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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

THE GOLDEN STAIRCASE AND THE GOLDEN LINE

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0. INTRODUCTION

An arrangement of squares dissected from the golden rectangle and placed along the positive X-axis creates a golden staircase, upon which sits the golden line. In this paper we consider some algebraic and geometric relationships expressed in this figure. An extended figure depicts infinite series and relationships involving Fibonacci numbers.

1. THE GOLDEN LINE

In a golden rectangle, the ratio of the length to the width is the same as the ratio of the sum of the length and the width to the length. This golden ratio ϕ has a value of $(1+\sqrt{5})/2$. Consider a golden rectangle with length ϕ and width 1. When a square with side 1 is inscribed as in Figure 1, the remaining rectangle has a length to width ratio of $1/(\phi-1)$, which simplifies to ϕ, establishing a nice relationship between ϕ and its reciprocal

$$\phi-1=\frac{1}{\phi}, \tag{1}$$

which is equivalent to $\phi^2=\phi+1$. When continued, this partitioning process generates the familiar infinite progression of spiraling squares, the first few of which are shown in Figure 2.

When all the squares are placed along the positive X-axis creating the golden staircase shown in Figure 3, the upper right corners of the squares are collinear and define the golden line, which has equation $y=(-1/\phi)x+\phi$. The equation of the line AB through the upper right corners of the first two squares is $y-1=\frac{(1/\phi-1)}{(\phi-1)}(x-1)$, which simplifies to

$$y=\frac{-1}{\phi}x+\phi. \tag{2}$$

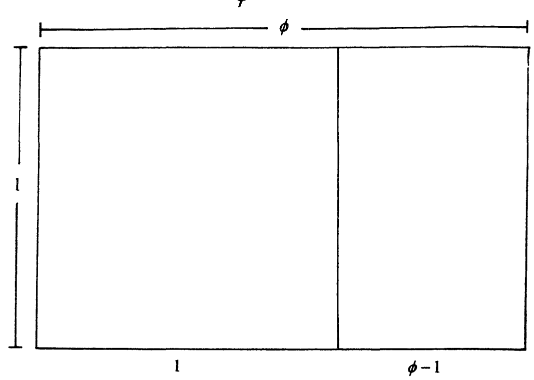

FIGURE 1

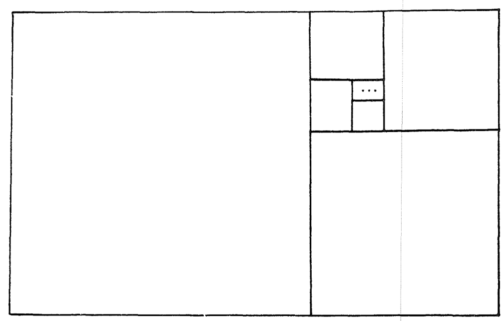

FIGURE 2

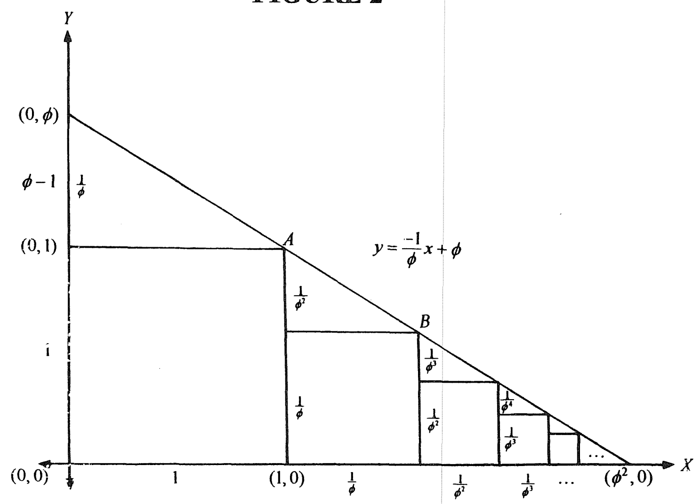

FIGURE 3

The slope of the line through the upper right corners of the n^{th} pair of adjacent squares is

$$\frac{\Delta y}{\Delta x}=\frac{(1/\phi^n-1/\phi^{n-1})}{(1/\phi^n)}=1-\phi=\frac{-1}{\phi},\quad n=1,2,3,\ldots,$$

making these corners collinear with A and B, the corners of the first pair of adjacent squares. The points $(\phi^2, 0)$ and $(0, \phi)$ satisfy (2) and lie on the golden line, so the sides of the squares on the X-axis provide

$$\sum_{n=0}^{\infty}\frac{1}{\phi^n}=\phi^2. \tag{3}$$

The right triangle under the golden line is a golden triangle and also half of a golden rectangle, since the ratio of its legs is ϕ.

One- and two-dimensional representations of ϕ and its reciprocal can be found in Figure 3. ϕ is the y-intercept of the golden line and, since $\phi^2=\phi+1$, the distance between $(1, 0)$ and the x-intercept of the golden line is ϕ. Also ϕ can be seen as the distance from the origin to the end of the second square. $1/\phi$ is the length of the second square and also the altitude of the first triangle. All of the squares fit exactly into the golden rectangle of Figure 2, so the sum of the areas of all of the squares is ϕ, the area of the golden rectangle. The areas of the squares also form a geometric sequence so that

$$\sum_{k=0}^{\infty}\left(\frac{1}{\phi}\right)^{2k} = \phi = \sum_{k=0}^{\infty}(\phi-1)^{2k}. \tag{4}$$

Since the area of the first square is 1, the sum of the areas of the squares beyond the first is $\phi-1$ or $1/\phi$. Thus, we have in one picture both a linear and a planar representation of ϕ and $1/\phi$, neatly sheltered beneath the golden line, $y=(-1/\phi)x+\phi$.

2. EXPANDING THE PICTURE

Above each square, construct a rectangle whose diagonal lies on the golden line shown in Figure 4. Each of these small horizontal rectangles is also a golden rectangle. Joining any such rectangle to the square below it creates a new, larger vertical golden rectangle. Figure 4 also shows the line $y=-x+\phi^2$ drawn through the corners of the golden rectangles.

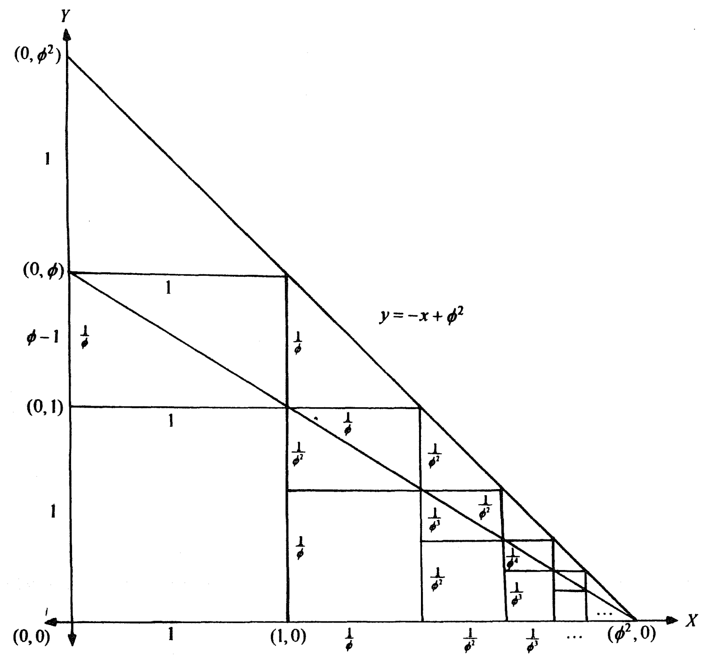

FIGURE 4

Let us focus our attention on the series of vertical and horizontal golden rectangles in Figure 4. After the first, each vertical rectangle is congruent to a horizontal rectangle. The sum ΣV_n of the areas of the vertical rectangles is the area of the largest vertical rectangle added to the sum ΣH_n of the areas of the horizontal rectangles. That is, $\Sigma V_n = \phi + \Sigma H_n$, where also $\Sigma V_n =$ $(\phi^2)\Sigma H_n$ by similarity. Since $\phi^2 = \phi+1$, $\Sigma V_n = \phi^2$, and $\Sigma H_n = 1$. Also, each vertical rectangle is a square added to a horizontal rectangle. If ΣS_n is the sum of the areas of the squares, $\Sigma V_n =$ $\Sigma S_n + \Sigma H_n$, which gives $\Sigma S_n = \phi$ as in (4). $\Sigma H_n = 1$ leads to

$$\sum_{k=0}^{\infty} \frac{1}{\phi^{2k+1}} = 1, \tag{5}$$

and all of the small rectangles will fit exactly into the first square.

In intercept form, the equation of the line through the upper corners of the horizontal golden rectangle is $(x/\phi^2)+(y/\phi^2)=1$; for the upper corners of the squares, $(x/\phi^2)+(y/\phi)=1$; and for the lower corners of the horizontal rectangles, $(x/\phi^2)+(y/1)=1$.

3. FINDING FIBONACCI

Where we find the golden ratio, we can expect to find Fibonacci, but first we need to set the stage for his entrance. From Figure 4, the length of the side of a square is the same as the length of the adjacent golden rectangle to the right. The length of that vertical golden rectangle is the sum of the length of a square and the width of a horizontal golden rectangle above the square. For example, with the third square and its adjoining golden rectangle, we have $(1/\phi^2)=(1/\phi^3)+(1/\phi^4)$, and for the $(k+1)^{\text{st}}$ square, $(1/\phi)^k=(1/\phi)^{k+1}+(1/\phi)^{k+2}$. This representation of a power of $1/\phi$ as the sum of the next two consecutive powers of $1/\phi$ allows Fibonacci to enter. For consecutive Fibonacci numbers F_j and F_{j+1},

$$\left(\frac{1}{\phi}\right)^k = F_{j+1}\left(\frac{1}{\phi}\right)^{k+j} + F_j\left(\frac{1}{\phi}\right)^{k+j+1}, \tag{6}$$

which can be proved by induction.

Expanding powers of $\phi-1$ and simplifying leads to another expression involving Fibonacci numbers and ϕ, which also can be proved by mathematical induction:

$$(\phi-1)^k = \left(\frac{1}{\phi}\right)^k = \begin{cases} F_k\phi - F_{k+1}, & \text{for } k \text{ odd,} \\ F_{k+1} - F_k\phi, & \text{for } k \text{ even.} \end{cases} \tag{7}$$

To all the beautiful patterns in mathematics, we may now add the golden staircase, the golden line, and all that they inspire.

ACKNOWLEDGMENT

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STRONGLY MAGIC SQUARES

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(Submitted April 1995–Final Revision June 1996)

I. INTRODUCTION

Consider the classical 4×4 magic square

$$M_{\text{Classical}}=\begin{bmatrix}16&2&3&13\\5&11&10&8\\9&7&6&12\\4&14&15&1\end{bmatrix}.$$

Among the 2×2 subsquares that can be formed within $M_{\text{Classical}}$, there are only five with the property that their entries add to the magic constant 34. These are the four corner squares and the central one:

$$\begin{bmatrix}16&2\\5&11\end{bmatrix},\begin{bmatrix}3&13\\10&8\end{bmatrix},\begin{bmatrix}9&7\\4&14\end{bmatrix},\begin{bmatrix}6&12\\15&1\end{bmatrix},\begin{bmatrix}11&10\\7&6\end{bmatrix}.$$

If wrap-arounds are allowed, one more such subsquare arises, namely,

$$\begin{bmatrix}1&4\\13&16\end{bmatrix},$$

built from the four corners.

Compare this to the square

$$M^*=\begin{bmatrix}9&16&5&4\\7&2&11&14\\12&13&8&1\\6&3&10&15\end{bmatrix}$$

which has the stronger property that all sixteen 2×2 subsquares (allowing wrap-arounds) have entries adding to 34. What other magic squares have this stronger property?

Suppose M is a 4×4 magic square. That is, the sixteen entries are a permutation of the set $[1,2,3,\dots,16]$ and all the row-sums and column-sums equal 34. Writing this as

$$M=\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{bmatrix},$$

we define M to be a *strongly magic square* if, in addition, $a_{m,n}+a_{m,n+1}+a_{m+1,n}+a_{m+1,n+1}=34$ whenever $1\le m,n\le 3$. We will derive several further properties of strongly magic squares. For example, it follows that all the wrap-around 2×2 subsquares, like

$$\begin{bmatrix}a_{14}&a_{11}\\a_{24}&a_{21}\end{bmatrix},$$

also have entries adding to 34. Moreover, we will classify all the strongly magic squares, showing that there are exactly 384 of them. We also define a group of transformations by which each strongly magic square can be transformed into all the other 383 strongly magic squares.

II. SPECIAL PROPERTIES OF STRONGLY MAGIC SQUARES

1. In a strongly magic square

$$M_s = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

$$a_{11}+a_{12}=a_{23}+a_{24}=a_{31}+a_{32}=a_{43}+a_{44}=A \tag{1}$$

and

$$a_{13}+a_{14}=a_{21}+a_{22}=a_{33}+a_{34}=a_{41}+a_{42}=34-A. \tag{2}$$

This property follows directly from the definition of strongly magic squares.

2. The 3×3 Square Property.

Consider any 3×3 square formed within a strongly magic square M_s. The sum of the four corners of this square is 34 and the sum of each diagonally opposite corner pair is 17.

Let $C_1, C_2, C_3,$ and C_4 be the corner elements of any 3×3 subsquare of M_s. Then

$$C_1+C_4=C_2+C_3=17.$$

Proof: Each of the corners $C_1, C_2, C_3,$ and C_4 of any 3×3 subsquare can be considered as a corner of a 2×2 subsquare, three of them being corner squares and one being the inner central square. For example, consider the 3×3 square

$$M_3' = \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}:$$

$$C_1 = a_{22}, \text{ a corner of} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \text{ (inner central square);}$$

$$C_2 = a_{24}, \text{ a corner of} \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \text{ (corner square);}$$

$$C_3 = a_{42}, \text{ a corner of} \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \text{ (corner square);}$$

$$C_4 = a_{44}, \text{ a corner of} \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} \text{ (corner square).}$$

The corners of any 3×3 square of M_s can therefore be written as

$$\begin{aligned} C_1 &= 34 - S_1, \\ C_2 &= 34 - S_2, \\ C_3 &= 34 - S_3, \\ C_4 &= 34 - S_4, \end{aligned}$$

where S_1, S_2, S_3, and S_4 are the sums of the other three elements of the respective 2×2 magic squares. For the particular 3×3 square M_3',

$$\begin{aligned}
S_1 &= a_{23}+a_{32}+a_{33},\\
S_2 &= a_{13}+a_{14}+a_{23},\\
S_3 &= a_{31}+a_{32}+a_{41},\\
S_4 &= a_{33}+a_{34}+a_{43}.
\end{aligned}$$

The sum of the corners can be written as

$$C_1+C_2+C_3+C_4 = 4\times 34-(S_1+S_2+S_3+S_4).$$

Regrouping the terms that constitute S_1, S_2, S_3, and S_4 in the sum $S_1+S_2+S_3+S_4$, it can be shown that $S_1+S_2+S_3+S_4$ is the sum of a row, a column, and a diagonal of the 4×4 square, each of which is equal to 34. For example, for M_3',

$$\begin{aligned}
S_1+S_2+S_3+S_4 &= (a_{31}+a_{32}+a_{33}+a_{34})+(a_{13}+a_{23}+a_{33}+a_{43})+(a_{14}+a_{23}+a_{32}+a_{41})\\
&= 34+34+34.
\end{aligned}$$

Therefore,

$$C_1+C_2+C_3+C_4 = 4\times 34-3\times 34 = 34.$$

This is true for the classical magic square also.

In the case of strongly magic squares, two of the 2×2 squares that contain C_1, C_2, C_3, and C_4 can be chosen to be those formed by the inner two rows and columns (which have the magic property in M_s and not in $M_{\text{Classical}}$). For example, in the 3×3 square M_3', the corners C_2 and C_3 can be considered a part of the 2×2 squares

$$\begin{bmatrix} a_{23} & a_{24}\\ a_{33} & a_{34}\end{bmatrix} \text{ and } \begin{bmatrix} a_{32} & a_{33}\\ a_{42} & a_{43}\end{bmatrix},$$

respectively, in this case,

$$\begin{aligned}
S_1 &= a_{23}+a_{32}+a_{33},\\
S_2 &= a_{23}+a_{33}+a_{34},\\
S_3 &= a_{32}+a_{33}+a_{43},\\
S_4 &= a_{33}+a_{34}+a_{43};
\end{aligned}$$

$$S_2+S_3 = S_1+S_4,$$

i.e.,

$$C_1+C_4 = C_2+C_3.$$

Since $C_1+C_2+C_3+C_4 = 34$, we have $C_1+C_4 = C_2+C_3 = 17$.

3. Triangular Property

Form any triangle each side of which is made of three numbers of the 4×4 strongly magic square. Examples of such triangles are shown below:

$$\begin{array}{ccc} a_{11} & a_{12} & a_{13}\\ & a_{22} & a_{23}\\ & & a_{33}\end{array} \qquad \begin{array}{ccc} & & a_{24}\\ & a_{33} & a_{34}\\ a_{42} & a_{43} & a_{44}\end{array}$$

In M_s the sum of the six numbers along the sides of the triangle is the same for all such triangles and equal to 51. This also can be shown to follow from the additional 2×2 magic property.

Proof: Of the six numbers that constitute the sum, three can be considered as a part of a row or column of the M_s, and the other three as part of a 2×2 magic square. Let S_1 be the sum of the three numbers that are part of the row or column and S_2 that of three numbers that are part of the 2×2 square.

Since the sum of the numbers of the row or column as well as the sum of the numbers of the 2×2 square are equal to 34,

$$S_1 = 34 - N_1,$$
$$S_2 = 34 - N_2,$$

where N_1 is the remaining element of the row/column and N_2 is the remaining element of the 2×2 magic square. It is easy to see that N_1 and N_2 always form the opposite corners of a 3×3 square whose sum is 17. Therefore, the sum of the sides of the triangle can be written as

$$S = S_1 + S_2 = 68 - (N_1 + N_2) = 68 - 17 = 51.$$

III. TRANSFORMATIONS THAT PRESERVE THE STRONGLY MAGIC PROPERTY

There exist several transformations which, when applied to a strongly magic square yields another strongly magic square.

Some of these transformations along with the notations we use to represent them later in the paper are given below.

1) Cycling of rows (cyc R) or columns (cyc C).

2) Interchange of columns 1 and 3 ($C_{1\leftrightarrow 3}$) or rows 1 and 3 ($R_{1\leftrightarrow 3}$).

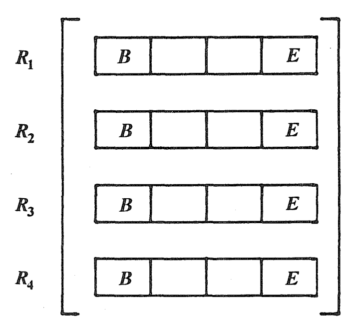
3) Interchange of columns 2 and 4 ($C_{2\leftrightarrow 4}$) or rows 2 and 4 ($R_{2\leftrightarrow 4}$).

4) Diagonal reflections (DRA on the ascending diagonal and DRD on the descending diagonal).

5) Replacement of every element x by $17 - x$.

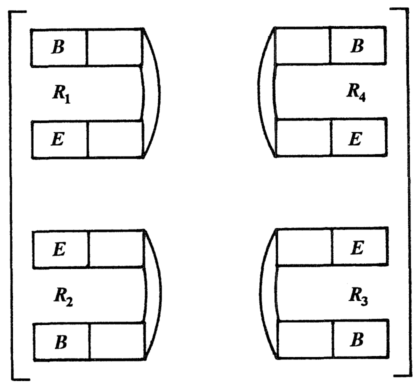
6) "Twisting" of rows (TWR) or columns (TWC) which is defined below:

The row twist of a square M is obtained by curling the first row of M into the upper left corner of TWR(M), the second row of M into the lower left corner of TWR(M), etc. Consider a strongly magic square with rows R_1, R_2, R_3, and R_4:



B and E are the beginning and end of the rows.

The "twisting" transformation is defined to yield the following square:



For example, the row-twisting transformation on M^* yields

$$\mathrm{TWR}(M^*) = \begin{bmatrix} 9 & 16 & 3 & 6 \\ 4 & 5 & 10 & 15 \\ 14 & 11 & 8 & 1 \\ 7 & 2 & 13 & 12 \end{bmatrix};$$

column-twisting M^* yields

$$\mathrm{TWC}(M^*) = \begin{bmatrix} 9 & 7 & 14 & 4 \\ 6 & 12 & 1 & 15 \\ 3 & 13 & 8 & 10 \\ 16 & 2 & 11 & 5 \end{bmatrix}.$$

It can be noted that $\mathrm{TWC}(M) = \mathrm{TWR}(M^{\mathrm{T}})$, where M^{T} is the transpose of the matrix M.

IV. THE TOTAL NUMBER OF DISTINCT STRONGLY MAGIC SQUARES

Only 384 distinct strongly magic squares can be formed from the set of numbers [1, 2, ..., 16].

Proof: We saw in Section II that

$$a_{11} + a_{12} = a_{23} + a_{24} = a_{31} + a_{32} = a_{43} + a_{44} = A \tag{1}$$

and

$$a_{13} + a_{14} = a_{21} + a_{22} = a_{33} + a_{34} = a_{41} + a_{42} = 34 - A. \tag{2}$$

Going through all possible values of A, it can be seen that A can take only eight values, namely, 9, 13, 15, 16, 18, 19, 21, and 25. Any other value of A would lead to at least two of the elements of the magic square being equal.

We now show that each value of A leads to 48 and only 48 strongly magic squares.

Consider the elements a_{12}, a_{24}, a_{31}, and a_{43} of a strongly magic square M_s. The strongly magic property implies that

$$\left.\begin{aligned} a_{13} &= 17 - a_{31} \\ a_{21} &= 17 - a_{43} \\ a_{34} &= 17 - a_{12} \\ a_{42} &= 17 - a_{24} \end{aligned}\right\} \text{ from the } 3\times 3 \text{ square property,}$$

$$\left.\begin{aligned} a_{11} &= A - a_{12} \\ a_{23} &= A - a_{24} \\ a_{32} &= A - a_{31} \\ a_{44} &= A - a_{43} \end{aligned}\right\},$$

$$\left.\begin{aligned} a_{14} &= 34 - A - a_{13} = 34 - A - (17 - a_{31}) = a_{31} - (A - 17) \\ a_{22} &= 34 - A - a_{21} = 34 - A - (17 - a_{43}) = a_{43} - (A - 17) \\ a_{33} &= 34 - A - a_{34} = 34 - A - (17 - a_{12}) = a_{12} - (A - 17) \\ a_{41} &= 34 - A - a_{42} = 34 - A - (17 - a_{24}) = a_{24} - (A - 17) \end{aligned}\right\} \text{ from equations (1) and (2).}$$

Thus, for a given value of A, M_s can be written as

$$M_s(A) = \begin{bmatrix} (A - a_{12}) & a_{12} & a'_{31} & (A - a_{31})' \\ a'_{43} & (A - a_{43})' & (A - a_{24}) & a_{24} \\ a_{31} & (A - a_{31}) & (A - a_{12})' & a'_{12} \\ (A - a_{24})' & a'_{24} & a_{43} & (A - a_{43}) \end{bmatrix},$$

where the notation x' means $17 - x$. Additionally, in order that all the elements are distinct and positive, further conditions have to be satisfied by the set $(a_{12}, a_{24}, a_{31}, a_{43})$. These conditions depend on the value of A. For example, when $A = 25$, a_{12}, a_{24}, a_{31}, and a_{43} can take on values between 9 and 16 only. Also, $a_{12} + a_{43} = a_{24} + a_{31} \neq 25$ because, if the sum is equal to 25, the elements of M_s (25) cannot be distinct.

By considering all possible number pair sets (a_{12}, a_{43}), (a_{24}, a_{31}) satisfying the above two conditions, for $A = 25$, it is seen that they are:

$$\begin{gathered} (16,11),(15,12);\ \ (16,10),(14,12);\ \ (15,9),(13,11); \\ (14,9),(13,10);\ \ (16,13),(15,14);\ \ (9,12),(10,11); \end{gathered}$$

and all the permutations possible within each set. From each of the above six sets, eight permutations are possible, leading to $8 \times 6 = 48$ possibilities for the set $[a_{12}, a_{24}, a_{31}, a_{43}]$.

Thus, we have proved that with $A = 25$, 48, and only 48, strongly magic squares can be obtained. Similarly, it is possible to obtain exactly 48 strongly magic squares from each of the other seven values of A. However, since this is a tedious procedure to prove directly, we follow a different approach. We show in the following that performing certain sequences of transformations on each of the 48 strongly magic squares for any value of A, we can get all strongly magic squares with the other seven values of A.

These transformations can be shown to be sequences of 3 basic transformations, namely,

$$\begin{aligned} T_1 &= C_{1\leftrightarrow 3}, \\ T_2 &= \text{ DRA}, \\ T_3 &= C_{1\leftrightarrow 3} + C_{2\leftrightarrow 4}. \end{aligned}$$

Applying T_1, T_2, and T_3 on any strongly magic square, we can form seven other strongly magic squares each with a distinct value of A. Let

$$M_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

$$\begin{aligned} M_2 &= T_1(M_1), \\ M_3 &= T_2(M_1), \\ M_4 &= T_1(M_3), \\ M_{5\text{ to }8} &= T_3(M_{1\text{ to }4}). \end{aligned}$$

The values of A for the squares $M_{1\text{ to }8}$ are given below:

$$\begin{aligned} A_1 &= a_{11} + a_{12}, \\ A_2 &= a'_{31} + a_{12}, \\ A_3 &= a'_{43} + (A_1 - a_{12}), \\ A_4 &= a_{31} + a'_{43}, \\ A_5 &= a'_{31} + (A_1 - a_{31})', \\ A_6 &= (A_1 - a_{12}) + (A_1 - a_{31})', \\ A_7 &= a_{31} + (A_1 - a_{24})', \\ A_8 &= (A_1 - a_{12}) + (A_1 - a_{24})', \end{aligned}$$

where $x' = 17 - x$.

Remembering that A can have only eight possibilities and that with the $A = 25$ we can have only 48 strongly magic squares; we can see that we get $48 \times 8 = 384$ strongly magic squares. Now, the above transformations applied to any strongly magic square with any other value of A will also yield strongly magic squares having the seven other possibilities for A which includes $A = 25$. If there are N possible magic squares with a certain value of $A \neq 25$, we can get $8N$ strongly magic squares by performing the transformations on each of the N squares. Thus, there will be N strongly magic squares for each value of A, including $A = 25$. We have already proved that there can be only 48 strongly magic squares with $A = 25$.

Thus, $N = 48$ for any value of A, and we have proved that there are exactly 384 strongly magic squares formed.

Equivalence of Strongly Magic Squares

Two strongly magic squares are defined as equivalent if one can be transformed to the other by a transformation or a sequence of transformations. It is shown below that each strongly magic square can be transformed into all the other 383 squares.

Take any strongly magic square,

$$M_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Then 24 distinct strongly magic squares with a_{11} as the first element can be formed from M_1 be applying some of the transformations mentioned in Section III in sequence:

$$\begin{aligned}
M_2 &= \mathrm{TWR}(M_1);\\
M_3 &= \mathrm{TWR}(M_2);\\
M_{4\text{ to }6} &= \mathrm{DRA}(M_{1\text{ to }3});\\
M_{7\text{ to }9} &= C_{2\leftrightarrow 4}(M_{1\text{ to }3});\\
M_{10\text{ to }12} &= C_{2\leftrightarrow 4}(M_{4\text{ to }6});\\
M_{13\text{ to }24} &= R_{2\leftrightarrow 4}(M_{1\text{ to }12}).
\end{aligned}$$

Note that a_{11} can be any of the 16 numbers from 1 to 16 because any of these numbers can be brought to the (1, 1) position by an appropriate sequence of row and column cycling. Each of these can then be transformed to 24 distinct strongly magic squares by the above mentioned transformations. Thus, one can obtain all the $384 = 16 \times 24$ strongly magic squares from any strongly magic square, i.e., all strongly magic squares are equivalent to all other strongly magic squares.

It is also clear from the above that there are 24 distinct strongly magic squares for any one position of a number in the square. It has already been shown that there are 24 strongly magic squares with any number occupying the (1, 1) position. Performing appropriate sequences of row and column cycling on these 24 squares, this number can be brought to the desired position, i.e., 24 strongly magic squares can be formed for a particular position of any number.

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FRACTAL CONSTRUCTION BY ORTHOGONAL PROJECTION USING THE FIBONACCI SEQUENCE

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1. INTRODUCTION

Let $\{G(j,k)\}_{k=1}^{\infty}$ denote the Fibonacci j-sequences such that $G(2,k)=F_k$, the k^{th} Fibonacci number and, for $j>2$,

Definition 1: $\{G(j,k)\}_{k=1}^{\infty}=\{G(j,k), k=1,2,\ldots. G(j,k)=G(j-1,k), k=1,2,\ldots,j.\ G(j,k)=G(j,k-1)+G(j,k-2)+\cdots+G(j,k-j), k>j\}$.

Thus, new elements of the set $\{G(j,k)\}_{k=1}^{\infty}$ for $j=2,3,\ldots$ are created by adding the previous j elements of the sequence, using as initial values the first j values of $\{G(j-1,k)\}_{k=1}^{\infty}$. Fibonacci j-sequences, satisfying the j^{th}-order linear recurrence relation in Definition 1, are also called j-bonacci, j-acci, or j-generalized Fibonacci numbers. They are a special case of a general linear recurrence relation studied by Levesque [10] and Tee [17]. The case $j=3$ yields so-called Tribonacci numbers (see Feinberg [6]). Table 1 gives the values $\{G(j,k)\}_{k=1}^{16}$ for $j=2,\ldots,7$.

TABLE 1. $\{G(j,k)\}_{k=1}^{16}$ **for** $j=2,\ldots,7$

j/k																
1/0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
3	1	1	2	4	7	13	24	44	81	149	274	504	927	1705	3136	5768
4	1	1	2	4	8	15	29	56	108	208	401	773	1490	2872	5536	10671
5	1	1	2	4	8	16	31	61	120	236	464	912	1793	3525	6930	13624
6	1	1	2	4	8	16	32	63	125	248	492	976	1936	3840	7617	15019
7	1	1	2	4	8	16	32	64	127	253	504	1004	2000	3984	7936	15808

There is associated with each sequence $\{G(j,k)\}_{k=1}^{\infty}$ the j^{th}-degree polynomial

$$F_j(x)=x^j-x^{j-1}-x^{j-2}-\cdots-x-1, \tag{1}$$

denoted in the present paper as Fibonacci j-polynomials. Let the real or complex number S_j denote the sum of the j^{th} powers of the roots of a polynomial of degree j. Then Newton's formula is given by (see Tee [18])

$$S_j=a_1S_{j-1}+a_2S_{j-2}+\cdots+a_{j-1}S_1+ja_j,\ S_1=1, \tag{2}$$

where the a_1 are coefficients of the monic polynomial $x^j-a_1x^{j-1}-a_2x^{j-2}-\cdots-a_{j-1}x-a_j=0$.

As an observation, referring to (1), $G(j, j+2) = 2^j - 1 = S_j$, $j \geq 2$ if $a_i = 1$, $\forall i$, which can be shown inductively. Godsil and Razen [8] derived the generating function for a self-generating sequence having parameters k, m, and r, denoted $\mathrm{SGS}(k, m, r)$, given by

$$F(x) = \frac{p_{k+r}(x)}{(1-x)^k - mx^{k+r}},$$

and showed no Fibonacci j-sequence was a SGS for $j \geq 4$, where p_{k+r} is a polynomial of degree at most $k+r$. The well-known generating function for Fibonacci j-sequences (see Philippou [15]),

$$Q_j(x) = \frac{x}{1 - x - x^2 - \cdots - x^{j-1} - x^j} = \sum_{k=1}^{\infty} G(j, k)x^k, \ |x| < 0.5, \tag{4}$$

which also appeared in the work of Godsil and Razen [8], is a special case of the generating formula of Levesque [10]. If x is replaced by η^{-1} and a factor of $\eta - 1$ is introduced, then (4) becomes

$$\sum_{k=1}^{\infty} \frac{G(j, k)}{\eta^k} = \frac{(\eta-1)\eta^{j-1}}{1 + (\eta-2)\eta^j}, \ |\eta| > 2. \tag{5}$$

The region of convergence of (5) is (see Tee [17]) $\{\eta : |\eta| > x_j\}$, where x_j is the largest real root of (1). The form of (5) is useful in the context of the present paper. A derivation of (5) is presented in the next section by an alternate method that also reveals several number theoretic properties of the sequences $\{G(j, k)\}_{k=1}^{\infty}$. Properties of the zeros of the Fibonacci j-polynomials $F_j(x)$ are restated, and several are proved by a different method.

Another result of the paper is a geometrical interpretation of $\{G(j, k)\}_{k=1}^{\infty}$ in terms of a sequence of sets such that the first set depends on the Fibonacci numbers and subsequent sets on the Fibonacci j-sequences. For $j = 2$, a fractal is given and it is shown that a sequence of compact sets exists such that the fractal dimension, counting, and tiling features depend on the Fibonacci j-sequences. An exact expression for the fractal dimension is derived which depends on the largest real zeros of the Fibonacci j-polynomials, x_j, $\forall j \geq 2$. Fractals are of interest in the mathematical sciences (see Mandelbrot [12]).

