The Fibonacci Quarterly

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The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

ON THE UNIQUENESS OF REDUCED PHI-PARTITIONS

Corey Powell
Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720
(*Submitted July 1994*)

1. PRELIMINARIES

For any positive integer k, let $P_k = k^{\text{th}}$ prime and define $s_k = \prod_{j=1}^{k-1} P_j$ A positive integer n is *simple* if $n = s_k$ for some positive integer k.

A ϕ*-partition* of n is a partition $n = d_1 + \cdots + d_i$, where i and $d_1, \ldots, d_i$ are positive integers, that satisfies the condition $\phi(n) = \phi(d_1) + \cdots + \phi(d_i)$, where ϕ is the Euler phi-function. In [1], Jones shows that the simple integers s_k have only the trivial ϕ-partition $s_k = s_k$, and so define a ϕ-partition of a positive integer n to be *reduced* if all the summands are simple.

Writing a partition as $\sum_{j=1}^{i} b_j * d_1 + \cdots + b_i * d_i$, means that d_j occurs b_j times in the partition for $j = 1, \ldots, i$.

Every positive integer n has a unique partition $\sum_{j=1}^{i} c_j * s_j$ satisfying the condition that $0 \le c_j < P_j$ for $j \ge 1$. This partition is a special type of *Cantor base representation* of n, which is a direct extension of the standard base 10 representation of n.

Throughout the paper, n will denote a positive integer. Let $p_1 < p_2 < \cdots < p_\ell$ be the primes dividing n and let $q_1 < q_2 < \cdots$ be the primes not dividing n.

2. THE ALGORITHM

Jones gives the following recursive algorithm for finding a reduced ϕ-partition for n:

1. If n is simple, then $n = 1 * n$ is a reduced ϕ-partition.
2. If $p^2 | n$ for some prime p, then $p * (n/p)$ is a ϕ-partition of n. Apply the algorithm to n/p to give a reduced ϕ-partition $\sum_{j=1}^{i} a_j * s_j$ for n/p; the desired reduced ϕ-partition for n is $\sum_{j=1}^{i}(a_j p) * s_j$.
3. If n is square-free and not simple, then let p be a prime divisor of n and let q be a prime such that $q < p$ and $q \nmid n$. Such p and q exist since n is not simple; p could be chosen to be the largest prime dividing n, and q could be chosen to be the smallest prime not dividing n. Then $(p-q) * (n/p) + 1 * (qn/p)$ is a ϕ-partition for n. Apply the algorithm to n/p and qn/p to give reduced ϕ-partitions $\sum_{j=1}^{i} a_j * s_j$ and $\sum_{j=1}^{i} a'_j * s_j$, respectively. The desired reduced ϕ-partition for n is $\sum_{j=1}^{i}[(p-q)a_j + a'_j] * s_j$.

At each step of the algorithm, it will be generally true that more than one prime or pair of primes can be chosen. The next section shows that the result of the algorithm is independent of these choices.

3. THE ALGORITHM GIVES A UNIQUE REDUCED ϕ-PARTITION

For any integer w, let $\phi_w(n) = n \prod_{\substack{p|n \\ p \text{ prime}}} (1 - w/p)$, so that $\phi_0(n) = n$ and $\phi_1(n) = \phi(n)$. Define a ϕ_w-partition and a reduced ϕ_w-partition analogously to a ϕ-partition and a reduced ϕ-partition, respectively.

If p is prime and $p^2 | n$, then $\phi_w(n) = p\phi_w(n/p)$, and so $p * (n/p)$ is a ϕ_w-partition of n. If p and q are primes, $p|n$, $p^2 \nmid n$, $q \nmid n$ and $p > q$, then

$$\begin{aligned}\phi_w(n) &= (p-w)\phi_w(n/p)\\ &= (p-q)\phi_w(n/p) + (q-w)\phi_w(n/p)\\ &= (p-q)\phi_w(n/p) + \phi_w(qn/p),\end{aligned}$$

so that $(p-q) * (n/p) + 1 * (qn/p)$ is a ϕ_w-partition of n. These facts together with an induction argument show that any reduced ϕ-partition given by the algorithm is also a reduced ϕ_w-partition for every integer w.

For the rest of the paper, let i be the unique positive integer such that $s_i \le n < s_{i+1}$; any reduced ϕ-partition of n can be written in the form $\sum_{j=1}^{i} b_j * s_j$.

Theorem 1: The reduced ϕ-partition for n given by the algorithm above is independent of the primes chosen at each step of the algorithm.

Proof: Suppose that $\sum_{j=1}^{i} a_j * s_j$ is a reduced ϕ-partition of n given by the algorithm. By replacing w with P_1, with P_2, ..., and finally with P_i, the following system of i equations and i unknowns is obtained:

$$\begin{gathered}a_1\phi_{P_1}(s_1) + a_2\phi_{P_1}(s_2) + \cdots + a_i\phi_{P_1}(s_i) = \phi_{P_1}(n),\\ a_1\phi_{P_2}(s_1) + a_2\phi_{P_2}(s_2) + \cdots + a_i\phi_{P_2}(s_i) = \phi_{P_2}(n),\\ \vdots\\ a_1\phi_{P_i}(s_1) + a_2\phi_{P_i}(s_2) + \cdots + a_i\phi_{P_i}(s_i) = \phi_{P_i}(n).\end{gathered}$$

This system of equations can be rewritten in the form $N\vec{a} = \vec{b}$, where the matrix N has entries

$$N_{\ell k} = \phi_{P_\ell}(s_k) = \prod_{j=1}^{k-1}(P_j - P_\ell).$$

If $\ell < k$, then $N_{\ell k} = 0$, so that N is lower-triangular, and if $\ell = k$, then $N_{\ell k} \ne 0$, so that N is invertible. It follows that the solution to this linear system uniquely determines the coefficients a_j in the reduced ϕ-partition given by the algorithm. □

The algorithm gives a unique reduced ϕ-partition for n, but frequently this is not the only reduced ϕ-partition that n has. The integer 8, for example, has 2 reduced ϕ-partitions: $4 * 2$ and $2 * 1 + 6$. Certain characteristics of the reduced ϕ-partition given by the algorithm are critical, however, in determining whether n has a unique reduced ϕ-partition. The following two theorems summarize these characteristics. Let $M(n) = n/\left(\prod_{\substack{p|n\\ p \text{ prime}}} p\right)$.

Theorem 2: Let k be the largest integer such that $s_k | n$ and let ℓ be the number of distinct prime factors of n. The algorithm above gives a reduced ϕ-partition for n of the form $\sum_{j=k}^{\ell+1} a_j * s_j$, where $a_{\ell+1} = M(n)$ and $a_k \ge a_{k+1} \ge \cdots \ge a_{\ell+1}$.

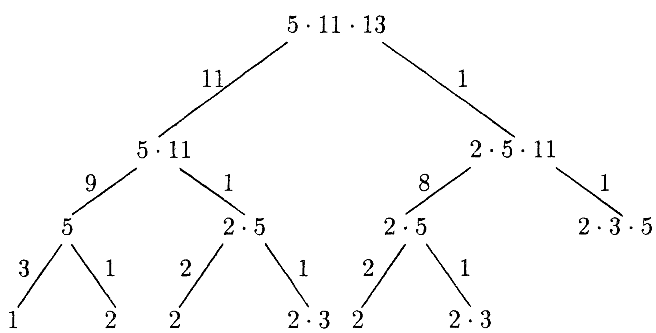
Proof: It follows that $a_j = 0$ for $1 \le j < k$ from examining the first $k-1$ equations of the linear system above and noting that $b_j = \phi_{P_j}(n) = 0$ for $1 \le j < k$. It is clear that $a_{\ell+1} = M(n)$ and

$a_j = 0$ for $\ell + 1 < j$ from the three cases presented in the algorithm together with an induction argument on n.

The claim that $a_k \geq a_{k+1} \geq \cdots \geq a_{\ell+1}$ will also be proven by induction on n. If $n = 1$, then $1 = 1 * 1$ is the reduced ϕ-partition. If $n > 1$ and $p^2 | n$ for some prime p, then establish the claim by using the ϕ-partition $n = p * (n / p)$ and applying the induction hypothesis to n / p.

If n is square-free, then the proof is divided into two cases. In the first case, $p_\ell < q_1$, so that $n = s_k$. If $p_\ell > q_1$ and there is a prime t such that $t \nmid n$ and $q_1 < t < p_\ell$, then the claim follows from using the ϕ-partition $n = (p_\ell - t) * (n / p_\ell) + 1 * (tn / p_\ell)$ and applying the induction hypothesis to n / p_ℓ and tn / p_ℓ. If $p_\ell > q_1$, but there is no prime t as described above, then $q_1 n / p_\ell = s_{\ell+1}$, since $q_1 n / p_\ell$ is simple and has the same number of prime factors as n. Prove the claim by using the ϕ-partition $n = (p_\ell - q_1) * (n / p_\ell) + 1 * s_{\ell+1}$ and applying the induction hypothesis to n / p_ℓ. □

By Theorem 1, the reduced ϕ-partition $n = \sum_{j=1}^{i} a_j * s_j$ given by the algorithm can be represented by a weighted binary tree as follows. As noted in part 3 of the algorithm, it is possible to choose p as the largest prime dividing n and choose q as the smallest prime dividing n. Assume without loss of generality that n is square-free; the algorithm will find a reduced ϕ-partition for $\prod_{\substack{p|n \\ p \text{ prime}}} p$ and incorporate this reduced ϕ-partition into the ϕ-partition $n = M(n) * \prod_{\substack{p|n \\ p \text{ prime}}} p$. If n is not simple, then the left branch has weight $p_\ell - q_1$ and the left child is n / p_ℓ, while the right branch has weight 1 and the right child is $q_1 n / p_\ell$. Apply this process recursively to n / p_ℓ and $q_1 n / p_\ell$, terminating only when all the leaves of the tree are simple integers. The example below gives the tree for $5 \cdot 11 \cdot 13$.



It is possible to determine a_j from the tree representation by taking the sum over all paths from n to s_j of the product of the weights along each path. The coefficient a_2 for $5 \cdot 11 \cdot 13$, for example, is $11 \cdot 9 \cdot 1 + 11 \cdot 1 \cdot 2 + 1 \cdot 8 \cdot 2$.

Let $n = p_1 \ldots p_\ell$, and suppose that m is a vertex at level $u = u(m)$ of the tree described above, where level 0 denotes the top of the tree. Let $L = L(m)$ and $R = R(m) = u = L(m)$ be the number of left and right branches, respectively, in the path from n to m, and define $t_0 < t_1 < \cdots < t_{L-1}$ to be the levels where the path branches to the left. An induction argument on the level u proves the following lemma.

Lemma 3: If m is a vertex at level u of the tree, then $m = p_1 \ldots p_{\ell-u} q_1 \ldots q_R$ and the product of the weights along the path from n to m is $\prod_{e=0}^{L-1} (p_{\ell - t_e} - q_{1+t_e-e})$.

Proof: The proof will be by induction on u. If $u=0$, then $m=n$ and the result is clear. Suppose that the lemma is true for m and its ancestors, and consider the children of m. Assume without loss of generality that m is not simple, since simple vertices have no children. To establish the lemma for the children of m, it suffices to show that the largest prime dividing m is $p_{\ell-u}$ and that the smallest prime not dividing m is q_{R+1}. The largest prime dividing m is clearly $\max(p_{\ell-u}, q_R)$, and the smallest prime not dividing m is clearly $\min(p_{\ell-u+1}, q_{R+1})$. These two facts reduce the two assertions above to the condition $p_{\ell-u} > q_{R+1}$. Divide the proof of this condition into two cases. In the first case, assume that m is the right child of $m' = p_1 \dots p_{\ell-u+1} q_1 \dots q_{R-1}$. It follows from the construction of m that $p_{\ell-u+1}$ is the largest prime dividing m' and that q_R is the smallest prime not dividing m'. It also follows that $p_{\ell-u+1} > q_R$, since m' would be simple otherwise. Hence $p_{\ell-u} > q_{R+1}$, because m is not simple. The proof when m is a left child is similar. □

Fix a path from n to s_j. A left child has one less prime divisor than its parent, while a right child has the same number of prime divisors as its parent. This implies that the path from n to s_j must branch to the left $\ell-j+1$ times, since s_j has $j-1$ prime factors. Let m be the vertex at level $t_{\ell-j}$. It follows from the proof of the previous lemma that $p_{\ell-t_{\ell-j}}$ is the largest prime dividing m, that $q_{1+t_{\ell-j}-(\ell-j)}$ is the smallest prime not dividing m, and that $p_{\ell-t_{\ell-j}} > q_{1+t_{\ell-j}-(\ell-j)}$.

Conversely, suppose that $0 \le t_0 < t_1 < \dots < t_{\ell-j} < \ell$ and that $p_{\ell-t_{\ell-j}} > q_{1+t_{\ell-j}-(\ell-j)}$. An induction argument on $\ell-j$ shows that there is a path from n to s_j that branches to the left at levels $t_0, \dots, t_{\ell-j}$. If $\ell-j=0$, then an induction on the level together with the previous lemma and the condition $p_{\ell-t_0} > q_{1+t_0}$ guarantees that there is a path branching to the right at levels $0, \dots, t_0-1$. This condition also guarantees that the vertex at level t_0 is not simple, since $p_{\ell-t_0}$ and q_{1+t_0} are, respectively, the largest prime dividing the vertex and the smallest prime not dividing the vertex. Branching to the left at level t_0 will give a vertex with $\ell-1$ prime factors, and so branching to the right at level t_0+1 and all higher levels will give a path that terminates at s_ℓ.

Assume that the claim above is true for $\ell-j-1$. This implies that there is a path from n to s_{j+1} that branches to the left at levels $t_0, \dots, t_{\ell-j-1}$, since $p_{\ell-t_{\ell-j-1}} > p_{\ell-t_{\ell-j}} > q_{1+t_{\ell-j}-(\ell-j)} \ge q_{1+t_{\ell-j-1}-(\ell-j-1)}$. Take this path from n to level $t_{\ell-j-1}$, and then construct a path from the vertex at this level to s_j by the same method as in the previous paragraph. This proves the claim above. Non-square-free n adds a factor of $M(n)$ to the calculations, as noted previously, and so the claim above combined with the previous lemma proves the following theorem.

Theorem 4: If $k \le j \le \ell$, then

$$a_j = M(n) \Sigma \prod_{e=0}^{\ell-j} (p_{\ell-t_e} - q_{1+t_e-e}),$$

where the sum is taken over all $0 \le t_0 < \dots < t_{\ell-j} < \ell$ with $p_{\ell-t_{\ell-j}} > q_{1+t_{\ell-j}-(\ell-j)}$. In particular,

$$a_k = M(n) \prod_{e=k}^{\ell} (p_e - q_1) \quad \text{and} \quad a_{k+1} \ge M(n) \prod_{e=k+1}^{\ell} (p_e - q_1).$$

The following section gives a necessary and sufficient criterion for determining if n has a unique reduced ϕ-partition by using the previous two theorems together with the specific Cantor base representation of n described in Section 1.

4. WHEN DOES n HAVE A UNIQUE REDUCED ϕ-PARTITION?

Theorem 5: A positive integer n has a unique reduced ϕ-partition if and only if $n=9$, n is prime, or the Cantor base representation for n is a reduced ϕ-partition.

By enumerating all possible partitions of 9 consisting of simple integers, one can verify that $3*1+3*2$ is the unique reduced ϕ-partition of 9. The following two lemmas will complete the proof of the *if* part of the theorem.

Lemma 6: Primes have a unique reduced ϕ-partition.

Proof: From Theorem 7 in [1], a ϕ-partition of a prime q must be of the form $(q-r)*1+r$, where r is prime and $r \le q$. The only simple prime number is 2, and so $(q-2)*1+1*2$ is the unique reduced ϕ-partition of q. □

Lemma 7: Let $\sum_{j=1}^{i} c_j * s_j$ be the Cantor base representation of n described in Section 1. If $\sum_{j=1}^{i} c_j * s_j$ is a reduced ϕ-partition, then it is the unique reduced ϕ-partition for n.

Proof: It suffices to show that for any other partition $\sum_{j=1}^{i} b_j * s_j$ of n into simple integers, $\sum_{j=1}^{i} c_j\phi(s_j) < \sum_{j=1}^{i} b_j\phi(s_j)$. Suppose that $\sum_{j=1}^{i} b_j * s_j$ is a counterexample with $\sum_{j=1}^{i} b_j$ minimal. There is an h such that $b_h \ge P_h$ since $\sum_{j=1}^{i} b_j * s_j$ is distinct from $\sum_{j=1}^{i} c_j * s_j$. Form a new partition $\sum_{j=1}^{i} b_j' * s_j$ by converting P_h of the s_h's into an s_{h+1}. This new partition is a counterexample that contradicts the minimality of $\sum_{j=1}^{i} b_j$, and hence proves the lemma. □

Suppose that $n \ne 9$, that n is composite, and that the Cantor base representation for n is not a reduced ϕ-partition. It then follows from Theorem 2 that $a_k \ge P_k$, where $\sum_{j=1}^{i} a_j * s_j$ is the reduced ϕ-partition given by the algorithm and k the largest positive integer such that $s_k | n$, as defined in Theorem 2. Theorem 4 gives the formula $a_k = M(n)\prod_{e=k}^{\ell}(p_e - P_k)$, since $q_1 = P_k$. It is clear from this formula that $a_k \ne P_k$, since P_k is prime, P_k does not divide n, and P_k does not divide $(p-P_k)$ for any primes p dividing n. If $k>1$, then apply the following lemma with $h=k$ to show that n has a second reduced ϕ-partition. If $k=1$, then n is odd, $q_1 = 2$, and the inequality $a_2 \ge M(n)\prod_{e=2}^{\ell}(p_e - 2)$ is a result of Theorem 4. It is a straightforward consequence of this inequality that $a_2 > 3$ if $n \ne 15$, and so it follows from the following lemma with $h=2$ that n has a second reduced ϕ-partition. The observation that $3+1+3*2+6$ and $1+7*2$ are reduced ϕ-partitions for 15 completes the proof of the theorem.

Lemma 8: Let $\sum_{j=1}^{i} b_j * s_j$ be a reduced ϕ-partition of n. If $b_h > P_h$ for some $h > 1$, then n has a second reduced ϕ-partition.

Proof: To prove the lemma, it will be necessary to show that for each $j>1$ there is a partition $\sum_{f=1}^{j-1} \beta_f * s_f$ of s_j such that $\sum_{f=1}^{j-1} \beta_f\phi(s_f) = 2\phi(s_j)$. This will be shown by induction on j. If $j=2$, then $2 = 2*1$ is the desired partition. If $j>2$, then, by the induction hypothesis, s_{j-1} has a partition $\sum_{f=1}^{j-2} \beta_f' * s_f$ with the desired property. Hence, $s_j = \sum_{f=1}^{j-2}[\beta_f'(P_{j-1}-2)] * s_f + 2 * s_{j-1}$ is a partition of s_j with the desired property.

Now, suppose h is a positive integer such as in the hypothesis of the lemma and let $\sum_{f=1}^{h-1}\beta_f * s_f$ be a partition of s_h with the above property. Construct a new reduced ϕ-partition for n by combining P_h of the s_h terms into one s_{h+1} term. There is a net loss of $\phi(s_h)$ when the sum of the ϕ values in the partition is taken, since $\phi(s_{h+1}) = (P_h - 1)\phi(s_h)$. Breaking up one of the remaining s_h terms compensates for this loss. The second reduced ϕ-partition for n is

$$\sum_{f=1}^{h-1}(b_f + \beta_f) * s_f + (b_h - P_h - 1) * s_h + (b_{h+1} + 1) * s_{h+1} + \sum_{f=h+2}^{i} b_f * s_f. \quad \square$$

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ON THE STRUCTURE OF QUADRATIC IRRATIONALS ASSOCIATED WITH GENERALIZED FIBONACCI AND LUCAS NUMBERS

Edward B. Burger and Christopher S. Kollett
Department of Mathematics, Williams College, Williamstown, MA 01267
(Submitted July 1994)

1. INTRODUCTION

In 1970 C. T. Long and J. H. Jordan completed a series of two papers, [3] and [4], in which they analyzed the arithmetical structure of certain classes of quadratic irrationals and the effects on their structure after multiplication by rational numbers. In particular, for a positive integer a, let $\mathscr{F}_n = \mathscr{F}_n(a)$ and $\mathscr{L}_n = \mathscr{L}_n(a)$ be the n^{th} *generalized Fibonacci* and *generalized Lucas numbers*, respectively. That is, $\mathscr{F}_0 = 0$, $\mathscr{F}_1 = 1$, $\mathscr{L}_0 = 2$, $\mathscr{L}_1 = a$ and, for $n > 1$, $\mathscr{F}_n = a\mathscr{F}_{n-1} + \mathscr{F}_{n-2}$, $\mathscr{L}_n = a\mathscr{L}_{n-1} + \mathscr{L}_{n-2}$. We denote the *generalized golden ratio* by φ_a. Thus,

$$\varphi_a = \frac{a+\sqrt{a^2+4}}{2} = [a, a, \dots] = [\overline{a}],$$

where the last expression denotes the (simple) continued fraction expansion for φ_a and the bar indicates the periodicity. It follows that $\lim_{n\to\infty}\mathscr{F}_{n+1}/\mathscr{F}_n = \varphi_a$. We note that in the case in which $a = 1$ we have $\mathscr{F}_n = F_n$, $\mathscr{L}_n = L_n$, and $\varphi_a = \varphi$.

Among their other interesting results, Long and Jordan investigated and compared the continued fraction expansions of $\frac{r}{s}\varphi_a$ and $\frac{s}{r}\varphi_a$ when r and s are consecutive generalized Fibonacci numbers or consecutive generalized Lucas numbers. These results led them to consider the structure of numbers of the form $\frac{r}{s}\varphi_a$ and $\frac{s}{r}\varphi_a$ where $r = \mathscr{F}_n$ and $s = \mathscr{L}_n$. They wrote (in the present notation) [4]:

"In view of the preceding results, one would expect an interesting theorem concerning the simple continued fraction of

$$\frac{\mathscr{F}_n}{\mathscr{L}_n}\varphi_a \quad \text{and} \quad \frac{\mathscr{L}_n}{\mathscr{F}_n}\varphi_a$$

but we were unable to make a general assertion value for all a. To illustrate the difficulty, note that, when $a = 2$ and $\varphi_2 = 1+\sqrt{2}$, we have

$$\frac{\mathscr{F}_4}{\mathscr{L}_4}\varphi_2 = \left[0, 1, \overline{5, 1, 3, 5, 1, 7}\right],$$

$$\frac{\mathscr{F}_5}{\mathscr{L}_5}\varphi_2 = \left[0, 1, \overline{5, 1, 5, 3, 1, 4, 1, 7}\right],$$

$$\frac{\mathscr{F}_6}{\mathscr{L}_6}\varphi_2 = \left[0, 1, \overline{5, 1, 4, 1, 3, 5, 1, 4, 1, 7}\right]."$$

They do, however, discover the following two beautiful identities for the case in which $a = 1$. We state them here as

Theorem 1: For $n \geq 4$,

$$\frac{F_n}{L_n}\varphi = \left[0, 1, \overline{2, \underbrace{1, 1, \ldots, 1, 1}_{n-4 \text{ times}}, 3, \underbrace{1, 1, \ldots, 1, 1}_{n-3 \text{ times}}, 4}\right] \tag{1.1}$$

and

$$\frac{L_n}{F_n}\varphi = \left[3, \overline{\underbrace{1, 1, \ldots, 1, 1}_{n-3 \text{ times}}, 3, \underbrace{1, 1, \ldots, 1, 1}_{n-4 \text{ times}}, 2, 4}\right].$$

Our first objective here is to extend their result to the more general case for arbitrary a. We begin with the elementary observation that

$$\lim_{n\to\infty}\frac{F_n}{L_n}\varphi = \frac{5+\sqrt{5}}{10} = [0, 1, 2, \overline{1}],$$

and, thus, the initial string of 1's in (1.1) is not surprising. It is the beautiful near mirror symmetry of the interior portion of the periodic part of (1.1) that is unexpected. More generally, one has

$$\lim_{n\to\infty}\frac{\mathcal{F}_n}{\mathcal{L}_n}\varphi_a = \frac{a^2+4+a\sqrt{a^2+4}}{2(a^2+4)} = \left[0, 1, a^2+1, \overline{1, a^2}\right].$$

Thus, for large n, we would expect the continued fraction expansion for $(\mathcal{F}_n / \mathcal{L}_n)\varphi_a$ to begin with $[0, 1, a^2+1, 1, a^2, 1, a^2, 1, \ldots]$. As Long and Jordan remark, however, in this case we appear to lose the symmetry. In fact, the near mirror symmetry in (1.1) is somewhat deceptive. Perhaps it is better to view (1.1) as a "recursive system" in the following sense. We define the strings or "words" $\vec{W}_n = W_n$ for $n \geq 4$ by $W_4 = (3, 1)$ and, for $n > 4$, $W_n = (\overleftarrow{W}_{n-1}, 1, 1)$, where $\overleftarrow{W}_{n-1}$ is the word W_{n-1} read *backwards*. For example, $W_5 = (1, 3, 1, 1)$ and $W_6 = (1, 1, 3, 1, 1, 1)$. Thus, we may now reformulate (1.1) as

$$\frac{F_n}{L_n}\varphi = \left[0, 1, \overline{2, \vec{W}_n, 4}\right].$$

We note that the continued fraction expansions given above for $(\mathcal{F}_n / \mathcal{L}_n)\varphi_a$ obey a similar recursive behavior. This leads to our first result.

Theorem 2: Let $\mathcal{F}_n = \mathcal{F}_n(a)$ and $\mathcal{L}_n = \mathcal{L}_n(a)$ be the n^{th} generalized Fibonacci and Lucas numbers, respectively. Let $\vec{\mathcal{W}}_4 = \mathcal{W}_4(a) = (1, a^2-1, a^2+1, 1)$ and, for $n > 4$, let $\vec{\mathcal{W}}_n = \mathcal{W}_n(a) = (\overleftarrow{\mathcal{W}}_{n-1}, a^2, 1)$. Then, for $n \geq 4$,

$$\frac{\mathcal{F}_n}{\mathcal{L}_n}\varphi_a = \left[0, 1, \overline{a^2+1, \vec{\mathcal{W}}_n, a^2+3}\right], \tag{1.2}$$

and

$$\frac{\mathcal{L}_n}{\mathcal{F}_n}\varphi_a = \left[a^2+2, \overline{\overleftarrow{\mathcal{W}}_n, a^2+1, a^2+3}\right]. \tag{1.3}$$

We remark that for $a = 1$,

$$[\mathscr{W}_4]=1+\cfrac{1}{0+\cfrac{1}{2+\cfrac{1}{1}}}=1+2+\frac{1}{1}=3+\frac{1}{1}=[W_4],$$

thus, Theorem 1 and Theorem 2 are equivalent when $a=1$.

One may define the words $\vec{\mathscr{W}}_n$ occurring in Theorem 2 explicitly rather than iteratively. In particular, a simple induction argument reveals that, for $n \geq 4$ even,

$$\vec{\mathscr{W}}_n = \left(\{1, a^2\}^{(n-4)/2}, 1, a^2-1, a^2+1, 1, \{a^2, 1\}^{(n-4)/2}\right), \tag{1.4}$$

and, for $n>4$ odd,

$$\vec{\mathscr{W}}_n = \left(\{1, a^2\}^{(n-5)/2}, 1, a^2+1, a^2-1, 1, \{a^2, 1\}^{(n-3)/2}\right),$$

where by $\{1, a^2\}^n$ we mean the word $(1, a^2)$ repeated n times.

As Long and Jordan implicitly note with respect to Theorem 1, Theorem 2 immediately implies that $(\mathscr{F}_n / \mathscr{L}_n)\varphi_a$ and $(\mathscr{L}_n / \mathscr{F}_n)\varphi_a$ are not equivalent numbers. Recall that two real numbers are said to be *equivalent* if, from some point on, their continued fraction expansions agree (see [5]).

Next, we extend Theorem 1 in a different direction. We wish to analyze the structure of quadratic irrationals of the form $(\mathscr{F}_{mn} / \mathscr{L}_n)\varphi_a$. If m is even, then $\mathscr{F}_{mn} / \mathscr{L}_n$ is an integer (see [7]); thus, we consider only the case in which m is odd. We first state an extension of Theorem 1 in this context for the case $m=3$.

Theorem 3: For $n \geq 4$, if n is even, then

$$\frac{F_{3n}}{L_n}\varphi = \left[F_{2n+1}-1, \overline{3, \underbrace{1, 1, \ldots, 1, 1}_{n-2 \text{ times}}, L_{2n}-2, 2, \underbrace{1, 1, \ldots, 1, 1}_{n-4 \text{ times}}, 2, 1, L_{2n}-2}\right].$$

If $n>4$ is odd, then

$$\frac{F_{3n}}{L_n}\varphi = \left[F_{2n+1}, \overline{1, 2, \underbrace{1, 1, \ldots, 1, 1}_{n-4 \text{ times}}, L_{2n}, \underbrace{1, 1, \ldots, 1, 1}_{n-2 \text{ times}}, 3, L_{2n}}\right].$$

The general formulation of Theorem 3 appears to be more complicated and requires us to define several useful sums. For odd integers m, we let

$$\mathbf{F}_1(m) = \sum_{k=1}^{(m-1)/2} (-1)^k \mathscr{F}_{2k+1}, \qquad \mathbf{F}(m) = \sum_{k=1}^{(m-1)/2} \mathscr{F}_{2k+1},$$

$$\mathbf{L}_1(m) = \sum_{k=1}^{(m-1)/2} (-1)^k \mathscr{L}_{2k}, \qquad \mathbf{L}(m) = \sum_{k=1}^{(m-1)/2} \mathscr{L}_{2k}.$$

We remark that $\mathbf{F}_1(m)$ and $\mathbf{L}_1(m)$ are positive integers if and only if $m \equiv 1 \bmod 4$. We believe that one may generalize the proof of Theorem 3 to prove the following conjecture.

Conjecture 4: Let $\mathscr{F}_n = \mathscr{F}_n(a)$ and $\mathscr{L}_n = \mathscr{L}_n(a)$ be the n^{th} generalized Fibonacci and Lucas numbers, respectively, and m an odd integer. Suppose $m \geq 3$ is an odd integer and $n \geq 4$. For n even, let $\vec{\mathcal{U}}_n = \mathcal{U}_n(a) = (1, a^2+1, 1, \{a^2, 1\}^{(n-4)/2}, a^2-1, 1)$ and $\vec{\mathcal{V}}_n = \mathcal{V}_n(a) = (\{a^2, 1\}^{(n-2)/2}, a^2+2)$; for n odd, let $\vec{\mathcal{W}}_n = \mathcal{W}_n(a) = (\{a^2, 1\}^{(n-5)/2})$. If n is odd, then

$$\frac{\mathcal{F}_{mn}}{\mathcal{L}_n}\varphi_a = \left[\mathbf{F}(m), \overline{1, a^2+1, 1, \vec{\mathcal{W}}_n, a^2+1, a^2\mathbf{L}(m)+a^2-1, \vec{\mathcal{W}}_{n+2}, 1, a^2+2, \mathbf{L}(m)}\right].$$

If n is even and $m \equiv 1 \bmod 4$, then

$$\frac{\mathcal{F}_{mn}}{\mathcal{L}_n}\varphi_a = \left[\mathbf{F}_1(m), \overline{\vec{\mathcal{U}}_n, \mathbf{L}_1(m), \vec{\mathcal{V}}_n, \mathbf{L}_1(m)}\right].$$

If n is even and $m \equiv 3 \bmod 4$, then

$$\frac{\mathcal{F}_{mn}}{\mathcal{L}_n}\varphi_a = \left[-\mathbf{F}_1(m)-1, \overline{\vec{\mathcal{V}}_n, -\mathbf{L}_1(m)-2, \vec{\mathcal{U}}_n, -\mathbf{L}_1(m)-2}\right].$$

One may also find analogous expansions for $(\mathcal{L}_{mn}/\mathcal{F}_n)\varphi_a$. For example, one may adopt the method of proof of Theorem 3 to deduce

Theorem 5: For $n \ge 4$, if n is even, then

$$\frac{L_{3n}}{F_n}\varphi = \left[5F_{2n+1}+3, \overline{3, \underbrace{1, 1, \dots, 1, 1}_{n-3 \text{ times}}, 2, L_{2n}, \underbrace{1, 1, \dots, 1, 1}_{n-3 \text{ times}}, 2, 5L_{2n}+4}\right].$$

If $n > 4$ is odd, then

$$\frac{L_{3n}}{F_n}\varphi = \left[5F_{2n+1}-4, \overline{2, \underbrace{1, 1, \dots, 1, 1}_{n-3 \text{ times}}, L_{2n}-2, 2, \underbrace{1, 1, \dots, 1, 1}_{n-3 \text{ times}}, 5L_{2n}-6}\right].$$

Long and Jordan [4] concluded their investigation by proving the surprising result that, for any positive integers m and n, $(\mathcal{F}_m/\mathcal{F}_n)\varphi_a$ and $(\mathcal{F}_n/\mathcal{F}_m)\varphi_a$ are equivalent numbers. They remarked, however, that it is not always the case that $(\mathcal{L}_m/\mathcal{L}_n)\varphi_a$ and $(\mathcal{L}_n/\mathcal{L}_m)\varphi_a$ are equivalent numbers. To illustrate this, they noted that

$$\frac{L_2}{L_4}\varphi = [0, 1, \overline{2, 3, 1, 4}] \quad \text{and} \quad \frac{L_4}{L_2}\varphi = [3, \overline{1, 3, 2, 4}].$$

We observe that, in their example, the indices 2 and 4 are *not* relatively prime. Here we prove that this is the only possible case in which two such numbers are not equivalent. In particular, we prove

Theorem 6: If $\mathcal{L}_n = \mathcal{L}_n(a)$ is the n^{th} generalized Lucas number, then for relatively prime positive integers m and n,

$$\frac{\mathcal{L}_m}{\mathcal{L}_n}\varphi_a \quad \text{and} \quad \frac{\mathcal{L}_n}{\mathcal{L}_m}\varphi_a$$

are equivalent numbers.

