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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980) and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

SELF-GENERATING PYTHAGOREAN QUADRUPLES AND *N*-TUPLES

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(*Submitted December 1993*)

1. INTRODUCTION

In a rectangular solid, the length of an interior diagonal is determined by the formula

$$a^2+b^2+c^2=d^2, \tag{1}$$

where a, b, c are the dimensions of the solid and d is the diagonal.

When a, b, c, and d are integral, a **Pythagorean quadruple** is formed.

Mordell [1] developed a solution to this Diophantine equation using integer parameters (m, n, and p), where $m+n+p\equiv 1 \pmod 2$ and $(m, n, p)=1$. The formulas are:

$$\begin{aligned} a&=2mp & c&=p^2-(n^2+m^2),\\ b&=2np & d&=p^2+(n^2+m^2). \end{aligned} \tag{2}$$

However, the Pythagorean quadruple $(36, 8, 3, 37)$ cannot be generated by Mordell's formulas since c must be the smaller of the odd numbers and

$$\begin{aligned} 3&=p^2-(n^2+m^2)\\ \underline{37}&\underline{=p^2+(n^2+m^2)}\\ 40&=2p^2\\ 20&=p^2; \qquad \text{so } p \text{ is not an integer.} \end{aligned}$$

This quadruple, however, can be generated from Carmichael's formulas [2], using $(m, n, p, q)=(1, 4, 2, 4)$, that is,

$$\begin{aligned} a&=2mp+2nq & c&=p^2+q^2-(n^2+m^2),\\ b&=2np-2mq & d&=p^2+q^2+(n^2+m^2). \end{aligned} \tag{3}$$

By using an additional parameter, the Carmichael formulas generate a wider set of solutions where $m+n+p+q\equiv 1 \pmod 2$ and $(m, n, p, q)=1$.

In the formulas above, either three or four variables are needed to generate four other integers (a, b, c, d). In this paper, we present 2-parameter Pythagorean quadruple formulas where the two integral parameters are also part of the solution set. We shall call them **self-generating formulas**. These formulas will then be generalized to give **Pythagorean *N*-tuples** when a set of $(n-2)$ integers is given.

2. THE SELF-GENERATING QUADRUPLE FORMULAS

We use a and b to designate the two integer parameters that will generate the Pythagorean quadruples. The following theorem deals with the three possible cases arising from parity conditions imposed upon a and b.

Theorem 1: For positive integers a and b, where a or b or both are even, there exist integers c and d such that $a^2+b^2+c^2=d^2$. When a and b are both odd, no such integers c and d exist.

<u>**Case 1**</u>. If a and b are of opposite parity, then

$$c=(a^2+b^2-1)/2 \quad \text{and} \quad d=(a^2+b^2+1)/2. \tag{4}$$

Proof:

$$\begin{aligned} d^2-c^2 &= (d+c)(d-c) \\ &= \left[\left(\frac{a^2+b^2+1}{2}\right)+\left(\frac{a^2+b^2-1}{2}\right)\right]\left[\left(\frac{a^2+b^2+1}{2}\right)-\left(\frac{a^2+b^2-1}{2}\right)\right] \\ &= \left[\frac{2(a^2+b^2)}{2}\right]\left[\frac{2}{2}\right] \\ &= a^2+b^2. \end{aligned}$$

Therefore, $d^2=a^2+b^2+c^2$.

Since a and b differ in parity, c and d in (4) are integers.

Corollary: From (4), we see that c and d are consecutive integers. Therefore, $(a,b,c,d)=1$, even when $(a,b)\neq 1$.

<u>**Case 2**</u>. If a and b are both even, then

$$c=\left(\frac{a^2+b^2}{4}\right)-1 \quad \text{and} \quad d=\left(\frac{a^2+b^2}{4}\right)+1. \tag{5}$$

Proof:

$$\begin{aligned} 16(d^2-c^2) &= (a^2+b^2+4)^2-(a^2+b^2-4)^2 \\ &= 16(a^2+b^2). \end{aligned}$$

Therefore, $d^2=a^2+b^2+c^2$.

Since a and b are both even, c and d in (5) are integers.

Corollary: If $a-b\equiv 0 \pmod 4$, $(a^2+b^2)/4$ is even and c and d are consecutive odd integers, so $(a,b,c,d)=1$. But, if $a-b\equiv 2 \pmod 4$, $(a^2+b^2)/4$ is odd, c and d are consecutive even integers, and $(a,b,c,d)\neq 1$.

<u>**Case 3**</u>. If a and b are both odd, then $a^2\equiv b^2\equiv 1 \pmod 4$.

Since $c^2\equiv 0 \pmod 4$ or $c^2\equiv 1 \pmod 4$, and similarly for d^2, we have:

$$a^2+b^2+c^2\equiv 2 \pmod 4 \not\equiv d^2 \text{ for any integer } d;$$

or

$$a^2+b^2+c^2\equiv 3 \pmod 4 \not\equiv d^2 \text{ for any integer } d.$$

Hence, no Pythagorean quadruple exists in this case.

3. SELF-GENERATING PYTHAGOREAN N-TUPLES

The ideas and methods of proof for the self-generating quadruples can be generalized to the N-tuple case. We need to find formulas for generating integer N-tuples $(a_1, a_2, \dots, a_n)$ when given a set of integer values for the $(n-2)$ members of the "parameter set" $S = (a_1, a_2, \dots, a_{n-2})$. Analogous to the parity conditions imposed on the self-generating quadruple formulas, we introduce the variable T. Proofs of the formulas are left to the reader; they are similar to those for Theorem 1.

Theorem 2: Let $S = (a_1, a_2, \dots, a_{n-2})$, where a_1 is an integer, and let $T =$ "# of odd integers in S."

If $T \not\equiv 2 \pmod 4$, then there exist integers a_{n-1} and a_n such that

$$a_1^2 + a_2^2 + \cdots + a_{n-1}^2 = a_n^2. \tag{6}$$

<u>**Case 1.**</u> Let $T \equiv 1 \pmod 2$, which implies that $T \equiv 1 \pmod 4$ or $T \equiv 3 \pmod 4$. Then, setting

$$a_{n-1} = [a_1^2 + a_2^2 + \cdots + a_{n-2}^2 - 1]/2,$$

and

$$a_n = [a_1^2 + a_2^2 + \cdots + a_{n-2}^2 + 1]/2, \tag{7}$$

we have

$$\begin{aligned} a_n^2 - a_{n-1}^2 &= (a_n + a_{n-1})(a_n - a_{n-1}) \\ &= [2(a_1^2 + a_2^2 + \cdots + a_{n-2}^2)/2][2/2] \\ &= a_1^2 + a_2^2 + \cdots + a_{n-2}^2 \end{aligned}$$

as required.

<u>**Case 2.**</u> Let $T \equiv 0 \pmod 4$. Then, setting

$$a_{n-1} = [a_1^2 + a_2^2 + \cdots + a_{n-2}^2]/4 - 1$$

and

$$a_n = [a_1^2 + a_2^2 + \cdots + a_{n-2}^2]/4 + 1, \tag{8}$$

we have

$$\begin{aligned} a_n^2 - a_{n-1}^2 &= (a_n + a_{n-1})(a_n - a_{n-1}) \\ &= [2(a_1^2 + a_2^2 + \cdots + a_{n-2}^2)/4][2] \\ &= a_1^2 + a_2^2 + \cdots + a_{n-2}^2 \end{aligned}$$

as required.

<u>**Case 3.**</u> Suppose $T \equiv 2 \pmod 4$. Then $a_1^2 + a_2^2 + \cdots + a_{n-2}^2 \equiv 2 \pmod 4$ And since either $a_{n-1}^2 \equiv 0 \pmod 4$ or $a_{n-1}^2 \equiv 1 \pmod 4$, we have

$$a_1^2 + a_2^2 + \cdots + a_{n-2}^2 + a_{n-1}^2 \equiv 2 \pmod 4 \not\equiv a_n^2 \text{ for any integer } a_n,$$

or

$$a_1^2 + a_2^2 + \cdots + a_{n-2}^2 + a_{n-1}^2 \equiv 3 \pmod 4 \not\equiv a_n^2 \text{ for any integer } a_n.$$

Hence, no Pythagorean quadruple N-tuple exists in this case.

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COUSINS OF SMITH NUMBERS: MONICA AND SUZANNE SETS

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(*Submitted February 1994*)

1. INTRODUCTION

For any $x \in \mathbb{N}$, x may be expressed as the sum $a_0 + a_1(10^1) + \cdots + a_d(10^d)$, where each $a_i \in \{0, 1, 2, \ldots, 9\}$.

Suppose that $x \in \mathbb{N}$ is a composite number. Then $x = p_1 p_2 \ldots p_m$, where each p_k is a prime number. We can then formally define two functions $S(x)$ and $S_p(x)$ as

$$S(x) = \sum_{j=0}^{d} a_j \quad \text{and} \quad S_p(x) = \sum_{i=1}^{m} S(p_i).$$

That is, $S(x)$ is the digital sum of x and $S_p(x)$ is the digital sum of the prime factors of x. Wilansky [2] defines a Smith number as a composite integer where $S(x) = S_p(x)$. This paper deals with two kinds of sets related to Smith numbers. These sets are called Monica sets and Suzanne sets.

Definition 1.1: The n^{th} Monica set $\mathbf{M}_n$ consists of those composite numbers x for which $n | S(x) - S_p(x)$ [we write $S(x) \equiv_n S_p(x)$].

Definition 2.1: The n^{th} Suzanne set $\mathbf{S}_n$ consists of those composite numbers x for which $n | S(x)$ and $n | S_p(x)$.

It should be noted that because I developed this concept from Smith numbers, I consider it to be akin to Smith's. Therefore, I have named these sets after my cousins, Monica and Suzanne Hammer.

2. ON THE POPULATION OF MONICA AND SUZANNE SETS

The following theorems give indications of what sort of integers belong to Monica and Suzanne sets.

Theorem 2.1: If x is a Smith number, then $x \in \mathbf{M}_n$, $\forall n \in \mathbb{N}$.

Theorem 2.2: $x \in \mathbf{S}_n \Rightarrow x \in \mathbf{M}_n$.

Note that the converse of Theorem 2.2 is *not* true; for example, $10 = 5 \times 2$, thus $S(10) = 1$ and $S_p(10) = 7$. $10 \in \mathbf{M}_6$ since $6 | 1 - 7$, but $10 \notin \mathbf{S}_6$ since $6 \nmid 1$.

Theorem 2.3: For any integer $\mathbf{k} > 1$, if x is a $\mathbf{k}$-Smith, then $x \in \mathbf{M}_{k-1}$.

Proof: McDaniel [1] defines a $\mathbf{k}$-Smith as a composite number x such that $\mathbf{k}S(x) = S_p(x)$. Thus, $S(x) - S_p(x)$ is divisible by $\mathbf{k} - 1$. Therefore, $x \in \mathbf{M}_{k-1}$. $\square$

3. RELATIONS BETWEEN SETS OF MONICAS AND SETS OF SUZANNES

There are some rather simple properties of Monica and Suzanne sets that may be useful in later studies.

Theorem 3:

(a) If $p, q \in \mathbb{N}$ and $p|q$, then $x \in \mathbf{M}_q$ implies $x \in \mathbf{M}_p$;

(b) If $p, q \in \mathbb{N}$ and $p|q$, then $x \in \mathbf{S}_q$ implies $x \in \mathbf{S}_p$;

(c) If $p, q \in \mathbb{N}$ and $p|q$ are relatively prime, then $x \in \mathbf{M}_p$ and $x \in \mathbf{M}_q$ implies $x \in \mathbf{M}_{pq}$;

(d) If $p, q \in \mathbb{N}$ and $p|q$ are relatively prime, then $x \in \mathbf{S}_p$ and $x \in \mathbf{S}_q$ implies $x \in \mathbf{S}_{pq}$.

4. INFINITE ELEMENTS IN EACH MONICA AND SUZANNE SET

The most interesting property of Monica and Suzanne sets is that every Monica set and every Suzanne set has an infinite number of elements. McDaniel [1] proves that there is an infinite number of Smiths; this implies, by Theorem 2.1, that every Monica set has an infinite number of elements.. The proof that there is an infinite number of elements in each Suzanne set is more complicated.

Theorem 4: All Suzanne sets have infinitely many elements.

Proof: Consider $\mathbf{S}_1$. For any composite number x, $1|S(x)$ and $1|S_p(x)$.

For $\mathbf{S}_n$, where $n > 1$, we need to construct a candidate integer r such that $S(r) = n$. Let r be an n-digit *Repunit*, that is, a string of n ones (see [3]). Let $z = \alpha r$, where α is determined by the following table:

$$\begin{array}{lll}
S_p(r) \equiv_7 0 & \text{then} \quad \alpha = 1 & \text{since } S_p(1r) = S_p(r) \\
S_p(r) \equiv_7 1 & \text{then} \quad \alpha = 9 & \text{since } S_p(9) = 6 \\
S_p(r) \equiv_7 2 & \text{then} \quad \alpha = 5 & \text{since } S_p(5) = 5 \\
S_p(r) \equiv_7 3 & \text{then} \quad \alpha = 4 & \text{since } S_p(4) = 4 \\
S_p(r) \equiv_7 4 & \text{then} \quad \alpha = 3 & \text{since } S_p(3) = 3 \\
S_p(r) \equiv_7 5 & \text{then} \quad \alpha = 2 & \text{since } S_p(2) = 2 \\
S_p(r) \equiv_7 6 & \text{then} \quad \alpha = 15 & \text{since } S_p(15) = 8 \equiv_7 1
\end{array}$$

From the table it should be obvious that $7|S_p(r) + S_p(\alpha)$, and thus $7|S_p(z)$. Note that $S(z) = S(r)S(\alpha)$ because of our choice of r, so $n|S(z)$.

Let m be an integer such that $n|(S_p(z) + 7m)$ and let $y = z * 10^m$. Clearly, $S_p(y) = S_p(z) + S_p(10^m)$ and $S_p(10^m) = 7m$; thus, $n|S_p(y) = S_p(z) + S_p(10^m)$.

Note that $S(y) = S(z)$, so $n|S(y)$; thus, $\alpha r * 10^m = y \in \mathbf{S}_n$ for all m such that $n|S_p(\alpha r) + 7m$. Clearly, there are infinitely many choices for m. □

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TRIANGULAR NUMBERS IN THE PELL SEQUENCE

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1. INTRODUCTION

In 1989, Luo Ming [5] proved Vern Hoggatt's conjecture that the only triangular numbers in the Fibonacci sequence $\{F_n\}$ are 1, 3, 21, and 55. It is our purpose, in this paper, to show that 1 is the only triangular number in the Pell sequence $\{P_n\}$.

Aside from the proof itself, Ming's unique contribution in his paper was his development and use of the value of the Jacobi symbol $(8F_{2kg+m}+1|L_{2k})$, where $\{L_n\}$ is the sequence of Lucas numbers, g is odd, and $k \equiv \pm 2 \pmod 6$. In other papers involving similar arguments, the value of the Jacobi symbol $(f(2kg+m)|L_t)$, for certain functions $f(n)$ of F_n and/or L_n, has often been obtained for t a divisor of k, but not for t equal to $2k$ (e.g., [1], [2], [3], and [7]).

It is immediate, from the definition, that an integer k is a triangular number iff $8k+1$ is a perfect square >1. We shall employ an argument similar to that used by Ming to show that, for every integer $n \neq \pm 1$, there exists an integer $w(n)$, such that $8P_n+1$ is a quadratic nonresidue modulo $w(n)$.

We require the sequence of "associated" Pell numbers defined by $Q_0 = 1$, $Q_1 = 1$ and, for all integers $n \geq 0$, $Q_{n+2} = 2Q_{n+1} + Q_n$. The first few Pell and associated Pell numbers are

$$\{P_n\} = \{0, 1, 2, 5, 12, \dots\} \quad \text{and} \quad \{Q_n\} = \{1, 1, 3, 7, 17, \dots\}.$$

2. SOME IDENTITIES AND PRELIMINARY LEMMAS

The following formulas and identities are well known. For all integers n and m,

$$P_{-m} = (-1)^{m+1}P_m \quad \text{and} \quad Q_{-m} = (-1)^m Q_m, \tag{1}$$

$$P_{m+n} = P_m P_{n+1} + P_{m-1}P_n, \tag{2}$$

$$P_{m+n} = 2P_m Q_n - (-1)^n P_{m-n}, \tag{3}$$

$$P_{2^t m} = P_m(2Q_m)(2Q_{2m})(2Q_{4m})\cdots(2Q_{2^{t-1}m}), \tag{4}$$

$$Q_m^2 = 2P_m^2 + (-1)^m, \tag{5}$$

$$Q_{2m} = 2Q_m^2 - (-1)^m. \tag{6}$$

If $d = \gcd(m, n)$, then

$$\begin{cases} \gcd(P_m, Q_n) = Q_d & \text{if } m/d \text{ is even,} \\ \gcd(P_m, Q_n) = 1 & \text{otherwise (see [4]).} \end{cases} \tag{7}$$

We note that (6) readily implies that, if $t > 1$, then $Q_{2^t} \equiv 1 \pmod 8$.

Lemma 1: Let $k = 2^t, t \geq 1, g > 0$ be odd and m be any integer. Then,

(i) $P_{2kg+m} \equiv -P_m \pmod{Q_k}$, and
(ii) $P_{2kg} \equiv \pm P_{2k} \pmod{Q_{2k}}$.

Proof: ***(i)*** is known [and can be easily proved using (1) and (3)]. If $n = 2kg = 2k(g-1)+2k$, then, using (3), ***(ii)*** readily follows from

$$P_n = 2P_{2k(g-1)}Q_{2k} - (-1)^{2k}P_{2k(g-2)} \equiv -P_{2k(g-2)} \pmod{Q_{2k}}.$$

Lemma 2: If $k = 2^t, t \geq 1$, then $(8P_{2k}+1|Q_{2k}) = (-8P_{2k}+1|Q_{2k})$.

Proof: We first observe that each Jacobi symbol is defined. Indeed, if $d = \gcd(8P_{2k}+1, Q_{2k})$ or $\gcd(-8P_{2k}+1, Q_{2k})$, then, using (5), d divides

$$(8P_{2k}+1)\cdot(-8P_{2k}+1) = 1-64P_{2k}^2 = 1-32(Q_{2k}^2-1) = 33-2Q_{2k}^2.$$

Hence, $d|33$. But $33|P_{12}$ which implies $d=1$, since, by (7), $\gcd(P_{12}, Q_{2k}) = 1$.

To establish the lemma, note that

$$(8P_{2k}+1|Q_{2k})(-8P_{2k}+1|Q_{2k}) = (1-64P_{2k}^2|Q_{2k}) = (33|Q_{2k}) = (Q_{2k}|33).$$

Now, by (6), $Q_4 = 17$, $Q_8 = 2Q_4^2-1 = 2\cdot 17^2-1 \equiv 16 \pmod{33}$, and by induction, $Q_{2k} \equiv 16$ (mod 33) if $t \geq 2$. Hence, if $t \geq 1$, $(Q_{2k}|33) = +1$.

Lemma 3: If $k = 2^t, t \geq 2$, then $(8P_k+Q_k|33) = -1$.

Proof: $Q_2 = 3$, $Q_4 = 17 \equiv -16 \pmod{33}$, and as observed in the proof of Lemma 2, $Q_{2^j} \equiv 16$ (mod 33), if $j \geq 3$. Hence, by (4), if $t \geq 2$,

$$8P_k = 8P_2(2Q_2)(2Q_4)\cdots(2Q_{2^{t-1}}) \equiv 8\cdot 2\cdot 6\cdot(\pm 1) \equiv \pm 3 \pmod{33},$$

so, $8P_k+Q_k \equiv \pm 13$ or ± 19 (mod 33) and both $(\pm 13|33)$ and $(\pm 19|33) = -1$.

From a table of Pell numbers (e.g., [6], p. 59), we find that $P_{24} \equiv 0 \pmod 9$ and $P_{25} \equiv 1$ (mod 9). Using (2),

$$P_{n+24} = P_nP_{25} + P_{n-1}P_{24} \equiv P_n \pmod 9,$$

and we have, immediately,

Lemma 4: If $n \equiv m \pmod{24}$, then $P_n \equiv P_m \pmod 9$.

3. THE MAIN THEOREM

Theorem: The term P_n of the Pell sequence is a triangular number iff $n = \pm 1$.

Proof: If $n = \pm 1$, P_n is the triangular number 1. By (1), if n is an even negative integer, then $8P_n+1$ is negative, and if n is odd, then $P_{-n} = P_n$; hence, it suffices to show that $8P_n+1$ is not a square for $n > 1$. Let $n = 2kg+m, k = 2^t, t \geq 1, g \geq 1$ odd, and assume P_n is a triangular number. [Then $(8P_n+1|N) = +1$ for all odd integers N.]

Case 1. n odd. Since $n \equiv \pm 1 \pmod 4$, $8P_n + 1 = 8P_{2kg\pm 1} + 1 \equiv -7 \pmod{Q_k}$, by Lemma 1(i) and (1). But it is readily shown, using (6), that $Q_k \equiv 3 \pmod 7$. Hence,

$$(8P_n + 1 | Q_k) = (-7 | Q_k) = (Q_k | 7) = (3|7) = -1,$$

a contradiction.

Case 2 (mod 4): $n \equiv 2$. It is easily seen that $\{P_n\}$ has period 6 modulo 7, and that, for n even, $(8P_n + 1|7) = +1$ only if $n \equiv 0 \pmod 6$. Hence, $n \equiv \pm 6 \pmod{24}$. By Lemma 4,

$$8P_n + 1 \equiv \pm 8P_6 + 1 \equiv 3 \text{ or } 8 \pmod 9.$$

But 3 and 8 are quadratic nonresidues of 9, so $8P_n + 1$ is not a square.

Case 3. $n \equiv 0 \pmod 4$. By Lemma 1(ii) and Lemma 2,

$$(8P_n + 1 | Q_{2k}) = (8P_{2k} + 1 | Q_{2k}).$$

If $k = 2(t = 1)$, $(8P_{2k} + 1 | Q_{2k}) = (97|17) = -1$. Assume $t \geq 2$. Now,

$$8P_{2k} + 1 \equiv 8P_{2k} + (2Q_k^2 - Q_{2k}) \equiv 2Q_k(8P_k + Q_k) \pmod{Q_{2k}}.$$

Let $s_k = 8P_k + Q_k$ [note that $s_k \equiv 1 \pmod 8$]. Then, using properties (5) and (6),

$$\begin{aligned}(8P_{2k} + 1 | Q_{2k}) &= (Q_k | Q_{2k})(s_k | Q_{2k}) = (Q_{2k} | Q_k)(Q_{2k} | s_k) \\ &= (2Q_k^2 - 1 | Q_k)(2P_k^2 + Q_k^2 | s_k) = (+1)(2P_k^2 + (s_k - 8P_k)^2 | s_k) \\ &= (66P_k^2 | s_k) = (33 | s_k) = (s_k | 33) = (8P_k + Q_k | 33) = -1,\end{aligned}$$

by Lemma 3, and the proof is complete.

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VAROL'S PERMUTATION AND ITS GENERALIZATION

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INTRODUCTION

Let S_n be the set of $n!$ permutations $\Pi = a_1a_2\ldots a_n$ of $\mathbb{Z}_n = \{1, 2, \ldots, n\}$. For $\Pi \in S_n$, we write $\Pi = a_1a_2\ldots a_n$, where $\Pi(i) = a_i$.

Definition 1: $P_n := \{\Pi \in S_n \mid a_{i+1} \neq a_i + 1 \text{ for all } i, 1 \leq i < n\}$, $|P_n| = p_n$;
$\overline{P}_n := \{\Pi \in P_n \mid a_n = n\}$, $|\overline{P}_n| = \overline{p}_n$;
$P'_n = \{\Pi \in S_n \mid a_{i+1} \neq a_i - 1 \text{ for all } i,\ 1 \leq i < n\}$, $|P'_n| = p'_n$;
$\overline{P}'_n = \{\Pi \in P'_n \mid a_n = n\}$, $|\overline{P}'_n| = \overline{p}'_n$.

Definition 2:
$$T_n := \left\{\Pi \in P_n \,\middle|\, \sum_{j=1}^{i} a_j > \frac{i(i+1)}{2} \text{ for any } i, 1 \leq i < n\right\},$$
$$T_1 = \phi;$$
$$T'_n := \left\{\Pi \in P'_n \,\middle|\, \sum_{j=1}^{i} a_j > \frac{i(i+1)}{2} \text{ for any } i, 1 \leq i < n\right\},$$
$$T'_1 = \phi,\ |T_n| = t_n,\ |T'_n| = t'_n.$$

Definition 3: $R_n := P_n \cap P'_n$, $G_n := T_n \cap T'_n$, $|G_n| = r_n$, $|G_n| = g_n$.

From Computer Science, Varol first studied T_n and obtained the recurrence for t_n (see [1]). In [2], R. Luan discussed the enumeration of T'_n and G_n. This paper deals with the above problems in a way that is different from [1] and [2]. A series of new formulas of enumeration for t_n, t'_n, and g_n (Theorems 1-9) has been derived.

1. ENUMERATION OF t_n AND t'_n

Lemma 1:
$$P_n = (n-1)!\sum_{k=0}^{n-1}(-1)^k \frac{n-k}{k!} = D_n + D_{n-1}, \tag{1.1}$$

where $D_n = n!\sum_{j=0}^{n}\frac{(-1)^j}{j!}$ is the number of derangement of $\{1, 2, \ldots, n\}$ (see [3]).

Proof: Consider the set $S' = \{(1, 2); (2, 3); \ldots; (n-1, n)\}$. We say that an element $(j, j+1)$ of S' is in a permutation Π if $\Pi(i) = j, \Pi(i+1) = j+1$ for some i.

Let W_k be the number of permutations in S_n containing at least k elements of S'. The number of ways of taking k elements from S' is $\binom{n-1}{k}$. Suppose the k elements have j digits in common. Then these k elements form $(k-j)$ continuous sequences of natural numbers, each of which is called a *block*. Thus, the number of remaining elements in $\mathbb{Z}_n$ is $(n-2k+j)$. The number of permutations of $(n-2k+j)$ elements and $(k-j)$ blocks is $[(n-2k+j)+(k-j)]! = (n-k)!$. Hence, $W_k = \binom{n-1}{k}(n-k)!$.

By the principle of inclusion and exclusion (see [3]):

$$P_n = \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(n-k)! = (n-1)!\sum_{k=0}^{n}(-1)^k\frac{n-k}{k!} = D_n + D_{n-1},$$

where $D_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$ is the number of derangement of $\{1, 2, \ldots, n\}$ (see [3], p. 59). □

Lemma 2:

$$\overline{p_n} = \sum_{j=0}^{n-1}(-1)^{n+1-j}p_j, \text{ where } p_0 = 1. \tag{1.2}$$

Proof: It is easy to see that $\overline{p_n} = p_{n-1} - \overline{p_{n-1}}$. Applying the above recurrence repeatedly, we get (1.2). □

Let

$$P(x) = \sum_{n=0}^{\infty} p_n x^n. \tag{1.3}$$

Theorem 1:

$$t_n = \sum_{i=0}^{n}(-1)^{n-i}p_i - \sum_{i=1}^{n-2}p_i t_{n-i}. \tag{1.4}$$

Proof: Consider the following subset P_n^0 of P_n.

$$P_n^0 = \Big\{(a_1a_2\ldots a_i\ldots a_n) \in P_n \,|\, \text{for some } i,\ 1 \le i < n,$$
$$a_1 + a_2 + \cdots + a_i = \frac{i(i+1)}{2}, \text{ but for } i < j < n,$$
$$a_1 + a_2 + \cdots + a_j > \frac{j(j+1)}{2}\Big\}.$$

If $i < n-1$, then the number of such permutations is $p_i t_{n-i}$; if $i = n-1$, the number is $\overline{p_{n-1}}$. Thus,

$$|P_n^0| = \sum_{i=1}^{n-2} p_i t_{n-i} + \overline{p_{n-1}}.$$

Hence,

$$t_n = p_n - |P_n^0| = p_n - \sum_{i=1}^{n-2} p_i t_{n-i} - \overline{p_{n-1}}.$$

Substituting (1.2) into the above formula, we have (1.4). □

To simplify (1.4), we establish a lemma as follows.

Lemma 3: If

$$t_n = a_n - \sum_{i=1}^{n-2} b_i t_{n-i}, \quad n \ge 2,$$

then

$$t_n = \begin{vmatrix} 1 & & & & & a_2 \\ b_1 & 1 & & \mathbf{0} & & a_3 \\ b_2 & b_1 & 1 & & & a_4 \\ \cdots & \cdots & \ddots & & & \cdots \\ \cdots & \cdots & & b_1 & 1 & \cdots \\ b_{n-2} & b_{n-3} & \cdots & \cdots & b_1 & a_n \end{vmatrix}. \tag{1.5}$$

Proof: This follows from the expansion of the determinant along the bottom row. □

Hence, we can write (1.4) as

Theorem 2:

$$t_n = \begin{vmatrix} 1 & & & & & p_2 - p_1 + 1 \\ p_1 1 & & & \mathbf{0} & & p_3 - p_2 + p_1 - 1 \\ p_2 & p_1 & 1 & & & p_4 - p_3 + p_2 - p_1 + 1 \\ \cdots & \cdots & \ddots & & & \cdots \\ \cdots & \cdots & & p_1 & 1 & \cdots \\ p_{n-2} & p_{n-3} & \cdots & \cdots & p_1 & p_n - p_{n-1} + p_{n-2} - \cdots + (-1)^n \end{vmatrix}, \quad n \geq 2, \tag{1.6}$$

Example 1:

$$t_5 = \begin{vmatrix} 1 & 0 & 0 & p_2 - p_1 + 1 \\ p_1 & 1 & 0 & p_3 - p_2 + p_1 - 1 \\ p_2 & p_1 & 1 & p_4 - p_3 + p_2 - p_1 + 1 \\ p_3 & p_2 & p_1 & p_5 - p_4 + p_3 - p_2 + p_1 - 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 9 \\ 3 & 1 & 1 & 44 \end{vmatrix} = 33.$$

Let $T(x) = \sum_{n=0}^{\infty} t_n x^n$, $t_0 = 0$. Then we have

Theorem 3:

$$T(x) = (1+x)^{-1} - \frac{1}{P(x)}, \text{ where } P(x) = \sum_{n=0}^{\infty} p_n x^n. \tag{1.7}$$

Proof: From (1.4),

$$\begin{aligned} P(x)T(x) &= \sum_{n=0}^{\infty} p_n x^n \sum_{n=0}^{\infty} t_n x^n = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} p_i t_{n-i} \right) x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} (-1)^{n-i} p_i \right) x^n = \sum_{m=0}^{\infty} (-1)^m x^m \sum_{n=0}^{\infty} p_n x^n - 1 = (1+x)^{-1} P(x) - 1. \end{aligned}$$

Hence,

$$T(x) = (1+x)^{-1} - \frac{1}{P(x)}. \quad \square$$

Now we shall consider t'_n.

Lemma 4:

$$p'_n = p_n. \tag{1.8}$$

Proof: If $(a_1 a_2 \ldots a_n) \in P_n$, then $(a_n a_{n-1} \ldots a_2 a_1) \in P'_n$. The above correspondence is one-to-one; thus, $|P_n| = |P'_n|$. □

Arguing as in the proof of Theorem 1, we get

Theorem 4:

$$t'_n = p_n - \sum_{i=1}^{n-1} p_i t'_{n-i}. \tag{1.9}$$

By recurrence (1.9) and Lemma 4, we have an explicit formula for t'_n as follows.

Theorem 5:

$$t_n = \begin{vmatrix} 1 & & & & & p_1 \\ p_1 & 1 & & 0 & & p_2 \\ p_2 & p_1 & 1 & & & p_3 \\ \cdots & \cdots & & \ddots & & \cdots \\ \cdots & \cdots & & & 1 & p_{n-1} \\ p_{n-1} & p_{n-2} & \cdots & \cdots & p_1 & p_n \end{vmatrix}. \tag{1.10}$$

Let the generating function for t'_n be $T'(x) = \sum_{n=0}^{\infty} t'_n x^n$, $t'_n = 0$.

Theorem 6:

$$T'(x) = 1 - \frac{1}{P(x)}. \tag{1.11}$$

Proof: By (1.9), $\sum_{i=0}^{n} p_i t'_{n-i} = p_n, n \geq 1$. Thus, $P(x)T'(x) = P(x) - 1$, and we have

$$T'(x) = 1 - \frac{1}{P(x)}, \text{ as required. } \square$$

Lemma 5:

$$t_n = t'_n + (-1)^n. \tag{1.12}$$

Proof: Since $T(x) = 1/(1+x) - 1/[P(x)]$, $T'(x) = 1 - 1/[P(x)]$; hence,

$$T(x) - T'(x) = -\frac{x}{1+x},$$

i.e.,

$$\sum_{n=0}^{\infty} (t_n - t'_n)x^n = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Comparing the coefficients of x^n, we have $t_n - t'_n = (-1)^n$, i.e., (1.12). □

According to (1.12) and (1.10), we have a simple expression for t_n as follows:

Theorem 7:

$$t_n = \begin{vmatrix} 1 & & & & & p_1 \\ p_1 & 1 & & 0 & & p_2 \\ p_2 & p_1 & 1 & & & p_3 \\ \cdots & \cdots & \cdots & \ddots & & \cdots \\ \cdots & \cdots & \cdots & & 1 & p_{n-1} \\ p_{n-1} & \cdots & \cdots & p_2 & p_1 & p_n \end{vmatrix} + (-1)^n. \tag{1.13}$$

For Example 1, we can count t_5 by the above formula:

$$t_5 = \begin{vmatrix} 1 & 0 & 0 & 0 & p_1 \\ p_1 & 1 & 0 & 0 & p_2 \\ p_2 & p_1 & 1 & 0 & p_3 \\ p_3 & p_2 & p_1 & 1 & p_4 \\ p_4 & p_3 & p_2 & p_1 & p_5 \end{vmatrix} + (-1)^5 = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 3 \\ 3 & 1 & 1 & 1 & 11 \\ 11 & 3 & 1 & 1 & 53 \end{vmatrix} - 1 = 33.$$

Since the enumeration for p_n is simple (we may look it up in a table of values of D_n) counting t_n by (1.1) and (1.13) is easier than the method of [2].

2. ENUMERATION OF g_n

Definition 4: $R_{n,i} := \{\Pi = (a_1a_2...a_n) \in S_n \mid \Pi$ contains either $(i, i+1)$ or $(i+1, i)$, but for $\forall j < i$, Π contains neither $(j, j+1)$ nor $(j+1, j)\}$. Let $|R_{n,i}| = r_{n,i}$.