2. CONVERGENCE PROPERTIES

Miller [13] showed that the zeros of the polynomials $F_j(x)$ are distinct, all but one lies in the unit disk and the latter is real and lies in the interval $(1, 2)$. Flores [7] showed that $x_j \to 2$ as $j \to +\infty$. The monotonic properties of the sequence $\{x_j\}_{j=1}^{+\infty}$ are indicated in

Lemma 1:

$$1 < x_j < x_{j+1} < 2, \ j = 2, 3, \ldots, \tag{6}$$

$$x_j \to 2 \text{ monotonically as } j \to +\infty. \tag{7}$$

Proof: Referring to (1), for each j, $F_j(1) = -(j-1)$, $F_j(2) = 1$. Thus, there is a real zero, denoted x_j. Since $F_j(x) - F_{j-1}(x) = x^{j-1}(x-2)$, it follows by continuity that

$$F_j(x) < F_{j-1}(x), \ 0 < x < 2,$$

$$F_j(x) > F_{j-1}(x), \ 2 < x < +\infty,$$

which implies (6). Note that x_j is largest in magnitude among zeros, since $F_j(x) > F_2(x) > 1$ if $x > 2, j > 2$. To show (7), write

$$F_j(x) = F_2(x) + x^2(x-2)\frac{1-x^{j-2}}{1-x}.$$

Suppose that $\sup\{x_j : F_j(x_j) = 0\} = \varepsilon < 2$. If $\delta_j \to 0^+$ as $j \to +\infty$ then, for some positive sequence $\{\delta_j\}_{j=1}^{+\infty}$, it follows by continuity of F_j that the δ_j may be chosen small enough that $|F_j(\varepsilon - \delta_j)| < 1/j$. Thus, noting that $\varepsilon > 1$, it follows that

$$\lim_{j\to+\infty} F_j(\varepsilon-\delta_j) = \lim_{j\to+\infty}\left(F_2(\varepsilon-\delta_j) + (\varepsilon-\delta_j)^2(\varepsilon-\delta_j-2)\frac{1-(\varepsilon-\delta_j)^{j-2}}{1-\varepsilon+\delta_j}\right) = -\infty.$$

The previous statement is a contradiction which proves the lemma.

A result of Flores [7] is the following theorem.

Theorem 1: For sufficiently large *k*, $\exists$ a constant $c > 0$ such that

$$G(j,k) \approx cx_j^k \quad \text{and} \quad \lim_{k\to+\infty}\frac{G(j,k+1)}{G(j,k)} = x_j,\ j = 2, 3, \ldots. \tag{8}$$

The following numerical examples were calculated on a *TI*–85©:

$$x_3 = 1.839286755,\ x_4 = 1.927561975,\ x_{10} = 1.999018633,\ x_{20} = 1.999999046.$$

The exponential growth of the Fibonacci *j*-sequences is evident in

Corollary 1: Let $M > 0, n \in Z^+$. Then $\forall_j > 2,\ 1 \le i < j-1,\ \exists k$ such that

$$|G(j,k_0) - MG(j-i,k_0)| > n,\ k_0 > k. \tag{9}$$

Proof: By Theorem 1 and the definition of limit, there is a constant $C > 0$ that depends on i, j so that, for large enough *k*, $G(j,k)/G(j-i,k) > C(x_j/x_{j-i})^k \to +\infty$ as $k \to +\infty$.

In [17] Tee showed convergence of the infinite series in the following theorem as a special case of a more general result if $|\eta| > x_j$, $j = 2, 3, \ldots$, for which a proof is also given in the present paper.

Theorem 2:

$$G_j(\eta) = \sum_{k=1}^{\infty}\frac{G(j,k)}{\eta^k} = \frac{(\eta-1)\eta^{j-1}}{1+(\eta-2)\eta^j},\ |\eta| > x_j,\ j = 2, 3, \ldots, \tag{10}$$

such that (10) diverges at $\eta = \pm x_j$ and

$$\lim_{j\to+\infty} G_j(\eta) = \begin{cases} (\eta-1)/((\eta-2)\eta), & \text{if } \eta > 2,\ \eta \le -2, \\ +\infty, & \text{if } \eta = 2. \end{cases} \tag{11}$$

Proof: The theorem is proved first for $\eta = 2$, $j \ge 2$. A sketch of the proof, which is essentially the same as that for $\eta = 2$, is indicated for values of η other than 2. Parallel results are given for $\eta < -x_j$, for which the same method is applicable. A sequence of lemmas establishes the theorem for values of $j \geq 2$. Define the infinite sequence

$$H(1, j, k) = G(j, k+3) - \left(\sum_{i=1}^{k+1} G(j, i) + 1\right), \ k \geq 1. \tag{12}$$

The significance of the 1 in the argument of H will become apparent. Then

Lemma 2:

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{G(j,k)}{2^k} = 1 + \sum_{k=1}^{\infty}\frac{H(1,j,k)}{2^{k+3}}. \tag{13}$$

$$\frac{5}{6}\sum_{k=1}^{\infty}\frac{G(j,k)(-1)^k}{2^k} = -\frac{1}{3} + \sum_{k=1}^{\infty}\frac{H(1,j,k)(-1)^{k+1}}{2^{k+3}}. \tag{14}$$

Proof: Equation (14) corresponds to $\eta = -2$. To prove (13), expand

$$\sum_{k=1}^{\infty}\frac{G(j,k)}{2^k} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{G(j,k)}{2^k} + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \cdots + \frac{G(j,3)-1}{8} + \frac{G(j,4)-1}{16} + \cdots + \frac{G(j,k)-1}{2^k} + \cdots$$

$$= \sum_{k=1}^{\infty}\frac{1}{2^k} + \left(\frac{1}{8} + \frac{1}{16} + \cdots\right)G(j,1) + \frac{G(j,3)-1-G(j,1)}{8} + \frac{G(j,4)-1-G(j,1)}{16} + \cdots$$

$$+ \frac{G(j,k)-1-G(j,1)}{2^k} + \cdots$$

$$= 1 + \frac{1}{2}\frac{G(j,1)}{2} + \left(\frac{1}{16} + \frac{1}{32} + \cdots\right)G(j,2) + \frac{G(j,4)-1-G(j,1)-G(j,2)}{16} + \cdots$$

$$+ \frac{G(j,k)-1-G(j,1)-G(j,2)}{2^k} + \cdots$$

$$= \cdots = 1 + \frac{1}{2}\frac{G(j,1)}{2} + \frac{1}{2}\frac{G(j,2)}{4} + \frac{1}{2}\frac{G(j,3)}{8} + \frac{H(1,j,1)}{16} + \frac{H(1,j,2)}{32} + \cdots$$

$$+ \frac{G(j,k)-1-G(j,1)-G(j,2)-G(j,3)}{2^k} + \cdots \forall k > 5$$

$$= \cdots = 1 + \frac{1}{2}\frac{G(j,1)}{2} + \frac{1}{2}\frac{G(j,2)}{4} + \frac{1}{2}\frac{G(j,3)}{8} + \cdots + \frac{1}{2}\frac{G(j,k+1)}{2^{k+1}}$$

$$+ \frac{H(1,j,1)}{16} + \frac{H(1,j,2)}{32} + \cdots + \frac{H(1,j,k)}{2^{k+3}}$$

$$+ \frac{G(j,k')-1-G(j,1)-G(j,2)-\cdots-G(j,k+1)}{2^{k'}} + \cdots,$$

for every $k' > k+3$. Denote the last term in the above expression by $I(k, j, k')/2^{k'}$, $\forall k' > k+3$, $k \geq 1$.

By the definition of $G(j,k)$ and $H(1,j,k)$

$$G(j,k')/2^{k'} > I(k,j,k')/2^{k'} > H(1,j,k)/2^{k'} \geq 0, \; j \geq 2. \tag{15}$$

Thus, by Theorem 1 and (15), the last term approaches 0 as k and $k' \to +\infty$ and (13) follows. The proof of (14) is similar, and one obtains

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{G(j,k)(-1)^k}{2^k} &= \frac{-1}{2}+\frac{1}{4}+\cdots+\frac{(-1)^k G(j,k)}{2^k}+\cdots \\ &= \frac{-1}{3}-\frac{1}{6}\frac{G(j,1)}{2}+\frac{1}{6}\frac{G(j,2)}{4}-\frac{1}{6}\frac{G(j,3)}{8}+\cdots+\frac{(-1)^{k+1}}{6}\frac{G(j,k+1)}{2^{k+1}} \\ &\quad +\frac{H(1,j,1)}{16}-\frac{H(1,j,2)}{32}+\cdots+\frac{(-1)^{k+1}H(1,j,k)}{2^{k+3}} \\ &\quad +(-1)^{k'}\frac{G(j,k')-1-G(j,1)-G(j,2)-\cdots-G(j,k+1)}{2^{k'}}+\cdots.\end{aligned}$$

Noting that $H(1,2,k)=0$, $\forall k \geq 1$, which follows from (12), and the identity $F_1+F_2+\cdots+F_n=F_{n+2}-1$, define the following infinite sequences depending on j and k:

$$H(i,j\,k)=H(i-1,j,k+1)-G(j,k+2), \; i=2,\ldots,j-2, \; j\geq 4, \tag{16}$$

$$H(j-1,j,k)=H(j-2,j,k+3)-G(j,k+4), \; k\geq 1, \; j\geq 3. \tag{17}$$

Note that (16) begins at $j=4$ and (17) begins at $j=3$. Then, for $j\geq 3$, we have

Lemma 3:

$$\begin{aligned}H(j-1,j,k) &= G(j,k+j+3)-\left(\sum_{i=1}^{k+j+1}G(j,i)+1\right) \\ &\quad -G(j,k+j+1)-G(j,k+j)-\cdots-G(j,4)=H(1,j,k).\end{aligned} \tag{18}$$

Proof: By (16) and (17) [one can also use the identity $G(j,k+j+3)=2G(j,k+j+2)-G(j,k+2)$],

$$\begin{aligned}H(j-1,j,k) &= H(j-2,j,k+3)-G(j,k+4) \\ &= H(j-3,j,k+4)-G(j,k+5)-G(j,k+4) \\ &= \cdots = H(j-i,j,k+i+1)-G(j,j,k+i+2)-\cdots-G(j,k+4) \\ &= \cdots = H(1,j,k+j)-G(j,k+j+1)-\cdots-G(j,k+4), \; (i=j-1), \\ &= G(j,k+j+3)-\left(\sum_{i=1}^{k+j+1}G(j,i)+1\right)-G(j,k+j+1)-\cdots-G(j,k+4) \\ &= H(1,j,k)+G(j,k+j+3)-2(G(j,k+j+1)+G(j,k+j)+\cdots \\ &\quad +G(j,k+4))-G(j,k+3)-G(j,k+2) \\ &= H(1,j,k)+G(j,k+j+2)-(G(j,k+j+1)+G(j,k+j)+\cdots \\ &\quad +G(j,k+3)+G(j,k+2)) \\ &= H(1,j,k).\end{aligned}$$

From (16)-(18) and (12),

$$H(i, j, k) = \begin{cases} H(i+1, j, k-1) + G(j, k+1), & \text{if } k > 1, i = 1, \dots, j-3, \\ G(j, k+1) = 1, & \text{if } k = 1, \end{cases} \tag{19}$$

$$H(j-2, j, k) = \begin{cases} H(1, j, k-3) + G(j, k+1), & \text{if } k \geq 4, \\ G(j, k+1) = 1, 2, 4, & \text{if } k = 1, 2, 3, \text{ resp.} \end{cases} \tag{20}$$

The second result of (19) is shown as follows. By (12), $H(1, j, 1) = 1$. For $2 \leq m \leq j-2$, $j \geq 4$,

$$\begin{aligned} H(m, j, 1) &= H(m-1, j, 2) - G(j, 3) \\ &= H(m-2, j, 3) - G(j, 4) - G(j, 3) \\ &= \cdots = H(1, j, m) - G(j, m+1) - \cdots - G(j, 3) \\ &= G(j, m+3) - \left(\sum_{i=1}^{m+1} G(j, i) + 1 \right) - G(j, m+1) - \cdots - G(j, 3) \\ &= G(j, m+3) - (1 + G(j, 1) + G(j, 2) + 2G(j, 3) + \cdots + 2G(j, m+1)) \\ &= 2^{j-i-1} - 2(1 + 2 + \cdots + 2^{j-i-3}) - 1, \ 0 \leq i \leq j-4, \\ &= 2^{j-i-1} - (1 + 2 + \cdots + 2^{j-i-2}) = 1. \end{aligned}$$

The second result of (20) follows by a similar method. From (19) and (20), one obtains

Lemma 4:

$$\sum_{k=1}^{\infty} \frac{H(i, j, k)}{2^{k+2+i}} = \sum_{k=2}^{\infty} \frac{G(j, k)}{2^{k+1+i}} + \sum_{k=1}^{\infty} \frac{H(i+1, j, k)}{2^{k+3+i}}, \ i = 1, \dots, j-3, \tag{21}$$

$$\sum_{k=1}^{\infty} \frac{H(j-2, j, k)}{2^{k+j}} = \sum_{k=2}^{\infty} \frac{G(j, k)}{2^{k+j-1}} + \sum_{k=1}^{\infty} \frac{H(1, j, k)}{2^{k+3+j}}. \tag{22}$$

Proof: The equalities (21) and (22) follow by summing (19) and (20) and adjusting the summation subscripts after division, respectively, by 2^{k+2+i} and 2^{k+j}.

Returning now to the proof of Theorem 2 when $\eta = 2$, applying (21) and (22) in Lemma 4 recursively, it follows that

$$\frac{2^j - 1}{2^j} \sum_{k=1}^{\infty} \frac{H(1, j, k)}{2^{k+3}} = \sum_{k=2}^{\infty} \frac{G(j, k)}{2^{k+2}} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{j-3}} \right). \tag{23}$$

Taking this and Lemma 2, one obtains

Lemma 5:

$$\sum_{k=1}^{\infty} \frac{G(j, k)}{2^k} = 2^{j-1}, \quad \sum_{k=1}^{\infty} \frac{G(j, k)(-1)^k}{2^k} = \frac{2^{j-1} 3(-1)^j}{1 - 4(-2)^j}. \tag{24}$$

Proof: Substituting (13) into (23) yields

$$\frac{2^j-1}{2^j}\left(\frac{1}{2}\sum_{k=1}^{\infty}\frac{G(1,k)}{2^k}-1\right)=\frac{1}{4}\sum_{k=2}^{\infty}\frac{G(j,k)}{2^k}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{j-3}}\right)$$

$$\Leftrightarrow\sum_{k=1}^{\infty}\frac{G(j,k)}{2^k}\left[\frac{2^j-1}{2^{j+1}}-\frac{1}{2}\left(1-\frac{1}{2^{j-2}}\right)\right]=\frac{2^j-1}{2^j}-\frac{1}{4}\left(1-\frac{1}{2^{j-2}}\right)$$

$$\Leftrightarrow\sum_{k=1}^{\infty}\frac{G(j,k)}{2^k}\left[\frac{2^j-1+4-2^j}{2^{j+1}}\right]=\frac{2^j-1-2^{j-2}+1}{2^j}$$

$$\Leftrightarrow\sum_{k=1}^{\infty}\frac{G(j,k)}{2^k}\frac{3}{2^{j+1}}=\frac{2^{j-2}3}{2^j}\Leftrightarrow\sum_{k=1}^{\infty}\frac{G(j,k)}{2^k}=2^{j-1}.$$

For $\eta=-2$ (and $\eta<-x_j$), the analogs of (21) and (22) are multiplied by $(-1)^k$ and $(-1)^{k+1}$ on the left- and right-hand sides, respectively. Similarly, by Lemmas 2 and 4,

$$\frac{2^j-(1)^j}{2^j}\sum_{k=1}^{\infty}\frac{H(1,j,k)(-1)^k}{2^{k+3}}=\sum_{k=2}^{\infty}\frac{G(j,k)(-1)^k}{2^{k+2}}\left(-1+\frac{1}{2}+\cdots+\frac{(-1)^j}{2^{j-3}}\right)$$

$$\Rightarrow\sum_{k=1}^{\infty}\frac{G(j,k)(-1)^k}{2^k}=\frac{2^{j-1}3(-1)^j}{1-4(-2)^j}.$$

A similar analysis yields the theorem for other values of η and these details are briefly outlined. Lemma 2 becomes

Lemma 6:

$$\left(1-\frac{1}{\eta(\eta-1)}\right)\sum_{k=1}^{\infty}\frac{G(j,k)}{\eta^k}=\frac{1}{\eta-1}+\sum_{k=1}^{\infty}\frac{H(1,j,k)}{\eta^{k+3}},\ \eta>x_j, \tag{25}$$

$$\left(1-\frac{1}{\eta(\eta+1)}\right)\sum_{k=1}^{\infty}\frac{G(j,k)(-1)^k}{\eta^k}=\frac{-1}{\eta+1}+\sum_{k=1}^{\infty}\frac{H(1,j,k)(-1)^{k+1}}{\eta^{k+3}},\ \eta>x_j. \tag{26}$$

Proof: This result follows by straightforward application of geometric series. The interval of convergence follows by an argument similar to that given for Lemma 2.

Lemma 4 and (23) have 2 everywhere replaced by η. By Theorem 1 and the ratio test for absolute convergence, (10) diverges at the endpoints $\eta=\pm x_j$ and, therefore, diverges for $\{\eta:\eta\le|x_j|\}$. To prove (11), observe that $G_j(2)=2^{j-1}$ which, as $j\to+\infty$, implies the second part of (11). The first part follows directly by factoring η^{j-1} from $G_j(\eta)$ and letting $j\to+\infty$. The infinite series (10) is absolutely convergent for all values of j. Table 2 gives the values of $\{H(i,j,k)\}_{k=1}^{16}$, $i=1,2,3,4$, and $G(j,k)$ for $j=5$. The sequences $\{H(i,j,k)\}_{k=1}^{\infty}$ appear as periodic differences, as defined in (16) and (17).

In the next section, a brief introduction to fractals and fractal dimension is given along with several examples of fractals. A fractal is presented with counting features depending on the Fibonacci numbers.

TABLE 2. $\{G(5,k),\ H(i,j,k)\}_{k=1}^{16}$ **for** $i = 1, 2, 3, 4$

$i/j/k$																
$k=$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$G(5,k)$	1	1	2	4	8	16	31	61	120	236	464	912	1793	3525	6930	13624
$i=1$	1	3	7	14	28	56	111	219	431	848	1668	3280	6449	12679	24927	49006
2	1	3	6	12	25	50	99	195	384	756	1487	2924	5749	11303	22222	43678
3	1	2	4	9	19	38	75	148	292	575	1131	2224	4373	8598	16894	33223
4	1	3	7	14	28	56	111	219	431	848	1668	3280	6449	12679	24927	49006

3. FIBONACCI NUMBERS AND FRACTALS

By definition, a fractal is a self-similar (self-affine) structure such that the topological dimension is strictly less than the Hausdorff-Besicovitch dimension (see Mandelbrot [12]). The topological "covering" dimension D_T or a set X has the property that any open covering of X has an open refinement with at most $D_T + 1$ open sets intersecting (see Hastings & Sugihara [9]). $D_T = 2$ in $\Re^2$, since any open covering has a refinement with at most three open sets intersecting.

Another important concept in fractals is the Box dimension

$$D = \lim_{r \to 0} \frac{-\log N(r)}{\log r} \tag{27}$$

where, for each $r > 0$, $N(r)$ is the smallest number of open balls having radius r which also cover X (see [9]). D is also denoted as the Hausdorff dimension when the dimensions are equal, including the fractal in the present paper. The value of D is computed for simple geometrical objects using the concept of scale factor and scaling dimension. Suppose X is reconstructed into n scaled copies of itself, each diminished in size by a factor k. Then

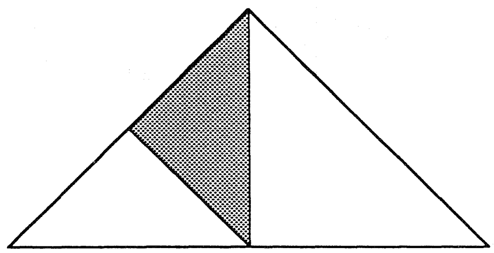
$$D = \frac{\log n}{\log 1/k}. \tag{28}$$

In certain fractals the scaling, Hausdorff, and Box dimension are all equal, including the fractals in the present work, since the Hausdorff dimension of a self-similar set with scaling ratio $1/k$ satisfies (28) also (see Crownover [5]).

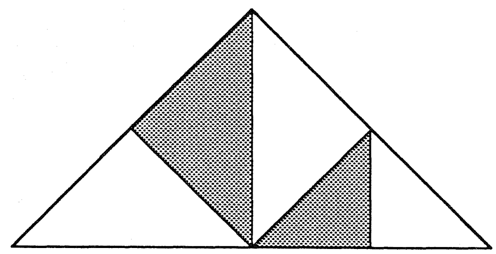
Fractals are generated mathematically and have a geometric structure in Euclidean space. They are used as mathematical models for natural objects such as length of shorelines, leaf or fern patterns, Brownian motion, chaos, cause and effect such as minimization of energy to create fractal-like mud-flats, more exotically, minimization of scalar fields in the self-reproducing inflationary universe. The artist M. C. Escher was a precursor to many geometric ideas, having created drawings of self-similar structures (see *Scientific American* [16]). Fractals have also been studied by Pietgen, Jürgens, and Saupe (see [14]) who give an interesting introduction to the subject.

The concept of a self-similar (self-affine) structure is intrinsic to a fractal, although not all self-similar structures are fractals, i.e., continually subdividing a square into four sub-squares does not create a fractal because $D = D_T = \log 2^{2n} / \log 2^n = 2$.

The Cantor set is defined by removing the middle third of a given set of intervals, starting with the unit interval. The Cantor set has the cardinality of the unit interval although it is a totally disconnected set. In the present work a self-affine, two-dimensional structure is created by beginning with a right triangle and then orthogonally projecting onto the sides in clockwise direction [Figs. 1(a), 1(b), 2(a), and 2(b)].

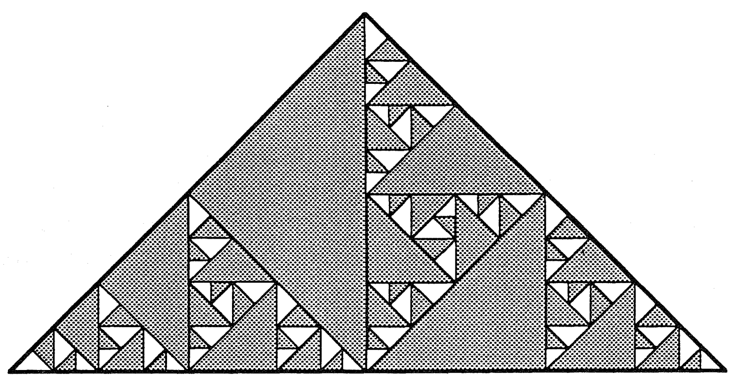


(a) Fractal Generation, Level 1, $j = 2$

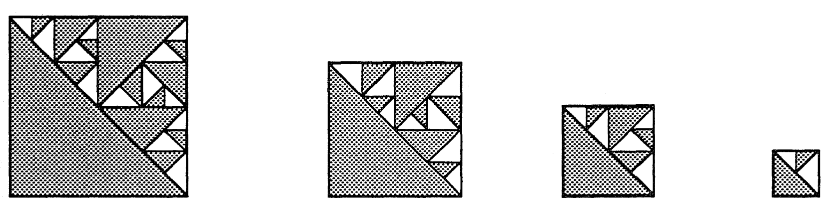


(b) Fractal Generation, Level 2, $j = 2$

FIGURE 1



(a) Fractal Generation, Level 8, $j = 2$



(b) Self-Similar Pieces

FIGURE 2

In the present paper the geometrical meaning of "orthogonally project" (on the sides of an isosceles right triangle) is as follows: begin with the vertex of the right angle and draw a line that is orthogonal to and meets the opposite side at the midpoint. Proceeding clockwise, from the midpoint of this side draw another line orthogonally to the midpoint of the opposite side. The new boundaries form a right triangle that is similar to the original triangle but rotated 90 degrees counterclockwise. Likewise, two other similar triangles are formed, one on either side of the new triangle. It is left only to decide which triangles the process is applied to at every stage.

To give an example: begin with a right triangle with vertices at $(0, 1)$, $(-1, 0)$, $(1, 0)$. Let this be the 0^{th} stage. At stage $k = 1$, orthogonally project onto the sides of the triangle in a clockwise direction. This forms a triangle (grey-shaded) and two new (unshaded) triangles. Continue this process by orthogonally projecting onto the *largest* of the unshaded triangles at level k, $k = 1, 2, \ldots$.

Lemma 7: This process forms a Fibonacci sequence such that the total number of unshaded triangles at level k is F_{k+2} of which F_{k+1} are largest. The total number of shaded triangles at stage k is $F_{k+2} - 1$. The respective side lengths of the largest unshaded triangles is reduced in side length for consecutive stages by scale factor $1/\sqrt{2}$.

Proof: By inspection of Figure 1(a), the induction hypothesis is true for $k = 1$, since $F_3 = 2$. At level k, there are F_{k+2} unshaded triangles of which F_{k+1} are largest and scaled in length size by a factor of $1/\sqrt{2}$ with respect to stage $k-1$. Projecting on the F_{k+1} largest triangles results in $2F_{k+1} + F_k = F_{k+3}$ unshaded triangles of which $F_k + F_{k+1} = F_{k+2}$ are largest, since the scale factor of the largest to the smallest at any stage is $1/\sqrt{2}$. Thus, the small unshaded triangles at stage k become some of the large unshaded triangles at stage $k+1$. That the number of shaded triangles is $F_{k+2} - 1$ follows by induction also, since at level $k+1$ the number of old and new shaded triangles is $(F_{k+2} - 1) + F_{k+1} = F_{k+3} - 1$.

The large unshaded triangles in Figures 1 and 2 and described in Lemma 7 are generated by an affine transformation of the form

$$\mathcal{T}(x, y) = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos\frac{3\pi}{4} & -\sin\frac{3\pi}{4} \\ \sin\frac{3\pi}{4} & \cos\frac{3\pi}{4} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a_{k,i} \\ b_{k,i} \end{bmatrix},$$

where k is the level of the projection and the integer i depends on k. Determining the precise values for $a_{k,i}$ and $b_{k,i}$ are not considered in the present paper. The small unshaded triangles in Figures 1 and 2 and described in Lemma 7 are generated by the following affine transformation:

$$\mathcal{U}(x, y) = \frac{1}{2}\begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c_{k,i} \\ d_{k,i} \end{bmatrix},$$

for real numbers $c_{k,i}$, $d_{k,i}$, $k = 1, 2, 3, \ldots$. For example, $a_{1,1} = b_{1,1} = 1/2$, $c_{1,1} = -1/2$, $d_{1,1} = 0$.

Construct a compact set E_2 as follows: at each stage k, for a projection on a given triangle, delete the shaded triangle, leaving two open unshaded triangles and their boundaries. Let $\mathcal{W}_k$ denote the union of all of the unshaded triangles with their boundaries at level k. Then set $E_2 = \bigcap_{k=1}^{+\infty} \mathcal{W}_k$. Clearly, E_2 is self-similar by construction, that is, E_2 is scale invariant and has topological dimension 1, i.e., no open subsets of $\mathfrak{R}^2$. Let L_{k+1} denote the perimeters of the large unshaded triangles in Lemma 7 corresponding to level k, which have hypotenuse length $1/\sqrt{2}^{k-2}$ and base=height length $1/\sqrt{2}^{k-1}$. It follows that $L_{k+1} = 2/\sqrt{2}^{k-1} + 1/\sqrt{2}^{k-2}$. Hence, observing that $\mathcal{W}_k \subseteq \mathcal{W}_{k-1}$, $\forall k \geq 2$, the total length, denoted L, of the set E_2 is given by

Lemma 8: $L = \lim_{k\to+\infty} (F_{k+1}L_{k+1} + F_k L_k) = +\infty$.

Proof: The proof follows from Theorem 1 and the fact that $\sqrt{2} < x_j$, $\forall j \geq 2$.

It is noted that Figure 2(a) contains eight basic shapes, a right, isosceles triangle T_0, Figure 4 rotated counterclockwise through 135°. The shapes are generated with the affine transformations $\mathcal{T}$ and $\mathcal{U}$.

Theorem 3: The compact set E_2 is a self-similar set with Box=scaling=Hausdorff (dimension) $D = 1.38848$.

Proof: Such sets are normally called fractals in the literature. The Hausdorff and Box dimensions both equal the scaling dimension, since E_2 is self-similar with two scaling ratios, $1/\sqrt{2}$ and $1/2$, as observed by the geometry of Figure 2(a). This is also evident in the transformations $\mathcal{T}(x, y)$ and $\mathcal{U}(x, y)$. By Lemma 7, there are F_k large triangles at level $k-1$, and the k^{th} Fibonacci number is almost linearly proportional to x_2^k for large k so that the scaling dimension

$$D = \lim_{k\to+\infty} \frac{\log cx_2^k}{\log\sqrt{2}^k} = \lim_{k\to+\infty} \frac{\log c + k\log x_2}{k\log\sqrt{2}}$$

$$= \frac{2\log x_2}{\log 2} = \frac{2(\log(1+\sqrt{5}) - \log 2)}{\log 2} = 1.38848.$$

For completeness, D is calculated by the Box Counting Theorem (see Barnsley [3]).

Theorem 4 (The Box Counting Theorem): Cover $\mathfrak{R}^2$ by boxes of size Cr^n, $C > 0$. $0 < r < 1$, where C and r are fixed real numbers. Let $\mathcal{N}_n$ denote the number of boxes of side length Cr^n that intersect any compact set $\mathcal{H} \subseteq \mathfrak{R}^2$. Then $\mathcal{H}$ has fractal dimension

$$D = \lim_{n\to+\infty} \frac{\log \mathcal{N}_n}{\log Cr^n}.$$

By inspection of Figure 2(a) and generalizing to all values of k, one finds that

Lemma 9: For $k = 1, 2, 3, \ldots, F_{k+4}$ squares of side length $1/2\sqrt{2}^{k-1}$ cover E_2.

Proof: To prove this lemma, we observe several facts: the triangles are all oriented with respect to the x–y plane so that either the sides or diagonals of any triangle (shaded or unshaded) are parallel to the y-axis. Hence, any covering or tiling of E_2 or a subset of E_2. can be done in two ways, which is countable if the covering boxes are aligned with the boundaries of the triangles. A right isosceles triangle of hypotenuse length $1/\sqrt{2}^{k-2}$ can be covered by two squares of side length $1/2\sqrt{2}^{k-2}$ or three squares of side length $1/2\sqrt{2}^{k-1}$ (see Fig. 3). By inspection of Figure 1(b), the two large unshaded triangles of hypotenuse length 1 can be covered by three boxes of side length $1/2\sqrt{2}$. The small unshaded triangle of hypotenuse length $1/\sqrt{2}$ can be covered by two boxes of the same side length ($1/2\sqrt{2}$). The total number of boxes is 8. For this value of k, the boxes can be oriented 45°, with diagonals parallel to the y-axis. Proceeding inductively, assume that F_{k+1} large and F_k small unshaded triangles of hypotenuse lengths $1/\sqrt{2}^{k-2}$ and $1/\sqrt{2}^{k-1}$, respectively, are generated that can be covered by $3F_{k+1}+2F_k=F_{k+4}$ squares of side length $1/2\sqrt{2}^{k-1}$. In applying the inductive hypothesis to $k+1$, we apply the identity $3F_{k+2}+2F_{k+1}=F_{k+5}$, the new unshaded triangles have hypotenuse length $1/\sqrt{2}^{k-1}$, $1/\sqrt{2}^{k}$, and the fact that $E_2 \subseteq \mathcal{W}_k$, $\forall k \geq 1$. There is no overlap of covering boxes on adjacent unshaded triangles, since opposite to the hypotenuse of any unshaded triangle is the boundary or a shaded triangle of equal or greater area. The fractal dimension is given by the Box Counting Theorem:

$$D=\lim_{k\to+\infty}\frac{\log F_{k+4}}{\log 2\sqrt{2}^{k-1}}=1.38848.$$

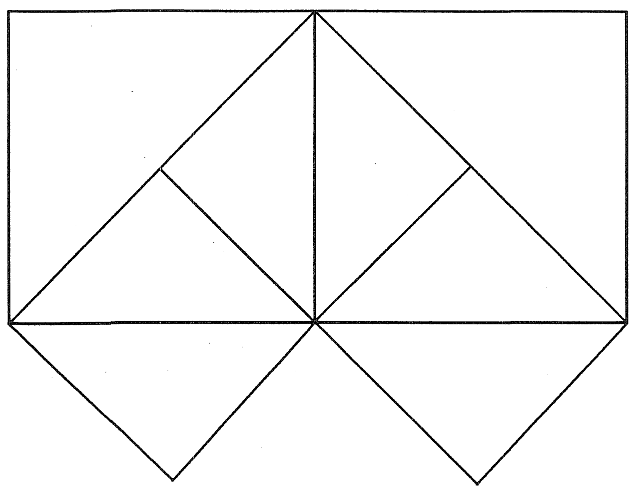


FIGURE 3. Covering the Right Isosceles Triangle by Squares

In the next section, triangles T_{j-2}, $j\geq 2$, are defined in the x–y plane. A theorem is given related to tiling $\bigcup_{i=0}^{j-2} T_i$, $j\geq 2$, with the triangles of the tiling enumerated by a Fibonacci j-sequence. E_j, $j\geq 2$, is characterized precisely, in terms of the union of a set of points that is contained in the set $T_0\cup T_1\cup\cdots\cup T_{j-2}=\bigcup_{i=0}^{j-2} T_i$. For a particular tiling of $\bigcup_{i=0}^{j-2} T_i$, it is shown that E_j is compact. In this case, the geometric object E_2 is translated, contracted in size, and rotated to create sets E_j, $j\geq 3$.

4. SETS E_j WITH FRACTAL DIMENSION

Consider the line $y=x+1$ and the ordinates $\{-1,1,3,7,\ldots,2^n-1,\ldots\}$ (Fig. 4). Denote by T_1, $T_3,\ldots,T_{2n-1}$ the triangles with boundaries formed by the set of vertices given, respectively, by

$$\{\{(1,0),(1,2),(0,1)\},\ \{(3,0),(3,4),(1,2)\},\ \{(7,0),(7,8),(3,4)\},\ldots,$$
$$\{(2^n-1,0),(2^n-1,2^n),(2^{n-1}-1,2^{n-1})\},\ldots\}.$$

Similarly, denote by $T_0, T_2, \ldots, T_{2n}$ the interlocking triangles with boundaries formed by the set of vertices given, respectively, by

$$\{\{(-1,0),(0,1),(1,0)\},\ \{(1,0),(1,2),(3,0)\},\ \{(3,0),(3,4),(7,0)\},\ldots,\\ \{(2^n-1,0),(2^n-1,2^n),(2^{n+1}-1,0)\},\ldots\}.$$

Note that T_0 does not follow the pattern given by the general triple of vertices.

