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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

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COUNTING THE NUMBER OF EQUIVALENCE CLASSES OF (*m*, *F*) SEQUENCES AND THEIR GENERALIZATIONS

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1. INTRODUCTION

K. T. Atanassov and others, in [3], [1], and [2], introduced (2, *F*) and (3, *F*) sequences which were pairs and triples of sequences defined by two or three simultaneous Fibonacci-like recurrences, respectively, for which the exact definition will be given at the end of this section.

There are four (2, *F*) sequences, among which one is a pair of (1, *F*) sequences defined by the original Fibonacci recurrence and the other three are essential. As we are interested in the solutions of the systems of recurrence equations with the general initial conditions rather than the resulting sequences for some particular initial conditions, we call such a system a "(2, *F*) system." The (2, *F*) system consisting of two (1, *F*) recurrences is called a "separable (2, *F*) system," and the other three are called "inseparable (2, *F*) systems."

In the case of three sequences, some of the thirty-six (3, *F*) systems of simultaneous recurrence equations give the same triple of sequences apart from their order provided appropriate initial conditions. K. T. Atanassov [2] and W. R. Spickerman et al. [5] studied equivalence classes of (3, *F*) systems of recurrences which give essentially the same sequences and determined eleven classes. One of them consists of three (1, *F*) recurrence equations and three of them are separated into one (1, *F*) recurrence and an inseparable (2, *F*) system of recurrence equations. Therefore, we have seven classes of inseparable (3, *F*) systems of recurrence equations, for which the definition will be given in Section 4.

The purposes of this paper are to establish the method of counting the number of equivalence classes of (*m*, *F*) systems consisting of *m* Fibonacci-like recurrences and the number of classes of inseparable (*m*, *F*) systems, and give their values for small *m*. Furthermore, we apply the same method to $(m, F^{(f)})$ systems where the Fibonacci-like recurrences in (*m*, *F*) systems are replaced with f^{th}-order recurrences of type (1). More precisely, an $(m, F^{(f)})$ system is defined as follows.

Definition 1: A set of *m* recurrence equations

$$F_{n+1}^{(f)}(k) = F_n^{(f)}(\sigma_1(k)) + F_{n-1}^{(f)}(\sigma_2(k)) + \cdots + F_{n-f+1}^{(f)}(\sigma_f(k)) \quad (\text{for } n \geq f), \tag{1}$$

where $k = 1, 2, \ldots, m$ and $\sigma_1, \sigma_2, \ldots, \sigma_f$ are permutations belonging to the symmetric group S_m of order *m* is called an $(m, F^{(f)})$ system, and a set of *m* sequences $\{F_n^{(f)}(k)\}$, where $k = 1, 2, \ldots, m$ and $n = 1, 2, \ldots, \infty$, or a sequence of *m*-dimensional vectors that can be determined as the solutions of this system with given initial values $\{F_n^{(f)}(k)\}$, where $k = 1, 2, \ldots, m$ and $n = 1, 2, \ldots, f$, is called an $(m, F^{(f)})$ sequence. In particular, in the case $f = 2$, it is called an (*m*, *F*) sequence.

2. PREPARATION FROM GROUP THEORY

First, we recall a counting theorem given by Burnside.

Burnside's Theorem: Let G be a finite group of order $|G|$ operating on a finite set M. Then the number of distinct orbits associated with G is given by

$$\frac{1}{|G|}\sum_{g\in G}\lambda_1(g),$$

where $\lambda_1(g)$ is the number of fixed points in M by g.

The proof can be found, for instance, in [4] and will be omitted here.

Now, let p_m denote the number of conjugate classes in S_m, and let $b_i = |B_i|$ be the number of elements of the conjugate class B_i for $i = 1, 2, \ldots, p_m$. Each $\sigma \in S_m$ can be represented as the product of disjoint cycles uniquely up to their order. If σ is represented as the product of λ_1 cycles of length 1, λ_2 cycles of length 2, ..., λ_m cycles of length m, we say that it has the cycle type

$$1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m}, \tag{2}$$

where λ_1, λ_2, ..., λ_m are nonnegative integers satisfying

$$1\cdot\lambda_1 + 2\cdot\lambda_2 + \cdots + m\cdot\lambda_m = m. \tag{3}$$

Two permutations in S_m are conjugate if and only if they have the same cycle type since an element $\eta \in S_m$ satisfies $\eta\sigma\eta^{-1} = \sigma$ if and only if it does not change each cycle of σ or just make some permutations of the cycles of the same length. Since this gives also the condition that $\eta \in S_m$ satisfies $\eta\sigma = \sigma\eta$, the centralizers of the elements of B_i in S_m must have the same order, which will be denoted by c_i. Since all permutations in B_i have the same cycle type, we can represent it by (2). Then we have $b_i = m!/(\lambda_1!\lambda_2!\ldots\lambda_m!1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m})$ and

$$c_i = \lambda_1!\lambda_2!\ldots\lambda_m!1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m} \tag{4}$$

so that the relation

$$b_ic_i = |S_m| = m! \quad (\text{for } i = 1, 2, \ldots, p_m) \tag{5}$$

always holds.

The conjugate classes of cycle types of S_m bijectively correspond to the integer partitions of m, and an algorithm for listing them can be found, for example, in D. Stanton and D. White [6].

3. THE EQUIVALENCE CLASSES OF $(m, F^{(f)})$ SYSTEMS

First, we consider $(m, F^{(f)})$ systems. Following the manner that K. T. Atanassov did for $m = 2$ and 3, for each $m > 0$, an $(m, F^{(f)})$ system is defined by m simultaneous recurrence equations $F_{n+1}(k) = F_n(\sigma_1(k)) + F_{n-1}(\sigma_2(k))$, for $n \geq 3$, where $k = 1, 2, \ldots, m$ and σ_1 and σ_2 are permutations in S_m. This is the special case of $(m, F^{(f)})$ systems of recurrence equations defined by (1) for $f = 2$. If we give any initial values $F_n(k)$, where $k = 1, 2, \ldots, m$ and $n = 1, 2$, then an (m, F) sequence $\{F_n(k)\}$, where $k = 1, 2, \ldots, m$ and $n = 1, 2, \ldots, \infty$, will be determined by these recurrences. Since this (m, F) system is determined depending only on σ_1 and σ_2, it will be denoted by $S(\sigma_1, \sigma_2)$.

Definition 2: Two (m, F) systems $S(\sigma_1, \sigma_2)$ and $S(\tau_1, \tau_2)$ are said to be equivalent if there is an $\eta \in S_m$ such that $\eta\sigma_1\eta^{-1} = \tau_1$ and $\eta\sigma_2\eta^{-1} = \tau_2$ are satisfied.

It is shown in W. R. Spickerman et al. [5] that two $(3, F)$ systems are equivalent if and only if they define the same triple of sequences up to their order by choosing appropriate initial values of one of them for the given initial values of the other.

We define the operation of $\eta \in S_m$ on the system $S(\sigma_1, \sigma_2)$ by

$$\eta(S(\sigma_1, \sigma_2)) = S(\eta\sigma_1\eta^{-1}, \eta\sigma_2\eta^{-1}). \tag{6}$$

Assuming that the group acts on the set $M = \{S(\sigma_1, \sigma_2) \mid \sigma_1, \sigma_2 \in S_m\}$ in this manner, we apply Burnside's theorem.

Let η be an element of S_m. Then η leaves $S(\sigma_1, \sigma_2)$ fixed if and only if $\eta\sigma_1\eta^{-1} = \sigma_1$ and $\eta\sigma_2\eta^{-1} = \sigma_2$, or $\eta\sigma_1 = \sigma_1\eta$ and $\eta\sigma_2 = \sigma_2\eta$. If $\eta \in B_i$, the number of such σ_1 and σ_2 are both c_i, so that c_i^2 of $S(\sigma_1, \sigma_2)$ will be fixed by η. Since we have b_i permutations in B_i, the number of systems fixed by permutations in a conjugate class B_i sums to $b_i c_i^2$. If we denote the number of distinct orbits in M associated with S_m, i.e., the number of equivalence classes in M by $N(m, F)$, using Burnside's theorem and relation (5), we can represent it as

$$N(m, F) = (\Sigma b_i c_i^2)/|S_m| = \Sigma c_i, \tag{7}$$

where the summation is taken over all the conjugate classes of S_m, and we can evaluate this value by (4).

We can easily generalize this result to the $(m, F^{(f)})$ system $S(\sigma_1, \sigma_2, \ldots, \sigma_f)$ which is defined by the recurrences (1).

Definition 3: Two $(m, F^{(f)})$ systems $S(\sigma_1, \sigma_2, \ldots, \sigma_f)$ and $S(\tau_1, \tau_2, \ldots, \tau_f)$ are said to be equivalent if there is an $\eta \in S_m$ such that $\eta\sigma_1\eta^{-1} = \tau_1$, $\eta\sigma_2\eta^{-1} = \tau_2, \ldots,$ and $\eta\sigma_f\eta^{-1} = \tau_f$ are satisfied.

Using the operation of $\eta \in S_m$ on $(m, F^{(f)})$ systems defined by

$$\eta(S(\sigma_1, \sigma_2, \ldots, \sigma_f)) = S(\eta\sigma_1\eta^{-1}, \eta\sigma_2\eta^{-1}, \ldots, \eta\sigma_f\eta^{-1}) \tag{8}$$

instead of (6), we will have the formula for the number of equivalence classes of $(m, F^{(f)})$ systems $N(m, F^{(f)})$, in a manner similar to the case of (m, F) systems as

$$N(m, F^{(f)}) = (\Sigma b_i c_i^f)/|S_m| = \Sigma c_i^{f-1}.$$

Thus, we have the following theorem.

Theorem 1: The number, $N(m, F^{(f)})$, of equivalence classes of the set of $(m, F^{(f)})$ systems $S(\sigma_1, \sigma_2, \ldots, \sigma_f)$ defined by the recurrences (1) is given by $N(m, F^{(f)}) = \Sigma c_i^{f-1}$, where $c_i = \lambda_1!\lambda_2!\ldots\lambda_m!1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m}$, and the summation is taken over p_m congruent classes in S_m corresponding to the sets of nonnegative integers $\lambda_1, \lambda_2, \ldots, \lambda_m$ satisfying (3). In particular, for $f = 2$, we have $N(m, F) = \Sigma c_i$.

For $f = 1$, the value of $N(m, F^{(1)})$ represents the number p_m of congruent classes in S_m, which is also the number of integer partitions of m. This number can be calculated by any algorithm for finding all the cycle types in S_m.

If $p_k(r)$ denotes the number of integer partitions of k into exactly r parts, we can also calculate the value of p_m directly using the following properties:

(i) For $k > 0$, $p_k(1) = p_k(k) = 1$, and $p_k(r) = 0$ if $r > k$.

(ii) If $k > r > 0$, $p_k(r) = p_{k-r}(1) + p_{k-r}(2) + \cdots + p_{k-r}(r)$.

(iii) $p_m = p_m(1) + p_m(2) + \cdots + p_m(m)$.

The values of $N(m, F^{(f)})$ for small m and f are shown in Table 1.

TABLE 1

f \ m	1	2	3	4	5	6	7
1	1	2	3	5	7	11	15
2	1	4	11	43	161	901	5579
3	1	8	49	681	14721	524137	25471105
4	1	16	251	14491	1730861	373486525	128038522439

4. THE NUMBER OF INSEPARABLE EQUIVALENCE CLASSES

As we have stated for the case $m = 2, 3$ and $f = 2$, some of the $(m, F^{(f)})$ systems can be separated into smaller systems.

Definition 4: An $(m, F^{(f)})$ system $S = S(\sigma_1, \sigma_2, \ldots, \sigma_f)$ is separable if there exists a nonempty proper subset M' of $M = \{1, 2, .., m\}$ such that M' is stable (mapped into itself) by the permutations $\sigma_1, \sigma_2, \ldots, \sigma_f$. Then the system (1) can be partitioned into an $(m', F^{(f)})$ system and an $(m'', F^{(f)})$ system corresponding to M' and its relative complement $M'' = M - M'$, where $|M'| = m'$ and $|M''| = m''$, and S is separated into an $(m', F^{(f)})$ system $S'(\sigma_1', \sigma_2', \ldots, \sigma_f')$ and an $(m'', F^{(f)})$ system $S''(\sigma_1'', \sigma_2'', \ldots, \sigma_f'')$, where σ_s' and σ_s'' are restrictions of σ_s on M' and M'', respectively, for $s = 1, 2, \ldots, f$. Otherwise, S is said to be inseparable.

Definition 5: An $(m, F^{(f)})$ system S is said to have type $T = 1^{\lambda_1}2^{\lambda_2} \ldots m^{\lambda_m}$, if it can be divided into $\lambda_1(1, F^{(f)})$ systems, $\lambda_2(2, F^{(f)})$ systems, ..., and $\lambda_m(m, F^{(f)})$ systems that are inseparable, where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are nonnegative integers satisfying (3). If $\lambda_t > 0$, S has a subsystem of type t^{λ_t} consisting of λ_t inseparable $(t, F^{(f)})$ systems, which is referred to as the t-part of S. If $\lambda_t = 0$, we say that the t-part of S is empty.

Besides the symbol $N(m, F^{(f)})$ defined above, we need the following notations.

<u>**Notations**</u>

$S(m, F^{(f)})$: The number of equivalence classes of separable $(m, F^{(f)})$ systems.

$I(m, F^{(f)})$: The number of equivalence classes of inseparable $(m, F^{(f)})$ systems.

$N[T, F^{(f)}]$: The number of equivalence classes of $(m, F^{(f)})$ systems of type T.

When we discuss a fixed f, we sometimes abbreviate the above symbols as $N(m)$, $S(m)$, $I(m)$, and $N[T]$, omitting $F^{(f)}$.

$H(n,r)$: The number of r-combinations with repetition of n distinct things, which is given by

$$H(n,r)=\binom{n+r-1}{r}=\frac{(n+r-1)!}{(n-1)!\,r!},$$

where we use the convention $H(n,0)=1$ for $n>0$ as usual.

Using the notations defined above, we can state the next theorem.

Theorem 2: The numbers $N[1^{\lambda_1}2^{\lambda_2}\dots m^{\lambda_m}]$, $S(m)$, and $I(m)$ are given by the following formulas:

$$N[1^{\lambda_1}2^{\lambda_2}\dots m^{\lambda_m}]=\Pi H(I(t),\lambda_t), \tag{9}$$

where the product is taken over $t=1,2,\dots,m$;

$$S(m)=\Sigma N[1^{\lambda_1}2^{\lambda_2}\dots(m-1)^{\lambda_{m-1}}] \quad\text{and}\quad I(m)=N[m^1]=N(m)-S(m),$$

where the summation is taken over all the integer partitions of m into more than one part or all of the $(m-1)$-tuples of nonnegative integers $\lambda_1,\lambda_2,\dots,\lambda_{m-1}$ satisfying

$$1\cdot\lambda_1+2\cdot\lambda_2+\dots+(m-1)\lambda_{m-1}=m.$$

Proof: Let $S=S(\sigma_1,\sigma_2,\dots,\sigma_f)$ be an $(m,F^{(f)})$ system defined by (1). A system $\eta S(\sigma_1,\sigma_2,\dots,\sigma_f)$ equivalent to S, which is defined by (8), is given by replacing functions $F_s^{(f)}(x)$ in all terms of (1) with $F_s^{(f)}(\eta(x))$ for $s=n+1,n,n-1,\dots,n-f+1$, and rearranging the m equations so that $\eta(k)$'s of the left-hand side become increasing in order. If the $(m,F^{(f)})$ system S is separable, then the nonempty subsets M' and M'' in Definition 4, which are stable by $\sigma_1,\sigma_2,\dots,\sigma_f$, are mapped onto $\eta(M')$ and $\eta(M'')$, which are complements of each other and are stable by $\eta\sigma_1\eta^{-1},\eta\sigma_2\eta^{-1},\dots,\eta\sigma_f\eta^{-1}$. Therefore, it is clear that two equivalent systems have the same type and two systems of the same type are equivalent if and only if their t-parts are equivalent for $t=1,2,\dots,m$.

The equivalence class of the t-part of S will be determined by the classes of $I(t)$ to which $\lambda_t(t,F^{(f)})$ subsystems of S belong, not depending on the location or the variables used in them. Therefore, the number of equivalence classes of the t-part with type t^{λ_t} of $(m,F^{(f)})$ systems is the number of λ_t-combinations with repetition taken from $I(t)$, which is denoted by $H(I(t),\lambda_t)$. Since different choices of an equivalence class for any t-part give different equivalence classes of $(m,F^{(f)})$ systems of type $T=1^{\lambda_1}2^{\lambda_2}\dots m^{\lambda_m}$, their number will be represented by (9).

Since $N(m,F^{(f)})$ is the sum of expression (9) for all the solutions of equation (3), and the only solution of (3) with $\lambda_m>0$ is given by $\lambda_1=\lambda_2=\dots=\lambda_{m-1}=0$ and $\lambda_m=1$, and the type of an inseparable $(m,F^{(f)})$ system is m^1, we have

$$S(m)=N(m)-I(m)=N(m)-N[m^1]=\Sigma N[1^{\lambda_1}2^{\lambda_2}\dots(m-1)^{\lambda_{m-1}}],$$

and the proof is completed.

Since we have only one equivalence class for $(1,F^{(f)})$ system, the number of equivalence classes of $(m,F^{(f)})$ systems of type $1^{\lambda_1}2^{\lambda_2}\dots(m-1)^{\lambda_{m-1}}$ for which $\lambda_1>0$ must be equal to the number of equivalence classes of $(m-1,F^{(f)})$ systems of type $1^{\lambda_1}2^{\lambda_2}\dots(m-1)^{\lambda_{m-1}}$, so the total number of equivalence classes with nonempty 1-parts of $(m,F^{(f)})$ systems is equal to $N(m-1)$.

Since an $(m, F^{(f)})$ system with an empty 1-part cannot have an $(m-1)$-part, we have another expression for $S(m)$ and $I(m)$ that is useful for inductive calculation.

Corollary: $S(m) = N(m-1) + \Sigma H(I(2), \lambda_2) H(I(3), \lambda_3) \ldots H(I(m-2), \lambda_{m-2})$, where summation is taken over all the nonnegative integers $\lambda_2, \lambda_3, \ldots, \lambda_{m-2}$ satisfying $2 \cdot \lambda_2 + 3 \cdot \lambda_3 + \cdots + (m-2) \cdot \lambda_{m-2} = m$ and $I(m) = N[m^1] = N(m) - S(m)$.

The numbers of equivalence classes $N[T]$ for T with small values of m and $f = 2$ and 3 are given in Table 2, where the number $I(m) = N[m^1]$ of equivalence classes of inseparable $(m, F^{(f)})$ systems can be found in the right-most column for each m.

TABLE 2

$m = 2$

$f \backslash T$	1^2	2^1
2	1	3
3	1	7

$m = 3$

$f \backslash T$	1^3	$1^1 2^1$	3^1
2	1	3	7
3	1	7	41

$m = 4$

$f \backslash T$	1^4	$1^2 2^1$	$1^1 3^1$	2^2	4^1
2	1	3	7	6	26
3	1	7	41	28	604

$m = 5$

$f \backslash T$	1^5	$1^3 2^1$	$1^2 3^1$	$1^1 2^2$	$1^1 4^1$	$2^1 3^1$	5^1
2	1	3	7	6	26	21	97
3	1	7	41	28	604	287	13753

$m = 6$

$f \backslash T$	1^6	$1^4 2^1$	$1^3 3^1$	$1^2 2^2$	$1^2 4^1$	$1^1 2^1 3^1$	$1^1 5^1$	2^3	$2^1 4^1$	3^2	6^1
2	1	3	7	6	26	21	97	10	78	28	624
3	1	7	41	28	604	287	13753	84	4228	861	504243

$m = 7$

$f \backslash T$	1^7	$1^5 2^1$	$1^4 3^1$	$1^3 2^2$	$1^3 4^1$	$1^2 2^1 3^1$	$1^2 5^1$	$1^1 2^3$	$1^1 2^1 4^1$	$1^1 3^2$	$1^1 6^1$	$2^2 3^1$	$2^1 5^1$	$3^1 4^1$	7^1
2	1	3	7	6	26	21	97	10	78	28	624	42	291	182	4163
3	1	7	41	28	604	287	13753	84	4228	861	504243	1148	96271	24764	24824785

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ON MULTIPLICITY SEQUENCES

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The concept of divisibility sequence is quite popular in the mathematical literature. Starting from [1], where Marshall Hall called a sequence g of rational integers a **divisibility sequence** iff

$$\forall_{m,n\in N} \quad m|n \Rightarrow g(m)|g(n), \qquad \textbf{(DS)}$$

numerous papers appeared (see, e.g., [6], [7]). Another study of such sequences was initiated by Kimberling who in [2] called g a **strong divisibility sequence** iff

$$\forall_{m,n\in N} \quad \text{G.C.D.}(g(m), g(n)) = g(\text{G.C.D.}(m, n)). \qquad \textbf{(SDS)}$$

It is obvious that $\mathbf{SDS} \Rightarrow \mathbf{DS}$. If we take a sequence g defined by $g(2^k(2m+1)) = 2^{k(2m+1)}$, we get a **DS** sequence which is not a **SDS** sequence.

The problem of characterizing polynomial **DS** sequences was taken up in [3] and [4]. It was proved in [4] that polynomial **DS** sequences are exactly those of the form $g(n) = an^k$.

As the concept of LCM of rational integers is "parallel" with the GCD of rational integers, it is natural to introduce the following definition: g is a **multiplicity sequence** iff

$$\forall_{m,n\in N} \quad \text{L.C.M.}(g(m), g(n)) = g(\text{L.C.M.}(m, n)). \qquad \textbf{(MS)}$$

The sequence of the Fibonacci numbers is a **SDS** sequence but not a **MS** sequence. Another example of **SDS** not **MS** sequence is $g(n) = 2^n - 1$.

Theorem: $\mathbf{MS} \Rightarrow \mathbf{SDS}$.

Proof:

First step. We shall assume that g is multiplicative $(\text{G.C.D.}(m, n) = 1 \Rightarrow g(mn) = g(m)g(n))$. In this case, we actually have $\mathbf{MS} \Leftrightarrow \mathbf{SDS}$. In fact, let us note that for the multiplicative sequence g we have

$$g(m)g(n) = g(\text{G.C.D.}(m, n))g(\text{L.C.M.}(m, n)) \qquad (1)$$

for any $m, n \in N$. So, if g is **MS**, then by (1) we get

$$g(\text{G.C.D.}(m, n)) = \frac{g(m)g(n)}{g(\text{L.C.M.}(m, n))} = \frac{g(m)g(n)}{\text{L.C.M.}(g(m), g(n))} = \text{G.C.D.}(g(m), g(n)).$$

Analogously, we can show that $\mathbf{SDS} \Rightarrow \mathbf{MS}$.

Second step. Suppose g is a **MS** sequence. Thus,

$$g(m)|\text{L.C.M.}(g(m), g(n)) = g(\text{L.C.M.}(m, n)),$$
$$g(n)|\text{L.C.M.}(g(m), g(n)) = g(\text{L.C.M.}(m, n)),$$

and $g(m)g(n) = cg(\text{L.C.M.}(m, n))$. Therefore, if $\text{G.C.D.}(m, n) = 1$, then

$$g(m)g(n) = cg(mn). \qquad (2)$$

Sequences (functions) that satisfy (2) are called **quasi-multiplicative** (see [5]). We note that $c = g(1)$ and $G(n) = \frac{g(n)}{g(1)}$ is **MS**, which is also multiplicative. Hence,

$$\begin{aligned} \text{G.C.D.}(g(m), g(n)) &= \text{G.C.D.}(g(1)G(m), g(1)G(n)) \\ &= g(1)\text{G.C.D.}(G(m), G(n)) \\ &= g(1)G(\text{G.C.D.}(m, n)) \\ &= g(\text{G.C.D.}(m, n)). \end{aligned}$$

Remark: It follows from the Theorem and from Monzingo's result [4] that if $g(n)$ is a polynomial **MS** sequence, then $g(n) = an^k$.

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RETIREMENT OF SUBSCRIPTION MANAGER

When Richard Vine retired as an administrator at Lockheed Corporation some years ago, The Fibonacci Association was the lucky winner because Richard brought all of his very able talents to his job as Subscription Manager of *The Fibonacci Quarterly*. Richard also belonged to a local tennis club where he was active on the court as well as with administrative duties. Furthermore, Richard had an extremely beautiful voice and sang as a professional actor in such plays as "Paint Your Wagon." Frequently, when conversing with Richard over the phone or while he was visiting with a local member of the Board of Directors concerning a Fibonacci chore, he would tell the story of the week from the tennis club. To wit: What do you get when you cross a pitbull with a collie?...A dog that bites you and then goes for help.

After 17 years of taking subscription and book orders with an extra bit of special care and flair, Richard Vine has decided to retire as our Subscription Manager. Richard, the members of the Board of Directors of the Fibonacci Association and the Editor of *The Fibonacci Quarterly*, who never could have done his job so well without your help, want to offer you a big **thank you** for a job splendidly done. You shall definitely be missed.

MODIFIED DICKSON POLYNOMIALS

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1. INTRODUCTION AND PRELIMINARIES

The aim of this paper is to extend the previous work [1] by considering the polynomials

$$Z_n(y) \stackrel{\text{def}}{=} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} y^{\lfloor n/2 \rfloor - j} \quad (n \geq 1) \tag{1.1}$$

in the indeterminate y, where the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function. It can be seen that

$$Z_n(y) = \begin{cases} p_n(y^{1/2}, 1) & (n \text{ even}), \\ y^{-1/2} p_n(y^{1/2}, 1) & (n \text{ odd and } y \neq 0), \end{cases} \tag{1.2}$$

where $p_n(y, 1)$ are the *Dickson polynomials* in y with the parameter $c = 1$ (e.g., see (1.1) of [1]). Because of the relation (1.2), the quantities $Z_n(y)$ will be referred to as *modified Dickson polynomials*. Information on theoretical aspects and practical applications of (usual) Dickson polynomials can be found through the exhaustive list of references reported in [1], where an extension of them has been studied.

In this article we are concerned with modified Dickson polynomials taken at nonnegative integers. In fact, it is the purpose of this article to establish basic properties of the elements of the sequences of integers $\{Z_n(k)\}_0^\infty$ $(k = 0, 1, 2, \ldots)$. More precisely, in Section 2 closed-form expressions for $Z_n(k)$ are found which, for $k = 2, 3$, and 4, give rise to three supposedly new combinatorial identities. Several identities involving $Z_n(k)$ are exhibited in Section 3, while some congruence properties of these numbers are established in Section 4.

To obtain the results presented in Sections 2 and 3, we make use of the main properties of the generalized Fibonacci numbers $U_n(x)$ and the generalized Lucas numbers $V_n(x)$ (e.g., see [2], [8]) defined by

$$U_n(x) = xU_{n-1}(x) + U_{n-2}(x), \quad [U_0(x) = 0, U_1(x) = 1], \tag{1.3}$$

$$V_n(x) = xV_{n-1}(x) + V_{n-2}(x), \quad [V_0(x) = 2, V_1(x) = x], \tag{1.4}$$

where x is an arbitrary (possibly complex) quantity. Recall that closed-form expressions (Binet forms) for $U_n(x)$ and $V_n(x)$ are

$$\begin{cases} U_n(x) = (\alpha_x^n - \beta_x^n) / \Delta_x, \\ V_n(x) = \alpha_x^n + \beta_x^n, \end{cases} \tag{1.5}$$

where

$$\begin{cases} \Delta_x = \sqrt{x^2 + 4}, \\ \alpha_x = (x + \Delta_x) / 2, \\ \beta_x = (x - \Delta_x) / 2. \end{cases} \tag{1.6}$$

As an illustration, the numbers $Z_n(k)$ are displayed in Table 1 for the first few values of k and n. From (1.1), we can observe that $Z_0(k)$ yields the indeterminate form $0/0$. For the sake of completeness, we assume that

$$Z_0(k) \overset{\text{def}}{=} 2 \ \forall k. \tag{1.7}$$

It can be checked readily that all the results established throughout the paper are consistent with the assumption (1.7).

TABLE 1. The Numbers $Z_n(k)$ for $0 \le n, k \le 8$

n \ k	0	1	2	3	4	5	6	7	8
0	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1
2	−2	−1	0	1	2	3	4	5	6
3	−3	−2	−1	0	1	2	3	4	5
4	2	−1	−2	−1	2	7	14	23	34
5	5	1	−1	−1	1	5	11	19	29
6	−2	2	0	−2	2	18	52	110	198
7	−7	1	1	−1	1	13	41	91	169
8	2	−1	2	−1	2	47	194	527	1154

2. CLOSED-FORM EXPRESSIONS FOR $Z_n(k)$

The following identity (see [3]) plays a crucial role in the proofs of the results established in this article.

$$Z_n(\Delta_x^2) = \begin{cases} V_n(x) & (n \text{ even}), \\ U_n(x) & (n \text{ odd}). \end{cases} \tag{2.1}$$

As particular cases of (2.1), we have

$$Z_n(5) = \begin{cases} V_n(1) = L_n & (n \text{ even}), \\ U_n(1) = F_n & (n \text{ odd}), \end{cases} \tag{2.2}$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively, and

$$Z_n(8) = \begin{cases} V_n(2) = Q_n & (n \text{ even}), \\ U_n(2) = P_n & (n \text{ odd}), \end{cases} \tag{2.3}$$

where P_n and Q_n are the n^{th} Pell and Pell-Lucas numbers, respectively (e.g., see [6]).

2.1 Results

A closed-form expression for $Z_n(k)$ which is valid for all k can be obtained readily from (2.1), (1.5), and (1.6). Namely, we get

$$Z_n(k) = \begin{cases} V_n(\sqrt{k-4}) & (n \text{ even}), \\ U_n(\sqrt{k-4}) & (n \text{ odd}). \end{cases} \tag{2.4}$$

It is worth mentioning that using (2.4) along with an interesting result established by Melham and Shannon [9, (5.1)-(5.3)] allows us to state that the terms of $\{Z_n(k)\}$ are generated by the powers of the 2-by-2 matrix $\mathbf{M}_k$ defined as

$$\mathbf{M}_k = \begin{bmatrix} k-2 & \sqrt{k-4} \\ \sqrt{k-4} & 2 \end{bmatrix}. \tag{2.5}$$

More precisely, it can be seen that the lower-right entry of $\mathbf{M}_k^n$ equals $k^{\lfloor n/2 \rfloor} Z_{n-1}(k)$.

As we shall see in the following, for $k = 1, 2, 3$, and 4, the corresponding value of $Z_n(k)$ is periodic, and (2.4) produces some interesting combinatorial identities. The proofs of these results are given in subsection 2.2.

The trivial case $k = 0$ is treated here only for the sake of completeness. This can be solved readily on the basis of the usual convention (e.g., see [10, p. 147])

$$0^h = \begin{cases} 1, & \text{if } h = 0, \\ 0, & \text{if } h > 0. \end{cases} \tag{2.6}$$

In fact, from (1.1) and (2.6), we have

$$Z_n(0) = \frac{n}{n - \lfloor n/2 \rfloor} \binom{n - \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} (-1)^{\lfloor n/2 \rfloor} = \begin{cases} 2(-1)^{n/2} & (n \text{ even}), \\ n(-1)^{(n-1)/2} & (n \text{ odd}). \end{cases} \tag{2.7}$$

The case $k = 1$ gives rise to a particularly interesting combinatorial identity. Its solution (credited to Hardy, 1924), which is reported in [10, p. 77], contains several misprints. In [4] we proved that

$$Z_n(1) = \begin{cases} 2(-1)^n, & \text{if } n \equiv 0 \pmod 3, \\ (-1)^{n+1}, & \text{otherwise}. \end{cases} \tag{2.8}$$

In this article we give a simpler proof of (2.8) which is obtained by using the Binet forms for $U_n(x)$ and $V_n(x)$, and certain trigonometric identities. For $2 \le k \le 4$, we get the identities

$$Z_n(2) = \begin{cases} -2, & \text{if } n \equiv 4, \\ -1, & \text{if } n \equiv \pm 3, \\ 0, & \text{if } n \equiv \pm 2 \quad (\text{mod } 8), \\ 1, & \text{if } n \equiv \pm 1, \\ 2, & \text{if } n \equiv 0, \end{cases} \tag{2.9}$$

$$Z_n(3) = \begin{cases} -2, & \text{if } n \equiv 6, \\ -1, & \text{if } n \equiv \pm 4 \text{ or } \pm 5, \\ 0, & \text{if } n \equiv \pm 3 \qquad (\text{mod } 12) \\ 1, & \text{if } n \equiv \pm 1 \text{ or } \pm 2, \\ 2, & \text{if } n \equiv 0, \end{cases} \tag{2.10}$$

and

$$Z_n(4) = \begin{cases} 1 & (n \text{ odd}), \\ 2 & (n \text{ even}). \end{cases} \tag{2.11}$$

2.2 Proofs

The proofs of (2.8)-(2.11) are similar so that, for the sake of brevity, we prove only (2.8) and (2.9).

