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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

ON SOME PROPERTIES OF GENERALIZED HERMITE POLYNOMIALS

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(*Submitted March 1994*)

1. INTRODUCTION

The Hermite polynomials belong to the system of classical orthogonal polynomials (see [3], [6]). The following properties of these polynomials are well known: the orthogonal property, differential equation, Rodrigues representation, three-term recurrence relation. In 1990, P. R. Subramanian [5] studied a class of Hermite polynomials $H_n(x)$ in the sense that one of the above-mentioned four properties implies the other three.

In [4], H. M. Srivastava defined a class of generalized Hermite polynomials $\{\gamma_n^m(x)\}_{n=0}^{\infty}$ by the generating function

$$e^{mxt-t^m}=\sum_{n=0}^{\infty}\gamma_n^m(x)t^n.$$

2. THE POLYNOMIALS $h_{n,m}(x)$

In this paper, we consider the polynomials $\{h_{n,m}(x)\}_{n=0}^{\infty}$ defined by $h_{n,m}(x)=\gamma_n^m(2x/m)$. Their generating function is given by

$$F(x,t)=e^{2xt-t^m}=\sum_{n=0}^{\infty}h_{n,m}(x)t^n. \tag{2.1}$$

Note that $h_{n,2}(x)=H_n(x)/n!$ (Hermite polynomials).

Expanding the left-hand side of (2.1), we obtain the following explicit formula:

$$h_{n,m}(x)=\sum_{k=0}^{[n/m]}(-1)^k\frac{(2x)^{n-mk}}{k!(n-mk)!}. \tag{2.2}$$

By differentiating (2.1) with respect to t and comparing the corresponding coefficients, we obtain the following three-term relation:

$$nh_{n,m}(x)=2xh_{n-1,m}(x)-mh_{n-m,m}(x),\quad n\ge m\ge 1. \tag{2.3}$$

The starting polynomials are

$$h_{n,m}(x)=\frac{(2x)^n}{n!},\quad n=0,1,\dots,m-1. \tag{2.4}$$

By differentiating (2.2) with respect to x, one by one, s times, we get

$$D^sh_{n,m}(x)=2^sh_{n-s,m}(x),\quad n\ge s\ge 1,\ \ D^s\equiv d^s/dx^s. \tag{2.5}$$

For $s=1$, (2.5) is

$$Dh_{n,m}(x)=2h_{n-1,m}(x),\quad n\ge 1. \tag{2.6}$$

For $s = m-1$, (2.5) becomes

$$D^{m-1}h_{n,m}(x) = 2^{m-1}h_{n+1-m,m}(x), \quad n \geq m-1. \tag{2.7}$$

Now, from (2.3) and (2.7), we obtain

$$nh_{n,m}(x) = \left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]h_{n-1,m}(x), \quad n \geq 1, \tag{2.8}$$

where D is the differential operator d/dx.

If $m = 2$, the relation (2.8) becomes (see [1]) $H_n(x) = (2x - D)H_{n-1}(x)$, $n \geq 1$.
A very interesting relation now follows:

$$h_{n,m}(x) = \frac{f^{-1}}{n!}\left[2x - \frac{m}{2^{m-1}}D^{m-1} + \frac{m}{2^{m-1}}\sum_{k=0}^{m-2}(m-k)_k D^{m-1-k}(f)\sum_{j=0}^{k}\frac{D^{k-j}(f^{-1})}{j!(k-j)!}D^j\right]^n f, \tag{2.9}$$

where $f(x)$ is any differentiable function not identically zero, $D^s \equiv d^s/dx^s$, and $(\lambda)_n = \lambda(\lambda+1)$ ... $(\lambda+n-1)$ is the Pochhammer symbol (see [2], [3]).

3. EQUIVALENCE OF (2.9) AND OTHER RELATIONS

First, we shall prove the relation (2.9). Let $f(x)$ be any differentiable function not identically zero. From (2.8), we find:

$$\begin{aligned} fh_{n,m}(x) &= \frac{f}{n}\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]h_{n-1,m}(x) \\ &= \frac{1}{n}\left[2x - \frac{m}{2^{m-1}}D^{m-1} + \frac{m}{2^{m-1}}\sum_{k=0}^{m-2}(m-k)_k D^{m-1-k}(f)\sum_{j=0}^{k}\frac{D^{k-j}(f^{-1})}{j!(k-j)!}D^j\right]\{fh_{n-1,m}(x)\}. \end{aligned} \tag{3.1}$$

Iteration of (3.1) yields

$$fh_{n,m}(x) = \frac{1}{n!}\left[2x - \frac{m}{2^{m-1}}D^{m-1} + \frac{m}{2^{m-1}}\sum_{k=0}^{m-2}(m-k)_k D^{m-1-k}(f)\sum_{j=0}^{k}\frac{D^{k-j}(f^{-1})}{j!(k-j)!}D^j\right]^n f, \quad n \geq 1, \tag{3.2}$$

since $h_{0,m}(x) = 1$. However, (3.2) is also true for $n = 0$. The relation (2.9) follows immediately.

From (2.9) and $f(x) = 1$, we get the following beautiful relation:

$$h_{n,m}(x) = \frac{1}{n!}\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]^n 1, \quad n \geq 0. \tag{3.3}$$

If $m = 2$, (3.3) becomes (see [1]) $H_n(x) = [2x - D]^n 1$, $n \geq 0$. If $m = 3$, then (2.9) becomes

$$h_{n,3}(x) = \frac{f^{-1}}{n!}\left[2x - \frac{3}{4}D^2 + \frac{3}{4}f^{-1}\{D^2 f\} + \frac{3}{2}\{Df\}\{Df^{-1}\} + \frac{3}{2}f^{-1}\{Df\}D\right]^n f, \quad n \geq 0. \tag{3.4}$$

If $f(x) = e^{-x^3}$, relation (3.4) yields

$$h_{n,3}(x) = \left(-\frac{1}{4}\right)^n \frac{e^{x^3}}{n!}[3D^2 + 10x + 27x^4 + 18x^2 D]^n e^{-x^3}, \quad n \geq 0.$$

Now, we shall show that (2.9) is a spring for developing the properties of $h_{n,m}(x)$. First, we prove (2.8), starting from (2.9):

$$h_{n,m}(x) = \frac{f^{-1}}{n!}\left[2x - \frac{m}{2^{m-1}}D^{m-1} + \frac{m}{2^{m-1}}\sum_{k=0}^{m-2}(m-k)_k D^{m-1-k}(f)\sum_{j=0}^{k}\frac{D^{k-j}(f^{-1})}{j!(k-j)!}D^j\right]^n f$$

$$= \frac{1}{n!}\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]^n 1 = \frac{1}{n}\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]h_{n-1,m}(x), \quad n \geq 1.$$

Hence, we get (2.8).

From (2.3) with $n+1$ substituted for n, and using (2.5), (2.6), and (2.8), we find

$$(n+1)Dh_{n+1,m}(x) = D\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]h_{n,m}(x)$$

$$= 2h_{n,m}(x) + \left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]Dh_{n,m}(x) = 2(n+1)h_{n,m}(x).$$

Thus, we obtain the following differential recurrence relation:

$$Dh_{n+1,m}(x) = 2h_{n,m}(x). \tag{3.5}$$

Now, we shall prove the three-term recurrence relation (2.3). From (2.7) and (2.8), we get

$$nh_{n,m}(x) = 2xh_{n-1,m}(x) - \frac{m}{2^{m-1}}D^{m-1}h_{n-1,m}(x)$$

$$= 2xh_{n-1,m}(x) - mh_{n-m,m}(x), \quad n \geq m \geq 1.$$

The relation (2.3) follows from the last equality.

By differentiating (2.8) with $n+1$ substituted for n, and using (3.5), we obtain

$$\begin{aligned}(n+1)Dh_{n+1,m}(x) &= D\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]h_{n,m}(x)\\ &= 2h_{n,m}(x) + \left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]Dh_{n,m}(x).\end{aligned} \tag{3.6}$$

Next, from (3.6), we get the following differential equation:

$$\left[\frac{m}{2^{m-1}}D^m - 2xD + 2n\right]h_{n,m}(x) = 0. \tag{3.7}$$

For $m = 2$ or 3, equation (3.7) becomes

$$[D^2 - 2xD + 2n]H_n(x) = 0, \quad n \geq 0,$$

and

$$\left[\frac{3}{4}D^3 - 2xD + 2n\right]h_{n,3}(x) = 0, \quad n \ge 0.$$

Note that the first equation above is the Hermite differential equation.

Now, we show that (2.1) can be derived from the recurrence relation as follows (see [5]). Assume the existence of a generating function of the form

$$F(x,t) = \sum_{n=0}^{\infty} h_{n,m}(x)t^n. \tag{3.8}$$

Differentiate $F(x,t)$ with respect to t and, using (2.3) and (2.4), develop the following first-order differential equation for $F(x,t)$:

$$F^{-1}(\partial F / \partial t) = 2x - mt^{m-1}. \tag{3.9}$$

Now, we integrate both sides of (3.9) with respect to t, from 0 to t, to obtain

$$F(x,t) = F(x,0)e^{2xt-t^m}. \tag{3.10}$$

Since $F(x,0) = h_{0,m}(x) = 1$, by (2.4), it follows that $F(x,t) = e^{2xt-t^m}$.

Finally, we shall prove in this section that the polynomial $h_{n,m}(x)$ is a solution of the differential equation (3.7).

Assume that the polynomial $y = \sum_{k=0}^{n} a_k \cdot x^{n-k}$ is a solution of equation (3.7). Then,

$$Dy = \sum_{k=0}^{n-1} (n-k)a_k \cdot x^{n-1-k}, \tag{3.11}$$

and

$$D^m y = \sum_{k=0}^{n-m} (n+1-m-k)_m \cdot a_k \cdot x^{n-m-k}. \tag{3.12}$$

If we substitute (3.11) and (3.12) into equation (3.7), we get

$$\frac{m}{2^{m-1}} \sum_{k=m}^{n} (n+1-k)_m \cdot a_{k-m} \cdot x^{n-k} - 2\sum_{k=0}^{n} (n-k)a_k \cdot x^{n-k} + 2n\sum_{k=0}^{n} a_k \cdot x^{n-k} = 0. \tag{3.13}$$

From (3.13), we obtain

$$\sum_{k=m}^{n} \left[\frac{m}{2^{m-1}}(n+1-k)_m \cdot a_{k-m} + 2ka_k\right] x^{n-k} + \sum_{k=0}^{m-1} 2ka_k \cdot x^{n-k} = 0. \tag{3.14}$$

Next, from (3.14), we find

$$ka_k = 0, \quad k = 0, 1, 2, \ldots., m-1, \tag{3.15}$$

and

$$a_k = -\frac{m(n+1-k)_m}{2^m k} a_{k-m}, \quad k \ge m. \tag{3.16}$$

Finally, from (3.15), (3.16), and $a_0 = 2^n / n!$, using induction, we can show that the polynomial $y = \sum_{k=0}^{n} a_k \cdot x^{n-k}$ has the following form:

$$y = \sum_{k=0}^{[n/m]} (-1)^k \frac{(2x)^{n-mk}}{k!(n-mk)!}. \tag{3.17}$$

Comparing (3.17) with (2.2), we see that the polynomial y is the generalized Hermite polynomial $h_{n,m}(x)$.

4. RELATION $h_{n,m}(x) = (2^n / n!)\{\exp[-D^m / 2^m]\}x^n$

In this section, we prove the following relation

$$h_{n,m}(x) = \frac{2^n}{n!}\{\exp[-D^m / 2^m]\}\, x^n. \tag{4.1}$$

Note that the operator $\exp[-D^m / 2^m]$ has the following expansion:

$$\exp[-D^m / 2^m] = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{D^{ms}}{2^{ms}}. \tag{4.2}$$

Since

$$D^{ms}x^n = \begin{cases} [n!/(n-ms)!]x^{n-ms}, & n \geq ms \quad (s \leq [n/m]), \\ 0, & n < ms. \end{cases} \tag{4.3}$$

The relation (4.1) follows from (2.2), using (4.2) and (4.3).

For $m = 2$, (4.1) has the form (see [2])

$$H_n(x) = 2^n\{\exp[-D^2 / 4]\}x^n;$$

for $m = 3$, (4.1) becomes

$$h_{n,3}(x) = \frac{2^n}{n!}\{\exp[-D^3 / 8]\}x^n.$$

Remark: We can classify the starting points into two distinct groups (see [5]): (a) full self-contained springs and (b) associated springs. The generating function (2.1) and the relations (2.2), (2.9), (3.3), and (4.1) belong to category (a). These springs completely specify the generalized Hermite polynomials $h_{n,m}(x)$. The differential equation (3.7), the recurrence relation (2.3), the differential recurrence relation (3.5), and the relation (2.8) belong to category (b) because they require supplementary conditions to specify the generalized Hermite polynomials $h_{n,m}(x)$ fully.

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MINMAX POLYNOMIALS

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1. INTRODUCTION

Background

MinMax numbers $\{M_n\}$, and their subsidiary numbers $\{N_n\}$, for Pell numbers $\{P_n\}$ were studied in some detail in [2]. They are those positive integers for which the minimal and maximal representations by Pell numbers coincide.

Analogous results for the MinMax numbers $\{\mathcal{Q}_n\}$, and their subsidiary numbers $\{\mathcal{R}_n\}$, for the modified Pell numbers $\{q_n\}$ have been obtained in [3]. ($q_n = \frac{1}{2}Q_n$, where Q_n are the Pell-Lucas numbers [4].)

Our motivation in this paper is to extend these MinMax number systems to their algebraic polynomial counterparts $\{M_n(x)\}, \{N_n(x)\}, \{\mathcal{Q}_n(x)\}$, and $\{\mathcal{R}_n(x)\}$, and to analyze their properties. When $x = 1$, the MinMax numbers $\{M_n(1)\} = \{M_n\}$, etc., are naturally specified.

Pell polynomials $\{P_n(x)\}, n \geq 0$, are defined recursively [4] by

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x), \quad P_0(x) = 0, \; P_1(x) = 1, \tag{1.1}$$

while the modified Pell polynomials $\{q_n(x)\}, n \geq 0$, are similarly defined by

$$q_{n+2}(x) = 2xq_{n+1}(x) + q_n(x), \quad q_0(x) = 1, \; q_1(x) = x. \tag{1.2}$$

A useful connective between (1.1) and (1.2) is $q_n(x) = xP_n(x) + P_{n-1}(x)$.

Detailed information on the properties of, and interrelations between, $\{P_n(x)\}$ and $\{Q_n(x) = 2q_n(x)\}$ appear in [4] and [5], including lists of some of these polynomials. To conserve space, we assume that these data are accessible to the reader.

Just as there is the connection [2] between M_n and P_n, so there is the polynomial nexus

$$M_n(x) = \sum_{i=1}^{n} P_i(x). \tag{1.3}$$

The Sequence $\{q_n^*(x)\}$

Allied to $\{q_n(x)\}$ is the polynomial sequence $\{q_n^*(x)\}$ defined by the recurrence relation

$$q_{n+2}^*(x) = 2xq_{n+1}^* + q_n^*(x), \quad q_0^*(x) = 1, \; q_1^*(x) = 1. \tag{1.4}$$

Whereas $q_1(x) = x$, here $q_1^*(x) = 1$. Consult Table 1.

Putting $x = 1$ in $q_n^*(x)$, we find that $q_n^* = q_n$. Expressed otherwise, both $\{q_n^*(x)\}$ and $\{q_n(x)\}$ are polynomial generalizations of the modified Pell numbers $\{q_n\}$.

By standard methods, we derive the generating function

$$[1-(1-2x)y][1-(2xy+y^2)]^{-1} = \sum_{i=0}^{\infty} q_i^* y^i \tag{1.5}$$

and the Binet form

$$q_n^*(x) = \frac{(1-\beta)\alpha^n - (1-\alpha)\beta^n}{\alpha - \beta} \tag{1.6}$$

where [2]

$$\begin{cases} \alpha = x + \Delta \\ \beta = x - \Delta \end{cases} \quad \text{where } \Delta = \sqrt{x^2+1} \tag{1.7}$$

leading to

$$q_n^*(x) = P_n(x) + P_{n-1}(x). \tag{1.8}$$

For convenience, in (1.7) we employ the abbreviated symbolism $\alpha \equiv \alpha(x), \beta \equiv \beta(x), \Delta \equiv \Delta(x)$.

Using (1.7) and (1.8) in conjunction with the Binet form and Simson formula for $\{P_n(x)\}$, we have eventually the Simson analogue for $\{q_n^*(x)\}$:

$$q_{n+1}^* q_{n-1}^* - (q_n^*(x))^2 = (-1)^{n-1} 2x. \tag{1.9}$$

TABLE 1

$q_0^*(x) = 1$	$q_0(x) = 1$
$q_1^*(x) = 1$	$q_1(x) = x$
$q_2^*(x) = 2x+1$	$q_2(x) = 2x^2+1$
$q_3^*(x) = 4x^2+2x+1$	$q_3(x) = 4x^3+3x$
$q_4^*(x) = 8x^3+4x^2+4x+1$	$q_4(x) = 8x^4+8x^2+1$
$q_5^*(x) = 16x^4+8x^3+12x^2+4x+1$	$q_5(x) = 16x^5+20x^3+5x$
$q_6^*(x) = 32x^5+16x^4+32x^3+12x^2+6x+1$	....
$q_7^*(x) = 64x^6+32x^5+80x^4+32x^3+24x^2+6x+1$	
$q_8^*(x) = 128x^7+64x^6+192x^5+80x^4+80x^3+24x^2+8x+1$	
....	

2. MINMAX POLYNOMIALS $\{M_n(x)\}$

Define the polynomials $\{M_n(x)\}$, $n \ge 0$, by the recurrence relation

$$M_{n+2}(x) = 2xM_{n+1}(x) + M_n(x) + 1, \quad M_0(x) = 0, \ M_1(x) = 1. \tag{2.1}$$

Polynomials $\{M_n(x)\}$ may be called the *MinMax polynomials* for the Pell numbers.

Letting $x = 1$ gives us the MinMax numbers $\{M_n\}$ for the Pell numbers [2].

Table 2 displays the first few polynomials of $\{M_n(x)\}$.

That (1.3) and (2.1) are in conformity may be deduced by exploiting the defining recurrence relation (1.1) for $\{P_n(x)\}$ and recalling (1.1) that $P_1(x) = 1$.

It is a straightforward procedure by a standard technique to obtain the generating function for $\{M_n(x)\}$:

$$[1-(2x+1)y+(2x-1)y^2+y^3]^{-1} = \sum_{i=1}^{\infty} M_i(x) y^{i-1}. \tag{2.2}$$

From (1.3) and [4, (2.15)], we may express $M_n(x)$ explicitly by means of the double summation

$$M_n(x) = \sum_{i=1}^{n}\left(\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{i-k-1}{k}(2x)^{i-2k-1}\right). \tag{2.3}$$

Illustration of (2.3):

$$\begin{aligned}M_{11}(x) = {} & 1024x^{10} + 512x^9 + 2560x^8 + 1152x^7 + 2304x^6 \\ & + 896x^5 + 896x^4 + 280x^3 + 140x^2 + 30x + 6\end{aligned}$$

(on calculation), which is readily verifiable from Table 2 and (2.1).

TABLE 2. The MinMax Polynomials $M_n(x)$ ($n = 0, 1, 2, \ldots, 10$)

$$\begin{aligned}
M_0(x) &= 0\\
M_1(x) &= 1\\
M_2(x) &= 2x+1\\
M_3(x) &= 4x^2+2x+2\\
M_4(x) &= 8x^3+4x^2+6x+2\\
M_5(x) &= 16x^4+8x^3+16x^2+6x+3\\
M_6(x) &= 32x^5+16x^4+40x^3+16x^2+12x+3\\
M_7(x) &= 64x^6+32x^5+96x^4+40x^3+40x^2+12x+4\\
M_8(x) &= 128x^7+64x^6+224x^5+96x^4+120x^3+40x^2+20x+4\\
M_9(x) &= 256x^8+128x^7+512x^6+224x^5+336x^4+120x^3+80x^2+20x+5\\
M_{10}(x) &= 512x^9+256x^8+1152x^7+512x^6+896x^5+336x^4+280x^3+80x^2+30x+5
\end{aligned}$$

....

Combining (1.3) and [4, (2.11)], we derive

$$M_n(x) = \frac{P_{n+1}(x)+P_n(x)-1}{2x} = \frac{q^*_{n+1}(x)-1}{2x} \tag{2.4}$$

yielding the Binet form

$$M_n(x) = \frac{\alpha^n(1+\alpha)-\beta^n(1+\beta)-\Delta}{2x\Delta}. \tag{2.5}$$

A characteristic feature of $\{M_n(x)\}$ is the Simson formula

$$M_{n+1}(x)M_{n-1}(x) - M_n^2(x) = \frac{P_n(x)-P_{n+1}(x)+(-1)^n}{2x}. \tag{2.6}$$

Other derivations of interest in the MinMax theory include

$$M_n(x) - M_{n-1}(x) = P_n(x), \tag{2.7}$$

$$M_n(x) - M_{n-2}(x) = q^*_n(x) \quad \text{by (2.7), (1.8),} \tag{2.8}$$

$$\begin{aligned} M_n(x)+M_{n-1}(x) &= \frac{P_{n+1}(x)+(1-x)P_n(x)-1}{x} \quad &\text{by (2.4), (1.1)}\\ &(= \mathcal{Q}(x) &\text{by (4.3)),}\end{aligned} \tag{2.9}$$

$$M_n(x)+M_{n-2}(x)=\frac{(1+x)P_n(x)+(1-x)P_{n-1}-1}{x} \quad \text{by (2.7), (2.9)}$$
$$(=N_{n-1} \quad \text{by (3.1)}). \tag{2.10}$$

3. THE SUBSIDIARY MINMAX POLYNOMIALS $\{N_n(x)\}$

Next, we introduce a sequence of polynomials $\{N_n(x)\}$ associated with $\{M_n(x)\}$ which we define thus ($n \geq 1$):

$$N_n(x)=M_{n+1}(x)+M_{n-1}(x), \quad N_0(x)=1. \tag{3.1}$$

These polynomials $\{N_n(x)\}$ may be called the *subsidiary polynomials* of $\{M_n(x)\}$ for the Pell numbers. Table 3 lists the first few of them. See also (2.10).

TABLE 3. The Subsidiary MinMax Polynomials $N_n(x)$ ($n=0,1,2,\ldots,9$)

$N_0(x)=1$
$N_1(x)=2x+1$
$N_2(x)=4x^2+2x+3$
$N_3(x)=8x^3+4x^2+8x+3$
$N_4(x)=16x^4+8x^3+20x^2+8x+5$
$N_5(x)=32x^5+16x^4+48x^3+20x^2+18x+5$
$N_6(x)=64x^6+32x^5+112x^4+48x^3+56x^2+18x+7$
$N_7(x)=128x^7+64x^6+256x^5+112x^4+160x^3+56x^2+32x+7$
$N_8(x)=256x^8+128x^7+576x^6+256x^5+432x^4+160x^3+120x^2+32x+9$
$N_9(x)=512x^9+256x^8+1280x^7+576x^6+1120x^5+432x^4+400x^3+120x^2+50x+9$
....

When $x=1$, the numerical specializations are the *subsidiary numbers* $\{N_n\}$ investigated in [2].

For the criterion $N_0(x)=1$ to prevail in (3.1), we necessarily have $M_{-1}(x)=0$, obtainable by extension of (2.1) to the value of $n=-1$.

Immediately from (3.1) with (2.1) flows the consequence

$$N_{n+2}(x)=2xN_{n+1}(x)+N_n(x)+2 \quad (n \geq 0), \tag{3.2}$$

which is the recurrence relation for $\{N_n(x)\}$.

The generating function for $\{N_n(x)\}$ is, from (2.2) and (3.1),

$$(1+y^2)[1-(2x+1)y+(2x-1)y^2+y^3]^{-1}=\sum_{i=1}^{\infty} N_i(x)y^{i-1}. \tag{3.3}$$

Explicitly, from (2.3) and (3.1),

$$N_n(x)=2\sum_{i=1}^{n-1}\underbrace{\left(\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{i-k-1}{k}(2x)^{i-2k-1}\right)}_{A}+\sum_{i=n}^{n+1}\left[A+\binom{i-\left[\frac{n+1}{2}\right]-1}{\left[\frac{n+1}{2}\right]}(2x)^{i-2\left[\frac{n+1}{2}\right]-1}\right], \tag{3.4}$$

in which A stands for the second summation in the double summation, as represented symbolically.

Perseverance in calculation with (3.5) leads us to, for instance,

$$N_{10}(x) = 1024x^{10} + 512x^9 + 2816x^8 + 1280x^7 + 2816x^6 + 1120x^5 \\ + 1232x^4 + 400x^3 + 220x^2 + 50x + 11,$$

which may be readily checked from Table 3 and (3.2), or directly from (3.1) in conjunction with Table 2. Recall the expression for $M_{11}(x)$ in the illustration of (2.3).

Equations (2.4) and (3.1) produce

$$\begin{cases} N_n(x) = \dfrac{(1+x)P_{n+1}(x) + (1-x)P_n(x) - 1}{x} \\ \qquad\;\; = \dfrac{Q_{n+1}(x) + Q_n(x) - 2}{2x} \end{cases} \tag{3.5}$$

(where $Q_n(x) = P_{n+1}(x) + P_{n-1}(x)$ [4, (2.1)]), leading to the Binet form, see (1.7),

$$N_n(x) = \frac{\alpha^n(1+\alpha) + \beta^n(1+\beta) - 2}{2x}. \tag{3.6}$$

Suitable algebraic manipulation, involving (3.5) and [4], reveals in due course the Simson formula for $\{N_n(x)\}$,

$$N_{n+1}(x)N_{n-1}(x) - N_n^2(x) = \frac{Q_n(x) - Q_{n+1}(x) + 2\Delta^2(-1)^{n+1}}{x}. \tag{3.7}$$

Furthermore, (3.5) with (1.2), in which $Q_n(x) = 2q_n(x)$, reveals that

$$N_n(x) - N_{n-1}(x) = Q_n(x), \tag{3.8}$$

whence

$$N_n(x) - N_{n-2}(x) = Q_{n+1}(x) + Q_n(x). \tag{3.9}$$

Moreover,

$$N_n(x) + N_{n-1}(x) = \frac{Q_{n+1}(x) + (1-x)Q_n(x) - 2}{x} \quad \text{by (3.5)} \tag{3.10}$$

and

$$N_n(x) + N_{n-2}(x) = \frac{(1+x)Q_n(x) + (1-x)Q_{n-1}(x) - 2}{x} \quad \text{by (3.8), (3.10).} \tag{3.11}$$

Perusing the polynomial properties in Sections 2 and 3, one is struck by the harmonious balance of those results for $\{M_n(x)\}$ relating to $\{P_n(x)\}$ and similar ones for $\{N_n(x)\}$ relating to $\{Q_n(x)\}$ ("sweet harmony of contrasts"), e.g., compare (2.10) and (3.11).

This mathematical symbiosis does not really transfer to the polynomials to be discussed in Sections 4 and 5, though.

Notice, however, the same direct nexus between results for $\{M_n(x)\}$ in relation to $P_n(x)$ and those for $\{\mathfrak{Q}_n(x)\}$ in relation to $\{q_n^*\}$, e.g., contrast (2.10) and (4.10).

Comparison, e.g., of (3.11) and (4.10) shows the balance between results for $\{N_n(x)\}$ in relation to $\{Q_n(x)\}$ and those for $\{\mathfrak{Q}_n(x)\}$ in relation to $\{q_n^*(x)\}$, thus completing the third "side" of a "triangle" of relationships, i.e., (2.10), (3.11), and (4.10).

Second-order expressions (excepting the Simson analogues) are generally less manageable. Difference of squares such as $M_n^2(x) - M_{n-1}^2(x) = (M_n(x) + M_{n-1}(x))(M_n(x) - M_{n-1}(x))$, etc., are readily derivable from (2.7) and (2.9), but direct calculations and simplifications would otherwise be onerous. Coming to the sum of squares, we discover that

$$M_n^2(x) + M_{n-1}^2(x) = \frac{1}{2x^2}\left[\left\{\frac{1}{2}Q_n(x) + P_{2n}(x) - Q_n(x) - 2P_n(x)\right\} + 1\right] \tag{2.11}$$

and

$$N_n^2(x) - N_{n-1}^2(x) = \frac{1}{2x^2}\left[\left\{\frac{1}{2}Q_{(2n)}(x) + P_{(2n)}(x) - Q_{(n)}(x) - 2P_{(n)}(x)\right\} + 4\right] - 2P_n(x) \tag{3.12}$$

where, in the latter equation, we have used the symbolism, e.g.,

$$P_{(n)}(x) = P_{n+1}(x) + 2P_n(x) + P_{n-1}(x). \tag{3.13}$$

4. THE MINMAX POLYNOMIALS $\{\mathcal{Q}_n(x)\}$

Instead of the MinMax polynomials for the Pell numbers, we now consider the analogous polynomials for the modified Pell numbers.

Define the *MinMax polynomials* $\{\mathcal{Q}_n(x)\}, n \geq 0$, for the modified Pell numbers $\{q_n\}$—see (1.2)—by the recurrence relation

$$\mathcal{Q}_{n+2}(x) = 2x\mathcal{Q}_{n+1}(x) + \mathcal{Q}_n(x) + 2, \quad \mathcal{Q}_0(x) = 0,\ \mathcal{Q}_1(x) = 1. \tag{4.1}$$

Table 4 records the simplest of these polynomials.

TABLE 4. The MinMax Polynomials $\mathcal{Q}_n(x)$ ($n = 0, 1, 2, \ldots, 7$)

$\mathcal{Q}_0(x) = 0$
$\mathcal{Q}_1(x) = 1$
$\mathcal{Q}_2(x) = 2x + 2$
$\mathcal{Q}_3(x) = 4x^2 + 4x + 3$
$\mathcal{Q}_4(x) = 8x^3 + 8x^2 + 8x + 4$
$\mathcal{Q}_5(x) = 16x^4 + 16x^3 + 20x^2 + 12x + 5$
$\mathcal{Q}_6(x) = 32x^5 + 32x^4 + 48x^3 + 32x^2 + 18x + 6$
$\mathcal{Q}_7(x) = 64x^6 + 64x^5 + 112x^4 + 80x^3 + 56x^2 + 24x + 7$
....

Letting $x = 1$, we obtain the MinMax numbers $\{\mathcal{Q}_n\}$ for $\{q_n\}$ given in [3], namely, $\{\mathcal{Q}_n(x)\} = \{1, 4, 11, 28, 69, 168, 407, \ldots\}$.

Without difficulty, we establish the generating function

$$(1+y)[1-(2x+1)y+(2x-1)y^2+y^3]^{-1} = \sum_{i=1}^{\infty} \mathcal{Q}_i(x) y^{i-1}. \tag{4.2}$$

Immediately, we have from (4.2) with (2.2) that [cf. (2.9)]

$$\mathfrak{Q}_n(x) = M_n(x) + M_{n-1}(x). \tag{4.3}$$

With (2.5) substituted in (4.3), there results the Binet form for $\mathfrak{Q}_n(x)$, see (1.7),

$$\mathfrak{Q}_n(x) = \frac{\alpha^{n-1}(1+\alpha)^2 - \beta^{n-1}(1+\beta)^2 - 2\Delta}{2x\Delta} \tag{4.4}$$

which gives, with (1.6),

$$\mathfrak{Q}_n(x) = \frac{q^*_{n+1} + q^*_n(x) - 2}{2x}. \tag{4.5}$$

Compare this with the form for $N_n(x)$ in (3.5).