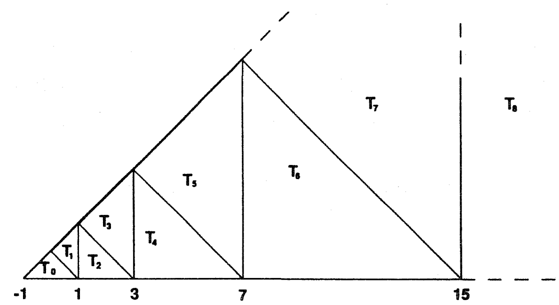


FIGURE 4. Fractal Boundaries

It follows that (if $\cup$ denotes the union of geometric objects) $T_0 \cup T_1$ is a reflection of T_0 about the line $y = 1 - x$, union with T_0; $T_0 \cup T_1 \cup T_2$ is a reflection of $T_0 \cup T_1$ about $x = 1$, union with $T_0 \cup T_1$; recursively, $T_0 \cup T_1 \cup \cdots \cup T_{2n-1}$ is a reflection of $T_0 \cup T_1 \cup \cdots \cup T_{2n-2}$ about the line $y = 2^n - 1 - x$, union with $T_0 \cup T_1 \cup \cdots \cup T_{2n-2}$; $T_0 \cup T_1 \cup \cdots \cup T_{2n}$ is a reflection of $T_0 \cup T_1 \cup \cdots \cup T_{2n-1}$ about the line $x = 2^n - 1$, union with $T_0 \cup T_1 \cup \cdots \cup T_{2n-1}$.

Theorem 5: A right isosceles triangle of area 2^{j-2} can be subdivided into (tiled by) similar triangles enumerated by the Fibonacci j-sequences $G(j, k)$, $\forall j \geq 2$. The total number of triangles of a given area forms a sequence $1, 1, 2, \ldots, G(j, k), \ldots$. The numbers in the sequence correspond with the number of similar triangles of area, respectively, 1/4, 1/8, ..., $G(j,k)/2^{k+1}, \ldots$.

Proof: Consider a right isosceles triangle of area 2^{j-2}, $j \geq 2$, for example $\cup_{i=0}^{j-2} T_i$, which can be subdivided into 2^j congruent subtriangles each having area 1/4. Figure 5 is a tiling of T_0 with 16 subtriangles each having area 1/16.

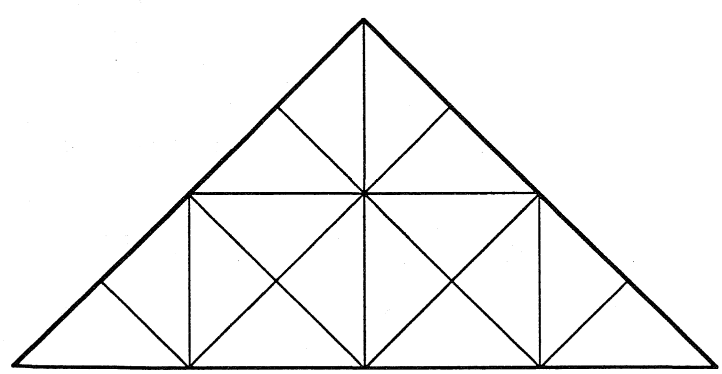


FIGURE 5. A Tiling of T_0

Construct another tiling of $\bigcup_{i=0}^{j-2} T_i$ as follows. The first element or triangle of the tiling consists of any one, $G(j,1)$, of the 2^j subtriangles. Then subdivide the remaining 2^j-1 triangles into $2^{j+1}-2$ similar subtriangles of area 1/8, by bisecting the right angle of each triangle. The second element or triangle of the tiling consists of any one, $G(j,2)$, of the $2^{j+1}-2$ subtriangles. Then subdivide the remaining $2^{j+1}-2-1$ into $2^{j+2}-4-2$ subtriangles each of area 1/16. Continue this procedure so that there are

Lemma 10:

$$G(j, j+k) = 2^{j+k-2} - 2^{k-2}G(j,1) - \cdots - 2G(j,k-2) - G(j,k-1) \tag{29}$$

subtriangles to be subdivided into triangles of area $1/2^{k+1}$, $k \geq 2$.

Proof: The proof follows by induction on j and k. For example,

$$\begin{aligned} 773 = G(4,12) = 2^{10} - 2^6G(4,1) - 2^5G(4,2) - 2^4G(4,3) \\ - 2^3G(4,4) - 2^2G(4,5) - 2^1G(4,6) - G(4,7). \end{aligned}$$

Hence, we see that the number of unchosen triangles is a Fibonacci j-sequence. From (29) and Lemma 5, we find that

$$\frac{G(j,j+k)}{2^{j+k-2}} = 1 - \sum_{i=1}^{k-1} \frac{G(j,i)}{2^{j+i-1}} \to 0,\ k \to +\infty,$$

which simply states that all of the area of the triangle $\bigcup_{i=0}^{j-2} T_i$ is tiled by this procedure.

This concept can be illustrated more formally in set-theoretic language. it is shown below that in the limit this procedure gives a tiled area equal to the area of the triangle $\bigcup_{i=0}^{j-2} T_i$. However, it is not clear that $\bigcup_{i=0}^{j-2} T_i$ is the union of all of these tiles. For example, when constructing the standard middle 1/3's Cantor set, the interval $[0,1]$ is not equal to the union of the middle 1/3's that are removed.

For given j, and $k \geq 1$, denote the set $\mathcal{F}_{j,k} = \{f(i, G(j,j)) : i = 1, \ldots, G(j,k)\}$ having as elements the $G(j,k)$ congruent subtriangles described by the k^{th} step of the procedure above. By construction, for given j, and each $k \geq 1$, the triangles $f(i, G(j,k))$ are pair-wise disjoint except for boundaries. Moreover, for each $j \geq 2$,

$$\begin{aligned} \text{area}\{\textstyle\bigcup_{k=1}^{+\infty} \mathcal{F}_{j,k}\} &= \text{area}\{\textstyle\bigcup_{k=1}^{+\infty} \bigcup_{i=1}^{G(j,k)} f(i, G(j,k))\} \\ &= \sum_{k=1}^{+\infty} \sum_{i=1}^{G(j,k)} \text{area}\{f(i, G(j,k))\} = 2^{j-2} = \sum_{k=1}^{+\infty} \frac{G(j,k)}{2^{k+1}}. \end{aligned}$$

This completes the proof of the theorem. It is observed that the theorem may be generalized by replacing $G(j,k)$ by an increasing sequence of positive integers $n(j,k)$ with the property that

$$\sum_{k=1}^{+\infty} \frac{n(j,k)}{2^{k+1}} = 2^{j-2},\ 0 < n(j,k) < 2^{j+k-1},\ k \geq 1.$$

For convenience in what follows, take the triangles $f(i, G(j,k))$ as open triangles, without boundary, thus $\text{interior}(f(i, G(j,k))) = f(i, G(j,k))$. Even though $\bigcup_{k=1}^{+\infty} \mathcal{F}_{j,k}$ is not necessarily the same set as $\bigcup_{i=0}^{j-2} T_i$, we have the following, where an overbar represents closure of a set.

Lemma 11: $\overline{\bigcup_{k=1}^{+\infty}\mathcal{F}_{j,k}} = \overline{\bigcup_{k=1}^{+\infty}\overline{\mathcal{F}_{j,k}}} = T_0 \cup T_1 \cup \cdots \cup T_{j-2} = \bigcup_{i=0}^{j-2} T_i$.

Proof: We denote $\bigcup_{i=0}^{j-2} T_i$ as right triangles previously defined together with the boundaries. Let $x \in \bigcup_{i=0}^{j-2} T_i$, then there is a sequence $\{x_k\} \in \bigcup_{k=1}^{+\infty}\mathcal{F}_{j,k}$ such that $x_k \neq x, |x-x_k| \to 0$ as $k \to +\infty$. Otherwise, there is $\varepsilon > 0$ such that

$$|x - x_\alpha| > \varepsilon,\ \forall x_\alpha \in \bigcup_{k=1}^{+\infty}\mathcal{F}_{j,k},$$

where α is the index of an uncountable set containing any possible sequence. Hence,

$$2^{j-2} = \text{area}\left\{\bigcup_{k=1}^{+\infty}\mathcal{F}_{j,k}\right\} \leq 2^{j-2} - \pi\varepsilon^2\delta,$$

where δ is the proportion of the ε-disk that intersects the triangle $T_0 \cup T_1 \cup \cdots \cup T_{j-2}$, a contradiction, so $x \in \overline{\bigcup_{k=1}^{+\infty}\mathcal{F}_{j,k}}$, i.e., x is an accumulation point of $\bigcup_{k=1}^{+\infty}\mathcal{F}_{j,k}$ (see Apostol [2]). Each x is arbitrary, which shows that $\bigcup_{i=0}^{j-2} T_i \subseteq \overline{\bigcup_{k=1}^{+\infty}\mathcal{F}_{j,k}}$. Equality holds, since the opposite set inclusion is true, that is, $\overline{f(i, G(j,k))} \subseteq \bigcup_{i=0}^{j-2} T_i, \forall i, k$. By a similar argument, $\overline{\bigcup_{k=1}^{+\infty}\overline{\mathcal{F}_{j,k}}} = \bigcup_{i=0}^{j-2} T_i$. The set $\bigcup_{i=0}^{j-2} T_i \setminus \bigcup_{k=1}^{+\infty}\mathcal{F}_{j,k}$ consists of points and straight line segments, i.e., contains the union of the boundary lines of the triangles $f(i, G(j,k))$ and $\bigcup_{i=0}^{j-2} T_i$. We note that $\bigcup_{i=0}^{j-2} T_i$ is closed and $\bigcup_{k=1}^{+\infty}\mathcal{F}_{j,k}$ is open, so that $\bigcup_{i=0}^{j-2} T_i \setminus \bigcup_{k=1}^{+\infty}\mathcal{F}_{j,k}$ is closed and bounded and, hence, compact.

Lemma 12: There is a "tiling" $\{f(i, G(2,k))\}$; $i = 1, \ldots, G(2,k)$; $k = 1, 2, \ldots$ of T_0 so that $E_2 \subseteq T_0 \setminus \bigcup_{k=1}^{+\infty}\mathcal{F}_{2,k}$, where the "tiling" has the same area as T_0 but is not necessarily equal to T_0.

Proof: To prove this, we let the open shaded triangles in the generation of E_2 be denoted by the triangles $f(i, G(2,k))$ which, by Theorem 5, tile T_0 and, hence, have the same area as T_0. To prove the first part of the lemma, it follows that if $x \in E_2$ then $x \notin f(i, G(2,k))$ for any i, k; thus, $x \notin \bigcup_{k=1}^{+\infty}\mathcal{F}_{2,k}$, and so $x \in T_0 \setminus \bigcup_{k=1}^{+\infty}\mathcal{F}_{2,k}$ since $x \in T_0$.

We note that $x \in E_2$ is not necessarily on a straight line segment or even a vertex of a triangle in $T_0 \setminus \bigcup_{k=1}^{+\infty}\mathcal{F}_{2,k}$. Analogously, the endpoints of the deleted intervals in the construction of the Cantor set are not the entire Cantor set.

Define $V\{T\}$ to be the set of vertices of a triangle denoted T. Then we find that

Theorem 6: $E_2 \supseteq \bigcup_{k=1}^{+\infty}\bigcup_{i=1}^{G(2,k)} V\{f(i, G(2,k))\}$.

Proof: That E_2 contains this set follows by the nature of the construction of E_2 and noting that two new vertices are added at level k, for each large unshaded triangle, which are the midpoint of the hypotenuse and the midpoint of an adjacent side. It is clear that no vertices are deleted once accumulated in E_2 by this process.

E_2 also has the property that

Corollary 2: Each point x in E_2 is the accumulation point of some countable sequence $\{x_k\}$ in E_2, such that $x_k \neq x$, $|x - x_k| \to 0$, $k \to +\infty$. That is, there are no isolated points in E_2.

Proof: Any vertex of $f(i, G(2, k))$ for some i, k is always a vertex of successively smaller triangles $f(i, G(2, k_0))$, $k_0 > k$, having vertices belonging to E_2 and limiting side lengths tending to 0 as $k \to +\infty$.

Example of a set E_j with dimension $2\log x_j / \log 2$

Define, for $j > 2$, the set $\widetilde{E}_j = \bigcup_{k=1}^{+\infty} \bigcup_{i=1}^{G(j,k)} V\{f(i, G(j, k))\}$ for an arbitrary tiling of $\bigcup_{i=0}^{j-2} T_i$. In words, this is the set of all possible vertices of the tiled triangles $f(i, G(j, k))$. This definition applies equally well to $j = 2$. For the sake of convenience, include $j = 2$ in the following analysis.

$\widetilde{E}_j$ is not necessarily closed, but is bounded. Let $\{x\}$ denote the set of accumulation points of $\widetilde{E}_j$ that lie in $\bigcup_{i=0}^{j-2} T_i \setminus \bigcup_{k=1}^{+\infty} \mathcal{F}_{j,k}$. Define $E_j = \widetilde{E}_j \cup \{x\}$ so that E_j contains all its accumulation points and is closed and bounded, so that $E_j \subseteq \Re^2$ as in Theorem 4.

To calculate the Box dimension, we observe that $(G(j, k))$ triangles $f(i, G(j, k))$, $i = 1, \ldots, G(j, k)$ are tiled at each level k. By induction, it can be shown that the number of ways to position the $G(j, k)$ unshaded triangles is $2G(j, k + j) = G(j, k) + G(j, k + j + 1)$, $k \geq 1$, as in Lemma 10 (except $k \geq 2$). We have

Theorem 7: $\forall j \geq 2$, E_j has Box dimension $= 2\log x_j / \log 2$.

Proof: For each k, the number of unshaded triangles forms twice a Fibonacci j-sequence, $2G(j, k + j)$, $\forall k \geq 1$ having hypotenuse length $1/\sqrt{2}^{k-1}$, each of which by Lemma 9 can be covered by two or three squares of side length $1/2\sqrt{2}^{k-1}$ or $1/2\sqrt{2}^{k}$, respectively. If we consider the latter, then at least one square is nonintersecting, except for boundaries, with other triangles. Hence, squares that overlap on different triangles cannot exceed $4G(j, k + j)$. Thus, it follows that the number of covering squares of size $1/2\sqrt{2}^{k}$ is at least $2G(j, k + j)$ and at most $6G(j, k + j)$, that is, the number of squares of side length $1/2\sqrt{2}^{k}$ that intersect E_j is bounded between the two scaled multiples of $G(j, k + j)$. By taking the limit as in Lemma 9 and applying a sandwich technique and Theorem 1, one obtains

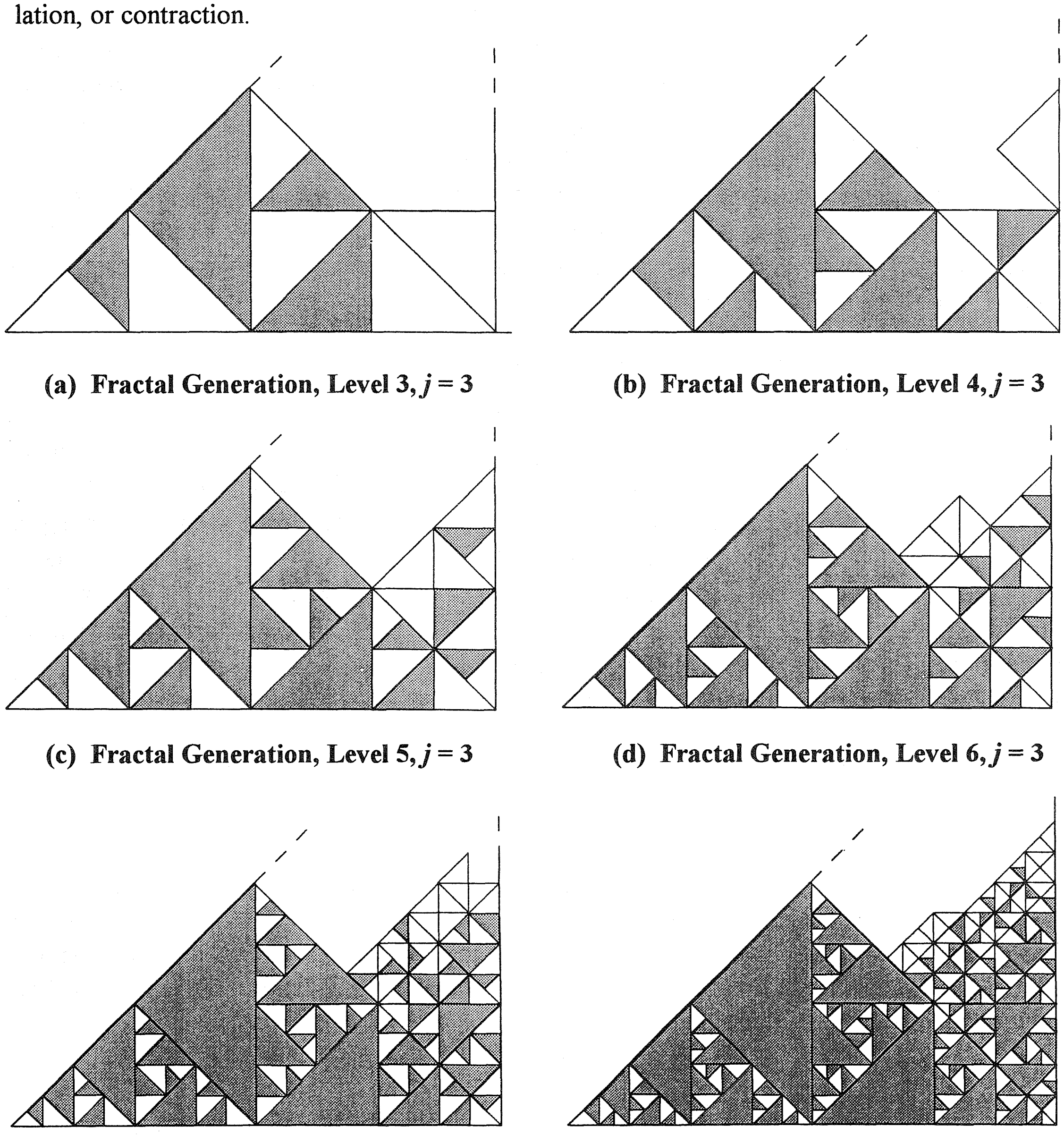
$$\frac{2\log x_j}{\log 2} = \lim_{k \to +\infty} \frac{\log 2G(j, k + j)}{\log 2\sqrt{2}^{k}} \leq D \leq \lim_{k \to +\infty} \frac{\log 6G(j, k + j)}{\log 2\sqrt{2}^{k}} = \frac{2\log x_j}{\log 2}.$$

This completes the proof of the theorem.

The construction of E_2 suggests that compact structures are formed by reflecting triangles of suitable size into an adjacent triangle. The affine transformations $\mathcal{T}, \mathcal{U}$ can be applied to form subsequent projections on the reflected triangles. E_j, $j \geq 3$, can be constructed with countably many copies of T_0, since the sequences $\{G(j, k)\}$ are the sum of countably many "shifted" sequences F_k. For example, for $j = 3$,

$$\begin{aligned}\{G(3, k)\} &= \{1, 1, 2, 4, 7, 13, 24, \ldots, G(3, k), \ldots\}\\ &= \{1, 1, 2, 3, 5, 8, 13, \ldots, F_k, \ldots\} + \{0, 0, 0, 1, 1, 2, 3, \ldots, F_k, \ldots\}\\ &\quad + \{0, 0, 0, 0, 1, 1, 2, \ldots, F_k, \ldots\} + \cdots + \{0, 0, \ldots, 0, 1, 1, 2, 3, 5, \ldots, F_k, \ldots\} + \cdots.\end{aligned}$$

Each of the sequences above corresponds to a scaled in size, tiled copy of T_0 which contains the fractal E_2 such that two copies of T_0 have the same area if the same number of zeros appear in the sequence. In the above, the sequences on the right-hand side correspond with triangles of area 1, 1/8, 1/16, ..., respectively, illustrated in Figures 6(a)-6(f), which show one of the possible ways to construct E_3, by reflecting unshaded triangles, projecting on these reflected triangles to create shaded triangles, and reflecting the new unshaded, smaller in area by 1/2, triangles so that their number forms a sequence $\{G(3, k)\}$. By Theorem 5, this process tiles $T_0 \cup T_1$ and generates E_3 with each point in E_3 on a translated, rotated, and contracted copy of E_2, and hence an accumulation point of E_3. We also note that the fractal dimension of E_2 is invariant under rotation, translation, or contraction.



(a) Fractal Generation, Level 3, $j = 3$

(b) Fractal Generation, Level 4, $j = 3$

(c) Fractal Generation, Level 5, $j = 3$

(d) Fractal Generation, Level 6, $j = 3$

(e) Fractal Generation, Level 7, $j = 3$

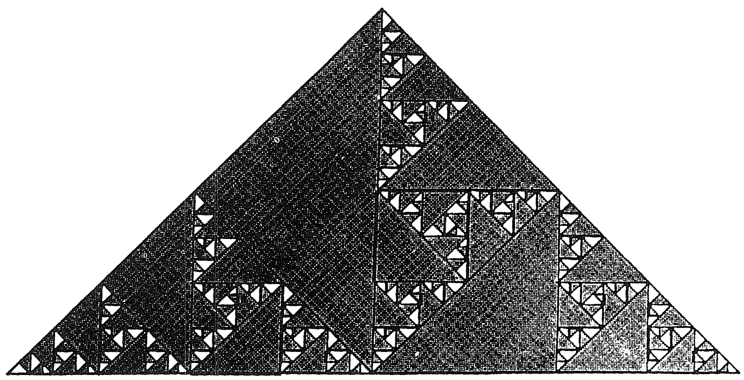
(f) Fractal Generation, Level 8, $j = 3$

FIGURE 6

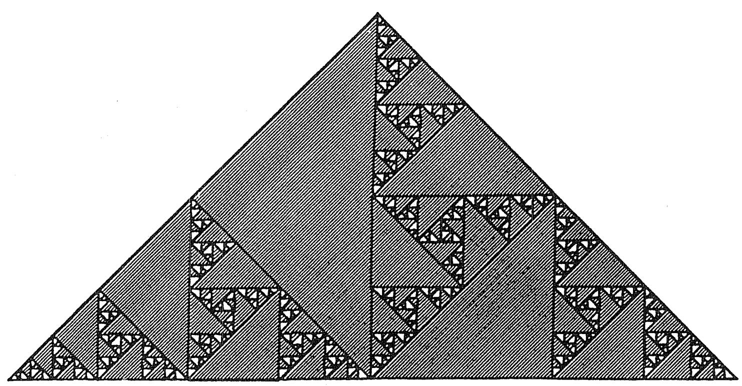
Figures 7(a) and 7(b) represent approximations of the fractal E_2 described in Section 3, and were constructed using *Logo* from a program by Robert G. Clason (see also [4]). The following dimensions were calculated on a *TI*–85©,

$$D_3 = 1.758292843,\ D_4 = 1.893554493,\ D_{10} = 1.998583839,\ D_{20} = 1.999998624.$$

As an interesting note, the projections on right triangles may be viewed as projections onto hyperplanes in $\mathcal{R}^2$. This idea was also investigated in a more general setting in the manuscript of Angelos et al. [1].



(a) Fractal Generation, Level 10, $j = 2$



(b) Fractal Generation, Level 11, $j = 2$

FIGURE 7

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SOME IDENTITIES INVOLVING THE FIBONACCI NUMBERS

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1. INTRODUCTION AND RESULTS

As usual, a second-order linear recurrence sequence $U=(U_n)$, $n=0,1,2,\dots$, is defined by integers a, b, U_0, U_1 and by the recursion

$$U_{n+2}=bU_{n+1}+aU_n \tag{1}$$

for $n\geq 0$. We suppose that $ab\neq 0$ and not both U_0 and U_1 are zero. If α and β denote the roots of the characteristic polynomial x^2-bx-a of the sequence U and α/β is not a root of unity, then U is called a nondegenerate sequence. In this case, as is well known (see [2]), the terms of the sequence U can be expressed as $U_n=p\alpha^n-q\beta^n$ for $n=0,1,2,\dots$, where

$$p=\frac{U_1-U_0\beta}{\alpha-\beta} \quad\text{and}\quad 1=\frac{U_1-U_0\alpha}{\alpha-\beta}.$$

If $U_0=0$, $a=b=U_1=1$, then the sequence U is called the Fibonacci sequence, and we shall denote it by $F=(F_n)$.

The various properties of second-order linear recurrence sequences were investigated by many authors. For example, Duncan [1] and Kuipers [3] proved that $(\log F_n)$ is uniformly distributed mod 1. Robbing [4] studied the Fibonacci numbers of the forms $px^2\pm 1$, $px^3\pm 1$, where p is a prime. The main purpose of this paper is to study how to calculate the summation of one class of second-order linear recurrence sequences, i.e.,

$$\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}U_{a_2}\dots U_{a_k} \tag{2}$$

where the summation is over all n-tuples with positive integer coordinates $(a_1,a_2,\dots,a_k)$ such that $a_1+a_2+\cdots+a_k=n$.

Regarding (2), it seems that it has not yet been studied; at least this author has not seen expressions like (2) before. The problem is interesting because it can help us to find some new convolution properties. In this paper we use the generating function of the sequence U and its derivative to study the evaluation of (2) and give an interesting identity for any fixed positive integer k. That is, we shall prove the following two propositions.

Proposition 1: Let $U=(U_n)$ be defined by (1). If $U_0=0$, then, for any positive integer $k\geq 2$, we have

$$\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}U_{a_2}\dots U_{a_k}=\frac{U_1^{k-1}}{(b^2+4a)^{k-1}(k-1)!}[g_{k-1}(n)U_{n-k+1}+h_{k-1}(n)U_{n-k}],$$

where $g_{k-1}(x)$ and $h_{k-1}(x)$ are two effectively computable polynomials of degree $k-1$, their coefficients depending only on a, b, and k.

Proposition 2: Under the condition of Proposition 1, we have the following identities:

(i) $\sum_{a+b=n} U_a U_b = \frac{U_1}{b^2+4a}[b(n-1)U_n + 2anU_{n-1}];$

(ii) $$\sum_{a+b+c=n} U_a U_b U_c = \frac{U_1^2}{2(b^2+4a)^2}\{[(b^3+4ab)n^2-(3b^3+6ab)n+(2b^3-4ab)]U_{n-1} + [(b^2a+4a^2)n^2-3b^2an+(2b^2a-4a^2)]U_{n-2}\};$$

(iii) $$\begin{aligned}\sum_{a+b+c+d=n} U_a U_b U_c U_d = \frac{U_1^3}{6(b^2+4a)^3}\{&[(b^5+7b^3a+12ba^2)n^3 \\ &-(6b^5+30b^3a+24ba^2)n^2+(11b^5+17b^3a-48ba^2)n \\ &-(6b^5-30b^3a-36ba^2)]U_{n-2}+[(b^4a+6b^2a^2+8a^3)n^3 \\ &-(6b^4a+24a^2b^2)n^2+(11b^4a+6b^2a^2-32a^3)n-(6b^4a-36a^2b^2)]U_{n-3}\}.\end{aligned}$$

Taking $U_1 = a = b = 1$, then $U_n = F_n$ is the Fibonacci sequence, i.e., $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, ... Thus, from Proposition 2, we obtain Corollaries 1 and 2.

Corollary 1: Let (F_n) be the Fibonacci sequence. Then we have:

(i) $\sum_{a+b=n} F_a F_b = \frac{1}{5}[(n-1)F_n + 2nF_{n-1}],\ n \geq 1;$

(ii) $\sum_{a+b+c=n} F_a F_b F_c = \frac{1}{50}[(5n^2-9n-2)F_{n-1}+(5n^2-3n-2)F_{n-2}],\ n \geq 2;$

(iii) $\sum_{a+b+c+d=n} F_a F_b F_c F_d = \frac{1}{150}[(4n^3-12n^2-4n+12)F_{n-2}+(3n^3-6n^2-3n+6)F_{n-3}],\ n \geq 3.$

Corollary 2: We have the following congruences:

(i) $(n-1)F_n + 2nF_{n-1} \equiv 0 \pmod 5,\ n \geq 1;$

(ii) $(5n^2-9n-2)F_{n-1}+(5n^2-3n-2)F_{n-2} \equiv 0 \pmod{50},\ n \geq 2;$

(iii) $(4n^3-12n^2-4n+12)F_{n-2}+(3n^3-6n^2-3n+6)F_{n-3} \equiv 0 \pmod{150},\ n \geq 3.$

2. PROOF OF THE PROPOSITIONS

In this section, we shall give the proof of the propositions. First, we recall some known results on the second-order linear recurrence sequences and prove two lemmas that will be used in the proof of the propositions.

Let $U = (U_n)$ be a nondegenerate second-order linear recurrence sequence defined by (1). If $U_0 = 0$, then the generating function of U is

$$G(x) = xF(x) = \frac{U_1 x}{1-bx-ax^2} = \sum_{n=0}^{\infty} U_n x^n, \tag{3}$$

where $U_n = G^{(n)}(0)/n!$ and $G^{(k)}(x)$ denotes the k^{th} derivative of $G(x)$.

For $F(x) = G(x)/x = \sum_{n=1}^{\infty} U_n x^{n-1}$, we have the following lemma.

Lemma 1: If $F(x)$ is defined by (3), then $F(x)$ satisfies:

(i) $F^2(x) = \frac{U_1}{b^2+4a}[F'(x)(b+2ax)+4aF(x)];$

(ii) $F^3(x) = \dfrac{U_1^2}{2(b^2+4a)^2}[F''(x)(b+2ax)^2 + 14aF'(x)(b+2ax) + 32a^2F(x)];$

(iii) $F^4(x) = \dfrac{U_1^3}{6(b^2+4a)^3}[F'''(x)(b+2ax)^3 + 30aF''(x)(b+2ax)^2$
$+ 228a^2F'(x)(b+2ax) + 384a^3F(x)].$

Proof: Using the definition of $F(x)$ and the derivative of the function $F(x)(b+2ax)$, we get

$$\begin{aligned}[F(x)(b+2ax)]' = F'(x)(b+2ax) + 2aF(x) &= \left[\frac{U_1(b+2ax)}{1-bx-ax^2}\right]' \\ &= \frac{U_1(b^2+2a+2abx+2a^2x^2)}{(1-bx-ax^2)^2} = \frac{U_1(b^2+4a)}{(1-bx-ax^2)^2} - \frac{2aU_1}{1-bx-ax^2} \\ &= \frac{b^2+4a}{U_1}F^2(x) = 2aF(x),\end{aligned}$$

so that

$$\frac{b^2+4a}{U_1}F^2(x) = F'(x)(b+2ax) + 4aF(x). \tag{4}$$

This gives the conclusion (i) of Lemma 1.

Differentiating in (4), we have

$$\frac{b^2+4a}{U_1}\cdot 2F(x)F'(x) = F''(x)(b+2ax) + 6aF'(x).$$

So

$$\frac{b^2+4a}{U_1}\cdot 2F(x)F'(x)(b+2ax) = F''(x)(b+2ax)^2 + 6aF'(x)(b+2ax).$$

Applying (4) again, we have

$$\frac{b^2+4a}{U_1}\cdot 2F(x)\left[\frac{b^2+4a}{U_1}F^2(x) - 4aF(x)\right] = F''(x)(b+2ax)^2 + 6aF'(x)(b+2ax).$$

Thus,

$$\begin{aligned}\frac{2(b^2+4a)^2}{U_1^2}F^3(x) &= F''(x)(b+2ax)^2 + 6aF'(x)(b+2ax) + \frac{8a(b^2+4a)}{U_1}F^2(x) \\ &= F''(x)(b+2ax)^2 + 14aF'(x)(b+2ax) + 32a^2F(x).\end{aligned} \tag{5}$$

Conclusion (ii) of Lemma 1 now follows from (5).

Similarly, differentiating in (5) and applying (5), we can also obtain

$$\begin{aligned}\frac{3!(b^2+4a)^3}{U_1^3}F^4(x) &= F'''(x)(b+2ax)^3 + 30aF''(x)(b+2ax)^2 \\ &\quad + 228a^2F'(x)(b+2ax) + 384a^3F(x).\end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2: Let $k \geq 2$ be an integer. Then there exist $k-1$ effectively computable positive integers $c_1, c_2, \ldots, c_{k-1}$ such that

$$\frac{(k-1)!(b^2+4a)^{k-1}}{U_1^{k-1}}F^k(x) = F^{(k-1)}(x)(b+2ax)^{k-1} + c_1 a F^{(k-2)}(x)(b+2ax)^{k-2} + \cdots + c_{k-2}a^{k-2}F'(x)(b+2ax) + c_{k-1}a^{k-1}F(x), \tag{6}$$

where $F^{(i)}(x)$ denotes the i^{th} derivative of $F(x)$.

Proof: This formula can be obtained via Lemma 1 and induction.

Now we complete the proof of the propositions. First, we prove Proposition 1. Equating the coefficients of x^{n-k} on both sides of (6), we obtain

$$\frac{(k-1)!(b^2+4a)^{k-1}}{U_1^{k-1}} \sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}U_{a_2}\ldots U_{a_k}$$
$$= \sum_{i=0}^{k-1} c_i a^i \sum_{j=0}^{k-1-i} \binom{k-1-i}{k-1-i-j} \cdot \frac{(n-i-j)!}{(n-k-j)!} b^{k-1-i-j}(2a)^j U_{n-1-i-j}.$$

Substituting $U_{n-m} = bU_{n-m-1} + aU_{n-m-2}$ $(1 \le m \le k-1)$ repeatedly in the above formula gives

$$\frac{(k-1)!(b^2+4a)^{k-1}}{U_1^{k-1}} \sum_{a_1+a_2+\cdots+a_k=n} U_{a_1}U_{a_2}\ldots U_{a_k} = g_{k-1}(n)U_{n-k+1} + h_{k-1}(n)U_{n-k},$$

where $g_{k-1}(x)$ and $h_{k-1}(x)$ are two effectively computable polynomials with their coefficients depending only on a, b and k. This completes the proof of Proposition 1.

