The Fibonacci Quarterly

Dedicated to Herta Taussig Freitag

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

MULTIVARIATE SYMMETRIC IDENTITIES

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(*Submitted October 1994*)

1. INTRODUCTION

As an application of Lagrange inversion, Riordan [9] gave the following expansions:

$$\exp(bz)=\sum_{k=0}^{\infty}\frac{b(ak+b)^{k-1}}{k!}(z\exp(-az))^k; \tag{1.1}$$

$$\frac{\exp(bz)}{1-az}=\sum_{k=0}^{\infty}\frac{(ak+b)^k}{k!}(z\exp(-az))^k; \tag{1.2}$$

$$(1-z)^b=\sum_{k=0}^{\infty}\frac{b}{ak+b}\binom{ak+b}{k}\left(\frac{z}{(1+z)^a}\right)^k; \tag{1.3}$$

$$\frac{(1+z)^b}{1-\frac{az}{1+z}}=\sum_{k=0}^{\infty}\binom{ak+b}{k}\left(\frac{z}{(1+z)^a}\right)^k. \tag{1.4}$$

Starting from these identities, Gould ([3], [4]) obtained various convolution identities. The multivariate case of (1.2) and (1.4) was obtained by Carlitz [1] using MacMahon's "master theorem."

Using other methods, Cohen and Hudson [2] gave bivariate generalizations of (1.1) and (1.2) that are different from those of Carlitz.

Krattenthaler [5] showed that the preceding formulas are a consequence of his bivariate version of Lagrange inversion; furthermore, he has generalized (1.3) and (1.4).

One must note that we do not need to use Lagrange inversion in two variables to prove these types of identities, as Krattenthaler did, but need only use Lagrange interpolation, which is a much simpler tool.

Lagrange interpolation must be considered as describing the properties of a linear operator sending a function of one variable to a symmetric function. It can be written as a summation on a set or as a product of divided differences; it is this latter version that we shall use here. In fact, in Section 2 we give the four Lagrange interpolation formulas, (2.1)-(2-4), that contain many of Krattenthaler's identities as special cases. In Section 3 we show how our Lagrange interpolation formulas can even be used to derive q-analogs of these identities.

2. MULTIPLE INTERPOLATION

Let $A=\{a_1,a_2,a_3,\ldots\}$ and $B=\{b_1,b_2,b_3,\ldots\}$ be two alphabets and let κ, ϕ, and ζ be three bivariate functions of x and y. For all positive integers m and n, put

$$K_1(m,n,x,y)=\kappa(x,y)(\phi(x,y))^m(\zeta(x,y))^n,$$
$$K_2(m,n,x,y)=\kappa(x,y)(\phi(x,y))_m(\zeta(x,y))_n,$$
$$K_3(m,n,x,y)=\kappa(x,y)(\phi(x,y))^m(\zeta(x,y))_n,$$
$$K_4(m,n,x,y)=\kappa(x,y)(\phi(x,y))_m(\zeta(x,y))^n,$$

where $(a)\ln = a(a-1)(a-2)\cdots(a-n+1)$.

Lagrange interpolation generally stated that, for any function of one variable x, and "interpolation points" b, c, d, ...

$$\frac{f(x)}{(x-b)(x-c)(x-d)\cdots}=\frac{f(b)}{(x-b)(b-c)(b-d)\cdots}+\frac{f(c)}{(x-c)(c-b)(c-d)\cdots}$$
$$+\frac{f(d)}{(x-d)(d-b)(d-c)\cdots}+\textit{remainder}.$$

We shall only need the Lagrange interpolation formula, but written in a symmetrical manner. It is more satisfactory to consider the set $A=\{x,b,c,\dots,d\}$ and write

$$\sum_{a\in A}\frac{f(a)}{R(a,\,A\setminus a)}=\textit{remainder},$$

where $R(a,\,A\setminus a)$ is the product $\prod_{a'\neq a}(a-a')$.

In other words, Lagrange interpolation amounts to considering properties of the linear operator $f\to\sum_{a\in A}f(a)/R(a,\,A\setminus a)$. This operator sends a polynomial of degree k to a symmetric polynomial in A of degree $k-n$, with $\mathrm{card}(A)=n+1$. In particular, it annihilates polynomials of degree $<n$, and maps $f(x)=x^n$ to the constant 1.

These properties suffice to characterize the Lagrange operator.

If $\phi(x,y,z,\dots)$ is a polynomial, the difference $\phi(x,y,z,\dots)-\phi(y,x,z,\dots)$ is divisible by $x-y$. Following Newton, for any pair of variables (x,y), one defines a divided difference operator, ∂_{xy}, acting on the ring of polynomials as

$$\phi(x,y,\dots)\to\partial_{xy}\phi(x,y,\dots)=\frac{\phi(x,y,z,\dots)-\phi(y,x,z,\dots)}{x-y}.$$

It is clear that the product (now we need to order $A_{n+1}:=\{a_1,a_2,\dots,a_{n+1}\}$)

$$\Lambda(A_{m+1}):=\partial_{a_n,a_{n+1}}\cdot\partial_{a_{n-1},a_n}\cdot\cdots\cdot\partial_{a_1,a_2}$$

also satisfies the same properties and, therefore, coincides with the Lagrange operator (see [7]). Thus, we have

$$\Lambda(A_{m+1})\phi(a_1)=\sum_{k=1}^{m}\frac{\phi(a_{k+1})}{R(a_{k+1},\,A_{m+1}\setminus a_{k+1})}.$$

One can note that divided differences are also the main ingredient in the Newton interpolation formula, and by relating their properties to the symmetric group one can extend Newton interpolation to multivariable functions (see [6] and [8]).

For our purpose, we shall use Lagrange interpolation for two independent alphabets and functions of two variables:

$$\Lambda(A_{m+1})\Lambda(B_{n+1})\phi(a_1, b_1) = \sum_{k=0}^{m}\sum_{p=0}^{n} \frac{\phi(a_{k+1}, b_{p+1})}{R(a_{k+1}, A_{m+1} \setminus a_{k+1})R(b_{p+1}, B_{n+1} \setminus b_{p+1})}.$$

We deduce, without difficulties, the following theorem.

Theorem: Let $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$ be two infinite alphabets. Then we have:

$$\begin{aligned}&\sum_{m\geq 0, n\geq 0} \Lambda(A_{m+1})\Lambda(B_{n+1})K_1(m, n, a_1, b_1)t^m z^n \\ &= \sum_{k\geq 0, p\geq 0} \kappa(a_{k+1}, b_{p+1}) \frac{(\phi(a_{k+1}, b_{p+1}))^k (\zeta(a_{k+1}, b_{p+1}))^p}{R(a_{k+1}, A_k)R(b_{p+1}, B_p)} t^k z^p \\ &\times \sum_{m\geq 0, n\geq 0} \frac{(\phi(a_{k+1}, b_{p+1}))^m (\zeta(a_{k+1}, b_{p+1}))^n}{R(a_{k+1}, A_{m+k+1} \setminus A_{k+1})R(b_{p+1}, B_{n+p+1} \setminus B_{p+1})} t^m z^n;\end{aligned} \tag{2.1}$$

$$\begin{aligned}&\sum_{m\geq 0, n\geq 0} \Lambda(A_{m+1})\Lambda(B_{n+1})K_2(m, n, a_1, b_1)t^m z^n \\ &= \sum_{k\geq 0, p\geq 0} \kappa(a_{k+1}, b_{p+1}) \frac{(\phi(a_{k+1}, b_{p+1}))_k (\zeta(a_{k+1}, b_{p+1}))_p}{R(a_{k+1}, A_k)R(b_{p+1}, B_p)} t^k z^p \\ &\times \sum_{m\geq 0, n\geq 0} \frac{(\phi(a_{k+1}, b_{p+1}))_m (\zeta(a_{k+1}, b_{p+1}))_n}{R(a_{k+1}, A_{m+k+1} \setminus A_{k+1})R(b_{p+1}, B_{n+p+1} \setminus B_{p+1})} t^m z^n;\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\sum_{m\geq 0, n\geq 0} \Lambda(A_{m+1})\Lambda(B_{n+1})K_3(m, n, a_1, b_1)t^m z^n \\ &= \sum_{k\geq 0, p\geq 0} \kappa(a_{k+1}, b_{p+1}) \frac{(\phi(a_{k+1}, b_{p+1}))^k (\zeta(a_{k+1}, b_{p+1}))_p}{R(a_{k+1}, A_k)R(b_{p+1}, B_p)} t^k z^p \\ &\times \sum_{m\geq 0, n\geq 0} \frac{(\phi(a_{k+1}, b_{p+1}))^m (\zeta(a_{k+1}, b_{p+1}))_n}{R(a_{k+1}, A_{m+k+1} \setminus A_{k+1})R(b_{p+1}, B_{n+p+1} \setminus B_{p+1})} t^m z^n;\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\sum_{m\geq 0, n\geq 0} \Lambda(A_{m+1})\Lambda(B_{n+1})K_4(m, n, a_1, b_1)t^m z^n \\ &= \sum_{k\geq 0, p\geq 0} \kappa(a_{k+1}, b_{p+1}) \frac{(\phi(a_{k+1}, b_{p+1}))_k (\zeta(a_{k+1}, b_{p+1}))^p}{R(a_{k+1}, A_k)R(b_{p+1}, B_p)} t^k z^p \\ &\times \sum_{m\geq 0, n\geq 0} \frac{(\phi(a_{k+1}, b_{p+1}))_m (\zeta(a_{k+1}, b_{p+1}))^n}{R(a_{k+1}, A_{m+k+1} \setminus A_{k+1})R(b_{p+1}, B_{n+p+1} \setminus B_{p+1})} t^m z^n.\end{aligned} \tag{2.4}$$

We shall use the above theorem in the case of different specializations κ, ϕ, and ζ for which the divided difference is easily calculable. The simple fact that the operator $\Lambda(A_{m+1})$ decreases the total degree in A by m implies the following identities.

Lemma (2.5): If we specialize $\kappa(x,y)\to 1$, $\phi\to\phi_1(x,y)=\frac{\lambda_1+\mu_1 x}{\lambda_2+\mu_2 y}$ and $\zeta\to\zeta_1(x,y)=\frac{\lambda_2+\mu_2 y}{\lambda_1+\mu_1 x}$, we obtain

$$\begin{aligned}\Lambda(A_{m+1})\Lambda(B_{n+1})K_1(m,n,a_1,b_1) &= \Lambda(A_{m+1})\Lambda(B_{n+1})K_2(m,n,a_1,b_1)\\ &= \Lambda(A_{m+1})\Lambda(B_{n+1})K_3(m,n,a_1,b_1)\\ &= \begin{cases}\left(\frac{\mu_1}{\lambda_2+\mu_2 b_1}\right)^m & \text{if } n=0,\\ \left(\frac{\mu_2}{\lambda_1+\mu_1 a_1}\right)^n & \text{if } m=0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Lemma (2.6): If we put $\kappa(x,y)\to\kappa_1(x,y)=\frac{1}{\lambda_1+\mu_1 x}$, $\phi\to\phi_1(x,y)=\frac{\lambda_1+\mu_1 x}{\lambda_2+\mu_2 y}$, and $\zeta\to\zeta_2(x,y)=\frac{\lambda_2+\mu_2 y}{\lambda_1+\mu_1 x}(\alpha_1+\beta_1 x)$, we obtain

$$\begin{aligned}\Lambda(A_{m+1})\Lambda(B_{n+1})K_1(m,n,a_1,b_1) &= \Lambda(A_{m+1})\Lambda(B_{n+1})K_2(m,n,a_1,b_1)\\ &= \Lambda(A_{m+1})\Lambda(B_{n+1})K_3(m,n,a_1,b_1)\\ &= \Lambda(A_{m+1})\Lambda(B_{n+1})K_4(m,n,a_1,b_1)\\ &= \begin{cases}\frac{1}{\lambda_1+\mu_1 a_1}\left(\frac{\beta_1+\alpha_1 a_1}{\lambda_1+\mu_1 a_1}\right)^m & \text{if } m=0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Lemma (2.7): If we put $\kappa(x,y)\to\kappa_2(x,y)=\frac{\lambda_1\alpha_2}{(\lambda_1+\mu_1 x)(\alpha_2+\beta_2 y)}+\frac{\lambda_2\alpha_1}{(\lambda_2+\mu_2 y)(\alpha_1+\beta_1 x)}-\frac{\alpha_1\alpha_2}{(\alpha_1+\beta_1 x)(\alpha_2+\beta_2 y)}$, $\phi\to\phi_2(x,y)=\frac{\lambda_1+\mu_1 x}{\lambda_2+\mu_2 y}(\alpha_2+\beta_2 y)$, and $\zeta\to\zeta_2(x,y)=\frac{\lambda_2+\mu_2 y}{\lambda_1+\mu_1 x}(\alpha_1+\beta_1 x)$, we obtain

$$\begin{aligned}\Lambda(A_{m+1})\Lambda(B_{n+1})K_1(m,n,a_1,b_1) &= \Lambda(A_{m+1})\Lambda(B_{n+1})K_2(m,n,a_1,b_1)\\ &= \Lambda(A_{m+1})\Lambda(B_{n+1})K_3(m,n,a_1,b_1)\\ &= \begin{cases}\kappa_2(a_1,b_1) & \text{if } m=n=0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Lemma (2.8): If we put $\kappa(x,y)\to\kappa_3(x,y)=\frac{1}{(\lambda_1+\mu_1 x)(\lambda_2+\mu_2 y)}$, $\phi\to\phi_2(x,y)=\frac{\lambda_1+\mu_1 x}{\lambda_2+\mu_2 y}(\alpha_2+\beta_2 y)$, and $\zeta\to\zeta_2(x,y)=\frac{\lambda_2+\mu_2 y}{\lambda_1+\mu_1 x}(\alpha_1+\beta_1 x)$, we obtain

$$\begin{aligned}&\Lambda(A_{m+1})\Lambda(B_{n+1})K_1(m,n,a_1,b_1)\\ &= \begin{cases}\frac{1}{\mu_1\mu_2}\left(\alpha_1-\frac{\beta_1}{\mu_1}\lambda_1\right)^m\left(\alpha_2-\frac{\beta_2}{\mu_2}\lambda_2\right)^n\left(\prod_{j=0}^{m}\left(\frac{\lambda_1}{\mu_1}+a_{j+1}\right)\right)^{-1}\left(\prod_{j=0}^{n}\left(\frac{\lambda_2}{\mu_2}+b_{j+1}\right)\right)^{-1} & \text{if } m=n,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The identities (2.1)-(2.10) and (3.16) of Krattenthaler [5] arise as different specializations of the functions κ,ϕ, and ζ considered in Lemmas (2.5)-(2.8) above, and to the case in which $A=B=\{0,1,2,\ldots\}$. For each of the four cases given in our Theorem, we give the formulas when A and B are general. We then specialize to the case where A and B are sequences of "q-integers."

3. APPLICATION OF IDENTITY (2.1)

Formula (2.1) and the previous lemmas provide the following identities:

I) In the case where A and B are general alphabets, we have

$$\left(1-\frac{\mu_1}{\lambda_1+\mu_1 a_1\lambda_2+\mu_2 b_1}tz\right)\left(1-\frac{\mu_1}{\lambda_2+\mu_2 b_1}t\right)^{-1}\left(1-\frac{\mu_2}{\lambda_1+\mu_1 a_1}z\right)^{-1}$$
$$=\sum_{k\geq 0,\, p\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^k(\zeta_1(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p \tag{3.1}$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^m(\zeta_1(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n,$$

$$\frac{1}{(\lambda_1+\mu_1 a_1)-(\alpha_1+\beta_1 a_1)\mu_2 z}=\sum_{k\geq 0,\, p\geq 0}\frac{1}{\lambda_1+\mu_1 a_{k+1}}\frac{(\phi_1(a_{k+1},b_{p+1}))^k(\zeta_2(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^m(\zeta_2(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n, \tag{3.2}$$

$$\kappa_2(a_1,b_1)=\sum_{k\geq 0,\, p\geq 0}\kappa_2(a_{k+1},b_{p+1})\frac{(\phi_2(a_{k+1},b_{p+1}))^k(\zeta_2(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_2(a_{k+1},b_{p+1}))^m(\zeta_2(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n, \tag{3.3}$$

$$\frac{1}{\mu_1\mu_2}\sum_{j=0}\left(\left(\alpha_1-\frac{\beta_1}{\mu_1}\lambda_1\right)\left(\alpha_2-\frac{\beta_2}{\mu_2}\lambda_2\right)tz\right)^j\left(\prod_{s=0}^{j}\left(\frac{\lambda_1}{\mu_1}+a_{s+1}\right)\right)^{-1}\left(\prod_{s=0}^{j}\left(\frac{\lambda_2}{\mu_2}+b_{s+1}\right)\right)^{-1}$$
$$=\sum_{k\geq 0,\, p\geq 0}\frac{1}{(\lambda_1+\mu_1 a_{k+1})(\lambda_2+\mu_2 b_{p+1})}\frac{(\phi_2(a_{k+1},b_{p+1}))^k(\zeta_2(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}(-t)^k(-z)^p \tag{3.4}$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_2(a_{k+1},b_{p+1}))^m(\zeta_2(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}(-t)^m(-z)^n.$$

II) Let $[n]=\frac{1-q^n}{1-q}$, $[n]!=[n][n-1]\cdots[1]$, $\exp_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n]!}$. Then in the case $A=\{0,[1],[2],\dots\}$, $B=\{0,[1],[2],\dots\}$, we obtain q-analogs for Krattenthaler's identities (2.1), (2.4), (2.8), and (3.6) in [5]. For example, from (3.1), we obtain the following q-analog of (2.1) in [5].

$$\left(1-\frac{\mu_1}{\lambda_1}\frac{\mu_2}{\lambda_2}tz\right)\left(1-\frac{\mu_1}{\lambda_2}t\right)^{-1}\left(1-\frac{\mu_2}{\lambda_1}z\right)^{-1}=\sum_{k\geq 0,\, p\geq 0}\frac{(\phi_1([k],[p]))^k}{q^{k(k-1)/2}[k]!}\frac{((\zeta_1([k],[p]))^p}{q^{p(p-1)/2}[p]!}t^k z^p$$
$$\times\exp_q\left(-\frac{\phi_1([k],[p])}{q^k}t\right)\exp_q\left(-\frac{\zeta_1([k],[p])}{q^p}z\right). \tag{3.5}$$

4. APPLICATION OF IDENTITY (2.2)

I) In the case where A and B are general alphabets, we have

$$\left(1-\frac{\mu_1}{\lambda_1+\mu_1 a_1}\frac{\mu_2}{\lambda_2+\mu_2 b_1}tz\right)\left(1-\frac{\mu_1}{\lambda_2+\mu_2 b_1}t\right)^{-1}\left(1-\frac{\mu_2}{\lambda_1+\mu_1 a_1}z\right)^{-1}$$
$$=\sum_{k\geq 0,\, p\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))_k(\zeta_1(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p \tag{4.1}$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))_m(\zeta_1(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n,$$

$$\frac{1}{(\lambda_1+\mu_1 a_1)-(\alpha_1+\beta_1 a_1)\mu_2 z}=\sum_{k\geq 0,\, p\geq 0}\frac{1}{\lambda_1+\mu_1 a_{k+1}}\frac{(\phi_1(a_{k+1},b_{p+1}))_k(\zeta_2(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))_m(\zeta_2(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n, \tag{4.2}$$

$$\kappa_2(a_1,b_1)=\sum_{k\geq 0,\, p\geq 0}\kappa_2(a_{k+1},b_{p+1})\frac{(\phi_2(a_{k+1},b_{p+1}))_k(\zeta_2(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_2(a_{k+1},b_{p+1}))_m(\zeta_2(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n, \tag{4.3}$$

II) For example, in the case $A=\{0,[1],[2],\dots\}, B=\{0,[1],[2],\dots\}$ we obtain from (4.2) the following q-analog of (2.5) in [5].

$$\frac{1}{\lambda_1-\alpha_1\mu_2 z}=\sum_{k\geq 0,\, p\geq 0}\frac{1}{\lambda_1+\mu_1[k]}\frac{(\phi_1([k],[p]))_k}{q^{k(k-1)/2}[k]!}\frac{((\zeta_2([k],[p]))_p}{q^{p(p-1)/2}[p]!}t^k z^p$$
$$\times\sum_{m\geq 0,\, n\geq 0}(-1)^{m+n}\frac{(\phi_1([k],[p])-k)_m}{q^{mk}[m]!}\frac{(\zeta_2([k],[p])-p)_n}{q^{np}[n]!}t^m z^n. \tag{4.4}$$

5. APPLICATION OF IDENTITY (2.3)

I) In the case where A and B are general alphabets, we have

$$\left(1-\frac{\mu_1}{\lambda_1+\mu_1 a_1}\frac{\mu_2}{\lambda_2+\mu_2 b_1}tz\right)\left(1-\frac{\mu_1}{\lambda_2+\mu_2 b_1}t\right)^{-1}\left(1-\frac{\mu_2}{\lambda_1+\mu_1 a_1}z\right)^{-1}$$
$$=\sum_{k\geq 0,\, p\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^k(\zeta_1(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p \tag{5.1}$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^m(\zeta_1(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n,$$

$$\begin{aligned}\frac{1}{(\lambda_1+\mu_1 a_1)-(\alpha_1+\beta_1 a_1)\mu_2 z} &= \sum_{k\geq 0,\, p\geq 0}\frac{1}{\lambda_1+\mu_1 a_{k+1}}\frac{(\phi_1(a_{k+1},b_{p+1}))^k(\zeta_2(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p \\ &\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^m(\zeta_2(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n,\end{aligned} \tag{5.2}$$

$$\begin{aligned}\kappa_2(a_1,b_1) &= \sum_{k\geq 0,\, p\geq 0}\kappa_2(a_{k+1},b_{p+1})\frac{(\phi_2(a_{k+1},b_{p+1}))^k(\zeta_2(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p \\ &\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_2(a_{k+1},b_{p+1}))^m(\zeta_2(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n,\end{aligned} \tag{5.3}$$

II) For example, in the case $A=\{0,[1],[2],\ldots\}$, $B=\{0,[1],[2],\ldots\}$ we obtain from (5.3) the following q-analog of (2.10) in [5].

$$\begin{aligned}1 &= \sum_{k\geq 0,\, p\geq 0}\kappa_2([k],[p])\frac{(\phi_2([k],[p]))^k}{q^{k(k-1)/2}[k]!}\frac{((\zeta_2([k],[p]))_p}{q^{p(p-1)/2}[p]!}t^k z^p \\ &\times \exp_q\left(-\frac{\phi_2([k],[p]))}{q^k}t\right)\sum_{n\geq 0}(-1)^n\frac{(\zeta_2([k],[p])-p)_n}{q^{np}[n]!}z^n.\end{aligned} \tag{5.4}$$

6. APPLICATION OF IDENTITY (2.4)

I) In the case where A and B are general alphabets, we have

$$\begin{aligned}\frac{1}{(\lambda_1+\mu_1 a_1)-(\alpha_1+\beta_1 a_1)\mu_2 z} &= \sum_{k\geq 0,\, p\geq 0}\frac{1}{\lambda_1+\mu_1 a_{k+1}}\frac{(\phi_1(a_{k+1},b_{p+1}))_k(\zeta_2(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p \\ &\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))_m(\zeta_2(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n.\end{aligned} \tag{6.1}$$

II) For example, in the case $A=\{0,[1],[2],\ldots\}$, $B=\{0,[1],[2],\ldots\}$ we obtain from (6.1) the following q-analog of (2.7) in [5].

$$\begin{aligned}\frac{1}{\lambda_1-\alpha_1\mu_2 z} &= \sum_{k\geq 0,\, p\geq 0}\frac{1}{\lambda_1+\mu_1[k]}\frac{(\phi_1([k],[p]))_k}{q^{k(k-1)/2}[k]!}\frac{((\zeta_2([k],[p]))^p}{q^{p(p-1)/2}[p]!}t^k z^p \\ &\times \exp_q\left(-\frac{\zeta_2([k],[p]))}{q^k}z\right)\sum_{m\geq 0}(-1)^m\frac{(\phi_1([k],[p])-p)_m}{q^{mp}[n]!}t^m.\end{aligned} \tag{6.2}$$

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ON THE K^{th}-ORDER DERIVATIVE SEQUENCES OF FIBONACCI AND LUCAS POLYNOMIALS

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1. INTRODUCTION

The Fibonacci polynomials $u_n = u_n(x)$ and the Lucas polynomials $v_n = v_n(x)$ are defined by the second-order linear recurrence relations

$$u_n = xu_{n-1} + u_{n-2} \quad (u_0 = 0, u_1 = 1),$$
and
$$v_n = xv_{n-1} + v_{n-2} \quad (v_0 = 2, v_1 = x), \tag{1.1}$$

where x is an indeterminate. Their k^{th}-order derivative sequences are defined as

$$u_n^{(k)} = u_n^{(k)}(x) = \frac{d^k}{dx^k} u_n(x) \quad \text{and} \quad v_n^{(k)} = v_n^{(k)}(x) = \frac{d^k}{dx^k}(x).$$

Denote $f_n = u_n(1), \ell_n = v_n(1), f_n^{(k)} = u_n^{(k)}(1), \ell_n^{(k)} = v_n^{(k)}(1)$. P. Filipponi and A. F. Horadam ([1], [2]) considered $f_n^{(k)}$ and $\ell_n^{(k)}$ for $k = 1, 2$ and obtained a series of results. By the end of [2], seven conjectures were presented for arbitrary k. In this paper we shall consider the more general cases, $u_n^{(k)}$ and $v_n^{(k)}$, for arbitrary k. Our results will be generalizations of the results in [1] and [2]. As special cases of our results, the seven conjectures in [2] will be proved.

Following the symbols in [1] and [2], denote $\Delta = \sqrt{x^2+4}$, $\alpha = (x+\Delta)/2$, $\beta = (x-\Delta)/2$, so that $\alpha+\beta = x, \alpha\beta = -1, \alpha - \beta = \Delta$. It is well known that

$$u_n = (\alpha^n - \beta^n)/\Delta, \quad v_n = \alpha^n + \beta^n. \tag{1.2}$$

2. EXPRESSIONS FOR $u_n^{(k)}$ AND $v_n^{(k)}$ IN TERMS OF FIBONACCI AND LUCAS POLYNOMIALS

Theorem 2.1:

$$u_n^{(k)} = \frac{k!}{2\Delta^{2k}}(a_{n,k}u_n + b_{n,k}v_n), \tag{2.1}$$

where

$$a_{n,k} = \sum_{\substack{i=0 \\ 2|k-i}}^{k} \binom{k-i+n}{k-i} \Delta^{k-i}(c_{k,i} + d_{k,i}) + \sum_{\substack{i=0 \\ 2\nmid k-i}}^{k} \binom{k-i+n}{k-i} \Delta^{k-i}(c_{k,i} - d_{k,i}), \tag{2.2}$$

and

$$b_{n,k} = \sum_{\substack{i=0 \\ 2|k-i}}^{k} \binom{k-i+n}{k-i} \Delta^{k-1-i}(c_{k,i} - d_{k,i}) + \sum_{\substack{i=0 \\ 2\nmid k-i}}^{k} \binom{k-i+n}{k-i} \Delta^{k-1-i}(c_{k,i} + d_{k,i}), \tag{2.3}$$

where $c_{k,i}$ and $d_{k,i}$ $(i = 0, 1, \ldots, k)$ satisfy the systems of linear equations

$$c_{k,i}+\binom{k+1}{1}\beta c_{k,i-1}+\cdots+\binom{k+1}{i}\beta^i c_{k,0}=(-1)^i\binom{k+1}{i}\Delta^i \tag{2.4}$$

and

$$d_{k,i}+\binom{k+1}{1}\alpha d_{k,i-1}+\cdots+\binom{k+1}{i}\alpha^i d_{k,0}=\binom{k+1}{i}\Delta^i. \tag{2.5}$$

Furthermore, for $i=0,1,\ldots,k$, there exist polynomials $p_{k,i}$ and $q_{k,i}$ in x, with integer coefficients, which satisfy

$$c_{k,i}=p_{k,i}\alpha+q_{k,i} \quad\text{and}\quad d_{k,i}=p_{k,i}\beta+q_{k,i}. \tag{2.6}$$

Proof: Let the generating functions of $\{u_n\}$ and $\{u_n^{(k)}\}$ be $U(t)=U(t,x)=\sum_{n=0}^{\infty}u_nt^n$ and $U_k(t)=U_k(t,x)=\sum_{n=0}^{\infty}u_n^{(k)}t^n$, respectively. It is well known that $U(t)=t/(1-xt-t^2)$, hence,

$$U_k(t)=\frac{\partial^k}{\partial x^k}U(t)=k!\,t^{k+1}/(1-xt-t^2)^{k+1}. \tag{2.7}$$

By partial fractions we have

$$t^{k+1}/(1-xt-t^2)^{k+1}=\sum_{i=0}^{k}Q_{k,i}/(1-\alpha t)^{k+1-i}+\sum_{i=0}^{k}R_{k,i}/(1-\beta t)^{k+1-i}, \tag{2.8}$$

where $Q_{k,i}$ and $R_{k,i}$ are independent of t. Multiplying by $\alpha^{k+1}(1-\beta t)^{k+1}$, we obtain

$$(\alpha t)^{k+1}/(1-\alpha t)^{k+1}=(\alpha+t)^{k+1}\sum_{i=0}^{k}Q_{k,i}/(1-\alpha t)^{k+1-i}+\varphi(t), \tag{2.9}$$

where the function $\varphi(t)$ is analytic at the point $t=\alpha^{-1}$ under the condition that t is considered as a complex variable (while x is a real constant). Since $(\alpha t)^{k+1}/(1-\alpha t)^{k+1}=[(1-\alpha t)^{-1}-1]^{k+1}$ and $(\alpha+t)^{k+1}=[\Delta+\beta(1-\alpha t)]^{k+1}$, we can rewrite (2.9) as

$$\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}(1-\alpha t)^{-(k+1-i)}=\sum_{i=0}^{k+1}\binom{k+1}{i}\Delta^{k+1-i}\beta^i(1-\alpha t)^i\cdot\sum_{i=0}^{k}Q_{k,i}(1-\alpha t)^{-(k+1-i)}+\varphi(t).$$

Because of the uniqueness of the Laurent series [4] at the point $t=\alpha^{-1}$ for the function $(\alpha t)^{k+1}/(1-\alpha t)^{k+1}$, we can compare the coefficients of $(1-\alpha t)^{-(k+1-i)}$ $(i=0,1,\ldots,k)$ of the two sides in the last equality to get

$$\sum_{j=0}^{i}\binom{k+1}{j}\Delta^{k+1-j}\beta^jQ_{k,i-j}=(-1)^i\binom{k+1}{i}. \tag{2.10}$$

Let

$$Q_{k,i}=\Delta^{-(k+1+i)}c_{k,i}\quad(i=0,1,\ldots,k) \tag{2.11}$$

and substitute it into (2.10); then we get (2.4). For the same reason, it follows that

$$\sum_{j=0}^{i}\binom{k+1}{j}(-\Delta)^{k+1-j}\alpha^jR_{k,i-j}=(-1)^i\binom{k+1}{i}. \tag{2.12}$$

Let

$$R_{k,i}=(-\Delta)^{-(k+1+i)}d_{k,i}\quad(i=0,1,\ldots,k) \tag{2.13}$$

and substitute it into (2.12); then we get (2.5).

Now we shall prove (2.6). From (2.4) and (2.5), $c_{k,0} = d_{k,0} = 1$; hence, the conclusion holds for $i = 0$. Suppose the conclusion holds for $0, 1, \ldots, i-1$. Then, from (2.4) and (2.5), we have

$$c_{k,i} = (-1)^i \binom{k+1}{i} \Delta^i - \sum_{j=1}^{i} \binom{k+1}{j} \beta^j (p_{k,i-j}\alpha + q_{k,i-j}) \tag{2.14}$$

and

$$d_{k,i} = \binom{k+1}{i} \Delta^i - \sum_{j=1}^{i} \binom{k+1}{j} \alpha^j (p_{k,i-j}\beta + q_{k,i-j}). \tag{2.15}$$

From (1.2), it is easy to show that $\beta^j = -u_j\alpha + u_{j+1}$; hence,

$$\begin{aligned}\beta^j(p_{k,i-j}\alpha + q_{k,i-j}) &= -p_{k,i-j}\beta^{j-1} + q_{k,i-j}\beta^j \\ &= (p_{k,i-j}u_{j-1} - q_{k,i-j}u_j)\alpha + (q_{k,i-j}u_{j+1} - p_{k,i-j}u_j).\end{aligned}$$

For the same reason, we have

$$\alpha^j(p_{k,i-j}\beta + q_{k,i-j}) = (p_{k,i-j}u_{j-1} - q_{k,i-j}u_j)\beta + (q_{k,i-j}u_{j+1} - p_{k,i-j}u_j).$$

We can see that Δ^i is a polynomial in x with integer coefficients for $2|i$, but $\Delta^i = \Delta^{i-1}(x-2\beta)$ and $(-\Delta)^i = \Delta^{i-1}(x-2\alpha)$ for $2\nmid i$. By substituting the above results into (2.14) and (2.15), and by the inductive hypothesis, the conclusion is proved.