Lemma 6: $r_{n,i} = 2(n-1)!$, $n \geq 2$; $r_{1,1} = 1$.

Proof: By definition, $R_{n,1}$ is all permutations of S_n containing either $(1, 2)$ or $(2, 1)$. If we regard $(1, 2)$ as an element, then $(1, 2)$ and the remaining $(n-2)$ elements of $\mathbb{Z}_n$ form $(n-1)!$ permutations. Note that there are two permutations of $\{1, 2\}$. Thus, $r_{n,1} = 2(n-1)!$. □

Lemma 7: $r_{n,2} = 2(n-1)! - 2(n-2)!$.

Proof: Let us count the number of permutations containing either $(2, 3)$ or $(3, 2)$ but neither $(1, 2)$ nor $(2, 1)$.

Arguing as in Lemma 6, we know that the number of permutations containing either $(2, 3)$ or $(3, 2)$ is $2(n-1)!$. We have to eliminate those permutations containing $(1, 2, 3)$ or $(3, 2, 1)$, the number of which is $2(n-2)!$ by an argument analogous to Lemma 6. Thus, we have proved Lemma 7. □

Lemma 8:
$$r_{n,i} - r_{n,i-1} - r_{n-1,i-1} - r_{n-1,i-2}, \text{ where } 2 \leq i < n,\ r_{n,0} = 0. \tag{2.1}$$

Proof: Each permutation of $R_{n,i}$ contains $(i, i+1)$ or $(i+1, i)$. If $(i, i+1)$ is followed by $i-1$ or if $(i+1, i)$ is preceded by $i-1$, then $(i+1)$ is removed, and we subtract 1 from every digit that is greater than $i+1$. Thus, we get an element of $R_{n-1,i-1}$. Conversely, given a permutation of $R_{n-1,i-1}$, we add 1 to each element greater than i and then interpose an $i+1$ between $(i, i-1)$ or $(i-1, i)$. This yields an element of $R_{n,i}$.

If $(i, i+1)$ is not followed by $i-1$ or preceded by $(i+1, i)$, we regard $(i, i+1)$ or $(i+1, i)$ as a single element and subtract 1 from every digit greater than $i+1$. This yields an element $S_{n-1} - R_{n-1,1} \cup R_{n-1,2} \cup \cdots \cup R_{n-1,i-1}$. Thus,

$$r_{n,i} = r_{n-1,i-1} + 2\left[(n-1)! - \sum_{j=1}^{i-1} r_{n-1,j}\right].$$

Hence,

$$r_{n,i} = 2\left[(n-1)! - \sum_{j=1}^{i-2} r_{n-1,j}\right] - r_{n-1,i-1}, \tag{2.2}$$

i.e.,

$$r_{n,i} = (r_{n,i-1} - r_{n-1,i-2}) - r_{n-1,i-1}. \quad \square$$

Using Lemmas 6, 7, and 8 above, we can express $r_{n,k}$ in terms of $r_{n,1}$:

$$r_{n,2} = r_{n,1} - r_{n-1,1} = 2(n-1)! - 2(n-2)!,$$
$$r_{n,3} = r_{n,1} - 3r_{n-1,1} + r_{n-2,1} = 2(n-1)! - 6(n-2)! + 2(n-3)!,$$
$$r_{n,4} = r_{n,1} - 5r_{n-1,1} + 5r_{n-2,1} - r_{n-3,1} = 2(n-1)! - 10(n-2)! + 10(n-3)! - 2(n-4)!,$$
$$\ldots$$

In general, let

$$r_{n,k} = a_{k,1}2(n-1)! - a_{k,2}2(n-2)! + \cdots + a_{k,k}(-1)^{k+1}2(n-k)!.$$

Obviously $a_{k,i}$ is independent of n. It only depends on k. We can prove

Lemma 9:

$$\begin{aligned} &a_{k,1} = 1, \\ &a_{k,j} = a_{k-1,j} + a_{k-1,j-1} + a_{k-2,j-1}, \quad 1 < j < k, \\ &a_{k,k} = 1. \end{aligned} \tag{2.3}$$

Proof: Since $r_{n,k} = \sum_{j=1}^{k} a_{k,j}2(n-j)!(-1)^{j+1}$ by (2.1), we have:

$$\begin{aligned} r_{n,k} &= r_{n,k-1} - r_{n-1,k-1} - r_{n-1,k-2} \\ &= \sum_{j=1}^{k-1}(-1)^{j+1}a_{k-1,j}2(n-j)! - \sum_{j=1}^{k-1}(-1)^{j+1}a_{k-1,j}2(n-1-j)! - \sum_{j=1}^{k-2}(-1)^{j+1}a_{k-2,j}2(n-1-j)! \\ &= a_{k-1,1}2(n-1)! + \sum_{j=2}^{k-1}(-1)^{j+1}(a_{k-1,j} + a_{k-1,j-1} + a_{k-2,j-1})2(n-j)! + (-1)^{k+1}a_{k-1,k-1}2(n-k)!. \end{aligned}$$

Comparing the two formulas above, we obtain relations for $a_{n,k}$ as follows:

$$\begin{aligned} &a_{k,1} = a_{k-1,1}, \text{ thus, } a_{k,1} = a_{k-1,1} = a_{k-2,1} = \cdots = a_{1,1} = 1, \\ &a_{k,j} = a_{k-1,j} + a_{k-1,j-1} + a_{k-2,j-1}, \quad 1 < j < k, \\ &a_{k,k} = a_{k-1,k-1}, \text{ thus, } a_{k,k} = a_{k-1,k-1} = \cdots = a_{1,1} = 1. \quad \square \end{aligned}$$

Lemma 10:

$$a_{k,j} = a_{k,k+1-j}. \tag{2.4}$$

Proof: We prove the lemma by induction on k. For $k = 1, 2,$ or 3, this is straightforward. Suppose that (2.3) holds for $k-1$. By (2.2),

$$a_{k,j} = a_{k-1,j} + a_{k-1,j-1} + a_{k-2,j-1} = a_{k-1,k-j} + a_{k-1,k-j+1} + a_{k-2,k-j} = a_{k,k+1-j}.$$

By (2.3) and (2.4), we easily obtain the expression for $r_{n,k}$:

$$r_{n,1} = 2(n-1)!,$$
$$r_{n,2} = 2(n-1)! - 3\cdot 2(n-2)! + 2(n-3)!,$$
$$r_{n,3} = 2(n-1)! - 5\cdot 2(n-2)! + 5\cdot 2(n-3)! - 2(n-4)!,$$
$$r_{n,4} = 2(n-1)! - 7\cdot 2(n-2)! + 13\cdot 2(n-3)! - 7\cdot 2(n-4)! + 2(n-5)!.$$

For $a_{k,j}$, using

$$\begin{array}{c} a_{k-2,j-1} \\ + \\ a_{k-1,j-1} + a_{k-1,j} \\ \qquad\qquad \| \\ \qquad\qquad a_{k,j}, \end{array}$$

we obtain the above formulas one by one. Now, using (2.1), we get the table for $r_{n,k}$ shown below.

TABLE 1. $r_{n,k}$ $(k < n)$, $r_{n,n} := r_n$

n \ k	0	1	2	3	4	5	6
0	0						
1	0	1					
2	0	2	0				
3	0	4	2	0			
4	0	12	8	2	2		
5	0	48	36	16	6	14	
6	0	240	192	108	56	34	90
7	0	1440	1200	768	468	304	214

Setting $f_n(x) = \sum_{k=0}^{n-1} r_{n,k} x^k$, and letting $i = n-1$ in (2.2), we have

Corollary:

$$r_{n,n-1} = 2(n-1)! + r_{n-1,n-2} - 2f_{n-1}(1). \tag{2.5}$$

Lemma 11:

$$r_n = \tfrac{1}{2}(r_{n+1,n} - r_{n,n-1}). \tag{2.6}$$

Proof: Denote the set of permutations containing neither $(n-1, n+1, n)$ nor $(n, n+1, n-1)$ in $R_{n+1,n}$ by $R^*_{n+1,n} \subset R_{n+1,n}$.

For any $\alpha \in R_n$, inserting $n+1$ to the left or right of n in α, we get $\alpha' \in R^*_{n+1,n}$. Conversely, if $\alpha' \in R^*_{n+1,n}$, then eliminating $n+1$ yields $\alpha \in R_n$. Hence, $2r_n = |R^*_{n+1,n}|$.

Now we count $|R^*_{n+1,n}|$. It is sufficient to subtract the number of permutations containing $(n-1, n+1, n)$ or $(n, n+1, n-1)$ in $R_{n+1,n}$ from $r_{n+1,n}$. Regard $(n+1, n)$ as a single element. Then $\frac{1}{2}r_{n,n-1}$ is the number of permutations containing no $(n-1, n+1, n)$.

By a similar argument, the number of permutations containing no $(n, n+1, n-1)$ is $\frac{1}{2}r_{n,n-1}$. Thus, $|R^*_{n+1,n}| = r_{n+1,n} - \frac{1}{2}r_{n,n-1} - \frac{1}{2}r_{n,n-1}$. Since $2r_n = |R^*_{n+1,n}|$, we get (2.6). □

Lemma 12:

$$r_n = (n-1)(n-1)! + f_{n-1}(1) - f_n(1) + r_{n-1}, \tag{2.7}$$

$$r_n = \sum_{i=1}^{n} (n-i)(n-i)! - f_n(1), \tag{2.8}$$

where $0 \cdot 0! = 1$.

Proof: Substituting (2.5) into (2.6), we obtain

$$r_n = \tfrac{1}{2}[2\cdot n! + r_{n,n-1} - 2f_n(1) - 2(n-1)! - r_{n-1,n-2} + 2f_{n-1}(1)].$$

Using (2.6), we get (2.7). Applying (2.7) repeatedly, we have

$$r_n = \sum_{i=1}^{n-2}(n-i)(n-i)! - f_n(1) + r_2 + f_2(1).$$

Since $r_2 = 0$ and $f_n(1) = 2$, we obtain (2.8). □

By (2.6), it is easy to get r_n from $r_{n,k}$. In Table 1 we denote $r_{n,n} = r_n$. Thus, by (2.6), the values of r_n on the principal diagonal can be obtained as half the difference between the two adjacent elements on the secondary diagonal. Unfortunately, we cannot count r_n until we complete Table 1. But (2.7) and (2.8) can do that, namely, both $r_{n,k}$ and r_n are counted without $r_{n+1,k}$.

If we set $f_{n+1}(x) = \sum_{k=0}^{n} r_{n+1,k}x^k$, we have

Lemma 13:

$$f_n(x) = \frac{x}{1-x}[2(n-1)! - 2r_{n-1}x^{n-1} - (1+x)f_{n-1}(x)]. \tag{2.9}$$

Proof: By (2.1), we have

$$\sum_{k=2}^{n-1} r_{n,k}x^k = \sum_{k=2}^{n-1} r_{n,k-1}x^k - \sum_{k=2}^{n-1} r_{n-1,k-1}x^k - \sum_{k=2}^{n-1} r_{n-1,k-2}x^k,$$

$$f_n(x) - r_{n,1}x = xf_n(x) - r_{n,n-1}x^n - xf_{n-1}(x) - x^2 f_{n-1}(x) + r_{n-1,n-2}x^n.$$

By (2.6), we have

$$(1-x)f_n(x) = 2(n-1)!\,x - 2r_{n-1}x^n - x(1+x)f_{n-1}(x). \quad \square$$

Example 2: Since $f_4(x) = 12x + 8x^2 + 2x^3$, $r_4 = 2$. We get

$$\begin{aligned} f_5(x) &= \frac{x}{1-x}[2\cdot 4! - 2\cdot 2x^4 - (1+x)(12 + 8x^2 + 2x^3)] \\ &= \frac{x}{1-x}[48 - 12 - 20x^2 - 10x^3 - 6x^4] \\ &= 48x + 36x^2 + 16x^3 + 6x^4, \end{aligned}$$

i.e., $r_{5,1} = 48$, $r_{5,2} = 36$, $r_{5,3} = 16$, $r_{5,4} = 6$. From (2.9), we may obtain

$$\begin{aligned} f_n(x) = \frac{2}{(1-x)^{n-2}}\Bigg[&\sum_{i=1}^{n-2}(-1)^{i-1}(n-i)!\,x^i(1+x)^{i-1}(1-x)^{n-i-2} \\ &+ x^n\sum_{i=1}^{n-2}(-1)^i r_{n-i}(1+x)^{i-1}(1-x)^{n-i-2} + (-1)^n x^{n-1}(1+x)^{n-2}\Bigg]. \end{aligned} \tag{2.10}$$

The application of (2.10) is not as convenient as that of (2.9), but it provides the following information: $r_{n,k}$ must be even. It coincides with the expression of $r_{n,k}$, i.e.,

$$r_{n,k}=\sum_{j=1}^{n}a_{k,j}\cdot 2(n-j)!(-1)^{j+1}.$$

Theorem 7:

$$g_n=\sum_{j=1}^{n}(-1)^{n-j}r_j-\sum_{j=1}^{n-1}r_jg_{n-j}+(-1)^n. \tag{2.11}$$

Proof: By a method similar to Theorem 1, it is easy to show that

$$g_n=r_n-\sum_{j=1}^{n-1}r_jg_{n-j}-\overline{r_{n-1}}, \tag{2.12}$$

where $\overline{r_{n-1}}$ is the number of permutations in R_{n-1} whose right-most entry is $n-1$.

Similar to Theorem 1, we have

$$\overline{r_{n-1}}=\sum_{j=1}^{n-1}(-1)^{n-1-j}r_j+(-1)^{n+1}. \tag{2.13}$$

Now, substituting (2.13) into (2.12), we obtain (2.11) as required. □

According to (2.11), we can count g_n by recurrence. Using (2.11) and noticing that $g_1=g_2=g_3=0$, we get an explicit formula for g_n.

Theorem 8:

$$g_n=\begin{vmatrix}1 & & & & & r_4\\ 1 & 1 & & & & r_5-r_4\\ r_2 & 1 & 1 & \ddots & & r_6-r_5+r_4\\ \cdots & & \ddots & & & \cdots\\ \cdots & & & & 1 & r_{n-1}-\cdots+(-1)^{n-1}r_4\\ r_{n-4} & r_{n-5} & \cdots & r_2 & 1 & r_n-r_{n-1}+\cdots+(-1)^nr_4\end{vmatrix},\ n\geq 4. \tag{2.14}$$

Example 3:

$$g_6=\begin{vmatrix}1 & 0 & r_4\\ 1 & 1 & r_5-r_4\\ r_2 & 1 & r_6-r_5+r_4\end{vmatrix}=\begin{vmatrix}1 & 0 & 2\\ 1 & 1 & 14-2\\ 0 & 1 & 90-14+2\end{vmatrix}=68.$$

Let $G(x)=\sum_{n=0}^{\infty}g_nx^n$, $R(x)=\sum_{n=0}^{\infty}r_nx^n$, $r_0=0$.

Theorem 9: $G(x)=(1+x)^{-1}-(R(x)+1)^{-1}$.

Proof: By (2.11),

$$\sum_{j=1}^{n-1}r_jg_{n-j}+g_n=\sum_{j=1}^{n}(-1)^{n-j}r_j+(-1)^n.$$

Noticing that $g_1=0$, we have

$$G(x)\cdot R(x)+G(x)=(1+x)^{-1}R(x)+(1+x)^{-1}-1;$$

thus, $G(x)=(1+x)^{-1}-(R(x)+1)^{-1}$. □

Corollary:

$$g_n = \frac{1}{n!}\frac{d^n}{dx^n}\left[(1+x)^{-1} - (R(x)+1)^{-1}\right]\Big|_{x=0}.$$

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ON 2-NIVEN NUMBERS AND 3-NIVEN NUMBERS

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A Niven number [3] is a positive integer that is divisible by the sum of its digits. Various papers have appeared concerning digital sums and properties of the set of Niven numbers. In 1993, Cooper and Kennedy [1] proved that there does not exist a sequence of more than 20 consecutive Niven numbers; they also proved that this bound is the best possible by producing an infinite family of sequences of 20 consecutive Niven numbers. They used a computer to help solve systems of linear congruences, the smallest such sequence they found has **44363342786** digits. In 1994 Grundman [2] generalized the problem to n-Niven numbers with the following definition: For any integer $n \geq 2$, an n-Niven number is a positive integer that is divisible by the sum of its digits in the base n expansion. He proved that no more than $2n$ consecutive n-Niven numbers is possible. He also conjectured that there exists a sequence of consecutive n-Niven numbers of length $2n$ for each $n \geq 2$. In this paper, by solving some congruent equations of higher degree, we obtain the following theorem without the use of a computer.

Theorem: For $n = 2$ or 3, there exists an infinite family of sequences of consecutive n-Niven numbers of length $2n$.

Let $s_n(x)$ denote the digital sum of the positive integer in base n. Consider

$$x = 3^{k_1} + 3^{k_2} + \cdots + 3^{k_8} + 3^3, \quad k_1 > k_2 > \cdots > k_8 > 3,$$

since $s_3(x) = 9$, $s_3(x+1) = 10$, $s_3(x+2) = 11$, $s_3(x-1) = 14$, $s_3(x-2) = 13$, $s_3(x-3) = 12$, the set $\{x-3, x-2, x-1, x, x+1, x+2\}$ is 6 consecutive 3-Niven numbers if and only if the following congruences are satisfied:

$$x_0 + 3 \equiv 0 \pmod{5} \tag{1}$$
$$x_0 + 7 \equiv 0 \pmod{11} \tag{2}$$
$$x_0 + 5 \equiv 0 \pmod{7} \tag{3}$$
$$x_0 + 12 \equiv 0 \pmod{13} \tag{4}$$
$$x_0 \equiv 0 \pmod{4} \tag{5}$$

where $x_0 = 3^{k_1} + 3^{k_2} + \cdots + 3^{k_8}$. Noting that the orders of 3 modulo 5, 11, 7, 13, 4 are 4, 5, 6, 3, 2, respectively, and $[4, 5, 6, 3, 2] = 60$, if the set $\{x-3, x-2, x-1, x, x+1, x+2\}$ is 6 consecutive 3-Niven numbers, then all of the sets $\{x'-3, x'-2, x'-1, x', x'+1, x'+2\}$ with

$$x' = x'(m_1, m_2, \ldots, m_8) = 3^{k_1+60m_1} + 3^{k_2+60m_2} + \cdots + 3^{k_8+60m_8}, m_1, m_2, \ldots, m_8 \geq 0$$

are 6 consecutive 3-Niven numbers.

Note that $3^k \equiv 3 \pmod{4}$ iff $k \equiv 1 \pmod{2}$, $3^k \equiv 1 \pmod{4}$ iff $k \equiv 0 \pmod{2}$. Let x_1 and x_2 denote the number of odd k_i and even k_i, respectively. Then from (5) one has

$$\begin{aligned} x_1 + x_2 &= 8 \\ 3x_1 + x_2 &\equiv 0 \pmod{4} \end{aligned} \tag{5'}$$

with particular solutions $(x_1, x_2) = (8, 0), (6, 2), (4, 4)$, or $(2, 6)$.

Similarly, $3^k \equiv 3 \pmod{13}$ iff $k \equiv 1 \pmod 3$, $3^k \equiv 9 \pmod{13}$ iff $k \equiv 2 \pmod 3$, $3^k \equiv 1$ (mod 3) iff $k \equiv 0 \pmod 3$. Let x_1, x_2, and x_3 denote the number of k_i $(1 \le i \le 8)$ in the form $3m+1$, $3m+2$, or $3m$, respectively. Then from (4) one has

$$\begin{aligned} &x_1 + x_2 + x_3 = 8 \\ &3x_1 + 9x_2 + x_3 + 12 \equiv 0 \pmod{13} \end{aligned} \tag{4'}$$

with particular solutions $(1, 7, 0), (3, 0, 5), (4, 3, 1)$, and $(3, 2, 3)$.

Also, $3^k \equiv 3, 2, 6, 4, 5, 1 \pmod 7$ iff $k \equiv 1, 2, 3, 4, 5, 0 \pmod 6$, respectively. Let x_j $(0 \le j \le 5)$ denote the number of k_i $(1 \le i \le 8)$ satisfying $k \equiv j \pmod 6$. Then from (3) one has

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 + x_5 + x_0 = 8 \\ &3x_1 + 2x_2 + 6x_3 + 4x_4 + 5x_5 + x_0 + 5 \equiv 0 \pmod 7. \end{aligned} \tag{3'}$$

There are many solutions to this system. We find some which also satisfy equations (4′) and (5′). That is,

$$(x_1 + x_3 + x_5, x_2 + x_4 + x_0) = (8, 0), (6, 2), (4, 4), \text{ or } (2, 6);$$
$$(x_1 + x_4, x_2 + x_5, x_3 + x_0) = (1, 7, 0), (3, 0, 5), (4, 3, 1), \text{ or } (3, 2, 3).$$

For example,

$$(x_1, x_2, x_3, x_4, x_5, x_0) = (0, 3, 0, 4, 0, 1), (3, 2, 0, 1, 1, 1), \ldots.$$

Noting that $3^k \equiv 3, 4, 2, 1 \pmod 5$ iff $k \equiv 1, 2, 3, 0 \pmod 4$, respectively, and $3^k \equiv 3, 9, 5, 4, 1$ (mod 11) iff $k \equiv 1, 2, 3, 4, 0 \pmod 5$, respectively. Let x_j $(0 \le j \le 3)$ and x_j $(0 \le j \le 4)$ denote the number of k_i $(1 \le i \le 8)$ satisfying $k \equiv j \pmod 4$ and $k \equiv j \pmod 5$, respectively. Then from equations (1) and (2) one has

$$\begin{aligned} &x_1 + x_2 + x_3 + x_0 = 8 \\ &3x_1 + 4x_2 + 2x_3 + x_4 + 3 \equiv 0 \pmod 5 \end{aligned} \tag{1'}$$

and

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 + x_0 = 8 \\ &3x_1 + 9x_2 + 5x_3 + 4x_4 + x_0 + 7 \equiv 0 \pmod{11}. \end{aligned} \tag{2'}$$

Let us first consider the solution $(3, 2, 0, 1, 1, 1)$ of equations (3′)-(5′), we make an adjustment so that it also satisfies (1′) and (2′), and obtain

$$x = 10000000010000000110000000000110000000110100,$$

that is,

$$3^3 + 3^5 + 3^6 + 3^{13} + 3^{14} + 3^{25} + 3^{26} + 3^{34} + 3^{43}$$

or

$$328273647965397560259.$$

So the smallest 6 consecutive 3-Niven numbers we obtained has **21** digits. Similarly, from the solution $(0, 3, 0, 4, 0, 1)$ of (3′)-(5′), we obtain $x = 3^3 + 3^4 + 3^{48} + 3^{62} + 3^{64} + 3^{122} + 3^{124} + 3^{182} + 3^{184}$, which has 88 digits.

For the case $n = 2$, we may consider

$$x = 2^{k_1} + 2^{k_2} + 2^{k_3} + 2^4, \; k_1 > k_2 > k_3 > 4.$$

Since $s_2(x)=4$, $s_2(x+1)=5$, $s_2(x-1)=7$, $s_2(x-2)=6$, the set $\{x-2, x-1, x, x+1\}$ is 4 consecutive 2-Niven numbers if and only if

$$x_0+1 \equiv 0 \pmod 5$$
$$x_0-1 \equiv 0 \pmod 7$$
$$x_0-2 \equiv 0 \pmod 3$$

are satisfied, where $x_0=2^{k_1}+2^{k_2}+2^{k_3}$. Noting that the orders of 2 modulo 5, 6, 3 are 4, 3, 2, respectively, $[4,3,2]=12$. Therefore, if the set $\{x-2, x-1, x, x+1\}$ is 4 consecutive 2-Niven numbers, all of the sets $\{x'-2, x'-1, x', x'+1\}$ are 4 consecutive 2-Niven numbers, where

$$x'=x'(m_1, m_2, m_3)=2^{k_1+12m_1}+2^{k_2+12m_2}+2^{k_3+12m_3}.$$

We omit the rest of the process. The smallest such sequence we found is (6222, 6223, 6224, 6225) with $6224=2^4+2^6+2^{11}+2^{12}$. Other sequences we found are (33102, 33103, 33104, 33105) with $33104=2^4+2^6+2^8+2^{15}$ and (53262, 53263, 53264, 53265) with $53264=2^4+2^{12}+2^{14}+2^{15}$.

Also we may consider

$$x=2^{k_1}+2^{k_2}+\cdots+2^{k_7}+2^4, \ k_1>k_2>\cdots>k_7>4.$$

The smallest such sequence we found is $(x-2, x-1, x, x+1)$, where

$$x=1100578832=2^4+2^{15}+2^{16}+2^{19}+2^{20}+2^{23}+2^{24}+2^{30}.$$

In principle, this method could be used to find n-Niven numbers of length $2n$ for larger base n. For example, for $n=4$, we may consider $x=4^{k_1}+4^{k_2}+\cdots+4^{k_{15}}+4^{36}$ and, for $n=5$, we may consider $5^{k_1}+5^{k_2}+\cdots+5^{k_{24}}+5^{90}$. But it will be more and more difficult to find a suitable $\{k_1\}$ while n is getting larger.

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CLASSES OF IDENTITIES FOR THE GENERALIZED FIBONACCI NUMBERS $G_n = G_{n-1} + G_{n-c}$ FROM MATRICES WITH CONSTANT VALUED DETERMINANTS

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The generalized Fibonacci numbers $\{G_n\}$, $G_n = G_{n-1} + G_{n-c}$, $n \ge c$, $G_0 = 0$, $G_1 = G_2 = \cdots = G_{c-1} = 1$, are the sums of elements found on successive diagonals of Pascal's triangle written in left-justified form, by beginning in the left-most column and moving up $(c-1)$ and right 1 throughout the array [1]. Of course, $G_n = F_n$, the n^{th} Fibonacci number, when $c = 2$. Also, $G_n = u(n-1; c-1, 1)$, where $u(n; p, q)$ are the generalized Fibonacci numbers of Harris and Styles [2]. In this paper, elementary matrix operations make simple derivations of entire classes of identities for such generalized Fibonacci numbers, and for the Fibonacci numbers themselves.

1. INTRODUCTION

Begin with the sequence $\{G_n\}$, such that

$$G_n = G_{n-1} + G_{n-3},\ n \ge 3,\ G_0 = 0,\ G_1 = G_2 = 1. \tag{1.1}$$

For the reader's convenience, the first values are listed below:

n	0	1	2	3	4	5	6	7	8	9	10
G_n	0	1	1	1	2	3	4	6	9	13	19
n	11	12	13	14	15	16	17	18	19	20	21
G_n	28	41	60	88	129	189	277	406	595	872	1278

These numbers can be generated by a simple scheme from an array which has 0, 1, 1 in the first column, and which is formed by taking each successive element as the sum of the element above and the element to the left in the array, except that in the case of an element in the first row we use the last term in the preceding column and the element to the left:

$$\begin{array}{cccccccc} 0 & 1 & 4 & 13 & [41]\downarrow & 129 & \cdots \\ 1 & 2 & 6 & [19]\rightarrow & \underline{60} & 189 & \cdots \\ 1 & 3 & 9 & 28 & 88 & 277 & \cdots \end{array} \tag{1.2}$$

If we choose a 3×3 array from any three consecutive columns, the determinant is 1. If any 3×4 array is chosen with 4 consecutive columns, and row reduced by elementary matrix methods, the solution is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{pmatrix}. \tag{1.3}$$

We note that, in any row, any four consecutive elements d, e, f, and g are related by

$$d - 3e + 4f = g. \tag{1.4}$$

Each element in the third row is one more than the sum of the $3k$ elements in the k preceding columns; i.e., $9 = (3+2+1+1+1+0)+1$. Each element in the second row satisfies a "column property" as the sum of the three elements in the preceding column; i.e., $60 = 28+19+13$ or, alternately, a "row property" as each element in the second row is one more than the sum of the element above and all other elements in the first row; i.e., $60 = (41+13+4+1+0)+1$. Each element in the third row is the sum of the element above and all other elements to the left in the second row; i.e., $28 = 19+6+2+1$. It can be proved by induction that

$$G_1 + G_2 + G_3 + \cdots + G_n = G_{n+3} - 1, \tag{1.5}$$

$$G_3 + G_6 + G_9 + \cdots + G_{3k} = G_{3k+1} - 1, \tag{1.6}$$

$$G_1 + G_4 + G_7 + \cdots + G_{3k+1} = G_{3k+2}, \tag{1.7}$$

$$G_2 + G_5 + G_8 + \cdots + G_{3k+2} = G_{3k+3}; \tag{1.8}$$

which compare with

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1, \tag{1.9}$$

$$F_1 + F_3 + F_5 + \cdots + F_{2k+1} = F_{2k+2}, \tag{1.10}$$

$$F_2 + F_4 + F_6 + \cdots + F_{2k} = F_{2k+1} - 1, \tag{1.11}$$

for Fibonacci numbers.

The reader should note that forming a two-rowed array analogous to (1.2) by taking 0, 1 in the first column yields Fibonacci numbers, while taking 0, 1, 1, ..., with an infinite number of rows, forms Pascal's triangle in rectangular form, bordered on the top by a row of zeros. We also note that all of these sequences could be generated by taking the first column as all 1's or as 1, 2, 3, ..., or as the appropriate number of consecutive terms in the sequence. They all satisfy "row properties" and "column properties." The determinant and matrix properties observed in (1.2) and (1.3) lead to entire classes of identities in the next section.

2. IDENTITIES FOR THE FIBONACCI NUMBERS AND FOR THE CASE $c = 3$

Write a 3×3 matrix $A_n = (a_{ij})$ by writing three consecutive terms of $\{G_n\}$ in each column and taking $a_{11} = G_n$, where $c = 3$ as in (1.1):

$$A_n = \begin{pmatrix} G_n & G_{n+p} & G_{n+q} \\ G_{n+1} & G_{n+p+1} & G_{n+q+1} \\ G_{n+2} & G_{n+p+2} & G_{n+q+2} \end{pmatrix}. \tag{2.1}$$

We can form matrix A_{n+1} by applying (1.1), replacing row 1 by (row 1 + row 3) in A_n followed by two row exchanges, so that

$$\det A_n = \det A_{n+1}. \tag{2.2}$$

Let $n = 1, p = 1, q = 2$ in (2.1) and find $\det A_1 = -1$. Thus, $\det A_n = -1$ for

$$A_n = \begin{pmatrix} G_n & G_{n+1} & G_{n+2} \\ G_{n+1} & G_{n+2} & G_{n+3} \\ G_{n+2} & G_{n+3} & G_{n+4} \end{pmatrix}. \tag{2.3}$$

As another special case of (2.1), use 9 consecutive elements of $\{G_n\}$ to write

$$A_n = \begin{pmatrix} G_n & G_{n+3} & G_{n+6} \\ G_{n+1} & G_{n+4} & G_{n+7} \\ G_{n+2} & G_{n+5} & G_{n+8} \end{pmatrix}, \tag{2.4}$$

which has $\det A_n = 1$.

These simple observations allow us to write many identities for $\{G_n\}$ effortlessly. We illustrate our procedure with an example. Suppose we want an identity of the form

$$\alpha G_n + \beta G_{n+1} + \gamma G_{n+2} = G_{n+4}.$$

We write an augmented matrix A_n^*, where each column contains three consecutive elements of $\{G_n\}$ and where the first row contains G_n, G_{n+1}, G_{n+2}, and G_{n+4}:

$$A_n^* = \begin{pmatrix} G_n & G_{n+1} & G_{n+2} & G_{n+4} \\ G_{n+1} & G_{n+2} & G_{n+3} & G_{n+5} \\ G_{n+2} & G_{n+3} & G_{n+4} & G_{n+6} \end{pmatrix}.$$

Then take a convenient value for n, say $n = 1$, and use elementary row operations on the augmented matrix A_1^*,

$$A_1^* = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

to obtain a generalization of the "column property" of the introduction,

$$G_n + G_{n+1} + G_{n+2} = G_{n+4}, \tag{2.5}$$

which holds for any n.

While we are using matrix methods to solve the system

$$\begin{cases} \alpha G_n + \beta G_{n+1} + \gamma G_{n+2} = G_{n+4}, \\ \alpha G_{n+1} + \beta G_{n+2} + \gamma G_{n+3} = G_{n+5}, \\ \alpha G_{n+2} + \beta G_{n+3} + \gamma G_{n+4} = G_{n+6}, \end{cases}$$

notice that each determinant that would be used in a solution by Cramer's rule is of the form $\det A_n = \det A_{n+1}$ from (2.1) and (2.2), and, moreover, the determinant of coefficients equals -1 so that there will be integral solutions. Alternately, by (1.1), notice that (α, β, γ) will be a solution of $\alpha G_{n+3} + \beta G_{n+4} + \gamma G_{n+5} = G_{n+7}$ whenever (α, β, γ) is a solution of the system above for any $n \geq 0$ so that we solve all such equations whenever we have a solution for any three consecutive values of n.

We could make one identity at a time by augmenting A_n with a fourth column beginning with G_{n+w} for any pleasing value for w, except that $w < 0$ would force extension of $\{G_n\}$ to negative subscripts. However, it is not difficult to solve

$$A_n^* = \begin{pmatrix} G_n & G_{n+1} & G_{n+2} & G_{n+w} \\ G_{n+1} & G_{n+2} & G_{n+3} & G_{n+w+1} \\ G_{n+2} & G_{n+3} & g_{n+4} & G_{n+w+2} \end{pmatrix}$$

by taking $n = 0$ and elementary row reduction, since $G_{w+2} - G_{w+1} = G_{w-1}$ by (1.1), and

$$A_0^* = \begin{pmatrix} 0 & 1 & 1 & G_w \\ 1 & 1 & 1 & G_{w+1} \\ 1 & 1 & 2 & G_{w+2} \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 & G_w \\ 1 & 0 & 0 & G_{w-2} \\ 0 & 0 & 1 & G_{w-1} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & G_{w-2} \\ 0 & 1 & 0 & G_{w-3} \\ 0 & 0 & 1 & G_{w-1} \end{pmatrix},$$

so that

$$G_{n+w} = G_{w-2}G_n + G_{w-3}G_{n+1} + G_{w-1}G_{n+2}. \tag{2.6}$$

For the Fibonacci numbers, we can use the matrix A_n,

$$A_n = \begin{pmatrix} F_n & F_{n+q} \\ F_{n+1} & F_{n+q+1} \end{pmatrix},$$

for which $\det A_n = (-1)\det A_{n+1}$. Of course, when $q = 1$, $\det A_n = (-1)^{n+1}$ where, also, $\det A_n = F_nF_{n+2} - F_{n+1}^2$, giving the well-known

$$(-1)^{n+1} = F_nF_{n+2} - F_{n+1}^2. \tag{2.7}$$

Solve the augmented matrix A_n^* as before,

$$A_n^* = \begin{pmatrix} F_n & F_{n+1} & F_{n+w} \\ F_{n+1} & F_{n+2} & F_{n+w+1} \end{pmatrix},$$

by taking $n = -1$,

$$A_{-1}^* = \begin{pmatrix} 1 & 0 & F_{w-1} \\ 0 & 1 & F_w \end{pmatrix},$$

to obtain

$$F_{w-1}F_n + F_wF_{n+1} = F_{n+w}. \tag{2.8}$$

Identities of the type $\alpha G_n + \beta G_{n+2} + \gamma G_{n+4} = G_{n+6}$ can be obtained as before by row reduction of

$$A_n^* = \begin{pmatrix} G_n & G_{n+2} & G_{n+4} & G_{n+6} \\ G_{n+1} & G_{n+3} & G_{n+5} & G_{n+7} \\ G_{n+2} & G_{n+4} & G_{n+6} & G_{n+8} \end{pmatrix}.$$

If we take $n = 0$, $\det A_0 = 1$, and we find $\alpha = 1, \beta = 2, \gamma = 1$, so that

$$G_{n+6} = G_n + 2G_{n+2} + G_{n+4}. \tag{2.9}$$

In a similar manner, we can derive

$$G_{n+9} = G_n - 3G_{n+3} + 4G_{n+6}, \tag{2.10}$$

$$G_{n+12} = G_n - 2G_{n+4} + 5G_{n+8}, \tag{2.11}$$

where we compare (1.4) and (2.10).