To prove Proposition 2, comparing the coefficients of x^{n-2}, x^{n-3}, and x^{n-4} on both sides of Lemma 1, we get the following convolution product formulas:

$$\left(\frac{U_1}{b^2+4a}\right)^{-1} \cdot \left(\sum_{a+b=n} U_a U_b\right) = [b(n-1)U_n + 2anU_{n-1}]; \tag{7}$$

$$\left(\frac{U_1^2}{2(b^2+4a)^2}\right)^{-1} \cdot \left(\sum_{a+b+c=n} U_a U_b U_c\right) = b^2(n^2-3n+2)U_n + ab(4n^2-6n-4)U_{n-1} + 4a^2(n^2-1)U_{n-2}; \tag{8}$$

and

$$\left(\frac{U_1^3}{6(b^2+4a)^3}\right)^{-1} \cdot \left(\sum_{a+b+c+d=n} U_a U_b U_c U_d\right) = b^3(n^3-6n^2+11n-6)U_n + b^2a(6n^3-24n^2+6n+36)U_{n-1} + ba^2(12n^3-24n^2-48n+36)U_{n-2} + a^3(8n^3-32n)U_{n-3}. \tag{9}$$

Substituting $U_n = bU_{n-1} + aU_{n-2}$ in (8), we have

$$\sum_{a+b+c=n} U_a U_b U_c = \frac{U_1^2}{2(b^2+4a)^2}[((b^3+4ab)n^2 - (3b^3+6ab)n + (2b^3-4ab))U_{n-1} + ((b^2a+4a^2)n^2 - 3b^2an + (2b^2a-4a^2))U_{n-2}]. \tag{10}$$

Finally, substituting $U_{n-1} = bU_{n-2} + aU_{n-3}$ and $U_n = bU_{n-1} + aU_{n-2} = (b^2\_a)U_{n-2} + abU_{n-3}$ in (9), we get the identity

$$\sum_{a+b+c+d=n} U_a U_b U_c U_d = \frac{U_1^3}{6(b^2+4a)^3}[((b^5+7b^3a+12ba^2)n^3-(6b^5+30b^3a+24ba^2)n^2$$
$$+(11b^5+17b^3a-48ba^2)n-(6b^5-30b^3a-36ba^2))\,U_{n-2} \tag{11}$$
$$+((b^4a+6b^2a^2+8a^3)n^3-(6b^4a+24b^2a^2)n^2$$
$$+(11b^4a+6b^2a^2-32a^3)n-(6b^4a-36a^2b^2))\,U_{n-3}].$$

Proposition 2 now follows from (7), (10), and (11).

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The author expresses his gratitude to the anonymous referee for very helpful and detailed comments that improved the presentation of this paper.

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AMS Classification Numbers: 11B37, 11B39

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LETTER TO THE EDITOR

Dear Professor Bergum:

The Fibonacci Quarterly readers will be interested in yet another natural occurrence of the Golden Ratio. This occurrence is described in the current issue of *The College Mathematics Journal* (Vol. 28, No. 3, May 1997). On page 205, Peter Schumer (schumer@middlebury.edu) of Middlebury College in Middlebury VT provides an interesting variant on the classical problem of showing that the rectangle with fixed perimeter and maximum area is a square.

Schumer notes that texts often present this problem as the dilemma of a farmer who has a fixed length of fencing and wants to build the most efficient animal pen for grazing. It is a simple calculus problem. The problem is complicated somewhat when the farmer has a fixed length of fencing and is using one side of a barn for all or part of one side of the animal pen. Schumer provides a neat analysis of the optimum pen shape when the length of fencing is some multiple of the length of the barn side used.

When the length of fencing available is $\sqrt{5}$ times the length of the side of barn used, the optimum pen shape is a golden rectangle. This is a neat result, simply derived, of interest to *FQ* readers, and which I have not seen before.

Best regards,

Harvey J. Hindin
Vice-President, Emerging Technologies Group, Inc.

ON CERTAIN IDENTITIES INVOLVING FIBONACCI AND LUCAS NUMBERS

M. N. S. Swamy
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(Submitted January 1996)

In a recent article [3], Jennings established the following three theorems expressing $F_{(2q+1)n}$ as a polynomial in F_n as well as expressing F_{mn}/F_n as a polynomial in L_n, where F_n and L_n are, respectively, the n^{th} Fibonacci and Lucas numbers.

Theorem 1: $F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q+1}{q+k+1} 5^k \binom{q+k+1}{2k+1} F_n^{2k}, \quad n, q \ge 0.$

Theorem 2: $F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \binom{q+k}{2k} L_n^{2k}, \quad n, q \ge 0.$

Theorem 3: $F_{2qn} = F_n \sum_{k=1}^{q} (-1)^{(n+1)(q+k)} \binom{q+k-1}{2k-1} L_n^{2k-1}, \quad n \ge 0, q \ge 1.$

In a later article [2], Filipponi derived Theorem 1 very simply by letting $X = \alpha^n$, $Y = -\beta^n$, and $m = (2q+1)$ in Waring's formula, given by

$$X^m + Y^m = \sum_{k=0}^{[m/2[} (-1)^k \frac{m}{m-k} \binom{m-k}{k} (XY)^k (X+Y)^{m-2k}, \quad m \ge 1. \tag{1}$$

By letting $m = 2q$ in the above formula, he also established the following theorem, which expresses L_{2qn} as a polynomial in F_n.

Theorem 4: $L_{2qn} = \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q}{q+k} \binom{q+k}{2k} 5^k F_n^{2k}, \quad n, q \ge 0.$

In the same article, Filipponi derived another result by letting $X = \alpha^n$ and $Y = \beta^n$ in the identity given by (1). This result, which expresses L_{mn} in powers of L_n, is given by the following two theorems, wherein the notation of Jennings has been used:

Theorem 5: $L_{2qn} = \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \frac{2q}{q+k} \binom{q+k}{2k} L_n^{2k}, \quad n, q \ge 0.$

Theorem 6: $L_{(2q+1)n} = L_n \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \frac{2q+1}{q+k+1} \binom{q+k+1}{2k+1} L_n^{2k}, \quad n, q \ge 0.$

In this short article, we will first derive Theorems 2 and 3 of Jennings in a very simple manner by utilizing the following identity, which has been used by Carlitz in 1963 (see [1]) to obtain some results concerning certain Fibonacci arrays:

$$\frac{X^m - Y^m}{X - Y} = \sum_{k=0}^{[(m-1)/2]} (-1)^k \binom{m-k-1}{k} (XY)^k (X+Y)^{m-2k-1}, \quad m \geq 1. \tag{2}$$

Using this identity, we will establish two other theorems that express $L_{(2q+1)n} / L_n$ and F_{2qn} / L_n in powers of F_n.

Now, letting $X = \alpha^n$ and $Y = \beta^n$ in identity (2), we obtain $X + Y = \alpha^n + \beta^n = L_n$ and $XY = (\alpha\beta)^n = (-1)^n$. Thus, we have

$$\alpha^{mn} - \beta^{mn} = (\alpha^m - \beta^m) \sum_{k=0}^{[(m-1)/2]} (-1)^k \binom{m-k-1}{k} (-1)^{nk} L_n^{m-2k-1}, \quad n \geq 0, m \geq 1,$$

or

$$F_{mn} = F_n \sum_{k=0}^{[(m-1)/2]} (-1)^{(n+1)k} \binom{m-k-1}{k} L_n^{m-2k-1}, \quad n \geq 0, m \geq 1. \tag{3}$$

Setting $m = 2q + 1$,

$$F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{(n+1)k} \binom{2q-k}{k} L_n^{2q-2k}, \quad n, q \geq 0.$$

Changing k to $q - k$, we may rewrite the above as

$$F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \binom{q+k}{2k} L_n^{2k}, \quad n, q \geq 0,$$

which proves Theorem 2. Similarly, by setting $m = 2q$ in (3), we establish Theorem 3.

We now state and prove the following two new theorems.

Theorem 7: $F_{2qn} = F_n L_n \sum_{k=0}^{q-1} (-1)^{n(q+k+1)} \binom{q+k}{2k+1} 5^k F_n^{2k}, \quad n, q \geq 0.$

Proof: Let $X = \alpha^n$, $Y = -\beta^n$, and $m = 2q$ in (2). Then we have $X + Y = \alpha^n - \beta^n = \sqrt{5}\, F_n$, $XY = -(\alpha\beta)^n = (-1)^{n+1}$, and

$$\alpha^{2qn} - \beta^{2qn} = (\alpha^n + \beta^n) \sum_{k=0}^{q-1} (-1)^k \binom{2q-k-1}{k} (-1)^{(n+1)k} 5^{(2q-2k-1)/2} F_n^{2q-2k-1}, \quad n, q \geq 0.$$

Thus

$$F_{2qn} = L_n \sum_{k=0}^{q-1} (-1)^{nk} \binom{2q-k-1}{k} 5^{q-k-1} F_n^{2q-2k-1}, \quad n, q \geq 0.$$

Changing k to $q - 1 - k$, we may rewrite the above as

$$F_{2qn} = F_n L_n \sum_{k=0}^{q-1} (-1)^{n(q+k+1)} \binom{q+k}{2k+1} 5^k F_n^{2k}, \quad n, q \geq 0,$$

and hence the theorem.

Theorem 8: $L_{(2q+1)n} = L_n \sum_{k=0}^{q} (-1)^{n(q+k)} \binom{q+k}{2k} 5^k F_n^{2k}, \quad n, q \geq 0.$

This theorem may be established by letting $X=\alpha^n$, $Y=-\beta^n$, and $m=(2q+1)$ in (2), and following the same steps used to prove Theorem 7.

Finally, it may be mentioned that similar results can be established for the Pell and Pell-Lucas numbers P_n and Q_n using the identities given in (1) and (2).

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Announcement

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A COMPOSITE OF MORGAN-VOYCE GENERALIZATIONS

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1. RATIONALE

Two recent papers, [1] and [3], detailed properties of

(i) a generalization $\{P_n^{(r)}(x)\}$ of the familiar Morgan-Voyce polynomials $B_n(x)$ and $b_n(x)$, and

(ii) an associated set $\{Q_n^r(x)\}$ of generalized polynomials.

Here, we amalgamate these two sets of polynomials into one more embracing class of polynomials $\{R_n^{(r,u)}(x)\}$.

In fact,

$$R_n^{(r,1)}(x) = P_n^{(r)}(x) \tag{1.1}$$

and

$$R_n^{(r,2)}(x) = Q_n^{(r)}(x). \tag{1.2}$$

Hopefully, the reader will have access to [1], [2], and [3]. However, the following summary may be helpful for reference purposes (in our notation):

$$P_n^{(0)}(x) = b_{n+1}(x), \tag{1.3}$$

$$P_n^{(1)}(x) = B_{n+1}(x), \tag{1.4}$$

$$P_n^{(2)}(x) = c_{n+1}(x), \tag{1.5}$$

$$Q_n^{(0)}(x) = C_n(x), \tag{1.6}$$

where $C_n(x)$ and $c_{n+1}(x)$ are polynomials related to the Morgan-Voyce polynomials. It may be mentioned that the polynomial $C_n(x)$ has already been defined by Swamy in [4], where it has been used in the analysis of Ladder networks. Knowledge of the definitions of the Fibonacci polynomials $\{F_n(x)\}$ and the Lucas polynomials $\{L_n(x)\}$ is assumed. When $x = 1$, the Fibonacci numbers F_n and the Lucas numbers L_n emerge.

Only the skeletal structure of the simple deductions from the definitions (2.1) and (2.2) germane to [1] and [3] will be displayed. This procedure follows the patterns in [1] and [3].

For internal consistency in my papers, I shall interpret symbolism in [1] in the notation adopted in [2] and [3]. Throughout, $n \geq 0$ except for the explicitly stated value $n = -1$.

Much of the material and approach offered in this paper appears to be new.

2. OUTLINE OF BASIC PROPERTIES OF $\{r_n^{(r,u)}(x)\}$

Definition

Define

$$R_n^{(r,u)}(x) = (x+2)R_{n-1}^{(r,u)}(x) - R_{n-2}^{(r,u)}(x) \quad (n \geq 2), \tag{2.1}$$

with

$$R_0^{(r,u)}(x) = u, \quad R_1^{(r,u)}(x) = x + r + u, \tag{2.2}$$

where r, u are integers. Then

$$R_n^{(r,u)}(x) = \sum_{k=0}^{n} c_{n,k}^{(r,u)} x^k, \tag{2.2}$$

with

$$c_{n,n}^{(r,u)} = 1 \text{ if } k \geq 1. \tag{2.4}$$

Recurrences

Clearly, from (2.1), $c_{n,0}^{(r,u)} = R_n^{(r,u)}(0)$ satisfies the recurrence

$$c_{n,0}^{(r,u)} = 2c_{n-1,0}^{(r,u)} - c_{n-2,0}^{(r,u)} \quad (n \geq 2), \tag{2.5}$$

with

$$\left.\begin{aligned} c_{0,0}^{(r,u)} &= u \\ c_{1,0}^{(r,u)} &= r+u \end{aligned}\right\}, \tag{2.6}$$

whence

$$c_{n,0}^{(r,u)} = nr + u, \tag{2.7}$$

$$\left.\begin{aligned} c_{n,0}^{(0,u)} &= u \\ c_{n,0}^{(r,0)} &= nr \end{aligned}\right\}, \tag{2.8}$$

$$\left.\begin{aligned} c_{n,0}^{(1,u)} &= n+u \\ c_{n,0}^{(r,1)} &= nr+1 \end{aligned}\right\}. \tag{2.9}$$

Comparison of coefficients of x^k in (2.1) reveals the recurrence ($n \geq 2$, $k \geq 1$)

$$c_{n,k}^{(r,u)} = 2c_{n-1,k}^{(r,u)} - c_{n-2,k}^{(r,u)} + c_{n-1,k-1}^{(r,u)}. \tag{2.10}$$

The Coefficients $c_{n,k}^{(r,u)}$

Table 1 sets out some of the simplest of the coefficients $c_{n,k}^{(r,u)}$. For visual convenience in this table, we choose u to precede r.

From Table 1, [1], and [3], one may spot empirically the binomial formula

$$c_{n,k}^{(r,u)} = \binom{n+k-1}{2k-1} + r\binom{n+k}{2k+1} + u\binom{n+k-1}{2k} \tag{2.11}$$

$$= \binom{n+k}{2k} + r\binom{n+k}{2k+1} + (u-1)\binom{n+k-1}{2k}, \tag{2.12}$$

by Pascal's Theorem.

Multiply (2.12) throughout by x^k and sum. Accordingly,

Theorem 1: $R_n^{(r,u)}(x) = P_n^{(r)}(x) + (u-1)b_n(x)$.

Special cases:

$$R_n^{(0,1)}(x) = b_{n+1}(x) \quad \text{by (1.3), [2],} \tag{2.13}$$

$$R_n^{(1,1)}(x) = B_{n+1}(x) \quad \text{by (1.4), [2],} \tag{2.14}$$

$$R_n^{(2,1)}(x) = c_{n+1}(x) \quad \text{by (1.5), [2],} \tag{2.15}$$

$$R_n^{(0,2)}(x) = b_{n+1}(x) + b_n(x) = C_n(x) \quad \text{by [2].} \tag{2.16}$$

Furthermore,

$$R_n^{(0,0)}(x) = b_{n+1}(x) - b_n(x) = xB_n(x) \quad \text{by [2].} \tag{2.17}$$

TABLE 1. The Coefficients $c_{n,k}^{(r,u)}$

$n \backslash k$	0	1	2	3	4	5	6
0	u						
1	$u+r$	1					
2	$u+2r$	$2+u+r$	1				
3	$u+3r$	$3+3u+4r$	$4+u+r$	1			
4	$u+4r$	$4+6u+10r$	$10+5u+6r$	$6+u+r$	1		
5	$u+5r$	$5+10u+20+r$	$20+15u+21r$	$21+7u+8r$	$8+u+r$	1	
6	$u+6r$	$6+15u+35r$	$35+35u+56r$	$56+28u+36r$	$36+9u+10r$	$10+u+r$	1
...	...	...	...	...	...	...	...

3. FIBONACCI AND LUCAS NUMBERS

Substitute $x = 1$ in Theorem 1. Then, with $F_n(1) = F_n$ and $L_n(1) = L_n$,

$$\begin{aligned} R_n^{(r,u)}(1) &= F_{2n+1} - F_{2n-1} + rF_{2n} + uF_{2n-1} \\ &= (1+r)F_{2n} + uF_{2n-1}. \end{aligned} \tag{3.1}$$

For example, $R_4^{(r,u)}(1) = 21 + 21r + 13u = (1+r)F_8 + uF_7$, as may be verified quickly in Table 1.

Special cases:

$$R_n^{(0,1)}(1) = F_{2n+1}, \tag{3.2}$$

$$R_n^{(1,1)}(1) = F_{2n+2}, \tag{3.3}$$

$$R_n^{(2,1)}(1) = L_{2n+1}, \tag{3.4}$$

$$R_n^{(0,2)}(1) = L_{2n}. \tag{3.5}$$

Also,

$$R_n^{(0,0)}(1) = F_{2n}. \tag{3.6}$$

Relationships between the Fibonacci and Lucas numbers, and the Morgan-Voyce polynomials when $x = 1$, are specified in [2].

4. CHEBYSHEV POLYNOMIALS

Write

$$\frac{x+2}{2} = \cos t \quad (-4 < t < 0). \tag{4.1}$$

In [2], it is shown that

$$B_n(x) = U_n\left(\frac{x+2}{2}\right), \tag{4.2}$$

$$b_n(x) = U_n\left(\frac{x+2}{2}\right) - U_{n-1}\left(\frac{x+2}{2}\right), \tag{4.3}$$

$$c_n(x) = U_n\left(\frac{x+2}{2}\right) + U_{n-1}\left(\frac{x+2}{2}\right), \tag{4.4}$$

$$C_n(x) = 2T_n\left(\frac{x+2}{2}\right), \tag{4.5}$$

where $U_n(x)$ and $T_n(x)$ are Chebyshev polynomials.

Empirically, (4.2)-(4.5), taken with (2.13)-(2.16), suggest a more general formula connecting $R_n^{(r,u)}(x)$ with the Chebyshev polynomials.

Theorem 2: $R_n^{(r,u)}(x) = U_{n+1}\left(\frac{x+2}{2}\right) + (r+u-2)U_n\left(\frac{x+2}{2}\right) - (u-1)U_n\left(\frac{x+2}{2}\right)$.

Thus, in particular

$$R_n^{(r,1)}(x) = U_{n+1}\left(\frac{x+2}{2}\right) + (r-1)U_n\left(\frac{x+2}{2}\right) \tag{4.6}$$

and

$$R_n^{(r,2)}(x) = 2T_n\left(\frac{x+2}{2}\right) + rU_n\left(\frac{x+2}{2}\right). \tag{4.7}$$

Zeros and orthogonality properties of $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$ may be found in [5], [4], and [2]. *En passant*, the zeros of $R_3^{(0,0)}(x)$, say, are, by (2.17), the zeros of $xB_3(x)$, namely, $0, -1, -2$.

5. THREE IMPORTANT PROPERTIES

Roots $\alpha(x) = \alpha$ and $\beta(x) = \beta$ of the characteristic equation for (2.1), namely,

$$\lambda^2 - (x+2)\lambda + 1 = 0, \tag{5.1}$$

are

$$\begin{cases} \alpha = \dfrac{x+2+\sqrt{x^2+4}}{2}, \\ \beta = \dfrac{x+2-\sqrt{x^2+4}}{2}, \end{cases} \tag{5.2}$$

whence

$$\begin{cases} \alpha\beta = 1, \\ \alpha+\beta = x+2, \\ \alpha-\beta = \sqrt{x^2+4x}. \end{cases} \tag{5.3}$$

The Binet form for $B_n(x)$ is, by [2],

$$B_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{5.4}$$

Moreover, by [2],

$$(x+1)B_n(x) - B_{n-1}(x) = b_{n+1}(x), \tag{5.5}$$

$$(x+2)B_n(x) - B_{n-1}(x) = B_{n+1}(x), \tag{5.6}$$

$$(x+3)B_n(x) - B_{n-1}(x) = c_{n+1}(x), \tag{5.7}$$

$$(x+2)B_n(x) - 2B_{n-1}(x) = C_n(x). \tag{5.8}$$

Standard methods involving (2.1) and (2.2) yield the *Binet form* for $R_n^{(r,u)}(x)$.

Theorem 3: $R_n^{(r,u)}(x) = \dfrac{(x+r+u)(\alpha^n - \beta^n) - u(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}$

$$= (x+r+u)B_n(x) - uB_{n-1}(x), \qquad \text{by (5.4).}$$

Use of Theorem 3 in conjunction with (5.5)-(5.8) returns us to (2.13)-(2.16). Next, we record that, from (2.1) and (2.2),

$$R_{-1}^{(r,u)} = (u-1)x + u - r, \tag{5.9}$$

whence, by (2.13)-(2.16), $B_0(x) = 0$, $b_0(x) = 1$, $c_0(x) = -1$, and $C_{-1}(x) = x+2$.

Successive applications of the Binet form (Theorem 3) eventually give, on simplification and use of (2.2), (5.4), and (5.9), the *Simson formula*

Theorem 4: $\left.\begin{aligned} R_{n+1}^{(r,u)}(x)R_{n-1}^{(r,u)}(x) - [R_n^{(r,u)}(x)]^2 &= (x+r+u)[(u-1)x+u-r] - u^2 \\ &= R_1^{(r,u)}(x)R_{-1}^{(r,u)}(x) - [R_0^{(r,u)}(x)]^2. \end{aligned}\right\}$

Familiar techniques produce the *generating function* (Theorem 5) to complete our trilogy of salient features of $R_n^{(r,u)}(x)$.

Theorem 5: $\left.\begin{aligned} \sum_{i=0}^{\infty} R_i^{(r,u)}(x)y^i &= \frac{u - \{(u-1)x+u-r\}y}{1-(x+2)y+y^2} \\ &= \frac{R_0^{(r,u)}(x) - R_{-1}^{(r,u)}(x)y}{1-(x+2)y+y^2} \quad \text{by (2.2), (5.9).} \end{aligned}\right\}$

Special cases of Theorems 4 and 5:

r	u	$R_n^{(r,u)}(x)$	R.H.S. of Th. 4	Numerator in Th. 5
0	1	$b_{n+1}(x)$	x	$1-y$
1	1	$B_{n+1}(x)$	-1	1
2	1	$c_{n+1}(x)$	$-(x+4)$	$1+y$
0	2	$C_n(x)$	$x(x+4)$	$2-(2+x)y$

Observe that, in the third column, (i) row 1 × row 3 = row 2 × row 4, (ii) row 3 = $-\frac{(\alpha-\beta)^2}{x}$, (iii) row 4 = $(\alpha-\beta)^2$.

6. RISING DIAGONAL FUNCTIONS

Imagine, in the mind's eye, a set of parallel upward-slanting diagonal lines in Table 1 that delineate the *rising diagonal functions* $\mathfrak{R}_n^{(r,u)}(x)$ [= $\mathfrak{R}_n(x)$ for brevity] defined by

$$\mathfrak{R}_n(x) = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} c_{n+1-k,k}^{(r,u)}(x)x^k \tag{6.1}$$

with

$$\mathfrak{R}_0(x) = r+u, \ \mathfrak{R}_1(x) = x+2r+u, \tag{6.2}$$

where the values of the coefficients of x^k in (6.1) are given in (2.5)-(2.12).

Thus, for example,

$$\mathfrak{R}_4(x) = c_{5,0}^{(r,u)} + c_{4,1}^{(r,u)}x + c_{3,2}^{(r,u)}x^2 = (5r+u) + (4+10r+6u)x + (4+r+u)x^2$$

as may be checked in Table 1.

Choosing $\mathfrak{R}_0(x) = r+u$ involves a slightly subtle point. If one allows negative subscripts of $\mathfrak{R}_n(x)$, then the diagonal function $\mathfrak{R}_{-1}(x)$ is not equal merely to u, but to a more complicated expression.

Some intriguing results of a fundamental nature for $\{\mathfrak{R}_n^{(r,u)}(x)\}$ now emerge. First, we discover the recurrence relation. For this we need, by (2.11),

$$c_{\frac{n}{2}+1,\frac{n}{2}}^{(r,u)} = n+r+u \quad (n \text{ even}). \tag{6.3}$$

Theorem 6: $\mathfrak{R}_n(x) = 2\mathfrak{R}_{n-1}(x) + (x-1)\mathfrak{R}_{n-2}(x) \quad (n \geq 2)$.

Proof: Use (6.1). Sum for each power of x for $k = 0, 1, 2, \ldots, [\frac{n-1}{2}]$ and simplify according to (2.4), (2.7), (2.10), (2.11), and (6.3). Then

$$\begin{aligned} 2\mathfrak{R}_{n-1}(x) + (x-1)\mathfrak{R}_{n-2}(x) &= 2\sum_{k=0}^{[\frac{n}{2}]} c_{n-k,k,}x^k - \sum_{k=0}^{[\frac{n-1}{2}]} c_{n-1-k,k}x^k + \sum_{k=0}^{[\frac{n-1}{2}]} c_{n-1-k,k}x^{k+1} \\ &= c_{n+1,0} + c_{n,1}x + \cdots + c_{n+1-m,m}x^m + \cdots + \begin{cases} n+r+u, & n \text{ even,} \\ 1 & n \text{ odd,} \end{cases} \\ &= \mathfrak{R}_n(x). \end{aligned}$$

Corollary 1: $\mathfrak{R}_n(1) = 2^{n-1}(1+2r+u)$.

Proof:

$$\begin{aligned} \mathfrak{R}_n(1) &= 2\mathfrak{R}_{n-1}(1) && \text{by Th. 6,} \\ &= 2^2\mathfrak{R}_{n-2}(1) && \text{by Th. 6 again,} \\ &\cdots \\ &= 2^{n-1}\mathfrak{R}_1(1) && \text{by repeated use of Th. 6,} \\ &= 2^{n-1}(1+2r+u) && \text{by (6.2).} \end{aligned}$$

Special cases: Substituting in Corollary 1 the values of r and u appropriate to $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$, we obtain the corresponding values for the diagonal functions of these polynomials when $x = 1$, as stated in the concluding segment of [2].

From Theorem 6, the characteristic equation for $\mathfrak{R}_n^{(r,u)}(x)$ is $\lambda^2 - 2\lambda - (x-1) = 0$ with roots $\gamma(x) = \gamma$, $\delta(x) = \delta$ expressed by

$$\begin{cases} \gamma = 1+\sqrt{x}, \\ \delta = 1-\sqrt{x}, \end{cases} \tag{6.4}$$

so that

$$\begin{cases} \gamma+\delta = 2, \\ \gamma\delta = 1-x, \\ \gamma-\delta = 2\sqrt{x}. \end{cases} \tag{6.5}$$

In the standard process for the derivation of the generating function of $\mathfrak{R}_n(x)$, a fine nuance presents itself, namely, the recognition that, by (6.2),

$$\mathfrak{R}_3(x) - 2\mathfrak{R}_2(x) = x+2r+u-2(r+u) = x-u. \tag{6.6}$$

Applying Theorem 6 and (6.4), our treatment creates the following generating function.

Theorem 7: $\sum_{i=0}^{\infty} \mathfrak{R}_i(x)y^i = \{r+u+(x-u)y\}[1-(2y+(x-1)y^2)]^{-1}$.

Straightforward techniques yield the Binet form

Theorem 8: $\mathfrak{R}_n(x) = \dfrac{\{\mathfrak{R}_1(x)-\delta\mathfrak{R}_0(x)\}\gamma^n - \{\mathfrak{R}_1(x)-\gamma\mathfrak{R}_0(x)\}\delta^n}{\gamma-\delta}$.

Finally, by Theorem 8, we derive the Simson formula

Theorem 9: $\mathfrak{R}_{n+1}(x)\mathfrak{R}_{n-1}(x)-\mathfrak{R}_n^2(x) = (-1)^n(x-1)^{n-1}\{(r+x)^2-x(r+u)^2\}$.

It is clear from Theorem 9, or from Corollary 1, that

$$\mathfrak{R}_{n+1}(1)\mathfrak{R}_{n-1}(1) = \mathfrak{R}_n^2(1). \tag{6.7}$$

The particular situations for $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$ in relation to Theorems 6-9 may be readily deduced.

7. CONCLUDING THOUGHTS

There does seem to be scope for further developments. One such advance, for instance, might be the extension of the theory through negative subscripts of $\mathfrak{R}_n^{(r,u)}(x)$. Recall (5.9) for $n=-1$.

Another innovation is the consideration of the replacement of $x+2$ by $x+k$ (k integer). And what of interest might eventuate if $k=r$? $k=u$?

Possibly, some worthwhile differential equations could be hidden among the $\mathfrak{R}_n^{(r,u)}(x)$. Experience teaches us that this is often the case when exploring diagonal functions.

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ON CYCLIC STRINGS WITHOUT LONG CONSTANT BLOCKS

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Given integers k, w, and n. *How many n-letter cyclic strings with marked first letter are there, over an alphabet of k letters, which contain no constant substring of length $> w$?* Let $L_w^k(n)$ denote the number of such strings. We remark that by itself the phrase "cyclic string with marked first letter" is the same as "linear string." The difference between our problem and a similar one for linear strings lies in the phrase "no constant substring." In our problem this constant substring can lie on the circular, rather than only on the linear string.

This problem was solved for $k=2$ in [3]. Here we present the solution for arbitrary k, in the form of a surprisingly explicit formula.

Theorem 1: There is an integer $n_0 = n_0(k, w)$ and an algebraic number $\beta = \beta(k, w)$ such that for all $n \ge n_0$ the number of such strings is given by the (exact, not asymptotic) formula

$$L_w^k(n) = \langle \beta^n \rangle + \begin{cases} w(k-1), & \text{if } (w+1)/n, \\ -(k-1), & \text{otherwise,} \end{cases}$$

where "$\langle \cdot \rangle$" is the "nearest integer" function, β is the positive real root of the equation $x^w = (k-1)(1+x+x^2+\cdots+x^{w-1})$, and n_0 can be taken as

$$n_0 = n_0(w, k) = \max\left(w+1, \left\lceil \frac{k^w \log 2w}{(k-1)} \right\rceil\right).$$

This will follow from an analysis of the generating function, which is contained in the following theorem.

Theorem 2: Let $F_w^k(x) = \sum_{n=0}^{\infty} L_w^k(n) x^n$ be the generating function for $\{L_w^k(n)\}$. Then

$$F_w^k(x) = \frac{1-x^w}{1-x}\left(kx + (k-1)x\left(\frac{w+1-wkx}{1-kx+(k-1)x^{w+1}} - \frac{w+1}{1-x^{w+1}}\right)\right).$$

We use the following notation. Our alphabet $\mathcal{A} = \mathcal{A}_k$ will be the set of residues $\{0, 1, \ldots, k-1\}$ modulo k. If $\mathbf{y} = y_1 \ldots y_n$ is a string, then its *sum* is $\sum_j y_j$ modulo k. The set of n-letter cyclic strings with marked first letter, over this k-letter alphabet, will be denoted by $\mathrm{CS}(n, k)$, and those which also have no constant substrings of length $>w$ will be denoted by $\mathrm{CS}(n, k, w)$. The subset of $\mathrm{CS}(n, k)$ that consists of just those strings whose sum is 0 modulo k will be denoted by $\mathrm{CS}_0(n, k)$, and the subset of those which also have no *zero* substring of length $>w$ will be $\mathrm{CS}_0(n, k, w)$. Finally, $\mathrm{LS}_0(n, k, w)$ will be the set of *linear* n-strings over $\mathcal{A}$ that contain no zero substring of length $>w$.

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1. PROOF OF THEOREM 2

We will reduce our problem to counting $\mathrm{LS}_0(n,k,w)$, which is much easier, as we will see later.

Step 1. First, we will show that our problem can be reduced to a simpler problem of counting certain strings in $\mathrm{CS}(n, k)$ without zero substrings of length $> (w-1)$.

We consider a map $T: \mathrm{CS}(n, k) \to \mathrm{CS}(n, k)$ defined as follows:

$$T\mathbf{x} = \{x_{i+1} - x_i \pmod k\}_{i=1}^{n} \quad (\text{where } x_{n+1} := x_1).$$

This is a generalization of the map defined in [1] for $k=2$.

Clearly, $T(\mathrm{CS}(n,k)) \subseteq \mathrm{CS}_0(n,k)$. Note also that all maximal j-letter constant substrings of nonconstant strings are mapped onto zero substrings of length $j-1$, and constant strings in $\mathrm{CS}(n, k)$ are mapped onto $\mathbf{0} \in \mathrm{CS}(n, k)$ if $n = w$.

Given a string $\mathbf{y} = y_1 \dots y_n \in \mathrm{CS}_0(n,k)$ and any letter $a \in \mathcal{A}$, we can uniquely determine a string $\mathbf{x} = x_1 \dots x_n$ such that $T(\mathbf{x}) = \mathbf{y}$ and $x_1 = a$ since

$$(\forall i \in \{1, \dots, n\}): x_{i+1} \equiv x_i + y_i \pmod k \quad (\text{again, } x_{n+1} := x_1).$$

Therefore, T is a $k{:}1$ map onto $\mathrm{Im}\, T = \mathrm{CS}_0(n,k)$. Furthermore, $T(\mathrm{CS}_0(n,k,w)) = \mathrm{CS}_0(n,k,w-1)$, together with $\mathbf{0} \in \mathrm{CS}(n,k)$ if $n = w$.

Let $\dot{L}_w^k(n) = |\mathrm{CS}_0(n,k,w)|$ Then we have that

$$L_w^k(n) = \begin{cases} k\dot{L}_{w-1}^k(n), & \text{for } n \neq w, \\ k(\dot{L}_{w-1}^k(n)+1), & \text{for } n = w, \end{cases}$$

where 1 accounts for the string $\mathbf{0} \in \mathrm{CS}(n,k)$ in the case $n = w$.

Step 2. Consider a $\mathrm{CS}_0(n,k,w-1)$ string that ends on a nonzero letter. [We will denote the number of such strings by $\Lambda_{w-1}^k(n)$ and, of those, the strings whose first letter is nonzero will be denoted by $\tilde{\Lambda}_{w-1}^k(n)$.] This string has $\le w-1$ zeros at the beginning, so if we remove the zeros we will get a unique $\mathrm{CS}_0(n-i,k,w-1)$ string, where $0 \le i \le w-1$, whose first and last letters are nonzero. Clearly, we can also perform the inverse operation, i.e., obtain a unique $\mathrm{CS}_0(n,k,w-1)$ string whose last letter is nonzero, given $i \in \{1, \dots, n\}$ and a $\mathrm{CS}_0(n-i,k,w-1)$ string whose first and last letters are nonzero, by adding i zeros at the beginning. Hence, we see that

$$\Lambda_{w-1}^k(n) = \sum_{i=0}^{w-1} \tilde{\Lambda}_{w-1}^k(n-i).$$

Step 3. Let us now look at the linear strings of the type $\mathrm{LS}_0(n,k,w-1)$. We define a map from $\mathrm{LS}_0(n-1,k,w-1)$ to the set of those strings in $\bigcup_{i=0}^{w}\mathrm{CS}_0(n-i,k,w-1)$ strings (where $0 \le i \le w$) whose last letter is nonzero, plus the empty string if $n \le w$.