Using (4.5) along with (1.4) and (1.8), we discover the Simson formula

$$\mathfrak{Q}_{n+1}(x)\mathfrak{Q}_{n-1}(x) - \mathfrak{Q}_n^2(x) = (-1)^{n-1} - \frac{(q^*_{n+1}(x) - q^*_n(x))}{x}. \tag{4.6}$$

It readily follows from (4.5) and (1.4) that

$$\mathfrak{Q}_n(x) - \mathfrak{Q}_{n-1}(x) = q^*_n(x), \tag{4.7}$$

whence

$$\mathfrak{Q}_n(x) - \mathfrak{Q}_{n-2}(x) = q^*_{n+1}(x) + q^*_n(x). \tag{4.8}$$

Also,

$$\mathfrak{Q}_n(x) + \mathfrak{Q}_{n-1}(x) = \frac{1}{x}\{q^*_{n+1}(x) + (1-x)q^*(x) - 2\} \quad \text{by (4.5), (1.4),} \tag{4.9}$$

giving

$$\begin{aligned} \mathfrak{Q}_n(x) + \mathfrak{Q}_{n-2}(x) &= \frac{1}{x}\{(1+x)q^*_n(x) + (1-x)q^*_{n-1} - 2\} && \text{by (4.7)} \\ &(= \mathfrak{R}_{n-1}(x) && \text{by (5.1)).} \end{aligned} \tag{4.10}$$

An important result is

$$\mathfrak{Q}_n(x) = \sum_{i=1}^{n} q^*_i(x). \tag{4.11}$$

Proof of (4.11)

$$\begin{aligned} \sum_{i=1}^{n} q^*_i(x) &= \frac{1}{\Delta}\left\{(1-\beta)\sum_{i=1}^{n}\alpha^i - (1-\alpha)\sum_{i=1}^{n}\beta^i\right\} \quad \text{by (1.6)} \\ &= \frac{1}{\Delta}\left\{(1-\beta)\alpha\cdot\frac{1-\alpha^n}{1-\alpha} - (1-\alpha)\beta\cdot\frac{1-\beta^n}{1-\beta}\right\} \\ &= \frac{1}{\Delta}\left\{(1+\alpha)\left(\frac{1-\alpha^n}{1-\alpha}\right) - (1+\beta)\left(\frac{1-\beta^n}{1-\beta}\right)\right\} \\ &= \frac{(1+\alpha)(1-\beta)(1-\alpha^n) - (1-\alpha)(1+\beta)(1-\beta^n)}{\Delta(1-\alpha)(1-\beta)} \end{aligned}$$

$$= \frac{(2+\Delta)(1-\alpha^n)-(2-\Delta)(1-\beta^n)}{-2x\Delta} \quad \text{by (1.7)}$$
$$= \frac{2+\Delta-2+\Delta-[\alpha^n(2+\alpha+\alpha^{-1})-\beta^n(2+\beta+\beta^{-1})]}{-2x\Delta}$$
$$= \frac{\alpha^{n-1}(1+\alpha)^2-\beta^{n-1}(1+\beta)^2-2\Delta}{2x\Delta}$$
$$= \mathcal{Q}_n(x) \quad \text{by (4.4).}$$

So,

$$\mathcal{Q}_n(x) = \sum_{i=1}^{n} q_i^*(x) \neq \sum_{i=1}^{n} q_i(x).$$

5. THE SUBSIDIARY MINMAX POLYNOMIALS $\{\mathfrak{R}_n(x)\}$

We now introduce a sequence of polynomials $\{\mathfrak{R}_n(x)\}$ which bears the same relationship to $\{\mathcal{Q}_n(x)\}$ for $\{q_n(x)\}$ as $\{N_n(x)\}$ bears to $\{M_n(x)\}$ for $P_n(x)$.

Define the *subsidiary MinMax polynomials* $\{\mathfrak{R}_n(x)\}$ of $\{\mathcal{Q}_n(x)\}$ for $\{q_n(x)\}$ by

$$\mathfrak{R}_n(x) = \mathcal{Q}_{n+1}(x) + \mathcal{Q}_{n-1}(x), \quad \mathfrak{R}_0(x) = 0, \tag{5.1}$$

whence, by (4.1),

$$\mathfrak{R}_{n+2}(x) = 2x\mathfrak{R}_{n+1}(x) + \mathfrak{R}_n(x) + 4. \tag{5.2}$$

For the definition to apply for $n \geq 0$, we must have $\mathcal{Q}_{-1}(x) = -1$. Some of the most elementary of these polynomials are displayed in Table 5.

TABLE 5. The Subsidiary MinMax Polynomials $\mathfrak{R}_n(x)$ ($n = 0, 1, 2, \ldots, 5$)

$$\begin{aligned}
&\mathfrak{R}_0(x) = 0\\
&\mathfrak{R}_1(x) = 2x+2\\
&\mathfrak{R}_2(x) = 4x^2+4x+4\\
&\mathfrak{R}_3(x) = 8x^3+8x^2+10x+6\\
&\mathfrak{R}_4(x) = 16x^4+16x^3+24x^2+16x+8\\
&\mathfrak{R}_5(x) = 32x^5+32x^4+56x^3+40x^2+26x+10\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Putting $x = 1$ gives $\mathfrak{R}_n(1) = 4M_n(1)$, i.e., $\mathfrak{R}_n = 4M_n$ as in [3].

It is a relatively effortless exercise to determine the generating function

$$2[(1+x)+(1-x)y][1-(2x+1)y+(2x-1)y^2+y^3]^{-1} = \sum_{i=1}^{\infty} \mathfrak{R}_i(x)y^{i-1} \tag{5.3}$$

leading to

$$\mathfrak{R}_n(x) = 2[(1+x)M_n(x)+(1-x)M_{n-1}(x)], \tag{5.4}$$

where we have invoked (2.2).

After a little simplification involving (2.1) in (5.4), this is reducible to [cf. (3.10)]

$$\mathfrak{R}_n(x) = N_n(x) + N_{n-1}(x) \tag{5.5}$$

which by (3.6) ensures the Binet form, see (1.7),

$$\mathcal{R}_n(x) = \frac{\alpha^{n-1}(1+\alpha)^2 + \beta^{n-1}(1+\beta)^2 - 4}{2x}. \tag{5.6}$$

Equations (5.1) and (4.1) together produce

$$\begin{aligned}\mathcal{R}_n(x) &= \frac{\mathcal{Q}_{n+2}(x) - \mathcal{Q}_{n-2}(x) - 4}{2x} \\ &= \frac{Q_{n+1}(x) + 2Q_n(x) + Q_{n-1}(x) - 4}{2x} \quad \text{by (5.5) and (3.5).}\end{aligned} \tag{5.7}$$

Next,

$$\begin{aligned}\mathcal{R}_n(x) - \mathcal{R}_{n-1}(x) &= q^*_{n+1}(x) + q^*_{n-1}(x) && \text{by (5.1), (4.7)} \\ &= Q_n(x) + Q_{n-1}(x) && \text{by (1.8), [4, (2.1)]} \\ &= 2(q_n(x) + q_{n-1}(x)) && \text{since } Q_n(x) = 2q_n(x) \\ &= N_n(x) - N_{n-2}(x) && \text{by (5.5),}\end{aligned} \tag{5.8}$$

leading to expressions for $\mathcal{R}_n(x) - \mathcal{R}_{n-2}(x)$.

Moreover,

$$\mathcal{R}_n(x) + \mathcal{R}_{n-1}(x) = \frac{\alpha^{n-2}(1+\alpha)^3 + \beta^{n-2}(1+\beta)^3 - 8}{2x} \quad \text{by (5.6)} \tag{5.9}$$

whence

$$\mathcal{R}_n(x) + \mathcal{R}_{n-1}(x) = \frac{Q_{n-2}(x) + 3Q_{n-1}(x) + 3Q_n(x) + Q_{n+1}(x) - 8}{2x} \tag{5.10}$$

on using the Binet form [4, (3.31)] for $\{Q_n(x)\}$. An expression for $\mathcal{R}_n(x) + \mathcal{R}_{n-2}(x)$ follows by joining (5.8) to (5.10), $n \to n-1$ in the latter equation.

6. MISCELLANEOUS REMARKS

Determinantal Values

Computation gives us the pleasing and somewhat unexpected result,

$$\begin{vmatrix} M_n(x) & M_{n+1}(x) & M_{n+2}(x) \\ M_{n+1}(x) & M_{n+2}(x) & M_{n+3}(x) \\ M_{n+2}(x) & M_{n+3}(x) & M_{n+4}(x) \end{vmatrix} = (-1)^n, \tag{6.1}$$

which is clearly independent of x. Establishing (6.1) requires (2.1) and the Simson formula for $P_n(x)$ [4, (3.30)], together with some routine determinantal manipulations.

Similarly, the appropriate algebraic maneuvering leads to

$$\begin{vmatrix} N_n(x) & N_{n+1}(x) & N_{n+2}(x) \\ N_{n+1}(x) & N_{n+2}(x) & N_{n+3}(x) \\ N_{n+2}(x) & N_{n+3}(x) & N_{n+4}(x) \end{vmatrix} = 8(-1)^{n+1}\Delta^2, \tag{6.2}$$

which is **not** independent of x [cf. (6.1)].

An investigation into a corresponding determinantal value for $\{\mathcal{Q}_n(x)\}$ led to some unlovely algebra which was abandoned. However, to compensate for this disappointment, our general endeavors are rewarded with a result such as (6.1).

Diagonal Functions

When analyzing the structure of a set of polynomials, it is sometimes instructive to consider the rising (and descending) diagonal functions which, in the inward eye, are inherent in the system along upward (downward) slanting "lines." See, for instance, [1].

While such new polynomial sets can create some interest (e.g., the existence of certain differential equations—partial or ordinary), preliminary efforts with polynomials exposed in this paper do not seem particularly promising. But for "Time's winged chariot hurrying near," one could be encouraged to persevere with this challenge.

Other MinMax Systems

MinMax numbers for the Fibonacci numbers are exhibited in [2]. From these one may construct corresponding Fibonacci polynomials. Likewise, for the Lucas numbers and their polynomials. Experience suggests that an interconnected theory for these polynomials and for Lucas polynomials analogous to that established in the preceding treatment might be possible.

One does not have to be psychic to expect that similar developments might be worthwhile involving polynomials abstracted from other number sequences, e.g., Jacobsthal numbers.

The Tables

Of passing aesthetic appreciation is the varying pattern of constants in the polynomials listed in Tables 1-5, e.g., in Table 3 the sequence $\{1, 1, 3, 3, 5, 5, 7, 7, 9, 9, \ldots\}$ for $\{N_n(x)\}$.

7. CONCLUSION

It should be noted that numerical, i.e., nonpolynomial, recurrences specialized from (2.1), (3.2), (4.1), and (5.2)—along with other recurrences with a fixed additive constant—have recently been investigated in [2].

One wonders, *en passant*, what opportunities for discovery might exist from the innovative invention of polynomial sequences of the kind defined by $p_n(x) = xq_n(x) + q_{n-1}(x)$, or perhaps $p_n^*(x) = xq_n^*(x) + q_{n-1}^*(x)$.

There are further possible variations on our theme. Among these is the extension of our polynomials to negatively-subscripted symbols, e.g., $\{M_{-n}(x)\}$, but our ambitions are tempered by the sobering reminder of Longfellow that "Art is long and Time is fleeting."

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Announcement

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ON THE APPROXIMATION OF IRRATIONAL NUMBERS WITH RATIONALS RESTRICTED BY CONGRUENCE RELATIONS

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(*Submitted May 1994*)

1. INTRODUCTION

It is a well-known theorem of A. Hurwitz that for any real irrational number ξ there are infinitely many integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| \le \frac{1}{\sqrt{5}v^2}.$$

Usually this theorem is proved by using continued fractions; see Theorem 193 in [2].

S. Hartman [3] has restricted the approximating numbers $\frac{u}{v}$ to those fractions, where u and v belong to fixed residue classes a and b with respect to some modulus s. He proved the following:

For any irrational number ξ, any $s \ge 1$, and integers a and b, there are infinitely many integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| < \frac{2s^2}{v^2} \tag{1}$$

and

$$u \equiv a \mod s, \quad v \equiv b \mod s.$$

The special case $a = b = 0$ shows that the exponent 2 of s^2 is best-possible. In what follows, we are interested in the case where a and b are not both divisible by s; and we allow the denominators v to be negative. Using these conditions, S. Uchiyama [10] has published the following result:

For any irrational number ξ, any $s > 1$, and integers a and b, there are infinitely many integers u and $v \ne 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| < \frac{s^2}{4v^2} \tag{2}$$

and

$$u \equiv a \mod s, \quad v \equiv b \mod s,$$

provided that it is not simultaneously $a \equiv 0 \mod s$ and $b \equiv 0 \mod s$.

Years before, J. F. Koksma [4] had proved a slightly weaker theorem. From the case $s = 2$ and Theorem 3.2 in L. C. Eggan's paper [1], it is clear that the constant $\frac{1}{4}$ in (2) is best-possible. It is proved by Eggan that for any $\sigma > 0$ and for any choice of the three types *odd/odd*, *odd/even*, or *even/odd* of the fractions $\frac{u}{v}$ there is an irrational number ξ so that no fraction of the chosen type satisfies

$$\left|\xi - \frac{u}{v}\right| < \frac{1-\sigma}{v^2}.$$

The case $s=2$ has been studied by other authors, see, e.g., [5], [6], [8], and [9]. In this paper we prove a smaller bound for $|\xi-\frac{u}{v}|$, assuming $u\equiv v \bmod s$ for some prime s. Actually, the result is a bit stronger.

Theorem 1: Let $0<\varepsilon\le 1$, and let p be a prime with

$$p>\left(\frac{2}{\varepsilon}\right)^2;$$

h denotes any integer that is not divisible by p. Then, for any real irrational number ξ, there are infinitely many integers u and $v>0$ satisfying

$$\left|\xi-\frac{u}{v}\right|\le\frac{(1+\varepsilon)p^{3/2}}{\sqrt{5}v^2} \tag{3}$$

and

$$u\equiv hv\not\equiv 0 \mod p.$$

To prove Theorem 1, we will apply the methods of S. Hartman [3] and S. Uchiyama [10]. It will be convenient to use the same notations as in Uchiyama's paper, but this is done for another reason: there is a small gap in the proof of Uchiyama's result stated above in (2). In what follows, we are concerned with the same difficulty at this point, and we will fill the gap.

2. AUXILIARY RESULTS

Apart from Hartman's method, we need two lemmas.

Lemma 1: Let $0<\varepsilon\le 1$, and let p be a prime, and let w_1 and w_2 be integers with

$$p>\left(\frac{2}{\varepsilon}\right)^2, \tag{4}$$

$$0<w_1<p,\quad 0<w_2<p. \tag{5}$$

Then there are integers b, g_1, and g_2 satisfying

$$|b|<p, \tag{6}$$

$$0<|g_1|\le(1+\varepsilon)p^{1/2},\quad 0<|g_2|\le(1+\varepsilon)p^{1/2}, \tag{7}$$

and

$$bw_1\equiv g_1 \mod p,\quad bw_2\equiv g_2 \mod p. \tag{8}$$

Lemma 2: There are no integers x and y with $xy>0$ satisfying simultaneously the following conditions:

$$\frac{1}{xy}\ge\frac{1}{\sqrt{5}}\left(\frac{1}{x^2}+\frac{1}{y^2}\right),\quad \frac{1}{x(x+y)}\ge\frac{1}{\sqrt{5}}\left(\frac{1}{x^2}+\frac{1}{(x+y)^2}\right).$$

Proof of Lemma 1: We try to solve a linear system of equations

$$\left.\begin{aligned} w_1x-y_1+py_2&=0\\ w_2x-y_3+py_4&=0\end{aligned}\right\} \tag{9}$$

with integers $x, y_1, y_2, y_3,$ and y_4, where

$$|x| < p,$$

$$0 < |y_1|, |y_3| \le (1+\varepsilon)p^{1/2},$$

and

$$0 \le |y_2|, |y_4| \le 2kp.$$

Hence, $b = x$, $g_1 = y_1$, and $g_2 = y_3$ satisfy (6), (7), and (8) in our lemma. But first we do need an auxiliary inequality:

From $p > \left(\frac{2}{\varepsilon}\right)^2$, we conclude that

$$\frac{1}{\sqrt{p}} + \frac{1}{p} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For $0 \le t \le \frac{1}{2}$, we have $\frac{1}{\sqrt{1-t}} \le 1+t$; hence,

$$\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{1-\frac{1}{p}}} < 1+\varepsilon \quad (p \ge 2).$$

This is equivalent to

$$p^2 < (p-1)\left(2\left(\frac{1+\varepsilon}{2}\sqrt{p} - 1\right) + 1\right)^2.$$

It follows that

$$p^2 < (p-1)\left(2\left[\frac{1+\varepsilon}{2}\sqrt{p}\right] + 1\right)^2, \tag{10}$$

where $[\alpha]$ denotes the integral part of α for nonnegative real numbers α. The integers X_1, X_2, and X_3 are given by

$$X_1 = \frac{p-1}{2}, \quad X_2 = \left[\frac{1+\varepsilon}{2}p^{1/2}\right], \quad X_3 = kp, \tag{11}$$

where k is a sufficiently large positive integer satisfying

$$\frac{\left(p(p-1) + 2\left[\frac{1+\varepsilon}{2}p^{1/2}\right] + 2kp^2 + 1\right)^2}{(2kp+1)^2} \le p^2 \frac{p}{p-1}.$$

By (10) and (11), this implies

$$\frac{(2pX_1 + 2X_2 + 2pX_3 + 1)^2}{(2X_3+1)^2} < p(2X_2+1)^2$$

or

$$(2(pX_1 + X_2 + pX_3) + 1)^2 < (2X_1+1)(2X_2+1)^2(2X_3+1)^2. \tag{12}$$

There are $(2X_1+1)(2X_2+1)^2(2X_3+1)^2$ different sets of integers $x, y_1, y_2, y_3,$ and y_4 with

$$-X_1 \le x \le X_1, \quad -X_2 \le y_1, y_3 \le X_2, \quad -X_3 \le y_2, y_4 \le X_3.$$

We denote the left-hand sides of the two forms in (9) by f_1 and f_2; for each such set of integers, we have

$$-(pX_1+X_2+pX_3) \le f_1, f_2 \le pX_1+X_2+pX_3.$$

Here we have applied (5). There are at most $(2(pX_1+X_2+pX_3)+1)^2$ different sets f_1 and f_2. But now, by (12) and the box principle, there must be two distinct sets of five numbers x, y_1, y_2, y_3, and y_4 that correspond to the same set f_1 and f_2; their difference gives a nontrivial solution of (9) where, by (11):

$$0 \le |x| < p, \tag{13}$$

$$0 \le |y_1|, |y_3| \le (1+\varepsilon)p^{1/2}, \tag{14}$$

and

$$0 \le |y_2|, |y_4| \le 2kp.$$

To finish the proof, we must show that $y_1 \ne 0$ and $y_3 \ne 0$. We assume the contrary for y_1, which gives $w_1 x \equiv 0 \bmod p$ from the first equation in (9). Since p is a prime and $0 < w_1 < p$ by (5), this holds if and only if $x \equiv 0 \bmod p$. This means, by (13), that $x = 0$. Thus, the first equation in (9) becomes $py_2 = 0$ or $y_2 = 0$ and the second one becomes $-y_3 + py_4 = 0$ or $y_3 \equiv 0 \bmod p$. Now we apply the condition $\sqrt{p} > 2$ from (4), which yields

$$|y_3| \overset{(14)}{\le} (1+\varepsilon)p^{1/2} \le 2p^{1/2} < p;$$

hence, $y_3 = 0$.

We have obtained $x = y_1 = y_2 = y_3 = y_4 = 0$, which contradicts our construction of a nontrivial solution of (9). We proceed in the same way if we assume $y_3 = 0$. Thus, the proof of Lemma 1 is complete.

Proof of Lemma 2: (See "Hilfssatz 2.2", Ch. 10, in [7].) Without loss of generality, we may assume $x > 0$ and $y > 0$; hence, from the two inequalities stated in Lemma 2, we have

$$0 \ge x^2 + y^2 - xy\sqrt{5} \quad \text{and} \quad 0 \ge (2-\sqrt{5})(x^2+xy) + y^2.$$

The sum of these inequalities gives

$$0 \ge 2\left(\frac{3-\sqrt{5}}{2}x^2 + (1-\sqrt{5})xy + y^2\right) = 2\left(\frac{\sqrt{5}-1}{2}x - y\right)^2.$$

It follows that $2y = (\sqrt{5}-1)x$, which is impossible for $x \ne 0$.

3. PROOF OF THEOREM 1

Any real irrational number ξ is represented by its continued fraction expansion, that is, $\xi = [a_0; a_1, a_2, \ldots]$, where $a_0 \in \mathbb{Z}$, $a_n \in \mathbb{Z}_{>0}$ $(n \ge 1)$. p_n and q_n from the n^{th} convergent $\frac{p_n}{q_n}$ with

$$\left|\xi - \frac{p_n}{q_n}\right| \le \frac{1}{q_n^2} \tag{15}$$

satisfy the recurrences

$$p_{-1}=1,\quad p_0=a_0,\quad p_n=a_np_{n-1}+p_{n-2}\qquad (n\geq 1),$$
$$q_{-1}=0,\quad q_0=1,\quad q_n=a_nq_{n-1}+q_{n-2}\qquad (n\geq 1).$$

It is well known that

$$p_{n-1}q_n-p_nq_{n-1}=(-1)^n \tag{16}$$

holds for all integers $n\geq 1$. According to the usual notations, we have to distinguish carefully the notation p_m (with index m) and p, which denotes the prime modulus in Theorem 1.

Now, following the idea of Hartman, we consider for $n\geq 1$ a small system of congruences

$$\left.\begin{aligned} p_nx+p_{n-1}y&\equiv a \mod s,\\ q_nx+q_{n-1}y&\equiv b \mod s,\end{aligned}\right\} \tag{17}$$

where $s\geq 2$ is some positive integer and a and b are fixed integers such that there is not simultaneously $a\equiv 0 \mod s$ and $b\equiv 0 \mod s$. It is easily proved by (16) that a solution of (17) is given by $x=t_{n-1}$ and $y=t_n$, where the integers t_{n-1} and t_n are determined by

$$t_m\equiv(-1)^m(aq_m-bp_m) \mod s. \tag{18}$$

In what follows, we consider only the sequence of all *even* integers $n>0$. In (17) and (18), we put $s=p$ and $a=hb$ and so, for even integers n, we compute t_n and t_{n-1} by

$$\left.\begin{aligned} t_n&\equiv b(hq_n-p_n) \mod p,\\ t_{n-1}&\equiv b(p_{n-1}-hq_{n-1}) \mod p.\end{aligned}\right\} \tag{19}$$

If the sequence $a_0,a_1,a_2,\ldots$ from $\xi=[a_0;a_1,a_2,\ldots]$ is unbounded, there is an unbounded subset N of all positive integers such that, for certain integers $0\leq w_1<p$ and $0\leq w_2<p$, we have, for all $n\in N$:

$$a_n>2\sqrt{p}+1$$

and

$$\left.\begin{aligned} hq_n-p_n&\equiv w_1 \mod p,\\ p_{n-1}-hq_{n-1}&\equiv w_2 \mod p.\end{aligned}\right\} \tag{20}$$

Without loss of generality, we may assume that all integers from N are *even*; in the case in which $a_0,a_1,a_2,\ldots$ has an unbounded subsequence only with odd indices, the arguments are the same apart from a change of sign in most of the subsequent formulas.

Moreover, if the sequence $a_0,a_1,a_2,\ldots$ is bounded, it is obvious that there is an unbounded subset N of all *even* positive integers satisfying (20) for all $n\in N$ with certain integers w_1 and w_2.

If it is $w_1=0$ or $w_2=0$, we have $p_n\equiv hq_n \mod p$ or $p_{n-1}\equiv hq_{n-1} \mod p$ for all $n\in N$; thus, the theorem is already proved in this case by taking the convergents $\frac{p_n}{q_n}$ or $\frac{p_{n-1}}{q_{n-1}}$ according to $w_1=0$ or $w_2=0$. The inequality (3) holds by (15); it remains to check the condition

$$p_n\equiv hq_n\not\equiv 0 \mod p \quad\text{or}\quad p_{n-1}\equiv hq_{n-1}\not\equiv 0 \mod p \quad (n\in N)$$

from the theorem. Assuming the contrary for $\frac{p_n}{q_n}$, we get

$$p_n\equiv 0\equiv hq_n \mod p.$$

From $h \not\equiv 0 \mod p$ then follows $(p_n, q_n) \geq p$, a contradiction to a well-known fact. In the same way, one sees that $p_{n-1} \equiv hq_{n-1} \equiv 0 \mod p$ for $n \in N$ is impossible.

It remains to treat the case in which $0 < w_1 < p$ and $0 < w_2 < p$. Now conditions (4) and (5) of Lemma 1 hold; hence, there are integers b, g_1, and g_2 satisfying (6), (7), and (8). By (8), (19), and (20), we may put

$$t_n = g_1, \quad t_{n-1} = g_2 \quad (n \in N). \tag{21}$$

We define, for $n \in N$,

$$\left.\begin{aligned} u_n &= p_n g_2 + p_{n-1} g_1, \\ v_n &= q_n g_2 + q_{n-1} g_1. \end{aligned}\right\} \tag{22}$$

By (17), for all $n \in N$, these integers u_n and v_n satisfy

$$u_n \equiv hb, \quad v_n \equiv b \mod p, \tag{23}$$

and $b \not\equiv 0 \mod p$ is a consequence of (8) and $0 < |g_1| < p$. In particular, we conclude that $u_n v_n \neq 0$.

Furthermore, we put, for $n \in N$,

$$\left.\begin{aligned} u_n(\alpha, \beta) &= p_n \alpha + p_{n-1} \beta, \\ v_n(\alpha, \beta) &= q_n \alpha + q_{n-1} \beta. \end{aligned}\right\} \tag{24}$$

This means that $u_n(g_2, g_1) = u_n$ and $v_n(g_2, g_1) = v_n$.

In the next step of our proof, we will follow Uchiyama [10]. From (7) we know that $g_1 \neq 0$ and $g_2 \neq 0$; therefore, we have to distinguish several cases according to the signs of g_1 and g_2. We always assume $n \in N$; in particular, we know that n is even.

Some additional arguments are necessary when $g_1 g_2 < 0$ to show that the sequence v_n from (22) is unbounded for certain subsets of N. This also fills the gap in Uchiyama's paper.

Case 1. $g_1 g_2 > 0$

From (16) it is clear that, for all even n, we have

$$\frac{p_n}{q_n} < \xi < \frac{p_{n-1}}{q_{n-1}}. \tag{25}$$

From this inequality and (22), we get

$$\frac{p_n}{q_n} < \frac{u_n}{v_n} < \frac{p_{n-1}}{q_{n-1}}. \tag{26}$$

The relationship between ξ and $\frac{u_n}{v_n}$ now gives occasion to consider two subcases according to $\xi < \frac{u_n}{v_n}$ or $\xi > \frac{u_n}{v_n}$:

Case 1.1. $\frac{p_n}{q_n} < \xi < \frac{u_n}{v_n}$ for infinitely many $n \in N_1 \subseteq N$

Let θ denote the sign of g_1 (resp. g_2). For integers $j \geq 0$, we define integers

$$\left.\begin{aligned} u_{n,j} &= u_n(g_2 + \theta jp, g_1) \overset{(22),(24)}{=} u_n + \theta p p_n j, \\ v_{n,j} &= v_n(g_2 + \theta jp, g_1) \overset{(22),(24)}{=} v_n + \theta p q_n j. \end{aligned}\right\} \tag{27}$$

Now we keep n fixed and, by straightforward computation, show that the fractions $\frac{u_{n,j}}{v_{n,j}}$ monotonically decrease as j increases, and that

$$\lim_{j\to\infty}\frac{u_{n,j}}{v_{n,j}}=\frac{p_n}{q_n}.$$

Hence, by the assumption of Case 1.1, there exists some unique integer $k\geq 1$ such that

$$\frac{u_{n,k}}{v_{n,k}}<\xi<\frac{u_{n,k-1}}{v_{n,k-1}}. \tag{28}$$

We also have

$$\frac{u_{n,k}}{v_{n,k}}<\frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}}<\frac{u_{n,k-1}}{v_{n,k-1}}, \tag{29}$$

since both inequalities in (29) are equivalent to

$$u_{n,k-1}v_{n,k}-v_{n,k-1}u_{n,k}>0;$$

this holds because

$$\begin{aligned} u_{n,k-1}v_{n,k}-v_{n,k-1}u_{n,k} &= \theta p(q_n u_n - p_n v_n)\\ &\overset{(22)}{=} \theta p(p_{n-1}q_n - p_n q_{n-1})g_1\\ &\overset{(16)}{=} (-1)^n\theta g_1 p>0. \end{aligned} \tag{30}$$

Again two subcases arise from (28) and (29):

If we have

$$\frac{u_{n,k}}{v_{n,k}}<\xi<\frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}}<\frac{u_{n,k-1}}{v_{n,k-1}},$$

we assume that the following three inequalities hold simultaneously:

$$\xi-\frac{u_{n,k}}{v_{n,k}}\geq\frac{w}{\sqrt{5}v_{n,k}^2}, \tag{31}$$

$$\frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}}-\xi\geq\frac{w}{\sqrt{5}(v_{n,k}+v_{n,k-1})^2}, \tag{32}$$

$$\frac{u_{n,k-1}}{v_{n,k-1}}-\xi\geq\frac{w}{\sqrt{5}v_{n,k-1}^2}, \tag{33}$$

where $w=\theta g_1 p$. We sum up (31) and (32) and also (31) and (33); after some calculations and application of (30), we get

$$\frac{1}{v_{n,k}(v_{n,k}+v_{n,k-1})}\geq\frac{1}{\sqrt{5}}\left(\frac{1}{v_{n,k}^2}+\frac{1}{(v_{n,k}+v_{n,k-1})^2}\right)$$

and

$$\frac{1}{v_{n,k}v_{n,k-1}}\geq\frac{1}{\sqrt{5}}\left(\frac{1}{v_{n,k}^2}+\frac{1}{v_{n,k-1}^2}\right).$$

From (22) and (27), from the definition of θ, and since $g_1g_2 > 0$, we know that $v_{n,k}v_{n,k-1} > 0$ and, finally, all together contradict Lemma 2. Hence, at least one of the three inequalities—(31), (32), (33)—does not hold, and because each of the left-hand sides of these inequalities is positive, we have, for some

$$\frac{u}{v} \in \left\{\frac{u_{n,k}}{v_{n,k}}, \frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}}, \frac{u_{n,k-1}}{v_{n,k-1}}\right\},$$

$$\left|\xi - \frac{u}{v}\right| < \frac{w}{\sqrt{5}v^2} = \frac{p|g_1|}{\sqrt{5}v^2} \overset{(7)}{\leq} \frac{(1+\varepsilon)p^{3/2}}{\sqrt{5}v^2}. \tag{34}$$

But if ξ satisfies

$$\frac{u_{n,k}}{v_{n,k}} < \frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}} < \xi < \frac{u_{n,k-1}}{v_{n,k-1}},$$

we assume instead of (31), (32), (33),

$$\xi - \frac{u_{n,k}}{v_{n,k}} \geq \frac{w}{\sqrt{5}v_{n,k}^2}, \tag{35}$$

$$\xi - \frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}} \geq \frac{w}{\sqrt{5}(v_{n,k}+v_{n,k-1})^2}, \tag{36}$$

$$\frac{u_{n,k-1}}{v_{n,k-1}} - \xi \geq \frac{w}{\sqrt{5}v_{n,k-1}^2}. \tag{37}$$

Now we sum up (35) and (37), (36) and (37), which leads in the same way to a contradiction of Lemma 2. Hence, (34) holds in this subcase, too.