For the Fibonacci numbers, solve

$$A_n^* = \begin{pmatrix} F_n & F_{n+2} & F_{n+4} \\ F_{n+1} & F_{n+3} & F_{n+5} \end{pmatrix}$$

by taking $n = 1$,

$$A_1^* = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 3 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix},$$

so that

$$F_{n+4} = -F_n + 3F_{n+2}. \tag{2.12}$$

Similarly,

$$F_{n+6} = F_n + 4F_{n+3}, \tag{2.13}$$

$$F_{n+8} = -F_n + 7F_{n+4}. \tag{2.14}$$

In the Fibonacci case, we can solve directly for F_{n+2p} from

$$A_n^* = \begin{pmatrix} F_n & F_{n+p} & F_{n+2p} \\ F_{n+1} & F_{n+p+1} & F_{n+2p+1} \end{pmatrix}$$

by taking $n = -1$,

$$A_{-1}^* = \begin{pmatrix} 1 & F_{p-1} & F_{2p-1} \\ 0 & F_p & F_{2p} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & (F_pF_{2p-1} - F_{p-1}F_{2p}) / F_p \\ 0 & 1 & F_{2p} / F_p \end{pmatrix} \to \begin{pmatrix} 1 & 0 & (-1)^{p-1} \\ 0 & 1 & L_p \end{pmatrix},$$

since $F_pF_{2p-1} - F_{p-1}F_{2p} = (-1)^{p-1}F_p$ and $F_{2p} = F_pL_p$ are known identities for the Fibonacci and Lucas numbers. Thus,

$$F_{n+2p} = (-1)^{p-1}F_n + L_pF_{n+p}. \tag{2.15}$$

Returning to (2.9), we can derive identities of the form $\alpha G_n + \beta G_{n+2} + \gamma G_{n+4} = G_{n+2w}$ from (2.1) with $p = 2, q = 4$, taking the augmented matrix A_n^* with first row containing G_n, G_{n+2}, G_{n+4}, G_{n+2w}. It is computationally advantageous to take $n = -1$; notice that we can define $G_{-1} = 0$. We make use of $G_{2w} - G_{2w-1} = G_{2w-3}$ from (1.1) to solve

$$A_{-1}^* = \begin{pmatrix} 0 & 1 & 1 & G_{2w-1} \\ 0 & 1 & 2 & G_{2w} \\ 1 & 1 & 3 & G_{2w+1} \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 & G_{2w-1} \\ 0 & 0 & 1 & G_{2w} - G_{2w-1} \\ 1 & 0 & 1 & G_{2w+1} - G_{2w} \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 & G_{2w-1} \\ 0 & 0 & 1 & G_{2w-3} \\ 1 & 0 & 1 & G_{2w-2} \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 1 & 0 & G_{2w-1} - G_{2w-3} \\ 0 & 0 & 1 & G_{2w-3} \\ 1 & 0 & 0 & G_{2w-2} - G_{2w-3} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & G_{2w-5} \\ 0 & 1 & 0 & G_{2w-1} - G_{2w-3} \\ 0 & 0 & 1 & G_{2w-3} \end{pmatrix},$$

obtaining

$$G_{n+2w} = G_{2w-5}G_n + (G_{2w-1} - G_{2w-3})G_{n+2} + G_{2w-3}G_{n+4}. \tag{2.16}$$

In the Fibonacci case, taking $n = -1$,

$$A_n^* = \begin{pmatrix} F_n & F_{n+2} & F_{n+2w} \\ F_{n+1} & F_{n+3} & F_{n+2w+1} \end{pmatrix} \to A_{-1}^* = \begin{pmatrix} 1 & 1 & F_{2w-1} \\ 0 & 1 & F_{2w} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -F_{2w-2} \\ 0 & 1 & F_{2w} \end{pmatrix},$$

we have

$$F_{n+2w} = -F_{2w-2}F_n + F_{2w}F_{n+2}. \tag{2.17}$$

Returning to (2.6) and (2.16), the same procedure leads to

$$G_{n+3w} = G_{3w-6}G_n + (G_{3w} - 4G_{3w-3})G_{n+3} + G_{3w-3}G_{n+6}. \tag{2.18}$$

The Fibonacci case, derived by taking $n = -1$,

$$A_n^* = \begin{pmatrix} F_n & F_{n+3} & F_{n+3w} \\ F_{n+1} & F_{n+4} & F_{n+3w+1} \end{pmatrix},$$

gives us

$$F_{n+3w} = F_nF_{3w-3}/2 + F_{n+3}F_{3w}/2, \tag{2.19}$$

where $F_{3m}/2$ happens to be an integer for any m. Note that $\det A_n = (-1)^{n+1}2$, and hence, $\det A_n \neq \pm 1$. We cannot make a pleasing identity of the form $\alpha G_n + \beta G_{n+4} + \gamma G_{n+8} = G_{n+4w}$ for arbitrary w because $\det A_n \neq \pm 1$, leading to nonintegral solutions. However, we can find an identity for $\{G_n\}$ analogous to (2.15). We solve

$$\begin{cases} \alpha G_{-1} + \beta G_{p-1} + \gamma G_{2p-1} = G_{3p-1}, \\ \alpha G_0 + \beta G_p + \gamma G_{2p} = G_{3p}, \\ \alpha G_1 + \beta G_{p+1} + \gamma G_{2p+1} = G_{3p+1}, \end{cases}$$

for (α, β, γ) by Cramer's rule. Note that the determinant of coefficients D is given by $D = G_{2p}G_{p-1} - G_pG_{2p-1}$. Then $\alpha = A/D$, where A is the determinant

$$A = \begin{vmatrix} G_{3p-1} & G_{p-1} & G_{2p-1} \\ G_{3p} & G_p & G_{2p} \\ G_{3p+1} & G_{p+1} & G_{2p+1} \end{vmatrix}.$$

After making two column exchanges in A, we see from (2.1) and (2.2) that $A = D$, so $\alpha = 1$. Then $\beta = B/D$, where B is the determinant

$$B = \begin{vmatrix} 0 & G_{3p-1} & G_{2p-1} \\ 0 & G_{3p} & G_{2p} \\ 1 & G_{3p+1} & G_{2p+1} \end{vmatrix} = G_{2p}G_{3p-1} - G_{3p}G_{2p-1}.$$

Similarly, $\gamma = C/D$, where $C = G_3G_{p-1} - G_pG_{3p-1}$. Thus,

$$G_{n+3p} = G_n + G_{n+p}(G_{3p-1}G_{2p} - G_{3p}G_{2p-1})/D + G_{n+2p}(G_{3p}G_{p-1} - G_{3p-1}G_p)/D,$$

where $D = (G_{2p}G_{p-1} - G_pG_{2p-1})$. The coefficients of G_{n+p} and G_{n+2p} are integers for $p = 1, 2, \ldots, 9$, and it is conjectured that they are always integers.

As an observation before going to the general case, notice that identities such as (2.9), (2.10), and (2.11) generate more matrices with constant valued determinants. For example, (2.9) leads to matrix B_n,

$$B_n = \begin{pmatrix} G_n & G_{n+p} & G_{n+q} \\ G_{n+2} & G_{n+p+2} & G_{n+q+2} \\ G_{n+4} & G_{n+p+4} & G_{n+q+4} \end{pmatrix},$$

where $\det B_n = \det B_{n+2}$.

3. THE GENERAL CASE: $G_n = G_{n-1} + G_{n-c}$

The general case for $\{G_n\}$ is defined by

$$G_n = G_{n-1} + G_{n-c},\ n \ge c, \text{ where } G_0 = 0,\ G_1 = G_2 = \cdots = G_{c-1} = 1. \tag{3.1}$$

To write the elements of $\{G_n\}$ simply, use an array of c rows with the first column containing 0 followed by $(c-1)$ 1's, noting that 1, 2, 3, ..., c will appear in the second column, analogous to the array of (1.2). Take each term to be the sum of the term above and the term to the left, where we drop below for elements in the first row as before. Any $c \times c$ array formed from any c consecutive columns will have a determinant value of ± 1. Each element in the c^{th} row is one more than the sum of the ck elements in the k preceding columns, i.e.,

$$G_1 + G_2 + G_3 + \cdots + G_{ck} = G_{c(k+1)} - 1, \tag{3.2}$$

which can be proved by induction. It is also true that

$$G_1 + G_2 + G_3 + \cdots + G_n = G_{n+c} - 1. \tag{3.3}$$

Each array satisfies the "column property" of (2.5) in that each element in the $(c-1)^{\text{st}}$ row is the sum of the c elements in the preceding column and, more generally, for any n,

$$G_{n+c-2} = G_{n-c} + G_{n-(c-1)} + \cdots + G_{n-2} + G_{n-1} \quad (c \text{ terms}). \tag{3.4}$$

Each array has "row properties" such that each element in the i^{th} row, $3 \le i \le c$, is the sum of the element above and all other elements to the left in the $(i-1)^{\text{st}}$ row, while each element in the second row is one more than the sum of the elements above and to the left in the first row, or

$$G_0 + G_c + G_{2c} + G_{3c} + \cdots + G_{ck} = G_{ck+1} - 1, \tag{3.5}$$

$$G_m + G_{c+m} + G_{2c+m} + \cdots + G_{ck+m} = G_{ck+m+1},\ m = 1, 2, \ldots, c-1, \tag{3.6}$$

for a total of c related identities reminiscent of (1.6), (1.7), and (1.8).

The matrix properties of Section 2 also extend to the general case. Form the $c \times c$ matrix $A_{n,c} = (a_{ij})$, where each column contains c consecutive elements of $\{G_n\}$ and $a_{11} = G_n$. Then, as in the case $c = 3$,

$$\det A_{n,c} = (-1)^{c-1} \det A_{n+1,c}, \tag{3.7}$$

since each column satisfies $G_{n+c} = G_{n+c-1} + G_n$. We can form $A_{n+1,c}$ from $A_{n,c}$ by replacing row 1 by (row 1 + row c) followed by $(c-1)$ row exchanges.

When we take the special case in which the first row of $A_{n,c}$ contains c consecutive elements of $\{G_n\}$, then $A_{n,c} = \pm 1$. The easiest way to justify this result is to observe that (3.1) cam be used to extend $\{G_n\}$ to negative subscripts. In fact, in the sequence $\{G_n\}$ extended by recursion (3.1), $G_1 = 1$ and G_1 is followed by $(c-1)$ 1's and preceded by $(c-1)$ 0's. If we write the first row of $A_{n,c}$ as $G_n, G_{n-1}, G_{n-2}, \ldots, G_{n-(c-1)}$, then, for $n = 1$, the first row is 1, 0, 0, ..., 0. If each column contains c consecutive increasing terms of $\{G_n\}$, then G_n appears on the main diagonal in every row. Thus, $A_{1,c}$ has 1's everywhere on the main diagonal with 0's everywhere above, so that det $A_{1,c} = 1$. That $\det A_{n,c} = \pm 1$ is significant, however, because it indicates that we can write identities following the same procedures as for $c = 3$, expecting integral results when solving systems as before. Note that $\det A_{n,c} = \pm 1$ if the first row contains c consecutive elements of $\{G_n\}$, but order does not matter. Also, we have the interesting special case that $\det A_{n,c} = \pm 1$ whenever $A_{n,c}$ contains c^2 consecutive terms of $\{G_n\}$, taken in either increasing or decreasing order, $c \geq 2$. Det $A_{n,c} = 0$ only if two elements in row 1 are equal, since any c consecutive germs of G_n are relatively prime [2].

Again, solving an augmented matrix $A^*_{n,c}$ will make identities of the form

$$G_{n+w} = \alpha_0 G_n + \alpha_1 G_{n+1} + \alpha_2 G_{n+2} + \cdots + \alpha_{c-1} G_{n+c-1}$$

for different fixed values of c, or other classes of identities of your choosing. As examples, we have:

$$\begin{array}{ll}
c=2 & F_{n+w} = F_n F_{w-1} + F_{n+1} F_w, \\
c=3 & G_{n+w} = G_n G_{w-2} + G_{n+1} G_{w-3} + G_{n+2} G_{w-1}, \\
c=4 & G_{n+w} = G_n G_{w-3} + G_{n+1} G_{w-4} + G_{n+2} G_{w-5} + G_{n+3} G_{w-2}, \\
c=5 & G_{n+w} = G_n G_{w-4} + G_{n+1} G_{w-5} + G_{n+2} G_{w-6} + G_{n+3} G_{w-7} + G_{n+4} G_{w-3}, \\
\ldots & \\
c=c & G_{n+w} = G_n G_{w-c+1} + G_{n+1} G_{w-c} + G_{n+2} G_{w-c-1} + \cdots + G_{n+c-1} G_{w-c+2};
\end{array}$$

$$\begin{array}{ll}
c=2 & F_{n+3} = F_n + F_{n+1} + F_{n+1}, \\
c=3 & G_{n+4} = G_n + G_{n+1} + G_{n+2}, \\
c=4 & G_{n+5} = G_n + G_{n+1} + G_{n+3}, \\
c=5 & G_{n+6} = G_n + G_{n+1} + G_{n+4}, \\
\ldots & \\
c=c & G_{n+c+1} = G_n + G_{n+1} + G_{n+c-1}.
\end{array}$$

So many identities, so little time!

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ON FIBONACCI HYPERBOLIC TRIGONOMETRY AND MODIFIED NUMERICAL TRIANGLES

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1. INTRODUCTION

One of the most satisfactory methods for modeling the physical reality consists in arriving at a suitable differential system which describes, in appropriate terms, the features of the phenomenon investigated. The problem is relatively uncomplicated in the finite dimensional setting but becomes very challenging when various partial differential equations, such as the wave, heat, electromagnetic, and other equations, become involved in the more specific description of the system.

When it is difficult or even impossible to obtain an exact solution of the partial differential equations governing a continuous plant, the mathematical model is almost always reduced to a discrete form. Then the plant is represented by an appropriate connection of lumped-parameter elements and it may vibrate only in combinations of a certain set of assumed modes.

In modeling continuous-time systems that are continuous or discrete in space, such classic trigonometric functions as sine, cosine, tangent, and cotangent, as well as corresponding hyperbolic functions, are widely used. As is well known, these functions are based on two irrational numbers: $\pi = 3.14156926\ldots$ and $e = 2.7182818\ldots$.

In this paper we shall be concerned with a new class of hyperbolic functions that are defined on the basis of the irrational number $\phi = \frac{1+\sqrt{5}}{2} \sim 1.618033\ldots$, also known as the golden ratio.

We shall introduce new functions called "Fibonacci hyperbolic functions" and show how they result from suitable application of modified numerical triangles. We shall also establish a set of suitable properties of Fibonacci hyperbolic functions such as symmetry, shifting, and links with the classic trigonometric and hyperbolic functions, respectively. Some examples illustrating pos-sible applications of the involved functions in mathematical modeling of physical plants are also presented.

2. THE FIBONACCI TRIGONOMETRY

Recently, studies and applications of discrete functions based on the irrational number $\phi = \frac{1+\sqrt{5}}{2} \sim 1.618033\ldots$ have received considerable attention, especially in the theory of recurrence equations, of the Fibonacci sequence, their generalizations and applications (e.g., see [1], [2], [4], [5], [6], [9], and [10]).

In this section we shall present fundamentals of a new class of functions called Fibonacci hyperbolic functions.

Definition 1: Let

$$\psi = 1 + \phi = \frac{3+\sqrt{5}}{2} \sim 2.618033\ldots \tag{1}$$

be given, where ϕ denotes the golden ratio.

For $x \in (-\infty, \infty)$, we define by analogy to the classic hyperbolic functions: *chx*, *shx*, *thx*, *cthx*, continuous functions

$$sFh(x) = \frac{\psi^x - \psi^{-x}}{\sqrt{5}}, \quad cFh(x) = \frac{\psi^{\left(x+\frac{1}{2}\right)} + \psi^{-\left(x+\frac{1}{2}\right)}}{\sqrt{5}},$$
$$tFh(x) = \frac{sFh(x)}{cFh(x)}, \quad ctFh(x) = \frac{cFh(x)}{sFh(x)}, \tag{2}$$

as the Fibonacci hyperbolic sine, cosine, tangent, and cotangent, respectively.

Diagrams representing the above-defined Fibonacci sine and cosine are presented in Figure 1. Respective diagrams can easily be established for the Fibonacci tangent and cotangent. They are omitted here for the sake of presentation simplicity.

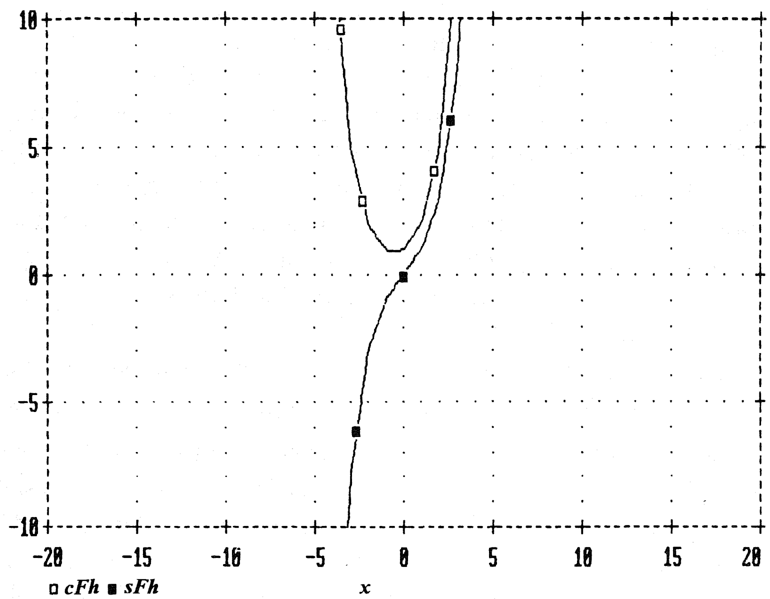

FIGURE 1. Diagrams of $cFh(x)$ and $sFh(x)$

It is worth noting that function $sFh(x)$ is odd-symmetric with respect to the coordinate origin but function $cFh(x)$, while asymmetric with respect to the vertical coordinate $x = 0$, is even-symmetric with respect to $-\frac{1}{2}$.

On the basis of the above definition relations, we are able to establish a set of important properties of Fibonacci hyperbolic functions. In the sequel we shall focus attention on $sFh(x)$ and $cFh(x)$ only.

First, they can be expressed in terms of the golden division ratio ϕ as follows. Using the well-known identity

$$\phi^2 = 1 + \phi \tag{3}$$

and substituting it into expressions (2), we obtain

$$sFh(x) = \frac{\phi^{2x} - \phi^{-2x}}{\sqrt{5}}, \quad cFh(x) = \frac{\phi^{(2x+1)} + \phi^{-(2x+1)}}{\sqrt{5}}. \tag{4}$$

Second, it is easy to demonstrate that when, instead of the continuous independent variable x, we use a discrete variable $k \in I$ (a set of all integer numbers $k = \ldots, -2, -1, 0, 1, 2, \ldots$) we can express functions $sFh(x)$ and $cFh(x)$ in terms of the corresponding elements of the Fibonacci sequence

$$f(k+1) = f(k) + f(k-1), \quad k = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \tag{5}$$

with $f(0) = 0$ and $f(1) = 1$ as follows:

$$sFh(k) = f(2k), \quad cFh(k) = f(2k+1). \tag{6}$$

Next, applying the well-known Binet formula (see [1], [2]) to the right-hand sides of expressions (6) yields

$$\begin{aligned} sFh(k) &= \frac{1}{2^{2k-1}}\left[\binom{2k}{1} + 5\binom{2k}{3} + 5\binom{2k}{5} + \cdots + 5^r\binom{2k}{2r+1} + \cdots\right] \\ &= \frac{1}{2^{2k-1}}\left[\sum_{r=0}^{\infty} 5^r \binom{2k}{2r+1}\right] \end{aligned} \tag{7}$$

and

$$\begin{aligned} cFh(k) &= \frac{1}{2^{2k}}\left[\binom{2k+1}{1} + 5\binom{2k+1}{3} + 5\binom{2k+1}{5} + \cdots + 5^r\binom{2k+1}{2r+1} + \cdots\right] \\ &= \frac{1}{2^{2k}}\left[\sum_{p=0}^{\infty} 5^p \binom{2k+1}{2p+1}\right]. \end{aligned} \tag{8}$$

Note that the right-hand sides of expressions (6) and (8) do not represent an infinite series but are finite sums, since their general term vanishes for $2k < 2r+1$ and $2k < 2p$, respectively. For instance, at $k = 8$, the first vanishing term corresponds to $2r > 15$ for the $sFh(k)$ and to $p > 8$ for the $cFh(k)$. Thus, the calculations of $sFh(k)$ and $cFh(k)$ $(k \in I)$ are reduced to easily computed sums involving simple binomial coefficients, $\binom{n}{m}$.

Finally, it is possible to establish links of the Fibonacci hyperbolic functions $sFh(k)$ and $cFh(k)$ $(k \in I)$ with the classic hyperbolic functions $\sinh(x)$ and $\cosh(x)$, but they are based on transition from an expression through its natural logarithm. For this purpose, we calculate the logarithm of the irrational number ϕ, namely,

$$\alpha = \ln\phi = \ln\frac{1+\sqrt{5}}{2} \sim 0.4812118\ldots. \tag{9}$$

Next, we calculate exponential functions

$$e^{\alpha} = \frac{1+\sqrt{5}}{2} = \phi, \quad e^{-\alpha} = \frac{\sqrt{5}-1}{2} = \phi^{-1}, \tag{10}$$

and the corresponding hyperbolic functions

$$\cosh\alpha = \frac{\sqrt{5}}{2}, \quad \sinh\alpha = \frac{1}{2}. \tag{11}$$

Substituting values (11) raised to the $2k^{\text{th}}$ power into (7) and (8), we get

$$sFh(k)=\frac{2}{\sqrt{5}}\sinh(2k\alpha),\quad cFh(k)=\frac{2}{\sqrt{5}}\cosh[(2k+1)\alpha]. \tag{12}$$

Thus, we have one operation only for calculating $sFh(k)$ or $cFh(k)$, i.e., the multiplication of the known hyperbolic function of the argument $2k\alpha$ or $(2k+1)\alpha$, respectively, by the coefficient $2/\sqrt{5}=0.8944271\ldots$. For example,

$$sFh(8)=\frac{2}{\sqrt{5}}sh(16\alpha)=0.8944271\cdot 1103.6922=987$$

and

$$cFh(8)=\frac{2}{\sqrt{5}}ch(17\alpha)=0.8944271\cdot 1785.5002=1597.$$

In a similar manner we can establish links of Fibonacci hyperbolic functions with such trigonometric functions as sine and cosine with respective arguments.

3. PROPERTIES OF FIBONACCI HYPERBOLIC FUNCTIONS

Taking into account the expressions presented in the preceding section, we can derive a set of important properties and relations which come into existence in Fibonacci hyperbolic trigonometry.

First, it is possible to demonstrate on the basis of (6) that the following equalities hold:

$$sFh(-k)=-sFh(k),\quad cFh(-k)=cFh(k-1). \tag{13}$$

Thus, $sFh(k)$ is odd-symmetric with respect to the coordinate origin but $cFh(k)$ is even-symmetric with respect to the vertical line $k=-\frac{1}{2}$. Note that $cFh(-\frac{1}{2})=\frac{2}{\sqrt{5}}=0.8944271\ldots$, which means that the minimum of $cFh(k)$ appears at $k=-\frac{1}{2}$ and differs from that for the classic hyperbolic $ch(x)$ which equals $\min(\cosh(x))$ at $x=0$. On the other hand, for $k=0$, function $cFh(k)$ takes the value $cFh(0)=1$.

It is easy to prove the remaining important properties of functions $sFh(k)$ and $cFh(k)$. Some of these are given below:

1. $sFh(k)+cFh(k)=sFh(k+1)$,
2. $sFh^2(k)+cFh^2(k)=cFh(2k)$,
3. $cFh^2(k)-sFh^2(k)=1+sFh(k)cFh(k)$,
4. $sFh(k)+sFh(\ell)=\sqrt{5}sFh\left(\frac{k+\ell}{2}\right)cFh\left(\frac{k-\ell-1}{2}\right)$,
5. $sFh(k)-sFh(\ell)=\sqrt{5}sFh\left(\frac{k-\ell}{2}\right)cFh\left(\frac{k+\ell-1}{2}\right)$,
6. $cFh(k)+cFh(\ell)=\sqrt{5}cFh\left(\frac{k+\ell}{2}\right)cFh\left(\frac{k+\ell-1}{2}\right)$,

7. $cFh(k) - cFh(\ell) = \sqrt{5}sFh\left(\frac{k-\ell}{2}\right)sFh\left(\frac{k+\ell-1}{2}\right)$,

8. $sFh(2k) = \sqrt{5}sFh(k)cFh\left(k-\frac{1}{2}\right)$,

9. $cFh(2k) = \sqrt{5}cFh(k)cFh\left(k-\frac{1}{2}\right)+1$,

10. $cFh(k)cFh(k-1) - sFh^2(k) = 1$.

For the sake of presentation compactness, the corresponding proofs are omitted here. It is worth noting that the above properties also remain valid for continuous arguments $x \in (-\infty, \infty)$ and $y \in (-\infty, \infty)$, respectively.

4. RELATIONSHIPS BETWEEN FIBONACCI HYPERBOLIC FUNCTIONS AND MODIFIED NUMERICAL TRIANGLES

Some advantages in calculating Fibonacci hyperbolic functions follow from the structure and properties of modified numerical triangles (see [5], [9], [10]). To facilitate their demonstration, we shall briefly discuss these triangles and their main characteristics.

The first modified numerical triangle (MNT1) contains elements corresponding to coefficients of polynomials in q defined by the recurrence expression

$$T_{k+1}(q) = (2+q)T_k(q) - T_{k-1}(q),\ T_0(q) = 1,\ T_1(q) = 1+q, \tag{14}$$

with q as, in a general case, a complex parameter and $k = 0, \pm 1, \pm 2, \pm 3, \dots$.

Coefficients of the above polynomials for successive values of k belong to MNT1, which takes the form

MNT1

k \ m	0	1	2	3	4	5	6	…
0	1							
1	1	1						
2	1	3	1					
3	1	6	5	1				
4	1	10	15	7	1			
5	1	15	35	28	9	1		
6	1	21	70	84	45	11	1	
…	…	…	…	…	…	…		

The second modified numerical triangle (MNT2) corresponds to polynomials in q defined by the expression

$$P_{k+1}(q) = (2+q)P_k(q) - P_{k-1}(q),\ P_0(q) = 0,\ P_1(q) = 1, \tag{15}$$

with $k = 0, \pm 1, \pm 2, \pm 3, \dots$.

Coefficients of these polynomials belong to MNT2, which takes the form

MNT2

$k \backslash m$	0	1	2	3	4	5	6	$\cdots$
0	0							
1	1							
2	2	1						
3	3	4	1					
4	4	10	6	1				
5	5	20	21	8	1			
6	6	35	56	36	10	1		
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		

The above polynomials fulfill a set of important relations, and some examples are as follows:

$$T_k(q) - T_{k-1}(q) = qP_k(q), \quad P_k - P_{k-1} = T_{k-1}(q). \tag{16}$$

It was demonstrated in [9] that, for $q = 1$, the following relations hold:

$$T_k(1) = f(2k+1), \quad P_k = f(2k), \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots . \tag{17}$$

Thus, taking into account expressions (6), we have

$$cFh(k) = T_k(1), \quad sFh(k) = P_k(1). \tag{18}$$

It is worth noting that the modified numerical triangles can be used effectively to determine values of corresponding Fibonacci hyperbolic functions.

5. ILLUSTRATION EXAMPLES

Let us now proceed to illustrate possible applications of Fibonacci hyperbolic trigonometry for solving problems arising from biology, physics, or technics. We shall demonstrate these applications through suitable examples.

Example 1: A microwave system usually contains such an essential part as a junction. It consists of two or more microwave components or transmission lines connected together (see [3], [8]). The propagation of electromagnetic signals along each component is described by the transmission line equation,

$$\frac{d^2V}{dx^2} = ZYV, \tag{19}$$

where V is the Laplace transform of the voltage at point $x \in (0, \ell)$ of the space variable in the direction of propagation and Z and Y denote the per unit length impedance and admittance of the line, respectively.

In a general case, the solution for the voltage as a function of time is difficult; for this and other reasons, recourse to an approximate approach is needed. Following this line of reasoning and applying the well-known second-order difference approximation yields

$$V(k+1) - (2+q)V(k) + V(k-1) = 0, \; V(0) = V_0, \; V(1) = (1+q)V(0) \tag{20}$$

with

$$q = Z_0 Y_0, \tag{21}$$

where $Z_0 = Z \cdot h$ and $Y_0 = Y \cdot h$ denote the impedance and admittance per distance $h = \Delta x$ of the space coordinate discretization.

Solving equation (20) with respect to $V(k)$ gives

$$V(k) = T_k(q)V_0 + Z_0 P_k(q) I_0, \tag{22}$$

where V_0 and I_0 denote the Laplace transforms of the voltage and current at $x = 0$, i.e., for $k = 0$.

On the other hand, following the general method of solution of difference equation (20) yields

$$V(k) = q^{-k}[cFh_q(k)V_0 + Z_0 sFh_q(k)I_0], \tag{23}$$

where $sFh_q(k)$ and $cFh_q(k)$ denote generalized Fibonacci hyperbolic sinus and cosinus, respectively. They are defined as follows.

Definition 2: If q denotes, in the general case, a complex parameter, then the following expressions,

$$\begin{aligned} sFh_q(k) &= \frac{1}{\sqrt{q^2+4q}}\left[\left(\frac{q+2+\sqrt{q^2+4q}}{2}\right)^{2k} - \left(\frac{-q-2+\sqrt{q^2+4q}}{2}\right)^{-2k}\right], \\ cFh_q(k) &= \frac{1}{\sqrt{q^2+4q}}\left[\left(\frac{q+2+\sqrt{q^2+4q}}{2}\right)^{2k+1} + \left(\frac{-q-2+\sqrt{q^2+4q}}{2}\right)^{-(2k+1)}\right], \end{aligned} \tag{24}$$

define the so-called generalized Fibonacci hyperbolic sinus and cosinus, respectively. Using the above expressions, we can easily establish the generalized Fibonacci hyperbolic tangent and cotangent. For the sake of presentation compactness, corresponding expressions are omitted here.

Thus, comparing solutions (22) and (23) and referring to (24) gives

$$cFh_q(k) = q^k T_k(q), \quad sFh_q(k) = q^k P_k(q). \tag{25}$$

Moreover, it is easily seen that fixing $q = 1$ we obtain the usual Fibonacci hyperbolic functions $cFh(k)$ and $sFh(k)$, so that we have

$$cFh_q(k)|_{q=1} = cFh(k), \quad sFh_q(k)|_{q=1} = sFh(k). \tag{26}$$

Now it is evident that the above presented Fibonacci hyperbolic functions and modified numerical triangles can be very useful for practical problems studies.

Example 2: The filter design problem at microwave frequencies, where distributed parameter elements must be used, is extremely complicated, and no complete theory or synthesis procedure exists for solving the problem. The complex behavior of microwave circuit elements makes it impossible to develop a general and complete synthesis procedure [7]. However, a procedure based on the Fibonacci hyperbolic trigonometry appears as useful technique for studies of microwave filters. The effect of lossy elements or quarter-wave transformers can easily be considered.

The latter case is represented by the network shown in Figure 2. It contains a number of quarter-wave transformers loaded by the lumped parameter elements characterized by impedance Z_1. The voltage and current distributions along the system are described by the matrix equation

$$\begin{bmatrix} U(k+1) \\ I(k+1) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} U(k) \\ I(k) \end{bmatrix}, \; k = 0, 1, 2, \ldots, \tag{27}$$

where a, b, c, and d denote, in a general case, complex parameters fulfilling the relation

$$ad - bc = 1. \tag{28}$$

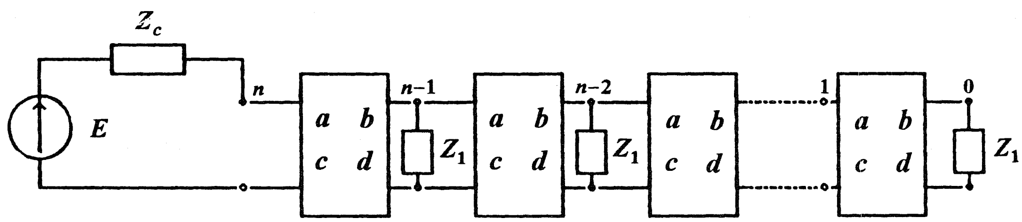

FIGURE 2. Ladder of Two Ports

In the sequel we shall limit our attention to a system having the following parameters:

$$a = 1, \; b = -jZ_c, \; c = \frac{-j}{Z_c}, \; d = 0, \tag{29}$$

where Z_c is the characteristic impedance of each one of two port elements in the system and $j = \sqrt{-1}$.