Now substituting (2.11), (2.13), and (2.6) into (2.8), then into (2.7), we get

$$\begin{aligned}U_k(t) &= \frac{k!}{\Delta^{2k}}\left[\sum_{i=0}^{k} c_{k,i}\Delta^{k-1-i}/(1-\alpha t)^{k+1-i} + \sum_{i=0}^{k} d_{k,i}(-\Delta)^{k-1-i}/(1-\beta t)^{k+1-i}\right] \\ &= \frac{k!}{\Delta^{2k}}\left[\sum_{2|k-i}(c_{k,i}/(1-\alpha t)^{k+1-i} - d_{k,i}/(1-\beta t)^{k+1-i})\Delta^{k-1-i}\right. \\ &\qquad \left. + \sum_{2\nmid k-i}(c_{k,i}/(1-\alpha t)^{k+1-i} + d_{k,i}/(1-\beta t)^{k+1-i})\Delta^{k-1-i}\right].\end{aligned}$$

Expanding the right side of the last expression into power series in t and using (2.6), we obtain

$$u_n^{(k)} = \frac{k!}{\Delta^{2k}}\left[\sum_{2|k-i}\binom{k-i+n}{k-i}\Delta^{k-i}(p_{k,i}u_{n+1} + q_{k,i}u_n) + \sum_{2\nmid k-i}\binom{k-i+n}{k-i}\Delta^{k-1-i}(p_{k,i}v_{n+1} + q_{k,i}v_n)\right]. \tag{2.16}$$

It is easy to prove that $u_{n+1} = (xu_n + v_n)/2$, $v_{n+1} = (\Delta^2 u_n + xv_n)/2$; hence,

$$\begin{aligned}p_{k,i}u_{n+1} + q_{k,i}u_n &= ((p_{k,i}x + 2q_{k,i})u_n + p_{k,i}v_n)/2 \\ &= ((c_{k,i} + d_{k,i})u_n + (c_{k,i} - d_{k,i})\Delta^{-1}v_n)/2,\end{aligned} \tag{2.17}$$

$$\begin{aligned}p_{k,i}v_{n+1} + q_{k,i}v_n &= (p_{k,i}\Delta^2 u_n + (p_{k,i}x + 2q_{k,i})v_n)/2 \\ &= ((c_{k,i} - d_{k,i})\Delta u_n + (c_{k,i} + d_{k,i})v_n)/2.\end{aligned} \tag{2.18}$$

Substitute (2.17) and (2.18) into (2.16) and we are done. □

As an example, when $k = 3$ and 4, Theorem 2.1 gives the following results:

$$c_{30} = d_{30} = 1, \qquad c_{31} = -4\Delta - 4\beta,\ d_{31} = 4\Delta - 4\alpha,$$
$$c_{32} = 6\Delta^2 + 16\beta\Delta + 10\beta^2, \qquad d_{32} = 6\Delta^2 - 16\alpha\Delta + 10\alpha^2,$$
$$c_{33} = -4\Delta^3 - 24\beta\Delta^2 - 40\beta^2\Delta - 20\beta^3, \qquad d_{33} = 4\Delta^3 - 24\alpha\Delta^2 + 40\alpha^2\Delta - 20\alpha^3,$$
$$c_{30} + d_{30} = 2, \qquad c_{31} + d_{31} = -4x,$$
$$c_{32} + d_{32} = 6x^2 + 4, \qquad c_{33} + d_{33} = -4x^3 + 4x,$$
$$c_{30} - d_{30} = 0, \qquad c_{31} - d_{31} = -4\Delta,$$
$$c_{32} - d_{32} = 6x\Delta, \qquad c_{33} - d_{33} = (-4x^2 + 4)\Delta,$$

$$a_{n3} = \binom{2+n}{2}\Delta^2(-4x) + \binom{0+n}{0}(-4x^3 + 4x) + \binom{3+n}{3}\Delta^3 \cdot 0 + \binom{1+n}{1}\Delta \cdot 6x\Delta$$
$$= -2(n^2+1)x^3 - 4(2n^2 - 3)x,$$

$$b_{n3} = \binom{2+n}{2}\Delta(-4\Delta) + \binom{0+n}{0}\Delta^{-1}(-4x^3 + 4)\Delta + \binom{3+n}{3}\Delta^2 \cdot 2 + \binom{1+n}{1}(6x^2 + 4)$$
$$= \frac{1}{3}n(n^2 + 11)x^2 + \frac{4}{3}n(n^2 - 4),$$

$$u_n^{(3)} = [-(6(n^2+1)x^3 + 12(2n^2-3)x)u_n + (n(n^2+11)x^2 + 4n(n^2-4))v_n] / \Delta^6; \tag{2.19}$$

in particular,

$$f_n^{(3)} = (n^2 - 1)(n\ell_n - 6f_n) / 25. \tag{2.20}$$

$$c_{40} = d_{40} = 1, \qquad c_{41} = -5\Delta - 5\beta,\ d_{41} = 5\Delta - 5\alpha,$$
$$c_{42} = 10\Delta^2 + 25\beta\Delta + 15\beta^2, \qquad d_{42} = 10\Delta^2 - 25\alpha\Delta + 15\alpha^2,$$
$$c_{43} = -10\Delta^3 - 50\beta\Delta^2 - 75\beta^2\Delta - 35\beta^3, \qquad d_{43} = 10\Delta^3 - 50\alpha\Delta^2 + 75\alpha^2\Delta - 35\alpha^3,$$
$$c_{44} = 5\Delta^4 + 50\beta\Delta^3 + 150\beta^2\Delta^2 + 175\beta^3\Delta + 70\beta^4,$$
$$d_{44} = 5\Delta^4 - 50\alpha\Delta^3 + 150\alpha^2\Delta^2 - 175\alpha^3\Delta + 70\alpha^4,$$
$$c_{40} + d_{40} = 2, \qquad c_{41} + d_{41} = -5x,$$
$$c_{42} + d_{42} = 10x^2 + 10, \qquad c_{43} + d_{43} = -10x^3 - 5x,$$
$$c_{44} + d_{44} = 5x^4 - 15x^2, \qquad c_{40} - d_{40} = 0,$$
$$c_{41} - d_{41} = -5\Delta, \qquad c_{42} - d_{42} = 10x\Delta,$$
$$c_{43} - d_{43} = (-10x^2 + 5)\Delta, \qquad c_{44} - d_{44} = (5x^3 - 15x)\Delta,$$

$$a_{n4} = \binom{4+n}{4}\Delta^4 \cdot 2 + \binom{2+n}{2}\Delta^2(10x^2 + 10) + \binom{0+n}{0}(5x^4 - 15x^2)$$
$$+ \binom{3+n}{3}\Delta^3(-5\Delta) + \binom{1+n}{1}\Delta(-10x^2 + 5)\Delta$$
$$= \frac{1}{12}(n^4 + 35n^2 + 24)x^4 + \frac{1}{3}(2n^4 + 25n^2 - 72)x^2 + \frac{4}{3}(n^4 - 10n^2 + 9),$$

$$\begin{aligned} b_{n4} &= \binom{4+n}{4}\Delta^3\cdot 0+\binom{2+n}{2}\Delta(10x\Delta)+\binom{0+n}{0}\Delta^{-1}(5x^3-15x)\Delta \\ &\quad +\binom{3+n}{3}\Delta^2(-5x)+\binom{1+n}{1}(-10x^3-5x) \\ &= -\frac{5}{6}n(n^2+5)x^3-\frac{5}{3}n(2n^2-11)x, \end{aligned}$$

$$\begin{aligned} u_n^{(4)} &= [((n^4+35n^2+24)x^4+4(2n^4+25n^2-72)x^2+16(n^4-10n^2+9))u_n \\ &\quad -(10n(n^2+5)x^3+20n(2n^2-11)x)v_n]/\Delta^8; \end{aligned} \tag{2.21}$$

in particular,

$$f_n^{(4)} = [(5n^4-5n^2-24)f_n - 2n(5n^2-17)\ell_n]/125. \tag{2.22}$$

We observe that (2.6) can be verified by using the above results.

From $v_n^{(k)} = mu_n^{(k-1)}$ (see 1° of Theorem 3.1 in the next section) and Theorem 2.1, we can obtain the expression for $v_n^{(k)}$ in terms of u_n and v_n.

3. SOME IDENTITIES INVOLVING $u_n^{(k)}$ AND $v_n^{(k)}$

If we differentiate certain identities involving u_n and v_n, we can get the corresponding identities involving $u_n^{(k)}$ and $v_n^{(k)}$.

Theorem 3.1:

1°. $v_n^{(k)} = mu_n^{(k-1)}$; (3.1)

2°. $u_n^{(k)} = xu_{n-1}^{(k)}+u_{n-2}^{(k)}+ku_{n-1}^{(k-1)}$, $v_n^{(k)} = xv_{n-1}^{(k)}+v_{n-2}^{(k)}+kv_{n-1}^{(k-1)}$; (3.2)

3°. $v_n^{(k)} = u_{n+1}^{(k)}+u_{n-1}^{(k)}$, (3.3)

$\Delta^2 u_n^{(k)}+2kxu_n^{(k-1)}+k(k-1)u_n^{(k-2)} = v_{n+1}^{(k)}+v_{n-1}^{(k)}$; (3.4)

4°.
$$u_{m+n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}(u_{m+1}^{(k-i)}u_n^{(i)}+u_m^{(k-i)}u_{n-1}^{(i)}), \tag{3.5}$$

$$v_{m+n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}(v_{m+1}^{(k-i)}u_n^{(i)}+v_m^{(k-i)}u_{n-1}^{(i)}), \tag{3.6}$$

$$u_{m-n}^{(k)} = (-1)^n\sum_{i=0}^{k}\binom{k}{i}(u_m^{(k-i)}u_{n+1}^{(i)}-u_{m+1}^{(k-i)}u_n^{(i)}), \tag{3.7}$$

$$v_{m-n}^{(k)} = (-1)^n\sum_{i=0}^{k}\binom{k}{i}(v_{m+1}^{(k-i)}u_n^{(i)}-u_m^{(k-i)}v_{n+1}^{(i)}); \tag{3.8}$$

in particular,

$$u_{-n}^{(k)} = (-1)^{n-1}u_n^{(k)}; \tag{3.9}$$

$$v_{-n}^{(k)} = (-1)^n v_n^{(k)}; \tag{3.10}$$

$$u_{2n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}u_n^{(k-i)}v_n^{(i)}; \tag{3.11}$$

$$v_{2n}^{(k)} = 2\sum_{i=0}^{k-1}\binom{k-1}{i} v_n^{(k-i)} v_n^{(i)}; \tag{3.12}$$

$$u_{2n+1}^{(k)} = \sum_{i=0}^{k}\binom{k}{i} u_{n+1}^{(k-i)} v_n^{(i)}; \tag{3.13}$$

$$v_{2n+1}^{(k)} = \sum_{i=0}^{k}\binom{k}{i} v_{n+1}^{(k-i)} v_n^{(i)} - (-1)^n \delta_{k,1} \ \ (\delta \text{ is the Kronecker function}); \tag{3.14}$$

5°. $$u_{m+n}^{(k)} + (-1)^n u_{m-n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i} u_m^{(k-i)} v_n^{(i)}; \tag{3.15}$$

$$u_{m+n}^{(k)} - (-1)^n u_{m-n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i} v_m^{(k-i)} u_n^{(i)}; \tag{3.16}$$

$$v_{m+n}^{(k)} + (-1)^n v_{m-n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i} v_m^{(k-i)} v_n^{(i)}; \tag{3.17}$$

$$v_{m+n}^{(k)} - (-1)^n v_{m-n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i} u_m^{(k-i)} (v_{n+1}^{(i)} + v_{n-1}^{(i)}); \tag{3.18}$$

6°. $$x v_n^{(k)} = (n-k+1) v_n^{(k-1)} - 2(v_{n-1}^{(k)} + u_{n-1}^{(k-1)}). \tag{3.19}$$

Proof: **1°**. This can be obtained by differentiating the identity $v_n^{(1)} = nu_n$, which had been proved in [1].

2°. By differentiating (1.1).

3° ~ 5°. By differentiating the following identities, which can be seen in [5] or can be derived from (1.2):

$$\begin{array}{ll}
v_n = u_{n+1} + u_{n-1}, & \Delta^2 u_n = v_{n+1} + v_{n-1}, \\
u_{m+n} = u_{m+1}u_n + u_m u_{n-1}, & v_{m+n} = v_{m+1}u_n + v_m u_{n-1}, \\
u_{m-n} = (-1)^n (u_m u_{n+1} - u_{m+1} u_n), & v_{m-n} = (-1)^n (u_{m+1} v_n - u_m v_{n+1}), \\
u_{m+n} + (-1)^n u_{m-n} = u_m v_n, & u_{m+n} - (-1)^n u_{m-n} = v_m u_n, \\
v_{m+n} + (-1)^n v_{m-n} = v_m v_n, & v_{m+n} - (-1)^n v_{m-n} = \Delta^2 u_m u_n = u_m (v_{n+1} + v_{n-1}), \\
u_{-n} = (-1)^{n-1} u_n, & v_{-n} = (-1)^n v_n, \\
u_{2n} = u_n v_n, & v_{2n} = v_n^2 - 2(-1)^n, \\
u_{2n+1} = u_{n+1} v_n - (-1)^n, & v_{2n+1} = v_{n+1} v_n - (-1)^n x.
\end{array}$$

6°. From the well-known identity $v_n = xu_n + 2u_{n-1}$, we get $xnu_n = nv_n - 2((n-1)u_{n-1} + u_{n-1})$, that is, $xv_n^{(1)} = nv_n - 2(v_{n-1}^{(1)} + u_{n-1})$, and the proof is finished by differentiating the last expres sion. □

Let $x = 1$ in 1°, 2°, 3°, and 6° of Theorem 3.1; then Conjectures 1-5 in [2] and [3] are proved.

4. SOME CONGRUENCE RELATIONS AND MODULAR PERIODICITIES

First, we introduce some concepts and lemmas. Set polynomials

$$g(t) = t^k - a_1 t^{k-1} - \cdots - a_{k-1} t - a_k \tag{4.1}$$

and

$$\tilde{g}(t) = 1 - a_1 t - \cdots - a_{k-1} t^{k-1} - a_k t^k. \tag{4.2}$$

Obviously, $g(t)=t^k\tilde{g}(1/t)$ and $\tilde{g}(t)=t^k g(1/t)$. The set of homogeneous linear recurrence sequences $\{g_n\}$ of order k [each of which has $g(t)$ as its characteristic polynomial] defined by

$$g_{n+k}=a_1g_{n+k-1}+\cdots+a_{k-1}g_{n+1}+a_kg_n \tag{4.3}$$

is denoted by $\Omega(g(t))=\Omega(a_1,\ldots,a_k)$. The sequence $\{w_n\}\in\Omega(g(t))$ is called **the principal sequence** in $\Omega(g(t))$ if it has the initial values $w_0=w_1=\cdots=w_{k-2}=0,\ w_{k-1}=1$.

Lemma 4.1: Let $\{w_n\}$ be the principal sequence in $\Omega(g(t))$; then its generating function is

$$W(t)=t^{k-1}/\tilde{g}(t) \tag{4.4}$$

(see [6], p. 137).

In the following discussions, we suppose that $a_1,\ldots,a_k$ are all integers. Let $\{g_n\}$ be an integer sequence in $\Omega(g(t))$ and m be an integer greater than one. Denote the period of $\{g_n\}$ modulo m by $P(m, g_n)$. If there exists a positive integer λ such that

$$t^\lambda\equiv 1\ (\mathrm{mod}\, m, g(t)), \tag{4.5}$$

then the least positive integer λ such that (4.5) holds is called **the period of $g(t)$ modulo m** and is denoted by $P(m, g(t))$.

We point out that

$$P(m, g(t))=P(m,\tilde{g}(t)) \text{ for } \gcd(m, a_k)=1. \tag{4.6}$$

To show (4.6), it is sufficient to show that $g(t)|(t^\lambda-1)$ (mod m) iff $\tilde{g}(t)|(t^\lambda-1)$ (mod m). Assume that $\tilde{g}(t)|(t^\lambda-1)$ (mod m). Then we have $t^\lambda-1=h(t)\tilde{g}(t)+m\cdot r(t)$, where $h(t)$ and $r(t)\in Z(t)$ (the set of polynomials with integer coefficients). Replacing t with $1/t$, we obtain $(1/t)^\lambda-1=h(1/t)\tilde{g}(1/t)+m\cdot r(1/t)$. Multiplying by t^λ, we then have $-(t^\lambda-1)=t^{\lambda-k}h(1/t)g(t)+m\cdot t^\lambda r(1/t)$. Since $\gcd(m, a_k)=1$, the degree of $\tilde{g}(t)$ (mod m) is k. This leads to $t^{\lambda-k}h(1/t)$ and $t^\lambda r(1/t)\in Z(t)$. Hence, $g(t)|(t^\lambda-1)$ (mod m). The converse can be proved in the same way.

Let $B(t)=1/\tilde{g}(t)=\sum_{n=0}^\infty b_nt^n$. Let $\{w_n\}$ be the principal sequence in $\Omega(g(t))$. Then, from (4.4), we have $w_n=b_{n-k+1}$; and therefore, $P(m, w_n)=P(m, b_n)$. Corollary 2 in [7] means that $P(m, b_n)=P(m,\tilde{g}(t))$.* Therefore,

$$P(m, w_n)=P(m,\tilde{g}(t)). \tag{4.7}$$

From (4.6) and (4.7), we obtain

Lemma 4.2: Let $\{w_n\}$ be the principal sequence in $\Omega(g(t))=\Omega(a_1,\ldots,a_k)$, $\gcd(m, a_k)=1$. Then

$$P(m, w_n)=P(m, g(t)). \tag{4.8}$$

Using the footnote and (4.6), Theorems 17, 21, and 15 in [7] can be rewritten as Lemmas 4.3, 4.4, and 4.5, respectively.

---

* In [7] the period of $\{b_n\}$ modulo m is referred to as the period of its generating function $B(t)=1/\tilde{g}(t)$ modulo m. Hence, the concept "the period of $1/\tilde{g}(t)$ modulo m" stated in [7] should be translated into "$P(m,\tilde{g}(t))$" in this paper.

Lemma 4.3: Let $\varphi(t)$ be a monic polynomial with integer coefficients, p be a prime, $p \nmid \varphi(0)$, and $\varphi(t)$ be irreducible modulo p; then, for $p^{r-1} < s \le p^r$ $(r \ge 1)$,

$$P(p^m, \varphi(t)^s) = p^{m+r-1} \cdot P(p, \varphi(t)). \tag{4.9}$$

Lemma 4.4: Let $\varphi(t)$ be a monic polynomial with integer coefficients, p be an odd prime, $p \nmid \varphi(0)$, and $\varphi(t)$ be irreducible modulo p. Assume $h_\tau(t) = \prod_{i=1}^{\tau} \Psi_i(t)$, where $\Psi_i(t) \equiv \varphi(t)^s$ (mod p) $(i = 1, \dots, \tau)$. For fixed $s, r \ge 1$, if there exists an integer $T > 1$ such that

$$(T-1)s \le p^{r-1} < Ts < (T+1)s \le p^r, \tag{4.10}$$

then, for every τ satisfying $p^{r-1} < \tau s \le p^r$, it follows that

$$P(p^m, h_\tau(t)) = P(p^m, \varphi(t)^{\tau s}) = p^{m+r-1} \cdot P(p, \varphi(t)). \tag{4.11}$$

Lemma 4.5: Let $\varphi(t)$ be a monic polynomial with integer coefficients, p be an odd prime, $p \nmid \varphi(0)$. If $P(p, \varphi(t)) = P(p^2, \varphi(t)) = \cdots = P(p^i, \varphi(t)) \ne P(p^{i+1}, \varphi(t))$, then $m > i$ leads to

$$P(p^m, \varphi(t)) = p^{m-i} \cdot P(p^i, \varphi(t)). \tag{4.12}$$

Lemma 4.6: Let p be an odd prime, for $j = 1, 2$, $\varphi_j(t)$ be a monic polynomial with integer coefficients, $p \nmid \varphi_j(0)$, and $\varphi_j(t)$ be irreducible modulo p. Assume $h_\tau(t) = \prod_{i=1}^{\tau} \Psi_i(t)$, where $\Psi_i(t) \equiv \varphi_1(t)^s \varphi_2(t)^s$ (mod p) $(i = 1, \dots, \tau)$, $\gcd(\varphi_1(t), \varphi_2(t)) = 1$ (mod p). For fixed $s, r \ge 1$, if there exists an integer $T > 1$ such that (4.10) holds, then for every τ satisfying $p^{r-1} < \tau s \le p^r$ it follows that

$$P(p^m, h_\tau(t)) = P(p^m, \varphi_1(t)^{\tau s} \varphi_2(t)^{\tau s}) = p^{m+r-1} \cdot \operatorname{lcm}\{P(p, \varphi_1(t)), P(p, \varphi_2(t))\}. \tag{4.13}$$

Proof: Denote $P(p, \varphi_j(t)) = \lambda_j$ $(j = 1, 2)$, $\operatorname{lcm}\{\lambda_1, \lambda_2\} = \lambda$. Since $h_\tau(t) \equiv \varphi_1(t)^{\tau s} \varphi_2(t)^{\tau s}$ (mod p), $\gcd(\varphi_1(t), \varphi_2(t)) = 1$ (mod p), we have $P(p, h_\tau(t)) = \operatorname{lcm}\{P(p, \varphi_1(t)^{\tau s}), P(p, \varphi_2(t)^{\tau s})\}$. By Lemma 4.3, $P(p, \varphi_j(t)^{\tau s}) = p^r \lambda_j$; hence, $P(p, h_\tau(t)) = p^r \lambda$.

Because T is the least τ satisfying $p^{r-1} < \tau s \le p^r$ from (4.10), we get $h_T(t) | h_\tau(t)$; therefore, $P(p^m, h_T(t)) | P(p^m, h_\tau(t))$. By Lemma 4.5, $P(p^m, h_\tau(t)) | p^{m-1} \cdot P(p, h_\tau(t)) = p^{m+r-1}\lambda$. By the same lemma, if we can show $P(p^2, h_T(t)) \ne P(p, h_T(t)) = p^r \lambda$, then $P(p^m, h_T(t)) = p^{m+r-1}\lambda$ and (4.13) holds.

Now we can rewrite $\Psi_i(t) = \varphi_1(t)^s \varphi_2(t)^s - p\theta_i(t)$, $i = 1, \dots, T$. Hence,

$$h_T(t) \equiv \varphi_1(t)^{sT} \varphi_2(t)^{sT} - p\varphi_1(t)^{s(T-1)} \cdot \varphi_2(t)^{s(T-1)} \cdot \zeta(t) \pmod{p^2}, \text{ where } \zeta(t) = \sum_{i=0}^{T} \theta_i(t).$$

Then $h_T(t)[\varphi_1(t)^s \varphi_2(t)^s + p\zeta(t)] \equiv \varphi_1(t)^{sT+s} \varphi_2(t)^{sT+s}$ (mod p^2). Therefore,

$$\frac{t^{p^r\lambda} - 1}{h_T(t)} \equiv \frac{t^{p^r\lambda} - 1}{\varphi_1(t)^{sT} \varphi_2(t)^{sT}} + \frac{p(t^{p^r\lambda} - 1)\zeta(t)}{\varphi_1(t)^{sT+s} \varphi_2(t)^{sT+s}} \pmod{p^2}. \tag{4.14}$$

From (4.10) and Lemma 4.3, we know that $P(p, \varphi_j(t)^{sT+s}) = p^r \cdot P(p, \varphi_j(t)) = p^r \lambda_j$; thus, $\varphi_j(t)^{sT+s} | (t^{p^r\lambda} - 1)$ (mod p). From $\gcd(\varphi_1(t), \varphi_2(t)) = 1$ (mod p), it follows that

$$\varphi_1(t)^{sT+s} \varphi_2(t)^{sT+s} | (t^{p^r\lambda} - 1) \pmod{p},$$

and so

$$\varphi_1(t)^{sT+s}\varphi_2(t)^{sT+s}\,|\,p(t^{p^r\lambda}-1) \pmod{p^2}.$$

Assume that $P(p^2, h_T(t)) = p^r\lambda$, then $h_T(t)\,|\,(t^{p^r\lambda}-1) \pmod{p^2}$. From equation (4.14), we get $\varphi_j(t)^{sT}\,|\,(t^{p^r\lambda}-1) \pmod{p^2}$; this leads to $P(p^2, \varphi_j(t)^{sT})\,|\,p^r\lambda$. But from Lemma 4.3 we have $P(p^2, \varphi_j(t)^{sT}) = p^{r+1}\lambda_j$. This leads to the contradiction that $p^{r+1}\lambda\,|\,p^r\lambda$. □

In the following discussions of this section when the divisibilities of $u_n^{(k)}$ and $v_n^{(k)}$ are considered, we assume x takes integer values only.

Theorem 4.1:

$$u_n^{(k)} \equiv v_n^{(k)} \equiv 0 \pmod{k!}. \tag{4.15}$$

Proof: Denote

$$F_k(t) = (t^2 - xt - 1)^{k+1}. \tag{4.16}$$

Let $\{w_n\}$ be the principal sequence in $\Omega(F_k(t))$. From Lemma 4.1, the generating function of $\{w_n\}$ is

$$W(t) = t^{2k+1}/(1 - xt - t^2)^{k+1}. \tag{4.17}$$

Comparing (2.7) to (4.17), we get

$$u_n^{(k)} = k!\,w_{n+k}. \tag{4.18}$$

Because $\{w_n\}$ is an integer sequence, we have $u_n^{(k)} \equiv 0 \pmod{k!}$, and from (3.3) we get $v_n^{(k)} \equiv 0 \pmod{k!}$. □

Theorem 4.2:

$$v_n^{(k)} \equiv 0 \pmod{n}\ \ (k \geq 1). \tag{4.19}$$

This follows from (3.1).

The results of the last two theorems are generalizations of the results of Conjectures 6-7 in [2].

Theorem 4.3: Let p be an odd prime, $p > k$.

1°. If $p \nmid \Delta^2$, then

$$P(p^m, u_n^{(k)}) = P(p^m, v_n^{(k)}) = p^m \cdot P(p, u_n) = p^m \cdot P(p, v_n). \tag{4.20}$$

2°. If $p\,|\,\Delta^2$ and $p^{r-1} < 2k+2 < p^r$ $(r = 1 \text{ or } 2)$, then

$$P(p^m, u_n^{(k)}) = 4p^{m+r-1}. \tag{4.21}$$

3°. If $p\,|\,\Delta^2$ and $p^{r-1} < 2k < p^r$ $(r = 1 \text{ or } 2)$, then

$$P(p^m, v_n^{(k)}) = 4p^{m+r-1}. \tag{4.22}$$

Proof: Denote $f(t) = t^2 - xt - 1$. From Lemma 4.2, (4.18), and (4.16), for $p > k$, we have $P(p, u_n) = P(p, f(t))$ and $P(p^m, u_n^{(k)}) = P(p^m, F_k(t))$.

1°. Let $p \nmid \Delta^2$. From $v_n = u_{n+1} + u_{n-1}$ and $\Delta^2 u_n = v_{n+1} + v_{n-1}$, it follows that $P(p, u_n) = P(p, v_n)$ $= \lambda$.

When $f(t)$ is irreducible modulo p, the conclusion $P(p^m, u_n^{(k)}) = p^m\lambda$ can be proved by letting $\varphi(t) = f(t), s = k+1, r = 1$ in Lemma 4.3. When $f(t) \equiv (t-a)(t-b), a \not\equiv b \pmod p$, the same conclusion can be proved by letting $\varphi_1(t) = t-a, \varphi_2(t) = t-b, s = r = 1, \tau = k+1$ in Lemma 4.6.

We now prove $P(p^m, v_n^{(k)}) = p^m\lambda$. From (3.3), we can see that $P(p^m, v_n^{(k)}) | P(p^m, u_n^{(k)})$. On the other hand, from $u_n = (v_{n+1} + v_{n-1}) / \Delta^2$, by differentiating, we can obtain

$$u_n^{(k)} = \sum_{i=0}^{k} \binom{k}{i} (v_{n+1}^{(k-i)} + v_{n-1}^{(k-i)}) M_i(x) / \Delta^{2i+2}, \tag{4.23}$$

where $M_i(x)$ is a polynomial in x with integer coefficients that are independent of n. We see that (3.2) implies $P(p^m, v_n^{(i-1)}) | P(p^m, v_n^{(i)})$. Hence, for $i = 0, 1, \ldots, k$, $P(p^m, v_n^{(k-i)}) | P(p^m, v_n^{(k)})$. From (4.23), it follows that $P(p^m, u_n^{(k)}) | P(p^m, v_n^{(k)})$. Thus, $P(p^m, v_n^{(k)}) = P(p^m, u_n^{(k)}) = p^m\lambda$.

2°. Let $p | \Delta^2$, then $f(t) \equiv (t - x/2)^2 \pmod p$. From $x^2 \equiv -4$, we get $(x/2)^2 \equiv -1 \pmod p$. Hence, $P(p, t - x/2) = \text{ord}_p(x/2) = 4$.* In Lemma 4.4, if we take $\varphi(t) = t - x/2$, $h_\tau(t) = F_k(t) \equiv \varphi(t)^{2k+2} \pmod p$, $s = 2, r = 1$ or $2, \tau = k+1$, then we get the required result.

3°. Using the result of 2°, it follows that $P(p^m, v_n^{(k)}) = P(p^m, nu_n^{(k-1)}) | \text{lcm}\{P(p^m, n), P(p^m, u_n^{(k-1)})\} = 4p^{m+r-1}$ when $p^{r-1} < 2k < p^r$ ($r = 1$ or 2). Since $v_n = \alpha^n + \beta^n \equiv 2(x/2)^n \pmod p$, then $4 = P(p, v_n) | P(p^m, v_n^{(k)})$, and we have $P(p^m, v_n^{(k)}) = 4p^M$. We want to show that $M = m + r - 1 = m+1$ for $r = 2$, or $= m$ for $r = 1$. First, let $r = 2$. If it would not be the case, that is, if $M \le m$, then if we replace n by $n + 4p^m$ in (3.19) we have

$$xv_n^{(k)} \equiv (n + 4p^m - k + 1) v_n^{(k-1)} - 2[v_{n-1}^{(k)} + u_{n+4p^m-1}^{(k-1)}] \pmod{p^m}.$$

Subtracting this from $xv_n^{(k)} \equiv (n-k+1)v_n^{(k-1)} - 2[v_{n-1}^{(k)} + u_{n-1}^{(k-1)}] \pmod{p^m}$, we get $u_{n+4p^m-1}^{(k-1)} - u_{n-1}^{(k-1)} \equiv 2p^m v_n^{(k-1)} \equiv 0 \pmod{p^m}$. This means that $P(p^m, u_n^{(k-1)}) | 4p^m$ for $r = 2$. But, by 2°, we should have $P(p^m, u_n^{(k-1)}) = 4p^{m+1}$ for $r = 2$. A contradiction!

Next, let $r = 1$. The least k satisfying $1 < 2k < p$ is 1. Recalling that $P(p^m, v_n^{(1)}) | P(p^m, v_n^{(k)})$, we need only prove that $M = m$ for $k = 1$. On the contrary, suppose $M \le m-1$. then

$$v_{n+4p^{m-1}}^{(1)} - v_n^{(1)} = (n + 4p^{m-1}) u_{n+4p^{m-1}} - nu_n \equiv 0 \pmod{p^m}.$$

Expanding u_n in (1.2) into the polynomial in x, Δ, and noting $p | \Delta^2$, we obtain

$$nu_n = n \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} (x/2)^{n-2i-1} (\Delta/2)^{2i} \equiv n \sum_{i=0}^{m-1} \binom{n}{2i+1} (x/2)^{n-2i-1} (\Delta/2)^{2i} \pmod{p^m} \tag{4.24}$$

and

$$(n + 4p^{m-1}) u_{n+4p^{m-1}} \equiv (n + 4p^{m-1}) \sum_{i=0}^{m-1} \binom{n+4p^{m-1}}{2i+1} (x/2)^{n+4p^{m-1}-2i-1} (\Delta/2)^{2i} \pmod{p^m}. \tag{4.25}$$

When $m > 1$, since

---

* Let m and a be integers greater than one, $\gcd(m, a) = 1$. The least positive integer λ satisfying $a^\lambda \equiv 1 \pmod m$ is called the order of a modulo m and is denoted by $\text{ord}_m(a)$. Since $t^\lambda - 1 = [(t-a) + a]^\lambda - 1 \equiv a^\lambda - 1 \pmod{(t-a)}$, we have $P(m, t-a) = \text{ord}_m(a)$.

$$(n+4p^{m-1})\binom{n+4p^{m-1}}{2i+1} \equiv n\binom{n}{2i+1} \pmod{p^{m-1}} \quad \text{and} \quad p\mid\Delta^{2i} \text{ for } i \geq 1,$$

and furthermore, $(x/2)^4 = 1 \pmod{p}$ implies $(x/2)^{4p^{m-1}} \equiv 1 \pmod{p^m}$, (4.25) can be reduced to

$$(n+4p^{m-1})u_{n+4p^{m-1}} \equiv (n+4p^{m-1})^2(x/2)^{n-1} + n\sum_{i=1}^{m-1}\binom{n}{2i+1}(x/2)^{n-2i-1}(\Delta/2)^{2i} \pmod{p^m}. \quad (4.26)$$

Subtract (4.24) from (4.26) to get

$$(n+4p^{m-1})u_{n+4p^{m-1}} - nu_n \equiv 8np^{m-1}(x/2)^{n-1} \not\equiv 0 \pmod{p^m} \ \text{ for } p\nmid n.$$

This is a contradiction!

When $m=1$, from (4.24) and (4.25), we obtain

$$(n+4)u_{n+4} - nu_n \equiv 8(n+2)(x/2)^{n-1} \not\equiv 0 \pmod{p}$$

for $n \not\equiv -2 \pmod{p}$. This is also a contradiction! □

From Theorem 4.3, we can obtain many specific congruences. For this, we introduce another concept. Let $\{g_n\}$ be an integer sequence. If there exists a positive integer s, a nonnegative integer n_0, and an integer c, $\gcd(m, c) = 1$, such that

$$g_{n+s} \equiv cg_n \pmod{m} \ \text{ iff } \ n \geq n_0, \quad (4.27)$$

then the least positive integer s satisfying (4.27) is called **the constrained period of $\{g_n\}$ modulo m** and is denoted by $s = P'(m, g_n)$. The number c is called the **multiplier**.

Lemma 4.7: Let $\{w_n\}$ be the principal sequence in $\Omega(F_k(t))$, where $F_k(t)$ is denoted by (4.16). Then $P'(m, w_n) = s$ exists and the multiplier c is equal to w_{s+2k+1} (mod m). Furthermore, if $r = \text{ord}_m(c)$, then $P(m, w_n) = sr$, and the structure of $\{w_n \pmod{m}\}$ in a period is as follows:

$$\begin{cases} 0, \ \ldots, \ 0, \ 1, & w_{2k+2}, \quad w_{2k+3}, \quad \ldots, \ w_{s-1}, \\ 0, \ \ldots, \ 0, \ c, & cw_{2k+2}, \quad cw_{2k+3}, \quad \cdots, \ cw_{s-1}, \\ \cdots & \\ 0, \ \ldots, \ 0, \ c^{r-1}, & c^{r-1}w_{2k+2}, \ c^{r-1}w_{2k+3}, \ \ldots, \ c^{r-1}w_{s-1}. \end{cases} \quad (4.28)$$

Proof: Because $\{w_n\}$ is periodic, it must be constrained periodic [in the most special case, the multiplier c may be equal to 1 (mod m)]. We have $w_0 = \cdots = w_{2k} = 0$ and $w_{2k+1} = 1$. Replacing n by $2k+1$ in the expression

$$w_{n+s} \equiv cw_n \pmod{m}, \quad (4.29)$$

we obtain $c \equiv w_{s+2k+1} \pmod{m}$. By induction, from (4.29), we can get

$$w_{n+js} \equiv c^j w_n \pmod{m}. \quad (4.30)$$

If $j = r = \text{ord}_m(c)$, then (4.30) becomes $w_{n+rs} \equiv w_n \pmod{m}$. This means that $P(m, w_n) = sr$. In (4.30), let j be $0, 1, \ldots, r-1$ and n be $0, 1, \ldots, s-1$; then (4.28) follows. □

From Lemma 4.7, (4.18), and (3.1), we obtain

Theorem 4.4: Let $\{w_n\}$ be the principal sequence in $\Omega(F_k(t))$, where $F_k(t)$ is denoted by (4.16), and let p be an odd prime, $p > k$, $P'(p^m, w_n) = s$. If $w_n \equiv 0 \pmod{p^m}$ for $n \equiv i \pmod{s}$, then

$$u_n^{(k)} \equiv 0 \pmod{p^m} \text{ for } n \equiv i-k \pmod{s}$$

and

$$v_n^{(k+1)} \equiv 0 \pmod{p^m} \text{ for } n \equiv i-k \pmod{s} \text{ or } n \equiv 0 \pmod{p^m}.$$

Furthermore, if $\lambda p^r \equiv i-k \pmod{s}$, then $v_{\lambda p^r}^{(k+1)} \equiv 0 \pmod{p^{m+r}}$.

