag{11}$$

$$c_p c_{\alpha(1)} \dots c_{\alpha(k-1)} c_{\alpha(k)-2} \supset c_{\beta(1)} \dots c_{\beta(k)};\ v_k. \tag{12}$$

The factor $c_{\alpha(1)} \dots c_{\alpha(k-1)}$ in (7), (8), and (12) does not appear if $k = 1$.

Proof: Lemma 3.1 follows immediately from the statements of Lemmas 3.2 and 3.3 which are proved below.

Lemma 3.2: If $k = 1$, then one of the three cases listed in Lemma 3.1 holds.

Proof: There are four cases to consider, according to the values of $\alpha(1)$ and $\beta(1)$.

Case (a). $\alpha(1) = p+2$ and $\beta(1) = p+1$

We show that case (iii) holds with $u_1 = c_p$ and $v_1 = c_{p-2}$. Clearly (9) holds. By Lemma 1.2(iii), (10) is satisfied. Alignment (11) follows from Lemma 1.2(i), while alignment (12) follows from Lemma 1.2(vi).

Case (b). $\alpha(1) = p+2$ and $\beta(1) = p+2$

We show that (i) holds with $u_1 = 1$. Then (5) follows from Lemma 1.2(ii) and (6) follows from Lemma 1.2(v).

Case (c). $\alpha(1) = p+3$ and $\beta(1) = p+1$

We show that (ii) holds with $u_1 = 1$. Then (7) follows from Lemma 1.2(ii) and (8) follows from Lemma 1.2(iv).

Case (d). $\alpha(1) = p+3$ and $\beta(1) = p+2$

We show that (iii) holds with $u_1 = c_{p+1}$ and $v_1 =$. Clearly (9) holds. Lemma 1.2(iii) implies equation (10), alignment (11) follows from Lemma 1.2(i), and (12) follows from Lemma 1.2(iii).

Lemma 3.3: Suppose, for some integer $k \geq 1$, case (iii) of Lemma 3.1 holds. Then, for $k+1$, one of the three cases of Lemma 3.1 holds.

Proof: First, note that, by (9), $\beta(k+1) \in \{\alpha(k), \alpha(k)+1\}$. There are now four cases to consider, according to the values of $\alpha(k+1)$ and $\beta(k+1)$.

Case (a). $\alpha(k+1) = \alpha(k)+1$ and $\beta(k+1) = \alpha(k)$

Let

$$u_{k+1} = u_k c_{\alpha(k)-1} \text{ and } v_{k+1} = v_k c_{\alpha(k)-3}. \tag{13}$$

We show that (iii) holds with $k+1$ replacing k. Clearly $\beta(k+1) = \alpha(k+1)-1$. By Lemma 1.2(iii) and (10), we have

$$\begin{aligned} v_{k+1} c_{\alpha(k+1)-1} &= v_k c_{\alpha(k)-3} c_{\alpha(k)} = v_k c_{\alpha(k)-3} c_{\alpha(k)-2} c_{\alpha(k)-1} \\ &= v_k c_{\alpha(k)-1} c_{\alpha(k)-1} = c_p u_k c_{\alpha(k)-1} = c_p u_{k+1}. \end{aligned}$$

This demonstrates that (10) holds with k replaced by $k+1$.

To prove that (11) holds with $k+1$ replacing k, we concatenate the following two alignments: (11) as is, with k and $c_{\alpha(k+1)} \supset c_{\beta(k+1)}$; $c_{\alpha(k)-1}$, the last alignment following from Lemma 1.2(i).

To prove that (12) holds with $k+1$ replacing k, we concatenate the following two alignments: (12) as is, with k and $c_{\alpha(k)-1}c_{\alpha(k+1)-2} \supset c_{\beta(k+1)}$; $c_{\alpha(k)-3}$, the last alignment following from Lemma 1.2(vi) with $n=\alpha(k)$. Alignment (12) with $k+1$ replacing k then holds since, by Lemma 1.2(iii), $c_{\alpha(k)-2}c_{\alpha(k)-1} = c_{\alpha(k)}$.

Case (b). $\alpha(k+1) = \alpha(k)+1$ and $\beta(k+1) = \alpha(k)+1$

Let $u_{k+1} = u_k$. We prove that (i) holds with $k+1$ replacing k.

To prove that (5) holds with $k+1$ replacing k, we concatenate the following two alignments: (11) and $c_{\alpha(k+1)} \supset c_{\beta(k+1)}$; 1, this last alignment holding by Lemma 1.2(ii).

To prove (6) with $k+1$ replacing k, we concatenate the following two alignments: (12) and $c_{\alpha(k)-1}c_{\alpha(k+1)} \supset c_{\beta(k+1)}$); $c_{\alpha(k)-1}$, the last alignment following from Lemma 1.2(v). Alignment (6) with $k+1$ replacing k then follows from (10) and Lemma 1.2(iii) with $n=\alpha(k)$.

Case (c). $\alpha(k+1) = \alpha(k)+2$ and $\beta(k+1) = \alpha(k)$

Let $u_{k+1} = u_k$. We show that (ii) holds with $k+1$ replacing k.

To prove (7) with $k+1$ replacing k, we concatenate the following two alignments: (11) and $c_{\alpha(k+1)-2} \supset c_{\beta(k+1)}$; 1, the last alignment following from Lemma 1.2(ii).

To prove (8) with $k+1$ replacing k, we concatenate the following two alignments: (12) and $c_{\alpha(k)-1}c_{\alpha(k+1)-2} \supset c_{\beta(k+1)}$; $c_{\alpha(k)-1}$, the last alignment following from Lemma 1.2(iii) and (i). Alignment (8) with $k+1$ replacing k then follows from (10) and Lemma 1.2(iii) with $n=\alpha(k)$.

Case (d). $\alpha(k+1) = \alpha(k)+2$ and $\beta(k+1) = \alpha(k)+1$

Let $u_{k+1} = u_k c_{\alpha(k)}$ and let $v_{k+1} = v_k$. We show that (iii) holds with $k+1$ replacing k. Clearly (10) with $k+1$ replacing k follows from (10) as is and Lemma 1.2(iii).

To prove (11) with $k+1$ replacing k, we concatenate the following two alignments: (11) as is and $c_{\alpha(k+1)} \supset c_{\beta(k+1)}$; $c_{\alpha(k)}$, the last alignment following from Lemma 1.2(i).

To prove (12) with $k+1$ replacing k, we concatenate the following two alignments: (12) as is and $c_{\alpha(k)-1}c_{\alpha(k+1)-2} \supset c_{\beta(k+1)}$; 1, the last alignment following from Lemma 1.2(iii) and (i).

As already noted, Lemmas 3.2 and 3.3 provide an inductive proof to Lemma 3.1.

Lemma 3.4:

(i) If cases (i) and (iii) of Lemma 3.1 do not hold for any k, then eventually (for all $k \geq r$) we are in case (a) of Lemma 3.3.

(ii) In such a case, v_k (resp. u_k) is a proper left factor of v_{k+1} (resp. u_{k+1}).

Proof: By the hypothesis of this lemma, Lemma 3.2, and Lemma 3.3, case (iii) of Lemma 3.1 must hold for all k. By the hypothesis at the beginning of the section, $\alpha(k+1) = \alpha(k)+1$ for all $k \geq r$. Hence, of the four cases of Lemma 3.3, case (d) cannot hold and, clearly, cases (b) and (c) also do not hold. This proves assertion (i).

Assertion (ii) follows from equation (13).

We are now in a position to state the main lemma.

Lemma 3.5: Assume that the notations and assumptions stated at the beginning of this section hold. If $s \supset t;\, u$, then $c_p s \supset t;\, c_p u$.

Proof: The proof of Lemma 3.5 follows directly from the proof of Lemmas 3.6 and 3.7 below.

Lemma 3.6: If, for some k, case (i) or (ii) of Lemma 3.1 holds, then Lemma 3.5 is true.

Proof: Let

$$s' = c_{\alpha(k+1)} c_{\alpha(k+2)} \cdots,$$
$$t' = c_{\beta(k+1)} c_{\beta(k+2)} \cdots.$$

Then

$$s = c_{\alpha(1)} \cdots c_{\alpha(k)} s',$$
$$t = c_{\beta(1)} \cdots c_{\beta(k)} t'.$$

If (i) holds, then define u' so that $s' \supset t';\, u'$. Note that u' exists because s' and t' each have an infinite number of "a"s and "b"s. By concatenating this alignment with (5) and (6), respectively, we obtain

$$s \supset t;\; u_k u'$$
$$c_p s \supset t;\; c_p u_k u'.$$

Hence, $u_k u' = u$ by uniqueness of extracted strings, $c_p u_k u' = c_p u$ and we are done.

If (ii) holds, let $c_{\alpha(k)-1} s' \supset t';\, u'$. Then

$$s \supset t;\; u_k u'$$
$$c_p s \supset t;\; c_p u_k u'$$

with $u_k u' = u$, $c_p u_k u' = c_p u$ and again we are done.

Lemma 3.7: If cases (i) and (ii) of Lemma 3.1 do not hold for any k, then Lemma 3.5 is true.

Proof: By Lemma 3.4(ii), both $v = \lim v_k$ and $u_0 = \lim u_k$ are infinite strings. Taking the limits of (11) and (12) as k goes to infinity, it is clear that

$$s \supset t;\; u_0$$
$$c_p s \supset t;\; v.$$

By uniqueness of extracted strings, we have $u = u_0$. By Lemma 3.4 and (13), we have

$$v_{k+2} = v_{k+1} c_{\alpha(k+1)-3} = v_k c_{\alpha(k)-3} c_{\alpha(k)-2} = v_k c_{\alpha(k)-1} = c_p u_k \quad (k \geq r).$$

Consequently, $v = \lim v_{k+2} - \lim c_p u_k = c_p \lim u_k = c_p u$.

Remark: Lemma 3.5 also holds when $p = 1$. The proof for this case is straightforward and is left for the reader.

4. THE ALIGNMENTS $x_m \supset x_n$; $y_{m,n}$

Theorem 4.1: Either the two extracted strings $y_{m,n}$ and $y_{m+1,n+1}$ are equal or else they differ by the first letter only. Here, $y_{m,n}$ is defined by (2).

Proof: Let $x_m = c_{\alpha(1)}c_{\alpha(2)} \ldots$ and $x_n = c_{\beta(1)}c_{\beta(2)} \ldots$ be the canonical representations of x_m and x_n, respectively. By Corollary 1.5(i), we have three cases to consider, according to the values of $\alpha(1)$ and $\beta(1)$.

Case (i). $\alpha(1) = \beta(1)$

Clearly $y_{m,n} = y_{m+1,n+1}$ in this case.

Case (ii). $\alpha(1) = 2$ and $\beta(1) = 1$

By Corollary 1.5(i), there are three subcases to consider:

(a) If $x_m = c_2 s,\ x_n = c_1 c_2 t$ and $s \supset c_2 t;\, u$, then $y_{m,n} = au$ and $y_{m+1,n+1} = bu$.

(b) If $x_m = c_2 c_3 s,\ x_n = c_1 c_3 t$ and $s \supset t;\, u$, then $y_{m,n} = au$ and $y_{m+1,n+1} = bu$.

(c) If $x_m = c_2 c_4 s,\ x_n = c_1 c_3 t$ and $s \supset t;\, u$, then $y_{m,n} = ac_2 u$ and $y_{m+1,n+1} = c_3 u = bc_2 u$ by Lemma 3.5.

Case (iii). $\alpha(1) = 1$ and $\beta(1) = 2$

(a) If $x_m = c_1 c_2 s,\ x_n = c_2 t$ and $s \supset t;\, u$, then $y_{m,n} = bu$ and $y_{m+1,n+1} = au$.

(b) If $x_m = c_1 c_3 s,\ x_n = c_2 t$ and $s \supset t;\, u$, then $y_{m,n} = bbu$ and $y_{m+1,n+1} = c_2 u = abu$ by Lemma 3.5.

This theorem, together with Theorems 2.4 and 2.5 (the modified Hofstadter's conjecture) imply the followiing result.

Corollary 4.2: Let m and n be two nonnegative integers.

(a) If $m > n$, then $y_{m,n}$ is an infinite string; for $m \geq n+2$ (resp. $m = n+1$) the strings $y_{m,n}$ and x_{m-n-2} (resp. ax) differ by at most the first letter.

(b) If $n > m$, then $y_{m,n}$ is a finite string with length $n-m$; the strings $y_{m,n}$ and $R(s_{n-m})$ differ by at most the first letter.

The above corollary motivates determining the first letters of the strings $y_{m,n}$ $(m \neq n)$, x_{m-n-2} $(m \geq n+2)$, and $R(s_{n-m})$ $(n > m)$, where m and n are nonnegative integers.

Lemma 4.3:

(a) Let $m \geq n+2$. Let $m-n-2 = \sum_{j=1}^{\infty} \varepsilon_j F_{j+1}$ be the Zeckendorf representation of $m-n-2$. Then the first letter of x_{m-n-2} is an "a" or "b" depending on whether ε_1 equals 1 or 0, respectively.

(b) Let $n > m$. Let $n-m-1 = \sum_{j=1}^{\infty} \varepsilon_j F_{j+1}$ be the Zeckendorf representation of $n-m-1$. Then the first letter of $R(s_{n-m})$ is an "a" or "b" depending on whether ε_1 equals 1 or 0, respectively.

(c) Let $m \neq n$. Let $m = \sum_{j=1}^{\infty} \varepsilon_j F_{j+1}$ and $n = \sum_{j=1}^{\infty} \delta_j F_{j+1}$ be the Zeckendorf representations of m and n, respectively. Let k be the smallest positive integer such that $\varepsilon_k \neq \delta_k$. Then the first letter of $y_{m,n}$ is an "a" iff either $\varepsilon_k = 0$ with k even or $\varepsilon_k = 1$ with k odd.

Proof: (a) and (b) follow from [8, p. 85]. A similar proof holds for (c) after noting that, by Lemma 1.4, the following statements are true:

If $\varepsilon_k = 0$ (resp. 1) and $\delta_k = 1$ (resp. 0), then $x_m = uc_k s$ (resp. $uc_{k+1}s$) and $x_n = uc_{k+1}t$ (resp. $uc_k t$) for some strings u, s, and t.

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ON EVEN PSEUDOPRIMES

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A composite number n is called a pseudoprime if $n|2^n-2$. Until 1950 only odd pseudoprimes were known. So far, little is known about even pseudoprimes. D. H. Lehmer (see Erdös [5]) found the first even pseudoprime: $161038 = 2\cdot 73\cdot 1103$. In 1951 Beeger [2] showed the existence of infinitely many even pseudoprimes and found the following three even pseudoprimes: $2\cdot 23\cdot 31\cdot 151$, $2\cdot 23\cdot 31\cdot 1801$, and $2\cdot 23\cdot 31\cdot 100801$. Later Maciąg (see Sierpiński [9], p. 131) found the following two other even pseudoprimes:

$$2\cdot 73\cdot 1103\cdot 2089 \text{ and } \frac{2(2^{23}-1)(2^{29}-1)}{47} = 2\cdot 233\cdot 1103\cdot 2089\cdot 178481.$$

The first-named author in his book [8] put forward the following problems: Does there exist a pseudoprime of the form 2^n-2? (problem #22) and: Do there exist infinitely many even pseudoprime numbers which are the products of three primes? (problem #51).

In 1989 McDaniel [4] gave an example of a pseudoprime which is itself of the form $2^n-2 = 2(2^{pq}-1)$ by showing that $2^{\mathcal{N}}-2$ is a pseudoprime if $\mathcal{N} = 465794 = 2\cdot 7^4\cdot 97$, $p = 37$, and $q = 12589$.

In connection with the second problem, McDaniel [4] found the following even pseudoprimes: $2\cdot 178481\cdot 154565233$ and $2\cdot 1087\cdot 164511353$.

In 1965 (see [7], [6]) the first-named author proved the following two theorems:

1. The number pq, where p and q are different primes is a pseudoprime if and only if the number $(2^p-1)(2^q-1)$ is a pseudoprime.
2. For every prime number p $(7 < p \neq 13)$, there exists a prime q such that $(2^p-1)(2^q-1)$ is a pseudoprime. For p = 2, 3, 5, 7, and 13, there is no prime q for which $(2^p-1)(2^q-1)$ is a pseudoprime.

If the number $2(2^p-1)$, where p is a prime, is a pseudoprime, then $2^p-1|2^{2^{p+1}-3}-1$; hence, $2^{p+1}\equiv 3 \pmod{p}$, which is impossible. McDaniel [4] showed that, if n satisfies the congruence $2^{n+1}\equiv 3 \pmod{n}$, then $2(2^n-1)$ is an even pseudoprime for $n = p_1p_2$ if $2^{p_1+1}\equiv 3 \pmod{p_2}$ and $2^{p_2+1}\equiv 3 \pmod{p_1}$. Here we shall prove the following theorem.

Theorem: Let p and q be primes and d be a divisor of $(2^p-1)(2^q-1)$. If d is coprime to p and q and not divisible by either 2^p-1 or 2^q-1, then $\frac{2(2^p-1)(2^q-1)}{d}$ is an even pseudoprime if and only if $\frac{2(2^{pq}-1)}{d}$ is an even pseudoprime.

Proof: Let $M = (2^p - 1)(2^q - 1)$, $N = 2^{pq} - 1$, where p and q are distinct primes. Suppose d is a divisor of M that is coprime to pq and which is divisible by neither $2^p - 1$ nor $2^q - 1$. First note that $M \equiv N \pmod{pq}$. Indeed, $M \equiv 2^q - 1 \equiv N \pmod{p}$ and, similarly, $M \equiv N \pmod{q}$, so that the assertion follows. Next let $\ell(m)$ denote the exponent to which 2 belongs modulo the odd natural number m, so that $2m$ is an even pseudoprime if and only if $\ell(m)|2m-1$. Now it is easy to see that, if d has the stated properties, then $\ell(\frac{M}{d}) = \ell(\frac{N}{d}) = pq$. Thus, $\frac{2M}{d}$ is an even pseudoprime if and only if $pq|\frac{2M}{d}-1$ if and only if $pq|\frac{2N}{d}-$ [since $M \equiv N \pmod{pq}$ and $(pq, d) = 1$] if and only if $\frac{2N}{d}$ is an even pseudoprime. Q.E.D.

Example: Since 47 is coprime to $23 \cdot 29$, from Maciąg's pseudoprime $\frac{2(2^{23}-1)(2^{29}-1)}{47}$, by the Theorem, we get the pseudoprime $\frac{2^{668}-2}{47}$.

For $d = 1$, we get the following corollary from the Theorem.

Corollary: The number $2(2^p - 1)(2^q - 1)$ is a pseudoprime if and only if the number $2(2^{pq} - 1)$ is a pseudoprime.

Example: By the Corollary, from McDaniel's [4] pseudoprime $2(2^{37 \cdot 12589} - 1)$, we get the pseudoprime $2(2^{37} - 1)(2^{12589} - 1)$.

Using the method presented in the paper of McDaniel [4] and the tables in [3], we found the following 24 even pseudoprimes with 3, 4, 5, 6, 7, and 8 prime factors:

$2\cdot 311\cdot 79903$, $2\cdot 1319\cdot 288313$, $2\cdot 4721\cdot 459463$, $2\cdot 7\cdot 359\cdot 601$, $2\cdot 23\cdot 271\cdot 631$,
$2\cdot 31\cdot 233\cdot 631$, $2\cdot 127\cdot 199\cdot 3191$, $2\cdot 127\cdot 599\cdot 1289$, $2\cdot 73\cdot 631\cdot 3191$, $2\cdot 7\cdot 191\cdot 153649$,
$2\cdot 47\cdot 311\cdot 68449$, $2\cdot 7\cdot 79\cdot 7555991$, $2\cdot 151\cdot 383\cdot 201961$, $2\cdot 73\cdot 271\cdot 2940521$,
$2\cdot 89\cdot 337\cdot 11492353$, $2\cdot 23\cdot 31\cdot 151\cdot 991$, $2\cdot 73\cdot 631\cdot 991\cdot 3191$,
$2\cdot 233\cdot 1103\cdot 2089\cdot 12007\cdot 178481$, $2\cdot 233\cdot 1103\cdot 2089\cdot 178481\cdot 458897$,
$2\cdot 233\cdot 1103\cdot 2089\cdot 178481\cdot 88039999$, $2\cdot 233\cdot 1103\cdot 2089\cdot 12007\cdot 178481\cdot 458897$,
$2\cdot 233\cdot 1103\cdot 2089\cdot 12007\cdot 178481\cdot 88039999$, $2\cdot 233\cdot 1103\cdot 2089\cdot 178481\cdot 458897\cdot 88039999$,
$2\cdot 233\cdot 1103\cdot 2089\cdot 12007\cdot 178481\cdot 458897\cdot 88039999$.

Beeger's [2] proof of the existence of an infinite number of even pseudoprimes has been based on the fact that, for every even pseudoprime $a_1 = 2n$, there exists a prime p such that $a_2 = pa_1$ is also a pseudoprime. We shall repeat it shortly. By a theorem of Bang [1], it follows that there exists a prime p (called a primitive prime factor of $2^{2n-1} - 1$) for which holds $2^{2n-1} \equiv 1 \pmod{p}$, $2^x \not\equiv 1 \pmod{p}$, $1 \le x < 2n-1$, and $p \equiv 1 \pmod{2(2n-1)}$, which leads to the fact that pa_1 is a pseudoprime. We can take instead of a primitive prime factor of $2^{2n-1} - 1$ any other factor of the same number that is $\equiv 1 \pmod{2(2n-1)}$ and coprime with a_1 if it exists. So the infinite sequence $a_1, a_2, \ldots$, has the property $2 < a_1 | (a_i, a_j)$ for $i \ne j$. Thus, the following problem arises:

1. Does there exist an infinite sequence $a_1, a_2, \ldots$ of even pseudoprimes such that $(a_i, a_j) = 2$ for every $i \ne j$?

It is easy to see that if the problem #51 mentioned at the beginning of the present paper has an affirmative answer then there is a positive answer to problem 1, but problem 1 seems to be easier.

We also do not know the answer to the following question:

2. Does there exist an integer n such that n and $n+1$ are pseudoprimes?

It would be of interest to investigate the case of n even or odd separately.

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GENERALIZATIONS OF SOME SIMPLE CONGRUENCES

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1. INTRODUCTION

Over many years in this journal there have appeared results concerning congruence and divisibility in relation to the Fibonacci and Lucas numbers. Here we take four such results and translate them to sequences which generalize the Fibonacci and Lucas sequences.

We hope that the nature of our results will demonstrate to the beginning Fibonacci enthusiast that there is scope to obtain further generalizations of a similar nature.

2. THE SEQUENCES

In the notation of Horadam [7] write

$$W_n = W_n(a, b; p, q), \tag{2.1}$$

meaning that

$$W_n = pW_{n-1} - qW_{n-2}, \; W_0 = a, W_1 = b, \; n \geq 2. \tag{2.2}$$

We assume throughout that a, b, p, q are integers.

The auxiliary equation associated with (2.2) is

$$x^2 - px + q = 0, \tag{2.3}$$

whose roots

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2}, \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2} \tag{2.4}$$

are assumed distinct. We write

$$\Delta = (\alpha - \beta)^2 = p^2 - 4q. \tag{2.5}$$

We shall be concerned with specializations of the following two sequences:

$$\begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p; p, q). \end{cases} \tag{2.6}$$

The sequences $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences, respectively, generated by (2.2). They are natural generalizations of the Fibonacci and Lucas sequences and have been studied extensively, particularly by Lucas [11]. Further information can be found, for example, in [1], [7], and [10].

The Binet forms for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2.7}$$

$$V_n = \alpha^n + \beta^n. \tag{2.8}$$

These sequences can be extended to negative subscripts by the use of the recurrence (2.2) or the Binet forms.