Introducing characteristic parameter

$$p = \frac{Z_1}{Z_c}, \tag{30}$$

and solving equation (27) with respect to $U(k)$, $k = 0, 1, 2, \ldots$, we get the second-order difference equation with complex coefficients, that is,

$$U(k+1) + jpU(k) + U(k-1) = 0, \; U(0) = U_0, \; U(1) = -jpU(0). \tag{31}$$

Now, comparing respective coefficients in equations (22) and (31) yields

$$U(2k) = (-j)^{2k} T_k(p^2), \quad U(2k+1) = (-j)^{2k+1} P_k(p^2), \tag{32}$$

where $T_k(x)$ and $P_k(x)$ are the polynomials in $x = p^2$ with coefficients from MNT1 and MNT2, respectively.

Taking into account the relationship between Fibonacci hyperbolic functions and polynomials $T_k(x)$ and $P_k(x)$, we can transform relations (32) into the following forms:

$$U(2k) = \left(\frac{1}{jp^2}\right)^{2k} cFh_{p^2}(k), \quad U(2k+1) = \left(\frac{1}{jp^2}\right)^{2k+1} sFh_{p^2}(k), \text{ with } k = 0, 1, 2, \ldots . \tag{33}$$

Thus, a set of suitable expressions has been established which gives much more facility and improvement with respect to up-to-date available ones in the design of microwave filters. It must be stressed that no assumption has been made on the lumped parameter elements; therefore, the presented approach is quite general.

Example 3: One of the fundamental problems in botany lies in suitable descriptions of leaf growings [12]. The geometry of leaf growing is characterized by a spiral-symmetry structure. Bio-organisms draw images on the surface of the leaves forming left- and right-turning spiral lines with crossings at respective points. The symmetry order of the leaf-grilles are determined by a number of spiral lines in respective patterns. During leaf growing, these spiral lines can be transformed into moving hyperboles with cross-points determined by the coordinates expressed in terms of Fibonacci hyperbolic functions as follows:

$$u_k = a \cdot sFh(k), \quad u_{k-1} = a \cdot cFh(k-1), \tag{34}$$

where $k = 0, 1, 2, \dots$ is the index of the cross-point in the leaf-grille and a denotes the scale coefficient of the moving hyperbole with respect to parameters of a unit hyperbole.

If the grille is square, then the coordinates of the cross-points take integer values that fulfill the relation

$$u_{k+1}u_k - u_{k+1}^2 + u_k^2 = \text{ const}. \tag{35}$$

The structure-symmetry order of the logarithmic grille is determined by the parameter

$$q_\ell = q^D, \tag{36}$$

where q denotes the similarity coefficient and D is the angle divergence.

For tree foliage, leaf growing is determined in terms of the Fibonacci sequence and fulfills the equation

$$|f^2(k) + f(k)f(k+1) - f^2(k+1)| = 1, \tag{37}$$

and at the limit $k \to \infty$, the angle divergence is equal to

$$\lim_{k\to\infty} D = \frac{\sqrt{5}-1}{2} = \phi^{-1} \sim 0.618033\dots . \tag{38}$$

Other cases of leaf growing are governed by similar expressions. Following a more general line of reasoning, it is possible to prove that there are general principles in pattern formation on the plants.

6. CONCLUSIONS

In this paper we have presented the new ideas and concepts concerning hyperbolic trigonometry. It has been shown that many problems appearing in mathematical modeling of physical plants can be solved successfully by applying such new functions as Fibonacci hyperbolic sine and/or cosine. The concepts presented in this paper have the following features: a) they produce analytic expressions for both continuous and discrete arguments; b) in the discrete case, there exist respective links with the classic Fibonacci sequence; c) important simplifications in calculus can be achieved by using modified numerical triangles.

The application of Fibonacci hyperbolic functions has been illustrated by suitable examples.

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LOCAL MINIMAL POLYNOMIALS OVER FINITE FIELDS

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1. INTRODUCTION

Let F_q denote the finite field of order $q = p^e$, where q is an odd prime. If $f(x)$ is a polynomial of degree $d \geq 1$ over F_q, then it is clear that

$$\left[\frac{q-1}{d}\right]+1 \leq V(f) = \left|\{f(x) : x \in F_q\}\right| \leq q,$$

where $[w]$ denotes the greatest integer less than or equal to w. We say that $f(x)$ permutation polynomial if $V(f) = q$, and $f(x)$ is a minimal value set polynomial if

$$V(f) = \left[\frac{q-1}{d}\right]+1.$$

A polynomial $f(x, y)$ with coefficients in F_q is a local permutation (minimal value set) polynomial over F_q if $f(a, x)$ and $f(x, b)$ are permutation (minimal value set) polynomials in x for all a and b in F_q. Local permutation polynomials have been studied by Mullen in [5] and [6].

In this note we will consider local minimal polynomials of small degree ($<\sqrt{q}$) on both x and y. We will show that there are only five classes of local minimal polynomials. Namely,

(a) $f(x, y) = aX^mY^n + bX^m + cY^n + d, m, n|(q-1)$,

(b) $f(x, y) = (aX + bY + c)^m + d, m|(q-1)$,

(c) $f(x, y) = aX^2Y^n + bX^2 + cX + dY^n + e, n|(q-1)$,

(d) $f(x, y) = aX^mY^2 + bY^2 + cY + dX^m + e, m|(q-1)$, and

(e) $f(x, y) = aX^2Y^2 + bX^2 + cY^2 + dX + eY + gXY + h$.

where $X = (x - x_0)$ and $Y = (y - y_0)$ with x_0, y_0 in F_q.

2. THEOREM AND PROOF

Minimal value set polynomials have been studied by several authors. L. Carlitz, D. J. Lewis, W. H. Mills, and E. Strauss [2] showed that, when q is a prime and $d = \deg(f) < q$, all minimal value set polynomials with $V(f) \geq 3$ have the form $f(x) = a(x+b)^d + c$ with d dividing $q-1$. Later, W. H. Mills [4] gave a complete characterization of minimal value set polynomials over arbitrary finite fields with $d < \sqrt{q}$. A weakened form of Mills's results can be stated as follows:

Lemma 1 (Mills): If F_q is a finite field with q elements and $f(x)$ is a monic polynomial over F_q of degree d prime to q, then

$$d < \sqrt{q} \quad \text{and} \quad V(f) = \left[\frac{q-1}{d}\right] + 1$$

imply

$$d \mid (q-1) \quad \text{and} \quad f(x) = (x+b)^d + c.$$

For other related results, see [1] and [3]. We are now ready for our result.

Theorem 2: Let F_q denote a finite field of order $q = p^e$, where p is an odd prime. Let

$$f(x,y) = \sum_{i=0}^{n} a_i(x) y^i = \sum_{j=0}^{m} b_j(y) x^j$$

denote a polynomial with coefficients in F_q. Assume that m, n, $n-1$, and $m-1$ are relatively prime to q and $1 < m, n < \sqrt{q}$. Assume $a_n(x) b_m(y) \neq 0$ for all x, y in F_q. Then $f(x,y)$ is a local minimal polynomial if and only if $f(x,y)$ has one of the following forms:

(a) $f(x,y) = aX^mY^n + bX^m + cY^n + d, m, n \mid (q-1)$,

(b) $f(x,y) = (aX + bY + c)^m + d, m \mid (q-1)$,

(c) $f(x,y) = aX^2Y^n + bX^2 + cX + dY^n + e, n \mid (q-1)$,

(d) $f(x,y) = aX^mY^2 + bY^2 + cY + dX^m + e, m \mid (q-1)$, and

(e) $f(x,y) = aX^2Y^2 + bX^2 + cY^2 + dX + eY + gXY + h$.

where $X = (x - x_0)$ and $Y = (y - y_0)$ with x_0, y_0 in F_q.

Proof: If $f(x,y)$ is one of the forms (a)–(e), then it is easy to see that $f(x,y)$ is a local minimal value set polynomial. Now, let

$$f(x,y) = \sum_{i=0}^{n} a_i(x) y^i = \sum_{j=0}^{m} b_j(y) x^j$$

denote a local minimal value set polynomial over F_q satisfying:

(i) $1 < m, n < \sqrt{q}$,

(ii) $(mn(m-1)(n-1), q) = 1$,

(iii) $a_n(x) b_m(y) \neq 0$ for all x, y in F_q.

Also, and without loss of generality, assume that $m \leq n$ and $n \geq 3$ [$n = 2$ gives form (e)]. Then, by Lemma 1,

$$f(x,y) = a_n(x)\left(y + \frac{a_{n-1}(x)}{na_n(x)}\right)^n + a_0(x) - \frac{a_{n-1}^n(x)}{n^n a_n^{n-1}(x)}, \tag{1}$$

$$= b_m(y)\left(x + \frac{b_{m-1}(y)}{mb_m(y)}\right)^m + b_0(y) - \frac{b_{m-1}^m(y)}{m^m b_m^{m-1}(y)}, \tag{2}$$

for all x, y in F_q and $m, n|(q-1)$. Hence,

$$b_m^{m-1}(y)\left[\left(a_n(x)y+\frac{a_{n-1}(x)}{n}\right)^n+a_0(x)a_n^{n-1}(x)-\frac{a_{n-1}^n(x)}{n^n}\right]$$
$$=a_n^{n-1}(x)\left[\left(b_m(y)x+\frac{b_{m-1}(y)}{m}\right)^m+b_0(y)b_m^{m-1}(y)-\frac{b_{m-1}^m(y)}{m^m}\right] \quad (3)$$

for all x, y in F_q. Further, since $1<m\leq n<\sqrt{q}$, equation (3) also establishes the equality of the polynomials. Therefore,

$$b_m^{m-2}(y)\left[a_n^{n-1}(x)y^n+a_n^{n-2}(x)a_{n-1}(x)y^{n-1}+\cdots+\frac{a_{n-1}^{n-1}(x)}{n^{n-2}}y+a_0(x)a_n^{n-2}(x)\right]$$
$$=a_n^{n-2}(x)\left[b_m^{m-1}(y)x^m+b_m^{m-2}(y)b_{m-1}(y)x^{m-1}+\cdots+\frac{b_{m-1}^{m-1}(y)}{m^{m-2}}x+b_0(y)b_m^{m-2}(y)\right].$$

Hence,

$$a_n^{n-2}(x) \text{ divides } \binom{n}{2}\frac{a_n^{n-3}(x)a_{n-1}^2(x)}{n^2}y^{n-3}+\cdots+\frac{a_{n-1}^{n-1}(x)}{n^{n-2}}$$

and, consequently, $a_n^{n-2}(x)$ divides $a_{n-1}^{n-1}(x)$. Now, if $g(x)$ is an irreducible factor of $a_n(x)$ so that $g^c(x)|a_n(x)$ but $g^{c+1}(x)\nmid a_n(x)$, then $g^e(x)$ divides $a_{n-1}(x)$ for some integer e such that $1< c(n-2)\leq(n-1)e$. Therefore, since $\deg(g(x))\geq 2$, $e\leq c-1$ implies $c(n-2)\leq(n-1)(c-1)$ or $n-1\leq c\leq\frac{m}{2}\leq\frac{n}{2}$, a contradiction. Thus, $a_n(x)$ divides $a_{n-1}(x)$.

Case 1. $a_{n-1}(x)=0$. Then, by (1),

$$f(x,y)=a_n(x)y^n+a_0(x)=\left(\sum_{i=0}^m a_{ni}x^i\right)y^n+\sum_{i=0}^m a_{0i}x^i=\sum_{i=0}^m(a_{ni}y^n+a_{0i})x^i$$
$$=(a_{nm}y^n+a_{0m})\left(x+\frac{a_{nm-1}y^n+a_{0m-1}}{m(a_{nm}y^n+a_{0m})}\right)^m+a_{n0}y^n+a_{00}-\frac{(a_{nm-1}y^n+a_{0m-1})^m}{m^m(a_{nm}y^n+a_{0m})^{m-1}}.$$

Hence, $f(x,y)$ has the form (c) or $m\geq 3$ and

$$(a_{nm}y^n+a_{0m})\binom{m}{i}\left(\frac{a_{nm-1}y^n+a_{0m-1}}{m(a_{nm}y^n+a_{0m})}\right)^{m-i}=a_{ni}y^n+a_{0i}$$

or

$$\binom{m}{i}\left(\frac{a_{nm-1}y^n+a_{0m-1}}{m}\right)^{m-i}=(a_{nm}y^n+a_{0m})^{m-i-1}(a_{ni}y^n+a_{0i}) \quad (4)$$

for all y in F_q and $i=1,2,\ldots,m$. So, if $a_{nm}=0$, then $a_{nm-1}=0$ and we obtain

$$f(x,y)=a_{0m}\left(x+\frac{a_{0m-1}}{ma_{0m}}\right)^m+a_{n0}y^n+a_{00}-\left(\frac{a_{0m-1}}{ma_{0m}}\right)^m a_{0m},$$

where $a_{0m}a_{n0} \neq 0$. On the other hand, if $a_{nm} \neq 0$, then, again by (4),

$$\frac{a_{0m}}{a_{nm}} = \frac{a_{0m-1}}{a_{nm-1}}.$$

Therefore, either $f(x,y)$ has the form (c) or

$$\begin{aligned} f(x,y) &= (a_{nm}y^n + a_{0m})\left(x + \frac{a_{nm-1}y^n + a_{0m-1}}{m(a_{nm}y^n + a_{om})}\right)^m + a_{n0}y^n + a_{00} - \frac{(a_{nm-1}y^n + a_{0m-1})^m}{m^m(a_{nm}y^n + a_{0m})^{m-1}} \\ &= (a_{nm}y^n + a_{0m})\left(x + \frac{a_{nm-1}}{ma_{nm}}\right)^m + a_{n0}y^n + a_{00} - \left(\frac{a_{nm-1}}{ma_{nm}}\right)^m (a_{nm}y^n + a_{0m}) \end{aligned}$$

and $f(x,y)$ has the form (a).

Case 2. $a_n(x) | a_{n-1}(x) \neq 0$. Then, by (1),

$$\deg(a_n(x)) + (n-1)\deg\left(\frac{a_{n-1}(x)}{a_n(x)}\right) \leq m.$$

Hence, either $\deg(\frac{a_{n-1}(x)}{a_n(x)}) = 0$ or $\deg(\frac{a_{n-1}(x)}{a_n(x)}) = 1$ and $\deg(a_n(x)) = 0$. First, we assume that $\deg(\frac{a_{n-1}(x)}{a_n(x)}) = 1$ and $\deg(a_n(x)) = 0$. Thus, $n-1 \leq m \leq n$ and

$$f(x,y) = A_1(y + a_1x + c_1)^n + g(x),$$

where $A_1a_1 \neq 0$ and $g(x)$ denotes a polynomial of degree less than or equal to n. Now, $m = n-1$ gives $b_m(y) = b_{n-1}(y) = na_1^{n-1}(y + c_1) + c_2$, a contradiction to (iii). Thus, $b_m(y) = b_n(y)$ is a constant polynomial, $\deg(\frac{b_{m-1}(y)}{b_m(y)}) = 1$ and

$$f(x,y) = A_2(x + a_2y + c_2)^m + h(y),$$

where $A_2a_2 \neq 0$ and $h(y)$ denotes a polynomial of degree less than or equal to $n = m$. Therefore, there exist constants A_3, a_3, and c_3 such that

$$A_3(x + a_3y + c_3)^n + \sum_{i=0}^{n} r_ix^i = A_2(x + a_2y + c_2)^n + \sum_{i=0}^{n} s_iy^i, \tag{5}$$

where $g(x) = \sum_{i=0}^n r_ix^i$ and $h(y) = \sum_{i=0}^n s_iy^i$. Now we compare the coefficients of $x^{n-i}y^i$ in (5) to obtain

$$A_3\binom{n}{i}a_3^i = A_2\binom{n}{i}a_2^i$$

for $i = 1, \ldots, n-1$. Since $(n-1, q) = 1$, it follows that $A_2 = A_3$ and $a_2 = a_3$. Thus, comparing the coefficients of $x^{n-2}y$, $c_2 = c_3$. Therefore, $g(x) = h(y) = d$ for some constant d, and

$$f(x,y) = A(x + ay + c)^n + d$$

which has the form (b).

Now we assume that $\deg(\frac{a_{n-1}(x)}{a_n(x)}) = 0$. Then

$$f(x,y) = a_n(x)(y + \alpha)^n + g(x)$$

for some $\alpha \in F_q$. Therefore, $f(x, y-\alpha) = a_n(x)y^n + g(x)$, which is a polynomial already considered in Case 1. This completes Case 2 and the proof for $m \le n$. If $n < m$, then a similar argument will provide form (d).

The next example illustrates the necessity of the condition $(n-1, q) = 1$.

Example: For a in F_{81}, let $f(x, y)$ denote the polynomial

$$f(x,y) = 2x^4 + x^3y + xy^3 + y^4 + 2ax^3 + ay^3 + 2a^3x + a^3y.$$

Then

$$\begin{aligned} f(x,y) &= (x+y+a)^4 + x^4 + ax^3 + a^3x + 2a^4 \\ &= 2(x+2y+a)^4 + 2y^4 + a^4. \end{aligned}$$

Therefore, since $4|80$, $f(x, y)$ is a local minimal polynomial that is not in the list (a)–(e).

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ON k-SELF-NUMBERS AND UNIVERSAL GENERATED NUMBERS*

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1. INTRODUCTION

In 1963, D. R. Kaprekar [1] introduced the concept of self-numbers. Let $k>1$ be an arbitrary integer. A natural number m is said to be a k-self-number iff the equation

$$m = n + d_k(n)$$

has no solution in an integer $n>0$, where $d_k(n)$ denotes the sum of digits of n while represented in the base k. Otherwise, we say that m is a k-generated number. And m is said to be a universal generated number if it is generated in every base. For example, 2, 10, 14, 22, 38, etc. are universal generated numbers. The number 12 is 4-generated by 9, but it is a 6-self-number.

In 1973, V. S. Joshi [2] proved that "if k is odd, then m is a k-self-number iff m is odd," i.e., every even number in an odd base is a generated number.

In 1991, R. B. Patel ([3], M.R. 93b:11011) tested for self-numbers in an even base k. What he proved is: $2ki$, $4k+2$, k^2+2k+1 are k-self-numbers in every even base $k \geq 4$.

In the present paper, we first prove some new results on self-numbers in an even base k.

Theorem 1: Suppose

$$m = b_0 + b_1 k,\ 0 \leq b_0 < k,\ 0 < b_1 < k,\ 2|k,\ k \geq 4.$$

Then m is a k-self-number iff $b_0 - b_1 = -2$.

In particular, $2k$, $3k+1$, $4k+2$, $5k+3$, etc. are k-self-numbers.

Theorem 2: Suppose

$$m = b_0 + b_1 k + b_2 k^2,\ 0 \leq b_0 < k,\ 0 \leq b_1 < k,\ 0 < b_2 < k,\ 2|k,\ k \geq 4.$$

Then m is a k-self-number iff b_0, b_1, and b_2 satisfy one of the following conditions: $b_1 = 0$, $b_0 - b_1 - b_2 = -4$ or $k-3$; $b_1 = 1$, $b_0 - b_1 - b_2 = -2$ or -4; $b_1 = 2$ or 3, $b_0 - b_1 - b_2 = -2$; $b_1 \geq 4$, $b_0 - b_1 - b_2 = -2$ or $-k-3$.

In particular, k^2+k, k^2+2k+1, k^2+3k+2, $2k^2+k+1$, $2k^2+2k+2$, $3k^2+k+2$, $5k^2+1$ $(k \geq 6)$, $4k^2+k+1$ $(k \geq 6)$, $5k^2-k$ $(k \geq 6)$, k^3-k^2+4k, etc. are k self-numbers.

Secondly, we study the number $G(x)$ of universal generated numbers $m \leq x$. It is not known if $G(x) \to \infty$ but, as an ingenious application of Theorem 1, we prove that $G(x) \leq 2\sqrt{x}$. As a matter of fact, we obtain

Theorem 3: Every universal generated number can be represented in only one way, in the form $2^s n + 2^{s-1} - 2$, with $s \geq 3$, $n \leq 2^{s-2}$. Moreover, for all $x > 1$, one has $G(x) \leq 2\sqrt{x}$.

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2. PROOF OF THEOREM 1

If possible, let m be k-generated by some n, where

$$n=\sum_{i=0}^{t} a_i k^i, \quad 0 \le a_i < k, \ 0 \le i \le t.$$

Then

$$d_k(n)=\sum_{i=0}^{t} a_i \quad \text{and} \quad m=n+d_k(n)=\sum_{i=0}^{t} a_i(k_i+1).$$

Since $m=b_0+b_1k<k+(k-1)k=k^2$, we have $a_i=0$ for $i\ge 2$, i.e.,

$$b_0+b_1k=2a_0+a_1(k+1), \quad 0\le a_0, \ a_1<k. \tag{1}$$

Here $a_1>b_1$ or $a_1<b_1-2$ is impossible, so that $a_1=b_1-i, \ 0\le i\le 2$.

(A) If $i=0$, then (1) holds iff $b_0-b_1\ge 0$ is even;

(B) If $i=1$, then (1) holds iff $b_0-b_1\le k-3$ is odd;

(C) If $i=2$, then (1) holds iff $b_0-b_1\le -4$ is even.

Hence, m is a k-self-number iff $b_0-b_1=-2$ or $k-1$. The latter is impossible because $b_0\le k-1$. This completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

If possible, let m be k-generated by some n. As in the proof of Theorem 1, we have

$$b_0+b_1k+b_2k^2=2a_0+a_1(k+1)+a_2(k^2+1), \tag{2}$$

with $b_2-1\le a_2\le b_2$.

Case I. $a_2=b_2$. From (2), we see that $a_1\le b_1$. Taking $a_1=b_1-j, \ j\ge 0$, we have

$$b_0-b_1-b_2+j(k+1)=2a_0. \tag{3}$$

Noting that $0\le a_0<k$, one has:

(A) If $j=0$, then (3) holds iff $b_0-b_1-b_2\ge 0$ is even;

(B) If $j=1$, then (3) holds iff $b_0-b_1-b_2\ge -k-1$ is odd and $b_1\ge 1$;

(C) If $j=2$, then (3) holds iff $b_0-b_1-b_2\le -4$ is even and $b_1\ge 2$;

(D) If $j=3$, then (3) holds iff $b_0-b_1-b_2\le k-5$ is odd and $b_1\ge 3$;

(E) If $j\ge 4$, then (3) never holds.

Case II. $a_2=b_2-1$. Taking $a_1=k-j, \ j\ge 1$, it follows from (2) that

$$(b_1+j-1)k=2a_0-j-1+b_2-b_0$$

or

$$b_0-b_2+(b_1+j-1)k+j-1=2a_0. \tag{4}$$

Since $2a_0-j-1+b_2-b_0\le 3(k-1)$, one has $b_1+j-1\le 2$. Noting that $0\le a_0\le k-1$, one has:

(A)′ If $b_1=0, j=1$, then (4) holds iff $b_0-b_2\ge -2$ is even;

(B)′ if $b_1=0, j=2$, then (4) holds iff $b_0-b_2\le k-5$ is odd;

(C)′ If $b_1 = 0$, $j = 3$, then (4) holds iff $b_0 - b_2 \le -6$ is even;

(D)′ If $b_1 = 1$, $j = 1$, then (4) holds iff $b_0 - b_2 \le k - 4$ is even;

(E)′ If $b_1 = 1$, $j = 2$, then (4) holds iff $b_0 - b_2 \le -5$ is odd;

(F) If $b_1 = 2$, $j = 1$, then (4) holds iff $b_0 - b_2 \le -4$ is even;

(G) If $b_1 \ge 3$, then (4) never holds.

Thus, (A)′, (B)′, and (C)′ together imply that if $b_1 = 0$, (4) does not hold iff $b_0 - b_2 = -4$ or $k - 3$, i.e., $b_0 - b_1 - b_2 = -4$ or $k - 3$. According to Case I, (2) has no solution iff $b_0 - b_1 - b_2 = -4$ or $k - 3$.

If $b_1 = 1$, (D)′ and (E)′ together imply that (4) does not hold iff $b_0 - b_2 > k - 4$ or $k - 4 > b_0 - b_2 > -5$ is odd, i.e., $b_0 - b_1 - b_2 > k$ or $k - 5 > b_0 - b_1 - b_2 > -6$ is even. According to Case I, (2) has no solution iff $b_0 - b_1 - b_2 = -2$ or -4.

If $b_1 = 2$, then from (F) (4) does not hold iff $b_0 - b_2 > -4$ or is odd, i.e., $b_0 - b_1 - b_2 > -6$ or is odd. According to Case I, (2) has no solution iff $b_0 - b_1 - b_2 = -2$.

If $b_1 \ge 3$, (4) never holds. According to Case I, (2) has no solution iff $b_0 - b_1 - b_2 = -2$ or $-k - 3$. For the latter, $b_1 \ge 4$. This completes the proof of Theorem 2.

4. PROOF OF THEOREM 3

Let $f_s(n)$ denote $2^s n + 2^{s-1} - 2$, where $s \ge 1$ and $n \ge 1$. Then $f_1(n) = 2n - 1$, $f_2(n) = 4n$, $f_3(n) = 8n + 2$, $f_4(n) = 16n + 6$, ... Noting that $f_s(n) = f_{s_1}(n_1)$ iff $n - n_1$, $s = s_1$, one has from the fundamental theorem of arithmetic: every positive integer can be represented in only one way, in the form $2^s n + 2^{s-1} - 2$. If $s = 1$, $n \ge 2$, it is clear that $f_1(n) = 2n - 1$ is not generated by $2n$. If $s \ge 2$, taking $b_0 = 2^{s-1} - 2$, $b_1 = 2^{s-1}$, $k = 2n$, and applying Theorem 1 we see that $f_s(n)$ is a k-self-number, i.e., it is not a universal generated number if $n > 2^{s-2}$. Moreover,

$$G(x) \le \sum_{\substack{1 \le 2^s n + 2^{s-1} - 2 \le x \\ s \ge 1,\, n \le 2^{s-2}}} 1 \le \sum_{s \ge 1} \min\{2^{s-2}, x/2^s\} \le \sum_{s \le (1/2)\log_2 x + 1} 2^{s-2} + \sum_{s > (1/2)\log_2 x + 1} x/2^s \le 2\sqrt{x}.$$

This completes the proof of Theorem 3.

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ON THE LEAST SIGNIFICANT DIGIT OF ZECKENDORF EXPANSIONS

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1. INTRODUCTION

A well-known digital expansion is the so-called Zeckendorf number system [7], where every positive integer n can be written as

$$n = \sum_{k=0}^{L} \varepsilon_k F_k, \tag{1.1}$$

where F_k denotes the sequence of Fibonacci numbers given by $F_{k+2} = F_{k+1} + F_k$, $F_0 = 1$, and $F_1 = 2$ (cf. [5]). The digits ε_k are 0 or 1, and $\varepsilon_k \varepsilon_{k+1} = 0$. Using the same recurrence relation but the initial values $L_0 = 3$ and $L_1 = 4$, the sequence L_k of Lucas numbers is defined. In a recent volume of *The Fibonacci Quarterly*, P. Filipponi proposed the following conjectures (Advanced Problem H-457, cf. [2]).

Conjecture 1: Let $f(N)$ denote the number of 1's in the Zeckendorf decomposition of N. For given positive integers k and n, there exists a minimal positive integer $R(k)$ (depending on k) such that $f(kF_n)$ has a constant value for $n \geq R(k)$.

Conjecture 2: For $k \geq 6$, let us define

(i) μ, the subscript of the smallest odd-subscripted Lucas number such that $k \leq L_\mu$,
(ii) ν, the subscript of the largest Fibonacci number such that $k > F_\nu + F_{\nu-6}$.

Then $R(k) = \max(\mu, \nu) + 2$.

We note that we have chosen different initial values compared to [5] and [2] (the so-called "canonical" initial values, cf. [4]) which seem to be more suitable for defining digital expansions and yield an index translation by 2. In [3] we have proved that the first conjecture is true in a much more general situation, i.e., for digital expansions with respect to linear recurrences with nonincreasing coefficients. As in [3], let $U(k)$ be the smallest index u such that

$$kF_u = \sum_{\ell=0}^{L(k)} \varepsilon_\ell F_\ell \quad \text{and} \quad kF_n = \sum_{\ell=0}^{L(k)} \varepsilon_\ell F_{\ell+n-u} \; \forall n \geq u. \tag{1.2}$$

We prove an explicit formula for $U(k)$ in terms of Lucas numbers that is an improved version of Conjecture 1. Note that Filipponi's Conjecture 1 has been proved by Bruckman in [1] and for the more general case of digital expansions with respect to linear recurrences in [3]. We have also obtained a weak formulation of Conjecture 2 which only yields an upper bound for $U(k)$. However, Bruckman's proof of a modification of Filipponi's Conjecture 2 is false because his proof does not guarantee the minimality of $R(k)$; this was pointed out in a personal communication by Piero Filipponi. We apologize here for referring in [3] to this erroneous proof instead of presenting our own proof of the original Conjecture 2. It is the aim of this note to provide a complete proof of Conjecture 2.

2. PROOF OF CONJECTURE 2

In the following, let $V(k) = L(k) - U(k)$ be the largest power of the golden ratio $\beta = \frac{1+\sqrt{5}}{2}$ in Parry's β-expansion of k, see [6]. Obviously, $V(k) = \lfloor \log_\beta k \rfloor$. For proving Conjecture 2, let us intro-duce some special notation. By Zeckendorf's theorem, every nonnegative integer n can be written uniquely as

$$n = F_{k_r} + \cdots + F_{k_2} + F_{k_1}, \; k_r \gg \cdots \gg k_2 \gg k_1, \; r \geq 0, \tag{2.2}$$

where $k' \gg k''$ means that $k' \geq k'' + 2$ [compare to (1.1)].

It will be convenient to have the sequences of Fibonacci and Lucas numbers extended for negative indices. Let $F_{-2} = 0, F_{-1} = 1, F_{-n-2} = (-1)^{n+1} F_{n-2}$ and $L_{-2} = 2, L_{-1} = 1, L_{-n-2} = (-1)^n L_{n-2}$ for positive integers n. In this way, the definitions of μ and ν hold for all integers. We need the following well-known lemmas which can be shown by induction.

Lemma 1: For integers m and n, we have $L_m F_n = (-1)^m F_{n-m} + F_{n+m}$.

Lemma 2: Let m and n be integers, $n > m$ and $m \equiv n \bmod 2$. Then

$$F_n - F_m = \sum_{i=1}^{\frac{n-m}{2}} F_{m+2i-1}.$$

Theorem 1: For all positive integers k there exist uniquely determined integers $c_1, \ldots, c_t$ such that, for all integers n,

$$kF_n = \sum_{i=1}^{t} F_{n+c_i} \tag{2.3}$$

with

$$-U(k) = c_1 \ll c_2 \ll \cdots \ll c_{t-1} \ll c_t = V(k), \tag{2.4}$$

where $U(k) \geq 2$ are even numbers defined by $L_{U(k)-3} < k \leq L_{U(k)-1}$.

Proof: We consider the following partition of the set of natural numbers $\mathbb{N} = \bigcup_{j=-1}^{\infty} \mathbb{L}_j$, where $\mathbb{L}_{-1} = \{1\}$ and $\mathbb{L}_j = \{n \in \mathbb{N} \mid L_{2j-1} < n \leq L_{2j+1}\}$ for $j \geq 0$. The proof will proceed by induction on j.

If $j = -1$, i.e., $k = 1$, then the assertion is satisfied with $t = 1$ and $c_1 = 0$. Suppose that (2.3) and (2.4) hold for $j \geq 0$ for each i with $-1 \leq i \leq j-1$ and all $k \in \mathbb{L}_i$. Then we have to show (2.3) and (2.4) hold for all $k \in \mathbb{L}_j$. Three cases will be distinguished.

Case 1: $L_{2j-1} < k < L_{2j}$

From Lemma 2 with $m = 2j+1$ and by $-F_{n-2j+1} = F_{n-2j} - F_{n-2j+2}$, we have

$$kF_n = F_{n-2j} - F_{n-2j+2} + (k - L_{2j-1})F_n + F_{n+2j-1}. \tag{2.5}$$

Since $1 \le k - L_{2j-1} < L_{2j-2}$, by the induction hypothesis we obtain from (2.5),

$$kF_n = F_{n-2j} - F_{n-2j+2} + \sum_{i=1}^{\bar{t}} F_{n+\bar{c}_i} + F_{n+2j-1} \tag{2.6}$$

with $\bar{c}_1 \ge -2(j-1)$, $\bar{c}_{\bar{t}} \le 2(j-1)-1$, and $\bar{c}_1 \ll \cdots \ll \bar{c}_{\bar{t}}$. Write (2.6) in the form

$$kF_n = F_{n-2j} + F_{n+\bar{c}_1} - F_{n-2j+2} + \sum_{i=1}^{\bar{t}} F_{n+\bar{c}_i} + F_{n+2j-1}. \tag{2.7}$$

If $\bar{c}_1 = -2j+2$, then by $\bar{c}_1 \ll \bar{c}_2$ we have $-2j+4 \le \bar{c}_2$. Letting $t = \bar{t}+1$, $c_1 = -2j$, and $c_2 = \bar{c}_2$, ..., $c_{t-1} = \bar{c}_{\bar{t}}$, then $c_1 \le c_2 - 4$. Thus, $c_1 \ll c_2$ and, by the induction hypothesis, $c_2 \ll \cdots \ll c_{t-1}$. If $\bar{c}_1 > -2(j-1)$, then Lemma 2 applies for $F_{n+\bar{c}_1} - F_{n-2j+2}$ since, by the induction hypothesis, $\bar{c}_1$ is a value of the even-valued function U. Hence, we get

$$kF_n = F_{n-2j} + \sum_{\ell=1}^{\hat{t}} F_{n-2j+2\ell+1} + \sum_{i=1}^{\bar{t}} F_{n+\bar{c}_i} + F_{n+2j-1} \tag{2.8}$$

with $\hat{t} = (\bar{c}_i - 2(j-1))/2$. Representation (2.8) is already in the form (2.3). Letting $t = \bar{t} + \hat{t} + 2$ and $c_1 = -2j$, $c_2 = -2j+3, \ldots, c_{\hat{t}+1} = \bar{c}_1 - 1, c_{\hat{t}+2} = \bar{c}_2, \ldots, c_{\hat{t}+\bar{t}+1} = \bar{c}_{\bar{t}}$, and using $c_2 = c_1 + 3$, $c_{i+1} \ge c_i + 2$ $(i = 2, \ldots, \hat{t})$, we get $c_1 \ll c_2 \ll \cdots \ll c_{\hat{t}+1}$. Applying the induction hypothesis yields $c_{\hat{t}+2} \ll \cdots \ll c_{t-1}$. Taking $c_t = 2j-1$, (2.4) is established.