Let such a string $\mathbf{y} = y_1 \dots y_{n-1} \in \mathrm{LS}_0(n-1,k,w-1)$ be given, and put $s = s(\mathbf{y}) = -\sum_j y_j$ (mod k). Then our map will take $\mathbf{y}$ to the string

$$\begin{cases} y_1 y_2 \dots y_{n-1} s, & \text{if } s \neq 0; \\ y_1 y_2 \dots y_{n-1-i}, & \text{if } s = 0,\ y_{n-1-i} \neq 0, \text{ and } y_{n-i} = \dots = y_{n-1} = 0,\ 0 \le i \le w-1; \\ \emptyset, & \text{if } \mathbf{y} \text{ is a zero string of length } n-1 \le w-1. \end{cases}$$

Clearly, this map is a bijection onto its image. Let $\ell_w^k(n) = |\mathrm{LS}_0(n, k, w)|$. Then we can conclude from the above that

$$\ell_{w-1}^k(n-1) = \sum_{i=0}^{w} \Lambda_{w-1}^k(n-i) + \begin{cases} 1, & \text{if } 1 \le n \le w, \\ 0, & \text{if } n \ge w+1. \end{cases}$$

Step 4. Now consider some string $\mathbf{y} \in \mathrm{CS}_0(n, k, w-1)$. Either $\mathbf{y}$ is a zero string of length $\le w-1$ or it has $\le w-1$ zeros as the (possibly empty) union of its initial and terminal blocks of zeros. If we remove these zeros, we will get either an empty string (if $n \le w-1$) or a $\mathrm{CS}_0(n-i, k, w-1)$ string (where $0 \le i \le w-1$) whose first and last letters are nonzero.

Conversely, given a string $\mathbf{y} \in \mathrm{CS}_0(n-i, k, w-1)$ (where $0 \le i \le w-1$) with nonzero first and last letters, add i zeros between the last and first letter and, in the resulting string, mark one of the added zeros or the first nonzero letter of $\mathbf{y}$ as the first letter of the resulting $\mathrm{CS}_0(n, k, w-1)$ string. There are $i+1$ choices for the new first letter. Let us show that this map is $1:(i+1)$ from $\mathrm{CS}_0(n-i, k, w-1) \setminus \{\mathbf{0}\}$ onto its image.

Suppose not. This means that, for some $i \in \{0, 1, \ldots, w-1\}$, we can obtain two identical nonzero $\mathrm{CS}_0(n, k, w-1)$ strings by adding i zeros

- **(a)** to two different nonzero $\mathrm{CS}_0(n-i, k, w-1)$ strings with first and last nonzero letters, or
- **(b)** to the same string in the above set and then marking different letters as the new first letter.

But (a) is clearly impossible, since it implies that by removing the i zeros (i.e., all the zeros) between the last and first nonzero letters, we can get two different $\mathrm{CS}_0(n-i, k, w-1)$ strings that begin and end on a nonzero letter.

Hence, (b) must be true, i.e., there must exist a nonzero $\mathbf{x} \in \mathrm{CS}_0(n, k, w-1)$ such that

1) $(\exists s \ne 0)(\forall r)(x_r = x_{r+s})$, and
2) $x_1 = \cdots = x_s = 0$.

But it is easy to see that 1) and 2) imply that $\mathbf{x} = \mathbf{0}$. This is a contradiction, so our map must be $1:(i+1)$ from $\mathrm{CS}_0(n-i, k, w-1) \setminus \{\mathbf{0}\}$ onto its image.

Therefore, it follows that $\dot{L}_{w-1}^k(0) = 0$ [since $L_{w-1}^k(0) = 0$], and

$$\dot{L}_{w-1}^k(n) = \sum_{i=0}^{w-1} (i+1)\tilde{\Lambda}_{w-1}^k(n-i) + \begin{cases} 1, & \text{if } 1 \le n \le w, \\ 0, & \text{if } n \ge w. \end{cases}$$

We can summarize the developments so far by giving the following set of equations that have been proved:

$$L_w^k(n) = \begin{cases} k\dot{L}_{w-1}^k(n), & \text{for } n \ne w, \\ k(\dot{L}_{w-1}^k(n)+1), & \text{for } n = w. \end{cases}$$

$$\Lambda_{w-1}^k(n) = \sum_{i=0}^{w-1} \tilde{\Lambda}_{w-1}^k(n-i),$$

$$\ell_{w-1}^k(n-1) = \begin{cases} 1 + \sum_{i=0}^{w} \Lambda_{w-1}^k(n-i), & \text{for } 1 \le n \le w, \\ \sum_{i=0}^{w} \Lambda_{w-1}^k(n-i), & \text{for } n \ge w+1. \end{cases}$$

$$\dot{L}^k_{w-1}(n) = \begin{cases} 1+\sum_{i=0}^{w-1}(i+1)\tilde{\Lambda}^k_{w-1}(n-i), & \text{for } 1 \le n \le w-1, \\ \sum_{i=0}^{w-1}(i+1)\tilde{\Lambda}^k_{w-1}(n-i), & \text{for } n \ge w. \end{cases}$$

In [3] is it shown that

$$\ell^2_{w-1}(n) = \begin{cases} 2^n, & \text{for } 0 \le n \le w-1, \\ \sum_{i=1}^{w} \ell^2_{w-1}(n-i), & \text{for } n \ge w. \end{cases}$$

Generalizing the proof of this fact for a k-letter alphabet, it is easy to show that

$$\ell^2_{w-1}(n) = \begin{cases} k^n, & \text{for } 0 \le n \le w-1, \\ (k-1)\left(\sum_{i=1}^{w} \ell^k_{w-1}(n-i)\right), & \text{for } n \ge w. \end{cases} \tag{1}$$

It is also a special case of Example 6.4 on pages 1102-1103 of [4] (for k-letter, instead of binary, strings). [Of course, it is assumed that if $n<0$ or $w<0$ in any of the above formulas, then $L^k_w(n) = \dot{L}^k_w(n) = \Lambda^k_w(n) = \tilde{\Lambda}^k_w(n) = \ell^k_w(n) = 0$.]

Define the generating functions

$$\dot{F}^k_w(x) = \sum_{n=0}^{\infty} \dot{L}^k_w(n)x^n, \qquad \Phi^k_w(x) = \sum_{n=0}^{\infty} \Lambda^k_w(n)x^n,$$

$$\tilde{\Phi}^k_w(x) = \sum_{n=0}^{\infty} \tilde{\Lambda}^k_w(n)x^n, \qquad f^k_w(x) = \sum_{n=0}^{\infty} \ell^k_w(n)x^n.$$

Then we have

$$F^k_w(x) = k\dot{F}^k_{w-1}(x) + kx^w,$$

$$\Phi^k_{w-1}(x) = (1+x+\cdots+x^{w-1})\,\tilde{\Phi}^k_{w-1}(x) = \frac{1-x^w}{1-x}\tilde{\Phi}^k_{w-1}(x),$$

$$\begin{aligned} xf^k_{w-1}(x) &= x+x^2+\cdots+x^w+(1+x+\cdots+x^w)\,\Phi^k_{w-1}(x) \\ &= \frac{x(1-x^w)}{1-x}+\frac{1-x^{w+1}}{1-x}\Phi^k_{w-1}(x), \end{aligned}$$

$$\begin{aligned} \dot{F}^k_{w-1}(x) &= x+x^2+\cdots+x^{w-1}+(1+2x+\cdots+wx^{w-1})\,\tilde{\Phi}^k_{w-1}(x) \\ &= \frac{x-x^w}{1-x}+\frac{1-(w+1)x^w+wx^{w+1}}{(1-x)^2}\tilde{\Phi}^k_{w-1}(x), \end{aligned}$$

and, finally, from (1) above,

$$f^k_{w-1}(x) = \frac{1+x+\cdots+x^{w-1}}{1-(k-1)x-\cdots-(k-1)x^w} = \frac{1-x^w}{1-kx+(k-1)x^{w+1}}.$$

Hence,

$$x\left(\frac{1-x^w}{1-kx+(k-1)x^{w+1}}\right) = \frac{x(1-x^w)}{1-x}+\frac{1-x^{w+1}}{1-x}\frac{1-x^w}{1-x}\tilde{\Phi}^k_{w-1}(x),$$

or, equivalently,

$$\tilde{\Phi}^k_{w-1}(x) = \frac{(k-1)x^2(1-x^w)}{1-kx+(k-1)x^{w+1}}\frac{1-x}{1-x^{w+1}},$$

so

$$\begin{aligned}\dot{F}^k_{w-1}(x) &= \frac{x-x^w}{1-x} + \frac{1-(w+1)x^w+wx^{w+1}}{(1-x)^2}\frac{(k-1)x^2(1-x^w)}{1-kx+(k-1)x^{w+1}}\frac{1-x}{1-x^{w+1}} \\ &= \frac{x-x^w}{1-x} + \frac{1-x^w}{1-x}(k-1)x^2\frac{1-(w+1)x^w+wx^{w+1}}{(1-kx+(k-1)x^{w+1})(1-x^{w+1})},\end{aligned}$$

and thus,

$$\begin{aligned}F^k_w(x) &= k\left(x^w + \frac{x-x^w}{1-x}\right) + k\frac{1-x^w}{1-x}(k-1)x^2\frac{1-(w+1)x^w+wx^{w+1}}{(1-kx+(k-1)x^{w+1})(1-x^{w+1})} \\ &= \frac{1-x^w}{1-x}\left(kx+(k-1)x\left(\frac{w+1-wkx}{1-kx+(k-1)x^{w+1}} - \frac{w+1}{1-x^{w+1}}\right)\right),\end{aligned}$$

as asserted in the statement of the theorem.

Let us now find the coefficients of $F^k_w(n)$. We have that, for $n > 1$,

$$L^k_w(n) = (k-1)\sum_{i=1}^{w} A^k_w(n-i),$$

where

$$A^k_w(n) = [x^n]\left(\frac{w+1-wkx}{1-kx+(k-1)x^{w+1}} - \frac{w+1}{1-x^{w+1}}\right),$$

where "$[x^n](\cdot)$" means "the coefficient of x^n in $(\cdot)$". Let

$$B^k_w(n) = [x^n]\left(\frac{1}{1-kx+(k-1)x^{w+1}}\right),$$

then

$$A^k_w(n) = (w+1)B^k_w(n) - wkB^k_w(n-1) - \delta_n,$$

where

$$\delta_n = \begin{cases} w+1, & \text{if } (w+1)/n, \\ 0, & \text{otherwise.}\end{cases}$$

Let us find an exact formula for $B^k_w(n)$ that involves binomial coefficients. Then in the next section we will find the formula that is claimed in Theorem 1 above. We have that

$$\frac{1}{1-kx+(k-1)x^{w+1}} = \frac{1}{1-kx}\frac{1}{1+\dfrac{(k-1)x^{w+1}}{1-kx}} =$$

$$=\sum_{m=0}^{\infty}(-1)^m\frac{((k-1)x^{w+1})^m}{(1-kx)^{m+1}}=\sum_{m=0}^{\infty}\frac{(1-k)^m x^{m(w+1)}}{(1-kx)^{m+1}}$$
$$=\sum_{m=0}^{\infty}(1-k)^m x^{m(w+1)}\left(\sum_{j=0}^{\infty}\binom{m+j}{m}(kx)^j\right)$$
$$=\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\binom{m+j}{m}(1-k)^m k^j x^{m(w+1)+j},$$

so

$$B_w^k(n)=k^n\sum_{0\le m\le n/(w+1)}\binom{n-wm}{m}\left(\frac{1-k}{k^{w+1}}\right)^m=k^n\sum_m\binom{n-wm}{m}\left(\frac{1-k}{k^{w+1}}\right)^m,$$

and hence,

$$A_w^k(n)=k^n\sum_m\left[(w+1)\binom{n-wm}{m}-w\binom{n-1-wm}{m}\right]\left(\frac{1-k}{k^{w+1}}\right)^m-\delta_n$$
$$=k^n\sum_{m\ne n/w}\frac{n}{n-wm}\binom{n-wm}{m}\left(\frac{1-k}{k^{w+1}}\right)^m-\delta_n,$$

since

$$m=\frac{n}{w}\Rightarrow n-wm=0\Rightarrow(w+1)\binom{n-wm}{m}-w\binom{n-1-wm}{m}=0.$$

2. PROOF OF THEOREM 1

If we expand in partial fractions,

$$\frac{(1-x^w)(k-1)x(w+1-wkx)}{(1-x)(1-kx+(k-1)x^{w+1})}=\sum_{\alpha\ne1}\frac{C_\alpha}{1-x/\alpha}+D+\frac{C_1}{1-x},$$

where α runs over the zeros of the second factor in the denominator, then it is a simple exercise to verify that *all* $C_\alpha=1\,(\alpha\ne1)$, $C_1=w(k-1)$, and $D=-kw$. Hence, if we read off the coefficient of x^n in the last form of the generating function given in Theorem 2, we find that

$$L_w^k(n)=\sum_{\alpha\ne1}\frac{1}{\alpha^n}+w(k-1)+\begin{cases}-kw, & \text{if } n=0,\\ k, & \text{if } 1\le n\le w,\\ -(k-1)(w+1), & \text{if } w+1 \text{ does not divide } n,\\ 0, & \text{otherwise.}\end{cases}\tag{2}$$

Proposition 1: The roots of the equation $1-kx+(k-1)x^{w+1}=0$ consist of a root $x=1$, one positive real root <1, and $w-1$ other roots all of which have absolute values >1.

Proof: Indeed, the roots other than $x=1$ are the reciprocals of the roots of

$$\psi_{w,k}(x)=x^w-(k-1)(x^{w-1}+\cdots+1)=0.\tag{3}$$

Let β be the positive real zero of $\psi_{w,k}$. Then its remaining zeros are those of

$$\begin{aligned}\frac{\psi_{w,k}(x)}{x-\beta} &= x^{w-1}+(\beta-(k-1))x^{w-2}+(\beta^2-(k-1)\beta-(k-1))x^{w-1}+\cdots \\ &= \sum_{j=1}^{w-1}\psi_{w-j-1,k}(\beta)x^j.\end{aligned} \tag{4}$$

We claim that the coefficients of this last polynomial increase steadily, i.e., that $\{\psi_{j,k}(\beta)\}\downarrow_j$. Indeed, note first that, since $\psi_{w,k}(0)<0$ and $\psi_{w,k}(k)>0$, we have $\beta<k$. But then the claim is true, because $\psi_{j+1,k}(\beta)-\psi_{j,k}(\beta)=\beta^j(\beta-k)<0$. A theorem of Enestrom-Kakeya [2] holds that if the coefficients of a polynomial are positive real numbers and are increasing then all of the zeros of the polynomial lie inside the unit disk. This completes the proof of the proposition. □

We now investigate the quantity n_0 in the statement of Theorem 1, which requires sharper bounds on the roots of equation (3) above. First, we require a bound on β itself.

Proposition 2: We have $k-k/k^w\le\beta\le k-(k-1)/k^w$.

Proof: We note that $\psi_{w,k}(x)$ is negative in $0<x<\beta$ and positive in $\beta<x<k$. Hence, it suffices to show that $\psi_{w,k}(k-(k-1)/k^w)>0$ and $\psi_{w,k}(k-k/k^w)<0$. But

$$\begin{aligned}\psi_{w,k}\left(k-\frac{k-1}{k^w}\right) &= k^{w+1}\left\{\left(1-\frac{k-1}{k^{w+1}}\right)^{w+1}-\left(1-\frac{k-1}{k^{w+1}}\right)^{w}\right\}+(k-1)\\ &=(k-1)\left(1-\left(1-\frac{k-1}{k^{w+1}}\right)^{w}\right)>0\end{aligned}$$

and, similarly,

$$\begin{aligned}\psi_{w,k}\left(k-\frac{k}{k^w}\right) &= k^{w+1}\left\{\left(1-\frac{k}{k^{w+1}}\right)^{w+1}-\left(1-\frac{k}{k^{w+1}}\right)^{w}\right\}+(k-1)\\ &=k\left(1-\left(1-\frac{1}{k^w}\right)^{w}\right)-1\le k\left(1-\left(1-\frac{w}{k^w}\right)\right)-1=\frac{w}{k^{w-1}}-1\le 0,\end{aligned}$$

and the equality holds iff $w=1$. □

Next, we require a better bound for the roots of $\psi_{w,k}$ other than the root β. We know that these other roots have moduli <1, but the following proposition gives a sharper result.

Proposition 3: The zeros of $\psi_{w,k}(x)$, other than β, all lie in the disk $|x|\le 1-(k-1)/k^w$.

Proof: Observe that the zeros, other than 1 and β, are the set of all zeros of the polynomial displayed in the last member of (4) above. If we denote that polynomial by $g(x)$, then we claim that not only do the coefficients of g increase steadily, as shown above, but that if we choose $R=\beta-k+1$, then $0<R<1$ and the coefficients of $g(Rx)$ still increase steadily. If we can show this, then we will know that all zeros of g lie in the disk $|x|\le R<1$.

But is R is chosen so that the coefficient sequence of $g(Rx)$, viz. the sequence

$$\{\psi_{w-j-1,k}(\beta)R^j\}_{j=0}^{w-1},$$

increases with j, then the result will follow. Now R is surely large enough to achieve this if

$$\min_{1\le j\le w-1} \frac{\psi_{w-j-1,k}(\beta)R^j}{\psi_{w-j,k}(\beta)R^{j-1}} \ge 1,$$

i.e., if

$$R \ge R^* = \max_{1\le j\le w-1} \frac{\psi_{j,k}(\beta)}{\psi_{j-1,k}(\beta)}.$$

But, since $\psi_{j,k}(\beta) = \beta\psi_{j-1,k}(\beta) - (k-1)$, we have

$$R^* = \beta - \frac{(k-1)}{\max_{1\le j\le w-1}\psi_{j-1,k}(\beta)} = \beta - \frac{k-1}{\psi_{0,k}(\beta)} = \beta - k + 1.$$

Thus, all zeros of g lie in the disk $|x| \le \beta - k + 1$, and the result follows by Proposition 2 above. □

Now consider the exact formula (2) for the number $L_w^k(n)$ of strings. The first term will be the nearest integer to β^n as soon as the contribution of all of the other roots $\alpha \ne 1/\beta$ is <1/2. In view of Proposition 3, this contribution will be less than 1/2 if $n \ge k^w \log(2w)/(k-1)$, and the proof of Theorem 1 is complete. □

Notice, however, that, in order to obtain the estimate for n_0, we bounded the absolute value of the sum of powers of the small roots by the sum of their absolute values. Since this does not take into account many cancellations in $\sum_{\alpha\ne 1,\beta}\alpha^{-n}$, our estimate (which grows like k^{w-1}) is much greater than the actual n_0's. In fact, based on empirical data for small k and w, we conjecture that $n_0(k, w)$ grows polynomially in both k and w (specifically, slower than k^2w^3, but faster than kw^2).

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CONICS WHICH CHARACTERIZE CERTAIN LUCAS SEQUENCES

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1. INTRODUCTION

In the notation of Horadam [3], write

$$W_n = W_n(a, b; p, q), \tag{1.1}$$

meaning that

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a, \quad W_1 = b, \quad n \geq 2. \tag{1.2}$$

The sequence $\{W_n\}_{n=0}^{\infty}$ can be extended to negative subscripts using (1.2); we write simply $\{W_n\}$.

We shall be concerned with the sequences

$$\begin{cases} U_n = W_n(0, 1; P, -1), \\ V_n = W_n(2, P; P, -1), \end{cases} \tag{1.3}$$

where $P \neq 0$ is an integer, and

$$\begin{cases} u_n = W_n(0, 1; p, 1), \\ v_n = W_n(2, p; p, 1), \end{cases} \tag{1.4}$$

where $|p| > 2$ is also an integer.

For the sequences (1.3) and (1.4), we define $\Delta = P^2 + 4$ and $D = p^2 - 4$, respectively. Taking α and β to be the roots of $x^2 - Px - 1 = 0$, we have the well-known expressions (the Binet forms)

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n. \tag{1.5}$$

Similarly, if γ and δ are the roots of $x^2 - px + 1 = 0$, then

$$u_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad v_n = \gamma^n + \delta^n. \tag{1.6}$$

According to Dickson ([2], p. 405), Lucas proved that if x and y are consecutive Fibonacci numbers, then (x, y) is a lattice point on one of the hyperbolas

$$y^2 - xy - x^2 = \pm 1, \tag{1.7}$$

and Wasteels proved the converse in 1902. Interest in conics whose equations are satisfied by pairs of successive terms of linear recursive sequences has been rekindled. See, for example, [1], [4], and [5]. Recently McDaniel [6] has provided converses to several of the results of these writers. For example, he proved the following.

Theorem: Let x and y be positive integers. The pair (x, y) is a solution of $y^2 - Pxy - x^2 = \pm 1$ if and only if there exists a positive integer n such that $x = U_n$ and $y = U_{n+1}$.

The object of this paper is to generalize McDaniel's results and to obtain new ones.

2. SOME PRELIMINARY RESULTS

Throughout this paper, m and n denote integers. Also, Δ and D are as defined in Section 1. For the sequences (1.3), we record the following results, each of which can be proved using the Binet forms:

$$V_m^2 - 4 = \Delta U_m^2, \; m \text{ even}, \tag{2.1}$$

$$V_m^2 + 4 = \Delta U_m^2, \; m \text{ odd}, \tag{2.2}$$

$$U_n V_m + V_n U_m = 2U_{n+m}, \tag{2.3}$$

$$U_n V_m - V_n U_m = \begin{cases} 2U_{n-m}, & m \text{ even}, \\ -2U_{n-m}, & m \text{ odd}, \end{cases} \tag{2.4}$$

$$V_n V_m + \Delta U_n U_m = 2V_{n+m}, \tag{2.5}$$

$$V_n V_m - \Delta U_n U_m = \begin{cases} 2V_{n-m}, & m \text{ even}, \\ -2V_{n-m}, & m \text{ odd}. \end{cases} \tag{2.6}$$

We shall also need the following results:

Lemma 1: The integer solutions of $\Delta x^2 + 4 = z^2$ are precisely the pairs $(x, z) = (\pm U_{2n}, \pm V_{2n})$.

Lemma 2: The integer solutions of $\Delta x^2 - 4 = z^2$ are precisely the pairs $(x, z) = (\pm U_{2n+1}, \pm V_{2n+1})$.

These two lemmas constitute the first half of McDaniel's Corollary 1, a well-known result for which he provides an alternative proof.

Lemma 3: If Δ is square free, the integer solutions of $\Delta(x^2 - 4) = z^2$ are precisely the pairs $(x, z) = (\pm V_{2n}, \pm \Delta U_{2n})$.

Proof: Since Δ is square free and $\Delta | z^2$, then $\Delta | z$. Writing $z = \Delta z_0$ we obtain $\Delta z_0^2 + 4 = x^2$, and the use of Lemma 1 completes the proof. □

In a similar manner, using Lemma 2, we can prove

Lemma 4: If Δ is square free, the integer solutions of $\Delta(x^2 + 4) = z^2$ are precisely the pairs $(x, z) = (\pm V_{2n+1}, \pm \Delta U_{2n+1})$.

Results for the sequences (1.4) which parallel (2.1)-(2.6) are as follows:

$$v_m^2 - 4 = D u_m^2, \tag{2.7}$$

$$u_n v_m + v_n u_m = 2u_{n+m}, \tag{2.8}$$

$$u_n v_m - v_n u_m = 2u_{n-m}, \tag{2.9}$$

$$v_n v_m + D u_m u_n = 2v_{n+m}, \tag{2.10}$$

$$v_n v_m - D u_n u_m = 2v_{n-m}. \tag{2.11}$$

For completion, we state the following lemma, which is the second part of McDaniel's Corollary 1.

Lemma 5: The integer solutions of $Dx^2+4=z^2$ are precisely the pairs $(x, z)=(\pm u_n, \pm v_n)$.

Now, using Lemma 5, and following the method of proof of Lemma 3, it is easy to prove

Lemma 6: If D is square free, the integer solutions of $D(x^2-4)=z^2$ are precisely the pairs $(x, z)=(\pm v_n, \pm Du_n)$.

3. CONICS CHARACTERIZING THE SEQUENCES (1.3)

We now give a sequence of theorems concerning pairs of conics whose integer points are derived from the sequences (1.3). In the proofs we must recall that

$$\sqrt{a^2}=\begin{cases} a, & a\geq 0, \\ -a, & a<0. \end{cases}$$

Theorem 1: If m is a fixed even integer, then the points with integer coordinates on the conics $y^2-V_mxy+x^2\pm U_m^2=0$ are precisely the pairs $(x, y)=\pm(U_n, U_{n+m})$.

Proof: Consider first the conic $y^2-V_mxy+x^2+U_m^2=0$. Regarding this as a quadratic equation in y, and making use of (2.1), we obtain

$$y=\frac{V_mx\pm U_m\sqrt{\Delta x^2-4}}{2}.$$

From Lemma 2, integer points can arise only when $x=\pm U_{2n+1}$. Now, using (2.3) and (2.4), we see that the integer points are $(x, y)=\pm(U_{2n+1}, U_{2n+1+m})$ together with the points $(x, y)=\pm(U_{2n+1}, U_{2n+1-m})$, where n ranges over all integers. Since these sets coincide, we consider only the first.

Proceeding in the same manner, and making use of (2.1), Lemma 1, (2.3), and (2.4), we see that the integer points on the conic $y^2-V_mxy+x^2-U_m^2=0$ are $(x, y)=\pm(U_{2n}, U_{2n+m})$. This completes the proof. □

We now state three additional theorems, each of which can be proved using the results of Section 2. Since the proofs are similar to the proof of Theorem 1, we refrain from giving them here.

Theorem 2: If m is a fixed odd integer, then the points with integer coordinates on the conics $y^2-V_mxy-x^2\pm U_m^2=0$ are precisely the pairs $(x, y)=\pm(U_n, U_{n+m})$.

Theorem 3: If m is a fixed even integer and Δ is square free, then the points with integer coordinates on the conics $y^2-V_mxy+x^2\pm\Delta U_m^2=0$ are precisely the pairs $(x, y)=\pm(V_n, V_{n+m})$.

Theorem 4: If m is a fixed odd integer and Δ is square free, then the points with integer coordinates on the conics $y^2-V_mxy-x^2\pm\Delta U_m^2=0$ are precisely the pairs $(x, y)=\pm(V_n, V_{n+m})$.

We remark that Theorem 2 generalizes McDaniel's Theorem 1, and Theorem 4 generalizes McDaniel's Corollary 2.

4. CONICS CHARACTERIZING THE SEQUENCES (1.4)

Next, we state two theorems concerning conics whose integer points are derived from the sequences (1.4). Each can be proved by following the method of proof of Theorem 1, while making use of the appropriate results from Section 2.

Theorem 5: If m is any fixed integer, then the points with integer coordinates on the conic $y^2 - v_m xy + x^2 - u_m^2 = 0$ are precisely the pairs $(x, y) = \pm(u_n, u_{n+m})$.

Theorem 6: If m is any fixed integer and D is square free, then the points with integer coordinates on the conic $y^2 - v_m xy + x^2 + Du_m^2 = 0$ are precisely the pairs $(x, y) = \pm(v_n, v_{n+m})$.

We note that Theorem 5 generalizes McDaniel's Theorem 2, and Theorem 6 generalizes McDaniel's Corollary 3.

5. AN INTERESTING EXAMPLE

If Δ is not square free, it is easy to show by substitution, using Binet forms, that the stated solutions in Theorems 3 and 4 remain as solutions. The same is true of Theorem 6. However, as McDaniel observes, other solutions may occur. He cites the example

$$y^2 - 4xy - x^2 \pm 20 = 0. \tag{5.1}$$

The conics (5.1) provide an example of the conics in Theorem 4 where $P = 4$, $m = 1$, and $\Delta = 20 = 2^2.5$ is not square free. Now $(x, y) = (1, 7)$ is a solution of (5.1), but $V_n \neq 1$ for any n. Observe, however, that the conics (5.1) may be written as

$$y^2 - L_3 xy - x^2 \pm 5F_3^2 = 0. \tag{5.2}$$

This is an instance of Theorem 4 in which $P = 1$, $m = 3$, and $\Delta = 5$ is square free. Hence, the solutions are precisely $(x, y) = \pm(L_n, L_{n+3})$.

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SUMMATION FORMULAS FOR SPECIAL LEHMER NUMBERS

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1. INTRODUCTION AND PRELIMINARIES

The observation made in [2] brings to the attention of the reader the fact that an improper use of the geometric series formula (g.s.f.) for obtaining summation formulas for the well-known generalized sequences $\{W_n(0,1;p,q)\}$ and $\{W_n(2,p;p,q)\}$ (e.g., see [5]) leads to meaningless expressions when p and q assume certain special values. The same problem may arise if we seek summation formulas for *Lehmer numbers* (e.g., see [4] and [7] for recent studies on the properties of these numbers).

The closed-form expression (Binet form) for the n^{th} Lehmer number $U_n(p,q)$ (or simply U_n is no misunderstanding can arise) is

$$U_n = \begin{cases} (\alpha^n - \beta^n)/(\alpha^2 - \beta^2) & (n \text{ even}), \\ (\alpha^n - \beta^n)/(\alpha - \beta) & (n \text{ odd}), \end{cases} \tag{1.1}$$

α and β being the roots of the equation $x^2 - \sqrt{p}x + q = 0$, where p and q are integers. These roots are given by

$$\begin{cases} \alpha = (\sqrt{p} + \sqrt{p-4q})/2 = (\sqrt{p} + \Delta)/2, \\ \beta = (\sqrt{p} - \Delta)/2, \end{cases} \tag{1.2}$$

so that

$$\alpha + \beta = \sqrt{p}, \tag{1.3}$$

$$\alpha - \beta = \Delta, \tag{1.4}$$

$$\alpha\beta = q, \tag{1.5}$$

$$\alpha^2 + \beta^2 = (p + \Delta^2)/2 = p - 2q. \tag{1.6}$$

The numbers U_n obey (e.g., see [7]) the second-order recurrence relations

$$U_n = \begin{cases} U_{n-1} - qU_{n-2} & (n \geq 2 \text{ even}), \\ pU_{n-1} - qU_{n-2} & (n \geq 3 \text{ odd}), \end{cases} \tag{1.7}$$

with initial conditions

$$U_0 = 0 \quad \text{and} \quad U_1 = 1, \tag{1.7'}$$

whence it can be observed that $U_n(1,-1)$ is the n^{th} Fibonacci number. As done in [7], without loss of generality, we can assume that

$$p > 0, \tag{1.8}$$

$$p - 4q > 0, \tag{1.9}$$

$$q \neq 0. \tag{1.10}$$

The simplest summation formula for *normal* Lehmer numbers [that is, Lehmer numbers with arbitrary parameters p and q satisfying (1.8)-(1.10)] is

$$\sum_{n=0}^{N} U_n = \frac{q^2(U_N + U_{N-1}) - U_{N+1} - U_{N+2} + q + 2}{q^2 + 2q - p + 1}. \tag{1.11}$$

This formula can be obtained after some manipulation involving the use of (1.1) [cf. (2.6) and (2.7)], (1.5), (1.6), and the g.s.f. The relation $U_2 = U_1 = 1$ [see (1.7) and (1.7′)] must also be used.

One can immediately observe that (1.11) *does not have general validity*. In fact, if $p = k^2$ (k a positive integer) and

$$q = -1 \pm k, \tag{1.12}$$

then the denominator on the right-hand side of (1.11) vanishes. The same problem arises also in *general* summation formulas (that is, summations where the subscripts of the summands are in arithmetical progression with arbitrary difference) for normal Lehmer numbers.

The principal aim of this note is to establish general summation formulas for a subset of normal Lehmer numbers: the *special* Lehmer numbers $U_n(k^2, -1 \pm k)$. As a concluding remark, some simple properties of these numbers are pointed out in Section 5. To save space, the number of proofs has been kept to a minimum.

2. SUMMATION FORMULAS FOR NORMAL LEHMER NUMBERS: BASIC RELATIONS

For notational convenience, let us define

$$S_N(h, r) = \sum_{n=0}^{N} (\alpha^{hn+r} - \beta^{hn+r}), \tag{2.1}$$

$$\delta = 1/(\alpha - \beta), \tag{2.2}$$

$$\gamma = 1/(\alpha^2 - \beta^2). \tag{2.3}$$

The following relations are fundamental tools for establishing general summation formulas for normal Lehmer numbers. They can be proved readily by simply using (1.1) and will be used to obtain summation formulas for special Lehmer numbers.

$$\sum_{n=0}^{N} U_{hn+r} = \gamma S_N(h, r) \quad (h \text{ and } r \text{ even, } N \text{ arbitrary}), \tag{2.4}$$

$$= \delta S_N(h, r) \quad (h \text{ even, } r \text{ odd, } N \text{ arbitrary}), \tag{2.5}$$

$$= \gamma S_{N/2}(2h, r) + \delta S_{N/2-1}(2h, h+r) \quad (h \text{ odd, } r \text{ and } N \text{ even}), \tag{2.6}$$

$$= \gamma S_{(N-1)/2}(2h, r) + \delta S_{(N-1)/2}(2h, h+r) \quad (h \text{ and } N \text{ odd, } r \text{ even}), \tag{2.7}$$

$$= \gamma S_{N/2-1}(2h, h+r) + \delta S_{N/2}(2h, r) \quad (h \text{ and } r \text{ odd, } N \text{ even}), \tag{2.8}$$

$$= \gamma S_{(N-1)/2}(2h, h+r) + \delta S_{(N-1)/2}(2h, r) \quad (h, r, \text{ and } N \text{ odd}). \tag{2.9}$$

3. SUMMATION FORMULAS FOR $U_n(k^2, -1 \pm k)$

3.1 Summation Formulas for $U_n(k^2, k-1)$

If $p = k^2$ and

$$q = k - 1, \tag{3.1}$$

then conditions (1.8)-(1.10) imply that we must have

$$k^2 \geq 9 \tag{3.2}$$

which, by (3.1), implies that

$$q \geq 2. \tag{3.3}$$

If (3.1) holds, then from (1.2) we have

$$\begin{cases} \alpha = k - 1 = q, \\ \beta = 1. \end{cases} \tag{3.4}$$

By using (3.4), (2.4)-(2.9) and the g.s.f. *properly* (that is, taking the value of β into account), we get the following general summation formulas:

(i) For $h \geq 2$ even

$$\sum_{n=0}^{N} U_{hn+r} = \frac{1}{q^2-1}\left[\frac{U_{h(N+1)+r} - U_r}{U_h} - (N+1)\left(1 + q\frac{1-(-1)^r}{2}\right)\right]. \tag{3.5}$$

(ii) For h odd

$$\sum_{n=0}^{N} U_{hn+r} = \frac{1}{q^2-1}\left[\frac{U_{h(N+2)+r} - U_{h(N+1)+r} - U_{h+r} - U_r}{U_{2h}} - \frac{2(q+2)(N+1) - q(-1)^r[1+(-1)^N]}{4}\right]. \tag{3.6}$$

The proofs of (3.5) and (3.6) are easy but rather tedious. For the sake of brevity, only the proof of (3.6) (for r odd and N even) will be given. Observe that letting $h = 1$ and $r = 0$ in (3.6) yields the identity

$$\sum_{n=0}^{N} U_n = \frac{1}{q^2-1}\left[U_{N+2} + U_{N+1} - \frac{2[4+(q+2)N] + [1-(-1)^N]q}{4}\right], \tag{3.7}$$

which gives the correct closed-form expression for the left-hand side of (1.11) in this case. We do not exclude the possibility that more compact expressions for (3.5) and (3.6) can be found.