We will make use of the following well-known results which we state for easy reference:

$$q^n U_{-n} = -U_n, \tag{2.9}$$

$$q^n V_{-n} = V_n, \tag{2.10}$$

$$U_{2n} = U_n V_n, \tag{2.11}$$

$$\text{if } m|n \text{ then } U_m | U_n. \tag{2.12}$$

The following identities, which occur in Bergum and Hoggatt [1], will also be needed:

$$U_{n+k} + q^k U_{n-k} = U_n V_k, \tag{2.13}$$

$$U_{n+k} - q^k U_{n-k} = U_k V_n, \tag{2.14}$$

$$V_{n+k} + q^k V_{n-k} = V_n V_k, \tag{2.15}$$

$$V_{n+k} - q^k V_{n-k} = \Delta U_n U_k. \tag{2.16}$$

The sequences

$$\begin{cases} U_n = W_n(0, 1; p, -1), \\ V_n = W_n(2, p; p, -1), \end{cases} \tag{2.17}$$

are an important subclass of the sequences (2.6) and can be looked upon as an intermediate level of generalization of the Fibonacci and Lucas numbers in which $p = 1$. The specializations $p = 2$ and $p = 2x$ also yield cases of interest. For $p = 2$ see [4], [8], [15] and for $p = 2x$ see [9], [12], [13].

We use the U_n–V_n notation throughout to refer to the sequences (2.6) and to the sequences (2.17). There will be no ambiguity since we shall always indicate the set to which we are referring.

3. CONGRUENCE RESULT I

Singh [17] gives the following:

$$L_{2^n} \equiv 7 \pmod{40} \quad \text{for } n \geq 2. \tag{3.1}$$

Generalization: Let $\{U_n\}$ and $\{V_n\}$ be the sequences defined in (2.17). Then

$$V_{2^n} \equiv V_4 \pmod{\Delta U_6 U_{10}}, \quad n = 2, 4, 6, \ldots, \tag{3.2}$$

$$V_{2^n} \equiv V_8 \pmod{\Delta U_6 U_{20} V_2 (V_4 - 1)}, \quad n = 3, 5, 7, \ldots, \tag{3.3}$$

and so

$$V_{2^n} \equiv V_4 \pmod{\Delta U_2 U_6} \quad \text{for } n \geq 2. \tag{3.4}$$

Proof: We shall use the following, all of which can be proved using Binet forms:

$$V_{2^{k+1}} = V_{2^k}^2 - 2, \tag{3.5}$$

$$p(V_4 + 1) = U_6, \tag{3.6}$$

$$V_4 - 2 = \Delta p^2, \tag{3.7}$$

$$p(V_4^2 + V_4 - 1) = U_{10}, \tag{3.8}$$

$$p(V_8 + 1) = U_6(V_4 - 1), \tag{3.9}$$

$$\Delta p^2 V_2^2 = V_8 - 2, \tag{3.10}$$

$$pV_2(V_8^2 + V_8 - 1) = U_{20}. \tag{3.11}$$

Using (3.5) twice, we also obtain

$$V_{2^{k+2}} = V_{2^k}^4 - 4V_{2^k}^2 + 2. \tag{3.12}$$

Now (3.2) is true for $n = 2$ and if it is true for $n = k$ (even) then by (3.12) and the induction hypothesis

$$V_{2^{k+2}} = V_{2^k}^4 - 4V_{2^k}^2 + 2 \equiv V_4^4 - 4V_4^2 + 2 \pmod{\Delta U_6 U_{10}}.$$

But by (3.6)-(3.8),

$$(V_4^4 - 4V_4^2 + 2) - V_4 = (V_4 + 1)(V_4 - 2)(V_4^2 + V_4 - 1) = \Delta U_6 U_{10}.$$

This proves (3.2). Congruence (3.3) can be proved similarly by making use of (3.9)-(3.12).

From (2.16) we see that $V_8 - V_4 = \Delta U_2 U_6$, and (2.12) shows that $\Delta U_2 U_6$ divides both moduli in (3.2) and (3.3). This proves (3.4). □

Putting $V_n = L_n$ so that $U_n = F_n$, we see that (3.4) reduces to (3.1).

4. CONGRUENCE RESULT II

Berzsenyi [2] states that

$$F_{6n+1}^2 \equiv 1 \pmod{24}, \; n \text{ an integer}. \tag{4.1}$$

Generalization: Let $\{U_n\}$ and $\{V_n\}$ be the sequences defined in (2.17). Then

$$U_{6n+1}^2 \equiv 1 \pmod{U_4 U_6}, \; n \text{ an integer}. \tag{4.2}$$

Proof:

$$\begin{aligned} U_{6n+1}^2 - 1 &= (U_{6n+1} - U_1)(U_{6n+1} + U_1) \\ &= (U_{3n+1+3n} - U_{3n+1-3n})(U_{3n+1+3n} + U_{3n+1-3n}) \\ &= U_{3n} V_{3n} U_{3n+1} V_{3n+1}, \end{aligned} \tag{4.3}$$

where we have used (2.13) and (2.14) with $q = -1$.

Taking m to be an integer, we consider two cases:

Case 1. $n = 2m + 1$. Using (2.11), the right side of (4.3) becomes $U_{12m+6} U_{12m+8}$. Then by (2.12), $U_4 | U_{12m+8}$ and $U_6 | U_{12m+6}$ and (4.2) follows.

Case 2. $n = 2m$. Using (2.11), the right side of (4.3) becomes $U_{12m} U_{12m+2}$. Since $U_4 | U_{12m}$, $U_6 | U_{12m}$, and $(U_4, U_6) = U_2 | U_{12m+2}$, then $U_4 U_6 | U_{12m} U_{12m+2}$ and (4.2) follows.

This completes the proof of (4.2). □

5. CONGRUENCE RESULT III

Freitag [5] gives the following:

$$L_{2\mathrm{p}^k} \equiv 3 \pmod{10} \tag{5.1}$$

for all primes $\mathrm{p} \geq 5$ and natural numbers k. We caution against confusing the prime p with the parameter p.

Generalization: Let $\{U_n\}$ and $\{V_n\}$ be the sequences defined in (2.17). Then

$$V_{n+12} \equiv V_n \pmod{40}. \tag{5.2}$$

Proof: Using (2.16), we see that

$$V_{n+12} - V_n = (p^2+4)U_6U_{n+6} = p(p^2+1)(p^2+3)(p^2+4)U_{n+6}. \tag{5.3}$$

The right side of (5.3) is divisible by 40 since 5 divides either p, p^2+1, or p^2+4 and $8 \mid p(p^2+4)$ if p is even while $8 \mid (p^2+1)(p^2+3)$ if p is odd.

To see that (5.2) generalizes (5.1) we note that, as observed in Bruckman [3],

$$2\mathrm{p}^k \equiv 2 \text{ or } -2 \pmod{12} \tag{5.4}$$

for all primes $\mathrm{p} \geq 5$. Now, since $L_2 = L_{-2} = 3$, (5.1) follows from (5.2) and (5.4). □

6. A DIVISIBILITY RESULT

Grassi [6] gives the following:

$$12 \mid (F_{4n-2} + F_{4n} + F_{4n+2}), \tag{6.1}$$

$$168 \mid (F_{8n-4} + F_{8n} + F_{8n+4}). \tag{6.2}$$

Generalization: Let $\{U_n\}$ and $\{V_n\}$ be the sequences defined in (2.6). Then for $n \geq 0$, $k \geq 1$,

$$U_{2k-1}V_{4k-2}V_{6k-3} \mid (q^{4k-2}U_{(4k-2)(2n-1)} - q^{2k-1}U_{(8k-4)n} + U_{(4k-2)(2n+1)}), \tag{6.3}$$

$$V_{2k}V_{4k}U_{6k} \mid (q^{4k}U_{4k(2n-1)} + q^{2k}U_{8kn} + U_{4k(2n+1)}). \tag{6.4}$$

Proof: We prove (6.3) by using reasoning similar to Mana [14]. Fixing k and denoting the dividend by $G_n^{(k)}$ we have, by (2.9),

$$\begin{aligned} G_0^{(k)} &= q^{4k-2}U_{-(4k-2)} + U_{4k-2} \\ &= -U_{4k-2} + U_{4k-2} = 0. \end{aligned}$$

Also

$$\begin{aligned} G_1^{(k)} &= q^{4k-2}U_{4k-2} - q^{2k-1}U_{8k-4} + U_{12k-6} \\ &= q^{4k-2}U_{2k-1}V_{2k-1} + U_{2k-1}V_{10k-5} && \text{[by (2.11) and (2.14)]} \\ &= U_{2k-1}V_{6k-3}V_{4k-2} && \text{[by (2.15)].} \end{aligned}$$

Now $\{G_n^{(k)}\}$ can be regarded as the sum of three sequences each satisfying the same homogeneous linear second-order recurrence relation with integer coefficients (see Shannon and Horadam [16]).

Hence, $\{G_n^{(k)}\}$ also satisfies this second-order recurrence. Therefore, since $U_{2k-1}V_{4k-2}V_{6k-3}|G_0^{(k)}$ and $U_{2k-1}V_{4k-2}V_{6k-3}|G_1^{(k)}$, then $U_{2k-1}V_{4k-2}V_{6k-3}|G_n^{(k)}$ for all $n \geq 0$. Since k was arbitrary, the proof of (6.3) is complete. The proof of (6.4) is similar. □

Taking $\{U_n\} = \{F_n\}$, $\{V_n\} = \{L_n\}$ and putting $k = 1$, we see that (6.3) and (6.4) reduce to (6.1) and (6.2), respectively.

7. CONCLUDING COMMENTS

We have chosen an assortment of results requiring essentially different methods of proof. For the most part, the moduli or divisors in question are products of terms from the relevant sequences. We feel that with this observation there is scope for the beginner to discover generalizations of a similar nature.

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NONZERO ZEROS OF THE HERMITE POLYNOMIALS ARE IRRATIONAL

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1. INTRODUCTION

The Hermite polynomials belong to the system of classical orthogonal polynomials [7, 10] and they are defined by means of the following relation [1, 7]:

$$\exp(2st - t^2) = \sum_{L=0}^{\infty} H_L(s)t^L / L!. \tag{1}$$

All the zeros of the Hermite polynomials are real, distinct, and are located in the open interval $(-\infty, \infty)$ [7, 10]. The Hermite polynomials of even degree have no zero at the origin, while each Hermite polynomial of odd degree has a simple zero at the origin [7]. The purpose of this article is to present an *elementary* proof that *all* the *nonzero* zeros of the Hermite polynomials are necessarily *irrational.*

2. BASIS OF THE PROOF

Our proof of the irrationality of the nonzero zeros of the Hermite polynomial $H_n(s)$ is based on the following facts:

(a) All the coefficients of the Hermite polynomials [1] are integers.

(b) A factor $2^m s^r$ can be pulled out of $H_n(s)$, $n \geq 1$, such that the remaining factor $R_{2k}^{(r)}(s)$ is an even polynomial in s of degree $2k$, still containing only integers as coefficients. The nonnegative integers m, r, and k are given by

$$m = [(n+1)/2], \quad r = n - 2[n/2], \quad k = [n/2], \tag{2}$$

where $[t]$ is the greatest integer $\leq t$. Note that $m = r + k$, and that r is zero (unity) when n is even (odd).

(c) The constant term of $R_{2k}^{(r)}(s)$ is $(-1)^k(2k+2r-1)!!$, where

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1), \; n \geq 1. \tag{3}$$

We follow the convention that $(-1)!! = 1$. *All* the factors of the constant term of $R_{2k}^{(r)}(s)$ are *odd.*

(d) The leading coefficient (i.e., the coefficient of the highest power of s) in $R_{2k}^{(r)}(s)$ is 2^k, whose factors are of the form 2^c, $0 \leq c \leq k$, c being a nonnegative integer.

(e) The constant term of $R_{2k}^{(r)}(s)$ is *odd,* while all the other coefficients are *even* and nonzero.

(f) The zeros of $R_{2k}^{(r)}(s)$ are just the zeros of $H_n(s)$, $n \geq 1$, when n is even. If n is odd, the nontrivial zeros of $H_n(s)$ are simply the zeros of $R_{2k}^{(r)}(s)$, the trivial zero being the one located at the origin. The last result follows from the fact that $H_L(s)$ has a definite parity [10],

$$H_L(-s) = (-1)^L H_L(s), \ L \geq 0, \tag{4}$$

so that $H_{2M+1}(s)$, $M \geq 0$, is an odd polynomial in s and, hence, is zero at the origin.

(g) For all $p, q \in Z$, $2p \pm (2q+1) = 2(p \pm q) \pm 1 \neq 0$.

Thus (see [6], p. 81), for example,

$$\begin{aligned} H_6(s) &= 64s^6 - 480s^4 + 720s^2 - 120 \\ &= 8(8s^6 - 60s^4 + 90s^2 - 15), \end{aligned} \tag{5}$$

and

$$\begin{aligned} H_9(s) &= 512s^9 - 9216s^7 + 48384s^5 - 80640s^3 + 30240s \\ &= 32s(16s^8 - 288s^6 + 1512s^4 - 2520s^2 + 945). \end{aligned} \tag{6}$$

Using the power series expansion of the Hermite polynomials [1],

$$H_L(s) = \sum_{Q=0}^{[L/2]} \frac{(-1)^Q L!(2s)^{L-2Q}}{Q!(L-2Q)!}, \ L \geq 0, \tag{7}$$

and the relation

$$(2M)! = 2^M M!(2M-1)!!, \ M = 0, 1, 2, 3, \ldots, \tag{8}$$

we can prove the results (a)-(e) given above. In Section 5, we prove these results for the case in which n is even. Using similar arguments, one can easily establish the results for odd values of n also.

3. ZEROS CANNOT BE NONZERO INTEGERS

An immediate and interesting consequence of the result (e) of Section 2 is the fact that *no* nonzero integer (positive or negative) can be a zero of $H_n(s)$, since

$$R_{2k}^{(r)}(\text{integer}) = \text{even } \# \pm \text{odd } \# = \text{odd } \# \neq 0$$

(see result (g) of Section 2). Thus, $H_n(s)$, $n \geq 1$, is nonzero whenever s is an integer $\neq 0$. Hence, the zeros of $H_n(s)$ are neither positive nor negative integers.

4. NO RATIONAL ZEROS

If B/D, where B and D are integers, is a rational zero of a polynomial whose coefficients are all integers, and if B/D is in its lowest terms, then B must be a factor of the constant term and D must be a factor of the leading coefficient (see [5], Theorem 9-14, p. 303). Thus, a nontrivial rational zero of the Hermite polynomial, being a rational zero of $R_{2k}^{(r)}(s)$, whose coefficients are all integers, would be of the form B/D, where B = odd # and $D = 2^c$, where $c = 0, 1, 2, \ldots, k$. (Remember results (c) and (d) of Section 2.) The case $c = 0$ corresponds to an integer as a possible zero and, hence, can be ruled out (see Section 3). Using (7), it can be shown that

$$2^{n(c-1)} H_n(\text{odd } \#/2^c) = \text{odd } \# \neq 0, \ n \geq 1, \ c \geq 1. \tag{9}$$

For a proof, see Section 6. Since $2^{n(c-1)} \neq 0$, $s = \text{odd } \#/2^c$ cannot be a zero of $H_n(s)$. We conclude that the Hermite polynomial $H_n(s)$, $n \geq 1$, has *no* nonzero rational zeros.

5. PROOF OF CERTAIN RESULTS FROM SECTION 2

We now prove some of the statements given in Section 2 for the case in which n = even #. If the coefficient of s^{2N-2Q} in $H_{2N}(s)$, $N \geq 1$, is A_{2N-2Q}, then, from (7) with $L = 2N$,

$$A_{2N-2Q} = (-1)^Q (2N)! 2^{2N-2Q} / \{Q!(2N-2Q)!\}. \tag{10}$$

Now, when K is a positive integer, $(2K)! = (2K)(2K-1)(2K-2)(2K-3)\cdots 4\cdot 3\cdot 2\cdot 1$, and since $(-1)!! = 1$ (see result (c) of Section 2), we have

$$(2M)! = 2^M M!(2M-1)!!, \;\; M = 0, 1, 2, 3, \ldots. \tag{8}$$

It follows from (8) and (10) that

$$A_{2N-2Q} = 2^N (-1)^Q \binom{N}{Q} 2^{N-Q} \{(2N-1)!!/(2N-2Q-1)!!\}. \tag{11}$$

In (11), the binomial coefficient $\binom{N}{Q}$ is necessarily a positive *integer*; the expression within the braces $\{\cdots\}$ is essentially an odd positive *integer*, since $Q \leq N$, and both Q and N are nonnegative integers. The phase factor $(-1)^Q$ is an odd integer $(= \pm 1)$. The factor 2^{N-Q} is *even* as long as $Q \neq N$; for the constant term of $H_{2N}(s)$, this quantity is just unity and, hence, odd (as $Q = N$). The leading coefficient of $H_{2N}(s)$ is $A_{2N} = 2^{2N}$ (since $Q = 0$). It is now clear from (11) that *all* the coefficients of $H_{2N}(s)$, $N \geq 1$, are *integers*. Moreover, a factor 2^N can be pulled out from *all* the coefficients of $H_{2N}(s)$, but still the coefficients of $R_{2N}^{(0)}(s)$ are *all* integers. The constant term of $R_{2N}^{(0)}(s)$ is *odd*, the leading coefficient is 2^N (= even #) and all the other coefficients are *even* numbers. Incidentally,

$$H_{2N}(0) = A_0 = (-1)^N 2^N (2N-1)!! \neq 0, \tag{12}$$

and, hence, $H_{2N}(s)$ can never be zero at the origin. It follows from (12) that the constant term of $R_{2N}^{(0)}(s)$ is just $(-1)^N(2N-1)!! =$ odd # $\neq 0$.

6. PROOF OF RELATION (9)

Let us now present the proof of (9) when n = even #. The proof is similar for the case when n = odd #.

Using (7) and (11), we have, with $c \geq 1$, $N \geq 1$,

$$\begin{aligned} 2^{2N(c-1)} H_{2N}(\text{odd }\#/2^c) = \sum_{Q=0}^{N} (-1)^Q \binom{N}{Q} 2^{Q(2c-1)} (\text{odd }\#)^{2N-2Q} \\ \times \{(2N-1)!!/(2N-2Q-1)!!\}. \end{aligned} \tag{13}$$

In the right-hand side of (13), the first term in the summation (i.e., $Q = 0$ term) is *odd*. For all the other terms, $Q \geq 1$, $Q(2c-1) \geq 1$, and, hence, $2^{Q(2c-1)} =$ even # ≥ 2. Therefore, except for the first term, all the remaining terms are definitely *even*. Hence, $H_{2N}(\text{odd }\#/2^c) \neq 0$ and $s =$ odd #$/2^c$ cannot be a zero of $H_{2N}(s)$.

7. CONCLUSION

Zeros of the Hermite polynomials, if nonzero, are irrational. Using a computer program, we verified statements (b)-(e) of Section 2 and the results presented in Sections 3 and 4 for $n \leq 12$. Readers familiar with Gaussian quadrature are practically aware that the nonzero zeros of $H_n(s)$, $2 \leq n \leq 20$ (say), are irrational [3, 4].

Recently, in connection with our work [9], we have been informed by the Editor of *The Journal of Number Theory* that the collected works of Professor I. Schur [8] contain a proof that the zeros of the Hermite and Laguerre polynomials are irrational. However, we are unable to access this material, independent of which our work was done, for verification. In fact, we learned about Schur's work [8] and Gow's work [2] only after [9] had already been accepted for publication and was subsequently rejected. The proof due to Professor Schur [8] *should be* delightful and, hopefully, *distinct* from ours!

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A GENERALIZATION OF A RESULT OF D'OCAGNE

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1. INTRODUCTION

In this paper we consider some aspects of sequences generated by the m^{th} order homogeneous linear recurrence relation

$$R_n = \sum_{i=1}^{m} a_i R_{n-i} \quad \text{for } m \geq 2, \tag{1.1}$$

where $a_m \neq 0$ and the underlying field is the complex numbers. To generate a sequence $\{R_n\}_{n=0}^{\infty}$, we specify initial values $R_0, R_1, \ldots, R_{m-1}$. Indeed, this sequence can be extended to negative subscripts by using (1.1), and with this convention we simply write $\{R_n\}$.

For the case $m = 2$, we adopt the notation of Hordam [3] and write

$$W_n = W_n(a, b; p, q), \tag{1.2}$$

meaning that

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a,\ W_1 = b. \tag{1.3}$$

If $(R_0, \ldots, R_{m-2}, R_{m-1}) = (0, \ldots, 0, 1)$, we write $\{R_n\} = \{U_n\}$. The sequence $\{U_n\}$ is called the fundamental sequence generated by (1.1). It is "fundamental" in the sense that, if $\{R_n\}$ is any sequence generated by (1.1), then there exist complex numbers $b_0, \ldots, b_{m-1}$ depending upon $a_1, \ldots, a_m$ and $R_0, \ldots, R_{m-1}$ such that

$$R_n = \sum_{i=0}^{m-1} b_i U_{n+i} \quad \text{for all integers } n. \tag{1.4}$$

In this regard, see Jarden [4], p. 114 or Dickson [1], p. 409, where this result is attributed to D'Ocagne. In §2 we generalize this idea.

For the Fibonacci and Lucas numbers, it can be proved that

$$L_n^2 + L_{n+1}^2 = 5(F_n^2 + F_{n+1}^2). \tag{1.5}$$

More generally, for the second-order fundamental and primordial sequences of Lucas [5] defined by

$$\begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p; p, q), \end{cases} \tag{1.6}$$

where $\Delta = p^2 - 4q \neq 0$, we have

$$-qV_n^2 + V_{n+1}^2 = \Delta(-qU_n^2 + U_{n+1}^2). \tag{1.7}$$

In §3 we demonstrate the existence of a result analogous to (1.7) for any two sequences generated by (1.1).

2. A GENERALIZATION OF D'OCAGNE'S RESULT

Let $\{R_n\}$ and $\{S_n\}$ be any two sequences generated by (1.1). Define the $(m+1)\times(m+1)$ determinant D_n, for all integers n, by

$$D_n = \begin{vmatrix} R_n & S_n & S_{n+1} & \cdots & S_{n+m-1} \\ R_{m-1} & S_{m-1} & S_m & \cdots & S_{2m-2} \\ R_{m-2} & S_{m-2} & S_{m-1} & \cdots & S_{2m-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_0 & S_0 & S_1 & \cdots & S_{m-1} \end{vmatrix}.$$

Theorem 1: $D_n = 0$ for all integers n.

Proof: $D_0 = D_1 = \cdots = D_{m-1} = 0$ since, in each case, we have an $(m+1)\times(m+1)$ determinant with two identical rows. Now expanding D_n along the top row, we see that D_n is a linear combination of $R_n, S_n, \ldots, S_{n+m-1}$. Therefore, since each of the sequences $\{R_n\}, \{S_n\}, \ldots, \{S_{n+m-1}\}$ is generated by (1.1) then so is $\{D_n\}$. But $\{D_n\}$ has m successive terms that are zero and so all its terms are zero. This completes the proof. □

We now come to the main result of this section.

Corollary 1: There exist constants c and c_{oj}, $0 \le j \le m-1$, such that

$$cR_n = \sum_{j=0}^{m-1} c_{oj} S_{n+j} \quad \text{for all integers } n. \tag{2.1}$$

Proof: Expand D_n along the top row. □

Equation (2.1) generalizes D'Ocagne's result (1.4), where the b_i are normally specified without the use of determinants. If $\{S_n\} = \{U_n\}$, then c, which is the minor of R_n is unity and we obtain an equivalent form of D'Ocagne's result.

3. A RESULT CONCERNING SUMS OF SQUARES

From (2.1) we have, for any integer i,

$$cR_{n+i} = \sum_{j=0}^{m-1} c_{oj} S_{n+i+j}. \tag{3.1}$$

Using (1.1), the right side of (3.1) can be written in terms of $S_n, S_{n+1}, \ldots, S_{n+m-1}$. That is, for any integer i there exist constants c_{ij}, $0 \le j \le m-1$, such that

$$cR_{n+i} = \sum_{j=0}^{m-1} c_{ij} S_{n+j}. \tag{3.2}$$

Write $\ell = \binom{m}{2}$. Then, for parameters $d_0, d_1, \ldots, d_\ell$ we have, from (3.2),

$$c^2 \sum_{i=0}^{\ell} d_i R_{n+i}^2 = \sum_{j=0}^{m-1} S_{n+j}^2 \sum_{i=0}^{\ell} d_i c_{ij}^2 + 2 \sum_{0 \le j < k \le m-1} S_{n+j} S_{n+k} \sum_{i=0}^{\ell} d_i c_{ij} c_{ik}. \tag{3.3}$$

Consider the system of equations

$$\sum_{i=0}^{\ell} d_i c_{ij} c_{ik} = 0, \quad 0 \le j < k \le m-1, \tag{3.4}$$

in the unknowns $d_0, d_1, \dots, d_\ell$. Since (3.4) is a system of ℓ homogeneous linear equations in $\ell+1$ unknowns, there are an infinite number of solutions $(d_0, d_1, \dots, d_\ell)$. Choose any nontrivial solution and put

$$e_i = c^2 d_i, \qquad 0 \le i \le \ell,$$

$$f_j = \sum_{i=0}^{\ell} d_i c_{ij}^2, \quad 0 \le j \le m-1.$$

Making these substitutions in (3.3), we have succeeded in proving the following theorem.