Case 2: $L_{2j} < k < L_{2j+1}$

From Lemma 1 with $m = 2j$ we derive

$$kF_n = F_{n-2j} + (k - L_{2j})F_n + F_{n+2j}. \tag{2.9}$$

Since $1 \le k - L_{2j} < L_{2j-1}$, the induction hypothesis yields a representation of the form (2.3),

$$kF_n = F_{n-2j} + \sum_{i=1}^{\bar{t}} F_{n+\bar{c}_i} + F_{n+2j}, \tag{2.10}$$

with $\bar{c}_1 \ge -2(j-1), \bar{c}_{\bar{t}} \ge 2(j-1)$, and $\bar{c}_1 \ll \cdots \ll \bar{c}_{\bar{t}}$. Letting $t = \bar{t}+2, c_1 = -2j, c_t = 2j$, and $c_{i+1} = \bar{c}_i$ $(i = 1, \ldots, \hat{t})$, we obtain (2.4).

Case 3: $k = L_{2j}$

By Lemma 2 we have $L_{2j}F_n = F_{n-2j} + F_{n+2j}$. Thus, we can proceed without using the induction hypothesis, obtaining (2.3) and (2.4) with $t = 2, c_1 = -2j$, and $c_2 = 2j$.

Uniqueness of $c_1, \ldots, c_t$ is implied by the uniqueness of the Zeckendorf representation. □

Corollary 1: As an immediate consequence of Theorem 1, we get $R(k) \le U(k)$.

To prove Conjecture 2, we need an additional lemma.

Lemma 3: Let c_1 and c_2 be as in Theorem 1. Then $c_2 = c_1 + 2$ if and only if

$$k > 2L_{-c_1-1}. \tag{2.11}$$

Proof: By Theorem 1, we have $4F_n = F_{n-2} + F_n + F_{n+2}$; thus, $c_2 = c_1 + 2$. Also by Theorem 1, for $k \geq 5$, we obtain $c_2 \geq c_1 + 2$ and $c_1 = -2j$ for some integer $j \geq 1$. From the proof of that theorem, it is clear that $L_{2j-1} < k \geq L_{2j+1}$. If $L_{2j-1} < k < L_{2j}$, then $c_2 > c_1 + 2$. If $k = L_{2j}$, then $t = 2$ and $c_2 - c_1 = 4j > 2$. If $L_{2j} < k \leq L_{2j+1}$, then $0 < k - L_{2j} < L_{2j-1}$. Observing that (2.11) is equivalent to $k - L_{2j} > L_{2j-3}$, Theorem 1 yields $U(k - L_{2j}) > -2(j-1)$ if $0 < k - L_{2j} \leq L_{2(j-1)-1}$ and $U(k - L_{2j}) = -2(j-1)$ if $L_{2(j-1)-1} < k - L_{2j} \leq L_{2(j-1)+1}$. Thus, we conclude that $c_2 = -2j + 2$ if and only if (2.11) holds. □

Theorem 2: $R(1) = 0$, $R(2) = R(3) = 1$, and for $k \geq 4$ we have

$$R(k) = \begin{cases} 2j-1 & \text{if } L_{2j-3} < k \leq 2L_{2j-3}, \\ 2j & \text{if } 2L_{2j-3} < k \leq L_{2j-1}. \end{cases}$$

Proof: $R(1) = 0$ is immediate from the definitions. By the identities $2F_n = F_{n-2} + F_{n+1}$, $3F_n = F_{n-2} + F_{n+2}$ for integral n, and $2F_1 = F_0 + F_2$, $3F_1 = F_0 + F_3$ we obtain $R(2) \geq 1$ and $R(3) \geq 1$. Since $2F_0 = F_1$ and $3F_0 = F_2$, we get $R(2) = R(3) = 1$. Let $k \geq 4$. By Corollary 1, we have $R(k) \leq U(k)$ and $f(kF_n) = t$ for $n \geq U(k)$.

In the following, we distinguish two cases.

Case 1: $2L_{2j-3} < k \leq L_{2j-1}$

Let $n = U(k) - 1$. We show that in this case $f(kF_n) < t$; hence, $R(k) = U(k)$. Theorem 1 and Lemma 3 yield

$$kF_n = F_{-1} + F_1 + \sum_{i=3}^{t} F_{n+c_i} = F_2 + \sum_{i=3}^{t} F_{n+c_i}. \tag{2.12}$$

If $n + c_3 > 3$, then the right-hand side of (2.12) is a Zeckendorf representation and $f(kF_n) = t - 1$. If $n + c_3 = 3$, then let i_0 be the largest $i \geq 2$ such that $c_i = c_{i-2} + 2$; let $i_0 = 1$ if such i does not exist. Then the right-hand side of (2.12) can be written in the form of a Zeckendorf representation as

$$F_{n+c_{i_0}+1} + \sum_{i > i_0}^{t} F_{n+c_i}. \tag{2.13}$$

Thus, $f(kF_n) = t - i_0 + 1$.

Case 2: $L_{2j-3} < k \leq 2L_{2j-3}$

We show $f(kF_n) = t$ provided that $n = U(k) - 1$; however, $f(kF_n) = t - 1$ for $n = U(k) - 2$. Hence, we have $R(k) = U(k) - 1$. Let $n = U(k) - 1$. As a consequence of Theorem 1, we get

$$kF_n = F_{-1} + \sum_{i=2}^{t} F_{n+c_i} = F_0 + \sum_{i=2}^{t} F_{n+c_i}.$$

Applying Lemma 3, we derive $n + c_2 \geq 2$. Thus, the right-hand side is the Zeckendorf representation of kF_n and we obtain $f(kF_n) = t$. Let $n = U(k) - 2$. Theorem 1 yields

$$kF_n = F_{-2} + \sum_{i=2}^{t} F_{n+c_i} = \sum_{i=2}^{t} F_{n+c_i}. \tag{2.14}$$

The right-hand side of (2.14) is the Zeckendorf representation of kF_n; hence, $f(kF_n) = t-1$ and the proof is complete. □

Remark: To see that $R(k)$ is the same as in Filipponi's Conjecture 2, note that $\mu = 2j-1$ if $L_{2j-1} < k \le L_{2j+1}$ and if $F_\nu + F_{\nu-6}$ (in the definition of ν) can be replaced by $2L_{\nu-3}$.

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NUMBERS OF SUBSEQUENCES WITHOUT ISOLATED ODD MEMBERS

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We find the numbers of subsequences of $\{1, 2, \ldots, n\}$ in which every odd member is accompanied by at least one even neighbor. For example, 123568 is acceptable, but 123578 is not, since 5 has no even neighbor. The empty sequence is always acceptable. Preliminary calculations for $0 \le n \le 8$ yield the following values of $z_{n,c}$, the number of such subsequences of length c. It is convenient to define $z_{n,c} = 0$ if c is not in the interval $0 \le c \le n$ and $z_n = \sum_{c=0}^{n} z_{n,c}$ is the total number of such subsequences. x_n and $x_{n,c}$ are the corresponding numbers of subsequences from which n is **excluded**, whereas n **does** occur in the subsequences counted by y_n and $y_{n,c}$. Of course, $x_n + y_n = z_n$ with similar formulas for specific lengths.

The following tables suggest several simple relations, which are easily verified by considering the last two or three members of the relevant subsequences:

$$
\begin{array}{lll}
x_{n+1} = z_n, & x_{n+1,c} = z_{n,c} & (n \ge 0), \\
y_{2k+1} = y_{2k}, & y_{2k+1,c+1} = y_{2k,c} & (k \ge 0), \\
y_{2k} = z_{2k-1} + z_{2k-3}, & y_{2k,c+1} = z_{2k-1,c} + z_{2k-3,c-1} & (k \ge 0),
\end{array}
$$

where we adopt the conventions $z_{-1} = z_{-1,0} = 1$ and $x_{-2} = z_{-3} = z_{-3,-1} = -1$.

TABLE 1. $z_{n,c}$, $c = 0, 1, 2, \ldots, n$

n																		z_n
0									1									1
1								1		0								1
2							1		1		1							3
3						1		1		2		1						5
4					1		2		4		3		1					11
5				1		2		5		5		3		1				17
6			1		3		8		11		10		5		1			39
7		1		3		9		14		16		12		5		1		61
8	1		4		13		25		35		33		20		7		1	139

The corresponding arrays for the values of x and y are given in Tables 2 and 3, respectively.

TABLE 2. $x_{n,c}$, $c = 0, 1, 2, \ldots, n$

n										x_n
0	1									1
1	1	0								1
2	1	0	0							1
3	1	1	1	0						3
4	1	1	2	1	0					5
5	1	2	4	3	1	0				11
6	1	2	5	5	3	1	0			17
7	1	3	8	11	10	5	1	0		39
8	1	3	9	14	16	12	5	1	0	61

TABLE 3. $y_{n,c}$, $c = 0, 1, 2, \ldots, n$

n										y_n
0	0									0
1	0	0								0
2	0	1	1							2
3	0	0	1	1						2
4	0	1	2	2	1					6
5	0	0	1	2	2	1				6
6	0	1	3	6	7	4	1			22
7	0	0	1	3	6	7	4	1		22
8	0	1	4	11	19	21	15	6	1	78

We illustrate the last for the case $k = 4, c = 5$. Each subsequence counted by $y_{8,6}$ is of just one of the shapes $*****8$, $****68$, $***678$, or $****78$, where $*$ represents a member other than 6, 7, or 8. The first three of these are formed by appending 8 to a subsequence of length 5, counted by $z_{7,5}$, and the last by appending 78 to a subsequence of length 4, counted by $x_{6,4}$ (none of which end in 6; note the correspondence between the subsequences counted by $x_{6,4}$ and those counted by $z_{5,4}$):

$$z_{2k} = 2z_{2k-1} + z_{2k-3}, \quad z_{2k+1} = 3z_{2k-1} + 2z_{2k-3}.$$

This last recurrence, which again holds for $k \geq 0$ with the aforementioned conventions, can be solved in the classical manner to show that

$$z_{2k+1} = \left(\frac{17+7\sqrt{17}}{34}\right)\left(\frac{3+\sqrt{17}}{2}\right)^k + \left(\frac{17-7\sqrt{17}}{34}\right)\left(\frac{3-\sqrt{17}}{2}\right)^k;$$

the previous formula then gives

$$z_{2k} = \left(\frac{17+3\sqrt{17}}{34}\right)\left(\frac{3+\sqrt{17}}{2}\right)^k + \left(\frac{17-3\sqrt{17}}{34}\right)\left(\frac{3-\sqrt{17}}{2}\right)^k.$$

Since the second term in these formulas tends rapidly to zero, we find that

$$z_n \text{ is the nearest integer to } c\zeta^n,$$

where

$$\zeta = \frac{1}{2}(3+\sqrt{17}) = 3.561552812808830274910704927$$

and

$$\begin{cases} c = \dfrac{1}{34}(17+3\sqrt{17}) = 0.8638034375544994602783596931 & \text{if } n \text{ is even,} \\ c = \dfrac{1}{34}(17+7\sqrt{17}) = 1.348874687627165407316172617 & \text{if } n \text{ is odd.} \end{cases}$$

We searched in our preview copy of [3] without any success, and were surprised that there seemed to be no earlier occurrences of members of our arrays. A similar problem with a similar but not very closely related answer is discussed in [1]. We then tried the main sequence $\{z_n\}$ on Superseeker [2], which produced the generating function

$$\sum_{j=0}^{\infty} z_j t^j = (1+t+2t^3)(1-(3t^2+2t^4))^{-1}$$

which should, perhaps, be thought of as the sum of two generating functions, one for odd-ranking terms, the other for even.

As the sequence does not seem to have been calculated earlier, we give a fair number of terms in the table below.

TABLE 4

n	z_n	n	z_n	n	z_n
1	1	17	34921	33	904069513
2	3	18	79647	34	2061980415
3	5	19	124373	35	3219891317
4	11	20	283667	36	7343852147
5	17	21	442961	37	11467812977
6	39	22	1010295	38	26155517271
7	61	23	1577629	39	40843221565
8	139	24	3598219	40	93154256107
9	217	25	5618809	41	145465290649
10	495	26	12815247	42	331773802863
11	773	27	20011685	43	518082315077
12	1763	28	45642179	44	1181629920803
13	2753	29	71272673	45	1845177526529
14	6279	30	162557031	46	4208437368135
15	9805	31	253841389	47	6571697209741
16	22363	32	578955451		

As may be expected from sequences defined from recurrence relations, there are congruence and divisibility properties. The terms of odd rank are alternately congruent to 1 and 5 modulo 8, and those of even rank after the second are congruent to 3 and 7 modulo 8 alternately. Every fourth term, starting with z_2 is divisible by 3, every third of those (e.g., z_{10}) is divisible by 9, every third of those (e.g., z_{34}) is divisible by 27, and so on. The terms that are divisible by 5 are every twelfth, starting with z_3 among the odd ranks and with z_{10} among those with even rank. Every sixteenth term is divisible by 7, starting with z_9 and z_{14}. Every sixth term is divisible by 11, starting with z_4; but no odd-ranking terms are. Seventeen is special to this sequence and divides every thirty-fourth term starting with $z_5 = 17$ itself and with z_{32} among the even-ranking terms.

Note that if you use the recurrences to calculate earlier terms in the sequence, $z_{-1} = 1$ and $z_{-3} = -1$, as we have already assumed, $z_{-2} = 0$ (and so is divisible in particular by 17, 11, 7, 5, and any power of 3), $z_{-4} = \frac{1}{2}$, $z_{-5} = 2$, $z_{-6} = -\frac{3}{4}$, $z_{-7} = -\frac{7}{2}$, $z_{-8} = \frac{11}{8}$, $z_{-9} = \frac{25}{4}$, ..., and there are p-adic interpretations of the divisibility properties. For example, z_{-9} is divisible by 25, and we leave it to the reader to confirm that $25|z_{51}$ and that $5^4|z_{171}$.

Why shouldn't the even numbers get equal time? If we denote by w_n the number of subsequences whose **even** members all have at least one **odd** neighbor, then for even $n = 2k$ there is the obvious symmetry $w_{2k} = z_{2k}$. The values of w_n for odd rank are the averages of the even-ranking neighbors: $2w_{2k+1} = w_{2k} + w_{2k+2}$, whereas for the $\{z_n\}$ sequence, the roles are reversed: $2z_{2k} = z_{2k-1} + z_{2k+1}$. Both sequences satisfy the recurrence $u_n = 3u_{n-2} + 2u_{n-4}$, while the generating function for $\{w_n\}$ has numerator $1 + 2t + t^3$ in place of $1 + t + 2t^3$.

TABLE 5. Some Odd-Ranking Members of the $\{w_n\}$ Sequence

n	w_n	n	w_n	n	w_n
−7	5/16	7	89	21	646981
−5	−1/8	9	317	23	2304257
−3	1/4	11	1129	25	8206733
−1	1/2	13	4021	27	29228713
1	2	15	14321	29	104099605
3	7	17	51005	31	370756241
5	25	19	181657	33	1320467933

The special role of 17 is illustrated by the form of w_{17} (i.e., $51005 = 5 \cdot 101^2$) and in the formula

$$w_{2k+1} = \left(\frac{17+4\sqrt{17}}{17}\right)\left(\frac{3+\sqrt{17}}{2}\right)^k + \left(\frac{17-4\sqrt{17}}{17}\right)\left(\frac{3-\sqrt{17}}{2}\right)^k.$$

More investigative readers will discover the many corresponding congruence and divisibility properties.

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FIBONACCI EXPANSIONS AND "F-ADIC" INTEGERS

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I. INTRODUCTION

A Fibonacci expansion of a nonnegative integer n is an expression of n as a sum of Fibonacci numbers F_k with $k \geq 2$. It may be thought of as a partition of n into Fibonacci parts. The most commonly studied such expansion is the unique one in which the parts are all distinct and no two consecutive Fibonacci numbers appear. C. G. Lekkerkerker first showed this expansion was unique in 1952 [5]. There is also a unique dual form of this expansion in which no two consecutive Fibonacci numbers not exceeding n do *not* occur in the expansion [2]. Lekkerkerker's expansion is the only one I refer to in the remainder of this paper; from now on, I will call it the Fibonacci expansion of n, or fib(n) (I will give a precise definition in Part II). The Fibonacci expansion of nonnegative integers is similar in many ways to a fixed-base expansion (in fact, in some sense, it may be thought of as a base-τ expansion, where $\tau = \frac{1}{2}(1+\sqrt{5}) \approx 1.61803$ is the golden mean). First, in each case there are both "top-down" and "bottom-up" algorithms for obtaining the expansion of a nonnegative integer (see [3], pages 281-282). Second, there are mechanical rules for adding the expansions of two or more nonnegative integers [1]. Third, each case may be generalized by defining infinite expansions (p-adic or "F-adic" integers), both of which have interesting algebraic properties. One should be warned, however, that this analogy has its limitations. For instance, the p-adic integers form a ring, but the F-adic integers do not. My main result in this paper is that there is a 1-1 correspondence between the F-adic integers and the points on a circle, and that both of these sets share some important geometric properties.

II. FIBONACCI EXPANSIONS OF NONNEGATIVE INTEGERS

Definition: Let $n \in \omega = \{0, 1, 2, \dots\}$. Suppose there exists a sequence $(c_k)_0^\infty \in \{0, 1\}^\omega$ such that $c_k c_{k+1} = 0\ (\forall k)$ and $n = \sum_{k=0}^\infty c_k F_{k+2}$. Then (c_k) is called the *Fibonacci expansion of n* and is denoted fib(n). It is well known that every nonnegative integer has a unique Fibonacci expansion [5], so fib: $\omega \to \{0, 1\}^\omega$ is well defined.

In this paper, I use the convention of increasing coefficient indices in Fibonacci expansions going from left to right. Thus, for instance,

$$\begin{aligned} \text{fib}(5) &= 0001 \\ \text{fib}(10) &= 01001 \\ \text{fib}(100) &= 0010100001 \end{aligned}$$

where the rightmost 1 in each expansion is assumed to be followed by an infinite sequence of zeros.

The top-down algorithm for computing fib(n) is as follows (see [4], page 573). First, find the largest nonnegative integer k such that F_{k+2} does not exceed n, and let $c_k = 1$. Next, subtract F_{k+2} from n and repeat the above procedure for the difference. After a finite number of iterations, the difference will be zero; then we have obtained fib(n). This procedure is well known, and it is easy to check that the resulting expansion has the right form (see [3], [5]).

As an example, suppose $n = 10$. Since $F_6 = 8$ is the largest Fibonacci number not exceeding 10, we set $c_4 = 1$ and subtract 8 from 10, obtaining 2. Since $2 = F_3$, we set $c_1 = 1$; now our difference is $2 - 2 = 0$, so we stop. Thus, $\text{fib}(10) = 01001$.

III. BOTTOM-UP ALGORITHM

Both top-down and bottom-up algorithms for expanding a nonnegative integer in a fixed base are well known. For instance, to find the binary expansion of a nonnegative integer n, we could proceed by first finding the largest power k of 2 less than n, setting $c_k = 1$, subtracting 2^k from n, and repeating this procedure until $n = 0$. This is the top-down algorithm. Alternatively, we could first determine n mod 2, set this equal to c_i for $i = 0$, subtract c_i from n, divide n by 2, increase i by 1, and repeat until $n = 0$. This is the bottom-up algorithm. The top-down algorithm given for finding fib(n) is clearly analogous to the top-down procedure for finding the binary expansion of n. By analogy with the binary case, we look for a bottom-up algorithm for calculating fib(n). Such an algorithm does exist; moreover, this algorithm makes it clear how to extend the Fibonacci expansion to negative integers and, more generally, "F-adic" integers. The algorithm goes as follows:

Step 1: Let $i = 0$.
Step 2: Let x be the unique real number congruent to n mod τ^2 and lying in the interval $[-1, \tau)$. Determine whether x lies in the closed interval from $-(-\tau)^{-i-1}$ to $-(-\tau)^{-i}$. (Note that the intervals zero in on the origin as i increases.) If x lies in the subinterval, let $c_i = 0$ and increase i by 1; otherwise, let $c_i = 1$, $c_{i+1} = 0$, decrease n by F_{i+2}, and increase i by 2.
Step 3: If $n = 0$, stop. Otherwise, go to Step 2.

Again, I illustrate for $n = 10$. It is straightforward to check that the unique real number in the interval $[-1, \tau)$ congruent to 10 mod τ^2 is $x = 10 - 4\tau^2 \approx -0.47213$. Since $x \in [-1, \tau^{-1}]$, we have $c_0 = 0$. Thus, we leave n alone and increase i to 1. Now we check whether x lies in $[-\tau^{-2}, \tau^{-1}]$; it does not, since the lower limit is too high. Thus, we set $c_1 = 1$, decrease n by $F_3 = 2$ to 8, and increase i by 2 to 3. Now $8 - 3\tau^2 \approx 0.14590$, which lies in the interval $[-\tau^{-4}, \tau^{-3}]$. (As we will see shortly, by Lemma 1, we have $8 = F_6 \equiv \tau^{-4} \approx 0.14590$.) Thus, we set $x = \tau^{-4}$, $c_2 = c_3 = 0$, and increase i by 1, leaving $n = 8$ alone. Next, we check whether x lies in the interval $[-\tau^{-4}, \tau^{-5}]$. It does not, so we set $c_4 = 1$ and decrease n by $F_6 = 8$. But now $n = 0$, so we stop. Thus, we have again obtained $\text{fib}(10) = 01001$.

To show that the above algorithm works, we need a few lemmas.

Lemma 1: $F_k \equiv (-\tau)^{2-k} \bmod \tau^2 \ (\forall k \in \omega)$.

Proof: Since $F_0 = 0 \equiv \tau^2 \bmod \tau^2$ and $F_1 = 1 \equiv -\tau \bmod \tau^2$, the lemma holds for $k = 0$ and 1. Also, note that $(-\tau)^{2-k} + (-\tau)^{1-k} = (-\tau)^{-k}(\tau^2 - \tau) = (-\tau)^{-k}$. The lemma then follows by induction on k. □

Lemma 2: Let $\bar{n} = \sum_{k=0}^{\infty} c_k(-\tau)^{-k}$, where the c_k's are the coefficients of the Fibonacci expansion of n. Then $\bar{n}$ is the unique real number in the interval $[-1, \tau)$ congruent to n mod τ^2.

Proof: Let $n=\sum_{k=0}^{\infty}c_kF_{k+2}\equiv\sum_{k=0}^{\infty}c_k(-\tau)^{-k}=\overline{n}\ (\text{mod}\ \tau^2)$. Thus, it is enough to show that $-1\le\overline{n}<\tau$ (uniqueness then follows, since the interval $[-1,\tau)$ has length τ^2). The supremum of $\overline{n}$ is attained by setting $c_k=1$ for all even k and 0 for odd k; its value is $1+\tau^{-2}+\tau^{-4}+\cdots=\tau$. Similarly, the infimum of $\overline{n}$ is $(-\tau^{-1})+(-\tau^{-3})+(-\tau^{-5})+\cdots=-1$. □

Lemma 3: $c_k=0\ (\forall k<\ell)\Leftrightarrow\overline{n}\in\begin{cases}(-\tau^{-\ell},\tau^{1-\ell}) & \ell\text{ even},\\ (-\tau^{1-\ell},\tau^{-\ell}) & \ell\text{ odd}.\end{cases}$

Proof: First, I will prove the forward implication. Consider the case where ℓ is even (the case where ℓ is odd is similar). Clearly, $\overline{n}<\tau^{-\ell}+\tau^{-\ell-2}+\tau^{-\ell-4}+\cdots=\tau^{1-\ell}$. (The upper limit is approached by an arbitrarily long string of alternating 1's and 0's in fib(n) with the leading 1 in the ℓ^{th} position.) Similarly, $\overline{n}>-\tau^{-\ell-1}-\tau^{-\ell-3}-\cdots=-\tau^{-\ell}$.

The proof of the reverse implication goes as follows. Let k be the smallest integer such that $c_k=1$ and assume $k<\ell$. Let $n'=n-F_{k+2}$. Clearly, the first $k+1$ coefficients of fib(n') are zero, so by the first part of the proof (replacing ℓ by $k+2$), we get

$$\overline{n'}\in\begin{cases}(-\tau^{-(k+2)},\tau^{-(k+1)}) & k\text{ even},\\ (-\tau^{-(k+1)},\tau^{-(k+2)}) & k\text{ odd}.\end{cases}$$

First, suppose k is even. Then $\overline{n}=\overline{n'}+\tau^{-k}$ lies in $(\tau^{-(k+1)},\tau^{-(k-1)})$. But then $\overline{n}$ is too big to fall in any interval of the form $(-\tau^{-\ell},\tau^{-\ell})$ for $\ell>k$, so the inverse implication holds. A similar argument can be used in the case where k is odd. □

First, note that $n\equiv\overline{n}\bmod\tau^2$ by Lemma 1; thus, if there exists an integer m such that $n-m\tau^2$ lies in the closed interval from $(-\tau)^{-i-1}$ to $(-\tau)^{-i}$, then; $c_k=0$ for $k\le i$ by Lemma 2. This justifies the first if-then statement of Step 2 of the algorithm. If the condition in Step 2 is not met, c_i must be 1; in this case, we subtract F_{i+2} from n, obtaining a new n with $c_k=0$ for $k<i+2$; this justifies the second if-then statement of Step 2. Thus, the algorithm works.

The basis of the bottom-up algorithm is the fact that, if m mod τ^2 and n mod τ^2 are close (here, m and n are nonnegative integers), then the first few coefficients of fib(m) and fib(n) are the same. Figure 1 illustrates this. On the left side of the figure, n and fib(n) are plotted and tabulated against $\overline{n}$ (height along the figure) for $0\le n\le 21=F_8$. Although the figure is illustrated as a vertical line, it should be thought of as a circle with circumference τ^2. See Part V for an explanation of the right side of the figure.

IV. ADDITION OF FIBONACCI EXPANSIONS OF NONNEGATIVE INTEGERS

Here, I present an algorithm for adding two Fibonacci expansions of nonnegative integers (see [4], [5]); i.e., given $\text{fib}(m)=(a_k)$ and $\text{fib}(n)=(b_k)$, it finds $\text{fib}(m+n)=(c_k)$. The algorithm goes as follows. First, add the expansions coefficientwise, i.e., let $c_k=a_k+b_k$ for all k. The result will be a string of 0's, 1's, and 2's. To get rid of the 2's and consecutive 1's, apply the transformations

$$x+1,\,y+1,\,0\mapsto x,\,y,\,1$$
$$x,\,0,\,y+2,\,0\mapsto x+1,\,0,\,y,\,1$$

fib(n)	n	n	fib$_1(n)$	fib$_2(n)$
		$\overline{n} = \tau$		
		-1	$0101010\overline{10}$	$1010101\overline{01}$
101010	12			
		-9	$1010001\overline{01}$	$1010100\overline{10}$
101000	4			
101001	17			
		-17	$1010000\overline{10}$	$1010010\overline{01}$
		-4	$1000101\overline{01}$	$1010010\overline{10}$
100010	9			
		-12	$1000001\overline{01}$	$1000100\overline{10}$
100000	1			
100001	14			
		-20	$1000000\overline{10}$	$1000010\overline{01}$
		-7	$1000010\overline{10}$	$1001001\overline{01}$
100100	6			
100101	19			
		-15	$1001000\overline{10}$	$1001010\overline{01}$
		-2	$0010101\overline{01}$	$1001010\overline{10}$
001010	11			
		-10	$0010001\overline{01}$	$0010100\overline{10}$
001000	3			
001001	16			
		-18	$0010000\overline{10}$	$0010010\overline{01}$
		-5	$0000101\overline{01}$	$0010010\overline{10}$
000010	8			
		-13	$0000001\overline{01}$	$0000100\overline{10}$
000000	0			
000001	13			
		-21	$0000000\overline{10}$	$0000010\overline{01}$
		-8	$0000010\overline{10}$	$0001001\overline{01}$
000100	5			
000101	18			
		-16	$0001000\overline{10}$	$0001010\overline{01}$
		-3	$0001010\overline{10}$	$0100101\overline{01}$
010010	10			
		-11	$0100001\overline{01}$	$0100100\overline{10}$
010000	2			
010001	15			
		-19	$0100000\overline{10}$	$0100010\overline{01}$
		-6	$0100010\overline{10}$	$0101001\overline{01}$
010100	7			
010101	20			
		-14	$0101000\overline{10}$	$0101010\overline{01}$
		-1	$0101010\overline{10}$	$1010101\overline{01}$
		$\overline{n} = -1$		

FIGURE 1

to the rightmost applicable string. Continue until (c_k) has no 2's or consecutive 1's. These transformations are justified by the identities $F_{k+2} = F_{k+1} + F_k$ and $2F_k = F_{k+1} + F_{k-2}$. The algorithm must terminate after a finite number of steps, since each step increases the value of (c_k) viewed as a ternary number with the order of the digits reversed, and this cannot increase indefinitely because the last digit must correspond to a Fibonacci number that does not exceed $m+n$. However, it should be noted that, as presented, this algorithm is not complete, since it may yield an expansion (c_k) with a nonzero coefficient for $k=-1$ or $k=-2$. For instance, adding 1 and 1 gives the expansion 10.01 for 2 (coefficients with negative indices appear to the left of the decimal point). The case $k=-2$ is easy to deal with; simply eliminate this coefficient. This can be done because $F_0 = 0$. In the case where $c_{-1} = 1$, first set $c_{-1} = 0$ and $c_0 = 1$. (In this case, c_0 must have been 0 previously, since we have no two consecutive 1's at this stage.) Next, apply the first transformation repeatedly, this time starting on the left, until (c_k) is in the standard form. Again, only a finite number of applications is necessary, since each one decreases the number of 1's by 1.

V. NEGATIVE AND "F-ADIC" INTEGERS

One advantage of the bottom-up algorithm is that it allows a straightforward extension of fib to negative integers. We run the algorithm as stated, but must now allow for infinite expansions. For instance, applying the algorithm to -1, we get fib$(-1) = 01010101\ldots$. Note that, if we had used open instead of closed intervals in Step 2, we would have obtained fib$(-1) = 10101010\ldots$.

In fact, both expansions are valid, and these are the only ones. This dichotomy occurs for all negative integers. We use the notation fib_1 for the first case (open intervals) and fib_2 for the other case (closed intervals). Note that using closed intervals gives priority to the first 0 in the Fibonacci expansion where there is a choice between a 0 or a 1. Thus, the first coefficient that differs in $\text{fib}_1(n)$ and $\text{fib}_2(n)$ is a 0 in the former and a 1 in the latter. This can be seen in Figure 1 above.

Although the bottom-up algorithm as stated can be used to find $\text{fib}_1(n)$ for all integers n, it is not practical to use it directly for negative integers. A better method is as follows. First, find the smallest Fibonacci number $F_{k+2} \geq -n$. Set the first $k+1$ coefficients of $\text{fib}_1(n)$ equal to those of $\text{fib}(F_{k+2}+n)$. Finally, for $i>k$, set $c_i=0$ if i and k have the same parity; otherwise, set $c_i=1$. For example, say $n=-24$. Then the smallest Fibonacci number exceeding $-n$ is $F_9=34$, so we set the first eight coefficients of fib(-24) equal to those of $\text{fib}(34-24)=\text{fib}(10)$, i.e., 01001000. For $i \geq 8$, we set $c_i=0$ if i is odd; 1 otherwise. Thus, $\text{fib}_1(-24)=01001000\overline{10}$, where, as in the case for repeating decimal expansions, a line above a string of coefficients means that string is repeated endlessly.

What about $\text{fib}_2(n)$? It can be found by a simple modification of the above procedure. First, instead of finding the smallest Fibonacci number exceeding $-n$, find the next smallest; in the above example, this would be $F_{10}=55$. Again, calculate $\text{fib}(F_{k+2}+n)$, and set the first $k+1$ coefficients of fib(n) equal to these. (Now, however, k is one greater than last time.) Thus, returning to the example, $\text{fib}(55-24)=\text{fib}(31)=010010100$. The last step is exactly the same as before, but with k replaced by $k+1$; thus, $\text{fib}_2(-24)=01001010\overline{01}$ is the other expansion. Note that one expansion has $c_k=0$ for even k in the repeating portion of fib(n), and the other has $c_k=0$ for odd k. This is always the case. Also, note that the nonrepeating portions of the two expansions only differ in one place. In fact, for all negative integers except -1, the nonrepeating portions differ in one place. (The two expansions of -1 are both purely periodic.)

Let us refer again to the right side of Figure 1. Note that the negative integers lie on the borderline of regions where c_i is constant for $i \leq k$ for some k. Also note that the positions of the positive and negative integers are staggered and that, for negative n, the first six coefficients of $\text{fib}_1(n)$ and $\text{fib}_2(n)$ agree with the expansions of the two positive neighbors of n.

The bottom-up algorithm involves first calculating the residue of an integer mod τ^2. What if, instead, we start with an arbitrary real number x, calculate its residue class mod τ^2 and apply the bottom-up algorithm? Then we will, in general, obtain an infinite sequence of 0's and 1's with no two consecutive 1's. Let U be the set of all such sequences, and define the equivalence relation E by letting two distinct sequences in U be equivalent iff one is $\text{fib}_1(n)$ and the other is $\text{fib}_2(n)$ for some negative integer n. Define the *F-adic integers* to be the elements of the set U/E (they are analogous to p-adic integers).

VI. GEOMETRIC STRUCTURE OF "F-ADIC" INTEGERS

In Figure 1 I illustrated how the Fibonacci expansions of the integers have a nice interpretation as points on a circle of circumference τ^2. In this section I would like to make that analogy more precise and to extend it to all the F-adic integers, which I denote $\mathbf{Z}_F$. (In fact, one can show that $\mathbf{Z}_F$ is a topological group isomorphic to the circle group, but the proof is rather unenlightening.)

As indicated in Part V, the bottom-up algorithm may be applied to any arbitrary real number x to give an F-adic integer, which is unique up to the congruence class of x mod τ^2. Thus, there is a 1-1 correspondence between the F-adic integers and points on a circle of circumference τ^2. Furthermore, as can be seen in Figure 1, nearby points on the circle seem to correspond to "nearby" F-adic integers, where "nearby" roughly means having Fibonacci expansions agreeing to the first several places. I first need to make the notion of "nearby" F-adic integers more precise.

Definition: Let α and β be F-adic integers. Then α and β are *similar in k places* if there exists an F-adic integer γ and sequences $(a_i), (b_i), (c_i)$, and (c_i') in U such that $\alpha \sim (a_i)$, $\beta \sim (b_i)$, $(c_i) \sim (c_i')$, and for all $i < k$, we have $a_i = c_i$ and $b_i = c_i'$. Here, $\sim$ denotes the equivalence defined in Part V. If α and β are similar in m places but not in $m+1$ places, we say they are similar in *exactly m* places.