Proof of (2.8): Denoting the imaginary unit by i, from (2.4) write

$$Z_n(1) = \begin{cases} V_n(i\sqrt{3}) & (n \text{ even}), \\ U_n(i\sqrt{3}) & (n \text{ odd}), \end{cases} \tag{2.12}$$

whence, on using the Binet forms (1.5),

$$\begin{aligned} Z_n(1) &= [(i\sqrt{3}+1)/2]^n + (-1)^n[(i\sqrt{3}-1)/2]^n \\ &= \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + (-1)^n\left[\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right]. \end{aligned} \tag{2.13}$$

Using (2.13) along with the trigonometric identities

$$\sin\frac{n\pi}{3} = (-1)^{n+1}\sin\frac{2n\pi}{3} \tag{2.14}$$

and

$$\cos\frac{n\pi}{3} = \begin{cases} (-1)^n, & \text{if } 3|n, \\ (-1)^{n+1}/2, & \text{otherwise}, \end{cases} \tag{2.15}$$

yields identity (2.8). Q.E.D.

Proof of (2.9): From (2.4) write

$$Z_n(2) = \begin{cases} V_n(i\sqrt{2}) & (n \text{ even}), \\ U_n(i\sqrt{2}) & (n \text{ odd}), \end{cases} \tag{2.16}$$

whence, on using (1.5),

$$\begin{aligned} Z_n(2) &= \begin{cases} [(i\sqrt{2}+\sqrt{2})/2]^n + [(i\sqrt{2}-\sqrt{2})/2]^n & (n \text{ even}), \\ \frac{1}{\sqrt{2}}\{[(i\sqrt{2}+\sqrt{2})/2]^n - [(i\sqrt{2}-\sqrt{2})/2]^n\} & (n \text{ odd}), \end{cases} \\ &= \begin{cases} \cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4} & (n \text{ even}), \\ \frac{1}{\sqrt{2}}\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} - \cos\frac{3n\pi}{4} - i\sin\frac{3n\pi}{4}\right] & (n \text{ odd}). \end{cases} \end{aligned} \tag{2.17}$$

Using (2.17) along with the trigonometric identities

$$\sin\frac{n\pi}{4} = (-1)^{n+1}\sin\frac{3n\pi}{4} \tag{2.18}$$

and

$$\cos\frac{n\pi}{4} = (-1)^n \cos\frac{3n\pi}{4} = \begin{cases} -1, & \text{if } n \equiv 4, \\ -1/\sqrt{2}, & \text{if } n \equiv \pm 3, \\ 0, & \text{if } n \equiv \pm 2 \pmod 8, \\ 1/\sqrt{2}, & \text{if } n \equiv \pm 1, \\ 1, & \text{if } n \equiv 0, \end{cases} \tag{2.19}$$

yields identity (2.9). Q.E.D.

3. SOME IDENTITIES INVOLVING $Z_n(k)$

Some simple identities involving the numbers $Z_n(k)$ (or simply Z_n if no misunderstanding can arise) are exhibited in this section. Most of the proofs are left as an exercise for the interested reader. First, we get the recurrences

$$Z_{n+1} - Z_n = \begin{cases} -Z_{n-1} & (n \text{ even}), \\ Z_{n+2} & (n \text{ odd}). \end{cases} \tag{3.1}$$

Then, we observe that, for n even, identity (3.1) is a special case (namely, $m = 1$) of the more general identity

$$Z_{n+m} + Z_{n-m} = \begin{cases} kZ_nZ_m, & \text{if } n \text{ and } m \text{ are odd}, \\ Z_nZ_m, & \text{otherwise}, \end{cases} \tag{3.2}$$

which can be proved by using (2.4) and the identities (3)-(8) of [7, p. 94]. It is worth noting that letting n be a suitable function of m in (3.2) yields

$$Z_nZ_{n-1} = Z_{2n-1} + 1 \quad (n = m+1), \tag{3.3}$$

$$Z_{2n} = \begin{cases} Z_n^2 - 2 & (n \text{ even}) \\ kZ_n^2 - 2 & (n \text{ odd}) \end{cases} \quad (n = m), \tag{3.4}$$

$$\begin{aligned} Z_{3n} &= Z_n(Z_{2n} - 1) \quad (n = 2m) \\ &= \begin{cases} Z_n^3 - 3Z_n & (n \text{ even}) \\ kZ_n^3 - 3Z_n & (n \text{ odd}) \end{cases} \quad \text{[from (3.4)]}. \end{aligned} \tag{3.5}$$

More generally, for $h = 1, 2, 3, \ldots$, we get the multiplication formula

$$Z_{hn} = \sum_{j=0}^{\lfloor h/2 \rfloor} (-1)^j \frac{h}{h-j}\binom{h-j}{j}\begin{bmatrix} \cdot Z_n^{h-2j} & (n \text{ even}), \\ \cdot Z_n^{h-2j} k^{\lfloor h/2 \rfloor - j} & (n \text{ odd}). \end{bmatrix} \tag{3.6}$$

Induction on h provides the required proof. Observe that, for n even, Z_{hn} and the Dickson polynomial $p_h(Z_n, 1)$ coincide, whereas, for n odd and h even, $Z_{hn}(k) = Z_h(kZ_n^2(k))$.

The Simson formula analog for the sequence $\{Z_n(k)\}$ is

$$Z_n^2 - Z_{n-1}Z_{n+1} = \frac{(-1)^n(k-1)Z_{2n} + 2}{k} + \begin{bmatrix} 1 & (n \text{ even}) \\ 2 - k & (n \text{ odd}). \end{bmatrix} \tag{3.7}$$

Properties of the matrix $\mathbf{M}_k$ [see (2.5)] are useful tools for discovering combinatorial identities involving Z_n. For example, denoting by $\mathbf{I}$ the 2-by-2 identity matrix, we can expand the identity (see (5.8) of [9])

$$[k(\mathbf{M}_k - \mathbf{I})]^n = \mathbf{M}_k^{2n}, \tag{3.8}$$

and equate the lower-right entries on both sides to obtain

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} k^{\lfloor j/2 \rfloor} Z_{j-1} = (-1)^n Z_{2n-1}. \tag{3.9}$$

Remark: The assumption $Z_{-1} = 1 \, \forall k$ is implied by the definition $\mathbf{M}_k^0 = \mathbf{I}$. The same result can be obtained by using (3.1) and (1.7).

Analogously, after noting that $\mathbf{M}_k^{-1} = \mathbf{I} - \mathbf{M}_k / k$, we can expand the identity $\mathbf{M}_k^n \mathbf{M}_k^{-h} = \mathbf{M}_k^{n-h}$ to get the relation

$$\sum_{j=0}^{h} (-1)^j \binom{h}{j} k^{\lfloor (n-j)/2 \rfloor} Z_{n+j-1} = k^{\lfloor (n-h)/2 \rfloor} Z_{n-h-1} \quad (n \geq h) \tag{3.10}$$

which, for $n = h$, reduces to

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} k^{\lfloor (n-j)/2 \rfloor} Z_{n+j-1} = 1. \tag{3.11}$$

Let us conclude this section by stating the summation identity:

$$S_N(k) \stackrel{\text{def}}{=} \sum_{n=1}^{N} Z_n = \frac{Z_{N+2} + Z_{N+1} - Z_N - Z_{N-1}}{k-4} - 1 \quad (k \neq 4) \tag{3.12}$$

$$= \begin{cases} (Z_{N+2} - 2Z_{N-1}) / (k-4) - 1 & (N \text{ even}) \\ (2Z_{N+2} - Z_{N-1}) / (k-4) - 1 & (N \text{ odd}) \end{cases} \quad \text{[from (3.1)]}. \tag{3.12'}$$

Remarks:

(i) Assumption (1.7) is needed to get the obvious result $S_1(k) = Z_1 = 1$.

(ii) $$S_N(4) = \begin{cases} 3N/2 & (N \text{ even}) \\ (3N-1)/2 & (N \text{ odd}) \end{cases} \quad \text{[from (2.11)]}. \tag{3.13}$$

Proof of (3.12): First, consider N odd, and rewrite $S_N(k)$ as

$$\begin{aligned} S_N(k) &= \sum_{j=1}^{(N+1)/2} Z_{2j-1} + \sum_{j=1}^{(N-1)/2} Z_{2j} \\ &= \sum_{j=1}^{(N+1)/2} U_{2j-1}(\sqrt{k-4}) + \sum_{j=1}^{(N-1)/2} V_{2j}(\sqrt{k-4}) \quad \text{[from (2.4)]}. \end{aligned} \tag{3.14}$$

By using the Binet forms (1.3) and (1.4), and the geometric series formula, it can be readily seen that

$$\sum_{j=1}^{h} U_{2j-1}(x) = [U_{2h+1}(x) - U_{2h-1}(x)] / x^2 \tag{3.15}$$

and

$$\sum_{j=1}^{h} V_{2j}(x) = [V_{2h+2}(x) - V_{2h}(x) - x^2]/x^2, \tag{3.16}$$

whence, invoking (2.4) again, (3.14) reduces to (3.12). The proof for N even is analogous to that for N odd, and is omitted. Q.E.D.

4. CONGRUENCE PROPERTIES OF $Z_n(k)$

In this section we show some basic congruence properties of the numbers $Z_n(k)$. For reasons of space, only Proposition 2 is proved in detail. We have established the following:

$$Z_n(k) \equiv 0 \pmod 2 \text{ if } \begin{cases} n \equiv 0 \pmod 2 & (k \text{ even}) \\ n \equiv 0 \pmod 3 & (k \text{ odd}). \end{cases} \tag{4.1}$$

From (1.1), we clearly have that

$$Z_n(k) \equiv Z_n(0) \pmod k \quad (k \geq 1), \tag{4.2}$$

where $Z_0(k)$ is given by (2.7). From (4.2), (2.7), and (3.4), one can readily see that

$$\frac{Z_n(k) - Z_n(0)}{k} = \frac{Z_n(k) - 2}{k} = Z_{n/2}^2(k) \quad [n \equiv 2 \pmod 4]. \tag{4.3}$$

Observe that (4.2) and (2.7) imply the congruence

$$Z_k(k) \equiv 0 \pmod k \quad (k \text{ odd}). \tag{4.4}$$

From (4.4) and (2.4), one immediately gets the following (supposedly known) result.

Proposition 1: If m is an odd integer and $h = m^2 + 4$, then $U_h(m)$ is divisible by h.

Finally, let us state the following proposition.

Proposition 2: If p is an odd prime, then

$$Z_p(k) \equiv (k/p) \pmod p, \tag{4.5}$$

where (k/p) denotes the Legendre symbol.

It is worth noting that (2.4) and (4.5) constitute a simple proof of a well-known congruence property of the generalized Fibonacci numbers $U_n(s)$ (s an arbitrary integer) defined by (1.3). In fact, we get

$$U_p(s) \equiv (s^2 + 4/p) \pmod p. \tag{4.6}$$

Proof of Proposition 2: That $\binom{n-j}{j} \equiv 0 \pmod{n-j}$ if $j \geq 1$ and $\gcd(n, j) = 1$ is a well-known fact (e.g., see Lemma 1 of [5]). Consequently, from (1.1), we have

$$Z_p(k) \equiv k^{(p-1)/2} \pmod p, \tag{4.7}$$

whence

$$Z_p(k) \equiv 0 \pmod p \text{ if } k \equiv 0 \pmod p. \tag{4.8}$$

If $k \not\equiv 0 \pmod{p}$, by Fermat's little theorem we have the congruence $k^{p-1} \equiv 1 \pmod{p}$, whence we can write

$$(k^{(p-1)/2}+1)(k^{(p-1)/2}-1) \equiv 0 \pmod{p}. \tag{4.9}$$

Let $a\,(b)$ be the first (second) factor on the left-hand side of (4.9). Since $p \ge 3$ by definition, either a or b (not both) is divisible by p. If k is a quadratic residue (q.r.) (mod p) [i.e., if there exists z such that $k \equiv z^2 \pmod{p}$], then, by Fermat's little theorem, we have $k^{(p-1)/2} \equiv z^{2(p-1)/2} \equiv 1$ (mod p), that is, $b \equiv 0 \pmod{p}$. If k is not a q.r. (mod p), then we necessarily have $a \equiv 0 \pmod{p}$. Therefore, from (4.7), we can write

$$Z_p(k) \equiv \begin{cases} 1 \pmod{p} & \text{if } k \text{ is a q.r. } \pmod{p}, \\ -1 \pmod{p} & \text{otherwise.} \end{cases} \tag{4.10}$$

Congruences (4.8) and (4.10) prove the proposition. Q.E.D.

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IDENTITIES INVOLVING PARTIAL DERIVATIVES OF BIVARIATE FIBONACCI AND LUCAS POLYNOMIALS

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1. INTRODUCTION

The work of Filipponi and Horadam in [2] and [3] revealed that the first- and second-order derivative sequences of Lucas type polynomials defined by $u_{n+1}(x) = u_n(x) + u_{n-1}(x)$ yield some nice recurrence properties. More precisely, in [2] and [3], some identities involving first- and second-order derivative sequences of the Fibonacci polynomials $U_n(x)$ and the Lucas polynomials $V_n(x)$ are established. These results may also be extended to the k^{th} derivative case, as conjectured in [3] and recently confirmed in [7]. See also [4]. Furthermore, Filipponi and Horadam [5] considered the partial derivative sequences of bivariate second-order recurrence polynomials.

In this paper we shall extend some of the results established in [5] and derive some identities involving the partial derivative sequences of the bivariate Fibonacci polynomials $U_n(x, y)$ and the bivariate Lucas polynomials $V_n(x, y)$ defined respectively by (cf. [5])

$$U_n(x, y) = xU_{n-1}(x, y) + yU_{n-2}(x, y),\ n \geq 2,\ U_0(x, y) = 0, U_1(x, y) = 1, \tag{1}$$

$$V_n(x, y) = xV_{n-1}(x, y) + yV_{n-2}(x, y),\ n \geq 2,\ V_0(x, y) = 2, V_1(x, y) = x. \tag{2}$$

Moreover, we shall establish some convolution-type identities as counterparts to those given in [7]. As may be seen, these results, together with those in [6], explain in some sense the "heredity" of linearity under differentiation.

Throughout the paper we use U_n and V_n, respectively, to denote $U_n(x, y)$ and $V_n(x, y)$. The partial derivatives of U_n and V_n are defined by

$$U_n^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} U_n, \quad V_n^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} V_n,\ k \geq 0, j \geq 0. \tag{3}$$

Using an argument similar to that given in [1] or by induction, one may easily obtain the combinatorial expressions of U_n and V_n in terms of x and y. They are:

$$U_n = \sum_{i=0}^{[(n-1)/2]} \binom{n-i-1}{i} x^{n-2i-1} y^i,\ n \geq 1, \tag{4}$$

$$V_n = \sum_{i=0}^{[n/2]} \frac{n}{n-i} \binom{n-i}{i} x^{n-2i} y^i,\ n \geq 1, \tag{5}$$

where $[a]$ denotes the greatest integer not exceeding a.

The extension of the bivariate Fibonacci and Lucas polynomials through the negative subscripts yields

$$U_{-n} = -(-y)^{-n} U_n \quad \text{and} \quad V_{-n} = (-y)^{-n} V_n,\ n > 0. \tag{6}$$

2. SOME IDENTITIES INVOLVING $U_n^{(k,j)}$ AND $V_n^{(k,j)}$

Theorem 1: Let n be any integer and let $k, j \geq 0$. Then the following identities hold:

(i) $V_n^{(k,j)} = yU_{n-1}^{(k,j)} + jU_{n-1}^{(k,j-1)} + U_{n+1}^{(k,j)}$,

(ii) $U_n^{(k,j)} = xU_{n-1}^{(k,j)} + yU_{n-2}^{(k,j)} + kU_{n-1}^{(k-1,j)} + jU_{n-2}^{(k,j-1)}$,

(iii) $V_n^{(k,j)} = xV_{n-1}^{(k,j)} + yV_{n-2}^{(k,j)} + kV_{n-1}^{(k-1,j)} + jV_{n-2}^{(k,j-1)}$,

(iv) $V_n^{(k+1,j)} = nU_n^{(k,j)}$, $V_n^{(k,j+1)} = nU_{n-1}^{(k,j)}$. Hence, $U_n^{(k,j)} = U_{n-j}^{(k+j,0)}$, $V_n^{(k,j)} = nV_{n-j}^{(k+j,0)}/(n-j)$.

Proof:

(i) It is easy to show by induction that $V_n = yU_{n-1} + U_{n+1}$ for any integer n. Hence, we have

$$V_n^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}(yU_{n-1} + U_{n+1}) = \frac{\partial^j}{\partial y^j}(yU_{n-1}^{(k,0)}) + U_{n+1}^{(k,j)} = yU_{n-1}^{(k,j)} + jU_{n-1}^{(k,j-1)} + U_{n+1}^{(k,j)}.$$

(ii) From (1), we see that

$$\begin{aligned} U_n^{(k,j)} &= \frac{\partial^{k+j}}{\partial x^k \partial y^j}(xU_{n-1} + yU_{n-2}) = \frac{\partial^k}{\partial x^k}(xU_{n-1}^{(0,j)}) + \frac{\partial^j}{\partial y^j}(yU_{n-2}^{(k,0)}) \\ &= xU_{n-1}^{(k,j)} + kU_{n-1}^{(k-1,j)} + yU_{n-2}^{(k,j)} + jU_{n-2}^{(k,j-1)}. \end{aligned}$$

(iii) This result can be proved by a method similar to that shown in (ii).

(iv) We first prove the case $(k, j) = (1, 0)$. This can be done by induction on n. The identity trivially holds when $n = 0, 1$. Suppose that $V_{n-1}^{(1,0)} = (n-1)U_{n-1}$ and $V_{n-2}^{(1,0)} = (n-2)U_{n-2}$ for $n \geq 2$. Then

$$\begin{aligned} V_n^{(1,0)} &= \frac{\partial}{\partial x}(xV_{n-1} + yV_{n-2}) = xV_{n-1}^{(1,0)} + yV_{n-2}^{(1,0)} + V_{n-1} \\ &= (n-1)xU_{n-1} + (n-2)yU_{n-2} + yU_{n-2} + U_n = nU_n. \end{aligned}$$

From (6), it follows that

$$V_{-n}^{(1,0)} = \frac{\partial}{\partial x}((-y)^{-n}V_n) = (-y)^{-n}V_n^{(1,0)} = -nU_{-n}.$$

Similarly, we can prove that $V_n^{(0,1)} = nU_{n-1}$ for any integer n. Thus,

$$V_n^{(k+1,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}V_n^{(1,0)} = n\frac{\partial^{k+j}}{\partial x^k \partial y^j}U_n = nU_n^{(k,j)} \tag{7}$$

and

$$V_n^{(k,j+1)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}V_n^{(0,1)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}(nU_{n-1}) = nU_{n-1}^{(k,j)}. \tag{8}$$

It now follows from (7) and (8) that $U_n^{(k,j+1)} = V_n^{(k+1,j+1)}/n = U_{n-1}^{(k+1,j)}$. Hence, $U_n^{(k,j)} = U_{n-j}^{(k+j,0)}$ and $V_n^{(k,j)} = nU_n^{(k-1,j)} = nU_{n-j}^{(k+j-1,0)} = nV_{n-j}^{(k+j,0)}/(n-j)$ by (7). □

As expected, (i)-(iv) reduce to Identities 1-4 in [7] when $y = 1$ and $j = 0$.

3. CONVOLUTION-TYPE IDENTITIES INVOLVING $U_n^{(k,j)}$ AND $V_n^{(k,j)}$

Theorem 2: For any $k, j \geq 0$, we have:

(a) $\sum_{i=0}^{n} U_i^{(k,j)} U_{n-i} = \frac{1}{k+j+1} U_n^{(k+1,j)}$,

(b) $\sum_{i=0}^{n} U_i^{(k,j)} V_{n-i} = \frac{n+k+1}{k+j+1} U_n^{(k,j)}$,

(c) $\sum_{i=0}^{n} V_i^{(k,j)} U_{n-i} = \left[\delta(0, k+j) + \frac{n(k+j)+j}{(k+j+1)(k+j)}\right] U_n^{(k,j)}$,

(d) $\sum_{i=0}^{n} V_i^{(k,j)} V_{n-i} = [1+\delta(0, k+j)] V_n^{(k,j)} + \frac{(n-1)(k+j)+j}{(k+j+1)(k+j)} U_{n-1}^{(k,j)} + \frac{(n+1)(k+j)+j}{(k+j+1)(k+j)} U_{n+1}^{(k,j)}$,

where $\delta(s, r) = 1(0)$ if $s = (\neq)\, r$ is the Kronecker symbol.

Proof:

(a) Let $A_n^{(k)} = \Sigma_{i=0}^n U_i^{(k,0)} U_{n-i}$. Now it may be shown by an induction argument that U_i is a monic bivariate polynomial whose highest leading term is x^{i-1}, so that $U_0^{(k,0)} = U_1^{(k,0)} = \cdots = U_k^{(k,0)} = 0$ and $U_{k+1}^{(k,0)} = k!$ Therefore, $A_k^{(k)} = A_{k+1}^{(k)} = 0$ and $A_{k+2}^{(k)} = U_{k+1}^{(k,0)} U_1 = k! = U_{k+2}^{(k+1,0)} / (k+1)$. Assume $A_{n-1}^{(k)} = U_{n-1}^{(k+1,0)} / (k+1)$ and $A_{n-2}^{(k)} = U_{n-2}^{(k+1,0)} / (k+1)$ for $n \geq 2$. Then, from the assumption and Theorem 1(ii), we have

$$\begin{aligned} A_n^{(k)} &= \sum_{i=0}^{n} U_i^{(k,0)} U_{n-i} = \sum_{i=0}^{n-1} U_i^{(k,0)} (x U_{n-1-i} + y U_{n-2-i}) \\ &= x A_{n-1}^{(k)} + y A_{n-2}^{(k)} + U_{n-1}^{(k,0)} (y U_{-1}) \\ &= \frac{1}{k+1} (x U_{n-1}^{(k+1,0)} + y U_{n-2}^{(k+1,0)} + (k+1) U_{n-1}^{(k,0)}) = \frac{1}{k+1} U_n^{(k+1,0)}. \end{aligned} \tag{9}$$

From (9) and Theorem 1(iv), we have

$$\begin{aligned} \sum_{i=0}^{n} U_i^{(k,j)} U_{n-i} &= \sum_{i=0}^{n} U_{i-j}^{(k+j,0)} U_{n-i} = \sum_{r=0}^{n-j} U_r^{(k+j,0)} U_{n-j-r} \\ &= A_{n-j}^{(k+j)} = \frac{1}{k+j+1} U_{n-j}^{(k+j+1,0)} = \frac{1}{k+j+1} U_n^{(k+1,j)}. \end{aligned} \tag{10}$$

(b) Using (10) and the fact that $V_n = y U_{n-1} + U_{n+1}$ for any integer n [see the proof of Theorem 1(i)], we have

$$\begin{aligned} \sum_{i=0}^{n} U_i^{(k,j)} V_{n-i} &= \sum_{i=0}^{n} U_i^{(k,j)} (y U_{n-i-1} + U_{n+1-i}) \\ &= y \sum_{i=0}^{n-1} U_i^{(k,j)} U_{n-1-i} + U_n^{(k,j)} (y U_{-1}) + \sum_{i=0}^{n+1} U_i^{(k,j)} U_{n+1-i} \\ &= \frac{1}{k+j+1} (y U_{n-1}^{(k+1,j)} + U_{n+1}^{(k+1,j)}) + U_n^{(k,j)} \end{aligned}$$

$$= \frac{1}{k+j+1}(V_n^{(k+1,j)} - jU_{n-1}^{(k+1,j-1)}) + U_n^{(k,j)}$$

$$= \frac{1}{k+j+1}(nU_n^{(k,j)} - jU_n^{(k,j)}) + U_n^{(k,j)} = \frac{n+k+1}{k+j+1}U_n^{(k,j)}. \tag{11}$$

Using Theorem 1 and an argument similar to that of (a), it is easy to prove (c) and (d). Hence, the proofs are omitted here. □

Finally, we give two generalizations of identity (a) in Theorem 2. It is worth mentioning that (b)-(d) possess similar generalized forms.

Theorem 3: Let $k, j, r, s \geq 0$. Then

$$\sum_{i=0}^{n} U_i^{(k,j)} U_{n-i}^{(r,s)} = \left[(k+j+r+s+1)\binom{k+j+r+s}{r+s}\right]^{-1} U_n^{(k+r+1,j+s)}.$$

Proof: Let $A_n^{(k,j,r)} = \sum_{i=0}^{n} U_i^{(k,j)} U_{n-i}^{(r,0)}$. First, we show by induction on r that

$$A_n^{(k,j,r)} = \left[(k+j+r+1)\binom{k+j+r}{r}\right]^{-1} U_n^{(k+r+1,j)}. \tag{12}$$

The case $r = 0$ is just Theorem 2(a). Suppose the above identity is true for some $r \geq 0$. Since

$$\frac{\partial}{\partial x} A_n^{(k,j,r)} = A_n^{(k+1,j,r)} + A_n^{(k,j,r+1)} = \left[(k+j+r+1)\binom{k+j+r}{r}\right]^{-1} U_n^{(k+r+2,j)},$$

we get

$$A_n^{(k,j,r+1)} = \left\{\left[(k+j+r+1)\binom{k+j+r}{r}\right]^{-1} - \left[(k+1+j+r+1)\binom{k+1+j+r}{r}\right]^{-1}\right\} U_n^{(k+r+2,j)}$$

$$= \left[(k+j+r+2)\binom{k+j+r+1}{r+1}\right]^{-1}\left(\frac{k+j+r+2}{r+1} - \frac{k+j+1}{r+1}\right) U_n^{(k+r+2,j)}$$

$$= \left[(k+j+r+2)\binom{k+j+r+1}{r+1}\right]^{-1} U_n^{(k+r+2,j)}.$$

Therefore, we have

$$\sum_{i=0}^{n} U_i^{(k,j)} U_{n-i}^{(r,s)}$$

$$= \sum_{i=0}^{n} U_i^{(r,s)} U_{n-i}^{(k,j)} = \sum_{i=s}^{n} U_{i-s}^{(r+s,0)} U_{n-i}^{(k,j)} \quad \text{[from Theorem 1(iv)]}$$

$$= \sum_{i=0}^{n-s} U_i^{(r+s,0)} U_{n-s-i}^{(k,j)} = \sum_{i=0}^{n-s} U_i^{(k,j)} U_{n-s-i}^{(r+s,0)}$$

$$= \left[(k+j+r+s+1)\binom{k+j+r+s}{r+s}\right]^{-1} U_{n-s}^{(k+r+s+1,j)} \quad \text{[from (12)]}$$

$$= \left[(k+j+r+s+1)\binom{k+j+r+s}{r+s}\right]^{-1} U_n^{(k+r+1,j+s)} \quad \text{[from Theorem 2(iv)]}. \ \square$$

Theorem 4: Let $k, j \geq 0$ and $t \geq 2$. Then

$$\sum_{i_1+i_2+\cdots+i_t=n} U_{i_1}^{(k,j)} U_{i_2}^{(k,j)} \cdots U_{i_t}^{(k,j)} = \prod_{i=2}^{t}\left[(i\alpha-1)\binom{i\alpha-2}{\alpha}\right]^{-1} U_n^{(tk+t-1,tj)}$$

where $\alpha = k+j+1$.

The proof of Theorem 4 can be carried out by induction on t and is omitted here for the sake of brevity.

4. CONCLUDING REMARKS

The bivariate polynomials defined by (3) and (4) may be used to obtain identities for k^{th} derivative sequences of Pell and Pell-Lucas polynomials by taking $x = 2$, $y = 1$, and $j = 0$ [6]. It is likely that this kind of bivariate treatment may also be extended to the bivariate integration sequences $\iint U_n \, dx \, dy$ to parallel some identities found in [4].

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INITIAL VALUES FOR HOMOGENEOUS LINEAR RECURRENCES OF SECOND ORDER

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0. INTRODUCTION

A homogeneous linear recurrence of second order with constant coefficients is a sequence of equations

$$u_{n+2} = au_{n+1} + bu_n, \quad n \geq 0, \tag{0}$$

for fixed complex numbers $a, b \neq 0$. A solution $\{u_n\}_{n\geq 0}$ is completely determined by (0) and the two initial values u_0, u_1. C. Kimberling [1] raised the following problem: under what conditions on two nonnegative integers i, j does every complex pair u_i, u_j determine the whole recurrence sequence $\{u_n\}$ with (0)? In this article, I give two answers to this question (Theorems 1 and 2; the second corrects Theorems 2 and 6 of [1]) and apply them to the properties of the initial pairs. In Theorem 3 I discuss how they are distributed, while in Theorem 4 I discuss which initial values generate a periodic sequence.

1. A FIRST CRITERION FOR INITIAL PAIRS

Given a recurrence (0), we call two nonnegative numbers $i < j$ an "initial pair" if, for all complex numbers c_i, c_j, there exists one and only one solution $\{u_n\}$ of (0) with $u_i = c_i$, $u_j = c_j$. An initial pair is always $i, i+1$. Most pairs i, j will be initial, but there are exceptions: $0, 2$ is not an initial pair of $u_{n+2} = u_n$.

Theorem 1 ([1], Theorem 1): Given the recurrence (0) with $b \neq 0$, for every pair of nonnegative integers i, j with $i+1 < j$, the following two conditions are equivalent:

i, j is an initial pair for (0); (1)

the $(j-i-1)$-rowed matrix

$$D_{j-i} := \begin{pmatrix} a & -1 & 0 & & & \\ b & a & -1 & & & \\ 0 & b & a & & & \\ & & & \ddots & & \\ & & & b & a & -1 \\ & & & 0 & b & a \end{pmatrix} \tag{2}$$

is regular.

Proof: The pair $i, i+2$ is initial iff $a \neq 0$, since $au_{i+1} = u_{i+2} - bu_i$. So let $j > i+2$. If $u_i = c_i$ and $u_j = c_j$ are given, then the equations $bu_n + au_{n+1} - u_{n+2} = 0$, for $n = i, i+1, \ldots, j-2$, give us the system

$$
\begin{array}{rl}
au_{i+1} - u_{i+2} & = -bc_i \\
bu_{i+1} + au_{i+2} - u_{i+3} & = 0 \\
bu_{i+2} + au_{i+3} - u_{i+4} & = 0 \\
\ddots & \vdots \\
bu_{j-3} + au_{j-2} - u_{j-1} & = 0 \\
bu_{j-2} + au_{j-1} & = c_j .
\end{array}
$$

Now, i, j is an initial pair iff this system of $j-i-1$ linear equations has a unique solution $u_{i+1}, u_{i+2}, \dots, u_{j-1}$ (and hence all $u_n, n \geq 0$, are determined) for all c_i, c_j. A necessary and sufficient condition for this is that the associated homogeneous linear system is only trivially soluble, hence the regularity of the coefficient matrix D_{j-i}. $\square$

Remark: This criterion can be extended to sequences of higher order (see [1], Theorem 7). Condition (1) is equivalent to the following: the monoms z^i, z^j are a basis of the complex vector-space $\mathbb{C}[z]$ of polynomials modulo the subspace $\mathbb{C}[z](z^2 - az - b)$. This was generalized by M. Peter [2] to recurrences of several variables of higher order.