Theorem 2: Let $\{R_n\}$ and $\{S_n\}$ be any two sequences generated by the recurrence (1.1). Then there exist constants e_i, $0 \le i \le \ell = \binom{m}{2}$, and f_i, $0 \le i \le m-1$, not all zero such that, for all integers n,

$$\sum_{i=0}^{\ell} e_i R_{n+i}^2 = \sum_{i=0}^{m-1} f_i S_{n+i}^2. \tag{3.5}$$

Theorem 2 shows the existence of a result analogous to (1.7) for any two sequences generated by (1.1).

Example 1: Let $\{W_n\}$ and $\{S_n\}$ be any two sequences generated by the recurrence (1.3). Then, after some tedious algebra, we obtain the following determinantal identity:

$$\begin{vmatrix} \begin{vmatrix} S_n^2 & S_{n+1}^2 \\ \begin{vmatrix} W_2 & S_1 \\ W_1 & S_0 \end{vmatrix} & \begin{vmatrix} S_2 & W_1 \\ S_3 & W_2 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} W_n^2 & W_{n+1}^2 \\ \begin{vmatrix} S_2 & W_1 \\ S_3 & W_2 \end{vmatrix} & q^2 \begin{vmatrix} W_2 & S_1 \\ W_1 & S_0 \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} S_1 & S_2 \\ S_2 & S_3 \end{vmatrix} & -q \begin{vmatrix} W_1 & W_2 \\ W_2 & W_3 \end{vmatrix} \end{vmatrix} = 0. \tag{3.6}$$

Example 2: For a fixed integer k, consider the sequences $\{F_{kn}\}$ and $\{L_{kn}\}$. They both satisfy the recurrence (1.3) with $p = L_k$ and $q = (-1)^k$. Substitution into (3.6) yields

$$5(F_{kn}^2 + (-1)^{k-1} F_{k(n+1)}^2) = L_{kn}^2 + (-1)^{k-1} L_{k(n+1)}^2. \tag{3.7}$$

Example 3: In (1.1), taking $m = 3$ and $a_1 = a_2 = a_3 = 1$, we have

$$R_n = R_{n-1} + R_{n-2} + R_{n-3}. \tag{3.8}$$

Feinberg [2] referred to sequences generated by (3.8) as Tribonacci sequences.

For $(R_0, R_1, R_2) = (0, 0, 1)$ write $\{R_n\} = \{U_n\}$.
For $(R_0, R_1, R_2) = (3, 1, 3)$ write $\{R_n\} = \{V_n\}$.

Then $\{V_n\}$ bears the same relation to $\{U_n\}$ as does the Lucas sequence to the Fibonacci sequence (see [6], p. 300).

Now assuming a relationship between $\{U_n\}$ and $\{V_n\}$ of the form (3.5) and solving for the coefficients e_i and f_i yields

$$34V_n^2 - 30V_{n+1}^2 + V_{n+2}^2 + 9V_{n+3}^2 = -154U_n^2 + 176U_{n+1}^2 + 726U_{n+2}^2. \tag{3.9}$$

Alternatively, we have

$$46U_n^2 - 50U_{n+1}^2 - 114U_{n+2}^2 + 54U_{n+3}^2 = -7V_n^2 + 12V_{n+1}^2 - V_{n+2}^2. \tag{3.10}$$

4. OPEN QUESTION

Is there a result analogous to (3.5) for higher powers?

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GEOMETRIC DISTRIBUTIONS AND FORBIDDEN SUBWORDS

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In a recent paper [1] Barry and Lo Bello dealt with the moment generating function of the geometric distribution of order k. I want to draw the attention of the *Fibonacci Community* to several related papers that were apparently missed by the authors and also to provide a straightforward derivation of their result.

Since the moment generating function $M(t)$ is related to the probability generating function $f(z)$ by $M(t) = f(e^t)$, it is sufficient to consider $f(z)$.

We code a success trial by **1** and a failure by **0**, thereby obtaining a *word* consisting of the letters **0** and **1**. A sequence of n trials is thus represented by a *word* of length n over the *alphabet* $\{\mathbf{0}, \mathbf{1}\}$. In a natural way we attach a *weight* ω to each word x by replacing **1** by p and **0** by q and then multiplying as usual. For instance, the word **0110** has the weight p^2q^2. We consider *languages* (sets of words) L and their *generating function* $\ell(z)$. The latter is defined to be

$$\ell(z) = \sum_{x \in L} \omega(x) z^{|x|}, \tag{1}$$

where $|x|$ is the *length* (number of letters) of the word x. This generating function can be obtained simply by formally replacing the letter **1** by pz and **0** by qz in the language L and replacing the so-called *concatenation* of words by the usual product and the (disjoint) *union* by the usual addition so that, for instance, $L = \{\mathbf{0}, \mathbf{010}, \mathbf{0110}\}$ has the generating function $\ell(z) = qz + pq^2z^3 + p^2q^2z^4$.

Instead of considering $\mathbb{P}\{X = n\}$, it is easier to consider $\mathbb{P}\{X > n\}$; that means the probability that n trials did not produce k consecutive successes, or the probability that a random word of n letters does not contain the (contiguous) subword $\mathbf{1}^k$. We consider the language of these words. A compact notion of it is

$$(\mathbf{1}^{<k}\,\mathbf{0})^*\mathbf{1}^{<k}, \tag{2}$$

where $\mathbf{1}^{<k} = \{\varepsilon, \mathbf{1}, \mathbf{11}, \ldots, \mathbf{1}^{k-1}\}$, with ε being the empty word. This expresses the fact that words without the (contiguous) subword $\mathbf{1}^k$ can be written as several blocks of less than k ones, separated by zeros. Let us recall that the asterisk L^* describes sequences of L. More formally, $L^* = \bigcup_{n \ge 0} L^n$, and L^n means the concatenation of n copies of L, which can be defined recursively by $LL = \{xy \mid x \in L, y \in L\}$ and $L^n = L^{n-1}L$ and $L^0 = \{\varepsilon\}$. Quite nicely, the generating function of L^* is obtained by $\frac{1}{1-\ell(z)}$. Now, to the language $\mathbf{1}^{<k}\,\mathbf{0}$ the generating function

$$(1 + pz + (pz)^2 + \cdots + (pz)^{k-1}) \cdot qz = \frac{1 - p^k z^k}{1 - pz} qz \tag{3}$$

is associated, and thus we have, furthermore,

*This note was written while the author visited the University Paris 6; he is thankful for the warm hospitality he encountered there.

$$g(z)=\sum_{n\geq 0}\mathbb{P}\{X>n\}z^n=\frac{1}{1-qz\frac{1-p^kz^k}{1-pz}}\cdot\frac{1-p^kz^k}{1-pz}=\frac{1-p^kz^k}{1-z+qp^kz^{k+1}}. \tag{4}$$

From this we also obtain the probability generating function

$$\begin{aligned} f(z)&:=\sum_{n\geq 0}\mathbb{P}\{X=n\}z^n=\sum_{n\geq 0}\big(\mathbb{P}\{X>n-1\}-\mathbb{P}\{X>n\}\big)z^n\\ &=1+z\sum_{n\geq 1}\mathbb{P}\{X>n-1\}z^{n-1}-\sum_{n\geq 0}\mathbb{P}\{X>n\}z^n\\ &=1-(1-z)g(z)=\frac{1-z+qp^kz^{k+1}-1+p^kz^k+z-p^kz^{k+1}}{1-z+qp^kz^{k+1}}\\ &=\frac{p^kz^k(1-pz)}{1-z+qp^kz^{k+1}}. \end{aligned} \tag{5}$$

This derivation completely avoided unpleasant recursions. For such very useful combinatorial constructions and their automatic translation into generating functions, we refer to the survey [2] and a few earlier survey papers of Flajolet cited therein.

The probability generating function (5) appeared first in [10].

Guibas and Odlyzko in a series of papers ([3], [4], [5]) dealt with general forbidden subwords, not just $\mathbf{1}^k$. These papers were surveyed in [8] and [9]. Rewriting things accordingly, formula (6.44) in [9] gives

$$f(z)=\frac{(pz)^k}{(pz)^k+(1-z)C(z)}, \tag{6}$$

where the polynomial $C(z)$ (the "correlation polynomial") depends on the forbidden pattern and is

$$C(z)=1+(pz)+\cdots+(pz)^{k-1}=\frac{1-(pz)^k}{1-pz} \tag{7}$$

in this special instance.

Knuth used similar arguments in [7]. He considered strings of **0**, **1**, **2**, where **0** and **2** appear with probability 1/4 and **1** appears with probability 1/2 and the string $\mathbf{1}^k\mathbf{2}$ is forbidden. Also, he considered the zeros of the "auxiliary equation"

$$1-z+qp^kz^{k+1}=0. \tag{8}$$

For example, there is a "dominant" solution $\rho=\rho_k$ which can be approximated by "bootstrapping": Starting from $z=1+qp^kz^{k+1}$, a first approximation is $\rho\approx 1$. Inserting this on the right-hand side and expanding, we find $\rho\approx 1+qp^k$, and after one more step,

$$\rho\approx 1+qp^k+(k+1)q^2p^{2k}, \tag{9}$$

etc. Kirschenhofer and Prodinger also used this type of argument in [6].

With this dominant singularity it is also easy to find the asymptotics of $\mathbb{P}\{X=n\}$ for fixed k, as $n\to\infty$. We have

$$f(z)=\frac{p^kz^k(1-pz)}{1-z+qp^kz^{k+1}}\sim\frac{A_k}{1-z/\rho}\quad\text{as } z\to\rho. \tag{10}$$

This can be explained informally by saying that *locally* only one term of the *partial fraction decomposition* of the *rational function* $f(z)$ is needed to describe its behavior in a vicinity of the dominant singularity ρ.

Here, A_k is a constant that can be found by the traditional techniques to compute the partial fraction decomposition of a rational function.

Thus, the coefficient of z^n in $f(z)$ (i.e., $\mathbb{P}\{X=n\}$) behaves as $A_k \cdot \rho^{-n}$ (the coefficient of z^n in $\frac{A_k}{1-z/\rho}$). The constant A_k behaves as $A_k \approx qp^k$ for $k \to \infty$.

Such asymptotic considerations are to be found in many textbooks and survey articles, notably in [9].

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ON THE GENERAL LINEAR RECURRENCE RELATION

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The general m^{th}-order linear recurrence relation can be written as

$$R_n = \sum_{i=1}^{m} a_i R_{n-i}, \quad \text{for } m \geq 2, \tag{1}$$

where the a_i's are any complex numbers, with $a_m \neq 0$. If suitable initial values $R_{-(m-2)}$, $R_{-(m-3)}$, $\ldots, R_0, R_1$ are specified, the sequence $\{R_n\}$ is uniquely determined for all integral n.

The auxiliary equation of (1) is

$$x^m = \sum_{i=1}^{m} a_i x^{m-i}. \tag{2}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the m roots, assumed distinct, of (2) and define $\overline{\alpha}_j$ by

$$\overline{\alpha}_j = \prod_{\substack{i=1 \\ i \neq j}}^{m} (\alpha_j - \alpha_i).$$

Then the *fundamental* $\{U_n\}$ and *primordial* $\{V_n\}$ sequences that satisfy (1) are given by the following Binet formulas [1]. For any integer n, we have

$$U_n = \sum_{j=1}^{m} \frac{\alpha_j^{n+m-2}}{\overline{\alpha}_j} \quad \text{and} \quad V_n = \sum_{j=1}^{m} \alpha_j^n, \tag{3}$$

so that $U_{-(m-2)} = U_{-(m-3)} = \cdots = U_{-1} = U_0 = 0$ and $U_1 = 1$. Also $V_1 = a_1$ and

$$V_i = a_1 V_{i-1} + \cdots + a_{i-1} V_1 + i a_i, \quad \text{for } 1 \leq i \leq m. \tag{4}$$

In this paper we answer a question of Jarden, who in his book [2] (p. 88), see also [1], asked for the value of $U_{2n} - U_n V_n$ for the m^{th}-order linear recurrence relation. For example, when $m = 2$, where $a_1 = a_2 = 1$, $\{U_n\}$ and $\{V_n\}$ are the Fibonacci and Lucas sequences, respectively. In this case, we have

$$U_{2n} - U_n V_n = 0.$$

For the general third- and fourth-order linear recurrence relations we have, respectively,

$$U_{2n} - U_n V_n = a_3^n U_{-n} \quad \text{and} \quad U_{2n} - U_n V_n = (-1)^n a_4^n \{U_{-n} V_{-n} - U_{-2n}\}.$$

For the general m^{th}-order linear recurrence relation, we have the following, very appealing theorem.

Theorem: For any integer n, and $m \geq 2$, we have

$$U_{2n} - U_n V_n = (-1)^{(m+1)(n+1)} a_m^n \sum_{i=0}^{m-2} \sum_{\pi(i)} \frac{(-1)^k}{k_1!k_2!\dots k_i!1^{k_1}2^{k_2}\dots i^{k_i}} V_{-n}^{k_1} V_{-2n}^{k_2} \cdots V_{-in}^{k_i} U_{-(m-2-i)n},$$

where a_m is the constant term in the auxiliary equation and the inner summation is taken over all partitions of $i = 1k_1 + 2k_2 + \cdots + ik_i$ so that k_j is the number of parts of size j. Here, $k = k_1 + k_2 + \cdots + k_i$ is the total number of parts in the partition. The coefficient of $U_{-(m-2-i)n}$, inside the second summation sign, is taken to be 1 when $i = 0$.

In order to prove the above theorem, we use the following lemma.

Lemma: Using the above notation, we have

$$\sum_{\pi(i)} \frac{(-1)^k}{k_1!k_2!\dots k_i!1^{k_1}2^{k_2}\dots i^{k_i}} V_{-1}^{k_1} V_{-2}^{k_2} \dots V_{-i}^{k_i} = \frac{a_{m-i}}{a_m} \quad \text{for } 0 \leq i \leq (m-1),$$
$$= -\frac{1}{a_m} \quad \text{for } i = m.$$

Proof of Lemma: First, we note that

$$\exp\left\{-\left(\frac{V_{-1}}{1}x + \frac{V_{-2}}{2}x^2 + \frac{V_{-3}}{3}x^3 + \cdots\right)\right\}$$
$$= \sum_{i=0}^{\infty} x^i \sum_{\pi(i)} \frac{(-1)^k}{k_1!k_2!\dots k_i!1^{k_1}2^{k_2}\dots i^{k_i}} V_{-1}^{k_1} V_{-2}^{k_2} \dots V_{-i}^{k_i}. \tag{5}$$

Therefore, we need to evaluate the function,

$$f(x) = \sum_{i=1}^{\infty} \frac{V_{-i}}{i} x^i.$$

Using the fact that $\{V_n\}$ satisfies the recurrence relation (1), with the help of (4) it is not hard to see that the generating function $g(x) = \sum_{n=0}^{\infty} V_{-n} x^n$, for V_{-n}, is given by

$$g(x) = \frac{ma_m + (m-1)a_{m-1}x + (m-2)a_{m-2}x^2 + \cdots + 2a_2 x^{m-2} + a_1 x^{m-1}}{a_m + a_{m-1}x + \cdots + a_1 x^{m-1} - x^m}. \tag{6}$$

Letting

$$h(x) = 1 + \frac{a_{m-1}}{a_m}x + \frac{a_{m-2}}{a_m}x^2 + \cdots + \frac{a_1}{a_m}x^{m-1} - \frac{1}{a_m}x^m, \tag{7}$$

from (6) and (7) we have

$$g(x) = m - \frac{h'(x)}{h(x)}x. \tag{8}$$

Now, since $V_0 = m$, from (8) we have

$$-\sum_{n=1}^{\infty} V_{-n} x^{n-1} = \frac{m - g(x)}{x} = \frac{h'(x)}{h(x)}.$$

Integrating, and using $h(0)=1$ to eliminate the constant of integration, we have

$$-\sum_{n=1}^{\infty}\frac{V_{-n}}{n}x^n=\log h(x).$$

Therefore,

$$\exp\left\{-\sum_{n=1}^{\infty}\frac{V_{-n}}{n}x^n\right\}=h(x). \tag{9}$$

So, from (5) and (9) we have

$$h(x)=\sum_{i=0}^{\infty}x^i\sum\frac{(-1)^k}{k_1!k_2!\dots k_i!1^{k_1}2^{k_2}\dots i^{k_i}}V_{-1}^{k_1}V_{-2}^{k_2}\dots V_{-i}^{k_i}. \tag{10}$$

Using the expression for $h(x)$ given by (7), we can equate the coefficients of x in (10) to complete the proof of the lemma. □

Proof of Theorem: From the Binet formulas (3) for U_n and V_n, we have

$$\begin{aligned}U_{2n}-U_nV_n&=\left(\frac{\alpha_1^{2n+m-2}}{\overline{\alpha}_1}+\frac{\alpha_2^{2n+m-2}}{\overline{\alpha}_2}+\dots+\frac{\alpha_m^{2n+m-2}}{\overline{\alpha}_m}\right)\\&\quad-\left(\frac{\alpha_1^{n+m-2}}{\overline{\alpha}_1}+\frac{\alpha_2^{n+m-2}}{\overline{\alpha}_2}+\dots+\frac{\alpha_m^{n+m-2}}{\overline{\alpha}_m}\right)(\alpha_1^n+\alpha_2^n+\dots+\alpha_m^n)\\&=-\sum_{i\neq j}\frac{\alpha_j^{n+m-2}\alpha_i^n}{\overline{\alpha}_j},\end{aligned} \tag{11}$$

where the summation is taken over all $1\le i,j\le m$, such that $i\neq j$. Therefore, to prove the theorem, we need to show that the right-hand side of the theorem is given by the right-hand side of (11). First, we require some new notation. The a_i in (2) are given by

$$a_i=(-1)^{i+1}\sum\alpha_1\alpha_2\dots\alpha_i,$$

where α_i are the roots of (2) and the summation is taken over all possible distinct products of i distinct α_j's. Now define $a_i(n)$ and $c_i(n)$ by

$$a_i(n)=(-1)^{i+1}\sum\alpha_1^n\alpha_2^n\dots\alpha_i^n \quad\text{and}\quad c_i(n)=\sum\alpha_1^n\alpha_2^n\dots\alpha_i^n,$$

so that $a_i(n)=(-1)^{i+1}c_i(n)$. Now, by the lemma, for any integer n, we have

$$\begin{aligned}\sum_{\pi(i)}\frac{(-1)^k}{k_1!k_2!\dots k_i!1^{k_1}2^{k_2}\dots i^{k_i}}V_{-n}^{k_1}V_{-2n}^{k_2}\dots V_{-in}^{k_i}&=\frac{a_{m-i}(n)}{a_m(n)}\quad\text{for } 0\le \mathrm{i}\le(\mathrm{m}-1),\\&=-\frac{1}{a_m(n)}\quad\text{for } i=m.\end{aligned} \tag{12}$$

Using (12), we can rewrite the theorem as

$$U_{2n}-U_nV_n=(-1)^{(m+1)(n+1)}a_m^n\sum_{i=0}^{m-2}\frac{a_{m-i}(n)}{a_m(n)}U_{-(m-2-i)n}. \tag{13}$$

Since

$$a_m^n = (-1)^{(m+1)n} c_m(n),$$
$$a_{m-i}(n) = (-1)^{m+i+1} c_{m-i}(n), \tag{14}$$

and

$$a_m(n) = (-1)^{m+1} c_m(n),$$

we have, from (13) and (14),

$$U_{2n} - U_n V_n = (-1)^{m+1} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) U_{-(m-2-i)n}\,. \tag{15}$$

By the Binet formula,

$$U_{-(m-2-i)n} = \sum_{j=1}^{m} \frac{\alpha_j^{in-mn+2n+m-2}}{\overline{\alpha}_j},$$

which, when inserted into (15), gives

$$\begin{aligned} U_{2n} - U_n V_n &= (-1)^{m+1} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) \sum_{j=1}^{m} \frac{\alpha_j^{in-mn+2n+m-2}}{\overline{\alpha}_j} \\ &= (-1)^{m+1} \sum_{j=1}^{m} \frac{\alpha_j^{2n+m-2}}{\overline{\alpha}_j} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) \alpha_j^{(i-m)n}. \end{aligned} \tag{16}$$

Now we note that

$$\begin{aligned} \left(x + \frac{1}{\alpha_1^n}\right)\left(x + \frac{1}{\alpha_2^n}\right) \cdots \left(x + \frac{1}{\alpha_m^n}\right) &= \sum_{i=0}^{m} \frac{c_i(n)}{c_m(n)} x^i \\ &= \sum_{i=0}^{m} \frac{c_{m-i}(n)}{c_m(n)} x^{m-i} \end{aligned} \tag{17}$$

So if we let $x = -1/\alpha_j^n$ in (17), for any $j = 1, 2, \ldots, m$, we have

$$\sum_{i=0}^{m} (-1)^i c_{m-i}(n) \alpha_j^{(i-m)n} = 0. \tag{18}$$

From (18), we easily obtain

$$(-1)^{m+1} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) \alpha_j^{(i-m)n} = -c_1(n) \alpha_j^{-n} + c_0(n). \tag{19}$$

Now we note that $c_0(n) = 1$ and $c_1(n) = \sum_{i=1}^m \alpha_i^n$. Therefore, using (19) in (16), we have

$$U_{2n} - U_n V_n = \sum_{j=1}^{m} \frac{\alpha_j^{2n+m-2}}{\overline{\alpha}_j} \left\{ -\sum_{i=1}^{m} \alpha_i^n \alpha_j^{-n} + 1 \right\} = -\sum_{j=1}^{m} \sum_{i=1}^{m} \frac{\alpha_j^{n+m-2} \alpha_i^n}{\overline{\alpha}_j} + \sum_{j=1}^{m} \frac{\alpha_j^{2n+m-2}}{\overline{\alpha}_j} = -\sum_{i \neq j} \frac{\alpha_j^{n+m-2} \alpha_i^n}{\overline{\alpha}_j}.$$

Which agrees with the right-hand side of (11). Hence, the theorem is proved. □

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AMS Classification Numbers: 11B37, 11B39

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Announcement

SEVENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

July 14-July 19, 1996
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Call for Papers

The SEVENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Technische Universität Graz from July 14 to July 19, 1996. This conference will be sponsored jointly by the Fibonacci Association and Technische Universität Graz.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

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ON A PROBABILISTIC PROPERTY OF THE FIBONACCI SEQUENCE

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Let $\eta_1, \ldots, \eta_n, \ldots$ be a sequence of independent integer-valued random variables. Let $S_n = \eta_1 + \cdots + \eta_n$, $A_n = ES_n$, $B_n^2 = \operatorname{var} S_n$, $P_n(m) = P(S_n = m)$, and $f(t, \eta_j)$ denote the characteristic function of the random variable η_j.

The local limit theorem (LLT) is formulated as $P_n(m) = (2\pi B_n^2)^{-1/2} \cdot \exp\{-(m - A_n)^2 / 2B_n^2\} + o(B_n^{-1})$ when $n \to \infty$ uniformly for m.

The first results on the normal approximation of binomial distributions belong to de Moivre, Laplace, and Poisson. Very general theorems on the LLT were obtained by von Mises in [1]. Assuming additionally that the summands are i.i.d. and have a finite variance, B. Gnedenko [2] derived necessary and sufficient conditions for the LLT. The next step, for not i.i.d. but uniformly bounded variables, was made by Yu. V. Prohorov in [3]. Besides those mentioned above, the LLT problem was investigated by W. Feller [4] and C. Stone [5]. More complete bibliographical information can be found in [6].