For example, suppose $\alpha = 4$ and $\beta = 20$. Then $(a_i) = \text{fib}(4) = 101\overline{0}$ and $(b_i) = \text{fib}(20) = 010101\overline{0}$. Let $\gamma = -1$ and let $(c_i) = \text{fib}_2(-1) = \overline{10}$ and $(c_i') = \text{fib}_1(-1) = \overline{01}$. Then $a_i = c_i$ and $b_i = c_i'$ for $i < 4$ so α and β are similar in four places.

Now I will state and prove the main theorem of my paper.

Theorem: There exists a bijection $\phi: \mathbf{Z}_F \to \mathbf{R} / \tau^2\mathbf{Z}$ for which, given any pair of F-adic integers α and β which are similar in k places, there exists a real number $x \equiv \phi(\alpha) - \phi(\beta) \pmod{\tau^2}$ such that $|x| \le 2\tau^{2-k}$. Conversely, if $\alpha, \beta \in \mathbf{Z}_F$ are such that there exists a real number $x \equiv \phi(\alpha) - \phi(\beta)$ $\pmod{\tau^2}$ such that $|x| \le \tau^{-k}$, then α and β are similar in k places.

Proof: Consider the map

$$\hat{\phi}: U \to \mathbf{R} / \tau^2\mathbf{Z}$$

$$(c_i)_0^\infty \mapsto \sum_{i=0}^{\infty} c_i(-\tau)^{-i} \pmod{\tau^2}.$$

Note that the inverse of $\hat{\phi}$ is just the bottom-up algorithm, and that this inverse is unique except when (c_i) corresponds to a negative integer. Thus, we may define $\phi(x)$ to be $\hat{\phi}(\bar{x})$, where $\bar{x}$ is the equivalence class of x in $\mathbf{Z}_F$.

Now let α and β be F-adic integers that are similar in k places. Then there exist sequences $\alpha \sim (a_i)$ and $\beta \sim (b_i)$ and an F-adic integer $\gamma \sim a_0 a_1 \ldots a_{k-1} c_k c_{k+1} \ldots \sim b_0 b_1 \ldots b_{k-1} c_k' c_{k+1}' \ldots$. Now let $\alpha' = a_0 \ldots a_{k-1}\overline{0}$ and $\beta' = b_0 \ldots b_{k-1}\overline{0}$. By Lemma 3, both $\phi(\alpha) = \phi(\alpha')$ and $\phi(\gamma) - \phi(\alpha')$ lie in a fixed interval of the form $\pm[-\tau^{-k}, \tau^{1-k}]$, so their difference, $\phi(\alpha) - \phi(\gamma)$, has absolute value not exceeding the length of the interval, τ^{2-k}. Similarly, $|\phi(\beta) - \phi(\gamma)| \le \tau^{2-k}$. Thus, by the triangle inequality, $|\phi(\alpha) - \phi(\beta)| \le 2\tau^{2-k}$.

To prove the converse, suppose α and β are as in the statement of the second half of the theorem. Say $\alpha \sim (a_i)$ and $\beta \sim (b_i)$. Suppose (a_i) and (b_i) agree to exactly ℓ places, so that $a_i = b_i = c_i$ for $i < \ell$, and $a_\ell \ne b_\ell$. Without loss of generality, we may assume $a_\ell = 0$ and $b_\ell = 1$. Note that $c_{\ell-1} = 0$ since, otherwise, (b_i) would contain two consecutive 1's. Let n be the unique negative integer such that $\text{fib}_1(n)$ agrees with (a_i) to $\ell+1$ places and $\text{fib}_2(n)$ agrees with (b_i) also to $\ell+1$ places. Now we have

$$\mathrm{fib}_1(n) = c_0c_1c_2\ldots c_{\ell-1}0\overline{01}$$
$$\text{and } \mathrm{fib}_2(n) = c_0c_1c_2\ldots c_{\ell-1}10\overline{01}.$$

Now suppose α and β are similar *through* n in exactly m places, where $m > \ell$, i.e., γ may be replaced by n in the definition of similarity. Then there are two possibilities for what (a_i) and (b_i) look like, depending on whether (a_i) or (b_i) has the first discrepancy from $\mathrm{fib}_1(n)$ or $\mathrm{fib}_2(n)$, respectively. In the first case, we have

$$(a_i) = c_0c_1c_2\ldots c_{\ell-1}001\ldots0100a_{m+1}a_{m+2}\ldots,$$
$$(b_i) = c_0c_1c_2\ldots c_{\ell-1}1001\ldots010b_{m+1}b_{m+2}\ldots,$$

where the second string of dots in each expansion stands for a finite repeating string of the form $01\ldots01$. Note that in this case, (b_i) necessarily agrees with $\mathrm{fib}_2(n)$ to at least $m+1$ places and that $m \equiv \ell \pmod 2$. From the definition of ϕ, we have

$$\begin{aligned}(-1)^m(\phi(n)-\phi(\alpha)) &\equiv \tau^{-m} + a_{m+1}\tau^{-m-1} \\ &\quad + (1-a_{m+2})\tau^{-m-2} + a_{m+3}\tau^{-m-3} + \cdots \\ &\geq \tau^{-m}\end{aligned}$$

and

$$\begin{aligned}(-1)^m(\phi(\beta)-\phi(n)) &\equiv (1-b_{m+1})\tau^{-m-1} + b_{m+2}\tau^{-m-2} \\ &\quad + (1-b_{m+3})\tau^{-m-3} + b_{m+4}\tau^{-m-4} + \cdots \\ &\geq 0,\end{aligned}$$

where the congruence is modulo τ^2.

In the second case, we have

$$(a_i) = c_0c_1c_2\ldots c_{\ell-1}001\ldots01010a_{m+1}a_{m+2}\ldots,$$
$$(b_i) = c_0c_1c_2\ldots c_{\ell-1}1001\ldots0100b_{m+1}b_{m+2}\ldots.$$

This time, we see that (a_i) necessarily agrees with $\mathrm{fib}_1(n)$ to at least $m+1$ places and that $\ell \not\equiv m \pmod 2$. Now we find

$$\begin{aligned}(-1)^m(\phi(n)-\phi(\alpha)) &\equiv (1-a_{m+1})\tau^{-m-1} + a_{m+2}\tau^{-m-2} \\ &\quad + (1-a_{m+3})\tau^{-m-3} + a_{m+4}\tau^{-m-4} + \cdots \\ &\geq 0\end{aligned}$$

and

$$\begin{aligned}(-1)^m(\phi(\beta)-\phi(n)) &\equiv \tau^{-m} = b_{m+1}\tau^{-m-1} \\ &\quad + (1-b_{m+2})\tau^{-m-2} + b_{m+3}\tau^{-m-3} + \ldots \\ &\geq \tau^{-m}.\end{aligned}$$

Again, the congruence is modulo τ^2. In each case, since $\phi(\gamma)$ is between $\phi(\alpha)$ and $\phi(\beta)$, we conclude

$$|\phi(\alpha)-\phi(\beta)| = |\phi(\beta)-\phi(\gamma)| + |\phi(\gamma)-\phi(\alpha)| \geq \tau^{-m},$$

where the above absolute values refer to the minimal such absolute values of real numbers belonging to the congruence class of the expression inside modulo τ^2. But since we are assuming $|x| \leq \tau^{-k}$ for some real number $x \equiv (\phi(\alpha)-\phi(\beta)) \pmod \tau$, we conclude that $m \geq k$. Thus, α and β are similar in k places. □

As I indicated earlier, the F-adic integers have more structure than I have presented. For instance, the map ϕ may be used to define addition on $\mathbf{Z}_F$. This addition makes $\mathbf{Z}_F$ into an additive group isomorphic to the circle group [i.e., by requiring that $\phi(\alpha+\beta)=\phi(\alpha)+\phi(\beta)$.] The map ϕ also turns out to be a topological group isomorphism.

VII. GENERALIZATIONS

There are many ways to generalize the above procedure to other types of sequences. Perhaps the simplest (see [4]) is to consider sequences of the form $S_{k+1}=aS_k+bS_{k-1}$; $S_1=S_2=1$, where a and b are positive integers with $a\geq b$. The corresponding expansion is $E(n)=(e_k)_{k=0}^{\infty}$, where $n=\sum_{k=0}^{\infty}e_kS_{k+2}$, where now $0\leq e_k\leq a$ and $e_k=a$ implies $e_{k+1}<b$. Let $\lambda=\frac{1}{2}(a+\sqrt{a^2+4b})$ and $\overline{\lambda}=\frac{1}{2}(a-\sqrt{a^2+4b})$; then it is easy to check that $S_{k+2}\equiv\overline{\lambda}^k \bmod \lambda$. Since $|\overline{\lambda}|<1$, $S_k\to 0 \bmod \lambda$; thus, we should again have a bottom-up algorithm for determining $E(n)$. It looks like the same analysis should carry through for these more general sequences. In particular, if we again define the analogous infinite sequences of coefficients (e_k), they should again form an additive group isomorphic to $\mathbf{R}/\mathbf{Z}$. One can also carry out this procedure for a much more general class of sequences. The reader is invited to try his hand with the sequence 1, 10, 100, 1000, ...

Another way to generalize is to define "F-adic numbers," the analog of p-adic numbers. At first, this does not seem feasible, for the F_n are integers for negative as well as positive n, so we gain nothing by considering sums of the form $\sum_{k=\ell}^{\infty}c_kF_{k+2}$, where $\ell<0$. The solution is to just consider formal sequences of the form $(c_k)_{\ell}^{\infty}$, where $c_k=0$ or 1, $c_kc_{k+1}=0$ for all k, and $\ell\in\mathbf{Z}$. We treat these sequences as before, but simplify the addition algorithm so as not to worry about fixing coefficients with negative indices. The resulting group seems to be isomorphic to $\mathbf{R}$. It should be noted (see [5]) that an ordinary integer n will, in general, have a *different* expansion of this type than fib(n).

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GENERALIZED FIBONACCI NUMBERS AND THE PROBLEM OF DIOPHANTUS

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1. INTRODUCTION

Let n be an integer. A set of positive integers $\{a_1, \ldots, a_m\}$ is said to have *the property of Diophantus of order n*, symbolically $D(n)$ if, for all $i, j = 1, \ldots, m$, $i \neq j$, the following holds: $a_i a_j + n = b_{ij}^2$, where b_{ij} is an integer. The set $\{a_1, \ldots, a_m\}$ is called *a Diophantine m-tuple*.

In this paper we construct several Diophantine quadruples whose elements are represented using generalized Fibonacci numbers. It is a generalization of the following statements (see [8], [12], [6]): The sets

$$\{F_{2n}, F_{2n+2}, F_{2n+4}, 4F_{2n+1}F_{2n+2}F_{2n+3}\} \quad \text{and} \quad \{n, n+2, 4n+4, 4(n+1)(2n+1)(2n+3)\}$$

have the property $D(1)$; the set

$$\{2F_{n-1}, 2F_{n+1}, 2F_n^3F_{n+1}F_{n+2}, 2F_{n+1}F_{n+2}F_{n+3}(2F_{n+1}^2 - F_n^2)\}$$

has the property $D(F_n^2)$ for all positive integers n.

These results are applied to the Pell numbers and are used to obtain explicit formulas for quadruples with the property $D(\ell^2)$, where ℓ is an integer.

2. PRELIMINARIES

2.1 The Problem of Diophantus

The Greek mathematician Diophantus of Alexandria noted that the numbers x, $x+2$, $4x+4$, and $9x+6$, where $x = 1/16$, have the following property: the product of any two of them increased by 1 is a square of a rational number (see [3]). The French mathematician Pierre de Fermat first found a set with the property $D(1)$, and it was $\{1, 3, 8, 120\}$. Later, Davenport and Baker [2] showed that if there is a set $\{1, 3, 8, d\}$ with the property $D(1)$, then d has to be 120.

In [5], the problem of the existence of Diophantine quadruples with the property $D(n)$ was considered for an arbitrary integer n. The following result was proved: if an integer n is not of the form $4k+2$ and $n \notin \{3, 5, 8, 12, 20, -1, -3, -4\}$, then there exists a quadruple with the property $D(n)$.

Nonexistence of Diophantine quadruples with the property $D(4k+2)$ was proved in [1] and [5].

The sets with the property $D(\ell^2)$ were particularly discussed in [5]. It was proved that for any integer ℓ and any set $\{a, b\}$ with the property $D(\ell^2)$, where ab is not a perfect square, there exists an infinite number of sets of the form $\{a, b, c, d\}$ with the property $D(\ell^2)$. We would like to describe the construction of those sets.

Let $ab+\ell^2=k^2$ and let s and t be positive integers satisfying the Pellian equation $S^2-abT^2=1$ (s and t exist since ab is not a perfect square). Two double sequences $y_{n,m}$ and $z_{n,m}$, $n,m\in Z$, can be defined as follows (see [5]):

$$y_{0,0}=\ell,\ z_{0,0}=\ell,\ y_{1,0}=k+a,\ z_{1,0}=k+b,$$
$$y_{-1,0}=k-a,\ z_{-1,0}=k-b,$$
$$y_{n+1,0}=\tfrac{2k}{\ell}y_{n,0}-y_{n-1,0},\ z_{n+1,0}=\tfrac{2k}{\ell}z_{n,0}-z_{n-1,0},\ n\in Z,$$
$$y_{n,1}=sy_{n,0}+atz_{n,0},\ z_{n,1}=bty_{n,0}+sz_{n,0},\ n\in Z,$$
$$y_{n,m+1}=2sy_{n,m}-y_{n,m-1},\ z_{n,m+1}=2sz_{n,m}-z_{n,m-1},\ n,m\in Z.$$

Let us write

$$x_{n,m}=(y_{n,m}^2-\ell^2)/a. \tag{1}$$

According to Theorem 2 of [5], if $x_{n,m}$ and $x_{n+1,m}$ are positive integers, then the set $\{a,b,x_{n,m},x_{n+1,m}\}$ has the property $D(\ell^2)$. It is also proved that the sets $\{a,b,x_{0,m},x_{1,m}\}$, $m\in Z\setminus\{-2,-1,0\}$, and $\{a,b,x_{-1,m},x_{0,m}\}$, $m\in Z\setminus\{-1,0,1\}$, have the property $D(\ell^2)$. So, it is sufficient to find one positive integer solution of the Pellian equation $S^2-abT^2=1$ to extend a set $\{a,b\}$ with the property $D(\ell^2)$ to a set $\{a,b,c,d\}$ with the same property.

2.2 Generalized Fibonacci Numbers

In [9], the generalized Fibonacci sequence of numbers (w_n) was defined by Horadam as follows: $w_n=w_n(a,b;p,q)$, $w_0=a$, $w_1=b$, $w_n=pw_{n-1}-qw_{n-2}$ $(n\ge 2)$, where $a,b,p,$ and q are integers. The properties of that sequence were discussed in detail in [10], [11], and [13]. The following identities have been proved:

$$w_nw_{n+2r}-eq^nU_r=w_{n+r}^2, \tag{2}$$

$$4w_nw_{n+1}^2w_{n+2}+(eq^n)^2=(w_nw_{n+2}+w_{n+1}^2)^2, \tag{3}$$

$$w_nw_{n+1}w_{n+3}w_{n+4}=w_{n+2}^4+eq^n(p^2+q)w_{n+2}^2+e^2q^{2n+1}p^2, \tag{4}$$

$$\begin{aligned}4w_nw_{n+1}w_{n+2}w_{n+4}w_{n+5}w_{n+6}+e^2q^{2n}(w_nU_4U_5-w_{n+1}U_2U_6-w_nU_1U_8)^2\\=(w_{n+1}w_{n+2}w_{n+6}+w_nw_{n+4}w_{n+5})^2.\end{aligned} \tag{5}$$

Here $e=pab-qa^2-b^2$ and $U_n=w_n(0,1;p,q)$. Identity (5) is due to Morgado [13].

Our purpose is to apply the above identities to constructing Diophantine quadruples. Considering the construction described in §2.1, we will restrict our attention to two special cases. For simplicity of notation, these are

$$u_n=u_n(p)=w_n(0,1;p,-1),\quad p\ge 1,$$
$$g_n=g_n(p)=w_n(0,1;p,1),\quad p\ge 2.$$

The Fibonacci sequence $F_n=u_n(1)$, the Pell sequence $P_n=u_n(2)$, the Fibonacci numbers of even subscript $F_{2n}=g_n(3)$, and $g_n(2)=n$ are important special cases of the above sequences.

Apart from the sequences (u_n) and (g_n), we also wish to investigate joined sequences (v_n) and (h_n), which are defined by $v_n=u_{n-1}+u_{n+1}$, $h_n=g_{n+1}-g_{n-1}$. It is easy to check that $v_n=w_n(2,p;p,-1)$ and $h_n=w_n(2,p;p,1)$.

3. QUADRUPLES WITH PROPERTIES $D(p^2u_n^2)$ AND $D(h_n^2)$

For every positive integer n,

$$4u_nu_{n+2}+(pu_{n+1})^2=v_{n+1}^2. \tag{6}$$

Indeed, $v_{n+1}^2-(pu_{n+1})^2=(u_n+u_{n+2})^2-(u_{n+2}-u_n)^2=4u_nu_{n+2}$. From the above, it follows that the sets $\{2u_n, 2u_{n+2}\}$, $\{u_n, 4u_{n+2}\}$, and $\{4u_n, u_{n+2}\}$ have the property $D(p^2u_{n+1}^2)$. In order to extend these sets to the quadruples with the property $D(p^2u_{n+1}^2)$ by applying the construction described in §2.1, it is necessary to find a solution to the Pellian equation $S^2-4u_nu_{n+2}T^2=1$. One solution of this equation can be obtained from the identity

$$4u_nu_{n+1}^2u_{n+2}+1=(u_{n+1}^2+u_nu_{n+2})^2, \tag{7}$$

which is the direct consequence of (2). Therefore, we will set $s=u_{n+1}^2+u_nu_{n+2}$, $t=u_{n+1}$. Now, applying the construction from §2.1, we obtain an infinite number of sets with the property $D(p^2u_{n+1}^2)$. In particular, we have

Theorem 1: Let n and p be positive integers. Then the six sets

$$\{2u_n, 2u_{n+2}, 2p^2u_{n+1}^3(u_{n+1}-u_n)(u_{n+2}-u_n), 2p^2u_{n+1}^3(u_n+u_{n+1})(u_{n+1}+u_{n+2})\},$$

$$\begin{gathered}\{2u_n, 2u_{n+2}, 2p^2u_{n+1}^3(u_n+u_{n+1})(u_{n+1}+u_{n+2}),\\ 2(u_n+u_{n+1})(u_{n+1}+u_{n+2})(u_n+2u_{n+1}+u_{n+2})(u_nu_{n+1}+2u_nu_{n+2}+u_{n+1}u_{n+2})\},\end{gathered}$$

$$\begin{gathered}\{u_n, 4u_{n+2}, (u_{n+1}-u_n)(u_{n+2}-u_{n+1})(2u_{n+2}-u_n-u_{n+1})(2u_{n+1}u_{n+2}-u_nu_{n+1}-u_nu_{n+2}),\\ p^2u_{n+1}^3(u_n+2u_{n+1})(u_{n+1}+2u_{n+2})\},\end{gathered}$$

$$\begin{gathered}\{u_n, 4u_{n+2}, p^2u_{n+1}^3(u_n+2u_{n+1})(u_{n+1}+2u_{n+2}),\\ (u_n+u_{n+1})(u_{n+1}+u_{n+2})(u_n+3u_{n+1}+2u_{n+2})(u_nu_{n+1}+3u_nu_{n+2}+2u_{n+1}u_{n+2})\},\end{gathered}$$

$$\begin{gathered}\{4u_n, u_{n+2}, (u_{n+1}-u_n)(u_{n+1}+u_{n+2}-2u_n)(u_nu_{n+2}+u_{n+1}u_{n+2}-2u_nu_{n+1}),\\ p^2u_{n+1}^3(2u_n+u_{n+1})(2u_{n+1}+u_{n+2})\},\end{gathered}$$

and

$$\begin{gathered}\{4u_n, u_{n+2}, p^2u_{n+1}^3(2u_n+u_{n+1})(2u_{n+1}+u_{n+2}),\\ (u_n+u_{n+1})(u_{n+1}+u_{n+2})(2u_n+3u_{n+1}+u_{n+2})(2u_nu_{n+1}+3u_nu_{n+2}+u_{n+1}u_{n+2})\}\end{gathered}$$

have the property $D(p^2u_{n+1}^2)$.

Proof: The main idea of the proof is to show that the six sets in Theorem 1 are of the form $\{a, b, x_{0,1}, x_{1,1}\}$ or $\{a, b, x_{-1,1}, x_{0,1}\}$. Combining (6) with (7), we obtain $\ell=pu_{n+1}$, $k=v_{n+1}$, $s=u_{n+1}^2+u_nu_{n+2}$, $t=u_{n+1}$. To simplify notation, we write $u_{n+2}=A$, $u_{n+1}=B$. Hence, according to (2), $A^2-pAB-B^2=(-1)^{n+1}$, and that gives

$$(A^2-pAB-B^2)^2=1. \tag{8}$$

We arrange the proof in three parts, each part relating to two of the six sets.

Part 1. $a=2u_n$, $b=2u_{n+2}$

Using the notation of §2.1, we have

$$y_{0,0} = z_{0,0} = pu_{n+1}, \; y_{1,0} = 3u_n + u_{n+2}, \; z_{1,0} = u_n + 3u_{n+2},$$
$$y_{-1,0} = pu_{n+1}, \; z_{-1,0} = -pu_{n+1}.$$

From this, we obtain

$$\begin{aligned} y_{0,1} &= pB[A^2 + (2-p)AB - (2p-1)B^2], \\ y_{1,1} &= 4A^3 + (8-7p)A^2B + (3p^2 - 10p + 4)AB^2 + p(2p-3)B^3, \\ y_{-1,1} &= pB[A^2 - (p+2)AB + (2p+1)B^2]. \end{aligned}$$

Relation (8) will be used to represent expressions of $x_{i,1}$, $i = -1, 0, 1$, obtained by putting $y_{i,1}$ in (1), as homogeneous polynomials in two variables A and B. When those polynomials are factored, we have

$$\begin{aligned} x_{0,1} &= 2p^2B^3\{A-(p-1)B](A+B) = 2p^2u_{n+1}^3(u_n + u_{n+1})(u_{n+1} + u_{n+2}), \\ x_{1,1} &= 2[A-(p-1)B]A+B)[2A-(p-2)B][2A^2 - 2(p-1)AB - pB^2] \\ &= 2(u_n + u_{n+1})(u_{n+1} + u_{n+2})(u_n + 2u_{n+1} + u_{n+2})(u_nu_{n+1} + 2u_nu_{n+2} + u_{n+1}u_{n+2}), \\ x_{-1,1} &= 2p^2B^3[(p+1)B - A](A-B) = 2p^2u_{n+1}^3(u_{n+1} - u_n)(u_{n+2} - u_{n+1}). \end{aligned}$$

Part 2. $a = u_n$, $b = 4u_{n+2}$

We now have

$$y_{0,0} = z_{0,0} = pu_{n+1}, \; y_{1,0} = 2u_n + u_{n+2}, \; z_{1,0} = u_n + 5u_{n+2},$$
$$y_{-1,0} = u_{n+2}, \; z_{-1,0} = u_n - 3u_{n+2}.$$

Hence

$$\begin{aligned} y_{0,1} &= pB[A^2 - (p-1)AB - (p-1)B^2], \\ y_{1,1} &= 3A^3 - (5p-6)A^2B + (2p^2 - 7p + 3)AB^2 + p(p-2)B^3, \\ y_{-1,1} &= A^3 - (p+2)A^2B + (p+1)AB^2 + pB^3, \end{aligned}$$

and, from (1) and (8),

$$\begin{aligned} x_{0,1} &= p^2B^3(A+2B)[2A-(p-1)B] = p^2u_{n+1}^3(u_n + 2u_{n+1})(u_{n+1} + 2u_{n+2}), \\ x_{1,1} &= [A-(p-1)B](A+B)[3A-(p-3)B][3A^2 - 3(p-1)AB - pB^2] \\ &= (u_n + u_{n+1})(u_{n+1} + u_{n+2})(u_n + 3u_{n+1} + 2u_{n+2})(u_nu_{n+1} + 3u_nu_{n+2} + 2u_{n+1}u_{n+2}), \\ x_{-1,1} &= [A-(p-1)B][A-(p+1)B](A-B)[A^2 - (p+1)AB - pB^2] \\ &= (2u_{n+2} - u_n - u_{n+1})(u_{n+1} - u_n)(u_{n+2} - u_{n+1})(2u_{n+1}u_{n+2} - u_nu_{n+1} - u_nu_{n+2}). \end{aligned}$$

Part 3. $a = 4u_n$, $b = u_{n+2}$

In this case,

$$y_{0,0} = z_{0,0} = pu_{n+1}, \; y_{1,0} = 5u_n + u_{n+2}, \; z_{1,0} = u_n + 2u_{n+2},$$
$$y_{-1,0} = u_{n+2} - 3u_n, \; z_{-1,0} = u_n.$$

Accordingly,

$$y_{0,1} = pB[A^2 - (p-4)AB - (4p-1)B^2],$$
$$y_{1,1} = 6A^3 - (11p-12)A^2B + (5p^2 - 16p + 6)AB^2 + p(4p-5)B^3,$$
$$y_{-1,1} = -2A^3 + (5p+4)A^2B - (3p^2 + 8p + 2)AB^2 + p(4p+3)B^3,$$

and, finally,

$$x_{0,1} = p^2B^3(A+2B)[2A - (2p-1)B] = p^2u_{n+1}^3(2u_{n+1} + u_{n+2})(2u_n + u_{n+1}),$$
$$x_{1,1} = [A - (p-1)B](A+B)[3A - (2p-3)B][3A^2 - 3(p-1)AB - 2pB^2]$$
$$= (u_n + u_{n+1})(u_{n+1} + u_{n+2})(2u_n + 3u_{n+1} + u_{n+2})(2u_nu_{n+1} + 3u_nu_{n+2} + u_{n+1}u_{n+2}),$$
$$x_{-1,1} = [A - (p+1)B][A - (2p+1)B](A-B)[A^2 - (p+1)AB + 2pB^2]$$
$$= (u_{n+1} - u_n)(u_{n+2} - u_{n+1})(u_{n+1} + u_{n+2} - 2u_n)(u_nu_{n+2} + u_{n+1}u_{n+2} - 2u_nu_{n+1}). \quad \square$$

Using the identities $4g_ng_{n+2} + h_{n+1}^2 = p^2g_{n+1}^2$ and $4g_ng_{n+1}^2g_{n+2} + 1 = (g_{n+1}^2 + g_ng_{n+2})^2$, we find the following theorem may be proved in much the same way as Theorem 1.

Theorem 2: Let $n \geq 1$ and $p \geq 2$ be integers. Then the six sets

$$\{2g_n, 2g_{n+2}, 2g_{n+1}h_{n+1}^2(g_{n+1} - g_n)(g_{n+2} - g_{n+1}), 2g_{n+1}h_{n+1}^2(g_n + g_{n+1})(g_{n+1} + g_{n+2})\},$$

$$\{2g_n, 2g_{n+2}, 2g_{n+1}h_{n+1}^2(g_n + g_{n+1})(g_{n+1} + g_{n+2}),$$
$$2(p+2)g_{n+1}(g_n + g_{n+1})(g_{n+1} + g_{n+2})(g_ng_{n+1} + 2g_ng_{n+2} + g_{n+1}g_{n+2})\},$$

$$\{g_n, 4g_{n+1}, (g_{n+1} - g_n)(g_{n+2} - g_{n+1})(2g_{n+2} - g_n - g_{n+1})(2g_{n+1}g_{n+2} - g_ng_{n+1} - g_ng_{n+2}),$$
$$g_{n+1}h_{n+1}^2(g_n + 2g_{n+1})(g_{n+1} + 2g_{n+2})\},$$

$$\{g_n, 4g_{n+2}, g_{n+1}h_{n+1}^2(g_n + 2g_{n+1})(g_{n+1} + g_{n+2}),$$
$$(g_n + g_{n+1})(g_{n+1} + g_{n+2})(g_n + 3g_{n+1} + 2g_{n+2})(g_ng_{n+1} + 3g_ng_{n+2} + 2g_{n+1}g_{n+2})\}$$

$$\{4g_n, g_{n+2}, (g_{n+1} - g_n)(g_{n+2} - g_{n+1})(g_{n+1} + g_{n+2} - 2g_n)(g_ng_{n+2} + g_{n+1}g_{n+2} - 2g_ng_{n+1}),$$
$$g_{n+1}h_{n+1}^2(2g_n + g_{n+1} + g_{n+2})\},$$

and

$$\{4g_n, g_{n+2}, g_{n+1}h_{n+1}^2(2g_n + g_{n+1})(2g_{n+1} + g_{n+2}),$$
$$(g_n + g_{n+1})(g_{n+1} + g_{n+2})(2g_n + 3g_{n+1} + g_{n+2})(2g_ng_{n+1} + 3g_ng_{n+2} + g_{n+1}g_{n+2})\}$$

have the property $D(h_{n+1}^2)$.

4. THE MORGADO IDENTITY

We are now going to use the Morgado identity (5). It is easy to check that

$$w_nU_4U_5 - w_{n+1}U_2U_6 - w_nU_1U_8 = U_2U_3(w_{n+4} - qw_{n+2}),$$
$$w_{n+1}w_{n+2}w_{n+6} + w_nw_{n+4}w_{n+5} = w_{n+3}(eq^nU_2^2U_3 + 2w_{n+2}w_{n+4}).$$

If we restrict the discussion to the sequences (u_n) and (g_n), the Morgado identity can be used as a base for constructing quadruples with the properties $D((u_2u_3v_{n+3})^2)$ and $D((g_2g_3h_{n+3})^2)$.

We are again going to use the construction described in §2.1. This time it is not necessary to use the solutions of the Pellian equation. We will try to choose the numbers a and b in the manner that the solution of the problem can be obtained using only the sequence $(x_{n,0})$. According to §2.1, if $x_{2,0} \in N$ or $x_{-2,0} \in N$, then, respectively, $\{a, b, x_{1,0}, x_{2,0}\}$ and $\{a, b, x_{-1,0}, x_{-2,0}\}$ are Diophantine quadruples.

Since $y_{2,0} = \frac{2k}{\ell}(k+a) - \ell$, $y_{-2,0} = \frac{2k}{\ell}(k-a) - \ell$, we have

$$x_{2,0} = \frac{y_{2,0}^2 - \ell^2}{a} = \frac{4k(k+a)(k+b)}{\ell^2} = \frac{4k}{\ell^2}(kx_{1,0} - \ell^2),$$

$$x_{-2,0} = \frac{y_{-2,0}^2 - \ell^2}{a} = \frac{-4k(k-a)(k-b)}{\ell^2} = \frac{4k}{\ell^2}(kx_{-1,0} + \ell^2).$$

Theorem 3: Let n and p be positive integers and $k = u_{n+3}[2u_{n+2}u_{n+4} - (-1)^n p^2(p^2+1)]$. Then the three sets

$$\left\{2u_n u_{n+1} u_{n+2},\, 2u_{n+4}u_{n+5}u_{n+6},\, 2(p^2+1)^2 u_{n+3}u_{n+3}^2,\, 4k\left(\frac{2ku_{n+3}}{p^2} - 1\right)\right\},$$

$$\left\{2u_n u_{n+1} u_{n+4},\, 2u_{n+2}u_{n+5}u_{n+6},\, 2p^2 u_{n+3}v_{n+3}^2,\, 4k\left(\frac{2ku_{n+3}}{(p^2+1)^2} + 1\right)\right\},$$

and

$$\left\{2u_n u_{n+2} u_{n+5},\, 2u_{n+1}u_{n+4}u_{n+6},\, 2u_{n+3}v_{n+3}^2,\, 4k\left(\frac{2ku_{n+3}}{p^2(p^2+1)^2} - 1\right)\right\}$$

have the property $D(p^2(p^2+1)^2 v_{n+3}^2)$.

Proof: The proof is by applying the construction from §2.1 to identity (5) for $w_n = u_n$. Three cases need to be considered.

Case 1. $a = 2u_n u_{n+1} u_{n+2}$, $b = 2u_{n+4}u_{n+5}u_{n+6}$

Hence, $a + b = 2(p^2+2)u_{n+3}[(p^2+1)(u_{n+2}^2 + u_{n+4}^2) + (p^2-1)u_{n+2}u_{n+4}]$. This gives

$$x_{1,0} = a + b + 2k = 2(p^2+1)^2 u_{n+3}(u_{n+2} + u_{n+4})^2 = 2(p^2+1)^2 u_{n+3} v_{n+3}^2,$$

$$x_{2,0} = 4k\left(\frac{k \cdot 2(p^2+1)^2 u_{n+3} v_{n+3}^2}{p^2(p^2+1)^2 v_{n+3}^2} - 1\right) = 4k\left(\frac{2ku_{n+3}}{p^2} - 1\right).$$

Case 2. $a = 2u_n u_{n+1} u_{n+4}$, $b = 2u_{n+2}u_{n+5}u_{n+6}$

Now we have $a + b = 2u_{n+3}[(p^2+1)(p^2+4)u_{n+2}u_{n+4} - u_{n+2}^2 - u_{n+4}^2]$ and

$$x_{-1,0} = a + b - 2k = 2p^2 u_{n+3} v_{n+3}^2,$$

$$x_{-2,0} = 4k\left(\frac{k \cdot 2p^2 u_{n+3} v_{n+3}^2}{p^2(p^2+1)^2 v_{n+3}^2} + 1\right) = 4k\left(\frac{2ku_{n+3}}{(p^2+1)^2} + 1\right).$$

Case 3. $a = 2u_n u_{n+2} u_{n+5}$, $b = 2u_{n+1}u_{n+4}u_{n+6}$

We have $a + b = 2(p^2+2)u_{n+3}[u_{n+2}^2 + u_{n+4}^2 - (p^2+1)u_{n+2}u_{n+4}]$ and

$$x_{1,0} = 2u_{n+3}v_{n+3}^2,$$
$$x_{2,0} = 4k\left(\frac{2ku_{n+3}}{p^2(p^2+1)^2} - 1\right).$$

It remains to prove that all elements of the sets from this theorem are integers. It is sufficient to prove that the number $8k^2u_{n+3}/p^2(p^2+1)^2$ is an integer for all positive integers n. That is the direct consequence of the relation

$$\frac{8k^2u_{n+3}}{p^2(p^2+1)^2} = \frac{8u_{n+3}^3[p^4(p^2+1)^2 - (-1)^n 4p^2(p^2+1)u_{n+2}u_{n+4} + 4u_{n+2}^2u_{n+4}^2]}{u_2^2u_3^2}$$

and the fact that $u_2 \mid u_{2m}$ and $u_3 \mid u_{3m}$ for all $m \in N$, which is easy to prove by induction. □

The following theorem can be proved in much the same way as Theorem 3.