Example 1: Let $x = 1$, $p = 3$. Then $\Delta^2 = 5$, $p \nmid \Delta^2$. Hence, from (4.20), we obtain $P(3^m, f_n^{(k)}) = P(3^m, \ell_n^{(k)}) = 3^m \cdot P(3, f_n) = 8 \cdot 3^m$ for $k = 1, 2$.

Example 2: Let $x = 1$, $p = 5$. Then $p \mid \Delta^2 = 5$. Hence, from (4.21), we get $P(5^m, f_n^{(k)}) = 4 \cdot 5^{m+1}$ for $k = 2, 3, 4$, or $4 \cdot 5^m$ for $k = 1$ and, from (4.22), we get $P(5^m, \ell_n^{(k)}) = 4 \cdot 5^{m+1}$ for $k = 3, 4$ or $4 \cdot 5^m$ for $k = 1, 2$.

Example 3: We show that $f_n^{(2)} = 0 \pmod{10}$ iff $n \equiv 0, \pm 1, \pm 2 \pmod{25}$, and $\ell_n^{(3)} \equiv 0 \pmod{30}$ iff $n \equiv \pm 1, \pm 2 \pmod{25}$ or $n \equiv 0 \pmod{5}$.

Proof of Example 3: We have $F_2(t) = (t^2 - t - 1)^3 = t^6 - 3t^5 + 5t^3 - 3t - 1 \equiv t^6 - 3t^5 - 3t - 1 \pmod{5}$ for $x = 1$. Let $\{w_n\}$ be the principal sequence in $\Omega(F_2(t))$. Then $w_{n+6} \equiv 3w_{n+5} + 3w_{n+1} + w_n \pmod{5}$.

Calculate $\{w_n \pmod{5}\}_0^\infty$ according to the last congruence:

$$0,0,0,0,0,1,-2,-1,2,1,1,-2,-1,2,1,2,1,-2,-1,2,-2,-1,2,1,-2,0,0,0,0,0,-2,\ldots \pmod{5}.$$

This implies that $s = P'(5, w_n) = 25$ and $w_n \equiv 0 \pmod{5}$ iff $n \equiv 0, 1, 2, 3, 4 \pmod{25}$. Hence, the example is proved by Theorem 4.1 and Theorem 4.4.

5. EVALUATION OF SOME SERIES INVOLVING $u_n^{(k)}$ AND $v_n^{(k)}$

Lemma 5.1:

1°. $\displaystyle\sum_{i=0}^{n} u_i = (u_{n+1} + u_n - 1)/x \quad (x \neq 0).$ (5.1)

2°. $\displaystyle\sum_{i=0}^{n} v_i = (v_{n+1} + v_n - 2)/x + 1 \quad (x \neq 0).$ (5.2)

3°. $\displaystyle\sum_{i=0}^{n} \binom{n}{i} x^i h_{i+r} = h_{2n+r}$, where $\{h_n\}$ is $\{u_n\}$ or $\{v_n\}$. (5.3)

4°. $\displaystyle\sum_{i=0}^{n} (-1)^i \binom{n}{i} h_{2i+r} = (-1)^n x^n h_{n+r}$, where $\{h_n\}$ is $\{u_n\}$ or $\{v_n\}$. (5.4)

5°. $\displaystyle\sum_{i=0}^{n} \binom{n}{i} u_{2i+r} = (x^2+4)^{n/2} u_{n+r}$ for $2 \mid n$, or $(x^2+4)^{(n-1)/2} v_{n+r}$ for $2 \nmid n$. (5.5)

6°. $\displaystyle\sum_{i=0}^{n} \binom{n}{i} v_{2i+r} = (x^2+4)^{n/2} v_{n+r}$ for $2 \mid n$, or $(x^2+r)^{(n+1)/2} u_{n+r}$ for $2 \nmid n$. (5.6)

Proof: We prove only 2° and 5°. The rest can be proved in the same way.

2°. $\sum_{i=0}^{n} v_i = \sum_{i=0}^{n}(\alpha^i + \beta^i) = (1-\alpha^{n+1})/(1-\alpha) + (1-\beta^{n+1})/(1-\beta)$

$$= (1-\alpha^{n+1} - \beta - \alpha^n + 1 - \beta^{n+1} - \alpha - \beta^n)/(-x) = (v_{n+1} + v_n - 2)/x + 1.$$

5°. We have

$$\sum_{i=0}^{n}\binom{n}{i}\alpha^{2i} = (1+\alpha^2)^n = (-\alpha\beta + \alpha^2)^n = \Delta^n\alpha^n.$$

For the same reason

$$\sum_{i=0}^{n}\binom{n}{i}\beta^{2i} = (-1)^n\Delta^n\beta^n.$$

Hence,

$$\sum_{i=0}^{n}\binom{n}{i}u_{2i+r} = \sum_{i=0}^{n}\binom{n}{i}(\alpha^{2i+r} - \beta^{2i+r})/\Delta = \Delta^n[\alpha^{n+r} - (-1)^n\beta^{n+r})/\Delta$$

$$= \Delta^n[\alpha^{n+r} - \beta^{n+r})/\Delta = (x^2+4)^{n/2}u_{n+r} \quad \text{for } 2\,|\,n,$$

$$\text{or} \qquad = \Delta^{n-1}(\alpha^{n+r} + \beta^{n+r}) = (x^2+4)^{(n-1)/2}v_{n+r} \quad \text{for } 2\nmid n. \quad \square$$

Theorem 5.1:

$$\sum_{i=0}^{n} u_i^{(k)} = \sum_{i=0}^{k}(-1)^i(k)_i[u_{n+1}^{(k-i)} + u_n^{(k-i)} - \delta_{k,i}]/x^{i+1} \quad (x \neq 0); \tag{5.7}$$

$$\sum_{i=0}^{n} v_i^{(k)} = \sum_{i=0}^{k}(-1)^i(k)_i[v_{n+1}^{(k-i)} + v_n^{(k-i)} - 2\delta_{k,i}]/x^{i+1} \quad (x \neq 0); \tag{5.8}$$

$$\sum_{i=0}^{n}\binom{n}{i}x^i h_{i+r}^{(k)} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(n)_i h_{2n-i+r}^{(k-i)}, \tag{5.9}$$

where $\{h_n^{(i)}\}$ is $\{u_n^{(i)}\}$ or $\{v_n^{(i)}\}$ $(i = 0, \ldots, k)$;

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}h_{2i+r}^{(k)} = (-1)^n\sum_{i=0}^{k}\binom{k}{i}(n)_i x^{n-i}h_{n+r}^{(k-i)}, \tag{5.10}$$

where $\{h_n^{(i)}\}$ is $\{u_n^{(i)}\}$ or $\{v_n^{(i)}\}$ $(i = 0, \ldots, k)$;

$$\sum_{i=0}^{n}\binom{n}{i}u_{2i+r}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}u_{n+r}^{(k-i)}\frac{d^i}{dx^i}(x^2+4)^{n/2} \quad \text{for } 2\,|\,n,$$

$$\text{or} \qquad = \sum_{i=0}^{k}\binom{k}{i}v_{n+r}^{(k-i)}\frac{d^i}{dx^i}(x^2+4)^{(n-1)/2} \quad \text{for } 2\nmid n; \tag{5.11}$$

$$\sum_{i=0}^{n}\binom{n}{i}v_{2i+r}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}v_{n+r}^{(k-i)}\frac{d^i}{dx^i}(x^2+4)^{n/2} \quad \text{for } 2\,|\,n,$$

$$\text{or} \qquad = \sum_{i=0}^{k}\binom{k}{i}u_{n+r}^{(k-i)}\frac{d^i}{dx^i}(x^2+r)^{(n+1)/2} \quad \text{for } 2\nmid n. \tag{5.12}$$

Proof: Every one of (5.7), (5.8), (5.10)-(5.12) can be proved straightforwardly by differentiating the corresponding one of (5.1), (5.2), (5.4)-(5.6). The proof of (5.9) is as follows.

Let

$$g_{n,k,r} = g_{n,k,r}(x) = \sum_{i=0}^{k} \binom{n}{i} x^i h_{i+r}^{(k)}. \tag{5.13}$$

Then

$$g'_{n,k,r} = \sum_{i=0}^{k} \binom{n}{i} x^i h_{i+r}^{(k+1)} + \sum_{i=1}^{n} \binom{n}{i} i x^{i-1} h_{i+r}^{(k)} = g_{n,k+1,r} + n \cdot g_{n-1,k,r+1}.$$

So

$$g_{n,k+1,r} = g'_{n,k,r} - n \cdot g_{n-1,k,r+1}. \tag{5.14}$$

When $k = 0$, from (5.3), we can see that (5.9) holds. Assume that (5.9) holds for k; then from (5.14), we have

$$g_{n,k+1,r} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n)_i h_{2n-i+r}^{(k+1-i)} - n \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n-1)_i h_{2n-1-i+r}^{(k-i)}.$$

The second summation in the right side of the last expression can be rewritten as

$$\begin{aligned} &-n \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (n-1)_i h_{2n-1-i+r}^{(k-i)} - n(-1)^k \cdot (n-1)_k h_{2n-1-k+r} \\ &= \sum_{i=1}^{k} (-1)^i \binom{k}{i-1} (n)_i h_{2n-i+r}^{(k+1-i)} + (-1)^{k+1} (n)_{k+1} h_{2n-(k+1)+r}. \end{aligned}$$

From this, it follows that

$$g_{n,k+1,r} = \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (n)_i h_{2n-i+r}^{(k+1-i)},$$

that is, (5.9) also holds for $k+1$, and we are done. □

It is known that the generating function of $\{u_n^{(k)}\}$ is expressed by (2.7). It is well known that the generating function of $\{v_n\}$ is

$$V(t) = (2 - xt) / (1 - xt - t^2). \tag{5.15}$$

Differentiating (5.15), we can know that the generating function of $\{v_n^{(k)}\}$ is

$$V_k(t) = k!\, t^k (1 + t^2) / (1 - xt - t^2)^{k+1} \quad (k \geq 1). \tag{5.16}$$

Obviously, the following identities hold:

$$U_k(t) \cdot U_r(t) = \frac{k!\, r!}{(k+r+1)!} U_{k+r+1}(t);$$

$$V_k(t) \cdot V_r(t) = \frac{k!\, r!}{(k+r+1)!} (t + t^{-1}) V_{k+r+1}(t) \quad (k, r \geq 1);$$

$$U_k(t) \cdot V_r(t) = \frac{k!\, r!}{(k+r+1)!} V_{k+r+1}(t) \quad (r \geq 1);$$

$$U_k(t) \cdot V(t) = \frac{1}{k+1} (2t^{-1} - x) U_{k+1}(t);$$

$$V_k(t)\cdot V(t)=\frac{1}{k+1}(2t^{-1}-x)V_{k+1}(t)\quad (k\geq 1).$$

Equalizing the coefficients of t^n of the two sides in each of the above identities, we have

Theorem 5.2:

$$\sum_{i=0}^{n}u_i^{(k)}u_{n-i}^{(r)}=\frac{k!r!}{(k+r+1)!}u_n^{(k+r+1)};\tag{5.17}$$

$$\sum_{i=0}^{n}v_i^{(k)}v_{n-i}^{(r)}=\frac{k!r!}{(k+r+1)!}(v_{n-i}^{(k+r+1)}+v_{n+1}^{(k+r+1)})\quad (k,r\geq 1);\tag{5.18}$$

$$\sum_{i=0}^{n}u_i^{(k)}v_{n-i}^{(r)}=\frac{k!r!}{(k+r+1)!}v_n^{(k+r+1)}\quad (r\geq 1);\tag{5.19}$$

$$\sum_{i=0}^{n}u_i^{(k)}v_{n-i}=\frac{1}{k+1}(2u_{n+1}^{(k+1)}-xu_n^{(k+1)});\tag{5.20}$$

$$\sum_{i=0}^{n}v_i^{(k)}v_{n-i}=\frac{1}{k+1}(2v_{n+1}^{(k+1)}-xv_n^{(k+1)})\quad (k\geq 1).\tag{5.21}$$

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ON THE EXISTENCE OF COUPLES OF SECOND-ORDER LINEAR RECURRENCES WITH RECIPROCAL REPRESENTATION PROPERTIES FOR THEIR FIBONACCI SEQUENCES

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The aim of this note is to show that for any given second-order linear recurrence on the complex field

$$r_{n+2} - ar_{n+1} + br_n = 0, \tag{R1}$$

where $\Delta = a^2 - 4b \neq 0$ and $b \neq 0$, another one exists such that it is possible to represent the generalized Fibonacci numbers of any of them with sums of the generalized Fibonacci numbers of the other one, with a set of coefficients to be detailed later.

To establish this property, we need the following lemmas.

Lemma 1: Let $U_n(a, b)$ denote the n^{th} generalized Fibonacci number of the (R1) recursion. That is, $U_{n+2} - aU_{n+1} + bU_n = 0$, $U_0 = 0$, $U_1 = 1$, where $\Delta = a^2 - 4b \neq 0$ and $b \neq 0$. Let $\sqrt{b}$ denote any of the roots of the equation $z^2 = b$. Then

$$U_{n+1}(a, b) = \sum_{p=0}^{n} \binom{2n+1-p}{p} (a - 2\sqrt{b})^{n-p} (\sqrt{b})^p. \tag{F1}$$

Lemma 2: If S is the set of all the couples of complex numbers (u, v), their order being indifferent [that is, $(u, v) = (v, u)$], and if T is the transformation defined on all S by

$$T(u, v) = \left(\frac{u+v}{2} + \sqrt{uv},\ \frac{u+v}{2} - \sqrt{uv} \right),$$

then $T^2(u, v) = (u, v)$, where T^2 is the second iterate of T.

Proof of Lemma 1: Edouard Lucas [1] proved that if $U_n(t, s)$ is the n^{th} generalized Fibonacci number of the recursion defined on the complex field by $r_{n+2} - tr_{n+1} + sr_n = 0$, then

$$U_{n+1}(t, s) = \sum_{p=0}^{[n/2]} \binom{n-p}{p} (t)^{n-2p} (-s)^p.$$

Throughout the rest of this paper, when we refer to the *characteristic roots* of a linear recursion we mean the roots of its auxiliary algebraic equation.

Now let α and β be the characteristic roots (supposed *distinct*) of the recursion (R1), let $\sqrt{\alpha}$ be any root of the equation $z^2 = \alpha$, and let $-\sqrt{\beta}$ be any root of the equation $z^2 = \beta$.

If Y_n is the n^{th} generalized Fibonacci number of the second-order linear recursion whose characteristic roots are $\sqrt{\alpha}$ and $-\sqrt{\beta}$ then, using Lucas' formula, we obtain

$$Y_{n+1} = \sum_{p=0}^{[n/2]} \binom{n-p}{p} (\sqrt{\alpha} - \sqrt{\beta})^{n-2p} (\sqrt{\alpha}\sqrt{\beta})^p.$$

Now, using the usual Binet form, we easily obtain

$$Y_{2(n+1)} = (\sqrt{\alpha} - \sqrt{\beta})U_{n+1}(a, b),$$

whence

$$U_{n+1}(a, b) = \sum_{p=0}^{n} \binom{2n+1-p}{p} (\sqrt{\alpha} - \sqrt{\beta})^{2(n-p)} (\sqrt{\alpha}\sqrt{\beta})^p .$$

But we have $(\sqrt{\alpha} - \sqrt{\beta})^2 = \alpha + \beta - 2\sqrt{\alpha}\sqrt{\beta} = a - 2\sqrt{\alpha}\sqrt{\beta}$.

Since $\alpha\beta = b$, it is obvious from the above definitions of $\sqrt{\alpha}$ and $\sqrt{\beta}$ that we can replace $\sqrt{\alpha}\sqrt{\beta}$ by any of the roots of $z^2 = b$. This completes the demonstration.

Since $\sqrt{b}$ may be *any* of the roots of $z^2 = b$, the following formula is also true:

$$U_{n+1}(a, b) = \sum_{p=0}^{n} \binom{2n+1-p}{p} (a + 2\sqrt{b})^{n-p} (-\sqrt{b})^p . \tag{F2}$$

Proof of Lemma 2: The proof is immediate by directly computing

$$T\left(\frac{u+v}{2} + \sqrt{uv},\ \frac{u+v}{2} - \sqrt{uv}\right).$$

Now, to the recursion (R1), let us associate the recursion (R2), whose characteristic roots are $a/2 + \sqrt{b}$ and $a/2 - \sqrt{b}$, that is, the one defined by:

$$r_{n+2} - ar_{n+1} + (\Delta / 4)r_n = 0, \tag{R2}$$

where Δ is the discriminant of (R1).

It is immediate that the couple of roots of (R2) are obtained by applying the T transformation to the couple of roots of (R1). Therefore, by applying the same transformation to the couple of roots of (R2), we obtain the couple of roots of (R1), according to Lemma 2. Then the associate recursion for (R2) is (R1).

Now we may write (F1) and (F2) as follows:

$$U_{n+1}(a, b) = U_{n+1} = \sum_{p=0}^{n} \binom{2n+1-p}{p} 2^{n-p} (a/2 - \sqrt{b})^{n-p} (\sqrt{b})^p ,$$

$$U_{n+1}(a, b) = U_{n+1} = \sum_{p=0}^{n} \binom{2n+1-p}{p} 2^{n-p} (a/2 + \sqrt{b})^{n-p} (-\sqrt{b})^p .$$

Letting (Φ_n) be the generalized Fibonacci sequence of (R2), we may write the following formulas which are easily obtained by induction:

$$(a/2 - \sqrt{b})^{n-p} = \Phi_{n-p+1} - \Phi_{n-p}(a/2 + \sqrt{b}),$$
$$(a/2 + \sqrt{b})^{n-p} = \Phi_{n-p+1} - \Phi_{n-p}(a/2 - \sqrt{b}).$$

By substitutions in the previous formulas, we obtain

$$U_{n+1} = \sum_{p=0}^{n} \binom{2n+1-p}{p} 2^{n-p} (\sqrt{b})^p (\Phi_{n-p+1} - \Phi_{n-p}(a/2 + \sqrt{b})),$$

$$U_{n+1}=\sum_{p=0}^{n}\binom{2n+1-p}{p}2^{n-p}(-\sqrt{b})^p(\Phi_{n-p+1}-\Phi_{n-p}(a/2-\sqrt{b})),$$

and, summing both relations, we have

$$\begin{aligned}U_{n+1}=&\sum_{\substack{p\text{ even}\\ p\le n}}\binom{2n+1-p}{p}2^{n-p}(b)^{p/2}(\Phi_{n-p+1}-a/2\Phi_{n-p})\\ &-\sum_{\substack{p\text{ odd}\\ p\le n}}\binom{2n+1-p}{p}2^{n-p}(b)^{(p+1)/2}\Phi_{n-p}.\end{aligned}\tag{S1}$$

Since the associate recursion for (R2) is (R1), we have, symmetrically,

$$\begin{aligned}\Phi_{n+1}=&\sum_{\substack{p\text{ even}\\ p\le n}}\binom{2n+1-p}{p}2^{n-p}\left(\frac{\Delta}{4}\right)^{p/2}(U_{n-p+1}-a/2U_{n-p})\\ &-\sum_{\substack{p\text{ odd}\\ p\le n}}\binom{2n+1-p}{p}2^{n-p}\left(\frac{\Delta}{4}\right)^{(p+1)/2}U_{n-p},\end{aligned}\tag{S2}$$

because the fact that $4b\neq 0$, $4b$ being the discriminant of (R2), allows the same treatment for Lucas' formula for Φ_{n+1} as the one for U_{n+1}.

Remarks:

1. Do there exist recursions which are their own associates? (R1) will be so if and only if $b=\Delta/4 \Leftrightarrow b=(a^2)/8$. Therefore, a necessary and sufficient condition for a recursion to be its own associate is to assume the form

$$r_{n+2}-ar_{n+1}+(a^2)/8\,r_n=0,$$

where a is an arbitrary nonzero complex number. Its characteristic roots are $a\sqrt{2}(\sqrt{2}+1)/4$ and $a\sqrt{2}(\sqrt{2}-1)/4$. Within the first pair of parentheses is the greatest root of the Pell recurrence, $r_{n+2}-2r_{n+1}-r_n=0$, while within the second pair is the opposite of the remaining root of the Pell recurrence. This allows us to obtain sum formulas specific for Pell and Pell-Lucas numbers, thanks to (S1).

2. To any second-order linear recursion, we may also associate the auxiliary polynomial of its associate recursion. That is, to the recursion defined by $r_{n+2}-ar_{n+1}+br_n=0$, associate the polynomial $x^2-ax+\Delta/4$. With this meaning, it appears that the associate polynomial for the general second-order linear recursion has been mentioned in the literature at least once, because Richard André-Jeannin [2] proved the following orthogonality property (with our notations):

$$\int_{-a-2\sqrt{b}}^{-a+2\sqrt{b}}\sqrt{x^2+2ax+\Delta}\,U_n(a+x,b)U_p(a+x,b)\,dx=0$$

for $n\neq p$, and it is obvious that the polynomial under the radical is equal to $4p\left(-\frac{x}{2}\right)$, where $p(x)$ is the associate polynomial for the recursion (R1).

With a trivial change of variable, the orthogonality relation may be written as

$$\int_h^k \sqrt{p(x)}\, U_n(a-2x,b)\, U_p(a-2x,b)\, dx = 0$$

where h and k are the roots of $p(x)$: $a/2-\sqrt{b}$ and $a/2+\sqrt{b}$.

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AVERAGE NUMBER OF NODES IN BINARY DECISION DIAGRAMS OF FIBONACCI FUNCTIONS*

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(*Submitted January 1995*)

1. INTRODUCTION

A binary decision diagram (BDD) is a directed graph representation of a switching function $f(x_1, x_2, \ldots, x_n)$. Subfunctions of f correspond to nodes in the BDD; f itself is represented by a source node, i.e., a node with no incoming arcs. Attached to this node are two outgoing arcs, labeled 0 and 1, that go to descendent nodes representing $f(x_1, x_2, \ldots, 0)$ and $f(x_1, x_2, \ldots, 1)$, respectively. Attached to each of these nodes are descendent nodes, where x_{n-1} is replaced by 0 and 1, etc. This process is repeated until all variables are assigned values. The last assigned functions are a constant **0** and **1**, which correspond to sink nodes, i.e., nodes with no outgoing arcs. If two nodes represent the same function, they are merged into one node, and if the descendents of one node η are the same, η is removed. If $f = 1$ (0) for some assignment of values to $x_1, x_2, \ldots,$ and x_n, then there is a path in the BDD for f from the source node to the sink node **1** (**0**) for that assignment. Figure 1(a) shows the BDD of the OR function on four variables. As is usual, the arrows are omitted; all arcs are assumed to be directed down. As can be seen, there is a path from the source node to the node labeled **1** if and only if at least one variable is 1. Figure 1(b) shows the BDD of the AND function of four variables, which is the mirror image of the OR function BDD.

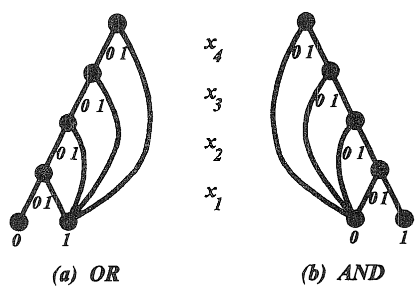

FIGURE 1. BDD's of the OR and AND Function on Four Variables

There is significant work on this topic dating back to 1959 [5]. In spite of this, there are few enumerations of nodes in BDD's of useful classes of functions. Symmetric functions, which are unchanged by a permutation of variables, have received some attention. The worst case number of nodes is known [3], [6], [7], as well as the average number of nodes [1].

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We demonstrate another class of functions and characterize its BDD. A *threshold function*, $f(x_1, x_2, \ldots, x_n)$, has the property that $f = 1$ if and only if $w_n x_n + w_{n-1}x_{n-1} + \cdots + w_1 x_1 \geq T$, where w_i and T are integers and the logic values, 0 and 1, of x_i are viewed as integers. The value of $w_n x_n + w_{n-1}x_{n-1} + \cdots + w_1 x_1$, for some assignment of values to $x_1, x_2, \ldots,$ and x_n, is called the *weighted sum*. A threshold function is completely specified by a *weight-threshold* vector $(w_n, w_{n-1}, \ldots, w_1; T)$. For example, the four-variable OR and AND functions have weight-threshold vectors $(1,1,1,1; T)$, where $T = 1$ and 4, respectively. A *Fibonacci function* is a threshold function with weight-threshold vector $(F_n, F_{n-1}, \ldots, F_2, F_1; T)$, where F_i is the i^{th} Fibonacci number and $0 < T < F_{n+2}$. For example, the Fibonacci functions associated with weight-threshold vectors $(3, 2, 1, 1; 1)$ and $(3, 2, 1, 1; 7)$ correspond to the OR and AND function, respectively. on four variables. The *BDD of a Fibonacci function* is a BDD in which a path from the source node to a sink node is a sequence of arcs associated with variables of descending weights. Figure 2 shows the BDD's of all of the other four-variable Fibonacci functions, which have a weight-threshold vector $(3, 2, 1, 1; T)$, for $1 < T < 7$; thus, Figures 1 and 2 represent the entire set of seven four-variable Fibonacci function BDD's.

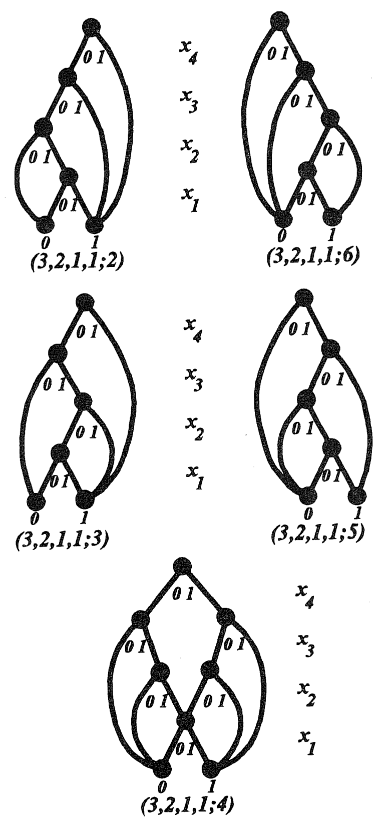

FIGURE 2. BDD's of Other Fibonacci Functions on Four Variables

The representation of a Fibonacci function by a BDD is related to the representation of integers by the Fibonacci number system, for which there exist many papers (see, e.g., [2], [4]). That is, every positive integer N can be represented as $N = \alpha_n F_n + \cdots + \alpha_2 F_2 + \alpha_1 F_1$, where F_i is a

Fibonacci number and $\alpha_i \in \{0, 1\}$. In a BDD, there is a path from the source node to **1** for all assignments of values to α_i, for $0 \le i \le n$, such that $N \ge T$.

2. STRUCTURE OF THE BDD'S OF FIBONACCI FUNCTIONS

In preparation for the calculations of the average number and variance of nodes in BDD's of Fibonacci functions, we consider the structure of such BDD's. Figure 3 shows how the structure near the source node depends on the threshold. Specifically, it shows that the destination of arcs emanating from the source node depends on the value of x_n. Figure 3a shows the *Type a* structure. As shown, if $0 < T \le F_n$, the arc corresponding to $x_n = 1$ goes to **1**. That is, for this range of T and this value of x_n, the weighted sum exceeds or equals the threshold, and $f = 1$. If $x_n = 0$, then the weighted sum exceeds or equals the threshold if and only if the Fibonacci function corresponding to the weight-threshold vector $(F_{n-1}, F_{n-2}, \dots, F_1; T)$ is 1. The latter is represented by a node that is the 0 descendent of the source node.

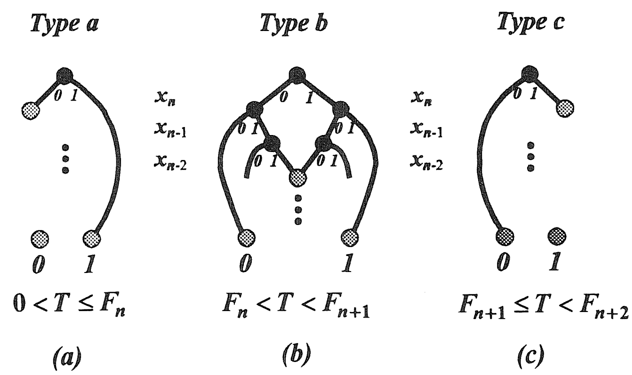

FIGURE 3. Structure of the BDD of a Fibonacci Function

A similar analysis of the case $F_{n+1} \le T < F_{n+2}$, which corresponds to a *Type c* structure, shows that there is mirror image symmetry with a Type a structure, as can be seen by comparing Figure 3(c) with 3(a).

Consider the remaining values of T, which correspond to a *Type b* structure. Figure 3(b) shows that, for this structure, both $x_n = 0$ and $x_n = 1$ yield nodes at the next lower level. If $x_n x_{n-1} = 11$, the weighted sum is at least $F_n + F_{n-1} = F_{n+1}$, and this equals or exceeds the threshold regardless of the values of the remaining variables. Thus, there is a path from the source node to **1** for $x_n x_{n-1} = 11$. If $x_n x_{n-1} = 00$, the weighted sum can be no greater than $F_{n-2} + F_{n-3} + \cdots + F_1 = F_n - 1$. Thus, the threshold is neither equaled nor exceeded, and there is a path from the source node to **0**. If $x_n x_{n-1} = 01$, the weighted sum ranges from a minimum of F_{n-1} to a maximum of $F_{n-1} + F_{n-2} + \cdots + F_1 = F_{n+1} - 1$, for which $f = 0$ and 1, respectively. It follows that there is a path from the source node to a non-sink node corresponding to $x_n x_{n-1} = 01$. A similar analysis shows that there is a non-sink node corresponding to $x_n x_{n-1} = 10$. Similarly, non-sink nodes exist for $x_n x_{n-1} x_{n-2} = 011$ and for $x_n x_{n-1} x_{n-2} = 100$. Indeed, since $F_{n-1} + F_{n-2} = F_n$, the weights are the same for the last two cases, and they correspond to the same node.

A fourth type of structure, the *Type d* structure, consists of a node that has as descendents the two sink nodes **0** and **1**. This represents the Fibonacci function with weight-threshold vector

$(1; 1)$. Indeed, all threshold functions contain this structure. As can be seen in Figures 1 and 2, it is part of all BDD's of Fibonacci functions on four variables.

Composing BDD's of Fibonacci Functions

Consider combining structures. If a BDD has a Type a structure, as shown in Figure 3(a), and the weight-threshold vector associated with the Fibonacci function of the source node is $(F_n, F_{n-1}, \ldots, F_1; T)$, where $0 < T \leq F_n$, then the node that is the 0 descendent of the source node corresponds to a Fibonacci function with weight-threshold vector $(F_{n-1}, F_{n-2}, \ldots, F_1; T)$. Further, the 0 descendent can also have a Type a structure, in which case the node at $x_n x_{n-1} = 00$ is associated with the weight-threshold vector $(F_{n-2}, F_{n-3}, \ldots, F_1; T)$. Indeed, this process can be repeated until the last variable, which has a Type d structure. Represent this composition as $a^i d$, for $i \geq 1$, and the set of all such compositions as aa^*d. Here, $a^* = \{\lambda, a, aa, aaa, \ldots\}$, where λ is the *null* structure. Thus, aa^*d represents the concatenation of one or more Type a structures followed by a Type d structure. By this convention, the right to left sequence in the string representation corresponds to the top to bottom sequence in the BDD. Such compositions occur only when $T = 1$, which is the OR function. For example, the BDD in Figure 1(a) is described by a^3d and corresponds to the weight-threshold vector $(3, 2, 1, 1; 1)$.

In a similar manner, repeated use of the Type c structure corresponds to a BDD described by $c^i d$, for $i \geq 1$, producing a mirror image of $a^i d$. Such compositions occur only when $T = F_{n+2} - 1$, which is the AND function. For example, the BDD in Figure 1(b) is described by $c^3 d$ and corresponds to the weight-threshold vector $(3, 2, 1, 1, 7)$.

Consider combining Types a and c. For example, let the source node have a Type a structure and its 0 descendent have a Type c structure. Thus, the 0 descendent of the source node is associated with weight-threshold vector $(F_{n-1}, F_{n-2}, \ldots, F_1; T_1)$, where $0 < T_1 = T \leq F_n$. But, because it is a Type c structure, we have $F_n \leq T_1 < F_{n+1}$. Since there is only one value of T_1 that satisfies both inequalities, it follows that $T = T_1 = F_n$. It follows that the weight-threshold vector of the 1 descendent of the 0 descendent of the source node is $(F_{n-2}, F_{n-3}, \ldots, F_1; F_{n-2})$, since $F_{n-2} = F_n - F_{n-1}$. Thus, this node has a Type a structure whose 0 descendent has a Type c structure, etc., until all variables are exhausted. The resulting compositions are described by $ac(ac)^*(a+\lambda)d$, where + is set union. A similar result occurs if the source node has a Type c structure, in which case the resulting compositions are described by $ca(ca)^*(\lambda + c)d$. These observations have important implications in the composition of the BDD's of Fibonacci functions.

- A BDD can consist of a sequence of one or more Type a structures followed by an alternating sequence of Type c and Type a structures, as described by $a^*(ca)^*(\lambda + c)d$. Similarly, a BDD can consist of a sequence of one or more Type c structures followed by an alternating sequence of Type a and Type c structures, as described by $c^*(ac)^*(a+\lambda)d$. As an example, see the BDD's in Figures 1 and 2 corresponding to thresholds $T = 1, 2, 3, 5, 6$, and 7.
- A "crest" pattern of the form shown in Figure 3(b) can only occur after a sequence of Type a structures exclusively or Type c structures exclusively. On the contrary, if both types occur, we have a situation as described immediately above, in which case, no crest can occur anywhere in the BDD.

Consider the composition of the BDD's of Fibonacci functions involving the crest pattern; i.e., Type b structures. Figure 4 shows how the BDD structure depends on T in the range $F_n < T < F_{n+1}$. Here, the top node of the crest pattern is the source node of the BDD. It is interesting how the structure changes at the boundary between ranges and that Fibonacci numbers define these boundaries. In the BDD for $T = F_n + F_{n-3}$ and $F_n + F_{n-2}$, the bottom node of the crest corresponds to a weight-threshold vector where the threshold is F_{n-3} and F_{n-2}, respectively. From the discussion above, this part of the BDD consists of a sequence of structures chosen alternatively as Type a and Type c. Again, the mirror image symmetry of the BDD's of Fibonacci functions is evident.