It is well known that for uniformly distributed random variables the LLT is equivalent to the central limit theorem [9], [10]. Hence, it is reasonable to ask whether this holds in general. The answer is negative. Using the Fibonacci sequence, we will construct below another sequence of independent asymptotically uniformly distributed random variables which satisfies the central limit theorem, has the uniform asymptotic negligibility (UAN) property but for which the local limit theorem fails to be valid.

Let $[1; 1, \ldots, 1, \ldots]$ be a continued fraction representation of the number $\varphi = (1 + \sqrt{5})/2$. Denote by P_j / Q_j the convergents of the continued fraction of φ, which can be represented by the table below.

j					1	2	3	...
P_j	0	1	1	2	3	5	8	...
Q_j	1	0	1	1	2	3	5	...

It follows from the table that P_j $(j = 0, 1, 2, \ldots)$ is the Fibonacci sequence and $P_{j-1} = Q_j$ for $j \geq 1$.

Let us now consider a sequence of independent integer-valued random variables represented by

$$\begin{array}{ll} 1. & \xi_1, \ldots, \xi_{n_1}, \\ 2. & \xi_{n_1+1}, \ldots, \xi_{n_1+n_2}, \\ & \ldots \\ j. & \xi_{n_1+\cdots+n_{j-1}+1}, \ldots, \xi_{n_1+\cdots n_j}, \\ & \ldots \\ k. & \xi_{n_1+\cdots+n_{k-1}+1}, \ldots, \xi_{n_1+\cdots+n_k}. \end{array} \tag{1}$$

Each value of the line j is assumed to take the values 0, Q_j, P_j with respective probability values of $(P_j - 2)/P_j, 1/P_j, 1/P_j$. Thus, if ξ_r is in row j, then

$$f(t, \xi_r) = \frac{P_j - 2 + e^{itQ_j} + e^{itP_j}}{P_j},$$

$$|f(t, \xi_r)|^2 = \frac{(P_j - 2)^2 + 2}{P_j^2} + \frac{2}{P_j^2}\cos t(P_j - Q_j) + \frac{2(P_j - 2)}{P_j^2}(\cos tQ_j + \cos tP_j),$$

$$E\xi_r = \frac{(P_j + Q_j)}{P_j} = \frac{P_{j+1}}{P_j},$$

and

$$\operatorname{var}\xi_r = \frac{P_j^2 + Q_j^2}{P_j} - \frac{(P_j + Q_j)^2}{P_j^2}.$$

Notice that

$$\operatorname{var}\xi_r > (1 - 1/P_j)\frac{P_j^2 + Q_j^2}{P_j} > \frac{1}{3}\frac{P_j^2 + Q_j^2}{P_j}.$$

We will take n_j as

$$n_j = \left[P_j^{3/2}\right] + 1, \tag{2}$$

where $[a]$ represents the integer part of a.

Let $N_k = n_1 + \cdots + n_k$ and

$$B_{N_k}^2 = \operatorname{var} S_{N_k} = \sum_1^k \left(\left[P_j^{3/2}\right] + 1\right)\operatorname{var}\xi_{N_k} = O(P_k^{5/2}).$$

First, we will verify that the sequence has the UAN property. For an arbitrary n, we can choose a number k such that $N_{k-1} < n \le N_k$. Hence,

$$\max_{1\le j\le n}\left|\xi_j - E\xi_j\right| \le P_k \text{ and } B_{N_k}^2 \ge 3^{-1}\sum_1^k (P_j^2 + Q_j^2)n_j / P_j.$$

Therefore,

$$\max_{1\le j\le n}\left|\xi_j - E\xi_j\right| \Big/ B_n \le c / P_{k-1}^{1/4} \to 0 \text{ as } n \to \infty. \tag{3}$$

Here and in what follows c denotes a positive constant. However, the same symbol c may also stand for different constants. The preceding limit result is the UAN property. One may also check that Liapunov's condition,

$$\left(\sum_{j=1}^n E\left|\xi_j - E\xi_j\right|^{2+\delta}\right)^{1/(2+\delta)} \Big/ B_n \to 0,$$

for some $\delta > 0$, holds.

Next, we will investigate the property of the sequence being asymptotically uniformly distributed. We use the Dvoretzky-Wolfowitz test [8], which states that this is so if, for an arbitrary fixed $h > 0$ and $z = 1, 2, \ldots, h-1$, the characteristic function of the sums of the independent random variables tends to zero at the rational point $2\pi\alpha$, where $\alpha = z / h$.

It will be assumed, without loss of generality, that z and h are mutually prime numbers. Clearly,

$$\left| f(2\pi z / h, \xi_{N_j}) \right| \le 1 / P_j + \left| (P_j - 2) / P_j + \exp(2\pi i Q_j z / h) / P_j \right|.$$

Assume Q_j is not a multiple of h. We can then write $zQ_j = mh + k, 1 \le k \le h-1$. Hence,

$$\begin{aligned}\left| (P_j - 2) / P_j + \exp(2\pi i z Q_j / h) / P_j \right| &\le \max_{1 \le k \le h-1} \left| (P_j - 2) / P_j + \exp(2\pi i k / h) / P_j \right| \\ &= \max_{1 \le k \le h-1} \left| (P_j - 3) / P_j + (1 + \exp(2\pi i k / h)) / P_j \right| \\ &= (P_j - 3) / P_j + \max_{1 \le k \le h-1} \left| 1 + \exp(2\pi i k / h) \right| / P_j \\ &\le (P_j - 1 - \rho) / P_j,\end{aligned}$$

where $\rho = \rho(h) = 2(1 - \cos(\pi / h))$. That is,

$$\left| f(2\pi z / h, \xi_{N_j}) \right| \le 1 - \rho / P_j.$$

Choosing n_j, by (2), we obtain

$$\left| f(2\pi z / h, \xi_{N_j}) \right|^{2n_j} \le (1 - \rho / P_j)^{2P_j} \le \exp(-2\rho).$$

The latter inequality holds only when Q_j is not a multiple of h. Let us count the number of such Q_j. Since $P_{j-1}Q_j - P_jQ_{j-1} = \pm 1$, it follows that Q_{j-1} and Q_j are not simultaneously multiples of h. Therefore, there are at least $[k/2]$ members of the sequence $Q_1, \ldots, Q_n$ that are not multiples of h. Thus,

$$\prod_{j=1}^{k} \left| f(2\pi z / h, \xi_{N_j}) \right|^{2n_j} \le \exp\{-k\rho\} \to 0 \text{ as } k \to \infty.$$

Therefore, the Dvoretsky-Wolfowitz test is satisfied.

It should be noted that in [7] we find the following necessary condition for the LLT:

$$I_n = B_n \int_{\varepsilon_n \le t \le 2\pi} \prod_{k=1}^{n} \left| f(t, \xi_j) \right|^2 dt \to 0$$

for any positive ε_n which tends to zero as $n \to \infty$. We will show that this condition does not hold.

Using Taylor's expansion when $|t - 2\pi / \varphi| \le 1 / B_{N_k}$, we write

$$\begin{aligned}\left| f(t, \xi_{N_j}) \right|^2 = \left| f(2\pi / \varphi, \xi_{N_j}) \right|^2 &+ \left| t - 2\pi / \varphi \right| \left[\left| f(t, \xi_{N_j}) \right|^2 \right]'_{t = 2\pi / \varphi} \\ &+ \left| t - 2\pi / \varphi \right|^2 \left[\left| f(t, \xi_{N_j}) \right|^2 \right]''_{t = \theta} \Big/ 2,\end{aligned} \tag{4}$$

where $t < \theta < 2\pi / \varphi$.

Next, each term from (4) is estimated separately in the following manner:

$$\left|f(2\pi/\varphi,\xi_{N_j})\right|^2=\left|f(t_j,\xi_{N_j})\right|^2+\left|t_j-2\pi/\varphi\right|\left[\left|f(t,\xi_{N_j})\right|^2\right]'_{t=t_j}$$

$$+\left|t_j-2\pi/\varphi\right|^2\left[\left|f(t,\xi_{N_j})\right|^2\right]''_{t=\theta}\Big/2,$$

where $t_j=2\pi Q_{j-1}/Q_j$.

Using $P_{j-1}Q_j-P_jQ_{j-1}=\pm1$ and the elementary inequality $\cos x\geq 1-x^2/2$, we may write

$$\begin{aligned}\left|f(t_j,\xi_{N_j})\right|^2&\geq\frac{(P_j-2)^2+2+2(P_j-2)}{P_j^2}+\frac{2+2(P_j-2)}{P_j^2}(1-1/2(2\pi/Q_j)^2)\\&=1-\frac{(P_j-1)}{P_j^2}(2\pi/Q_j)^2>1-(2\pi/Q_j)^2/P_j,\quad(j\geq2).\end{aligned}\tag{5}$$

We then have

$$\begin{aligned}\left[\left|f(t,\xi_{N_j})\right|^2\right]'_{t=t_j}&=-\frac{2(P_j-Q_j)}{P_j^2}\sin2\pi\frac{Q_{j-1}}{Q_j}(P_j-Q_j)-\frac{2Q_j(P_j-2)}{P_j^2}\\&\quad\times\sin2\pi Q_{j-1}-\frac{2P_j(P_j-2)}{P_j}\sin2\pi\frac{Q_{j-1}}{Q_j}P_j\\&=\left(\frac{2(P_j-Q_j)}{P_j^2}+\frac{2P_j(P_j-2)}{P_j^2}\right)\sin\left((-1)^{j-1}\frac{2\pi}{Q_j}\right)\leq\left(\frac{2(P_j-Q_j)}{P_j^2}+\frac{2P_j(P_j-2)}{P_j^2}\right)\frac{2\pi}{Q_j}\\&=\frac{4\pi}{Q_j}(1-1/P_j-Q_j/P_j^2)\leq4\pi/Q_j\end{aligned}\tag{6}$$

and

$$\begin{aligned}\left[\left|f(t,\xi_{N_j})\right|^2\right]''&=2\left|(P_j-2)\cos P_jt+\frac{(P_j-2)Q_j^2}{P_j^2}\cos Q_jt+\frac{(P_j-Q_j)^2}{P_j^2}\cos(P_j-Q_j)t\right|<\left|f''(0,\xi_{N_j})\right|\\&=2\left((P_j-2)+\frac{(P_j-2)Q_j^2}{P_j^2}+\frac{(P_j-Q_j)^2}{P_j^2}\right)\\&=2\left(P_j-1+\frac{Q_j^2-2Q_j}{P_j}-\frac{Q_j^2}{P_j^2}\right)<2(P_j^2+Q_j^2)/P_j.\end{aligned}$$

Using $\left|t_j-2\pi/\varphi\right|\leq2\pi/Q_j^2$ and taking into consideration the estimations (5) and (6), we have

$$\left|f(2\pi/\varphi,\xi_{N_j})\right|^2\geq1-(2\pi/Q_j)^2/P_j-8\pi^2/Q_j^3-(2\pi/Q_j^2)^2(P_j^2+Q_j^2)/P_j.$$

Furthermore,

$$\left[\left|f(t,\xi_{N_j})\right|^2\right]'_{t=2\pi/\varphi} \le \left[\left|f(t_j,\xi_{N_j})\right|^2\right]' + \left|t_j - 2\pi/\varphi\right| \left[\left|f(t_j,\xi_{N_j})\right|^2\right]''$$
$$\le 4\pi/Q_j + 4\pi(P_j^2+Q_j^2)/Q_j^2P_j$$

and

$$\left[\left|f(t,\xi_{N_j})\right|^2\right]''_{t=\theta} \le 2(P_j^2+Q_j^2)/P_j.$$

Taking the above estimations into account for expansion (4), we have

$$\left|f(t,\xi_{N_j})\right|^2 \ge 1-(2\pi/Q_j)^2P_j - 8\pi^2/Q_j^3 - 8\pi^2(P_j^2+Q_j^2)/Q_j^4P_j$$
$$-(4\pi/Q_j + 4\pi(P_j^2+Q_j^2)/Q_j^2P_n)/B_{N_k} - 2(P_j^2+Q_j^2)/B_{N_k}^2P_j.$$

By a simple transformation, we obtain

$$\left|f(t,\xi_{N_j})\right|^2 \ge 1 - c/P_j^3 - c/B_{N_k}P_j - cP_j/B_{N_k}^2.$$

Using the elementary inequality $\exp(-cx) < 1-x$ for $0 < x < 1/2$, and $c > \ln 4$, we have

$$\prod_{j=1}^{k}\left|f(t,\xi_{N_j})\right|^{2n_j} \ge \exp\left\{-\sum_{j=1}^{k} n_j(P_j^{-3} + (B_{N_k}P_j)^{-1} + P_jB_{N_k}^{-2})\right\}.$$

Hence, we conclude that, if k is sufficiently large, then

$$I_{N_k} > B_{N_k}\int\limits_{|t-2\pi/\varphi|\le B_{N_k}^{-1}} \prod_{j=1}^{n}\left|f(t,\xi_j)\right|^{2n_j}dt > B_{N_k}\int\limits_{|t-2\pi/\varphi|\le B_{N_k}^{-1}} \exp(-c)dt = 2e^{-c}.$$

So we have shown that the sequence (1) of independent integer valued random variables constructed by using the Fibonacci sequence is asymptotically uniformly distributed, satisfies the central limit theorem, and has the UAN property, but the local limit theorem fails to be valid for (1).

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PIERCE EXPANSIONS OF RATIOS OF FIBONACCI AND LUCAS NUMBERS AND POLYNOMIALS

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INTRODUCTION

The connection between the Euclidean algorithm for determining the greatest common divisor of two positive integers a and b and the continued fraction expansion of the rational number a/b is well known. As Lamé [9] observed, two successive Fibonacci numbers F_{n-1} and F_n provide a pair of integers for which the Euclidean algorithm takes as long as possible to terminate, in the sense that (F_{n-1}, F_n) takes as long or longer than any pair (a, b) with $b > a > 1$ and $b \le F_n$. Analogous results hold if arithmetic is done in $\mathbb{Q}[x]$ or $\mathbb{F}_q[x]$, for $\mathbb{F}_q[x]$ the finite field with $q = p^a$ elements [4], [6], [7], [10].

One can view the continued fraction expansion more generally as an association with a real number of a sequence of positive integers, the sequence being finite if and only if the real number is rational. Other methods exist to accomplish the same task. Two in particular of interest are the so-called Engel expansion and the Pierce expansion. Each arises from an iterated division algorithm, but the roles of successive dividends and divisors are played by different elements than in the Euclidean algorithm.

In particular, for $1 \le a \le b$ integers, the Pierce expansion of a/b is the unique representation

$$\frac{a}{b} = \frac{1}{x_1} - \frac{1}{x_1 x_2} + \frac{1}{x_1 x_2 x_3} - \cdots + \frac{(-1)^{n-1}}{x_1 x_2 \ldots x_n}, \tag{1}$$

where the x_i are integers with $1 \le x_1 < x_2 < \cdots < x_n$. Successive x_i may be obtained via the division algorithm. Write $b = qa + r$ with q and r nonnegative integers and $r < a$. Then $a = \frac{b-r}{q}$ and so $\frac{a}{b} = \frac{1}{q} - \frac{1}{q}\left(\frac{r}{b}\right)$. Thus, $x_1 = q$, and the procedure may be applied again to the fraction r/b. The iteration stops when $r = 0$, which must happen after at most a steps (see [11]). A convenient notation for this expansion is

$$\frac{a}{b} = \{x_1, x_2, x_3, \ldots, x_n\}.$$

The Engel expansion is a similar expansion with all positive terms. Thus, for $1 \le a \le b$ integers, it is the unique representation of the form

$$\frac{a}{b} = \frac{1}{y_1} + \frac{1}{y_1 y_2} + \frac{1}{y_1 y_2 y_3} + \cdots + \frac{1}{y_1 y_2 \cdots y_n}, \tag{2}$$

where the y_i are integers with $1 \le y_1 \le y_2 \le \cdots \le y_n$. Here one would iterate the version of the division algorithm with negative remainders. Thus, $b = qa + r = (q+1)a - (a-r)$ and, hence, $a = \frac{b+(a-r)}{q+1}$ gives $\frac{a}{b} = \frac{1}{q+1} + \frac{1}{q+1}\left(\frac{a-r}{b}\right)$. The procedure is applicable again until $r = 0$. This expansion is frequently denoted

$$\frac{a}{b} = (y_1, y_2, y_3, \ldots, y_n).$$

Maximal lengths and other properties of Pierce and Engel expansions have been studied in [2], [11], [13], and [14].

In the case of polynomial rings, the appropriate measure of the size of the remainder is given by its degree, so that signs are no longer relevant and there is no distinguishing the Pierce and Engel expansions. For the Fibonacci polynomials [1] defined by

$$F_1(x) = 1,\ F_2(x) = x,\ F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \ \text{ for } n \geq 2,$$

and the Lucas polynomials given by

$$L_0(x) = 2,\ L_1(x) = x,\ L_{n+1}(x) = xL_n(x) + L_{n-1}(x) \ \text{ for } n \geq 1,$$

there are some especially attractive continued fraction expansions. In particular,

$$\frac{F_{n-1}(x)}{F_n(x)} = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{\ddots + \cfrac{1}{x}}}}, \tag{3}$$

where there are $n-1$ occurrences of x in (3), and

$$\frac{L_{n-1}(x)}{L_n(x)} = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{\ddots + \cfrac{1}{x/2}}}}, \tag{4}$$

where the continued fraction in (4) has n x's in its expansion.

Motivated by these expansions, we consider the Pierce-Engel expansions of these rational functions. In contrast to the longest possible expansions in their continued fraction expansions, the Pierce-Engel expansions are predictably short. For some special values of n they are especially short and elegant.

There are also regularities to note in the Pierce expansions of the rational numbers F_{n-1}/F_n and L_{n-1}/L_n. One such follows from a general result of Shallit [13], and we establish others in the last section.

EXPANSIONS OF FIBONACCI AND LUCAS POLYNOMIAL QUOTIENTS

We are most interested in the quotients $F_{n-1}(x)/F_n(x)$ and $L_{n-1}(x)/L_n(x)$, although the theorems we use apply more generally. Since in the limit we have

$$\lim_{n\to\infty} \frac{F_{n-1}(1/x)}{F_n(1/x)} = \frac{-1+\sqrt{1+4x^2}}{2x},$$

as n increases there are ever more terms incorporated in the infinite Pierce/Engel expansion of this function, shown in [3] to be

$$\frac{1}{L_1(z)}-\frac{1}{L_1(z)L_2(z)}-\frac{1}{L_1(z)L_2(z)L_4(z)}-\frac{1}{L_1(z)L_2(z)L_4(z)L_8(z)}-\cdots,$$

where $z = x^{-1}$. This particular expansion is also a concrete example of the Engel-type expansions for power series developed in [8]. This limiting case sets the pattern for the finite expansions of rational functions in the variable x. Using the notation $(a, b, c, d, \ldots)$ introduced earlier for the expansion

$$\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}+\frac{1}{abcd}+\cdots,$$

the finite expansion beginning

$$\frac{1}{L_1(x)}-\frac{1}{L_1(x)L_2(z)}-\frac{1}{L_1(x)L_2(x)L_4(x)}-\frac{1}{L_1(x)L_2(x)L_4(x)L_8(x)}-\cdots$$

can be written more compactly as $(L_1, -L_2, L_4, L_8, \ldots)$. Later we also allow more complicated expressions involving Lucas polynomials as entries.

It is possible to write an alternate representation in terms of the Chebyshev polynomials $C_n(x) = 2T_n(x/2)$, where

$$T_0(x) = 1,\ T_1(x) = x,\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ for } n \ge 1,$$

since $C_n(x) = (-i)^n L_n(ix)$.

The form of the expansions follows from two general results in [1], which we state as lemmas.

Lemma 1: Whenever a Fibonacci polynomial $F_m(x)$ is divided by a Fibonacci polynomial $F_{m-k}(x)$, $m \ne k$, of lesser or equal degree, the remainder is always a Fibonacci polynomial or the negative of a Fibonacci polynomial, and the quotient is a sum of Lucas polynomials whenever the division is not exact. Explicitly, for $p \ge 1$:

(i) the remainder is $\pm F_{(2p-1)m-2kp}(x)$ when

$$\frac{2p|m|}{2p+1} > |k| > \frac{(2p-2)|m|}{2p-1};$$

(ii) the quotient is $\pm L_k(x)$ when $|k| < 2|m|/3$;

(iii) the quotient is given by

$$Q_p(x) = \sum_{i=0}^{p-1} (-1)^{i(m-k)} L_{(2i+1)k-2im}(x)$$

for m, k, and p as in (i), and by $Q_p(x) + (-1)^{p(m-k)}$ if $k = 2pm/(2p+1)$;

(iv) the division is exact when $k = 2pm/(2p+1)$ or $k = (2p-1)m/2p$.

Lemma 2: Whenever a Lucas polynomial $L_m(x)$ is divided by a Lucas polynomial $L_{m-k}(x)$, $m \neq k$, of lesser degree, a nonzero remainder is always a Lucas polynomial or the negative of a Lucas polynomial. Explicitly:

(i) nonzero remainders have the form $\pm L_{(2p-1)m-2pk}(x)$ when

$$\frac{2p|m|}{2p+1} > |k| > \frac{(2p-2)|m|}{2p-1};$$

(ii) if $|k| < 2|m|/3$, the quotient is $\pm L_k(x)$;

(iii) the division is exact when $k = 2pm/(2p+1)$, $p \neq 0$.

These lemmas apply to give

Theorem 3: Any quotient of Fibonacci polynomials or Lucas polynomials has a (finite) Pierce-Engel expansion in which every entry is expressible as a linear combination of Lucas polynomials with coefficients 0 or ± 1. In the case $F_{n-1}(x)/F_n(x)$ or $L_{n-1}(x)/L_n(x)$, there are at least $m = \lfloor \log_2 n \rfloor$ entries, and the first $\lfloor \log_2 n \rfloor$ entries are $(L_1, -L_2, L_4, \ldots, L_{2^{m-1}})$.

Proof: The Pierce-Engel expansion for quotients of Fibonacci polynomials comes from the sequence of identities

$$\begin{aligned} F_n(x) &= L_k(x)F_{n-k}(x) + (-1)^{k+1}F_{n-2k}(x) \\ F_n(x) &= L_{2k}(x)F_{n-2k}(x) - F_{n-4k}(x) \\ F_n(x) &= L_{4k}(x)F_{n-4k}(x) - F_{n-8k}(x) \\ &\vdots, \end{aligned}$$

which may be continued as long as the last subscript remains nonnegative. These identities may be read as special cases of Lemma 1. Lemma 2 provides similar identities for Lucas polynomials. A negative subscript is replaced by a positive subscript via the identity $F_m(x) = (-1)^{m+1}F_{-m}(x)$. Then

$$\begin{aligned} \frac{F_{n-k}(x)}{F_n(x)} &= \frac{1}{L_k(x)}\left(1 + (-1)^k \frac{F_{n-2k}(x)}{F_n(x)}\right) \\ &= \frac{1}{L_k(x)}\left(1 + \frac{(-1)^k}{L_{2k}(x)}\left(1 - \frac{F_{n-4k}(x)}{F_n(x)}\right)\right) \\ &= \cdots \\ &= \left(L_k(x), (-1)^k L_{2k}(x), L_{4k}(x), \ldots\right). \end{aligned}$$

Table 1 on the following page gives Pierce-Engel expansions of some rational functions for small values of n.

The next theorem was obtained in [3] and [16]. The technique of proof can be modified to provide several other similar relations, which are collected in the theorem thereafter.