More recently, Long [2] studied the arithmetical structure of classes of quadratic irrationals involving generalized Fibonacci and Lucas numbers of the form

$$\frac{aS_n + T_m\sqrt{a^2+4}}{2}, \tag{1.5}$$

where S_n is either $\mathcal{F}_n$ or $\mathcal{L}_n$ and T_m is either $\mathcal{F}_m$ or $\mathcal{L}_m$. For example, he investigated

$$\frac{a\mathcal{F}_n + \mathcal{L}_m\sqrt{a^2+4}}{2}. \tag{1.6}$$

For numbers of the form (1.5), Long showed that their continued fraction expansions have the general shape

$$\left[a_0, \overline{a_1, a_2, \ldots, a_r}\right],$$

where the a_0 and a_r were explicitly computed. He also proved that $(a_1, a_2, \ldots, a_{r-1})$ is a palindrome, but was unable to determine the precise value of a_n for $0 < n < r$. Long also observed that the period length r appeared always to be even and that the value of r appeared not to be bounded as a function of a, m, and n. Here we claim that the continued fraction for such numbers may be completely determined. As an illustration, in Section 6 we provide the precise formula for the continued fraction expansion for numbers of the form (1.6). As the expansion is somewhat complicated in general, we do not state it here in the introduction; instead, we state it explicitly in Section 6 as Theorems 7 and 8. As a consequence of our results, we are able to prove that Long's first observation is true while his second observation is false.

2. BASIC IDENTITIES AND CONTINUED FRACTIONS

We begin with a list of well-known identities involving Fibonacci and Lucas numbers that will be utilized in our arguments (for proofs, see, e.g., [7]). For $n \geq 1$,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \tag{2.1}$$

$$F_n = F_{n-1} + F_{n-2},\ L_n = L_{n-1} + L_{n-2},\ L_n = F_{n+1} + F_{n-1}, \tag{2.2}$$

$$F_{n+m} - (-1)^m F_{n-m} = F_m L_n, \tag{2.3}$$

$$L_{n+m} + (-1)^m L_{n-m} = L_m L_n, \tag{2.4}$$

$$L_n^2 + 4(-1)^{n+1} = 5F_n^2, \tag{2.5}$$

$$F_{2n} = F_n L_n. \tag{2.6}$$

If $\mathcal{F}_n = \mathcal{F}_n(a)$ and $\mathcal{L}_n = \mathcal{L}_n(a)$ denote the n^{th} generalized Fibonacci and Lucas numbers, respectively, then for $n \geq 1$,

$$\begin{pmatrix} a^2+1 & a^2 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} \mathcal{F}_{2n+1} & a\mathcal{F}_{2n} \\ a^{-1}\mathcal{F}_{2n} & \mathcal{F}_{n-1} \end{pmatrix}, \tag{2.7}$$

$$\mathcal{L}_n = a\mathcal{F}_n + 2\mathcal{F}_{n-1}, \tag{2.8}$$

$$\mathcal{L}_n^2 = 4\mathcal{F}_{n+1}\mathcal{F}_{n-1} + a^2\mathcal{F}_n^2, \tag{2.9}$$

$$\mathcal{F}_n\mathcal{F}_{n+2} - \mathcal{F}_{n+1}^2 = (-1)^{n+1}, \tag{2.10}$$

$$\mathcal{F}_{n+2}\varphi_a + \mathcal{F}_{n+1} = \varphi_a(\mathcal{F}_{n+1}\varphi_a + \mathcal{F}_n), \tag{2.11}$$

$$(a^2+4)\mathcal{F}_n^2 + 4(-1)^n = \mathcal{L}_n^2. \tag{2.12}$$

For a real number α, we write $\alpha = [a_0, a_1, \ldots]$ for the simple continued fraction expansion of α. That is,

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

where all the a_n are integers and $a_n > 0$ for all $n > 0$ (for further details, see [5]). Basic to our method is a fundamental connection between 2×2 matrices and formal continued fractions. This connection has been popularized recently by Stark [6] and by van der Poorten [8] and [9]. Let $c_0, c_1, \ldots, c_N$ be real numbers. Then the *fundamental correspondence* may be stated as follows: If

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_N & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_N & p_{N-1} \\ q_N & q_{N-1} \end{pmatrix},$$

then

$$\frac{p_N}{q_N} = [c_0, c_1, \ldots, c_N].$$

We remark that since $c_0, c_1, \ldots, c_N$ are real numbers, p_N / q_N may not necessarily be rational.

3. THE PROOF OF THEOREM 2

We first consider the case in which n is even. Let α be the quadratic irrational defined by

$$\alpha = \left[\overline{a^2+1, \{1, a^2\}^{(n-4)/2}, 1, a^2-1, a^2+1, 1, \{a^2, 1\}^{(n-4)/2}, a^2+3}\right].$$

We will compute α via the fundamental correspondence between matrices and continued fractions. Thus, if we express the following matrix product as

$$\begin{pmatrix} a^2+1 & 1 \\ 1 & 0 \end{pmatrix}\left\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a^2 & 1 \\ 1 & 0 \end{pmatrix}\right\}^{(n-4)/2}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a^2-1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a^2+1 & 1 \\ 1 & 0 \end{pmatrix}\cdot$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\left\{\begin{pmatrix} a^2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right\}^{(n-4)/2}\begin{pmatrix} a^2+3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix},$$

then it follows that $\alpha = r/t$. In view of (2.7), we may express the above as

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a^2+1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathcal{F}_{n-3} & a^{-1}\mathcal{F}_{n-4} \\ a\mathcal{F}_{n-4} & \mathcal{F}_{n-5} \end{pmatrix}\begin{pmatrix} a^2 & 1 \\ a^2-1 & 1 \end{pmatrix}\cdot$$

$$\begin{pmatrix} a^2+2 & a^2+1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{F}_{n-3} & a\mathcal{F}_{n-4} \\ a^{-1}\mathcal{F}_{n-4} & \mathcal{F}_{n-5} \end{pmatrix}\begin{pmatrix} a^2+3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}.$$

The functional equation $\mathcal{F}_k = a\mathcal{F}_{k-1} + \mathcal{F}_{k-2}$ enables us to simplify the above product and carry out the multiplication to deduce

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{n+1}^2 + 2\mathcal{F}_n^2 & \mathcal{F}_n^2 \\ (a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2 & \mathcal{F}_{n-1}^2 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (\mathcal{F}_{n+1}^2 + 2\mathcal{F}_n^2)\alpha + \mathcal{F}_n^2 & \mathcal{F}_{n+1}^2 + 2\mathcal{F}_n^2 \\ ((a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2)\alpha + \mathcal{F}_{n-1}^2 & (a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2 \end{pmatrix}$$

Thus, we have

$$\alpha = \frac{r}{t} = \frac{(\mathcal{F}_{n+1}^2 + 2\mathcal{F}_n^2)\alpha + \mathcal{F}_n^2}{((a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2)\alpha + \mathcal{F}_{n-1}^2}$$

or, equivalently,

$$((a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2)\alpha^2 + (\mathcal{F}_{n-1}^2 - 2\mathcal{F}_n^2 - \mathcal{F}_{n+1}^2)\alpha - \mathcal{F}_n^2 = 0. \tag{3.1}$$

For ease of exposition, we make the following change of variables: let

$$A = (a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2, \quad B = \mathcal{F}_{n-1}^2 - 2\mathcal{F}_n^2 - \mathcal{F}_{n+1}^2, \quad C = -\mathcal{F}_n^2.$$

Since $\alpha > 0$, equation (3.1) gives

$$\alpha = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

Next, if we let $x = [0, 1, \alpha] = \alpha / (\alpha + 1)$, then

$$x = \frac{2C - B + \sqrt{B^2 - 4AC}}{2(A - B + C)}.$$

The expressions $2C - B$, $B^2 - 4AC$, and $A - B + C$ may be simplified slightly by successive applications of the functional equation for $\mathcal{F}_n$. It is an algebraically complicated but straightforward task to verify that

$$x = \frac{\mathcal{F}_n}{2}\left(\frac{a(a\mathcal{F}_n + 2\mathcal{F}_{n-1}) + \sqrt{(a^2+4)(4\mathcal{F}_{n+1}\mathcal{F}_{n-1} + a^2\mathcal{F}_n^2)}}{4\mathcal{F}_{n+1}\mathcal{F}_{n-1} + a^2\mathcal{F}_n^2}\right). \tag{3.2}$$

Finally, by (2.8) and (2.9), we have

$$\mathcal{L}_n^2 = (a\mathcal{F}_n + 2\mathcal{F}_{n-1})^2 = 4\mathcal{F}_{n+1}\mathcal{F}_{n-1} + a^2\mathcal{F}_n^2,$$

and therefore, (3.2) implies

$$x = \frac{\mathcal{F}_n\left(a\mathcal{L}_n + \mathcal{L}_n\sqrt{a^2+4}\right)}{2\mathcal{L}_n^2} = \frac{\mathcal{F}_n\left(a + \sqrt{a^2+4}\right)}{2\mathcal{L}_n} = \frac{\mathcal{F}_n}{\mathcal{L}_n}\varphi_a,$$

which, by (1.4), is precisely equation (1.2) for even n.

The proof of (1.2) for n odd is similar to the even case given above. In particular, for n odd, we let

$$\alpha = \left[\overline{a^2+1, \{1, a^2\}^{(n-5)/2}, 1, a^2+1, a^2-1, 1, \{a^2, 1\}^{(n-3)/2}, a^2+3}\right].$$

Thus, in the language of matrices, we have

$$\begin{pmatrix} a^2+1 & 1 \\ 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^2 & 1 \\ 1 & 0 \end{pmatrix} \right\}^{(n-5)/2} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^2+1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^2-1 & 1 \\ 1 & 0 \end{pmatrix} \cdot$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} a^2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}^{(n-3)/2} \begin{pmatrix} a^2+3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

with $\alpha = r/t$. Simplifying the matrix product, as in the case for n even, reveals

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{n+1}^2 + 2\mathcal{F}_n^2 & \mathcal{F}_n^2 \\ (a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2 & \mathcal{F}_{n-1}^2 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}.$$

Equation (1.2) for n odd now follows from the previous argument.

Equation (1.3) follows immediately from (1.2) and Theorem 11 of [4], which completes the proof.

4. THE PROOF OF THEOREM 3

We essentially adopt the argument used in the proof of Theorem 2. First, we consider the case in which n is even. Let α be the quadratic irrational defined by

$$\alpha = \left[\overline{3, \{1\}^{n-2}, L_{2n}-2, 2, \{1\}^{n-4}, 2, 1, L_{2n}-2}\right].$$

By the fundamental correspondence between matrices and continued fractions, we observe that if we express the following matrix product as

$$\begin{gathered} \left(\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} L_{2n}-2 & 1 \\ 1 & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-4} \cdot \right. \\ \left. \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} L_{2n}-2 & 1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}, \end{gathered} \tag{4.1}$$

then we have $\alpha = r/t$. Using (2.1), (2.2), (2.3), and (2.4) together with the fact that n is even, we multiply and simplify the products within the parentheses to produce

$$\begin{aligned} \begin{pmatrix} r & s \\ t & u \end{pmatrix} &= \begin{pmatrix} L_{3n}-L_{n-2} & L_n \\ F_{3n-1}+2F_{n-2} & F_{n-1} \end{pmatrix} \begin{pmatrix} L_{3n}-F_{n-1} & L_n \\ L_{3n-2}+F_{n+3}+L_{n-3} & L_{n-2} \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} k_1 &= L_{6n} - F_{4n-1} - F_{2n+1} + 1, \\ k_2 &= L_{3n}L_n, \\ k_3 &= (F_{3n-1}+2F_{n-2})(L_{3n}-F_{n-1}) + F_{n-1}(L_{3n-2}+F_{n+3}+L_{n-3}), \\ k_4 &= F_{4n-1} + F_{2n+1} - 1. \end{aligned} \tag{4.3}$$

We note that identities (4.1) and (4.2) lead to a complicated, but useful, identity involving Fibonacci and Lucas numbers. In particular, we observe that

$$\det\left(\begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}\right) = (-1)^{2n},$$

and thus we have

$$k_1k_4 - k_2k_3 = 1. \tag{4.4}$$

By identity (4.2), we have

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} k_1\alpha + k_2 & k_1 \\ k_3\alpha + k_4 & k_2 \end{pmatrix},$$

and since $\alpha = r/t$, this implies

$$\alpha = \frac{k_1\alpha + k_2}{k_3\alpha + k_4}.$$

Therefore, $k_3\alpha^2 + (k_4 - k_1)\alpha - k_2 = 0$, hence,

$$\alpha = \frac{k_1 - k_4 + \sqrt{(k_4 - k_1)^2 + 4k_2k_3}}{2k_3}.$$

If we now let $x = [F_{n+1} - 1, \alpha]$, then

$$\begin{aligned} x &= F_{n+1} - 1 + \frac{2k_3}{k_1 - k_4 + \sqrt{(k_4 - k_1)^2 + 4k_2k_3}} \\ &= F_{n+1} - 1 + \frac{k_1 - k_4 - \sqrt{(k_4 - k_1)^2 + 4k_2k_3}}{-2k_2}. \end{aligned} \tag{4.5}$$

By (4.3), we note that $k_1 + k_4 = L_{6n}$. This, together with (4.4), reveals that

$$\begin{aligned} (k_4 - k_1)^2 + 4k_2k_3 &= (k_4 + k_1)^2 - 4k_1k_4 + 4k_2k_3 \\ &= L_{6n}^2 - 4(k_1k_4 - k_2k_3) = L_{6n}^2 - 4. \end{aligned}$$

In view of (2.5) and the fact that n is even, we may express the above as $(k_4 - k_1)^2 + 4k_2k_3 = 5F_{6n}^2$. This, along with (4.5), (2.2), (2.3), (2.4), and (2.6), yields

$$\begin{aligned} [F_{2n+1} - 1, \alpha] &= F_{2n+1} - 1 + \frac{L_{6n} - 2F_{4n-1} - 2F_{2n+1} + 2 - \sqrt{5}F_{6n}}{-2(L_{3n}L_n)} \\ &= \frac{-2F_{2n+1}L_{3n}L_n + 2L_{3n}L_n + L_{6n} - 2F_{4n-1} - 2F_{2n+1} + 2 + \sqrt{5}F_{6n}}{-2(L_{3n}L_n)} \\ &= \frac{-F_{6n} - \sqrt{5}F_{6n}}{-2L_{3n}L_n} = \frac{L_{3n}F_{3n} + L_{3n}F_{3n}\sqrt{5}}{2L_{3n}L_n} = \frac{F_{3n} + F_{3n}\sqrt{5}}{2L_n} = \frac{F_{3n}}{L_n}\varphi, \end{aligned}$$

which completes the proof for n even.

The proof for n odd is similar to the even case given above with the exception that the change of variables of (4.3) is replaced by

$$\begin{aligned} k_1 &= L_{6n} - F_{4n+1} - F_{2n-1} - 1, \\ k_2 &= L_{3n}L_n, \\ k_3 &= F_{3n+1}(L_{3n} - F_{n+1}) + F_{n+1}(L_{3n-1} + L_{n+1} + F_{n-2}), \\ k_4 &= F_{4n+1} - F_{2n-1} + 1. \end{aligned}$$

5. THE PROOF OF THEOREM 6

It is a classical result from the theory of continued fractions that α and β are equivalent numbers if and only if there exist integers a, b, c, and d so that $\alpha = (a\beta + b)/(c\beta + d)$ with $ad - bc = \pm 1$ (see [5]). Since m and n are relatively prime positive integers, we may find positive integers x and y so that $nx - my = 1$. Thus, if we let $k = 2ym$, then we also have $k = 2xn - 2$. We now define

$$a = \frac{\mathcal{L}_m \mathcal{F}_{k+2}}{\mathcal{L}_n}, \quad b = c = \mathcal{F}_{k+1}, \quad d = \frac{\mathcal{L}_n \mathcal{F}_k}{\mathcal{L}_m}.$$

As we remarked in the introduction, since $\mathcal{F}_{k+2} = \mathcal{F}_{(2x)n}$ and $\mathcal{F}_k = \mathcal{F}_{(2y)m}$, $\mathcal{F}_{k+2} / \mathcal{L}_n$ and $\mathcal{F}_k / \mathcal{L}_m$ are both integers. Thus, a, b, c, and d are all integers. Also, by (2.10), we note that

$$ad - bc = \mathcal{F}_k \mathcal{F}_{k+2} - \mathcal{F}_{k+1}^2 = (-1)^{k+1} = \pm 1.$$

Next, in light of (2.11), we have

$$\frac{a\left(\dfrac{\mathcal{L}_n}{\mathcal{L}_m}\varphi_a\right) + b}{c\left(\dfrac{\mathcal{L}_n}{\mathcal{L}_m}\varphi_a\right) + d} = \frac{\mathcal{L}_m}{\mathcal{L}_n}\left(\frac{\mathcal{F}_{k+2}\varphi_a + \mathcal{F}_{k+1}}{\mathcal{F}_{k+1}\varphi_a + \mathcal{F}_k}\right) = \frac{\mathcal{L}_m}{\mathcal{L}_n}\varphi_a.$$

Hence, $(\mathcal{L}_m / \mathcal{L}_n)\varphi_a$ and $(\mathcal{L}_n / \mathcal{L}_m)\varphi_a$ are equivalent numbers.

6. A RELATED CLASS OF QUADRATIC IRRATIONALS

For integers $n > 2$ and $m > 0$, we define the quadratic irrational $\mathfrak{R}(n, m) = \mathfrak{R}(a; n, m)$ by

$$\mathfrak{R}(n, m) = \frac{a\mathcal{F}_n + \mathcal{L}_m \sqrt{a^2 + 4}}{2}.$$

It will also be useful to define the integer $N = N(n, m)$ to be $N = a\mathcal{F}_n + (a^2 + 4)\mathcal{F}_m$. We now examine the continued fraction expansion for $\mathfrak{R}(n, m)$. We consider separately the case of N even and the case of N odd. As will be evident, the case of N odd is substantially more complicated that the case of N even.

Theorem 7: If N is even, then

(i) if m is even,

$$\mathfrak{R}(n, m) = \left[N/2, \overline{\mathcal{F}_m, (a^2 + 4)\mathcal{F}_m}\right];$$

(ii) if m is odd and $\mathcal{F}_m > 2$,

$$\mathfrak{R}(n, m) = \left[(N-2)/2, \overline{1, \mathcal{F}_m - 2, 1, (a^2 + 4)\mathcal{F}_m - 2}\right].$$

Theorem 8: If N is odd, then

(i) if m is even, $\mathcal{F}_m \equiv 0 \bmod 4$ and $\mathcal{F}_m > 4$,

$$\mathfrak{R}(n, m) = \left[(N-1)/2, \overline{1, 1, (\mathcal{F}_m - 4)/4, 1, 1, (a^2 + 4)\mathcal{F}_m - 1}\right];$$

(ii) if m is even, $\mathcal{F}_m \equiv 1 \bmod 4$ and $\mathcal{F}_m > 5$,

$$\mathcal{R}(n,m) = \left[(N-1)/2, \overline{\overleftarrow{\mathcal{W}}_1, 4\mathcal{F}_m, \mathcal{W}_1, (a^2+4)\mathcal{F}_m - 1}\right],$$

where $\mathcal{W}_1 = \left(1, 1, (\mathcal{F}_m - 5)/4, 1, 3, ((a^2+4)\mathcal{F}_m - 1)/4\right)$;

(iii) if m is even, $\mathcal{F}_m \equiv 3 \bmod 4$ and $\mathcal{F}_m > 3$,

$$\mathcal{R}(n,m) = \left[(N-1)/2, \overline{\overleftarrow{\mathcal{W}}_2, 4\mathcal{F}_m, \mathcal{W}_2, (a^2+4)\mathcal{F}_m - 1}\right],$$

where $\mathcal{W}_2 = \left(1, 1, (\mathcal{F}_m - 3)/4, 3, 1, ((a^2+4)\mathcal{F}_m - 3)/4\right)$;

(iv) if m is odd, $\mathcal{F}_m \equiv 1 \bmod 4$ and $F_m > 5$,

$$\mathcal{R}(n,m) = \left[(N-1)/2, \overline{\overleftarrow{\mathcal{W}}_3, \mathcal{F}_m - 1, \mathcal{W}_3, (a^2+4)\mathcal{F}_m - 1}\right],$$

where $\mathcal{W}_3 = \left(2, (\mathcal{F}_m - 5)/4, 1, 2, 1, ((a^2+4)\mathcal{F}_m - 5)/4, 2\right)$;

(v) if m is odd, $\mathcal{F}_m \equiv 2 \bmod 4$ and $F_m > 6$,

$$\mathcal{R}(n,m) = \left[(N-1)/2, \overline{\overleftarrow{\mathcal{W}}_4, 4(a^2+4)\mathcal{F}_m - 2, \mathcal{W}_4, (a^2+4)\mathcal{F}_m - 1}\right],$$

where $\mathcal{W}_4 = (2, (\mathcal{F}_m - 6)/4, 1)$.

Since the proof of Theorem 8 involves the same ideas as the proof of Theorem 7, we include only the (less complicated) proof of Theorem 7. Before proceeding with the proof of Theorem 7, we make three remarks.

First, it may appear that Theorem 8 is not complete in the sense that three cases seem to be missing; in particular, the cases: m even, $\mathcal{F}_m \equiv 2 \bmod 4$; m odd, $\mathcal{F}_m \equiv 0 \bmod 4$; m odd, $\mathcal{F}_m \equiv 3 \bmod 4$. It is a straightforward calculation to verify that none of these cases can occur when N is odd. For example, one has that $\mathcal{F}_m \equiv 2 \bmod 4$ only if either $a \equiv 1 \bmod 4$ and $m \equiv 3 \bmod 6$ or $a \equiv 3 \bmod 4$ and $m \equiv 3 \bmod 6$ or $a \equiv 2 \bmod 4$ and $m \equiv 2 \bmod 4$. In the first two cases, m is odd, and in the third case a is even; thus, N must be even. So if $\mathcal{F}_m \equiv 2 \bmod 4$, then we cannot have both m even and N odd. Similarly, the other two remaining cases may be shown not to occur. Therefore, Theorem 8 gives the complete situation for odd N. Our second remark involves the numbers $((a^2+4)\mathcal{F}_m - 1)/4$, $((a^2+4)\mathcal{F}_m - 3)/4$, and $((a^2+4)\mathcal{F}_m - 5)/4$ occurring in cases (ii), (iii), and (iv), respectively. Of course, we must require that these be integers. It is easy to see that each is an integer in the appropriate case if and only if a is odd. However, again, if a were even, then N would be even and Theorem 7 would apply. Hence, if N is odd, then a is also odd; there-fore, the three numbers above are indeed integers as required. Third, we note that the period length for $\mathcal{R}(n,m)$ is either 2, 4, 6, 8, 14, or 16. This proves an observation made by Long [2] that the period is always even, but it also shows that the period length is, in fact, a bounded func-tion of a, n, and m which Long believed not to be the case.

Proof of Theorem 7: We consider first the case of m even and let

$$\alpha = \left[N/2, \overline{\mathcal{F}_m, (a^2+4)\mathcal{F}_m}\right].$$

We now examine the corresponding matrix product:

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} N/2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{F}_m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (a^2+4)\mathcal{F}_m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (a^2+4)\mathcal{F}_m + \alpha - N/2 & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows that $\alpha = r/t$, in particular,

$$\alpha = \frac{(N\mathcal{F}_m/2+1)((a^2+4)\mathcal{F}_m + \alpha - N/2) + N/2}{\mathcal{F}_m((a^2+4)\mathcal{F}_m + \alpha - N/2) + 1}.$$

Thus, we have

$$\begin{aligned} &\mathcal{F}_m\alpha^2 + (\mathcal{F}_m((a^2+4)\mathcal{F}_m - N/2) - N\mathcal{F}_m/2)\alpha \\ &\quad + (N/2)^2\mathcal{F}_m - (a^2+4)\mathcal{F}_m - N(a^2+4)\mathcal{F}_m^2/2 = 0 \end{aligned}$$

which, together with our definition of N yields

$$\alpha = \frac{a\mathcal{F}_n + \sqrt{((a^2+4)\mathcal{F}_m^2 + 4)(a^2+4)}}{2}.$$

As m is even, identity (2.12) becomes $(a^2+4)\mathcal{F}_m^2 + 4 = \mathcal{L}_m^2$; hence, $\alpha = \mathcal{R}(n, m)$.

If m is odd, we again let

$$\alpha = \left[(N-2)/2, \overline{1, \mathcal{F}_m - 2, 1, (a^2+4)\mathcal{F}_m - 2}\right],$$

and proceed in a similar manner to deduce

$$\alpha = \frac{N - (a^2+4)\mathcal{F}_m + \sqrt{((a^2+4)\mathcal{F}_m^2 - 4)(a^2+4)}}{2}.$$

In view of identity (2.12) with m odd, together with the definition of N, we have $\alpha = \mathcal{R}(n, m)$, which completes the proof.

As a consequence of the two previous theorems and a result of Long [2], we are able to deduce immediately the continued fraction expansion for numbers of the form

$$\mathcal{S}(n, m) = \frac{a\,\mathcal{L}_n + \mathcal{L}_m\sqrt{a^2+4}}{2}.$$

Long proved (Theorem 8, [2]) that the continued fraction expansions of $\mathcal{R}(n, m)$ and $\mathcal{S}(n, m)$ are identical after the first partial quotient. In view of the two theorems of this section, it appears clear that one may explicitly express the continued fraction expansion for

$$\frac{a\mathcal{F}_n + \mathcal{L}_m\sqrt{a^2+4}}{2}$$

and, thus, by Theorem 9 of [2], the expansion for

$$\frac{a\mathcal{F}_n + \mathcal{F}_m\sqrt{a^2+4}}{2}.$$

It seems very reasonable to conjecture that these period lengths will again be even and bounded.

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AN EXTENSION OF STIRLING NUMBERS

David Branson
Department of Mathematics, University of Essex, Wivenhoe Park, Colchester CO4 3SQ, England
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1. INTRODUCTION

Stirling numbers may be defined as the coefficients in an expansion of positive integral powers of a variable in terms of factorial powers, or vice-versa:

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k, \quad n \geq 0, \tag{1.1}$$

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \quad n \geq 0, \tag{1.2}$$

where

$$\begin{aligned}(x)_n &= x(x-1)\cdots(x-n+1), \quad n \geq 1,\\ (x)_0 &= 1.\end{aligned} \tag{1.3}$$

The numbers $s(n, k)$ and $S(n, k)$ are, in the notation of Riordan [6], Stirling numbers of the first and second kind, respectively.

Writing (1.3) as

$$(x)_n = \Gamma(x+1)/\Gamma(x-n+1) \tag{1.4}$$

we may generalize factorial powers to negative integral values of n:

$$(x)_{-n} = \frac{1}{(x+1)(x+2)\cdots(x+n)}, \quad n \geq 1.$$

The question then arises as to whether we can extend Stirling numbers to negative integral values of one or both of their arguments. Several authors have discussed the case where both n and k are negative integers (for a brief history, see Knuth [4]), and we briefly discuss this case in Section 3. We shall refer to such numbers as Negative-Negative Stirling Numbers (NNSN) whereas we call the numbers defined by (1.1) and (1.2) Positive-Positive Stirling Numbers (PPSN). It is clearly impossible to have an expansion of the form (1.1) or (1.2), where n is a positive integer and k is summed over negative values: in the first case, the left-hand side is a polynomial whereas the right-hand side has a singularity at $x = 0$; in the second case, taking the limit as x goes to infinity, the left-hand side goes to infinity whereas the right-hand side goes to zero (for a proof of the uniform convergence required to take the limit term-by-term, see, for example, Milne-Thomson [5]). However, it is possible to extend (1.1) and (1.2) to the case where n is a negative integer and k is summed over positive integers. We call the resulting coefficients Negative-Positive Stirling Numbers (NPSN), and the purpose of this article is to discuss these numbers and some of their properties.

In Section 2 we summarize some well-known properties of PPSN. In Section 3 we describe four urn models: the first two illustrate PPSN and the second two demonstrate the connection between NNSN and PPSN. We define NPSN in Section 4 and obtain explicit expressions for them by means of two further urn models. Finally, in Section 5 we give some alternative representations of NPSN, tabulate some values and derive some properties.

2. PROPERTIES OF POSITIVE-POSITIVE STIRLING NUMBERS

In this section we list, for future reference, some well-known properties of PPSN. For more details, see, for example, Jordan [3], Chapter IV.

By using the definition (1.1) and the identity

$$(x)_{n+1} = (x)_n(x-n), \tag{2.1}$$

and by equating coefficients of powers of x, we may derive the recurrence relation

$$s(n+1, k) = s(n, k-1) - ns(n, k). \tag{2.2}$$

Similarly, the definition (1.2) and the identity

$$x^{n+1} = x^n(x-k+k), \tag{2.3}$$

and the equating of factorial powers of x, leads to

$$S(n+1, k) = S(n, k-1) + kS(n, k). \tag{2.4}$$

Numerical values for PPSN may readily be generated using (2.2) or (2.4) together with the boundary values [which follow immediately from (1.1) and (1.2)]

$$s(n, n) = 1, \quad s(n+1, 0) = 0, \quad n \geq 0, \tag{2.5}$$

$$S(n, n) = 1, \quad S(n+1, 0) = 0, \quad n \geq 0. \tag{2.6}$$

We may define a generating function for $S(n, k)$ (with respect to n) by

$$A_k(t) = \sum_{n=k}^{\infty} t^n S(n, k).$$

By using (2.4), we can obtain a first-order linear difference equation for A_k which, with the initial condition $A_0(t) = 1$, can be solved to give

$$\frac{t^k}{(1-t)(1-2t)\cdots(1-kt)} = \sum_{n=k}^{\infty} t^n S(n, k), \quad k \geq 1. \tag{2.7}$$

The left-hand side of (2.7) can be expressed in partial fractions:

$$\frac{1}{k!}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\frac{1}{1-rt}.$$

Expanding the last factor by the binomial theorem and comparing with (2.7) gives the following representation, known as Stirling's formula:

$$S(n, k) = \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}r^n. \tag{2.8}$$

3. URN MODELS AND STIRLING NUMBERS

In all the models of this section and the next, an urn contains white balls and black balls. Some, but not necessarily all, of the balls are of unit mass. Balls are drawn one at a time with replacement, and the probability that a particular ball is drawn is proportional to its mass.