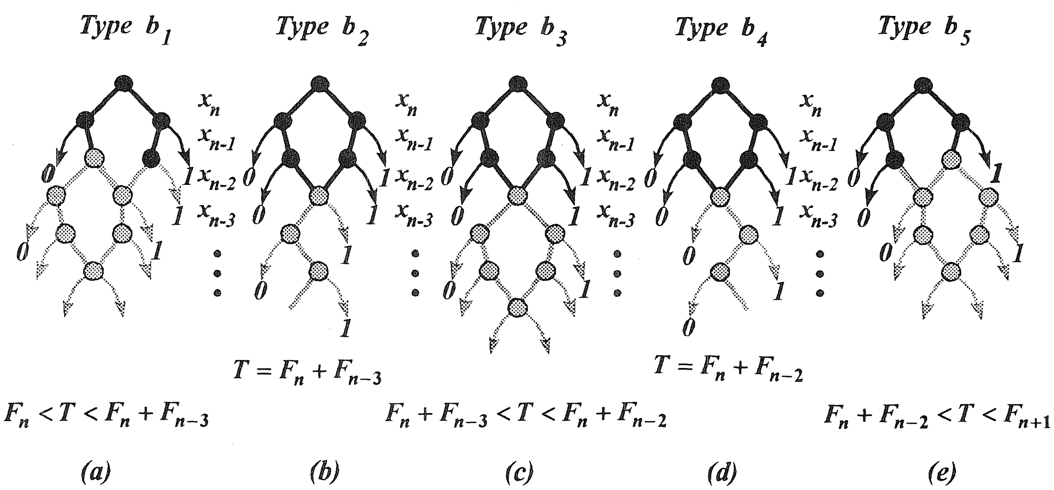

FIGURE 4. Structure of the BDD of a Fibonacci Function in the Range $F_n < T < F_{n+1}$

3. THE AVERAGE NUMBER OF NODES IN BDD'S OF FIBONACCI FUNCTIONS

Let $T(x, y)$ be the ordinary generating function for the number of BDD's of Fibonacci functions, where x tracks the number of variables and y tracks the number of nodes. Let $t_{n,i}$ be the number of BDD's of n-variable Fibonacci functions that have i nodes. From the results in the previous section, it follows that if $t_{n,i} > 0$, then $i \geq n+2$, since there is at least one node for every variable and two sink nodes **0** and **1**. Thus, a term in $T(x, y)$ is

$$T(x, y) = \cdots + x^n(t_{n,n+2}y^{n+2} + t_{n,n+4}y^{n+4} + \cdots) + \cdots. \tag{1}$$

Note that $t_{n,n+2i+1} = 0$ for $i = 1, 2, \ldots,$ since additional nodes beyond the minimum number $n+2$ occur because each crest pattern contributes two *additional* nodes to the node count. If we differentiate (1) with respect to y and set y equal to 1, the resulting coefficient of x^n is the total number of nodes in all BDD's of Fibonacci functions on n variables. Dividing by the number of BDD's of such functions yields the average number of nodes.

To derive $T(x, y)$, we use the classification given in Figure 3. That is,

$$T(x, y) = T_a(x, y) + T_b(x, y) + T_c(x, y) + xy^3, \tag{2}$$

where $T_a(x, y)$, $T_b(x, y)$, and $T_c(x, y)$ are the generating functions for Type a, b, and c structures, respectively, and xy^3 is the generating function for the Type d structure. By symmetry,

$$T_c(x, y) = T_a(x, y). \tag{3}$$

We can derive $T_a(x, y)$ by observing that there are two types of BDD's counted in $T_a(x, y)$—those that contain at least one crest pattern (but not at the very top, which are Type b structures) and those that do not. BDD's of the first type are enumerated by $x^i y^i T_b(x, y)$ for $i \geq 1$. Recall that the top crest pattern is preceded by a sequence of Type a structures. BDD's of the second type are enumerated by $ix^{i+1}y^{i+3}$ for $i \geq 1$. That is, this type of structure consists of a sequence of i Type a structures ending with a Type d structure or followed by Type c structures alternating with Type a structures ending with a Type d structure. The string representation for this is $a^i(ca)^*(\lambda + c)d$. The factor x^{i+1} counts the variables involved, and the factor y^{i+3} counts the nodes involved, including the two sink nodes **0** and **1**. Therefore,

$$T_a(x, y) = xyT_b(x, y) + x^2y^2T_b(x, y) + \cdots + x^i y^i T_b(x, y)$$
$$+ \cdots + x^2y^4 + 2x^3y^5 + \cdots + ix^{i+1}y^{i+3} + \cdots,$$

which can be written as

$$T_a(x, y) = \frac{xy}{1 - xy} T_b(x, y) + \frac{x^2y^4}{(1 - xy)^2}. \tag{4}$$

We can calculate $T_b(x, y)$ by observing that BDD's of Fibonacci functions containing a crest at the source node can be completed in three ways. Figure 4(c) shows that the bottom node of the crest is the top node of a Type b structure. The number of ways to choose a Type b structure is counted by the generating function $T_b(x, y)$. The contribution of the crest itself to the variable and node count is expressed as x^3y^5. Thus, the total contribution to the variable and node count is expressed as $x^3y^5T_b(x, y)$. Figures 4(b) and 4(d) show that the bottom node can also be the source node of a BDD with one node per variable expressed as $(ac)^*(a + \lambda)d$ and $(ca)^*(\lambda + c)d$, respectively. The contribution of these nodes is expressed as $2x^2y^4 + 2x^3y^5 + 2x^4y^6 + \cdots$. The coefficient 2 occurs because of the two ways this part of the BDD can occur [Figures 4(b) and 4(d)]. The superscript of x counts variables and the superscript of y counts nodes, including the two sink nodes **0** and **1**. The generating function for this power series is $2x^2y^4 / (1 - xy)$. A sub-BDD consisting of just the lowest variable and the three nodes, including two sink nodes **0** and **1** (i.e., a Type d structure) should also be included, and this is expressed as xy^3. Figures 4(a) and 4(e) show that more than one crest can also be cascaded so that each adjacent pair of crests share an arc and two nodes. In this case, the top BDD contributes two variables and four nodes. Since there are two ways for this to happen, the contribution is described by $2x^2y^4T_b(x, y)$. Considering all three ways to form a Type b BDD, we have

$$T_b(x, y) = x^3y^5\left[T_b(x, y) + xy^3 + \frac{2x^2y^4}{1 - xy}\right] + 2x^2y^4T_b(x, y). \tag{5}$$

Solving for $T_b(x, y)$ in (5) yields

$$T_b(x, y) = \frac{x^4y^8(1 + xy)}{(1 - xy)(1 - x^3y^5 - 2x^2y^4)}. \tag{6}$$

From (2), (3), (4), and (6), we can write

$$T(x, y) = \frac{x^3y^5 - 2x^3y^7 + xy^3}{(1-xy)^2(1-x^3y^5-2x^2y^4)}. \tag{7}$$

Recall that a typical term in (7) is given in (1). We can find the total number of nodes by differentiating (7) with respect to y and setting y to 1. Doing this yields

$$T(x) = \frac{3+2x}{(1-x-x^2)^2} - \frac{7+3x}{1-x-x^2} + \frac{1}{(1-x)^2} + \frac{3}{1-x}. \tag{8}$$

The number of n-variable BDD's is calculated as follows. There are as many BDD's as there are integer threshold functions from 1 to the largest threshold. The largest threshold is the same as the largest weighted sum, $1+1+2+3+\cdots+F_n = F_{n+2}-1$. Note that we exclude BDD's corresponding to $T=0$ and F_{n+2}, which are trivial. Therefore, the average number of nodes is the coefficient t_n of the power series expansion of (8) divided by $F_{n+2}-1$. Table 1 shows the average number of nodes as calculated in this way.

TABLE 1. The Average Number of Nodes in BDD's of Fibonacci Functions of n Variables

Number of Variables n	Average Number of Nodes	Standard Deviation on the Number of Nodes
1	3.000	0.000
2	4.000	0.000
3	5.000	0.000
4	6.286	0.700
5	7.667	0.943
6	9.200	1.327
7	10.818	1.585
8	12.519	1.853
9	14.273	2.049
10	16.070	2.224
11	17.897	2.354
12	19.745	2.462
13	21.608	2.543
14	23.481	2.609
15	25.361	2.659
∞	$1.8944\,n$	$0.2540\sqrt{n}$

Asymptotic Approximation

Consider now the average number of nodes in BDD's of Fibonacci functions when the number of variables is large. We can factor the quadratic denominators in the partial fraction expansion (8), forming a partial fraction expansion in which denominators involve linear factors only. That is, we can rewrite (8) as

$$T(x)=\frac{\frac{11+5\sqrt{5}}{10}}{\left(1-\frac{2}{\sqrt{5}-1}x\right)^2}-\frac{\frac{61+31\sqrt{5}}{10\sqrt{5}}}{\left(1-\frac{2}{\sqrt{5}-1}x\right)}+\cdots, \tag{9}$$

where $\cdots$ represents terms whose contributions to t_n, the coefficient of x_n in the power series expansion of $T(x)$, are negligible for large n compared to the contributions from the terms shown. Specifically, missing terms have denominators that are powers of $(1+(2/\sqrt{5}+1)x)$ and $(1+x)$. Indeed, the second term in (9) is negligible for large n compared to the first term; we include it for a reason that will become clear in the next section. The contribution to t_n from these terms is

$$\left(\frac{11+5\sqrt{5}}{10}(n+1)-\frac{61+31\sqrt{5}}{10\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}-1}\right)^n. \tag{10}$$

The number of BDD's of n-variable Fibonacci functions, $F_{n+2}-1$, is approximated by

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}, \tag{11}$$

when n is large. Dividing (10) by (11) yields the following asymptotic approximation to the average number of nodes in BDD's of Fibonacci functions on n variables,

$$\frac{5+2\sqrt{5}}{5}n-\frac{2+6\sqrt{5}}{5}\approx 1.8944n-3.0832, \tag{12}$$

which is asymptotic to $1.8944n$, for large n. As can be seen from Table 1, 3.0832 is significant for the values of n shown here.

4. THE VARIANCE OF THE NUMBER OF NODES IN BDD'S OF FIBONACCI FUNCTIONS

We can calculate the variance on the number of nodes in BDD's of Fibonacci functions using the generating function for the distribution of nodes given in (7). That is, if X is a random variable, then the variance $\sigma^2(X)$ of X is given as

$$\sigma^2(X)=E(X^2)-E^2(X),$$

where $E(X^2)$ is the expected value of X^2 and $E(X)$ is the expected value of X. $E(X)$ was calculated in the previous section. $E(X^2)$ can be calculated by differentiating (7) with respect to y, multiplying by y, differentiating with respect to y again, and setting y to 1. In the resulting expression, the coefficient of x^n is ΣX^2. Dividing this by the number of BDD's of Fibonacci functions yields $E(X^2)$. Differentiating (7) with respect to y, multiplying by y, differentiating with respect to y again, and setting y to 1 yields

$$\frac{16+10x}{(1-x-x^2)^3}-\frac{49+16x}{(1-x-x^2)^2}+\frac{49+25x}{1-x-x^2}+\frac{6}{(1-x)^3}-\frac{1}{(1-x)^2}-\frac{23}{1-x}+\frac{2}{1+x}. \tag{13}$$

The coefficient of x^n in the power series expansion of (13) is decreased by $E^2(X)$ and the result divided by the number of BDD's of Fibonacci functions on n variables, $F_{n+2}-1$, to get the variance on the number of nodes for n-variable BDD's of Fibonacci functions. This yields $\sigma^2(X)$. Table 1 shows the standard deviation, $\sigma(X)$, of the number of nodes, as calculated in this way.

Asymptotic Approximation

Consider the standard deviation on the number of nodes in BDD's of Fibonacci functions when the number of variables is large. We can rewrite (13) as

$$\frac{\frac{47+21\sqrt{5}}{5\sqrt{5}}}{\left(1-\frac{2}{\sqrt{5}-1}x\right)^3}-\frac{\frac{691+277\sqrt{5}}{50}}{\left(1-\frac{2}{\sqrt{5}-1}x\right)^2}+\frac{\frac{2131+881\sqrt{5}}{50\sqrt{5}}}{\left(1-\frac{2}{\sqrt{5}-1}x\right)}+\cdots, \tag{14}$$

where the contribution to ΣX^2 for large n from other terms is negligible compared to the contribution from the terms shown. The contribution of these three terms is indeed

$$\left[\frac{47+21\sqrt{5}}{10\sqrt{5}}(n^2+3n+2)-\frac{691+277\sqrt{5}}{50}(n+1)+\frac{2131+881\sqrt{5}}{50\sqrt{5}}\right]\left[\frac{2}{\sqrt{5}-1}\right]^n. \tag{15}$$

Dividing this result by (11), an approximation to the number of BDD's of Fibonacci functions, yields $E(X^2)$. Subtracting from this the square of the average number of BDD's of Fibonacci functions, as given in (12), yields the following asymptotic approximation to the variance on the number of nodes in BDD's of Fibonacci functions

$$\frac{100-44\sqrt{5}}{25}n+\frac{228-28\sqrt{5}}{25}\approx 0.0645n+6.6156. \tag{16}$$

Note that there is no n^2 term in (16); the n^2 term in $E(X^2)$ has been canceled by an identical term in $E^2(X)$. Therefore, terms of order n are needed in the asymptotic expressions for $E(X^2)$ and $E^2(X)$. This is why we included in (10) and (12) an asymptotically insignificant term.

Equation (16) is an expression for $\sigma^2(X)$. The standard deviation $\sigma(X)$ is then

$$\sqrt{\frac{100-44\sqrt{5}}{25}n+\frac{228-28\sqrt{5}}{25}}\approx\sqrt{0.0645n+6.6156}, \tag{17}$$

which is asymptotic to $0.2540\sqrt{n}$, for large n. As can be seen from Table 1, 6.6156 is significant for the values of n shown.

5. DISTRIBUTION OF THE NUMBER OF NODES IN BDD'S OF FIBONACCI FUNCTIONS

Figure 5 shows the distribution of nodes in the BDD's of Fibonacci functions, as computed from (7). Here, the number of variables and the number of nodes in BDD's are plotted horizontally, while the number of Fibonacci functions is plotted vertically. A vertical line represents the number of Fibonacci functions whose BDD's have the number of variables and the number of nodes as specified by the coordinates in the horizontal plane. The vertical axis shows the *log* of the number of functions. Note the linear increase in the log of number of functions with the number of nodes in BDD's for a fixed number of variables, which corresponds to an exponential increase in the number of functions.

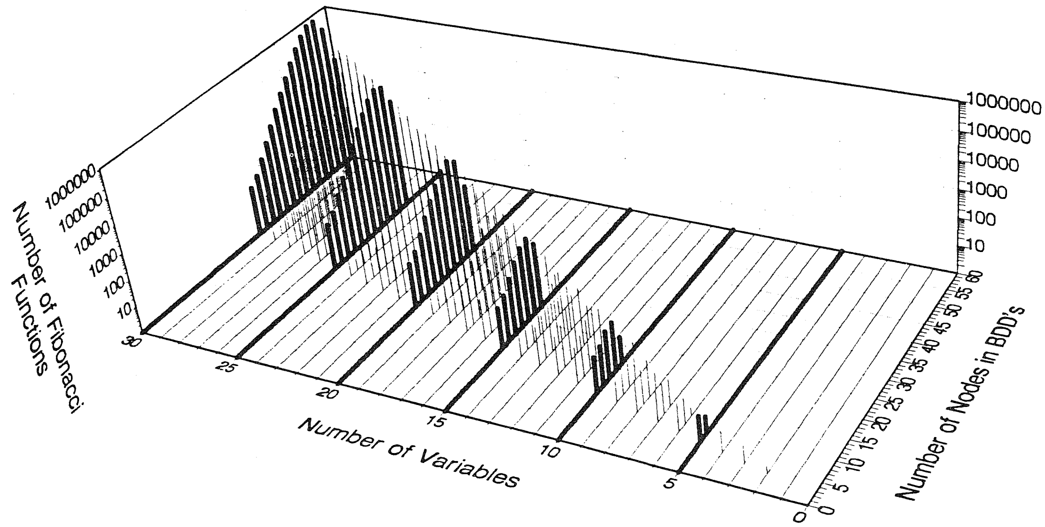

FIGURE 5. Distribution of Fibonacci Functions by Nodes and Variables

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ON TRIANGULAR AND BAKER'S MAPS WITH GOLDEN MEAN AS THE PARAMETER VALUE

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(Submitted January 1995)

1. INTRODUCTION

We discuss the triangular map and also Baker's map [2], [4] with the parameter μ chosen with the value of the golden mean: $(1+\sqrt{5})/2 \approx 1.618$. For an arbitrary parameter value in the range $0 \le \mu \le 2$, the starting graph $(x, f(x))$ in the range $0 \le x \le 1$ is a line segment for a triangular map, and two line segments for Baker's map (see Fig. 1 and Fig. 4). We are interested in the graph of $f^{[n]}$, where $n > 1$. Since the starting graph contains a set of line segments, the proceeding graphs $(x, f^{[n]}(x))$ are then obtained from iterates of the beginning line segments in the starting graph. It is therefore important to discuss iterating these two maps on line segments. Since these two maps are piecewise linear maps, it can then be shown that the graph of $f^{[n]}$ is a composition of line segments (see Fig. 3 and Fig. 6). These two maps are simple for μ in the range $0 \le \mu \le 1$ because the number of line segments does not increase under the action of mapping; the graph of $f^{[n]}$ is therefore simple. Yet, they are often complicated in the range $\mu > 1$, as the action of mapping on starting line segments will generate more line segments of which the lengths in general are different. The graph of $f^{[n]}$ is then a set of line segments with irregular shape which becomes very complicated when n is large. It is then difficult to determine the graph of $f^{[n]}$. However, we can show that when μ is chosen with specific values, for instance, the golden mean, the graphs $(x, f^{[n]}(x))$ are again simple. There are only a few types of line segments in each graph and, interestingly, the numbers of line segments of the graphs are those of the Fibonacci numbers. Nature shows that Fibonacci numbers occur quite frequently in various areas; therefore, it is interesting to know that Fibonacci numbers and, in fact, Fibonacci numbers of degree m, can be generated from a simple dynamical system [3]. In this work, we contain some reviews of [3], show the similarity of these two maps when a specific parameter value is chosen, derive geometrically a well-known identity in Fibonacci numbers, and show that some invariant sequences can be obtained.

2. THE TRIANGULAR MAP WITH $\mu = (1+\sqrt{5})/2$

First, we discuss the general triangular T_μ map, which is defined by

$$T_\mu(x) = 1 - \mu|x|, \tag{1}$$

or

$$x_{n+1} = 1 - \mu|x_n|, \tag{2}$$

where μ is the parameter. We restrict the ranges to: $-1 \le x \le 1$ and $0 \le \mu \le 2$, so that T_μ maps from the interval $[-1, 1]$ to $[-1, 1]$. Figure 1 shows three graphs of T_μ for, respectively, $\mu = 0.6, 1,$

and $(1+\sqrt{5})/2$. We define $x_1 \equiv \mathrm{T}_\mu(x)$ as the first iterate of x for T_μ, and $x_n = \mathrm{T}_\mu(x_{n-1}) \equiv \mathrm{T}_\mu^{[n]}(x)$ as the n^{th} iterate of x for T_μ. Since all the graphs $(x, \mathrm{T}_\mu^{[n]}(x))$ are symmetrical for $x>0$ and $x<0$, we henceforth consider these graphs in only the region of $x \geq 0$. The starting graph, $\mathrm{T}_\mu(x) = 1-\mu x$, in the range $x \geq 0$ is then a line segment from point (0, 1) to point $(1, 1-\mu)$ with slope $-\mu$. We call this the **starting line segment**. Iterating this map on this starting line segment then generates all the proceeding $\mathrm{T}_\mu^{[n]}$ graphs.

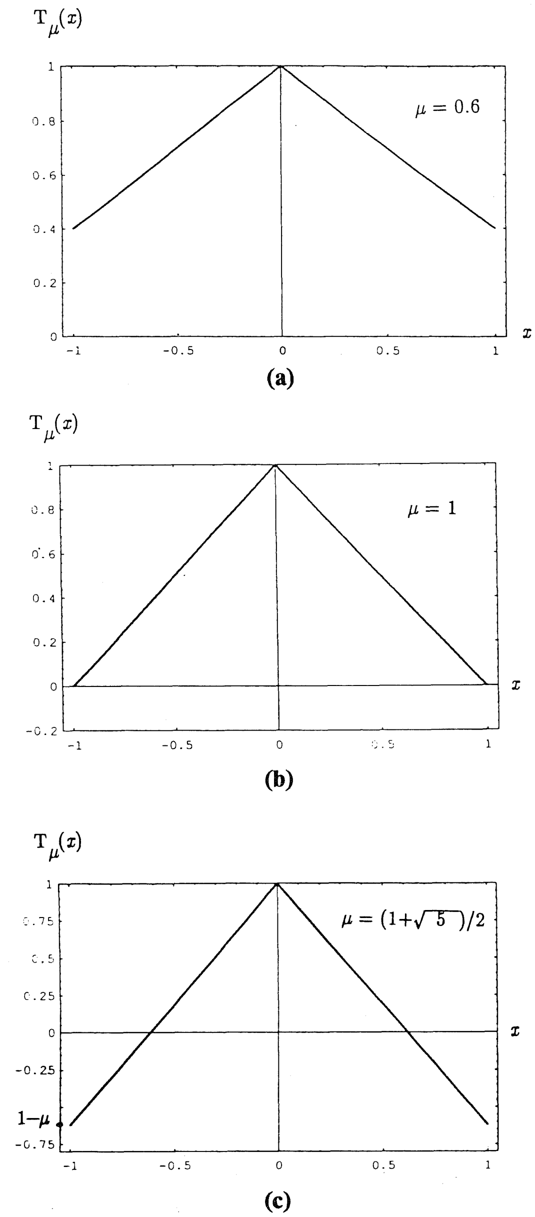

FIGURE 1

For a general discussion on an iterate of a line segment, we consider an arbitrary line segment described by $y = g(x) = a + bx$, where $0 \le x \le 1$ and $-1 \le g(x) \le 1$. The upper piece of this line segment, for which $g(x) \ge 0$, after one iteration, goes to a line segment described by $y = \mathrm{T}_\mu(g(x)) = 1 - a\mu - b\mu x$. Hence, the slope and the length of the upper piece have been rescaled, the sign of the slope alters as well; the slope rescaling factor is seen to be $-\mu$. The lower piece, for which $g(x) \le 0$, goes to another line segment described by $y = \mathrm{T}_\mu(g(x)) = 1 + a\mu + b\mu x$. Hence, the slope and the length of the lower piece have been rescaled, the sign of the slope is not altered; the slope rescaling factor is μ. Since the slope rescaling factors are different for these two pieces, a line segment after one iteration will be folded into two connected line segments if this line segment intersects the x-axis. A line segment that does not intersect the x-axis will only change its slope and length but not be folded. These are useful in graphical analysis of iterates.

For μ in the range $0 \le \mu < 1$, the T_μ map is simple, as there is one stable fixed point $x^* = 1/(1+\mu)$, and the basin of attraction of x^* consists of all $x \in [-1, 1]$, hence the graph of $\mathrm{T}_\mu^{[n]}$, as $n \to \infty$, will finally approach a flat line segment with height x^*. Or, we may see this from iterates of the starting line segment. Since the starting line segment does not intersect the x-axis and the slope rescaling factor is $-\mu$ for which $|\mu| < 1$, the starting line segment under iterations then remains a line segment and getting flatter but with one point fixed. The graph of $\mathrm{T}_\mu^{[n]}$, as $n \to \infty$, then approaches a flat line segment with a height which can only be of the value of the fixed point, that is, x^*. This map with $\mu < 1$ is therefore simple.

For $\mu = 1$, we have $\mathrm{T}_\mu(x) = 1 - x$ and $\mathrm{T}_\mu^{[2]}(x) = x$, hence $\{x, 1-x\}$ is a 2-cycle for each x, and we have, therefore, only two shapes which are the graphs $(x, \mathrm{T}_\mu(x))$ and $(x, \mathrm{T}_\mu^{[2]}(x))$. Or, since the starting line segment, with slope -1, does not pass through the x-axis and the slope rescaling factor is -1; therefore, the starting line segment under iterations remains a line segment but with slope 1 and -1, alternatively. This map with $\mu = 1$ is also a simple map.

For $1 < \mu \le 2$, there exist no stable fixed points for T_μ, $\mathrm{T}_\mu^{[2]}$, and in fact for $\mathrm{T}_\mu^{[n]}$ with n any positive integer. This is because the |slope| at each fixed point of $\mathrm{T}_\mu^{[n]}$ equals μ^n which is greater than one; hence, there are no stable fixed points for $\mathrm{T}_\mu^{[n]}$. In such a region without any stable cycles, the iterative behaviors in general are complicated. Indeed, as the starting line segment now intersects the x-axis, the action of mapping will keep on folding line segments and, therefore, producing more and more line segments of which the lengths in general are different; thus, the line segments in each graph are of many types. Each graph is then the connection of line segments of many types and, therefore, has an irregular zigzag shape. The complication of the graph $\mathrm{T}_\mu^{[n]}$ increases with n, and it is hard to predict what the final result will be. Interestingly, there are cases that are easier to analyze. We may consider focusing on some particular values of μ such that the iterative behaviors are simple.

To choose the proper values of μ in the range $1 < \mu \le 2$, the consideration is upon the particular point of $x = 0$ at which the function $\mathrm{T}_\mu(x)$ has a maximum height of 1. We require that this point be a periodic point of the map. The orbit of $x = 0$ is 0, $\mathrm{T}_\mu(0)$, $\mathrm{T}_\mu^{[2]}(0)$, $\mathrm{T}_\mu^{[3]}(0)$, ... etc. We can easily see that $\mathrm{T}_\mu^{[2]}(0) = 1 - \mu$ and $\mathrm{T}_\mu^{[3]}(0) = 1 - \mu(\mu - 1) = 1 + \mu - \mu^2$. The required parameter value for $x = 0$ becoming a period-2 point is determined from the equation $\mathrm{T}_\mu^{[2]}(0) = 0$. The solution is $\mu = 1$. As discussed above, it is a simple map in this case. Requiring $x = 0$ to be a period-3 point, we should have $\mathrm{T}_\mu^{[3]}(0) = 0$; that is, $\mu^2 - \mu - 1 = 0$. For $\mu > 1$, the solution is $\mu = (1+\sqrt{5})/2$, which is the well-known golden mean. With these basic arguments, we then have the following results when $\mu = (1+\sqrt{5})/2$.

Proposition 2.1: The point $x = 0$ is a period-3 point of the triangular T_μ map.

Proof: This is obvious, as we see that the 3-cycle is $\{0, 1, 1-\mu\}$; $1-\mu = -1/\mu$.

In what follows, we will discuss the graph $y = \mathrm{T}_\mu^{[n]}(x)$ in the range $0 \le x \le 1$. It is important to discuss an iterate of a line segment. We first define two types of line segments. We denote by L a **long line segment** connecting points $(x_1, 1-\mu)$ and $(x_2, 1)$, where $0 \le x_1,\ x_2 \le 1$ and by S a **short line segment** connecting points $(x_3, 0)$ and $(x_4, 1)$, where $0 \le x_3,\ x_4 \le 1$. The four graphs of Figure 2 show some examples of line segments of these two types, where the subscripts + and – label line segments with positive and negative slope, respectively.

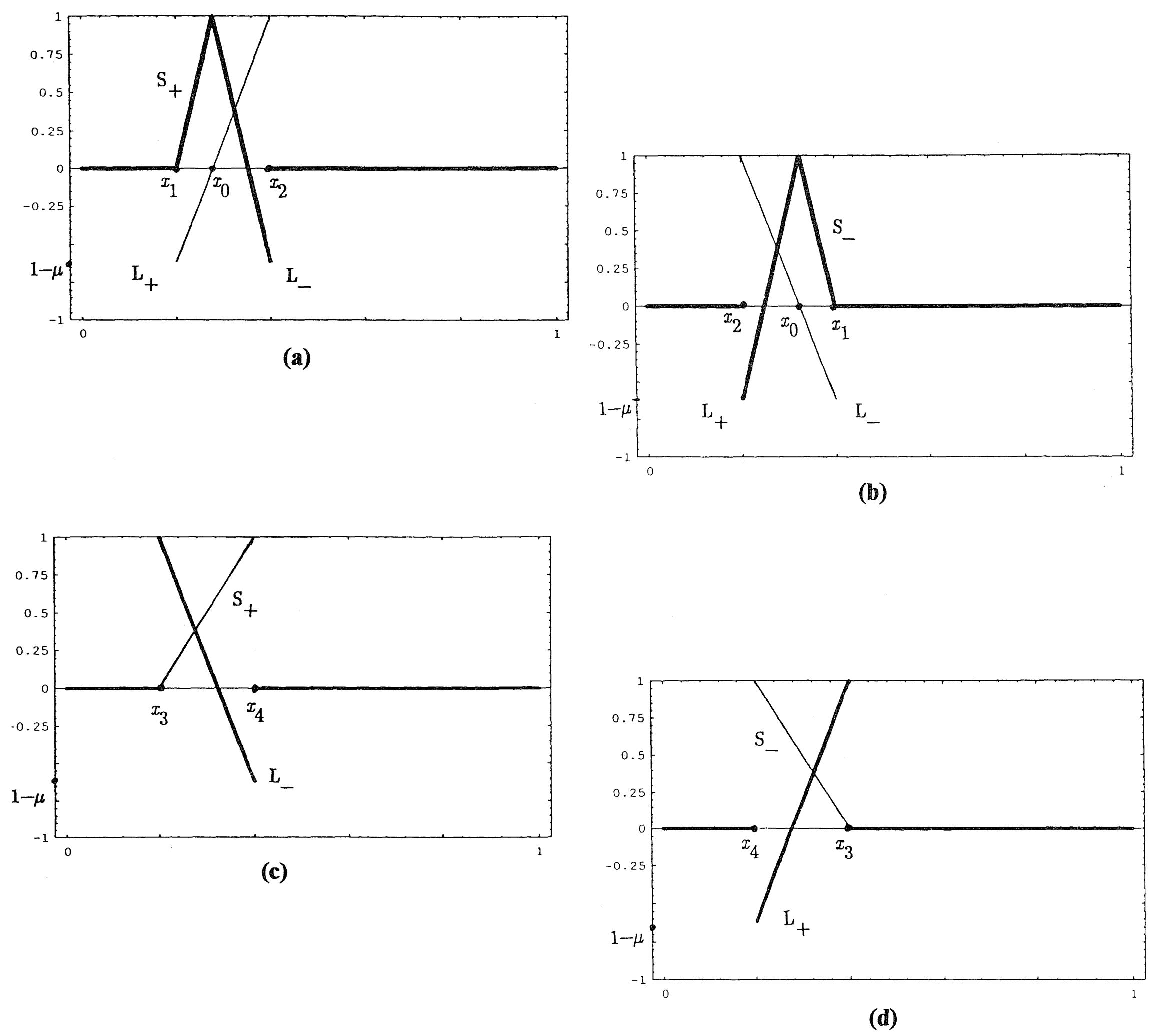

FIGURE 2

In fact, as will be shown, we need only consider an iterate of line segments of type L and type S; hence, we now discuss $T_\mu(L(x))$ and $T_\mu(S(x))$, where $L(x)$ and $S(x)$ are linear functions whose graphs are **long** and **short** segments. Using $T_\mu(0)=1$, $T_\mu(1)=1-\mu$, and $T_\mu(1-\mu)=0$, Figure 2(a) then shows that a line segment L_+ connecting points $(x_1, 1-\mu)$ and $(x_2, 1)$, after one iteration, is folded into two line segments, of which one connects points $(x_1, 0)$ and $(x_0, 1)$, i.e., an S_+, and the other connects points $(x_0, 1)$ and $(x_2, 1-\mu)$, i.e., an L_-, where $(x_0, 0)$ is the intersection point of the line segment L_+ with the x-axis. Therefore, an L_+ after one iteration goes to an S_+ and an L_-, we denote this by $T_\mu: L_+ \to S_+L_-$. The graphs of Figure 2(b), 2(c), and 2(d) show an iterate of a line segment of type L_-, S_+, and S_-, respectively. From these, we conclude that the action of this map on line segments of these four types is described as:

$$\begin{aligned} T_\mu: L_+ &\to S_+L_- \\ L_- &\to L_+S_- \\ S_+ &\to L_- \\ S_- &\to L_+ \end{aligned} \tag{3}$$

T_μ then acts as a discrete map for L and S. If we are only to count the number of line segments in a graph—we need not distinguish L_+ from L_-, S_+ from S_-, and LS from SL—then (3) can be expressed more simply as:

$$\begin{aligned} T_\mu: L &\to LS \\ S &\to L \end{aligned} \tag{4}$$

From (4), we have the following results.

Proposition 2.2: The graph of $y=T_\mu^{[n]}(x)$ in the range $0 \le x \le 1$ contains line segments of only two types, type L and type S, and the total number of these line segments is F_{n+1}.

Proof: Since the starting graph is a line segment connecting points $(0, 1)$ and $(1, 1-\mu)$ [see Fig. 1(c)], it is thus a line segment of type L. From (4), we see that, starting from an L, line segments generated from iterates of that are, therefore, only of two types: type L and type S. From (4), we also see that line segments of type L and S, respectively, are similar to those rabbits of type large and small in the original Fibonacci problem; hence, the numbers of line segments of all the graphs $T_\mu^{[n]}$ would be those of the Fibonacci numbers. Therefore, we have shown an interpretation of the Fibonacci sequence from the point of view of a simple iterated map. Although we start from a functional map T_μ on a finite interval of x, however, if we take line segments as the entities, then T_μ acts as a discrete map for these entities, and the mechanism of generating line segments from the action of this discrete map is now the same as the breeding of the Fibonacci rabbits. We now let $L(n)$ and $S(n)$ represent, respectively, the numbers of L's and S's in the graph of $T_\mu^{[n]}$. Then, from (4), we have

$$\begin{aligned} L(n) &= L(n-1)+S(n-1), \\ S(n) &= L(n-1). \end{aligned} \tag{5}$$

Equation (5) shows that $L(n)=L(n-1)+L(n-2)$ and $S(n)=S(n-1)+S(n-2)$. Thus, both sequences $\{L(n)\}$ and $\{S(n)\}$ are the Fibonacci-type sequences. Since we start from an L with slope $-\mu$, we have $L(1) = 1$ and $S(1) = 0$. According to (5), we then have $L(n) = F_n$ and $S(n) = F_{n-1}$, where F_n is the n^{th} Fibonacci number, and the |slope| of each line segment in the graph of $T_\mu^{[n]}$ is μ^n. Therefore, the total number of line segments in the graph of $T_\mu^{[n]}$ is $L(n) + S(n) = F_n + F_{n-1} = F_{n+1}$. Figure 3 shows the graph of $T_\mu^{[5]}$, from which we can count the number of line segments as being $F_6 = 8$.

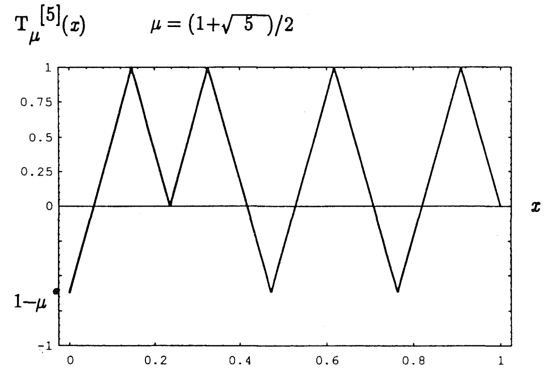

FIGURE 3

In [1] there is an interesting theorem stating that $\lim_{n\to\infty} \log C(n, \mu)/n = \log \mu$, where $C(n, \mu)$ is the number of line segments of the graph $y = f^{[n]}(x, \mu)$ and μ is an arbitrary parameter value. In our case, $C(n, \mu) = F_{n+1}$ and $\mu = (1+\sqrt{5})/2$, so using the well-known formula $F_n = (\mu^n - (1-\mu)^n)/\sqrt{5}$ we can easily calculate that $\lim_{n\to\infty} \log C(n, \mu)/n$ is indeed the value $\log \mu$.