Theorem 4: For $n \geq 1$, $\dfrac{F_{2^n-1}(x)}{F_{2^n}(x)} = (L_1, -L_2, L_4, \ldots, L_{2^{n-1}})$

TABLE 1. Expansions of Quotients of Fibonacci Polynomials

n	Pierce-Engel Expansion of $F_{n-1}(x)\,/\,F_n(x)$
2	L_1
3	$L_1, -(L_2-1)$
4	$L_1, -L_2$
5	$L_1, -L_2, L_4-L_2+1$
6	$L_1, -L_2, L_4+1$
7	$L_1, -L_2, L_4, L_6-L_4+L_2-1$
8	$L_1, -L_2, L_4$
9	$L_1, -L_2, L_4, L_8-L_6+L_4-L_2+1$
10	$L_1, -L_2, L_4, L_8+L_4+1$
11	$L_1, -L_2, L_4, L_8-L_2, -(L_{10}-L_8+L_6-L_4+L_2-1)$
12	$L_1, -L_2, L_4, L_8+1$
13	$L_1, -L_2, L_4, L_8, L_{10}-L_4, -(L_{12}-L_{10}+L_8-L_6+L_4-L_2+1)$
14	$L_1, -L_2, L_4, L_8, -(L_{12}+L_8+L_4+1)$
15	$L_1, -L_2, L_4, L_8, L_{14}-L_{12}+L_{10}-L_8+L_6-L_4+L_2-1$
16	$L_1, -L_2, L_4, L_8$
17	$L_1, -L_2, L_4, L_8, L_{16}-L_{14}+L_{12}-L_{10}+L_8-L_6+L_4-L_2+1$
18	$L_1, -L_2, L_4, L_8, L_{16}+L_{12}+L_8+L_4+1$

Theorem 5: For $n \geq 1$, $\dfrac{F_{3\cdot 2^n-1}(x)}{F_{3\cdot 2^n}(x)} = (L_1, -L_2, L_4, \ldots, L_{2^n}, L_{2^{n+1}}+1)$.

For $n \geq 2$, $\dfrac{F_{2^n}(x)}{F_{2^n+1}(x)} = \left(L_1, -L_2, L_4, \ldots, L_{2^{n-1}}, \sum_{i=0}^{2^{n-1}}(-1)^i L_{2i}-1\right)$.

For $n \geq 3$, $\dfrac{F_{2^n-2}(x)}{F_{2^n-1}(x)} = \left(L_1, -L_2, L_4, \ldots, L_{2^{n-1}}, -\sum_{i=0}^{2^{n-1}-1}(-1)^i L_{2i}+1\right)$.

There are, in addition, dual results for Lucas polynomials. A brief table of Lucas polynomial expansions follows (see Table 2), and a general theorem (Theorem 6) makes explicit some of the patterns apparent in the table. Other patterns may be noted in the tables as well.

Theorem 6: For $n \geq 2$, $\dfrac{L_{2^n-1}(x)}{L_{2^n}(x)} = (L_1, -L_2, L_4, \ldots, L_{2^{n-1}}, L_{2^n}/2)$.

For $n \geq 2$, $\dfrac{L_{2^n}(x)}{L_{2^n+1}(x)} = \left(L_1, -L_2, L_4, \ldots, L_{2^{n-1}}, \sum_{i=0}^{2^{n-1}} L_{2i}-1\right)$.

$$\text{For } n \geq 3,\ \frac{L_{2^n-2}(x)}{L_{2^n-1}(x)} = \left(L_1, -L_2, L_4, \ldots, L_{2^{n-1}}, -\sum_{i=0}^{2^{n-1}-2} L_{2i} + 1\right).$$

$$\text{For } n \geq 1,\ \frac{L_{3\cdot 2^n-1}(x)}{L_{3\cdot 2^n}(x)} = (L_1, -L_2, L_4, \ldots, L_{2^n}, L_{2^{n+1}} - 1).$$

It is interesting to note the $L_{2^n}/2$ entry, in light of the last convergent of (4).

TABLE 2. Expansions of Quotients of Lucas Polynomials

n	Pierce-Engel Expansion of $L_{n-1}(x)/L_n(x)$
1	$L_1/2$
2	$L_1, -L_2/2$
3	$L_1, -(L_2+1)$
4	$L_1, -L_2, L_4/2$
5	$L_1, -L_2, L_4+L_2+1$
6	$L_1, -L_2, L_4-1$
7	$L_1, -L_2, L_4, -(L_6+L_4+L_2+1)$
8	$L_1, -L_2, L_4, L_8/2$
9	$L_1, -L_2, L_4, L_8+L_6+L_4+L_2+1$
10	$L_1, -L_2, L_4, L_8-L_4+1$
11	$L_1, -L_2, L_4, L_8+L_2, L_{10}+L_8+L_6+L_4+L_2+1$
12	$L_1, -L_2, L_4, L_8-1$
13	$L_1, -L_2, L_4, L_8, -(L_{10}+L_4), -(L_{12}+L_{10}+L_8+L_6+L_4+L_2+1)$
14	$L_1, -L_2, L_4, L_8, L_{12}-L_8+L_4-1$
15	$L_1, -L_2, L_4, L_8, -(L_{14}+L_{12}+L_{10}+L_8+L_6+L_4+L_2+1)$
16	$L_1, -L_2, L_4, L_8, L_{16}/2$
17	$L_1, -L_2, L_4, L_8, L_{16}+L_{14}+L_{12}+L_{10}+L_8+L_6+L_4+L_2+1$
18	$L_1, -L_2, L_4, L_8, L_{16}-L_{12}+L_8-L_4+1$

PIERCE EXPANSIONS OF QUOTIENTS OF FIBONACCI NUMBERS

The limiting value of F_{n-1}/F_n or L_{n-1}/L_n is the same: $(\sqrt{5}-1)/2$. Hence, Engel expansions eventually begin with the pattern of numbers in the Engel expansion of $(\sqrt{5}-1)/2$:

$$2, 5, 6, 13, 16, 16, 38, 48, 58, 104, 177, 263, \ldots,$$

i.e.,

$$\frac{\sqrt{5}-1}{2} = \frac{1}{2} + \frac{1}{2\cdot 5} + \frac{1}{2\cdot 5\cdot 6} + \cdots.$$

There is no pattern apparent in this sequence. In contrast, Pierce expansions begin

$$1, 2, 4, 17, 19, 5777, 5779, 192900153617, \ldots,$$

corresponding to

$$\frac{\sqrt{5}-1}{2}=1-\frac{1}{2}+\frac{1}{2\cdot 4}-\frac{1}{2\cdot 4\cdot 17}+\cdots.$$

The Pierce expansion has been analyzed before [13]. it is convenient to express it as

$$\{1, c_0-1, c_0+1, c_1-1, c_1+1, \ldots\},$$

where $c_0, c_1, c_2, \ldots = 3, 18, 5778, \ldots$ is the sequence given by the recurrence

$$c_0=3,\ \ c_{n+1}=c_n^3-3c_n \quad \text{for } n\geq 0.$$

For F_{n-1}/F_n or L_{n-1}/L_n, any particular choice of n gives a rational number and, hence, a finite Pierce expansion, and it often happens that the form of the finite expansion can be given conveniently in terms of the elements of $\{c_i\}$. It turns out to be powers of three that govern the patterns arising, and there are similar results for the Fibonacci and Lucas sequences.

Shallit [13] observed that, for $k\geq 0$,

$$c_k=\left(\frac{3+\sqrt{5}}{2}\right)^{3^k}+\left(\frac{3-\sqrt{5}}{2}\right)^{3^k}.$$

This relates $\{c_k\}$ to the well-known formulas

$$F_n=(\phi^n-\hat{\phi}^n)/\sqrt{5},\quad L_n=\phi^n+\hat{\phi}^n,$$

where $\phi=(1+\sqrt{5})/2$ and $\hat{\phi}=(1-\sqrt{5})/2$.

Theorem 7: For $k\geq 1$, $F_{3^k-1}/F_{3^k}=(1, c_0-1, c_0+1, c_1-1, c_1+1, \ldots, c_{k-1}-1)$.

We prove this with the aid of several lemmas. The lemmas may be of independent interest for the factorizations they provide for certain Fibonacci and Lucas numbers.

Lemma 8: $c_k=L_{2\cdot 3^k},\ k\geq 0$.

Proof:

$$c_k=\left(\frac{3+\sqrt{5}}{2}\right)^{3^k}+\left(\frac{3-\sqrt{5}}{2}\right)^{3^k}=(\phi^2)^{3^k}+(\hat{\phi}^2)^{3^k}=L_{2\cdot 3^k}.$$

A similar sequence, introduced by Shallit in [12], provides a formula for the 3^kth Lucas number.

Lemma 9: $F_{3^k}=(c_0-1)(c_1-1)\cdots(c_{k-1}-1),\ k\geq 1$.

Proof: For $k=1$, $F_3=2=c_0-1$. Now, using induction on *k*,

$$\begin{aligned}(c_0-1)(c_1-1)\cdots(c_k-1)&=F_{3^k}(c_k-1)\\&=(\phi^{3^k}-\hat{\phi}^{3^k})(\phi^{2\cdot 3^k}+\hat{\phi}^{2\cdot 3^k}-1)/\sqrt{5}\quad\text{(by Lemma 8)}\\&=(\phi^{3^{k+1}}-\hat{\phi}^{3^{k+1}})/\sqrt{5}\quad(\text{since } \phi\hat{\phi}=-1)\\&=F_{3^{k+1}}.\end{aligned}$$

Lemma 10: $L_{3^k} = (c_0+1)(c_1+1)\cdots(c_{k-1}+1),\ k\geq 1$.

Proof: Again induct on k.

Lemma 11: $F_{2\cdot 3^k} = (c_0-1)(c_0+1)\cdots(c_{k-1}-1)(c_{k-1}+1),\ k\geq 1$.

Proof: $F_{2\cdot 3^k} = F_{3^k}L_{3^k}$, and the result follows from Lemmas 9 and 10.

Lemma 12: $c_k = L_{3^k}^2+2,\ k\geq 0$.

Proof: $L_{3^k}^2+2 = (\phi^{3^k}+\hat{\phi}^{3^k})^2 = \phi^{2\cdot 3^k}+\hat{\phi}^{2\cdot 3^k}+2(\phi\hat{\phi})^{3^k}+2 = \phi^{2\cdot 3^k}+\hat{\phi}^{2\cdot 3^k} = c_k$.

By Lemma 8, this says $L_{2\cdot 3^k} = L_{3^k}^2+2$, so Lemma 12 also follows from the identity $L_{4n-2} = L_{2n-1}^2+2,\ n\geq 1$.

Lemma 13: $F_{3^{k+1}-1} = F_{3^k-1}(L_{3^k}^2+1)+L_{3^k},\ k\geq 1$.

Proof: The left-hand side may be written as $(\phi^{3^{k+1}-1}-\hat{\phi}^{3^{k+1}-1})/\sqrt{5}$. Write the right-hand side as $(\phi^{3^k-1}-\hat{\phi}^{3^k-1})(\phi^{2\cdot 3^k}+\hat{\phi}^{2\cdot 3^k}-1)/\sqrt{5}+\phi^{3^k}+\hat{\phi}^{3^k}$ by applying Lemma 12. This may be expanded as

$$(\phi^{3^{k+1}-1}-\hat{\phi}^{3^{k+1}-1}+\phi^{3^k-1}\hat{\phi}^{2\cdot 3^k}-\hat{\phi}^{3^k-1}\phi^{2\cdot 3^k}-\phi^{3^k-1}+\hat{\phi}^{3^k-1}+\sqrt{5}\phi^{3^k}+\sqrt{5}\hat{\phi}^{3^k})/\sqrt{5}$$
$$= F_{3^{k+1}-1}+((\phi\hat{\phi})^{3^k}(\hat{\phi}^{3^k}\phi^{-1}-\phi^{3^k}\hat{\phi}^{-1})-\phi^{3^k}\phi^{-1}+\hat{\phi}^{3^k}\hat{\phi}^{-1}+\sqrt{5}\phi^{3^k}+\sqrt{5}\hat{\phi}^{3^k})/\sqrt{5}$$
$$= F_{3^{k+1}-1}+(\hat{\phi}^{3^k}(-\phi^{-1}+\hat{\phi}^{-1}+\sqrt{5})+\phi^{3^k}(\hat{\phi}^{-1}-\phi^{-1}+\sqrt{5}))/\sqrt{5}.$$

But this is just $F_{3^{k+1}-1}$, since $\hat{\phi}^{-1}-\phi^{-1}+\sqrt{5}=0$.

Proof of Theorem 7: The proof is by induction on k. For $k=1$, $F_2/F_3 = 1/2 = (1, c_0-1)$. Now assume the theorem holds for k, and consider

$$(1, c_0-1, c_0+1, \ldots, c_k-1) = \frac{F_{3^k-1}}{F_{3^k}}+\frac{1}{(c_0-1)(c_o+1)\cdots(c_{k-1}-1)}\left(\frac{1}{c_{k-1}+1}-\frac{1}{(c_{k-1}+1)(c_k-1)}\right)$$
$$= \frac{F_{3^k-1}}{F_{3^k}}+\frac{1}{(c_0-1)(c_0+1)\cdots(c_{k-1}-1)}\frac{c_k-2}{(c_{k-1}+1)(c_k-1)}$$
$$= \frac{1}{F_{3^k}}\left(F_{3^k-1}+\frac{c_k-2}{L_{3^k}(c_k-1)}\right) \quad \text{by Lemmas 9 and 10}$$
$$= \frac{F_{3^k-1}L_{3^k}(c_k-1)+c_k-2}{F_{3^k}L_{3^k}(c_k-1)}$$
$$= \frac{F_{3^k-1}(c_k-1)+(c_k-2)/L_{3^k}}{F_{3^{k+1}}} \quad \text{by Lemma 9}$$

$$= \frac{F_{3^k-1}(L_{3^k}^2+1)+L_{3^k}}{F_{3^{k+1}}} \quad \text{by Lemma 12}$$

$$= \frac{F_{3^{k+1}-1}}{F_{3^{k+1}}} \quad \text{by Lemma 13.}$$

We note that the Pierce expansion considered by Shallit [13] is similar to but not the same as that of Theorem 7 or Theorem 14 below.

Theorem 14: For $k \geq 1$,

$$\frac{L_{3^k-1}}{L_{3^k}} = (1, c_0-1, c_0+1, c_1-1, c_1+1, \ldots, c_{k-2}-1, c_{k-2}+1, c_{k-1}+1).$$

Proof: By Theorem 7,

$$\begin{aligned}
&(1, c_0-1, c_0+1, c_1-1, c_1+1, \ldots, c_{k-2}-1, c_{k-2}+1, c_{k-1}+1)\\
&= \frac{F_{3^k-1}}{F_{3^k}} + \frac{1}{(c_0-1)(c_0+1)\cdots(c_{k-2}+1)}\left(\frac{1}{c_{k-1}-1} - \frac{1}{c_{k-1}+1}\right)\\
&= \frac{F_{3^k-1}}{F_{3^k}} + \frac{2}{F_{3^k}L_{3^k}} = \frac{F_{3^k-1}L_{3^k}+2}{L_{3^k}F_{3^k}} = \frac{L_{3^k-1}}{L_{3^k}}.
\end{aligned}$$

The last step follows because

$$\begin{aligned}
F_{3^k-1}L_{3^k}+2 &= (\phi^{3^k-1}-\hat{\phi}^{3^k-1})(\phi^{3^k}+\hat{\phi}^{3^k})/\sqrt{5}+2\\
&= (\phi^{2\cdot 3^k-1}-\hat{\phi}^{2\cdot 3^k-1}+\hat{\phi}^{-1}-\phi^{-1}+2\sqrt{5})/\sqrt{5}\\
&= (\phi^{2\cdot 3^k-1}-\hat{\phi}^{2\cdot 3^k-1}+\phi^{-1}-\hat{\phi}^{-1})/\sqrt{5}\\
&= (\phi^{3^k-1}+\hat{\phi}^{3^k-1})(\phi^{3^k}-\hat{\phi}^{3^k})/\sqrt{5} = L_{3^k-1}F_{3^k}.
\end{aligned}$$

There are many related identities that can be noted. We close with the omnibus theorem below, indicating several patterns that we have observed. The proofs are omitted, since the identities may be derived in the same way as the paradigms in Theorems 7 and 14.

Theorem 15: For $n = 2\cdot 3^k$, $k \geq 1$,

$$\frac{F_{n-1}}{F_n} = (1, c_0-1, c_0+1, \ldots, c_{k-1}-1, c_{k-1}+1).$$

For $n = 4\cdot 3^k$, $k \geq 1$,

$$\frac{F_{n-1}}{F_n} = (1, c_0-1, c_0+1, \ldots, c_{k-1}-1, c_{k-1}+1, c_k).$$

For $n = 8\cdot 3^k$, $k \geq 1$,

$$\frac{F_{n-1}}{F_n} = (1, c_0-1, c_0+1, \ldots, c_k-1, c_k+1, c_k+c_{k+1}).$$

For $n=5\cdot 3^k$, $k\geq 1$,

$$\frac{F_{n-1}}{F_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k+1, (c_k-1)c_k-1).$$

For $n=7\cdot 3^k$, $k\geq 1$,

$$\frac{F_{n-1}}{F_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k+1, ((c_k-1)c_k-1)c_k-(c_k-1)).$$

For $n=2\cdot 3^k$, $k\geq 1$,

$$\frac{L_{n-1}}{L_n}=(1, c_0-1, c_0+1, \ldots, c_{k-1}-1, c_{k-1}+1, c_k/2).$$

For $n=4\cdot 3^k$, $k\geq 1$,

$$\frac{L_{n-1}}{L_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k+2, c_k^2/2-1).$$

For $n=8\cdot 3^k$, $k\geq 1$,

$$\frac{L_{n-1}}{L_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k+1, c_{k+1}-c_k, c_kc_{k+1}/2-(c_k^2/2-1)).$$

For $n=5\cdot 3^k$, $k\geq 1$,

$$\frac{L_{n-1}}{L_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k, c_k+2, c_k^2+c_k-1).$$

For $n=7\cdot 3^k$, $k\geq 1$,

$$\frac{L_{n-1}}{L_n}=(1, c_0-1, c_0+1, \ldots, c_k-1, c_k+1, (c_k^2+c_k-2)c_k-1).$$

We note finally that nonlinear recurrence relations also arise in the expansions of certain rational numbers by means of other related algorithms (see [5], [15]).

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NEW EDITORIAL POLICIES

The Board of Directors of The Fibonacci Association during their last business meeting voted to incorporate the following two editorial policies effective January 1, 1995

1. All articles submitted for publication in *The Fibonacci Quarterly* will be blind refereed.
2. In place of Assistant Editors, *The Fibonacci Quarterly* will change to utilization of an Editorial Board.

EXPONENTIAL GROWTH OF RANDOM FIBONACCI SEQUENCES

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1. INTRODUCTION

There are various ways in which the standard Fibonacci sequence can be generalized. Examples are:

1. Choose arbitrary starting values.
2. Introduce extra terms, for example, the "Tribonacci" sequence, $T_n = T_{n-1} + T_{n-2} + T_{n-3}$.
3. Introduce multipliers, for example, $x_n = ax_{n-1} + bx_{n-2}$, where a and b are positive integers or, more generally, positive (real) numbers.

A natural question to ask is: What is the rate of growth of the sequence? This could be tackled by investigating whether $x_n \sim K\phi^n$ for some constants K and ϕ, or the weaker condition, the convergence of $\frac{1}{n}\ln(x_n)$ as $n \to \infty$. If $\frac{1}{n}\ln(x_n)$ converges to ψ, then ψ is the rate of exponential growth in the sense that, for every $\delta > 0$,

$$\frac{x_n}{e^{(\psi+\delta)n}} \to 0 \quad \text{and} \quad \frac{x_n}{e^{(\psi-\delta)n}} \to \infty.$$

In this paper a further generalization of the Fibonacci sequence is considered. Instead of using fixed multipliers, choose pairs (a_n, b_n) at random, according to some specified probability distribution, and let

$$x_0 = 0, \; x_1 = 1, \; x_n = a_n x_{n-1} + b_n x_{n-2}, \quad n \geq 2.$$

$\{x_n\}$ is now a sequence of random variables.

A simple example is to choose a_n to be either 1 or 2 with probability $\frac{1}{2}$ (and independently of the previous a's) and to take all b_n's equal to 1.

We will show that, subject to certain conditions on the probability distribution of the multipliers, $\frac{1}{n}\ln(x_n)$ converges to a constant ψ for every sequence except for those in a set which together have zero probability of occurring.

2. MAIN RESULT

Let $\{a_n, b_n\}_{n\geq 1}$ be a sequence of pairs of random variables that satisfy the following conditions:

1. a_n and b_n are strictly positive.
2. (a_n, b_n) are independent pairs, that is, for every n and $k \geq 1$, and for all $0 \leq c_{n+j} < d_{n+j} < \infty$ and $0 \leq e_{n+j} < f_{n+j} < \infty$,

$$\begin{aligned} &P(c_n < a_n \leq d_n,\, e_n < b_n \leq f_n, \ldots, c_{n+k} < a_{n+k} \leq d_{n+k},\, e_{n+k} < b_{n+k} \leq f_{n+k}) \\ &= P(c_n < a_n \leq d_n,\, e_n < b_n \leq f_n) \cdots P(c_{n+k} < a_{n+k} \leq d_{n+k},\, e_{n+k} < b_{n+k} \leq f_{n+k}). \end{aligned}$$

This means that the probability distribution of (a_n, b_n) is not affected by knowing the values of the previous a's and b's.

3. $P(c < a_n \le d, e < b_n \le f) = P(c < a_1 \le d, e < b_1 \le f)$ for all n and for all $0 \le c < d < \infty$ and $0 \le e < f < \infty$.
4. $-\infty < E(\ln(a_1)) < \infty$ and $-\infty < E(\ln(b_1)) < \infty$.

$$\left[E(\ln(a_1)) = \int_0^\infty \ln(x) F(dx), \text{ where } F(x) = P(a_1 \le x). \text{ Similarly for } E(\ln(b_1)).\right]$$

Let

$$w_n = a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots\, \cfrac{}{a_{n-1} + \cfrac{b_{n-1}}{a_n}}}}}.$$

w_n is a finite continued fraction (see Hardy and Wright [4] for basic properties).

Definition 1: To say that a condition holds on a sequence of random variables $\{z_n\}$ *almost surely* (a.s.) means that the sequences for which it does not hold form a set which has probability (measure) 0.

We will show that the sequence $\{w_n\}$ converges almost surely. Let w denote the limiting random variable.

Theorem 1: $\frac{1}{n}\ln(x_n) \xrightarrow{\text{a.s.}} \psi$, where $\psi = E(\ln(w))$.

Note: Since $a_1 < w < a_1 + \frac{b_1}{a_2}$, condition 4 implies that $E(\ln(w))$ is finite.

We note that the same method is used by Billingsley ([1], Ch. 1, §4) to prove a result of a similar nature involving the rate of growth of the "convergents" to a number by Diophantine approximation.

For $n \ge 2$,

$$x_n = a_n x_{n-1} + b_n x_{n-2} \quad \text{or} \quad \frac{x_n}{x_{n-1}} = a_n + b_n \frac{x_{n-2}}{x_{n-1}}.$$

Let

$$\begin{aligned} y_n &= \frac{x_n}{x_{n-1}}, \qquad n \ge 2, \\ &= a_n + \cfrac{b_n}{a_{n-1} + \cfrac{b_{n-1}}{\ddots\, \cfrac{b_3}{a_2 + \cfrac{b_2}{a_1}}}} \end{aligned}$$

Let $y_1 = x_1$, then

$$\frac{1}{n}\ln(x_n) = \frac{1}{n}\sum_{k=1}^{n}\ln(y_i).$$

Proposition 1: $\{w_n\}$ converges almost surely.

Proof:

$$w_2 - w_1 = \frac{b_1}{a_2}$$

$$w_3 - w_2 = \frac{-b_1 b_2}{a_2(a_2 a_3 + b_2)}$$

$$w_4 - w_3 = \frac{b_1 b_2 b_3}{(a_2 a_3 + b_2)(a_2 a_3 a_4 + a_4 b_2 + a_2 b_3)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Let

$$c_2 = 1,\ d_2 = a_2,$$
$$c_{n+1} = d_n,$$
$$d_{n+1} = a_{n+1}d_n + b_n c_n.$$

Then

$$w_n - w_{n-1} = \frac{(-1)^n b_1 b_2 \cdots b_{n-1}}{c_n d_n}.$$

A well-known property of continued fractions is that $\{w_{2n}\}$ is monotone decreasing and $\{w_{2n+1}\}$ is monotone increasing.