For every fraction $\frac{u}{v}$ satisfying (34), we know from (23) and (27) that either

$$u \equiv hv \equiv hb \mod p$$
or
$$u \equiv hv \equiv 2hb \mod p,$$

where $hb \not\equiv 0 \mod p$ implies $2hb \not\equiv 0 \mod p$ for all odd primes p. At last we note that $|v_n|$ tends to infinity for increasing $n \in N_1$; this follows from (22) and from $g_1g_2 > 0$. By (27) this means that, independently from k defined in (28), each of the numbers $|v_{n,k}|, |v_{n,k-1}|$, and $|v_{n,k}+v_{n,k-1}|$ tends to infinity for increasing $n \in N_1$. Thus, we have proved that there are infinitely many fractions $\frac{u}{v}$ satisfying (34), $u \equiv hv \mod p$ and, without loss of generality, $v > 0$, provided the assumption of Case 1.1 holds for an unbounded subset N_1 of N.

Case 1.2. $\frac{u_n}{v_n} < \xi < \frac{p_{n-1}}{q_{n-1}}$ **for infinitely many** $n \in N_2 \subseteq N$

In this case we define, for integers $j \geq 0$,

$$u_{n,j} = u_n(g_2, g_1 + \theta jp) \overset{(22),(24)}{=} u_n + \theta pp_{n-1}j,$$

$$v_{n,j} = v_n(g_2, g_1 + \theta jp) \overset{(22),(24)}{=} v_n + \theta pq_{n-1}j.$$

We proceed with similar arguments as in Case 1.1:

For any fixed n, the fractions $\frac{u_{n,j}}{v_{n,j}}$ monotonously increase with j; and from

$$\lim_{j\to\infty}\frac{u_{n,j}}{v_{n,j}}=\frac{p_{n-1}}{q_{n-1}}$$

we conclude that there is some unique integer $k\geq 1$ satisfying

$$\frac{u_{n,k-1}}{v_{n,k-1}}<\xi<\frac{u_{n,k}}{v_{n,k}}.$$

Again we consider the mediant $\frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}}$, which lies between $\frac{u_{n,k-1}}{v_{n,k-1}}$ and $\frac{u_{n,k}}{v_{n,k}}$, and distinguish two subcases according as ξ is greater or smaller than the mediant. Instead of (30), we now have

$$v_{n,k-1}u_{n,k}-u_{n,k-1}v_{n,k}=(-1)^n\theta g_2 p>0,$$

and as in Case 1.1 we get infinitely many fractions $\frac{u}{v}$ with

$$\left|\xi-\frac{u}{v}\right|<\frac{p|g_2|}{\sqrt{5}v^2}\overset{(7)}{\leq}\frac{(1+\varepsilon)p^{3/2}}{\sqrt{5}v^2},\tag{38}$$

where $u\equiv hv\not\equiv 0 \mod p$, provided the assumption of Case 1.2 holds for an unbounded subset N_2 of N.

Case 2. $g_1>0$, $g_2<0$

Let $q_n>p$ and assume $v_n=0$. From $(q_n,q_{n-1})=1$ and (22), we have $q_n|g_1$, which is impossible because $0<|g_1|<p<q_n$. Hence, for sufficiently large n, we know that $v_n\neq 0$, and we distinguish two subcases according as $v_n>0$ or $v_n<0$. We may repeat all the arguments from Case 1 [with the exception of the infinity of the rationals $\frac{u}{v}$ in (34) or (38)]; we leave the details to the reader. We only state the definitions of the fractions $\frac{u_{n,j}}{v_{n,j}}$ corresponding to the subcases.

Case 2.1. $v_n>0$ for infinitely many $n\in N_3\subseteq N$

$$u_{n,j}=u_n(g_2+jp,g_1),$$
$$v_{n,j}=v_n(g_2+jp,g_1),\qquad \text{for } j\geq 0.$$

Case 2.2. $v_n<0$ for infinitely many $n\in N_4\subseteq N$

$$u_{n,j}=u_n(g_2,g_1-jp),$$
$$v_{n,j}=v_n(g_2,g_1-jp),\qquad \text{for } j\geq 0.$$

Case 3. $g_1<0$, $g_2>0$

Case 3.1. $v_n>0$ for infinitely many $n\in N_5\subseteq N$

$$u_{n,j}=u_n(g_2,g_1+jp),$$
$$v_{n,j}=v_n(g_2,g_1+jp),\qquad \text{for } j\geq 0.$$

Case 3.2. $v_n < 0$ for infinitely many $n \in N_6 \subseteq N$

$$u_{n,j} = u_n(g_2 - jp, g_1),$$
$$v_{n,j} = v_n(g_2 - jp, g_1), \qquad \text{for } j \geq 0.$$

It remains to show that in each of these four subcases there are *infinitely many* fractions $\frac{u}{v}$ satisfying (34) or (38). We treat only subcase 2.1; there are no essential differences in the other cases. In this last part of the proof of Theorem 1, we also complete some details in Uchiyama's paper [10].

It suffices to show that the sequence of integers $v_n = q_n g_2 + q_{n-1} g_1$ is not bounded if n takes all values from N_3. We assume the contrary, and from the assumptions of Case 2 we conclude that there is some positive real number C satisfying

$$0 < q_{n-1}|g_1| - q_n|g_2| \leq C \;\; (n \in N_3) \quad \text{or} \quad 0 < \left|\frac{g_1}{g_2}\right| - \frac{q_n}{q_{n-1}} \leq \frac{C}{|g_2|q_{n-1}} \to 0 \;\; \text{for } n \in N_3, n \to \infty$$

[note (7) and $v_n > 0$]. Hence:

$$\text{The sequence } \left(\frac{q_n}{q_{n-1}}\right)_{n \in N_3} \text{ tends to the positive rational number } \left|\frac{g_1}{g_2}\right|. \tag{39}$$

Let us first assume that the sequence $a_0, a_1, a_2, \ldots$ is unbounded; we recall from the definition of N that

$$n \in N_3 \subseteq N \Rightarrow a_n > 2\sqrt{p} + 1. \tag{40}$$

From the recurrence relation for q_n, we conclude

$$\frac{q_n}{q_{n-1}} = a_n + \frac{q_{n-2}}{q_{n-1}} \;\; (n \in N_3); \tag{41}$$

and by $q_{n-2} < q_{n-1}$ it follows for all sufficiently large integers $n \in N_3$ that

$$a_n = \left[\frac{q_n}{q_{n-1}}\right] \leq \frac{q_n}{q_{n-1}} \overset{(39)}{\leq} \left|\frac{g_1}{g_2}\right| + 1 \leq |g_1| + 1 \overset{(7)}{\leq} 2\sqrt{p} + 1,$$

which contradicts (40).

Now we treat the more interesting case where the sequence $a_0, a_1, a_2, \ldots$ is bounded. In what follows, we assume that $n \in N_3$ is sufficiently large. We denote the continued fraction expansion of $|\frac{g_1}{g_2}|$ for some integer r and $c_r > 1$ by*

$$\left|\frac{g_1}{g_2}\right| = [c_0; c_1, c_2, \ldots, c_r].$$

It is a well-known fact from the elementary theory of continued fractions that

$$\frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \ldots, a_2, a_1]. \tag{42}$$

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* In the case $r = 0$, $c_0 = 1$, $g_1 = -g_2$, it is clear that $|v_n| = |g_1|(a_n q_{n-1} + q_{n-2} - q_{n-1}) \geq |g_1| q_{n-2}$ tends to infinity.

On the other hand we have, from (39) for all sufficiently large integers $n \in N_3$,

$$\frac{q_n}{q_{n-1}} = [c_0; c_1, \ldots, c_r + \delta(n)],$$

where

$$0 \neq \delta(n) \to 0 \text{ for } n \in N_3, n \to \infty. \tag{43}$$

For $0 < \delta(n) < 1$ we have, from (42),

$$a_{n-r} = c_r,$$
$$a_{n-r-1} = \left[\frac{1}{\delta(n)}\right]. \tag{44}$$

$-1/2 < \delta(n) < 0$ implies

$$a_{n-r} = c_r - 1,$$
$$a_{n-r-1} = \left[\frac{1}{1+\delta(n)}\right] = 1,$$
$$a_{n-r-2} = \left[\frac{1}{\frac{1}{1+\delta(n)} - 1}\right] = \left[-\frac{1}{\delta(n)} - 1\right]. \tag{45}$$

By (43), (44), and (45), a certain unbounded subsequence of $a_0, a_1, a_2, \ldots$ is given, a contradiction to our assumption.

The proof of Theorem 1 is now complete.

4. CONCLUDING REMARKS

The application of Lemma 2 in the proof of Theorem 1 will lead to some nice results by the way, if we put $s = 2$ instead of $s = p$ in (17). The following result is, in some sense, a supplement of Scott's and Robinson's theorems (see [9] and [8]).

Theorem 2: For any irrational number ξ, there are infinitely many pairs of integers u_1 and $v_1 > 0$, respectively u_2 and $v_2 > 0$, satisfying

(i) $$\left|\xi - \frac{u_1}{v_1}\right| \leq \frac{2}{\sqrt{5}v_1^2} \quad \text{and} \quad u_1 \equiv v_1 \bmod 2;$$

(ii) $$\left|\xi - \frac{u_2}{v_2}\right| \leq \frac{2}{\sqrt{5}v_2^2} \quad \text{and} \quad v_2 \equiv 0 \bmod 2, \text{ respectively.}$$

To prove (i), put $a = 1$ and $b = 1$ in (17); for (ii), let $a = 1$ and $b = 0$. It is obvious that we do not need Lemma 1.

Now let us consider such a real number ξ where $a_n \equiv 0 \bmod 2$ for all $n \geq 0$ in the continued fraction expansion of ξ. From the recurrence relations, it can easily be seen that this implies

$$p_n + q_n \equiv 1 \bmod 2 \quad (n \geq 1).$$

For instance $\xi = 1+\sqrt{2} = [2; \overline{2}]$ belongs to these numbers. We derive the following corollary from Theorem 2(i).

Corollary 1: There is an uncountable set of real numbers such that, for every number ξ from this set, there are infinitely many Dirichlet-approximants $\frac{u}{v}$ satisfying

$$\left|\xi - \frac{u}{v}\right| \le \frac{2}{\sqrt{5}v^2},$$

such that no fraction $\frac{u}{v}$ belongs to $\left\{\frac{p_n}{q_n} : n \ge 1\right\}$.

To appreciate this corollary, we refer to Theorem 184 in [2], which states that

$$\frac{u}{v} \in \left\{\frac{p_n}{q_n} : n \ge 1\right\}, \text{ if } \left|\xi - \frac{u}{v}\right| < \frac{1}{2v^2}.$$

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THE BRAHMAGUPTA POLYNOMIALS

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1. INTRODUCTION

In this paper we define the Brahmagupta matrix [see (1), below] and show that it generates a class of homogeneous polynomials x_n and y_n in x and y satisfying a host of relations; the polynomials contain as special cases the well-known Fibonacci, Lucas, and Pell sequences, and the sequences observed by Entringer and Slater [1], while they were investigating the problem of information dissemination through telegraphs; x_n and y_n also include the Fibonacci polynomials, the Pell and Pell-Lucas polynomials [5], [7], and the Morgan-Voyce polynomials in Ladder Networks and in Electric Line Theory [6], [9]. We also extend some series and convolution properties that hold for the Fibonacci and Lucas sequences and the Pell and Pell-Lucas polynomials to x_n and y_n [7], [5].

2. THE BRAHMAGUPTA MATRIX

To solve the indeterminate equation $x^2 = ty^2 \pm m$ in integers, where t is square free, the Indian astronomer and mathematician Brahmagupta (*ca.* 598) gave an iterative method of deriving new solutions from the known ones by his *samasa-bhavana*, the principle of composition: If (x_1, y_1, m_1) and (x_2, y_2, m_2) are trial solutions of the indeterminate equation, then the triple $(x_1x_2 \pm ty_1y_2, x_1y_2 \pm y_1x_2, m_1m_2)$ is also a solution of the indeterminate equation which can be expressed using the multiplication rule for a 2 by 2 matrix. Notice that if we set

$$B(x, y) = \begin{bmatrix} x & y \\ \pm ty & \pm x \end{bmatrix}, \tag{1}$$

and $m = \det B$, then the results

$$B(x_1, y_1)B(x_2, y_2) = B(x_1x_2 \pm ty_1y_2, x_1y_2 \pm y_1x_2), \tag{2}$$

$$\det[B(x_1, y_1)B(x_2, y_2)] = \det B(x_1, y_1) \det B(x_2, y_2) = m_1m_2, \tag{3}$$

give the Brahmagupta rule. Equation (3) is usually referred to as the *Brahmagupta Identity* and appears often in the history of number theory [10].

Let **M** denote the set of matrices of the form

$$B = \begin{bmatrix} x & y \\ ty & x \end{bmatrix}, \tag{4}$$

where t is a fixed real number and x and y are variables. Define B to be the Brahmagupta matrix. **M** satisfies the following properties:

1. **M** is a field for $x, y, t \in R$ and $t < 0$; in particular, if $t = -1$, then we have the well-known one-to-one correspondence between the set of matrices and the complex numbers $x + iy$:

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \leftrightarrow x + iy.$$

2. The following eigenrelations, in which T denotes the transpose, hold:
$$B[1, \pm\sqrt{t}]^T = (x \pm y\sqrt{t})[1, \pm\sqrt{t}]^T,$$
and these relations imply
$$B^n[1, \pm\sqrt{t}]^T = (x \pm y\sqrt{t})^n[1, \pm\sqrt{t}]^T$$
and
$$\begin{bmatrix} x & y \\ ty & x \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{t}{2}} & -\sqrt{\frac{t}{2}} \end{bmatrix} \begin{bmatrix} x + y\sqrt{t} & 0 \\ 0 & x - y\sqrt{t} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2t}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2t}} \end{bmatrix}.$$
Define
$$B^n = \begin{bmatrix} x & y \\ ty & x \end{bmatrix}^n = \begin{bmatrix} x_n & y_n \\ ty_n & x_n \end{bmatrix} = B_n.$$
3. Then the following recurrence relations are satisfied:
$$x_{n+1} = xx_n + tyy_n,\ y_{n+1} = xy_n + yx_n, \tag{5}$$
with $x_n = x$ and $y_n = y$.
4. Using the above eigenrelations, we derive the following Binet forms for x_n and y_n:
$$x_n = \frac{1}{2}[(x + y\sqrt{t})^n + (x - y\sqrt{t})^n], \tag{6}$$
$$y_n = \frac{1}{2\sqrt{t}}[(x + y\sqrt{t})^n - (x - y\sqrt{t})^n], \tag{7}$$
and $x_n \pm \sqrt{t}y_n = (x \pm \sqrt{t}y)^n$.
5. Let $\xi_n = x_n + y_n\sqrt{t}$, $\eta_n - y_n\sqrt{t}$, and $\beta_n = x_n^2 - ty_n^2$, with $\eta_n = \eta$, $\xi_n = \xi$, and $\beta_n = \beta$; we then have $\xi_n = \xi^n$, $\eta_n = \eta^n$, and $\beta_n = \beta^n$. To show the last equality, consider $\beta^n = (x^2 - ty^2)^n = \xi^n\eta^n = \xi_n\eta_n = (x_n^2 - ty_n^2) = \beta_n$. Notice that $\beta = \det B$.
6. The recurrence relations (5) also imply that x_n and y_n satisfy the difference equations:
$$x_{n+1} = 2x\, x_n - \beta x_{n-1},\ y_{n+1} = 2x\, y_n - \beta y_{n-1}. \tag{8}$$
Conversely, if $x_0 = 1$, $x_1 = x$, and $y_0 = 0$, $y_1 = y$, then the solutions of the difference equations (8) are indeed given by the Binet forms (6) and (7).
7. Notice that if we set $x = 1/2 = y$ and $t = 5$, then $\beta = -1$ and $2y_n = F_n$ is the Fibonacci sequence, while $2x_n = L_n$ is the Lucas sequence, where $n > 0$. For the number-theoretic properties of F_n and L_n, the reader is referred to [2] and [3].
8. In particular, if $x = y = 1$ and $t = 2$, then both x_n and y_n satisfy $x_n^2 - 2y_n^2 = (-1)^n$ and they generate the Pell sequences:
$$x_n = 1, 1, 3, 7, 17, 41, 99, 239, 577, \ldots,\ y_n = 0, 1, 2, 5, 12, 29, 70, 169, 408, \ldots.$$
It is interesting to note that if we set
$$a = 2(x_n + y_n)y_n\ \ b = x_n(x_n + 2y_n),\ c = x_n^2 + 2x_ny_n + 2y_n^2,$$

we then obtain integral solutions of the Pythagorean relation $a^2+b^2=c^2$, where a and b are consecutive integers [8].

9. If $t=1$, then $x_n+y_n=(x+y)^n$, and if $t=-1$, then $x_n+iy_n=(x+iy)^n$. Also, for every square free integer t, the set of matrices **M** is isomorphic to the set $\{x+y\sqrt{t}\,|\,x,y\in Z\}$, where Z is the set of integers.

10.
$$e^B=\frac{1}{4}\begin{bmatrix} e^{\xi}+e^{\eta} & \frac{1}{\sqrt{t}}(e^{\xi}-e^{\eta}) \\ \sqrt{t}(e^{\xi}-e^{\eta}) & e^{\xi}+e^{\eta}\end{bmatrix},\quad \det e^B=e^{2x}.$$

To show these results, let us write $2x_k=\xi^k+\eta^k$, $2\sqrt{t}y_k=\xi^k-\eta^k$. Since

$$e^B=\sum_{k=0}^{\infty}\frac{B^k}{k!}\quad\text{and}\quad\frac{B^k}{k!}=\frac{1}{k!}\begin{bmatrix} x_k & y_k \\ ty_k & x_k\end{bmatrix},$$

we express x_k and y_k in terms of ξ and η and obtain the desired results.

11. x_n and y_n can be extended to negative integers by defining $x_{-n}=x_n\beta^{-n}$ and $y_{-n}=-y_n\beta^{-n}$. We will then have

$$B^{-n}=\begin{bmatrix} x & y \\ ty & x\end{bmatrix}^{-n}=\begin{bmatrix} x_{-n} & y_{-n} \\ ty_{-n} & x_{-n}\end{bmatrix}=B_{-n};$$

here we have used the property

$$\left(\begin{bmatrix} x & y \\ ty & x\end{bmatrix}^{-1}\right)^n=\left(\frac{1}{\beta}\begin{bmatrix} x & -y \\ -ty & x\end{bmatrix}\right)^n=\frac{1}{\beta^n}\begin{bmatrix} x_n & -y_n \\ -ty_n & x_n\end{bmatrix}.$$

All the recurrence relations extend to the negative integers, also. Notice that $B^0=I$, the identity matrix.

3. THE BRAHMAGUPTA POLYNOMIALS

1. Using the Binet forms (6) and (7), we can deduce a number of results. Write x_n and y_n as polynomials in x and y using the binomial expansion:

$$x_n=x^n+t\binom{n}{2}x^{n-2}y^2+t^2\binom{n}{4}x^{n-4}y^4+\cdots,$$
$$y_n=nx^{n-1}y+t\binom{n}{3}x^{n-3}y^3+t^2\binom{n}{5}x^{n-5}y^5+\cdots.$$

Notice that x_n and y_n are homogeneous in x and y. The first few polynomials are

$$x_0=1,\ x_1=x,\ x_2=x^2+ty^2,\ x_3=x^3+3tyx^2,\ x_4=x^4+6tx^2y^2+t^2y^4,\ldots,$$
$$y_0=0,\ y_1=y,\ y_2=2xy,\ y_3=3x^2y+ty^3,\ y_4=4x^3y+4txy^3,\ldots.$$

2. If $t>0$, then x_n and y_n satisfy

$$\lim_{n\to\infty}\frac{x_n}{y_n}=\sqrt{t},\quad \lim_{n\to\infty}\frac{x_n}{x_{n-1}}=\lim_{n\to\infty}\frac{y_n}{y_{n-1}}=x+\sqrt{t}y.$$

3.
$$\frac{\partial x_n}{\partial x} = \frac{\partial y_n}{\partial y} = nx_{n-1},$$
$$\frac{\partial x_n}{\partial y} = t\frac{\partial y_n}{\partial y} = nty_{n-1}.$$

From the above relations, we infer that x_n and y_n are the polynomial solutions of the wave equation
$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{t}\frac{\partial^2}{\partial y^2}\right)U = 0.$$

4. If $\beta = -1$, then $ty^2 = x^2 + 1$, then the difference equations (8) become
$$x_{n+1} = 2x\,x_n + x_{n-1}, \quad y_{n+1} = 2xy_n + y_{n-1}. \tag{9}$$

5. If $2x = \alpha$ and $\beta = 1$, $x_1 = 1$, $x_2 = \alpha$, and $y_1 = 1$, $y_2 = \alpha - 1$, then x_n and y_n generate Morgan-Voyce polynomials [6], [9].

4. RECURRENCE RELATIONS

1. From the Binet forms (6) and (7), we can derive the following recurrence relations:

$$\begin{array}{rl}
\text{(i)} & x_{m+n} = x_m x_n + ty_m y_n, \\
\text{(ii)} & y_{m+n} = x_m y_n + y_m x_n, \\
\text{(iii)} & \beta^n x_{m-n} = x_m x_n - ty_m y_n, \\
\text{(iv)} & \beta^n y_{m-n} = x_n y_m - x_m y_n, \\
\text{(v)} & x_{m+n} + \beta^n x_{m-n} = 2x_m x_n, \\
\text{(vi)} & y_{m+n} + \beta^n y_{m-n} = 2x_n y_m, \\
\text{(vii)} & x_{m+n} - \beta^n x_{m-n} = 2ty_m y_n, \\
\text{(viii)} & y_{m+n} - \beta^n y_{m-n} = 2x_m y_n, \\
\text{(ix)} & 2(x_m^2 - x_{m+n}x_{m-n}) = \beta^{m-n}(\beta^n - x_{2n}), \\
\text{(x)} & x_{2m} - 2ty_{m+n}y_{m-n} = \beta^{m-n}x_{2n}.
\end{array} \tag{10}$$

2. Put $m=n$ in (i) and (ii) above; then we see that
$$x_{2n} = x_n^2 + ty_n^2, \quad y_{2n} = 2x_n y_n,$$
and these relationships imply that
(a) x_{2n} is divisible by $x_n \pm i\sqrt{t}y_n$, if $t > 0$,
(b) x_{2n} is divisible by $x_n \pm i\sqrt{t}y_n$, if $t < 0$,
(c) y_{2n} is divisible by x_n and y_n; also, if r divides s, then x_{rn} and y_{rn} are divisors of y_{sn}.

3. Let $\sum_{k=1}^{n} = \sum$. Then, using the Binet forms, we can also derive the following relations:

$$
\begin{aligned}
&\text{(i)} && \sum x_k = \frac{\beta x_n - x_{n+1} + x - \beta}{\beta - 2x + 1}, \\
&\text{(ii)} && \sum y_k = \frac{\beta y_n - y_{n+1} + y}{\beta - 2x + 1}, \\
&\text{(iii)} && \sum x_k^2 = \frac{\beta^2 x_{2n} - x_{2n+2} + x_2 - \beta^2}{2(\beta^2 - 2x_2 + 1)} + \frac{\beta(\beta^n - 1)}{2(\beta - 1)}, \\
&\text{(iv)} && \sum y_k^2 = \frac{\beta^2 x_{2n} - x_{2n+2} + x_2 - \beta^2}{2t(\beta^2 - 2x_2 + 1)} - \frac{\beta(\beta^n - 1)}{2t(\beta - 1)}, \\
&\text{(v)} && 2\sum x_k x_{n+1-k} = n x_{n+1} + \frac{\beta y_n}{y}, \\
&\text{(vi)} && 2t\sum y_k y_{n+1-k} = n x_{n+1} - \frac{\beta y_n}{y}, \\
&\text{(vii)} && 2\sum x_k y_{n-k+1} = 2\sum y_k x_{n-k+1} = n y_{n+1}.
\end{aligned}
\tag{11}
$$

4. Now we show an interesting result which generalizes a property that holds between F_n and L_n, namely, $e^{L(x)} = F(x)$, where

$$F(x) = F_1 + F_2 x + F_3 x^2 + \cdots + F_{n+1} x^n + \cdots,$$

and

$$L(x) = L_1 x + \frac{L_2}{2} x^2 + \frac{L_3}{3} x^2 + \cdots + \frac{L_n}{n} x^n + \cdots$$

(see [4]). Let X and Y be generating functions of x_n and y_n, respectively; that is,

$$X = \sum_1^\infty \frac{x_n}{n} s^n, \quad Y = \sum_1^\infty y_n s^n,$$

then $Y(s) = sye^{2X(s)}$. To prove this result, consider $Y(s) = y_1 s + y_2 s^2 + y_3 s^3 + \cdots + y_n s^n + \cdots$. Then $sY(s) = y_1 s^2 + y_2 s^3 + y_3 s^4 + \cdots + y_n s^{n+1} + \cdots$, and $s^2 Y(s) = y_1 s^3 + y_2 s^4 + \cdots + y_n s^{n+2} + \cdots$. Substituting the power series for $Y(s)$ into the expression $Y(x) - 2xsY(s) + \beta s^2 Y(s)$, we obtain

$$[1 - 2xs + \beta s^2] Y(s) = ys + \sum_{k=1}^\infty [y_{k+1} - 2xy_k + \beta y_{k-1}] s^{k+1},$$

where we have put $y_1 = y$. Now, using the property $y_{k+1} - 2xy_k + \beta y_{k-1} = 0$ in equation (8), we find that the above expression reduces to

$$[1 - 2xs + \beta s^2] Y(s) = ys. \tag{12}$$

Now consider the series

$$X(s) = x_1 s + \frac{x_2}{2} s^2 + \frac{x_3}{3} s^3 + \cdots + \frac{x_n}{n} s^n + \cdots,$$

and in it express x_n in terms of ξ_n and η_n to get

$$X(s) = \frac{1}{2}(\zeta + \eta)s + \frac{1}{2}\left[\frac{1}{2}(\xi^2 + \eta^2)\right] s^2 + \frac{1}{3}\left[\frac{1}{2}(\xi^3 + \eta^3)\right] s^3 + \cdots + \frac{1}{n}\left[\frac{1}{2}(\xi^n + \eta^n)\right] s^n + \cdots,$$

which can be rewritten in the form

$$X(s)=\frac{1}{2}\left[\xi s+\frac{1}{2}\xi^2s^2+\frac{1}{3}\xi^3s^3+\cdots\right]+\frac{1}{2}\left[\eta s+\frac{1}{2}\eta^2s^2+\frac{1}{3}\eta^3s^3+\cdots\right].$$

Therefore,

$$X(s)=-\frac{1}{2}\ln(1-\xi s)-\frac{1}{2}\ln(1-\eta s)=-\frac{1}{2}\ln[(1-\xi s)(1-\eta s)].$$

Since $(1-\xi s)(1-\eta s)=1-2xs+\beta s^2$, we have

$$2X(s)=-\ln[1-2xs+\beta s^2]. \tag{13}$$

Now compare (12) and (13) and obtain the desired result: $Y(s)=sye^{2X(s)}$.

5. SERIES SUMMATION INVOLVING RECIPROCALS OF x_n AND y_n

Let us look at some infinite series summations involving x_n and y_n and extend some of the infinite series results that are known for F_n and L_n [4] and for $P_n(x)$ and $Q_n(x)$ [7] to x_n and y_n. First, we shall show that

1. $$\sum_{k=1}^{\infty}\frac{1}{x_{k+1}}\left(\frac{2x}{x_{k-1}}-\frac{\beta+1}{x_k}\right)=\frac{1}{x}.$$

To show the above result, consider

$$\frac{1}{x_{k-1}x_k}-\frac{1}{x_kx_{k+1}}=\frac{x_{k+1}-x_{k-1}}{x_{k-1}x_kx_{k+1}}$$
$$=\frac{2xx_k-\beta x_{k-1}-x_{k-1}}{x_{k-1}x_kx_{k+1}}=\frac{2x}{x_{k-1}x_{k+1}}-\frac{\beta+1}{x_kx_{k+1}},$$

where we have used the property $x_{k+1}=2xx_k-\beta x_{k-1}$. Therefore,

$$\sum_{k=1}^{\infty}\frac{1}{x_{k+1}}\left(\frac{2x}{x_{k-1}}-\frac{\beta+1}{x_k}\right)=\sum_{k=1}^{\infty}\left(\frac{1}{x_{k-1}x_k}-\frac{1}{x_kx_{k+1}}\right)=\frac{1}{x_0x_1}=\frac{1}{x}.$$

In particular, if $ty^2=1+x^2$ and $y=1$, then $\beta=-1$, and the above result reduces to

$$\sum_{k=1}^{\infty}\frac{1}{x_{k-1}x_{k+1}}=\frac{1}{2x^2},$$

where x_k is given by equation (9). Similarly, we can show that

2. $$\sum_{k=r+1}^{\infty}\left(\frac{2x}{x_{k-1}x_{k+1}}-\frac{\beta+1}{x_{k+1}x_k}\right)=\frac{1}{x_rx_{r+1}}.$$

For the special case $ty^2=1+x^2$ and $y=1$, then $\beta=-1$ and the above result becomes

$$\sum_{k=r+1}^{\infty}\frac{1}{x_{k-1}x_{k+1}}=\frac{1}{2xx_rx_{r+1}},$$

where x_k is given by equation (9). Following a similar argument, we can show

3. $$\sum_{k=r+1}^{\infty}\left(\frac{2x}{y_{k-1}y_{k+1}}-\frac{\beta+1}{y_k y_{k+1}}\right)=\frac{1}{y_r y_{r+1}}.$$

Again using $2xx_k = x_{k+1}+\beta x_{k-1}$, we can derive

4. $$\sum_{k=r+1}^{\infty}\frac{2xx_k}{x_{k-1}x_{k+1}}=\sum_{k=r+1}^{\infty}\left(\frac{1}{x_{k-1}}+\frac{\beta}{x_{k+1}}\right).$$

If $ty^2 = 1+x^2$ and $y=1$, then we have

$$\sum_{k=r+1}^{\infty}\frac{x_k}{x_{k-1}x_{k+1}}=\frac{1}{2x}\left(\frac{1}{x_r}+\frac{1}{x_{r+1}}\right),$$

where x_k is given by equation (9). Similarly, from the recurrence relation $2xy_k = y_{k+1}+\beta y_{k-1}$, we have

5. $$\sum_{k=r+1}^{\infty}\frac{2xy_k}{y_{k-1}y_{k+1}}=\sum_{k=r+1}^{\infty}\left(\frac{1}{y_{k-1}}+\frac{\beta}{y_{k+1}}\right).$$

In particular, if $ty^2 = 1+x^2$ and $y=1$, then $\beta=-1$ and the above result becomes

$$\sum_{k=r+1}^{\infty}\frac{y_k}{y_{k-1}y_{k+1}}=\frac{1}{2x}\left(\frac{1}{y_r}+\frac{1}{y_{r+1}}\right),$$

where y_k is now given by equation (9).