Model A

An urn originally contains only white balls of total mass x $(> n-1)$ including at least $n-1$ balls of unit mass. We successively draw n balls with replacement. After a white ball is drawn (and replaced) and before the next ball is drawn, we substitute a unit mass white ball in the urn by a unit mass black ball. Then the probability that the number of white balls drawn is k is equal to

$$P_n^A(k) = \Sigma \frac{x}{x}\left(\frac{1}{x}\right)^{a_1} \frac{x-1}{x}\left(\frac{2}{x}\right)^{a_2} \frac{x-2}{x} \cdots \left(\frac{k-1}{x}\right)^{a_{k-1}} \frac{x-k+1}{x}\left(\frac{k}{x}\right)^{a_k}, \quad a \le k \le n, \tag{3.1}$$

where the sum is over all nonnegative integers a_i satisfying $a_1 + \cdots + a_k = n-k$. It is clear that, for $n \ge 1$, $P_n^A(0) = 0$, so (3.1) may be rewritten as

$$P_n^A(k) = \frac{(x)_k}{x^n} b(n, k), \tag{3.2}$$

with

$$b(n, k) = \Sigma u_1 u_2 \cdots u_{n-k}, \quad 1 \le k < n, \tag{3.3}$$

where the sum is over all integers u_i $(i = 1, 2, \ldots, n-k)$ satisfying

$$1 \le u_1 \le u_2 \le \cdots \le u_{n-k} \le k;$$
$$b(n, 0) = 0, \qquad b(n, n) = 1.$$

In terms of the urn model, if $n = 0$, then certainly $k = 0$, so (3.2) continues to hold if we put $b(0, 0) = 1$.

From the condition $\Sigma_{k=0}^n P_n^A(k) = 1$ and equation (3.2), we deduce

$$x^n = \sum_{k=0}^{n} (x)_k b(n, k).$$

Comparing this expression with (1.2), we conclude that a representation for $S(n, k)$ is provided by

$$S(n, k) = b(n, k). \tag{3.4}$$

Model B

An urn originally contains only white balls of total mass y. We successively draw n balls with replacement. After each ball is drawn (and replaced) and before the next ball is drawn, we add one black ball of unit mass to the urn. The probability that the number of white balls drawn is k is equal to

$$P_n^B(k) = \frac{y^k \Sigma v_1 v_2 \cdots v_{n-k}}{y(y+1) \cdots (y+n-1)}, \quad 1 \le k < n, \tag{3.5}$$

where the sum is over all integers v_i $(i = 1, 2, \ldots, n-k)$ satisfying

$$1 \le v_1 < v_2 < \cdots < v_{n-k} \le n-1. \tag{3.6}$$

The factors in the denominator of (3.5) represent the total mass of the balls in the urn at successive drawings; the factor y^k arises from drawing a white ball on k occasions, and the factor v_i arises from the drawing of a black ball at a time when the urn contains v_i black balls.

Note that, for $n \geq 1$, $P_n^B(0) = 0$ and $P_n^B(n) = y^n / [y(y+1) \cdots (y+n-1)]$, so that (3.5) may be written as

$$P_n^B(k) = \frac{y^k c(n, k)}{y(y+1) \cdots (y+n-1)}, \quad 0 \leq k \leq n, \tag{3.7}$$

with

$$c(n, k) = \sum v_1 v_2 \cdots v_{n-k}, \quad 1 \leq k < n, \tag{3.8}$$

under the restriction (3.6); $c(n, 0) = 0$, $c(n, n) = 1$.

Putting $y = -x$, (3.7) may be written as

$$P_n^B(k) = \frac{x^k}{(x)_n} (-1)^{n+k} c(n, k). \tag{3.9}$$

As in Model A, if $n = 0$ then $k = 0$, so (3.9) holds if $c(0, 0) = 1$. The probabilities in (3.9) must add to one; hence,

$$(x)_n = \sum_{k=0}^{n} x^k (-1)^{n+k} c(n, k).$$

Comparing this with (1.1), we see that

$$s(n, k) = (-1)^{n+k} c(n, k) \tag{3.10}$$

where $c(n, k)$ is given by (3.8).

The PPSN representations derived above by means of Model A and Model B are, of course, well known and have been derived by other methods (see, for example, Jordan [3]). We now consider two urn models of relevance to Negative-Negative Stirling Numbers. By analogy with (1.1) and (1.2) the NNSN of the first and second kind, respectively, are defined by

$$(x)_{-n} = \sum_{k=0}^{\infty} s(-n, -k) x^{-k}, \quad n \geq 0, \tag{3.11}$$

$$x^{-n} = \sum_{k=0}^{\infty} S(-n, -k)(x)_{-k}, \quad n \geq 0. \tag{3.12}$$

Model C

The situation here is the same as in Model B except that now we continue to draw balls until we have drawn $n+1$ white balls. The probability that the total number of balls drawn is $k+1$ is equal to

$$P_{n+1}^C(k+1) = \frac{y^{n+1} \sum v_1 v_2 \cdots v_{k-n}}{y(y+1) \cdots (y+k)}, \quad 1 \leq n < k, \tag{3.13}$$

where $1 \leq v_1 < v_2 < \cdots < v_{k-n} \leq k-1$. If all the balls drawn are white, we have

$$P_{n+1}^C(n+1) = y^{n+1} / [y(y+1) \cdots (y+n)]$$

whereas, if $n = 0$, then certainly $k = 0$. It follows from (3.8) that (3.13) may be written as

$$P_{n+1}^C(k+1) = y^n (y)_{-k} c(k, n), \quad 0 \leq n \leq k. \tag{3.14}$$

Suppose that after the $(i+1)^{\text{th}}$ ball is drawn all the remaining balls drawn are black. The probability of this event is given by the infinite product

$$\prod_{m=1}^{\infty}\frac{i+m}{y+i+m}=\prod_{m=1}^{\infty}\left(1-\frac{y}{y+i+m}\right)=0$$

(see, for example, Ferrar [2], p. 147). It follows that the probability distribution (3.14) is proper and sums to one. Hence,

$$y^{-n}=\sum_{k=n}^{\infty}(y)_{-k}c(k,n).$$

The coefficients in an expansion such as this are unique (see Milne-Thomson [5], p. 288), so, by comparison with (3.12) and using (3.10), we see that

$$S(-n,-k)=c(k,n)=(-1)^{n+k}s(k,n),\quad 0\le n\le k \tag{3.15}$$

[and $S(-n,-k)=0$ for $0\le k<n$].

Model D

This is the same as Model A, except that the white balls originally in the urn have total mass x ($>n$) and include at least n balls of unit mass. We now continue to draw balls until we have drawn $n+1$ white balls. The probability that the total number of balls drawn is $k+1$ is equal to

$$P_{n+1}^{D}(k+1)=\Sigma\frac{x}{x}\left(\frac{1}{x}\right)^{a_1}\frac{x-1}{x}\left(\frac{2}{x}\right)^{a_2}\frac{x-2}{x}\cdots\left(\frac{n}{x}\right)^{a_n}\frac{x-n}{x},\quad 0\le n\le k,$$

where $a_1+\cdots+a_n=k-n$.

Using the notation of (3.3), we can write

$$P_{n+1}^{D}(k+1)=\frac{x(x-1)\cdots(x-n)b(k,n)}{x^{k+1}},\quad 0\le n\le k. \tag{3.16}$$

The probability that after the i^{th} white ball is drawn all the remaining balls drawn are black is

$$\lim_{m\to\infty}\left(\frac{i}{x}\right)^{m}=0,$$

so the probability distribution (3.16) is proper and sums to one.

Putting $x=-y$, (3.16) may be written as

$$P_{n+1}^{D}(k+1)=\frac{(-1)^{n+k}}{(y)_{-n}y^{k}}b(k,n),\quad 0\le n\le k.$$

If we sum over k and rearrange this equation, we obtain

$$(y)_{-n}=\sum_{k=n}^{\infty}y^{-k}(-1)^{n+k}b(k,n).$$

Comparing this with (3.11) (using the uniqueness of Laurent series coefficients) and using (3.4), we conclude

$$s(-n,-k)=(-1)^{n+k}b(k,n)=(-1)^{n+k}S(k,n),\quad 0\le n\le k, \tag{3.17}$$

[and $s(-n,-k)=0$ for $0\le k<n$].

Equations (3.15) and (3.17) show the interesting fact, noted by other authors (see Knuth [4]), that, apart from a sign, NNSN of the first (second) kind are obtained from PPSN of the second (first) kind by a reflection in the line $n = -k$.

4. NEGATIVE-POSITIVE STIRLING NUMBERS

Negative-Positive Stirling Numbers of the first and second kind may be defined by the obvious modifications of (1.1) and (1.2), respectively:

$$(x)_{-n} = \sum_{k=0}^{\infty} s(-n, k)x^k, \quad n \geq 1, \tag{4.1}$$

$$x^{-n} = \sum_{k=1}^{\infty} S(-n, k)(x)_k, \quad n \geq 0. \tag{4.2}$$

(The reason for the limits on n and k will become clear later.)

Two further urn models allow us to give explicit representations for these NPSN.

Model E

Originally the urn contains white balls of mass x (< 1) and black balls of mass $1-x$. Balls are drawn one at a time with replacement. After a black ball is drawn (and replaced) and before the next ball is drawn, we add one black ball of unit mass to the urn. We continue until $n+1$ white balls have been drawn. For $n \geq 0$, the probability that the number of black balls drawn is $k-1$ is equal to

$$P_{n+1}^E(k-1) = \Sigma\left(\frac{x}{1}\right)^{a_1} \frac{1-x}{1}\left(\frac{x}{2}\right)^{a_2} \frac{2-x}{2} \cdots \left(\frac{x}{k-1}\right)^{a_{k-1}} \frac{k-1-x}{k-1}\left(\frac{x}{k}\right)^{a_k+1}, \quad k \geq 2,$$

where the sum is over all nonnegative integers a_i satisfying $a_1 + \cdots + a_k = n$; if all the balls drawn are white, then $P_{n+1}^E(0) = x^{n+1}$. We may therefore write

$$P_{n+1}^E(k-1) = x^n(x)_k(-1)^{k-1}a(n, k)/k! \tag{4.3}$$

with

$$a(n, k) = \Sigma\frac{1}{u_1 u_2 \cdots u_n}, \quad n \geq 1, \tag{4.4}$$

where the sum is over all integers u_i $(i = 1, 2, \ldots, n)$ satisfying

$$1 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq k;$$
$$a(0, k) = 1.$$

The probability that after the i^{th} black ball is drawn all the remaining balls drawn are black is

$$\prod_{m=1}^{\infty}\frac{i+m-x}{i+m} = \prod_{m=1}^{\infty}\left(1 - \frac{x}{i+m}\right) = 0.$$

We conclude that the probabilities in (4.3) sum to one, and hence,

$$x^{-n} = \sum_{k=1}^{\infty}(x)_k(-1)^{k-1}a(n, k)/k!. \tag{4.5}$$

An expansion of x^{-n} of this form in terms of factorial powers $(x)_k$ has unique coefficients. The reason for this is that we have effectively put the coefficient of $(x)_0$ equal to zero (see Milne-Thomsom [5], pp. 305-06).

Comparison of (4.5) and (4.2) gives

$$S(-n,k)=(-1)^{k-1}a(n,k)/k!, \tag{4.6}$$

with $a(n,k)$ given by (4.4).

Model F

The rules for this model are the same as for Model E, except that now we continue until n *black* balls have been drawn. For $n \geq 1$, the probability that the number of white balls drawn is k is equal to

$$P_n^F(k)=\Sigma\left(\frac{x}{1}\right)^{a_1}\frac{1-x}{1}\left(\frac{x}{2}\right)^{a_2}\frac{2-x}{2}\cdots\left(\frac{x}{n}\right)^{a_n}\frac{n-x}{n}, \quad k\geq 0,$$

where $a_1+\cdots+a_n=k$.

Using (4.4), we may write

$$P_n^F(k)=x^k(1-x)\cdots(n-x)a(k,n)/n!. \tag{4.7}$$

The probability that after the $(i-1)^{\text{th}}$ black ball is drawn all the remaining balls drawn are white is

$$\lim_{m\to\infty}\left(\frac{x}{i}\right)^m=0$$

so, again, the probabilities must add to one.

Putting $x=-y$, (4.7) becomes

$$P_n^F(k)=\frac{y^k}{(y)_{-n}}(-1)^k a(k,n)/n!.$$

Summing over k gives

$$(y)_{-n}=\sum_{k=0}^{\infty}y^k(-1)^k a(k,n)/n!,$$

and a comparison with (4.1) gives

$$s(-n,k)=(-1)^k a(k,n)/n!. \tag{4.8}$$

5. PROPERTIES OF NEGATIVE-POSITIVE STIRLING NUMBERS

At the end of Section 3 we noted that

$$|S(-n,-k)|=|s(k,n)| \tag{5.1}$$

whenever n and k have the same sign. Equations (4.6) and (4.8) show that NPSN are related by the same reflection in the line $n=-k$, that is, (5.1) continues to hold if n is positive and k is negative. The consequence is that NPSN of the first and second kinds are (apart from a sign) the same set of numbers, but differently indexed, so that any property of NPSN of one kind can be immediately expressed as a property of the other kind. Explicitly,

$$S(-n, k) = (-1)^{n+k-1} s(-k, n). \tag{5.2}$$

Different Representations

If we regard (4.1) as a Taylor series, it follows, using (5.2), that

$$S(-n, k) = \frac{(-1)^{n+k-1}}{n!}\left[\frac{d^n}{dx^n}\left(\frac{1}{(x+1)(x+2)\cdots(x+k)}\right)\right]_{x=0}. \tag{5.3}$$

It is not difficult to show directly the equivalence of (4.6) and (5.3).

If we expand (5.3) by partial fractions, perform the n-fold differentiation, and put $x=0$, we obtain

$$S(-n, k) = \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}r^{-n}. \tag{5.4}$$

Thus, we note that Stirling's formula (2.8) continues to hold if n is negative.

Recurrence Formulas and Table of Values

The recurrence relations for PPSN, (2.2) and (2.4), were derived from the identities (2.1) and (2.3). These identities hold whatever the sign of n, and it follows that (2.2) and (2.4) continue to hold for NPSN. These relations can therefore be used, together with appropriate boundary values, to generate numerical values. For NPSN of the first kind, we may rewrite (2.2) as

$$s(-n+1, k) = s(-n, k-1) + ns(-n, k), \quad n \geq 2,\ k \geq 1. \tag{5.5}$$

On putting, respectively, $n=1$ and $x=0$ in the definition (4.1), we obtain

$$s(-1, k) = (-1)^k, \qquad s(-n, 0) = 1/n!. \tag{5.6}$$

Combining (5.5) with (5.6), we can generate the values for $s(-n, k)$ given in Table 1. Values for $S(-n, k)$ are then given by (5.2).

TABLE 1. Values of $s(-n, k)$

$n \backslash k$	0	1	2	3
1	1	-1	1	-1
2	$\frac{1}{2}$	$-\frac{3}{4}$	$\frac{7}{8}$	$-\frac{15}{16}$
3	$\frac{1}{6}$	$-\frac{11}{36}$	$\frac{85}{216}$	$-\frac{575}{1296}$
4	$\frac{1}{24}$	$-\frac{25}{288}$	$\frac{415}{3456}$	$-\frac{5845}{41472}$

Generating Functions

We derive a number of generating functions for NPSN. They can easily be translated into generating functions for the other kind of NPSN by use of (5.2).

The definition (4.1) itself provides a generating function for $s(-n, k)$. An exponential generating function can be obtained from (5.4):

$$\sum_{n=0}^{\infty} S(-n, k)\frac{y^n}{n!} = \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}\exp\left(\frac{y}{r}\right).$$

Two double generating functions can be obtained from (4.1) as follows:

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} s(-n, k)x^k y^n = \sum_{n=1}^{\infty}\frac{\Gamma(x+1)}{\Gamma(x+n+1)}y^n \qquad \text{by (4.1) and (1.4)}$$

$$= M(1, 1+x, y) - 1,$$

where M is a confluent hypergeometric function (see [1], p. 504). Similarly,

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} s(-n, k)x^k \frac{y^n}{n!} = {}_0F_1(1+x, y) - 1$$

$$= [\Gamma(x+1)I_x(2\sqrt{y}) / y^{x/2}] - 1,$$

where ${}_0F_1$ and I_x are, respectively, a generalized hypergeometric function and a modified Bessel function (see [1], pp. 556, 374, 377).

Another pair of double generating functions can be obtained from (4.2):

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} S(-n, k)(x)_k y^n = \frac{x}{x-y},$$

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} S(-n, k)(x)_k \frac{y^n}{n!} = \exp(y/x).$$

Asymptotic Behavior

If n is taken to infinity in (5.4), only the $r = 1$ term survives. Hence,

$$\lim_{n\to\infty} S(-n, k) = (-1)^{k-1}/(k-1)!.$$

The definition (4.4) implies that we can express $a(n, k)$ as $h_n(1, \frac{1}{2}, \ldots, \frac{1}{k})$, where h_n is a homogeneous product sum symmetric function (see Riordan [6], p. 47). Riordan shows that $n!h_n$ can be expressed as a (Bell) polynomial Y_n in the variables s_i $(i = 1, \ldots, n)$, where

$$s_i = 1 + \frac{1}{2^i} + \cdots + \frac{1}{k^i}.$$

As $k \to \infty$, all s_i tend to a finite limit apart from s_1 which behaves like $\ln k$. It is clear that the term involving the highest power of s_1 in Y_n is s_1^n. Hence, as $k \to \infty$, $Y_n \sim (\ln k)^n$. From (4.6), we conclude that

$$|S(-n, k)| \sim \frac{(\ln k)^n}{k!n!} \text{ as } k \to \infty.$$

Orthogonality and Other Relations

For $n \geq m \geq 1$, we have

$$(x)_n / (x)_m = (x-m)(x-m-1)\cdots(x-n+1) = (x-m)_{n-m},$$

and

$$\frac{1}{(x)_m} = \frac{1}{x(x-1)\cdots(x-m+1)} = \frac{(-1)^{m-1}}{x(-x+1)\cdots(-x+m-1)} = \frac{(-1)^{m-1}(-x)_{-m+1}}{x}.$$

Hence,

$$(x-m)_{n-m} = (-1)^{m-1}(x)_n(-x)_{-m+1}/x. \tag{5.7}$$

If, for $m \geq 2$, we expand the factorial powers by (1.1) and (4.1) we obtain

$$\sum_{k=0}^{n-m} s(n-m, k)(x-m)^k = (-1)^{m-1}x^{-1}\sum_{p=0}^{n} s(n, p)x^p \sum_{q=0}^{\infty} s(-m+1, q)(-x)^q. \tag{5.8}$$

Expanding the term $(x-m)^k$ on the left-hand side by the binomial theorem, and equating coefficients of like powers of x, gives, for $0 \leq r \leq n-m$, the following relation between PPSN and NPSN

$$\sum_{k=r}^{n-m} (-1)^k \binom{k}{r} s(n-m, k)m^{k-r} = \sum_{p=0}^{r+1} (-1)^{m+p} s(n, p)s(-m+1, r+1-p), \tag{5.9}$$

and for $r > n-m$, the orthogonality relation

$$\sum_{p=0}^{\min(r+1,\, n)} (-1)^p s(n, p)s(-m+1, r+1-p) = 0. \tag{5.10}$$

When $m > n \geq 1$, the left-hand side of (5.7) is replaced by

$$\frac{1}{(x-n)(x-n-1)\cdots(x-m+1)}. \tag{5.11}$$

If we express this function in partial fractions and then expand each term as a power series in x, we can again equate the coefficients of powers of x with those on the right-hand side of (5.8), obtaining, for $r \geq 0$,

$$\sum_{k=n}^{m-1} \frac{(-1)^k}{(k-n)!(m-1-k)!\,k^{r+1}} = \sum_{p=0}^{\min(r+1,\, n)} (-1)^{p+r} s(n, p)s(-m+1, r+1-p). \tag{5.12}$$

It is possible to obtain equivalent results involving NNSN [and hence, by (3.17), PPSN] instead of NPSN by using (3.11) to expand the term $(-x)_{-m+1}$ in (5.7). For $n \geq m \geq 1$ and $0 \leq r \leq n-m$, the only difference from (5.9) is that the sum on the right-hand side now goes from $p = m+r$ to $p = n$; if $r < 0$, the sum on the left-hand side of (5.10) now goes from $p = \max(m+r, 0)$ to $p = n$. Similarly, if $m > n > 0$, we expand the terms in the partial fraction version of (5.11) as power series in $(1/x)$ and equate coefficients of like powers. For $r \leq -(m-n)$, the right-hand side of (5.12) acquires an extra factor of (-1) and the sum now goes from $p = \max(m+r, 0)$ to n.

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THE ASYMPTOTIC BEHAVIOR OF THE GOLDEN NUMBERS

Helmut Prodinger
Department of Algebra and Discrete Mathematics, Technical University of Vienna, Austria
(*Submitted August 1994*)

In [2] the "Golden polynomials"

$$G_{n+2}(x) = xG_{n+1}(x) + G_n(x), \quad G_0(x) = -1, \; G_1(x) = x-1,$$

and their maximal real root g_n (the "golden numbers") were investigated. It was observed that, as $n \to \infty$, $g_n \to 3/2$; furthermore, it was suggested there might be a more precise formula, since numerical experiments seemed to indicate a dependency on the parity of n of the lower order terms.

This open question will be solved in the present paper.

Solving the recursion for the Golden polynomials by standard methods, we get the explicit formula

$$G_n(x) = A\lambda^n + B\mu^n,$$

with

$$\lambda = \frac{x+\sqrt{x^2+4}}{2}, \quad \mu = \frac{x-\sqrt{x^2+4}}{2},$$

$$A = \frac{1}{2\sqrt{x^2+4}}\left(3x-2-\sqrt{x^2+4}\right), \quad B = -\frac{1}{2\sqrt{x^2+4}}\left(3x-2+\sqrt{x^2+4}\right).$$

Everything is much nicer when we substitute

$$x = u - \frac{1}{u}.$$

$G_n(x) = 0$ can be rephrased as $-B/A = (\lambda/\mu)^n$, or

$$\frac{(2u+1)(u-1)}{(u+1)(u-2)} = (-u^2)^n.$$

Now it is plain to see that, for large n, this equation can only hold if u is either close to 2 or to $u = -1/2$. In both cases, this would mean x is close to 3/2. Let us assume that u is close to 2. It is clear that the cases when n is even or odd have to be distinguished. We start with $n = 2m$ and rewrite the equation as

$$u - 2 = \frac{(2u+1)(u-1)}{(u+1)} u^{-4m}.$$

We get the asymptotic behavior of the desired solution by a process known as "bootstrapping" which is explained in [1]. First, we set $u = 2+\delta$, insert $u = 2$ into the right-hand side, and get an approximation for δ. Then we insert $u = 2+\delta$ into the right-hand side, expand, and get the next term. This procedure can be repeated to get as many terms as needed. In this way, we get

$$\delta \sim \frac{5}{3}\cdot 16^{-m},$$

and with

$$u = 2 + \frac{5}{3} \cdot 16^{-m} + \varepsilon,$$

we find

$$\varepsilon \sim -\frac{25}{6} m \cdot 256^{-m}.$$

From

$$u \sim 2 + \frac{5}{3} \cdot 16^{-m} - \frac{50}{9} m \cdot 256^{-m}$$

we find by substitution

$$x \sim \frac{3}{2} + \frac{25}{12} \cdot 16^{-m} - \frac{125}{18} m \cdot 256^{-m}.$$

Now let us consider the case n is odd, $n = 2m+1$. Then our equation is

$$u - 2 = -\frac{(2u+1)(u-1)}{(u+1)u^2} u^{-4m},$$

and we find as above

$$u \sim 2 - \frac{5}{12} \cdot 16^{-m} - \frac{25}{72} m \cdot 256^{-m},$$

and also

$$x \sim \frac{3}{2} - \frac{25}{48} \cdot 16^{-m} - \frac{125}{288} m \cdot 256^{-m}.$$

Confining ourselves to two terms, we write our findings in a single formula as

$$g_n \sim \frac{3}{2} + (-1)^n \frac{25}{12} \cdot 4^{-n},$$

which matches perfectly with the empirical data from [2].

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DIGRAPHS FROM POWERS MODULO p

Caroline Lucheta
Box 1121 GCC, 100 Campus Drive, Grove City, PA 16127

Eli Miller
PO Box 410, Sumneytown, PA 18084

Clifford Reiter
Department of Mathematics, Lafayette College, Easton, PA 18042-1781

(Submitted August 1994)

1. INTRODUCTION

Given any function f defined modulo m, we can consider the digraph that has the residues modulo m as vertices and a directed edge (a, b) if and only if $f(a) \equiv b \pmod{m}$. This digraph can be thought of as a geometric representation of all the sequences generated by iterating f modulo m. The digraph associated with squaring modulo p, a prime, has been studied in [1]. In that paper the cycle lengths and the number of cycles appearing were characterized. The structure of the trees attached to cycle elements was also completely described. Our paper will generalize those results to the digraph associated with the function x^k modulo a prime p with k any positive integer. Since zero is an isolated cycle for all p and k, we will consider the digraph generated by the nonzero residues. Hence, the vertex set of the digraph is equal to Z_p^*. We will let G_p^k denote the digraph on the nonzero residues modulo p with edges given by $x^k \pmod{p}$. For example, G_{53}^3 is shown in Figure 1 and G_{41}^4 is shown in Figure 2. Note that, when $p=2$, G_p^k consists of the vertex 1 in a loop. Thus, we need only consider G_p^k when p is an odd prime. We will use p to denote an odd prime throughout this paper.

Elementary results about these digraphs are described in Section 2. In particular, we see that each component contains a single cycle and we can determine when there are noncycle vertices. Section 3 characterizes the cycle lengths that appear. Section 4 explores the relationship of geometric subsets of the digraph to subgroups of the group of units modulo p. Section 5 considers some special cases where long cycles occur. Section 6 returns to the basic structure of the digraph and shows that all the forests appearing must be isomorphic and characterizes their heights. Section 7 explores the simplifications of the structures that appear when k is prime.

We begin by enumerating six well-known elementary theorems which will be used. Proofs can be found in standard texts.

Theorem 1: If $a \not\equiv 0$, there are 0 or $\gcd(k, p-1)$ solutions to $x^k \equiv a \pmod{p}$.

Proof: See [3], p. 47. □

Theorem 2: If d is a positive integer such that $d \mid p-1$, then there are exactly $\phi(d)$ incongruent residues of order d modulo p.

Proof: See [3], p. 48. □

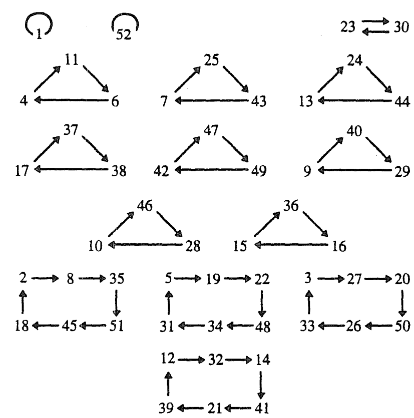

FIGURE 1. The Digraph of G_{53}^3

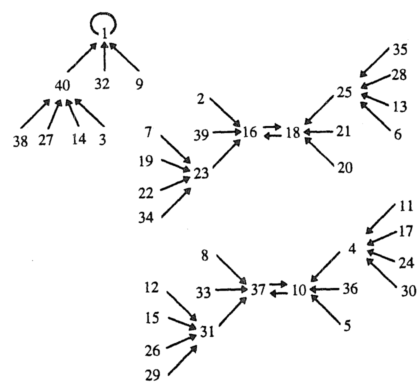

FIGURE 2. The Digraph of G_{41}^4

Theorem 3: If n is a positive integer, then

$$\sum_{d|n,\, d>0} \Phi(d) = n.$$

Proof: See [5], p. 83. □

Theorem 4: If a is an integer such that $\gcd(a, m) = 1$ and i is a positive integer, then

$$\mathrm{ord}_m a^i = \frac{\mathrm{ord}_m a}{\gcd(i, \mathrm{ord}_m a)}.$$

Proof: See [5], p. 132. □

Theorem 5: A primitive root modulo m exists if and only if m is of the form 2, 4, p^n, or $2p^n$, where p is an odd prime.

Proof: See [3], p. 49. □

Theorem 6: If a and b are elements of Z_p^* such that $\alpha = \mathrm{ord}_p a$, $\beta = \mathrm{ord}_p b$, and $\gcd(\alpha, \beta) = 1$, then $\mathrm{ord}_p ab = \alpha\beta$.

Proof: See [4], p. 46. □

2. BASIC PROPERTIES

The following lemmas are easy to prove but fundamental to the understanding of the digraph structure of G_p^k. We will see that, for all G_p^k, each graph component contains a unique cycle which may have forest structures attached to it.

Lemma 7: The outdegree of any vertex in G_p^k is one.

Proof: The function $x^k \pmod{p}$ maps the vertex a to a^k and only a^k. □

Lemma 8: Given any element in G_p^k, repeated iteration of $x^k \pmod{p}$ will eventually lead to a cycle.

Proof: Because there are $p-1$ vertices in G_p^k, iterating $x^k \pmod{p}$ must eventually produce a repeated value. □

Lemma 9: Every component of G_p^k contains exactly one cycle.

Proof: Suppose a component has more than one cycle; then, somewhere along the undirected path connecting any two cycles, there exists a vertex with outdegree at least 2, which is impossible. □

Lemma 10: The set of noncycle vertices leading to a fixed cycle vertex forms a forest.

Proof: Since each component contains exactly one cycle, the vertices leading to a cycle vertex cannot contain a cycle; thus, they are a forest. □

Lemma 11: The indegree of any vertex in G_p^k is 0 or $\gcd(k, p-1)$.

Proof: This result is an immediate application of Theorem 1. □

Lemma 12: Every component of G_p^k is cyclical if and only if $\gcd(k, p-1) = 1$.

Proof: ($\Rightarrow$) If all digraph components are cyclical both the indegree and outdegree are one, which implies from Lemma 11 that $\gcd(k, p-1) = 1$.

($\Leftarrow$) Conversely, if $\gcd(k, p-1) = 1$, the indegree of every vertex is 0 or 1. If some component were not cyclical, there would exist a cycle vertex with indegree ≥ 2, a contradiction. □

For example, each component of G_{53}^3 is cyclical because $\gcd(3, 52) = 1$; this is apparent in Figure 1. Likewise, G_{41}^4 has vertices outside the cycles because $\gcd(4, 40) = 4$ (see Figure 2). We will refer to a *child* of a vertex a as a vertex v that satisfies the equation $v^k \equiv a \pmod{p}$. These are the predecessors of a in G_p^k. Note that our child vertices are children in the sense of the forest structure but not in the standard sense of direction. Predecessors that are not in a cycle will be called *noncycle* children. For example, in G_{41}^4, 37 has the three noncycle children 8, 31, and 33.

Lemma 13: Any cycle vertex, c, has $\gcd(k, p-1) - 1$ noncycle children.

Proof: From Lemma 11, the indegree of c is 0 or $\gcd(k, p-1)$. Since c is a cycle vertex, the indegree is not zero but $\gcd(k, p-1)$. The number of noncycle children is $\gcd(k, p-1) - 1$. □

Above we saw some of the basic results about G_p^k. Notice that if we fix k and vary p the number of vertices changes and infinitely many different digraphs result. However, if p is fixed, only finitely many distinct digraphs result as k varies. The next theorem identifies the powers that result in identical digraphs.

Theorem 14: $k_1 \equiv k_2 \pmod{p-1}$ if and only if $G_p^{k_1} = G_p^{k_2}$.

Proof: ($\Rightarrow$) Suppose $k_1 \equiv k_2 \pmod{p-1}$ and without loss of generality $k_1 \geq k_2$. If a is any vertex in the reduced residue set, $\operatorname{ord}_p a \mid (p-1) \mid (k_1 - k_2)$, so $a^{k_1 - k_2} \equiv 1$ implies $a^{k_1} \equiv a^{k_2} \pmod{p}$. Hence, $G_p^{k_1} = G_p^{k_2}$.