Theorem 4: Let $n \geq 1$ and $p \geq 2$ be integers and $k = g_{n+3}[2g_{n+2}g_{n+4} - p^2(p^2-1)]$. Then the three sets

$$\left\{2g_ng_{n+1}g_{n+2},\ 2g_{n+4}g_{n+5}g_{n+6},\ 2(p^2-1)^2 g_{n+3}h_{n+3}^2,\ 4k\left(\frac{2kg_{n+3}}{p^2} + 1\right)\right\},$$

$$\left\{2g_ng_{n+1}g_{n+4},\ 2g_{n+2}g_{n+5}g_{n+6},\ 2p^2 g_{n+3}h_{n+3}^2,\ 4k\left(\frac{2kg_{n+3}}{(p^2-1)^2} - 1\right)\right\},$$

and

$$\left\{2g_ng_{n+2}g_{n+5},\ 2g_{n+1}g_{n+4}g_{n+6},\ 2g_{n+3}h_{n+3}^2,\ 4k\left(\frac{2kg_{n+3}}{p^2(p^2-1)^2} + 1\right)\right\}$$

have the property $D(p^2(p^2-1)^2h_{n+3}^2)$.

We now want to show that the sequence (g_n) possesses another interesting property based on the identity

$$g_ng_{n+1}g_{n+3}g_{n+4} + [(p \pm 1)g_{n+2}]^2 = (g_{n+2}^2 \pm p)^2. \tag{9}$$

Now, the construction described in §2.1 can be applied on the relation (9). We have $a = g_ng_{n+1}$, $b = g_{n+3}g_{n+4}$, $k = g_{n+2}^2 \pm p$, which gives

$$x_{\mp 1,0} = a + b \mp 2k = (p^3 - 3p \mp 2)g_{n+2}^2 = (p \pm 1)^2(p \mp 2)g_{n+2}^2,$$
$$x_{\mp 2,0} = 4(g_{n+2}^2 \pm p)(g_{n+1} \mp g_n)(g_{n=4} \mp g_{n+3}).$$

Thus, we have proved

Theorem 5: Let $n \geq 1$ and $p \geq 2$ be integers. Then the set

$$\{g_ng_{n+1},\ g_{n+3}g_{n+4},\ (p+1)^2(p-2)g_{n+2}^2,\ 4(g_{n+2}^2 + p)(g_{n+1} - g_n)(g_{n+4} - g_{n+3})\}$$

has the property $D((p+1)^2g_{n+2}^2)$, and the set

$$\{g_ng_{n+1},\ g_{n+3}g_{n+4},\ (p-1)^2(p+2)g_{n+2}^2,\ 4(g_{n+2}^2 - p)(g_{n+1} + g_n)(g_{n+3} + g_{n+4})\}$$

has the property $D((p-1)^2g_{n+2}^2)$.

5. GENERALIZATION OF A RESULT OF BERGUM

Hoggatt and Bergum [8] have proved that the set

$$\{F_{2n}, F_{2n+2}, F_{2n+4}, 4F_{2n+1}F_{2n+2}F_{2n+3}\} \tag{10}$$

has the property $D(1)$ for every positive integer n. It has been proved in [4] that the set

$$\{F_{2n}, F_{2n+4}, 5F_{2n+2}, 4L_{2n+1}F_{2n+2}L_{2n+3}\} \tag{11}$$

also has the property $D(1)$. In [5], quadruples with the properties $D(4)$, $D(9)$, and $D(64)$ have been found using Fibonacci numbers. We now want to extend these results to the sequences (u_n) and (g_n) starting from identity (2). Applying (2) to the sequence (u_n), we get

$$u_{2n} \cdot u_{2n+2r} + u_r^2 = u_{2n+r}^2. \tag{12}$$

Therefore, the sets $\{u_{2n}, u_{2n+2}\}$ and $\{u_{2n}, u_{2n+4}\}$ have, respectively, the properties $D(1)$ and $D(p^2)$ for every positive integer n. It was shown in §4 that, if a, b, k, and ℓ are the positive integers such that $ab + \ell^2 = k^2$ and if the number $\pm 4k(k \pm a)(k \pm b) / \ell^2$ is a positive integer, then the set $\{a, b, a+b \pm 2k, \pm 4k(k \pm a)(k \pm b) / \ell^2\}$ has the property $D(\ell^2)$. According to this, we have

Theorem 6: Let n and p be positive integers. Then the sets

$$\{u_{2n}, u_{2n+2}, 2u_{2n} + (p-2)u_{2n+1}, 4u_{2n+1}[(p-2)u_{2n+1}^2 + 2u_{2n}u_{2n+1} + 1]\}$$

and

$$\{u_{2n}, u_{2n+2}, 2u_{2n} - (p-2)u_{2n+1}, 4u_{2n+1}[2u_{2n+1}u_{2n+2} - (p-2)u_{2n+1}^2 - 1]\}$$

have the property $D(1)$ and the set

$$\{u_{2n}, u_{2n+4}, p^2 u_{2n+2}, 4u_{2n+1}u_{2n+2}u_{2n+3}\}$$

has the property $D(p^2)$.

For the sequence (g_n), we can prove an even stronger result, namely, from (2) we have

$$g_n \cdot g_{n+2r} + g_r^2 = g_{n+r}^2 \tag{13}$$

for every (not just even) positive integer n. Starting from the sets $\{g_n, g_{n+2}\}$ and $\{g_n, g_{n+4}\}$ with the properties $D(1)$ and $D(p^2)$, respectively, we find that the following result may be proved in much the same way as Theorem 6.

Theorem 7: Let $n \geq 1$ and $p \geq 2$ be integers. Then the sets

$$\{g_n, g_{n+2}, (p-2)g_{n+1}, 4g_{n+1}[(p-2)g_{n+1}^2 + 1]\}$$

and

$$\{g_n, g_{n+2}, (p+2)g_{n+1}, 4g_{n+1}[(p+2)g_{n+1}^2 - 1]\}$$

have the property $D(1)$, and the set

$$\{g_n, g_{n+4}, p^2 g_{n+2}, 4g_{n+1}g_{n+2}g_{n+3}\}$$

has the property $D(9)$.

6. APPLICATION TO THE PELL NUMBERS AND POLYNOMIALS

In this section we apply the results discussed in the previous sections to some special cases of the sequences (u_n) and (g_n). The case of the Fibonacci sequence $F_n = u_n(1)$ and the case of the joined Lucas sequence $L_n = v_n(1)$ are studied in detail in [6].

Let us first examine the Pell sequence $P_n = u_n(2)$ and the Pell-Lucas sequence $Q'_n = v_n(2)$. All elements of the sequence (Q'_n) are even numbers, so we can write $Q'_n = 2Q_n$. The numbers P_n and Q_n are the solutions of the Pellian equation $x^2 - 2y^2 = \pm 1$. Namely, it is true that

$$Q_n^2 - 2P_n^2 = (-1)^n.$$

The sequences (P_n) and (Q_n) are related by relation $P_n + P_{n+1} = Q_{n+1}$. Applying this relation to Theorem 1, we get

Corollary 1: For every positive integer n, the sets

$$\{P_n, P_{n+2}, 4P_{n+1}^3 Q_n Q_{n+1}, 4P_{n+1}^3 Q_{n+1} Q_{n+2}\}$$

and

$$\{P_n, P_{n+2}, 4P_{n+1}^3 Q_{n+1} Q_{n+2}, 4P_{n+2} Q_{n+1} Q_{n+2}[P_{n+1}P_{n+2} - (-1)^n]\}$$

have the property $D(P_{n+1}^2)$.

In [6], quadruples with the property $D(L_{n+2}^2)$ are constructed using the following identities:

$$4F_nF_{n+4} + L_{n+2}^2 = 9F_{n+2}^2, \tag{14}$$

$$4F_nF_{n+2}^2F_{n+4} + 1 = (F_{n+2}^2 + F_nF_{n+4})^2. \tag{15}$$

For the sequences (u_n), the following analogs of the above identities are valid:

$$4u_nu_{n+4} + (pv_{n+2})^2 = [(p^2+2)u_{n+2}]^2, \tag{16}$$

$$4u_{n+4}u_{n+2}^2u_{n+4} + p^4 = (u_{n+2}^2 + u_nu_{n+4})^2. \tag{17}$$

Unfortunately, existence of the term p^4 in (17) makes it impossibie to apply the construction for finding quadruples with the property $D(p^2v_{n+2}^2)$ from §2.1. But in the case $p = 2$, the solution of the equation $S^2 - abT^2 = 4$ can be obtained from relation (17). Thus, we can apply the modified construction described in Remark 1 of [5].

Theorem 8: For every positive integer n, the sets

$$\{P_n, P_{n+4}, 4P_{n+1}P_{n+2}P_{n+3}Q_{n+2}^2, 4P_{n+2}Q_{n+1}Q_{n+2}^2Q_{n+3}\}$$

and

$$\{P_n, P_{n+4}, 4P_{n+2}Q_{n+1}Q_{n+2}^2Q_{n+3}, 16P_{n+2}Q_{n+1}Q_{n+3}(2P_{n+2}^2 - P_{n+1}P_{n+3})\}$$

have the property $D(4Q_{n+2}^2)$.

Proof: The sets from Theorem 8 are easily seen to be of the forms $\{a, b, x'_{-1,1}, x'_{0,1}\}$ and $\{a, b, x'_{0,1}, x'_{1,1}\}$, respectively, where the sequence $(x'_{n,m})$ is constructed as described in Remark 1 of [5], that is, by setting $a = P_n$, $b = P_{n+4}$, $s' = P_{n+2}^2 + P_nP_{n+4}$, $t' = P_{n+2}$. □

In distinction from the identities (16) and (17), the construction from §2.1 can be applied directly to the following identities:

$$Q_n Q_{n+2} + Q_{n+1}^2 = 4P_{n+1}^2, \tag{18}$$

$$Q_n Q_{n+1}^2 Q_{n+2} + 1 = 4P_{n+1}^4. \tag{19}$$

We have thus proved

Theorem 9: For every positive integer n, the sets

$$\{Q_n, Q_{n+2}, 4P_nP_{n+1}Q_{n+1}^3, 4P_{n+1}P_{n+2}Q_{n+1}^3\}$$

and

$$\{Q_n, Q_{n+2}, 4P_{n+1}P_{n+2}Q_{n+1}^3, 4P_{n+1}P_{n+2}Q_{n+2}(P_{n+1}P_{n+3} - P_nP_{n+2})\}$$

have the property $D(Q_{n+1}^2)$.

Obviously, Theorems 3 and 6 can also be applied to the sequence (P_n). However, applying Theorem 6, as it is done for Fibonacci numbers in Theorem 3 of [5], gives us more.

Corollary 2: For every positive integer n, the sets

$$\{P_{2n}, P_{2n+2}, 2P_{2n}, 4P_{2n+1}Q_{2n}Q_{2n+1}\}$$

and

$$\{P_{2n}, P_{2n+2}, 2P_{2n+2}, 4P_{2n+1}Q_{2n+1}Q_{2n+2}\}$$

have the property $D(1)$, the sets

$$\{P_{2n}, P_{2n+4}, 4P_{2n+2}, 4P_{2n+1}P_{2n+2}P_{2n+3}\}$$

and

$$\{P_{2n}, P_{2n+4}, 8P_{2n+2}, 4P_{2n+2}Q_{2n+1}Q_{2n+3}\}$$

have the property $D(4)$, and the set

$$\{P_{2n}, P_{2n+8}, 36P_{2n+4}, P_{2n+2}P_{2n+4}P_{2n+6}\}$$

has the property $D(144)$.

In this paper only the quadruples with the property $D(n)$, where n is a perfect square, have been examined. However, let us mention that the set

$$\{1, P_{2n+1}(3P_{2n+1} - 2), 3P_{2n+1}^2 - 1, P_{2n+1}(3P_{2n+1} + 2)\}$$

has the property $D(-Q_{2n+1}^2)$ for every positive integer n.

Since $g_n(2) = n$, the results from this paper can be used to obtain the sets with the property of Diophantus whose elements are polynomials. For example, from Theorem 7, we get the Jones result that the set $\{n, n+2, 4(n+1), 4(n+1)(2n+1)(2n+3)\}$ has the property $D(1)$ for every positive integer n (see [12]).

The following interesting property of the binomial coefficients can be obtained as a consequence of the results from §4 above.

For every positive integer $n \geq 4$, the sets

$$\left\{\binom{n-1}{3}, \binom{n+3}{3}, 6n, \frac{2n(n^2-7)(n^2-3n+1)(n^2+3n-1)}{3}\right\}$$

and

$$\left\{\binom{n-1}{3}, \binom{n+3}{3}, \frac{2n(n^2+2)}{3}, \frac{2n(n^2-7)(n^3-3n^2+2n-3)(n^3+3n^2+2n+3)}{27}\right\}$$

have the property $D(1)$. Note that $h_n(2)=2$.

Finally, let us mention that, using these results, the explicit formulas for quadruples with the property $D(\ell^2)$, for a given integer ℓ, can be obtained. Of course, only the sets with at least one element that is not divisible by ℓ are of any interest to us here.

Corollary 3: Let ℓ be an integer. The sets

$$\{(\ell-1)(\ell-2), (\ell+1)(\ell+2), 4\ell^2, 2(2\ell-3)(2\ell+3)(\ell^2-2)\}, \text{ for } \ell \geq 3, \tag{20}$$

and

$$\{1, \ell^4-3\ell^2, \ell^2(\ell^2-1), 4\ell^2(\ell^2-1)(\ell^2-2)\}, \text{ for } \ell \geq 2, \tag{21}$$

have the property $D(\ell^2)$.

Proof: We can get set (20) by putting $p=2$ and $n+2=\ell$ in the second set of Theorem 5. Since $g_1(p)=1$, $g_3(p)=p^2-1$, $g_5(p)=p^4-3p^2+1$, set (21) can be obtained by putting $n=1$ and $p=\ell$ in the third set of Theorem 7. □

Remark 1: One question still unanswered is whether any of the Diophantine quadruples from this paper can be extended to the Diophantine quintuple with the same property. In this connection, let us mention that it is proved in [7] that, for every integer ℓ and every set $\{a,b,c,d\}$ with the property $D(\ell^2)$, where $abcd \neq \ell^4$, there exists a rational number r, $r \neq 0$, such that the set $\{a,b,c,d,r\}$ has the property that the product of any two of its elements increased by ℓ^2 is a square of a rational number.

For example, if the method from [7] is applied to the second set in Corollary 3, we get

$$r = \frac{8\ell(\ell-1)(\ell+1)(\ell^2-2)(2\ell^2-3)(2\ell^4-4\ell^2+1)(2\ell^4-6\ell^2+3)}{[4(\ell-1)^2(\ell+1)^2(\ell^2-2)(\ell^2-\ell-1)(\ell^2+\ell-1)-1]^2}.$$

From this, for $\ell=2$, we have the set $\{89760, 128881, 644405, 1546572, 12372576\}$ with the property $D(4\cdot 359^4)$.

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Professor Steven Vajda

Steven Vajda, well known to readers of *The Fibonacci Quarterly* as the author of *Fibonacci & Lucas Numbers, and the Golden Section,* Ellis Horwood, 1989, died on December 10, 1995, at the age of 94. He was born in Budapest on August 20, 1901. He was Professor of Operational Research at the University of Birmingham, England, from 1965 to 1968 and subsequently a senior research fellow at the University of Sussex, England. Steven Vajda was best known for his work in communicating the early developments in the field of linear programming, as in his book *Readings in Linear Programming,* Pitman, 1958.

ON REPETITIONS IN FREQUENCY BLOCKS OF THE GENERALIZED FIBONACCI SEQUENCE $u(3,1)$ WITH $u_0 = u_1 = 1$

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Let $u_0 = u_1 = 1$ and define the generalized Fibonacci sequence $(u_n) = u(3,1)$ to satisfy the recurrence relation $u_n = 3u_{n-1} + u_{n-2}$ for $n \geq 2$. For an integer $m > 1$, let $(\overline{u}_n)$ denote the sequence (u_n) considered modulo m. It is known that $(\overline{u}_n)$ is purely periodic [7], that is, there exists a positive integer r such that $\overline{u}_{n+r} = \overline{u}_n$ for all $n = 0, 1, \ldots$. Define $h(m)$ to be the length of a shortest period of $(\overline{u}_n)$, and $S(m)$ to be the set of residue frequencies within any full period of $(\overline{u}_n)$, as well as $A(m,d)$ to denote the number of times the residue d appears in a full period of $(\overline{u}_n)$ ([5], [6]). Hence, for a fixed m, the range of $A(m,d)$ is the set $S(m)$, that means

$$\{A(m,d) : 0 \leq d < m\} = S(m).$$

We say (u_n) is uniformly distributed modulo m if all residues modulo m occur with the same frequency in any full period. In this case, the length of any period will be a multiple of m; moreover, $|S(m)| = 1$ and $A(m,d)$ is a constant function [4].

For a fixed $m \geq 2$, form a number block $B_m \in N^m$ to consist of the frequency values of the residue d when d runs through the complete residue system modulo m. This number block, B_m, will be called the *frequency block* modulo m, which has properties like $(qB_m)^r = q(B_m)^r$ and $((B_m)^r)^s = (B_m)^{rs}$ with

$$(B_m)^r = (\underbrace{B_m, \ldots, B_m}_{r \text{ times}})$$

and $q, r, s \in N$. Here are some examples for B_m together with the period length $h(m)$.

$$\begin{array}{ll}
B_2 = (1,2) & h(2) = 3 \\
B_3 = (0,1,0) & h(3) = 1 \\
B_4 = (1,3,1,1) & h(4) = 6 \\
B_5 = (0,3,3,3,3) & h(5) = 12 \\
B_8 = (0,3,0,1,2,3,2,1) & h(8) = 12 \\
B_9 = 2(B_3)^3 & h(9) = 6 \\
B_{16} = (0,3,0,1,2,3,0,1,0,3,0,1,2,3,4,1) & h(16) = 24 \\
B_{18} = (0,2,0,0,1,0,0,1,0,0,0,0,0,1,0,0,1,0) & h(18) = 6 \\
B_{26} = 4(B_2)^{13} & h(26) = 156 \\
B_{27} = 2(B_3)^9 & h(27) = 18 \\
B_{32} = (B_{16})^2 & h(32) = 48 \\
B_{52} = 2(B_4)^{13} & h(52) = 156
\end{array}$$

$B_{54} = (B_{18})^3$ $\qquad$ $h(54) = 18$

$B_{65} = (B_5)^{13}$ $\qquad$ $h(65) = 156$

$B_{81} = 2(B_3)^{27}$ $\qquad$ $h(81) = 54$

All but the first few examples show a certain kind of repetition in the frequency blocks, that means such frequency blocks can be produced by repetition of their first few elements a whole number of times. For a given m, this repetition is possible only in the case for which there exists an integer $1 < c < m$ such that $c|m$ and $h(c)|h(m)$. Moreover, the first few repeating elements of B_m are the elements of B_c or some multiple of them. Letting $0 \le x < c$, $0 \le y < m$, and $y \equiv x \pmod{c}$, this fact can be expressed by $A(m, y) = q \cdot A(c, x)$ for some positive integer q. A similar result in connection with the uniform distribution was found in [3] for the Fibonacci numbers. The considered sequence (u_n) is uniformly distributed modulo 13^k for $k \ge 1$ (see [1]). Thus, the above examples show that the repetition in the frequency blocks does not occur exclusively in connection with the uniform distribution.

To search for repetition possibilities in the frequency blocks of the sequence (u_n), we made a computer run for moduli $m \le 1000$. However, we did not consider moduli m with $13|m$ because we wanted to investigate the repetition possibilities that had no direct connections with the uniform distribution.

Making use of the above-mentioned notation $A(m, y) = q \cdot A(c, x)$ with $0 \le x < c$, $0 \le y < m$, $y \equiv x \pmod{c}$, and $1 \le q \in N$, we discovered the following:

D1: $A(3^{k+1}, y) = 2 \cdot A(3, x)$ for $k \ge 1$.

D2: $A(3^k c, y) = A(c, x)$ for $k \ge 1$ and $c \in \{18, 21, 33, 36, 45, 51, 57, 69, 72, 87, 90, 93, 111, 123, 126, 144, 147, 159, 180, 198, 201, 219, 231, 237, 252, 291, 303, 306, 315, 321, 327\}$.

D3: $A(p^{k+1}, y) = A(p, x)$ for $k \ge 1$ and $p \in \{11, 17, 29\}$.

D4: $A(11c, y) = A(c, x)$ for $c \in \{22, 33, 44, 55, 66, 77, 88\}$.

D5: $A(17c, y) = A(c, x)$ for $c \in \{34, 51\}$.

D6: $A(2c, y) = A(c, x)$ for $c \in \{16, 48, 144, 368\}$.

D7: $A(6c, y) = A(c, x)$ for $c = 144$.

Now it is natural to ask how the above discoveries could be proved. We will give proofs for some of them.

We note that in this paper (a, b) and $[a, b]$ will denote the greatest common divisor and the least common multiple of the integers a and b, respectively.

Lemma 1: The sequence (u_n) is purely periodic mod 3^r with the exact period length $h(3^r) = 1$ for $r = 1$ and $h(3^r) = 2 \cdot 3^{r-1}$ for $r > 1$. Let w be a fixed integer with $0 \le w < h(3^r)$. If u_w leaves the remainder x mod 3^r $(0 \le x < 3^r)$, then the numbers $u_{w+jh(3^r)}$ $(0 \le j \le 2)$ leave the remainders $x + i \cdot 3^r$ $(0 \le i \le 2)$ mod 3^{r+1} in a certain ordering.

Proof: The fact that (u_n) is purely periodic mod 3^r with period length $h(3^r) = 1$ if $r = 1$ and $h(3^r) = 2 \cdot 3^{r-1}$ if $r > 1$ follows by arguments similar to those given by Wall in Theorems 1, 4, 5, 10, and 12 of [7]. The remainder of Lemma 1 follows from results in the preprint "Bounds for Frequencies of Residues in Second-Order Recurrences Modulo p^r" by Lawrence Somer.

Lemma 2: For $2 \le c \in N$, $(c, 3) = 1$, and $1 \le k \in N$, let $q = h(3^{k+1}c)/3h(3^k c)$. Then $q = 1/3$, $2/3$, or 1 if $k = 1$, and $q = 1/3$ or 1 if $k > 1$.

Proof: Since $(c, 3) = 1$, we have

$$q = \frac{[h(3^{k+1}), h(c)]}{3[h(3^k), h(c)]}.$$

The case $k = 1$ yields

$$q = \frac{[h(9), h(c)]}{3[h(3), h(c)]} = \frac{[6, h(c)]}{3[1, h(c)]} = \frac{2}{(6, h(c))} = \begin{cases} 2 & \text{if } (6, h(c)) = 1, \\ 1 & \text{if } (6, h(c)) = 2, \\ 2/3 & \text{if } (6, h(c)) = 3, \\ 1/3 & \text{if } (6, h(c)) = 6. \end{cases}$$

Now, using the known facts that $h(2) = 3, h(3) = 1, h(6) = 3$, and $h(c)$ is even for $c > 3$ and $c \ne 6$, we obtain $(6, h(c)) = 1$ iff $c = 1$ or $c = 3$, which are excluded in Lemma 2. Moreover, if $(c, 3) = 1$, then $(6, h(c)) = 3$ iff $c = 2$.

In the case $k > 1$, we have by Lemma 1 that

$$q = \frac{[2\cdot 3^k h(3), h(c)]}{3[2\cdot 3^{k-1}h(3), h(c)]} = \frac{[2\cdot 3^k, h(c)]}{3[2\cdot 3^{k-1}, h(c)]}$$

$$= \frac{(2\cdot 3^{k-1}, h(c))}{(2\cdot 3^k, h(c))} = \begin{cases} 1/3 & \text{if } 3^k | h(c), \\ 1 & \text{if } 3^{t-1} | h(c) \text{ and } 3^t \nmid h(c), \text{ where } 1 \le t \le k. \end{cases}$$

For some $1 \le b \in N$, let $v_3(b)$ denote the exact power of 3 such that $3^{v_3(b)} | b$ but $3^{v_3(b)+1} \nmid b$.

Corollary 1: For $2 \le c \in N$, $(c, 3) = 1$, and $1 \le k \in N$, $q = h(3^{k+1}c)/3h(3^k c)$ is an integer iff $v_3[h(c)] \le k - 1$. In this case, the only possible value for q is $q = 1$.

Corollary 2: For $2 \le c = 3^r s \in N$, $r \in N$, $1 \le s \in N$, $(s, 3) = 1$, we have:

$$r = 0 \Rightarrow h(3c) = h(c),$$

$$r = 1 \Rightarrow h(3c) = \begin{cases} 6h(c) & \text{if } s = 1, \\ 3h(c) & \text{if } s > 2 \text{ and } 3 \nmid h(s), \\ 2h(c) & \text{if } s = 2, \\ h(c) & \text{if } s > 2 \text{ and } 3 | h(s). \end{cases}$$

$$r > 1 \Rightarrow h(3c) = \begin{cases} h(c) & \text{if } 3^r | h(s), \\ 3h(c) & \text{otherwise.} \end{cases}$$

Hence, the value of $q = h(3c)/3h(c)$ with $c = 3^r s$, $r \in N$, $1 \le s \in N$, and $(s, 3) = 1$ cannot be an integer if $r = 0$, or if $r = 1$ and $s \ge 2$ and $3 | h(s)$, or if $r > 1$ and $s > 2$ and $3^r | h(s)$. These cases can be omitted from here on.

Corollary 3: For $2 < c = 3^r s \in N$, $1 \le r$, $s \in N$, and $(s, 3) = 1$, $q = h(3c)/3h(c)$ is an integer iff $r \ge 1$ and $v_3[h(s)] \le r - 1$. Now suppose that q is an integer. If $r = 1$ and $s = 1$, then $q = 2$; if $r = 1$, $s > 1$, and $(s, 3) = 1$, then $q = 1$; if $r > 1$, $s \ge 1$, and $(s, 3) = 1$, then again $q = 1$.

Theorem 1: For $2 < c = 3^r s \in N$, $1 \le r$, $s \in N$, $(s, 3) = 1$, $v_3[h(s)] \le r-1$, and $q = h(3c)/3h(c)$, we have $B_{3c} = q(B_c)^3$.

Proof: **Case 1.** $r = 1$

Now $c = 3s$, $(s, 3) = 1$, and $v_3[h(s)] \le 0 \Rightarrow 3 \nmid h(s)$.

If $s = 1$, then $q = h(9)/3h(3) = 2$. Thus, $B_{3^2} = 2(B_3)^3$ can be checked by computation.

If $s > 1$, then $q = h(3c)/3h(c) = 1$. Thus, we need to prove that $B_{3^2 s} = (B_{3s})^3$.

Since $h(3c) = 3h(c)$ and $h(3s) = h(s)$, we need to show that, for any $w \in N$ and $j \in \{0,1,2\}$, the three values of $u_{w+jh(s)}$ are pairwise different modulo 9, and hence also modulo $9s$. Let $j_1, j_2 \in \{0,1,2\}$ with $1 \le |j_1 - j_2| \le 2$. For a fixed $w \in N$, let z_1 and z_2 be the residues of the numbers $w + j_1 h(s)$ and $w + j_2 h(s)$ mod $h(9)$, respectively. This means $0 \le |z_1 - z_2| < h(9) = 6$. The consequence of $s > 1$ and $3 \nmid h(s)$ is that $s \ge 7$; therefore, $h(s)$ is even. This yields

$$2 \le h(s) \le h(s)|j_1 - j_2| = |z_1 - z_2| \not\equiv 0 \pmod 6,$$

so that z_1 and z_2 are different mod 6 and, in addition, are not consecutive numbers; whence, u_{z_1} (mod 9) and u_{z_2} (mod 9) also have two different values that can be checked using the following table:

n	0	1	2	3	4	5	6	$\cdots$
u_n (mod 9)	1	1	4	4	7	7	1	$\cdots$

Case 2. $r > 1$

Now $c = 3^r s$, $(s, 3) = 1$, and $v_3[h(s)] \le r-1 \Rightarrow q = h(3c)/3h(c) = 1$. Thus, we must prove that $B_{3c} = (B_c)^3$.

We need to show that, for any fixed $w \in N$ and $j \in \{0,1,2\}$, the numbers $u_{w+jh(c)}$ are pairwise different modulo $3c$. Since $(s, 3) = 1$ and $v_3[h(s)] \le r-1$, we have $h(c) = h(3^r s) = [h(3^r), h(s)] = h(3^r)z$ with some $1 \le z \in n$ and $3 \nmid z$. Hence, for a fixed $w \in N$ and $j \in \{0,1,2\}$, the numbers $w + jh(c)$ and $w + jh(3^r)$ are always in the same residue class modulo $h(3^r)$. Therefore, the numbers $u_{w+jh(c)}$ and $u_{w+jh(3^r)}$ are also in the same residue class modulo 3^r. But the numbers $u_{w+jh(3^r)}$ are pairwise different mod 3^{r+1} because of Lemma 1. Thus, the numbers $u_{w+jh(c)}$ are again pairwise different mod 3^{r+1}, and thereby also mod $3c$.

Theorem 2: For $1 \le k \in N$ and $q = h(3^{k+1})/3h(3^k)$, we have $B_{3^{k+1}} = q(B_{3^k})^3$.

Proof: We proceed by induction on k. For $k = 1$, we go back to Case 1 of Theorem 1, whence $q = 2$ and $B_{3^2} = 2(B_3)^3$.

Assume the statement is true for $k > 1$. As a consequence of Case 2 of Theorem 1, we can take $q = 1$. Thus, $B_{3^{(k+1)+1}} = B_{3(3^{k+1})} = 1 \cdot (B_{3^{k+1}})^3 = (q(B_{3^k})^3)^3 = q((B_{3^k})^3)^3 = q(B_{3^{k+1}})^3$.

Corollary 4: For $1 \le k \in N$ and $q = h(3^{k+1})/3h(3^k)$, we have $B_{3^{k+1}} = 2(B_3)^{3^k}$.

Proof: $k = 1 \Rightarrow q = 2$ and $B_{3^2} = 2(B_3)^3$. $k > 1 \Rightarrow q = 1$ and $B_{3^{k+1}} = (B_{3^k})^3 = (B_{3(3^{k-1})})^3 = ((B_{3^{k-1}})^3)^3 = (B_{3^{k-1}})^{3^2} = \cdots = (B_{3^2})^{3^{k-1}} = (2(B_3)^3)^{3^{k-1}} = 2(B_3)^{3^k}$.

Corollary 5: For any $1 \le k \in N$, we have $|S(3^k)| = |S(3)| = 2$.

Thus, we have a complete proof for D1. The statement in D2 is a direct consequence of Theorem 1. The proof of D7 can be done using D2 and D6 as follows:

$$B_{6c} = B_{3(2c)} = (B_{2c})^3 = ((B_c)^2)^3 = (B_c)^6.$$

The proofs of the other discoveries can, for the most part, be carried out in a similar manner, so they are left to the interested reader.

The only reason for considering the above specific problem was Corollary 2 in [1], where it was proved that the sequences $u(3, 1)$ with $u_0 = 1$ and $u_1 \in \{1,3,5\}$ are uniformly distributed mod 13^k for all $k \ge 1$. The reader should consider the related more general sequences $u(p, 1)$ satisfying the recursion relation $u_n = pu_{n-1} + u_{n-2}$ for $n \ge 2$ with $u_0 = u_1 = 1$ and p a fixed odd prime. It can be proved by similar methods that $B_{p^{k+1}} = 2(B_p)^{p^k}$ is also valid for these recurrences; here, B_p refers to the frequency block defined above. The reader might consider proving this result, and possibly other results similar to those found in this paper. In the meantime, it is advisable to remember the fundamental fact that the recurrences $u(p, 1)$ with $u_0 = u_1 = 1$ are irregular modulo p, that is, the vectors (u_0, u_1) and (u_1, u_2) are linearly dependent modulo p.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Proposers should inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-808 ***Proposed by Paul S. Bruckman, Jalmiya, Kuwait***

Years after Mr. Feta's demise at Bellevue Sanitarium, a chance inspection of his personal effects led to the discovery of the following note, scribbled in the margin of a well-worn copy of Professor E. P. Umbugio's "22 / 7 Calculated to One Million Decimal Places":

> To divide "n-choose-one" into two other non-trivial "choose one's", "n-choose-two", or in general, "n-choose-m" into two non-trivial "choose-m's", for any natural m is always possible, and I have assuredly found for this a truly wonderful proof, but the margin is too narrow to contain it.

Because of the importance of this result, it has come to be known as Mr. Feta's Lost Theorem. We may restate it in the following form:

Solve the Diophantine equation $x^{\underline{m}} + y^{\underline{m}} = z^{\underline{m}}$ for $m \le x \le y \le z$, $m = 1, 2, 3, \ldots$, where $X^{\underline{m}} = X(X-1)(X-2)\cdots(X-m+1)$. Was Mr. Feta crazy?

B-809 ***Proposed by Pentti Haukkanen, University of Tampere, Finland***

Let k be a positive integer. Find a recurrence consisting of positive integers such that each positive integer occurs exactly k times.

B-810 ***Proposed by Herta Freitag, Roanoke, VA***

Let $\langle H_n \rangle$ be a generalized Fibonacci sequence defined by $H_{n+2} = H_{n+1} + H_n$ for $n > 0$ with initial conditions $H_1 = a$ and $H_2 = b$, where a and b are integers. Let k be a positive integer. Show that

$$\begin{vmatrix} H_n & H_{n+1} \\ H_{n+k+1} & H_{n+k+2} \end{vmatrix}$$

is always divisible by a Fibonacci number.

B-811 ***Proposed by Russell Euler, Maryville, MO***

Let n be a positive integer. Show that:

(a) if $n \equiv 0 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots - L_2 + 1$;
(b) if $n \equiv 1 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots - L_3 + 1$;
(c) if $n \equiv 2 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots + L_2 - 1$;
(d) if $n \equiv 3 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots + L_3 - 1$.

B-812 ***Proposed by John C. Turner, University of Waikato, Hamilton, New Zealand***

Let P, Q, and R be three points in space with coordinates $(F_{n-1}, 0, 0)$, $(0, F_n, 0)$, and $(0, 0, F_{n+1})$, respectively. Prove that twice the area of ΔPQR is an integer.