2. A SECOND CRITERION FOR INITIAL PAIRS

Let $n := j - i$. We compute $d_n := \det D_n$ by expanding the determinant of D_{n+2} *à la* Laplace:

$$d_{n+2} = ad_{n+1} + bd_n, \quad d_0 := 0, \quad d_1 := 1. \tag{3}$$

Let

$$\zeta_1 := \tfrac{1}{2}(a + \sqrt{a^2 + 4b}) \quad \text{and} \quad \zeta_2 := \tfrac{1}{2}(a - \sqrt{a^2 + 4b})$$

be the zeros of the companion polynomial $z^2 - az - b$ of (0), then the solution of the initial problem (3) has the Binet representation

$$d_n = \begin{cases} \dfrac{1}{\zeta_i - \zeta_2}(\zeta_1^n - \zeta_2^n) & \text{if } \zeta_1 \neq \zeta_2, \\ n\left(\dfrac{a}{2}\right)^{n-1} & \text{if } \zeta_1 = \zeta_2, \end{cases} \tag{4}$$

for all $n \in \mathbb{N}$. Hence we get $d_n = 0 \Leftrightarrow \zeta_1 \neq \zeta_2, \quad \zeta_1^n = \zeta_2^n$. The last condition is equivalent to

$$\exists\, 1 \leq m \leq n-1: \quad \zeta_1 = \exp\left(2\pi i \frac{m}{n}\right)\zeta_2.$$

We compute

$$
\begin{aligned}
\zeta_1 = \exp\left(2\pi i \frac{m}{n}\right)\zeta_2 &\Leftrightarrow a + \sqrt{a^2 + 4b} = \exp\left(2\pi i \frac{m}{n}\right)(a - \sqrt{a^2 + 4b}) \\
&\Leftrightarrow \sqrt{a^2 + 4b}\left(\exp\left(2\pi i \frac{m}{n}\right) + 1\right) = a\left(\exp\left(2\pi i \frac{m}{n}\right) - 1\right) \\
&\Leftrightarrow \sqrt{a^2 + 4b}\, \cos\left(\pi \frac{m}{n}\right) = -ia\, \sin\left(\pi \frac{m}{n}\right).
\end{aligned}
$$

This finally means

$$\exists\, 1 \le m \le n-1: \;\; a^2 = -4b \cos^2\left(\pi \frac{m}{n}\right).$$

Combining this with Theorem 1, we have

Theorem 2: Suppose we have a recurrence (0) with $b \neq 0$ and a pair of nonnegative integers $i < j$. Then the following three properties are equivalent:

i, j is an initial pair of (0); (1)

if ζ_1 and ζ_2 are the zeros of the polynomial $z^2 - az - b$, then $\zeta_1 = \zeta_2$ or $\zeta_1^{j-i} \neq \zeta_2^{j-i}$; (5)

$\dfrac{a^2}{4b} \neq -\cos^2\left(\pi \dfrac{m}{j-i}\right)$ for every $1 \le m < j-i$. (6)

Examples (cf. [1], Theorems 2-5): For each of the following cases, a necessary and sufficient condition that $i < j$ is an initial pair of (0) is

i) $a = 0$: $j - i \not\equiv 0 \mod 2$;
ii) $a^2 = -b$: $j - i \not\equiv 0 \mod 3$;
iii) $a^2 = -2b$: $j - i \not\equiv 0 \mod 4$;
iv) $a^2 = -3b$: $j - i \not\equiv 0 \mod 6$.

If $a^2 = -kb$ with $k \in \mathbb{Z} - \{0, 1, 2, 3\}$, then every pair $i < j$ is initial.

3. DISTRIBUTION OF INITIAL PAIRS IN RESIDUE CLASSES

In the examples of initial pairs $i < j$ given above, $j - i$ lies outside of some residue class. The next theorem explains why.

Theorem 3:

a) Suppose that the recurrence (0) with $b \neq 0$ has a pair that is not initial, then there exists an integer $m \ge 2$ such that, for every pair $i < j$ of nonnegative integers, we have that i, j is initial for (0) $\Leftrightarrow j - i \not\equiv 0 \mod m$.

b) For every natural number $m \ge 2$, there is a recurrence (0) such that $0, j$ is initial for (0) $\Leftrightarrow j \not\equiv 0 \mod m$.

Proof:

a) By Theorem 1, there exists a natural number $n \ge 2$ with $d_n = 0$. Let $m := \min\{n \ge 2: d_n = 0\}$ and $\delta := d_{m+1}$. From (4), we deduce that $d_{qm+r} = \delta^q d_r$ for all $q \in \mathbb{N}_0$, $0 \le r < m$. Furthermore, since $\delta \neq 0$, we have $d_n = 0 \Leftrightarrow n \equiv 0 \mod m$.

Using Theorem 1, we see that this is equivalent to our first assertion.

b) Let $\zeta := \exp(2\pi i / m)$, $a := \zeta + 1$, $b := -\zeta$, then $d_j = (\zeta^j - 1)/(\zeta - 1)$, $j \in \mathbb{N}$, so that $d_j = 0 \Leftrightarrow j \equiv 0 \mod m$.

Theorem 3 is proved. □

4. PERIODIC SEQUENCES

If i, j is an initial pair for (0), we now seek conditions under which two complex numbers c_i, c_j generate a *periodic* recurrence sequence $\{u_n\}$ with $u_i = c_i$ and $u_j = c_j$.

Theorem 4: Given a recurrence (0) with $b \neq 0$, a pair i, j in $\mathbb{N}_0$ with $i < j$, complex numbers c_i, c_j not both zero, and $m \in \mathbb{N}$, then the following two conditions are equivalent:

$$i, j \text{ is an initial pair for (0) and the solution } \{u_n\}_{n\geq 0} \text{ of (0) with } u_i = c_i,\ u_j = c_j \text{ has period } m. \tag{7}$$

One of these four cases is valid:

$$\left.\begin{array}{ll} \text{(a)} & \zeta_1^m = 1,\ \zeta_1^{j-i} \neq \zeta_2^{j-i},\ c_j = c_i\zeta_1^{j-i}; \\ \text{(b)} & \zeta_2^m = 1,\ \zeta_1^{j-i} \neq \zeta_2^{j-i},\ c_j = c_i\zeta_2^{j-i}; \\ \text{(c)} & \zeta_1^m = \zeta_2^m = 1,\ \zeta_1^{j-i} \neq \zeta_2^{j-i}; \\ \text{(d)} & \left(\frac{a}{2}\right)^2 = -b,\ \left(\frac{a}{2}\right)^m = 1,\ c_j = c_i\left(\frac{a}{2}\right)^{j-i}. \end{array}\right\} \tag{8}$$

Here again, ζ_1, ζ_2 are the zeros of $z^2 - az - b$.

Proof: Because of Theorem 2, each of the four conditions implies that i, j is an initial pair for (0). Hence, it suffices to show under which condition the unique solution $\{u_n\}$ of (0) with $u_i = c_i$ and $u_j = c_j$ has period m.

1) $\zeta_1 \neq \zeta_2$. In this case,

$$u_n = \frac{1}{\zeta_1^{j-i} - \zeta_2^{j-i}}[(c_j - c_i\zeta_2^{j-i})\zeta_1^{n-i} - (c_j - c_i\zeta_1^{j-i})\zeta_2^{n-i}], \quad n \geq 0.$$

However, the property $u_{n+m} = u_n$, $n \geq 0$, is equivalent to

$$(c_j - c_i\zeta_2^{j-i})\zeta_1^{n+m-i} - (c_j - c_i\zeta_1^{j-i})\zeta_2^{n+m-i} = (c_j - c_i\zeta_2^{j-i})\zeta_1^{n-i} - (c_j - c_i\zeta_1^{j-i})\zeta_2^{n-i}, \ \forall n \in \mathbb{N}$$

$$\Leftrightarrow (c_j - c_i\zeta_2^{j-i})(\zeta_1^m - 1)\zeta_1^{n-i} = (c_j - c_i\zeta_1^{j-i})(\zeta_2^m - 1)\zeta_2^{n-i}, \ \forall n \in \mathbb{N}$$

$$\Leftrightarrow \begin{cases} (c_j - c_i\zeta_2^{j-i})(\zeta_1^m - 1) = 0 \\ (c_j - c_i\zeta_1^{j-i})(\zeta_2^m - 1) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{(a)} & \zeta_1^m = 1,\ c_j = c_i\zeta_1^{j-i}, \\ \text{(b)} & \zeta_2^m = 1,\ c_j = c_i\zeta_2^{j-i}, \\ \text{(c)} & \zeta_1^m = \zeta_2^m = 1, \\ \text{(d)} & c_j = c_i\zeta_1^{j-i} = c_i\zeta_2^{j-i}. \end{cases}$$

Since $\zeta_1^{j-i} \neq \zeta_2^{j-i}$, case (d) is impossible.

2) $\zeta_1 = \zeta_2$. Here $u_n = \frac{1}{j-i}\left[(n-i)c_j + (j-n)c_i\left(\frac{a}{2}\right)^{j-i}\right]\left(\frac{a}{2}\right)^{n-j}$.

One can easily compute

$$u_{n+m} = u_n, \ \forall n \geq 0 \Leftrightarrow \left(\frac{a}{2}\right)^m = 1, \ c_j = c_i\left(\frac{a}{2}\right)^{j-i},$$

which is the case (d) of (8), and Theorem 4 is proved. □

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AMS Classification Numbers: 11A25, 11B37, 11B39

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Announcement

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THE CONSTANT FOR FINITE DIOPHANTINE APPROXIMATION

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Let x be an irrational number. In 1891, Hurwitz [3] proved that there are infinitely many rational numbers p/q such that p and q are coprime integers and $|x-p/q|<1/(\sqrt{5}q^2)$. Hurwitz' theorem has been extensively investigated (see [6]).

In 1948, following Davenport's suggestion, Prasad [4] initiated the study of finite Diophantine approximation. He proved that, for any given irrational number x, and any given positive integer m, there is a constant C_m such that the inequality $|x-p/q|<1/(C_m q^2)$ has at least m rational solutions p/q. In [4], the structure of C_m has been mentioned, and $C_1=(3+\sqrt{5})/2$ has been calculated, but the values of C_m as a function of m is still unknown.

In this note we will use the Fibonacci sequence to prove that

$$C_m=\sqrt{5}+\frac{\sqrt{5}}{\left(\frac{7+3\sqrt{5}}{2}\right)^m-1}. \tag{1}$$

Theorem 1: Let x be an irrational number. If m is a given positive integer, then there are at least m rational numbers p/q such that p and q are coprime integers and $|x-p/q|<1/(C_m q^2)$, where C_m is as shown in formula (1). The constants C_m cannot be replaced by a smaller number.

Proof: Let $x=[a_0; a_1, a_2, \ldots, a_n, \ldots]$ be the expansion of x in a simple continued fraction. Let $p_n/q_n=[a_0; a_1, \ldots, a_n]$ be the n^{th} convergent, then p_n and q_n are coprime integers. It is well known that (see [5])

$$|x-p_n/q_n|=1/(M_n q_n^2),$$

where $M_n=a_{n+1}+[0; a_{n+2}, a_{n+3}, \ldots]+[0; a_n, a_{n-1}, \ldots, a_1]$.

By Legendre's theorem [5], $|x-p/q|<1/(2q^2)$ implies that p/q must be a convergent p_n/q_n for some n. Thus, we need only discuss the rational solutions of $|x-p/q|<1/(C_m q^2)$ among the convergents p_n/q_n.

We discuss the following possible cases on the partial quotients a_n. It is easily seen that $C_m \le C_1 < 8/3 < 3$.

Suppose there are infinitely many $a_n \ge 3$, then $M_{n-1} \ge a_n \ge 3 \ge C_m$ for all positive integer m. Hence, we need only consider the case in which there are only finitely many $a_n \ge 3$. That is to say, there is a positive integer N_1 such that $n \ge N_1$ implies $a_n \le 2$. We consider two cases.

Case 1. There are infinitely many a_n such that $a_n=2$. Then, for these n, $n>N_1+2$ implies $M_{n-1} \ge 2+[0; 2, 1]+[0; 2, 1]=8/3>C_m$ for all positive integers m.

Case 2. There are finitely many $a_n=2$. Thus, there is a positive integer $N_2 \ge N_1$ such that $n \ge N_2$ implies $a_n=1$.

Let $N=\max\{n, a_n \neq 1\}$. Then $a_N \geq 2$, $a_{N+1}=a_{N+2}=\cdots=1$. Therefore, if we use $[0; (1)_k]$ to denote $[0; 1, \ldots, 1]$ with k consecutive 1's, the following inequalities are true because $a_{N+1}=a_{N+2}=\cdots=a_{N+2m-1}=1$; there are $2m-1$ consecutive 1's.

$$\begin{aligned} M_{N+2m-1} &= a_{N+2m}+[0; \overline{1}]+[0; 1, \ldots, 1, a_N, a_{N-1}, \ldots, a_1] \\ &\geq 1+[0; \overline{1}]+[0; (1)_{2m-1}]. \end{aligned} \tag{2}$$

Similarly, we have

$$\begin{aligned} M_{N+2m+1} &= a_{N+2m+2}+[0; \overline{1}]+[0; 1, \ldots, 1, a_N, a_{N-1}, \ldots, a_1] \\ &\geq 1+[0; \overline{1}]+[0; (1)_{2m+1}], \end{aligned}$$

$\ldots,$

$$\begin{aligned} M_{N+4m-3} &= a_{N+4m-2}+[0; \overline{1}]+[0; 1, \ldots, 1, a_N, a_{N-1}, \ldots, a_1] \\ &\geq 1+[0; \overline{1}]+[0; (1)_{4m-3}]. \end{aligned}$$

It is easily seen that $M_{N+2m-1}<M_{N+2m+1}<\cdots<M_{N+4m-3}$. Denoting $C_m=M_{N+2m-1}$, then the inequality $|x-p/q|<1/(C_m q^2)$ has at least m rational solutions p_n/q_n.

Now we calculate C_m with the help of the Fibonacci sequence.

Let $F_1=1, F_2=1, F_n=F_{n-1}+F_{n-2}$ be the Fibonacci sequence. We are going to find a formula for $[0; (1)_{2m-1}]$ by mathematical induction.

It is easily seen that $[0; (1)_1]=[0; 1]=1/1=F_1/F_2$. Suppose $[0; (1)_{2k-1}]=F_{2k-1}/F_{2k}$, then we have $[0; (1)_{2(k+1)-1}]=[0; 1, 1, (1)_{2k-1}]=1/(1+(1+F_{2k-1}/F_{2k}))=F_{2k+1}/F_{2k+2}$. Thus, $[0; (1)_{2m-1}]=F_{2m-1}/F_{2m}$.

By Binet's formula for the Fibonacci sequence [1], i.e., $F_n=((1+\sqrt{5})^n-(1-\sqrt{5})^n)/(2^n\sqrt{5})$, we can find F_{2m-1}/F_{2m} as follows:

$$\begin{aligned} \frac{F_{2m-1}}{F_{2m}} &= \frac{2((1+\sqrt{5})^{2m-1}-(1-\sqrt{5})^{2m-1})}{(1+\sqrt{5})^{2m}-(1-\sqrt{5})^{2m}} = \frac{\sqrt{5}((1+\sqrt{5})^{2m}+(1-\sqrt{5})^{2m})}{2((1+\sqrt{5})^{2m}-(1-\sqrt{5})^{2m})}-\frac{1}{2} \\ &= \frac{\sqrt{5}(1+(\sqrt{5}-3)/2)^{2m}}{2(1-(\sqrt{5}-3)/2)^{2m}}-\frac{1}{2} = \frac{\sqrt{5}}{2}\left(1+\frac{2((\sqrt{5}-3)/2)^{2m}}{1-((\sqrt{5}-3)/2)^{2m}}\right)-\frac{1}{2} \\ &= \frac{\sqrt{5}}{2}\left(1+\frac{2}{((3+\sqrt{5})/2)^{2m}-1}\right)-\frac{1}{2}. \end{aligned}$$

Notice that because $[0; \overline{1}]=(\sqrt{5}-1)/2$ we have, by formula (2), that

$$C_m=M_{N+2m-1}=1+(\sqrt{5}-1)/2+F_{2m-1}/F_{2m},$$

which gives formula (1).

The constants C_m cannot be replaced by smaller numbers since, for $x=[0; \overline{1}]$, we have exactly $C_m=M_{2m-1}=1+[0; \overline{1}]+[0; (1)_{2m-1}]$. □

Corollary 1: $C_1=(3+\sqrt{5})/2=2.6180,$
$C_2=(7+3\sqrt{5})/6=2.2847,$
$C_3=(9+4\sqrt{5})/8=2.2430.$

Corollary 2: $\lim_{m \to \infty} C_m = \sqrt{5} = 2.2361.$

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ON A RECURRENCE RELATION IN TWO VARIABLES

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1. INTRODUCTION

Gupta [3] considers the array $\{c(n,k)\}$, which is defined by the recurrence relation

$$c(n+1,k)=c(n,k)+c(n,k-1) \tag{1}$$

with initial values $c(n,0)=a(n)$, $c(1,k)=b(k)$, $n,k\geq 1$, where $\{a(n)\}$ and $\{b(k)\}$ are given sequences of numbers. In particular, if $a(n)\equiv 1$ and $b(1)=1$, $b(k)=0$ for $k\geq 2$, then $\{c(n,k)\}$ is the classical binomial array. The array $\{c(n,k)\}$ also has applications, for example, in the theory of partitions of integers ([1], [2]). The main object of Gupta [3] is to handle $\{c(n,k)\}$ with the aid of generating functions. Wilf [7, §§1.5 and 1.6] handles $\{c(n,k)\}$ and an analog of $\{c(n,k)\}$, namely, the array of the Stirling numbers of the second kind, with the aid of generating functions.

In this paper we consider a further analog of the array $\{c(n,k)\}$, namely, the array $\{L(n,k)\}$ defined by the recurrence relation

$$L(n,k)=L(n-1,k-1)-L(n,k-1) \tag{2}$$

with initial values $L(n,0)=a(n)$, $L(0,k)=b(k)$, $L(0,0)=a(0)=b(0)$, $n,k\geq 1$, where $\{a(n)\}$ and $\{b(k)\}$ are given sequences of numbers. We derive an expression for the numbers $L(n,k)$ in terms of the initial values using the method of generating functions. We motivate the study of the array $\{L(n,k)\}$ by providing a concrete example of this kind of array from the theory of stack filters.

2. AN EXPRESSION FOR THE NUMBERS $L(n, k)$

Let $L_k(x)$ denote the generating function of the sequence $\{L(n,k)\}_{n=0}^{\infty}$, that is,

$$L_k(x)=\sum_{n=0}^{\infty}L(n,k)x^n. \tag{3}$$

The promised expression for the numbers $L(n,k)$ comes out as follows: We use the recurrence relation (2) to obtain a recurrence relation for the generating function $L_k(x)$. This recurrence relation yields an expression for the generating function $L_k(x)$, which gives an expression for the numbers $L(n,k)$.

Theorem 1: Let $\{L(n,k)\}$ be the array given in (2). Then

$$L(n,k)=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}a(n-j)+\sum_{j=n}^{k-1}\binom{j}{n}(-1)^{j-n}[b(k-j)-b(k-j-1)]. \tag{4}$$

Proof: If $k=0$, then (4) holds. Let $k\geq 1$. Then, by (3) and (4),

$$\begin{aligned} L_k(x) &= L(0,k) + \sum_{n=1}^{\infty}[L(n-1,k-1) - L(n,k-1)]x^n \\ &= L(0,k) + [xL_{k-1}(x) - (L_{k-1}(x) - L(0,k-1))] \end{aligned}$$

or

$$L_k(x) = (x-1)L_{k-1}(x) + b(k) + b(k-1).$$

Let $d(k) = b(k) + b(k-1)$. Proceeding by induction on k, we obtain

$$L_k(x) = (x-1)^k L_0(x) + \sum_{j=0}^{k-1}(x-1)^j d(k-j). \tag{5}$$

Here

$$\begin{aligned} (x-1)^k L_0(x) &= \left(\sum_{n=0}^{\infty}\binom{k}{n}(-1)^{k-n}x^n\right)\left(\sum_{n=0}^{\infty}a(n)x^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{k}{j}(-1)^{k-j}a(n-j)\right)x^n \end{aligned} \tag{6}$$

and

$$\begin{aligned} \sum_{j=0}^{k-1}(x-1)^j d(k-j) &= \sum_{j=0}^{k-1}\sum_{n=0}^{j}\binom{j}{n}x^n(-1)^{n-j}d(k-j) \\ &= \sum_{n=0}^{k-1}\left(\sum_{j=n}^{k-1}\binom{j}{n}(-1)^{j-n}d(k-j)\right)x^n. \end{aligned} \tag{7}$$

Now, combining (3), (5), (6), and (7), we obtain (4).

Remark: We considered above the generating function of the array $\{L(n,k)\}$ with respect to the variable n. We could also consider generating functions with respect to the variable k and with respect to both the variables n and k. These considerations, however, would be more laborious and are not presented here (cf. [3], [7]).

3. AN EXAMPLE FROM THE THEORY OF STACK FILTERS

Consider a stack filter (for definition, see [4], [6]) with continuous i.i.d. inputs having distribution function $\Phi(\cdot)$ and with window size N. The γ-order moment about the origin of the output can be written as

$$\alpha^\gamma = E\{Y_{\text{out}}^\gamma\} = \sum_{k=0}^{N-1}A_k M(\Phi,\gamma,N,k),$$

where

$$M(\Phi,\gamma,N,k) = \int_{-\infty}^{\infty}x^\gamma \frac{d}{dx}\left((1-\Phi(x))^k\Phi(x)^{N-k}\right)dx,\ k = 0,1,\ldots,N-1, \tag{8}$$

and where the coefficients A_k, $k = 0,1,\ldots,N-1$, have a certain natural interpretation (see [4], [5]). By using the output moments about the origin, we easily obtain output central moments, denoted by $\mu^\gamma = E\{(Y_{\text{out}} - E\{Y_{\text{out}}\})^\gamma\}$, for example, the second-order central output moment equals

$$\mu^2 = \sum_{k=0}^{N-1} A_k M(\Phi, 2, N, k) - \left(\sum_{k=0}^{N-1} A_k M(\Phi, 1, N, k) \right)^2. \tag{9}$$

The second-order central output moment is quite often used as a measure of the noise attenuation capability of a filter. It quantifies the spread of the input samples with respect to their mean value. Equation (9) gives an expression for the second-order central output moment. In this expression, the numbers $M(\Phi, \gamma, N, k)$ play a crucial role.

Kuosmanen [4] and Kuosmanen & Astola [5] studied the properties of the numbers $M(\Phi, \gamma, N, k)$ under certain conditions on $\Phi(\cdot)$. They showed, among other things, that the numbers $M(\Phi, \gamma, N, k)$ satisfy the recurrence relation

$$M(\Phi, \gamma, N, k) = M(\Phi, \gamma, N-1, k-1) - M(\Phi, \gamma, N, k-1), \; 1 \le k \le N,$$

with initial values

$$M(\Phi, \gamma, N, 0) = \int_{-\infty}^{\infty} x^\gamma \frac{d}{dx}\left(\Phi(x)^N\right) dx, \; N \ge 0.$$

This means that the numbers $M(\Phi, \gamma, N, k)$ satisfy recurrence relation (2). As $M(\Phi, \gamma, N, k) = 0$ if $N = 0$, that is, as $b(k) \equiv 0$ in (2), application of Theorem 1 gives

$$M(\Phi, \gamma, N, k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} M(\Phi, \gamma, N-j, 0). \tag{10}$$

Note that Kuosmanen [4] and Kuosmanen & Astola [5] derived (10) directly from (8) using the binomial theorem.

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ON PSEUDOPRIMES RELATED TO GENERALIZED LUCAS SEQUENCES

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1. INTRODUCTION

In this paper we consider the general sequences U_n and V_n satisfying the recurrences

$$U_{n+2} = mU_{n+1} + U_n, \quad V_{n+2} = mV_{n+1} + V_n, \tag{1.1}$$

where m is a given positive integer, and $U_0 = 0, U_1 = 1, V_0 = 2, V_1 = m$.

We shall occasionally refer to these sequences as $U(m)$ and $V(m)$ to emphasize their dependence on the parameter m. They can be represented by the *Binet forms*

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta), \quad V_n = \alpha^n + \beta^n, \tag{1.2}$$

where $\alpha + \beta = m$ and $\alpha\beta = -1$, and we define $\Delta = \delta^2 = (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = m^2 + 4$. When $m = 1$, these sequences reduce to F_n and L_n, with $\Delta = 5$.

Using (1.2), we can derive the identities (1.3) through (1.7), which correspond to well-known formulas that are proved, for instance, in [11]:

$$U_{2n} = U_n V_n, \tag{1.3}$$

$$V_{2n} = V_n^2 - 2(-1)^n, \tag{1.4}$$

$$\Delta U_n^2 = V_n^2 - 4(-1)^n, \tag{1.5}$$

$$2U_{n+s} = U_n V_s + V_n U_s, \tag{1.6}$$

$$2V_{n+s} = V_n V_s + \Delta U_n U_s. \tag{1.7}$$

When n is a prime p, we have

$$V_p = \alpha^p + \beta^p \equiv (\alpha + \beta)^p = m^p \pmod{p},$$

and using Fermat's "little theorem," this gives the well-known result

$$V_p \equiv m \pmod{p}, \text{ when } p \text{ is prime.} \tag{1.8}$$

Any composite numbers n satisfying the corresponding equation

$$V_n \equiv m \pmod{n} \tag{1.9}$$

are called pseudoprimes. Di Porto and Filipponi [7] have called such numbers *Fibonacci Pseudoprimes* of the m^{th} kind (m-F.Psps), whereas Bruckman [2] has called them *Lucas Pseudoprimes*. As a compromise, we shall call them V(m)-pseudoprimes or V(m)-psp. In the case of $m = 1$, when V_n becomes L_n, it has been proved that all V(1)-psp's are *odd*: see {14], [5], and [3]. In the more general case, since the interest in V(m)-psp's relates to tests for primality, only *odd* V(m)-psp's will be considered, as in [7], and we shall restrict the definition of pseudoprimes to *odd* composite n satisfying (1.9).

Suppose now that n satisfies (1.9). Then, for any prime factor p of n,

$$V_n \equiv m \pmod{p}, \tag{1.10}$$

or, if the factorization of n contains a prime power p^e, with $e \geq 1$,

$$V_n \equiv m \pmod{p^e}. \tag{1.11}$$

Now it is well-known that the sequence V modulo a prime power p^e is periodic. We shall denote the corresponding period of repetition by $R(p^e)$ or R, defined as the smallest positive integer R for which we have $V_R \equiv 2$ *and* $V_{R+1} \equiv m \pmod{p^e}$. Since $V_{2R} \equiv V_R \equiv V_0 = 2 \pmod{p^e}$, (1.4) shows that, if p is odd, the period R must be even. The sequence U modulo p^e is also periodic, and is known to have the same period R as the corresponding V-sequence, except when $\Delta \equiv 0 \pmod{p}$.

We also note that the entry point Z of p^e in the sequence U is defined as the smallest positive Z such that $U_Z \equiv 0 \pmod{p^e}$. It is well known that p^e divides U_n if and only if Z divides n. Also U_r divides U_n if and only if r divides n. Vinson [12] established the relationship between Z and the period R of the sequence U for the case $m = 1$, and we can easily prove that, for odd p, the same holds for any m, namely:

$$\text{if } Z \text{ is odd, then } R = 4Z; \tag{1.12}$$

$$\text{if } Z \equiv 2 \pmod{4}, \text{ then } R = Z; \tag{1.13}$$

$$\text{if } Z \equiv 0 \pmod{4}, \text{ then } R = 2Z. \tag{1.14}$$

In Sections 2 and 4, we shall derive relationships between n and R giving necessary and sufficient conditions for a number n to be a pseudoprime. In Section 3, we shall find conditions for the occurrence of square factors in such pseudoprimes, and present some numerical examples. Finally, in Section 5, we shall prove certain theorems concerning special forms of V(m)-pseudoprimes, Theorems 7-10 being generalizations of results proved by Di Porto and Filipponi for the case $m = 1$ in [7].

2. PSEUDOPRIMES AND THE PERIODICITY OF THE LUCAS SEQUENCE

Using (1.2), we can define U and V with negative subscripts, giving

$$U_{-n} = -(-1)^n U_n \quad \text{and} \quad V_{-n} = (-1)^n V_n. \tag{2.1}$$

Putting $s = 1$ and then $s = -1$, equation (1.7) gives the identities

$$2V_{n+1} = \Delta U_n + mV_n \quad \text{and} \quad 2V_{n-1} = \Delta U_n - mV_n, \tag{2.2}$$

and multiplying the two parts of (2.2) and using (1.5) gives the well-known identity

$$V_{n+1}V_{n-1} = V_n^2 - \Delta(-1)^n. \tag{2.3}$$

We shall now derive an important identity. Using (2.3) when n is *odd*, we have

$$\begin{aligned}(V_{n+1}+2)(V_{n-1}-2) &= V_{n+1}V_{n-1} - 2(V_{n+1} - V_{n-1}) - 4 \\ &= V_n^2 + \Delta - 2mV_n - 4,\end{aligned}$$

and since $\Delta = m^2 + 4$ this reduces to

$$(V_{n+1}+2)(V_{n-1}-2)=(V_n-m)^2 \ \ (n \text{ odd}); \tag{2.4}$$

we shall call this the *key identity*, as it provides the basis for the proofs of several theorems in this paper. Our first theorem examines at what points of the periodic cycle we might find an *odd* n satisfying $V_n \equiv m \pmod{p^e}$, and is a generalization of the result proved for the case $e=1$ by Di Porto in [6].

Theorem 1: If n is odd and $V_n \equiv m \pmod{p^e}$, where $p>2$ is prime and $e \geq 1$, and if $R=R(p^e)$ is the period of the sequence $V(m)$ modulo p^e, then we have

$$\textit{either } n \equiv 1 \pmod{R} \quad \textit{or} \quad n \equiv \tfrac{1}{2}R-1 \pmod{R}; \tag{2.5}$$

since n is odd, the second alternative can occur only when $\frac{1}{2}R$ is even.

Proof: Putting $V_n \equiv m \pmod{p^e}$ in the key identity (2.4), we find that the right side of the identity is divisible by $(p^e)^2$, and it follows that at least one of the two factors on the left must be divisible by p^e. Thus, we have

$$\textit{either } V_{n-1} \equiv 2 \pmod{p^e} \quad \textit{or} \quad V_{n+1} \equiv -2 \pmod{p^e}. \tag{2.6}$$

Taking the first alternative, we have $V_{n-1} \equiv 2$ and $V_n \equiv m \pmod{p^e}$, showing that $n-1$ is a multiple of the period R in this case. Taking the second alternative, we have $V_n \equiv m$, together with $V_{n+1} \equiv -2$, so that the recurrence relation (1.1) gives $V_{n+2} \equiv -m \pmod{p^e}$. It follows from (1.1) that

$$V_{n+1+t} \equiv -V_t, \quad \text{for } t=0,1,2,\ldots,$$

showing that in this case $n+1$ is an odd multiple of half the period. Therefore, one or the other of the alternatives in (2.5) is true. Q.E.D.

Theorem 2: Let n be an odd $V(m)$-psp divisible by a prime power p^e, and let R be the period of the sequence $V(m)$ modulo p^e, $e \geq 1$. Then, for each such R, we have

$$\textit{either } n \equiv 1 \pmod{R} \quad \textit{or} \quad n = \tfrac{1}{2}R-1 \pmod{R}. \tag{2.7}$$

Note: If p is an odd prime and if R is the period of $V(m)$ modulo p, then using (1.8) and Theorem 1 with $n=p$ and $e=1$ gives

$$\textit{either } p \equiv 1 \pmod{R} \quad \textit{or} \quad p = \tfrac{1}{2}R-1 \pmod{R}; \tag{2.8}$$

this is equivalent to the well-known result that R divides either $p-1$ or $2(p+1)$ when $(\Delta, p)=1$; our derivation shows that (2.8) remains true also when $\Delta \equiv 0 \pmod{p}$.

3. ON THE OCCURRENCE OF SQUARE FACTORS IN A V(*m*)-PSEUDOPRIME

Theorem 3: If n is an odd $V(m)$-psp divisible by a prime power p^e, where $e>1$, then the periods of the sequence $V(m)$ modulo p^e and modulo p are the same.

Proof: Let $R(p^e)$ be the period modulo p^e, and $R(p)$ the period modulo p. Then $R(p^e) = p^f R(p)$, with $0 \leq f < e$, as was proved by E. Lucas [11], and for $m=1$ by Wall [13]. But

Theorem 2 shows that $R(p^e)$ and n have no common factor; therefore, p does not divide $R(p^e)$. Hence, $f = 0$ and $R(p^e) = R(p)$.