($\Leftarrow$) Suppose $G_p^{k_1} = G_p^{k_2}$, and assume $k_1 \geq k_2$. Then $a^{k_1} \equiv a^{k_2} \pmod{p}$ implies $\operatorname{ord}_p a \mid (k_1 - k_2)$ for all vertices a. So $(p-1) \mid (k_1 - k_2)$, and the conclusion follows. □

The 12 different digraphs for G_{13}^k are shown in Figure 3. Notice that some have only cycles and some have forest structures. This theorem gives a condition for equality of digraphs, but does not settle the question of when two digraphs can be *isomorphic* for different values of k. For example, $G_{11}^2 \neq G_{11}^8$ but $G_{11}^2 \approx G_{11}^8$.

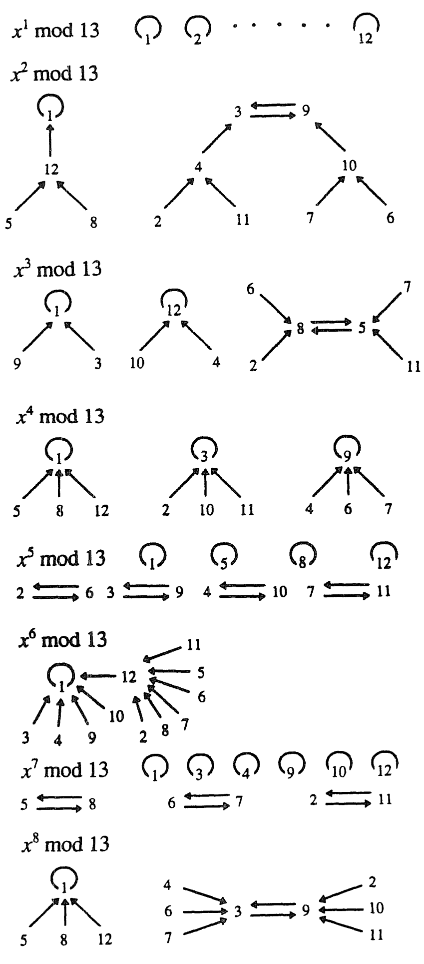

FIGURE 3. All Possible Digraphs of G_{13}^k

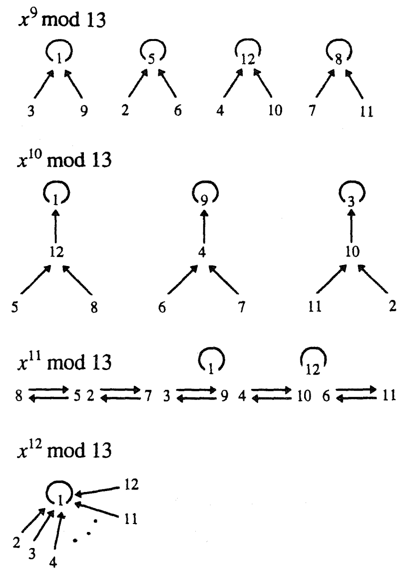

FIGURE 3. All Possible Digraphs of G_{13}^k (continued)

3. CHARACTERIZING CYCLES

When considering the cycle structure of G_p^k, it is convenient to factor $p-1$ as wt, where t is the largest factor of $p-1$ relatively prime to k. So $\gcd(k,t)=1$ and $\gcd(w,t)=1$. For example, if $p=41$ and $k=6$, then $p-1=2^35$, so $w=8$ and $t=5$. Similarly, if $p=47$ and $k=4$, then $w=2$ and $t=23$; also, if $p=19$ and $k=6$, then $w=18$ and $t=1$. In all the theorems below, we will be considering the digraph G_p^k with $p-1=wt$ as described.

Theorem 15: The vertex c is a cycle vertex if and only if $\mathrm{ord}_p c \mid t$.

Proof: ($\Rightarrow$) Since c is in a cycle there exists some $x \ge 1$ such that $c^{k^x} \equiv c$ and thus $c^{k^x-1} \equiv 1$ (mod p). Hence, $\mathrm{ord}_p c \mid k^x-1$, which implies that $\gcd(\mathrm{ord}_p c, k)=1$, so $\gcd(\mathrm{ord}_p c, w)=1$ also. We know that $\mathrm{ord}_p c \mid p-1=wt$, and so $\mathrm{ord}_p c$ must divide t.

($\Leftarrow$) Suppose $c \in G_p^k$ and $\mathrm{ord}_p c \mid t$; therefore, $\gcd(\mathrm{ord}_p c, k)=1$. On repeated iteration, c must eventually end up in a cycle. If y is the number of steps to reach the cycle and x is the cycle length, then $c^{k^y(k^x-1)} \equiv 1$ (mod p). Therefore, $\mathrm{ord}_p c \mid k^y(k^x-1)$, but since $\gcd(\mathrm{ord}_p c, k)=1$, $\mathrm{ord}_p c \mid k^x-1$. Hence, $c^{k^x} \equiv c$ (mod p), which implies that c is a cycle vertex. □

Corollary 16: There are t vertices in cycles.

Proof: From Theorem 15, the total number of cycle vertices is $\sum_{d \mid t} N(d)$, where $N(d)$ is the number of elements of order d (mod p). Theorems 2 and 3 imply that this is t. □

Theorem 17: Vertices in the same cycle have the same order (mod p).

Proof: Assume a and b are in the same cycle. Hence, there exists an e such that $a^e \equiv b$ (mod p). Let $\alpha = \mathrm{ord}_p a$ and $\beta = \mathrm{ord}_p b$. It follows that $b^\alpha \equiv a^{e\alpha} \equiv 1$ (mod p) and thus $\beta \mid \alpha$. Similarly, $\alpha \mid \beta$; hence, the orders are equal. □

Theorem 17 shows that the order (mod p) of vertices in the same cycle are equal. Hence, the notion of the *order of a cycle* is well defined. We now look at the relationship between the order of a cycle and its length.

Theorem 18: Let x be the length of a cycle of order d (mod p), then $k^x - 1$ is the smallest number of the form $k^n - 1$ divisible by d.

Proof: Let c be a vertex in the cycle. Since the cycle is of length x, $c^{k^x - 1} \equiv 1$ (mod p). It follows that $d = \mathrm{ord}_p c$ divides $k^x - 1$. If $\mathrm{ord}_p c \,|\, k^s - 1$ for some $s < x$, then $c^{k^s} \equiv c$, a contradiction. □

Theorem 18 shows that the length of a cycle depends entirely on its order. If we let $\ell(d)$ denote the *length of cycles* with order d, we get the following theorem.

Theorem 19: Let a, b, and d be orders of cycles in G_p^k. Then:

(i) $\ell(d) = \mathrm{ord}_d k$.

(ii) There are $\phi(d) / \ell(d)$ cycles of order d.

(iii) $\ell(\mathrm{lcm}(a, b)) = \mathrm{lcm}(\ell(a), \ell(b))$.

(iv) The longest cycle length is $\ell(t) = \mathrm{ord}_t k$.

Proof:

(i) By Theorem 18, $\ell(d) = \min\{n : d \,|\, k^n - 1\}$; hence, $\ell(d) = \mathrm{ord}_d k$.

(ii) By Theorem 2, there are $\phi(d)$ elements of order d, and there are $\ell(d)$ in each cycle by(i), hence the result.

(iii.a) By (i), $k^{\ell(a)} \equiv 1$ (mod a) and $k^{\ell(b)} \equiv 1$ (mod b); thus, $k^{\mathrm{lcm}(\ell(a), \ell(b))} \equiv 1$ (mod a) and $k^{\mathrm{lcm}(\ell(a), \ell(b))} \equiv 1$ (mod b). It follows that $k^{\mathrm{lcm}(\ell(a), \ell(b))} \equiv 1$ (mod lcm(a, b)). Therefore, $\ell(\mathrm{lcm}(a, b)) \,|\, \mathrm{lcm}(\ell(a), \ell(b))$.

(iii.b) We know that $k^{\ell(\mathrm{lcm}(a, b))} \equiv 1\,(\mathrm{mod}\ \mathrm{lcm}(a, b))$. Therefore, $k^{\ell(\mathrm{lcm}(a, b))} \equiv 1$ (mod a) and $k^{\ell(\mathrm{lcm}(a, b))} \equiv 1$ (mod b). Thus, $\ell(a) \,|\, \ell(\mathrm{lcm}(a, b))$ and $\ell(b) \,|\, \ell(\mathrm{lcm}(a, b))$, which implies that $\mathrm{lcm}(\ell(a), \ell(b)) \,|\, \ell(\mathrm{lcm}(a, b))$.

Putting (iii.a) and (iii.b) together gives $\ell(\mathrm{lcm}(a, b)) = \mathrm{lcm}(\ell(a), \ell(b))$.

(iv) All orders of cycles divide t and if $d \,|\, t$ then $\ell(t) = \ell(\mathrm{lcm}(t, d)) = \mathrm{lcm}(\ell(t), \ell(d))$, which implies $\ell(d) \,|\, \ell(t)$. Thus, $\ell(t)$ is the maximal cycle length. □

We are now in a position to identify the number of cycles of every length appearing in the digraph of G_p^k. For example, consider G_{53}^3 (Fig. 1); in this case $p - 1 = 52$ so $w = 1$ and $t = 52$. The possible orders of the cycle elements are the divisors of t: 1, 2, 4, 13, 26, and 52. There are $\phi(52) = 24$ elements of order 52, and these 24 elements are in cycles of length $\ell(52) = \mathrm{ord}_{52} 3 = 6$, contributing four cycles of length 6. Similarly, the elements of order 26 appear in 4 cycles of length 3; and those of order 13 are in 4 cycles of length 3. There are 2 elements of order 4 in one cycle and two cycles of length one with orders 1 and 2. Table 1 gives some details about cycles in G_p^k for selected p and k.

TABLE 1. Cycle Lengths in G_p^k

k	2			3			4			5			6		
p	d	$\ell(d)$	#	d	$\ell(d)$	#	d	$\ell(d)$	#	d	$\ell(d)$	#	d	$\ell(d)$	#
41	1 5	1 4	1 1	1 2 4 5 8 10 20 40	1 1 2 4 2 4 4 4	1 1 1 1 2 1 2 4	1 5	1 2	1 2	1 2 4 8	1 1 1 2	1 1 2 2	1 5	1 1	1 4
43	1 3 7 21	1 2 3 6	1 1 2 2	1 2 7 14	1 1 6 6	1 1 1 1	1 3 7 21	1 1 3 3	1 2 2 4	1 2 3 6 7 14 21 42	1 1 2 2 6 6 6 6	1 1 1 1 1 1 2 2	1 7	1 2	1 3
47	1 23	1 11	1 2	1 2 23 46	1 1 11 11	1 1 2 2	1 23	1 11	1 2	1 2 23 46	1 1 22 22	1 1 1 1	1 23	1 11	1 2
53	1 13	1 12	1 1	1 2 4 13 26 52	1 1 2 3 3 6	1 1 1 4 4 4	1 13	1 6	1 2	1 2 4 13 26 52	1 1 1 4 4 4	1 1 2 3 3 6	1 13	1 12	1 1

4. SUBGROUPS OF Z_p^* IN G_p^k

We now consider orders of elements throughout G_p^k. We will be able to associate elements of various orders with subgroups of Z_p^*, which allows for the identification of certain subgroups of Z_p^* with geometric subsets of the digraph. In later sections we will return to characterizing the cycle and forest structure of these digraphs.

Lemma 20: If H_d is the set of residues with orders dividing d, $d \geq 1$, then H_d is a cyclic subgroup of Z_p^*.

Proof: Since Z_p^* is a finite cyclic group, we need only show that H_d is nonempty and closed under multiplication. Clearly H_d is not empty because it has the identity. To show closure, suppose that $a, b \in H_d$ and let $\alpha = \text{ord}_p a$ and $\beta = \text{ord}_p b$. Since $(ab)^{\text{lcm}(\alpha,\beta)} \equiv 1 \pmod{p}$, $\text{ord}_p ab$ divides $\text{lcm}(\alpha, \beta)$ which in turn divides d. Therefore, H_d is a subgroup of Z_p^*. □

For instance, if we consider the group Z_{41}^*, the elements of order 1 and 2 form the subgroup H_2, while H_{10} contains those elements of order 1, 2, 5, and 10.

We will now introduce a notation for the forest originating from any given cycle vertex. Let F_c^n represent the set of vertices in the n^{th} level of the forest originating from the cycle vertex c. Of course, F_c^n depends on G_p^k. For example, in G_{41}^4 (Fig. 2), $F_{16}^1 = \{2, 23, 39\}$ and $F_{16}^2 = \{7, 19, 22, 34\}$. Similarly, F^n refers to the vertices in the n^{th} level of all forests and F_c refers to all forest vertices associated with the cycle vertex c at all levels $n \geq 1$. Note that the cycle vertices are not a part of the forests but for convenience we will denote the set of cycle elements by F^0 and also $F_c^0 = \{c\}$ but $c \notin F_c$.

The next theorem and corollary explain the subgroup structures present in the digraphs. First, it will be shown that the order of an element is constrained by its height in the forest structure.

Theorem 21: Let $a \in F_c$ and $\text{ord}_p c = d \mid t$. Then $\text{ord}_p a \mid k^h d$ if and only if $a \in F_c^x$, where $x \leq h$.

Proof: ($\Rightarrow$) Suppose $\text{ord}_p a \mid k^h d$. Then $(a^{k^h})^d \equiv 1 \pmod{p}$. Now, using Theorem 4,

$$1 = \text{ord}_p 1 = \text{ord}_p (a^{k^h})^d = \frac{\text{ord}_p a^{k^h}}{\gcd(d, \text{ord}_p a^{k^h})}.$$

Since $\gcd(d, \text{ord}_p a^{k^h}) = \text{ord}_p a^{k^h}$, $\text{ord}_p a^{k^h}$ divides d which divides t. From Theorem 15, a^{k^h} must be a cycle element, and hence $a \in F_c^x$, where $x \leq h$.

($\Leftarrow$) If $a \in F_c^x$ and $x \leq h$, then a^{k^h} is a cycle element of order d. Furthermore,

$$d = \text{ord}_p a^{k^x} = \frac{\text{ord}_p a}{\gcd(k^x, \text{ord}_p a)};$$

hence, $d \cdot \gcd(k^x, \text{ord}_p a) = \text{ord}_p a$, and thus $\text{ord}_p a \mid d \cdot k^x \mid d \cdot k^h$. □

From Theorem 21, it can be ascertained that various geometric subsets of the digraph form subgroups of Z_p^*, as stated in the next corollary.

Corollary 22: For all d dividing t and all h, $\bigcup_{\substack{0 \leq x \leq h \\ \text{ord}_p c \mid d}} F_c^x$ is a subgroup of Z_p^*, namely $H_{k^h d}$.

Proof: The union over all c such that $\text{ord}_p c \mid d$ and over all $x \leq h$ contains all $a \in Z_p^*$ such that $\text{ord}_p a \mid k^h d$ by Theorem 21. This is the subgroup $H_{k^h d}$ of Z_p^* by Lemma 20. □

Corollary 22 indicates that the union of cycle vertices with orders dividing a fixed d and vertices in their associated forest structures up to a fixed height form a subgroup of Z_p^*. In particular, if $d = 1$, all the vertices in F_1 up to any fixed level (along with 1) form a subgroup. On the other extreme, if $d = t$, all the vertices in all the components up to a fixed height form subgroups. Examining the digraph of G_{41}^4 (Fig. 2), one finds the following subgroups:

$d = 1, h = 0$: $F_1^0 = \{1\} = \text{H}_1$;

$d = 1, h = 1$: $F_1^0 \cup F_1^1 = \{1, 9, 32, 40\} = \text{H}_4$;

$d = 1, h = 2$: $F_1^0 \cup F_1^1 \cup F_1^2 = \{1, 3, 9, 14, 27, 32, 38, 40\} = \text{H}_{16}$;

$d = 5, h = 0$: $F^0 = \{1, 10, 16, 18, 37\} = \text{H}_5$;

$d = 5, h = 1$: $F^0 \cup F^1 = \{1, 2, 4, 5, 8, 9, 10, 16, 18, 20, 21, 23, 25, 31, 32, 33, 36, 37, 39, 40\} = \text{H}_{20}$;

$d = 5, h = 2$: $F^0 \cup F^1 \cup F^2 = \{1, \ldots, 40\} = Z_{41}^* = \text{H}_{80}$.

These algebraic properties imply that the highest level of the digraph, which will be referred to as the *canopy*, must contain at least half of the vertices. For example, in G_{41}^4 (Fig. 2) the canopy is F^2.

Corollary 23: If h_0 is the maximal height attained by the forest elements in G_p^k, then $|F^{h_0}| \geq (p-1)/2$.

Proof:

(i) If $h_0 = 0$, then all vertices are in cycles, so $|F^0| = p-1 \geq (p-1)/2$.

(ii) If $h_0 \geq 1$, then from Corollary 22, $\mathrm{H}_{k^{h_0-1}t} = \bigcup_{0 \leq n \leq h_0 - 1} F^n$ is a proper subgroup of Z_p^*. The number of elements in $\mathrm{H}_{k^{h_0-1}t}$ must be a proper divisor of $|Z_p^*| = p-1$. Since the largest proper divisor of $p-1$ is $(p-1)/2$, there are at least $(p-1)/2$ vertices remaining in the canopy. □

We also see a relationship between forest elements, cycle elements, and their products in the following corollary.

Corollary 24: The product of a forest element and a cycle element is a forest element.

Proof: The cycle elements of G_p^k form a closed multiplicative subgroup of Z_p^*; hence, the product of a forest element and a cycle element must be a forest element. □

5. OCCURRENCE OF LONG CYCLES

Control over the lengths of cycles is highly desirable. This is essential for applications to pseudo-random number generation and data encryption. The first theorem below provides an upper bound for the cycle lengths appearing in G_p^k. Special cases where long cycles can be guaranteed are then considered.

Theorem 25: Let $p > 5$ be prime. Then the length of the longest cycle in G_p^k is less than or equal to $(p-3)/2$.

Proof: Consider two cases depending on $\gcd(k, p-1)$.

(i) Suppose $\gcd(k, p-1) \neq 1$. By Lemma 12, G_p^k is not entirely cyclical and by Corollary 23 it has a forest structure with at least $(p-1)/2$ vertices in the canopy. Therefore, there are at most $(p-1)/2$ vertices in cycles. Since $p > 5$, we know we are not interested in longest cycles of length 1, and since 1 is in a loop, it is not part of any longest cycle of interest; hence, the maximal length cannot exceed $[(p-1)/2] - 1 = (p-3)/2$.

(ii) Suppose $\gcd(k, p-1) = 1$; thus, G_p^k consists entirely of cycles. From Theorem 19, the longest cycle length is associated with the elements of order $t = p-1$, which can be factored as $2^s\tau$, where $s \geq 1$ and τ is odd.

(a) If $\tau \neq 1$, then the number of elements of order $p-1$ is $\phi(p-1) = \phi(2^s\tau) = 2^{s-1}\phi(\tau) < 2^{s-1}\tau = (p-1)/2$. Now, since $\phi(p-1)$ is an integer, $\phi(p-1) \leq (p-3)/2$. Hence, even if all elements of order $p-1$ were together in one cycle, the length could not exceed $(p-3)/2$.

(b) If $\tau = 1$, then $p = 2^s + 1$ is a Fermat prime larger than 5, so $s > 2$. By Theorem 19, the length of the longest cycle is $\ell(t) = \ell(2^s) = \mathrm{ord}_{2^s} k$. However, $Z_{2^s}^*$ does not have a primitive root for $s > 2$ (Theorem 5). Thus, $\text{ord}_{2^s}k < \phi(2^s) = (p-1)/2$; hence, $\ell(t) \le (p-3)/2$ in this case as well. □

While Theorem 25 gives an upper bound for the cycle lengths in G_p^k, it does not specify whether this bound is ever attained. The next theorem shows that, for Sophie Germain primes, these maximal cycle lengths can be attained.

Theorem 26: Let $p = 2q+1$, where q is an odd Sophie Germain prime. If k is a primitive root mod q, then G_p^k contains a cycle of length $(p-3)/2$.

Proof: Because $\gcd(k, q) = 1$ and $q \mid t$, the elements with order q are in cycles of length $\text{ord}_q k = q - 1 = (p-3)/2$. □

Corollary 27: Let $p = 2q+1$, where q is an odd Sophie Germain prime. If k is an odd primitive root mod q, then G_p^k contains two cycles of length $(p-3)/2$.

Proof: Since k is odd, $t = 2q$ and the graph is entirely cyclical. The elements of order q are in a cycle of length $\text{ord}_q k = q - 1 = (p-3)/2$. The elements of order $t = 2q$ are, by Theorem 19, in the longest cycles of the digraphs. So the elements of order $2q$ are also in a cycle of length $(p-3)/2$. □

For example, consider G_{23}^2 (Fig. 4); $p = 23 = 2(11)+1$, and 2 is a primitive root mod 11. As expected, G_{23}^2 contains one cycle of length 10. Corollary 27 is illustrated by G_{47}^5, where $p = 47 = 2(23)+1$, 5 is a primitive root mod 23, and the digraph has two cycles of length 22.

Given a prime of the form $2q+1$, where k is a primitive root mod q, it is simple to iterate through a cycle of length $(p-3)/2$. Any residue between 2 and $p-2$ is either in a long cycle or is one step away. Beginning with any such residue, we can iterate x^k (mod p) to produce $(p-3)/2$ incongruent values. For example, consider the prime $9887 = 2(4943)+1$, where 4943 is also prime. Since 7 is a primitive root of 4943, iteration of x^7 (mod 9887) beginning with any x from 2 to 9885 will yield 4942 incongruent values.

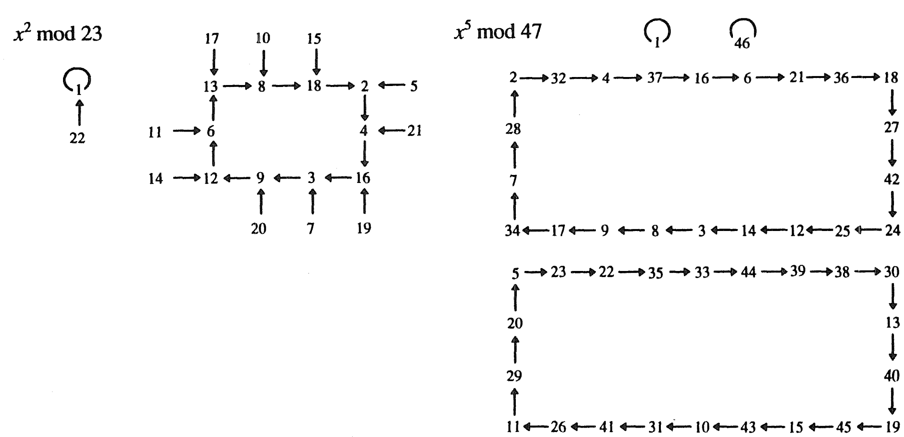

FIGURE 4. The Digraphs G_{23}^2 and G_{47}^5

6. CHARACTERIZING FORESTS

Having completely characterized the cycles for the digraph generated by $x^k \pmod p$, we turn our attention to characterizing the noncyclical elements of G_p^k. In each of our examples, we notice that the forests in any particular digraph are isomorphic. This turns out to be true in general, and will be proved by constructing a one-to-one correspondence between F_1 and F_c. The next lemma gives the essence of how the correspondence will be constructed.

Lemma 28: If $a \in \mathrm{F}_1^h$ and c is a cycle vertex, then $ac \in \mathrm{F}_{c^{k^h}}^h$.

Proof: Using Corollary 24, it follows immediately that $ac \notin \mathrm{F}^0$. Furthermore, $(ac)^{k^h} \equiv c^{k^h} \pmod p$ is a cycle element but $(ac)^{k^{h-1}}$ is a forest element, which implies that $ac \in \mathrm{F}_{c^{k^h}}^h$. □

Theorem 29: Let c be a cycle element, then $\mathrm{F}_1 \approx \mathrm{F}_c$.

Proof:

(i) First, we show that there exists a one-to-one correspondence between the vertices of F_1^h and F_c^h for all heights h, and hence between F_1 and F_c. Let h be fixed and let c_h denote the unique *cycle* element such that $c_h^{k^h} \equiv c \pmod p$. Define $f_h : \mathrm{F}_1^h \to \mathrm{F}_c^h$ by $f_h(a) \equiv a \cdot c_h \pmod p$. Next, we check that f_h is one-to-one and onto. Let $b \in \mathrm{F}_c^h$. Then $(b \cdot c_h^{-1})^{k^h} \equiv b^{k^h}(c_h^{k^h})^{-1} \equiv c \cdot c^{-1} \equiv 1 \in \mathrm{F}_1^0$ and $(b \cdot c_h^{-1})^{k^{h-1}} \notin \mathrm{F}^0$ because $b^{k^{h-1}} \notin \mathrm{F}^0$ and $c_h^{-k^{h-1}} \in \mathrm{F}^0$. It follows that $b \cdot c_h^{-1} \in \mathrm{F}_1^h$. Furthermore, $f_h(b \cdot c_h^{-1}) \equiv b \cdot c_h^{-1} \cdot c_h \equiv b \pmod p$, so f_h is onto F_c^h Suppose $f_h(a_1) \equiv f_h(a_2) \pmod p$ for $a_1, a_2 \in \mathrm{F}_1^h$. Then $a_1 \cdot c_h \equiv a_2 \cdot c_h$ implies $a_1 \equiv a_2 \pmod p$. Thus, f_h is one-to-one.

(ii) It remains to be shown that there exists a one-to-one correspondence between the edges of F_1 and F_c. We want to define $g : \mathrm{E}(\mathrm{F}_1) \to \mathrm{E}(\mathrm{F}_c)$ by $g(a, a^k) = (f_h(a), f_{h-1}(a^k))$, where h is the height of a in F_1. If $(f_h(a), f_{h-1}(a^k))$ is in fact in $\mathrm{E}(\mathrm{F}_c)$, then g will inherit the one-to-one and onto properties from f_h and f_{h-1}. We have an edge $(f_h(a), f_{h-1}(a^k))$ if and only if $(f_h(a))^k \equiv f_{h-1}(a^k) \pmod p$. Now $f_h(a) \equiv a \cdot c_h \pmod p$, where $c_h^{k^h} \equiv c$ and $f_{h-1}(a) \equiv a \cdot c_{h-1} \pmod p$, where $c_{h-1}^{k^{h-1}} \equiv c \pmod p$; thus, $c_h^{k^h} \equiv c \Rightarrow (c_h^k)^{k^{h-1}} \equiv c \pmod p$. By the uniqueness of c_h, $c_{h-1} \equiv c_h^k$ and $f_{h-1}(a) \equiv a \cdot c_h^k \pmod p$. Now $(f_h(a))^k \equiv (a \cdot c_h)^k \equiv a^k \cdot c_h^k \equiv f_{h-1}(a^k) \pmod p$. Hence, $(f_h(a), f_{h-1}(a^k)) \in \mathrm{E}(\mathrm{F}_c)$ and, by the argument above, the edges and vertices are in one-to-one correspondence, so $\mathrm{F}_1 \approx \mathrm{F}_c$. □

There is another property of this mapping that can be addressed. Consider $a \in \mathrm{F}_1$ and $c \in \mathrm{F}^0$; the order of the element $ac \pmod p$ will be $(\mathrm{ord}_p a)(\mathrm{ord}_p c)$. That is, the isomorphism "preserves" orders between F_1 and F_c in that the orders of corresponding elements in F_c are multiplied by the order of c.

Theorem 30: If $a \in \mathrm{F}_1$ and $b \in \mathrm{F}_c$ with c_h the cycle element such that $b \equiv a \cdot c_h \pmod p$, then $\mathrm{ord}_p b = (\mathrm{ord}_p a)(\mathrm{ord}_p c)$.

Proof: By Theorem 21, $\mathrm{ord}_p a | k^h$ and thus $\gcd(\mathrm{ord}_p a, t) = 1$. By Theorem 15, $\mathrm{ord}_p c = \mathrm{ord}_p c_h | t$, hence $\gcd(\mathrm{ord}_p a, \mathrm{ord}_p c) = 1$, and applying Theorem 6 gives the desired result. □

Some examples of the isomorphism described in Theorem 29 and Theorem 30 can be seen in Table 2.

TABLE 2. Some Orders and Products in G_{41}^4

a	c	ac	$\text{ord}_p\, a$	$\text{ord}_p\, c$	$\text{ord}_p\, ac$
1	10	10	1	5	5
9	37	5	4	5	20
3	10	30	8	5	40

Finally, we prove a result that determines the height of the forests.

Theorem 31: If h_0 is the minimal h such that $p-1|k^h t$, then h_0 is the height of the forests in G_p^k.

Proof:

(i) If $\gcd(k, p-1) = 1$, then $h_0 = 0$ and all the vertices are in cycles.

(ii) If $\gcd(k, p-1) \neq 1$, then $h_0 \geq 1$. Let a be a vertex of order $p-1$. From Theorem 21, a must be at height h_0 because $\text{ord}_p a | k^{h_0} t$ but $\text{ord}_p a \nmid k^{h_0 - 1} t$. There are no vertices at a greater height since, for any vertex b in G_p^k, $\text{ord}_p b | p-1 | k^{h_0} t$; thus, b is at level h_0 or lower in G_p^k. □

For example, in G_{41}^4 we see that $t = 5$ and $41 - 1 = 40 | 4^2 5$, which implies that the height of the forest is 2. This value is apparent in Figure 2.

7. PRIME POWERS

In the special case where the powers are prime, many of our results simplify. In particular, while we were able to prove that the forest structures were isomorphic with a general exponent k, we can completely characterize that structure if the exponent is prime. We will consider the digraph associated with a prime exponent q, letting $p - 1 = q^s t$, where t is relatively prime to q.

Corollary 32: The indegree of a vertex in G_p^q is $\begin{cases} 0 \text{ or } q & \text{if } p \equiv 1 \pmod{q}, \\ 1 & \text{otherwise.} \end{cases}$

Proof: This follows from Lemma 11 and Lemma 12 with $k = q$. □

This result implies that all the digraph components are cyclical if and only if $p \not\equiv 1 \pmod{q}$.

Corollary 33: If $p \equiv 1 \pmod{q}$, then any cycle vertex has $q-1$ noncycle children.

Proof: This follows from Lemma 13. □

Theorem 34: If $p \equiv 1 \pmod{q}$, the $q-1$ noncycle children of each cycle element are roots of complete q-nary trees.

Proof: Since the t forests in G_p^q are isomorphic and there are $q^s t$ elements in the digraph, $|\text{F}_1| = q^s - 1$. Furthermore, Theorem 31 implies that the height of F_1 is s. If F_1 is not composed of

$q-1$ complete q-nary trees, there exists a vertex that has indegree 0 but is not at height s. This would imply that $|F_1| < q^s - 1$, a contradiction. Since all the forests are isomorphic to F_1, and F_1 consists of complete q-nary trees, all the forests in G_p^q are complete q-nary trees as well. □

As an example, consider F_1 in the digraph G_{109}^3, as shown in Figure 5. Since $109-1=3^3(4)$ and $45^3 \equiv 63^3 \equiv 1 \pmod{109}$, 45 and 63 are roots of complete ternary trees with height $3-1=2$.

When the power is prime, we can say more about the orders of elements in the digraph G_p^q.

Theorem 35: A vertex $a \in F_1^h$ if and only if $\mathrm{ord}_p a = q^h$.

Proof: Consider Theorem 21 with $k=q$ and $c=1$, and hence $d=1$. Then $\mathrm{ord}_p a \,|\, q^h$ if and only if $a \in F_1^x$ for $x \le h$. Having elements of order q^h in a level x less than h would imply that $q^h \,|\, q^x$, where $x < h$, a contradiction. □

Returning to G_{109}^3 (Fig. 5), one can check that the orders of 3, 9, and 27 correspond to F_1^1, F_1^2, and F_1^3.

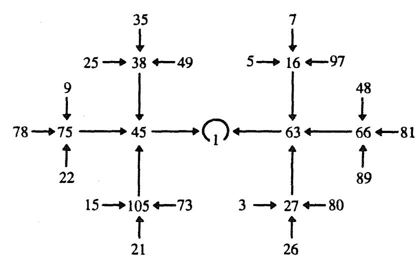

FIGURE 5. The Forest F_1 in G_{109}^3

Corollary 36: If $a \in F_c^h$ then $\mathrm{ord}_p a = q^h \mathrm{ord}_p c$.