6. Now we generalize the results of items 2 and 3 of this section; we shall show that

$$\sum_{k=2}^{\infty}\frac{1}{x_{(k+1)r}}\left(\frac{2x_r}{x_{(k-1)r}}-\frac{\beta^r+1}{x_{kr}}\right)=\frac{1}{x_r x_{2r}}.$$

To show this, we consider the left-hand side of the above result:

$$\sum_{k=2}^{\infty}\frac{1}{x_{(k+1)r}}\left(\frac{2x_r x_{kr}-\beta^r x_{(k-1)r}-x_{(k-1)r}}{x_{(k-1)r}x_{kr}}\right).$$

The above result can be simplified by using property (v) of (10), with $m=rk$, $n=r$, that is, $x_{rk+r}+\beta^r x_{rk-r}=2x_{rk}x_r$. Then the above expression becomes

$$\sum_{k=2}^{\infty}\frac{1}{x_{(k+1)r}}\left(\frac{x_{(k+1)r}-x_{(k-1)r}}{x_{(k-1)r}x_{kr}}\right),$$

which reduces to

$$\sum_{k=2}^{\infty}\left(\frac{1}{x_{(k-1)r}x_{kr}}-\frac{1}{x_{kr}x_{(k+1)r}}\right),$$

which, when summed over k, reduces to $1/(x_r x_{2r})$. Similarly, we show that

$$\sum_{k=2}^{\infty}\frac{1}{y_{(k+1)r}}\left(\frac{2x_r}{y_{(k-1)r}}-\frac{\beta^r+1}{y_{kr}}\right)=\frac{1}{y_r y_{2r}}.$$

7. Let us now generalize a property that holds for Fibonacci series [4]. For $t>0$, consider the series:

$$S=\sum_{k=2}^{\infty}\frac{\beta^{2^{k-1}-2}y_2}{y_{2^k}}=\frac{y_2}{y_4}+\frac{\beta^2y_2}{y_8}+\frac{\beta^6y_2}{y_{16}}+\cdots.$$

Denote

$$S_n=\frac{y_2}{y_4}+\frac{\beta^2y_2}{y_8}+\frac{\beta^6y_2}{y_{16}}+\cdots+\frac{\beta^{2^{n-1}-2}y_2}{y_{2^n}}.$$

By induction we shall show that

$$S_n=\frac{y_{2^n-2}}{y_{2^n}}. \tag{14}$$

Note that $y_{2^n}=0$ implies that either $x=0$ or $y=0$ from the Binet form for y_{2^n}. Therefore, we shall assume that $y_{2^n}\neq 0$. Observe also that equation (12) is true for $n=2,3$. Consider

$$S_{n+1}=S_n+\frac{\beta^{2^n-2}y_2}{y_{2^{n+1}}}=\frac{y_{2^n-2}}{y_{2^n}}+\frac{\beta^{2^n-2}y_2}{y_{2^{n+1}}}=\frac{y_{2^{n+1}}y_{2^n-2}+y_{2^n}\beta^{2^n-2}y_2}{y_{2^n}y_{2^{n+1}}}.$$

Use the property $y_{2m}=2x_my_m$, with $m=2^{n+1}=2(2^n)$, to get

$$S_{n+1}=\frac{2x_{2^n}y_{2^n}y_{2^n-2}+y_{2^n}\beta^{2^n-2}y_2}{y_{2^n}y_{2^{n+1}}}=\frac{2x_{2^n}y_{2^n-2}+\beta^{2^n-2}y_2}{y_{2^{n+1}}}.$$

Now recall property (viii) of equation (10), $y_{m+p}-\beta^py_{m-p}=2x_my_p$, and in it set $m=2^n$, $p=2^n-2$. We then have

$$S_{n+1}=\frac{y_{2^{n+1}-2}}{y_{2^{n+1}}},$$

which completes the induction. Therefore, for $t>0$, we have

$$S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{y_{2^n-2}}{y_{2^n}}=\xi^{-2}=\frac{1}{(x+y\sqrt{t})^2}.$$

6. CONVOLUTIONS FOR x_n AND y_n

Given two homogeneous polynomial sequences $a_n(x,y)$ and $b_n(x,y)$ in two variables x and y, where n is an integer ≥ 1, their *first convolution sequence* is defined by

$$(a_n*b_n)^{(1)}=\sum_{j=1}^{n}a_jb_{n+1-j}=\sum_{j=1}^{n}b_ja_{n+1-j}.$$

In the above definition, we have written $a_n=a_n(x,y)$ and $b_n=b_n(x,y)$. Denote $x_n*x_n=X_n^{(1)}$, $y_n*y_n=Y_n^{(1)}$, $2x_n*y_n=y_n^{(1)}$, and $X_n^{(1)}+tY_n^{(1)}=x_n^{(1)}$. To determine these convolutions, we use the matrix properties of B, namely,

$$\begin{bmatrix}x & y\\ ty & x\end{bmatrix}^{n+1}=\begin{bmatrix}x_{n+1} & y_{n+1}\\ ty_{n+1} & x_{n+1}\end{bmatrix}=B^{n+1}=B^jB^{n+1-j}=\begin{bmatrix}x_j & y_j\\ ty_j & x_j\end{bmatrix}\begin{bmatrix}x_{n+1-j} & y_{n+1-j}\\ ty_{n+1-j} & x_{n+1-j}\end{bmatrix}.$$

Let

$$B_n^{(1)} = \sum_{j=1}^{n} B_j B_{n+1-j} = \sum_{j=1}^{n} B^{n+1}.$$

Note that $B^n = B_n$. We prefer using the subscript notation. Since $\sum_{j=1}^{n} B_{n+1} = nB_{n+1}$, we have

$$nB_{n+1} = \sum_{j=1}^{n} \begin{bmatrix} x_j & y_j \\ ty_j & x_j \end{bmatrix} \begin{bmatrix} x_{n+1-j} & y_{n+1-j} \\ ty_{n+1-j} & x_{n+1-j} \end{bmatrix}.$$

Let $\sum_{j=1}^{n} = \sum$, then the above result can be written

$$nB_{n+1} = \begin{bmatrix} \sum x_j x_{n+1-j} + t\sum y_j y_{n+1-j} & \sum x_j y_{n+1-j} + \sum y_j x_{n+1-j} \\ t\left(\sum x_j y_{n+1-j} + \sum y_j x_{n+1-j}\right) & \sum x_j x_{n-j} + t\sum y_j y_{n+1-j} \end{bmatrix},$$

or

$$nB_{n+1} = \begin{bmatrix} x_n * x_n + ty_n * y_n & 2x_n * y_n \\ 2tx_n * y_n & x_n * x_n + ty_n * y_n \end{bmatrix} = \begin{bmatrix} x_n^{(1)} & y_n^{(1)} \\ ty_n^{(1)} & x_n^{(1)} \end{bmatrix} = B_n^{(1)}.$$

Therefore, we have

$$x_n^{(1)} = nx_{n+1}, \quad y_n^{(1)} = ny_{n+1}.$$

The above result can be extended to the k^{th} convolution by defining

$$B_n^{(k)} = \sum_{j=1}^{n} B_j (B^{(k-1)})_{n+1-j}.$$

Now we shall show that

$$B_n^{(k)} = \binom{n+k-1}{k} B_{n+k}.$$

We shall prove the result by induction on k. Since $B^{(1)} = nB_{n+1}$, the result is true for $k = 1$. Now consider

$$\begin{aligned} B_n^{(k+1)} &= \sum B_j B_{n+1-j}^{(k)} = \sum B_{n+1-j} (B^{(k)})_j \\ &= \sum B_{n+1-j} \binom{j+k-1}{k} B_{j+k} = B_{n+k+1} \sum \binom{j+k-1}{k} = \binom{n+k}{k+1} B_{n+k+1}, \end{aligned}$$

which completes the induction.

From the above results, we write the k^{th} convolution of x_n and y_n:

$$x_n^{(k)} = \binom{n+k-1}{k} x_{n+k}, \quad y_n^{(k)} = \binom{n+k-1}{k} y_{n+k}. \tag{15}$$

Also, from properties (v) and (vi) of (10), we have

$$2X^{(1)} = nx_{n+1} + \frac{\beta y_n}{y}, \quad 2tY^{(1)} = nx_{n+1} - \frac{\beta y_n}{y}, \tag{16}$$

which can be written in the form

$$2X^{(1)} = x_n^{(1)} + \frac{\beta y_n}{y}, \quad 2tY^{(1)} = x_n^{(1)} - \frac{\beta y_n}{y}.$$

We can also extend the above result to the k^{th} convolution of x_n and y_n, namely,

$$x_n * x_n^{(k)},\ y_n * y_n^{(k)},\ x_n * y_n^{(k)},\ y_n * x_n^{(k)}.$$

Using the results (10)(v), (15), and (16), and some computation, we obtain

$$2x_n * x_n^{(k)} = \binom{n+k}{k+1} x_{n+k+1} + \sum_{j=1}^{n} \binom{j+k-1}{k} \beta^{j+k} x_{n+1-2j-k}.$$

Similarly, we have

$$2ty_n * y_n^{(k)} = \binom{n+k}{k+1} x_{n+k+1} - \Sigma \binom{j+k-1}{k} \beta^{j+k} x_{n+1-2j-k},$$

$$2x_n^{(k)} * y_n = \binom{n+k}{k+1} y_{n+k+1} + \Sigma \binom{j+k-1}{k} \beta^{j+k} y_{n+1-2j-k},$$

$$2x_n * y_n^{(k)} = \binom{n+k}{k+1} y_{n+k+1} - \Sigma \binom{j+k-1}{k} \beta^{j+k} y_{n+1-2j-k}.$$

What we have seen here is but a sample of the properties displayed by the versatile matrix B. We are sure there are many more.

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JACOBSTHAL REPRESENTATION NUMBERS

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1. INTRODUCTION

Two sequences of numbers concern us, namely, the *Jacobsthal sequence* $\{J_n\}$ (see [4]) defined by

$$J_{n+2} = J_{n+1} + 2J_n, \qquad J_0 = 0, J_1 = 1, \ n \geq 0, \tag{1.1}$$

and the *Jacobsthal-Lucas sequence* $\{j_n\}$ defined by

$$j_{n+2} = j_{n+1} + 2j_n, \qquad j_0 = 2, j_1 = 1, \ n \geq 0. \tag{1.2}$$

Applications of these two sequences to curves are given in [4]. Sequence (1.1) appears in [11], but (1.2) does not.

From (1.1) and (1.2) we thus have the following tabulation for the *Jacobsthal numbers* J_n and the *Jacobsthal-Lucas numbers* j_n:

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
J_n	0	1	1	3	5	11	21	43	85	171	341	$\cdots$
j_n	2	1	5	7	17	31	65	127	257	511	1025	$\cdots$

(1.3)

When required, we can extend these sequences through negative values of n by means of the recurrences (1.1) and (1.2). Observe that all the J_n and j_n (except j_0) are odd, by virtue of the definitions.

Recurrences (1.1) and (1.2) involve the characteristic equation

$$x^2 - x - 2 = 0 \tag{1.4}$$

with roots

$$\alpha = 2, \quad \beta = -1 \tag{1.5}$$

so that

$$\alpha + \beta = 1, \quad \alpha\beta = -2, \quad \alpha - \beta = 3. \tag{1.6}$$

Wherever it is sensible to do so, we will replace α, β by $2, -1$, respectively.

Explicit closed form expressions for J_n and j_n are $(n \geq 1)$

$$J_n = \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-r}{r} 2^r \tag{1.7}$$

(see [3]) and

$$j_n = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-r} \binom{n-r}{r} 2^r. \tag{1.8}$$

Induction on n provides the required proofs.

In the theory of minimal and maximal representations of nonnegative integers by elements of a sequence $\{a_n\}$ (e.g., Fibonacci or Pell numbers; see [2], [7], [8]), we discover the importance of a new sequence whose members are those integers that can be represented both minimally and maximally by a sum of elements of $\{a_n\}$ for which the coefficients are all unity.

It is the object of this article to investigate the corresponding new sequences associated with $\{J_n\}$ and $\{j_n\}$.

But first we establish a few basic properties of the sequences (1.1) and (1.2), some of which will find subsequent application as our theme develops.

2. BASIC PROPERTIES OF THE JACOBSTHAL NUMBERS

Initially, these properties enter our mathematical Noah's Ark in pairs, as did (1.7) and (1.8). Standard techniques may be used to generate them and their numerous progeny, the most handsome of which is (2.9).

Generating functions

$$\sum_{i=1}^{\infty} J_i x^{i-1} = (1-x-2x^2)^{-1} \quad (\text{cf. } [3]), \tag{2.1}$$

$$\sum_{i=1}^{\infty} j_i x^{i-1} = (1+4x)(1-x-2x^2)^{-1}. \tag{2.2}$$

Binet forms

$$J_n = \frac{\alpha^n - \beta^n}{3} = \frac{1}{3}(2^n - (-1)^n), \tag{2.3}$$

$$j_n = \alpha^n + \beta^n = 2^n + (-1)^n. \tag{2.4}$$

Simson formulas

$$J_{n+1}J_{n-1} - J_n^2 = (-1)^n 2^{n-1}, \tag{2.5}$$

$$j_{n+1}j_{n-1} - j_n^2 = 9(-1)^{n-1}2^{n-1} = -9(J_{n+1}J_{n-1} - J_n^2). \tag{2.6}$$

Summation formulas

$$\sum_{i=2}^{n} J_i = \frac{J_{n+2}-3}{2}, \tag{2.7}$$

$$\sum_{i=1}^{n} j_i = \frac{j_{n+2}-5}{2}. \tag{2.8}$$

The significance of the lower bound for i in the useful formulas (2.7) and (2.8) will become apparent later in Sections 4 and 5.

Interrelationships

$$j_n J_n = J_{2n}, \tag{2.9}$$

$$j_n = J_{n+1} + 2J_{n-1}, \tag{2.10}$$

$$9J_n = j_{n+1} + 2j_{n-1}, \tag{2.11}$$

$$j_{n+1} + j_n = 3(J_{n+1} + J_n) = 3.2^n, \tag{2.12}$$

$$j_{n+1} - j_n = 3(J_{n+1} - J_n) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1}, \tag{2.13}$$

$$j_{n+1} - 2j_n = 3(2J_n - J_{n+1}) = 3(-1)^{n+1}. \tag{2.14}$$

$$2j_{n+1} + j_{n-1} = 3(2J_{n+1} + J_{n-1}) + 6(-1)^{n+1}, \tag{2.15}$$

$$j_{n+r} + j_{n-r} = 3(J_{n+r} + J_{n-r}) + 4(-1)^{n-r} = 2^{n-r}(2^{2r} + 1) + 2(-1)^{n-r}, \tag{2.16}$$

$$j_{n+r} - j_{n-r} = 3(J_{n+r} - J_{n-r}) = 2^{n-r}(2^{2r} - 1), \tag{2.17}$$

$$j_n = 3J_n + 2(-1)^n \quad \text{[cf. (2.12)]}, \tag{2.18}$$

$$3J_n + j_n = 2^{n+1}, \tag{2.19}$$

$$J_n + j_n = 2J_{n+1}, \tag{2.20}$$

$$\lim_{n\to\infty}\left(\frac{J_{n+1}}{J_n}\right) = \lim_{n\to\infty}\left(\frac{j_{n+1}}{j_n}\right) = 2, \tag{2.21}$$

$$\lim_{n\to\infty}\left(\frac{j_n}{J_n}\right) = 3, \tag{2.22}$$

$$j_{n+2}j_{n-2} - j_n^2 = -9(J_{n+2}J_{n-2} - J_n^2) = 9(-1)^n 2^{n-2}, \tag{2.23}$$

$$J_m j_n + J_n j_m = 2J_{m+n} \quad [m = n \to (2.9)], \tag{2.24}$$

$$j_m j_n + 9J_m J_n = 2j_{m+n}, \tag{2.25}$$

$$j_n^2 + 9J_n^2 = 2j_{2n} \quad [m = n \text{ in } (2.25)], \tag{2.26}$$

$$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}, \tag{2.27}$$

$$j_m j_n - 9J_m J_n = (-1)^n 2^{n+1} j_{m-n}, \tag{2.28}$$

$$j_n^2 - 9J_n^2 = (-1)^n 2^{n+2} \quad [m = n \text{ in } (2.28)]. \tag{2.29}$$

Economies of space (and cost!) preclude the addition of further properties which may be of lesser interest and value. Observe, however, that (2.9) is an important feature of $\{J_n\}$ and $\{j_n\}$, being analogous to $F_nL_n = F_{2n}$ and $P_nQ_n = P_{2n}$ for Fibonacci and Lucas numbers, and Pell and Pell-Lucas numbers, respectively. One might remark, in passing that the infinite limit of our $\frac{1}{2}Q_n / P_n$ [cf. (2.22)] is mentioned in [12] in dealing with irrationality.

Associated Sequences

Invoking [6], we define the k^{th} *associated sequences* $\{J_n^{(k)}\}$ and $\{j_n^{(k)}\}$ of $\{J_n\}$ and $\{j_n\}$ to be, respectively, given by ($k \geq 1$)

$$J_n^{(k)} = J_{n+1}^{(k-1)} + 2J_{n-1}^{(k-1)} \tag{2.30}$$

and

$$j_n^{(k)} = j_{n+1}^{(k-1)} + 2j_{n-1}^{(k-1)}, \tag{2.31}$$

where $J_n^{(0)} = J_n$, $j_n^{(0)} = j_n$. Accordingly,

$$J_n^{(1)} = j_n \quad \text{by (2.10)} \tag{2.32}$$

and

$$j_n^{(1)} = 9J_n \quad \text{by (2.11)} \tag{2.33}$$

are the generic members of the *first associated sequences* $\{J_n^{(1)}\}$ and $\{j_n^{(1)}\}$.

Deducing the following neat results is an easy matter on appeal to (2.10) and (2.11):

$$J_n^{(2m)} = 3^{2m} J_n, \tag{2.34}$$

$$J_n^{(2m+1)} = 3^{2m} j_n, \tag{2.35}$$

$$j_n^{(2m)} = 3^{2m} j_n, \tag{2.36}$$

$$j_n^{(2m-1)} = 3^{2m} J_n. \tag{2.37}$$

Expressed succinctly,

$$\left.\begin{array}{l} J_n^{(2m)} = j_n^{(2m-1)} \\ j_n^{(2m)} = J_n^{(2m+1)} \end{array}\right\}. \tag{2.38}$$

Analogous results to (2.34)-(2.37) for Fibonacci and Lucas numbers are stated in [6]. Pairs of results like these can be incorporated into a more general system for polynomials that extends to negative values of m and n. Material on this research has been submitted for publication.

3. JACOBSTHAL REPRESENTATION SEQUENCES

Later, in Section 4, the significance of the summations (2.7) and (2.8) in representation theory will be manifested.

Irrespective of this representation application, however, each of the two sequences (2.7) and (2.8)—now (3.1) and (3.2)—merits some discussion *per se*. Neither sequence appears in [11].

Write, for convenience,

$$\mathcal{J}_n = \sum_{i=2}^{n} J_i, \quad \mathcal{J}_0 = 0, \ \mathcal{J}_1 = 1 \tag{3.1}$$

and

$$\hat{j}_n = \sum_{i=1}^{n} j_i, \quad \hat{j}_0 = 0. \tag{3.2}$$

Consequently, we have the following tabulation for $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$ (in both of which the elements are alternatively odd and even):

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\mathcal{J}_n$	0	1	4	9	20	41	84	169	340	681	1364	$\cdots$
$\hat{j}_n$	0	1	6	13	30	61	126	253	510	1021	2046	$\cdots$

(3.3)

Simple detective work readily enables us to spot the recurrences (3.4) and (3.5) in (3.3), which we expect to be modeled on (1.1) and (1.2). As with J_n and j_n in Section 2, we arrange the basic features of $\mathcal{J}_n$ and $\hat{j}_n$ in pairs.

Recurrence relations

$$\mathcal{J}_{n+2} = \mathcal{J}_{n+1} + 2\mathcal{J}_n + 3, \tag{3.4}$$

$$\hat{j}_{n+2} = \hat{j}_{n+1} + 2\hat{j}_n + 5. \tag{3.5}$$

Generating functions

$$\sum_{i=1}^{\infty} \mathcal{J}_i x^{i-1} = (1+2x)(1-2x-x^2+2x^3)^{-1}, \tag{3.6}$$

$$\sum_{i=1}^{\infty} \hat{j}_i x^{i-1} = (1+4x)(1-2x-x^2+2x^3)^{-1}. \tag{3.7}$$

Binet forms

$$\mathcal{J}_n = \frac{J_{n+3}-3}{2} = \frac{2^{n+3}+(-1)^n-9}{6}, \tag{3.8}$$

$$\hat{j}_n = \frac{j_{n+2}-5}{2} = \frac{2^{n+2}+(-1)^n-5}{2}. \tag{3.9}$$

Simson formulas

$$\begin{aligned} \mathcal{J}_{n+1}\mathcal{J}_{n-1} - \mathcal{J}_n^2 &= 2^n\{(-1)^{n-1}-1\}+(-1)^n \\ &= -1 \text{ when } n \text{ is odd,} \end{aligned} \tag{3.10}$$

$$\begin{aligned} \hat{j}_{n+1}\hat{j}_{n-1} - \hat{j}_n^2 &= 2^{n-1}\{9(-1)^{n+1}-5\}+5(-1)^n \\ &= 2^{n+1}-5 \text{ when } n \text{ is odd.} \end{aligned} \tag{3.11}$$

Summations

$$\sum_{i=1}^{n} \mathcal{J}_i = \frac{\mathcal{J}_{n+2}-1-3(n+1)}{2}, \tag{3.12}$$

$$\sum_{i=1}^{n} \hat{j}_i = \frac{\hat{j}_{n+2}-1-5(n+1)}{2}. \tag{3.13}$$

Interrelationships

$$\mathcal{J}_{n+1} + 2\mathcal{J}_{n-1} = \hat{j}_{n+1} - 2, \tag{3.14}$$

$$\hat{j}_{n+1} + 2\hat{j}_{n-1} = 3(3\mathcal{J}_{n-1}+2), \tag{3.15}$$

$$\mathcal{J}_{2n} = 4J_{2n} = 4J_n j_n \quad \text{by (2.9)}, \tag{3.16}$$

$$\mathcal{J}_{2n+1} = 4J_{2n+1} - 3, \tag{3.17}$$

$$\hat{j}_{2n} = 2j_{2n} - 4 = 6J_{2n} \quad \text{by (2.18)}, \tag{3.18}$$

$$\hat{j}_{2n+1} = 2j_{2n+1} - 1, \tag{3.19}$$

$$\mathcal{J}_{n+1} + \mathcal{J}_n = \begin{cases} \hat{j}_{n+1} & n \text{ even,} \\ \hat{j}_{n+1} - 1 & n \text{ odd.} \end{cases} \tag{3.20}$$

$$\hat{j}_{n+1} + \hat{j}_n = 3.2^{n+1} - 5, \tag{3.21}$$

$$\mathcal{J}_n - \mathcal{J}_{n-1} = J_{n+1}, \tag{3.22}$$

$$\hat{j}_n - \hat{j}_{n-1} = j_n, \tag{3.23}$$

$$\mathcal{J}_n - \mathcal{J}_{n-2} = 2^n, \tag{3.24}$$

$$\hat{j}_n - \hat{j}_{n-2} = 3 \cdot 2^{n-1}, \tag{3.25}$$

$$\hat{j}_{n+r} - \hat{j}_{n-r} = \frac{3}{2}(\mathcal{J}_{n+r} - \mathcal{J}_{n-r}) = 6(J_{n+r} - J_{n-r}) = 2(j_{n+r} - j_{n-r}), \tag{3.26}$$

$$\mathcal{J}_{n+2}\mathcal{J}_{n-2} - \mathcal{J}_n^2 = 2^{n-1}\{(-1)^n - 9\}, \tag{3.27}$$

$$\hat{j}_{n+2}\hat{j}_{n-2} - \hat{j}_n^2 = 2^{n-2} \cdot 9\{(-1)^n - 5\}, \tag{3.28}$$

$$\lim_{n\to\infty}\left(\frac{\mathcal{J}_{n+1}}{\mathcal{J}_n}\right) = \lim_{n\to\infty}\left(\frac{\hat{j}_{n+1}}{\hat{j}_n}\right) = 2, \tag{3.29}$$

$$\lim_{n\to\infty}\left(\frac{\hat{j}_n}{\mathcal{J}_n}\right) = \frac{3}{2}, \tag{3.30}$$

$$3\mathcal{J}_n - 2\hat{j}_n = \begin{cases} 0 & n \text{ even,} \\ 1 & n \text{ odd,} \end{cases} \tag{3.31}$$

$$3\mathcal{J}_{2n} = 2\hat{j}_{2n} \quad \text{by (3.31)} \tag{3.32}$$

$$\mathcal{J}_n - J_n = \begin{cases} j_n - 1 & n \text{ odd,} \\ j_n - 2 & n \text{ even,} \end{cases} \tag{3.33}$$

$$\hat{j}_n - J_n = \frac{5}{2}(J_{n+1} - 1), \tag{3.34}$$

$$\mathcal{J}_n - j_n = \begin{cases} J_n - 1 & n \text{ odd,} \\ J_n - 2 & n \text{ even,} \end{cases} \tag{3.35}$$

$$\hat{j}_n - \mathcal{J}_n = J_{n+1} - 1. \tag{3.36}$$

Determinantal evaluations

$$\Delta_{\mathcal{J}} = \begin{vmatrix} \mathcal{J}_n & \mathcal{J}_{n+1} & \mathcal{J}_{n+2} \\ \mathcal{J}_{n+1} & \mathcal{J}_{n+2} & \mathcal{J}_{n+3} \\ \mathcal{J}_{n+2} & \mathcal{J}_{n+3} & \mathcal{J}_{n+4} \end{vmatrix} = 3(-1)^{n+1}2^{n+2} \tag{3.37}$$

and

$$\Delta_{\hat{j}} = \begin{vmatrix} \hat{j}_n & \hat{j}_{n+1} & \hat{j}_{n+2} \\ \hat{j}_{n+1} & \hat{j}_{n+2} & \hat{j}_{n+3} \\ \hat{j}_{n+2} & \hat{j}_{n+3} & \hat{j}_{n+4} \end{vmatrix} = 45(-1)^{n+1}2^{n+1} = \frac{15}{2}\Delta_{\mathcal{J}} \quad \text{by (3.37)}, \tag{3.38}$$

for which are required *inter alia*

$$\mathcal{J}_n \mathcal{J}_{n+3} - \mathcal{J}_{n+1}\mathcal{J}_{n+2} = 2^{n+1}\{(-1)^n - 3\} \tag{3.39}$$

and

$$\hat{j}_n\hat{j}_{n+3} - \hat{j}_{n+1}\hat{j}_{n+2} = 2^n \cdot 3\{3(-1)^n - 5\}. \tag{3.40}$$

With similar notation, it follows obviously from (1.1) and (1.2) that $\Delta_J = \Delta_j = 0$.

Our selection of properties of $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$ in (3.4)-(3.40) does not exhaust the many pleasant features of these research-friendly sequences. However, they do give a "flavor" to $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$. It might be noted that, on calculation,

$$\mathcal{J}_n\hat{j}_n \neq \mathcal{J}_{2n}. \tag{3.41}$$

[Because $\{\hat{j}_n\}$ is not a Lucas-type sequence as $\{j_n\}$ is, i.e., $\hat{j}_0 \neq 2$, the "classical" relation of the type (2.9) cannot hold. Indeed, the left-hand side of (3.41) is rather unlovely.] Divisibility properties of (3.16) and (3.18) might also be observed.

Associated Sequences

With notation for *associated sequences* of $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$ similar to that for $\{J_n\}$ and $\{j_n\}$ in (2.30)-(2.33), we derive

$$\mathcal{J}_n^{(1)} = \hat{j}_{n+1} - 2 \quad \text{by (3.14)} \tag{3.42}$$

and

$$\hat{j}_n^{(1)} = 3(3\mathcal{J}_{n-1} + 2) \quad \text{by (3.15)}. \tag{3.43}$$

Invoking (3.14) and (3.15), we have, eventually,

$$\mathcal{J}_n^{(2m)} = 3^{2m}\mathcal{J}_n, \tag{3.44}$$

$$\mathcal{J}_n^{(2m+1)} = 3^{2m}(\hat{j}_{n+1} - 2), \tag{3.45}$$

$$\hat{j}_n^{(2m)} = 3^{2m}\hat{j}_n, \tag{3.46}$$

$$\hat{j}_n^{(2m-1)} = 3^{2m-1}(3\mathcal{J}_{n-1} + 2). \tag{3.47}$$

More briefly,

$$\left.\begin{aligned} \mathcal{J}_n^{(2m)} &= \hat{j}_{n+1}^{(2m-1)} - 2\cdot 3^{2m-1} \\ \hat{j}_n^{(2m)} &= \mathcal{J}_{n-1}^{(2m+1)} + 2\cdot 3^{2m} \end{aligned}\right\}. \tag{3.48}$$

Both $\mathcal{J}_n^{(k)}$ and $\hat{j}_n^{(k)}$ are also expressible in terms of J_n and j_n, but this alternative produces slightly less attractive formulas.

Each of the sequences $\{\mathcal{J}_n^{(1)}\}$ and $\{\hat{j}_n^{(1)}\}$ in (3.42) and (3.43) may be regarded as a separate individual entity with a mathematical life of its own, as for $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$, leading *inter alia* to Binet forms, generating functions, Simson formulas, recurrence relations, summation formulas, and miscellaneous interrelationships of varying importance.

Graphs

Suppose we label a pair of rectangular Cartesian axes $\hat{j}$ $(= y)$ and $\mathcal{T}$ $(= x)$. Then (3.30), as n takes on its permissible values, the coordinates $\{\mathcal{T}_n, \hat{j}_n\}$ cluster about the line $y = \frac{3}{2}x$, appearing alternately on opposite sides of this line. Likewise (2.22), in a changed notation, the points (J_n, j_n) as n varies approximate to the line $y = 3x$.

4. JACOBSTHAL REPRESENTATION OF POSITIVE INTEGERS: $\{J_n\}$

Primarily, our concern now is to answer the question: "Can a positive integer N be represented as a sum of Jacobsthal numbers?"

Considerations of minimality and maximality of a representation do not enter into the argument at this stage. Nor does the possibility of uniqueness. Of course, for any minimal representation of N in terms of $\{J_n\}$, we should need

$$N = \sum_{i=2}^{\infty} \Pi_i J_i \quad (\Pi_i = 0, 1, 2) \tag{4.1}$$

subject to the criterion

$$\Pi_i = 2 \Rightarrow \Pi_{i+1} = 0 \tag{4.2}$$

by virtue of (1.1). (Cf. the corresponding Pell condition for minimality [7].)

Why the lower bound $i = 2$ in (4.1)?

Recall from (1.3) that $J_1 = J_2 = 1$. To avoid problems with this two-fold designation of 1, we will omit J_1 from our deliberations and therefore deal only with $\{J_n\}_{n\geq 2}$.

Accordingly, write

$$J_n' = J_{n+1} \tag{4.3}$$

(i.e., $J_1' = J_2 = 1, \ldots,$ with $J_0' = 1$) and

$$\Pi_i' = \Pi_{i+1}. \tag{4.4}$$

One has from (2.14), adjusted by (4.3), that

$$2J_n' = J_{n+1}' - 1 < J_{n+1}', \quad n \text{ odd}, \tag{2.14a}$$

$$2J_n' = J_{n+1}' + 1 > J_{n+1}', \quad n \text{ even}. \tag{2.14b}$$

For the set $\{S_k\}$ of digits 0, 1, 2 of length k,

$$(\Pi_1', \Pi_2', \ldots, \Pi_k'), \tag{4.5}$$

let us use the following symbolism:

$$\left.\begin{aligned} N_k^{\max} &= \text{the largest integer in } S_k \\ N_k^{\min} &= \text{the smallest integer in } S_k \\ R_k &= \text{the range of integers in } S_k \\ I_k &= \text{the number of integers in } S_k \end{aligned}\right\}. \tag{4.6}$$

Now (Table 2), in each block of k coefficient digits, the smallest number is necessarily given by

$$(0, 0, 0, \ldots, 0, 1) \tag{4.7}$$

i.e.,

$$N_k^{\min} = J'_k \quad \text{by (4.7)}, \tag{4.8}$$

and the largest number by either

$$(0, 0, 0, \dots, 0, 2), \quad k \text{ odd}, \tag{4.9}$$

or

$$(1, 1, 1, \dots, 1, 1), \quad k \text{ even}. \tag{4.10}$$

Clearly, then,

$$N_k^{\max} = 2J'_k = J'_{k+1} - 1 \quad \text{by (4.9), (2.14a)}, \quad k \text{ odd}, \tag{4.11}$$

or

$$\begin{aligned} N_k^{\max} &= \sum_{i=1}^{k} J'_i = \mathcal{T}_k \quad \text{by (4.10), (3.1)}, \quad k \text{ even}, \\ &= \frac{J'_{k+2} - 3}{2} \quad \text{by (2.7)} \\ &= J'_{k+1} - 1 \quad \text{by (1.1), (2.14b)}, \end{aligned} \tag{4.12}$$

i.e.,

$$N_k^{\max} = J'_{k+1} - 1 \quad \text{for all } k. \tag{4.13}$$

From (4.8) and (4.13), we derive

$$\begin{aligned} I_k &= (J'_{k+1} - 1) - (J'_k - 1) \quad \text{obviously} \\ &= J'_{k+1} - J'_k \\ &= 2J'_{k-1} \quad \text{by (1.1)}. \end{aligned} \tag{4.14}$$

Thus, by (4.8) and (4.14),

Lemma 1:

$$J'_k \le N \le J'_{k+1} - 1. \tag{4.15}$$

For example, $J'_{10}\ (= 683) \le N = 1{,}000 \le J'_{11} - 1\ (= 1{,}367 - 1 = 1{,}366)$.