Proof of (3.6) (for r odd and N even): First, use (2.8), (2.1)-(2.3), and (3.4), along with the g.s.f. to write

$$\begin{aligned}\sum_{n=0}^{N} U_{hn+r} &= \frac{1}{q^2-1}\sum_{n=0}^{N/2-1}(q^{(2n+1)h+r} - 1) + \frac{1}{q-1}\sum_{n=0}^{N/2}(q^{2nh+r} - 1) \\ &= \frac{1}{q^2-1}\left(q^{h+r}\frac{q^{hN}-1}{q^{2h}-1} - \frac{N}{2}\right) + \frac{1}{q-1}\left(q^r\frac{q^{h(N+2)}-1}{q^{2h}-1} - \frac{N+2}{2}\right).\end{aligned} \tag{3.8}$$

Then, take into account the parity of N and r, and use (1.1) and (3.4) to rewrite (3.8) as

$$\sum_{n=0}^{N} U_{nh+r} = \frac{U_{h(N+2)+r} + U_{h(N+1)+r} - U_{h+r} - U_r}{q^{2h}-1} - \frac{N}{2(q^2-1)} - \frac{(q+1)(N+2)}{2(q+1)(q-1)}$$
$$= \frac{U_{h(N+2)+r} + U_{h(N+1)+r} - U_{h+r} - U_r}{(q^2-1)U_{2h}} - \frac{qN+2q+2(N+1)}{2(q^2-1)}$$
$$= idem - \frac{(q+2)(N+1)+q}{2(q^2-1)}. \quad \text{Q.E.D.}$$

3.2 Summation Formulas for $U_n(k^2, -k-1)$

If $p = k^2$ and

$$q = -k-1, \tag{3.9}$$

then conditions (1.8)-(1.10) imply that we must have

$$k^2 \geq 1 \tag{3.10}$$

which, by (3.9), implies that

$$q \leq -2. \tag{3.11}$$

If (3.9) holds, then from (1.2) we have

$$\begin{cases} \alpha = k+1 = -q, \\ \beta = -1. \end{cases} \tag{3.12}$$

Remark 1: By replacing (3.4) and (3.12) in (1.1), one can observe that $U_n(k^2, -1+k)$ and $U_n(k^2, -1-k)$ have the same form as functions of q. Hence, we obtain summation formulas that are identical to (3.5) and (3.6).

4. OTHER SUMMATION FORMULAS

Of course, other kinds of summation formulas for $U_n(k^2, -1 \pm k)$ may be of interest. As a minor example, we show the following closed-form expressions:

$$\sum_{n=0}^{N} (-1)^n U_n = \frac{1}{q^2-1}\left[(-1)^N (U_{N+2} - U_{N+1}) + \frac{2qN + (q+2)[1-(-1)^N]}{4}\right], \tag{4.1}$$

$$\sum_{n=0}^{N} \binom{N}{n} U_n = \frac{(q+2)[(q+1)^N - 2^N] - q(1-q)^N}{2(q^2-1)}, \tag{4.2}$$

and

$$\sum_{n=0}^{N} nU_n = \begin{cases} \dfrac{N(A_{N+4} - A_{N+2}) - 2q^2(A_{N+1}-1) + (q^2-1)(U_{N+1}-1)}{(q^2-1)^2} - X_N & (N \text{ even}), \\[2ex] \dfrac{(N-1)(A_{N+4} - A_{N+2}) - 2q^2(A_N - 1) + (q^2-1)(U_{N+2}-1)}{(q^2-1)^2} - Y_N & (N \text{ odd}), \end{cases} \tag{4.3}$$

where

$$A_N = U_N + U_{N-1}, \tag{4.4}$$

$$X_n = N[(q+2)N+2]/[4(q^2-1)], \tag{4.5}$$

$$Y_N = (N+1)[(q+2)N+q]/[4(q^2-1)]. \tag{4.6}$$

The (partial) proof of (4.2) is given below, whereas the proofs of (4.1) and (4.3) are left as an exercise for the interested reader. We confine ourselves to mentioning that the proof of (4.3) involves the use of (3.1) of [1].

It has to be noted that the summation formulas (3.5), (3.6), and (4.1)-(4.3) also apply for negative values of h and/or r. Obviously, the extension of (normal) Lehmer numbers through negative values of the subscripts may be required. In fact, from (1.1) and (1.5), we readily get

$$U_{-n} = -U_n / q^n. \tag{4.7}$$

Proof of (4.2) $[U_n \equiv U_n(k^2, k-1), N$ **even**$]$**:** By using (2.6), (3.4), and the identities available in [6; Ex. 4, p. 133], the left-hand side of (4.2) can be written as

$$\frac{1}{q^2-1}\sum_{n=0}^{N/2}\binom{N}{2n}q^{2n} + \frac{1}{q-1}\sum_{n=0}^{N/2-1}\binom{N}{2n+1}q^{2n+1} - \frac{1}{q^2-1}\left[\sum_{n=0}^{N/2}\binom{N}{2n} + (q+1)\sum_{n=0}^{N/2-1}\binom{N}{2n+1}\right]$$

$$= \frac{1}{2(q^2-1)}[(1+q)^N + (1-q)^N] + \frac{1}{2(q-1)}[(1+q)^N - (1-q)^N] - \frac{2^N + q2^{N-1}}{q^2-1}$$

$$= \frac{(q+2)[(1+q)^N - 2^N] - q(1-q)^N}{2(q^2-1)}. \quad \text{Q.E.D.}$$

Remark 2: By virtue of Remark 1, the case $U_n \equiv U_n(k^2, -k-1)$ (N even) is also covered by the above proof.

5. CONCLUDING REMARKS

Let us conclude this note by pointing out some simple properties of the special Lehmer numbers that might be of some interest. For notational convenience, put

$$U_n(k^2, k-1) \stackrel{\text{def}}{=} U_n^+(k) = \begin{cases} [(k-1)^n - 1]/[k(k-2)] & (n \text{ even}) \\ [(k-1)^n - 1]/(k-2) & (n \text{ odd}) \end{cases} \quad (k \geq 3), \tag{5.1}$$

$$U_n(k^2, -k-1) \stackrel{\text{def}}{=} U_n^-(k) = \begin{cases} [(k+1)^n - 1]/[k(k+2)] & (n \text{ even}) \\ [(k+1)^n + 1]/(k+2) & (n \text{ odd}) \end{cases} \quad (k \geq 1). \tag{5.2}$$

(i) From (5.1) and (5.2), the following identities can easily be derived.

$$U_{2n}^-(k) = U_{2n}^+(k+2), \tag{5.3}$$

$$U_{2n+1}^-(k) = [kU_{2n+1}^+(k+2) + 2]/(k+2), \tag{5.4}$$

$$U_n^+(k) = [U_{n+2}^+(k) + (k-1)^2 U_{n-2}^+(k)]/[k(k-2)+2], \tag{5.5}$$

$$U_n^-(k) = [U_{n+2}^-(k) + (k+1)^2 U_{n-2}^-(k)]/[k(k+2)+2], \tag{5.6}$$

$$U_{2n}^{+}(k) = [U_{2n+1}^{+}(k) - U_{2n+2}^{+}(k)]/(k-1) \tag{5.7}$$

$$= -1 + [U_{2n+1}^{+}(k) - (k-1)(k-2)U_{2n-1}^{+}(k) + 1]/k, \tag{5.7'}$$

$$U_{2n}^{-}(k) = [U_{2n+2}^{-}(k) - U_{2n+1}^{-}(k)]/(k+1) \tag{5.8}$$

$$= [U_{2n+1}^{-}(k) - (k+1)U_{2n-1}^{-}(k)]/k^2. \tag{5.8'}$$

(ii) As a final remark, we observed that the numbers $U_n^{+}(k)$ seem to be related to the *central factorial numbers of the second kind* $T(h,n)$ (e.g., see [3]). More precisely, we found the following identities, the proofs of which are based on (1.1), (3.4), (4.7), and the definition of $T(h,n)$.

Proposition: For n an arbitrary integer, we have

$$U_{2n}^{+}(3) = T(2n+2, 4), \tag{5.9}$$

$$U_{2n}^{+}(4) = 4^{n-1}T(2n+1, 3), \tag{5.10}$$

$$U_{2n+1}^{+}(5) = T(4n+4, 4). \tag{5.11}$$

A possible generalization of (5.9)-(5.11) will be the object of a future study.

ACKNOWLEDGMENTS

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DYNAMICS OF THE MÖBIUS MAPPING AND FIBONACCI-LIKE SEQUENCES

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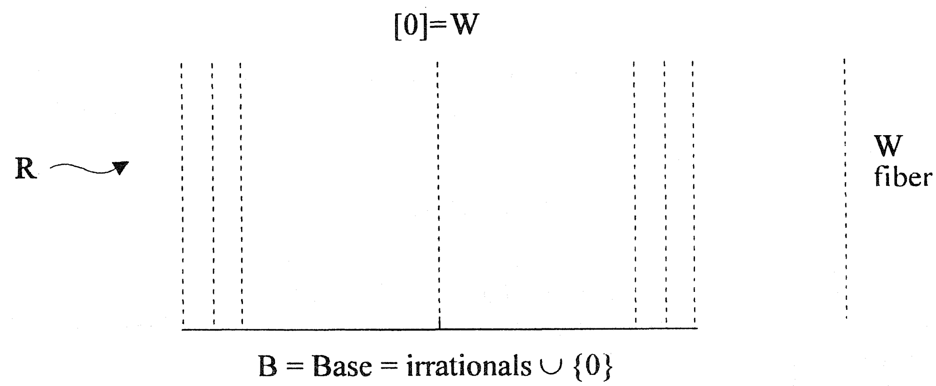
Any two numbers $\varsigma, \eta \in \mathbf{R}$ are equivalent ($\varsigma \sim \eta$) if and only if there exists

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2, \mathbf{Z}) \equiv \{A \in M_2(\mathbf{Z}); |\det A| = 1\},$$

such that

$$\varsigma = f_A(\eta) \equiv \frac{a\eta + b}{c\eta + d}.$$

It is well known [4] that the above equivalence relation "~" provides us with the following fibration of **R:**



Consider now the dynamical system $(\mathbf{R}, f_A)$ with the specially chosen Möbius mapping $f_A : \mathbf{R} \to \mathbf{R}$; $A \in U(2, \mathbf{Z})$. One sees then that f_A acts along fibers. That is,

$$\forall b \in B : [b] \ni x \to f_A(x) \in [b] \Rightarrow \forall n \in \mathbf{N} : \left\{f_A^k(x)\right\}_{k=1}^{n} \subset [b].$$

(Naturally, $f_A^n = f_{A^n}$.)

An example of such dynamics is $(\mathbf{R}, f_A)$ with $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in U(2, \mathbf{Z})$. This was investigated in [3].

In this note, the authors give a concise presentation of the dynamics generated by iteration of the arbitrary Möbius transformation $f_{\hat{A}}$; $\hat{A} = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$; $\det \hat{A} \equiv -t \neq 0$.

In view of the Cayley-Hamilton theorem, it is enough to consider the matrices of the form $A = \left(\begin{smallmatrix} s & t \\ 1 & 0 \end{smallmatrix}\right)$, where $s = \operatorname{Tr} \hat{A}$ and $t = -\det \hat{A}$; $\hat{A} \in \mathrm{GL}(2, \mathbf{R})$.

Naturally,

$$\hat{A}^2 = s\hat{A} + t\mathbf{1};\ \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\hat{A}^{n+1} = H_{n+1}\hat{A} + tH_n\mathbf{1}, \tag{1}$$

where

$$H_{n+2} = sH_{n+1} + tH_n,\ H_0 = 0, H_1 = 1;\ n \in \mathbf{N} \cup \{0\} \tag{2}$$

and

$$H_n(s,t) \equiv H_n.$$

It is also easy to see that when $A = \left(\begin{smallmatrix} s & t \\ 1 & 0 \end{smallmatrix}\right)$,

$$A^n = \begin{pmatrix} H_{n+1} & tH_n \\ H_n & tH_{n-1} \end{pmatrix};\quad n \in \mathbf{N}. \tag{3}$$

The singular point of the transformation f_A is 0. However, this point is never reached unless one chooses $x_0 \in S_A$ (or $x_0 = 0$) as a starting point, where

$$S_A = \left\{ v_n \in \mathbf{R};\ v_n = f_A^{-n}(0);\ n \in \mathbf{N} \right\} \Rightarrow S_A = \left\{ v_n;\ v_n = -t\frac{H_n}{H_{n+1}};\ n \in \mathbf{N} \right\}.$$

Note, however, that for $A \notin U(2, \mathbf{Z})$ the trajectories $\{f_A^n(x);\ x \notin S_A;\ n \in \mathbf{N}\}$ run across $[b] \sim \mathbf{W}$ fibers of $\mathbf{R}$.

It is also useful to note the following. Let us call $(\mathbf{R}, f_{\hat{A}})$ and $(\mathbf{R}, f_{\hat{B}})$ equivalent and write $(\mathbf{R}, f_{\hat{A}}) \sim (\mathbf{R}, f_{\hat{B}})$ if and only if $\exists U \in \mathrm{GL}(2, \mathbf{R});\ \hat{B} = U^{-1}\hat{A}U$. Then the characteristic points of the dynamical system, that is, the set $S_{\hat{B}}$ (see the definition of S_A), the attracting (stable) fixed point as well as the unstable fixed point of the $(\mathbf{R}, f_{\hat{B}})$ system are just the corresponding characteristic points of $(\mathbf{R}, f_{\hat{A}})$ shifted by f_U Möbius transformation. For example,

$$\left(\mathbf{R}, f_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)}\right) \text{ of [3] is equivalent to } \left(\mathbf{R}, f_{\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)}\right) \text{ with } U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As far as these characteristic points of the dynamic system $(\mathbf{R}, f_{\hat{A}})$ are concerned, the overall picture of all dynamics is the same as in [3] under the condition that there are two fixed points of f_A, that is, we have

$$x_\pm^* = \frac{s \pm \sqrt{s^2 + 4t}}{2} \quad \text{where } s^2 + 4t > 0 \tag{4}$$

and

$$\left| \frac{d}{dx} f_A(x) \right|_{x = x_+^*} < 1, \tag{5}$$

$$\left| \frac{d}{dx} f_A(x) \right|_{x = x_-^*} > 1. \tag{6}$$

Conditions (5) and (6) impose calculable restrictions on the s and t parameters. If these are satisfied, then x_+^* is a stable attracting point. That is, the sequence $x_n = f_A^n(x_0)$, $x_0 \notin S_A$ converges to x_+^* (almost regardless of the choice of starting point x_0). The x_-^* is then the unstable fixed point. When $x_0 \neq x_-^*$, the sequence x_n converges to x_-^* if and only if $\exists N;\ \forall n > N;\ x_n = x_-^*$. One proves this via *a contratio* reasoning (see [2]). Explicitly, one has, for any unstable fixed point

$$\forall x_0 \in \mathbf{U}_A;\ x_n \to x_-^*,$$

where

$$\mathbf{U}_A = \left\{\chi_n;\ \chi_n = f_A^{-n}(x_-^*)\ n \in \mathbf{N}\right\} \Rightarrow \mathbf{U}_A = \left\{\chi_n;\ \chi_n = t\frac{\Xi_n}{\Xi_{n+1}}\ n \in \mathbf{N}\right\}, \tag{7}$$

where

$$\Xi_{n=2} = s\cdot\Xi_{n+1} - \Xi_n,\ \Xi_0 = -x_-^*,\ \Xi_1 = 1. \tag{8}$$

That is, apart from the set S_A another characteristic set $\mathbf{U}_A$ is attributed to the dynamical system$(\mathbf{R}, f_A)$.

However, conditions (5) and (6) need not be met. For example, $f_A;\ A = \left(\begin{smallmatrix}2 & -1\\ 1 & 0\end{smallmatrix}\right)$ has only one fixed point $x_+^* = x_-^* \equiv x^*$, and

$$\left|\frac{d}{dx} f_A(x^*)\right| = 1.$$

It is easy to see that, for all x_0, $x_0 \notin S_A$, $f_A^n(x_0) \xrightarrow{n\to\infty} 1$.

However, starting at $x_0 = 1 - \varepsilon$ ($\varepsilon > 0$; ε small) the iterates x_n move away from 1. Hence, x^* is not an attracting fixed point. Note the difference from (6); the argument giving rise to the set $\mathbf{U}_A$ necessitates an inequality $\left|\frac{d}{dx} f_A(x^*)\right| > 1$ for an unstable fixed point (see [2]).

Following [2], one states that, to any single fixed point $x^* = x_+^* = x_-^*$, there corresponds a set $\{f_A;\ \beta \neq 0\}$ of Möbius maps where

$$A = \begin{pmatrix} x^* & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & \beta\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & -x^*\end{pmatrix};\ \beta \neq 0.$$

Since $s = \operatorname{Tr} A = 2$ and $t = -\det A = -1$, the above f_A Möbius transformation with $A = \left(\begin{smallmatrix}2 & -1\\ 1 & 0\end{smallmatrix}\right)$ is representative of the whole class of equivalent dynamical systems $\{(\mathbf{R}, f_{\hat{A}});\ \operatorname{Tr}\hat{A} = 2, \det\hat{A} = 1\}$. (Note that f_A acts along $\mathbf{W}$-fibers of $\mathbf{R}$.)

In conclusion, we state that the general features of

$$\left(\mathbf{R}, f_{\left(\begin{smallmatrix}0 & 1\\ 1 & 1\end{smallmatrix}\right)}\right)$$

dynamics described in [3] are typical for the dynamics $(\mathbf{R}, f_A)$ when $s^2 + 4t > 0$, where $\operatorname{Tr} A = s$, $\det A = -t$. For $s^2 + 4t > 0$, one has one stable attracting point x_+^* and one unstable repelling point x_-^*, as then

$$A \sim \begin{pmatrix} x_+^* & 0 \\ 0 & x_-^* \end{pmatrix} \equiv D.$$

That is, $(\mathbf{R}, f_A) \sim (\mathbf{R}, f_D)$ and $f_U(0) = x_-^*$, while $f_U(\infty) = x_+^*$;

$$U = \begin{pmatrix} 1 & 1 \\ 1/x_+^* & 1/x_-^* \end{pmatrix}.$$

The fixed point x_-^* is therefore the repelling one even for

$$\left|\frac{d}{dx} f_A(x_-^*)\right| = 1 \Leftrightarrow \{s^2 + 4t = 1 \vee 2s^2 = 1 + \sqrt{1+16t}\}.$$

The general features of the $(\mathbf{R}, f_A)$ dynamical system apart from fixed points consist of two descending sequences of intervals

$$\{[\mu_n, \mu_{n+1}]\};\ \mu_n = \frac{H_{n+1}}{H_n};\ \text{ and }\ \{[\nu_n, \nu_{n+1}]\};\ \nu_n = -t\frac{H_n}{H_{n+1}};$$

which, by virtue of (3), converge correspondingly to x_+^* and $x_-^* \equiv -t/x_+^*$.

In this note, we also notice that $\mathbf{U}_A$, the set of points defined by (7) and (8), is attributed to the $(\mathbf{R}, f_A)$ dynamical system with an unstable fixed point x_-^*.

The detailed behavior of the $(\mathbf{R}, f_A)$ iterative system is then finally established by the following sequence of bijections (for $t > 0$, $s > 0$):

$$\begin{aligned}
&f_A\colon (\nu_2, 0) \to (-\infty, \nu_1),\\
&f_A\colon (-\infty, \nu_1) \to (0, \infty),\\
&f_A\colon (\nu_{2n+2}, \nu_{2n}) \to (\nu_{2n-1}, \nu_{2n+1}),\\
&f_A\colon (\nu_{2n+1}, \nu_{2n+3}) \to (\nu_{2n+2}, \nu_{2n}),\\
&f_A\colon (0, x_+^*] \to [x_+^*, \infty),\\
&f_A\colon [x_+^*, \infty) \to (0, x_+^*].
\end{aligned}$$

The above shows that any point $x_0 \in \mathbf{R}$ (such that $x_0 \notin \mathbf{U}_A$ and $x_0 \notin S_A$) escapes from any vicinity of x_-^* and runs to x_+^*. This is also illustrated in the figures presented below.

The case of $s^2 + 4t = 0$ is the limit case. Thus, one has

$$x^* = x_+^* = x_-^* = \frac{s}{2};\ \mu_n = \frac{s}{2}\left(1 + \frac{1}{n}\right) \to x^*$$

and

$$S_A = \left\{\nu_n = -t\frac{H_n}{H_{n+1}} = \frac{s}{2}\cdot\frac{n}{n+1};\ n \in \mathbf{N}\right\}$$

because the Fibonacci-like sequence $\{H_n\}$ is now given by $H_n = (s/2)^{n-1}\cdot n$; $n \in \mathbf{N}$, and $H_0 = 0$.

As in the case of $A = \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ considered above, we have for all $x_0 \in \mathbf{R}$; $x_0 \notin S_A \cup \{0\}$ $(s^2 + 4t = 0)$:

$$f_A^n(x_0) \xrightarrow{n\to\infty} \frac{s}{2}.$$

One also easily sees from

$$f_A^n(x^* + \varepsilon) = x^* \frac{x^* + (n+1)\varepsilon}{x^* + n\varepsilon}$$

that, for small ε, the first iterates $x_n \equiv f_A^n(x^* + \varepsilon)$ are attracted or repelled, depending on whether x^* and ε are of the same sign or not. The fixed point $s/2$ is therefore neither attracting nor repelling.

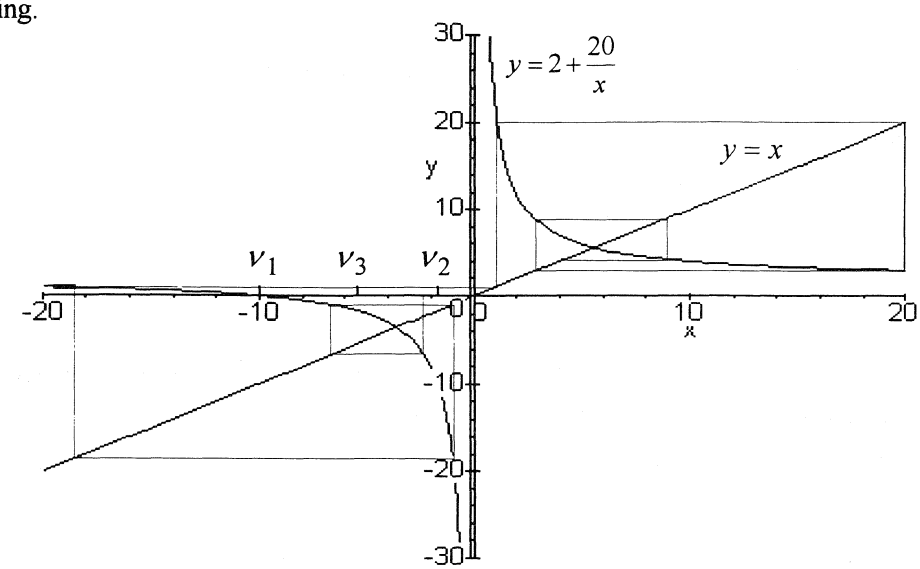


FIGURE 1. Illustration of the General Behavior of the Dynamical System with Two Fixed points ($s = 1$; $t = 20$)

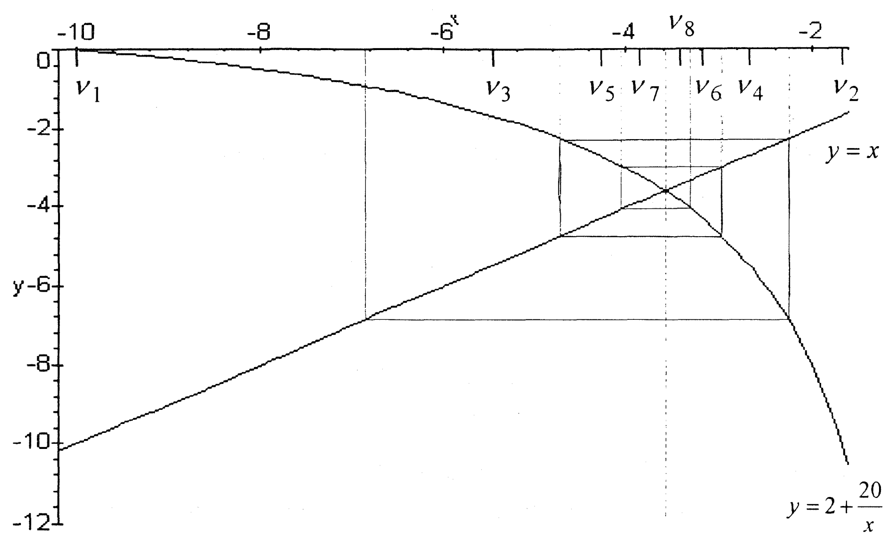


FIGURE 2. Magnification of the x_-^* Neighborhood from Figure 1

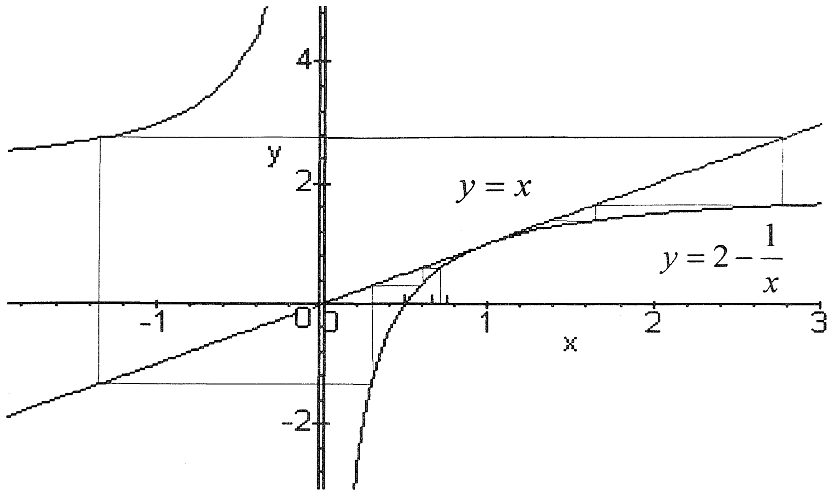


FIGURE 3. Illustration of the General Behavior of the Dynamical System with One Fixed Point ($s = 2$; $t = -1$)

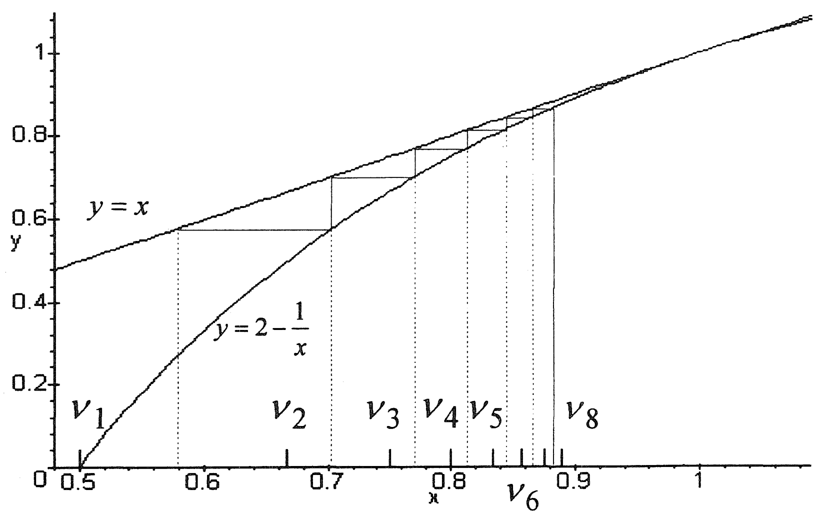


FIGURE 4. Magnification of the x_-^* Neighborhood from Figure 3

In the case of $s^2 + 4t = 0$, $s > 0$, the detailed behavior of the $(\mathbf{R}, f_A)$ iterative system is established again through the following sequence of bijections:

$$\begin{aligned}
f_A&: (s/2, \infty) \to (s/2, \infty),\\
f_A&: (-\infty, 0) \to (s/2, \infty),\\
f_A&: (v_1, 0) \to (-\infty, 0),\\
f_A&: (v_1, v_2) \to (0, v_1),\\
f_A&: (v_{n+1}, v_{n+2}) \to (v_n, v_{n+1}).
\end{aligned}$$

The cases $s = 0$ and $s^2 + 4t < 0$ (that is, without *real* fixed points) are easily treated, too (see [2]). In this case, one may encounter also finite periodic orbits (as, for example,

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^6 = 1 \text{ or } \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}^3 = 1,$$

etc.) if

$$\exists n \in N; \left(\frac{x_+^*}{x_-^*}\right)^n = 1;$$

otherwise, orbit forms a dense subset of an interval.

The presented investigation also provides one with some general insights that are useful for describing the $(\mathscr{C}, f_A)$ dynamical system, where $\mathscr{C}$ stands for Clifford algebra and f_A is a corresponding Möbius transformation in $\mathbf{R}^n$ (see [1]). There, the Clifford numbers' valued Fibonacci-like sequences play a role similar to that of the $\{H_n\}_0^\infty$ and $\{\Xi_n\}$ sequences in the $\mathbf{R}$ case.

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SOME IDENTITIES INVOLVING GENERALIZED SECOND-ORDER INTEGER SEQUENCES

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1. INTRODUCTION

In the notation of Horadam [2], write

$$W_n = W_n(a,b;\, p,q),$$

so that

$$W_n = pW_{n-1} - qW_{n-2},\quad W_0 = a,\ W_1 = b,\quad n \ge 2. \tag{1.1}$$

If α and β, assumed distinct, are the roots of $\lambda^2 - p\lambda + q = 0$, we have the Binet form [2]:

$$W_n = A\alpha^n + B\beta^n, \tag{1.2}$$

where $A = \frac{b-a\beta}{\alpha-\beta}$ and $B = \frac{a\alpha-b}{\alpha-\beta}$.

The sequence $\{W_n\}$ has been studied in the recent papers of Melham and Shannon [4], [5]. The purpose of this article is to establish some new identities involving W_n by using the method of Carlitz and Ferns [1].

Throughout this paper, the symbol $\binom{n}{i,j}$ is defined by $\binom{n}{i,j} = \frac{n!}{i!\,j!\,(n-i-j)!}$.

2. THE MAIN RESULTS

Carlitz and Ferns [1] have given a large number of interesting Fibonacci and Lucas identities. By adapting their method to the sequence $\{W_n\}$, we have obtained the following results.

Theorem 2.1:

$$W_{2n+k} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} p^j q^{n-j} W_{j+k}. \tag{2.1}$$

Lemma: Let $u = \alpha$ or β, then

(i) $-pq + (p^2 - q)u = u^3,$ (2.2)

(ii) $-q^3 + pq^2u + u^6 = (p^2 - 2q)u^4,$ (2.3)

(iii) $-q^5 + pq^4u + u^{10} = (p^4 - 4p^2q + 2q^2)u^6,$ (2.4)

(iv) $-q^9 + pq^8u + u^{18} = \Delta u^{10},$ (2.5)

where $\Delta = p^8 - 8p^6q + 20p^4q^2 - 16p^2q^3 + 2q^4$.

Theorem 2.2:

$$(p^2 - q)W_{k+1} - pqW_k = W_{k+3}, \tag{2.6}$$

$$-q^3W_k + pq^2W_{k+1} + W_{k+6} = (p^2 - 2q)W_{k+4}, \tag{2.7}$$

$$-q^5W_k + pq^4W_{k+1} + W_{k+10} = (p^4 - 4p^2q + 2q^2)W_{k+6}, \tag{2.8}$$

$$-q^9W_k + pq^8W_{k+1} + W_{k+18} = \Delta W_{k+10}. \tag{2.9}$$

Theorem 2.3:

$$W_{3n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{i+s} p^{2j+s} q^{i+s} W_{i+j+k}, \tag{2.10}$$

$$W_{n+k} = (-q)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j p^{2j+s} q^s W_{3i+j+k}. \tag{2.11}$$

Theorem 2.4:

$$W_{n+k} = (pq^2)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j q^{3s} (p^2-2q)^i W_{4i+6j+k}, \tag{2.12}$$

$$W_{4n+k} = (p^2-2q)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^s p^j q^{2j+3s} W_{6i+j+k}, \tag{2.13}$$

$$W_{6n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j p^j q^{3s+2j} (p^2-2q)^i W_{4i+j+k}. \tag{2.14}$$

Theorem 2.5:

$$W_{n+k} = (pq^4)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j q^{5s} (p^4-4p^2q+2q^2)^i W_{6i+10j+k}, \tag{2.15}$$

$$W_{6n+k} = (p^4-4p^2q+2q^2)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^s p^j q^{4j+5s} W_{10i+j+k}, \tag{2.16}$$

$$W_{10n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j p^j q^{5s+4j} (p^4-4p^2q+2q^2)^i W_{6i+j+k}. \tag{2.17}$$

Theorem 2.6:

$$W_{n+k} = (pq^8)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j q^{9s} \Delta^i W_{10i+18j+k}, \tag{2.18}$$

$$W_{10n+k} = \Delta^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^s p^j q^{8j+9s} W_{18i+j+k}, \tag{2.19}$$

$$W_{18n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j p^j q^{9s+8j} \Delta^i W_{10i+j+k}. \tag{2.20}$$

3. THE PROOFS OF THE MAIN RESULTS

Since α and β are roots of $\lambda^2 - p\lambda + q = 0$, then

$$\alpha^2 = p\alpha - q, \tag{3.1}$$

$$\beta^2 = p\beta - q. \tag{3.2}$$

Now, by the binomial theorem, we have

$$\alpha^{2n} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} p^j q^{n-j} \alpha^j, \tag{3.3}$$

$$\beta^{2n} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} p^j q^{n-j} \beta^j. \tag{3.4}$$

Theorem 2.1 follows if we multiply both sides of (3.3) and (3.4) by α^k and β^k, respectively, and use the Binet form (1.2).