Proof: Theorem 35 and the multiplying principle of Theorem 30 give the desired result. □

CONCLUSIONS

We have seen that many of the features of the digraph G_p^k can be determined in terms of properties of p and k. In particular, we have seen that the digraphs consist of components with exactly one cycle per component and that the forest structures associated with each cycle vertex throughout the digraph are isomorphic. The cycle lengths depend on the orders of the elements. We can also determine the height of the forests. In special cases, long cycles can be found and complete q-nary trees can be guaranteed.

While we have found a very rich structure for the digraphs associated with $x^k \bmod p$, it is natural to ask what other digraphs arising from functions such as these have a rich structure. The function $x^k \bmod m$, where m is not prime, will have a much different digraph since 0 will not necessarily be in a trivial cycle and primitive roots may not exist. In [2], the authors start to investigate this problem. On the other hand, looking at x^k in a finite field where 0 must be trivial and primitive roots always exist ought to lead to a theory like that seen in this paper. Digraphs from functions such as $x^k+1 \pmod p$ will be difficult to handle because we cannot lean on the theory of orders of elements as in this paper. It would be interesting to know what kind of control on the digraphs can be obtained in such cases.

ACKNOWLEDGMENT

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CONSTRUCTION OF SMALL CONSECUTIVE NIVEN NUMBERS

Brad Wilson
Department of Mathematics, SUNY College at Brockport, Brockport, NY 14420
(*Submitted August 1994*)

1. DEFINITIONS AND NOTATION

A Niven number is a number divisible by its digital sum. In [1] it is shown there can exist at most twenty consecutive Niven numbers; moreover, an infinite family of such is constructed where the first example requires over 4 billion digits. Here we get a lower bound on the number of digits in each of twenty consecutive Niven numbers and construct an example with nearly this few digits.

We start by recalling

Definition: A positive integer is called a *Niven number* if it is divisible by its digital sum.

Example: The number 12 is a Niven number since $(1+2)|12$. The number 11 is not a Niven number since $(1+1)\nmid 11$.

For $n \in \mathbb{Z}$, let $s(n)$ denote the digital sum of n. Examples of 20 consecutive Niven numbers have a large number of digits. To represent these numbers nicely, we concatenate digits or blocks of digits. If a and b are blocks of digits, let ab be the concatenation of these blocks. To denote multiplication, we use $a * b$. Finally, a_k denotes the concatenation of k copies of a.

2. PRELIMINARY LEMMA

In [1], Cooper and Kennedy show that any sequence of twenty consecutive Niven numbers begins with a number congruent to 90 modulo 100. We push this idea further.

Definition: Let $n \in \mathbb{Z}^+$. A positive integer is a 10^n*-mark* if it is congruent to $9_{n-1}0$ modulo 10^n but not congruent to 9_n0 modulo 10^{n+1}.

What Cooper and Kennedy actually show is that a sequence of twenty consecutive Niven numbers must begin with a 10^{2*n_1}-mark for some $n_1 \in \mathbb{Z}^+$.

Lemma 1: A sequence of twenty consecutive Niven numbers begins with a 10^{280*n_2}-mark for some $n_2 \in \mathbb{Z}^+$.

Proof: Since twenty consecutive Niven numbers begin with a 10^{2*n_1}-mark for some $n_1 \in \mathbb{Z}^+$, the digital sums of the first ten Niven numbers in our sequence are consecutive, as are the digital sums of the second ten Niven numbers in our sequence. Therefore, there is at least one number, say m, in the first ten Niven numbers whose digital sum is divisible by 2^3. By the definition of a Niven number, this means m is divisible by 2^3. Similarly, there exists such a number, say m', among the second ten. We can take m and m' to differ by exactly 8. Then

$$s(m)-s(m') = 9*2*n_1-8$$

as $2*n_1$ nines are converted to zeros upon crossing from the first to the second ten numbers if we start at a 10^{2*n_1}-mark. Since $9*2*n_1-8=s(m)-s(m')\equiv 0 \bmod 8$, we see that $2*n_1$ is a multiple of 8.

Using the same method, we see $2*n_1$ must also be a multiple of 5 and of 7. Taken together, this means that the first of our twenty consecutive Niven numbers is a 10^{280*n_2}-mark with $n_2=\frac{2*n_1}{280}\in\mathbb{Z}^+$. □

3. CONGRUENCE RESTRICTIONS OF THE DIGITAL SUM

In this section we develop congruence restrictions on the digital sum of our first Niven number. To do this, we assume β is the first of twenty consecutive Niven numbers, that it is a 10^{280*n_2}-mark for fixed $n_2\in\mathbb{Z}^+$, and that $s(\beta)=\alpha$. Then

$$(\alpha+i)|(\beta+i) \quad \text{for } i=0,1,\dots,9,$$

$$(\alpha+j-2520*n_2)|(\beta+j) \quad \text{for} j=10,11,\dots,19.$$

Let

$$\gamma=\operatorname{lcm}(\alpha,\alpha+1,\dots,\alpha+9)$$

and

$$\gamma'=\operatorname{lcm}(\alpha+10-2520*n_2,\alpha+11-2520*n_2,\dots,\alpha+19-2520*n_2).$$

This gives us

$$\beta\equiv\alpha \bmod \gamma$$

and

$$\beta\equiv\alpha-2520*n_2 \bmod \gamma'.$$

By the Chinese remainder theorem, these are consistent if and only if

$$\gcd(\gamma,\gamma')|2520*n_2. \tag{1}$$

This condition reduces to congruence conditions on the prime divisors of $A=\{2520*n_2-19, 2520*n_2-18,\dots,2520*n_2-1\}$, the set of possible differences between $\{\alpha,\alpha+1,\dots,\alpha+9\}$ and $\{\alpha+10-2520*n_2,\dots,\alpha+19-2520*n_2\}$. Let P be the set of prime divisors of the numbers in A. For $p\in P$, let $v(p)$ be such that $p^{v(p)}|2520*n_2$ but $p^{v(p)+1}\nmid 2520*n_2$. Then condition (1) implies $p^{v(p)+1}\nmid\gcd(\gamma,\gamma')$, so $p^{v(p)+1}\nmid\gamma$ or $p^{v(p)+1}\nmid\gamma'$, which can be restated as

$$\begin{aligned}\alpha&\equiv 1,2,\dots,p^{v(p)+1}-10 \bmod p^{v(p)+1} \quad \text{or}\\ \alpha+10-2520*n_2&\equiv 1,2,\dots,p^{v(p)+1}-10 \bmod p^{v(p)+1}.\end{aligned} \tag{2}$$

Conversely, condition (2) assures $p^{v(p)+1}\nmid\gcd(\gamma,\gamma')$ and $p|\gcd(\gamma,\gamma')$ implies $p\in P$, so we get $\gcd(\gamma,\gamma')|2520*n_2$. Further, α satisfies some additional congruences modulo powers of 2 and 5. Since β is a 10^{280*n_2}-mark, $\beta\equiv 990 \bmod 1000$. Then $\beta\equiv 6 \bmod 8$, so

$$\alpha\equiv 6 \bmod 8. \tag{3}$$

Similarly $\beta\equiv 0 \bmod 5$ means

$$\alpha\equiv 0 \bmod 5. \tag{4}$$

These lead to

Lemma 2: A sequence of twenty consecutive Niven numbers must begin at a 10^{560*n_3}-mark for some $n_3 \in \mathbb{Z}^+$.

Proof: Suppose not, i.e., suppose we have twenty consecutive Niven numbers beginning at a 10^{280*n_2}-mark with n_2 odd. Then by (2) we have

$$\alpha \equiv 1, 2, \ldots, 6 \bmod 16 \quad \text{or}$$
$$\alpha + 10 - 2520 * n_2 \equiv 1, 2, \ldots, 6 \bmod 16,$$

so $\alpha \equiv 0, 1, \ldots, 6, 15 \bmod 16$. By (3) this means $\alpha \equiv 6 \bmod 16$. Further, $\beta \equiv 9990 \bmod 10000$, so $\beta \equiv 6 \bmod 16$. Then $\alpha + 18 - 2520 * n_2 \equiv 0 \bmod 16$, so $16|(\beta + 18)$, which is a contradiction. □

As a result of Lemma 2, we see that $2520 * n_2 \equiv 0 \bmod 16$, so the digital sums of $\beta, \beta+1, \ldots, \beta+19$ are consecutive modulo 16. Since $\beta \equiv 6 \bmod 16$, we get

$$\alpha \equiv 6 \bmod 16. \tag{5}$$

4. OUTLINE OF METHOD

Assume $n_2 \in 2\mathbb{Z}^+$ fixed and α satisfies (2), (4), and (5). We outline how to construct the first of twenty consecutive Niven numbers β so that $s(\beta) = \alpha$. By our choice of α, we can find a solution to

$$x \equiv \alpha - 2520 * n_2 \bmod \gamma' \tag{6}$$

and

$$x \equiv \alpha \bmod \gamma. \tag{7}$$

In fact, we can find infinitely many solutions differing from each other by multiples of $\delta = \text{lcm}(\gamma, \gamma')$. Let b be the least positive solution. We modify b by adding multiples of δ so that the resulting number, b', still satisfies (6) and (7) and is a 10^{280*n_2}-mark. Finally, we may be able to modify b' by adding multiples of δ so that the resulting number, β, still satisfies (6) and (7), is still a 10^{280*n_2}-mark, and has a digital sum α. Such a β is the first of twenty consecutive Niven numbers.

5. AN EXAMPLE OF A SEQUENCE OF SMALL NIVEN NUMBERS

We construct a 10^{280*4}-mark. This means $n_2 = 4$. We can solve the congruences (2), (4), and (5) for α modulo $p \in P$ to get $\alpha = 15830$. This leads to

$$\begin{aligned}\delta &= \text{lcm}(\alpha, \alpha+1, \ldots, \alpha+19-10080) \\ &= 304883065587843789022679986681660 3\\ &\quad\ 216269482282665704639500236070208 0.\end{aligned}$$

Solving for x in (6) and (7), we get

$$\begin{aligned}b = x = {} & 36346620873326536780272919778661 48\\ & 01904361423373711756818904629695 0.\end{aligned}$$

Adding a suitable multiple of δ (to get a 10^{1120}-mark), we get

$$\begin{aligned}b' = {} & 212221855965415386709173598107865 34\\ & 26215175822736105351778251310 59_{1119}0.\end{aligned}$$

Continue to add multiples of δ so as not to disturb the terminal 1121 digits of b'. In short, add multiples of $5^6 * 10^{1114} * \delta$. Doing this, we get

$$\beta = 4981497945879639583073518757993538244 7\\ 764488580550605587252791407 29_{601}59_{1119}0.$$

It is easy to check that this is a number with 1788 digits and digital sum 15830 and that β is the first of twenty Niven numbers.

6. LOWER BOUNDS ON THE NUMBER OF DIGITS

Theorem 1: The smallest sequence of twenty consecutive Niven numbers begins with a 10^{1120}-mark of digital sum 15830.

Proof: Let β be the first of twenty consecutive Niven numbers and let $\alpha = s(\beta)$. Suppose β has fewer than 1789 digits (i.e., no more than in our example in the previous section). Since β is a 10^{280*n_2}-mark with n_2 even, $n_2 = 2, 4$, or 6. A computer search shows there is no α less than $9*1789$ satisfying (2), (4), and (5) with $n_2 = 2, 4$, or 6 other than $\alpha = 15830$ for $n_2 = 4$. □

In the last section we saw an example with 1788 digits. This need not be the smallest, but it is close to the smallest.

Theorem 2: The smallest sequence of twenty consecutive Niven numbers begins with a number having at least 1760 digits.

Proof: By Theorem 1, we know the digital sum of the first number is 15830. Given that the terminal digit is a zero, there are at least 1+15830/9 digits. □

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SUMS OF POWERS OF INTEGERS VIA GENERATING FUNCTIONS

F. T. Howard
Wake Forest University, Winston-Salem, NC 27109
(*Submitted August 1994*)

1. INTRODUCTION

Let

$$S_{k,n}(a,d) = a^k + (a+d)^k + (a+2d)^k + \cdots + [a+(n-1)d]^k, \tag{1.1}$$

where k and n are nonnegative integers with $n > 0$, and a and d are complex numbers with $d \neq 0$. We shall use the notation

$$S_{k,n} = S_{k,n}(1,1) = 1^k + 2^k + \cdots + n^k. \tag{1.2}$$

Similarly, let $T_{k,n}(a,d)$ be the alternating sum

$$T_{k,n}(a,d) = a^k - (a+d)^k + (a+2d)^k - \cdots + (-1)^{n-1}[a+(n-1)d]^k, \tag{1.3}$$

and let

$$T_{k,n} = T_{k,n}(1,1) = 1^k - 2^k + 3^k - \cdots + (-1)^{n-1}n^k. \tag{1.4}$$

It is, of course, well known that

$$S_{k,0} = n; \quad S_{k,1} = n(n+1)/2; \quad S_{k,2} = n(n+1)(2n+1)/6;$$

and so on. Many different writers have worked on the problem of finding simple formulas for $S_{k,n}$, and many different methods have been used; see [3], [5], [6], [9], [10] for just a small sampling of recent articles. The formulas for $T_{k,n}$ are certainly less well known.

In the present paper we use generating functions to find new recurrences for $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$. We also show how $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$ can be determined from $S_{k-1,n}(a,d)$ and $T_{k-1,n}(a,d)$, respectively, and we show how $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$ can be expressed in terms of Bernoulli numbers. One of the main results is a new "lacunary" recurrence formula for $S_{k,n}$ with gaps of 6 (Theorem 3.1); that is, we can use the formula to find $S_{m,n}$ for $m = 0, 1, \ldots, 5$; then, using only $S_{m,n}$, we can find $S_{6+m,n}$; then, using only $S_{m,n}$ and $S_{6+m,n}$, we can find $S_{12+m,n}$, and so on. There is a similar recurrence for $T_{k,n}$.

There are several motivations for this paper: (1) A recent article by Wiener [10] dealt with equation (1.1) and generalized some well-known properties of $S_{k,n}$. We show how the formulas of [10] can be derived very quickly and how they can be extended. (2) In a recent article by Howard [4], formulas were found which connected Bernoulli numbers to $T_{k,n}$. Evidently, the properties of $T_{k,n}$ are not well known, so the results of [4] are a stimulus to study $T_{k,n}$ and $T_{k,n}(a,d)$ in some detail. (3) The new lacunary recurrences mentioned above are useful and easy to use, and (in the writer's opinion) they are of considerable interest. In Section 3 we illustrate the formulas by computing $S_{6,n}$, $S_{12,n}$, and $S_{18,n}$; in Section 6 we compute $T_{4,n}$, $T_{10,n}$, and $T_{16,n}$. (4) Perhaps the main purpose of the paper is to show how generating functions provide a simple, unified approach to the study of sums of powers of integers. The many, and often repetitious articles on $S_{k,n}$ that have appeared in the last twenty-five years seem to indicate a need for such a unified approach.

2. RECURRENCES FOR $S_{k,n}(a,d)$

We first note that by (1.1) the exponential generating function for $S_{k,n}(a,d)$ is

$$\sum_{k=0}^{\infty} S_{k,n}(a,d)\frac{x^k}{k!} = e^{ax} + e^{(a+d)x} + \cdots + e^{[a+(n-1)d]x} = \frac{e^{(a+nd)x} - e^{ax}}{e^{dx}-1}. \tag{2.1}$$

We can use (2.1) to prove the next two theorems in a very direct way.

Theorem 2.1: Let $k \geq 0$ and $n > 0$. We have the following recurrences for $S_{k,n}(a,d)$:

$$\sum_{j=0}^{k}\binom{k+1}{j} d^{k+1-j} S_{j,n}(a,d) = (a+nd)^{k+1} - a^{k+1}, \tag{2.2}$$

$$\sum_{j=0}^{k}(-1)^j\binom{k+1}{j} d^{k+1-j} S_{j,n}(a,d) = (-1)^k[(a+nd-d)^{k+1} - (a-d)^{k+1}]. \tag{2.3}$$

Proof: From (2.1) we have

$$(e^{dx}-1)\sum_{k=0}^{\infty} S_{k,n}(a,d)\frac{x^k}{k!} = e^{(a+nd)x} - e^{ax};\ \text{i.e.,}$$

$$\sum_{j=1}^{\infty} d^j \frac{x^j}{j!}\cdot\sum_{k=0}^{\infty} S_{k,n}(a,d)\frac{x^k}{k!} = \sum_{j=0}^{\infty}[(a+nd)^j - a^j]\frac{x^j}{j!}. \tag{2.4}$$

If we examine both sides of (2.4) and equate coefficients of $x^{k+1}/(k+1)!$, we have (2.2). Now if we replace x by $-x$ in (2.1), we have, after simplification,

$$\sum_{k=0}^{\infty}(-1)^k S_{k,n}(a,d)\frac{x^k}{k!} = \frac{e^{(d-a)x} - e^{(d-a-nd)x}}{e^{dx}-1}. \tag{2.5}$$

Multiplying both sides of (2.5) by $e^{dx}-1$ and equating coefficients of $x^{k+1}/(k+1)!$, we have (2.3). This completes the proof.

Formulas (2.2) and (2.3) generalize known formulas for $a = d = 1$ [8, p. 159]. We note that (2.2) was found by Wiener [10]; see also [2, p. 169]. Bachmann [1, p. 28] found a recurrence for $S_{k,n}(a,d)$ involving only $S_{j,n-1}$ for $j = 1, \ldots, k$.

We now add (2.2) and (2.3) to obtain the next theorem.

Theorem 2.2: For $k = 1, 2, 3, \ldots$, we have

$$2\sum_{j=0}^{k-1}\binom{2k}{2j} d^{2k-2j} S_{2j,n}(a,d) = (a+nd)^{2k} - (a+nd-d)^{2k} + (a-d)^{2k} - a^{2k},$$

and for $k = 2, 3, \ldots$, we have

$$2\sum_{j=1}^{k-1}\binom{2k-1}{2j-1}d^{2k-2j}S_{2j-1,n}(a,d)=(a+nd)^{2k-1}-(a+nd-d)^{2k-1}+(a-d)^{2k-1}-a^{2k-1}.$$

Theorem 2.2 can be compared to results of Wiener [10] and Riordan [8, p. 160].

3. RECURRENCES WITH INDICES $6K+M$

We now show how to find formulas of the type (2.2) where the index j varies only over integers of the form $6k+m$, with $0\le m\le 5$. To the writer's knowledge, these formulas are new. After stating Theorem 3.1 and its corollary, we give some applications; the proof of Theorem 3.1 is given at the end of this section.

Let θ be the complex number $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, so

$$\theta^3=1 \quad\text{and}\quad \theta^2+\theta+1=0, \tag{3.1}$$

and define the sequence $\{w_j\}$ in the following way:

$$\begin{aligned} &w_j=1+(-1)^j(\theta^j+\theta^{6-j}) && \text{for } j=0,1,\dots,5,\\ &w_j=w_{6+j} && \text{for } j=0,\pm1,\pm2,\dots. \end{aligned} \tag{3.2}$$

For example, $w_{-1}=w_5=2$. The values of w_j for $j=0,1,\dots,5$ are given in the following table:

j	0	1	2	3	4	5
w_j	3	2	0	−1	0	2

(3.3)

Theorem 3.1 Let w_j be defined by (3.2) and (3.3). Then, for $m=0,1,\dots,5$ and $n>0$, $k\ge 0$, we have:

$$6\sum_{j=0}^{k}\binom{6k+m+3}{6j+m}d^{6k-6j+3}S_{6j+m,n}(a,d)$$
$$=\sum_{j=1}^{6k+m+1}\binom{6k+m+3}{j}d^{6k+m+3-j}w_{m-j}[(a-d+nd)^j-(a-d)^j].$$

Corollary: For $m=0,1,\dots,5$ and $n>0$, $k\ge 0$, we have:

$$\binom{6k+m+3}{3}S_{6k+m,\,n}=-\sum_{j=0}^{k-1}\binom{6k+m+3}{6j+m}S_{6j+m,n}+\frac{1}{6}\sum_{j=1}^{6k+m+1}\binom{6k+m+3}{j}w_{m-j}n^j.$$

The corollary gives us an easy way to write $S_{k,n}$ as a polynomial in n of degree $k+1$. In particular, we have for $m=0,1,\dots,5$:

$$\binom{m+3}{3}S_{m,n}=\frac{1}{6}\sum_{j=1}^{m+1}\binom{m+3}{j}w_{m-j}n^j.$$

Using the corollary, we easily compute $S_{0,n}=n$ and

$$\binom{9}{3}S_{6,n} = -n + \frac{1}{6}\sum_{j=1}^{7}\binom{9}{j}w_{-j}n^{n},$$

By (3.2) and (3.3), the numbers w_{-j} are easy to find. We have $w_{-j} = 2$ for $j = 1, 5, 7$; $w_{-j} = 0$ for $j = 2$ and 4; $w_{-j} = -1$ for $j = 3$; $w_{-j} = 3$ for $j = 6$. Thus, we have

$$S_{6,n} = \frac{1}{42}n - \frac{1}{6}n^3 + \frac{1}{2}n^5 + \frac{1}{2}n^6 + \frac{1}{7}n^7.$$

Continuing in the same way, we have

$$\binom{15}{3}S_{12,n} = -n - \binom{15}{6}S_{6,n} + \frac{1}{6}\sum_{j=1}^{13}\binom{15}{j}w_{-j}n^{j},$$

so

$$S_{12,n} = -\frac{691}{2730}n + \frac{5}{3}n^3 - \frac{33}{10}n^5 + \frac{22}{7}n^7 - \frac{11}{6}n^9 + n^{11} + \frac{1}{2}n^{12} + \frac{1}{13}n^{13}.$$

It is easy to keep going:

$$\binom{21}{3}S_{18,n} = -n - \binom{21}{6}S_{6,n} - \binom{21}{12}S_{12,n} + \frac{1}{6}\sum_{j=1}^{19}\binom{21}{j}w_{-j}n^{j},$$

which gives

$$\begin{aligned} S_{18,n} = {} & \frac{43867}{798}n - \frac{3617}{10}n^3 + 714n^5 - \frac{23494}{35}n^7 + \frac{1105}{3}n^9 - \frac{663}{5}n^{11} \\ & + 34n^{13} - \frac{34}{5}n^{15} + \frac{3}{2}n^{17} + \frac{1}{2}n^{18} + \frac{1}{19}n^{19}. \end{aligned}$$

Another application of Theorem 3.1, involving Bernoulli numbers, is given in Section 4.

Proof of Theorem 3.1: Let

$$A(x) = \frac{e^{(a+nd)x} - e^{ax}}{e^{dx} - 1} = \sum_{k=0}^{\infty} S_{k,n}(a, d)\frac{x^k}{k!}, \tag{3.4}$$

and define $A_0(x)$, $A_1(x)$, and $A_2(x)$ as follows:

$$A_0(x) = \frac{1}{3}[A(x) + A(\theta x) + A(\theta^2 x)] = \sum_{k=0}^{\infty} S_{3k,n}(a, d)\frac{x^{3k}}{(3k)!}, \tag{3.5}$$

$$A_1(x) = \frac{1}{3}[(A(x) + \theta^2 A(\theta x) + \theta A(\theta^2 x)] = \sum_{k=0}^{\infty} S_{3k+1,n}(a, d)\frac{x^{3k+1}}{(3k+1)!}, \tag{3.6}$$

$$A_2(x) = \frac{1}{3}[A(x) + \theta A(\theta x) + \theta^2 A(\theta^2 x)] = \sum_{k=0}^{\infty} S_{3k+2,n}(a, d)\frac{x^{3k+2}}{(3k+2)!}. \tag{3.7}$$

The equalities on the extreme right of (3.5), (3.6), and (3.7) follow from (3.1) and (3.4). Using (3.4) and the equalities on the extreme left of (3.5), (3.6), and (3.7), we can write

$$A_m(x) = \frac{N_m}{3D_m} \quad (m = 0, 1, 2),$$

with $D_0 = D_1 = D_2 = (e^{dx}-1)(e^{\theta dx}-1)(e^{\theta^2 dx}-1)$. Using (3.1), it is easy to compute

$$D_0 = D_1 = D_2 = 6\sum_{k=0}^{\infty} d^{6k+3}\frac{x^{6k+3}}{(6k+3)!}.$$

The formulas for N_0, N_1, and N_2 are more complicated, but they are easy to work out from (3.4), (3.5), (3.6), and (3.7). We first note that, for $m = 0, 1, 2$, we have

$$A_m(x) = \frac{1}{3}[A(x) + \theta^{3-m}A(\theta x) + \theta^m A(\theta^2 x)] = \sum_{k=0}^{\infty} S_{3k+m,n}(a,d)\frac{x^{3k+m}}{(3k+m)!}.$$

Thus, we have:

$$\begin{aligned} N_m &= (e^{(a+nd)x} - e^{ax})(e^{-dx} - e^{d\theta x} - e^{d\theta^2 x} + 1) \\ &+ \theta^{3-m}(e^{(a+nd)\theta x} - e^{a\theta x})(e^{-d\theta x} - e^{dx} - e^{d\theta^2 x} + 1) \\ &+ \theta^m(e^{(a+nd)\theta^2 x} - e^{a\theta^2 x})(e^{-d\theta^2 x} - e^{d\theta x} - e^{dx} + 1). \end{aligned} \tag{3.8}$$

We now multiply, regroup, and expand the terms in (3.8). For example, we have

$$\begin{aligned} e^{(a+nd-d)x} + \theta^{3-m}e^{(a+nd-d)\theta x} + \theta^m e^{(a+nd-d)\theta^2 x} &= \sum_{j=0}^{\infty}(a+nd-d)^j(1+\theta^{3-m+j}+\theta^{m+2j})\frac{x^j}{j!} \\ &= 3\sum_{j=0}^{\infty}(a+nd-d)^{3j+m}\frac{x^{3j+m}}{(3j+m)!}. \end{aligned}$$

Regrouping and expanding the other terms in (3.8), we have, for $m = 0, 1,$ and 2,

$$\begin{aligned} \frac{1}{3}N_m = \sum_{j=0}^{\infty}[(a+nd+d)^{3j+m} + (a+nd)^{3j+m} + (a+nd-d)^{3j+m} - (a+d)^{3j+m} - a^{3j+m} \\ - (a-d)^{3j+m}]\frac{x^{3j+m}}{(3j+m)!} - 3\sum_{j=0}^{\infty}d^{3j}\frac{x^{3j}}{(3j)!}\cdot\sum_{j=0}^{\infty}[(a+nd)^{3j+m} - a^{3j+m}]\frac{x^{3j+m}}{(3j+m)!}. \end{aligned}$$

Since

$$D_m\sum_{k=0}^{\infty}S_{3k+m,n}(a,d)\frac{x^{3k+m}}{(3k+m)!} = \frac{1}{3}N_m, \tag{3.9}$$

we can equate coefficients of $x^{3k+m}/(3k+m)!$ in (3.9) and state the following: For $m = 0, 1, 2$,

$$\begin{aligned} &3\sum_{j=0}^{k}[1+(-1)^{k-j+1}]\binom{3k+m}{3j+m}d^{3k-3j}S_{3j+m,n}(a,d) \\ &= (a+nd+d)^{3k+m} + (a+nd)^{3k+m} + (a+nd-d)^{3k+m} - (a+d)^{3k+m} \\ &\quad - a^{3k+m} - (a-d)^{3k+m} - 3\sum_{j=0}^{k}\binom{3k+m}{3j+m}[(a+nd)^{3j+m} - a^{3j+m}]d^{3k-3j}. \end{aligned} \tag{3.10}$$

At this point we observe how the sums in (3.10) can be simplified. By using properties of θ and the binomial theorem, we see that

$$
\begin{aligned}
&3\sum_{j=0}^{k}\binom{3k+m}{3j+m}(a+nd)^{3j+m}d^{3k-3j}\\
&=(d+a+nd)^{3k+m}+\theta^{2m}(d+a\theta+nd\theta)^{3k+m}+\theta^{m}(d+a\theta^2+nd\theta^2)^{3k+m}\\
&=(d+a+nd)^{3k+m}+\theta^{2m}[-d\theta^2+\{(a-d)+nd\}\theta]^{3k+m}+\theta^{m}[-d\theta+\{(a-d)+nd\}\theta^2]^{3k+m}\\
&=(d+a+nd)^{3k+m}+\sum_{j=0}^{3k+m}\binom{3k+m}{j}(-1)^{k+m-j}d^{3k+m-j}(\theta^{m-j}+\theta^{2m+j})(a-d+nd)^j\\
&=(d+a+nd)^{3k+m}+\sum_{j=0}^{3k+m}\binom{3k+m}{j}(-1)^{k}d^{3k+m-j}(-1+w_{m-j})(a-d+nd)^j. \qquad (3.11)
\end{aligned}
$$

We substitute (3.11) [and also (3.11) with $n=0$] into (3.10), and we consider the two cases of k even and k odd. Then using the binomial theorem and the fact that $w_{m-j}=0$ when $j=6k+m+2$, we can easily simplify (3.10) to get Theorem 3.1. This completes the proof.

4. $S_{k,n}(a,d)$ IN TERMS OF BERNOULLI NUMBERS

The Bernoulli polynomial $B_k(x)$ may be defined by means of the generating function

$$\frac{xe^{xz}}{e^x-1}=\sum_{k=0}^{\infty}B_k(z)\frac{x^k}{k!}. \qquad (4.1)$$

When $z=0$, we have the ordinary Bernoulli number B_k, i.e., $B_k(0)=B_k$. It is well known [2, pp. 48-49] that $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, and $B_{2m+1}=0$ for $m>0$. It follows from (4.1) that

$$B_k(z)=\sum_{j=0}^{k}\binom{k}{j}B_{k-j}z^j. \qquad (4.2)$$

Comparing (4.1) and (1.2), we see

$$S_{k,n}(a,d)=\frac{d^k}{k+1}\left[B_{k+1}\left(\frac{a}{d}+n\right)-B_{k+1}\left(\frac{a}{d}\right)\right]. \qquad (4.3)$$

Now, for fixed a and d, suppose we write $S_{k,n}(a,d)$ as a polynomial in $(a-d+nd)$; i.e.,

$$
\begin{aligned}
S_{k,n}(a,d)&=S_{k,n-1}(a,d)+[a+(n-1)d]^k\\
&=v_{k,0}+v_{k,1}(a-d+nd)+v_{k,2}(a-d+nd)^2+\cdots+v_{k,k+1}(a-d+nd)^{k+1}.
\end{aligned} \qquad (4.4)
$$

By (4.2) and (4.3), we have the following result.

Theorem 4.1: If $S_{k,n}(a,d)$ is written as a polynomial in $(a-d+nd)$ and $v_{k,j}$ is defined by (4.4) for $j=1,2,\ldots,k+1$, then

$$v_{k,j}=\frac{1}{k+1}\binom{k+1}{j}d^{k-j}B_{k+1-j}\quad(1\le j\le k-1),$$

$$v_{k,k}=\frac{1}{2},\ v_{k,k+1}=\frac{1}{d(k+1)},\ v_{k,0}=\frac{d^k}{k+1}\left[B_{k+1}-B_{k+1}\left(\frac{a}{d}\right)\right].$$

When $a=d=1$, Theorem 4.1 gives the well-known result [2, pp. 154-55]:

$$S_{k,n}=S_{k,n}(1,1)=n^k+\frac{1}{k+1}\sum_{j=1}^{k+1}\binom{k+1}{j}B_{k+1-j}n^j.$$

Here, we can give another application of Theorem 3.1. Since $v_{k,1}=B_k$ for $k>1$, and $a=d=1$, we see from the corollary to Theorem 3.1 that, for $m=0,\dots,5$ and $6k+m>1$,

$$\binom{6k+m+3}{3}B_{6k+m}=-\sum_{j=1}^{k-1}\binom{6k+m+3}{6j+m}B_{6j+m}+\frac{1}{6}(6k+m+3)w_{m-1}. \tag{4.5}$$

Formula (4.5) is a lacunary recurrence for the Bernoulli numbers that is equivalent to a formula of Ramanujan [7, pp. 3-4]. See also [8, pp. 136-37].