Lemma 2: k is uniquely determined by N.

For instance, $N = 1{,}000 \Rightarrow k = 10$.

Therefore, it has been shown that

Theorem 1: Every positive integer N has a representation of the form

$$N = \sum_{i=1}^{\infty} \Pi'_i J'_i \tag{4.16}$$

where $\Pi'_i = 0, 1, 2$, and $\Pi'_i = 2 \Rightarrow \Pi'_{i+1} = 0$.

Details of the discussion encapsulated in Theorem 1 are assembled, in the symbolism of (4.6), in Table 1.

TABLE 1. Data for Representations Involving $\{J_n\}_{n\geq 2}$

k	S_k	R_k	$N_k^{\min}$	$N_k^{\max}$	I_k
1	S_1	$1, 2$	J_1'	$J_2' - 1$	$2\ (= 2J_0')$
2	S_2	$3, 4$	J_2'	$J_3' - 1$	$2\ (= 2J_1')$
3	S_3	$5, \ldots, 10$	J_3'	$J_4' - 1$	$6\ (= 2J_2')$
4	S_4	$11, \ldots, 20$	J_4'	$J_5' - 1$	$10\ (= 2J_3')$
5	S_5	$21, \ldots, 42$	J_5'	$J_6' - 1$	$22\ (= 2J_4')$
6	S_6	$43, \ldots, 84$	J_6'	$J_7' - 1$	$42\ (= 2J_5')$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	S_k	$J_k', \ldots, J_{k+1}'$	J_k'	$J_{k+1}' - 1$	$2J_{k-1}'$

Specific information for the representation summarized in Table 1 is provided in Table 2. Recall the notation (4.3).

While based on the minimality criterion (4.2), our representation is by no means unique. As a simple illustration, $N = 6$ and $N = 7$ are also given by the sequences of coefficients $(0, 2)$ and $(1, 2)$, respectively. But our choice of representation of an integer N consistently includes the greatest $J_n' : J_n' \leq N$, e.g., $6 = J_1' + J_3'\ (= 1 + 5)$ rather than $6 = 2J_2'\ (= 2 \times 3)$ and $7 = 2J_1' + J_3'$ $(= 2 + 5)$ rather than $7 = J_1' + 2J_2'\ (= 1 + 6)$. Infinitely many similar situations exist, along with variations of them.

Our chosen representation in Table 2 has the virtues of simplicity and methodical structure. Because of the usual patterns apparent in Table 2, we may refer to this representation as a *patterned representation*.

For a detailed, but different approach to the representation of integers by means of Jacobsthal numbers, one might consult [1], which investigates a "special" sequence. This sequence is indeed our Jacobsthal sequence, though this cognomen is never alluded to.

5. JACOBSTHAL REPRESENTATION OF POSITIVE INTEGERS: $\{j_n\}$

Turning now to $\{j_n\}$, we may generally parallel the arguments used in Section 4, though here we need to commence the sequence with $j_0\ (= 2)$, for otherwise there is no representation possible for the numbers 3 and 4.

Key results corresponding to (2.14a) and (2.14b) are, from (2.14),

$$2j_n = j_{n+1} - 3 < j_{n+1}, \quad n \text{ odd}, \tag{2.14c}$$

and

$$2j_n = j_{n+1} + 3 > j_{n+1}, \quad n \text{ even}. \tag{2.14d}$$

Symbolism used in Section 4 for $\{J_n\}$ will now, for $\{j_n\}$, be replaced by non-capital letters. However, the set $\{s_k\}$ of digits 0, 1, 2 analogous to (4.5) must now become

$$(\pi_0, \pi_1, \pi_2, \ldots, \pi_k), \tag{5.1}$$

which is of length $k + 1$.

Adapting the notation in (4.6), we may proceed to establish and arrange the data in Table 3, using methods similar to those in the previous section.

TABLE 2. A Representation of Integers $1 \le N \le 100$ by Jacobsthal Numbers J_n

	J_2	J_3	J_4	J_5	J_6	J_7	J_8		J_2	J_3	J_4	J_5	J_6	J_7	J_8
1	1							51		1	1			1	
2	2							52	1	1	1			1	
3		1						53			2			1	
4	1	1						54				1		1	
5			1					55	1			1		1	
6	1		1					56	2			1		1	
7	2		1					57		1		1		1	
8		1	1					58	1	1		1		1	
9	1	1	1					59			1	1		1	
10			2					60	1		1	1		1	
11				1				61	2		1	1		1	
12	1			1				62		1	1	1		1	
13	2			1				63	1	1	1	1		1	
14		1		1				64					1	1	
15	1	1		1				65	1				1	1	
16			1	1				66	2				1	1	
17	1		1	1				67		1			1	1	
18	2		1	1				68	1	1			1	1	
19		1	1	1				69			1		1	1	
20	1	1	1	1				70	1		1		1	1	
21					1			71	2		1		1	1	
22	1				1			72		1	1		1	1	
23	2				1			73	1	1	1		1	1	
24		1			1			74			2		1	1	
25	1	1			1			75				1	1	1	
26			1		1			76	1			1	1	1	
27	1		1		1			77	2			1	1	1	
28	2		1		1			78		1		1	1	1	
29		1	1		1			79	1	1		1	1	1	
30	1	1	1		1			80			1	1	1	1	
31			2		1			81	1		1	1	1	1	
32				1	1			82	2		1	1	1	1	
33	1			1	1			83		1	1	1	1	1	
34	2			1	1			84	1	1	1	1	1	1	
35		1		1	1			85							1
36	1	1		1	1			86	1						1
37			1	1	1			87	2						1
38	1		1	1	1			88		1					1
39	2		1	1	1			89	1	1					1
40		1	1	1	1			90			1				1
41	1	1	1	1	1			91	1		1				1
42					2			92	2		1				1
43						1		93		1	1				1
44	1					1		94	1	1	1				1
45	2					1		95			2				1
46		1				1		96				1			1
47	1	1				1		97	1			1			1
48			1			1		98	2			1			1
49	1		1			1		99		1		1			1
50	2		1			1		100	1	1		1			1

TABLE 3. Data for Representations Involving $\{j_n\}_{n\geq 0}$

k	s_k	r_k	$N_k^{\min}$	$N_k^{\max}$	i_k
1	s_1	$1, \ldots, 4$	j_1	$j_2 - 1$	$4\,(=2j_0)$
2	s_2	$5, 6$	j_2	$j_3 - 1$	$2\,(=2j_1)$
3	s_3	$7, \ldots, 16$	j_3	$j_4 - 1$	$10\,(=2j_2)$
4	s_4	$17, \ldots, 30$	j_4	$j_5 - 1$	$14\,(=2j_3)$
5	s_5	$31, \ldots, 64$	j_5	$j_6 - 1$	$34\,(=2j_4)$
6	s_6	$65, \ldots, 126$	j_6	$j_7 - 1$	$62\,(=2j_5)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	s_k	$j_k, \ldots, j_{k+1} - 1$	j_k	$j_{k+1} - 1$	$2j_{k-1}$

Now (Table 4), in each block of $k+1$ coefficient digits, the smallest number must be given by

$$(0, 0, 0, \ldots, 0, 0, 1), \tag{5.2}$$

i.e.,

$$N_k^{\min} = j_k \quad \text{by (5.2)}, \tag{5.3}$$

and the largest by either

$$(1, 0, 0, \ldots, 0, 0, 2), \quad k \text{ odd}, \tag{5.4}$$

or

$$(0, 1, 1, 1, \ldots, 1, 1, 1), \quad k \text{ even}. \tag{5.5}$$

Then

$$\begin{aligned} N_k^{\max} &= 2j_k + 2, \quad k \text{ odd}, \\ &= j_{k+1} - 3 + 2 \quad \text{by (2.14c)} \\ &= j_{k+1} - 1, \end{aligned} \tag{5.6}$$

while

$$\begin{aligned} N_k^{\max} &= \sum_{i=1}^{k} j_i = \hat{j}_k \quad \text{by (3.2)}, \quad n \text{ even}, \\ &= \frac{j_{k+2} - 5}{2} \quad \text{by (2.8)} \\ &= \frac{2j_{k+1} + 3 - 5}{2} \quad \text{by (1.2), (2.14d)} \\ &= j_{k+1} - 1, \end{aligned} \tag{5.7}$$

i.e., for all k,

$$N_k^{\max} = j_{k+1} - 1. \tag{5.8}$$

Thus,

Lemma 3: $j_k \leq N \leq j_{k+1} - 1$.

Lemma 4: k is uniquely determined by N.

Examples: $j_9\,(=511) \leq N = 1{,}000 \leq j_{10} - 1\,(=1{,}025 - 1 = 1{,}024)$; $N = 1{,}000 \Rightarrow k = 9$.

TABLE 4. A Representation of Integers $1 \le N \le 100$ by Jacobsthal-Lucas Numbers j_n

	j_0	j_1	j_2	j_3	j_4	j_5	j_6		j_0	j_1	j_2	j_3	j_4	j_5	j_6
1		1						51	1	1			1	1	
2	1							52	1	2			1	1	
3	1	1						53			1		1	1	
4	1	2						54		1	1		1	1	
5			1					55				1	1	1	
6		1	1					56		1		1	1	1	
7				1				57	1			1	1	1	
8		1		1				58	1	1		1	1	1	
9	1			1				59	1	2		1	1	1	
10	1	1		1				60			1	1	1	1	
11	1	2		1				61		1	1	1	1	1	
12			1	1				62						2	
13		1	1	1				63		1				2	
14				2				64	1					2	
15		1		2				65							1
16	1			2				66		1					1
17					1			67	1						1
18		1			1			68	1	1					1
19	1				1			69	1	2					1
20	1	1			1			70			1				1
21	1	2			1			71		1	1				1
22			1		1			72				1			1
23		1	1		1			73		1		1			1
24				1	1			74	1			1			1
25		1		1	1			75	1	1		1			1
26	1			1	1			76	1	2		1			1
27	1	1		1	1			77			1	1			1
28	1	2		1	1			78		1	1	1			1
29			1	1	1			79				2			1
30		1	1	1	1			80		1		2			1
31						1		81	1			2			1
32		1				1		82					1		1
33	1					1		83		1			1		1
34	1	1				1		84	1				1		1
35	1	2				1		85	1	1			1		1
36			1			1		86	1	2			1		1
37		1	1			1		87			1		1		1
38				1		1		88		1	1		1		1
39		1		1		1		89				1	1		1
40	1			1		1		90		1		1	1		1
41	1	1		1		1		91	1			1	1		1
42	1	2		1		1		92	1	1		1	1		1
43			1	1		1		93	1	2		1	1		1
44		1	1	1		1		94			1	1	1		1
45				2		1		95		1	1	1	1		1
46		1		2		1		96						1	1
47	1			2		1		97		1				1	1
48					1	1		98	1					1	1
49		1			1	1		99	1	1				1	1
50	1				1	1		100	1	2				1	1

Theorem 2: Every positive integer N has a representation of the form

$$N = \sum_{i=1}^{\infty} \pi_i j_i, \tag{5.9}$$

where $\pi_i = 0, 1, 2,$ and $\pi_i = 2 \Rightarrow \pi_{i+1} = 0$.

Actual details of the j_n-representations are supplied in Table 4 above. As in the case of $\{J_n\}$, these representations contain the criterion for minimality [i.e., condition (4.2) adjusted to π_i], but our chosen representation is nonunique, being selected for convenience to demonstrate that a representation does exist. For instance, we may also have the following representations (cf. Table 5), in which dots denote zeros:

TABLE 5

$N=$ 34	$\cdot\ \cdot\ \cdot\ \cdot$ 2	45	$\cdot$ 1 2 $\cdot$ 2
35	$\cdot$ 1 $\cdot\ \cdot$ 2	46	1 $\cdot$ 2 $\cdot$ 2
36	1 $\cdot\ \cdot\ \cdot$ 2	48	1 1 2 $\cdot$ 2
$\cdot$	$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$	48	$\cdot\ \cdot\ \cdot\ \cdot$ 1 1

The tabulation in Table 4 again expresses a *patterned representation.*

6. FINALE

A mild investigation into the possibility of maximum representations was essayed, but no conclusions are offered here. Nevertheless, we reiterate that both $\{J_n\}$ and $\{\hat{j}_n\}$ correspond to the *MinMax sequences* for Pell numbers that were introduced and examined in [7].

Our presentation of some of the basic features of Jacobsthal representations is meant to whet the appetite for further analyses of their properties. Among the opportunities available for exploration are, at least, the following three:

(a) polynomials $\{\mathcal{J}_n(x)\}$ and $\{\hat{j}_n(x)\}$ which generalize $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$,
(b) generalizations of (3.4) and (3.5) when the additive constant is k, and
(c) negatively-subscripted Jacobsthal numbers $\{\mathcal{J}_{-n}\}$ and $\{\hat{j}_{-n}\}$.

Preliminary studies of these topics have been completed by the author, and papers prepared.

For a selection of references relevant to our treatment of representations, one may consult [5]. (Reference [10], though not strictly germane to this paper, is included to remedy an omission in the choice in [5].)

Historical Note

The origins of Jacobsthal numbers (1), 1, 3, 5, 11, 21, ..., where the first term in (1.3) does not occur, predate Jacobsthal's article [9]. Indeed [11], they and their *loi de récurrence* (and Binet form) are traceable, in a trigonometrical setting, to *Nouvelle Correspondance Mathématique*, Vol. 6 (1880), page 146, being there associated with the name of Brocard.

Another, but much later, reference [11] is to page 12 of Vol. 26 (1963) of *Eureka*, the journal of the Archimedeans (Cambridge University Mathematical Society). Here, the first term 1 in (1.3) is given; however, the occurrence of the Jacobsthal numbers is in a purely recreational context, namely: given the first six nonzero terms of (1.3), determine the next two numbers in the sequence.

Jacobsthal polynomials [3], [9] are natural algebraic extensions of their numerical counterparts. Knowing the long history of many mathematical ideas, we should be mildly surprised if the first use of the Jacobsthal numbers did not antedate the year 1880.

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DIAGONALIZATION OF THE BINOMIAL MATRIX

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1. INTRODUCTION

The results of this paper assume a familiarity with linear algebra. A good reference for the results assumed here is [1].

As is well known, the Fibonacci numbers may be generated in the following manner. Let $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Then

$$A^h \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{h+1} \\ F_h \end{bmatrix}.$$

If we diagonalize A as

$$A = BDB^{-1} = \begin{bmatrix} \gamma & -1/\gamma \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & -1/\gamma \end{bmatrix} \begin{bmatrix} \gamma & -1/\gamma \\ 1 & 1 \end{bmatrix}^{-1},$$

where $\gamma = \frac{1+\sqrt{5}}{2}$ is the golden ratio, then from

$$\begin{bmatrix} F_{h+1} \\ F_h \end{bmatrix} = A^h \begin{bmatrix} 1 \\ 0 \end{bmatrix} = BD^hB^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

one obtains the formula

$$F_h = \frac{\gamma^h - (-1/\gamma)^h}{\sqrt{5}}. \tag{1.1}$$

More generally, if $f(x) = x^m - s_1 x^{m-1} - \cdots - s_m$ is a polynomial with distinct roots α_i, and C is the companion matrix of $f(x)$,

$$C = \begin{bmatrix} s_1 & s_2 & \cdots & s_{m-1} & s_m \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

then $v_h = C^h v_0$ generates the recurrence sequence with initial values given by v_0 and recurrence polynomial $f(x)$. Again, we can diagonalize $C = BDB^{-1}$ and obtain the formula

$$a_h = \sum A_i \alpha_i^h, \tag{1.2}$$

for some $A_i \in \mathbb{C}$.

In fact, there is nothing special about companion matrices here. If M is any square matrix over $\mathbb{Z}$ (say) and $v_h = M^h v_0$, then, as we shall prove in the next section, each component of v_h is a recurrence sequence with recurrence polynomial equal to the characteristic polynomial of M.

Now let us examine some generalizations of the relation above for the Fibonacci numbers. One way to generalize the matrix A above is to the binomial matrix. For example, consider

$$A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

By following the above method, we find that the characteristic polynomial of A_3 is $x^3 - 2x^2 - 2x + 1$; the eigenvalues of A_3 are $1/\gamma^2, -1, \gamma^2$; and

$$A_3^h v_0 = \begin{bmatrix} F_{h+1}^2 \\ F_h F_{h+1} \\ F_h^2 \end{bmatrix}. \tag{1.3}$$

We prove in this article a generalization of this observation. We find that the eigenvalues of the n-by-n binomial matrix are powers of the golden ratio. As a consequence, we shall derive the generalization of (1.3) above. Moreover, we show how explicitly to diagonalize this binomial matrix, and we give recurrence relations for the characteristic polynomials.

More precisely, let $\gamma = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Let $A_n = [a_{i,j}]$ be the "inverted" (or upside-down) binomial matrix (Pascal's triangle):

$$a_{ij} = \begin{cases} 0 & \text{if } i+j > n+1, \\ \binom{n-i}{j-1} & \text{otherwise.} \end{cases}$$

Let $W_n = \{\gamma^{n-1-i}(-1/\gamma)^i\}_{i=0}^{n-1}$ and let $Q_n(x) = \prod_{w \in W_n}(x-w)$. Let D_n be the diagonal matrix whose diagonal entries are the elements of W_n listed in decreasing order according to size of the absolute value. Let E_n be the eigenvector matrix of A_n with column vectors listed in decreasing order of absolute value of the corresponding eigenvalues, and with its columns scaled so that the bottom row is all 1's. So, for example, for $n = 5$ we have

$$A_5 = \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_5 = \begin{bmatrix} \gamma^4 & 0 & 0 & 0 & 0 \\ 0 & -\gamma^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\gamma^{-2} & 0 \\ 0 & 0 & 0 & 0 & \gamma^{-4} \end{bmatrix},$$

$$E_5 = \begin{bmatrix} \gamma^4 & -\gamma^2 & 1 & -\gamma^{-2} & \gamma^{-4} \\ \gamma^3 & -\gamma^{-1}/4 & -1/2 & \gamma/4 & -\gamma^{-3} \\ \gamma^2 & \gamma/2 & -1/6 & -\gamma^{-1}/2 & \gamma^{-2} \\ \gamma & \gamma^3/4 & 1/2 & -\gamma^{-3}/4 & -\gamma^{-1} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The main result follows.

Theorem 1.4: The eigenvalues of A_n are exactly the set of values in W_n, so that the characteristic polynomial of A_n is $Q_n(x)$. Moreover, an explicit recursive method can be given for generating E_n and E_n^{-1} so that we can diagonalize A_n explicitly as

$$E_n^{-1}A_nE_n = D_n.$$

In addition, the coefficients of the characteristic polynomial of A_n can be generated recursively.

In Section 2 we present some background on recurrence sequences and derive a simple self-contained proof of the first statement (Theorem 2.8). In Section 3 we give recurrence relations for the characteristic polynomial array (Definition 3.4, Proposition 3.5, and Corollary 3.11) and in Section 4 we give an explicit diagonalization of A_n (Theorem 4.3). As a consequence, we obtain a second proof that the characteristic polynomial of A_n is $Q_n(x)$. In Section 5 we give a generalization (Theorem 5.10). This approach demonstrates the explicit recursive method (Corollaries 5.8 and 5.9) for generating the eigenvector matrices. As a consequence of this approach, we obtain a third proof that the characteristic polynomial of A_n is $Q_n(x)$. However, slightly more algebra is required for this approach.

The first proof is based on elementary facts from vector recurrences, for which we provide a quick review. We give an overview of the second proof. We define recursively an array of numbers $b_{n,m}$ (Definition 3.4). From the n^{th} row of this array of numbers $b_{n,m}$, we form a polynomial $P_n(x)$. We show inductively that the roots of $P_n(x)$ form the set W_n, whence $P_n(x) = Q_n(x)$. Finally, we demonstrate that the companion matrix of $P_n(x)$ is similar to A_n, giving our result, since similar matrices have the same eigenvalues. The similarity computation requires the auxiliary matrices that we define in Section 4.

2. REVIEW OF RECURRENCE SEQUENCES

We present a review of recurrence sequences and, as a consequence, obtain a quick proof of the first statement of Theorem 1.4. Moreover, we find an interesting characterization of recurrence sequences generated by $Q_n(X)$ (Theorem 2.8) using some of the results developed in later sections. See [3] for generalities regarding recurrence sequences.

Definition 2.1: A sequence (a_h) satisfying a linear recursion

$$a_h = \sum_{k=1}^{n} s_k a_{h-k}$$

is called a (linear) *recurrence sequence*. We call the polynomial $x^n - \sum_{k=1}^{n} s_k x^{n-k}$ the *recurrence polynomial* for (a_h), and we say it generates (a_h). We call (a_h) *degenerate* if it is also generated by a polynomial of smaller degree.

If $f(x)$ has m roots α_k, then, as in (1.2), it is easy to show that

$$a_h = \sum_{k=1}^{m} A_k(h)\alpha_k^h, \tag{2.2}$$

where the $A_k(h)$ is a polynomial whose degree is the multiplicity of α_k in $f(x)$. Moreover, any such *generalized power sum* is a recurrence sequence with recurrence polynomial $f(x)$. Hence, it follows that the set of all recurrence sequences with recurrence polynomial $f(x)$ is a vector space of dimension n.

We shall make use of the following proposition in Section 4 below.

Proposition 2.3: Let x_h be a recurrence sequence of degree s with recurrence polynomial $p(x) = \prod_{k=1}^{s}(x-\alpha_k)$ with the α_i distinct and let y_h be a recurrence sequence of degree t with recurrence polynomial $q(x) = \prod_{\ell=1}^{t}(x-\beta_\ell)$ with the β_ℓ distinct. Let W be the distinct set of numbers of the form $\alpha_k\beta_\ell$ with $w = |W|$. Then the sequence x_hy_h is a recurrence sequence of degree w with recurrence polynomial $\prod_{\lambda\in W}(x-\lambda)$.

Proof: The vector space of sequences with recurrence polynomial $p(x)$ is spanned by the sequences α_k^h for $k = 1, \ldots, s$. Thus, we can write $x_h = \sum_{k=1}^{s} u_k\alpha_i^h$ for some u_k. Similarly, we can write $y_h = \sum_{\ell=1}^{t} v_\ell\beta_\ell^h$ for some v_ℓ. Multiplying yields

$$x_hy_h = \sum_{k,\ell} u_kv_\ell(\alpha_k\beta_\ell)^h.$$

Thus, x_hy_h is in the span of the sequences λ^h for $\lambda \in W$ and, hence, has recurrence polynomial as above. □

It is easy to characterize the space of sequences generated by a polynomial.

Proposition 2.4: The sequence (a_h) is a nondegenerate recurrence sequence generated by $f(x)$ of degree n, if and only if the matrix

$$A = \begin{bmatrix} a_{n-1} & a_n & \cdots & a_{2n-1} \\ a_{n-2} & a_{n-1} & \cdots & a_{2n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}$$

is invertible. In this case, the n sequences $(a_{h+k})_{k=0}^{n-1}$ generate the space of recurrence sequences generated by $f(x)$.

Proof: If A had a nontrivial element in its kernel, then so would

$$C^hA = \begin{bmatrix} a_{h+n-1} & a_{h+n} & \cdots & a_{h+2n-1} \\ a_{h+n-2} & a_{h+n-1} & \cdots & a_{h+2n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_h & a_{h+1} & \cdots & a_{h+n-1} \end{bmatrix},$$

where C is the companion matrix for $f(x)$. This is true if and only if (a_h) is a degenerate recurrence sequence. □

Next, we consider recurrence sequences that arise from matrices. This generalization is quite simple.

Definition 2.5: Let M be an n-by-n matrix and let v_0 be an n-dimensional column vector. The sequence of vectors (v_h) defined by $v_h = M^hv_0$ is called a *vector recurrence sequence*.

If M is the companion matrix for $f(x)$,

$$M = C = \begin{bmatrix} s_1 & s_2 & \cdots & s_{m-1} & s_m \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

then our situation is closely related to recurrence sequences. Let

$$v_0 = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix},$$

and let (a_h) be the corresponding recurrence sequences generated by $f(x)$. Then

$$v_h = C^h v_0 = \begin{bmatrix} a_{n+h-1} \\ \vdots \\ a_h \end{bmatrix}.$$

Even in a more general case, this picture is only altered with a change of basis.

Proposition 2.6: Let M be an n-by-n matrix with characteristic polynomial $f(x)$. Suppose further that $f(x)$ is actually the minimal polynomial of M, so that we have the similarity relation $M = BCB^{-1}$, where C is the companion matrix of $f(x)$. Let (v_h) be the vector recurrence sequence generated by M with initial value v_0. Then the i^{th} component of (v_h) forms a recurrence sequence (v_h^i) with recurrence polynomial $f(x)$. Moreover, the recurrence sequence generated by $f(x)$ with initial values given by $B^{-1}v_0$ is nondegenerate if and only if the n-by-n matrix $[v_0 \ \cdots \ v_{n-1}]$ is invertible. Hence, in this case, the (v_n^i) form a basis of the space of recurrence sequences generated by $f(x)$.

Remark: The condition that $f(x)$ is the minimal polynomial of M is not necessary; however, the statement becomes more complicated and the conclusion weaker.

Proof: Using the similarity relation, we find that

$$C \cdot \left[(B^{-1}v_0) \ \cdots \ (B^{-1}v_{n-1})\right] = \left[(B^{-1}v_1) \ \cdots \ (B^{-1}v_n)\right].$$

Thus, $(B^{-1}v_h)$ is a vector recurrence sequence for the matrix C. In other words,

$$B^{-1}[v_0 \ \cdots \ v_{n-1}] = A = \begin{bmatrix} a_{n-1} & a_n & \cdots & a_{2n-1} \\ a_{n-2} & a_{n-1} & \cdots & a_{2n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix},$$

where (a_h) is a recurrence sequence generated by $f(x)$. Thus,

$$M^h v_0 = BC^h(B^{-1}v_0) = B\begin{bmatrix} a_{n+h-1} \\ \vdots \\ a_h \end{bmatrix}.$$

This implies that the i^{th} component (v_h^i) is a linear combination of recurrence sequences generated by $f(x)$; hence, (v_h^i) itself is generated by $f(x)$.

The last statement then follows by applying Proposition 2.4 to $A = B^{-1}[v_0 \ \cdots \ v_{n-1}]$, and noting that B is invertible. □

We apply the above development to our binomial matrices. We shall consider the sequences $(F_{h+1}^{n-i}F_h^{i-1})$ for $1 \le i \le n$ as column vectors for fixed h.

Proposition 2.7: For all $h \ge 0$, $A_n[F_{h+1}^{n-i}F_h^{i-1}] = [F_{h+2}^{n-i}F_{h+1}^{i-1}]$. In other words, $(F_{h+1}^{n-i}F_h^{i-1})$ is a recurrence sequence generated by the characteristic polynomial of A_n for all i, $1 \le i \le n$.

Proof: This follows trivially by induction. One must only check the top entry, which follows from the relation

$$\sum_{j=1}^{n}\binom{n-1}{j-1}F_{h+1}^{n-j}F_h^{j-1} = (F_{h+1}+F_h)^{n-1} = F_{h+2}^{n-1}. \ \square$$

Now we use (1.1):

$$F_{h+1}^{n-i}F_h^{i-1} = \left(\frac{\gamma^{h+1}-(-1/\gamma)^{h+1}}{\sqrt{5}}\right)^{n-i}\left(\frac{\gamma^{h}-(-1/\gamma)^{h}}{\sqrt{5}}\right)^{i-1} = \sum_{w\in W_n} A_w w^h.$$

Thus, by (2.2), the polynomial $Q_n(x) = \prod_{w\in W_n}(x-w)$ generates each of the sequences $(F_{h+1}^{n-i}F_h^{i-1})$. Clearly, if we can show that $(F_{h+1}^{n-i}F_h^{i-1})$ is nondegenerate of degree n, then we must have

$$\text{char poly}(A_n) = Q_n(x).$$

Theorem 2.8: The eigenvalues of A_n are W_n; hence, the characteristic polynomial of A_n is $Q_n(x)$. Moreover, the sequences $(F_{h+1}^{n-i}F_h^{i-1})_{i=1}^{n}$ form a basis for the space of recurrence sequences generated by $Q_n(x)$.

Proof: In light of the above, it only remains to show that the matrix $[F_{j+1}^{n-i}F_j^{i-1}]$ is invertible. If we scale the j^{th} column by dividing by F_{j+1}^{n-1}, then we obtain the Vandermonde matrix $[(F_j/F_{j+1})^{i-1}]$, and as is well known, this determinant is nonzero. □

3. THE CHARACTERISTIC POLYNOMIAL

We set out some well-known (and easily proved) facts about the Fibonacci and Lucas numbers to refer to later.

$$\gamma^h = \frac{L_h + F_h\sqrt{5}}{2}. \tag{3.1}$$

$$\gamma^{-h} = (-1)^h\frac{L_h - F_h\sqrt{5}}{2}. \tag{3.2}$$

$$F_hL_k + F_kL_h = 2F_{h+k}. \tag{3.3}$$

We shall see that the following array of numbers gives the coefficients of the characteristic polynomial of A_n.

Definition 3.4: Define the array of numbers $b_{n,m}$ for $n, m \ge 0$ as follows. Let $b_{n,0} = 1$ for all $n \ge 0$ and let $b_{n,m} = 0$ for $m > n$. For $0 < m \le n$, we define $b_{n,m}$ recursively by

$$b_{n,m} = b_{n-1,m-1}\frac{F_n}{F_m}(-1)^m.$$

Proposition 3.5: For $m \le n$, we have

$$|b_{n,m}| = \frac{F_n F_{n-1} \cdots F_{n-m+1}}{F_m \cdots F_1}.$$

Moreover, $|b_{n,m}| = |b_{n,n-m}|$, and we have the relation

$$b_{n,m} = b_{n,m-1}\frac{F_{n-m+1}}{F_m}(-1)^m.$$

The proof is obvious. The first several rows of the $b_{n,m}$ array are given in Table 3.6, where n indexes the rows and m indexes the columns.

TABLE 3.6. Coefficients of the Characteristic Polynomial

	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	−1	0	0	0	0	0	0	0
2	1	−1	−1	0	0	0	0	0	0
3	1	−2	−2	1	0	0	0	0	0
4	1	−3	−6	3	1	0	0	0	0
5	1	−5	−15	15	5	−1	0	0	0
6	1	−8	−40	60	40	−8	−1	0	0
7	1	−13	−104	260	260	−104	−13	1	0
8	1	−21	−273	1092	1820	−1092	−273	21	1

Definition 3.7: Let $P_n(x) = \sum_{j=0}^{n} b_{n,n-j}x^j$. The first few $P_n(x)$ (which can be read from Table 3.6) and W_n (defined in Section 1) are:

$$\begin{array}{ll}
P_1(x) = x-1 & W_1 = \{1\} \\
P_2(x) = x^2 - x - 1 & W_2 = \{\gamma, -\gamma^{-1}\} \\
P_3(x) = x^3 - 2x^2 - 2x + 1 & W_3 = \{\gamma^2, -1, \gamma^{-2}\} \\
P_4(x) = x^4 - 3x^3 - 6x^2 + 3x + 1 & W_4 = \{\gamma^3, -\gamma, \gamma^{-1}, -\gamma^{-3}\}
\end{array}$$

We note that the $(n-1)$ column of the $b_{n,m}$ array is just the coefficients of the formal power series $(-1)^n / P_n((-1)^{n-1}x)$. This is equivalent to

$$\sum_{k=0}^{n}(-1)^{k(n-1)}b_{n,k}b_{k+j,n-1} = 0$$

for all $j \ge n-1$. Although we do not use this fact here, we record it as Corollary 3.11 to Theorem 3.8.