The Lemma can be proved by using (3.1) and (3.2). We prove only (2.3) since the proofs of (2.2), (2.4), and (2.5) are similar.

Proof of (2.3): Using (3.1) and (3.2), we have

$$\begin{aligned}-q^3+pq^2u+u^6 &= q^2(pu-q)+u^4(pu-q)\\ &= q^2u^2+pu^5-qu^4 = q^2u^2+pu^3(pu-q)-qu^4\\ &= (p^2-q)u^4+q^2u^2-pqu^3 = (p^2-q)u^4-qu^2(pu-q) = (p^2-2q)u^4.\end{aligned}$$

This completes the proof of (2.3).

Theorem 2.2 can be proved by using the results of the Lemma and proceeding in the same manner as the proof of Theorem 2.1.

The proofs of Theorems 2.3-2.6 are similar. Therefore, we prove only Theorem 2.4.

Proof of Theorem 2.4: By using (2.3) and the multinomial theorem, we have

$$(pq^2)^n u^n = \sum_{i+j+s=n} \binom{n}{i,\,j}(-1)^j q^{3s}(p^2-2q)^i u^{4i+6j},$$

$$(p^2-2q)^n u^{4n} = \sum_{i+j+s=n} \binom{n}{i,\,j}(-1)^s p^j q^{2j+3s} u^{6i+j},$$

$$u^{6n} = \sum_{i+j+s=n} \binom{n}{i,\,j}(-1)^j p^j q^{3s+2j}(p^2-2q)^i u^{4i+j}.$$

If we multiply both sides in the preceding identities by u^k and use the Binet form (1.2), we obtain (2.12), (2.13), and (2.14), respectively. This completes the proof of Theorem 2.4.

4. SOME CONGRUENCE PROPERTIES

From (2.12), (2.15), and (2.18), by using the decomposition

$$\sum_{i+j+s=n} = \sum_{\substack{i+j+s=n\\ i=0}} + \sum_{\substack{i+j+s=n\\ i\neq 0}},$$

we obtain

Theorem 4.1:

$$p^n q^{2n} W_{n+k} - \sum_{j=0}^{n}\binom{n}{j}(-1)^j q^{3n-3j} W_{6j+k} \equiv 0 \pmod{p^2-2q}, \tag{4.1}$$

$$p^n q^{4n} W_{n+k} - \sum_{j=0}^{n}\binom{n}{j}(-1)^j q^{5n-5j} W_{10j+k} \equiv 0 \pmod{p^4-4p^2q+2q^2}, \tag{4.2}$$

$$p^n q^{8n} W_{n+k} - \sum_{j=0}^{n}\binom{n}{j}(-1)^j q^{9n-9j} W_{18j+k} \equiv 0 \pmod{\Delta}. \tag{4.3}$$

From (2.14), (2.17), and (2.20), by also using the above decomposition and Theorem 2.1, we get the following result:

Theorem 4.2:

$$W_{6n+k} - (-1)^n q^{2n} W_{2n+k} \equiv 0 \pmod{p^2 - 2q}, \tag{4.4}$$

$$W_{10n+k} - (-1)^n q^{4n} W_{2n+k} \equiv 0 \pmod{p^4 - 4p^2q + 2q^2}, \tag{4.5}$$

$$W_{18n+k} - (-1)^n q^{8n} W_{2n+k} \equiv 0 \pmod{\Delta}. \tag{4.6}$$

5. A REMARK

Some of the results in this paper are not as "practical" as others. For example, if we put $n = 10$ and $k = 0$ in (2.13), then we seek to find W_{40}. However, on the right-hand side, we need to know $W_6, W_{12}, W_{18}, \ldots, W_{60}$ (and many other terms) in order to find W_{40}. In contrast, (2.14) is more practical since, in order to find W_{60}, we need to know the value of terms whose subscripts are much less than 60.

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NEW EDITOR

On August 31, 1998, after eighteen years of continuous service, the current editor of *The Fibonacci Quarterly* will retire. He will be replaced by:

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DUCCI-PROCESSES OF 4-TUPLES

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INTRODUCTION

The aim of this note is to investigate some properties of special sequences of 4-tuples. These sequences were first examined by Wong [7] and are called *Ducci-processes*. Wong defines them as follows ([7], pp. 97, 102):

The successive iterations of a function f are called a Ducci-process if f satisfies the following conditions:

1. There exists a function $g(x, y)$ whose domain is the set of pairs of nonnegative integers and whose range is the set of nonnegative integers.
2. $f(x_1, x_2, \ldots, x_n) = (g(x_1, x_2), g(x_2, x_3), \ldots, g(x_{n-1}, x_n), g(x_n, x_1))$.
3. The n entries of $f^k(x_1, x_2, \ldots, x_n)$ are bounded for all k. The bound depends on the initial choice of $x_1, x_2, \ldots, x_n$.

For $g(x, y) = |x - y|$ we obtain so-called *Ducci-sequences* of n-tuples and so Ducci-processes are generalized Ducci-sequences. Since Ducci-sequences were introduced in the 30s (see Ciamberlini & Marengoni [1]), they have been extensively examined (for references, see Meyers [6] or Ehrlich [2]). Most studies dealt with the following questions:

- Does every sequence of n-tuples lead to (0, ..., 0)?
- How many steps in the sequence of a given n-tuple are necessary to reach (0, ..., 0) or a cycle of n-tuples?
- What can be said about the length of the cycles?

It seems that there have been no further studies about Ducci-processes. Only Engel [3] uses them for a computer exercise for school children. He asks them to find properties of cycles of the Ducci-processes of 4-tuples for $g(x, y) = (x + y) \bmod m$.

We want to answer the above questions for this Ducci-process of 4-tuples.

STABILITY

Before giving an answer to the first question, we need some definitions. Many techniques that are applied for studying Ducci-sequences transfer in a quite obvious way to our problem. So we will use similar notation to [2] as far as possible. We denote our 4-tuples by (a, b, c, d).

Definition 1: Let $\mathfrak{D}_m$ be the operator on 4-tuples over $\mathbb{Z}$, which is defined as follows:

$$\mathfrak{D}_m(a, b, c, d) = ((a+b) \bmod m, (b+c) \bmod m, (c+d) \bmod m, (d+a) \bmod m).$$

It is clear from the definition of $\mathfrak{D}_m$ that we can choose the entries of the 4-tuples under investigation from $\mathbb{Z}_m$. As we are always—if not otherwise stated—computing over $\mathbb{Z}_m$ for some m, we will omit "mod m."

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Since the number of 4-tuples in $\mathbb{Z}_m$ is bounded, we reach a cycle of 4-tuples after a finite number of applications of $\mathcal{D}_m$.

Definition 2: Let A be a given 4-tuple. Then the smallest natural number k satisfying $\mathcal{D}_m^{k+\ell} A = \mathcal{D}_m^k A$ for some $\ell \in \mathbb{N}$ is called the *life span of* A and will be denoted as $\mathcal{L}_m(A)$.

Thus, $\mathcal{L}_m(A)$ is the number of applications of $\mathcal{D}_m$ needed to reach the cycle produced by A.

Definition 3: For a given 4-tuple A, we call the smallest natural number $\ell > 0$ satisfying $\mathcal{D}_m^{k+\ell} A = \mathcal{D}_m^k A$ for every $k \geq \mathcal{L}_m(A)$ the length of the cycle generated by A.

Considering the cycles that are produced by all possible 4-tuples with entries in $\mathbb{Z}_m$, we find at least one cycle of maximum length. We use $\ell(m)$ for this maximum length.

Definition 4: A Ducci-process is called *stable* if the cycle generated by every 4-tuple contains only one 4-tuple, i.e., $\ell(m) = 1$ (see [7]).

Obviously, the first question breaks down into two parts now:

1. For which m is the Ducci-process produced by $\mathcal{D}_m$ stable?
2. Which 4-tuples can be in a cycle of length 1?

The first part has been answered by Wong ([7], 3.(1)).

Theorem 1: The Ducci-process produced by $\mathcal{D}_m$ is stable if and only if $m = 2^r$ for some $r \in \mathbb{N}$.

As with Ducci-sequences, only one 4-tuple can be contained in a *trivial* cycle, i.e., a cycle of length 1.

Lemma 1: The 4-tuple $(0, 0, 0, 0)$ is the only 4-tuple contained in a trivial cycle and so a 4-tuple A leads to a trivial cycle if and only if $\mathcal{D}_m^k A = (0, 0, 0, 0)$ for some k.

Proof: Let $A = (a, b, c, d)$ such that $\mathcal{D}_m A = A$. Then

$$\mathcal{D}_m A = (a+b, b+c, c+d, d+a) = (a, b, c, d) = A.$$

Comparing the first entries, we deduce that $b = 0$. The other entries show that $c = 0$, $d = 0$, and $a = 0$. $\square$

Thus, every 4-tuple in a Ducci-process produced by $\mathcal{D}_m$ leads to $(0, 0, 0, 0)$ if and only if $m = 2^r$.

Theorem 1 also shows that $\ell(m) = 1$ if and only if $m = 2^r$. Consequently, for every m that is not a power of 2 there are *nontrivial* cycles, i.e., cycles of length greater than 1.

CYCLES OF 4-TUPLES

In order to determine a special 4-tuple that produces a nontrivial cycle for every $m \neq 2^r$, we introduce a very helpful symbol.

Definition 5: Let $A = (a, b, c, d)$. Set $S(A) = a+b+c+d$ (mod m) and call $S(A)$ the *sum of* A.

We set $A_0 = (1, 0, 0, 0)$ (as with Ducci-sequences, the cyclic permutations of a given n-tuple all behave alike so they are not considered separately) and $A_k = \mathcal{D}_m^k A_0$.

Lemma 2: If $m \neq 2^r$ for any r, then $A_0 = (1, 0, 0, 0)$ leads to a nontrivial cycle.

Proof: Let $B = (a, b, c, d)$ and so $S(B) = a + b + c + d$. Obviously, we have $S(\mathcal{D}_m B) = 2S(B)$ and it follows by induction that $S(\mathcal{D}_m^k B) = 2^k S(B)$.

For A_0 we get $S(A_0) = 1$ and so $S(\mathcal{D}_m^k A_0) = 2^k$. But, as m does not equal a power of 2, it follows that $2^k \not\equiv 0 \bmod m$ for every $k \in \mathbb{N}$. Thus, $(0, 0, 0, 0)$ cannot be found in the sequence produced by A_0. □

The 4-tuple A_0 also gives rise to a cycle of maximum length.

Theorem 2: The length of the cycle produced by A_0 equals $\ell(m)$ for every m and the length of the cycle produced by any 4-tuple divides $\ell(m)$.

Proof: We observe that $\mathcal{D}_m$ is a linear operator and that every 4-tuple can be written as a linear combination of the cyclic permutations of A_0. Let ℓ be the length of the cycle produced by A_0, k such that $\mathcal{D}_m^k A_0$ is in the cycle, and $B = (a, b, c, d)$ a given 4-tuple. Then $B = a(1, 0, 0, 0) + b(0, 1, 0, 0) + c(0, 0, 1, 0) + d(0, 0, 0, 1)$ and

$$\begin{aligned}\mathcal{D}_m^{\ell+k} B &= a\mathcal{D}_m^{\ell+k}(1, 0, 0, 0) + b\mathcal{D}_m^{\ell+k}(0, 1, 0, 0) + c\mathcal{D}_m^{\ell+k}(0, 0, 1, 0) + d\mathcal{D}_m^{\ell+k}(0, 0, 0, 1) \\ &= a\mathcal{D}_m^{k}(1, 0, 0, 0) + b\mathcal{D}_m^{k}(0, 1, 0, 0) + c\mathcal{D}_m^{k}(0, 0, 1, 0) + d\mathcal{D}_m^{k}(0, 0, 0, 1) = \mathcal{D}_m^k B.\end{aligned}$$

Thus, the cycle produced by A_0 has maximum length and the length of the cycle produced by B must divide $\ell(m)$. □

Here we have a close relation to the cycles of Ducci-sequences. The n-tuple $(1, 0, \ldots, 0)$ produces a cycle of maximum length in a Ducci-sequence for every n and it is not contained in a cycle itself (see [2], Corollary 2). The second statement is also valid for our 4-tuple A_0.

Lemma 3: The 4-tuple $A_0 = (1, 0, 0, 0)$ is not contained in any cycle.

Proof: Assume that A_0 is contained in a cycle. Then there is a $B = (a, b, c, d)$ such that $\mathcal{D}_m B = A_0$. Consequently,

$$a + b = 1,\ b + c = 0,\ c + d = 0,\ d + a = 0.$$

Thus, $b = -c, -c = d, d = -a$, and $b = -a$. But then $a + b = a - a = 0$, which is a contradiction to the equation for the first entry. □

In the next theorem, we use a well-known fact from number theory: $\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{t_1}} \oplus \cdots \oplus \mathbb{Z}_{p_r^{t_r}}$ if $p_1^{t_1} \cdot \ldots \cdot p_r^{t_r}$ is the decomposition of m into prime numbers, where $\oplus$ denotes the "usual" direct sum.

Theorem 3: Let $m = p_1^{t_1} \cdot \ldots \cdot p_r^{t_r}$. Then $\ell(m) = \operatorname{lcm}\{\ell(p_1^{t_1}), \ldots, \ell(p_r^{t_r})\}$ (lcm denotes the least common multiple.

Proof: We consider a sequence with A_0 as the first 4-tuple. There is a k_l for every l so that $\mathcal{D}_l^{k_l} A_0$ is contained in a cycle.

Let $m = p_1^{t_1} \cdot \ldots \cdot p_r^{t_r}$ and k be the maximum of $\{k_{p_1^{t_1}}, \ldots, k_{p_r^{t_r}}, k_m\}$. Then $\mathcal{D}_m^k A_0 = (a, b, c, d)$ lies in a cycle over $\mathbb{Z}_m$ as well as over each of the $\mathbb{Z}_{p_i^{t_i}}$. Since $\mathcal{D}_m$ is linear, we obtain

$$\underbrace{(a,b,c,d)}_{\in\mathbb{Z}_m^4} \cong (\underbrace{(a_1,b_1,c_1,d_1)}_{\in\mathbb{Z}^4_{p_1^{t_1}}},\dots,\underbrace{(a_r,b_r,c_r,d_r)}_{\in\mathbb{Z}^4_{p_r^{t_r}}}).$$

Further, $\mathcal{D}_m(a,b,c,d)\cong(\mathcal{D}_{p_1^{t_1}}(a_1,b_1,c_1,d_1),\dots,\mathcal{D}_{p_r^{t_r}}(a_r,b_r,c_r,d_r))$. Let $B=\mathcal{D}_m^k A_0$. From the above construction, it follows that $\mathcal{D}_m^h B = B$ over $\mathbb{Z}_m$ for some minimal h if and only if $\mathcal{D}_m^h B = B$ over all $\mathbb{Z}_{p_i^{t_i}}$. Clearly, h is the least common multiple of the $\ell(p_i^{t_i})$ and $\ell(m)=h$. □

Corollary 1: Let m be odd. Then $\ell(2^r m)=\ell(m)$.

Proof: The proof is obvious since $\ell(2^r)=1$ by Theorem 1. □

For our further investigation, we have to examine four special 4-tuples more closely. Let

$$X_1=(1,-1,1,-1),\ X_2=(1,1,1,1),\ X_3=(1,-1,-1,1),\ X_4=(1,1,-1,-1).$$

If p is an odd prime, these 4-tuples are linearly independent over $\mathbb{Z}_p$, so every 4-tuple can be written as a linear combination of the X_i over $\mathbb{Z}_p$ in exactly one way. Further, the 4-tuples X_i have some special properties:

$$\begin{aligned}
\mathcal{D}_p X_1 &= (0,0,0,0),\\
\mathcal{D}_p X_2 &= 2X_2,\\
\mathcal{D}_p X_3 &= X_3 - X_4,\\
\mathcal{D}_p^2 X_3 &= -2X_4,\\
\mathcal{D}_p X_4 &= X_3 + X_4,\\
\mathcal{D}_p^2 X_4 &= 2X_3.
\end{aligned}$$

We consider $A_1=(1,0,0,1)=\mathcal{D}_m A_0$. If m is an odd prime, we can write A_1 as

$$A_1 = 2^{-1}((1,1,1,1)+(1,-1,-1,1)) = 2^{-1}(X_2+X_3).$$

By induction, we deduce the following set of equations (the powers of 2 still have to be reduced modulo p):

$$\mathcal{D}_p^{8k} A_1 = 2^{-1}(2^{8k}X_2 + 2^{4k}X_3), \tag{1}$$
$$\mathcal{D}_p^{8k+1} A_1 = 2^{-1}(2^{8k+1}X_2 + 2^{4k}(X_3 - X_4)), \tag{2}$$
$$\mathcal{D}_p^{8k+2} A_1 = 2^{-1}(2^{8k+2}X_2 - 2^{4k+1}X_4), \tag{3}$$
$$\mathcal{D}_p^{8k+3} A_1 = 2^{-1}(2^{8k+3}X_2 - 2^{4k+1}(X_3 + X_4)), \tag{4}$$
$$\mathcal{D}_p^{8k+4} A_1 = 2^{-1}(2^{8k+4}X_2 - 2^{4k+2}X_3), \tag{5}$$
$$\mathcal{D}_p^{8k+5} A_1 = 2^{-1}(2^{8k+5}X_2 - 2^{4k+2}(X_3 - X_4)), \tag{6}$$
$$\mathcal{D}_p^{8k+6} A_1 = 2^{-1}(2^{8k+6}X_2 + 2^{4k+3}X_4), \tag{7}$$
$$\mathcal{D}_p^{8k+7} A_1 = 2^{-1}(2^{8k+7}X_2 + 2^{4k+3}(X_3 + X_4)), \tag{8}$$
$$\mathcal{D}_p^{8(k+1)} A_1 = 2^{-1}(2^{8(k+1)}X_2 - 2^{4(k+1)}X_3). \tag{9}$$

Since 2 is in the group of units $\mathbb{Z}_m^*$ if and only if m is odd, these equations also hold for every such m. If m is even, the equations cannot be used, as 2 is not a unit in $\mathbb{Z}_m$ and 2^{-1} does not exist.

The above set of equations is the cornerstone of the following proofs. Before fully exploiting these equations, we need one more definition.

Definition 6: Let m be an odd number. Then we denote the order of 2 in the group of units of $\mathbb{Z}_m$ as $O_m(2)$.

Lemma 4: If m is odd, then A_1 is contained in the cycle produced by A_0.

Proof: We use equation (1):

$$\begin{aligned}\mathfrak{D}_m^{8O_m(2)}A_1 &= 2^{-1}(2^{8O_m(2)}X_2 + 2^{4O_m(2)}X_3)\\ &= 2^{-1}((2^{O_m(2)})^8 X_2 + (2^{O_m(2)})^4 X_3)\\ &= 2^{-1}(X_2 + X_3) = A_1. \ \square\end{aligned}$$

Corollary 2: If m is odd, then $\ell(m)|8O_m(2)$.

Theorem 4: For every odd m, $O_m(2)|\ell(m)$.

Proof: By Theorem 2 and Lemma 4, A_1 is in the cycle of maximum length for every odd m. Obviously, $S(A_1) = 2$. Since $S(\mathfrak{D}_m^{\ell(m)}A_1) = S(A_1) = 2$ and $S(\mathfrak{D}_m C) = 2S(C)$ for every 4-tuple C, it follows that $S(\mathfrak{D}_m^{\ell(m)-1}A_1) = 1$.

On the other hand, using $S(\mathfrak{D}_m C) = 2S(C)$, we can conclude by induction that $S(\mathfrak{D}_m^{\ell(m)-1}A_1) = 2^{\ell(m)-1}S(A_1) = 2^{\ell(m)}$; thus, $2^{\ell(m)} \equiv 1 \bmod m$. Euler's well-known theorem completes the proof. $\square$

Now we can give a characterization of $\ell(p)$ for every prime p.

Theorem 5: Let p be an odd prime. Then

$$\ell(p) = \begin{cases} O_p(2) & : \ 4|O_p(2),\ 8\nmid O_p(2),\\ 2O_p(2) & : \ 8|O_p(2),\\ 4O_p(2) & : \ 2|O_p(2),\ 4\nmid O_p(2),\\ 8O_p(2) & : \ 2\nmid O_p(2).\end{cases}$$

Proof: Corollary 2 shows $\ell(p)|8O_p(2)$. On the other hand, we know from Theorem 4 that $O_p(2)|\ell(p)$. Thus, we only have to check $O_p(2), 2O_p(2)$, and $4O_p(2)$ as possible values for $\ell(p)$.

1. $4|O_p(2),\ 8\nmid O_p(2)$: We can write $O_p(2) = 4(2s+1) = 8s+4$ for an $s \in \mathbb{N}_0$. Equation (5) shows:

$$\begin{aligned}\mathfrak{D}_p^{O_p(2)}A_1 &= 2^{-1}(2^{8s+4}X_2 - 2^{4s+2}X_3)\\ &= 2^{-1}(2^{O_p(2)}X_2 - 2^{\frac{O_p(2)}{2}}X_3)\\ &= 2^{-1}(X_2 + X_3) = A_1.\end{aligned}$$

Thus, $\ell(p) = O_p(2)$. Here we have used the fact that $(2^{[O_p(2)]/2})^2 = 2^{O_p(2)} \equiv 1 \bmod p$ and, since $\mathbb{Z}_p$ is a field, the equation $x^2 \equiv 1 \bmod p$ has the two solutions 1 and -1. From the definition of $O_p(2)$, it follows that $2^{[O_p(2)]/2} \equiv -1 \bmod p$.

2. $8|O_p(2)$: Assume that $\ell(p) = O_p(2)$. Since $O_p(2) = 8(2s+1)$, we can use equation (9):

$$\begin{aligned}\mathfrak{D}_p^{O_p(2)}A_1 &= 2^{-1}(2^{8(2s+1)}X_2 + 2^{4(2s+1)}X_3)\\ &= 2^{-1}(X_2 - X_3) \neq A_1.\end{aligned}$$

Using equation (9) again, we can conclude that $\mathfrak{D}_p^{2O_p(2)}A_1 = A_1$ and thus $\ell(p) = 2O_p(2)$.

3. $2|O_p(2)$, $4\nmid O_p(2)$: We consider $4O_p(2)$. Obviously, $8|4O_p(2)$ and so, by equation (1),

$$\begin{aligned}\mathfrak{D}_p^{4O_p(2)}A_1 &= 2^{-1}(2^{4O_p(2)}X_2 + 2^{2O_p(2)}X_3)\\ &= 2^{-1}(X_2 + X_3) = A_1.\end{aligned}$$

Now assume $\ell(p) = 2O_p(2)$. Since $O_p(2) = 2(2s+1)$, we can use equation (5):

$$\begin{aligned}\mathfrak{D}_p^{2O_p(2)}A_1 &= 2^{-1}(2^{2O_p(2)}X_2 - 2^{O_p(2)}X_3)\\ &= 2^{-1}(X_2 - X_3) \neq A_1.\end{aligned}$$

So $\ell(p) = 4O_p(2)$.

4. $2\nmid O_p(2)$: Now we can write $4O_p(2) = 4(2s+1)$ and, using basically the same calculations as in the case above, we see that $\ell(m)$ cannot equal $4O_p(2)$ or one of its divisors. □

Corollary 3: If p is a prime and $p \equiv -1 \bmod 4$, then

$$\ell(p) = \begin{cases} 4O_p(2) & : \ 2|O_p(2), \\ 8O_p(2) & : \ 2\nmid O_p(2). \end{cases}$$

Proof: By Euler's formula, $O_p(2)|(p-1)$. But $p-1 \equiv -2 \bmod 4$; thus, neither $p-1$ nor $O_p(2)$ is divisible by 4. □

Before stating another consequence of Theorem 5, we want to mention an easy way to determine whether $O_p(2)$ is even or odd.

Lemma 5: If p is a prime and $p \equiv -1 \bmod 4$, then $O_p(2)$ is odd if and only if $(p+1)/4$ is even.

For further details and the proof, see Lemma 13 in [4].

Corollary 4: Let p be a prime. If $p \equiv -1 \bmod 4$, then $\ell(p)|4(p-1)$. If $p \equiv 1 \bmod 4$, then $\ell(p)|2(p-1)$.

Proof: We treat the case $p \equiv -1 \bmod 4$ first. Obviously, $p-1$ is even. If $O_p(2)$ is odd, then $O_p(2)\left|\frac{p-1}{2}\right.$ and so $8O_p(2)\left|8\frac{p-1}{2}\right.$. If $O_p(2)$ is even, the result is obvious.

The proof for $p \equiv 1 \bmod 4$ runs along the same lines. □

Remark: If $p \equiv 1 \bmod 4$, then $\ell(p)$ is even a divisor of $p-1$. This can be shown using some techniques of Ehrlich [2] and writing $\mathfrak{D}_p$ as a sum of two operators.

We have shown that every $\ell(m)$ can be computed if the decomposition of m into prime numbers and $\ell(p^r)$ for $p^r|m$ are known. We have determined $\ell(p)$ [in terms of $O_p(2)$] but have not yet investigated powers of primes. In this case, we can give only a partial solution.

Theorem 6: Let $m = p^r$ for some odd prime p. Then

1. $\ell(p)|\ell(m)$,

2. $\ell(m)|p^{r-1}\ell(p)$.

Proof:

1. Obviously, $\mathcal{D}_m^{\ell(m)} A_1 = A_1$ and so $\mathcal{D}_p^{\ell(m)} A_1 = A_1$. Thus, $\ell(p) | \ell(m)$.
2. From $\mathcal{D}_p^{\ell(p)} A_1 = A_1$, we deduce, by induction, that $\mathcal{D}_{sp}^{s\ell(p)} A_1 = A_1$ for every odd s and, consequently, $\mathcal{D}_m^{p^{r-1}\ell(p)} A_1 = \mathcal{D}_{p^{r-1}p}^{p^{r-1}\ell(p)} A_1 = A_1$. Thus, $\ell(m) | p^{r-1}\ell(p)$. □

Remark: There are cases in which $\ell(p^r) < p^{r-1}\ell(p)$, e.g., for $p = 1093$,

$$\ell(p) = \frac{p-1}{3} = \ell(p^2).$$

We will end this section with a final observation.

Corollary 5: If m is odd, then $4 | \ell(m)$.

Proof: From Theorem 5, we deduce that $4 | \ell(p)$ for every prime p. Thus, $4 | \ell(p^r)$ by Theorem 6 and $4 | \ell(m)$ by Theorem 3. □

THE LIFE SPAN

As we have seen above, A_0 produces a cycle of maximum length. It also has the highest possible life span.

Lemma 6: Let B be a 4-tuple. Then $\mathcal{L}_m(B) \le \mathcal{L}_m(A_0)$.

Proof: B can be written as a linear combination of the cyclic permutations of A_0 (see the proof of Theorem 2). If $\mathcal{D}_m^k A_0 = (0, 0, 0, 0)$ for some k, then $\mathcal{D}_m^k C = (0, 0, 0, 0)$, where C is any cyclic permutation of A_0. Thus, $\mathcal{D}_m^k B = (0, 0, 0, 0)$. □

Therefore, we can limit our investigation to A_0. Before stating our last theorem, we need some further notations and a rather technical lemma.

Notations: Let $\mathcal{D}$ and $\mathcal{H}$ be the operators on 4-tuples over $\mathbb{Z}$ defined by $\mathcal{D}(a, b, c, d) = (a+b, b+c, c+d, d+a)$ and $\mathcal{H}(a, b, c, d) = (b, c, d, a)$. Obviously, $\mathcal{D}A \equiv \mathcal{D}_m A \bmod m$ for every 4-tuple A with entries from $\mathbb{Z}$. If every entry of A is divisible by $r \in \mathbb{N}$, we write $A \equiv 0 \bmod r$.

Lemma 7: Let $B = (b-2, b-1, b, b-1)$, where $b \ge 3$ is odd. Then $\mathcal{D}B \not\equiv 0 \bmod 2$ and $\mathcal{D}^2 B = 2\mathcal{H}C$, where $C = (c-2, c-1, c, c-1)$ and c is odd.

Proof:

$$\begin{aligned}\mathcal{D}^2 B &= \mathcal{D}(2b-3, 2b-1, 2b-1, 2b-3)\\ &= (4b-4, 4b-2, 4b-4, 4b-6)\\ &= 2(2b-2, 2b-1, 2b-2, 2b-3)\\ &= 2(c-1, c, c-1, c-2),\end{aligned}$$

where $c = 2b-1$. □

Theorem 7: Let $m \ge 2$, $m = 2^r k$ for some $r \in \mathbb{N}_0$ and k an odd natural number. Then

$$\mathcal{L}_m(A_0) = \begin{cases} 1 & : r = 0, \\ 2r+2 & : r \ge 1. \end{cases}$$

Proof:

- Let $r=0$, i.e., m is odd. Lemma 4 shows that $A_1=\mathscr{D}_m A_0$ is in a cycle and Lemma 6 completes the proof.
- Let $r\geq 1$, i.e., m is even. As in Theorem 3, we can compute over $\mathbb{Z}_{p_1^{t_1}}\oplus\cdots\oplus\mathbb{Z}_{p_s^{t_s}}$. Since $\mathscr{D}_{p^r}A_0$ is in a cycle for every odd prime p, we have to consider only the case $p_i^{t_i}=2^r$. We compute $\mathscr{D}^k A_0$:

$$\begin{aligned}A_0&=(1,0,0,0),\\A_1&=(1,0,0,1),\\A_2&=(1,0,1,2),\\A_3&=(1,1,3,3),\\A_4&=(2,4,6,4).\end{aligned}$$

Obviously, only the entries of A_4 are all divisible by 2. We can write A_4 as $A_4=2\cdot(3-2,3-1,3,3-1)$. Thus, we can apply the preceding lemma, and it follows by induction that $A_{2r+2}\equiv 0 \bmod 2^r$ and $A_k\not\equiv 0 \bmod 2^r$ for $k<2r+2$. Therefore, $A_l\equiv 0 \bmod 2^r$ if and only if $\ell\geq 2r+2$ and $\mathscr{L}_m(A_0)=2r+2$. □

ACKNOWLEDGMENTS

I am indebted to Mr. Herbert Glaser for introducing me to this interesting problem and giving me a number of useful hints. I would also like to thank the referee for a number of valuable suggestions.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Proposers should inform us of the history of the problem if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-831 ***Proposed by the editor***

Find a polynomial $f(x, y)$ with integer coefficients such that $f(F_n, L_n) = 0$ for all integers n.

B-832 ***Proposed by Andrew Cusumano, Great Neck, NY***

Find a pattern in the following numerical identities and create a formula expressing a more general result.

$$\begin{aligned}
3^5+2^5+1^5+1^5 &= 5\cdot 3^4-128\\
5^5+3^5+2^5+1^5+1^5 &= 8\cdot 5^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
8^5+5^5+3^5+2^5+1^5+1^5 &= 13\cdot 8^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
&\quad -3\cdot5\cdot8(19+2\cdot3\cdot5+2\cdot5\cdot8)\\
13^5+8^5+5^5+3^5+2^5+1^5+1^5 &= 21\cdot 13^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
&\quad -3\cdot5\cdot8(19+2\cdot3\cdot5+2\cdot5\cdot8)\\
&\quad -5\cdot8\cdot13(19+2\cdot3\cdot5+2\cdot5\cdot8+2\cdot8\cdot13)\\
21^5+13^5+8^5+5^5+3^5+2^5+1^5+1^5 &= 34\cdot 21^4-128-2\cdot3\cdot5(19+2\cdot3\cdot5)\\
&\quad -3\cdot5\cdot8(19+2\cdot3\cdot5+2\cdot5\cdot8)\\
&\quad -5\cdot8\cdot13(19+2\cdot3\cdot5+2\cdot5\cdot8+2\cdot8\cdot13)\\
&\quad -8\cdot13\cdot21(19+2\cdot3\cdot5+2\cdot5\cdot8+2\cdot8\cdot13+2\cdot13\cdot21)
\end{aligned}$$

B-833 ***Proposed by Al Dorp, Edgemere, NY***

For n a positive integer, let $f(x)$ be the polynomial of degree $n-1$ such that $f(k) = L_k$ for $k = 1, 2, 3, \ldots, n$. Find $f(n+1)$.

B-834 ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia***

For x a real number and n an integer larger than 1, prove that

$$(x+1)F_1+(x+2)F_2+\cdots+(x+n)F_n<2^n\sqrt{\frac{n(n+1)(2n+1+6x)+nx^2}{6}}.$$

B-835 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY***

In a sequence of coin tosses, a *single* is a term (H or T) that is not the same as any adjacent term. (For example, in the sequence HHTHHHTH, the singles are the terms in positions 3, 7, and 8.) Let $S(n,r)$ be the number of sequences of n coin tosses that contain exactly r singles. If $n\geq 0$ and p is a prime, find the value modulo p of $\frac{1}{2}S(n+p-1,p-1)$.

NOTE: The Elementary Problems column is in need of more *easy*, yet elegant and nonroutine problems.

SOLUTIONS

Perfect Squares

B-814 ***(Corrected) Proposed by M. N. Deshpande, Institute of Science, Nagpur, India (Vol. 34, no. 4, August 1996)***

Show that, for each positive n, there exists a constant C_n such that $F_{2n+2i}F_{2i}+C_n$ and $F_{2n+2i+1}F_{2i+1}-C_n$ are both perfect squares for all positive integers i.

Solution by Paul S. Bruckman, Seattle, WA

It is easy to show (for example, by using the Binet forms) that

$$F_{2n+j}F_j+(-1)^jF_n^2=F_{n+j}^2$$

holds for all integers n and j. Thus, $C_n=F_n$ is the solution.

Also solved by Brian D. Beasley, Russell Euler and Jawad Sadek, Herta T. Freitag, Hans Kappus, Harris Kwong, Carl Libis, David E. Manes, Bob Prielipp, H.-J. Seiffert, I. Strazdins, and the proposer.