Corollary: If a V(m)-psp is divisible by p^e, where $e > 1$, and if p does not divide Δ, then p^e and p have the same entry point Z in the sequence $U(m)$.

Proof: This follows from Theorem 3 by Vinson's rules, as stated in (1.12)-(1.14). Note that when p^e divides Δ the period of $V(m)$ modulo p^e is 4, whereas that of $U(m)$ is $4p^e$.

Note: Bruckman [1] has proved a result equivalent to Theorem 3 for the case $m = 1$. Furthermore, it has also been shown that, for $m = 1$, $R(p^2) = pR(p)$ for all $p < 10^4$ by Wall [13], for all $p < 10^6$ by Dresel [10], and for all $p < 10^9$ by H. C. Williams [15]. It then follows from Theorem 3 that any V(1)-psp less than 10^{18} must be square-free.

The situation for $m > 1$ is rather different. Thus, for $m = 2$, we obtain the *Pell* sequence with $U_7 = P_7 = 169 = 13^2$, while P_{30} is divisible by 31^2. Correspondingly, we find that among the first seven V(2)-psp's there are three containing square factors, namely, $13^2, 31^2$, and $13^2 \times 29$.

Let us call a prime p *divalent* in $U(m)$ if the entry points of p and p^2 in the $U(m)$-sequence are the same. For most of the values of $m \leq 25$, we can find examples of divalent primes with $p < 300$, the exceptional cases being $m = 1, 8, 10, 11, 16$, and 17. In the case of $m = 24$, we have five such primes, namely, 7, 11, 17, 37, and 41, and among the first 21 V(24)-psp's there are ten containing square factors, namely, $7^2, 11^2, 17^2, 7^3, 7^2 \times 17, 3 \times 17^2, 7^2 \times 23, 11^3, 37^2$, and 41^2.

4. SUFFICIENT CONDITIONS FOR A V(m)-PSEUDOPRIME

We shall use the key identity (2.4) to prove the following lemma.

Lemma 1: If R is the period of the sequence V modulo p^e, where $p > 2$ is prime and $e \geq 1$, and if p^c is the highest power of p that divides Δ, where $0 \leq c \leq e$, then

(i) $V_R \equiv 2 \pmod{p^{2e-c}}$, and

(ii) conversely, if $V_{2t} \equiv 2 \pmod{p^{2e-c}}$, then R divides $2t$.

(iii) If, further, $\frac{1}{2}R$ is even, then we also have $V_{\frac{1}{2}R} \equiv -2 \pmod{p^{2e-c}}$ and $V_{\frac{1}{2}R-1} \equiv m \pmod{p^e}$.

Proof: By the definition of R, we have $V_R \equiv 2$ and $V_{R+1} \equiv m \pmod{p^e}$. Since R is even, putting $n = R + 1$ in the key identity (2.4), we obtain

$$(V_R - 2)(V_{R+2} + 2) \equiv 0 \pmod{p^{2e}} \tag{4.1}$$

while

$$(V_{R+2} + 2) - (V_R - 2) = mV_{R+1} + 4 \equiv m^2 + 4 = \Delta \pmod{p^e}. \tag{4.2}$$

(i) Since p^e divides $(V_R - 2)$ and p^c divides Δ, (4.2) shows that p^c is the highest power of p dividing $(V_{R+2} + 2)$; hence, (4.1) gives $V_R \equiv 2 \pmod{p^{2e-c}}$.

(ii) Given $V_{2t} \equiv 2 \pmod{p^{2e-c}}$ and putting $n = 2t$ in (1.5), we obtain $\Delta(U_{2t})^2 \equiv 0 \pmod{p^{2e-c}}$ and, therefore, $(\Delta U_{2t})^2 \equiv 0 \pmod{p^{2e}}$, giving $\Delta U_{2t} \equiv 0 \pmod{p^e}$. Finally, substituting in (2.2), we obtain $2V_{2t+1} \equiv 2m = 2V_1 \pmod{p^e}$, so that $2t$ is a multiple of the period R modulo p^e.

(iii) If $\frac{1}{2}R$ is even, then (1.4) together with (i) above gives

$$(V_{\frac{1}{2}R})^2 = V_R + 2 \equiv 4 \pmod{p^{2e-c}};$$

hence, $V_{\frac{1}{2}R} \equiv -2 \pmod{p^{2e-c}}$, since $V_{\frac{1}{2}R} \equiv 2$ would contradict (ii). Then (1.5) gives $\Delta(U_{\frac{1}{2}R})^2 \equiv 0 \pmod{p^{2e-c}}$ and, therefore, $(\Delta U_{\frac{1}{2}R})^2 \equiv 0 \pmod{p^{2e}}$, giving $\Delta U_{\frac{1}{2}R} \equiv 0 \pmod{p^e}$; finally, (2.2) gives $V_{\frac{1}{2}R-1} \equiv m \pmod{p^e}$.

Note: If $c = e$, so that p^e divides Δ, we have $V_2 = m^2 + 2 \equiv -2 \pmod{p^e}$ and $V_3 \equiv -m$, giving $R = 4$, and we have both $V_R \equiv 2$ and $V_{R+2} \equiv -2 \pmod{p^e}$.

We shall now prove the converse of Theorem 2, namely,

Theorem 4: Let n be odd and composite, and let R be the period of the sequence $V(m)$ modulo a prime power p^e, $e \geq 1$. If, for each p^e dividing n, we have

$$\textit{either } n \equiv 1 \pmod{R} \quad \textit{or} \quad n \equiv \tfrac{1}{2}R - 1 \pmod{R}, \tag{4.3}$$

then n is a V(m)-psp.

Proof: If the first alternative in (4.3) is true, then by definition of R we have $V_n = V_{kR+1} \equiv V_1 \equiv m \pmod{p^e}$; if the second alternative in (4.3) applies, then by Lemma 1(iii) we again have $V_n \equiv V_{\frac{1}{2}R-1} \equiv m \pmod{p^e}$. Thus, (1.11) is satisfied for each prime power p^e which divides n. Hence, (1.9) is true, showing that n is a V(m)-psp.

Note: Theorems 2 and 4 together give necessary and sufficient conditions for n to be a V(m)-psp and provide the basis for the proofs given in the next section. A different approach by Di Porto, Filipponi, and Montolivo [9] gives a sufficient (but not a necessary) condition expressed in terms of the prime factors of n.

We shall now prove a converse of Theorem 3, namely,

Theorem 5: If there is an odd prime p for which the sequence $V(m)$ has the same period R modulo p and p^e, $e > 1$, then p^e is a V(m)-psp.

Proof: By (2.8), we have *either* $p \equiv 1 \pmod{R}$ *or* $p \equiv \frac{1}{2}R - 1 \pmod{R}$. If the first alternative applies, we have $p^e \equiv 1 \pmod{R}$, and Theorem 4 shows that p^e is a V(m)-psp. If the second alternative applies, we have $\frac{1}{2}R$ even (as p is odd) and $p^e \equiv (\frac{1}{2}R - 1)^e \pmod{R}$, so that $p^e \equiv 1 \pmod{R}$ if e is even, and $p^e \equiv (\frac{1}{2}R - 1) \pmod{R}$ if e is odd. Since $R = R(p^e)$, Theorem 4 completes the proof.

Examples: For $m = 2$, we have 13^2 and 31^2 as V(2)-psp's.

Corollary: If $e > 1$ and p^e divides Δ, then p^e is a V(m)-psp.

Proof: If p^e divides Δ, $V(m)$ has the period 4 both modulo p and modulo p^e, so that the conditions of Theorem 5 are satisfied.

Examples: For $m = 11$, we have $\Delta = m^2 + 4 = 125 = 5^3$, and both 25 and 125 are V(11)-psp's. Similarly, for $m = 14$, $\Delta = 200$, so that 25 is a V(14)-psp.

Note: Theorem 5 may be regarded as a special case of Theorem 6 below.

5. SOME SPECIAL FORMS OF V(m)-PSEUDOPRIMES

Theorem 6: If n is odd, composite, and such that all its prime or prime power factors have the same period R in the sequence $V(m)$, then n is a V(m)-psp.

Proof: If $\frac{1}{2}R$ is odd, then by (2.8) n is the product of primes p_j satisfying $p_j = Rk_j + 1$. It is easily seen that the product of two or more such primes satisfies $n = kR + 1$, and the result then follows from Theorem 4. In the same way, if $\frac{1}{2}R$ is even, n is the product of primes of the form $p_j = Rk_j + 1$ *or* of the form $q_i = \frac{1}{2}Rh_i - 1$, where h_i is odd. The product of such primes is again of one or other of these forms, depending on whether the number of primes of the form q_i is even or odd. The result then follows from Theorem 4 as before.

Example: The sequence $V(2)$ has the same period 40 modulo the primes 19 and 59; therefore, their product 1121 is a V(2)-psp.

Corollary: If n is an odd composite number dividing Δ, then n is a V(m)-psp.

Proof: The period of $V(m)$ modulo any prime or p^e that divides Δ is 4. Q.E.D.

We shall use Theorems 4 and 6 to show that certain expressions are V(m)-psp, thus generalizing some results proved for the special case $m = 1$ by Di Porto and Filipponi in [7], and by Bruckman in [2], [4]. First, we shall state some basic facts.

Lemma 2: ***(i)*** U_n and V_n have no odd common factors.

(ii) If p is an odd prime dividing Δ, then $(p, V_n) = 1$ for all n.

These well-known results are easily proved by *reductio ad absurdum* from (2.2).

Lemma 3: For all m, we have

$$U_{2s} \equiv sm \pmod{m^3} \quad \text{and} \quad U_{2s+1} \equiv 1 \pmod{m^2}, \tag{5.1}$$

$$V_{2s} \equiv 2 \pmod{m^2} \quad \text{and} \quad V_{2s+1} \equiv (2s+1)m \pmod{m^3}. \tag{5.2}$$

This is easily proved by induction on s.

Lemma 4: ***(i)*** When m is odd, then U_n and V_n are odd if and only if 3 does not divide n.

(ii) When m is even, then U_n and V_n / m are odd if n is odd.

Theorem 7: If $q > 3$ (*or*, when m is even, $q \geq 3$) is prime and $(\Delta, q) = 1$, and if U_q is composite, then U_q is a V(m)-psp.

Proof: Since q is prime, all the factors of U_q have q as their entry point in the sequence $U(m)$ and, by (1.12), their period is $4q$. Since $(\Delta, q) = 1$, they have the same period in the $V(m)$-sequence. Also, by Lemma 4, U_q is odd. Hence, Theorem 6 applies.

Examples: For $m=1$, see [8]; for $m=2$, the following Pell numbers are V(2)-psp's: $U_7 = 169 = 13^2$, $U_{17} = 137 \times 8297$, $U_{19} = 37 \times 179057$, and $U_{23} = 229 \times 982789$.

Theorem 8: ***(i)*** If $T = 2^k$, $k \geq 1$, and m is odd, then V_T, if composite, is a V(m)-psp.
(ii) If m is even, and $T = 2^k$, $k \geq 1$, then $V_T / 2$, if composite, is a V(m)-psp.

Proof: If m is odd, then V_T is odd since $T = 2^k$ is not divisible by 3. But if m is even, (5.2) gives $V_T \equiv 2 \pmod{m^2}$, so that $V_T / 2$ is odd. Next, consider any odd prime p that divides V_T; then, by Lemma 2, neither U_T nor Δ are divisible by p. Also, $U_{2T} = U_T V_T$; therefore, any odd p or p^e that divides V_T has the entry point $2T$ in the U-sequence and, therefore, the period $4T$ by (1.14). Since $(p, \Delta) = 1$, the period is the same for the V-sequence, and the results then follow from Theorem 6.

Examples: For $m = 11$, $V_2 = 123 = 3 \times 41$ and $V_4 = 15127 = 7 \times 2161$ are V(11)-psp. For $m = 24$, $V_2 / 2 = 289 = 17^2$ and $V_4 / 2 = 167041 = 7^3 \times 487$ are V(24)-psp.

Theorem 9: If $q > 3$ (*or*, when m is even, $q \geq 3$) is prime and $(m, q) = 1$, and if V_q / m is composite, then V_q / m is a V(m)-psp.

Proof: We have $U_{2q} = U_q V_q$, and V_q and U_q have no odd common factor. Hence, any odd prime p which divides V_q has entry point 2 or $2q$ in the U-sequence. But $U_2 = m$, and (5.2) gives $V_q / m \equiv q \pmod{m^2}$, so that V_q / m is odd and prime to m. Therefore, any p or p^e dividing V_q / m has entry point $2q$. By (1.13), the corresponding period is $R = 2q$, and this is also the period in the $V(m)$-sequence, since $(p, \Delta) = 1$ by Lemma 2(ii). Hence, Theorem 6 applies.

Note: By (2.8), any factor of V_q / m would be of the form $2qk + 1$.

Example: When $m = 2$, $V_{11} / 2 = 8119 = 23 \times 353$ is a V(2)-psp.

Theorem 10: If n is a V(m)-psp which is odd (and not divisible by 3 when m is odd), and if $(n, m) = 1$, then the same is true for $N = V_n / m$.

Proof: We have $U_{2n} = U_n V_n$; therefore, any odd prime p or p^e which divides V_n also divides U_{2n} but not U_n, and by (1.13) the corresponding period R divides $2n$. Since n is V(m)-psp, $V_n \equiv m \pmod{n}$, and since $(n, m) = 1$, we have $V_n / m \equiv 1 \pmod{n}$. But V_n / m is odd by Lemma 4, hence $V_n / m \equiv 1 \pmod{2n}$. Since R divides $2n$, we have $V_n / m \equiv 1 \pmod{R}$; furthermore, since n is the product of odd numbers, say $n = pq$, V_n is divisible by V_p so that V_n / m is divisible by V_p / m and, therefore, V_n / m is composite. Hence, Theorem 4 shows that $N = V_n / m$ is a V(m)-psp. It remains to show that N satisfies $(N, m) = 1$ and that $(N, 3) = 1$ if $(n, 3) = 1$.

Since n is odd, (5.2) shows that $V_n / m \equiv n \pmod{m^2}$, and since $(n, m) = 1$, it follows that V_n / m also is prime to m. Furthermore, the entry point of 3 in $U(m)$ is 2 if 3 divides m and 4 otherwise. In the first case, since $V_n / m \equiv n \pmod{m^2}$, 3 does not divide V_n / m if it does not divide n. In the second case, since $2n$ is not divisible by 4, it follows that $U_n V_n$ is not divisible by 3; therefore, $(N, 3) = 1$. Q.E.D.

Corollary: Given one V(m)-psp satisfying the conditions of this theorem, we can find infinitely may such V(m)-psp.

Example: Since 169 is a V(2)-psp, there are infinitely many V(2)-psp's.

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ON LUCASIAN NUMBERS

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1. INTRODUCTION

Let $u(r, s)$ and $v(r, s)$ be Lucas sequences satisfying the same second-order recursion relation

$$w_{n+2} = rw_{n+1} + sw_n \tag{1}$$

and having initial terms $u_0 = 0, u_1 = 1, v_0 = 2, v_1 = r$, respectively, where r and s are integers. We note that $\{F_n\} = u(1, 1)$ and $\{L_n\} = v(1, 1)$. Associated with the sequences $u(r, s)$ and $v(r, s)$ is the characteristic polynomial

$$f(x) = x^2 - rx - s \tag{2}$$

with characteristic roots α and β. Let $D = (\alpha - \beta)^2 = r^2 + 4s$ be the discriminant of both $u(r, s)$ and $v(r, s)$. By the Binet formulas

$$u_n = (\alpha^n - \beta^n) / (\alpha - \beta) \tag{3}$$

and

$$v_n = \alpha^n + \beta^n. \tag{4}$$

We say that the recurrences $u(r, s)$ and $v(r, s)$ are *degenerate* if $\alpha\beta = -s = 0$ or α / β is a root of unity. Since α and β are the zeros of a quadratic polynomial with integer coefficients, it follows that α / β can be an n^{th} root of unity only if $n = 1, 2, 3, 4$, or 6. Thus, $u(r, s)$ and $v(r, s)$ can be degenerate only if $r = 0$, $s = 0$, or $D \le 0$.

We say that the integer m is a *divisor* of the recurrence $w(r, s)$ satisfying the relation (1) if $m | w_n$ for some $n \ge 1$. Carmichael [2, pp. 344-45], showed that, if $(m, s) = 1$, then m is a divisor of $u(r, s)$. Carmichael [1, pp. 47, 61, and 62], also showed that if $(r, s) = 1$, then there are infinitely many primes which are not divisors of $v(r, s)$. In particular, Lagarias [4] proved that the set of primes which are divisors of $\{L_n\}$ has density $2/3$. Given the Lucas sequence $v(r, s)$, we say that the integer m is *Lucasian* if m is a divisor of $v(r, s)$. In Theorems 1 and 2, we will show that, if $u(r, s)$ and $v(r, s)$ are nondegenerate, then u_n is not Lucasian for all but finitely many positive integers n. We will obtain stronger results in the case for which $(r, s) = 1$ and $D > 0$.

A related question is to determine all a and b such that v_a divides u_b. Using the identity $u_a v_a = u_{2a}$, one sees that v_a always divides u_{2a}. Since $u_{2a} | u_b$ if $2a | b$, we have that $v_a | u_b$ if $2a | b$. We will show later that if $rs \ne 0$, $(r, s) = 1$, $|v_a| \ge 3$, and $v_a | u_b$, then $2a | b$.

Theorem 1: Consider the Lucas sequences $u(r, s)$ and $v(r, s)$. Suppose that $rs \neq 0$, $(r, s) = 1$, and $D > 0$. Let a and b be positive integers. Then $u_a | v_b$ if and only if one of the following conditions holds:

(i) $a = 1$;
(ii) $|r| = 1$ or 2 and $a = 2$;
(iii) $|r| \geq 3, a = 2$, and b is odd;
(iv) $|r| = 1, s = 1, a = 3$, and $3|b$;
(v) $|r| = 1, a = 4$, and $2|b$ oddly, where $m|n$ oddly if n/m is an odd integer.

In particular, $u_n, n \geq 5$, is not Lucasian.

Theorem 2: Consider the nondegenerate Lucas sequences $u(r, s)$ and $v(r, s)$. If $(r, s) = 1$ and $D < 0$, then u_n is not Lucasian for $n > e^{452}2^{68}$. If $(r, s) > 1$, then there exists a constant $N(r, s)$ dependent on r and s such that u_n is not Lucasian for $n \geq N(r, s)$.

2. NECESSARY LEMMAS AND THEOREMS

The following lemmas and theorems will be needed for the proofs of Theorems 1 and 2.

Lemma 1: $u_{2n} = u_n v_n$.

Proof: This follows from the Binet formulas (3) and (4) and is proved in [6, p. 185] and [3, Section 5]. □

Lemma 2:

$$u_n(-r, s) = (-1)^{n+1} u_n(r, s). \tag{5}$$
$$v_n(-r, s) = (-1)^n v_n(r, s). \tag{6}$$

Proof: Equations (5) and (6) follow from the Binet formulas (3) and (4) and can be proved by induction. □

Lemma 3: Let $u(r, s)$ and $v(r, s)$ be Lucas sequences such that $rs \neq 0$ and $D = r^2 + 4s > 0$. Then $|u_n|$ is strictly increasing for $n \geq 2$. Moreover, if $|r| \geq 2$, then $|u_n|$ is strictly increasing for $n \geq 1$. Furthermore, $|v_n|$ is strictly increasing for $n \geq 1$.

Proof: By Lemma 2, we can assume that $r \geq 1$. The results for $|u_n|$ and $|v_n|$ clearly hold if $s \geq 1$. We now assume that $r \geq 1$ and $s \leq -1$. Since $D > 0$, we must have that $-r^2/4 < s \leq -1$, which implies that $r \geq 3$. We will show by induction that, if $w(r, s)$ is any recurrence satisfying the recursion relation (1) for which $w_0 \geq 0$, $w_1 \geq 1$, and $w_1 \geq (r/2)w_0$, then $w_n \geq 1$ and $w_n \geq (r/2)w_{n-1}$ for all $n \geq 1$. Our results for $u(r, s)$ and $v(r, s)$ will then follow. Assume that $n \geq 1$, and that $w_n \geq 1$, $w_{n-1} \geq 0$, $w_n \geq (r/2)w_{n-1}$. Then $w_{n-1} \leq (2/r)w_n$. By the recursion relation defining $w(r, s)$, we now have

$$w_{n+1} = rw_n + sw_{n-1} > rw_n - (r^2/4)(2/r)w_n = (r/2)w_n,$$

so that $w_{n+1} \geq 1$ and the lemma follows. □

Lemma 4: Consider the Lucas sequences $u(r, s)$ and $v(r, s)$. Then $u_n | u_{in}$ for all $i \geq 1$ and $v_n | v_{(2j+1)n}$ for all $j \geq 0$.

Proof: These results follow from the Binet formulas (3) and (4). □

Lemma 5: Consider the Lucas sequences $u(r, s)$ and $v(r, s)$ for which $(r, s) = 1$ and r and s are both odd. Then u_n even $\Leftrightarrow v_n$ even $\Leftrightarrow 3|n$.

Proof: Both sequences are congruent modulo 2 to the Fibonacci sequence, for which the result is trivial. □

For the Lucas sequence $u(r, s)$, the *rank of apparition** of the positive integer m, denoted by $\omega(m)$, is the least positive integer n, if it exists, such that $m|u_n$. The rank of apparition of m in $v(r, s)$, denoted by $\overline{\omega}(m)$, is defined similarly.

Lemma 6: Consider the Lucas sequences $u(r, s)$ and $v(r, s)$. Let p be an odd prime such that $p \nmid (r, s)$. If $\omega(p)$ is odd, then $\overline{\omega}(p)$ does not exist and p is not Lucasian.

Proof: This was proved by Carmichael [1, p. 47] for the case in which $(r, s) = 1$. The proof extends to the case in which $p \nmid (r, s)$. □

Lemma 7: Consider the Lucas sequences $u(r, s)$ and $v(r, s)$. Suppose that p is an odd prime such that $p \nmid (r, s)$ and $\omega(p) = 2n$. Then $\overline{\omega}(p) = n$.

Proof: This is proved in Proposition 2(iv) of [10]. □

We let $[n]_2$ denote the 2-valuation of the integer n, that is, the largest integer k such that $2^k | n$.

Lemma 8: Consider the Lucas sequence $v(r, s)$. Suppose that m is Lucasian and that p and q are distinct odd prime divisors of m such that $pq \nmid (r, s)$. Then $[\overline{\omega}(p)]_2 = [\overline{\omega}(q)]_2$.

Proof: This is proved in Proposition 2(ix) of [10]. □

Theorem 3: Let $u(r, s)$ and $v(r, s)$ be Lucas sequences such that $rs \neq 0$ and $(r, s) = 1$. Let a and b be positive integers and let $d = (a, b)$.

(i) $(u_a, u_b) = u_d$;

(ii) $(v_a, v_b) = \begin{cases} v_d & \text{if } [a]_2 = [b]_2, \\ 1 \text{ or } 2 & \text{otherwise;} \end{cases}$

(iii) $(u_a, v_b) = \begin{cases} v_d & \text{if } [a]_2 > [b]_2, \\ 1 \text{ or } 2 & \text{otherwise.} \end{cases}$

Proof: This is proved in [7] and [3, Section 5]. □

Remark: It immediately follows from the formula for (v_a, u_b) that if $rs \neq 0$, $(r, s) = 1$, and $|v_a| \geq 3$, then $v_a | u_b$ if and only if $2a | b$. Noting that $v_2 = r^2 + 2s$, we see by Lemma 3 that if $rs \neq 0$ and $D = r^2 + 4s > 0$, then $|v_a| \geq 3$ for $a \geq 2$.

We say that the prime p is a primitive prime divisor of u_n if $p | u_n$ but $p \nmid u_i$ for $1 \leq i < n$.

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* Plainly, "apparition" is an intended English translation of the French "apparition." Thus, "appearance" would have been a better term, since no ghostly connotation was intended!

Theorem 4 (Schinzel and Stewart): Let the Lucas sequence $u(r, s)$ be nondegenerate. Then there exists a constant $N_1(r,s)$ dependent on r and s such that u_n has a primitive odd prime divisor for all $n \geq N_1(r, s)$. Moreover, if $(r, s) = 1$, then u_n has a primitive odd prime divisor for all $n > e^{452}2^{67}$.

Proof: The fact that the constant $N_1(r,s)$ exists for all nondegenerate Lucas sequences $u(r, s)$ was proved by Lekkerkerker [5] for the case in which $D > 0$ and by Schinzel [8] for the case in which $D < 0$. The fact that if $u(r, s)$ is a nondegenerate Lucas sequence for which $(r, s) = 1$, then an absolute constant N, independent of r and s, exists such that u_n has a primitive odd prime divisor if $n > N$ was proved by Schinzel [9]. Stewart [11] showed that N can be taken to be $e^{452}2^{67}$. □

3. PROOFS OF THE MAIN THEOREMS

We are now ready for the proofs of Theorems 1 and 2.

Proof of Theorem 1

By Lemma 4 and inspection, it is evident that any of conditions (i)-(iv) implies that $u_a | v_b$. Now suppose that $|r| \geq 3, a = 2$, and $u_a | v_b$. Then $|u_a| = |v_1| = |r| \geq 3$. By Theorem 3(ii), we see that b is odd. By Lemma 5, if $r = \pm 1$, $s = 1, u_a | v_b$, and $a = 3$, then $3 | b$. Suppose next that $|r| = 1$, $a = 4$, and $u_a | v_b$. Since $D = r^2 + 4s > 0$, we must have that $s \geq 1$. Then, by Lemma 1, $|u_a| = |v_2| = 2s + 1 \geq 3$. By Theorem 3(ii), it follows that $2 | b$ oddly.

We now note that if $D > 0$ and $rs \neq 0$, then $|u_a| \leq 2$ if and only if $a = 1$, *or* $|r| \leq 2$ and $a = 2$, *or* $|r| = 1, s = 1$, and $a = 3$. Thus it remains to prove that

$$\begin{aligned} &\textit{if } u_a | v_b \text{ and } |u_a| \geq 3, \text{ then either} \\ &\quad |r| \geq 3 \text{ and } a = 2, \text{ or} \\ &\quad |r| = 1 \text{ and } a = 4. \end{aligned} \tag{7}$$

We prove (7) by first proving a lemma which is, in fact, a weaker statement, namely,

Lemma 9: If $D > 0$, $rs \neq 0$, $(r, s) = 1$, $|u_a| = |v_b|$, and $|u_a| \geq 3$, then either $|r| \geq 3$ and $a = 2$, *or* $|r| = 1$ and $a = 4$.

Proof of Lemma 9: Since $|u_a| = |v_b| \geq 3$, $(u_a, v_b) = |v_b| \geq 3$. Thus, by Theorem 3(iii), we conclude that $[a]_2 > [b]_2$; hence, $(u_a, v_b) = |v_d|$, where $d = (a, b)$. Thus, $|v_b| = |v_d|$; but by Lemma 3, $|v_n|$ is an increasing function of n for n positive. Therefore, $b = d$ and $b | a$. Since $[a]_2 > [b]_2$, we have that $2b | a$ and so, by Lemmas 1 and 4, $v_b | u_{2b} | u_a$. But $|u_a| = |v_b|$. Hence, by Lemma 1, $|u_{2b}| = |v_b| = |v_b u_b|$, and so $|u_b| = 1$. Since $|u_n|$ is an increasing function of n for $n \geq 2$ by Lemma 3, we see that $b = 1$ or 2. We can only have that $b = 2$ if $|r| = 1$. However, $|v_b| \geq 3$, so either $b = 1$ and $|u_a| = |v_b| = |r| \geq 3$, implying that $a = 2$, *or* $b = 2$, $|r| = 1$, $s \geq 1$, and $|u_a| = |v_b| = 2s + 1 \geq 3$, which implies that $a = 4$.

Proof of (7): Since $u_a | v_b$ and $|u_a| \geq 3$, we have that $(u_a, v_b) = |u_a| \geq 3$. Using Theorem 3(iii), we infer as in the proof of Lemma 9 that $|u_a| = |v_d|$, where $d = (a, b)$. Hence, by Lemma 9, either $|r| \geq 3$ and $a = 2$, *or* $|r| = 1$ and $a = 4$. □

Proof of Theorem 2

First, suppose that $(r, s) = 1$. Now suppose that $n > 3^{452}2^{68}$ and n is odd. By Theorem 4, u_n has a primitive odd prime divisor p. By Lemma 6, p is not Lucasian and hence u_n is not Lucasian. Now suppose that $n > 3^{452}2^{68}$ and n is even. Then, by Theorem 4, $u_{n/2}$ has a primitive odd prime divisor p_1, and u_n has a primitive odd prime divisor p_2. By Lemma 8, $p_1 p_2$ is not Lucasian. Since $u_{n/2} \mid u_n$ by Lemma 4, we see that u_n is not Lucasian.

Now suppose that $(r, s) > 1$. By Theorem 4, there exists a constant $N_1(r, s) > 2$, dependent on r and s, such that if $n > N_1(r, s)$, then u_n has a primitive odd prime divisor. We note that if p is a prime and $p \mid (r, s)$, then $\omega(p) = 2$. Taking $N(r, s) = 2N_1(r, s)$, we complete our proof by using a completely similar argument to the one above. □

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BINOMIAL GRAPHS AND THEIR SPECTRA

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1. INTRODUCTION

Pascal's triangle with entries reduced modulo 2 has been the object of a variety of investigations, including number theoretical questions on the parity of binomial coefficients [4] and geometrical explorations of the self-similarity of the Sierpinski triangle [7]. Graph theory has also entered the scene as a consequence of various binary (that is, $\{0, 1\}$) matrix constructions that exploit properties of Pascal's triangle. For example, in [2] reference is made to *Pascal graphs* of order n whose (symmetric) adjacency matrix has zero diagonal and the first $n-1$ rows of Pascal's triangle, modulo 2, in the off diagonal elements. Constructions such as these are of special interest when the corresponding graphs unexpectedly reveal or reflect properties intrinsic to Pascal's triangle.

This is the case with *binomial graphs*, the subject of this paper. The adjacency matrices of these graphs are also related to Pascal's triangle, modulo 2. The graphs are found to exhibit a number of interesting properties including a graph property that relates to the Fibonacci sequence. Recall that the n^{th} Fibonacci number F_n appears in Pascal's triangle as the sum:

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}.$$

Other properties of binomial graphs relate to the golden mean, to the Lucas numbers, and to several other features associated with Pascal's triangle.

2. BINOMIAL GRAPHS

For each nonnegative integer n, we define the *binomial graph* B_n to have vertex set $V_n = \{v_j : j = 0, 1, \ldots, 2^n - 1\}$ and edge set $E_n = \{\{v_i, v_j\} : \binom{i+j}{j} \equiv 1 \pmod 2\}$. We define $\binom{0}{0} = 1$; thus, each binomial graph has a loop at v_0, but is otherwise a simple graph (that is, has no other loop and no multiedge). The binomial graph B_3 and its adjacency matrix $A(B_3)$ are depicted in Figure 1.

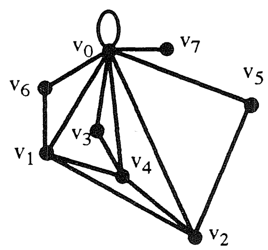
Obviously, $|V_n| = 2^n$. Also, for each $k = 0, 1, \ldots, n-1$, B_n has $\binom{n}{k}$ vertices of degree 2^k and a single vertex, v_0, of degree $2^n + 1$. Thus, the sum of the degrees of vertices in B_n is

$$\sum_{k=0}^{n-1} \binom{n}{k} 2^k + (2^n + 1) = 1 + \sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n + 1.$$

Consequently, $|E_n| = \frac{1}{2}(3^n + 1)$.

The adjacency matrix $A(B_n)$ exhibits a self-similarity. In this form, it can be described in terms of a Kronecker product of matrices. Recall that if $A = [a_{ij}]$ is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $mp \times nq$ matrix, $A \otimes B = [a_{ij}B]$.