Ignoring all terms with two or more a's,

$$d_n \geq b_2 b_4 b_6 \cdots b_{n-1} \qquad \text{if } n \text{ is odd, } n \geq 3, \text{ while}$$

$$\begin{aligned} d_n \geq\ & a_n b_{n-2} b_{n-4} \cdots b_2 + a_{n-2} b_{n-1} b_{n-4} \cdots b_2 \\ & + a_{n-4} b_{n-1} b_{n-3} b_{n-6} \cdots b_2 + \cdots \\ & + a_2 b_{n-1} b_{n-3} \cdots b_3 \qquad \text{if } n \text{ is even, } n \geq 4. \end{aligned}$$

Hence, for n even, $|w_n - w_{n-1}|$ and $|w_{n+1} - w_n|$ are bounded by

$$\frac{1}{\frac{a_2}{b_1} + \frac{a_4}{b_3}\frac{b_2}{b_1} + \frac{a_6}{b_5}\frac{b_2 b_4}{b_1 b_3} + \cdots + \frac{a_n}{b_{n-1}}\frac{b_2 b_4 \cdots b_{n-2}}{b_1 b_3 \cdots b_{n-3}}}.$$

If every $b_n = 1$, this becomes $\frac{1}{a_2 + a_4 + a_6 + \cdots + a_n}$, which tends to 0 almost surely.

Otherwise, $\ln\left(\frac{b_2 b_4 \cdots b_{n-2}}{b_1 b_3 \cdots b_{n-3}}\right)$ is a symmetric random walk and, with probability one, will take values $\geq k$, for every k, for some value n. Thus, since the sequence $\left\{\frac{a_n}{b_{n-1}}\frac{b_2 b_4 \cdots b_{n-2}}{b_1 b_3 \cdots b_{n-3}}\right\}$ is unbounded almost surely, the denominator diverges almost surely

$$\text{or,} \quad |w_n - w_{n-1}| \xrightarrow{\text{a.s.}} 0.$$

Together with the fact that $\{w_{2n}\}$ and $\{w_{2n+1}\}$ are monotone, this implies that $\{w_n\}$ converges almost surely.

The ergodic theorem appears in many forms. In a probabilistic context it usually involves "stationary" sequences of random variables (see Billingsley [1], or Breiman [2], Ch. 6).

Definition 2: A sequence $\{z_n\}_{n\geq 1}$ of random variables is called *stationary* if $(z_1, z_2, \ldots, z_k)$ and $(z_{n+1}, z_{n+2}, \ldots, z_{n+k})$ have the same probability distribution for every $k \geq 1$ and $n \geq 1$.

The sequence $\{z_n\}$ determines a probability measure P on $(R^{\mathbb{Z}^+}, \mathscr{F})$, where $\mathscr{F}$ is the σ-field of events generated by $\{z_n\}$ (Breiman [2], Ch. 2).

Definition 3: A *tail event* A is one that does not depend on the values of $z_1, z_2, \ldots, z_n$ for any n. [For example, $A = (\{z_n\} : z_n$ converges) is a tail event.]

If every tail event has probability 0 or 1, then $\{z_n\}$ is *ergodic* (Breiman [2], Prop. 6.3.2 and Def. 6.30).

Consider now the doubly infinite sequence $\{a_n, b_n\}_{n\in\mathbb{Z}}$. For $n \geq 1$, let

$$z_n = \ln\left(a_n + \cfrac{b_n}{a_{n-1} + \cfrac{b_{n-1}}{a_{n-2} \cdots}} \right).$$

Proposition 2: $\{z_n\}$ is ergodic.

Proof: Stationarity is an immediate consequence of conditions 2 and 3.

A tail event for $\{z_n\}$ corresponds to an event that does not involve $\ldots (a_0, b_0), \ldots, (a_n, b_n)$ for every $n \geq 1$.

Since (a_n, b_n) are independent pairs, it can be deduced from Kolmogorov's 0–1 law (Breiman [2], Th. 3.12) that all tail events for $\{z_n\}$ have probability 0 or 1.

Theorem 2 (Ergodic Theorem):

$$\frac{1}{n}\sum_{i=1}^{n} z_i \xrightarrow{\text{a.s.}} E(z_1).$$

Proof of Theorem 1:

$$\frac{1}{n}\ln(x_n) = \frac{1}{n}\sum_{i=1}^{n}\ln(y_i) \xrightarrow{\text{a.s.}} E(z_1)$$

$$\text{since } \frac{1}{n}\sum_{1}^{n} z_i \xrightarrow{\text{a.s.}} E(z_1)$$

$$\text{and } |\ln(y_n) - z_n| \xrightarrow{\text{a.s.}} 0.$$

3. EXAMPLE

Let $a_n = 1$, with probability 1/2 ; $a_n = 2$, with probability 1/2 (a_n's are chosen independently); $b_n \equiv 1$. Examples of possible sequences are:

(i) 0, 1, 1, 3, 4, 7, 18, 25, 68, ...,
(ii) 0, 1, 3, 7, 10, 17, 44, 105, 149, ...

w represents a "randomly" chosen number whose continued fraction expansion contains only 1's and 2's (every possible sequence in the first n places is equally likely, for every n).

$E(\ln(w))$ is easily approximated by

$$\frac{1}{2^n} \sum_{a=1 \text{ or } 2} \ln \left(a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \; a_{n-1} + \cfrac{1}{a_n}}}} \right)$$

and is, to three decimal places, .673.

Hence, almost surely, such sequences grow at the rate $e^{.673} = 1.960$.

This compares with a result of Davison [3] which was recently brought to the author's attention.

Let x be an irrational number in (0, 2).

Define $b_n = 1 + ([nx] \bmod 2)$ ($[x]$ = integer part of x)

($\{b_n\}$ is a sequence of 1's and 2's).

Let $x_n = b_n x_{n-1} + x_{n-2}$.

Then $\lim_{n\to\infty} x_n^{1/n}$ always lies between 1.93 and 1.976.

4. CONCLUDING REMARKS

The conditions on (a_n, b_n) are not meant to be optimal. Any improvement, however, would result in greater complexity both of the results and proofs.

An interesting feature of the above results is that while individual sequences grow at a rate e^{ψ}, the average value of x_n. [$E(x_n)$, the expectation value], in general, grows at a different rate, since the sequence $\{E(x_n)\}$ satisfies $E(x_n) = E(a_n)E(x_{n-1}) + E(b_n)E(x_{n-2})$; hence, $\frac{1}{n}\ln(E(x_n)) \to \ln\phi$, where ϕ is the positive root of $x^2 - E(a_1)x - E(b_1) = 0$ [assuming $E(a_1)$ and $E(b_1)$ are finite].

(For the example in Section 3, $\phi = 2$.)

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A DIFFERENCE-OPERATIONAL APPROACH TO THE MÖBIUS INVERSION FORMULAS

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1. INTRODUCTION

Worth noticing is that the well-known Möbius inversion formulas in the elementary theory of numbers (cf. e.g., [2] and [3]),

$$f(n)=\sum_{d|n} g(d) \tag{1}$$

and

$$g(n)=\sum_{d|n} f(d)\mu(n/d)=\sum_{d|n} f(n/d)\mu(d), \tag{2}$$

may be viewed precisely as a discrete analog of the following Newton-Leibniz fundamental formulas

$$F(x_1,\ldots,x_s)=\int_{c_1}^{x_1}\cdots\int_{c_s}^{x_s} G(t_1,\ldots,t_s)dt_s\ldots dt_1 \tag{3}$$

and

$$G(x_1,\ldots,x_s)=\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_s}F(x_1,\ldots,x_s), \tag{4}$$

wherein the summations of (1) and (2) are taken over all the divisors d of n, and $G(t_1,\ldots,t_s)$ is an integrable function so that $F(x_1,\ldots,x_s)=0$ when there is some $x_i=c_i$ $(1\le i\le s)$. This will be made clear in what follows.

Let us use the prime factorization forms for n and d, say $n=p_1^{x_1}\cdots p_s^{x_s}$ and $d=p_1^{t_1}\cdots p_s^{t_s}$, p_i being distinct primes, x_i and t_i being nonnegative integers with $0\le t_i\le x_i$ $(i=1,\ldots,s)$, and replace $f(n)$ and $g(d)$ of (1) by $f((x))\equiv f(x_1,\ldots,x_s)$ and $g((t))\equiv g(t_1,\ldots,t_s)$, respectively. Then one may rewrite (1) and (2) as multiple sums of the following:

$$f(x_1,\ldots,x_s)=\sum_{0\le t_i\le x_i} g(t_1,\ldots,t_s) \tag{5}$$

and

$$g(x_1,\ldots,x_s)=\sum_{0\le t_i\le x_i} f(x_i-t_1,\ldots,x_s-t_s)\mu_1(t_1,\ldots,t_s), \tag{6}$$

where each summation is taken over all the integers t_i $(i=1,\ldots,s)$ such that $0\le t_i\le x_i$, and $\mu_1((t))\equiv\mu_1(t_1,\ldots,t_s)$ is defined by

$$\mu_1((t))=\begin{cases}(-1)^{t_1+\cdots+t_s}, & \text{if all } t_i\le 1,\\ 0, & \text{if there is a } t_i\ge 2.\end{cases} \tag{7}$$

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Evidently $\mu_1((t)) = \mu(d)$ is just the classical Möbius function defined for positive integers d with $\mu(1) = 1$ (cf. [4]).

Now we introduce the backward difference operator $\underset{x}{\Delta}$ and its inverse $\underset{x}{\Delta}^{-1}$ by the following:

$$\underset{x}{\Delta} f(x) = f(x) - f(x-1), \quad \underset{x}{\Delta}^{-1} g(x) = \sum_{0 \le t \le x} g(t) \tag{8}$$

so that $\underset{x}{\Delta}\underset{x}{\Delta}^{-1} g(x) = g(x)$, $\underset{x}{\Delta}^{-1}\underset{x}{\Delta} f(x) = f(x)$, and we may denote $\underset{x}{\Delta}\underset{x}{\Delta}^{-1} = \underset{x}{\Delta}^{-1}\underset{x}{\Delta} = I$ with $If(x) \equiv f(x)$, where we assume that $f(x) = g(x) = 0$ for $x < 0$. Thus, (5) and (6) can be expressed as

$$f((x)) = \underset{x_1}{\Delta}^{-1} \cdots \underset{x_s}{\Delta}^{-1} g((x)) \tag{9}$$

and

$$g((x)) = \underset{x_1}{\Delta} \cdots \underset{x_s}{\Delta} f((x)), \tag{10}$$

where it is always assumed that $f((x)) = g((x)) = 0$ whenever there is some $x_i < 0$ $(1 \le i \le s)$, s being any positive integer.

Apparently, the reciprocal pair (9) $\Leftrightarrow$ (10) is just a discrete analog of the inverse relations (3) $\Leftrightarrow$ (4). This is what we claimed in the beginning of this section.

2. A GENERALIZATION OF (9) $\Leftrightarrow$ (10)

Difference operators of higher orders may be defined inductively as follows:

$$\underset{x}{\Delta}^r = \underset{x}{\Delta}\underset{x}{\Delta}^{r-1}, \; \underset{x}{\Delta}^{-r} = \underset{x}{\Delta}^{-1}\underset{x}{\Delta}^{-(r-1)}, \; (r \ge 2), \; \Delta^0 = I.$$

Lemma 1: For any positive integer r, we have $\underset{x}{\Delta}^r \underset{x}{\Delta}^{-r} = \underset{x}{\Delta}^{-r}\underset{x}{\Delta} = I$.

Proof: (By induction.) The case $r = 1$ has been noted previously. If it holds for the case $r = k \ge 1$, then, for any given $f(x)$,

$$\underset{x}{\Delta}^{k+1}\underset{x}{\Delta}^{-k-1} f(x) = \underset{x}{\Delta}^{k}\underset{x}{\Delta}\underset{x}{\Delta}^{-1}\underset{x}{\Delta}^{-k} f(x) = \underset{x}{\Delta}^{k} I \underset{x}{\Delta}^{-k} f(x) = \underset{x}{\Delta}^{k}\underset{x}{\Delta}^{-k} f(x) = f(x),$$

and, consequently, $\underset{x}{\Delta}^{k+1}\underset{x}{\Delta}^{-(k+1)} = I$. Hence, $\underset{x}{\Delta}^r\underset{x}{\Delta}^{-r} = I$ holds for all $r \ge 1$. Similarly, $\underset{x}{\Delta}^{-r}\underset{x}{\Delta}^r = I$ may also be verified by induction. □

In what follows, we always assume that every function $f((x))$ or $g((x))$ will vanish whenever there is some $x_i < 0$ $(1 \le i \le s)$.

Lemma 2: For every given $(r) \equiv (r_1, \ldots, r_2)$ with $r_i \ge 1$, we have the following pair of reciprocal relations:

$$f((x)) = \left(\prod_{i=1}^{s} \underset{x_i}{\Delta}^{-r_i} \right) g((x)) \tag{11}$$

and

$$g((x)) = \left(\prod_{i=1}^{s} \underset{x_i}{\Delta}^{r_i} \right) f((x)). \tag{12}$$

Proof: This is easily verified by repeated application of Lemma 1. In fact, the implication $(11) \Rightarrow (12)$ follows from the identity

$$\left(\prod_{i=1}^{s} \Delta_{x_i}^{r_i}\right)\left(\prod_{i=1}^{s} \Delta_{x_i}^{-r_i}\right) = I. \tag{13}$$

Similarly, we have $(12) \Rightarrow (11)$. □

Evidently, the reciprocal pair $(11) \Leftrightarrow (12)$ implies $(1) \Leftrightarrow (2)$ with $r_i = 1$ $(i = 1, \ldots, s)$, since (1) and (2) are equivalent to (9) and (10), respectively.

3. AN EXPLICIT FORM

It is not difficult to find some explicit expressions for the right-hand sides of (11) and (12). For the case $s = 1$, write $f((x)) = f(x)$. By mathematical induction, we easily obtain, for $r \geq 2$,

$$\Delta_x^r f(x) = \sum_{0 \leq t \leq r} (-1)^t \binom{r}{t} f(x-t), \tag{14}$$

$$\Delta_x^{-r} g(x) = \sum_{0 \leq t \leq t_1 \leq \cdots \leq t_{r-1} \leq x} g(t) \tag{15}$$

$$\Delta_x^{-r} g(x) = \sum_{0 \leq t \leq x} \binom{x-t+r-1}{r-1} g(t) = \sum_{0 \leq t \leq x} \binom{t+r-1}{r-1} g(x-t), \tag{16}$$

where the summation contained in (15) is taken over all the r-tuples of integers $(t, t_1, \ldots, t_{r-1})$ such that $0 \leq t \leq t_1 \leq \cdots \leq t_{r-1} \leq x$. It is readily seen that, for each fixed $t \geq 0$, the number of all such r-tuples is given by $\binom{x-t+r-1}{r-1}$, so that (16) follows from (15).

As may be verified, the explicit forms given by (14) and (16) can be used to produce another proof of Lemma 1 and of Lemma 2, with the aid of the combinatorial identity

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} \binom{n-j+r-1}{r-1} = \begin{cases} 1 & \text{when } n = 0, \\ 0 & \text{when } n \geq 1. \end{cases}$$

Actually, this identity follows at once from comparing the coefficients of z^n on both sides of the product of the following expansions:

$$(1-z)^r = \sum_{j \geq 0} (-1)^j \binom{r}{j} z^j, \quad (1-z)^{-r} = \sum_{j \geq 0} \binom{j+r-1}{r-1} z^j.$$

In what follows, we denote $(x)-(t) \equiv (x_1 - t_1, \ldots, x_s - t_s)$ with $(x) \equiv (x_1, \ldots, x_s)$ and $(t) \equiv (t_1, \ldots, t_s)$ as before. Also, we use $(0) \leq (t) \leq (x)$ to denote the conditions $0 \leq t_i \leq x_i$ $(i = 1, \ldots, s)$, etc. As the right-hand sides of (11) and (12) consist of only repeated sums, we see that Lemma 2 together with (14) and (16) for $r = r_i$, $x = x_i$ $(i = 1, \ldots, s)$ imply the following

Theorem: For any given $(r) \equiv (r_1, \ldots, r_s)$ with all $r_i \geq 1$, there hold the reciprocal relations

$$f((x)) = \sum_{(0) \leq (t) \leq (x)} \mu_{(r)}^{-1}((t)) g((x)-(t)) \tag{17}$$

and

$$g((x)) = \sum_{(0)\le(t)\le(r)} \mu_{(r)}((t)) f((x)-(t)), \tag{18}$$

where $\mu_{(r)}((t))$ and $u_{(r)}^{-1}((t))$ are defined by the following:

$$\mu_{(r)}((t)) = \prod_{i=1}^{s} \binom{r_i}{t_i}(-1)^{t_i}, \quad \mu_{(r)}^{-1}((t)) = \prod_{i=1}^{s} \binom{t_i + r_i - 1}{r_i - 1}. \tag{19}$$

Note that for the case $(r) \equiv (1, \ldots, 1)$ the function $\mu_{(r)}((t))$ becomes the ordinary Möbius function, so that (17) and (18) constitute a generalized pair of Möbius inversions. Accordingly, $\mu_{(r)}^{-1}((t))$ may be called the inverse Möbius function with given $(r) \equiv (r_1, \ldots, r_s)$ as a parametric vector. Moreover, it may be observed that the condition $(0) \le (t) \le (r)$ under the summation of (18) may also be replaced by $(0) \le (t) \le (x)$ inasmuch as $g((x)-(t)) = 0$ whenever there is some $x_i - t_i < 0$. Consequently, (17) and (18) may be expressed as "convolutions":

$$f((x)) = \mu_{(r)}^{-1} * g((x)), \quad g((x)) = \mu_{(r)} * f((x)). \tag{20}$$

Remark: Reversing the ordering relations in the summation process, one may find that there are dual forms corresponding to (17) and (18). Suppose that $(m) \equiv (m_1, \ldots, m_s)$ is a fixed s-tuple of positive integers and that we are considering such functions $f^*((x))$ and $g^*((x))$ with the property that $f^*((x)) = g^*((x)) = 0$ whenever there is some $x_i > m_i$ $(1 \le i \le s)$. Then the dual forms of (17)–(18) are given by

$$f^*((x)) = \sum_{(x)\le(t)\le(m)} \mu_{(r)}^{-1}((t)-(x)) g^*((t)) \tag{21}$$

and

$$g^*((x)) = \sum_{(x)\le(t)\le(m)} \mu_{(r)}((t)-(x)) f^*((t)), \tag{22}$$

where the summations are taken over all (t) such that $x_i \le t_i \le m_i$ $(i = 1, \ldots, s)$. This reciprocal pair $(21) \Leftrightarrow (22)$ has certain applications to the Probability Theory of Arbitrary Events. For instance, the case $(r) \equiv (1, \ldots, 1)$ may be used to yield a generalization of Poincaré's formula for the calculus of probabilities (cf. [1]).

4. A CONSEQUENCE OF THE THEOREM

Returning now to the theory of numbers, let us denote by $\partial(p|d)$ the highest power of the prime number p that divides d. Thus, for $d = p_1^{t_1} \cdots p_s^{t_s}$, we have $\partial(p_i|d) = t_i$. Also, we define $\partial(1|d) = 0$.

Notice that the functions $f(n) = f(p_1^{x_1} \cdots p_s^{x_s})$ and $g(d) = g(p_1^{t_1} \cdots p_s^{t_s})$ may be mapped to the corresponding functions $\tilde{f}((x))$ and $\tilde{g}((t))$, respectively. Thus, making use of the theorem with $r_i = r$ $(i = 1, \ldots, s)$, we easily get a pair of reciprocal relations, as follows,

$$f(n) = \sum_{d|n} g\left(\frac{n}{d}\right) v_r(d) \tag{23}$$

and

$$g(n) = \sum_{d|n} f\left(\frac{n}{d}\right)\mu_r(d), \tag{24}$$

where $\nu_r(d)$ and $\mu_r(d)$ are defined by the following:

$$\nu_r(d) = \prod_{p|d} \binom{\partial(p|d)+r-1}{r-1}, \quad \mu_r(d) = \prod_{p|d} \binom{r}{\partial(p|d)}(-1)^{\partial(p|d)}.$$

Obviously, the classical pair (1)–(2) is a particular case of (23)–(24) with $r = 1$. Moreover, for the case $r = 2$, we have

$$\nu_2(d) = \prod_{p|d} (\partial(p|d)+1) \stackrel{\text{def}}{=} \delta(d),$$

where $\delta(d)$ stands for the divisor function that represents the number of divisors of d. Consequently, (23)–(24) imply the following reciprocal pair as the second interesting case:

$$f(n) = \sum_{d|n} g\left(\frac{n}{d}\right)\delta(d); \tag{25}$$

$$g(n) = \sum_{d|n} f\left(\frac{n}{d}\right)\mu_2(d). \tag{26}$$

Surely (25)–(26) may be used to obtain various relations between special number sequences by taking $g(n)$ or $f(n)$ to be special number-theoretic functions.

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ON THE k^{th} DERIVATIVE SEQUENCES OF FIBONACCI AND LUCAS POLYNOMIALS*

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1. INTRODUCTION

As in [1] and [2], the Fibonacci polynomials $U_n(x)$ and the Lucas polynomials $V_n(x)$ (or simply U_n and V_n, when no misunderstanding can arise) are defined by the second-order linear recurrence relations

$$U_n = xU_{n-1} + U_{n-2} \quad (U_0 = 0,\ U_1 = 1), \tag{1}$$

and

$$V_n = xV_{n-1} + V_{n-2} \quad (V_0 = 2,\ V_1 = x), \tag{2}$$

where x is an indeterminate. Then the k^{th} derivatives of $U_n(x)$ and $V_n(x)$ are

$$U_n^{(k)} = \frac{d^k}{dx^k}U_n \quad \text{and} \quad V_n^{(k)} = \frac{d^k}{dx^k}V_n,$$

respectively. For convenience, we write $U_n^{(0)} = U_n$ and $V_n^{(0)} = V_n$.

Since $U_{-n} = (-1)^{n+1}U_n$ and $V_{-n} = (-1)^n V_n$, it can easily be deduced that the recurrence relations (1) and (2) hold for any integer n, and

$$U_{-n}^{(k)} = (-1)^{n+1}U_n^{(k)}, \tag{3}$$

$$V_{-n}^{(k)} = (-1)^n V_n^{(k)}. \tag{4}$$

The sequences $\{F_n^{(k)}\}$ and $\{L_n^{(k)}\}$ are defined as $F_n^{(k)} = [U_n^{(k)}(x)]_{x=1}$ and $L_n^{(k)} = [V_n^{(k)}(x)]_{x=1}$.

For $k = 1$ and 2, the sequences $\{U_n^{(k)}\}$, $\{V_n^{(k)}\}$, $\{F_n^{(k)}\}$, and $\{L_n^{(k)}\}$ were considered in [1] and [2], respectively. For any $k > 0$, the following conjectures were made in [2]:

Conjecture 1: $L_n^{(k)} = nF_n^{(k-1)}$.

Conjecture 2: $L_n^{(k)} = (n-k+1)L_n^{(k-1)} - 2(L_{n-1}^{(k)} + F_{n-1}^{(k-1)})$.

Conjecture 3: $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + kF_{n-1}^{(k-1)}$.

Conjecture 4: $L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + kL_{n-1}^{(k-1)}$.

Conjecture 5: $F_{n-1}^{(k)} + F_{n+1}^{(k)} = L_n^{(k)}$.

Conjecture 6: $F_n^{(k)} \equiv L_n^{(k)} \equiv 0 \pmod 2$ for $k \ge 2$.

Conjecture 7: $L_n^{(k)} \equiv 0 \pmod n$ for $k \ge 1$.

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The goal of this paper is to establish some identities and congruences involving the polynomials $U_n^{(k)}$ and $V_n^{(k)}$. For the sake of brevity, we shall not list the corresponding identities involving $F_n^{(k)}$ and $L_n^{(k)}$. The reader can easily obtain them by letting $x=1$ in the general identities. The validity of the above conjectures emerges from the results established in Section 2. Observe that all results have been obtained by making no use of the explicit expressions for $U_n(x)$ and $V_n(x)$ which one can get by taking the k^{th} derivatives with respect to x of the sums (1.6) and (1.7) of [1], respectively.