B-813 ***Proposed by Peter Jeuck, Mahwah, NJ***

Let $\langle X_n\rangle$, $\langle Y_n\rangle$, and $\langle Z_n\rangle$ be three sequences that each satisfy the recurrence $W_n = pW_{n-1} + qW_{n-2}$ for $n > 1$, where p and q are fixed integers. (The initial conditions need not be the same for the three sequences.) Let a, b, and c be any three positive integers. Prove that

$$\begin{vmatrix} X_a & X_b & X_c \\ Y_a & Y_b & Y_c \\ Z_a & Z_b & Z_c \end{vmatrix} = 0.$$

SOLUTIONS

A Floored Sum

B-781 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 33, no. 1, February 1995)

Let $F(j) = F_j$. Find a closed form for $\sum_{k=0}^{n} F\left(k - \lfloor \sqrt{k} \rfloor^2\right)$.

Solution by Graham Lord, Princeton, NJ

If we write $k = m^2 + \ell$, where $\ell = 0, 1, 2, \ldots, 2m$, then $k - (\lfloor \sqrt{k} \rfloor)^2 = \ell$. This follows from the fact that $m^2 \le k < (m+1)^2$ or $m \le \sqrt{k} < m+1$, so that $m = \lfloor \sqrt{k} \rfloor$.

Hence, if $n = N^2 + P$, where $0 \le P \le 2N$, then

$$\begin{aligned}\sum_{k=0}^{n} F\left(k - \lfloor \sqrt{k} \rfloor^2\right) &= F_0 + (F_0 + F_1 + F_2) + (F_0 + F_1 + F_2 + F_3 + F_4) + \cdots \\ &\quad + (F_0 + F_1 + \cdots + F_{2N-2}) + (F_0 + F_1 + \cdots + F_P) \\ &= (F_2 - 1) + (F_4 - 1) + (F_6 - 1) + \cdots + (F_{2N} - 1) + (F_{P+2} - 1) \\ &= (F_0 + F_1) + (F_2 + F_3) + (F_4 + F_5) + \cdots + (F_{2N-2} + F_{2N-1}) + F_{P+2} - N\end{aligned}$$

$$= F_{2N+1} + F_{P+2} - N - 2$$
$$= F(2\lfloor\sqrt{n}\rfloor + 1) + F(n - \lfloor\sqrt{n}\rfloor^2 + 2) - \lfloor\sqrt{n}\rfloor - 2.$$

Above, we have made repeated use of the identity $F_1 + F_2 + \cdots + F_i = F_{i+2} - 1$, which is Identity ($I_1$) from [1].

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

The proposer generalized to Lucas numbers. He showed that, if $L(j) = L_j$, *then*

$$\textstyle\sum_{k=0}^{n} L(k - \lfloor\sqrt{k}\rfloor^2) = L(2\lfloor\sqrt{n}\rfloor + 1) + L(n - \lfloor\sqrt{n}\rfloor^2 + 2) - \lfloor\sqrt{n}\rfloor - 2.$$

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Carl Libis, Igor O. Popov, David Zeitlin, and the proposer.

Sum of Three Squares

B-782 ***Proposed by László Cseh, Stuttgart, Germany, & Imre Merény, Budapest, Hungary (Vol. 33, no. 1, February 1995)***

Express $(F_{n+h}^2 + F_n^2 + F_h^2)(F_{n+h+k}^2 + F_{n+k}^2 + F_k^2)$ as the sum of three squares.

Solution by H.-J. Seiffert, Berlin, Germany

An easy calculation shows that, for all numbers a_1, a_2, a_3, b_1, b_2, and b_3,

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) = (a_1b_1 + a_2b_2 + a_3b_3)^2 + (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2.$$

(This is known as Lagrange's Identity. –Ed.) Now take $a_1 = F_{n+h}$, $a_2 = F_n$, $a_3 = F_h$, $b_1 = F_{n+k}$, $b_2 = -F_{n+h+k}$, and $b_3 = (-1)^{n+1}F_k$. Since (see [1], formula 3.32, or [2], formula 20a)

$$F_{n+h}F_{n+k} - F_nF_{n+h+k} = (-1)^n F_hF_k,$$

we have $a_1b_1 + a_2b_2 + a_3b_3 = 0$. Thus, by Lagrange's Identity, we find that

$$(F_{n+h}^2 + F_n^2 + F_h^2)(F_{n+h+k}^2 + F_{n+k}^2 + F_k^2)$$
$$= (F_nF_{n+k} + F_{n+h}F_{n+h+k})^2 + (F_hF_{n+h+k} - (-1)^n F_nF_k)^2 + (F_hF_{n+k} + (-1)^n F_{n+h}F_k)^2.$$

References

1. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.
2. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester, England: Ellis Horwood Ltd., 1989.

Zeitlin found the general identity

$$(F_{n+h}^2 + H_n^2 + H_h^2)(F_{n+h+k}^2 + H_{n+k}^2 + F_k^2)$$
$$= (H_nH_{n+k} + F_{n+h}F_{n+h+k})^2 + (H_hF_{n+h+k} - (-1)^n H_nF_k)^2 + (H_hF_{n+k} + (-1)^n F_{n+h}F_k)^2$$

where $\langle H_n \rangle$ is any sequence satisfying $H_{n+2} = H_{n+1} + H_n$. No solver gave a general procedure for writing a Fibonacci expression as a sum of squares.

Also solved by Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, Norbert Jensen, David Zeitlin, and the proposers.

Crazed Rational Functions

B-783 ***Proposed by David Zeitlin, Minneapolis, MN (Vol. 33, no. 1, February 1995)***

Find a rational function $P(x, y)$ such that

$$P(F_n, F_{2n}) = \frac{105n^5 - 1365n^3 + 1764n}{25n^6 + 175n^4 - 5600n^2 + 5904}$$

for $n = 0, 1, 2, 3, 4, 5, 6$.

Solution by C. Georghiou, University of Patras, Greece

The values taken by the given expression when $n = 0, 1, 2, \ldots, 6$ are, respectively, 0, 1, 1/3, 1/2, 3/7, 5/11, and 4/9; or equivalently: 0, 1/1, 1/3, 2/4, 3/7, 5/11, and 8/18; i.e., F_n / L_n. Since $F_{2n} = F_n L_n$, it follows that $P(x, y) = x^2 / y$.

Several solvers pointed out that, in this solution, we have to handwave the case when $n = 0$ because x^2 / y is not defined at $(0, 0)$. The editor was hoping that some solver would find a function such as

$$P(x, y) = \frac{y^2 + 6054y + 5850x}{30079y - 35364x^2 + 4026x + 13164},$$

but no solver came up with such a crazed function.

Also solved by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Jay Hendel, Norbert Jensen, Joseph J. Koštál, Bob Prielipp, H.-J. Seiffert, and the proposer.

Lucas in Disguise

B-784 ***Proposed by Herta Freitag, Roanoke, VA (Vol. 33, no. 2, May 1995)***

Show that, for all n, $\alpha^{n-1}\sqrt{5} - L_{n-1} / \alpha$ is a Lucas number.

Solution by Thomas Leong, Staten Island, NY

Since $\beta = -1 / \alpha$ and $\sqrt{5} - 1 / \alpha = \sqrt{5} + \beta = \alpha$, we have

$$\begin{aligned}\alpha^{n-1}\sqrt{5} - L_{n-1} / \alpha &= \alpha^{n-1}\sqrt{5} - (\alpha^{n-1} + \beta^{n-1}) / \alpha \\ &= \alpha^{n-1}(\sqrt{5} - 1 / \alpha) - \beta^{n-1} / \alpha \\ &= \alpha^n + \beta^n = L_n.\end{aligned}$$

Haukkanen found the formulas $\alpha^{n-1} - F_{n-1} / \alpha = F_n$, $\beta^{n-1} - F_{n-1} / \beta = F_n$, $-\beta^{n-1}\sqrt{5} - L_{n-1} / \beta = L_n$. Redmond generalized to an arbitrary second-order linear recurrence.

Also solved by Michel Ballieu, Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Andrej Dujella, Russell Euler, Peter Gilbert, Pentti Haukkanen, Norbert Jensen, Joseph J. Koštál, Can. A. Minh, Bob Prielipp, Don Redmond, H.-J. Seiffert, Tony Shannon, Sahib Singh, Lawrence Somer, M. N. S. Swamy, and the proposer.

It's a Multiple of $a_n a_{n+1}$

B-785 *Proposed by Jane E. Friedman, University of San Diego, CA (Vol. 33, no. 2, May 1995)*

Let $a_0 = a_1 = 1$ and let $a_n = 5a_{n-1} - a_{n-2}$ for $n \geq 2$. Prove that $a_{n+1}^2 + a_n^2 + 3$ is a multiple of $a_n a_{n+1}$ for all $n \geq 1$.

Solution by Andrej Dujella, University of Zagreb, Croatia

We will prove by induction that

$$a_{n+1}^2 + a_n^2 + 3 = 5a_n a_{n+1} \quad \text{for all } n \geq 1.$$

For $n = 1$, we have: $16+1+3 = 5 \cdot 1 \cdot 4$. Let us suppose that the assumption holds for some positive integer n. Then,

$$\begin{aligned} a_{n+2}^2 + a_{n+1}^2 + 3 &= (5a_{n+1} - a_n)^2 + a_{n+1}^2 + 3 \\ &= 25a_{n+1}^2 - 10a_n a_{n+1} + a_n^2 + a_{n+1}^2 + 3 \\ &= 25a_{n+1}^2 - 10a_n a_{n+1} + 5a_n a_{n+1} \\ &= 5a_{n+1}(5a_{n+1} - a_n) \\ &= 5a_{n+1}a_{n+2}. \end{aligned}$$

Gilbert found that if $a_n = ka_{n-1} - a_{n-2}$, then $a_{n+1}^2 + a_n^2 + [ka_0a_1 - a_0^2 - a_1^2] = ka_na_{n+1}$. Redmond found that if $a_n = ka_{n-1} - ra_{n-2}$, then $a_{n+1}^2 + ra_n^2 + [ka_0a_1 - ra_0^2 - a_1^2]r^n = ka_na_{n+1}$.

Also solved by Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Russell Euler, Herta T. Freitag, Peter Gilbert, Norbert Jensen, Thomas Leong, Bob Prielipp, Don Redmond, H.-J. Seiffert, Tony Shannon, Sahib Singh, Lawrence Somer, M. N. S. Swamy, and the proposer.

Finding Coefficients of an Identity

B-786 *Proposed by Jayantibhai M. Patel, Bhavan's R. A. Col. of Sci., Gujarat State, India (Vol. 33, no. 2, May 1995)*

If $F_{n+2k}^2 = aF_{n+2}^2 + bF_n^2 + c(-1)^n$, where a, b, and c depend only on k but not on n, find a, b, and c.

Solution 1 by Paul S. Bruckman, Jalmiya, Kuwait

Given that the indicated relation must be valid for all n, set $n = -2$, -1, and 0, respectively. This yields the following three equations:

$$\begin{aligned} b + c &= F_{2k-1}^2; \\ a + b - c &= F_{2k-1}^2; \\ a + c &= F_{2k}^2. \end{aligned}$$

Solving for the three unknowns a, b, and c and simplifying yields

$$a = F_{4k}/3, \quad b = -F_{4k-4}/3, \quad \text{and} \quad c = 2F_{2k}F_{2k-2}/3.$$

It is a trite but straightforward exercise to verify that these values do indeed make the given equation an identity, as claimed.

Solution 2 by Stanley Rabinowitz, Mathpro Press, Westford, MA

All problems of this nature can be solved by the following method. We want to find when the expression $E = F_{n+2k}^2 - aF_{n+2}^2 - bF_n^2 - c(-1)^n$ is identically 0 in the variable n. First, apply the reduction formula $F_{n+x} = (F_nL_x + L_nF_x)/2$ to isolate n in subscripts. Then apply the formula $F_n^2 = (L_n^2 - 4(-1)^n)/5$ to remove any powers of F_n. The result is

$$20E = [36a + 16b - 20c - 4L_{2k}^2](-1)^n + [-30a + 10F_{2k}L_{2k}]F_nL_n + [-14a - 4b + 5F_{2k}^2 + L_{2k}^2]L_n^2.$$

This is known as the canonical form of the expression (considering n a variable and k a constant). There is a theorem that says that a polynomial expression in F_n and L_n is identically 0 if and only if its canonical form is 0. (For more details, see my paper "Algorithmic Manipulation of Fibonacci Identities" in *Proceedings of the Sixth International Conference on Fibonacci Numbers and Their Applications.*) Thus, E will be identically 0 if and only if each of the above coefficients in square brackets is 0. That is, if and only if

$$\begin{aligned} 36a + 16b - 20c - 4L_{2k}^2 &= 0, \\ -30a + 10F_{2k}L_{2k} &= 0, \\ -14a - 4b + 5F_{2k}^2 + L_{2k}^2 &= 0. \end{aligned}$$

Solving these equations simultaneously for the unknowns a, b, and c in terms of the constant k shows that E is 0 (identically in n) if and only if

$$a = \frac{F_{2k}L_{2k}}{3}, \quad b = \frac{5F_{2k}^2}{4} - \frac{7F_{2k}L_{2k}}{6} + \frac{L_{2k}^2}{4}, \quad \text{and} \quad c = F_{2k}^2 - \frac{F_{2k}L_{2k}}{3}.$$

Also solved by Brian D. Beasley, Andrej Dujella, Peter Gilbert, Norbert Jensen, Can. A. Minh, Don Redmond, H.-J. Seiffert, Tony Shannon, Sahib Singh, and the proposer.

Addenda: Igor O. Popov was inadvertently omitted as a solver of Problems B-779 and B-780. C. Georghiou was inadvertently omitted as a solver of Problems B-760, B-761, B-763, and B-765.

Dr. Dresel has informed us of the passing away of Steven Vajda, whose book, *Fibonacci & Lucas Numbers, and the Golden Section*, is often quoted in this column. Dr. Vajda was born in Budapest on August 20, 1901, and died in Sussex on December 10, 1995. An obituary can be found in *The Times* (London, January 3, 1996).

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-509 ***Proposed by Paul S. Bruckman, Salmiya, Kuwait***

The *continued fractions (base k)* are defined as follows:

$$[u_1, u_2, \ldots, u_n]_k = u_1 + \frac{k}{u_{2^+}} \frac{k}{u_{3^+}} \cdots \frac{k}{u_n}, \quad n = 1, 2, \ldots, \tag{1}$$

where k is an integer $\neq 0$ and $(u_i)_{i=1}^{\infty}$ is an arbitrary sequence of real numbers.

Given a prime p with $\left(\frac{-k}{p}\right) = 1$ (Legendre symbol) and $k \not\equiv 0 \pmod p$, let h be the solution of the congruence

$$h^2 \equiv -k \pmod p, \quad \text{with } 0 < h < \tfrac{1}{2}p. \tag{2}$$

Suppose a symmetric continued fraction (base k) exists, such that

$$\frac{p}{h} = [a_1, a_2, \ldots, a_{n+1}, a_{n+1}, \ldots, a_1]_k, \tag{3}$$

where the a_i's are integers, n is even, and $k \mid a_i, i = 2, 4, \ldots, n$. Show that the integers x and y exist, with g.c,d.$(x, y) = 1$, given by

$$\frac{x}{y} = [a_{n+1}, \ldots, a_1]_k \tag{4}$$

which satisfy

$$x^2 + ky^2 = p. \tag{5}$$

H-510 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the Pell numbers by $P_0 = 0$, $P_1 = 1$, $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Show that, for $n = 1, 2, \ldots,$

$$P_n = \sum_{k \in A_n} (-1)^{[(3k-2n-1)/4]} 2^{[3k/2]} \binom{n+k}{2k+1},$$

where [ ] denotes the greatest integer function and

$$A_n = \{k \in \{0, 1, \ldots, n-1\} \mid 3k \not\equiv 2n \pmod 4\}.$$

H-511 ***Proposed by M. N. Deshpande, Aurangabad, India***

Find all possible pairs of positive integers m and n such that $m(m+1)=n(m+n)$. [Two such pairs are: $m=1, n=1$; and $m=9, n=6$.]

H-512 ***Proposed by Paul S. Bruckman, Salmiya, Kuwait***

The *Fibonacci pseudoprimes* (or FPP's) are those composite n with g.c.d.$(n, 10)=1$ such that $n \mid F_{n-\varepsilon_n}$ where ε_n is the Jacobi symbol $\left(\frac{5}{n}\right)$. Suppose $n=p(p+2)$, where p and $p+2$ are "twin primes." Prove that n is a FPP if and only if $p \equiv 7 \pmod{10}$.

H-508 (Corrected) ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n \geq 2$. Show that, for all complex numbers x and y and all positive integers n,

$$F_n(x)F_n(y)=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}(x+y)^k F_{k+1}\left(\frac{xy-4}{x+y}\right). \tag{1}$$

As special cases of (1), obtain the following identities:

$$F_n(x)F_n(x+1)=n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}F_{k+1}(x^2+x+4); \tag{2}$$

$$F_n(x)F_n(4/x)=n\sum_{k=0}^{[\frac{n-1}{2}]}\frac{1}{2k+1}\binom{n+2k}{4k+1}\left(\frac{x^2+4}{x}\right)^{2k}, \quad x \neq 0; \tag{3}$$

$$F_n(x)^2=n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}(x^2+4)^k; \tag{4}$$

$$F_n(x)^2=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}\frac{x^{2k+2}-(-4)^{k+1}}{x^2+4}; \tag{5}$$

$$F_{2n-1}(x)=(2n-1)\sum_{k=0}^{2n-2}\frac{(-1)^2}{k+1}\binom{2n+k-1}{2k+1}x^k F_{k+1}(4/x). \tag{6}$$

SOLUTIONS

Probably

H-493 ***Proposed by Stefano Mascella and Piero Filipponi, Rome, Italy***
(Vol. 33, no. 1, February 1995)

Let $P_k(d)$ denote the probability that the k^{th} digit (from left) of an ℓ digit ($\ell \geq k$) Fibonacci number F_n (expressed in base 10) whose subscript is randomly chosen within a *large* interval $[n_1, n_2]$ $(n_1 \gg n_2)$ is d.

That the sequence $\{F_n\}$ obeys Benford's law is a well-known fact (e.g., see [1] and [2]). In other words, it is well known that $P_1(d)=\log_{10}(1+1/d)$.

Find an expression for $P_2(d)$.

References

1. P. Filipponi. "Some Probabilistic Aspects of the Terminal Digits of Fibonacci Numbers." *The Fibonacci Quarterly* (to appear).
2. L. C. Washington. "Benford's Law for Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **19.2** (1981):175-77.

Solution by Norbert Jensen, Kiel, Germany

Let $d \in \{0, 1, \ldots, 9\}$. For each $i \in \mathbb{N}$, let $A_{i,d}$ be the set of those $n \in \mathbb{N}$ for which $F_n \geq 10^{i-1}$ and the i^{th} digit (from the left) of F_n equals d. For all $n_1, n_2 \in \mathbb{N}$ with $n_1 \leq n_2$, let $I(n_1, n_2)$ denote the set of all integers n with $n_1 \leq n \leq n_2$. Let $p := \log_{10}((1+1/(1+d\cdot 10^{-1}))(1+1/(2+d\cdot 10^{-1}))\ldots(1+1/(9+d\cdot 10^{-1})))$.

Let $n_1 \in \mathbb{N}$. We show that

$$\frac{|A_{2\cdot d} \cap I(n_1, n_2)|}{|I(n_1, n_2)|} \to p \text{ as } n_2 \text{ tends to infinity.}$$

This proves that $P_2(d)$ is approximately equal to p for a given interval $I(n_1, n_2)$, provided that n_2 is large enough.

[Note that, in general, it is **not** true that $P_2(d) = p$ for all $d \in \{0, 1, \ldots, 9\}$ for a finite interval $I(n_1, n_2)$ with a certain minimum of members. If we had one, we could add $n_2 + 1$ to it. Suppose, without loss of generality, that the second digit of F_{n_2+1} is $\neq d$. Then

$$|A_{2\cdot d} \cap I(n_1, n_2+1)| < p|I(n_1, n_2+1)|.$$

A similar argument applies to $P_1(d)$ and $\log_{10}(1+1/d)$.]

Proof: Step (0). $\log_{10}(\alpha)$ is irrational.

Proof: Suppose it is rational. Then we find $a \in \mathbb{Z}, b \in \mathbb{N}$ such that $\log_{10}(\alpha) = a/b$. Hence, $\log_{10}(\alpha^b) = b\cdot\log_{10}(\alpha) = a$ and $F_b\alpha + F_{b-1} = \alpha^b = 10^a$, whence $\sqrt{5} \in \mathbb{Q}$, a contradiction. Q.E.D. Step (0).

Step (1). $\log_{10}(F_n) = n\cdot\log_{10}(\alpha) + \log_{10}(1-(-1)^n\beta^{2n}) - \log_{10}(\sqrt{5})$ for all $n \in \mathbb{N}$.

Proof:

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} = \alpha^n(1-(\beta/\alpha)^n)/\sqrt{5} = \alpha^n(1-(-\beta^2)^n)/\sqrt{5} = \alpha^n(1-(-1)^n\beta^{2n})/\sqrt{5}.$$

Q.E.D. Step (1).

For any $x \in \mathbb{R}$, let $\langle x\rangle$ denote the purely fractional part of x, i.e., $\langle x\rangle = x - [x]$.

Step (2). The sequence $(\langle\log_{10}(F_n)\rangle)$ is uniformly distributed modulo 1.

Proof: By (0) and according to Example 2.1 on page 8 of [1], the sequence $(\langle n\log_{10}(\alpha)\rangle)$ is uniformly distributed modulo 1. Since $\log_{10}(1-(-1)^n\beta^{2n})$ converges (to zero), the sequence $(\langle n\log_{10}(\alpha) + \log_{10}(1-(-1)^n\beta^{2n}) - \log_{10}\sqrt{5}\rangle)$ is uniformly distributed (see [1], Theorem 1.2, p. 3). Thus, $(\langle\log_{10}(F_n)\rangle)$ is uniformly distributed modulo 1 by Step (1). Q.E.D. Step (2).

Step (3). Let $Z_1 \in \{1, 2, \ldots, 9\}$, $Z_2 \in \{0, 1, \ldots, 9\}$. Let $n \in \mathbb{N}$. Let $t = [\log_{10}(F_n)]$. We have the following equivalences:

$\Leftrightarrow$ There is an $R \in \mathbb{N}_0$ with $R < 10^{t-1}$ such that $F_n = Z_1 \cdot 10^t + Z_2 \cdot 10^{t-1} + R$.

$\Leftrightarrow$ There is an $R \in \mathbb{N}_0$ with $R < 10^{t-1}$ such that

$$\langle \log_{10}(F_n) \rangle = \log_{10}(F_n) - [\log_{10}(F_n)] = \log_{10}(Z_1 + Z_2 \cdot 10^{-1} + R \cdot 10^{-t}).$$

$\Leftrightarrow$ $\langle \log_{10}(F_n) \rangle \in [\log_{10}(Z_1 + Z_2 \cdot 10^{-1}), \log_{10}(Z_1 + (Z_2 + 1) \cdot 10^{-1})]$.

So, by the definition of "uniform distribution" ([1], p. 1), we have that

$$\frac{|A_{1,Z_1} \cap A_{2,Z_2} \cap I(n_1, n_2)|}{|I(n_1, n_2)|}$$

converges to the length of the interval $[\log_{10}(Z_1 + Z_2 \cdot 10^{-1}), \log_{10}(Z_1 + (Z_2 + 1) \cdot 10^{-1})]$, namely, to $\log_{10}(1 + 1/(Z_1 + Z_2 \cdot 10^{-1}))$, when n_2 tends to infinity. Since the intervals are disjoint for different pairs of digits (Z_1, Z_2), it is clear that we can fix $Z_2 = d$ and take the sum over $Z_1 = 1, 2, \ldots, 9$. Q.E.D.

Remarks:

1. The above proof can be abridged by using Washington's theorem [2] for the base $b = 10^2$.

2. We even have the following more general result: For each $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that, for all $n_1 \in \mathbb{N}$ and all $n_2 \in \mathbb{N}$ with $n_2 \geq n_1 + n_0$, we have

$$\left| \frac{|A_{2,d} \cap I(n_1, n_2)|}{|I(n_1, n_2)|} - p \right| < \varepsilon.$$

In other words: We have uniform convergence. The quality of the approximation depends only on the cardinality of $|I(n_1, n_2)|$, not on the choice of n_1.

Proof of Remark 2: By Weyl's criterion, the sequence $(\langle n \log_{10}(\alpha) \rangle)$ is well distributed modulo 1 (see [1], p. 40, p. 42, Example 5.2). This implies that $(\langle \log_{10}(F_n) \rangle)$ is well distributed (see [1], Theorem 5.4, p. 43). Modifying the arguments of (3) with respect to n_1, we obtain the assertion. Q.E.D.

References

1. L. Kuipers & H. Niederreiter. *Uniform Distribution of Sequences*. New York, 1974.
2. L. C. Washington. "Benford's Law for Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **19.2** (1981).

Also solved by P. Bruckman.

Apparently

H-494 ***Proposed by David M. Bloom, Brooklyn College, New York, NY (Vol. 33, no. 1, February 1995)***

It is well known that if $P(p)$ is the Fibonacci entry point ("rank of apparition") of the odd prime $p \neq 5$, then $P(p)$ divides $p + e$ where $e = \pm 1$. In [1] it is stated without proof [Theorem 5(b)] that the integer $(p+e)/P(p)$ has the same parity as $(p-1)/2$. Give a proof.

Reference

1. D. Bloom. "On Periodicity in Generalized Fibonacci Sequences." *Amer. Math. Monthly* **72** (1965):856-61.

Solution by H.-J. Seiffert, Berlin, Germany

It is well known that $\varepsilon = -(5/p)$, where $(5/p)$ denotes Legendre's symbol. In 1930, D. H. Lehmer (see [1], p. 325, Lemma 5) proved that

$$p \mid F_{(p+\varepsilon)/2} \text{ if and only if } p \equiv 1 \pmod 4. \tag{1}$$

Let $k = (p+\varepsilon)/P(p)$. If k is even, then $p \mid F_{(k/2)P(p)} = F_{(p+\varepsilon)/2}$, since $P(p) \mid (k/2)P(p)$ and $p \mid F_{P(p)}$. Thus, we have $p \equiv 1 \pmod 4$, by (1), so that $k \equiv 0 \equiv (p-1)/2 \pmod 2$. Now, suppose that k is odd. Assuming that $p \equiv 1 \pmod 4$, we would have $p \mid F_{(p+\varepsilon)/2} = F_{kP(p)/2}$, again by (1). This would imply that $P(p)$ is even, that $k \geq 3$, and that $p \mid L_{P(p)/2}$, since p divides $F_{P(p)} = F_{P(p)/2}L_{P(p)/2}$, but does not divide $F_{P(p)/2}$. Now, from

$$F_{kP(p)/2} = L_{P(p)/2}F_{(k-1)P(p)/2} - (-1)^{P(p)/2}F_{(k-2)P(p)/2},$$

it then would follow that $p \mid F_{(k-2)P(p)/2}$. Repeating this argument, we would arrive at the contradiction that $p \mid F_{P(p)/2}$. Thus, we must have $p \equiv 3 \pmod 4$, so that $k \equiv 1 \equiv (p-1)/2 \pmod 2$. This completes the solution.

Reference

1. Lawrence Somer. "The Divisibility Properties of Primary Lucas Recurrences with Respect to Primes." *The Fibonacci Quarterly* **18.4** (1980):316-34.

Also solved by P. Bruckman, A. Dujella, N. Jensen, and the proposer.

Achieve Parity

<u>H-495</u> ***Proposed by Paul S. Bruckman, Salmiya, Kuwait***
(Vol. 33, no. 1, February 1995)

Let p be a prime $\neq 2, 5$ and let $Z(p)$ denote the *Fibonacci entry-point* of p (i.e., the smallest positive integer m such that $p \mid F_m$). Prove the following "Parity Theorem" for the Fibonacci entry-point:

A. If $p \equiv 11$ or $19 \pmod{20}$, then $Z(p) \equiv 2 \pmod 4$;
B. If $p \equiv 13$ or $17 \pmod{20}$, then $Z(p)$ is odd;
C. If $p \equiv 3$ or $7 \pmod{20}$, then $4 \mid Z(p)$.

Solution by the proposer

We employ two well-known results, stated as lemmas without proof.

Lemma 1: If $p \neq 2, 5$ and $p' = \frac{1}{2}\left(p - \left(\frac{5}{p}\right)\right)$, then (i) $p \mid F_{p'}$ if $p \equiv 1 \pmod 4$, or (ii) $p \mid L_{p'}$ if $p \equiv -1 \pmod 4$.

An equivalent formulation of Lemma 1 is restated as

Lemma 1': If $p \neq 2, 5$ and $q = \frac{1}{2}(p-1)$, then (i) $p \mid F_q$ if $p \equiv 1$ or $9 \pmod{20}$; (ii) $p \mid L_q$ if $p \equiv 11$ or $19 \pmod{20}$; (iii) $p \mid F_{q+1}$ if $p \equiv 13$ or $17 \pmod{20}$; (iv) $p \mid L_{q+1}$ if $p \equiv 3$ or $7 \pmod{20}$.

Lemma 2: $Z(p)$ is even for all primes $p > 2$ if and only if $p \mid L_n$ for some n.

Lemma 2 implies that if $p > 2$ and $p \mid L_n$, then $Z(p) = 2n/r$ for some odd integer r dividing n.

Proof of A: By Lemma 1′(ii), $p|L_q$. Then $Z(p)|2q$ and $Z(p)$ must be even, by Lemma 2. Since $2q = p-1 \equiv 2 \pmod 4$ in this case, it follows that $Z(p) \equiv 2 \pmod 4$.
Proof of B: By Lemma 1′(iii), $p|F_{q+1}$. Then $Z(p)|(q+1)$. In this case, $q+1 = \frac{1}{2}(p+1) \equiv 7$ or $9 \pmod{10}$, an odd integer. Therefore, $Z(p)$ must be odd.
Proof of C: By Lemma 1′(iv), $p|L_{q+1}$. Then $Z(p) = 2(q+1)/r = (p+1)/r$, where r is odd, and $r|(p+1)$. Since $p+1 \equiv 0 \pmod 4$ in this case, we see that $4|Z(p)$.

Note: No inference may be made about the parity of $Z(p)$ if $p \equiv 1$ or $9 \pmod{20}$.

Also solved by D. Bloom, A. Dujella, N. Jensen, and H.-J. Seiffert.

FLUPPS and ELUPPS

H-496 ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 33, no. 2, May 1995)

Let n be a positive integer >1 with g.c.d.$(n, 10) = 1$, and $\delta = (5/n)$, a Jacobi symbol. Consider the following congruences:
(1) $F_{n-\delta} \equiv 0 \pmod n$, $L_n \equiv 1 \pmod n$;
(2) $F_{\frac{1}{2}(n-\delta)} \equiv 0 \pmod n$ if $n \equiv 1 \pmod 4$, $L_{\frac{1}{2}(n-\delta)} \equiv 0 \pmod n$ if $n \equiv 3 \pmod 4$.
Composite n which satisfy (1) are called *Fibonacci-Lucas pseudoprimes,* abbreviated "FLUPPS." Composite n which satisfy (2) are called *Euler-Lucas pseudoprimes with parameters* $(1,-1)$, abbreviated "ELUPPS." Prove that FLUPPS and ELUPPS are equivalent.

Solution by Andrej Dujella, University of Zagreb, Croatia

(1) $\Rightarrow$ (2): It is easy to check that, for $\delta \in \{-1, 1\}$, it holds: $2L_n - 5F_{n-\delta} = \delta L_{n-\delta}$. Considering that, from (1), it follows that $L_{n-\delta} \equiv 2\delta \pmod n$. From the identity $L_{2n} + 2\cdot(-1)^n = L_n^2$ [see S. Vajda, *Fibonacci & Lucas Numbers, and the Golden Section* (Chichester: Halsted, 1989), (17c)], we have $L^2_{\frac{1}{2}(n-\delta)} = L_{n-\delta} + 2\cdot(-1)^{\frac{1}{2}(n-\delta)} \equiv 2\delta + 2\cdot(-1)^{\frac{1}{2}(n-\delta)} \pmod n$.

If $n \equiv 3 \pmod 4$, then $2\delta + 2\cdot(-1)^{(n-\delta)/2} = 2\delta + 2\cdot(-1)^{(1+\delta)/2} = 0$; therefore, $L_{\frac{1}{2}(n-\delta)} \equiv 0 \pmod n$.

If $n \equiv 1 \pmod 4$, then $2\delta + 2\cdot(-1)^{(n-\delta)/2} = 2\delta + 2\cdot(-1)^{(1+\delta)/2} = 4\delta$, and using g.c.d.$(F_m, L_m) \le 2$ and $F_{n-\delta} = F_{(n-\delta)/2}L_{(n-\delta)/2} \equiv 0 \pmod n$, we have $F_{\frac{1}{2}(n-\delta)} \equiv 0 \pmod n$.

(2) $\Rightarrow$ (1): From $F_{n-\delta} = F_{(n-\delta)/2}L_{(n-\delta)/2}$ and (2), it follows that $F_{n-\delta} \equiv 0 \pmod n$. Now, from $2L_n - 5F_{n-\delta} = \delta L_{n-\delta}$ it may be concluded that $2\delta L_n \equiv L_{n-\delta} \pmod n$.

If $n \equiv 3 \pmod 4$, we have $2\delta L_n \equiv L^2_{\frac{1}{2}(n-\delta)} - 2\cdot(-1)^{\frac{1}{2}(1+\delta)} \equiv 2\cdot(-1)^{\frac{1}{2}(1+\delta)} \equiv 2\delta \pmod n$; therefore, $L_n \equiv 1 \pmod n$.

If $n \equiv 1 \pmod 4$, we have $2\delta L_n \equiv 5F^2_{\frac{1}{2}(n-\delta)} + 2\cdot(-1)^{\frac{1}{2}(1-\delta)} \equiv 2\cdot(-1)^{\frac{1}{2}(1-\delta)} \equiv 2\delta \pmod n$, and again $L_n \equiv 1 \pmod n$.

Also solved by A. G. Dresel, H.-J. Seiffert, and the proposer.

Editorial Note: *The editor will appreciate it if all proposals and solutions are submitted in typed format.*

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau, Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

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A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

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Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger. FA, 1993.

Fibonacci Entry Points and Periods for Primes 100,003 through 415,993 by Daniel C. Fielder and Paul S. Bruckman.

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