5. FINDING $S_{k,n}(a,d)$ FROM $S_{k-1,n}(a,d)$

Several writers, like Khan [6], have pointed out that when $a=d=1$, if we know just $S_{k-1,n}$, we can evaluate $S_{k,n}$. Using (2.1), it is easy to prove this and to generalize it. First of all, we can use mathematical induction on (2.2) to prove that $S_{k,n}(a,d)$ is a polynomial in n of degree $k+1$, with constant term equal to 0. (That also follows from Section 4.) Thus, for fixed a and d, we can write

$$S_{k,n}(a,d)=c_{k,1}n+c_{k,2}n^2+\cdots+c_{k,k+1}n^{k+1}. \tag{5.1}$$

Theorem 5.1: For fixed a and d, let $c_{k,j}$ be defined by (5.1) for $j=1,\dots,k+1$. Then, for $k\ge 1$, we have

$$c_{k,j}=\frac{kd}{j}c_{k-1,j-1}\quad (j=2,\dots,k+1), \tag{5.2}$$

$$c_{k,1}=a^k-c_{k,2}-c_{k,3}-\cdots-c_{k,k+1}. \tag{5.3}$$

Proof: Define the polynomial $P_k(z)$ by means of the generating function

$$\sum_{k=0}^{\infty}P_k(z)\frac{x^k}{k!}=\frac{e^{(a+zd)x}-e^{ax}}{e^{dx}-1}, \tag{5.4}$$

so $P_k(n)=S_{k,n}(a,d)$. This implies that

$$P_k(z)=c_{k,1}z+c_{k,2}z^2+\cdots+c_{k,k+1}z^{k+1} \tag{5.5}$$

for all positive integers z, and hence for all complex numbers z. Now we differentiate both sides of (5.4) with respect to z to obtain

$$\sum_{k=0}^{\infty}P_k'(z)\frac{x^k}{k!}=\frac{dxe^{(a+zd)x}}{e^{dx}-1}=\frac{dx(e^{(a+zd)x}-e^{ax})}{e^{dx}-1}+\frac{dxe^{ax}}{e^{dx}-1}. \tag{5.6}$$

We recall the definition of the Bernoulli polynomials, formula (4.1), and we see that

$$\frac{dxe^{ax}}{e^{dx}-1}=\sum_{k=0}^{\infty}d^kB_k\left(\frac{a}{d}\right)\frac{x^k}{k!}, \tag{5.7}$$

and we note that $B_k(\frac{a}{d})$ is independent of z. From (5.4), (5.6), and (5.7), we have

$$\sum_{k=0}^{\infty} P_k'(z)\frac{x^k}{k!} = d\sum_{k=0}^{\infty} P_k(z)\frac{x^{k+1}}{k!} + \sum_{k=0}^{\infty} d^k B_k\left(\frac{a}{d}\right)\frac{x^k}{k!}. \tag{5.8}$$

Equating coefficients of $x^k / k!$ in (5.8), we have

$$P_k'(z) = dkP_{k-1}(z) + d^k B_k\left(\frac{a}{d}\right). \tag{5.9}$$

Thus, by using (5.5) and equating coefficients of z^{j-1} in (5.9), we have

$$c_{k,j} = \frac{kd}{j}c_{k-1,j-1} \quad (j = 2, \ldots, k+1).$$

Also, by (5.1) and the fact that $S_{k,1}(a, d) = a^k$, we have

$$d^k B_k\left(\frac{a}{d}\right) = c_{k,1} = a^k - c_{k,2} - c_{k,3} - \cdots - c_{k,k+1},$$

and the proof is complete.

Thus, if we know the coefficients of $S_{k-1,n}(a, d)$, we can determine the coefficients $c_{k,j}$ of $S_{k,n}(a, d)$ for $j = 2, \ldots, k+1$ from (5.2) and then compute $c_{k,1}$ from (5.3). For example,

$$S_{0,n}(a, d) = c_{0,1}n = n,$$

so, by (5.2) and (5.3), we have $c_{1,2} = \frac{d}{2}c_{0,1} = \frac{d}{2}$, and $c_{1,1} = a - \frac{d}{2}$; that is,

$$S_{1,n}(a, d) = c_{1,1}n + c_{1,2}n^2 = \left(a - \frac{d}{2}\right)n + \frac{d}{2}n^2. \tag{5.10}$$

Equivalently, by (5.9), we can integrate to find $P_k(z)$:

$$P_k(z) = dk\int P_{k-1}(z)\,dz + d^k B_k\left(\frac{a}{d}\right)z, \tag{5.11}$$

[The dz in (5.11) should not be confused with the complex variable d.] The constant of integration is 0, and $c_{k,1} = d^k B_k(\frac{a}{d})$ can be found by means of (5.3). When $a = d = 1$ and $k > 1$, $B_k(\frac{a}{d})$ is the k^{th} Bernoulli number. We illustrate (5.11) by finding $S_{2,n}(a, d)$. From (5.10) and (5.11) we have, after integrating $S_{1,n}(a, d)$ with respect to n and multiplying by $3d$,

$$S_{2,n}(a, d) = d^2 B_2\left(\frac{a}{d}\right)n + \left(ad - \frac{d^2}{2}\right)n^2 + \frac{d^2}{3}n^3,$$

so by (5.3) we have

$$S_{2,n}(a, d) = \left(a^2 - ad + \frac{d^2}{6}\right)n + \left(ad - \frac{d^2}{2}\right)n^2 + \frac{d^2}{3}n^3.$$

6. RECURRENCES FOR $T_{k,n}(a, d)$

Let $T_{k,n}(a,d)$ be defined by (1.3), with $k \geq 0$, $n > 0$, and $d \neq 0$. This type of sum is discussed briefly by Bachmann [1, pp. 27-29] and Turner [9]. We note that $T_{k,n}(a,d)$ can be expressed in terms of $S_{k,n}(a,d)$ in the following ways:

$$T_{k,2n+1}(a,d) = S_{k,n+1}(a,2d) - S_{k,n}(a+d,2d),$$
$$T_{k,2n}(a,d) = S_{k,n}(a,2d) - S_{k,n}(a+d,2d).$$

Also, if $a = d = 1$, then

$$T_{k,2n} = S_{k,2n} - 2^{k-1}S_{k,n}, \tag{6.1}$$

which makes some of the formulas for $T_{k,n}$ trivial in light of the results of Sections 1-5.

In the remainder of the paper, we find formulas for $T_{k,n}(a,d)$ that correspond to the ones for $S_{k,n}(a,d)$. The essential tool is the generating function

$$\sum_{k=0}^{\infty} T_{k,n}(a,d)\frac{x^k}{k!} = e^{ax} - e^{(a+d)x} + \cdots + (-1)^{n-1}e^{(a+nd-d)x} = \frac{(-1)^{n-1}e^{(a+nd)x} + e^{ax}}{e^{dx}+1}. \tag{6.2}$$

The following three theorems are analogs of Theorems 2.1, 2.2, and 3.1, and they are proved in exactly the same way as those earlier theorems. The proofs, which use (6.2) instead of (2.1), are omitted.

Theorem 6.1: We have the following two recurrences for $T_{k,n}(a,d)$: For $k \geq 1, n > 0$,

$$2T_{k+1,n}(a,d) + \sum_{j=0}^{k}\binom{k+1}{j}d^{k+1-j}T_{j,n}(a,d) = (-1)^{n-1}(a+nd)^{k+1} + a^{k+1}, \tag{6.3}$$

$$2T_{k+1,n}(a,d) + \sum_{j=0}^{k}\binom{k+1}{j}(-d)^{k+1-j}T_{j,n}(a,d) = (-1)^{n+1}(a+nd-d)^{k+1} + (a-d)^{k+1}. \tag{6.4}$$

Formula (6.3) generalizes a formula of Turner [9].

Theorem 6.2: For $k = 1, 2, 3, \ldots,$ we have

$$2\sum_{j=0}^{k-1}\binom{2k-1}{2j}d^{2k-2j-1}T_{2j,n}(a,d)$$
$$= (-1)^{n-1}(a+nd)^{2k-1} + (-1)^n(a+nd-d)^{2k-1} - (a-d)^{2k-1} + a^{2k-1},$$

and

$$2\sum_{j=1}^{k}\binom{2k}{2j-1}d^{2k-2j+1}T_{2j-1,n}(a,d)$$
$$= (-1)^{n-1}(a+nd)^{2k} + (-1)^n(a+nd-d)^{2k} - (a-d)^{2k} + a^{2k}.$$

Theorem 6.3: Let w_j be defined by (3.2) and (3.3). Then, for $m = 0, 1, \ldots, 5$, and $n > 0, k \geq 0$, we have

$$\begin{aligned}8T_{6k+m,n}(a,d) = &-6\sum_{j=0}^{k-1}\binom{6k+m}{6j+m}d^{6k-6j}T_{6j+m,n}(a,d)\\ &+\sum_{j=0}^{6k+m-1}\binom{6k+m}{j}d^{6k+m-j}w_{m-j}[(-1)^{n-1}(a-d+nd)^j+(a-d)^j]\\ &+4[(-1)^{n-1}(a-d+nd)^{6k+m}+(a-d)^{6k+m}].\end{aligned}$$

Corollary: Let $n>0$, $k\geq 0$. For $m=0,1,\ldots,5$, and m and k not both 0, we have

$$\begin{aligned}8T_{6k+m,n} = &-6\sum_{j=0}^{k-1}\binom{6k+m}{6j+m}T_{6j+m,n}+(-1)^{n-1}\sum_{j=0}^{6k+m-1}\binom{6k+m}{j}w_{m-j}n^j\\ &+4(-1)^{n-1}n^{6k+m}+[1+(-1)^{n-1}]w_m.\end{aligned}$$

Note that, for $m=1,2,\ldots,5$, we have

$$8T_{m,n}=(-1)^{n-1}\sum_{j=1}^{m-1}\binom{m}{j}w_{m-j}n^j+4(-1)^{n-1}n^m+[1+(-1)^{n-1}]w_m.$$

To illustrate Theorem 6.3, we first calculate $T_{4,n}$:

$$8T_{4,n}=(-1)^{n-1}\left[4n^4+\binom{4}{3}2n^3+\binom{4}{2}0n^2+\binom{4}{1}(-1)n+0\right];$$

so

$$T_{4,n}=(-1)^{n-1}\left(-\frac{1}{2}n+n^3+\frac{1}{2}n^4\right).$$

Then

$$8T_{10,n}=-6\binom{10}{4}T_{4,n}+(-1)^{n-1}\sum_{j=1}^{9}\binom{10}{j}w_{4-j}n^j+(-1)^{n-1}4n^{10},$$

which gives us

$$T_{10,n}=(-1)^{n-1}\left(\frac{155}{2}n-\frac{255}{2}n^3+63n^5-15n^7+\frac{5}{2}n^9+\frac{1}{2}n^{10}\right).$$

Continuing in the same way, we have

$$8T_{16,n}=-6\binom{16}{4}T_{4,n}-6\binom{16}{10}T_{10,n}+(-1)^{n-1}\sum_{j=1}^{15}\binom{16}{j}w_{4-j}n^j+(-1)^{n-1}4n^{16},$$

which gives us

$$\begin{aligned}T_{16,n}=(-1)^{n-1}\Big(&-\frac{929569}{2}n+764540n^3-377286n^5+88660n^7\\ &-12155n^9+1092n^{11}-70n^{13}+4n^{15}+\frac{1}{2}n^{16}\Big).\end{aligned}$$

$T_{k,n}(a, d)$ IN TERMS OF GENOCCHI NUMBERS

The Euler polynomial $E_k(z)$ may be defined by the generating function [2, pp. 48-49]

$$\frac{2e^{xz}}{e^x+1}=\sum_{k=0}^{\infty}E_k(z)\frac{x^k}{k!}. \tag{7.1}$$

For $z=1$, we have

$$E_k(1)=\frac{2(2^{k+1}-1)}{k+1}B_{k+1}=-\frac{1}{k+1}G_{k+1}, \tag{7.2}$$

where B_{k+1} is a Bernoulli number and G_{k+1} is called a Genocchi number [2, p. 49]. The Genocchi numbers are integers such that $G_{2m+1}=0$ for $m>0$; the first few are $G_0=0$, $G_1=1$, $G_2=-1$, $G_4=1$, $G_6=-3$. It follows from (7.1) that

$$E_k(z)=\sum_{j=0}^{k}\binom{k}{j}\frac{G_{k-j+1}}{k-j+1}z^j. \tag{7.3}$$

Comparing (7.1) and (6.2), we see that

$$T_{k,n}(a,d)=\frac{d^k}{2}\left[(-1)^{n-1}E_k\left(\frac{a}{d}+n\right)+E_k\left(\frac{a}{d}\right)\right]. \tag{7.4}$$

By (7.3) and (7.4), we have the following result. For fixed a and d, if we write $T_{k,n}(a,d)$ as a polynomial in $(a-d+nd)$, i.e.,

$$\begin{aligned}T_{k,n}(a,d)&=T_{k,n-1}(a,d)+(a-d+nd)^k\\&=u_{k,0}+u_{k,1}(a-d+nd)+u_{k,2}(a-d+nd)^2+\cdots+u_{k,k}(a-d+nd)^k,\end{aligned} \tag{7.5}$$

then we have explicit formulas for the coefficients $u_{k,j}$ in terms of Genocchi numbers.

Theorem 7.1: If $T_{k,n}(a,d)$ is written as a polynomial in $(a-d+nd)$ and $u_{k,j}$ is defined by (7.5) for $j=0,1,\ldots,k$, then

$$u_{k,j}=\frac{(-1)^n d^{k-j}}{2(k-j+1)}\binom{k}{j}G_{k-j+1}\quad(1\le j\le k-1),$$

$$u_{k,0}=\frac{(-1)^n d^k}{2(k+1)}G_{k+1}+\frac{d^k}{2}E_k\left(\frac{a}{d}\right),\quad u_{k,k}=\frac{(-1)^{n-1}}{2}.$$

When $a=d=1$, we have

$$T_{k,n}=T_{k,n}(1,1)=\frac{(-1)^{n-1}}{2}n^k+\frac{(-1)^n}{2}\sum_{j=0}^{k-1}\binom{k}{j}\frac{G_{k-j+1}}{k-j+1}n^j-\frac{G_{k+1}}{2(k+1)}. \tag{7.6}$$

When n is even, (7.6) follows from (6.1) and the formulas in Section 6 [1, p. 27].

For example, $T_{3,n}=1^3-2^3+\cdots+(-1)^{n-1}n^3$

$$=\frac{(-1)^{n-1}}{2}n^3+\frac{3(-1)^n}{4}G_2n^2+\frac{(-1)^n}{2}G_3n+\frac{[(-1)^n-1]}{8}G_4=\frac{(-1)^{n-1}}{8}[4n^3+6n^2-1+(-1)^n].$$

An application of Theorem 6.3 that is analogous to (4.5) is the following. If n is odd and $a = d = 1$, by Theorem 7.1 we have $u_{k,0} = -G_{k+1}/(k+1)$. Thus, by Theorems 7.1 and 6.3 we have, for $m = 1, 2, \ldots, 5$,

$$8G_{6k+m+1} = -6\sum_{j=0}^{k-1}\binom{6k+m+1}{6j+m+1}G_{6j+m+1} - 2(6k+m+1)w_m,$$

which is equivalent to a formula of Ramanujan [7, p. 12].

8. FINDING $T_{k,n}(a,d)$ FROM $T_{k-1,n}(a,d)$

We proceed as we did for $S_{k,n}(a,d)$. By using induction on (6.3), we can prove that $T_{k,n}(a,d)$ is a polynomial in n of degree k, and the constant term is 0 if n is even. That also follows from the results of Section 7. Thus, for fixed a and d, we can write

$$T_{k,n}(a,d) = \begin{cases} t_{k,1}n + t_{k,2}n^2 + \cdots + t_{k,k}n^k & (n \text{ even}), \\ h_{k,0} + h_{k,1}n + \cdots + h_{k,k}n^k & (n \text{ odd}). \end{cases} \tag{8.1}$$

Using the generating function (6.2), we prove the next theorem just as we proved Theorem 5.1.

Theorem 8.1: For fixed a and d, let $t_{k,j}$ and $h_{k,j}$ be defined by (8.1) for $j = 0, \ldots, k$. Then for $k \geq 1$ we have

$$t_{k,j} = \frac{kd}{j}t_{k-1,j-1} \quad (j = 2, \ldots, k); \qquad h_{k,j} = \frac{kd}{j}h_{k-1,j-1} \quad (j = 2, \ldots, k);$$

$$t_{k,1} = -\frac{kd^k}{2}E_{k-1}\left(\frac{a}{d}\right) = \frac{1}{2}[a^k - (a+d)^k] - (2t_{k,2} + 2^2t_{k,3} + \cdots + 2^{k-1}t_{k,k});$$

$$h_{k,0} = a^k - h_{k,1} - \cdots - h_{k,k};$$

$$h_{k,1} = -\frac{kd^k}{2}E_{k-1}\left(\frac{a}{d}\right) + kd\,h_{k-1,0} = t_{k,1} + kd\,h_{k-1,0}.$$

Thus, if we know $T_{k-1,n}(a,d)$, we can use Theorem 8.1 to find $T_{k,n}(a,d)$. For example,

$$T_{0,n}(a,d) = \begin{cases} 0 & (n \text{ even}), \\ 1 & (n \text{ odd}); \end{cases} \qquad T_{1,n}(a,d) = \begin{cases} -(d/2)n & (n \text{ even}), \\ (a - d/2) + (d/2)n & (n \text{ odd}). \end{cases}$$

Then, for n even, we have $T_{2,n}(a,d) = t_{2,1}n + t_{2,2}n^2$, with

$$t_{2,2} = dt_{1,1} = -d^2/2; \qquad t_{2,1} = \frac{1}{2}[a^2 - (a+d)^2] + d^2 = -ad + d^2/2.$$

Thus,

$$T_{2,n}(a,d) = \left(\frac{1}{2}d^2 - ad\right)n - \frac{1}{2}d^2n^2 \quad (n \text{ even}).$$

For n odd, $T_{2,n}(a,d) = h_{2,0} + h_{2,1}n + h_{2,2}n^2$, with

$$h_{2,2} = dh_{1,1} = d^2/2; \quad h_{2,1} = t_{2,1} + 2dh_{1,0} = ad - d^2/2; \quad h_{2,0} = a^2 - h_{2,1} - h_{2,2} = a^2 - ad.$$

Thus,

$$T_{2,n}(a,d) = (a^2 - ad) + \left(ad - \frac{1}{2}d^2\right)n + \frac{1}{2}d^2n^2 \quad (n \text{ odd}).$$

Equivalently, by Theorem 8.1, we can integrate to find $S_{k,n}(a,d)$:

$$S_{k,n}(a,d) = kd\int S_{k-1,n}(a,d)\,dn - \frac{kd^k}{2}E_{k-1}\left(\frac{a}{d}\right)n + h_k(n),$$

where $E_{k-1}(\frac{a}{d})$ can be found by means of Theorem 8.1, and

$$h_k(n) = \begin{cases} 0 & (n \text{ even}), \\ h_{k,0} = a^k - h_{k,1} - h_{k,2} - \cdots - h_{k,k} & (n \text{ odd}). \end{cases}$$

9. FINAL COMMENTS

In summary, we have used generating functions to prove and generalize some of the basic formulas for sums of powers of integers. In particular, we have used the generating function technique to find: recurrence relations for $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$; explicit formulas (involving Bernoulli numbers) for $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$ if they are written as polynomials in n; methods for finding $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$ from $S_{k-1,n}(a,d)$ and $T_{k-1,n}(a,d)$. Some of the results are old (and scattered in the literature), and most of the proofs are straightforward. However, the writer believes that many of the generalizations are new, and he believes that Theorems 3.1 and 6.1 give us new recurrence formulas that are of interest.

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A FIBONACCI MODEL OF INFECTIOUS DISEASE

F. Dubeau
Département de mathématiques et d'informatique, Faculté des Sciences,
Université de Sherbrooke, 2500 Boul. Université, Sherbrooke (Québec), Canada, J1K 2R1

A. G. Shannon
School of Mathematical Sciences, University of Technology, Sydney, 2007, Australia
(*Submitted September 1994*)

1. INTRODUCTION

The Fibonacci rabbit population model is often regarded as one of the first studies of population growth using mathematics. Later, an analytic model of population dynamics was introduced by Voltera (Deakin & McElwain [2]). Systematic epidemic modeling in age-structured populations was first carried out in this century by Hoppensteadt [6].

Dubeau [3], in revisiting the Fibonacci rabbit growth model, has developed an approach that can be applied to population dynamics and epidemiology where censoring occurs either by inability to procreate or by death. It is the purpose of this note to apply Dubeau's method to Fibonacci's model of infectious diseases which was developed by Makhmudov [9] and to combine it with the approach of Shannon et al. [12] who attempted to refine the work of Makhmudov.

2. THE MODEL

Following Makhmudov [9], three epidemiological stages in the process of spreading infectious diseases are postulated:

(i) an initial (incubation) stage of r periods (periods 0, 1, 2, ..., $r-1$) during which those who are ill with the disease do not affect others,

(ii) a mature (infectious) stage of t periods (periods $r, r+1, \dots, r+t-1$) when each person infects s healthy people, and

(iii) a removal stage of m periods (periods $r+t, \dots, r+t+m-1$) when those who have been infected are no longer infectious.

An example might be the common cold which, on average, takes about two days to develop ($r=2$), a person is then infectious for about three days ($t=3$), and the symptoms persist for about seven days ($r+t+m=7$, hence $m=2$). In general, s is variable, but we shall treat it as a constant in the absence of other information. For background material on the structure of general epidemic models, the reader is referred to Billiard & Zhen Zhao [1].

In terms of a modification to Fibonacci's rabbit problem, these correspond in turn to

(i) the infancy stage,

(ii) the reproductive stage, and

(iii) the post reproductive stage,

respectively, and instead of infectives we have male-female pairs of rabbits. For the original Fibonacci model, we take $r=2, s=1,$ and $t=m=+\infty$.

2.1 A Direct Approach

Following Dubeau [3], let

u_n be the total number of disease carriers at the n^{th} period, and

v_n^i be the number of i-period old disease carriers at the n^{th} period.

More precisely, v_n^i represents the disease carriers in the

(i) initial stage for $i = 0, \ldots, r-1$,
(ii) mature stage for $i = r, \ldots, r+t-1$,
(iii) removal reproductive stage for $i = r+t, \ldots, r+t+m-1$,

and for $i = r+t+m, \ldots$, v_n^i represents the disease carriers who have been infected in the past but have already recovered.

It will be convenient to define u_n and v_n^i for all $n \in Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ and $i \in N = \{0, 1, 2, 3, \ldots\}$. We consider the following initial conditions on v_n^i:

$$v_n^i = \begin{cases} 0 & \text{for } \begin{cases} n < 0 \text{ and } i = 0, 1, 2, \ldots, \\ n = 0 \text{ and } i = 1, 2, 3, \ldots, \end{cases} \\ 1 \text{ (or } v_0^0) & \text{for } n = 0 \text{ and } i = 0. \end{cases}$$

As a consequence, for any $n \in Z$,

$$v_n^i = v_{n-1}^{i-1} \quad \text{for } i > 0,$$

and

$$v_n^0 = \begin{cases} 1 & \text{for } n = 0, \\ s\{v_n^r + \cdots + v_n^{r+t-1}\} & \text{for } n \neq 0. \end{cases}$$

We obtain

$$v_n^0 = v_{n-1}^0 + s\{v_{n-r}^0 - v_{n-r-t}^0\} \tag{1}$$

for $n > 1$. From the definition we have, for any $n \in Z$,

$$u_n = \sum_{i=0}^{r+t+m-1} v_n^i.$$

It follows that

$$u_n = \begin{cases} 0 & \text{for } n < 0, \\ u_0 + s\sum_{k=r}^{r+t-1} u_{n-k} & \text{for } n = 0, \ldots, r+t+m-1, \\ s\sum_{k=r}^{r+t-1} u_{n-k} & \text{for } n \geq r+t+m, \end{cases} \tag{2}$$

or

$$u_n = u_{n-1} - \delta_{n,\, r+t+m} u_0 + s\{u_{n-r} - u_{n-r-t}\} \tag{3}$$

for $n \geq 1$, where $\delta_{i,j} = 0$ if $i \neq j$, or 1 if $i = j$.

Example 1: Table 1 contains values of v_n^i and u_n for $r=2, t=3, m=2,$ and $s=1$.

TABLE 1. $r=2, t=3, m=2$, and $s=1$

	u_n							
$n \backslash i$	0	1	2	3	4	5	6	u_n
0	1	0	0	0	0	0	0	1
1	0	1	0	0	0	0	0	1
2	1	0	1	0	0	0	0	2
3	1	1	0	1	0	0	0	3
4	2	1	1	0	1	0	0	5
5	2	2	1	1	0	1	0	7
6	4	2	2	1	1	0	1	11
7	5	4	2	2	1	1	0	15
8	8	5	4	2	2	1	1	23
9	11	8	5	4	2	2	1	33
10	17	11	8	5	4	2	2	49
11	24	17	11	8	5	4	2	71
12	36	24	17	11	8	5	4	105
13	52	36	24	17	11	8	5	153
14	77	52	36	24	17	11	8	225
15	112	77	52	36	24	17	11	329
16	165	112	77	52	36	24	17	483
17	241	165	112	77	52	36	24	707
18	354	241	165	112	77	52	36	1037
19	518	354	241	165	112	77	52	1519
20	760	518	354	241	165	112	77	2227

Remark—The Effect of m: Let $\{u_n\}_{n=0}^{+\infty}$ and $\{\widetilde{u}_n\}_{n=0}^{+\infty}$ be the sequences generated with m and $m+1$ for the same values of r, t, and s. From (3) we have, for $n \geq 1$,

$$u_n = u_{n-1} - \delta_{n,\, r+t+m} u_0 + s\{u_{n-r} - u_{n-r-t}\}$$

and

$$\widetilde{u}_n = \widetilde{u}_{n-1} - \delta_{n,\, r+t+m+1} u_0 + s\{\widetilde{u}_{n-r} - \widetilde{u}_{n-r-t}\}.$$

Let $\Delta_n u = u_n - \widetilde{u}_n$, then from (3)

$$\Delta_n u = \Delta_{n-1} u + (\delta_{n,\, r+t+m} - \delta_{n, r+t+m+1}) u_0 + s\{\Delta_{n-r} u - \Delta_{n-r-t} u\}.$$

It follows from (1) that $\Delta_{r+t+m+n} u = v_n^0$ for $n \geq 0$.

2.2 A Generating Function Approach

Following Weland [13], Hoggatt [4], Hoggatt & Lind [5], and Parberry [11], we can use the generating function method to obtain the recurrence relation (1), (2), or (3).

Let us define the generating function for the sequence $\{u_n\}_{n=0}^{+\infty}$:

$$U(x) = \sum_{n=0}^{+\infty} u_n x^n.$$

The function $U(x)$ can be expressed in terms of (i) the generating function of the infectious process (a polynomial)

$$B(x) = \sum_{n=0}^{+\infty} b_n x^n \quad (b_0 = 0),$$

where b_n indicates the number of infected healthy people by an n-period old disease carrier, and (ii) the "total recovering" polynomial $D(x) = x^{r+t+m}$.

Let

$$V(x) = \sum_{n=0}^{+\infty} v_n^0 x^n$$

be the generating function associated to the sequence $\{v_n^0\}_{n=0}^{+\infty}$, where

$$\begin{aligned} &v_0^0 = 1, \\ &v_1^0 = b_0 v_1^0 + b_1 v_0^0, \\ &v_2^0 = b_0 v_2^0 + b_1 v_1^0 + b_2 v_0^0 \\ &\text{etc.,} \end{aligned}$$

and, in general,

$$v_n^0 = \sum_{j=0}^{n} b_j v_{n-j}^0$$

for $n \geq 1$. It follows that

$$V(x) = \frac{1}{1 - B(x)}.$$

Let u_n^* be the number of disease carriers at the n^{th} period, assuming no recovery, and

$$U^*(x) = \sum_{n=0}^{+\infty} u_n^* x^n.$$

Then

$$u_n^* = \sum_{j=0}^{n} v_j^0$$

and we obtain

$$U^*(x) = \frac{1}{(1-x)}V(x) = \frac{1}{(1-x)(1-B(x))}.$$

If we now allow for recoveries, since each disease carrier recovers $r+t+m$ periods after its infection, the number h_n of recovering people at the n^{th} period is given by $h_n = v^0_{n-(r+t+m)}$. Therefore,

$$H(x) = \sum_{n=0}^{+\infty} h_n x^n = D(x)V(x) = \frac{D(x)}{1-B(x)}.$$

Let r_n be the total number of people who recovered up to the n^{th} period, then

$$r_n = \sum_{j=0}^{n} h_j$$

and

$$R(x) = \sum_{n=0}^{+\infty} r_n x^n = \frac{1}{1-x}H(x) = \frac{D(x)}{(1-x)(1-B(x))}.$$

Now, $u_n = u^*_n - r_n$ $(n \geq 0)$, so that

$$U(x) = U^*(x) - R(x) = \frac{1-D(x)}{(1-x)(1-B(x))}.$$

From the model, we have

$$B(x) = s\sum_{n=r}^{r+t-1} x^n, \qquad D(x) = x^{r+t+m},$$

and

$$U(x) = \frac{1-x^{r+t+m}}{1-x-sx^r+sx^{r+t}}.$$

It follows that $u_0 = 1$ and, for $n \geq 1$, $u_n = u_{n-1} - \delta_{n,r+t+m}u_0 + s\{u_{n-r} - u_{n-(r+t)}\}$. Moreover, since

$$V(x) = \frac{1}{1-s\sum_{n=r}^{r+t-1} x^n},$$

it follows that $v^0_0 = 1$, and $v^0_n = s\{v^0_{n-r} + \cdots + v^0_{n-(r+t-1)}\}$ for $n \geq 1$.

2.3 A Matrix Approach

Following Klarner [8], let us consider the sequence of $(r+t+m)$-vectors $\{v_n\}_{n=0}^{+\infty}$:

$$v_n = [v^0_n, v^1_n, \ldots, v^{r+t+m-1}_n] \quad (n = 0, 1, 2, \ldots).$$

They are related by the equation

$$v_{n+1} = v_n F = \cdots = v_0 F^{n+1}, \tag{4}$$

where $v_0 = [1, 0, \ldots, 0]$ and $F = (f_{ij})$ is a square matrix of order $r+t+m$ with entries f_{ij} $(i = 0, \ldots, r+t+m-1;\ j = 0, \ldots, r+t+m-1)$ such that

$$v_{n+1}^j = \sum_{i=0}^{r+t+m-1} v_n^i f_{ij}.$$

For our problem,

$$f_{i0} = \begin{cases} s & \text{for } i = r, \ldots, r+r-1, \\ 0 & \text{elsewhere,} \end{cases}$$

and for $j = 1, \ldots, r+t+m-1$,

$$f_{ij} = \begin{cases} 1 & \text{if } i = j-1, \\ 0 & \text{elsewhere.} \end{cases}$$

The characteristic polynomial of F is $\det(xI - F) = x^{r+t+m} - s(x^{t+m} + \cdots + x^{1+m}) = c_F(x)$. From the Cayley-Hamilton theorem, the matrix F satisfies its characteristic equation, and we have $c_F(F) = 0$. Hence, $F^n c_F(F) = 0$ for any $n \ge 0$. It follows that $F^n - s(F^{n-r} + \cdots + F^{n-(r+t-1)}) = 0$ for $n \ge r+t+m$. Finally, from (4), we have

$$v_n - s(v_{n-r} + \cdots + v_{n-(r+t-1)}) = 0$$

and, since $u_n = v_n 1$, where $1 = [1, \ldots, 1]^{\mathrm{T}}$,

$$u_n - s(u_{n-r} + \cdots + u_{n-(r+t-1)}) = 0$$

for $n \ge r+t+m$.

3. A RELATED ARRAY

Let $w_n^i(k)$ be the number of i-period old disease carriers of the k^{th} generation at the n^{th} period, and $w_n(k)$ be the number of disease carriers of the k^{th} generation at the n^{th} period. We have

$$w_n(k) = \sum_{i=0}^{r+t+m-1} w_n^i(k)$$

and, for $i \ge r+t+m$, $w_n^i(k)$ indicates the number of people of the k^{th} generation at the n^{th} period infected i periods ago and who have already recovered.