B_3



$A(B_3)$

	j = 0	1	2	3	4	5	6	7
i = 0	1	1	1	1	1	1	1	1
1	1	0	1	0	1	0	1	0
2	1	1	0	0	1	1	0	0
3	1	0	0	0	1	0	0	0
4	1	1	1	1	0	0	0	0
5	1	0	1	0	0	0	0	0
6	1	1	0	0	0	0	0	0
7	1	0	0	0	0	0	0	0

FIGURE 1. The Binomial Graph B_3 and Its Adjacency Matrix

Thus, if we take $A(B_0) = [1]$, then, for each $n \geq 1$, the adjacency matrix of the binomial graph B_n is

$$A(B_n) = \begin{bmatrix} A(B_{n-1}) & A(B_{n-1}) \\ A(B_{n-1}) & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(B_{n-1}) = A(B_1) \otimes A(B_{n-1}).$$

3. SPECTRA OF BINOMIAL GRAPHS

The *eigenvalues of a graph* G are the eigenvalues of $A(G)$, the adjacency matrix of G. The *spectrum of a graph* is the sequence (or multiset) of its eigenvalues. We denote the spectrum of graph G by $\Lambda(G)$.

To obtain the spectrum of the binomial graph B_n, we exploit the following result concerning Kronecker products.

Lemma 1 (see [1]): Let A be an $n \times n$ matrix with (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and eigenvalues $x_1, x_2, \ldots, x_n$. Let B be an $m \times m$ matrix with eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$ and eigenvectors $y_1, y_2, \ldots, y_m$. Then the Kronecker product $A \otimes B$ has nm eigenvalues $\lambda_i \mu_j$ and eigenvectors $x_i \otimes y_j$ for each $i = 1, 2, \ldots, n$ and each $j = 1, 2, \ldots, m$. □

We use this lemma to establish that the eigenvalues of binomial graphs are powers of the golden mean, as are the entries in the corresponding eigenvectors.

Theorem 1: Let $\varphi = \frac{1}{2}(1-\sqrt{5})$. For each $n \geq 0$, the binomial graph B_n has $n+1$ distinct eigenvalues, specifically, $(-1)^j \varphi^{n-2j}$, for each $j = 0, 1, \ldots, n$. Each of these eigenvalues occurs with multiplicity $\binom{n}{j}$, so that the spectrum of B_n is

$$\Lambda(B\ ) = [((-1)^j \varphi^{n-2j})^{\binom{n}{j}} : j = 0, 1, \ldots, n],$$

where $\lambda^{(m)}$ means that the eigenvalue λ has multiplicity m. Furthermore, for $n \geq 1$, 2^n linearly independent eigenvectors of B_n are scalar multiples of the columns in the Kronecker product $X(B_n) = X(B_1) \otimes X(B_{n-1})$, where $X(B_n) = [x_1, x_2, \ldots, x_{2^n}]$ is the matrix of eigenvectors of B_n with

$$X(B_0) = [1], \quad X(B_1) = \begin{bmatrix} 1 & 1 \\ 1/\varphi & -\varphi \end{bmatrix}.$$

Finally, the characteristic polynomial of B_n is

$$\mathcal{P}(B_n; x) = \prod_{j=0}^{n} [x - (-1)^j \varphi^{n-2j}]^{\binom{n}{j}}.$$

Proof: Since $A(B_0) = [1]$, obviously $\Lambda(B_0) = [1]$ and $\mathcal{P}(B_0; x) = x - 1$. Since

$$A(B_1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

then

$$\mathcal{P}(B_1; x) = \det \begin{bmatrix} x-1 & -1 \\ -1 & x \end{bmatrix} = x^2 - x - 1,$$

so that $\Lambda(B_1) = [\varphi, -\frac{1}{\varphi}]$, as required by the theorem. Furthermore, the two eigenvectors are $x_1^{\mathrm{T}} = [1, \varphi^{-1}]$ and $x_2^{\mathrm{T}} = [1, -\varphi]$ (or scalar multiples thereof), so that

$$X(B_1) = \begin{bmatrix} 1 & 1 \\ 1/\varphi & -\varphi \end{bmatrix}.$$

Since, for each $n > 1$

$$A(B_n) = A(B_1) \otimes A(B_{n-1}) = \underbrace{A(B_1) \otimes A(B_1) \otimes \cdots \otimes A(B_1)}_{n \text{ factors}},$$

then, by Lemma 1, the spectrum $\Lambda(B_n)$ consists of the n-fold (Cartesian) product of eigenvalues from the spectrum $\Lambda(B_1) = [\varphi, -\frac{1}{\varphi}]$. That is, the j^{th} distinct eigenvalue λ_j of B_n is the coefficient of $\binom{n}{j} t^j$ in the expansion of

$$\left(\varphi - \frac{t}{\varphi}\right)^n = \Sigma(-1)^j \binom{n}{j} \varphi^{n-2j} t^j,$$

and the multiplicity of λ_j is $\binom{n}{j}$. Furthermore, also by Lemma 1, $X(B_n) = X(B_1) \otimes X(B_{n-1})$. $\square$

4. CHARACTERISTIC POLYNOMIALS OF BINOMIAL GRAPHS

A polynomial of degree n, $P(x) = \sum_{k=0}^{n} c_k x^k$, $c_0 \neq 0$, is called *palindromic* if, for each $k = 0, \ldots, n$, $|c_k| = |c_{n-k}|$ (see [3] and [6]). Some interest attaches to graphs whose characteristic polynomials are palindromic. A palindromic polynomial is said to be *exactly palindromic* if, for each k, $c_k = c_{n-k}$ and *skew palindromic* if $c_k = -c_{n-k}$. A palindromic polynomial of even degree is called *even pseudo palindromic* if, for each k, $c_k = (-1)^k c_{n-k}$ and *odd pseudo palindromic* if $c_k = -(-1)^k c_{n-k}$.

By expressing the characteristic polynomials of binomial graphs as products of simple (unit) quadratic factors involving the Lucas numbers, we show that the binomial graphs are palindromic with respect to their characteristic polynomials.

From Theorem 1, $\mathcal{P}(B_0; x) = x - 1$ is obviously skew palindromic. For even $n > 0$,

$$\begin{aligned}\mathcal{P}(B_n; x) &= \prod_{j=0}^{n}[x-(-1)^j\varphi^{n-2j}]^{\binom{n}{j}}\\ &= (x-(-1)^{n/2})^{\binom{n}{n/2}}\prod_{j=0}^{(n-2)/2}[x^2-(-1)^j(\varphi^{n-2j}+\hat{\varphi}^{n-2j})x+(-1)^n]^{\binom{n}{j}},\end{aligned}$$

where $\hat{\varphi} = -\frac{1}{\varphi}\left(=\frac{1-\sqrt{5}}{2}\right)$. Since, for even $n > 0$, the central binomial coefficient $\binom{n}{n/2}$ is even, then

$$\mathcal{P}(B_n; x) = (x^2-(-1)^{n/2}L_0x+1)^{\frac{1}{2}\binom{n}{n/2}}\prod_{j=0}^{(n-2)/2}[x^2-(-1)^jL_{n-2j}x+1]^{\binom{n}{j}},$$

where L_k is the k^{th} Lucas number for $k \geq 1$ and $L_0 = 2$. Consequently, for even $n > 0$, $\mathcal{P}(B_n; x)$ is a product of exact palindromic (quadratic) polynomials; hence, see Lemma 2.2 in [3], $\mathcal{P}(B_n; x)$ is exact palindromic.

For n odd, B_n has no eigenvalue of unit magnitude, but similarly,

$$\mathcal{P}(B_n; x) = \prod_{j=0}^{(n-1)/2}[x^2-(-1)^jL_{n-2j}x-1]^{\binom{n}{j}},$$

so that $\mathcal{P}(B_n; x)$ is a product of 2^{n-1} odd pseudo palindromic polynomials. Obviously $\mathcal{P}(B_n; x)$ is odd pseudo palindromic but (see [3], Lemma 2.2), for each odd $n > 1$, $\mathcal{P}(B_n; x)$ is even pseudo palindromic.

Note that for each binomial graph B_n with $n > 1$, the characteristic polynomial $\mathcal{P}(B_n; x)$ can be expressed as a product of unit quadratic factors whose central coefficients are Lucas numbers L_k with $k \equiv n \pmod 2$.

5. CLOSED WALKS IN BINOMIAL GRAPHS

As was observed by P. W. Kasteleyn [5], the characteristic polynomial $\mathcal{P}(G; x)$ of a graph G can be applied to determine the number of closed walks of fixed length in G. We state this result as

Lemma 2: The total number of closed walks of length k in a graph G is the coefficient of t^k in the generating function

$$W(G; t) = \frac{\mathcal{P}'(G; \frac{1}{t})}{t\mathcal{P}(G; \frac{1}{t})}, \text{ where } \mathcal{P}'(G; x) = \frac{d}{dx}\mathcal{P}(G; x). \quad \square$$

By applying this lemma to the graphs B_n, we obtain a connection between binomial graphs and the Lucas numbers.

Theorem 2: The (ordinary) generating function for the total number of closed walks of length k in the binomial graph B_n is

$$W(B_n;t)=\sum_{k=0}^{\infty}L_k^n t^k,$$

where L_k is the k^{th} Lucas number for $k\ge 1$ and $L_0=2$.

Proof: By Lemma 2,

$$W(B_n;t)=\frac{\mathcal{P}'(B_n;\frac{1}{t})}{t\mathcal{P}(B_n;\frac{1}{t})},$$

where, from Theorem 1,

$$\mathcal{P}(B_n;x)=\prod_{j=0}^{n}[x-(-1)^j\varphi^{n-2j}]^{\binom{n}{j}}.$$

Setting $\hat{\varphi}=-\frac{1}{\varphi}\left(=\frac{1-\sqrt{5}}{2}\right)$, we can write

$$\mathcal{P}(B_n;x)=\prod_{j=0}^{n}(x-\hat{\varphi}^j\varphi^{n-j})^{\binom{n}{j}}.$$

Taking the logarithm of both sides and differentiating with respect to x yields

$$\frac{\mathcal{P}'(B_n;x)}{\mathcal{P}(B_n;x)}=\sum_{j=0}^{n}\frac{\binom{n}{j}}{x-\hat{\varphi}^j\varphi^{n-j}}.$$

It follows that

$$W(B_n;t)=\sum_{j=0}^{n}\frac{\binom{n}{j}}{1-\hat{\varphi}^j\varphi^{n-j}t}=\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{\infty}\hat{\varphi}^{jk}\varphi^{(n-j)k}t^k$$

$$=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\hat{\varphi}^{jk}\varphi^{(n-j)k}\right)t^k=\sum_{k=0}^{\infty}(\varphi^k+\hat{\varphi}^k)^n t^k=\sum_{k=0}^{\infty}L_k^n t^k.\ \square$$

Consider now the number of closed walks of length k in B_n with initial (and final) vertex v_0. Let $W_0(B_n;t)$ denote the generating function for this sequence. To determine the coefficients of this generating function, we first need the following lemma.

Lemma 3: Let $v_j\in V(B_n)$ with the vertices labeled in natural order $\{0,1,\ldots,2^n-1\}$ and let $w_n(j)$ denote the representation of the natural number j as a binary word of length n. Then $\{v_i,v_j\}\in E(B_n)$ if and only if $w_n(i)$ and $w_n(j)$ have no 1-bit in common.

Proof: The lemma is an immediate consequence of the fact that

$$\binom{i+j}{i}=\binom{i+j}{j}\equiv 1 \pmod 2$$

if and only if $w_n(i)$ and $w_n(j)$ have no 1-bit in common. $\square$

Theorem 3: The number of closed walks of length k with initial vertex v_0 in B_n is the coefficient of t^k in the generating function

$$W_0(B_n;t)=\sum_{k=0}^{\infty}F_{k+1}^n t^k$$

where F_k is the k^{th} Fibonacci number.

Proof: The statement is easily verified for $k = 0$ or 1: the number of closed walks starting at v_0 in B_n is equal to 1 in each case. For $k \geq 2$, a walk of length k in B_n can be described as an ordered list of $k+1$ vertices. Let each vertex v_j $(j = 0, 1, \ldots, 2^n - 1)$ be labeled with the corresponding binary word, $w_n(j)$, of length n. Then a walk of length k in B_n can be described as an ordered list of $k+1$ binary words each of length n and such that no two consecutive words have a 1-bit in common. Obviously, for a closed walk commencing at vertex v_0, the first and last binary word is the zero word $w_n(0)$.

Consider the $(k-1) \times n$ matrix $\boldsymbol{M}$, whose *rows* in sequence are the binary words describing a closed walk in B_n starting at v_0, with the first and last word $w_n(0)$ deleted. Now the *columns* of $\boldsymbol{M}$ can be viewed as n independent and ordered $\{0, 1\}$-sequences of length $k-1$, with the property that no two 1-bits are adjacent. Since there are exactly F_{k+1} such sequences, where F_k is the k^{th} Fibonacci number, it follows that there are F_{k+1}^n binary words of length n in which no two consecutive words have a 1-bit in common. That is, the number of closed walks of length k from v_0 in B_n is F_{k+1}^n. □

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ON PERIODS MODULO A PRIME OF SOME CLASSES OF SEQUENCES OF INTEGERS

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In [2] and [3] we used the T transformation of sequences of integers (u_n), defined by $T(u_n) = xu_{n+k} - u_n$, to prove in a simple way properties of periodicity modulo a given prime p for (u_n) satisfying several types of second-order linear recurrences.

The aim of this note is to extend these early results to more general forms of the transformation and of the sequence (u_n).

Theorem 1: Let $u_n, n \geq 0$, be the general term of a given sequence of integers and define the transformation $T_{(x,y,k)}(u_n)$ as $T_{(x,y,k)}(u_n) = xu_{n+k} + yu_n$ for every $n \geq 0$, k being a positive integer.

Then, if x and y are nonzero integers and there exists a positive prime number p which divides $T_{(x,y,k)}(u_n)$ for every $n \geq 0$ and is relatively prime to x, the distribution of the residues of (u_n) modulo p is either constant or periodic with period $k(p-1)$.

Proof: If $(T(u_n))^{(m)}$ denotes the m^{th} iterate of the transformation $T_{(x,y,k)}$ on (u_n) for given x, y, and k, it is quite easy to prove by induction that, for any n and m,

$$(T(u_n))^{(m)} = \sum_{r=0}^{m} \binom{m}{r} (x)^r (y)^{m-r} u_{n+rk}.$$

Put $m = p$ in this formula. Since p is prime, the binomial coefficients are all divisible by p, except the two extreme ones (see [1], p. 417). Therefore,

$$(T(u_n))^{(p)} \equiv x^p u_{n+pk} + y^p u_n \pmod{p}.$$

Since by construction $(T(u_n))^{(p)}$ is a sum of terms all supposedly divisible by p, this entails that $x^p u_{n+pk} + y^p u_n \equiv 0 \pmod{p}$.

Since p is prime, by Fermat's little theorem, $x^p \equiv x \pmod{p}$ and $y^p \equiv y \pmod{p}$, and the previous congruence becomes $xu_{n+pk} + yu_n \equiv 0 \pmod{p}$.

By hypothesis, for any n, $xu_{n+k} + yu_n \equiv 0 \pmod{p}$, and from the difference with the previous congruences we obtain $x(u_{n+pk} - u_{n+k}) \equiv 0 \pmod{p}$. Since, by hypothesis, p and x are relatively prime, this implies $u_{n+pk} - u_{n+k} \equiv 0 \pmod{p}$ for any n. This proves Theorem 1.

Examples:

(1) Theorem 1 contains known properties for particular second-order linear sequences. For instance, let us consider the following one, with a and b being arbitrary nonzero integers:

$$u_{n+2} - au_{n+1} + bu_n = 0. \tag{R1}$$

An equivalent form of this recursion is $u_{n+2} + bu_n = au_{n+1}$.

If we take arbitrary integral values for u_0 and u_1, all u_n are integers; therefore, if p divides a, Theorem 1 may be applied with $x = 1$, $y = b$, and $k = 2$, which proves that the distribution of the residues of (u_n) modulo p is either constant or periodic with period $2(p-1)$. This was shown in [4] by Lawrence Somer, for a particular case of (u_n). The reader is also referred to [5] and [6] for other results about the periods of residues modulo a prime on examples of second-order (u_n) more restricted than ours but with more detailed results.

(2) The scope of Theorem 1 is not limited to *second*-order linear recursions (not even to *linear* ones). For instance, let us consider the third-order recursion

$$u_{n+3}+au_{n+2}+bu_{n+1}+cu_n=0$$

with nonzero integers as coefficients and initial values. If the prime p divides both a and b, then, by Theorem 1, the distribution of the residues of (u_n) modulo p is either constant or periodic with period $3(p-1)$. For p dividing both a and c, the corresponding period will be $2(p-1)$; it will be $p-1$ for p dividing both b and c.

(3) The T transformation allows a fresh look at the fundamental recursion (R1) and helps to provide an easy demonstration on a periodicity modulo a prime p property of sequences of the type $(2u_{n+1}-au_n)$.

If $\Delta=a^2-4b$, we may replace b in (R1) by $(a^2-\Delta)/4$ and, after simple computation, we obtain $\Delta u_n=4u_{n+2}-4au_{n+1}+a^2u_n$, where we recognize the right-hand side to be $T^2_{(2,-a,1)}(u_n)$, which is the result of the first iteration of the transformation $T_{(2,-a,1)}$. Therefore, by applying Theorem 1 to the sequence $(2u_{n+1}-au_n)=(T_{(2,-a,1)}(u_n))$, with $k=1$, $x=2$, and $y=-a$, we see that if p is any *odd* positive prime divisor of Δ, the discriminant of (R1), supposed nonzero, the distribution of the residues of $(2u_{n+1}-au_n)$ modulo p is either constant or periodic with period $p-1$ for any (u_n) satisfying (R1) and made up of integers. (In that case, the condition that p be odd is necessary to insure that p and $x=2$ are relatively prime.) The interesting fact here is that *any* member of the set of the sequences $(2u_{n+1}-au_n)$ exhibits the same periodicity property with regard to *any number* in the set of odd prime divisors of Δ.

As a more concrete example of application, let (U_n) and (V_n) be, respectively, the *generalized* Fibonacci and Lucas sequences of (R1). If $u_n=U_n$, then, by a well-known formula, we get $2u_{n+1}-au_n=V_n$. This proves that the distribution of the residues of V_n modulo any odd prime divisor p of Δ is either constant or periodic with period $p-1$.

(4) We may generalize this set to set relationship by studying the composition of two T transformations with different integral parameters. For any sequence (u_n), we have

$$T_{(v,w,1)}(T_{(x,y,1)}(u_n))=vxu_{n+2}+(vy+wx)u_{n+1}+wyu_n,$$

which proves that the composition of these transformations is commutative.

If (u_n) satisfies (R1), this expression is equal to $(vy+wx+avx)u_{n+1}+(wy-bvx)u_n$, and by applying Theorem 1 we prove that if p is any positive prime divisor of the gcd of $vy+wx+avx$ and $wy-bvx$ (if one exists), and is relatively prime with both x and v, then the sequences of the residues modulo p of $(xu_{n+1}+yu_n)$ and $(vu_{n+1}+wu_n)$ are either constant or periodic with period $p-1$.

Here we have two different sets of sequences that display the same behavior, in terms of periodicity, regarding a given set of prime numbers (the prime divisors of the gcd of $vy+wx+avx$ and $wy-bvx$).

(5) The period provided by Theorem 1 is not necessarily the *shortest* one, as shown in [3]. The following example shows how this situation may occur. Let us suppose that we have a sequence (u_n) of integers satisfying the recursion (R1), and two nonzero integers x and y such that $xu_{n+2}+yu_n$ is divisible by a prime number p for every n, p being prime with both x and a. The application of Theorem 1 to this situation yields $2(p-1)$ as the corresponding period. But $xu_{n+2}+yu_n=axu_{n+1}+(y-bx)u_n$, which means that the right-hand side is also divisible by p for every n; this time, applying Theorem 1 to this situation yields $p-1$ as the corresponding period. This proves, with the result of Example 1, that the primes p for which there exist integers x and y, x prime with p, such that p divides every $xu_{n+2}+yu_n$, and the distribution of the residues of (u_n) mod p has a corresponding *shortest* period of $2(p-1)$, are necessarily divisors of a.

Therefore, when $a=\pm 1$, for any prime p such that there exist integers x and y such that $xu_{n+2}+yu_n\equiv 0 \pmod{p}$ for every n, x prime with p, the corresponding shortest period is $p-1$ or less. For instance, if (L_n) and (F_n) are, respectively, the classifal Lucas and Fibonacci sequences, the shortest period mod 5 for (L_n) is precisely $p-1=4$, in accordance with the fact that $L_{n+2}+L_n$ is divisible by 5 for every n and $a=1$.

For (F_n), the shortest period mod 5 is 20, which means that, when $0<k<5$, integers x and y, x prime with 5 and such that $xF_{n+k}+yF_n$ is divisible by 5 for every n, do not exist because, in that case, $k(p-1)=4k<20$.

For $k=5$, we easily find that $F_{n+5}+2F_n$ is divisible by 5 for every n, and the corresponding period is $k(p-1)=20$.

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AN OBSERVATION ON SUMMATION FORMULAS FOR GENERALIZED SEQUENCES

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1. PRELIMINARIES

For a, b, p, and q arbitrary integers, in the notation of Horadam [2] write

$$W_n = W_n(a,b;\, p,q) \tag{1.1}$$

so that

$$W_0 = a,\ W_1 = b,\ W_n = pW_{n-1} - qW_{n-2} \quad \text{for } n \geq 2. \tag{1.2}$$

In particular, we write

$$\begin{cases} U_n = W_n(0,1;\, p,q), \\ V_n = W_n(2,p;\, p,q). \end{cases} \tag{1.3}$$

The Binet forms for U_n and V_n are

$$U_n = (\alpha^n - \beta^n)/\sqrt{\Delta}, \tag{1.4}$$

$$V_n = \alpha^n + \beta^n, \tag{1.5}$$

where

$$\Delta \doteq p^2 - 4q, \tag{1.6}$$

and

$$\alpha = (p+\sqrt{\Delta})/2 \quad \text{and} \quad \beta = (p-\sqrt{\Delta})/2 \qquad (1.7$$

are the roots, assumed distinct, of the equation $x^2 - px + q = 0$. Observe that (1.7) yields the two identities

$$\alpha + \beta = p \quad \text{and} \quad \alpha\beta = q. \tag{1.8}$$

As done in [3], throughout this note it is assumed that

$$\Delta > 0, \tag{1.9}$$

so that α, β, and $\sqrt{\Delta}$ are real and $\alpha \neq \beta$. We also assume that

$$q \neq 0 \tag{1.10}$$

to warrant that (1.2) is a second-order recurrence relation. Finally, observe that the particular case $p = 0$ yields

$$U_n = \begin{cases} 0 & (n \text{ even}), \\ (-q)^{(n-1)/2} & (n \text{ odd}), \end{cases} \quad \text{and} \quad V_n = \begin{cases} 2(-q)^{n/2} & (n \text{ even}), \\ 0 & (n \text{ odd}). \end{cases} \tag{1.11}$$

Throughout our discussion, the special sequences (1.11) will not be considered, that is, we shall assume that

$$p \neq 0. \tag{1.12}$$

2. MOTIVATION OF THIS NOTE

Some months ago, I had the opportunity of reviewing (for the American Mathematical Society) an article [3] in which the author establishes several summation formulas for U_n and V_n by using the Binet forms (1.4) and (1.5) and the geometric series formula (g.s.f.).

As usual, I began my review by checking the results numerically. Without intention, I chose the values $p=4$ and $q=3$ which satisfy (1.9), (1.10), and (1.12) and, to my great surprise, noticed that the formulas in [3] do not work for these values of p and q because certain denominators vanish. On the other hand, I ascertained that they work perfectly for many other values of these parameters.

The aim of this note is to bring to the attention of the reader a fact that seems to have passed unnoticed in spite of its simplicity: if $q=p-1$, then either α or β [see (1.7)] equals 1, whereas if $q=-(p+1)$, then either α or β equals -1. Consequently, for obtaining summation formulas for U_n and V_n, the g.s.f. must be used *properly* to avoid getting meaningless expressions.

The example given in Section 4 will clarify our statement.

3. BINET FORMS FOR U_n AND V_n IN THE SPECIAL CASES $q=p-1$ AND $q=-(p+1)$

The Binet forms for U_n and V_n in the cases $q=p-1$ and $q=-(p+1)$ obviously play a crucial role throughout our discussion.

3.1 The case $q=p-1$

If

$$q=p-1 \tag{3.1}$$

then the expression (1.6) becomes

$$\Delta=p^2-4p+4 \tag{3.2}$$

whence, to fulfill (1.9), we must impose the condition

$$p\neq 2. \tag{3.3}$$

Remark 1: Conditions (3.1), (1.12), and (3.3) imply that

$$q\neq \pm 1. \tag{3.4}$$

Since we assumed that $\sqrt{\Delta}$ is positive [see (1.9)], (3.2) also implies that

$$\sqrt{\Delta}=\begin{cases} p-2, & \text{if } p>2,\\ 2-p, & \text{if } p<2,\end{cases} \tag{3.5}$$

whence [see (1.7)]

$$\alpha=\begin{cases} p-1=q \text{ (and } \beta=1), & \text{if } p>2,\\ 1 \text{ (and } \beta=q), & \text{if } p<2.\end{cases} \tag{3.6}$$

From (1.4), (1.5), (3.6), (3.5), and (3.1), it can be seen readily that the Binet forms for U_n and V_n are

$$U_n = \frac{q^n - 1}{q - 1} \quad \text{[cf. (3.4)]} \tag{3.7}$$

and

$$V_n = q^n + 1. \tag{3.8}$$

Remark 2: By virtue of condition (1.10), the Binet forms (3.7) and (3.8) also have meaning for negative values of n.

3.2 The Case $q = -(p+1)$

If

$$q = -(p+1), \tag{3.9}$$

then expression (1.6) becomes

$$\Delta = p^2 + 4p + 4 \tag{3.10}$$

whence, to fulfill (1.9), we must impose the condition

$$p \neq -2 \tag{3.11}$$

which, due to (3.9) and (1.12), implies (3.4) as well.

Since we assumed that $\sqrt{\Delta}$ is positive, (3.10) also implies that

$$\sqrt{\Delta} = \begin{cases} p+2, & \text{if } p > -2, \\ -(p+2), & \text{if } p < -2, \end{cases} \tag{3.12}$$

whence [see (1.7)]

$$\alpha = \begin{cases} p+1 = -q \text{ (and } \beta = -1), & \text{if } p > -2, \\ -1 \text{ (and } \beta = -q), & \text{if } p < -2. \end{cases} \tag{3.13}$$

From (1.4), (1.5), (3.13), (3.12), and (3.9), it can be seen readily that the Binet forms for U_n and V_n are

$$U_n = (-1)^n \frac{q^n - 1}{1 - q} \quad \text{[cf. (3.4)]}, \tag{3.14}$$

and

$$V_n = (-1)^n (q^n + 1). \tag{3.15}$$

Observe that Remark 2 also applies to the Binet forms (3.14) and (3.15).

4. SUMMATION FORMULAS THAT DO NOT HAVE GENERAL VALIDITY

Here we clarify the malfunctioning of the summation formulas in [3] by means of the following example. By using (1.5) and the g.s.f. {*without realizing that, if* $q = p-1$, *then* α (*or* β) $= 1$, *and if* $q = -(p+1)$, then α (*or* β) $= -1$ [see (3.6) and (3.13), respectively]}, after some simple manipulation involving the use of (1.8), one gets

$$\sum_{k=0}^{n} V_{km+r} = \frac{q^m (V_{mn+r} - V_{r-m}) + V_r - V_{m(n+1)+r}}{q^m - V_m + 1} \quad (m \neq 0). \tag{4.1}$$

Remark 3: The right-hand side of (4.1) may involve the use of the extension

$$V_{-m} = V_m / q^m, \tag{4.2}$$

which can be obtained immediately from (1.8).

Warning: Formula (4.1) works for all values of p and q except for those values for which either (3.1) (m arbitrary) or (3.9) (m even) holds. In fact, in these cases, from (3.8) [or (3.15)] we have $q^m - V_m + 1 = 0$. More precisely, it can be proved that the right-hand side of (4.1) assumes the indeterminate form 0/0. Analogous summation formulas yield the same indeterminate form.

If (3.1) holds, the correct closed-form expression for the left-hand side of (4.1) is

$$\begin{aligned}\sum_{k=0}^{n} V_{km+r} &= \sum_{k=0}^{n} (q^{km+r} + 1) \quad \text{[from (3.8)]} \\ &= n + 1 + q^r \frac{q^{m(n+1)} - 1}{q^m - 1} = n + 1 + q^r \frac{V_{m(n+1)} - 2}{V_m - 2} \quad (m \neq 0).\end{aligned} \tag{4.3}$$

If (3.9) holds and m is even, from (3.15), the correct closed-form expression for the left-hand side of (4.1) is readily found to be

$$\sum_{k=0}^{n} V_{km+r} = (-1)^r (n+1) + (-q)^r \frac{V_{m(n+1)} - 2}{V_m - 2} \quad (m \neq 0, \text{ even}). \tag{4.4}$$

Observe that, if (3.9) holds and m is odd, the expression

$$\sum_{k=0}^{n} V_{km+r} = \begin{cases} (-1)^r + (-q)^r V_{m(n+1)} / V_m & (n \text{ even}), \\ (-q)^r (V_{m(n+1)} - 2) / V_m & (n \text{ odd}), \end{cases} \tag{4.5}$$

obtainable from (3.15), is nothing but a compact form for expression (4.1) which, in this case, works as well.

5. SUMMATION FORMULAS FOR U_n AND V_n WHEN $q = p - 1$

We conclude this note by giving a brief account of the various kinds of summation formulas for U_n and V_n that are valid when (3.1) and (3.4) hold. Since their proofs are straightforward, they are omitted for brevity. We confine ourselves to mentioning that the proofs of (5.4)-(5.5) and (5.6)-(5.7) involve the use of the identities—see (3.1) and (3.4) of [1]—

$$\sum_{i=0}^{h} iy^i = \frac{hy^{h+2} - (h+1)y^{h+1} + y}{(y-1)^2} \quad \text{and} \quad \sum_{i=0}^{h} \binom{h}{i} iy^i = hy(y+1)^{h-1},$$

respectively.

$$\sum_{k=0}^{n} U_{km+r} = \frac{q^r U_{m(n+1)}}{(q-1)U_m} - \frac{n+1}{q-1} \quad (m \neq 0), \tag{5.1}$$

$$\sum_{k=0}^{n} \binom{n}{k} U_{km+r} = \frac{q^r V_m^n - 2^n}{q-1}, \tag{5.2}$$

$$\sum_{k=0}^{n} \binom{n}{k} V_{km+r} = q^r V_m^n + 2^n, \tag{5.3}$$

$$\sum_{k=0}^{n} kU_{km+r} = q^r \frac{nU_{m(n+2)} - (n+1)U_{m(n+1)} + U_m}{[(q-1)U_m]^2} - \frac{n(n+1)}{2(q-1)} \quad (m \neq 0), \tag{5.4}$$

$$\sum_{k=0}^{n} kV_{km+r} = q^r \frac{nV_{m(n+2)} - (n+1)V_{m(n+1)} + V_m}{[(q-1)U_m]^2} - \frac{n(n+1)}{2} \quad (m \neq 0), \tag{5.5}$$

$$\sum_{k=0}^{n} k\binom{n}{k} U_{km+r} = \frac{n}{q-1}(q^{m+r}V_m^{n-1} - 2^{n-1}), \tag{5.6}$$

$$\sum_{k=0}^{n} k\binom{n}{k} V_{km+r} = n(q^{m+r}V_m^{n-1} + 2^{n-1}). \tag{5.7}$$

It is obvious that summations (5.1)-(5.7) can be expressed simply in terms of powers of q. Doing so, we sometimes obtain more compact expressions. For example, we get

$$\sum_{k=0}^{n} kV_{km+r} = q^{m+r} \frac{q^{mn}[n(q^m - 1) - 1] + 1}{(q^m - 1)^2} + \frac{n(n+1)}{2} \quad (m \neq 0). \tag{5.5'}$$

Finally, we give the following example pertaining to alternate sign summations:

$$\sum_{k=0}^{n} k\binom{n}{k}(-1)^k V_{km+r} = \begin{cases} 0, & \text{if } n = 0, \\ -V_{m+r}, & \text{if } n = 1, \\ n(-1)^n q^{m+r}[(q-1)U_m]^{n-1}, & \text{if } n > 1. \end{cases} \tag{5.8}$$

The interested reader is urged to work out analogous summation formulas for the case in which $q = -(p+1)$ and m is even.