Theorem 3.8: The set of roots of the polynomial $P_n(x)$ is exactly W_n: $P_n(x) = Q_n(x)$.

Proof: We use induction. Since $W_n = (-1/\gamma)W_{n-1} \cup \{\gamma^{n-1}\}$, we have the relation

$$Q_n(x) = \frac{1}{(-\gamma)^{n-1}} P_{n-1}(-\gamma x)\cdot(x - \gamma^{n-1}). \tag{3.9}$$

We rewrite this as

$$Q_n(x) = (x + (-1)^n(-\gamma)^{n-1})\sum_{j=0}^{n-1} b_{n-1,\,n-1-j}(-\gamma)^{j-n+1}x^j$$
$$= x^n + (-1)^n b_{n-1,\,n-1} + \sum_{j=1}^{n-1}[b_{n-1,\,n-j}(-\gamma)^{j-n} + (-1)^n b_{n-1,\,n-1-j}(-\gamma)^j]x^j.$$

Thus, we need to show the relation

$$b_{n,\,n-j} = b_{n-1,\,n-j}(-\gamma)^{j-n} + (-1)^n b_{n-1,\,n-1-j}(-\gamma)^j. \tag{3.10}$$

By equations (3.1) and (3.2), this can be written as

$$b_{n,\,n-j} = b_{n-1,\,n-j}\left(\frac{L_{n-j} - F_{n-j}\sqrt{5}}{2}\right) + (-1)^{n+j} b_{n-1,\,n-j-1}\left(\frac{L_j + F_j\sqrt{5}}{2}\right).$$

By Proposition 3.5, this becomes

$$b_{n,\,n-j} = (-1)^{n+j} b_{n-1,\,n-j-1}\left[\frac{F_j}{F_{n-j}}\left(\frac{L_{n-j} - F_{n-j}\sqrt{5}}{2}\right) + \frac{L_j + F_j\sqrt{5}}{2}\right]$$
$$= (-1)^{n+j} b_{n-1,\,n-j-1}\left[\frac{F_j L_{n-j} + F_{n-j}L_j}{2F_{n-j}}\right].$$

By equation (3.3) above, this simplifies to

$$b_{n,\,n-j} = (-1)^{n+j} b_{n-1,\,n-j-1}\frac{F_n}{F_{n-j}},$$

which follows from Definition 3.4. □

Corollary 3.11: The $(n-1)$ column of the array $b_{n,m}$ forms the coefficients of the formal power series $(-1)^n / P_n((-1)^{n-1}x)$. More precisely, we have

$$\frac{(-1)^n}{P_n((-1)^{n-1}x)} = \sum_{k=0}^{\infty} b_{k+n-1,\,n-1}x^k.$$

Proof: Repeated application of (3.10) gives the following:

$$b_{n,\,m} = \gamma^{n+1}\sum_{k=1}^{n-m+1}(-1)^{km}\gamma^{-k(m+1)}b_{n-k,m-1}.$$

By changing the order of summation and relabeling the indices, this is equivalent to

$$b_{k+n-1,\,n-1} = (-1)^{n-1}\sum_{s=0}^{k} b_{s+n-2,\,n-2}\gamma^s(-\gamma)^{(k-s)(1-n)}.$$

But this just expresses the power series identity

$$\sum_{k=0}^{\infty} b_{k+n-1,\,n-1} x^k = (-1)^{n-1}\left(\sum_{s=0}^{\infty} b_{s+n-2,\,n-2}(\gamma x)^s\right)\left(\sum_{t=0}^{\infty}\left(\frac{x}{(-\gamma)^{n-1}}\right)^t\right). \tag{3.12}$$

Now, applying induction, the inverse of the right-hand side of (3.12) is just

$$(-1)^{n-1}\left(\frac{P_{n-1}((-1)^{n-2}\gamma x)}{(-1)^{n-1}}\right)\left(1-\frac{x}{(-\gamma)^{n-1}}\right)=\frac{(-1)^n}{(-\gamma)^{n-1}}P_{n-1}((-1)^{n-2}\gamma x)((-1)^{n-1}x-\gamma^{n-1}).$$

Substituting $y=(-1)^{n-1}x$, we have

$$\frac{(-1)^n}{(-\gamma)^{n-1}}P_{n-1}(-\gamma y)(y-\gamma^{n-1})=(-1)^n P_n(y),$$

by (3.9) and Theorem 3.8. Thus, as required, the left-hand side of (3.12) is

$$\sum_{k=0}^{\infty} b_{k+n-1,\,n-1}x^k=\frac{(-1)^n}{P_n((-1)^{n-1}x)}. \quad \square$$

4. EXPLICIT DIAGONALIZATION

We define the following integral matrices that will be used to diagonalize A_n in Theorem 4.3 explicitly.

Definition 4.1: Let $n>1$. Let C_n be the companion matrix for $P_n(x)$, $C_n=[c_{ij}]$, where

$$\begin{cases} c_{i,\,i+1}=1 & \text{for } i=1,\dots,n-1,\\ c_{n,\,j}=-b_{n,\,n+1-j} & \text{for } j=1,\dots,n,\\ c_{i,\,j}=0 & \text{otherwise.}\end{cases}$$

Let $R_n=[r_{ij}]$, where $r_{ij}=\binom{n-1}{j-1}F_{i-2}^{j-1}F_{i-1}^{n-j}$. Let $M_n=[m_{ij}]$, where $m_{ij}=\binom{n-1}{j-1}F_{i-1}^{j-1}F_i^{n-j}$.

Observe that the i^{th} row of R_n gives the terms in $(F_{i-2}+F_{i-1})^{n-1}$ and that the i^{th} row of M_n gives the terms in $(F_{i-1}+F_i)^{n-1}$. These matrices will be used in Theorem 4.3 to prove that A_n is similar to C_n. The matrix R_n arose originally by observing the relation $R_nE_n=V_n$ (see Definition 4.2). From here, it is natural to bring in the companion matrix C_n, since V_n is the eigenvector matrix for C_n.

We illustrate Definition 4.1 for $n=5$:

$$C_5=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&-5&-15&15&5\end{bmatrix},\qquad R_5=\begin{bmatrix}0&0&0&0&1\\1&0&0&0&0\\1&4&6&4&1\\16&32&24&8&1\\81&216&216&96&16\end{bmatrix},$$

$$M_5=\begin{bmatrix}1&0&0&0&0\\1&4&6&4&1\\16&32&24&8&1\\81&216&216&96&16\\625&1500&1350&540&81\end{bmatrix}.$$

Definition 4.2: For each $n>1$, let V_n be the Vandermonde matrix which is the eigenvector matrix for C_n with eigenvectors listed in decreasing order of the absolute values of the corresponding eigenvalues.

Thus, for example, we have

$$V_5=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \gamma^4 & -\gamma^2 & 1 & -\gamma^{-2} & \gamma^{-4} \\ \gamma^8 & \gamma^4 & 1 & \gamma^{-4} & \gamma^{-8} \\ \gamma^{12} & -\gamma^6 & 1 & -\gamma^{-6} & \gamma^{-12} \\ \gamma^{16} & \gamma^8 & 1 & \gamma^{-8} & \gamma^{-16} \end{bmatrix}.$$

Theorem 4.3: For all n, we have the relation $M_n=C_nR_n=R_nA_n$. Moreover, $P_n(x)$ is the characteristic polynomial of A_n, R_n is invertible, and the eigenvector matrix of A_n is given by

$$E_n=R_n^{-1}V_n.$$

Proof: Multiplying the first $n-1$ rows of C_n by R_n clearly gives the first $n-1$ rows of M_n. For the last row, for each j, $1\le j\le n$, we must show the relation

$$\sum_{k=1}^{n}-b_{n,\,n+1-k}F_{k-2}^{j-1}F_{k-1}^{n-j}=F_{n-1}^{j-1}F_n^{n-j}. \tag{4.4}$$

Now $P_2(x)$ is the recurrence polynomial for the sequence F_k (and, hence, for F_{k-1}). Thus, using the fact that $W_uW_v=W_{u+v-1}$, we can apply Proposition 2.3 repeatedly to find that $P_m(x)$ is the recurrence polynomial for the sequence whose h^{th} entry is a product of $m-1$ factors, each chosen from the set $\{F_h, F_{h-1}\}$. In other words, $P_m(x)$ is the recurrence polynomial for the sequence $F_{h-1}^{j-1}F_h^{m-j}$ for any j, $1\le j\le m$. Explicitly, this means that

$$\sum_{k=1}^{m}-b_{m,\,m+1-k}F_{r+k-m-2}^{j-1}F_{r+k-m-1}^{m-j}=F_{r-1}^{j-1}F_r^{m-j}.$$

Equation (4.4) now follows, since it is just this same recurrence relation for $m=n$ at the $r=n$ term. This proves $M_n=C_nR_n$.

To prove $M_n=R_nA_n$ it is equivalent to show that, for all i, j with $1\le i, j\le n$, we have

$$\sum_{k=0}^{n-j}\binom{n-1}{k}F_{i-2}^{k}F_{i-1}^{n-k-1}\cdot\binom{n-k-1}{j-1}=\binom{n-1}{j-1}F_{i-1}^{j-1}F_i^{n-j}.$$

Combining the binomials and dividing by F_{i-1}^{j-1}, this is equivalent to showing

$$\sum_{k=0}^{n-j}\binom{n-j}{k}F_{i-2}^{k}F_{i-1}^{n-k-j}=F_i^{n-j}.$$

But, by the binomial theorem, this is just $(F_{i-2}+F_{i-1})^{n-j}=F_i^{n-j}$, which is just the Fibonacci recursion. This proves that $C_nR_n=R_nA_n$.

The fact that R_n is invertible was actually proved previously in the proof of Theorem 2.8; again, we can scale R_n to obtain a Vandermonde matrix which has a nonzero determinant. Hence, C_n and A_n satisfy the similarity relation $A_n=R_n^{-1}C_nR_n$. Thus, they have the same characteristic polynomial $P_n(x)$. Since V_n is the eigenvector matrix for C_n, the similarity relation shows that $R_n^{-1}V_n$ is the eigenvector matrix for A_n. □

5. A GENERALIZATION

In this section we give an alternative development of Theorem 1.4. As a result, we obtain a recursive method of generating the eigenvector matrix. Moreover, we find a nice explanation for the eigenvalue behavior. Our methods yield the following generalization: if any matrix is generated in the same way as the A_n, then it must be essentially binomial.

Definition 5.1: Let B be a 2-by-2 matrix, $B=\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$. Define an n-by-n matrix $S_n(B)$ as follows: the j^{th} entry of the i^{th} row of $S_n(B)$ is given by

$$S_n(B)(i,j) = \text{the coefficient of } x^{n-j}y^{j-1} \text{ in } (ax+by)^{n-i}(cx+dy)^{i-1}.$$

Then, for $A=A_2=\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0\end{smallmatrix}\right]$, by the binomial theorem we have $A_n=S_n(A)$. For general B as in Definition 5.1, we let $B_n=S_n(B)$. Thus, we have

$$B_2=B=\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \qquad B_3=\begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2\end{pmatrix},$$

$$B_4=\begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & 2abc+a^2d & 2abd+b^2c & b^2d \\ ac^2 & 2acd+bc^2 & 2bcd+ad^2 & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3\end{pmatrix}.$$

Lemma 5.2: Let $B=B_2$ be a 2-by-2 matrix, $B=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ and let $B_n=S_n(B)$. Then B_n may be generated recursively from B_{n-1} by

$$B_n(i,j)=bB_{n-1}(i,j-1)+aB_{n-1}(i,j),$$
$$B_n(n,j)=\binom{n-1}{j-1}c^{n-j}d^{j-1},$$

with the convention that $B_n(i,j)=0$ for $j<1$ or $j>n$.

Proof: Induction on n. □

In order to prove the next lemmas, we need to define some notation.

Definition 5.3: Let $R=\mathbb{C}[x,y]$ be the ring over $\mathbb{C}$ in two indeterminates. Define V_n to be the $\mathbb{C}$-vector space of homogeneous polynomials in R of degree $n-1$. A basis for V_n is $\{x^{n-1}, x^{n-2}y, \ldots, y^{n-1}\}$, so V_n is of dimension n. Any 2-by-2 matrix $B=\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$ induces a ring homomorphism, $\phi_B: R\to R$, by sending x to $ax+by$ and sending y to $cx+dy$. Since ϕ_B is degree-preserving and linear in x and y, it induces a linear transformation on V_n. We denote this linear transformation by $\phi_{B,n}$.

Lemma 5.4: If we write an element V_n as a row vector with respect to the basis $\{x^{n-1}, x^{n-2}y, \ldots, y^{n-1}\}$, then the action of $\phi_{B,n}$ on V_n is given by multiplication by $S_n(B)$ on the right.

Proof: The i^{th} basis vector of V_n, namely, the vector $x^{n-i}y^{i-1}$, goes to $(ax+by)^{n-i}(cx+dy)^{i-1}$ under the linear transformation $\phi_{B,n}$, which just forms the i^{th} row of $S_n(B)$. □

Lemma 5.5: Let B and C be 2-by-2 matrices. Then $S_n(BC) = S_n(B)S_n(C)$.

Proof: The matrix equation $BC = B \cdot C$ gives rise to the ring homomorphism equation $\phi_{BC} = \phi_C \circ \phi_B$ (note that the matrices act on the right; hence, ϕ_B is applied first). Since the ring homomorphisms act the same on V_n, we obtain the equality of linear transformations $\phi_{BC,n} = \phi_{C,n}\phi_{B,n}$. Now, by Lemma 5.4, we obtain $S_n(BC) = S_n(B)S_n(C)$ □

Theorem 5.6: Let $G_n = S_n(E_2)$, where $E_2 = \begin{bmatrix} \gamma & -1/\gamma \\ 1 & 1 \end{bmatrix}$ is the eigenvector matrix for A_2. Then

$$A_n G_n = G_n D_n.$$

Hence, G_n is the eigenvector matrix for A_n (scaled so that the bottom row of G_n is the top row of A_n) and D_n is the diagonalization of A_n, giving the eigenvalues of A_n to be $(-1)^{n-i}\gamma^{2i-n+1}$ as i ranges from 0 to $n-1$.

Proof: As we have observed after Definition 5.1, we have

$$S_n(A_2) = A_n. \tag{5.7}$$

If we start with the matrix equation $A_2E_2 = E_2D_2$ and apply the operator S_n, then, from Lemma 5.5 and equation (5.7), we obtain

$$A_n S_n(E_2) = S_n(E_2)S_n(D_2).$$

The action of D_2 on V_2 sends x to γx and y to $-\gamma^{-1}y$. Therefore, $S_n(D_2)$ sends $x^i y^{n-1-i}$ to $(-1)^{n-1-i}\gamma^{2i-n+1}x^i y^{n-1-i}$, which is exactly the action of D_n. Thus, $D_n = S_n(D_2)$ and, consequently, $S_n(E_2)$ must be the eigenvector matrix. This gives the result. □

Remark: These results can be interpreted in terms of the symmetric algebra of $\mathbb{C}^2$, denoted Sym $\mathbb{C}^2$ (see [2], p. 141). If e_1 and e_2 are a basis for $\mathbb{C}^2$, the ring R above is isomorphic to the symmetric algebra of $\mathbb{C}^2$ by sending x to e_1 and y to e_2. The set of homogeneous polynomials of degree n of R is just the $(n-1)$-fold symmetric tensor product of $\mathbb{C}^2$, denoted $\text{Sym}^{n-1}\mathbb{C}^2$. As we have observed above, the linear transform A_2 acting on $\mathbb{C}^2$ induces an action on Sym $\mathbb{C}^2$.

Lemma 5.2 gives an explicit means for computing the eigenvector matrix. Since $G_n = S_n(E_2)$, we have the following recursive method for computing the eigenvector matrix.

Corollary 5.8:
$$G_n(i,j) = -1/\gamma G_{n-1}(i, j-1) + \gamma G_{n-1}(i,j),$$
$$G_n(n,j) = \binom{n-1}{j-1}.$$

Similarly, we can compute the inverse of the eigenvalue matrix so that the explicit diagonalization of A_n can be given. We have

$$H_2 = G_2^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 1/\gamma \\ -1 & \gamma \end{bmatrix}.$$

Thus, since $H_n = G_n^{-1} = S_n(G_2^{-1})$, we obtain the following result from Lemma 5.2.

Corollary 5.9:

$$H_n(i,j) = \frac{1}{\sqrt{5}}[1/\gamma B_{n-1}(i,j-1) + B_{n-1}(i,j)],$$

$$H_n(n,j) = \sqrt{5}^{n-1}\binom{n-1}{j-1}(-1)^{n-j}\gamma^{j-1}.$$

We note that the proof of Theorem 5.6 actually shows the following generalization.

Theorem 5.10: Let B be a 2-by-2 matrix with distinct nonzero eigenvalues α and α'. Then eigenvalues of $S_n(B)$ are $\alpha^{(n-i)}\alpha'^{(i-1)}$, where i ranges from 1 to n. Moreover, if E is the eigenvector matrix for B, then the eigenvector matrix for $S_n(B)$ is $S_n(E)$.

However, we also note that if the set of matrices $S_n(B)$ comes from a single array of numbers as the inverted binomial matrices (the A_n) do, then the array of numbers must be essentially binomial.

Theorem 5.11: Let B be a 2-by-2 matrix. Suppose that the entries of the matrix $S_n(B)$ come from a single array of numbers for each $n > 1$. Then B must be of the form $\left[\begin{smallmatrix} a & b \\ 1 & 0 \end{smallmatrix}\right]$. In this case, the entries of the i^{th} row of $S_n(B)$ are just the coefficients of $(ax+by)^{n-i}$.

Proof: Assume $B = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Then

$$S_3(B) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix},$$

so that we must have $c^2 = c, 2cd = d, ac = a, ad+bc = b$. These imply that $c = 1$ and $d = 0$. Then the entries of the i^{th} row of $S_n(B)$ are the coefficients of $(ax+by)^{n-i}$ and, hence, just the binomial matrix scaled by powers of a and b. □

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EXTENSION OF A SYNTHESIS FOR A CLASS OF POLYNOMIAL SEQUENCES

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1. MOTIVATION

Featured in [2] was a pair of generalized functions

$$W_n^{(k)}(x) = \frac{\Delta^k(x)\{\alpha^n(x) - (-1)^k \beta^n(x)\}}{\Delta(x)} \tag{1.1}$$

and

$$\mathcal{W}_n^{(k)}(x) = \Delta^k(x)\{\alpha^n(x) + (-1)^k \beta^n(x)\}, \tag{1.2}$$

where

$$\begin{cases} \alpha(x) = \dfrac{p(x) + \sqrt{p^2(x) + 4q(x)}}{2}, \\ \beta(x) = \dfrac{p(x) - \sqrt{p^2(x) + 4q(x)}}{2}, \end{cases} \tag{1.3}$$

giving

$$\begin{cases} \alpha(x) + \beta(x) = p(x), \\ \alpha(x)\beta(x) = -q(x), \\ \alpha(x) - \beta(x) = \sqrt{p^2(x) + 4q(x)} = \Delta(x). \end{cases} \tag{1.4}$$

Observe that (1.1) and (1.2) lead to

$$W_0^{(k)}(x) = \Delta^k(x)\frac{\{1 - (-1)^k\}}{\Delta(x)} \tag{1.5}$$

and

$$\mathcal{W}_0^{(k)}(x) = \Delta^k(x)\{1 + (-1)^k\}. \tag{1.6}$$

Hence

$$W_0^{(0)}(x) = 0 \qquad (W_0^{(0)}(0) = 0) \tag{1.7}$$

and

$$\mathcal{W}_0^{(0)}(x) = 2 \qquad (\mathcal{W}_0^{(0)}(0) = 2). \tag{1.8}$$

When $k = 1$, clearly (1.1) and (1.2) in conjunction compress to

$$W_n^{(1)}(x) = \mathcal{W}_n^{(0)}(x) = \mathcal{W}_n(x), \tag{1.9}$$

and

$$\mathcal{W}_n^{(1)}(x) = \Delta^2(x) W_n^{(0)}(x) = \Delta^2(x) W_n(x). \tag{1.10}$$

Generally,

$$W_n^{(k)}(x) = \mathcal{W}_n^{(k-1)}(x). \tag{1.11}$$

Properties of (1.1) and (1.2) were developed in [2] and then related to several special cases of pairs of polynomial sequences, and to numerical particularizations of them arising when $x=1$ or $x=\frac{1}{2}$ as appropriate.

For a partial description, from a different viewpoint, of some of the material in [2], the reader is referred to [1].

Special cases of (1.1) and (1.2) to which we will refer are [2] by polynomial symbolism and name [with corresponding $p(x)$-value and $q(x)$-value]:

$$\begin{cases} & & & & p(x) & q(x) \\ F_n(x): & \text{Fibonacci} & L_n(x): & \text{Lucas} & x & 1 \\ P_n(x): & \text{Pell} & Q_n(x): & \text{Pell-Lucas} & 2x & 1 \\ J_n(x): & \text{Jacobsthal} & j_n(x): & \text{Jacobsthal-Lucas} & 1 & 2x \\ \mathcal{F}_n(x): & \text{Fermat} & f_n(x): & \text{Fermat-Lucas} & 3x & -2 \\ U_n(x): & \leftarrow \text{Chebyshev} \rightarrow & :T_n(x) & & 2x & -1 \\ \mathcal{U}_n(x): & \leftarrow \text{Hyperbolic} \rightarrow & :\mathcal{V}_n(x) & & 2x & -1 \end{cases} \tag{1.12}$$

For the Chebyshev polynomials, we have $x=\cos\theta$, whereas in the case of the hyperbolic functions we know that $x=\cosh t$.

Toward the end of [2] it was suggested that one of the many extensions of that research was an investigation of the numerical values of (1.1) and (1.2) when k and/or n are negative.

Here, we propose to examine the general theory of polynomials (1.1) and (1.2) for negative k and n. A smooth transition from positive to negative is usually effected. Our endeavors bring into being a collection of Theorems $\mathrm{A}', \mathrm{B}', \ldots, \mathrm{F}'$ paralleling those in [2]. Of these, the last incorporates the desired synthesis.

2. NEGATIVE SUBSCRIPTS AND SUPERSCRIPTS

Negative Subscripts

After a little calculation using (1.1) and (1.2), we deduce that

$$W_{-n}^{(k)}(x) = -(-1)^k(-q)^{-n}W_n^{(k)}(x) \tag{2.1}$$

and

$$\mathcal{W}_{-n}^{(k)}(x) = (-1)^k(-1)^{-n}\mathcal{W}_n^{(k)}(x), \tag{2.2}$$

showing the connection between positive and negative subscripts.

More particularly, when $k=0$,

$$W_{-n}(x) = -(-q)^{-n}W_n(x) \tag{2.3}$$

and

$$\mathcal{W}_{-n}(x) = (-q)^{-n}\mathcal{W}_n(x). \tag{2.4}$$

Special Cases

Combining (1.12), (2.1), and (2.2), we derive

$$\begin{cases} F_{-n}^{(k)}(x) = -(-1)^{k-n} F_n^{(k)}(x), & L_{-n}^{(k)}(x) = (-1)^{k-n} L_n^{(k)}(x), \\ P_{-n}^{(k)}(x) = -(-1)^{k-n} P_n^{(k)}(x), & Q_{-n}^{(k)}(x) = (-1)^{k-n} Q_n^{(k)}(x), \\ J_{-n}^{(k)}(x) = -(-1)^{k}(-2x)^{-n} J_n^{(k)}(x), & j_{-n}^{(k)}(x) = (-1)^{k}(-2x)^{-n} j_n^{(k)}(x), \\ \mathcal{F}_{-n}^{(k)}(x) = -(-1)^{k} 2^{-n} \mathcal{F}_n^{(k)}(x), & f_{-n}^{(k)}(x) = (-1)^{k} 2^{-n} f_n^{(k)}(x), \\ U_{-n}^{(k)}(x) = -(-1)^{k} U_n^{(k)}(x), & T_{-n}^{(k)}(x) = (-1)^{k} T_n^{(k)}(x), \\ \mathcal{U}_{-n}^{(k)}(x) = -(-1)^{k} \mathcal{U}_n^{(k)}(x), & \mathcal{V}_{-n}^{(k)}(x) = (-1)^{k} \mathcal{V}_n^{(k)}(x). \end{cases} \tag{2.5}$$

Putting $k = 0$ in (2.5), we have the standard simplifications [refer to (2.3), (2.4)].

Examples

$$J_{-5}(x) = \frac{4x^2+6x+1}{32x^5} = -\frac{1}{(-2x)^5} J_5(x),$$
$$F_{-3}^{(2)}(x) = x^4+5x^2+4 = (x^2+4)(x^2+1) = F_3^{(2)}(x),$$
$$T_{-4}^{(3)}(x) = -64x(x^2-1)^2(2x^2-1) = -T_4^{(3)}(x) \quad (x = \cos\theta).$$

Differentiation

As in [2], when $k = 0$,

$$\frac{d}{dx}\mathcal{W}_{-n}(x) = \begin{cases} -np'(x)W_{-n}(x) & \text{for } p'(x) \neq 0,\ q'(x) = 0, \\ -nq'(x)W_{-n-1}(x) & \text{for } p'(x) = 0,\ q'(x) \neq 0, \end{cases} \tag{2.6}$$

where the superscript dash (′) denotes differentiation with respect to x.

Thus,

$$\frac{d}{dx} f_{-3}(x) = \frac{d}{dx}\left(\frac{27x^3-18x}{8}\right) = -3.3\frac{(-9x^2+2)}{8} = -3.3\mathcal{F}_{-3}(x),$$

$$\frac{d}{dx} j_{-4}(x) = \frac{d}{dx}\left(\frac{8x^2+8x+1}{16x^4}\right) = \frac{-4.2(4x^2+6x+1)}{32x^5} = -4.2J_{-5}(x).$$

Negative Superscripts

What meaning can be attached to a symbol with a negative superscript? From (1.1), (1.2),

$$W_n^{(-k)}(x) = \Delta^{-k}(x)\frac{\{\alpha^n(x) - (-1)^k \beta^n(x)\}}{\Delta(x)} \tag{2.7}$$

and

$$\mathcal{W}_n^{(-k)}(x) = \Delta^{-k}(x)\{\alpha^n(x) + (-1)^k \beta^n(x)\} \tag{2.8}$$

with obvious extensions when n is replaced by $-n$ (i.e., both subscript and superscript are negative). Refer back to (2.1), (2.2).

For instance,

$$P_2^{(-5)}(x) = (4x^2+2)(4x^2+4)^{-3},$$
$$f_3^{(-4)}(x) = 9x(3x^2-2)(9x^2-8)^{-2}.$$

Some Generalized Products

Without difficulty, one may establish the following multiplicative identities, which were omitted from [2]:

$$\begin{aligned} W_m^{(h)}(x)W_n^{(k)}(x) = \frac{\Delta^{h+k}(x)}{\Delta^2(x)}\{&\alpha^{m+n}(x)+(-1)^{h+k}\beta^{m+n}(x) \\ &-(-q(x))^n[(-1)^k\alpha^{m-n}(x)+(-1)^h\beta_{m-n}(x)]\}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} W_m^{(h)}(x)\mathcal{W}_n^{(k)}(x) = \frac{\Delta^{h+k}(x)}{\Delta(x)}\{&\alpha^{m+n}(x)-(-1)^{h+k}\beta^{m+n}(x) \\ &+(-q(x))^n[(-1)^k\alpha^{m-n}(x)-(-1)^h\beta^{m-n}(x)]\}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \mathcal{W}_m^{(h)}(x)\mathcal{W}_n^{(k)}(x) = \Delta^{h+k}(x)\{&\alpha^{m+n}(x)+(-1)^{h+k}\beta^{m+n}(x) \\ &+(-q(x))^n[(-1)^k\alpha^{m-n}(x)+(-1)^k\alpha^{m-n}(x)]\}. \end{aligned} \tag{2.11}$$

Various combinations of the above involving $\pm h, \pm k, \pm m, \pm n$ might be investigated. For example, (2.10) with (1.10) leads to

$$W_n^{(k)}(x)\mathcal{W}_n^{(-k)}(x) = W_n^{(-k)}(x)\mathcal{W}_n^{(k)}(x) = W_{2n}(x). \tag{2.12}$$

Another pleasing deduction flows from (2.11), namely,

$$\mathcal{W}_n^{(k)}(x)\mathcal{W}_{-n}^{(-k)}(x) - \mathcal{W}_{-n}^{(k)}(x)\mathcal{W}_n^{(-k)}(x) = 0 \tag{2.13}$$

with a similar conclusion for $W_n^{(k)}(x)$.

Again, applying (2.9) and (2.11) in tandem, we obtain

$$\mathcal{W}_n^{(k)}(x)\mathcal{W}_n^{(-k)}(x) - \Delta^2(x)W_n^{(k)}(x)W_n^{(-k)}(x) = 4(-1)^k(-q(x))^n. \tag{2.14}$$

3. BASIC UNIFYING THEOREMS

Theorems A-F in [2] can now be paralleled. Except that we now use (2.1) and (2.2), of course, the proofs follow those in [2].

Our homologous theorems will be labeled Theorem A′, ..., Theorem F′. Enunciations of these theorems are given below.

Theorem A′: $W_{-n}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x) = W_{-2n}^{(2k)}(x)$.

Theorem B′(a): $W_{-m}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x) + W_{-n}^{(k)}(x)\mathcal{W}_{-m}^{(k)}(x) = 2W_{-(m+n)}^{(2k)}(x)$.

If $m = n$, then Theorem B′(a) reduces to Theorem A′.

Replacing m by $-m$, we derive

Corollary B′(a): $W_m^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x) + W_{-n}^{(k)}(x)\mathcal{W}_m^{(k)}(x) = 2W_{m-n}^{(2k)}(x)$,

$$\begin{cases} = 2W_0^{(2k)}(x) & \text{if } m = n, \\ = 0 & \text{by (1.5).} \end{cases}$$

Theorem B'(b): $\mathcal{W}_{-m}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x)+\Delta^2(x)W_{-m}^{(k)}(x)W_{-n}^{(k)}(x)=2\mathcal{W}_{-(m+n)}^{(2k)}(x)$.

If $m=n$, then Theorem B'(b) contracts to a sum of squares on the left-hand side.
Making the transformation $m\to -m$ gives

Corollary B'(b): $\mathcal{W}_{m}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x)+\Delta^2(x)W_{m}^{(k)}(x)W_{-n}^{(k)}(x)=2\mathcal{W}_{m-n}^{(2k)}(x)$

$$\begin{cases} =2\mathcal{W}_0^{(2k)}(x) & \text{if } m=n,\\ =4\Delta^{2k}(x) & \text{by (1.6).}\end{cases}$$

Theorem C'(a): $W_{-m}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x)-W_{-n}^{(k)}(x)\mathcal{W}_{-m}^{(k)}(x)=2(-1)^k(-q(x))^{-n}W_{-(m-n)}^{(2k)}(x)$.

Putting $m=n$ yields the trivial identity $0=0$, by (1.5).
Other considerations are: (i) $m=-n$, (ii) interchange m, n.