We also have

$$\begin{aligned} w_n^i(k) &= 0 && \text{for } n < 0 \text{ or } i < 0, \\ w_n^i(k) &= \delta_{ni} && \text{for } n \ge 0 \text{ and } i \ge 0, \end{aligned}$$

and, for $k \ge 1, n \ge 0$, and $i \ge 0$,

$$w_n^i(k) = \begin{cases} s\{w_n^r(k-1) + \cdots + w_n^{r+t-1}(k-1)\} & \text{for } i = 0, \\ w_{n-1}^{i-1}(k) & \text{for } i \ge 1. \end{cases}$$

We can deduce that $w_n^0(k) \ne 0$ for $kr \le n \le k(r+t-1), k = 0, 1, 2, \ldots$. It follows that $w_n(k) \ne 0$ only for $k \ge 0$ and $n \ge 0$ such that $kr = n_L(k) \le n \le n_U(k) = (r+t+m-1) + k(r+t-1)$. Then, for a given n, let

$$k_L(n) = \min\left\{k \in N \mid k \geq 0 \text{ and } k \geq \frac{n-(r+t+m-1)}{r+t-1}\right\},$$
$$k_U(n) = \max\left\{k \in N \mid k \leq \frac{n}{r}\right\};$$

hence, $w_n(k) = 0$ for $k < k_L(n)$ and $k > k_U(n)$. To relate the $w_n(k)$ and $w_n^i(k)$ to the u_n and v_n^i, we have

$$u_n = \sum_{k=0}^{+\infty} w_n(k) = \sum_{k=k_L(n)}^{k_U(n)} w_n(k),$$
$$v_n^i = \sum_{k=0}^{+\infty} w_n^i(k) = \sum_{k=0}^{k_U(n)} w_n^i(k).$$

Also, for $k \geq 1$,

$$\begin{aligned} w_n(k) &= \sum_{i=0}^{r+t+m-1} w_n^i(k) = \sum_{i=0}^{r+t+m-1} w_{n-i}^0(k) \\ &= \sum_{i=0}^{r+t+m-1} s \sum_{\ell=r}^{r+t-1} w_{n-i}^{\ell}(k-1) = s \sum_{\ell=r}^{r+t-1} \sum_{i=0}^{r+t+m-1} w_{n-i}^{\ell}(k-1) \\ &= s \sum_{\ell=r}^{r+t-1} \sum_{i=0}^{r+t+m-1} w_{n-\ell}^{i}(k-1) = s \sum_{\ell=r}^{r+t-1} w_{n-\ell}(k-1). \end{aligned}$$

As a consequence, using the generating function, we have

$$\begin{aligned} G_k(x) &= \sum_{n=0}^{+\infty} w_n(k) x^n = s \sum_{n=0}^{+\infty} \left(\sum_{\ell=r}^{r+t-1} w_{n-\ell}(k-1) \right) x^n \\ &= s \sum_{n=0}^{+\infty} \left(\sum_{\ell=n-(r+t-1)}^{n-r} w_{\ell}(k-1) \right) x^n = s(x^r + \cdots + x^{r+t-1}) \sum_{n=0}^{+\infty} w_n(k-1) x^n \\ &= s(x^r + \cdots + x^{r+t-1}) G_{k-1}(x). \end{aligned}$$

Also,

$$G_0(x) = \sum_{n=0}^{+\infty} w_n(0) x^n = 1 + x + \cdots + x^{r+t+m-1},$$

thus

$$G_k(x) = s^k \left[x^r (1 + x + \cdots + x^{t-1})\right]^k G_0(x)$$

and

$$G_k(1) = (st)^k (r+t+m) = \sum_{n=n_L(k)}^{u_U(k)} w_n(k).$$

Example 2: Table 2 illustrates the values of $w_n^i(k)$ and $w_n(k)$ for $r = 2, t = 3, m = 2$, and $s = 1$.

TABLE 2. $r = 2, t = 3, m = 2$, **and** $s = 1, n_L(k) = 2k, n_U(k) = 4k + 6$,
$k_L(n) = \min\left\{k \in N \mid k \geq 0 \text{ and } k \geq \frac{n-6}{4}\right\}$, $k_U(n) = \max\left\{k \in N \mid k \leq \frac{n}{2}\right\}$

		$w_n^i(k)$							
		i							
n	$k_L(n) \leq k \leq k_U(n)$	0	1	2	3	4	5	6	$w_n(k)$
0	$k_L(0) = 0 = k_U(0)$	1	0	0	0	0	0	0	1
1	$k_L(1) = 0 = k_U(1)$	0	1	0	0	0	0	0	1
2	$k_L(2) = 0$	0	0	1	0	0	0	0	1
	$1 = k_U(2)$	1	0	0	0	0	0	0	1
3	$k_L(3) = 0$	0	0	0	1	0	0	0	1
	$1 = k_U(3)$	1	1	0	0	0	0	0	2
4	$k_L(4) = 0$	0	0	0	0	1	0	0	1
	1	1	1	1	0	0	0	0	3
	$2 = k_U(4)$	1	0	0	0	0	0	0	1
5	$k_L(5) = 0$	0	0	0	0	0	1	0	1
	1	0	1	1	1	0	0	0	3
	$2 = k_U(5)$	2	1	0	0	0	0	0	3
6	$k_L(6) = 0$	0	0	0	0	0	0	1	1
	1	0	0	1	1	1	0	0	3
	2	3	2	1	0	0	0	0	6
	$3 = k_U(6)$	1	0	0	0	0	0	0	1
7	$k_L(7) = 1$	0	0	0	1	1	1	0	3
	2	2	3	2	1	0	0	0	8
	$3 = k_U(7)$	3	1	0	0	0	0	0	4
8	$k_L(8) = 1$	0	0	0	0	1	1	1	3
	2	1	2	3	2	1	0	0	9
	3	6	3	1	0	0	0	0	10
	$4 = k_U(8)$	1	0	0	0	0	0	0	1
9	$k_L(9) = 1$	0	0	0	0	0	1	1	2
	2	0	1	2	3	2	1	0	9
	3	7	6	3	1	0	0	0	17
	$4 = k_U(9)$	4	1	0	0	0	0	0	5

4. LIMIT OF RATIOS u_{n+1}/u_n

We consider the linear difference equation (2) of order $r+t-1$:

$$u_n = s\sum_{k=r}^{r+t-1} u_{n-k} \quad (n \geq r+t+m).$$

The sequence $\{u_n\}_{n=r+t+m}^{+\infty}$ is completely defined if we assume that the values u_{m+1}, u_{m+2}, ..., $u_{r+t+m-1}$ are known.

For our model (2) or (3), we observe that the finite sequence $\{u_n\}_{n=0}^{r+t+m-1}$ is a sequence of non-decreasing integers with $u_0 = 1$ (or any initial value $u_0 > 0$).

We consider two cases for the analysis of the ratios u_{n+1}/u_n: the case $t = 1$ and the case $t > 1$.

4.1 The Case $t = 1$

We have $u_n = su_{n-r}$ $(n \geq r+m+1)$. It follows that

$$\frac{u_{n+r+1}}{u_{n+r}} = \frac{u_{n+1}}{u_n} \quad (n \geq m+1)$$

and the sequence of ratios u_{n+1}/u_n is a sequence of length r repeated infinitely many times. It is completely characterized by the finite sequence

$$\frac{u_{n+1}}{u_n} \quad \text{for } n = m+1, \ldots, m+r.$$

Using (2), for the initial value $u_0 = 1$, we have

$$u_n = \sum_{i=0}^{k} s^i \quad \text{for} \begin{cases} n \leq r+m, \text{ and} \\ kr \leq n < (k+1)r \end{cases}$$

and

$$u_n = \sum_{i=0}^{\lfloor \frac{n}{r} \rfloor} s^i \quad \text{for } n = 0, \ldots, r+m.$$

Let

$$\rho_U = \left\lfloor \frac{r+m}{r} \right\rfloor \quad \text{and} \quad \rho_L = \left\lfloor \frac{1+m}{r} \right\rfloor,$$

then $\rho_U = \rho_L$ or $\rho_L + 1$. Hence, the sequence $\left\{\frac{u_{n+1}}{u_n}\right\}_{n=m+1}^{m+r}$ is such that

$$\frac{u_{n+1}}{u_n} = \begin{cases} 1 & r-2 \text{ times,} \\ s\sum_{i=0}^{\rho_L} s^i \Big/ \sum_{i=0}^{\rho_U} s^i & 1 \text{ time,} \\ \sum_{i=0}^{\rho_U} s^i \Big/ \sum_{i=0}^{\rho_L} s^i & 1 \text{ time.} \end{cases} \tag{6}$$

It can be shown that the set

$$\left\{ \frac{u_{n+1}}{u_n} \,\middle|\, n = m+1, \ldots, m+r \right\}$$

converges to the set $\{1, s\}$ when m goes to $+\infty$.

Example 3: Table 3 contains the values of (6) for $r = 4$, $t = 1$, and $s = 2$.

TABLE 3. $r = 4$, $t = 1$, and $s = 2$

$\frac{u_{m+i+1}}{u_{m+i}}$	m								
i	1	2	3	4	5	6	7	8	9
1	1	3	1	1	1	7/3	1	1	1
2	3	1	1	1	7/3	1	1	1	15/7
3	1	1	1	7/3	1	1	1	15/7	1
4	2/3	2/3	2	6/7	6/7	6/7	2	14/15	14/15

4.2 The Case $t > 1$

Let $K = r + t - 1$. The linear difference equation (5) is equivalent to the following linear difference equation of order K,

$$u_{n+K} = s\sum_{k=0}^{t-1} u_{n+k} \quad (n \geq 0)$$

if the sequence $\{u_n\}_{n=0}^{K-1}$ for (7) corresponds to the sequence $\{u_n\}_{n=m+1}^{m+K}$ for (5). Hence, the limit of u_{n+1} / u_n is the same for both equations.

Let us recall some definitions and results about linear difference equations of the form

$$u_{n+K} - b_1 u_{n+K-1} - \cdots - b_{K-1}u_{n+1} - b_K u_n = 0 \quad (n \geq 0). \tag{8}$$

Definitions:

(a) The polynomial $\varphi(\lambda) = \lambda^K - b_1\lambda^{K-1} - \cdots - b_K$ is called the characteristic polynomial for (8).

(b) The equation $\varphi(\lambda) = 0$ is the characteristic equation for (8).

(c) The solutions $\lambda_1, \ldots, \lambda_\ell$ of the characteristic equation are the characteristic roots.

The first result is a standard result about the general solution of (8).

Theorem 1: Suppose (8) has characteristic roots $\lambda_1, \ldots, \lambda_k$ with multiplicities $j_1, \ldots, j_k$, respectively. Then (8) has n independent solutions $n^j\lambda_\ell^n$, $j = 0, \ldots, j_\ell - 1$; $\ell = 1, \ldots, k$. Moreover, any solution of (8) is of the form

$$u_n = \sum_{\ell=1}^{k} \sum_{j=0}^{j_\ell - 1} \beta_{\ell,j} n^j \lambda_\ell^n \quad (n \geq 0),$$

where the $\beta_{\ell,j}$ are obtained from the values of u_n for $n = 0, \ldots, K-1$.

Proof: See, for example, Kelley & Peterson [7]. □

The next two results depend on the form of (8).

Theorem 2: Assume the b_i are nonnegative in (8).

(a) If at least one b_i is strictly positive, then (8) has a unique simple characteristic root $\sigma > 0$ and all other characteristic roots of (8) have moduli not greater than σ.

(b) If the indices of the b_i that are strictly positive have the common greatest divisor 1, then (8) has a unique simple characteristic root $\sigma > 0$, and the moduli of all other characteristic roots of (8) is strictly less than σ.

Proof: See Ostrowski [10, pp. 91-92]. □

Theorem 3: If in (8) the b_i are nonnegative and $\{u_n\}_{n=0}^{+\infty}$ is a sequence satisfying (8) such that u_0, u_1, ..., u_{K-1} are strictly positive, then we have $u_n \geq \alpha\sigma^n$ $(n \geq 0)$, where $\alpha > 0$ is given by

$$\alpha = \min\left\{\frac{u_n}{\sigma^n} \,|\, n = 0, \ldots, K-1\right\}.$$

Proof: See Ostrowski [10, p. 93]. □

Since

$$b_i = \begin{cases} 0 & \text{for } i = 1, \ldots, r-1, \\ s & \text{for } i = r, \ldots, r+t-1, \end{cases}$$

and the common greatest divisor of $r, \ldots, r+t-1$ is 1 for $t > 1$, it follows from Theorem 2(b) that (7) as a unique simple characteristic root $\sigma > 0$ and the moduli of all other characteristic roots are less than σ.

Let $\lambda_1, \ldots, \lambda_k$ and σ be the characteristic roots of (7), then, from Theorem 1,

$$u_n = \beta\sigma^n + \sum_{\ell=1}^{k}\sum_{j=0}^{j_\ell - 1} \beta_{\ell,j} n^j \lambda_\ell^n.$$

Moreover, since $u_0 \geq 1$ and $\{u_n\}_{n=0}^{K-1}$ is a nondecreasing sequence, we obtain, from Theorem 3, $u_n \geq \alpha\sigma^n$ for

$$\alpha = \min\left\{\frac{u_n}{\sigma^n} \,|\, n = 0, \ldots, K-1\right\}.$$

It follows that

$$\alpha \leq \frac{u_n}{\sigma^n} = \beta + \sum_{\ell=1}^{k}\sum_{j=0}^{j_\ell - 1} \beta_{\ell,j} n^j \left(\frac{\lambda_\ell}{\sigma}\right)^n,$$

and taking the limit on both sides we have $\lim_{n\to+\infty} u_n / \sigma^n = \beta \geq \alpha > 0$ as a consequence of the following lemma.

Lemma: If $|\rho| < 1$, then $\lim_{n\to+\infty} n^\alpha \rho^n = 0$ for any $\alpha = 0, 1, 2, \ldots$. □

Finally,

$$\frac{u_{n+1}}{u_n} = \sigma \frac{u_{n+1} / \sigma^{n+1}}{u_n / \sigma^n}$$

and we obtain $\lim_{n\to+\infty} u_{n+1}/u_n = \sigma$, where σ is the unique positive root of

$$\varphi(x) = x^{r+t-1} - s\sum_{i=0}^{t-1} x^i \quad (t > 1).$$

5. A MORE REALISTIC MODEL

In any real population, the epidemiological status of members is as follows: (i) susceptibles, (ii) infected, and (iii) resistants. Thus, there is not an unlimited supply of susceptibles.

Let

N be the total population,
S_n be the number of susceptibles at the n^{th} period,
U_n be the number of infected and carriers at the n^{th} period, and
R_n be the number of resistants at the n^{th} period.

Then $N = S_n + U_n + R_n$, and the initial conditions are $S_0 = N-1$, $U_0 = 1$, and $R_0 = 0$. Using the notation of Section 2, we have

$$U_n = \sum_{i=0}^{r+t+m-1} v_n^i$$

$$R_n = \sum_{i=r+t+m}^{+\infty} v_n^i = \sum_{i=r+t+m}^{+\infty} v_{n-i}^0 = \begin{cases} 0 & \text{if } n < r+t+m, \\ \displaystyle\sum_{i=r+t+m}^{n} v_{n-i}^0 & \text{if } n \geq r+t+m, \end{cases}$$

and

$$S_n = N - U_n - R_n.$$

However, the number of susceptibles is limited, so

$$v_n^0 = \min\left\{S_{n-1}, s\sum_{i=r}^{r+t-1} v_n^i\right\}$$

and

$$S_n = S_{n-1} - v_n^0,$$

$$U_n = U_{n-1} + v_n^0 - v_n^{r+t+m},$$

$$R_n = R_{n-1} + v_n^{r+t+m}.$$

For R_n we have

$$\begin{aligned} R_n &= R_{n-(r+t+m)} + \sum_{i=0}^{r+t+m-1} v_{n-i}^{r+t+m} \\ &= R_{n-(r+t+m)} + \sum_{i=0}^{r+t+m-1} v_{n-(r+t+m)}^i \\ &= R_{n-(r+t+m)} + U_{n-(r+t+m)} = N - S_{n-(r+t+m)}. \end{aligned}$$

It follows that

$$S_n - S_{n-(r+t+m)} + U_n = 0$$

or

$$U_n = S_{n-(r+t+m)} - S_n$$

and, if $S_n = 0$, then $U_n = S_{n-(r+t+m)}$.

Example 4: Table 4 illustrates this model for $r = 2, t = 3, m = 2, s = 1$, and $N = 200$.

TABLE 4. $r = 2, t = 3, m = 2, s = 1$, and $N = 200$

	v_n^i									
$n \diagdown i$	0	1	2	3	4	5	6	U_n	R_n	S_n
0	1	0	0	0	0	0	0	1	0	199
1	0	1	0	0	0	0	0	1	0	199
2	1	0	1	0	0	0	0	2	0	198
3	1	1	0	1	0	0	0	3	0	197
4	2	1	1	0	1	0	0	5	0	195
5	2	2	1	1	0	1	0	7	0	193
6	4	2	2	1	1	0	1	11	0	189
7	5	4	2	2	1	1	0	15	1	184
8	8	5	4	2	2	1	1	23	1	176
9	11	8	5	4	2	2	1	33	2	165
10	17	11	8	5	4	2	2	49	3	148
11	24	17	11	8	5	4	2	71	5	124
12	36	24	17	11	8	5	4	105	7	88
13	52	36	24	17	11	8	5	153	11	36
14	36	52	36	24	17	11	8	184	16	0
15	0	36	52	36	24	17	11	176	24	0
16	0	0	36	52	36	24	17	165	35	0
17	0	0	0	36	52	36	24	148	52	0
18	0	0	0	0	36	52	36	124	76	0
19	0	0	0	0	0	36	52	88	112	0
20	0	0	0	0	0	0	36	36	164	0
21	0	0	0	0	0	0	0	0	200	0

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ON THE FIBONACCI NUMBERS WHOSE SUBSCRIPT IS A POWER

Piero Filipponi
Fondazione Ugo Bordoni, Via B. Castiglione 59, I-00142 Rome, Italy
(*Submitted October 1994*)

1. AIM OF THE PAPER

The well-known identity (e.g., see [1], p. 127)

$$F_{2^n} = \prod_{i=1}^{n-1} L_{2^i} \tag{1.1}$$

led us to investigate the Fibonacci numbers of the form F_{k^n} with k and n positive integers. The principal aim of this note is to present some new identities involving F_{k^n}, some of which generalize (1.1). This is done in Sections 2 and 3. In Section 4, a first-order recurrence relation for F_{k^n} is established which involves certain combinatorial quantities whose properties will be investigated in a future paper. A glimpse of analogous results concerning the Lucas numbers L_{k^n} is caught in Section 5.

The formulas established in this note encompass the trivial case $n = 1$ under the usual assumptions

$$\prod_{i=a}^{b} f(i) = 1 \text{ if } b < a \tag{1.2}$$

and

$$\sum_{i=a}^{b} f(i) = 0 \text{ if } b < a. \tag{1.3}$$

2. MAIN RESULT

Proposition 1: If k is even and $n \geq 1$, then

$$F_{k^n} = F_k \prod_{i=1}^{n-1} \left[\sum_{j=1}^{k/2} L_{(2j-1)k^i} \right]. \tag{2.1}$$

We can immediately observe that, if $k = 2$, identity (2.1) reduces to (1.1).

Proof of Proposition 1: Write

$$F_{k^n} = F_k \cdot \frac{F_{k^2}}{F_k} \cdot \frac{F_{k^3}}{F_{k^2}} \cdot \ldots \cdot \frac{F_{k^n}}{F_{k^{n-1}}} = F_k \prod_{i=1}^{n-1} \frac{F_{k^{i+1}}}{F_{k^i}} \tag{2.2}$$

whence, following the notation used in [2] [namely, $R_s(t) = F_{st} / F_t$], we can rewrite (2.2) as

$$F_{k^n} = F_k \prod_{i=1}^{n-1} R_k(k^i). \tag{2.3}$$

Using (2.3) along with (2.1) of [2] yields

$$F_{k^n} = F_k \prod_{i=1}^{n-1} \left\{ \sum_{j=1}^{k/2} [F_{k^i(2j-1)-1} + F_{k^i(2j-1)+1}] \right\}. \tag{2.4}$$

The right-hand side of (2.4) clearly equals that of (2.1). Q.E.D.

Proposition 2: If k is odd and $n \geq 1$, then

$$F_{k^n} = (-1)^{(n-1)(k-1)/2} F_k \prod_{i=1}^{n-1} \left[1 + \sum_{j=1}^{(k-1)/2} (-1)^j L_{2jk^i} \right]. \tag{2.5}$$

Observe that, if $k = 3$, identity (2.5) reduces beautifully to

$$F_{3^n} = 2\prod_{i=1}^{n-1} (L_{2\cdot 3^i} - 1). \tag{2.6}$$

In order to prove Proposition 2, we need the identity

$$\sum_{i=1}^{r} (-1)^i L_{ai} = \frac{(-1)^{a+r} L_{ar} + (-1)^r L_{a(r+1)} - L_a - 2(-1)^a}{L_a + 1 + (-1)^a}, \tag{2.7}$$

which can readily be proved by using the Binet form for Lucas numbers and the geometric series formula. We also need the identity

$$L_{a+b} - (-1)^b L_{a-b} = 5F_a F_b, \tag{2.8}$$

which can be obtained from identities $I_{21} - I_{24}$ of [3].

Proof of Proposition 2: The proof has to be split into two cases according to the residue of k modulo 4.

Case 1. $k \equiv 1 \pmod 4$ [i.e., $(k-1)/2$ is even]
By (2.2), we can write

$$\begin{aligned}
F_{k^n} &= F_k \prod_{i=1}^{n-1} \frac{F_{k^{i+1}}}{F_{k^i}} = F_k \prod_{i=1}^{n-1} \left[1 + \frac{5F_{k^{i+1}} F_{k^i}}{5F_{k^i}^2} - 1 \right] \\
&= F_k \prod_{i=1}^{n-1} \left[1 + \frac{L_{k^{i+1}-k^i} + L_{k^{i+1}+k^i}}{L_{2k^i} + 2} - 1 \right] \quad (\text{by (2.8) and } I_{17} \text{ of [3]}) \\
&= F_k \prod_{i=1}^{n-1} \left[1 + \frac{L_{2k^i(k-1)/2} + L_{2k^i(k+1)/2} - L_{2k^i} - 2}{L_{2k^i} + 2} \right] \\
&= F_k \prod_{i=1}^{n-1} \left[1 + \sum_{h=1}^{(k-1)/2} (-1)^h L_{2hk^i} \right] \quad [\text{by (2.7)}].
\end{aligned}$$

Observe that, since $(k-1)/2$ is even by hypothesis, the above expression does not vary if we multiply it by $(-1)^{(n-1)(k-1)/2}$.

Case 2. $k \equiv 3 \pmod 4$ [i.e., $(k-1)/2$ is odd]
Analogously, we can write

$$F_{k^n} = (-1)^{n-1} F_k \prod_{i=1}^{n-1} \left[-\frac{5F_{k^{i+1}}F_{k^i}}{5F_{k^i}^2} - 1 + 1 \right]$$

$$= (-1)^{n-1} F_k \prod_{i=1}^{n-1} \left[1 + \frac{-L_{2k^i(k-1)/2} - L_{2k^i(k+1)/2} - L_{2k^i} - 2}{L_{2k^i} + 2} \right]$$

$$= (-1)^{n-1} F_k \prod_{i=1}^{n-1} \left[1 + \sum_{h=1}^{(k-1)/2} (-1)^h L_{2hk^i} \right].$$

Observe that, since $(k-1)/2$ is odd by hypothesis, the factor $(-1)^{n-1}$ in the above expression can be rewritten as $(-1)^{(n-1)(k-1)/2}$. Q.E.D.

3. RELATED RESULTS

Some results related to those established in Section 2 can be obtained readily. Observe that, if the exponent n in (2.1) is composite (say, $n = st$), then F_{k^n} (k even) can be expressed as

$$F_{k^n} = F_{k^{st}} = F_{k^s} \prod_{i=1}^{t-1} \left[\sum_{j=1}^{k^s/2} L_{(2j-1)k^{si}} \right] \quad (k \text{ even}), \tag{3.1}$$

where s and t can obviously be interchanged. For example, by (1.1) and (3.1), we have

$$F_{2^{2n}} = F_{4^n} = \prod_{i=1}^{2n-1} L_{2^i} = F_{2^n} \sum_{j=1}^{2^{n-1}} L_{(2j-1)2^n} = 3 \prod_{i=1}^{n-1} (L_{4^i} + L_{3 \cdot 4^i}). \tag{3.2}$$

For k odd, the analog of (3.1) can be obtained immediately.

An expression analogous to (2.1) can be established for F_{mk^n} when mk is even. If mk is odd, the corresponding expression is somewhat unattractive and its presentation is omitted.

Proposition 3: If $n \geq 1$,

$$F_{mk^n} = F_{mk} \prod_{i=1}^{n-1} \left[\sum_{j=1}^{k/2} L_{(2j-1)mk^i} \right] \quad (k \text{ even}, m \text{ arbitrary}) \tag{3.3}$$

and

$$F_{mk^n} = F_{mk} \prod_{i=1}^{n-1} \left[1 + \sum_{j=1}^{(k-1)/2} L_{2jmk^i} \right] \quad (k \text{ odd}, m \text{ even}). \tag{3.4}$$

Proof: Write

$$F_{mk^n} = F_{mk} \prod_{i=1}^{n-1} \frac{F_{mk^{i+1}}}{F_{mk^i}} = F_{mk} \prod_{i=1}^{n-1} R_k(mk^i)$$

and use (2.1) and (2.2) of [2]. Q.E.D.

If we let $m = k$ in (3.3), we see that an equivalent form for (2.1) is

$$F_{k^n} = F_{k^2} \prod_{i=1}^{n-2} \left[\sum_{j=1}^{k/2} L_{(2j-1)k^{i+1}} \right] \quad (k \text{ even}, n \geq 2). \tag{3.5}$$

Moreover, if we let $n = u + v$ $(u, v \geq 1)$ and $m = k^u$ in (3.3), we get the relation

$$F_{k^n} = F_{k^{u+v}} = F_{k^u k^v} = F_{k^{u+1}} \prod_{i=1}^{v-1} \left[\sum_{j=1}^{k/2} L_{(2j-1)k^{u+i}} \right] \quad (k \text{ even}), \tag{3.6}$$

which generalizes (3.5).

4. A RECURRENCE RELATION FOR F_{k^n}

A problem [6] that appeared in this journal led us to discover the first-order nonlinear homogeneous recurrence relation

$$F_{3^{n+1}} = 5F_{3^n}^3 - 3F_{3^n} \quad (n \geq 0). \tag{4.1}$$

The aim of this section is to obtain an analogous relation valid for all positive subscripts k^{n+1} with k *odd* and n an arbitrary nonnegative integer.

Proposition 4: If k is a positive odd integer and n is a nonnegative integer, then

$$F_{k^{n+1}} = 5^{(k-1)/2} F_{k^n}^k - \sum_{i=0}^{(k-3)/2} 5^i C_{i,k} F_{k^n}^{2i+1}, \tag{4.2}$$

where the coefficients $C_{i,k}$ are given by

$$C_{i,k} = (-1)^{(k+1)/2+i} \binom{(k+1)/2+i}{2i+1} \frac{k}{(k+1)/2+i} \quad [0 \leq i \leq (k-3)/2]. \tag{4.3}$$

As an example, for $k = 3, 5, 7$, and 9, (4.2) gives (4.1),

$$F_{5^{n+1}} = 25F_{5^n}^5 - 25F_{5^n}^3 + 5F_{5^n}, \tag{4.4}$$

$$F_{7^{n+1}} = 125F_{7^n}^7 - 175F_{7^n}^5 + 70F_{7^n}^3 - 7F_{7^n}, \tag{4.5}$$

and

$$F_{9^{n+1}} = 625F_{9^n}^9 - 1125F_{9^n}^7 + 675F_{9^n}^5 - 150F_{9^n}^3 + 9F_{9^n}, \tag{4.6}$$

respectively.

Proof of Proposition 4: First, let us write

$$F_{k^n}^k = \frac{1}{(\sqrt{5})^k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \beta^{jk^n} \alpha^{(k-j)k^n}, \tag{4.7}$$

where $\alpha = 1 - \beta = (1+\sqrt{5})/2$. After several simple but tedious manipulations involving the use of the Binet forms for Fibonacci and Lucas numbers, and the relation $\alpha\beta = -1$, (4.7) yields the following identities which may be of some interest *per se*:

$$F_{k^n}^k = \frac{1}{5^{(k-1)/2}}\left[F_{k^{n+1}} + \sum_{j=1}^{(k-1)/2}\binom{k}{j}F_{(k-2j)k^n}\right] \quad (k \text{ odd, } n \ge 0), \tag{4.8}$$

$$F_{k^n}^k = \frac{1}{5^{k/2}}\left[(-1)^{k/2}\binom{k}{k/2} + \sum_{j=0}^{(k-2)/2}(-1)^j\binom{k}{j}L_{(k-2j)k^n}\right] \quad (k \text{ even, } n \ge 1). \tag{4.9}$$

By (4.8), we immediately obtain

$$F_{k^{n+1}} = 5^{(k-1)/2}F_{k^n}^k - \sum_{j=1}^{(k-1)/2}\binom{k}{j}F_{(k-2j)k^n} \quad (k \text{ odd}). \tag{4.10}$$

Then, using (4.10) along with Theorem 1 of [4] leads to the expression

$$\begin{aligned} F_{k^{n+1}} &= 5^{(k-1)/2}F_{k^n}^k \\ &- \sum_{j=1}^{(k-1)/2}\sum_{i=0}^{(k-1)/2-j}(-1)^{(k-1)/2+i-j}\frac{k-2j}{(k+1)/2+i-j}5^i\binom{k}{j}\binom{(k+1)/2+i-j}{2i+1}F_{k^n}^{2i+1} \end{aligned} \tag{4.11}$$

which, after reversing the summation order, can be rewritten as

$$F_{k^{n+1}} = 5^{(k-1)/2}F_{k^n}^k - \sum_{i=0}^{(k-1)/2}5^i A_{i,k}F_{k^n}^{2i+1}, \tag{4.12}$$

where

$$A_{i,k} = (-1)^i\sum_{j=1}^{(k-1)/2-i}(-1)^{(k-1)/2-j}\frac{k-2j}{(k+1)/2+i-j}\binom{k}{j}\binom{(k+1)/2+i-j}{2i+1}. \tag{4.13}$$

Since $A_{(k-1)/2,k} = 0$ by (1.3), expression (4.12) becomes

$$F_{k^{n+1}} = 5^{(k-1)/2}F_{k^n}^k - \sum_{i=0}^{(k-3)/2}5^i A_{i,k}F_{k^n}^{2i+1}. \tag{4.14}$$

Now it remains to show that the numbers $A_{i,k}$ [defined by (4.13)] and the numbers $C_{i,k}$ [defined by (4.3)] coincide. To do this, consider the combinatorial identity

$$\begin{aligned} &\sum_{j=1}^{m}(-1)^j\frac{k-2j}{k-m-j}\binom{k}{j}\binom{k-m-j}{m-j} \\ &= -\frac{k}{k-m}\binom{k-m}{m} \quad [1 \le m \le (k-1)/2], \end{aligned} \tag{4.15}$$

which can be obtained by [5, p. 58], and replace m by $(k-1)/2-i$ in (4.15) to obtain the desired result $C_{i,k} = A_{i,k}$. Q.E.D.