Proposition 2.3: A simple identity, $\mu F_n + F_{n-1} = \mu^n$.

Proof: In the graph of $T_\mu^{[n]}$, we have F_n long line segments and F_{n-1} short line segments, and the |slope| of each line segment is μ^n. We denote by $d(L, n)$ and $d(S, n)$, respectively, the projection length of a line segment of type L and type S on the x-axis in the graph of $T_\mu^{[n]}$. We then have $d(L, n) = (\mu)^{1-n}$ and $d(S, n) = (\mu)^{-n}$. Since the total projection length of these F_{n+1} line segments in the x-axis should be 1, we have

$$F_n d(L, n) + F_{n-1} d(S, n) = 1 \tag{6}$$

or

$$\mu F_n + F_{n-1} = \mu^n. \tag{7}$$

This is a well-known identity in Fibonacci numbers; here we have derived it geometrically.

Proposition 2.4: There are three infinite LS sequences that are invariant under three iterations of the map (3).

Proof: We consider the shape of the graph of $y = T_\mu^{[n]}(x)$ in the range $0 \le x \le 1$. Since all the shapes are from the connection of line segments of type L_+, L_-, S_+, and S_-; therefore, we can describe each shape in terms of an LS sequence. We let $LS[T^n]$ represent the LS sequence describing the shape of the graph of $y = \mathrm{T}_{\mu}^{[n]}(x)$ in the range $0 \le x \le 1$. The starting graph is simply L_-. From (3), we have:

$$\begin{aligned}
&\mathrm{LS}[\mathrm{T}^1] = \mathrm{L}_- \\
&\mathrm{LS}[\mathrm{T}^2] = \mathrm{L}_+\mathrm{S}_- \\
&\mathrm{LS}[\mathrm{T}^3] = \mathrm{S}_+\mathrm{L}_-\mathrm{L}_+ \\
&\mathrm{LS}[\mathrm{T}^4] = \mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_- \\
&\mathrm{LS}[\mathrm{T}^5] = \mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_- \\
&\mathrm{LS}[\mathrm{T}^6] = \mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+ \\
&\dots \\
&\mathrm{LS}[\mathrm{T}^n] = \mathrm{LS}[\mathrm{T}^{n-3}]\mathrm{LS}[\mathrm{T}^{n-2}]\mathrm{LS}[\mathrm{T}^{n-4}]\mathrm{LS}[\mathrm{T}^{n-3}]
\end{aligned} \tag{8}$$

The last formula in (8) is for general n and can be proved easily by induction: starting from $n = 5$, we have $\mathrm{LS}[\mathrm{T}^5] = \mathrm{LS}[\mathrm{T}^2]\mathrm{LS}[\mathrm{T}^3]\mathrm{LS}[\mathrm{T}^1]\mathrm{LS}[\mathrm{T}^2]$, and then, after one iteration, we have $\mathrm{LS}[\mathrm{T}^6] = \mathrm{LS}[\mathrm{T}^3]\mathrm{LS}[\mathrm{T}^4]\mathrm{LS}[\mathrm{T}^2]\mathrm{LS}[\mathrm{T}^3]$, ... etc. The length of the $\mathrm{LS}[\mathrm{T}^n]$ sequence is F_{n+1}. Equation (8) shows that, after three iterations, an LS sequence will get longer but the original LS sequence remains. By taking $n \to \infty$, we then have an infinite LS sequence which is invariant after three iterations of the map (3), since from (8) we have

$$\lim_{n\to\infty} \mathrm{LS}[\mathrm{T}^n] = \lim_{n\to\infty} \mathrm{LS}[\mathrm{T}^{n-3}].$$

We denote by $\{\mathrm{T}_1^\infty\}$ the first invariant infinite LS sequence obtained from iterates of an L_-, that is,

$$\{\mathrm{T}_1^\infty\} = \lim_{n\to\infty} \mathrm{T}_{\mu}^{[3n]}(\mathrm{L}_-).$$

We see that $\{\mathrm{T}_1^\infty\}$ is invariant after three iterations of the map (3). There are two other invariant infinite sequences which we denote by $\{\mathrm{T}_2^\infty\}$ and $\{\mathrm{T}_3^\infty\}$, where $\{\mathrm{T}_2^\infty\}$ is obtained from an iterate of $\{\mathrm{T}_1^\infty\}$, i.e., $\{\mathrm{T}_2^\infty\} = \mathrm{T}_\mu\{\mathrm{T}_1^\infty\}$, and $\{\mathrm{T}_3^\infty\}$ is from an iterate of $\{\mathrm{T}_2^\infty\}$, i.e., $\{\mathrm{T}_3^\infty\} = \mathrm{T}_\mu\{\mathrm{T}_2^\infty\} = \mathrm{T}_\mu^{[2]}\{\mathrm{T}_1^\infty\}$. Therefore, there are at least three infinite LS sequences that are invariant after three iterations of the map (3). Since

$$\{\mathrm{T}_2^\infty\} = \mathrm{T}_\mu\{\mathrm{T}_1^\infty\} = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n+1]}(\mathrm{L}_-) = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n]}(\mathrm{L}_+\mathrm{S}_-) = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n]}(\mathrm{L}_+),$$

this means that $\{\mathrm{T}_2^\infty\}$ can be obtained from iterates of an L_+. Finally, since

$$\{\mathrm{T}_3^\infty\} = \mathrm{T}_\mu\{\mathrm{T}_2^\infty\} = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n+1]}(\mathrm{L}_+) = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n]}(\mathrm{S}_+\mathrm{L}_-) = \lim_{n\to\infty} \mathrm{T}_\mu^{[3n]}(\mathrm{S}_+),$$

this means that $\{\mathrm{T}_3^\infty\}$ can be obtained from iterates of an S_+. Therefore, from the first few iterates of L_+, L_-, and S_+, we have the following first few elements of these three infinite sequences:

$$\begin{aligned}
\{\mathrm{T}_1^\infty\} &= \mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_- \dots \\
\{\mathrm{T}_2^\infty\} &= \mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+ \dots \\
\{\mathrm{T}_3^\infty\} &= \mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-\mathrm{S}_+ \dots
\end{aligned} \tag{9}$$

There is no invariant sequence with S_- as the first element since, after three iterations, S_- goes to $\mathrm{L}_-\mathrm{L}_+\mathrm{S}_-$, the first element is now L_- instead of S_-. If a sequence were an invariant sequence, its length must be infinite and its first element can only be an L_+ or an L_- or an S_+, and that would just correspond to the infinite sequences $\{T_1^\infty\}$, $\{T_2^\infty\}$, and $\{T_3^\infty\}$; hence, there are only three infinite LS sequences that are invariant after three iterations of the map (3). As a result, to arrange LS sequences that are invariant after three iterations of the map (3), the L_+'s, L_-'s, S_+'s, and S_-'s should be arranged according to the orders described in (9). This interesting phenomenon may have applications in genetics. The general rule for deciding the n^{th} entry in each of these three sequences is complicated; we will discuss this in the next section, where we treat a similar but simpler case.

3. BAKER'S MAP WITH $\mu = (1+\sqrt{5})/2$

We now consider the easier Baker B_μ map, which is defined by

$$B_\mu(x) = \begin{cases} \mu x & \text{for } 0 \le x \le 1/2, \\ \mu(x-1/2) & \text{for } 1/2 < x \le 1, \end{cases} \tag{10}$$

where μ is the parameter. We restrict the ranges to $0 \le \mu \le 2$ and $0 \le x \le 1$, so that B_μ maps from the interval [0, 1] to [0, 1]. Figure 4 shows the graph of B_μ for $\mu = (1+\sqrt{5})/2$.

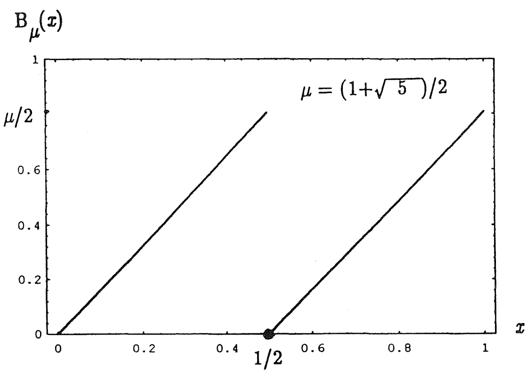

FIGURE 4

The interesting point of x now is $x = 1/2$. We require that $x = 1/2$ be a period-2 point of the map. The parameter value in this condition is easily determined to be $\mu = (1+\sqrt{5})/2$. Hence, μ is again the value of the golden mean. Using this parameter value, we have the following results.

Proposition 3.1: The point $x = 1/2$ is a period-2 point of the B_μ map.

Proof: We easily see that the 2-cycle is $\{1/2, \mu/2\}$ when $\mu = (1+\sqrt{5})/2$.

The starting graph in the range $0 \le x \le 1$ is $y = B_\mu(x)$ (see Fig. 4). It consists of two parallel line segments: one connecting points (0, 0) and $(1/2, \mu/2)$; the other connecting points $(1/2, 0)$ and $(1, \mu/2)$. We denote by L a **long line** segment connecting points $(x_1, 0)$ and $(x_2, \mu/2)$, where $x_2 > x_1$. and by S a **short line** segment connecting points $(x_3, 0)$ and $(x_4, 1/2)$, where $x_4 > x_3$. Figure 5 shows some examples of line segments of these two types.

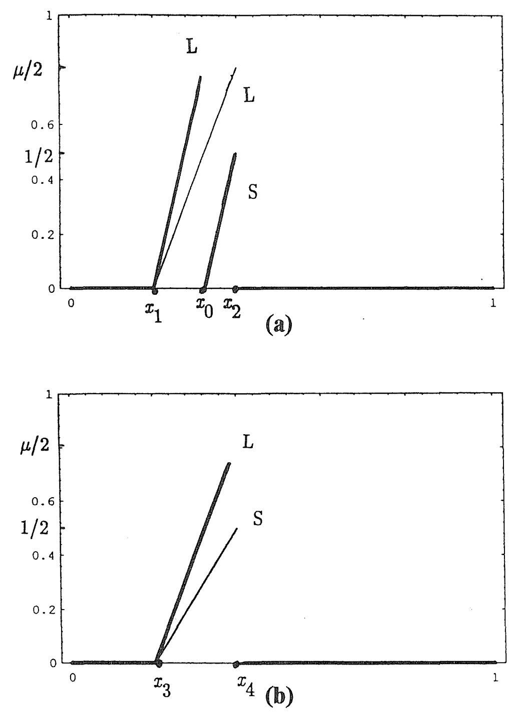

FIGURE 5

The graph of $y = \mathrm{B}_\mu(x)$ then contains two parallel line segments of type L with slope μ. Using $\mathrm{B}_\mu(0) = 0$, $\mathrm{B}_\mu(1/2) = \mu/2$, and $\mathrm{B}(\mu/2) = 1/2$, Figure 5(a) shows that a long line segment connecting points $(x_1, 0)$ and $(x_2, \mu/2)$, after one iteration, splits into two line segments, of which one is a long segment connecting points $(x_1, 0)$ and $(x_2, \mu/2)$ and the other is a short segment connecting points $(x_0, 0)$ and $(x_2, 1/2)$, where $x_1 < x_0 < x_2$ and $\mathrm{B}_\mu(x_0) = 1/2$. Figure 5(b) shows that a short line segment connecting points $(x_3, 0)$ and $(x_4, 1/2)$, after one iteration, goes to a long segment connecting points $(x_3, 0)$ and $(x_4, \mu/2)$. From these, we conclude that the action of this map on line segments of these two types is described as:

$$\begin{aligned} \mathrm{B}_\mu : \mathrm{L} &\to \mathrm{LS} \\ \mathrm{S} &\to \mathrm{L} \end{aligned} \tag{11}$$

B_μ is then also a discrete map for L and S. From (11), Proposition 3.2 follows immediately.

Proposition 3.2: The graph of $y = \mathrm{B}_\mu^{[n]}(x)$ in the range $0 \le x \le 1$ contains line segments of only two types, type L and type S, and the number of these line segments is$2F_{n+1}$.

Figure 6 shows the graph of $\mathrm{B}_\mu^{[4]}$. We can count the number of line segments in this graph to be $2F_5 = 10$.

Proposition 3.3: A simple identity, $\mu F_n + F_{n-1} = \mu^n$.

Proof: The proof is similar to the triangular T_μ map with $\mu = (1+\sqrt{5})/2$ in Proposition 2.3.

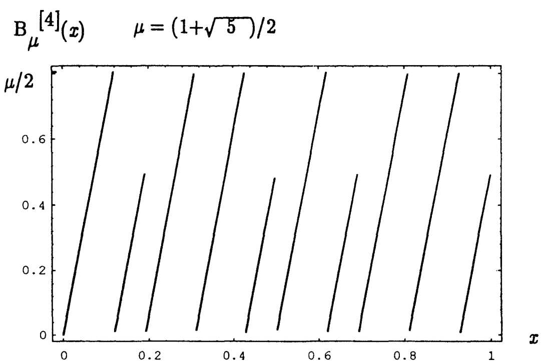

FIGURE 6

Proposition 3.4: There is an infinite LS sequence which is invariant under the action of map (11).

Proof: We denote by LS[B^n] the LS sequence describing the shape of the graph $y = B_\mu^{[n]}(x)$ in the range $0 \le x \le 1/2$. From (11), we have the following results:

$$
\begin{aligned}
&\mathrm{LS}[\mathrm{B}^1] = \mathrm{L} && \\
&\mathrm{LS}[\mathrm{B}^2] = \mathrm{LS} && \\
&\mathrm{LS}[\mathrm{B}^3] = \mathrm{LSL} && = \mathrm{LS}[\mathrm{B}^2]\mathrm{LS}[\mathrm{B}^1] \\
&\mathrm{LS}[\mathrm{B}^4] = \mathrm{LSLLS} && = \mathrm{LS}[\mathrm{B}^3]\mathrm{LS}[\mathrm{B}^2] \\
&\mathrm{LS}[\mathrm{B}^5] = \mathrm{LSLLSLSL} && = \mathrm{LS}[\mathrm{B}^4]\mathrm{LS}[\mathrm{B}^3] \\
&\dots && \\
&\mathrm{LS}[\mathrm{B}^n] = \mathrm{LS}[\mathrm{B}^{n-1}]\mathrm{LS}[\mathrm{B}^{n-2}] &&
\end{aligned}
\tag{12}
$$

The length of the LS[B^n] sequence is F_{n+1}. Equation (12) shows that, after one iteration, an LS sequence gets longer, but the original LS sequence remains. By taking $n \to \infty$, we then have an infinite LS sequence which is invariant under the action of map (11) since, from (12), we have

$$\lim_{n\to\infty} \mathrm{LS}[\mathrm{B}^n] = \lim_{n\to\infty} \mathrm{LS}[\mathrm{B}^{n-1}].$$

We denote by $\{B^\infty\}$ the infinite LS sequence obtained from iterates of an L, that is,

$$\{\mathrm{B}^\infty\} = \lim_{n\to\infty} \mathrm{B}_\mu^{[n]}(\mathrm{L}).$$

We see that $\{B^\infty\}$ is invariant under the action of map (11). It is impossible to have an invariant sequence with S as the first element. Therefore, we have only one infinite LS sequence that is invariant under the action of map (11). Therefore, from the first few iterates of L, we have the following first few elements of this infinite sequence:

$$\{\mathrm{B}^\infty\} = \mathrm{LSLLSLSLLSLLSLSLLSLSL}\ \dots \tag{13}$$

Thus, to have an invariant LS sequence, the L's and S's should be arranged according to the order described in (13). We shall discuss this special symbol sequence in more detail. Denoting by $B(k)$ the k^{th} element of this sequence, then, from (13), we have $B(k) = L$ for k = 1, 3, 4, 6, 8, 9, 11, 12, 14, ..., and $B(k) = S$ for k = 2, 5, 7, 10, 13, 15, ... It would be interesting to find a way of determining that B(k) is an L or an S for a given k. So far, we have not obtained a simple formula for this, except for the following descriptions that are based on the following theorem.

Theorem: For $k \geq 3$, $\mathrm{B}(k) = \mathrm{B}(d)$, where $k = d + F_{n(k)}$ and $F_{n(k)}$ is the largest Fibonacci number that is less than k.

Proof: Suppose $F_{n(k)}$ is the largest Fibonacci number that is less than k, and let $d = k - F_{n(k)}$. Then $F_{n(k)} < k \leq F_{n(k)+1}$. Using the property that the length of the LS[B^n] sequence is F_{n+1}, this k^{th} element is then in the LS[$\mathrm{B}^{n(k)}$] sequence. Since $F_{n(k)} < k$, and using also the property that

$$\mathrm{LS}[\mathrm{B}^n] = \mathrm{LS}[\mathrm{B}^{n-1}]\mathrm{LS}[\mathrm{B}^{n-2}] = (\mathrm{LS}[\mathrm{B}^{n-2}]\mathrm{LS}[\mathrm{B}^{n-3}])\,\mathrm{LS}[\mathrm{B}^{n-2}],$$

we find that the k^{th} element in the LS[$\mathrm{B}^{n(k)}$] sequence is equivalent to the $(k - F_{n(k)})^{\text{th}}$ element in the LS[$\mathrm{B}^{n(k)-2}$] sequence. We then have the result that $\mathrm{B}(k) = \mathrm{B}(k - F_{n(k)}) = \mathrm{B}(d)$.

The determination of B(k) is then reduced to the determination of B(d); we call this the **reduction rule**. According to this reduction rule, we may reduce the original number k down to a final number d_k with $d_k = 1$ or 2, that is,

$$k - \sum_{n=3}^{n(k)} c_n F_n = d_k, \text{ where } c_n = 0 \text{ or } 1, \text{ and } d_k = 1 \text{ or } 2. \tag{14}$$

Equation (14) means that, for a given number $k \geq 3$, we subtract successively appropriate different Fibonacci numbers from k, until the final reduced number, d_k, is one of the two values $\{1, 2\}$. B(k) is then the same as B(d_k). We call d_k the residue of the number k. We conclude that

$$\mathrm{B}(k) = \mathrm{B}(d_k). \tag{15}$$

Since B(1) = L and B(2) = S, we have

$$\begin{cases} \mathrm{B}(k) = \mathrm{L}, & \text{if } d_k = 1, \\ \mathrm{B}(k) = \mathrm{S}, & \text{if } d_k = 2. \end{cases} \tag{16}$$

For example, if we are to determine B(27), then as $27 = F_8 + 6$ and $6 = F_5 + 1$, we have B(27) = B(6) = B(1) = L. The method of reducing a number k down to d_k mentioned above is unique. We note that if B(k) = S, then $\mathrm{B}(k-1) = \mathrm{B}(k+1) = \mathrm{L}$, since if $d_k = 2$, then the coefficient c_3 in (14) must be zero; otherwise, we would have $d = 4$, and then $d_k = 1$ from the reduction rule. This contradicts $d_k = 2$; hence, $k = \sum_{n=4}^{n(k)} c_n F_n + 2$. Therefore,

$$k - 1 = \sum_{n=4}^{n(k)} c_n F_n + 1 \quad \text{and} \quad k + 1 = \sum_{n=4}^{n(k)} c_n F_n + 3 = \sum_{n=3}^{n(k)} c_n F_n + 1;$$

hence, $\mathrm{B}(k-1) = \mathrm{B}(1) = \mathrm{L}$ and $\mathrm{B}(k+1) = \mathrm{B}(1) = \mathrm{L}$. As a result, the two neighbors of an S in the sequence must be L's, or there are no two successive elements that are both S's. On the other hand, if $\mathrm{B}(k) = \mathrm{L}$, it is possible that $\mathrm{B}(k+1) = \mathrm{L}$. This occurs when k can be reduced to 3, that is $k = \sum_{n=5}^{n(k)} c_n F_n + 3$, we see that $\mathrm{B}(k) = \mathrm{B}(3) = \mathrm{L}$ and $\mathrm{B}(k+1) = \mathrm{B}(4) = \mathrm{L}$. Now, since $k + 2 = \sum_{n=3}^{n(k)} c_n F_n + 2$, we have $\mathrm{B}(k+2)$ = B(2) = S; therefore, it is impossible to have three successive elements that are all L's. We present the following table:

d_k	B(k)	k														
$(d_k = 1)$	L|	1	3	4	6	8	9	11	12	14	16	17	19	21	22	…
$(d_k = 2)$	S|	2		5	7		10		13	15		18	20		23	…

Since the two neighbors of an S are L's, it is better that we use the first three elements as the set of residues. That is, we start from B(1) = L, B(2) = S, B(3) = L, and then we have

$$\begin{cases} \mathrm{B}(k) = \mathrm{L}, & \text{if } d_k = 1 \text{ or } 3, \\ \mathrm{B}(k) = \mathrm{S}, & \text{if } d_k = 2. \end{cases} \tag{17}$$

For the case $\mathrm{B}(k) = \mathrm{S}$, we would expect that $\mathrm{B}(k-1) = \mathrm{L}$, with $d_{k-1} = 1$, and $\mathrm{B}(k+1) = \mathrm{L}$, with $d_{k+1} = 3$; however, this is not so. For example, when $k = 5$, $\mathrm{B}(5) = \mathrm{S}$, and according to the reduction rule, we have $4 = F_4 + 1$, so B(4) = L and $d_4 = 1$; but $6 = F_5 + 1$, so B(6) = L and $d_6 = 1$ not 3. Instead, if we now write $6 = F_4 + 3$ and assign $d_6 = 3$, then we would obtain the same result: B(6) = B(3) = L. This shows that we may assign d_6 as either 1 or 3. Numbers whose residue can be assigned as either 1 or 3 are

$$k = \sum_{n=6} c_n F_n + 6, \quad c_n = 0 \text{ or } 1. \tag{18}$$

This enables us to present the following table:

d_k	B(k)	k												
$(d_k = 1)$	L|	1	4	6	9	12	14	17	19	22	25	27	30	…
$(d_k = 2)$	S|	2	5	7	10	13	15	18	20	23	26	28	31	…
$(d_k = 3)$	L|	3	6	8	11	14	16	19	21	24	27	29	32	…

The first, second, and third lines are, respectively, for those k with $d_k = 1, 2$, and 3. We note that numbers, such as 6, 14, 19, 27, ... etc., appear both in the first and third lines. From (17), we see that there are two cases, $d_k = 1$ or $d_k = 3$, for $\mathrm{B}(k)$ being an L, and only one case, $d_k = 2$, for $\mathrm{B}(k)$ being an S. Therefore, naively, we would think that the ratio of the total number of k with $\mathrm{B}(k) = \mathrm{L}$ to that with $\mathrm{B}(k) = \mathrm{S}$ is 2; however, this is due to the fact that we have made the double counting on those numbers of the form (18). Without the double counting, the ratio should be less than 2; indeed, the actual ratio is the golden mean $\approx 1.618 < 2$. It can be seen from (12) that the length of the LS[n] sequence is F_{n+1}, and that among the elements in this sequence there are F_n L's and F_{n-1} S's. Therefore, the asymptotic ratio is

$$\lim_{n\to\infty} (F_n / F_{n-1}) = (1+\sqrt{5})/2.$$

We also have the following results:

(1) If $\mathrm{B}(k) = \mathrm{S}$, then $\mathrm{B}(k-1) = \mathrm{B}(k+1) = \mathrm{L}$.

(2) If $\mathrm{B}(k) = \mathrm{L}$ and k is of the form (18), then $\mathrm{B}(k+1) = \mathrm{B}(k-1) = \mathrm{S}$.

(3) If $\mathrm{B}(k) = \mathrm{L}$ and k is not of the form (18), then $\begin{cases} \text{for } d_k = 1, \ \mathrm{B}(k+1) = \mathrm{S} \text{ and } \mathrm{B}(k-1) = \mathrm{L}, \\ \text{for } d_k = 3, \ \mathrm{B}(k+1) = \mathrm{L} \text{ and } \mathrm{B}(k-1) = \mathrm{S}. \end{cases}$

4. A FINAL REMARK

In a triangular map, requiring that $x = 0$ be a period-3 point determines $\mu = (1+\sqrt{5})/2 \equiv \Sigma_3$, and we find the Fibonacci numbers in this map. This is not an accident, since Σ_3 is not an arbitrary number but, in fact, equals $\lim_{n\to\infty}(F_n / F_{n-1})$, the limit ratio of the Fibonacci sequence. We can generalize the above results by requiring that $x = 0$ be a period-m point [3]. This then determines a unique parameter value $\mu = \Sigma_m$ in the range $\Sigma_{m-1} < \mu < 2$. Correspondingly, the numbers of line segments in a triangular map with $\mu = \Sigma_m$ are those of the Fibonacci numbers of degree m. These are not accidents either, since Σ_m in fact equals $\lim_{n\to\infty} F_n^{(m)} / F_{n-1}^{(m)}$, the limit ratio of the corresponding generalized Fibonacci sequence, where $F_n^{(m)}$ is the n^{th} Fibonacci number of degree m. The same applies also to Baker's map with $\mu = \Sigma_m$. Therefore, following the same methods used in this paper, one would also obtain a simple identity and some invariant sequences from these two maps with $\mu = \Sigma_m$.

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OPTIMAL COMPUTATION, BY COMPUTER, OF FIBONACCI NUMBERS

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1. INTRODUCTION

For a given index n, this paper presents an alternative way of computing F_n. Instead of using addition only, F_n. can be computed using addition, multiplication, and formula (2) below. The formula has a parameter k, and this paper finds such k that minimizes the number of arithmetic operations required to calculate F_n. In this paper, it is assumed that n is large enough that F_n is not stored in a single computer word but, rather, is represented as an array of BITS.

2. BASIC PROPERTIES

In this section we present some relevant features of Fibonacci numbers.

Lemma 1:

$$F(i,j,k) = L_k * F(i,j-1,k) + (-1)^{(k+1)} * F(i,j-2,k), \tag{1}$$

where $F(i,j,k)$ denotes the value of the Fibonacci number that occupies the (i,j) position of a matrix when the Fibonacci numbers are arranged, k elements in a column, or

$$F_{n+k} = F_n * L_k + (-1)^{k+1} * F_{n-k}, \tag{2}$$

where k denotes the number of elements in one such column, L_k represents the Lucas numbers, and n is arbitrary.

Proof: From [1], we have

$$F_n = (\alpha^n - \beta^n)/\sqrt{5}, \tag{3}$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. It is obvious that $\beta = -1/\alpha$; thus,

$$F_n = (\alpha^n - (-1)^n * \alpha^{-n})/\sqrt{5}. \tag{4}$$

Also, it is known that

$$L_k = (\alpha^k + \beta^k) = \alpha^k + (-1)^k * (\alpha)^{-k}. \tag{5}$$

Therefore,

$$F_{n+k} = (\alpha^{n+k} - (-1)^{n+k} * \alpha^{-n-k})/\sqrt{5}, \tag{6}$$

$$\begin{aligned} F_n * L_k + (-1)^{k+1} * F_{n-k} &= (\alpha^{n+k} - (-1)^n * \alpha^{k-n} + (-1)^k \alpha^{n-k} - (-1)^{n+k} \alpha^{-n-k} \\ &\quad - (-1)^k \alpha^{n-k} + (-1)^{n-k+k} \alpha^{-(n-k)})/\sqrt{5} \\ &= (\alpha^{n+k} - (-1)^{n+k} * \alpha^{-n-k})/\sqrt{5} = F_{n+k}. \end{aligned} \tag{7}$$

Q.E.D.

The importance of Lemma 1 is that F_n can now be calculated by using multiplication along with addition. In order to minimize the effort of calculation, we have to calculate the addition effort and the multiplication effort. This is done in Lemmas 2 and 3.

Lemma 2: $O(X+Y) = \delta * MIN(LOG_2(X), LOG_2(Y))$, where X and Y are arrays of binary digits, δ is at most 7, and $O(X+Y)$ denotes the number of arithmetic operations that are performed when adding X to Y.

Proof: The addition algorithm is:

```
CARRY = 0.
    For i = 1 to MIN(LOG2(X), LOG2(Y)),
    TEMP = (X(i) XOR Y(i)) XOR CARRY               (2 operations)      (8)
    CARRY = (X(i) AND Y(i)) OR (X(i) AND CARRY)                        (9)
              OR (Y(i) AND CARRY)                  (5 operations)
    Y(i) = TEMP
    END
```

At the end of the process, the result is found in Y. The number of operations in the loop (8), (9) is 7. This will be referred to later as δ. The loop is executed $MIN(LOG_2(X), LOG_2(Y))$ times. This immediately proves Lemma 2. Q.E.D.

Lemma 3: $O(X * Y) = \delta * LOG_2(X) * (One\,(Y) - 1)$, where X and Y are arrays of binary digits, $One\,(X)$ denotes the number of 1's in the binary representation of X, and $O(X * Y)$ is the number of operations that are performed when multiplying X by Y.

Proof: The following algorithm performs the multiplication:

```
Set Z to ZERO    (Z is the result array)
Set SHIFT to ZERO
For i = 1 to LOG2(Y)
    WHILE Y(i) = 0
        i = i + 1
        SHIFT = SHIFT + 1
    END
    IF SHIFT > 0
        ADD(X, Z, SHIFT)     [see (12)]                                (10)
        SHIFT = SHIFT +1                                               (11)
    END
END
```

ADD(X, Z, SHIFT) is an addition algorithm that adds X to Z, (12)

such that X(1) is added to Z(SHIFT), X(2) is added to Z(SHIFT + 1), etc. The result is stored in Z.

Since the main arithmetic effort of the multiplication (10) is executed (One (Y) −1) times, and each execution "costs," according to Lemma 2, $\delta * \mathrm{LOG}_2(\mathrm{X})$ operations. The number of operations is $\mathrm{O(X * Y)} = \delta * \mathrm{LOG}_2(\mathrm{X}) * (\mathrm{One\ (Y)} - 1)$. Q.E.D.

Lemma 4 computes $\log_2(F_n)$ while Conjecture 1 computes $\mathrm{One}(L_k)$.

Lemma 4: $\mathrm{Log}_2(F_n) = (\log_2 \alpha) * n$, where $\alpha = (1+\sqrt{5})/2$.

Later, we shall refer to $\log \alpha$ as 2λ (approximately 0.69).

Proof: $F_n = \sim\sim \alpha ** n/\sqrt{5}$ ([1], p. 82; Eq. (15) below); $\log_2(F_n) = \sim\sim \log_2 \alpha * n = 2\lambda * n$. Q.E.D.

Conjecture 1: $\mathrm{One}(L_n) = \sim\sim \lambda n$. For example:

n	1192	1193	1194	1195	1196	1197	1198	1199	1200
$\mathrm{One}(L_n)$	402	429	408	412	437	448	435	417	406

This paper claims that the conjecture is true, without proof; it was found by calculating F_n for large numbers using a computer. We can see that $\mathrm{One}(L_n)$ has "Ups" and "Downs" but, in general, it fits the above formula.

2. MINIMIZING $O(F_n)$

Assume, for simplicity, that $n = k * 1$, $1 = n/k$.* According to (2), the following algorithm computes F_n:

Compute F_0 up to F_{k+1}. In this way, L_k is already computed.

[*The required number of operations for this computation is*

$$\sum_{j=1}^{k} 2\lambda\delta j = \lambda\delta(k^2+k)$$

(see Lemma 2 and Lemma 4).]

For $i = 1$ to $\ell - 1$,

Temp = $F_{ik} * L_k$.

[*The required number of operations for this computation is*

$$2k^2\lambda^2\delta\sum_{i=1}^{\ell-1} i - 2k\lambda\delta\sum_{i=1}^{\ell-1} i = \lambda^2\delta n(n-k) - \lambda\delta n(n-k)/k$$

(see Lemma 3 and Conjecture 1)].

---

* Although the setting $n = k * 1$ seems arbitrary, it was found by running a computer program that the optimal k is $(1/6)n$, which is very close to our result $\lambda n/2 = \sim\sim 0.17n$.

$F_{(i+1)k} = \text{Temp} + (-1)^{k+1} * F_{(i-1)k}$ [see (2)].

[*The required number of operations for this calculation is*

$$\begin{aligned}\delta\sum_{i=1}^{\ell-1}\log_2 \text{Temp} &= \delta\sum_{i=1}^{\ell-1}\log_2 F_{ik} + \delta\sum_{i=1}^{\ell-1}\log_2 L_k\\ &= \delta\lambda n(n-k)/k + \delta\lambda k(n-k)/k\\ &= \delta\lambda n(n-k)/k + \delta\lambda n - \delta\lambda k.]\end{aligned}$$

END.

The total number of operations is given by

$$O(F_n) = \lambda\delta(k^2+k) + \lambda^2\delta n(n-k) + \delta\lambda n - \lambda\delta k. \tag{13}$$

In order to find the optimal k, we derive, by (13),

$$\lambda\delta(2k+1) - \lambda^2\delta n - \lambda\delta = 0;$$

$$k = \lambda n \backslash 2. \tag{14}$$

3. COMPARISON

Substituting the optimal $k : \lambda n \backslash 2$ in (13) yields

$$\begin{aligned}O(F_n) &= \lambda\delta(\lambda^2 n^2/4) + \lambda^2\delta n(n-\lambda n/2) + \delta\lambda n - \lambda\delta n^2/2\\ &= \delta\lambda^3 n^2/4 + \delta\lambda^2 n/2 + \boldsymbol{\delta\lambda^2 n^2} - \delta\lambda^3 n/2 + \delta\lambda n - \delta\lambda^2 n/2\\ &=^{\sim\sim} \boldsymbol{\delta\lambda^2 n^2},\end{aligned} \tag{15}$$

where the bold terms represent the significant terms.

On the other hand, the computation of F_n in the regular way needs

$$\begin{aligned}O(F_n) &= 2\delta\lambda(1+2+3+\cdots+n-1)\\ &= 2\delta\lambda(1+n-1)(n-1)/2\\ &=^{\sim\sim} \boldsymbol{\delta\lambda n^2}\end{aligned} \tag{16}$$

(see Lemma 2).

The ratio between the expression in (15) and that in (16) is λ. This means that the proposed method is $1/\lambda$ (approximately 2.88) times more efficient than the regular method.

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ON THE POSSIBILITY OF PROGRAMMING THE GENERAL 2-BY-2 MATRIX ON THE COMPLEX FIELD

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For more than thirty years, several 2-by-2 matrices have been used to discover properties of classical or generalized Fibonacci and Lucas sequences. The references [1]-[6] and their bibliographies provide a few landmarks for this area of study.

However, no one seems to have addressed the following problem: Given a linear second-order sequence (u_n) defined by an arbitrary recursion on the complex field,

$$u_{n+2} - pu_{n+1} + qu_n = 0, \tag{R1}$$

and arbitrary initial values u_0 and u_1, is it possible to program the general 2-by-2 matrix on the complex field, that is, to set the entries of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in such a way that at least one of the entries of

$$A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

be u_n for any $n > 0$?