Theorem C'(b): $\mathcal{W}_{-m}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x)-\Delta^2(x)W_{-m}^{(k)}(x)W_{-n}^{(k)}(x)=2(-1)^k(-q(x))^{-n}\mathcal{W}_{-(m-n)}^{(2k)}(x)$.

Variations: (i) $m=n$, (ii) $m\to -m$, (iii) m, n interchanged.

Theorem D': $W_{-n+1}^{(k)}(x)+q(x)W_{-n-1}^{(k)}(x)=\mathcal{W}_{-n}^{(k)}(x)$.

Theorem E': $\mathcal{W}_{-n+1}^{(k)}(x)+q(x)\mathcal{W}_{-n-1}^{(k)}(x)=\Delta^2(x)W_{-n}^{(k)}(x)$.

Illustrations

(A'): $\mathcal{F}_{-2}^{(1)}(x)f_{-2}^{(1)}(x)=\dfrac{-3x(9x^2-8)(9x^2-4)}{16}=\mathcal{F}_{-4}^{(2)}(x)$.

(B'(a)): $F_{-1}^{(2)}(x)L_{-2}^{(2)}(x)+F_{-2}^{(2)}(x)L_{-1}^{(2)}(x)=2(x^2+1)(x^2+4)^2=2F_{-3}^{(4)}(x)$.

(B'(b)): $Q_{-1}^{(1)}(x)Q_{-2}^{(1)}(x)+4(x^2+1)P_{-1}^{(1)}(x)P_{-2}^{(1)}(x)=-16x(x^2+1)(4x^2+3)=2Q_{-3}^{(2)}(x)$.

(C'(a)): $U_{-1}^{(1)}(x)T_{-2}^{(1)}(x)-U_{-2}^{(1)}(x)T_{-1}^{(1)}(x)=-8(x^2-1)=-2U_1^{(2)}(x)$.

(C'(b)): $\mathcal{V}_{-1}^{(1)}(x)\mathcal{V}_{-2}^{(1)}(x)-4(x^2-1)\mathcal{U}_{-1}^{(1)}(x)\mathcal{U}_{-2}^{(1)}(x)=-16x(x^2-1)=-2\mathcal{V}_1^{(2)}(x)$.

(D'): $\mathcal{F}_{-1}^{(3)}(x)-2\mathcal{F}_{-3}^{(3)}(x)=-\dfrac{3x}{4}(9x-8)^2=f_{-2}^{(3)}(x)$.

(E'): $j_{-2}^{(4)}(x)+2xj_{-4}^{(4)}(x)=\dfrac{(2x+1)(8x+1)^3}{8x^3}=(8x+1)J_{-3}^{(4)}(x)$.

In (C'(a)), $x=\cos\theta\,(\neq 1)$.

In (C'(b)), $x=\cosh t\,(\neq 1)$.

4. SYNTHESIS

Elementary algebraic calculations in (1.1), (1.2) when m and n are positive or negative allows us to assert the following synopsis of the relationships connecting $W_n^{(k)}(x)$ and $\mathcal{W}_n^{(k)}(x)$.

Theorem F': For all integers m and n,

$$\begin{cases} W_n^{(2m)}(x) = \mathcal{W}_n^{(2m-1)}(x) = \Delta^{2m}(x)W_n(x), \\ \mathcal{W}_n^{(2m)}(x) = W_n^{(2m+1)}(x) = \Delta^{2m}(x)\mathcal{W}_n(x). \end{cases}$$

Examples

$$F_{-3}^{(6)}(x) = L_{-3}^{(5)}(x) = (x^2+4)^3(x^2+1),$$

$$j_4^{(-4)}(x) = J_4^{(-3)}(x) = (8x^2+8x+1)(8x+1)^{-2},$$

$$f_{-3}^{(-5)}(x) = \mathcal{F}_{-3}^{(-4)}(x) = -\frac{(9x^2-2)(9x^2-8)^{-2}}{8},$$

$$T_{-4}^{(-3)}(x) = U_{-4}^{(-2)}(x) = -x(2x^2-1)(x^2-1)^{-1}.$$

This synthesis extends and complements that presented in [2].

Numerical Specializations

Throughout this paper it is useful to make appropriate numerical substitutions in theory. So,

$$F_{-3}^{(6)}(1) = L_{-3}^{(5)}(1) = 250,$$

$$j_4^{(-4)}(1) = J_4^{(-3)}(1) = \frac{17}{81},$$

$$f_{-3}^{(-5)}(1) = \mathcal{F}_{-3}^{(-4)}(1) = -\frac{7}{8},$$

$$T_{-4}^{(-3)}\left(\frac{1}{2}\right) = U_{-4}^{(-2)}\left(\frac{1}{2}\right) = -\frac{1}{3}.$$

5. A CONCLUDING MISCELLANY

Simson Formulas

Analogs of *Simson's formula* are readily established by means of (1.1), (1.2) for $k > 0$, with immediate extension when $k \to -k$:

$$W_{n+1}^{(k)}(x)W_{n-1}^{(k)}(x) - \{W_n^{(k)}(x)\}^2 = (-1)^{k+1}(-q(x))^{n-1}\Delta^{2k}(x), \tag{5.1}$$

and

$$\mathcal{W}_{n+1}^{(k)}(x)\mathcal{W}_{n-1}^{(k)}(x) - \{\mathcal{W}_n^{(k)}(x)\}^2 = (-1)^{k}(-q(x))^{n-1}\Delta^{2k+2}(x). \tag{5.2}$$

Similar results apply when $n \to -n$.

Variations of these orthodox Simson formulas (Simsonic variations!) include the "*inverted*" *Simson formulas*

$$W_n^{(k+1)}(x)W_n^{(k-1)}(x) - \{W_n^{(k)}(x)\}^2 = 4(-1)^{k}(-q(x))^{n}\Delta^{2k-2}(x), \tag{5.3}$$

and

$$\mathcal{W}_n^{(k+1)}(x)\mathcal{W}_n^{(k-1)}(x) - \{\mathcal{W}_n^{(k)}(x)\}^2 = 4(-1)^{k+1}(-q(x))^{n}\Delta^{2k}(x) \tag{5.4}$$

in which the roles of subscript and superscript in (5.1) and (5.2) have been reversed.

Hybrid Results

Use of (1.1), (1.2) produces the "hybrid Simson formulas"

$$W_{n+1}^{(k)}(x)W_{n-1}^{(k)} - \Delta^{-2}(x)\{\mathcal{W}_n^{(k)}(x)\}^2 = -(-1)^k p^2(x)(-q(x))^{n-1}\Delta^{2k-2}(x), \tag{5.5}$$

and

$$\Delta^{-2}(x)\mathcal{W}_{n+1}^{(k)}(x)\mathcal{W}_{n-1}^{(k)}(x) - \{W_n^{(k)}(x)\}^2 = (-1)^k p^2(x)(-q(x))^{n-1}\Delta^{2k-2}(x). \tag{5.6}$$

Clearly,

$$(5.5)+(5.6)=0 \qquad (i).$$

This is also confirmed by looking at

$$(5.1)+\Delta^{-2}(5.2)=0 \qquad (ii),$$

since the left-hand sides of (i), (ii) are merely re-arrangements of each other.

Further formulas arise when $k \to -k$ and/or $n \to -n$.

Searching for new results involving the data in this paper is an extremely pleasurable activity. Readers may wish to reflect on some of the possibilities.

Surveying the material in this paper and in [2], one is left wondering whether there may be other sets of polynomial-pairs whose major properties may be assembled by means of a synthesis of some kind.

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ON THE EXISTENCE OF EVEN FIBONACCI PSEUDOPRIMES WITH PARAMETERS P AND Q

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(Submitted June 1994)

1. INTRODUCTION

Let us consider the generalized Lucas sequence $\{V_n\}$ defined by the recurrence relation

$$V_n = PV_{n-1} - QV_{n-2};\ V_0 = 2,\ V_1 = P,$$

where P is a positive integer and $Q = \pm 1$. A Fibonacci pseudoprime with parameters P and Q [(P, Q)-FPSP] is a composite number n such that

$$V_n \equiv P \pmod{n}.$$

Recently (see [1], [2], and [5]), the following theorem was proved.

Theorem 1: There do not exist even Fibonacci pseudoprimes with parameters $P = 1$ and $Q = -1$.

In this paper, our aim is to investigate the existence of the even (P, Q)-FPSP. We shall prove the following result.

Theorem 2: If $(P, Q) \neq (1, -1)$ and $(P, Q) \neq (1, 1)$, then there exists at least one even Fibonacci pseudoprime with parameters P and Q.

Theorem 2 is a consequence of Theorem 1 and of the following propositions.

Proposition 1: There do not exist even Fibonacci pseudoprimes with parameters $P = Q = 1$.

Proposition 2: $n = 2^k, k \geq 2$. is a (P, Q)-FPSP, $Q = \pm 1$, if and only if $P \equiv 2 \pmod{2^k}$ or $P \equiv -1 \pmod{2^k}$.

Proposition 3: If $P \equiv 0 \pmod 4$ or $P \equiv 1 \pmod 4$ (with $P \neq 1$) and if $(P, Q) \neq (5, 1)$, then there exists an odd prime number p such that $n = 2p$ is an even (P, Q)-FPSP.

Proposition 4: There exist odd prime numbers p and q, with $p \neq q$, such that $n = 2pq$ is an even $(5, 1)$-FPSP.

2. PRELIMINARIES

In this section, we shall gather some lemmas which will be needed in the sequel.

Lemma 1: If $P \equiv 0 \pmod 4$ and $Q = \pm 1$, then the number $A = P^2 - P - 2Q$ admits an odd prime divisor.

Proof: We have $A \equiv 2 \pmod 4$ since $P \equiv 0 \pmod 4$, whence A admits an odd prime divisor, unless $A = \pm 2$, which is clearly impossible.

Lemma 2: If $P \equiv 1 \pmod 4$ and $Q = \pm 1$, then the number $A = P^2 - P - 2Q$ admits an odd prime divisor $p \neq 3$, unless $(P, Q) = (1, -1)$, $(P, Q) = (1, 1)$, or $(P, Q) = (5, 1)$.

Proof: We have $A \equiv 2 \pmod 4$ since $P \equiv 1 \pmod 4$, whence A admits an odd prime divisor, unless $A = \pm 2$. We consider two possibilities:

(a) Assuming first that $Q = -1$, we see that $A = P^2 - P + 2 = \pm 2$ if and only if $P = 1$. Moreover, $A \equiv \pm 1 \pmod 3$. Thus, A admits an odd prime divisor $p \neq 3$ when $P \neq 1$.

(b) Supposing now that $Q = 1$, we see that $A = P^2 - P - 2 = \pm 2$ if and only if $P = 1$. Moreover, $A \equiv 0 \pmod 3$ only if $P \equiv 2 \pmod 3$. Thus, A admits an odd prime divisor $p \neq 3$, except possibly when $P \equiv 1 \pmod 4$ and $P \equiv 2 \pmod 3$; in other words, when $P \equiv 5 \pmod{12}$. If $P = 5$, then $A = 18 = 2 \cdot 3^2$. If $P > 5$, we put $P = 12k + 5\,(k \geq 1)$ and we get that $A = 18(2k+1)(4k+1)$ and at least one of the factors $(2k+1)$ or $(4k+1)$ contains an odd prime divisor $p \neq 3$, since $\text{g.c.d.}(2k+1, 4k+1) = 1$. This completes the proof.

Lemma 3: Let $\{a_k\}$ be a sequence of integers defined by the recurrence relation

$$a_{k+1} = a_k^2 - 2, \quad k \geq 1. \tag{2.1}$$

If a_1 is even, then $a_k \equiv 2 \pmod{2^k}$, $k \geq 1$, and if a_1 is odd, then $a_k \equiv -1 \pmod{2^k}$, $k \geq 1$.

Proof: The statements clearly hold for $k = 1$. Let us suppose that $a_k \equiv \alpha \pmod{2^k}$, where $k \geq 1$ and $\alpha = -1$ or $\alpha = 2$ (notice that $\alpha^2 - 2 = \alpha$). Thus, we have

$$a_k = \alpha + \lambda 2^k, \text{ where } \lambda \text{ is an integer}$$

and

$$\begin{aligned} a_{k+1} &= a_k^2 - 2 = \alpha^2 - 2 + 2^{k+1}(\alpha\lambda + \lambda^2 2^{k-1}) \\ &\equiv \alpha^2 - 2 = \alpha \pmod{2^{k+1}}. \end{aligned}$$

This completes the proof.

3. PROOFS

Proof of Proposition 1: Let us consider the sequence

$$V_n = V_{n-1} - V_{n-2},\; V_0 = 2,\; V_1 = 1.$$

It is clear that the sequence $\{V_n\}$ is periodic, with period 6 and that

$$V_{6k} = 2, \; k \geq 0, \quad \text{and} \quad V_{6k \pm 2} = -1, \; k \geq 0,$$

which implies that there does not exist an even (1, 1)-FPSP.

Proof of Proposition 2: It is well known and readily proven [4] that, for every $n \geq 0$, $V_{2n} = V_n^2 - 2Q^n$, and thus that

$$V_{2^{k+1}} = V_{2^k}^2 - 2(\pm 1)^{2^k} = V_{2^k}^2 - 2, \text{ for } k \geq 1.$$

Hence, the sequence $a_k = V_{2^k}$ satisfies the recurrence relation (2.1). Noticing that $a_1 = V_2 = P^2 - 2Q \equiv P^2 \equiv P \pmod 2$, we see by Lemma 3 that $V_{2^k} \equiv 2 \pmod{2^k}$ if $k \geq 1$ and P is even, and that

$V_{2^k} \equiv -1 \pmod{2^k}$ if $k \geq 1$ and P is odd. Hence 2^k $(k > 1)$ is a (P, Q)-FPSP if and only if $P \equiv 2$ (mod 2^k) or $P \equiv -1 \pmod{2^k}$.

Remark: The proof for P odd positive and $Q = -1$ can be found on page 175 in [2].

Corollary: $n = 4$ is a (P, Q)-FPSP if and only if $P \equiv 2 \pmod 4$ or $P \equiv -1 \pmod 4$.

Proof of Proposition 3: We first recall some well-known properties:

(i) If P is even, then $V_n \equiv 0 \pmod 2$ for every $n \geq 0$.
(ii) If P is odd, then $V_n \equiv 0 \pmod 2$ if and only if $n \equiv 0 \pmod 3$.
(iii) If p is a prime number, then $V_{np} \equiv V_n \pmod p$ for every $n \geq 0$.

For a proof of (iii), the reader may wish to consult [3] or [4]. Let us now suppose that $P \equiv 0$ (mod 4) or $P \equiv 1 \pmod 4$ and that p is an odd prime number. The congruence $V_{2p} \equiv P \pmod{2p}$ is equivalent to the system

$$V_{2p} \equiv P \pmod 2 \tag{3.1}$$

and

$$V_{2p} \equiv P \pmod p. \tag{3.2}$$

By (i) and (ii), the congruence (3.1) holds for every odd prime number p if $P \equiv 0 \pmod 4$ and for every prime number $p > 3$ if $P \equiv 1 \pmod 4$. By (iii), we see that (3.2) is equivalent to $V_2 \equiv P$ (mod p) which can also be written

$$P^2 - P - 2Q \equiv 0 \pmod p. \tag{3.3}$$

If $P \equiv 0 \pmod 4$, we see by Lemma 1 that there exists an odd prime number p such that (3.3) and, thus, (3.2) hold. If $P \equiv 1 \pmod p$ and $P > 5$, we see by Lemma 2 that the same result holds (with $p > 3$), so the proof is complete.

Remark: If $(P, Q) = (5, 1)$, we see by Lemma 2 that there does not exist an odd prime number p such that $n = 2p$ is a $(5, 1)$-FPSP. Actually, we see by (3.3) that $p = 3$ and by (ii) we have $V_{2p} = V_6 \equiv 0 \not\equiv 5 \pmod 2$.

Proof of Proposition 4: Let us suppose that $(P, Q) = (5, 1)$. We shall prove that $n = 6554 = 2 \cdot 29 \cdot 113$ is an even $(5, 1)$-FPSP. Let $N(p)$ be the period of the sequence $\{V_n\}$ modulo p. By direct computation, one can see that $N(2) = 3$, $N(29) = 5$, and $N(113) = 57$. We also see that $6554 \equiv -1$ [mod $N(p)$], where $p = 2$, $p = 29$, or $p = 113$. Hence,

$$V_{6554} = V_{kN(p)-1} \equiv V_{N(p)-1} = 5V_{N(p)} - V_{N(p)+1} = 5 \pmod p,$$

and therefore,

$$V_{6554} \equiv 5 \pmod{6554}.$$

This completes the proof.

Remark: One can also verify that the numbers $11026 = 2 \cdot 37 \cdot 149$, $26506 = 2 \cdot 29 \cdot 457$, and $119074 = 2 \cdot 29 \cdot 2053$ are even $(5, 1)$-FPSP. This can be easily checked, noticing that $N(37) = 9$, $N(149) = 75$, and $N(457) = N(2053) = 57$.

ACKNOWLEDGMENT

I am grateful to the referee for pointing out a generalization of an earlier version of Proposition 2.

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REFEREES

In addition to the members of the Board of Directors and our Assistant Editors, the following mathematicians, engineers, and physicists have assisted *The Fibonacci Quarterly* by refereeing papers during the past year. Their special efforts are sincerely appreciated, and we apologize in advance for any names that inadvertently may have been overlooked or misspelled.

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VOLUME 5
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ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-802 ***Proposed by Al Dorp, Edgemere, NY***

For $n > 0$, let $T_n = n(n+1)/2$ denote the n^{th} triangular number. Find a formula for T_{2n} in terms of T_n.

B-803 ***Proposed by Herta T. Freitag, Roanoke, VA***

For n even and positive, evaluate

$$\sum_{i=0}^{n/2} \binom{n}{i} L_{n-2i}.$$

B-804 ***Proposed by the editor***

Find integers a, b, c, and d (with $1 < a < b < c < d$) that make the following an identity:

$$F_n = F_{n-a} + 9342F_{n-b} + F_{n-c} + F_{n-d}.$$

B-805 ***Proposed by David Zeitlin, Minneapolis, MN***

Solve the recurrence $P_{n+6} = P_{n+5} + P_{n+4} - P_{n+2} + P_{n+1} + P_n$, for $n \ge 0$, with initial conditions $P_0 = 1, P_1 = 1, P_2 = 2, P_3 = 3, P_4 = 4,$ and $P_5 = 7$.

B-806 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN***

(a) Show that the coefficient of every term in the expansion of $\frac{x}{1-2x+x^3}$ is the difference of two Fibonacci numbers.

(b) Show that the coefficient of every term in the expansion of $\frac{x}{1-2x-2x^2+x^3}$ is the product of two consecutive Fibonacci numbers.

B-807 ***Proposed by R. André-Jeannin, Longwy, France***

The sequence $\langle W_n \rangle$ is defined by the recurrence $W_n = PW_{n-1} - QW_{n-2}$, for $n \geq 2$, with initial conditions $W_0 = a$ and $W_1 = b$, where a and b are integers and P and Q are odd integers. Prove that, for $k \geq 0$,

$$W_{n+3\cdot 2^k} \equiv W_n \pmod{2^{k+1}}.$$

SOLUTIONS

An Integral Ratio

B-772 ***Proposed by Herta Freitag, Roanoke, VA***
(Vol. 32, no. 5, November 1994)

Prove that

$$\frac{L_n^2 + L_{n+a}^2}{F_n^2 + F_{n+a}^2}$$

is always an integer if a is odd. How should this problem be modified if a is even?

Solution by C. Georghiou, University of Patras, Patras, Greece

Identity (I_{12}) of [1] reads $5F_n^2 = L_n^2 - 4(-1)^n$. It follows that, for a odd,

$$L_n^2 + L_{n+a}^2 = 5(F_n^2 + F_{n+a}^2);$$

whereas, for a even,

$$L_n^2 - L_{n+a}^2 = 5(F_n^2 - F_{n+a}^2);$$

and the integer, in both cases, is 5. In other words, for $a \neq 0$, we have

$$\frac{L_n^2 - (-1)^a L_{n+a}^2}{F_n^2 - (-1)^a F_{n+a}^2} = 5.$$

When a is even, Zeitlin and Filipponi (independently) found the formula

$$\frac{L_n^2 + L_{n+a}^2 - 8(-1)^n}{F_n^2 + F_{n+a}^2} = 5.$$

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Also solved by Michel A. Ballieu, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, Russell Jay Hendel, Norbert Jensen, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, M. N. S. Swamy, David C. Terr, David Zeitlin, and the proposer.

Zecky Would Be Proud

B-773 ***Proposed by Herta Freitag, Roanoke, VA***
(Vol. 32, no. 5, November 1994)

Find the number of terms in the Zeckendorf representation of $S_n = \sum_{i=1}^n F_i^2$.

[The Zeckendorf representation of an integer expresses that integer as a sum of distinct nonconsecutive Fibonacci numbers.]

Solution by David C. Terr, University of California, Berkeley, CA

We claim that

$$S_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} F_{2n-4k-1}$$

is the Zeckendorf representation of S_n. It consists of $\lfloor (n+1)/2 \rfloor$ terms.

Proof: Clearly this equation holds for $n=1$ and $n=2$. Thus, it is enough to show that $S_n - S_{n-2} = F_{2n-1}$ for $n>1$. But $S_n - S_{n-2} = F_n^2 + F_{n-1}^2$ and $F_n^2 + F_{n-1}^2 = 5F_{2n-1}$ by identity (I_{11}) of [1]. Thus, the result follows.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Santa Clara, CA: The Fibonacci Association, 1979.

The proposer also found that the number of terms in the Zeckendorf representation of $\sum_{i=1}^{n} L_i^2$ is n. A very nice result. No other reader came up with any generalization of this problem.

Also solved by Michel A. Ballieu, Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Joseph J. Koštál, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, David Zeitlin, and the proposer.

A Congruence for the Period

B-774 ***Proposed by Herta Freitag, Roanoke, VA***
(Vol. 32, no. 5, November 1994)

Let $\langle H_n \rangle$ be any sequence of integers such that $H_{n+2} = H_{n+1} + H_n$ for all n. Let p and m be positive integers such that $H_{n+p} \equiv H_n \pmod{m}$ for all integers n. Prove that the sum of any p consecutive terms of the sequence is divisible by m.

Solution by Leonard A. G. Dresel, Reading, England

Let $S(a)$ be the sum of p consecutive terms of the sequence starting with H_a. Using the recurrence relation $H_n = H_{n+2} - H_{n+1}$, we have

$$\begin{aligned} S(a) &= H_a + H_{a+1} + \cdots + H_{a+p-1} \\ &= (H_{a+2} - H_{a+1}) + (H_{a+3} - H_{a+2}) + \cdots + (H_{a+p+1} - H_{a+p}) \\ &= H_{a+p+1} - H_{a+1} \quad \text{(since all other terms cancel)} \\ &\equiv 0 \pmod{m}, \end{aligned}$$

for any value of a. Therefore, the sum of any p consecutive terms is divisible by m.

Also solved by Michel Ballieu, Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Joseph J. Koštál, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, M. N. S. Swamy, David Zeitlin, and the proposer.

Golden Powers

B-775 ***Proposed by Herta Freitag, Roanoke, VA***
(Vol. 32, no. 5, November 1994)

Let $g = \alpha + 2$. Express g^{17} in the form $p\alpha + q$, where p and q are integers.

Solution by Norbert Jensen, Kiel, Germany

We show that

$$g^{2n} = 5^n(F_{2n}\alpha + F_{2n-1})$$
$$g^{2n+1} = 5^n(L_{2n+1}\alpha + L_{2n})$$

for all integers n. It then follows that $g^{17} = 5^8 L_{17}\alpha + 5^8 L_{16} = 1394921875\alpha + 862109375$.

Proof of the general assertion. We use the following identities:

(i) $$g = \alpha + 2 = \alpha^2 + 1 = \alpha(\alpha - \beta) = \alpha\sqrt{5}.$$

(ii) $$\alpha^n = F_n\alpha + F_{n-1}.$$

(iii) $$F_n + F_{n+2} = L_{n+1}$$

Both (ii) and (iii) are easily proved by induction on n (or they can be found in [1] on pages 52 and 24, respectively).

Applying (i) and (ii), we obtain

$$g^{2n} = 5^n\alpha^{2n} = 5^n(F_{2n}\alpha + F_{2n-1}).$$

Therefore, by (iii), $g^{2n+1} = 5^n(F_{2n}\alpha + F_{2n-1})(\alpha + 2) = 5^n[(3F_{2n} + F_{2n-1})\alpha + F_{2n} + 2F_{2n-1}] = 5^n[(F_{2n} + F_{2n+2})\alpha + F_{2n-1} + F_{2n+1}] = 5^n(L_{2n+1}\alpha + L_{2n})$.

Reference

1. S. Vajda. Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications. Chichester, England: Ellis Horwood Ltd., 1989.

Generalization 1 by David Zeitlin, Minneapolis, MN

Let a and b be distinct roots of $x^2 = Px - Q$. Let $\langle U_n\rangle$ and $\langle V_n\rangle$ be sequences defined by the recurrences $U_{n+2} = PU_{n+1} - QU_n$, $U_0 = 0, U_1 = 1$, and $V_{n+2} = PV_{n+1} - QV_n$, $V_0 = 2, V_1 = P$. We will show that

$$(Pa - 2Q)^{2n+1} = (P^2 - 4Q)^n(aV_{2n+1} - QV_{2n})$$

for all nonnegative integers n. (In our problem, $P = 1, Q = -1, a = \alpha, V_n = L_n$, and $n = 8$.)

To prove this identity, we note that, since $Q = ab$, $Pa - 2Q = (Pa - Q) - Q = a^2 - ab = a(a - b) = a(P^2 - 4Q)^{1/2}$. Thus, $(Pa - 2Q)^{2n+1} = a^{2n+1}(P^2 - 4Q)^{n+1/2}$. Since $a^k = aU_k - QU_{k-1}$,

$$\begin{aligned}(P^2 - 4Q)^{1/2}a^{2n+1} &= (P^2 - 4Q)^{1/2}(aU_{2n+1} - QU_{2n})\\ &= a(a^{2n+1} - b^{2n+1}) - Q(a^{2n} - b^{2n})\\ &= a^{2n+2} - Qb^{2n} - Qa^{2n} + Qb^{2n} = a^{2n+2} - Qa^{2n}\\ &= (aU_{2n+2} - QU_{2n+1}) - Q(aU_{2n} - QU_{2n-1})\\ &= a(U_{2n+2} - QU_{2n}) - Q(U_{2n+1} - QU_{2n-1})\\ &= aV_{2n+1} - QV_{2n}, \quad \text{since } V_n = U_{n+1} - QU_{n-1}.\end{aligned}$$

This proves the result. In the same manner, we can also prove

$$(Pa - 2Q)^{2n} = (P^2 - 4Q)^n(aU_{2n} - QU_{2n-1}),$$
$$(Pb - 2Q)^{2n+1} = (P^2 - 4Q)^n(bV_{2n+1} - QV_{2n}),$$
$$(P_b - 2Q)^{2n} = (P^2 - 4Q)^n(bU_{2n} - QU_{2n-1}).$$

Generalization 2 by Murray S. Klamkin, University of Alberta, Canada

We will show that, if m and r are given integers (with $r > 0$) and if we let $g = \alpha + m$, then g^r can be written as $p\alpha + q$ for some integers p and q. Let $g^r = p_r\alpha + q_r$. Then

$$\begin{aligned} g^{r+1} &= (p_r\alpha + q_r)(\alpha + m) = p_{r+1}\alpha + q_{r+1} \\ &= p_r\alpha^2 + (mp_r + q_r)\alpha + mq_r = p_r(\alpha + 1) + (mp_r + q_r)\alpha + mq_r. \end{aligned}$$

Hence, $p_{r+1} = (m+1)p_r + q_r$ and $q_{r+1} = p_r + mq_r$. It then follows that

$$p_{r+2} = (2m+1)p_{r+1} - (m^2 + m - 1)p_r \quad \text{and} \quad q_{r+2} = (2m+1)q_{r+1} - (m^2 + m - 1)q_r$$

with $p_0 = 0, q_0 = 1, p_1 = 1,$ and $q_1 = m$. Thus, p_r and q_r are integers for all positive integers r.

Klamkin went on to give explicit formulas for p_r and q_r. Anderson found the same generalization as Klamkin.

Also solved by Mark Anderson, Michel Ballieu, Glenn A. Bookout, Scott H. Brown, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, F. J. Flanigan, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, M. N. S. Swamy, David C. Terr, and the proposer.

An Even Sum

B-776 ***Proposed by Herta Freitag, Roanoke, VA (Vol. 32, no. 5, November 1994)***

Find all values of n for which $S_n = \sum_{k=1}^{n} kF_k$ is even.

Solution by Paul S. Bruckman, Edmonds, WA

The periodic sequences $\langle n \pmod 2\rangle$ and $\langle F_n \pmod 2\rangle$ for $n > 0$ have periods 2 and 3, respectively. Therefore, the sequence $\langle nF_n \pmod 2\rangle$ has period 6, as does the sequence $\langle S_n \pmod 2\rangle$. We may then form the following table:

n	F_n	$nF_n \pmod 2$	$S_n \pmod 2$
1	1	1	1
2	1	0	1
3	2	0	1
4	3	0	1
5	5	1	0
6	8	0	0

By the foregoing comments, and by inspection of the table above, we conclude that S_n is even if and only if n is congruent to 0 or 5 modulo 6.

Also solved by Charles Ashbacher, Michel Ballieu, Charles K. Cook, Leonard A. G. Dresel, Piero Filipponi, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Joseph J. Koštál, Carl Libis, H.-J. Seiffert, Sahib Singh, Lawrence Somer, M. N. S. Swamy, David Zeitlin, and the proposer.

A Tricky Congruence Criterion

B-777 ***Proposed by Herta Freitag, Roanoke, VA (Vol. 32, no. 5, November 1994)***

Find all integers a such that $n \equiv a \pmod 4$ if and only if $L_n \equiv a \pmod 5$.

Solution by Paul S. Bruckman, Edmonds, WA

As is easily verified, the periodic sequence $\langle L_n \pmod 5\rangle_{n\ge 0} = (2,1,3,4,2,1,3,4,\ldots)$ has period equal to 4. The solutions of our problem are therefore those values of a modulo 20 such that any of the following conditions holds:

(i) $a \equiv 0 \pmod 4$ and $a \equiv L_0 = 2 \pmod 5$;
(ii) $a \equiv 1 \pmod 4$ and $a \equiv L_1 = 1 \pmod 5$;
(iii) $a \equiv 2 \pmod 4$ and $a \equiv L_2 = 3 \pmod 5$;
(iv) $a \equiv 3 \pmod 4$ and $a \equiv L_3 = 4 \pmod 5$.

The solutions of (i)-(iv), respectively, are as follows: $a \equiv 12, 1, 18$, or $19 \pmod{20}$. Therefore, $n \equiv a \pmod 4$ if and only if $L_n \equiv a \pmod 5$, whenever $a \equiv 1, 12, 18$, or $19 \pmod{20}$, and for no other values of a.

A related fact: $n \equiv a \pmod 4$ if and only if $L_n \equiv L_a \pmod 5$, appeared in this Quarterly **32.3** (1994):245.

Also solved by Charles Ashbacher, Norbert Jensen, H.-J. Seiffert, Lawrence Somer, David C. Terr, and the proposer. Six incorrect solutions were received.

Fibonacci's Last Theorem

B-778 ***Proposed by Eliot Jacobson, Ohio University, Athens, OH (Vol. 33, no. 1, February 1995)***

Show that the equation $x^n + y^n = z^n$ has no nontrivial solutions consisting entirely of Fibonacci numbers, for $n \ge 2$.