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ON A KIND OF GENERALIZED ARITHMETIC-GEOMETRIC PROGRESSION

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1. INTRODUCTION

Let $(t|h)_p$ denote the generalized falling factorial of t of degree p and increment h, namely, $(t|h)_0 = 1$ and

$$(t|h)_p = \prod_{j=0}^{p-1}(t-jh), \quad p = 1, 2, 3, \ldots.$$

In particular, $(t|1)_p$ is denoted by $(t)_p$. The main purpose of this note is to establish an explicit summation formula for a kind of generalized arithmetic-geometric progression of the form

$$S_{a,h,p}(n) = \sum_{k=1}^{n} a^k (k|h)_p,$$

where a and h are real or complex numbers, and p a positive integer. It is always assumed that $a \neq 0$ and $a \neq 1$.

It is known that the sum $S_{a,0,p}(n) = \sum_{k=1}^{n} a^k k^p$ has been investigated with different methods by de Bruyn [1] and Gauthier [6]. De Bruyn developed some explicit formulas by using certain determinant expressions derived from Cramer's rule, and Gauthier made repeated use of the differential operator $D = x(d/dx)$ to express $S_{a,0,p}(n)$ as a^n times a polynomial of degree p in n, plus an n-independent term in which the coefficients are determined recursively. In this note we shall express the general sum $S_{a,h,p}(n)$ explicitly in terms of the degenerate Stirling numbers due to Carlitz [2]. In particular, an explicit formula for $S_{a,0,p}(n)$ will be given via Stirling numbers of the second kind. Finally, as other applications of our Lemma 1, some combinatorial sums involving generalized factorials will be presented.

2. AN EXPLICIT SUMMATION FORMULA

We will make use of the degenerate Stirling numbers $S(n, k|\lambda)$ first defined by Carlitz [2] using the generating function

$$\frac{1}{k!}((1+\lambda x)^{1/\lambda} - 1)^k = \sum_{n=k}^{\infty} S(n, k|\lambda)\frac{x^n}{n!}. \tag{1}$$

The ordinary Stirling numbers of the second kind are given by

$$S(n, k) := S(n, k|0) = \lim_{\lambda \to 0} S(n, k|\lambda).$$

Also, Carlitz proved among other things (cf. [2])

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$$(t|h)_p = \sum_{j=0}^{p} S(p, j|h)(t)_j \tag{2}$$

$$S(p, j|h) = \frac{1}{j!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}(k|h)_p. \tag{3}$$

Note that the so-called C-numbers extensively studied by Charalambides and others (see [3] and [4]) are actually related to the degenerated Stirling numbers in such a way that

$$S(p, j|h) = h^p C(p, j, 1/h), \ (h \neq 0).$$

The following lemma may be regarded as a supplement to the simple summation rule proposed in [7].

Lemma 1: Let $F(n, k)$ be a bivariate function defined for integers $n, k \geq 0$. If there can be found a formula such as

$$\sum_{k=0}^{n} F(n, k)\binom{k}{j} = \psi(n, j), \ (j \geq 0), \tag{4}$$

then for every integer $p \geq 0$ we have a summation formula

$$\sum_{k=0}^{n} F(n, k)(k|h)_p = \sum_{j=0}^{p} \psi(n, j) j! S(p, j|h) \tag{5}$$

with its limiting case for $h \to 0$,

$$\sum_{k=0}^{n} F(n, k)k^p = \sum_{j=0}^{p} \psi(n, j) j! S(p, j). \tag{6}$$

Proof: Plainly, (5) may be verified at once by substituting expression (2) with $t = k$ into the left-hand side of (5) and by changing the order of summation and using (4), namely,

$$\sum_{k=0}^{n} F(n, k)\sum_{j=0}^{p} S(p, j|h) j!\binom{k}{j} = \sum_{j=0}^{p} S(p, j|h) j! \psi(n, j).$$

The fact that (5) implies (6) with $h \to 0$ is also evident. □

Lemma 2: For $a \neq 0$ and $a \neq 1$, let $\phi(n, j) \equiv \phi(n, j; a)$ be defined by

$$\phi(n, j) = \sum_{k=0}^{n} a^k\binom{k}{j}, \ (j \geq 0). \tag{7}$$

Then $\phi(n, j)$ satisfies the recurrence relations

$$a\phi(n, j-1) + (a-1)\phi(n, j) = a^{n+1}\binom{n+1}{j} \tag{8}$$

with $\phi(n, 0) = (a^{n+1} - 1)/(a-1)$ and $j = 1, 2, \ldots$.

Proof: Evidently, we have

$$\begin{aligned} a^{n+1}\binom{n+1}{j} &= \phi(n+1, j) - \phi(n, j) \quad (j \geq 1) \\ &= \sum_{k=0}^{n} a^{k+1}\binom{k+1}{j} - \phi(n, j) \\ &= a\sum_{k=0}^{n} a^{k}\left[\binom{k}{j} + \binom{k}{j-1}\right] - \sum_{k=0}^{n} a^{k}\binom{k}{j} \\ &= (a-1)\phi(n, j) + a\phi(n, j-1), \end{aligned}$$

where the initial condition is given by $\phi(n, 0) = \sum_{k=0}^{n} a^k = (a^{n+1}-1)/(a-1)$. □

In what follows we will occasionally make use of the forward difference operator Δ, defined by $\Delta f(x) = f(x+1) - f(x)$ and $\Delta^j = \Delta\Delta^{j-1}, (j \geq 2)$.

Proposition: The following summation formulas hold:

$$S_{a,h,p}(n) = \sum_{j=1}^{p} \phi(n, j; a) j!\, S(p, j \mid h); \tag{9}$$

$$S_{a,0,p}(n) = \sum_{j=1}^{p} \phi(n, j; a) j!\, S(p, j). \tag{10}$$

These formulas may also be written as

$$S_{a,h,p}(n) = \sum_{j=1}^{p} \phi(n, j; a)\left[\Delta^j (t \mid h)_p\right]_{t=0} \tag{11}$$

and

$$S_{a,0,p}(n) = \sum_{j=1}^{p} \phi(n, j; a)\left[\Delta^j t^p\right]_{t=0}, \tag{12}$$

where $\phi(n, j; a)$ $(1 \leq h \leq p)$ and the higher differences involved have explicit expressions, viz.,

$$\phi(n, j; a) = \frac{1}{1-a}\left[\left(\frac{a}{1-a}\right)^j - a^{n+1}\sum_{r=0}^{j}\binom{n+1}{j-r}\left(\frac{a}{1-a}\right)^r\right], \tag{13}$$

$$\left[\Delta^j (t \mid h)_p\right]_{t=0} = \sum_{r=0}^{j} (-1)^{j-r}\binom{j}{r}(r \mid h)_p, \tag{14}$$

$$\left[\Delta^j t^p\right]_{t=0} = \Delta^j 0^p = \sum_{r=0}^{j} (-1)^{j-r}\binom{j}{r} r^p. \tag{15}$$

Proof: Comparing (7) with (4), in which $F(n, k)$ corresponds to a^k (not depending on n), we see that (9) and (10) are merely consequences of Lemma 1. Thus, it suffices to verify (11) and (13). Actually (14) and (15) are well-known expressions from the calculus of finite differences. Consequently, the equivalence between (9) and (11) and that between (10) and (12) both follow from the relations (3) and (14).

Moreover, Lemma 2 implies the following:

$$\phi(n, j) = \frac{a^{n+1}}{a-1}\sum_{r=0}^{j-1}\binom{n+1}{j-r}\left(\frac{a}{a-1}\right)^r + \left(\frac{a}{1-a}\right)^j \phi(n, 0)$$
$$= \frac{a^{n+1}}{a-1}\sum_{r=0}^{j}\binom{n+1}{j-r}\left(\frac{a}{1-a}\right)^r + \frac{1}{1-a}\left(\frac{a}{1-a}\right)^j.$$

This is precisely equivalent to (13). □

Remark: According to the terminology adopted in Comtet's book [5], we may say that both (11) and (12) provide summation formulas of rank 3 as they both consist of triple sums after having substituted (13), (14), and (15) into the right-hand sides of (11) and (12), respectively. The number of terms involved in each formula is, obviously,

$$\sum_{j=1}^{p}(j+2)(j+1) = \frac{1}{3}(p+3)(p+2)(p+1) - 2.$$

Surely these formulas are of practical value for computation when $n \gg p$.

Let us rewrite (10) in the form

$$\sum_{k=1}^{n} a^k k^p = \sum_{j=1}^{p}\frac{j!S(p, j)}{1-a}\left[\left(\frac{a}{1-a}\right)^j - a^{n+1}\sum_{r=0}^{j}\binom{n+1}{j-r}\left(\frac{a}{1-a}\right)^r\right], \tag{10'}$$

where $j!S(p, j) = \Delta^j 0^p$ are given by (15). Notice that for $|a| < 1$ the limit of $a^{n+1}\binom{n+1}{r}$ $(0 \le r \le j)$ is zero when $n \to \infty$. Thus, using (10') and passing to limit, we easily obtain the following convergent series:

$$\sum_{k=1}^{\infty} a^k k^p = \sum_{j=1}^{p} j!S(p, j)\frac{a^j}{(1-a)^{j+1}}, \quad (|a| < 1).$$

3. EXAMPLES

Example 1: Let the Fibonacci numbers F_k and the Lucas numbers L_k be given by the Binet forms

$$F_k = (\alpha^k - \beta^k)/\sqrt{5} \quad \text{and} \quad L_k = \alpha^k + \beta^k$$

with $\alpha = (a+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then it is easily seen that the following sums, i.e.,

$$\Phi(n) = \sum_{k=1}^{n}(k|h)_p F_k \quad \text{and} \quad \Lambda(n) = \sum_{k=1}^{n}(k|h)_p L_k,$$

can be computed by means of (9) or (11). Indeed, we have

$$\Phi(n) = \frac{1}{\sqrt{5}}\sum_{j=1}^{p}(\phi(n, j; \alpha) - \phi(n, j; \beta))j!S(p, j|h),$$

and

$$\Lambda(n) = \sum_{j=1}^{p}(\phi(n, j; \alpha) + \phi(n, j; \beta))j!S(p, j|h),$$

where $\phi(n, j; \alpha)$ and $\phi(n, j; \beta)$ are given by (13) with $a = \alpha$ and $a = \beta$.

Example 2: Given real numbers h and θ with $0<\theta<2\pi$. It is easily found that the sums

$$C(n)=\sum_{k=1}^{n}(k\,|\,h)_p\cos k\theta \quad\text{and}\quad S(n)=\sum_{k=1}^{n}(k\,|\,h)_p\sin k\theta$$

can also be computed via (9) or (11). To see this, let us take $a=e^{i\theta}$ with $i=\sqrt{-1}$. Evidently, the sums $C(n)$ and $S(n)$ are given by the real and imaginary parts of the sum formula for $S_{a,h,p}(n)$, namely,

$$C(n)=\operatorname{Re}\sum_{j=1}^{p}\phi(n,j;e^{i\theta})j!S(p,j\,|\,h) \quad\text{and}\quad S(n)=\operatorname{Im}\sum_{j=1}^{p}\phi(n,j;e^{i\theta})j!S(p,j\,|\,h).$$

Example 3: Let $\phi(n,j)\equiv\phi(n,j;a)$ be given by (13). Then, using the values of $S(p,j)$ for $j\le p\le 5$, we can immediately write down several special formulas for the sum $S_{a,p}(n)=\sum_{k=1}^{n}a^kk^p$, as follows:

$$S_{a,1}(n)=\phi(n,1),$$
$$S_{a,2}(n)=\phi(n,1)+2\phi(n,2),$$
$$S_{a,3}(n)=\phi(n,1)+6\phi(n,2)+6\phi(n,3),$$
$$S_{a,4}(n)=\phi(n,1)+14\phi(n,2)+36\phi(n,3)+24\phi(n,4),$$
$$S_{a,5}(n)=\phi(n,1)+30\phi(n,2)+150\phi(n,3)+240\phi(n,4)+120\phi(n,5).$$

As may be verified, the first two equalities given above do agree with the two explicit expressions displayed in de Bruyn [1].

4. OTHER APPLICATIONS OF LEMMA 1

It is clear that the key step necessary for applying the summation rule given by Lemma 1 is to find an available form of $\psi(n,j)$ with respect to a given $F(n,k)$. Now let us take $F(n,k)$ to be the following forms, respectively,

$$1,\ \binom{n}{k},\ \binom{n}{k}^2,\ \binom{n}{2k},\ \binom{n}{2k+1},\ \binom{s+k}{s},\ H_k,$$

where $H_k=1+1/2+\cdots+1/k$ are harmonic numbers. Then the corresponding $\psi(n,j)$'s may be found easily by using some known combinatorial identities, and, consequently, we obtain the following sums (with $p\ge 1$) via (5):

$$\sum_{k=1}^{n}(k\,|\,h)_p=\sum_{j=1}^{p}\binom{n+1}{j+1}j!S(p,j\,|\,h), \tag{16}$$

$$\sum_{k=1}^{n}\binom{n}{k}(k\,|\,h)_p=\sum_{j=1}^{p}\binom{n}{j}2^{n-j}j!S(p,j\,|\,h), \tag{17}$$

$$\sum_{k=1}^{n}\binom{n}{k}^2(k\,|\,h)_p=\sum_{j=1}^{p}\binom{2n-j}{n}(n)_jS(p,j\,|\,h), \tag{18}$$

$$\sum_{k=1}^{[n/2]}\binom{n}{2k}(k\,|\,h)_p=\sum_{j=1}^{p}2^{n-2j}\binom{n-j}{j}\frac{n}{n-j}j!S(p,j\,|\,h), \tag{19}$$

$$\sum_{k=1}^{[n/2]}\binom{n}{2k+1}(k|h)_p=\sum_{j=1}^{p}2^{n-2j}\binom{n-j}{j}j!S(p,j|h), \tag{20}$$

$$\sum_{k=1}^{n}\binom{s+k}{s}(k|h)_p=\sum_{j=1}^{p}\binom{n+1}{j}\binom{n+1+s}{s}\frac{n+1-j}{s+1+j}j!S(p,j|h), \tag{21}$$

$$\sum_{k=1}^{n}(k|h)_pH_k=\sum_{j=1}^{p}\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right)j!S(p,j|h). \tag{22}$$

These sums with $h=0$ will reduce to the cases displayed in [7]. Actually, (19) and (20) follow from an application of the pair of Moriarty identities, (21) from that of Knuth's identity, and (22) is obviously implied by (4)-(5) and the well-known relation

$$\sum_{k=j}^{n}\binom{k}{j}H_k=\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right) \text{ (see §2 of [7]).}$$

Evidently, (21) implies the classical formula for $\sum_{k=1}^{n}k^p$ when $s=0$ and $h=0$.

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SUMMATION OF RECIPROCALS IN CERTAIN SECOND-ORDER RECURRING SEQUENCES

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1. INTRODUCTION

We consider the sequence $\{W_n\} = \{W_n(a,b;P,Q)\}$ of integers defined by

$$W_0 = a,\ W_1 = b,\ W_n = PW_{n-1} - QW_{n-2} \quad (n \geq 2), \tag{1.1}$$

where a, b, P, and Q are integers, with $PQ \neq 0$. Particular cases of $\{W_n\}$ are the sequences $\{U_n\}$ of Fibonacci and $\{V_n\}$ of Lucas defined by $U_n = W_n(0,1;P,Q)$ and $V_n = W_n(2,P;P,Q)$. In the sequel we shall suppose that $\Delta = P^2 - 4Q > 0$. It is readily proven [6] that

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.2}$$

where $\alpha = (P+\sqrt{\Delta})/2$, $\beta = (P-\sqrt{\Delta})/2$, $A = b - \beta a$, and $B = b - \alpha a$. Following Horadam [6], we define the number e_w by $e_w = AB = b^2 - Pab + Qa^2$. It is clear that $e_u = 1$ and $e_v = -\Delta = -(\alpha-\beta)^2$, where e_u and e_v are associated with the Fibonacci and Lucas sequences. By means of the Binet form (1.2), one can easily prove the Catalan relation

$$W_n^2 - W_{n-1}W_{n+1} = e_w Q^{n-1}. \tag{1.3}$$

Notice that

$$\alpha > 1 \quad \text{and} \quad \alpha > |\beta|, \ \text{if } P > 0, \tag{1.4}$$

and that

$$\beta < -1 \quad \text{and} \quad |\beta| > |\alpha|, \ \text{if } P < 0. \tag{1.5}$$

By (1.4) and (1.5), it is clear that $U_n \neq 0$ for $n \geq 1$ and that $V_n \neq 0$ for $n \geq 0$. More generally, there exists an integer p such that $W_p = 0$ if and only if $W_n = W_{p+1}U_{n-p}$ for every integer n. By (1.4) and (1.5), we obtain

$$W_n \simeq \frac{A}{\alpha-\beta}\alpha^n, \ \text{if } P > 0 \quad \text{and} \quad W_n \simeq \frac{-B}{\alpha-\beta}\beta^n, \ \text{if } P < 0. \tag{1.6}$$

The purpose of this paper is to investigate the infinite sums

$$S_k = \sum_{n=1}^{+\infty} \frac{Q^n}{W_n W_{n+k}} \quad \text{and} \quad T_k = \sum_{n=1}^{+\infty} \frac{1}{W_n W_{n+k}},$$

where k is a positive integer. We shall suppose that $W_n \neq 0$ for $n \geq 1$ (see the remark above) and that $e_w = AB \neq 0$ (which means that $\{W_n\}$ is not a purely geometric sequence). By (1.4) and (1.5), use of the ratio test shows that the series S_k and T_k are absolutely convergent. Notice that $S_k = T_k$, when $Q = 1$.

More generally, let $\pi(n) = m + sn$ be an arithmetical progression, with $m \geq 0$ and $s \geq 1$. We shall examine the sums

$$S_{k,\pi} = \sum_{n=1}^{+\infty} \frac{Q^{\pi(n)}}{W_{\pi(n)} W_{\pi(n+k)}} \quad \text{and} \quad T_{k,\pi} = \sum_{n=1}^{+\infty} \frac{1}{W_{\pi(n)} W_{\pi(n+k)}}.$$

By the way, we shall also obtain a symmetry property (Theorem 1) that generalizes a recent result of Good [5].

Remark 1: Notice that $S_{k,\pi} = T_{k,\pi}$ when $Q = 1$ and that $S_{k,\pi} = (-1)^m T_{k,\pi}$ when $Q = -1$ and s is even.

2. MAIN RESULTS

Theorem 1: We have

$$U_k \sum_{n=1}^{m} \frac{Q^n}{W_n W_{n+k}} = U_m \sum_{n=1}^{k} \frac{Q^n}{W_n W_{n+m}},$$

where k and m are nonnegative integers.

Theorem 2: If $P > 0$, then

$$S_k = \frac{1}{e_w U_k} \left[\sum_{r=1}^{k} \frac{W_{r+1}}{W_r} - k\alpha \right]. \tag{2.1}$$

If $P < 0$, replace α by β in the right member.

Theorem 2': If $P > 0$ or if $P < 0$ and s is even, then

$$S_{k,\pi} = \frac{1}{e_w U_s U_{sk}} \left[\sum_{r=1}^{k} \frac{W_{\pi(r+1)}}{W_{\pi(r)}} - k\alpha^s \right]. \tag{2.2}$$

If $P < 0$ and s is odd, replace α^s by β^s in the right member.

Theorem 3: If $P > 0$, then

$$AU_k T_k = (1 - Q^k) \sum_{r=1}^{+\infty} \frac{1}{\alpha^r W_r} + Q^k \sum_{r=1}^{k} \frac{1}{\alpha^r W_r}. \tag{2.3}$$

If $P < 0$, replace A by B in the left member and α by β in the right member.

Corollary 1: If $Q = -1$, then

$$T_{2k} = \frac{1}{U_{2k}} \sum_{r=1}^{k} \frac{1}{W_{2r} W_{2r-1}} \tag{2.4}$$

and

$$T_{2k+1} = \frac{1}{U_{2k+1}} \left[T_1 - \sum_{r=1}^{k} \frac{1}{W_{2r} W_{2r+1}} \right]. \tag{2.5}$$

Corollary 2: If $Q = -1$ and s is odd, then

$$T_{2k,\pi} = \frac{U_s}{U_{2ks}} \sum_{r=1}^{k} \frac{1}{W_{\pi(2r)} W_{\pi(2r-1)}} \tag{2.6}$$

and

$$T_{2k+1,\pi}=\frac{U_s}{U_{(2k+1)s}}\left[T_{1,\pi}-\sum_{r=1}^{k}\frac{1}{W_{\pi(2r)}W_{\pi(2r+1)}}\right]. \tag{2.7}$$

Remark 2: If $Q=-1$, $k=1$, and $W_n=U_n$ or V_n, then Theorem 3 is Lemma 2 in [1].

Remark 3: Theorem 1 shows that S_k is a rational number if and only if α is rational or, equivalently, if and only if Δ is a perfect square. Corollary 1 shows that, in the case $Q=-1$, T_{2k} is rational, while T_{2k+1} is rational if and only if T_1 is rational. Notice that, even in the usual case $W_n=W_n(0,1;1,-1)=F_n$, the value and the arithmetical nature of T_1 is unknown. One can obtain similar results for the numbers $S_{k,\pi}$ and $T_{k,\pi}$.

Theorem 1 is given by Good [5] in the case $Q=-1$. Theorem 2′ was first obtained by Lucas [8, p. 198] in the case $k=1$, $W_n=U_n$ or V_n. The same results were rediscovered by Popov [11]. Brousseau [3] proved Theorem 2 for $W_n=F_n$ and he gave numerical examples of Corollary 1. Good [5] proved Theorem 2 in the case $Q=-1$. In [2], [7], and [9], one can find variants of Theorem 2′ applied to Fibonacci, Lucas, Pell, and Chebyshev polynomials.

3. PRELIMINARIES

In the sequel, we shall need the following lemmas.

Lemma 1: For integers $n\ge 0$ and $k\ge 0$

$$\begin{cases} W_{n+k}-\beta^k W_n = A\alpha^n U_k, & (3.1)\\ W_{n+k}-\alpha^k W_n = B\beta^n U_k. & (3.2)\end{cases}$$

Proof: Using Binet form (1.2), the result is immediate.

Lemma 2: For integers $k\ge 1$,

$$\sum_{r=1}^{k}\frac{\beta^r}{W_r}=\frac{1}{B}\left[\sum_{r=1}^{k}\frac{W_{r+1}}{W_r}-k\alpha\right], \tag{3.3}$$

$$\sum_{r=1}^{k}\frac{\alpha^r}{W_r}=\frac{1}{A}\left[\sum_{r=1}^{k}\frac{W_{r+1}}{W_r}-k\beta\right]. \tag{3.4}$$

Proof: We prove only (3.3); the proof of (3.4) is similar. By (3.2), where $n=r$ and $k=1$, we have

$$\sum_{r=1}^{k}\frac{\beta^r}{W_r}=\frac{1}{B}\sum_{r=1}^{k}\frac{W_{r+1}-\alpha W_r}{W_r}=\frac{1}{B}\left[\sum_{r=1}^{k}\frac{W_{r+1}}{W_r}-k\alpha\right].$$

Lemma 3: If $Q=-1$, we have, for $k\ge 1$,

$$\sum_{r=1}^{k}\frac{1}{\alpha^r W_r}=A\sum_{r=1}^{k}\frac{1}{W_{2r}W_{2r-1}}, \tag{3.5}$$

$$\sum_{r=2}^{2k+1} \frac{1}{\alpha^r W_r} = A \sum_{r=1}^{k} \frac{1}{W_{2r} W_{2r+1}}. \tag{3.6}$$

One can obtain two similar formulas by replacing α by β and A by B.

Proof: We prove only (3.5). Since $Q = -1$, we have $\alpha^r \beta^r = (-1)^r$ for $k \geq 1$; thus,

$$\begin{aligned}
\sum_{r=1}^{2k} \frac{1}{\alpha^r W_r} &= \frac{1}{B} \sum_{r=1}^{2k} \frac{(-1)^r \beta^r B}{W_r} = \frac{1}{B} \sum_{r=1}^{2k} (-1)^r \frac{W_{r+1} - \alpha W_r}{W_r}, \quad \text{by (3.2)} \\
&= \frac{1}{B} \sum_{r=1}^{2k} (-1)^r \frac{W_{r+1}}{W_r} = \frac{1}{B} \sum_{r=1}^{k} \left(\frac{-W_{2r}}{W_{2r-1}} + \frac{W_{2r+1}}{W_{2r}} \right) \\
&= \frac{1}{B} \sum_{r=1}^{k} \frac{W_{2r+1} W_{2r-1} - W_{2r}^2}{W_{2r} W_{2r-1}} = \frac{1}{B} \sum_{r=1}^{k} \frac{-e_w (-1)^{2r-1}}{W_{2r} W_{2r-1}}, \quad \text{by (1.3)} \\
&= A \sum_{r=1}^{k} \frac{1}{W_{2r} W_{2r-1}}, \quad \text{since } e_w = AB.
\end{aligned}$$

Lemma 4: Let $\{a_n\}$ be a sequence of numbers and $\{b_{n,k}\}$ be the sequence defined by

$$b_{n,k} = a_n - a_{n+k}, \quad k \geq 0. \tag{3.7}$$

For every $m \geq 0$ and $k \geq 0$, we then have

$$\sum_{n=1}^{m} b_{n,k} = \sum_{n=1}^{k} b_{n,m}. \tag{3.8}$$

Proof: Without loss of generality, we assume $m > k$. By (3.7) we get

$$\begin{aligned}
\sum_{n=1}^{m} b_{n,k} &= (a_1 + \cdots + a_m) - (a_{k+1} + \cdots + a_{m+k}) \\
&= (a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_m) - (a_{k+1} + \cdots + a_m) - (a_{m+1} + \cdots + a_{m+k}) \\
&= (a_1 + \cdots + a_k) - (a_{m+1} + \cdots + a_{m+k}) = \sum_{n=1}^{k} b_{n,m}.
\end{aligned}$$

4. PROOF OF THEOREMS 1, 2, AND 2′

We get by (3.1) that

$$\frac{\beta^n}{W_n} - \frac{\beta^{n+k}}{W_{n+k}} = \frac{AQ^n U_k}{W_n W_{n+k}}. \tag{4.1}$$

Putting $a_n = \beta^n / W_n$ and $b_{n,k} = AQ^n U_k / W_n W_{n+k}$, we see by (4.1) that $b_{n,k} = a_n - a_{n+k}$. Theorem 1 follows immediately by this and Lemma 4.

Assuming now that $P > 0$ and letting $n = 1, 2, \ldots, N$, where $N \geq k$, we obtain

$$AU_k \sum_{n=1}^{N} \frac{Q^n}{W_n W_{n+k}} = \sum_{r=1}^{k} \frac{\beta^r}{W_r} - \sum_{r=N+1}^{N+k} \frac{\beta^r}{W_r}.$$

Now, by (1.6) we have

$$\frac{\beta^r}{W_r} \simeq \frac{\alpha-\beta}{A}\left(\frac{\beta}{\alpha}\right)^r,$$

and since $\alpha > |\beta|$, the last sum in the right member vanishes as $N \to +\infty$. Thus, by (3.3),

$$AU_k \sum_{n=1}^{+\infty} \frac{Q^n}{W_n W_{n+k}} = \sum_{r=1}^{k} \frac{\beta^r}{W_r} = \frac{1}{B}\left[\sum_{r=1}^{k} \frac{W_{r+1}}{W_r} - k\alpha\right],$$

and the conclusion follows from this, since $e_w = AB$. If $P < 0$, replace β by α in the left member of (4.1) and A by B in the right member. Using (3.2) and (3.4) and recalling that $|\beta| > |\alpha|$ in this case, the end of the proof is similar.

Let us examine some particular cases. If $W_n = U_n$ (respectively V_n) and since $e_u = 1$ (respectively $e_v = -\Delta$), we get that

$$\sum_{n=1}^{+\infty} \frac{Q^n}{U_n U_{n+k}} = \frac{1}{U_k}\left[\sum_{r=1}^{k} \frac{U_{r+1}}{U_r} - k\alpha\right] \tag{4.2}$$

and

$$\sum_{n=1}^{+\infty} \frac{Q^n}{V_n V_{n+k}} = \frac{1}{\Delta U_k}\left[k\alpha - \sum_{r=1}^{k} \frac{V_{r+1}}{V_r}\right] \tag{4.3}$$

when $P > 0$.

If $P < 0$, replace α by β in the above formulas.

We turn now to the proof of Theorem 2′. Let us consider a second-order recurring sequence $\{W_n'\}$ (see [4] and [10]) satisfying

$$W_n' = P'W_{n-1}' - Q'W_{n-2}', \; n \ge 2, \tag{4.4}$$

where $P' = \alpha^s + \beta^s = V_s$ and $Q' = \alpha^s \beta^s = Q^s$. Notice that $P' > 0$ if and only if $P > 0$ or if $P < 0$ and s is even. The Fibonacci sequence associated with the recurrence (4.4) is defined by

$$U_n' = \frac{\alpha^{sn} - \beta^{sn}}{\alpha^s - \beta^s} = \frac{U_{sn}}{U_s}. \tag{4.5}$$

On the other hand, we have

$$W_{\pi(n)} = W_{m+sn} = \frac{A'\alpha^{sn} - B'\beta^{sn}}{\alpha - \beta},$$

where $A' = A\alpha^m$ and $B' = B\beta^m$. If $\{W_n'\}$ is the solution of (4.4) defined by $W_n' = \frac{A'\alpha^{sn} - B'\beta^{sn}}{\alpha^s - \beta^s}$, we have

$$W_n' = \frac{W_{\pi(n)}}{U_s}. \tag{4.6}$$

It follows by Theorem 2 applied to $\{W_n'\}$ that, if $P' > 0$,

$$\sum_{n=1}^{+\infty} \frac{Q^{sn}}{W_n' W_{n+k}'} = \frac{1}{e_{w'} U_k'}\left[\sum_{r=1}^{k} \frac{w_{r+1}'}{w_r'} - k\alpha^s\right]. \tag{4.7}$$

Using (4.5) and (4.6) and noticing that $e_{w'} = A'B' = AB\alpha^m\beta^m = e_w Q^m$, we easily deduce (2.2) from (4.7). If $P' < 0$, replace α^s by β^s in the right member of (4.7).

5. PROOF OF THEOREM 3 AND COROLLARIES 1 AND 2

Supposing first that $P > 0$, we get by (3.1) that

$$\frac{1}{\alpha^n W_n} - \frac{Q^k}{\alpha^{n+k} W_{n+k}} = \frac{AU_k}{W_n W_{n+k}}. \tag{5.1}$$

Letting $n = 1, 2, \ldots, N$, where $N \geq k$, and summing, we obtain

$$\begin{aligned} AU_k \sum_{n=1}^{N} \frac{1}{W_n W_{n+k}} &= \sum_{r=1}^{k} \frac{1}{\alpha^r W_r} + (1-Q^k) \sum_{r=k+1}^{N} \frac{1}{\alpha^r W_r} - Q^k \sum_{r=N+1}^{N+k} \frac{1}{\alpha^r W_r} \\ &= (1-Q^k) \sum_{r=1}^{N} \frac{1}{\alpha^r W_r} + Q^k \sum_{r=1}^{k} \frac{1}{\alpha^r W_r} - Q^k \sum_{r=N+1}^{N+k} \frac{1}{\alpha^r W_r}. \end{aligned}$$

The first sum in the right member converges as $N \to +\infty$ since $\alpha^r W_r \simeq \frac{A}{\alpha - \beta} \alpha^{2r}$, where $\alpha > 1$. We also see that the last sum vanishes when $N \to +\infty$. This concludes the proof of Theorem 3 when $P > 0$. If $P < 0$, the proof is similar.

Notice that the first term in the right member of (2.3) vanishes if and only if $Q = 1$ (in which case $S_k = T_k$) or $Q = -1$ and k is even. The series $\sum_{r=1}^{+\infty} \frac{1}{\alpha^r W_r}$ seems difficult to evaluate. If $Q = -1$ and if $W_n = U_n$ or $W_n = V_n$, this series can be expressed with the help of the Lambert series [1, Lemma 3]. If $Q = 1$, it does not appear in (2.3). This fact explains why Melham and Shannon [9, p. 199] obtain formulas that do not involve Lambert series.

If $Q = -1$ and k is even, then (2.3) becomes

$$AU_{2k} T_{2k} = \sum_{r=1}^{2k} \frac{1}{\alpha^r W_r} = A \sum_{r=1}^{k} \frac{1}{W_{2r} W_{2r-1}}$$

by (3.5), when $P > 0$. This concludes the proof of (2.4). If $P < 0$, the proof is similar.