This could be useful to study the general sequence (u_n) as it was for the classical or generalized Fibonacci and Lucas sequences recently in [5] or [6].

This note proves that the answer to this question with such a general scope is "no," and also shows that by introducing some slight restrictions, it is possible to program A in such a way that both the entries a_n and d_n bear a close relationship to u_n.

Lemma 1: If we set $p = a + d$ and $q = ad - bc$ in (R1), then

$$A^n = \begin{pmatrix} \frac{1}{2}L_n + \frac{1}{2}(a-d)F_n & bF_n \\ cF_n & \frac{1}{2}L_n - \frac{1}{2}(a-d)F_n \end{pmatrix}, \tag{F1}$$

(L_n) and (F_n) being, respectively, the Lucas and the Fibonacci sequences for (R1).

Formula (F1) is easily proved by first showing by induction that

$$A^n = \begin{pmatrix} F_{n+1} - dF_n & bF_n \\ cF_n & F_{n+1} - aF_n \end{pmatrix}$$

and then using $L_n = 2F_{n+1} - pF_n = 2F_{n+1} - (a+d)F_n$ to obtain the form (F1).

Obviously, when considered as functions of n, all the entries are sequences satisfying (R1). From this formula, it is clear that:

1. whatever the coefficients p and q, we can always find a, b, c, and d as to obtain $p = a + d$ and $q = ad - bc$. In fact, there are infinitely many solutions for a, b, c, and d;

2. we cannot obtain at will an arbitrary sequence for the entries (b_n) or (c_n), since they must be proportional to the Fibonacci sequence for (R1);

In the remainder of this paper, we shall assume that p and q are arbitrarily chosen. As for the other entries, we have

Lemma 2: If (u_n) is a sequence satisfying (R1) and such that $u_0 \neq 0$, and if a, b, c, and d satisfy the following equalities,

$$a = u_1 / u_0, \quad d = p - a, \quad bc = ad - q, \tag{E}$$

then for any $n > 0$ we have the formula:

$$A^n = \begin{pmatrix} \frac{u_n}{u_0} & bF_n \\ cF_n & L_n - \frac{u_n}{u_0} \end{pmatrix}. \tag{F2}$$

Proof: The proof is necessary only for a_n since, for the other entries, the result will be the consequence of (F1). Since both (a_n) and $(u_n)/u_0$ satisfy (R1), it is sufficient to show that they coincide for two consecutive values of n. For $n = 1$, they coincide by construction. For $n = 2$, we have $a_2 = a^2 + bc = a^2 + a(p-a) - q = pa - q$; but since $a = u_1 / u_0$, we find by applying (R1) that $a_2 = u_2 / u_0$. Q.E.D.

For a given recurrence (R1) and a given sequence (u_n) satisfying the conditions for the theorem, there are infinitely many corresponding matrices A since b and c are required only to satisfy $bc = ad - q$.

We could also have programmed d_n similarly by exchanging a and d in the above set of equalities (E).

These results may be summarized into the following theorem, which is the main aim of this paper.

Theorem 1:

(a) Let (R1) be the general second-order linear recurrence on the complex field, $u_{n+2} - pu_{n+1} + qu_n = 0$, and (u_n) any sequence satisfying (R1) and such that $u_0 \neq 0$. Then a necessary and sufficient condition for all the entries of A^n considered as functions of $n > 0$ to satisfy (R1) and for the entry (a_n) to be $(u_n)/u_0$ for any $n > 0$ is that

$$a = u_1 / u_0, \quad d = p - a, \quad bc = ad - q.$$

(b) The other entries of A^n are then determined, once b and c have been individually chosen in accordance with the above equalities, by

$$b_n = bF_n, \quad c_n = cF_n, \quad d_n = L_n - u_n / u_0,$$

where L_n and F_n are, respectively, the generalized Lucas and Fibonacci sequences of (R1).

APPLICATIONS

As applications of (F2), we shall derive two formulas concerning the general sequence (u_n) satisfying (R1).

1. Separation of variables for u_{m+n}.

By writing that for any positive integers m and n, $A^{m+n} = A^m A^n$, and by equating the upper left entries on both sides, we obtain $u_0 u_{m+n} = u_m u_n + (u_0)^2 bcF_m F_n$.

By taking $m = n = 1$, we get $(u_0)^2 bc = u_0 u_2 - (u_1)^2$ and the formula

$$u_0 u_{m+n} = u_m u_n + \{u_0 u_2 - (u_1)^2\} F_m F_n. \tag{F3}$$

This formula has the following applications to the study of the sequence of the residues of (u_n) modulo a prime when u_n is an integer for any $n \geq 0$.

Let us assume that D is a positive prime that divides $u_0 u_2 - (u_1)^2$. Then D also divides $u_0 u_{m+n} - u_m u_n$ for any m and $n \geq 0$, and therefore divides $u_0 u_{n+1} - u_1 u_n$ for any $n \geq 0$. Now define the T transformation as: $T(u_n) = u_0 u_{n+1} - u_1 u_n$ for any n. Then, by iterating this transformation $D-1$ times, we shall prove the following theorem by exactly the same method as in [7].

Theorem 2: If D is a positive prime that divides $u_0 u_2 - (u_1)^2$ and is relatively prime to u_0, then the sequence of the residues of (u_n) modulo D is either constant or periodic with period $D-1$.

Thanks to the formula, $u_{n+m+1} = u_{n+1} F_{m+1} - q u_n F_m$ (which can be proved by an easy recursion), and using the method of the iterated T transformation given above, we can prove

Theorem 3: If D is a positive prime that divides F_m, then the sequence of the residues of (u_n) modulo D is either constant or periodic with period $m(D-1)$.

This latter property is shared by any sequence of integers satisfying (R1), since we made no assumption on the values of any u_n, which was not the case for the former property. So there exist at least two kinds of periods for the sequences of the residues of (u_n) modulo a given prime: the **universal** periods depending only on the value of the prime and which exist for any (u_n), and **particular** periods depending also on the initial values of the sequence considered. For instance, if (L_n) and (F_n) are the classical Lucas and Fibonacci sequences, the shortest period modulo 5 for (L_n) is 4, in accordance with the fact that $L_0 L_2 - (L_1)^2 = 5$, while the shortest period modulo 5 for (F_n) is 20, this number also being a period [not necessarily the shortest one, as shown by (L_n)] for any sequence (u_n), in accordance with the fact that for $m = 5$ we have $D = 5$ as divisor of F_m and that $m(D-1) = 20$ in that case.

2. Since the n^{th} power of the determinant of a square matrix is the determinant of its n^{th} power, and the determinant of A is q, we obtain, from (F2):

$$\{u_0 u_2 - (u_1)^2\}(F_n)^2 + (u_n)^2 + q^n (u_0)^2 = u_0 u_n L_n. \tag{F4}$$

This proves, for instance, that if, for any $n \geq 0$, all the sequences involved in this formula are made up of integers, and if, for a given n, the integer D divides both u_0 and F_n, then it also divides u_n. Since, for any integer $m \geq 1$, F_n divides F_{mn}, D also divides F_{mn} and therefore u_{mn}.

Theorem 4: Any divisor D of u_0 generates a sequence of zero residues of (u_n) mod D with a period equal to the entry point of D in (F_n).

[This does not mean that all the zero residues of (u_n) mod D are located in this sequence.]

Formula (F4) also shows that any common divisor of $u_0u_2-(u_1)^2$ and u_n, if also relatively prime to u_0, is a divisor of q. Therefore, if $q=\pm 1$ for any $n\geq 0$, $u_0u_2-(u_1)^2$ and u_n are either prime to each other or share a common divisor with u_0. Then if (L_n) is the generalized Lucas sequence of (R1), since we have $L_0L_2-(L_1)^2=\Delta=$ the discriminant of (R1), and $L_0=2$ always, we can state the next theorem.

Theorem 5: If $q=\pm 1$ for any $n\geq 0$, then L_n is relatively prime to the discriminant of its recursion (R1), provided that this discriminant is odd.

This generalizes the well-known property of the classical Lucas sequence regarding 5.

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ON THE ZECKENDORF FORM OF F_{kn} / F_n

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1. INTRODUCTION

Filipponi and Freitag [1] obtained the Zeckendorf representation F_{kn} / F_n for $k, n \geq 1$ and showed that its form depends on the parity of n and the congruence class of k modulo 4. These representations were not deduced directly, but were conjectured from numerical evidence. The purpose of this note is to present a constructive proof of these results. An important intermediate step in our proof, which is of interest in its own right, is to obtain the Zeckendorf form for the difference of two Fibonacci numbers.

2. A SUBTRACTION ALGORITHM

Theorem: For $n, k \geq 1$,

$$F_{n+k} - F_n = \sum_{r=1}^{[k/2]} F_{n+k+1-2r} + \begin{cases} 0, & k \text{ even}, \\ F_{n-1}, & k \text{ odd}, \end{cases} \tag{1}$$

where $[x]$ denotes the greatest integer not greater than x and, when $k = 1$, the empty sum denotes zero.

Proof: We will show by induction on k that (1) holds for $k \geq 1$ and all $n \geq 1$. First we note that (1) holds for $k = 1$ and for $k = 2$ and all $n \geq 1$. We now write

$$F_{n+k+2} - F_n = F_{n+k+1} + (F_{n+k} - F_n).$$

Thus, from (1),

$$F_{n+k+2} - F_n = F_{n+k+1} + \sum_{r=1}^{[k/2]} F_{n+k+1-2r} + \begin{cases} 0, & k \text{ even}, \\ F_{n-1}, & k \text{ odd}, \end{cases}$$

and hence

$$F_{n+k+2} - F_n = \sum_{r=1}^{[k/2]+1} F_{n+k+3-2r} + \begin{cases} 0, & k \text{ even}, \\ F_{n-1}, & k \text{ odd}. \end{cases}$$

We have shown that if (1) holds for some $k \geq 1$ it also holds for $k + 2$ and, since (1) holds for $k = 1$ and $k = 2$, it holds for all $k \geq 1$. □

3. THE MAIN RESULT

Let us replace F_{kn} and F_n by their Binet forms to give

$$\frac{F_{kn}}{F_n} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha^n - \beta^n} = \sum_{s=1}^{k} \alpha^{(k-s)n} \beta^{(s-1)n}. \tag{2}$$

We may combine the terms of the latter sum in pairs and use the relation $\alpha\beta = -1$, writing

$$\alpha^{(k-r)n}\beta^{(r-1)n} + \alpha^{(r-1)n}\beta^{(k-r)n} = (-1)^{(r-1)n}(\alpha^{(k-2r+1)n} + \beta^{(k-2r+1)n}) = (-1)^{(r-1)n} L_{(k-2r+1)n}$$

for $1 \le r \le [k/2]$, to express the right side of (2) in terms of Lucas numbers. The result clearly depends on the parity of k, and we obtain

$$\frac{F_{kn}}{F_n} = \sum_{r=1}^{[k/2]} (-1)^{(r-1)n} L_{(k-2r+1)n} + \begin{cases} (-1)^{(k-1)n/2}, & k \text{ odd}, \\ 0, & k \text{ even}, \end{cases} \tag{3}$$

for $k \ge 2$. This combines two formulas quoted in Vajda [2, p. 182]. We now derive the Zeckendorf form of F_{kn} / F_n from (3) as follows. First, for n even we have, on replacing each L_m in (3) by $F_{m+1} + F_{m-1}$,

$$\frac{F_{kn}}{F_n} = \sum_{r=1}^{[k/2]} (F_{(k-2r+1)n+1} + F_{(k-2r+1)n-1}) + \begin{cases} F_2, & k \text{ odd}, \\ 0, & k \text{ even}, \end{cases} \tag{4}$$

which is in Zeckendorf form. (See Filipponi & Freitag [1, formulas (2.1) and (2.2)].)

For n odd, more effort is required. First we obtain from (3) that

$$\frac{F_{kn}}{F_n} = \sum_{r=1}^{[k/2]} (-1)^{(r-1)} L_{(k-2r+1)n} + \begin{cases} (-1)^{(k-1)/2}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases} \tag{5}$$

Because of the alternating signs in (5), we need to group the terms in pairs, as far as possible. With $k \ge 4$, the first pair in (5) is

$$L_{(k-1)n} - L_{(k-3)n} = F_{(k-1)n+1} + F_{(k-1)n-1} - F_{(k-3)n+1} - F_{(k-3)n-1}. \tag{6}$$

On using the above "subtraction algorithm" to combine the second and third terms on the right of (6), we derive

$$L_{(k-1)n} - L_{(k-3)n} = F_{(k-1)n+1} + \left(\sum_{s=1}^{n-1} F_{(k-1)n-2s} \right) - F_{(k-3)n-1},$$

from which we obtain the Zeckendorf form

$$L_{(k-1)n} - L_{(k-3)n} = F_{(k-1)n+1} + \left(\sum_{s=1}^{n-2} F_{(k-1)n-2s} \right) + F_{(k-3)n+1} + F_{(k-3)n-2}.$$

For a general pair of terms on the right of (5), we have the Zeckendorf form

$$L_{(k-4r-1)n} - L_{(k-4r-3)n} = F_{(k-4r-1)n+1} + \left(\sum_{s=1}^{n-2} F_{(k-4r-1)n-2s} \right) + F_{(k-4r-3)n+1} + F_{(k-4r-3)n-2}.$$

Thus, for n odd, it is clear that the transformation of (5) into Zeckendorf form depends on whether $[k/2]$ is odd or even; that is, the final form depends on the residue class of k modulo 4. First, for $k = 4m$ and n odd, we obtain

$$\frac{F_{kn}}{F_n} = \sum_{r=1}^{m} (L_{(k-4r+3)n} - L_{(k-4r+1)n})$$

and thus

$$\frac{F_{kn}}{F_n} = S_{k,n}, \tag{7}$$

say, where

$$S_{k,n} = \sum_{r=0}^{[k/4]-1} \left(F_{(k-4r-1)n+1} + \left(\sum_{s=1}^{n-2} F_{(k-4r-1)n-2s} \right) + F_{(k-4r-3)n+1} + F_{(k-4r-3)n-2} \right). \tag{8}$$

We similarly work through the other cases, where n is odd and $k \equiv 1, 2,$ and 3 (mod 4). In each case, the "most significant" part of the Zeckendorf form is $S_{k,n}$, defined by (8). The precise Zeckendorf form is

$$\frac{F_{kn}}{F_n} = S_{k,n} + e_{k,n}, \tag{9}$$

where the least significant part of the Zeckendorf sum is

$$e_{k,n} = \begin{cases} 0, & k \equiv 0 \pmod 4, \\ F_2, & k \equiv 1 \pmod 4, \\ F_{n+1} + F_{n-1}, & k \equiv 2 \pmod 4, \\ F_{2n+1} + \sum_{r=1}^{n-1} F_{2n-2r}, & k \equiv 3 \pmod 4. \end{cases} \tag{10}$$

Thus the Zeckendorf representation of F_{kn} / F_n is given by (4) for n even and by (9) and (10) for n odd.

Note added in proof: The relation (1) above appeared earlier in Filipponi [3].

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ALGORITHMIC MANIPULATION OF THIRD-ORDER LINEAR RECURRENCES

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1. INTRODUCTION

In [12] we showed how to algorithmically prove all polynomial identities involving a certain class of elements from second-order linear recurrences with constant coefficients. In this paper, we attempt to extend these results to third-order linear recurrences.

Let $\langle S_n \rangle$ be a sequence defined by the third-order linear recurrence

$$S_n = pS_{n-1} + qS_{n-2} + rS_{n-3}, \tag{1}$$

where $r \neq 0$. We will consider three special such sequences, $\langle X_n \rangle$, $\langle Y_n \rangle$, and $\langle Z_n \rangle$, given by the following initial conditions:

$$\begin{array}{lll} X_0 = 0, & X_1 = 0, & X_2 = 1; \\ Y_0 = 0, & Y_1 = 1, & Y_2 = 0; \\ Z_0 = 1, & Z_1 = 0, & Z_2 = 0. \end{array} \tag{2}$$

These initial conditions were chosen so that the three sequences form a basis for the set of all third-order linear recurrences with constant coefficients, and because they will allow us (in a future paper) to generalize our results to higher-order recurrences. These three sequences also have nice Binet forms.

Given any sequence $\langle S_n \rangle$ that satisfies recurrence (1), we can write its elements as a linear combination of X_n, Y_n, and Z_n, namely,

$$S_n = S_2 X_n + S_1 Y_n + S_0 Z_n. \tag{3}$$

Thus, it suffices to show that we can algorithmically prove any identity involving X_n, Y_n, and Z_n.

The sequence $\langle S_n \rangle$ can be defined for negative values of n by using recurrence (1) to extend the sequence backwards or, equivalently, by using the recurrence

$$S_{-n} = (-qS_{-n+1} - pS_{-n+2} + S_{-n+3}) / r. \tag{4}$$

A short table of values for X_n, Y_n, and Z_n for small values of n is given below:

n	-2	-1	0	1	2	3	4	5
X_n	$-q/r^2$	$1/r$	0	0	1	p	p^2+q	$p^3+2pq+r$
Y_n	$(pq+r)/r^2$	$-p/r$	0	1	0	q	$pq+r$	$p^2q+pr+q^2$
Z_n	$(q^2-pr)/r^2$	$-q/r$	1	0	0	r	pr	$r(p^2+q)$

The characteristic equation for recurrence (1) is

$$x^3 - px^2 - qx - r = 0. \tag{5}$$

Let the roots of this equation be r_1, r_2, and r_3, which we shall assume are distinct. The condition that these roots are distinct is that Δ, the discriminant, is nonzero. That is,

$$\Delta^2 = (r_1 - r_2)^2 (r_2 - r_3)^2 (r_3 - r_1)^2 = p^2 q^2 - 27r^2 + 4q^3 - 4p^3 r - 18pqr > 0. \tag{6}$$

The Binet forms for our sequences are given by:

$$\begin{aligned} X_n &= A_1 r_1^n + B_1 r_2^n + C_1 r_3^n, \\ Y_n &= A_2 r_1^n + B_2 r_2^n + C_2 r_3^n, \\ Z_n &= A_3 r_1^n + B_3 r_2^n + C_3 r_3^n, \end{aligned} \tag{7}$$

where

$$\begin{aligned} A_1 &= \frac{1}{(r_1 - r_2)(r_1 - r_3)}, & B_1 &= \frac{1}{(r_2 - r_3)(r_2 - r_1)}, & C_1 &= \frac{1}{(r_3 - r_1)(r_3 - r_2)}; \\ A_2 &= \frac{-(r_2 + r_3)}{(r_1 - r_2)(r_1 - r_3)}, & B_2 &= \frac{-(r_3 + r_1)}{(r_2 - r_3)(r_2 - r_1)}, & C_2 &= \frac{-(r_1 + r_2)}{(r_3 - r_1)(r_3 - r_2)}; \\ A_3 &= \frac{r_2 r_3}{(r_1 - r_2)(r_1 - r_3)}, & B_3 &= \frac{r_3 r_1}{(r_2 - r_3)(r_2 - r_1)}, & C_3 &= \frac{r_1 r_2}{(r_3 - r_1)(r_3 - r_2)}. \end{aligned} \tag{8}$$

Another sequence of interest is

$$W_n = X_{n+2} + Y_{n+1} + Z_n = pX_{n+1} + 2qX_n + 3rX_{n-1} = (p^2 + 2q)X_n + pY_n + 3Z_n$$

because W_n has the Binet form

$$W_n = r_1^n + r_2^n + r_3^n. \tag{9}$$

We can solve the equations in (7) for the r_i^n. We get

$$\begin{aligned} r_1^n &= r_1^2 X_n + r_1 Y_n + Z_n, \\ r_2^n &= r_2^2 X_n + r_2 Y_n + Z_n, \\ r_3^n &= r_3^2 X_n + r_3 Y_n + Z_n. \end{aligned} \tag{10}$$

This idea was suggested by Murray Klamkin. It also follows from Lemma 1 of [11]. These equations let us convert an expression involving powers of r_i, where a variable n occurs in the exponents, to expressions involving X_n, Y_n, and Z_n.

From the relationship between the roots and coefficients of a cubic, we have

$$\begin{aligned} r_1 + r_2 + r_3 &= p, \\ r_1 r_2 + r_2 r_3 + r_3 r_1 &= -q, \\ r_1 r_2 r_3 &= r. \end{aligned} \tag{11}$$

Thus, any symmetric polynomial involving r_1, r_2, and r_3 can be expressed in terms of p, q, and r. An algorithmic method (Waring's Algorithm) for performing this transformation can be found on page 14 in [5].

An explicit formula for X_n in terms of p, q, and r was given in [13], namely,

$$X_{n+2} = \sum_{a+2b+3c=n} \binom{a+b+c}{a\ b\ c} p^a q^b r^c. \tag{12}$$

Similar formulas for Y_n and Z_n can be obtained from the fact that $Y_n = X_{n+1} - pX_n$ and $Z_n = rX_{n-1}$.

Matrix formulations were given in [17] and [20]:

$$\begin{pmatrix} S_{n+2} \\ S_{n+1} \\ S_n \end{pmatrix} = \begin{pmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} S_2 \\ S_1 \\ S_0 \end{pmatrix}, \tag{13}$$

$$\begin{pmatrix} X_n \\ Y_n \\ Z_n \end{pmatrix} = \begin{pmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{14}$$

and

$$\begin{pmatrix} X_{n+2} & Y_{n+2} & Z_{n+2} \\ X_{n+1} & Y_{n+1} & Z_{n+1} \\ X_n & Y_n & Z_n \end{pmatrix} = \begin{pmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n. \tag{15}$$

2. THE BASIC ALGORITHMS

Algorithm "TribEvaluate"

Given an integer constant n, to evaluate X_n, Y_n, or Z_n numerically, apply the following algorithm:

Step 1. [Make subscript positive.] If $n < 0$, apply Algorithm "TribNegate" given below.

Step 2. [Recurse.] If $n > 2$, apply the recursion: $S_n = pS_{n-1} + qS_{n-2} + rS_{n-3}$. This reduces the subscript by 1, so the recursion must eventually terminate. If n is 0, 1, or 2, use the values in display (2).

Note: While this may not be the fastest way to evaluate X_n, Y_n, and Z_n, it is nevertheless an effective algorithm.

The key idea to algorithmically proving identities involving polynomials in X_{an+b}, Y_{an+b}, and Z_{an+b} is to first reduce them to polynomials in X_n, Y_n, and Z_n. To do that, we need reduction formulas for X_{m+n}, Y_{m+n}, and Z_{m+n}. Such formulas can be obtained from equations (7), (8), (10), and (11).

From (10), we can compute r_i^{n+m} by multiplying together r_i^n and r_i^m. Then (7) gives us X_{m+n}. Therefore, $X_{n+m} = A_1(r_1^2X_n + r_1Y_n + Z_n)(r_1^2X_m + r_1Y_m + Z_m) + B_1(r_2^2X_n + r_2Y_n + Z_n)(r_2^2X_m + r_2Y_m + Z_m) + C_1(r_3^2X_n + r_3Y_n + Z_n)(r_3^2X_m + r_3Y_m + Z_m)$. Substituting in the values of the A_1, B_1, and C_1 from (8) gives us an expression that is symmetric in r_1, r_2, and r_3. Applying Waring's Algorithm allows us to express this in terms of p, q, and r using (11). We can do the same for Y_{n+m} and Z_{n+m}. The results obtained are given by the following algorithm.

Algorithm "TribReduce" To Remove Sums in Subscripts

Use the identities

$$\begin{aligned} X_{m+n} &= (p^2+q)X_mX_n + p(X_nY_m + X_mY_n) + X_nZ_m + X_mZ_n + Y_mY_n, \\ Y_{m+n} &= (pq+r)X_mX_n + q(X_nY_m + X_mY_n) + Y_nZ_m + Y_mZ_n, \\ Z_{m+n} &= \qquad prX_mX_n + r(X_nY_m + X_mY_n) + Z_mZ_n. \end{aligned} \tag{16}$$

These are also known as the addition formulas.

From the table of initial values, we find that the reduction formulas can also be written in the form

$$\begin{aligned} X_{m+n} &= X_4 X_m X_n + X_3(X_n Y_m + X_m Y_n) + X_n Z_m + X_m Z_n + Y_m Y_n, \\ Y_{m+n} &= Y_4 X_m X_n + Y_3(X_n Y_m + X_m Y_n) + Y_n Z_m + Y_m Z_n, \\ Z_{m+n} &= Z_4 X_m X_n + Z_3(X_n Y_m + X_m Y_n) + Z_m Z_n. \end{aligned} \tag{17}$$

The matrix formulation is

$$X_{m+n} = \begin{pmatrix} X_m \\ Y_m \\ Z_m \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} X_4 & X_3 & X_2 \\ X_3 & X_2 & X_1 \\ X_2 & X_1 & X_0 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \\ Z_n \end{pmatrix} \tag{18}$$

with similar expressions for Y_{m+n} and Z_{m+n}.

If we allow subscripts on the right other than "n" and "m", simpler forms of the reduction formula can be found. For example, [18] gives the following:

$$S_{n+m} = X_m S_{n+2} + Y_m S_{n+1} + Z_m S_n. \tag{19}$$

Similar expressions can be found in [7] and [17]. In matrix form, they can be expressed as

$$\begin{pmatrix} S_{n+m} \\ S_{n+m-1} \\ S_{n+m-2} \end{pmatrix} = \begin{pmatrix} X_{m+1} & Y_{m+1} & Z_{m+1} \\ X_m & Y_m & Z_m \\ X_{m-1} & Y_{m-1} & Z_{m-1} \end{pmatrix} \begin{pmatrix} S_{n+1} \\ S_n \\ S_{n-1} \end{pmatrix}. \tag{20}$$

These formulations come from [18] and [20].

Algorithm "TribReduce" allows us to replace any term of the form S_{an+b}, where a and b are positive integers by terms of the form S_n. To allow a and b to be negative integers as well, we can also use equation (16); however, then we will obtain expressions of the form S_{-n}. Since we would like to express these in the form S_n, we must find formulas for S_{-n}. The same procedure we used before works again. For example, from (10), we can compute r_i^{-n} as $1/r_i^n$. Equation (7) then gives $X_{-n} = A_1/(r_1^2 X_n + r_1 Y_n + Z_n) + B_1/(r_2^2 X_n + r_2 Y_n + Z_n) + C_1/(r_3^2 X_n + r_3 Y_n + Z_n)$. Again we apply Waring's Algorithm and we get the following result.

Algorithm "TribNegate" To Remove Negative Subscripts

Use the identities

$$\begin{aligned} X_{-n} &= \frac{pX_n Y_n - qX_n^2 + Y_n^2 - X_n Z_n}{r^n}, \\ Y_{-n} &= \frac{(pq+r)X_n^2 - p^2 X_n Y_n - pY_n^2 - Y_n Z_n}{r^n}, \\ Z_{-n} &= \frac{(q^2-pr)X_n^2 - (pq+r)X_n Y_n - qY_n^2 + (p^2+2q)X_n Z_n + pY_n Z_n + Z_n^2}{r^n}. \end{aligned} \tag{21}$$

If we allow subscripts on the right other than "n", simpler forms can be found. For example,

$$\begin{aligned} X_{-n} &= (X_{n+1}Y_n - X_n Y_{n+1})/r^n, \\ Y_{-n} &= (X_n Y_{n+2} - X_{n+2} Y_n)/r^n, \\ Z_{-n} &= (X_{n+2} Y_{n+1} - X_{n+1} Y_{n+2})/r^n. \end{aligned} \tag{22}$$

3. THE FUNDAMENTAL IDENTITY CONNECTING X, Y, AND Z

The Fibonacci and Lucas numbers are connected by the fundamental identity

$$L_n^2 = 5F_n^2 + 4(-1)^n. \tag{23}$$

Furthermore, it can be shown that, if $f(F_n, L_n)$ is any nonconstant polynomial [with coefficients that are constants or of the form $(-1)^n$] that is 0 for all integral values of n, then this polynomial must be divisible by $L_n^2 - 5F_n^2 - 4(-1)^n$. That is, (23) is the unique identity connecting F_n and L_n.

A similar result holds for arbitrary second-order linear recurrences. For third-order linear recurrences, we believe there is also exactly one fundamental identity connecting X_n, Y_n, and Z_n. In this section, we will find such an identity, but we do not prove that this identity is unique.

To obtain an identity connecting X_n, Y_n, and Z_n, we can multiply together the equations in display (10). The result is a symmetric polynomial in r_1, r_2, and r_3 and can thus be expressed in terms of p, q, and r. The result is the following.

The Fundamental Identity:

$$\begin{aligned} r^n = r^2X_n^3 + rY_n^3 + Z_n^3 + (q^2 - 2pr)X_n^2Z_n - qrX_n^2Y_n + prX_nY_n^2 \\ + (p^2 + 2q)X_nZ_n^2 - qY_n^2Z_n + pY_nZ_n^2 - (pq + 3r)X_nY_nZ_n. \end{aligned} \tag{24}$$

If we allow subscripts on the right other than "n", simpler forms of the fundamental identity can be found. For example, [15] gives the following equivalent formulation:

$$\begin{vmatrix} X_{n+2} & X_{n+1} & X_n \\ Y_{n+2} & Y_{n+1} & Y_n \\ Z_{n+2} & Z_{n+1} & Z_n \end{vmatrix} = r^n. \tag{25}$$

4. THE SIMPLIFICATION ALGORITHM

Let us be given a polynomial function of elements of the form X_w, Y_w, and Z_w, where the subscripts of X, Y, and Z are of the form $a_1n_1 + a_2n_2 + \cdots + a_kn_k + b$, where b and the a_i are integer constants and the n_i are variables. To put this expression in "canonical form," we apply the following algorithm.

Algorithm "TribSimplify" To Transform an Expression to Canonical Form

Step 1. [Remove sums in subscripts.] Apply Algorithm "TribReduce" to remove any sums (or differences) in subscripts.

Step 2. [Make multipliers positive.] All subscripts are now of the form cn, where c is an integer. For any term in which the multiplier c is negative, apply Algorithm "TribNegate".

Step 3. [Remove multipliers.] All subscripts are now of the form cn, where c is a positive integer. For any term in which the multiplier c is not 1, apply Algorithm "TribReduce" successively until all subscripts are variables.

Step 4. [Remove powers of Z.] If any term involves an expression of the form Z_n^k, where $k > 2$, reduce the exponent by 1 by replacing Z_n^3 by its equivalent value as given by the fundamental identity (24), namely,

$$\begin{aligned} Z_n^3 = r^n - r^2 X_n^3 - rY_n^3 - (q^2 - 2pr)X_n^2 Z_n + qrX_n^2 Y_n - prX_n Y_n^2 \\ - (p^2 + 2q)X_n Z_n^2 + qY_n^2 Z_n - pY_n Z_n^2 + (pq + 3r)X_n Y_n Z_n. \end{aligned} \tag{26}$$

Continue doing this until no Z_n term has an exponent larger than 2.

Proving Identities

To prove that an expression is identically 0, it suffices to apply Algorithm "TribSimplify". If the resulting canonical form is 0, then the expression is identically 0. We believe that the converse is true as well; that is, an expression is identically 0 if and only if Algorithm "TribSimplify" transforms it to 0. A formal proof can probably be given along the lines of [18]; however, we do not do so. Suffice it to say that Algorithm "TribSimplify" was checked on about 100 identities culled from the literature and it worked every time. A selection of these identities is given in the appendix. See also [6] for a related algorithm for trigonometric polynomials.

5. OTHER ALGORITHMS

These algorithms can be verified by applying Algorithm "TribSimplify."

Algorithm "ConvertToX" To Change *Y*'s and *Z*'s to *X*'s

Use the identities

$$\begin{aligned} Y_n &= -pX_n + X_{n+1}, \\ Z_n &= rX_{n-1}. \end{aligned} \tag{27}$$

Algorithm "ConvertToY" To Change *Z*'s and *X*'s to *Y*'s

Use the identities

$$\begin{aligned} Z_n &= (rY_{n+1} - qrY_{n-1}) / (pq + r), \\ X_n &= (pY_{n+1} + rY_{n-1}) / (pq + r). \end{aligned} \tag{28}$$

Algorithm "ConvertToZ" To Change *X*'s and *Y*'s to *Z*'s

Use the identities

$$\begin{aligned} X_n &= Z_{n+1} / r, \\ Y_n &= Z_{n-1} + qZ_n / r. \end{aligned} \tag{29}$$

Algorithm "Removepqr" To Remove *p*'s, *q*'s, and *r*'s

Use the identities

$$\begin{aligned} p &= (X_{n+1} - Y_n) / X_n, \\ q &= (Y_{n+1} - Z_n) / X_n, \\ r &= Z_{n+1} / X_n. \end{aligned} \tag{30}$$

Algorithm "TribShiftDown1" To Decrease a Subscript by 1

Use the identities

$$\begin{aligned} X_{n+1} &= pX_n + Y_n, \\ Y_{n+1} &= qX_n + Z_n, \\ Z_{n+1} &= rX_n. \end{aligned} \tag{31}$$

These can be found in [10].

Algorithm "TribShiftUp1" To Increase a Subscript by 1

Use the identities

$$\begin{aligned} X_{n-1} &= Z_n / r, \\ Y_{n-1} &= X_n - pZ_n / r, \\ Z_{n-1} &= Y_n - qZ_n / r. \end{aligned} \tag{32}$$

Subtraction Formulas

Use the identities

$$\begin{aligned} X_{m-n} &= (rX_n(X_nY_m - X_mY_n) - (qX_n + Z_n)(X_nZ_m - X_mZ_n) \\ &\quad + (pX_n + Y_n)(Y_nZ_m - Y_mZ_n)) / r^n, \\ Y_{m-n} &= (r(pX_n + Y_n)(X_mY_n - X_nY_m) + (pq + r)X_n(X_nZ_m - X_mZ_n) \\ &\quad - (p(p+1)X_n - Z_n)(Y_nZ_m - Y_mZ_n)) / r^n, \\ Z_{m-n} &= (r^2X_mX_n^2 - qrX_n^2Y_m + prX_nY_mY_n + rY_mY_n^2 + q^2X_n^2Z_m - prX_n^2Z_m \\ &\quad - pqX_nY_nZ_m - rX_nY_nZ_m - qY_n^2Z_m - prX_mX_nZ_n - rX_nY_mZ_n \\ &\quad - rX_mY_nZ_n + p^2X_nZ_mZ_n + 2qX_nZ_mZ_n + pY_nZ_mZ_n + Z_mZ_n^2) / r^n. \end{aligned} \tag{33}$$

If we allow subscripts on the right other than simple variables, simpler subtraction formulas can be found. For example, [2] gives the following equivalent formulation:

$$\begin{aligned} X_{m-n} &= \begin{vmatrix} Z_m & Y_m & X_m \\ Z_n & Y_n & X_n \\ Z_{n+1} & Y_{n+1} & X_{n+1} \end{vmatrix} / r^n, \\ Y_{m-n} &= \begin{vmatrix} Z_m & Y_m & X_m \\ Z_n & Y_n & X_n \\ Z_{n+2} & Y_{n+2} & X_{n+2} \end{vmatrix} / r^n, \\ Z_{m-n} &= \begin{vmatrix} Z_m & Y_m & X_m \\ Z_{n+1} & Y_{n+1} & X_{n+1} \\ Z_{n+2} & Y_{n+2} & X_{n+2} \end{vmatrix} / r^n. \end{aligned} \tag{34}$$

Double Argument Formulas

Letting $m=n$ in equation (16) gives us the following:

$$\begin{aligned} X_{2n} &= (p^2+q)X_n^2 + 2pX_nY_n + Y_n^2 + 2X_nZ_n, \\ Y_{2n} &= (pq+r)X_n^2 + 2qX_nY_n + 2Y_nZ_n, \\ Z_{2n} &= prX_n^2 + 2rX_nY_n + Z_n^2. \end{aligned} \tag{35}$$

To Remove Scalar Multiples of Arguments in Subscripts

An expression of the form S_{kn}, where k is a positive integer, can be thought of as being of the form $S_{n+n+\cdots+n}$, where there are k terms in the subscript. This can be expanded out in terms of S_n by $k-1$ repeated applications of the reduction formula (16). For example, for $k=3$, we get the following identities:

$$
\begin{aligned}
X_{3n} &= (p^4+3p^2q+q^2+2pr)X_n^3+3(p^3+2pq+r)X_n^2Y_n+3(p^2+q)X_nY_n^2 \\
&\quad + pY_n^3+3(p^2+q)X_n^2Z_n+6pX_nY_nZ_n+3Y_n^2Z_n+3X_nZ_n^2, \\
Y_{3n} &= (p^3q+2pq^2+p^2r+2qr)X_n^3+3(p^2q+q^2+pr)X_n^2Y_n+3(pq+r)X_nY_n^2 \\
&\quad + qY_n^3+3(pq+r)X_n^2Z_n+6qX_nY_nZ_n+3Y_nZ_n^2, \\
Z_{3n} &= (p^3r+2pqr+r^2)X_n^3+3r(p^2+q)X_n^2Y_n+3prX_nY_n^2+rY_n^3 \\
&\quad + 3prX_n^2Z_n+6rX_nY_nZ_n+Z_n^3.
\end{aligned}
$$

In general, we have

$$
S_{kn} = \sum_{a+b+c=k} \binom{k}{a\ \ b\ \ c} S_{2a+b} X_n^a Y_n^b Z_n^c, \tag{36}
$$

where $\binom{k}{a\ b\ c}$ denotes the trinomial coefficient $\frac{k!}{a!b!c!}$. Formula (36) can be proven by induction on k.