Solution by the proposer

Let $z = F_c$, with $c \ge 3$. Using the fact that $F_cF_{c-2} = F_{c-1}^2 + (-1)^{c-1}$, which is Identity ($I_{13}$) from [1], observe that

$$\begin{aligned} F_c^2 &= F_c(F_{c-1} + F_{c-2}) \\ &= F_cF_{c-1} + F_cF_{c-2} \\ &\ge (1 + F_{c-1})F_{c-1} + F_cF_{c-2} \\ &= F_{c-1} + F_{c-1}^2 + F_{c-1}^2 + (-1)^{c-1} \\ &> F_{c-1}^2 + F_{c-1}^2. \end{aligned}$$

Since $F_c > F_{c-1}$, it follows that $F_c^n > F_{c-1}^n + F_{c-1}^n$, so the equality can never hold.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Several solvers noted that it suffices to consider positive Fibonacci numbers in light of the identity $F_{-k} = (-1)^{k+1}F_k$. Seiffert also proved Lucas's Last Theorem: If $n \ge 2$ is an integer, then the equation $x^n + y^n = z^n$ has no solution in Lucas numbers, x, y, and z. Prielipp noted that this problem is equivalent to Theorem 6 in L. Carlitz, "A Note on Fibonacci Numbers," this Quarterly **2.1** *(1964):15-28.*

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, C. Georghiou, Norbert Jensen, Bob Prielipp, and H.-J. Seiffert. Andrew Wiles could not be reached for comment.

Find the Identity

B-779 *Proposed by Andrew Cusumano, Great Neck, NY*
(Vol. 33, no. 1, February 1995)

Find integers a, b, c, and d (with $1<a<b<c<d$) that make the following an identity:

$$F_n = F_{n-a} + 6F_{n-b} + F_{n-c} + F_{n-d}.$$

Editorial Comment: Most solvers pulled the answer out of a hat:

$$F_n = F_{n-2} + 6F_{n-5} + F_{n-6} + F_{n-8}.$$

The proof by induction is then straightforward. Only Bruckman and Georghióu submitted a proof that showed how to find a, b, c, and d; but their methods do not seem to generalize. So to test your prestidigitation abilities, the editor has concocted a related problem (see Problem B-804 in this issue). Let's see who can pull a rabbit out of *that* hat.

Solved by Paul S. Bruckman, Leonard A. G. Dresel, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Bob Prielipp, H.-J. Seiffert, David Zeitlin, and the proposer.

Production Inequality

B-780 *Proposed by Zdravko F. Starc, Vršac, Yugoslavia*
(Vol. 33, no. 1, February 1995)

Prove that:

(a) $$F_1\cdot F_2\cdot F_3\cdot\cdots\cdot F_n \le \exp(F_{n+2}-n-1);$$
(b) $$F_1\cdot F_3\cdot F_5\cdot\cdots\cdot F_{2n-1} \le \exp(F_{2n}-n);$$
(c) $$F_2\cdot F_4\cdot F_6\cdot\cdots\cdot F_{2n} \le \exp(F_{2n+1}-n-1).$$

Solution by David Zeitlin, Minneapolis, MN

All proofs are by mathematical induction. Since $1+x\le \exp(x)$ for $x>0$, we get (letting $y=x+1$): $y\le\exp(y-1)$ for $y>1$. We will use this inequality repeatedly, below. The results are easily seen to be true for $n=1$ and $n=2$, so we need only give the induction step.

Proof of (a): $$\begin{aligned}(F_1F_2F_3\ldots F_n)F_{n+1} &\le F_{n+1}\cdot\exp(F_{n+2}-n-1)\\ &\le \exp(F_{n+1}-1)\cdot\exp(F_{n+2}-n-1)\\ &= \exp(F_{n+1}-1+F_{n+2}-n-1) = \exp(F_{n+3}-(n+1)-1).\end{aligned}$$

Proof of (b): $$\begin{aligned}(F_1F_3F_5\ldots F_{2n-1})F_{2n+1} &\le F_{2n+1}\cdot\exp(F_{2n}-n)\\ &\le \exp(F_{2n+1}-1)\cdot\exp(F_{2n}-n)\\ &= \exp(F_{2n+1}-1+F_{2n}-n) = \exp(F_{2(n+1)}-(n+1)).\end{aligned}$$

Proof of (c): $$\begin{aligned}(F_2F_4F_6\ldots F_{2n})F_{2n+2} &\le F_{2n+2}\cdot\exp(F_{2n+1}-n-1)\\ &\le \exp(F_{2n+2}-1)\cdot\exp(F_{2n+1}-n-1)\\ &= \exp(F_{2n+2}-1+F_{2n+1}-n-1) = \exp(F_{2(n+1)+1}-(n+1)-1).\end{aligned}$$

Generalization 1 by H.-J. Seiffert, Berlin, Germany

We shall prove that for all positive integers n,

(a′) $$F_1\cdot F_2\cdot F_3\cdot\cdots\cdot F_n \le 2^{F_{n+2}-n-1};$$
(b′) $$F_1\cdot F_3\cdot F_5\cdot\cdots\cdot F_{2n-1} \le 2^{F_{2n}-n};$$
(c′) $$F_2\cdot F_4\cdot F_6\cdot\cdots\cdot F_{2n} \le 3^{(F_{2n+1}-n-1)/2}.$$

Since $F_{2n+1} > F_{2n} \geq n$, $F_{n+2} \geq n+1$, $n \geq 1$; and $\sqrt{3} < 2 < e$, it is obvious that these inequalities imply the proposed ones. In (a′) and (b′), the base 2 cannot be replaced by a smaller base, as is readily seen by setting $n=3$ and $n=2$, respectively. Similarly, in (c′), the base $\sqrt{3}$ is best possible, as one can see by taking $n=2$.

From identities (I_1), (I_5), and (I_6) of [1], we obtain

$$\sum_{k=1}^{n}(F_k - 1) = F_{n+2} - n - 1, \tag{1}$$

$$\sum_{k=1}^{n}(F_{2k-1} - 1) = F_{2n} - n, \tag{2}$$

and

$$\sum_{k=1}^{n}(F_{2k} - 1) = F_{2n+1} - n - 1. \tag{3}$$

It is easily seen that $m \leq 2^{m-1}$ for all positive integers m. Thus, (a′) follows by considering the product of the inequalities $F_k \leq 2^{F_k - 1}$, $k = 1,2,3,\ldots,n$, and applying (1). Similarly, (b′) follows from the inequalities $F_{2k-1} \leq 2^{F_{2k-1}-1}$, $k = 1,2,3,\ldots,n$, and (2). For the proof of (c′), we note that $m \leq 3^{(m-1)/2}$ for all positive integers m such that $m \neq 2$. Since no Fibonacci number with even subscript equals 2, (c′) follows from $F_{2k} \leq 3^{(F_{2k}-1)/2}$, $k = 1,2,3,\ldots,n$, and (3).

Generalization 2 by H.-J. Seiffert, Berlin, Germany

We will show that

$$\prod_{k=1}^{n} F_{2k} \leq \begin{cases} F_{n+1}^{n}, & \text{for odd } n, \\ (L_{n+1}/\sqrt{5})^n, & \text{for even } n. \end{cases}$$

[Also: lower bounds are $(2/e)^n$ times the upper bounds.] This follows by taking $q = \beta^2/\alpha^2$ in the inequality

$$(2/e)^n(1-q^{(n+1)/2})^n \leq \prod_{k=1}^{n}(1-q^k) \leq (1-q^{(n+1)/2})^n,$$

which holds for all $q \in (0, 1)$ and all positive integers n. This inequality comes from [2].

References

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.
2. H.-J. Seiffert. "Problem 4406." *School Science and Mathematics* **94.1** (1994):54.

Also solved by Šefket Arslahagić, Paul S. Bruckman, Charles K. Cook, L. A. G. Dresel, C. Georghiou, Pentti Haukkanen, Russell J. Hendel, Norbert Jensen, Joseph J. Koštál, Can. A. Minh, Bob Prielipp, H.-J. Seiffert, and the proposer.

Addendum: Adam Stinchcombe was inadvertently omitted as a solver of Problems B-769 and B-771.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-506 ***Proposed by Paul S. Bruckman, Highwood, IL***

Let

$$A=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{5n+1}+\frac{1}{5n+4}\right) \quad \text{and} \quad B=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{5n+2}+\frac{1}{5n+3}\right).$$

Evaluate A and B, showing that $A=\alpha B$.

H-507 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN***

Prove that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{n}\sum_{k=1}^{m}(-1)^i 2^{-(k+1)(i+j)}\left(\frac{n(i+1)(i+2)\cdots(i+n-1)}{j!(n-j)!}\right)(F_k)^{i+j}=1.$$

H-508 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n\geq 2$. Show that, for all complex numbers x and y and all positive integers n,

$$F_n(x)F_n(y)=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}(x+y)^k F_{k+1}\left(\frac{xy-4}{x+y}\right). \tag{1}$$

As special cases of (1), obtain the following identities:

$$F_n(x)F_n(x+1)=n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}F_{k+1}(x^2+x+4); \tag{2}$$

$$F_n(x)F_n(4/x)=n\sum_{k=0}^{[\frac{n-1}{2}]}\frac{1}{2k+1}\binom{n+2k}{4k+1}\left(\frac{x^2+4}{x}\right)^{2k}, \quad x\neq 0; \tag{3}$$

$$F_n(x)^2=n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}(x^2+4)^k; \tag{4}$$

$$F_n(x)^2 = n\sum_{k=0}^{n-1} \frac{1}{k+1}\binom{n+k}{2k+1}\frac{x^{2k+2}-(-4)^{k+1}}{x^2+4}; \tag{5}$$

$$F_{2n-1}(x) = \sum_{k=0}^{2n-2} \frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}x^k F_{k+1}(4/x). \tag{6}$$

SOLUTIONS

Get Hyper

H-491 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 32, no. 5, November 1994)

Prove the following identities:

$$F_{2n} = 2\binom{2n}{n}^{-1}\sum_{k=0}^{n-1}\binom{n-\frac{1}{2}}{k}\binom{n-\frac{1}{2}}{n-1-k}5^k, \quad n = 1, 2, \ldots; \tag{a}$$

$$F_{2n+1} = \binom{2n}{n}^{-1}\sum_{k=0}^{n}\binom{n-\frac{1}{2}}{k}\binom{n+\frac{1}{2}}{n-k}5^k, \quad n = 0, 1, 2, \ldots. \tag{b}$$

Solution by the proposer

Proof of Part (a): Let θ_n denote the sum given in the right member of the statement of part (a). It is more convenient to evaluate θ_{n+1}. Thus:

$$\theta_{n+1} = 2\binom{2n+2}{n+1}^{-1}\sum_{k=0}^{n}\binom{n+\frac{1}{2}}{k}\binom{n+\frac{1}{2}}{n-k}5^k, \quad n = 0, 1, \ldots. \tag{1}$$

Now

$$2\binom{2n+2}{n+1}^{-1} = 2(n+1)^2/(2n+2)(2n+1)\binom{2n}{n} = (n+1)/(2n+1)(-4)^n\binom{-\frac{1}{2}}{n} = \frac{n+1}{(-4)^n\binom{-\frac{3}{2}}{n}};$$

also,

$$\binom{n+\frac{1}{2}}{n-k}\binom{-\frac{3}{2}}{k} = (-1)^k\binom{n+\frac{1}{2}}{n-k}\binom{k+\frac{1}{2}}{k} = (-1)^k\binom{n+\frac{1}{2}}{n}\binom{n}{k} = (-1)^{n+k}\binom{-\frac{3}{2}}{n}\binom{n}{k}.$$

Therefore, after some simplification, θ_{n+1} is transformed to the following expression:

$$\theta_{n+1} = (n+1)\cdot 4^{-n}\sum_{k=0}^{n}\binom{n+\frac{1}{2}}{k}\binom{n}{k}\binom{-\frac{3}{2}}{k}^{-1}(-5)^k. \tag{2}$$

We recognize the last expression as a special case of the Hypergeometric Function. The Hypergeometric Function ${}_2F_1[{a,b \atop c}; z]$ is defined if $c-a-b>0$ and $|z|<1$, as follows:

$${}_2F_1[{a,b \atop c}; z] = \sum_{n\ge 0}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}. \tag{3}$$

It may happen that the series in (3) terminates after a finite number of terms, i.e., is a polynomial in z, in which case the restriction $|z|<1$ may be removed.

Since

$$\binom{n+\frac{1}{2}}{k}\binom{n}{k}\binom{-\frac{3}{2}}{k}^{-1}(-5)^k = (-n-\tfrac{1}{2})_k\,(-n)_k\,(\tfrac{3}{2})_k^{-1}\frac{5^k}{k!},$$

we see that

$$\theta_{n+1} = (n+1)\cdot 4^{-n}\cdot {}_2F_1\left[\begin{matrix}-n-\frac{1}{2}, -n\\ \frac{3}{2}\end{matrix}; 5\right]. \tag{4}$$

(Note that this expression is well-defined, since θ_{n+1} is a finite sum).

The following formula is given as Formula 15.1.10 in [1]:

$${}_2F_1\left[\begin{matrix}a, a+\frac{1}{2}\\ \frac{3}{2}\end{matrix}; z^2\right] = (2z)^{-1}(1-2a)^{-1}[(1+z)^{1-2a}-(1-z)^{1-2a}]. \tag{5}$$

Setting $a=-n-\frac{1}{2}$, $z=\sqrt{5}$ in (5), and comparing with (4) yields:

$$\begin{aligned}\theta_{n+1} &= (n+1)\cdot 4^{-n}(2\sqrt{5})^{-1}(2n+2)^{-1}[(1+\sqrt{5})^{2n+2}-(1-\sqrt{5})^{2n+2}]\\ &= 2^{2n+2}\cdot 4^{-n-1}\cdot 5^{-\frac{1}{2}}(\alpha^{2n+2}-\beta^{2n+2}) = F_{2n+2}.\end{aligned}$$

Then $\theta_n = F_{2n}$.

Proof of Part (b): Let Φ_n denote the sum given in the right member of the statement of part (b). Then, performing manipulations similar to those used in the proof of part (a), we obtain:

$$\Phi_n = (2n+1)\cdot 4^{-n}\sum_{k=0}^{n}\binom{n-\frac{1}{2}}{k}\binom{n}{k}\binom{-\frac{3}{2}}{k}^{-1}(-5)^k. \tag{6}$$

Then, as in the proof of part (a), Φ_n may be expressed in terms of the Hypergeometric Function as follows:

$$\Phi_n = (2n+1)\cdot 4^{-n}\cdot {}_2F_1\left[\begin{matrix}-n, -n+\frac{1}{2}\\ \frac{3}{2}\end{matrix}; 5\right]. \tag{7}$$

This time, we set $a=-n$, $z=\sqrt{5}$ in (5), which yields:

$$\begin{aligned}\Phi_n &= (2n+1)\cdot 4^{-n}(2\sqrt{5})^{-1}(2n+1)^{-1}[(1+\sqrt{5})^{2n+1}-(1-\sqrt{5})^{2n+1}]\\ &= 4^{-n}(2\sqrt{5})^{-1}2^{2n+1}(\alpha^{2n+1}-\beta^{2n+1}) = F_{2n+1}. \quad \text{Q.E.D.}\end{aligned}$$

Reference

1. *Handbook of Mathematical Functions*, ed. M. Abramowitz & I. A. Stegun, National Bureau of Standards, Ninth Printing, November, 1970.

Also solved by N. Jensen and H.-J. Seiffert.

More Sums

H-492 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 32, no. 5, November 1994)***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n\geq 2$. Show that, for all complex numbers x and y and for all nonnegative integers n,

$$\sum_{k=0}^{[n/2]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y) = z^{n-1}F_n(xy/z), \tag{1}$$

where $z=(x^2+y^2+4)^{1/2}$. [ ] denotes the greatest integer function.

As special cases of (1), obtain the following identities:

$$\sum_{k=0}^{[n/2]}\binom{n}{k}F_{n-2k}^2=(3^n-(-2)^n)/5, \tag{2}$$

$$\sum_{k=0}^{n}\binom{2n+1}{n-k}F_{2k+1}=5^n, \tag{3}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_{2k}F_{4k}=5^{n-1}(4^n-1) \tag{4}$$

$$\sum_{k=0}^{n}\binom{2n+1}{n-k}F_{2k+1}F_{4k+2}=5^n(2^{2n+1}+1), \tag{5}$$

$$\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}F_{2n-4k}P_{n-2k}=F_n(6), \tag{6}$$

where $P_j=F_j(2)$ is the j^{th} Pell number,

$$\sum_{\substack{k=0\\(5,\,n-2k)=1}}^{[n/2]}(-1)^{[(n-2k+2)/5]}\binom{n}{k}=F_n. \tag{7}$$

The latter equation is the one given in H-444.

Solution by Norbert Jensen, Kiel, Germany

0. Note that the term on the right side of equation (1) is not defined for $z=0$, which can occur if $x, y\in\mathbb{C}$. However, the singularity is removable. For instance, it is easy to prove by induction that, for each $n\in\mathbb{N}_0$, there is a polynomial function $g_n:\mathbb{C}\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ such that $z^{n-1}F_n(xy/z)=g_n(x,y,z^2)$. [The start of the induction is trivial. Then

$$g_{n+2}=z^{n+1}F_{n+2}(xy/z)=z^{n+1}((xy/z)F_{n+1}(xy/z)+F_n(xy/z))=xyg_{n+1}(x,y,z^2)+z^2g_n(x,y,z^2).]$$

1. Let $x\in\mathbb{R}$. Define

$$D(x)=x^2+4,\ d(x)=\sqrt{D(x)},$$
$$\alpha(x)=\frac{1}{2}(x+d(x)),\ \beta(x)=\frac{1}{2}(x-d(x)).$$

The explicit formula for the polynomials $F_n(x)$ is

$$F_n(x)=\frac{\alpha(x)^n-\beta(x)^n}{d(x)}\quad\text{for all } n\in\mathbb{N}_0.$$

2. Identity (1) will be derived from the following: Let $a, b, c, d\in\mathbb{R}$ be such that $ab=cd$. We prove, for all $n\in\mathbb{N}_0$,

2.1 $$\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{k}(ab)^k[a^{n-2k}+b^{n-2k}-c^{n-2k}-d^{n-2k}]=(a+b)^n-(c+d)^n.$$

Proof: ***(i)*** Let $\lambda \in \mathbb{R}$. For all $n \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{k} \lambda^k (1+\lambda^{n-2k}) = \begin{cases} (1+\lambda)^n, & \text{if } n \text{ is odd,} \\ (1+\lambda)^n - \binom{n}{n/2}\lambda^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Proof of (i): Let $s = [(n-1)/2]$.

Case 1. n is odd. Then $s = (n-1)/2$. We have

$$\sum_{k=s+1}^{n} \binom{n}{k} \lambda^k = \sum_{k=s+1}^{n} \binom{n}{n-k} \lambda^k = \sum_{k=0}^{s} \binom{n}{k} \lambda^{n-k}.$$

Hence, the binomial theorem implies

$$(1+\lambda)^n = \sum_{k=0}^{s} \binom{n}{k} \lambda^k + \sum_{k=0}^{s} \binom{n}{k} \lambda^{n-k} = \sum_{k=0}^{s} \binom{n}{k} \lambda^k (1+\lambda^{n-2k}).$$

Case 2. n is even. Then $s = (n/2)-1$. The proof is similar. The expression on the left side contains just the first s and the last s of the $s+1$ terms of the binomial sum.

Hence, we have to subtract the term $\binom{n}{n/2}\lambda^{n/2}$ on the right side. Q.E.D.

(ii) Let $b \neq 0$. Substituting λ by a/b and homogenizing the expression, we obtain

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{k} (ab)^k (a^{n-2k} + b^{n-2k}) = \begin{cases} (a+b)^n, & \text{if } n \text{ is odd,} \\ (a+b)^n - \binom{n}{n/2}(ab)^{n/2}, & \text{if } n \text{ is even.} \end{cases}$$

It is easy to check that the above equation is true for $b = 0$ as well.

(iii) Substituting c for a and d for b in the above equation and subtracting the equation obtained in this way from the one above we get, for all $n \in \mathbb{N}_0$:

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{k} (ab)^k [a^{n-2k} + b^{n-2k} - c^{n-2k} - d^{n-2k}] = (a+b)^n - (c+d)^n.$$

If n is odd, then $[(n-1)/2] = [n/2]$; if n is even, then $[(n-1)/2] = n/2 - 1$. However, the term $a^{n-2k} + b^{n-2k} - c^{n-2k} - d^{n-2k} = 0$ for $k = n/2$, so we can replace $[(n-1)/2]$ by $[n/2]$ in both cases. Q.E.D.

3. We prove equation (1) for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}_0$.

3.1 Let $x, y \in \mathbb{R}$, and let

$$\begin{aligned} a &= (x+d(x))(y+d(y)), \\ b &= (x-d(x))(y-d(y)), \\ c &= (x+d(x))(y-d(y)), \\ d &= (x-d(x))(y+d(y)). \end{aligned}$$

Note that $ab = (x^2 - D(x))(y^2 - D(y)) = cd = 16$.

3.2 We have $d(x)d(y) = zd(xy/z)$.

Proof: $D(x)D(y) = (xy)^2 + 4x^2 + 4y^2 + 16 = (xy)^2 + 4z^2 = z^2[(xy/z)^2 + 4] = z^2 D(xy/z)$ and $d(x)d(y) = zd(xy/z)$. Q.E.D.

3.3 For each $k \in \{0, 1, \ldots, [n/2]\}$, we have

$$a^{n-2k} + b^{n-2k} - c^{n-2k} - d^{n-2k} = 4^{n-2k} d(x)d(y)F_{n-2k}(x)F_{n-2k}(y).$$

Proof:

$$\begin{aligned}
&a^{n-2k} + b^{n-2k} - c^{n-2k} - d^{n-2k}\\
&= (x+d(x))^{n-2k}(y+d(y))^{n-2k} + (x-d(x))^{n-2k}(y-d(y))^{n-2k}\\
&\quad - (x+d(x))^{n-2k}(y-d(y))^{n-2k} - (x-d(x))^{n-2k}(y+d(y))^{n-2k})\\
&= [(x+d(x))^{n-2k} - (x-d(x))^{n-2k}](y+d(y))^{n-2k}\\
&\quad - [(x+d(x))^{n-2k} - (x-d(x))^{n-2k}](y-d(y))^{n-2k}\\
&= [(x+d(x))^{n-2k} - (x-d(x))^{n-2k}][(y+d(y))^{n-2k} - (y-d(y))^{n-2k}]\\
&= 4^{n-2k} d(x)d(y)F_{n-2k}(x)F_{n-2k}(y). \quad \text{Q.E.D.}
\end{aligned}$$

3.4 We have $a + b = 4z\alpha(xy/z)$, $c + d = 4z\beta(xy/z)$.

Proof:

$$\begin{aligned}
a + b &= (x+d(x))(y+d(y)) + (x-d(x))(x-d(y))\\
&= 2(xy + d(x)d(y)) = 2z(xy/z + d(xy/z)) = 4z\alpha(xy/z)
\end{aligned}$$

and

$$c + d = 2(xy - d(x)d(y)) = 2z(xy/z - d(xy/z)) = 4z\beta(xy/z). \quad \text{Q.E.D.}$$

3.5 As by 3.1, we have $ab = (x^2 - D(x))(y^2 - D(y)) = cd = 16$ and we can apply equation **2.1** for a, b, c, and d chosen as above. Using 3.3 and 3.4, we obtain

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} 16^k 4^{n-2k} zd(xy/z)F_{n-2k}(x)F_{n-2k}(y) = 4^n z^n(\alpha(xy/z)^n - \beta(xy/z)^n).$$

Dividing by $zd(xy/z)$ gives

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} F_{n-2k}(x)F_{n-2k}(y) = z^{n-1}\frac{\alpha(xy/z)^n - \beta(xy/z)^n}{d(xy/z)} = z^{n-1}F_n(xy/z).$$

As the term on the left side is, by the recursion formula, a polynomial expression in the variables x, y and the right side is a polynomial in x, y, z^2 by 0, the equation is true for all $x, y \in \mathbb{C}$. Here z can be any of the at most two possible roots of $x^2 + y^2 + 4$.

4. Proof of identities (2)-(7).

4.1 Proof of identity (2): Take $x = y = 1$. From the recursion formula, we obtain

4.1.1 $F_j(1) = F_j$ for each $j \in \mathbb{N}_0$. Now calculation shows: $z = \sqrt{6}$, $d(xy/z) = d(1/\sqrt{6}) = 5/\sqrt{6}$, $\alpha(xy/z) = \alpha(1/\sqrt{6}) = \sqrt{6}/2$, and $\beta(xy/z) = \beta(1/\sqrt{6}) = -2/\sqrt{6}$. Hence, we obtain

4.1.2 $$z^{n-1}F_n(xy/z) = (\sqrt{6})^{n-1}F_n(1/\sqrt{6}) = (\sqrt{6})^{n-1}\frac{\left(\frac{\sqrt{6}}{2}\right)^n - \left(-\frac{2}{\sqrt{6}}\right)^n}{5/\sqrt{6}} = \frac{3^n - (-2)^n}{5}.$$

Now substitute 4.4.1, and 4.1.2 in (1). Q.E.D.

4.2 Proof of identity (3): Take $x=1$, $y=0$. Let $n=2m+1$ (and, finally, substitute m by n). We have $F_j(0)=(1^j-(-1)^n)/2=1$ or 0, according to whether j is odd or even for each $j\in\mathbb{N}_0$. Thus,

$$\textbf{4.2.1}\quad \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y)=\sum_{k=0}^{m}\binom{2m+1}{k}F_{2(m-k)+1}(1)F_{2(m-k)+1}(0)=\sum_{k=0}^{m}\binom{2m+1}{m-k}F_{2k+1}.$$

Calculation shows: $z=\sqrt{5}$, $d(xy/z)=d(0)=2$, $\alpha(xy/z)=\alpha(0)=1$, and $\beta(xy/z)=\beta(0)=-1$. Hence, we obtain

$$\textbf{4.2.2}\quad z^{n-1}F_n(xy/z)=(\sqrt{5})^{n-1}F_n(0)=(\sqrt{5})^{2m}F_{2m+1}(0)=5^m.$$

Now substitute 4.2.1 and 4.2.2 in (1). Q.E.D.

4.3 Proof of identity (4): Let $x=1$, $y=\sqrt{5}$. Let $n=2m$. Since $d(y)=d(\sqrt{5})=\sqrt{(5+4)}=3$, $\alpha(y)=(\sqrt{5}+3)/2=\alpha^2$, and $\beta(y)=-\beta^2$, we have $F_{2j}(y)=(\alpha^{4j}-(-\beta^2)^{2j})/3=(\sqrt{5}/3)F_{4j}$ for each $j\in\mathbb{N}_0$. Hence,

$$\textbf{4.3.1}\quad \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y)=\sum_{k=0}^{m}\binom{2m}{k}F_{2(m-k)}(x)F_{2(m-k)}(y)$$
$$=\sum_{k=0}^{m}\binom{2m}{m-k}F_{2k}(x)F_{2k}(y)=\sum_{k=0}^{m}\binom{2m}{m-k}F_{2k}F_{4k}(\sqrt{5}/3).$$

Now calculation shows: $z=\sqrt{10}$, $d(xy/z)=d(1/\sqrt{2})=3/\sqrt{2}$, $\alpha(xy/z)=\alpha(1/\sqrt{2})=\sqrt{2}$, and $\beta(xy/z)=\beta(1/\sqrt{2})=-1/\sqrt{2}$. Thus,

$$\textbf{4.3.2}\quad z^{n-1}F_n(xy/z)=(\sqrt{10})^{2m-1}F_{2m}(1/\sqrt{2})=\frac{10^m}{\sqrt{10}}\frac{2^m-(1/2)^m}{3/\sqrt{2}}=\frac{5^m}{\sqrt{5}}\frac{4^m-1}{3}.$$

Now substitute 4.3.1 and 4.3.2 in (1). Multiplying the equation by$3/\sqrt{5}$ completes the proof of identity (4). Q.E.D.

4.4 Proof of identity (5): Take $x=1, y=\sqrt{5}$ and let $n=2m+1$. Then $F_{2j+1}(y)=(\alpha^{4j+2}+\beta^{4j+2})/3=L_{4j+2}/3$ for each $j\in\mathbb{N}_0$. Hence

$$\textbf{4.4.1}\quad \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y)=\sum_{k=0}^{m}\binom{2m+1}{k}F_{2(m-k)+1}L_{r(m-k)+2}/3$$
$$=\sum_{k=0}^{m}\binom{2m+1}{m-k}F_{2k+1}L_{4k+2}/3.$$

Also

$$\textbf{4.4.2}\quad z^{n-1}F_n(xy/z)=z^{2m}F_{2m+1}(xy/z)=10^m\frac{\sqrt{2}^{2m+1}-(-1/\sqrt{2})^{2m+1}}{3/\sqrt{2}}=5^m\frac{2^{2m+1}+1}{3}.$$

Now substitute 4.4.1 and 4.4.2 in (1). Multiplying all equations by 3 completes the proof of identity (5). Q.E.D.

4.5 Proof of identity (6): Take $x = 3i$, where $i = \sqrt{-1}$, $y = 2$. Then $d(x) = i\sqrt{5}$, $\alpha(x) = (3i + i\sqrt{5})/2 = i\alpha^2, \beta(x) = i\beta^2$, $F_j(x) = ((i\alpha^2)^j - (i\beta^2)^j)/i\sqrt{5} = i^{j-1}F_{2j}$, and $F_j(y) = P_j$, for each $j \in \mathbb{N}_0$. Then

$$\textbf{4.5.1} \quad \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} F_{n-2k}(x) F_{n-2k}(y) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} i^{n-2k-1} F_{2n-4k} P_{n-2k}.$$

Calculation shows: $z = i$, $d(xy/z) = d(6) = 2\sqrt{10}$, $\alpha(xy/z) = \alpha(6) = 3 + \sqrt{10}$, and $\beta(xy/z) = \beta(6) = 3 - \sqrt{10}$. Hence

$$\textbf{4.5.2} \quad z^{n-1} F_n(xy/z) = i^{n-1} F_n(6).$$

Now substitute 4.5.1 and 4.5.2 in (1). Dividing all equations by i^{n-1} completes the proof of identity (6). Q.E.D.

4.6 Proof of identity (7): Let $x = i\alpha$, $y = i\beta$, where again $i = \sqrt{-1}$. We apply the recursion formula for the sequences $(F_j(i\alpha))$ and $(F_j(i\beta))$. In this way, we calculate $F_0(i\alpha)$, $F_1(i\alpha)$, ..., $F_{20}(i\alpha)$ and $F_0(i\beta), F_1(i\beta), \ldots, F_{20}(i\beta)$ and realize that both sequences $(F_j(i\alpha))$ and $(F_j(i\beta))$ have period 20. Thus

$$F_j(i\alpha)F_j(i\beta) = \begin{cases} 0, & \text{if } j \equiv 0 \pmod{5}, \\ 1, & \text{if } j \equiv 1, 2, 8, 9 \pmod{10}, \\ -1, & \text{if } j \equiv 3, 4, 6, 7 \pmod{10}. \end{cases} \quad \text{for each } j \in \mathbb{N}_0.$$

In particular, $F_j(i\alpha)F_j(i\beta) = (-1)^{[(j+2)/5]}$ if j and 5 are coprime. Hence,

$$\textbf{4.6.1} \quad \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} F_{n-2k}(i\alpha) F_{n-2k}(i\beta) = \sum_{\substack{k=0 \\ (5,\, n-2k)=1}} (-1)^{[(n-2k+2)/5]} \binom{n}{k}.$$

Also, $z = \sqrt{(-\alpha^2 - \beta^2 + 4)} = 1$, $xy/z = xy = 1$, so

$$\textbf{4.6.2} \quad z^{n-1} F_n(xy/z) = F_n(1) = F_n.$$

Now substitute 4.6.1 and 4.6.2 in (1). Q.E.D.

Also solved by P. Bruckman and the proposer.

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