Ternary Cubic Forms

B-815 ***Proposed by Paul S. Bruckman, Highwood, IL (Vol. 34, no. 4, August 1996)***

Let $K(a,b,c)=a^3+b^3+c^3-3abc$. Show that, if x_1, x_2, x_3, y_1, y_2, and y_3 are integers, then there exist integers z_1, z_2, and z_3 such that

$$K(x_1,x_2,x_3)\cdot K(y_1,y_2,y_3)=K(z_1,z_2,z_3).$$

Solution by H.-J. Seiffert, Berlin, Germany

Let

$$M(a,b,c)=\begin{pmatrix} a & b & c\\ c & a & b\\ b & c & a\end{pmatrix}.$$

A straightforward but tedious calculation shows that

$$M(x_1, x_2, x_3) \cdot M(y_1, y_2, y_3) = M(z_1, z_2, z_3),$$

where

$$\begin{aligned} z_1 &= x_1y_1 + x_2y_3 + x_3y_2, \\ z_2 &= x_1y_2 + x_2y_1 + x_3y_3, \\ z_3 &= x_1y_3 + x_2y_2 + x_3y_1. \end{aligned}$$

It is easily verified that $\det M(a,b,c) = K(a,b,c)$. Thus, $K(x_1, x_2, x_3) \cdot K(y_1, y_2, y_3) = K(z_1, z_2, z_3)$, where z_1, z_2, and z_3 are as given above.

Clary found this result in [1].

Reference

1. G. Chrystal. *Textbook of Algebra*, Part I, exercise 21, p. 84. New York: Dover Publications, 1961.

Also solved by Brian D. Beasley, Stuart Clary, Hans Kappus, Bob Prielipp, Adam Stinchcombe, David C. Terr, and the proposer.

Triple Rational Inequality

B-816 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN (Vol. 34, no. 4, August 1996)***

Let i, j, and k be any three positive integers. Show that

$$\frac{F_jF_k}{F_i + F_iF_jF_k} + \frac{F_kF_i}{F_j + F_iF_jF_k} + \frac{F_iF_j}{F_k + F_iF_jF_k} < 2.$$

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC

We prove the following generalization: Given $n \geq 3$, let $x_1, x_2, \ldots, x_n$ be positive integers and let $x = x_1x_2\cdots x_n$. Then

$$S = \sum_{i=1}^{n} \frac{x/x_i}{x_i + x} < n-1.$$

We begin the proof by assuming, without loss of generality, that $x_1 \leq x_2 \leq \cdots \leq x_n$. If $x_{n-1} \geq 2$, then

$$\begin{aligned} S &= \frac{x/x_1}{x_1+x} + \frac{x/x_2}{x_2+x} + \cdots + \frac{x/x_n}{x_n+x} < \frac{x/x_1}{x} + \frac{x/x_2}{x} + \cdots + \frac{x/x_n}{x} \\ &= \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \leq (n-2)(1) + 1/2 + 1/2 = n-1. \end{aligned}$$

If $x_{n-1} < 2$, then $x_1 = x_2 = \cdots = x_{n-1} = 1$, so $x = x_n$. Thus,

$$S = (n-1)\frac{x_n}{1+x_n} + \frac{1}{2x_n} = \frac{2(n-1)x_n^2 + x_n + 1}{2x_n(1+x_n)} < \frac{2(n-1)x_n^2 + 2(n-1)x_n}{2x_n^2 + 2x_n} = n-1$$

since $n \geq 3$ implies that $(2n-3)x_n > 1$.

The result in Problem B-816 now follows by taking $n = 3$ and $x_1 + 1 = F_i$, $x_2 = F_j$, and $x_3 = F_k$.

Also solved by Michel A. Ballieu, Paul S. Bruckman, H.-J. Seiffert, Adam Stinchcombe, and the proposer.

Radical Integer

B-817 ***Proposed by Kung-Wei Yang, Western Michigan University, Kalamazoo, MI (Vol. 34, no. 4, August 1996)***

Show that

$$\sqrt[k]{\sum_{i=0}^{k}\binom{k}{i}F_{ni-1}F_{n(k-i)+1}-\sum_{j=1}^{k-1}\binom{k}{j}F_{nj}F_{n(k-j)}}$$

is an integer for all positive integers k and n.

Solution by Paul S. Bruckman, Seattle, WA

We use the identity $F_{u-1}F_{v+1}-F_uF_v=(-1)^vF_{u-v-1}$. Set $u=ni$ and $v=n(k-i)$ and note that the second sum under the radical in the statement of the problem may include the terms $j=0$ and $j=k$. Replacing j by i, the given expression under the radical sign is transformed as follows:

$$\begin{aligned}\sum_{i=0}^{k}\binom{k}{i}(-1)^{n(k-i)}F_{n(2i-k)-1} &= 5^{-1/2}\sum_{i=0}^{k}\binom{k}{i}(-1)^{n(k-i)}[\alpha^{n(2i-k)-1}-\beta^{n(2i-k)-1}]\\ &= 5^{-1/2}[\alpha^{-nk-1}(\alpha^{2n}+(-1)^n)^k-\beta^{-nk-1}(\beta^{2n}+(-1)^n)^k]\\ &= 5^{-1/2}[-\beta(\alpha^n+\beta^n)^k+\alpha(\alpha^n+\beta^n)^k]\\ &= (\alpha^n+\beta^n)^k=L_n^k.\end{aligned}$$

Therefore, the given expression reduces to L_n, which is, of course, an integer (independent of k).

Also solved by H.-J. Seiffert, David Zeitlin, and the proposer.

Binomial Harmonic Sum

B-818 ***Proposed by L. C. Hsu, Dalian University of Technology, Dalian, China (Vol. 34, no. 4, August 1996)***

Let $H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. Find a closed form for

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{2k}.$$

Editorial Note: *This problem also appeared as Problem 60 in the November 1996 issue of the journal* Math Horizons. *I apologize for the duplication. The problem column editor for* Math Horizons *asked me if I had any problems they could use, and since this problem (which was originally submitted here) did not involve Fibonacci numbers, I released it to them with the author's permission. Unfortunately, I forgot to delete it from my computer files, so when we started running low on problems, I inadvertently used it.*

Solution by Hans Kappus, Rodersdorf, Switzerland

Let us tackle the more general sum

$$S(p,n)=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{pk},$$

where p and n are positive integers.

In a first attempt, one may try to look for a recurrence. Thus,

$$S(p,n+1)=\sum_{k=1}^{n}(-1)^{k-1}\left[\binom{n}{k}+\binom{n}{k-1}\right]H_{pk}+(-1)^n H_{p(n+1)}$$
$$=S(p,n)+H_p+\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_{p(k+1)}.$$

But

$$H_{p(k+1)}=H_p+\sum_{i=1}^{p}\frac{1}{pk+i}.$$

Hence,

$$S(p,n+1)=\sum_{i=1}^{p}J_p(i,n),$$

where

$$J_p(i,n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{pk+i}.$$

Using

$$\frac{1}{pk+i}=\int_0^1 x^{pk+i-1}\,dx$$

and the Binomial Theorem leads to

$$J_p(i,n)=\int_0^1 x^{i-1}(1-x^p)^n\,dx;\quad 1\le i\le p.$$

Obviously, $J_p(i,0)=1/i$ and, using integration by parts, we find

$$J_p(i,n)=\frac{pn}{i}J_p(p+i,n-1).$$

The explicit formula

$$J_p(i,n)=p^n n!\prod_{j=0}^{n}\frac{1}{jp+i}$$

can now be proved by an easy induction. Writing n instead of $n+1$ again, our final result reads:

$$S(p,n)=p^{n-1}(n-1)!\sum_{i=1}^{p}\prod_{j=0}^{n-1}\frac{1}{jp+i}.$$

For the special case $p=2$, we get the neat formula

$$S(2,n)=\frac{1}{2n}+\frac{2^{2n-1}}{n\binom{2n}{n}}.$$

Lord found the related formula: $\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_k=\frac{1}{n}$.

Also solved by Paul S. Bruckman, Carl Libis, Graham Lord, David E. Manes, H.-J. Seiffert, David Zeitlin, and the proposer.

Finding a Pellian Identity

B-819 ***Proposed by David Zeitlin, Minneapolis, MN***
(Vol. 34, no. 4, August 1996)

Find integers a, b, c, and d (with $1 < a < b < c < d$) that make the following an identity:

$$P_n = P_{n-a} + 444P_{n-b} + P_{n-c} + P_{n-d},$$

where P_n is the Pell sequence, defined by $P_{n+2} = 2P_{n+1} + P_n$, for $n \geq 0$, with $P_0 = 0$, $P_1 = 1$.

Solution by H.-J. Seiffert, Berlin, Germany

If Q_n is the Pell-Lucas sequence, defined by the recurrence $Q_{n+2} = 2Q_{n+1} + Q_n$, with initial conditions $Q_0 = Q_1 = 2$, then (see [1], page 12, equations 3.22 and 3.24)

$$Q_r P_m = P_{m+r} + (-1)^r P_{m-r}$$

for all integers r and m. From this equation, it easily follows that

$$P_n = P_{n-u+v} + (Q_u - Q_v)P_{n-u} + P_{n-u-v} + P_{n-2u}$$

if u is odd and v is even.

Taking $u = 7$ and $v = 4$, and noting that $Q_4 = 34$ and $Q_7 = 478$, we find

$$P_n = P_{n-3} + 444P_{n-7} + P_{n-11} + P_{n-14},$$

showing that $a = 3$, $b = 7$, $c = 11$, and $d = 14$ work.

Reference

1. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.

Also solved by Paul S. Bruckman, Curtis Cooper, Daina Krigens, Carl Libis, David E. Manes, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-529 ***Proposed by Paul S. Bruckman, Highwood, IL***

Let ρ denote the set of Pythagorean triples (a, b, c) such that $a^2+b^2=c^2$. Find all pairs of integers $m, n > 0$ such that $(a, b, c) = (F_mF_n, F_{m+1}F_{n+2}, F_{m+2}F_{n+1}) \in \rho$.

H-530 ***Proposed by Andrej Dujella, University of Zagreb, Croatia***

Let $k(n)$ be the period of a sequence of Fibonacci numbers $\{F_i\}$ modulo n. Prove that $k(n) \le 6n$ for any positive integer n. Find all positive integers n such that $k(n) = 6n$.

H-531 ***Proposed by Paul S. Bruckman, Highwood, IL***

Consider the sum $S = \sum_{n=1}^{\infty} t(n)/n^2$, where $t(1) = 1$ and $t(n) = \prod_{p|n}(1-p^{-2})^{-1}$, $n > 1$, the product taken over all prime p dividing n. Evaluate S and show that it is rational.

SOLUTIONS

Comment by H.-J. Seiffert

Correction: The identity of Problem H-510 should read

$$P_n = \sum_{k \in A_n} (-1)^{[(3k-2n+3)/4]} 2^{[3k/2]} \binom{n+k}{2k+1}.$$

The proposer's solution, however, is correct. The mistake arose in the very last step, when replacing n by $n-1$. Indeed, H-510 is the proposer's first (incorrect) version of H-476.

Continued

H-509 ***Proposed by Paul S. Bruckman, Salmiya, Kuwait***
(Vol. 34, no. 2, May 1996)

The *continued fractions* (*base k*) are defined as follows:

$$[u_1, u_2, \ldots, u_n]_k = u_1 + \frac{k}{u_{2^+}} \frac{k}{u_{3^+}} \cdots \frac{k}{u_n}, \quad n = 1, 2, \ldots, \tag{1}$$

where k is an integer $\ne 0$ and $(u_i)_{i=1}^{\infty}$ is an arbitrary sequence of real numbers.

Given a prime p with $\left(\frac{-k}{p}\right) = 1$ (Legendre symbol) and $k \not\equiv 0 \pmod{p}$, let h be the solution of the congruence

$$h^2 \equiv -k \pmod{p}, \text{ with } 0 < h < \tfrac{1}{2}p. \tag{2}$$

Suppose a symmetric continued fraction (base k) exists, such that

$$\frac{p}{h} = [a_1, a_2, \ldots, a_{n+1}, a_{n+1}, \ldots, a_1]_k, \tag{3}$$

where the a_i's are integers, n is even, and $k \mid a_i$, $i = 2, 4, \ldots, n$. Then show that integers x and y exist, with $\text{g.c.d.}(x, y) = 1$, given by

$$\frac{x}{y} = [a_{n+1}, \ldots, a_1]_k \tag{4}$$

that satisfy

$$x^2 + ky^2 = p. \tag{5}$$

Solution by the proposer

Let $[u_1, u_2, \ldots, u_n]_k = p_n / q_n, n = 1, 2, \ldots,$ define the n^{th} convergent of the c.f. (base k), assuming that the u_i's are integers. The p_n's and q_n's satisfy the common recurrence

$$z_n = u_n z_{n-1} + k z_{n-2},\ n = 3, 4, \ldots. \tag{6}$$

Also, $p_1 / q_1 = [u_1]_k = u_1 / 1$ and $p_2 / q_2 = [u_1, u_2]_k = u_1 + k / u_2 = (u_1 u_2 + k) / u_2$, which yields the initial conditions

$$p_1 = u_1,\ q_1 = 1;\ \ p_2 = u_1 u_2 + k,\ q_2 = u_2. \tag{7}$$

First, we need some results concerning c.f.'s (base k), which we state as lemmas and prove by induction.

Lemma 1: Let p_n / q_n and p_{n+1} / q_{n+1} denote successive convergents of a c.f. (base k). Let $w_n = p_n q_{n+1} - p_{n+1} q_n$, $n = 1, 2, \ldots.$ Then

$$w_n = (-k)^n. \tag{8}$$

Proof: Let S_1 denote the set of positive integers n satisfying (8). Now $w_1 = u_1 \cdot u_2 - (u_1 u_2 + k) \cdot 1 = -k = (-k)^1$; hence, $1 \in S_1$.

Suppose $n \in S_1$. Then we get $w_{n+1} = p_{n+1} q_{n+2} - p_{n+2} q_{n+1} = p_{n+1}(u_{n+2} q_{n+1} + k q_n) - (u_{n+2} p_{n+1} + k p_n) q_{n+1} = -k(p_n q_{n+1} - p_{n+1} q_n) = -k w_n = -k(-k)^n$ (by the inductive hypothesis), or $w_{n+1} = (-k)^{n+1}$. Thus, $n \in S_1 \Rightarrow (n+1) \in S_1$. The result follows by induction. □

Lemma 2: Let $p_n / q_n = [u_1, u_2, \ldots, u_n]_k$, where the u_i's are integers with $k \mid u_i$, $i = 2, 4, 6, \ldots,$ for $n = 1, 2, \ldots.$ Furthermore, suppose the p_n's and q_n's are the integers naturally produced in the c.f. (base k) expansion, applying the recurrence relation in (6) and the initial conditions in (7). Then, for all even n,

$$\text{g.c.d.}(p_{n-1}, q_{n-1}) = |k|^{\frac{1}{2}n-1}, \tag{9}$$

$$\text{g.c.d.}(p_n, q_n) = |k|^{\frac{1}{2}n}. \tag{10}$$

Proof: Let S_2 denote the set of even positive integers n for which (9) and (10) are valid. Clearly, $\text{g.c.d.}(p_1, q_1) = 1$, since $q_1 = 1$. Note that $1 = |k|^{2/2-1}$. Also, since $k \mid u_2$, it follows that

$k|(u_1u_2+k)$. Thus, $1\cdot(u_1u_2/k+1)-u_1u_2/k=1$, which implies g.c.d.$(p_2/k,q_2/k)=1$; hence, g.c.d.$(p_2,q_2)=|k|=|k|^{2/2}$. We thus see that $2\in S_2$.

Suppose $n\in S_2$ and $p_{n-1}=(-k)^{\frac{1}{2}n-1}p'_{n-1}, q_{n-1}=(-k)^{\frac{1}{2}n-1}q'_{n-1}$, $p_n=(-k)^{\frac{1}{2}n}p'_n$, $q_n=(-k)^{\frac{1}{2}n}q'_n$, where g.c.d.$(p'_{n-1},q'_{n-1})=$ g.c.d.$(p'_n,q'_n)=1$. Then we have $p_{n+1}=u_{n+1}p_n+kp_{n-1}=(-k)^{\frac{1}{2}n}p'_{n+1}$, where $p'_{n+1}=u_{n+1}p'_n-p'_{n-1}$; similarly, $q_{n+1}=(-k)^{\frac{1}{2}n}q'_{n+1}$, where $q'_{n+1}=u_{n+1}q'_n-q'_{n-1}$. Therefore, $p_nq_{n+1}-p_{n+1}q_n=(-k)^n(p'_nq'_{n+1}-p'_{n+1}q'_n)=(-k)^n$ (using Lemma 1), so $p'_nq'_{n+1}-p'_{n+1}q'_n=1$. Then g.c.d.$(p'_{n+1},q'_{n+1})=1$, which implies g.c.d.$(p_{n+1},q_{n+1})=|k|^{\frac{1}{2}n}=|k|^{\frac{1}{2}(n+2)-1}$. This is the statement of (9) for $(n+2)$.

Again supposing $n\in S_2$, let $u_{n+2}=-ku'_{n+2}$ (since $k|u_{n+2}$). Then we get $p_{n+2}=u_{n+2}p_{n+1}+kp_n=(-k)(-k)^{\frac{1}{2}n}u'_{n+2}p'_{n+1}-(-k)(-k)^{\frac{1}{2}n}p'_n=(-k)^{1+\frac{1}{2}n}p'_{n+2}$, where $p'_{n+2}=u'_{n+2}p'_{n+1}-p'_n$; similarly, $q_{n+2}=(-k)^{1+\frac{1}{2}n}q'_{n+2}$, where $q'_{n+2}=u'_{n+2}q'_{n+1}-q'_n$. Then $p_{n+1}q_{n+2}-p_{n+2}q_{n+1}=(-k)^{\frac{1}{2}n}(-k)^{1+\frac{1}{2}n}(p'_{n+1}q'_{n+2}-p'_{n+2}q'_{n+1})=(-k)^{n+1}$ (using Lemma 1), so $p'_{n+1}q'_{n+2}-p'_{n+2}q'_{n+1}=1$. Therefore, g.c.d.$(p'_{n+2},q'_{n+2})=1$, which implies g.c.d.$(p_{n+2},q_{n+2})=|k|^{1+\frac{1}{2}n}=|k|^{\frac{1}{2}(n+2)}$. This is the statement of (10) for $(n+2)$. Thus, $n\in S_2\Rightarrow(n+2)\in S_2$. Since $2\in S_2$, the results follow by induction. □

Lemma 3: If $p_n/q_n=[u_1,u_2,\ldots,u_n]_k$, $n=1,2,\ldots$, then

$$[u_n,u_{n-1},\ldots,u_2]_k=q_n/q_{n-1} \text{ and } [u_n,u_{n-1},\ldots,u_1]_k=p_n/p_{n-1},\ n=2,3,\ldots. \tag{11}$$

Proof: Let S_3 denote the integers $n\ge 2$ for which (11) is valid. Note that $[u_2]_k=u_2/1=q_2/q_1$ and $[u_2,u_1]_k=u_2+k/u_1=(u_1u_2+k)/u_1=p_2/p_1$ [using (7)]. Therefore, $2\in S_2$.

Suppose $n\in S_3$. Then we get $[u_{n+1},u_n,\ldots,u_2]_k=u_{n+1}+k/[u_n,\ldots,u_2]_k=u_{n+1}+k/(q_n/q_{n-1})=(u_{n+1}q_n+kq_{n-1})/q_n=q_{n+1}/q_n$ [using (6)]. Also $[u_{n+1},u_n,\ldots,u_1]_k=u_{n+1}+k/[u_n,\ldots,u_1]_k=u_{n+1}+k/(p_n/p_{n-1})=(u_{n+1}p_n+kp_{n-1})/p_n=p_{n+1}/p_n$. Thus, $n\in S_3\Rightarrow(n+1)\in S_3$. Since $2\in S_3$, the result follows by induction. □

Also, we will make use of the following identity:

$$(a^2+kb^2)(c^2+kd^2)=(ac+kbd)^2+k(ad-bc)^2. \tag{12}$$

Now suppose $p_i/q_i=[a_1,a_2,\ldots,a_i]_k$, $i=1,2,\ldots,n+1$, in the sense described in the hypothesis of Lemma 2. Then $p_n=(-k)^{\frac{1}{2}n}p'_n$, $q_n=(-k)^{\frac{1}{2}n}q'_n$, $p_{n+1}=(-k)^{\frac{1}{2}n}p'_{n+1}$, and $q_{n+1}=(-k)^{\frac{1}{2}n}q'_{n+1}$, where g.c.d.$(p'_n,q'_n)=$ g.c.d.$(p'_{n+1},q'_{n+1})=1$. Moreover, $p'_nq'_{n+1}-p'_{n+1}q'_n=1$. Also, using Lemma 3, $[a_{n+1},\ldots,a_2]_k=q_{n+1}/q_n$ and $[a_{n+1},\ldots,a_1]_k=p_{n+1}/p_n$. The n^{th} and $(n+1)^{\text{st}}$ convergents of the c.f. (base k) given by (3) are p_n/q_n and p_{n+1}/q_{n+1}, respectively; the "remainder" of this c.f. is equal to p_{n+1}/p_n, which assumes the role of u_{n+2}. Thus, the value of the c.f. (base k) in (3) is given by

$$\frac{(p_{n+1}/p_n)p_{n+1}+kp_n}{(p_{n+1}/p_n)q_{n+1}+kq_n}=\frac{p_{n+1}^2+kp_n^2}{p_{n+1}q_{n+1}+kp_nq_n}=N/D,$$

where $N=(p'_{n+1})^2+k(p'_n)^2$ and $D=p'_{n+1}q'_{n+1}+kp'_nq'_n$ [dividing throughout by the common factor $(-k)^n$]. Therefore, $p/h=N/D$. Now set $a=p'_{n+1}$, $b=p'_n$, $c=q'_{n+1}$, and $d=q'_n$ in (12) and let $Q=(q'_{n+1})^2+k(q'_n)^2$. That identity then becomes

$$D^2+k=NQ. \tag{13}$$

Let $g = \text{g.c.d.}(N, D)$. We see from (13) that $g|k$. Since $N = pg$ and $\text{g.c.d.}(p, k) = 1$(by hypothesis), it follows that $g = 1$, so $N = p$ and $D = h$. However, we know that $[a_{n+1}, \ldots, a_1]_k = p_{n+1} / p_n = p'_{n+1} / p'_n$. Setting $x = p'_{n+1}$ and $y = p'_n$ completes the proof of (4) and (5).

Summary: Given the minimal positive solution of the congruence in (2), we have indicated an algorithm for generating solutions of (5). This construction involves a special type of c.f. (base k), as defined by (1). The conditions in (3) might, at first glance, seem unduly restrictive. It may be shown, however, that p/h may always be put into the desired c.f. form in (3), provided that integers x and y exist that satisfy (5). The proof of this assertion is left to the interested reader.

Setting $k = 1$ in the problem yields Serret's construction (1848), one of several known in the literature for finding the *unique* x and y such that $p = x^2 + y^2$, provided p is a prime with $p \equiv 1$ (mod 4). Also, for $k = 1$, the identity in (12) reduces to an identity attributable to Leonardo of Pisa (a.k.a. Fibonacci), such identity appearing in his *Liber Abaci* (1202).

Two examples illustrate the construction's applicability.

Example 1: Let $k = 3$ and $p = 757$. Note that

$$\left(\frac{-3}{757}\right) = \left(\frac{-3 + 4\cdot 757}{757}\right) = \left(\frac{3025}{757}\right) = \left(\frac{55^2}{757}\right) = 1.$$

Hence, the minimal positive solution of the congruence $h^2 \equiv -3 \pmod{757}$ is $h = 55$. Without disclosing the logic of the following expansion, we may at least verify its accuracy:

$$\begin{aligned}
&757/55 = 13 + 42/55 = 13 + 3/\theta_1;\\
&\theta_1 = 55/14 = 3 + 13/14 = 3 + 3/\theta_2;\\
&\theta_2 = 42/13 = 0 + 3/\theta_3;\\
&\theta_3 = 13/14 = 0 + 3/\theta_4;\\
&\theta_4 = 42/13 = 3 + 3/13 = 3 + 3/\theta_5;\\
&\theta_5 = 13.
\end{aligned}$$

Thus, $757/55 = [13, 3, 0, 0, 3, 13]_3$, which is of the desired form, with $n = 2$. Then the solutions of $x^2 + 3y^2 = 757$ are found by $x/y = [0, 3, 13]_3$. We find the successive convergents of this c.f.: $0/1$, $3/3$, and $39/42$. Hence, $x/y = 39/42 = 13/14$, so $x = 13$ and $y = 14$. As we may verify, $13^2 + 3\cdot 14^2 = 757$.

Example 2: Let $k = -2$ and $p = 193$. Since

$$\left(\frac{2}{193}\right) = \left(\frac{2 + 193\cdot 14}{193}\right) = \left(\frac{2704}{193}\right) = \left(\frac{52^2}{193}\right) = 1$$

we see that $h = 52$ is the minimal positive solution of the congruence $h^2 \equiv 2 \pmod{193}$. We may expand $193/52$ as follows:

$$\begin{aligned}
&193/52 = 5 - 67/52 = 5 - 2/\theta_1;\\
&\theta_1 = 104/67 = 2 - 30/67 = 2 - 2/\theta_2;\\
&\theta_2 = 67/15 = 5 - 8/15 = 5 - 2/\theta_3;\\
&\theta_3 = 15/4 = 5 - 5/4 = 5 - 2/\theta_4;\\
&\theta_4 = 8/5 = 2 - 2/5 = 2 - 2/\theta_5;\\
&\theta_5 = 5.
\end{aligned}$$

Thus, $193/52 = [5, 2, 5, 5, 2, 5]_{-2}$, which is of the desired form, with $n = 2$. Therefore, solutions of $x^2 - 2y^2 = 193$ are found from $x/y = [5, 2, 5]_{-2}$. This yields the convergents: $5/1$, $8/2$, and $30/8$, so $x = 15$ and $y = 4$. Q.E.D.

Searching for Pairs

H-511 *Proposed by M. N. Deshpande, Aurangabad, India*
(Vol. 34, no. 2, May 1996)

Find all possible pairs of positive integers m and n such that $m(m+1) = n(m+n)$. [Two such pairs are: $m = 1, n = 1$; $m = 9, n = 6$.]

Solution by H.-J. Seiffert, Berlin, Germany

The pairs $(m, n) \in N^2$ asked for are $(m, n) = (F_{2k}^2, F_{2k-1}F_{2k})$, where k is a positive integer. It is easily verified that, for these pairs, the considered equation is indeed satisfied.

Below we will use the well-known result that all solutions $(a, b) \in N^2$ of the Pell equation $a^2 - 5b^2 = -4$ are given by $(a, b) = (L_{2k-1}, F_{2k-1})$, $k \in N$. In particular, we have $a \geq b$.

Let $(m, n) \in N^2$ such that $m(m+1) = n(m+n)$. Write $m = rp$ and $n = rq$, where $p, q, r \in N$ such that $\gcd(p, q) = 1$. Then the given equation becomes $p(rp+1) = rq(p+q)$, which shows that r divides p. Letting $p = rs$, $s \in N$, we get $s(r^2 s + 1) = q(rs + q)$. From $p = rs$, $\gcd(p, q) = 1$, and $s|q^2$, it follows that $s = 1$. Now, the resulting equation $r^2 + 1 = q(r+q)$ may be written as $(2r - q)^2 - 5q^2 = -4$. Hence, $(2r - q, q) = (L_{2k-1}, F_{2k-1})$ for some $k \in N$. It readily follows that $r = F_{2k}$, so that we have $(m, n) = (F_{2k}^2, F_{2k-1}F_{2k})$.

Also solved by P. Bruckman, L. A. G. Dresel, A. Dujella, C. Georghiou, and the proposer.

FPP's

H-512 *Proposed by Paul S. Bruckman, Highwood, IL*
(Vol. 34, no. 2, May 1996)

The *Fibonacci pseudoprimes* (or FPP's) are those composite n with g.c.d.$(n, 10) = 1$ such that $n | F_{n-\varepsilon_n}$, where ε_n is the Jacobi symbol $(\frac{5}{n})$. Suppose $n = p(p+2)$, where p and $p+2$ are "twin primes." Prove that n is a FPP if and only if $p \equiv 7 \pmod{10}$.

Solution by Lawrence Somer, Catholic University of America, Washington DC

We first suppose that $p \equiv 7 \pmod{10}$. Then $p+2 \equiv 9 \pmod{10}$. By quadratic reciprocity, we see that $(\frac{5}{p}) = -1$ and $(\frac{5}{p+2}) = 1$. Hence, $(\frac{5}{p(p+2)}) = (\frac{5}{p})(\frac{5}{p+2}) = (-1)(1) = -1$. We want to show that $p(p+2) | F_{p(p+2)+1}$. It is well known that $F_n | F_{kn}$ for any positive integer k. Since both p and $p+2$ are primes, $p | F_{p-\varepsilon_p} = F_{p+1}$ and $p+2 | F_{p+2-\varepsilon_{p+2}} = F_{p+1}$. Further, since $p(p+2)+1 = (p+1)^2$, $F_{p+1} | F_{(p+1)^2}$, and g.c.d.$(p, p+2) = 1$, we see that $p(p+2) | F_{p(p+2)+1}$.

Now suppose that $n = p(p+2)$ is a FPP. We must have $p \equiv 1, 3, 5, 7$, or $9 \pmod{10}$. If $p \equiv 5 \pmod{10}$, then g.c.d.$(n, 10) \neq 1$. If $p \equiv 3 \pmod{10}$, then $p+2 \equiv 5 \pmod{10}$ and, again, g.c.d.$(n, 10) \neq 1$. Suppose $p \equiv 1 \pmod{10}$. Then $p+2 \equiv 3 \pmod{10}$. By quadratic reciprocity, $(\frac{5}{p}) = 1$ and $(\frac{5}{p+2}) = -1$. Hence, $\varepsilon_n = (\frac{5}{p(p+2)}) = (\frac{5}{p})(\frac{5}{p+2}) = (1)(-1) = -1$, so $n - \varepsilon_n = p(p+2)+1 = p^2 + 2p + 1$. Thus, $p(p+2) | F_{p^2+2p+1}$. It is well known that $(F_a, F_b) = F_{(a,b)}$, where (a, b) denotes the g.c.d. of a and b. We note that $p | F_{p-\varepsilon_p} = F_{p-1}$. Now, $p^2 + 2p - 3 = (p-1)(p+3)$. Hence,

$p|F_{p^2+2p-3}$. Therefore, $p|(F_{p^2+2p+1}, F_{p^2+2p-3})$, which implies that $p|F_{(p^2+2p+1,\, p^2+2p-3)}$. However, $(p^2+2p+1, p^2+2p-3)|(p^2+2p+1)-(p^2+2p-3)=4$, so $p|F_4=3$. Thus, $p=3$, which is a contradiction since $p\equiv 1 \pmod{10}$. Thus, $p\not\equiv 1 \pmod{10}$. Now suppose that $p\equiv 9 \pmod{10}$. Then $p+2\equiv 1 \pmod{10}$. By quadratic reciprocity, $(\frac{5}{p})=(\frac{5}{p+2})=1$. Therefore, $\varepsilon_n=(\frac{5}{p(p+2)})=(\frac{5}{p})(\frac{5}{p+2})=(1)(1)=1$, so $n-\varepsilon_n=p(p+2)-1=p^2+2p-1$. Now, $p|F_{p-\varepsilon_p}=F_{p-1}$. Thus, as in our above argument, $p|F_{p^2+2p-3}$. Hence, $p|(F_{p^2+2p-1}, F_{p^2+2p-3})=F_{(p^2+2p-1,\, p^2+2p-3)}$. However, $(p^2+2p-1, p^2+2p-3)|(p^2+2p-1)-(p^2+2p-3)=2$. Thus, $p|F_2=1$, which is a contradiction. Therefore, $p\not\equiv 9 \pmod{10}$; hence, $p\equiv 7 \pmod{10}$.

Also solved by L. A. G. Dresel, A. Dujella, H.-J. Seiffert, D. Terr, and the proposer.

Sum Product

H-513 ***Proposed by Paul S. Bruckman, Highwood, IL (Vol. 34, no. 4, August 1996)***

Define the following quantities:

$$A=\sum_{n\geq 0}\frac{1}{(n!)^2},\quad B=\sum_{n\geq 0}\frac{1}{n!(n+1)!},\quad C=\sum_{n\geq 0}\frac{(2n)!}{(n!)^4},\quad D=\sum_{n\geq 0}\frac{(2n+2)!}{n!((n+1)!)^2(n+2)!}.$$

Prove that $A^2D=B^2C$.

Solution by the proposer

Clearly, the series defining A and B are convergent. Using Stirling's formula, $\binom{2n}{n}\sim 4^n(n\pi)^{-1/2}$ as $n\to\infty$. Thus, the convergence of the series defining C is comparable to that of the series

$$\sum_{n\geq 1}\frac{4^n}{n^{1/2}(n!)^2};$$

since the latter series is clearly convergent, so is the series defining C. Also, D is defined by a series that is comparable to the series

$$\sum_{n\geq 1}\frac{4}{n^2}\cdot\frac{(2n)!}{(n!)^4},$$

and so the series defining D is convergent. Clearly, all quantities are positive quantities.

We recognize the Modified Bessel Functions of integer order, defined as follows:

$$I_n(z)=\left(\tfrac{1}{2}z\right)^n\sum_{k\geq 0}\frac{\left(\frac{1}{4}z^2\right)^k}{k!(n+k)!},\text{ an entire function of } z,\ n=0,1,2,\ldots. \tag{1}$$

See, e.g., *Handbook of Mathematical Functions*, ed. M. Abramowitz & I. A. Stegun (9th prtg., §9. Washington, D.C.: National Bureau of Standards, 1970). We then see that $A=I_0\equiv I_0(2)$ and $B=I_1\equiv I_1(2)$. It is also indicated in this source that the following relation holds:

$$I_m(z)I_n(z)=\left(\tfrac{1}{2}z\right)^{m+n}\sum_{k\geq 0}\frac{(2k+m+n)!\left(\frac{1}{4}z^2\right)^k}{(k+m)!(k+n)!k!},\quad m,n=0,1,2,\ldots. \tag{2}$$

It follows from (2) that $C=(I_0)^2$ and $D=(I_1)^2$. Then $A^2D=B^2C=(I_0I_1)^2$.

Also Solved by C. Georghiou.

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BOOKS AVAILABLE
THROUGH THE FIBONACCI ASSOCIATION

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