On the other hand, put $Q = -1$ and replace k by $2k + 1$ in (2.2) to obtain

$$AU_{2k+1} T_{2k+1} = 2 \sum_{r=1}^{+\infty} \frac{1}{\alpha^r W_r} - \sum_{r=1}^{2k+1} \frac{1}{\alpha^r W_r},$$

and, using (3.6), we deduce from this

$$AU_{2k+1} T_{2k+1} - AU_1 T_1 = -\sum_{r=1}^{2k+1} \frac{1}{\alpha^r W_r} = -A \sum_{r=1}^{k} \frac{1}{W_{2r} W_{2r+1}}.$$

This concludes the proof of (2.5) when $P > 0$. The case in which $P < 0$ is similar.

Using (4.5) and (4.6) and applying Corollary 1 to the sequence $\{W_n'\}$, one can easily obtain the proof of Corollary 2 when noticing that $Q^s = -1$, since s is odd.

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❖❖❖

FIBONACCI NUMBERS AND ALGEBRAIC STRUCTURE COUNT OF SOME NON-BENZENOID CONJUGATED POLYMERS

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1. INTRODUCTION

The algebraic structure count (ASC) of a graph G is

$$\mathrm{ASC}\{G\} \stackrel{\mathrm{def}}{=} \sqrt{|\det A|},$$

where A is the adjacency matrix of G. This quantity has noteworthy applications in chemistry (see below), provided that the graph G represents the carbon-atom skeleton of a molecule of a class of hydrocarbons, the so-called *conjugated hydrocarbons* [9]. Therefore, we call this graph by the same name as the respective hydrocarbon.

Of particular importance for chemical applications are graphs that are connected, bipartite, and planar and which, when considered as plane graphs, have the property that every face-boundary (cell) is a circuit of length of the form $4s+2$ $(s=1,2,\ldots)$ [2]. We refer to these graphs as *benzenoid*, noting, however, that the actual definition of benzenoid systems is slightly more complicated [8]. Molecular graphs that are connected, bipartite, and planar, but in which some face-boundaries are circuits of length of the form $4s$ $(s=1,2,\ldots)$ will be referred to as *non-benzenoid*. The graphs studied in the present work belong to this latter class.

In the case of benzenoid graphs, the ASC-value coincides with the number of perfect matchings (1-factors), which is a result of crucial importance for chemical applications. Chemists call the 1-factors *Kekulé structures* [3], and these objects play significant roles in various chemical theories [8]. The enumeration of 1-factors in benzenoid graphs is not too difficult a task [3] and can be accomplished by various recursive methods. Consequently, the calculation of ASC of benzenoid graphs is easy.

In the case of non-benzenoid graphs, the relation between ASC and the number of perfect matchings is less simple and is given below (Theorem 2). Contrary to the former case, in this case the determination of the ASC-value is a nontrivial task because no efficient recursive graphical technique is known for computing ASC [5]. A systematic study of the ASC-values of non-benzenoid conjugated systems was recently initiated by one of the present authors (see [6], [7]). Among others, in [7], the linear [n]phenylene [Fig. 1(a)] and the angular [n]phenylene [Fig. 1(b)] are considered.

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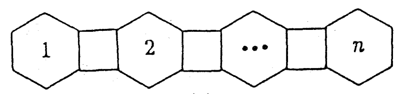
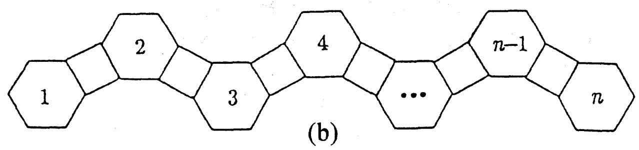


(a)



(b)

FIGURE 1

The Linear [n]Phenylene (a) and the Angular [n]Phenylene (b); Algebraic Structure Counts of Angular Phenylenes are Fibonacci Numbers

It is established that the ASC-value of the angular [n]phenylene is equal to the $(n+2)^{\text{th}}$ Fibonacci number ($F_0=0,\ F_1=1,\ F_2=1,\dots$). It has been known for a long time [3] that the number of 1-factors (K-value) of the zig-zag chain $A(n)$ of n hexagons (circuits of length 6) (Fig. 2) is equal to the same number, i.e.,

$$K\{A(n)\}=F_{n+2}. \tag{1}$$

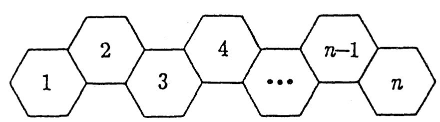


FIGURE 2

The Zig-Zag Hexagonal Chain $A(n)$; Number of 1-Factors of $A(n)$ Is a Fibonacci Number

In this paper we show that the ASC-value of a class of non-benzenoid hydrocarbons can be expressed by means of Fibonacci numbers. This structure $B_n\equiv B_n(A(m_1),A(m_2),\dots,A(m_n))$ (which will be described in detail later) consists of n zig-zag chains concatenated by $(n-1)$ squares (circuits of length 4). The manner of concatenation of two zig-zag chains by a square depends on *the types* (I-IV) of these zig-zag chains (which will also be defined later). The main result is the following statement.

Theorem 1: If the graph B_n consists of n zig-zag chains of the same length m $(m>2)$ concatenated in the same manner, i.e., all zig-zag chains are of the same type, then

$$\mathrm{ASC}\{B_n\}=\frac{1}{2^{n+1}}\left[\left(1+\frac{F_{m+2}+F}{D}\right)(F_{m+2}-F+D)^n+\left(1-\frac{F_{m+2}+F}{D}\right)(F_{m+2}-F-D)^n\right],$$

where

$$D=\begin{cases}\sqrt{(F_{m+2}-F_{m-2})^2+4\cdot(-1)^{m+1}}, & \text{if } A(m) \text{ is of type I},\\ \sqrt{(F_{m+2}-F_{m-1})^2+4\cdot(-1)^{m}}, & \text{if } A(m) \text{ is of type II or III},\\ \sqrt{F_{m+1}^2+4\cdot(-1)^{m+1}}, & \text{if } A(m) \text{ is of type IV},\end{cases}$$

and

$$F = \begin{cases} F_{m-2}, & \text{if } A(m) \text{ is of type I,} \\ F_{m-1}, & \text{if } A(m) \text{ is of type II or III,} \\ F_m, & \text{if } A(m) \text{ is of type IV.} \end{cases}$$

Before proving the validity of Theorem 1, we wish to mention a few more chemical aspects of the ASC-concept [4].

First of all, ASC is a well-defined quantity only for molecular graphs that are bipartite. There are two basic applications of ASC. First, if ASC = 0, then the respective conjugated hydrocarbon is predicted to have unpaired electrons. In practice, this means that this hydrocarbon is extremely reactive and usually does not exist. Second, thermodynamic stability of conjugated hydrocarbons is related to, and is a monotone increasing function of the ASC-value of the underlying molecular graph. In practice, this means that among two isomeric conjugated hydrocarbons, the one having greater ASC will be more stable. (Recall that the molecular graphs of isomeric hydrocarbons have an equal number of vertices and an equal number of edges.)

In the case of benzenoid hydrocarbons, the above remains true if ASC is interchanged by K, the number of 1-factors [8]. In particular, not a single benzenoid hydrocarbon with $K = 0$ is known, whereas many hundreds of such hydrocarbons with $K > 0$ exist.

In the case of benzenoid hydrocarbons, the following example (kindly suggested by the anonymous referee) illustrates another aspect of the role of 1-factors. Consider two isomers, A and B, consisting of n fused benzene rings (i.e., hexagons). Compound A consists of a linear arrangement of hexagons, and possesses $n+1$ Kekulé structures (1-factors). Compound B consists of a zig-zag arrangement of n hexagons (see Fig. 2); it possesses F_{n+2} Kekulé structures. The electron distribution in compounds A and B can be (as a reasonable approximation) obtained by averaging of the Kekulé structures [8]. By means of this approach, one finds that compound A has very nearly double bonds at its ends (i.e., bonds the order of which is about 2), which implies a relatively high reactivity in this region of the molecule. In the case of compound B, the same averaging results in bond orders 1.618 (the golden ratio) at the terminal bonds, implying a significantly greater chemical stability of B relative to A.

Readers interested in further details of the chemical applications of 1-factors (including the theory of ASC) should consult the references quoted.

2. COUNTING THE ASC-VALUE OF A BIPARTITE GRAPH

Consider a bipartite graph G with $n+n$ vertices, i.e., a graph all of whose circuits are of even length. Define a binary relation ρ in the set of all 1-factors of G in the following way.

Definition 1: The 1-factors k_1 and k_2 are in relation ρ if and only if the union of the sets of edges of k_1 and k_2 contains an even number of circuits whose lengths are all multiples of 4.

It can be proved that this binary relation is an equivalence relation and subdivides the set of 1-factors into two equivalence classes [2]. In [2] this relation is called "being of the same parity" and the numbers of these classes are denoted by K_+ and K_-. The following theorem by Dewar and Longuet-Higgins [2] connects the ASC-value of G and the numbers K_+ and K_-.

Theorem 2: $\det A = (-1)^n (K_+ - K_-)^2$.

This theorem implies $\mathrm{ASC}\{G\} = \sqrt{|\det A|} = |K_+ - K_-|$.

In the case of benzenoid hydrocarbons all 1-factors are in the same class, i.e., one of the numbers K_+ or K_- is equal to zero. This follows directly from Definition 1. Hence, $\mathrm{ASC}\{G\} = K\{G\}$. It does not hold in the case of non-benzenoid hydrocarbons. In this case, the following theorem can be useful for evaluating the ASC-value.

Theorem 3: Two 1-factors k_1 and k_2 are in distinct classes (of opposite parity) if one is obtained from the other by cyclically rearranging an even number of edges. In other words, two 1-factors k_1 and k_2 are in distinct classes if the union of the sets of edges of k_1 and k_2 contains just a single circuit, and the length of this circuit is a multiple of 4.

Proof: Theorem 3 follows directly from Definition 1.

3. THE STRUCTURE OF THE CONSIDERED GRAPH

The graph $B_n \equiv B_n(X_1, X_2, \ldots, X_n)$ considered in this paper is obtained from the linear [n]phenylene [Fig. 1(b)] by replacing its i^{th} hexagon with a zig-zag chain, labeled by X_i ($X_i = A(m_i)$) for $i = 1, \ldots, n$ [Fig. 3(a)]. The places of concatenation are the edges $f_i \equiv p_i q_i$ ($i = 2, \ldots, n$) and $g_i \equiv r_i s_i$ ($i = 1, 2, \ldots, n-1$) which belong to the terminal hexagons of the zig-zag chain. In the graph B_n, the valencies of the vertices p_i, q_i, r_{i-1}, and s_{i-1} ($i = 2, 3, \ldots, n$) are equal to 3. Recall that the notation $B_n(A(m_1), A(m_2), \ldots, A(m_n))$ does not uniquely determine a graph, because for a unique characterization the places of concatenations also need to be specified (as discussed in detail below).

Figure 3(b) shows one of the possible structures of the graph $B_4(A(3), A(4), A(4), A(2))$.

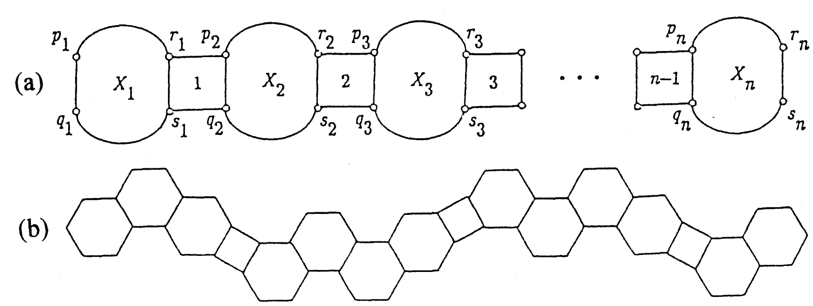


FIGURE 3

The Graph $B_n \equiv B_n(X_1, X_2, \ldots, X_n)$ (a) and Its Special Case $B_4(A(3), A(4), A(4), A(2))$ (b); Note that the Symbol $B_4(A(3), A(4), A(4), A(2))$ Does Not Specify a Unique Graph

The present authors considered in [1] the generalization of the structure of the type B_n [Fig. 3(a)], where X_i are arbitrary bipartite graphs all of whose 1-factors are of equal parity, i.e., $\mathrm{ASC}\{X_i\} = K\{X_i\}$

Consider now a zig-zag segment $A(m_i)$ of B_n. Let $m_i = 1$. Then (see Fig. 4) there are two possible choices of the edges f_i and g_i depending on whether these edges are parallel or not (note that we can always represent squares and hexagons as regular k-gons).

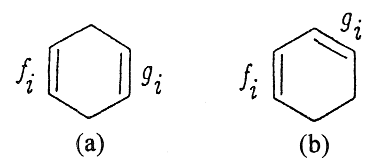


FIGURE 4

Two Ways of Choosing the Edges f_i and g_i in a Hexagon

If $m_i \geq 2$, then there are four *types* of the zig-zag chains $A(m_i)$, depending on the choices of the edges f_i and g_i, according to whether or not these edges are parallel to the edge in common of the hexagon containing the considered edge and its adjacent hexagon (see Fig. 5).

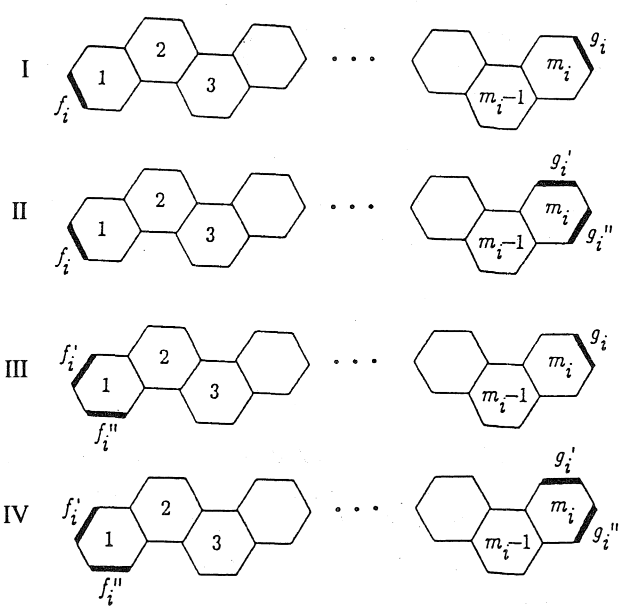


FIGURE 5

Four Ways of Choosing the Edges f_i and g_i in a Zig-Zag Chain $A(m_i)$

Note that in the type II and type III we have two possibilities for the edge pair f_i, g_i and in the type IV we have four such possibilities.

4. A METHOD FOR CALCULATING THE ASC-VALUE OF THE GRAPH B_n

Observe that edges belonging to a 1-factor of B_n (marked by double lines in Fig. 6) can be arranged in and around a four-membered circuit in exactly five different ways. This is the consequence of the fact that the fragments of B_n lying on the left- and right-hand side of the four-membered circuit (not shown in Fig. 6) both possess an even number of vertices. If the number of vertices in these fragments would be odd (which, according to the way in which B_n is constructed, is impossible), then every 1-factor of B_n would contain one horizontal edge of the four-membered circuit, and would not contain the other.

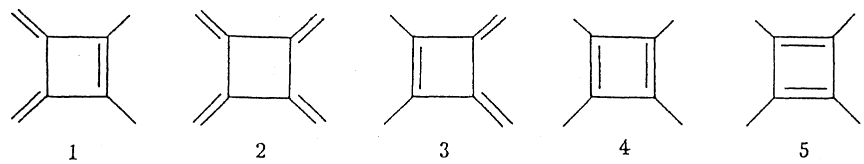


FIGURE 6

Arrangements of Edges in 1-Factors of B_n

The modes 4 and 5 are interconverted by rearranging two (an even number) edges of the 1-factor. According to Theorem 4, modes 4 and 5 are of opposite parity. Consequently, they need not be taken into account when the algebraic structure count is evaluated. The 1-factors of B_n that do not contain arrangements of mode 4 or 5 are called *good*, and their number is denoted by $\kappa\{B_n\}$. Note that the horizontal edges of squares are never in a good 1-factor. Hence, the edges of good 1-factors can be rearranged only within each fragment X_i. This implies that all good 1-factors of B_n are of equal parity, i.e.,

$$\mathrm{ASC}\{B_n\} = \kappa\{B_n\}. \tag{2}$$

We now enumerate the good 1-factors of B_n using the so-called *transfer matrix method* [2]. For that purpose, we define auxiliary subgraphs $X_{i,j}$ in the following way:

$$\left.\begin{aligned} X_{i,1} &= X_i - (f_i) - (g_i), \\ X_{i,2} &= X_i - (f_i) - g_i, \\ X_{i,3} &= X_i - f_i - (g_i), \\ X_{i,4} &= X_i - f_i - g_i. \end{aligned}\right\} \tag{3}$$

(The subgraph $G-e$ is obtained from G by deleting the edge e and the subgraph $G-(e)$ is obtained from G by deleting both the edge e and its terminal vertices.) To simplify the notation, denote $K\{X_{i,j}\}$ by $K_{i,j}$. Observe that $K_{i,1}$-$K_{i,4}$ are equal to the number of 1-factors of X_i in which one of the following four conditions is fulfilled. In particular:

- $K_{1,i}$ counts the 1-factors containing both edges f_i and g_i
  (we say that this class of the 1-factors of X_i is *assigned to the graph* $X_{i,1}$);
- $K_{i,2}$ counts the 1-factors containing the edge f_i and not containing the edge g_i
  (this class is *assigned to the graph* $X_{i,2}$);
- $K_{i,3}$ counts the 1-factors that do not contain f_i and do contain g_i
  (this class is *assigned to the graph* $X_{i,3}$);
- $K_{i,4}$ counts the 1-factors containing neither f_i nor g_i
  (*assigned to the graph* $X_{i,4}$).

Evidently, the above four cases cover all possibilities.

Now, we associate with each 1-factor of B_n a word $j_1 j_2 \ldots j_n$ of the alphabet $\{1, 2, 3, 4\}$ in the following way: *If the considered 1-factor induces in X_i a 1-factor that is assigned to the graph $X_{i,j}$, then $j_i = j$.* Note that by choosing the edges of the 1-factor in X_i and X_{i+1} (i.e., by choosing subwords $j_i j_{i+1}$) we must not generate one of modes 4 or 5 of arrangements of the 1-factor in the square between X_i and X_{i+1}, i.e., the subwords $j_i j_{i+1}$ must not belong to the set $\{11, 12, 31, 32\}$.

Let $\mathcal{T}_n$ be the set of all words in $\{1, 2, 3, 4\}^n$, in which the subwords 11, 12, 31, and 32 are forbidden. Then, according to (2), we have

$$\mathrm{ASC}\{B_n\} = \sum_{j_1 j_2 \cdots j_n \in \mathcal{T}_n} K_{1, j_i} K_{2, j_2}, \ldots, K_{n, j_n}. \tag{4}$$

It can be shown that the set $\mathcal{T}_n$ has $4 \cdot 3^{n-1}$ elements. Hence, there are $4 \cdot 3^{n-1}$ summands on the right-hand side of (4).

Let

$$M_i = \begin{bmatrix} 0 & 0 & K_{i,3} & K_{i,4} \\ K_{i,1} & K_{i,2} & K_{i,3} & K_{i,4} \\ 0 & 0 & K_{i,3} & K_{i,4} \\ K_{i,1} & K_{i,2} & K_{i,3} & K_{i,4} \end{bmatrix}.$$

Keeping (4) in mind, we see that the ASC-value of B_n is equal to the sum of all elements of the last row in the product of transfer matrices $\prod_{i=1}^{n} M_i$, i.e.,

$$\mathrm{ASC}\{B_n\} = \sum_{k=1}^{4} (M_1 \cdot M_2 \cdot \cdots \cdot M_n)_{4,k}. \tag{5}$$

In our case, the subgraphs X_i are zig-zag chains and the value $\mathrm{ASC}\{B_n\}$ is equal to the sum of the products of some Fibonacci numbers, i.e., the following statement holds.

Lemma 1: For every type of zig-zag chain $A(m_i)$, the quantities $K_{i,1}$-$K_{i,4}$ are equal to some Fibonacci numbers.

Proof: Note that all 1-factors of $A(m)$ can be divided into four classes (a)-(d) according to which edges in the terminal hexagons belong to the 1-factor (see Fig. 7). Observe that the numbers of elements in classes (a)-(d) are Fibonacci numbers F_{m-2}, F_{m-1}, F_{m-1}, and F_m, respectively. Evidently, their sum is equal to $K\{A(m_i)\}$, i.e., F_{m+2}.

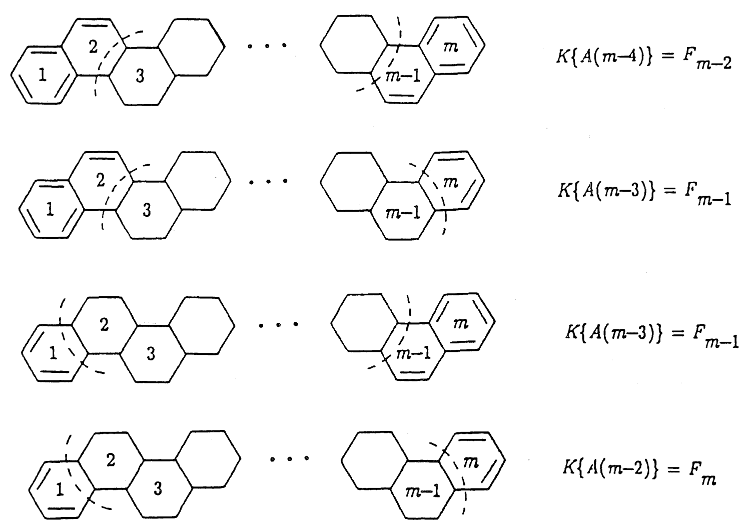


FIGURE 7
Classes of 1-Factors of $A(m)$

Now, if the zig-zag chain is of type I (Fig. 5), then the four numbers $K_{i,1}$-$K_{i,4}$ represent a cardinal number of classes (a)-(d) in Figure 7, respectively. Therefore, we can write

$$[K_{i,1},\ K_{i,2},\ K_{i,3},\ K_{i,4}] = [F_{m_i-2},\ F_{m_i-1},\ F_{m_i-1},\ F_{m_i}]\text{—for type I.} \tag{6}$$

Similarly, we obtain:

$$[K_{i,1},\ K_{i,2},\ K_{i,3},\ K_{i,4}] = [F_{m_i-1},\ F_{m_i-2},\ F_{m_i},\ F_{m_i-1}]\text{—for type II;} \tag{7}$$

$$[K_{i,1},\ K_{i,2},\ K_{i,3},\ K_{i,4}] = [F_{m_i-1},\ F_{m_i},\ F_{m_i-2},\ F_{m_i-1}]\text{—for type III;} \tag{8}$$

$$[K_{i,1},\ K_{i,2},\ K_{i,3},\ K_{i,4}] = [F_{m_i},\ F_{m_i-1},\ F_{m_i-1},\ F_{m_i-2}]\text{—for type IV.} \tag{9}$$

5. PROOF OF THEOREM 1

Let all zig-zag chains $A(m_i)$ be of the same length m $(m = m_1 = m_2 = \cdots = m_n)$ and of fixed type. Then we can write K_j instead of $K_{i,j}$ $(j = 1, \ldots, 4)$ and M instead of M_i. Then expression (5) reduces to $\text{ASC}\{B_n\} = \sum_{k=1}^{4}(M^n)_{4,k}$, where

$$M = \begin{bmatrix} 0 & 0 & K_3 & K_4 \\ K_1 & K_2 & K_3 & K_4 \\ 0 & 0 & K_3 & K_4 \\ K_1 & K_2 & K_3 & K_4 \end{bmatrix}.$$

The characteristic equation of M is

$$\lambda^4 - [K_2 + K_3 + K_4]\lambda^3 + [K_2K_3 - K_1K_4]\lambda^2 = 0, \tag{10}$$

and its eigenvalues are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = [L-D]/2$, and $\lambda_4 = [L+D]/2$, where

$$L = K_2 + K_3 + K_4 \ \&\ D = \sqrt{L^2 + 4(K_1K_4 - K_2K_3)}. \tag{11}$$

Note that $D = \sqrt{(K_2 - K_3)^2 + K_4(4K_1 + 2K_2 + 2K_3 + K_4)}$ and is equal to 0 if and only if $K_2 = K_3$ and $K_4 = 0$. Further, recall the Cayley-Hamilton theorem which says that each square matrix satisfies its own characteristic equation. If we label with $m_{i,j}(n)$ the (i,j)-entry in the matrix M^n, then using the mentioned theorem we obtain that the sequence $m_{i,j}(n)$ [and, consequently, the sequence $\text{ASC}\{B_n\} = \sum_{j=1}^{4} m_{4,j}(n)$] satisfies a difference equation. The coefficients of this difference equation are obtained from the characteristic polynomial of M^n, i.e., from equation (10). Thus, the desired recurrence relation for the algebraic structure count of the graph B_n is

$$\text{ASC}\{B_n\} = (K_2 + K_3 + K_4)\text{ASC}\{B_{n-1}\} + (K_1K_4 - K_2K_3)\text{ASC}\{B_{n-2}\} \tag{12}$$

with initial conditions

$$\begin{aligned}\text{ASC}\{B_1\} &= K_1 + K_2 + K_3 + K_4 \ \&\ \text{ASC}\{B_2\} \\ &= (K_1 + K_3)(K_3 + k_4) + (K_2 + K_4)(K_1 + K_2 + K_3 + K_4).\end{aligned}$$

Solving the recurrence relation (12), we obtain the following general solution:

$$\text{ASC}\{B_n\} = \begin{cases} (nK_1 + (n+1)K_2)K_2^{n-1} & \text{if } K_2 = K_3 \ \&\ K_4 = 0, \\ \frac{1}{2^{n+1}}\left[\left(1 + \frac{2K_1+L}{D}\right)(L+D)^n + \left(1 - \frac{2K_1+L}{D}\right)(L-D)^n\right] & \text{otherwise,} \end{cases} \tag{13}$$

where L and D are given by (11). We arrive at the generating function for the sequence $\mathrm{ASC}\{B_n\}$ by standard methods of the theory of difference equation

$$\mathrm{ASC}(x) = \frac{1+K_1x}{(K_2K_3 - K_1K_4)x^2 - (K_2+K_3+K_4)x+1}.$$

Since $\mathrm{ASC}(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \mathrm{ASC}\{B_n\}x^n$, this implies

$$\mathrm{ASC}\{B_n\} = \frac{1}{n!}\mathrm{ASC}^{(n)}(0).$$

In our case, the fragments X_i are zig-zag chains $A(m)$ with a fixed number of hexagons (m) of fixed type.

- The case $m=1$ was considered in [7]. For the sake of completeness, we recall that

$$\mathrm{ASC}\{B_n\} = n+1, \quad \text{if } A(1) \text{ is of type I [Fig. 4(a)] (the linear phenylene)},$$
$$\mathrm{ASC}\{B_n\} = F_{n+2}, \quad \text{if } A(1) \text{ is of type II [Fig. 4(b)] (the angular phenylene)}.$$

- In the case $m=2$, we have to use both terms in (13). So, using relations (6)-(9) in (13), we obtain

$$\mathrm{ASC}\{B_n\} = \begin{cases} \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+1}\right], & \text{if } A(2) \text{ is of type I,} \\ \frac{1}{2}\left[(1+\sqrt{2})^{n+1} + (1-\sqrt{5})^{n+1}\right], & \text{if } A(2) \text{ is of type II or III,} \\ 2n+1, & \text{if } A(2) \text{ is of type IV.} \end{cases}$$

- For the case $m>2$, all K_i $(i=1,\dots,4)$ numbers are positive, so we apply only the second term in (13). Bearing in mind relations (6)-(9) again and the well-known relations for the Fibonacci numbers, $F_{m-1}^2 - F_mF_{m-2} = (-1)^m$ and $F_{m-2} + 2F_{m-1} + F_m = F_{m+2}$, the required statement of Theorem 1 follows.

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AMS Classification Numbers: 05C70, 05B50, 05A15

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THE FIBONACCI CONFERENCE IN GRAZ

Herta T. Freitag

The Seventh International Research Conference on Fibonacci Numbers and Their Applications was held at the Technische Universität in Graz, Austria, July 15-19, 1996. It was sponsored by the Austrian Federal Ministry of Science, the Governor of Styria, the Mayor of Graz, the Technische Universität in Graz, the Austrian Academy of Sciences, the European Mathematical Society, and the Fibonacci Association. We all wish to express our deep gratitude to these sponsors.

How befitting that Graz was chosen as the site. This old university town (*die alte Universität* was founded in 1845) radiates the charm of old-worldliness combined with the spirit of progressive modernism and technology. What an atmosphere for thought and reflection—mathematical or otherwise! Enriched by new and happy experiences, all crammed into but a few days, we once again felt what a unifying force our mathematics is. Being the international language *par excellence*, it bridges nationalities, customs, ideas. Colorfully different accents but enhance the fact that our discipline is understood by all its devotees. And loved by them.

A record number of 95 papers was presented: the U.S.A. provided 27 of them; Austria 11; Italy and Japan tied with nine each; France and Germany with eight. Three speakers came from Canada, and also from Russia; one or two speakers hailed from each of the remaining countries. Significantly justifying the fact that our Conference is truly an **International** one, a count of the nationalities on the roster revealed the stunning number of 32—among them Australia, the Republic of Belarus, Cyprus, New Zealand, and South Africa.

These large numbers bespeak the growing magnetism of our "Fibonacci-type mathematics," and—maybe—Austria's popularity. (May I, a former Viennese, be accused of bias?) Hence, it was with considerable reluctance that it became necessary to resort to double sessions. We, indeed, wanted to hear it all.

We did work hard. The sessions started at 9:00 A.M. and extended to the early evening, followed by enjoyable social events, planned by the Local Committee. Even just listening to the titles of the presentations, no one could doubt that there is more imagination in the mind of a mathematician than, possibly, in that of a poet.

The ties of old friendships were strengthened; new ones were kindled. Many of these became fertile soil for joint authorship research. Predictably, the "Goddess Mathesis," as Howard Eves calls her, smiles benevolently upon this phenomenon. I was saddened by the absence of one of my co-authors, George M. Phillips, who, through illness, was unable to attend.

Our deep thanks go to Gerald E. Bergum, the **very soul** of the Fibonacci Association; to our Robert Tichy, who ever-so-amiably coped with all the work; and, indeed, also to the other Committee members, both local and international. Nor will we ever forget Verner E. Hoggatt, Jr., who created The Fibonacci Association; or Andreas N. Philippou, who launched the idea of a Fibonacci Conference. Our appreciation, however, also goes out to all the participants of the Conference. The presentations mirrored their intense mathematical involvement and enthusiasm.

Finally, our *Wiedersehen* in Graz had to come to an end. Now, however, in another two years—**Rochester, here we come!** And may our Conferences always be so very fruitful and enjoyable.

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-814 (Corrected) ***Proposed by M. N. Deshpande, Institute of Science, Nagpur, India***

Show that, for each positive integer n, there exists a constant C_n such that $F_{2n+2i}F_{2i} + C_n$ and $F_{2n+2i+1}F_{2i+1} - C_n$ are both perfect squares for all positive integers i.

B-821 ***Proposed by L. A. G. Dresel, Reading, England***

Consider the rectangle with sides of lengths F_{n-1} and F_{n+1}. Let A_n be its area, and let d_n be the length of its diagonal. Prove that $d_n^2 = 3A_n \pm 1$.

B-822 ***Proposed by Anthony Sofo, Victoria University of Technology, Australia***

For $n > 0$, simplify $\sqrt[n]{\alpha F_n + F_{n-1}} + (-1)^{n+1}\sqrt[n]{F_{n+1} - \alpha F_n}$.

B-823 ***Proposed by Pentti Haukkanen, University of Tampere, Finland***

It is easy to see that the solution of the recurrence relation $A_{n+2} = -A_{n+1} + A_n$, $A_0 = 0$, $A_1 = 1$, can be written as $A_n = (-1)^{n+1}F_n$.

Find a solution to the recurrence $A_{n+2} = -A_{n+1} + A_n$, $A_0 = 1$, $A_1 = 1$, in terms of F_n and L_n.