5. CONCLUDING REMARKS

Some properties of the numbers F_{k^n} have been investigated in this note. In particular, expressions for these numbers in terms of products involving Lucas numbers have been established. Analogous expressions for L_{k^n} appeared to be rather unpleasant, so we confine ourselves to show some partial results whose proofs are left to the perseverance of the reader. In particular, we show the identity

$$L_{k^n} = 2 + (L_{k^2} - 2)\prod_{i=1}^{n-2}\left[\sum_{j=1}^{k/2} L_{(2j-1)k^{i+1}/2}\right]^2 \quad (k \text{ even},\ n \geq 2) \tag{5.1}$$

which, for $k = 2$, reduces to

$$L_{2^n} = 2 + 5\prod_{i=1}^{n-2} L_{2^i}^2 \quad (n \geq 2). \tag{5.2}$$

We also have

$$L_{3^n} = 4\prod_{i=1}^{n-1}(L_{2\cdot 3^i} + 1) \quad (n \geq 1). \tag{5.3}$$

Observe that the identity

$$F_{2\cdot 3^n} = 8\prod_{i=1}^{n-1}(L_{2\cdot 3^i}^2 - 1) = 8\prod_{i=1}^{n-1}(L_{4\cdot 3^i} + 1) \quad (n \geq 1) \tag{5.4}$$

can be obtained either by (2.6) and (5.3), or by (3.4).

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ON THE DIOPHANTINE EQUATION $\left(\frac{x(x-1)}{2}\right)^2 = \frac{y(y-1)}{2}$

Ming Luo
Department of Mathematics, Chongqing Teachers' College
Sichuan Province, 630047, People's Republic of China
(*Submitted October 1994*)

Positive integers of the form $\frac{1}{2}m(m-1)$ are called *triangular numbers*. The Diophantine equation

$$\left(\frac{x(x-1)}{2}\right)^2 = \frac{y(y-1)}{2} \tag{1}$$

corresponds to the question: For which triangular numbers are their squares still triangular [1]? In 1946, Ljunggren [2] solved this problem when he proved the following.

Theorem: The Diophantine equation (1) has only the following solutions in positive integers: $(x, y) = (1, 1)$, $(2, 2)$, and $(4, 9)$.

That is, only the two triangular numbers 1 and 36 can be represented as squares of numbers of the same form. However, Ljunggren used his knowledge of the biquadratic field $Q(2^{1/4})$ and the p-adic method, so his proof is somewhat complex. In 1965, Cassels [3] gave a much simpler proof, but he also used his knowledge of the biquadratic field $Q((-2)^{1/4})$. In 1989, Cao [4] conjectured that (1) could be solved by the method of recurrent sequences. We verify his conjecture in this paper by giving the theorem and an elementary as well as simple proof by the method of recurrent sequences without using anything deeper than reciprocity.

Proof of the Theorem: Let $X = 2x-1$ and $Y = 2y-1$, then equation (1) may be reduced to $Y^2 - 2(\frac{X^2-1}{4})^2 = 1$. Since $u + v\sqrt{2} = u_n + v_n\sqrt{2} = (1+\sqrt{2})^n$ gives the general solution of the Pellian equation $u^2 - v^2 = (-1)^n$, where $1+\sqrt{2}$ is its fundamental solution and n is an arbitrary integer (see, e.g., [1]), we get

$$X^2 = 4v_n + 1, \quad 2|n. \tag{2}$$

The following relations may be derived easily from the general solution of Pell's equation:

$$u_{n+2} = 2u_{n+1} + u_n, \qquad u_0 = 1,\ u_1 = 1; \tag{3}$$

$$v_{n+2} = 2v_{n+1} + v_n, \qquad v_0 = 0,\ v_1 = 1; \tag{4}$$

$$u_{2n} = u_n^2 + 2v_n^2, \qquad v_{2n} = 2u_nv_n; \tag{5}$$

$$v_{-n} = (-1)^{n+1}v_n; \tag{6}$$

$$v_{n+2k} \equiv (-1)^{k+1}v_n \pmod{u_k}. \tag{7}$$

If $n < 0$, then $4v_n + 1 < 0$, and (2) is impossible. Hence, it is necessary that $n \ge 0$. We shall prove that (2) cannot hold for any $n > 4$ by showing that $4v_n + 1$ is a quadratic nonresidue modulo some positive integer.

First, we consider the following three cases:

Case 1. If $n \equiv 0 \pmod 6$ and $n > 0$, then we write $n = 3^r(6k \pm 2)$, $4 \geq 1$. Let $m = 3^r$, then, by (6) and (7), we get $v_n \equiv v_{\pm 2m} \equiv \pm v_{2m} \pmod{u_{3m}}$. Since $u_{3m} = u_m(u_m^2 + 6v_m^2)$, we obtain $4v_n + 1 \equiv \pm 4v_{2m} + 1 \pmod{u_m^2 + 6v_m^2}$.

Note that $2 \nmid m$ implies $u_m^2 + 6v_m^2 \equiv 7 \pmod 8$ and $v_m \equiv 1 \pmod 4$ so, by (5), we obtain

$$\begin{aligned}\left(\frac{4v_{2m}+1}{u_m^2+6v_m^2}\right) &= \left(\frac{8u_m v_m - u_m^2 + 2v_m^2}{u_m^2+6v_m^2}\right) = \left(\frac{8v_m(u_m+v_m)}{u_m^2+6v_m^2}\right) \\ &= \left(\frac{(u_m+v_m)/2^s}{u_m^2+6v_m^2}\right) \qquad \text{(where } 2^s \| u_m + v_m\text{)} \\ &= \left(\frac{-1}{(u_m+v_m)/2^s}\right)\left(\frac{u_m^2+6_m^2}{(u_m+v_m)/2^s}\right) \\ &= \left(\frac{-1}{(u_m+v_m)/2^s}\right)\left(\frac{7}{(u_m+v_m)/2^s}\right) = \left(\frac{u_m+v_m}{7}\right).\end{aligned}$$

Similarly,

$$\left(\frac{-4v_{2m}+1}{u_m^2+6v_m^2}\right) = \left(\frac{-u_m+v_m}{7}\right).$$

Equations (3) and (4) modulo 7 yield two residue sequences with the same period of 6. Since $m \equiv 3 \pmod 6$, we have $\pm u_m + v_m \equiv 5 \pmod 7$, so that

$$\left(\frac{4v_n+1}{u_m^2+6v_m^2}\right) = \left(\frac{5}{7}\right) = -1,$$

and (2) cannot hold.

Case 2. If $n \equiv 2 \pmod 4$ and $n > 2$, then we write $n = 2 + 2\cdot 3^r \cdot m$, where $r \geq 0$, $m \equiv \pm 2 \pmod 6$. By (7), we have $4v_n + 1 \equiv -4v_2 + 1 \equiv -7 \pmod{u_m}$,

$$\left(\frac{4v_n+1}{u_m}\right) = \left(\frac{-7}{u_m}\right) = \left(\frac{u_m}{7}\right) = \left(\frac{3}{7}\right) = -1,$$

so that (2) cannot hold.

Case 3. If $n \equiv 4 \pmod{60}$ and $n > 4$, then we write $n = 4 + 2\cdot 3\cdot 5\cdot k\cdot 2^r$, where $r \geq 1$, $2 \nmid k$. Let $m = 2^r$ or $3\cdot 2^r$ or $15\cdot 2^r$ (to be determined). By (7), we have $4v_n + 1 \equiv -4v_4 + 1 \equiv -47 \pmod{u_m}$,

$$\left(\frac{4v_n+1}{u_m}\right) = \left(\frac{-47}{u_m}\right) = \left(\frac{u_m}{47}\right).$$

The residue sequence of (3) modulo 47 has period 46. The period, with respect to r, of the residue sequence of $\{2^r\}$ modulo 46 is 11. We determine our choice of m as follows:

$$m = \begin{cases} 2^r, & \text{if } r \equiv 3, 5, 6, 7, 8, 9 \pmod{11}, \\ 3 \cdot 2^r, & \text{if } r \equiv 0, 1, 10 \pmod{11}, \\ 15 \cdot 2^r & \text{if } r \equiv 2, 4 \pmod{11}, \end{cases}$$

from which we obtain the following table.

$r \pmod{11}$	0	1	2	3	4	5	6	7	8	9	10
$2^r \pmod{46}$				8		32	18	36	26	6	
$3\cdot 2^r \pmod{46}$	26	6									36
$15\cdot 2^r \pmod{46}$			14		10						
$u_m \pmod{47}$	35	5	33	13	26	33	15	26	35	5	26

It is easy to verify that each of the u_m in this table is a quadratic nonresidue modulo 47, from which it follows that (2) is impossible.

The three cases above tell us that, for (2) to hold, n must satisfy one of the following conditions:

$$n = 0, 2, 4;$$

or

$$n \equiv 8, 16, 20, 28, 32, 40, 44, 52, 56 \pmod{60}. \tag{8}$$

We now exclude all residues in (8) by considering some moduli of the sequence $\{4v_n + 1\}$.

First, consider modulo 5. The residue sequence of $\{4v_n + 1\}$ has period 12. If $n \equiv 8$ (mod 12), then $4v_n + 1 \equiv 3$ (mod 5), which implies that (2) is impossible. Thus, we exclude $n \equiv 8$, 20, 32, 44, 56 (mod 60) in (8).

Second, consider modulo 31. We get the residue sequence of $\{4v_n + 1\}$ having a period 30. If $n \equiv 10, 16, 22, 28 \pmod{30}$, then $4v_n + 1 \equiv 27, 17, 12, 24 \pmod{31}$, respectively. However, all of these are quadratic nonresidue modulo 31; thus, (2) cannot hold. Hence, we can exclude in (8) the other four residue classes of $n \equiv 16, 28, 40, 52 \pmod{60}$.

Finally, we look at the three values of $n = 0, 2, 4$, which give $X = 1, 3, 7$, respectively, in (2). Therefore, we see that all positive integer solutions of (1) are $(x, y) = (1, 1), (2, 2)$, and $(4, 9)$ and the proof is complete.

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GENERALIZED HOCKEY STICK IDENTITIES AND N-DIMENSIONAL BLOCKWALKING

Charles H. Jones
P.O. Box 1716, Rosamond, CA 93560
(Submitted October 1994)

1. INTRODUCTION

Traditionally, the word "abracadabra" was encrypted onto amulets and other magical paraphernalia to help ward off evil. George Pólya ([8], [9]) provided the cryptic form of this word shown in Figure 1 and asked how many ways abracadabra can be spelled out using this diagram. If we replace the diagram in Figure 1 with the grid shown in Figure 2 where letters in the original diagram are placed at points of intersection of the grid, then the question is equivalent to asking how many paths there are from the top of the grid to the spade at the bottom of the grid where each step is either down and left or down and right. One such path that spells out abracadabra is shown. By considering this grid as a city street map, we can think of such paths as walking city blocks—or *blockwalking*.

```
          A
        B   B
      R   R   R
    A   A   A   A
  C   C   C   C   C
A   A   A   A   A   A
  D   D   D   D   D
    A   A   A   A
      B   B   B
        R   R
          A
```

FIGURE 1. How many ways can you spell ABRACADABRA in the above diagram?

We can represent a blockwalking path by a series of R's (right turns) and L's (left turns). For example, the abracadabra path illustrated in Figure 2 can be represented by *LRLRLRLRLR*. It is easy to see that any path which spells abracadabra will have 5 R's and 5 L's. More generally, we can see that any corner of the map can be determined uniquely by how many L's and R's it takes to get there. If there are n total steps in the path to get to a corner and there are k L's, then there are $n-k$ R's. The number of paths to that corner is thus the number of combinations of k L's and $n-k$ R's. As is well known, this is the binomial coefficient $\binom{n}{k}=\frac{n!}{k!(n-k)!}$. In particular, the number of ways abracadabra can be spelled out in the diagram is $\binom{10}{5}=252$. Thus, we use the binomial coefficients to label the corners, and it should be apparent that we are constructing Pascal's triangle. When looking at the binomial coefficient for a particular corner, remember that the n's indicate the row of the triangle whereas the k's count the number of lefts taken. For an excellent history of Pascal's triangle, see [3], which includes an English translation of Pascal's original treatise [7].

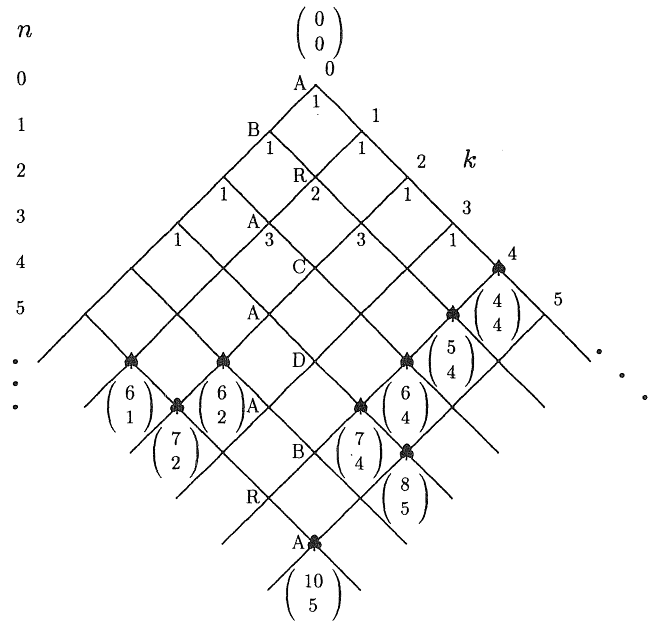

FIGURE 2. The Hockey Stick Identity gets its name from the shape of a blockwalking argument in Pascal's triangle.

Pascal's triangle is often created by first placing a 1 at each corner along the outside edges of the triangle. Second, an entry not on the edge is calculated by adding the two entries immediately above it. The top four rows generated using this method are shown in Figure 2, where the fourth row is 1 3 3 1. In terms of binomial coefficients, the second part of this construction is equivalent to the recurrence relation

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}. \tag{1}$$

Surprisingly, it appears that Pólya never published a description of how blockwalking can be used to prove combinatorial identities—although it has been described elsewhere (e.g., [5], [13]). We introduce this technique by providing a blockwalking proof of (1). Partition the paths to $\binom{n}{k}$ depending on whether the last step is an L or an R. The number of paths with last step L is $\binom{n-1}{k-1}$ and the number of paths with last step R is $\binom{n-1}{k}$. Summing these gives (1). Conceptually, this is equivalent to setting up a *sieve* in the streets so that we must pass through the sieve to our destination and so that once we have passed through the sieve there is only one path to our final destination. Specifically, a sieve is defined to be a set of corners which partitions the paths to a particular corner into equivalence classes. For example, as shown in Figure 2, in order to get to the club at $\binom{7}{2}$, we must pass through one of the spades at $\binom{6}{1}$ or $\binom{6}{2}$. Also, there is only one way to get to $\binom{7}{2}$ from either of these spades. Thus, the corners indicated by these two spades form a sieve for the corner indicated by the club. (Note that identities involving products of binomial coefficients can be shown using sieves for which there is more than one path to the destination upon exiting a sieve.)

A different sieve leads us to the *Hockey Stick Identity*:

$$\sum_{i=k}^{n}\binom{i}{k}=\binom{n+1}{k+1}. \tag{2}$$

Consider that for any path there must be a last L after which all other choices are R's. Thus, we may partition paths to $\binom{n+1}{k+1}$ into classes depending on how many consecutive R's are at the end of the path. The sum of the sizes of these classes thus equals the number of paths to $\binom{n+1}{k+1}$. This proves (2). Again, the equivalence classes can be associated with a sieve of corners. For example, the four spades from $\binom{4}{4}$ to $\binom{7}{4}$ form a sieve for $\binom{8}{5}$. This example also illustrates the use of the name *Hockey Stick Identity*. (This is also called the *Stocking Identity*. Does anyone know who first used these names?)

The following sections provide two distinct generalizations of the blockwalking technique. They are illustrated by proving distinct generalizations of the Hockey Stick Identity. We will be using the standard Hockey Stick Identity several times to prove these generalized forms. When we wish to do so by referencing its pictorial representation, we will refer to "summing the spades into the club."

2. MULTITIERED BLOCKWALKING

A somewhat obvious variation of the Hockey Stick Identity is to multiply the binomial coefficient inside the summation by i. Surprisingly, the author has not been able to find a reference for this variation (including [4] and [12]). We will give a generalized blockwalking argument to provide a closed form for this variation which we will call the *Extended Hockey Stick Identity*:

$$\sum_{i=k}^{n} i\binom{i}{k}=n\binom{n+1}{k+1}-\binom{n+1}{k+2}. \tag{3}$$

As illustrated in Figure 3, stack copies of Pascal's triangle on top of each other in tiers. The multitiered blockwalking technique follows these rules:

1. On each tier, you may walk blocks in the normal 2-dimensional fashion.
2. Your starting point may be at the apex on any tier.
3. At the end of a walk on your starting tier, you may step into an elevator and be raised a given number of tiers.
4. After having been elevated, you are at your final destination.

Any such path may be considered a series of L's and R's followed by some number of U's (ups). (The alternative interpretation, which allows moving up at *any* point along a path, will be discussed under multinomial blockwalking in the next section.)

As with the standard single-tiered case, we label a corner with the number of paths to that corner. Using the specified rules, the $\binom{n}{k}$ corner on Tier t counts $t\binom{n}{k}$ paths; $\binom{n}{k}$ paths for each of t tiers. Figure 3 shows the path *LRLLRRLUUU* which starts on Tier 3 and ends at $6\binom{7}{3}$. (For pictorial convenience, Figures 3 and 4 have the k's count R's instead of L's. This is symmetric to the standard representation of Pascal's triangle in Figure 2.)

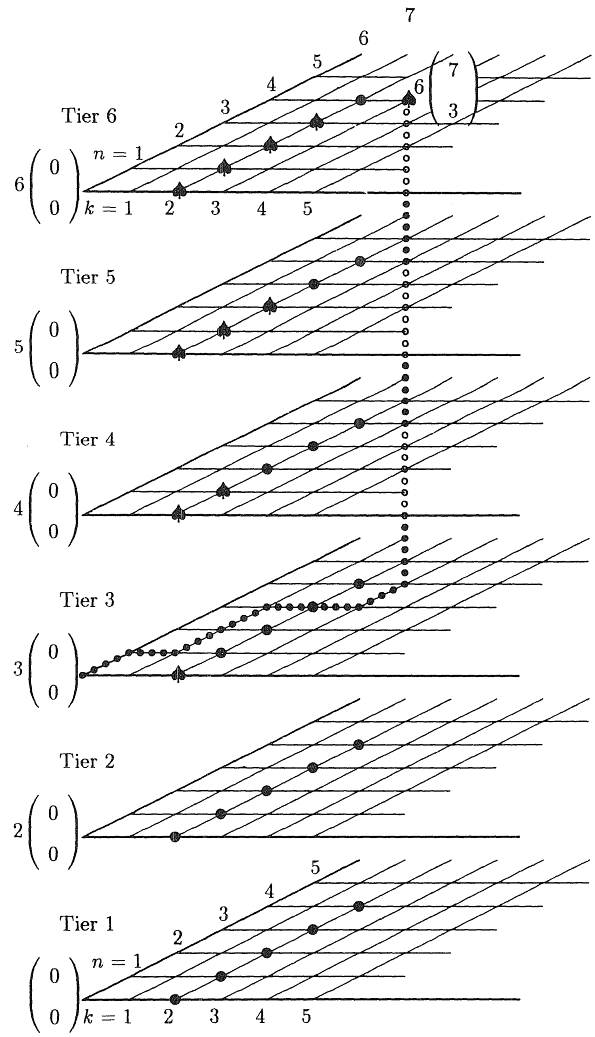

FIGURE 3. Multitiered Blockwalking

We can partition the paths to $n\binom{n+1}{k+1}$ into equivalence classes based on how many U's are at the end and how many consecutive L's precede the U's. We illustrate this with $n=6$ and $k=2$ in Figure 3. The sieve for $6\binom{7}{3}$ is shown using spades and filled circles. This illustrates:

$$n\binom{n+1}{k+1} = n\sum_{i=k}^{n}\binom{i}{k} = \sum_{i=k}^{n} n\binom{i}{k}. \tag{4}$$

By splitting the corners in the sieve appropriately, we illustrate:

$$\sum_{i=k}^{n} n\binom{i}{k} = \sum_{i=k}^{n} i\binom{i}{k} + \sum_{i=k}^{n}(n-i)\binom{i}{k}. \tag{5}$$

The filled circles in Figure 3 represent the first summation on the right-hand side of (5) and the spades represent the second summation. By rearranging (5) and applying (4), we get

$$\sum_{i=k}^{n} i\binom{i}{k} = n\binom{n+1}{k+1} - \sum_{i=k}^{n}(n-i)\binom{i}{k}. \tag{6}$$

In order to complete the proof of (3), we need to find a closed form for the subtrahend in (6).

A copy of Pascal's triangle can be formed by slicing a plane through the tiers as shown in Figure 4. That is, consider the plane that slices in front of the circles and through the last spade on each tier. This newly formed Pascal triangle consists of one street from each tier starting on Tier $t = k+1$. In the example, the corner labeled $3\binom{0}{0}$ is the apex of the new slicing Pascal triangle. Other nodes on this new Pascal triangle are labeled the same as they originally were except that the binomial coefficient is not multiplied by the tier number.

Using the standard Hockey Stick Identity on the spades on each tier (i.e., sum the spades into the clubs in Figure 4), we see that

$$\sum_{i=k}^{n}(n-i)\binom{i}{k} = \sum_{i=k}^{n-1}\sum_{j=k}^{i}\binom{j}{k} = \sum_{i=k}^{n-1}\binom{i+1}{k+1}.$$

Applying the standard Hockey Stick Identity on the slicing plane (i.e., summing the clubs into the circle at $\binom{7}{4}$ in Figure 4) gives

$$\sum_{i=k}^{n}(n-i)\binom{i}{k} = \sum_{i=k}^{n-1}\binom{i+1}{k+1} = \binom{n+1}{k+2}. \tag{7}$$

Combining (6) with (7) gives (3).

In personal correspondences to the author William Webb proved (3) using the "Snake Oil Method" of [14] and Bruce Berndt proved (3) using Vandermonde's theorem involving hypergeometric series (see [1]). However, a telescoping series gives the *Generalized Extended Hockey Stick Identity*:

$$\begin{aligned}\sum_{i=k}^{n} i^p\binom{i}{k} &= k^p\sum_{i=k}^{n}\binom{i}{k} + ((k+1)^p - k^p)\sum_{i=k+1}^{n}\binom{i}{k} + \cdots + (n^p-(n-1)^p)\sum_{i=n}^{n}\binom{i}{k} \\ &= k^p\binom{n+1}{k+1} + \sum_{i=k+1}^{n}(i^p-(i-1)^p)\left[\binom{n+1}{k+1} - \binom{i}{k+1}\right].\end{aligned}$$

Note that $i^p - (i-1)^p$ is a polynomial of degree $p-1$. Thus, the p^{th} case can be determined recursively.

Let us consider this generalized form in terms of multitiered blockwalking. When $p = 1$, the structure used to slice through the multitiered sieve is a line (between the spades and circles). Similar (though much more complicated) arguments could be provided for the p^{th} case by slicing the sieve with a curve of the form i^p.

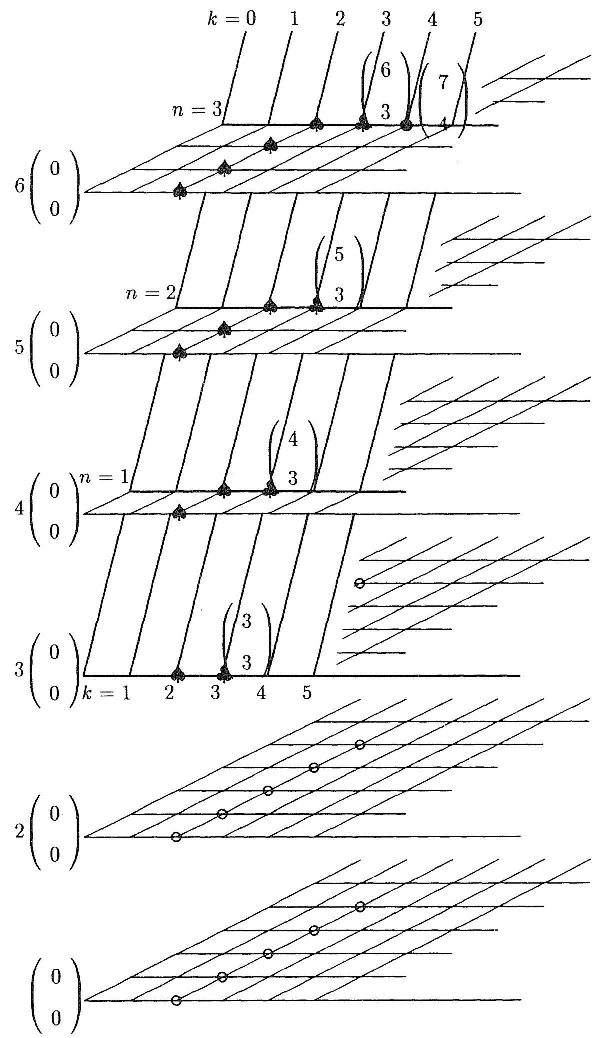

FIGURE 4. Slicing through the tiers and between the spades and circles.

3. MULTINOMIAL BLOCKWALKING

Perhaps a more natural generalization of the blockwalking technique to 3 dimensions is to allow moving up at *any* point along the path rather than just at the end. Also, it seems more natural to start at only one place—the origin. (Such a 3-dimensional version of the abracadabra concept is presented in [6].) This is easily generalized to j dimensions. That is, when blockwalking in j dimensions, not only can we walk along the standard 2 dimensions, but we can invoke our magic abracadabra amulet to take steps in all other $j-2$ dimensions. The grid we are walking on thus becomes the nonnegative j-dimensional lattice with corners labeled by multinomials

$$\binom{n}{n_1, n_2, \ldots, n_j} = \frac{n!}{n_1!n_2!\ldots n_j!}, \text{ where } n_1+n_2+\cdots+n_j=n.$$

This is also called a *Pascal hyperpyramid* or a *Pascal polytope*. Generalizations of Pascal's original results to hyperpyramids can be found in [10] and [11]. An excellent survey and an extensive bibliography of Pascal hyperpyramids can be found in [2].

Figure 5 illustrates blockwalking in 5 dimensions:

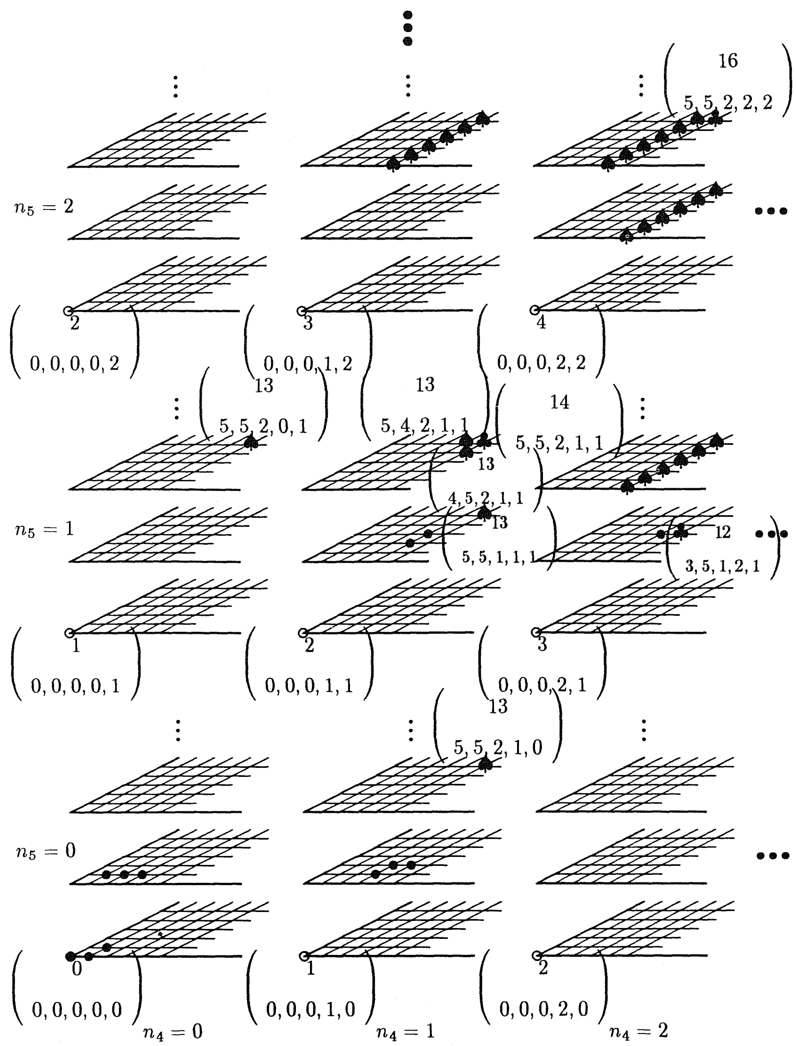

FIGURE 5. Blockwalking in 5 dimensions

Here, each tier represents a 2-dimensional blockwalk; tiers stacked on top of each other represent the 3rd dimension; the 4th dimension is represented by a row of 3-dimensional tiers going from left to right; the 5th dimension is represented by copies of the 4-dimensional rows stacked one above another. In 5 dimensions, we consider combinations of L's, R's, U's, H's (hyperspaces), and W's (warpdrives). The path *RLURRHLRWLHR* is illustrated by circles in Figure 5. This path starts at the origin and ends at

$$\binom{12}{3, 5, 1, 2, 1}.$$

The recurrence relation (1) for Pascal's triangle generalizes for hyperpyramids:

$$\binom{n}{n_1, n_2, \dots, n_j} = \sum_{k=1}^{j} \binom{n-1}{n_1, \dots, n_k - 1, \dots, n_j}.$$

The blockwalking argument also generalizes. That is, partition the paths depending on what the last step is. For example, in Figure 5, the club at

$$\binom{14}{5, 5, 2, 1, 1}$$

has a sieve of five spades—each of which is one step away in one of five dimensions.

This brings us to the *Multinomial Hockey Stick Identity*:

$$\binom{n}{n_1, n_2, \dots, n_j} = \sum_{k=2}^{j} \sum_{i=n_2}^{n_1+n_2} \binom{i + n_3 + n_4 + \cdots + n_j - 1}{i - n_2, n_2, n_3, \dots, n_k - 1, \dots, n_j}.$$

The sieve for this identity has $j-1$ copies of the sieve for the standard Hockey Stick Identity. This is illustrated in Figure 5 (with $j = 5$) by the four rows of spades that form a sieve for the club at

$$\binom{16}{5, 5, 2, 2, 2}.$$

4. COMMENTS

The beauty of Pólya's blockwalking is that it provides a geometric interpretation of algebraic equations and, in so doing, gives a way of visualizing the concepts involved. The attempt here has been to utilize this idea to provide geometric and visual aids for more complex equations and higher dimensions. In developing the presentation of this visualization, the author conceived of the 2-dimensional representation of 5 dimensions shown in Figure 5. Although the 4-dimensional tesseract is often drawn in 2 dimensions, the author has never seen an attempt to represent 5 dimensions on a flat surface. The final comments consider how the representation in Figure 5 can be generalized to more dimensions.

The basic concept is to represent the n^{th} dimension by making copies of the $(n-1)^{\text{st}}$ dimension. Thus, to visualize the 6^{th} dimension, we make copies of Figure 5, say, by stacking pages on top of each other. Notice that this gives us a 3-dimensional array of 3-dimensional arrays. We can visualize further dimensions by considering larger and larger 3-dimensional arrays. For example, we can consider our 6-dimensional stack of pages as a ream of paper. Obtain the next 3 dimensions by considering a 3-dimensional array of reams. Continuing in this fashion, call the array of reams a box; consider a 3-dimensional array of boxes, call this a pallet; consider a 3-dimensional array of pallets; etc. The structure just described, up through a 3-dimensional array of pallets, represents a 15-dimensional nonnegative lattice. The limiting factor in this method is our ability to establish a mnemonically reasonable ordering of containers to describe each level of 3-dimensional arrays. This approach can be compared to a Mandelbrot set in that we are maintaining the same visual picture only at larger and larger scales.

The visualization just described is aided by the discrete nature of the lattice (i.e., the blocks of the blockwalking) and by restricting the picture to the nonnegative "hyperquadrant" of the lattice. In order to eliminate the discrete nature of our picture, instead of making discrete copies of the $(n-1)$-dimensional representation, move the $(n-1)$-dimensional representation in a continuous motion. To remove the restriction of only visualizing the nonnegative hyperquadrant of n-space, consider 2^n copies of the hyperquadrant to represent the 2^n hyperquadrants that exist in n-space.

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AMS Classification Numbers: 05A19, 05A10, 11B65

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A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

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Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger. FA, 1993.

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