2. SOME IDENTITIES AND CONGRUENCES INVOLVING $U_n(x)$ AND $V_n(x)$

The following four identities are most basic.

Identity 1: $U_{n-1}^{(k)}+U_{n+1}^{(k)}=V_n^{(k)}$ for $k\geq 0$.

Identity 2: $U_n^{(k)}=xU_{n-1}^{(k)}+U_{n-2}^{(k)}+kU_{n-1}^{(k-1)}$ for $k\geq 0$.

Identity 3: $V_n^{(k)}=xV_{n-1}^{(k)}+V_{n-2}^{(k)}+kV_{n-1}^{(k-1)}$ for $k\geq 0$.

Identity 4: $V_n^{(k)}=nU_n^{(k-1)}$ for $k\geq 1$.

Proof of Identity 1: That $U_{n-1}+U_{n+1}=V_n$ is a well-known fact. Take the k^{th} derivative (with respect to x) of both sides of this identity. □

Proof of Identity 2 (by induction on **k*):*** The identity clearly holds for $k=0$. Suppose it holds for a certain $k-1\geq 1$, that is, suppose that $U_n^{(k-1)}=xU_{n-1}^{(k-1)}+U_{n-2}^{(k-1)}+(k-1)U_{n-1}^{(k-2)}$. Take the first derivative of both sides of this identity. □

Identity 3 can be proved in a similar way.

Proof of Identity 4: Clearly, it suffices to prove that it holds for $k=1$. This has been done in [1, formula (2.4)]. □

The following variety of identities can be regarded as generalizations of Identity 1.

Identity 5: $U_{n+m}^{(k)}+(-1)^m U_{n-m}^{(k)}=\dfrac{d^k}{dx^k}(U_nV_m)=\sum_{i=0}^{k}\binom{k}{i}U_n^{(i)}V_m^{(k-i)}$ for $k\geq 0$.

Identity 6: $U_{n+m}^{(k)}-(-1)^m U_{n-m}^{(k)}=\dfrac{d^k}{dx^k}(V_nU_m)=\sum_{i=0}^{k}\binom{k}{i}V_n^{(i)}U_m^{(k-i)}$ for $k\geq 0$.

Identity 7: $V_{n+m}^{(k)}+(-1)^m V_{n-m}^{(k)}=\dfrac{d^k}{dx^k}(V_nV_m)=\sum_{i=0}^{k}\binom{k}{i}V_n^{(i)}V_m^{(k-i)}$ for $k\geq 0$.

Identity 8: $V_{n+m}^{(k)}-(-1)^m V_{n-m}^{(k)}=\dfrac{d^k}{dx^k}(U_nW_m)=\dfrac{d^k}{dx^k}(W_nU_m)$ for $k\geq 0$.

Here and in the sequel to this paper, we let $W_n=V_{n-1}+V_{n+1}=(x^2+4)U_n$.

Evidently, the above four identities follow immediately from the case $k=0$ for which we have the following well-known results.

Identity 5': $U_{n+m}+(-1)^m U_{n-m}=U_n V_m$.

Identity 6': $U_{n+m}-(-1)^m U_{n-m}=V_n U_m$.

Identity 7': $V_{n+m}+(-1)^m V_{n-m}=V_n V_m$.

Identity 8': $V_{n+m}-(-1)^m V_{n-m}=U_n W_m=W_n U_m$.

To prove Conjectures 1-7, we shall establish another identity and two congruences.

Identity 9: $xV_n^{(k)}=(n-k+1)V_n^{(k-1)}-2(V_{n-1}^{(k)}+U_{n-1}^{(k-1)})$.

Proof: Using Identities 4, 1, and 3, we have

$$\begin{aligned}
&(n-k+1)V_n^{(k-1)}-2(V_{n-1}^{(k)}+U_{n-1}^{(k-1)})\\
&=(n+1)V_n^{(k-1)}-kV_n^{(k-1)}-2nU_{n-1}^{(k-1)}\\
&=(n+1)V_n^{(k-1)}+xV_n^{(k)}+V_{n-1}^{(k)}-V_{n+1}^{(k)}-2nU_{n-1}^{(k-1)}\\
&=xV_n^{(k)}+(n+1)V_n^{(k-1)}+(n-1)U_{n-1}^{(k-1)}+(n+1)U_{n+1}^{(k-1)}-2nU_{n-1}^{(k-1)}\\
&=xV_n^{(k)}+(n+1)(V_n^{(k-1)}-U_{n-1}^{(k-1)}-U_{n+1}^{(k-1)})\\
&=xV_n^{(k)}.
\end{aligned}$$

Congruence 1: $U_n^{(k)}\equiv V_n^{(k)}\equiv 0 \pmod{k!}$.

Proof: If we take the k^{th} derivative with respect to x of the combinatorial sums which give U_n and V_n (e.g., see [1, (1.6) and (1.7)]), we see that each of their summands contains the product of k consecutive integers. It follows that all of them are divisible by $k!$. □

Congruence 2: $V_n^{(k)}\equiv 0 \pmod{n}$ for $k\geq 1$.

Proof: It is an immediate consequence of Identity 4.

Letting $x=1$ in the above stated identities and congruences yields the following corollary.

Corollary: Conjectures 1-7 are all true.

3. SOME CONVOLUTION IDENTITIES INVOLVING $U_n(x)$ AND $V_n(x)$

In this section we discuss some finite series involving $U_n^{(k)}$ and $V_n^{(k)}$ that have simple closed-form expressions for their sums.

Proposition 1: $\displaystyle\sum_{i=0}^{n} U_i^{(k)}U_{n-i}=\frac{1}{k+1}U_n^{(k+1)}$.

Proposition 2: $\displaystyle\sum_{i=0}^{n} U_i^{(k)}V_{n-i}=\frac{1}{k+1}V_n^{(k+1)}+U_n^{(k)}$.

Proposition 3: $\displaystyle\sum_{i=0}^{n} V_i^{(k)}U_{n-i}=\frac{1}{k+1}V_n^{(k+1)}+\delta(0,k)U_n$.

Proposition 4: $\sum_{i=0}^{n} V_i^{(k)} V_{n-i} = \frac{1}{k+1} W_n^{(k+1)} + (1+\delta(0,k)) V_n^{(k)}$.

Here, $\delta(0, k)$ is Kronecker's symbol which equals 1 if $k = 0$, and equals 0 otherwise.

Proofs of Propositions 1-4: Let $A_n^{(k)} = \sum_{i=0}^{n} U_i^{(k)} U_{n-i}$. Since U_j $(j \geq 1)$ is a monic polynomial of degree $j-1$ (cf. [2, (1.6)]), we have that $U_0^{(k)} = U_1^{(k)} = \cdots = U_k^{(k)} = 0$ and $U_{k+1}^{(k)} = k!$, so that $A_k^{(k)} = A_{k+1}^{(k)} = 0$ and $A_{k+2}^{(k)} = U_{k+1}^{(k)} U_1 = k! = \frac{1}{k+1} U_{k+2}^{(k+1)}$. Suppose that $A_{n-1}^{(k)} = \frac{1}{k+1} U_{n-1}^{(k+1)}$ and $A_{n-2}^{(k)} = \frac{1}{k+1} U_{n-2}^{(k+1)}$ for $n \geq 2$. Then

$$\begin{aligned} A_n^{(k)} &= \sum_{i=0}^{n} U_i^{(k)} U_{n-i} = \sum_{i=0}^{n-1} U_i^{(k)} (x U_{n-1-i} + U_{n-2-i}) = x A_{n-1}^{(k)} + A_{n-2}^{(k)} + U_{n-1}^{(k)} U_{-1} \\ &= \frac{1}{k+1}\left(x U_{n-1}^{(k+1)} + U_{n-2}^{(k+1)} + (k+1) U_{n-1}^{(k)}\right) = \frac{1}{k+1} U_n^{(k+1)}. \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{n} U_i^{(k)} V_{n-i} &= \sum_{n=0}^{n} U_i^{(k)} (U_{n-1-i} + U_{n+1-i}) = A_{n-1}^{(k)} + A_{n+1}^{(k)} + U_n^{(k)} U_{-1} \\ &= \frac{1}{k+1}\left(U_{n-1}^{(k+1)} + U_{n+1}^{(k+1)}\right) + U_n^{(k)} = \frac{1}{k+1} V_n^{(k+1)} + U_n^{(k)}. \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{n} V_i^{(k)} U_{n-i} &= \sum_{i=0}^{n} (U_{i-1}^{(k)} + U_{i+1}^{(k)}) U_{n-i} = \sum_{i=1}^{n} U_{i-1}^{(k)} U_{n-i} + U_{-1}^{(k)} U_n + \sum_{i=0}^{n} U_{i+1}^{(k)} U_{n-i} \\ &= A_{n-1}^{(k)} + A_{n+1}^{(k)} + U_1^{(k)} U_n = \frac{1}{k+1}(U_{n-1}^{(k+1)} + U_{n+1}^{(k+1)}) + \delta(0,k) U_n \\ &= \frac{1}{k+1} V_n^{(k+1)} + \delta(0,k) U_n. \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{n} V_i^{(k)} V_{n-i} &= \sum_{n=0}^{n} V_i^{(k)} (U_{n-1-i} + U_{n+1-i}) = \sum_{i=0}^{n-1} V_i^{(k)} U_{n-1-i} + V_n^{(k)} U_{-1} + \sum_{i=0}^{n+1} V_i^{(k)} U_{n+1-i} \\ &= \frac{1}{k+1}(V_{n-1}^{(k+1)} + V_{n+1}^{(k+1)}) + \delta(0,k)(U_{n-1} + U_{n+1}) + V_n^{(k)} \\ &= \frac{1}{k+1} W_n^{(k+1)} + (1+\delta(0,k)) V_n^{(k)}. \end{aligned}$$

Furthermore, for any $k, j \geq 0$, we have

Proposition 5: $\sum_{i=0}^{n} U_i^{(k)} U_{n-i}^{(j)} = \left[(k+j+1)\binom{k+j}{j}\right]^{-1} U_n^{(k+j+1)}$.

Proposition 6: $\sum_{i=0}^{n} V_i^{(k)} U_{n-i}^{(j)} = \left[(k+j+1)\binom{k+j}{j}\right]^{-1} V_n^{(k+j+1)} + \delta(0,k) U_n^{(j)}$.

Proposition 7: $\sum_{i=0}^{n} V_i^{(k)} V_{n-i}^{(j)} = \left[(k+j+1)\binom{k+j}{j}\right]^{-1} W_n^{(k+j+1)} + (\delta(0,k) + \delta(0,j)) V_n^{(k+j)}$.

For the sake of brevity, we shall prove only Proposition 5.

Proof of Proposition 5 (by induction on j): By virtue of Proposition 1, the statement holds for $j=0$. Suppose it holds for some $j \geq 1$. Since

$$\frac{d}{dx}\left(\sum_{i=0}^{n} U_i^{(k)} U_{n-i}^{(j)}\right) = \sum_{i=0}^{n} U_i^{(k+1)} U_{n-i}^{(j)} + \sum_{i=0}^{n} U_i^{(k)} U_{n-i}^{(j+1)} = \left[(k+j+1)\binom{k+j}{j}\right]^{-1} U_n^{(k+j+2)},$$

we can write

$$\begin{aligned}\sum_{i=0}^{n} U_i^{(k)} U_{n-i}^{(j+1)} &= \left[(k+j+1)\binom{k+j}{j}\right]^{-1} U_n^{(k+j+2)} - \left[(k+1+j+1)\binom{k+1+j}{j}\right]^{-1} U_n^{(k+1+j+1)} \\ &= \left[(k+j+2)\binom{k+j+1}{j+1}\right]^{-1}\left[\frac{k+j+2}{j+1} - \frac{k+1}{j+1}\right] U_n^{(k+j+2)} \\ &= \left[(k+j+2)\binom{k+j+1}{j+1}\right]^{-1} U_n^{(k+j+2)}.\end{aligned}$$

ACKNOWLEDGMENT

The author is very grateful to the referee for his various suggestions that led to an improvement of this paper.

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AMS Classification Numbers: 11B83, 11B39, 26A24

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A NOTE ON CHOUDHRY'S RESULTS

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(*Submitted August 1993*)

1. INTRODUCTION

In [4] Fell, Graz, and Paasche proved that if the equation

$$x^n + y^n = z^n, \tag{1}$$

where $n \geq 2$ is an integer, has a solution in positive integers $x < y < z$, then

$$x^2 > 2y + 1. \tag{2}$$

In 1969 M. Perisastri (see [7], p. 226) proved that

$$x^2 > z. \tag{3}$$

In [1] it was proved that

$$x^2 > 2z + 1. \tag{4}$$

A. Choudhry (in [3]) improved the inequality (4) to the form

$$x^{n/(n-1)} > z. \tag{5}$$

In fact, from the proof given by Choudhry [3], it follows that

$$z < C(n) x^{n/(n-1)}, \tag{6}$$

where

$$C(n) = 2^{1/n} / n^{1/(n-1)}, \quad n > 1. \tag{7}$$

In [2] we improved the constant (7) to the form

$$C_1(n) = 2^{1/2n} / n^{1/(n-1)} < C(n). \tag{8}$$

In this note, we shall prove the following

Theorem: Let $C(j, k; n) = j^{1/n} / k^{1/(n-1)}$ and let equation (1) have a solution in positive integers $x < y < z$, then

$$z < \begin{cases} C(2, n; n) x^{n/(n-1)}, & \text{if } z - y = 1, \\ C(\sqrt{2}, 2n; n) x^{n/(n-1)}, & \text{if } z - y = 2, \\ C(\sqrt{2}, 2^n; n) x^{n/(n-1)}, & \text{if } z - y > 2. \end{cases}$$

Proof of the Theorem: Suppose equation (1) has a solution in positive integers $x < y < z$. Then we have

$$x^n = z^n - y^n = (z - y)(z^{n-1} + z^{n-2} y + \cdots + y^{n-1}). \tag{9}$$

We note that

$$z^{n-1} + z^{n-2} y + \cdots + y^{n-1} > n(zy)^{(n-1)/2}. \tag{10}$$

On the other hand, if $x<y<z$, we have, by (1),

$$y>(1/2)^{1/n}z. \tag{11}$$

From (10) and (11), we obtain

$$z^{n-1}+z^{n-2}y+\cdots+y^{n-1}>(n/2^{(n-1)/2n})z^{n-1}. \tag{12}$$

It is well-known (see [7], Ch. 11) that if n, x, y, z are positive integers with $x<y<z$ and $(x, y, z)=1$ such that (1) holds, then there exist $\delta\in\{0,1\}$ and positive integers a, d with $d|n$ such that

$$z-y=2^{\delta}d^{-1}a^{n}. \tag{13}$$

From (13), it follows that if $z-y>2$ then

$$z-y\geq 2^{n}/n \quad (cf.[5]). \tag{14}$$

From (12) and (9), we obtain

$$x^{n}>(z-y)(n/2^{(n-1)/2n})z^{n-1}. \tag{15}$$

From (15) and (14), we have

$$x^{n}>C_2(n)/2^{(n-1)/2n}z^{n-1}, \tag{16}$$

where

$$C_2(n)=\begin{cases} n, & \text{if } z-y=1,\\ 2n, & \text{if } z-y=2,\\ 2, & \text{if } z-y>2.\end{cases}$$

Now, by (16), the Theorem follows. □

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AMS Classification Number: 11D41

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Proposers should inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-784 ***Proposed by Herta Freitag, Roanoke, VA***

Show that for all n, $\alpha^{n-1}\sqrt{5} - L_{n-1}/\alpha$ is a Lucas number.

B-785 ***Proposed by Jane E. Friedman, University of San Diego, San Diego, CA***

Let $a_0 = a_1 = 1$ and let $a_n = 5a_{n-1} - a_{n-2}$ for $n \ge 2$. Prove that $a_{n+1}^2 + a_n^2 + 3$ is a multiple of $a_n a_{n+1}$ for all $n \ge 1$.

B-786 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. College of Science, Gujarat State, India***

If $F_{n+2k}^2 = aF_{n+2}^2 + bF_n^2 + c(-1)^n$, where a, b, and c depend only on k but not on n, find a, b, and c.

B-787 ***Proposed by H.-J. Seiffert, Berlin, Germany***

For $n \ge 0$ and $k > 0$, it is known that F_{kn}/F_k and P_{kn}/P_k are integers. Show that these two integers are congruent modulo $R_k - L_k$.

[**Note:** P_n and $R_n = 2Q_n$ are the Pell and Pell-Lucas numbers, respectively, defined by $P_{n+2} = 2P_{n+1} + P_n$, $P_0 = 0$, $P_1 = 1$ and $Q_{n+2} = 2Q_{n+1} + Q_n$, $Q_0 = 1$, $Q_1 = 1$.]

B-788 ***Proposed by Russell Jay Hendel, University of Louisville, Louisville, KY***

(a) Let $G_n = F_{n^2}$. Prove that $G_{n+1} \sim L_{2n+1}G_n$.

[**Note:** $f(n) \sim g(n)$ means that *f* is asymptotic to *g*, that is, $\lim_{n\to\infty} f(n)/g(n) = 1$.]

(b) Find the error term. More specifically, find a constant C such that $G_{n+1} \sim L_{2n+1}G_n + CG_{n-1}$.

B-789 ***Proposed by Richard André-Jeannin, Longwy, France***

The Lucas polynomials, $L_n(x)$, are defined by $L_0 = 2$, $L_1 = x$, and $L_n = xL_{n-1} + L_{n-2}$, for $n \ge 2$.

Find a differential equation satisfied by $L_n^{(k)}$, the k^{th} derivative of $L_n(x)$, where k is a nonnegative integer.

SOLUTIONS

Inequality for All

B-752 ***Proposed by Richard André-Jeannin, Longwy, France***
(Vol. 31, no. 4, November 1993)

Consider the sequences $\langle U_n\rangle$ and $\langle V_n\rangle$ defined by the recurrences $U_n = PU_{n-1} - QU_{n-2}$, $n \ge 2$, with $U_0 = 0, U_1 = 1$, and $V_n = PV_{n-1} - QV_{n-2}$, $n \ge 2$, with $V_0 = 2, V_1 = P$, where P and Q are real numbers with $P > 0$ and $\Delta = P^2 - 4Q > 0$. Show that, for $n \ge 0, U_{n+1} \ge (P/2)U_n$ and $V_{n+1} \ge (P/2)V_n$.

Solution by A. N. 't Woord, Eindhoven Univ. of Tech., Eindhoven, The Netherlands

Let $\langle W_n\rangle$ be any sequence that satisfies $W_n = PW_{n-1} - QW_{n-2}$ for $n \ge 2$ and $W_1 \ge (P/2)W_0 \ge 0$. Using induction on *n*, we shall show that $W_{n+1} \ge (P/2)W_n \ge 0$. We already know this for $n = 0$, so suppose the inequality holds for $n-1$. Then

$$\begin{aligned} W_{n+1} &= PW_n - QW_{n-1} \ge PW_n - (2Q/P)W_n \\ &= (P/2)W_n + (P/2 - 2Q/P)W_n \\ &= (P/2)W_n + (\Delta/2P)W_n \ge (P/2)W_n \ge 0. \end{aligned}$$

This gives the required result for both the sequences $\langle U_n\rangle$ and $\langle V_n\rangle$.

Note that the same style proof shows that strict inequality holds for $n > 0$.

Also solved by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, C. Georghiou, Norbert Jensen, Hans Kappus, H.-J. Seiffert, Lawrence Somer, J. Suck, and the proposer.

An Old Determinant

B-753 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. Col. of Sci., Gujarat State, India***
(Vol. 31, no 4, November 1993)

Prove that, for all positive integers *n*,

$$\begin{vmatrix} F_{n-1}^3 & F_n^3 & F_{n+1}^3 & F_{n+2}^3 \\ F_n^3 & F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 \\ F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 \\ F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 & F_{n+5}^3 \end{vmatrix} = 36.$$

Comment by J. Suck, Essen, Germany

"Surely you must be joking! . . . The world's leading previously published problems surveyor . . . was taken in?" This problem is the same as problem H-25 which was proposed by Erbacher and Fuchs in *The Fibonacci Quarterly* in 1964.

The editor apologizes for repeating this problem. Many readers pointed out this duplication. See the simple solution by C. R. Wall that was originally printed in this Quarterly **2.3** *(Oct. 1964):207.*

Generalization by H.-J. Seiffert, Berlin, Germany

For the positive integer p, let

$$A_n(p) = \begin{vmatrix} F_{n-1}^{p-1} & F_n^{p-1} & \cdots & F_{n+p-2}^{p-1} \\ F_n^{p-1} & F_{n+1}^{p-1} & \cdots & F_{n+p-1}^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+p-2}^{p-1} & F_{n+p-1}^{p-1} & \cdots & F_{n+2p-3}^{p-1} \end{vmatrix}_{p\times p},$$

where n is an arbitrary integer. According to a result of D. Jarden (see [1], p. 85, exercise 30), we have

$$\sum_{k=0}^{p} \begin{bmatrix} p \\ k \end{bmatrix} (-1)^{\lceil (p-k)/2 \rceil} F_{N+k}^{p-1} = 0 \tag{1}$$

for all integers N, where $\left[\begin{smallmatrix} p \\ k \end{smallmatrix}\right]$ is the Fibonomial coefficient defined by

$$\begin{bmatrix} p \\ k \end{bmatrix} = \frac{F_p F_{p-1} \cdots F_{p-k+1}}{F_k F_{k-1} \cdots F_1}$$

with $\left[\begin{smallmatrix} p \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} p \\ p \end{smallmatrix}\right] = 1$. Here $\lceil x \rceil$ denotes the least integer greater than or equal to x. Letting $N = n - 2 + j$ in equation (1) gives

$$F_{n+p-2+j}^{p-1} = -(-1)^{\lceil p/2 \rceil} F_{n-2+j}^{p-1} - \sum_{k=1}^{p-1} \begin{bmatrix} p \\ k \end{bmatrix} (-1)^{\lceil (p-k)/2 \rceil} F_{n+k-2+j}^{p-1}$$

for all $j = 0, 1, 2, \ldots, p-1$. Thus, we have $A_n(p) = (-1)^{p+\lceil p/2 \rceil} A_{n-1}(p)$ for all integers n, which implies that

$$A_n(p) = (-1)^{(n-m)(p+\lceil p/2 \rceil)} A_m(p)$$

for all integers m and n. Letting $m = 2 - p$ allows us to calculate $A_n(p)$. The results are given in the following table.

p	1	2	3	4	5	6
$A_n(p)$	1	$(-1)^n$	$2(-1)^n$	36	13824	$324000000(-1)^n$

Reference

1. Donald E. Knuth. *The Art of Computer Programming.* Vol. 1. Reading, Mass.: Addison-Wesley, 1973.

Dresel found that if the Fibonacci numbers are replaced by Lucas numbers in the original proposal, then the value of the determinant obtained is 562,500. He also showed that if the original determinant is enlarged to be $r \times r$ for $r > 4$, then the value of the determinant is 0.

This follows from the identity $F_{n+2}^3 - 3F_{n+1}^3 - 6F_n^3 + 3F_{n-1}^3 + F_{n-2}^3 = 0$, *which shows that the rows are linearly dependent.*

Also solved by Marjorie Bicknell-Johnson, Paul S. Bruckman, Leonard A. G. Dresel, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Samih A. Obaid, H.-J. Seiffert, J. Suck, A. N. 't Woord, David Zeitlin, and the proposer.