CHANGE OF BASIS (Shift Formulas)

Algorithm "TribShift" To Transform an Expression Involving X_n, Y_n, Z_n Into One Involving X_{n+a}, Y_{n+b}, Z_{n+c}

Use identities such as

$$
X_n = \frac{1}{D}\left(\begin{vmatrix} qX_b+Z_b & Y_b \\ rX_c & Z_c \end{vmatrix} X_{n+a} - \begin{vmatrix} pX_a+Y_a & X_a \\ rX_c & Z_c \end{vmatrix} Y_{n+b} + \begin{vmatrix} pX_a+Y_a & X_a \\ qX_b+Z_b & Y_b \end{vmatrix} Z_{n+c} \right),
$$

where

$$
D = \begin{vmatrix} (p^2+q)X_a+pY_a+Z_a & pX_a+Y_a & X_a \\ (pq+r)X_b+qY_b & qX_b+Z_b & Y_b \\ prX_c+rY_c & rX_c & Z_c \end{vmatrix}, \tag{37}
$$

which can be obtained by solving the linear equations

$$
\begin{aligned}
X_{n+a} &= (p^2+q)X_aX_n+p(X_nY_a+X_aY_n)+X_nZ_a+X_aZ_n+Y_aY_n, \\
Y_{n+b} &= (pq+r)X_bX_n+q(X_nY_b+X_bY_n)+Y_nZ_b+Y_bZ_n, \\
Z_{n+c} &= prX_cX_n+r(X_nY_c+X_cY_n)+Z_cZ_n,
\end{aligned}
$$

for X_n, Y_n, and Z_n.

One can change from the basis (X_n, Y_n, Z_n) to the basis $(X_{n+a}, X_{n+b}, X_{n+c})$ in a similar manner. Other combinations can be found in the same way. To change from one arbitrary basis to another, apply Algorithm "TribReduce" to transform the given expression to the basis (X_n, Y_n, Z_n). Then use one of the above formulas.

6. TURNING SQUARES INTO SUMS

For Lucas numbers, there is the well-known formula,

$$
L_n^2 = L_{2n} - 2(-1)^n, \tag{38}
$$

which allows us to replace the square of a term with a sum of terms. To find an analog for third-order recurrences, we can proceed as follows.

Combining equations (21) and (35) gives us six equations in the six variables X_nY_n, Y_nZ_n, X_nZ_n, X_n^2, Y_n^2, and Z_n^2. We can then solve these equations for X_n^2, Y_n^2, and Z_n^2 in terms of X_{2n}, Y_{2n}, Z_{2n}, X_{-n}, Y_{-n}, and Z_{-n}. We get the following (computer-generated) result.

Algorithm "TribExpandSquares" To Turn Squares into Sums

$$\begin{aligned} dX_n^2 &= r^n[2(p^4+5p^2q+4q^2+6pr)X_{-n}+2(p^3+4pq+9r)Y_{-n}+2(p^2+3q)Z_{-n}] \\ &\quad +2(3pr-q^2)X_{2n}+(pq+9r)Y_{2n}-2(p^2+3q)Z_{2n}, \end{aligned} \tag{39}$$

$$\begin{aligned} dY_n^2 &= r^n[2(p^6+6p^4q+8p^2q^2+8p^3r+16pqr+9r^2)X_{-n} \\ &\quad +2(p^5+5p^3q+4pq^2+7p^2r+3qr)Y_{-n}+2(p^4+4p^2q+q^2+6pr)Z_{-n}] \\ &\quad +(9r^2-p^2q^2-2q^3+2p^3r+4pqr)X_{2n}+(p^3q+3pq^2+p^2r+3qr)Y_{2n} \\ &\quad -2(p^4+4p^2q+q^2+6pr)Z_{2n}, \end{aligned} \tag{40}$$

$$\begin{aligned} dZ_n^2 &= r^n[2r(p^5+6p^3q+8pq^2+7p^2r+12qr)X_{-n}+2r(p^4+5p^2q+4q^2+6pr)Y_{-n} \\ &\quad +2r(p^3+4pq+9r)Z_{-n}]-2r^2(p^2+3q)X_{2n}+r(p^2q+4q^2-3pr)Y_{2n} \\ &\quad +(9r^2-p^2q^2-4q^3+2p^3r+10pqr)Z_{2n}, \end{aligned} \tag{41}$$

where $d=27r^2-p^2q^2-4q^3+4p^3r+18pqr$.

These formulas are a bit outrageous. Are there any simpler formulas? Can these be put in simpler form? To be more specific, we ask the following.

Query: Is there a simpler formula than formula (41) that allows us to express Z_n^2 as a linear combination of terms, each of the form X_{an+b}, Y_{an+b}, or Z_{an+b}? The coefficients may include the constants p, q, and r as well as the nonlinear expression r^n.

7. TURNING PRODUCTS INTO SIMPLER PRODUCTS

For Lucas numbers, there is the well-known formula,

$$L_mL_n = L_{m+n}+(-1)^nL_{m-n}, \tag{42}$$

which allows us to turn products into sums. For third-order recurrences, there probably is no corresponding formula. However, there is a formula that allows us to turn products of three or more terms into sums of products consisting of just two terms.

To find a formula for $X_mX_nX_s$, we can proceed as follows. From equation (7), we have

$$X_mX_nX_s=(A_1r_1^m+A_2r_2^m+A_3r_3^m)(A_1r_1^n+A_2r_2^n+A_3r_3^n)(A_1r_1^s+A_2r_2^s+A_3r_3^s).$$

After expanding this out, replace any term of the form $r_1^ar_2^br_3^c$ (with $a,b,c>0$) by $r^sr_1^{a-s}r_2^{b-s}r_3^{c-s}$, which is equivalent because $r_1r_2r_3=r$. Since one of a, b, or c is equal to s, this substitution turns this term into one involving the product of only two powers of the r_i. Use equation (10) to convert powers of r_1, r_2, and r_3 back to expressions involving X, Y, and Z. Then use Waring's Algorithm and equations (8) and (11) to replace A_1, A_2, A_3, r_1, r_2, and r_3 by p, q, and r. We get the following (computer-generated) result.

$$
\begin{aligned}
X_m X_n X_s = [&-c_8 X_{m+n} X_s - c_8 X_n X_{m+s} - c_8 X_m X_{n+s} + c_6 X_{m+n+s} - c_7 X_{n+s} Y_m \\
&- c_7 X_{m+s} Y_n - c_3 X_s Y_{m+n} - c_7 X_{m+n} Y_s - c_6 Y_{m+n} Y_s - c_3 X_n Y_{m+s} \\
&- c_6 Y_n Y_{m+s} - c_3 X_m Y_{n+s} - c_6 Y_m Y_{n+s} - c_5 Y_{m+n+s} - c_6 X_{n+s} Z_m \\
&+ c_5 Y_{n+s} Z_m - c_6 X_{m+s} Z_n + c_5 Y_{m+s} Z_n - c_2 X_s Z_{m+n} + c_5 Y_s Z_{m+n} \\
&- c_6 X_{m+n} Z_s + c_5 Y_{m+n} Z_s + 3c_1 Z_{m+n} Z_s - c_2 X_n Z_{m+s} + c_5 Y_n Z_{m+s} \\
&+ 3c_1 Z_n Z_{m+s} - c_2 X_m Z_{n+s} + c_5 Y_m Z_{n+s} + 3c_1 Z_m Z_{n+s} \\
&- 3c_1 Z_{m+n+s} - r^s(-2c_8 X_{m-s} X_{n-s} + c_9 X_{n-s} Y_{m-s} \\
&+ c_9 X_{m-s} Y_{n-s} - 2c_6 Y_{m-s} Y_{n-s} + 2c_4 X_{n-s} Z_{m-s} + 2c_5 Y_{n-s} Z_{m-s} \\
&+ 2c_4 X_{m-s} Z_{n-s} + 2c_5 Y_{m-s} Z_{n-s} + 6c_1 Z_{m-s} Z_{n-s})] / d^2,
\end{aligned}
$$

where

$$
\begin{aligned}
c_1 &= p^2q^2 + 4q^3 - 4p^3r - 18pqr - 27r^2, \\
c_2 &= -2p^4q^2 - 13p^2q^3 - 20q^4 + 8p^5r + 56p^3qr + 90pq^2r + 54p^2r^2 + 135qr^2, \\
c_3 &= p^3q^3 + 4pq^4 - 4p^4qr - 12p^2q^2r + 24q^3r - 24p^3r^2 - 135pqr^2 - 162r^3, \\
c_4 &= p^4q^2 + 6p^2q^3 + 8p^4 - 4p^5r - 27p^3qr - 36pq^2r - 27p^2r^2 - 54qr^2, \\
c_5 &= pc_1, \\
c_6 &= qc_1, \\
c_7 &= -3c_1 r, \\
c_8 &= -p^2q^4 - 4q^5 + 6p^3q^2r + 26pq^3r - 8p^4r^2 - 36p^2qr^2 + 27q^2r^2 - 54pr^3, \\
c_9 &= -p^3q^3 - 4pq^4 + 4p^4qr + 15p^2q^2r - 12q^3r + 12p^3r^2 + 81pqr^2 + 81r^3,
\end{aligned}
$$

and

$$
d = 27r^2 - p^2q^2 - 4q^3 + 4p^3r + 18pqr.
$$

These formulas can be simplified. Using the first formula in display (16), we can remove any terms of the form $Y_m Y_n$. Using the second formula in display (16), we can remove any terms of the form $Y_n Z_m + Y_m Z_n$. Using the third formula in display (16), we can remove any terms of the form $Z_m Z_n$. Upon doing this, we get the following:

$$
\begin{aligned}
dX_m X_n X_s = 2(q^2 - 3pr)&[X_s X_{m+n} + X_n X_{s+m} + X_m X_{n+s} - 2r^s X_{m-s} X_{n-s}] \\
&- 2q[X_{m+n+s} - r^s X_{m+n-2s}] + 2p[Y_{m+n+s} - r^s Y_{m+n-2s}] + 6[Z_{m+n+s} - r^s Z_{m+n-2s}] \\
&- (pq + 9r)[X_s Y_{m+n} + X_n Y_{s+m} + X_m Y_{n+s} - r^s(X_{m-s} Y_{n-s} + X_{n-s} Y_{m-s})] \\
&+ 2(p^2 + 3q)[X_s Z_{m+n} + X_n Z_{s+m} + X_m Z_{n+s} - r^s(X_{m-s} Z_{n-s} + X_{n-s} Z_{m-s})].
\end{aligned}
\tag{43}
$$

This can also be expressed in the following form:

Algorithm "TribShortenProducts" To Turn Products of Many Terms into Products of Two Terms

$$
\begin{aligned}
X_m X_n X_s = [&X_s C_{m+n} + X_n C_{s+m} + X_m C_{n+s} - r^s(X_{m-s} C_{n-s} + X_{n-s} C_{m-s}) \\
&- 2qX_{m+n+s} + 2pY_{m+n+s} + 6Z_{m+n+s} - r^s(-2qX_{m+n-2s} + 2pY_{m+n-2s} + 6Z_{m+n-2s})] / d,
\end{aligned}
\tag{44}
$$

where $d = 27r^2 - p^2q^2 - 4q^3 + 4p^3r + 18pqr$ and

$$
C_n = 2(q^2 - 3pr)X_n - (pq + 9r)Y_n + 2(p^2 + 3q)Z_n.
$$

For products of three terms not all involving X's, first apply Algorithm "ConvertToX", formula (27), to change any Y or Z terms to X terms. For products of more than three terms, this procedure can be repeated, three terms at a time, until only products of two terms remain.

Formula (44) is still pretty messy. Can it be simplified? Can it be made to look symmetric under permutations of (m, n, s)?

8. SIMSON'S FORMULA

For Fibonacci numbers, there is the well-known Simson formula, $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = -(-1)^{n-1}. \tag{45}$$

The generalization of this to third-order recurrences is

$$\begin{vmatrix} X_{n+2} & X_{n+1} & X_n \\ X_{n+1} & X_n & X_{n-1} \\ X_n & X_{n-1} & X_{n-2} \end{vmatrix} = -r^{n-2}, \tag{46}$$

which can be further generalized to

$$\begin{vmatrix} S_{n+4} & S_{n+3} & S_{n+2} \\ S_{n+3} & S_{n+2} & S_{n+1} \\ S_{n+2} & S_{n+1} & S_n \end{vmatrix} = r^n \begin{vmatrix} S_4 & S_3 & S_2 \\ S_3 & S_2 & S_1 \\ S_2 & S_1 & S_0 \end{vmatrix}. \tag{47}$$

These formulas come from [15].

9. SUMMATIONS

We can perform indefinite summations of expressions involving X_n, Y_n, and Z_n any time we can perform such summations with a^n instead since, by (7), these terms are actually exponentials with bases r_1, r_2, and r_3.

First, the expression is converted to exponential form using equation (7). Then it is summed. The result is converted back to X's, Y's, and Z's by using equation (10). Then r_1, r_2, and r_3 are converted to p, q, and r using equation (11). The following summations were found using this method.

$$\sum_{k=1}^{n} x^k X_k = \frac{-x^2 + x^{n+1}(X_{n+1} + xY_{n+1} + x^2 Z_{n+1})}{-1 + px + qx^2 + rx^3}, \tag{48}$$

$$\begin{aligned} \sum_{k=0}^{n} X_{ak+b} = {} & [(Y_{a+b} - Y_{(n+1)a+b})\{rX_a^2 + (pX_a + Y_a)(Z_a - 1)\} \\ & + (X_{a+b} - X_{(n+1)a+b})\{(Z_a - 1)^2 - rZ_aY_a \\ & + qX_a(Z_a - 1)\} + (Z_{a+b} - Z_{(n+1)a+b})\{(pX_a + Y_a)Y_a - qX_a^2 \\ & - X_a(Z_a - 1)\}] / [r^2X_a^3 + rY_a^3 + (Z_a - 1)^3 - qY_a^2(Z_a - 1) \\ & + X_a^2((q^2 - 2pr)(Z_a - 1) - qrY_a) + pY_a(Z_a - 1)^2 \\ & + X_a((p^2 + 2q)(Z_a - 1)^2 + prY_a^2 - Y_a(pq + 3r)(Z_a - 1))], \end{aligned} \tag{49}$$

$$\begin{aligned}\sum_{k=1}^{n} kX_k = [2-p+r-(n+1)(2r+q+1)X_{n+1}+n(2r+q+1)X_{n+2} \\ +(n+1)(p-r-2)Y_{n+1}-n(p-r-2)Y_{n+2} \\ +(n+1)(2p+q-3)Z_{n+1}-n(2p+q-3)Z_{n+2}]/(p+q+r-1)^2,\end{aligned} \tag{50}$$

$$\begin{aligned}\sum_{k=1}^{n} k^2X_k = [(1+3q-pq+7r-3pr+r^2)\{-(n+1)^2X_{n+1}+(2n^2+2n-1)X_{n+2}-n^2X_{n+3}\} \\ +(3-3p+p^2+q+6r-3pr-qr)\{-(n+1)^2Y_{n+1}+(2n^2+2n-1)Y_{n+2}-n^2Y_{n+3}\} \\ +(6-8p+3p^2-3q+3pq+q^2+3r-pr)\{-(n+1)^2Z_{n+1} \\ +(2n^2+2n-1)Z_{n+2}-n^2Z_{n+3}\}]/(p+q+r-1)^3,\end{aligned} \tag{51}$$

$$\begin{aligned}\sum_{k=0}^{n} X_kX_{n-k} = [-(n+1)prX_n+(9r-npq-3nr)X_{n+1}+q(n-1)X_{n+2}-3r(n+1)Y_n \\ +(np^2-p^2-3q+nq)Y_{n+1}-p(n-1)Y_{n+2}+(n+1)(p^2+4q)Z_n \\ +2npZ_{n+1}-3(n-1)Z_{n+2}]/(p^2q^2+4q^3-27r^2-4p^3r-18pqr).\end{aligned} \tag{52}$$

Most of the above formulas are special cases of formula (5.2) in [22].

10. THE TRIBONACCI SEQUENCE

The Tribonacci sequence, $\langle T_n\rangle$, may be defined by

$$T_n = T_{n-1}+T_{n-2}+T_{n-3}, \tag{53}$$

with initial conditions $T_0=0,\ T_1=1,$ and $T_2=1$. A basis can be formed from (T_n, T_{n+1}, T_{n+2}).

For this sequence, we have $T_n = X_{n+1}$ with $p=q=r=1$. To convert X's, Y's, and Z's to T's, use the identities

$$\begin{aligned}X_n &= T_{n+2}-T_{n+1}-T_n,\\ Y_n &= 2T_n+T_{n+1}-T_{n+2},\\ Z_n &= 2T_{n+1}-T_{n+2}.\end{aligned} \tag{54}$$

The reduction formulas are

$$\begin{aligned}T_{n+m} = T_n(2T_{m+1}-T_{m+2})+T_{n+1}(2T_m+T_{m+1}-T_{m+2}) \\ -T_{n+2}(T_m+T_{m+1}-T_{m+2})\end{aligned} \tag{55}$$

and

$$\begin{aligned}T_{n-m} = T_n(T_{m+1}^2-T_mT_{m+2})+T_{n+1}(T_{m+2}^2-T_mT_{m+1}-T_{m+2}T_m-T_{m+2}T_{m+1}) \\ +T_{n+2}(T_m^2+T_mT_{m+1}+T_{m+1}^2-T_{m+1}T_{m+2}).\end{aligned} \tag{56}$$

A form of the addition formula was first found by Agronomof in 1914 [1].

The double argument formula is

$$T_{2n} = T_{n+2}^2+T_{n+1}^2+4T_nT_{n+1}-2T_nT_{n+2}-2T_{n+1}T_{n+2}. \tag{57}$$

A form of this can also be found in [1].

The negation formula is

$$T_{-n} = T_{n+2}^2+T_{n+1}^2+T_n^2-T_{n+2}(2T_{n+1}+T_n). \tag{58}$$

The fundamental identity connecting T_n, T_{n+1}, and T_{n+2} is

$$T_n^3 + 2T_{n+1}^3 + T_{n+2}^3 + 2T_nT_{n+1}(T_n + T_{n+1}) + T_nT_{n+2}(T_n - T_{n+2} - 2T_{n+1}) - 2T_{n+1}T_{n+2}^2 = 1. \tag{59}$$

The formula to expand squares is

$$T_n^2 = (5T_{2n+2} - 3T_{2n+1} - 4T_{2n} + 4T_{-n} + 10T_{-n-1} - 2T_{-n-2}) / 22. \tag{60}$$

The analog of Simson's formula is

$$\begin{vmatrix} T_{n+2} & T_{n+1} & T_n \\ T_{n+1} & T_n & T_{n-1} \\ T_n & T_{n-1} & T_{n-2} \end{vmatrix} = -1, \tag{61}$$

which was found by Miles [9] along with generalizations to higher-order recurrences.

Miles [9] also generalized the relationship between Fibonacci numbers and binomial coefficients from Pascal's triangle,

$$F_{n+1} = \sum_{a+2b=n} \binom{a+b}{a},$$

to the following formula which relates Tribonacci numbers and trinomial coefficients from Pascal's pyramid:

$$T_{n+1} = \sum_{a+2b+3c=n} \binom{a+b+c}{a \;\; b \;\; c}. \tag{62}$$

The following summation was found using the methods of Section 9:

$$\sum_{k=1}^{n} T_k^2 = [1 + 4T_nT_{n+1} - (T_{n+1} - T_{n-1})^2] / 4. \tag{63}$$

APPENDIX 1: SELECTED IDENTITIES

We now present some selected identities culled from the literature. All these identities were successfully checked by Algorithm "TribSimplify". Recall that W_n is defined by equation (9).

The following six identities come from Jarden [7]:

$$\begin{gathered} S_{n+m} = rX_mS_{n-1} + X_{m+1}(S_{n+1} - pS_n) + X_{m+2}S_n, \\ X_{2n} = (2rX_{n-1} + qX_n)X_n + X_{n+1}^2, \\ X_{2n+1} = rX_n^2 + (2X_{n+2} - pX_{n+1})X_{n+1}, \\ X_{2n} = X_nW_n + r^nX_{-n}, \\ W_{2n} = W_n^2 - 2r^nW_{-n}, \\ X_{2n+1} = X_{n+1}W_n + r^nX_{1-n}. \end{gathered}$$

The following three identities come from Yalavigi [21]:

$$\begin{gathered} 2W_{3n} = W_n(3W_{2n} - W_n^2) + 6r^n, \\ W_{4n} = W_nW_{3n} - W_{2n}(W_n^2 - W_{2n}) / 2 + r^nW_n, \\ W_{4n+4m} - W_{4n} = W_{n+m}W_{3n+3m} - W_nW_{3n} - W_{2n+2m}(W_{n+m}^2 - 2W_{2n+2m}) / 2 \\ + W_{2n}(W_n^2 - 2W_{2n}) / 2 + r^n(W_{n+m} - W_n). \end{gathered}$$

The following three identities come from Yalavigi [20]:

$$S_{m+n} = X_{m+2}S_n + Y_{m+2}S_{n-1} + Z_{m+2}S_{n-2},$$
$$S_{2n} = X_{n+2}S_n + Y_{n+2}S_{n-1} + Z_{n+2}S_{n-2},$$
$$S_{m+n} = X_{m+h+2}S_{n-h} + Y_{m+h+2}S_{n-h-1} + Z_{m+h+2}S_{n-h-2}.$$

The following two identities come from Shannon and Horadam [14]:

$$(S_nS_{n+4})^2 + (2(S_{n+1}+S_{n+2})S_{n+3})^2 = (S_n^2 + 2(S_{n+1}+S_{n+2})S_{n+3})^2,$$
$$4(S_{n+2}S_{n-1} - S_{n+2}^2) = S_{n-1}^2 - S_{n+3}^2.$$

The following identity comes from Shannon and Horadam [15]:

$$Y_n = qX_{n-1} + rX_{n-2}.$$

The following ten identities come from Carlitz [4] (both ρ_n and σ_n satisfy third-order linear recurrences with $r=1$ and the same p and q with initial conditions $\rho_0 = 1, \rho_1 = \rho_2 = 0,\ \sigma_0 = 3,$ $\sigma_1 = p,\ \sigma_2 = p^2 + 2q$. In particular, with $r = 1$, we have $\sigma_n = W_n$ and $\rho_n = Z_n$):

$$2\rho_m\rho_n - \rho_{m+1}\rho_{n-1} - \rho_{m-1}\rho_{n+1} = \sigma_{m-3}\sigma_{n-3} - \sigma_{m+n-6} - \sigma_{m-3}\rho_{m-3} - \sigma_{n-3}\rho_{m-3} + 2\rho_{m+n-6},$$
$$\sigma_{m+3n} - \sigma_{m+2n}\sigma_n + \sigma_{m+n}\sigma_{-n} - \sigma_m = 0,$$
$$\sigma_{2n} = \sigma_n^2 - 2\sigma_{-n},$$
$$\sigma_{3n} = \sigma_n^3 - 3\sigma_n\sigma_{-n} + 3,$$
$$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \rho_{3-n},$$
$$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \rho_{2n-6} - \rho_{n-3}\sigma_{n-3} + \sigma_{3-n},$$
$$\rho_m\sigma_n = \rho_{m+n} + \rho_{m-n}\sigma_{-n} - \rho_{m-2n},$$
$$\sigma_m\sigma_n = \sigma_{m+n} + \sigma_{m-n}\sigma_{-n} - \sigma_{m-2n},$$
$$\rho_{2n} = \rho_n\sigma_n - \sigma_{-n} + \rho_{-n},$$
$$\rho_{3n} = \rho_n\sigma_n^2 - \sigma_n\sigma_{-n} + \rho_{-n}\sigma_n - \rho_n\sigma_{-n} + 1.$$

The following nine identities come from Waddill [17] (in their notation, $U_n = X_{n+1}$):

$$S_{n+m} = U_{n-k}S_{m+k+1} + Y_{n-k+1}S_{m+k} + rU_{n-k-1}S_{m+k-1},$$

$$S_{n+m} = U_{m-k}S_{n+k+1} + Y_{m-k+1}S_{n+k} + rU_{m-k-1}S_{n+k-1},$$

$$S_n^2 + qS_{n-1}^2 + 2rS_{n-1}S_{n-2} = S_2S_{2n-2} + (qS_1 + rS_0)S_{2n-3} + rS_1S_{2n-4},$$

$$U_{2n-1} = U_n^2 + qU_{n-1}^2 + 2rU_{n-1}U_{n-2},$$

$$U_{2n-1} = U_{n+1}U_{n-1} + rU_{n-1}U_{n-2} + U_n^2 - pU_nU_{n-1},$$

$$\begin{aligned} qU_{2n-1} = U_{n+1}^2 - pU_{n+1}U_n + (r-pq)U_nU_{n-1} + qU_n^2 - pr(U_nU_{n-2} + U_{n-1}^2) \\ - qrU_{n-1}U_{n-2} - r^2(U_{n-1}U_{n-3} + U_{n-2}^2), \end{aligned}$$

$$\begin{aligned} U_{3n-1} = U_{n-1}(U_{n+1}^2 + Y_{n+2}U_n + rU_{n-1}U_n) + Y_n(U_nU_{n+1} + Y_{n+1}U_n + rU_{n-1}^2) \\ + rU_{n-2}(U_{n-1}U_{n+1} + Y_nU_n + rU_{n-2}U_{n-1}), \end{aligned}$$

$$\begin{vmatrix} S_{n+m+h} & S_{n+j+h} & S_{n+h} \\ S_{n+m+k} & S_{n+j+k} & S_{n+k} \\ S_{n+m} & S_{n+j} & S_n \end{vmatrix} = r^n \begin{vmatrix} U_{h-1} & U_h \\ U_{k-1} & U_k \end{vmatrix} \cdot \begin{vmatrix} S_{m+2} & S_{m+1} & S_m \\ S_{j+2} & S_{j+1} & S_j \\ S_2 & S_1 & S_0 \end{vmatrix},$$

$$\begin{vmatrix} S_{5n} & S_{4n} & S_{3n} \\ S_{4n} & S_{3n} & S_{2n} \\ S_{3n} & S_{2n} & S_n \end{vmatrix} = r^n \begin{vmatrix} U_{2n-1} & U_{2n} \\ U_{n-1} & U_n \end{vmatrix} \cdot \begin{vmatrix} S_{2n+2} & S_{2n+1} & S_{2n} \\ S_{n+2} & S_{n+1} & S_n \\ S_2 & S_1 & S_0 \end{vmatrix}.$$

The following five identities were found by Zeitlin [23]:

$$\begin{aligned} S_{n+6}^2 &= (p^2+q)S_{n+5}^2 + (q^2+qp^2+rp)S_{n+4}^2 + (2r^2+rp^3+4pqr-q^3)S_{n+3}^2 \\ &\quad + (r^2p^2-rpq^2-r^2q)S_{n+2}^2 + (r^2q^2-r^3p)S_{n+1}^2 - r^4S_n^2, \end{aligned}$$

$$S_{2n+6} - (p^2+2q)S_{2n+4} + (q^2-2rp)S_{2n+2} - r^2S_{2n} = 0,$$

$$r^nS_{-n} = S_0(W_n^2 - W_{2n})/2 - W_nS_n + S_{2n},$$

$$(n-1)X_{n+1} = p\sum_{j=0}^{n+2} X_jX_{n+2-j} + 2q\sum_{j=0}^{n+1} X_jX_{n+1-j} + 3r\sum_{j=0}^{n} X_jX_{n-j},$$

$$\sum_{k=0}^{n} X_kX_{n-k} = \frac{(9r+pq)(n-1)X_{n+1} - (6q+2p^2)nY_{n+1} + (4q^2-3pr+p^2q)(n+1)X_n}{27r^2 - p^2q^2 - 4q^3 + 4p^3r + 18pqr}.$$

See [19] for other identities.

APPENDIX 2. SELECTED TRIBONACCI IDENTITIES

We present below selected identities from the literature in which $p = q = r = 1$. All these identities were successfully checked by Algorithm "TribSimplify".

The following three identities come from Agronomof [1]:

$$T_{n+m} = T_{m+1}T_m + (T_m + T_{m-1})T_{n-1} + T_mT_{n-2},$$
$$T_{2n} = T_{n-1}^2 + T_n(T_{n+1} + T_{n-1} + T_{n-2}),$$
$$T_{2n-1} = T_n^2 + T_{n-1}(T_{n-1} + 2T_{n-2}).$$

The following three identities come from Lin [8] (in their notation, we have $U_n = Y_{n+2}$, with $p = q = r = 1$):

$$U_{4n+1}U_{4n+3} + U_{4n+2}U_{4n+4} = T_{4n+4}^2 - T_{4n+2}^2,$$
$$U_{n+1}^2 + U_{n-1}^2 = 2(T_n^2 + T_{n+1}^2),$$
$$T_{n+1}^2 - T_n^2 = U_{n+1}U_{n-1}.$$

The following five identities were found by Zeitlin [23]:

$$\begin{aligned} T_{n+6+a}T_{n+6+b} &= 2T_{n+5+a}T_{n+5+b} + 3T_{n+4+a}T_{n+4+b} + 6T_{n+3+a}T_{n+3+b} \\ &\quad - T_{n+2+a}T_{n+2+b} - T_{n+a}T_{n+b}, \end{aligned}$$

$$\begin{aligned}-(1-2x-3x^2-6x^3+x^4+x^6)\sum_{k=0}^{n}T_k^2x^k &= T_{n+1}^2x^{n+1}+(T_{n+2}^2-2T_{n+1}^2)x^{n+2}\\ &\quad+(T_{n+3}^2-2T_{n+2}^2-3T_{n+1}^2)x^{n+3}\\ &\quad+(T_{n+4}^2-2T_{n+3}^2-3T_{n+2}^2-6T_{n+1}^2)x^{n+4}\\ &\quad-T_{n-1}^2x^{n+5}-T_n^2x^{n+6}-x+x^2+x^3+x^4,\end{aligned}$$

$$8\sum_{k=0}^{n}T_k^2 = T_{n+5}^2-T_{n+4}^2-4T_{n+3}^2-10T_{n+2}^2-9T_{n+1}^2-T_n^2+2,$$

$$T_{-n}=-W_nT_n+T_{2n},$$

$$22\sum_{j=0}^{n-2}T_jT_{n-2-j}=5(n-1)T_n-2(n-1)T_{n-1}-4nT_{n-2}.$$

The following eleven identities come from Waddill and Sacks [16] (in their notation, we have $K_n=X_{n+1}$, $L_n=Y_{n+1}$, and $R_n=S_{n-1}+S_{n-2}$, with $p=q=r=1$):

$$L_n=K_{n-1}+K_{n-2},$$

$$S_{n+h}=K_{h+1}S_n+L_{h+1}S_{n-1}+K_hS_{n-2},$$

$$S_{2n}=K_{n+1}S_n+L_{n+1}S_{n-1}+K_nS_{n-2},$$

$$S_{2n-1}=K_nS_n+(K_{n-1}+K_{n-2})S_{n-1}+K_{n-1}S_{n-2},$$

$$S_{n+h}=K_{h+m+1}S_{n-m}+L_{h+m+1}S_{n-m-1}+K_{h+m}S_{n-m-2},$$

$$S_n^2+S_{n-1}^2+2S_{n-1}S_{n-2}=S_2S_{2n-2}+R_2S_{2n-3}+S_1S_{2n-4},$$

$$\begin{vmatrix}S_n & S_{n+h} & S_{n+h+k}\\ S_{n+t} & S_{n+h+t} & S_{n+h+k+t}\\ S_{n+m} & S_{n+h+m} & S_{n+h+k+m}\end{vmatrix}=\begin{vmatrix}K_h & K_{h+k}\\ L_{h+1} & L_{h+k+1}\end{vmatrix}\cdot\begin{vmatrix}S_t & S_{t+1} & S_{t+2}\\ S_m & S_{m+1} & S_{m+2}\\ S_0 & S_1 & S_2\end{vmatrix},$$

$$\begin{vmatrix}K_n & K_{n+h} & K_{n+h+k}\\ K_{n+t} & K_{n+h+t} & K_{n+h+k+t}\\ K_{n+m} & K_{n+h+m} & K_{n+h+k+m}\end{vmatrix}=\begin{vmatrix}K_h & K_{h-1}\\ K_{h+k} & K_{h+k-1}\end{vmatrix}\cdot\begin{vmatrix}K_m & K_t\\ K_{m-1} & K_{t-1}\end{vmatrix},$$

$$\begin{vmatrix}K_{n+1} & K_n & K_{n+h}\\ K_{n+h+1} & K_{n+h} & K_{n+2h}\\ K_{n+2h+1} & K_{n+2h} & K_{n+3h}\end{vmatrix}=K_{h-1}\cdot\begin{vmatrix}K_h & K_{h-1}\\ K_{2h} & K_{2h-1}\end{vmatrix},$$

$$\begin{vmatrix}K_n & K_{n+h} & K_{n+m}\\ K_{n+h} & K_{n+2h} & K_{n+h+m}\\ K_{n+m} & K_{n+h+m} & K_{n+2m}\end{vmatrix}=-\begin{vmatrix}K_h & K_m\\ K_{h-1} & K_{m-1}\end{vmatrix}^2,$$

$$\begin{vmatrix}S_{n+h+k+t} & S_{n+h+k} & S_{n+h+k+m}\\ R_{n+h+t} & R_{n+h} & R_{n+h+m}\\ S_{n+t} & S_n & S_{n+m}\end{vmatrix}=\begin{vmatrix}K_{h+k-1} & K_{h+k}\\ L_{h-1} & L_h\end{vmatrix}\cdot\begin{vmatrix}S_t & S_{t+1} & S_{t+2}\\ S_m & S_{m+1} & S_{m+2}\\ S_0 & S_1 & S_2\end{vmatrix}.$$

Errata: Computer verification of the various identities encountered in the references consulted during this research revealed a number of typographical errors in the literature. We list the corrections below to set the record straight.