B-824 ***Proposed by Brian D. Beasley, Presbyterian College, Clinton, SC***

Fix a nonnegative integer m. Solve the recurrence $A_{n+2} = L_{2m+1}A_{n+1} + A_n$, for $n \geq 0$, with initial conditions $A_0 = 1$ and $A_1 = L_{2m+1}$, expressing your answer in terms of the Fibonacci and/or Lucas numbers.

B-825 ***Proposed by Lawrence Somer, The Catholic Univ. of America, Washington, D.C.***

Let $\langle V_n \rangle$ be a sequence defined by the recurrence $V_n = PV_{n+1} - QV_n$, where P and Q are integers and $V_0 = 2$, $V_1 = P$. The integer d is said to be a divisor of $\langle V_n \rangle$ if $d | V_n$ for some $n \geq 1$.

(a) If P and Q are both even, show that 2^m is a divisor of $\langle V_n \rangle$ for any $m \geq 1$.

(b) If P or Q is odd, show that there exists a fixed nonnegative integer k such that 2^k is a divisor of $\langle V_n \rangle$ but 2^{k+1} is not a divisor of $\langle V_n \rangle$. If exactly one of P or Q is even, show that $2^k | V_1$; if P and Q are both odd, show that $2^k | V_3$.

NOTE: The Elementary Problems Column is in need of more *easy*, yet elegant and nonroutine problems.

SOLUTIONS

Double, Double, Triangular Numbers and Trouble

B-802 ***Proposed by Al Dorp, Edgemere, NY***
(Vol. 34, no. 1, February 1996)

Let $T_n = n(n+1)/2$ denote the n^{th} triangular number. Find a formula for T_{2n} in terms of T_n.

Editorial Note: I loved this problem because of all the varied solutions. It is amazing how resourceful our readers can be, and I enjoyed seeing some ingenious solutions.

Solutions by various solvers

Many solvers:	$T_{2n} = 4T_n - n$.
Marjorie Bicknell-Johnson:	$T_{2n} = n^2 + 2T_n$.
Many solvers:	$T_{2n} = \dfrac{8T_n + 1 - \sqrt{8T_n + 1}}{2}$.
Joseph J. Koštál:	$T_{2n} = n\sqrt{8T_n + 1}$.
Paul S. Bruckman:	$T_{2n} = \dbinom{\sqrt{8T_n + 1}}{2}$.
Herta T. Freitag:	$T_{2n} = \frac{1}{2}n^2(n+1)(2n+1)/T_n$.
Many solvers:	$T_{2n} = 3T_n + T_{n-1}$.
David Zeitlin:	$T_{2n} = 6T_n - 3T_{n+1} + T_{n+2}$.

One reader came up with the wondrous formula $T_{2n} = (T_n^2 + 101T_n - 12)/30$. *Unfortunately, this formula was too good to be true.*

Generalization by David Terr, University of California, Berkeley, CA, and Daina A. Krigens, Encinatas, CA (independently)

$$T_{kn} = k^2 T_n - nT_{k-1}.$$

Generalization by Marjorie Bicknell-Johnson, Santa Clara, CA

$$T_{kn+p} = [(2p+1)k - 4T_p]T_n + (n+1)^2 T_p + n^2 T_{k-p-1}.$$

Also solved by Mohammad K. Azarian, Brian D. Beasley, Charles K. Cook, M. N. Deshpande, Leonard A. G. Dresel, Steve Edwards, Russell Euler, Thomas M. Green, Russell Jay Hendel, Gerald Heuer, Harris Kwong, Carl Libis, Bob Prielipp, John A. Schumaker, R. P. Sealy, H.-J. Seiffert, Lawrence Somer, and the proposer.

Half a Lucas Sum

B-803 ***Proposed by Herta Freitag, Roanoke, VA (Vol. 34, no. 1, February 1996)***

For n even and positive, evaluate

$$\sum_{i=0}^{n/2}\binom{n}{i}L_{n-2i}.$$

Solution by L. A. G. Dresel, Reading, England

Let $A=\sum_{i=0}^{n}\binom{n}{i}L_{n-2i}$ and $B=\sum_{i=0}^{n/2}\binom{n}{i}L_{n-2i}$. Since $\binom{n}{n-i}=\binom{n}{i}$ and $L_{-2t}=L_{2t}$ and n is even, we have $A+\binom{n}{n/2}L_0=2B$, and

$$A=\sum_{i=0}^{n}\binom{n}{i}(\alpha^{n-2i}+\beta^{n-2i})=\alpha^{-n}(\alpha^2+1)^n+\beta^{-n}(\beta^2+1)^n=(\alpha+\alpha^{-1})^n+(\beta+\beta^{-1})^n.$$

But $\alpha+\alpha^{-1}=\sqrt{5}$ and $\beta+\beta^{-1}=-\sqrt{5}$. Therefore, $A=2(5^{n/2})$ and $B=5^{n/2}+\binom{n}{n/2}$.

Haukkanen found a corresponding formula for Fibonacci numbers:

$$\sum_{i=0}^{(n-1)/2}\binom{n}{i}F_{n-2i}=5^{(n-1)/2},\ n \text{ odd}.$$

Seiffert gave the generalization,

$$\sum_{i=0}^{n/2}(-1)^{(k+1)i}\binom{n}{i}L_{k(n-2i)}=5^{n/2}F_k^n+(-1)^{(k+1)n/2}\binom{n}{n/2},$$

which comes from [1] and can also be traced back to Lucas [3]. Haukkanen found a generalization for the sequences defined by $U_n=mU_{n-1}+U_{n-2}$, $U_0=0, U_1=1$, and $V_n=mV_{n-1}+V_{n-2}$, $V_0=2$, $V_1=m$, which comes from [2]:

$$\sum_{i=0}^{n/2}\binom{n}{i}V_{k(n-2i)}=(m^2+4)^{n/2}U_k^n+\binom{n}{n/2},\ k \text{ odd},\ n \text{ even}.$$

References

1. The Citadel Problem Solving Group. "Problem 519: A Linear Combination of Lucas Numbers." *The College Mathematics Journal* **26.1** (1995):70.
2. P. Filipponi. "Waring's Formula, the Binomial Formula, and Generalized Fibonacci Matrices." *The Fibonacci Quarterly* **30.3** (1992):225-31.
3. Edouard Lucas. *The Theory of Simply Periodic Numerical Functions.* Santa Clara, CA: The Fibonacci Association, 1969.

Also solved by Paul Bruckman, M. N. Deshpande, Russell Euler, Pentti Haukkanen, Russell Jay Hendel, R. P. Sealy, H.-J. Seiffert, David Zeitlin, and the proposer.

Finding an Identity without a Crystal Ball

B-804 ***Proposed by the editor***
(Vol. 34, no. 1, February 1996)

Find integers a, b, c, and d (with $1 < a < b < c < d$) that make the following an identity:

$$F_n = F_{n-a} + 9342F_{n-b} + F_{n-c} + F_{n-d}.$$

Solution by L. A. G. Dresel, Reading, England

We note that $9342 = 9349 - 7 = L_{19} - L_4$. Using the identities I_{23} and I_{21} of [1], we have

$$F_{m+19} - F_{m-19} = F_m L_{19} \quad \text{and} \quad F_{m+4} + F_{m-4} = F_m L_4.$$

Subtracting, we obtain $F_{m+19} - F_{m-19} - F_{m+4} - F_{m-4} = F_m(L_{19} - L_4)$. Putting $n = m+19$ and rearranging, gives the identity $F_n = F_{n-15} + 9342F_{n-19} + F_{n-23} + F_{n-38}$. We conclude that $a = 15$, $b = 19$, $c = 23$, and $d = 38$ provides a solution to this problem.

Bruckman, Dresel, Johnson, and Seiffert all found the general identity

$$F_n = F_{n-u+v} + (L_u - L_v)F_{n-u} + F_{n-u-v} + F_{n-2u}$$

with u odd and v even. Zeitlin found the identity $F_n = F_{n-2} + 9349F_{n-20} + F_{n-40} + F_{n-41}$.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by Paul S. Bruckman, Russell Jay Hendel, Marjorie Bicknell-Johnson, Daina A. Krigens, H.-J. Seiffert, David Zeitlin, and the proposer.

A Slightly Perturbed Fibonacci Sequence

B-805 ***Proposed by David Zeitlin, Minneapolis, MN***
(Vol. 34, no. 1, February 1996)

Solve the recurrence $P_{n+6} = P_{n+5} + P_{n+4} - P_{n+2} + P_{n+1} + P_n$, for $n \geq 0$, with initial conditions $P_0 = 1, P_1 = 1, P_2 = 2, P_3 = 3, P_4 = 4,$ and $P_5 = 7$.

Editorial composite of the solutions received:

Let
$$A_n = \begin{cases} 0, & \text{if } n \equiv 1, 5 \pmod 8, \\ 1, & \text{if } n \equiv 0, 2, 3 \pmod 8, \\ -1, & \text{if } n \equiv 4, 6, 7 \pmod 8. \end{cases}$$

Then note that A_n satisfies the given recurrence. Also note that F_n satisfies the given recurrence. Thus, any linear combination of F_n and A_n satisfies the given recurrence. Now, one only needs to find the linear combination that meets the initial conditions. The solution is $P_n = (A_n + F_{n+3}) / 3$.

This can also be written in the form $P_n = \left\lfloor \frac{1+F_{n+3}}{3} \right\rfloor$.

Also solved by Brian D. Beasley, Paul Bruckman, Leonard A. G. Dresel, Pentti Haukkanen, Russell Jay Hendel, Gerald A. Heuer, Harris Kwong, H.-J. Seiffert, and the proposer. A partial solution was obtained by Jackie Roehl.

Power Series with Fibonacci Features

B-806 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN (Vol. 34, no. 1, February 1996)***

(a) Show that the coefficient of every term in the expansion of $\frac{x}{1-2x+x^3}$ is the difference of two Fibonacci numbers.

(b) Show that the coefficient of every term in the expansion of $\frac{x}{1-2x-2x^2+x^3}$ is the product of two consecutive Fibonacci numbers.

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

(a) It is well known that $\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$. Hence,

$$\frac{x}{1-2x+x^3} = \frac{x}{1-x-x^2}\cdot\frac{1}{1-x} = \left(\sum_{n=0}^{\infty} F_n x^n\right)\left(\sum_{n=0}^{\infty} x^n\right)$$

generates $c_n = \sum_{k=0}^{n} F_k = F_{n+2} - F_2$.

(b) Routine calculation reveals that

$$\frac{x}{1-3x+x^2} = \frac{x}{(1-\alpha^2 x)(1-\beta^2 x)} = \sum_{n=0}^{\infty}\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha^2-\beta^2}\right)x^n = \sum_{n=0}^{\infty} F_{2n}x^n.$$

Thus,

$$\frac{x}{1-2x-2x^2+x^3} = \frac{x}{1-3x+x^2}\cdot\frac{1}{1+x} = \left(\sum_{n=0}^{\infty} F_{2n}x^n\right)\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)$$

generates $d_n = F_{2n} - F_{2n-2} + \cdots + (-1)^n F_0$, which satisfies the recurrence relation $d_{n+1} + d_n = F_{2n+2}$. Using $F_{2n+2} = L_{n+1}F_{n+1} = (F_{n+2}+F_n)F_{n+1} = F_{n+1}F_{n+2} + F_nF_{n+1}$ and induction, we find that $d_n = F_nF_{n+1}$.

Also solved by Paul S. Bruckman, Charles K. Cook, M. N. Deshpande, Steve Edwards, Russell Euler, Pentti Haukkanen, Russell Jay Hendel, Carl Libis, H.-J. Seiffert, David Zeitlin, and the proposer.

Generalized Mod Squad

B-807 ***Proposed by R. André-Jeannin, Longwy, France (Vol. 34, no. 1, February 1996)***

The sequence $\langle W_n\rangle$ is defined by the recurrence $W_n = PW_{n-1} - QW_{n-2}$, for $n \ge 2$, with initial conditions $W_0 = a$ and $W_1 = b$, where a and b are integers and P and Q are odd integers. Prove that, for $k \ge 0$, $W_{n+3\cdot 2^k} \equiv W_n \pmod{2^{k+1}}$.

Solution by Lawrence Somer, The Catholic Univ. of America, Washington, D.C.

Let $\langle U_n\rangle$ and $\langle V_n\rangle$ be sequences satisfying the same recursion as $\langle W_n\rangle$ with initial terms $U_0 = 0$, $U_1 = 1$, and $V_0 = 2, V_1 = P$, respectively. It can be proved by the Binet formulas and induction that

$$W_{n+m} = -QW_{n-1}U_m + W_nU_{m+1}, \tag{1}$$

$$U_{2n} = U_nV_n, \tag{2}$$

and

$$U_{2n+1} = -QU_n^2 + U_{n+1}^2. \tag{3}$$

Since P and Q are odd, the sequence $\langle U_n \rangle$, modulo 2, is 0, 1, 1, 0, 1, 1, ... By inspection, one sees that both $\langle U_n \rangle$ and $\langle V_n \rangle$ are purely periodic modulo 2 with periods equal to 3. Moreover, one sees by inspection that $U_n \equiv 0 \pmod 2$ if and only if $3|n$, and $V_n \equiv 0 \pmod 2$ if and only if $3|n$.

We now show by induction that $U_{3\cdot 2^k} \equiv 0 \pmod{2^{k+1}}$ for $k \geq 0$. If $k = 0$, one sees by what was stated above that $U_3 \equiv 0 \pmod 2$. Suppose the result is true up to k. Then

$$U_{3\cdot 2^{k+1}} = U_{3\cdot 2^k} V_{3\cdot 2^k} \tag{4}$$

by formula (2). Since $U_{3\cdot 2^k} \equiv 0 \pmod{2^{k+1}}$ by our induction hypothesis, and $V_{3\cdot 2^k} \equiv 0 \pmod 2$, we see from (4) that $U_{3\cdot 2^{k+1}} \equiv 0 \pmod{2^{k+2}}$, and our induction is complete.

We next show by induction that $U_{3\cdot 2^k+1} \equiv 1 \pmod{2^{k+1}}$ for $k \geq 0$. If $k = 0$, then, by what was stated earlier, $U_4 \equiv 1 \pmod 2$. Assume the result is true up to k. Then

$$U_{3\cdot 2^{k+1}+1} = -QU_{3\cdot 2^k}^2 + U_{3\cdot 2^k+1}^2 \tag{5}$$

by formula (3). Since $U_{3\cdot 2^k} \equiv 0 \pmod{2^{k+1}}$ by our above result, $U_{3\cdot 2^k}^2 \equiv 0 \pmod{2^{k+2}}$. Since $U_{3\cdot 2^k+1} = 1 + r2^{k+1}$ for some integer r by our induction hypothesis, we see that $U_{3\cdot 2^k+1}^2 \equiv 1 \pmod{2^{k+2}}$. Then, by (5), we have $U_{3\cdot 2^{k+1}+1} \equiv 1 \pmod{2^{k+2}}$ and our induction is complete.

We now see that

$$W_{n+3\cdot 2^k} = -QW_{n-1}U_{3\cdot 2^k} + W_n U_{3\cdot 2^k+1} \equiv (-QW_{n-1}) \cdot 0 + W_n \cdot 1 \equiv W_n \pmod{2^{k+1}}.$$

The proposer mentions that this problem came about by his efforts to generalize problem B-732.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Harris Kwong, H.-J. Seiffert, and the proposer.

Addenda: The editor wishes to apologize for misplacing some solutions that were sent in on time. We therefore acknowledge solutions from the following solvers:

Brian D. Beasley—B-790, 791;	Russell J. Hendel—B-784;
L. A. G. Dresel—B-796, 797, 798, 799, 801;	Gerald A. Heuer—B-792, 793;
C. Georghiou—B-796, 797, 798, 799, 800, 801;	Igor O. Popov—B-796, 797;
Pentti Haukkanen—B-793;	Dorka O. Popova—B-799.

Errata: In the solution to B-773 (Feb. 1996), identity (I_{11}) should read $F_n^2 + F_{n-1}^2 = F_{2n-1}$. In the solution to B-798 (Nov. 1996), the comment by Bloom should read: If p is odd, $k \geq 1$, and F_n is divisible by p^k but not by p^{k+1}, then F_{np} is divisible by p^{k+1} but not by p^{k+2}.

David Zeitlin

I have been informed by David Zeitlin's niece that Dr. Zeitlin passed away on Nov. 5, 1996. She wrote that her uncle's entire life was devoted to mathematics. Readers of this column are no doubt familiar with his writings, since he has been an active contributor to this column and to this journal since it began back in 1963. His papers have been way ahead of their time. Readers who go back to some of the early issues of this Quarterly will find some amazing results buried deep within his papers. I will miss his scrawled handwriting and the incredible formulas that arrive in the mail in response to some of the problems in this column.

—Editor

ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-521 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let ρ denote any zero of the Riemann Zeta Function $\zeta(z)$ lying in the strip $S = \{z \in C: 0 < \mathrm{Re}(z) < 1\}$. Prove the following:

(1) $\sum_{\rho \in S} (\rho - \frac{1}{2})^{-1} = 0$;

(2) $\sum_{\rho \in S} \rho^{-1} = 1 + \frac{1}{2}\gamma - \frac{1}{2}\log 4\pi$, where γ is Euler's Constant.

H-522 ***Proposed by N. Gauthier, Royal Military College, Kingston, Ontario, Canada***

Let A and B be the following 2×2 matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Show that, for $m \geq 1$,

$$\sum_{n=0}^{m-1} 2^n A^{2^n} (A^{2^n} + B^{2^n})^{-1} = c_{2^m} C_{2^m} - (A + B),$$

where

$$c_m = m / (F_{m+1} + F_{m-1} - 2) \text{ and } C_m = \begin{pmatrix} F_{m+1} - 1 & F_m \\ F_m & F_{m-1} - 1 \end{pmatrix};$$

F_m is the m^{th} Fibonacci number.

H-523 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let $Z(n)$ denote the "Fibonacci entry-point" of n, i.e., $Z(n)$ is the smallest positive integer m such that $n | F_m$. Given any odd prime p, let $q = \frac{1}{2}(p-1)$; for any integer s, define $g_p(s)$ as follows:

$$g_p(s) = \sum_{k=1}^{q} \frac{s^k}{k}.$$

Prove the following assertion:

$$Z(p^2) = Z(p) \text{ iff } g_p(1) \equiv g_p(5) \pmod{p}. \tag{*}$$

H-524 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Let p be a prime with $p \equiv 1$ or 9 (mod 20). It is known that $a := (p-1)/Z(p)$ is an even integer, where $Z(p)$ denotes the entry-point in the Fibonacci sequence [1]. Let $q := (p-1)/2$. Show that

(1) $(-1)^{a/2} \equiv (-5)^{q/2} \pmod{p}$ if $p \equiv 1 \pmod{20}$,

(2) $(-1)^{a/2} \equiv -(-5)^{q/2} \pmod{p}$ if $p \equiv 9 \pmod{20}$.

Reference

1. P. S. Bruckman. "Problem H-515." *The Fibonacci Quarterly* **34.4** (1996):379.

H-525 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let p be any prime $\neq 2, 5$. Let

$$q = \frac{1}{2}(p-1), \quad e = \left(\frac{5}{p}\right), \quad r = \frac{1}{2}(p-e).$$

Let $Z(p)$ denote the entry-point of p in the Fibonacci sequence. Given that $2^{p-1} \equiv 1 \pmod{p}$ and $5^q \equiv e \pmod{p}$, let

$$A = \frac{1}{p}(2^{p-1} - 1), \quad B = \frac{1}{p}(5^q - e), \quad C = \sum_{k=1}^{q} \frac{5^{k-1}}{2k-1}.$$

Prove that $Z(p^2) = Z(p)$ if and only if $eA - B \equiv C \pmod{p}$.

SOLUTIONS

Another Complex Problem

H-504 ***Proposed by Z. W. Trzaska, Warsaw, Poland***
(Vol. 33, no. 5, November 1995)

Given a sequence of polynomials in complex variable $z \in C$ defined recursively by

(i) $$R_{k+1}(z) = (3+z)R_k(z) - R_{k-1}(z), \quad k = 0, 1, 2, \ldots,$$

with $R_0(z) = 1$ and $R_1(z) = (1+z)R_0$.

Prove that

(ii) $$R_k(0) = F_{2k+1},$$

where F_ℓ, $\ell = 0, 1, 2, \ldots$, denotes the ℓ^{th} term of the Fibonacci sequence.

Solution by Paul S. Bruckman, Edmonds, WA

The correct expression for $R_k(0)$ is F_{2k-1}, not F_{2k+1}.

Proof: Let $R_k(0) = S_k$, $k = 0, 1, \ldots$. The given recurrence reduces to the following one with constant coefficients, by setting $z = 0$:

$$S_{k+2} - 3S_{k+1} + S_k = 0, \quad k = 0, 1, \ldots; \tag{1}$$

also

$$S_0 = S_1 = 1. \tag{2}$$

The characteristic equation of this recurrence is

$$z^2 - 3z + 1 = 0, \tag{3}$$

which has the roots α^2 and β^2. Therefore, $S_k = AF_{2k} + BL_{2k}$, for appropriate constants A and B. Setting $k = 0$ and $k = 1$ yields $S_0 = 1 = 2B$ and $S_1 = 1 = A + 3B$, whence $A = -\frac{1}{2}$ and $B = \frac{1}{2}$. Then

$$S_k = \tfrac{1}{2}(L_{2k} - F_{2k}) = \tfrac{1}{2}(F_{2k+1} + F_{2k-1} - F_{2k})$$

or

$$R_k(0) = S_k = F_{2k-1}. \text{ Q.E.D.} \tag{4}$$

Also solved by L. A. G. Dresel, A. Dujella, J. Koštál, and the proposer.

Sum Formulae!

H-505 ***Proposed by Juan Pla, Paris, France***
(Vol. 33, no. 5, November 1995)

Edouard Lucas once noted: "On ne connaît pas de formule simple pour la somme des cubes du binôme" [No simple formula is known for the sum of the cubes of the binomial coefficients] (see Edouard Lucas, *Théorie des Nombres*, Paris, 1891, p. 133, as reprinted by Jacques Gabay, Paris, 1991).

The following problem is designed to find *closed*, if not quite "simple," formulas for the sum of the cubes of all the coefficients of the binomial $(1+x)^n$.

1) Prove that

$$\sum_{p=0}^{p=n} \binom{n}{p}^3 = \frac{2^n}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \{1 + \cos\varphi + \cos\theta + \cos(\varphi + \theta)\}^n d\theta\, d\varphi.$$

2) Prove that

$$\sum_{p=0}^{p=n} \binom{n}{p}^3 = \frac{8^n}{\pi^2} \int_0^{\pi} \int_0^{\pi} \{\cos\varphi \cos\theta \cos(\varphi + \theta)\}^n d\theta\, d\varphi.$$

Solution by Paul S. Bruckman, Edmonds, WA

Given $n = 0, 1, 2, \ldots$, define

$$S_n = \sum_{k=0}^{n} \binom{n}{k}^3, \tag{1}$$

$$A_n = \frac{2^n}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} [1 + \cos x + \cos y + \cos(x+y)]^n dx\, dy, \tag{2}$$

$$B_n = \frac{8^n}{\pi^2} \int_0^{\pi} \int_0^{\pi} [\cos u \cdot \cos v \cdot \cos(u+v)]^n du\, dv. \tag{3}$$

Note that

$$1 + \cos x + \cos y + \cos(x+y) = (1 + \cos x)(1 + \cos y) - \sin x \cdot \sin y$$

$$= 4\cos^2\frac{x}{2}\cdot\cos^2\frac{y}{2} - 4\sin\frac{x}{2}\cos\frac{x}{2}\sin\frac{y}{2}\cos\frac{y}{2}$$
$$= 4\cos\frac{x}{2}\cos\frac{y}{2}\left(\cos\frac{x}{2}\cos\frac{y}{2} - \sin\frac{x}{2}\sin\frac{y}{2}\right)$$
$$= 4\cos\frac{x}{2}\cos\frac{y}{2}\cos\left(\frac{x+y}{2}\right).$$

Therefore,

$$A_n = \frac{2^n}{4\pi^2}\cdot 4^n\int_0^{2\pi}\int_0^{2\pi}\left[\cos\frac{x}{2}\cdot\cos\frac{y}{2}\cdot\cos\left(\frac{x+y}{2}\right)\right]^n dx\,dy$$
$$= \frac{8^n}{4\pi^2}\cdot 4\int_0^{\pi}\int_0^{\pi}[\cos u\cdot\cos v\cdot\cos(u+v)]^n du\,dv;$$

thus,

$$A_n = B_n. \tag{4}$$

Now, it suffices to prove that $S_n = B_n$. Toward this end, we employ the following identity and integral (the latter valid for all integers m):

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \text{ for all complex } z; \tag{5}$$

$$\frac{1}{\pi}\int_o^{\pi} e^{2itm}dt = \delta_{m:0} = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \neq 0. \end{cases} \tag{6}$$

Note that

$$B_n = \frac{8^n}{\pi^2}\int_o^{\pi}(\cos v)^n C_n(v)\,dv, \text{ where } C_n(v) = \int_0^{\pi}[\cos u\cdot\cos(u+v)]^n du.$$

Then

$$C_n(v) = \int_0^{\pi} 4^{-n}[(e^{iu} + e^{-iu})(e^{i(u+v)} + e^{-i(u+v)})]^n du$$
$$= 4^{-n}\int_0^{\pi}\sum_{a=0}^{n}\sum_{b=0}^{n}\binom{n}{a}\binom{n}{b}e^{iu(n-2a)+i(u+v)(n-2b)}du$$
$$= 4^{-n}\sum_{a=0}^{n}\sum_{b=0}^{n}\binom{n}{a}\binom{n}{b}e^{iv(n-2b)}\int_0^{\pi}e^{2iu(n-a-b)}du.$$

Thus, using (6),

$$C_n(v) = 4^{-n}\sum_{a=0}^{n}\sum_{b=0}^{n}\binom{n}{a}\binom{n}{b}e^{iv(n-2b)}\cdot\pi\delta_{n-a-b:0}$$

or

$$C_n(v) = \pi\cdot 4^{-n}\sum_{b=0}^{n}\binom{n}{b}^2 e^{iv(n-2b)}. \tag{7}$$

Then

$$B_n = \frac{2^n}{\pi}\int_0^{\pi}2^{-n}(e^{iv} + e^{-iv})^n\sum_{b=0}^{n}\binom{n}{b}^2 e^{iv(n-2b)}dv$$

$$= \frac{1}{\pi}\int_0^{\pi} \sum_{b=0}^{n} \sum_{c=0}^{n} \binom{n}{b}^2 \binom{n}{c} e^{iv(n-2b)+iv(n-2c)} dv$$

$$= \frac{1}{\pi}\sum_{b=0}^{n} \sum_{c=0}^{n} \binom{n}{b}^2 \binom{n}{c} \int_0^{\pi} e^{2iv(n-b-c)} dv$$

$$= \sum_{b=0}^{n} \sum_{c=0}^{n} \binom{n}{b}^2 \binom{n}{c} \delta_{n-b-c:0} = \sum_{b=0}^{n} \binom{n}{b}^3 = S_n. \text{ Q.E.D.}$$

Also solved by the proposer.

Sum Figuring

H-506 ***Proposed by Paul S. Bruckman, Edmonds, WA (Vol. 34, no. 1, February 1996)***

Let

$$A = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{5n+1}+\frac{1}{5n+4}\right) \text{ and } B = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{5n+2}+\frac{1}{5n+3}\right).$$

Evaluate A and B, showing that $A = \alpha B$.

Solution by C. Georghiou, University of Patras, Patras, Greece

Since, for $|x| < 1$,

$$\frac{1}{1+x^5} = 1 - x^5 + x^{10} - x^{15} + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{5n}.$$

we let, for $-1 < x < 1$,

$$A(x) = \int_0^x \frac{1+u^3}{1+u^5} du = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{5n+1}}{5n+1}+\frac{x^{5n+4}}{5n+4}\right)$$

and

$$B(x) = \int_0^x \frac{u+u^2}{1+u^5} du = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{5n+2}}{5n+2}+\frac{x^{5n+3}}{5n+3}\right).$$

By Abel's Limit Theorem, we have $A = A(1)$ and $B = B(1)$. But

$$\frac{1+x^3}{1+x^5} = \frac{x^2-x+1}{x^4-x^3+x^2-x+1} = \frac{1}{\alpha-\beta}\left(\frac{2\cos(\pi/5)}{x^2+2\cos(2\pi/5)x+1} + \frac{2\cos(2\pi/5)}{x^2-2\cos(\pi/5)x+1}\right)$$

and

$$\frac{x+x^2}{1+x^5} = \frac{x}{x^4-x^3+x^2-x+1} = \frac{1}{\alpha-\beta}\left(\frac{1}{x^2-2\cos(\pi/5)x+1} - \frac{1}{x^2+2\cos(2\pi/5)x+1}\right).$$

Now it is easy to verify that, for $0 < \gamma < \pi$,

$$\int_0^x \frac{du}{u^2 \pm 2u\cos\gamma + 1} = \frac{1}{\sin\gamma}\tan^{-1}\frac{x\sin\gamma}{1 \pm x\cos\gamma}$$

and, therefore,

$$\int_0^1 \frac{dx}{x^2+2x\cos(2\pi/5)+1} = \frac{1}{\sin(2\pi/5)}\tan^{-1}\frac{\sin(2\pi/5)}{1+\cos(2\pi/5)} = \frac{1}{\sin(2\pi/5)}\frac{\pi}{5}$$

and

$$\int_0^1 \frac{dx}{x^2-2x\cos(\pi/5)+1} = \frac{1}{\sin(\pi/5)}\tan^{-1}\frac{\sin(\pi/5)}{1-\cos(\pi/5)} = \frac{1}{\sin(\pi/5)}\frac{2\pi}{5}.$$

Finally, we find

$$A = \frac{\pi/5}{\sin(\pi/5)} = \frac{2\pi}{5\sqrt{3-\alpha}} \quad \text{and} \quad B = \frac{\pi/5}{\sin(2\pi/5)} = \frac{2\pi}{5\alpha\sqrt{3-\alpha}},$$

where we used the fact that $\alpha = 2\cos(\pi/5)$ and $\beta = -2\cos(2\pi/5)$.

Also solved by K. Davenport, H. Kappus, H.-J. Seiffert, D. Terr, and the proposer.

Triple Threat

H-507 (Corrected) ***Proposed by Mohammad K. Azarian, Univ. of Evansville, Evansville, IN (Vol. 34, no. 1, February 1996)***

Prove that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{n}\sum_{k=1}^{m}(-1)^i 2^{-(k+1)(i+j)}\left(\frac{n(i+1)(i+2)\cdots(i+n-1)}{j!(n-j)!}\right)(F_k)^{i+j} = m.$$

Solution by H.-J. Seiffert, Berlin, Germany

Let $x_1, \ldots, x_m \in (-1, 1)$. Then

$$\begin{aligned}
S &= \sum_{i=0}^{\infty}\sum_{j=0}^{n}\sum_{k=1}^{m}(-1)^i\left(\frac{n(i+1)(i+2)\cdots(i+n-1)}{j!(n-j)!}\right)x_k^{i+j} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{n}\sum_{k=1}^{m}\binom{-n}{i}\binom{n}{j}x_k^{i+j} = \sum_{k=1}^{m}\sum_{i=0}^{\infty}\binom{-n}{i}x_k^i\sum_{j=0}^{n}\binom{n}{j}x_k^j \\
&= \sum_{k=1}^{m}\sum_{i=0}^{\infty}\binom{-n}{i}x_k^i(1+x_k)^n = \sum_{k=1}^{m}(1+x_k)^n\sum_{i=0}^{\infty}\binom{-n}{i}x_k^i \\
&= \sum_{k=1}^{m}(1+x_k)^n(1+x_k)^{-n},
\end{aligned}$$

or $S = m$. Since $0 < F_k < \alpha^k < 2^k < 2^{k+1}$, $k \ge 1$, we may take $x_k = 2^{-k-1}F_k$, $k = 1, \ldots, m$. From the above, it follows that the sum in question has the value m; we note the mistake in the proposal.

Also solved by P. Bruckman.

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau, Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book, Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965

A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Applications of Fibonacci Numbers, Volumes 1-6. Edited by G.E. Bergum, A.F. Horadam and A.N. Philippou

Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger. FA, 1993.

Fibonacci Entry Points and Periods for Primes 100,003 through 415,993 by Daniel C. Fielder and Paul S. Bruckman.

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