A Summing of Pells

B-754 ***Proposed by Joseph J. Kostal, University of Illinois at Chicago, IL (Vol. 32, no. 1, February 1994)***

Find closed form expressions for

$$\sum_{k=1}^{n} P_k \quad \text{and} \quad \sum_{k=1}^{n} Q_k.$$

The Pell numbers P_n and their associated numbers Q_n are defined by

$$P_{n+2} = 2P_{n+1} + P_n, \quad P_0 = 0,\ P_1 = 1;$$
$$Q_{n+2} = 2Q_{n+1} + Q_n,\ Q_0 = 1,\ Q_1 = 1.$$

Solution by Glenn A. Bookhout, Durham, NC and by H.-J. Seiffert, Berlin, Germany (independently)

Let $\langle G_n \rangle$ be any sequence that satisfies the recurrence $G_{n+2} = 2G_{n+1} + G_n$. Then

$$\sum_{k=1}^{n} G_{k+2} = 2\sum_{k=1}^{n} G_{k+1} + \sum_{k=1}^{n} G_k.$$

Thus,

$$\sum_{k=1}^{n} G_k + G_{n+1} + G_{n+2} - G_1 - G_2 = 2\sum_{k=1}^{n} G_k + 2G_{n+1} - 2G_1 + \sum_{k=1}^{n} G_k.$$

Hence,

$$\sum_{k=1}^{n} G_k = \frac{1}{2}(G_{n+2} - G_{n+1} - G_2 + G_1) = \frac{1}{2}(G_{n+1} + G_n - G_2 + G_1).$$

Several solvers pointed out that this and similar problems can be solved using the Binet forms and the formula for the sum of a finite geometric progression: $\sum_{k=1}^{n} x^k = (x - x^{n+1})/(1-x)$. *Some of the other equivalent answers obtained were:* $\sum P_k = (P_{n+1} + P_n - 1)/2 = (Q_{n+1} - 1)/2$ *and* $\sum Q_k = (Q_{n+1} + Q_n)/2 - 1 = P_{n+1} - 1$. *Haukkanen points out that Horadam showed in this Quarterly* **3.2** *(1965):161-77 that, if the sequence* w_n *is defined by* $w_{n+2} = cw_{n+1} - dw_n$ *for* $n \geq 0$ *with* $w_0 = a$ *and* $w_1 = b$ *and* $c \neq d + 1$, *then*

$$\sum_{k=0}^{n} w_k = \frac{w_{n+2} - b - (c-1)(w_{n+1} - a)}{c - d - 1}.$$

Gauthier found that for any integers s and t,

$$\sum_{k=1}^{n} x^n G_{sk+t} = \frac{(-1)^s x^{n+1} G_{sn+t} - x^n G_{s(n+1)+t} + G_{s+t} + (-1)^{s+1} x G_t}{1 - 2x(G_s + G_{s-1}) + (-1)^s x^2}.$$

Also solved by Seung-Jin Bang, Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Bill Correll, Jr., Steve Edwards, Russell Euler, Herta T. Freitag, N. Gauthier, C. Georghiou, Pentti Haukkanen, Russell Jay Hendel, Hans Kappus, H. K. Krishnapriyan, Carl Libis, Bob Prielipp, Sahib Singh, David C. Terr, and the proposer.

An Interleaving of Pells

B-755 *Proposed by Russell Jay Hendel, Morris College, Sumter, SC (Vol. 32, no. 1, February 1994)*

Find all nonnegative integers m and n such that $P_n = Q_m$.

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC

The only solutions are $(m, n) = (0, 1)$ or $(1, 1)$. First, n cannot be 0, since Q_m is never 0. Next, for $n = 1$, we obtain the two solutions listed above. Finally, for $n > 1$, it is straightforward to show that $Q_n = P_{n-1} + P_n = P_{n+1} - P_n$. Since $\langle P_n \rangle$ is a strictly increasing sequence of positive integers for $n > 1$, this yields $P_n < Q_n < P_{n+1}$, so Q_m cannot equal P_n for any $n > 1$.

Also solved by Paul S. Bruckman, Charles K. Cook, Bill Correll, Jr., Steve Edwards, C. Georghiou, Hans Kappus, Murray S. Klamkin, Wayne L. McDaniel, H.-J. Seiffert, Sahib Singh, David C. Terr, and the proposer.

A Fibonacci Formula for P_n

B-756 *Proposed by the editor (Vol. 32, no. 1, February 1994)*

Find a formula expressing the Pell number P_n in terms of Fibonacci and/or Lucas numbers.

Editorial Note: *Although some very ingenious solutions were submitted, none had the elegance that might be expected of our distinguished panel of solvers. This problem will thus be kept open for another six months.*

Fibonacci-Pell Congruences

B-757 *Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 32, no. 1, February 1994)*

Show that for $n > 0$,

(a) $$P_{3n-1} \equiv F_{n+2} \pmod{13},$$

(b) $$P_{3n+1} \equiv (-1)^{\lfloor (n+1)/2 \rfloor} F_{4n-1} \pmod{7}.$$

Solution by Bill Correll, Jr., Student, Denison University, Granville, OH

(b) Modulo 7, the Pell numbers P_n repeat in the sequence 0, 1, 2, 5, 5, 1, ... Thus, P_{3n+1} repeats in the sequence 1, 5, 1, 5, ... for $n = 0, 1, 2, \ldots$. Similarly, the Fibonacci numbers (mod 7) repeat every 16 terms and F_{4n-1} repeats in the sequence 1, 2, 6, 5, ... for $n = 0, 1, 2, \ldots$. Thus, we have the following table:

$n \pmod 4$	0	1	2	3
$P_{3n+1} \pmod 7$	1	5	1	5
$(-1)^{\lfloor (n+1)/2 \rfloor} F_{4n-1} \pmod 7$	1	−2	−6	5

Since the given congruence holds in each case, it is true in general.

(a) A similar analysis proves part (a). Considering the sequences P_n and F_n modulo 13 suggests considering n modulo 28. A table of values for $P_{3n-1} \pmod{13}$ and $F_{n+2} \pmod{13}$ show that they repeat every 28 terms and the corresponding values are congruent.

Seiffert also found that $P_{6n-4} \equiv (-1)^{\lfloor (n-1)/2 \rfloor} F_{5n+2} \pmod{11}$. *He notes that many such congruences seem to exist.*

Also solved by Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, David C. Terr, David Zeitlin, and the proposer.

Another Pell Sum

B-758 ***Proposed by Russell Euler, Northwest Missouri State University, Maryville, MO (Vol. 32, no. 1, February 1994)***

Evaluate $\sum_{k=0}^{\infty} \frac{k2^k Q_k}{5^k}$.

Solution by Hans Kappus, Rodersdorf, Switzerland

Consider more generally

$$f(x) = \sum_{k=0}^{\infty} kQ_k x^k, \quad \text{for } |x| < \sqrt{2} - 1.$$

We will use the formula ([1], p. 21, formula 1.113)

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}, \quad \text{for } |x| < 1.$$

The Binet form for Q_k is $Q_k = (p^k + q^k)/2$, where $p = 1 + \sqrt{2}$ and $q = 1 - \sqrt{2}$. Substituting in the Binet form gives

$$f(x) = \frac{1}{2}\left\{\frac{px}{(1-px)^2} + \frac{qx}{(1-qx)^2}\right\} = \frac{x(1+2x-x^2)}{(1-2x-x^2)^2}.$$

In particular, since $2/5 < \sqrt{2} - 1$, the sum in question equals $f(2/5) = 410$.

Reference

1. I. S. Gradshteyn & I. M. Ryzhik. *Table of Integrals, Series, and Products.* San Diego, Calif.: Academic Press, 1980.

Also solved by Seung-Jin Bang, Glenn Bookhout, Paul S. Bruckman, Charles K. Cook, Bill Correll, Jr., Steve Edwards, Piero Filipponi, N. Gauthier, C. Georghiou, Pentti Haukkanen, Russell Jay Hendel, Joseph J. Kostal, H. K. Krishnapriyan, Bob Prielipp, H.-J. Seiffert, Sahib Singh, David Zeitlin, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-490 ***Proposed by A. Stuparu, Valcea, Romania (corrected)***

Prove that the equation $S(x)=p$, where p is a given prime number, has just $D((p-1)!)$ solutions, all of them in between p and $p!$ [$S(n)$ is the Smarandache Function: the smallest integer such that $S(n)!$ is divisible by n, and $D(n)$ is the number of positive divisors of n.]

H-496 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let n be a positive integer >1 with $\gcd(n,10)=1$, and $\delta=(5/n)$, a Jacobi symbol. Consider the following congruences:

(1) $F_{n-\delta}\equiv 0 \pmod{n}$, $L_n\equiv 1 \pmod{n}$;

(2) $F_{\frac{1}{2}(n-\delta)}\equiv 0 \pmod{n}$ if $n\equiv 1 \pmod{4}$, $L_{\frac{1}{2}(n-\delta)}\equiv 0 \pmod{n}$ if $n\equiv 3 \pmod{4}$.

Composite n which satisfy (1) are called *Fibonacci-Lucas pseudoprimes*, which is abbreviated as "FLUPPS." Composite n which satisfy (2) are called *Euler-Lucas pseudoprimes with parameters (1, −1)*, abbreviated as "ELUPPS." Prove that FLUPPS and ELUPPS are equivalent.

H-497 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN***

Solve the recurrence relation

$$\sum_{i=0}^{k}\left(\prod_{j=0}^{k}\frac{x_{n-j}}{x_{n-i}}\right)^r+\left(\prod_{t=0}^{k}x_{n-t}\right)^r=0,$$

where r is any nonzero real number, $n>k\geq 1$, and $x_m\neq 0$ for all m.

H-498 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let $u=u_e=L_{2^e}$, $e=2,3,\ldots$. Show that if u is composite it is both a Fibonacci pseudoprime (or "FPP") and a Lucas pseudoprime (or "LPP"). Specifically, show that $u\equiv 7 \pmod{10}$, $F_{u+1}\equiv 0 \pmod{u}$, and $L_u\equiv 1 \pmod{u}$.

SOLUTIONS

Quite Prime

H-483 ***Proposed by James Nicholas Boots (deceased) & Lawrence Somer, The Catholic University of America, Washington, D.C.***
(Vol. 32, no. 1, February 1994)

Let $m \geq 2$ be an integer such that

$$L_m \equiv 1 \pmod{m}. \tag{1}$$

It is well known (see [1], p. 44) that if m is a prime, then (1) holds. It has been proved by H. J. A. Duparc [3] that there exist infinitely many composite integers, called Fibonacci pseudoprimes, such that (1) holds. It has also been proved in [2] and [4] that every Fibonacci pseudoprime is odd.

(i) Prove that $L_{m-1}^2 + L_{m-1} - 6 \equiv 0 \pmod{m}$.

In particular, conclude that if m is prime, then $L_{m-1} \equiv 2$ or $-3 \pmod{m}$.

(ii) Prove that $F_{m-2} - L_{m-1}F_{m-1} \equiv 1 \pmod{m}$.

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3. H. J. A. Duparc. *On Almost Primes of the Second Order*, pp. 1-13. Amsterdam: Rapport ZW, 1955-013, Math. Center, 1955.
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Solution by the Proposer

(i) If $m = 2$, then

$$L_{m-1}^2 + L_{m-1} - 6 = L_1^2 + L_1 - 6 = 1^2 + 1 - 6 = -4 \equiv 0 \pmod{2}$$

and

$$L_{m-1} = L_1 = 1 \equiv -3 \pmod{2}.$$

Now assume that $m > 2$. Then m is odd. It is well-known that

$$L_{2n} = L_n^2 - 2(-1)^n. \tag{2}$$

Thus,

$$L_{2m} = L_m^2 - 2(-1)^m \equiv 1^2 - 2(-1) \equiv 3 \pmod{m}. \tag{3}$$

Further, it follows by identity (I_{31}) on page 59 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr., that

$$L_{2m-1} = L_m L_{m-1} - (-1)^{m-1} \equiv (1)L_{m-1} - 1 \equiv L_{m-1} - 1. \tag{4}$$

By (2),

$$L_{2m-2} = L_{m-1}^2 - 2(-1)^{m-1} \equiv L_{m-1}^2 - 2(1) \equiv L_{m-1}^2 - 2 \pmod{m}. \tag{5}$$

Since $L_{2m} = L_{2m-1} + L_{2m-2}$, it follows by (3), (4), and (5) that

$$3 \equiv L_{m-1} - 1 + L_{m-1}^2 - 2 \pmod{m}, \tag{6}$$

which implies that

$$L_{m-1}^2 + L_{m-1} - 6 \equiv 0 \pmod{m}. \tag{7}$$

Since

$$L_{m-1}^2 + L_{m-1} - 6 = (L_{m-1} - 2)(L_{m-1} + 3), \tag{8}$$

it follows from (7) and (8) that $L_{m-1} \equiv 2$ or $-3 \pmod{m}$ if m is prime.

(ii) If $m = 2$, then

$$F_{m-2} - L_{m-1}F_{m-1} = F_0 - L_1F_1 = 0 - (1)(1) = -1 \equiv 1 \pmod{2}.$$

Now assume that $m > 2$. Then m is odd. We will first prove by induction that

$$L_{m-k} \equiv (-1)^k (F_{k-1} - L_{m-1}F_k) \pmod{m} \tag{9}$$

for $k \geq 0$. If $k = 0$, then

$$L_{m-k} = L_m \equiv 1 \equiv (-1)^0 (F_{-1} - L_{m-1}F_0) \equiv (1)(1 - L_m \cdot 0) \equiv 1 \pmod{m}.$$

If $k = 1$, then

$$L_{m-k} = L_{m-1} \equiv (-1)^1 (F_0 - L_{m-1}F_1) \equiv (-1)(0 - L_{m-1}(1)) \equiv L_{m-1} \pmod{m}.$$

Now assume that (9) holds up to $k = r$. Then

$$\begin{aligned} L_{m-(r+1)} &= L_{m-(r-1)} - L_{m-r} \\ &\equiv (-1)^{m-(r-1)} (F_{r-2} - L_{m-1}F_{r-1}) - (-1)^{m-r} (F_{r-1} - L_{m-1}F_r) \\ &\equiv (-1)^{m-(r+1)} ((F_{r-2} + F_{r-1}) - L_{m-1}(F_{r-1} + F_r)) \\ &\equiv (-1)^{m-(r+1)} (F_r - L_{m-1}F_{r+1}) \pmod{m}. \end{aligned}$$

Thus, (9) holds for $k \geq 0$. Now let $k = m-1$. Since m is odd, it follows by (9) that

$$L_{m-(m-1)} = L_1 = 1 \equiv (-1)^{m-1} (F_{m-2} - L_{m-1}F_{m-1}) \equiv F_{m-2} - L_{m-1}F_{m-1} \pmod{m}.$$

Also solved by P. Bruckman, L. Dresel, and H. Seiffert.

Strictly Monotone

H-484 ***Proposed by J. Rodriguez, Sonora, Mexico***
(Vol. 32, no. 1, February 1994)

Find a strictly increasing infinite series of integer numbers such that, for any consecutive three of them, the Smarandache Function is neither increasing nor decreasing.

*Find the largest strictly increasing series of integer numbers for which the Smarandache Function is strictly decreasing.

Solution by Paul S. Bruckman, Edmonds, WA

Solution to Part 1: For a given natural n, the *Smarandache Function* of n, denoted by $S(n)$, is defined to be the smallest natural m such that $n|m!$.

The following results ensue from the definition:

$$S(n) = \max_{p^e \| n}\{S(p^e)\}; \tag{1}$$

$$S(p^e) = ep, \quad \text{if } p \ge e; \tag{2}$$

$$S(n!) = n. \tag{3}$$

Given m natural, we define $U(m)$ to be the set of natural n such that $S(n) = m$ for all $n \in U(m)$. Then $n \in U(m)$ iff $n|m!$ and $n\nmid(m-1)!$ We may easily show from this that

$$U(m) = \bigcup_{p|m,\ p^e\|(m-1)!} \{p^{e+1}d : d \,|\, p^{-e-1}\cdot m!\} \tag{4}$$

In particular, if m is equal to p, a prime,

$$U(p) = \{pd : d\,|(p-1)!\}. \tag{5}$$

For example, $U(2) = \{2\}$, $U(3) = \{3, 6\}$, $U(5) = \{5, 10, 15, 20, 30, 40, 60, 120\}$, etc.

Thus, the smallest element of $U(p)$ is p, while the largest is $p!$. The number of elements of $U(p)$ is $\tau((p-1)!)$, which increases rapidly with increasing p.

Using these facts, we may construct an infinite sequence $X = \{x_n\}_{n\ge1}$ with the properties required in part 1 of the problem. Incidentally, the wording of the problem, in both parts, should be changed to substitute the word "sequence" for "series."

We let $\{p_n\}_{n\ge1} = \{2, 3, 5, 7, \dots\}$ denote the sequence of primes. Our first step is to define an infinite sequence $E = \{e_n\}_{n\ge1}$ of positive integers as follows:

$$e_{4u} = 2u+2,\ u = 1, 2, \dots;\ e_{4u+1} = 2u+1,\ e_{4u+2} = 2u+3,\ e_{4u+3} = 2u+2,\ u = 0, 1, \dots . \tag{6}$$

Thus, $E = \{1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6, 8, 7, 9, 8, 10, \dots\}$.

Next, we define the sequence of primes Q as follows:

$$Q = \{p_{e_n}\}_{n\ge1}. \tag{7}$$

Thus, $Q = \{2, 5, 3, 7, 5, 11, 7, 13, 11, 17, 13, 19, 17, 23, 19, 29, \dots\}$.

Each distinct value of terms in E and Q occurs exactly twice, except the first and third values, which occur only once. Observe that no three consecutive terms of E are increasing or decreasing, since the values alternate in magnitude; the same is true of Q, since the primes form an increasing sequence.

We now set each term p_{e_n} of Q equal to $S(x_n)$ and seek to find x_n such that $X = \{x_n\}_{n\ge1}$ is an increasing sequence of positive integers. For definiteness, we define x_n to be the *smallest* positive integer such that $x_n > x_{n-1}$, beginning with $x_1 = 2$. Using the result in (5), we may thus uniquely determine $x_n \in S^{-1}(Q)$ such that $x_n > x_{n-1}$, with $x_1 = 2$. We may illustrate by displaying the first 20 terms of X in the table below. Note that x_n is a multiple of p_{e_n} in all cases; indeed x_n is the smallest multiple of p_{e_n} satisfying the requirement that X is an increasing sequence. The process may be continued ad infinitum, yielding X, a solution to part 1 of the problem.

n	e_n	$p_{e_n} = S(x_n)$	x_n	n	e_n	$p_{e_n} = S(x_n)$	x_n
1	1	2	2	11	6	13	39
2	3	5	5	12	8	19	57
3	2	3	6	13	7	17	68
4	4	7	7	14	9	23	69
5	3	5	10	15	8	19	76
6	5	11	11	16	10	29	87
7	4	7	14	17	9	23	92
8	6	13	26	18	11	31	93
9	5	11	33	19	10	29	116
10	7	17	34	20	12	37	148

Solution to Part 2: Using the fact that $p|\binom{p}{n}$ for all $n \in \{1, 2, \ldots, p-1\}$, where p is prime, we see that $S(\binom{p}{n}) = p$ for these values. Moreover,

$$\binom{p}{1} < \binom{p}{2} < \cdots < \binom{p}{\frac{1}{2}(p-1)}, \; p \geq 5.$$

These facts enable us to construct a strictly increasing sequence of natural numbers, beginning with an arbitrary prime, for which the Smarandache Function is strictly decreasing.

Let $\{p_n\}_{n\geq 1} = \{2, 3, 5, \ldots\}$ denote the sequence of primes. Given $n > 1$, we may construct a sequence of binomial coefficients

$$V(p_n) = \left\{\binom{p_n}{m_1}, \binom{p_{n-1}}{m_2}, \ldots, \binom{p_{n-r+1}}{m_r}\right\},$$

where the m_i's are chosen to be the minimum natural numbers subject to $1 = m_1 < m_2 < \cdots < m_r \leq \frac{1}{2}(p_{n-r+1} - 1)$, such that

$$\binom{p_n}{m_1} < \binom{p_{n-1}}{m_2} < \cdots < \binom{p_{n-r+1}}{m_r}.$$

We may choose $m_i = i$ for $i \leq s$, say, but require $m_i > i$ for all $i > s$. The number of terms in the sequence, namely the integer r, depends solely on n. The sequence $V(p_n)$ is finite because, for some r,

$$\binom{p_{n-r+2}}{m} < \binom{p_{n-r+1}}{m_r}$$

for all m. Note that

$$S\left(\binom{p_n}{m_1}\right) = p_n, \; S\left(\binom{p_{n-1}}{m_2}\right) = p_{n-1}, \ldots, S\left(\binom{p_{n-r+1}}{m_r}\right) = p_{n-r+1};$$

thus, $S(V(p_n))$ is a strictly decreasing sequence, as required.

We illustrate with two examples. If $n = 26$, we take $p_n = 101$. We may then take

$$\begin{aligned} V(101) &= \left\{\binom{101}{1}, \binom{97}{2}, \binom{89}{3}, \binom{83}{4}, \binom{79}{5}, \binom{73}{6}, \binom{71}{7}, \binom{67}{8}, \binom{61}{9}, \binom{59}{10}, \binom{53}{11}, \binom{47}{13}, \binom{43}{15}, \binom{41}{17}\right\} \\ &= \{x_n\}_{n=1}^{14}, \end{aligned}$$

say. We easily check that $x_1 < x_2 < x_3 < \cdots < x_{14}$, however, $101 > 97 > \cdots > 41$, i.e., $S(x_1) > S(x_2) > \cdots > S(x_{14})$.

For our second example, we take $n = 51$ (hence, $p_n = 233$). In this case, we take

$$V(233) = \left\{\binom{233}{1}, \binom{229}{2}, \binom{227}{3}, \binom{223}{4}, \binom{211}{5}, \binom{199}{6}, \binom{197}{7}, \binom{193}{8}, \binom{191}{9}, \binom{181}{10}, \binom{179}{11}, \binom{173}{12}, \binom{167}{13}, \binom{163}{14}, \binom{157}{15}, \binom{151}{16}, \binom{149}{17}, \binom{139}{18}, \binom{137}{19}, \binom{131}{20}, \binom{127}{21}, \binom{113}{23}, \binom{109}{24}, \binom{107}{25}, \binom{103}{26}, \binom{101}{27}, \binom{97}{29}, \binom{89}{34}\right\}.$$

As we may verify, the sequence given above is an increasing sequence. The sequence terminates at the 28th term, since $\binom{83}{41} < \binom{89}{34}$.

Clearly, we may construct a sequence $V(p)$ in this fashion for all primes p of arbitrary size. The number of terms of $V(p)$ clearly grows with p in some fashion; apparently, $|V(p)| = 0(p/\log p)$ as $p \to \infty$, but this has not been established.

Also solved by H. Seiffert and the proposer.

Ghost from the Past

H-459 ***Proposed by Stanley Rabinowitz, Westford, MA (Vol. 29, no. 4, November 1991)***

Prove that, for all $n > 3$,

$$\frac{13\sqrt{5}-19}{10} L_{2n+1} + 4.4(-1)^n$$

is very close to the square of an integer.

Solution by H.-J. Seiffert, Berlin, Germany

We shall prove that

$$(5F_{n-1} - F_{n-3})^2 - \left(\frac{13\sqrt{5}-19}{10} L_{2n+1} + 4.4(-1)^n\right) = -2.6\sqrt{5}\beta^{2n+1}. \tag{1}$$

Since (β^{2n}) is a strictly decreasing sequence of positive reals, a simple calculation gives $0 < A_n \leq 2.6(85 - 38\sqrt{5})$ for $n > 3$, where A_n denotes the left side of (1). Noting that $2.6(85 - 38\sqrt{5}) \sim 0.076492$, we see that the statement of the proposal is reasonable.

To prove (1), we use the following easily verifiable equations:

$$5F_{n-1}^2 = L_{2n-2} + 2(-1)^n;\ 5F_{n-1}F_{n-3} = L_{2n-4} + 3(-1)^n;$$
$$5F_{n-3}^2 = L_{2n-6} + 2(-1)^n = 3L_{2n-4} - L_{2n-2} + 2(-1)^n;\ L_{2n+1} = 5L_{2n-2} - 2L_{2n-4}.$$

Now, a straightforward calculation yields $10A_n = 13((11 - 5\sqrt{5})L_{2n-2} + 2(2 - \sqrt{5})L_{2n-4})$ or, by $2 - \sqrt{5} = \beta^3$ and $11 - 5\sqrt{5} = 2\beta^5$ and the Binet form of the Lucas numbers,

$$10A_n = 26(\beta^4 - 1)\beta^{2n-1} = 26(\beta^2 - \alpha^2)\beta^{2n+1} = -26\sqrt{5}\beta^{2n+1}.$$

This proves (1).

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