In [4], equation (1.15) should be the same as equation (4.1). Also, equation (1.16) should be the same as equation (3.14).

In [10], equation (2.1) should read "$J_{n+1} = PJ_n + K_n$".

In [13], in equation (1.4), "$t_2 = P^2 + Q$" should be "$t_2 = P^2 + 2Q$". Equation (2.2) should read "$t_n = Ps_{n-1} + 2Qs_{n-2} + 3Rs_{n-3}$".

In [16], the last term of equation (21) should be "$K_{h+k}P_{n-2}$", not "$K_{n+k}P_{n-1}$". Also, the final subscript in equation (41) should be "$h-1$", not "$n-1$". In equation (42), "P_{n+h+m}" should be "R_{n+h+m}" and "K_{n+k}" should be "K_{h+k}".

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A SIMPLER GRAMMAR FOR FIBONACCI NUMBERS

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Fibonacci numbers can be regarded as a formal language in a very natural way: The language $\mathcal{F}_b$ consists of words encoding each Fibonacci number in b-ary notation over alphabet $\{0, 1, \ldots, b-1\}$. This works for any base $b > 1$. The language $\mathcal{F}_1$ of *unary* Fibonacci numbers is defined as $\{0^{F_n} \mid n \geq 1\} = \{0, 00, 000, 00000, \ldots\}$.

The Chomsky hierarchy of formal languages consists of four levels: Regular, context-free, context-sensitive, and recursively enumerable languages (see Hopcroft and Ullman [1] for details).

It was shown by Moll and Venkatesan [2] that $\mathcal{F}_b$ is not context-free for any base $b > 1$. It is clear also that $\mathcal{F}_1$ is not context-free. If $\mathcal{F}_1$ were context-free, the Fibonacci numbers would be a semi-linear set according to Parikh's Theorem [4], i.e., a finite union of linear sets $\{nk + \ell \mid n \geq 0\}$.

Mootha [3] presented a context-sensitive grammar for $\mathcal{F}_1$, demonstrating that unary Fibonacci numbers are context-sensitive, thus placing them optimally in the Chomsky hierarchy. However, the grammar is complicated and includes 53 symbols and 80 rules. The following simpler grammar $G = (N, T, P, S)$ with nonterminals $N = \{S, A, B, B_r, C\}$, terminals $T = \{0\}$, axiom S, and productions

$$\begin{aligned} S &\to CS \mid B_r \\ CA &\to BC \\ CB &\to ABC \\ CB_r &\to AB_r \\ A &\to 0 \\ B &\to 0 \\ B_r &\to 0 \end{aligned}$$

also generates exactly the unary Fibonacci numbers.

It works as follows: First, a sentential form $CC \ldots CB_r$ is generated using only the rules $S \to CS$ and $S \to B_r$. Let us call a sentential form $C \ldots C\alpha B_r$ *basic*, if α contains only A's and B's. $\#_A(\beta)$ denotes the number of A's and $\#_B(\beta)$ the number of B's and B_r's in the word β.

If β and γ are basic sentential forms, $\beta \overset{*}{\Rightarrow} \gamma$, and β contains one more C than γ, then

$$\#_A(\gamma) = \#_B(\beta) \quad \text{and} \quad \#_B(\gamma) = \#_A(\beta) + \#_B(\beta).$$

From a basic sentential form $C \ldots CB_r$ of length $n-1$, a basic sentential form of length F_n will finally be derived, which itself leads to the word 0^{F_n}. Note that the length of the derivation is exactly $F_{n+2}+n-3$ and that the number of productions and the cardinality of N and T are Fibonacci numbers, too.

Observe that b-ary Fibonacci numbers are also context-sensitive. The context-sensitive languages are a superset of all languages that are recognizable with linear space by a deterministic Turing machine. Since addition of numbers in b-ary notation can be done in linear space, all Fibonacci numbers can be generated by use of the recurrence that defines Fibonacci numbers.

Finally, we note that if we change the production for nonterminal S to $S \to C_\ell B_r \mid B_r$ and add the productions

$$C_\ell A \to C_\ell BC \mid BC$$
$$C_\ell B \to C_\ell ABC \mid ABC$$
$$C_\ell B_r \to C_\ell AB_r \mid AB_r$$

then the word 0^{F_n} can be derived in exactly $F_{n+2}-1$ steps.

REFERENCES

1. J. E. Hopcroft & J. D. Ullman. *Introduction to Automata Theory, Languages and Computation*. New York: Addison-Wesley, 1979.
2. R. J. Moll & S. M. Venkatesan. "Fibonacci Numbers Are Not Context-Free." *The Fibonacci Quarterly* **29.1** (1991):59-61.
3. V. K. Mootha. "Unary Fibonacci Numbers Are Context-Sensitive." *The Fibonacci Quarterly* **31.1** (1993):41-43.
4. R. J. Parikh. "On Context-Free Languages." *Journal of the ACM* **13.4** (1966):570-81.

AMS Classification Numbers: 68Q50, 68Q45

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This issue of *The Fibonacci Quarterly* is dedicated to

Herta Taussig Freitag

as she enters her 89th year, in recognition of her years of outstanding service and achievement in the mathematics community through excellence in teaching, problem solving, lecturing, and research.

During Dr. Freitag's years at Hollins College, she earned many honors, among them the prestigious Algeron Sidney Sullivan Award. She was the first faculty member to receive the Hollins Medal, and the first recipient of the Virginia College Mathematics Teacher of the Year Award. She was the first woman to become President of the Virginia, Maryland, and District of Columbia Section of the Mathematical Association of America, after having served as Vice-President and Secretary.

Although she officially retired in 1971, Dr. Freitag continues her professional activities—research, publishing, and lecturing—throughout the region and abroad. Of her many accomplishments, she is perhaps most proud of her perfect attendance at the seven International Conferences of the Fibonacci Association. Herta has presented at least one paper at each conference and considers participants as not merely mathematical colleagues, but virtual family members. The problem section is the first page Herta turns to in *The Fibonacci Quarterly*, and here is a story she often tells: When two non-mathematicians meet on the street and one says, "I've got problems," the other answers, "I'm so sorry for you." When two mathematicians meet and one says, "I've got problems," the other says, "Oh, goody!"

We would like to take the opportunity here to thank Herta in this small way for her innumerable contributions to the mathematics community.

Photo courtesy of Colin Paul Spears

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEM PROPOSED IN THIS ISSUE

B-820 ***Proposed by the editor; dedicated to Herta T. Freitag***

Find a recurrence (other than the usual one) that generates the Fibonacci sequence.

[The usual recurrence is a second-order linear recurrence with constant coefficients. Can you find a first-order recurrence that generates the Fibonacci sequence? Can you find a third-order linear recurrence? a nonlinear recurrence? one with nonconstant coefficients? etc.]

SOLUTIONS

A Disguise for Zero

B-795 ***Proposed by Wray Brady, Jalisco, Mexico (Vol. 33, no. 4, August 1995)***

Evaluate

$$E = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!} L_{2n}.$$

Solution by L. A. G. Dresel, Reading, England

Since $L_n = \alpha^n + \beta^n$, the given expression is

$$E = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!} (\alpha^n + \beta^n).$$

Comparing this with the well-known power series for $\cos(x)$, we have $E = \cos(\pi\alpha) + \cos(\pi\beta)$. Since $\alpha + \beta = 1$, we have $\cos\pi\alpha = \cos(\pi - \pi\beta) = -\cos(\pi\beta)$, giving $E = 0$.

Seiffert showed that $\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!} L_{2kn} = 0$ *if and only if k is not divisible by 3.*

Also solved by Glenn Bookhout, Paul S. Bruckman, Charles K. Cook, Andrej Dujella, Russell Euler, C. Georghiou, Russell Jay Hendel, Hans Kappus, Joseph J. Koštál, Can. A. Minh, Bob Prielipp, R. P. Sealy, H.-J. Seiffert, Sahib Singh, and the proposer.

A Disguise for Five

B-796 ***Proposed by M. N. S. Swamy, St. Lambert, Quebec, Canada (Vol. 33, no. 5, November 1995)***

Show that

$$\frac{L_n^2 + L_{n+1}^2 + L_{n+2}^2 + \cdots + L_{n+a}^2}{F_n^2 + F_{n+1}^2 + F_{n+2}^2 + \cdots + F_{n+a}^2}$$

is always an integer if a is odd.

Solution by Sahib Singh, Clarion University of Pennsylvania, PA

We prove that the value of the given expression is 5. The result follows from identity (I_{12}) of [1]: $L_n^2 = 5F_n^2 + 4(-1)^n$. Using this identity yields

$$\begin{aligned} L_n^2 + L_{n+1}^2 &= 5(F_n^2 + F_{n+1}^2), \\ L_{n+2}^2 + L_{n+3}^2 &= 5(F_{n+2}^2 + F_{n+3}^2), \\ &\cdots \\ L_{n+a-1}^2 + L_{n+a}^2 &= 5(F_{n+a-1}^2 + F_{n+a}^2). \end{aligned}$$

Addition yields $L_n^2 + L_{n+1}^2 + \cdots + L_{n+a}^2 = 5(F_n^2 + F_{n+1}^2 + \cdots + F_{n+a}^2)$, from which the result follows.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, Calif.: The Fibonacci Association, 1979.

Also solved by Charles Ashbacher, Wray Brady, Paul S. Bruckman, Andrej Dujella, Russell Euler, Herta T. Freitag, Russell Jay Hendel, Joseph J. Koštál, Carl Libis, Can. A. Minh, Bob Prielipp, H.-J. Seiffert, and the proposer.

Decimal Congruence

B-797 ***Proposed by Andrew Cusumano, Great Neck, NY (Vol. 33, no. 5, November 1995)***

Let $\langle H_n \rangle$ be any sequence that satisfies the recurrence $H_{n+2} = H_{n+1} + H_n$. Prove that $7H_n \equiv H_{n+15} \pmod{10}$

Solution by Russell Euler, Northwest Missouri State University, Maryville, MO

From formula (8) of [1], we have $H_{n+15} = F_{14}H_n + F_{15}H_{n+1}$. So $H_{n+15} = 377H_n + 610H_{n+1} \equiv 7H_n \pmod{10}$.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Haukkanen showed how to generate many sets of integers c, k, and m, such that $cH_n \equiv H_{n+k}$ *(mod m) for all n.*

Also solved by Charles Ashbacher, Brian D. Beasley, David M. Bloom, Wray Brady, Paul S. Bruckman, Andrej Dujella, Herta T. Freitag, Pentti Haukkanen, Russell Jay Hendel, Gerald A. Heuer, Joseph J. Koštál, Carl Libis, Can. A. Minh, Bob Prielipp, R. P. Sealy, H.-J. Seiffert, Sahib Singh, and the proposer.

Powers of 5

B-798 ***Proposed by Seung-Jin Bang, Ajou University, Suwon, Korea (Vol. 33, no. 5, November 1995)***

Prove that, for n a positive integer, F_{5^n} is divisible by 5^n but not by 5^{n+1}.

Comment by the editor: Several readers pointed out that this is a duplicate of problem B-248. Sorry about that. How could I have missed this? See this Quarterly (1973)553 for the solution.

Bloom mentioned the stronger result that, if p is odd and $k \geq 1$, *then* F_{np} *is divisible by* p^{k+1} *but not* p^{k+2}. *This was proven by Lucas in 1876; see page 396 in [1].*

Reference

1. L. E. Dickson. *History of the Theory of Numbers.* Vol. 1. New York: Chelsea, 1971.

Also proposed by V. E. Hoggatt, Jr. Also solved by Brian D. Beasley, David M. Bloom, Wray Brady, Paul S. Bruckman, Warren Cheeves, Andrej Dujella, Russell Euler, Herta T. Freitag, Pentti Haukkanen, Can. A. Minh, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Gregory Wulczyn, and the proposers.

A Recurrence

B-799 ***Proposed by David Zeitlin, Minneapolis, MN (Vol. 33, no. 5, November 1995)***

Solve the recurrence $A_{n+2} = 4A_{n+1} + A_n$, for $n \geq 0$, with initial conditions $A_0 = 1$ and $A_1 = 4$; expressing your answer in terms of Fibonacci and/or Lucas numbers.

Solution by David M. Bloom, Brooklyn College, NY

From formula (15a) of [1], we have that, for all a and b, $F_{a+b} + (-1)^b F_{a-b} = F_a L_b$. Setting $b = 3$ gives $F_{a+3} - F_{a-3} = 4F_a$. Hence, if we let $A_n = \frac{1}{2}(F_{3n+1} + F_{3n+2})$, we have $A_{n+1} - A_{n-1} = \frac{1}{2}(F_{3n+4} + F_{3n+5} - F_{3n-2} - F_{3n-1}) = \frac{1}{2}(4F_{3n+1} + 4F_{3n+2}) = 4A_n$. Clearly, $A_0 = 1$ and $A_1 = 4$, so these A's coincide with the A's of the problem. Hence, $A_n = \frac{1}{2}(F_{3n+1} + F_{3n+2}) = \frac{1}{2}F_{3n+3}$.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Seiffert found the generalization that, if k is a positive integer, and the sequence $A_n(k)$ satisfies the recurrence $A_{n+2}(k) = L_k A_{n+1}(k) - (-1)^k A_n(k)$, for $n \geq 0$, with initial conditions $A_0(k) = 1$ and $A_1(k) = L_k$, then we have $A_n(k) = F_{nk+k} / F_k$. This is an immediate consequence of the result of problem B-748, see this Quarterly 33.1(1995):88.

Also solved by Brian D. Beasley, Paul S. Bruckman, Andrej Dujella, Russell Euler, Herta T. Freitag, Pentti Haukkanen, Russell Jay Hendel, Joseph J. Koštál, Daina A. Krigens, Carl Libis, Graham Lord, Can. A. Minh, Bob Prielipp, V. Ravoson & R. Caboz (jointly), H.-J. Seiffert, Sahib Singh, M. N. S. Swamy, and the proposer.

Pell/Fibonacci Inequality

B-800 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 33, no. 5, November 1995)***

Define the Pell numbers by the recurrence $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$, with initial conditions $P_0 = 0$ and $P_1 = 1$.

Show that, for all integers $n \geq 4$, $P_n < F_{k(n)}$, where $k(n) = \lfloor (11n+2)/6 \rfloor$.

Solution by Paul S. Bruckman, Highwood, IL

Given $u = 1 + \sqrt{2}$, we may easily verify that $u^6 = 70u + 29$ and $u^{12} = 13860u + 5741$; hence, $u^{12} = 198u^6 - 1$. From this we may deduce that $P_{n+12} = 198P_{n+6} - P_n$ for all integers n. Likewise, it is straightforward to show that $F_{n+22} = 199F_{n+11} + F_n$.

Our next observation is the following:

$$k(n+6) = k(n) + 11.$$

We will use these identities to establish the desired result by induction.

Let S denote the set of integers $n \geq 4$ for which $P_n < F_{k(n)}$. Our first step is to construct a table of the first few values of P_n and $F_{k(n)}$. These are given below.

n	$k(n)$	P_n	$F_{k(n)}$
4	7	12	13
5	9	29	34
6	11	70	89
7	13	169	233
8	15	408	610
9	16	985	987
10	18	2378	2584
11	20	5741	6765
12	22	13860	17711
13	24	33461	46368
14	26	80782	121393
15	27	195025	196418

Note that $P_3 = 5 = F_5 = F_{k(3)}$, justifying the restriction $n \geq 4$. We see from the table that $n \in S$ for $4 \leq n \leq 15$. Assume that $n \in S$ and $(n+6) \in S$. Then $P_n < F_{k(n)}$ and $P_{n+6} < F_{k(n+6)} = F_{k(n)+11}$. Then $P_{n+12} = 198P_{n+6} - P_n < 198F_{k(n)+11} < 199F_{k(n)+11} + F_{k(n)} = F_{k(n)+22}$, or $P_{n+12} < F_{k(n+12)}$. Thus, $n \in S$ and $(n+6) \in S$ implies that $(n+12) \in S$, and the proof by induction is complete.

Also solved by Andrej Dujella, Russell Jay Hendel, and the proposer. One incorrect solution was received.

Congruences mod 40

B-801 ***Proposed by Larry Taylor, Rego Park, NY***
(Vol. 33, no. 5, November 1995)

Let $k \geq 2$ be an integer and let n be an odd integer. Prove that

(a) $F_{2^k} \equiv 27 \cdot 7^k \pmod{40}$;

(b) $F_{n2^k} \equiv 7^k F_{16n} \pmod{40}$.

Solution by H.-J. Seiffert, Berlin, Germany

(a) From [2], we know that

$$L_{2^j} \equiv 7 \pmod{40}, \text{ for } j \geq 2. \tag{1}$$

Repeated application of the equation (I_7) of [1], $F_{2m} = F_m L_m$, gives

$$F_{2^k} = F_4 \prod_{j=2}^{k-1} L_{2^j}, \quad k \geq 2,$$

so that, by $F_4 = 3$ and (1), we have $F_{2^k} \equiv 3 \cdot 7^{k-2} \equiv 27 \cdot 7^k \pmod{40}$, for $k \geq 2$.

(b) Let $k \geq 2$. We shall prove that

$$F_{n2^k} \equiv 7^k F_{16n} \pmod{40}, \text{ for all integers } n. \tag{2}$$

It suffices to prove (2) for $n \geq 0$, since $F_{-2m} = -F_{2m}$. Since $F_{16} = 987 \equiv 27 \pmod{40}$, (2) is true for $n = 1$, by part (a). Clearly, it is also true for $n = 0$. Suppose that (2) holds for all $j \in \{0, 1, 2, \ldots, n\}$, $n \geq 1$. Then, by equation (I_{21}) of [1] and (1),

$$\begin{aligned} F_{(n+1)2^k} &= L_{2^k} F_{n2^k} - F_{(n-1)2^k} \equiv 7 \cdot 7^k F_{16n} - 7^k F_{16(n-1)} \\ &= 7^k (7F_{16n} - F_{16(n-1)}) \equiv 7^k (L_{16} F_{16n} - F_{16(n-1)}) \\ &= 7^k F_{16(n+1)} \pmod{40}. \end{aligned}$$

This completes the induction proof of (2).

References

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Santa Clara, Calif.: The Fibonacci Association, 1979.
2. S. Singh. Problem B-694. *The Fibonacci Quarterly* **30.3** (1992):276.

Also solved by Paul S. Bruckman, Andrej Dujella, Russell Jay Hendel, Joseph J. Koštál, and Bob Prielipp.

NOTE: The Elementary Problems Column is in need of more *easy*, yet elegant and non-routine problems.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-517 ***Proposed by Paul S. Bruckman, Seattle, WA***

Given a positive integer n, define the sums $P(n)$ and $Q(n)$ as follows:

$$P(n)=\sum_{d\mid n}\mu\left(\frac{n}{d}\right)L_d, \qquad Q(n)=\sum_{d\mid n}\Phi\left(\frac{n}{d}\right)L_d,$$

where μ and Φ and the Möbius and Euler functions, respectively. Show that $n\mid P(n)$ and $n\mid Q(n)$.

H-518 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n\ge 2$. Show that, for all complex numbers x and y and all positive integers n,

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)F_k(y)=(x-y)^{n-1}F_n\left(\frac{xy+4}{x-y}\right). \tag{1}$$

As special cases of (1), obtain the following identities:

$$\sum_{\substack{k=0\\ 5\nmid 2n-k-1}}^{2n-1}(-1)^{[(2n-k+1)/5]}\binom{4n-2}{k}=5^{n-1}L_{2n-1}; \tag{2}$$

$$\sum_{\substack{k=0\\ 5\nmid 2n-k}}^{2n}(-1)^{[(2n-k+2)/5]}\binom{4n}{k}=5^{n}F_{2n}; \tag{3}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_{3k}P_k=2^nF_n(6), \text{ where } P_k=F_k(2) \text{ is the } k^{\text{th}} \text{ Pell number}; \tag{4}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)F_k(x+1)=F_n(x^2+x+4); \tag{5}$$

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_k(x)F_k(4/x)=\frac{1-(-1)^n}{2}\left(\frac{x^2+4}{x}\right)^{n-1},\ x\neq 0; \tag{6}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)^2=(x^2+4)^{n-1};\tag{7}$$

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_k(x)^2=\frac{4^n-(-x^2)^n}{4+x^2};\tag{8}$$

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{2n}{n-2k-1}F_{2k+1}(x)=x^{n-1}F_n(4/x).\tag{9}$$

The latter equation is the one given in H-500. **Hint:** Deduce (1) from the main identity of H-492.

H-519 ***Proposed by Paul S. Bruckman, Seattle, WA***

Let p denote a prime $\equiv 1 \pmod 4$.

(a) Prove that, for all $p \not\equiv 1 \pmod{24}$, there exist positive integers k, u, and v such that
   (i) $k \mid u^2$;
   (ii) $p+4k=(4u-1)(4v-1)$.

(b) Prove or disprove the conjecture that the restriction $p \not\equiv 1 \pmod{24}$ in part (a) may be removed, i.e., part (a) is true for all $p \equiv 1 \pmod 4$.

H-520 ***Proposed by Andrej Dujella, University of Zagreb, Croatia***

Let n be an integer. Prove that there exist an infinite set $D \subseteq \mathbf{N}$ with the property that for all $c, d \in D$ the integer $cd+n$ is not square free.

SOLUTIONS

Complex Situation

H-502 ***Proposed by Zdzislaw W. Trzaska, Warsaw, Poland***
(Vol. 33, no. 4, August 1995)

Given two sequences of polynomials in the complex variable $z \in C$ defined recursively as

$$T_k(z)=\sum_{m=0}^{k}a_{km}z^m,\quad k=0,1,2,\ldots,\tag{i}$$

with $T_0(z)=1$ and $T_1(z)=(1+z)T_0$, and

$$P_k(z)=\sum_{m=0}^{k}b_{km}z^m,\quad k=0,1,2,\ldots,\tag{ii}$$

with $P_0(z)=0$ and $P_1(z)=1$.

Prove that, for all $z \in C$ and $k=0,1,2,\ldots$, the equality

$$P_k(z)T_{k-1}(z)-T_k(z)P_{k-1}(z)=1\tag{iii}$$

holds.

Solution by the proposer

From (i), we have

$$a_{kp} = \left.\frac{\partial^p T_k(z)}{\partial z^p}\right|_{z=0}, \quad p = 0, 1, 2, \ldots, \tag{1}$$

so that we can write

$$T_2(z) = a_{20} + a_{21}z + a_{22}z^2 \tag{2}$$

with

$$a_{20} = 1, \quad a_{21} = 3, \quad a_{22} = 1. \tag{3}$$

Thus, the polynomial $T_k(z)$ fulfills the relation

$$T_{k+1}(z) = (2+z)T_k(z) - T_{k-1}(z), \quad k = 1, 2, \ldots. \tag{4}$$

Similarly, we can write

$$P_{k+1}(z) = (2+z)P_k(z) - P_{k-1}(z), \quad k = 0, 1, 2, \ldots, \tag{5}$$

with $P_0(z) = 0$ and $P_1(z) = 1$.

Note that coefficients of both polynomials belong to modified numerical triangles MNT1 and MNT2, respectively (see [1]).

Substituting the above results into LHS of (iii) gives

$$\begin{aligned} \text{LHS(iii)} &= P_k(z)T_{k+1}(z) - [(2+z)T_{k-1}(z) - T_{k-1}(z)]P_{k-1}(z) \\ &= [P_k(z) - (2+z)P_{k-1}(z)]T_{k-1}(z) + P_{k-1}(z)T_{k-2}(z). \end{aligned} \tag{6}$$

Next, using (2) yields

$$\text{LHS(iii)} = -P_{k-2}(z)T_{k-1}(z) + P_{k-1}(z)T_{k-2}(z). \tag{7}$$

Thus, repeating the above procedure $(k-1)$ times, we finally get

$$\text{LHS(iii)} = P_1(z)T_0(z) - T_1(z)P_0(z). \tag{8}$$

But, from (i) and (ii), we obtain

$$\text{LHS(iii)} = 1, \tag{9}$$

which means that LHS(iii) = RHS(iii), thus completing the proof.

Note that another proof can be presented by using the mathematical induction approach.

Reference

1. Z. Trzaska. "On Numerical Triangles Showing Links with Chebyshev Polynomials." *C. Lanczos Int. Cent. Conf.*, December 12-17, 1993, NCSU, Raleigh, NC.

A Complex Product

H-503 ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 33, no. 5, November 1995)

Let $\mathcal{S}$ be the set of functions $F: C^3 \to C$ (C is the complex plane) satisfying the following formal properties:

$$xyz\,F(x, x^3y, x^3y^2z) = F(x, y, z); \tag{1}$$

$$F(x^{-1}, y, z^{-1}) = F(x, y, z). \tag{2}$$

Formally define the functions U and V as follows:

$$U(x,y,z)=\Sigma x^{n^3}y^{n^2}z^n \quad \text{(summed over all integers } n\text{);} \tag{3}$$

$$V(x,y,z)=\prod_{n=1}^{\infty}(1-y^{2n}A(x))(1+x^{3n^2-3n+1}y^{2n-1}z)(1+x^{-3n^2+3n-1}y^{2n-1}z^{-1}), \tag{4}$$

where

$$A(x)=\frac{\Sigma x^{3m}o_m}{\Sigma x^{3m}e_m} \quad \text{(summed over all integers } m\text{),} \tag{5}$$

$$o_m=\tfrac{1}{2}(1-(-1)^m), \qquad e_m=\tfrac{1}{2}(1+(-1)^m). \tag{6}$$

Show that, at least formally,

$$U\in\mathcal{S}, \qquad V\in\mathcal{S}; \tag{7}$$

$$A(1)=1; \tag{8}$$

$$U(1,y,z)=V(1,y,z). \tag{9}$$

Prove or disprove that $U(x,y,z)\equiv V(x,y,z)$ identically. Can $U(x,y,z)$ be factored into an infinite product?

Solution by the proposer

Proof that U satisfies (1): $xyzU(x,x^3y,x^3y^2z)=\Sigma x^{n^3+3n^2+3n+1}y^{n^2+2n+1}z^{n+1}=\Sigma x^{(n+1)^3}y^{(n+1)^2}z^{n+1}=\Sigma x^{n^3}y^{n^2}z^n=U(x,y,z)$.

Proof that U satisfies (2): $U(x^{-1},y,z^{-1})=\Sigma x^{-n^3}y^{n^2}z^{-n}=\Sigma x^{n^3}y^{(-n)^2}z^n=U(x,y,z)$. Therefore, $U\in\mathcal{S}$.

Proof that V satisfies (1): Let

$$P_n=P_n(x,y)=1-y^{2n}A(x),\ \ Q_n=Q_n(x,y,z)=1+x^{3n^2-3n+1}y^{2n-1}z,\ \ \overline{Q}_n=Q_n(x^{-1},y,z^{-1}).$$

Note that $x^{6n}A(x)=[\Sigma x^{3(m+2n)}o_m]\div[\Sigma x^{3m}e_m]=[\Sigma x^{3m}o_{m-2n}]\div[\Sigma x^{3m}e_m]=A(x)$; therefore, we have $P_n(x,x^3y)=1-y^{2n}x^{6n}A(x)=1-y^{2n}A(x)$, or

$$P_n(x,y)=P_n(x,x^3y). \tag{10}$$

Next,

$$\prod_{n=1}^{\infty}Q_n(x,x^3y,x^3y^2z)=\prod_{n=1}^{\infty}[1+x^{3n^2-3n+1+6n-3+3}y^{2n-1+2}z]$$
$$=\prod_{n=1}^{\infty}[1+x^{3n^2+3n+1}y^{2n+1}z]=\prod_{n=2}^{\infty}[1+x^{3n^2-3n+1}y^{2n-1}z],$$

or

$$\prod_{n=1}^{\infty}Q_n(x,x^3y,x^3y^2z)=(1+xyz)^{-1}\prod_{n=1}^{\infty}Q_n(x,y,z). \tag{11}$$

Also

$$\prod_{n=1}^{\infty}\overline{Q}_n(x,x^3y,x^3y^2z)=\prod_{n=1}^{\infty}[1+x^{-3n^2+3n-1+6n-3-3}y^{2n-1-2}z^{-1}]$$

$$= \prod_{n=1}^{\infty}[1 + x^{-3n^2+9n-7}y^{2n-3}z^{-1}] = \prod_{n=0}^{\infty}[1 + x^{-3n^2+3n-1}y^{2n-1}z^{-1}],$$

or

$$\prod_{n=1}^{\infty}\overline{Q}_n(x, x^3y, x^3y^2z) = (xyz)^{-1}(1+xyz)\prod_{n=1}^{\infty}\overline{Q}_n(x, y, z). \tag{12}$$

Combining (10), (11), and (12), we see that (at least formally),

$$V(x, x^3y, x^3y^2z) = \prod_{n=1}^{\infty} P_n(x, x^3y)Q_n(x, x^3y, x^3y^2z)\overline{Q}_n(x, x^3y, x^3y^2z)$$

$$= (xyz)^{-1}\prod_{n=1}^{\infty} P_n(x, y)Q_n(x, y, z)\overline{Q}_n(x, y, z),$$

or

$$xyzV(x, x^3y, x^3y^2z) = V(x, y, z). \tag{13}$$

Proof that V satisfies (2): Clearly $A(x^{-1}) = A(x)$, so $P_n(x^{-1}, y) = P_n(x, y)$; also, $\overline{Q}_n(x^{-1}, y, z^{-1}) = Q_n(x, y, z)$. It follows that $V(x^{-1}, y, z^{-1}) = V(x, y, z)$. Therefore, $V \in \mathcal{S}$.

Proof of (8): Let $A_n(x) = \sum_{-n}^{n} x^{3m}o_m / \sum_{-n}^{n} x^{3m}e_m$. We readily find that $\sum_{-n}^{n} o_m = n + o_n$ and $\sum_{-n}^{n} e_m = n + e_n$. Thus, $A_n(1) = (n+o_n)/(n+e_n)$. Taking limits, we have $A(1) = \lim_{n\to\infty} A_n(1) = 1$.

Proof of (9): Setting $x = 1$ in (3) and (4) [using (8)], we find that $U(1, y, z) = \Sigma y^{n^2}z^n$ and $V(1, y, z) = \prod_{n=1}^{\infty}(1-y^{2n})(1+y^{2n-1}z)(1+y^{2n-1}z^{-1})$.

From this, we recognize that (9) is merely a statement of the famous triple-product identity of Jacobi. This is intimately connected with the theory of elliptic functions and, in particular, the Theta-functions studied by Jacobi.

Although the above results seem quite interesting and seem to imply some relationship between U and V, this relationship appears to be illusory, except for certain values of x. For one thing, the above results have only been demonstrated formally; a more rigorous treatment leads one to the conclusion that the series defining U and the product defining V are divergent unless $|x| = 1$. Setting $x = \exp i\theta$, where θ is real, we may show that

$$A_n(x) = \left\{\frac{\sin 3(n+1)\theta}{\sin 3(n-1)\theta}\right\}^{(-1)^n}. \tag{14}$$

Otherwise, $A_n(x)$ has two distinct cluster points, namely, x^6 and x^{-6} (assuming $x \neq 0$), and therefore has no limit point, as $n \to \infty$. Even if $|x| = 1$, however, the expression in (14) has no limit points, except for a finite set of values of x. In fact, it is not difficult to deduce the following result from (14):

$$\begin{array}{l} A(x) = (-1)^{[\frac{1}{4}(k+3)]-[\frac{1}{4}k]}, \text{ if } x = \exp(ki\pi/6),\, k \text{ integral;} \\ \text{otherwise, } A(x) \text{ is undefined.} \end{array} \tag{15}$$

Thus, $A(x) = 1$ if $x^3 = 1$, in which case $P_n = 1 - y^{2n}$. Also, since $n^3 \equiv n \pmod 6$, we see that, if $x^6 = 1$, $U(x, y, z) = U(1, y, xz)$ and $V(x, y, z) = V(1, y, xz)$. Jacobi's identity states that these last two quantities must be equal.

Experience suggests that if two expressions such as U and V are equal for certain special values of x (e.g., for which $|x|=1$), one should be able to employ analytic continuation to extend the equality for $|x| \neq 1$. However, as we have found, this extension is impossible; thus, $U \not\equiv V$ identically. Another way to show this is as follows: Since $(1+y/xz)$ is a factor of V, it follows that V has a zero (qua function of z) at $z=-y/x$. However, if we set $z=-y/x$ in the series defining U, this yields an expression that does *not* vanish identically, namely, $U(x,y,-y/x)= \Sigma(-1)^n x^{n^3-n} y^{n^2+n} = -y^2(1-x^{-6})+y^6(x^6-x^{-24})-y^{12}(x^{24}-x^{-60})+\cdots$; however, this *does* vanish at the special values for which $x^6=1$.

The factorization of U, if any such exists (and this seems doubtful), must be exotic indeed, and remains an open question. Toward this end, it would seem desirable, if possible, to replace $A_n(x)$ by some other function that is better behaved, while still satisfying the appropriate criteria. Also, the functions Q_n and $\overline{Q}_n$ might, conceivably, by replaced by other, more esoteric expressions that still satisfy the desired conditions of the factorization problem. The general problem may be stated in the following way. Given $U(x,y,z)$ as defined by (3), find a factorization as follows:

$$U(x,y,z) \equiv \prod_{n=1}^{\infty} S_n(x,y,z) \text{ (valid for appropriate convergence criteria),} \tag{16}$$

where the S_n's satisfy the conditions:

$$S_n(x^{-1}, y, z^{-1}) = S_n(x,y,z); \tag{17}$$

$$xyz\prod_{n=1}^{\infty} S_n(x, x^3y, x^3y^2z) = \prod_{n=1}^{\infty} S_n(x,y,z); \tag{18}$$

$$S_n(1,y,z) = (1-y^{2n})(1+y^{2n-1}z)(1+y^{2n-1}z^{-1}). \tag{19}$$

The proposer of this research problem is indebted to his former mentor, A. O. L. Atkin at the University of Illinois in Chicago, for providing helpful hints and suggestions. Moreover, Dr. Atkin suggested another area of possible extension, namely, to work with the sum $\Sigma w^{n^4} x^{n^3} y^{n^2} z^n$; this latter sum has fewer convergence problems than the sum proposed in this problem but it has the undesirable quality of being more complicated. This is as far as this proposer took this problem. All comments are invited from the readers.

Also solved by A. Dujella.

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